NEW RESULTS ON SIMPLEX-CLUSTERS IN SET SYSTEMS

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A $d$-simplex is defined to be a collection $A_1,\ldots,A_{d+1}$ of subsets of size $k$ of $[n]$ such that the intersection of all of them is empty, but the intersection of any $d$ of them is non-empty. Furthermore, a $d$-cluster is a collection of $d+1$ such sets with empty intersection and union of size $\leq 2k$, and a $d$-simplex-cluster is such a collection that is both a $d$-simplex and a $d$-cluster. The Erdős–Chvátal $d$-simplex Conjecture from 1974 states that any family of $k$-subsets of $[n]$ containing no $d$-simplex must be of size no greater than $\binom{n-1}{k-1}$. In 2011, Keevash and Mubayi extended this conjecture by hypothesizing that the same bound would hold for families containing no $d$-simplex-cluster. In this paper, we resolve Keevash and Mubayi’s conjecture for all $4 \leq d+1 \leq k$ and $n \geq 2k-d+2$, which in turn resolves all remaining cases of the Erdős–Chvátal Conjecture except when $n$ is very small (i.e. $n<2k-d+2$).

1. Introduction

For positive integers $n,k$ we define $[n] := \{1,\ldots,n\}$ and use $\binom{X}{k}$ to denote the set of all $k$-element subsets of a set $X$. Furthermore, we refer to a set $\mathcal{F} \subseteq \binom{[n]}{k}$ as a family, and if every element of $\mathcal{F}$ contains some $S \in \binom{[n]}{s}$, we say that $\mathcal{F}$ is an $s$-star centered at $S$. If $S = \{x\}$, we say simply that $\mathcal{F}$ is a star centered at $x$. Observe that $s$-stars can be no bigger than $\binom{n-s}{k-s}$. The following theorem, commonly known as the Erdős-Ko-Rado (EKR) theorem, is a foundational result in extremal combinatorics.

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Theorem 1. Let \( n \geq 2k \) and suppose \( F \subseteq \binom{[n]}{k} \). Furthermore, if \( A \cap B \neq \emptyset \) for all \( A, B \in F \), then
\[
|F| \leq \binom{n-1}{k-1},
\]
where, if \( n > 2k \), equality holds only if \( F \) is a maximum-sized star.

Families \( F \) that satisfy this condition are sometimes referred to as pairwise intersecting. Since its original publication in 1961 [4], EKR has seen numerous applications and has been proven using a wide array of different combinatorial and algebraic techniques. It has even generated a whole subfield of extremal combinatorics known as intersection problems, in which one considers the maximum size of a family with a certain forbidden subfamily, where the subfamily is defined according to some intersection or union constraints. For an introduction to this field, we direct readers to a recent survey of Frankl and Tokushige [9]. One of the more heavily-studied problems in this area involves the notion of a \( d \)-simplex.

Definition 1. A \( d \)-simplex is a set of \( d+1 \) elements \( \{A_1, A_2, \ldots, A_{d+1}\} \) of \( \binom{[n]}{k} \) with the properties that \( \bigcap_{i=1}^{d+1} A_i = \emptyset \) and for any \( 1 \leq j \leq d+1 \) we have \( \bigcap_{i \neq j} A_i \neq \emptyset \).

In 1971, Erdős conjectured that a family \( F \subseteq \binom{[n]}{k} \) that contains no 2-simplex (also known as a triangle) must adhere to the same bound of \( |F| \leq \binom{n-1}{k-1} \). In 1974, Chvátal extended this conjecture to the following.

Conjecture 1 ([2]). Let \( 3 \leq d+1 \leq k \) and \( n \geq \frac{d+1}{d} k \), and suppose \( F \subseteq \binom{[n]}{k} \) contains no \( d \)-simplex. Then
\[
|F| \leq \binom{n-1}{k-1},
\]
with equality only if \( F \) is a star.

