SINGULAR SCALAR CURVATURE EQUATIONS

KAI ZHENG

Abstract. We develop estimates for the equation of scalar curvature of singular metrics with cone angle \( \beta > 1 \), in a big and semi-positive cohomology class on a \( \text{Kähler} \) manifold. We further derive the Laplacian estimate for the scalar curvature equation of degenerate \( \text{Kähler} \) metrics. We then have several applications of these estimates on the singular constant scalar curvature \( \text{Kähler} \) metrics, which also include the singular \( \text{Kähler-Einstein} \) metrics.

1. Introduction

Let \((X, \omega)\) be a closed \( \text{Kähler} \) manifold. Calabi initiated the study of \( \text{Kähler} \) metrics with constant scalar curvature (cscK) in a given \( \text{Kähler} \) class \([\omega]\) on \( X \). Recently, Chen-Cheng [7] solved the geodesic stability conjecture and the properness conjecture on existence of the cscK metrics, via establishing new a priori estimates for the 4th order cscK equation [8]. However, the Yau-Tian-Donaldson conjecture expects the equivalence between the existence of cscK metrics and algebraic stabilities, which requires further understood of the possible singularity/degeneration developing from the cscK problem. In this article, we work on a priori estimates for the scalar curvature equation of both the singular and degenerate \( \text{Kähler} \) metrics.

Before we move further, we fix some notations. We are given an ample divisor \( D \) in \( X \), together with its associated line bundle \( L_D \). We let \( s \) be a holomorphic section of \( L_D \) and \( h \) be a Hermitian metric on \( L_D \). We set \( \Omega \) to be a big and semi-positive cohomology class.

The singular scalar curvature equation in \( \Omega \) was introduced in [26] c.f. Definition 4.15. Within the same article, we derive estimation of its solution when \( 0 < \beta \leq 1 \), which results in existence of cscK metrics on some normal varieties. If we further let \( \Omega \) to be \( \text{Kähler} \), the singularities is called cone singularities and the solution of the singular scalar curvature equation is called \textit{cscK cone metric}. The uniqueness, existence and the necessary direction of the corresponding log version of the Yau-Tian-Donaldson conjecture were proven in [1,15,17,26,28]. In this article, we continue studying the singular scalar curvature equation, when

\[
\beta > 1.
\]

The singular scalar curvature equation has the expression

\[
(1.1) \quad \omega^n_\varphi = (\omega_{sr} + i\partial \bar{\partial} \varphi)^n = e^F \omega^n_0, \quad \triangle_\varphi F = \text{tr}_\varphi \theta - R,
\]
where, $\omega_{sr}$ is a smooth representative in the class $\Omega$ and $R$ is a real-valued function.

We introduce two parameters to approximate the singular equation (1.1), $t$ for the lose of positivity and $\epsilon$ for the degeneration, see Definition 4.17. We say a solution $(\varphi_{t,\epsilon}, F_{t,\epsilon})$ to the approximate equation (4.12) is **almost admissible**, if it has uniform weighted estimates independent of $t, \epsilon$, including the $L^\infty$-estimates of both $\varphi_{t,\epsilon}$ and $F_{t,\epsilon}$, the gradient estimate of $\varphi_{t,\epsilon}$ and the $W^{2,p}$-estimate. The precise definition is given in Definition 4.22.

**Theorem 1.1.** The solution to the approximate singular scalar curvature equation (4.12) with bounded entropy is almost admissible.

The detailed statement of these estimates will be given in the $L^\infty$-estimates (Theorem 5.1), the gradient estimate of $\varphi$ (Theorem 6.1), the $W^{2,p}$-estimate (Theorem 7.1).

The singular scalar curvature equation extends the singular Monge-Ampère equation. Actually, the reference metric $\omega_\theta$ in the singular scalar curvature equation (1.1), is defined to be a solution to the singular Monge-Ampère equation

$$
\omega_n^\theta = (\omega_{sr} + i\partial\bar{\partial}\varphi_{\theta})^n = |s|^{23-2}e^{b_\theta}\omega^n.
$$

(1.2)

The precise definition is given in Definition 4.6 in Section 4.2.

**Remark 1.2.** There is a large literature on the singular Monge-Ampère equations [3, 5, 10, 13, 22, 25] and the corresponding Kähler-Ricci flow [2, 14, 20, 21], see also the references therein. However, more effort is required to tackle the new difficulties arising from deriving estimates for the scalar curvature equation (1.1), because of its 4th order nature. The tool we apply here is the integration method, instead of the maximum principle.

**Remark 1.3.** We extend Chen-Cheng’s estimates [7] to the singular scalar curvature equation (1.1).

If we focus on the degeneration, setting $\Omega$ to be a Kähler class, then the singular equation (1.1) is named the **degenerate scalar curvature equation**. The accurate definition is given in Definition 2.13 and the corresponding approximation is stated in Definition 2.24. The almost admissible estimates (Definition 2.21) for the approximate solution $\varphi_t$ are obtained from Theorem 1.1, immediately.

In Section 3, we further show metric equivalence from the volume ratio bound, i.e. to prove the Laplacian estimate for the degenerate scalar curvature equation, Theorem 3.1.

An almost admissible function is called **$\gamma$-admissible**, if it admits a weighted Laplacian estimate, which is defined in Definition 2.9.
Theorem 1.4. The almost admissible solution to the approximate degenerate scalar curvature equation (2.23) with bounded $\|\partial F_\epsilon\|_{\psi_\epsilon}$ is

- admissible, i.e. $\text{tr}_\omega \omega_\epsilon \leq C$, when $\beta > \frac{n+1}{2}$.
- $\gamma$-admissible for any $\gamma > 0$, when $1 < \beta \leq \frac{n+1}{2}$.

Remark 1.5. The Laplacian estimate holds for the smooth cscK metric $\beta = 1$ in [7, Theorem 1.7], the cscK cone metric when $0 < \beta < 1$ in [26, Theorem 5.25] and see also [26, Question 4.42] for further development. However, when $\beta > 1$, the Laplacian estimate is quite different from the one when the cone angle is not larger than 1.

For the degenerate scalar curvature equation, the Laplacian estimates is more involved. Our new approach is an integration method with weights. The proof is divided into seven main steps.

The first step is to transform the approximate degenerate scalar curvature equation (2.23) into an integral inequality with weights. The second step is to deal with the trouble term containing $\Delta F$ by integration by parts. Unfortunately, this causes new difficulties, because of the loss of the weights, see Remark 3.9.

The third step is to apply the Sobolev inequality to the integral inequality to conclude a rough iteration inequality. However, weights loss in this step again, see Remark 3.11. Both the fourth and fifth step are designed to overcome these difficulties. The fourth step aims to construct several inverse weighted inequalities, with the help of introducing two useful parameters $\sigma$ and $\gamma$. While, the fifth step is set for weighted inequalities, by carefully calculating the sub-critical exponent for weights.

As last, with all these efforts, we arrive at an iteration inequality and proceed the iteration argument to conclude the Laplacian estimates.

After we establish the estimation, in Section 4.7, we have two quick applications on the singular/degenerate cscK metrics, which were introduced in [26] as the $L^1$-limit of the approximate solutions, Definition 4.25.

One is to show regularities of the singular/degenerate cscK metrics. Precision statements are given in Theorem 4.30 and Theorem 4.31. We see that that the potential function of the degenerate cscK metric is almost admissible and the convergence is smooth outside the divisor $D$. We also find that the volume form $\omega^n$ is shown to be prescribed with the given degenerate type along $D$ as $|s_h|^{2\beta-2} \omega^n$. The volume ratio bound could be further improved to be a global metric bound, under the assumption that the volume ratio of the approximate sequence has bounded gradient. Consequently, the degenerate cscK metric has weighted complex Hessian on the whole manifold $X$. In particular, when the cone angle is larger than half of $n + 1$, the complex Hessian is globally bounded.
The other application is to revisit Yau’s result on degenerate Kähler-Einstein (KE) metrics in the seminal work [23]. A special situation of the degenerate scalar curvature equation is the degenerate Kähler-Einstein equation
\[
\omega^n = |s|^{2\beta - 2} e^{\lambda \varphi + c} \omega^n.
\]
(1.3)
In Section 2.2, the variational structure of the degenerate Kähler-Einstein metrics will be presented. We will show that they are all critical solutions of a logarithm variant of Ding’s functional. Ding functional was introduced in [11].

A direct corollary from Theorem 1.1 and Theorem 1.4 is

**Corollary 1.6** (Yau [23]). For any \( \beta \geq 1 \), the solution \( \varphi_\epsilon \) to the degenerate Kähler-Einstein equation (1.3) has \( L^\infty \)-estimate and bounded Hessian.

**Remark 1.7.** The accurate statement is given in Theorem 3.21.

For the degenerate Monge-Ampère problem, the difficult fourth order nonlinear term \( \Delta F \) in the scalar curvature problem becomes the second order term \( \Delta \varphi \), which could be controlled directly, comparing Proposition 3.6 in Section 3.2 and (3.31) in Section 3.7.

**Remark 1.8.** As a result, Yau’s theorem for degenerate Kähler-Einstein metrics is recovered, that is when \( \lambda \leq 0 \), the degenerate Monge-Ampère equation (1.3) admits a solution with bounded complex Hessian and being smooth outside the divisor \( D \). But, the method we apply to derive the Laplacian estimates is the integration method, which is different from Yau’s maximum principle.

As a further matter, the integration method with weights is applied to derive the gradient estimates in Section 6 as well.

**Remark 1.9.** Our results also extend the gradient and the Laplacian estimate for the non-degenerate Monge-Ampère equation, which were obtained by the integral method [9].

At last, as a continuation of Question 1.14 in [28] and Question 1.9 in [26], we could propose the following uniqueness question.

**Question 1.10.** whether the \( \gamma \)-admissible degenerate cscK metric constructed above is unique (up to automorphisms)?

It is worth to compare this question to its counterpart for the cone metrics [4, 27, 28].

**Remark 1.11.** On Riemannian surfaces, there are intensive study on constant curvature metrics with large cone angles, see [6, 18] and references therein.
2. Degenerate scalar curvature problems

We denote the curvature form of $h$ to be $\Theta_D = -i\partial\bar{\partial}\log h$, which represents the class $C_1(L_D)$. The Poincaré-Lelong equation gives us that

\[ 2\pi[D] = i\partial\bar{\partial}\log |s|^2_h + \Theta_D = i\partial\bar{\partial}\log |s|^2. \]  

In this section, we assume that the given Kähler class $\Omega$ is proportional to $C_1(X,D) := C_1(X) - (1 - \beta)C_1(L_D)$.

We let $\lambda$ be a constant such that

\[ C_1(X,D) = \lambda\Omega. \]  

This cohomology condition (2.2) implies the existence of a smooth function $h_\omega$ such that

\[ \text{Ric}(\omega) = \lambda\omega + (1 - \beta)\Theta_D + i\partial\bar{\partial}h_\omega. \]  

2.1. Critical points of the log Ding functional. The log Ding functional is a modification of the Ding functional [11] by adding the weight $|s|^2_h - 2h_\omega$.

Definition 2.1. For all $\varphi$ in $\mathcal{H}_\Omega$, we define the log Ding functional to be

\[ F_\beta(\varphi) = -D_\omega(\varphi) - \frac{1}{V} \log\left( \frac{1}{V} \int_X e^{h_\omega - \lambda\varphi} |s|^{2\beta - 2}\omega^n \right). \]

Here $D_\omega(\varphi) := \frac{1}{V} \sum_{j=0}^n \int_X \varphi^j \omega^{n-j}$.

We compute the critical points of the log Ding functional.

Proposition 2.2. The 1st variation of the log Ding functional at $\varphi$ is

\[ \delta F_\beta(u) = -\frac{1}{V} \int_X u\omega^n + \left( \int_X e^{h_\omega - \lambda\varphi} |s|^{2\beta - 2}\omega^n \right)^{-1} \int_X u e^{h_\omega - \lambda\varphi} |s|^{2\beta - 2}\omega^n. \]

The critical point of the log Ding functional satisfies the following equation

\[ \omega^n_\varphi = |s|^{2\beta - 2}_h \frac{V \cdot e^{h_\omega - \lambda\varphi}}{\int_X e^{h_\omega - \lambda\varphi} |s|^{2\beta - 2}\omega^n}. \]

Definition 2.3 (Log Kähler-Einstein metric). We call the solution of (2.4), log Kähler-Einstein metric. When $0 < \beta < 1$, it is called Kähler-Einstein cone metric, c.f. [16, 24].

Further computation shows that the log Kähler-Einstein metric satisfies the following identity

\[ \text{Ric}(\omega_\varphi) = \lambda\omega_\varphi + 2\pi(1 - \beta)[D]. \]
Proposition 2.4. The 2nd variation of the log Ding functional at $\varphi$ is

$$\delta^2 F_\beta(u, v) = \frac{1}{V} \int_X (\partial u, \partial v) \varphi \omega^n + \lambda \left\{ \frac{\int_X u e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|}{\int_X e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|^2} - \frac{\int_X u v e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|}{\int_X e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|} \right\}.$$

Proof. We continue the proceeding computation to see that

$$\delta^2 F_\beta(u, v) = -\frac{1}{V} \int_X u \Delta \varphi v \omega^n - \left( \int_X e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n| \right)^{-2} \int_X (-\lambda v) e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n| \int_X u e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|$$

$$+ \left( \int_X e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n| \right)^{-1} \int_X u(-\lambda v) e^{h_\omega - \lambda \varphi} |s_h^{2\beta - 2} \omega^n|.$$

So the conclusion follows directly. \qed

Then inserting the critical point equation (2.4) into the 2nd variation of the log Ding functional, we have the following corollaries.

Corollary 2.5. At a log Kähler-Einstein metric, the 2nd variation of the log Ding functional becomes

$$\delta^2 F_\beta(u, v) = \frac{1}{V} \int_X (\partial u, \partial v) \varphi \omega^n + \lambda \left\{ \frac{1}{V^2} \int_X u \omega^n \int_X v \omega^n - \frac{1}{V} \int_X u v \omega^n \right\}.$$

Corollary 2.6. The log Ding functional is convex at its critical points, if one of the following condition holds

- $C_1(X, D) \leq 0$,
- $X$ is a projective Fano manifold, $D$ stays in the linear system of $|K_X|^{-1}$ and $\Omega = C_1(X)$, $0 < \beta \leq 1$.

Proof. Since $\lambda \Omega = C_1(X, D)$ by (2.2), we have $\lambda \leq 0$. The lemma follows from applying the Cauchy-Schwarz inequality to Lemma 2.5.

From the assumptions of the second statement, we have $C_1(X, D) = \beta C_1(X) = \beta \Omega$ and then $\lambda = \beta$ by (2.2). Thus convexity follows from the lower bound of the first eigenvalue of the Laplacian for a Kähler-Einstein cone metric [16]. \qed

2.2. Reference metric and prescribed Ricci curvature problem.

We choose a smooth $(1,1)$-form $\theta \in C_1(X, D)$. Then there exists a smooth function $h_\theta$ such that

$$\text{Ric}(\omega) = \theta + (1 - \beta) \Theta_D + i \partial \bar{\partial} h_\theta.$$

Definition 2.7 (Reference metric). A reference metric $\omega_\theta$ is defined to be the solution of the degenerate Monge-Ampère equation

$$\omega_\theta^n = |s_h^{2\beta - 2} e^{h_\theta} \omega^n|.$$
under the normalisation condition
\begin{equation}
\int_X |s|_h^{2\beta - 2} e^{h_\theta} \omega^n = V. \tag{2.8}
\end{equation}

The reference metric $\omega_\theta = \omega + i\partial \bar{\partial} \varphi_\theta$ satisfies the equation for prescribing the Ricci curvature problem
\begin{equation}
\text{Ric}(\omega_\theta) = \theta + 2\pi (1 - \beta)[D]. \tag{2.9}
\end{equation}

The equation (2.7) is a special case of the degenerate complex Monge-Ampère equation (2.4).

**Lemma 2.8.** With the reference metric $\omega_\theta$ in (2.7), the log Ding functional is rewritten as
\begin{equation}
F_\beta(\varphi) = -D(\varphi) - \frac{1}{\lambda} \log( \frac{1}{V} \int_X e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta ).
\end{equation}

The critical point satisfies the following equation
\begin{equation}
\omega^n_\varphi = \frac{V \cdot e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta}{\int_X e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta }. \tag{2.10}
\end{equation}

From the log Kähler-Einstein equation (2.4) and the reference metric equation (2.7), we have
\begin{equation}
e^{h_\omega - \lambda \varphi} \omega^n_\theta = e^{h_\omega - \lambda \varphi} |s|_h^{2\beta - 2} e^{h_\theta} \omega^n = e^{h_\theta} \omega^n_\varphi.
\end{equation}

The 2nd variation of the log Ding functional at $\varphi$ is
\begin{equation}
\delta^2 F_\beta(u, v) = \frac{1}{V} \int_X (\partial u, \partial v) \omega^n_\varphi
+ \lambda \left\{ \frac{\int_X u e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta \int_X v e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta}{(\int_X e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta)^2} - \frac{\int_X u v e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta}{\int_X e^{h_\omega - h_\theta - \lambda \varphi} \omega^n_\theta} \right\}. \tag{2.11}
\end{equation}

**2.3. Degenerate Monge-Ampère equations.** The degenerate Monge-Ampère equations (2.4) and (2.7) could be summarised as
\begin{equation}
\omega^n_\varphi = |s|_h^{2\beta - 2} e^{F(x, \varphi)} \omega^n.
\end{equation}

When the cone angle $\beta > 1$, we define a solution of (2.11) as following.

**Definition 2.9.** A function $\varphi$ is said to be $\gamma$-admissible for some $\gamma \geq 0$, if
- $\text{tr}_\omega \omega_\varphi \leq C |s|_h^{-2\gamma}$, on $X$,
- $\varphi$ is smooth on $M$.

Moreover, we say a function $\varphi$ is a $\gamma$-admissible solution to the equation (2.11), if it is a $\gamma$-admissible function and satisfies (2.11) on $M$. When $\gamma = 0$, the function $\varphi$ is said to be admissible.

Yau initiated the study of these degenerate Monge-Ampère equations in his seminal article [23] on Calabi conjecture. For the Kähler-Einstein cone metrics with cone angle $0 < \beta \leq 1$, see [16] and references therein.
2.3.1. Approximation of the degenerate Monge-Ampère equations. In this section, we discuss some properties of the approximation of the reference metric. We let

\[ h_\epsilon := (\beta - 1) \log S_\epsilon, \quad S_\epsilon := |s^2_h + \epsilon|. \]

We define \( \omega_{\theta_\epsilon} \) to be the approximation of the reference metric (2.7),

\[ \omega_{\theta_\epsilon}^n = e^{h_\theta + h_\epsilon + c_\epsilon \omega} \text{ with } \int_X e^{h_\theta + h_\epsilon + c_\epsilon \omega} = V. \]

By (2.6), its Ricci curvature has the formula.

**Lemma 2.10.**

\[ \text{Ric}(\omega_{\theta_\epsilon}) = \text{Ric}(\omega) - i\partial \bar{\partial} h_\theta - i\partial \bar{\partial} h_\epsilon \]

\[ = \theta + (1 - \beta)\Theta + (1 - \beta)i\partial \bar{\partial} \log S_\epsilon. \]

**Lemma 2.11.** There exists a nonnegative constant \( C \) such that

\[ CS_\epsilon^{-1} \omega \geq i\partial \bar{\partial} \log S_\epsilon \geq -|s^2_h| \Theta_D \geq -C\omega. \]

**Proof.** It follows from the calculation

\[ i\partial \bar{\partial} \log S_\epsilon = \frac{i\partial \bar{\partial} |s^2_h|}{S_\epsilon} - \frac{i\partial |s^2_h \bar{\partial} |s^2_h|}{S_\epsilon^2}. \]

The upper bound is direct to see, by using \(|\partial S_\epsilon|_\omega \leq CS_\epsilon^{\frac{1}{2}}\), that

\[ i\partial \bar{\partial} \log S_\epsilon \leq CS_\epsilon^{-1}. \]

While, making use of the identity

\[ |s^2_h i\partial \bar{\partial} \log |s|^2_h| = |s^2_h i\partial \bar{\partial} |s|^2_h| - i\partial |s^2_h \bar{\partial} |s|^2_h|, \]

we have

\[ i\partial \bar{\partial} \log S_\epsilon = \frac{i\partial \bar{\partial} |s^2_h|}{S_\epsilon} + \frac{|s^2_h i\partial \bar{\partial} \log |s|^2_h| - |s^2_h i\partial \bar{\partial} |s|^2_h|}{S_\epsilon^2} \]

\[ = \frac{\epsilon i\partial \bar{\partial} |s^2_h|}{S_\epsilon^2} + \frac{|s^2_h i\partial \bar{\partial} \log |s|^2_h|}{S_\epsilon^2}. \]

Replaced by the identity above again, it is further reduced to

\[ = \frac{\epsilon |s^2_h i\partial \bar{\partial} \log |s|^2_h|}{S_\epsilon^2} + \frac{|s^2_h i\partial \bar{\partial} \log |s|^2_h|}{|s^2_h S_\epsilon^2|} \geq -\frac{|s^2_h \Theta_D}{S_\epsilon}. \]

\( \square \)
2.3.2. Approximation of the log KE metrics. Similarly, the approximation for the log KE metric is defined to be

$$\omega_n^\varphi = e^{h\varphi - \lambda \varphi} e^{c_\epsilon \omega_n}, \quad \int_X e^{h\varphi - \lambda \varphi} e^{c_\epsilon \omega_n} = V.$$  

(2.15)

They are critical points of the approximation of the log Ding functional

$$F_\beta^\epsilon(\varphi) = -D_\omega(\varphi) - \frac{1}{\lambda} \log \left[ \frac{1}{V} \right] \int_X e^{h\varphi - \lambda \varphi} S_{\epsilon}^{3-1} \omega_n.$$  

(2.16)

When $\beta > 1$, we say an admissible solution to (2.4) is a degenerate KE metric. If $0 < \beta < 1$, the Hölder space $C^{2,\alpha,\beta}$ was introduced in [12]. A KE cone metric is a $C^{2,\alpha,\beta}$ solution to (2.4) and smooth on $M$.

Corresponding to Corollary 2.6, we have

**Proposition 2.12.** The following statements of existence of log KE metric and its smooth approximation hold.

- When $\lambda > 0$ and $0 < \beta \leq 1$, we have $F_\beta^\epsilon \geq F_\beta$. Furthermore, if $F_\beta$ is proper, then there exists a KE cone metric of cone angle $\beta$, which has smooth approximation (2.15).

- When $\lambda < 0$ and $\beta > 1$, we have $F_\beta^\epsilon \geq F_\beta$. Furthermore, there exists a degenerate KE metric, which has smooth approximation (2.15).

2.4. Degenerate scalar curvature equation. In this section, we assume $n \geq 2$ and consider the degenerate case when $\beta > 1$. Recall that $\theta$ is a smooth $(1,1)$-form in $C^1(X,D)$ and $\omega_\theta$ is the reference metric defined in (2.7). We also set $h := -(1-\beta) \log |s|^2_h$ and define

$$S_\beta = \frac{C_1(X,D)[\omega]^{n-1}}{[\omega]^n}.$$  

(2.16)

**Definition 2.13.** The degenerate scalar curvature equation is defined as

$$\omega_n^\varphi = e^{F} \omega_\theta^\varphi, \quad \triangle_F \varphi = \text{tr}_\varphi \theta - R.$$  

(2.17)

Here, the reference metric $\omega_\theta$ is introduced in Definition (2.7). When $R = S_\beta$, a solution to the degenerate scalar curvature equation is called a degenerate cscK metric.

Direct computation shows that

**Lemma 2.14.** The scalar curvature of the degenerate scalar curvature equation satisfies that

$$S(\omega_\varphi) = R$$  

on $M$.

(2.18)

At last, we close this section by making use of the reference metric (2.7) then rewriting (2.17) as below with respect to the smooth Kähler metric $\omega$. We let

$$f = -h_\theta - h, \quad \tilde{F} = F - f.$$
Lemma 2.15. The degenerate scalar curvature equation satisfies the following equations
\begin{equation}
\omega^n_{\varphi} = e^F \omega^n, \quad \triangle_{\varphi} \hat{F} = \text{tr}_{\varphi}(\theta - i\partial\bar{\partial}f) - R.
\end{equation}

2.5. Log $K$-energy. Motivated from the log $K$-energy for the cone angle $\beta$ in $(0,1]$ in [28], we define the log $K$-energy for large cone angle $\beta > 1$ to be

Definition 2.16. The log $K$-energy is defined as
\begin{equation}
\nu_{\beta}(\varphi) := E_{\beta}(\varphi) + J_{-\theta}(\varphi) + \frac{1}{V} \int_M (\mathfrak{h} + h_{\theta}) \omega^n, \quad \forall \varphi \in \mathcal{H}_\Omega.
\end{equation}

Here $\omega_0$ is the reference metric, which is the admissible solution to the degenerate complex Monge-Ampère equation (2.7). Also, the log entropy is defined in terms of the reference metric $\omega_0$ as
\[ E_{\beta}(\varphi) = \frac{1}{V} \int_M \log \frac{\omega^n_{\varphi}}{\omega^n_{\theta}}, \]
and the $j_\chi$- and $J_\chi$-functionals are defined to be
\[ j_\chi(\varphi) := \frac{1}{V} \int_X \varphi \sum_{j=0}^{n-1} \omega^j \wedge \omega_{\varphi}^{n-1-j} \wedge \chi, \quad J_\chi(\varphi) := j_\chi(\varphi) - D_\omega(D_\varphi). \]

Corollary 2.17. For all $\varphi$ in $\mathcal{H}_\Omega$, we have
\[ \nu_{\beta}(\varphi) = \nu_1(\varphi) + (1 - \beta) \frac{1}{V} \int_X \log |s|^2_H(\omega^n_{\varphi} - \omega^n_\theta) + (1 - \beta) J_{\Theta_D}(\varphi) \]
\[ = \nu_1(\varphi) + (1 - \beta) \cdot \left[ D_{\omega,D}(\varphi) - \frac{\text{Vol}(D)}{V} \cdot D_\omega(D_\varphi) \right]. \]

Here $\nu_1(\varphi) = E_1(\varphi) + J_{-\text{Ric}}(\varphi)$ is the Mabuchi $K$ energy, the entropy of $\omega^n_{\varphi}$ is
\[ E_1(\varphi) = \frac{1}{V} \int_X \log \frac{\omega^n_{\varphi}}{\omega^n_\theta}, \]
and, the corresponding volume and the normalisation functional on the divisor $D$ are defined to be
\[ \text{Vol}(D) = \int_D \Omega^{n-1}, \quad D_{\omega,D}(\varphi) = \frac{n}{V} \int_0^1 \int_D \partial_t \varphi \omega^{n-1}_{\varphi} dt. \]

Corollary 2.18. Writing in terms of the smooth background metric $\omega$, we have
\begin{equation}
\nu_{\beta}(\varphi) = E_1(\varphi) + J_{-\theta}(\varphi) + \frac{1}{V} \int_M (\mathfrak{h} + h_{\theta})(\omega^n - \omega^n_{\varphi}),
\end{equation}
for all $\varphi \in \mathcal{H}_\Omega$.

We see that the last term has $\beta$ involved,
\begin{equation}
-(1 - \beta) \frac{1}{V} \int_M \log |s|^2_H(\omega^n - \omega^n_{\varphi}).
\end{equation}
2.6. Approximate degenerate scalar curvature equation. We first define the approximate degenerate cscK equation.

**Definition 2.19.** We say \( \omega_\epsilon \varphi_\epsilon \) is an approximation of the degenerate scalar curvature equation, if it satisfies the following PDEs

\[
\omega^n_{\varphi_\epsilon} = e^{\tilde{F}_\epsilon} \omega^n_\theta, \quad \triangle_{\varphi_\epsilon} F_\epsilon = \text{tr}_{\varphi_\epsilon} \theta - R.
\]

Particularly, when \( R \) is a constant, we call it the approximate degenerate cscK equation.

Then we write the approximate equation (2.23) in terms of the smooth background metric \( \omega \).

**Lemma 2.20.** The approximate degenerate scalar curvature equation satisfies the equations

\[
\omega^n_{\varphi_\epsilon} = e^{\tilde{F}_\epsilon} \omega^n_\theta, \quad \triangle_{\varphi_\epsilon} \tilde{F}_\epsilon = \text{tr}_{\varphi_\epsilon} (\theta - i\partial \overline{\partial} \tilde{f}_\epsilon) - R.
\]

Here \( \tilde{F}_\epsilon = F_\epsilon - \tilde{f}_\epsilon \), \( \tilde{f}_\epsilon = -h_\theta - \theta_\epsilon - c_\epsilon \).

