Spontaneous Polarization of the Kondo problem associated with the higher-spin six-vertex model

Nobuhisa Fukushima

Department of Mathematics
School of Science and Engineering, Waseda University, 3-4-1, Okubo Shinjuku-ku, Tokyo 169, JAPAN

Takeo Kojima

Department of Mathematics
College of Science and Technology, Nihon University, 1-8, Kanda-Surugadai, Chiyoda-ku, Tokyo 101, JAPAN

Abstract

We study the multi-channel Kondo model associated with an integrable higher-spin analog of the anti-ferroelectric 6-vertex model, which is constructed by inserting a spin $\frac{1}{2}$ to spin 1 lines: $\cdots \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cdots$.

We formulate the problem in terms of representation theory of quantum affine algebra $U_q(\widehat{sl}_2)$. We derive an exact formula of the spontaneous staggered polarization for our model, which corresponds to Baxter’s formula for the 6-vertex model.
1 Introduction

R. Baxter [2] studied spontaneous staggered polarization of the 6-vertex model in 1973. He derived an exact formula of this quantity by the Transfer Matrix Method:

\[
\frac{(q^2; q^2)_\infty^2}{(-q^2; q^2)_\infty^2}.
\]  
(1)

Here we have used the standard notation

\[(z; p)_\infty = \prod_{n=0}^{\infty} (1 - p^n z).\]

In 1976 R. Baxter [3] invented the Corner Transfer Matrix Method. The calculation of the spontaneous staggered polarization is reduced to counting the multiplicities of the eigenvalues of the Corner Transfer Matrix. It was recognized that in many interesting cases the eigenvalue of the Corner Transfer Matrix can be described in terms of the characters of affine Lie algebras. Kyoto-school [4], [1] gave the mathematical explanations of the Corner Transfer Matrix Method, and at the same time they invented the representation theoretical approach to solvable lattice models. Kyoto-school’s approach reproduces Baxter’s formula (1) and makes it possible to calculate the quantities which cannot be calculated by the Corner Transfer Matrix Method. Kyoto-school’s methods have been applied to various problems [5], [6], [7], [8]. A. Nakayashiki [9] introduced new-type vertex operators and gave the mathematical formulation of the usual Kondo model.

In this article we consider the multi-channel Kondo problem [10] associated with an integrable higher-spin analog of the anti-ferroelectric 6-vertex model, which is constructed by inserting a spin $\frac{1}{2}$ to spin 1 lines:

\[
\ldots \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \ldots .
\]

This problem has quantum affine symmetry $U_q(\widehat{sl}_2)$. Our main result is an exact formula of spontaneous staggered polarization:

\[
- \frac{1}{1 - q^4} \frac{(q^{16}; q^{16})_\infty}{(q^4; q^4)_\infty} \left\{ (1 + q^4) \frac{(-q^4; q^8)_\infty}{(-q^4; q^4)_\infty^2} - 2q^2 \frac{(-q^8; q^{16})_\infty^2}{(-q^2; q^4)_\infty^2} - 4q^4 \frac{(-q^{16}; q^{16})_\infty^2}{(-q^2; q^4)_\infty^2} \right\}.
\]

Now a few words about the organization of the paper. In section 2 we set the problem and state the main result. In section 3 we derive an exact formula of spontaneous staggered polarization.

2 Problem and Result

The purpose of this section is to set the problem and summarize the main result.
2.1 Quantum affine algebra $U_q(\widehat{sl}_2)$

We follow the notation of [1]. We give definitions of quantum affine Lie algebras $U_q(\widehat{sl}_2)$, highest weight modules, and principal evaluation modules.

Consider a free abelian group on the letters $\Lambda_0, \Lambda_1, \delta$:

$$P = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \mathbb{Z} \delta.$$ 

Define the simple roots $\alpha_0, \alpha_1$ and an element $\rho$ by

$$\alpha_0 + \alpha_1 = \delta, \quad \Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}, \quad \rho = \Lambda_0 + \Lambda_1.$$ 

Let $(h_0, h_1, d)$ be an basis of $P^* = \text{Hom}(P, \mathbb{Z})$ dual to $(\Lambda_0, \Lambda_1, \delta)$. Define a symmetric bilinear form by

$$(\Lambda_0, \Lambda_0) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1,$$

$$(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0.$$ 

Regarding $P^* \subset P$ via this bilinear form we have the identification

$$h_0 = \alpha_0, \quad h_1 = \alpha_1, \quad d = \Lambda_0.$$ 

We use the symbol

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$ 

The quantum affine algebra $U_q(\widehat{sl}_2)$ is an algebra with 1 over $\mathbb{C}$, defined on the generators $e_0, e_1, f_0, f_1$ and $q^h$ ($h \in P^*$) through the defining relations:

$$q^h q^{h'} = q^{h+h'}, \quad q^0 = 1,$$

$$q^h e_i q^{-h} = q^{(\alpha_i, h)} e_i, \quad q^h f_i q^{-h} = q^{-(\alpha_i, h)} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_j}{q - q^{-1}},$$

$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j),$$

$$f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j).$$

Here $t_i = q^{\alpha_i}$. We write $U_q'(\widehat{sl}_2)$ for subalgebra of $U_q(\widehat{sl}_2)$ generated by $e_0, e_1, f_0, f_1$, $q^{h_0}, q^{h_1}$, and $U_q(\widehat{sl}_2)$ by $e_1, f_1, q^{h_1}$. We define the coproduct $\Delta$ by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$ 

We define the irreducible highest weight module. Set $P_+ = \mathbb{Z}_{\geq 0} \Lambda_0 \oplus \mathbb{Z}_{\geq 0} \Lambda_1$. For $\lambda \in P_+$, a $U_q(\widehat{sl}_2)$ module $V(\lambda)$ is called an irreducible highest weight module with highest weight $\lambda$ if the following conditions are satisfied: there exists a nonzero vector $|\lambda\rangle \in V(\lambda)$, called the highest weight vector, such that $q^h |\lambda\rangle = q^{(\lambda, h)} |\lambda\rangle (h \in P^*)$, $e_i |\lambda\rangle = f_i^{1, (\lambda, h_i) + 1} |\lambda\rangle = 0 (i = 0, 1)$, and $V(\lambda) =$
in the following. The Boltzmann weights of our model are specified by the spin $q$ and level $k$.

Let $V$ be a module of $U'_q(\widehat{sl}_2)$. We equip $V_\zeta$ with a $U'_q(\widehat{sl}_2)$-module structure by setting

$$e_0(v_t \otimes \zeta^m) = (f_1 v_t) \otimes \zeta^{m+1}, \quad e_1(v_t \otimes \zeta^m) = (e_1 v_t) \otimes \zeta^{m+1},$$
$$f_0(v_t \otimes \zeta^m) = (e_1 v_t) \otimes \zeta^{m-1}, \quad f_1(v_t \otimes \zeta^m) = (f_1 v_t) \otimes \zeta^{m-1},$$
$$t_0 = t_1^{-1}, \quad t_1(v_t \otimes \zeta^m) = (t_1 v_t) \otimes \zeta^m.$$

### 2.2 R-matrix and Lattice Model

In this subsection we will define our two-dimensional lattice model, and summarize the main result. Let $V_\zeta^{(1)} \simeq \mathbb{C}^3$ and $V_\zeta^{(1\frac{1}{2})} \simeq \mathbb{C}^2$ be the $U_q(\widehat{sl}_2)$ principal modules. We fix real numbers $q$ and $\zeta$ as

$$-1 < q < 0, \quad 1 < \zeta < (-q)^{-1},$$

in the following. The Boltzmann weights of our model are specified by the spin $(1, 1)$ R-matrix intertwiner $R^{(1,1)}(\zeta)$ and the spin $(1\frac{1}{2}, 1)$ R-matrix intertwiner $R^{(1\frac{1}{2},1)}(\zeta)$. The spin $(1, 1)$ R-matrix intertwiner $R^{(1,1)}(\zeta_1/\zeta_2) : V_{\zeta_1}^{(1)} \otimes V_{\zeta_2}^{(1)} \to V_{\zeta_2}^{(1)} \otimes V_{\zeta_1}^{(1)}$ is given by

$$R^{(1,1)}(\zeta) = \frac{1}{\kappa^{(1,1)}(\zeta)} \begin{pmatrix}
a_1 & a_3 & a_5 & a_6 \\
a_2 & a_4 & a_5 & a_6 \\
a_3 & a_2 & a_4 & a_6 \\
a_5 & a_7 & a_5 \\
a_6 & a_5 & a_3 \\
a_7 & a_2 & a_3 \\
a_4 & a_2 & a_1
\end{pmatrix}. \quad (2)$$

