Rethinking Parameter Counting in Deep Models: Effective Dimensionality Revisited

Wesley J. Maddox*  Gregory Benton*  Andrew Gordon Wilson
New York University

Abstract

Neural networks appear to have mysterious generalization properties when using parameter counting as a proxy for complexity. Indeed, neural networks often have many more parameters than there are data points, yet still provide good generalization performance. Moreover, when we measure generalization as a function of parameters, we see double descent behaviour, where the test error decreases, increases, and then again decreases. We show that many of these properties become understandable when viewed through the lens of effective dimensionality, which measures the dimensionality of the parameter space determined by the data. We relate effective dimensionality to posterior contraction in Bayesian deep learning, model selection, double descent, and functional diversity in loss surfaces, leading to a richer understanding of the interplay between parameters and functions in deep models.

1. Introduction

Parameter counting is often used as a proxy for model complexity to reason about generalization (e.g., Zhang et al., 2017; Shazeer et al., 2017; Belkin et al., 2019a), but it can be a poor description of both model flexibility and inductive biases. One can easily construct degenerate cases, such as predictions being generated by a sum of parameters, where the number of parameters is divorced from the statistical properties of the model. When reasoning about generalization, overparametrization is besides the point: what matters is how the parameters combine with the functional form of the model.

Indeed, the practical success of convolutional neural networks (CNNs) for image recognition tasks is almost entirely about the inductive biases of convolutional filters, depth, and sparsity, for extracting local similarities and hierarchical representations, rather than flexibility (LeCun et al., 1989; Szegedy et al., 2015). Convolutional neural networks have far fewer parameters than fully connected networks, yet can provide much better generalization. Moreover, width can provide flexibility, but it is depth that has made neural networks distinctive in their generalization abilities.

In this paper, we argue that we should move beyond simple parameter counting, and show how the generalization properties of neural networks become interpretable through the lens of effective dimensionality (MacKay, 1992a). Effective dimensionality was originally proposed to measure how many directions in parameter space had been determined in a Bayesian neural network, by computing the eigenspectrum of the Hessian on the training loss (Eq. (3), Section 2). We provide explicit connections between effective dimensionality, posterior contraction, model selection, loss surfaces, and generalization behaviour in modern deep learning.

Consider Figure 1, where we see that once a model has achieved low training loss, the effective dimensionality, computed from training data alone, closely tracks the mysterious double descent behaviour for neural networks. Indeed,
models with increasing width actually have lower effective dimensionality, and better generalization. Such models can be viewed as providing a better compression of the data, despite having more parameters. Alternatively, consider Figure 2, where we see that width and depth determine effective dimensionality in different ways, though both are related to numbers of parameters. Remarkably, for models with low training loss (above the green partition), the effective dimensionality closely tracks generalization performance. Low training loss and high effective dimensionality suggest that subspace and ensembling methods could be improved through the avoidance of expensive computations within degenerate parameter regimes.

• In Section 6, we show that effective dimensionality provides a compelling mechanism for model selection, resolving generalization behaviour in deep learning that appears mysterious from the perspective of simple parameter counting. Given two models with the same training error, the one with lower effective dimensionality, but not necessarily fewer parameters, should be preferred.

We make code available at https://github.com/g-benton/hessian-eff-dim.

2. Posterior Contraction, Effective Dimensionality, and the Hessian

We consider a model, typically a neural network, \( f(x; \theta) \), with inputs \( x \) and parameters \( \theta \in \mathbb{R}^k \). We define the Hessian as the \( k \times k \) matrix of second derivatives of the loss, \( \mathcal{H}_\theta = -\nabla \nabla \theta \mathcal{L}(\theta, \mathcal{D}) \). Often the loss used to train a model by optimization is taken to be the negative log posterior \( \mathcal{L} = -\log p(\theta | \mathcal{D}) \).

To begin, we describe posterior contraction, effective dimensionality, and connections to the Hessian.

2.1. Posterior Contraction

Definition 2.1. We define posterior contraction of a set of parameters, \( \theta \), as the difference in the trace of prior and posterior covariance.

\[
\Delta_{\text{post}}(\theta) := \text{tr}(\text{Cov}_{p(\theta)}(\theta)) - \text{tr}(\text{Cov}_{p(\theta|\mathcal{D})}(\theta)),
\] (1)
where \(p(\theta)\) is the prior distribution and \(p(\theta|D)\) is the posterior distribution given data, \(D\).

With increases in data the posterior distribution of parameters becomes increasingly concentrated around a single value (e.g., van der Vaart, 1998, Chapter 10). Therefore Eq. (1) serves to measure the increase in certainty about the parameters under the posterior as compared to the prior.

2.2. Parameter Space and Function Space

When combined with the functional form of a model, a distribution over parameters \(p(\theta)\) induces a distribution over functions \(p(f(x; \theta))\). The parameters are of little direct interest — what matters for generalization is the distribution over functions. Figure 3 provides both parameter- and function-space viewpoints. As parameter distributions concentrate around specific values, we expect to generate less diverse functions.

We show in Appendix C that the posterior contraction for Bayesian linear regression, \(y \sim \mathcal{N}(f = \Phi^T \beta, \sigma^2 I)\), with isotropic Gaussian prior, \(\beta \sim \mathcal{N}(0, \alpha^2 I_N)\), is given by

\[
\Delta_{\text{post}}(\theta) = \alpha^2 \sum_{i=1}^{N} \frac{\lambda_i}{\lambda_i + \alpha^{-2}},
\]

where \(\lambda_i\) are the eigenvalues of \(\Phi^T \Phi\). This quantity is distinct from the posterior contraction in function space (also shown in Appendix C). We refer to the summation in Eq. (2) as the effective dimensionality of \(\Phi^T \Phi\).

2.3. Effective Dimensionality

Definition 2.2. The effective dimensionality of a symmetric matrix \(A \in \mathbb{R}^{k \times k}\) is defined as

\[
N_{\text{eff}}(A, z) = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_i + z},
\]

in which \(\lambda_i\) are the eigenvalues of \(A\) and \(z > 0\) is a regularization constant (MacKay, 1992a).

Typically as neural networks are trained we observe a gap in the eigenspectrum of the Hessian of the loss (Sagun et al., 2017); a small number of eigenvalues become large while the rest take on values near 0. In this definition of effective dimensionality, eigenvalues much larger than \(z\) contribute a value of approximately 1 to the summation, and eigenvalues much smaller than \(z\) contribute a value of approximately 0.