This section is devoted to prove the Laplacian estimates of the smooth solution \( (\varphi_\epsilon, F_\epsilon) \) for the approximate degenerate scalar curvature equations (2.23) or (2.24).

2.7. Almost admissible solutions. In order to clarify the idea of the Laplacian estimates. We introduce the following definition.

**Definition 2.21** (Almost admissible solution for degenerate equations). We say \( \varphi_\epsilon \) is an almost admissible solution to the approximate degenerate scalar curvature equation (2.24), if the following uniform estimates independent of \( \epsilon \) hold

- **L∞-estimates** in Theorem 5.3:

\[
\|\varphi_\epsilon\|_\infty, \quad \|F_\epsilon\|_\infty \leq C; \tag{2.25}
\]

- **gradient estimate** of \( \varphi \) in Theorem 6.1:

\[
\Sigma^\beta_\epsilon \|\partial \varphi\|_{L^\infty(\omega)}^2 \leq C, \quad 1 > \sigma_D^1 > \max\{1 - \frac{2\beta}{n+2}, 0\}; \tag{2.26}
\]

- **W2,p-estimate** in Theorem 7.3: for any \( p \geq 1 \),

\[
\int_X (\text{tr}_\omega \omega_{\varphi_\epsilon})^p \Sigma^\beta_\epsilon \omega^n \leq C(p), \quad \sigma_D^2 := (\beta - 1) \frac{n-2}{n-1+p^{-1}}. \tag{2.27}
\]

**Remark 2.22.** These three estimates will be obtained for more general equation, that is the singular scalar curvature equation introduced in Section 4, see Theorem 5.1, Theorem 6.1 and Theorem 7.1.

This definition of the almost admissible solution further gives us the following estimates of \( \tilde{F}_\epsilon \),
Lemma 2.23. Assume the $L^\infty$-estimate of $F_\epsilon$ holds. Then there exists a uniform nonnegative constant $C$ depending on $\theta$, $\| h_\theta + c_\epsilon \|_{C^2(\omega)}$ and $\Theta_D$ such that
\[
\tilde{F}_\epsilon \leq C, \quad \triangle \tilde{F}_\epsilon \geq \triangle F_\epsilon - C,
\]
\[
C[1 + (\beta - 1) S_\epsilon^{-1}] \text{tr}_{\varphi_\epsilon} \omega - R \geq \triangle \varphi_\epsilon \tilde{F}_\epsilon \geq -C \text{tr}_{\varphi_\epsilon} \omega - R.
\]

Proof. The upper bound of $\tilde{F}_\epsilon$ follows from the $L^\infty$-estimate of $F_\epsilon$ and
\[
\tilde{F}_\epsilon = F_\epsilon + h_\theta + (\beta - 1) \log S_\epsilon + c_\epsilon.
\]
The rest results are obtained from the following identities
\[
\triangle \varphi_\epsilon \tilde{F}_\epsilon = \text{tr}_{\varphi_\epsilon} [\theta + i \partial \partial (h_\theta + h_\epsilon)] - R, \quad i \partial \partial \tilde{F}_\epsilon = i \partial \partial (F_\epsilon + h_\theta + h_\epsilon)
\]
and the lower bound of $i \partial \partial h_\epsilon = (\beta - 1) i \partial \partial \log S_\epsilon$ from Lemma 2.11. □

Remark 2.24. This lemma tells us that the directions of the inequalities above when $\beta > 1$, are exactly opposite to their counterparts when $0 < \beta < 1$, where $\tilde{F}_\epsilon$ has lower bound and both $\triangle \varphi_\epsilon \tilde{F}_\epsilon$ and $\triangle \tilde{F}_\epsilon$ have upper bound.

Lemma 2.25. Assume the gradient estimate of $F_\epsilon$ holds i.e.
\[
\| \partial F_\epsilon \|_{L^\infty(\omega_{\varphi_\epsilon})} \leq C.
\]
Then there exists a uniform constant $C$ depending on $\| h_\theta + c_\epsilon \|_{C^1(\omega)}$ and $\sup_X S_\epsilon^{-\frac{1}{2}}|\partial S_\epsilon|_\omega$ such that
\[
|\partial F_\epsilon|^2 \leq C \text{tr}_\omega \omega_{\varphi_\epsilon}, \quad |\partial \tilde{F}_\epsilon|^2 \leq C[1 + \text{tr}_\omega \omega_{\varphi_\epsilon} + \frac{(\beta - 1)^2}{S_\epsilon}].
\]

Proof. We also make use of the inequality
\[
|\partial F_\epsilon|^2 \leq |\partial F_\epsilon|_{\omega_{\varphi_\epsilon}}^2 \cdot \text{tr}_\omega \omega_{\varphi_\epsilon}.
\]
We compute and see that
\[
|\partial \tilde{F}_\epsilon|^2 = |\partial (F_\epsilon + h_\theta + (\beta - 1) \log S_\epsilon)|^2
\]
\[
\leq C[1 + |\partial F_\epsilon|^2 + (\beta - 1)^2 \frac{|\partial S_\epsilon|^2}{S_\epsilon^2}] 
\]
\[
\leq C[1 + \text{tr}_\omega \omega_{\varphi_\epsilon} + \frac{(\beta - 1)^2}{S_\epsilon}].
\]
In the last inequality, we use $|\partial S_\epsilon| \leq C_2.4 |S_\epsilon|^{\frac{1}{2}}$. □

2.8. Approximation of log $K$-energy. In this subsection, we consider the degenerate cscK equation.

Definition 2.26.
\[
(2.29) \quad \omega_{\varphi_\epsilon}^n = e^{\tilde{F}_\epsilon} \omega^n, \quad \triangle_{\varphi_\epsilon} \tilde{F}_\epsilon = \text{tr}_{\varphi_\epsilon} (\theta - i \partial \partial \tilde{F}_\epsilon) - S_\beta.
\]
Definition 2.27 (Approximate log K-energy). The approximate log K-energy is defined as

\[(2.30) \quad \nu_{\beta}^{\epsilon}(\varphi) = E_1(\varphi) + J_{-\theta}(\varphi) - \frac{1}{V} \int_X \tilde{f}_\epsilon(\omega^n - \omega^n_{\varphi}), \quad \forall \varphi \in H_{\Omega}.\]

Here \(\tilde{f}_\epsilon = - (\beta - 1) \log S_\epsilon - h_{\theta} - c_\epsilon.\)

Lemma 2.28 ([26] Lemma 3.12). The first derivative of the approximate log K-energy is

\[\partial_t \nu_{\beta}^{\epsilon} = \frac{1}{V} \int_X \varphi_t \left[ \Delta_{\varphi} \log \frac{\omega^n_{\varphi}}{\omega^n_{\varphi}} - \text{tr}_{\varphi} (\theta - i\partial\bar{\partial} \tilde{f}_\epsilon) + S_\beta \right] \omega^n_{\varphi}.\]

Its critical point satisfies the approximate degenerate cscK equation (2.29).

Lemma 2.29. Let \(\varphi_\epsilon\) be the solution to the approximate degenerate cscK equation (2.29) with bounded entropy \(E_{\beta,\epsilon} = \frac{1}{V} \int_X F_{\epsilon}^{\omega^n_{\varphi}}.\) Then

\[(2.31) \quad \nu_{\beta}^{\epsilon}(\varphi_\epsilon) \geq \nu_{\beta}(\varphi_\epsilon) - C.\]

Proof. Comparing the log K-energy with its approximation (2.30), we have

\[\nu_{\beta}^{\epsilon} - \nu_{\beta} = \frac{1}{V} \int_M [((\beta - 1)(\log S_\epsilon - \log |s|^2_{\beta}) + c_\epsilon](\omega^n - \omega^n_{\varphi}).\]

We see that

\[\frac{\beta - 1}{V} \int_X (\log S_\epsilon - \log |s|^2_{\beta}) \omega^n \geq 0.\]

We use the volume ratio bound from Corollary 2.23 and the \(L^\infty\)-estimate of \(F\) in Theorem 5.1 to show that

\[\frac{\beta - 1}{V} \int_X (\log S_\epsilon - \log |s|^2_{\beta}) \omega^n = \frac{\beta - 1}{V} \int_X (\log S_\epsilon - \log |s|^2_{\beta}) e^{\tilde{f}_\epsilon} \omega^n \leq C_1 \frac{\beta - 1}{V} \int_X (\log S_\epsilon - \log |s|^2_{\beta}) \omega^n \leq C_2.\]

In which, the constants \(C_1, C_2\) are independent of \(\epsilon\). Thus the lemma is proved. \(\square\)

We compute the second variation of the approximate log K-energy and obtain the upper bound of \(\nu_{\beta}^{\epsilon}(\varphi_\epsilon)\).

Proposition 2.30. The second derivative of the approximate log K-energy at the critical point is

\[\delta^2 \nu_{\beta}^{\epsilon}(u, v) = \frac{1}{V} \int_X (\partial u, \partial v) \omega^n_{\varphi} + \frac{1}{V} \int_X [\text{Ric}(\omega) - \theta + i\partial\bar{\partial} \tilde{f}_\epsilon](\partial u, \partial v) \omega^n_{\varphi}.\]

Note that, by (2.13), we have \(i\partial\bar{\partial} \log S_\epsilon \geq - \frac{|s|^2_{\epsilon}}{S_\epsilon} \Theta_D\) and

\[\text{Ric}(\omega) - \theta + i\partial\bar{\partial} \tilde{f}_\epsilon = \text{Ric}(\omega_{\theta_\epsilon}) - \theta = (1 - \beta) \Theta_D + (1 - \beta)i\partial\bar{\partial} \log S_\epsilon.\]
When $\beta \leq 1$, we have
\[
\text{Ric}(\omega) - \theta + i\partial\bar{\partial}\tilde{f}_\epsilon \geq (1 - \beta)\Theta_D - (1 - \beta)\frac{|s|_h}{S_\epsilon} \Theta_D = (1 - \beta)\frac{\epsilon}{S_\epsilon} \Theta_D.
\]

**Corollary 2.31.** The approximate log $K$-energy is convex at its critical points, when $\beta = 1$, or $0 < \beta < 1$ and $\Theta_D \geq 0$.

### 3. Laplacian estimate for degenerate scalar curvature equation

**Theorem 3.1** (Laplacian estimate). Suppose that $\varphi_\epsilon$ is an almost admissible solution to (2.24) and the gradient estimate $\|\partial F_\epsilon\|_{L^\infty(\omega_\varphi)}$ holds. Assume $\beta > 1$ and the degenerate exponent satisfies that
\[
\begin{align*}
\sigma_D &= 0, \text{ when } \beta > \frac{n + 1}{2}; \\
\sigma_D &> 0, \text{ when } \beta \leq \frac{n + 1}{2}.
\end{align*}
\]

Then there exists a uniform constant $C$ such that
\[
\text{tr}_\omega \omega_\varphi \varphi \leq C \cdot S_\epsilon^{1 - \sigma_D}.
\]

The uniform constant $C$ depends on the gradient estimate of $\varphi_\epsilon$, the $W^{2,p}$-estimate, $\beta, \sigma_D, c_\epsilon, n$ and
\[
\inf_{i \neq j} R_{iijj}(\omega), \quad C_S(\omega), \quad \Theta_D, \quad \sup_X S_\epsilon, \quad \sup_X S_\epsilon^{-\frac{1}{2}}|\partial S_\epsilon|_\omega, \quad \|h_\theta + c_\epsilon\|_{C^2(\omega)}.
\]

**Proof.** In this proof, we let $C_1$ to be a constant determined in (3.7). We denote the degenerate exponent $\sigma_D$ by $\gamma$ and also set
\[
v := \text{tr}_\omega \omega_\varphi, \quad w := e^{-C_1 \gamma} v, \quad u := S_\epsilon^\gamma w = S_\epsilon^{\gamma - C_1 \gamma} \text{tr}_\omega \omega_\varphi.
\]

We will omit the lower index $\epsilon$ for convenience. We will apply the integration method and the proof of this theorem will be divided into several steps as following. \(\boxdot\)

#### 3.1. Step 1: integral inequality. We now transform the approximate degenerate scalar curvature equations (2.24) into an integral inequality.

**Proposition 3.2** (Differential inequality). Assume that $\gamma$ is a nonnegative real number. Then there exist constants $C_{1.4}$ (3.7) and $C_{1.5}$ (3.9) such that the following differential inequality holds,
\[
\Delta_\varphi u \geq C_{1.4} u \text{tr}_\omega \omega + e^{-C_1 \gamma} \Delta F \cdot S_\epsilon^\gamma - C_{1.5}(1 + u).
\]

**Proof.** According to Yau’s computation,
\[
\Delta_\varphi \log(\text{tr}_\omega \omega_\varphi) \geq \frac{g^{kl} R_{iijj}(\omega) g_{eij} - S(\omega) + \tilde{F}}{\text{tr}_\omega \omega_\varphi}
\]
\[
\geq -C_{1.1} \text{tr}_\omega \omega + \frac{\Delta \tilde{F}}{\text{tr}_\omega \omega_\varphi}.
\]
The constant $-C_{1,1}$ is the lower bound of the bisectional curvature of $\omega$, i.e. \( \inf_{i \neq j} R_{iijj}(\omega) \). Moreover,

\[
\Delta_\varphi \log w \geq (C_1 - C_{1,1}) \text{tr}_\varphi \omega + \frac{\Delta \tilde{F}}{v} - C_1 n.
\]

(3.4)

Also, due to Lemma 2.11, there exists $C_{1,2}$ depending on $\Theta_D$ such that

\[
\Delta_\varphi \log S_\epsilon \geq -C_{1,2} \text{tr}_\varphi \omega.
\]

Also, due to Lemma 2.11, there exists $C_{1,2}$ depending on $\Theta_D$ such that

\[
\Delta_\varphi \log S_\epsilon \geq -C_{1,2} \text{tr}_\varphi \omega.
\]

Adding together, we establish the differential inequality for $u = w S_\epsilon^\gamma$,

\[
\Delta_\varphi \log u \geq (C_1 - C_{1,1} - \gamma C_{1,2}) \text{tr}_\varphi \omega + \frac{\Delta \tilde{F}}{v} - C_1 n.
\]

Note that we need the positivity of $\gamma$, i.e. $\gamma \geq 0$.

The lower bound of $\Delta \tilde{F} \geq \Delta F - C_{1,3}$ follows from Corollary 2.23. The constant $C_{1,3}$ is

\[
-C_{1,3} := \inf_X [\Delta (h_\theta + h_\epsilon)] = \inf_X \Delta h_\theta + (\beta - 1) \inf_X \Delta \log S_\epsilon
\]

\[
= \inf_X \Delta h_\theta - (\beta - 1) C_{1,2} n.
\]

Choosing sufficiently large nonnegative $C_1$ such that

\[
C_{1,4} := C_1 - C_{1,1} - \gamma \cdot C_{1,2} \geq 1,
\]

we find

\[
\Delta_\varphi \log u \geq C_{1,4} \text{tr}_\varphi \omega + \frac{\Delta F - C_{1,3}}{v} - C_1 n.
\]

Therefore, it is further written in the form

\[
\Delta_\varphi u = u \cdot |\Delta_\varphi \log u + |\partial \log u_i^2| \geq u \Delta_\varphi \log u
\]

\[
\geq C_{1,4} u \text{tr}_\varphi \omega + e^{-C_1 \varphi} (\Delta F - C_{1,3}) S_\epsilon^\gamma - C_1 n u.
\]

(3.8)

Setting

\[
C_{1,5} := 2 \max \{C_{1,3} e^{-C_1 \inf_X \varphi} \sup_X S_\epsilon^\gamma, C_1 n \},
\]

and inserting it to (3.8), we thus obtain the expected differential inequality.

\[\square\]

We introduce a notion $\tilde{u} = u + K$ for some nonnegative constant $K$ and set

\[
LHS_1 := \int_X |\partial \tilde{u}^p|^2 S_\epsilon^\gamma \omega_\varphi^n, \quad LHS_2 := \int_X \tilde{u}^{2p} u^\frac{n}{p-1} S_\epsilon^\gamma \omega_\varphi^n.
\]

**Proposition 3.3** (Integral inequality). There exists constants $C_{1,6}(C_1, \sup_X \varphi)$, and $C_{1,7}$ depending on $C_{1,4}$, $\inf_X \varphi$, $\sup_X F$, $\sup_X h_\theta$, $C_1$, $c_\epsilon$ such that

\[
\frac{2 C_{1,6}}{p} LHS_1 + C_{1,7} LHS_2 \leq RHS := \mathcal{N} + C_{1,5} \int_X \tilde{u}^{2p+1} \omega_\varphi^n,
\]

\[
\mathcal{N} := - \int_X e^{-C_1 \varphi} \tilde{u}^{2p} \Delta F S_\epsilon^\gamma e^F \omega_\varphi^n.
\]

(3.10)
Proof. Multiplied with $\tilde{u}^{2p}$, $p \geq 1$ and integrated with respect to $\omega^n_\varphi$ on $X$, the differential inequality in Proposition 3.2 yields

$$
\frac{2}{p} \int_X \tilde{u} |\partial \tilde{u}|^{2}_\varphi \omega^n_\varphi = - \int_X \tilde{u}^{2p} \Delta_\varphi \tilde{u} \omega^n_\varphi
\leq - \int_X \tilde{u}^{2p} [C_{1.4} u \text{tr}_\varphi \omega + e^{-C_{1.7}} \Delta F S^n_\gamma - C_{1.5} \tilde{u}] \omega^n_\varphi.
$$

Replacing $u$ by $e^{-C_{1.7}} v S^n_\gamma$ and using the relation between two norms $| \cdot |_\varphi$ and $| \cdot |$, i.e. $v | \cdot |_\varphi \geq | \cdot |$, we observe the lower bound of the gradient term

$$
\frac{2}{p} \int_X \tilde{u} |\partial \tilde{u}|^{2}_\varphi \omega^n_\varphi \geq \frac{2}{p} \int_X u |\partial u|^{2}_\varphi \omega^n_\varphi = \frac{2}{p} \int_X e^{-C_{1.7}} v S^n_\gamma |\partial \tilde{u}|^{2}_\varphi \omega^n_\varphi 
\geq C_{1.6} (C_1, \sup_{X} \varphi) \frac{2}{p} \int_X |\partial \tilde{u}|^{2}_\varphi S^n_\gamma \omega^n_\varphi.
$$

Consequently, we regroup positive terms to the left hand side of the integral inequality to conclude that

$$
\frac{2C_{1.6}}{p} \int_X |\partial \tilde{u}|^{2}_\varphi S^n_\gamma \omega^n_\varphi + C_{1.4} \int_X \tilde{u}^{2p} \text{tr}_\varphi \omega \omega^n_\varphi \leq \text{RHS}.
$$

Substituting $\tilde{F} = F + h_\theta + (\beta - 1) \log S_\epsilon + c_\epsilon$ into the fundamental inequality $\text{tr}_\varphi \omega \geq \frac{1}{e^\frac{1}{n-1}}$, we see that

$$
\text{tr}_\varphi \omega \geq u \frac{1}{e^{\frac{-1}{n-1}}} S^{\frac{1}{n-1}} e^{-F + h_\theta + (\beta - 1) \log S_\epsilon + c_\epsilon} \geq C_{1.7} u \frac{1}{e^{\frac{-1}{n-1}}} S^{\frac{1}{n-1}} \omega^n_\varphi.
$$

Thus

$$
\int_X \tilde{u}^{2p} u \text{tr}_\varphi \omega \omega^n_\varphi \geq C_{1.7} \int_X \tilde{u}^{2p} u \frac{n}{n-1} S^{\frac{-1}{n-1}} \omega^n_\varphi.
$$

The constant $C_{1.7}$ depends on $n, C_1, \inf \varphi, \sup F, \sup h_\theta, c_\epsilon$. Therefore, (3.10) is obtained by inserting this inequality to (3.12).

$\square$

Remark 3.4. The constant $C_{1.3}$ depends on the cone angle $\beta \geq 1$.

Remark 3.5. Actually, we could choose $K = 0$ and derive the integral inequality for $u$,

$$
\frac{2C_{1.6}}{p} \int_X |\partial u|^{2}_\varphi S^n_\gamma \omega^n_\varphi + C_{1.4} \int_X u^{2p+1} \text{tr}_\varphi \omega \omega^n_\varphi \leq \text{RHS}.
$$

We denote $k := (2p+1)\gamma + (\beta - 1)$. With the help of the $L^\infty$-estimates of $\varphi$ and $F$, this inequality is simplified to be

$$
\int_X u^{2p+1} \text{tr}_\varphi \omega \omega^n_\varphi \geq C_{1.7} \int_X u^{2p+1} \frac{n}{n-1} S^{\frac{-k-\beta-1}{n-1}} \omega^n.
$$
3.2. **Step 2: nonlinear term containing \( \Delta F \).** The term \( \mathcal{N} \) containing \( \Delta F \) in the RHS of the integral inequality (3.10) requires further simplification by integration by parts.

**Proposition 3.6 (Integral inequality).** There exists a constant \( C_{2.5} \) depending on \( \inf X \varphi, \sup X F, \| \partial \varphi \|_{L^\infty(\omega)}, \| \partial F \|_{L^\infty(\omega_F)} \), the constant in Lemma 2.23 and Lemma 2.25, and the constants \( C_{1.5}, C_{1.6}, C_{1.7}, \beta \) such that

\[
(3.14) \quad p^{-1} \text{LHS}_1 + \text{LHS}_2 \leq \text{RHS} \leq pC_2 \cdot \text{RHS}_1, \\
(3.15) \quad \text{RHS}_1 := \int_X \tilde{u}^{2p}[u + 1 + \frac{1}{5} S^2_e] \varphi + \frac{\gamma - \beta}{\gamma - 2} \omega^n. 
\]

**Proof.** By integration by parts, we split the nonlinear term \( \mathcal{N} \) into four sub-terms,

\[
(3.16) \quad \mathcal{N} := I + II + III + IV = -C_1 \int_X e^{-C_1 \varphi} (\partial \varphi, \partial F) \tilde{u}^{2p} S^2_e e^{\tilde{F}} \omega^n \\
+ 2p \int_X e^{-C_1 \varphi} \tilde{u}^{2p-1} (\partial \tilde{u}, \partial F) S^2_e e^{\tilde{F}} \omega^n \\
+ \int_X e^{-C_1 \varphi} \tilde{u}^{2p} (\partial F, \partial S^2_e) e^{\tilde{F}} \omega^n + \int_X e^{-C_1 \varphi} \tilde{u}^{2p} (\partial F, \partial \tilde{F}) S^2_e e^{\tilde{F}} \omega^n.
\]

The assumption on the gradient estimate of the volume ratio \( F \) and Lemma 2.25 give us that

\[
|\partial F| \leq Ce^{\frac{1}{2} \varphi} S^{-\frac{n}{2}}_e u^\frac{1}{n}.
\]

Inserting it with the gradient estimate of \( \varphi \) in (2.26). i.e.

\[
|\partial \varphi| \leq CS^{-\frac{n}{2}}_e, \quad \sigma^{-1}_D < 1
\]

to the first term, we see that

\[
I \leq C_1 \int_X e^{-C_1 \varphi} |\partial \varphi| |\partial F| \tilde{u}^{2p} S^2_e \omega^n \leq C_{2.1} \int_X e^{-C_1 \varphi} \tilde{u}^{2p} \varphi u^\frac{1}{2} S^\frac{n}{2} \omega^n,
\]

where, the constant \( C_{2.1} \) depends on \( \|S^\frac{1}{2}_e \partial \varphi\|_{L^\infty(\omega)}, \|\partial F\|_{L^\infty(\omega_F)} \).

The second term is nothing but using Hölder’s inequality and the bound of \( |\partial F|^2 \),

\[
II = 2 \int_X e^{-C_1 \varphi} \tilde{u}^p (\partial \tilde{u}^p, \partial F) S^2_e \omega^n \\
\leq \frac{C_{1.6}}{p} \int_X |\partial \tilde{u}^p|^2 S^2_e \omega^n + \frac{p}{C_{1.6}} \int_X e^{2C_1 \varphi} \tilde{u}^{2p} |\partial F|^2 S^2_e \omega^n \\
\leq \frac{C_{1.6}}{p} \int_X |\partial \tilde{u}^p|^2 S^2_e \omega^n + pC_{2.2} \int_X e^{C_1 \varphi} \tilde{u}^{2p} \omega^n.
\]
In order to bound the third term, we use \( |\partial F| \) again and the fact that
\[
|\partial S^\gamma_\epsilon| \leq \gamma C_{2,3} |S_\epsilon|^{\gamma - \frac{1}{2}}.
\]
As a result, we have
\[
III \leq \gamma C_{2,3} \int_X e^{-\frac{C_1}{2} \varphi} \tilde{u}^{2p} u^{\frac{1}{2}} S_\epsilon^{\frac{\gamma - 1}{2}} \omega^n.
\]
Due to Lemma 2.25 again, we get
\[
|\partial \tilde{F}|^2 \leq C [1 + e^{C_1 \varphi} S_\epsilon^{1/2} u^{1/2} + (\beta - 1) S_\epsilon^{1/2}].
\]
Then the fourth term is bounded by
\[
IV \leq \int_X e^{-\frac{C_1}{2} \varphi} \tilde{u}^{2p} \omega^n |\partial \tilde{F}| S_\epsilon^{\frac{\gamma - 1}{2}} \omega^n
\]
\[
\leq C_{2,3} \int_X e^{-\frac{C_1}{2} \varphi} \tilde{u}^{2p} u^{\frac{1}{2}} S_\epsilon^{\frac{\gamma - 1}{2}} \omega^n + \int_X \tilde{u}^{2p} u \omega^n
\]
\[
+ (\beta - 1) \int_X e^{-\frac{C_1}{2} \varphi} \tilde{u}^{2p} u^{1/2} S_\epsilon^{\frac{\gamma - 1}{2}} \omega^n].
\]
Inserting them back to (3.16), we have the bound of \( N \). Substituting \( N \) in (3.12) and note that \( \sigma_1^k < 1 \), we have thus proved (3.14). \( \square \)

**Remark 3.7.** When \( \gamma = 0 \), the third term \( III = 0 \).

**Remark 3.8.** When \( \beta = 1 \), \( \partial \tilde{F} = \partial F + \partial h_\theta \). Then the fourth term
\[
IV = \int_X e^{-C_1 \varphi} u^{2p} (\partial F, \partial \tilde{F}) S_\epsilon^\gamma e^{\varphi} \omega^n.
\]

**Remark 3.9.** In the third and the fourth term, the power of \( S_\epsilon \) loses \( \frac{1}{2} \), which cause troubles.

### 3.3. Step 3: rough iteration inequality.

We will apply the Sobolev inequality to the gradient term
\[
LHS_1 = \int_X |\partial \tilde{u}^p|^2 S_\epsilon^\gamma \omega^n
\]
in (3.14). We set
\[
k_\gamma := \gamma + \beta - 1 + \sigma, \quad \chi := \frac{n}{n - 1}, \quad \tilde{\mu} := S^{k_\gamma \chi} \omega^n.
\]
It is direct to see that \( k + \sigma = k_\gamma + 2p_\gamma \) and
\[
\|\tilde{u}\|_{L^{2p_\gamma}(\tilde{\mu})}^2 = \int_X (\tilde{u}^{2p_\gamma} S_\epsilon^{k_\gamma})^{\chi} \omega^n = \int_X \tilde{u}^{2p_\gamma} \tilde{\mu}.
\]

**Proposition 3.10** (Rough iteration inequality). There exists a constant \( C_3 \) depending on \( C_{1.6}, C_{1.7} \), the dependence in \( C_2, \inf_X F, \inf_X h_\theta, c_\epsilon \) and the Sobolev constant \( C_S(\omega) \) such that
\[
\|\tilde{u}\|_{L^{2p_\gamma}(\tilde{\mu})}^{2p_\gamma} + pLHS_2 \leq C_3 (p^2 RHS_1 + RHS_2 + 1)
\]
where RHS$$_1$$ is given in (3.15) and

\begin{equation}
(3.19) \quad \text{RHS}_2 := \int_X (u^2 \hat{u}^{2p-2} S^\gamma_\epsilon + C_{3.1} u^2 \hat{u}^{2p-2} S^\gamma_\epsilon + \sigma) \omega^n.
\end{equation}

Proof. We need to deal with the weights. Recall the equations of the volume form \(\omega^n\) from (2.23) and the volume form of the approximate reference metric \(\omega^n_\epsilon\) by (2.12),

\[ \omega^n_\epsilon = e^{\tilde{f}_\epsilon} \omega, \quad e^{\tilde{f}_\epsilon} = e^{F_\epsilon + h_\theta + c_\epsilon S^\beta_\epsilon - 1}. \]

We assume \(S_\epsilon \leq 1\). We see that there exists a constant \(C_{3.0}\) depending on \(\inf_X F\), \(\inf_X h_\theta\) and \(c_\epsilon\) such that

\[ \text{LHS}_1 \geq C_{3.0} \int_X |\partial \hat{u} F| S^\gamma_\epsilon + \beta - 1 \omega^n \geq C_{3.0} \int_X |\partial \hat{u} F| S^{k_\gamma_\epsilon} \omega^n, \quad \sigma \geq 0. \]

Using Lemma 3.13 with \(p_1 = 1\), \(p_2 = p - 1\), we have

\[ \text{LHS}_1 \geq C_{3.0} \int_X |\partial (u \hat{u}^{p-1})| S^{k_\gamma_\epsilon} \omega^n. \]

Further calculation shows that

\begin{equation}
(3.20) \quad = C_{3.0} \int_X |\partial (u \hat{u}^{p-1} S^k_\epsilon \omega^\gamma)|^2 \omega^n - \int_X u^2 \hat{u}^{2p-2} |\partial S^k_\epsilon \omega^\gamma|^2 \omega^n].
\end{equation}

We now make use of the Sobolev inequality to the first term in (3.20) with \(f = u \hat{u}^{p-1} S^k_\epsilon \omega^\gamma\), which states

\[ \|f\|_{L^2(\omega)} \leq C_S (\|\partial f\|_{L^2(\omega)} + ||f||_{L^2(\omega)}), \]

that is

\[ \int_X |\partial (u \hat{u}^{p-1} S^k_\epsilon \omega^\gamma)|^2 \omega^n \geq C_{3.0}^{-1} \left( \int_X u \hat{u}^{p-1} S^k_\epsilon \omega^\gamma \right)^2 \omega^n - \int_X u^2 \hat{u}^{2p-2} S^{k_\gamma_\epsilon} \omega^n. \]

Note that the power of the weight is increasing from \(k_\gamma\) to \(k_\gamma \chi\).