Here we set

$$\kappa^{(1,1)}(\zeta) = \zeta^2 \frac{1 - q^2 \zeta^{-2}}{1 - q^2 \zeta^2},$$

and

$$a_1 = 1, \quad a_2 = (1 - \zeta^2)q^2/d_4, \quad a_3 = (1 - q^4)\zeta/d_4,$$
$$a_4 = (1 - \zeta^2)(q^2 - \zeta^2)q^2/d_2d_4, \quad a_5 = (1 - \zeta^2)(1 - q^4)q\zeta/d_2d_4,$$
$$a_6 = (1 - q^2)(1 - q^4)\zeta^2/d_2d_4,$$
$$a_7 = a_2 + a_6, \quad d_2 = 1 - q^2 \zeta^2, \quad d_4 = 1 - q^4 \zeta^2.$$
It is the Boltzmann weight $a_6$ that dominates at low temperature, i.e., when $q$ is nearly equal to 0. The R-matrix $R^{(1,1)}(\zeta)$ satisfies unitarity and crossing-symmetry:

$$R^{(1,1)}(\zeta)R^{(1,1)}(\zeta^{-1}) = I, \quad R^{(1,1)}(-q^{-1}\zeta_{k,t}^{k',t'}) = R^{(1,1)}(\zeta^{-1})_{2-k,t}^{2-k',t'}.$$

Let us define the spin $(\frac{1}{2}, 1)$ R-matrix intertwiner $R^{(\frac{1}{2}, 1)}(\zeta_1/\zeta_2) : V^{(\frac{1}{2})}_{\zeta_1} \otimes V^{(1)}_{\zeta_2} \to V^{(1)}_{\zeta_2} \otimes V^{(\frac{1}{2})}_{\zeta_1}$ by

$$R^{(\frac{1}{2}, 1)}(\zeta) = \frac{1}{\kappa^{(\frac{1}{2}, 1)}(\zeta)} \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_3 & b_4 & b_1 & b_2 \\ b_4 & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_3 & b_4 \end{pmatrix}. \quad (3)$$

Here we set

$$\kappa^{(\frac{1}{2}, 1)}(\zeta) = \frac{\zeta q^3 \zeta^2; q^4 q^3 \zeta^{-2}; q^4}{(q^3 \zeta^2; q^4)(q^3 \zeta^{-2}; q^4)},$$

and

$$b_1 = 1, \quad b_2 = \frac{(1 - q^2)q}{1 - q^3 \zeta^2}, \quad b_3 = \frac{(q - \zeta^2)q}{1 - q^3 \zeta^2}, \quad b_4 = \sqrt{1 + q^2} \frac{(1 - q^2)\zeta}{1 - q^3 \zeta^2}.$$

It is the Boltzmann weight $b_4$ that dominates at low temperature, i.e., when $q$ is nearly equal to 0. The R-matrix $R^{(\frac{1}{2}, 1)}(\zeta)$ satisfies unitarity and crossing-symmetry:

$$R^{(\frac{1}{2}, 1)}(\zeta) R^{(\frac{1}{2}, 1)}(\zeta^{-1}) = I, \quad R^{(\frac{1}{2}, 1)}(-q^{-1}\zeta_{k,t}^{k',t'}) = R^{(\frac{1}{2}, 1)}(\zeta^{-1})_{1-k,t}^{1-k',t'}.$$

A lattice vertex associated with the interaction of a spin 1 and spin 1 line has spin variables $i, i' = (0, 1, 2)$ and $j, j' = (0, 1, 2)$, and spectral parameters $\zeta_1, \zeta_2 \in \mathbb{C}$. A Boltzmann weight $R^{(1,1)}(\zeta_1/\zeta_2)_{i,j}^{i',j'}$ is attached to the configuration of these variables shown in figure 1. A lattice vertex associated with the interaction of a spin $\frac{1}{2}$ and spin 1 line has spin variables $i, i' = (0, 1)$ and $j, j' = (0, 1, 2)$, and spectral parameters $\zeta_1, \zeta_2 \in \mathbb{C}$. A Boltzmann weight $R^{(\frac{1}{2}, 1)}(\zeta_1/\zeta_2)_{i,j}^{i',j'}$ is attached to the configuration of these variables shown in figure 2.
Now we consider the finite lattice in figure 3 under special boundary conditions.

Our model has $2N + 1$ vertical lines with spectral parameter $\zeta$ and $2N$ horizontal lines with spectral parameter 1, where $N \in \mathbb{N}$. The boundary conditions $a_j, b_j, c_j, d_j$ ($j = 1, 2, \cdots, 2N$) are fixed as following 4 cases, and their ground states are shown in figure 4.
1. \((\Lambda_0 + \Lambda_1, 2\Lambda_0)\)-Case :
   \[ a_j = 1 + (-1)^{N+j+1}, \quad b_j = 1, \quad c_j = 1, \quad d_j = 1 + (-1)^N, \]

2. \((\Lambda_0 + \Lambda_1, 2\Lambda_1)\)-Case :
   \[ a_j = 1 + (-1)^{N+j}, \quad b_j = 1, \quad c_j = 1, \quad d_j = 1 + (-1)^{N+1}, \]

3. \((2\Lambda_0, \Lambda_0 + \Lambda_1)\)-Case :
   \[ a_j = 1, \quad b_j = 1 + (-1)^{N+1}, \quad c_j = 1 + (-1)^{N+j}, \quad d_j = 1, \]

4. \((2\Lambda_1, \Lambda_0 + \Lambda_1)\)-Case :
   \[ a_j = 1, \quad b_j = 1 + (-1)^N, \quad c_j = 1 + (-1)^{N+j+1}, \quad d_j = 1. \]
Let us set a configuration $C$ be an assignment of spins. Hence there are $4N^2 + 4N + 1$ configurations for each boundary condition $(\lambda, \mu)$. We introduce a probability measure in the set of all configurations, assigning a statistical weight $W_N^{(\lambda, \mu)}(C)$ to each configuration $C$ attached to the boundary condition $(\lambda, \mu)$. The weight $W_N^{(\lambda, \mu)}(C)$ is given as the product over all vertices

$$W_N^{(\lambda, \mu)}(C) = \prod_{\text{vertex}} P^{(1,1)}(\zeta)^{ij} \prod_{\text{vertex}} P^{(4,1)}(\xi)^{ijkl}.$$ 

Here we multiply $R$-matrices under the boundary condition $(\lambda, \mu)$. The probability for the configuration $C$ to take place is $\frac{1}{Z_N^{(\lambda, \mu)}} W_N^{(\lambda, \mu)}(C)$, where

$$Z_N^{(\lambda, \mu)} = \sum_C W_N^{(\lambda, \mu)}(C).$$

This normalization factor $Z_N^{(\lambda, \mu)}$ is called the Partition function. The probability that the vertex of the center of our lattice takes value $\epsilon$ is given as follows

$$P_\epsilon^{(\lambda, \mu)}(N) = \frac{\sum_{(s.t.(\epsilon(C) = \epsilon)} W_N^{(\lambda, \mu)}(C)}{Z_N^{(\lambda, \mu)}}. \quad (4)$$

Here the suffix $(\lambda, \mu)$ represents the boundary conditions. In this article we are interested in the probability functions in the infinite volume limit defined by

$$P_\epsilon^{(\lambda, \mu)} = \lim_{N \to \infty} P_\epsilon^{(\lambda, \mu)}(N). \quad (5)$$

We consider the infinite volume limit in the region given by

$$-1 < q < 0, \quad 1 < \zeta < (-q)^{-1}.$$

From the symmetry arguments, we have the relations between the probability functions:

$$P_\epsilon^{(\lambda_0 + \Lambda_1, 2\lambda_0)} = P_\epsilon^{(\lambda_0 + \Lambda_1, \lambda_0 + \Lambda_1)} = P_\epsilon^{(2\lambda_1, \lambda_0 + \Lambda_1)} = P_\epsilon^{(2\lambda_0, \lambda_0 + \Lambda_1)} = P_\epsilon^{(\lambda_0 + \Lambda_1, 2\lambda_0)} , \quad (\epsilon = 0, 1).$$

We will show that the probability functions have the following formulae.

$$P_0^{(\lambda_0 + \Lambda_1, 2\lambda_0)} = \frac{1}{2} \left( q^4; q^2 \right)_\infty - \frac{1}{1 - q^2} \left( q^{16}; q^{16} \right)_\infty \left\{ \frac{1}{2} \left( -1 - q^2 + \frac{4q^4}{1 - q^4} \right) \frac{(-q^8; q^{16})^2}{(-q^2; q^{16})^2} + \left( -q^2 - q^4 + 4q^6 \right) \frac{(-q^8; q^{16})^2}{(-q^2; q^{16})^2} \right\}, \quad (6)$$

and

$$P_1^{(\lambda_0 + \Lambda_1, 2\lambda_0)} = \frac{1}{2} \left( q^4; q^2 \right)_\infty - \frac{1}{1 - q^2} \left( q^{16}; q^{16} \right)_\infty \left\{ \frac{1}{2} \left( -1 - q^2 + \frac{4q^4}{1 - q^4} \right) \frac{(-q^8; q^{16})^2}{(-q^2; q^{16})^2} + \left( -q^2 - q^4 + 4q^6 \right) \frac{(-q^8; q^{16})^2}{(-q^2; q^{16})^2} \right\}. \quad (7)$$
The following is the main result of our paper, which is a direct consequence of relations (6) and (7).

