A NOTE ON 2-PLECTIC HOMOGENEOUS MANIFOLDS

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Abstract. In this note we study the existence of 2-plectic structures on homogeneous spaces. In particular we show that $S^5 \cong SU(3)/SU(2)$, $SU(3)/S^1$, and $SU(3)/T^2$ admit a 2-plectic structure. Furthermore, if $G$ is a Lie group with Lie algebra $g$ and $R$ is a closed Lie subgroup of $G$ corresponding to the nilradical of $g$, then $G/R$ is a 2-plectic manifold.

1. Introduction. Let $V$ be a real vector space. A 3-form $\omega \in \wedge^3 V^*$ is called a 2-plectic form if $\omega$ is nondegenerate in the sense that $\iota_v \omega = 0$ if and only if $v = 0$. If $\omega$ is a 2-plectic form on $V$, the pair $(V, \omega)$ is called a 2-plectic vector space. A smooth manifold $M$ is called a 2-plectic manifold if there is a closed 3-form $\omega$ on $M$ such that $(T_x M, \omega_x)$ is a 2-plectic vector space, for all $x \in M$. 2-plectic structures (and in general multisymplectic structures) in the above sense, appeared in [3] for the first time. In the same paper, the authors introduced three important classes of 2-plectic manifolds as follows:

1. Compact semisimple Lie groups with the 2-plectic structure induced by the Killing form.
2. The bundle of exterior 2-forms $E$ on a smooth manifold $M$ with the 2-plectic structure $\omega = d\Theta$, where $\Theta$ is the canonical 2-form on $E$, characterized by $\alpha^*(\Theta) = \alpha$, for all 2-forms $\alpha$ on $E$.
3. Cosymplectic manifolds $(M, \theta, \eta)$ of dimension $2n + 1$ with the 2-plectic structure $\omega = \theta \wedge \eta$, where $\theta$ is a closed 2-form and $\eta$ is a closed 1-form on $M$ such that $\theta^n \wedge \eta \neq 0$.

Use of multisymplectic structures in the covariant Hamiltonian formulation of classical field theories have already been considered extensively ([6, 5, 4, 2]). In particular, 2-plectic manifolds are used to describe a classical string ([1, 9]). However, geometrically, 2-plectic geometry, in contrast to symplectic case, have not been considered. A possible reason is that the Darboux theorem does not hold in this case ([10, 8]). Another reason, can be the fact that 2-plectic manifolds are not, in comparison to symplectic manifolds, do not enjoy variety. In this note, using homogeneous spaces, we try to introduce new 2-plectic manifolds.

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2. Existence of 2-plectic structures on homogenous manifolds. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. We recall that for a $p$-form $\alpha \in \Lambda^p \mathfrak{g}^*$, $\delta(\alpha)$ is defined by

$$
\delta(\alpha)(X_1, \ldots, X_{p+1}) = \frac{1}{p+1} \sum_{i < j} (-1)^{i+j+1} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).
$$

For a $p$-form $\alpha$ on $\mathfrak{g}$, the kernel of $\alpha$ is defined by

$$
\mathfrak{k} = \text{Ker}\alpha = \{ X \in \mathfrak{g} : \iota_X \alpha = 0 \}.
$$

If $\alpha$ is closed, i.e $\delta(\alpha) = 0$, then $\mathfrak{k}$ is a sub lie algebra of $\mathfrak{g}$. In this case $\alpha$, as a left invariant form induces a foliation on $G$ with leaves diffeomorphic to $K$, where $K$ is the connected Lie group corresponding to $\mathfrak{k}$. Furthermore if $K$ is closed then $\alpha$ induces a nondegenerate $p$-form $\tilde{\alpha}$ on $\mathcal{G}/K$, the space of left cosets of $K$, which is invariant under left action of $G$ on $\mathcal{G}/K$ and satisfying $\pi^*\tilde{\alpha} = \alpha$, where $\pi : G \to \mathcal{G}/K$ is the canonical projection. Using this fact, in this section we are interested to construct 2-plectic structures on some homogenous manifolds. In this section, at first we prove some general results about such structures. These results are similar to the results which have been proved in [11] for symplectic case. Then some existence results will be proved.

