STABILITY OF INVERSE NODAL PROBLEM FOR ENERGY-DEPENDENT
STURM-LIOUVILLE EQUATION
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Abstract: Inverse nodal problem on diffusion operator is the problem of finding the potential functions and parameters in the boundary conditions by using nodal data. In particular, we solve the reconstruction and stability problems using nodal set of eigenfunctions. Furthermore, we show that the space of all potential functions \( q \) is homeomorphic to the partition set of all asymptotically equivalent nodal sequences induced by an equivalence relation. To show this stability which is known Lipschitz stability, we have to construct two metric spaces and a map \( \Phi_{diff} \) between these spaces. We find that \( \Phi_{diff} \) is a homeomorphism when the corresponding metrics are magnified by the derivatives of \( q \). Basically, this method is similar to [1] and [2] which is given for Sturm-Liouville and Hill operators, respectively and depends on the explicit asymptotic expansions of nodal points and nodal lengths.

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1. Introduction
Inverse spectral problems is about recovering operators by using their spectral characterististics as spectrum, norming constants and nodal points. Such problems have been an important research issue in mathematics and have many applications in natural sciences. Inverse spectral problems are divided into two parts. One of them is inverse eigenvalue problem and the other one is inverse nodal problem. Inverse eigenvalue problems have been studied for along time by many authors [3], [4], [5], [6], [7], [8]. Inverse nodal problem was first posed and solved by McLaughlin [9] who showed that the knowledge of a dense set of nodal points of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. Independently, Shen and his coworkers studied the relation between the nodal points and the density function of the string equation [10]. Recently, many authors have studied inverse nodal problem for different operators [11], [12], [13], [14], [15], [16].

Consider the following boundary value problem generated by the differential equation

\[ Ly = y'' + \left[ \lambda^2 - 2\lambda p(x) - q(x) \right] y = 0, \quad x \in (0, \pi), \] (1.1)

and boundary conditions

\[ y'(0) - hy(0) = 0, \] (1.2)
\[ y'(\pi) + Hy(\pi) = 0. \] (1.3)

where \( \lambda \) is spectral parameter and \( h, H \in \mathbb{R}; q \in W^1_2[0, \pi], p \in W^2_2[0, \pi] \) [17], [18]. Since \( L \) is determined by its potential functions, we identify \( L \) with \( L_{p,q} \). Problem (1.1)-(1.3) are called quadratic pencils of Schrödinger operator (or diffusion operator). Some versions of eigenvalue
Let $\lambda_n$ be the $n$–th eigenvalue, $y(x, \lambda_n)$ the eigenfunction corresponding to the eigenvalue $\lambda_n$ and $0 < x_1^{(n)} < x_2^{(n)} < \ldots < x_{n-1}^{(n)} < \pi$ be the nodal points of the $n$–th eigenfunction $y(x, \lambda_n)$. In other words, $y(x_j^{(n)}, \lambda_n) = 0$, $j = 1, 2, \ldots, n$. Also let $I_j^{(n)} = [x_j^{(n)}, x_{j+1}^{(n)}]$ be the $j$–th nodal domain of the $n$–th eigenvalue and let $l_j^{(n)} = x_{j+1}^{(n)} - x_j^{(n)}$ be the nodal length. Define $x_0^{(n)} = 0, x_n^{(n)} = \pi$. We also define the function $j_n(x)$ to be the largest index $j$ such that $0 \leq x_{j}^{(n)} \leq x$. Thus, $j = j_n(x)$ if and only if $x \in [x_j^{(n)}, x_{j+1}^{(n)}]$. Define $X = \{x_j^{(n)}\}, n \geq 0, j = 1, 2, \ldots, n-1$. $X$ is called the set of nodal points of (1.1)-(1.3).

Inverse nodal problem for diffusion operator is to determine potential functions and parameters in the boundary conditions. This type problems have been studied by many authors [27], [28], [29], [30], [31].

This study is organized as follows: in section 2, we mention some physical and spectral properties of diffusion operator. In section 3, we give a reconstruction formula for potential function and some important results for the problem (1.1)-(1.3). Finally, we solve Lipschitz and high order Lipschitz stability problems for diffusion operator in sections 4 and 5, respectively.

2. Some Physical and Spectral Properties of Diffusion Equation

Jaulent and Jean [32] stated the actual background of diffusion operators and discussed the inverse problem for the diffusion equation. Also, Gasymov and Guseinov studied the spectral theory of diffusion operator [17].

The problem of describing the interactions between colliding particles is of fundamental interesting in physics. In many cases, a description can be carried out through a well known theoretical model. In particular, one is interested in collisions of two spinless particles, and it is supposed that the $s$–wave binding energies and $s$–wave scattering matrix are exactly known from collision experiments. $s$–wave Schrödinger equation with a radial static potential $V$ can be written as

$$y'' + \left[E - V(x)\right]y = 0, \ x \geq 0 \tag{2.1}$$

where the potential function depends on energy in some way and has the following form of energy dependence

$$V(x, E) = U(x) + 2\sqrt{EQ(x)}. \tag{2.2}$$

$U(x)$ and $Q(x)$ are complex-valued functions. (2.1) reduces to the Klein-Gordon $s$–wave equation with the static potential $Q(x)$, for a particle of zero mass and the energy $\sqrt{E}$ with the additional condition $U(x) = -Q^2(x)$ [32].

