No-π Theorem for Euclidean Massless Correlators

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We provide the reader with a (very) short review of recent advances in our understanding of the π-dependent terms in massless (Euclidean) 2-point functions as well as in generic anomalous dimensions and β-functions. We extend the considerations of [1] by one more loop, that is for the case of 6-loop correlators and 7-loop renormalization group (RG) functions.

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1. Introduction and Preliminaries

Since the seminal calculation of the Adler function at order $\alpha_s^3$ \cite{Adler:1973 pylab} it has been known that p-functions demonstrate striking regularities in terms proportional to $\pi^{2n}$, with $n$ being positive integer. Here by p-functions we understand (\MSbar-renormalized) Euclidean Green functions\footnote{Like quark-quark-gluon vertex in QCD with the external gluon line carrying no momentum.} or 2-point correlators or even some combination thereof, expressible in terms of massless propagator-like Feynman integrals (to be named p-integrals below).

To describe these regularities we need to introduce a few notations and conventions. (In what follows we limit ourselves by the case of QCD considered in the Landau gauge). Let

$$ F_n(a, \ell_\mu) = 1 + \sum_{1 \leq i \leq n} g_{i,j} (\ell_\mu)^j a^i $$

(1.1)

be a p-function, where $a = \frac{\alpha_s(\mu)}{4\pi}$, $\ell_\mu = \ln \frac{\mu^2}{Q^2}$ and $Q$ is an (Euclidean) external momentum. The integer $n$ stands for the (maximal) power of $\alpha_s$ appearing in the p-integrals contributing to $F_n$. The $F$ without $n$ will stand as a shortcut for a formal series $F_\infty$.

In terms of bare quantities\footnote{We assume the use of the dimensional regularization with the space-time dimension $D = 4 - 2\varepsilon$.}

$$ F = Z F_B(a_B, \ell_\mu), \quad Z = 1 + \sum_{1 \leq i \leq n} Z_{a,j} \frac{a^i}{\varepsilon^{i-j}}, $$

(1.2)

with the bare coupling constant and the corresponding renormalization constant being

$$ a_B = \mu^{2\varepsilon} Z_a a, \quad Z_a = 1 + \sum_{i \geq 1} \left( \frac{a}{\varepsilon} \right)_{i,j} Z_{i,j} \frac{a^i}{\varepsilon^{i-j}}, $$

(1.3)

with the anomalous dimension (AD)

$$ \gamma(a) = \sum_{i \geq 1} \gamma_i a^i, \quad \gamma_i = -iZ_{a,1}. $$

(1.5)

The coefficients of the $\beta$-function $\beta_i$ are related to $Z_a$ in the standard way:

$$ \beta_i = i (Z_a)_{i,1}. $$

(1.6)

A p-function $F$ is called scale-independent if the corresponding AD $\gamma \equiv 0$. If $\gamma \neq 0$ then one can always construct a scale-invariant object from $F$ and $\gamma$, namely:

$$ F_{n+1}^{si}(a, \ell_\mu) = \frac{\partial}{\partial \ell_\mu} (\ln F)_{n+1} \equiv \frac{\left( \frac{\gamma(a) - \beta(a) a^2}{\sigma_0} \right) F_n}{F_n}.$$

(1.7)

Note that $F_{n+1}^{si}(a, \ell_\mu)$ starts from the first power of the coupling constant $a$ and is formally composed from $O(\alpha_s^{n+1})$ Feynman diagrams. In the same time is can be completely restored from $F_n$ and the $(n+1)$-loop AD $\gamma$.

An (incomplete) list of the currently known regularities\footnote{For discussion of particular examples of $\pi$-dependent contributions into various p-functions we refer to works \cite{Chetyrkin:2013wga, Chetyrkin:2013vna, Chetyrkin:2015wma, Chetyrkin:2016wuc}.} includes the following cases.
1. Scale-independent p-functions $F_n$ and $F_{n+1}^\pi$ with $n \leq 4$ are free from $\pi$-dependent terms.

