On the temperature of lowest order inner bremsstrahlung

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(Dated: November 3, 2022)

The relativistic thermodynamics of classical radiation from a single accelerating electron is investigated. The temperature characterizing the system highlights the dependence on acceleration. As the electron moves along a particular accelerated trajectory a deep infrared analysis of the soft radiation emitted is corroborated by experimental observation of lowest order inner bremsstrahlung during beta decay. In the context of acceleration radiation, the dynamic Casimir effect with Planck-distributed photons, and thermal black hole evaporation, we provide supporting analytic consistency for equilibrium thermal radiation. The motion is ultra-relativistic with a specific time-dependent Lorentz-invariant proper acceleration possessing long-lasting constant local acceleration.

PACS numbers: 41.60.-m (Radiation by moving charges), 04.70.Dy (Quantum aspects of black holes)

Keywords: moving mirrors, beta decay, black hole evaporation, acceleration radiation, dressed electrons

Introduction. - Black holes, with surface gravity $\kappa = c^4/4GM$, have ‘quantum’ ($h$) temperature [1],

$$T_{BH} = \frac{\hbar \kappa}{2\pi \epsilon_0 c B}, \quad (1)$$

because, in part, the radiated particles in equilibrium are frequency distributed with a Planck factor, and the power emitted scales according to $P \sim T^2$ substantiating black holes as one-dimensional information channels [2].

In this note we help make the case and present some detail supporting the idea of a classical moving point charge analog to Eq. (1). The proposal concerns the temperature of an electron’s radiation: when the power emitted is uniform and classical equilibrium thermodynamics applies, the emission has a temperature proportional to the acceleration, $\kappa$, of the electron. One finds the ‘classical’ (no $h$) temperature,

$$T_{\text{electron}} = \frac{\mu_0 e^2 \kappa}{2\pi \epsilon_0 c}, \quad (2)$$

only when the radiated particles are commensurate with constant power emission [3]. Interestingly, this occurs during Planck-distributed radiation from an analog moving mirror accelerated along the same specific trajectory (given therein). A horizontal leveling of the power is visually seen at extremely ultra-relativistic final speeds (given therein). A horizontal leveling of the power is

Analog bridge. - First, let us consider the simple action-correspondence, between Eq. (1) and Eq. (2),

$$h \rightarrow \frac{e^2}{\epsilon_0 c} = \mu_0 e^2, \quad (3)$$

where the reduced Planck’s constant is, as usual,

$$h = 1.054 \times 10^{-34} \text{ J s}, \quad (4)$$

and the smaller action (or angular momentum) classical quantity is,

$$\mu_0 e^2 = 9.671 \times 10^{-36} \text{ J s}. \quad (5)$$

That is, with fine structure constant $\alpha$, notice that,

$$\frac{h}{\mu_0 e^2} = \frac{1}{4\pi \alpha} \approx 10.91. \quad (6)$$

For a given acceleration scale $\kappa$, the classical temperature, Eq. (2) is about a magnitude order colder than the quantum temperature Eq. (1). This analog ‘substitution’, Eq. (3), if you will, can help us bridge the connection between the elementary particle and black hole in an easy-to-use way; i.e. substituting $h \rightarrow \mu_0 e^2$ in Eq. (1) gives Eq. (2). We will help justify and generalize this in the following sections.

Extension bridge. - This bridge, Eq. (3), is not limited to Eq. (1) and Eq. (2). It proves useful as an action correspondence in general between the quantum moving mirror model ($q$) and the classical moving point charge model ($c$). This is seen, respectively, in the power, see e.g. [4] and [5], where $\alpha$ is the proper acceleration of the mirror or electron,

$$P_q = \frac{\hbar \alpha^2}{6\pi c^2}, \quad P_c = \frac{\mu_0 e^2 \alpha^2}{6\pi c}, \quad (7)$$

and self-force, where the prime indicates the derivative with respect to proper time, see e.g. [6] and [7],

$$F_q = \frac{\hbar \alpha'}{6\pi c^2}, \quad F_c = \frac{\mu_0 e^2 \alpha'}{6\pi c}, \quad (8)$$

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for any limited, horizon-less trajectory whose acceleration is asymptotically zero (asymptotic inertia).

