Quantum Mechanics from Classical Logic

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Abstract. Although quantum mechanics is generally considered to be fundamentally incompatible with classical logic, it is argued here that the gap is not as great as it seems. Any classical, discrete, time reversible system can be naturally described using a quantum Hilbert space, operators, and a Schrödinger equation. The quantum states generated this way resemble the ones in the real world so much that one wonders why this could not be used to interpret all of quantum mechanics this way. Indeed, such an interpretation leads to the most natural explanation as to why a wave function appears to “collapse” when a measurement is made, and why probabilities obey the Born rule. Because it is real quantum mechanics that we generate, Bell’s inequalities should not be an obstacle.

1. Introduction

Quantum Mechanics is usually presented as being a totally new paradigm, complete with its own rules of logic. This, indeed, is how quantum mechanics works in practice. The rules of quantum logic, for instance, categorically disallow certain questions from being asked; in particular, one should not even try to ask “what is really going on?” when calculations are made concerning some quantum phenomenon. If a photon is sent through two tiny slits, and an interference pattern arises, one should never ask through which slit the photon is going, or whether the photon follows any particular path at all – or even what a photon “actually” is.

However, very occasionally, quantum mechanics is encountered in altogether different settings. Not always do we have to modify our “classical” sense of logic. Quantum mechanics can also be used as a tool to compute something. A question is then asked about a mathematical problem that itself has nothing to do with quantum mechanics. Nor does the answer to the question refer to quantum mechanics as a special doctrine. Only it so happens that the path from question to answer requires an intermediate stage where quantum technology, including its logical interpretation, is indispensable. My most cherished example is the Two-Dimensional Ising Problem. It is the following:

In statistical physics, one may consider atoms forming a rectangular grid, and each atom may have a spin pointing either up or down. Questions of the following form may then arise, although this particular question is a very special one:

Consider a two-dimensional rectangular grid with $N$ rows and $M$ columns, $N$ and $M$
both being very large. For simplicity one may consider periodic boundary conditions. Fill the entire lattice with 1’s and 0’s. The total number of 1’s and 0’s are equal: $NM/2$. The total number of boundaries is $2NM$. Of these, a fraction $x$ is such that a 0 is adjacent to a 1. The remaining boundaries, the fraction being $1-x$, are either between two 1’s or between two 0’s. Question: in how many different ways can we position the 1’s and the 0’s?

The one-dimensional version of this question is easy to solve for an undergraduate student, but the two-dimensional variant is quite hard. The solution was found by L. Onsager, and written down in two beautiful papers with his student, B. Kaufman[1]. In statistical physics, the problem can easily be rephrased in terms of a matrix (the transfer matrix) that describes the interactions between two adjacent rows. What has to be calculated is the trace of this matrix raised to the $N^{th}$ power. For large $N$, only the lowest eigenvalue of the transfer matrix contributes. The transfer matrix can be rewritten as $e^{-\beta H}$, where $\beta$ is related to the given fraction $x$, and $H$ in every respect resembles a quantum mechanical hamiltonian.

What Onsager and Kaufman discovered is that this hamiltonian is the one of a quantum field theory describing fermions. These fermions indeed are anticommuting and have spin $\frac{1}{2}$. The zero eigenvalue belongs to the ground state of this system. So, they had to calculate the energy of the vacuum. It so happens that, in this special case, the fermions are free fermions – their interactions vanish. So, the energy of the vacuum could be computed exactly. Most notably, vacuum expectation values of spins exactly correspond to the ‘average’ values in the statistical system, and the correlation functions in the statistical system are exactly the connected Feynman diagrams of the quantum field theory.

This is what we now wish to emphasize: quantum mechanics was just being used as a tool, a smart tool indeed, but never in the argument was there any need to replace the logic of statistical physics into anything unconventional. There are no questions that are disallowed. Use was made of some good mathematics, and that’s all there is to it.

In the world of atoms and sub-nuclear particles, we also have quantum mechanics, and their properties are contained in quantum field theories. Could it not be that these quantum field theories are also nothing but the answers to questions asked, about a system that itself is not quantum mechanical at all? We can confirm that the answer to the question indeed works correctly: it is what we call quantum mechanics. But now, what is the question? More precisely, what is the statistical system and what are its laws of nature, that also require the quantum mechanical method to arrive at an answer?

The Ising model is not quantum mechanical. The question could just as well have been asked about checker boards or beer crates. Similarly, when we consider dynamical laws, we can also use quantum notation to describe big, non quantum mechanical things. Take, for instance, the planets orbiting the sun, the prototype of a classical, deterministic system. It does not forbid the introduction of operators that look quantum mechanical. An example is the Earth-Mars exchange operator. It puts Earth where Mars was, and sticks Mars to Earth’s previous position. The velocities of the two planets are exchanged accordingly. One can now ask the mathematically relevant question: how will this system continue to evolve? Will the planets, through their mutual interactions, settle for new stable orbits?

In the jargon of quantum mechanics, we can confirm that the position operators for Earth and Mars do not commute with this displacement operator. It is legitimate to ask how this operator will evolve with time. Its eigenvalues are easy to calculate: they are $\pm 1$, but the time evolution of this operator is complicated.

