On Permuting Cut with Contraction

Abstract

This paper presents a cut-elimination procedure for intuitionistic propositional logic in which cut is eliminated directly, without introducing the multiple-cut rule mix, and in which pushing cut above contraction is one of the reduction steps. The presentation of this procedure is preceded by an analysis of Gentzen’s mix-elimination procedure, made in the perspective of permuting cut with contraction. It is also shown that in the absence of implication, pushing cut above contraction doesn’t pose problems for directly eliminating cut.

1 Introduction

The structural rule of contraction poses special problems for cut elimination. It is because of contraction that in the cut-elimination procedure of [1935] Gentzen replaced his rule

\[
\frac{\Gamma \vdash \Theta, A \quad A, \Delta \vdash A}{\Gamma, \Delta \vdash \Theta, \Lambda} \quad \text{(Gentzen’s cut)}
\]

by a rule derived from cut, contraction and interchange, called mix (Mischung in German),

\[
\frac{\Gamma \vdash \Theta \quad \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Theta^*, \Lambda}
\]

where \( \Theta \) and \( \Delta \) are sequences of formulae in each of which occurs at least once a formula \( A \), called the mix-formula, and \( \Theta^* \) and \( \Delta^* \) are obtained from, respectively, \( \Theta \) and \( \Delta \) by deleting all occurrences of \( A \). That cut can be eliminated is then demonstrated by eliminating mix.

Mix also solves a problem involving the structural rule of interchange. Namely, we cannot permute (Gentzen’s cut) with an interchange above the cut involving the cut formula \( A \), because Gentzen required that the cut formula be the last formula in the sequence on the right-hand side of the left premise and the first formula on the left-hand side of the right premise. However, this problem is easily solved by replacing (Gentzen’s cut) with

\[
\frac{\Gamma \vdash \Theta_1, A, \Theta_2 \quad \Delta_1, A, \Delta_2 \vdash \Lambda}{\Delta_1, \Gamma, \Delta_2 \vdash \Theta_1, A, \Theta_2} \quad \text{(cut)}
\]

which in the context of classical and intuitionistic logic doesn’t represent an essential departure from the original systems of Gentzen. We can always obtain the effect of the rule (cut) with the help of (Gentzen’s cut) and interchanges preceding and following this cut.

The special problem brought for cut elimination by contraction, because of which mix is introduced, occurs when we have to permute a cut with contraction above the cut involving the cut formula \( A \), i.e. when we push a cut above such a contraction. If a figure with a topmost cut

\[
\frac{A, A, \Delta \vdash \Lambda}{\Gamma \vdash \Theta, A \quad A, \Delta \vdash \Lambda \quad \text{contraction}}
\]

\[
\frac{\Gamma, \Delta \vdash \Theta, A}{\Gamma \vdash \Theta, A \quad \text{cut}}
\]

(*)
is replaced by the figure

\[
\begin{array}{c}
\Gamma \vdash \emptyset, A \\
\hline
\Gamma, A \vdash \emptyset, A \\
\hline
\Gamma, \Delta \vdash \Theta, \emptyset, A \\
\hline
\Gamma, A, \Delta \vdash \Theta, A \\
\hline
\end{array}
\]

\text{cut}

we have two cuts with the same cut formula \( A \) replacing a single cut with this cut formula. Of these two cuts, the upper cut has lower rank and can be eliminated by the induction hypothesis, but after this elimination is made, the remaining, lower, cut, which has now become topmost, need not have lower rank than the original cut.

On the other hand, if the following figure with a topmost mix

\[
\begin{array}{c}
\Gamma \vdash \emptyset, A \\
\hline
\Gamma, A \vdash \emptyset, A, A \\
\hline
\Gamma, \emptyset \vdash \emptyset, A, A \\
\hline
\Gamma, \Delta \vdash \Theta, \emptyset, A \\
\hline
\end{array}
\]

\text{cut}

is replaced by the figure

\[
\begin{array}{c}
\Gamma \vdash \emptyset, A \\
\hline
\Gamma, A \vdash \emptyset, A, A \\
\hline
\Gamma, \Delta \vdash \Theta, \emptyset, A \\
\hline
\end{array}
\]

\text{mix}

then the new, single, application of mix is topmost and has lower rank than the original mix.

It is sometimes assumed that Gentzen’s cut-elimination procedure is based on replacing \((\ast)\) by \((\ast\ast)\) (see [Carbone 1997], p. 285). When mix is reconstructed in terms of cut and other structural rules, Gentzen’s procedure does indeed involve pushing cut above contraction, but only as part of more complicated steps, as we shall show in Section 2 below. It might even be taken that in some of these steps cut is pushed below contraction, in the opposite direction.

When some forty years ago Lambek undertook in [1958] to eliminate cut in a contractionless sequent system, he didn’t need to bother with mix, and could eliminate cut directly. Of course, one also need not rely on mix in other contractionless systems of substructural logics that have been introduced since: namely, systems of BCK logic and linear logic.

In [1978], Szabo attempted to systematize the cut-elimination algorithm so that it can apply to a number of systems, with and without contraction. In this algorithm, when contraction is present, cut is permuted with contraction by passing from a figure like \((\ast)\) to a figure like \((\ast\ast)\) (see [Szabo 1978], Appendix C, C.19.3, p. 234, C.38.3, p. 239). To demonstrate that the figure of \((\ast\ast)\) is somehow simpler, Szabo introduced in [1978] (pp. 242-243) a measure of complexity counting the number of contractions above a cut. However, Szabo’s measure fails to show that the lower cut in \((\ast\ast)\) will have a smaller measure of complexity, as can be seen in a counterexample presented in detail in the last section of [B. 1997].

Actually, one cannot push a cut above both a contraction on the left and a contraction on the right, as the following simple counterexample shows. The figure

\[
\begin{array}{c}
\Gamma \vdash \emptyset, A, A \\
\hline
\Gamma, A \vdash \emptyset, A, A \\
\hline
\Gamma, \Delta \vdash \Theta, \emptyset, A \\
\hline
\end{array}
\]

\text{cut}

is not replaceable by a figure where all the cuts will be above all contractions. Szabo doesn’t eschew problems posed by this figure, though he requires in [1978] (p. 234) that the right rank of

\footnote{We are grateful to Andreja Prijatelj for pointing a long time ago to one of us that Szabo’s treatment of the matter is unsatisfactory.}
the cut be \( I \) if we want to diminish the left rank. Because, after permuting the cut in the figure with the contraction on the left above the right premise, we obtain

\[
\text{cut} \quad \begin{array}{ccc}
\Gamma \vdash \Theta, A, A & \text{contraction} & \Gamma \vdash \Theta, A, A \\
\Gamma \vdash \Theta, A & \text{contraction} & \Gamma \vdash \Theta, A \\
\Gamma, A, \Delta \vdash \Theta, A & \text{cut} & A, A, \Delta \vdash \Lambda \\
\Gamma, \Gamma, \Delta \vdash \Theta, \Theta, \Lambda & \text{interchanges and contractions} & \\
\Gamma, \Delta \vdash \Theta, \Lambda & \text{cut} & \\
\end{array}
\]

where the upper cut may be of the lowest possible right rank. When we next permute this upper cut with contraction on the right over the left premise, we obtain the figure

\[
\text{cut} \quad \begin{array}{ccc}
\Gamma \vdash \Theta, A, A & \text{cut} & A, A, \Delta \vdash \Lambda \\
\Gamma, A, \Delta \vdash \Theta, A, \Lambda & \text{interchanges and contractions} & A, A, \Delta \vdash \Lambda \\
\Gamma, A, \Delta, A, \Delta \vdash \Theta, A, \Lambda, \Lambda & \text{cut} & \\
\Gamma, A, \Delta \vdash \Theta, A, \Lambda & \text{interchanges and contractions} & \\
\Gamma, \Gamma, \Delta \vdash \Theta, \Theta, \Lambda & \text{cut} & \\
\Gamma, \Delta \vdash \Theta, \Lambda & \text{cut} & \\
\end{array}
\]

where the lowest cut is in the same position as the initial one.

However, this does not exclude that Szabo’s complexity measure could be replaced by another measure, presumably more complicated, which would show that the algorithm he envisaged would terminate if we have only contraction on the left in a system close to Gentzen’s \( LJ \) of [1935], i.e., in a system for intuitionistic logic. (In [Girard et al. 1992] something like this measure is computed, but the elimination of all cuts is not sought: in particular, some difficult cuts with contracted cut formulae are not eliminated.) These matters are very much tied to the particular formulation of a system. Zucker shows in [1974] (§7) that if in a system for intuitionistic logic one replaces Gentzen’s “additive”, i.e. lattice, rules for disjunction by “multiplicative” rules, a procedure such as envisaged by Szabo would not terminate.

Actually, it is not difficult to find such a measure in the absence of implication, as we show below, in Section 4. The presence of implication poses special problems, for which we shall devise a cut-elimination procedure that involves permuting contractions with other rules, and not only with cut. Such permutations of contraction were studied in [Kleene 1952], [Zucker 1974], [Minc 1996] and [Dyckhoff & Pinto 1997], but we are not aware that they have been integrated before into a cut-elimination procedure. (Among these papers only Zucker’s envisages permuting contraction with cut.)

