Newton type method for nonlinear Fredholm integral equations

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Abstract

This paper presents a numerical method for solving nonlinear Fredholm integral equations. The method is based upon Newton type approximations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key words: Fredholm integral equation, Newton type method, reliable space, sub-differential.

MSC: Primary 45G10; 47H99; 65J15.

1. Introduction and some preliminaries

The solution of the nonlinear Fredholm integral equation has been of considerable concern. This equation arises in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic and various physical and biological problems. A thorough discussion of the formulation of these models and the scientific structure are given in [7, 22, 23] and the references therein.

The nonlinear Fredholm integral equation is given in [23] as

\[ h(x) = f(x) + \lambda \int_0^1 G(x, t, h(t)) dt, \] (1)

where \( h(x) \) is an unknown function, the functions \( f(x) \) and \( G(x, t, h(t)) \) are analytic on \([0, 1]\). The Existence and uniqueness results for Eq. (1) may be
found in [7, 11, 14]. However, few numerical methods for Eq. (1) are known in the literature [23]. For the linear case, the time collocation method was introduced in [17] and projection method was presented in [12, 13]. In [2], the results of [17] have been extended to nonlinear Volterra-Hammerstein integral equations. In [15, 23], a technique based on the Adomian decomposition method was used for the solution of Eq. (1).

The next contexts from [19] will be used later on. Recall that the lower sub differential of a function $f : X \to [-\infty, +\infty]$ on a Banach space $X$ at $x \in D(f)$, is given by

$$\partial f(x) = \{x^* \in X^* : \forall \nu \in X < x^*, \nu \leq f'(x, \nu)\},$$

where $X^*$ is the dual space of $X$ and $f'$ is the lower sub derivative of $f$ at $x$, given by

$$f'(x, \nu) = \liminf_{t \to 0} \frac{f(x + tu) - f(x)}{t}, \quad \nu \in X.$$ 

And $\partial f$ is given by

$$\partial f = \{(x, x^*) : x \in D(f), \ x^* \in \partial f(x)\}.$$ 

A Banach space $X$ is said to be a reliable space, in short an R-space, if for any lower semi continuous function $f : X \to [-\infty, +\infty]$, for any convex Lipschitzian function $g$ on $X$, for any $x \in D(f)$ at which $f + g$ attains its infimum, and for any $\varepsilon > 0$, one has $0 \in \partial f(u) + \partial g(v) + \varepsilon B^*$, for some $u, v \in B_x$ such that $|f(u) - f(x)| < \varepsilon$.

We say that a function $f : X \to (-\infty, +\infty)$ on a Banach space $X$ is lower $T^1$ or of class $LT^1$ if it is lipschitzian, Gâteaux differentiable, and if its derivative $f' : X \to X^*$ is continuous when $X^*$ is endowed with the weak* topology.

A Banach space $X$ is said to be $LT^1$-bumpable if there exists a nonnull function of class $LT^1$ on $X$ with bounded support. Any Banach space whose norm is Gâteaux differentiable off 0 is $LT^1$-bumpble. The fact that any $LT^1$-bumpable Banach space is reliable is used in the following.

**Theorem 1.1.** Let $f : X \to [-\infty, +\infty]$ be a lower semi continuous on an R-space. If there exists a constant $k \geq 0$ such that for any $(x, x^*) \in \partial f$ one has $\|x^*\| \leq k$, then $f$ is Lipschitzian with rate $k$ on its domain $D(f)$.
2. Newton type method

Consider the nonlinear Fredholm integral equation is given in [23] as

\[ h(x) = f(x) + \lambda \int_0^1 G(x, t, h(t))dt. \]

Let \( f \) be continuous on \([0, 1]\), and continuous function \( G \) have continuous, bounded derivative on its third component.

We define operator \( F : C[0, 1] \to C[0, 1] \) by

\[ F(h)(x) = h(x) - f(x) - \lambda \int_0^1 G(x, t, h(t))dt, \quad h \in C[0, 1], \ x \in [0, 1], \]

and for each \( u \in C[0, 1] \), define operator \( T_{F,u} : C[0, 1] \to C[0, 1] \) as:

\[ T_{F,u}(h)(x) = \lim_{\varepsilon \to 0} \frac{F(h + \varepsilon u)(x) - F(h)(x)}{\varepsilon} \quad h \in C[0, 1], \ x \in [0, 1]. \]

So we have

\[ T_{F,u}(h)(x) = u(x) - \lambda \int_0^1 \frac{\partial G(x, t, h(t))}{\partial h}u(t)dt. \]

**Note.** The Operator \( T_{F,u} \) is continuous with respect to \( h \) and \( u \). Put

\[ H_u(h)(x) = h(x) - \frac{F(h)(x)}{T_{F,u}(h)(x)}, \quad h, u \in C[0, 1], x \in [0, 1]. \]

We say that \( F \) is \( u \)-smooth, if \( T_{F,u}(h)(x) \) is nonzero for each \( h \in C[0, 1] \) and \( x \in [0, 1] \). If \( F \) is \( u \)-smooth then the operator \( H_u : C[0, 1] \to C[0, 1] \) is well-defined.

