Fractons emerge as charges with reduced mobility in a new class of gauge theories. Here, we generalise fractonic theories of $U(1)$ type to what we call $(k,n)$-fractonic Maxwell theory, which employs symmetric order-$n$ tensors of $k$-forms (rank-$k$ antisymmetric tensors) as “vector potentials”. The generalisation has two key manifestations. First, the objects with mobility restrictions extend beyond simple charges to higher order multipoles (dipoles, quadrupoles, . . .) all the way to $n^{th}$-order multipoles. Second, these fractonic charges themselves are characterized by tensorial densities of $(k − 1)$-dimensional extended objects. The source-free sector exhibits ‘photonic’ excitations with dispersion $\omega \sim q^k$.

The obvious question of where the fracton phases fit in the classification has motivated further work. A natural starting point is to ask for a field theoretical description of such phases that obtains both the excitation physics and the ground state degeneracy. Ref. [18], starting from the X-cube model, uses the BF-formulation to obtain a field theory that contains the key fractonic physics. Other approaches[19] combine the Chern-Simons/BF formulation with symmetric tensor (see below) gauge formulations to obtain the fracton characterization. All this clearly points to a rich panorama of possibilities.

Concurrently with these developments, fractons have appeared in an apparently different context in the study of spin liquids described by symmetric tensor gauge theories[20, 21]. Such $U(1)$ symmetric-tensor gauge theories support a gapless “photon” excitation much like Maxwell electrodynamics, which is described by a vector gauge field. Sources(charges) of such a symmetric-tensor gauge theory have a crucial additional character – an isolated charge of a symmetric-tensor gauge theory is immobile. This arises from the fact that charge conservation in such theories also forces the conservation of dipole moment. As a consequence, motion of an isolated charge, as it changes the dipole moment, is forbidden, endowing the charge with a fracton character. An isolated dipole (total charge zero) does not suffer such constraints: dipoles are free to move in an unconstrained fashion (to be compared with the dipole excitations in the X-cube model discussed above). Charge conservation is enforced by the continuity equation

$$\partial_t \rho + \partial_i J^0_i = 0,$$

where $\rho$ is the charge density, $t$ is time, $\partial_i$ is time, and $\partial_i$ the spatial derivative along the $i$-th Cartesian coordinate direction, and $J^0_i$ is the charge current vector. The fractonic nature of charge in symmetric-tensor gauge theories follows from the fact that the charge current vector is itself the divergence of the dipole current tensor $J_{ij}$, i. e.,

$$J^0_i = \partial_j J^i_j.$$

Several interesting aspects of such symmetric-tensor gauge theories, including their connection to gravity, have been explored [22, 23]. While these developments are interesting and encouraging, future research will have to reveal if such
symmetric-tensor gauge theories form a generic framework to describe gapless fracton phases (as noted above, gapped variants starting from symmetric-tensor descriptions are discussed in ref. [19]).

It is compelling to look for generalization of fractonic theories. In the symmetric-tensor gauge theories, charge conservation forces dipole moment conservation. It is then natural to ask if a hierarchy of such conservation laws can be constructed, where conservation of charge implies conservation of all multipole-moments of the charges, say up to some n-th moment (i.e., $2^n$-pole). Physically, this would imply mobility restrictions not only for individual charges, but also on all higher-moment objects such as dipoles, quadrupoles, ..., $2^n$-pole – we call this the rank-$n$ fractonic character. Another desirable generalization of the symmetric-tensor gauge theories is from mobility restricted “scalar” charges to “extended (k – 1) dimensional” objects.

In this paper, we achieve this by formulating what we call $(k, n)$-fracton theory. These have tensor charge densities of rank-$k$-1-dimensional extended objects with generalized mobility restrictions on all multipoles up to $n$-th order. This is accomplished by constructing a new gauge structure – gauge fields are order $n$ symmetric tensors of $k$-forms (anti-symmetric tensors) in arbitrary spatial dimension $d$. This leads naturally to a sequence of theories under the heading of $(k, n)$-fractonic Maxwell theory, which for $(k = 1, n = 1)$ reduces to standard electromagnetism.

