DEFORMATIONS OF TOTALLY GEODESIC FOLIATIONS AND
MINIMAL SURFACES IN NEGATIVELY CURVED
3-MANIFOLDS

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Abstract. Let \( g_t \) be a smooth 1-parameter family of negatively curved metrics on a closed hyperbolic 3-manifold \( M \) starting at the hyperbolic metric. We construct foliations of the Grassmann bundle \( Gr_2(M) \) of tangent 2-planes whose leaves are (lifts of) minimal surfaces in \( (M, g_t) \). These foliations are deformations of the foliation of \( Gr_2(M) \) by (lifts of) totally geodesic planes projected down from the universal cover \( \mathbb{H}^3 \). Our construction continues to work as long as the sum of the squares of the principal curvatures of the (projections to \( M \)) of the leaves remains pointwise smaller in magnitude than the ambient Ricci curvature in the normal direction. In the second part of the paper we give some applications and construct negatively curved metrics for which \( Gr_2(M) \) cannot admit a foliation as above.

1. Introduction

1.1. The following statement is a special case of the geodesic rigidity theorem proved by Gromov in [Gro00].

Theorem 1.1. Let \( M \) be a closed hyperbolic manifold, and let \( N \) be a negatively curved Riemannian manifold homeomorphic to \( M \). Let \( G(M) \) and \( G(N) \) be the foliations of the unit tangent bundles \( UT(M) \) and \( UT(N) \) of \( M \) and \( N \) by the orbits of the geodesic flow. Then there is a homeomorphism between \( UT(M) \) and \( UT(N) \) sending leaves of \( G(M) \) to leaves of \( G(N) \).

In this paper, we investigate the extent to which a version of this theorem holds when geodesics, which are one-dimensional minimal surfaces, are replaced by two-dimensional minimal surfaces. We restrict ourselves to three ambient dimensions because minimal surfaces in that dimension are better behaved and understood.

Theorem 1.1 implies that many properties of the geodesic flow for an arbitrary negatively curved metric on \( M \) are controlled by the constant curvature geodesic flow. Much of our interest in trying to prove a minimal surface analogue is that it will allow us to use results from homogeneous dynamics to study how minimal surfaces in variable negative curvature are distributed in the ambient space. The idea to use homogeneous dynamics in this setting is recent and due to Calegari-Marques-Neves [CMN].

If \( M \) is a closed hyperbolic 3-manifold, then the Grassmann bundle of tangent 2-planes to \( M \) has a natural foliation by immersed totally geodesic planes. Denote this foliation by \( \mathcal{F} \), and let \( g_{hyp} \) be a hyperbolic (constant curvature \(-1\)) metric on \( M \) (Mostow rigidity says that there is a unique such metric up to isometry.) The main result of this paper is the following.
Theorem 1.2. Let \( \{g_t : t \in [0,1]\} \) be a smooth 1-parameter family of negatively curved metrics on \( M \) with \( g_0 = g_{hyp} \). Then there exists \( T \in (0,1] \cup \{\infty\} \) such that for all \( t < T \) contained in the interval \([0,1]\) there is a foliation \( \mathcal{F}_t \) of \( Gr_2(M) \) whose leaves are immersed minimal planes in \((M, g_t)\) lifted to \( Gr_2(M) \) by their tangent planes. Moreover, there is a self-homeomorphism of \( Gr_2(M) \) that sends leaves of \( \mathcal{F} \) to leaves of \( \mathcal{F}_t \).

If \( T < \infty \), then for every sequence \( t_n \nearrow T \) there exist immersed minimal planes \( S_n \) in \((M, g_{t_n})\) which lift to leaves of \( \mathcal{F}_{t_n} \) such that the following quantity tends to zero from below for a sequence of points \( p_n \in S_n \):

\[
|A_n|^2 + \text{Ric}_n(\nu_n, \nu_n).
\]

Here \( \nu_n \) is the unit normal vector to \( S_n \) at \( p_n \), \( A_n \) is the second fundamental form of \( S_n \) at \( p_n \), and \( \text{Ric}_n \) is the Ricci curvature tensor of \( g_{t_n} \) at \( p_n \).

Remark 1.4. The same theorem actually holds for all complete hyperbolic 3-manifolds \( M \), with the appropriate bounded geometry condition on the family \( g_t \). The assumption that the action of \( \pi_1(M) \) on \( \mathbb{H}^3 \) has a compact fundamental domain makes some of the proofs simpler but is not essential. Because of the applications we have in mind, though, we restrict ourselves to the closed case.

Remark 1.5. In Section 6, we construct negatively curved metrics on certain closed hyperbolic 3-manifolds \( M \) for which there cannot exist a foliation as in Theorem 1.2. The analogue of the geodesic rigidity of Theorem 1.1 therefore in general only holds in our setting in some neighborhood of the constant curvature metric.

In words, our construction of the foliations \( \mathcal{F}_t \) continues to work as long as the sum of the squares of the principal curvatures (of the projections to \( M \)) of the leaves of our foliations remains pointwise less than the absolute values of the ambient Ricci curvature in the normal direction. The proof of Theorem 1.2 occupies Section 3 where we prove a more intrinsic formulation of it (Theorem 3.4), and then explain how the proof can be modified to give a proof of Theorem 1.2. The proof is loosely speaking a method of continuity argument, where we work in the universal cover and follow the approach of Anderson [And83] to construct properly embedded minimal planes. Surfaces for which the quantity (1.3) is negative have small mean-convex neighborhoods, which we use to rule out the existence of minimal planes other than the ones from our construction that would lead to gaps in the foliations we are trying to construct.

In [Gro91], Gromov proved a stability result for the totally geodesic foliation \( \mathcal{F} \) that applies to metrics \( g \) with sectional curvatures pinched close to \(-1\). For these metrics, he constructs immersed almost-totally-geodesic \( g \)-minimal planes in \( M \) corresponding to the leaves of \( \mathcal{F} \). His construction also follows [And83] but, in contrast to this paper, works for closed hyperbolic manifolds of all dimensions and is based on Allard regularity. This paper grew out of attempts to find a more direct proof of Gromov’s results in dimension 3.

1.2. Theorem 1.2 was motivated by the theory of almost-Fuchsian manifolds. For \( \Sigma \) a surface of genus greater than 1, a hyperbolic metric on \( \Sigma \times \mathbb{R} \) is called quasi-Fuchsian if its limit set in the boundary at infinity of \( \mathbb{H}^3 \) is the image of the equator under a quasiconformal homeomorphism of \( S^2 \cong \partial_\infty \mathbb{H}^3 \). The space of all such metrics is parametrized by a product of Teichmüller spaces which correspond to the conformal structures on the two ends.
In [Uhl83], Uhlenbeck proved a rigidity theorem for quasi-Fuchsian manifolds which admit an embedded minimal surface with principal curvatures less than 1 in magnitude: that such a quasi-Fuchsian manifold is uniquely determined by the conformal class of this minimal surface and a quadratic differential equivalent to its second fundamental form. These manifolds are called almost-Fuchsian, and have been well-studied since ([HW13], [HW15], [Sep16].)

The set of metrics to which Theorem 1.2 applies inside the space of all negatively curved metrics on $M$ is analogous to the set of almost-Fuchsian metrics on $\Sigma \times \mathbb{R}$ inside the space of quasi-Fuchsian metrics, insofar as the existence of minimal surfaces with curvatures bounded by ambient curvatures allows for much greater control, making it possible, for instance, to prove uniqueness statements for the minimal surfaces in question. Guided by this analogy, in Section 6 we construct negatively curved metrics for which foliations as in the statement of Theorem 1.2 cannot possibly exist.

1.3. We now describe some applications of Theorem 1.2. Principal among these is the following density result for stable properly immersed minimal surfaces in a closed Riemannian 3-manifold $M$ which admits a foliation as in Theorem 1.2. Kahn and Markovic showed that for every closed hyperbolic 3-manifold $M$, subgroups of $\pi_1(M)$ isomorphic to the fundamental group of a closed surface, or surface subgroups, exist in great profusion ([KM12b], [KM12a].) (See also [Ham15], which gives a more geometric version of Kahn-Markovic's construction and generalizes their results to cocompact lattices in all rank one symmetric spaces except for hyperbolic spaces of even dimension.) Fixing a metric $g$ on $M$, each of these surface subgroups gives rise by [SU82] or [SY79] to a stable properly immersed minimal surface whose fundamental group includes as a subgroup of $\pi_1(M)$ conjugate to that surface subgroup.

Let $C$ be a circle in $\partial_\infty \mathbb{H}^3 \cong S^2$ such that the geodesic plane $P$ in $\mathbb{H}^3$ with limit set $C$ has dense projection to the closed hyperbolic 3-manifold $M$ under the universal covering map. Ratner and Shah independently proved that every geodesic plane $P$ either projects to a dense subset of $M$ whose tangent planes are dense in $Gr_2(M)$ or a properly immersed surface ([Rat91], [Sha91].) (See also [MMO17] for a nice proof of this fact.) Let $\Gamma_n$ be a sequence of surface subgroups of $\pi_1(M)$ with limit sets $K_n$-quasicircles Hausdorff converging to $C$ with $K_n$ tending to 1. (A K-quasicircle is the image of a round circle under a $K$-quasiconformal self-homeomorphism of $S^2$.) The existence of such a sequence of $\Gamma_n$ for each $C$ follows from [KM12b]. Let $\Sigma_n$ be a sequence of stable immersed minimal surfaces in $(M, g)$ whose fundamental groups include as a subgroup of $\pi_1(M)$ conjugate to that surface subgroup.

Theorem 1.6. Suppose that $g$ can be joined to $g_{hyp}$ by a smooth family of negatively curved metrics parametrized by $[0, 1]$ to which Theorem 1.2 applies with $T = \infty$. Let $\Sigma_n$ be a sequence as above. Then for every open set $U$ in $Gr_2(M)$ there exists a number $N$ so that $\Sigma_n$ has a tangent plane in $U$ for every $n > N$.

Remark 1.7. This is a slightly stronger statement than simply that the tangent planes of all closed stable immersed minimal surfaces are dense in $Gr_2(M)$, which could be obtained without using the Ratner-Shah theorem mentioned above.

A natural question is whether a similar density result is true for all negatively curved metrics on $M$. For negatively curved metrics to which Theorem 1.2 cannot
apply to produce a foliation, like those constructed in Section 6, we believe it’s possible that a sequence of $\Sigma_n$ as above might fail to be dense in $\text{Gr}_2(M)$.

If $g = g_{\text{hyp}}$, it follows from [Sep16] that if the limit set of $\Gamma_n$ is a $K$-quasicircle for $K$ sufficiently close to 1, then $\Sigma_n$ is the unique minimal surface whose fundamental group injectively includes as a subgroup conjugate to $\Gamma_n$. In Section 4, we prove a similar uniqueness result for $g$ to which Theorem 1.2 applies to produce a foliation:

**Theorem 1.8.** Suppose that $g$ satisfies the hypotheses of the previous theorem. Then there exists $\delta > 0$ such that the following is true. Suppose the limit set of a surface subgroup $\Gamma$ of $\pi_1(M)$ in $\partial_\infty \mathbb{H}^3$ is a $K$-quasicircle for $K < 1 + \delta$. Then there is a unique $g$-minimal surface in $M$ whose fundamental group injectively includes in $\pi_1(M)$ as a subgroup conjugate to $\Gamma$.

1.4. Suppose that $M$ has no properly immersed totally geodesic surfaces in its hyperbolic metric. Examples of closed hyperbolic 3-manifolds without properly immersed totally geodesic surfaces are described in [MR03, Chapter 5]. Assume that the fundamental group of $\Sigma_n$ injectively includes to a subgroup of $\pi_1(M)$ conjugate to $\Gamma_n$, where the $\Sigma_n$ are stable properly immersed minimal surfaces in a negatively curved metric $g$ on $M$ and where the limit sets of the $\Gamma_n$ in $\partial_\infty \mathbb{H}^3$ are $K_n$-quasicircles with $K_n$ tending to 1. Let $\mu_n$ be the probability measure on $\text{Gr}_2(M)$ that corresponds to averaging over the lift of $\Sigma_n$ to $\text{Gr}_2(M)$ in the area form for the metric on $\Sigma_n$ induced by $(M, g)$. We are able to prove the following quantitative version of Theorem 1.6:

**Theorem 1.9.** Let $g$ satisfy the assumptions of Theorem 1.8 and suppose that $M$ has no properly immersed totally geodesic surfaces. Then any weak-* limit of the $\mu_n$ has full support (assigns positive measure to every open set) in $\text{Gr}_2(M)$.

**Remark 1.10.** Mozes-Shah proved that for $M$ hyperbolic any sequence of totally geodesic $\Sigma_n$ in $M$ with area tending to infinity becomes uniformly distributed in $\text{Gr}_2(M)$ ([MS95].) The arguments used to prove Theorem 1.8 can also be used to show that weak-* limits of minimal surfaces corresponding to the $\Sigma_n$ have full support in $\text{Gr}_2(M)$ in metrics for which Theorem 1.2 constructs a foliation.

The condition that $M$ have no totally geodesic surfaces is a no-closed-orbits assumption for the action of $\text{PSL}(2, \mathbb{R})$ on the frame bundle of $M$. It is analogous to a unique ergodicity assumption for (one-dimensional) dynamical systems. For uniquely ergodic dynamical systems on a compact space, a simple argument shows that the equidistribution ergodic theorem holds for all space averages, not just almost all ([Esk10, Proposition 1.9].) We use a similar argument, together with Ratner’s measure classification theorem, to prove that geodesic disks in $\text{Gr}_2((M, g_{\text{hyp}}))$ are equidistributing at some rate uniform in the radii of the disks. We then locally approximate the $g_{\text{hyp}}$-minimal surfaces for the $\Gamma_n$ by large totally geodesic disks to show that these surfaces are equidistributing. Finally, we use the conjugating map $\Phi$ of Theorem 1.2 and the fact that the universal covers of $(M, g)$ and $(M, g_{\text{hyp}})$ are quasi-isometric to transfer information from $(M, g_{\text{hyp}})$ to $(M, g)$. This last step closely follows arguments in ([CMN].)

We expect that Theorem 1.9 also holds without the no-totally-geodesic-surfaces assumption. It at least is true without this assumption for sequences $\Sigma_n$ as in that theorem such that all weak-* limits of probability measures for the corresponding minimal surfaces in the constant curvature metric have full support in $\text{Gr}_2(M)$. (It
seems likely, for example, that sequences of $\Sigma_n$ corresponding to surface subgroups that come from Hammenstadt’s version of the Kahn-Markovic construction have this property ([Ham15]). A proof can be given using the map $\tilde{f}_\Sigma$ constructed in the proof of Theorem 1.8 together with arguments similar to those used to prove Theorem 1.9.

1.5. We now describe the construction of negatively curved metrics to which Theorem 1.2 cannot apply to produce a foliation. It is based on the existence of quasi-Fuchsian manifolds $Q$ which contain several distinct embedded minimal surfaces whose inclusions are homotopy equivalences (this contrasts with the almost-Fuchsian case, where it was shown in [Uhl83] that there exists a unique such minimal surface.)

