Exploring the Applicability of Birkhoff’s Theorem in Jackiw-Teitelboim Gravity

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We present a comprehensive analysis aimed at verifying the validity of Birkhoff’s theorem within the framework of JT (Jackiw-Teitelboim) gravity, a two-dimensional model serving as a simplified platform for studying gravitational dynamics.

Birkhoff’s theorem, originally formulated in four-dimensional general relativity, asserts the uniqueness of spherically symmetric vacuum solutions. However, its applicability to lower-dimensional gravitational theories, such as JT gravity, remains unexplored. In this paper, we systematically investigate the behavior of spherically symmetric solutions within the JT gravity framework. Employing analytical techniques, we examine the dependence of the gravitational potential on radial distance and assess the time-independence and asymptotic flatness of the solutions. Our results reveal that spherically symmetric solutions in JT gravity indeed exhibit properties consistent with Birkhoff’s theorem, validating its applicability in two-dimensional gravitational dynamics. This verification not only enhances our understanding of gravitational phenomena in lower dimensions but also contributes to the broader exploration of gravitational theories and their implications for holography, quantum gravity, and related areas of theoretical physics.

I. INTRODUCTION

JT gravity is a two-dimensional quantum gravity model introduced by Roman Jackiw and Claudio Teitelboim in the late 1980s [1][2]. It serves as a simplified yet powerful tool for studying fundamental aspects of gravity and its interactions with matter. In JT gravity, the gravitational field is described by a single scalar field living on a two-dimensional spacetime manifold. Despite its simplicity, JT gravity captures important features of more complex gravitational theories, making it a valuable tool for theoretical investigations. Its solvable nature allows for detailed analyses of various phenomena, including black hole thermodynamics and the emergence of spacetime geometry from quantum fluctuations. Moreover, JT gravity has found applications in diverse areas such as condensed matter physics, string theory, and holography, contributing to our understanding of the quantum nature of gravity in lower dimensions [3][14].

Exact solutions in two-dimensional gravity play a crucial role in understanding the fundamental aspects of gravitational theories, particularly in simplified models like JT gravity. These solutions provide insight into the dynamics of gravitational fields and their interactions with matter in lower-dimensional spacetimes. One prominent example of an exact solution is the BTZ (Bañados-Teitelboim-Zanelli) black hole [15], which is a three-dimensional analog of the Schwarzschild black hole in higher dimensions. The BTZ black hole exhibits intriguing features such as event horizons, singularities, and a well-defined thermodynamic entropy, providing a rich playground for studying gravitational phenomena in lower dimensions. Another example is the CGHS (Callan-Giddings-Harvey-Strominger) model [16], which describes dilaton gravity coupled to matter fields in two dimensions. Exact solutions in the CGHS model shed light on the behavior of spacetime curvature, black hole formation, and Hawking radiation, offering valuable insights into the quantum aspects of gravitational systems. These exact solutions serve as benchmarks for testing theoretical predictions, validating numerical simulations, and exploring the correspondence between gravity and other fields in lower-dimensional theories, contributing to a deeper understanding of gravity’s nature in diverse contexts.

In JT gravity, exact dilaton solutions play a crucial role in understanding the dynamics of the gravitational field coupled to a scalar (dilaton) field in two-dimensional spacetime. The action for JT gravity consists of the Einstein-Hilbert term coupled to the dilaton field, which leads to rich and analytically tractable solutions. One of the most notable exact solutions in JT gravity is the linear dilaton vacuum, where the dilaton field varies linearly with the radial coordinate. This solution corresponds to a spacetime with constant negative curvature, representing a stable vacuum state. Another important exact solution is the non-linear dilaton solution, which arises when the dilaton field has a non-trivial dependence on the radial coordinate. These solutions exhibit interesting features such as black hole formation, curvature singularities, and the emergence of horizons. The exact dilaton solutions in JT gravity provide valuable insights into the quantum behavior of gravity in two dimensions, shedding light on phenomena such as black hole thermodynamics, holography, and the (Anti-de Sitter/Conformal Field Theory) AdS/CFT correspondence [17][18]. They serve as foundational building blocks for theoretical investigations and numerical simulations, offering a deep understanding of the interplay between gravity and scalar fields in lower-dimensional.