2. Scale-independent p-functions $F_{5}^\pi$ are free from $\pi^6$ and $\pi^2$ but do depend on $\pi^4$.

3. The QCD $\beta$-function starts to depend on $\pi$ at 5 loops only \cite{7, 8, 9} (via $\zeta_4 = \pi^4/90$). In addition, there exits a remarkable identity \cite{1}

$$\beta_5^\pi = \frac{9}{8} \beta_4 \beta_4^\pi, \quad \text{with} \quad F_5^\pi = \lim_{\zeta \to 0} \frac{\partial}{\partial \zeta_i} F.$$ 

4. If we change the $\overline{\text{MS}}$-renormalization scheme as follows:

$$a = \bar{a} (1 + c_1 \bar{a} + c_2 \bar{a}^2 + c_3 \bar{a}^3 + \frac{1}{3} \frac{\beta_2}{\beta_1} \bar{a}^4), \quad (1.8)$$

with $c_1, c_2$ and $c_3$ being any rational numbers, then the function $F_{5}^\pi(\bar{a}, \ell, \mu)$ and the (5-loop) $\beta$-function $\bar{\beta}(\bar{a})$ both loose any dependence on $\pi$. This remarkable fact was discovered in \cite{5}.

It should be stressed that eventually every separate diagram contributing to $F_n$ and $F_{n+1}$ contains the following set of irrational numbers: $\zeta_3, \zeta_4, \zeta_5, \zeta_6$ and $\zeta_7$ for $n = 4$, $\zeta_3, \zeta_4$ and $\zeta_5$ for $n = 3$. Thus, the regularities listed above are quite nontrivial and for sure can not be explained by pure coincidence.

2. Hatted representation of p-integrals and its implications

The full understanding and a generic proof of points 1,2 and 3 above have been recently achieved in our work \cite{1}. The main tool of the work was the so-called “hatted” representation of transcendental objects contributing to a given set of p-integrals. Let us reformulate the main results of \cite{1} in an abstract form.

We will call the set of all $L$-loop p-integrals $\mathcal{P}_L$ a $\pi$-safe one if the following is true.

(i) All p-integrals from the set can be expressed in terms of $(M+1)$ mutually independent (and $\varepsilon$-independent) transcendental generators

$$\mathcal{T} = \{t_1, t_2, \ldots, t_{M+1}\} \quad \text{with} \quad t_{M+1} = \pi. \quad (2.1)$$

This means that any p-integral $F(\varepsilon)$ from $\mathcal{P}_L$ can be uniquely$^4$ presented as follows

$$F(\varepsilon) = F(\varepsilon, t_1, t_2, \ldots, \pi) + \mathcal{O}(\varepsilon), \quad (2.2)$$

where by $F$ we understand the exact value of the p-integral $F$ while the combination $\varepsilon^k F(\varepsilon, t_1, t_2, \ldots, t_M, \pi)$ should be a rational polynomial$^5$ in $\varepsilon, t_1, \ldots, t_M, \pi$. Every such polynomial is a sum of monomials $T_i$ of the generic form

$$\sum_\alpha r_\alpha T_\alpha, \quad T_\alpha = \varepsilon^n \prod_{i=1, M+1} t_i^{m_i}, \quad (2.3)$$

$^4$We assume that $F(\varepsilon, t_1, t_2, \ldots, \pi)$ does not contain terms proportional to $\varepsilon^n$ with $n \geq 1$.

$^5$That is a polynomial having rational coefficients.
with \( n \leq L, n_i \) and \( r_\alpha \) being some non-negative integers and rational numbers respectively. A monomial \( T_\alpha \) will be called \( \pi \)-dependent and denoted as \( T_{\pi, \alpha} \) if \( n_{M+1} > 0 \). Note that a generator \( t_i \) with \( i \leq M \) may still include explicitly the constant \( \pi \) in its definition, see below.

(ii) For every \( t_i \) with \( i \leq M \) let us define its hatted counterpart as follows:

\[
\hat{t}_i = t_i + \sum_{j=1}^{M} h_j(\varepsilon) T_{\pi,j},
\]

with \( \{h_j\} \) being rational polynomials in \( \varepsilon \) vanishing in the limit of \( \varepsilon = 0 \) and \( T_{\pi,j} \) are all \( \pi \)-dependent monomials as defined in (2.3). Then there should exist a choice of both a basis \( \mathcal{T} \) and polynomials \( \{h_j\} \) such that for every \( L \)-loop p-integral \( F(\varepsilon, t_i) \) the following equality holds:

\[
F(\varepsilon, t_1, t_2, \ldots, t_{M+1}) = F(\varepsilon, \hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M, 0) + O(\varepsilon).
\]

We will call \( \pi \)-free any polynomial (with possibly \( \varepsilon \)-dependent coefficients) in \( \{t_i, i = 1, \ldots, M\} \).

