Endpoint Strichartz estimates for magnetic wave equations on \( \mathbb{H}^2 \)

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Abstract In this paper, we prove that Kato smoothing effects for magnetic half wave operators can yield the endpoint Strichartz estimates for linear wave equation with magnetic potential on two dimensional hyperbolic spaces. This result serves as a cornerstone for the author’s work [27] and collaborative work [29] in the study of asymptotic stability of harmonic maps for wave maps from \( \mathbb{R} \times \mathbb{H}^2 \) to \( \mathbb{H}^2 \).

1 Introduction

The motivation of this problem is the study of asymptotic stability of harmonic maps for wave maps from \( \mathbb{R} \times \mathbb{H}^2 \) to \( \mathbb{H}^2 \). In fact, the heat tension field \( \phi_s \) which provides a natural measure for the distance between the solution of wave maps and the limit harmonic map satisfies a master semilinear wave equation under Tao’s caloric gauge. After separating the limit part of connections and differential fields and applying “dynamic separation”, the linear part of the master equation becomes a wave equation with magnetic potential:

\[
\begin{aligned}
\Box u + B(x)u + 2(A, du) - (d^* A)u &= F, \\
u(0, x) &= u_0 : \mathbb{H}^2 \rightarrow \mathbb{R}^2, \partial_t u(0, x) = u_1(x) : \mathbb{H}^2 \rightarrow \mathbb{R}^2
\end{aligned}
\]

(1.1)

where \( u \) is a \( \mathbb{R}^2 \)-valued field defined on \( \mathbb{R} \times \mathbb{H}^2 \), \( A = A_i dx^i \) is a real anti-symmetric \( 2 \times 2 \) matrix valued one form defined on \( \mathbb{H}^2 \), \( B \) is a real symmetric \( 2 \times 2 \) matrix defined on \( \mathbb{H}^2 \). \( (A, du) \) denotes the metric of one forms. Since \( A \) is a matrix valued one form and \( du \) is a \( \mathbb{R}^2 \) valued one form, \( (A, du) \) is \( \mathbb{R}^2 \) valued as well. And \( \Box = -\partial^2_t + \Delta_{\mathbb{H}^2} \) is the D’Alembertian on \( \mathbb{R} \times \mathbb{H}^2 \).

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Integration by parts shows \( W u := B_0(x) u + 2(A, du) - (d^* A) u \) is formally symmetric in \( L^2(\mathbb{H}^2; C^2) \). In \([27]\), \( A = A_i dx^i \) is indeed the connection one form on the pullback bundle \( Q^*(T\mathbb{H}^2) \) with \( Q : \mathbb{H}^2 \to \mathbb{H}^2 \) denoting the limit harmonic map. As in the Euclidean case, we may use the tight notation \( W u = V u + Xu \) to denote the potential part, where \( V := B(x) - (d^* A) \) is a matrix valued function defined on \( \mathbb{H}^2 \) and \( X := 2 h^{ij} A_i \frac{\partial}{\partial x^j} \) is a matrix valued vector field defined on \( \mathbb{H}^2 \).

The wave map equation on flat spacetimes known as the nonlinear \( \sigma \)-model, arises as a model problem in general relativity and particle physics, see for instance \([30, 1]\). The Cauchy and dynamic problems for wave maps on flat spacetimes have been a fruitful field with plenty of works, see for instance \([20, 40, 46, 45, 43, 44]\). The dynamics for the wave map equations on curved spacetimes were less understood until now. We mention the work of Shatah, Tahvildar-Zadeh \([38]\) on the \( S^2 \times \mathbb{R} \) background and Lawrie, Oh, Shahshahani \([23, 25, 26, 24]\) on the \( \mathbb{H}^n \times \mathbb{R} \) background.

In this paper, we focus on the endpoint Strichartz estimates for the magnetic wave equation \((1.1)\). The Strichartz estimates for magnetic Schrödinger equations (MS), magnetic Dirac equations (MD) and magnetic wave equations (MW) on flat spaces were intensively studied in decades, for instance \([17, 2, 10, 12, 7, 8, 9]\). In the fundamental work of Rodnianski, Schlag \([36]\), they showed that the Kato smoothing effects imply the non-endpoint Strichartz estimates for MS. This idea was further developed to MS with large potential by Erdogan, Goldberg, Schlag \([10]\) and MW, MD by D’Ancona, Fanelli \([7]\). The endpoint Strichartz estimates for free wave and Schrödinger equations were obtained first by Keel, Tao \([21]\). With a key lemma of Ionescu, Kenig \([15]\) whose proof is based on \([21]\), D’Ancona, Fanelli, Vega, Visciglia \([8]\) obtained endpoint Strichartz estimates for MS in the small potential case. Strichartz estimates for free Schrödinger, wave and Klein Gordon equations on \( \mathbb{H}^n \) were obtained by Tataru \([35]\), Metcalfe, Taylor \([34, 33]\), Anker, Pierfelice, Vallarino \([3, 4]\) and see Metcalfe, Tataru \([32]\) for small perturbations of flat spacetimes. And the study of resolvent estimates, spectral measures, scattering, analytic continuation, degenerate elliptic operators, etc. on hyperbolic/asymptotic hyperbolic spaces has become an active field, see the works \([34, 47, 48, 6]\) for instance. The dispersive estimates of Schrödinger operators with electric potential on \( \mathbb{H}^d \) were obtained by Borthwick, Marzuola \([5]\) for \( t \geq 1 \).

Our main theorems consist of two parts. The first result shows that the Kato smoothing effect estimates for magnetic half wave operators on \( \mathbb{H}^2 \) imply both the non-endpoint Strichartz and endpoint Strichartz estimates. Second, we prove the Kato smoothing effect estimates in the small potential case. Thus, by our first result the endpoint Strichartz estimates hold in the small potential case, which is useful for \([29]\). And we remark that for the special magnetic Schrödinger operator appearing in the study of wave maps from \( \mathbb{R} \times \mathbb{H}^2 \) to \( \mathbb{H}^2 \), the Kato smoothing effect can also be established for arbitrary large potentials, see \([27]\).

Let \( \mathbb{D} \) denote the Poincare disk model for \( \mathbb{H}^2 \). Let \( r = d(x, O) \) be the geodesic distance between \( x \in \mathbb{D} \) and the origin point \( O \) in \( \mathbb{D} \). Recall \( V := B(x) - d^* A \). Our main theorems are as follows.
**Theorem 1.1.** Suppose that $B, A$ satisfy for some $\varrho > 0$

$$\|Ve^{\varrho \theta}\|_{L^\infty_x} + \|e^{\varrho \theta}|A||_{L^\infty_x} < \infty,$$  \hspace{1cm} (1.2)

and the Schrödinger operator $H = -\Delta + V + X$ is strictly positive\(^\dagger\), i.e., there exists some positive constant $c > 0$ such that the spectrum of $H$ in $L^2(\mathbb{H}^2, \mathbb{C}^2)$ is contained in $(c, \infty)$. Assume further that for some $0 < \alpha < 2\varrho$,

$$\|H^{1/2}f\|_{L^2} \lesssim \|Df\|_{L^2} + \|f\|_{L^2},$$  \hspace{1cm} (1.3)

$$\|Df\|_{L^2} \lesssim \|H^{1/2}f\|_{L^2} + \|f\|_{L^2},$$  \hspace{1cm} (1.4)

$$\|e^{-\alpha r} \nabla f\|_{L^2} \lesssim \|e^{-\alpha r} H^{1/2}f\|_{L^2} + \|e^{-\alpha r} f\|_{L^2},$$  \hspace{1cm} (1.5)

provided the right hand sides are finite. Then if the Kato smoothing effect

$$\|e^{-\alpha r} e^{\pm it\sqrt{H}} f\|_{L^2_t L^2} \lesssim \|f\|_{L^2}$$

holds, we have the following endpoint Strichartz estimates for (1.1): Let $u$ solve (1.1), then for any $p \in (2, 6)$

$$\|D^{1/2}u\|_{L^2_t L^p_x} + \|e^{-\alpha r} \nabla u\|_{L^2_t L^2_x} + \|\partial_t u\|_{L^2_t L^2_x} + \|\nabla u\|_{L^2_t L^2_x} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \|F\|_{L^1_t L^2_x}.$$

**Remark 1.1** Generally (1.3)-(1.5) hold if $H$ is a bounded perturbation of $-\Delta$. Even if $H$ has discrete spectrum, we can still expect (1.3)-(1.5) to be right. But if one expects the exact equivalence without the zero order term $\|f\|_{L^2}$, the discrete spectrum of $H$ must be eliminated.

**Remark 1.2** If $H$ has discrete spectrum, the Kato smoothing estimates can only hold in the continuous spectrum part of $H$.

The following corollary will show that the Kato smoothing estimates hold for small potentials. Moreover, in the large potential case considered in [27], we can prove the Kato smoothing estimates via choosing a suitable frame on the bundle $Q^+(T\mathbb{H}^2)$. In fact, the one form $A$ in (1.1) indeed depends on the frame fixed on $Q^+(T\mathbb{H}^2)$. Then using the geometric setting of the Schrödinger operator $H$ and the Coulomb gauge, we can prove no discrete spectrum, no bottom resonance and no embedded eigenvalue exist, which are the enemies in the low frequency and mediate frequency. Moreover, the negative sectional curvature property of the target $\mathbb{H}^2$ is very important to make the electric potential part be a non-negative operator. Finally, the Kato smoothing effect follows by the decay estimates for the high frequency via choosing a suitable weight and energy arguments. See [27] for more details.

As a corollary we have the endpoint Strichartz estimates for magnetic wave equations in the small potential case.

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\(^\dagger\) By Kato's perturbation theorem, (1.2) shows $H$ is self-adjoint in $L^2$. See Lemma 3.3.

\(^\dagger\) Since $H$ is self-adjoint, $H^{1/2}$ is defined by spectrum theorem. $D := (-\Delta)^{1/2}$ and can be defined by the Fourier transform on hyperbolic spaces, see Section 2.
Corollary 1.1. Suppose that the Schrödinger operator $H = -\Delta + V + X$ satisfy for some $\rho > 0$

$$\|e^{\rho V}\|_{L^\infty} + \|V\|_{L^2} + \|e^{\rho |A|}\|_{L^\infty} + \|A\|_{L^2} \leq \mu_1. \quad (1.6)$$

Assume $0 < \mu_1 \ll 1$, $0 < \alpha \ll 1$ and $0 < \alpha < 3\rho$. If $u$ solves (1.1), then for any $p \in (2, 6)$, there holds

$$\|e^{-\alpha r} \nabla u\|_{L^2_t L^2_x} + \|D^\frac{3}{2} u\|_{L^2_t L^p_x} + \|\partial_t u\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L^\infty_t L^2_x} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \|F\|_{L^1_t L^2_x}.$$ 

And as a byproduct, for $s \in [0, \frac{1}{2}], p \in [2, \infty)$, we have

$$\|D^{2s} f\|_{L^p} \sim \|H^s f\|_{L^p}.$$ \quad (1.7)

Remark 1.3 (1.7) is useful for studying the well-posedness and scattering of semilinear dispersive equations with magnetic potentials particularly because no chain rule and Leibnitz rule are available for magnetic Schrödinger operators $H$.

The key for the proof of Theorem 1.1 is the weighted Morawetz estimate in Lemma 3.3 and the endpoint weighted Strichartz estimate for free wave equations on $\mathbb{H}^2$ in Lemma 3.6 inspired by [15, 8]. The proof of Lemma 3.6 depends on the bilinear argument of [21], complex interpolation and the frequency decomposition. It is important that Theorem 1.1 is essentially suitable to potentials of any size. The key point to involve large potentials in Theorem 1.1 is the three advantages of the hyperbolic background compared with the flat case, i.e., the exotic Strichartz estimates for free wave equations on $\mathbb{H}^2$, the Kunze-Stein phenomenon, the exponential decay of the spherical functions. In fact, as noticed by [16, 4], Strichartz estimates of dispersive equations on hyperbolic spaces own more Strichartz pairs than the Euclidean one, see [Corollary 1.3, [21]] and Lemma 2.2 below. For the $\mathbb{H}^2$ background studied here, an $L^2_t$-type Strichartz estimate is available, which is essential for Lemma 3.6 and unavailable in the $\mathbb{R}^2$ case.

The key for the proof of Corollary 1.1 is to prove (1.3)-(1.5). The Kato smoothing effect in the small potential case is respectively easy. For (1.3)-(1.5), we apply an almost equivalence technique of our previous paper [28] instead of the heat semigroup techniques usually used in the flat case. In fact, in the Euclidean case (1.3) and (1.4) are usually proved by Simon’s heat semigroup method with Kato’s strong Trotter formula, see [9, 18, 39]. The convenience of our almost equivalence arguments here is that we need neither the commutation property nor the special structure of $H$, which seems to fail for the hyperbolic backgrounds due to the non-vanishing connection coefficients. In fact, in our argument, it suffices to prove the $L^p_t L^q_x$ and weighted $L^2_x$ resolvent estimates on the half-line ($-\infty, 1/4$). These resolvent estimates can be proved by carefully bounding the resolvent kernel and applying the

* The Strichartz pair $(p, q)$ in the norm $L^p_t L^q_x$ for wave equations in $\mathbb{R}^2$ requires $p \geq 4$ at least, see [Corollary 1.3, [21]].
Kunze-Stein phenomenon. Moreover, due to the spectrum gap of $-\Delta$, we find this almost equivalence argument directly yields the exact equivalence.

**Notation** Let $D = (-\Delta)^{\frac{1}{2}}$. The square of $H$ is denoted by $H^\frac{1}{4}$ or $\sqrt{H}$. And denote the shifted derivative by $\tilde{D} = (-\Delta - \frac{i}{4} + \kappa^2)^{\frac{1}{2}}$ with $\kappa > \frac{1}{2}$.

The resolvent of an operator $L$ from one function space to the other is always denoted by $R_L(z) = (L - z)^{-1}$ for simplicity, for instance

$$R_H(z) = (H - z)^{-1}, \ R_D(z) = (D - z)^{-1}, \ R_{\sqrt{H}}(z) = (\sqrt{H} - z)^{-1}.$$  

In order to coincide with the notions in [5], we introduce $\mathcal{R}_0(s)$ defined by

$$\mathcal{R}_0(s) = (-\Delta - (1-s)z)^{-1}.$$  

Notice that $s(1-s)$ ranges over all the complex plane if $s$ ranges over the half plane $\{\frac{1}{2} + z : \Re z \geq 0\}$. And if $s$ takes values in the critical line $\{\frac{1}{2} + i\lambda : \lambda \in \mathbb{R}\}$, then $s(1-s)$ lies in $[1/4, \infty)$ which is the continuous spectrum of $-\Delta$. Similarly, we introduce the notations: $\mathcal{R}_D(z) = (D - z(1-z))^{-1}$, and

$$R_{\sqrt{H}}(z) = (H - z(1-z))^{-1}, \ R_{\sqrt{H}}(z) = (\sqrt{H} - z(1-z))^{-1}.$$  

Let $S(t)$ be any function defined in $\mathbb{R}$, the notation $S(t) \leq t^{-\infty}$ introduced in [3] means for any positive integer $n$, there exists some constant $C(n) > 0$ such that $S(t) \leq C(n)|t|^{-n}$ when $|t| \to \infty$. Similarly, for any function $S(\cdot)$ defined on integers, $S(j) \leq j^{-\infty}$ means for any positive integer $n$, there exists some constant $C(n) > 0$ such that $S(j) \leq C(n)|j|^{-n}$ when $|j| \to \infty$.

