On elliptic Lax systems on the lattice and a compound theorem for hyperdeterminants

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Received 15 July 2014, revised 13 October 2014
Accepted for publication 17 October 2014
Published 22 December 2014

Abstract
A general elliptic $N \times N$ matrix Lax scheme is presented, leading to two classes of elliptic lattice systems, one which we interpret as the higher-rank analogue of the Landau–Lifschitz equations, while the other class we characterize as the higher-rank analogue of the lattice Krichever–Novikov equation (or Adler’s lattice). We present the general scheme, but focus mainly on the latter type of models. In the case $N = 2$ we obtain a novel Lax representation of Adler’s elliptic lattice equation in its so-called 3-leg form. The case of rank $N = 3$ is analyzed using Cayley’s hyperdeterminant of format $2 \times 2 \times 2$, yielding a multi-component system of coupled 3-leg quad-equations.

Keywords: hyperdeterminants, elliptic Lax systems, compound theorem, elliptic lattice systems

PACS numbers: 02.30.Ik, 02.30.Jr, 05.20.-y, 05.50.+q

1. Introduction

Adler’s lattice equation, [2], is an integrable lattice version of the Krichever–Novikov (KN) equation, [20], i.e. of the nonlinear evolution equation

$$u_t = \frac{1}{4} \left( u_{xx} + \frac{3}{2} \frac{r(u) - u_x^2}{u_x} \right),$$

(1.1)
in which $r(u) = 4u^3 - g_2u - g_3$ is the polynomial associated with a Weierstrass elliptic curve (or more generally an arbitrary quartic polynomial). This lattice equation, which was obtained as the permutability condition for the Bäcklund transformations for (1.1), can be written in the form:

$$A[(u-b)(\tilde{u} - b) - (a-b)(c-b)] + B[(u-a)(\tilde{u} - a) - (b-a)(c-a)] = ABC(a-b),$$

(1.2)

see [27], where $u = u(n, m)$ is the dependent variable, with the shifted variables $\tilde{u} = u(n+1, m)$ and $\tilde{\tilde{u}} = u(n+1, m+1)$ defining the different values of $u$ at the vertices around an elementary plaquette, see figure 1. The $a, b$ in figure 1 are lattice parameters associated with the grid size, and in this elliptic equation they are points $a = (a, A), b = (b, B)$, together with $\tilde{\tilde{u}} = (c, C)$, on a Weierstrass elliptic curve, i.e.

$$A^2 = r(a) \equiv 4a^3 - g_2a - g_3, \quad B^2 = r(b), \quad C^2 = r(c),$$

(1.3)

which can be parametrized in terms of the Weierstrass $\wp$-function as follows:

$$(a, A) = (\wp(a), \wp'(a)), \quad (b, B) = (\wp(b), \wp'(b)), \quad (c, C) = (\wp(c), \wp'(c)),$$

(1.4)

where $a$ and $b$ are the corresponding uniformizing parameters and $\gamma = b - a$. The parameters $a$, $b$ and $c$ are related through the addition formulae on the elliptic curve:

$$A(c - b) = C(a - b) - B(c - a),$$

$$a + b + c = \frac{1}{4} \left( \frac{A + B}{a - b} \right)^2.$$  

(1.5)

Furthermore, we use the notation for the lattice shifts

$$u \xrightarrow{a} \tilde{u}, \quad u \xrightarrow{b} \tilde{\tilde{u}}$$

being the elementary shifts on a quadrilateral lattice, each being associated with the lattice parameters $(a, A)$ respectively $(b, B)$, with the equation (1.2) expressing the condition for commutativity of these shifts as expressed through the diagram:

A Lax pair for Adler’s equation was given in [27], and the equation reemerged in [4] as the top equation in the ABS list of affine–linear quadrilateral equations, where it was renamed

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5 Note that in the original paper [2] the equation was written in a slightly different form with rather complicated expressions for the coefficients given in terms of the moduli $g_2$ and $g_3$ of the Weierstrass curve.
Q4. The key integrability characteristic of Adler’s equation is its multidimensional consistency [11, 26], which in the case of Adler’s equation can be made manifest through its so-called 3-leg form, see [4]:

\[
\frac{\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi - \beta)\sigma(\xi + \xi + \beta)}{\sigma(\xi - \xi - \alpha)\sigma(\xi + \xi + \alpha)\sigma(\xi + \xi - \beta)} = \frac{\sigma(\xi - \xi - \gamma)\sigma(\xi + \xi + \gamma)}{\sigma(\xi - \xi + \gamma)\sigma(\xi + \xi - \gamma)},
\]

in which the uniformizing variable \( \xi = \xi(n, m) \) is now the dependent variable of the equation, related to the original variable \( u \) of the rational form (1.2) of the equation through the identification \( u = \wp(\xi) \). The connection between rational and elliptic form of the equation parallels that of the KN equation, which in its (original) elliptic form reads:

\[
\xi_{\xi \xi \xi} + \frac{3}{2} \left( \frac{1 - \xi^2}{\xi_x^2} \right) - 6\wp(2\xi)\xi_s^3 = 0,
\]

We note in passing that there are alternative forms for Adler’s equation based on different choices of the underlying elliptic curve. Thus, if one could consider (1.2) to be the Weierstrass form of the equation (with parameters on a Weierstrass elliptic curve (1.3)), the equation in Jacobi form (due to Hietarinta [17]) reads:

\[
Q(v, v, v', v') = p(vv' + vv') - q(vv' + vv') - r(vv' + vv') + pqr(1 + vv'vv') = 0,
\]

where the dependent variable \( v \) is related to \( u' \) of (1.2) through a fractional linear transformation, and the parameters \( (p, P), (q, Q) \) and \( (r, R) \) are now points on a Jacobi type elliptic curve:

\[
\Gamma: \quad X^2 \equiv x^4 - yx^2 + 1, \quad y^2 = k + 1/k,
\]

with modulus \( k \). They can be parametrized in terms of Jacobi elliptic function as follows:

\[
\begin{align*}
p &= (p, P) = \left( \sqrt{k} \ \text{sn}(\alpha; k), \ \text{sn}'(\alpha; k) \right), \\
q &= (q, Q) = \left( \sqrt{k} \ \text{sn}(\beta; k), \ \text{sn}'(\beta; k) \right), \\
r &= (r, R) = \left( \sqrt{k} \ \text{sn}(\alpha - \beta; k), \ \text{sn}'(\alpha - \beta; k) \right).
\end{align*}
\]

Many interesting results were established for the latter form of the equation, notably explicit expressions for the (doubly elliptic) N-soliton solutions [7] and singular-boundary solutions [8], however for the sake of the present paper we shall concentrate once again on the Weierstrass form of the equation.

In the present paper we propose a general elliptic Lax scheme of rank \( N \), which is inspired by a novel Lax representation of Adler’s lattice equation. This Lax scheme leads to two distinct classes of systems which we coin as being ‘of Landau–Lifschitz (LL) type’ (or spin-non-zero case) and as ‘of KN type’ (or spin-zero case). We present general results for both classes in section 2, but then focus in the remainder of the paper on the KN class of Lax systems. In that case for \( N = 2 \) we show that the scheme amounts to a novel Lax representation for Adler’s lattice equation, which yields the equation directly in 3-leg form (this in contrast with the lax pair constructed in [27] from multidimensional consistency). Notably in the rank \( N = 3 \) case the analysis of the compatibility condition exploits a (to our knowledge novel) compound theorem for Caley’s hyperdeterminants of format \( 2 \times 2 \times 2 \), [13], a result which may have some significance in its own right. We conjecture that the resulting rank 3
lattice system may be regarded as a discrete analogue of a rank 3 KN type of differential system that was constructed by Mokhov in [29].

2. General elliptic Lax scheme

Consider the Lax pair of the form:

\[
\begin{align*}
\chi' &= L\chi, \quad (2.1a) \\
\tilde{\chi}' &= M\tilde{\chi}, \quad (2.1b)
\end{align*}
\]

defining horizontal and vertical shifts of the vector function \(\chi\), according to the diagram in figure 2: where the vectors \(\chi\) are located at the vertices of the quadrilateral and in which the matrices \(L\) and \(M\) are attached to the edges linking the vertices. The matrices \(L\) and \(M\) can be taken of the form:

\[
\begin{align*}
(\Phi_{\xi}(\xi_j - \xi_i - \alpha))h_j, \\
(\Phi_{\xi}(\xi_j - \xi_i - \beta))k_j,
\end{align*}
\]

where \(\Phi_{\xi}(\xi_j - \xi_i - \alpha)\) and \(\Phi_{\xi}(\xi_j - \xi_i - \beta)\) are Lamé functions with \(\alpha\) and \(\beta\) denoting the uniformized lattice parameters (as in (1.4)), while \(\kappa\) is the (uniformized) spectral parameter. In (2.2), the coefficients \(h_j\) and \(k_j\) are functions of the variables \(\xi_i\), and their shifts, that remain to be determined. The compatibility condition between (2.1a) and (2.1b) is given by the lattice zero-curvature condition:

\[
\tilde{L}\chi = \tilde{\chi}M. \quad (2.4)
\]

Using the addition formula

\[
\Phi_{\xi}(\xi)\Phi_{\xi}(\eta) = \Phi_{\xi}(\xi + \eta)[\zeta(\kappa) + \zeta(\kappa + \xi + \eta) - \zeta(\kappa + \xi + \eta)],
\]

where \(\zeta(\kappa) = \frac{1}{\kappa} \ln \sigma(\kappa)\) is the Weierstrass zeta function, the consistency relation (2.4) gives rise to
\[
\sum_{i=1}^{N} \tilde{h}_i \tilde{k}_j \left[ \zeta (\xi_j - \xi_i - \alpha) + \zeta (\xi_i - \xi_j - \beta) + \zeta (N\kappa - \xi_i - \xi_j - \alpha - \beta) \right] \\
= \sum_{i=1}^{N} \tilde{k}_i h_j \left[ \zeta (\xi_j - \xi_i - \beta) + \zeta (\xi_i - \xi_j - \alpha) + \zeta (N\kappa - \xi_i - \xi_j - \alpha - \beta) \right] \\
(i, j = 1, ..., N). \tag{2.6}
\]