**Main Result**  
The Spontaneous Staggered Polarization of our model has the following infinite product formula

\[
P(\Lambda_0 + 2\Lambda_0, 2\Lambda_0) - P_1^{(\Lambda_0 + 2\Lambda_0, 2\Lambda_0)} = -\frac{1}{1 - q^4} \left( \frac{q^{16}; q^{16}}{(q^4; q^4)_\infty} \right) \left\{ (1 + q^4) \frac{(-q^8; q^8)_\infty}{(-q^4; q^4)_\infty^2} - 2q^2 \frac{(-q^8; q^{16})_\infty^2}{(-q^2; q^4)_\infty^2} - 4q^4 \frac{(-q^{16}; q^{16})_\infty^2}{(-q^2; q^4)_\infty^2} \right\}.
\]

In fact, the Spontaneous Staggered Polarization is independent of the spectral parameter \( \zeta \).

**Remark.** From the relation (9) and the trace formulae (10), we get

\[
P_0^{(\Lambda_0 + \Lambda_1, 2\Lambda_0)} + P_1^{(\Lambda_0 + \Lambda_1, 2\Lambda_0)} = 1.
\]

In the sequel we explain how to derive this formula.

## 3 Derivation

The purpose of this section is to show Main Result.

### 3.1 Infinite Volume Limit

We consider the infinite volume limit \( N \to \infty \). For simplicity, we concentrate ourselves to the boundary condition \((\Lambda_0 + \Lambda_1, 2\Lambda_0)\).

A path is defined as a sequence of 0, 1, 2, denoted by \( |p\rangle = \{p(j)\}_{j \geq 1} \). For weights \( \lambda = 2\Lambda_0, \Lambda_0 + \Lambda_1, 2\Lambda_1 \), consider the set of paths \( P_{2\Lambda_0}, P_{\Lambda_0 + \Lambda_1}, P_{2\Lambda_1} \) by

\[
P_{2\Lambda_0} = \{ |p\rangle | p(j) = 1 + (-1)^j, \text{for } j \gg 0 \},
\]

\[
P_{\Lambda_0 + \Lambda_1} = \{ |p\rangle | p(j) = 1, \text{for } j \gg 0 \},
\]

\[
P_{2\Lambda_1} = \{ |p\rangle | p(j) = 1 + (-1)^{j+1}, \text{for } j \gg 0 \}.
\]

The infinite lattice so defined may be split into six pieces, consisting of four corners and two half columns(see figure 5). The associated Corner Transfer Matrix are labelled \( A(\zeta), B(\zeta), C(\zeta) \) and \( D(\zeta) \). Two lines are labelled \( \Phi_{UP, \epsilon}(\zeta) \) and \( \Phi_{LOW, \epsilon}(\zeta) \).
Following Baxter we define the Corner Transfer Matrix $O^{(1)}(\zeta), O^{(2)}(\zeta)$ in the infinite volume limit $N \to \infty$, by the sum over the spin configurations in the interior, $$(O^{(b)}(\zeta))_{|p\rangle}^{\langle p'|} = \sum_{\text{interior edges}} \prod_{e_{1,2}} r_{e_{1,2}}^{(b)},$$
where we take summations with the following boundary conditions related to the suffix $b = 1, 2$. For $b = 1$, the paths $|p\rangle, |p'|$ belong to the set of paths $P_{\Lambda_0+\Lambda_1}$, and North-West boundary is fixed by $b = 1$(see figure 6). For $b = 2$, the paths $|p\rangle, |p'|$ belong to the set of paths $P_{2\Lambda_0}$, and North-West boundary is fixed by $b = 2$(see figure 7). The Corner Transfer Matrix $O^{(1)}(\zeta)$ and $O^{(2)}(\zeta)$ act on the path spaces $P_{\Lambda_0+\Lambda_1}$ and $P_{2\Lambda_0}$, respectively.
Define the operators $S : P_{2\Lambda_0} \cup P_{\Lambda_0+\Lambda_1} \cup P_{2\Lambda_1} \to P_{2\Lambda_0} \cup P_{\Lambda_0+\Lambda_1} \cup P_{2\Lambda_1}$ by $p(j) \to 2 - p(j)$. The Corner Transfer Matrix $A(\zeta), D(\zeta)$ act on the path space $P_{2\Lambda_0}$. The Corner Transfer Matrix $B(\zeta), C(\zeta)$ act on the path space $P_{\Lambda_0+\Lambda_1}$. Using the crossing symmetry of R-matrix, we can write

$$A(\zeta) = O^{(2)}(-q^{-1} \zeta^{-1}) \cdot S|_{P_{2\Lambda_0}}, \quad B(\zeta) = O^{(1)}(\zeta)|_{P_{\Lambda_0+\Lambda_1}}, \quad C(\zeta) = S \cdot O^{(1)}(-q^{-1} \zeta^{-1})|_{P_{\Lambda_0+\Lambda_1}}, \quad D(\zeta) = S \cdot O^{(2)}(\zeta) \cdot S|_{P_{2\Lambda_0}}.$$ 

Baxter’s argument \cite{Baxter} implies that $O^{(1)}(\zeta) = \text{const.} \zeta^{-H_{CTM}|_{P_{\Lambda_0+\Lambda_1}}}$, and $O^{(2)}(\zeta) = \text{const.} \zeta^{-H_{CTM}|_{P_{2\Lambda_0}}}$, where $H_{CTM}|_{P_{\lambda}}$ does not depend on the spectral parameter $\zeta$. Kyoto-school’s conjecture is to identify the path spaces $P_{2\Lambda_0}, P_{\Lambda_0+\Lambda_1}$ and $P_{2\Lambda_1}$ with the highest weight modules of $U_q(\hat{sl}_2), V(2\Lambda_0), V(\Lambda_0+\Lambda_1)$ and $V(2\Lambda_1)$, which has been proved at $q = 0$ by crystal base argument. Under this identification the degree operator $H_{CTM}|_{P_{\lambda}}$ is realized as $H_{CTM}|_{P_{\lambda}} = D|_{V(\lambda)} = -\rho + (\rho, \lambda)$, where $\lambda = 2\Lambda_0, \Lambda_0 + \Lambda_1$ and $2\Lambda_1$. The semi-infinite chain $\Phi_{UP,\epsilon}(\zeta)$ is identified with the Type-I Vertex operator $\Phi^{2\Lambda_0+\Lambda_1}_{2\Lambda_0,\epsilon}(\zeta)$ defined by

$$\Phi^{2\Lambda_0+\Lambda_1}_{2\Lambda_0}(\zeta) = \sum_{\epsilon = 0, 1} \Phi^{2\Lambda_0+\Lambda_1}_{2\Lambda_0,\epsilon}(\zeta) \otimes v_{\epsilon},$$

where $U_q(\hat{sl}_2)$-intertwiner $\Phi^{2\Lambda_0+\Lambda_1}_{2\Lambda_0}(\zeta)$ is defined by

$$\Phi^{2\Lambda_0+\Lambda_1}_{2\Lambda_0}(\zeta) : V(2\Lambda_0) \longrightarrow V(\Lambda_0 + \Lambda_1) \otimes V^{(1)}_{\zeta}.$$ 

The semi-infinite chain $\Phi_{LOW,\epsilon}(\zeta)$ is identified with the Type-I Vertex operator

$$\Phi_{LOW,\epsilon}(\zeta) = S \cdot \Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1,1-\epsilon}(\zeta) \cdot S.$$ 