## 2 Preliminaries

Some preliminaries on the geometric notions and the Fourier analysis on the hyperbolic planes are recalled in this section. Most materials are standard and can be found in Helgason [13], while some are folk and we will contain some proofs if necessary.

Let $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 < 1\}$ be the Poincare model of the hyperbolic plane $\mathbb{H}^2$ with the metric tension

$$4 \frac{dx_1^2 + dx_2^2}{(1 - |x_1|^2 - |x_2|^2)^2}.$$  

In the polar coordinates $(r, \theta)$, the metric tension of $\mathbb{D}$ is $dr^2 + \sinh^2 r d\theta^2$.  

* Since [3] also considered shifted wave equations, they introduce the operator $\tilde{D}$ to eliminate the singularity at zero of the corresponding symbols when applying the Fourier transform. The coincidence of $\|D^s \cdot \|_{L^p_\mathbb{T}}$ and $\|\tilde{D}^s \cdot \|_{L^p_\mathbb{T}}$ for $p \in (1, \infty)$ makes the use of $\tilde{D}$ safe for the final estimates and beneficial due to the elimination of singularity at zero. In our case, since the symbol of $(-\Delta)^{-1}$ is $(\lambda^2 + 1/4)^{-1}$ and has no singularity, it is generally not necessary to introduce $\tilde{D}$. We keep this notation for reader’s convenience of contrasting these papers.
The Laplace-Beltrami operator on \( \mathbb{D} \) is
\[
\Delta = \partial_r^2 + \coth r \partial_r + \sinh^{-2} r \partial_\theta.
\]
The spherical functions \( \varphi_\lambda \) with \( \lambda \in \mathbb{C} \) on \( \mathbb{D} \) are normalized radial eigenfunctions of \( \Delta \):
\[
\begin{aligned}
\{ \Delta \varphi_\lambda &= -(\lambda^2 + \frac{1}{4}) \varphi_\lambda \\
\varphi_\lambda(0) &= 1
\end{aligned} \tag{2.1}
\]
For any \( \lambda \in \mathbb{R}, \ r \geq 0 \), the spherical functions are of exponential decay:
\[
|\varphi_\lambda(r)| \leq \varphi_0(r) \lesssim (1 + r)e^{-r^2}.
\]
A horocycle for \( \mathbb{D} \) is a circle in \( \mathbb{D} \) tangential to the boundary \( \mathbb{B} = \partial \mathbb{D} \). Given \( b \in \mathbb{B} \) and \( z \in \mathbb{D} \), denote the horocycle through \( b \) and \( z \) by \( \xi(z, b) \). Then we put
\[
[z, b] = \text{distance from O to } \xi(z, b) \text{ (with sign; to be taken negative if O lies inside } \xi(z, b)).
\]
If \( f \) is a complex-valued function on \( \mathbb{D} \), the Fourier transform is defined by
\[
\tilde{f}(\lambda, b) = \int_{\mathbb{D}} f(z)e^{-(i\lambda+1)[z, b]}dz, \tag{2.2}
\]
for all \( \lambda \in \mathbb{C}, \ b \in \mathbb{B} \), if this integral exists. The inverse Fourier transform is defined by
\[
f(z) = \text{const. } \int_{0}^{\infty} \left( \int_{\mathbb{B}} \tilde{f}(\lambda, b)e^{(i\lambda+1)[z, b]}db \right) |c(\lambda)|^{-2}d\lambda, \tag{2.3}
\]
where \( c(\lambda) \) is the Harish-Chandra’s c-function. The Plancherel formula is
\[
\|f\|_{L^2}^2 = \text{const. } \int_{0}^{\infty} \int_{\mathbb{B}} |\tilde{f}(\lambda, b)|^2|c(\lambda)|^{-2}dbd\lambda. \tag{2.4}
\]
Any function \( m : \mathbb{R} \to \mathbb{C} \) induces a Fourier multiplier operator \( m(-\Delta) \) by the formula \( m(-\Delta)f(\lambda, b) = m(\lambda^2 + \frac{1}{4})\tilde{f}(\lambda, b) \). Thus, the symbol of the operator \( m(D) \) is \( m(\sqrt{\lambda^2 + \frac{1}{4}}) \). Consider the group
\[
\text{SU}(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}
\]
with the action on \( \mathbb{D} \) defined by the map
\[
g : z \to \frac{az + b}{bz + a}, \ (z \in \mathbb{D}).
\]
Then we have the identification \( \mathbb{D} = \text{SU}(1, 1)/\text{SO}(2) \). Let \( d\mu(g) \) denote the
Lemma 2.1. If $H$ coincides with Sobolev spaces denoted by $H^s$, for any Fourier multiplier operator $f$ on manifolds, see for instance Hebey [14]. It is known that one has the same Young’s convolution inequality as the Euclidean space. Furthermore, hyperbolic planes have the so-called Kunze-Stein phenomenon, see Lemma 3.2 below. The convolution operation $f_1 * f_2$ has an equivalent form if one of $f_1, f_2$ is radial. In fact, suppose that $f_2(x) = f_2(d(x, O))$ is radial, then

$$f_1 * f_2(z) = \int_{\mathbb{D}} f_1(x) f_2(d(x, z)) dx. \quad (2.7)$$

Indeed, since $f_2$ is radial, $f_2(g^{-1} \cdot z) = f_2(d(g^{-1} \cdot z, O))$. And since $G$ keeps the Riemannian structure of $\mathbb{D}$, we have $d(g^{-1} \cdot z, O) = d(z, g \cdot O)$. Then (2.6) reduces to

$$f_1 * f_2(z) = \int_{G} f_1(g \cdot O) f_2(d(z, g \cdot O)) d\mu(g). \quad (2.8)$$

Considering $f_1(\cdot) f_2(d(\cdot, \cdot))$ as a function in $\mathbb{D}$, we have (2.7) by (2.8) and (2.5). For any Fourier multiplier operator $m(-\Delta)$, we say the radial function $k(r)$ is the corresponding kernel if $m(-\Delta)f = f * k$. And by (2.7), the function $k(d(x, y))$ defined on $\mathbb{D} \times \mathbb{D}$ is exactly the Schwartz kernel of $m(-\Delta)$.

Let $H^{k,p}(\mathbb{D}; \mathbb{R}^2)$ be the usual Sobolev space for scalar functions defined on manifolds, see for instance Hebey [14]. It is known that $C^\infty_c(\mathbb{D}; \mathbb{R}^2)$ is dense in $H^{k,p}(\mathbb{D}; \mathbb{R}^2)$. We also recall the norm of $H^{k,p}$:

$$\|f\|_{H^{k,p}}^p = \sum_{l=1}^{k} \|\nabla^l f\|_{L_x^p}^p,$$

where $\nabla^l f$ is the covariant derivative, $p \in [1, \infty)$. The fractional power Sobolev spaces denoted by $H^{s,p}$ are defined by $\{f : D^s f \in L^p\}$. And it coincides with $H^{k,p}$ if $s = k$ is an integer. $H^{s,2}$ are usually written by $H^s$ for simplicity.

We now recall the Sobolev inequalities.

**Lemma 2.1.** If $f \in C^\infty_c(\mathbb{D}; \mathbb{R})$, then for $1 < p < \infty$, $p \leq q \leq \infty$, $1 < r < 2,$
$r \leq l < \infty$, $\alpha > 1$ the following inequalities hold

$$\|f\|_{L^2} \lesssim \|\nabla f\|_{L^2}; \quad \|\nabla f\|_{L^p} \sim \|Df\|_{L^p}; \quad \|f\|_{L^1} \lesssim \|\nabla f\|_{L^r} \quad \text{when } \frac{1}{r} - \frac{1}{2} = \frac{1}{l};$$

$$\|f\|_{L^\infty} \lesssim \|D^\alpha f\|_{L^2}.$$

For the proof, we refer to [16, 14, 41], see also [25].

The dispersive estimates and Strichartz estimates for free wave equations on $\mathbb{H}^d$ were considered by [45, 34, 33, 3, 4]. Theorem 5.2 and Remark 5.5 of Anker, Pierfelice [3] obtained the Strichartz estimates for linear wave/Klein-Gordon equations. Recall $\tilde{D} = (-\Delta - \frac{1}{4} + \kappa^2)^{\frac{1}{2}}$ for some $\kappa > \frac{1}{2}$.

**Lemma 2.2** ([3]). Let $(p, q)$ and $(\tilde{p}, \tilde{q})$ be two admissible couples, i.e.,

$$\left\{(p^{-1}, q^{-1}) \in (0, \frac{1}{2}) \times (0, \frac{1}{2}) : \frac{1}{p} > \frac{1}{2} \left(\frac{1}{2} - \frac{1}{q}\right)\right\} \cup \left\{0, \frac{1}{2}\right\},$$

and similarly for $(\tilde{p}, \tilde{q})$. Meanwhile assume that

$$\sigma \geq \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \tilde{\sigma} \geq \frac{3}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right),$$

then the solution $u$ to the linear wave equation

$$\begin{align*}
\partial^2_t u - \Delta u &= F \\
u(0, x) &= u_0(x), \partial_t |_{t=0} u(t, x) = u_1(x)
\end{align*}$$

satisfies the Strichartz estimate

$$\left\|\tilde{D}^{\sigma - \frac{1}{2}}_x u\right\|_{L_t^p L_x^q} + \left\|\tilde{D}^{\tilde{\sigma} - \frac{1}{2}}_x \partial_t u\right\|_{L_t^p L_x^q} \lesssim \left\|\tilde{D}^{\frac{1}{2}}_x u_0\right\|_{L^2} + \left\|\tilde{D}^{-\frac{1}{2}}_x u_1\right\|_{L^2} + \left\|\tilde{D}^{\tilde{\sigma} - \frac{1}{2}} F\right\|_{L_t^p L_x^q}.$$  

**Remark** For all $\sigma \in \mathbb{R}$, $p \in (1, \infty)$, $\|\tilde{D}^\sigma f\|_p$ is equivalent to $\|D^\sigma f\|_p$, see [Page 5618, 4].

### 3 Proof of Theorem 1.1

**Lemma 3.1.** Let $B, A$ satisfy ([1,2]), then $H$ is a self-adjoint operator in $L^2(\mathbb{D}; \mathbb{C}^2)$ with domain $D(H) = H^2$.

**Proof.** Since we will work with $\mathbb{C}^2$-valued functions, the operators $d, d^*$ are assumed to act on $\Omega^p(\mathbb{D}) \otimes \mathbb{C}^2$ and $\Omega^p(\mathbb{D}) \otimes g(2, \mathbb{C})$. First, $-\Delta$ is self-adjoint in $L^2$ with domain $H^2$ (see for instance [11]). Second, $H$ is symmetric in $L^2(\mathbb{D}; \mathbb{C}^2)$ with domain $H^2$ by integration by parts. In fact, denote the inner product in $\mathbb{C}^2$ by $\langle \cdot, \cdot \rangle$. Given, $f, g \in C_c^\infty(\mathbb{D}; \mathbb{C}^2)$. Since $B$ is symmetric and real, it is obvious that

$$\int_{\mathbb{D}} \langle Bf, g \rangle dz = \int_{\mathbb{D}} \langle f, Bg \rangle dz.$$  

(3.1)
Let $K$ where $B$ spaces with (3.2), (3.3) implies Thus the Poincare inequality

$$\|f\|_{L^2(B)} \leq \epsilon \|\Delta f\|_{L^2(B)} + C_1(\epsilon, K)\|f\|_{L^2(B)}$$

where $B_K$ denotes the geodesic ball with center $O$ of radius $K$. By taking $K \gg 1$, the exponential decay of $V$ and $|A|$ (see (1.2)) yields

$$\|(V + X)f\|_{L^2(B_K^c)} \lesssim e^{-\epsilon K} \|f\|_{L^2} + e^{-\epsilon K} \|\nabla f\|_{L^2}.$$  

Thus the Poincare inequality $\|f\|_{L^2} + \|\nabla f\|_{L^2} \lesssim \|\Delta f\|_{L^2}$ for hyperbolic spaces with (3.2), (3.3) implies

$$\|(V + X)f\|_{L^2(\mathbb{D})} \lesssim (e^{-\epsilon K} + \epsilon)\|\Delta f\|_{L^2} + C_1(\epsilon, K)\|f\|_{L^2}.$$  

Let $K$ be sufficiently large, by Kato’s perturbation theorem, $H$ is self-adjoint.

Since $H$ is assumed to be positive, one can define its fractional power $H^s$ for any $s \in \mathbb{R}$ via the spectrum theorem.

For reader’s convenience, we recall the following lemma of [4] whose proof is based on the Kunze-Stein phenomenon.

**Lemma 3.2** (Lemma 5.1, [4]). * There exists a constant $C > 0$ so that for any radial function $h$ on $\mathbb{D}$, any $2 \leq m, k < \infty$ and $g \in L^k(\mathbb{D})$,

$$\|g * h\|_{L^m} \lesssim \|g\|_{L^k} \left( \int_0^\infty \sinh r(\varphi_0(r))^P |h(r)|^Q dr \right)^{1/Q},$$

where $P = \frac{2\min\{m,k\}}{m+k}$, $Q = \frac{mk}{k+m}$, and $\varphi_0$ is the spherical function defined in [2.7].

* The JDE version of [Lemma 5.1,[4]] has some misprints. And we take this Lemma from its arxiv version which is correct.
The following weighted Strichartz estimate will be important to prove the endpoint Strichartz estimates. Its proof is an application of the smoothing effects. Recall $\rho(x) = e^{-d(x,0)}$.

**Lemma 3.3.** Let $H$ satisfy assumptions in Theorem 1.1. Assume that $u$ solves
\begin{align}
\begin{cases}
\partial_t^2 u + Hu = F \\
u(0, x) = u_0, \partial_t u(0, x) = u_1
\end{cases}
\end{align}
Then we have
\begin{equation}
\|\rho^{\alpha} \nabla u\|_{L^2_t L^2_x} \lesssim \|F(t)\|_{L^1_t L^2_x} + \|\nabla u_0\|_{L^2_x} + \|u_1\|_{L^2_x}.
\end{equation}

**Proof.** The proof is an easy application of the Kato’s smoothing effect of $e^{\pm it\sqrt{H}}$. In fact, by Duhamel principle,
\begin{equation}
u(t) = \cos\left(t\sqrt{H}\right) u_0 + \sin\left(t\sqrt{H}\right) u_1 + \int_0^t \sin\left((t-s)\sqrt{H}\right) F(s) ds.
\end{equation}
By the Christ-Kiselev lemma, for the inhomogeneous term it suffices to prove
\begin{equation}
\left\|\int_\mathbb{R} \rho^{\alpha} \nabla H^{-\frac{1}{2}} \sin\left((t-s)\sqrt{H}\right) F(s) ds\right\|_{L^2_t L^2_x} \lesssim \|F\|_{L^1_t L^2_x}.
\end{equation}
By (1.5), the Kato’s smoothing effect and Minkowski,
\begin{align}
\left\|\int_\mathbb{R} \rho^{\alpha} \nabla H^{-\frac{1}{2}} \sin\left((t-s)\sqrt{H}\right) F(s) ds\right\|_{L^2_t L^2_x} &\lesssim \int_\mathbb{R} \|\rho^{\alpha} \sin\left((t-s)\sqrt{H}\right) F(s)\|_{L^2_t L^2_x} ds \\
&+ \int_\mathbb{R} \|\rho^{\alpha} H^{-\frac{1}{2}} \sin\left((t-s)\sqrt{H}\right) F(s)\|_{L^2_t L^2_x} ds \\
&\lesssim \int_\mathbb{R} \|F(s)\|_{L^2_x} ds + \int_\mathbb{R} \|H^{-\frac{1}{2}} F(s)\|_{L^2_x} ds.
\end{align}
Meanwhile, the strict positiveness of the self-adjoint operator $H$ and the spectrum theorem imply
\begin{equation}
\|H^{-\frac{1}{2}} F\|_{L^2_x}^2 = \langle H^{-1} F, F \rangle \leq c \|F\|_{L^2_x}^2.
\end{equation}
Hence the estimates for the inhomogeneous term follow by (3.8) and (3.7). Similarly, the two homogeneous terms are bounded by $\|\nabla u_0\|_{L^2_x} + \|u_1\|_{L^2_x}$ by applying (1.5), Kato’s smoothing effect for $e^{\pm it\sqrt{H}}$ and the standard Poincare inequality $\|f\|_{L^2_x} \lesssim \|\nabla f\|_{L^2}$.

