Convergent perturbative nuclear effective field theory

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Abstract

We consider the nuclear effective field theory including pions in the two-nucleon sector in the S waves up to including the next-to-next-to-leading order (NNLO) terms according to the power counting suggested by the Wilsonian renormalization group analysis done in a previous paper. We treat only the leading contact interaction nonperturbatively, and the rest, including the long-distance part of pion exchange, are treated as perturbations. To define the long-distance part, it is important to introduce a separation scale, or a cutoff. We employ a hybrid regularization, in which the loops with only contact interactions are regularized with Power Divergence Subtraction (PDS), while the loops with (long-distance part of) pion exchange are regularized with a Gaussian damping factor (GDF), to simplify the (nonperturbative) leading-order amplitudes. The scale introduced by PDS is identified with the cutoff of GDF up to a numerical factor. We emphasize that the introduction of the GDF requires a careful definition of the coupling constant for the pion exchange. We obtain the analytic expressions for the phase shifts for the $^1S_0$ and $^3S_1$-$^3D_1$ channels. By fitting them to the Nijmegen partial wave analysis data, it is shown that the effective theory expansion with perturbative long-distance part of pion exchange is converging.
INTRODUCTION

Since the seminal papers by Weinberg \cite{1,2}, nuclear effective field theory (NEFT), an effective field theory describing systems with more than one nucleons at low energies, has attracted much attention. See Ref. \cite{3} for a recent review. With a vast number of papers written over twenty years, one might think that it has become a matured discipline. In fact, there are $N^3LO$ calculations \cite{4,5} in the literature.

However, a very fundamental issue is still under discussions: the power counting and renormalization. Since an effective field theory contains an infinitely many operators, a power counting rule is necessary to organize calculations to achieve a certain order of accuracy. Cutoff dependence of physical quantities must be absorbed in the coupling constants order by order, that is, consistently with the power counting. The original Weinberg’s power counting, which is widely used in most of numerical calculations, is known to be inconsistent, i.e., cutoff dependence arising at a certain order can be absorbed only by terms of higher orders \cite{6}. Several authors (e.g. \cite{7}) consider variants of the Weinberg’s scheme and discuss the nonperturbative renormalization of them.

An alternative, consistent power counting scheme (known as “KSW power counting”) was proposed by Kaplan, Savage and Wise \cite{8,9}, and independently by van Kolck \cite{10}. In their scheme, pion exchanges are treated as perturbation. Fleming, Mehen, and Stewart \cite{11} however showed that the effective theory expansion fails to converge at NNLO, due to the strong tensor force of pion exchange at short distances. Beane, Bedaque, Savage, and van Kolck \cite{12} proposed a remedy, in which the $1/r^3$ part of pion exchange is treated nonperturbatively.

In a series of papers \cite{13–16}, we advocate that the power counting should be based on the scaling dimensions obtained by Wilsonian renormalization group (RG) analysis. (See Refs. \cite{17–19} for related works.) The idea is that power counting is an order of magnitude estimate based on the dimensional analysis, and that the quantum notion of the dimension of an operator is the scaling dimension. A nonperturbative RG analysis is necessary because in the S waves the physical two-nucleon system is governed by a nontrivial fixed point which is inaccessible in perturbation theory.

In a previous paper \cite{16}, we perform the Wilsonian RG analysis for the nucleon-nucleon scattering in the S waves in the NEFT including pions. We emphasize that it is important
to divide pion exchange into its short-distance part (S-OPE) and the long-distance part (L-OPE) separated by a cutoff scale, because they play different roles. The S-OPE is represented as contact terms. A part of the S-OPE is relevant in the RG sense and thus should be treated nonperturbatively while the L-OPE is treated as perturbation. It turns out that the power counting for the contact interactions and the L-OPE is very similar to the KSW power counting.

In this paper, we consider the NEFT including pions in the two-nucleon sector in the S waves with the power counting suggested by the Wilsonian RG analysis mentioned above. In order to separate the pion exchange into two parts, we introduce an explicit separation scale, or a cutoff, \( \lambda \), and propose a new hybrid regularization in which the diagrams with only contact interactions are regularized with Power Divergence Subtraction (PDS)\[8, 9\], but those with potential pion exchange (L-OPE) are regularized with a Gaussian damping factor (GDF). We obtain the analytic expressions for the phase shifts for \( ^1S_0 \) and \( ^3S_1 \) at NNLO and fit them to Nijmegen partial wave analysis (PWA) data to determine the low-energy constants (LECs). All the technical details together with the lengthy analytic expressions will be given elsewhere.