2.4. The Hessian and the Posterior Distribution

We provide a simple example involving posterior contraction, effective dimensionality, and their connections to the Hessian. Figure 3 shows the prior and posterior distribution for a Bayesian linear regression model with a single parameter, with predictions generated by parameters drawn from these distributions. As expected from Sections 2.1 and 2.3, we see that the variance of the posterior distribution is significantly reduced from that of the prior — what we refer to here as posterior contraction.

We can see from Figure 3 that the arrival of data increases the curvature of the loss (negative log posterior) at the optimum. This increase in curvature of the loss that accompanies certainty about the parameters leads to an increase in the eigenvalues of the Hessian of the loss in the multivariate case. Thus, growth in eigenvalues of the Hessian of the loss corresponds to increased certainty about parameters, leading to the use of the effective dimensionality of the Hessian of the loss as a proxy for the number of parameters that have been determined.\(^1\)

We often desire models that are both consistent with data, but as simple as possible in function space, embodying Occam’s razor and avoiding overfitting. The effective dimensionality explains the number of parameters that have been determined by the data, which corresponds to the number of parameters the model is using to make predictions. Therefore in comparing models of the same parameterization that achieve low loss on the training data, we expect models with lower effective dimensionality to generalize better — which is empirically verified in Figures 1 and 2.

We can further connect the Hessian and the posterior distribution by considering a Laplace approximation as in MacKay (1992a;b). Here we assume that the distribution of parameters \(\theta\) is multivariate normal around the maximum a posteriori (MAP) estimate, \(\theta_{\text{MAP}} = \arg\max_{\theta} p(\theta|D)\), and the Hessian of the negative log posterior, \(\mathcal{H}_\theta + A\),\(^2\) serves as the precision matrix. The approximating distribution is then \(\mathcal{N}(\theta_{\text{MAP}}, (\mathcal{H}_\theta + A)^{-1})\). The intuition built using Figure 3 carries through to this approximation: as the eigenvalues of the Hessian increase, the eigenvalues of the covariance matrix in our approximation to the posterior distribution shrink.

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\(^1\)Empirically described in Appendix A.

\(^2\)\(A = -\nabla\nabla_\theta \log p(\theta)\) is the Hessian of the log prior.
further indicating contraction around the MAP estimate. We demonstrate this property algebraically in Appendix B, where we also connect the effective dimensionality to the bias-variance tradeoff (Dobriban & Wager, 2018) and to the Hilbert space norm (Rasmussen & Williams, 2008).

2.5. Practical Computations And Parameterizations

For deep and wide neural networks the Hessian of the loss is large, and thus computing the eigenvalues and eigenvectors is nontrivial. We employ an efficient implementation of the Lanczos method for determining eigenvalues provided in GPyTorch, allowing for the rapid computation of approximate eigenvalue, eigenvector pairs (Gardner et al., 2018). In practice, we estimate effective dimensionality by selectively computing the leading eigenvalues, since many of the eigenvalues are typically close to zero and do not significantly contribute to the estimate.

In general, the Hessian of the loss (and its effective dimension) is not invariant to re-parameterizations (e.g. ReLU rescaling and batch normalization) (MacKay, 2003, Chapter 27). For this reason we assume a fixed parameterization, as is typically the case in practice, and compare only between models of the same parameterization.

3. Related Work

Cleveland (1979) introduced effective dimensionality into the splines literature as a measure of goodness of fit, while Hastie & Tibshirani (1990, Chapter 3) used it to assess generalized additive models. Gull (1989) first applied effective dimensionality in a Bayesian setting for an image reconstruction task, while MacKay (1992ab) used it to compute posterior contraction in Bayesian neural networks. Moody (1992) argued for the usage of the effective dimensionality as a proxy for generalization error, while Moody (1991) suggested that effective dimensionality could be used for neural network architecture selection. Zhang (2005) and Caponnetto & Vito (2007) studied the generalization abilities of kernel methods in terms of the effective dimensionality.

Friedman et al. (2001, Chapter 7) use the effective dimensionality (calling it the effective degrees of freedom) to compute the expected generalization gap for regularized linear models. Dobriban & Wager (2018) specifically tied the bias variance decomposition of predictive risk in ridge regression (e.g. the finite sample predictive risk under Gaussian priors) to the effective dimensionality of the feature matrix, $\Phi^T \Phi$. Hastie et al. (2019), Muthukumar et al. (2019), Bartlett et al. (2019), Mei & Montanari (2019), and Belkin et al. (2019b) studied risk and generalization in over-parameterized linear models, including under model misspecification. Bartlett et al. (2019) also introduced the concept of effective rank of the feature matrix, which has a similar interpretation to effective dimensionality.

Sagun et al. (2017) found that the eigenvalues of the Hessian increase through training, while Papan et al. (2018) and Ghorbani et al. (2019) studied the eigenvalues of the Hessian for a range of modern neural networks. Suzuki (2018) produced generalization bounds on neural networks via the effective dimensionality of the covariance of the functions at each hidden layer. Fukumizu et al. (2019) embedded narrow neural networks into wider neural networks and studied the flatness of the resulting minima in terms of their Hessian via a PAC-Bayesian approach. Achille & Soatto (2018) argue that flat minima have low information content (many small magnitude eigenvalues of the Hessian) by connecting PAC-Bayesian approaches to information theoretic arguments, before demonstrating that low information functions learn invariant representations of the data. Dziugaite & Roy (2017) optimize a PAC Bayesian bound to both encourage flatness and to compute non-vacuous generalization bounds, while Jiang et al. (2019) recently found that PAC Bayesian measures of flatness, in the sense of insensitivity to random perturbations, perform well relative to other generalization bounds.

Moreover, MacKay (2003) and Smith & Le (2017) provide an Occam factor perspective linking flatness and generalization. Related minimum description length perspectives can be found in MacKay (2003) and Hinton & Van Camp (1993). Other works also link flatness and generalization (e.g., Hochreiter & Schmidhuber, 1997; Keskar et al., 2017; Chaudhari et al., 2019; Izmailov et al., 2018), with Izmailov et al. (2018) and Chaudhari et al. (2019) developing optimization procedures to select for flat regions of the loss.