We first present a straightforward generalization of the symmetric tensor gauge theories to make $(1, n)$-fracton theories with higher conserved multipole moments of “scalar” charges, followed by the development of the full $(k, n)$-fracton theory. We close with an extended outlook.

**Notation:** We use a notation that brings out the physical character of our theory that is not designed for “relativistic covariance”, with $x$ denoting a point in $d$-dimensional Cartesian space, and $t$, time. This simple mathematical notation uses symmetric and anti-symmetric tensors. A symmetric tensor of rank-$n$, i.e., with $n$-indices $i_1, i_2, ..., i_n$ is denoted by $S_{i_1i_2...i_n}$ where the $(\ )$ used to explicitly indicate the symmetric nature. On the other hand an anti-symmetric tensor of rank $k$ with indices $i_1, ..., i_k$ is denoted by $T_{[i_1,...,i_k]}$ where $[\ ]$ makes the anti-symmetric character of the tensor $T$ explicit. Note that $S_{i_1i_2} = S_{i_2i_1}$ and $T_{[i_1]} = T_1$ is to be understood. A collection of $s$ indices $i_1i_2...i_s$ is abbreviated into a composite index $I_{sr}$

$$I_{sr} = i_1i_2...i_s.$$  

Further, $P_s$ of $I_{sr}$ permutes $1 ... s$,

$$P_sI_{sr} = i_{P_s(1)}i_{P_s(2)}...i_{P_s(s)}.$$  

The sign of the permutation is denoted by $(-1)^{P_s}$.

**$(1, n)$-Fracton Theories:** We first generalize the symmetric tensor gauge theories of ref. [20] to rank-$n$ theories, which impose the generalized higher multipole mobility constraints. Consider a gauge field described by $(\phi(x, t), A_{\phi(x, t)}(x, t))$ where the vector potential of the usual Maxwell theory is generalized to a symmetric tensor of rank $n$. We define the electric field tensor (symmetric tensor of rank $n$) as

$$E_{\phi(x, t)} = -\partial_t \phi - \partial_i A_{\phi(x, t)}.$$  

(5)

The magnetic field tensor is defined as

$$B_{(i_1i_2...i_s)} = \sum_{(P_s)} \left[ \prod_{r=1}^{n} (-1)^{P_{sr}} \partial_{i_{P_s(r)}} A_{i_{P_s(r)}i_{P_s(r+1)}...i_{P_s(s)}} \right]$$  

(6)

where $I_{sr} = i_{P_s(1)}...i_{P_s(s)}$, is a composite index and $P_s$ permutes $I_{sr}$ in the notation introduced above. Note that the magnetic field tensor, with a rather unconventional structure, is symmetric in exchange of $I_{sr}$ with $I_{sr}$ composite indices. It is anti-symmetric upon exchange of the two indices within any $I_{sr}$.

These fields can be sourced by suitable charges and currents. The charge density $\rho$ and a symmetric $n$-tensor current density $J_{(i_1i_2...i_s)}$ couple minimally to the gauge fields. The system is described by a Lagrangian density

$$L = L_E - L_B + (-1)^{P_s} \rho \phi + J_{(i_1i_2...i_s)} A_{(i_1i_2...i_s)}$$  

(7)

where repeated indices are contracted ignoring the parentheses (which simply remind us of the symmetric nature of the tensors involved). Here $L_E$ and $L_B$ are electric and magnetic energy densities defined by

$$L_E = \frac{1}{2} \epsilon E_{i(j,...,n)} E_{i(j,...,n)}$$  

(8)

where $\epsilon$ is a suitable positive definite dielectric symmetric tensor (in interchange of the set $i$ with $j$) tensor and

$$L_B = \frac{1}{2} \kappa B_{i_1i_2...i_s} B_{i_1i_2...i_s}$$  

(9)

with $\kappa$ an inverse permeability tensor, again symmetric positive definite. The electric and magnetic fields are invariant under the gauge transformation involving the function $\psi(x, t)$:

$$\phi \rightarrow \phi + \partial_t \psi;$$  

$$A_{i_1i_2...i_s} \rightarrow A_{i_1i_2...i_s} - \partial_i \psi.$$  

The gauge invariance of the action (integral of the Lagrangian density) produces the continuity equation

$$\partial_t \rho + \partial_i \rho \partial_x \phi \rho \partial_i \rho \partial_x \phi \rho = 0.$$  