Due to the arbitrariness of the spectral parameter \(\kappa\) the equations (2.6) separate into two parts, namely

\[
\left\{ \sum_{i=1}^{N} \tilde{h}_i \right\} k_j = \left( \sum_{i=1}^{N} \tilde{h}_i \right) h_j, \quad (j = 1, ..., N), \tag{2.7a}
\]

\[
\left\{ \sum_{i=1}^{N} \tilde{h}_i \left[ \zeta (\xi_j - \xi_i - \alpha) + \zeta (\xi_i - \xi_j - \beta) \right] \right\} k_j \\
= \left\{ \sum_{i=1}^{N} \tilde{k}_i \left[ \zeta (\xi_j - \xi_i - \beta) + \zeta (\xi_i - \xi_j - \alpha) \right] \right\} h_j \\
(i, j = 1, ..., N). \tag{2.7b}
\]

Now there are two scenarios which we refer to as the ‘LL type’ (or physically, the spin non-zero) case and the ‘KN type’ (spin zero) cases respectively:

**Case #1** Discrete LL type case : \(\sum_i h_i \neq 0\), in which case we have that the variables \(h_i, k_j\) are proportional to each other, \(k_j = \rho h_j\), and after summing (2.7a) we obtain the conservation law:

\[
\sum_{i=1}^{N} \tilde{h}_i = \sum_{i=1}^{N} \tilde{k}_i, \tag{2.8}
\]

and in which case equations (2.7b) reduce to:

\[
\sum_{i=1}^{N} \left[ \zeta (\xi_j - \xi_i - \alpha) \rho \tilde{h}_i - \zeta (\xi_i - \xi_j - \beta) \tilde{k}_i \right] \\
= \sum_{i=1}^{N} \left[ \zeta (\xi_j - \xi_i + \beta) \rho h_i - \zeta (\xi_i - \xi_j + \alpha) \tilde{k}_i \right] \\
(i, j = 1, ..., N). \tag{2.9}
\]

This system of equations can be reduced under the condition:

\[
\Xi + \tilde{\Xi} = \tilde{\Xi} + \Xi, \quad \Xi \equiv \sum_{i=1}^{N} \xi_i, \tag{2.10}
\]

which is a conservation law for the centre of mass motion. In fact, (2.10) follows from the determinant of the relation (2.4) and using the Frobenius determinant formula (see appendix B).

**Case #2** KN type case : \(\sum_i h_i = \sum_i k_i = 0\), in which case (2.7a) becomes vacuous. In this case we seek further reductions by the additional constraint \(\Xi = \sum_i \xi_i = 0\) (modulo the period lattice of the elliptic functions).
In this paper we will focus primarily on the class of models in \# 2, but we will conclude this section by presenting the general structure of the systems that emerge from the Lax system in both cases, and then in the ensuing sections present an alternative analysis for the Lax system of class \# 2 for the cases $N = 2$ and $N = 3$.

In order to proceed with the general analysis of (2.9) we use a trick that was employed in [25], based on an elliptic version of the Lagrange interpolation formula (see appendix B) in order to identify the variables $h_l$, $k_l$. Consider the following elliptic function, where as a consequence of the conservation law (2.10) for the variables $\xi$ the Lagrange interpolation (B.6) of appendix B is applicable, leading to the following identity:

$$F(\xi) = \prod_{l=1}^{N} \frac{\sigma(\xi - \xi_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \xi_l - \alpha)\sigma(\xi - \xi_l - \beta)},$$

$$= \sum_{l=1}^{N} \left[ \xi(\xi - \xi_l - \alpha) - \xi(\eta - \xi_l - \alpha) \right] H_l$$

$$+ \sum_{l=1}^{N} \left[ \xi(\xi - \xi_l - \beta) - \xi(\eta - \xi_l - \beta) \right] K_l,$$  \hspace{1cm} (2.11)

which holds for any four sets of variables $\xi_i, \xi_j, \xi_k, \xi_l$ such that (2.10) holds. In (2.11) $\eta$ can be any one of the zeroes of $F(\xi)$, i.e. $\xi_l$ or $\xi_l + \alpha + \beta$, and the coefficients $H_l, K_l$ are given by:

$$H_l = \prod_{k=1}^{N} \frac{\sigma(\xi_l - \xi_k + \alpha)\sigma(\xi_l - \xi_k - \beta)}{\prod_{k \neq l}^{N} \sigma(\xi_l - \xi_k)},$$ \hspace{1cm} (2.12a)

$$K_l = \prod_{k=1}^{N} \frac{\sigma(\xi_l - \xi_k + \beta)\sigma(\xi_l - \xi_k - \alpha)}{\prod_{k \neq l}^{N} \sigma(\xi_l - \xi_k)}.$$ \hspace{1cm} (2.12b)

Furthermore, the coefficients obey the identity:

$$\sum_{l=1}^{N} (H_l + K_l) = 0.$$ \hspace{1cm} (2.13)

Taking $\xi = \xi_l$, $\eta = \xi_l + \alpha + \beta$ in (2.11) and comparing with (2.7b), we can make the identifications:

$$tH_l = \rho \tilde{H}_l, \quad tK_l = -\rho \tilde{K}_l, \quad l = 1, \ldots, N,$$ \hspace{1cm} (2.14)

with a function $t$ being an arbitrary proportionality factor. Thus in this case (case \#1) by eliminating $h_l$ from (2.14) we get the set of equations

$$\frac{\tilde{t}}{\rho} \tilde{H}_l + \frac{\tilde{t}}{\rho} \tilde{K}_l = 0, \quad l = 1, \ldots, N,$$ \hspace{1cm} (2.15)

which, by inserting the expressions (2.12a) for $H_l$ and $K_l$, is a system of $N$ equations for $N + 2$ unknowns $\xi_l, (l = 1, \ldots, N)$, and $\rho$ and $t$. Rewriting this system in explicit form, we obtain the system of $N$ 7-point equations:
\begin{equation}
\prod_{k=1}^{N} \frac{\sigma(\xi_i - \xi_k + \alpha) \sigma(\xi_i - \xi_k - \beta) \sigma(\xi_i - \xi_k + \gamma)}{\sigma(\xi_i - \xi_k + \beta) \sigma(\xi_i - \xi_k - \alpha) \sigma(\xi_i - \xi_k - \gamma)} = p
\end{equation}

for $N+1$ variables $\xi_i$ ($i = 1, ..., N$) and $p = -\mathcal{L} \rho / (\mathcal{L} \rho)$, supplemented with (2.10) which fixes the discrete dynamics of the centre of mass $\Xi$. In (2.16) the under-accents $\sim$ and $\hat{\cdot}$ denote reverse lattice shifts, i.e., $\xi_{i,j}(n, m) = \xi_{i,j}(n-1, m)$ and $\xi_{i,j}(n, m) = \xi_{i,j}(n, m-1)$ respectively. These equations and their rational forms will be investigated in more detail in a future publication. The implicit system of PDEs arises as Euler–Lagrange equation from the following Lagrangian:

\begin{equation}
\mathcal{L} = \sum_{i,j=1}^{N} \left[ f(\xi_i - \xi_j + \alpha) - f(\xi_i - \xi_j + \beta) - f(\xi_i - \xi_j + \alpha - \beta) \right] \ln |p| \Xi,
\end{equation}

in which the function $f$ is the elliptic dilogarithm $f(x) = \int_{0}^{x} \ln \sigma(x) \sigma(1/x) dx$, with respect to variations of the dependent variables $\xi_i$ ($i = 1, 2, ..., N$). The one-step periodic reduction, $\chi_{\lambda} \chi = \sim \kappa$, leads to an implicit system of ordinary difference equations which amounts to the time-discretization of the Ruijsenaars (relativistic Calogero–Moser) model, given in [25]. We consider the system (2.16) to be ‘of LL class’ although a precise connection with the LL equation remains still to be established. Lattice versions of the LL equation were given in the papers [1, 3, 24]. However, not only the connection of (2.16) with these earlier models remains unclear at this stage, but also the relation between these various discretizations of the LL equation have remained obscure to this date. In the remainder of the paper we will concentrate on the case $N = 2$ which, as we show for $N = 2$, leads to Adler’s lattice equation in 3-leg form, and for higher rank of $N$ ($N \geq 3$) is expected to lead to higher rank version of Adler’s equation. For that case, we will perform a different kind of analysis.

3. Elliptic Lax pairs for 3-leg lattice systems

In this section we shall focus on case $N = 2$ of general elliptic Lax systems introduced in the previous section, corresponding to the ‘spin-zero’ case (where $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i = 0$). We will first demonstrate, in the case $N = 2$ of this system, how the 3-leg form of Adler’s equation arises in a natural way from this Lax pair. In fact, it turns out that the elaboration of the compatibility conditions for this Lax pair immediately produces the required equations, and is far less laborious than of the consistency-around-the-cube Lax pair of [27] yielding the corresponding rational form of Q4. Next we will analyze the much more generic case of $N = 3$, and produce a novel system of elliptic lattice equations, which constitutes the main result of this paper. We also present the structure of the lattice system arising form the scheme for general $N$, based on similar ingredients as the ones used in the case $N = 2$ elaborated in the previous section, but subject to slightly different conditions.