The goal of this paper is to present this cut-elimination procedure for intuitionistic propositional logic, in which cut is directly eliminated, without passing via mix, and in which pushing cut above contraction, i.e. passing from \((*)\) to \((**)\), is a reduction step. The cut-elimination procedure of [B. 1997] also eliminates cut directly, and it involves pushing cut above contraction, but it is different and more entangled than the procedure we are going to present here. In a procedure envisaged by [Carbone 1997], reminiscent of Curry’s mix-elimination procedure (see [Curry 1963], Chapter 5, D2), cut should be directly eliminated, but without pushing it above contraction.

Although we suppose our procedure could be extended to the whole of intuitionistic predicate logic, we restrict ourselves to the propositional case, to make the exposition simpler. Anyway, our result is rather of theoretical, and not of practical, interest. If one is just interested in eliminating
cut, and does not care how exactly this is done, Gentzen’s solution based on mix is simpler. It is probably optimal.

However, our procedure may perhaps come in handy in studies of complexity of proofs. It exhibits more clearly than Gentzen’s procedure that contraction is the culprit for the hyperexponential growth of proofs in cut elimination.

Anyway, it seems worth knowing that cut can be eliminated by pushing it above contraction. If for nothing else, then to block inept criticism that would confuse “I don’t know how to eliminate cut by pushing it above contraction” with “Cut cannot be so eliminated”.

Our procedure consists of three phases. In the first phase we push contractions below all rules, including cut, except for the rule of introduction of implication on the right. Proofs where this has been accomplished are called “W-normal”. In the second phase, to reduce the rank, we push cuts above other rules, among which, because of W-normality, we don’t have any more troublesome applications of contraction, like those in (\(*\)). This phase involves essentially permuting cuts with cuts, which is a matter only implicitly and incompletely present in Gentzen’s procedure (see the comments below (2*) in Section 2, and cases (2.4), (3.7) and (3.8) in the proof of Theorem 6.1; see also the passage from (3P) to (3*P) in Section 2, the end of Section 2 and the beginning of Section 7). However, this permuting is prominent in categorial proof theory: it corresponds to associativity of composition and to bifunctoriality equalities. In the third phase, we reduce cuts to cuts of lower degree. Then we reenter the first phase of W-normalizing, and then again we go into the second phase, etc. The last phase will be a second phase where only cuts with axioms remain, and these are then eliminated.

Before describing precisely this procedure we consider in the next section (Section 2) what Gentzen’s mix-elimination procedure has to say about permuting cut with contraction when the mix rule is reconstructed in terms of cut, contraction and interchange. In Section 3 we introduce formally our variant of Gentzen’s sequent system \(LJ\) of intuitionistic propositional logic, which we call \(G\). The main difference between \(LJ\) and \(G\) is that in the latter we have rules like (cut) above, instead of (Gentzen’s cut). In Section 4 we show by a simple argument that in implicationless \(G\) we can eliminate cut by freely pushing cuts above contractions. Perhaps, as Szabo supposed, such a free policy of pushing cut above contraction leads to cut elimination in \(G\) even in the presence of implication, but we have been unable to show that indeed it does.

In the last two sections we present our cut-elimination procedure. Section 5 is devoted to W-normalizing, and Section 6 to the remaining phases of the procedure. In Section 7 we make some concluding comments.

2 Cut elimination via mix elimination in \(LJ\)

Gentzen’s mix rule is derivable in the presence of the structural rules of cut, contraction and interchange. However, for any mix of Gentzen’s system \(LJ\) of [1935]

\[
\frac{\Gamma \vdash A \quad \Delta \vdash A}{\Gamma, \Delta^* \vdash A} \quad (\text{mix})
\]

where \(A\) happens to occur in \(\Delta\) more than once, there is no unique way to reconstruct it in terms of cut, contraction and interchange. For example, the following instance of (mix)

\[
\frac{B, C \vdash A \quad A, A, D, A \vdash E}{B, C, D \vdash E} \quad \text{mix}
\]
can be reconstructed either as a number of cuts and interchanges followed by contractions:

\[
\begin{align*}
B, C & \vdash A & A, A, D, A & \vdash E \\
B, C & \vdash A & A, A, D, A & \vdash E \\
& & \vdots & \text{interchanges} \\
& & B, C, B, C, D, A & \vdash E \\
& & B, C, B, C, D, A & \vdash E \\
& & \vdots & \text{interchanges} \\
& & B, C, B, C, B, C, D & \vdash E \\
& & B, C, B, C, B, C, D & \vdash E \\
& & \vdots & \text{interchanges and contractions} \\
& & B, C, D & \vdash E
\end{align*}
\]

or as interchanges and contractions followed by a single cut:

\[
\begin{align*}
A, A, D, A & \vdash E & \text{interchange} \\
A, A, A, D & \vdash E \\
& & \vdots & \text{contractions} \\
B, C & \vdash A & A, D & \vdash E \\
& & \text{LJ cut} \\
& & B, C, D & \vdash E
\end{align*}
\]

or in many other ways intermediate between these two extremes, as for instance

\[
\begin{align*}
B, C & \vdash A & A, A, D, A & \vdash E & \text{contraction} \\
B, C & \vdash A & A, D, A & \vdash E \\
& & \text{LJ cut} \\
& & B, C, D, A & \vdash E \\
& & \vdots & \text{interchanges} \\
B, C & \vdash A & A, B, C, D & \vdash E \\
& & \text{LJ cut} \\
& & B, C, B, C, D & \vdash E \\
& & \vdots & \text{interchanges and contractions} \\
& & B, C, D & \vdash E
\end{align*}
\]

We call the first of these reconstructions, with many cuts, polytomic, while the second, with a single cut, will be monotomic. Note that in the polytomic reconstruction, and in the intermediate third reconstruction, the left premise of \text{mix} appears more than once. To pass from such reconstructions to the \text{mix} reconstructed, we have to apply a contraction principle of higher level, which permits to omit repetitions among the sequents that make the premises of a rule.

Note also that the polytomic reconstruction of a \text{mix} is not unique: one such reconstruction may be obtained from another by introducing interchanges and by permuting \text{LJ} cuts with other \text{LJ} cuts. The order of contractions in the bottom of the reconstruction is also not uniquely determined. It is possible to make this reconstruction unique by introducing an order among the rules involved in the reconstruction, the shortest way being to attack first the leftmost formula. However, there is something arbitrary in this order.

Whether Gentzen’s mixes of \text{LJ} will be reconstructed polytomically, monotomically or in some other, intermediate, way is a matter of choice. This choice is of no consequence if the goal is
just to eliminate cut by whatever means. However, if we are interested in describing exactly the cut-elimination procedure, and wish to reconstruct this procedure from the mix-elimination procedure, we will not end up by the same algorithm if we reconstruct mix always polytomically or always monotomically.

Let us now investigate when cut has to be pushed above contraction involving the cut formula in the uniform polytomic and uniform monotomic reconstructions; namely, in the reconstruction where mixes are always reconstructed polytomically and in the reconstruction where mixes are always reconstructed monotomically. We shall only consider these uniform reconstructions. (Note that passing from the monotomic to the polytomic reconstruction of a mix may itself be conceived as obtained by pushing cut above contraction.)

If the right rank of a mix is equal to 1, then this mix is just an $LJ$ cut. So we have only to consider cases where the right rank of the mix is greater than 1 (see [Gentzen 1935], Section III.3.121). The first interesting case for us is when we have

$$\Gamma \vdash A \quad \frac{A, A, \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Lambda} \text{ contraction}$$

and there are $n$ occurrences of $A$ in $\Delta$. Polytomic, (1) is reconstructed as

$$\Gamma \vdash A \quad \frac{A, A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Lambda} \text{ contraction}$$

$$\Gamma, \Delta \vdash \Lambda \quad \frac{\Gamma, \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Lambda} \text{ $LJ$ cut}$$

and interchanges and $n-1$ applications of $LJ$ cut

$$(1P) \quad \Gamma \vdash A \quad \frac{A, \Gamma, \ldots, \Gamma, \Delta^* \vdash \Lambda}{\Gamma, \Gamma, \ldots, \Gamma, \Delta^* \vdash \Lambda} \text{ $LJ$ cut}$$

and monotomically as

$$\Gamma \vdash A \quad \frac{A, A, \Delta \vdash \Lambda}{A, \Delta \vdash \Lambda} \text{ contraction}$$

$$\frac{\ldots}{\Gamma, \Delta^* \vdash \Lambda} \text{ $LJ$ cut}$$

and interchanges and contractions

$$(1M) \quad \Gamma \vdash A \quad \frac{A, \Delta^* \vdash \Lambda}{\Gamma, \Delta^* \vdash \Lambda} \text{ $LJ$ cut}$$

In [1935] (III.3.121.21) Gentzen transforms (1) into

$$\Gamma \vdash A \quad \frac{A, A, \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Lambda} \text{ mix}$$

$$(1^*)$$
Polytomically, \((1^*)\) is reconstructed as

\[
\begin{array}{c}
\Gamma \vdash A \quad A, A, \Delta \vdash \Lambda \\
\hline
\Gamma, A, \Delta \vdash \Lambda \\
\hline
\Gamma, A, \Delta \vdash \Lambda \\
\hline
\Gamma, A, \Delta \vdash \Lambda \\
\hline
\vdots
\end{array}
\]

