Conformal $\beta$-change in Finsler spaces

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Abstract. We investigate what we call a conformal $\beta$-change in Finsler spaces, namely

$$L(x, y) \rightarrow *L(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$$

where $\sigma$ is a function of $x$ only and $\beta(x, y)$ is a given 1-form.

This change generalizes various types of changes: conformal changes, Randers changes and $\beta$-changes.

Under this change, we obtain the relationships between some tensors associated with $(M, L)$ and the corresponding tensors associated with $(M, *L)$. We investigate some $\sigma$-invariant tensors. This investigation allows us to give an answer to the question: Are the properties of $C$-reducibility, $S_3$-likeness and $S_4$-likeness invariant under a conformal $\beta$-change?

1. Introduction and Notations

Let $(M, L)$ be a Finsler space, where $M$ is an $n$-dimensional differentiable manifold equipped with a fundamental function $L$. Given a function $\sigma$, the change

$$L(x, y) \rightarrow e^{\sigma(x)}L(x, y)$$

is called a conformal change. The conformal theory of Finsler spaces has been initiated by M.S. Kneblman [5] in 1929 and has been deeply investigated by many authors: [1], [3], [4],... etc.

In 1941, Randers [9] has introduced the Finsler change

$$^rL(x, y) \rightarrow ^rL(x, y) + \beta(x, y)$$

where $^rL$ is a Riemanian structure and $\beta$ is a 1-form on $M$. The resulting space is a Finsler space. This change has been studied by several authors: [7], [12],... etc.

The Randers change has been generalized by Shibata [10] to what is called a $\beta$-change

$$L(x, y) \rightarrow L(x, y) + \beta(x, y)$$

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where $L$ a fundamental Finslerian function. The resulting space known as a generalized Randers space was studied in [13], [4], [7], [11] and [8], etc.

In this paper, we construct a theory which generalizes all the above mentioned changes. In fact, we consider a change of the form

$$L(x, y) \rightarrow *L(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y),$$

where $\sigma$ is a function of $x$ and $\beta(x, y) = b_i(x)y^i$ is a 1-form on $M$, which we call a conformal $\beta$-change. This change generalizes various type of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a $\beta$-change and consequently to a Randers change.

We obtain the relationships between some tensors associated with $(M, L)$ (the fundamental tensor, the $h(hv)$ - torsion and the third curvature tensor) and the corresponding tensors associated with $(M, *L)$.

Under the conformal $\beta$ - change, we investigate some $\sigma$- invariant tensors (a tensor $K$ is $\sigma$- invariant if $*K(x, y) = e^\sigma K(x, y)$).

This investigation leads us to find out necessary and /or sufficient conditions for the properties of C-reducibility, $S_3$-likeness and $S_4$-likeness to be invariant under a conformal $\beta$ - change (cf. theorems A, B, and C).

More investigation and development of this theory will be the object of forthcoming papers.

Throughout the present paper, $(x^i)$ denotes the coordinates of a point of the base manifold $M$ and $(y^i)$ the supporting element (\dot{x}^i).

We use the following notations:

- $l_i := \dot{\partial}_i L = \frac{\partial L}{\partial y^i}$: the normalized supporting element,
- $h_{ij} := \dot{L}l_i l_j = Ll_{ij}$: the angular metric tensor,
- $g_{ij} := \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$: the fundamental tensor,
- $c_{ijk} := \dot{\partial}_k (g_{ij}/2)$: the(h) hv -torsion tensor ,
- $c_i := g^{jk} c_{ijk}$: the torsion vector,
- $c^k := g^{jk} c_i$ : , $c^2 = c_i c^i$,
- $S_{hijk} := c_{ijr} c_{h}^r k - c_{ikr} c_{h}^r j$: the components of the third curvature tensor.

2. Conformal $\beta$-change

We firstly introduce the following definition

**Definition 1.** A change of Finsler metric defined by

$$L(x, y) \rightarrow *L(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$$

where $\sigma = \sigma(x)$ is a function of $x$ and $\beta(x, y) = b_i(x)y^i$ is a 1-form, will be called a conformal $\beta$-change.
This change generalizes various changes studied by Randers [9], Matsumoto [7], Shibata [10]...etc.

We assume that $^*L(x, y)$ enjoys the same properties possessed by $L(x, y)$.

As the Finsler space associated to $L$ is denoted by $(M, L)$, we denote the Finsler space associated to the conformal $\beta$–change by $(M, ^*L)$.

Throughout the whole paper, the geometric objects associated with $^*L(x, y)$ will be asterisked.