There do seem to be features that quantum mechanics has, but planetary systems have not.
In planetary systems, operators such as the Earth-Mars exchange operators seem to be mostly useless. In conventional quantum systems, however, we frequently see how observables such as positions of things, evolve and mix with typical exchange operators. For instance, magnetic fields can rotate the $z$-spin observable $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ into the $z$-spin exchange operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The reasons for this could be a very simple one:

In our quantum mechanical theories for the atomic scale, we have not yet identified those observables that truly correspond to the actual states of a system. All operators we use at the atomic scale are superpositions of primary observables ('beables') and typical permutation operators ('changeables').

There is reason to suspect that, in the atomic and sub-atomic world, we are confusing the true 'beables' with the 'changeables' (such as the Earth-Mars exchange operator). It so happens that both of these evolve by the same rules: the linear Schrödinger equation.

2. Examples of quantum systems allowing for classical interpretations

One of the simplest examples of a quantum system that allows for a deterministic interpretation is the Zeeman atom, a spin 1 atom in a homogeneous magnetic field. The eigenvalues for the hamiltonian are $E = (\mu B, 0, -\mu B)$. Consider the time step $t_0 = \frac{2\pi}{3\mu B}$; the evolution operator for this time step, $U(t_0)$, obeys:

$$U(t_0) = e^{-it_0H}, \quad U(3t_0) = U(t_0)^3 = 1,$$

and in another basis (the discrete Fourier transform), this operator can be written as

$$U(t_0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

since it has the same eigenvalues. In this basis, $U$ is simply a permutation operator, and it describes a completely deterministic system that permutes three states in every time step of length $t_0$.

In the limit where we have infinitely many energy eigenstates, all equally spaced: $E_n = (n+\frac{1}{2})\hbar\omega, \ n = 0, 1, \cdots$, we find that the time unit $t_0$ becomes infinitesimal. This is the quantum harmonic oscillator. The deterministic permutation operator now becomes an infinitesimal rotation over a circle. Thus, the quantum harmonic oscillator is seen to be mathematically mapped onto a deterministic particle running around in a circle, with a period $T$ that equals the period of the oscillator.

A third example is that of massless, chiral, second-quantized, non-interacting neutrinos. Performing a similar analysis, identifying the observables that commute at all times, one finds that these “neutrinos” can be regarded as infinite, totally flat membranes moving along a vector orthogonal to the membranes, with the speed of light. The two directions in which a membrane can travel correspond to the two values of the spin.

The most important example however is that of the cellular automaton. It is the first example where the dynamics is non-trivial. It is discussed in the next section.

1 but they do qualify as observable operators just as in quantum mechanics!
3. The time-reversible cellular automaton

Consider a cellular automaton[3][4]. Space and time[5] are both discrete here: we have a $D$ dimensional space, where positions are indicated by integers: $\vec{x} = (x^1, x^2, \cdots, x^D)$, with $x^i \in \mathbb{Z}$. Also time $t$ will be indicated by integers, and time evolution takes place stepwise. The physical variables $F(\vec{x}, t)$ in the model could be assumed to take a variety of forms, but the most convenient choice is to take these to be integers modulo some integer $N$. We now write down an explicit model[3], where these physical degrees of freedom are defined to be attached only to the even lattice sites:

$$\sum_{i=1}^{D} x^i + t = \text{even}. \tag{3.3}$$

Furthermore, the data can be chosen freely at two consecutive times, so for instance at $t = 0$, we can choose the initial data to be $\{F(\vec{x}, t = 0), F(\vec{x}, t = 1)\}$.

The dynamical equations of the model can be chosen in several ways, provided that they are time reversible. To be explicit, we choose them to be as follows:

$$F(\vec{x}, t + 1) = F(\vec{x}, t - 1) + Q \left( F(x^1 \pm 1, x^2, \cdots, x^D, t), \cdots, F(x^1, \cdots, x^D \pm 1, t) \right) \text{ Mod } N, \tag{3.4}$$

where the integer $Q$ is some arbitrary given function of all variables indicated: all nearest neighbors of the site $\vec{x}$ at time $t$. This is time reversible because we can find $F(\vec{x}, t - 1)$ back from $F(\vec{x}, t + 1)$ and the neighbors at time $t$. Assuming $Q$ to be a sufficiently irregular function, one generally obtains quite non-trivial cellular automata this way. Indeed, this category of models have been shown to contain examples that are computationally universal [6]. Models of this sort are often considered in computer animations.

We now discuss the mathematics of this model using Hilbert space notation. We switch from the Heisenberg picture, where states are fixed, but operators such as the beables $F(\vec{x}, t)$ are time dependent, to the Schrödinger picture. Here, we call the operators $F$ on the even sites $X(\vec{x})$, and the ones on the odd sites $Y(\vec{x})$. As a function of time $t$, we alternatingly update $X(\vec{x})$ and $Y(\vec{x})$, so that we construct the evolution operator over two time steps. Keeping the time parameter $t$ even:

$$U(t, t - 2) = A \cdot B, \tag{3.5}$$

where $A$ updates the data $X(\vec{x})$ and $B$ updates the data $Y(\vec{x})$.