In the same paper Chvátal also resolved the case of \( k = d+1 \). Conjecture 1 is now sometimes referred to as the Erdős-Chvátal simplex conjecture, and since its inception there have been a number of partial results. The conjecture was first proven for a wide range of \( n, k \) and \( d \) by Frankl in [6], and the case of \( n > n_0(k,d) \) was resolved by Frankl and Füredi in [8]. In 2005, Mubayi and Verstraëte solved completely the case of \( d = 2 \) [17], and in 2010, Keevash and Mubayi solved the case where both \( k/n \) and \( n/2 - k \) are bounded away from zero [11]. Very recently, Keller and Lifshitz showed that the conjecture holds for all \( n > n_0(d) \) [12]. A related notion, known as a \( d \)-cluster, is defined as follows.
**Definition 2.** A *d-cluster* is a set of $d+1$ elements $\{A_1, A_2, \ldots, A_{d+1}\}$ of $\binom{[n]}{k}$ with the properties that $\bigcap_{i=1}^{d+1} A_i = \emptyset$ and $|\bigcup_{i=1}^{d+1} A_i| \leq 2k$. If $\{A_1, \ldots, A_{d+1}\}$ is both a $d$-simplex and a $d$-cluster, we say that it is a *d-simplex-cluster*.

As with simplices, it was conjectured [14] that a family $\mathcal{F} \subseteq \binom{[n]}{k}$ containing no $d$-cluster would have to obey the bound $|\mathcal{F}| \leq \binom{n-1}{k-1}$. This problem also had a long history (see [7,14,15,16,11] for some of the more significant developments) and was completely resolved recently in a paper of the author [3]. In 2010, Keevash and Mubayi extended both conjectures by hypothesizing that the same bound would hold for any $\mathcal{F} \subseteq \binom{[n]}{k}$ containing no $d$-simplex-cluster, and very recently Lifshitz answered their question in the affirmative for all $n > n_0(d)$ in [13].

The primary goal of this paper is to show that Keevash and Mubayi’s conjecture holds for all $n \geq 2k-d+2$ when $d \geq 3$.

**Theorem 2.** Suppose that $4 \leq d+1 \leq k$ and $n \geq 2k-d+2$, and that $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no $d$-simplex-cluster. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1},$$

with equality only if $\mathcal{F}$ is a maximum sized star.

A family containing no $d$-simplex must also contain no $d$-simplex-cluster. Thus, we get as an immediate corollary (when combined with results from [17] for the case $d = 2$) that Conjecture 1 holds for all values of $d, k$ and $n$ except for the very small values of $n$, where $n < 2k-d+2$.

We take a moment here to discuss the range $n < 2k$. Intersection problems of this type tend to have a slightly different flavor when considered for these values of $n$. There are sometimes obvious reasons for this - the Erdős-Ko-Rado theorem in its original form, for example, does not make much sense when considered in this context because any two $k$-sets will intersect. As another example, the problem of clusters is different in this range because the union condition holds automatically, so a $d$-cluster-free family is simply a family with no $d+1$ sets that have empty intersection. There is, however, some history of results for problems of this type in the range $n < 2k$. The most notable example is perhaps the Complete Intersection Theorem of Ahlswede and Khachatrian [1], which gives us a characterization of families where all elements intersect each other in at least $t$ places. For simplices, there are two results known to the author. The first is for $d = 2$, when the full range of $n \geq 3k/2$ was shown in [17]. The other example (and the only for general $d$) is from [5], where the case of $n < k\frac{d}{d-1}$ is resolved.
In our proof of Theorem 2, we will use as one of our primary tools the following theorem of the author, which was used in [3] to resolve the question of $d$-clusters. We include the proof for completeness.

**Theorem 3.** Let $n \geq 2k$, and suppose $\mathcal{F}^* \subseteq \mathcal{F} \subseteq \binom{[n]}{k}$ are such that $A, B \in \mathcal{F}^*$ for any $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$. Then

$$|\mathcal{F}| \leq \left(\frac{n-1}{k-1}\right) + \frac{n-k}{n}|\mathcal{F}^*|,$$

where, for $n > 2k$, equality is achieved only if $\mathcal{F}^* = \mathcal{F} = \binom{[n]}{k}$ or if $\mathcal{F}^* = \emptyset$ and $\mathcal{F}$ is a maximum sized star.

Note that this theorem is itself a generalization of EKR; if one sets $\mathcal{F}^* = \emptyset$, then Theorem 3 says simply that a pairwise intersecting family has size at most $\left(\frac{n-1}{k-1}\right)$.