The Type-I Vertex operator $\Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1,\epsilon}(\zeta)$ is defined as the same manner

$$\Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1}(\zeta) = \sum_{\epsilon = 0, 1} \Phi^{2\Lambda_0}_{\Lambda_0+\Lambda_1,\epsilon}(\zeta) \otimes v_{\epsilon},$$
where \( U_q(\mathfrak{sl}_2) \)-intertwiner \( \Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0} (\zeta) \) is defined by
\[
\Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0} (\zeta) : V(\alpha_0 + \alpha_1) \longrightarrow V(2 \alpha_0) \otimes V(\zeta) .
\]
We assume, along the line of XXZ-chain \([3]\), that the Vertex operators satisfy the Homogeneity condition,
\[
\xi^{-D} \cdot \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) \cdot \xi^{D} = \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta / \xi).
\]
From this condition we have
\[
P_{\epsilon}^{(\alpha_0 + \alpha_1, 2 \alpha_0)}(\alpha_0 + \alpha_1, \epsilon) = \frac{\text{tr}_{V(2 \alpha_0)} \left( q^{2 D} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0 + \alpha_1 - \epsilon} (-q^{-1} \zeta) \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0}(\zeta) \right)}{\sum_{\epsilon = 0, 1} \text{tr}_{V(2 \alpha_0)} \left( q^{2 D} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0 + \alpha_1 - \epsilon} (-q^{-1} \zeta) \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0}(\zeta) \right)} .
\]
We adopt the normalizations
\[
\langle \alpha_0 + \alpha_1 | \Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0 + \alpha_1}(\zeta) | 2 \alpha_0 \rangle = 1 , \quad \langle 2 \alpha_0 | \Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0 + \alpha_1}(\zeta) | \alpha_0 + \alpha_1 \rangle = 1 .
\]
The Vertex operator \( \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (-q^{-1} \zeta) \) is identified with the dual Vertex operator
\[
\Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (-q^{-1} \zeta) = \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} .
\]
The dual vertex operator \( \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) \) is defined by
\[
\Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) | v \rangle = \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) | v \rangle \otimes | v \rangle .
\]
where \( U_q(\mathfrak{sl}_2) \)-intertwiner \( \Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0} (\zeta) \) is defined by
\[
\Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0} (\zeta) : V(\alpha_0 + \alpha_1) \otimes V(\zeta) \longrightarrow V(2 \alpha_0) .
\]
We adopt the normalization.
\[
\langle 2 \alpha_0 | \Phi_{\alpha_0 + \alpha_1}^{2 \alpha_0 \epsilon} (\zeta) | \alpha_0 + \alpha_1 \rangle = 1 .
\]
Because the operator \( \sum_{\epsilon = 0, 1} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) \) commutes with \( U_q(\mathfrak{sl}_2) \) on the irreducible module \( V(2 \alpha_0) \), it becomes a constant \( g_{2 \alpha_0}^{-1} \) on \( V(2 \alpha_0) \). The constant \( g_{2 \alpha_0}^{-1} \) can be determined by solving the q-KZ equation for variables \( \zeta_1 / \zeta_2 \), which is satisfied by the vacuum expectation value \( \langle 2 \alpha_0 | \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta_1) \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta_2) | 2 \alpha_0 \rangle \)

\[
g_{2 \alpha_0} \sum_{\epsilon = 0, 1} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0} (\zeta) = \text{id} .
\]
We get the following formulæ
\[
P_{\epsilon}^{(\alpha_0 + \alpha_1, 2 \alpha_0)} = \frac{\text{tr}_{V(2 \alpha_0)} \left( q^{2 D} \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0 \epsilon} (\zeta) \Phi_{\alpha_0 + \alpha_1, \epsilon}^{2 \alpha_0}(\zeta) \right)}{g_{2 \alpha_0}^{-1} \text{tr}_{V(2 \alpha_0)} \left( q^{2 D} \right)} .
\]
Here we have used
\[ g_{2\Lambda_0}^{-1} = (1 + q^2) \frac{(q^{12}; q^8)_\infty (q^{10}; q^4)_\infty (q^8; q^4)_\infty}{(q^8; q^8)_\infty (q^{12}; q^4)_\infty}, \]
and
\[ \text{tr}_{V(2\Lambda_0)} (q^{2D}) = (-q^2; q^2)_\infty (-q^4; q^4)_\infty. \]

Here we have used the notation
\[ (z; p_1, p_2, \ldots, p_k)_\infty = \prod_{m_1, m_2, \ldots, m_k = 0}^{\infty} (1 - p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} z). \]

By the same arguments we have the followings formulae for the boundary conditions \((\mu, \lambda) = (\Lambda_0 + \Lambda_1, 2\Lambda_1)\), \((2\Lambda_0, \Lambda_0 + \Lambda_1)\) or \((2\Lambda_1, \Lambda_0 + \Lambda_1)\).

\[
P^\mu(\mu, \lambda) = \frac{\text{tr}_{V(\lambda)} (q^{2D} \Psi_{\Phi}^\mu \Psi^{*}(\zeta) \Phi_{\lambda}^{\mu}(\zeta))}{g_{\lambda}^{-1} \text{tr}_{V(\lambda)} (q^{2D})}. \tag{11}
\]

Here we have used
\[ g_{2\Lambda_1}^{-1} = (1 + q^2) \frac{(q^{12}; q^8)_\infty (q^{10}; q^4)_\infty (q^8; q^4)_\infty}{(q^8; q^8)_\infty (q^{12}; q^4)_\infty}, \]
and
\[ \text{tr}_{V(2\Lambda_1)} (q^{2D}) = (-q^2; q^2)_\infty (-q^4; q^4)_\infty, \]
\[ \text{tr}_{V(\Lambda_0 + \Lambda_1)} (q^{2D}) = (-q^2; q^2)_\infty (-q^4; q^4)_\infty. \]

The vertex operators are defined as the same manner. From the cyclic property of trace, we obtain the following relations
\[ P^{(2\Lambda_0, \Lambda_0 + \Lambda_1)}_{\lambda_1, \lambda_2} = P^{(\Lambda_0 + \Lambda_1, 2\Lambda_0)}_{\lambda_1, \lambda_2}, \quad P^{(2\Lambda_1, \Lambda_0 + \Lambda_1)}_{\lambda_1, \lambda_2} = P^{(\Lambda_0 + \Lambda_1, 2\Lambda_1)}_{\lambda_1, \lambda_2}. \]

From symmetries we easily know the following relations
\[ P^{(2\Lambda_0, \Lambda_0 + \Lambda_1)}_{\lambda_1, \lambda_2} = P^{(2\Lambda_1, \Lambda_0 + \Lambda_1)}_{\lambda_1, \lambda_2}, \quad P^{(\Lambda_0 + \Lambda_1, 2\Lambda_0)}_{\lambda_1, \lambda_2} = P^{(\Lambda_0 + \Lambda_1, 2\Lambda_1)}_{\lambda_1, \lambda_2}. \]

From the commutation relation of Vertex operators \([2]\) and the cyclic property of trace, we can write down q-difference equation for parameter \(\zeta_1/\zeta_2\), which the trace \(\text{tr}_{V(2\Lambda_0)} (q^{2D} \Psi_{\lambda_0 + \Lambda_1} \Psi^{*}(\zeta_1) \Phi_{\lambda_0 + \Lambda_1}^{\mu}(\zeta_2))\) satisfies. However we cannot solve this q-difference equation, now. In order to get exact formulae of the probability functions, we will use another method - free field realizations.
3.2 Free field realization

In order to calculate the trace of vertex operators

\[ \text{tr}_{V(2\Lambda_0)} \left( q^{2D} \Phi_{\Lambda_0 + \Lambda_1, \epsilon}(\zeta) \Phi_{2\Lambda_0, \epsilon}(\zeta) \right), \]

we use the free field realization obtained by Y. Hara \[13\]. For readers’ convenience, we summarize his result. The formulae in this paper are slightly different from Hara’s paper, because his paper includes a few mistakes, which is serious for our purpose. We use current type generators of \( U'_q(\hat{sl}_2) \) introduced by Drinfeld. Let \( A \) be an algebra generated by \( x^\pm (m \in \mathbb{Z}), a_m (m \in \mathbb{Z} \neq 0), \gamma \) and \( K \) with relations

\[ \gamma : \text{central}, \]

\[ [a_m, a_n] = \delta_{m+n,0} \frac{2m}{m} \gamma^m - \gamma^{-m}, \]

\[ [a_m, K] = 0, \]

\[ K x_m^+ K^{-1} = q^{+2} x_m^+, \]

\[ [a_m, x_n^+] = \pm \frac{2m}{m} \gamma^{-m} x_{m+n}^+, \]

\[ x_{m+1}^+ x_n^+ - q^{+2} x_n^+ x_{m+1}^+ = q^{+2} x_m^+ x_{n+1}^+ - x_{n+1}^+ x_m^+, \]

\[ [x_m^+, x_n^-] = \frac{1}{q - q^{-1}} (\gamma^{\frac{1}{2}(m-n)} \psi_{m+n} - \gamma^{-\frac{1}{2}(m-n)} \varphi_{m+n}), \]

where

\[ \sum_{m=0}^{\infty} \psi_m z^{-m} = K \exp \left[ (q - q^{-1}) \sum_{m=1}^{\infty} a_m z^{-m} \right], \]

\[ \sum_{m=0}^{\infty} \varphi_m z^m = K^{-1} \exp \left[ -(q - q^{-1}) \sum_{m=1}^{\infty} a_m z^m \right], \]

and \( \psi_m = \varphi_m = 0 \) for \( m > 0 \). Drinfeld showed that the algebra \( A \) is isomorphic to \( U'_q(\hat{sl}_2) \). The Chevalley generators are given by the identification

\[ t_0 = \gamma K^{-1}, \ t_1 = K, \ e_1 = x_0^+, \ f_1 = x_0^-, \ e_0 = x_1^- K^{-1}, \ f_0 = K x_{-1}^+. \]