To the best of our knowledge, Opper et al. (1989) and Bös et al. (1993) introduced the idea that generalization capability can be non-monotonic for neural networks in terms of the number of parameters (e.g. the double descent curve) while Belkin et al. (2019a) re-introduced the idea into the modern machine learning community by demonstrating its existence on a wide variety of machine learning problems. Nakkiran et al. (2019) found generalization gains as neural networks become highly overparameterized, showing the double descent phenomenon that occurs as the width parameter of both residual and convolutional neural networks is increased.

4. Posterior Contraction and Function-Space Homogeneity in Bayesian Models

In this section, we demonstrate that effective dimensionality of both the posterior parameter covariance and the Hessian of the loss provides insights into how a model adapts to data during training. We derive an analytic relationship between effective dimensionality and posterior contraction.
for models where inference is exact, and demonstrate this relationship experimentally for deep neural networks.

4.1. Posterior Contraction of Bayesian Linear Models

**Theorem 4.1** (Posterior Contraction in Bayesian Linear Models). Let \( \Phi = \Phi(x) \in \mathbb{R}^{n \times k} \) be a feature map of \( n \) data observations, \( x \), with \( n < k \) and assign isotropic prior \( \beta \sim \mathcal{N}(0, \alpha^2 I_k) \) for parameters \( \beta \in \mathbb{R}^k \). Assuming a model of the form \( y \sim \mathcal{N}(\Phi \beta, \sigma^2 I_n) \) the posterior distribution of \( \beta \) has a \( k - n \) directional subspace in which the variance is identical to the prior variance.

We prove Theorem 4.1 in Appendix D.1, in addition to an equivalent result for generalized linear models. Theorem 4.1 demonstrates why parameter counting often makes little sense: for a fixed data set of size \( n \), only \( \min(n, k) \) parameters can be determined, leaving many dimensions in which the posterior is unchanged from the prior when \( k \gg n \).

**Empirical Demonstration for Theorem 4.1.** We construct \( \Phi(x) \) with each row as an instance of a 200 dimensional feature vector consisting of sinusoidal terms for each of 500 observations: \( \Phi(x) = [\cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \ldots] \). We assign the coefficient vector \( \beta \) a prior \( \beta \sim \mathcal{N}(0, I) \), and draw ground truth parameters \( \beta^* \) from this distribution. The model takes the form \( \beta \sim \mathcal{N}(0, I) \) and \( y \sim \mathcal{N}(\Phi \beta, \sigma^2 I) \).

We randomly add data points one at a time, tracking the posterior covariance matrix at each step. We compute the effective dimensionality, \( N_{\text{eff}} \) (\( \Sigma_{\beta|D, \sigma}, \alpha \)), where \( \Sigma_{\beta|D, \sigma} \) is the posterior covariance of \( \beta \).

In Figure 4 we see that the effective dimensionality of the posterior covariance decreases linearly with an increase in available data until the model becomes overparameterized, at which point the effective dimensionality of the posterior covariance of the parameters slowly approaches 0, while the effective dimensionality of the Hessian of the loss increases towards an asymptotic limit. As the parameters become more determined (e.g. the effective dimensionality of the posterior covariance decreases), the curvature of the loss increases (the effective of the Hessian increases). In the Bayesian linear model setting, the Hessian of the loss is the inverse covariance matrix and the trade-off between the effective dimensionality of the Hessian and the parameter covariance can be determined algebraically (see Appendix B.1).

![Figure 4. Left: Bayesian linear regression. Right: Bayesian neural network. Both: The effective dimensionality of the posterior covariance over parameters and the function-space posterior covariance. Red indicates the under-parameterized setting, yellow the critical regime with \( p \approx n \), and green the over-parameterized regime. In both models we see the expected increase in effective dimensionality in parameter space and decrease in effective dimensionality of the Hessian.](https://github.com/pyro-ppl/numpyro/blob/master/examples/bnn.py)

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1Here we use \( \alpha = 5 \), however the results remain qualitatively the same as this parameter changes.

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4.2. Posterior Contraction of Bayesian Neural Networks

While much effort has been spent grappling with the challenges of marginalizing a high dimensional parameter space for Bayesian neural networks, the practical existence of subspaces where the posterior variance has not collapsed from the prior suggests that both computational and approximation gains can be made from ignoring directions in which the posterior variance is unchanged from the prior. This observation helps explain the success of subspace based techniques that examine the loss in a lower dimensional space such as Izmailov et al. (2019). Alternatively, by working directly in function space, as in Sun et al. (2019), the redundancy of many parameters could be avoided.

For Bayesian linear models, the effective dimensionality of the parameter covariance is the inverse of the Hessian, and as the effective dimensionality of the parameter covariance decreases the effective dimensionality of the Hessian increases. We hypothesize that a similar statement holds for Bayesian neural networks — as the number of data points grows, the effective dimensionality of the posterior covariance should decrease while the effective dimensionality of the Hessian should increase.

To test this hypothesis, we generate a nonlinear function of the form, \( y = w_1 x + w_2 x^2 + w_3 x^3 + (0.5 + x^2)^2 + \sin(4x^2) + \epsilon \), with \( w_i \sim \mathcal{N}(0, I) \) and \( \epsilon \sim \mathcal{N}(0, 0.05^2) \), and de-mean and standardize the inputs. We then construct a Bayesian neural network with two hidden layers each with 20 units, no biases, and \( \tanh \) activations, placing independent Gaussian
5. Loss Surfaces and Function Space Representations

Recent works have discussed the desirability of finding solutions corresponding to flat optima in the loss surface, arguing that such parameter settings lead to better generalization (Izmailov et al., 2018; Keskar et al., 2017). There are multiple notions of flatness in loss surfaces, relating to both the volume of the basin in which the solution resides and the rate of increase in loss as one moves away from the found solution. As both definitions correspond to low curvature in the loss surface, it is standard to use the Hessian of the loss to examine structure in the loss surface (Madras et al., 2019; Keskar et al., 2017).

The effective dimensionality of the Hessian of the loss indicates the number of parameters that have been determined by the data. In highly over-parameterized models we hypothesize that the effective dimensionality is substantially less than the number of parameters, i.e., \( N_{\text{eff}}(\mathcal{H}, \alpha) \ll p \), since we should be unable to determine many more parameters than we have data observations.

Recall from Section 2.3 the large eigenvalues of the Hessian have eigenvectors corresponding to directions in which parameters are determined. Eq. (3) dictates that low effective dimensionality (in comparison to the total number of parameters) would imply that there are many directions in which parameters are not determined, and the Hessian has eigenvalues that are near zero, meaning that in many directions the loss surface is constant. We refer to directions in parameter space that have not been determined as degenerate for two reasons: (1) degenerate directions in parameter space provide minimal structure in the loss surface, shown in Section 5.1; (2) parameter perturbations in degenerate
directions do not provide diversity in the function-space representation of the model, shown in Section 5.2. We refer to the directions in which parameters have been determined, directions of high curvature, as determined.

