GLOBAL SOLUTIONS FOR A HYPERBOLIC-PARABOLIC SYSTEM OF CHEMOTAXIS

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ABSTRACT. We study a hyperbolic-parabolic model of chemotaxis in dimensions one and two. In particular, we prove the global existence of classical solutions in certain dissipation regimes.

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1. INTRODUCTION

In this note we study the following system of partial differential equations

\begin{align}
\partial_t u &= -\Lambda^\alpha u + \nabla \cdot (u q), \text{ for } x \in \mathbb{T}^d, \ t \geq 0, \\
\partial_t q &= \nabla f(u), \text{ for } x \in \mathbb{T}^d, \ t \geq 0,
\end{align}

where \( u \) is a non-negative scalar function, \( q \) is a vector in \( \mathbb{R}^d \), \( \mathbb{T}^d \) denotes the domain \([-\pi, \pi]^d\) with periodic boundary conditions, \( d = 1, 2 \) is the dimension, \( f(u) = u^2/2, 0 < \alpha \leq 2 \) and \( (-\Delta)^{\alpha/2} = \Lambda^\alpha \) is the fractional Laplacian.

This system was proposed by Othmers & Stevens [21] based on biological considerations as a model of tumor angiogenesis. In particular, in the previous system, \( u \) is the density of vascular endothelial cells and \( q = \nabla \log(v) \) where \( v \) is the concentration of the signal protein known as vascular endothelial growth factor (VEGF) (see Bellomo, Li, & Maini [1] for more details on tumor modelling). Similar hyperbolic-dissipative systems arise also in the study of compressible viscous fluids or magnetohydrodynamics (see S. Kawashima [8] and the references therein).
Equation (1) appears as a singular limit of the following Keller-Segel model of aggregation of the slime mold *Dictyostelium discoideum* [9] (see also Patlak [20])

\[
\begin{aligned}
\partial_t u &= -\Lambda^\alpha u - \chi \nabla \cdot (u \nabla G(v)), \\
\partial_t v &= \nu \Delta v + (f(u) + \lambda) v,
\end{aligned}
\]

when \( G(v) = \log(v) \) and the diffusion of the chemical is negligible, i.e. \( \nu \to 0 \).

Similar equations arising in different context are the Majda-Biello model of Rossby waves [18] or the magnetohydrodynamic-Burgers system proposed by Fleischer & Diamond [3].

Most of the results for (1) correspond to the case where \( d = 1 \). Then, when the diffusion is local i.e. \( \alpha = 2 \), (1) has been studied by many different research groups. In particular, Fan & Zhao [2], Li & Zhao [13], Mei, Peng & Wang [19], Li, Pan & Zhao [12], Jun, Jixiong, Huijiang & Changjiang [7], Li & Wang [16] and Zhang & Zhu [25] studied the system (1) when \( \alpha = 2 \) and \( f(u) = u \) under different boundary conditions (see also the works by Jin, Li & Wang [6], Li, Li & Wang [14], Wang & Hillen [22] and Wang, Xiang & Yu [23]). The case with general \( f(u) \) was studied by Zhang, Tan & Sun [26] and Li & Wang [17].

Equation (1) in several dimensions has been studied by Li, Li & Zhao [11], Hau [5] and Li, Pan & Zhao [15]. There, among other results, the global existence for small initial data in \( H^s \), \( s > 2 \) is proved.

To the best of our knowledge, the only result when the diffusion is non-local, i.e. \( 0 < \alpha < 2 \), is [4]. In that paper we obtained appropriate lower bounds for the fractional Fisher information and, among other results, we proved the global existence of weak solution for \( f(u) = u^{r}/r \) and \( 1 \leq r \leq 2 \).

In this note, we address the existence of classical solutions in the case \( 0 < \alpha \leq 2 \). This is a challenging issue due to the hyperbolic character of the equation for \( q \). In particular, \( u \) verifies a transport equation where the velocity \( q \) is one derivative more singular than \( u \) (so \( \nabla \cdot (uq) \) is two derivatives less regular than \( u \)).