Substituting \(u = \hat{u} - K\) and \(\tilde{u} = S^{k_\gamma_\epsilon} \omega^n\) in the main term, we get

\[ \int_X |u \hat{u}^{p-1} S^k_\epsilon \omega^\gamma|^{2} \omega^n = \int_X |\hat{u} - K|^{2} \hat{u}^{2(p-1)2x} \tilde{u}, \]

\[ \geq C(n) \left[ \int_X \hat{u}^{2px} \tilde{u} - K^{2x} \int_X \hat{u}^{2^{(p-1)x} \tilde{u}}. \right. \]

With the help of Young’s inequality

\[ \hat{u}^{2^{(p-1)x}} \leq \frac{p-1}{p} \hat{u}^{2px} + 1 \leq \hat{u}^{2px} + 1, \]

choosing \(0 < K < K^{2x} \geq \frac{1}{2}\), we obtain

\[ \int_X |u \hat{u}^{p-1} S^k_\epsilon |^{2} \omega^n \geq C(n) \left[ \frac{1}{2} \int_X \hat{u}^{2px} \tilde{u} - K^{2x} \right] = \frac{C(n)}{2} \left[ \int_X \hat{u}^{2px} \tilde{u} - 1 \right]. \]
Using \(|\partial S|^{2} \leq C_{2.4}S_{\epsilon}\), we can estimate the second term in (3.20),

\[
\int_{X} u^{2} \tilde{u}^{2p-2} |\partial S_{\epsilon}^{k_{\gamma}}|^{2} \omega^{n} = \frac{k^{2}}{4} \int_{X} u^{2} \tilde{u}^{2p-2} S_{\epsilon}^{k_{\gamma} - 2} |\partial S_{\epsilon}|^{2} \omega^{n}
\]

(3.21)

\[
\leq C_{3.1} \int_{X} u^{2} \tilde{u}^{2p-2} S_{\epsilon}^{k_{\gamma} - 1} \omega^{n}.
\]

At last, we add these inequalities together to see that

\[
LHS_{1} \geq C_{3.2}\left\{ C_{S}^{-1} \left( \int_{X} \tilde{u}^{2p\chi} \tilde{\mu} - 1 \right)^{\chi^{-1}} - \int_{X} u^{2} \tilde{u}^{2p-2} S_{\epsilon}^{k_{\gamma}} \omega^{n} \right. - C_{3.1} \int_{X} u^{2} \tilde{u}^{2p-2} S_{\epsilon}^{k_{\gamma} - 1} \omega^{n} \right\}.
\]

Inserting this inequality to the integral inequality (3.14), we obtain the rough iteration inequality (3.18).

**Remark 3.11.** We observe that 1 is subtracted from the weight of \(S_{\epsilon}\) in the second term of \(RHS_{2}\) (3.19), which causes difficulties presented in the weighted inequality, Proposition 3.14. We will solve this problem by making use of the inverse weighted inequalities, Proposition 3.15.

**Remark 3.12.** When \(\beta = 1\) and \(\gamma = \sigma = 0\), the trouble term (3.21) vanishes.

We end this section by computing the auxiliary inequality.

**Lemma 3.13.** We write \(p = p_{1} + p_{2}\) with \(p_{1}, p_{2} \geq 0\).

\[
\int_{X} |\partial \tilde{\mu}|^{2} S_{\epsilon}^{k_{\gamma}} \omega^{n} \geq \int_{X} |\partial (u^{p_{1}} \tilde{\mu}^{p_{2}})|^{2} S_{\epsilon}^{k_{\gamma}} \omega^{n}.
\]

**Proof.** It is a direct computation

\[
\partial (u^{p_{1}} \tilde{\mu}^{p_{2}}) = \partial u^{p_{1}} \tilde{\mu}^{p_{2}} + u^{p_{1}} \partial (\tilde{\mu}^{p_{2}}) = p_{1} u^{p_{1} - 1} \partial u^{p_{2}} + u^{p_{1} - 1} \tilde{\mu}^{p_{2}} \partial u
\]

\[
= \partial u (u^{p_{1} - 1} \tilde{\mu}^{p_{2}} + p_{2} \tilde{\mu}) \leq p_{2} \partial \tilde{\mu} u^{p_{1} - 1} = \partial (\tilde{\mu} u),
\]

where we use \(u \leq \tilde{u}\). \(\square\)

3.4. **Step 4: weighted inequality.** We compare the left term \(\|\tilde{\mu}\|_{L^{2p\chi}(\tilde{\mu})}^{2p}\) of the rough iteration inequality, Proposition 3.10, with the right terms in \(RHS_{1}\) and \(RHS_{2}\), which are of the form

\[
\int_{X} \tilde{\mu}^{2p} S_{\epsilon}^{k_{\gamma} + \sigma - k_{\gamma}'} \omega^{n}.
\]

**Proposition 3.14** (Weighted inequality). Assume that \(n \geq 2\) and \(k' < 1\). Then there exists \(1 < a < \chi = \frac{a}{n-1}\) such that

\[
\int_{X} \tilde{\mu}^{2p} S_{\epsilon}^{k_{\gamma} + \sigma - k_{\gamma}'} \omega^{n} \leq C \int_{X} \tilde{\mu}^{2p} S_{\epsilon}^{k_{\gamma} - k_{\gamma}'} \omega^{n} \leq C_{4.1}\|\tilde{\mu}\|_{L^{2p\chi}(\tilde{\mu})}^{2p}
\]

where \(C_{4.1} = \|S_{\epsilon}^{k_{\gamma} - k_{\gamma}'}\|_{L^{\infty}(\tilde{\mu})}\) is finite for some \(c > n\).
Proof. From \( \tilde{\mu} = S^{k\chi} \omega^n = S^{(\gamma + \beta - 1 + \sigma)\chi} \omega^n \), we compute
\[
\int_X \tilde{u}^{2p} S^{k-k'} \omega^n = \int_X \tilde{u}^{2p} S^{k\gamma - k\chi - k'} \tilde{\mu}.
\]
By the generalisation of Hölder’s inequality with \( \frac{1}{a} + \frac{1}{c} = 1 \), this term is dominated by
\[
\| \tilde{\mu} \|_{L^{2p}(\tilde{\mu})} (\int_X S^{(k\gamma - k\chi - k')c} \tilde{\mu})^{\frac{1}{c}}.
\]
In order to make sure the last integral is finite, it is sufficient to ask
\[
2(2k\gamma - k\gamma \chi + k\chi + 2n) > 0,
\]
which is equivalent to
\[
c < n - \frac{k\gamma + n - 1}{k\gamma + k'(n - 1)} := c_0.
\]
Since \( k' < 1 \), we have \( c_0 > n \). Then, we could choose \( c \) between \( n \) and \( c_0 \) such that \( a < \frac{n}{n - 1} \).

3.5. Step 5: inverse weighted inequality. Our tour to bound each term in \( RHS_1 \) (3.15) and \( RHS_2 \) (3.19)
\[
RHS_1 = \int_X \tilde{u}^{2p} [u + u^\frac{1}{2} S^{\frac{n-\beta}{2}} S^{\frac{n-\beta}{2}}] \omega^n,
\]
\[
RHS_2 = \int_X (u^2 \tilde{u}^{2p-2} S^{\gamma+\sigma} + C_3 u^2 \tilde{u}^{2p-2} S^{\gamma+\sigma-1}) \omega^n,
\]
is via applying Young’s inequality repeatedly, with the help of the positive term
\[
LHS_2 = p \int_X \tilde{u}^{2p} u^{\frac{n}{n-1 - \frac{n-\beta-1}{n-1}}} \omega^n.
\]
According to Proposition 3.14, the second term in \( RHS_2 \) is the trouble term.

Proposition 3.15 (Inverse weighted inequality). Assume that the parameters \( \sigma \) and \( \gamma \) satisfy \( \sigma + \gamma < 1 \) and
\[
\begin{align*}
\sigma = \gamma = 0, & \text{ when } \beta > n; \\
\frac{1}{2} \geq \sigma > \frac{n - \beta}{n - 1}, & \text{ when } \frac{n + 1}{2} < \beta \leq n; \\
\sigma < \frac{1}{n + 1}, & \text{ when } \beta \leq \frac{n + 1}{2}.
\end{align*}
\]
Then there exists an exponent \( k' < 1 \) such that
\[
\| \tilde{u} \|_{L^{2p}(\tilde{\mu})}^{2p} \leq C_5 [p^3 \int_X \tilde{u}^{2p} S^{\gamma+\sigma-k'} \omega^n + 1].
\]
Proof. The proof of the inverse weighted inequality is divided into Lemma 3.16 for RHS1, Lemma 3.17 for RHS2, and Lemma 3.18 for examining the criteria. Adding the resulting inequalities of RHS1 and RHS2, we have

\[ p^2 \text{RHS}_1 + \text{RHS}_2 \leq \tau p \text{LHS}_2 + p^3 C(\tau) \int_X \tilde{u}^{2p} S^{\gamma + \sigma - \max\{k'_2, k'_5\}} \omega^n \]
\[ + \tau p \text{LHS}_2 + \frac{C(\tau)}{p} \int_X \tilde{u}^{2p} S^{\gamma + \sigma - k'_7} \omega^n. \]

We set a new \( k' \) to be \( \max\{k'_2, k'_5, k'_7\} \). Inserting this inequality to the rough iteration inequality (3.18) and choosing sufficiently small \( \tau \), we therefore obtain the asserted inequality.

\[ \Box \]

Lemma 3.16. Assume that

\[ \sigma < \frac{\beta}{n + 1}, \quad \gamma + \sigma < 1. \]  

Let \( k' = \max\{1 + \sigma - \frac{\beta}{n + 1}, \gamma + \sigma\} \). Then

\[ \text{RHS}_1 \leq \frac{\tau}{p} \text{LHS}_2 + p C(\tau) \int_X \tilde{u}^{2p} S^{\gamma + \sigma - \max\{k'_2, k'_5\}} \omega^n. \]

The exponents \( k'_2, k'_5 \) are given in the following proof, see (3.25) and (3.24), respectively.

Proof. We now establish the estimates for the five terms in \( \text{RHS}_1 \). The 1st one is decomposed as

\[ \int_X \tilde{u}^{2p} u \omega^n = \int_X \left( \tilde{u}^{2p} u^{\frac{n}{n - 1}} S^{\gamma - \frac{\beta - 1}{n - 1}} \right) \tilde{u}^{\frac{1}{n}} \frac{1}{n!} \frac{1}{u^{\frac{n}{n - 1}} a_1^{\frac{1}{n}} S^{\frac{\beta}{n - 1}} a_1} \omega^n. \]

By Young’s inequality with small \( \tau \), it implies that

\[ \int_X \tilde{u}^{2p} u \omega^n \leq \frac{\tau}{4p} \text{LHS}_2 + p C \int_X \tilde{u}^{2p} u^{(1 - \frac{\beta - 1}{n - 1}) b_1} S^{\frac{\gamma + \beta - 1}{n - 1} b_1} \omega^n. \]

The conjugate exponents we choose are \( a_1 = \frac{n}{n - 1} \) and \( b_1 = n \). Accordingly, we have the exponent over \( u \) is zero and

\[ k_1 := \gamma + \beta - 1 \frac{b_1}{a_1} = \gamma + \beta - 1. \]

The estimate of the 3rd term is proceeded in the same way,

\[ \int_X \tilde{u}^{2p} u^{\frac{1}{2}} S^{\gamma} \omega^n = \int_X \left( \tilde{u}^{2p} u^{\frac{n}{2}} S^{\gamma - \frac{\beta - 1}{n - 1}} \right) \tilde{u}^{\frac{1}{2}} u^{\frac{1}{2}} S^{\frac{\beta - 1}{n - 1} b_2} \omega^n \]
\[ \leq \frac{\tau}{4p} \text{LHS}_2 + p C \int_X \tilde{u}^{2p} u^{(1 - \frac{\beta - 1}{n - 1}) b_2} S^{\gamma + \frac{\beta - 1}{n - 1} b_2} \omega^n. \]
The exponent \( a_2 \) is set to be \( \frac{2n}{n+1} \). Hence, the exponent above \( u \) vanishes. Moreover, \( b_2 = \frac{2n}{n+1} \) and

\[
k_2 := \left( \frac{\gamma}{2} + \frac{\gamma + \beta - 1}{n-1} \frac{1}{a_2} \right) b_2 = \gamma + \frac{\beta - 1}{n+1}.
\]

The 4th and 5th terms are treated by Young’s inequality with small \( \tau \), as well. The estimate for the 4th term is

\[
\int_X \tilde{u}^{2p} \omega_\varphi^{\gamma - \frac{\sigma_2}{2}} S_\varphi \omega_\varphi^n \leq \frac{\tau}{4p} LHS_2 + pC \int_X \tilde{u}^{2p} \left( \frac{\gamma + \beta - 1}{n-1} \frac{1}{a_2} \right) b_2 S_\varphi k_4 \omega_\varphi^n
\]

and the exponent is

\[
k_4 := \left( \frac{\gamma - \frac{\sigma_2}{2}}{2} + \frac{\gamma + \beta - 1}{n-1} \frac{1}{a_2} \right) b_2 = \gamma + \frac{\beta - 1 - n\sigma_2}{n+1}.
\]

While, the estimate for the 5th term is

\[
\int_X \tilde{u}^{2p} \omega_\varphi^{\frac{\gamma - 1}{2}} S_\varphi \omega_\varphi^n \leq \frac{\tau}{4p} LHS_2 + pC \int_X \tilde{u}^{2p} \left( \frac{\gamma}{2} + \frac{\gamma + \beta - 1}{n-1} \frac{1}{a_2} \right) b_2 S_\varphi k_5 \omega_\varphi^n,
\]

with the exponent satisfying

\[
k_5 := \left( \frac{\gamma - 1}{2} + \frac{\gamma + \beta - 1}{n-1} \frac{1}{a_2} \right) b_2 = \gamma + \frac{\beta - 1 - n}{n+1}.
\]

We let \( k_i' \) satisfy \( k_i = \gamma + \sigma - k_i' \). Then we summary the exponents from the above estimates to see that

\[
k_i' := \gamma + \sigma - (\gamma + \beta - 1) = \sigma - (\beta - 1),
\]

\[
k_3' := \gamma + \sigma - (\gamma + \frac{\beta - 1}{n+1}) = \sigma - \frac{\beta - 1}{n+1},
\]

\[
k_4' := \gamma + \sigma - (\gamma + \frac{\beta - 1 - n\sigma_2}{n+1}) = \sigma - \frac{\beta - 1 - n\sigma_2}{n+1},
\]

\[
k_5' := \gamma + \sigma - (\gamma + \frac{\beta - 1 - n}{n+1}) = \sigma - \frac{\beta - 1 - n}{n+1}.
\]

Since \( \sigma_2 \leq 1 \), we observe that \( k_5' \) is the largest one among these four exponents. We further compute that

\[
k_5' - 1 = \sigma - \frac{\beta}{n+1},
\]

which is negative under the hypothesis of our lemma, i.e. \( \sigma < \frac{\beta}{n+1} \).

The integrand of the 2nd term is decomposed as,

\[
\int_X \tilde{u}^{2p} \omega_\varphi^n = \int_X \tilde{u}^{2p} S_\varphi^{\gamma + \sigma - k_2} \omega_\varphi^n,
\]

where we choose the exponent

\[
k_2' := \gamma + \sigma < 1.
\]

Therefore, we set the exponent to be \( \max\{k_2', k_5'\} \) and obtain the required inequality for \( RHS_1 \).
We then derive the estimates for the terms in $RHS_2$.

**Lemma 3.17.** Assume the following condition holds
\begin{equation}
\gamma > 1 - \frac{\beta}{n} - \sigma(1 - \frac{1}{n}) := \gamma_0.
\end{equation}

Then
$$RHS_2 \leq \tau p LHS_2 + \frac{C(\tau)}{p} \int_X \tilde{u}^{2p} S_\epsilon^{\gamma_0 + \sigma - k'} \omega^n,$$

where $k' < 1$ is given in (3.26).

*Proof.* The first term in $RHS_2$ is a good term. On the other hand, the trouble term is treated with the help of $LHS_2$,
$$RHS_2 := \int_X u^2 \tilde{u}^{2p - 2} S_\epsilon^{\gamma_0 + \sigma - 1} \omega^n,$$

by applying an argument analogous to the proof of $RHS_1$. By Young’s inequality, we get
$$RHS_2 \leq \tau LHS_2 + C \int_X \tilde{u}^{2p - 2b_7} u^{2b_7 - \frac{n}{n-1} b_7} S_\epsilon^{k'} \omega^n.$$

Using $\tilde{u} \geq K$ and choosing $a_7 = \frac{n}{2(n-1)}$,
$$RHS_2 \leq \tau LHS_2 + C \int_X \tilde{u}^{2p} S_\epsilon^{k'} \omega^n.$$

The exponent
$$k' := [\gamma + \sigma - 1 + \frac{\gamma + \beta - 1}{n-1}] b_7.$$

Direct computation shows that
\begin{equation}
k' := [\gamma + \sigma - 1 + \frac{\gamma n - \sigma(n-1) + a_7(n-1) - \beta + 1}{(a_7 - 1)(n-1)}] b_7.
\end{equation}

Thus the conclusion $k' < 1$ holds, under the condition (3.26).

At last, we examine the conditions (3.23) and (3.26).

**Lemma 3.18.** Assume that the parameters $\sigma$ and $\gamma$ satisfy $\sigma + \gamma < 1$ and (3.22). Then the condition (3.23) and (3.26) hold.

*Proof.* We check that, when $\beta > n$ and $\sigma = \gamma = 0$, the condition (3.23) is satisfied, i.e.
$$\sigma = 0 < \frac{n}{n+1} < \frac{\beta}{n+1}.$$
and Meanwhile, the condition (3.26) is satisfies, too. That is
\[\gamma_0 = 1 - \frac{\beta}{n} - \sigma(1 - \frac{1}{n}) = 1 - \frac{\beta}{n} < 0 = \gamma.\]

Similarly, the second criterion implies that
\[\frac{\beta}{n + 1} > \frac{1}{2} \geq \sigma.\]
and
\[\gamma_0 = 1 - \frac{\beta}{n} - \sigma(1 - \frac{1}{n}) < 1 - \frac{\beta}{n} - \frac{n - \beta}{n - 1}(1 - \frac{1}{n}) = 0 = \gamma.\]

The third criterion also deduces that
\[\frac{\beta}{n + 1} \geq \frac{1}{n + 1} > \sigma.\]
and
\[\gamma_0 = 1 - \frac{\beta}{n} - \sigma(1 - \frac{1}{n}) \leq (1 - \sigma)\frac{n - 1}{n} < \gamma.\]

Therefore, any one of the three criteria in (3.26) guarantees both the conditions (3.23) and (3.26).

\[\square\]

Remark 3.19. Using Young’s inequality similar to the proof above, we could also obtain a $W^{2,p}$ estimate for $\text{tr}_\omega \omega \phi$. However, this bound relies on the bound of $\partial F$. Alternatively, we will obtain an accurate $W^{2,p}$ estimate in Theorem 7.1 in Section 7, without any condition on $\partial F$.

3.6. Step 6: iteration. Combining the weighted inequality Proposition 3.14 with the weighted inequality Proposition 3.15, we obtain that

Proposition 3.20 (Iteration inequality). Assume that the parameters $\sigma$ and $\gamma$ satisfy $\sigma + \gamma < 1$ and (3.22). Then it holds

(3.28) \[\|\tilde{u}\|^{2p}_{L^{2p}((\bar{\mu}))} \leq C_0[p^3\|\tilde{u}\|^{2p}_{L^{2p}((\bar{\mu}))} + 1].\]

We normalise the measure to be one. The norm $\|\tilde{u}\|_{L^{2p}((\bar{\mu}))}$ is increasing in $p$. We assume $\|\tilde{u}\|_{L^{2p_0}((\bar{\mu}))} \geq 1$ for some $p_0 \geq 1$, otherwise it is done.

We develop the iterating process from the iteration inequality

(3.29) \[\|\tilde{u}\|_{L^{2p_{i+1}}((\bar{\mu}))} \leq p^3\|\tilde{u}\|_{L^{2p_{i+1}}((\bar{\mu}))} + \].

Setting
\[\chi_a := \frac{n - 1}{a} > 1, \quad p = \chi_a^i, \quad i = 0, 1, 2, \ldots\]
and iterating (3.29) with $p = \chi_a^m$,
\[\|\tilde{u}\|_{L^{2\chi_a^m \frac{n}{n - 1}}((\bar{\mu}))} \leq \chi_a^3\chi_a^m C_0 \frac{1}{\chi_a^m} \|\tilde{u}\|_{L^{2\chi_a^m \frac{n}{n - 1}((\bar{\mu}))}}.\]
which is

\[ \chi_a^m \chi_a^{-m} C_7^1 \| \tilde{u} \|_{L^2 \chi_a^{-1} m (\tilde{\mu})}. \]

We next apply (3.29) again with \( p = \chi^m \),

\[ \leq \chi_a^m + \frac{1}{2} (m-1) \chi_a^{-1} \chi_a^{-n} \int \| \tilde{u} \|_{L^2 \chi_a^{-1} m (\tilde{\mu})}. \]

We choose \( i_0 \) such that \( \tilde{p}_0 = \chi_a^{i_0} \geq p_0 \). Repeating the argument above, we arrive at

\[ \leq \chi_a^m \sum_{i=i_0}^\infty \chi_a^{-i} C_7^1 \sum_{i=i_0}^\infty \chi_a^{-i} \| \tilde{u} \|_{L^2 \tilde{p}_0 (\tilde{\mu})}. \]

Since these two series \( \sum_{i=i_0}^\infty \chi_a^{-i} \) and \( \sum_{i=i_0}^\infty \chi_a^{-i} \) are convergent, we take \( m \to \infty \) and conclude that

\[ \| \tilde{u} \|_{L^\infty} \leq C \| \tilde{u} \|_{L^2 \tilde{p}_0 (\tilde{\mu})} \leq C \| u \|_{L^2 \tilde{p}_0 (\tilde{\mu})} + \| K \|_{L^2 \tilde{p}_0 (\tilde{\mu})}. \]

In order to obtain the uniform bound of \( \text{tr}_\omega \omega \beta \cdot S^\gamma \), we apply the \( L^\infty \) bound of \( \varphi \) and \( L^2 \tilde{p}_0 \) bound of \( \text{tr}_\omega \omega \beta \) from Definition 2.21, it is left to compare the exponent of the weight in the integral

\[ \int_X e^{-C_1 (2a \tilde{p}_0 + 1) \gamma + \beta - 1 + \sigma} \frac{n}{n-1} \omega^\beta \]

with the exponent \( \sigma_D^2 \) in (2.27),

\[ [(2a \tilde{p}_0 + 1) \gamma + \beta - 1 + \sigma] \frac{n}{n-1} \geq \sigma_D^2 = \beta - 1 \frac{n-2}{n-1 + (2a \tilde{p}_0)^{-1}}. \]

In conclusion, we obtain the Laplacian estimate

\[ \nu_{\varphi} := \text{tr}_\omega \omega \beta \leq C \nu. \]

Moreover, \( \gamma = 0 \), when \( \beta > \frac{n+1}{2} \).

3.7. **Laplacian estimate for degenerate KE equation.** We apply the weighted integration method developed for the degenerate scalar curvature equation to the degenerate KE problem, that provides an alternative proof of Yau’s Laplacian estimate for the approximate degenerate KE equation (2.15),

\[ \omega_{\varphi}^n = e^{h_n + h_n - \lambda \varphi_n + c_n \omega_n}. \]

Comparing with the approximate degenerate scalar curvature equation (2.24), we have \( \theta = \lambda \omega, \ R = \bar{S} = \lambda n \) and

\[ F_\epsilon = -\lambda \varphi_\epsilon, \ \tilde{f}_\epsilon = -h_\omega - \eta_\epsilon - c_\epsilon, \ \tilde{F}_\epsilon = F_\epsilon - \tilde{f}_\epsilon. \]

We also have

\[ \Delta F_\epsilon = -\lambda \Delta \varphi_\epsilon = -\lambda (\text{tr}_\omega \omega \beta - n). \]
**Theorem 3.21.** Suppose \( \varphi_\varepsilon \) is a solution to the approximate KE equation (3.30). Then there exists a uniform constant \( C \) such that

\[
\text{tr}_\omega \omega_\varphi \leq C \text{ on } X
\]

where the uniform constant \( C \) depends on

\[
\|\varphi_\varepsilon\|_\infty, \|h_\omega\|_\infty, \inf_X \Delta h_\omega, \inf_{i\neq j} R_{ij\bar{j}}(\omega), C_S(\omega), \Theta_D,
\]

and \( \lambda, \beta, c_\varepsilon, n \).

**Proof.** Substituting (3.31) into the nonlinear term (3.10), we have

\[
N = \lambda \int_X e^{-C_1 \varphi_\varepsilon} \tilde{u}^{2p}(v - n) S_\varepsilon \omega_\varphi^n = \lambda \int_X \tilde{u}^{2p} u_\omega \omega_\varphi - n \lambda \int_X e^{-C_1 \varphi_\varepsilon} \tilde{u}^{2p} S_\varepsilon \omega_\varphi^n.
\]

The integral inequality is reduced immediately to

\[
2 C_1 \frac{6}{p} \|LHS_1\|_2 + C_1 \|LHS_2\|_2 \leq (\lambda + C_1) \int_X \tilde{u}^{2p+1} \omega_\varphi^n.
\]

Modified the constants, it becomes

\[
p^{-1} \|\tilde{u}\|_{L^{2p}(\tilde{h})} + p \|LHS_2\|_2 \leq C_2 \|RHS_1\|_2 := C_2 \int_X \tilde{u}^{2p+1} \omega_\varphi^n.
\]

Following the argument in Proposition 3.10, we have the rough iteration inequality

\[
\|\tilde{u}\|_{L^{2p}(\tilde{h})} \leq C_3(p \|RHS_1\|_2 + \|RHS_2\|_2 + 1).
\]

Examining the estimates from Lemma 3.16 for \( \|RHS_1\|_2 \), Lemma 3.17 for \( \|RHS_2\|_2 \), we deduce the inverse weighted inequalities

\[
\|\tilde{u}\|_{L^{2p}(\tilde{h})} \leq C_3 \|p \|RHS_1\|_2 + \|RHS_2\|_2 + 1\| \text{ for some } k' < 1,
\]

under the conditions

\[
\begin{cases}
\sigma - (\beta - 1) < 1; \\
\sigma + \gamma < 1; \\
\gamma > 1 - \frac{\beta}{n} - \sigma(1 - \frac{1}{n}),
\end{cases}
\]

which are actually automatically satisfied from the criteria

\[
\gamma = 0, \quad 1 > \sigma > \frac{n - \beta}{n - 1}.
\]

Therefore, we could apply the weighted inequality, Proposition 3.14, to derive the iteration inequality. Then we further employ the iteration techniques as Section 3.6 to conclude the \( L^\infty \) norm of

\[
\tilde{u} = e^{-C_1 \varphi_\varepsilon} \text{tr}_\omega \omega_\varphi + K
\]

in terms of the \( W^{2,p} \)-estimate, which could be obtained from Theorem 7.1 in a similar way, or from the argument in Remark 3.19. \( \square \)
4. SINGULAR CSCK METRICS

In [28], we introduced the singular cscK metrics and proved several existence results, where we focus on the case $0 < \beta < 1$. In this section, we consider $\beta > 1$ and obtain the $L^\infty$ estimate, the gradient estimate and the $W^{2,p}$ estimate for the singular cscK metrics.