To empirically test our hypotheses regarding degenerate directions in loss surfaces and function space diversity, we train a neural network classifier on 1000 points generated from the two-dimensional Swiss roll data, with a similar setup to Huang et al. (2019), using Adam with a learning rate of 0.01 (Kingma & Ba, 2015). The network is fully connected, consisting of 5 hidden layers each 20 units wide (plus a bias term), and uses ELU activations with a total of 2181 parameters. We choose a small model with two-dimensional inputs so that we can both tractably compute all the eigenvectors and eigenvalues of the Hessian and visualize the functional form of the model. To demonstrate the breadth of these results, we provide comparable visualizations in the Appendix E, but for a convolutional network trained on CIFAR-10.

5.1. Loss Surfaces as Determined by the Hessian

To examine the loss surface more closely, we visualize low dimensional projections. To create the visualizations, we first define a basis given by a set of vectors, then choose a two random vectors, \( u \) and \( \tilde{v} \), within the span of the basis. We use Gram-Schmidt to orthogonalize \( \tilde{v} \) with respect to \( u \), ultimately giving \( u \) and \( v \) with \( u \perp v \). We then compute the loss at parameter settings \( \theta \) on a grid surrounding the optimal parameter set, \( \theta^* \), which are given by

\[
\theta \leftarrow \theta^* + \alpha u + \beta v
\]

for various \( \alpha \) and \( \beta \) values such that all points on the grid are evaluated.

By selecting the basis in which \( u \) and \( v \) are defined we can specifically examine the loss in determined and degenerate directions. Figure 6 shows that in determined directions, the optimum appears extremely sharp. Conversely, in all but the most determined directions, the loss surface loses all structure and appears constant. Even in degenerate directions, if we deviate from the optimum far enough the loss will eventually become large. However to observe this increase in loss requires perturbations to the parameters that are significantly larger in norm than \( \theta^* \).

5.2. Degenerate Parameters Lead to Homogeneous Models

In this section we show that degenerate parameter directions do not contain diverse models. This result is not at odds with the notion that flat regions in the loss surface can lead to diverse but high performing models. Rather, we find that there is a subspace in which the loss is constant and one cannot find model diversity, noting that this subspace

\[
\theta \leftarrow \theta^* + s \frac{B v}{||B v||_2}
\]

where \( B \in \mathbb{R}^{k \times d} \) is a \( d \)-dimensional basis in which we wish to perturb \( \theta^* \), and \( v \sim \mathcal{N}(0, I_d) \), giving \( B v \) as a random vector from within the span of some specified basis (i.e. the dominant or minimal eigenvectors). The value \( s \) is chosen to determine the scale of the perturbation, i.e. the length of the random vector by which the parameters are perturbed.

Experimentally, we find that in a region near the optimal parameters \( \theta^* \), i.e. \( s \leq ||\theta^*||_2/2 \) the function-space diversity of the model is contained within the subspace of determined directions. While the degenerate directions contain wide ranges of parameter settings with low loss, the models are equivalent in function space.

Figure 7 shows the trained classifier and the differences in function-space between the trained classifier and those generated from parameter perturbations. We compare perturbations of size \( ||\theta^*||_2/2 \approx 10 \) in the direction of the 500 minimal eigenvectors and perturbations of size 0.1 in the directions of the 3 maximum eigenvectors. A perturbation from the trained parameters in the directions of low curvature (center plot in Figure 7) still leads to a classifier that labels all points identically. A perturbation roughly 100 times smaller the size in directions in which parameters have been determined leads to a substantial change in the decision boundary of the classifier.

However, the change in the decision boundary resulting from perturbations in determined directions is not necessarily desirable. One need not perturb parameters in either determined or degenerate directions to perform a downstream task such as ensembling. Here, we are showcasing
Figure 7. Swiss roll data. Left: Adam trained feed-forward, fully connected classifier. Center: Differences in original and perturbed classifier when parameters are perturbed by in low curvature, degenerate directions. Right: Differences in the original and perturbed classifier when parameters are perturbed in high curvature directions. Note the perturbation in the center plot is approximately 100 times the size of that of the plot on the right.

the degeneracy of the subspace of parameter directions that have not been determined by the data. This result highlights that despite having many parameters the network could be described by relatively low dimensional subspace.

6. Double Descent and Effective Dimensionality

In practice we typically desire parsimony in a model — a model that is as simple as possible while still fitting the training data. By the same token we ought to desire a model with low effective dimensionality. Low effective dimensionality indicates that the model is making full use of a smaller number of parameters, yielding better compression of the data, and thus likely better generalization. The phenomenon of double descent of the generalization performance in both linear and deep models has attracted recent attention (Nakkiran et al., 2019; Belkin et al., 2019a); here, we consider the relationship of double descent to the effective dimensionality of the model.

We find that for models in which the training loss converges to near zero, the effective dimensionality corresponds remarkably well to generalization performance, despite having been determined only from training data. For models that are only just able to interpolate the training data, but generalize poorly due to overfitting, the effective dimension is high. In these cases high effective dimensionality is due the sensitivity of the fit to the precise settings of the parameters. As the model changes and grows, there exist a greater variety of subspaces which provide more effective compressions of the data, and thus we achieve a lower effective dimensionality. We demonstrate the correspondence of effective dimensionality to generalization performance in the regime with near-zero training loss for both linear models and deep neural networks.

Figure 8. Demonstration of double descent for linear models with an increasing number of features. We plot the effective dimensionality of the Hessian of the loss. In the regime with near-zero train error, the test error is almost entirely explained by the effective dimensionality, which is computed on the train set alone. The red region corresponds to underparameterized models, yellow to critically parameterized models, and green to overparameterized models.

6.1. Double Descent on Linear Models

Although double descent is often associated with neural networks, we here demonstrate similar behaviour with a linear model with a varying number of features: first drawing 200 data points $y \sim \mathcal{N}(0, 1)$ and then drawing 20 informative features $y + \epsilon$, where $\epsilon \sim \mathcal{N}(0, 1)$, before drawing $k - 20$ features that are also just random Gaussian noise, where $k$ is the total number of features in the model.5 For the test set, we repeat the generative process. In Figure 8 we show a pronounced double descent curve in the test error as we increase the number of features, which is mirrored by the effective dimensionality.

