On the equivalence between the cell-based smoothed finite element method and the virtual element method

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SUMMARY

We revisit the cell-based smoothed finite element method (SFEM) for quadrilateral elements and extend it to arbitrary polygons and polyhedra in 2D and 3D. We highlight the equivalence between the SFEM and the virtual element method (VEM). Based on the VEM, we propose a new stabilization approach to the SFEM when applied to arbitrary polygons and polyhedra. The accuracy and the convergence properties of the SFEM are studied with a few benchmark problems in 2D and 3D linear elasticity. Later, the SFEM is combined with the scaled boundary finite element method to problems involving singularity within the framework of linear elastic fracture mechanics in 2D. Copyright \textcopyright{} 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The finite element method (FEM) relies on discretizing the domain with non-overlapping regions, called ‘elements’. In the conventional FEM, the topology of the elements is restricted to triangles and quadrilaterals in 2D or tetrahedrals and hexahedrals in 3D. The use of such standard shapes, simplifies the construction of the approximation over the elements, however, this may require sophisticated (re-) meshing algorithms to either generate high-quality meshes or to capture topological changes. Moreover, the accuracy of the solution depends on the quality of the element employed: Lee and Bathe \cite{1} observed that the shape functions lose their ability to reproduce the displacement fields when the mesh is distorted. In an effort to overcome the limitations of the FEM, research has been focussed on:

- De-coupling geometry and analysis, for example, meshfree methods \cite{2,3}, PU enrichment \cite{4,5}, Immersed boundary method \cite{6}.
- Improving the element formulations
  - Strain smoothing \cite{7,8,9,10}
  - Unsymmetric formulations \cite{11,12}
  - hybrid Trefftz FEM \cite{13,14}
  - Polygonal FEM \cite{15}
- Coupling geometry and analysis, for example, isogeometric analysis \cite{16,17}, isogeometric boundary element method \cite{18,19}.

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• Boundary based methods, for example, boundary element method [20, 21], scaled boundary finite element method [22].
• Advanced mesh generators [23, 24, 25]

In this study, we focus and attempt to bridge the gap between two classes of methods which both focus on relaxing somewhat the constraints posed on the mesh used in finite element analysis, the strain smoothing technique [26] and the virtual element method [27].

1.1. Background

Liu et al. [26], extended the concept of stabilized conforming nodal integration (SCNI) [28] to finite element approximations and coined the resulting method the Smoothed Finite Element Method (SFEM). Liu et al. formulated a series of SFEM models: cell-based SFEM (CSFEM) [7], node based SFEM (NSFEM) [29], edge-based SFEM (ESFEM) [30], face-based SFEM (FSFEM) [31] and alpha-FEM [32]. All these SFEM use finite element meshes with linear interpolants, because the idea of the method is to improve the behaviour of simplex elements (triangles, tetrahedra) for which meshes are easier to generate automatically. Only one attempt to use smoothing for higher order approximation is known [10], which is the only counter example. In the CSFEM, the elements are divided into smoothing cells, over which the standard (compatible) strain field is smoothed. The method may also be seen as dividing the domain into smoothing domains, which may be constructed independently of the mesh. Yet, from a practical view point, it is simpler to use the mesh data structure to generate the smoothing domain, either from the element interior (cell-based), the nodes (node-based), the edges or face of the elements (edge/face based). This smoothing allows the volume integration to be transformed into a surface integration by employing the divergence theorem and hence the computation of the stiffness matrix requires only information on the boundary of the subcells. It should be noted that only the CSFEM employs quadrilateral elements, whilst all other SFEM models usually rely on simplex elements as reference mesh. When the CSFEM is used with triangular elements, the resulting stiffness matrix is identical to the conventional FEM. The convergence, stability, accuracy and important computational aspects of this method were studied in detail in [8]. The method was further extended to treat various problems in solid mechanics such as plates [33], shells [34], nearly incompressible elasticity [35, 36] to name but a few. Recently, the strain smoothing method was combined with enrichment methods in [10, 37] to model problems with strong discontinuities and singularities. However, certain difficulties still exist as discussed in [10].

On another related front, polygonal finite element methods (PFEM) have been receiving increasing attention. In PFEM, the domain can be discretized without needing to maintain a particular element topology (see Figure (1)). Moreover, this is advantageous in adaptive mesh refinement, where a straightforward subdivision of individual elements usually results in hanging nodes (see Figure (1)). Conventionally, this is eliminated by introducing additional edges/faces to retain conformity. This can be avoided if we can directly compute the stiffness matrices on polyhedral meshes with hanging nodes. Polygonal/polyhedral elements allow to treat all elements within a quad-tree/oct-tree mesh within a single paradigm. For example, elements of class (1) quadrilaterals, (2) pentagons and (3) hexagons can be assembled within a single routine.

In 1971, Wachspress [38] developed a method based on rational basis approximations for elements with an arbitrary number of sides. However, these elements were not easily used because of difficulties in constructing the basis functions, performing numerical integration, but more importantly generating polyhedral meshes on arbitrary domains. Thanks to advancements in mathematical software, viz., Mathematica © and Maple © and the pioneering work of Alwood and Cornes [15], Sukumar and Tabarraei [16], Dasgupta [17], to name a few and the recent drive from the computer graphics/science community [23, 24, 25], the discretization of the domain with finite elements having arbitrary number of sides/edges is possible for realistic application.

Once the mesh is generated and the basis functions are constructed, the conventional Galerkin procedure is normally employed to solve the governing equations over the polygonal/polyhedral meshes. However, the numerical integration of the terms in the stiffness matrix over polygonal/polyhedral meshes requires special techniques and often is complicated. Improving
Figure 1. Domain discretized with finite elements: (a) discretization with arbitrary polygons and (b) adaptive refinement leading to a quadtree mesh, where the ‘dotted’ line represents an inner boundary and the ‘highlighted’ elements are the elements with hanging nodes.

Numerical integration over polytopes has thus been the focus of significant attention in the recent literature [39, 40, 41, 42, 43]. The strain smoothing technique is another alternative to integrate over arbitrary polygons. Dai et al., [44] observed that on an arbitrary polygon with $n > 4$ (where $n$ is the number of sides of the polygon) must have a minimum of $n$ subcells to ensure stability. Recently, Natarajan et al., [45] and Ooi et al., [46] employed the scaled boundary finite element method over arbitrary polygons for linear elasticity and linear elastic fracture mechanics, respectively.

The virtual element method (VEM) was recently discovered in [27, 47, 48, 49]. The VEM has evolved from the mimetic difference methods [27]. It is a Galerkin approach, however, unlike the conventional FEM, the VEM does not require an explicit form for the basis functions to compute the stiffness matrix. Moreover, the VEM can be employed over arbitrary polygons and polyhedra. Within the framework of the VEM, the space within an element of decomposition contains certain polynomials that guarantee accuracy and additional functions for stability. The VEM also alleviates the numerical integration difficulty encountered in the conventional polygonal FEM. As the method does not require the knowledge of the shape functions in explicit form, the implementation is computationally less intensive.

1.2. Objective

We saw above that the strain smoothing technique can be applied to arbitrary polygons. However, it was observed in [45] that the strain smoothing technique over arbitrary polytopes yield less accurate solution when compared to other techniques, such as the conventional polygonal finite element method. Moreover, like the conventional finite elements, the polygonal elements requires special techniques, such as enrichment methods to treat problems with strong discontinuities and singularities. On the other hand, the virtual element method enables accurate and stable solution for completely arbitrary polygons. It therefore seems sensible to investigate the eventual connections between both classes of methods. More specifically, the main objectives of the paper are:

- to revisit the strain smoothing technique, in particular, the cell based SFEM and extend it to arbitrary polygons and polyhedra.
- investigate connections between the strain smoothing technique and the VEM.
- propose a new stabilization for the SFEM with one subcell inspired from the VEM, and which is applicable to arbitrary polygons.

There is no restriction on the shape of the element in VEM
• to study the accuracy and the convergence properties of the SFEM with the new stabilization technique.
• to couple the SFEM with the scaled boundary finite element method (SBFEM) (see Section 5.3 for a detailed discussion on the SBFEM) to study problems with strong discontinuities and singularities.