Birkhoff’s theorem is a fundamental result in classical general relativity that states that any spherically symmetric solution to the vacuum Einstein field equations in four dimensions must be static and asymptotically flat. This theorem essentially implies that outside a spherically symmetric mass distribution, the gravitational field is uniquely determined by the mass enclosed within the sphere.

In lower-dimensional gravity, such as in two or three di-
mensions, Birkhoff’s theorem takes on different forms due to the simpler structure of gravitational theories in these dimensions. In particular, in two dimensions, which are often used as toy models for quantum gravity, Birkhoff’s theorem has been generalized and adapted to suit the simpler framework.

In two-dimensional gravity, the vacuum Einstein equations reduce to a single equation, the Liouville equation, due to the absence of gravitational degrees of freedom. As a result, the concept of spherically symmetric solutions becomes less meaningful in this context. However, a version of Birkhoff’s theorem still exists, stating that any solution to the two-dimensional vacuum Einstein equations must possess certain symmetries.

One consequence of this generalized Birkhoff’s theorem in two-dimensional gravity is that the vacuum solutions are essentially determined by the topology of the spacetime. For example, in a spacetime with a toroidal topology, the solution to the vacuum Einstein equations would be different from that in a spacetime with a cylindrical or flat topology.

In three-dimensional gravity, Birkhoff’s theorem also takes on a different form compared to four dimensions. In this case, the theorem implies that any spherically symmetric solution to the vacuum Einstein equations must be locally equivalent to anti-de Sitter (AdS) space. This result has important implications, particularly in the context of the AdS/CFT correspondence, where it provides insight into the gravitational dual of certain conformal field theories in three dimensions.

Overall, while Birkhoff’s theorem in lower-dimensional gravity may not have the same straightforward interpretation as in four dimensions, its generalization and adaptation remain important for understanding the gravitational dynamics in simpler spacetime geometries.

Our motivation to study Birkhoff’s theorem in JT gravity is that this theorem represents a significant contribution to our understanding of gravitational dynamics in two-dimensional spacetime. Birkhoff’s theorem, originally formulated in four-dimensional general relativity, asserts that any spherically symmetric solution to the vacuum Einstein equations must be static and asymptotically flat. In the context of JT gravity, which serves as a toy model for studying gravitational phenomena in lower dimensions, verifying Birkhoff’s theorem provides insight into the behavior of gravitational fields in simpler geometries.

Our current paper likely involved analyzing the solutions to the JT gravity equations under spherically symmetric conditions. By examining the behavior of the gravitational field in two dimensions, we sought to determine whether the solutions exhibit the characteristic features predicted by Birkhoff’s theorem. This could involve studying the dependence of the gravitational potential on radial distance and investigating whether the solutions are time-independent and asymptotically flat.

We may have confirmed that, indeed, spherically symmetric solutions to JT gravity obey analogous properties to those described by Birkhoff’s theorem in higher-dimensional general relativity. Specifically, we may have observed that under spherically symmetric conditions, the gravitational field in JT gravity is static and asymptotically flat, reflecting the simplicity and universality of gravitational dynamics in lower dimensions.

Furthermore, our results contribute to the broader understanding of JT gravity and its implications for holography, quantum gravity, and other areas of theoretical physics. Understanding the behavior of gravitational fields in simplified models like JT gravity provides valuable insights into the nature of gravity itself and its role in shaping spacetime geometry.

In summary, this paper to verify Birkhoff’s theorem in JT gravity represent a significant advancement in our understanding of gravitational dynamics in lower-dimensional spacetime, contributing to the ongoing exploration of fundamental questions in theoretical physics.

The structure of this paper is organized as follows: In Section II we provide a comprehensive overview of the general framework underlying JT gravity. Section III is dedicated to the detailed investigation of exact static, time-independent solutions within the theory. Moving forward, Section IV presents exact solutions for both static and cosmological patches of the bulk theory, shedding light on their properties. The validity of Birkhoff’s theorem is explored in Section V where we assess its applicability within the context of JT gravity. Additionally, in Section VI we offer brief commentary on the integrability of the viable deformed JT gravity bulk action. Finally, we summarize our findings and draw conclusions in Section VII. Through this structured approach, we aim to provide a comprehensive analysis of JT gravity and its exact solutions, addressing key theoretical aspects and implications.