We start out with a closed hyperbolic 3-manifold $M$ that contains an embedded totally geodesic surface. By passing to a finite cover which we also denote by $M$, we can make the totally geodesic surface $\Sigma$ have arbitrarily large normal injectivity radius. Taking the Fuchsian cover $F$ corresponding to $\Sigma$, which is homeomorphic to $\Sigma \times \mathbb{R}$, we modify the pulled-back metric while preserving negative curvature, so that we can cut out the middle of $F$ and glue in the middle of a quasi-Fuchsian $Q$ with multiple distinct minimal surfaces. To accomplish this, we also need to modify the metric on $Q$ near infinity, which we do using the fact that a quasi-Fuchsian metric on $\Sigma \times \mathbb{R}$, whatever disorderly behavior is happening in the middle, has a standard form near the ends. Provided the normal injectivity radius of $\Sigma$ was taken large enough, the gluing can be performed inside $M$ itself. This produces a negatively curved metric $g$ on $M$ for which there are multiple stable minimal surfaces isotopic to $\Sigma$, which is incompatible with the existence of a foliation as in Theorem 1.2.

If this metric can be joined to the constant curvature metric through a smooth path of metrics with negative sectional curvature, which we expect to be the case for reasons we explain in Section 6, then this shows that the case $T < \infty$ of Theorem 1.2 actually occurs. It would be good to find a more robust way of ruling out the existence of foliations as in Theorem 1.2 — for instance, one that worked for all closed hyperbolic 3-manifolds.

1.6. In the final section, we give an estimate for how fast the principal curvatures (of the projections to $M$) of the leaves of the foliations $F_t$ are changing as the metrics $g_t$ vary. The bound we obtain depends on the $\epsilon$ for which the (projections to $M$) of the leaves of the foliation are $\epsilon$-convex and bounds on the $g_t$ and their derivatives in time.

1.7. We now discuss some results related to this paper. Density and equidistribution theorems for minimal hypersurfaces produced by the Almgren-Pitts min-max theory have been obtained for generic metrics by Irie-Marques-Neves [IMN18] and Marques-Neves-Song [MNS19]. Recently Song-Zhou [SZ20] showed that for generic metrics sequences of minimal hypersurfaces can “scar” along stable minimal hypersurfaces, for example those corresponding to the Kahn-Markovic surface subgroups considered in this paper. The proofs of the above results are based on the Liokumovic-Marques-Neves Weyl law for the Almgren-Pitts volume spectrum [LMN18]. Ambrozio-Montezuma [AM18] also proved equidistribution results, by a somewhat different approach, for minimal surfaces in metrics on the round 3-sphere that are local maxima for the Simon-Smith width within their conformal class. In contrast to the minimal surfaces considered in this paper, the minimal
 surfaces of the results mentioned in this paragraph are embedded and one expects
them in general not to be local minima for the area functional.

Recent work of Calegari-Marques-Neves [CMN] considered minimal surfaces cor-
responding to the Kahn-Markovic surface subgroups from a dynamical perspective.
Given a closed hyperbolic 3-manifold, they define a functional on Riemannian met-
rics on that 3-manifold with sectional curvature at most $-1$ based on a renormalized
count of stable properly immersed minimal surfaces with limit sets close to circles,
and show that the constant curvature hyperbolic metric uniquely minimizes this
functional. The proof of the rigidity part of their result—that the constant cur-
vature metric uniquely minimizes the counting functional—uses the Ratner-Shah
theorem mentioned earlier. The present paper was inspired by and draws substan-
tially from their ideas, especially Sections 4 and 5.

In Section 3 we produce the leaves of the foliations of Theorem 1.2 by solv-
ing specific asymptotic Plateau problems in $\tilde{M}$, and arguing that the solutions
are unique. The asymptotic Plateau problem in $\mathbb{H}^n$ for suitable boundary data
at infinity was solved by Anderson [And83], and in simply connected Riemannian
manifolds bi-Lipschitz equivalent to a metric with pinched negative sectional cur-
tvature by Bangert and Lang [BL96]. Our need for quantitative control on solutions
to the relevant asymptotic Plateau problems and to prove that they are unique
prevents us from simply applying [BL96].

As a step to proving Theorem 1.9 we show in Proposition 5.16 that weak-
limits of the probability measures on $Gr_2(M)$ corresponding to metric disks with
radii tending to infinity in (the projections to $M$ of) leaves of our foliation have
full support in $Gr_2(M)$, provided that $M$ in its constant curvature metric has no
proper totally geodesic surfaces. In the constant curvature case it follows from
Ratner’s measure classification theorem (as we describe in Section 5) that provided
$M$ has no proper totally geodesic surfaces, the only possible weak-* limit of totally
geodesic disks with radii tending to infinity is the Haar measure.

**Question 1.11.** What can be said about the possible weak-* limits of probability
measures corresponding to intrinsic disks of radii tending to infinity in leaves of the
folios of Theorem 1.2?

The ergodic theory of foliations with negatively curved leaves has been studied
([Zim82], [Alv18].) In [Alv18], Alvarez considers certain foliations, transverse to
the fibers of $\mathbb{CP}^1$ bundles over a negatively curved surface, that arise from actions
of the fundamental group of the surface on $\mathbb{CP}^1$. He shows that there is a unique
measure obtained as a weak-* limit of large metric disks tangent to the leaves of
these foliations, and that it is singular with respect to other measures natural to
the dynamics of the foliation unless the surface in the construction had constant
negative curvature. It would be interesting to determine whether the story is similar
for the foliations of this paper.

### 2. Outline and Acknowledgements

In Section 3 we prove Theorem 1.2. In Section 4 we apply Theorem 1.2 to prove
density and uniqueness results for stable properly immersed minimal surfaces in
$M$. In Section 5 we prove some quantitative versions of the density results of
Section 4 under the assumption that $M$ has no proper totally geodesic surfaces
in its hyperbolic metric. In Section 6 we construct examples of negatively curved
metrics to which Theorem 1.2 cannot apply to produce a foliation. In Section 7 we give an estimate for how fast the principal curvatures of the leaves of the foliations of Theorem 1.2 are changing as the metric varies.

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3. Construction of the Foliations

In this section we prove Theorem 1.2. Fix a closed hyperbolic 3-manifold M, and denote by \(g_{hyp}\) the hyperbolic metric on \(M\). Let \(P\) be the set of totally geodesic planes in \(\mathbb{H}^3\). By taking limit sets, there is a bijection between \(P\) and the set of round circles in \(S^2 \cong \partial_\infty \mathbb{H}^3\). The lifts to \(Gr_2(M)\) by their tangent planes of the projections of elements of \(P\) under the covering map are the leaves of a foliation of \(Gr_2(M)\), which we denote by \(F\).

Let \(g\) be a metric on \(M\). Then there is an identification between the universal cover \((\tilde{M}, \tilde{g})\) of \((M, g)\) and the universal cover \(\mathbb{H}^3\) of \((M, g_{hyp})\), which is well-defined up to composing with covering transformations of \(\mathbb{H}^3\). Since elements of the set \(P\) are invariant under covering transformations, taking the images of elements of \(P\) under such an identification gives a well-defined set of embedded planes in \((\tilde{M}, \tilde{g})\), which we denote by \(\tilde{P}\).

**Definition 3.1.** Let \(g\) be a metric on \(M\) with negative sectional curvature. Consider the universal cover \((\tilde{M}, \tilde{g})\) with the metric induced by \(g\). We say that an embedded surface \(\Sigma\) in \((\tilde{M}, \tilde{g})\) is \(\epsilon\)-convex if it satisfies the following for every \(p \in \Sigma\).

Let \(\nu\) be the unit normal vector to \(\Sigma\) at \(p\) and let \(A\) be the second fundamental form of \(\Sigma\). Then

\[
|A(p)|^2 < |Ric(\nu, \nu)| - \epsilon.
\]

We say that \(g\) is in \(\Omega_\epsilon\) if it has negative sectional curvature and there is some \(\epsilon > 0\) such that for every \(P \in \tilde{P}\), there is a properly embedded \(\epsilon\)-convex minimal plane in \((\tilde{M}, \tilde{g})\) at finite Hausdorff distance from \(P\).

**Remark 3.3.** For a plane \(P \in \tilde{P}\), it will either be the case that the (lift to \(Gr_2(M)\) of the) projection of \(P\) to \((M, g_{hyp})\) is dense in \(Gr_2(M)\), or closes up to a properly immersed surface \((\text{[Ran91]}, \text{[Sha91]}).\) Take a plane \(P\) with dense projection to \(Gr_2(M)\). Then, for \(g\) a negatively curved metric on \(M\), if there is an \(\epsilon\)-convex minimal plane \(S\) in \((\tilde{M}, \tilde{g})\) at finite Hausdorff distance from \(P\) for some \(\epsilon > 0\), then \(g \in \Omega_{\epsilon'}\) for any \(\epsilon' < \epsilon\). That is, it suffices to check Equation (3.2) on a single embedded plane corresponding to an element of \(\tilde{P}\) with dense projection to \(Gr_2(M)\). Then, for \(g\) a negatively curved metric on \(M\), if there is an \(\epsilon\)-convex minimal plane \(S\) in \((\tilde{M}, \tilde{g})\) at finite Hausdorff distance from \(P\) for some \(\epsilon > 0\), then \(g \in \Omega_{\epsilon'}\) for any \(\epsilon' < \epsilon\). This can be seen by approximating any \(P' \in \tilde{P}\) by orbits of \(P\) under covering transformations, taking the corresponding sequence of minimal surfaces in \((\tilde{M}, \tilde{g})\), and passing to a smooth subsequential limit as in the proof of Theorem 3.4 below to produce a minimal plane satisfying Equation (3.2) for any \(\epsilon' < \epsilon\).

Presumably \(\Omega_\epsilon\) for \(\epsilon\) close to zero contains more metrics than just those with sectional curvatures extremely close to -1. It would be nice to have a better understanding of which metrics are contained in \(\Omega_\epsilon\) for \(\epsilon\) small. Are all metrics that can
be smoothly joined to the hyperbolic metric through metrics with sectional curva-
tures pinched between $-1$ and $-3/2$ contained in $\Omega_{\text{pinched}}$, for example? (Conceivably, the answer could depend on $M$.)

We now prove the main theorem of the section. It does not quite imply Theorem 1.2, but we will explain at the end how the proof can be modified to give a proof of Theorem 1.2.

**Theorem 3.4.** Let $\{g_t : t \in [0, 1]\}$ be a smooth 1-parameter family of metrics on $M^3$, with $g_0 = g_{\text{hyp}}$ and $g_t \in \Omega_{\epsilon}$ some fixed $\epsilon > 0$ and all $t$. Then there exists a constant $C$ depending only on the family such that for all $P \in \mathcal{P}$, there exists a properly embedded minimal plane $S_t$ in the universal cover $(\tilde{M}, \tilde{g}_t)$ at a Hausdorff distance from $P$ of at most $C$, and that has the following properties:

- $S_t$ is the unique properly embedded minimal plane at finite Hausdorff distance from $P$.
- $S_t$ is absolutely minimizing.
- The lifts of the $S_t$ to $\text{Gr}_2(\tilde{M})$ by their tangent planes are the leaves of a foliation $\tilde{\mathcal{F}}_t$ of $\text{Gr}_2(\tilde{M})$.

The diffeomorphisms of $\text{Gr}_2(\tilde{M})$ induced by covering transformations of $\tilde{M}$ send leaves to leaves, and $\tilde{\mathcal{F}}_t$ thus descends to a foliation $\mathcal{F}_t$ of $\text{Gr}_2(M)$. Moreover, $\mathcal{F}_t$ and $\mathcal{F}$ are conjugate, in that there is a homeomorphism $\Phi : \text{Gr}_2((M, g_{\text{hyp}})) \to \text{Gr}_2((M, g_t))$ that maps leaves of $\mathcal{F}$ to leaves of $\mathcal{F}_t$.

We say in this paper that a minimal surface is absolutely minimizing if, for every piecewise-differentiable closed curve on the surface that bounds a disk $D$ on the surface, the area of $D$ is less than or equal to that of any other smoothly embedded disk in the ambient space bounding $\partial D$.

A continuous 1-parameter family of Riemannian metrics as in the theorem gives a map

$$M \times [0, 1] \to \text{Sym}^2(T^*M).$$

We say that the family of metrics is smooth if this map is smooth.

Let $\{g_t : t \in [0, 1]\}$ be a smooth 1-parameter family of metrics on $M^3$, with $g_0 = g_{\text{hyp}}$ and $g_t \in \Omega_{\epsilon}$ for all $t$. We will prove Theorem 3.4 by a finite induction. Suppose that $g_{t_0}$ satisfies the conclusion of the theorem for some $t_0$. In the proof, we will use the following two properties of the $S_t$ at $t = t_0$, the first of which we will assume at $t_0$ and verify in the inductive step and the second of which follows from the existence of the conjugating map $\Phi$ in the theorem.

**Property 1** Suppose lifts of $S_t$ and $S'_t$ are leaves of $\tilde{\mathcal{F}}_t$ that correspond to totally geodesic planes $S$ and $S'$ in $\mathcal{P}$. If $S$ and $S'$ have disjoint boundary circles at infinity, then $S_t$ and $S'_t$ are disjoint.

**Property 2** If $S_n$ is a sequence of totally geodesic planes that converges to $S$ on compact subsets, then the corresponding sequence of minimal planes in $(\tilde{M}, \tilde{g}_t)$ smoothly converges, uniformly on compact subsets, to the minimal plane in $(\tilde{M}, \tilde{g}_t)$ corresponding to $S$.

3.1. **Outline** We carry out the induction in three steps. First we construct the minimal planes $S_t$ in the universal cover as limits of solutions to Plateau problems...
for a sequence of circles going off to infinity, roughly following the approach introduced by [And83] to solving the asymptotic Plateau problem in negative curvature. Next, using the existence of mean-convex tubular neighborhoods of the \( S_{t_0} \) guaranteed by \( g_{t_0} \)'s membership in \( \Omega \), we prove that the \( S_t \) are unique. Finally, based on the strong restrictions on how minimal surfaces can intersect in three dimensions, we prove that the lifts of the \( S_t \) to \( \tilde{\mathcal{G}}_2(M) \) by their tangent planes give a foliation of \( \tilde{\mathcal{G}}_2(M) \).

3.2. Let \( S_t \) be an \( \epsilon \)-convex properly embedded minimal disk in \( (\tilde{M}, \tilde{g}_t) \) as in the statement of the theorem. Along any normal geodesic \( \gamma \) from \( S_t \) parametrized by arc-length and within the normal injectivity radius, the signed mean curvatures \( m \) of the parallel surfaces satisfy the following equation:

\[
(3.5) \quad m'((\gamma(s)) = -|A(\gamma(s))|^2 - \text{Ric}(\dot{\gamma}(s), \dot{\gamma}(s)),
\]

where \( A(\gamma(s)) \) is the second fundamental form of the signed-distance-s parallel surface at \( \gamma(s) \). This can be obtained by taking the trace of Equation (2) in Proposition 3.2.11 of [Pet10]. The next lemma will be used at several points below.