The proof of non-endpoint Strichartz estimates is quite standard. The non-endpoint homogeneous Strichartz estimates are given below.
Lemma 3.4. Let \((p, q)\) be an admissible pair with \(p > 2\), then
\[
\|D_\pm e^{\pm it\sqrt{H}} f\|_{L^p_t L^q_x} \lesssim \|Df\|_{L^2_x}. \tag{3.9}
\]
\[
\|D_\pm e^{\pm it\sqrt{H}} f\|_{L^p_t L^q_x} \lesssim \|H^{\frac{3}{2}} f\|_{L^2_x}. \tag{3.10}
\]

Proof. We follow the framework of \([7]\). Recall that \(W = V + X\) is the potential part of \(H\). Denote \(e^{it\sqrt{H}} f = u\), then
\[
e^{it\sqrt{H}} f = \cos (tD \pm i\sin D) \sqrt{H} f - \int_0^t \sin ((t - s)D) \sqrt{H} f - \int_0^t \sin ((t - s)D) W u ds. \tag{3.11}
\]

Lemma 2.2, (1.3) and Lemma 2.1 show
\[
\|\sin (tD) \sqrt{H} f\|_{L^p_t L^q_x} \lesssim \|\sqrt{H} f\|_{L^2_x} \lesssim \|Df\|_{L^2_x}.
\]

Thus the homogeneous estimate is done. The rest is to handle the inhomogeneous term. As a preparation, we first prove
\[
\left\| \int_0^t \frac{\sin ((t - s)D)}{D^\frac{3}{2}} W u ds \right\|_{L^p_t L^q_x} \lesssim \|\rho^\alpha H^{\frac{3}{2}} u\|_{L^2_x} + \|\rho^\alpha u\|_{L^2_x}. \tag{3.12}
\]

Since \(p > 2\), by the Christ-Kiselev lemma, to verify (3.12) it suffices to prove
\[
\left\| \int_{\mathbb{R}} \frac{e^{\pm i(t-s)D}}{D^\frac{3}{2}} W u ds \right\|_{L^p_t L^q_x} \lesssim \|\rho^\alpha H^{\frac{3}{2}} u\|_{L^2_x} + \|\rho^\alpha u\|_{L^2_x}. \tag{3.13}
\]

Recall the Kato smoothing effect for \(e^{iDt}\): for any \(g \in L^2\) there holds
\[
\|\rho^\alpha e^{\pm itD} g\|_{L^2_t L^2_x} \lesssim \|g\|_{L^2_x}.
\]

The dual version is
\[
\left\| \int_{\mathbb{R}} e^{\pm iD} F(s) ds \right\|_{L^2_x} \lesssim \|\rho^{-\alpha} F\|_{L^2_x}. \tag{3.14}
\]

(3.14) and Lemma 2.2 give
\[
\left\| \int_{\mathbb{R}} D^\frac{1}{2} e^{\pm i(t-s)D} W u ds \right\|_{L^p_t L^q_x} \lesssim \left\| \int_{\mathbb{R}} e^{-isD} W u ds \right\|_{L^2_x} \lesssim \|\rho^{-\alpha} (V + X) u\|_{L^2_t L^2_x}.
\]
Thus by (1.2) and (1.5), one deduces
\[ \left\| \int_0^t e^{i\sqrt{H}(t-s)} F \, ds \right\|_{L^2_t L^2_x} \]
\[ \lesssim \left( \left\| \nabla \rho^{-2\alpha} \right\|_{L^\infty} + \left\| |A| \rho^{-2\alpha} \right\|_{L^\infty} \right) \left( \left\| \rho^\alpha u \right\|_{L^2_t L^2_x} + \left\| \rho^\alpha \nabla u \right\|_{L^2_t L^2_x} \right) \]
\[ \lesssim \left\| \rho^\alpha u \right\|_{L^2_t L^2_x} + \left\| \rho^\alpha H^\frac{1}{2} u \right\|_{L^2_t L^2_x}. \]

Hence (3.12) has been proved. Since \( H^\frac{1}{2} \) commute with \( e^{\pm i \sqrt{H}} \), (3.12), the Kato’s smoothing effect for \( e^{\pm i \sqrt{H}} \) and (1.3)-(1.4) yield
\[ \int_0^t \left\| \rho^\alpha \right\|_{L^2_t L^2_x} + \left\| H^\frac{1}{2} f \right\|_{L^2_x} \lesssim \left\| \nabla f \right\|_{L^2_x}. \]

Thus we have obtained (3.9). (3.10) follows by (3.9) and (3.8).

\[ \square \]

**Proposition 3.1.** Let \( H \) satisfy the assumptions in Theorem 1.1. Then we have the non-endpoint Strichartz estimates for the magnetic wave equation: If \( u \) solves the equation
\[ \begin{cases} \partial_t^2 u + Hu = F \\ u(0,x) = u_0, \partial_t u(0,x) = u_1 \end{cases} \quad (3.15) \]
then it holds for any admissible pair \((p,q)\) with \( p > 2, q \in (2,6] \)
\[ \left\| D^\frac{1}{2} u \right\|_{L^p_t L^q_x} + \left\| D^\frac{1}{2} \partial_t u \right\|_{L^p_t L^q_x} + \left\| \partial_t u \right\|_{L^p_t L^2_x} + \left\| \nabla u \right\|_{L^p_t L^2_x} \]
\[ \lesssim \left\| \nabla u_0 \right\|_{L^2} + \left\| u_1 \right\|_{L^2} + \left\| F \right\|_{L^1_t L^2_x}. \]

\[ \square \]

**Proof.** By Duhamel principle,
\[ u(t) = \cos \left( t \sqrt{H} \right) u_0 + \frac{1}{\sqrt{H}} \sin \left( t \sqrt{H} \right) u_1 + \int_0^t \sin \left( s \sqrt{H} \right) \frac{F(s)}{\sqrt{H}} \, ds. \]

The homogenous estimates follow directly by Lemma 3.4, (1.3)-(1.4), and the inequality \( \left\| \nabla f \right\|_{L^2} \lesssim \left\| (-\Delta)^s f \right\|_{L^2} \) for any \( s \in (0,1) \). It remains to deal with the inhomogeneous term. By the Christ-Kiselev lemma, it suffices to prove
\[ \left\| \int_0^t D^\frac{1}{2} H^{-\frac{1}{2}} \sin \left( (t-s) \sqrt{H} \right) F(s) \, ds \right\|_{L^p_t L^q_x} \lesssim \left\| F \right\|_{L^1_t L^2_x}. \]

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This is an immediate corollary of (1.3)-(1.4) and (3.10). In fact, we have by (3.10) and Minkowski inequality
\[
\int_{\mathbb{R}} D^{1/2} H^{-1/2} \sin \left( (t-s) \sqrt{H} \right) F(s) ds \lesssim \int_{\mathbb{R}} D^{1/2} H^{-1/2} e^{\pm i (t-s) \sqrt{H}} F(s) ds
\]
\[
\lesssim \int_{\mathbb{R}} \left| H^{1/2} H^{-1/2} e^{\pm is \sqrt{H}} F(s) \right| ds \lesssim \|F(s)\|_{L^1_\sigma L^2_x}.
\]
The estimate of \(\partial_t u\) is similar. \(\square\)

Recall \(\tilde{D} = (-\Delta - \frac{1}{4} + \kappa^2)^{1/2}\) in Section 1. Let \(\chi_\infty(r)\) be a cutoff function which equals one when \(r \geq 3/2\) and vanishes near zero. In the vertical strip \(0 \leq \Re \sigma \leq 3/2\), we define an analytic family of operators
\[
\tilde{W}^{\sigma,\infty}_t = \frac{e^{\sigma^2}}{\Gamma(2 - \sigma)} \chi_\infty(D) \tilde{D}^{-\sigma} e^{itD}
\]
Denote its kernel by \(\tilde{w}^{\sigma,\infty}_t(r)\). It is easily seen that \(\tilde{W}^{\sigma,\infty}_t\) is the high frequency truncation of \(\tilde{D}^{-\sigma} e^{itD}\). The Gamma function added to (3.16) allows us to handle the boundary \(\Re \sigma = \frac{3}{2}\) (see [3]). For \(\sigma \in \mathbb{R}\), define the low frequency truncation of \(\tilde{D}^{-\sigma} e^{itD}\) to be
\[
W^{\sigma,0}_t = \tilde{D}^{-\sigma} e^{itD}(I - \chi_\infty(D)).
\]
Denote its kernel by \(w^{\sigma,0}_t(r)\). We collect some results from [Section 3, [3]] for reader’s convenience.

**Lemma 3.5 ([3])**. The kernel \(w^{\sigma,0}_t(r)\) satisfies the point-wise estimates for \(\sigma \in \mathbb{R}, |t| \geq 2\)
\[
|w^{\sigma,0}_t(r)| \lesssim \begin{cases} |t|^{-3/2} (1 + r) \varphi_0(r), & 0 \leq r \leq \frac{1}{2} |t| \\ \varphi_0(r), & r \geq \frac{1}{2} |t| \end{cases}
\]
And for \(\sigma \in \mathbb{C}\) with \(\Re \sigma = \frac{3}{2}\), the kernel \(\tilde{w}^{\sigma,\infty}_t(r)\) satisfies
\[
|w^{\sigma,\infty}_t(r)| \lesssim \begin{cases} |t|^{-\infty} \varphi_0(r), & 0 \leq r \leq \frac{1}{2} |t| \\ e^{-\frac{1}{2} r t}, & r \geq \frac{1}{2} |t| \end{cases}
\]

**Remark 3.1.** ([3],[18]) can be found in Theorem 3.1 of [3]. (3.19) is contained in the proof of Theorem 3.2 in [3].

**Lemma 3.5** has several corollaries.

**Corollary 3.1.** Assume that \(p \in (2, 6), q \in (2, 6)\), then for \(t \geq 2\)
\[
\left\| f \ast w^{\sigma,0}_t(r) \right\|_{L^p} \lesssim t^{-3/2} \|f\|_{L^q'}.
\]
Moreover, for $\Re \sigma > 3(\frac{1}{2} - \frac{1}{p})$, $\Re \sigma > 3(\frac{1}{2} - \frac{1}{q})$, $t \geq 2$, one has

$$\| f \ast \tilde{w}_{t}^{\sigma, \infty} \|_{L^{p}} \lesssim t^{-\infty} \| f \|_{L^{p'}}.$$  \hspace{1cm} (3.21)

**Proof.** First, we prove the $w_{t}^{\sigma, 0}(r)$ part in (3.20). When $r \leq \frac{1}{2} |t|$ the desired estimate in (3.20) follows by applying Lemma 3.2 and (3.18). When $r \geq \frac{1}{2} |t|$, since $\varphi_{0}(r) \lesssim (1 + r)e^{-\frac{1}{2}r}$, choosing arbitrary $0 < \epsilon < 1$, we obtain $\varphi_{0}(r) \lesssim t^{-\frac{3}{2}} e^{-\frac{1}{2} - \epsilon r}$. Then the desired estimate in (3.20) follows by applying Lemma 3.2 as well.

Second, we deal with the $\tilde{w}_{t}^{\sigma, \infty}(r)$ part in (3.21). This is achieved by interpolation. In fact, for $\Re \sigma = 0$

$$\| f \ast \tilde{w}_{t}^{\sigma, \infty} \|_{L^{2}} \leq C \| f \|_{L^{2}}.$$  \hspace{1cm} (3.22)

For $\Re \sigma = 3/2$, as in the first step, (3.19) with Lemma 3.2 yields for any $2 < m < \infty$, $k \in [2, \infty)$

$$\| f \ast \tilde{w}_{t}^{\sigma, \infty} \|_{L^{k}} \leq C t^{-\infty} \| f \|_{L^{m'}}.$$  \hspace{1cm} (3.23)

Interpolating (3.22) with (3.23), for $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{k}$, $\frac{1}{q'} = \frac{\theta}{2} + \frac{1-\theta}{m'}$, and $\Re \sigma = \frac{3}{2}(1 - \theta)$, we have

$$\| f \ast \tilde{w}_{t}^{\sigma, \infty} \|_{L^{p}} \leq C t^{-\infty} \| f \|_{L^{q'}}.$$  \hspace{1cm} (3.24)

In this case, by checking the relations $k = \frac{2 \sigma}{3} / (\frac{1}{p} - \frac{1}{2} + \frac{1}{q'} \sigma)$, $m' = \frac{2 \sigma}{3} / (\frac{1}{q'} - \frac{1}{2} + \frac{1}{q} \sigma)$, and $2 < m < \infty$, $k \in [2, \infty)$, we conclude that for $\Re \sigma > 3(\frac{1}{2} - \frac{1}{p})$, $\Re \sigma > 3(\frac{1}{2} - \frac{1}{q})$, $p \in (2, 6)$, $q \in (2, 6)$, there holds

$$\| f \ast \tilde{w}_{t}^{\sigma, \infty} \|_{L^{p}} \leq C(p, q) t^{-\infty} \| f \|_{L^{q'}}.$$  \hspace{1cm} (3.25)

**Lemma 3.6.** Let $u$ solve the linear wave equation

$$\begin{cases}
\partial_{t}^{2} u - \Delta u = F \\
u(0, x) = 0, \partial_{t} u(0, x) = 0
\end{cases}$$  \hspace{1cm} (3.26)

Let $\alpha > 0$, then for $q \in (2, 6)$

$$\left\| D^{\frac{3}{2} + \alpha} u \right\|_{L^{2}_{t} L^{2}_{x}} \leq \left\| \rho^{-\alpha} F \right\|_{L^{2}_{t} L^{2}_{x}}.$$  \hspace{1cm} (3.27)

**Proof.** Step 1. A Non-endpoint Result. Although the Christ-Kiselev
Lemma is not available here, we can firstly prove a non-endpoint result, i.e.,

$$\left\| D^{1/2} u \right\|_{L^p_t L^q_x} \lesssim \left\| \rho^{-\alpha} F \right\|_{L^r_t L^s_x},$$  \hspace{1cm} (3.26)

where \((p, q)\) is an admissible pair and \(p > 2\). The proof of (3.26) follows from the Christ-Kiselev lemma, Lemma 2.2 and the Kato’s smoothing effect.