Throughout this paper, SFEM stands for the cell-based smoothed finite element method unless mentioned otherwise.

1.3. Outline

The paper is organized as follows. Section 2 revisits the basics of the cell-based smoothed finite element method as applied to 2D and 3D elasticity. Section 3 briefly reviews the virtual element method. The similarity between the SFEM and the VEM is discussed in Section 4. Some numerical examples are analyzed to demonstrate the accuracy and the convergence properties of the cell-based smoothed finite element method in Section 5 with a few problems taken from linear elasticity. The SFEM is combined with the SBFEM and the accuracy of the approach is demonstrated through benchmark problems in linear elastic fracture mechanics. The major conclusions and future research directions are summarised in the final section.

2. OVERVIEW OF THE SMOOTHED FINITE ELEMENT METHOD

2.1. Background

The strain-smoothing method (SSM) was proposed in [28] where the strain is written as the divergence of a spatial average of the standard (compatible) strain field –i.e. symmetric gradient of the displacement field. In the cell-based SFEM, the elements are divided into subcells as shown in Figure 2. The strain field $\tilde{\varepsilon}_{ij}^h$, used to compute the stiffness matrix is computed by a weighted average of the standard strain field $\varepsilon_{ij}^h$. At a point $x_C$ in an element $\Omega^h$,

![Figure 2. Subdivision of an element into subcells: (a) quadrilateral element and (b) arbitrary polygon.](image-url)
Figure 3. Calculation of the smoothed discretized gradient operator.

\[
\tilde{\varepsilon}_{ij}^h(x_C) = \int_{\Omega_h} \varepsilon_{ij}^h(x) \Phi(x - x_C) dx
\]  

(1)

where \( \Phi \) is a smoothing function that generally satisfies the following properties \[50\]

\[
\Phi \geq 0 \quad \text{and} \quad \int_{\Omega_h} \Phi(x) dx = 1
\]  

(2)

\[
\Phi = \frac{1}{A_C} \quad \text{in} \quad \Omega_C \quad \text{and} \quad \Phi = 0 \quad \text{elsewhere}
\]  

(3)

To use Equation (1), the subcell containing point \( x_C \) must first be located in order to compute the correct value of the weight function \( \Phi \).

The discretised strain field is computed, through the so-called smoothed discretised gradient operator \( \tilde{\mathbf{B}} \), defined by (see Figure 3 for a schematic representation of the construction)

\[
\tilde{\varepsilon}_{ij}^h(x_C) = \tilde{\mathbf{B}}_{ij}(x_C)q
\]  

(4)

where \( q \) contains unknown displacements coefficients defined at a node of a finite element. This definition is similar to the conventional FEM. The smoothed element stiffness matrix for element \( e \) is computed by the sum of the contributions of the subcells (Figure 3): \[5\]

\[
\tilde{\mathbf{K}}^e = \sum_{C=1}^{ne} \Omega_C \int_{\Omega_C} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C d\Omega = \sum_{C=1}^{ne} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_C \int_{\Omega_C} d\Omega = \sum_{C=1}^{ne} \tilde{\mathbf{B}}_C^T \mathbf{D} \tilde{\mathbf{B}}_CA_C
\]  

(5)

where \( ne \) is the number of the smoothing cells of the element. The strain displacement matrix \( \tilde{\mathbf{B}}_C \) is constant over each \( \Omega_C \) and is of the following form

\[
\tilde{\mathbf{B}}_C = \begin{bmatrix}
\tilde{\mathbf{B}}_{C1} & \tilde{\mathbf{B}}_{C2} & \tilde{\mathbf{B}}_{C3} & \cdots & \tilde{\mathbf{B}}_{Cn}
\end{bmatrix}
\]  

(6)

where for all shape functions \( I \in \{1, \ldots, n\} \), the 3 × 2 submatrix \( \tilde{\mathbf{B}}_{CI} \) represents the contribution to the strain displacement matrix associated with shape function \( I \) and cell \( C \) and writes (see Figure

\[5\]
∀I ∈ \{1, 2, \ldots, n\}, ∀C ∈ \{1, 2, \ldots, nc\}
\tilde{\mathbf{B}}_{CI} = \frac{1}{A_C} \int_{S_C} \mathbf{n}^T(x)N_I(x)dS
= \frac{1}{A_C} \int_{S_C} \begin{bmatrix} n_x & 0 \\ 0 & n_y \\ n_y & n_x \end{bmatrix} (x)N_I(x)dS \quad (7)

Note that since Equation (7) is computed on the boundary of Ω_C and one Gauß point is sufficient for an exact integration:
\tilde{\mathbf{B}}_{CI}(x_C) = \frac{1}{A_C} \sum_{b=1}^{nb} \begin{bmatrix} N_I(x^G_b) n_x & 0 \\ 0 & N_I(x^G_b) n_y \\ N_I(x^G_b) n_y & N_I(x^G_b) n_x \end{bmatrix} l^C_b \quad (8)

where x^G_b and l^C_b are the center point (Gauß point) and the length of Γ^C_b, respectively. Until now, no assumption was made on the shape of the element. The procedure outlined so far is general and is applicable to polygons of arbitrary shapes [44, 45]. Due to the process of strain smoothing, only the shape function is involved in the calculation of the field gradients and hence the stiffness matrix. In this study, we employ the simple averaging technique to compute the shape functions over arbitrary polygons. The construction of shape function is as follows: for a general polygonal element, the central point O is located by:
(x_o, y_o) = \frac{1}{n} \sum_i^n (x_i, y_i) \quad (9)

where n is the number of nodes of the polygonal element. The shape function at point O is given by [1/n · · · 1/n] with size 1 × n.

2.2. Extension to 3D

When the strain smoothing is used over three-dimensional domains, the volume integral is transferred to a surface integral. This surface integral is to be performed over the polygonal surfaces that build up the polyhedron. As in the 2D case, the smoothed element stiffness matrix is the sum over the subcells of the contribution from each subcell [§] (see Figure (4)), which is constant:
\tilde{K}^e = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \tilde{\mathbf{D}} \tilde{\mathbf{B}}_C d\Omega = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \tilde{\mathbf{D}} \tilde{\mathbf{B}}_C \int_{\Omega_C} d\Omega = \sum_{C=1}^{nc} \tilde{\mathbf{B}}_C^T \tilde{\mathbf{D}} \tilde{\mathbf{B}}_C V_C \quad (10)

The strain-displacement matrix \tilde{\mathbf{B}}_C is constant over each Ω_C and is of the following form:
\tilde{\mathbf{B}}_C = \begin{bmatrix} \tilde{\mathbf{B}}_{C1} & \tilde{\mathbf{B}}_{C2} & \tilde{\mathbf{B}}_{C3} & \cdots & \tilde{\mathbf{B}}_{Cn} \end{bmatrix} \quad (11)

\text{§Note that in 3D, the subcell is a volume}
where for all shape functions $I \in \{1, \ldots, n\}$, the $6 \times 3$ submatrix $\tilde{B}_{CI}$ represents the contribution to the strain displacement matrix associated with shape function $I$ and cell $C$ and writes:

$$\forall I \in \{1, 2, \ldots, n\}, \forall C \in \{1, 2, \ldots, nc\} \tilde{B}_{CI} = \frac{1}{V_C} \int_{S_C} n^T(x)N_I(x) dS$$

As in 2D, due to the process of strain smoothing, only the shape function is involved in the calculation of the field gradient and hence the stiffness matrix. In this study, over an arbitrary polygonal surface, we employ Wachspress interpolants [38]. In computing the strain-displacement matrix given by Equation (12) and the stiffness matrix, only the shape functions associated with the polygonal surface contribute to the integral. To evaluate the integral in Equation (12), two schemes are adopted [51]: (a) nodal quadrature and (b) conforming interpolant quadrature.