The Klein Gordon equation is considered one of the most important mathematical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomena in general. In addition, it also appears in nonlinear optics and plasma physics. The Klein-Gordon equation arise in physics in linear and nonlinear forms [33].

Now, we will consider Klein Gordon wave equation. After some straightforward computations, it turns diffusion equation which is given in (1.1). This form of Klein Gordon equation was first given in [34]. But here we shall improve our understanding. Let consider following Klein Gordon wave equation

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{1}{\cosh^2} \right] y(x, \lambda_n) = 0.$$
This equation represents a spinless particle of charge $e$ and mass $m$ in a scalar potential $\phi$ and vector potential $\vec{A}(r, t)$ where natural units $\hbar = c = 1$ have been used. In case of $\vec{A}(r, t) = 0$ and the scalar potential to be time independent, equation (2.3) is reduced the following form

$$\left[ \nabla^2 - \frac{\partial^2}{\partial t^2} - 2ie\phi \frac{\partial}{\partial t} + e^2 \phi^2 \right] \Psi = m^2 \Psi. \quad (2.4)$$

If we set $\Psi(r, t) = \varphi(r)e^{-iEt}$, equation (2.4) takes the form

$$\left[ \nabla^2 + (E - e\phi)^2 \right] \varphi(r) = m^2 \varphi(r), \quad (2.5)$$

where

$$\Psi_t(r, t) = -\varphi(r)iEe^{-iEt}$$

$$\Psi_{tt}(r, t) = -\varphi(r)E^2e^{-iEt}.$$ 

This equation is a well-known form of Klein-Gordon equation. The wave equation (2.5) with a spherically symmetric potential energy may be written in spherical coordinates

$$\left\{ \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + (E - e\phi)^2 \right\} \varphi = m^2 \varphi. \quad (2.6)$$

We first separate the radial and angular parts by substituting

$$\varphi(r, \theta, \phi) = R(r)Y(\theta, \phi),$$

into equation (2.6) and dividing through by $\varphi$, we get

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left[ (E - e\phi)^2 - m^2 \right] R = -\frac{1}{Y} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right]. \quad (2.7)$$

Since the left side of equation (2.7) depends only on $r$, and the right side depends only on $\theta$ and $\phi$, both sides must be equal to a constant that we call $l(l + 1)$ where $l$ is orbital quantum number. Then, the equation (2.7) gives us a radial equation with the scalar potential $V = e\phi$ for the Klein Gordon equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left[ (E^2 - m^2) + V(V - 2E) - \frac{l(l + 1)}{r^2} \right] R = 0 \quad (2.8)$$

where $l = 0, 1, 2, ...$ Substituting $R = \frac{\psi(r)}{r}$ in equation (2.8), we obtain

$$\psi''(r) + \left[ K^2 + V(V - 2E) - \frac{l(l + 1)}{r^2} \right] \psi(r) = 0,$$

where $E^2 = K^2 + m^2$. In case of $l = m = 0$, we get

$$\psi''(r) + [V^2 - 2KV] \psi(r) = -K^2 \psi(r). \quad (2.9)$$
There are many important spectral properties of eigenvalues and eigenfunctions for the problem (1.1)-(1.3). We collect some of these in the lemmas below.

**Lemma 2.1.** The eigenvalues of the operator $L_{p,q}$ are simple.

**Proof:** It can be easily proved by using similar way with [35].

**Lemma 2.2.** The operator $L_{p,q}$ defined by (1.1)-(1.3) is in fact symmetric on space $L^2[0,\pi]$.

**Proof:** Let $u$ and $v$ be twice differentiable functions which satisfy the boundary conditions (1.2)-(1.3). This lemma can be easily proved by using integration by parts on these two functions $u$ and $v$. Considering the definition of inner product on $L^2[0,\pi]$, we get

$$< L_{p,q}[u], v > = \int_0^\pi \{ -u'' + [q(x) + 2\lambda p(x)] u \} v dx$$

$$= hu(0)v(0) + v(\pi)Hu(\pi) - hu(0)v(0) + \int_0^\pi \{ -v'' + [q(x) + 2\lambda p(x)] v \} u dx$$

$$= < u, L_{p,q}[v] > .$$

**Lemma 2.3.** The eigenfunctions corresponding to different eigenvalues are orthogonal for the operator $L_{p,q}$.

**Proof:** Let $\phi_m$ and $\phi_n$ be the eigenfunctions corresponding to the eigenvalues $\lambda_m$ and $\lambda_n$, $(\lambda_m \neq \lambda_n)$ respectively. If $\phi_k$ denotes the $k$–th eigenfunction, we can integrate the identity

$$-\phi_m'' \phi_n + \phi_n'' \phi_m + 2p(x)\phi_m \phi_n (\lambda_m - \lambda_n) = (\lambda_m^2 - \lambda_n^2) \phi_m \phi_n,$$

to obtain

$$\int_0^\pi [2p(x) - \lambda_m - \lambda_n] \phi_m \phi_n dx = 0,$$

from which the result follows.