We start from the basic setup of the singular scalar curvature equation we introduced in [28].

**Definition 4.1.** A real $(1,1)$-cohomology class $\Omega$ is called big, if it contains a Kähler current, which is a closed positive $(1,1)$-current $T$ satisfying $T \geq t \omega_K$ for some $t > 0$. A big class $\Omega$ is defined to be semi-positive, if it admits a smooth closed $(1,1)$-form representative.

Recall that $\omega$ is a Kähler metric on $X$. We let $\omega_{sr}$ be a smooth representative in the big and semi-positive class $\Omega$.

### 4.1. Perturbed Kähler metrics

In order to modify $\omega_{sr}$ to be a Kähler metric, one way is to perturb $\omega_{sr}$ by adding another Kähler metric,

$$\omega_t := \omega_{sr} + t \cdot \omega \in \Omega_t := \Omega + t[\omega], \quad \text{for all } t > 0. $$

The other way is to apply Kodaira’s Lemma, namely there exists a sufficiently small number $a_0$ and an effective divisor $E$ such that $\Omega - a_0[E]$ is ample and

$$\omega_K := \omega_{sr} + i\partial \overline{\partial} \phi_E > 0, \quad \phi_E := a_0 \log h_E$$

is a Kähler metric. In which, $h_E$ is a smooth Hermitian metric on the associated line bundle of $E$ and $s_E$ is the defining section of $E$.

We also write

$$\tilde{\omega}_t := \omega_K + t \omega. $$

**Lemma 4.2.** The three Kähler metrics $\omega, \omega_K, \tilde{\omega}_t$ are all equivalent,

$$\begin{align*}
\omega_K &\leq \tilde{\omega}_t \leq \omega_K + \omega, \\
C_{1}^{-1} \omega &\leq \omega_K \leq C_K \omega.
\end{align*}$$

With the help of these three Kähler metrics, we are able to measure the $(1,1)$-form $\theta$. As Lemma 5.6 in [26], we define the bound of $\theta$, which is independent of $t$, by using $\tilde{\omega}_t$ as the background metric.

**Definition 4.3.** We write

$$\begin{align*}
C_1 \cdot \tilde{\omega}_t &\leq \theta \leq C_u \cdot \tilde{\omega}_t,
\end{align*}$$

where $C_1 := \min\{0, \inf_{\chi, \chi} \theta\}$, $C_u := \max\{0, \sup_{\chi, \chi} \theta\}$.

**Lemma 4.4.** The given $(1,1)$-form $\theta$ has the lower bound

$$\begin{align*}
\theta &\geq C_1 \omega_{\theta_{1,s}} - i\partial \overline{\partial} \phi_t, \quad \phi_t := C_1(\varphi_{\theta_{1,s}} - \phi_E) \\
\theta &\leq C_u \omega_{\theta_{1,s}} - i\partial \overline{\partial} \phi_u, \quad \phi_u := C_u(\varphi_{\theta_{1,s}} - \phi_E).
\end{align*}$$
Remark 4.5. In particular, if the big and semi-positive class $\Omega$ is propositional to $C_1(X, D)$, i.e. $\Omega = \lambda C_1(X, D)$, $\lambda = -1, 1$, then we have $C_t = 0$, when $\lambda = 1$, and $C_u = 0$, when $\lambda = -1$.

Actually, it even holds $\theta = \lambda \omega_{sr}$.

4.2. Reference metrics.

Definition 4.6 (Reference metric). The reference metric $\omega_{\theta}$ is defined to be the solution of the following singular Monge-Ampère equation

$$\omega^n_{\theta} := (\omega_{sr} + i\partial\bar{\partial}\varphi_\theta)^n = |s|^{2\beta - 2}e^{h_{\theta}}\omega^n$$

where $h_{\theta}$ is defined in (2.6).

4.2.1. $t$-perturbation. Replacing $\omega_{sr}$ by the Kähler metric $\omega_t$, we introduce the following perturbed metric.

Definition 4.7. The $t$-perturbed reference metric $\omega_{\theta_t} \in \Omega_t$ is defined to be a solution to the following degenerate Monge-Ampère equation

$$\omega^n_{\theta_t} := (\omega_t + i\partial\bar{\partial}\varphi_{\theta_t})^n = |s|^{2\beta - 2}e^{h_{\theta} + c_t}\omega^n,$$

under the condition

$$Vol(\Omega_t) = \int_X |s|^{2\beta - 2}e^{h_{\theta} + c_t}\omega^n.$$

Using the Poincaré-Lelong formula (2.1) and the Ricci curvature of $\omega$ from (2.6), we compute the Ricci curvature of $\omega_{\theta_t}$.

Lemma 4.8. The Ricci curvature equation for (4.6) reads

$$Ric(\omega_{\theta_t}) = \theta + 2\pi(1 - \beta)[D].$$

4.2.2. ($t, \epsilon$)-approximation. Similar to Section 2.3.1, we further approximate (4.6) by a family of smooth equation for $\omega_{\theta_{t, \epsilon}} \in \Omega_t$.

Definition 4.9. We define the ($t, \epsilon$)-approximate reference metric

$$\omega_{\theta_{t, \epsilon}} := \omega_t + i\partial\bar{\partial}\varphi_{\theta_{t, \epsilon}}$$

satisfy the equation

$$\omega^n_{\theta_{t, \epsilon}} = S_{\epsilon}^{\beta - 1}e^{h_{\theta} + c_t}\omega^n.$$

The constant $c_{t, \epsilon}$ is determined by the normalised volume condition

$$Vol(\Omega_t) = \int_X S_{\epsilon}^{\beta - 1}e^{h_{\theta} + c_t}\omega^n.$$

The volume is uniformly bounded independent of $t, \epsilon$. In the following, when we say a constant or an estimate is uniform, it means it is independent of $t$ or $\epsilon$.

Lemma 4.10. The Ricci curvature equation for (4.7) reads

$$Ric(\omega_{\theta_{t, \epsilon}}) = \theta + (1 - \beta)[2\pi D] - i\partial\bar{\partial}(\log |s|^{2}e^{h_{\theta}} - \log S_{\epsilon}).$$
Proof. Taking $-i\partial\bar{\partial}\log$ on (4.7), we get
\[
\text{Ric}(\omega_{\theta,t}) = - (\beta - 1) i\partial\bar{\partial}\log S_t - i\partial\bar{\partial} h + \text{Ric}(\omega).
\]
Then we have by (2.6) that
\[
\text{Ric}(\omega_{\theta,t}) = - (\beta - 1) i\partial\bar{\partial}\log S_t + \theta + (1 - \beta)\Theta_D.
\]
The conclusion thus follows from (2.1). □

4.2.3. Estimation of the reference metric. We write
\[
\tilde{\varphi}_{\theta,t,\epsilon} := \varphi_{\theta,t,\epsilon} - \phi_E.
\]
Accordingly, we see that
\[
\tilde{\omega}_t = \omega_t + i\partial\bar{\partial} \varphi_E, \quad \omega_{\theta,t,\epsilon} = \tilde{\omega}_t + i\partial\bar{\partial} \tilde{\varphi}_{\theta,t,\epsilon}.
\]
By substitution into (4.7), we rewrite the $(t,\epsilon)$-approximation (4.7) of the reference metric as

Lemma 4.11.
\[
\omega_{\theta,t,\epsilon}^n = (\tilde{\omega}_t + i\partial\bar{\partial} \tilde{\varphi}_{\theta,t,\epsilon})^n = e^{-f_{t,\epsilon}} \tilde{\omega}_t^n,
\]
where
\[
f_{t,\epsilon} := (\beta - 1) \log S_t + h + c_{t,\epsilon} + \log \frac{\omega^n}{\tilde{\omega}_t^n}.
\]

Lemma 4.12. There exists a uniform constant $C$ such that
\[
\begin{align*}
 f_{t,\epsilon} &\ge C, \quad e^{-f_{t,\epsilon}} \le C, \\
 |\partial f|^2 &\le C[1 + (\beta - 1)^2 S_t^{-1}] \\
 - C[(\beta - 1) S_t^{-1} + 1] \omega &\le i\partial\bar{\partial} f_{t,\epsilon} \le C\omega.
\end{align*}
\]
Proof. Since the expression of $e^{-f_{t,\epsilon}}$ is
\[
e^{-f_{t,\epsilon}} = S_t^{\beta - 1} e^{h + c_{t,\epsilon} + \log \frac{\omega^n}{\tilde{\omega}_t^n}},
\]
we obtain its upper bound from $\beta \ge 1$. Moreover, its derivatives are estimated by
\[
|\partial f|^2 = |\partial[(\beta - 1) \log S_t + h + c_{t,\epsilon} + \log \frac{\omega^n}{\tilde{\omega}_t^n}]|^2 \\
\le C[(\beta - 1)^2 |\partial s|^2 S_t^{-2} + 1] \le C[(\beta - 1)^2 S_t^{-1} + 1].
\]
and
\[
i\partial\bar{\partial} f_{t,\epsilon} = -i\partial\bar{\partial}[(\beta - 1) \log S_t + h + c_{t,\epsilon} + \log \frac{\omega^n}{\tilde{\omega}_t^n}].
\]
Thus this lemma is a consequence of Lemma 2.11. □

Lemma 4.13. Suppose that $\varphi_{\theta,t,\epsilon}$ is a solution to the approximate equation (4.7). Then there exists a uniform constant $C$ independent of $t,\epsilon$ such that
\[
\|\varphi_{\theta,t,\epsilon}\|_{\infty} \le C, \quad C^{-1} S_t^{\beta - 1} |s_E|^n |s_E|^{a(n - 1)} \omega \le \omega_{\theta,t,\epsilon} \le C |s_E|^{-a} \omega.
\]
Moreover, the sequence $\varphi_{\theta, t, \epsilon}$ converges to $\varphi_\theta$, which is the unique solution to the equation (4.5). The solution $\varphi_\theta \in C^0(X) \cap C^\infty(X \setminus (E \cup D))$.

Proof. We outline the proof for readers' convenience. The $L^\infty$ estimate is obtained, by applying Theorem 2.1 and Proposition 3.1 in [13], since

$$(|s|^2 + \epsilon)^{\beta - 1}e^{h_{\theta} + ct, \epsilon}$$

is uniformly in $L^p(\omega_{sr})$ for $p > 1$.

For the Laplacian estimate, we denote $v = tr \omega_{\theta, t, \epsilon}$, $w = e^{-C_1^1 \varphi_{\theta, t, \epsilon} v}$, and obtain from (3.4) that

$$(4.10) \quad e^{C_1^1 \varphi_{\theta, t, \epsilon}} \triangle \omega \geq v^{1 + \frac{1}{n-1}}e^{-\frac{\varphi_{\theta, t, \epsilon}}{n-1}} + \triangle \tilde{F}_{\theta, t, \epsilon} - C_1^1 nv.$$  

The constant $C_1^1$ is taken to be $C_1^1 + 1$ and $C_1^1$ is the lower bound the bisectional curvature of $\tilde{\omega}_t$.

We write $\tilde{F}_{\theta, t, \epsilon} = -f_{t, \epsilon}$ and use Lemma 4.12 to show that there exists a uniform constant $C$, independent of $t$ and $\epsilon$ such that

$$\tilde{F}_{\theta, t, \epsilon} \leq C, \quad \triangle \tilde{F}_{\theta, t, \epsilon} \geq -C.$$  

Then the argument of maximum principle applies. At the maximum point $p$ of $w$, $v(p)$ is bounded above from (4.10). While, at any point $x \in X$, $w(x) \leq w(p)$, which means

$$v(x) \leq e^{C_1^1 \varphi_{\theta, t, \epsilon}(x) - \varphi_{\theta, t, \epsilon}(p)}v(p) \leq e^{C_1^1 (\varphi_{\theta, t, \epsilon}(x) - \varphi_{\theta, t, \epsilon}(p))}s_{E}^{-C_1^{1a_0}}(x).$$

Moreover, $\triangle \tilde{\omega}_t \phi_E$ is uniformly bounded independent of $t, \epsilon$. inserting them into

$$\triangle \tilde{\omega}_t \varphi_{\theta, t, \epsilon} = tr \omega_{\theta, t, \epsilon} - n + \triangle \tilde{\omega}_t \phi_E = v - n + \triangle \tilde{\omega}_t \phi_E,$$

we obtain the Laplacian estimate

$$\triangle \tilde{\omega}_t \varphi_{\theta, t, \epsilon} \leq C|s_{E}|^{-a}_{h_{E}}$$

by taking $a = C_1^1 a_0$. The metric bound is thus obtained from the following Lemma 4.14, namely

$$C^{-1}|s_{E}|^{a(n-1)}_{h_{E}} \varphi_{\theta, t, \epsilon} \leq \omega_{\theta, t, \epsilon} \leq C|s_{E}|^{-a}_{h_{E}} \omega_{t}^{n}$$

together with the equivalence of the reference metrics Lemma 4.2.  

Lemma 4.14. Assume we have the Laplacian estimate

$$\triangle \tilde{\omega}_t \varphi_{\theta, t, \epsilon} \leq A.$$  

Then the metric bound holds

$$CS_{t}^{\beta - 1}(n + A)^{-(n-1)} \omega_{\theta, t, \epsilon} \leq \varphi_{\theta, t, \epsilon} \leq (n + A) \omega_{t}.$$
Proof. The upper bound of $\omega_{t,\epsilon}$ is obtained from
\[
\text{tr}_{\omega_{t,\epsilon}} \omega_{t,\epsilon} = n + \Delta_{\omega_{t,\epsilon}} \varphi_{t,\epsilon} \leq n + A.
\]
The lower bound follows directly from the fundamental inequality and the equation (4.7) of the volume ratio,
\[
\text{tr}_{\omega_{t,\epsilon}} \omega_{t,\epsilon} \leq \left( \frac{\omega_{n,\epsilon}}{\omega_{t,\epsilon}} \right)^{-1} (\text{tr}_{\omega_{t,\epsilon}} \omega_{t,\epsilon})^{-1} \leq CS_{\epsilon}^{1-\beta} (n + A)^{-1}.
\]
□

4.3. Singular scalar curvature equation.

**Definition 4.15.** Let $\Omega$ be a big and semi-positive class and $\omega_{sr}$ is a smooth representative in $\Omega$. The singular scalar curvature equation in $\Omega$ is defined to be
\[
\omega_{\varphi}^n = (\omega_{sr} + i\partial \bar{\partial} \varphi)^n = e^{F_{\theta}} \omega_{\theta}^n, \quad \Delta_{\varphi} F = \text{tr}_{\varphi} \theta - R.
\]
The reference metric $\omega_{\theta}$ is introduced in (4.5) and $R$ is a real-valued function. In particular, when considering the singular cscK equation, we have
\[
R = S_{\beta} = \frac{nC_1(X,D)\Omega^{n-1}}{\Omega^n}.
\]
Inserting the expression of $\omega_{\theta}$, i.e. $\omega_{\theta}^n = e^{-f} \omega^n$, into (4.11), we get

**Lemma 4.16.**
\[
(\omega_{sr} + i\partial \bar{\partial} \varphi)^n = e^{\tilde{F}} \omega^n, \quad \Delta_{\varphi} \tilde{F} = \text{tr}_{\varphi}(\theta - i\partial \bar{\partial} f) - R,
\]
where $\tilde{F} = F - f, \quad f = -(\beta - 1) \log |s|^2_h - h_{\theta}$.

4.4. Approximate singular scalar curvature equation. We define an analogue of the $t$-perturbation and the $(t,\epsilon)$-approximation of the singular scalar curvature equation. In general, we consider the perturbed equations.

**Definition 4.17.** The perturbed singular scalar curvature equation is defined as
\[
\omega_{\varphi_t}^n = (\omega_t + i\partial \bar{\partial} \varphi_t)^n = e^{F_t} \omega_{\theta_t}^n, \quad \Delta_{\varphi_t} F_t = \text{tr}_{\varphi_t} \theta - R.
\]
While, the approximate singular scalar curvature equation is set to be
\[
\omega_{\varphi_{t,\epsilon}}^n = (\omega_t + i\partial \bar{\partial} \varphi_{t,\epsilon})^n = e^{F_{t,\epsilon}} \omega_{\theta_{t,\epsilon}}^n, \quad \Delta_{\varphi_{t,\epsilon}} F_{t,\epsilon} = \text{tr}_{\varphi_{t,\epsilon}} \theta - R.
\]

**Remark 4.18.** The function $R$ is also needed to be perturbed. Since their perturbation are bounded and does not effect the estimates we will derive, we still use the same notation in the following sections for convenience.
Definition 4.19. We approximate the singular cscK equation by
\[ \omega_{\varphi_t}^n = (\omega_t + i\partial\bar{\partial}\varphi_t)^n = e^{\tilde{F}_t} \omega_{\varphi_t}^n, \quad \Delta_{\varphi_t} F_t = \text{tr}_{\varphi_t} \theta - R_t \]
with the constant to be
\[ R_t = \frac{nC_1(X, D)(\Omega + t[\omega])^{n-1}}{(\Omega + t[\omega])^n}. \]

By the formulas of Ric(\omega_{\varphi_t}) from Lemma 4.8 and Ric(\omega_{\varphi_t, \epsilon}) by Lemma 4.10, the scalar curvature of both approximations are given as below.

Lemma 4.20. On the regular part \( M = X \setminus D \), we have
\[ S(\omega_{\varphi_t}) = R, \quad S(\omega_{\varphi_t, \epsilon}) = R + (1 - \beta)\Delta_{\varphi_t, \epsilon}(\log S_{\epsilon} - \log |s|_h^2). \]

Proof. They are direct obtained by inserting the Ricci curvature equations \( \text{Ric}(\omega_{\varphi_t}) = \theta + 2\pi(1 - \beta)[D] \) and \( \text{Ric}(\omega_{\varphi_t, \epsilon}) = \theta + (1 - \beta)\{2\pi[D] - i\partial\bar{\partial}(\log |s|_h^2 - \log S_{\epsilon})\} \) into the scalar curvature equations
\[ S(\omega_{\varphi_t}) = -\Delta_{\varphi_t} F_t + \text{Ric}(\omega_{\vartheta_t}), \quad S(\omega_{\varphi_t, \epsilon}) = -\Delta_{\varphi_t, \epsilon} F_{t, \epsilon} + \text{Ric}(\omega_{\vartheta_t, \epsilon}). \]

We will derive a priori estimates for the smooth solutions to the approximate equation (4.12). We set
\[ \tilde{\varphi}_{t, \epsilon} := \varphi_{t, \epsilon} - \phi_E, \quad \tilde{F}_{t, \epsilon} := F_{t, \epsilon} - f_{t, \epsilon}. \]
Then it follows \( \omega_{\varphi_{t, \epsilon}} = \omega_t + i\partial\bar{\partial}\varphi_{t, \epsilon} = \tilde{\omega}_t + i\partial\bar{\partial}\tilde{\varphi}_{t, \epsilon} \). Hence, (4.12) is rewritten for the couple \((\Phi_{t, \epsilon}, \tilde{F}_{t, \epsilon})\) as following equations regarding to the smooth Kähler metric \( \tilde{\omega}_t \).

Lemma 4.21.
\[ (\tilde{\omega}_t + i\partial\bar{\partial}\tilde{\varphi}_{t, \epsilon})^n = e^{\tilde{F}_{t, \epsilon}} \tilde{\omega}_t^n, \quad \Delta_{\varphi_{t, \epsilon}} \tilde{F}_{t, \epsilon} = \text{tr}_{\varphi_{t, \epsilon}}(\theta - i\partial\bar{\partial} f_{t, \epsilon}) - R. \]

According to Lemma 4.12, we see that \( f_{t, \epsilon} \) have uniform lower bound and \( i\partial\bar{\partial} f_{t, \epsilon} \) has uniform upper bound, when \( \beta > 1 \). It is different from the the cone metric \( 0 < \beta < 1 \) or the smooth metric \( \beta = 1 \), whose \( f_{t, \epsilon} \) have uniform upper bound and \( i\partial\bar{\partial} f_{t, \epsilon} \) have uniform lower bound.

The a priori estimates in [7] were extended to the cone metrics in the big and semi-positive class in [26]. In the following sections, including Section 5, Section 6 and Section 7, we obtain estimation for the approximate singular scalar curvature equation (4.12) with \( \beta > 1 \).

Definition 4.22 (Almost admissible solution for singular equations). We say \( \varphi_{t, \epsilon} \) is an almost admissible solution to the approximate singular scalar curvature equation (4.12), if there are uniform constants independent of \( t, \epsilon \) such that the following estimates hold
• $L^\infty$-estimates in Theorem 5.1:
\[
\|\varphi_{t,\epsilon}\|_{\infty} \leq C, \quad \|e^{\hat{F}_{t,\epsilon}}\|_{p;\tilde{\omega}_t^n}, \quad \|e^{\hat{F}_{t,\epsilon}}\|_{p;\tilde{\omega}_t^n} \leq C(p), \quad p \geq 1;
\]
\[
\sup_X (F_{t,\epsilon} - \sigma_s \phi_E), \quad -\inf_X [F_{t,\epsilon} - \sigma_i \phi_E] \leq C;
\]
\[
\sigma_s := C_1 - \tau, \quad \sigma_i := C_u + \tau, \quad \forall \tau > 0.
\]

• gradient estimate of $\varphi$ in Theorem 6.1:
\[
|s_E|^{2n_2 \sigma_2 E} \int_X |\partial_\varphi_{t,\epsilon}|^2 \leq C, \quad \sigma_1^D \geq 1
\]
where the singular exponent $\sigma_1^E$ satisfies (6.9) and (6.11);

• $W^{2,p}$-estimate in Theorem 7.1:
\[
\int_X (tr_{\omega_t} \omega_\varphi)^p |s_E|^{\sigma_2 E} \GG_{E_{t,\epsilon}}^{\omega_t} \leq C(p), \quad \forall p \geq 1
\]
where $\sigma_2^D > (\beta - 1)\frac{n-2-2np^{-1}}{n-1+p^{-1}}$ and $\sigma_2^E$ is given in (7.1).

**Theorem 4.23.** The family of solutions to the approximate singular scalar curvature equation (4.12) with bounded entropy is almost admissible.

Suppose $\Omega$ is Kähler, the singular equation is reduced to the degenerate equation (2.17).

**Theorem 4.24.** Assume $\{\varphi_\epsilon\}$ is a family of solutions to the approximate degenerate scalar curvature equation (2.23) with bounded entropy. Then $\varphi_\epsilon$ is almost admissible, Definition 2.21.

4.5. **Singular metrics with prescribed scalar curvature.** In this section, we will prove an existence theorem for singular metrics with prescribed scalar curvature. The idea is to construct solution to the singular equation by taking limit of the solutions to the approximate equation. The proof is an adaption of Theorem 4.40 in our previous article [26].

We recall the definition of singular canonical metric satisfying particular scalar curvature, see [26, Definition 4.39], which is motivated from the study of canonical metrics on connected normal complex projective variety.

**Definition 4.25** (Singular metrics with prescribed scalar curvature). In a big cohomology class $[\omega_{sr}]$, we say $\omega := \omega_{sr} + i\partial \bar{\partial} \varphi$ is a singular metric with prescribed scalar curvature, if $\varphi \in \text{PSH}(\omega_{sr})$ is a $L^1$-limit of a sequence of Kähler potentials solving the approximate scalar curvature equation (4.12).

In addition, we say the singular metric $\omega_\varphi$ is bounded, if $\varphi \in \mathcal{E}_1(\omega_{sr}) \cap L^\infty$.

When $\Omega$ is Kähler, we name it the degenerate metric instead.
4.6. Estimation of approximate solutions. We need to consider the existence of the solution to the approximate singular scalar curvature equation (4.12). Restricted to the case for log KE metrics, we have seen the smooth approximation in Proposition 2.12. For the cscK cone metrics, $0 < \beta \leq 1$, the smooth approximation is shown in [26, Proposition 4.37]. We collect the results as below. We say a real $(1,1)$-cohomology class $\Omega$ is nef, if the cohomology class $\Omega_t := \Omega + t[\omega]$ is Kähler for all $t > 0$.

**Proposition 4.26.** Let $\Omega$ be a big and nef class on a Kähler manifold $(X,\omega)$, whose automorphism group $\text{Aut}(M)$ is trivial. Suppose that $\Omega$ satisfies the cohomology condition

\[
\left\{ \begin{array}{l}
0 \leq \eta < \frac{n+1}{n} \alpha_\beta, \\
C_1(X,D) < \eta \Omega,
\end{array} \right.
\]

\[
(-n\frac{C_1(X,D)\cdot \Omega^{n-1}}{t^n} + \eta)\Omega + (n-1)C_1(X,D) > 0.
\]

Then $\Omega$ has the cscK approximation property, which precisely asserts that, for small positive $t$, we have

1. the log $K$-energy is $J$-proper in $\Omega_t$;
2. there exists a cscK cone metric $\omega_t$ in $\Omega_t$;
3. the cscK cone metric $\omega_t$ a smooth approximation $\omega_{t,\epsilon}$ in $\Omega_t$.

**Question 4.27.** We may ask the question whether the approximate degenerate cscK equation (2.23) has a smooth solution, if the log $K$-energy (2.20) is proper.

It is different from the case $0 < \beta \leq 1$, when the approximate log $K$-energy $\nu_\beta$ dominates the log $K$-energy $\nu_\beta$. So the properness of $\nu_\beta$ follows from the properness of $\nu_\beta$ and then implies existence of the approximate solutions by [8].

4.7. Convergence and regularity. Now we further consider the convergence problem of the family $\{\varphi_{t,\epsilon}\}$ is consisting of smooth approximation solutions for (4.12) and regularity of the convergent limit. We normalise the family to satisfy $\sup_X \varphi_{t,\epsilon} = 0$. According to the Hartogs’s lemma, a sequence of $\omega_{sr}$-psh functions has $L^1$-weak convergent subsequence.

**Lemma 4.28.** The family $\{\varphi_{t,\epsilon}\}$ converges to $\varphi$ in $L^1$, as $t, \epsilon \to 0$. Moreover, $\varphi$ is an $\omega_{sr}$-psh function with $\sup_X \varphi = 0$.

**Lemma 4.29.** Assumed the uniform bound of $\|\varphi_{t,\epsilon}\|_\infty$ and $\|e^{F_{t,\epsilon}}\|_{\omega_{sr}^p}$, the limit $\varphi$ belongs to $E^1(X,\omega_{sr}) \cap L^\infty(X)$.

**Proof.** The proof is included in the proof of [26, Proposition 4.41]. □

In conclusion, we apply the gradient and $W^{2,p}$ estimates in Theorem 4.23 to obtain
Theorem 4.30. Assume that the family $\{\varphi_{t,\epsilon}\}$ is the almost admissible solution to the approximate singular scalar curvature equation (4.12). Then there exists a bounded singular metric $\omega_\varphi$ with prescribed scalar curvature, moreover, $\omega_\varphi$ has the gradient estimate and the $W^{2,p}$-estimate, as stated in Theorem 4.23.

The convergence behaviour is improved for the degenerate metrics.

Theorem 4.31. Assume that $\Omega$ is Kähler and the family $\{\varphi_\epsilon\}$, consisting of the approximate degenerate scalar curvature solutions (2.23), has bounded entropy. Then there exists degenerate metric $\omega_\varphi$ with prescribed scalar curvature, which is bounded, and has the gradient estimate and the $W^{2,p}$-estimate as stated in Theorem 4.24. Moreover, $\omega_\varphi$ is smooth and satisfies the degenerate scalar curvature equation (2.13) outside $D$.

Furthermore, if $\{\varphi_\epsilon\}$ has bounded gradients of the volume ratios $\|\partial F_\epsilon\|_\varphi$, then the degenerate metric $\omega_\varphi$ has Laplacian estimate, namely it is admissible when $\beta > \frac{n+1}{2}$, and $\gamma$-admissible for any $\gamma > 0$ when $1 < \beta < \frac{n+1}{2}$.