### 2. Statement of the results

For the sake of clarity, let us state some notation: we define the mean as

\[ \langle g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(x) dx. \]

Also, from this point onwards, we write \( H^s \) for the \( L^2 \)-based Sobolev space of order \( s \) endowed with the norm

\[ \| u \|^2_{H^s} = \| u \|^2_{L^2} + \| u \|^2_{H^s}, \quad \| u \|_{H^s} = \| \Lambda^s u \|_{L^2}. \]

For \( \beta \geq 0 \), we consider the following energies \( E_\beta \) and dissipations \( D_\beta \),

\[ E_\beta(t) = \| u \|^2_{H^{\beta}} + \| q \|^2_{H^{\beta}}, \quad D_\beta(t) = \| u \|^2_{H^{\beta+\alpha/2}}. \]

Recall that the lower order norms verify the following energy balance [4]

\[ \frac{1}{2} (\| u(t) \|^2_{L^2} + \| q(t) \|^2_{L^2}) + \int_0^t \| u(s) \|^2_{H^{\alpha/2}} ds = \frac{1}{2} (\| u_0 \|^2_{L^2} + \| q_0 \|^2_{L^2}). \]
2.1. On the scaling invariance. Notice that the equations (1)-(2) verify the following scaling symmetry: for every $\lambda > 0$

$$u_\lambda(x,t) = \lambda^{\alpha-1} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x,t) = \lambda^{\alpha-1} q(\lambda x, \lambda^\alpha t).$$

This scaling serves as a zoom in towards the small scales. We also know that

$$\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$$

is the strongest (known) quantity verifying a global-in-time bound. Then, in the one dimensional case, the $L^2$ norms of $u$ and $q$ are invariant under the scaling of the equations when $\alpha = 1.5$. That makes $\alpha = 1.5$ the critical exponent for the global estimates known. Equivalently, if we define the rescaled (according to the scaling of the strongest conserved quantity $\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$) functions

$$u_\gamma(x,t) = \gamma^{0.5} u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x,t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that $u_\gamma$ and $q_\gamma$ solve

$$\partial_t u_\gamma = -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5} \partial_x (u_\gamma q_\gamma),$$
$$\partial_t q_\gamma = \gamma^{\alpha-1.5} \partial_x u_\gamma.$$

Larger values of $\alpha$ form the subcritical regime where the diffusion dominates the drift in small scales. Smaller values of $\alpha$ form the supercritical regime where the drift might be dominant at small scales.

Similarly, the two dimensional case has critical exponent $\alpha = 2$.

Remark 1. Notice that the equations (1)-(2) where $f(u) = u$ have a different scaling symmetry but the same critical exponent $\alpha = 1.5$. In this case, the scaling symmetry is given by

$$u_\lambda(x,t) = \lambda^{2\alpha-2} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x,t) = \lambda^{\alpha-1} q(\lambda x, \lambda^\alpha t),$$

while the conserved quantity is $\|u(t)\|_{L^1} + \|q(t)\|_{L^2}^2$. Thus, if we define the rescaled (according to the scaling of the conserved quantity) functions

$$u_\gamma(x,t) = \gamma u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x,t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that $u_\gamma$ and $q_\gamma$ solve

$$\partial_t u_\gamma = -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5} \partial_x (u_\gamma q_\gamma),$$
$$\partial_t q_\gamma = \gamma^{\alpha-1.5} \partial_x u_\gamma.$$

A global existence result when $\alpha$ is the range $1.5 \leq \alpha < 2$ for the problem where $f(u) = u$ is left for future research.