**Lemma 2.4.** [17], [30] Let $\lambda_n, n \in \mathbb{Z} - \{0\}$ be the spectrum of the problem (1.1)-(1.3). It is well known that the sequence $\{\lambda_n : n = 1, 2, \ldots\}$ satisfies the following asymptotic formula

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \quad (2.10)$$

where

$$c_0 = \frac{1}{\pi} \int_0^\pi p(x) dx, \quad c_1 = \frac{1}{\pi} \left[ h + H + \frac{1}{2} \int_0^\pi [q(x) + p^2(x)] dx \right], \quad \sum_{n=1}^\infty |c_{1,n}|^2 < \infty.$$

Let consider the equation (1.1) with the initial conditions

$$y(0) = 0, \quad y'(0) = 1, \quad (2.11)$$

We will denote by $\varphi(x, \lambda)$ the solution of (1.1) satisfying the initial condition (1.2) and by $\psi(x, \lambda)$...
Lemma 2.5. [27] The solutions of the problems (1.1)-(1.3) and (1.1),(1.3),(2.11) have the following forms,

\[ \varphi(x, \lambda) = \cos(\lambda x) - \frac{h}{\lambda} \sin(\lambda x) + \int_0^x \frac{\sin[\lambda (x-t)]}{\lambda} [q(t) + 2\lambda p(t)] \varphi(t, \lambda) dt, \quad (2.12) \]

\[ \psi(x, \lambda) = \frac{\sin(\lambda x)}{\lambda} + \int_0^x \frac{\sin[\lambda (x-t)]}{\lambda} [q(t) + 2\lambda p(t)] \psi(t, \lambda) dt, \quad (2.13) \]

respectively.

Lemma 2.6. [27] Suppose that \( q \in L_1[0, \pi] \) and \( p \in W_2^2[0, \pi] \). Then, the nodal points and nodal lengths has the following asymptotic forms, as \( n \to \infty \)

(a) For the problem (1.1)-(1.3),

\[ x_j^n = \frac{(j - \frac{1}{2}) \pi}{\lambda_n} - \frac{h}{2\lambda_n^2} + \frac{1}{2\lambda_n^2} \int_0^{x_j^n} [1 + \cos(2\lambda_n t)] [q(t) + 2\lambda_n p(t)] dt + o \left( \frac{1}{\lambda_n^3} \right), \quad (2.14) \]

\[ l_j^n = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n^2} \int_{x_j^n}^{x_{j+1}^n} [1 + \cos(2\lambda_n t)] [q(t) + 2\lambda_n p(t)] dt + o \left( \frac{1}{\lambda_n^3} \right). \quad (2.15) \]

(b) For the problem (1.1), (1.3), (2.11),

\[ x_j^n = \frac{j \pi}{\lambda_n} + \frac{1}{2\lambda_n^2} \int_0^{x_j^n} [1 - \cos(2\lambda_n t)] [q(t) + 2\lambda_n p(t)] dt + o \left( \frac{1}{\lambda_n^3} \right), \quad (2.16) \]

\[ l_j^n = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n^2} \int_{x_j^n}^{x_{j+1}^n} [1 - \cos(2\lambda_n t)] [q(t) + 2\lambda_n p(t)] dt + o \left( \frac{1}{\lambda_n^3} \right). \quad (2.17) \]

Lemma 2.7. Suppose that \( q \in L_1[0, \pi] \). Then, for almost every \( x \in (0, \pi) \), with \( j = j_n(x) \)

\[ \lim \limits_{n \to \infty} \lambda_n \int_{x_j^n}^{x_{j+1}^n} q(t) dt = q(x). \]

Proof: It can be proved by using similar way with [1].

The above equalities are still valid even if \( x_j^n \) and \( x_{j+1}^n \) are replaced by \( X_j^n \) and \( X_{j+1}^n \).

3. A Reconstruction Formula for Potential Function and some Important Results

In this section, we will give a reconstruction formula for the potential function of diffusion operator and some results. Then, we shall give definitions of \( d_5 \) and \( d_0 \). We will denote \( L_n \) (grid
the space $\Omega_{\text{dif}}$ as a collection of all diffusion operators $L_{p,q}$ and the space $\Sigma_{\text{dif}}$ as a collection of all admissible double sequences of nodes such that corresponding functions are convergent in $L_1$. A pseudometric $d_{\Sigma_{\text{dif}}}$ will be defined on $\Sigma_{\text{dif}}$. Essentially, $d_{\Sigma_{\text{dif}}}(X, \overline{X})$ is so close to
\[ d_0(X, \overline{X}) = \lim_{n \to \infty} n^2 \pi \sum_{k=1}^{n-1} |L^n_k - \overline{L^n}_k| + n \int_0^\pi |p(x) - \overline{p}(x)| \, dx, \tag{3.1} \]
where $X, \overline{X} \in \Sigma_{\text{dif}}$, $L^n_k = X^n_{k+1} - X^n_k$ and $\overline{L^n}_k$ is defined similarly.