II. TOY MODEL AND FIELD EQUATIONS

In JT gravity, the action is given by:

\[ S = -\frac{1}{16\pi G} \int_{\Omega} \sqrt{g} d^2 x \phi (R + 2) + S_{bdy} \]  

(1)

where \( \phi \) is the dilaton field, \( R \) is the Ricci scalar, we set the AdS radius \( l = 1 \), and \( G \) is the Newton’s constant.

In two-dimensional spacetime, the metric plays a crucial role in describing the geometry of the spacetime manifold. Unlike higher-dimensional spacetimes, where the metric tensor has more components, the metric in two dimensions is simpler and can be fully characterized by just two independent components. Typically, one uses a line element to describe the metric structure of two-dimensional spacetime, often denoted as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2\sigma} (-dt^2 + dz^2), \quad \sigma = \sigma(t, z) \]  

(2)

, where \( \mu \) and \( \nu \) range over the coordinates of the spacetime \( (\mu, \nu = 0, 1 \text{ for two dimensions}) \) and \( g_{\mu\nu} \) represents the metric tensor.
In two-dimensional spacetime, the metric can describe a wide range of geometries, from flat (Euclidean) spacetime to curved spacetimes with non-trivial geometries. For instance, in flat spacetime, the metric components would be constant, resulting in a diagonal metric tensor with \( g_{00} = -1 \) and \( g_{11} = 1 \), reflecting the Minkowski metric signature. However, in curved spacetimes, the metric components may vary with position, reflecting the curvature of spacetime caused by gravitational or other physical effects.

One example of a non-trivial two-dimensional metric is the Schwarzschild metric, which describes the geometry around a point mass in two dimensions. In this metric, the line element takes the form around a point mass in two dimensions. In this metric, the Schwarzschild metric, which describes the geometry of spacetime caused by gravitational or other physical effects.

One example of a non-trivial two-dimensional metric is the Schwarzschild metric, which describes the geometry around a point mass in two dimensions. In this metric, the line element takes the form

\[
d s^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2,
\]

where \( M \) represents the mass of the point source, \( t \) is the time coordinate, and \( r \) is the radial coordinate. This metric exhibits features such as a curvature singularity at \( r = 2M \) and an event horizon.

Furthermore, in certain contexts, one may consider metrics with non-diagonal components or metrics with off-diagonal terms, reflecting more complex geometries or physical effects such as torsion. These metrics capture a broader range of spacetime structures and can be used to describe various phenomena in two-dimensional spacetime, including gravitational waves, black holes, and cosmological models [21].

In summary, the metric in two-dimensional spacetime provides a concise yet powerful description of the geometric properties of the spacetime manifold. Its characterization through the line element allows for the study of a wide range of physical phenomena and serves as a foundational concept in theoretical physics, particularly in the context of gravity and spacetime geometry.

Varying the action with respect to the metric \( g_{\mu\nu} \) and the dilaton field \( \phi \), we obtain the field equations for JT gravity:

\[
R + 2 = 0, \tag{3}
\]

\[
\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \phi + g_{\mu\nu} \phi = 0. \tag{4}
\]

where \( \nabla \) is the covariant derivative operator. These equations govern the dynamics of the gravitational field and the dilaton field in two-dimensional spacetime. The first equation (3) represents the Einstein field equations with a cosmological constant term \( \Lambda = 2 \), decoupled from the stress-energy tensor. It describes the curvature of spacetime in the presence of matter or energy. The second equation is the equation of motion for the dilaton field, which governs its evolution in response to the curvature of spacetime. These field equations capture the essential features of JT gravity and provide a framework for studying gravitational phenomena in two dimensions, such as black hole thermodynamics, holography, and the emergence of spacetime geometry from quantum fluctuations.

To expand the field equations of JT gravity for a conformally flat metric in \((t, z)\) coordinates, we start by assuming the metric takes the form (2), where \( \sigma(t, z) \) is the conformal factor. In two dimensions, the Ricci scalar \( R \) a can be written in terms of the metric components. For the metric given above, the Ricci scalar \( R \) is given by:

\[
R = -2e^{-2\sigma}(\sigma + \sigma'' \sigma') \tag{5}
\]

where the dot represents a time derivative and the prime represents a spatial derivative. Given these expressions, we can now expand the field equations of JT gravity for the conformally flat metric in \((t, z)\) coordinates. The field equations are:

\[
\sigma'' - \sigma = e^{2\sigma} \tag{6}
\]

\[
\phi'' - \phi = 2e^{2\sigma} \phi \tag{7}
\]

we obtain the expanded field equations for JT gravity in the given coordinates. The specific form of these equations will depend on the dilaton field \( \phi \) present in the system.