6.2. Double Descent for Deep Models

We finally demonstrate and explain double descent as a function of the effective dimensionality for deep neural networks. Nakkiran et al. (2019) demonstrated that double descent can also occur for modern deep neural networks including transformers, CNNs, ResNets. Following their experiments, we train ResNet18s (He et al., 2016) with varying width parameters, reproducing the double descent curve shown by Nakkiran et al. (2019).6 We compute the effective dimensionality of the model using 100 eigenvalues as calculated from Lanczos (Gardner et al., 2018). In Figure 1; we can see that the effective dimensionality tracks remarkably well with generalization — displaying the double descent curve that is seen in the test error. We emphasize again that the effective dimensionality is computed using solely the training data, supporting the hypothesis that the eigenvalues of the Hessian matrix can provide a good proxy for generalization performance. In Appendix F, we test small neural networks on a problem for which we can compute all of the eigenvalues replicating a similar finding to Figure 1 as

5From https://github.com/ORIE4741/demos/blob/master/double-descent.ipynb.
6See Appendix G for training details.
width increases.

### 6.3. Networks of Varying Width and Depth

Double descent experiments typically only consider increases in width. However, it is depth which has endowed neural networks with distinctive generalization properties.

In Figure 2, we consider varying both the width and depth of a convolutional neural network on the CIFAR-100 dataset. We measure effective dimensionality, training loss, and testing loss. The yellow curves show networks with a constant number of parameters, indicating the simple parameter counting is not a good proxy for generalization. However, in the region of near-zero training loss, separated by the green curve, we see effective dimensionality closely matches generalization performance. Moreover, wide but shallow models tend to overfit, providing low training loss, but high effective dimensionality and test loss. On the other hand, deeper models have lower test loss and lower effective dimensionality, showing that depth enables a better compression of the data.

### 7. Conclusion

We have shown how effective dimensionality can be used to gain insight into a range of phenomena, including double descent, posterior contraction, loss surface structure, and function-space diversity of models. As we have seen, simple parameter counting can be a misleading proxy for model complexity and generalization performance; models with many parameters combined with a particular functional form can give rise to simple explanations for data. Indeed, we have seen how depth and width have different effects on generalization performance, regardless of the total number of parameters in the model. In all cases, effective dimensionality tracks generalization performance for models with comparable training loss, helping to explain behaviour that appears mysterious when measured against simple parameter counting. Moving forward, we hope our work will help inspire a continued effort to capture the nuanced interplay between the statistical properties of parameter space and function space in understanding generalization behaviour.

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A. The Hessian and Effective Dimensionality over the Course of Training

One possible limitation of using the Hessian as a measurement for posterior contraction for (Bayesian) deep learning
B. Further Statements on Effective Dimensionality

In this section, we provide further results the effective dimensionality, including its connection to both the bias-variance decomposition of predictive risk (Geman et al., 1992; Dobriban & Wager, 2018) as well as the Hilbert space norm of the induced kernel (Rasmussen & Williams, 2008).

B.1. Effective Dimensionality of the Inverse of A

We show that

$$\text{rank}(A) - N_{eff}(A, \alpha) = N_{eff}(A^+, 1/\alpha), \quad (6)$$

formalizing the idea that as the effective dimensionality of the covariance increases, the effective dimensionality of the inverse covariance decreases. This statement is alluded to in the analysis of MacKay (1992b) but is not explicitly shown.

We assume that $A$ has rank $r$ and that $\alpha \neq 0$; we also assume that $A^+$ is formed by inverting the non-zero eigenvalues of $A$ and leaving the zero eigenvalues fixed in the eigendecomposition of $A$ (i.e. the Moore-Penrose pseudo-inverse).

With $\lambda_i$ as the eigenvalues of $A$, we can see that

$$r - N_{eff}(A, \alpha) = \sum_{i=1}^{r} \frac{\lambda_i + \alpha - \lambda_i}{\lambda_i + \alpha} = \alpha \sum_{i=1}^{r} \frac{1}{\lambda_i + \alpha} = \sum_{i=1}^{r} \frac{1}{\alpha \lambda_i + \alpha} = \sum_{i=1}^{r} \frac{1}{\lambda_i + 1} = \sum_{i=1}^{r} \frac{1}{1 + \lambda_i/(\lambda_i + 1)} = N_{eff}(A^+, 1/\alpha).$$

When $A$ is invertible, the result reduces to $k - N_{eff}(A, \alpha) = N_{eff}(A^{-1}, 1/\alpha)$ for $A \in \mathbb{R}^{k \times k}$.

B.2. Predictive Risk for Bayesian Linear Models

Dobriban & Wager (2018) and Hastie et al. (2019) have extensively studied over-parameterized ridge regression. In particular, Theorem 2.1 of Dobriban & Wager (2018) gives the predictive risk (e.g. the bias-variance decomposition of Geman et al. (1992)) as a function of effective dimensionality and intrinsic noise. The critical aspect of their proof is to decompose the variance of the estimate into the effective dimensionality and a second term which then cancels with the limiting bias estimate. For completeness, we restate Theorem 2.1 of Dobriban & Wager (2018) theorem for fixed feature matrices, $\Phi$, and an explicit prior on the parameters, $\beta \sim \mathcal{N}(0, \alpha^2 I)$, leaving the proof to the original work.

**Theorem B.1** (Predictive Risk of Predictive Mean for Ridge Regression). Under the assumption of model correct specification, $y = \Phi \beta + \epsilon$, with $\beta$ drawn from the prior and $\epsilon \sim \mathcal{N}(0, I_n)$, and defining $\hat{f} = \Phi \hat{\beta}$, with $\beta = (\Phi^T \Phi + \alpha^{-2} I)^{-1} \Phi^T y$ (the predictive mean under the prior specification), then

$$R(\Phi) = \mathbb{E}(||Y - \hat{f}||_2^2) = 1 + \frac{1}{n} N_{eff}(\Phi \Phi^T, \alpha^{-2}). \quad (7)$$