3.1. Case $N = 2$: Elliptic Lax pair for the Adler 3-leg lattice equation

Let $\xi = \xi_{n,m}$ be a function of the discrete independent variables $n, m$ for which we want to derive a lattice equation from the following Lax pair:
in which the coefficients $\lambda$ and $\mu$ are functions $\lambda = \lambda(\xi, \tilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \tilde{\xi}; \beta)$, respectively. Its explicit form will be derived subsequently, but these forms will actually not be relevant for the determination of the resulting lattice equation, which is Adler’s system in 3-leg form. The discrete zero-curvature condition (2.4) can, once again, be analyzed using the addition formula (2.5) for the Lamé function $\Phi$ and analyzed entry-by-entry. Applying this to each entry of both the left-hand side and right-hand side of (2.4) we observe that in all four entries a common factor containing the spectral parameter $\kappa$ will drop out and that we are left with the following four relations:

\[
\begin{align*}
\hat{\lambda} & \left[ \zeta(\xi - \tilde{\xi} - \alpha) + \zeta(\xi - \tilde{\xi} - \beta) - \zeta(\xi + \tilde{\xi} - \alpha) + \zeta(\xi + \tilde{\xi} + \beta) \right] \\
= & \tilde{\mu} \left[ \zeta(\xi - \tilde{\xi} - \alpha) + \zeta(\xi + \tilde{\xi} - \alpha) - \zeta(\xi + \tilde{\xi} + \beta) + \zeta(\xi + \tilde{\xi} + \beta) \right] \\
\hat{\lambda} & \left[ \zeta(-\xi - \tilde{\xi} - \alpha) + \zeta(-\xi - \tilde{\xi} + \beta) - \zeta(-\xi + \tilde{\xi} - \alpha) + \zeta(-\xi + \tilde{\xi} + \beta) \right] \\
= & \tilde{\mu} \left[ \zeta(-\xi - \tilde{\xi} - \alpha) + \zeta(-\xi + \tilde{\xi} - \alpha) - \zeta(-\xi + \tilde{\xi} + \beta) + \zeta(-\xi + \tilde{\xi} + \beta) \right] \\
\hat{\lambda} & \left[ \zeta(-\xi + \tilde{\xi} + \alpha) + \zeta(-\xi + \tilde{\xi} + \beta) - \zeta(-\xi - \tilde{\xi} + \alpha) + \zeta(-\xi - \tilde{\xi} + \beta) \right] \\
= & \tilde{\mu} \left[ \zeta(-\xi + \tilde{\xi} + \alpha) + \zeta(-\xi + \tilde{\xi} + \beta) - \zeta(-\xi - \tilde{\xi} + \alpha) + \zeta(-\xi - \tilde{\xi} + \beta) \right] \\
\end{align*}
\]

Using the identity (2.5) these four relations can be rewritten as:

\[
\begin{align*}
\hat{\lambda} & \frac{\sigma(2\xi)\sigma(\xi + \tilde{\xi} + \beta - \alpha)}{\sigma(\xi + \tilde{\xi} - \alpha)\sigma(\xi + \tilde{\xi} + \alpha)\sigma(\xi - \tilde{\xi} - \beta)\sigma(\xi + \tilde{\xi} + \beta)} \\
= & \tilde{\mu} \left[ \frac{\sigma(2\xi)\sigma(\xi + \alpha + \beta)}{\sigma(\xi - \tilde{\xi} - \beta)\sigma(\xi + \tilde{\xi} - \alpha)\sigma(\xi - \tilde{\xi} + \beta)\sigma(\xi + \tilde{\xi} + \beta)} \right], \\
\end{align*}
\]
Eliminating $\lambda$ and $\mu$, simply by dividing pairwise the relations over each other, we obtain directly the 3-leg formulae. In fact, we obtain two seemingly different-looking equations for $\xi$, namely:

\[ \frac{\sigma(2\xi)\sigma(\xi - \xi + \beta - \alpha)}{\sigma(\xi - \xi - \alpha)\sigma(\xi + \xi - \alpha)\sigma(\xi - \xi + \beta)\sigma(\xi + \xi - \beta)} = \frac{\sigma(2\xi)\sigma(\xi - \xi + \alpha)}{\sigma(\xi - \xi + \beta)\sigma(\xi + \xi + \alpha)\sigma(\xi - \xi - \alpha)\sigma(\xi + \xi + \alpha)}, \quad (3.3b) \]

\[ \frac{\sigma(2\xi)\sigma(\xi + \xi - \beta + \alpha)}{\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi + \alpha)\sigma(\xi - \xi - \beta)\sigma(\xi + \xi - \beta)} = \frac{\sigma(2\xi)\sigma(\xi + \xi - \alpha + \beta)}{\sigma(\xi - \xi + \beta)\sigma(\xi + \xi + \beta)\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi + \alpha)}, \quad (3.3c) \]

\[ \frac{\sigma(2\xi)\sigma(\xi + \xi + \beta + \alpha)}{\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi + \alpha)\sigma(\xi - \xi - \beta)\sigma(\xi + \xi - \beta)} = \frac{\sigma(2\xi)\sigma(\xi + \xi - \alpha + \beta)}{\sigma(\xi - \xi + \beta)\sigma(\xi + \xi + \beta)\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi + \alpha)}, \quad (3.3d) \]

in which as before $\gamma = \beta - \alpha$ and

\[ \frac{\sigma(\xi - \xi + \alpha)\sigma(\xi + \xi - a)}{\sigma(\xi - \xi - a)\sigma(\xi + \xi + a)\sigma(\xi - \xi + \beta)\sigma(\xi + \xi - \beta)} = \frac{\sigma(\xi - \xi + \gamma)\sigma(\xi + \xi + \gamma)}{\sigma(\xi - \xi - \gamma)\sigma(\xi + \xi - \gamma)}, \quad (3.4a) \]

but actually these two equations are equivalent. The first equation (3.4a) is identical to (1.6), namely the 3-leg form of the Adler lattice equation. The second equation (3.4b) is obtained from the first by interchanging $\xi \leftrightarrow \tilde{\xi}$, $\alpha \leftrightarrow \beta$, which is a symmetry of the equation. The equivalence between these two forms is made manifest by passing to the rational form (1.2) of the equation, and the latter connection can be seen to be a consequence of an interesting identity given in the following statement.

**Proposition 3.1.1.** For arbitrary (complex) variables $X, Y, \text{and } Z$, we have the following identity

\[ J. \text{Phys. A: Math. Theor.} 48 (2015) 035206 \]
\begin{align*}
& (X - \wp(\xi + \alpha))(Y - \wp(\xi + \beta))(Z - \wp(\xi - \alpha + \beta)) \\
& - t^2(X - \wp(\xi - \alpha))(Y - \wp(\xi + \beta))(Z - \wp(\xi + \alpha - \beta)) \\
& = s\{A[(\wp(\xi) - b)(Y - b) - (a - b)(c - b)][(X - b)(Z - b) \\
& - (a - b)(c - b)] + B[(\wp(\xi) - a)(X - a) - (b - a)(c - a)] \\
& \times [(Y - a)(Z - a) - (b - a)(c - a)] - ABC(a - b)\},
\end{align*}

in which

\begin{equation}
\begin{aligned}
t &= \frac{\sigma(\xi + \alpha)\sigma(\xi + \beta)\sigma(\xi + \alpha - \beta)}{\sigma(\xi + \alpha)\sigma(\xi - \beta)\sigma(\xi - \alpha + \beta)}, \\
s &= \frac{1 - t^2}{(A + B)\wp(\xi) - Ab - aB}
\end{aligned}
\end{equation}

and where \((a, A), (b, B)\) and \((c, C)\) are given as before.

A (computational) proof of the proposition 3.1.1 is given in appendix A. Identifying
\(u = \wp(\xi), X = \bar{u} = \wp(\bar{\xi}), Y = \bar{u} = \wp(\bar{\xi})\) and \(Z = \bar{u} = \wp(\bar{\xi})\), and using

\begin{equation}
\wp(\xi) - \wp(\eta) = \frac{\sigma(\eta + \xi)\sigma(\eta - \xi)}{\sigma^2(\eta)\sigma^2(\xi)},
\end{equation}

It can be readily seen that the elliptic identity (3.5) relates the rational form of Adler’s equation in the Weierstrass case (1.2) with the 3-leg (3.4a). Since the Adler system (1.2) is manifestly invariant under the replacements \(u \leftrightarrow \bar{u}, \alpha \leftrightarrow -\beta\)—whilst not interchanging \(\bar{u}\) and \(\wp\)—(this being a particular aspect of the \(D_\infty\)-symmetry of the equation), the 3-leg form (3.4a) is also invariant under the parallel exchange on the level of the uniformizing variables: \(\xi \leftrightarrow \bar{\xi}, \alpha \leftrightarrow \bar{\beta}\). This is the symmetry that connects the two forms (3.4a) and (3.4b), which are hence equivalent.

Remark 1. The coefficients \(\lambda\) and \(\mu\) are determined by the condition for which the dynamical equation for the determinants of the Lax matrices \(\kappa_L, \kappa_M\) needs to be trivially satisfied. Thus a possible choice for \(\lambda\) and \(\mu\) is to determine these factors such that \(\kappa_L, \kappa_M\) are proportional to constants (i.e. independent of \(\xi\)), which leads to the following expressions

\begin{equation}
\lambda = \left(\frac{H(u, \bar{u}, a)}{AU\bar{U}}\right)^{1/2}, \quad \mu = \left(\frac{H(u, \bar{u}, b)}{BU\bar{U}}\right)^{1/2},
\end{equation}

where \(u = \wp(\xi), U = r(u) = \wp'(\xi)\), and similarly \(\bar{u} = \wp(\bar{u}), \bar{U} = r(\bar{u}) = \wp'(\bar{\xi})\), and \(\bar{u} = \wp(\bar{\xi}), \bar{U} = r(\bar{u}) = \wp'(\bar{\xi})\). The symmetric triquadratic function \(H\) is given by

\begin{equation}
H(u, v, a) \equiv \left(u v + a u + a v + \frac{g_4}{4}\right)^2 - \left(4a uv - g_3\right)(u + v + a),
\end{equation}

and which can be obtained in the following form in terms of \(\sigma\)-function

\begin{equation}
H(u, v, a) = (u - v)^2 \frac{1}{4} \left(\frac{U - V}{u - v}\right)^2 - (u + v + a) \left[\frac{1}{4} \left(\frac{U + V}{u - v}\right)^2 - (u + v + a)\right]
\end{equation}

\begin{equation}
= \frac{\sigma(\xi + \eta + \alpha)\sigma(\xi + \eta - \alpha)\sigma(\xi - \eta + \alpha)\sigma(\xi - \eta - \alpha)}{\sigma^4(\xi)\sigma^4(\eta)\sigma^4(\alpha)},
\end{equation}

in which \(U^2 \equiv r(u), V^2 \equiv r(v)\). We also have the expression in terms of the polynomial of the curve:
\[ \left[ r(u) + r(a) - 4(u - a)^2(u + v + a) \right]^2 - 4r(u)r(a) = 16(u - a)^2 H(u, v, a). \]  

(3.11)

We further note at this point that the discriminant of the triquadratic in each argument factorizes:

\[ H^2 - 2H_{uv} = r(a)r(u). \]  

(3.12)

In [5] the discriminant properties of affine–linear quadrilaterals and their relation with the corresponding biquadratics and their discriminants, were exploited to tighten the classification result of [4].