\(LJ\) cut

interchanges

\[
\begin{array}{c}
\Gamma \vdash A \\
\hline
A, \Gamma, \Delta \vdash \Lambda \\
\hline
\Gamma, \Gamma, \Delta \vdash \Lambda \\
\hline
\Gamma, \Gamma, \Delta \vdash \Lambda \\
\hline
\vdots
\end{array}
\]

\(LJ\) cut

interchanges and \(n-1\) applications of \(LJ\) cut

\[
\begin{array}{c}
\Gamma \vdash A \quad A, \Gamma, \Gamma, \ldots, \Gamma, \Delta^* \vdash \Lambda \\
\hline
\Gamma, \Gamma, \Gamma, \ldots, \Gamma, \Delta^* \vdash \Lambda \\
\hline
\Gamma, \Delta^* \vdash \Lambda \\
\hline
\Gamma, \Delta^* \vdash \Lambda \\
\hline
\Gamma, \Delta^* \vdash \Lambda \\
\hline
\vdots
\end{array}
\]

\(LJ\) cut

interchanges and contractions

Transforming \((1P)\) into \((1^*P)\) involves pushing cut above contraction. The monotomic reconstruction \((1^*M)\) of \((1^*)\) is obtained from \((1M)\) by permuting interchanges with contractions, and transforming \((1M)\) into \((1^*M)\) does not involve pushing cut above contraction.

The next interesting case is when we have

\[
\begin{array}{c}
\Psi, \Delta \vdash A_1 \quad R \\
\hline
A, A \vdash A_2 \\
\hline
\Psi, A, \Delta \vdash A_2 \\
\hline
\Psi, \Delta \vdash A_2 \\
\hline
\end{array}
\]

\(mix\)

where \(A\) does not occur in \(\Gamma\) and either \(R\) is introduction of \(\land\) on the left, in which case \(A\) is of the form \(A_1 \land A_2\), while \(\Psi\) is either \(A_1\) or \(A_2\), and \(A_1\) is equal to \(A_2\), or \(R\) is introduction of \(\neg\) on the left, in which case \(A\) is of the form \(\neg A_1\), while \(\Psi\) and \(A_2\) are empty and \(A_1\) is \(A_1\). Polytomically, \((2)\) is reconstructed as

\[
\begin{array}{c}
\Psi, \Delta \vdash A_1 \quad R \\
\hline
A, A \vdash A_2 \\
\hline
\Psi, A, \Delta \vdash A_2 \\
\hline
\Psi, \Delta \vdash A_2 \\
\hline
\end{array}
\]

\(LJ\) cut

interchanges and \(LJ\) cuts

\[
\begin{array}{c}
\Gamma \vdash A \quad A, \Gamma, \ldots, \Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Gamma, \ldots, \Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\vdots
\end{array}
\]

\(LJ\) cut

interchanges and contractions

and monotonically as

\[
\begin{array}{c}
\Psi, \Delta \vdash A_1 \\
\hline
A, A \vdash A_2 \\
\hline
\Psi, A, \Delta \vdash A_2 \\
\hline
\Psi, \Delta \vdash A_2 \\
\hline
\end{array}
\]

\(LJ\) cut

interchanges and contractions

\[
\begin{array}{c}
\Gamma \vdash A \quad A, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Delta^* \vdash A_2 \\
\hline
\vdots
\end{array}
\]
In [1935] (III.3.121.22 and 3.121.222) Gentzen transforms (2) into

\[
\begin{array}{c}
\Gamma \vdash A \quad \Psi, \Delta \vdash A_1 \\
\hline
\Gamma, \Psi, \Delta^* \vdash A_1 \\
\vdots \\
\Psi, \Gamma, \Delta^* \vdash A_1 \\
\hline
A, \Gamma, \Delta^* \vdash A_2 \\
\hline
\Gamma, \Gamma, \Delta^* \vdash A_2 \\
\end{array}
\]

mix

\[\text{thinning or interchanges}\]

\[\text{mix (i.e. LJ cut)}\]

\[\text{interchanges and contractions}\]

When the upper mix of (2*) is reconstructed polytomically, the result of the reconstruction being called (2*P), transforming (2P) into (2*P) involves permuting LJ cuts with LJ cuts and with R. (This permuting of cut with cut corresponds to (3.8) of the proof of Theorem 6.1 below, and not to (2.4) and (3.7).) It also involves pushing contraction above cut, but it does not involve pushing cut above contraction.

When, on the other hand, the upper mix of (2*) is reconstructed monotonically, the result of the reconstruction being called (2*M), transforming (2M) into (2*M) involves, among other things, pushing cut above contraction.

The final interesting case is when we have

\[
\begin{array}{c}
\Delta \vdash B \quad C, \Theta \vdash A \\
\hline
\Gamma \vdash A \\
\hline
B \rightarrow C, \Delta, \Theta \vdash A \\
\hline
\Gamma, (B \rightarrow C)^*, \Delta^*, \Theta^* \vdash A \\
\end{array}
\]

where \(A\) does not occur in \(\Gamma\), while \((B \rightarrow C)^*\) stands either for the empty sequence or for \(B \rightarrow C\), according as \(A\) is \(B \rightarrow C\) or not, and \(A\) occurs in both \(\Delta\) and \(\Theta\). Polytomically, (3) is reconstructed as

\[
\begin{array}{c}
\Delta \vdash B \quad C, \Theta \vdash A \\
\hline
\Gamma \vdash A \\
\hline
B \rightarrow C, \Delta, \Theta \vdash A \\
\hline
\Gamma, (B \rightarrow C)^*, \Delta^*, \Theta^* \vdash A \\
\end{array}
\]

\[\rightarrow L\]

\[\text{interchanges}\]

\[\text{LJ cut}\]

\[\text{interchanges and LJ cuts}\]

\[\text{interchanges and contractions}\]
and monotonically as

\[
\frac{\Delta \vdash B \quad C, \Theta \vdash \Lambda}{\vdash B \rightarrow C, \Delta, \Theta \vdash \Lambda} \rightarrow L
\]

\[\frac{\cdots}{\vdash A, \Theta \vdash \Lambda} \text{interchanges and contractions}
\]

\[
\frac{\Gamma \vdash A \quad A, (B \rightarrow C)^*, \Theta^* \vdash \Lambda}{\vdash \Gamma, (B \rightarrow C)^*, \Delta^*, \Theta^* \vdash \Lambda} \text{LJ cut}
\]

In [1935] (III.3.121.233.1) Gentzen transforms (3) into

\[
\frac{\Gamma \vdash A \quad C, \Theta \vdash \Lambda}{\vdash \Gamma, C^*, \Theta^* \vdash \Lambda} \text{mix}
\]

\[
\frac{\Gamma \vdash A \quad \Delta \vdash B}{\vdash \Gamma, \Delta^* \vdash B} \text{mix}
\]

\[
\frac{\vdash \cdots \quad \vdash C, \Gamma, \Theta^* \vdash \Lambda}{\vdash \Gamma, \Delta^*, \Theta^* \vdash \Lambda} \rightarrow L)
\]

which if \(B \rightarrow C\) is \(A\), is continued by

\[
\frac{\Gamma \vdash A \quad B \rightarrow C, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}{\vdash \Gamma, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda} \text{mix (i.e. LJ cut)}
\]

\[
\frac{\vdash \cdots \quad \vdash \Gamma, \Delta^*, \Theta^* \vdash \Lambda}{\vdash \Gamma, \Delta^*, \Theta^* \vdash \Lambda} \text{interchanges and contractions}
\]

and if \(B \rightarrow C\) is not \(A\), is continued by

\[
\frac{B \rightarrow C, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}{\vdash \cdots \quad \vdash \Gamma, B \rightarrow C, \Delta^*, \Theta^* \vdash \Lambda} \text{interchanges and contractions}
\]

When the two top mixes of (3*) are reconstructed polytomically, the result of the reconstruction being called (3*P), transforming (3P) into (3*P), which is analogous to the transformation of (2P) into (2*P), does not involve pushing cut above contraction. When, on the other hand, the two top mixes of (3*) are reconstructed monotonically, the result of the reconstruction being called (3*M), transforming (3M) into (3*M) involves pushing cut above contraction.

So we can conclude that in the polytomic reconstruction pushing cut above contraction is involved in the first case, while in the monotonic reconstruction in the second and third case. If in the first case we favour the monotonic reconstruction, while nonuniformly, in the second and third case we favour the polytomic reconstruction, we shall never have to push cut above contraction in order to perform the steps of Gentzen’s procedure, but we shall need this pushing to pass from a monotonic reconstruction to the corresponding polytomic reconstruction.

It is worth remarking that in the polytomic reconstruction, in the second and third case we don’t only lack pushing cut above contraction, but instead, in the opposite direction, we push contraction above cut.

In the second case and in the third case when the mix formula \(A\) is \(B \rightarrow C\), let us call the lowest mix in (2*) and (3*). which is in fact an LJ cut, the critical mix of the transformation. The specificity of the critical mix is that it is the lowest mix in the figure and that its right rank is 1. In the monotonic reconstruction, the critical mix, i.e. LJ cut, originates from one of the two cuts obtained by pushing a cut above a contraction. In this pushing, which is the relativization to LJ...
of the transformation of (*) into (**) of Section 1, we must ensure that the critical mix originates in the lower cut of (**) . Otherwise, we would need to permute also cut with cut to ensure that the critical mix ends up as being the lowest cut.