If we define $X \sim \overline{X}$ if and only if $d_{\Sigma_{\text{dif}}}(X, \overline{X}) = 0$, then $\sim$ is an equivalence relation on $\Sigma_{\text{dif}}$ and $d_{\Sigma_{\text{dif}}}$ would be a metric for the partition set $\Sigma^*_{\text{dif}} = \Sigma_{\text{dif}}/\sim$. Let $\Sigma_{\text{dif}1} \subset \Sigma_{\text{dif}}$ be the subspace of all asymptotically equivalent nodal sequences and let $\Sigma^*_{\text{dif}1} = \Sigma_{\text{dif}1}/\sim$. Let $\Phi_{\text{dif}}$ be the homeomorphism between the spaces $\Omega_{\text{dif}}$ and $\Sigma^*_{\text{dif}1}$. We call $\Phi_{\text{dif}}$ as a nodal map for diffusion operator.

**Theorem 3.1.** Let $q \in L_1[0, \pi]$ and $p \in W^2_2[0, \pi]$. Define $F_n$ by
\[ F_n = n \left[ \sum_{k=1}^{n-1} 2n^2 L^n_k - 2n\pi - 2p(x) \right], \]
for the problem (1.1)-(1.3). Then, $F_n$ converges to $q$ pointwise almost everywhere and also in $L_1$ sense. Furthermore, pointwise convergence holds for all the continuity points of $q$.

**Proof:** We shall consider the asymptotic formulas for the nodal lengths of the problem (1.1)-(1.3). Observe that by Lemma 2.6, we have
\[ l^n_j \frac{\lambda_n}{\pi} - 1 = \frac{1}{2\lambda_n \pi} \int_{x^n_j}^{x^n_{j+1}} \left[ 1 + \cos(2\lambda_n t) \right] [q(t) + 2\lambda_n p(t)] \, dt + o\left( \frac{1}{\sqrt{n}} \right). \]

And, after some algebraic computations, we get
\[ 2\lambda_n^2 (l^n_j \lambda_n - \pi) - 2\lambda_n p(x) = \lambda_n \int_{x^n_j}^{x^n_{j+1}} q(t) \, dt + o(1). \]

If we consider the left side of this equality with the asymptotic formula of $\lambda_n$, it yields
\[ 2\lambda_n^2 (l^n_j \lambda_n - \pi) - 2\lambda_n p(x) = \lambda_n (2\lambda_n^2 l^n_j - 2\lambda_n \pi - 2p(x)) = \left[ n + o\left( \frac{1}{n} \right) \right] \left\{ 2 \left[ n + o\left( \frac{1}{n} \right) \right] l^n_j - 2 \left[ n + o\left( \frac{1}{n} \right) \right] \pi - 2p(x) \right\} = n \left( 2n^2 l^n_j - 2n\pi - 2p(x) \right) + o(1). \]

Hence, to prove Theorem 3.1., it suffices to show Theorem 3.2. (b).

**Remark:** $F_n$ can be obtained similarly for the problem (1.1),(1.3),(2.11).

**Theorem 3.2.** Suppose that $X \in \Sigma_{\text{dif}}$ is asymptotically nodal to $L_{p,q} \in \Omega_{\text{dif}}$. Then, we have...
(a) For the problem (1.1)-(1.3),

\[ h = \lim_{n \to \infty} 2\lambda_n \pi \left( j - \frac{1}{2} - \frac{\lambda_n}{\pi} X_j^n \right). \]

(b) For almost every \( x \in [0, \pi] \)

\[ q(x) = \lim_{n \to \infty} 2\lambda_n \left[ \lambda_{n,j}^2 - \lambda_n \pi - p(x) \right]. \]

where \( q \in L_1[0, \pi] \).

Proof:

(a) From the Lemma 2.6, we know the following asymptotic formula for the problem (1.1)-(1.3),

\[ X_j^n = \frac{\left( j - \frac{1}{2} \right) \pi}{\lambda_n} - \frac{h}{2\lambda_n^2} + \frac{1}{2\lambda_n^2} \int_0^{X_j^n} [1 + \cos (2\lambda_n t)] [q(t) + 2\lambda_n p(t)] \, dt + o \left( \frac{1}{\lambda_n^3} \right). \] (3.2)

After some computations in (3.2), we get

\[ \lambda_n \pi \left( j - \frac{1}{2} - \frac{\lambda_n}{\pi} X_j^n \right) = \frac{h}{2} - \frac{1}{2} \int_0^{X_j^n} [1 + \cos (2\lambda_n t)] [q(t) + 2\lambda_n p(t)] \, dt + o \left( \frac{1}{\lambda_n} \right). \]

Since \( X_j^n \) goes to zero as \( n \to \infty \), we obtain

\[ h = \lim_{n \to \infty} 2\lambda_n \pi \left( j - \frac{1}{2} - \frac{\lambda_n}{\pi} X_j^n \right). \]

(b) Part (b) can be proved by using similar procedure with in [27].