2.2. Results in the one-dimensional case $d = 1$. One of our main results is

Theorem 1. Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $\alpha \geq 1.5$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0,T;H^2(\mathbb{T})) \cap L^2(0,T;H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0,T;H^2(\mathbb{T})).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0,\infty), H^1(\mathbb{T})) \times C([0,\infty), H^1(\mathbb{T})).$$
In the case where the strength of the diffusion, $\alpha$, is even weaker, we have the following global existence result for small data:

**Theorem 2.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(T) \times H^2(T)$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. There exists $C_\alpha$ such that if $1.5 > \alpha > 1$

$$\|u_0\|_{H^{\alpha/2}}^2 + \|q_0\|_{H^{\alpha/2}}^2 \leq C_\alpha$$

then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(T)) \cap L^2(0, T; H^{2+\alpha/2}(T)), q \in L^\infty(0, T; H^2(T)).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^{\alpha/2}}^2 + \|q(t)\|_{H^{\alpha/2}}^2 \leq \|u_0\|_{H^{\alpha/2}}^2 + \|q_0\|_{H^{\alpha/2}}^2.$$

**Corollary 1.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(T) \times H^2(T)$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $1 \geq \alpha \geq 0.5$ and

$$\|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2 < \frac{4}{9C_S^2}$$

where $C_S$ is defined in (7). Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(T)) \cap L^2(0, T; H^{2+\alpha/2}(T)), q \in L^\infty(0, T; H^2(T)).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2.$$

2.3. **Results in the two-dimensional case** $d = 2$. In two dimensions the global existence read

**Theorem 3.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(T^2) \times H^2(T^2)$ be the initial data such that $0 \leq u_0$, $\langle q_0 \rangle = 0$ and $\text{curl} q_0 = 0$. Assume that $\alpha = 2$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(T^2)) \cap L^2(0, T; H^3(T^2)), q \in L^\infty(0, T; H^2(T^2)).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0, \infty), H^1(T^2)) \times C([0, \infty), H^1(T^2)).$$

**Corollary 2.** Fix $T$ an arbitrary parameter and let $(u_0, q_0) \in H^2(T^2) \times H^2(T^2)$ be the initial data such that $0 \leq u_0$, $\langle q_0 \rangle = 0$ and $\text{curl} q_0 = 0$. Assume that $2 > \alpha \geq 1$ and

$$\|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2 < C$$

where $C$ is a universal constant. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(T^2)) \cap L^2(0, T; H^{2+\alpha/2}(T^2)), q \in L^\infty(0, T; H^2(T^2)).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|\nabla \cdot q(t)\|_{L^2}^2 \leq \|u_0\|_{H^1}^2 + \|\nabla \cdot q_0\|_{L^2}^2.$$
Remark 2. In the case where the domain is the one-dimensional torus, \( T \), local existence of solution for (1)-(2) was proved in [4] for a more general class of kinetic function \( f(u) \). The local existence of solution for (1)-(2) for the domain is the two-dimensional torus \( T^d \) with \( d = 2 \) follows from the local existence result in [4] with minor modifications. Consequently, we will focus on obtaining global-in-time a priori estimates.

2.4. Discussion. Due to the hyperbolic character of the equation for \( q \), prior available global existence results of classical solution for equation (1) impose several assumptions. Namely,

- either \( d = 1 \) and \( \alpha = 2 \) [26, 17],
- or \( d = 2, 3 \), \( \alpha = 2 \) and the initial data verifies some smallness condition on strong Sobolev spaces \( H^s, s \geq 2 \) [24, 27].

Our results removed some of the previous conditions. On the one hand, we prove global existence for arbitrary data in the cases \( d = 1 \) and \( \alpha \geq 1 \) and \( d = 2 \) and \( \alpha = 2 \). On the other hand, in the cases where we have to impose size restrictions on the initial data, the Sobolev spaces are bigger than \( H^2 \) (thus, the norm is weaker). Finally, let us emphasize that our results can be adapted to the case where the spatial domain is \( \mathbb{R}^d \).

A question that remains open is the trend to equilibrium. From (5) is clear that the solution \((u(t), q(t))\) tends to the homogeneous state, namely \((\langle u_0 \rangle, 0)\). However, the rate of this convergence is not clear.