**Nodal quadrature** In this case, the surface integral of the shape function $N_I$ over any face of the polyhedral element is given by:

$$\int_{S_C} N_I(x) dS = N_I(x_I)A_I = A_I$$

where $A_I$ is the nodal weight of the node $I$, which is the area of the quadrilateral formed by the node, the centroid of the face and the mid-points of the edges containing the node. This is shown in Figure (5), however, this scheme is applicable only to the elements where star convexity is satisfied.

**Conforming Interpolant quadrature** In this case, to evaluate the surface integral of the shape function $N_I$, we adopt an interpolation scheme. For this purpose, the knowledge of the shape functions within the polygonal surface is required and in this study, we employ Wachspress interpolants [38] over the polygonal surface. To integrate the terms in the strain-displacement matrix,
the polygonal surface is mapped onto a regular polygon. The regular polygon is sub-divided into triangles and triangular quadrature rules over each triangle are employed to numerically integrate the terms in Equation (12) (see Figure (6)). This process involves a two level iso-parametric mapping of the surface and relies on the positivity of the Jacobian matrix involved in the transformation. The other possible approaches include: (a) complex mappings such as the Schwarz-Christoffel conformal mapping [52, 40]; (b) adaptively weighted numerical integration scheme [53]; (c) generalized Gaussian quadrature rules [41] and (d) Guass-Green cubature [54], to name a few.

Figure 5. Nodal quadrature, where \( I \) is any node, \( A_I \) is the area formed by the node, the centroid \( C \) and the mid-points of the edges containing the node \( I \).

Figure 6. Surface integration scheme. The polygonal surface of a polyhedron is mapped onto a regular polygon. The regular polygon is then sub-divided into triangles and each of those sub-triangles is then mapped onto a standard triangle. Quadrature rules over the triangles are the used for numerical integration. ‘Filled’ circles denote the nodes.
3. BASICS OF THE VIRTUAL ELEMENT METHOD

The virtual element method can be seen as a generalization of the finite element method to arbitrary polygons and polyhedra. The VEM does not require quadrature formulae to compute the stiffness matrix nor an expression for the basis functions. The explicit computation of the basis functions is actually not needed and this is the reason of the word ‘Virtual’ in the VEM \[55\]. The important ingredient is the operator $\Pi^V$ that relates to the bilinear form of the problem. Once this is known, the local element stiffness matrix can be computed. It is beyond the scope of this paper to discuss the details of the VEM as applied to scalar and elasticity problems. Only important equations pertaining to the computation of the stiffness matrix associated to an arbitrary polygon/polyhedron is given in this section. Interested readers are referred to the work of Beirão Da Veiga et al., \[55\] for scalar problems and Beirão Da Veiga et al., \[56\] and Gain et al., \[51\] for three dimensional elasticity, where the method is dealt with in great detail. In this section, we only present the final expression to compute the stiffness matrix by the VEM in to three dimensional linear elasticity. For more detailed derivation and discussion, interested readers are referred to the work of Gain et al., \[51\].

The expression for the stiffness matrix can be written as:

$$K^E_h = \frac{|E|W_C W_C^T}{\text{Consistency term}=K_{\text{const}}} + \frac{(I - P_p)^T S^E (I - P_p)}{\text{Stability term}=K_{\text{stab}}}$$ \hspace{1cm} (14)

where $|E|$ is the measure of the polygon and

$$P_p = P_R + P_C$$ \hspace{1cm} (15)

and

$$P_R = N_R W_R^T$$
$$P_C = N_C W_C^T$$ \hspace{1cm} (16)

The block $3I - 2$: $3I$ rows of $N_R$ and $N_C$ are expressed as:

$$N_R(3I - 2: 3) = \begin{bmatrix} 1 & 0 & 0 & (x_I - \overline{x})_2 & 0 & (x_I - \overline{x})_3 \\ 0 & 1 & 0 & -(x_I - \overline{x})_1 & (x_I - \overline{x})_3 & 0 \\ 0 & 0 & 1 & 0 & -(x_I - \overline{x})_2 & (x_I - \overline{x})_1 \end{bmatrix}$$

$$N_C(3I - 2: 3) = \begin{bmatrix} (x_I - \overline{x})_1 & 0 & 0 & (x_I - \overline{x})_2 & 0 & (x_I - \overline{x})_3 \\ 0 & (x_I - \overline{x})_2 & (x_I - \overline{x})_1 & (x_I - \overline{x})_3 & 0 \\ 0 & 0 & (x_I - \overline{x})_3 & 0 & (x_I - \overline{x})_2 & (x_I - \overline{x})_1 \end{bmatrix}$$ \hspace{1cm} (17)

where $x_I$ is the coordinate of the node and $\overline{x}$ is the polyhedron centroid. The block $3I - 2$: $3I$ rows of $W_R$ and $W_C$ are expressed as:

$$W_R(3I - 2: 3) = \begin{bmatrix} 1/n & 0 & 0 & (q_I)_2 & 0 & -(q_I)_3 \\ 0 & 1/n & 0 & -(q_I)_1 & (q_I)_3 & 0 \\ 0 & 0 & 1/n & 0 & -(q_I)_2 & (q_I)_1 \end{bmatrix}$$

$$W_C(3I - 2: 3) = \begin{bmatrix} 2(q_I)_1 & 0 & 0 & (q_I)_2 & 0 & (q_I)_3 \\ 0 & 2(q_I)_2 & 0 & (q_I)_1 & (q_I)_3 & 0 \\ 0 & 0 & 2(q_I)_3 & 0 & (q_I)_2 & (q_I)_1 \end{bmatrix}$$ \hspace{1cm} (18)

where the subscript indicates the component of the associated vector and

$$q_I = \frac{1}{2|E|} \int_{\partial E} N_I n \ d\Gamma$$
where \( S^E = \alpha I \) and \( \alpha = \alpha^* \text{trace}(|E|W_CDW^T_C) \) is a scaling coefficient and \( n \) is the unit outward normal vector. It can be seen that the computation of the stiffness matrix involves computing the matrices \( N_R, N_C, W_R \) and \( W_C \). The calculation of the matrices \( W_R \) and \( W_C \) involves computing the surface integral of the basis functions. This can be computed by employing one of the techniques discussed in the previous section.

4. EQUIVALENCE BETWEEN THE SFEM AND THE VEM

In this section, we shall attempt to demonstrate the equivalence between the SFEM and the VEM, first by presenting a generic expression and then by computing the stiffness matrix of a generic polygonal element. We do this for the following model problem in two dimensions:

\[
\Delta u = f \text{ in } \Omega; \quad u = 0 \text{ on } \partial \Omega
\]  

where \( \Omega \in \mathbb{R}^2 \). The corresponding variational formulation reads: find \( u \in V : = H^1_0(\Omega) \) such that:

\[
\forall v \in V: a(u, v) = (f, v)
\]

where \((\cdot, \cdot)\) represents the scalar inner product in \( L^2 \) and \( a(u, v) = (\nabla u, \nabla v) \). The VEM for the above bilinear form starts by defining a projection operator that follows the orthogonality condition [48]:

\[
\forall p_k \in P(E) \quad (\nabla p_k, \nabla (\Pi_v v_h - v_h)) = 0
\]

where \( v_h \in V(E) \) is the finite element space and \( P(E) \) is the space of polynomials over a polygonal element \( E \) which has the basis \( m_\alpha \), where

\[
m_\alpha := \left( \frac{x - x_D}{h_D} \right)^\alpha
\]

where \( \alpha = (\alpha_1, \alpha_2), |\alpha| \leq k, k \) is the degree of the polynomial and \( x = (x, y) \), \( x_D \) and \( h_D \) is the centroid and the diameter of the element, respectively. When \( k = 1 \), Equation (21), becomes [48]

\[
\nabla p_1 \cdot \nabla (\Pi_v v_h) = \frac{1}{|E|} \int_E \nabla v_h
\]

\[
= \frac{1}{|E|} \int_E \nabla v_h = \nabla (\Pi_v v_h) = : g(v_h)
\]

where we have used \( p_1 = x_1 \) and \( p_1 = x_2 \). Hence,

\[
\Pi_v v_h = x \cdot g(v_h) + \text{constant}
\]