Updating the even sites only, is an operation that consists of many parts, each defined on an even space coordinate $\vec{x}$, and all commuting with one another:

$$A = \prod_{\vec{x} \text{ even}} A(\vec{x}), \quad [A(\vec{x}), A(\vec{x}')] = 0, \tag{3.6}$$

whereas the $B$ operator refers only to the odd sites,

$$B = \prod_{\vec{x} \text{ odd}} B(\vec{x}), \quad [B(\vec{x}), B(\vec{x}')] = 0. \tag{3.7}$$

Note however, that the operators $A(\vec{x})$ and $B(\vec{x}')$ do not all commute. If $\vec{x}$ and $\vec{x}'$ are neighbors, then

$$\vec{x} - \vec{x}' = \vec{e}, \quad |\vec{e}| = 1 \quad \rightarrow \quad [A(\vec{x}), B(\vec{x}')] \neq 0. \tag{3.8}$$
It is important to observe here that both the operators \( A(\vec{x}) \) and \( B(\vec{x}) \) only act in finite subspaces of Hilbert space, and they are all unitary, so we can easily write them as follows:

\[
A(\vec{x}) = e^{-i a(\vec{x})}, \quad B(\vec{x}) = e^{-i b(\vec{x})}.
\] (3.9)

In general, \( a(\vec{x}) \) and \( b(\vec{x}) \) are hermitean. We can write

\[
a(\vec{x}) = P_x(\vec{x}) Q(\{Y\}), \quad b(\vec{x}) = P_y(\vec{x}) Q(\{X\}),
\] (3.10)

where \( P_x(\vec{x}) \) is the generator for a one-step displacement of \( X(\vec{x}) : \)

\[
e^{i P_x(\vec{x})} |X(\vec{x})\rangle \stackrel{\text{def}}{=} |X(\vec{x}) - 1 \text{ Mod } N\rangle,
\] (3.11)

and, similarly, \( P_y(\vec{x}) \) generates one step displacement of the function \( Y(\vec{x}) \).

As an example, we give the matrix \( P \) for the case \( N = 5 \). They are easily obtained by discrete Fourier transformation. Defining the numerical coefficients \( \alpha = 2 \sin(\pi/5) + \sin(2\pi/5) \) and \( \beta = 2 \sin(2\pi/5) - \sin(\pi/5) \), we have

\[
P = \frac{4\pi i}{25} \begin{pmatrix}
0 & -\alpha & \beta & -\beta & \alpha \\
-\beta & \alpha & 0 & -\alpha & \beta \\
\beta & -\beta & \alpha & 0 & -\alpha \\
-\alpha & \beta & -\beta & \alpha & 0
\end{pmatrix}; \quad e^{iP} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (3.12)

We see that

\[
[a(\vec{x}), a(\vec{x}')] = 0, \quad [b(\vec{x}), b(\vec{x}')] = 0, \quad \forall (\vec{x}, \vec{x}');
\] (3.13)

\[
[a(\vec{x}), b(\vec{x}')] = 0 \quad \text{only if} \quad |\vec{x} - \vec{x}'| > 1.
\] (3.14)

A consequence of Eqs. (3.13) is that also the products \( A \) in Eq. (3.6) and \( B \) in Eq. (3.7) can be written as

\[
A = e^{-i \sum_{\vec{x} \text{ even}} a(\vec{x})}, \quad B = e^{-i \sum_{\vec{x} \text{ odd}} b(\vec{x})}.
\] (3.15)

However, now \( A \) and \( B \) do not commute. Nevertheless, we wish to compute the total evolution operator \( U \) for two consecutive time steps, writing it as

\[
U = A \cdot B = e^{-ia} e^{-ib} = e^{-2iH}.
\] (3.16)

For this calculation, we could use the power expansion given by the Baker-Campbell-Hausdorff formula[7],

\[
e^P e^Q = e^R,
\]

\[
R = P + Q + \frac{1}{2}[P,Q] + \frac{1}{12}[P,[P,Q]] + \frac{1}{12}[P,Q,[P,Q]] + \frac{1}{72}[P,[P,[P,Q]]], Q] + \cdots.
\] (3.17)

a series that continues exclusively with commutators[7]. Replacing \( P \) by \(-ia\), \( Q \) by \(-ib\) and \( R \) by \(-2iH\), we find a series for the ‘hamiltonian’ \( H \) in the form of an infinite sequence of
commutators. Now note that the commutators of the local operators $a(\vec{x})$ and $b(\vec{x}')$ are non-vanishing only if $\vec{x}$ and $\vec{x}'$ are neighbors, $|\vec{x} - \vec{x}'| = 1$. Consequently, if we insert the sums (3.15) into Eq. (3.17), we obtain again a sum:

$$H = \sum_{\vec{x}} \mathcal{H}(\vec{x}) ,$$

$$\mathcal{H}(\vec{x}) = \frac{1}{2} a(\vec{x}) + \frac{1}{2} b(\vec{x}) + \mathcal{H}_2(\vec{x}) + \mathcal{H}_3(\vec{x}) + \cdots ,$$