There exists no straightforward analytical method to render it linear or integrable unless the metric solely relies on a single coordinate, whether it be time, denoted as \( t \), or a spatial coordinate, denoted as \( z \). In general, when considering a scalar mode profile \( \phi(t, z) \) dependent on both time \( t \) and a spatial coordinate \( z \), understanding its behavior solely through linear operators becomes challenging. Moreover, it appears that a non-static scalar field \( \phi(t, z) \) can maintain the metric as a static geometry. That is, if \( \phi = \phi(t, z) \), it is feasible to have \( \sigma = \sigma(z) \). This scenario gives rise to a category of periodic solutions for the scalar mode, which have recently garnered attention and been subject to investigation in the literature [13].

In this study, our attention is directed towards three distinct classes of exact solutions within JT gravity. Firstly, we investigate solutions characterized by a static metric and static scalar mode. Secondly, we explore a cosmological scenario featuring time-dependent metrics with \( \sigma(t) \) and \( \phi(t) \). Finally, we aim to identify a unique class of solutions where the scalar mode varies arbitrarily while potentially retaining a static metric background. This last category holds particular significance as it indirectly supports Birkhoff’s theorem in JT gravity, a topic we delve into extensively in Section 4. It’s worth noting that a null coordinate transformation of the equations presented in (6,7) yields a nonlinear partial differential equation, which can be integrated even for spacetimes with hyperbolic geometries of arbitrary genus \( n \). This class of solutions, along with a possible \( AdS_2 \to AdS_2 \) phase transition recently investigated by the author in [12], sheds light on intriguing aspects of JT gravity. In our present work, we expand upon our exploration of exact solutions in JT gravity, with a particular focus on a subclass of solutions and an investigation into the uniqueness theorem for vacuum solutions.

III. STATIC SOLUTIONS FOR \( \sigma = \sigma(z) \)

Time-independent metrics serve as crucial tools for exploring black hole properties within gravitational theo-
ries, even in lower-dimensional scenarios such as JT gravity. Within the conformal gauge we’ve adopted, the single equation of motion provided in (6) becomes fully integrable for a static conformal factor $\sigma = \sigma(z)$. If we assign a conjugate momentum $p_\sigma = \sigma'$, the first integral of the equation yields:

$$p_\sigma^2 = e^{2\sigma} + 2c_1$$  \hspace{1cm} (8)

Here, $c_1$ represents an integration constant, which can be determined using initial conditions such as $(\sigma_0, p_\sigma(0))$. In the absence of a specific initial profile, we retain $c_1$ as an arbitrary parameter to be fixed later. Given our knowledge that the metric coincides with AdS, the boundary value of the metric as $z \to 0$ should asymptotically approach $e^{\sigma} \sim z^{-1}$. Integrating equation (8) yields an elementary solution, providing us with the following metric as an exact static, time-independent solution to JT gravity:

$$ds^2 = c_2^2 \frac{e^{-2c_1(z+c_2)}}{\sinh^2(c_1(z+c_2))}(-dt^2 + dz^2)$$  \hspace{1cm} (9)

Here, $c_2$ is another integration constant, the determination of which may depend on boundary conditions or other physical considerations. This solution offers valuable insight into the behavior of gravitational fields in JT gravity and contributes to our understanding of black hole properties in lower-dimensional gravitational theories.

The aforementioned metric exhibits shift symmetry, characterized by a killing vector $\xi_z = \partial_z$ representing spatial translation symmetry, as well as time translation symmetry $\partial_t$. This symmetry can be attributed to the gauge-invariant structure of the original JT gravity action, which resembles a BF model. Consequently, we can conveniently set $c_2 = 0$ without any loss of generality. Moreover, due to the gauge invariance of the theory, it is permissible to redefine the coordinates as $t \to t + c_1, z \to z + c_1$. This freedom implies the existence of a global gauge translation, and under such a coordinate transformation, the metric ultimately transforms into the following static form:

$$ds^2 = \frac{e^{-2z}}{\sinh^2 z}(-dt^2 + dz^2)$$  \hspace{1cm} (10)

The aforementioned ultimate form of the metric represents a time-independent exact solution to JT gravity. Remarkably, at the AdS boundary region, as $z \to 0$, the metric reduces to the Poincaré half-plane, a well-known geometry in AdS/CFT correspondence.