Theorem 3.3. Given \( L_{p,q} \) in \( \Omega_{dif} \), the set of nodal points \( \{ x_k^n \} \) is also asymptotically nodal to \( L_{p,q} \) itself.

Proof: We can easily prove by using similar way with in [1].

4. Lipschitz Stability of Inverse Nodal Problem for Diffusion Operator

In this section, we solve a Lipschitz stability problem for diffusion operator. Lipschitz stability is about a continuity between two metric spaces. To show this continuity, we will use a homeomorphism between these two spaces. Stability problems were studied by many authors [1], [2], [36], [37], [38].

Definition 4.1. Let \( \mathbb{N}' = \mathbb{N} - \{1, 2\} \).

(i) \( \Omega_{dif} = \{ q \in L_1(0, \pi) : q \) is the potential function of the diffusion equation} \)

\( \Sigma_{dif} = \) The collection of the all double sequences defined as
for each \( n \in \mathbb{N} \).

(ii) Let \( X \in \Sigma_{dif} \) and define \( X = \{X^n_n\} \) where \( L^n_k = X^n_{k+1} - X^n_k \) and \( P^n_k = (X^n_k, X^n_{k+1}) \). We say \( X \) is quasinodal to some \( q \in \Omega_{dif} \) if \( X \) is an admissible sequence of nodes and satisfies (I) and (II) below:

(I) \( X \) has the following asymptotics uniformly for \( k \), as \( n \to \infty \)

\[
X^n_k = \frac{(k - \frac{1}{2}) \pi}{n} + O \left( \frac{1}{n^2} \right), k = 1, 2, ..., n
\]

and the sequence

\[
F_n = n \left[ \sum_{k=1}^{n-1} 2n^2 L^n_k - 2n \pi - 2p(x) \right],
\]

converges to \( q \) in \( L_1 \) for the problem (1.1)-(1.3).

(II) \( X \) has the following asymptotics uniformly for \( k \), as \( n \to \infty \)

\[
X^n_k = \frac{k \pi}{n} + O \left( \frac{1}{n^2} \right), k = 1, 2, ..., n.
\]

Similarly, for this case \( F_n \) converges to \( q \) in \( L_1 \) for the problem (1.1), (1.3), (2.11).

**Definition 4.2.** Suppose that \( X, \overline{X} \in \Sigma_{dif} \) with \( L^n_k \) and \( \overline{L}^n_k \) as their respective grid lengths. Let

\[
S_n \left( X, \overline{X} \right) = n^2 \pi \sum_{k=1}^{n-1} \left| L^n_k - \overline{L}^n_k \right| + n \int_0^\pi |p(x) - \overline{p}(x)| \, dx.
\]

(4.1)

Define

\[
d_0 \left( X, \overline{X} \right) = \lim_{n \to \infty} S_n \left( X, \overline{X} \right) \quad \text{and} \quad d_{\Sigma_{dif}} \left( X, \overline{X} \right) = \lim_{n \to \infty} \frac{S_n \left( X, \overline{X} \right)}{1 + S_n \left( X, \overline{X} \right)}.
\]

This definition was first made by [1] for Sturm-Liouville operator. Since the function \( f(x) = \frac{x}{1 + x} \) is monotonic, we get

\[
d_{\Sigma_{dif}} \left( X, \overline{X} \right) = \frac{d_0 \left( X, \overline{X} \right)}{1 + d_0 \left( X, \overline{X} \right)} \in [0, \pi].
\]

Conversely,

\[
d_0 \left( X, \overline{X} \right) = \frac{d_{\Sigma_{dif}} \left( X, \overline{X} \right)}{1 - d_{\Sigma_{dif}} \left( X, \overline{X} \right)}.
\]

This equalities can be obtained easily.

**Lemma 4.1.** Let \( X, \overline{X} \in \Sigma_{dif} \).

(a) \( d_{\Sigma_{dif}} \) is a pseudometric on \( \Sigma_{dif} \).

(b) If \( X \) and \( \overline{X} \) belong to different cases, then \( d_{\Sigma_{dif}} \left( X, \overline{X} \right) = 1 \).