(10)

This theory thus encodes mobility restrictions not only on charges, but also on any $2^p$-pole (a collection of “closeby” charges with leading non vanishing $p$-th multipole moment), $p \leq n$, as eqn. (11) implies that

$$\partial_t \int_V d^d x \rho = 0$$  

$$\partial_i \int_V d^d x \rho = 0$$  

$$\partial_t \int_V d^d x \rho = 0$$  

(12)

$$\partial_i \int_V d^d x \rho = 0.$$
all multipoles up to order \( n \) are conserved in this theory (\( V \) is the \( d \)-volume of the system). The current \( J_{[i_1 \ldots i_d]} \) thus has a natural meaning of the 2\(^n\)-pole current. While charge current is the divergence of the dipole current, the dipole current is in turn that of the quadrupole current, and so on. In this sense, the theory describes rank-\( n \) fractonic physics of point charges whose density is \( \rho \).

**Theories with extended sources:** The natural next question is if there are generalizations of the theory where the charges are more complex objects. To explore this idea, we first review known Maxwellian theories where charges are extended objects, a well developed subject in itself (cf. [24, 25]), casting this theory in our notation. Such gauge theories are described by a gauge field \( (\phi_{[i_1 \ldots i_d]}(x, t), A_{[i_1 \ldots i_d]}(x, t)) \) where, following our convention, \( \phi \) and \( A \) are fully anti-symmetric tensors (\( A_{[i_1 \ldots i_d]} \) is usually called a \( k \)-form). The electric field is also an anti-symmetric tensor defined by

\[
E_{[i_1 \ldots i_d]} = -\frac{1}{(k-1)!} \sum_{P_k} (-1)^{P_k} \partial_{[i_1} \phi_{[i_2 \ldots i_d]} - \partial_{i_1} A_{[i_2 \ldots i_d]}.
\]

The magnetic field here is obtained as

\[
B_{[i_1 \ldots i_d+1]} = \frac{1}{k!} \sum_{P_{k+1}} (-1)^{P_{k+1}} \partial_{[i_1} A_{i_2 \ldots i_{d+1}]}.
\]

The extended charged sources\(^1\) are described by densities \( \rho_{[i_1 \ldots i_d]} \) and currents \( J_{[i_1 \ldots i_d]} \). One can now define a Lagrangian density for this theory as

\[
L_E = L_B = \rho \phi_{[i_1 \ldots i_d]} + \frac{1}{k!} A_{[i_1 \ldots i_d]},
\]

where repeated indices are summed over. The energy densities

\[
L_E = \frac{1}{2k} E_{[i_1 \ldots i_d]} E_{[i_1 \ldots i_d]} \quad \text{and} \quad L_B = \frac{1}{2(k+1)} B_{[i_1 \ldots i_d+1]} B_{[i_1 \ldots i_d+1]}
\]

have suitably defined positive definite dielectric, \( \epsilon \), and inverse permeability, \( \kappa \), tensors which are both symmetric under the swapping of primed and unprimed indices (with their internal order kept fixed) – this is the meaning of their superscripts \((\{i \ldots \} [i' \ldots \} \)\). Again, the electric and magnetic fields are gauge invariant under

\[
\phi_{[i_1 \ldots i_d]} = \phi_{[i_1 \ldots i_d]} + \partial_{i_1} \psi_{[i_2 \ldots i_d]} \\
A_{[i_1 \ldots i_d]} = A_{[i_1 \ldots i_d]} - \frac{1}{(k-1)!} \sum_{P_k} (-1)^{P_k} \partial_{[i_1} \psi_{[i_2 \ldots i_d]}.
\]

where \( \psi_{[i_1 \ldots i_d]}(x, t) \) denotes the field that characterizes the gauge transformation. Further, gauge invariance leads to (summing repeated indices):

\[
\partial_{i_1} \rho_{[i_1 \ldots i_d]} + \partial_{i_1} J_{[i_1 \ldots i_d]} = 0. \tag{19}
\]