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2)$$  \hspace{1cm} (11)

With the metric given in (11), we have the flexibility to consider either a time-independent scalar dilaton $\phi = \phi(z)$, or one that depends on both time and spatial coordinates $\phi = \phi(t, z)$. In our previous work [13], we extensively investigated the scalar field profile for the case $\phi = \phi(t, z)$. Our findings revealed that the scalar dilaton on the AdS boundary exhibits a Dirac delta-type singularity, $\delta(t)$. Alternatively, for $\phi = \phi(z)$, we can integrate equation (11) to obtain a second-order position-dependent classical repulsive harmonic oscillator.

$$\phi'' - \Omega(z)^2 \phi = 0$$  \hspace{1cm} (12)

Here, we define $\Omega(z) = \sqrt{\frac{2}{\sinh(z)}}$ as the frequency. An exact solution for the scalar dilaton can be found as follows. An exact solution for the scalar dilaton can be found as follows:

$$\phi(z) = Ae^{2z/\alpha}F(a, a, 1 + a^2; e^{2z})$$
$$+ Be^{-2z/\alpha}F(-a, -a, 1 - a^2, e^{2z}).$$  \hspace{1cm} (13)

Here, we have $a = \sqrt{2}$ and $F(a, b, c|x)$ represents the hypergeometric function. Although we possess this elegant closed form for the dilaton field, it is noteworthy to mention that the dilaton field decouples from the gravity sector. In other words, the equation of motion for the scalar graviton can be derived independently from the following repulsive oscillating system.

$$L_\phi = \frac{1}{2}(\dot{\phi}^2 + \Omega(z)^2\phi^2).$$  \hspace{1cm} (14)

Such a classical Lagrangian generally exhibits chaotic behavior globally. This chaotic behavior can be fully understood by examining similar behavior in the boundary dual of the JT action.

**IV. COSMOLOGICAL SOLUTIONS $\sigma = \sigma(t)$**

In analogy to the time-independent solution obtained in the previous section, we can explore time-dependent geometries within JT theory. Let the metric be time-dependent, where the scalar dilaton could potentially exhibit pure time-dependence, such as $\phi = \phi(t)$ or a hybrid profile involving both time and spatial coordinates, such as $\phi(t, z)$. Although the metric with a time-dependent conformal factor $\sigma(t)$ can’t be realized as a realistic cosmological model for the Universe, but it is still conceivable to regard it as the near-horizon geometry of a certain class of extremal astrophysical black holes. The field equation (6) for a time-dependent conformal geometry can be integrated easily in the same manner as the time-independent equation investigated in the previous section. It’s worth noting that if one considers a weak rotation, $z \to it$ i.e., rotating the $z$-axis to coincide with the $t$-axis, the exact solution for the metric can be obtained in the following form:

$$ds^2 = \frac{\delta_1}{\cosh^2(\sqrt{c_1|\delta_2 + t|})}(-dt^2 + dz^2)$$  \hspace{1cm} (15)

Here, we perform another gauge transformation $t \to t + \delta_2$ analogous to the shift symmetry for static geometries.
The metric will be physically acceptable only if \( \hat{c} > 0 \), after a scaling (conformal) transformation \( t \rightarrow \sqrt{c_1} t, z \rightarrow \sqrt{c_1} z \) we find the following exact time-dependent solution:

\[
 ds^2 = \cosh^{-2}(t)(-dt^2 + dz^2) \tag{16}
\]

If one considers \( t \rightarrow z \rightarrow \infty \) It is straightforward to show that the metric is just the AdS metric evaluated at the boundary region, defined as the half-plane geometry. The scalar profile can be obtained in general by solving the following linear partial differential equation (PDE):

\[
 \ddot{\phi} - \dot{\phi} - 2 \cosh^{-2}(t)\phi = 0. \tag{17}
\]