Moreover, the bridge also occurs specifically between the spectral radiance of a particular moving mirror model and lowest order inner bremsstrahlung (IB) during beta decay [3]. This is seen, respectively, in the frequency independence of the spectral energy per unit bandwidth, see e.g. [8] and [4],

$$I_q = \frac{h}{2\pi^2} \left( \frac{\eta}{s} - 1 \right), \quad I_c = \frac{\mu_0 c e^2}{2\pi^2} \left( \frac{\eta}{s} - 1 \right), \quad (9)$$

where $$s = \tanh \eta$$ is the final speed of the mirror or electron, and $$\eta$$ is the rapidity.

**UV cut-offs.** - For some orientation, consider now, re-expressing the temperature of the electron radiation in terms of the maximum appreciable energy emitted,

$$T = \frac{24\alpha E_\gamma}{\pi k_B}, \quad (10)$$

where the fine structure constant is $$\alpha = e^2/4\pi\varepsilon_0 \hbar c \approx 1/137$$. The energy range of the detected photons are UV/IR limited by $$E_\gamma = \hbar \omega_\gamma = \hbar (\omega_{\text{max}} - \omega_{\text{min}})$$. The sensitivity of maximum energy detected can be written succinctly with $$\beta = 1/k_BT$$ as,

$$E_\gamma = \frac{\pi}{24\alpha \beta} = \frac{\pi}{24\alpha} k_BT \approx 18 k_BT. \quad (11)$$

This gives some perspective of the dependence of the system on the cut-offs when in thermal equilibrium.

Including these UV/IR limits, experimental evidence of lowest order IB energy emitted during beta decay confirms the consistency of the theoretically derived frequency independence of the spectral energy per unit bandwidth, Eq. (9), see e.g. [9]. In the following sections, we support the physical notion of temperature in this context by providing corroborative analytic results confirming the mathematical validity of Eq. (2).

**Total energy emitted.** - To obtain the energy per unit bandwidth from Eq. (9), one associates the UV/IR scale of the system with the acceleration scale $$\kappa$$,

$$\int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega = \omega_{\text{max}} - \omega_{\text{min}} \equiv \omega_\gamma \equiv \frac{\pi \kappa}{12c}, \quad (12)$$

such that, using the first equation of Eq. (9), the quantum spectral energy per unit bandwidth,

$$I_q = \frac{dE_q}{d\omega} \rightarrow E_q = \frac{\hbar \kappa}{24\pi c} \left( \frac{\eta}{s} - 1 \right), \quad (13)$$

or the second equation of Eq. (9), the classical spectral energy per bandwidth,

$$I_c = \frac{dE_c}{d\omega} \rightarrow E_c = \frac{\mu_0 e^2 \kappa}{24\pi} \left( \frac{\eta}{s} - 1 \right). \quad (14)$$

This demonstrates, with the clarity of SI units, the two different models have analogous energy emission scaling.

It is now convenient to work with unit charge, allowing frequencies to be UV/IR-limited by cutoffs $$\omega_\gamma = \omega_{\text{max}} - \omega_{\text{min}}$$ such that $$\omega_\gamma = \pi \kappa/12$$ sets the acceleration scale $$\kappa$$ in natural units. The energy radiated by the electron or the mirror is finite in this frequency interval,

$$E = \frac{\kappa}{24\pi} \left( \frac{\eta}{s} - 1 \right) \equiv \frac{\kappa}{48\pi} \left[ \frac{1}{s} \ln \left( \frac{1 + s}{1 - s} \right) - 2 \right], \quad (15)$$

where again, $$s$$ is the final constant speed of the electron and $$\eta$$ is the rapidity from $$s = \tanh \eta$$. As we shall see, for thermality, ultra-relativistic speeds are required, $$s \sim 1$$. Only the classical version of Eq. (15), that is, Eq. (14) [10] or lowest order IB energy [11], has been observed, see e.g. [9].