We will give explicit constructions of level 2 irreducible highest weight modules. Let us put \( \gamma = q^2 \) since we want to construct level 2 modules. In the sequel we use a parameter \( x = -q \) for our convenience. Commutation and Anti-commutation relations of bosons and fermions are given by

\[ [a_m, a_n] = \delta_{m+n,0} \frac{2m}{m}, \]

\[ \{ \phi_m, \phi_n \} = \delta_{m+n,0} \eta_m, \]

\[ \eta_m = x^{-2m} + x^{2m}, \]
where \( m, n \in \mathbb{Z} + \frac{1}{2} \) or \( \in \mathbb{Z} \) for Neuveu-Schwartz-sector or Ramond-sector respectively. Fock spaces and vacuum vectors are denoted as \( F^a, F^{\phi^NS}, F^{\phi^R} \) and \( |\text{vac}\rangle, |NS\rangle, |R\rangle \) for boson and Neuveu-Schwartz and Ramond fermion respectively. Fermion currents are defined as

\[
\phi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi^{NS}_n z^{-n}, \quad \phi^R(z) = \sum_{n \in \mathbb{Z}} \phi^R_n z^{-n},
\]

Let us set the degree of the monomial of fermions, \( \phi^{NS}_{n_1} \phi^{NS}_{n_2} \cdots \phi^{NS}_{n_s} \) as \( n_1 + n_2 + \cdots + n_s \), and \( \phi^R_{n_1} \phi^R_{n_2} \cdots \phi^R_{n_r} \) as \( n_1 + n_2 + \cdots + n_r \). \( Q = \mathbb{Z}\alpha \) is the root lattice of \( sl_2 \) and \( F[Q] \) is the group algebra. We use \( \partial \) as

\[
[\partial, \alpha] = 2.
\]

The irreducible highest weight module \( V(2\Lambda_0) \) is identified with the Fock space

\[
F_+^{(0)} = F^a \otimes \left( F^{\phi^NS}_+ \otimes F[2Q] \right) \oplus \left( F^{\phi^NS}_- \otimes e^\alpha F[2Q] \right),
\]

where \( F^{\phi^NS}_+ \) represents the subspace of Fermion Fock space which is spanned by even degree fermions, and \( F^{\phi^NS}_- \) that by odd ones. The highest weight vector is \( |\text{vac}\rangle \otimes |NS\rangle \otimes 1 \). The irreducible highest weight module \( V(2\Lambda_1) \) is identified with the Fock space

\[
F_-^{(0)} = F^a \otimes \left( F^{\phi^NS}_+ \otimes e^\alpha F[2Q] \right) \oplus \left( F^{\phi^NS}_- \otimes F[2Q] \right).
\]

The highest weight vector is \( |\text{vac}\rangle \otimes |NS\rangle \otimes e^\alpha \). We define the actions of Drinfeld generators as

\[
\gamma = q^2, \quad K = q^\theta, \quad x^\pm(z) = \sum_{m \in \mathbb{Z}} x^\pm_m z^{-m} = E^\pm(z) \phi^{NS} \cdot e^\pm z^\frac{1}{2} \partial,
\]

where

\[
E^\pm(z) = \exp \left( \pm \sum_{m > 0} \frac{a_m}{2m} q^{m} z^{-m} \right), \quad E^\pm(z) = \exp \left( \pm \sum_{m > 0} \frac{a_m}{2m} q^{-m} z^{-m} \right).
\]

The irreducible highest weight module \( V(\Lambda_0 + \Lambda_1) \) is identified with the Fock space

\[
F^{(1)} = F^a \otimes F^{\phi^R} \otimes e^\frac{\theta}{2} F[Q],
\]

where

\[
\phi^R \mid R \rangle = \mid R \rangle.
\]

The highest weight vector is \( |\text{vac}\rangle \otimes |R\rangle \otimes e^\frac{\theta}{2} \). For the actions of Drinfeld generators, we just replace \( \phi^{NS}(z) \) with \( \phi^R(z) \) in (12). The free field realizations of vertex operators [13] are constructed by

\[
\Phi_{2\Lambda_0 + \Lambda_1}^\pm(\zeta) = \zeta^{1-\epsilon} \Phi_\epsilon(\zeta),
\]

\[
\Phi_{2\Lambda_0 + \Lambda_1, e}^\pm(\zeta) = -\alpha \zeta^{\frac{1}{2} - \epsilon} \Phi_\epsilon(\zeta).
\]
Here we have set

\[
\begin{align*}
\Phi_1(\zeta) &= B_{1,}(\zeta) B_{1,}(\zeta) \Omega_{\text{NS}}^R(\zeta) e^{\frac{x \partial}{\partial \zeta}} x^2 \zeta^2, \\
\Phi_0(\zeta) &= \oint \frac{dw}{2\pi i} B_{1,}(\zeta) E_<(w) B_{1,}(\zeta) E_<(w) \Omega_{\text{NS}}^R(\zeta) \phi_{\text{NS}}(w) e^{-\frac{x \partial}{\partial \zeta}} x^2 \zeta^2 w^{-\frac{3}{2}}.
\end{align*}
\]

(13)

\[
\begin{align*}
\times x^{-2} &\zeta^{-1} w^{-\frac{3}{2}} \left( \frac{-w}{x^3 \zeta^2} ; x^4 \right) \left( \frac{-w}{x^3 \zeta^2} ; x^4 \right) \infty \left\{ \frac{w}{1 + \frac{w}{x^5 \zeta^2}} + \frac{x^5 \zeta^2}{1 + \frac{x^5 \zeta^2}{w}} \right\},
\end{align*}
\]

(14)

where

\[
\begin{align*}
B_{1,}(\zeta) &= \exp \left( \sum_{n=1}^{\infty} \frac{n! a_{-n}}{[2n]^2} (-x^5 \zeta^2)^n \right), \\
B_{1,}(\zeta) &= \exp \left( -\sum_{n=1}^{\infty} \frac{n! a_{-n}}{[2n]^2} (-x^3 \zeta^2)^{n-1} \right).
\end{align*}
\]

Note that the sign above differs from the one in [13]. The integrand of \(\Phi_0(\zeta)\) has poles only at \(w = -x^5 \zeta^2, -x^3 \zeta^2\) except for \(w = 0, \infty\) and the contour of integration encloses \(w = 0, -x^5 \zeta^2\) as in figure 8.

![Figure 8](image-url)

For those of \(\Phi^2_{\Lambda_0 + \Lambda_1 + \Lambda_2}(\zeta)\) we just replace \(\Omega_{\text{NS}}^R(\zeta), \phi_{\text{NS}}(w)\) with \(\Omega_{\text{NS}}^R(\zeta), \phi_{\text{R}}(w)\) in (13), (14). The fermion part \(\Omega(\zeta)'s\) are intertwiners between different fermion sectors and satisfy

\[
\phi_{\text{NS}}(w) \Omega_{\text{NS}}^R(\zeta) = x^2 \zeta w^{-\frac{3}{2}} \left( \frac{-w}{x^3 \zeta^2} ; x^4 \right) \left( \frac{-x^5 \zeta^2}{w} ; x^4 \right) \infty \Omega_{\text{NS}}^R(\zeta) \phi_{\text{NS}}(w),
\]

and exactly the same equation except subscripts for fermion sectors are exchanged. The homogeneity condition of the fermion parts is given in [13]

\[
\xi^d \cdot \Omega_{\text{NS}}^R(\zeta) \cdot \xi^{-d_{\text{NS}}} = \Omega_{\text{NS}}^R(\zeta/\xi).
\]

(15)

The fermion parts \(\Omega_{\text{NS}}^R(\zeta), \Omega_{\text{NS}}^R(\zeta)\) are identified with Type-I Vertex operators of the two-dimensional Ising model \(\Phi_{\text{NS}}^R(\zeta), \Phi_{\text{NS}}^R(\zeta)\) which were investigated in details [14].