3 A sequent system for intuitionistic propositional logic

Our propositional language will have the propositional constant ⊥ and the binary connectives ∧, ∨ and → . We use A, B, C, . . . , A₁, . . . as schematic letters for formulae and Γ, Δ, Θ, . . . , Γ₁, . . . as schematic letters for finite, possibly empty, sequences of formulae. As usual, ¬A can be defined as A → ⊥ . Sequents are expressions of the form Γ ⊢ A.

The sequent system G has as postulates the postulates of the sequent system Gᵣ below with all superscripts omitted. The postulates of G are named by the same names as in Gᵣ save that the superscript r is always omitted. (It would be wasteful to write these postulates twice, once for G without superscripts, and once again, just a little bit further down, for Gᵣ with the superscripts added.)

In Gentzen’s original rules of [1935] the sequence Θ in the postulates of G is always empty, both in the structural rules and in the rules for connectives. Our, more general, rules are derivable from Gentzen’s rules in the presence of the structural rule of interchange. We already replaced (Gentzen’s cut) by the present form of cut in Section 1, in order to be able to permute cut with the structural rule of interchange. We replace likewise the other rules of Gentzen by the present more general forms to be able to permute contraction with other rules, and, also, for the sake of uniformity.

As usual, we call an application of (cut) in a proof of G a cut. With this form of speech it should be kept in mind that our cuts are applications of the rule (cut) of G, and not of (Gentzen’s cut).

The degree of a cut is, as usual, the number of binary connectives in the cut formula A. The degree of a proof in G is the maximal degree among the degrees of the cuts in this proof. A proof of degree 0 can have only cuts whose cut formulae are atomic. A proof without cuts has degree 0.

To compute the rank of a cut we introduce an auxiliary sequent system we call Gᵣ. In the sequents of Gᵣ we don’t have ordinary formulae, but indexed formulae Aᵣ where A is an ordinary formula and the rank index n ≥ 1 is a natural number. To formulate the postulates of Gᵣ we introduce the following conventions. If Γᵢ is a sequence of indexed formulae, then Γᵢ₊₁ is the sequence of indexed formulae obtained by increasing by 1 every rank index in Γᵢ. (Note that here the subscript i does not stand for a single natural number: it is a schema for any natural number in the rank indices of the sequence Γᵢ.) We use Γᵢ and Γⱼ for sequences of indexed formulae that may differ only in the rank indices. When for Γᵢ and Γⱼ we write i ≤ j, that means that if in Γᵢ we find Aᵣ and at the same place in Γⱼ we find Aᵣ, then n ≤ m. Starting from Γᵢ and Γⱼ we obtain the sequence of indexed formulae Γᵢ⁺⁺₁ in the following manner: if in Γᵢ we find Aᵣ and at the same place in Γⱼ we find Aᵣ, then at the same place in Γᵢ⁺⁺₁ we put Aᵣ⁺⁺₁.

We can now give the postulates of Gᵣ, which are just indexed variants of the postulates of G:

**axioms**

(Γᵣ) A¹ ⊢ A¹

(⊥ᵣ) ⊥¹ ⊢ A¹

**structural rules**

(Cᵣ) \[ \Deltaᵢ⁻¹, Aᵣ, Bᵣ, ☐ᵢ₊₁, Aᵣ₊₁, ☐ᵢ₊₁ ⊢ Cᵣ₊₁ \]

(Wᵣ) \[ ☐ᵢ, Aᵣ, Aᵣ, ☐ᵢ₊₁, Aᵣ₊₁ ⊢ Cᵣ₊₁ \]

(Kᵣ) \[ ☐ᵢ₊₁, Aᵣ⁺⁺₁, ☐ᵢ₊₁ ⊢ Cᵣ⁺⁺₁ \]

(cutᵣ) \[ \Deltaᵢ⁺⁺₁, ☐ᵢ, ☐ᵢ₊₁, Aᵣ⁺⁺₁ ⊢ Cᵣ⁺⁺₁ \]
The left rank \(G_1997\), the indices of \(G\) above contraction. To describe this procedure we introduce the following auxiliary implicationless procedure of cut elimination, which eliminates cut directly, not via mix, and involves pushing cut below) the more general notion of rank we have just introduced, which applies to any cut, and not of rank for topmost cuts. We need, however, for our cut-elimination procedure (see Section 6 below) the more general notion of rank we have just introduced, which applies to any cut, and not only topmost cuts.

\[ \text{rules for connectives} \]

\[
\frac{\Theta^i, A^n, \Gamma^j \vdash C^k}{\Theta^{i+1}, A \land B^j, \Gamma^{j+1} \vdash C^{k+1}} \quad \frac{\Theta^i, B^n, \Gamma^j \vdash C^k}{\Theta^{i+1}, A \land B^j, \Gamma^{j+1} \vdash C^{k+1}}
\]

\[
\frac{\Gamma^i \vdash A^n}{\Gamma^{i+1} \vdash A \lor B^1} \quad \frac{\Gamma^{i+1} \vdash A \lor B^1}{\Gamma^i \vdash A^n}
\]

\[
\frac{\Delta^i \vdash A^n}{\Theta^i, B^m, \Gamma^h \vdash C^k} \quad \frac{\Theta^i, B^m, \Gamma^h \vdash C^q}{\Theta^{i+1}, A \lor B^1, \Gamma^{j+1} \vdash C^{\max(k,q)+1}}
\]

\[
\frac{\rightarrow L'}{\Theta^i, B^m, \Gamma^h \vdash C^k} \quad \frac{\Theta^i, A^n, \Gamma^j \vdash C^k}{\Theta^{i+1}, \Delta^{i+1}, A \rightarrow B^1}
\]

\[
\frac{\rightarrow R'}{\Theta^i, A^n, \Gamma^j \vdash B^m} \quad \frac{\rightarrow R'}{A^n, \Gamma^i \vdash B^m}
\]

Take a cut in \(G\), and do again in \(G^*\) the proofs of the two premises \(\Delta \vdash A\) and \(\Theta, A, \Gamma \vdash C\) of this cut exactly as they are done in \(G\), save that in \(G^*\) rank indices are taken into account. Let these two proofs in \(G^*\) prove \(\Delta^i \vdash A^n\) and \(\Theta^i, A^n, \Gamma^h \vdash C^k\). Then the rank of our cut is \(n + m\). The left rank of this cut is \(n\), and the right rank is \(m\).

A cut in \(G\) is topmost if there are no cuts above it. Gentzen computed rank only for topmost mixes, i.e. those above which there are no mixes, and his notion of rank coincides with our notion of rank for topmost cuts. We need, however, for our cut-elimination procedure (see Section 6 below) the more general notion of rank we have just introduced, which applies to any cut, and not only topmost cuts.

4 Cut elimination in implicationless \(G\)

For every proof in \(G\) in which the connective of implication \(\rightarrow\) does not occur, there is a simple procedure of cut elimination, which eliminates cut directly, not via mix, and involves pushing cut above contraction. To describe this procedure we introduce the following auxiliary implicationless sequent system called \(G^*\). (The index \(z\) stands for “Zucker”, from whose indexing of sequents in [1974], the indices of \(G^*\) are derived; a measure analogous to these indices may be found in [B. 1997].)

On the left-hand sides of the sequents of \(G^*\) we don’t have ordinary formulae, but indexed formulæ \(A^n\) where \(A\) is an ordinary implicationless formula and the contraction index \(\alpha \geq 1\) is a natural number. To formulate the postulates of \(G^*\) we use conventions analogous to those we used for \(G^*\) in the preceding section.

The postulates of \(G^*\) are the following indexed variants of the postulates of \(G\) minus the rules for implication:

\[\text{axioms}\]

\[\text{structural rules}\]

\[
A^n \vdash A \\
A^n \vdash A
\]

\[
\frac{\Delta^i, A^n, B^j, \Gamma^j \vdash C}{\Delta^i, A^n + B^j, \Gamma^j \vdash C} \quad \frac{\Theta^i, A^n, B^j, \Gamma^j \vdash C}{\Theta^i, A^n + B^j, \Gamma^j \vdash C}
\]

\[
\frac{\Theta^i, A^n, \Gamma^j \vdash C}{\Theta^i + A^n, \Gamma^j \vdash C} \quad \frac{\Theta^i, A^n, \Gamma^j \vdash C}{\Theta^i + A^n, \Gamma^j \vdash C}
\]

\[
\frac{\Delta^i \vdash A}{\Theta^i, A^n, \Gamma^j \vdash C} \quad \frac{\Theta^i, A^n, \Gamma^j \vdash C}{\Theta^i + A^n, \Gamma^j \vdash C}
\]
rules for connectives

\[
\begin{align*}
(\land L^z) & \quad \frac{\Theta^i, A^\alpha, \Gamma^j \vdash C}{\Theta^i, A \land B^\alpha, \Gamma^j \vdash C} \\
(\land R^z) & \quad \frac{\Gamma^j \vdash A}{\Gamma^{\max(i,j)} \vdash A \land B} \\
(\lor L^z) & \quad \frac{\Theta^i, A^\alpha, \Gamma^j \vdash C}{\Theta^i, B^\beta, \Gamma^h \vdash C} \\
(\lor R^z) & \quad \frac{\Gamma^i \vdash A}{\Gamma^i \vdash A \lor B}
\end{align*}
\]

As we did for applications of (cut) in \(G\), we now call cuts applications of (cut^z) in \(G^z\).