The local stiffness matrix of the virtual element method is written as:

\[
(K^{VEM})_{IJ} = (\nabla \Pi_v \phi_I, \nabla \Pi_v \phi_J) + \text{additional terms for stability}
\]

The first term in the above equation is called the consistency term, which by using Equation (23) can be written as:

\[
(\nabla \Pi_v \phi_I, \nabla \Pi_v \phi_J) = |E| g(\phi_I) \cdot g(\phi_J)
\]

when \( \phi_I \) is linear on the edge of the element,

\[
g(\phi_I) = \frac{1}{|E|} \int_E \nabla \phi_I = \frac{1}{2|E|} (\ell_{I-1} n_{I-1} + \ell_I n_I)
\]
In case of the SFEM, the smoothed gradient matrix is given by:

\[
\mathbf{B} = \begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix}
\]

\[
\tilde{\mathbf{B}} = \frac{1}{AC} \int_{S_C} \mathbf{n}^T N_I(x) \, dS
\]  

(25)

where the integration is over the subcell boundary. When \( S_C = 1 \) and when the shape functions are linear on the boundary, the gradient matrix is given by:

\[
\tilde{\mathbf{B}} = \frac{1}{2AC} \left\{ \ell_{i-1} n_{i-1}^x + \ell_i n_i^x \right\} \\
\frac{1}{2AC} \left\{ \ell_{i-1} n_{i-1}^y + \ell_i n_i^y \right\}
\]

(26)

By comparing the above equation with Equation (24), we can easily see that the gradient matrix obtained by the smoothing technique is identical to the consistency term of the VEM.

Next, we demonstrate the equivalence between the methods with two worked examples by computing the stiffness matrix of (a) a quadrilateral element and (b) a pentagon. For comparison, we also present the stiffness matrix computed by conventional FEM by employing Gaussian quadrature. The following expressions are used for the respective methods, viz., the FEM, the SFEM and the VEM, to compute the stiffness matrix:

- \( \mathbf{K}^{\text{FEM}} = \int \mathbf{B}^T \mathbf{B} \, d\Omega \).
- \( \mathbf{K}^{\text{SFEM}} = \sum_{C=1}^{nc} \int_{\Omega_C} \tilde{\mathbf{B}}^T \tilde{\mathbf{B}} C \, d\Omega \).
- \( \mathbf{K}^{\text{VEM}} = \frac{\mathbf{R}\mathbf{R}^T}{|E|} + (\mathbf{I} - \Pi)^T (\mathbf{I} - \Pi) \)

where \( \mathbf{B} \) is the gradient matrix and \( \tilde{\mathbf{B}} \) is the smoothed gradient matrix given by:

\[
\mathbf{B} = \begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix}
\]

\[
\tilde{\mathbf{B}} = \frac{1}{AC} \int_{S_C} \mathbf{n}^T N_I(x) \, dS
\]  

(27)

and \( \Pi = \Pi + \Pi_o (\mathbf{I} - \tilde{\mathbf{I}}) \) and \( |E|\tilde{\mathbf{I}} = \mathbf{N}\mathbf{R}^T \). The matrices \( \mathbf{R} \) and \( \mathbf{N} \) are given by \([49, 48]\):

\[
\mathbf{R} = \frac{1}{2} \begin{bmatrix}
\ell_n n_n + \ell_1 n_1 \\
\ell_1 n_1 + \ell_2 n_2 \\
\vdots \\
\ell_{n-1} n_{n-1} + \ell_n n_n
\end{bmatrix} \\
\mathbf{N} = \begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\vdots & \vdots \\
x_n & y_n
\end{bmatrix}
\]

where \( \ell_i (i = 1, \cdots, n) \) is the length of edge \( i \), \( x_n, y_n \) are the coordinates of the vertex of the polygon, \( |E| \) is the measure of the polygon.

4.1. Stiffness matrix for the unit square

In this case, consider a unit square \([0, 1] \times [0, 1]\). We compute the stiffness matrix using the conventional FEM with bilinear shape functions, the VEM and the SFEM with both one and several subcells.

Finite element The stiffness matrix computed from the classical bilinear finite elements with reduced integration (i.e., one Gauss point at the center of the element) and full integral (four Gauss
Virtual element  In this case, we use the order of the monomial \( k = 1 \). This implies that the shape functions on the boundary of the elements are linear. As noted in Section 3, the stiffness matrix computed from the VEM has two parts (see Equation (14)): (a) the consistency term and (b) the stability term. The consistency and the stability terms are given by:

\[
\begin{align*}
K_{\text{const}}^{\text{VEM}} &= \frac{1}{2} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} \\
K_{\text{stab}}^{\text{VEM}} &= \frac{1}{4} \begin{bmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1
\end{bmatrix}
\end{align*}
\]  

(29)

and the final stiffness matrix is computed by adding the consistency term and the stability term:

\[
K^{\text{VEM}} = \frac{1}{12} \begin{bmatrix}
9 & -3 & -3 & -3 \\
-3 & 9 & -3 & -3 \\
-3 & 9 & -3 & -3 \\
-3 & -3 & -3 & 9
\end{bmatrix}
\]  

(30)

SFEM  In this case, we use one and two subcells. The stiffness matrices with one subcell and two subcells are given by:

\[
\begin{align*}
K_{\text{SC1Q4}}^{\text{SFEM}} &= \frac{1}{2} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} \\
K_{\text{SC2Q4}}^{\text{SFEM}} &= \frac{1}{16} \begin{bmatrix}
9 & -1 & -7 & -1 \\
-1 & 9 & -1 & -7 \\
-7 & -1 & 9 & -1 \\
-1 & -7 & -1 & 9
\end{bmatrix}
\end{align*}
\]  

(31)

4.2. Stiffness matrix for the pentagon

The coordinates of the pentagon are: \([(0, 0), (3, 0), (3, 2), (3/2, 4), (0, 4)]\). The consistency and the stability term for the VEM are given by:

\[
\begin{align*}
K_{\text{const}}^{\text{VEM}} &= \begin{bmatrix}
0.5952 & 0.0238 & -0.4881 & -0.4048 & 0.2738 \\
0.0238 & 0.3095 & 0.0833 & -0.1190 & -0.2976 \\
-0.4881 & 0.0833 & 0.4345 & 0.2976 & -0.3274 \\
-0.4048 & -0.1190 & 0.2976 & 0.3095 & -0.0833 \\
0.2738 & -0.2976 & -0.3274 & -0.0833 & 0.4345
\end{bmatrix} \\
K_{\text{stab}}^{\text{VEM}} &= \begin{bmatrix}
0.7422 & -0.1966 & -0.3412 & -0.2578 & 0.0534 \\
-0.1966 & 0.7422 & -0.3412 & -0.1354 & -0.0690 \\
-0.3412 & -0.3412 & 0.9896 & 0.0364 & -0.3437 \\
-0.2578 & -0.1354 & 0.0364 & 0.8646 & -0.5078 \\
0.0534 & -0.0690 & -0.3437 & -0.5078 & 0.8672
\end{bmatrix}
\end{align*}
\]  

(32)

and the stiffness matrix computed by employing the smoothing technique over the pentagon with one subcell is:

\[
\begin{align*}
K_{\text{onecell}}^{\text{SFEM}} &= \begin{bmatrix}
0.5952 & 0.0238 & -0.4881 & -0.4048 & 0.2738 \\
0.0238 & 0.3095 & 0.0833 & -0.1190 & -0.2976 \\
-0.4881 & 0.0833 & 0.4345 & 0.2976 & -0.3274 \\
-0.4048 & -0.1190 & 0.2976 & 0.3095 & -0.0833 \\
0.2738 & -0.2976 & -0.3274 & -0.0833 & 0.4345
\end{bmatrix}
\end{align*}
\]  

(33)
From the two examples, presented above, it can be observed that the consistency term of the VEM, stiffness matrix using SFEM with one subcell coincides with the conventional FEM with reduced integration. However, after the addition of the stability term in the VEM or increasing the number of subcells in the SFEM, we observe that the stiffness matrix computed from these approaches are different. It is also noted that the consistency term of the VEM is similar to the SFEM with one subcell. This observation is also true in the case of the pentagon.