(c) If \( X \) belongs to case (I) or case (II), then

\[
L^n_k = \frac{\pi}{n} + O \left( \frac{1}{n^2} \right), k = 1, 2, ..., n.
\]
i) The interval \( \delta_{n,k} \) between the points \( X^n_k \) and \( X^n_{k+1} \) has length \( O(\frac{1}{n^2}) \), as \( n \to \infty \).

ii) For all \( x \in (0, \pi) \), define \( J_n(x) = \max\{k : X^n_k \leq x\} \) so that \( k = J_n(x) \) if and only if \( x \in [X^n_k, X^n_{k+1}) \). Then, \( |J_n(z) - J_n(z)| \leq 1 \) for sufficiently large \( n \).

**Proof:** It can be proved similar to [1], [2].

After following theorem, we can say that inverse nodal problem for diffusion operator is Lipschitz stable.

**Theorem 4.1.** The metric spaces \((\Omega_{dif}, \| \cdot \|_1)\) and \((\Sigma_{dif}/\sim, d_{\Sigma_{dif}})\) are homeomorphic to each other. Here \( \sim \) is the equivalence relation induced by \( d_{\Sigma_{dif}} \). Furthermore

\[
\|q - \overline{q}\|_1 = \frac{2d_{\Sigma_{dif}}(X, \overline{X})}{1 - d_{\Sigma_{dif}}(X, \overline{X})},
\]

where \( d_{\Sigma_{dif}}(X, \overline{X}) < 1 \).

**Proof:** In view of Lemma 4.1., we only need to consider when \( X, \overline{X} \in \Sigma_{dif} \) belong to same case. Without loss of generality, let \( X, \overline{X} \) belong to case I. We have to show

\[
\|q - \overline{q}\|_1 = 2d_0(X, \overline{X}).
\]

According to the Theorem 3.1., \( F_n \) and \( \overline{F}_n \) convergence to \( q \) and \( \overline{q} \), respectively. If we use definition of norm on \( L_1 \) for the potential functions, we get

\[
\|q - \overline{q}\|_1 \leq 2n^3 \int_0^\pi |L^n_{J_n(x)} - \overline{L}^n_{J_n(x)}| \, dx + 2n^3 \int_0^\pi |p(x) - \overline{p}(x)| \, dx + o(1)
\]

and after some algebraic operations

\[
\|q - \overline{q}\|_1 \leq 2n^3 \int_0^\pi |L^n_{J_n(x)} - \overline{L}^n_{J_n(x)}| \, dx + 2n^3 \int_0^\pi |L^n_{J_n(x)} - \overline{L}^n_{J_n(x)}| \, dx + 2n^3 \int_0^\pi |p(x) - \overline{p}(x)| \, dx + o(1).
\]

Here, the integrals in the second and first terms can be written as

\[
\int_0^\pi |\overline{L}^n_{J_n(x)} - \overline{L}^n_{J_n(x)}| \, dx = o\left(\frac{1}{n^3}\right),
\]

and

\[
\int_0^\pi |L^n_{J_n(x)} - \overline{L}^n_{J_n(x)}| \, dx = \frac{n}{2} \sum_{k=1}^{n-1} |L^n_k - \overline{L}^n_k|,
\]

respectively. If we consider these equalities in (4.2), we get

\[
\|q - \overline{q}\|_1 \leq 2n^3 \left( \frac{1}{2} + c \right) \left[ \frac{n}{2} \sum_{k=1}^{n-1} |L^n_k - \overline{L}^n_k| \right] + 2n^3 \int_0^\pi |p(x) - \overline{p}(x)| \, dx + o(1).
\]
and

\[
\|q - \overline{q}\|_1 \leq 2n^2\pi \sum_{k=1}^{n-1} |L^n_k - \overline{L^n_k}| + 2n \int_0^\pi |p(x) - \overline{p}(x)| \, dx + o(1). \tag{4.3}
\]

Similarly, we can easily obtain

\[
\|q - \overline{q}\|_1 \geq 2n^2\pi \sum_{k=1}^{n-1} |L^n_k - \overline{L^n_k}| + 2n \int_0^\pi |p(x) - \overline{p}(x)| \, dx + o(1). \tag{4.4}
\]

Considering (4.3) and (4.4) together, it yields

\[
\|q - \overline{q}\|_1 = 2n^2\pi \sum_{k=1}^{n-1} |L^n_k - \overline{L^n_k}| + 2n \int_0^\pi |p(x) - \overline{p}(x)| \, dx.
\]

The proof is complete after by taking limit as \(n \to \infty\).

5. High order Lipschitz stability

In this section, we will solve a high order stability problem for diffusion operator. For \(n \in \mathbb{N}\),

\[
\Omega_{\text{dif}}(N) = \{ q \in L_1[0,\pi] : q \in C^{N+1}[0,\pi] \}.
\]

It has been proved in [31] that \(m\)-th derivative of potential function \(q\) for Diffusion operator can be approximated by some difference quotient of nodal length \(\delta^m l^n j\) where \(\delta^m\) is \(m\)-th order difference quotient operator defined as [1]

\[
\delta a_j^{(n)} = \frac{a_j^{(n)} - a_{j+1}^{(n)}}{a_j^{(n)}} \quad \text{and} \quad \delta^m a_j^{(n)} = \frac{\delta^{m-1} a_j^{(n)} - \delta^m a_j^{(n)}}{a_j^{(n)}}.
\]

Note that \(\delta\) and \(\delta^m\) operators depend on the double sequence \({a_i^{(n)}}\).