We shall prove the following theorem by eliminating cut directly and by pushing cut above contractions.

**Theorem 4.1** Every proof of \(\Pi^i \vdash C\) in \(G^z\) can be reduced to a cut-free proof of \(\Pi^j \vdash C\) where \(j \leq i\).

**Proof:** We proceed by an induction on triples \(\langle d, z, r \rangle\), lexicographically ordered, where \(d\) is the degree of a cut, \(z\) is the contraction index of the cut formula in the right premise of (cut^z) and \(r\) is the rank of the cut (rank is defined for \(G^z\) as it is defined for \(G\), via \(G^r\)). We show that every proof of \(\Pi^i \vdash C\) with a single cut, which is the last rule of the proof, can be reduced to a cut-free proof of \(\Pi^j \vdash C\) where \(j \leq i\).

(1) Suppose the rank of our cut is 2. Then our cut is covered by at least one of the following cases

\[
\begin{align*}
(1.1) \quad \frac{\pi}{A^1 \vdash A} & \quad \Theta^i, A^\alpha, \Gamma^h \vdash C \\
& \quad \Theta^i, A^\alpha, \Gamma^h \vdash C \quad \text{cut}^z
\end{align*}
\]

Then we replace this proof by the cut-free proof \(\pi\) of the right premise of cut^z.

\[
\begin{align*}
(1.2) \quad \frac{\pi}{\bot^1 \vdash A} & \quad \Theta^i, A^\alpha, \Gamma^h \vdash C \\
& \quad \Theta^i, \bot^\alpha, \Gamma^h \vdash C \quad \text{cut}^z
\end{align*}
\]

Then we replace this proof by

\[
\begin{align*}
\pi & \quad \frac{\bot^1 \vdash C}{\Theta^i, \bot^\alpha, \Gamma^1 \vdash C} \quad \text{applications of (K^z)} \\
& \quad \Theta^i, \bot^1, \Gamma^1 \vdash C
\end{align*}
\]

\[
\begin{align*}
(1.3) \quad \frac{\pi}{\Gamma^i \vdash C} & \quad C^1 \vdash C \\
& \quad \Gamma^i \vdash C \quad \text{cut}^z
\end{align*}
\]

Then we replace this proof by the cut-free proof \(\pi\) of the left premise of cut^z.

\[
\begin{align*}
\pi_2 & \quad \frac{\pi_1}{\Theta^h, \Delta^j \vdash C} \quad \text{K}^z \\
& \quad \Theta^h, \Gamma^i, \Delta^j \vdash C \quad \text{cut}^z
\end{align*}
\]

(1.4)
Then we replace this proof by the following proof

\[
\begin{array}{c}
\pi_2 \\
\Theta^h, \Delta^j \vdash C \\
\hline
\Theta^h, \Gamma^j, \Delta^j \vdash C
\end{array}
\]

(1.5)

Then we replace this proof by

\[
\begin{array}{c}
\pi_1 \hspace{1cm} \pi_2 \\
\Gamma^i \vdash A \\
\Gamma^j \vdash B \\
\hline
\Gamma^{\max(i,j)} \vdash A \land B
\end{array}
\]

\[
\begin{array}{c}
\pi \hspace{1cm} \pi \hspace{1cm} \pi \\
\Theta^h, A^\alpha, \Delta^i \vdash C \\
\Theta^h, A \land B^\alpha, \Delta^i \vdash C \\
\hline
\Theta^h, \Gamma^{\max(i,j)}, \Delta^j \vdash C
\end{array}
\]

\[
\begin{array}{c}
\land \mathbb{R}^z \\
\land \mathbb{L}^z
\end{array}
\]

Then we replace this proof by

\[
\begin{array}{c}
\pi_1 \hspace{1cm} \pi \\
\Gamma^i \vdash A \\
\hline
\Theta^h, A^\alpha, \Delta^i \vdash C
\end{array}
\]

\[
\begin{array}{c}
\pi_1 \\
\Theta^h, \Gamma^i, \Delta^i \vdash C
\end{array}
\]

(1.6)

Then we replace this proof by

\[
\begin{array}{c}
\pi \hspace{1cm} \pi_1 \hspace{1cm} \pi_2 \\
\Gamma^i \vdash A \\
\hline
\Theta^h, A^\alpha, \Delta^i \vdash C
\end{array}
\]

\[
\begin{array}{c}
\pi_2 \\
\Theta^u, B^\beta, \Delta^v \vdash C \\
\hline
\Theta^h, B^\alpha, \Delta^i \land \max(l,v) \vdash C
\end{array}
\]

\[
\begin{array}{c}
\land \mathbb{R}^z \\
\land \mathbb{L}^z
\end{array}
\]

Then we replace this proof by

\[
\begin{array}{c}
\pi_1 \\
\Theta^h, \Gamma^i, \Delta^i \vdash C
\end{array}
\]

(2.1)

We proceed analogously when \( A^\alpha \) is replaced by \( B^\alpha \).

(2)  Suppose the left rank of our cut is greater than 1. Then we have the following cases.

\[
\begin{array}{c}
\pi \\
\Delta^i \vdash D \\
\hline
\Phi^k \vdash D
\end{array}
\]

\[
\begin{array}{c}
\pi_1 \hspace{1cm} \pi_2 \\
\Theta^h, \Phi^k, \Gamma^h \vdash C \\
\Theta^j, D^\gamma, \Gamma^h \vdash C \\
\hline
\Theta^j, \Phi^k \gamma, \Gamma^h \vdash C
\end{array}
\]

\[
\begin{array}{c}
\land \mathbb{R} \\
\land \mathbb{L}^z
\end{array}
\]

where \( \mathbb{R} \) is \( C^z \), \( W^z \), \( K^z \) or \( \land \mathbb{L}^z \). Then we replace this proof by

\[
\begin{array}{c}
\pi_2 \\
\Theta^h, D^\gamma, \Gamma^h \vdash C \\
\hline
\Theta^j, \Delta^i \gamma, \Gamma^h \vdash C
\end{array}
\]

\[
\begin{array}{c}
\pi \\
\Delta^i \vdash D \\
\hline
\Theta^j, \Delta^i \gamma, \Gamma^h \vdash C
\end{array}
\]

\[
\begin{array}{c}
\Theta^j, \Phi^k \gamma, \Gamma^h \vdash C \\
\hline
\Theta^j, \Phi^k \gamma, \Gamma^h \vdash C
\end{array}
\]

\[
\begin{array}{c}
\land \mathbb{R} \\
\land \mathbb{L}^z
\end{array}
\]

13
When $R$ is $C^z$, $W^z$ or $\land L^z$, then $l = k_\gamma$, and when $R$ is $K^z$, then one index $\gamma$ of $\Phi^{k_\gamma}$ is replaced by 1 in $\Phi^I$.

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Theta^i, A^o, \Gamma^j \vdash D & \quad \Theta^i, B^\beta, \Gamma^h \vdash D \\
\Theta^{\max(i,l)}, A \lor B^{\max(a,\beta)}, \Gamma^{\max(j,h)} \vdash D & \quad \land L^z \\
\Delta^u, \Theta^{\max(i,l)\gamma}, A \lor B^{\max(a,\beta)\gamma}, \Gamma^{\max(j,h)\gamma}, \Xi^v \vdash C & \quad \Delta^u, D^\gamma, \Xi^v \vdash C \quad \text{cut}^z \\
\end{align*}
\]

(2.2) Then we replace this proof by

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Theta^i, A^o, \Gamma^j \vdash D & \quad \Delta^u, D^\gamma, \Xi^v + C \\
\Delta^u, \Theta^{\gamma}, A^{\alpha\gamma}, \Gamma^{\gamma\gamma}, \Xi^v + C & \quad \Delta^u, \Theta^{\gamma}, B^{\beta\gamma}, \Gamma^{h\gamma}, \Xi^v + C \\
\Delta^{\max(u,u)}, \Theta^{\gamma}, A^{\alpha\gamma}, \Gamma^{\gamma\gamma}, \Xi^v + C & \quad \Delta^{\max(u,u)}, \Theta^{\gamma}, B^{\beta\gamma}, \Gamma^{h\gamma}, \Xi^v + C \quad \land L^z \\
\end{align*}
\]

(3.1) Suppose the right rank of our cut is greater than 1. Then we have the following cases.

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Theta^i, A^o, \Gamma^j \vdash D & \quad \Delta^i + D \\
\Delta^i, \Theta^{\gamma}, B^{\beta}, \Gamma^{h} \vdash E & \quad \Theta^i, D^\gamma, \Gamma^{h} \vdash E \\
\Phi^I, \Delta^i + D & \quad \Phi^I, \Delta^i, \Xi^k + C \\
R & \quad \land L^z \\
\end{align*}
\]

save when $R$ is $C^z$, and when in the transformed proof $R$ can be a number of applications of ($C^z$).