Let $H$ be a closed Lie subgroup of $G$. A 2-plectic form $\tilde{\omega}$ on $\mathcal{G}/H$ is called $G$-invariant if $\tilde{\omega}$ is invariant under the action of $G$ on $\mathcal{G}/H$. Notice that if $\omega$ is a closed form on $\mathfrak{g}$ with kernel $\mathfrak{h}$, then the 2-plectic form $\tilde{\omega}$ on $\mathcal{G}/K$ induced by $\omega$ as above is $G$-invariant. In fact:

**Lemma 2.1.** Let $\omega$ be a closed left invariant 3-form on $G$ and $H$ be a closed Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. Then $\omega$ induces a $G$-invariant 2-plectic form $\tilde{\omega}$ on $\mathcal{G}/H$, with $\pi^*\tilde{\omega} = \omega$, if and only if $\text{Ker}\omega = \mathfrak{h}$.

Consider the decomposed Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_n$ (it means that $\mathfrak{g}$ is the direct sum of Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$) and let $p_i : \mathfrak{g} \to \mathfrak{g}_i, q_i : \mathfrak{g} \to \mathfrak{g}_i$ be the corresponding projection and injection of Lie algebras respectively. The 3-form $\omega$ on $\mathfrak{g}$ is called decomposable if $\omega = (q_1 \circ p_1)^*\omega + \ldots + (q_n \circ p_n)^*\omega$. It is easy to see that $\omega$ is decomposable if and only if $\omega(\mathfrak{g}_i, \mathfrak{g}_j, \mathfrak{g}_k) = 0$, whenever $i \neq j \neq k$.

**Lemma 2.2.** ([11]) Let the Lie algebra $\mathfrak{g}$ of the Lie group $G$ has a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_n$ and $\omega$ be a decomposable 3-form on $\mathfrak{g}$. Then the Lie algebra $\mathfrak{h}$ of the Lie subgroup $H = H_\omega = \{ x \in G : \text{Ad}_x^* \omega = \omega \}$ has a decomposition $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \ldots \oplus \mathfrak{h}_n$, $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$, $i = 1, \ldots, n$. Furthermore if $G$ decomposes into a corresponding direct product $G = G_1 \times \ldots \times G_n$, then $H$ decomposes into $H = H_1 \times \ldots \times H_n$ with $H_i = H \cap G_i$.

**Lemma 2.3.** Let $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{c}$ be ideals of $\mathfrak{g}$ with a semi simple and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cap \mathfrak{c} = \mathfrak{b} \cap \mathfrak{c} = 0$. Then for any closed 3-form $\omega$ on $\mathfrak{g}$, $\omega(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = 0$.

**Proof.** Since $\omega$ is closed, then

$$
\omega([X, Y], Z, W) = \omega([X, Z], Y, W) - \omega([X, W], Y, Z) + \omega([Y, Z], X, W)
$$

$$
-\omega([Y, W], X, Z) + \omega([Z, W], X, Y),
$$

for all $X, Y, Z, W \in \mathfrak{g}$. Now let $X, Y \in \mathfrak{a}, Z \in \mathfrak{b}$ and $W \in \mathfrak{c}$. Hypothesis imply that $[\mathfrak{a}, \mathfrak{b}] = [\mathfrak{a}, \mathfrak{c}] = [\mathfrak{b}, \mathfrak{c}] = 0$. Thus the right hand side of the equation vanishes, and since $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$, the result holds. \qed
The following result is a direct consequence of Lemma 2.3.

Corollary 1. If \( g = g_1 \oplus g_2 \oplus ... \oplus g_n \) is a semi simple Lie algebra, \( g_i \) simple ideal for \( i = 1, ..., n \), then any closed 3-form on \( g \) is decomposable.

Corollary 2. Let \( G \) be a connected semi simple Lie group with Lie algebra \( g = g_1 \oplus g_2 \oplus ... \oplus g_n \), and \( G_i \) be the connected Lie subgroup corresponding to \( g_i \), \( i = 1, ..., n \). If \( \omega \) is a closed 3-form on \( h \) and \( H = H_\omega \), then \( G = \bigoplus G_i \) and \( G_i = H \cap G_i \) are 2-plectic manifolds with the 2-plectic forms induced by \( \omega \) and \( (q_i \circ p_i)^* \omega \), respectively.

This Corollary is an extension of Lemma 1 and it is an immediate consequence of Lemmas 1 and 2.