**Proof.** We will proceed by the Katona cycle method [10]. First, we let $C(n)$ denote the set of all cyclic permutations on $n$ elements. Then, if we have $(a_0, \ldots, a_{n-1}) = \sigma \in C(n)$ and $\mathcal{G} \subset \binom{[n]}{k}$, we define (with all subscripts henceforth taken mod $n$)

$$S_\sigma(\mathcal{G}) := \{A \in \mathcal{G}: A = \{a_i, a_{i+1}, \ldots, a_{i+(k-1)}\} \text{ for some } i \in [0, n-1]\}.$$

Observe trivially that $|S_\sigma(\mathcal{G})| \leq n$. Furthermore, for any such $A = \{a_i, \ldots, a_{i+(k-1)}\}$, we say that $A$ has starting point $i$ in $\sigma = (a_0, \ldots, a_{n-1})$.

Now, we wish to prove the following:

(i) $|S_\sigma(\mathcal{F} \setminus \mathcal{F}^*)| \leq k$ for all $\sigma \in C(n)$

(ii) if $S_\sigma(\mathcal{F} \setminus \mathcal{F}^*) \neq \emptyset$, then $|S_\sigma(\mathcal{F}^*)| \leq 2(k - |S_\sigma(\mathcal{F} \setminus \mathcal{F}^*)|)$ for all $\sigma \in C(n)$.

Let $\sigma = (a_0, \ldots, a_{n-1})$ as before, suppose $S_\sigma(\mathcal{F} \setminus \mathcal{F}^*) \neq \emptyset$, and take $A \in S_\sigma(\mathcal{F} \setminus \mathcal{F}^*)$. Furthermore, suppose without loss of generality that $A = \{a_0, \ldots, a_{k-1}\}$. Then, let $A' \in (S_\sigma(\mathcal{F}) \setminus \{A\})$ and observe that since $A \cap A' \neq \emptyset$, it follows that $A'$ has starting point in either $[n - (k-1), n-1]$ or $[1, k-1]$. Suppose then that we have $A_1, A_2 \in (S_\sigma(\mathcal{F}) \setminus \{A\})$ with starting points $i_1 \in [n - (k-1), n-1]$ and $(i_1 + k) \in [1, k-1]$ in $\sigma$, respectively. Since $n \geq 2k$ this implies $A_1 \cap A_2 = \emptyset$ and thus that $A_1, A_2 \in \mathcal{F}^*$. Because only one element of $\mathcal{F}$ may have a given starting point in $\sigma$, we can combine these facts to get both (i) and (ii). Now, we define subsets of $C(n)$

$$C_j := \{\sigma \in C(n): |S_\sigma(\mathcal{F} \setminus \mathcal{F}^*)| = j\},$$

and using (i) we observe that $C_0, C_1, \ldots, C_k$ partition $C(n)$. Using (i) and (ii), and since every $A \in \mathcal{F}$ is in $S_\sigma(\mathcal{F})$ for precisely $k!(n-k)!$ different $\sigma \in C(n)$
we get
\[ |F \setminus F^*| k!(n-k)! = \sum_{\sigma \in C(n)} |S_\sigma(F \setminus F^*)| = \sum_{1 \leq i \leq k} |C_i|i \]
and
\[ |F^*| k!(n-k)! = \sum_{\sigma \in C(n)} |S_\sigma(F^*)| \leq n|C_0| + \sum_{1 \leq i \leq k} |C_i|2(k-i). \]

Combining these yields
\[ (1) \]
\[ |F^*| + \left( \frac{n}{k} \right) |F \setminus F^*| \leq \frac{n|C_0| + \sum_{i=1}^k 2(k-i)|C_i| + (n/k)\left( \sum_{i=1}^k i|C_i| \right)}{k!(n-k)!} = \frac{n|C_0| + n|C_k| + \sum_{i=1}^{k-1} \frac{in + 2k(k-i)}{k}|C_i|}{k!(n-k)!}. \]

A quick calculation gives us that, for all \( 1 \leq i \leq k - 1 \)
\[ (2) \]
\[ \frac{in + 2k(k-i)}{k} \leq \frac{in + n(k-i)}{k} = n, \]
with equality only if \( n = 2k \). Combining (1) and (2), since \( |C_0| + \cdots + |C_k| = |C(n)| = (n-1)! \), we get
\[ \left( \frac{k-n}{k} \right) |F^*| + \left( \frac{n}{k} \right) |F| = |F^*| + \frac{n}{k} |F \setminus F^*| \leq \frac{n(|C_0| + \cdots + |C_k|)}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \]