**Remark 2.** An alternative derivation of the \( N = 2 \) case can be given by using the system of equations (2.9). In this case the variables \( H_i \) and \( K_i \) take on the following forms, setting \( \xi_1 = -\xi_2 = \xi \):

\[ H_1 = \frac{\sigma(\xi - \xi + a)\sigma(\xi + \xi + a)\sigma(\xi - \xi - \beta)\sigma(\xi + \xi - \beta)}{\sigma(\xi - \xi - \gamma)\sigma(\xi + \xi - \gamma)\sigma(2\xi^2)}, \]  

(3.13a)

\[ H_2 = \frac{\sigma(-\xi - \xi + a)\sigma(-\xi + \xi + a)\sigma(-\xi - \xi - \beta)\sigma(-\xi + \xi - \beta)}{\sigma(-\xi - \xi - \gamma)\sigma(-\xi + \xi - \gamma)\sigma(-2\xi^2)}, \]  

(3.13b)

\[ K_1 = \frac{\sigma(\xi - \xi + \beta)\sigma(\xi + \xi + \beta)\sigma(\xi - \xi - a)\sigma(\xi + \xi - a)}{\sigma(\xi - \xi - \gamma)\sigma(\xi + \xi - \gamma)\sigma(2\xi^2)}, \]  

(3.13c)

\[ K_2 = \frac{\sigma(-\xi - \xi + \beta)\sigma(-\xi + \xi + \beta)\sigma(-\xi - \xi - a)\sigma(-\xi + \xi - a)}{\sigma(-\xi - \xi - \gamma)\sigma(-\xi + \xi - \gamma)\sigma(-2\xi^2)}. \]  

(3.13d)

The identity \( H_1 + H_2 = 0 \) upon inserting the above expressions yield the equation:

\[ \begin{vmatrix} \sigma(\xi + \xi + a)\sigma(\xi - \xi - a) \\ \sigma(\xi + \xi - a)\sigma(\xi - \xi + a) \end{vmatrix} \begin{vmatrix} \sigma(\xi + \xi - \beta)\sigma(\xi - \xi - \beta) \\ \sigma(\xi + \xi + \beta)\sigma(\xi - \xi + \beta) \end{vmatrix} \]  

\[ = \frac{\sigma(\xi + \xi + \gamma)\sigma(\xi - \xi + \gamma)}{\sigma(\xi - \xi - \gamma)\sigma(\xi + \xi - \gamma)}. \]  

(3.14)

which is equivalent to the elliptic lattice system (1.2) under the same changes of variables as discussed before. In fact, (3.14) can be obtained from (3.4a) by interchanging: \( \xi \leftrightarrow \xi \) and \( \hat{\xi} \leftrightarrow \hat{\xi} \). Similarly, the identity \( K_1 + K_2 = 0 \) upon inserting the expressions (3.13c) and (3.13d) for \( K_1 \) and \( K_2 \) yields a similar equation to (3.14) which can be obtained from (3.4a) by interchanging: \( \xi \leftrightarrow \xi \) and \( \hat{\xi} \leftrightarrow \hat{\xi} \). Thus, we recover from the scheme proposed in the previous section the Adler system in the various 3-leg forms based at different vertices of the elementary quadrilateral.

**3.2. Case \( N = 3 \)**

To generalize the results in the previous subsection to the rank 3 case, we consider the following form of a Lax representation on the lattice:
\[ \tilde{\chi} = \left( \begin{array}{ccc} h_1 \Phi_{3e} \left( \xi_1 - \xi_1 - \alpha \right) & h_2 \Phi_{3e} \left( \xi_2 - \xi_2 - \alpha \right) & h_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \alpha \right) \\ h_1 \Phi_{3e} \left( \xi_2 - \xi_1 - \alpha \right) & h_2 \Phi_{3e} \left( \xi_3 - \xi_2 - \alpha \right) & h_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \alpha \right) \\ h_1 \Phi_{3e} \left( \xi_3 - \xi_1 - \alpha \right) & h_2 \Phi_{3e} \left( \xi_3 - \xi_2 - \alpha \right) & h_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \alpha \right) \end{array} \right) \chi. \]  
\tag{3.15a}

\[ \tilde{\chi} = \left( \begin{array}{ccc} k_1 \Phi_{3e} \left( \xi_1 - \xi_1 - \beta \right) & k_2 \Phi_{3e} \left( \xi_2 - \xi_2 - \beta \right) & k_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \beta \right) \\ k_1 \Phi_{3e} \left( \xi_2 - \xi_1 - \beta \right) & k_2 \Phi_{3e} \left( \xi_3 - \xi_2 - \beta \right) & k_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \beta \right) \\ k_1 \Phi_{3e} \left( \xi_3 - \xi_1 - \beta \right) & k_2 \Phi_{3e} \left( \xi_3 - \xi_2 - \beta \right) & k_3 \Phi_{3e} \left( \xi_3 - \xi_3 - \beta \right) \end{array} \right) \chi. \]  
\tag{3.15b}

subject to \( \sum_{j=1}^{3} h_j = \sum_{j=1}^{3} k_i = 0 \), and where the coefficients \( h_j, k_i \) are some functions of the variables \( \xi_j \), and their shifts. The compatibility conditions (2.4) of this Lax pair results in a coupled set of Lax equations in terms of the three variables \( \xi_j \) as we shall demonstrate by performing a similar type of analysis as in the case \( N=2 \), which in this case is understandably more involved.

Eliminating\(^6\) \( h_3 = -h_1 - h_2 \) and \( k_3 = -k_1 - k_2 \) we obtain from (2.7b) the following system of equations:

\[ \sum_{j=1}^{2} \hat{h}_j k_j \left[ \zeta \left( \xi_1 - \xi_1 + \alpha \right) + \zeta \left( \xi_2 - \xi_2 + \beta \right) - \zeta \left( \xi_3 - \xi_3 + \alpha \right) - \zeta \left( \xi_1 - \xi_2 + \alpha \right) - \zeta \left( \xi_2 - \xi_3 + \alpha \right) - \zeta \left( \xi_1 - \xi_2 + \alpha \right) \right] \]
\[ = \sum_{j=1}^{2} \hat{k}_j h_j \left[ \zeta \left( \xi_1 - \xi_1 - \beta \right) + \zeta \left( \xi_2 - \xi_2 - \beta \right) - \zeta \left( \xi_3 - \xi_3 - \beta \right) - \zeta \left( \xi_1 - \xi_2 - \beta \right) - \zeta \left( \xi_2 - \xi_3 - \beta \right) - \zeta \left( \xi_1 - \xi_2 - \beta \right) \right] \quad \forall \ i, j = 1, 2, 3. \]  
\tag{3.16}

and using the addition formula (2.5) we next get:

\[ \sum_{j=1}^{2} \hat{h}_j k_j \frac{\sigma \left( \xi_1 - \xi_1 + \alpha \right) \sigma \left( \xi_2 - \xi_2 + \beta \right) \sigma \left( \xi_3 - \xi_3 + \alpha \right)}{\sigma \left( \xi_1 - \xi_1 + \alpha \right) \sigma \left( \xi_2 - \xi_2 + \beta \right) \sigma \left( \xi_3 - \xi_3 + \alpha \right)} \sigma \left( \xi_1 - \xi_1 - \beta \right) \sigma \left( \xi_2 - \xi_2 - \beta \right) \sigma \left( \xi_3 - \xi_3 - \beta \right) \]
\[ = \sum_{j=1}^{2} \hat{k}_j h_j \frac{\sigma \left( \xi_1 - \xi_1 - \beta \right) \sigma \left( \xi_2 - \xi_2 - \beta \right) \sigma \left( \xi_3 - \xi_3 - \beta \right)}{\sigma \left( \xi_1 - \xi_1 - \beta \right) \sigma \left( \xi_2 - \xi_2 - \beta \right) \sigma \left( \xi_3 - \xi_3 - \beta \right)} \sigma \left( \xi_1 - \xi_1 + \alpha \right) \sigma \left( \xi_2 - \xi_2 + \alpha \right) \sigma \left( \xi_3 - \xi_3 + \alpha \right) \sigma \left( \xi_1 - \xi_2 - \beta \right) \sigma \left( \xi_2 - \xi_3 - \beta \right) \sigma \left( \xi_1 - \xi_2 - \beta \right) \]
\[ \forall \ i, j = 1, 2, 3. \]  
\tag{3.17}

To write (3.17) in a more concise way, we denote the coefficients on the lhs and rhs of the equation as \( A_{ij} \equiv A_{ij} \left( \xi_1, \xi_2, \xi_3, \alpha, \beta \right) \) and \( B_{ij} \equiv B_{ij} \left( \xi_1, \xi_2, \xi_3, \alpha, \beta \right) \) respectively. Noting the common factors \( \hat{h}_j / \hat{k}_j (j = 1, 2, 3) \) in these equations, we next derive the system of six equations:

\[ \frac{h_j}{k_j} = \frac{A_{11} \hat{h}_1 + A_{12} \hat{h}_2}{B_{11} \hat{k}_1 + B_{12} \hat{k}_2} = \frac{A_{21} \hat{h}_1 + A_{22} \hat{h}_2}{B_{21} \hat{k}_1 + B_{22} \hat{k}_2} = \frac{A_{31} \hat{h}_1 + A_{32} \hat{h}_2}{B_{31} \hat{k}_1 + B_{32} \hat{k}_2} \quad (j = 1, 2, 3). \]  
\tag{3.18}