**Definition 2.** A geometric object $K$ is said to be $\sigma$–invariant if it is invariant, up to a factor $e^{\sigma(x)}$, under a conformal $\beta$–change: $^*K = e^{\sigma(x)}K$.

It follows from (1) that

$$^*l_i(x, y) = e^{\sigma(x)}l_i(x, y) + b_i(x), \quad ^*l_{ij}(x, y) = e^{\sigma(x)}l_{ij}(x, y).$$

The angular metric tensor $h_{ij}$ is given in terms of $h_{ij}$ by

$$^*h_{ij} = ^*L^*l_{ij} = ^*Le^{\sigma}l_{ij} = \tau h_{ij}, \quad \tau = e^{\sigma(x)}\frac{^*L}{L}.$$  \hspace{1cm} (3)

Then we have the following

**Lemma 1.** $\frac{h_{ij}}{L}$ is $\sigma$–invariant under a conformal $\beta$–change:

$$\frac{^*h_{ij}}{^*L} = e^{\sigma} \frac{h_{ij}}{L}.$$ \hspace{1cm} (4)

As $h_{ij} = g_{ij} - l_i l_j$, equations (3) give us a relation between the fundamental tensors $g_{ij}$ and $^*g_{ij}$:

$$^*g_{ij} = \tau(g_{ij} - l_i l_j) + ^*l_i ^*l_j.$$ \hspace{1cm} (5)

The relation between the corresponding covariant components is obtained in the form

$$^*g^{ij} = \tau^{-1}g^{ij} + \mu l^i l^j - \tau^{-2}(l^i b^j + l^j b^i),$$ \hspace{1cm} (6)

where $\mu = (e^{\sigma}Lb^2 + \beta)/L^2$, $b^2 = b^ib^j$, $b^i = g^{ij}b_j$.

Let us introduce the $\pi$–vector field

$$\overline{m} = \overline{B} - \frac{\beta}{L^2} \overline{m}, \quad m^i = b^i - \frac{\beta}{L} l^i, \quad m^2 = m_i m^i.$$

**Lemma 2.** For a conformal $\beta$–change which is not conformal (i.e $\beta \neq 0$), $\overline{m} \neq 0.$
In fact, if \( \overline{m} = 0 \), then \( m_i = 0 \) for all \( i \), and consequently \( b_i = \frac{2}{L} l_i \) which implies \( \beta = e^{\psi(x)} L \), for some function \( \psi(x) \).

**Lemma 3.** The (h) hv- torsion tensor \( ^*c_{ijk} \) associated to \( ^*F \) is given by

\[
^*c_{ijk} = \tau [c_{ijk} + \frac{1}{2^*L} h_{ijk}],
\]

where

\[
h_{ijk} = h_{ij} m_k + h_{jk} m_i + h_{ki} m_j.
\]

From the tensor \( ^*c_{ijk} \), we obtain the following important tensors:

\[
^*c^r_{ij} = c^r_{ij} + \frac{1}{2^*L} (h_{ij} m^r + h_{jr} m_i + h_{rj} m_j) - \tau^{-1} c_{ij} l^r b^s - \frac{1}{2^*L} (2 m_i m_j + m^2 h_{ij}) l^r,
\]

\[
^*c_i = c_i + \frac{n + 1}{2^*L} m_i,
\]

\[
^*c^k = \tau^{-2} [\tau c^k - c^r l^k + \frac{n + 1}{2^*L} (\tau m^k - m^2 l^k)],
\]

\[
^*c^2 = \tau^{-1} [c^2 + \frac{n + 1}{2^*L} A^2],
\]

\[
^*c^\beta = c^\beta + \frac{n + 1}{2^*L} m^2
\]

where \( A^2 = c^\beta + \frac{n + 1}{2^*L} m^2 \), \( c^\beta = c_i b^i \).

**Proof.**

- Equation (7) is deduced from the definition of \( c_{ijk} \) together with (6)
- Equation (9) is deduced by raising the index \( k \) in (7), using (6)
- Equation (10) is obtained by contracting the subscript \( i \) and the superscript \( r \) in (9)
- Equation (11) follows from (10) by raising its subscript, using (6)
- Equation (12) follows directly from (10) and (11) by contracted multiplication
- Equation (13) is obtained easily from (10) and the definition of \( c^\beta \) by contracted multiplication. □
Lemma 4.