Lemma 2.4. Let \( a, b \) and \( c \) be ideals of \( g \) and \( n \) be its nilradical. If \( H_{a,b,c} = \{ x \in G : \text{Ad}_x^* \omega_{a,b,c} = \omega_{a,b,c} \} \), where \( \omega_{a,b,c} \) denotes the restriction of \( \omega \) to \( a \times b \times c \), then the Lie algebra \( h_{a,b,c} \) contains \( n \), i.e., \( \text{ad}_X^* \omega_{a,b,c} = 0 \) for \( X \in n \). Furthermore if \( a \cap b = 0 = a \cap c \) and \( \omega \) is closed, then

\[
\omega(n,[b,c],a) = 0.
\]

Proof. The first statement is proved similar to Lemma 4.1 of [11]. To prove the second statement, let \( X \in n, Y \in b, Z \in c \) and \( W \in a \). Since \( \omega \) is closed, then

\[
\omega(X,[Y,Z],W) = \omega([X,Z],Y,W) - \omega([X,W],Y,Z) - \omega([X,Y],Z,W)
\]

\[
-\omega([Y,W],X,Z) + \omega([Z,W],X,Y),
\]

Now the first statement implies that the first three terms of the right hand side are zero. Since \( a \cap b = 0 = a \cap c \), therefore the last two terms are also zero. Thus the result holds.

Theorem 2.5. Let \( G \) be a Lie group with Lie algebra \( g \) and \( \tau \) be the radical of \( g \). If the connected Lie subgroup \( R \) corresponding to \( \tau \) is closed then \( G/\tau \) is a 2-plectic manifold.

Proof. Consider a Levi decomposition \( g = \tau \oplus s \) for \( g \) and let \( B \) denote the Killing form on \( s \). Define the 3-form \( \omega_0 \) on \( s \) by

\[
\omega_0(X,Y,Z) = B(X,[Y,Z]).
\]

\( \omega_0 \) is closed on \( s \). Define the 3-form \( \omega \) on \( g \) with \( \omega = p^* \omega_0 \), where \( p : g \to s \) is the projection. \( \omega \) is closed with \( ker \omega = \tau \). Indeed, since

\[
\delta(\omega)(X,Y,Z,W) = -\omega([X,Y],Z,W) + \omega([X,Z],Y,W)
\]

\[
-\omega([X,W],Y,Z) + \omega([Y,Z],X,W) - \omega([Y,W],X,Z) + \omega([Z,W],X,Y),
\]

then, if at least one of the vectors \( X, Y, Z \) and \( W \) is in \( \tau \), the right hand side is zero. Otherwise, \( \delta(\omega)(X,Y,Z,W) = \delta_0(\omega)(X,Y,Z,W) = 0 \). Thus \( \omega \) induces a \( G \)-invariant 2-plectic form on \( G/\tau \).

Theorem 2.6. Let \( G \) be a Lie group with Lie algebra \( g = g_1 \oplus g_2 \oplus ... \oplus g_n \), \( g_i \) semi simple, and \( G_i \) be the connected Lie subgroup corresponding to \( g_i \), \( i = 1, ..., n \). Furthermore, let \( J \subset \{1, ..., n\} \) and \( G_j \) be closed for \( j \in J \). Then \( \bigcap_{i \in J} G_i \) is a 2-plectic manifold.
Proof. Let the 3-form $\omega_j$ be defined on $g_j$ by its Killing form as Theorem 2.6 for $j \in J^c$. Put $\omega = p^*(\oplus_{j \in J^c} \omega_j)$, where $p : g \to \prod_{j \in J^c} g_j$ is the projection. $\omega$ is a closed 3-form on $g$ with $\text{Ker}\omega = \oplus_{i \in J^c} g_i$. Thus $\omega$ induces a 2-plectic structure on $\prod_{j \in J^c} g_j$.

3. Examples. In this section we consider the homogeneous spaces of $SU(3)$ and $SO(4)$. In particular we show that $S^5$ is a 2-plectic manifold.

3.1. The $SU(3)$-homogeneous 2-plectic manifolds. Consider the Gell-Mann matrices as follows

$$e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$

The set $E = \{e_1, ..., e_8\}$ is a basis for the Lie algebra $su(3)$. Let $\{\theta^1, ..., \theta^8\}$ be the corresponding dual basis and $\Theta^i$ be the left invariant 1-form on $SU(3)$ induced by $\theta^i$, $i = 1, ..., 8$.

Theorem 3.1. The homogeneous spaces $SU(3)/SU(2)$, $SU(3)/S^5$ and $SU(3)/T^2$ are 2-plectic manifolds.

Proof. Consider one of the following closed 3-forms on $SU(3)$

$\omega_1 = d(\Theta^1 \wedge \Theta^4 - \Theta^2 \wedge \Theta^3), \omega_2 = d(\Theta^1 \wedge \Theta^6 + \Theta^2 \wedge \Theta^7), \omega_3 = d(\Theta^1 \wedge \Theta^7 - \Theta^2 \wedge \Theta^6)$.