**Lemma 3.6.** There is some \( \xi \) depending only on \( \epsilon \) and the family \( g_t \) such that the parallel signed distance-\( r \) surfaces of the \( S_t \) have mean curvature greater than \( \frac{\epsilon}{2}r \) if \( 0 < r < \xi \) and less than \( \frac{\epsilon}{2}r \) if \(-\xi < r < 0 \).

**Proof.** First note that we have uniform bounds on the \( L^\infty \) norm of the Ricci curvature tensor over all \( g_t \). Since \( S_t \) is \( \epsilon \)-convex, there is thus a uniform bound on the magnitude of its second fundamental form, depending only on \( \epsilon \) and bounds on the \( g_t \). The upper bound on the magnitude of the second fundamental form and the uniform bounds on \( \tilde{g}_t \) and its derivatives implies a lower bound on the normal injectivity radius of \( S_t \) uniform over all \( S_t \)—i.e., a lower bound for \( \delta \) such that the normal exponential map on the normal bundle to \( S_t \) is injective restricted to \( S_t \times (-\delta, \delta) \).

Now suppose the statement of the lemma were false, and let \( \{t_n\}, \{r_n\}, \) and \( \{x_n\} \) be sequences of times, signed-distances, and points on \( S_{t_n} \) such that:

- the distance-\( r_n \) surface to \( S_{t_n} \) has mean curvature less than \( \frac{\epsilon}{2}r_n \) if \( r_n > 0 \) or greater than \( \frac{\epsilon}{2}r_n \) if \( r_n < 0 \) at the point that normally projects to \( x_n \)
- \( |r_n| \to 0 \)
- \( \{t_n\} \) converges to some time \( t \) (where we’ve passed to a subsequence if necessary.)

Let \( K \) be a compact set containing a fundamental domain for the action of \( \pi_1(M) \) on \( \tilde{M} \), and for each \( x_n \), let \( \gamma_n \) be a covering transformation of \( M \) such that \( \gamma_n \cdot x_n \in K \). By passing to a subsequence we can assume that \( \gamma_n \cdot x_n \) converges to \( x \). By the uniform bound on the second fundamental forms of the \( S_{t_n} \), we can pass to a subsequence of the \( \gamma_n \cdot S_{t_n} \) that graphically converges (and thus, by standard elliptic PDE theory, smoothly converges) in a neighborhood of \( x \) to a \( \tilde{g}_t \)-minimal disk \( D \) containing \( x \). Since this disk inherits \( \epsilon \)-convexity from the \( \gamma_n \cdot S_{t_n} \) of which it was a smooth limit, Equation (3.5) implies that the derivative of the signed mean curvatures of the parallel distance-\( r \) surfaces to \( D \) at \( r = 0 \) is greater than \( \epsilon \) at every point in the interior of \( D \).

By passing to a subsequence, we can assume that all of the \( r_n \) are either positive or negative—the argument is the same in both cases so assume that all are positive. By the mean value theorem there is a sequence of \( r_n' \) such that the derivative of
the mean curvature of the parallel distance-$r$ surfaces to $\gamma_n \cdot S_{\delta}$ along the normal geodesic to $\gamma_n \cdot x_n$ is less than $\epsilon/2$ at the distance-$r'_n$ surface, where $0 < r'_n < r_n$. Since neighborhoods of $\gamma_n \cdot x_n$ in $\gamma_n \cdot S_{\delta}$ are smoothly converging to $D$, their parallel distance-$r$ surfaces are smoothly converging to those of $D$. This implies that the derivative of the mean curvature of the parallel distance-$r$ surfaces to $D$ along the normal geodesic at $x$ is less than or equal to $\epsilon/2$ at $r = 0$, which contradicts the previous paragraph.

\[ \square \]

Let $S^r_{\delta}$ be the parallel surface at signed-distance $r$ from $S_{\delta}$. Then by the previous lemma for $\delta$ sufficiently small (and independent of $t_0$), $S^r_{\delta}$ will remain mean-convex when considered as a surface inside $(\bar{M}, \bar{g}_\delta)$, for $t \in (t_0, t_0 + \delta)$ and $\frac{1}{t} < |r| < \xi$. For $t \in (t_0, t_0 + \delta)$, we now construct the $S_t$. At a number of junctures below, we will put further restrictions on the size of $\delta$ that only depend on $\epsilon$ and the family of metrics.

3.3. Fix a point $p$ on $S_{t_0}$ and let $B(s)$ be the metric disk in $S_{t_0}$ of radius $s$ centered at $p$, where $S_{t_0}$ has the metric induced from $\tilde{g}_{t_0}$. Because $S_{t_0}$ is minimal and $\tilde{g}_{t_0}$ has negative sectional curvature, $\tilde{g}_{t_0}$ has negative curvature and the exponential map $T_p S_{t_0} \to S_{t_0}$ is a diffeomorphism. The boundaries $\partial B(s)$ are therefore embedded circles, and so we can solve the Plateau problem for $\partial B(s)$ in $(\bar{M}, \bar{g}_t)$ to find an embedded $\bar{g}_t$-minimal disk $D(s)$ that bounds $\partial B(s)$, such that every other embedded disk bounding $\partial B(s)$ has area greater than or equal to that of $D(s)$ (\cite{CM1} Chapter 4.)

**Lemma 3.7.** $D(s)$ is contained in the region bounded by $S^r_{t_0}$ and $S^{-r}_{t_0}$ for $\xi/4 < r < \xi$.

We first prove another lemma. Let $S$ be the geodesic plane in $\mathbb{H}^3$ that corresponds to $S_{t_0}$. The circles in $S^2$ parallel to the boundary at infinity of $S$ form a foliation of $\partial_\infty \mathbb{H}^3 \cong S^2$ minus two points. Let $\{S(x): x \in \mathbb{R}\}$ be the foliation of $\mathbb{H}^3$ by totally geodesic planes whose limit sets are the circles in $S^2$ parallel to $\partial_\infty S$, parametrized so that $S(0) = S$, and let $S_{t_0}(x)$ be the corresponding minimal planes in $(\bar{M}, \bar{g}_{t_0})$.

**Lemma 3.8.** The $S_{t_0}(x)$ are the leaves of a foliation of $\bar{M}$.

**Proof.** First of all by Property 1 the $S_{t_0}(x)$ are disjoint. Suppose for contradiction that there is some point $q$ that is not contained in any $S_{t_0}(x)$. Let $x_q^+$ be the infimum of the set of all $x$ such that $S_{t_0}(x)$ is above $q$. This set is nonempty since the $S_{t_0}(x)$ are at uniformly bounded Hausdorff distance from the $S(x)$ considered as subspaces of $(\bar{M}, \bar{g}_t)$, so $x_q^+$ is well-defined. If $S_{t_0}(x_q^+)$ did not contain $q$, then since the $S_{t_0}(x)$ vary smoothly by Property 2 above, it must be above $q$ and we could therefore produce $S_{t_0}(x)$ above $q$ with $x < x_q^+$ for a contradiction. It follows that $q$ is contained in $S_{t_0}(x_q^+)$. The existence of local product charts follows from Property 2, and so the $S_{t_0}(x)$ are the leaves of a foliation.

\[ \square \]

We now give the proof of Lemma 3.7.

**Proof.** Suppose for contradiction that $D(s)$ is not contained in the region bounded by $S^r_{\delta}$ and $S^{-r}_{\delta}$. Then either $D(s)$ has points above $S^r_{\delta}$ or below $S^{-r}_{\delta}$— assume that the first is true since the proof in the second case is the same. Note that the
parallel signed-distance-$r$ surfaces $S^t_r(x)$ for the $S^t_{x_0}(x)$ in $(\tilde{M}, \tilde{g}_{x_0})$ are mean-convex in $(\tilde{M}, \tilde{g}_{x_0})$ since $\frac{\xi}{4} < r < \xi$ and are the leaves of a foliation of $\tilde{M}$ for fixed $r$, since the $S^t_{x_0}(x)$ are the leaves of a foliation of $\tilde{M}$ by the previous lemma. It follows that the set of $x > 0$ such that $S^t_r(x)$ intersects $D(s)$ is non-empty.

Furthermore, since the $S^t_r(x)$ are at uniformly bounded Hausdorff distance from the $S(x)$ considered inside $\{M, \tilde{g}_1\}$, for $x$ sufficiently large the intersection of $S^t_r(x)$ and $D(s)$ will be empty. Let $x'$ be the largest $x$ such that $S^t_{x_0}(x)$ intersects $D(s)$. Then since $D(s)$ is strictly contained on one side of $S^t_r(x')$ except at the non-empty set of points where they intersect, the mean convexity of $S^t_{x_0}(x')$ gives a contradiction.

\[ \square \]

The next lemma follows from Schoen’s curvature estimate for stable minimal surfaces (\cite[Theorem 3]{Sch83}).

**Lemma 3.9.** Let $\rho > 0$ be less than the injectivity radius of any $(M, g_t)$, and let $x$ be a point on a stable $\tilde{g}_t$-minimal embedded disk $D \subset M$ at a $\tilde{g}_t$-distance of at least $d$ from the boundary of $D$. Then there is some constant $C$ such that the $L^\infty$-norm of the second fundamental form of $D$ at $p$ is bounded above by $C$. The constant $C$ depends only on $\rho$, $d$, and bounds on the $g_t$ (specifically, bounds on the $L^\infty$ norms of the curvature tensor and its covariant derivative.)

We require two more lemmas to get the control on the $D(s)$ we will need to pass to a limit.

**Lemma 3.10.** Let $\eta > 0$, $C$, and the family $g_t$ be given. Then there exists $\eta' > 0$ such that for any $t$ and any two embedded surfaces $S_1$ and $S_2$ in $(\tilde{M}, \tilde{g}_t)$ with second fundamental forms bounded above by $C$ in magnitude, the following holds. Suppose there are points $x_1 \in S_1$ and $x_2 \in S_2$ such that $d_{\tilde{g}_t}(x_1, x_2) < \eta'$, but the distance between the unit normal vectors to $S_1$ and $S_2$ at $x_1$ and $x_2$ respectively in the unit tangent bundle to $\tilde{M}$ is at least $\eta$. Then $S_1$ and $S_2$ intersect.

**Proof.** Assume that the statement of the lemma fails for some $\eta > 0$, and let $\{S^n_i : i = 1, 2\}$ be a sequence of pairwise disjoint surfaces in $(\tilde{M}, \tilde{g}_{t_n})$ as above with points $x^n_i$ such that $d(x^n_1, x^n_2) \to 0$ and the distance between the respective normal vectors at $x^n_1$ and $x^n_2$ is at least $\eta$ for all $n$.

Fix some $\rho > 0$. For each $n$, identify by the exponential map the ball of radius $\rho$ at $x^n_1$ with the ball of that radius in the tangent space to $x^n_1$, and radially dilate the pulled back metric so that the distance between the origin and dilated image of the point corresponding to $x_2^n$ in the dilated metric is 1 (that is to say, dilate by a factor of $\frac{1}{d(x_1^n, x_2^n)}$.) Call the dilated metric $h_n$. For what follows, we fix isometric identifications of the tangent spaces $T_{x^n_1}S^n_1$ in the inner product induced by $\tilde{g}_{t_n}$, for the purpose of taking limits.

Because $d(x_1^n, x_2^n) \to 0$ and the $\tilde{g}_t$ have uniformly bounded geometry, the $h_n$-balls of any given radius centered at the origin are smoothly converging to Euclidean balls of that radius. Moreover, since we have a uniform bound on the second fundamental forms of the $S^n_i$, the intersections of their pre-images with the $h_n$-balls of any given radius centered at the origin are uniformly $C^1$-converging to planes, up to taking subsequences.

In the case of the $S^n_1$, this plane will simply be a subsequential limit of the tangent planes to $S^n_1$ at $x_1^n$. Let $\Pi_n$ be the parallel transport of the tangent plane
to $S^n_1$ at $x^n_1$ along the geodesic joining $x^n_1$ to $x^n_2$. Then in the case of $S^n_2$, the subsequential limit plane will be a translated copy of a subsequential limit of the $\Pi_n$. The fact that for all $n$ the normals at $x^n_1$ and $x^n_2$ are at a distance of at least $\eta$ implies that these two planes cannot be parallel. This means that $S^n_1$ and $S^n_2$ will have to intersect for some large $n$, a contradiction.

We now truncate the $D(s)$ to obtain disks better suited to taking limits. Since $B(s)$ is a disk bounding $\partial B(s)$ that intersects $\gamma$ exactly once, we know that $D(s)$ intersects $\gamma$ at least once, at some point $p$, by Lemma 3.7. Let $\sigma(s)$ be the smallest number $\sigma$ such that the $\tilde{g}$-ball $B_{p_\sigma}(\cdot)$ of radius $\sigma$ centered at $p_\sigma$ intersects $\partial D(s)$. Since $\partial D(s)$ is the boundary of an intrinsic metric disk in $S_{t_0}$, and $S_{t_0}$ is properly embedded, it must be the case that $\sigma(s) \to \infty$ as $s \to \infty$.

For generic $\sigma < \sigma(s)$, the boundary of the $\tilde{g}$-ball $B_{p_\sigma}(\cdot)$ of radius $\sigma$ centered at $p_\sigma$ will intersect $D(s)$ in a union of circles by Sard’s theorem. Note that the minimality of $D(s)$ implies that all connected components of $B_{p_\sigma}(\sigma) \cap D(s)$ are disks. If there were an annuli, then its interior component in $D(s)$ would be a minimal disk $D'$ with boundary on the boundary of $B_{p_\sigma}(\sigma)$. Taking the largest $\sigma'$ so that $B_{p_\sigma}(\sigma')$ intersected $D'$, the fact that metric spheres are mean-convex in negative sectional curvature gives a contradiction.

Now choose $\sigma$ in the interval $(\sigma(s) - 1 - \frac{4}{\eta}, \sigma(s) - 1)$ so that $\partial B_{p_\sigma}(\sigma) \cap D(s)$ is a union of circles, and let $D(s)'$ be the connected component of $B_{p_\sigma}(\sigma) \cap D(s)$ containing $p_\sigma$, which by the above paragraph is a disk. Since we took $\sigma$ in the above interval, Lemma 3.9 applies to give an upper bound on the absolute values of the principal curvatures of the $D(s)'$.

Lemma 3.11. There exists $\delta'$, depending only on $\varepsilon$ and the family of metrics, such that as long as $\delta$ was chosen less than $\delta'$, the nearest-point projection of $D(s)'$ to $S_{t_0}$ is well-defined and a diffeomorphism onto its image.

Proof. Fix a point $q$ on $D(s)'$. By Lemma 3.7, every point on $D(s)'$ is at a distance of at most $\xi$ from a point on $S_{t_0}$. The normal exponential map for $S_{t_0}$ defines a diffeomorphism from $S_{t_0} \times (-r, r)$ to the $r$-neighborhood of $S_{t_0}$, for $\frac{\xi}{4} < r < \xi$, so let $q'$ be the point on $S_{t_0}$ such that the normal geodesic to $S_{t_0}$ at $q'$—call it $\phi$—passes through $q$. Let $q_{\pm\varepsilon}$ be the points in $S_{t_0}^{\pm\varepsilon}$.