**Constant power emission.** - As expected for thermal equilibrium, a stable emission period of constant power is measured by a far away observer. This is best represented as the change of energy with respect to retarded time $$u = t - r$$, and written as Larmor power $$P = \frac{\kappa^2 s^2 W^2}{6(1 + s)^2 }$$, where $$P = \alpha^2/6\pi$$. Here $$\alpha$$ is the proper acceleration and $$\beta$$ is the velocity. To avoid excessive mathematical detail, we directly use the trajectory in [3], compute $$P(u)$$, formulated in terms of retarded time $$u$$, and present the analytic expression,

$$P_c = \frac{k^2 s^2 W^2 (W + 1 - s)}{6(1 + s)^2}, \quad (16)$$

where $$W = W[e^{\kappa u} (1 - s)]$$ is the product log argument. This result has a plateau when the final speed of the electron is near the causal limit. Consider analytically, two separate limits, of high speeds and late times, which reveals, using $$T = \kappa/2\pi$$,

$$P_c \equiv \lim_{u \to \infty} \lim_{s \to 1} P(u) = \frac{\kappa^2}{48\pi} = \frac{\pi}{12} T^2. \quad (17)$$

A plot of $$P(u)$$ at high final asymptotic speeds $$s \sim 1$$ illustrates the constant power plateau indicative of thermal emission. See a plot of the power plateau in Figure 1.

Momentarily switching to SI units, we keep in mind, that we are working with classical (3+1) dimensional radiation of an electron. Therefore, we notice that Eq. (17) is a (1+1) dimensional classical power-temperature relation,

$$P_c = \frac{\mu_0 e^2 \kappa^2}{48\pi c} = \frac{\pi}{12} \frac{k_B^2}{\mu_0 c^2} T^2, \quad (18)$$

with scaling identical to the standard quantum (1+1) dimensional Stefan-Boltzmann law [12] which describes (3+1) dimensional black hole power radiance, see e.g. [2],

$$P_q = \frac{\pi k_B^2}{12h} T^2. \quad (19)$$

In the same way that a single spatial dimensional Planck distribution yields Eq. (19), a trans-Planckian distribution, $$J$$, or spectral energy density in angular frequency
space, where Eq. (3), $h \to \mu_0 ce^2$, has been applied,
\[
J(\omega) = \frac{1}{2\pi} \frac{\mu_0 ce^2 \omega}{e^{\mu_0 ce^2 \omega/k_BT} - 1},
\]
integrated over angular frequency,
\[
\int_0^\infty J(\omega) d\omega = \frac{\pi}{12} \frac{k_B^2}{\mu_0 ce^2} T^2,
\]
results in Eq. (18). It is natural to suppose a distribution $J(\omega)$, Eq. (20), might be responsible for Eq. (18). Such a distribution could lend support for the action correspondence, Eq. (3), but also corroborate the temperature Eq. (2). The distribution would only characterize the radiation during a long-lived constant power emission phase at sufficiently high speeds $s \sim 1$. Nevertheless, independent of any $J(\omega)$ supposition, the power emission, Eq (16), possesses a plateau consistent with Eq. (17).

**Constant radiation reaction.** Reinstating natural units and having seen the power plateau in $\bar{P}(u)$ originating from $P = \alpha^2/6\pi$, we now turn to the self-force, $F = \alpha'(\tau)/6\pi$ and the associated power which we call ‘Feynman power’ [13], $\bar{F}(u) = F/\alpha' = F/\beta(1-\beta)$, as a function of retarded time $u$,
\[
\bar{F} = \frac{\kappa^2 sW(s-W-1)(2(s+1)W^2+s+W-1)}{6\pi(W+1)^4((s+1)W-s+1)^3}.
\]
Here $\tau$ is the proper time, and we have used advanced coordinate $v = t+r$ and retarded time coordinate $u = t-r$. Taking the same two separate consecutive limits of high speeds and late times, as done for Larmor power in Eq. (17), reveals,
\[
\lim_{u \to \infty} \lim_{s \to 1} \bar{F}(u) = -\frac{\kappa^2}{48\pi} = -\frac{\pi}{12} T^2.
\]

See a plot of the period of constant Feynman power in Figure 2. It, like the Larmor power $\bar{P}$, also exhibits a constant period during which the electron emits particles in thermal equilibrium. Eq. (23) substantiates Eq. (2). 