3. Proof of Theorem 1

Step 1; \( H^1 \) estimate: Testing the first equation in (1) against \( \Lambda^2 u \), integrating by parts and using the equation for \( q \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^1} + \|\partial_x u\|^2_{H^{\alpha/2}} = -\int_T \partial_x (uq) \partial_x^2 u \, dx
\]

\[
= \frac{1}{2} \int_T \partial_x q (\partial_x u)^2 \, dx - \int_T \partial_x qu \partial_x^2 u \, dx
\]

\[
= \frac{1}{2} \int_T \partial_x q (\partial_x u)^2 \, dx - \int_T \partial_x q (\partial_t \partial_x q - (\partial_x u)^2) \, dx,
\]

so

\[
\frac{1}{2} \frac{d}{dt} (\|u\|^2_{H^1} + \|q\|^2_{H^1}) + \|u\|^2_{H^{1+\alpha/2}} = \frac{3}{2} \int_T \partial_x q (\partial_x u)^2 \, dx.
\]

Denoting

\[
I = \frac{3}{2} \int_T \partial_x q (\partial_x u)^2 \, dx,
\]

and using Sobolev embedding and interpolation, we have that

\[
I \leq \frac{3}{2} \|q\|_{H^1} \|\partial_x u\|^2_{L^4} \leq \frac{3}{2} C_S \|q\|_{H^1} \|\partial_x u\|^2_{H^{0.25}},
\]

where \( C_S \) is the constant appearing in the embedding

\[
\|g\|_{L^4} \leq C_S \|g\|_{H^{0.25}}.
\]

Using the interpolation

\[
H^{1+\alpha/2} \subset H^{1.25} \subset H^{0/2},
\]
and Poincaré inequality (if $\alpha > 1.5$) we conclude

$$I \leq c\|q\|_{H^1}\|\Lambda^{\alpha/2} u\|_{L^2}\|u\|_{H^{1+\alpha/2}}.$$  

Using (4), we have that

$$\frac{d}{dt}E_1 + D_1 \leq c\|u\|_{H^{\alpha/2}}^2 E_1.$$  

Using Gronwall’s inequality and the estimate (5), we have that

$$\sup_{0 \leq t < \infty} E_1(t) \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}),$$

and we conclude using Gronwall’s inequality.

**Step 2; $H^2$ estimate:** Now we prove that the solutions satisfying the previous bounds for $E_1$ and $D_1$ also satisfy the corresponding estimate in $H^2$. We test the equation for $u$ against $\Lambda^4 u$. We have that

$$\frac{1}{2} \frac{d}{dt}\|u\|_{H^2}^2 + \|u\|_{H^{2+\alpha/2}}^2 = -\int_T \partial_x^2 (uq) \partial_x^3 u dx$$

$$= \int_T \partial_x q \left(\frac{5(\partial_x^2 u)^2}{2}\right) dx - \int_T \partial_x q (\partial_x \partial_x^2 q - 5\partial_x u \partial_x^2 u) dx,$$

so

$$\frac{1}{2} \frac{d}{dt}(\|u\|_{H^2}^2 + \|q\|_{H^2}^2) + \|u\|_{H^{2+\alpha/2}}^2 = \frac{5}{2} \int_T \partial_x q (\partial_x^2 u)^2 dx + 5 \int_T \partial_x^2 q \partial_x^2 u \partial_x u dx.$$  

We define

$$J_1 = \frac{5}{2} \int_T \partial_x q (\partial_x^2 u)^2 dx, \quad J_2 = 5 \int_T \partial_x^2 q \partial_x^2 u \partial_x u dx.$$  

Then, we have that

$$J_1 \leq c\|\partial_x q\|_{L^\infty} \|\partial_x^2 u\|_{L^2}^2 \leq c\|\partial_x^2 q\|_{L^2}^{0.5} \|u\|_{H^{1+\alpha/2}} \|u\|_{H^{2+\alpha/2}}^{2-\alpha},$$

so, using Young’s inequality,

$$J_1 \leq c\|\partial_x^2 q\|_{L^2}^{0.5} \|u\|_{H^{1+\alpha/2}}^2 + \frac{1}{4} \|u\|_{H^{2+\alpha/2}}^2.$$  