(3.18)

where

$$\mathcal{H}_2(\vec{x}) = -\frac{1}{4} i \sum_{\vec{y}} [a(\vec{x}), b(\vec{y})] ,$$

$$\mathcal{H}_3(\vec{x}) = -\frac{1}{24} \sum_{\vec{y}_1, \vec{y}_2} [a(\vec{x}) - b(\vec{x}) , [a(\vec{y}_1), b(\vec{y}_2)]] , \text{ etc.}$$

(3.19)

All these commutators are only non-vanishing if the coordinates $\vec{y}$, $\vec{y}_1$, $\vec{y}_2$, etc., are all neighbors of the coordinate $\vec{x}$. It is true that, in the higher order terms, next-to-nearest neighbors may enter, but still, one may observe that these operators are all local functions of ‘field operators’ $\Phi(\vec{x}, t)$, and thus we arrive at a hamiltonian $H$ that can be regarded as the sum over $D$-dimensional space of a Hamilton density $\mathcal{H}(\vec{x})$, which has the property that

$$[\mathcal{H}(\vec{x}), \mathcal{H}(\vec{x}')] = 0 , \text{ if } |\vec{x} - \vec{x}'| \gg 1 .$$

(3.20)

The $\gg$ symbol here means that at the $n^{th}$ order in the BCH series, $\vec{x}$ and $\vec{x}'$ must be further than $n$ steps away from one another.

At every finite order of the series, the Hamilton density $\mathcal{H}(\vec{x})$ is a finite-dimensional Hermitean matrix, and therefore, it will have a lowest eigenvalue $\hbar$. In a large but finite volume $V$, the total hamiltonian $H$ will therefore also have a lowest eigenvalue, obeying

$$E_0 > \hbar V .$$

(3.21)

The associated eigenstate $|0\rangle$ might be identified with the ‘vacuum’. This vacuum is stationary, even if the automaton itself may have no stationary solution. The next-to-lowest eigenstate may be a one-particle state. In a Heisenberg picture, the fields $F(\vec{x}, t)$ may create a one-particle state out of the vacuum. Thus, we arrive at something that resembles a genuine quantum field theory. The states are quantum states in complete accordance with a Copenhagen interpretation. The fields $a(\vec{x}, t)$ and $b(\vec{x}, t)$ should obey the Wightman axioms.

4. The real world

There are three ways, however, in which this theory differs from conventional quantum field theories. One is, of course, that space and time are discrete. Well, maybe there is an interesting ‘continuum limit’, in which the particle mass(es) is(are) considerably smaller than the inverse of the time quantum.

Actually, the existence of large scale difference between the region where our model is defined (the Planck scale) and the scale up to where its quantum behavior continues to dominate (the atomic scale), is one of Nature’s greater mysteries, called the Hierarchy problem. This mystery is still completely unresolved. Be this as it may, Nature’s degrees of freedom at the atomic scale relate to those of the Planck scale by a complicated renormalization group relation: the running
parameters of the Standard Model undergo a complete metamorphosis, which may well explain why the beables and the changeables of the theory get thoroughly mixed at the atomic scale.

Secondly, no attempt has been made to arrive at Lorentz invariance, or even Galilei invariance. Thus, the dispersion relations for these particles, if they obey any at all, may be nothing resembling conventional physical particles. Do note, however, that no physical information can travel faster than velocity one in lattice units. This is an important constraint that the model still has in common with special relativity.

But the third difference is more profound. It was tacitly assumed that the Baker-Campbell-Hausdorff formula converges. This is often not the case. In Ref [3], it is argued that the series will converge well only if sandwiched between two eigenstates $|E_1\rangle$ and $|E_2\rangle$ of $H$, where $E_1$ and $E_2$ are the eigenvalues, that obey

$$2|E_1 - E_2| < 2\pi\hbar/\Delta t,$$

where $\Delta t$ is the time unit of our clock, and the first factor 2 is the one in Eq. (3.16). (“Planck’s constant”, $\hbar$, has been inserted merely to give time and energy the usual physical dimensions.)

This may seem to be a severe restriction, but, first, one can argue that $2\pi\hbar/\Delta t$ here is the Planck energy, and in practice, when we do quantum mechanics, we only look at energies, or rather energy differences, that indeed are much smaller than the Planck energy.

This argument will not convince skeptics. We have not proven that the cellular automaton can generate theories closely resembling the Standard Model. But this is not the aim of this investigation. What we wish to demonstrate is that the mathematical machinery borrowed from real quantum mechanics applies to deterministic systems such as a cellular automaton. Its states evolve in accordance with a Schrödinger equation. The hamiltonian shows much resemblance with that of genuine quantum systems. If we try to diagonalize the states of the hamiltonian that we find, the original ontological states are rearranged in terms of eigenstates of a hamiltonian, and as such they may well exhibit delicate forms of quantum entanglement.