Physically, this implies that the rate of change of the “density of the extended charge” is the divergence of its current. Note that this does not yet have any fractonic character; the extended object has no mobility constraints within this theory. In our nomenclature, this is a \((k, 1)\)-fracton theory. We now turn to the question how to endow such extended objects with constrained mobility. In any spatial dimension \( d \), we can let \( 1 \leq k \leq (d-1) \).

\((k, n)\)-fracton theory: The desideratum of endowing charges with a \((k-1)\)-dimensional structure with a rank-\( n \) fractonic character is achieved by constructing a suitable gauge structure. We here develop a natural extension of \( U(1) \) Maxwell electrodynamics towards this end. The gauge field for \((k, n)\)-fracton theory that we propose is of the form

\[
(\phi_{[i_1 \ldots i_d]} \phi_{[i_1 \ldots i_d]} [i' \ldots i'_d], A_{[i_1 \ldots i_d]}. \]

Any field \( T_{[i_1 \ldots i_d]} \) is fully anti-symmetric under the action of \( P_k \) on any \( B_k \), i.e.,

\[
T_{[i_1 \ldots i_d]} = (-1)^P T_{[i_1 \ldots i_d]}
\]

Further, the meaning of the subscript \(([i_1 \ldots i_d]]) \) i.e., \( [i_1 \ldots i_d] \) enclosed in () is that it is symmetric under the exchange of any two \( i_s \). In other words,

\[
T_{([i_1 \ldots i_d]])} = T_{([i'_1 \ldots i'_d]])}
\]

for any permutation \( P_s \) of numbers 1 \ldots \( n \). Note that the notation does not imply any symmetry in interchanging \( i_s \) with \( i'_s \), i.e., two particular indices of \([i_1 \ldots i_d]\) and \([i'_1 \ldots i'_d]\) (\( r \neq r' \)). In other words, the gauge fields that we posit are “symmetric tensors of anti-symmetric tensors (forms)”. We now define fields in a natural fashion as

\[
E_{([i_1 \ldots i_d] \ldots [i'_{d-1}])} = -\frac{1}{[(k-1)!]^n} \sum_{P_{d}} \prod_{r=1}^{n} (-1)^{P_r} \phi_{[i_1 \ldots i_d] \ldots [i'_{d-1}])} \phi_{[i'_1 \ldots i'_{d-1}])}
\]

for the electric field. We have used the notation that

\[
P_{d} \phi = \phi_{[i_1 \ldots i_d]} \phi_{[i_1 \ldots i_d]} \ldots \phi_{[i_1 \ldots i_d]}
\]

where \( \phi_{[i_1 \ldots i_d]} \) i.e., a composite index with \( k-1 \) members. The magnetic field is

\[
B_{([i_1 \ldots i_d] \ldots [i'_{d-1})]} = \frac{1}{[k!]^n} \sum_{P_{d}} \prod_{r=1}^{n} (-1)^{P_r} \partial_{[i_1 \ldots i_d]} \phi_{[i_1 \ldots i_d]} \phi_{[i_1 \ldots i_d] \ldots [i'_{d-1}])}
\]

\[\]

\[^1\] In a discrete lattice, these can be viewed as objects defined using links, plaquettes etc., emanating from a lattice point.\[26\]
The gauge transformation described by

\[ \phi_{\{I^1_k]\ldots\{I^n_k\}} \rightarrow \phi_{\{I^1_k\ldots\{I^n_k\}} + \partial_\gamma \phi_{\{I^1_k\ldots\{I^n_k\}} \]