Similarly, using Poincaré inequality and $\alpha \geq 0.5$,

$$J_2 \leq c\|\partial_x u\|_{L^4} \|\partial_x^2 u\|_{L^4} \|\partial_x^2 q\|_{L^2} \leq c\|u\|_{H^{1+\alpha/2}} \|u\|_{H^{2+\alpha/2}} \|\partial_x^2 q\|_{L^2},$$

and

$$J_2 \leq c\|u\|_{H^{1+\alpha/2}}^2 \|\partial_x^2 q\|_{L^2}^2 + \frac{1}{4} \|u\|_{H^{2+\alpha/2}}^2.$$  

Finally,

$$\frac{d}{dt}E_2(t) + D_2(t) \leq c\|u\|_{H^{1+\alpha/2}}^2 (E_2(t) + 1)$$

and we conclude using Gronwall’s inequality.
4. Proof of Theorem 2

**Step 1; \(H^{\alpha/2}\) estimate:** Testing the first equation in (1) against \(\Lambda^\alpha u\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^{\alpha/2}}^2 + \|u\|_{H^\alpha}^2 = \int_T \partial_x (uq) \Lambda^\alpha u dx
\]

\[
= - \int_T \Lambda^\alpha (uq) \partial_x u dx
\]

\[
= - \int_T (\Lambda^\alpha (uq) - u \Lambda^\alpha q) \partial_x u dx - \int_T \Lambda^\alpha qu \partial_x u dx,
\]

so

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^{\alpha/2}}^2 + \|q\|_{H^{\alpha/2}}^2 \right) + \|u\|_{H^\alpha}^2 \leq - \int_T [\Lambda^\alpha, u] q \partial_x u dx.
\]

We define

\[
K = - \int_T [\Lambda^\alpha, u] q \partial_x u dx.
\]

Using the classical Kenig-Ponce-Vega commutator estimate \([10]\) and Sobolev embedding, we have that

\[
\|[\Lambda^\alpha, u] q\|_{L^2} \leq c \left( \|\partial_x u\|_{L^{2+}}, \|\Lambda^{\alpha-1} q\|_{L^{\frac{4+2}{3}}} + \|\Lambda^\alpha u\|_{L^2} \|q\|_{L^\infty} \right)
\]

\[
\leq c \left( \|u\|_{H^{\frac{3}{2}}} \|\Lambda^{\alpha-1} q\|_{H^{\frac{3}{2}}} + \|u\|_{H^{\alpha}} \|q\|_{H^{\frac{3}{2}}} \right).
\]

Thus, taking \(\epsilon\) such that

\[1 + \frac{\epsilon}{4 + 2\epsilon} = \alpha, \; i.e. \; \epsilon = \frac{4\alpha - 4}{3 - 2\alpha}\]

Equation (8) reads

\[
\|[\Lambda^\alpha, u] q\|_{L^2} \leq c \left( \|u\|_{H^{\alpha}} \|q\|_{H^{\frac{3}{2}}} + \|u\|_{H^{\alpha}} \|q\|_{H^{\frac{3}{2}}} \right)
\]

Using (9) and Poincaré inequality, we have that

\[K \leq c \|u\|_{H^{\alpha}}^2 \|q\|_{H^{\frac{3}{2}}} \]

Then, we have that

\[
\frac{d}{dt} E_2 + D_2 \leq c \sqrt{E_2} D_2.
\]

Thus, due to the smallness restriction on the initial data, we obtain

\[E_2(t) + \delta \int_0^t D_2(s) ds \leq E_2(0)\]

for \(0 < \delta\) small enough.