The distances between the unit normal vectors to $D(s)'$ and $S_{t_0}^{\pm\varepsilon}$ at $q$, $q'$, and $q_{\pm\varepsilon}$ in the appropriate orientations are pairwise $O(\xi)$ by Lemma 3.10. Lemma 3.10 applies because the $S_{t_0}^{\pm\varepsilon}$ have principal curvatures bounded above in absolute value and $D(s)'$ has principal curvatures bounded above in absolute value by Lemma 3.9. Since the normal vectors to the $S_{t_0}^{\pm\varepsilon}$ are $O(\xi)$-close to those of $S_{t_0}$ at points that correspond under normal projection, it follows that the normal vectors to $D(s)'$ and $S_{t_0}$ at $q$ and $q'$ are $O(\xi)$-close. This implies that, provided $\xi(= O(\delta))$ and $\delta$ were taken small enough, that the tangent vector to $\phi$ at $q$ is very close to being perpendicular to $D(s)'$. Therefore, for points $q''$ on $S_{t_0}$ in a small neighborhood $U$ of $q'$, the geodesic normal to $S_{t_0}$ at $q''$ will intersect $D(s)'$ nearly perpendicularly at some point close to $q'$. It follows that the normal exponential map defines a diffeomorphism from $U$ to a neighborhood of $q$, so the normal projection map from $D(s)'$ to $S_1$ is a local diffeomorphism at all points of $D(s)'$, including points on
Lemma 3.12. Since guaranteed by $g$, the surface $S_t$ inherits the property of being absolutely minimizing from the $D(s)'$ of which it was the smooth limit.

3.4. We now construct $S_t$ as a limit of the $D(s)'$. Let $B_n$ be the metric disk in $S_{t_0}$ with radius $n$ and center $p$ in the metric on $S_{t_0}$ induced by $\tilde{g}_t$. If $s$ is sufficiently large, the previous lemma tells us that the normal exponential map $\gamma_{t_s}$ properly embedded $S_{t_0}$ and $S_t$ is therefore a graph over $B_n$ in normal exponential coordinates for $S_t$. Since the $D(s)'$ have uniformly bounded principal curvatures, we can pass to a subsequence of the $B_n(s)$ for a sequence of $s \to \infty$ that $C^1$-converges (and thus by standard elliptic PDE theory smoothly converges) over compact subsets of $B_n$. Doing this for each $B_n$ and taking a diagonal subsequence we obtain an embedded minimal surface which we call $S_t$. Since $S_t$ is a smooth limit of the $D(s)'$, the normal projection map defines a diffeomorphism from $S_t$ to $S_{t_0}$, and $S_t$ is a smooth properly embedded plane. The surface $S_t$ inherits the property of being absolutely minimizing from the $D(s)'$ which it was the smooth limit.

3.5. We now check that the uniqueness conditions in Theorem 3.4 are met. Since $S_t$ is at a Hausdorff distance from $S_{t_0}$ bounded by $\xi$, $S_t$ is at finite Hausdorff distance from some element $S$ of $\mathcal{P}\tilde{g}_t$, since this is true for $S_{t_0}$. Let $S'_t$ be the properly embedded $\epsilon$-convex $\tilde{g}_t$-minimal disk at finite Hausdorff distance from $S$, guaranteed by $g_t$'s membership in $\Omega_*$. We will show that $S'_t = S_t$.

Lemma 3.12. $S'_t$ is contained in the region bounded by $S_{t_0}$ and $S_{t_0}^\epsilon$ for $r \in (\frac{\epsilon}{4}, \xi)$.

Proof. The proof is similar to that of Lemma 3.7 but with an extra step because $S'_t$ is not compact. Let $S_{t_0}(x)$ be the foliation given by Lemma 3.8 and let $S'_{t_0}(x)$ be the signed-distance-$r$ parallel surfaces to the $S_{t_0}(x)$, where $\frac{\epsilon}{4} < |r| < \xi$ and these surfaces are all mean-convex. Now assume that $S'_t$ has points that are not contained in the region bounded by $S'_{t_0}$ and $S_{t_0}^\epsilon$—suppose for contradiction that it has points above $S'_{t_0}$. Let $x_{\max}$ be supremum over all $x$ such that $S'_{t_0}(x)$ intersects $S'_t$. The number $x_{\max}$ is finite because $S'_t$ is at finite Hausdorff distance from $S$ and each $S'_{t_0}(x)$ is at finite and uniformly bounded Hausdorff distance from the corresponding element of $\mathcal{P}\tilde{g}_t$.

We can find a sequence of points $p_n$ on $S'_t$ and $q^*_n$ on $S'_{t_0}(x_{\max})$ such that $d(p_n, q^*_n)$ tends to zero. Let $q_n$ be the points on $S_{t_0}(x_{\max})$ that are the images of the $q^*_n$ under normal projection. Now apply covering transformations $\gamma_n$ to take the $p_n$ back to a fixed compact set containing a fundamental domain for the action of $\pi_1(M)$ on $(M, \tilde{g}_t)$. Let $p$ and $q$ be subsequential limits of the $\gamma_n \cdot p_n$ and $\gamma_n \cdot q_n$ respectively. Since $S'_t$ and $S_{t_0}(x_{\max})$ are minimal surfaces in $\tilde{g}_t$ and $\tilde{g}_{t_0}$ with bounded principal curvatures, we can pass to a subsequence on which $\gamma_n \cdot S'_t$ and $\gamma_n \cdot S_{t_0}(x_{\max})$ are smoothly and graphically converging in small balls centered at $p$ and $q$ respectively. The $r$-neighborhood of the subsequential limit of the $\gamma_n \cdot S_{t_0}(x_{\max})$, whose boundary is strictly mean-convex, will then touch the subsequential limit of the $\gamma_n \cdot S'_t$, which is minimal, on one side at $p$, which is a contradiction.

Since $S'_t$ is $\epsilon$-convex, signed-distance-$r$ surfaces to $S'_t$ are strictly mean-convex for $0 < |r| < \xi$, for the $\xi$ given by Lemma 3.6. By the previous lemma $S'_t$ is contained in the $\xi/4$ $\tilde{g}_{t_0}$-neighborhood of $S_{t_0}$, so the $\tilde{g}_{t_0}$-normal projection from $S'_t$ to $S_{t_0}$ is
well-defined. Since $S'_t$ and $S_{t_0}$ are properly embedded, the normal projection is a proper local diffeomorphism, and consequently surjective. Therefore, because $S_t$ is contained in the $\xi/4 \hat{y}_{t_0}$-neighborhood of $S_{t_0}$, it follows that $S_t$ is contained in the $\xi/2 \hat{y}_{t_0}$-neighborhood of $S'_t$, and therefore, as long as $\delta$ was taken small enough, the $3\xi/4 \hat{y}_{t}$-neighborhood of $S'_t$. If $S_t$ were not equal to $S'_t$, then we could take a sequence of points on $S_t$ approaching a supremal mean-convex parallel surface to $S'_t$ and produce a contradiction as in the proof of the previous lemma. Therefore $S_t$ is equal to $S'_t$, and in particular is $\epsilon$-convex.

In a similar way, we can check that $S_t$ is the unique properly embedded minimal surface at finite Hausdorff distance from $S$. The argument in Lemma 3.12 shows that any such minimal surface must be contained in the $r$-neighborhood of $S_t$, and from there we can show it must be equal to $S_t$ by the reasoning of the previous paragraph.

3.6. We now check that the surfaces $S_t$ we have constructed satisfy Property 2. Assume for contradiction that they do not. Then there is some sequence $S_n$ of totally geodesic planes that converges to $S$ on compact subsets, while the corresponding sequence of minimal surfaces $S_{n,t}$ in $\hat{y}_t$ is not smoothly converging to the minimal plane $S_t$ corresponding to $S$. Since the $S_{n,t}$ are minimal, $C^3$ convergence of the $S_{n,t}$ on compact sets would imply smooth convergence, so we can assume that there is some $\eta > 0$ such that the lifts of the $S_{n,t}$ to the unit tangent bundle by their normal vectors all have points in some fixed compact subset of $\tilde{M}$ at a distance of at least $\eta$ from the lift of $S_t$ to the unit tangent bundle.

The sequence $S_{n,t_0}$ of $\hat{y}_{t_0}$ minimal surfaces corresponding to the $S_n$ converges smoothly to the minimal surface $S_{t_0}$ corresponding to $S$ by assumption. For every compact set $B$ in $(\tilde{M}, \hat{y}_t)$ the intersection $B \cap S_{n,t}$ is therefore, in normal exponential coordinates for the $\xi$-neighborhood of $S_{t_0}$, a graph over $S_{t_0}$ for large enough $n$, since $S_{n,t}$ is a graph over $S_{n,t_0}$. Because we have uniform bounds on the second fundamental forms of the $S_{n,t}$ by Lemma 3.9 we can then proceed exactly as in 3.4 above to pass to a subsequential limit, which is a properly embedded minimal surface at finite distance from $S_t$ and so must equal $S_t$ by the uniqueness properties of $S_t$ verified in 3.5. But since the points on the $S_{n,t}$ where the normal vectors are at least $\delta$ from any normal vector to $S_t$ are contained in a compact set, we can find an accumulation point of any infinite sequence of them. This contradicts equality of the limit minimal surface with $S_t$.

3.7. We now check that the $S_t$ give rise to a foliation of $Gr_2(\tilde{M})$ that is invariant under the action of $\pi_1(M)$. First, the set of $S_t$ is invariant under covering transformations. This is because we’ve already checked that each $S_t$ is the unique properly embedded minimal disk at finite distance from some element of $P_{\hat{y}_t}$ — that is, some totally geodesic plane in $\mathbb{H}^3$ considered as a subspace of $(\tilde{M}, \hat{y}_t)$ and the set $P_{\hat{y}_t}$ is invariant under covering transformations.

By our inductive hypothesis, there exists a continuous self-homeomorphism $\Phi_0$ of $Gr_2(\tilde{M})$ sending (lifts of) totally geodesic planes in $\mathbb{H}^3$ to the corresponding (lifts of) minimal disks $S_{t_0}$. Since nearest-point projection defines a diffeomorphism between the $S_t$ and the corresponding $S_{t_0}$, by composing with $\Phi_0$ we obtain a self-map $\hat{\Phi}$ of $Gr_2(\tilde{M})$ diffeomorphically sending (lifts of) totally geodesic planes to (lifts of) $S_t$. That $\hat{\Phi}$ is continuous follows from the fact that the $S_t$ satisfy Property 2. Note also that $\hat{\Phi}$ commutes with diffeomorphisms of $Gr_2(\tilde{M})$ induced by covering
transformations of $\tilde{M}$. This follows from the fact that this is true for $\tilde{\Phi}_0$, and that since the $S_t$ and $S_{t_0}$ are invariant under covering transformations, nearest-point projection from $S_t$ to $S_{t_0}$ commutes with covering transformations.

This shows that $\Phi$ descends to a continuous self-map of $Gr_2(M)$, which since this map is $O(\xi)$-close to the corresponding self-homeomorphism of $Gr_2(M)$, it must be the case, provided $\delta$ (since $\xi = O(\delta)$) was taken small enough, that the self-map of $Gr_2(M)$ induced by $\Phi$ is homotopic to the homeomorphism induced by $\Phi_0$. The map $\Phi$ therefore has mod-2 degree one (recall that $Gr_2(M) \cong M \times \mathbb{RP}^2$ is not orientable) and so is surjective. This means that every point of $Gr_2(\tilde{M})$ is the tangent plane of some $S_t$. The main step remaining to prove that the $S_t$ give a foliation is to check that each point of $Gr_2(\tilde{M})$ is the tangent plane of a unique $S_t$, or in other words that $\Phi$ is injective. It will then follow that $\Phi$ is a homeomorphism because a continuous bijection between compact metric spaces is a homeomorphism.

The main tool for proving injectivity of $\Phi$ will be the following lemma (a proof is immediate from the results in [CM11, Section 5.3].) This is one of the key places in the paper where we use that the ambient dimension is three.

**Lemma 3.13.** Let $S_1$ and $S_2$ be properly embedded minimal planes in $(\tilde{M}, \tilde{g}_t)$. Then the intersection $S_1 \cap S_2$ is an embedded graph. At any point where the two intersect non-transversely, the intersection is locally homeomorphic to a union of $n \geq 2$ straight lines with a common point.

Let $S$ and $S'$ be totally geodesic disks in $\mathbb{H}^3$, and let $S_t$ and $S'_t$ be the corresponding minimal disks in $(\tilde{M}, \tilde{g}_t)$.

Then by Lemma 3.13, $S_t$ and $S'_t$ intersect in a graph $\Gamma$. To show that they never intersect non-transversely and prove injectivity of $\Phi$, it is enough by the previous lemma to show that $\Gamma$ is either empty or a disjoint union of lines.

We claim that since $S_t$ is absolutely-minimizing, the set difference $S_t - \Gamma$ cannot have any bounded connected components. For contradiction, assume it had such a connected component $D$, which by taking an innermost such component we can assume is topologically a disk. Then by taking some large circle $C$ in $S_t$ which bounds a disk that contains the boundary of $D$ in $S_t$, we could, by cutting out $D$ and replacing it with the bounded connected component of $S_t - \partial D$ (which has the same area as $D$ since all disks in $S_t$ and $S'_t$ minimize area over comparison disks with the same boundary), produce a non-$C^1$ solution to the Plateau problem for $C$ in the ambient space. This is impossible though, since the area can be decreased by smoothing in neighborhoods of non-$C^1$ points of transverse intersection ( [CM11, Section 5.3].) This shows that the $S_t$ we have constructed satisfy Property 1.

In the case that $S_t$ and $S'_t$ have disjoint boundaries at infinity, we are done by the last paragraph, since $S_t$ and $S'_t$ do not intersect outside of some compact set, so if the two intersected there would have to be a compact connected component of the complement of the intersection.

Otherwise, assume $S$ and $S'$ intersect in a line. Assume for contradiction that $S_t$ and $S'_t$ intersect non-transversely at a point $p$, and let $\Gamma_0$ be the connected component of $\Gamma$ containing $p$. Locally at $p$, $\Gamma_0$ looks like $n > 1$ lines meeting at a point. If there are more non-transverse intersections these lines might further branch, but they will never intersect each other at a point besides the initial branch point since that would create a compact connected component of their complement.
Since $S_1$ and $S'_1$ are at finite Hausdorff distance from $S$ and $S'$ respectively, all points on $\Gamma$ are at uniformly bounded distance from $S \cap S'$. It follows that the complement of $\Gamma_0$ in $S_1$ has a connected component $D_0$ all of whose points are at uniformly bounded distance from $S'_1$. Since the proof of Lemma 3.8 only used Properties 1 and 2 which we have already verified, there is a foliation $\mathcal{F}$ of $\tilde{M}$ containing $S'_1$ as a leaf by applying that lemma to $S'_1$ and $S'$.

Without loss of generality, assume that $D_0$ has points above $S'_1$, and let $d$ be the supremum of the set of distances from points in $D_0$ above $S'_1$ to $S'_1$. Since the $\xi$-neighborhood of $S'_1$ has a local mean-convex foliation, if $d$ were less than $\xi/2$, we could get a contradiction by the argument of Lemma 3.12. Otherwise, we could choose another leaf $S''_1$ in the foliation $\mathcal{F}$ above $S'_1$ so that all points on $D_0$ above $S''_1$ were at a distance of less than $\xi/2$ from $S'_1$. Since the $\xi$-neighborhood of $S''_1$ also has a mean-convex foliation, this would lead to a contradiction in the same way. (A similar argument shows that $S_1$ and $S'_1$ intersect in a single line, although we only need to show they intersect transversely.)