**Step 2. Bilinear Argument for Endpoint.** Next, we prove the endpoint case. The proof is based on the bilinear argument of [21].

**Step 2.1. Reduction of Time Support.** By Duhamel principle,

$$D^{1/2} u(t) = \int_0^t \frac{\sin(D(t-s))}{D^{1/2}} F(s) ds.$$  \hspace{1cm} (3.27)

Meanwhile, the endpoint Strichartz estimates for the free wave equation in Lemma 2.2 and the dual Kato smoothing effect for \(e^{\pm itD} \) show

$$\left\| \int_{-\infty}^0 \frac{\sin(D(t-s))}{D^{1/2}} F(s) ds \right\|_{L^r_t L^s_x} \lesssim \left\| \int_{-\infty}^0 e^{\pm isD} F(s) ds \right\|_{L^r_t L^s_x} \lesssim \left\| \rho^{-\alpha} F \right\|_{L^r_t L^s_x}.$$

Hence, (3.25) reduces to prove

$$\left\| \int_{-\infty}^t \frac{\sin(D(t-s))}{D^{1/2}} F(s) ds \right\|_{L^r_t L^s_x} \lesssim \left\| \rho^{-\alpha} F \right\|_{L^r_t L^s_x}.$$  \hspace{1cm} (3.28)

Consider the bilinear form of (3.28):

$$\left\| \int_{-\infty}^t \int_{-\infty}^t e^{i(t-s)D} D^{-\frac{1}{2}} F(s) G(t) ds dt \right\| \lesssim \left\| \rho^{-\alpha} F(t) \right\|_{L^r_t L^s_x} \left\| G(t) \right\|_{L^r_t L^s_x}.$$  \hspace{1cm} (3.29)

Split the time integrand domain into the following dyadic subintervals:

$$\int_{-\infty}^t \int_{-\infty}^t e^{i(t-s)D} D^{-\frac{1}{2}} F(s) G(t) ds dt = \sum_{j \in \mathbb{Z}} \int_{-\infty}^t \int_{-\infty}^t e^{i(t-s)D} D^{-\frac{1}{2}} F(s) G(t) ds dt.$$  \hspace{1cm} (3.30)

For any fixed \(j \in \mathbb{Z}\), divide \(F(s), G(t)\) further into \(F(s) = \sum_{k \in \mathbb{Z}} F^j_k(s), G(t) = \sum_{n \in \mathbb{Z}} G^j_n(t)\) with

$$F^j_k(s) = F(s) 1_{s \in [k2^j, (k+1)2^j)}, \quad G^j_n(t) = G(t) 1_{t \in [n2^j, (n+1)2^j)}.$$  

When \(s \in [k2^j, (k+1)2^j), k \in \mathbb{Z}\), the relation \(t - 2^j \leq s \leq t - 2^j - 1\) shows \(|n - k| \leq 2\). If we have proved there exists a constant \(\beta(q) > 0\) such that for
all \(k, j \in \mathbb{Z}\)
\[
\int_{\mathbb{R}} \int_{\mathbb{D}} \int_{t-2^j \leq s \leq t-2^j-1, |n-k| \leq 2} e^{i(t-s)D} D_{-\frac{1}{2}} F^j_k(s)G^j_n(t) dsdxdt 
\lesssim |j|^{-\beta} \|\rho^{-\alpha} F^j_k\|_{L^2 L^2_x} \|G^j_n\|_{L^1_t L^p_y}.
\] (3.31)

Then the Cauchy-Schwartz inequality and the restriction \(|n-k| \leq 2\) give
\[
|3.30| \lesssim \sum_{j \in \mathbb{Z}} |j|^{-\beta} \sum_{k, n \in \mathbb{Z}, |n-k| \leq 2} \|F^j_k\|_{L^2 L^2_x} \|G^j_n\|_{L^1_t L^p_y} 
\lesssim \sum_{j \in \mathbb{Z}} |j|^{-\beta} \left( \sum_{k \in \mathbb{Z}} \|F^j_k\|_{L^2 L^2_x}^2 \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} \|G^j_n\|_{L^1_t L^p_y}^2 \right)^{\frac{1}{2}} 
\lesssim \|F\|_{L^2_t L^p_x} \|G\|_{L^1_t L^p_y} \sum_{j \in \mathbb{Z}} |j|^{-\beta}.
\]

Thus it suffices to prove (3.31). Therefore, without loss of generality, we assume \(F\) and \(G\) are supported on a time interval of size \(2^j\) on \(\{(t, s) : t - 2^j \leq s \leq t - 2^j-1\}\).

**Step 2.2. Sum of Negative \(j\).** For \(j \leq 0, q \in (2, 6]\), choose \(m\) to be slightly larger than 2, then Hölder and (3.20) give for \(\frac{1}{m} > \frac{1}{2} - \frac{1}{q}\) (then \((m, q)\) is an admissible pair)
\[
\left| \int_{\mathbb{R}} \int_{\mathbb{D}} \int_{t-2^j \leq s \leq t-2^j-1} e^{i(t-s)D} D_{-\frac{1}{2}} F(s)G(t) dsdxdt \right| 
\lesssim \left\| \int_{t-2^j \leq s \leq t-2^j-1} e^{i(t-s)D} D_{-\frac{1}{2}} F(s) ds \right\|_{L^m_t L^q_y} \|G(t)\|_{L^1_t L^p_y'} 
\lesssim \|\rho^{-\alpha} F(t)\|_{L^2_t L^2_x} \|G(t)\|_{L^1_t L^p_y'} 
\lesssim \|\rho^{-\alpha} F(t)\|_{L^2_t L^2_x} \|G(t)\|_{L^2_t L^2_y} 2^{(\frac{1}{m} - \frac{1}{q})}.
\] (3.32)

where we used the time support of \(G(t)\) is of size \(2^j\) in the last line. Therefore, the negative \(j\) part of (3.30) is summable.

**Step 2.3. Sum of Positive \(j\).** For the positive \(j\), let us consider a multiple parameter analytic family of operators defined by
\[
T^\sigma_\sigma^1_j(F, G) = \int_{\mathbb{R}} \int_{\mathbb{D}} \int_{t-2^j \leq s \leq t-2^j-1} e^{i(t-s)D} D_{-\sigma} D_{-\sigma_1} F(s)G(t) dsdxdt.
\] (3.33)

By Remark 2.1, every estimate for \(T^\sigma_\sigma^1_j(F, G)\) will yield a corresponding estimate for \(T^\sigma_\sigma^0_j(F, G)\) if both \(\sigma\) and \(\sigma_1\) are real. Furthermore, we divide \(T^\sigma_\sigma^1_j\) into \(T^\sigma_\sigma^1_j + T^\sigma_\sigma^0_j\), where \(T^\sigma_\sigma^1_j = \chi_\infty(D)T^\sigma_\sigma^1_j\) denotes the high frequency part (see Lemma 3.5 for \(\chi_\infty\)), and \(T^\sigma_\sigma^0_j = T^\sigma_\sigma^1_j(I - \chi_\infty)(D)\) denotes
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the low frequency part.

**Step 2.3.1. High Frequency for Positive $j$.** We claim for any $\Re \sigma > \frac{3}{2}(\frac{1}{2} - \frac{1}{q})$, $\Re \sigma_1 > \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$, $(p, q) \in (2, 6) \times (2, 6)$

$$|T^{\sigma, \sigma_1}_{j, \infty}(F, G)| \lesssim j^{-\infty} \|F\|_{L^p_t L^q_x} \|G\|_{\tilde{L}^p_t \tilde{L}^q_x}.$$  \hspace{1cm} (3.34)

(3.34) is essentially contained in [4]. For convenience, we give the detailed proof in Lemma 5.2 below. Meanwhile, given $0 < \eta \ll 1$, for any $p, q \in (2, 6)$ satisfying

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{3} + \eta,$$  \hspace{1cm} (3.35)

and any $\theta \in (\frac{1}{2} - \frac{3}{2} \eta, \frac{1}{2})$, we can divide $\theta$ into $\theta_1 + \theta_2 = \theta$, such that $\theta_1 > \frac{3}{2}(\frac{1}{2} - \frac{1}{q})$, $\theta_2 > \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$. Thus by our claim (3.34),

$$|T^{\theta_1, \theta_2}_{j, \infty}(F, G)| \lesssim j^{-\infty} \|F\|_{L^p_t L^q_x} \|G\|_{\tilde{L}^p_t \tilde{L}^q_x}.$$  \hspace{1cm} (3.36)

For a given $\alpha > 0$, choose $p$ slightly larger than 2 such that

$$\frac{2p'}{2 - p'} > 1,$$  \hspace{1cm} (3.37)

then Hölder and (3.30) give

$$|T^{\theta_1, \theta_2}_{j, \infty}(F, G)| \lesssim j^{-\infty} \|\rho^\alpha\|_{L^{p'}_t L^q_x} \|F\|_{L^p_t L^q_x} \|G\|_{\tilde{L}^p_t \tilde{L}^q_x} \lesssim j^{-\infty} \|\rho^{-\alpha}\|_{L^p_t L^q_x} \|G\|_{\tilde{L}^p_t \tilde{L}^q_x}.$$  \hspace{1cm} (3.38)

Notice that for $\theta_1 + \theta_2 = \theta \in (\frac{1}{2} - \eta, \frac{1}{2})$, (3.37) and (3.35) imply that in order to obtain (3.38), $q$ should be restricted to $q \in (\frac{6}{1+6\eta+6\eta^2}, 6)$. The special case of (3.38) when $\theta_1 + \theta_2 = \frac{1}{2}$ corresponds to (3.30). The left range of $q$ will be considered later.

**Step 2.3.2. Low Frequency for Positive $j$.** Meanwhile, (3.20) and Hölder give for all $p, q \in (2, 6)$ and any $\sigma_2 \in \mathbb{R}$,

$$|T^{\theta, \sigma_2}_{j, 0}(F, G)| \lesssim \int_{\mathbb{R}} \int_{t-2^j \leq s \leq t-2^{j-1}} \left\|(I - \chi_\infty(D))\bar{D}^{-\sigma_2} e^{\pm i(t-s)D} F(s)\right\|_{L^q} \|G(s)\|_{L^{q'}} ds dt$$

$$\lesssim \int_{\mathbb{R}} \int_{t-2^j \leq s \leq t-2^{j-1}} (t-s)^{-\frac{3}{2}} \|F(s)\|_{L^{q'}} \|G(s)\|_{L^{q'}} ds dt$$

$$\lesssim 2^{-j/2} \|F\|_{L^p_t L^q_x} \|G\|_{\tilde{L}^p_t \tilde{L}^q_x}.$$  \hspace{1cm} (3.39)
Choosing $p$ to be slightly larger than 2 such that (3.37) holds, we have by (3.39) that
\[
|T_{j,0}^{0,\sigma_2}(F,G)| \lesssim 2^{-j/2} \|\rho^{-\alpha} F\|_{L^\infty_t L^2_x} \|G\|_{L^q_t L^r_x}.
\] (3.40)

The special case when $\sigma_2 = \frac{1}{2}$ corresponds to the low frequency part of (3.30).

**Step 2.4. Sum for Positive $j$.** Hence, (3.29) is summable when $j \geq 0$ by (3.38) and (3.39) for $q \in (\frac{6}{1+6\alpha+6\eta} \cdot 6)$. Thus we have proved (3.25) for $q \in (\frac{6}{1+6\alpha+6\eta} \cdot 6)$. It remains to prove (3.25) for the left $p$ in (2,6). Since the negative $j$ part of (3.30) is done, we separate the positive $j$ part by defining:
\[
T_{\geq 0}^{\gamma} F := \int_{-\infty}^{t-\frac{1}{2}} e^{\pm itD} D^{-\gamma} F(s) ds.
\] (3.41)

Then it suffices to prove
\[
\|T_{\geq 0}^{\frac{1}{2}} F\|_{L^2_t L^4_x} \lesssim \|\rho^{-\alpha} F\|_{L^2_t L^2_x}.
\] (3.42)

Denote the high frequency truncation of $T_{\geq 0}^{\gamma}$ by $T_{\geq 0,hi}^{\gamma}$ and its low frequency truncation by $T_{\geq 0,low}^{\gamma}$ respectively. And the corresponding bilinear form can be divided into dyadic subintervals and high/low frequency parts as above. The only difference is $j$ is forced to be $j \geq 0$. Step 2.3.2 shows the low frequency part $T_{\geq 0,low}^{\gamma} F$ satisfies
\[
\|T_{\geq 0,low}^{\gamma} F\|_{L^2_t L^4_x} \lesssim \|\rho^{-\alpha} F\|_{L^2_t L^2_x}.
\] (3.43)

for all $\gamma \in \mathbb{R}$ and $q \in (2,6)$. Step 2.3.1 shows the high frequency part $T_{\geq 0,hi}^{\gamma} F$ satisfies
\[
\|T_{\geq 0,hi}^{\gamma} F\|_{L^2_t L^4_x} \lesssim \|\rho^{-\alpha} F\|_{L^2_t L^2_x}.
\] (3.44)

for all $\gamma \in (\frac{1}{2} - \eta, \frac{1}{2}]$ and $q \in (\frac{6}{1+6\alpha+6\eta} \cdot 6)$. The key point is (3.33), (3.44) give some gain in derivatives. (But it seems that this gain only happens in the positive $j$ part.)

**Step 2.4.1. Derivatives of Low Order.** Consider $T_{\leq 0}^{1,0}$. The corresponding dyadic bilinear form is
\[
T_{j,\infty}^{1,0}(F,G) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{2^{-j} t - 1 \leq s \leq 2^{-j} t - 1} e^{i(t-s)D} D^{-1} \chi_{\infty} F(s) G(t) ds dx dt,
\] (3.45)
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and

$$T_{j,0}^{1,0}(F, G) = \int_{\mathbb{R}} \int_D \int_{t-2^j \leq s \leq t-2^{j-1}} e^{i(t-s)D} D^{-1}(I - \chi_\infty(D))F(s)G(t) ds dx dt,$$

(3.46)

In this case, $T_{j,0}^{1,0}$ has a full derivative. Then by directly applying Corollary 3.1, we obtain for all $p, q \in (2, 6)$

$$\left| T_{j,\infty}^{1,0}(F, G) \right| \lesssim \int_{\mathbb{R}} \int_{t-2^j \leq s \leq t-2^{j-1}} (t-s)^{-\infty} \| F(s) \|_{L^p} \| G(s) \|_{L^q} ds dt$$

$$\lesssim j^{-\infty} \| F(s) \|_{L^2_t L^p_x} \| G(s) \|_{L^q_t L^2_x}.$$

Then by choosing $p$ to be slightly larger than 2, we get for all $q \in (2, 6)$,

$$\left| T_{j,\infty}^{0,1}(F, G) \right| \lesssim j^{-\infty} \| \rho^{-\alpha} F \|_{L^2_t L^2_x} \| G \|_{L^q_t L^2_x}.$$  

(3.47)

The low frequency part $T_{j,0}^{1,0}$ in (3.46) for $j \geq 0$ follows by the same arguments as $D^\frac{1}{2}$ considered above. Hence, we have for all $q \in (2, 6)$

$$\| T_{\geq 0}^1 \|_{L^2_t L^2_x} \lesssim \| \rho^{-\alpha} F \|_{L^2_t L^2_x}.$$  

(3.48)

**Step 2.4.2. Full Range of $q$ by Interpolation.** It suffices to prove (3.42). By Gagliardo-Nirenberg inequality, for any $q \in (2, 6)$ there exists $q_1 = 2^+$, $q_2 = 6^-$ and $\gamma = \left(\frac{1}{2}\right)^-$ such that

$$\left\| T_{\geq 0}^1 F \right\|_{L^2_x} \lesssim \| T_{\geq 0}^1 F \|_{L^{q_1}_t L^{q_1}_x} \| T_{\geq 0}^1 F \|_{L^{q_2}_t L^{q_2}_x}^{1-\tau},$$

(3.49)

with $\tau \in (0, 1)$. Then (3.42) follows by (3.49), (3.44), (3.43) and Hölder.