16
\[ \Omega_{NS}^R(\zeta) = \Phi_{NS}^R \left( -\frac{i}{x^2 \zeta} \right), \quad \Omega_{NS}^N(\zeta) = \Phi_{NS}^N \left( -\frac{i}{x^2 \zeta} \right). \]

For readers’ convenience we summarize the definitions and properties of vertex operators \( \Phi_{NS}^R(\zeta) \) and \( \Phi_{NS}^N(\zeta) \), which will be used later. The Type-I Vertex operators of the two-dimensional Ising model are operators on Fock spaces
\[ \Phi_{NS}^R(\zeta) : F_{\phi NS} \rightarrow F_{\phi R}, \]
\[ \Phi_{NS}^N(\zeta) : F_{\phi R} \rightarrow F_{\phi NS}. \]

Define the subsectors as
\[ \Phi_{NS,\sigma}(\zeta) = \Phi_{NS}^\sigma(\zeta)_{V_{\phi NS}}, \quad \Phi_{R,\sigma}(\zeta) = P^\sigma \Phi_{R}^N(\zeta), \quad \text{for} \quad \sigma = \pm, \]
where \( P^\sigma \) denotes the projection onto subspace \( F_{\phi NS}^\sigma \). The intertwining relations are given by
\[ \phi^{NS}(w)\Phi_{NS,\sigma}(\zeta) = f(w \zeta^2)\Phi_{NS,-\sigma}(\zeta)\phi^R(w), \]
\[ \phi^R(w)\Phi_{NS,\sigma}(\zeta) = f(w \zeta^2)\Phi_{NS,-\sigma}(\zeta)\phi^{NS}(w). \]

Here we set
\[ f(z) = -\sqrt{\frac{2\pi x}{I}}(x^4; x^4)_\infty(-x^4; x^4)_\infty^2 \text{sn}(v), \]
where \( z = \exp(\pi iv/I) \) and \( \text{sn}(v) \) is Jacobi elliptic function with half periods \( I, iI' \). Because of the intertwining relations, the following relations hold.
\[ \sum_{\sigma} \Phi_{NS,\sigma}(x\zeta)\Phi_{NS,\sigma}(\zeta) = g^R \times id_{F^R_{\phi NS}}, \quad (16) \]
\[ \Phi_{NS,\sigma}(x\zeta)\Phi_{NS,\sigma}(\zeta) = g^NS \times id_{F^NS_{\phi NS}}, \quad (17) \]
where the constants are
\[ g^R = \frac{(x^4; x^4, x^8)_\infty^2}{(x^2; x^4, x^8)_\infty}, \quad g^NS = \frac{(x^8; x^4, x^8)_\infty^2}{(x^8; x^4, x^8)_\infty}. \]

We will use the following intertwining relations in the next section.
\[ \sigma \Phi_{NS,\sigma}(\zeta) = \Phi_{NS,\sigma}(\zeta)\psi_1^R(\zeta), \quad (18) \]
\[ \sigma \Phi_{NS,\sigma}(\zeta) = -i\Phi_{NS,-\sigma}(\zeta)\psi_1^{NS}(\zeta), \quad (19) \]
where we have set
\[ \psi_1^R(\zeta) = \oint \frac{dz}{2\pi iz} f^R_0(z)\phi^R(z/\zeta^2), \]
\[ \psi_1^{NS}(\zeta) = \oint \frac{dz}{2\pi iz} f^{NS}_0(z)\phi^{NS}(z/\zeta^2). \]
Here we have set
\[ f_0^{NS}(z) = 2\sqrt{x^4 - x^4} \exp(-x^4) \text{cn}(v), \quad f_0^R(z) = \sqrt{2I} \text{dn}(v), \]
where \( z = \exp(\pi iv/I) \) and \( \text{cn}(v), \text{dn}(v) \) are Jacobi elliptic functions with half periods \( I, iI'. \)

### 3.3 Integral Formulae

In this subsection we will calculate trace of a product of two vertex operators and derive an integral formula of the probability function. The free field realizations of the degree operators are given by
\[
D|V(\lambda) = -\rho = -2\bar{a}^a - 2\bar{\phi}^a + \frac{1}{4} \partial_a^2 - \frac{1}{2} \partial_a - \frac{(\lambda, \lambda)}{2}, (\lambda = 2\Lambda_0, 2\Lambda_1, \Lambda_0 + \Lambda_1).
\]

Here we set
\[
\bar{a}^a = \sum_{m=0}^{\infty} mN_m^a, \quad \bar{\phi}^a = \sum_{k>0} kN_k^\phi,
\]
where
\[
N_m^a = \frac{m}{[2m]^2} a_{-m} a_m, \quad N_k^\phi = \frac{1}{x^{2k} + x^{-2k} \phi - k \phi_k}.
\]

We divide trace on \( V(2\Lambda_0) \approx F_+^{(0)} \) into three parts:
\[
\text{tr}_{V(2\Lambda_0)}(x^{2D} \cdots) = \text{tr}_{F_+}(x^{-2\bar{a}^a} \cdots) \cdot \text{tr}_{F_+^{NS}}(x^{-4\bar{\phi}^a} \cdots) \cdot \text{tr}_{F[2Q]}(x^{1/2\partial_a - \partial_a} \cdots)
\]
\[
+ \text{tr}_{F_+}(x^{-4\bar{a}^a} \cdots) \cdot \text{tr}_{F_-^{NS}}(x^{-4\bar{\phi}^a} \cdots) \cdot \text{tr}_{e^{\alpha}}(x^{1/2\partial_a - \partial_a} \cdots)
\]

The fermion parts can be written as
\[
\text{tr}_{F_+^{NS}}(x^{-4\bar{\phi}^a} \cdots) = \frac{1}{2} \left( \text{tr}_{F_+^{NS}}(x^{-4\bar{\phi}^a} \cdots) \pm \text{tr}_{F_+^{NS}}(ix^{-4\bar{\phi}^a} \cdots) \right).
\]

Now we consider the trace of a product of two vertex operators
\[
\text{tr}_{V(2\Lambda_0)}(x^{2D} \Phi_\epsilon(x^{-1} \zeta) \Phi_1 \cdots), \quad \text{for} \ \epsilon = 0, 1.
\]

The trace taken over bosonic space is direct consequence of the following formulae.
\[
\text{tr}_{F_+}(y^{-2\bar{a}^a} \exp(\sum_{n=1}^{\infty} A_n a_{-n}) \exp(\sum_{n=1}^{\infty} B_n a_n))
\]
\[
= \exp\left( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y^{2mn} A_n B_m \frac{[2n]^2}{n} \right) \prod_{m=1}^{\infty} \frac{1}{1 - y^{2m}},
\]
and
\[
\left( x^6 \left( \frac{\zeta_1}{\zeta_2} \right)^2 ; x^4, x^4 \right)_\infty B_{I, >}(\zeta_2) B_{I, <}(\zeta_1) = \left( x^4 \left( \frac{\zeta_1}{\zeta_2} \right)^2 ; x^4, x^4 \right)_\infty \left( x^8 \left( \frac{\zeta_1}{\zeta_2} \right)^2 ; x^4, x^4 \right)_\infty B_{I, <}(\zeta_1) B_{I, >}(\zeta_2). \tag{21}
\]

The trace taken over bosonic space \( F^a \) can be written as infinite products.
\[
\text{tr}_{F^a} \left( y^{-2d} B_{I, <}(\zeta_2) B_{I, <}(\zeta_1) E_{<}^<(w) B_{I, >}(\zeta_2) B_{I, >}(\zeta_1) E_{>}^>(w) \right) = \left[ \left( x^4 y^2 ; x^4, x^4, y^2 \right)_\infty \left( x^8 y^2 ; x^4, x^4, y^2 \right)_\infty \right] \left( x^6 y^2 \left( \frac{\zeta_1}{\zeta_2} \right)^2 ; x^4, x^4, y^2 \right)_\infty \left( x^4 y^2 \left( \frac{\zeta_2}{\zeta_1} \right)^2 ; x^4, x^4, y^2 \right)_\infty \left[ \left( -x^9 y^2 \frac{\zeta_1^2}{\zeta_2^4} ; x^4, x^4, y^2 \right)_\infty \left( -x^7 y^2 \frac{\zeta_2^2}{\zeta_1^4} ; x^4, x^4, y^2 \right)_\infty \right] \times (\zeta_2 \leftrightarrow \zeta_1). \tag{22}
\]

The trace taken over lattice space have the following theta function formulae.
\[
\text{tr}_{F[2\Omega]} \left( x^L \partial_a^2 \partial_b \partial_a^b \right) = \sum_{l \in \mathbb{Z}} x^{8l^2 - 4l} f^{4l} = \Theta_{x^4}(-x^4 f^4), \tag{23}
\]
\[
\text{tr}_{v^n F[2\Omega]} \left( x^L \partial_a^2 \partial_b \partial_a^b \right) = \sum_{l \in \mathbb{Z}} x^{8l^2 + 4l} f^{4l+2} = f^2 \Theta_{x^4}(-x^{12} f^4). \tag{24}
\]

Here we have used the standard notation of the theta function defined as
\[
\Theta_\Psi(z) = (p; p)_\infty (z; p)_\infty (pz^{-1}; p)_\infty.
\]