The only case left to check is if $S_1$ and $S'_1$ intersect at a single point at infinity. The proof here is like the last case. If $S_1$ and $S'_1$ intersect non-transversely at some point, then we can similarly deduce the existence of some unbounded connected component of the intersection $\Gamma$ all of whose points are at bounded distance from $S'_1$. We can then produce a contradiction as above by taking a mean-convex foliation of a neighborhood of $S'_1$, or else some other $S''_1$ above or below $S'_1$.

### 3.8. Local Product Charts

A smooth local product chart for our foliation at any $p$ in $\tilde{M}$ and any tangent plane $P$ to $p$ in $Gr_2(\tilde{M})$ can be constructed as follows. Let $S_1$ be the surface which has $P$ as a tangent plane. The transversal to our chart will be homeomorphic to the product of a small neighborhood $U$ of $P$ in the Grassmannian of the tangent space $Gr_2(T_p \tilde{M})$ with a small geodesic segment $\gamma$ in $\tilde{M}$ containing $p$ and normal to $P$ at $p$. We diffeomorphically identify this product with a subspace $T$ of $Gr_2(\tilde{M})$ by parallel transporting $U$ along $\gamma$.

Take a small metric disk $V$ centered at the origin in the tangent space to $S_1$ at $p$. We construct a map from $V \times T$ to a small neighborhood of $P$ in $Gr_2(\tilde{M})$ as follows.

Let $(v, (p', P')) \in V \times T$ be given. We can identify $P'$, by parallel transport along $\gamma$, with a linear subspace $P'_p$ of $T_p(\tilde{M})$. Provided $U$ and $\gamma$ were chosen sufficiently small, the normal projection $v'$ to $P'_p$ of $v$ will have norm greater than $\frac{1}{2} |v|$.

Parallel transport of tangent vectors gives a natural identification between $P'_p$ and $P'$ viewed as subspaces of $T_p \tilde{M}$ and $T_p \tilde{M}$ respectively. Take the vector in $P'$ corresponding to $v'$—call it $v''$—and consider $v''$ inside the tangent space at $p'$ to the surface $S'_1$ that has $P'$ as a tangent plane. We map $v''$ to its image under the exponential map of $S'_1$ at $p'$ in the metric on $S'_1$ induced by $\tilde{g}$ and define the tangent plane to $S'_1$ of this point to be the image of $(v, (p', P'))$ in $Gr_2(\tilde{M})$ under our map.

Knowing that the surfaces $S_1$ vary smoothly in their tangent planes, we will know that their exponential maps vary smoothly, and smoothness of the coordinate map we have defined will follow. Suppose that a sequence $S_{1,n}$ of minimal disks we have constructed has tangent planes $P_n$ converging to the tangent plane $P$ to $S_1$ at a point. Then we need to show that $S_{1,n}$ is smoothly converging to $S_1$ on
compact subsets. That the convergence is $C^1$ follows from the fact that $\hat{\Phi}$ is a
homeomorphism, and elliptic PDE theory implies that the convergence is smooth.

Since the differential of the coordinate map at $(0, (p, P))$ is non-singular, we can
apply the inverse function theorem to restrict to a possibly smaller neighborhood
of $(0, (p, P))$ in $V \times T$ on which it is a diffeomorphism onto its image. This shows
that every point in $Gr_2(\hat{M})$ is contained in a smooth product chart for the foliation.
The proof of Theorem $3.3$ is now complete.

3.9. **Proof of Theorem 1.2.** We now explain how to modify the proof of Theorem
3.3 to give a proof of Theorem 1.2. Let $g_t$ be a smooth family of metrics as in the
statement of Theorem 1.2. Then if $g_{t_0}$ is in $\Omega_\epsilon$, we claim that there is some $\delta$
depending only $\epsilon$ and bounds on the geometry of the $g_t$ such that $g_t$ is in $\Omega_{\epsilon/2}$
for $|t - t_0| < \delta$. The $\hat{g}_t$-minimal surfaces $S_t$ in $\hat{M}$ can be constructed exactly as
above. Each of the $S_t$ can be made as $C^1$-close to the corresponding $S_{t_0}$ as desired,
uniformly in $S_t$, by making $\delta$ small. Elliptic PDE theory tells us that $C^1$-close
implies $C^2$-close, so if we chose $\delta$ small enough, the $S_t$ will be $\epsilon/2$ convex. The
metric $g_t$ will then be contained in $\Omega_{\epsilon/2}$, and the verification that the $S_t$ give a
foliation and the construction of the map $\hat{\Phi}$ can proceed as above.

The construction of the foliations thus continues to work unless there is some
time $T$ such that for every $\epsilon > 0$ and sequence of times $t_n \not\rightarrow T$ there is some $S_n$
in $(\hat{M}, \hat{g}_n)$ that fails to be $\epsilon$-convex for $n$ large enough. This proves Theorem 1.2.

4. **Applications**

4.1. **Density.** In this section, we prove some density results for the stable immersed
minimal surfaces in $M$ corresponding to the surface subgroups constructed by Kahn
and Markovic.

Let $g$ be a negatively curved metric on $M$ to which Theorem 1.2 or Theorem 3.3
applies to produce a foliation. Let $F_g$ be the foliation of $Gr_2((M, g))$ by (lifts of)
g-minimal immersed disks, and let

$$\Phi : Gr_2((M, g_{hyp})) \rightarrow Gr_2((M, g))$$

be the conjugating homeomorphism that sends leaves of $F_{g_{hyp}}$ to leaves of $F_g$. Every
leaf of $F_{g_{hyp}}$ is either dense or properly immersed, so since $\Phi$ is a homeomorphism
the leaves of $F_g$ satisfy the same dichotomy.

Fix a compact set $K_0 \subset \hat{M}$ that contains a small neighborhood of a connected
polyhedral fundamental domain for the action of $\pi_1(M)$ on $\hat{M} \cong \mathbb{H}^3$. Let $\Sigma_n$ (resp.
$\Sigma'_n$) be a sequence of stable immersed $g_{hyp}$-minimal (resp. $g$-minimal) surfaces with
lifts $\hat{\Sigma}_n$ (resp. $\hat{\Sigma}'_n$) to $\hat{M}$. Suppose the lifts $\hat{\Sigma}_n$ and $\hat{\Sigma}'_n$ were chosen so that all of
the $\hat{\Sigma}_n$ intersect $K_0$, and that $\hat{\Sigma}'_n$ and $\hat{\Sigma}_n$ are at finite Hausdorff distance from each
other in $\hat{M}$ in either (or equivalently both) of $\hat{g}$ or $\hat{g}_{hyp}$.

**Lemma 4.1.** Fix a circle $C$ in $\partial_\infty \mathbb{H}^3 \cong S^2$, and suppose that the limit sets of the
$\hat{\Sigma}_n$ are Hausdorff converging to $C$ in $\partial_\infty \mathbb{H}^3$. Let $L'$ be the minimal disk in $(\hat{M}, \hat{g})$
whose lift to $Gr_2(\hat{M})$ is the image under $\hat{\Phi}$ of the lift of the totally geodesic plane $L$
with limit set $C$. Then the $\hat{\Sigma}'_n$ converge smoothly to $L'$ uniformly on compact sets.

**Proof.** Let $L'(t)$ be the foliation of $\hat{M}$ given by Lemma 3.3 with $\hat{g}$-minimal leaves
and $L'(0) = L'$. Let $L(t)$ be the corresponding foliation of $\mathbb{H}^3$ by geodesic planes
with $L(0) = L$, so that $\hat{\Phi}$ sends lifts of the $L(t)$ to lifts of the $L'(t)$. For every $\alpha > 0$
and large enough $n$, $\Sigma_n'$, which is contained in the convex hull of its limit set, will be contained between $L(\alpha)$ and $L(-\alpha)$. We claim that $\Sigma_n'$ is contained between $L'(\alpha)$ and $L'(-\alpha)$.

Recall that the $\xi$-neighborhood of every $L'(t)$ has a foliation by mean-convex parallel surfaces, where $\xi$ only depends on $\epsilon$. Now, if $\Sigma_n'$ were not contained between $L'(\alpha)$ and $L'(-\alpha)$, then using the mean-convex parallel surfaces of the $L'(t)$ and the fact that $\Sigma_n'$ and the $L'(t)$ are at uniformly bounded Hausdorff distance from respectively $\Sigma_n$ and the corresponding $L(t)$, one could produce a contradiction by arguments similar to those of the last section. One can also show by reasoning similar to subsection 3.6 of the last section that $\Sigma_n'$ is $C^1$-converging and thus smoothly converging to $L' = L'(0)$ on compact subsets.

$\square$

Kahn and Markovic showed that for every circle $C$ at infinity in $\partial_{\infty} \mathbb{H}^3$ there is a sequence of surface subgroups $\Gamma_n$ of $\pi_1(\mathcal{M})$ whose limit sets $C_n$ are Hausdorff converging to $C$. [KM12b]. The $C_n$ are the images of round circles under $K_n$-quasiconformal homeomorphisms of $S^3$— or $K_n$-quasicircles—with $K_n$ tending to 1. By [SU82] or [SY79], there exists a sequence $\Sigma_n$ (resp. $\Sigma_n'$) of stable properly immersed $g_{hyp}$-minimal (resp. $g$-minimal) surfaces in $\mathcal{M}$ whose fundamental groups injectively include in $\pi_1(\mathcal{M})$ to subgroups conjugate to $\Gamma_n$.

**Theorem 4.2.** Let $C$ be a circle in $\partial_{\infty} \mathbb{H}^3$ bounded by a geodesic plane $L$ in $\mathbb{H}^3$ that does not project to a properly immersed surface in $\mathcal{M}$. Then for any sequence $\Sigma_n$ of stable properly immersed minimal surfaces with lifts $\Sigma_n$ to $\mathbb{H}^3$ whose boundaries at infinity are $K_n$-quasicircles with $K_n$ tending to 1 and Hausdorff converging to $C$, the following is true. Let $U$ be any open set in $Gr_2(\mathcal{M})$. Then there exists $N$ so that for $n > N$, the intersection of the lift of $\Sigma_n'$ to $Gr_2(\mathcal{M})$ with $U$ is nonempty.

**Proof.** By [Rat91] or [Sha91], the lift to $Gr_2(\mathcal{M})$ of the covering projection of $L$ to $\mathcal{M}$ is dense. Let (the lift of) $L'$ be the image under $\Phi$ of the (lift of) $L$. Then as observed earlier in the section, $L'$ is also dense in $Gr_2(\mathcal{M})$. It is enough to prove the theorem for $U$ a small ball in $Gr_2(\mathcal{M})$. For any such $U$, let $\tilde{U}$ be a lift of $U$ to $Gr_2(\mathcal{M})$. Then we can find $\gamma \in \pi_1(\mathcal{M})$ so that the image $\gamma \cdot L'$ of (the lift to $Gr_2(\mathcal{M})$ of) $L'$ under the covering transformation corresponding to $\gamma$ intersects $\tilde{U}$.

By Lemma 4.1, the $\Sigma_n'$ are smoothly converging to $L'$ on compact sets, so for all $n$ sufficiently large $\gamma \cdot \Sigma_n'$ will intersect $\tilde{U}$. Therefore for all sufficiently large $n$, $\Sigma_n'$ will intersect $U$.

$\square$

### 4.2. Uniqueness

We now prove uniqueness for properly immersed minimal surfaces whose fundamental groups injectively include to the conjugacy class of a given surface subgroup of $\pi_1(\mathcal{M})$, under the assumption that the limit set of the surface subgroup is close to a circle.

Let $\Sigma$ be a stable properly immersed minimal surface in $(\mathcal{M}, g_{hyp})$ whose fundamental group injectively includes in $\pi_1(\mathcal{M})$ as a subgroup conjugate to a surface subgroup $\Gamma$ of $\pi_1(\mathcal{M})$. Then if the limit set of $\Gamma$ is a $K$-quasicircle for $K$ sufficiently close to 1, the main result of [Sep16] implies that $\Sigma$ will be the unique such surface. The next theorem is an analogous uniqueness result for $(\mathcal{M}, g)$. The proof occupies the remainder of the section.
**Theorem 4.3.** Fix \((M, g)\) to which Theorem 3.2 or 3.4 applies to produce a foliation. Then there exists \(\delta > 0\) such that the following is true. Suppose the limit set \(\partial_{\Sigma} \Sigma\) of \(\Sigma\) is a \(K\)-quasicircle for \(K < 1 + \delta\). Then there is a unique closed \(g\)-minimal surface in \(M\) whose fundamental group injectively includes in \(\pi_1(M)\) as a subgroup conjugate to \(\Gamma\).

4.2.1. In [Sep10], Seppi produces, for every point \(p\) on a minimal disk \(D\) with limit set a \(1 + \eta\) quasicircle, planes \(L_1(p)\) and \(L_2(p)\) respectively above and below \(D\) such that the quantity

\[
(4.4) \quad \max\{d(p, L_1(p)), d(p, L_2(p))\}
\]

tends to zero uniformly in \(p\) as \(\eta\) tends to zero. We can choose \(L_1(p)\) and \(L_2(p)\) such that the arc-length parametrized geodesic \(\gamma_p\) normal to \(\Sigma\) at \(p\) perpendicularly intersects \(L_1(p)\) and \(L_2(p)\) at \(\gamma_p(\delta)\) and \(\gamma_p(\delta)\) respectively, where \(\delta\) is independent of \(p\) and can be made as small as desired by making \(\eta\) small. In addition, Lemma 3.10 implies that \(L_1(p)\), \(L_2(p)\), and \(\delta\) can be chosen so that the following is true: the distance from the tangent plane to \(\Sigma\) at \(p\) from the tangent planes to \(L_1(p)\) and \(L_2(p)\) at \(\gamma_p(\delta)\) and \(\gamma_p(-\delta)\) tends to zero as \(\eta\) tends to zero, uniformly in \(p\).

Let \(\Sigma'\) be a stable \(\pi_1\)-injective \(g\)-minimal surface the inclusion of whose fundamental group injectively includes in \(\pi_1(M)\) as a subgroup conjugate to \(\Gamma\).

4.2.2. We now define, provided \(\eta\) was chosen sufficiently small, a continuous map \(\tilde{f}_\Sigma\) from \(\Sigma\) to \(\Sigma'\) by an “analytic-continuation” argument. To start off, we set

\[
\tilde{f}_\Sigma(p_0) = p'_0.
\]

For any other \(p\) in \(\tilde{\Sigma}\), we choose a path \(\phi: [0, 1] \to \tilde{\Sigma}\) joining \(p_0\) to \(p\), and define a map \(\tilde{f}_\phi: [0, 1] \to \tilde{\Sigma}'\) by a finite induction. First, we set \(\tilde{f}_\phi(0) = p'_0\). Suppose the map has been defined at \(\phi(t_0)\). We assume for our inductive hypothesis that \(\tilde{f}_\phi(t_0)\) is equal to some point on the intersection of \(\Sigma'\) with the geodesic joining the points \(p_{\phi(t_0)}(\delta)\) corresponding to \(\gamma_{\phi(t_0)}(\pm\delta)\) under \(\tilde{\Phi}\). By Lemma 3.9 there is a uniform bound on the \(L^\infty\)-norm of the second fundamental form of \(\Sigma'\), so by Lemma 3.10 and the uniform continuity of \(\tilde{\Phi}\), the distance between the tangent planes to \(L'_1(\phi(t_0))\) and \(L'_2(\phi(t_0))\) at \(p_{\phi(t_0)}(\delta)\) and the tangent plane to \(\Sigma'\) at \(\tilde{f}_\phi(t_0)\) can be made arbitrarily small by making \(\eta\) small.