Proposition 3.1 with Lemma 3.6, Lemma 3.3 gives

**Lemma 3.7.** Let $H$ satisfy assumptions in Theorem 1.1, $0 < \alpha < 2\rho$, then we have the weighted Strichartz estimates for the magnetic wave equation: If $u$ solves the equation

$$\left\{ \begin{array}{l}
\partial_t^2 u + Hu = 0 \\
u(0, x) = u_0, \partial_t u(0, x) = u_1
\end{array} \right.$$  

then it holds for any $p \in (2, 6)$

$$\left\| D^\frac{1}{2} u \right\|_{L^2_t L^p_x} + \| \rho^{\alpha} \nabla u \|_{L^2_t L^2_x} + \| \partial_t u \|_{L^\infty_t L^2_x} + \| \nabla u \|_{L^\infty_t L^2_x} \lesssim \| \nabla u_0 \|_{L^2} + \| u_1 \|_{L^2}.$$  

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**Proof.** (3.11), Lemma 3.2 and Lemma 3.6 with \( p \in (2, 6) \) give

\[
\left\| \frac{D^\frac{1}{2}}{\sqrt{\lambda}} u \right\|_{L^2_t L^p_x} + \| u \|_{L^2_t L^p_x} \lesssim \| \nabla u_0 \|_{L^2_x} + \| u_1 \|_{L^2_x} + \| \rho^{-\alpha} W u \|_{L^2_t L^2_x}.
\]

Hence by (1.2), one obtains

\[
\left\| \frac{D^\frac{1}{2}}{\sqrt{\lambda}} u \right\|_{L^2_t L^p_x} + \| u \|_{L^2_t L^p_x} \lesssim \| \nabla u_0 \|_{L^2_x} + \| u_1 \|_{L^2_x} + \| \rho^{-\alpha} W u \|_{L^2_t L^2_x}.
\]

(3.50)

Lemma 3.3 shows

\[
\| \rho^{-\alpha} \nabla u \|_{L^2_t L^2_x} \lesssim \| \nabla u_0 \|_{L^2_x} + \| u_1 \|_{L^2_x}.
\]

(3.51)

By (3.50), (3.51) and the Kato smoothing effect for \( e^{\pm i t \sqrt{\lambda}} \), we get the endpoint homogeneous estimate

\[
\left\| \frac{D^\frac{1}{2}}{\sqrt{\lambda}} u \right\|_{L^2_t L^p_x} + \| u \|_{L^2_t L^p_x} \lesssim \| \nabla u_0 \|_{L^2_x} + \| u_1 \|_{L^2_x}.
\]

\( \square \)

The remaining inhomogeneous endpoint Strichartz estimates are given below.

**Proposition 3.2.** Let \( H \) satisfy assumptions in Theorem 1.1, \( 0 < \alpha < 2\beta \), then we have the weighted Strichartz estimates for the magnetic wave equation: If \( u \) solves the equation

\[
\begin{cases}
\partial_t^2 u + H u = F \\
u(0, x) = u_0, \partial_t u(0, x) = u_1
\end{cases}
\]

then it holds for any \( p \in (2, 6) \)

\[
\left\| \frac{D^\frac{1}{2}}{\sqrt{\lambda}} u \right\|_{L^2_t L^p_x} + \| \rho^{-\alpha} \nabla u \|_{L^2_t L^2_x} + \| \partial_t u \|_{L^\infty_t L^2_x} + \| \nabla u \|_{L^\infty_t L^2_x} \lesssim \| \nabla u_0 \|_{L^2_x} + \| u_1 \|_{L^2_x} + \| F \|_{L^1_t L^2_x}.
\]

**Proof.** It remains to prove the case when \( u_0 = u_1 = 0 \) by Lemma 3.7. In this case, by the Christ-Kiselev lemma, it suffices to prove

\[
\left\| \int_\mathbb{R} \frac{D^\frac{1}{2}}{\sqrt{\lambda}} H^{-\frac{1}{2}} e^{\pm i (t-s) \sqrt{\lambda}} F(s) ds \right\|_{L^p_t L^p_x} \lesssim \| F \|_{L^1_t L^2_x}.
\]

This follows immediately from Minkowski, Lemma 3.7 and (1.3)-(1.4). \( \square \)
4 Proof of Corollary 1.1

In this section we prove Corollary 1.1. The Kato smoothing effect will be proved first, the equivalence of $H^{1/2}$ and $D$ in various spaces will be proved then.

4.1 Kato smoothing estimates for wave equations with small potentials

In this subsection, we aim to prove the Kato smoothing estimates for the magnetic half wave operator $e^{it\sqrt{H}}$. The technique we use is on one hand quite eclectic, and usually statements are proved by combining ideas borrowed from different works. And on the other hand, some refinements and new ideas are introduced to estimate troublesome terms.

The self-adjoint operator $H$ is strictly positive due to the smallness assumption of potentials. In fact, (1.6) shows
\[
\langle Hf, f \rangle = \langle -\Delta f, f \rangle + O(\mu_1 \|\nabla f\|_{L^2}^2 + \mu_1 \|f\|_{L^2}^2).
\]
(4.1)

Then the standard Poincare inequality $\langle -\Delta f, f \rangle \geq \frac{1}{4} \|f\|_{L^2}^2$ implies there exists some $c > 0$ such that $\langle Hf, f \rangle \geq c \|f\|_{L^2}^2$ provided $\mu_1$ is sufficiently small.

We recall the Kato smoothing Theorem.

Theorem 4.1 (\[35\]). (Kato smoothing theorem) Let $M, N$ be two Hilbert spaces and $H : M \to N$ be a self-adjoint operator. Denote its resolvent by $(H - \lambda)^{-1}$. Let $U : M \to N$ be a closed densely defined operator. Assume that for any $f \in D(U^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there holds
\[
\|U(H - \lambda)^{-1}U^*f\|_N \leq C\|f\|_N.
\]
Then $e^{\pm itH}g \in D(U)$ for all $g \in M$ and a.e. $t$. Moreover, it holds
\[
\int_{-\infty}^{\infty} \|Ue^{\pm itH}g\|_N^2 \, dt \leq \frac{2}{\pi} C^2 \|g\|_M^2.
\]

First we give the estimates for the kernel of the free resolvent. (4.3), (4.2) were established by Corollary 3.2 and Lemma 3.3 in [5]. (4.3) and (4.4) are new here.

Lemma 4.1. Let $R_0(\frac{1}{2} + \sigma) = (-\Delta + \sigma^2 - \frac{1}{4})^{-1}$ be the free resolvent and denote its Schwartz kernel by $R_0(\frac{1}{2} + \sigma, x, y)$. Then for $\Re \sigma \geq 0$, $|\sigma| \leq 1$, $r \in (0, \infty)$, we have
\[
|R_0(\frac{1}{2} + \sigma, x, y)| \leq \begin{cases} 
C|\log r|, & |r| \leq 1 \\
C|\sigma|^{-\frac{1}{2}} e^{-(\frac{1}{2} + \Re \sigma)r}, & |r| \geq 1
\end{cases}
\]
(4.2)
And for $\Re \sigma \geq 0$, $|\sigma| \geq 1$, $r \in (0, \infty)$, we have

$$|\mathcal{R}_0(\frac{1}{2} + \sigma, x, y)| \leq \left\{ \begin{array}{ll}
C |\log r|, & |r\sigma| \leq 1 \\
C |\sigma|^{-\frac{1}{2}} e^{-(\frac{1}{2} + \Re \sigma) r}, & |r\sigma| \geq 1
\end{array} \right. \quad (4.3)$$

Furthermore, for $\Re \sigma \geq 0$, $|\sigma| \leq 1$, $r \in (0, \infty)$, we have

$$\left| \nabla_x \mathcal{R}_0(\frac{1}{2} + \sigma, x, y) \right| \leq \left\{ \begin{array}{ll}
Cr^{-2}(\sinh r)^2(\cosh^2 r - 1)^{-\frac{1}{2}}, & |r\sigma| \leq 1 \\
C|\sigma|^{\frac{1}{2}} e^{-(\frac{1}{2} + \Re \sigma) r}(\sinh r)^2(\cosh^2 r - 1)^{-\frac{1}{2}}, & |r\sigma| \geq 1
\end{array} \right. \quad (4.4)$$

And for $\Re \sigma \geq 0$, $|\sigma| \geq 1$, $r \in (0, \infty)$, we have

$$\left| \nabla_x \mathcal{R}_0(\frac{1}{2} + \sigma, x, y) \right| \leq \left\{ \begin{array}{ll}
Cr^{-2}(\sinh r)^2(\cosh^2 r - 1)^{-\frac{1}{2}}, & |r\sigma| \leq 1 \\
C|\sigma|^{\frac{1}{2}} e^{-(\frac{1}{2} + \Re \sigma) r}(\sinh r)^2(\cosh^2 r - 1)^{-\frac{1}{2}}, & |r\sigma| \geq 1
\end{array} \right. \quad (4.5)$$

**Proof.** Let $[^{n}\mathcal{R}]_0(s, x, y)$ denote the Schwartz kernel of the resolvent of the Laplace-Beltrami operator in $\mathbb{H}^{n+1}$, i.e., the kernel of $(-\Delta_{\mathbb{H}^{n+1}} - s(n-s))^{-1}$. Then $[^{n}\mathcal{R}]_0(s, x, y)$ can be written in terms of Legendre functions (42),

$$[^{n}\mathcal{R}]_0(s, x, y) = c(n)e^{-i\pi\mu}(\sinh r)^{-\mu}Q_{\nu}^0(\cosh r), \quad (4.6)$$

where $r = d(x, y)$, $\nu = s - \frac{n+1}{2}$ and $\mu = \frac{n-1}{2}$. Particularly, if choosing $s = \frac{n}{2} + \sigma$ for $n = 1, 3$ we have

$$[^{1}\mathcal{R}]_0(\frac{1}{2} + \sigma, x, y) = \mathcal{R}_0(\frac{1}{2} + \sigma, x, y) \quad (4.7)$$

$$[^{1}\mathcal{R}]_0(\frac{1}{2} + \sigma, x, y) = c(1)Q_{\sigma}^0(\cosh r), \quad (4.8)$$

$$[^{3}\mathcal{R}]_0(\frac{1}{2} + \sigma, x, y) = c(3)e^{-i\pi}(\sinh r)^{-1}Q_{\sigma}^1(\cosh r), \quad (4.9)$$

Recall the formula for the derivative of the second class Legendre functions (see for instance [49])

$$(z^2 - 1)^{\frac{1}{2}} \frac{d}{dz} Q_{\nu}^0(z) = Q_{\nu}^1(z). \quad (4.10)$$

Therefore, (4.7) combined with (4.10), (4.8) and (4.9) implies

$$\partial_r \mathcal{R}_0(\frac{1}{2} + \sigma, x, y) = c(1)\partial_r Q_{\kappa}(\cosh r) = c(1)(\cosh^2 r - 1)^{-\frac{1}{2}} \sinh r Q_{\kappa}^1(\cosh r)$$

$$= c(\cosh^2 r - 1)^{-\frac{1}{2}} \sinh^2 r[^{3}\mathcal{R}]_0(\frac{3}{2} + \sigma, x, y).$$

Hence, (4.5) and (4.4) follow from the fact $|\nabla r| = 1$ and the corresponding resolvent estimates for $[^{3}\mathcal{R}]_0$ in [Corollary 3.2, Lemma 3.3, [5]].
The free resolvent $R_0(z) = (-\Delta - z(1 - z))^{-1}$ has the following basic estimates in weighted $L^2$ spaces.

**Lemma 4.2.** For $z \in \mathbb{C}$ with $\Re z > 0$, we have for $\alpha > 0$ sufficiently small

$$
\|\rho^\alpha R_0(\frac{1}{2} + z)f\|_{L^2} \lesssim |z|^{-1}\|f\rho^{-\alpha}\|_{L^2}
$$

(4.11)

$$
\|\rho^\alpha R_0(\frac{1}{2} + z)f\|_{L^2} \lesssim \|f\rho^{-\alpha}\|_{L^2}
$$

(4.12)

$$
\|\rho^\alpha \nabla R_0(\frac{1}{2} + z)f\|_{L^2} \lesssim \|f\rho^{-\alpha}\|_{L^2}
$$

(4.13)

**Proof.** (4.11) follows by the same arguments in [5, Proposition 4.1] and the following wave operator expression for the free resolvent:

$$
(-\Delta - \frac{1}{4} - (\lambda + i\mu)^2)^{-1} = \Lambda(\lambda, \mu) \int_0^\infty e^{i(\text{sgn} \mu) \lambda t} e^{-|\mu| (\cos t \sqrt{-\Delta - \frac{1}{4}})} dt,
$$

(4.14)

where $\Lambda(\lambda, \mu) = \frac{i \text{sgn} \mu}{\lambda + i\mu}$. By (4.5), (4.4), the kernel of $\rho^\alpha \nabla R_0 \rho^\alpha$ is bounded by

$$
\begin{cases}
Ce^{-(\alpha + \frac{1}{2})r} r^{-\frac{1}{2}}, & |z| \leq 1 \\
Cr^{-\frac{1}{2}}, & |z| \geq 1, |zr| \leq 1 \\
C|z|^{\frac{1}{2}} e^{-(\alpha + \frac{1}{2})r} r^{-\frac{1}{2}}, & |z| \geq 1, |zr| \geq 1
\end{cases}
$$

Then, for $|z| \leq 1$ the Kunze-Stein phenomenon yields

$$
\|\rho^\alpha \nabla R_0(\frac{1}{2} + z)f\|_{L^2} \lesssim \|\rho^\alpha f\|_{L^2},
$$

(4.15)

which shows (4.13) in the case $|z| \leq 1$. By Sobolev embedding and Leibnitz rule,

$$
\left\|\rho^\alpha R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2} \leq C \left\|\nabla \rho^\alpha R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2} \leq C\alpha \left\|\rho^\alpha R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2} + C \left\|\rho^\alpha \nabla R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2}.
$$

(4.16)

Since (4.11) has shown the LHS of (4.16) is finite provided $\Re z > 0$, choosing $\alpha > 0$ to be sufficiently small, we get

$$
\left\|\rho^\alpha R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2} \leq C \left\|\rho^\alpha \nabla R_0(\frac{1}{2} + z)\rho^\alpha f\right\|_{L^2}.
$$

Hence (4.12) follows by (4.11) when $|z| \geq 1$ and by (4.15) when $|z| \leq 1$. The rest is to prove (4.13) for $|z| \geq 1$. Integration by parts and $|\nabla d(x, 0)| = 1$.
yield
\[
\left\| \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2 = \left\langle \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle
\]
\[
= - \left\langle \rho^{2\alpha} \Delta R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle
\]
\[
- \left\langle (\nabla \rho^{2\alpha}) \cdot \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle
\]
\[
= \left\langle \rho^{2\alpha} \left( -\Delta - \frac{1}{4} + z^2 \right) R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle
\]
\[
+ O \left( \alpha \left\langle \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, \rho^\alpha R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle \right)
\]
\[
+ \left( \frac{1}{4} - z^2 \right) \left\langle \rho^\alpha R_0\left( \frac{1}{2} + z \right) \rho^\alpha f, \rho^\alpha R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\rangle.
\]
Thus for \(|z| \geq 1\), (4.11) gives
\[
\left\| \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 \left\| \rho^\alpha R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2 + \alpha \left\| \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2
\]
\[
+ \left( |z|^2 + 1 \right) \left\| \rho^\alpha R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2
\]
\[
\lesssim \alpha \left\| \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2}^2 + \|f\|_{L^2}^2.
\]
Let \(\alpha > 0\) be sufficiently small, we obtain for \(|z| \geq 1\)
\[
\left\| \rho^\alpha \nabla R_0\left( \frac{1}{2} + z \right) \rho^\alpha f \right\|_{L^2} \lesssim \|f\|_{L^2},
\]
which combined with (4.15) gives (4.13).