As we will discuss below the sets \( \mathcal{P}_L \) with \( i = 3, 4, 5 \) are for sure \( \pi \)-safe while \( \mathcal{P}_6 \) highly likely shares the property. In what follows we will always assume that every (renormalized) \( L \)-loop p-function as well as (\( L+1 \))-loop \( \overline{\text{MS}} \) \( \beta \)-functions and anomalous dimensions are all expressed in terms of the generators \( t_1, t_2, \ldots, t_{M+1} \).

Moreover, for any polynomial \( P(t_1, t_2, \ldots, t_{M+1}) \) we define its hatted version as

\[
\hat{P}(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M) := P(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M, 0).
\]

Let \( F_L \) is a (renormalized, with \( \varepsilon \) set to zero) p-function, \( \gamma_L \) and \( \beta_L \) are the corresponding anomalous dimension and the \( \beta \)-function (all taken in the \( L \)-loop approximation). The following statements have been proved in [10] under the condition that the set \( \mathcal{P}_L \) is \( \pi \)-safe and that both the set \( \mathcal{T} \) and the polynomials \( \{h_i(\varepsilon)\} \) are fixed.

1. **No-\( \pi \) Theorem**

(a) \( F_L \) is \( \pi \)-free in any (massless) renormalization scheme for which corresponding \( \beta \)-function and AD \( \gamma \) are both \( \pi \)-free at least at the level of \( L+1 \) loops.

(b) The scale-invariant combination \( F_{L+1}^{(i)} \) is \( \pi \)-free in any (massless) renormalization scheme provided the \( \beta \)-function is \( \pi \)-independent at least at the level of \( L+1 \) loops.

2. **\( \pi \)-dependence of \( L \)-loop p-functions**

If \( F_L \) is renormalized in \( \overline{\text{MS}} \)-scheme, then all its \( \pi \)-dependent contributions can be expressed in terms of \( \hat{F}_{L-1}|_{\varepsilon=0}, \hat{\beta}_{L-1}|_{\varepsilon=0} \) and \( \hat{\gamma}_{L-1}|_{\varepsilon=0} \).

3. **\( \pi \)-dependence of \( L \)-loop \( \beta \)-functions and AD**

If \( \beta_L \) and \( \gamma_L \) are given in the \( \overline{\text{MS}} \)-scheme, then all their \( \pi \)-dependent contributions can be expressed in terms of \( \hat{\beta}_{L-1}|_{\varepsilon=0}, \hat{\beta}_{L-1}|_{\varepsilon=0}, \hat{\gamma}_{L-1}|_{\varepsilon=0} \) correspondingly.

3. **\( \pi \)-structure of 3,4,5 and 6-loop p-integrals**

A hatted representation of p-integrals is known for loop numbers \( L = 3 \) [10], \( L = 4 \) [11] and \( L = 5 \) [12]. In all three cases it was constructed by looking for such a basis \( \mathcal{T} \) as well as polynomials \( h_j(\varepsilon) \) (see eq. (2.4)) that eq. (2.5) would be valid for sufficiently large subset of \( \mathcal{P}_L \).
In principle, the strategy requires the knowledge of all (or almost all) L-loop master integrals. On the other hand, if we assume the π-safeness of the set \(\mathcal{P}_6\) we could try to fix polynomials \(h_j(\varepsilon)\) by considering some limited subset of \(\mathcal{P}_6\).

Actually, we do have at our disposal a subset of \(\mathcal{P}_6\) due to work \([13]\) where all 4-loop master integrals have been computed up to the transcendental weight 12 in their \(\varepsilon\) expansion. As every particular 4-loop p-integral divided by \(\varepsilon^n\) can be considered as a \((4+n)\) loop p-integral we have tried this subset for \(n=2\). Our results are given below (we use even the zetas \(\zeta_4 = \pi^2/90, \zeta_6 = \pi^6/945, \zeta_8 = \pi^8/9450\) and \(\zeta_{10} = \pi^{10}/93555\) instead of the corresponding even powers of \(\pi\)).