B.3. Expected RKHS Norm

Finally, we show another unexpected connection of the effective dimensionality — that the reproducing kernel Hilbert space (RKHS) norm is in expectation, under model correct specification, the effective dimensionality. We follow the definition of Gaussian processes of Rasmussen & Williams (2008) and focus on the definition of the RKHS given in Rasmussen & Williams (2008, Chapter 6), which is defined as

$$||f||_2^2 = \langle f, f \rangle_H = \sum_{i=1}^{N} f_i^2 / \lambda_i,$$

where $\lambda_i$ are the eigenvalues associated with the kernel operator, $K$, of the RKHS, $\mathcal{H}$. The kernel is the covariance matrix of the Gaussian process, and assuming that the response is drawn from the same
model, then \( y \sim \mathcal{N}(0, K + \sigma^2 I) \), then \( a = (K + \sigma^2 I)^{-1}y \), where \( a \) is the optimal weights of the function with respect to the kernel, e.g. \( f = \sum_{i=1}^{N} a_i K(x_i, \cdot) \). To compute the Hilbert space norm, we only need to compute the optimal weights and the eigenvalues of the operator. For finite (degenerate) Hilbert spaces this computation is straightforward:

\[
\mathbb{E}_{p(y)}(\|f\|_2^2) = \mathbb{E}_{p(y)}(a^\top K a) = \mathbb{E}_{p(y)}(y^\top (K + \sigma^2 I)^{-1} K (K + \sigma^2 I)^{-1} y) = \mathbb{E}_{p(y)}(y^\top (K + \sigma^2 I)^{-1} y) = \text{tr}((K + \sigma^2 I)^{-1} K (K + \sigma^2 I)^{-1}) = \text{tr}((K + \sigma^2 I)^{-1} K) = N_{\text{eff}}(K, \sigma^2)
\]

with the second equality coming by plugging in the optimal \( a \) (see Rasmussen & Williams (2008, Chapter 6) and Belkin et al. (2019a) as an example). As linear models with Gaussian priors are Gaussian processes with a degenerate feature expansion, the expected RKHS norm becomes \( N_{\text{eff}}(\Phi^\top \Phi, \sigma^2/\alpha^2) \), which is the same value as our definition of posterior contraction. Further research connecting these two ideas is needed.

\section*{C. Measuring Posterior Contraction in Bayesian Generalized Linear Models}

We first consider the over-parametrized case, \( k > n \):

\[
\Delta_{\text{post}}(\theta) = \text{tr}(\text{Cov}_{p(\theta)}(\theta)) - \text{tr}(\text{Cov}_{p(\theta|D)}(\theta)) = \sum_{i=1}^{k} \alpha^2 - \sum_{i=1}^{n} (\lambda_i + \alpha^{-2})^{-1} + \sum_{i=n+1}^{k} \alpha^2 = k\alpha^2 - (k-n)\alpha^2 - \sum_{i=1}^{n} (\lambda_i + \alpha^{-2})^{-1} = \sum_{i=1}^{n} \frac{1 - \alpha^2(\lambda_i + \alpha^{-2})}{\lambda_i + \alpha^{-2}} = \alpha^2 \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \alpha^{-2}} ;
\]

where we have used Theorem 4.1 to assess the eigenvalues of the posterior covariance. When \( n > k \), we have the simplified setting where the summation becomes to \( k \) instead of \( n \), giving us that all of the eigenvalues are shifted from their original values to become \( \lambda_i + \alpha^{-2} \), and so

\[
\Delta_{\text{post}}(\theta) = \alpha^2 \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_i + \alpha^{-2}} ;
\]

where \( \lambda_i \) is the \( i \)th eigenvalue of \( \Phi^\top \Phi/\sigma^2 \).

\subsection*{C.1. Contraction in Function Space}

We can additionally consider the posterior contraction in function space. For linear models, the posterior covariance on the training data in function space becomes

\[
\Phi \Sigma_{\beta|\mathcal{D}} \Phi^\top = \sigma^2 \Phi (\Phi^\top \Phi + \frac{\sigma^2}{\alpha^2} I_p)^{-1} \Phi^\top ,
\]

while the prior covariance in function space is given by \( \alpha^2 \Phi \Phi^\top \). We will make the simplifying assumption that the features are normalized such that \( \text{tr}(\Phi \Phi^\top) = \text{rank}(\Phi \Phi^\top) = r \). Now, we can simplify

\[
\Delta_{\text{post}}(f) = \text{tr}(\text{Cov}_{p(f)}(f)) - \text{tr}(\text{Cov}_{p(f|\mathcal{D})(f)}) = \alpha^2 r - \sum_{i=1}^{r} \frac{\lambda_i}{\lambda_i + \sigma^2/\alpha^2} = \alpha^2 \sum_{i=1}^{r} \frac{\lambda_i}{\lambda_i + \sigma^2/\alpha^2} - \sum_{i=1}^{r} \frac{\lambda_i}{\lambda_i + \sigma^2/\alpha^2} = (\alpha^2 - \sigma^2) \sum_{i=1}^{r} \frac{\lambda_i}{\lambda_i + \sigma^2/\alpha^2} + \sigma^2 \sum_{i=1}^{r} \frac{1}{\lambda_i + \sigma^2/\alpha^2}.
\]

Simplifying and recognizing these summations as the effective dimensionalities of \( \Phi \Phi^\top \) and \( (\Phi^\top \Phi)^+ \), we get that

\[
\Delta_{\text{post}}(f) = (\alpha^2 - \sigma^2) N_{\text{eff}}(\Phi^\top \Phi, \sigma^2/\alpha^2) + \sigma^2 N_{\text{eff}}((\Phi^\top \Phi)^+, \alpha^2/\sigma^2) \]

thereby showing that the posterior contraction in function space is explicitly tied to the effective dimensionality of the Gram matrix.

\section*{D. Posterior Contraction and Function-Space Homogeneity Proofs and Additional Theorems}

In this section we complete the proofs to Theorems 4.1 and 4.2 and extend the results from linear models to generalized linear models.

\subsection*{D.1. Proof and Extensions to Theorem 4.1}

\textbf{Theorem (Posterior Contraction in Bayesian Linear Models).} Let \( \Phi = \Phi(x) \in \mathbb{R}^{n \times k} \) be a feature map of \( n \) data observations, \( x \), with \( n < k \) and assign isotropic prior \( \beta \sim N(0, \Sigma_0 = \alpha^2 I_k) \) for parameters \( \beta \in \mathbb{R}^k \). Assuming a model of the form \( y \sim N(\Phi \beta, \sigma^2 I_n) \) the posterior distribution of \( \beta \) has an \( p - k \) directional subspace in which the variance is identical to the prior variance.