(a) The relation between \( *S_{hijk} \) and \( S_{hijk} \) takes the form

\[
* S_{hijk} = \tau S_{hijk} - \frac{\tau}{2* L} [h_{ik} H_{jh} + h_{jh} H_{ik} - h_{hk} H_{ij} - h_{ij} H_{hk}],
\]

(14)

where

\[
H_{ij} = c_{i}^{r} j m_{r} + \frac{1}{2* L} m_{i} m_{j} + \frac{1}{4* L} h_{ij} m^{2}.
\]

(15)

(b) The v- Ricci tensor \( *S_{ik} \) is written in the form

\[
* S_{ik} = S_{ik} - \frac{1}{2* L} [A_{\beta} h_{ik} + (n - 3) H_{ik}]
\]

(16)

(c) The v- scaler curvature tensor is written in the form

\[
* S = \tau^{-1} [S - \frac{n - 2}{* L} A_{\beta}]
\]

(17)

Proof.

(a) From equations (7) and (9) we have

\[
* c_{ijr} * c_{h k}^{r} = \tau [c_{ijr} + \frac{1}{2* L} h_{ijr}] [c_{h k}^{r} + \frac{1}{2* L} (h_{hk} m_{r} + h_{r}^{h} m_{k})]
\]

\[
- \tau^{-1} c_{h k}^{r} b^{r} - \frac{1}{2* L} \tau (2 m_{h} m_{k} + m_{h}^{2}) b^{r}
\]

\[
= \tau c_{ijr} c_{h k}^{r} + \frac{\tau}{2* L} [(c_{r}^{j} h_{hk} + c_{r}^{k} h_{ij}) m_{r} + (c_{ijk} m_{h} + c_{jkh} m_{i} + c_{khi} m_{j} + c_{hij} m_{k})]
\]

\[
+ \frac{\tau}{4* L^2} [h_{ij} h_{hk} m^{2} + 2 h_{hk} m_{i} m_{j} + 2 h_{ij} m_{h} m_{k} + h_{jh} m_{i} m_{k} + h_{jk} m_{i} m_{h} + h_{ik} m_{j} m_{h}].
\]

Similarly, one can obtain \( *c_{ikr} * c_{h j}^{r} \) (by interchange \( j \) and \( k \)). Hence the result.

(b) follows from (a) by contracted multiplication, using (11).

(c) is obtained from (b), using (11) again, by contracted multiplication. \( \square \)

Remark. The tensor \( H_{ij} \) defined by (15) has the properties:

1. \( H_{ij} \) is a symmetric tensor : \( H_{ij} = H_{ji} \),

2. \( H_{ij} \) is an indicatory tensor : \( H_{ij} y^{i} = 0 = H_{ij} y^{j} \),

3. \( g^{ij} H_{ij} = A_{\beta} \).
3. Geometrical properties of the conformal β-change

Definition 3. [6] A Finsler space \((M, L)\) of dimension \(n \geq 3\) is called a C-reducible space if the \(h(hv)\)-torsion tensor \(c_{ijk}\) has the form

\[
c_{ijk} = h_{ij}M_k + h_{kj}M_i + h_{ki}M_j, \quad M_i = \frac{c_i}{n + 1}.
\]

(18)

Define the tensor

\[
K_{ijk} = [c_{ijk} - (h_{ij}M_k + h_{kj}M_i + h_{ki}M_j)]/L.
\]

It is clear that \(K_{ijk}\) is a symmetric and indicatory tensor. Moreover \(K_{ijk}\) vanishes if and only if the Finsler space is C-reducible.

Proposition 1. Under a conformal β-change, the tensor \(K_{ijk}\) is \(σ\)-invariant:

\[
*K_{ijk} = e^σK_{ijk}.
\]

Proof. Using Equation (17) together with the definition of \(K_{ijk}\), we get

\[
*K_{ijk} = [c_{ijk} - (h_{ij}M_k + h_{kj}M_i + h_{ki}M_j)]/L
\]

\[
= τ[(c_{ijk} + \frac{1}{2L}h_{ijk}) - (h_{ij}M_k + h_{kj}M_i + h_{ki}M_j)]/L
\]

\[
= τ[c_{ijk} + \frac{1}{2L}(h_{ij}m_k + h_{kj}m_i + h_{ki}m_j) - \frac{1}{n + 1}(h_{ij}c_k + h_{kj}c_i + h_{ki}c_j)]/L
\]

\[
= τ[c_{ijk} + \frac{1}{n + 1}(h_{ij}c_k + h_{kj}c_i + h_{ki}c_j)]/L
\]

\[
= e^σ[c_{ijk} - (h_{ij}M_k + h_{kj}M_i + h_{ki}M_j)]/L = e^σK_{ijk} \quad □
\]

Now, Proposition 1 yields

Theorem A. Under a conformal β-change \(L \rightarrow *L\), the space \((M, L)\) is C-reducible if and only if the space \((M, *L)\) is C-reducible.