\[
\begin{align*}
\zeta_3 &:= \frac{\zeta_3}{\delta(\varepsilon)} + \frac{3\varepsilon}{2} \zeta_4 - \frac{5\varepsilon^3}{2} \zeta_6 + \frac{21\varepsilon^5}{2} \zeta_8 - \frac{153\varepsilon^7}{2} \zeta_{10}, \\
\zeta_5 &:= \frac{\zeta_5}{\delta(\varepsilon)} + \frac{5\varepsilon}{2} \zeta_6 - \frac{35\varepsilon^3}{4} \zeta_8 + \frac{63\varepsilon^5}{4} \zeta_{10}, \\
\zeta_7 &:= \frac{\zeta_7}{\delta(\varepsilon)} + \frac{7\varepsilon}{2} \zeta_8 - \frac{21\varepsilon^3}{2} \zeta_{10}, \\
\phi &::= \frac{\phi}{\delta(\varepsilon)} - 3\varepsilon \zeta_4 \zeta_5 + \frac{5\varepsilon}{2} \zeta_3 \zeta_6 - \frac{24\varepsilon^2}{47} \zeta_{10} + \varepsilon^3 \left( -\frac{35}{4} \zeta_1 \zeta_8 + 5 \zeta_5 \zeta_6 \right), \\
\zeta_9 &:= \frac{\zeta_9}{\delta(\varepsilon)} + \frac{3\varepsilon}{2} \zeta_{10}, \\
\zeta_{7,3} &:= \frac{\zeta_{7,3} - \frac{793}{34} \zeta_{10}}{\delta(\varepsilon)} + 3\varepsilon \left( -\zeta_4 \zeta_7 - 5 \zeta_5 \zeta_6 \right), \\
\zeta_{11} &:= \frac{\zeta_{11}}{\delta(\varepsilon)} \\
\zeta_{5,3,3} &:= \frac{\zeta_{5,3,3} + 45 \zeta_2 \zeta_9 + 3 \zeta_4 \zeta_7 - 2 \zeta_5 \zeta_6}{\delta(\varepsilon)}.
\end{align*}
\]

Here
\[
\phi := \frac{3}{5} \zeta_{5,3} + \zeta_1 \zeta_5 - \frac{29}{20} \zeta_8 = \zeta_{6,2} - \zeta_{3,5} \approx -0.1868414
\]

and multiple zeta values are defined as \([14]\)

\[
\zeta_{m_1, n_1} := \sum_{i > j > 0} \frac{1}{i^{m_1} j^{n_1}}, \quad \zeta_{m_1, n_2, n_3} := \sum_{i > j > k > 0} \frac{1}{i^{m_1} j^{n_2} k^{n_3}}.
\]

Some comments on these eqs. are in order.
The boxed entries form a set of \( \pi \)-independent (by definition!) generators for the cases of \( L = 3 \) (eq. (3.3)), \( L = 4 \) (eqs. (3.1)–(3.3)), \( L = 5 \) (eqs. (3.1)–(3.5)) and \( L = 6 \) (eqs. (3.1)–(3.8)).

For \( L = 5 \) we recover the hatted representation for the set \( \mathcal{P}_5 \) first found in [12].

We do not claim that the generators

\[
\zeta_3, \zeta_5, \zeta_7, \phi, \zeta_9, \zeta_{7,3} \big|_{\epsilon=0}, \hat{\zeta}_{5,3,3} \text{ and } \pi \tag{3.11}
\]

are sufficient to present the pole and finite parts of every 6-loop \( p \)-integral. In fact, it is not true [3, 16, 17]. However we believe that it is safe to assume that all missing irrational constants can be associated with the values of some convergent 6-loop \( p \)-integrals at \( \epsilon = 0 \).

4. \( \pi \)-dependence of 7-loop \( \beta \)-functions and AD

Using the approach of [1] and the hatted representation of the irrational generators (3.11) as described by eqs. (3.1)–(3.8), we can straightforwardly predict the \( \pi \)-dependent terms in the \( \beta \)-function and the anomalous dimensions in the case of any 1-charge minimally renormalized field model at the level of 7 loops.

Our results read (the combination \( F^{(1)}_{t_1 t_2 \ldots t_M} \) stands for the coefficient of the monomial \( (t_1 t_2 \ldots t_M) \) in \( F \); in addition, by \( F^{(1)} \) we understand \( F \) with every generator \( t_i \) from \( \{t_1, t_2, \ldots, t_M+1\} \) set to zero).