The mathematical description of the cellular automaton that we arrive at, will be practically indistinguishable from other quantum mechanical systems. However, its universe can be described in terms of a very specially chosen basis of Hilbert space, the ‘ontological basis’. In terms of this basis, the wave function has the following properties:

- it is 1 for the ‘true’ state, and 0 elsewhere;
- it does not spread;
- it does not collapse, that is, in the ontological basis it is as collapsed as it ever can be;
- it describes only one world;
- it always obeys a linear Schrödinger equation.

We claim that our world could be of the same sort; the ‘wave function of the universe’ could be exactly as trivial as this one, but only if we manage to identify the ontological basis.

Actually, we suspect that the macroscopic states, such as the states that describe dead or live cats, can be distinguished by studying the statistical distributions of the data in the deterministic underlying automaton. This would mean that the wave function of the universe is always automatically collapsed when used to describe Schrödinger’s cat. We return to this in Sections 6–8.
5. The Bell inequalities

One of the most difficult issues to understand in this context is the Bell inequalities[8][9]. These inequalities refer to gedanken experiments in which quantum mechanical objects are produced in some quantum entangled states, such as two photons in a state with total spin zero, and many possible variations of this theme[10].

As has been shown with abundant evidence, such states can indeed be produced in real experiments. Observers, separated by macroscopic distances away from one another, can choose which component(s) of the wave function to detect, and they can use “free will” to determine their choices. Bell’s inequalities appear to imply that the correlations then found cannot possibly be reconciled with a deterministic hidden variable theory. In the hidden variable theories that one then has in mind, the quantum particles are, somehow, accompanied by classical hidden variables that decide what the outcome of any of the possible measurements will be, even if the measurement is not made.

Clearly, Bell has shown that such hidden variable theories are unrealistic. We must conclude that our cellular automaton (CA) theory cannot be of this particular type. Yet, we had a classical system, and we claim that it reproduces quantum mechanics, with probabilities generated by the squared norm of wave functions. Quantum states, in particular entangled quantum states, are perfectly legitimate to describe statistical distributions. But how exactly to understand why Bell’s inequalities can be violated in spite of the fact that we do start off from a classical deterministic, discrete theory, (to be referred to as a CA), requires a more detailed explanation than the one that can be given here. At the risk of not being understood (see also Ref [11]), let me briefly summarize what we think the situation is.

Even though we work with wave functions that are quite complicated quantum superpositions of the “Standard Model” (SM) eigenstates, we also emphasized that quantum superpositions of “ontological states” themselves are not ontological. Thus, if in a Bell experiment one axis was chosen, say, for the measurement of a spin, any other axis where the spin variable would not commute with the previous one, is in principle forbidden. One would be tempted to conclude that Bell’s inequalities should be obeyed, so an important question is raised: how can it be that experiments showing violations could be performed so easily, not only in our imagination but also in real experiments[12]?

It must be emphasized that all commonly employed wave functions associated to atoms, fields, etc., do not correspond to the ontological states of the CA. So, if one “considers” a decaying atom with total spin zero (such that two entangled photons emerge), this only describes the CA in a statistical sense. Many of the CA states qualify to describe the initial condition.

As will be explained in the next section, many of the atoms in the environment will therefore be entangled with the system that is being studied. We emphasize that this includes vacuum fluctuations, which are rarely included in the standard treatments of the Bell inequalities. These vacuum fluctuations generate spacelike correlations, which in principle could be held responsible for the apparent non-locality of systems that violate Bell.

The next notion that will have to be scrutinized is the concept of “free will”[13][14]. Clearly, an observer who uses “free will” to choose the direction of a spin to be measured (or any other, more general basis for a quantum measurement), actually makes his or her decision depending of the outcome of Nature’s laws in his/her own system. This is inevitable in deterministic theories, so we do not have to worry about “free will” itself[15].

The most difficult scenario is the following. Suppose that the observer’s decision is made to depend not on “free will” but on fluctuations of the light from a distant quasar. If Alice and Bob participate in the experiment, they both use quasars that are located diametrically
opposite to one another. These quasars would both be spacelike separated from the device that produced an entangled particle state. The initial state of the entire system does not allow for quantum superpositions of these ontological states, but we were led to the apparently inevitable conclusion that the atomic objects in the quantum measurement in question must be entangled with both quasars in a very delicate sense, even if all of them are spacelike separated.

The best way then to summarize this explanation is to emphasize that cellular automata can be described by having their elementary states form basis vectors in a linear Hilbert space. The evolution equation in this space is the linear Schrödinger equation. Then, the ‘SM variables’ (atoms, fields, ...) form a totally different basis in this Hilbert space, all of these elements being highly entangled. Included in this entanglement are the vacuum fluctuations, and it may generate entangled correlations over vast spacelike distances. This is why the initial state will be an entangled one, and all states at later times, such as the ones used in an EPR experiment[8], continue to produce results that exhibit quantum entanglement.

6. Collapsing quantum amplitudes

In this meeting, it was repeatedly claimed that the collapse of the wave function cannot be reconciled with Schrödinger’s equation[16]–[20], and therefore has to be introduced as a separate axiom, while Schrödinger’s equation is modified, for instance with non-linear “corrections”, just in order to comply with this ‘bizarre’ axiom.