Consequently the C-reducibility property is invariant under this change.

It should be noticed that Theorem 4-1 and Corollary 4-1 of Shibata [10] result from the above Theorem as a very special case. Some results of Matsumoto [7] are also contained in the above Theorem.

Definition 4. [2] A Finsler space \((M, L)\) of dimension \(n > 4\) is called an \(S_4\)-like space if the vertical curvature tensor \(S_{hijk}\) has the form

\[
S_{hijk} = h_{jh}M_{ik} + h_{ik}M_{jh} - h_{hk}M_{ij} - h_{ij}M_{hk},
\]

(19)

where \(M_{ij}\) is the symmetric and indicatory tensor given by \(M_{ij} = \frac{1}{n-3}[S_{ij} - \frac{S_{hijk}}{2(n-2)}]\).
Define the tensor
\[ \eta_{hijk} = \frac{[S_{hijk} - (h_{jh}M_{ik} + h_{ik}M_{jh} - h_{hk}M_{ij} - h_{ij}M_{hk})] / L}{}, \]

It is clear that \( \eta_{hijk} \) vanishes if and only if the manifold \( (M, L) \) is an \( S_4 \)-like manifold.

It is not difficult to prove the following.

**Lemma 5.** The tensor \( *M_{ij} \) is given in terms of \( M_{ij} \) by
\[ *M_{ij} = M_{ij} - \frac{1}{2} \ast_L H_{ij}. \]

In fact, the result follows from (12) and (16).

**Proposition 2.** Under a conformal \( \beta \)-change, the tensor \( \eta_{hijk} \) is \( \sigma \)-invariant:
\[ *\eta_{hijk} = e^\sigma \eta_{hijk}. \]

**Proof.** Taking Lemma 4a and Lemma 5 into account, we get
\[
*L^*\eta_{hijk} = *S_{hijk} - (h_{jh} *M_{ik} + *h_{ik} *M_{jh} - *h_{hk} *M_{ij} - *h_{ij} *M_{hk}) \\
= \tau S_{hijk} - \frac{\tau}{2^*L}[h_{jk} H_{ih} + h_{ih} H_{jk} - h_{hk} H_{ij} - h_{ij} H_{hk}] \\
- \tau [h_{jk} (M_{ih} - \frac{1}{2} *L H_{ih}) + h_{ik} (M_{jh} - \frac{1}{2} *L H_{jh})] \\
- h_{hk} (M_{ij} - \frac{1}{2} *L H_{ij}) - h_{ij} (M_{hk} - \frac{1}{2} *L H_{hk})] \\
= \tau [S_{hijk} - (h_{jh} M_{ih} + h_{ih} M_{jk} - h_{hk} M_{ij} - h_{ij} M_{hk})] \\
= \tau L \eta_{hijk} = e^\sigma *L \eta_{hijk}.
\]

Hence the result. \( \square \)

Proposition (2) yields

**Theorem B.** Under a conformal \( \beta \)-change \( L \rightarrow *L \), the space \( (M, L) \) is \( S_4 \)-like if and only if the space \( (M, *L) \) is an \( S_4 \)-like.

Consequently, the \( S_4 \)-likeness property is invariant under this change.

The above result generalizes Theorem 4-5 (and its Corollary) of Shibata [10].

**Definition 5.** A Finsler space \( (M, L) \) of dimension \( n > 3 \) is called an \( S_3 \)-like space if the vertical curvature tensor \( S_{hijk} \) has the form
\[
S_{hijk} = \frac{S}{(n - 1) (n - 2)} [h_{ik} h_{jh} - h_{ij} h_{hk}].
\]
Define the tensor
\[ \zeta_{hijk} = \left[ S_{hijk} - \frac{S}{(n-1)(n-2)}(h_{ik}h_{jh} - h_{ij}h_{hk}) \right] lL \]

It is clear that \( \zeta_{hijk} \) vanishes if and only if the manifold \((M, L)\) is an \(S_3\)-like manifold.

**Proposition 3.** Under a conformal \(\beta\)-change, the tensor \(\zeta_{hijk}\) is \(\sigma\)-invariant if and only if \(H_{ij} = \frac{1}{n-1}A_\beta h_{ij}\).