Proof. The first part of the theorem is a direct corollary of Theorem 4.30. The smooth convergence outside $D$ is obtained, by using the local Laplacian estimate in Section 6 of the arXiv version of [7]. By applying the Evans-Krylov estimates and the bootstrap method, the solution is smooth on the regular part $X \setminus D$ and satisfies the equation there. The global Laplacian estimates for degenerate metrics are obtained from Theorem 3.1. \hfill \Box

In Section 3, we develop an integration method with weights for the general degenerate scalar curvature equation (2.23). In particular, we utilise our theorems to the degenerate Kähler-Einstein metrics. We see that the integration method we develop here provide an alternative method to obtain the Laplacian estimate for the approximate degenerate Kähler-Einstein equation (2.15), as shown in Section 3.7. While, Yau’s proof of the Laplacian estimate applies the maximum principle.

Corollary 4.32. When $\lambda \leq 0$ and $\beta > 1$, there exists a family of smooth approximate Kähler-Einstein metrics (2.15), which converges to an admissible degenerate Kähler-Einstein metric.

5. $L^\infty$-estimate

This section is devoted to obtain the $L^\infty$-estimate for singular scalar curvature equation (4.12) for both $\varphi$ and $F$.

Theorem 5.1 ($L^\infty$-estimate for singular equation). Assume $\varphi_{t,\epsilon}$ is a solution of the approximate singular scalar curvature equation (4.12). Then we have the following estimates.
(1) For any $p \geq 1$, there exists a constant $A_1$ such that
\[ \| \varphi_{t,\epsilon} \|_{\infty, X}, \sup_X [F_{t,\epsilon} - \sigma_s \phi_E], \| e^{\tilde{F}_{t,\epsilon}} \|_{p,\tilde{\omega}_t}, \| e^{\tilde{F}_{t,\epsilon}} \|_{p,\tilde{\omega}_t} \leq A_1. \]

The constant $\sigma_s = C_1 - \tau$ and $A_1$ depends on the entropy $E_{\beta t,\epsilon} = \frac{1}{V} \int_X F_{t,\epsilon} \omega_n^\varphi_t$, the alpha invariant $\alpha(\Omega_1)$, $\| e^{C_1 \phi_E} \|_{L^p(\tilde{\omega}_t)}$ for some $p_0 \geq 1$ and
\[ \inf_X f_{t,\epsilon}, C_l = \inf_{(X, \tilde{\omega}_t)} \theta, \sup_X R, n, p. \]

(2) There also holds the lower bound the volume ratio
\[ \inf_X [F_{t,\epsilon} - \sigma_i \phi_E] \geq A_2, \quad \forall \tau > 0. \]

The constant $\sigma_i = C_u + \tau$ and $A_2$ depends on $\sup_X F_{t,\epsilon}$ and
\[ \inf_X f_{t,\epsilon}, C_u = \sup_{(X, \tilde{\omega}_t)} \theta, \inf_X R, n. \]

Furthermore, since $\tilde{F}_{t,\epsilon}$ is dominated by $F + (\beta - 1) \log S_\epsilon$ up to a smooth function, the $L^\infty$ estimate of $F_{t,\epsilon}$ gives the volume ratio bound.

**Corollary 5.2** (Volume ratio).

\[ C^{-1} e^{\sigma_i \phi_E} S_\epsilon^{\beta - 1} \omega_t^n \leq \omega_{\varphi_{t,\epsilon}}^n = e^{\tilde{F}_{t,\epsilon}} \omega_t^n \leq C e^{\sigma_s \phi_E} S_\epsilon^{\beta - 1} \omega_t^n. \]

Before we start the proof, we state the corresponding estimate for the degenerate equation. If we further assume that $\Omega = [\omega_K]$ is Kähler, then $\sigma_E = 0$.

**Theorem 5.3** ($L^\infty$-estimate for degenerate equation). Assume $\varphi_{t,\epsilon}$ is a solution of the approximate degenerate scalar curvature equation (2.23). Then for any $p \geq 1$, there exists a constant $A_0$ such that
\[ \| \varphi_{t,\epsilon} \|_{\infty}, \| F_{t,\epsilon} \|_{\infty, X}, \| e^{F_{t,\epsilon}} \|_{p,\omega_t^n}, \| e^{\tilde{F}_{t,\epsilon}} \|_{p,\omega_t^n} \leq A_0. \]

Now we start the proof of Theorem 5.1. To proceed further, we omit the indexes $(t, \epsilon)$ of (4.12) for convenience, that is
\[ (5.2) \quad \omega^n_{\varphi} = (\omega_t + i \partial \bar{\partial} \varphi)^n = e^F \omega^n_\varphi, \quad \Delta_\varphi F = \text{tr}_\varphi \theta - R. \]

5.1. **General estimation.** In this section, we extend Chen-Cheng [7] to the singular setting. We will first summarise a machinery to obtain the $L^\infty$-estimates in Proposition 5.6 and Proposition 5.7. Then we apply this robotic method to conclude $L^\infty$-estimates under various conditions on $\theta$ in Section 5.2.

We let $\varphi_a$ be an auxiliary function defined later in (5.13) and set
\[ (5.3) \quad w := b_0 F + b_1 \varphi + b_2 \varphi_a. \]

The singular scalar curvature equation (5.2) gives the identity.
Lemma 5.4. We write $A_R := -b_0 R + b_1 n + b_2 \text{tr}_{\varphi} \omega_{\varphi_n}$. Then
\[
\triangle_{\varphi} w = b_0 (\text{tr}_{\varphi} \theta - R) + b_1 (n - \text{tr}_{\varphi} \omega_t) + b_2 (\text{tr}_{\varphi} \omega_{\varphi_n} - \text{tr}_{\varphi} \omega_t)
= b_0 \text{tr}_{\varphi} \theta - (b_1 + b_2) \text{tr}_{\varphi} \omega_t + A_R.
\]

Remark 5.5. There are two ways to deal with $\omega_t$, one is $\omega_t = \tilde{\omega}_t - i \partial \bar{\partial} \phi_E$, and the other one is $\omega_t = \omega_{sr} + t \omega$. Thus, we have
\[
\triangle_{\varphi} w = b_0 \text{tr}_{\varphi} \theta - (b_1 + b_2) \text{tr}_{\varphi} \tilde{\omega}_t + (b_1 + b_2) \triangle_{\varphi} \phi_E + A_R
= b_0 \text{tr}_{\varphi} \theta - (b_1 + b_2) \text{tr}_{\varphi} (\omega_{sr} + t \omega) + A_R.
\]

The constant $b_1$ will be chosen to be negative such that $-(b_1 + b_2)$ is positive. Accordingly, we see that $\triangle_{\varphi} w \geq b_0 \text{tr}_{\varphi} \theta - t(b_1 + b_2) \text{tr}_{\varphi} \omega + A_R$.

We introduce a weight function $H$ and add it to $u$
\[ u := w - H \]
and utilise various conditions on the $(1,1)$-form $\theta$, aiming to obtain a differential inequality from the identity in Lemma 5.4
\[
(5.4) \quad \triangle_{\varphi} u \geq A_\theta \text{tr}_{\varphi} \tilde{\omega}_t + A_R.
\]

Here, we choose $b_1$ such that $A_\theta > 0$. Then the maximum principle is applied near the maximum of $u$ to conclude the estimate of $u$.

Given a point $z \in X$ and a ball $B_d(z) \subset X$, we let $\eta$ be the local cutoff function $B_d(z)$ regarding to the metric $\tilde{\omega}_t$ such that $\eta(z) = 1$ and $\eta = 1 - b_3$ outside the half ball $B_{\frac{d}{2}}(z)$. Then we have the standard estimate of the local cutoff function as
\[
(5.5) \quad \triangle_{\varphi} \log \eta \geq -A_{b_3} \text{tr}_{\varphi} \tilde{\omega}_t, \quad A_{b_3} := \left[ \frac{2b_3}{d(1 - b_3)} \right]^2 + \frac{4b_3}{d^2(1 - b_3)}.
\]

Proposition 5.6. Assume that $u$ satisfy (5.4). We define
\[
(5.6) \quad v := b_4 u = b_4 (b_0 F - H + b_1 \varphi + b_2 \varphi_a)
\]
and set $\tilde{d} = 8d^{-2}$ and $0 < b_3 << 1$ satisfy
\[
(5.7) \quad b_3 = \frac{A_\theta}{\tilde{d} \cdot b_1^{-1} + A_\theta} \quad \text{such that} \quad A_{b_3} \leq \tilde{d} \frac{b_3}{1 - b_3} = b_4 A_\theta.
\]

Then the following estimates hold.

(i) The upper bound of $v$ is $b_4^{-1} \sup_X v \leq C(n) d I^{\frac{1}{2n}}$ where
\[
I := b_3^{-2n} \int_{A_R \leq 0} (A_R)^{2n} e^{2n + 2(F - f)} \omega_t^n.
\]

(ii) The upper bound of $b_0 F - H$ is
\[
(5.9) \quad b_0 F - H \leq C(n) d I^{\frac{1}{2n}} - b_2 \varphi_a.
\]
Proof. Now we combine the inequality for (5.12) to obtain

\[ \|e^{b_0 F - H}\|_{L^p(\Omega^n)} \leq e^{P b_4^{-1} \sup_X v} \int_X e^{-\alpha \varphi_a^\omega} \omega^1. \]

Then it holds

\[ \|e^{b_0 F - H}\|_{L^p(\omega^n)} \leq e^{P b_4^{-1} \sup_X v} \int_X e^{-\alpha \varphi_a^\omega} \omega^1. \]

(iv) When \( b_0 = 1 \), assume \( e^H \in L^p(\tilde{\omega}_1^n) \) for some \( p_H^+ \geq 1 \). Then for any \( \tilde{p} \leq \frac{p_H^+}{p + p_H^+} \), we have \( \|e^F\|_{L^\tilde{p}(\tilde{\omega}_1^n)} \leq e^{-\inf_X f} \|e^F\|_{L^\tilde{p}(\omega_1^n)} \) and

\[ \|e^F\|_{L^\tilde{p}(\omega_1^n)} \leq \|e^{b_0 F - H}\|_{L^p(\omega_1^n)} \|e^H\|_{L^p(\tilde{\omega}_1^n)}. \]

which implies \( \|\varphi\|_\infty, \|\varphi_a\|_\infty \leq C \). Consequently, (5.9) becomes

\[ F \leq H + C(n) dI \frac{1}{\omega^n} - b_2\|\varphi_a\|_\infty. \]

(v) When \( b_0 = 1 \), we assume \( e^{-H} \in L^p(\tilde{\omega}_1^n) \) and \( b_4 \) satisfies

\[ 0 < b_4 \leq \min\{(-4nb_1)^{-1} \alpha(\Omega_1), (4n)^{-1} p_H^+\}. \]

Then the estimate of \( I \) is given in Proposition 5.7.

Proof. Now we combine the inequality for \( u \) and the cutoff function \( \eta \). Inserting (5.4) and (5.5) to \( \Delta \varphi(v + \log \eta) \), we get

\[ \Delta \varphi(v + \log \eta) \geq (b_4 A_0 - A_{b_1}) \text{tr}_\varphi \tilde{\omega}_t + b_4 A_R \geq b_4 A_R. \]

Furthermore, it implies

\[ \Delta \varphi(e^n \eta) \geq b_4 A_R e^n \eta. \]

Therefore, applying the Aleksandrov maximum principle to this differential inequality, we obtain the estimate of \( v \) in (i).

Before we obtain the estimate of \( I \) in (v), which will be given in Proposition 5.7, we derive the upper bound of \( b_0 F - H \) and an \( L^p \) bound of \( e^{b_0 F - H} \).

Since \( \varphi, \varphi_a \) are \( \omega_t \)-psh functions, we could modify a constant such that \( \varphi, \varphi_a \leq 0 \). Hence, \( b_1 \varphi \geq 0 \) and the formula (5.6) of \( v \) gives

\[ b_0 F - H \leq b_4^{-1} \sup_X v - b_2 \varphi_a. \]

Thus the upper bound (5.9) of \( b_0 F - H \) in (ii) is obtained by inserting (5.8) to the inequality above.

Note that \( \omega_1 = \omega_{sr} + \omega \) and \( \omega_{sr} \geq 0 \), we have \( \omega \leq \omega_1 \) and \( \varphi_a \) is also psh with respect to \( \omega_1 = \omega_{sr} + \omega \). Accordingly, we apply the \( \alpha \)-invariant of \( \Omega_1 = [\omega_1] \) to conclude the \( L^p(\omega_1^n) \) bound of \( e^{b_0 F - H} \) in (iii).

The proof of (iv) is given as following. While, \( \omega \) is equivalent to \( \tilde{\omega}_t \), i.e. \( \tilde{\omega}_t \leq (C_K + t) \omega \), we could replace \( L^p(\omega_1^n) \) norm by \( L^p(\tilde{\omega}_t) \) norm,

\[ \|e^{b_0 F - H}\|_{L^p(\omega_1^n)} \leq \|e^{b_0 F - H}\|_{L^p(\omega_1^n)}. \]

Under the hypothesis \( e^H \in L^p(\tilde{\omega}_t^n) \), the estimate of \( \|e^F\|_{L^p(\tilde{\omega}_t^n)} \) is obtained by applying the Hölder inequality to (5.11). Moreover, the upper
bound of $e^{-f}$ from Lemma 4.12 implies the $L^p(\tilde{\omega}_n^t)$ bound of $e^F$. By the equation $\omega^n = e^{F}\tilde{\omega}_n^p$, the bound $\|e^F\|_{L^p(\tilde{\omega}_n^T)}$ further implies the uniform bound of $\varphi$, due to Lemma 4.13.

The auxiliary function $\varphi_a$ is defined to be a solution to the following approximation of the singular Monge-Ampère equation

$$\omega^n_a = E^{-1}\omega^n e^F \sqrt{F^2 + 1}, \quad E = V^{-1} \int_X e^F \sqrt{F^2 + 1} \omega^n.$$

(5.13)

Similarly, the volume element of the auxiliary function $\omega \varphi_a$ is also $L^p$, by applying $\|e^F\|_{L^p(\tilde{\omega}_n^t)}$. For the same reason, $\varphi_a$ is also uniformly bounded, too.

Inserting the resulting estimate of $\varphi_a$ into (5.9), we have the upper bound of $F - H$. □

The rest of the proof is to estimate $I$, (v) in Proposition 5.6.

**Proposition 5.7** (Estimation of $I$). In general, we have

$$I \leq e^{-2\inf_X f} \int_{A_R \leq 0} |b^{-1}_3(b_0 R - b_1 n)|^{2n} e^{-2nbH} e^{2nb_4b_1 \varphi} e^{2nb_4b_0 + 2} F e^{-2} \tilde{\omega}_n,$$

(5.14)

and in the integral domain $\{x \in X | A_R \leq 0\}$ of $I$,

$$F \leq \left(\frac{|b_0 R - b_1 n|}{b_2 n}\right)^n (E_{\lambda}^3 + 2 e^{-1} + 1).$$

(5.15)

If we further assume that $|b_0 R - b_1 n|$ is bounded, $e^{-H} \in L^{pH}(\tilde{\omega}_n^t)$ and $b_4$ satisfies (5.12). Then

$$I^2 \leq C^2 (C_K + 1)^n \|e^{-H}\|_{L^{pH}(\tilde{\omega}_n^t)} \int_X e^{-\alpha(\Omega_1)\varphi} \omega^n_1,$$

(5.16)

where $C = e^{-2\inf_X f} e^{2nb_4 + 2} \sup_{A_R \leq 0} F \|b^{-1}_3(b_0 R - b_1 n)\|^{2n}$.  

**Proof.** We use the lower bound of $f$ from Lemma 4.12 in (5.8),

$$I \leq e^{-2\inf_X f} \int_{A_R \leq 0} (A_R)^{2n} e^{2nv} e^{2F} \tilde{\omega}_n.$$  

Then we deal with each factors in the integrand. Since $\varphi_a$ is an $\omega_t$-psh function, we have $\varphi_a \leq 0$. Thus (5.6) tells us

$$e^{2nv} \leq e^{2nb_4b_0 F} e^{-2nbH} e^{2nb_4b_1 \varphi}.$$

Using the expression of $A_R$ in Lemma 5.4, we get $0 \geq A_R \geq -b_0 R + b_1 n$. So, we estimate

$$(A_R)^{2n} \leq |b_0 R - b_1 n|^{2n}.$$  

Inserting these estimates into the expression of $I$, we obtain (5.14).
The bound of $F$ is obtained in terms of the entropy $E_{t,e}^\beta$, by using the auxiliary function $\varphi_u$. Since $A_R \leq 0$, applying the geometric mean inequality to the expression of $A_R$ in Lemma 5.4, we have

$$b_2 n \left( \frac{\omega^n}{\varphi_{\varphi_n}} \right)^{1/n} \leq b_2 \tr \omega \varphi_n \leq b_0 R - b_1 n.$$  

Substituting (5.13) into the inequality above, we have

$$F \leq \sqrt{F^2 + 1} \leq \left( \frac{b_0 R - b_1 n}{b_2 n} \right)^n E.$$  

At last, we use that fact that $E$ is bounded by the entropy $E_{t,e}^\beta$ from Lemma 5.4 in [26], to conclude (5.15).

When $2nb_1b_0 + 2 \geq 0$, we let $C = e^{-2\inf_X f}b_3^{-1} e^{(2nb_1b_0+2)\sup_X F}$ and get

$$I \leq C \int_{A_R \leq 0} (b_0 R - b_1 n)^{2n} e^{-2nb_4H} e^{2nb_1\varphi \tilde{\omega}_1^n}.$$  

By Hölder inequality, it is further bounded by

$$C\|b_0 R - b_1 n\|_{\infty}^{2n} \int_X e^{-4nb_4H\tilde{\omega}_1^n} \int_X e^{4nb_1\varphi \tilde{\omega}_1^n}.$$  

with $p_R > 2n$. The integral $\int_X e^{-4nb_4H\tilde{\omega}_1^n}$ is finite, when $4nb_4 \leq p_H$. As Lemma 5.11, we use $\tilde{\omega}_1 \leq \omega_K + \omega \leq (C_K + 1)\omega$ by Lemma 4.2 and $\omega \leq \omega_1 = \omega_{sr} + \omega$ by semi-positivity of $\omega_{sr}$. So, the integral

$$\int_X e^{4nb_1\varphi \tilde{\omega}_1^n} \leq (C_K + 1)^n \int_X e^{4nb_1\varphi \omega_1^n}$$  

is also bounded, once $-4nb_1b_1 \leq \alpha(\Omega_1)$. \hfill $\Box$

**Remark 5.8.** In the proof, we could use $\int_X (b_0 R - b_1 n)^{p_R} \tilde{\omega}_1^n$, $p_R > 2n$ instead.

### 5.2. Applications

Now we are ready to make use of different properties on $\theta$ to estimate $\varphi$ and $F$, with the help of Proposition 5.6 and its corollaries. We clarify the steps in practice. Firstly, we derive (5.4) to write down the formulas of $A_\theta$, $A_R$ and $H$. Secondly, we ask $A_\theta$ to be strictly positive to determine the value of $b_1$. While, $b_2$ is chosen as (5.10) depending on the auxiliary function $\varphi_u$. Thirdly, we use the expression of $H$ to verify both conditions including $\|e^H\|_{L^p_H(\omega_1^n)}$ and $\|e^{-H}\|_{L^p_H(\omega_1^n)}$ in (iv) of Proposition 5.6 and Proposition 5.7, respectively. Consequently, $b_1$ is determined form $b_1$, $p_H$ in (5.12). At last, we could obtain the value of $b_3$ by (5.7) and compute $I$ in Proposition 5.7.

Now we start our applications. As we observe in [26] that the particular property of the given $(1,1)$-form $\theta \in C_1(X, D)$ leads to various differential inequalities. The most general bound on $\theta$ is Definition 4.3

$$C_1 \cdot \tilde{\omega}_t \leq \theta \leq C_u \cdot \tilde{\omega}_t.$$  

Proposition 5.9. Suppose that $e^{\overline{C} \phi E} \in L^{p_0} (\overline{\omega}_t^n)$ for some $p_0 \geq 1$. Then there exists a constant $C$ such that for all $p > p_0$, $\tau > 0$,
\begin{equation}
(5.17) \quad \|e^F\|_{L^p(\overline{\omega}_t^n)}, \quad \|e^F\|_{L^p(\overline{\omega}_t^n)}, \quad \|\varphi\|_\infty, \quad \sup_X (F - \sigma s \phi E) \leq C,
\end{equation}
where $\sigma_s = C_l - \tau$. The constant $C$ depends on $E|_{t \in E}, C_l, \alpha(\Omega_1), \sup_X R, \inf_X f, \sup_X \phi_E, n, p$.

Proof. We let $b_0 = 1$ and insert the lower bound (4.3) of $\theta$, i.e. $\theta \geq C_l \tilde{\omega}_t$, together with $\omega_t = \tilde{\omega}_t - i \partial \bar{\partial} \phi E$ to Lemma 5.4,
\[ \Delta \varphi w \geq C_l \text{tr}_\varphi \tilde{\omega}_t - (b_1 + b_2) (\text{tr}_\varphi \tilde{\omega}_t - i \partial \bar{\partial} \phi E) + A_R. \]
Comparing with (5.4), we read from this inequality that
\[ H = (b_1 + b_2) \phi E, \quad A_0 = C_l - (b_1 + b_2), \quad A_R = -R + b_1 n + b_2 \text{tr}_\varphi \omega_{\varphi_n}. \]

Given a fixed $p \geq 1$, by (5.10), we further take $b_2 = p^{-1} \alpha(\Omega_1)$. Letting $A_0 = t_0 > 0$, we have $b_1 = C_l - b_2 - t_0$. Also, from (5.7), we get $b_3 = \frac{\tau t_0}{\delta - d t_1 + t_0}$.

As a result, we see that $H = (C_l - t_0) \phi E$. Clearly, $-H$ is bounded above. Moreover, since $e^{\overline{C} \phi E} \in L^{p_0} (\overline{\omega}_t^n)$ and $e^{-t_0 \phi E} \in L^p (\overline{\omega}_t^n)$ as long as $t_0$ is small enough, we have
\[ e^H = e^{(C_l - t_0) \phi E} \in L^{p_0} (\overline{\omega}_t^n) \quad \text{for all } p_H^+ \leq \frac{p_0 p_1}{p_0 + p_1}. \]
We also choose $b_4 \leq (-4 n b_1)^{-1} \alpha$ by (5.12). According to Proposition 5.7, we could examine that the integral $I$ is finite and all estimates are independent of $t$ and $\epsilon$.

Therefore, the hypotheses of $H$ in (v) in Proposition 5.6 and (iv) of Proposition 5.6 are satisfied and we conclude the estimates of $\|e^F\|_{L^p(\overline{\omega}_t^n)}, \|e^F\|_{L^p(\overline{\omega}_t^n)}, \|\varphi\|_\infty$ and $\sup_X (F - H) \leq C$. \hfill \Box

Proposition 5.10. Under the assumption in Proposition 5.9, it holds for any $\tau > 0$,
\begin{equation}
(5.18) \quad \inf_X [F - \sigma_i \phi E] \geq -C, \quad \sigma_i = C_u + \tau.
\end{equation}
The constant $C$ depends on $\|\varphi\|_\infty, C_u, \inf_X R, \inf_X f, \sup_X \phi_E, n$.

Proof. We apply Proposition 5.6 again, taking
\[ w = -F + b_1 \varphi, \quad b_0 = -1, \quad b_2 = 0. \]
Substituting $\theta \leq C_u \tilde{\omega}_t$ into Lemma 5.4, we obtain
\[ \Delta \varphi w \geq -C_u \text{tr}_\varphi \tilde{\omega}_t - b_1 \text{tr}_\varphi \omega_t + A_R, \quad A_R = R + b_1 n. \]
By $\tilde{\omega}_t = \omega_t + i \partial \bar{\partial} \phi E$, it is reduced to
\[ \Delta \varphi w \geq -C_u \text{tr}_\varphi \tilde{\omega}_t - b_1 \text{tr}_\varphi (\tilde{\omega}_t - i \partial \bar{\partial} \phi E) + A_R. \]
Accordingly, $H = b_1 \phi E$ and $A_0 = -C_u - b_1$. 

\[ \frac{\partial \bar{\partial} \phi E}{\partial \bar{\partial} \phi E} = \frac{\partial \bar{\partial} \phi E}{\partial \bar{\partial} \phi E} \quad \text{for all } \partial \bar{\partial} \phi E. \]
We choose \( b_1 = -C_u - t_0 \) such that

\[
A_\theta = t_0, \quad H = -(C_u + t_0)\phi_E.
\]

We see that \( e^{-H} \) is bounded above.

From (5.9), \(-F - H \leq C(u)dt^H\). We verify the estimate of \( I \) in Proposition 5.7. We let \( b_4 = \frac{1}{n} \). Due to (5.7), we have \( b_3 = \frac{t_0}{8d^{-2}b_4^{-1} + t_0} \).

Then we get

\[
I \leq e^{-2\inf_X f} \int_{A_R \leq 0} |b_3^{-1}(R - b_1n)|^{2n} e^{-2H} e^{-2(C_u + t_0)\phi_E} \tilde{\omega}^n,
\]

which is finite, by using the upper bound of \(-H = (C_u + t_0)\phi_E\) and \( \inf_X \varphi \). Therefore, the upper bound of \(-F + (C_u + t_0)\phi_E\) is derived. \( \square \)

We observe that we could remove \( \tau \).

**Corollary 5.11.** Assume that \(|R - C|n| \leq A_s t\) for some constant \( A_s \). Then we have (5.17) with \( \sigma_s = C_l \).

**Proof.** The proof is identical to Proposition 5.9, but choosing \( b_0 = 1 \), \( A_\theta = t \) and \( H = (C_l - t)\phi_E \). We also take \( t = t \). Since \( t \to 0 \), we have \( pb_2 = pt \leq \alpha(\Omega_1) \), which is the condition (5.10). Then \( b_1 = C_l - b_2 - t = C_l - 2t \) and \( b_4 = \min\{\frac{1}{n}, \frac{\alpha}{4n b_1} \} \). Also,

\[
\frac{t}{8d^{-2}b_4^{-1} + 1} \leq b_3 = \frac{t}{8d^{-2}b_4^{-1} + t} \leq \frac{1}{8d^{-2}b_4^{-1}}.
\]

The scalar curvature assumption gives

\[
|R - b_1n| = |R - (C_l - 2t)n| \leq |R - C_l|n| + 2tn \leq (A_s + 2n)t.
\]

We have \( F \leq \left(\frac{|R - b_1n|}{b_2} \right)^n (E^n + 2e^{-1} + 1) \) is bounded in the domain \( A_R \leq 0 \). We insert these values to (5.14) in Proposition 5.7 to estimate

\[
I \leq e^{-2\inf_X f} \int_{A_R \leq 0} |b_3^{-1}(R - b_1n)|^{2n} e^{-2nb_4^2H} e^{2nb_4^2\phi_E} (2nb_4^2 + F) \tilde{\omega}^n.
\]

By further using the bound of \( e^{-H} \) and \( |b_3^{-1}(R - b_1n)| \), we have \( I \) is bounded, if \( b_4 \) is smaller than \( \min\{\frac{1}{n}, \frac{\alpha}{2nb_1q} \} \). Consequently, the estimates (5.17) follow from Proposition 5.6 and (iv) of Proposition 5.6. We find that all constants are independent of \( t \), so we could further take \( t \to 0 \) such that the weight \( \sigma_s = C_l \). \( \square \)

**Corollary 5.12.** Assume \(|R - C|n| \leq A_s t\) for some constant \( A_s \). Then (5.18) holds with \( \sigma_s = C_u \).

**Proof.** We learn from Proposition 5.10 that \( A_\theta = t, H = -(C_u + t)\phi_E \).
We have \( b_2 = 0 \) and \( b_4 = \frac{1}{n} \). Also, \( b_1 = -C_u - t \) is bounded when \( 0 \leq t \leq 1 \) and

\[
b_3 = \frac{t}{8d^{-2}b_4^{-1} + t} \geq Ct.
\]
Inserting the upper bound of $e^{-H}$ and $|b^{-1}_3(-R - b_1 n)|$ in (5.19), we get the estimate of $I$ is independent of $t$. Therefore, we conclude from Proposition 5.10 with $t \to 0$. \hfill \Box

**Remark 5.13.** For the cscK problem, the averaged scalar curvature $R_t$ (4.13) of the approximate singular cscK metric is close to $\bar{S}_\beta = \frac{C_1(X,D)\Omega^{n-1}}{\Omega^n}$, namely

$$R_t - \bar{S}_\beta \leq Ct.$$  

So, the assumption in both corollaries means some sort of pinching of the eigenvalues of the representative $\theta \in C_1(X, D)$.