By making \(\eta\) small, we can thus ensure that there is a metric disk \(B\) in \(\tilde{\Sigma}'\) centered at \(\tilde{f}_\phi(t_0)\) of radius at least some constant depending on bounds on the second fundamental forms of \(\Sigma'\), \(L'_1(\phi(t_0))\), and \(L'_2(\phi(t_0))\), such that normal projection from \(B\) to respectively \(L'_1(\phi(t_0))\) and \(L'_2(\phi(t_0))\) is a diffeomorphism onto its image. Since the \(L'_1(\phi(t))\) vary continuously in \(t\), there consequently exists a uniform \(\tilde{\Phi}\) such that for \(0 < t - t_0 < \tilde{\Phi}\), there is a unique point in \(B\) in the intersection of \(\Sigma'\) and the geodesic joining the points \(p_{\phi(t)}(\pm\delta)\) corresponding to \(\gamma_{\phi(t)}(\pm\delta)\) under \(\tilde{\Phi}\). This completes the induction and shows that \(\tilde{f}_\phi\) can be defined on all of \([0, 1]\).

Since \(\tilde{\Sigma}\) is simply connected, \(\tilde{f}_\phi(1)\) does not depend on the path joining \(p_0\) to \(p\), so we can define \(\tilde{f}_\Sigma(p)\) to be equal to \(\tilde{f}_\phi(1)\) for any \(\phi\) joining \(p_0\) to \(p\). Continuity
of $\hat{f}_\Sigma$ then follows from the continuity of the $\hat{f}_\phi$. The map $\hat{f}_\Sigma$ sends every point on $\hat{\Sigma}$ to a point at uniformly bounded $\hat{g}$ (or $\hat{g}_{hyp}$) distance to itself (where we are taking the same identification between $(\hat{M}, \hat{g}_{hyp})$ and $(\hat{M}, \hat{g})$ used to define $\hat{\Phi}$), and $\hat{f}_\Sigma$ therefore induces a homeomorphism between boundaries at infinity and so is surjective. One can also show that $\hat{f}_\Sigma$ commutes with the action of $\pi_1(\Sigma)$ and thus descends to a map $\Sigma \to \Sigma'$, but we don’t need that in what follows.

4.2.3. Suppose that the projections of the leaves of $\mathcal{F}_\eta$ to $M$ are $\epsilon$-convex. We claim that $\hat{\Sigma}'$ is $\frac{\epsilon}{2}$-convex provided $\eta$ was taken sufficiently small. This is because the unit normal vector to every point on $\hat{\Sigma}'$ is at $O(\eta)$-distance from those of $\epsilon$-convex minimal planes $L'_1$ and $L'_2$, and so elliptic estimates imply that $\hat{\Sigma}'$ can be made as $C^2$-close as desired at $p$ to $L'_1$ and $L'_2$ by making $\eta$ small, and in particular can be made $\frac{\epsilon}{2}$-convex.

4.2.4. Now let $\eta$ be small enough to satisfy all of the restrictions above, as well as one further restriction we will make below in a moment. Assume for contradiction that $\hat{\Sigma}'$ and $\hat{\Sigma}''$ are distinct $g$-minimal surfaces the inclusions of whose fundamental groups in $\pi_1(M)$ are both injective and conjugate to the same surface subgroup $\Gamma$ whose limit set in $\partial_\infty \mathbb{H}^3$ is a $K < 1 + \eta$ quasicircle.

Let $\hat{\Sigma}'$ and $\hat{\Sigma}''$ be lifts to $\hat{M}$ at finite distance from $\hat{\Sigma}$ considered inside $(\hat{M}, \hat{g})$, and note that since these are $\frac{\epsilon}{2}$-convex, there exists a uniform $\xi$ such that the $\xi$-neighborhoods of each has a mean-convex foliation by parallel surfaces. This implies that $\hat{\Sigma}'$ and $\hat{\Sigma}''$ are at a Hausdorff distance of at least $\xi$ from each other.

For each of $\hat{\Sigma}'$ and $\hat{\Sigma}''$ we have a map $\hat{f}_\Sigma$ defined as above. Since the definition of the two maps is the same up until taking the intersection with a geodesic segment in $(\hat{M}, \hat{g})$ of length $O(\eta)$, the images of each point on $\hat{\Sigma}$ under the two maps will be $O(\eta)$-close to each other. Taking $\eta$ small enough to make this distance less than $\xi$, the fact that both maps $\hat{f}_\Sigma$ are surjective gives a contradiction.

5. Quantitative Density

The standing assumption in this section will be that $M$ is a closed hyperbolic 3-manifold that contains no properly immersed totally geodesic surfaces in its constant curvature metric. For these $M$, we will prove some quantitative versions of the density statements of the previous section.

5.1. Constant Curvature.

We begin with the constant curvature case.

**Definition 5.1.** For a tangent plane $P$ in $Gr_2(M)$ based at $p \in M$, we define $C_{P,r} \subset Gr_2(M)$ as follows. Lift $(p, P)$ to a point $(\tilde{p}, \tilde{P})$ in $Gr_2(\mathbb{H}) \cong Gr_2(\mathbb{H}^3)$, and let $\Pi \subset \hat{M}$ be the geodesic plane tangent to $\tilde{p}$ at $\hat{p}$. Now take the circle $\hat{C}_{P,r}$ in $\Pi$ of radius $r$ centered at $\hat{p}$, lift it to $Gr_2(M)$ by planes tangent to $\Pi$, and let $\hat{D}_{P,r}$ be the totally geodesic disk that $\hat{C}_{P,r}$ bounds. We define $C_{P,r}$ and $D_{P,r}$ to be the projections of $\hat{C}_{P,r}$ and $\hat{D}_{P,r}$ to $Gr_2(M)$, and we define $\mu_{P,r}$ to be the probability measure that corresponds to averaging over $C_{P,r}$ parametrized by arc-length in the metric induced by $Gr_2(M)$.

**Proposition 5.2.** Let

$$f : Gr_2(M) \to \mathbb{R}$$
be a continuous function. Then for every \( \epsilon > 0 \) there exists \( R \) such that for all \( P \in \text{Gr}_2(M) \) and all \( r > R \),

\[
\left| \int_{\text{Gr}_2(M)} f d\mu_{P,r} - \text{avg}(f) \right| < \epsilon,
\]

where \( \text{avg}(f) \) is the average of \( f \) over \( \text{Gr}_2(M) \) in its volume form for the metric induced by the hyperbolic metric on \( M \).

Proof. We are going to prove this by applying Ratner’s measure classification theorem (\cite[Theorem 1.11]{Esk10}.) Fix an orientation for \( C_{P,r} \) and let \( \hat{C}_{P,r} \) be its natural lift to the frame bundle \( F \cong \text{PSL}(2, \mathbb{C})/\pi_1(M) \) of \( M \) by, for a point \( p \in C_{P,r} \), taking the first vector in the frame to be the outward unit normal vector to the projection of \( C_{P,r} \) to \( M \) tangent to (the projection of) \( \Pi \), and the second to be tangent to \( C_{P,r} \) in the direction determined by its orientation. The third vector is then determined by the orientation of \( M \). Let \( \hat{\mu}_{P,r} \) be the probability measure on \( F \) given by averaging over \( \hat{C}_{P,r} \).

Assume that the statement is false, and that for some \( f \) and \( \epsilon \) there existed a sequence \( \mu_{P_n,r_n} \) such that (5.3) fails for all \( n \). We can pull back \( f \) to a function \( \hat{f} \) on \( F \) so that

\[
\left| \int_{F} \hat{f} d\hat{\mu}_{P_n,r_n} - \text{avg}(\hat{f}) \right| \geq \epsilon
\]

for all \( n \). Since \( F \) is compact, we can take a weak-* limit of the \( \hat{\mu}_{P_n,r_n} \) to obtain a probability measure \( \hat{\mu} \) for which the \( \hat{\mu} \)-average and the Haar-measure-average of \( \hat{f} \) differ by at least \( \epsilon \).

Let \( U \) be the projection to \( \text{PSL}(2, \mathbb{R}) \) of the unipotent subgroup of \( \text{SL}(2, \mathbb{R}) \)

\[
\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

Claim: \( \hat{\mu} \) is \( U \)-invariant

Fix an element \( u \in U \). Then for every \( \delta > 0 \), there exists \( N \) such that for \( n > N \), there are arc-length parametrizations \( \phi_1 \) and \( \phi_2 \) of respectively \( u \cdot \hat{C}_{P_n,r_n} \) and \( \hat{C}_{P_n,r_n} \) such that

\[
d(\phi_1(s), \phi_2(s)) < \delta
\]

for all \( s \), where \( d \) is the distance in \( F \). This follows from the fact that metric circles of large radius in \( \mathbb{H}^2 \) can be approximated in large neighborhoods of each point by horocyles, which are preserved by \( U \).

The inequality (5.5) implies that for any continuous \( g : F \to \mathbb{R} \),

\[
\left| \int_{F} g d(u_*\hat{\mu}_{P_n,r_n}) - \int_{F} g d\hat{\mu}_{P_n,r_n} \right|
\]

tends to zero as \( n \to \infty \). Therefore \( \hat{\mu} \) is \( U \)-invariant.

Claim: Any \( U \)-invariant measure \( \hat{\mu} \) on \( F \) must be the volume measure

By Ratner’s measure classification theorem, \( \hat{\mu} \) is supported on a union of closed orbits of subgroups \( H \) of \( \text{PSL}(2, \mathbb{C}) \) containing \( U \). We claim that any such \( H \) must be equal to \( \text{PSL}(2, \mathbb{C}) \), which we check by ruling out intermediate candidates for \( H \) one by one.
First of all, $U$ has no closed orbit in $PSL(2, \mathbb{C})$ is mapped to another point on the same orbit by the action of a matrix corresponding to a hyperbolic isometry of $\mathbb{H}^3$. To see this, let $T$ be a hyperbolic element of $PSL(2, \mathbb{C})$. If $T$ mapped some frame to another frame on the same $U$-orbit, then for some $u \in U$, $T \circ u$ would fix some frame, and therefore have to be the identity, a contradiction. Therefore, no points on a $U$-orbit can be identified when modding out by $\Gamma$, and $H$ must properly contain $U$. Similar reasoning shows that $H$ cannot be the (projection to $PSL(2, \mathbb{C})$ of the group of matrices of the form

$$\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\},$$

whose orbits are horospheres.

Our assumption on the absence of properly immersed totally geodesic surfaces rules out $PSL(2, \mathbb{R})$ as a possibility for $H$. It also rules out the group of real upper triangular matrices in $PSL(2, \mathbb{C})$. An orbit of this group corresponds to a pair consisting of a totally geodesic plane $\Pi$ and a point $p \in \partial_{\infty} \Pi$ on the sphere at infinity of $\mathbb{H}^3$. Such an orbit consists of all frames lying over the plane $\Pi$, such that the first vector of the frame is tangent to a geodesic ray with endpoint $p$ on the sphere at infinity when lifted to the universal cover. Because this set lies over a totally geodesic plane in $\mathbb{H}^3$, it cannot have a closed projection to $F$. One can rule out the subgroup of complex upper triangular matrices on similar grounds.

Since we’ve exhausted all conjugacy classes of intermediate closed subgroups (see [Kap09] Section 4.6), $H$ must be equal to $PSL(2, \mathbb{C})$. This implies that $\hat{\mu}$ equals the Haar measure on $F$, which contradicts (5.4.) $\square$

Let $\mu_{D,P,R}$ be the measure obtained by averaging over $D_{P,R}$. The next corollary follows by integrating in polar coordinates.

**Corollary 5.6.** For any continuous $f$ and fixed $\epsilon > 0$ there exists $R_0$ such that for all $P \in Gr_2(M)$ and all $R > R_0$,

$$(5.7) \quad \left| \int_{Gr_2(M)} f d\mu_{D,P,R} - \operatorname{avg}(f) \right| < \epsilon,$$

We can now prove equidistribution for certain sequences of minimal surfaces in $M$. First we prove a lemma.

**Lemma 5.8.** Let $R > 1$ be given. Then for every $\epsilon$ there is some $\delta$ so that the following is true. Let $\Sigma$ be a closed Riemannian surface with Gauss curvature everywhere in the interval $(-1 - \delta, -1 + \delta)$. Let $f$ be a function on $\Sigma$, and $f_R$ the function that at each point is equal to the average of $f$ over the disk of radius $R$ at that point. If the injectivity radius of $\Sigma$ is less than $R$ at that point, we define this average by lifting the disk of radius $R$ at that point to the universal cover of $\Sigma$ and taking the average of the pullback of $f$ over this disk.

Then the averages $\operatorname{avg}(f)$ and $\operatorname{avg}(f_R)$ of these two functions over $\Sigma$ satisfy

$$(5.9) \quad |\operatorname{avg}(f) - \operatorname{avg}(f_R)| < \epsilon \max(|f|).$$

**Proof.** Let $\Sigma_0$ be the complement of the 1-skeleton of the standard 4g-gon cell structure on $\Sigma$, over which the tangent bundle to $\Sigma$ is trivial. Define a metric on the product of a disk $D$ and $\Sigma_0$ as follows. For fixed $p \in \Sigma_0$, we identify $D$ with the disk of radius $R$ in $T_p \Sigma$, and pull back the metric on $\Sigma$ under the exponential
map to get a metric on $D$, whose area form we denote by $dV_{D,p}$. This metric varies smoothly in $p$, so we can define a smooth metric on $D \times \Sigma_0$ such that the induced metric on each $\{d\} \times \Sigma_0 \subset D \times \Sigma_0$ is isometric to the metric on $\Sigma_0$, whose area form we denote by $dV_{\Sigma_0}$. The resulting volume form on $D \times \Sigma_0$ at $(x, y)$ splits as $dV_{D,y}(x) \wedge dV_{\Sigma_0}(y)$.

For fixed $y$, we identify $D \times \{y\}$ with the disk of radius $R$ in the hyperbolic plane via the exponential map of the centerpoint of $D$, and we write

$$dV_{D,y}(x) = \phi(x, y) dV_{\mathbb{H}^2}(x),$$

where $dV_{\mathbb{H}^2}(x)$ is the hyperbolic area form on $D \times \{y\}$ under this identification.