Now we concentrate ourselves to trace taken over the fermionic space:
\[
\text{tr}_{F^{\pm NS}} \left( x^{-4d^2} \Omega_R^{NS}(\zeta/x) \Omega_R^{NS}(\zeta) \phi^{NS}(w) \right) = \text{tr}_{F^{\pm NS}} \left( x^{-4d^2} \Phi^{NS, \pm} \left( -\frac{i}{x^2 \zeta} \right) \Phi^{NS, \mp} \left( -\frac{i}{x^2 \zeta} \right) \phi^{NS}(w) \right), \tag{25}
\]

where the symbols \( \Phi^{NS, \pm}_R(\zeta) \) and \( \Phi^{NS, \mp}_R(\zeta) \) are Type-I Vertex operators of two dimensional Ising model. From the relations (17) and (19), we deform the vertex.
operators in \( [23] \) to the fermion currents. Only the fermion currents and the degree operator appear in trace.

\[
\pm \frac{i}{2} g^{NS} \left\{ \text{tr}_{F^0NS} \left( x^{-4d^6} \psi_1^{NS} \left( -\frac{i}{x^4} \right) \phi^{NS}(w) \right) \right. \\
\left. \pm \text{tr}_{F^0NS} \left( (ix)^{-4d^6} \psi_1^{NS} \left( -\frac{i}{x^4} \right) \phi^{NS}(w) \right) \right\}. \tag{26}
\]

Here we have used

\[
\psi_1^{NS}(\zeta) = \oint \frac{dz}{2\pi i z} f_0^{NS}(z) \phi^{NS}(z/\zeta^2), \quad f_0^{NS}(z) = 2\sqrt{z(x^4; x^4)_\infty(-x^4; x^4)_\infty^2} \text{cn}(v).
\]

To take the trace of (26) we invoke the following simple relation

\[
\frac{\text{tr}_{F^0NS}(\xi^{-2d^6} \phi^{NS}(w_1) \phi^{NS}(w_2))}{\text{tr}_{F^0NS}(\xi^{-2d^6})} = \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{x^{2m} + x^{-2m}}{1 + \xi^{2m}} \left( \frac{w_2}{w_1} \right)^m.
\]

We calculate the trace taken over fermionic space and calculate integrals in (26), using the Fourier expansion of coefficient function \( f_0^{NS}(z) \) given by

\[
f_0^{NS}(z) = \frac{1}{\sqrt{x(x^4; x^4)_\infty(-x^4; x^4)_\infty^2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{x^{2r} + x^{-2r}} e^r.
\]

We have the following infinite sum formulae

\[
\pm \frac{i}{2} g^{NS} \frac{1}{\sqrt{x(x^4; x^4)_\infty(-x^4; x^4)_\infty^2}} \times \left\{ \text{tr}_{F^0NS} \left( x^{-4d^6} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{1}{x^{2m} + x^{-2m}} \left( -\frac{w}{x^2} \right)^m \right) \right. \\
\left. \mp \text{tr}_{F^0NS} \left( (ix)^{-4d^6} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{1}{x^{2m} - x^{-2m}} \left( -\frac{w}{x^2} \right)^m \right) \right\}.
\]

Using the following theta function’s identities

\[
\sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{1}{x^{2m} \pm x^{-2m}} z^m = \pm x \sqrt{z} \left( x^4; x^4 \right)_\infty \left( \frac{\Theta_{x^4}(\mp x^4 z)}{\Theta_{x^4}(x^2 z)} \right),
\]

we get the infinite product formulae of trace taken over fermionic space

\[
\text{tr}_{F^0NS} \left( x^{-4d^6} \Omega^{NS}_R(\zeta/x) \Omega^{NS}_L(\zeta) \phi^{NS}(w) \right) = \mp \frac{1}{2} \frac{x^4}{\zeta x^2} \left( x^8; x^4 \right)_\infty \left( x^8; x^4 \right)_\infty \times \frac{1}{\Theta_{x^4}(\mp x^4 z)} \left\{ \Theta_{x^4}(\mp x^4 z) \pm (x^2; x^4)_\infty \Theta_{x^4}(x^4) \right\}. \tag{27}
\]
Here we have used the character formulae of fermion Fock space;
\[ \text{tr}_F^{N_S} (x^{-4d^a}) = (-x^2; x^4)_\infty. \]

Combining the relations (21), (22), (23), (24) and (27), we get an integral formulae of the spontaneous staggered polarizations. We can summarize the conclusion just obtained as follows:

**The trace of a product of two vertex operators has following integral formulae.**

\[
\begin{align*}
\text{tr}_{V(2\Lambda_0)} \left( x^{2D} \Phi^{2\Lambda_0} \Phi^{\Lambda_0+\Lambda_1} (\xi) \Phi^{\Lambda_0+\Lambda_1} (\xi) \right) &= 1 \\
&\times \int_{C_{1+}} \frac{dw}{2\pi i w} \left\{ (1 + x^2) w + 2x^5 - 2x^3 \zeta^2 \right\} \Theta_x^4 \left( \frac{w}{x^2} \right) \Theta_{x^4} \left( \frac{w}{x^4} \right) \\
&- \left\{ \Theta_{x^4} \left( \frac{w}{x^2} \right) \left( -x^2 ; x^4 \right)_\infty - \Theta_{x^4} \left( \frac{w}{x^4} \right) \left( x^2 ; x^4 \right)_\infty \right\} \Theta_{x^4} \left( \frac{1}{x^2} \left( \frac{w}{\zeta^2} \right)^2 \right) \\
&\times \left\{ \Theta_{x^4} \left( \frac{w}{x^2} \right) \left( -x^2 ; x^4 \right)_\infty + \Theta_{x^4} \left( \frac{w}{x^4} \right) \left( x^2 ; x^4 \right)_\infty \right\} \Theta_{x^4} \left( \frac{w}{x^4} \right) \left( -x^6 \left( \frac{w}{\zeta^2} \right)^2 \right) \\
\end{align*}
\]

Here the contours encircle \( w = 0 \) in such a way that
\[
\begin{align*}
C_1 &: -x^5 \zeta^2 \text{ is inside and } -x^3 \zeta^2 \text{ is outside}, \\
C_2 &: -x^3 \zeta^2 \text{ is inside and } -x \zeta^2 \text{ is outside},
\end{align*}
\]

as in figure 9.

\[ \text{Figure 9} \]

### 3.4 Infinite Product Formulae

The purpose of this subsection is to calculate integral in (28) and derive an infinite product formula of spontaneous staggered polarization in Main Result.
Let us use the following abbreviations:

\[
p_1(x) = \frac{(x^{16}; x^{16})_\infty}{(x^4; x^4)_\infty} (-x^4; x^8)_\infty,
\]
\[
p_2(x) = \frac{(x^{16}; x^{16})_\infty}{(x^2; x^2)_\infty^2(x^4; x^4)_\infty} (-x^4; x^8)_\infty^2, (-x^8; x^{16})_\infty^2,
\]
\[
p_3(x) = \frac{(x^{16}; x^{16})_\infty}{(x^2; x^2)_\infty^2(x^4; x^4)_\infty} (-x^4; x^8)_\infty^2, (-x^{16}; x^{16})_\infty^2,
\]
\[
p_4(x) = \frac{(x^{16}; x^{16})_\infty}{(x^2; x^2)_\infty^2(x^4; x^4)_\infty} (-x^2; x^4)_\infty^2 (-x^4; x^8)_\infty.
\]

Now we consider the following integral

\[
\oint \frac{dw}{2\pi i w} \frac{\Theta_{x^4}(\frac{w}{x\xi^2}) \Theta_{x^{16}}\left(-x^6 \left(\frac{w}{x\xi^2}\right)^2\right)}{\Theta_{x^4}\left(-\frac{w}{x\xi^2}\right) \Theta_{x^4}\left(-\frac{w}{x\xi^2}\right)}
\]

The integrand function \( I(z) = \frac{\Theta_{x^4}(z) \Theta_{x^{16}}\left(-x^8 z^2\right)}{\Theta_{x^4}\left(-z\right) \Theta_{x^4}\left(-z/x^2\right)} (z = w/x\xi^2) \) is an elliptic function and has odd invariance;

\[
I(x^8 z) = I(z), \quad I(z) = -I(z^{-1}).
\]

Therefore, taking the residue of Cauchy’s principal value at \( z = -1 \), we have

\[
\oint_{|z|=1} \frac{dz}{2\pi i z} I(z) = -\frac{1}{2} \text{Res}_{z=-1} I(z).
\]

Taking the residues near \( w = -x\xi^2 \), we get the following formulae

\[
\oint_{C} \frac{dw}{2\pi i w} \frac{\Theta_{x^4}(\frac{w}{x\xi^2}) \Theta_{x^{16}}\left(-x^6 \left(\frac{w}{x\xi^2}\right)^2\right)}{\Theta_{x^4}\left(-\frac{w}{x\xi^2}\right) \Theta_{x^4}\left(-\frac{w}{x\xi^2}\right)} = x^3 \xi^2 \times \left\{ p_2(x) - p_4(x), \quad p_2(x), \quad C = C_1, \quad C = C_2. \right\} (29)
\]