\( ^6 \) Instead of \( h_3 \) and \( k_3 \) we could have eliminated \( h_1 \) or \( h_2 \) and \( k_1 \) or \( k_2 \) yielding equivalent results.
We can rewrite the resulting set of relation (3.18) as
\[
(A_{11j}B_{21j} - A_{21j}B_{11j})\hat{h}_1\hat{k}_1 + (A_{11j}B_{22j} - A_{21j}B_{12j})\hat{h}_1\hat{k}_2 = 0,
\]
\[
(A_{12j}B_{21j} - A_{22j}B_{11j})\hat{h}_2\hat{k}_1 + (A_{12j}B_{22j} - A_{22j}B_{12j})\hat{h}_2\hat{k}_2 = 0,
\]
\[
(A_{11j}B_{31j} - A_{31j}B_{11j})\hat{h}_1\hat{k}_1 + (A_{11j}B_{32j} - A_{31j}B_{12j})\hat{h}_1\hat{k}_2 = 0,
\]
\[
(A_{12j}B_{31j} - A_{32j}B_{11j})\hat{h}_2\hat{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\hat{h}_2\hat{k}_2 = 0,
\]
\[
(A_{21j}B_{31j} - A_{31j}B_{21j})\hat{h}_1\hat{k}_1 + (A_{21j}B_{32j} - A_{31j}B_{22j})\hat{h}_1\hat{k}_2 = 0,
\]
\[
(A_{22j}B_{31j} - A_{32j}B_{21j})\hat{h}_2\hat{k}_1 + (A_{22j}B_{32j} - A_{32j}B_{22j})\hat{h}_2\hat{k}_2 = 0,
\]

(j = 1, 2, 3),

\[(3.19)\]

where
\[
A_{ij} = \frac{\sigma(\hat{x}_i - \hat{x}_j - \hat{z}_j + \xi_j - \alpha + \beta)\sigma(\hat{x}_i - \hat{x}_j)}{\sigma(\hat{x}_i - \hat{x}_j - \alpha)\sigma(\hat{x}_i - \hat{x}_j - \beta)\sigma(\hat{x}_i - \hat{x}_j - \alpha)}.
\]

\[(3.20a)\]

\[
B_{ij} = \frac{\sigma(\hat{x}_i - \hat{x}_j + \xi_j + \alpha - \beta)\sigma(\hat{x}_i - \hat{x}_j)}{\sigma(\hat{x}_i - \hat{x}_j - \beta)\sigma(\hat{x}_i - \hat{x}_j - \alpha)\sigma(\hat{x}_i - \hat{x}_j - \alpha)}.
\]

\[(3.20b)\]

We observe that these homogeneous bilinear systems for the variables \(\hat{h}_1, \hat{k}_1, \hat{h}_2\) and \(\hat{k}_2\) can be resolved by using Cayley’s three-dimensional \(2 \times 2 \times 2\)-hyperdeterminant [13]. Let us recall the general statement (see also [16]):

**Definition 1.** The hyperdeterminant of \(2 \times 2 \times 2\) hyper-matrix \(A = (a_{ijk})\) \((i, j, k = 0, 1)\) is given by:

\[
\text{Det}(A) = \left[\begin{array}{c}
\det\begin{pmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{pmatrix} + \det\begin{pmatrix} a_{100} & a_{010} \\ a_{101} & a_{011} \end{pmatrix} \\
-4 \det\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix} \end{array}\right]^2.
\]

\[(3.21)\]

Its main property is the following:

**Proposition 3.2.1.** The hyper-determinant (3.21) vanishes identically iff the following set of bilinear equations with six unknowns
\[
\begin{align*}
a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 &= 0, \\
a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 &= 0, \\
a_{010}x_0z_0 + a_{011}x_0z_1 + a_{101}x_1z_0 + a_{111}x_1z_1 &= 0, \\
a_{000}x_0z_0 + a_{001}x_0z_1 + a_{010}x_1z_0 + a_{011}x_1z_1 &= 0, \\
a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 &= 0, \\
a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 &= 0.
\end{align*}
\]

\[(3.22)\]

has a non-trivial solution (i.e., for which none of the vectors \(x = (x_0, x_1), y = (y_0, y_1), z = (z_0, z_1)\) are equal to the zero vector).
A proof of this statement can be found in [32]. The cubic hyper-matrix $A$ can be illustrated by the diagram of entries as given in figure 3.

In the case at hand, the components $a_{ijk}$ can be readily identified by comparing (3.19) with the system (3.22) and the variables $x_i$, $y_j$ with the $h_i$ and $k_j$ respectively. Noting that these particular coefficients are all $2 \times 2$ determinants, it turns out that the following compound theorem for hyper-determinants is directly applicable.

**Lemma 1** (Compound theorem for $2 \times 2 \times 2$ hyper-determinants). The following identity holds for the compound hyper-determinants of format $2 \times 2 \times 2$:

\[
\begin{vmatrix}
  a & a' & a'' \\
  b & b' & c' & d' \\
  c & c' & d' & d'' \\
  b & b' & c' & d''
\end{vmatrix}^2 + 
\begin{vmatrix}
  a' & a'' & a' \\
  b' & b'' & d' \\
  c' & c'' & c' \\
  b' & b'' & d''
\end{vmatrix}^2
- 4 \begin{vmatrix}
  a & a' & a'' \\
  b & b' & c' & d' \\
  c & c' & d' & d'' \\
  b & b' & c' & d''
\end{vmatrix}^2 = 
\begin{vmatrix}
  a & a'' \\
  b & b' \\
  c & c' \\
  d & d''
\end{vmatrix}^2.
\] (3.23)

**Proof.** This can be established by direct computation. Assuming w.l.o.g. that the entries $a''$, $b''$, $c''$, $d''$ are all non-zero, we can take out the common product $(a''b''c''d'')^2$ from all terms on the left-hand side. Denoting all the ratios $a''a''$, $a''a''$ by capitals $A$, $A'$ etc, and noting that the $2 \times 2$ determinant

\[
\begin{vmatrix}
  a & a'' \\
  b & b''
\end{vmatrix}
\]

is simply given by $A - B$ (and in a similar way the other determinants occurring in the expression on the left-hand side), then the left-hand side of (3.23) is representable by

\[
ad^2 b^2 c^2 d^2 \left[ \left( \begin{vmatrix}
  A - B & A'' - D' \\
  C - B & C' - D'
\end{vmatrix} + \begin{vmatrix}
  A' - B' & A'' - D' \\
  C' - B' & C' - D'
\end{vmatrix} \right)^2 \right.

- 4 \begin{vmatrix}
  A - B & A'' - D' \\
  C - B & C' - D'
\end{vmatrix} \begin{vmatrix}
  A' - B' & A'' - D' \\
  C' - B' & C' - D'
\end{vmatrix}
\]

Computing the expression between brackets, we observe that it can be simplified to:

\[
 (A - C)(B' - D') + (D - B)(C' - A') \right] - 4(A - C)(B - D)(A' - C')(B' - D') = \begin{vmatrix}
  A - C & B - D \\
  A' - C' & B' - D'
\end{vmatrix}^2
\]

which leads to the desired result. \( \square \)

To the best of our knowledge this compound theorem is a new result in the theory of hyper-determinants. It seems intimately linked to the structure of the linear equations (the Lax relations) from which it originate in the present context, and there may be analogues for the case of higher rank hyper-determinants (this is currently under investigation). A connection between hyper-determinants and minors of symmetric matrices was established in [18], but it is not clear whether (and if so how) those results are related to the above proposition. Hyperdeterminants have also appeared in the context of integrable systems as reviewed in \[\text{J. Phys. A: Math. Theor.} \text{48 (2015) 035206 N Delice et al.} \]
[33], where it was pointed out that the vanishing of a $2 \times 2 \times 2$ Cayley hyperdeterminant can be interpreted as the lattice CKP equation of [19, 31]. The appearance of the hyperdeterminant in the present paper, however, is of a different nature.

Identifying the coefficients of the system of homogeneous equations (3.19) as entries of a $2 \times 2 \times 2$ hyper-determinant, we observe that the structure of this hyper-determinant is exactly of the form as given in lemma 1, and hence we have the following immediate corollary.

**Proposition 3.2.2.** Identifying the eight entries $(a_{ik})_{i,k=0,1}$ by comparing the first two equations of (3.22) with the system of equations (3.19), the hyper-determinant takes the form as given by the compound theorem, lemma 1, and hence reduces to a perfect square of the form:

\[
\begin{pmatrix}
A_{ij} & A_{i'j'} \\
A_{i'j} & A_{ij'}
\end{pmatrix}
\begin{pmatrix}
A_{ij} & A_{i'j'} \\
A_{i'j} & A_{ij'}
\end{pmatrix}
\begin{pmatrix}
B_{ij} & B_{i'j'} \\
B_{i'j} & B_{ij'}
\end{pmatrix}
\begin{pmatrix}
B_{ij} & B_{i'j'} \\
B_{i'j} & B_{ij'}
\end{pmatrix}
\]

\[
(j = 1, 2, 3),
\]

\[
(3.24)
\]

where

\[
\begin{pmatrix}
A_{ij} & A_{i'j'} \\
A_{i'j} & A_{ij'}
\end{pmatrix}
= \frac{\sigma(\xi_j - \xi_1)\sigma(\xi_j - \xi_2)\sigma(\xi_j - \xi_3)}{\sigma(\xi_j - \xi_i - \alpha)\sigma(\xi_j - \xi_i - \alpha)\sigma(\xi_j - \xi_i - \alpha)\sigma(\xi_j - \xi_i - \alpha)}
\]

\[
\times \left[\frac{\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)}{\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)\sigma(\xi_i - \xi_i - \alpha)}\right],
\]

\[
(3.25)
\]

in which we can set $i, i' = 1, 2, l, l' = 1, 2 \neq 3,$ and where we naturally should take $i'' = 3$.

A similar expression for the corresponding determinant in terms of the $B_{ij}$ as given (3.25) interchanging $\alpha$ and $\beta$ and the shifts $\sim$ and $\cdot$.

The form (3.25) of the relevant $2 \times 2$ determinants, using the expressions for the entries (3.20), is computed in appendix C.

Figure 3. Cayley cube.
We apply now the compound theorem, lemma 1, to the system of homogeneous equations (3.19). In fact from that system of equations it follows that the ratios $h_{ij}$ and $k_{ij}$ obey quadratic equations whose discriminant, by virtue of the compound theorem, is a perfect square. Thus, those ratios can be obtained in a rather simple form. We distinguish between the two cases: (i) the hyper-determinant in question, i.e. the determinant (3.24), vanishes, and (ii) the hyper-determinant is non-zero.