**Proof.** Using Equation (7) together with the definition of \(K_{ijk}\), we get
\[
*L^*\zeta_{hijk} = \left[ *S_{hijk} - \frac{*S}{(n-1)(n-2)}(*h_{ik} *h_{jh} - *h_{ij} *h_{hk}) \right] \\
= \left[ \tau S_{hijk} - \frac{\tau}{2 *L} (h_{ik}H_{jh} + h_{jh}H_{ik} - h_{hk}H_{ij} - h_{ij}H_{hk}) \right] \\
- \frac{\tau}{(n-1)(n-2)} \left( S - \frac{(n-2)}{*L} A_\beta \right) (h_{ik} h_{jh} - h_{ij} h_{hk}) \\
= \left[ \tau (S_{hijk} - \frac{S}{(n-1)(n-2)}(h_{ik}h_{jh} - h_{ij}h_{hk})) \right] \\
- \frac{\tau}{2 *L} \left[ (h_{ik} H_{jh} + h_{jh} H_{ik} - h_{hk} H_{ij} - h_{ij} H_{hk}) + \frac{1}{(n-1)} A_\beta (2h_{ik} h_{jh} - 2h_{ij} h_{hk}) \right] \\
= \tau L \zeta_{hijk} - \frac{\tau}{2 *L} \left[ (h_{ik} (H_{jh} - \frac{1}{(n-1)} A_\beta h_{jh})) + (h_{jh} (H_{ik} - \frac{1}{(n-1)} A_\beta h_{ik})) + \right. \\
+ (h_{hk} (H_{ij} - \frac{1}{(n-1)} A_\beta h_{ij})) + h_{ij} (H_{hk} - \frac{1}{(n-1)} A_\beta h_{hk})) \right]
\]

Now the tensor \(*\zeta_{hijk}\) is \(\sigma\)-invariant \((*\zeta_{hijk} = e^\sigma \zeta_{hijk})\) if and only if all terms of the forms 
\(H_{ij} - \frac{1}{(n-1)} A_\beta h_{ij}\) vanish; that is, if and only if the condition \(H_{ij} = \frac{1}{n-1} A_\beta h_{ij}\) holds. \(\square\)

Consequently we get

**Theorem C.** Under a conformal \(\beta\)-change \(L \longrightarrow *L\), the following two assertions are equivalent

1. The space \((M, L)\) is \(S_3\)-like,

2. The space \((M, *L)\) is \(S_3\)-like

if and only if the condition \(H_{ij} = \frac{1}{n-1} A_\beta h_{ij}\) holds.

Consequently, the \(S_3\)-likeness property is invariant under this change if and only if \(H_{ij} = \frac{1}{n-1} A_\beta h_{ij}\)
References

[1] M. Hashiguchi, On conformal transformation of Finsler metric, J. Math. Kyoto Univ.. 16(1976) pp. 25-50.

[2] F. Ikedo, On $S_3$-and $S_4$-like Finsler spaces with the T- tensor of a special form, Tensor, N.S.. 35(1981) pp. 345-351.

[3] H. Izumi, Conformal transformations of Finsler spaces I and II, Tensor, N.S.. 31 and 33(1977 and 1980) pp. 33-41 and 337-359.

[4] M. Kitayama, Geometry of transformations of Finsler metrics, Hokkaido University of Education, Kushiro Campus.. Japan(2000)

[5] M. S. Knebelman, Conformal geometry of generalized metric spaces, Proc.nat.Acad. Sci.USA.. 15(1929) pp. 33-41 and 376-379.

[6] M. Matsumoto, On C - reducible Finsler spaces, Tensor, N.S.. 24(1972) pp. 29-37.

[7] M. Matsumoto, On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ.. 14(1974) pp. 477-498.

[8] R. Miron, General Randers space, Lagrange and Finsler geometry, Ed. by P.L. Antonelli and R.Miron..76(1996) pp.123-140.

[9] G. Randers, On the asymmetrical metric in the four- space of general relativity, Phys. Rev..2(1941)59 pp. 195-199.

[10] C. Shibata, On invariant tensors of $\beta$ - change of Finsler metrics, J. Math. Kyoto Univ.. 24(1984) pp. 163-188.

[11] C. Shibata and M. Asuma, C-conformal invariant and tensors of Finsler metrics, Tensor, N.S.. 52(1993) pp. 76-81.

[12] C. Shibata and H. Shimada and M. Azuma and H. Yasda, On Finsler spaces with Randers metric, Tensor, N.S..31(1977) pp. 219-226.

[13] A. A. Tamim and N. L. Youssef, On generalized Randers manifold, Algebras, Groups and geometries..16(1999) pp.115-126.