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta^i + D & \quad \Theta^i, D^\gamma, \Gamma^{h} \vdash E \\
\Theta^i, \Delta^i, \Xi^k + C & \quad \Theta^i, \Delta^i, \Xi^k + C \\
\Theta^{\max(j,k)}, D^{\max(a,\beta)}, \Gamma^{\max(h,l)} + C_1 \land C_2 & \quad \land R^z \\
\end{align*}
\]

(3.2) Then we replace this proof by

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta^i + D & \quad \Theta^i, D^\gamma, \Gamma^{h} \vdash E \\
\Theta^i, \Delta^i, \Xi^k + C & \quad \Theta^i, \Delta^i, \Xi^k + C \\
\Theta^{\max(j,k)}, \Delta^i, \Xi^k + C_1 \land C_2 & \quad \land R^z \\
\end{align*}
\]

(3.3)
Then we replace this proof by

\[
\begin{align*}
\pi & \quad \pi_1 \\
\Delta^i & \vdash D \\
\Theta_{1}^{i_1}, A^\gamma, \Theta_{2}^{i_2}, D^\alpha, \Gamma^h & \vdash C \\
\Theta_{1}^{i_1}, A^\gamma, \Theta_{2}^{i_2}, \Delta^{i_1}, \Gamma^h & \vdash C \\
\Theta_{1}^{\max(i_1,k_1)}, A \lor B_{\max(\gamma,\delta)}, \Theta_{2}^{\max(i_2,k_2)} & \Delta_{\max(i_1,i_3)}, \Gamma_{\max(h,l)} \vdash C \\
\text{cut}^z & \quad \text{cut}^z \\
\forall L^2 & \\
\end{align*}
\]

We proceed analogously when \(\pi_1\) ends with \(\Theta^j, D^\alpha, \Gamma^h, A^\gamma, \Gamma^h \vdash C\) and \(\pi_2\) ends with \(\Theta^k, D^\beta, \Gamma^l, B^\delta, \Gamma^l \vdash C\).

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta^i & \vdash D \\
\Theta^j, D^\alpha, D^\beta, \Gamma^h & \vdash C \\
\Theta^j, D^\alpha, D^\beta, \Gamma^h & \vdash C \\
\Theta^j, \Delta^{i(i+\beta)}, \Gamma^h & \vdash C \\
\text{cut}^z & \quad \text{cut}^z \\
W^z & \quad \text{cut}^z \\
\end{align*}
\]

Then we replace this proof by

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta^i & \vdash D \\
\Theta^j, D^\alpha, D^\beta, \Gamma^h & \vdash C \\
\Theta^j, \Delta^{i\alpha}, \Delta^{i\beta}, \Gamma^h & \vdash C \\
\Theta^j, \Delta^{i\alpha}, \Delta^{i\beta}, \Gamma^h & \vdash C \\
\cdots & \quad \text{applications of } (C^z) \text{ and } (W^z) \\
\Theta^j, \Delta^{i\alpha+i\beta}, \Gamma^h & \vdash C \\
\text{cut}^z & \quad \text{cut}^z \\
\end{align*}
\]

In the transformed proof, the cut formula of the upper cut\(^z\) has the same degree as in the original cut, but it has a lower contraction index in the right premise (even the rank has decreased, but this is not now essential). Hence by the induction hypotheses we have a cut-free proof of \(\Theta^k, \Delta^i, D^\gamma, \Gamma^m \vdash C\) with \(k \leq j, \; l \leq i\alpha, \; \gamma \leq \beta\) and \(n \leq h\). So we obtain

\[
\begin{align*}
\pi_1 & \quad \pi \\
\Delta^i & \vdash D \\
\Theta^k, \Delta^i, D^\gamma, \Gamma^m & \vdash C \\
\Theta^k, \Delta^i, \Delta^{i\gamma}, \Gamma^m & \vdash C \\
\cdots & \quad \text{applications of } (C^z) \text{ and } (W^z) \\
\Theta^k, \Delta^{i+i\gamma}, \Gamma^m & \vdash C \\
\text{cut}^z & \\
\end{align*}
\]

where the cut formula of cut\(^z\) is again of the same degree as in the original cut, but has a lower contraction index in the right premise (its rank has perhaps increased). \(\square\)

The contraction indices of \(G^z\) are not the only possible indices we could have chosen. For example, we could replace \(\alpha + \beta\) by \(\max(\alpha, \beta) + 1\) in \((W^z)\). Whereas the original contraction index measures the number of contractions in the clusters, this new index would measure the height of clusters. (For the notion of cluster, in German Bund, see [Gentzen 1938], Section 3.41; see also [D. & P. 1999] and references therein.) A rationale for the maximum function in the indices of \(\varGamma\) and \(\Theta\) in \((\land R^z)\) and \((\forall L^z)\) may be found in the proofs of Lemma 5.3 and Theorem 5.5 below.

## 5 W-normal form

To formulate our new cut-elimination procedure for \(G\) we need to introduce the following notion of normal form. (Note that W is sometimes used as a label for thinning, also called "weakening",
while our use of this label for contraction is suggested by combinatory logic. So our terminology shouldn’t be confused with the terminology of some other authors, which may use the same terms to designate other things; cf., for example, [Mints 1996].

A proof in \( \mathcal{G} \) is called \( \text{W-normal} \) iff every application of \((W)\) in this proof is either the last rule of the proof, or it has only applications of \((W)\) below it, or it is the upper rule in the following contexts:

\[
\frac{\Theta, A, A, A, \Gamma \vdash C}{\Theta, A, A, \Gamma \vdash C} \quad \text{W}
\]

\[
\frac{\Theta, A, \Gamma \vdash C}{\Theta, A, \Gamma \vdash C} \quad \text{W}
\]

\[
\frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B} \quad \text{W}
\]

\[
\frac{\Gamma \vdash A \rightarrow B}{\rightarrow \text{R}}
\]

We also need the following terminology.

We say that an application of \((W)\) in a proof \(\pi\)

\[
\Gamma \vdash C
\]

is \textit{tied} to an occurrence \(G\) of a formula in \(\Gamma\) iff the principal formula (i.e. contracted formula) of this application of \((W)\) belongs to the cluster of \(G\) in \(\pi\). (For the notion of cluster see [Gentzen 1938], Section 3.41.)

A contraction in a proof \(\pi\) is \textit{engaged} iff it is tied to the cut formula of the right premise of some cut in \(\pi\). If the corresponding cut is immediately below the engaged contraction, then we call such a contraction \textit{directly engaged}. A contraction in \(\pi\) that is not engaged is called \textit{neutral}.

We shall prove now a series of lemmata leading to the proof of the theorem that every proof can be reduced to a \(\text{W-normal} \) proof of the same degree of the same sequent. This theorem covers the first phase of our cut-elimination procedure.

**Lemma 5.1**  

\(\text{Every segment of a proof } \pi \text{ of the form}
\)

\[
\Phi \vdash C
\]


de \(e + n\) applications of \((W)\) followed by applications of \((C)\)

\[
\Psi \vdash C
\]

\(\text{can be transformed into a segment of the form}
\)

\[
\Phi \vdash C
\]


de \(e + n\) applications of \((C)\) followed by \(e + n\)

\[
\Psi \vdash C
\]

\(\text{where } e \text{ is the number of engaged contractions of } \pi \text{ and } n \text{ the number of neutral contractions of } \pi \text{ that occur in the figures above. The degree of the transformed proof is the same as the degree of } \pi.\)

**Proof:**  

By induction on the lexicographically ordered couples \((e + n, i)\), where \(i\) is the number of applications of \((C)\) in the initial segment \(\mathcal{S}\).
(a) The segment $S$ is of the form

\[
\begin{array}{c}
\Phi \vdash C \\
\ldots \\
e + n - 1 \text{ applications of (W)} \\
\Gamma, A, A, D, \Delta \vdash C \\
\Gamma, A, D, \Delta \vdash C \\
\Gamma, D, A, \Delta \vdash C \\
\ldots \\
i - 1 \text{ applications of (C)} \\
\Psi \vdash C
\end{array}
\]

By transforming the segment beginning with $\Gamma, A, A, \Delta \vdash C$ and ending with $\Gamma, D, A, \Delta \vdash C$, the whole segment $S$ is transformed into

\[
\begin{array}{c}
\Phi \vdash C \\
\ldots \\
e + n - 1 \text{ applications of (W)} \\
\Gamma, A, A, D, \Delta \vdash C \\
\Gamma, A, D, A, \Delta \vdash C \\
\Gamma, D, A, A, \Delta \vdash C \\
\Gamma, D, A, \Delta \vdash C \\
\ldots \\
i - 1 \text{ applications of (C)} \\
\Psi \vdash C
\end{array}
\]

This transformation preserves the engagement or neutrality of the lowest contraction.