The kernel of each of these forms at $e$ is isomorphic to $su(2)$. Thus each of them induces a foliation on $SU(3)$ with leaves diffeomorphic to $SU(2)$. Hence they induce 2-plectic structures on $SU(3)/SU(2)$.

In the same way the closed 3-forms

$$\nu = d(\Theta^5 \wedge \Theta^6 + \Theta^5 \wedge \Theta^7 - \Theta^4 \wedge \Theta^6 + \Theta^4 \wedge \Theta^7),$$

$$\nu = d(\Theta^1 \wedge \Theta^2 - \Theta^4 \wedge \Theta^5 + \Theta^6 \wedge \Theta^7)$$

induce 2-plectic structures on $SU(3)/S^5$ and $SU(3)/T^2$, respectively.

Remark 1. 1. Notice that $SU(3)/SU(2)$ is diffeomorphic to $S^5$, thus $S^5$ has a $SU(3)$-invariant 2-plectic structure. The statement of the Theorem 3.1 has been treated before: See Example 5.7 of [7].

2. There is no 2-plectic structure on $SU(3)/U(2)$. In fact there is no 2-plectic manifold of dimension 4.
3.2. $SO(4)$-homogeneous 2-plectic manifolds. Consider the set $E = \{e_1, \ldots, e_6\}$ as a basis for Lie algebra $so(4)$ of the Lie group $SO(4)$, where

\[
e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

Let $\{\theta^1, \ldots, \theta^6\}$ be the corresponding dual basis and $\Theta^i$ be the left invariant 1-form on $SO(4)$ induced by $\theta^i$, $i = 1, \ldots, 6$.

**Theorem 3.2.** The homogeneous spaces $\frac{SO(4)}{SO(2)}$ and $\frac{SO(4)}{SO(3)} \simeq S^3$ admit $SO(4)$-invariant 2-plectic structures.

**Proof.** Consider the closed left invariant 3-form $\alpha = \Theta^3 \wedge \Theta^5 \wedge \Theta^6$ on $SO(4)$. Then

$$Ker\alpha_e = \{v \in so(4) : \iota_v \alpha = 0\} = \text{Span}\{e_1, e_2, e_4\},$$

where $e$ is the identity element of $SO(4)$. But

$$\text{Span}\{e_1, e_2, e_4\} = T_e(SO(3)) \subset T_e(SO(4)).$$

Thus $\alpha$ induces a foliation on $SO(4)$ with leaves diffeomorphic to $SO(3)$ and hence it induces a 2-plectic structure on $\frac{SO(4)}{SO(3)} \simeq S^3$. A similar argument works for $\frac{SO(4)}{SO(2)}$, when we consider the 3-form $\beta = d(\Theta^2 \wedge \Theta^4 - \Theta^3 \wedge \Theta^5)$. 

**REFERENCES**

[1] J. C. Baez, A. E. Hoffnung and C. L. Rogers, Categorified symplectic geometry and the classical string, *Comm. Math. Phys.*, 293 (2010), 701–725.

[2] F. Cantrijn, A. Ibort and M. DeLeon, Hamiltonian structures on multisymplectic manifolds *Rend. Sem. Mat. Univ. Pol. Torino*, 54 (1996), 225–236.

[3] F. Cantrijn, A. Ibort and M. DeLeon, On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. (Series A)*, 66 (1999), 303–330.

[4] J. F. Carinena, M. Crampin and L. A. Ibort, On the multisymplectic formalism for first order field theories, *Diff. Geom. Appl.*, 1 (1991), 345–374.

[5] M. Gotay, J. Isenberg, J. Marsden and R. Montgomery, Momentum maps and classical relativistic fields, Part I: Covariant field theory, *arXiv:Physics/9901019*.

[6] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, *Commun. Math. Phys.*, 30 (1973), 99–128.

[7] T. B. Madsen and A. Swann, Multi-Moment maps, *Adv. Math.*, 229 (2012), 2287–2309, *arXiv:1012.2048v2*.

[8] G. Martin, A Darboux theorem for multisymplectic manifolds, *Lett. Math. Phys.*, 16 (1988), 133–138.

[9] C. L. Rogers, Higher Symplectic Geometry, Ph.D thesis, University of California, Riverside, available as *arXiv:1106.4068v1*.

[10] M. Shafiee, On compact semisimple Lie groups as 2-plectic manifolds, *J. Geom.*, 105 (2014), 615–623.
[11] Ph. B. Zwart and W. M. Boothby, On compact homogeneous symplectic manifolds, *Ann. Inst. Fourier*, 30 (1980), 129–157.

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