An important argument that can be brought up in favor of this position is the following: suppose that a system starting off in a quantum state $|A⟩_0$ would, after some time $t$, lead to a collapsed state $|A⟩_t$, while a system starting off as $|B⟩_0$ would end up in the collapsed state $|B⟩_t$. Then, where would the state $λ|A⟩_0 + µ|B⟩_0$ end up? If this isn’t the state $λ|A⟩_t + µ|B⟩_t$, would this not imply a violation of Schrödinger’s equation?

Not only do we never explicitly observe that Schrödinger’s equation is violated anywhere, but the claim is also at odds with models that this author brought up to explain quantum mechanics as being the realization of statistical features of an underlying deterministic theory (see also [21]–[27]).

Similar statements are encountered concerning the Born interpretation of the wave function as being a description of probabilities. The probability of finding a system described by a wave function $ψ$ to be in a certain state $|x⟩$ when a measurement is made, is exactly equal to the square of the norm of the inner product $⟨x|ψ⟩$. This also appears to be a special, separate axiom. What, after all, do probabilities have to do with equations such as the Schrödinger equation?

It is important to clarify these issues. Once and for all? Some issues will not be completely settled with the arguments presented here, so that the discussions will doubtlessly continue. In particular, Bell’s inequalities will continue to raise questions, but this author is convinced of the basic correctness of the presentation given below. As for the apparent breakdown of the superposition principle as mentioned above, the answer is simple: in a deterministic theory, a system can be in a state $A$ or in a state $B$, but never in a superposition. More precisely, the sub-microscopic degrees of freedom that we suspect to be deterministic, may also serve to fix the macroscopic variables describing outcomes of measurements, in a classical statistical sense, which could be just a perfect explanation of why an apparent collapse takes place.

The degrees of freedom in terms of which we usually describe atoms, molecules, subatomic particles and their fields will be referred to as microscopical degrees of freedom. It is these that have to be described as superpositions of the sub-microscopic states, and in turn, the macroscopic states are superpositions of microscopically defined states. Perhaps the most accurate way to
describe the situation is that the states we use to describe atoms, quantum fields, etc., is to say that these serve as templates. A particle in the state \(|x\rangle\) or in the state \(|p\rangle\) or whatever, will nearly always be a superposition of many of the sub-microscopic states; as such, they evolve exactly according to Schrödinger equations. In contrast, the sub-microscopic states evolve classically. The macroscopic states also evolve classically, but the details of their evolution laws are far too complicated to follow, which is what we need the microscopic template states for.

7. Wave functions

If a theory is deterministic, we can also choose the initial condition deterministically, that is, there is exactly one state that is realized at \(t = 0\), and it describes “the universe”. Being deterministic, such a theory describes a single world at all times - there is never any interference, in terms of the \(\vec{q}\) variables. The wave function at \(t = 0\) is \(|\psi, 0\rangle\) and, in the continuum case, it could be written as

\[
\langle \vec{q}|\psi, 0 \rangle = N \delta (\vec{q} - \vec{q}_0), \\
\langle \vec{q}|\psi, t \rangle = N \delta (\vec{q} - \vec{q}_t),
\]

(7.23)

where \(N\) is a normalization constant, while in the discrete case,

\[
\langle \vec{q}|\psi, t \rangle = \delta_{\vec{q}, \vec{q}_i}.
\]

(7.24)

Of course, such “wave functions” do not spread. At all times \(t\), only one value of the ‘ontological’ variable \(\vec{q}\) is realized. This wave function always takes the form of Eqs. (7.23) or (7.24). Note however that, even if it does neither spread nor collapse, it fully obeys the Schrödinger equation based on the hamiltonian constructed as in Eq. (3.18).

Now look at our world. At first sight, the wave functions we use to describe it look very different. But then we have to realize that we do not know the operators \(\vec{q}(t)\). The operators that we do use, for instance in describing the Standard Model, actually refer to states close to the lowest eigenstates of \(H\), so that they contain low-energy projection operators. Therefore, when expressed in terms of Standard Model (SM) operators, the observables \(\vec{q}(t)\) always form non-trivial superpositions of eigenstates of SM operators, probably compounded by operators of as yet unknown particles and fields at higher energies (the “hidden” variables). Thus, the Standard Model describes the real world in terms of templates. We only have access to a small subset of all templates; those that describe objects beyond the Standard Model, are simply not yet known.

In terms of the eigenstates of SM operators, the eigenstates of \(\vec{q}(t)\) must seem to be highly entangled. This is what happens in some of our models[3], and it may well be the reason why, in terms of microscopic variables, even the wave functions (7.23) and (7.24) may seem to be complicated entangled ones.