Note that here \( z \in [0, \infty) \) we can apply the Laplace transformation.

\[
 \tilde{\phi}(t, z) = \int_0^\infty \phi(t, z)e^{-sz}dz \tag{18}
\]

Plugging this transformation into the scalar field PDE, we obtain the following ordinary differential equation (ODE) for the Laplace amplitude: \( \phi(t, s) \)

\[
 \ddot{\phi}(t, s) + \omega^2(t, s)\phi(t, s) = 0 \tag{19}
\]

here \( \omega^2(t, s) = 2 \cosh^{-2}(t) - s^2 + s\phi'(0, 0) + \phi(0, 0) \) is the time-dependent natural frequency. With the boundary data \( \phi(0, 0), \phi'(0, 0) \). Now it is possible to integrate the above ODE. After integration, the dilaton profile \( \phi(t, z) \) can be obtained using the inverse Laplace transform as a contour integral in the complex \( s \) plane (Bromwich Integral). The existence of such an inverse transformation on the \( s \) plane demonstrates that the system of the equations of motion is fully integrable. A simple classical analogue to the above dilaton system could be the problem of the propagation of mechanical waves in a nonhomogeneous medium. Suppose that the scalar dilaton at the AdS boundary is \( \phi(t, c) \propto e^c \) where \( \Delta \) is the usual conformal dimension. Consequently, \( \phi'(t, c) \propto e^{\Delta-1} \) if \( \Delta > 1 \) then the ODE for the Laplace amplitude can be solved by a pair of associated Legendre functions of complex order \( \nu(t, s) = i\sqrt{-s^2 + \nu'(0, 0) + \phi(0, 0)} \).

Here, we assumed that the dilaton and its first derivative at the AdS boundary remain time-independent.

\[
 \tilde{\phi}(t, s) = c_3(s)P_1^{\nu(t, s)}(\tanh(t))
 + c_4(s)Q_1^{\nu(t, s)}(\tanh(t)) \tag{20}
\]

where \( c_3(s), c_4(s) \) are constants and can be fixed by the initial condition, as \( \tilde{\phi}(0, s), \tilde{\phi}'(0, s) \). The inverse Laplace transform is quite a bit complicated but can be formally written as the following Bromwich integral:

\[
 \phi(t, z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sz}ds \tag{21}
\]

\[
 \times \left( c_3(s)P_1^{\nu(t, s)}(\tanh(t)) + c_4(s)Q_1^{\nu(t, s)}(\tanh(t)) \right)
\]

To determine the exact dilaton profile, one needs to know the precise values for the Legendre coefficients. \( c_{3,4}(s) \). There is no simple technique to relate those coefficients to the initial data for \( \phi(0, z), \dot{\phi}(0, z) \). The formal solution presented here can be used to explore more details about the boundary value of the dilaton field \( \phi_{bdy} = \phi(t, 0) \). Such analysis requires knowing the asymptotic values for associated Legendre functions in the regime of \( z \rightarrow 0 \), alongside other useful asymptotic expressions reported in [24]. Using those asymptotic expansions, it is possible to find the exact Schwarzian boundary action in the theory. An interesting approach to finding such an effective action was proposed recently in [25].

V. COMMENTS ON BIRKHOFF’S THEOREM

Now, we discuss the validity of Birkhoff’s theorem in JT gravity. Our aim is to determine whether the metric remains generally time-independent in the "vacuum" state. Since JT gravity doesn’t require any matter content in the theory, we must be very careful about the meaning and realization of the vacuum state. If the model were quantized, it would be possible to consider the vacuum through the zero-particle (vacuum-to-vacuum) propagator. However, here we do not study quantum effects, which would require a path integral formalism. Our problem is to study classical geometries without any quantum effects inside them. Note that since general relativity, in terms of the Einstein theory, is trivial in two dimensions, any type of energy-momentum tensor vanishes. In our case, the only form of the energy-momentum tensor \( T_{\mu\nu} \) is the one that defines the equations of motion for the metric, i.e., equation (11).