Let \(\Sigma_{\text{dif}}(N)\) be the set of asymptotically equivalent nodal sequences in \(\Omega_{\text{dif}}(N)\). Let \(d_{\Omega_{\text{dif}}(N)}\) and \(D_{\Sigma_{\text{dif}}(N)}\) be some metrics on \(\Omega_{\text{dif}}(N)\) and \(\Sigma_{\text{dif}}(N)\) respectively magnified by the \(L_1\) norms of derivatives of the potential functions. We find that the nodal map \(\Phi_{\text{dif}}\) is still a homeomorphism under these strengthened metrics.

If we define \(X \sim_N \overline{X}\) if and only if \(D_{\Sigma_{\text{dif}}(N)}(X, \overline{X}) = 0\), then \(\sim_N\) is an equivalence relation on \(\Sigma_{\text{dif}}(N)\). Hence \(d_{\Omega_{\text{dif}}(N)}\) is a metric on \(\Omega_{\text{dif}}(N)\).

Definition 5.1. Let \(X, \overline{X} \in \Sigma_{\text{dif}}\) and \(X_k^n = \overline{X}_k^n = \pi\) for \(k > n\). For \(m = 1, 2, ..., N\), let

\[
S_{m,n}(X, \overline{X}) = \lambda_n \sum_{k=1}^{n-m-2} |\delta^m L^n_k - \delta^m \overline{L^n_k}| + \lambda_n \int_{X^n} |\delta^m \overline{p}(x) - \delta^m p(x, j)| \, dx
\]

\[
+ \lambda_n \int_{X^n} |\delta^m (\overline{p}^{(m)}(x) - p^{(m)}(x))| \, dx.
\tag{5.1}
\]
Define
\[ d_m (X, \overline{X}) = \lim_{n \to \infty} S_{m,n} (X, \overline{X}) \quad \text{and} \quad d_{\Sigma_{dif}(m)} (X, \overline{X}) = \lim_{n \to \infty} \frac{S_{m,n} (X, \overline{X})}{1 + S_{m,n} (X, \overline{X})}. \]

We have to use the following reconstruction formula for the function \( q^{(m)} \) in Theorem 5.1. to prove Theorem 5.3.

**Theorem 5.1.** [31] Let \( q \) is real valued, \((n+1)\)-th order continuous function from the class \( L_1[0,\pi] \) for \( N \geq 1 \) in (1.1), and let \( j = j_n(x) \) for each \( x \in [0,\pi] \). Then, as \( n \to \infty \)
\[ q(x) = \lambda_n \left[ 2\lambda_n^2 j - 2\pi \lambda_n - 2p(x) \right] + O \left( \frac{1}{n} \right), \]
and, for all \( m = 1, 2, ..., N \)
\[ q^{(m)}(x) = \frac{2\lambda_n^{3/2}}{\pi} - \delta^m L_j^m - 2\lambda_n \delta^m p(x_j) - 2\lambda_n p^{(m)}(x) + O(1). \]

**Theorem 5.2.** The metric spaces \((\Omega_{\Sigma_{dif}}, d_{\Omega_{\Sigma_{dif}}})\) and \((\Sigma_{dif1}(N)/ \sim_N, D_{\Sigma_{dif}(N)})\) are homeomorphic to each other, where \( \sim_N \) is the equivalence relation induced by \( D_{\Sigma_{dif}(N)} \).

**Proof:** In view of Lemma 4.1. we only need to consider \( X, \overline{X} \in \Sigma_{dif} \) belong to same case. We shall show that \( \|q^{(m)} - \overline{q}^{(m)}\| = 2d_m(X, \overline{X}) \). Hence, to prove Theorem 5.2., it suffices to prove Theorem 5.3.

After Theorem 5.3., we can say that the inverse nodal problem for the diffusion operator is high order Lipschitz stable.

**Theorem 5.3.** Suppose that \( q \) and \( \overline{q} \) are both \( N \)th order continuous functions from the class \( L_1[0,\pi]; L_{p,q}, \overline{L}_{p,q} \) belong to same case. Let \( X \) and \( \overline{X} \) be the corresponding asymptotically equivalent nodal sequences. Then, for all \( m = 1, 2, ..., N \)
\[ \|q^{(m)} - \overline{q}^{(m)}\| = 2d_m(X, \overline{X}). \]