The SFEM starts with an assumption that the strain is constant within the subcell and then employs the divergence theorem to convert the domain integral into a surface integral. This suppresses the need to compute the derivatives of the shape functions and the stiffness matrix is computed from the information available on the boundary. When linear elements are employed on the boundary, this assumption holds true. However, for higher order elements, this assumption breaks down. This was observed in [10], when the strain smoothing technique was used for Q8 and Q9 and enriched approximations.

In the case of the VEM, no such assumption of constant strain is made over the element. However, the method starts by assuming the variation of the shape functions on the boundary of the element. The method then employs the divergence theorem after defining the projection operators [55, 48]. Hence, when linear variation is assumed, the VEM with consistency term and the SFEM with one subcell coincide. This is also true when the number of sides is greater than 4, as noted in the previous example. The stiffness matrix of the VEM has two parts: (a) the first term ensures consistency and this term must be computed exactly and (b) the second term ensures stability, this can be approximated. The important features of the stability term are: (a) it should scale like the consistency term and (b) should be positive definite. Different choices of stability terms are possible as discussed in [55, 56, 51]. However, for this study, we employ the following stability term based on the work of Beirão Da Veiga et al., [55, 56]:

$$K_2 = \alpha P$$ (34)

where $\alpha = \alpha^* \text{trace}(K_{\text{const}}^\text{VEM})$ and $P$ is the orthogonal projection operation and is chosen as:

$$P = I - T (T^T T)^{-1} T^T$$ (35)

and the matrix $T$ is the modified nodal coordinate matrix $N$ of dimension $3n \times 12$ in the case of 3D:

$$T_{3l-2: 3l} = \begin{bmatrix} 1 & 0 & 0 & y_l & 0 & -z_l & x_l & 0 & 0 & y_l & 0 & z_l \\ 0 & 1 & 0 & -x_l & z_l & 0 & 0 & y_l & 0 & x_l & z_l & 0 \\ 0 & 0 & 1 & 0 & -y_l & x_l & 0 & 0 & z_l & 0 & y_l & x_l \end{bmatrix}$$ (36)

where $\alpha^*$ is a scaling coefficient chosen based on a parametric study conducted in the next section. We conclude that the CSFEM is a special case of the more general VEM. Instead of increasing the number of subcells, we add to the one-subcell, the stability term borrowed from the VEM. In this present study, we employ the following form for the stiffness matrix:

$$K^h = K_1 + K_2$$ (37)

where

$$K_1 = \int_{\Omega_C} \hat{B}_C^T D \hat{B}_C \, d\Omega$$ (38)

is computed by employing the strain smoothing technique and

$$K_2 = \alpha P$$ (39)

Since the consistency term of the VEM and the SFEM with one subcell are identical, we have $\alpha = \alpha^* \text{trace}(K_1)$. This approach enables the use of the most attractive of all cell-based smoothed finite element method (with one subcell), whilst, guaranteeing stability.
5. NUMERICAL EXAMPLES

In the first part of this section, we employ the cell-based smoothed finite element method with the new stabilization approach to two-dimensional benchmark problems in linear elasticity. The results from the new approach are compared with analytical solution where available and with the conventional FEM. The SFEM with stabilization is applied to polygonal elements in 2D and the accuracy and the convergence properties are studied in detail. Later, the proposed SFEM with stabilization is extended to 3D problems with hexahedral and polyhedral elements. Again, the accuracy and the convergence properties of the proposed method are studied with a patch test and the new stabilization approach to two-dimensional benchmark problems in linear elasticity. The results are compared with available solutions in the literature.

The built-in Matlab® function voronoin and Matlab® functions in PolyTop [57] for building mesh-connectivity are used to create the polygonal meshes. For polyhedral meshes, the open source software Neper [58] is employed for building the mesh-connectivity. For the purpose of error estimation and convergence studies, the error, \( L^2 \) and \( H^1 \) norms are used. The displacement norm is given by:

\[
\|u - u^h\|_{L^2(\Omega)} = \sqrt{\int_{\Omega} [(u - u^h) \cdot (u - u^h)] \, d\Omega}
\]  

where \( u^h \) is the numerical solution and \( u \) is the analytical or a reference solution. The energy norm is given by:

\[
\|u - u^h\|_{H^1(\Omega)} = \sqrt{\int_{\Omega} [(\varepsilon - \varepsilon^h)D(\varepsilon - \varepsilon^h)] \, d\Omega}
\]  

5.1. Applications to two dimensional problems

5.1.1. Cantilever beam A two-dimensional cantilever beam subjected to a parabolic shear load at the free end is examined as shown in Figure (7). The geometry is: length \( L = 8m \), height \( D = 4m \). The material properties are: Young’s modulus, \( E = 3e^7 \) N/m², Poisson’s ratio \( \nu = 0.3 \) and the parabolic shear force \( P = 250N \). The exact solution for displacements are given by:

\[
u(x, y) = \frac{Py}{6EI} \left[ (9L - 3x)x + (2 + \nu) \left( y^2 - \frac{D^2}{4} \right) \right]
\]

\[
v(x, y) = -\frac{P}{6EI} \left[ 3\nu y^2 (L - x) + (4 + 5\nu) \frac{D^2x}{4} + (3L - x)x^2 \right]
\]  

where \( I = D^3/12 \) is the moment of inertia, \( \varepsilon = E, \nu = \nu \) and \( \varepsilon = E/(1 - \nu^2), \nu = \nu/(1 - \nu) \) for plane stress and plane strain, respectively. The domain is discretized with two different mesh types: (a) structured quadrilateral elements (8×4, 16×8, 32×16, 64×32) and (b) polygonal elements. Figure (8) shows a sample polygonal mesh used for this study. Before demonstrating the convergence and the accuracy of the SFEM with stabilization, we investigate the influence of the scaling parameter \( \alpha^* \). Figure (9) shows the influence of the scaling parameter \( \alpha^* \) on the relative error in the \( L^2 \) and \( H^1 \) norm. It is observed that the relative error attains a minimum value for \( \alpha^* = 0.1 \).

The numerical convergence of the relative error in the displacement norm and the relative error in the energy norm is shown in Figure (10) for structured quadrilateral elements. The problem is solved with conventional SFEM with one (SC1Q4) and two subcells (SC2Q4) and with the proposed SFEM with and without stabilization. It is observed that the SC1Q4 and SFEM with no stabilization yield similar results, as seen in the scalar example in Section 4. However, the SC2Q4 and SFEM with stabilization yield different results. This can be attributed to the choice of the scaling parameter \( \alpha^* \). It can be seen that with mesh refinement, all the approaches converge with optimal rate. For a proper choice of \( \alpha^* \), it is seen that SFEM with stabilization yields more accurate results than the SC2Q4.
Figure 11 shows the convergence of the displacement and the energy norm with mesh refinement. In this case, one subcell per polygonal element cannot be used, as this would lead to spurious energy modes [44]. Hence, in this study, for the conventional SFEM and the polygonal FEM, we sub-triangulate the polygonal element and integrate over each sub-triangle. In the case of the proposed approach, we employ one subcell and add the stabilization term (Equation (34)). It can be observed that the proposed SFEM with stabilization yields more accurate results compared to the conventional SFEM with triangulation. It is also seen that the polygonal FEM and the proposed approach converges at an optimal rate for both the displacement norm and the energy norm.

Figure 7. Cantilever beam: Geometry and boundary conditions.

Figure 8. Cantilever beam: Typical polygonal mesh employed in this study.
$S. NATARAJAN, S. BORDAS, E.T. OOI$

\begin{align*}
\sigma_{11}(r, \theta) &= 1 - \frac{a^2}{r^2} \left( \frac{3}{2} (\cos 2\theta + \cos 4\theta) \right) + \frac{3a^4}{r^4} \cos 4\theta \\
\sigma_{22}(r, \theta) &= -\frac{a^2}{r^2} \left( \frac{1}{2} (\cos 2\theta - \cos 4\theta) \right) - \frac{3a^4}{r^4} \cos 4\theta \\
\sigma_{12}(r, \theta) &= -\frac{a^2}{r^2} \left( \frac{1}{2} (\sin 2\theta + \sin 4\theta) \right) + \frac{3a^4}{r^4} \sin 4\theta
\end{align*}

where $a$ is the radius of the hole. Owing to symmetry, only one quarter of the plate is modeled.