(i) Case (3.24) = 0. In this case the resulting set of equations is given by the vanishing of the hyper-determinant, i.e. the set of equations:

\[
\begin{vmatrix}
A_{ij} & A_{i'j} & A_{i''j} & A_{i''j}
\end{vmatrix} = 0
\]

Inserting the explicit expression (3.25), and its counterpart in terms of the quantities $B_{ij}$, into (3.26) we obtain the relations

\[
\frac{\sigma(\xi_i + \xi_i' - \xi_i - \xi_i' - \xi_i + \xi_i' + \beta - 2\alpha)}{\sigma(\xi_i + \xi_i' - \xi_i' - \xi_i' - \xi_i + \xi_i' + \beta - 2\alpha)} \times \frac{\sigma(\xi_i' - \xi_i' - \alpha)\sigma(\xi_i - \xi_i' - \alpha)\sigma(\xi_i - \xi_i - \alpha)}{\sigma(\xi_i - \xi_i' - \alpha)\sigma(\xi_i' - \xi_i - \alpha)\sigma(\xi_i' - \xi_i - \alpha)} = \frac{\sigma(\xi_i' + \xi_i - \xi_i' - \xi_i' - \xi_i + \xi_i' + \alpha - 2\beta)}{\sigma(\xi_i' + \xi_i' - \xi_i' - \xi_i' - \xi_i + \xi_i' + \alpha - 2\beta)} \times \frac{\sigma(\xi_i' - \xi_i - \beta)\sigma(\xi_i' - \xi_i' - \beta)\sigma(\xi_i' - \xi_i - \beta)}{\sigma(\xi_i' - \xi_i - \beta)\sigma(\xi_i' - \xi_i' - \beta)\sigma(\xi_i' - \xi_i - \beta)},
\]

(j = 1, 2, 3),

(3.27)

where again we can set $i, i' = 1, 2, l, l' = 1, 2 \neq 3$, and where we naturally should take $i'' = 3$. The set of relations (3.27) is a coupled system of three quadrilateral equations (for $j = 1, 2, 3$) of 3-leg type, i.e. in terms of three independent variables which reside in the arguments of the Weierstrass $\sigma$-functions. We note that all three equations (for $j = 1, 2, 3$) have a common factor, which in the case of a further reduction $\xi_i + \xi_3 + \xi_3 = 0$ (mod period lattice) involves only the ‘long legs’ (i.e. the differences over the diagonal). Thus, this system of equations may be too simple to figure as a proper candidate for a higher-rank analogue of the Adler lattice equation.

(ii) Case (3.24) $\neq 0$. As a consequence of the compound theorem, lemma 1, the hyper-determinant in the case at hand is a perfect square. Thus, going back to the system (3.19), by first eliminating the ratio $h_i/h_j$, we obtain a quadratic for the ratio $k_{ij}$, $(i, j = 1, 2)$ from which the latter can be solved using the fact that the discriminant of the quadratic (which coincides with the hyper-determinant) is a perfect square. Thus, we get rather manageable expressions for the solutions of the mentioned ratios in terms of the $2 \times 2$ determinant when $\xi_i + \xi_3 + \xi_3 = 0$ (mod period lattice).
determinants involving the expressions $A_{ij}$ and $B_{ij}$. The result of this computation is the following:

**Proposition 3.2.3.** If the expression (3.24) is non-vanishing, we have the following solutions of the system (3.19) given in terms of the ratios (i.e., up to a common multiplicative factor)

\[\hat{h}_1 = -\frac{A_{12j}}{A_{31j}}, \quad \hat{k}_1 = -\frac{B_{32j}}{B_{31j}}, \quad \left(\text{either together with } \frac{\hat{h}_1}{h_2} = -\frac{A_{12j}}{A_{31j}}, \frac{\hat{k}_1}{k_2} = -\frac{B_{32j}}{B_{31j}},\right)\]  
\[\text{or } \frac{\hat{h}_1}{h_2} = -\frac{B_{1ij}}{A_{1ij}} \frac{A_{2ij}}{B_{2ij}} \frac{B_{3ij}}{A_{3ij}}, \quad \frac{\hat{k}_1}{k_2} = -\frac{A_{1ij}}{A_{31j}} \frac{A_{2ij}}{A_{32j}} \frac{B_{3ij}}{B_{32j}},\]  
\[(j = 1, 2, 3)\]  
(3.28a)

The proof, once again, is by direct computation and involves some determinantal manipulations.

The system of equations resulting from (3.28a), inserting the explicit expressions for the quantities $A$ and $B$ from (3.20) reads as follows

\[\frac{\hat{h}_1}{h_2} = -\frac{\sigma(\xi_j^2 - \xi_j - \xi_j^2 \xi_j \alpha - \beta)}{\sigma(\xi_j^2 - \xi_j - \xi_j^2 + \xi_j^2 \alpha - \beta)}\]  
(3.29a)

\[\frac{\hat{k}_1}{k_2} = -\frac{\sigma(\xi_j^2 - \xi_j - \xi_j^2 \xi_j \alpha - \beta)}{\sigma(\xi_j^2 - \xi_j - \xi_j^2 \alpha - \beta)}\]  
(3.29b)

Inserting the expressions of (3.20) into the system of equations (comprising the equations for different values of $j = 1, 2, 3$). The system of equations (3.29) for $j = 1, 2, 3$, we do not consider to be viable because it seems to be too overdetermined taking into account the common factors in (3.29a) and (3.29b). Furthermore, neither does it admit the natural solution $\xi_j(n, m) = \xi_j(0, 0) + n\alpha + m\beta$ nor does it admit the reduction $\xi_j + \xi_j + \xi_j = 0$ (mod period lattice). Thus, we reject this system of equations.

Turning now to the system given by (3.28b) for $j = 1, 2, 3$, this constitutes a more complicated system of quadrilateral elliptic 3-leg type of equations, which can be written as a set of equalities:
with the determinants expanded by means of the formulae:

\[ A_{ij} = \frac{\sigma (\xi_i - \xi_j)\sigma (\xi_i - \xi_3 + \xi_j - \alpha + \beta)\sigma (\xi_i - \xi_3)}{\sigma (\xi_i - \xi_1)\sigma (\xi_i - \xi_j - \beta)\sigma (\xi_i - \xi_3 - \alpha)\sigma (\xi_3 - \xi_j - \beta)} \], (3.31)

and

\[ A_{ij} = A_{ij} \]

\[ A_{ij} = A_{ij} \]

\[ A_{ij} = A_{ij} \]

\[ A_{ij} = A_{ij} \]

\[
\begin{vmatrix}
B_{111} & A_{121} & B_{121} \\
B_{211} & A_{221} & B_{221} \\
B_{311} & A_{321} & B_{321}
\end{vmatrix}
= \begin{vmatrix}
B_{112} & A_{122} & B_{122} \\
B_{212} & A_{222} & B_{222} \\
B_{312} & A_{322} & B_{322}
\end{vmatrix}
= \begin{vmatrix}
B_{113} & A_{123} & B_{123} \\
B_{213} & A_{223} & B_{223} \\
B_{313} & A_{323} & B_{323}
\end{vmatrix},
\]

(3.30a)

\[
\begin{vmatrix}
A_{111} & A_{121} & B_{121} \\
A_{211} & A_{221} & B_{221} \\
A_{311} & A_{321} & B_{321}
\end{vmatrix}
= \begin{vmatrix}
A_{112} & A_{122} & B_{122} \\
A_{212} & A_{222} & B_{222} \\
A_{312} & A_{322} & B_{322}
\end{vmatrix}
= \begin{vmatrix}
A_{113} & A_{123} & B_{123} \\
A_{213} & A_{223} & B_{223} \\
A_{313} & A_{323} & B_{323}
\end{vmatrix},
\]

(3.30b)

with the determinants obtained from these by interchanging \( \sim \) and \( \sim \) and \( \alpha \) and \( \beta \). Explicit forms of the equations (3.30a) and (3.30b) can be obtained by expanding the 3 \( \times \) 3 determinants along the \( A \)- and \( B \)-columns respectively using the expression (3.31) and (3.32) and their \( B \)-counterparts. These determinants, which are almost but not quite of Frobenius (i.e., elliptic Cauchy) type do not simplify very drastically, giving rise to a rather complicated system of four equations for three variables \( \xi_1, \xi_2, \xi_3 \). However, it seems that one of the four equations is redundant as was observed to be the case in the rational limit performing random numerical substitutions for initial values. Although we do not yet have a general proof of this redundancy in the elliptic case (due to the implicitness of the equations), it seems natural to expect this to be the case, and hence that the system effectively reduces to a coupled set of three quadrilateral lattice equations for the three dependent variables \( \xi_1, \xi_2, \xi_3 \). This system is given explicitly in appendix D. We further remark that this system allows for a trivial solution of type \( \xi_i = \xi_i + \alpha n + \beta m(i = 1, 2, 3) \). The problem of finding a rational form for the system of equations, as well as verifying their reducibility under the additional constraint \( \xi_1 + \xi_2 + \xi_3 = 0 \) (mod period lattice) is currently

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under investigation. If so, the latter system of equations can be duly regarded as a higher-rank version of Adler’s lattice equation in 3-leg form (1.6).

4. Discussion

There are, to date, several types of integrable systems that are associated with elliptic curves (containing parameters that can be identified as the corresponding moduli). Such systems include the elliptic version of the Calogero–Moser [12] and Ruijsenaars–Schneider [30] many-body systems, (their discrete-time and spin generalizations [21, 22, 25]), and the elliptic Toda lattice [23]. In this paper we have proposed and investigated a general class of higher-rank elliptic Lax representations for systems of partial difference equations on the 2D lattice. Distinguishing between what we called spin-zero (generalizations of Adler’s lattice equation) and spin-non-zero (generalized LL type) models, we gave the general structure of the resulting equations (from the compatibility conditions) for the latter, but concentrated mainly on the former case for $N = 2$ and $N = 3$. For $N = 2$ we have shown that the Lax systems leads indeed to Adler’s lattice equation in its 3-leg form (for the Weierstrass class) and we have analyzed how these results generalize to the case $N = 3$ (as a representative example for the higher-rank case). Having established the resulting systems of equations, generalizing Adler’s 3-leg form, further work is needed to properly identify those systems. Thus, in further study we will investigate their rational and hyperbolic degenerations, as well as their continuum limits. A possible outcome would be to establish a connection with a differential system obtained by Mokhov in the 1980s [29], arising from third order commuting differential operators defining rank 3 vector bundles over an elliptic curve, see [28]. Regarding the systems of LL type, special subcases remain to be explored, but we have already noted that the one-step periodic reduction leads to the discrete-time elliptic Ruijsenaars–Schneider model of [25].