Our approach is similar to that by Beane, Kaplan, and Vuorinen \[20\] (BKV) in the respect that a separation scale is introduced. There are however important differences: (i) We use the same regularization both for the \( ^1S_0 \) and \( ^3S_1-^3D_1 \) channels, though BKV introduce the separation scale only for the \( ^3S_1 \) channel. (ii) We use a GDF to regularize the pion potential, while BKV use a Pauli-Villars type regulator, which we find insufficient to render several diagrams convergent. (iii) We interpret the separation scale as an analog of the floating cutoff in the Wilsonian RG analysis so that it should not exceed the physical cutoff \( \Lambda_0 \approx 350 \text{ MeV} \) above which the effective field theory description does not hold, while BKV consider a rather large value in the range \( 600 \text{ MeV} \leq \lambda \leq 1000 \text{ MeV} \), although they consider it as a low-momentum scale of \( \mathcal{O}(Q) \). (iv) We interpret the “renormalization scale” \( \mu \) appeared in the PDS as the separation scale too, and take \( \mu \sim \mathcal{O}(\lambda) \). BKV consider that \( \mu \) is independent of \( \lambda \) and choose \( \mu = m_\pi \). (v) In our formulation, the separation scale \( \lambda \) is smaller than or equal to the physical cutoff \( \Lambda_0 \), but otherwise arbitrary. On the other hand, BKV tune the value of \( \lambda \) to optimize the perturbation expansion.
HYBRID REGULARIZATION

Since we consider that the introduction of the separation scale is essential, we could work only with a GDF even for the contact interactions, but such a scheme is very complicated because the leading order (relevant) operator becomes a linear combination of several operators when higher order operators are included. On the other hand, in the PDS regularization the leading operator stays the same even when the higher order operators are included. The hybrid regularization takes advantage of both regularizations.

Our interpretation of the “renormalization scale” $\mu$ introduced in the PDS as a separation scale equivalent to $\lambda$ might look strange. It however comes from the comparison of power-divergent loop integrations calculated in the PDS and those with an explicit cutoff. The power divergences of the integral are represented as polynomials of $\mu$ in the PDS. They are typically related as $\mu = \lambda/\sqrt{\pi}$, which is inferred by calculating the simplest one-loop diagram with two regularizations. Note that the interpretation is different from the usual one for the dimensional regularization such as $\overline{MS}$ scheme in the relativistic field theory, where only the logarithmic divergences are explicitly treated. There the scale $\mu$ is arbitrary, but to avoid the large logarithms, it is taken to be the typical scale of the process in question. In the present case, however, $\mu$ is not a low-momentum scale, and a typical momentum scale $p$ satisfies $p < \mu \lesssim \Lambda_0$.

We introduce a GDF for the pion exchange so that they represent the L-OPE. For the scalar part, we can explicitly extract the S-OPE from the decomposition

$$\frac{k^2}{k^2 + m^2_\pi} = 1 - \frac{m^2_\pi}{k^2 + m^2_\pi},$$

(1)

where the first term may be considered as a contact interaction and the loops containing this term are regularized with the PDS. The second term is the L-OPE, for which we introduce a GDF $e^{-k^2/\lambda^2}$. For the tensor part, on the other hand, because of the tensor structure, it is impossible to extract local operators in a similar procedure. We assume that the effects of the S-OPE is already encoded in the coupling constants of the contact interactions, and consider only the L-OPE.