\section{L-shaped domain under mode I loading}
In this example, consider the singular problem of a finite portion of an infinite domain with a reentrant corner. The model is loaded on part of the boundary, which is indicated by discontinuous thick lines in Figure 15. The tractions correspond to the first terms of the asymptotic expansion that describes the exact solution under mixed mode loading conditions around the singular vertex. The exact displacement and stress fields for this singular elasticity problem can be found in [59]. Exact values of the generalised stress intensity factors (GSIF) [59] under mode I were taken as $K_I = 1$ and $K_{II} = 0$. The material parameters are Young’s modulus $E = 1000$, and Poisson’s ratio $\nu = 0.3$ and the domain is discretized with polygonal elements. The problem is solved by the conventional polygonal FEM and the SFEM with stabilization. The convergence of the relative error in displacement with mesh refinement is shown in Figure 16. It is observed that both the approaches converge with mesh refinement. However,
Figure 10. Bending of thick cantilever beam: Convergence results for (a) the relative error in the displacement norm ($L^2$) and (b) the relative error in the energy norm. The rate of convergence is also shown, where $m$ is the average slope. The domain is discretized with Q4 elements.
Figure 11. Bending of thick cantilever beam: Convergence results for (a) the relative error in the displacement norm ($L^2$) and (b) the relative error in the energy norm. The rate of convergence is also shown, where $m$ is the average slope. The domain is discretized with arbitrary polygonal elements.
Figure 12. Infinite plate with a circular hole.

Figure 13. Plate with a circular hole: domain discretized with polygonal elements: (a) 100 elements; (b) 200 elements; (c) 400 elements and (d) 800 elements.
5.2. Applications in three dimensional problems

5.2.1. Stability condition Before we proceed to study the accuracy and the convergence of the proposed method, we first investigate the stability condition by computing the eigenvalues of the hexahedral element shown in Figure 17. The eigenvalues of the stiffness matrix computed by using the trilinear shape functions are \( \lambda = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9.92, 11.64, 13.60, 16.37, 16.62, 46.98\} \) and with stabilization (\( \alpha = 0.1 \)) are: \( \lambda = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9.32, 9.45, 10.42, 10.64, 11.30, 11.49, 11.51, 11.51, 11.51, 11.51, 11.51, 11.53, 12.41, 15.09, 17.36, 17.72, 47.45\} \) It can be seen that SFEM with stabilization, like the FEM can capture the six zero energy modes corresponding to the physical rigid body modes. This indicates that the stiffness matrix is full rank and does not have any spurious energy modes. However, with one subcell and without stabilization, the eigenvalues of the stiffness matrix has 12 additional zero energy modes, these are non-physical. This is identical to the FEM with one integration point. Traditionally, for hexahedral elements, these are suppressed by stabilization procedures. In [9, 60], the authors suppress the zero energy modes by adding the stiffness matrix computed with 8 subcells. In the present study, we add the stability term as given by Equation (34).

5.2.2. Patch Test A three dimensional patch test with warped elements suggested in [61] is considered. The patch of elements shown in Figure 18 is tested with the following displacement

...
Figure 15. L-shaped domain.

Figure 16. L-shaped domain under mode I loading: Convergence results for the relative error in the displacement norm ($L^2$). The rate of convergence is also shown, where $m$ is the average slope.
field applied on the outer boundary:

\[
\begin{align*}
    u(x, y, z) &= \frac{1}{2}(2x + y + z)10^{-3} \\
    v(x, y, z) &= \frac{1}{2}(x + 2y + z)10^{-3} \\
    w(x, y, z) &= \frac{1}{2}(x + y + 2z)10^{-3}
\end{align*}
\] (44)

The coordinates of the interior nodes are given in Table I. The interior nodes are enclosed within a unit cube. It is observed that both the FEM with 3×3×3 Gaussian quadrature and cell-based smoothing technique with stabilization pass the patch test to machine precision.

Table I. Three dimensional patch test: coordinates of inner nodes. The inner nodes are enclosed within a unit cube [61].

| Node number | x     | y     | z     |
|-------------|-------|-------|-------|
| 1           | 0.249 | 0.342 | 0.192 |
| 2           | 0.826 | 0.288 | 0.288 |
| 3           | 0.850 | 0.649 | 0.263 |
| 4           | 0.273 | 0.750 | 0.230 |
| 5           | 0.320 | 0.186 | 0.643 |
| 6           | 0.677 | 0.305 | 0.683 |
| 7           | 0.788 | 0.693 | 0.644 |
| 8           | 0.165 | 0.745 | 0.705 |

5.2.3 *Cantilever beam under shear load* Consider a cantilever beam loaded in shear. The domain \( \Omega \) for this problem is \([-1, 1] \times [-1, 1] \times [0, L] \). The material is assumed to be isotropic with Young’s modulus, \( E = 1 \text{ N/m}^2 \) and Poisson’s ratio \( \nu = 0.3 \). The beam is subjected to a shear force \( F \) at \( z = L \) and at any cross section of the beam, the following conditions are satisfied:

\[
\int_{-a}^{b} \int_{-a}^{b} \sigma_{yz} \, dx \, dy = F \quad \int_{-a}^{b} \int_{-a}^{b} \sigma_{zz} \, dx \, dy = Fz
\]

(45)
The Cauchy stress field is given by [62]:

\[
\begin{align*}
\sigma_{xx}(x, y, z) &= \sigma_{xy}(x, y, z) = \sigma_{yy}(x, y, z) = 0 \\
\sigma_{zz}(x, y, z) &= \frac{F}{I} yz \\
\sigma_{xz}(x, y, z) &= \frac{2a^2 \nu F}{\pi^2 I (1 + \nu)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \sin \left( \frac{n \pi x}{a} \right) \frac{\sinh \left( \frac{n \pi y}{a} \right)}{\cosh \left( \frac{n \pi b}{a} \right)} \\
\sigma_{yz}(x, y, z) &= \frac{(b^2 - y^2) F}{2 I} + \frac{\nu F}{I (1 + \nu)} \left[ \frac{3x^2 - a^2}{6} - \frac{2a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{n \pi x}{a} \right) \frac{\cosh \left( \frac{n \pi y}{a} \right)}{\cosh \left( \frac{n \pi b}{a} \right)} \right]
\end{align*}
\]

where \( E \) is the Young’s modulus, \( \nu \) is Poisson’s ratio and \( I = 4ab^3/3 \) is the second moment of area about the \( x \)-axis. Two types of meshes are considered: (a) a regular hexahedral mesh and (2) a random closed -pack Voronoi mesh. Four levels of mesh refinement are considered for both hexahedral meshes \((2 \times 2 \times 10, 4 \times 4 \times 20, 8 \times 8 \times 40, 16 \times 16 \times 80)\) and for the random Voronoi meshes. Figure (20) shows the random Voronoi mesh employed for this study. The length of the beam is \( L = 5 \) and the shear load is taken as \( F = 1 \). Analytical displacements given by Equation (47) are applied.
at \( z = L \), whilst the beam is loaded in shear at \( z = 0 \). All other faces are assumed to be traction free. Figure (21) shows the relative error in the displacement norm with mesh refinement. It can be seen that both formulations, viz., FEM and SFEM with stabilization converge to the analytical solution with mesh refinement. It is also observed that the SFEM with stabilization yields slightly more accurate results when compared to the FEM with full integration. Also, shown in Figure (21) is the convergence of the method with polyhedra meshes. Although, the formulation when applied to polyhedra elements, converges with mesh refinement, it is not as accurate as the hexahedral elements. This can attributed to the parameter \( \alpha^* \) employed in this study and this observation is consistent with the results reported in the literature [51].