\[
\gamma^5_7 = -\frac{1}{2} \beta_5^3 \gamma + \frac{3}{2} \beta_1 \gamma^5_5, \tag{4.1}
\]

\[
\gamma^7_5 = -\frac{3}{8} \beta_4^5 \gamma + \frac{3}{2} \beta_2 \gamma^5_7 - \beta_3^5 \gamma_2 + \frac{3}{2} \beta_1 \gamma^5_5, \tag{4.2}
\]

\[
\gamma^6_5 = -\frac{5}{8} \beta_4^5 \gamma + \frac{5}{2} \beta_1 \gamma^5_5, \tag{4.3}
\]

\[
\gamma^5_5 \gamma_5 = 0, \tag{4.4}
\]

\[
\gamma^6_5 = \frac{3}{2} \beta_5^{(1)} \gamma^3_5 - \frac{3}{10} \beta_5^2 \gamma  - \frac{3}{4} \beta_5^3 \gamma_2 + \frac{3}{2} \beta_2 \gamma^5_5 - \frac{3}{2} \beta_5^3 \gamma_3^{(1)} + \frac{3}{2} \beta_1 \gamma^5_5, \tag{4.5}
\]

\[
\gamma^5_6 = -\frac{1}{2} \beta_5^5 \gamma - \frac{5}{8} \beta_5^3 \gamma_2 + \frac{5}{2} \beta_2 \gamma^5_5 + \frac{5}{2} \beta_1 \gamma^5_5 + \frac{3}{2} \beta^2_5 \gamma_3^{(1)} - \frac{5}{2} \beta_1 \gamma^5_5, \tag{4.6}
\]

\[
\gamma^5_6 \gamma_5 = -\frac{3}{5} \beta_5^2 \gamma + 3 \beta_1 \gamma^5_5, \tag{4.7}
\]

\[
\gamma^6_5 = -\frac{7}{10} \beta_5^5 \gamma + \frac{7}{2} \beta_1 \gamma^5_5, \tag{4.8}
\]

\[
\gamma^6_6 \gamma_5 = \gamma^5_6 \gamma_5 = 0, \tag{4.9}
\]

\[
\gamma^7_7 = -\frac{1}{4} \beta_6^5 \gamma + \frac{3}{2} \beta_3^{(1)} \gamma^5_5 + \frac{3}{2} \beta_4^{(1)} \gamma^5_5 - \frac{3}{5} \beta_5^5 \gamma_2 - \frac{9}{8} \beta_4^{(1)} \gamma_3^{(1)} + \frac{3}{2} \beta_2 \gamma^5_5 - 2 \beta_3 \gamma^5_5 (1) + \frac{3}{2} \beta_1 \gamma^5_5, \tag{4.10}
\]
\[\gamma_7^{\xi} = -\frac{5}{12} \beta_6^\xi \gamma_4 + \frac{5}{2} \beta_2 \beta_4 \gamma_3 \beta_5 - \frac{15}{8} \beta_4^\xi \gamma_1^{(1)} + \frac{5}{2} \beta_2 \gamma_3^\xi + \frac{5}{2} \beta_1 \gamma_6^\xi \]
\[+ \frac{5}{2} \beta_1 \beta_5^\xi \beta_2 \beta_7 + \frac{5}{4} \beta_7 \beta_2^\xi \gamma_1 - \frac{15}{2} \beta_1^2 \beta_2 \gamma_3^\xi + 3 \beta_2^2 \beta_3^\xi \gamma_2 - \frac{5}{2} \beta_3 \gamma_4^\xi, \quad (4.11)\]
\[\gamma_9^{\xi} = -\frac{1}{2} \beta_6^\xi \gamma_2 - \frac{6}{5} \beta_5^\xi \gamma_3 + \frac{3}{8} \beta_4 \beta_3 \gamma_1^\xi + \frac{5}{2} \beta_4 \beta_2 \gamma_5^\xi + \frac{5}{2} \beta_1 \gamma_6^\xi, \quad (4.12)\]
\[\gamma_5^{\xi} = -\frac{7}{12} \beta_6^\xi \gamma_4 - \frac{7}{8} \beta_2 \gamma_3^\xi + \frac{7}{2} \beta_2 \gamma_7^\xi + \frac{7}{12} (\beta_2^\xi)^2 \gamma_1 + \frac{7}{2} \beta_1 \gamma_6^\xi - \frac{7}{8} \beta_1 \beta_5^\xi \gamma_4 + \frac{1}{2} \beta_1^2 \gamma_4^\xi\]
\[+ \frac{21}{8} \beta_1 \beta_5^\xi \gamma_3^\xi + \frac{35}{8} \beta_1 \beta_4 \gamma_5^\xi - \frac{35}{4} \beta_3 \gamma_5^\xi, \quad (4.13)\]
\[\gamma_3^{\xi} = -\frac{5}{12} \beta_6^\xi \gamma_4 - \frac{5}{5} \beta_6^\xi \gamma_3 + \frac{5}{2} \beta_4 \gamma_5^\xi + \frac{5}{2} \beta_1 \gamma_6^\xi + \frac{5}{2} \beta_1 \gamma_6^\xi, \quad (4.14)\]
\[\gamma_3^{\xi} = -\frac{1}{2} \beta_5^\xi \gamma_3 + \frac{3}{2} \beta_2 \gamma_5^\xi - 3 \beta_5^\xi \gamma_5^\xi + \frac{3}{2} \beta_1 \gamma_6^\xi - 3 \beta_1 \gamma_6^\xi, \quad (4.15)\]
\[\gamma_5^{\xi} = -\frac{3}{4} \beta_6^\xi \gamma_3 + \frac{9}{2} \beta_1 \gamma_6^\xi, \quad (4.16)\]
\[\gamma_7^{\xi} = -\frac{3}{4} \beta_6^\xi \gamma_3 + \frac{9}{2} \beta_1 \gamma_6^\xi, \quad (4.17)\]
\[\gamma_7^{\xi} = \gamma_7^{\xi} = \gamma_7^{\xi} = 0. \quad (4.18)\]