with

\[ L_E = \frac{1}{2k^n} E_a \epsilon_{a(\beta)} E_\beta, \quad L_B = \frac{1}{2(k + 1)^n} B_\gamma \kappa_{(\gamma)\delta} B_\delta \]

where \( \epsilon \) and \( \kappa \) are, again, suitable positive definite dielectric and permeability tensors, \( \alpha, \beta \) are indices of the type \( \{I^1_k\ldots\{I^n_k\} \) and \( \gamma, \delta \) are indices of the type \( \{I^1_k+1\ldots\{I^n_k\} \).

This achieves the description of the \( (k, n) \)-fracton excitations. In this context, cast-ffing symmetric tensors - leading to yet different forms may be worthwhile.

An interesting aspect of our construction is the nature of the \( (k, n) \)-fractonic charges. The character can be better understood by considering the \( (2, 2) \)-fracton example. Here the charges are of the type \( \rho_{(ij)}, \) i.e., a symmetric tensor of rank 2. In this theory, individual charges are immobile, but two charges with opposite charge tensors separated in space can move together as they preserve the net dipole moment. This is pictorially illustrated in fig. 1.

**Summary and outlook:** In this paper, we have aimed to develop gauge theories that describe extended charges/sources \( (k - 1 \text{-dimensional}) \) with mobility constraints of rank \( n \). We have concluded that a gauge theory with gauge fields that are "symmetric tensors of anti-symmetric tensors" provides the desired theory and also fixes the structure of the corresponding \( (k, n) \)-fracton density.

Our theory, building on [20], can be viewed as a natural generalization of \( U(1) \) Maxwell electromagnetism with freely mobile charges (the \( (1, 1) \)-fracton theory in our formulation) and constructing a sequence of theories with \( (k, n) \)-fractons as sources. A number of interesting questions follow immediately from these considerations.

First, our formulation may not be a unique generalization of Maxwell electromagnetism, and other routes - e.g. involving symmetric tensors - leading to yet different kinds of fracton excitations may be worth exploring. In this context, casting our results in the more aesthetical language of differential forms may be worthwhile.

Second, we have not mandated relativistic invariance for our construction. This is rather natural in condensed matter settings, where such an invariance can effectively emerge, but
generically does not, see e.g. [27–29]. Nonetheless, identifying possible relativistically covariant generalised fracton theories is surely a worthwhile aim.

Third, a quantum mechanical extension of generalised fracton theories could unearth novel properties of these theories and their excitations, e.g. concerning nature, quantum numbers and statistics of their elementary excitations. This would be embedded in the broader quest for understanding the topological properties of quantum locally constrained models such as quantum dimer models [30].

Fourth, what are some microscopic models having generalised fracton theories as effective low-energy description? Particularly noteworthy here is the realization that the symmetric tensor gauge theory [31] in $d = 2$ is dual to elasticity theory (see also [32, 33]). Following this direction, it will be interesting to explore theories dual to $(k,n)$-fracton theories to look for possible physical realizations.

Fifth, a related direction is the exploration of the phases of the $(k,n)$-fracton theory and/or their microscopic models of origin. A case in point is, again, the symmetric tensor gauge theory[34, 35] in $d = 2$ which has provided a fresh perspective into defect driven phase transitions.

Finally, it is interesting explore the connection of $(k,n)$-fracton theory to some of the discrete fracton phases (as reviewed in the introductory section) with subextensive ground-state degeneracies (see, for example, [36]), and possible generalization of lattice gerbe theories[26, 37–39] to lattice tensor-gerbe theories with fractonic physics. A Chern-Simons/BF formulation[19] of $(k,n)$-fracton theories could also prove fruitful in obtaining field theories of generalized discrete fracton models.