5.3. Application to linear elastic fracture mechanics

The SFEM with stabilization discussed above over arbitrary polygons and polyhedra can be applied to problems with strong discontinuity and singularity. However, to accurately capture the asymptotic fields at the crack tip, a very fine mesh in combination with singular elements at the crack tip is usually required. This poses additional difficulties when the crack evolves. Another possibility is to enrich the approximation space with functions that can capture the discontinuity and singularity [4, 5]. In the literature, the latter method is referred to as the Generalized FEM (GFEM)/extended finite element method (XFEM). In [10], the authors combined the strain smoothing with the XFEM. It was observed that in the case of enrichment schemes for linear elastic fracture mechanics, the method yields less accurate results compared to the conventional XFEM. However, for the elements that are completely intersected by the discontinuous surface, with the strain smoothing operation, further sub-division is not required.

In this study, we propose to couple the SFEM with the scaled boundary finite element method (SBFEM) to model problems with strong discontinuity and singularities. The SBFEM is a novel method that has the advantages of both the FEM and the boundary element method (BEM). As in the FEM, no fundamental solution is required and as in BEM, the problem dimension is reduced by one. The SBFEM is a semi-analytical method and relies on defining a ‘scaling centre’ from which the entire boundary is visible. This is similar to the concept of ‘star convexity’. The boundary is divided into conventional linear finite elements, whilst the solution is sought analytically in the radial direction [22]. Moreover, by exploiting the special characteristics of the scaling centre, the stress intensity factors can be computed directly. When modelling a crack/notched surface the scaling
centre is placed at the crack tip. The straight crack/notch edges are formed by scaling the nodes A and B on the boundary and the crack surfaces are not discretized (see Figure 22).

**Displacement approximation** The geometry of the element described by the coordinates on the boundary \(x_b(\eta)\) is expressed as:

\[
x_b(\eta) = N(\eta)x_b
\] (48)

where \(N(\eta)\) is the shape function matrix of the finite elements discretising the polygon boundary. The standard 1D Gauss-Lobatto shape functions or Lagrange shape functions can be used. In this study, we employ Lagrange shape functions. The displacements of a point in a polygon is approximated by:

\[
u(\xi, \eta) = N(\eta)u(\xi)
\] (49)

where \(u(\xi)\) are radial displacement functions. Substituting Equation (49) in the definition of strain-displacement relations, the strains \(\varepsilon(\xi, \eta)\) are expressed as:

\[
\varepsilon(\xi, \eta) = Lu(\xi, \eta)
\] (50)

where \(L\) is a linear operator matrix formulated in the scaled boundary coordinates as

\[
L = b_1(\eta) \frac{\partial}{\partial \xi} + \xi^{-1}b_2(\eta)
\] (51)
Figure 21. Beam with shear load: convergence of the relative error in the displacement.

Figure 22. Cracked polygon representation by the scaled boundary finite element method.

with

\[
\mathbf{b}_1(\eta) = \frac{1}{|\mathbf{J}(\eta)|} \begin{bmatrix} y_\eta(\eta),_\eta & 0 \\ 0 & -x_\eta(\eta),_\eta \end{bmatrix}
\]

\[
\mathbf{b}_2(\eta) = \frac{1}{|\mathbf{J}(\eta)|} \begin{bmatrix} -y_\eta(\eta) & 0 \\ 0 & x_\eta(\eta) \end{bmatrix}
\]

(52)
By following the procedure outlined in [64, 65], the following ODE is obtained:

\[ E_0 \xi^2 u(\xi)_{,\xi \xi} + (E_0 + E_1) \xi u(\xi)_{,\xi} - E_2 u(\xi) = 0 \]  

(53)

where \( E_0, E_1 \) and \( E_2 \) are coefficient matrices given by:

\[ E_0 = \int_{\eta} B_1(\eta)^T D B_1(\eta) |J(\eta)| d\eta, \]

\[ E_1 = \int_{\eta} B_2(\eta)^T D B_1(\eta) |J(\eta)| d\eta, \]

\[ E_2 = \int_{\eta} B_2(\eta)^T D B_2(\eta) |J(\eta)| d\eta. \]

(54)

Using Equation (50) and Hooke’s law \( \sigma = D \varepsilon \), the stresses \( \sigma(\xi, \eta) \) is expressed as

\[ \sigma(\xi, \eta) = D (B_1(\eta) u(\xi)_{,\xi} + \xi^{-1} B_2(\eta) u(\xi)) \]

(55)

where \( D \) is the material constitutive matrix and the determinant of the Jacobian matrix is:

\[ |J(\eta)| = x_b(\eta) y_b(\eta) - y_b(\eta) x_b(\eta) \]

(56)

The coefficient matrices are evaluated element-by-element on the polygon boundary and assembled over a polygon. This process is similar to the standard FE procedure of assemblage. Equation (53) is a homogeneous second-order ordinary differential equation. Its solution is obtained by introducing the variable \( \chi(\xi) \)

\[ \chi(\xi) = \begin{bmatrix} u(\xi) \\ q(\xi) \end{bmatrix} \]

(57)

where \( q(\xi) \) is the internal load vector

\[ q(\xi) = E_0 \xi u(\xi)_{,\xi} + E_1^T u(\xi) \]

(58)

The boundary nodal forces are related to the displacement functions by:

\[ f = q(\xi = 1) = (E_0 \xi u(\xi)_{,\xi} + E_1^T u(\xi))|_{\xi=1} \]

(59)

This allows Equation (53) to be transformed into a first order ordinary differential equation with twice the number of unknowns in an element as:

\[ \xi \chi(\xi)_{,\xi} = -Z \chi(\xi) \]

(60)

where \( Z \) is a Hamiltonian matrix

\[ Z = \begin{bmatrix} E_0^{-1} E_1^T & -E_0^{-1} \\ E_1 E_0^{-1} E_1^T - E_2 & -E_1 E_0^{-1} \end{bmatrix} \]

(61)

An eigenvalue decomposition of \( Z \) is performed. The blocks of eigenvalues and transformation matrices necessary are:

\[ Z \begin{bmatrix} \Phi_u \\ \Phi_q \end{bmatrix} = \begin{bmatrix} \Phi_u \\ \Phi_q \end{bmatrix} \Lambda_n \]

(62)

In Equation (62), \( \Lambda_n = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \) contains the eigenvalues with negative real part. \( \Phi_u \) and \( \Phi_q \) are the corresponding transformation matrices of \( \Lambda_n \). They represent the modal displacements and forces, respectively. The general solution of Equation (60) is given by:

\[ u(\xi) = \Phi_u \xi^{-\Lambda_n}e \]

(63)

\[ q(\xi) = \Phi_q \xi^{-\Lambda_n}e \]

(64)
where $c$ are integration constants that are obtained from the nodal displacements $u_h = u(\xi = 1)$ as:

$$c = \Phi_u^{-1} u_h$$

(65)

The complete displacement field of a point defined by the sector covered by a line element on the element is obtained by substituting Equation (64) into Equation (49) resulting in:

$$u(\xi, \eta) = N(\eta) \Phi_u \xi^{-\Lambda_n} c$$

(66)

Taking the derivative of $u(\xi)$ with respect to $\xi$ and substituting into Equation (55) the stress field $\sigma(\xi, \eta)$ can be expressed as:

$$\sigma(\xi, \eta) = \Psi_\sigma(\eta) \xi^{-\Lambda_n} c$$

(67)

where the stress mode $\Psi_\sigma(\eta)$ is defined as:

$$\Psi_\sigma(\eta) = D ( -B_1(\eta) \Phi_u \Lambda_n + B_2(\eta) \Phi_u )$$

(68)

The stiffness matrix of an element is obtained by first substituting Equation (65) into Equation (64) at $\xi = 1$. This results in:

$$f = \Phi_q \Phi_u^{-1} u_h$$

(69)

From Equation (69), the stiffness matrix $K$ can be identified to be given by the expression

$$K = \Phi_q \Phi_u^{-1}$$

(70)

Remark 5.1. The stiffness computed by employing the SBFEM is positive definite and symmetric. Hence, the stiffness matrix can be assembled in the conventional FEM approach. A simple Matlab function is given in [45] to compute the stiffness matrix using the SBFEM.