### 6. Gradient estimate of $\varphi$

In this section, we obtain the gradient estimate for the singular cscK metric, extending the results for non-degenerate cscK metrics [7]. We will use the singular exponent $\sigma_E$ to measure the singularity of the given big and semi-positive class $\Omega$. Meanwhile, we will use the degenerate exponent $\sigma_D$ to reflect how the degeneracy of the singular cscK equation (4.12) could effect the gradient estimate. It is surprising to see that, when the cone angle $\beta > \frac{n+2}{2}$, the degenerate exponent does not appear in the gradient estimate. That means the gradient estimates remains exactly the same to the one for the non-degenerate metric.

**Theorem 6.1** (Gradient estimate of $\varphi$). Suppose that $\varphi_\epsilon$ is a solution to the approximate singular scalar curvature equation (4.14).

1. Assume that $\Omega$ is big and semi-positive. Then there exists a constant $C$ such that

$$|s_E|^{2\sigma_{E}} S_E^{\sigma_{D}} |\partial \varphi|_{\Omega_t}^2 \leq A_3, \quad \sigma_D \geq 1.$$  

The singular exponent $\sigma_E$ is sufficiently large, and determined in (6.9) and (6.11).

2. If we further assume that $\Omega = [\omega_K]$ is Kähler, then $\sigma_E = 0$ and the degenerate exponent $\sigma_D$ is weaken to satisfy

$$\left\{ \begin{array}{l} \sigma_D = 0, \quad \text{when } \beta > \frac{n+2}{2}, \\
\sigma_D > 1 - \frac{2\beta}{n+2} \geq 0, \quad \text{when } \beta \leq \frac{n+2}{2}. \end{array} \right.$$  

Then there holds the gradient estimate

$$S_E^{\sigma_D} |\partial \varphi_\epsilon|_{\omega_K}^2 \leq A_4.$$  

The precise statements will be given in Theorem 6.10 and Theorem 6.11, respectively.

**Remark 6.2.** In the second conclusion, we see that $\sigma_D$ could be chosen to be zero, when $\beta > \frac{n+2}{2}$. 

Proof. We denote $\tilde{\phi}$ by $\psi$ and all the norms are taken with respect to the Kähler metric $\tilde{\omega}$ in this proof. We will use the approximate singular scalar curvature equation (4.14) and omit the lower index for convenience.

(6.2)  $(\tilde{\omega}_t + i\partial\bar{\partial}\psi)^n = e^F\tilde{\omega}_t^n, \quad \Delta_\psi F = \text{tr}_\psi(\theta - i\partial\bar{\partial}f) - R,$

where, $F = F - f, f = -(\beta - 1) \log S_t - h_\theta - c_t - \log S_{\tilde{\omega}_t}$. We will divide the proof into several steps in this section. □

6.1. **Differential inequality.** Let $K$ be a positive constant determined later and $H$ be an auxiliary function on $F$ and $\psi$. We set $v := |\partial\psi|^2 + K, \quad u := e^H v.$

Then we calculate that

**Lemma 6.3.**

$$u^{-1} \Delta_\psi u = \Delta_\psi H + |\partial H|_\psi^2 + 2\frac{H_i \psi \bar{\psi}_i + H_i \psi \bar{\psi}_j}{(1 + \psi_i) v} + 2 \frac{Re(F_i \psi \bar{\psi}_i)}{v}.$$  

(6.3)

Proof. We compute the Laplacian of $\log v$ under the normal coordinates with respect to the metric $\tilde{\omega}_t$,

$$\Delta_{\tilde{\omega}_t} \log v = -|\partial \log \varphi|^2 + \frac{R_{\tilde{\omega}_t}(\bar{\omega}_i) \psi_j \bar{\psi}_j + |\psi_j|^2 + |\psi_i|^2}{(1 + \psi_i) v} + 2 \frac{Re(F_i \psi_j)}{v}.$$  

Applying $\Delta_{\tilde{\omega}_t} u = \Delta_{\tilde{\omega}_t} H + \Delta_{\tilde{\omega}_t} \log v$, we get

$$u^{-1} \Delta_{\tilde{\omega}_t} u = |\partial \log u|_{\tilde{\omega}_t}^2 + \Delta_{\tilde{\omega}_t} \log u$$

$$= |\partial \log u|_{\tilde{\omega}_t}^2 + \Delta_{\tilde{\omega}_t} H - |\partial \log v|^2_{\tilde{\omega}_t}$$

$$+ \frac{R_{\tilde{\omega}_t}(\bar{\omega}_i) \psi_j \bar{\psi}_j + |\psi_j|^2 + |\psi_i|^2}{(1 + \psi_i) v} + 2 \frac{Re(F_i \psi_j)}{v}.$$  

We further calculate that

$$|\partial \log u|_{\tilde{\omega}_t}^2 - |\partial \log v|^2_{\tilde{\omega}_t} = (\partial H, 2\partial \log v + \partial H)_{\psi} = |\partial H|_{\psi}^2 + 2(\partial H, \partial v)_{\psi} v^{-1}$$

$$= |\partial H|_{\psi}^2 + 2 \frac{H_i \psi \bar{\psi}_i + H_i \psi \bar{\psi}_j}{(1 + \psi_i) v}.$$  

In summary, the desired identity is obtained by adding them together. □

The differential inequality is obtained from the identity (6.3), after dropping off the positive terms and carefully choosing the weight function $H$.

Firstly, we remove the positive terms in (6.3).
Lemma 6.4. Let $K \geq 1$ and $-C_{1,1} := \inf_X R_{i\bar{j}j}(\tilde{\omega}_t)$. Then it holds
\begin{equation}
   -1 \Delta_{\psi} u \geq \Delta_{\psi} H + 2 \text{Re}[(H_i + \tilde{F}_i)\psi_i]v^{-1} + [(-C_{1,1} - 1) + v^{-1}] \text{tr}_\psi \tilde{\omega}_t + (\text{tr}_\psi \omega_{\phi} - 2n)v^{-1}.
\end{equation}

Proof. By removing the positive terms
\[ \frac{|\partial H|^2}{\psi^2} + 2 \frac{H_i \psi_i \psi_j}{1 + \psi_i} + \left| \frac{\psi_i}{1 + \psi_i} \right| \geq 0, \]
and inserting the lower bound of the bisectional curvature $R_{i\bar{j}j}(\tilde{\omega}_t)\psi_j \psi_j \geq -C_{1,1}v$, into (6.3), we get
\[ e^{-H} \Delta_{\psi} u \geq v \Delta_{\psi} H + K|\partial H|^2 + 2 \frac{H_i \psi_i \psi_i}{1 + \psi_i} - C_{1,1}v \text{tr}_\psi \tilde{\omega}_t + \left| \frac{\psi_i}{1 + \psi_i} \right| + 2 \text{Re}(\tilde{F}_i \psi_i). \]

While, we compute
\[ 2 \frac{H_i \psi_i \psi_i}{1 + \psi_i} = 2H_i \psi_i - 2 \frac{H_i \psi_i}{1 + \psi_i}. \]
By Young's inequality, it follows
\[ - \frac{2H_i \psi_i}{1 + \psi_i} \geq -|\partial H|^2 - |\partial \psi|^2 \text{tr}_\psi \tilde{\omega}_t \geq -|\partial H|^2 - v \text{tr}_\psi \tilde{\omega}_t. \]
By substitution into the inequality above, we have
\[ e^{-H} \Delta_{\psi} u \geq v \Delta_{\psi} H + (K - 1)|\partial H|^2 + (-C_{1,1} - 1)v \text{tr}_\psi \tilde{\omega}_t \]
\[ + \left| \frac{\psi_i}{1 + \psi_i} \right|^2 + 2 \text{Re}[(H_i + \tilde{F}_i)\psi_i]. \]
Rewriting the positive term
\[ \left| \frac{\psi_i}{1 + \psi_i} \right|^2 = \left| \frac{\psi_i}{1 + \psi_i} - 1 + 1 \right|^2 = \sum_i \left[ \psi_i - 1 + \frac{1}{1 + \psi_i} \right] = \text{tr}_\psi \omega_{\phi} - 2n + \text{tr}_\psi \tilde{\omega}_t \]
and inserting it back to the inequality, we have obtained (6.4). □

Then we continue the proof of Proposition (6.7). In order to deal with the gradient term in (6.4), we further choose
\[ B = -\sigma_E \psi + e^{-\psi}, \quad H := -F + B + \sigma_D \log S_t. \]
In which, we see that
\[ e^{-\psi} = e^{-\varphi + \psi_E} = e^{-\varphi} |s_{E}|_{h_{E}}^{2\alpha}, \]
which is bounded by using the $L^\infty$-estimate of $\varphi$. 
Lemma 6.5.

\[ (6.5) \quad 2\text{Re}[\langle H_i + \tilde{F}_i \rangle \psi_i]v^{-1} \geq 2(\sigma_E - e^{-\psi})(1 - Kv^{-1}) + A_w v^{-\frac{1}{2}} \]

with the weight
\[ (6.6) \quad A_w := -2C_2 - 2(\beta - 1 + \sigma_D)|\partial \log S_\epsilon|. \]

Proof. We compute
\[ B_i = (-\sigma_E - e^{-\psi})\psi_i \text{ and } H + \tilde{F} = B - f + \sigma_D \log S_\epsilon \]
to get
\[ 2\text{Re}[\langle H_i + \tilde{F}_i \rangle \psi_i] = 2(\sigma_E - e^{-\psi})|\partial \psi|^2 + 2 \text{Re}[-f_i \psi_i + \sigma_D(\log S_\epsilon_i)\psi_i]. \]

Inserting \( f = -(\beta - 1)\log S_\epsilon - c_{\epsilon,\omega} - \log \frac{\omega_n}{\omega_t} \) into the identity above, we have that
\[ -f_i \psi_i + \sigma_D(\log S_\epsilon_i)\psi_i = (h_\theta + \log \frac{\omega_n}{\omega_t})\psi_i + (\beta - 1 + \sigma_D)(\log S_\epsilon_i)\psi_i \]
\[ \geq -[C_2 + (\beta - 1 + \sigma_D)|\partial \log S_\epsilon|] \cdot |\partial \psi|, \]

where \( C_2 \) depends on \( \| \partial(h_\theta + \log \frac{\omega_n}{\omega_t}) \|_{L^\infty(\omega_t)} \). Consequently, we have proved (6.5) by using \( |\partial \psi|^2 = v - K. \]

Meanwhile, we bound the Laplacian term \( \triangle_\psi H \) in (6.4).

Lemma 6.6.

\[ (6.7) \quad \triangle_\psi H \geq \tilde{A}_\theta \text{tr}_\psi \tilde{\omega}_t + e^{-\psi}|\partial \psi|^2_{\psi} + (\inf R - \sigma_E n - e^{-\psi} n). \]

where, \( C_{1.4} := \inf_{(X,\tilde{\omega}_t)} i\partial\tilde{\partial} \log S_\epsilon \) and
\[ \tilde{A}_\theta = -\sup_{(X,\tilde{\omega}_t)} \theta + \sigma_E + e^{-\psi} + \sigma_D C_{1.4}. \]

Proof. We take the Laplacian of \( B = -\sigma_E \psi + e^{-\psi} \),
\[ \triangle_\psi B = -\sigma_E \triangle_\psi \psi + e^{-\psi}|\partial \psi|^2_{\psi} - e^{-\psi} \triangle_\psi \psi, \]

Then we calculate the Laplacian of the auxiliary function \( H \) with the singular scalar curvature equation (6.2),
\[ \triangle_\psi H = -\triangle_\psi F + \triangle_\psi B + \sigma_D \triangle_\psi \log S_\epsilon \]
\[ = -\text{tr}_\psi \theta + R + e^{-\psi}|\partial \psi|^2_{\psi} + (\sigma_E - e^{-\psi})(n - \text{tr}_\psi \tilde{\omega}_t) + \sigma_D \triangle_\psi \log S_\epsilon, \]

which implies the lower bound
\[ \triangle_\psi H \geq [\sup_{(X,\tilde{\omega}_t)} \theta + \sigma_E + e^{-\psi}] \text{tr}_\psi \tilde{\omega}_t + \inf R + e^{-\psi}|\partial \psi|^2_{\psi} - (\sigma_E + e^{-\psi}) n \]
\[ + \sigma_D \triangle_\psi \log S_\epsilon. \]

Since \( \sigma_D \geq 0 \), the asserted inequality follows from the lower bound of \( i\partial\tilde{\partial} \log S_\epsilon \) from Lemma 2.11, namely
\[ \triangle_\psi \log S_\epsilon \geq C_{1.4} \text{tr}_\psi \tilde{\omega}_t. \]
We set three bounded quantities to be
\[ A_\theta := \tilde{A}_\theta - C_{1,1} - 1 = - \sup_{(X, \tilde{\omega})} \theta + \sigma_E + e^{-\psi} + \sigma_D C_{1,4} - C_{1,1} - 1, \]
(6.8)
\[ A_s := \inf R - (n + 2)(\sigma_E + e^{-\psi}), \quad A_c := 2K(\sigma_E + e^{-\psi}) - 2n. \]
As a result, inserting the functions \( A_s, A_c, A_w \), (6.5) and (6.7) back to (6.4), we obtain that
\[ u^{-1} \Delta \psi u \geq A_\theta \operatorname{tr} \tilde{\omega} + e^{-\psi} |\partial \psi|^2 + A_s + (\operatorname{tr} \tilde{\omega} \omega + A_c) v^{-1} + A_w v^{-\frac{1}{2}}. \]
In order to apply Lemma 6.8 to deduce the lower bound of the first two terms, we need to verify the assumptions
\[ A_\theta \geq C_K e^{-\psi}, \quad C_K := 1 + \frac{K}{n^2(n - 1)}. \]
They are satisfied, as long as we choose large \( \sigma_E \), i.e.
(6.9)
\[ \sigma_E \geq \frac{K}{n^2(n - 1)} e^{-\psi} + \sup_{(X, \tilde{\omega})} \theta - \sigma_D C_{1,4} + C_{1,1} + 1. \]
Consequently, we obtain the desired differential inequality asserted in the following proposition.

**Proposition 6.7.** It holds
\[ \Delta \psi u \geq \frac{1}{n - 1} e^{-\frac{\mu + \tilde{F}}{n} - \psi} u^{1 + \frac{1}{n}} \]
(6.10)
\[ + A_s u + e^H (\operatorname{tr} \tilde{\omega} \omega + A_c) + A_w e^{\frac{H}{2}} u^{\frac{1}{2}}. \]

At last, we close this section by proving the required lemma.

**Lemma 6.8.** For any given function \( K \), there holds
\[ G := C_K \operatorname{tr} \tilde{\omega} + |\partial \psi|^2 \geq \frac{1}{n - 1} e^{-\frac{\tilde{F}}{n}} v^\frac{1}{n}. \]

**Proof.** By the inequality of arithmetic and geometric means and the equation (4.14), we have
\[ \operatorname{tr} \tilde{\omega} \geq (\operatorname{tr} \tilde{\omega} \omega \cdot e^{-\tilde{F}})_{n-1}. \]
Jensen inequality for the concave function \((\cdot)^{\frac{1}{n-1}}\) further implies that
\[ (\operatorname{tr} \tilde{\omega} \omega)_{n-1} \geq \frac{1}{n} \sum_i (1 + \psi_{ii})^{\frac{1}{n-1}} n^{\frac{n}{n-1}}. \]
Inserting them into the expression of \((n - 1)G\), we get
\[ (n - 1)G \geq \sum_i \left[ \frac{n - 1}{n} (1 + \psi_{ii})^{\frac{1}{n-1}} e^{-\frac{\tilde{F}}{n-1} n^{\frac{n}{n-1}}} + \frac{K}{n^2 (1 + \psi_{ii})} + \frac{(n - 1)|\psi_{ii}|^2}{1 + \psi_{ii}} \right]. \]
By \( n \geq 2 \) and \( n - 1 \geq \frac{1}{n} \), it follows
\[
\geq \sum_{i} \left[ \frac{n - 1}{n} \left( 1 + \psi_i \right)^{n-1} e^{-\frac{\bar{F}}{n}} + \frac{|\psi_i|^2 + n^{-1}K}{1 + \psi_i} \right].
\]

Using Young’s inequality, we thus obtain the lower bound of \((n-1)G\)
\[
\geq \sum_{i} \left[ (1 + \psi_i)^{n-1} e^{-\frac{\bar{F}}{n}} \cdot \frac{|\psi_i|^2 + n^{-1}K}{1 + \psi_i} \right]^{\frac{1}{n}} = e^{-\frac{\bar{F}}{n}} (|\partial \psi|^2 + K)^{\frac{1}{n}}.
\]

\( \square \)

6.2. Computing weights. We are examining the coefficients of the inequality (6.10).

Proposition 6.9. Assume that
\[
\sigma_E \geq \sigma_i = C_{u} + \tau.
\]
Then there exists nonnegative constants \( C, C_1, C_2 \) such that
\[
\triangle \psi u \geq C_1 [A_m u_{\psi}^{1+\frac{1}{n}} + e^{H} tr_{\omega_{\psi}} \omega_{\psi}] - C_2 [u + 1 - A_{w} e^{\frac{H}{n}} u_{\psi}^{2}],
\]
\[
A_\psi := (n - 1)^{-1} e^{-\frac{H + F}{n} - \psi} \geq C |s_E|_{h_E}^{-2\alpha_0(\sigma_E) + 1} S_\varepsilon^{-\frac{\beta - 1 + \sigma_D}{n}},
\]
\[
A_{w} e^{\frac{H}{n}} \geq -C [1 + (\beta - 1 + \sigma_D) S_\varepsilon^{-\frac{1}{n}} F_{E}^2 |s_E|_{h_E}^{2\alpha_0(\sigma_E - \sigma)}].
\]

Proof. We compute the weights and coefficients in (6.7), including
\[
A_{s}, \ A_{c}, \ e^{H}, \ e^{-\frac{H + F}{n} - \psi}, \ A_{w}.
\]

Since \( e^{-\psi} \) is bounded, we see that both \( A_{s} \) and \( A_{c} \) in (6.8) are bounded.

Applying the uniform bounds of \( \varphi \) and \( F \) from Theorem 5.1, namely
\[
A_1 + \sigma_\varphi \varphi \geq F \geq \sigma_{E} \varphi + A_2
\]
to \( H = -F - \sigma_{E} \varphi + \sigma_{E} \varphi_{E} + e^{-\psi} + \sigma_{D} \log S_\varepsilon \), we get
\[
C_{1} e^{(\sigma_{E} - \sigma_{\psi}) \sigma_{E} S_\varepsilon^2 \sigma_{D}} \leq e^{H} \leq C e^{(\sigma_{E} - \sigma_{E}) \sigma_{E} S_\varepsilon^2 \sigma_{D}}.
\]

Since \( \sigma_{E} \geq \sigma_i \), we know that \( e^{(\sigma_{E} - \sigma_{E}) \sigma_{E}} \) is bounded above.

Inserting \(-F + \bar{F} = -f \) and \( f = -(\beta - 1) \log S_\varepsilon - h_\varphi - c_\tau - \log \frac{\bar{u}}{\bar{u}} \)
into \(-\frac{H + \bar{F}}{n} - \psi \), we have
\[
-\frac{H + \bar{F}}{n} - \psi = -f - \sigma_{E} \varphi + \sigma_{E} \varphi_{E} + e^{-\psi} + \sigma_{D} \log S_\varepsilon - \varphi + \phi_{E}
\]
\[
= -\frac{h_\varphi + c_\tau + \log \frac{\bar{u}}{\bar{u}} - (\sigma_{E} + 1)(\varphi - \phi_{E}) + e^{-\psi} + (\beta - 1 + \sigma_{D}) \log S_\varepsilon}{n},
\]

which has the lower bound
\[
-\frac{H + \bar{F}}{n} - \psi \geq -C \left( \frac{\sigma_{E}}{n} + 1 \right) \phi_{E} - \frac{\beta - 1 + \sigma_{D}}{n} \log S_\varepsilon.
\]

As a result, we obtain the strictly positive lower bound of \( A_{m} \) in (6.13), which depends on the upper bound of \( |s_E|_{h_E}^2 \) and \( S_\varepsilon \).
By the same proof of Lemma 4.12, we get
\[ A_w \geq -2C_2 - 2(\beta - 1 + \sigma_D)|\partial \log S| \geq -C[1 + (\beta - 1 + \sigma_D)S^{\frac{1}{2}}]. \]
Thus (6.14) is obtained by using (6.15).

6.3. **Maximum principle.** We continue the proof of Theorem 6.1. We have proved that, when \( \sigma_D \geq 1 \) and \( \sigma_E \geq \sigma_i \), it follows according to Proposition 6.9 that
\[ A_w e^{\frac{H}{\epsilon}} \geq -CS_\epsilon^{(\sigma_D-1)} |s_E|^{a_0(\sigma_E-\sigma_i)} \]
which is bounded below, near \( D \) and \( E \). Therefore, there exists a non-negative constant \( C_1, C_2 \) such that
\[ \Delta \psi u \geq C_1 u^{1+\frac{1}{2}} - C_2[u + u^{\frac{1}{2}} + 1]. \]
We are ready to apply the maximum principle to prove the gradient estimate, Theorem 6.1, when \( \sigma_D \geq 1 \).

**Theorem 6.10.** Assume that \( \Omega \) is big and semi-positive. Suppose that \( \varphi_\epsilon \) is a solution to the approximate singular scalar curvature equation (4.14). Then there exists a constant \( A_3 \) such that
\[ |s_E|^{2a_0(\sigma_E-\sigma_s)}S_\epsilon |\partial \psi|^{2} \leq A_3. \]
and \( \sigma_E \) is determined in (6.9) and (6.11). The constant \( A_3 \) depends on
\[
\begin{align*}
&\|\varphi_\epsilon\|_{\infty}, \quad \sup_X (F_{t,\epsilon} - \sigma_s \varphi_E), \quad \inf_X (F_{t,\epsilon} - \sigma_i \varphi_E), \\
&\inf_{i \neq j} R_{ij} \omega_i, \quad \inf_X R, \quad \sup_{(X,\omega)} \theta, \quad \|h_\theta + c_{t,\epsilon} + \log \frac{\omega_\theta}{\omega_{t \lambda}}\|C_{\epsilon}(\omega_i), \\
&\sup_X S_\epsilon, \quad \sup_X S_\epsilon^{\frac{1}{2}}|\partial S|_\omega, \quad \Theta_D, \quad \sup_X \varphi_E, \quad \beta, \quad n.
\end{align*}
\]

**Proof.** We proceed with the argument of the maximum principle. We assume that \( p \) is at the maximum point of \( Z \). Then the inequality (6.16) at \( p \) implies that \( u(p) \) is uniformly bounded above. But at any point \( x \in X \), \( u(x) \) is bounded above by the maximum value \( u(p) \), which means
\[ |\partial \psi|^2(x) + K \leq e^{-H(x)} u(p). \]
Therefore, using \( H = -F - \sigma_E \psi + e^{-\psi} + \sigma_D \log S \epsilon \) and the upper bound of \( F \), \( \|\varphi_\epsilon\|_{\infty}, \|e^{-\psi}\|_{\infty} \), we obtain the gradient estimate
\[
|\partial \psi|^2(x) + K \leq e^{F+\sigma_E \psi - \sigma_E \varphi_E - e^{-\psi} - \sigma_D \log S_\epsilon} u(p) \leq C|s_E|^{2(\sigma_s-\sigma_E) a_0} S_\epsilon^{\frac{1}{2}-\sigma_D}.
\]
At last, the desired weighted gradient estimate is obtained. \( \square \)
6.4. **Integration method.** In this section, we aims to improve \( \sigma_D \) to be less than 1. The weighted integral method is applied to obtain the gradient estimate. In order to explain the ideas well, we restrict ourselves in the Kähler class \( \Omega \).

**Theorem 6.11.** Assume that \( \Omega = [\omega_K] \) is Kähler. Suppose that \( \varphi_\varepsilon \) is a solution to the approximate singular scalar curvature equation (4.14). Assume that the degenerate exponent satisfies the condition (6.1), that is

\[
\sigma_D > \max\{1 - \frac{2\beta}{n + 2}, 0\}.
\]

Then there holds the gradient estimate

\[
S^\sigma_\varepsilon |\partial \varphi_\varepsilon|^2_{\omega_K} \leq A_4.
\]

The constant \( A_4 \) depends on the same dependence of \( A_3 \) in Theorem 6.10, and in additional, the Sobolev constant of \( \tilde{\omega}_t \).

**Remark 6.12.** Note \( \tilde{\omega}_t \) is equivalent to \( \omega_K \).

We first obtain the general integral inequality. Then we apply the weighted analysis similar to the proof for the Laplacian estimate in Section 3 to proceed the iteration argument.

**Proposition 6.13 (Integral inequality).** Assume that \( \Omega \) is big and semi-positive and \( \varphi_\varepsilon \) is a solution to the approximate singular scalar curvature equation (4.12). We set \( \tilde{u} := u + K_0 \). Then we have

\[
\begin{align*}
\int_X \tilde{u}^{p-1} u^{1 + \frac{1}{n} e^{[\sigma_i - (\sigma_E - 1)\phi E} S^\beta_\varepsilon - \frac{2}{n} \frac{2 + \sigma D}_n \tilde{\omega}_t^n
+ \int_X \tilde{u}^{p-2}[(p - 1)|\partial u|^2 + \tilde{u} e^{[\sigma_E - 1] \phi E} S^\sigma_\varepsilon tr_{\omega_t} \omega_t] e^{[\sigma D - 1] \phi E S^{\beta_\varepsilon - 1} \tilde{\omega}_t^n
\leq C \int_X \tilde{u}^{p-1} u^{1 + \frac{2}{n} e^{[\sigma_E - 1] \phi E} S^\beta_\varepsilon - \frac{2}{n} \frac{2 + \sigma D}_n \omega_t^n
\end{align*}
\]

The dependence of \( C \) is the same to what of \( A_3 \).

**Proof.** Substituting the differential inequality (6.12) into the identity

\[
(p - 1) \int_X \tilde{u}^{p-2} |\partial u|^2 \omega_\psi^n = \int_X \tilde{u}^{p-1} (-\Delta \psi) u \omega_\psi^n, \quad p \geq 2,
\]

we have the integral inequality

\[
\begin{align*}
\int_X \tilde{u}^{p-1} u^{1 + \frac{1}{n} e^{[\sigma_i - (\sigma_E - 1)\phi E} S^\beta_\varepsilon - \frac{2}{n} \frac{2 + \sigma D}_n \omega_\psi^n
+ \int_X \tilde{u}^{p-2}[(p - 1)|\partial u|^2 + \tilde{u} e^{[\sigma_E - 1] \phi E} tr_{\omega_t} \omega_t] \omega_\psi^n
\leq C \int_X \tilde{u}^{p-1} u^{1 + \frac{2}{n} e^{[\sigma_E - 1] \phi E} S^\beta_\varepsilon - \frac{2}{n} \frac{2 + \sigma D}_n \omega_\psi^n.
\end{align*}
\]
Inserting the volume ratio bound (5.1), \( \frac{1}{C} e^{\sigma \phi E} S_\epsilon^{\beta - 1} \leq \frac{\omega_0^n}{\omega'} \leq C e^{\sigma \phi E} S_\epsilon^{\beta - 1} \) into the inequality above, we obtain the integral inequality.

In the following sections, we focus on the degenerate scale curvature equation in a given Kähler class, where \( a_0 = t = 0 \) and \( \phi_E = 0 \). Notions introduced in Section 4.1 become

\[
\omega_{st} = \tilde{\omega}_{t=0} = \omega_{t=0} = \omega_K, \quad \omega_\psi = \omega_\varphi.
\]

The following corollary is obtained immediately from inserting the bound of \( e^H \) in (6.15) into Proposition 6.13.