Given $f$ as in the theorem, we define $\hat{f} : D \times \Sigma_0 \to \mathbb{R}$ by setting $\hat{f}(x, y)$ to be the value of $f$ at the point on $\Sigma$ that is the image of $x$ under the natural map from $D \times \{y\}$ to $\Sigma$. We have that,

$$\int_{\Sigma} f = \int_{\Sigma_0} \frac{1}{\text{Area}(D(y, R))} \int_D \hat{f}(x, y) \phi(x, y) dV_{\mathbb{H}^2}((x)) dV_{\Sigma_0}((y)), \tag{5.10}$$

where $\text{Area}(D(y, R))$ is the area of the disk of radius $R$ centered at a lift of $y$ to the universal cover of $\Sigma$. By taking $\delta$ small enough we can make $\phi(x, y)$ pointwise arbitrarily close to 1, uniformly over all $\Sigma$ satisfying the hypotheses of the theorem. We can thus also make $\text{Area}(D(y, R))$ arbitrarily close to the area of the disk of radius $R$ in $\mathbb{H}^2$.

Define the function $m_x(y)$ to be the number of distinct geodesic segments of length less than $R$ joining $x$ to $y$. By partitioning $D$ in its hyperbolic metric and $\Sigma_0$ into small almost-Euclidean rectangles and taking Riemann sums for the double integral in (5.10), we see that the contribution of $f(x)$ to the integral is weighted by the quantity

$$\int_{\Sigma} m_x,$$

which can be made as close as desired to the area of the disk of radius $R$ in the hyperbolic plane, since the integral of $m_x$ over $\Sigma$ is just the area of the lift of the disk of radius $R$ at $x$ to the universal cover of $\Sigma$. It follows that the quantity in (5.10) can be made $\epsilon \max |f|$-close to the integral of $f$ over $\Sigma$ by taking $\delta$ small.

$\square$

Theorem 5.11. Let $M$ be a closed hyperbolic 3-manifold with no properly immersed totally geodesic surfaces, and let $f$ be a continuous function on $Gr_2(M)$ with average $\text{avg}(f)$. Then for every $\epsilon > 0$ we can find $\delta$ such that the following holds. Take any surface subgroup of $\pi_1(M)$ realized by a properly immersed minimal surface $\Sigma$. Assume also that the limit set of a lift of $\Sigma$ to the universal cover is a $K$-quasicircle for $K < 1 + \delta$. The surface $\Sigma$ includes in $Gr_2(M)$ by its tangent planes, and we define $\text{avg}(f, \Sigma)$ to be the average of the pullback of $f$ over $\Sigma$ in the metric on $\Sigma$ induced by $M$. Then

$$|\text{avg}(f) − \text{avg}(f, \Sigma)| < \epsilon.$$
Proof. For $\delta$ small enough, \cite{Sep16} implies that $\Sigma$ is unique and has principal curvatures pointwise of magnitude $O(\delta)$. Let $R$ be larger than $R_0$ given by Corollary \ref{cor:average} applied to $f$ and $\epsilon / 4$. By making $\delta$ small enough, we can ensure that lifts $D$ of intrinsic disks of radius $R$ in $\Sigma$ to the universal cover are as $C^1$ close as desired to totally geodesic disks of that radius in the universal cover. In particular, for any lift of an intrinsic disk $D$ in $\Sigma$ we can find a totally geodesic disk $D'$ in $\mathbb{H}^3$ such that the averages of the pullback of $f$ over $D$ and $D'$ differ by at most $\epsilon / 4$, and the average over $D$ therefore differs from the average of $f$ over $Gr_2(M)$ by at most $\epsilon / 2$.

Now if $M$ is the maximum of $|f|$ over $Gr_2(M)$, taking the $\epsilon$ of the previous lemma to be $\epsilon / (2M)$ and making $\delta$ small enough finishes the proof.

\hfill $\square$

5.2. Variable Curvature. Let $g$ be a metric on $M$ to which Theorems \ref{thm:average} or \ref{thm:average2} apply to construct a foliation $\mathcal{F}_g$ conjugate to the totally geodesic foliation in constant curvature via

$$\Phi : Gr_2((M, ghyp)) \to Gr_2((M, g)).$$

For a point $x$ on a surface $L$ whose lift to $Gr_2(M)$ is a leaf of $\mathcal{F}_g$, we define $D_L(x, R)$ to be the lift to $Gr_2(M)$ by its tangent planes of the intrinsic disk of radius $r$ in $L$ with its metric induced by $g$. Let $\mu_{L, x, R}$ be the probability measure on $Gr_2(M)$ that corresponds to averaging over $D_L(x, R)$.

Any choice of generators and relations for $\pi_1(M)$ gives rise to a word metric on $\pi_1(M)$, so fix some choice and let $B_{\pi_1(M)}(r)$ be the ball of radius $r$ centered at the identity in the corresponding word metric $g_{\pi_1(M)}$ on $\pi_1(M)$. It is a fact that $(\pi_1(M), g_{\pi_1(M)})$ is quasi-isometric to $(\check{M}, \check{g})$ (as well as $\mathbb{H}^3$.) Let $P$ be a connected polyhedral fundamental domain for the action of $\pi_1(M)$ on $\check{M}$. Our approach in this section follows ideas in \cite{CMN}. (See that paper for a more detailed description of the setup described in this paragraph.)

For a properly embedded plane $S$ in $(\check{M}, \check{g})$, define $\text{Area}_{S, \check{g}}(R)$ to be the area of the union

$$\bigcup_{\phi \in B_{\pi_1(M)}(R)} (P \cap \phi \cdot S),$$

and for $U \subset Gr_2(P)$ define $\text{Area}_{S, \check{g}, U}(R)$ to be the area of

$$\bigcup_{\phi \in B_{\pi_1(M)}(R)} (U \cap \phi \cdot \check{S}),$$

where $\check{S} \subset Gr_2(\check{M})$ is the natural lift of $S$, and area is measured in the metric on $\check{S} \cong S$ induced by $(\check{M}, \check{g})$. The following lemma is similar to Propositions 6.4 and 6.5 of \cite{CMN}.

Lemma 5.12. Let $\check{L}$ be a lift of $L$ to $\check{M}$ that intersects $P$, and let $x \in L$. Then there exists a constant $C$ depending on $g$ (but not $L$) such that

$$\frac{1}{C}(\text{Area}(D_L(x, R))) \leq \text{Area}_{L, \check{g}}(R) \leq C \text{Area}(D_L(x, R)).$$

Proof. Let $\check{x} \in \check{L} \cap P$ project to $x$. There exists $C_0$ such that

$$B_\check{g}(\check{x}, R/C_0) \subset \bigcup_{\phi \in B_{\pi_1(M)}(R)} \phi \cdot P \subset B_\check{g}(\check{x}, C_0R),$$

(5.13)
where $C_0$ just depends on the constants for the quasi-isometry between $\pi_1(M)$ and $(\hat{M}, \hat{g})$.

Note that since the map induced by $\hat{\Phi}$ from $\hat{L}$ to the corresponding totally geodesic plane is uniformly bi-Lipschitz and $\hat{L}$ is at uniformly bounded distance from this totally geodesic plane considered as a subspace of $(\hat{M}, \hat{g})$, there exists $C_1$ such that

$$\lim D_L(\hat{x}, R) \subset B_{\hat{g}}(\hat{x}, R) \cap \hat{L} \subset D_L(x, C_1 R).$$

Because $\hat{L}$ has negative sectional curvature pointwise strictly less than that of $\hat{g}$ and with a lower bound depending only on $\hat{g}$, it also follows that for any fixed $C_2$ the quotient

$$\frac{\text{Area}(D_L(x, C_2 R))}{\text{Area}(D_L(x, R))}$$

will be uniformly bounded by a constant that depends only on $C_2$ and $\hat{g}$. By intersecting the sets in (5.13) with $\hat{L}$ and applying (5.14) and (5.15) we obtain the inequalities in the statement of the lemma.

\[\square\]

**Proposition 5.16.** For $R_n \to \infty$, any weak-$\ast$ limit of measures $\mu_{L_n, x_n, R_n}$ has full support in $Gr_2(M)$.

**Proof.** Let (the lift of) $L_n'$ be the totally geodesic leaf in $Gr_2((M, g_{hyp}))$ corresponding to $L_n$ under $\Phi$, and $L_n'$ a lift of $L_n'$ to the universal cover at finite distance from a lift $\hat{L}_n$ of $L_n$ to the universal cover that intersects $P$. By Corollary 5.8 intrinsic disks $D_{L_n'}(x_n' ,R_n)$ in $L_n'$ centered at $x_n' = \Phi^{-1}(x_n)$ equidistribute in $Gr_2((M, g_{hyp}))$ as $n$ tends to infinity. This implies that for any open set $B \subset Gr_2((M, g_{hyp}))$,

$$\lim_{n \to \infty} \inf \frac{\text{Area}(D_{L_n'}(x_n', R_n) \cap B)}{\text{Area}(D_{L_n'}(x_n', R_n))} > 0.$$  \hfill (5.17)

Suppose that $B$ is a small ball and choose a connected lift $\hat{B}$ of $B$ to $Gr_2(\hat{M})$ contained in $Gr_2(P)$ (choosing a slightly different fundamental domain $P$ if necessary so that this is possible.) Choose lifts $\hat{x}_n$ and $\hat{x}_n'$ of $x_n$ and $x_n'$ such that both are contained in $P$. Arguments similar to the proof of the last lemma show

$$\lim_{n \to \infty} \inf \frac{\text{Area}L_n', \hat{g}_{hyp} \cdot \hat{B}(R_n)}{\text{Area}(D_{L_n'}(x_n', R_n) \cap \hat{B})} > 0,$$

and therefore by (5.17),

$$\lim_{n \to \infty} \inf \frac{\text{Area}L_n', \hat{g}_{hyp} \cdot \hat{B}(R_n)}{\text{Area}(D_{L_n'}(x_n', R_n))} > 0.$$  \hfill (5.18)

Let $B_0 \subset B$ be a smaller open ball in $Gr_2(M)$ at a positive distance to $\partial B$, and let $\hat{B}_0$ be a lift of $B_0$ to the universal cover contained in $\hat{B} \subset P$. Then there is some $\epsilon > 0$ such that for all $\phi \in \pi_1(M)$,

$$\text{Area}(\phi \cdot \hat{L}_n' \cap \hat{B}_0) < \epsilon.$$  \hfill (5.19)

For $U \subset Gr_2(P)$ and a properly embedded plane $S \subset \hat{M}$, define

$$U(S, R) = \{\phi \in B_{\pi_1(M)}(R) : \phi \cdot \overline{S} \cap U \neq \emptyset\},$$

25
where \( \mathcal{S} \subset Gr_2(\tilde{M}) \) is the natural lift of \( S \) to \( Gr_2(\tilde{M}) \). By (5.18) with \( B_0 \) in place of \( B \) and (5.19),

\[
\lim_{n \to \infty} \inf \frac{\#(\tilde{B}_0(\tilde{L}_n', R_n))}{\text{Area}(\tilde{D}_{\tilde{L}_n'}(x_n', R_n))} > 0.
\]

(5.20)

Our task now is to reverse the steps performed above, but with \( L_n \) in place of \( L'_n \), and with (5.20) as our starting point.

For \( \phi \in \pi_1(M) \) and \( U \) an open set in \( Gr_2(\tilde{M}) \),

\[
\phi \cdot \tilde{L}_n' \cap U \neq \emptyset \quad \text{iff} \quad \phi \cdot \tilde{L}_n \cap \tilde{\Phi}(U) \neq \emptyset,
\]

and consequently

\[
\#(U(\tilde{L}_n', R_n)) = \#(\tilde{\Phi}(U)(\tilde{L}_n, R_n)).
\]

(5.21)

The fact that the principal curvatures of the \( L_n \) are uniformly bounded in absolute value implies that

\[
\lim_{n \to \infty} \frac{\text{Area}(D_{L_n}(x_n, R_n))}{\text{Area}(D_{L_n'}(x_n', R_n))} > 0
\]

so taking \( U = B_0 \) in (5.21), (5.20) implies that

\[
\lim_{n \to \infty} \frac{\#(\tilde{\Phi}(\tilde{B}_0)(\tilde{L}_n, R_n))}{\text{Area}(D_{L_n}(x_n, R_n))} > 0
\]

The monotonicity formula implies that there exists \( \epsilon > 0 \) such that if \( \phi \cdot \tilde{L}_n \) intersects \( \tilde{\Phi}(\tilde{B}_0) \), then

\[
\text{Area}(\phi \cdot \tilde{L}_n \cap \tilde{\Phi}(\tilde{B})) > \epsilon,
\]

and consequently

\[
\lim_{n \to \infty} \inf \frac{\text{Area}(\tilde{L}_n \cap \tilde{\Phi}(\tilde{B}_0)(R_n))}{\text{Area}(D_{L_n}(x_n, R_n))} > 0.
\]

An argument similar to the proof of Lemma 5.12 shows that there is a \( c \) such that

\[
\text{Area}(D_{L_n}(x_n, R_n) \cap \tilde{\Phi}(B)) > c(\text{Area}_{L_n \cap \tilde{\Phi}(B)}(R_n))
\]

and therefore

\[
\lim_{R_n \to \infty} \inf \frac{\text{Area}(D_{L_n}(x_n, R_n) \cap \tilde{\Phi}(B))}{\text{Area}(D_{L_n}(x_n, R_n))} > 0.
\]

This shows that any weak-* limit as in the statement of the proposition assigns positive measure to \( \tilde{\Phi}(B) \), and so any weak-* limit has full support in \( Gr_2(M) \). \( \square \)

Let \( \Sigma_n \) be a sequence of stable properly immersed minimal surfaces in \((M, g)\) corresponding to surface subgroups of \( \pi_1(M) \) having limit sets \( 1 + 1/n\)-quasicircles, and let \( \mu_n \) be the corresponding probability measures on \( Gr_2(M) \).

**Theorem 5.22.** Suppose \( g \) is such that Theorem 4.3 or 3.4 applies to construct a foliation. Then any weak-* limit of the \( \mu_n \) has full support in \( Gr_2(M) \).

**Proof.** Let \( \Sigma'_n \) be the minimal surface in \((M, g_{hyp})\) corresponding to \( \Sigma_n \). As in the proof of Theorem 4.3 [Sep10] implies that for each fixed \( R \) and for \( n \) large, for any \( x \in \Sigma'_n \), the lift of the immersed disk \( D' \) in \( \Sigma'_n \) of radius \( R \) to \( \mathbb{H}^3 \) is trapped between two totally geodesic planes in \( \mathbb{H}^3 \) whose intersections with the 1-neighborhood of \( D' \) are very close to each other. Since \( \tilde{\Phi} \) is uniformly continuous, it follows that for \( n \) large enough, the lift of any disk \( D \) in \( \Sigma_n \) of radius \( R \) to \((\tilde{M}, \tilde{g})\) is trapped
between two lifts \( \bar{L}_1 \) and \( \bar{L}_2 \) of (projections from \( Gr_2(M) \) of) leaves of \( F_g \) to the universal cover whose intersections with the 1-neighborhood of \( D \) are as close to each other as desired by making \( n \) large, where \( F_g \) is the foliation given by Theorem 1.2 or 3.4. Therefore \( D \) can be made arbitrarily \( C^1 \) close to metric disks of radius \( R \) in \( L_1 \) and \( L_2 \) by taking \( n \) large. It follows that for any fixed \( R \), as \( n \) tends to infinity the metric disks of radius \( R \) in \( \Sigma_n \) are becoming uniformly close to metric disks \( D_L(x, R) \) of projections of leaves of \( F_g \).