Calculation of the stress intensity factors A unique feature of the SBFEM is that stress singularities, if present, are analytically represented in the radial displacement functions $u(\xi)$. When a crack is modelled by a polygon with its scaling centre chosen at the crack tip in Figure (22), some of the eigenvalues $\Lambda_n^{(s)} \subset \Lambda_n$ satisfy $-1 < \Lambda_n^{(s)} < 0$. These eigenvalues lead to singular stresses at the crack tip. Using $\Lambda_n^{(s)}$, the singular stress field $\sigma^{(s)}(\xi, \eta)$ can be defined as [58]

$$\sigma^{(s)}(\xi, \eta) = \Psi^{(s)}(\eta(\theta)) \xi^{-\Lambda_n^{(s)}} c^{(s)}$$

(71)

where the singular stress mode $\Psi^{(s)}(\eta(\theta))$ is:

$$\Psi^{(s)}(\eta(\theta)) = \begin{bmatrix} \Psi_{\sigma_{\theta\xi}}(\eta(\theta)) & \Psi_{\sigma_{\theta\eta}}(\eta(\theta)) & \Psi_{\sigma_{\theta\theta}}(\eta(\theta)) \end{bmatrix}^T$$

(72)

In Equation (72) $\Phi_u^{(s)} \subset \Phi_u$ and $c^{(s)} \subset c$, contain the displacement modes and integration constants corresponding to $\Lambda_n^{(s)}$. It can be discerned from Equation (71) that $\Lambda_n^{(s)}$ leads to singular stresses at the crack tip. This enables the stress intensity factors to be computed directly from their definitions. The stress intensity factors for a crack that is aligned with the Cartesian coordinate axes shown in Figure (23) are defined as

$$\begin{bmatrix} K_I \\ K_{II} \end{bmatrix} = \lim_{r \to 0} \begin{bmatrix} \sqrt{2\pi r \sigma_{\theta\theta}} |_{\theta=0} \\ \sqrt{2\pi r \tau_{\theta\phi}} |_{\theta=0} \end{bmatrix}$$

(73)

Substituting the stress components in Equation (71) at angle $\theta = 0$ into Eq. (73) and using the relation $\xi = r/L_o$ at $\theta = 0$, the stress intensity factors are

$$\begin{bmatrix} K_I \\ K_{II} \end{bmatrix} = \sqrt{2\pi L_o} \begin{bmatrix} \Psi_{\sigma_{\theta\xi}}(\eta(0)) c^{(s)} \\ \Psi_{\sigma_{\theta\phi}}(\eta(0)) c^{(s)} \end{bmatrix}$$

(74)
5.3.1. Plate with double edge crack in tension

The plate with double edge crack subjected to a uniform tension at both ends as shown in Figure (24) is considered. In the computations, the ratio of the crack length, $a$, to the width of the plate, $H$, is $a/H = 0.25$. The material properties of the plate are: Young’s modulus, $E = 200$ GPa and Poisson’s ratio, $\nu = 0.3$. In this example, plane stress conditions are assumed. The empirical mode I SIF that is given by:

$$K_I^{\text{ref}} = C \sigma \sqrt{\pi a}$$

where $C$ is a correction factor. For $a/b > 0.4$, $b = H/2$, the correction factor is given by [66]:

$$C = 1.12 + 0.203 \left( \frac{a}{b} \right) - 1.197 \left( \frac{a}{b} \right)^2 + 1.930 \left( \frac{a}{b} \right)^3$$

The above factor corrects for an infinite plate with an accuracy of 2%. For the chosen parameters, the reference normalized SIF is $K_I/\sqrt{\pi a} = 1.1635$. The plate is discretized with a polygonal mesh. For the polygons containing the crack tip, we employ the SBFEM technique to capture the singularity. In this polygon, each edge is further discretized with 5 linear elements so that the angular variation of the SIF can be computed accurately [46]. For the elements that do not contain the crack tip, we employ the SFEM with stabilization to compute the stiffness matrix. The convergence of the mode I SIF with mesh refinement is give in Table II. It can be seen that the proposed method converges to the empirical relation with mesh refinement.

Table II. Plate with an edge crack in remote tension: convergence of mode I SIF.

| $h$  | Number of Polygons | Number of nodes | $K_I$   | $K_I/\sqrt{\pi a}$ |
|------|--------------------|----------------|--------|--------------------|
| 0.25 | 32                 | 111            | 1.0524 | 1.1875             |
| 0.125| 91                 | 247            | 1.0555 | 1.1910             |
| 0.0625| 332               | 777            | 1.0452 | 1.1794             |
| 0.03125| 1229             | 2659           | 1.0431 | 1.1770             |
| 0.015625| 4940            | 10245          | 1.0432 | 1.1772             |

5.3.2. Angled crack in an isotropic material

In this example, a plate with an angled crack subjected to far field bi-axial stress field, $\sigma$ (see Figure (25)) with $a/w = 0.1$, $\sigma_1 = 1$ and $\sigma_2 = 2$ is considered. In this example, the mode I and the mode II SIFs, $K_I$ and $K_{II}$, respectively, are obtained as a function...
of the crack angle $\beta$. For the loads shown, the analytical SIF for an infinite plate are given by [67]:

$$K_I = (\sigma_2 \sin^2 \beta + \sigma_1 \cos^2 \beta) \sqrt{\pi a}$$

$$K_{II} = (\sigma_2 - \sigma_1) \sin \beta \cos \beta \sqrt{\pi a}$$

(77)
The material properties of the plate are: Young’s modulus, \( E = 200 \, \text{GPa} \) and Poisson’s ratio, \( \nu = 0.3 \). In this example, the plate is discretized with polygon meshes (containing 300 polygons). The variations of the mode I and mode II SIF with the crack orientation \( \beta \) are presented in Table III. It can be observed that the results from the present method agree very well with the reference solution.

![Figure 25. Plate with an oblique crack: geometry and boundary conditions.](image)

Table III. Mode I and Mode II SIF for a plate with an inclined crack.

| \( \beta \) (°) | mode I SIF | mode II SIF |
|-----------------|-------------|-------------|
|                 | Equation (77) | Crack Tip A | Crack Tip B | Equation (77) | Crack Tip A | Crack Tip B |
| 0°              | 1.0000       | 1.0176      | 1.0168      | 0.0000         | 0.0000       | 0.0000       |
| 15°             | 1.0670       | 1.0937      | 1.0876      | 0.2500         | 0.2343       | 0.2453       |
| 30°             | 1.2500       | 1.2786      | 1.2786      | 0.4330         | 0.4380       | 0.4379       |
| 45°             | 1.5000       | 1.5281      | 1.5266      | 0.5000         | 0.5039       | 0.5053       |
| 60°             | 1.7500       | 1.7893      | 1.7893      | 0.4330         | 0.4427       | 0.4429       |
| 75°             | 1.9330       | 1.9855      | 1.9738      | 0.2500         | 0.2880       | 0.2669       |
| 90°             | 2.0000       | 2.0351      | 2.0336      | 0.0000         | 0.0018       | 0.0122       |
6. CONCLUSIONS

In this paper, we revisited the cell-based smoothed finite element method (SFEM) and constructed a one subcell polygon/polyhedra smoothed finite element which is stable, accurate and convergent compared to existing approaches. We also demonstrated the equivalence of the method with the virtual element method (VEM). We conclude that the SFEM can be seen as a special case of the more general VEM. By utilizing the concept of the stabilizing term from the VEM, we proposed a new stabilized smoothed finite element method. Instead of increasing the number of subcells, we add to the one-subcell, the stability term borrowed from the VEM. When applied to arbitrary polygons/polyhedra, with the proposed method, sub-triangulation of the polygon/polyhedra is not required to ensure stability. From the detailed numerical study, we observe that the SFEM with stabilization term yields more accurate results than the conventional SFEM with many subcells for much fewer numerical operations. To study problems with singularities, the proposed was combined with the scaled boundary finite element method. The proposed method is flexible, easy to implement and yields accurate results.

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