**Proof:** By Theorem 5.1 and Lemma 2.6, we can write
\[ q^{(m)} - \overline{q}^{(m)} = \frac{2\lambda_n^{3/2}}{\pi} \left[ \delta^m L_j^m - \delta^m \overline{L}_j^m \right] + 2\lambda_n \left[ \delta^m \overline{q}(x_j) - \delta^m p(x_j) \right] + 2\lambda_n \left[ \overline{p}^{(m)}(x) - p^{(m)}(x) \right] + o(1), \]
and
\[ |q^{(m)} - \overline{q}^{(m)}| = \left| \frac{2\lambda_n^{3/2}}{\pi} \left[ \delta^m L_j^m - \delta^m \overline{L}_j^m \right] + 2\lambda_n \left[ \delta^m \overline{q}(x_j) - \delta^m p(x_j) \right] + 2\lambda_n \left[ \overline{p}^{(m)}(x) - p^{(m)}(x) \right] + o(1) \right|. \]

Then, by using the definition of norm on \( L_1 \) space, we obtain
\[ \left\|q^{(m)} - \overline{q}^{(m)}\right\| = \int_{\Omega_{\Sigma_{dif}}} \left| q^{(m)}(x) - \overline{q}^{(m)}(x) \right| dx, \]
and
\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\|_{1} = \frac{2 \lambda_{n}^{3/2}}{\pi} \int_{X_{1}^{n}} \left| \delta^{m} L_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx + 2 \lambda_{n} \left[ \delta^{m} \overline{p}(x) - \delta^{m} p(x) \right] dx + o(1)
\]
when \( n \) is sufficiently large. By using the property of triangle inequality in (5.2), we get
\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\|_{1} \leq \frac{2 \lambda_{n}^{3/2}}{\pi} \int_{X_{1}^{n}} \left| \delta^{m} L_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx + 2 \lambda_{n} \int_{X_{1}^{n}} \left| \delta^{m} \overline{p}(x) - \delta^{m} p(x) \right| dx + o(1),
\]
and after some computations,
\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\|_{1} \leq \frac{2 \lambda_{n}^{3/2}}{\pi} \int_{X_{1}^{n}} \left| \delta^{m} L_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx + \frac{2 \lambda_{n}^{3/2}}{\pi} \int_{X_{1}^{n}} \left| \delta^{m} \overline{T}_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx + 2 \lambda_{n} \int_{X_{1}^{n}} \left| \delta^{m} \overline{p}(x) - \delta^{m} p(x) \right| dx + o(1).
\]

Now, by using [539, Lemma 2.1] and Lemma 4.1., we obtain
\[
\int_{X_{1}^{n}} \left| \delta^{m} L_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx = \sum_{k=1}^{n-2} \left| \delta^{m} L_{k}^{n} - \delta^{m} \overline{T}_{k}^{n} \right| \left| L_{k}^{n} \right|,
\]
and
\[
\int_{X_{1}^{n}} \left| \delta^{m} \overline{T}_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx = O \left( \frac{1}{n^{4}} \right).
\]
Moreover, \( L_{k}^{n} = \frac{\pi}{n} + O \left( \frac{1}{n^{2}} \right) \). Thus,
\[
\frac{2 \lambda_{n}^{3/2}}{\pi} \int_{X_{1}^{n}} \left| \delta^{m} L_{j}^{n} - \delta^{m} \overline{T}_{j}^{n} \right| dx \leq \frac{2 \lambda_{n}^{3/2}}{\pi} O \left( \frac{1}{n^{4}} \right) + \frac{2 \lambda_{n}^{3/2}}{\pi} \sum_{k=1}^{n-2} \left| \delta^{m} L_{k}^{n} - \delta^{m} \overline{T}_{k}^{n} \right| \left| \frac{\pi}{n} + O \left( \frac{1}{n^{2}} \right) \right|
\]
\[
= o \left( \frac{1}{n^{5/2}} \right) + 2n^{1/2} \sum_{k=1}^{n-2} \left| \delta^{m} L_{k}^{n} - \delta^{m} \overline{T}_{k}^{n} \right| + \frac{2}{\pi} n^{3/2} nO(n^{-2}) o \left( \frac{1}{n^{3}} \right).
\]
Therefore,

\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\| \leq 2n^{1/2} \sum_{k=1}^{n-m-2} \left| \delta^m L_k^n - \delta^m \overline{L}_k^n \right| + 2\lambda_n \int_{X_1^n}^{X_{n-m-1}^n} \left| \delta^m \overline{p}(x) - \delta^m p(x) \right| dx \\
\quad + 2\lambda_n \int_{X_1^n}^{X_{n-m-1}^n} \left| \overline{p}^{(m)}(x) - p^{(m)}(x) \right| dx + o(1).
\]  

(5.3)

If we take limit as \( n \to \infty \) and use the metric definition, we obtain

\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\| \leq 2d_m(X, \overline{X}).
\]  

(5.4)

Using similar procedure,

\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\| \geq 2d_m(X, \overline{X}).
\]  

(5.5)

Then, considering (5.4) and (5.5) together, we get

\[
\left\| q^{(m)} - \overline{q}^{(m)} \right\| = 2d_m(X, \overline{X}).
\]

This completes the proof.

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