Dividing both sides by \( n/k \) we get our desired inequality. Now, suppose \( n > 2k \) and we have equality. Note that in this case we do not have equality in (2) and so \( C(n) = C_0 \cup C_k \). Furthermore, if we take arbitrary \( A, A' \in F \), we can easily construct \( \sigma \in C(n) \) such that \( A, A' \in S_\sigma(F) \). Since either \( \sigma \in C_0 \) or \( \sigma \in C_k \), this implies that \( A, A' \in F^* \) or \( A, A' \in (F \setminus F^*) \). Since \( A \) and \( A' \) were arbitrary, we get that either \( F = F^* \) or \( F = (F \setminus F^*) \). If we assume the former, then \( |F| = |F^*| = \binom{n}{k} \) in which case \( F = \binom{n}{k} \). For the latter, we get that \( |F| = |F \setminus F^*| = \binom{n}{k-1} \) and \( F \) is pairwise intersecting, in which case Theorem 1 tells us that \( F \) is a star. This completes the proof.
In the remainder of the paper, we will use the following notation.

**Definition 3.** Suppose $\mathcal{F} \subseteq \left(\begin{array}{c} n \\ k \end{array}\right)$ and we have $A \in \mathcal{F}$ and $D \subseteq [n]$. Then, we define $
abla_{\mathcal{F}}(D) \subseteq \mathcal{F}$ and $\alpha^i_{\mathcal{F}}(A) \subseteq A$ to be

$$\nabla_{\mathcal{F}}(D) := \{ B \in \mathcal{F} : D \subseteq B \}$$

and

$$\alpha^i_{\mathcal{F}}(A) := \{ x \in A : \nabla_{\mathcal{F}}(A \setminus \{x\}) = i \}.$$

The first definition is related to the common combinatorial notion of link or trace - that is, $\nabla_{\mathcal{F}}(D)$ is all elements of $\mathcal{F}$ that contain $D$. The second definition can be thought of as a measure of the removability of the elements of $A \in \mathcal{F}$. By this we mean that, if we have $x \in \alpha^i_{\mathcal{F}}(A)$ for some $i \geq 2$, then we can remove $x$ from $A$ without increasing its size very much - that is, we can find $B \in \mathcal{F}$ such that $x \notin B$ but $|A \cup B|$ is small. Furthermore, if $i \geq 3$ we have greater flexibility in choosing $B$ that we will leverage later on. These are useful notions because they provide a way to construct $d$-simplex-clusters in a way that is both controlled and relatively easy to count.

The following lemmas will make this more precise. The first shows us that if we have $A, B \in \mathcal{F}$ such that $A \cap B$ satisfies certain size and removability conditions, then $\mathcal{F}$ must contain a $d$-simplex-cluster.

**Lemma 1.** Suppose $d+1 \leq k$ and $n \geq 2k-d$, and let $\mathcal{F} \subseteq \left(\begin{array}{c} n \\ k \end{array}\right)$. Then, if there exist $A, B \in \mathcal{F}$ such that $A \cap B \in \left(\begin{array}{c} A \setminus \alpha^1_d(A) \\ \alpha^2_d(A) \end{array}\right)$, then $\mathcal{F}$ must contain a $d$-simplex-cluster.

**Proof.** Let $A, B$ be as described, with $A \cap B = \{x_1, \ldots, x_d\}$, and suppose without loss of generality that $x_d \in \alpha^i_{\mathcal{F}}(A)$ for some $i \geq 3$. Then, for all $1 \leq j \leq d$, since $x_j \in \alpha^i_{\mathcal{F}}(A)$ for $i \geq 2$, there exists $B_j \in \mathcal{F}$ such that $A \cap B_j = A \setminus \{x_j\}$. Note at this point that $B_1, \ldots, B_d$ may have an element (at most one) of intersection outside of $A$. However, if this is the case, since $x_d \in \alpha^i_{\mathcal{F}}(A)$ for $i \geq 3$, we can re-choose $B_d$ such that the $B_1, \ldots, B_d$ have empty intersection outside of $A$. We claim that $B, B_1, \ldots, B_d$ is a $d$-simplex-cluster. Verifying first the intersection condition gives

$$B \cap B_1 \cap \cdots \cap B_d = (A \cap B) \cap B_1 \cap \cdots \cap B_d = \emptyset,$$

and furthermore

$$|B \cup B_1 \cup \cdots \cup B_d| \leq |A \cup B| + |B_1 \setminus A| + \cdots + |B_d \setminus A| \leq (2k-d) + d = 2k.$$