**Step 2; \(H^1\) estimate:** Our starting point is (6). Then we use the interpolation

\[\|g\|_{H^{0.25}} \leq c \|g\|_{L^2} \|g\|_{H^{0.5}},\]

to obtain

\[I \leq c \|q\|_{H^1} \|u\|_{H^1} \|u\|_{H^{1.5}} \leq c E_1 D_2 + \frac{D_1}{2}.\]

Collecting all the estimates, we have that

\[
\frac{d}{dt} E_1 + D_1 \leq c E_1 D_2,
\]
and we conclude using Gronwall’s inequality. The $H^2$ estimates follows as in the proof of Theorem 1.

5. Proof of Corollary 1

Using $\alpha \geq 0.5$ and the estimate (6), we have that
\[
I \leq \frac{3}{2} C_S \|q\|_{H^1} \|u\|_{H^{1,\alpha/2}}^2 \leq \frac{3}{2} C_S \sqrt{E_1 D_1}.
\]
Thus,
\[
\frac{1}{2} \frac{d}{dt} E_1 + D_1 \leq \frac{3}{2} C_S \sqrt{E_1 D_1}.
\]
Thus, due to the smallness restriction on the initial data, we obtain
\[
E_1(t) + \delta \int_0^t D_1(s) ds \leq E_1(0)
\]
for $0 < \delta$ small enough. Equipped with this estimates, we can repeat the argument as in Step 2 in Theorem 1.

6. Proof of Theorem 3

Recall that the condition
\[
\text{curl} q_0 = 0
\]
propagates in time, i.e.
\[
\text{curl} q(t) = \text{curl} q_0 + \frac{1}{2} \int_0^t \text{curl} \nabla u^2 ds = 0.
\]
Using Plancherel Theorem, we have that
\[
\| \nabla q \|_{L^2}^2 = C \sum_{\xi \in \mathbb{Z}^2} |\xi|^2 |\hat{q}(\xi)|^2
\]
\[
= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1^2 + \xi_2^2) (\hat{q}_1^2 + \hat{q}_2^2).
\]
Due to the irrotationality
\[
\xi \perp \cdot \hat{q} = 0.
\]
Then, we compute
\[
\| \nabla \cdot q \|_{L^2}^2 = C \sum_{\xi \in \mathbb{Z}^2} |\xi \cdot \hat{q}(\xi)|^2
\]
\[
= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi) + \xi_2 \hat{q}_2(\xi))^2
\]
\[
= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + 2\xi_1 \hat{q}_1(\xi)\xi_2 \hat{q}_2(\xi)
\]
\[
= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + (\xi_2 \hat{q}_1(\xi))^2 + (\xi_1 \hat{q}_2(\xi))^2.
\]
So, the vector field $q$ satisfies
\[
\| \nabla q \|_{L^2} \leq \| \nabla \cdot q \|_{L^2}.
\]
As a consequence of $\langle \partial_t q_i \rangle = 0$ and $\langle q_0 \rangle = 0$, every coordinate of $q$ satisfy
\[
\langle q_i(t) \rangle = 0,
\]
and the Poincaré-type inequality

\[ \|q\|_{L^2} \leq c\|\nabla \cdot q\|_{L^2}. \]  

(10)

Notice that in two dimensions we also have the energy balance (5). We test equation (1) against \( \Lambda^2 u \) and use the equation for \( q \). We obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|^2_{H^1} + \|\nabla \cdot q\|^2_{L^2} \right) = -\|u\|^2_{H^2} - \int_{\mathbb{T}^2} \nabla u \cdot q \Delta u dx + \int_{\mathbb{T}^2} |\nabla u|^2 \nabla \cdot q dx. \]