A diagram with pion exchange may be expanded in powers of $p/\lambda$, where $p$ is an external momentum or the pion mass. Thus a single diagram produces a series of contributions of different orders. For example, a diagram which appears at NLO may also contain NNLO contributions. This is a new feature of the hybrid regularization.
A very nontrivial point with the GDF regularization comes from the requirement that the definition of the pion exchange coupling constant should be independent of the cutoff. In order to satisfy the requirement, one needs to introduce an extra factor $e^{-m_\pi^2/\lambda^2}$. Including the coupling constant, the scalar part of the pion exchange may be written as

$$-i \frac{g_A^2}{2f^2} e^{-m_\pi^2/\lambda^2} \left[ 1_{\text{PDS}} - \frac{m_\pi^2}{k^2 + m_\pi^2} e^{-k^2/\lambda^2} \right],$$

(2)

where $1_{\text{PDS}}$ stands for a contact operator that should be treated with PDS. One can see that this definition of the coupling constant has several favorable features: (i) The coupling constant defined by the residue of the pole of the Yukawa potential becomes independent of the separation scale $\lambda$. (ii) The results of loop integrals including pion propagators with GDFs contain the factor $e^{m_\pi^2/\lambda^2}$, which is canceled by the extra factor. If the extra factor is not included, there would be (disastrous) nonlocal contributions to higher orders when expanded in powers of $m_\pi^2/\lambda^2$.

The calculated amplitudes depend on $\lambda$ (and $\mu$). All the dependence is actually eliminated by making the LECs $\lambda$-dependent, i.e., there is no nonlocal $\lambda$-dependence.

**NNLO FITTING TO THE NIJMEGEN DATA**

We calculate the LO, NLO, and NNLO amplitudes for the nucleon-nucleon scattering in the $^1S_0$ and $^3S_1-^3D_1$ channels analytically and fit the phase shifts obtained from the amplitudes to the Nijmegen PWA data. The Lagrangian is the same as that of Fleming et al. [11]. We include diagrams containing an $S-D$ mixing contact term together with the diagrams given in Ref. [11], and calculate with the hybrid regularization explained above. The so-called “radiation pion” and “soft pion” contributions, and the isospin breaking terms are not included. As input parameters, we set the nucleon mass $M = 938$ MeV, the pion mass $m_\pi = 138$ MeV, and the pion exchange coupling constant $g_A^2/2f^2 = 4.68 \times 10^{-5}$ MeV$^{-2}$. We choose $\lambda = \sqrt{\pi\mu} = 350$ MeV, which is about the physical cutoff. We have however checked that the quality of the fitting does not depend on the value of $\lambda$, though the fitted values of the LECs are different for different values of $\lambda$. A detailed account based on RG analysis will be given in a separated paper.

In order to fit the calculated phase shifts to the Nijmegen PWA data, we use the Mathematica command `NonlinearModelFit` with a weight function $\sim 1/p^4$, emphasizing the
FIG. 1. The scattering phase shift for the $^1S_0$ channel. The LO(dashed line), NLO(dash-dotted line), and NNLO(dotted line) results are plotted together with the Nijmegen PWA data(solid line).

low-momentum region. The range of the data varies with the order: the ranges of center-of-mass momentum $0 – 30$ MeV, $0 – 70$ MeV, and $0 – 250$ MeV are used for LO, NLO, and NNLO respectively for both the $^1S_0$ and $^3S_1$ phase shifts. If one dares to try to fit to an unreasonably wider range, the quality of the fit in the low-momentum region becomes worse. We should not do so because NEFT must describe the lower-momentum region better than the higher-momentum region. Given the order of the expansion, the range in which the fitting is successful may be viewed as the validity region.

We show the calculated scattering phase shifts by using the fitted values of the LECs in Fig. 1 for the $^1S_0$ channel, and in Fig. 2 for the $^3S_1$ channel. Most importantly, no breakdown of the effective theory expansion is observed. By going to higher orders, the effective field theory fits better and with wider validity regions. The NNLO results are excellently fitted up to $\sim 300$ MeV.

There are one, three and six (seven) coupling constants to be fitted in the LO, NLO and NNLO calculations respectively in the $^1S_0$ ($^3S_1,^3D_1$) channel. We renormalize only the $\lambda$-dependence before fitting the phase shifts to the data, in order to avoid the cancellations among large numbers arising from the terms with positive powers of $\lambda$. 
FIG. 2. The scattering phase shift for the $^3S_1$ channel. For the notation see Fig. 1.