**Corollary 6.14** (Integral inequality: degenerate equation). Assume that \( \Omega \) is Kähler. Then the integral inequality in Proposition 6.13 reduces to

\[
LHS_1 + LHS_2 \leq C \cdot RHS_1,
\]

where, we set

\[
LHS_1 := \int_X \tilde{u}^{p - 2}(p - 1)|\partial u|^2 + \tilde{u} S_\epsilon^{\sigma_D} \text{tr} \omega_\psi \omega_\psi^n,
\]

\[
LHS_2 := \int_X \tilde{u}^{p - 1} u^{1 + \frac{1}{n}} S_\epsilon^{\frac{\beta - 1 + \sigma_D}{n}} \omega_\psi^n,
\]

\[
RHS_1 := \int_X \tilde{u}^{p - 1}[u + 1 + u^{\frac{1}{2}} S_\epsilon^{\frac{\sigma_D - 1}{2}}] \omega_\psi^n.
\]

6.5. **Rough iteration inequality.** We now deal with the lower bound of \( LHS_1 \) by applying the Sobolev inequality. We introduce some notions for convenience,

\[
k := p\sigma_D + \beta - 1, \quad k_\sigma := \frac{\sigma_D}{2} + \beta - 1 + \sigma, \quad \tilde{\mu} := S_\epsilon^{k_\sigma} \omega_\psi^n.
\]

Clearly, \( k_\sigma = k + \sigma - q\sigma_D \) and

\[
||\tilde{u}||^{q}_{L^\infty(\tilde{\mu})} = [\int_X \tilde{u} q_{\tilde{\mu}}]^{\frac{1}{q}} = [\int_X (\tilde{u}^{p - \frac{4}{2}} S_\epsilon^{k_\sigma}) \omega_\psi^n]^{\frac{2n - 4}{2n - 2}}.
\]

**Proposition 6.15** (Rough iteration inequality). It holds

\[
LHS_0 + \sqrt{q} LHS_2 \leq C(\sqrt{q} RHS_1 + RHS_2),
\]

where, we denote \( LHS_0 := [\int_X (u \tilde{u}^{q - 1})^{\frac{1}{q}}]^{\frac{1}{q - 1}} \) and

\[
(6.17) \quad RHS_2 := \int_X u \tilde{u}^{q - 1} S_\epsilon^{\frac{\beta p - 1}{2} + \sigma} \omega_\psi^n + k_\sigma \int_X u \tilde{u}^{q - 1} S_\epsilon^{\frac{\beta p - 1 + \sigma}{2}} \omega_\psi^n.
\]

**Proof.** Applying Young’s inequality to the \( LHS_1 \) of the integral inequality, Corollary 6.14, we get

\[
LHS_1 \geq \sqrt{p - 1} \int_X \tilde{u}^{p - \frac{4}{2}} S_\epsilon^{\frac{\beta p}{2}} |\partial u| \omega_\psi^n = q^{-1} \sqrt{p - 1} \int_X |\partial u|^q S_\epsilon^{\frac{q p}{2}} \omega_\psi^n.
\]
Making use of the key Lemma 3.13, we get
\[ LHS_1 \geq q^{-1} \sqrt{p - 1} \int_X |\partial(u \tilde{u}^{q-1})| S_{k_\sigma}^{q-1} \tilde{\omega}_i^n. \]

The lower bound of the volume ratio \( F \) in (5.1) further gives
\[
LHS_1 \geq C q^{-1} \left( \frac{1}{2} \left( p - \frac{1}{2} \right) \right) \int_X |\partial(u \tilde{u}^{q-1})| S_{k_\sigma}^{\frac{q-1}{2}+\beta-1} \tilde{\omega}_i^n
\geq C q^{-\frac{1}{2}} \int_X |\partial(u \tilde{u}^{q-1})| S_{k_\sigma}^{\frac{q-1}{2}+\beta-1+\sigma} \tilde{\omega}_i^n
\geq C q^{-\frac{1}{2}} \left( \int_X |\partial(u \tilde{u}^{q-1} S_{k_\sigma}^{\tilde{\omega}_i^n})| \tilde{\omega}_i^n - \int_X u \tilde{u}^{q-1} |\partial S_{k_\sigma}^{\tilde{\omega}_i^n}| \tilde{\omega}_i^n \right).
\]

The Sobolev imbedding theorem estimates the first part,
\[
\int_X |\partial(u \tilde{u}^{q-1} S_{k_\sigma})| \tilde{\omega}_i^n \geq C S^{-1} \left[ \int_X (u \tilde{u}^{q-1} S_{k_\sigma}^{\tilde{\omega}_i^n}) \chi_{\tilde{\mu}} \right] \chi_{\tilde{\omega}_i^n} \left( \int_X u \tilde{u}^{q-1} S_{k_\sigma} \tilde{\omega}_i^n \right)
= C S^{-1} LHS_0 - \int_X u \tilde{u}^{q-1} S_{k_\sigma} \tilde{\omega}_i^n.
\]

Meanwhile, the second part is estimated by
\[-\int_X u \tilde{u}^{q-1} |\partial S_{k_\sigma}^{\tilde{\omega}_i^n}| \tilde{\omega}_i^n \geq -k_\sigma \int_X u \tilde{u}^{q-1} S_{k_\sigma}^{\tilde{\omega}_i^n - \frac{1}{2}} \tilde{\omega}_i^n.\]

In summary, we use the volume ratio bound (5.1) again and combine these inequalities together to get the lower bound
\[ LHS_1 \geq C q^{-\frac{1}{2}} \left\{ C_S^{-1} LHS_0 - RHS_2 \right\}. \]

Therefore, the required inequality is obtained by inserting the inequality of \( LHS_1 \) to the integral inequality (6.14), namely \( LHS_1 + LHS_2 \leq C \cdot RHS_1 \).

\[\square\]

**Corollary 6.16.**
\[ \|\tilde{u}\|_{L^q(\tilde{\mu})}^q + \sqrt{q} LHS_2 \leq C(\sqrt{q} RHS_1 + RHS_2 + 1). \]

**Proof.** It follows from \( u = \tilde{u} - K_0 \) that
\[ LHS_0^X = \int_X (u \tilde{u}^{q-1}) \chi \tilde{\mu} \geq C(n) \left\| \int_X \tilde{u}^{qX} \tilde{\mu} \right\| - K_0^X \left\| \int_X \tilde{u}^{(q-1)X} \tilde{\mu} \right\|. \]

By Young’s inequality, which states that
\[ \tilde{u}^{(q-1)X} \leq \frac{q - 1}{q} \tilde{u}^{qX} + \frac{1}{q} \tilde{u}^{qX} + 1, \]
if follows
\[ LHS_0^X = \int_X (u \tilde{u}^{q-1}) \chi \tilde{\mu} \geq C(n) \left( 1 - K_0^X \right) \left\| \int_X \tilde{u}^{qX} \tilde{\mu} - K_0^X \right\| \left\| \int_X \tilde{\mu} \right\|. \]
Without loss of generality, we normalise $\int_X \tilde{\mu} = 1$. Then picking small $K_0$ satisfying $K_0^{\chi} \leq \frac{1}{2}$, we get

$$LHS_0^k \geq C(n) \left[ \frac{1}{2} \int_X \tilde{u}^{q_K \tilde{\mu}} - K_0^{\chi} \right] \geq \frac{C(n)}{2} \left[ \int_X \tilde{u}^{q_K \tilde{\mu}} - 1 \right].$$

Inserting to the rough iteration inequality, Proposition 6.15, we obtain the desired result.

\[\square\]

6.6. \textit{L}^p \textit{control}. In this section, we derive the \textit{L}^p estimate of \textit{u} from the integral inequality, Corollary 6.14. We set

$$\text{LHS}_2 := \int_X \tilde{u}^{p-1} S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n}} \omega_\psi^n,$$

which will be used to bound \textit{RHS}_1.

\textbf{Proposition 6.17 (L}^p \textit{control)}. \textit{Assume that the degenerate exponent satisfies the condition (6.1). Then}

$$\text{LHS}_2 \leq \tilde{\text{LHS}}_2 \leq C \cdot \text{RHS}_1 \leq C(p), \quad \forall p \geq 2.$$

\textit{Proof}. We keep the first term on the left hand side of the integral inequality from Corollary 6.14,

$$\text{LHS}_2 = \int_X \tilde{u}^{p-1} u^{1+\frac{1}{n}} S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n-1}} \omega_\psi^n \leq CRH_1 = \int_X \tilde{u}^{p-1} [u + 1 + u \frac{1}{2} S_{\epsilon}^{\frac{\sigma_d}{n-1}}] \omega_\psi^n.$$

Replacing \(\tilde{u}\) with \(u + K_0\), we get

$$\tilde{\text{LHS}}_2 \leq C(K_0) [\text{LHS}_2 + \int_X \tilde{u}^{p-1} S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n}} \omega_\psi^n].$$

Then Young’s inequality with conjugate exponents $\frac{p+1}{p-1}$ gives

$$\int_X \tilde{u}^{p-1} S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n}} \omega_\psi^n \leq \tau \tilde{\text{LHS}}_2 + C(\tau) \int_X \tilde{u}^{p-1} S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n}} \omega_\psi^n.$$

The latter integral is bounded by a constant $C$ independent of $p$, since

$$-\frac{\beta - 1 + \sigma_d}{n} + 2(\beta - 1) + 2n > 0.$$

In conclusion, we shows that

(6.18) $\tilde{\text{LHS}}_2 \leq C(\text{LHS}_2 + 1) \leq C(\text{RHS}_1 + 1).$

In order to proceed further, we apply Young’s inequality to each term in \textit{RHS}_1. For the first term in \textit{RHS}_1, we pick the conjugate exponents $p_1 = \frac{p+1}{p}, q_1 = \frac{p+n-1}{n-1}$, then

$$\int_X \tilde{u}^p \omega_\psi^n = \int_X (\tilde{u}^p S_{\epsilon}^{\frac{\beta-1+p\sigma_d}{n-1}} \omega_\psi^n \leq \tau \tilde{\text{LHS}}_2 + C(p, \tau) \int_X S_{\epsilon}^{k_1} \omega_\psi^n.$$
The exponent $k_1$ is then computed as

$$k_1 := \frac{\beta - 1 + \sigma_D q_1}{n} = (\beta - 1 + \sigma_D)p \geq 0.$$  

Similarly, we estimate the second term as

$$\int_X \tilde{u}^{p-1} \omega_\psi^p = \int_X (\tilde{u}^{p-1} \epsilon^{\frac{\beta-1+\sigma_D}{np_2}}) \epsilon^{\frac{\beta-1+\sigma_D}{np_2}} \omega_\psi^n \leq \tau \overline{LHS}_2 + C(p, \tau) \int_X \epsilon^{k_2} \omega_\psi^n$$

with the conjugate exponents $p_2 = \frac{p+n-1}{p-1}$, $q_2 = \frac{p+n-1}{1+n}$, and

$$k_2 := \frac{(\beta - 1 + \sigma_D)(p - 1)}{n + 1} \geq 0.$$  

The third term reduces to

$$\int_X \tilde{u}^{p-1} \epsilon^{\frac{\sigma_D}{np_3}} \omega_\psi^n = \int_X (\tilde{u}^{p-1} \epsilon^{\frac{\beta-1+\sigma_D}{np_3}}) \epsilon^{\frac{\beta-1+\sigma_D}{np_3} + \frac{\sigma_D p_3 - 1}{2}} \omega_\psi^n$$

$$\leq \tau \overline{LHS}_2 + C(p, \tau) \int_X \epsilon^{k_3} \omega_\psi^n$$

with $p_3 = \frac{p+n-1}{p-1}$, $q_3 = \frac{p+n-1}{1+n}$, and

$$k_3 := \left(\frac{\beta - 1 + \sigma_D}{np_3}\right) + \frac{\sigma_D - 1}{2} q_3 = \frac{(\beta - 1)(2p - 1) - (np + 1)}{n + 2} + \sigma_D p.$$  

In order to make the last integrand $\epsilon^{k_3}$ integrable, we need

$$k_3 + 2(\beta - 1) + 2n > 0,$$

which is equivalent to

$$[(n + 2)\sigma_D + 2(\beta - 1)]p + (\beta - 1)(2n + 3) + 2n(n + 2) > np + 1.$$  

Clearly, it is sufficient to ask $(n + 2)\sigma_D + 2(\beta - 1) \geq n$. Therefore, we see that the integrable condition holds when $\sigma_D \geq 1 - \frac{2\beta}{n+2}$.  

Adding them together, we have obtained that there exists a constant $C(p)$ depending on $p$ such that

$$RHS_1 \leq 3\tau \overline{LHS}_2 + C(p, \tau) \int_X (\epsilon^{k_1} + \epsilon^{k_2} + \epsilon^{k_3}) \omega_\psi^n < C(p)$$

Inserting it into (6.18), we get

$$\overline{LHS}_2 \leq C[3\tau \overline{LHS}_2 + C(p, \tau) \int_X (\epsilon^{k_1} + \epsilon^{k_2} + \epsilon^{k_3}) \omega_\psi^n + 1]$$

Taking sufficiently small $\tau$, we obtain the estimate of $\overline{LHS}_2$ and $RHS_1$.  

\[ \square \]
6.7. Weighted inequality. In this section, we deal with the term
\[
\int_X \tilde{u}^q S_{\epsilon}^{\frac{\sigma D}{2} + \sigma - k'} \omega_n^n,
\]
which appears on the right hand side of the rough iteration inequality, Corollary 6.16. We determine the range of \(k'\) in this term such that it is bounded by \(\|u\|_{L^q(\tilde{\mu})}^q\) for some \(1 < a < \frac{2n}{2n-1}\).

**Proposition 6.18** (Weighted inequality). Assume \(n \geq 2\) and \(k' < \frac{1}{2}\). Then there exists \(1 < a < \frac{2n}{2n-1}\) such that
\[
\int_X \tilde{u}^q S_{\epsilon}^{\frac{\sigma D}{2} + \sigma - k'} \omega_n^n \leq C_5 \left\| S_{\epsilon}^{k_{s - k_s \chi - k'}} \right\|_{L^c(\tilde{\mu})}
\]
for some \(c > 2n\).

**Proof.** With the help of the bound of the volume ratio \(\tilde{F}\), we get
\[
\int_X \tilde{u}^q S_{\epsilon}^{\frac{\sigma D}{2} + \sigma - k'} \omega_n^n \leq C \int_X \tilde{u}^q S_{\epsilon}^{k_{s - k_s \chi - k'}} \omega_n^n = C \int_X \tilde{u}^q S_{\epsilon}^{k_{s - k_s \chi - k'}} \tilde{\mu}.
\]
By applying the generalisation of Hölder’s inequality with the conjugate exponents \(a, c\), it is
\[
\leq C\left\| \tilde{u} \right\|_{L^q(\tilde{\mu})}^q \left( \int_X S_{\epsilon}^{(k_{s - k_s \chi - k'})c} \tilde{\mu} \right)^{\frac{1}{c}}.
\]
As we have seen before, the last integral is finite, if we let
\[
2(k_{s - k_s \chi - k'})c + 2k_{s \chi} + 2n > 0,
\]
which is equivalent to
\[
c < 2n \frac{k_{s - k_s \chi - k'}}{k_{s \chi} + k'(2n - 1)} := c_0.
\]
The assumption \(k' < \frac{1}{2}\) infers that \(c_0 > 2n\). Consequently, the exponent \(c\) is chosen to be between \(2n\) and \(c_0\) such that \(a < \frac{2n}{2n-1}\). \(\square\)

6.8. Inverse weighted inequality. We further estimate the the right hand side of the rough iteration inequality, Corollary 6.16, which contains two parts, \(RHS_1\) and \(RHS_2\),
\[
RHS_1 = \int_X \tilde{u}^{\beta - 1} [u + 1 + u^\frac{1}{2} \omega_n^n] S_{\epsilon}^{\frac{\sigma D}{2}} \omega_n^n,
\]
\[
RHS_2 = \int_X \tilde{u}^\sigma u S_{\epsilon}^{\frac{\sigma D}{2} + \sigma} \omega_n^n + k_{s \chi} \int_X \tilde{u}^{\sigma D - 1} u S_{\epsilon}^{\frac{\sigma D}{2} + \sigma} \omega_n^n.
\]
By comparing the weights in each term on the both sides of the rough iteration inequality,
\[
-\frac{\beta - 1 + \sigma D}{n}, \quad \frac{\sigma D - 1}{2},
\]
roughly speaking, we observe that the critical cone angle is
\[
\frac{\beta - 1}{n} = \frac{1}{2}.
\]
We will see accurate proof in the next inverse weighted inequality.

We examine all five integrals in \( R\text{HS}_1 \) and \( R\text{HS}_2 \).

\[
I := \int_X \tilde{u}^{p-1} u \omega^n, \quad II := \int_X \tilde{u}^{p-1} \omega^n, \quad III := \int_X \tilde{u}^{p-1} \frac{1}{2} S_e^{\frac{n-1}{2}} \omega^n,
\]
\[
VI := \int_X \tilde{u}^{q-1} S_{e}^{\frac{\sigma D}{2} + \sigma} \omega^n, \quad V := \int_X \tilde{u}^{q-1} S_{e}^{\frac{\sigma D}{2}} \omega^n.
\]

**Proposition 6.19** (Inverse weighted inequality). Assume that the degenerate exponent satisfies \( \sigma_D < 1 \), \( \sigma = 0 \) and the condition (6.1), equivalently,

\[
\sigma_D = 0, \quad \text{when} \quad \frac{\beta - 1}{n} > \frac{1}{2};
\]
\[
\sigma_D > 1 - \frac{2\beta}{n + 2} \geq 0, \quad \text{when} \quad \frac{\beta - 1}{n} \leq \frac{1}{2}.
\]

Then there exists \( 0 < k' < \frac{1}{2} \) such that
\[
I, II, III, VI, V \leq \varepsilon L\text{HS}_2 + C(\varepsilon, n) \int_X \tilde{u}^{q} S_{e}^{\frac{\sigma D}{2} + \sigma - k'} \omega^n.
\]

**Proof.** We will apply the \( \varepsilon \)-Young’s inequality and proceed similar to the proof of Proposition 6.17, by using the positive term
\[
L\text{HS}_2 = \int_X \tilde{u}^{p-1} \frac{1}{n+2} S_{e}^{\frac{\beta - 1 + \sigma D}{n}} \omega^n.
\]
But the constant should be chosen to be independent of \( p \).

In order to estimate the first one, we decompose the first term as
\[
I = \int_X \tilde{u}^{p-1} u \omega^n
\]
\[
= \int_X (\tilde{u}^{p-1} u^{1 + \frac{1}{n+2}} S_{e}^{\frac{\beta - 1 + \sigma D}{n}})^{\frac{1}{2}} \tilde{u}^{p-1} u^{1 - (1 + \frac{1}{n+2}) S_{e}^{\frac{\beta - 1 + \sigma D}{n}}} \omega^n
\]
\[
\leq \varepsilon L\text{HS}_2 + C(\varepsilon, n) \int_X \tilde{u}^{p-1} u^{1 - \frac{b_1}{a_1} S_{e}^{k_1}} \omega^n.
\]

By the choice of the conjugate exponents \( a_1 := \frac{n+2}{n} \) and \( b_1 := \frac{n+2}{2} \), we get \( 1 - \frac{b_1}{a_1} = \frac{1}{2} \) and
\[
I \leq \varepsilon L\text{HS}_2 + C(\varepsilon, n) \int_X \tilde{u}^{q} S_{e}^{k_1} \omega^n.
\]
We further compute
\[
k_1 = \left( \frac{\beta - 1 + \sigma D}{n} \right) \frac{b_1}{a_1} = \frac{\beta - 1 + \sigma D}{2}.
\]
Due to the weighted inequality, Proposition 6.18, we need to put condition on
\[ k'_1 = \frac{\sigma_D}{2} + \sigma - k_1 = \frac{-(\beta - 1)}{2} + \sigma. \]

The second one is direct, by \( \tilde{u} \geq K_0 \),
\[ II = \int_X \tilde{u}^{p-1} \omega_{\psi}^n \leq \int_X \tilde{u}^q K_0^{-\frac{1}{2}} \omega_{\psi}^n. \]
So, the exponent is
\[ k'_2 = \frac{\sigma_D}{2} + \sigma. \]

The third term \( III \) is
\[ \int_X \tilde{u}^{p-1} u^{\frac{\sigma_D}{2}} S_\epsilon^\frac{\sigma_D-1}{n} \omega_{\psi}^n \]
\[ = \int_X (\tilde{u}^{p-1} u^{1 + \frac{1}{n}} S_\epsilon^{-\frac{\beta-1+\sigma_D}{n}} u^{\frac{\sigma_D}{2} - (1 + \frac{1}{n}) \frac{1}{a_3} S_\epsilon^{-\frac{\beta-1+\sigma_D}{n}}}) \tilde{u}^{\frac{b_3}{2} - \frac{1}{n} b_3} \omega_{\psi}^n. \]
By Young’s inequality,
\[ III \leq \epsilon LHS_2 + C(\epsilon, n) \int_X \tilde{u}^{p-1} u^{\frac{b_3}{2} - \frac{1}{n} b_3} S_\epsilon^{k_3} \omega_{\psi}^n. \]
We choose the exponents \( b_3 = \frac{3n+4}{2n+4} > 1 \) such that
\[ \frac{b_3}{2} - (1 + \frac{1}{n}) \frac{b_3}{a_3} = \frac{1}{4}. \]
Accordingly,
\[ III \leq \epsilon LHS_2 + C(\epsilon, n) \int_X \tilde{u}^{\frac{3}{2}} S_\epsilon^{k_3} \omega_{\psi}^n. \]
Since \( \tilde{u} \geq K_0 \), we have \( \tilde{u}^{-\frac{3}{2}} \leq K_0^{-\frac{3}{2}} \). Hence,
\[ III \leq \epsilon LHS_2 + C(\epsilon, n, K_0) \int_X \tilde{u}^{q} S_\epsilon^{k_3} \omega_{\psi}^n. \]
We calculate the exponent \( k_3 = [\frac{\sigma_D}{2} + (\frac{\beta-1+\sigma_D}{n}) a_3] b_3 \), which is
\[ k_3 = \frac{\sigma_D(n a_3 + 2) - 2 \sigma a_3 + 2(\beta - 1)}{2n(a_3 - 1)} = \frac{\sigma_D}{2} + \sigma - k'_3. \]
Consequently, we have
\[ k'_3 = \frac{-\sigma_D(n + 2) - 2(\beta - 1) + 2a_3}{2n(a_3 - 1)} + \sigma. \]
We estimate the fourth one,
\[ IV = \int_X \tilde{u}^{q-1} u^{\frac{\sigma_D}{2}} S_\epsilon^{\sigma_D} \omega_{\psi}^n \leq \int_X \tilde{u}^{q} S_\epsilon^{k_4} \omega_{\psi}^n. \]
where
\[ k_4 = \frac{\sigma_D}{2} + \sigma = \left(\frac{\sigma_D}{2} + \sigma\right) - k'_4, \quad k'_4 = 0. \]

The fifth term is then estimated by
\[ V \leq \epsilon LHS_2 + C(\epsilon, n) \int_X \tilde{u}^{q \sigma_S^{k_5}} \omega_n^{q}. \]

The exponent \( k_5 = [\left(\frac{\sigma_D - 1}{2} + \sigma + \beta \frac{1 + \sigma_D}{n} a_5\right)]b_5 \) is further reduced to
\[ k_5 = \frac{\sigma_D (na_5 + 2) + na_5 (2\sigma - 1) + 2(\beta - 1)}{2n(a_5 - 1)} = \left(\frac{\sigma_D}{2} + \sigma\right) - k'_5 \]
and
\[ k'_5 = \frac{-\sigma_D (n + 2) - 2(\beta - 1) + na_5}{2n(a_5 - 1)} - \frac{\sigma}{a_5 - 1}. \]

At last, we verify the conditions such that \( k'_i < \frac{1}{2}, i = 1, 2, 3, 4, 5 \).
From \( \sigma = 0 \) and \( \sigma_D < 1 \), we have
\[ k'_1 = -\left(\frac{\beta - 1}{2}\right) + \sigma < \frac{1}{2}, \quad k'_2 = \frac{\sigma_D}{2} + \sigma < \frac{1}{2}, \quad k'_4 = 0 < \frac{1}{2}. \]
By using the weight condition (6.19), namely \( \beta - 1 > \frac{n}{2} \), we have
\[ k'_3 = \frac{-2(\beta - 1) + na_3}{2n(a_3 - 1)} < \frac{-n + na_3}{2n(a_3 - 1)} = \frac{1}{2}. \]
Meanwhile, under the weight condition (6.20), i.e. \( \sigma_D > 1 - \frac{2\beta}{n+2} \), we see that
\[ k'_3 = \frac{-\sigma_D(n + 2) - 2(\beta - 1) + na_3}{2n(a_3 - 1)} < \frac{-n + na_3}{2n(a_3 - 1)} = \frac{1}{2}. \]
The expression of \( k'_5 \) is the same to \( k'_3 \) when \( \sigma = 0 \). So,
\[ k'_5 < \frac{1}{2}, \]
provided the weight conditions (6.19) and (6.20) respectively. In conclusion, simplify choosing \( k' = \max\{k'_1, k'_2, k'_3, k'_4, k'_5\} \), we arrive at our desired inverse weighted inequality. \( \square \)

6.9. Iteration inequality. In this section, we combined all inequalities in the previous sections to obtain the iteration inequality and complete the iteration scheme.

**Proposition 6.20** (Iteration inequality).
\[ \|\tilde{u}\|_{L^\chi(\tilde{\mu})} \leq Cq^{\frac{1}{\gamma}}\|\tilde{u}\|_{L^\omega(\tilde{\mu})}, \quad 1 < a < \chi. \]
Proof. Since \( q = p - \frac{1}{2} > 1 \), we obtain the rough iteration inequality Corollary 6.16, which states

\[
\|\widetilde{u}\|_{L^q(\tilde{\mu})} + \sqrt{q} LHS_2 \leq C \sqrt{q}(RHS_1 + RHS_2 + 1).
\]

Applying the inverse weighted inequality, Proposition 6.19, we have

\[
RHS_1 + RHS_2 \leq 4\tau LHS_2 + C(\tau, n) \int_X \widetilde{u}^q S_\epsilon^{\sigma - k'} \omega_\psi^n.
\]

Choosing sufficiently small \( \tau \) and inserting back to the rough iteration inequality, we get

\[
\|\widetilde{u}\|_{L^q(\tilde{\mu})}^q \leq C \sqrt{q} \left( \int_X \widetilde{u}^q S_\epsilon^{\sigma - k'} \omega_\psi^n + 1 \right).
\]

Then the weighted inequality, Proposition 6.18, implies the desired iteration inequality

\[
\|\widetilde{u}\|_{L^q(\tilde{\mu})}^q \leq C \sqrt{q}(\|\widetilde{u}\|_{L^q(\tilde{\mu})}^q + 1).
\]

Finally, we finish the proof of the gradient estimate, Theorem 6.1. We assume \( \|\widetilde{u}\|_{L^{m^a}(\tilde{\mu})} \geq 1 \) for some \( q_0 \geq \frac{3}{2} \) and rewrite the iteration inequality (6.21) by using \( \tilde{X} := \frac{\tilde{X}}{a} \),

\[
\|\widetilde{u}\|_{L^q(\tilde{\mu})} \leq C^{q-1} \tilde{q} \|\widetilde{u}\|_{L^{m^a}(\tilde{\mu})}.
\]

To proceed the iteration argument, we set \( q = \tilde{X}^m \), then we have

\[
\|\widetilde{u}\|_{L^{\tilde{X}^m}(\tilde{\mu})} \leq C \tilde{X}^{-m} \tilde{X}^{m^m} \|\widetilde{u}\|_{L^{m^a}(\tilde{\mu})} = C \tilde{X}^{-m} \tilde{X}^{-m} \|\widetilde{u}\|_{L^{m-1}(\tilde{\mu})} \leq C \tilde{X}^{-m+1} \tilde{X}^{m+1} \|\widetilde{u}\|_{L^{m^2}(\tilde{\mu})} \leq \ldots \leq C^{\sum_{i=0}^{n} \tilde{X}^{-i}} \|\widetilde{u}\|_{L^{m^{i+1}}(\tilde{\mu})}.
\]

At the final step, we choose sufficiently large \( m \) and let \( i_0 \) satisfy

\[
\tilde{q}_0 := \tilde{X}^{i_0} = q_0.
\]

Since two series in the coefficient are convergent, it remains to check the bound of the last integral \( \|\widetilde{u}\|_{L^{\tilde{q}_0}(\tilde{\mu})} \), which is equal to

\[
\|\widetilde{u}\|_{L^{\tilde{q}_0}(\tilde{\mu})} = \int_X \tilde{u}^{\tilde{q}_0} S_\epsilon^{\sigma} \omega_t^n = \int_X \tilde{u}^{\tilde{q}_0} S_\epsilon^{\sigma} \omega_t^n.
\]

In order to estimate the bound of \( \|\widetilde{u}\|_{L^{\tilde{q}_0}(\tilde{\mu})} \) from the \( L^p \) bound of \( u \) proved in Proposition 6.17, i.e.

\[
\int_X \tilde{u}^{\tilde{q}_0} S_\epsilon^{\sigma} \omega_t^n \leq C(\tilde{q}_0),
\]

we verify that \( \tilde{u}^{\tilde{q}_0} < \tilde{u}^{\tilde{q}_0 + \frac{1}{n}} \), \( \sigma = 0 \) and the weights

\[
\frac{\sigma}{2n} \frac{2n}{2n-1} > -\frac{\sigma}{n}, \quad (\beta - 1) \frac{2n}{2n-1} > \frac{n - 1}{n} (\beta - 1).
\]
Let \( m \to \infty \), we thus obtain the gradient estimate of \(|\partial \psi|^2\) from \( \tilde{u} = e^H(|\partial \psi|^2 + K) + K_0 \) and the bound of \( e^H \).