As the same arguments above, we get

\[
\oint_{C} \frac{dw}{2\pi i w} \frac{\Theta_{x^4}(\frac{w}{x\xi^2}) \Theta_{x^{16}}\left(-\frac{1}{x} \left(\frac{w}{x\xi^2}\right)^2\right)}{\Theta_{x^4}\left(-\frac{w}{x\xi^2}\right) \Theta_{x^4}\left(-\frac{w}{x\xi^2}\right)} = \left\{ -2x^2 p_3(x) + p_4(x), \quad -2x^2 p_3(x), \quad C = C_1, \quad C = C_2. \right\} (30)
\]

We consider the following integral

\[
\oint \frac{dw}{2\pi i w} \frac{\Theta_{x^{16}}\left(-x^6 \left(\frac{w}{x\xi^2}\right)^2\right)}{\Theta_{x^4}\left(-\frac{w}{x\xi^2}\right)}.
\]
The integrand function \( J(z) = \frac{1}{2} \frac{\Theta_{x^{16}}(z^2)}{\Theta_{x^4}(-x^2 z^2)} \) is an elliptic function and is composed of a product of two odd invariant functions \( J_1(z) = \frac{\Theta_{x^{16}}(z^2)}{\Theta_{x^4}(z^2)} \) and \( J_2(z) = \frac{1}{2} \frac{\Theta_{x^{16}}(z^2)}{\Theta_{x^4}(-x^2 z^2)} \).

\[
J(z) = J_1(z)J_2(z), \\
J_k(x^8 z) = -J_k(z), \\
J_k(z) = -J_k(z^{-1}) \text{ for } k = 1, 2.
\]

From the odd invariance property, the constant term of Fourier expansion for variable \( u \) such that \( z = e^{iu} \) becomes zero. Therefore we have

\[
\oint_{|z|=1} dz \frac{2\pi i z}{2} J_1(z)J_2(z) = 0.
\]

Taking the residue near \( w = -x^3/\xi^2 \), we get the following formulae

\[
\oint_C dw \frac{\Theta_{x^{16}}(-x^6 \left( \frac{w}{\xi} \right)^2)}{\Theta_{x^4}(-\frac{w}{x^3 \xi})} = \begin{cases} 0, & C = C_1, \\ x^3 \xi^2 p_1(x), & C = C_2. \end{cases} \tag{31}
\]

As the same arguments above, we get the following formulae

\[
\oint_C dw \frac{\Theta_{x^{16}}(-\frac{1}{x^4} \left( \frac{w}{\xi} \right)^2)}{\Theta_{x^4}(-\frac{w}{x^3 \xi^2})} = \begin{cases} p_1(x), & C = C_1, \\ 0, & C = C_2. \end{cases} \tag{32}
\]

We consider the following integral

\[
\oint_C dw \frac{\Theta_{x^{16}}(-x^6 \left( \frac{w}{\xi} \right)^2)}{\Theta_{x^4}(-\frac{w}{x^3 \xi})}.
\]

The integrand function \( K(z) = \frac{\Theta_{x^{16}}(-z^2)}{\Theta_{x^4}(-z/x^6)} \), \( z = x^3 w/\xi^2 \) satisfies the quasi-periodicity.

\[
K(z) = x^8 K(x^8 z).
\]

Therefore we have

\[
\oint_{|z|=1} dw \frac{2\pi i z}{2} K(z) = \frac{-1}{1-x^8} \left\{ \oint_{z=-x^2} + \oint_{z=-x^6} \right\} \frac{dz}{2\pi i z} K(z),
\]

where we take the residues near \( \infty \). Now we get the following formulae.

\[
\oint_C dw \frac{\Theta_{x^{16}}(-x^6 \left( \frac{w}{\xi} \right)^2)}{\Theta_{x^4}(-\frac{w}{x^3 \xi})} = \begin{cases} \frac{1}{1+x^4} p_1(x), & C = C_1, \\ -\frac{1}{1+x^4} p_1(x), & C = C_2. \end{cases} \tag{33}
\]
As the same arguments above we get the following formulae:

\[
\oint_C \frac{dw}{2\pi i w} \frac{1}{\Theta_{x^4} \left( -\frac{w}{\xi^2} \right)} \left( -\frac{1}{x^2} \left( \frac{w}{\xi^2} \right)^2 \right) = \frac{1}{x^3 \xi^2} \left\{ -\frac{1}{1 + x^2} p_1(x), \quad C = C_1, \right. \\
\left. \frac{1}{1 + x^4} p_1(x), \quad C = C_2, \right. \tag{34}
\]

and

\[
\oint_C \frac{dw}{2\pi i w} \frac{1}{\Theta_{x^4} \left( \frac{w}{\xi^2} \right) \Theta_{x^4} \left( -\frac{x}{\xi^2} \right)} \left( -\frac{1}{x^2} \left( \frac{w}{\xi^2} \right)^2 \right) = \frac{-1}{x \xi^2 (1 - x^8)} \\
\times \left\{ -2x^4 p_2(x) - 4x^2 p_3(x) + (x^2 + x^{-2}) p_4(x), \quad C = C_1, \right. \\
\left. -2x^4 p_2(x) - 4x^2 p_3(x) + (x^2 + x^6) p_4(x), \quad C = C_2, \right. \tag{35}
\]

and

\[
\oint_C \frac{dw}{2\pi i w} \frac{\Theta_{x^4} \left( \frac{w}{\xi^2} \right) \Theta_{x^4} \left( -\frac{x}{\xi^2} \right)}{\Theta_{x^4} \left( \frac{w}{\xi^2} \right) \Theta_{x^4} \left( \frac{x}{\xi^2} \right)} \left( -\frac{1}{x^2} \left( \frac{w}{\xi^2} \right)^2 \right) = \frac{-1}{1 - x^8} \\
\times \left\{ 2x^2 p_2(x) + 4x^8 p_3(x) - (1 + x^4) p_4(x), \quad C = C_1, \right. \\
\left. 2x^2 p_2(x) + 4x^8 p_3(x) - (x^4 + x^8) p_4(x), \quad C = C_2. \right. \tag{36}
\]

Inserting the relations (29), (30), (31), (32), (33), (34), (35) and (36) into integral formulae (28), we arrive at the formulae (11) and (12). Now we have proved Main Result in page 9.

Acknowledgements We want to thank to Professor Tetsuji Miwa and Professor Kimio Ueno for their encouragements. This work is partly supported by Waseda University Grant for Special Research Projects, the Grant from Research Institute of Science and Technology, Nihon University, and the Grant from the Ministry of Education (11740099).

References

[1] M. Jimbo and T. Miwa : Algebraic Analysis of Solvable Lattice Models, CBMS 85, Am. Math. Soc., (1993).
[2] R. Baxter : Spontaneous staggered polarization of F-model, J. Stat. Phys. 9, 145-182, (1973).
[3] R. Baxter : Corner Transfer Matrices of Eight-Vertex Model. I., J. Stat. Phys. 15, 485-503, (1976).
[4] B. Davis, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki : Diagonalization of the XXZ Hamiltonian by vertex operators, Commun. Math. Phys., 151, 89-153, (1993).
[5] S. Lukyanov and Y. Pugai : Multi-point local height probabilities in the integrable RSOS model, *Nucl. Phys. B* **473** [FS], 631-658, (1996).

[6] M. Idzumi : Level 2 irreducible representations of $U_q(\hat{sl}_2)$, Vertex operators and their correlations, *Int. J. Mod. Phys. A* **9**, No. 25, 4449-4484, (1994).

[7] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa : XXZ-chain with a boundary, *Nucl. Phys. B* **441** [FS], 437-470, (1995).

[8] M. Lashkevich and Y. Pugai : Free field construction for correlation functions of the eight-vertex model, *Nucl. Phys. B* **516**, 623-651, (1998).

[9] A. Nakayashiki : Fusion of q-Vertex Operators and its Application to Solvable Vertex Models, *Commun. Math. Phys.* **177**, 27-62, (1992).

[10] M. Fabrizio and A.O. Gogolin : Toulouse limit for the overscreened four-channel Kondo problem, *cond-mat* 9407104.

[11] R. Baxter : Exactly Solved Models in Statistical Mechanics. Academic Press, London, (1982).

[12] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima, and T. Tokihiro : Quantum Affine Symmetry in vertex model, *Int. J. Mod. Phys. A* **8**, 1479-1511, (1993).

[13] Y. Hara : Free-field realization of Vertex Operators for Level two modules of $U_q(\hat{sl}_2)$, *J. Phys. A: Math. Gen.* **31**, 8483-8494, (1998).

[14] O. Foda, M. Jimbo, T. Miwa, K. Miki and A. Nakayashiki : Vertex Operators in Solvable Lattice Models, *J. Math. Phys.* **35**, 13-46, (1994).