Finally, we see that $B_1 \cap \cdots \cap B_d = A \setminus B \neq \emptyset$ and that $x_j \in \left(\begin{array}{c} B \cap \left(\bigcap_{i \neq j} B_j\right) \end{array}\right)$ for all $1 \leq j \leq d$. Thus, $B, B_1, \ldots, B_d$ is a $d$-simplex-cluster, completing the proof.
Thus, our task reduces in some sense to proving that $\alpha_1^F(A)$ and $\alpha_2^F(A)$ are small for most $A \in F$. The following lemma will be used to show this.

**Lemma 2.** Suppose $k < n$ and that $F \subseteq \binom{[n]}{k}$ is such that $|F| \geq \binom{n-1}{k-1}$. Then

$$\sum_{A \in F} |\alpha_1^F(A)| + \frac{n-k-1}{2(n-k)}|\alpha_2^F(A)| \leq \binom{n-1}{k-1}.$$

**Proof.** To start, we define $F_C := \binom{[n]}{k} \setminus F$, and observe that

$$|F| + |F_C| = \binom{n}{k} = n \binom{n-1}{k-1}.$$

Using our assumption that $|F| \geq \binom{n-1}{k-1}$, we see that

$$(n-k)\binom{n-1}{k-1} \geq k|F_C|$$

$$= \sum_{A \in F_C} \sum_{1 \leq i \leq (n-k+1)} |\alpha_{i,F_C}^i(A)|$$

$$\geq \sum_{A \in F_C} |\alpha_{n-k,F_C}^n(A)| + |\alpha_{n-k-1,F_C}^{n-1}(A)|$$

$$= \sum_{A \in F} (n-k)|\alpha_1^F(A)| + \frac{n-k-1}{2}|\alpha_2^F(A)|,$$

which is the desired result.

Having shown that $\alpha_1^F(A)$ and $\alpha_2^F(A)$ are small for most $A \in F$, we will want to use this in conjunction with Lemma 1. However, Lemma 1 is a statement about $d$-subsets of $A$, while Lemma 2 is about single elements of $A$. To bridge the gap between these two results, we use the following counting lemma.

**Lemma 3.** Suppose $d+1 \leq k < n$ and $F \subseteq \binom{[n]}{k}$. Then, we have

$$\sum_{A \in F} \left( \binom{k}{d} - \left( \binom{|A \setminus \alpha_1^F(A)|}{d} + \binom{|\alpha_2^F(A)|}{d} \right) \right) \leq \binom{k-1}{d-1} \sum_{A \in F} \left( |\alpha_1^F(A)| + \frac{|\alpha_2^F(A)|}{d} \right).$$
Proof. We use here the fact that if $m_1, m_2, \ell \in \mathbb{N}$, then \(\binom{m_1}{\ell} - \binom{m_2}{\ell} = \binom{m_1 - 1}{\ell - 1} + \binom{m_2 - 1}{\ell - 1} + \cdots + \binom{m_2 - 1}{\ell - 1}\), as well as the fact that $|\alpha_1^1(A)|, |\alpha_2^2(A)| \leq k$ for all $A \in \mathcal{F}$. This yields

\[
\sum_{A \in \mathcal{F}} \left( \binom{k}{d} - \frac{|A \setminus \alpha_1^1(A)|}{d} + \frac{|\alpha_2^2(A)|}{d} \right)
\leq \sum_{A \in \mathcal{F}} \left( \sum_{i=1}^{|\alpha_1^1(A)|} \binom{k-i}{d-1} + \frac{|\alpha_2^2(A)|}{d} \binom{k}{d} \right)
\leq \sum_{A \in \mathcal{F}} \left( |\alpha_1^1(A)| \binom{k-1}{d-1} + \frac{|\alpha_2^2(A)|}{d} \binom{k-1}{d-1} \right),
\]

thus completing the proof.

We may now proceed with the proof of our main result.