**Lemma 4.3.** Recall \(R_H(z) = (H - z(1 - z))^{-1}\), then for all \(\Re z > 0\), one has when \(0 < \mu_1 \ll 1, 0 < \alpha \ll 1\)
\[
\|\rho^\alpha R_H(z + \frac{1}{2}) f\|_{L^2} \leq C \min(1, |z|^{-1}) \|\rho^{-\alpha} f\|_{L^2}.
\]  
(4.17)

**Proof.** Formally we have the identity
\[
R_H(z + \frac{1}{2}) = R_0(z + \frac{1}{2}) (I + W R_0(z + \frac{1}{2}))^{-1}.
\]  
(4.18)
First we prove the operator \((I + W R_0)^{-1}\) is well-defined and uniformly
bounded in \( L(\rho^\alpha L^2, \rho^\alpha L^2) \). By Lemma 4.2 choose \( 3\alpha < \varrho \),
\[
\|\rho^{-\alpha}X\mathcal{R}_0 f\|_{L^2} \lesssim \|A\rho^{-2\alpha}\|_{L^\infty} \|\rho^\alpha \nabla \mathcal{R}_0 f\|_{L^2} 
\lesssim \mu_1 \|\rho^\alpha \nabla \mathcal{R}_0 \rho^\alpha\|_{L^2 \to L^2} \|\rho^{-\alpha} f\|_{L^2}.
\]
Meanwhile, we have
\[
\|\rho^{-\alpha} V\mathcal{R}_0 f\|_{L^2} \lesssim \mu_1 \|\rho^\alpha \mathcal{R}_0 \rho^\alpha\|_{L^2 \to L^2} \|\rho^{-\alpha} f\|_{L^2}.
\]
Therefore, by Neumann series argument we conclude \( I + W\mathcal{R}_0 \) is invertible in \( \rho^\alpha L^2 \) with
\[
\|\rho^{-\alpha}(I + W\mathcal{R}_0)^{-1} f\|_{L^2} \leq C \|\rho^{-\alpha} f\|_{L^2},
\]
where \( C \) is independent of \( z \). Hence Lemma 4.2 implies
\[
\|\rho^\alpha \mathcal{R}_0 (I + W\mathcal{R}_0)^{-1} f\|_{L^2} \leq C \|\rho^\alpha \mathcal{R}_0 \rho^\alpha\|_{L^2 \to L^2} \|\rho^{-\alpha}(I + W\mathcal{R}_0)^{-1} f\|_{L^2}
\lesssim \|\rho^{-\alpha} f\|_{L^2}.
\]
The \(|z|^{-1}\) decay follows directly from (4.11).

**Corollary 4.1.** Let \( H \) satisfy the assumptions in Corollary 1.1. If \( 0 < \mu_1 \ll 1 \), then the spectrum of \( H \) is absolutely continuous and \( \sigma(H) = \left[\frac{1}{4}, \infty\right) \).

**Proof.** In the beginning of Section 2, we have shown \( H \) is self-adjoint. By [Theorem XIII.19, Page 137, 35] and the continuity of spectral projection operators, to prove the spectrum is absolutely continuous, it suffices to prove for any bounded interval \((c, d)\) and any \( f \in C_c^\infty \)
\[
\sup_{0 < \varepsilon < 1} \int_c^d |\Im (f, R_H(\tau + i\varepsilon) f)|^2 d\tau < \infty. \tag{4.19}
\]
Using (4.17) one has for \( \varepsilon > 0 \)
\[
|\langle f, R_H(\tau + i\varepsilon) f \rangle| = |\langle f \rho^{-\alpha}, \rho^\alpha R_H(\tau + i\varepsilon) \rho^\alpha \rho^{-\alpha} f \rangle| \lesssim \|f \rho^{-\alpha}\|_2^2 < \infty,
\]
which yields (4.19). Meanwhile, by Weyl’s criterion, \( \sigma_{\text{ess}}(H) = \left[\frac{1}{4}, \infty\right) \). Therefore, we obtain \( \sigma(H) = \sigma_{\text{ac}}(H) = \left[\frac{1}{4}, \infty\right) \).

**Lemma 4.4.** Let \( H \) satisfy the assumptions in Corollary 1.1. Let \( z \in \mathbb{C} \setminus \mathbb{R} \). If \( 0 < \mu_1 \ll 1, 0 < \alpha \ll 1 \), then there holds
\[
\left\|\rho^\alpha R^{\sqrt{H}}(z) f\right\|_{L^2} \leq C \|\rho^{-\alpha} f\|_{L^2}, \tag{4.20}
\]
where \( C \) is independent of \( z \). And thus by Theorem 4.1, the Kato smoothing effect
\[
\|\rho^\alpha e^{\pm i \sqrt{H} t} f\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2_x}
\]
holds for all \( 0 < \alpha \ll 1 \).
Thus we have
\begin{align*}
&\left\| (\sqrt{\mathcal{H}} - z) f \right\|_{L^2}^2 = \left\| \sqrt{\mathcal{H}} f \right\|_{L^2}^2 + |z|^2 \| f \|_{L^2}^2 - 2\Re z \left\langle \sqrt{\mathcal{H}} f, f \right\rangle \\
&\quad \geq \left\| \sqrt{\mathcal{H}} f \right\|_{L^2}^2 \geq \frac{1}{4} \| f \|_{L^2}^2.
\end{align*}

Thus we have
\begin{align*}
&\| \rho^\alpha (\sqrt{\mathcal{H}} - z)^{-1} f \|_{L^2} \leq \| (\sqrt{\mathcal{H}} - z)^{-1} f \|_{L^2} \leq C \| f \|_{L^2} \leq C \| \rho^{-\alpha} f \|_{L^2}.
\end{align*}

Hence (a) is done. For (b), we use
\begin{align*}
R_{\sqrt{\mathcal{H}}} (z) = (\sqrt{\mathcal{H}} - z)^{-1} = 2z R_H (z^2) + (\sqrt{\mathcal{H}} + z)^{-1}.
\end{align*}

Since \( \Re z > 0 \) and \( z \notin \mathbb{R} \) in case (b), we have \( z^2 \notin [\frac{1}{4}, \infty) \). Then the first term in (4.21) follows from (4.17) and the second follows from case (a). \( \square \)

In the following lemmas, we prove the equivalence of \( \| (-\Delta)^s f \|_{\rho^{-\beta} L^2} \) and \( \| H^s f \|_{\rho^{-\beta} L^2} \) for \( s = \frac{1}{2} \) and \( \beta = 0, \alpha \). As a preparation, we prove the \( L^p - L^q \) estimates for the free resolvent.

**Lemma 4.5.** Let \( 0 < \alpha < \frac{1}{2} \). For any \( 2 \leq p < q < \infty \), \( \sigma \geq \frac{1}{2} \) and any \( \omega > \frac{1}{p} - \frac{1}{q} \), we have
\begin{align*}
\| (-\Delta + \sigma^2 - 1/4)^{-1} \|_{L^p \rightarrow L^q} &\lesssim \min(1, \sigma^{-2+\omega}) \quad (4.22) \\
\| \nabla (-\Delta + \sigma^2 - 1/4)^{-1} \|_{L^p \rightarrow L^q} &\lesssim \min(1, \sigma^{-1+2\omega}) \quad (4.23) \\
\| (-\Delta + \sigma^2 - 1/4)^{-1} \|_{\rho^{-\alpha} L^2 \rightarrow \rho^{-\alpha} L^2} &\lesssim \min(1, \sigma^{-2}) \quad (4.24) \\
\| \nabla (-\Delta + \sigma^2 - 1/4)^{-1} \|_{\rho^{-\alpha} L^2 \rightarrow \rho^{-\alpha} L^2} &\lesssim \min(1, \sigma^{-1}). \quad (4.25)
\end{align*}

**Proof.** First, we prove (4.22) for \( \frac{1}{4} \leq \sigma \leq 1 \). By Lemma 4.1 and Young’s inequality, it suffices to prove for \( \frac{1}{2} \leq \sigma \leq 1 \) and \( m \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{1}{m} \), there holds
\begin{align*}
\int_0^1 |\log r|^m r dr &\lesssim 1; \quad \sigma^{-\frac{1}{2}} \int_1^\infty e^{-m(\frac{1}{2} + \sigma)r} r dr \lesssim 1.
\end{align*}

The first inequality on the LHS is obvious. Noticing that \( m = 1/(1 - \frac{1}{p} + \frac{1}{q}) \) is strictly larger than 1 due to \( p < q \), for \( \sigma \geq \frac{1}{2} \), we obtain
\begin{align*}
\sigma^{-\frac{1}{2}} \int_1^\infty e^{-m(\frac{1}{2} + \sigma)r} r dr \lesssim \frac{1}{m - 1}.
\end{align*}

Second, we prove (4.22) for \( \sigma \geq 1 \). By Lemma 4.1 and Young’s inequality,
it suffices to prove for \( \sigma \geq 1 \) there holds
\[
\left( \int_0^\infty |\log r|^m r^{1/m} dr \right) \lesssim \sigma^{-2+\omega} \quad (4.26)
\]
\[
\left( \int_\sigma^\infty \sigma^{-m(\frac{1}{2}+\sigma)r} \sinh r dr \right) \lesssim \sigma^{-2+\omega}. \quad (4.27)
\]

(4.26) follows by direct calculation. For (4.27), we divide it into two regimes: for \( \sigma \geq 1, r \in [\sigma^{-1}, 1] \), one has
\[
\left( \int_{\sigma^{-1}}^1 \sigma^{-m(\frac{1}{2}+\sigma)r} \sinh r dr \right) \lesssim \left( \int_{\sigma^{-1}}^1 \sigma^{-m(\frac{1}{2}+\sigma)r} \right) \lesssim \sigma^{-2+\omega}, \quad (4.28)
\]
and for \( \sigma \geq 1, r \in [1, \infty) \) we have
\[
\left( \int_1^\infty \sigma^{-m(\frac{1}{2}+\sigma)r} \sinh r dr \right) \lesssim \left( \int_1^\infty \sigma^{-m(1-m)(\frac{1}{2}+\sigma)r} \right) \lesssim \sigma^{-2+\omega}. \quad (4.29)
\]

Third, we prove (4.23) when \( \frac{1}{2} \leq \sigma \leq 1 \). This follows by the same arguments as above and the following inequalities for the corresponding kernel:
\[
\sigma^{\frac{1}{2}r-2} e^{-\frac{1}{2}(\frac{1}{2}+\sigma)^2}(\sinh^2 r - 1)^{-\frac{1}{2}} \lesssim \left\{ \begin{array}{ll}
r^{1-\frac{1}{2}}, & 0 \leq r \leq 1; \\
e^{-\frac{1}{2}(\frac{1}{2}+\sigma)r}, & r \geq 1.
\end{array} \right. \]

Forth, we prove (4.23) when \( \sigma \geq 1 \). By Lemma 4.1 it suffices to prove
\[
\sigma^{\frac{1}{2}} \left( \int_{\sigma^{-1}}^1 r^{-2m}(\sinh r)^{2m} (\cosh^2 r - 1)^{-\frac{1}{2}} \right) \lesssim \left\{ \begin{array}{ll}
r^{1-\frac{1}{2}}, & 0 \leq r \leq 1; \\
e^{-\frac{1}{2}(\frac{1}{2}+\sigma)r}, & r \geq 1.
\end{array} \right. \]

When \( \sigma \geq 1 \), the LHS of (4.32) is bounded by
\[
\sigma^{\frac{1}{2}} \left( \int_{\sigma^{-1}}^1 r^{-\frac{m}{2}} e^{-m(\frac{1}{2}+\sigma)r} dr \right) \lesssim \sigma^{\frac{1}{2}} \left( \int_{\sigma^{-1}}^1 r^{-\frac{m}{2}} e^{-m(\frac{1}{2}+\sigma)r} dr \right) \lesssim \sigma^{-1+2\omega}, \quad (4.32)
\]
which yields (4.32). When \( \sigma \geq 1 \), the LHS of (4.33) is bounded by
\[
\sigma^{\frac{1}{2}} \left( \int_1^\infty e^{(m+1)-m(\frac{1}{2}+\sigma)r} dr \right) \lesssim e^{-\frac{1}{2}r} \sigma^{\frac{1}{2}} (m-1)^{-\frac{1}{2}} \lesssim \sigma^{-1+2\omega},
\]
which yields (4.33).

Fifth, we prove (4.24). Instead of Young’s inequality, we will use the the
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Kunze-Stein phenomenon:

\[
\|f \ast k\|_{L^2(\mathbb{D})} \lesssim \|f\|_{L^2(\mathbb{D})} \left\{ \int_0^\infty k(r) \varphi_0(r) \sinh r dr \right\}, \tag{4.34}
\]

where \( k(r) \) is assumed to be radial. Lemma 4.1 shows the point-wise estimate:

\[
\left| \rho^{\text{ad}(x,0)}(-\Delta + \sigma^2 - 1/4)^{-1}(\rho^{-\alpha}\rho f) \right| \lesssim \int_{\mathbb{D}} e^{\text{ad}(x,y)} |\mathcal{R}_0(x,y)f(y)| dy, \tag{4.35}
\]

and

\[
\left| \rho^{\text{ad}(x,0)}\nabla(-\Delta + \sigma^2 - 1/4)^{-1}(\rho^{-\alpha}\rho f) \right| \lesssim \int_{\mathbb{D}} e^{\text{ad}(x,y)} |\nabla \mathcal{R}_0(x,y)f(y)| dy. \tag{4.36}
\]

Inserting the bounds in Lemma 4.1 to (4.35), (4.36) and applying (4.34) implies it suffices to prove for \( \sigma \in [\frac{1}{2}, 1] \)

\[
\sigma^{-\frac{1}{2}} \int_0^1 |\log r| r \varphi_0 dr + \sigma^{-\frac{1}{2}} \int_1^\infty e^{-\left(\frac{1}{2}+\sigma\right)r} e^{(1+\alpha)r} \varphi_0 dr \lesssim 1 \tag{4.37}
\]

and for \( \sigma \in [1, \infty) \)

\[
\sigma^{-\frac{1}{2}} \int_0^\frac{1}{\sigma} |\log r| e^{\alpha r} \varphi_0 r dr + \sigma^{-\frac{1}{2}} \int_{\sigma^{-1}}^\infty \sigma^{-\frac{1}{2}} e^{(\alpha-\frac{3}{2}-\sigma)r} \sinh r \varphi_0 dr \lesssim \sigma^{-2} \tag{4.39}
\]

\[
\sigma^\frac{1}{2} \int_{\sigma^{-1}}^1 r^{-2} (\sinh r)^2 (\cosh^2 r - 1)^{-\frac{1}{2}} e^{(\alpha-\frac{3}{2}-\sigma)r} r \varphi_0 dr \lesssim \sigma^{-1} \tag{4.40}
\]

\[
\sigma^\frac{1}{2} \int_1^\infty r^{-2} (\sinh r)^2 (\cosh^2 r - 1)^{-\frac{1}{2}} e^{(\alpha-\frac{3}{2}-\sigma)r} r \varphi_0 dr \lesssim \sigma^{-1}. \tag{4.41}
\]

Using the bound \( \varphi_0(r) \lesssim (1+r)e^{-\frac{3}{2}r} \) to absorb the \( e^{\alpha r} \) growth, (4.37)-(4.41) follow by the same calculations as above.