TABLE I. Dimensionless coupling constants fitted to the weighted Nijmegen PWA data in the $^1S_0$ channel.

|       | $\tilde{C}_0^{(1S_0)}$ | $\tilde{C}_2^{(1S_0)}$ | $\tilde{D}_2^{(1S_0)}$ | $\tilde{C}_4^{(1S_0)}$ | $\tilde{D}_4^{(1S_0)}$ | $\tilde{E}_4^{(1S_0)}$ |
|-------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| LO    | -0.958                 | —                      | —                      | —                      | —                      | —                      |
| NLO   | -0.964                 | 0.517                  | 0.479                  | —                      | —                      | —                      |
| NNLO  | -0.967                 | 0.205                  | 0.099                  | -0.055                 | 1.180                  | 1.203                  |

Tables I and II show how fitted values of the ($\lambda$-renormalized) dimensionless coupling constants change as we go higher orders. The dimensionless coupling constants $\tilde{X}_{2n}^{(s)}$ for $X_{2n}^{(s)}$ with $X = C, D, E$, which are used in Ref. [11], are defined as $\tilde{X}_{2n}^{(s)} = (M/4\pi)\mu^{2n+1}X_{2n}^{(s)}$, where $s$ stands for the channel. It is important to note that most of these coupling constants are of order one. It implies that our estimate of the magnitude of each term is correct. There are however large ambiguities in determining the values of coupling constants at NNLO. There seem to be “flat directions”; sets of considerably different values of the LECs give almost the same phase shifts.
TABLE II. Dimensionless coupling constants fitted to the weighted Nijmegen PWA data in the $^3S_1-^3D_1$ channel.

|       | $\tilde{C}_0^{(3S_1)}$ | $\tilde{C}_2^{(3S_1)}$ | $\tilde{D}_2^{(3S_1)}$ | $\tilde{C}_4^{(3S_1)}$ | $\tilde{D}_4^{(3S_1)}$ | $\tilde{E}_4^{(3S_1)}$ | $\tilde{C}_2^{(SD)}$ |
|-------|-----------------------|-----------------------|-------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| LO    | -1.22                 | —                     | —                 | —                     | —                     | —                     | —                     |
| NLO   | -1.31                 | 1.47                  | 0.68              | —                     | —                     | —                     | —                     |
| NNLO  | -1.42                 | 0.89                  | -0.11             | -11.72                | 4.28                  | 31.1                  | -4.80                 |

**SUMMARY AND DISCUSSIONS**

We perform NNLO calculations for the scattering amplitudes for the nucleon-nucleon scattering in the S waves with the power counting suggested by the Wilsonian RG analysis done in a previous paper, which is very similar to the KSW power counting. A novel hybrid regularization is employed to introduce the scale $\lambda$, which separates the pion exchange into the S-OPE and the L-OPE. We fit the calculated phase shifts to the Nijmegen PWA data. The fitted values of most of the coupling constants are of the natural size and the effective field theory expansion seems converging up to including the NNLO.

In our approach, the “renormalization scale” $\mu$ appearing in the PDS is identified with the scale $\lambda$ up to a numerical constant. The scale $\lambda$ plays an analogous role to that of the floating cutoff in the Wilsonian RG analysis. The low-momentum physical quantities should not depend on the values of $\lambda$. This requirement leads to a set of renormalization group equations (RGEs) for the coupling constants. The requirement is satisfied order by order. With the RGEs and their solutions, we have a complete control over the $\lambda$ dependence. The RGEs and their solutions will be given elsewhere.

We find that the introduction of the GDF requires a careful definition of the pion exchange coupling constant and an additional factor is necessary. This finding is possible only with analytic expressions and the RG analysis. We think that it is also important for numerical studies with similar damping factors.

Our approach will be applicable to other partial waves. The application to the P waves would be of particular interest, because BKV suspect that N$^3$LO calculations would be necessary to have a convergent result. (Gegelia also found serious problems in the P waves in his unpublished work.) Recently we have done a Wilsonian RG analysis for the P waves.
in a similar manner to that in Ref. [16], and have argued that the pion exchange in the P waves demotes to higher order so that the counterterms are present to absorb the cutoff dependence arising from the loops containing pion exchanges[21]. The calculations of the scattering phase shifts with hybrid regularization are now in progress.

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