The results for \(\pi\)-dependent contributions to a \(\beta\)-function are obtained from the above eqs. by a formal replacement of \(\gamma\) by \(\beta\) in every term. For instance, the 7-loop \(\pi\)-dependent contributions read:

\[\beta_7^{\xi} = \frac{3}{8} \beta_4^{\xi} \beta_3^{\xi} + \frac{3}{10} \beta_2 \beta_3 \gamma^\xi - \frac{1}{2} \beta_3^\xi \beta_4^{(1)} + \frac{5}{4} \beta_1 \gamma_6^\xi, \quad (4.19)\]
\[\beta_5^{\xi} = \frac{5}{8} \beta_4 \beta_4^{\xi} \beta_4^{(1)} + \frac{3}{2} \beta_2 \beta_5 \gamma^\xi + \frac{25}{12} \beta_1 \beta_6 \gamma^\xi - 2 \beta_2^2 \beta_3 \gamma^\xi - \frac{5}{4} \beta_3 \gamma_4^\xi, \quad (4.20)\]
\[\beta_3^{\xi} = \frac{9}{8} \beta_2 \gamma_4^\xi - \frac{1}{8} \beta_4 \gamma_4^\xi + \frac{5}{2} \beta_1 \gamma_6^\xi, \quad (4.21)\]
\[\beta_5^{\xi} = \frac{21}{10} \beta_2 \gamma_5^\xi + \frac{35}{12} \beta_1 \beta_6 \gamma^\xi - \frac{7}{24} \beta_1 (\beta_2^\xi)^2 + \frac{7}{4} \beta_1 \beta_5 \gamma^\xi - \frac{35}{8} \beta_1 \beta_4 \gamma^\xi, \quad (4.22)\]
\[\beta_1^{\xi} = \frac{5}{8} \beta_4 \gamma_5^\xi + \frac{25}{12} \beta_1 \gamma_6^\xi + \frac{25}{12} \beta_1 \gamma_6^\xi, \quad (4.23)\]
\[\beta_3^{\xi} = \frac{1}{2} \beta_3 \gamma_3^\xi + \frac{5}{4} \beta_4 \gamma_3^\xi - \frac{5}{2} \beta_1 \gamma_6^\xi, \quad (4.24)\]
\[\beta_7^{\xi} = \frac{15}{4} \beta_1 \gamma_6^\xi, \quad (4.25)\]
\[\beta_7^{\xi} = \frac{15}{4} \beta_1 \gamma_6^\xi, \quad (4.26)\]
\[\beta_7^{\xi} = \beta_7^{\xi} = \beta_7^{\xi} = 0. \quad (4.27)\]
4.1 Tests

With eqs. (4.1)–(4.27) we have been able to reproduce successfully all $\pi$-dependent constants appearing in the $\beta$-function and anomalous dimensions $\gamma$ and $\beta_2$ of the $O(n)\phi^4$ model which all are known at 7 loops from [17]. In addition, we have checked that the $\pi$-dependent contributions to the terms of order $n^6 \alpha_s^7$ in the the QCD $\beta$-function as well as to the terms of order $n^6 \alpha_s^7$ and of order $n^5 \alpha_s^7$ contributing to the quark mass AD (all computed in [18, 19, 20]) are in agreement with constraints (4.19)–(4.27) and (4.10)–(4.18) respectively.

Numerous successful tests at 4, 5 and 6 loops have been presented in [1].

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