\textbf{Proof.} The posterior distribution of \( \beta \) in this case is known
and given as
\[
\beta|D \sim \mathcal{N}(\mu|D), (\Sigma|D))
\]
\[
\mu|D = (\Phi^\top \Phi/\sigma^2 + S_0^{-1})^{-1} \Phi^\top y/\sigma^2
\]
\[
\Sigma|D = (\Phi^\top \Phi/\sigma^2 + S_0^{-1})^{-1}
\]

We want to examine the distribution of the eigenvalues of the posterior variance. Let \(\Phi^\top \Phi/\sigma^2 = V\Lambda V^\top\) be the eigendecomposition with eigenvalues \(\Lambda = \text{diag}(\gamma_1, \ldots, \gamma_n, 0, 0, \ldots, 0_k)\); \(k - n\) of the eigenvalues are 0 since the gram matrix \(\Phi^\top \Phi\) is at most rank \(n\) by construction.

Substitution into the posterior variance of \(\beta\) yields,
\[
(\Phi^\top \Phi/\sigma^2 + S_0^{-1})^{-1} = (V AV^\top + \alpha^{-2} I_k)^{-1}
\]
\[
= V (\Lambda + \alpha^{-2} I_k)^{-1} V^\top
\]
\[
= V TV^\top.
\]

The eigenvalues of the posterior covariance matrix are given by the entries of \(\Gamma, ((\gamma_1 + \alpha^{-2})^{-1}, \ldots, (\gamma_n + \alpha^{-2})^{-1}, \alpha^{-2}, \ldots, \alpha^{-2})\), where there are \(k - n\) eigenvalues that retain a value of \(\alpha^{-2}\).

Therefore the posterior covariance has \(p - n\) directions in which the posterior variance is unchanged and \(n\) directions in which it has contracted as scaled by the eigenvalues of the gram matrix \(\Phi^\top \Phi\).

Generalized linear models (GLMs) do not necessarily have a closed form posterior distribution. However, Neal & Zhang (2006) give a straightforward argument using the invariance of the likelihood of GLMs to orthogonal linear transformations in order to justify the usage of PCA as a feature selection step. We can adapt their result to show that overparameterized GLMs have a \(k - n\) dimensional subspace in which the posterior variance is identical to the prior variance.

**Theorem D.1** (Posterior Contraction in Generalized Linear Models). We specify a generalized linear model, \(E[Y] = g^{-1}(\Phi \beta)\) and \(\text{Var}(Y) = V(g^{-1}(\Phi \beta))\), where \(\Phi \in \mathbb{R}^{n \times k}\) is a feature matrix of \(n\) observations and \(k\) features and \(\beta \in \mathbb{R}^k\) are the model parameters. In the overparameterized setting with isotropic prior on \(\beta\), there exists a \(k - n\) dimensional subspace in which the posterior variance is identical to the prior variance.

**Proof.** First note that the likelihood of a GLM takes as argument \(\Phi \beta\), thus transformations that leave \(\Phi \beta\) unaffected leave the likelihood, and therefore the posterior distribution, unaffected.

Let \(R\) be an orthogonal matrix, \(R^\top R = RR^\top = I_p\), and \(\tilde{\beta} = R \beta \sim \mathcal{N}(0, \sigma^2 I)\). If we assign a standard isotropic prior, to \(\beta\) then \(\tilde{\beta} \sim \mathcal{N}(0, \sigma^2 R I_p R^\top = \sigma^2 I_k)\). If we also rotate the feature matrix, \(\tilde{\Phi} = \Phi R^\top \in \mathbb{R}^{n \times k}\) so that \(\tilde{\Phi} \tilde{\beta} = \Phi R^\top R \beta = \Phi \beta\), showing that the likelihood and posterior remain unchanged under such transformations.

In the overparameterized regime, \(k > n\), with linearly independent features we have that \(\Phi\) has rank at most \(k\), and we can therefore choose \(R\) to be a rotation such that \(\tilde{\Phi} \tilde{R}\) has exactly \(k - n\) columns that are all 0. This defines a \(k - n\) dimensional subspace of \(\beta \in \mathbb{R}^k\) in which the likelihood is unchanged. Therefore the posterior remains no different from the prior distribution in this subspace, or in other words, the posterior distribution has not contracted in \(k - n\) dimensions.

**D.2. Function-Space Homogeneity**

**Theorem** (Function-Space Homogeneity in Linear Models). Let \(\Phi = \Phi(x) \in \mathbb{R}^{n \times k}\) be a feature map of \(n\) data observations, \(x\), with \(n < k\) and assign isotropic prior \(\beta \sim \mathcal{N}(0_k, S_0 = \alpha^2 I_k)\) for parameters \(\beta \in \mathbb{R}^k\). The minimal eigenvectors of the Hessian define a \(k - n\) dimensional subspace in which parameters can be perturbed without changing the training predictions in function-space.

**Proof.** The posterior covariance matrix for the parameters is given by
\[
\Sigma_{\beta|D} = \left(\frac{\Phi^\top \Phi}{\sigma^2} + \alpha^{-2} I_k\right)^{-1},
\]
and therefore the Hessian of the log-likelihood is
\[
\left(\frac{\Phi^\top \Phi}{\sigma^2} + \alpha^{-2} I_k\right). \]
By the result in Theorem 4.1 there are \(k - n\) eigenvectors of the Hessian all with eigenvalue \(\alpha^{-2}\). If we have some perturbation to the parameter vector \(u\) that resides in the span of these eigenvectors we have
\[
\left(\frac{\Phi^\top \Phi}{\sigma^2} + \alpha^{-2} I_k\right) u = \alpha^{-2} u,
\]
which implies \(u\) is in the nullspace of \(\Phi^\top \Phi\). By the properties of gram matrices we have that the nullspace of \(\Phi^\top \Phi\) is the same as that of \(\Phi\), thus \(u\) is also in the nullspace of \(\Phi\).

Therefore any prediction using perturbed parameters takes the form \(\tilde{y} = \Phi(\beta + u) = \Phi \tilde{\beta}\), meaning the function-space predictions on training data under such perturbations are unchanged.