**Remark.** Let \( F \) be 1-smooth. If there exists \( p \in C[0, 1] \) such that \( F(p) = 0 \), then for each \( u \in C[0, 1] \) and \( x \in [0, 1] \), we have

\[
T_{H_{1,u}}(p)(x) = \lim_{\varepsilon \to 0} \frac{H_{1}(p + \varepsilon u)(x) - H_{1}(p)(x)}{\varepsilon} \\
= \lim_{\varepsilon \to 0} u(x) - \frac{F(p + \varepsilon u)(x)}{\varepsilon} \frac{1}{T_{F,1}(p + \varepsilon u)(x)} \\
= u(x) - T_{F,u}(p)(x) \frac{1}{T_{F,1}(p)(x)}. 
\]
So, if \( F \) is 1-smooth and \( F(p) = 0 \), then \( T_{H_1}(p) = 0 \). Hence the hypothesis of continuity, implies, there exist neighborhoods of \( p \) and 1, respectively denoted by \( B_r(p) \) and \( B_\delta(1) \), for suitable \( r \) and \( \delta \), such that

\[
\sup_{x \in [0,1]} |T_{H_1}(h)(x)| \leq \frac{1}{2},
\]

for each \( h \in B_r(p) \) and \( u \in B_\delta(1) \).

Now for each \( x \in [0,1] \), define \( \varphi(x) \) from \( C[0,1] \) to real line, with

\[
\varphi(x)(h) = H_1(h)(x),
\]

where \( h \) is in \( C[0,1] \), and consider the next Lemma.

**Lemma 2.1.** Let \( F \) be 1-smooth, \( x \in [0,1] \), \( h \in B_r(p) \subseteq C[0,1] \) and \( h^* \in (C[0,1])^* \) are fix. If for each \( g \in C[0,1] \) there exist a neighborhood \( B_\gamma(g) \) and \( \varepsilon > 0 \), such that

\[
h^*(g) \leq \frac{\varphi(x)(h + \varepsilon u) - \varphi(x)(h)}{\varepsilon},
\]

for each \( u \in B_\gamma(g) \), then \( ||h^*|| \leq \frac{1}{2} \).

**Proof.** By hypothesis, \( h^*(g) \leq \frac{H_1(h + \varepsilon u)(x) - H_1(h)(x)}{\varepsilon} \), implies by attending \( \varepsilon \) to zero, \( h^*(g) \leq T_{H_1}(h)(x) \), for each \( u \in B_\gamma(g) \). Now let \( g \in C[0,1] \) and \( ||g - 1|| < \frac{\delta}{2} \), there exists \( \gamma_1 > 0 \) such that for each \( u \in B_\gamma_1(g) \), \( h^*(g) \leq T_{H_1}(h)(x) \). Put \( \gamma = \min\{\gamma_1, \frac{\delta}{2}\} \), hence for each \( u \in B_\gamma(g) \), we have

\[
||u - 1|| \leq ||u - g|| + ||g - 1|| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

and before Remark implies that \( h^*(g) < T_{H_1}(h)(x) \leq \frac{1}{2} \). If \( ||g|| = 1 \) and \( ||g - 1|| < \delta \) then \( h^*(g) \leq \frac{1}{2} \). So we assume that \( \delta < ||g - 1|| \), then the linearity of \( h^* \), by using of the convexity of closed unit ball in \( C[0,1] \), implies that there exist \( \alpha \in (0,1) \) and \( g' \in C[0,1] \), with \( ||g' - 1|| < \delta \), \( h^*(g') \leq h^*(1) \), and \( g = \frac{1}{\alpha}g' + (1 - \frac{1}{\alpha})1 \). Hence

\[
h^*(g) = \frac{1}{\alpha}h^*(g') + (1 - \frac{1}{\alpha})h^*(1) \leq \frac{1}{\alpha}h^*(1) + (1 - \frac{1}{\alpha})h^*(1) = h^*(1).
\]

But \( 1 \in B_\delta(1) \) and \( h^*(g) \leq \frac{1}{2} \). Using this fact, by linearity of \( h^* \) implies that there exists \( g \in C[0,1] \), with \( ||g|| = 1 \) and \( |h^*(g)| \leq \frac{1}{2} \), so \( ||h^*|| \leq \frac{1}{2} \) and our proof is completed. \( \blacksquare \)
Note. By considering theorem 5.4 of [6], we know that the $C[0, w_0]$, it can be endowed with Frechet differentiable norm and hence it can be a $LT^1$-bumpable space. So it can be a $R$-space. Because each $LT^1$-bumpable space is reliable[18, 19].