Using Hölder inequality, Sobolev embedding and interpolation, we have that

\[ \frac{d}{dt} \left( \|u\|^2_{H^1} + \|\nabla \cdot q\|^2_{L^2} \right) + 2\|u\|^2_{H^2} \leq c \left( \|u\|^0_{H^1} \|q\|^0_{L^2} \|u\|^1_{H^2} + \|u\|^2_{H^2} \|\nabla \cdot q\|^2_{L^2} \right) \]

\[ \leq c\|u\|^0_{H^1} \|q\|^0_{L^2} \|q\|^1_{H^2} \|u\|^1_{H^2} + c\|u\|^0_{H^1} \|u\|^2_{H^2} \|\nabla \cdot q\|^2_{L^2}. \]

Using the Hodge decomposition estimate together with the irrotationality of \( q \) and (10), we have that

\[ \|q\|_{H^1} \leq c \left( \|q\|_{L^2} + \|\nabla \cdot q\|_{L^2} \right) \leq c\|\nabla \cdot q\|_{L^2}. \]

Due to (5), we obtain that

\[ \|q\|_{L^\infty(0,\infty,\mathbb{R}^2)} + \|\nabla \cdot q\|^2_{L^2(0,\infty,\mathbb{H}^1)} \leq C(\|u_0\|_{L^2}, \|q_0\|_{L^2}) \]

so,

\[ \frac{d}{dt} \left( \|u\|^2_{H^1} + \|\nabla \cdot q\|^2_{L^2} \right) + \|u\|^2_{H^2} \leq c\|u\|^0_{H^1} \|u\|^2_{H^2} \|\nabla \cdot q\|^2_{L^2}. \]

Using Gronwall’s inequality and the integrability of \( \|u\|^2_{H^1} \) (see (5)), we obtain

\[ E_1 \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}), \]

\[ \int_0^T D_1(s) ds \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \quad \forall 0 < T < \infty. \]

To obtain the \( H^2 \) estimates, we test against \( \Lambda^4 u \). Then, using the previous \( H^1 \) uniform bound and

\[ \|q\|_{L^\infty} \leq c\|q\|_{L^2} \|q\|_{H^2} \leq c\|q\|_{L^2} \|\Delta q\|_{L^2}, \]

we have that

\[ \frac{1}{2} \frac{d}{dt} \left( \|\Delta u\|^2_{L^2} + \|u\|^2_{H^3} \right) = -\int_{\mathbb{T}^2} \nabla \Delta u \nabla (\nabla u \cdot q) dx \]

\[ -\int_{\mathbb{T}^2} u \nabla \Delta u \cdot \nabla (\nabla \cdot q) dx - \int_{\mathbb{T}^2} \nabla u \cdot \nabla \Delta u \nabla \cdot q dx. \]

Due to the irrotationality of \( q \) and the identity

\[ \nabla \nabla \cdot q - \Delta q = \text{curl} \, (\text{curl} q), \]

we have

\[ \nabla \nabla \cdot q = \partial_t \nabla (\nabla \cdot q) = \nabla |\nabla u|^2 + u \nabla \Delta u + \nabla u \Delta u. \]

Applying Sobolev embedding and interpolation, we obtain that (12) can be estimated as

\[ \frac{d}{dt} \left( \|\Delta u\|^2_{L^2} + \|\Delta q\|^2_{L^2} \right) + \|u\|^2_{H^3} \leq c\|\Delta q\|^2_{L^2}, \]
so,

\[ \frac{d}{dt} E_2 + D_2 \leq c E_2, \]

and we conclude using Gronwall’s inequality.

7. PROOF OF COROLLARY 2

We test the equation (1) against \( \Lambda^2 u \). We obtain that

\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) = -\|u\|_{H^1}^2 + \int_{T^2} \nabla(\nabla u \cdot q) \nabla u dx + \int_{T^2} |\nabla u|^2 \nabla q dx. \]

After a short computation, using Hölder estimates, Sobolev embedding and interpolation, we obtain that

\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + \|u\|_{H^1}^2 \leq c \|u\|_{H^1}^2 \|q\|_{H^1}. \]

Using (11) and \( \alpha \geq 0 \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + D_1 \leq c D_1 \|\nabla \cdot q\|_{L^2}^2. \]

We conclude the result with the previous ideas.

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