**Constant local acceleration.** Direct corroboration of an extended period of thermal equilibrium is given by the not-so-well-known object $\bar{k}(u) = \partial_u \ln v'(u)$. This quantity is not without precedent in the literature involving thermal particle radiation. For instance, it has been used as a measure of what is called ‘local acceleration’ by Carlitz-Willey [14]. The result for IB is,
\[
\bar{k}(u) = \frac{2\kappa sW}{(W+1)^2(1+(s+1)W-s)}.
\]
In the limit of high speeds and late times one sees,
\[
\lim_{u \to \infty} \lim_{s \to 1} \bar{k}(u) = \kappa.
\]

The local acceleration, $\bar{k}(u)$, is related to the Lorentz invariant proper acceleration, $\alpha$, via the relations $\kappa = 2\alpha e^\eta$ or via the first derivative of rapidity with respect to retarded time, $\bar{k}(u) = 2\eta'(u)$.

A plot of the local acceleration is given in Figure 3. A quasi-constant local acceleration is in harmony with the equilibrium of a thermal distribution and constant power emission; however, it is important to underscore the fact that a constant local acceleration does not describe uniform proper acceleration of the electron.

**Planck spectrum.** In the moving mirror model (see e.g. [15, 16]), the beta Bogolubov coefficients corroborate radiative equilibrium via an explicit Planck distribution. For IB during beta decay the Planck distribution is explicitly manifest in Eq. (26). Accelerating boundaries radiate soft particles whose long wavelengths lack the capability to probe the internal structure of the source.
Likewise, considering high final speeds and the low frequency approximation, $\omega' \ll \omega$ switches the prime on the $\omega$’s, leading to the (see e.g. [19])

$$|\beta_{\omega'}|^2 = \frac{1}{2\pi \kappa \omega'} \frac{1}{e^{2\pi \omega'/\kappa} - 1}.$$  \hspace{1cm} (31)

demonstrating Planck-factor validity to either frequency approximation.

The spectrum plot of the moving mirror radiation (Figure 4) illustrates the explicit Planck factor which demonstrates the particles, $N(\omega) = \int d\omega' |\beta_{\omega'}|^2$, are distributed with a temperature given in Eq. (2). Thermal emission is not so surprising considering the Larmor power plateau (Figure 1), Feynman power plateau (Figure 2), and acceleration plateau (Figure 3); as well as the close analogy for quantum and classical quantities of powers [4, 5] and self-forces [6, 7] between mirrors and electrons.

**Stefan-Boltzmann law.** - It is natural to consider how the classical power scales according to the (1+1) dimensional Stefan-Boltzmann law [12],

$$P \sim T^2,$$  \hspace{1cm} (32)

rather than the (3+1) Stefan-Boltzmann law,

$$P \sim AT^4,$$  \hspace{1cm} (33)

which governs\(^1\) the power radiated from a black body in terms of its temperature. A first heuristic answer is the classical electron is a point particle with no area.

Ultimately, a more complete and analogous understanding is related to black hole radiance. The scaling likely occurs for the same reason black holes are one-dimensional information channels [2], whose power also scales according to $P \sim T^2$. In the context of Eq. (2), the electron’s constant power peaks at exactly $P = \pi T^2/12 = \kappa^2/48\pi$ which is the well-known all-time constant equilibrium emission of the quantum stress tensor for the eternal thermal Carlitz-Willey moving mirror [20] and the late-time Schwarzschild mirror [21]. Investigation concerning the entropy and information flow related to the quadratic temperature dependence of the electron’s power emission is a worthwhile study but outside the scope of this work.