Since $\langle e + n - 1, 2 \rangle < \langle e + n, i \rangle$, by the induction hypothesis the segment beginning with $\Phi \vdash C$ and ending with $\Gamma, D, A, A, \Delta \vdash C$ can be transformed so that our whole segment becomes

\[
\begin{array}{c}
\Phi \vdash C \\
\ldots \\
\Phi' \vdash C \\
\ldots \\
e + n - 1 \text{ applications of (W)} \\
\Gamma, D, A, A, \Delta \vdash C \\
\Gamma, D, A, \Delta \vdash C \\
\ldots \\
i - 1 \text{ applications of (C)} \\
\Psi \vdash C
\end{array}
\]

Since $\langle e + n, i - 1 \rangle < \langle e + n, i \rangle$, by the induction hypothesis the segment beginning with $\Phi' \vdash C$ and ending with $\Psi \vdash C$ can be transformed so that the whole segment is brought into the desired form.
(b) The segment \( S \) is of the form

\[
\Phi \vdash C \\
\vdots \\
\Gamma', A, \Delta' \vdash C \\
\Gamma', A, \Delta' \vdash C \\
\vdots \\
\Psi \vdash C
\]

By transforming the segment beginning with \( \Gamma, A, \Delta \vdash C \) and ending with \( \Gamma', A, \Delta' \vdash C \), the whole segment \( S \) is transformed into

\[
\Phi \vdash C \\
\vdots \\
\Gamma, A, \Delta \vdash C \\
\Gamma, A, \Delta \vdash C \\
\vdots \\
\Psi \vdash C
\]

The remaining steps are analogous to the steps in (a).

A W-normal proof is called tailless iff its last rule is not an application of (W). Let \( \pi_1 \) and \( \pi_2 \) be tailless. We define inductively as follows the class of proofs \( \mathcal{C}(\pi_1, \pi_2) \):

(i) The proof \( \pi_2 \) belongs to \( \mathcal{C}(\pi_1, \pi_2) \).

(ii) If \( \pi \) belongs to \( \mathcal{C}(\pi_1, \pi_2) \), then the proof

\[
\pi_1 \pi \\
\Phi \vdash B \\
\vdots \\
\Psi \vdash B
\]

belongs to \( \mathcal{C}(\pi_1, \pi_2) \), provided that there is no occurrence of a formula in a subproof \( \pi_1 \) of \( \pi \) that belongs to the cluster of the cut formula in the right premise of the cut noted in the figure.

(iii) If \( \pi \) belongs to \( \mathcal{C}(\pi_1, \pi_2) \), then \( \pi \) followed by an application of (W) belongs to \( \mathcal{C}(\pi_1, \pi_2) \).

The application of (W) in (iii) in the definition of \( \mathcal{C}(\pi_1, \pi_2) \) is called mobile. The height of a mobile application of (W) is the number of applications of (W) and (cut) below it in the proof (we don’t count applications of (C)).

It is easy to verify that an application of (W) in a tailless subproof of a proof cannot be engaged in this proof. This fact will be useful in the proof of the following lemma.

**Lemma 5.2** For every pair of tailless proofs \( \pi_1 \) and \( \pi_2 \), every proof from \( \mathcal{C}(\pi_1, \pi_2) \) can be transformed into a W-normal proof of the same degree.
Proof: By induction on the lexicographically ordered pairs \((\kappa, \lambda)\), where \(\kappa\) is the number of engaged applications of (W) in the proof and \(\lambda\) is the sum of the heights of all mobile applications of (W) in the proof.

By the definition of \(C(\pi_1, \pi_2)\), if there is no mobile application of (W) followed immediately by a cut in a proof from \(C(\pi_1, \pi_2)\), then this proof is W-normal.

(a) Suppose our proof is of the form

\[
\begin{align*}
\pi_1 & \quad \Theta, C, C, \Delta \vdash B \\
\Gamma \vdash C & \quad \Theta, C, \Delta \vdash B \\
& \quad \Theta, \Gamma, \Delta \vdash B \\
& \quad \cdots \\
& \quad \Xi \vdash B
\end{align*}
\]

W directly engaged

cut

for \(\pi\) in \(C(\pi_1, \pi_2)\).

By pushing the directly engaged application of (W) below cut, this proof is transformed into

\[
\begin{align*}
\pi_1 & \quad \Gamma \vdash C \\
& \quad \Theta, C, C, \Delta \vdash B \\
\Gamma \vdash C & \quad \Theta, \Gamma, C, \Delta \vdash B \\
& \quad \Theta, \Gamma, \Delta \vdash B \\
& \quad \cdots \\
& \quad \Xi \vdash B \\
& \quad \cdots
\end{align*}
\]

The neutral applications of (W) mentioned above, which contract formulae from \(\Gamma\), are neutral by the proviso in (ii) of the definition of \(C(\pi_1, \pi_2)\).

By Lemma 5.1, the segment beginning with \(\Theta, \Gamma, \Delta \vdash B\) and ending with \(\Xi \vdash B\) can be transformed so that our whole proof becomes

\[
\begin{align*}
\pi_1 & \quad \Gamma \vdash C \\
& \quad \Theta, C, C, \Delta \vdash B \\
\Gamma \vdash C & \quad \Theta, \Gamma, C, \Delta \vdash B \\
& \quad \Theta, \Gamma, \Delta \vdash B \\
& \quad \cdots \\
& \quad \Xi \vdash B
\end{align*}
\]

which belongs to \(C(\pi_1, \pi_2)\) and has one engaged application of (W) less than the original proof. (Here we use the fact that no application of (W) in \(\pi_1\) can be engaged in the proof above.) So the measure of the transformed proof is \((\kappa - 1, \lambda) < (\kappa, \lambda)\). By the induction hypothesis this proof
can be transformed into a W-normal proof.

(b) Suppose our proof is of the form

$\frac{\pi_1}{\pi} \leftarrow \frac{\Theta, C, \Delta \vdash B}{\Theta', C, \Delta' \vdash B}$

W neutral or not directly engaged

$\begin{array}{c}
\Theta', \Gamma, \Delta' \vdash B \\
\vdots \\
\Xi \vdash B
\end{array}$

cut

By pushing the distinguished application of (W), which immediately follows $\pi$, below cut, this proof is transformed into

$\frac{\pi_1}{\pi} \leftarrow \frac{\Theta, C, \Delta \vdash B}{\Theta', \Gamma, \Delta' \vdash B}$

W

$\begin{array}{c}
\Theta, \Gamma, \Delta \vdash B \\
\vdots \\
\Xi \vdash B
\end{array}$

cut

where the distinguished applications of (W) in the original figure and in the transformed figure are either both neutral or both engaged. By Lemma 5.1, the segment beginning with $\Theta, \Gamma, \Delta \vdash B$ and ending with $\Xi \vdash B$ can be transformed so that our whole proof becomes

$\frac{\pi_1}{\pi} \leftarrow \frac{\Theta, C, \Delta \vdash B}{\Theta, \Gamma, \Delta \vdash B}$

cut

$\begin{array}{c}
\Theta, \Gamma, \Delta \vdash B \\
\vdots \\
\Xi \vdash B
\end{array}$

This proof belongs to $C(\pi_1, \pi_2)$ and has the same number of engaged applications of (W), but its $\lambda$ has decreased by 1. Since $\langle \kappa, \lambda - 1 \rangle < \langle \kappa, \lambda \rangle$, by the induction hypothesis our proof can be transformed into a W-normal proof. $\square$

The following lemma is covered by Lemma 12 in [Kleene 1952]. However, Kleene’s sequent system is not quite the same: interchange is only implicit in it, and his proof doesn’t cover all details we need to cover. (In his proof on p. 24, in the third illustration, Kleene assumes that the $n_1$ contractions above $A, A, \Gamma \rightarrow \Theta$ are all tied to the first $A$, whereas we cannot assume that. We could assume it only after introducing a new reduction step that transforms sequences of contractions tied to the same occurrence of a formula.)
Lemma 5.3  A proof of the form

\[
\begin{array}{c}
\pi_1 \\
\Phi \vdash C \\
\ldots \\
\Theta', A, \Gamma' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\end{array}
\begin{array}{c}
\pi_2 \\
\Psi \vdash C \\
\ldots \\
\Theta', B, \Gamma' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\end{array}
\]

where \( \pi_1 \) and \( \pi_2 \) are tailless, can be transformed into a \( W \)-normal proof, of the same degree, of the form

\[
\begin{array}{c}
\pi \\
\Xi \vdash C \\
\ldots \\
\Theta', A \lor B, \Gamma \vdash C \\
\end{array}
\]

where \( \pi \) is tailless, and for every occurrence \( G \) of a formula in \( \Theta \), if above the left premise of \( \lor L \) in the former figure there are \( k_1 \) applications of \( (W) \) tied to \( G \), and if above the right premise of \( \lor L \) in the former figure there are \( k_2 \) applications of \( (W) \) tied to this same \( G \), then in the latter figure there are \( \max(k_1, k_2) \) applications of \( (W) \) tied to \( G \). The same holds for occurrences of formulae in \( \Gamma \).