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[1] X.-G. Wen, “Colloquium: Zoo of quantum-topological phases of matter,” Rev. Mod. Phys. 89, 041004 (2017).
[2] A. Kitaev, “Periodic table for topological insulators and superconductors,” AIP Conference Proceedings 1134, 22 (2009).
[3] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, “Topological insulators and superconductors: tenfold way and dimensional hierarchy,” New Journal of Physics 12, 065010 (2010).
[4] T. Senthil, “Symmetry-protected topological phases of quantum matter,” Annual Review of Condensed Matter Physics 6, 299 (2015).
[5] A. Kitaev, “Fault-tolerant quantum computation by anyons,” Annals of Physics 303, 2 (2003).
[6] C. Chamon, “Quantum glassiness in strongly correlated clean systems: An example of topological overprotection,” Phys. Rev. Lett. 94, 040402 (2005).
[7] S. Bravyi, B. Leemhuis, and B. M. Terhal, “Topological order in an exactly solvable 3d spin model,” Annals of Physics 326, 839 (2011).
[8] C. Castelnovo and C. Chamon, “Topological quantum glassiness,” Philosophical Magazine 92, 304 (2012).
[9] J. Haah, “Local stabilizer codes in three dimensions without string logical operators,” Phys. Rev. A 83, 042330 (2011).
[10] B. Yoshida, “Exotic topological order in fractal spin liquids,” Phys. Rev. B 88, 125122 (2013).
[11] S. Bravyi and J. Haah, “Quantum self-correction in the 3d cubic code model,” Phys. Rev. Lett. 111, 200501 (2013).
[12] S. Vijay, J. Haah, and L. Fu, “A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations,” Phys. Rev. B 92, 235136 (2015).
[13] S. Vijay, J. Haah, and L. Fu, “Fracton topological order, generalized lattice gauge theory, and duality,” Phys. Rev. B 94, 235157 (2016).
[14] D. J. Williamson, “Fractal symmetries: Unraveling the cubic code,” Phys. Rev. B 94, 155128 (2016).
[15] T. H. Hsieh and G. B. Halász, “Fractons from partons,” Phys. Rev. B 96, 165105 (2017).
[16] R. M. Nandkishore and M. Hermele, “Fractons,” Annual Review of Condensed Matter Physics 10, 295 (2019).
[17] K. Slagle and Y. B. Kim, “X-cube model on generic lattices: Fracton phases and geometric order,” Phys. Rev. B 97, 165106 (2018).
[18] K. Slagle and Y. B. Kim, “Quantum field theory of x-cube fracton topological order and robust degeneracy from geometry,” Phys. Rev. B 96, 195139 (2017).
[19] Y. You, T. Devakul, S. L. Sondhi, and F. J. Burnell, “Fractonic Chern-Simons and BF theories,” arXiv e-prints , arXiv:1904.11530 (2019), arXiv:1904.11530 [cond-mat.str-el].
[20] M. Pretko, “Subdimensional particle structure of higher rank $u(1)$ spin liquids,” Phys. Rev. B 95, 115139 (2017).
[21] M. Pretko, “Generalized electromagnetism of subdimensional particles: A spin liquid story,” Phys. Rev. B 96, 035119 (2017).
[22] M. Pretko, “Higher-spin witten effect and two-dimensional fracton phases,” Phys. Rev. B 96, 125151 (2017).
[23] M. Pretko, “Emergent gravity of fractons: Mach’s principle revisited,” Phys. Rev. D 96, 024051 (2017).
[24] M. Kalb and P. Ramond, “Classical direct interstring action,” Phys. Rev. D 9, 2273 (1974).
[25] M. Henneaux and C. Teitelboim, “p-form electrodynamics,” Foundations of Physics 16, 593 (1986).
[26] R. Savit, “Duality in field theory and statistical systems,” Rev. Mod. Phys. 52, 453 (1980).
[27] R. Moessner, S. L. Sondhi, and E. Fradkin, “Short-ranged resonating valence bond physics, quantum dimer models, and ising gauge theories,” Phys. Rev. B 65, 024504 (2001).
[28] C. Xu, “Gapless bosonic excitation without symmetry breaking: An algebraic spin liquid with soft gravitons,” Phys. Rev. B 74, 224433 (2006).
[29] O. Benton, L. D. C. Jaubert, H. Yan, and N. Shannon, “A spin-liquid with pinch-line singularities on the pyrochlore lattice,” Nature Communications 7, 11572 (2016).
[30] R. Moessner and S. L. Sondhi, “Resonating valence bond phase in the triangular lattice quantum dimer model,” Phys. Rev. Lett. 86, 1881 (2001).
[31] M. Pretko and L. Radzihovsky, “Fracton-elasticity duality,” Phys. Rev. Lett. 120, 195301 (2018).
[32] A. Gromov, “Fractional Topological Elasticity and Fracton Order,” arXiv e-prints (2019), arXiv:1712.06600 [cond-mat.str-
The gauge transformation is [33] A. Gromov and P. Surówka, “On duality between Cosserat elasticity and fractons,” arXiv e-prints, arXiv:1908.06984 (2019), arXiv:1908.06984 [cond-mat.str-el].