In our view, the significance of the results of this paper is not only to add a new class of elliptic type of integrable systems to our already substantial zoo of such systems, but to depart from the rather restrictive confinement of $2 \times 2$ systems to which all ABS type systems, [4], belong. To obtain good insights in the essential structures behind (discrete and continuous) integrable systems, such departures into the multi-component cases are necessary. In the present paper we concentrated mostly on the spin-zero case, while the elaboration of the spin non-zero case is the subject of a future publication, some initial results of which were already presented in [34]. As a direction for the future, establishing connections, if any, with the recently found master-solution of the quantum Yang–Baxter equations [9] and its multi-spin generalization [10], may be of interest.

Acknowledgment

ND is supported by the Turkish Ministry of National Education. FWN is thankful for the hospitality of the Department of Mathematics of Shanghai University, during a visit where the current paper was finalized. He was partially supported by the EPSRC responsive mode grants EP/I002294/1 and EP/I038683/1. SY is supported by Thailand Research Fund (TRF) under grant number: TRG5680081.
Appendix A. Proof of the Q4 3-leg identity

The proof of the elliptic identity (3.5) can be achieved directly by showing that the coefficients of each monomials $1$, $X$, $Y$, $Z$, $XY$, $XZ$, $YZ$ and $XYZ$ of the identity are equivalent. By expanding the left-hand side of the identity as

$$LHS := (1 - r^2)XYZ + (r^2\phi(\xi - \alpha) - \phi(\xi + \alpha))YZ + (r^2\phi(\xi + \beta) - \phi(\xi - \beta))XZ + (r^2\phi(i\xi + \alpha - \beta) - \phi(i\xi - \alpha + \beta))XY$$

$$+ (\phi(i\xi - \beta)\phi(\xi - \alpha + \beta) - r^2\phi(i\xi + \beta)\phi(\xi + \alpha - \beta))X$$

$$+ (\phi(i\xi + \alpha)\phi(\xi + \alpha + \beta) - r^2\phi(i\xi - \alpha)\phi(\xi + \alpha - \beta))Y$$

$$+ (\phi(i\xi - \alpha)\phi(\xi - \alpha - \beta) - r^2\phi(i\xi + \alpha)\phi(\xi + \alpha - \beta))Z$$

$$+ r^2\phi(i\xi - \alpha)\phi(i\xi + \beta)\phi(\xi + \alpha + \beta) - \phi(i\xi - \alpha)\phi(i\xi - \beta)\phi(\xi - \alpha + \beta),$$

(A.1)

it is obvious that the first term of line 1 is equal to the corresponding term on the right-hand side of (3.5). The rest of the equalities of the corresponding coefficients follow by the same method as explained below. First, we make use of the Frobenius–Stickelberger formula [14] stated in appendix B, in terms of the variables $(\xi, \alpha, -\beta)$

$$\begin{vmatrix}
1 & \phi(\xi) & \phi'(\xi) \\
1 & \phi(\alpha) & \phi'(\alpha) \\
1 & \phi(-\beta) & \phi'(-\beta)
\end{vmatrix} = 2\frac{\sigma(\xi + \alpha - \beta)\sigma(\xi - \alpha)\sigma(\alpha + \beta)\sigma(\xi + \beta)}{\sigma^3(\xi)\sigma^3(\alpha)\sigma^3(\beta)},$$

and a similar relation with $(\xi, -\alpha, \beta)$. If we divide the former determinant by the latter one, we obtained the following expression for $t$ and $s$ in (3.6)

$$t = \frac{\phi'(\xi)(b - a) - Ab - aB + \phi(\xi)(A + B)}{\phi'(\xi)(b - a) + Ab + aB - \phi(\xi)(A + B)},$$

$$s = \frac{4(a - b)\phi'(\xi)}{(\phi'(\xi)(b - a) + Ab + aB - \phi(\xi)(A + B))^2}.$$  

Applying the elliptic addition formulae of the form, notably:

$$\phi(\xi) + \phi(\eta) + \phi(\xi \pm \eta) = \frac{1}{4} \left( \frac{\phi'(\xi) \mp \phi'(\eta)}{\phi(\xi) - \phi(\eta)} \right)^2,$$  

(A.2)

on (A.1), we get on the one hand

$$LHS = (1 - r^2)XYZ + \left( a + \phi(\xi) - \frac{(\phi'(\xi) - A)^2}{4\phi(\xi) - a^2} \right)YZ$$

$$+ r^2\left( -d - \phi(\xi) + \frac{(\phi'(\xi) + A)^2}{4\phi(\xi) - a^2} \right)XZ$$

$$+ \left( b + \phi(\xi) - \frac{(\phi'(\xi) + B)^2}{4\phi(\xi) - b^2} \right)YZ$$

$$+ \left( d + \phi(\xi) - \frac{(\phi'(\xi) - B)^2}{4\phi(\xi) - b^2} \right)XZ$$

on the other hand.
The proof is completed by using the relations (1.5) and subsequently (1.3), (1.4) on the terms of (A.3) and as well as on the right-hand side of (3.5).

Appendix B. Frobenius–Stickelberger type identities

Here we collect a number of results related to elliptic determinantal formulae of Frobenius and Frobenius–Stickelberger type (i.e. elliptic Cauchy and Vandermonde determinants). The Frobenius–Stickelberger formula, [14] is given by

$$
\begin{vmatrix}
\wp(x_1) & \wp'(x_1) & \cdots & \wp^{(n-2)}(x_1) \\
\wp(x_2) & \wp'(x_2) & \cdots & \wp^{(n-2)}(x_2) \\
\wp(x_3) & \wp'(x_3) & \cdots & \wp^{(n-2)}(x_3) \\
\vdots & \vdots & \ddots & \vdots \\
\wp(x_n) & \wp'(x_n) & \cdots & \wp^{(n-2)}(x_n)
\end{vmatrix}
$$

$$(A.3)$$

$$(2,1)$$
\[
\begin{align*}
\phi(x_1) \phi'(x_1) & \cdots \phi^{(n-2)}(x_1) \\
\phi(x_2) \phi'(x_2) & \cdots \phi^{(n-2)}(x_2) \\
\phi(x_3) \phi'(x_3) & \cdots \phi^{(n-2)}(x_3) \\
\vdots & \quad \vdots & \quad \ddots & \quad \vdots \\
\phi(x_n) \phi'(x_n) & \cdots \phi^{(n-2)}(x_n)
\end{align*}
\]

Denoting the Frobenius–Stickelberger matrix \( P(x_1, \ldots, x_n) = P(x) \) by:

\[
P(x) = \begin{bmatrix}
1 & \phi(x_1) & \cdots & \phi^{(n-2)}(x_1) \\
1 & \phi(x_2) & \cdots & \phi^{(n-2)}(x_2) \\
1 & \phi(x_3) & \cdots & \phi^{(n-2)}(x_3) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \phi(x_n) & \cdots & \phi^{(n-2)}(x_n)
\end{bmatrix},
\]

we have by using Cramer’s rule the following factorization formula:

\[
\left[ P(x) \cdot P(y)^{-1} \right]_{i,j} = \frac{1}{\sigma^n(x_i)} \Phi_\Sigma(x_i - y_j) \sigma^n(y_j) \prod_{i \neq j} \sigma(x_i - y_j),
\]

in which \( \Sigma \equiv \Sigma^N = \sum_i y_i \). As a consequence we obtain from this the Frobenius determinantal formula [15]:

\[
\det \begin{bmatrix}
\phi_i(x_i - y_j)
\end{bmatrix}_{i,j=1,\ldots,N} = \frac{\sigma(x + \Sigma) \prod_{i \neq j} \sigma(x_i - x_j) \sigma(y_j - y_i)}{\prod_{i,j} \sigma(x_i - y_j)},
\]

where \( x \) denotes any of the zeroes \( x_i \), \( i = 1, \ldots, N \). Both (B.5) can be obtained from the Frobenius formula by a set of degenerate limits. The elliptic Lagrange interpolation formulae

\[
\prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \frac{\sigma(x_i - x_j)}{\prod_{j \neq i} \sigma(y_i - y_j)},
\]

which holds if \( \Sigma \neq 0 \), and if \( \Sigma = 0 \):

\[
\prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \left[ \frac{\sigma(\xi - y_i) - \sigma(\xi - y_j)}{\prod_{j \neq i} \sigma(y_i - y_j)} \right] \prod_{j=1}^N \frac{\sigma(y_i - x_j)}{\prod_{j \neq i} \sigma(y_j - y_i)},
\]

where \( x \) denotes any of the zeroes \( x_i \), \( i = 1, \ldots, N \). Both (B.5) can be obtained from the Frobenius formula [15] by row-or column expansions (adding an extra row and column to the Frobenius matrix, say with \( x_0 = \xi \) and \( y_0 = \eta \), and then expanding along that row or column) and (B.6) can subsequently be obtained from a limiting case of the latter.