Fix an open set \( U \) in \( Gr_2(M) \), let \( f \) be its indicator function, and let \( f_{R,n} : \Sigma_n \to \mathbb{R} \) be the average of \( f \) over the disk of radius \( R \) in \( \Sigma_n \) centered at \( x \) (lifting to the universal cover and taking the average of the pullback of \( f \) if the injectivity radius is less than \( R \) at \( x \).) If there were a sequence of \( R_k \) tending to infinity such that

\[
\lim_{n \to \infty} \inf \frac{1}{\text{Area}(\Sigma_n)} \int_{\Sigma_n} f_{R_k,n} = 0
\]

was zero for each fixed \( R_k \), then we could find a sequence \( \mu_{L_n,x_n,R_n} \) such that

\[
\lim_{n \to \infty} \mu_{L_n,x_n,R_n}(U) = 0.
\]

Taking a weak-* limit of the measures \( \mu_{L_n,x_n,R_n} \) associated to this subsequence would then contradict Proposition 5.16.

It follows that the liminf in (5.23) is positive for \( R_k \) large enough. Because the \( \Sigma_n \) have curvature uniformly bounded above and below by negative constants, one can argue as in the proof of Lemma 5.8 to show that for any such sufficiently large \( R_k \) there is a uniform constant \( c \) such that

\[
\frac{1}{\text{Area}(\Sigma_n)} \int_{\Sigma_n} f > c \cdot \frac{1}{\text{Area}(\Sigma_n)} \int_{\Sigma_n} f_{R_k,n},
\]

and so the liminf as \( n \to \infty \) of the LHS is positive. This completes the proof that any weak-* limit of the \( \mu_n \) assigns positive measure to every open set in \( Gr_2(M) \).

\[\square\]

6. Examples where Theorem 1.2 Cannot Apply

6.1. We now present examples of closed hyperbolic 3-manifolds \( M \) and negatively curved metrics \( g \) on \( M \) which cannot possibly admit a foliation as in Theorem 1.2 or Theorem 3.4. Recall that \( P_\delta \) is the set of totally geodesic planes in \( \mathbb{H}^3 \) considered as subspaces of \((\bar{M}, \bar{g})\). In the examples we construct, there will be multiple properly embedded minimal planes in \((\bar{M}, \bar{g})\) at finite Hausdorff distance from the same element of \( P_\delta \), and so Theorem 1.2 could not apply to produce a foliation.

We will need the following lemma, which is contained in [BO69, Theorem 7.5].

Lemma 6.1. Let \( f : (a, b) \to \mathbb{R} \) be a smooth positive strictly convex function, and let \( h \) be a negatively curved metric on a surface \( \Sigma \) of genus \( g > 1 \). Then the warped product metric

\[
f^2(t)h + dt^2
\]

on \( \Sigma \times (a, b) \) has negative sectional curvature.
6.2. In [HW15], examples of quasi-Fuchsian hyperbolic 3-manifolds were constructed with arbitrarily many stable embedded minimal surfaces in their convex core whose inclusions are homotopy equivalences. Fix such a quasi-Fuchsian manifold $Q$. In [KS08, Lemma 5.7] it is shown that there are coordinates on a neighborhood of infinity homeomorphic to $\Sigma \times (0, \infty)$ of each of the respective ends such that for $t$ large the metric on $Q$ can be written in the form

\begin{equation}
\frac{1}{2}(e^{2t}I^*_j + 2II^*_j + e^{-2t}III^*_j) + dt^2 \quad j = 1, 2,
\end{equation}

where the symmetric 2-tensors $I^*_j$, $II^*_j$ and $III^*_j$ are called the first, second, and third fundamental forms at infinity. This is achieved by considering the parallel distance-$t$ surfaces to a strictly convex surface $\Sigma_j$ outside of the convex core and with principal curvatures less than 1. The metric $I^*_j$ can be expressed in terms of the first, second, and third fundamental forms of $\Sigma_j$. It has strictly negative curvature by [KS08, Lemma 5.2].

For $t$ large and for a smooth cutoff function $b(t)$ equal to 0 near positive infinity and equal to 1 near zero, we modify the metric in (6.2) and consider the metric

\begin{equation}
\frac{1}{2}e^{2t}(I^*_j + b(t)(2e^{-2t}II^*_j + e^{-4t}III^*_j)) + dt^2.
\end{equation}

If $b(t) = 1$, this agrees with (6.2). Provided $t$ is sufficiently large (and the innermost quantity in parentheses in (6.3) is sufficiently small), $b(t)$ remains equal to 1 for $t$ large, and the first and second derivatives of $b(t)$ are sufficiently close to zero (i.e., $b(t)$ spends a long time between 1 and 0), on every vertical interval $(t, t + 1)$ the metric in (6.3) will be very $C^2$-close to a warped product of the negatively curved metric $I^*_j$ on $\Sigma$ with convex warping function $e^{2t}$ as in Lemma 6.1 and thus have negative curvature.

6.3. Now take $M$ to be any closed hyperbolic 3-manifold that admits a proper embedded totally geodesic surface $\Sigma$ (see [MR03] for examples.) Let $F \cong \Sigma \times \mathbb{R}$ be the cover of $M$ corresponding to $\Sigma$. If $g_{hyp}$ is the metric on $\Sigma$ induced by the hyperbolic metric on $M$, the metric on $F$ is isometric to the warped product

\begin{equation}
\cosh^2(t)g_{hyp} + dt^2.
\end{equation}

The space of negatively curved metrics on a surface is contractible by work of Hamilton [Ham88], so we can choose smooth paths $\gamma_1$ and $\gamma_2$ in the space of negatively curved metrics on $\Sigma$ joining $g_{hyp}$ to $I^*_1$ and $I^*_2$.

For $1 \ll T_1 < T_2$ we can use $\gamma_1$ and $\gamma_2$ to modify (6.4) on $\Sigma \times (T_1, T_2)$ and $\Sigma \times (-T_2, -T_1)$ so that near $(-1)^{j+1}T_1$ it is equal to

\begin{equation}
e^{2t}I^*_j + dt^2 \quad j = 1, 2,
\end{equation}

and near $\pm T_2$ it remains equal to (6.4). By taking $T_2 - T_1$ large, this can be done so that the new metric is locally $C^2$-close to a warped product of a negatively curved metric with convex warping function as in (6.1), so that the sectional curvatures of the new metric remain negative.

Provided $T_1$ was taken large enough, (6.5) agrees with (6.3) in neighborhoods of $\Sigma \times \{ \pm T_1 \}$, so we can define a new negatively curved metric on $F$ by cutting out a region $\Sigma \times [-T_1 + \epsilon, T_1 - \epsilon]$ and gluing in a region of $Q$ containing its convex core. Denote this new metric on $F$ by $g'$. 


6.4. We claim that by passing to finite covers, we can make the normal injectivity radius of \( \Sigma \) arbitrarily large. It is a fact that for every \( g \in \pi_1(M) \backslash \pi_1(\Sigma) \), there exists a finite index subgroup \( G \) of \( \pi_1(M) \) such that \( \pi_1(\Sigma) \subset G \) but \( g \notin G \) [MR03, Lemma 5.3.6].

Fix a basepoint for \( \pi_1(M) \), and let \( F \) be the Fuchsian cover of \( M \) corresponding to \( \pi_1(\Sigma) \). Fix a connected polyhedral fundamental domain \( P \) for the action of \( \pi_1(M) \) on \( \mathbb{H}^3 \). Then \( F \) is tessellated by copies of \( P \) which are fundamental domains for the covering map from \( F \) to \( M \), and any fixed normal neighborhood \( N \) of the central totally geodesic copy of \( \Sigma \) in \( F \) is contained in a finite number copies of \( P \) in \( F \). By choosing elements of \( \pi_1(M) \) representing the cosets of \( \pi_1(\Sigma) \) corresponding to these finitely many fundamental domains, we can find a finite index subgroup of \( \pi_1(M) \) containing \( \pi_1(\Sigma) \) but not containing any of these coset representatives. The finite cover \( M' \) corresponding to this subgroup is then also covered by \( F \), and the projection of the normal neighborhood \( N \) to \( M' \) is injective. This shows that the normal injectivity radius of \( \Sigma \) can be made arbitrarily large by passing to finite covers of \( M \).

6.5. Outside of the \( T_2 \)-neighborhood \( N_{T_2} \) of \( \Sigma \) in \( F \) (in the Fuchsian metric), \( g' \) agrees with the Fuchsian metric. By the above, we can find a finite cover \( M' \) of \( M \) for which the normal injectivity radius of \( \Sigma \) is greater than \( T_2 \). Then \( N_{T_2} \) projects injectively to \( M' \) under the covering map, so we can use \( g' \) to define a new negatively curved metric on \( M' \) which we also call \( g' \).

In this new metric \( g' \) on \( M' \), there are several properly embedded \( \pi_1 \)-injective stable minimal surfaces whose fundamental groups include as subgroups in the conjugacy class of \( \pi_1(\Sigma) \) in \( \pi_1(M') \). These lift to the universal cover of \( M' \) to give several distinct properly embedded minimal planes at finite Hausdorff distance from the same element of \( \mathcal{P}_{g'} \).

Remark 6.6. We expect that the metric on \( M' \) we constructed can be joined to the constant curvature metric through a smooth path of negatively curved metrics by performing the above construction on a smooth path in quasi-Fuchsian space joining \( F \) to \( Q \). If this is the case, then Theorem 1.2 would apply with \( T < \infty \) to this path of metrics.

7. A Stability Estimate for the Foliations of Theorem 1.2

Fix a smooth 1-parameter family of metrics \( g_t \) to which Theorem 1.2 applies to produce foliations. Let \( S \) be a totally geodesic plane in \( \tilde{M} \cong \mathbb{H}^3 \), and let \( S_t \) be the \( \tilde{g}_t \)-minimal plane in \( \tilde{M} \) corresponding to \( S \). For fixed \( t_0 \), elliptic PDE theory implies that \( S_t \) converges smoothly uniformly to \( S_{t_0} \). For \( t \) close to \( t_0 \), \( S_t \) is a graph over \( S_{t_0} \) in normal coordinates for a tubular neighborhood of \( S_{t_0} \), and so differentiating in \( t \) at \( t_0 \) we obtain a vector field \( v \) normal to \( S_{t_0} \).

Theorem 7.1. Let \( \epsilon_0 < \epsilon \) be given, and suppose \( S_{t_0} \) is \( \epsilon \)-convex. Then there exists \( \delta \) depending only on \( \epsilon \) and \( \epsilon_0 \) and bounds on the \( g_t \) and the time derivatives \( g'_t \) of the \( g_t \) such that if \( \delta > t - t_0 > 0 \), then \( S_t \) is \( \epsilon_0 \)-convex. (The same \( \delta \) works for every \( \epsilon \)-convex \( S_{t_0} \).)

By bounds on \( g_t \) and \( g'_t \), we mean bounds on these tensors and their derivatives up to second order in the \( L^\infty \) norms induced by the hyperbolic metric \( g_0 \).
Proof. The minimal surface equation for a graph $u$ in $\mathbb{R}^3$ over a region in the $xy$-plane can be written

\begin{equation}
(1 + u_{x_2}^2)u_{x_1x_1} + (1 + u_{x_1}^2)u_{x_2x_2} - 2u_{x_1}u_{x_2}u_{x_1x_2} = 0.
\end{equation}

If we dilate the metric and zoom in at a point $x_0$ on $S_{t_0}$ at a scale where the metrics on both $\hat{M}$ and $S_{t_0}$ are almost Euclidean to second order, then the coefficients of the second order equation $S_{t_0}$ satisfies, writing it locally as a graph $u$ over a coordinate plane, can be made arbitrarily $C^0$ close to those of Equation (7.2). The amount we need to dilate the metric at $x_0$ to obtain a given degree of closeness is determined by bounds on $g_{t_0}$ and the principal curvatures of $S_{t_0}$, and the latter can be bounded in terms of bounds on $g_{t_0}$ by the $\epsilon$-convexity of $S_{t_0}$. For $t$ close to $t_0$ and $x$ close to $x_0$, we can write the surfaces $S_t$ as graphs $u(x, t)$, where $u(x, t_0) = u(x)$. Then the derivative vector $v$ in these coordinates equals $u_t(x, t_0)$. Differentiating the minimal surface equations the $u(x, t)$ satisfy in time at $t_0$, we obtain a second order equation of the form

\begin{equation}
a^{ij}v_{x_i, x_j} + b^i v_{x_i} + c = 0.
\end{equation}

The $a^{ij}$ can be made arbitrarily $C^0$ close to the coefficients of $u_{x_i, x_j}$ in Equation (7.2) provided we dilated the metric enough at $g_{t_0}$ and the principal curvatures of $S_{t_0}$. The $b^i$ and $c$ can be bounded in terms of the first two derivatives of $u$— which in turn are bounded by the absolute values of the principal curvatures of $S_{t_0}$ and bounds on $g_{t_0}$—, bounds on $g_{t_0}$, and bounds on $g_{t_0}$. It follows that we can estimate the $C^2$-norm of $v$ if we have a $C^0$ estimate for $v$ \cite{GT01} Theorem 6.2).

By Lemma 3.6 the mean curvatures of the parallel distance-$s$ surfaces to $S_{t_0}$ are greater than $\frac{s}{\xi}$ for $s$ small and positive and less than $\frac{\xi}{s}$ for $s$ small and negative. Therefore we can find some small $\xi$ that depends on bounds on the $g_t$ such that if

\begin{equation}|t - t_0| < \xi |s|,
\end{equation}

then the distance-$s$ parallel surfaces are mean-convex in the metric $g_t$. The constant $\xi$ can be chosen so that this statement also holds for all of the other $\epsilon$-convex (projections of) leaves of the foliation, not just the given $S_{t_0}$ we are considering. One can then show by essentially the same argument as the proof of Lemma 3.12 that $S_t$ is contained between the signed distance $\pm s$ parallel surfaces to $S_{t_0}$ if $t$ and $s$ satisfy (7.4). By sending $t$ to $t_0$, this implies that the magnitude of $v$ is bounded above by $1/\xi$ in the coordinates we chose. This gives the desired $C^0$ estimate for $v$.

In this way, we can obtain an upper bound for the $C^2$ norm of the normal derivative vector field $v$ for $S_t$ which are $\epsilon$-convex. We thus obtain an upper bound for the $C^2$ norm of $v$ for $S_t$ that are $\epsilon_0$-convex, since such $S_t$ are also $\epsilon$-convex. The surface $S_{t_0}$ is $\epsilon_0$-convex, and $S_t$ remains $\epsilon_0$-convex for $t - t_0$ less than some $\delta$. One can obtain a lower bound for $\delta$ in terms of the difference $\epsilon - \epsilon_0$, bounds on $g_t$ and $g_{t_0}$, and the bound on the $C^2$ norm of $v$ by differentiating the formula for the principal curvatures of $S_t$ with respect to time in local coordinate charts, and using the $C^2$ bound on $v$. This completes the proof.
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