**Proof of Theorem 2.** Let $|\mathcal{F}| \geq \binom{n-1}{k-1}$ and suppose $\mathcal{F}$ contains no $d$-simplex-cluster. Then, for any $D \in \binom{[n]}{d}$, we define the following subset of $\nabla_{\mathcal{F}}(D)$,

$$
\nabla_{\mathcal{F}}^*(D) := \{ A \in \nabla_{\mathcal{F}}(D) : D \cap \alpha_1^1(A) \neq \emptyset \text{ or } D \subseteq \alpha_2^2(A) \}.
$$

Now, suppose we have $A_1, A_2 \in \nabla_{\mathcal{F}}(D)$ such that $A_1 \cap A_2 = D$ and observe that by Lemma 1 and the fact that $\mathcal{F}$ contains no $d$-simplex-cluster, we have $A_1, A_2 \in \nabla_{\mathcal{F}}^*(D)$. Thus, we may apply Theorem 3 with \{$(A \setminus D) : A \in \nabla_{\mathcal{F}}(D)$\} as $\mathcal{F}$ and \{$(A \setminus D) : A \in \nabla_{\mathcal{F}}^*(D)$\} as $\mathcal{F}^*$ to get

$$
|\nabla_{\mathcal{F}}(D)| \leq \binom{n-d-1}{k-d-1} + \frac{n-k}{n-d} |\nabla_{\mathcal{F}}^*(D)|.
$$

Summing over all $D \in \binom{[n]}{d}$, and using Lemmas 2 and 3, we obtain

\[
|\mathcal{F}| \binom{k}{d} = \sum_{D \in \binom{[n]}{d}} |\nabla_{\mathcal{F}}(D)|
\leq \binom{n-d-1}{k-d-1} \binom{n}{d} + \frac{n-k}{n-d} \sum_{D \in \binom{[n]}{d}} |\nabla_{\mathcal{F}}^*(D)|
\leq \binom{n-d-1}{k-d-1} \binom{n}{d}.
\]
which is our desired inequality. Note that in (4) we have also used that $d \geq 3$
and $n \geq 2k - d + 2 \geq k + 3$. Now, suppose that we have equality, and in
particular, that we have equality in (3). We wish to show that $\mathcal{F}$ is a star.
To start, for every $1 \leq \ell \leq k$, we define $\mathcal{G}_\ell \subseteq \left[\binom{n}{\ell}\right]$ as follows

$$\mathcal{G}_\ell := \{D \in \left[\binom{n}{\ell}\right] : \bigtriangledown \mathcal{F}(D) = \bigtriangledown\left[\binom{n}{\ell}\right](D)\}.$$  

The proof will proceed as follows: we will start by showing that $|\mathcal{G}_d| \geq \binom{n-1}{d-1}$
and use this to show that $|\mathcal{G}_{d+1}| \geq \binom{n-1}{d}$. Then, we will show that $\mathcal{G}_{d+1}$
is $d$-simplex-free, and use this to show that it is a star. This will show by
extension that $\mathcal{F}$ is a star.

We start by showing that $|\mathcal{G}_d| \geq \binom{n-1}{d-1}$. To see this, observe that since
$n - d > 2(k - d)$, equality in (3) implies that, for any $D \in \left[\binom{n}{d}\right]$, we have either
that $\bigtriangledown \mathcal{F}(D)$ is a maximum-sized $(d+1)$-star or all of $\bigtriangledown\left[\binom{n}{d}\right](D)$. In particular,
this implies that $|\bigtriangledown \mathcal{F}(D)| = \binom{n-d}{k-d}$ for all $D \in \mathcal{G}_d$ and $|\bigtriangledown \mathcal{F}(D)| = \binom{n-d-1}{k-d-1}$
for all $D \in \left[\binom{n}{d}\right] \setminus \mathcal{G}_d$. Suppose for the sake of contradiction that $|\mathcal{G}_d| < \binom{n-1}{d-1}$. Then, we get

$$|\mathcal{F}| \binom{k}{d} = |\mathcal{G}_d| \binom{n-d}{k-d} + \left(\binom{n}{d} - |\mathcal{G}_d|\right) \binom{n-d-1}{k-d-1}$$