**Appendix C. Proof of equation (3.25)**

Here, we present the proof of the determinant in (3.25). By definition of \( A_{ij} \) given in (3.20) we have
\[
\begin{vmatrix}
A_{ij} & A_{i'j'} \\
A_{i'j} & A_{i'i'j'}
\end{vmatrix} = \frac{\sigma(\xi_i - \xi_j)\sigma(\xi_{i'} - \xi_{j'})}{S(\xi_i)S(\xi_{i'})\sigma(\xi_i - \xi_j - \beta)\sigma(\xi_{i'} - \xi_{j'} - \beta)\sigma^2(\xi_{i'} - \xi_{j'} - \beta)}
\]

\[
\begin{align*}
&= \sigma(\xi_i - \xi_j - \beta)\sigma(\xi_{i'} - \xi_{j'} - \beta)\sigma^2(\xi_{i'} - \xi_{j'} - \beta) \\
&\times \sigma(\xi_i - \xi_j - \alpha)\sigma(\xi_{i'} - \xi_{j'} - \alpha) \\
&- \sigma(\xi_{i'} - \xi_{i'} - \alpha)\sigma(\xi_i - \xi_j - \alpha + \beta) \\
&\times \sigma(\xi_{i'} - \xi_{i'} - \alpha)\sigma(\xi_i - \xi_j - \alpha + \beta).
\end{align*}
\]

where

\[
S(\xi) = \sigma(\xi - \xi_i - \alpha)\sigma(\xi - \xi_{i'} - \alpha)\sigma(\xi - \xi_j - \alpha).
\] (C.2)

Noting that the difference in the bracket can be simplified by applying the three-term relation for the \(\sigma\)-function in the following form:

\[
\sigma(x - a)\sigma(y - b)\sigma(z - b)\sigma(w - a) - \sigma(x - a)\sigma(z - a)\sigma(w - b) = \sigma(z + y - a - b)\sigma(x - y)\sigma(x - z)\sigma(b - a),
\] (C.3)

in which \(x - y = z - w\). Making now the following choice for \(x, y, z, w, a\) and \(b\) in the identity (C.3):

\[
x = \xi_i - \xi_j + \xi_j - \alpha + \beta \\
y = \xi_{i'} - \xi_{j'} + \xi_{j'} - \alpha + \beta \\
z = \xi_{i'} - \alpha \\
w = \xi_i - \alpha \\
a = \xi_i \\
b = \xi_{i'}.
\]

the expression between brackets on the right-hand side of (C.1) simplifies to

\[
[\cdots] = \sigma(\xi_{j'3} + \xi_{j'3} + \beta)\sigma(\xi_{i'3} + \xi_i - \xi_{i'} - \xi_j - 2\alpha + \beta) \\
\times \sigma(\xi_{i'} - \xi_{j'})\sigma(\xi_{i'} - \xi_{j'}).
\] (C.4)

Substituting the right-hand side of (C.4) into (C.1) and canceling the first factor against the corresponding factor in the prefactor of (C.1), using the fact that \(\sigma\) is an odd function, we obtain the desired result given by the determinant in (3.25). In a similar way (or by making the obvious replacements \(\alpha \leftrightarrow \beta\) and \(\xi_i \leftrightarrow \xi_{i'}\)) the computation of the \(2 \times 2\) determinant \(B_{i'j'}\) can be verified.

**Appendix D. Higher-rank \(N = 3\) Elliptic lattice systems (3.30) in explicit form**

Firstly, two equations from the determinantal equality (3.30a) can be explicitly evaluated by using the expression for (3.32) and (3.31). The fist one is given as follows:
\[
\begin{align*}
\sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + \xi_1 + \alpha + 2\beta)\sigma(\xi_3 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_1)S(\xi_2)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
- \sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + \xi_1 + \alpha + 2\beta)\sigma(\xi_3 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_1)S(\xi_2)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
+ \sigma(\xi_2 - \xi_3)\sigma(\xi_2 + \xi_3 - \xi_2 - \xi_3 + \xi_2 + \xi_1 + 2\beta)\sigma(\xi_1 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_2)S(\xi_3)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
+ \sigma(\xi_2 - \xi_3)\sigma(\xi_2 + \xi_3 - \xi_2 - \xi_3 + \xi_2 + \xi_1 + 2\beta)\sigma(\xi_1 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_2)S(\xi_3)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
\frac{\sigma(\xi_2 - \xi_1)\sigma(\xi_2 + \xi_1 - \xi_2 - \xi_1 + \xi_2 + \xi_1 + a + 2\beta)\sigma(\xi_3 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha)}{S(\xi_1)S(\xi_2)} = \frac{\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a)}{S(\xi_2)S(\xi_3)} \tag{D.1a}
\end{align*}
\]

where

\[S(\xi) = \sigma(\xi - \xi_1 - \beta)\sigma(\xi - \xi_2 - \beta)\sigma(\xi - \xi_3 - \beta).\]

The second one can be obtained from the first equation (D.1a) by interchanging \(\xi_2\) and \(\xi_3\). Namely,}

\[
\begin{align*}
\sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + \xi_1 + \alpha + 2\beta)\sigma(\xi_3 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_1)S(\xi_2)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
- \sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + \xi_1 + \alpha + 2\beta)\sigma(\xi_3 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_1)S(\xi_2)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
+ \sigma(\xi_2 - \xi_3)\sigma(\xi_2 + \xi_3 - \xi_2 - \xi_3 + \xi_2 + \xi_1 + 2\beta)\sigma(\xi_1 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_2)S(\xi_3)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a) \\
+ \sigma(\xi_2 - \xi_3)\sigma(\xi_2 + \xi_3 - \xi_2 - \xi_3 + \xi_2 + \xi_1 + 2\beta)\sigma(\xi_1 - \xi_2 - \xi_3 + \xi_1 + \beta - \alpha) \\
S(\xi_2)S(\xi_3)\sigma(\xi_1 - \xi_2 - a)\sigma(\xi_3 - \xi_2 - a)
\end{align*}
\]
\[ + \sigma(\xi_2 - \xi_1) \sigma(\xi_3 + \xi_1 - \xi_2 - \xi_3 + \xi_3 + \xi_1 - 2\alpha) \sigma(\xi_3 - \xi_2 + \xi_3 + \xi_1 - \beta + a) \] 
\[ \frac{S(\xi_2) S(\xi_3) \sigma(\xi_3 - \xi_2 - a) \sigma(\xi_3 - \xi_2 - a)}{S(\xi_2) S(\xi_3) \sigma(\xi_1 - \xi_2 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a)} \]
\[ - \sigma(\xi_2 - \xi_3) \sigma(\xi_3 + \xi_1 - \xi_2 - \xi_3 + \xi_3 + \xi_1 - 2\alpha) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a) \] 
\[ \frac{S(\xi_2) S(\xi_3) \sigma(\xi_1 - \xi_2 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a)}{S(\xi_2) S(\xi_3) \sigma(\xi_1 - \xi_2 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a)} \]
\[ \times \sigma(\xi_2 - \xi_3) \sigma(\xi_3 + \xi_1 - \xi_2 - \xi_3 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a) \] 
\[ \frac{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)}{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)} \]
\[ \sigma(\xi_1 - \xi_3) \sigma(\xi_1 + \xi_3 - \xi_1 - \xi_3 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a) \] 
\[ \frac{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)}{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)} \]
\[ \sigma(\xi_1 - \xi_3) \sigma(\xi_1 + \xi_3 - \xi_1 - \xi_3 + \xi_3 + \xi_1 + a - 2\beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta - a) \] 
\[ \frac{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)}{S(\xi_1) S(\xi_3) \sigma(\xi_3 - \xi_1 - a) \sigma(\xi_3 - \xi_1 - a)} \], \text{(D.1b)}

where

\[ S(\xi) = \sigma(\xi - \xi_3 - \beta) \sigma(\xi - \xi_3 - \beta) \sigma(\xi - \xi_3 - \beta). \]

Explicit form of the third one arising from (3.30b) can be acquired from the first equality (D.1a) by interchanging \( \sim \) and \( \hat{\sim} \) and \( \alpha \) and \( \beta \). The first one is given as follows:

\[ \sigma(\xi_1 - \xi_3) \sigma(\xi_1 + \xi_3 - \xi_1 - \xi_3 + \xi_3 + \xi_1 - 2\alpha + \beta) \sigma(\xi_1 - \xi_3 + \xi_3 + \xi_1 - \beta + a) \] 
\[ \frac{K(\xi_1) K(\xi_3) \sigma(\xi_3 - \xi_2 - \beta) \sigma(\xi_3 - \xi_2 - \beta)}{K(\xi_1) K(\xi_3) \sigma(\xi_3 - \xi_2 - \beta) \sigma(\xi_3 - \xi_2 - \beta)} \]
\[ - \sigma(\xi_2 - \xi_3) \sigma(\xi_2 + \xi_3 - \xi_2 - \xi_3 + \xi_3 + \xi_1 - 2\alpha + \beta) \sigma(\xi_2 - \xi_3 - \xi_2 + \xi_3 + \xi_1 + \beta + a) \] 
\[ \frac{K(\xi_2) K(\xi_3) \sigma(\xi_3 - \xi_2 - \beta) \sigma(\xi_3 - \xi_2 - \beta)}{K(\xi_2) K(\xi_3) \sigma(\xi_3 - \xi_2 - \beta) \sigma(\xi_3 - \xi_2 - \beta)} \]
\[ + \sigma(\xi_3 - \xi_1) \sigma(\xi_3 + \xi_1 - \xi_3 - \xi_1 - \xi_2 - \xi_3 + \xi_3 - \xi_2 + \xi_3 + \xi_1 - \beta + a) \] 
\[ \frac{K(\xi_3) K(\xi_1) \sigma(\xi_1 - \xi_3 - \beta) \sigma(\xi_1 - \xi_3 - \beta)}{K(\xi_3) K(\xi_1) \sigma(\xi_1 - \xi_3 - \beta) \sigma(\xi_1 - \xi_3 - \beta)} \]
\[
\begin{align*}
\left[\sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + 2\alpha + \beta)\sigma(\xi_1 - \xi_2 + 2\alpha + \beta)\right] \\
K(\xi_1)K(\xi_2)\sigma(\xi_1 - \xi_2 - \beta)\sigma(\xi_1 - \xi_2 - \beta) \\
\left[\sigma(\xi_1 - \xi_2)\sigma(\xi_1 + \xi_2 - \xi_1 - \xi_2 + 2\alpha + \beta)\sigma(\xi_1 - \xi_2 + 2\alpha + \beta)\right] \\
K(\xi_1)K(\xi_2)\sigma(\xi_1 - \xi_2 - \beta)\sigma(\xi_1 - \xi_2 - \beta) \\
\left[\sigma(\xi_2 - \xi_1)\sigma(\xi_2 + \xi_1 - \xi_2 - \xi_1 + 2\alpha + \beta)\sigma(\xi_2 - \xi_1 + 2\alpha + \beta)\right] \\
K(\xi_2)K(\xi_1)\sigma(\xi_2 - \xi_1 - \beta)\sigma(\xi_2 - \xi_1 - \beta) \\
\left[\sigma(\xi_2 - \xi_1)\sigma(\xi_2 + \xi_1 - \xi_2 - \xi_1 + 2\alpha + \beta)\sigma(\xi_2 - \xi_1 + 2\alpha + \beta)\right] \\
K(\xi_2)K(\xi_1)\sigma(\xi_2 - \xi_1 - \beta)\sigma(\xi_2 - \xi_1 - \beta) \\
\end{align*}
\]
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