However, common sense gives us the suspicion that the macroscopic observables may be diagonal again in terms of the \(\vec{q}(t)\) operators. It is an interesting assumption, which we will adhere to. It indeed implies that the wave function of the universe will be collapsed in terms of the macroscopic observables at all times. According to our deterministic theories, these wave functions should nevertheless obey the theory’s Schrödinger equation. Now that we expressed our suspicion that wave functions can be constructed that always stay collapsed when macroscopic variables are considered, we can attempt to construct them more directly, starting from conventional theories of physics such as the Standard Model.
8. Schrödinger’s cat

The prototype example is the Schrödinger cat gedanken experiment\cite{28}. Let us start with an over simplified description that runs as follows. At \( t = 0 \), we have an unstable atom\(^2\) in the initial state \( |\psi(0)\rangle = |1\rangle \), and a cat, in a sealed box. A certain moment later, at \( t = t_1 \), there is, say, a 40\% chance that the atom decayed into state \( |2\rangle \), by emitting a photon \( \gamma \). The wave function is then \( |\psi(t_1)\rangle = \sqrt{0.6}|1\rangle + \sqrt{0.4}|2, \gamma\rangle \). If the atom at that moment has decayed, the cat is poisoned and dies, so, according to the simple argument, we then have a cat in the super-imposed state \( \sqrt{0.6}|live\ cat\rangle + \sqrt{0.4}|dead\ cat\rangle \). Then, the box is opened for inspection; a “measurement” takes place.

Will the cat really be in a superimposed state? Of course not. Decoherence takes place\cite{29}, and we expect that the cat will be either in the live or in the dead state. The question usually asked is how the wave function really evolves. Does it “collapse”?

To do this right, we have to take into account all those physical degrees of freedom that might be responsible for decoherence. Billions of atoms interact very weakly with the decaying atom and the cat. Each of these atoms can be in dozens of states, so the total dimensionality of the vector space spanned by these atoms is a huge number,

\[
N = \exp(C \kappa) ,
\]

where \( \kappa \) is the number of atoms and \( C \) a number of order one or larger, so indeed, \( N \) is astronomically large. As explained in the previous sections, we expect these environment states in general to be highly entangled, but in the first step of this argument, we consider simple, “pure” environment states in their energy eigenstates. These states will be referred to as \( |ES, t\rangle \).

Now, we can consider the above process more carefully. The initial state at \( t = 0 \) is

\[
|\psi, 0\rangle = |1\rangle |ES, 0\rangle .
\]

Then, at \( t = t_1 \), we may assume that the wave function is (apart from an overall phase rotation)

\[
|\psi, t_1\rangle = \left( \sqrt{0.6}|1\rangle + \sqrt{0.4}|2, \gamma\rangle \right) e^{i\varphi(ES)} |ES, t_1\rangle .
\]

Here, we take into account that the environment states may have caused a relative phase rotation \( \varphi(ES) \). The point is that total energy is conserved, but a small (positive or negative) part of it may have been absorbed by the environment, a part that will be different when the atom has decayed. Since we are not closely watching the environment while doing the experiment, we do not have perfect control over this phase difference.

In Eq. (8.27), for simplicity, the \( ES \) states were considered to be in some eigenstate of the hamiltonian. But this is not the wave function that we are interested in. Both in terms of the SM degrees of freedom, and in terms of the ontological states of Section 7, the environment will be time dependent. Also, the ontological states \( |O, k\rangle \) are expected to appear in the conventional quantum mechanical description as entirely entangled states, which we shall refer to as \( |EES, k\rangle \):

\[
|EES, k\rangle = \sum_{i=1}^{N} \alpha_i^{(k)} |ES_i\rangle .
\]

\(^2\) Historically, Schrödinger thought of an unstable atom, but a Stern-Gerlach experiment might be even more illustrative. For our present argument any quantum system can be used.
Let us reexpress the result in terms of a density matrix:

\[
|EES, k\rangle \langle EES, k| = \sum_{i,j=1}^{N} \alpha_i^{(k)} \alpha_j^{* (k)} \begin{pmatrix}
0.6 e^{i\varphi_i} \\
0.24 e^{i(\varphi_i - \varphi_j)}
\end{pmatrix} |ES_i, t_1\rangle \langle ES_j, t_1| ,
\]

where \(\varphi_i\) stands for \(\varphi(ES_i)\). Furthermore, \(N\) is the number (8.25) of environment states, and the index \(i\) labels them.

If all states \(|EES, k\rangle\) occur with (approximately) equal probability \(1/N\), we can use orthogonality,

\[
\sum_k \alpha_i^{(k)} \alpha_j^{* (k)} = \delta_{ij} ,
\]

and the density matrix becomes

\[
\sum \frac{1}{N} |EES, k\rangle \langle EES, k| = \frac{1}{N} \begin{pmatrix}
0.6 I \\
0.24 X^* \\
0.4 I
\end{pmatrix} ,
\]

where \(X\) stands for

\[
X = \sum_i e^{-i\varphi_i} |ES_i\rangle \langle ES_i| .
\]

The density matrix is that of a microcanonical ensemble where the total energy is fixed, apart from small variations that allow our states to depend slowly on time, so that all states that obey the restrictions dictated by the macroscopic description of the environment (including the total energy) have (approximately) equal probabilities. It is important to realize that this assumes that the \(EES\) states that we use all describe this subset of all macroscopic states. Indeed, this is what our ontological theory supposes, so there is no contradiction here.