**Theorem D.2** (Function-Space Homogeneity in Generalized Linear Models). We specify a generalized linear model, \(E[Y] = g^{-1}(\Phi \beta)\), where \(\Phi \in \mathbb{R}^{n \times k}\) is a feature matrix of \(n\) observations and \(k\) features and \(\beta \in \mathbb{R}^k\) are the model parameters. In the overparameterized setting with isotropic prior on \(\beta\), there exists a \(k - n\) dimensional subspace in which parameters can be perturbed without changing the training predictions in function-space or the value of the Hessian.
Figure A.9. **Left:** A visualization of the log-loss surface taken in the direction of the top two eigenvectors of the Hessian of the loss. **Right:** A visualization of a random projection of the log-loss surface in all parameter directions except the top 200 eigenvectors of the Hessian. We can see that in nearly all directions the loss is constant even as we move far from the optimal parameters. **Note** the scale difference, even as we increase the resolution of the degenerate loss surface we still see no structure.

**Proof.** The Hessian of the log-likelihood for GLMs can be written as a function of the feature map, \( \Phi \), and the product of the feature map and the parameters, \( \Phi \beta \), i.e. \( \beta \) only appears multiplied by the feature map (Nelder & Wedderburn, 1972). We can then write \( H_\beta = f(\Phi \beta, \Phi) \)
Additionally predictions are generated by \( y = g^{-1}(\Phi \beta) \).
Since \( \Phi \in \mathbb{R}^{n \times p} \) with \( n < p \) there is a nullspace of \( \Phi \) with dimension at least \( n - p \). Thus for any \( u \in \text{null}(\Phi) \) we have \( g^{-1}(\Phi(\beta + u)) = g^{-1}(\Phi \beta) = y \) and \( f(\Phi(\beta + u), \Phi) = f(\Phi \beta, \Phi) = H_\beta \), which shows that the training predictions and the Hessian remain unchanged.

**E. Perturbations on CIFAR-10**

To demonstrate that the results presented in Section 5 apply to larger architectures similar to those seen in practice we train a convolutional classifier provided by Pytorch on the CIFAR-10 dataset. The network has approximately 62000 parameters and is trained on 50000 images.

Figure A.9 shows the presence of degenerate directions in parameter space. We compute the top 200 eigenvectors of the Hessian of the loss and consider perturbations in the directions of the top 2 eigenvectors, as well as in all parameter directions except the top 200 eigenvectors of the Hessian. We see that even for larger networks and more complex datasets degenerate directions in parameter space are still present and comprise most possible directions.

Figure A.10 demonstrates that the degenerate directions in parameter space lead to models that are homogeneous in function space on both training and testing data. As increasingly large perturbations are made in degenerate parameter directions, we still classify more than 99% of both training and testing points the same as the unperturbed classifier.

**F. More Classifiers**

Figures A.11 and A.12 provide more examples of perturbations in high and low curvature directions and the effect of the scale of the perturbation on function-space predictions for the two-spirals experiment in Section 5.

**Figure A.11.** Classifiers as the parameters are shifted in random directions within the span of the bottom 1500 eigenvectors of the Hessian of the loss. Scales of the perturbation range from 0 (upper left) to 2 (lower right).

**Figure A.12.** Classifiers as the parameters are shifted in random directions within the span of the top 3 eigenvectors of the Hessian of the loss. Scales of the perturbation range from 0 (upper left) to 0.5 (lower right).
G. Deep Neural Network Training and Eigenvalue Computation

Training Details For the double descent experiments in Figures 1 and 2 we use neural network architectures from the following sources:

- **CNNs** from https://gitlab.com/harvard-machine-learning/double-descent/-/blob/master/models/mcnn.py but also include an option to vary the depth,
- **ResNet18** from https://gitlab.com/harvard-machine-learning/double-descent/-/blob/master/models/resnet18k.py,
- **PreResNet-56** from https://github.com/bearpaw/pytorch-classification/blob/master/models/cifar/preresnet.py.

Specifically, we train with SGD with a learning rate of $10^{-2}$, momentum of 0.9, weight decay of $10^{-4}$ (thus corresponding approximately to a Gaussian prior of with variance 1000) for 200 epochs with a batch size of 128. The learning rate decays to $10^{-4}$ following the piecewise constant learning rate schedule in Izmailov et al. (2018) and Maddox et al. (2019a), beginning to decay at epoch 100. We use random cropping and flipping for data augmentation — turning off augmentations to compute eigenvalues of the Hessian.

Lanczos Calculations We use GPU enabled Lanczos as implemented in Gardner et al. (2018) to compute the eigenvalues approximately, running 100 steps to compute approximately 100 of the top eigenvalues. We note that our estimates of the effective dimensionality are somewhat biased from not including all of the small eigenvalues. However, these small eigenvalues will contribute negligibly and Lanczos will converge to the true eigenvalues if we ran $k$ steps where $k$ is the rank of the Hessian.

H. Double Descent and Effective Dimensionality: Further Experiments

Finally, we consider several further experiments on the two spirals problem to test the effects of increasing depth and width to serve as a sanity check for our results on both ResNets and CNNs. In Figure A.13, we fix the number of data points to be 3000, and vary the depth of the neural network (20 hidden units at each layer, ELU activations) using between one and 15 hidden units, training for 4000 steps as before. Here, we run each experiment with 25 repetitions and compute all of the eigenvalues of the Hessian at convergence (the largest model contains 6000 parameters). In the left panel, we see a pronounced double descent curve with respect to both effective dimensionality and test loss as we vary depth. In the right panel, we use the same data points, but use three hidden layer networks, varying the width of each layer between one and 30 units per layer. Here, we see only a monotonic decrease in both test error and effective dimensionality with increasing width not helping that much in terms of test error — the effective dimensionality is highest for the models with smallest size and slowly decreases as the width is increased. These results serve as a sanity check on our large-scale Lanczos results in the main text.

Finally, in Figure A.14, we plot the effective dimensionality against the test error for the linear model example in Section 6. A clear linear-looking trend is observed, which corresponds to the models that have nearly zero training error. The bend near the origin is explained by models that do not have enough capacity to fit — therefore, their effective dimensionality is very small. We observe a similar trend for ResNets and CNNs.
**Figure A.13.** Left: Increasing depth on the two spirals problem. Clearly seen is a double descent curve where the test loss first increases before decreasing as a function of depth. The effective dimensionality follows the same trend. Right: Increasing width on the two spirals problem. Here, increased width produces constant test performance after the training loss reaches zero, and the effective dimensionality stays mostly constant. Shading represents two standard deviations as calculated by 25 random generations of the spirals data.

**Figure A.14.** Effective dimensionality plotted against generalization gap (test error - train error) for the linear model of Section 6. Note that all but the very smallest models with effective dimensionality track nearly linearly with generalization error.