[34] A. Kumar and A. C. Potter, “Symmetry-enforced fractonicity and two-dimensional quantum crystal melting,” Phys. Rev. B 100, 045119 (2019).

[35] M. Pretko, Z. Zhai, and L. Radzihovsky, “Crystal-to-Fracton Tensor Gauge Theory Dualities,” arXiv e-prints, arXiv:1907.12577 (2019), arXiv:1907.12577 [cond-mat.str-el].

[36] H. Ma, M. Hermle, and X. Chen, “Fracton topological order from the higgs and partial-confinement mechanisms of rank-two gauge theory,” Phys. Rev. B 98, 035111 (2018).

[37] F. J. Wegner, “Duality in generalized ising models and phase transitions without local order parameters,” Journal of Mathematical Physics 12, 2259 (1971).

[38] A. E. Lipstein and R. A. Reid-Edwards, “Lattice gerbe theory,” Journal of High Energy Physics 2014, 34 (2014).

[39] D. A. Johnston, “z2 lattice gerbe theory,” Phys. Rev. D 90, 107701 (2014).

(2,2)-fracton theory

We illustrate the results for the (2,2)-fracton theory. Gauge fields are

\[ (\phi_{ij}, A_{(ijkl)}) \] (32)

The electric field (eqn. (23))

\[ E_{(ijkl)} = -\partial_i \partial_j \phi_{(ij)} - \partial_j \partial_k \phi_{(jk)} - \partial_k \partial_l \phi_{(kl)} + \partial_l \partial_i \phi_{(il)} - \partial_i A_{(ijkl)} \] (33)

The magnetic field (eqn. (25))

\[ B_{(ijkl)[mn]} = \partial_m \partial_n A_{(ijkl)[mn]} + \partial_n \partial_m A_{(ijkl)[mn]} + \partial_m \partial_n A_{(ijkl)[mn]} + \partial_n \partial_m A_{(ijkl)[mn]} + \partial_m \partial_n A_{(ijkl)[mn]} + \partial_n \partial_m A_{(ijkl)[mn]} \] (34)

The gauge transformation is

\[ \phi_{ij} \rightarrow \phi_{ij} + \partial_i \psi_{(ij)} \]

\[ A_{(ijkl)} \rightarrow A_{(ijkl)} + \partial_j \psi_{(ij)} \] (35)

Under the gauge transformation

\[ E_{(ijkl)} \rightarrow E_{(ijkl)} + \partial_i \partial_j \phi_{(ij)} - \partial_j \partial_k \phi_{(jk)} - \partial_k \partial_l \phi_{(kl)} + \partial_l \partial_i \phi_{(il)} - \partial_i A_{(ijkl)} \]

where the terms in the () vanishes ensuring gauge invariant electric and magnetic fields. Gauge invariance of the action will enforce

\[ \partial_i \partial_j \phi_{(ij)} + \partial_i \partial_j A_{(ijkl)} = 0. \] (37)