And we also need the boundedness of Riesz transform on weighted \( L^2 \) space.

**Lemma 4.6.** Let \( 0 < \alpha < \frac{1}{2} \), then \( \nabla D^{-1} \) is bounded from \( \rho^{-\alpha}L^2 \) to \( \rho^{-\alpha}L^2 \).

**Proof.** The proof is adapted from [Theorem 6.1, [41]]. For \( f \in C_0^\infty \), we have

\[
D^{-1}f(x) = \int_{\mathbb{D}} E(d(x,z)) f(z) dz,
\]
where $E$ which denotes the Schwartz kernel for $D^{-1}$ is defined by:

\[ E(t) = \int_0^\infty (\lambda^2 + \frac{1}{4})^{-\frac{1}{2}} \varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda, \]

where $\phi_\lambda$ is the spherical function and $c(\lambda)$ is the Harish-Chandra $c$-function (see Section 2). Let $\chi(\tau)$ be a cutoff function which equals one when $|\tau| \leq 1$ and vanishes for $|\tau| \geq 2$. Split $\nabla D^{-1}f$ into the local and global parts:

\[
\nabla D^{-1}f = \int_D \chi(d(x,z))\nabla_x E(d(x,z))f(z)dz \quad \text{(4.42)}
\]

\[
+ \int_D (1 - \chi(d(x,z)))\nabla_x E(d(x,z))f(z)dz. \quad \text{(4.43)}
\]

The local part (4.42) is bounded on $L^p$ by [Theorem 4.7, [41]]. Meanwhile, since $d(x,y) \leq 1$ in the local part, one has $\rho^\alpha(x)\rho^-\alpha(z) \leq e$ in (4.42), and thus (4.42) is also bounded in $\rho^-\alpha L^2$. It remains to consider the global part (4.43). The proof of [Proposition 4.5, [37]] has shown that for $0 < \varepsilon \ll 1$, $\rho^\alpha(x)\rho^-\alpha(z)\rho^-\alpha(z)$ satisfies

\[
\int_0^\infty |K_\varepsilon(r)|^2 dr < \infty. \quad \text{[41], Theorem 6.1 pointed out the same bound (4.44) holds for (1 - \chi(d(x,z)))\nabla_x E(d(x,z)). Thus one has}
\]

\[
\left| \int_D \rho^\alpha(x) (1 - \chi(d(x,z))) \nabla_x E(d(x,z))f(z)dz \right| 
\leq \int_D \rho^\alpha(x)e^{-(1+\varepsilon)\frac{1}{2}d(x,z)} (1 + K_\varepsilon(d(x,z))) |f(z)| dz
\leq \int_D \rho^\alpha(x)e^{-(1+\varepsilon)\frac{1}{2}d(x,z)} (1 + K_\varepsilon(d(x,z))) \rho^-\alpha(z) |f(z)| dz.
\]

Since $\rho^\alpha(x)\rho^-\alpha(z)\rho^-\alpha(z) \leq 1$, for $\varepsilon' = \frac{1}{2}\varepsilon - \alpha$ we have

\[
\text{(4.43)} \leq \int_D e^{-(1+\varepsilon')\frac{1}{2}d(x,z)} (1 + K_\varepsilon(d(x,z))) \rho^\alpha(z) |f(z)| dz. \quad \text{(4.45)}
\]

Applying the Kunze-Stein phenomenon (4.34) and the bound $\varphi_\varepsilon(r) \lesssim (1 + r)e^{-\frac{1}{2}r}$, we obtain

\[
\| \text{(4.45)} \|_{L^2(\mathbb{S})} \lesssim \| \rho^\alpha f \|_{L^2(\mathbb{S})} \left\{ \int_0^\infty (\sinh r)(1 + r)e^{-(1+\varepsilon')\frac{1}{2}r}e^{-\frac{1}{2}r}dr \right\}.
\]

Hence, for $0 < \alpha < \frac{1}{2}$, $\nabla D^{-1}$ belongs to $L(\rho^-\alpha L^2; \rho^-\alpha L^2)$.

**Lemma 4.7.** Let $0 < \mu_1 \ll 1$, and $H$ satisfy the assumptions in Corollary 4.7. For any $s \in [0, \frac{1}{2}]$ and any $p \in [2, \infty)$, there exists some constant
C(p, s) > 0 such that for all \( f \in \mathcal{H}^{2s, p} \)

\[ \frac{1}{C} \| H^s f \|_{L^{p}} \leq \| (\Delta)^s f \|_{L^{p}} \leq C \| H^s f \|_{L^{p}} \]  \hfill (4.46)

\[ \| D f - H^{\frac{1}{2}} \|_{\rho^{-\alpha}L^{2}} \lesssim \mu_1 \| f \|_{\rho^{-\alpha}L^{2}}. \]  \hfill (4.47)

**Proof.** Case 1. 0 < s < \( \frac{1}{2} \). Given \( p \in [2, \infty) \), fix a constant \( q \) such that

\[ 2 \leq p < q < \infty, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{200} \left( \frac{1}{2} - s \right) \]  in the following proof. And the \( \omega \) defined in Lemma 4.3 is fixed to be \( \frac{1}{100} \left( \frac{1}{2} - s \right) \). Then we see \( 2s + 3\omega - 3 < -1 \). Recall \( W = V + X \) defined in Section 1. Balakrishnan’s formula for non-negative operators and direct calculations (see Lemma 5.1 in Appendix) give,

\[ (-\Delta)^s f - H^s f = c(s) \int_0^\infty \lambda^s (\lambda - \Delta)^{-1} W(\lambda - H)^{-1} f d\lambda. \]  \hfill (4.48)

Let \( \lambda = \sigma^2 - \frac{1}{4}, \) (4.48) yields

\[ (-\Delta)^s f - H^s f = 2c(s) \int_{\frac{1}{2}}^\infty (\sigma^2 - \frac{1}{4})^s (-\Delta + \sigma^2 - 1/4)^{-1} W(-H + \sigma^2 - 1/4)^{-1} f \sigma d\sigma. \]  \hfill (4.49)

For \( \sigma \geq \frac{1}{2}, \) \( \frac{1}{q} + \frac{1}{q} = \frac{1}{p}, \) (1.16), (4.22), (1.23) and Hölder show that

\[ \left\| V(-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \| V \|_{L^{p}} \left\| (-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \mu_1 \]

\[ \left\| X(-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \| A \|_{L^{p}} \left\| \nabla (-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \mu_1. \]

Hence we get \( \left\| (V + X)(-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \mu_1 \), by which it follows that

\[ \left\| (I + W(-\Delta + \sigma^2 - 1/4)^{-1})^{-1} \right\|_{L^p \to L^q} \leq 1. \]  \hfill (4.50)

Therefore, we conclude from the resolvent identity that

\[ \left\| (H + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \left\| (-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \left\| (I + W(-\Delta + \sigma^2 - 1/4)^{-1})^{-1} \right\|_{L^p \to L^p} \lesssim \min(1, \sigma^{-2+\omega}), \]  \hfill (4.51)

Consequently, for any \( 2 \leq k < p < q < \infty \) and \( \frac{1}{q} + \frac{1}{q} = \frac{1}{k} \), by Hölder we get

\[ \left\| (-\Delta + \sigma^2 - 1/4)^{-1} V(H + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^p} \lesssim \left\| (-\Delta + \sigma^2 - 1/4)^{-1} \right\|_{L^k \to L^p} \| V \|_{L^q} \left\| (H + \sigma^2 - 1/4)^{-1} \right\|_{L^p \to L^q} \lesssim \mu_1 \min(1, \sigma^{-4+2\omega}). \]  \hfill (4.52)
The left is to bound the magnetic part. By the formal resolvent identity and \((4.23), (4.50)\),
\[
\| \nabla (H + \sigma^2 - 1/4)^{\frac{1}{2}} \|_{L^p \to L^q} \lesssim \| \nabla (-\Delta + \sigma^2 - 1/4)^{-1} \|_{L^p \to L^q} \| (I + W (-\Delta + \sigma^2 - 1/4)^{-1})^{-1} \|_{L^p \to L^p} \lesssim \min(1, \sigma^{-1+2\omega}).
\]

Thus for \(2 \leq k < p < q < \infty\) and \(\frac{1}{k} + \frac{1}{q} = \frac{1}{p}\) we have
\[
\| (-\Delta + \sigma^2 - 1/4)^{-1} X (H + \sigma^2 - 1/4)^{-1} \|_{L^p \to L^p} \lesssim \| (-\Delta + \sigma^2 - 1/4)^{-1} \|_{L^k \to L^p} \| A \|_{L^1} \| \nabla (H + \sigma^2 - 1/4)^{-1} \|_{L^p \to L^q} \lesssim \mu_1 \min(1, \sigma^{-3+3\omega}).
\]

Therefore, \((4.53), (4.52)\) and \((4.49)\) yield for any \(p \geq 2\), there exists some \(C(p) > 0\) such that
\[
\| (-\Delta)^s f \|_{L^p} \lesssim C(p) \mu_1 \| f \|_{L^p} + \| H^s f \|_{L^p}.
\]

Thus by the inequality \(\| f \|_{L^p} \lesssim \| D^s f \|_{L^p}\) for any \(s > 0\), we absorb the \(\mu_1 \| f \|_{L^p}\) to the LHS by taking \(\mu_1\) to be sufficiently small. Therefore, we conclude
\[
\| (-\Delta)^s f \|_{L^p} \lesssim \| H^s f \|_{L^p}.
\]

\((4.53), (4.52)\) and \((4.49)\) also yield
\[
\| H^s f \|_{L^p} \lesssim \| \mu_1 \| f \|_{L^p} + \| (-\Delta)^s f \|_{L^p}.
\]

Thus \((4.54)\) and the inequality \(\| f \|_{L^p} \lesssim \| D^s f \|_{L^p}\) for any \(s > 0\) imply
\[
\| f \|_{L^p} \lesssim \| H^s f \|_{L^p}.
\]

Therefore, absorbing the term \(\mu_1 \| f \|_{L^p}\) to the LHS by letting \(\mu_1\) to be sufficiently small gives our lemma.

**Case 2.** \(s = \frac{1}{2}\). In this case, given \(q_1 \in [2, \infty)\), fix a constant \(p_1\) such that \(2 \leq p_1 < q_1 < \infty, \frac{1}{p_1} - \frac{1}{q_1} < \frac{1}{200}\) in the following proof. And the \(\omega\) defined in Lemma \(4.5\) is fixed to be \(\frac{1}{100}\). Instead of \((4.48)\), we use the following \"inverse\" direction identity: (see Lemma \(5.1\) below)
\[
(-\Delta)^{\frac{1}{2}} f - H^\frac{1}{2} f = -c(s) \int_0^\infty \lambda^{\frac{1}{2}} (\lambda - H)^{-1} W(\lambda - \Delta)^{-1} f d\lambda. \tag{4.55}
\]

Let \(\sigma^2 - \frac{1}{2} = \lambda\), \((4.55)\) becomes
\[
(-\Delta)^{\frac{1}{2}} f - H^{\frac{1}{2}} f
\]
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\[
= -2c(s) \int_{\frac{1}{2}}^{\infty} (\sigma^2 - 1/4)^{\frac{3}{2}} (\sigma^2 - 1/4 - H)^{-1} W(\sigma^2 - 1/4 - \Delta)^{-1} f \sigma d\sigma.
\]

The same arguments as proving (4.51) show
\[
\|(H + \sigma^2 - 1/4)^{-1}\|_{L^{p_1} \to L^{q_1}} \lesssim \min(1, \sigma^{-2+\omega}). \quad (4.57)
\]
And for $\frac{1}{l_1} + \frac{1}{q_1} = \frac{1}{p_1}$, the same arguments give
\[
\|V(-\Delta + \sigma^2 - 1/4)^{-1}\|_{L^{p_1} \to L^{q_1}} \lesssim \|V\|_{L^{1}} \|(-\Delta + \sigma^2 - 1/4)^{-1}\|_{L^{p_1} \to L^{q_1}} \lesssim \mu_1 \min(1, \sigma^{-2+\omega}). \quad (4.58)
\]
The key point is the way of dealing with $X(-\Delta + \sigma^2 - 1/4)^{-1}$ to avoid the loss of decay of $\sigma$. In fact, by the equivalence of $\|\nabla f\|_{L^p}$ and $\|Df\|_{L^p}$, we have
\[
\|X(-\Delta + \sigma^2 - 1/4)^{-1}\|_{L^{p_1} \to L^{q_1}} \lesssim \|A\|_{L^{1}} \|D(-\Delta + \sigma^2 - 1/4)^{-1}\|_{L^{p_1} \to L^{q_1}}.
\]
Since $D$ commutes with $D(-\Delta + \sigma^2 - 1/4)^{-1}$, we have for any $f \in \mathcal{H}^{\frac{1}{2}, p_1}$,
\[
\|X(-\Delta + \sigma^2 - 1/4)^{-1} f\|_{L^{q_1}} \lesssim \mu_1 \min(1, \sigma^{-2+\omega}) \|Df\|_{L^{p_1}}. \quad (4.59)
\]
Therefore, (4.57), (4.58), (4.59) and (4.56) show
\[
\|Df - H^{\frac{1}{2}} f\|_{L^{p_1}} \lesssim \mu_1 (\|Df\|_{L^{p_1}} + \|f\|_{L^{p_1}}) \int_{\frac{1}{2}}^{\infty} (\sigma^2 - 1/4)^{\frac{3}{2}} \min(1, \sigma^{-4+2\omega}) \sigma d\sigma. \quad (4.60)
\]
Thus by $\|f\|_{L^{p_1}} \lesssim \|Df\|_{L^{p_1}}$, for any $p_1 \in [2, \infty)$ and $0 < \mu_1 \ll 1$, there holds
\[
\|Df\|_{L^{p_1}} \lesssim \|H^{\frac{1}{2}} f\|_{L^{p_1}}. \quad (4.61)
\]
The inverse direction is easy by (4.60) and $\|f\|_{L^{p_1}} \lesssim \|Df\|_{L^{p_1}}$. Thus (4.46) has been obtained.