Theorem 2.2. If $x \in [0, 1]$, $(h, h^*) \in \partial \varphi(x)$ and $||h-p|| < r$, then $||h^*|| \leq \frac{1}{2}$.

proof. Since $(h, h^*) \in \partial \varphi(x)$, for each $g \in C[0, 1]$, we have $h^*(g) \leq \varphi'(x)(h, g)$ and before Lemma implies that $||h^*|| \leq \frac{1}{2}$. ■

Corollary 2.3. The function $\varphi(x)$ is Lipschitzian on $B_r(p)$ with constant $\frac{1}{2}$, for each $x \in [0, 1]$.

proof. Put $D\varphi(x) = B_r(p)$ as a subset of a reliable space. So theorem immediately implies the conclusion. ■

Now we are already to prove that the sequence

$$u_{n+1}(x) = u_n(x) - \frac{F(u_n)(x)}{T_{F,1}(u_n)}(x) \quad x \in [0, 1], \; n = 0, 1, 2,...$$

converges to the solution of Eq. (1).

Theorem 2.4. Let $F$ be 1-smooth, $p \in C[0, 1]$ and $F(p) = 0$, then there exists a neighborhood of $p$ in $C[0, 1]$ such that the above sequence $\{u_n\}$ is convergence to $p$, with each arbitrary started point $u_0$ of this neighborhood.

proof. Consider the neighborhood, which exposed to the before Remark. Now for each $u_0$ of this neighborhood, we have $||u_0 - p|| < r$, hence

$$||u_n - p|| \leq \sup_{x \in [0, 1]} |H_1(u_{n-1})(x) - H_1(p)(x)| = \sup_{x \in [0, 1]} |\varphi(x)(u_{n-1}) - \varphi(x)(p)| \leq \frac{1}{2} ||u_{n-1} - p|| \leq ... \leq \frac{1}{2}^n ||u_0 - p|| < r \frac{1}{2^n}$$

and the proof is completed. ■

3. Illustrative Examples

We applied the method presented in this paper and solved some test problems.
3.1. Example 1

Consider the nonlinear Fredholm integral equation

\[ u(x) = x^2 - \frac{1}{8}\cos(1) + \frac{1}{8} - \frac{1}{4}\int_0^1 t\sin(u(t))dt, \]

which has the exact solution \( u(x) = x^2 \).

Note that

\[ F(h)(x) = h(x) - x^2 + \frac{1}{8}\cos(1) - \frac{1}{8} + \frac{1}{4}\int_0^1 t\sin(h(t))dt, \]

\[ T_{F,1}(h)(x) = 1 + \frac{1}{4}\int_0^1 t\cos(h(t))dt \neq 0. \]

We applied the method presented in this paper and solved Eq. (2).

The corresponding Newton sequence is

\[ u_{n+1}(x) = u_n(x) - \frac{u_n(x) - (x^2 - \frac{1}{8}\cos(1) + \frac{1}{8} - \frac{1}{4}\int_0^1 t\sin(u_n(t))dt)}{1 + \frac{1}{4}\int_0^1 t\cos(u_n(t))dt} \]

and \( u_0(x) = 1. \)

In figure 1 the exact solution and approximate solution \( u_3(x) \) are plotted.

3.2. Example 2

Consider the nonlinear Fredholm integral equation

\[ u(x) = e^x - \frac{1}{2}x(\cos(1) - \cos(e)) + \frac{1}{2}\int_0^1 xe^t\sin(u(t))dt, \]

which has the exact solution \( u(x) = e^x \).

Note that

\[ F(h)(x) = h(x) - e^x + \frac{1}{2}x(\cos(1) - \cos(e)) - \frac{1}{2}\int_0^1 xe^t\sin(h(t))dt, \]

\[ T_{F,1}(h)(x) = 1 - \frac{1}{2}\int_0^1 xe^t\cos(h(t))dt \neq 0. \]

We applied the method presented in this paper and solved Eq. (3).
The corresponding Newton sequence is
\[ u_{n+1}(x) = u_n(x) - \frac{u_n(x) - (e^x - \frac{1}{2}x(\cos(1) - \cos(e)) + \frac{1}{2} \int_0^1 xe^t \sin(u_n(t)) dt}{1 - \frac{1}{2} \int_0^1 xe^t \cos(u_n(t)) dt} \]
and \( u_0(x) = 1 \).

In figure 2 the exact and approximate solution \( u_7(x) \) are plotted.

4. CONCLUSION

The properties of the Legendre wavelets together with the Gaussian integration method are used to reduce the solution of the mixed Volterra-Fredholm integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Moreover, only a small number of Legendre wavelets are needed to obtain a satisfactory result. The given numerical examples support this claim.
Figure 2: Exact and Approximate Solution of $u(x)$

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