7. \( W^{2,p} \)-estimate

**Theorem 7.1** \((W^{2,p}-estimate for singular equation).\) For any \( p \geq 1 \), there exits a constant \( A_5 \) such that

\[
\int_X (\tr_{\omega_K} \omega)^p S_{\beta}^{\sigma E} S_{\beta}^{\sigma D} \omega^n \leq A_5, \quad \sigma_D > (\beta - 1) \frac{n - 2 - 2np^{-1}}{n - 1 + p^{-1}}
\]

where, the singular exponent \( \sigma_E \) is defined as following

\[
(7.1) \quad \sigma_E > 2a_0 \frac{(n - 2) \sigma_s + (b_1 - \sigma_s)p(n - 1) - 2np^{-1}}{n - 1 + p^{-1}}.
\]

and \( b_1 \) is given in (7.6) and (7.12). The constant \( A_5 \) depends on the \( L^\infty \)-estimates

\[
\sup_X (F_{t,\ell} - \sigma_s \phi_E), \quad \inf_X (F - \sigma_s \phi_E), \quad \| \phi \|_\infty
\]

and the quantities of the background metric \( \tilde{\omega}_t \),

\[
- C_{1,1} = \inf_{i \neq j} R_{iij}(\tilde{\omega}_t), \quad \| e^{-f} \|_{p_0, \tilde{\omega}^t_p}, p_0 \geq p + 1, \quad \sup_{(X, \tilde{\omega}_t)} i \partial \bar{\partial} f,
\]

\[
\sup_{X} \phi_E, \quad \sup_{(X, (1 + \epsilon) \omega_K)} \theta, \quad \inf_X R, \quad Vol([\tilde{\omega}_t]), \quad n, \quad p.
\]

**Remark 7.2.** When \( n = 2 \), we have \( \sigma_D = 0 \).

We obtain the \( W^{2,p} \)-estimate for degenerate equation as below.

**Theorem 7.3.** Suppose that \( \Omega = [\omega_K] \) is Kähler. Then the singular exponent \( \sigma_E \) vanishes and

\[
\int_X (\tr_{\omega_K} \omega)^p S_{\beta}^{\sigma E} S_{\beta}^{\sigma D} \omega^n \leq A_5, \quad \sigma_D := (\beta - 1) \frac{n - 2}{n - 1 + p^{-1}}.
\]

Moreover, written in terms of the volume element \( \omega^n_{\hat{\phi}_E} \),

\[
(7.2) \quad \int_X (\tr_{\omega_K} \omega_{\hat{\phi}_E})^p S_{\beta}^{\sigma_E} S_{\beta}^{\sigma D} \omega^n_{\hat{\phi}_E} \leq A_6.
\]

In the proof, we omit the indexes as before. We also write \( \psi := \hat{\varphi} = \varphi - \phi_E \). We will use the following notations in this section,

\[
(7.3) \quad v := \tr_{\tilde{\omega}_t} \omega_{\hat{\varphi}}, \quad \tilde{v} := v + K, \quad K \geq 0.
\]

The proof is divided into the following steps.
7.1. Differential inequality.

**Lemma 7.4.** Let the constant $-C_{1.1} = \inf_{i \neq j} R_{iijj}(\tilde{\omega}_t)$. Then

$$\Delta_\varphi \log \tilde{v} \geq -C_{1.1} \operatorname{tr}_\varphi \tilde{\omega}_t + \frac{\tilde{\Delta} F}{\tilde{v}},$$

where $\tilde{\Delta}$ is the Laplacian operator regarding to the metric $\tilde{\omega}_t$.

**Proof.** The proof of the Laplacian of $\tilde{v} = \operatorname{tr}_{\tilde{\omega}_t} \varphi + K$ is slightly different from Yau’s computation for $v = \operatorname{tr}_{\tilde{\omega}_t} \varphi$. We include the proof of $\Delta_\varphi \tilde{v}$ as below and refer to Lemma 3.4 [4] and Proposition 2.22 [28] for more references. The Laplacian of $\tilde{v}$ is given by the identity

$$\Delta_\varphi \tilde{v} = g^{ij} g^{kl} \partial_i \partial_j \partial_k g_{\varphi ij} - \operatorname{tr}_\varphi \operatorname{Ric}(\varphi) + g^{kl} \operatorname{R}^j_{\, kl}(\tilde{\omega}_t) g_{\varphi ij}.$$

By the volume ratio $\omega^n_t = e^{\tilde{v}} \tilde{\omega}_t^n$, we have

$$\operatorname{Ric}(\omega_t) = \operatorname{Ric}(\tilde{\omega}_t) - i \partial \overline{\partial} \tilde{F}$$

and then

$$\Delta_\varphi \tilde{v} = g^{ij} g^{kl} \partial_i \partial_j \partial_k g_{\varphi ij} + \tilde{\Delta} F - S(\tilde{\omega}_t) + g^{kl} \operatorname{R}^j_{\, kl}(\tilde{\omega}_t) g_{\varphi ij}.$$

Actually, it holds under the normal coordinates that

$$g^{ij} g^{kl} \partial_i \partial_j \partial_k g_{\varphi ij} = \frac{1}{1 + \varphi_{kk} \varphi_{ii}} \frac{1}{1 + \varphi_{pp}} \varphi_{ijkl} \varphi_{ijkl}.$$

By $1 + \varphi_{pp} \leq \sum_{p}(1 + \varphi_{pp}) = v \leq \tilde{v}$, we have

$$g^{ij} g^{kl} \partial_i \partial_j \partial_k g_{\varphi ij} \geq v^{-1} |\partial v|^2_\varphi \geq \tilde{v}^{-1} |\partial \tilde{v}|^2_{\varphi} = \tilde{v} |\partial \log \tilde{v}|^2_{\varphi}.$$

The term involving curvature reduces to the following inequality, by M. Paun’s trick [19],

$$- S(\tilde{\omega}_t) + g^{kl} \operatorname{R}^j_{\, kl}(\tilde{\omega}_t) g_{\varphi ij} = R_{ikkk}(\tilde{\omega}_t) \sum_{1 \leq i, k \leq n} \left[ \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} - 1 \right]$$

$$= R_{ikkk}(\tilde{\omega}_t) \sum_{1 \leq i < k \leq n} \left[ \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} - 1 \right] + R_{ikkk}(\tilde{\omega}_t) \sum_{1 \leq k < i \leq n} \left[ \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} - 1 \right].$$

By symmetric of the Riemannian curvature, it becomes

$$= R_{ikkk} \sum_{1 \leq i < k \leq n} \left[ \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} + \frac{1 + \varphi_{kk}}{1 + \varphi_{ii}} - 2 \right],$$

which is nonnegative and leads to

$$\geq -C_{1.1} \sum_{1 \leq i < k \leq n} \left[ \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} + \frac{1 + \varphi_{kk}}{1 + \varphi_{ii}} - 2 \right]$$

$$\geq -C_{1.1} \operatorname{tr}_\varphi \tilde{\omega}_t \cdot v \geq -C_{1.1} \operatorname{tr}_\varphi \tilde{\omega}_t \cdot \tilde{v}.$$

Inserting all these inequalities to the identity

$$\Delta_\varphi \log \tilde{v} = \tilde{v}^{-1} \Delta_\varphi \tilde{v} - |\partial \log \tilde{v}|^2_{\varphi},$$
we obtain that the inequality of $\triangle_{\varphi} \log \tilde{v}$. □

Furthermore, we calculate the Laplacian of $u$, which multiplies $\tilde{v}$ with the weight $e^{-H}$, i.e.

$$u = e^{-H} \tilde{v}.$$  

Then $\triangle_{\varphi} \log u = -\triangle_{\varphi} H + \triangle_{\varphi} \log \tilde{v}$. Combining with (7.4), we have

**Lemma 7.5.** $\triangle_{\varphi} \log u \geq -\triangle_{\varphi} H - C_{1,1} \text{tr}_{\varphi} \tilde{\omega}_{t} + \frac{\tilde{\Delta} F}{\tilde{v}}$.

In particular, we define $H$ as

(7.4) \hspace{1cm} H := b_{0} F + b_{1} \psi + b_{2} f, \quad b_{0} \geq 0.

**Proposition 7.6** (Differential inequality).

(7.5) \hspace{1cm} \triangle_{\varphi} u \geq A_{\theta} u \text{tr}_{\varphi} \tilde{\omega}_{t} + (A_{R} - b_{2} \triangle_{\varphi} f) u + e^{-H} \tilde{\Delta} F,$

where we set the constants

(7.6) \hspace{1cm} A_{\theta} := -b_{0} C_{u} + b_{1} - C_{1,1}, \quad A_{R} := b_{0} \inf_{X} R - b_{1} n < 0,

and choose the positive $b_{1}$ sufficiently large such that $A_{\theta} \geq 1$.

**Proof.** We use the upper bound of $\theta$ (4.4), namely $\text{tr}_{\varphi} \theta \leq C_{u} \text{tr}_{\varphi} \tilde{\omega}_{t}$, to compute the Laplacian of the auxiliary function $H$,

$$\triangle_{\varphi}(-H) = -b_{0}(\text{tr}_{\varphi} \theta - R) - b_{1}(n - \text{tr}_{\varphi} \tilde{\omega}_{t}) - b_{2} \triangle_{\varphi} f$$
$$\geq (-b_{0} C_{u} + b_{1}) \text{tr}_{\varphi} \tilde{\omega}_{t} + b_{0} R - b_{1} n - b_{2} \triangle_{\varphi} f.$$  

Adding the inequalities for $\triangle_{\varphi}(-H)$ and $\triangle_{\varphi} \log \tilde{v}$ together, we arrive at an inequality for $\log u$,

$$\triangle_{\varphi} \log u \geq A_{\theta} \text{tr}_{\varphi} \tilde{\omega}_{t} + (A_{R} - b_{2} \triangle_{\varphi} f) + \frac{\tilde{\Delta} F}{\tilde{v}}.$$  

Alternatively, rewritten in the form of $u = e^{-H} v$, it reduces to the desired inequality for $u$. □

**7.2. Integral inequality.** We integrate the differential inequality (7.5) into an integral inequality, by multiplying it with $-u^{p-1}$, $\forall p \geq 1$ and integrating over $X$ with respect to $\omega^{n}_{\varphi}$,

(7.7) \hspace{1cm} \text{LHS}_{1} + \text{LHS}_{2} \leq \int_{X} [-A_{R} + b_{2} \triangle_{\varphi} f] u^{p} \omega^{n}_{\varphi} + II.

where, we denote $II := - \int_{X} w^{p-1} e^{-H} \tilde{\Delta} F \omega^{n}_{\varphi}$,

$$\text{LHS}_{1} := (p - 1) \int_{X} u^{p-2} |\partial u|^{2} \omega^{n}_{\varphi}, \quad \text{LHS}_{2} := \int_{X} A_{\theta} u^{p} \text{tr}_{\varphi} \tilde{\omega}_{t} \omega^{n}_{\varphi}.$$
Applying the fundamental inequality $\text{tr}_\varphi \tilde{\omega}_t \geq e^{-\frac{1}{p-1}v \frac{1}{p-1}}$, we have

$$\widetilde{LHS}_2 \geq LHS_2 := \int_X u^p e^{-\frac{1}{p-1}v \frac{1}{p-1}} \omega^n.$$

Now we deal with the second term $II$, which involves $\tilde{\Delta} \tilde{F}$.

**Proposition 7.7.** Take $b_0 = p$, then

$$\frac{3}{4} \text{LHS}_1 + \widetilde{\text{LHS}}_2 \leq - \int_X A_R u^p \omega^n + \text{RHS}_2$$

where

$$\text{RHS}_2 := b_2 \int_X \Delta \varphi f u^p \omega^n + \frac{b_0 + b_2}{b_0 - 1} \int_X u^{p-1} e^{-H} \tilde{\Delta} f \omega^n$$

$$+ \int_X \frac{b_1}{b_0 - 1} u^{p-1} e^{-H} v \omega^n.$$

**Proof.** By $\omega_n = e^{\tilde{F}} \tilde{\omega}_t^n$, we get $II = - \int_X u^{p-1} e^{\tilde{F} - H} \tilde{\Delta} \tilde{F} \tilde{\omega}_t^n$, which is decomposed to

$$II = II_1 + II_2 := \frac{1}{b_0 - 1} \int_X u^{p-1} e^{\tilde{F} - H} \tilde{\Delta}[(\tilde{F} - H) + (H - b_0 \tilde{F})] \tilde{\omega}_t^n.$$

By integration by parts, the first part $II_1$ becomes

$$= - \frac{p - 1}{b_0 - 1} \int_X u^{p-2} e^{\tilde{F} - H} (\partial u, \partial(\tilde{F} - H)) \tilde{\omega}_t^n$$

$$- \frac{1}{b_0 - 1} \int_X u^{p-1} e^{\tilde{F} - H} |\partial(\tilde{F} - H)|^2 \tilde{\omega}_t^n.$$

We choose $b_0 > 1$ such that the constant before the second integral is negative. Then Hölder inequality gives us the upper bound of $II_1$,

$$II_1 \leq \frac{(p - 1)^2}{4(b_0 - 1)} \int_X u^{p-3} e^{\tilde{F} - H} |\partial u|^2 \tilde{\omega}_t^n.$$

Using $|\partial u|^2 \leq v |\partial u|^2$ and $u = e^{-H} v$, we deduce that

$$II_1 \leq \frac{(p - 1)^2}{4(b_0 - 1)} \int_X u^{p-3} e^{\tilde{F} - H} |\partial u|^2 \tilde{\omega}_t^n = \frac{p - 1}{4(b_0 - 1)} \text{LHS}_1.$$

In order to estimate $II_2$, we calculate

$$H - b_0 \tilde{F} = b_0 F + b_1 \tilde{\varphi} - b_0 \tilde{F} + b_2 f = (b_0 + b_2) f + b_1 \tilde{\varphi}$$

and

$$\tilde{\Delta}(H - b_0 \tilde{F}) = (b_0 + b_2) \tilde{\Delta} f + b_1 v - b_1 n.$$

By substitution into the part $II_2$, we get

$$II_2 = \frac{1}{b_0 - 1} \int_X u^{p-1} e^{-H}[(b_0 + b_2) \tilde{\Delta} f + b_1 v - b_1 n] \omega^n.$$
If we further choose $b_0 > 1, b_1 > 0$, the negative term could be dropped immediately. Hence, $II_2$ reduces to
\[
\leq \frac{1}{b_0 - 1} \int_X u^{p-1} e^{-H} [(b_0 + b_2)\tilde{\Delta}f + b_1 v] \omega_n^\varphi.
\]

Inserting $II_1$ and $II_2$ back to (7.7) and choosing $b_0$ depending on $p$ such that
\[
1 - \frac{p - 1}{4(b_0 - 1)} > 0,
\]
we have arrives at the desired weighted inequality. □

In order to estimate $\tilde{\Delta}f$, we need to use the upper bound of $i\partial \bar{\partial} f$.

**Lemma 7.8.** If $b_2 = 0$, then $H = b_0 F + b_1 \psi$ and
\[
L H S_2 \leq C \int_X [u^p + u^{p-1} e^{-H} + u^{p-1} e^{-H} v] \omega_n^\varphi.
\]

**Proof.** Taking $b_2 = 0$ in Lemma 7.7, we get
\[
R H S_2 = \frac{b_0}{b_0 - 1} \int_X u^{p-1} e^{-H} \tilde{\Delta}f \omega_n^\varphi + \frac{b_1}{b_0 - 1} \int_X u^{p-1} e^{-H} v \omega_n^\varphi.
\]
We make use of the particular property of $i\partial \bar{\partial} f$ in the degenerate situation as shown in Lemma 4.12, i.e. it is bounded above. So, we obtain that
\[
R H S_2 \leq \frac{b_0 n \sup_{(X, \tilde{\omega}_t)} i\partial \bar{\partial} f}{b_0 - 1} \int_X u^{p-1} e^{-H} \omega_n^\varphi + \frac{b_1}{b_0 - 1} \int_X u^{p-1} e^{-H} v \omega_n^\varphi.
\]
\[
\square
\]

**Corollary 7.9.** We further take $K = 0$ and rewrite (7.8) in the form
\[
L H S_2 = \int_X v^{p+1} \frac{1}{n - 1} \mu \leq C \int_X (v^p + v^{p-1}) e^{\frac{\tilde{F}}{n - 1}} \mu
\]
where we denote
\[
L := -pH + \tilde{F} - \frac{\tilde{F}}{n - 1}, \quad \mu = e^L \tilde{\omega}_t^n.
\]

We further treat the terms on the right hand side.

**Proposition 7.10.**
\[
\int_X v^{p+1} \frac{1}{n - 1} \mu \leq C \int_X v^{p-1} e^h \mu, \quad e^h := e^{\frac{\tilde{F}}{n - 1}} e^{\sigma_s \phi_E + \frac{n}{n - 1}}.
\]
where the constant $C$ depends on
\[
\sup_X (F - \sigma_s \phi_E), \quad A_\theta, \quad A_R, \quad b_0(p), \quad b_1, \quad \sup_{(X, \tilde{\omega}_t)} i\partial \bar{\partial} f.
\]
Proof. Applying Young’s inequality with $a = \frac{n}{n-1}$ and $b = n$, we have

$$
\int_X v^p e^{\frac{p}{n-1} \mu} = \int_X v^{(p+\frac{p}{n-1})\frac{1}{n}} v^{\frac{b}{(n-1)n}} e^{\frac{p}{n-1} \mu} 
$$

$$
\leq \tau LHS_2 + \int_X v^{p-\frac{b}{(n-1)n}} e^{\frac{p}{n-1} \mu} = \tau LHS_2 + \int_X v^{p-1} e^{\frac{p}{n-1} \mu}.
$$

Inserting in to (7.9), we get

$$
LHS_2 \leq C \left[ \int_X v^{p-1} e^{\frac{p}{n-1} \mu} + \tau LHS_2 + \int_X v^{p-1} e^{\frac{n}{n-1} \mu} \right].
$$

Choosing sufficiently small $\tau$, we obtain

$$
LHS_2 \leq C \int_X v^{p-1} (e^{\frac{p}{n-1} \mu} + e^{\frac{n}{n-1} \mu}).
$$

Recall the volume ratio estimate from (5.1), $C^{-1} e^{\sigma_i \phi - f} \leq e^{\tilde{F}} \leq Ce^{\sigma_s \phi - f}$ and $e^{-f}$ is of the order $S^{\beta - 1}$. We could compare the weights $e^{\frac{n}{n-1} \mu}$ and $e^{\frac{n}{n-1} \mu}$ and observe that both of them are estimated by $e^b$. Thus we have obtained (7.11).

Now we are ready to derive the $W^{2,p}$-estimates from (7.11). We first compute the weight $\mu = e^L$.

Lemma 7.11. We have

$$
L \leq |pb_1 - \sigma_i(pb_0 - \frac{n-2}{n-1})| \phi_E - \frac{n-2}{n-1} f + C,
$$

$$
L \geq |pb_1 - \sigma_s(pb_0 - \frac{n-2}{n-1})| \phi_E - \frac{n-2}{n-1} f - C.
$$

The constant $C$ depends on $\|\varphi\|_{\infty}$, $\sup_X (F - \sigma_s \phi_E)$, $\inf_X (F - \sigma_i \phi_E)$.

Proof. We compute that

$$
L = -pb_0 F - pb_1 \psi + \frac{n-2}{n-1} \tilde{F}
$$

$$
= (-pb_0 + \frac{n-2}{n-1} F - \frac{n-2}{n-1} f - pb_1 \varphi + pb_1 \phi_E.
$$

Making use of the bound of $F$ namely,

$$
\sigma_i \phi_E - C \leq F \leq \sigma_s \phi_E + C
$$

and the bound of $\varphi$ from Theorem 5.1, we obtain the bound of $L$.

Then we estimate $LHS_2$.

Proposition 7.12. We choose

$$
b_1 > (b_0 - \frac{n-2}{p(n-1)})\sigma_i - (1 + \frac{1}{p(n-1)})\sigma_s - \frac{2n}{p}.
$$
Then
\[ \int_X v^{p+\frac{1}{n-1}} \mu \leq C. \]

**Proof.** We apply the Hölder inequality with \( a = \frac{p+1}{n-1} \) to (7.11)
\[ \int_X v^{p+\frac{1}{n-1}} \mu \leq C \int_X v^{p-1} e^{\frac{h}{n-1}} \mu \leq C ( \int_X v^{p+\frac{1}{n-1}} \mu )^\frac{n}{n} ( \int_X e^{\frac{ah}{n-1}} \mu )^{1-\frac{n}{n}}. \]

We now estimate the integral \( \int_X e^{\frac{ah}{n-1}} \mu \). We calculate that \( \frac{a}{n-1} = \frac{p(n-1)+1}{n} \). Then we insert \( h \) (7.11) and the estimate of \( \mu \) (7.10) from Lemma 7.11 to the last integrand
\[ e^{\frac{ah}{n-1}} \mu = e^{\frac{p(n-1)+1}{n} \sigma_i \phi \omega^n} e^{\frac{n(n-1)+1}{n} \sigma_s} e^L \tilde{\omega}^n. \]

We write
\[ e^{\frac{ah}{n-1}} \mu \leq e^{k_1 \phi \omega^n + k_2 (-f)} \]
and compare the coefficients
\[ k_1 = pb_1 - (pb_0 - \frac{n-2}{n-1}) \sigma_i + (p + \frac{1}{n-1}) \sigma_s, \]
\[ k_2 = \frac{p}{n} + \frac{1}{(n-1)n} + \frac{n-2}{n-1}. \]

Since \( k_2 > 0 \), we have \( e^{k_2 (-f)} \) is bounded above. Also, \( e^{k_1 \phi \omega^n} \) is integrable, if \( k_1 + 2n > 0 \).

\[ \square \]

**Remark 7.13.** We further expand \( k_1 \),
\[ k_1 = pb_1 - (pb_0 - \frac{n-2}{n-1}) C_u + (p + \frac{1}{n-1}) C_l - (pb_0 - \frac{n-3}{n-1} + p) \tau. \]

Since \( \tau \) could be very small, it is sufficient to ask
\[ pb_1 - (pb_0 - \frac{n-2}{n-1}) C_u + (p + \frac{1}{n-1}) C_l \geq 0. \]

Choosing large \( b_1 \) as in Proposition 7.6, for example,
\[ b_1 = C_u b_0 - \frac{n}{n-1} C_l + C_{1.1} + 1, \]
we also obtain
\[ k_1 \geq -\tau (pb_0 - \frac{n-3}{n-1} + p). \]

Hence, \( e^{k_1 \phi \omega^n} \) is integral once \( \tau \) is small enough.

Therefore, we complete the proof of the \( W^{2,p} \)-estimates as following.
Proof of Theorem 7.1. We set
\[ \tilde{L} = k_1 \phi_E + k_2(-f), \]
where \( k_1, k_2 \) are will be determined in the following argument. The Hölder inequality with \( a = \frac{p+1}{n} \) gives
\[
\int_X v^p e^{L} \tilde{\omega}_{\tilde{t}}^n = \int_X v^p e^{\frac{k_1}{p+1}L - \frac{k_1}{p+1} \tilde{\omega}_{\tilde{t}}^n} \leq \left( \int_X e^{\frac{k_1}{p+1}L} \tilde{\omega}_{\tilde{t}}^n \right)^{\frac{p}{p+1}}. 
\]
By Lemma 7.11, we get
\[
-L \leq \left[ -pb_1 + \sigma_s(pb_0 - \frac{n-2}{n-1}) \right] \phi_E + \frac{n-2}{n-1} f + C. 
\]
In order to guarantee the last integral to be finite, we need
\[
\frac{1}{a-1} [ak_1 - pb_1 + \sigma_s(pb_0 - \frac{n-2}{n-1})] + 2n > 0, \\
\frac{1}{a-1} (ak_2 - \frac{n-2}{n-1}) + 2n > 0. 
\]
We calculate that \( \frac{1}{a-1} = p(n-1) \). These are exactly the hypotheses in Theorem 7.1. \( \square \)

REFERENCES

[1] T. Aoi, Y. Hashimoto, and K. Zheng, On uniform log K-stability for constant scalar curvature Kähler cone metrics, arXiv:2110.02518.
[2] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties, J. Reine Angew. Math. 751 (2019), 27–89. MR3956691
[3] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205 (2010), no. 2, 199–262. MR2746347
[4] S. Calamai and K. Zheng, Geodesics in the space of Kähler cone metrics, I, Amer. J. Math. 137 (2015), no. 5, 1149–1208. MR3405866
[5] J. Cao, H. Guenancia, and M. Paun, Variation of singular Kähler-Einstein metrics: positive Kodaira dimension, J. Reine Angew. Math. 779 (2021), 1–36. MR4319060
[6] C.-C. Chen and C.-S. Lin, Mean field equation of Liouville type with singular data: topological degree, Comm. Pure Appl. Math. 68 (2015), no. 6, 887–947. MR3340376
[7] X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (I)—A priori estimates, J. Amer. Math. Soc. 34 (2021), no. 4, 909–936. MR4301557
[8] X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (II)—Existence results, J. Amer. Math. Soc. 34 (2021), no. 4, 937–1009. MR4301558
[9] X. Chen and W. He, The complex Monge-Ampère equation on compact Kähler manifolds, Math. Ann. 354 (2012), no. 4, 1583–1600. MR2993005
[10] J.-P. Demailly and N. Pali, Degenerate complex Monge-Ampère equations over compact Kähler manifolds, Internat. J. Math. 21 (2010), no. 3, 357–405. MR2647006
[11] W. Y. Ding, Remarks on the existence problem of positive Kähler-Einstein metrics, Math. Ann. 282 (1988), no. 3, 463–471. MR967024
SINGULAR SCALAR CURVATURE EQUATIONS 69

[12] S. K. Donaldson, Kähler metrics with cone singularities along a divisor, Essays in mathematics and its applications, 2012, pp. 49–79. MR2975584

[13] P. Eyssidieux, V. Guedj, and A. Zeriahi, Singular Kähler-Einstein metrics, J. Amer. Math. Soc. 22 (2009), no. 3, 607–639. MR2505296

[14] V. Guedj, C. H. Lu, and A. Zeriahi, Pluripotential Kähler-Ricci flows, Geom. Topol. 24 (2020), no. 3, 1225–1296. MR4157554

[15] L. Li, J. Wang, and K. Zheng, Conic singularities metrics with prescribed scalar curvature: a priori estimates for normal crossing divisors, Bull. Soc. Math. France 148 (2020), no. 1, 51–97. MR4088684

[16] L. Li and K. Zheng, Generalized Matsushima’s theorem and Kähler-Einstein cone metrics, Calc. Var. Partial Differential Equations 57 (2018), no. 2, Paper No. 31, 43. MR3761174

[17] L. Li and K. Zheng, Uniqueness of constant scalar curvature Kähler metrics with cone singularities. I: reductivity, Math. Ann. 373 (2019), no. 1-2, 679–718. MR3968885

[18] G. Mondello and D. Panov, Spherical surfaces with conical points: systole inequality and moduli spaces with many connected components, Geom. Funct. Anal. 29 (2019), no. 4, 1110–1193. MR3990195

[19] M. Păun, Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds, Chinese Ann. Math. Ser. B 29 (2008), no. 6, 623–630. MR2470619

[20] J. Song and G. Tian, Canonical measures and Kähler-Ricci flow, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353. MR2869020

[21] J. Song and G. Tian, The Kähler-Ricci flow through singularities, Invent. Math. 207 (2017), no. 2, 519–595. MR3595934

[22] H. Tsuji, Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type, Math. Ann. 281 (1988), no. 1, 123–133. MR944606

[23] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411. MR480350

[24] H. Yin and K. Zheng, Expansion formula for complex Monge-Ampère equation along cone singularities, Calc. Var. Partial Differential Equations 58 (2019), no. 2, Paper No. 50, 32. MR3911741

[25] Z. Zhang, On degenerate Monge-Ampère equations over closed Kähler manifolds, Int. Math. Res. Not. (2006), Art. ID 63640, 18. MR2233716

[26] K. Zheng, Existence of constant scalar curvature Kähler cone metrics, properness and geodesic stability, arXiv:1803.09506.

[27] K. Zheng, Kähler metrics with cone singularities and uniqueness problem, Current trends in analysis and its applications, 2015, pp. 395–408. MR3496771

[28] K. Zheng, Geodesics in the space of Kähler cone metrics II: Uniqueness of constant scalar curvature Kähler cone metrics, Comm. Pure Appl. Math. 72 (2019), no. 12, 2621–2701. MR4020314

University of Chinese Academy of Sciences, Beijing 100049, P.R. China; Tongji University, Shanghai 200092, P.R. China

Email address: KaiZheng@amss.ac.cn