**Proof of (4.47).** By Lemma (4.5), (4.56) and similar arguments as **Case 2**, one has
\[
\|Df - H^{\frac{1}{2}} f\|_{\rho^{-\alpha} L^2} \lesssim (\|A\|_{L^\infty} \|Df\|_{\rho^{-\alpha} L^2} + \|V\|_{L^\infty} \|f\|_{\rho^{-\alpha} L^2}) \int_{\frac{1}{2}}^{\infty} (\sigma^2 - 1/4)^{\frac{3}{2}} \min(1, \sigma^{-4}) \sigma d\sigma.
\]
Thus (4.47) follows by letting $\mu_1$ be sufficiently small. \qed

In the following lemma, we prove the equivalence between $D$ and $H^{\frac{1}{2}}$ in the weighted $L^2$ space.
Lemma 4.8. Let \( H \) satisfy the assumptions in Corollary 1.1. For \( 0 < \alpha \ll 1, 0 < \mu_1 \ll 1 \), we have
\[
\frac{1}{C} \left\| H^{\frac{1}{2}} f \right\|_{\rho^{-\alpha} L^2} \leq \| Df \|_{\rho^{-\alpha} L^2} \leq C \left\| H^{\frac{1}{2}} f \right\|_{\rho^{-\alpha} L^2}. \tag{4.62}
\]
\[
\| \nabla f \|_{\rho^{-\alpha} L^2} \leq C \left\| H^{\frac{1}{2}} f \right\|_{\rho^{-\alpha} L^2}. \tag{4.63}
\]

Proof. It is easy to see (4.63) follows directly from (4.62) and Lemma 4.6. Thus it suffices to prove (4.62). By \(|\nabla d(x, 0)| = 1\) and Leibnitz rule we have
\[
\| \rho^\alpha f \|_{L^2} \lesssim \| \nabla (\rho^\alpha f) \|_{L^2} \leq \| \rho^\alpha \nabla f \|_{L^2} + C\alpha \| \rho^\alpha f \|_{L^2}.
\]
Then the smallness of \( \alpha \) implies
\[
\| f \|_{\rho^{-\alpha} L^2} \leq C \| \nabla f \|_{\rho^{-\alpha} L^2}.
\]
By Lemma 4.6 we get
\[
\| f \|_{\rho^{-\alpha} L^2} \lesssim \| Df \|_{\rho^{-\alpha} L^2}. \tag{4.64}
\]
Meanwhile, (4.47) gives
\[
\| Df \|_{\rho^{-\alpha} L^2} \leq C\mu_1 \| f \|_{\rho^{-\alpha} L^2} + \left\| H^{\frac{1}{2}} f \right\|_{\rho^{-\alpha} L^2}. \tag{4.65}
\]
Since \( \mu_1 \) is sufficiently small, the RHS of (4.65) can be absorbed to the LHS as before. Thus the second inequality of (4.62) is done. The first inequality of (4.62) follows by (4.47), (4.64) and the second one. \( \square \)

4.2 Conclusion

By Lemma 4.7, Lemma 4.8, Lemma 4.4, Corollary 1.1, we obtain Corollary 1.1 by applying Theorem 1.1.

5 Appendix

Lemma 5.1. Let \( s \in (0, 2) \). \( H \) is the nonnegative self-adjoint operator in Corollary 1.1. The following identity holds for \( f \in \mathcal{H}^2 \):
\[
H^{\frac{s}{2}} = (-\Delta)^{\frac{s}{2}} + c(s) \int_0^\infty \lambda^{\frac{s}{2}} (\lambda - \Delta)^{-1} W(\lambda - H)^{-1} d\lambda
\]
\[
H^{\frac{s}{2}} = (-\Delta)^{\frac{s}{2}} - c(s) \int_0^\infty \lambda^{\frac{s}{2}} (\lambda - H)^{-1} W(\lambda - \Delta)^{-1} d\lambda.
\]

Proof. For \( s \in (0, 2) \), the A.V. Balakrishnan formula for nonnegative operators \( T \) is
\[
T^{\frac{s}{2}} f = c(s) \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau + T)^{-1} T f d\tau.
\]
Then we have (5.1) by direct calculations and the resolvent identity:

$((-\Delta - \tau)^{-1} - (-\Delta + W - \tau)^{-1} = (-\Delta - \tau)^{-1}W(-\Delta + W - \tau)^{-1}.$

See [28] for the concrete calculation. \hfill \Box

**Lemma 5.2.** Assume that the time supports of $F$ and $G$ are of size $2^j$. For $(p, q) \in (2, 6)$ and $\sigma_1 > \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$, $\sigma > \frac{1}{2}(\frac{1}{2} - \frac{1}{q})$, one has

$$\left|T_{j, \infty}^{\sigma, \sigma_1}(F, G)\right| \lesssim 2^{-\infty j}\|F\|_{L_t^p L_x^\sigma} \|G\|_{L_t^q L_x^{\sigma_1}}.$$  \hfill (5.1)

**Proof.** The lemma essentially belongs to [4]. As before, the results still hold after exchanging the two operators $D$ and $D$ due to the equivalence $\|D \cdot \|_{L^p} \sim \|D \cdot \|_{L^p}$ for $p \in (1, \infty)$. By complex interpolation it suffices to consider the following three cases

(a) $2 = q < p < 6$, $\Re \sigma = \frac{3}{2}(\frac{1}{2} - \frac{1}{q})$, $\Re \sigma_1 = 0$;

(b) $2 = p < q < 6$, $\Re \sigma = 0$, $\Re \sigma_1 = \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$;

(c) $2 < p = q < 6$, $\Re \sigma = \Re \sigma_1 > \frac{3}{2}(\frac{1}{2} - \frac{1}{q})$.

For the case (a), since $\chi_\infty(D) \in \mathcal{B}(L^2; L^2)$, the inhomogeneous estimate in Lemma 2.2 and Hölder imply for a non-endpoint admissible pair $(m, p)$

$$\left|T_{j, \infty}^{\sigma, \sigma_1}(F, G)\right| \lesssim \left\|\int_{t-2^j \leq s \leq t-2^{j-1}} e^{\pm iD(t-s)} D^{-\sigma} F(s) ds \right\|_{L_t^p L_x^\sigma} \|G\|_{L_t^q L_x^{\sigma_1}}$$

$$\lesssim \|F\|_{L_t^m L_x^p} \|G\|_{L_t^q L_x^{\sigma_1}}$$

$$\lesssim \|F\|_{L_t^1 L_x^p} \|G\|_{L_t^1 L_x^{\sigma_1}} 2^{j(\frac{1}{p} - \frac{1}{m})}. \hfill (5.1)$$

In the case (b), the same arguments as (a) yield

$$\left|T_{j, \infty}^{\sigma, \sigma_1}(F, G)\right| \lesssim \|F\|_{L_t^p L_x^{\sigma_1}} \|G\|_{L_t^q L_x^\sigma} 2^{j(\frac{1}{2} - \frac{1}{m})}. \hfill (5.2)$$

For the case (c), by Corollary 3.1, we have for $p \in (2, 6)$ and $q \in (2, 6)$

$$\left|T_{j, \infty}^{\sigma, \sigma_1}(F, G)\right| \lesssim \sup_t \left\|\int_{t-2^j \leq s \leq t-2^{j-1}} \chi_\infty(D) \tilde{D}^{-\sigma_1} D^{-\sigma} F(s) ds \right\|_{L_t^q L_x^{\sigma_1}}$$

$$\lesssim 2^{-j\infty} \sup_t \left(\int_{t-2^j \leq s \leq t-2^{j-1}} \|F(s)\|_{L_x^p} ds \right) \|G\|_{L_t^1 L_x^{\sigma_1}}$$

$$\lesssim 2^{-j\infty} \|F(s)\|_{L_t^1 L_x^p} \|G\|_{L_t^1 L_x^{\sigma_1}} \|G\|_{L_t^q L_x^{\sigma_1}}.$$
Hence Hölder gives
\[
\left| T_{j, \infty}^{\sigma, \sigma} (F, G) \right| \lesssim 2^{-j \infty} \| F(s) \|_{L_t^q L_x^{q'}} \| G \|_{L_t^q L_x^{q'}}.
\] (5.3)

Interpolating (5.1), (5.2), (5.3) gives our lemma. \qed

References

[1] L. Andersson, N. Gudapati and J. Szeftel. Global regularity for the 2+1 dimensional equivariant Einstein-wave map system. arXiv preprint 2015.

[2] J. Avron, I. Herbst, B. Simon. Schrödinger operators with magnetic fields. I. General interactions. Duke Math. J., 45(4), 847-883, 1978.

[3] J.P. Anker and V. Pierfelice. Wave and Klein-Gordon equations on hyperbolic spaces. Anal. PDE, 7(4), 953-995, 2014.

[4] J.P. Anker, V. Pierfelice and M. Vallarino. The wave equation on hyperbolic spaces. J. Differential Equations, 252(10), 5613-5661, 2012.

[5] D. Borthwick and J.L. Marzuola. Dispersive Estimates for Scalar and Matrix Schrödinger Operators on $H^{n+1}$. Math. Phys. Anal. Geom., 18, 22, 2015.

[6] X.Chen, A. Hassell. Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds I: Resolvent construction at high energy. Comm. Partial Differential Equations, 41(3), 515-578, 2016.

[7] P. D’Ancona and L. Fanelli. Strichartz and smoothing estimates for dispersive equations with magnetic potentials. Comm. Partial Differential Equations, 33(6), 1082-1112, 2008.

[8] P. D’Ancona, L. Fanelli, L. Vega and N. Visciglia. Endpoint Strichartz estimates for the magnetic Schrödinger equation. J. Funct. Anal., 258(10), 3227-3240, 2010.

[9] P. D’ancona , V. Pierfelice. On the wave equation with a large rough potential. J. Funct. Anal., 227(1), 30-77, 2005.

[10] M.B. Erdogan, M. Goldberg, W. Schlag. Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $R^3$. J. Eur. Math. Soc. (JEMS), 10(2), 507-531, 2008.

[11] M.B. Erdogan, M. Goldberg, W. Schlag. Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions. Forum Math., 21(4), 687-722, 2009.
[12] M. Goldberg. Strichartz estimates for Schrödinger operators with a non-smooth magnetic potential. Discrete Contin. Dyn. Syst., 31(1), 109-118, 2011.

[13] S. Helgason. Groups and geometric analysis: integral geometry, invariant differential operators and spherical functions. Academic Press, AMS, 1994.

[14] E. Hebey. Sobolev spaces on Riemannian manifolds, Volume 1635 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996.

[15] A.D. Ionescu, C.E. Kenig. Well-posedness and local smoothing of solutions of Schrödinger equations. Mathematical research letters, 12(2), 193-206, 2005.

[16] A. Ionescu, B. Pausader, G. Staffilani. On the global well-posedness of energy-critical Schrödinger equations in curved spaces. Anal. PDE, 5(4), 705-746, 2012.

[17] K. Yajima. Schrödinger evolution equations with magnetic fields. J. Anal. Math., 56(1), 29-76, 1991.

[18] T. Kato and K. Masuda. Trotter’s product formula for nonlinear semigroups generated by the subdifferentials of convex functionals. J. Math. Soc. Japan, 30(1), 169-178, 1978.

[19] S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math., 46(9), 1221-1268, 1993.

[20] J. Krieger and W. Schlag. Concentration Compactness for critical wave maps. EMS Monographs. European Mathematical Society, Zürich, 2012.

[21] M. Keel and T.Tao. Endpoint strichartz estimates. Amer. J. Math., 120(5), 955-980, 1998.

[22] A. Lawrie. The Cauchy problem for wave maps on a curved background. Calc. Var. Partial Differential Equations, 45(3-4), 505-548, 2012.

[23] A. Lawrie, S.J. Oh and S. Shahshahani. Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space. J. Func. Anal., 271(11): 3111-3161, 2016.

[24] A. Lawrie, S.J. Oh, and S. Shahshahani. Equivariant wave maps on the hyperbolic plane with large energy. Math. Research Letters, 24(2), 449-479, 2017.

[25] A. Lawrie, S.J. Oh and S. Shahshahani. The Cauchy problem for wave maps on hyperbolic space in dimensions $d \geq 4$. Int. Math. Res. Not. IMRN, rnw272, 2016.
[26] A. Lawrie, S.J. Oh, and S. Shahshahani. Stability of stationary equivariant wave maps from the hyperbolic plane. Amer. J. Math. 139(4), 1085-1147, 2017.

[27] Z. Li. Asymptotic stability of large energy harmonic maps under the wave map from 2D hyperbolic spaces to 2D hyperbolic spaces. arXiv preprint July 2017.

[28] Z. Li. Asymptotic stability of solitons to 1D Nonlinear Schrödinger Equations in subcritical case. 2015 arXiv preprint, submitted.

[29] Z. Li, X. Ma, L. Zhao. Asymptotic stability of harmonic maps between 2D hyperbolic spaces under the wave map equation. II. Small energy case. 2017, arXiv preprint, submitted.

[30] N.S. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004.

[31] R. Melrose, A.S. Barreto, A. Vasy. Analytic Continuation and Semiclassical Resolvent Estimates on Asymptotically Hyperbolic Spaces. Comm. in Partial Differential Equations, 39, 452-511, 2014.

[32] J. Metcalfe and D. Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. Math. Ann., 353(4), 1183-1237, 2012.

[33] J. Metcalfe and M. Taylor. Nonlinear waves on 3D hyperbolic space. Trans. Amer. Math. Soc., 363(7), 3489-3529, 2011.

[34] J. Metcalfe and M. Taylor. Dispersive wave estimates on 3D hyperbolic space. Proc. Amer. Math. Soc., 140(11), 3861-3866, 2012.

[35] M. Reed and B. Simon. Methods of modern mathematical physics, Vol. 4. Academic, 1978.

[36] I. Rodnianski, W. Schlag. Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math., 155(3), 451-513, 2004.

[37] R.J. Stanton and P.A. Tomas. Expansions for spherical functions on noncompact symmetric spaces. Acta Math., 140(1), 251-276, 1978.

[38] J. Shatah and A. Tahvildar-Zadeh. On the stability of stationary wave maps. Comm. Math. Phys., 185(1), 231-256, 1997.

[39] B. Simon, Maximal and minimal Schrödinger forms, J. Operator Theory. 1, 37-47, 1979.

[40] J. Sterbenz and D. Tataru. Regularity of wave maps in 2+1 dimensions. Comm. Math. Phys., 298(1), 231-264, 2010.

[41] R. S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal., 52(1), 48-79, 1983.
[42] M.E. Taylor, Partial differential equations II: Qualitative studies of linear equations, Applied Mathematical Sciences, vol. 116, Springer-Verlag, 1996.

[43] T. Tao. Global Regularity of Wave Maps II. Small Energy in Two Dimensions. Comm. Math. Phys., 224(2), 443-544, 2001.

[44] T. Tao. Global Regularity of Wave Maps III-VII. arXiv preprint. 2008-2009.

[45] D. Tataru. Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation. Trans. Amer. Math. Soc., 353(2), 795-807, 2001.

[46] D. Tataru. On global existence and scattering for the wave maps equation. Amer. J. Math., 123(1), 37-77, 2001.

[47] A. Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov) 194(2), 381-513, 2013.

[48] A. Vasy. The wave equation on asymptotically de Sitter-like spaces. Adv. Math. 223(1), 49-97, 2010.

[49] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1945.

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