If it weren’t for the phases \(\varphi_i\), the environment would merely contribute the identity as its density matrix. But now let us consider the phases in the off-diagonal part. Suppose decoherence takes place[29]. This means that the phases \(\varphi_i\) take all values, practically randomly, depending on the energy of the environment states. These energy eigenstates are delocalized. Therefore, if any matrix element is considered describing localized operators (using our templates), many of the \(ES\) states contribute, and their contributions are equal apart from the phases. Therefore, one expects the phases to cancel out. In short, if used only in combination of localized observables or operators, the off diagonal terms in our density matrix, the matrices \(X\), cancel out to zero. The density matrix is therefore

\[
\varrho = \frac{1}{N} \begin{pmatrix}
0.6 & 0 \\
0 & 0.4
\end{pmatrix} .
\]

Note that this density matrix was arrived at by performing probabilistic averages, not by demanding a wave function to collapse. It therefore completely agrees with the Schrödinger equation. But now it is of interest to see what it means in terms of the ontological states \(|O, k\rangle\). We stated that the probability for starting out with any of these states was equal: \(P_k = 1/N\). Suppose now that, at the start, we had one single pure state \(|O, k_0\rangle\). The suspicion that we explained in Section 7 is that this ontological state will either lead to a dead cat or a live one, but never to a superposition. This agrees with our density matrix (8.33) if indeed the probability
that state $|1\rangle$ was realized was 60% and the probability for $|2, \gamma\rangle$ was 40%. The probability for a superposition to arise is zero.\(^3\)

Throughout the process, the probabilities for any of the ontological states to be realized were conserved. Therefore, we must conclude that, from the very start, the ontological states consisted for 60% of states that would later evolve into a live cat and 40% of states that would evolve into a dead one. If indeed we had picked a state at random, the 60/40 distribution would be that of the probabilities.

This, we now claim, is the origin of Born’s rule. The ontological states only evolve either into pure states only describing a live cat, or into pure states only containing a dead cat, and never a superposition. The probabilities are simply in the number of ontological states with these properties. If one starts out picking one at random, then the probabilities will always be given by Born’s rule. Notice that this argument identifies the Born rule probabilities with the relative abundances of the initial states that could have been picked “at random”.

In deterministic physics, such as in the classical Van der Waals gas, the origin of probabilistic distributions can only be in the arbitrariness of the initial state (assuming infinitely precise equations of motion). If one assumes any kind of continuous distribution of positions and momenta of molecules at $t = 0$, then this determines the fate of the system, again in probabilistic terms. According to our ontological theory of quantum mechanics, the probabilities generated by Born’s rule, are to be interpreted exactly in the same terms. If we do not know the initial state with infinite accuracy then we won’t be able to predict the final state any better than that. The probabilistic distribution at $t = 0$ determines the probabilistic distribution at all later times.

It is only if we restrict ourselves to the usual templates of states containing only limited numbers of localized particles, that we are fooled into believing that the wave function suddenly collapsed when the cat’s door was opened, since we are confronted with the density matrix (8.33); in reality the ontic states of the underlying automaton correspond to templates that are highly entangled with the environment, so that the density matrix (8.33) emerges naturally, fully in accordance with the Schrödinger equation.

9. Discussion

The number $N$ of Eq. (8.25) stands for the dimensionality of the Hilbert space of environment states, and for the total number of allowed ontological states of the cellular automaton. In the latter terminology, superpositions are forbidden, so if states $|A\rangle$ and $|B\rangle$ are ontological states, then $|\psi\rangle = \lambda |A\rangle + \mu |B\rangle$, with $\lambda \neq 0$ and $\mu \neq 0$, is not such a state. This is how the states $|\text{live cat}\rangle$ and $|\text{dead cat}\rangle$ could emerge as ontological states, but not the state $\lambda |\text{live cat}\rangle + \mu |\text{dead cat}\rangle$. Thus, the initial state automatically collapses with the appropriate probabilities.

In our deterministic theory for QM, quantum superposition must be looked at as a property of the statistical approach to handling the extremely complex local equations of motion. Quantum wave functions were introduced for the convenience of the computation; linearity came as a handy tool for making calculations, but it so happens that quantum superpositions of ontological states themselves do not describe any real world, and this, as it turns out now, explains why we do not see quantum superpositions occurring in the macro world. By using the linearity of the

\(^3\) Note, that the important assumption mentioned in Section 7 was used here. It was assumed that, in the deterministic theory, the question whether the cat is dead or alive can be settled by careful statistical analysis of the state of the sub-microscopic degrees of freedom of the system. If the wave function is delta-peaked on one of the sub-microscopic states, it will be delta-peaked as either a live cat or a dead cat.
Schrödinger equation, we automatically adopt the Born interpretation of the squared norms as probabilities, because only this way the linear evolution equation for the density matrix (8.31) can assure probability conservation.

One may even conclude that the absence of superimposed states in the macroscopic world, which is usually mistaken to imply a collapsing wave function, is actually an important argument in favor of microscopic hidden variables.

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