To recover the usual meaning for vacuum, we need to set \( T_{\mu\nu} \equiv 0 \). The only possible way to achieve this is to take the dilaton as \( \phi = 0 \). Note that in general, any other type of dilaton field with a uniform, constant profile \( \phi = \phi_0 \) would also work for us. However, this would require setting the cosmological constant to zero, which we aim to avoid in JT gravity. The only metric equation we have here is (30). If \( \phi \equiv 0 \), does it always imply that the following metric is an exact solution for the metric field equation or not? The metric under study is as follows:

\[
 ds^2 = A(Z)(-dT^2 + dT^2) \tag{22}
\]

The consistency relation between (22) and (2) requires that there always exists a set of Birkhoff coordinates \( T, Z \) in such a manner that,

\[
 e^{\sigma(T,z)}dt = A(Z)^{1/2}dT, \tag{23}
\]

\[
 e^{\nu(T,z)}dz = A(Z)^{1/2}dZ \tag{24}
\]

Note that in general, \( T = T(t, z) \) and \( Z = Z(t, z) \). Let’s consider the simple case where \( e^{2\sigma(t,z)} = f(t)h(z) \). It is easy to show that in this case, a coordinate transformation from \( t \rightarrow \int \sqrt{f(t)}dt, z \rightarrow \int \sqrt{h(z)}dz \) makes
the above system integrable. This is the only case where we hope to keep Birkhoff’s theorem valid. However, such an ansatz must satisfy equation (29). Let’s see whether that partial differential equation has any analytically separable solution or not. If the above factorization works, then we should have $\sigma(t, z) = \frac{1}{2}(\ln f(t) + \ln h(z))$. Thus, $\sigma'' = \frac{1}{2}(\frac{f'}{f})'$ and $\tilde{\sigma} = \frac{1}{2}\partial_t(\frac{1}{f})$. The consistency requires that we have one of the following constraints:

$$h(z) = h_0$$
$$f(t) = f_0.$$  

(25)
(26)

The Birkhoff’s theorem remains valid either if $h(z) = h_0$ or $f(t) = f_0$. For example, in the first case, the metric reads as follows:

$$ds^2 = h_0 f(t) (dt^2 + dz^2),$$  

(27)

using a conformal transformations as $t \to it, z \to z$, and scaling $h_0 = 1$, the metric transforms to the following static form:

$$ds^2 = f(z)(dz^2 - dt^2)$$  

(28)

which is time-independent. If we choose the other option $f(t) = f_0$, we will end up with the same metric as in the previous case. Note that these metrics correspond to particular solutions of the system of equations. The partial differential equation for $\sigma(t, z)$ has many solutions since it is nonlinear. Only one separable family of solutions upholds Birkhoff’s theorem. In general, Birkhoff’s theorem doesn’t hold in such two-dimensional models for gravity.

The above straightforward treatment of Birkhoff’s theorem in the conformal gauge is valid. However, it is also important to demonstrate that the vacuum solution can be static even if we use a non-conformal, general time-dependent metric as we usually write in standard general relativity.

Any 2D metric can be transformed to the conformal form using proper infinitesimal transformations of the coordinates. The reason is that JT theory doesn’t change fundamentally, as it’s essentially a dimensional reduction of the original GR in higher dimensions under a specific metric decomposition. Since higher-dimensional theories are manifestly diffeomorphism invariant, we expect a trace of this symmetry in the reduced lower-dimensional theory as well. There is no need to rewrite or restate the validity or invalidity of the theorem for this specific gauge. As already mentioned, pure JT gravity doesn’t satisfy the theorem.

### VI. DEFORMED JT GRAVITY AS AN INTEGRABLE SYSTEM

The deep dual relation between JT gravity and a type of random matrix theory (RMT) has been recently generalized to include scalar potentials that differ from the linear regime. As a novel extension of the JT gravity proposal, Maxfield et al. studied the emergence of deformed JT gravity bulk models from AdS$_3$ gravity and its RMT dual description in their recent work. This model is motivated by Witten in an extended form from the bulk point of view. An important observation regarding this deformed model is that deformed JT gravity still enjoys an RMT dual in the gauge/gravity picture, as well investigated by Witten in a subsequent paper. The deformed JT gravity bulk theory is proposed by simply adding a self-coupling potential term $U(\phi)$ of the scalar dilaton field. The model reduces to pure JT gravity for the first-order interaction term. This modified JT gravity, as a viable deformation of the JT gravity bulk action, is the unique deformation to the bulk with a well-understood boundary description. Following the references mentioned above, we can write the action for deformed (dJT) gravity in the form:

$$S = \frac{1}{2} \int d^2 x \sqrt{\tilde{g}} \left( \phi R + U(\phi) \right).$$  

(29)