**Scale dependence.** - The analog between black hole temperature and electron radiation temperature has limitations. Black hole temperature, $T = \hbar c/2\pi k_B\kappa'$, varies dependent on the surface gravity, $\kappa = c^4/4GM$, of the black hole, while electron radiation temperature, $T = \mu c^2\kappa/2\pi k_B$, varies on the acceleration scale, $\kappa = 12\omega_G/\pi$ inherently a function of the UV/IV frequency

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\(^1\)In flat spacetime, this is the relevant contrasting expression for the energy transmission of a single photon polarization out of a closed hot black body surface with temperature $T$ and area $A$ into 3-D space.
width detector sensitivity: $\omega_c = \omega_{\text{max}} - \omega_{\text{min}}$. Hence, because the charge of every electron is the same, the fine structure does not change in this context, and the temperature of the electron’s acceleration radiation is UV/IR dependent, the two expression differ with respect to both intuition and scale. In this context, it is useful to consider the universality of the soft-factor [17] and the thermal character of the infinite zero-energy photons emitted in this regime. Indeed, the thermal here is connected to every scattering process in the deep infrared, at least in the instantaneous collision reference frame. Thus there is an argument for the relevance of Eq. (2) beyond the bremsstrahlung context.

To this end, we point out that Eq. (2) is relevant for Feddeev-Kulish dressed states, where equivalent particle count and energy results [22] suggest one can can derive a ‘cloud temperature’. Analog systems with corresponding results are also subject to thermal character. For instance, ‘mirror temperature’ is a useful assignment in the context of the dynamical Casimir effect [18], as we have directly demonstrated with the spectral computation Eq (26). Moreover, since the internal structure of the source cannot be discerned by long wavelengths, these results can necessarily be extended in analog spacetime final states [23] where ‘black hole temperature’ leading to a left-over remnant becomes a useful characterization of the system. We leave these extensions for future investigations.

**Definition discussion.** Temperature is a collective property and is almost always defined with an assemblage of particles. The usefulness of thermodynamics is particularly salient in the regime with a large numbers of particles (in this case, the large amount of radiated particles are infinite soft thermal photons).

We emphasize that what is meant by ‘the temperature of electron radiation’ is a temperature extracted by averaging the photon energy radiated over many realizations of the same decay experiment with a single asymptotically ultra-relativistic electron. Only in this context, does it makes sense to consider a single electron radiating photons with temperature that scales quadratic to the power.

Here the frequency-distribution is also analogous to the moving mirror particle production which is Planck-distributed. The connection to black hole temperature is limited in the sense that an explicit Planck-distribution has not been derived for classical electron radiation, unlike the moving mirror Planck-distribution which is a result of the beta Bogolubov coefficients originating from the quantum fields in curved space approach. However the connection is explicitly tethered by the power-temperature scaling of the (1+1) dimensional Stefan-Boltzmann law. Importantly, this notion of electron radiation temperature is dynamically useful because it signals a corresponding period of uniform local acceleration.

**Conclusion.** In this note, we have drawn an analogy between black hole temperature and electron radiation temperature, computed periods of constant power and radiation reaction, indicative of thermal equilibrium. Indeed, by analogy with the dynamical Casimir effect, we have demonstrated a useful notion of thermality by symmetry between frequency modes in a proposed analog spectrum for the radiation of an accelerated electron, which at ultra-relativistic speeds manifests an explicit Planck distribution. The temperature is consistent with the constant periods of power, self-force and acceleration.

**Acknowledgements.** Thanks is given to Ernazar Abdikamalov, Eric Linder, Morgan Lynch, and Daniele Malafarina for insightful discussion. Funding comes in part from the FY2021-SGP-1-STMM Faculty Development Competitive Research Grant No. 021220FD3951 at Nazarbayev University.

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