$$< \binom{n-1}{d-1} \binom{n-d}{k-d} + \binom{n-1}{d} \binom{n-d-1}{k-d-1}$$

$$= \binom{n-1}{k-1} \binom{n-1}{k-1},$$

which is a contradiction, so $|\mathcal{G}_d| \geq \binom{n-1}{k-1}$. Now, we will show that $|\mathcal{G}_{d+1}| \geq \binom{n-1}{k-1}$ by a double-counting argument. Observe that if $D \in \mathcal{G}_d$ and $x \in [n] \setminus D$, then $(D \cup \{x\}) \in \mathcal{G}_{d+1}$. Furthermore, as noted before, for every $D \in \left[\binom{n-1}{k-1}\setminus \mathcal{G}_d$,
we have that $\nabla_F(D)$ is a maximum-sized $(d+1)$-star. Thus, in this case there exists exactly one $x \in [n] \setminus D$ such that $(D \cup \{x\}) \in \mathcal{G}_{d+1}$. Finally, any element of $\mathcal{G}_{d+1}$ will be counted in this way precisely $d+1$ times, giving us

$$\left|\mathcal{G}_{d+1}\right| \geq \frac{\left|\mathcal{G}_d\right| (n-d) + \binom{n}{d} - \left|\mathcal{G}_d\right|}{d+1} \geq \frac{(n-1)(n-d) + \binom{n}{d} - \binom{n-1}{d-1}}{d+1} = \binom{n-1}{d}.$$

We show next that $\mathcal{G}_{d+1}$ must contain no $d$-simplex. To see this, suppose for the sake of contradiction that $\mathcal{G}_{d+1}$ contains a $d$-simplex $\{D_1, \ldots, D_{d+1}\}$. We observe first that $\{D_1, \ldots, D_{d+1}\}$ must in fact also be a $d$-cluster. To see the union condition, note that there must be $\{x_1, \ldots, x_{d+1}\} \subseteq [n]$ such that $x_j \in D_i$ for all $i \neq j$. By extension we see $|D_i \setminus \{x_1, \ldots, x_{d+1}\}| = 1$, and it follows easily that $|\bigcup_i D_i| \leq 2(d+1)$. Next, we choose (not necessarily distinct) $(k-d-1)$-sets $E_1, \ldots, E_{d+1} \subseteq [n] \setminus (\bigcup_i D_i)$ such that $\bigcap_i E_i = \emptyset$ and $|\bigcup_i E_i| \leq 2(k-d-1)$. Then, because $D_i \in \mathcal{G}_{d+1}$ for all $1 \leq i \leq d+1$, it follows that $(D_i \cup E_i) \in \mathcal{F}$. We claim that $(D_1 \cup E_1), \ldots, (D_{d+1} \cup E_{d+1})$ is a $d$-simplex-cluster. To verify this, we check first the union condition

$$\left|\bigcup_i (D_i \cup E_i)\right| = \left|\bigcup_i D_i\right| + \left|\bigcup_i E_i\right| \leq 2(d+1) + 2(k-d-1) = 2k,$$

and the first intersection condition

$$\bigcap_i (D_i \cup E_i) = \left(\bigcap_i D_i\right) \cup \left(\bigcap_i E_i\right) = \emptyset,$$

and finally the second intersection condition

$$\bigcap_{j \neq i} (D_i \cup E_i) = \left(\bigcap_{j \neq i} D_i\right) \cup \left(\bigcap_{j \neq i} E_i\right) \supseteq \left(\bigcap_{j \neq i} D_i\right) \neq \emptyset.$$

However, this contradicts our assumption that $\mathcal{F}$ is $d$-simplex-cluster-free, so $\mathcal{G}_{d+1}$ must be $d$-simplex-free. However, the $d+1=k$ case of Conjecture 1 was resolved by Chvátal in [2]. Since $|\mathcal{G}_{d+1}| \geq \binom{n}{d}$, this implies that $\mathcal{G}_{d+1}$ is a star centered at some $x \in [n]$. We now count the number of elements of $\mathcal{F}$ that contain $x$ by another double counting argument. For every $D \in \mathcal{G}_{d+1}$ there will be $\binom{n-d-1}{k-d-1}$ elements of $\mathcal{F}$ that contain it. Furthermore, for any
$A \in \mathcal{F}$ that contains $x$, it will have $\binom{k-1}{d}$ subsets of size $d+1$ that contain $x$. From this, we obtain

$$|\nabla \mathcal{F}(\{x\})| \geq \frac{|G_{d+1}|(n-d-1)}{(k-1)} \geq \frac{(n-1)(n-d-1)}{(k-1)} = \binom{n-1}{k-1}.$$ 

Thus, $\mathcal{F}$ is a star centered at $x$. This completes the proof.

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