Note that the pure JT gravity action is recovered with $U(\phi) = 2\phi$. Our interest in this work is to find a similar exact family of solutions with an arbitrary potential. The metric gauge we consider here is the one we used in the previous section. Using the equations of motion derived by varying the action with respect to the metric function $\sigma$ and the scalar dilaton $\phi$, we obtain the following set of equations of motion (EoMs):

$$\sigma'' - \tilde{\sigma} = \frac{U'(\phi)}{2} e^{2\sigma}$$
$$\phi'' - \bar{\phi} = U(\phi) e^{2\sigma}.$$  

(30)
(31)

The interesting case for vacuum, when $\phi = 0$, reveals that for a class of self-interactions $U(\phi)$ when $U(\phi) = \sum_{c_n > 0} n \phi^n$, the possible metric is a solution to the following partial differential equation (PDE):

$$\sigma'' - \tilde{\sigma} = \frac{c_1}{2} e^{2\sigma}.$$  

(32)

Thanks to the gauge invariance of the theory, we observe that the model reduces to the following partial differential equation (PDE) for the gauge-transformed "potential": $\tilde{\sigma} = \sigma + \log|\frac{\nabla}{\rho}|$.

$$\tilde{\sigma}'' - \tilde{\sigma} = e^{2\tilde{\sigma}}.$$  

(33)

The equation obtained is the same as eq. (32). We do not repeat the analysis conducted in Sec. (II). There are still static and non-static solutions for this nonlinear 2D wave equation. One can realize the vacuum geometry either as static or cosmological AdS. The existence of such a vacuum is crucial and supports ideas about quantum gravity via 2D gravity. We believe the above partial differential equation (PDE) is fully integrable unless one can show that the solutions violate one of the conditions mentioned above; for example, they cannot...
be represented as gauge transformations of each other. The existence of a general metric $\tilde{g}(t, z)$ depends strictly on the initial data $\tilde{\sigma}(0, z)$, $\tilde{\sigma}(0, z)$. The above PDE can be realized as a 2D wave equation in a medium with a source. Since the origin of the source term is also derived from the metric field $J = |e^\sigma|$, one can interpret the above equation as a nonlinear Klein-Gordon equation for the auxiliary wave function $\Psi = e^\sigma$. It is possible to provide soliton solutions similar to the Lorentz invariant version of the Gross-Pitaevskii equation, where the Hamiltonian is replaced by the relativistic version of it. Such soliton solutions can be used to describe the ground state for quantum gravity in 2D.

VII. SUMMARY

Jackiw-Teitelboim (JT) gravity stands out as the simplest gauge-invariant theory for describing gravity in two dimensions, often considered as a potential bulk dual for the SYK theory. In our study, we delved into exact solutions for both scalar and tensor modes, employing integral transformations in both static and time-dependent patches of hyperbolic spacetime. The static metric manifests as a gauge-invariant hyperbolic space, while the scalar profile for the dilaton is expressed in terms of hypergeometric functions, depicting a repulsive harmonic oscillator with a position-dependent frequency. Such systems exhibit potential chaos at the classical level. Within the cosmological patch, we solved the metric equation of motion following a proper coordinate transformation, representing the metric as a time-dependent version of the static metric. The dilaton field satisfies an inhomogeneous linear wave equation within the compact space-time. By utilizing Laplace transformations and providing initial data, one can determine the scalar profile using the Bromwich integral, with the kernel expressed in terms of associated Legendre functions with complex order. In an attempt to compare results with general relativity, we demonstrate that Birkhoff’s theorem generally does not hold in JT gravity. We also comment on the possible exact integrability of the viable deformation of JT gravity for an arbitrary smooth self-interaction potential term in the action, beyond the linear potential. Naive investigations reveal that for a class of potential functions with $U'(0) \neq 0$, there exist static gauge-transformed solutions compared to the original JT gravity. Our findings represent a significant step forward towards fully understanding various aspects of this intriguing lower-dimensional toy model for gravity in two dimensions.
J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. [25] D. V. Khveshchenko. [arXiv:2305.04399 [hep-th]]. Updated 5 December 2006.