Proof:  Let \( n \) be the number of applications of \( (W) \) tied to \( A \) in the left premise of \( \lor L \) and \( m \) be the number of applications of \( (W) \) tied to \( B \) in the right premise of \( \lor L \) in the figure of the initial proof. We prove the lemma by induction on \( n + m \). Our proof is first transformed into

\[
\begin{array}{c}
\pi_1 \\
\Phi \vdash C \\
\ldots \\
\Theta', A, \ldots, A, \Gamma' \vdash C \\
\Theta', A, \Gamma' \vdash C \\
\Theta', A \lor B, \Gamma' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\end{array}
\begin{array}{c}
\pi_2 \\
\Psi \vdash C \\
\ldots \\
\Theta', B, \ldots, B, \Gamma' \vdash C \\
\Theta', B, \Gamma' \vdash C \\
\Theta', A \lor B, \Gamma' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\end{array}
\]

Note that this step involves permuting applications of \( (W) \) one with another. In the sequence of applications of \( (W) \) below the sequent \( \Theta', A \lor B, \Gamma' \vdash C \) there are \( \max(k_1, k_2) \) applications of \( (W) \) tied to \( G \) from \( \Theta \), where \( G, k_1 \) and \( k_2 \) are as in the formulation of the lemma.

If \( n + m = 0 \), then this proof is \( W \)-normal.
If \( n > 0 \), then our proof is transformed into

\[
\frac{\pi_1}{\Phi \vdash C} \quad \frac{\Psi \vdash C}{\ldots} \quad \text{applications of \((K)\)} \\
\frac{\Theta', A, \ldots, A, \Gamma' \vdash C}{\ldots} \quad \frac{n - 1 \text{ appl. of \((W)\)}}{\Theta', A, A, \Gamma' \vdash C} \quad \Theta', A, A, \Gamma' \vdash C \\
\Theta', A, A \lor B, \Delta' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C
\]

Consider the subproof whose endsequent is \( \Theta', A, A \lor B, \Gamma' \vdash C \). Its measure is \( n_2 + m \), where \( n_2 \) is the number of applications of \((W)\) tied to the right \( A \) in the left premise of the last rule of this subproof. The number of applications of \((W)\) tied to the left \( A \) of the same sequent is \( n_1 \) and we have \( n_1 + n_2 = n - 1 \). We apply the induction hypothesis to this subproof, and therefore our proof is transformed into

\[
\frac{\pi_2}{\Psi \vdash C} \quad \frac{\Theta', B, \ldots, B, \Gamma' \vdash C}{\ldots} \quad \text{applications of \((K)\)} \\
\frac{\Theta', B, \ldots, B, A \lor B, \Gamma' \vdash C}{\ldots} \quad \frac{m \text{ appl. of \((W)\)}}{\Theta', B, A \lor B, \Gamma' \vdash C} \quad \Theta', B, A \lor B, \Gamma' \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C \\
\Theta, A \lor B, \Gamma \vdash C
\]

where in the subproofs whose endsequent is \( \Theta', A, A \lor B, \Gamma' \vdash C \) there are only \( n_1 \) applications of \((W)\) tied to \( A \) and there are no applications of \((W)\) tied to any occurrence of a formula in \( \Theta' \) and \( \Gamma' \). Consider the subproof whose endsequent is \( \Theta', A \lor B, \Gamma \vdash C \). Its measure is \( n_1 + m \), and by applying the induction hypothesis, our proof is transformed into a \( W \)-normal proof with \( \max(k_1, k_2) \) applications of \((W)\) tied to \( \Gamma \) from \( \Theta \) in its endsequent.

We proceed quite analogously if \( m > 0 \). \( \square \)

**Lemma 5.4** A proof of the form

\[
\frac{\pi_1}{\Delta' \vdash A} \quad \frac{\pi_2}{\Phi \vdash C} \quad \frac{\Delta \vdash A}{\Theta, \Delta, A \rightarrow B, \Gamma \vdash C} \rightarrow L
\]

\( \rightarrow L \)
where $\pi_1$ and $\pi_2$ are tailless, can be transformed into a $W$-normal proof, of the same degree, of the form

$$
\begin{array}{c}
\pi \\
\Xi \vdash C \\
\vdots \\
\Theta, \Delta, A \rightarrow B, \Gamma \vdash C
\end{array}
$$

where $\pi$ is tailless, and for every occurrence $G$ of a formula in $\Theta$ if above the right premise of $\rightarrow L$ in the former figure there are $k$ applications of $(W)$ tied to $G$, then in the latter figure there are $k$ applications of $(W)$ tied to $G$. The same holds for occurrences of formulae in $\Gamma$.

**Proof:** Let $n$ be the number of applications of $(W)$ tied to $B$ in the right premise of $\rightarrow L$ in the figure of the initial proof, and let the total number of applications of $(W)$ above this premise be $l$. We prove the lemma by induction on $n$. Our proof is first transformed into

$$
\begin{array}{c}
\pi_2 \\
\Phi \vdash C \\
\vdots \\
\Theta', B, B, \Gamma' \vdash C
\end{array}
$$

Note that this step involves permuting applications of $(W)$ one with another. The transformed proof is next transformed into

$$
\begin{array}{c}
\pi_2 \\
\Phi \vdash C \\
\vdots \\
\Theta', B, B, \Gamma' \vdash C
\end{array}
$$

By the induction hypothesis, there is a $W$-normal proof

$$
\begin{array}{c}
\pi \\
\Psi \vdash C \\
\vdots \\
\Theta', \Delta', A \rightarrow B, B, \Gamma' \vdash C
\end{array}
$$

where $\pi$ is tailless, and where there are $m \leq n - 1$ applications of $(W)$ tied to $B$ in endsequent, and no application of $(W)$ tied to formulae in $\Theta'$ and $\Gamma'$. 

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We apply again the induction hypothesis to

\[
\pi \\
\Psi \vdash C \\
\ldots \text{ applications of (W)} \\
\Delta' \vdash A \\
\Theta', \Delta', A \rightarrow B, B, \Gamma' \vdash C \\
\Theta', \Delta', A \rightarrow B, \Delta', A \rightarrow B, \Gamma' \vdash C \\
\rightarrow \text{L}
\]

and we use Lemma 5.1 to obtain a W-normal proof of \(\Theta, \Delta, A \rightarrow B, \Gamma \vdash C\). In the final proof there are still only \(l - n\) applications of (W) tied to formulae in \(\Theta\) and \(\Gamma\).

\[\square\]

We can now prove the theorem that covers the first phase of our cut-elimination procedure.

**Theorem 5.5** Every proof of a sequent in \(\mathcal{G}\) can be reduced to a W-normal proof of the same degree of the same sequent.

**Proof:** We proceed by induction on the length of the proof of our sequent in \(\mathcal{G}\).

If our proof is just an axiom, then this proof is W-normal.

If our sequent is proved by the following proof

\[
\pi \\
\Delta, A, B, \Gamma \vdash C \\
\Delta, B, A, \Gamma \vdash C
\]

then, by the induction hypothesis, there is a W-normal proof

\[
\pi' \\
\Lambda \vdash C \\
\ldots \text{ applications of (W)} \\
\Delta, A, B, \Gamma \vdash C
\]

where \(\pi'\) is tailless. Then we apply Lemma 5.1.

If our sequent is proved by the following proof

\[
\pi \\
\Theta, A, A, \Gamma \vdash C \\
\Theta, A, \Gamma \vdash C
\]

then, by the induction hypothesis, there is a W-normal proof

\[
\pi' \\
\Theta, A, A, \Gamma \vdash C \\
\Theta, A, \Gamma \vdash C
\]

If our sequent is proved by the following proof

\[
\pi \\
\Theta, \Gamma \vdash C \\
\Theta, A, \Gamma \vdash C
\]

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then, by the induction hypothesis, there is a W-normal proof

\[
\pi'
\quad \frac{\Lambda \vdash C}{\Theta', \Gamma \vdash C}
\]

\[
\ldots \quad \text{applications of (W)}
\]

where \( \pi' \) is tailless. Applications of (W) can be permuted with K as follows:

\[
\frac{\Theta, \Gamma \vdash C}{\Theta', \Gamma' \vdash C}
\]

\[
W
\]

\[
\frac{\Theta, A, \Gamma' \vdash C}{\Theta', A, \Gamma' \vdash C}
\]

\[
K
\]

\[
\frac{\Theta', A, \Gamma' \vdash C}{\Theta', A, \Gamma' \vdash C}
\]

And, by induction on the number of applications of (W) below \( \pi' \), we prove that there is a W-normal proof of \( \Theta, A, \Gamma \vdash C \).

If our sequent is proved by the following proof

\[
\pi
\quad \frac{\Delta \vdash A}{\Theta, \Delta, \Gamma \vdash C}
\]

\[
\rho
\quad \frac{\Theta, A, \Gamma \vdash C}{\Theta, \Delta, \Gamma \vdash C}
\]

\[\text{cut}\]

then, by the induction hypothesis, we have a proof

\[
\pi'
\quad \frac{\Lambda \vdash A}{\Theta, \Delta, \Gamma \vdash C}
\]

\[
\rho'
\quad \frac{\Phi \vdash C}{\Theta, A, \Gamma \vdash C}
\]

\[\text{cut}\]

\[
\ldots \quad \text{applications of (W)}
\]

\[
\ldots \quad \text{applications of (W)}
\]

where \( \pi' \) and \( \rho' \) are tailless. We push below cut all the applications of (W) below \( \pi' \) so as to obtain

\[
\rho'
\quad \frac{\Phi \vdash C}{\Theta, A, \Gamma \vdash C}
\]

\[\text{cut}\]

\[
\ldots \quad \text{applications of (W)}
\]

\[
\ldots \quad \text{applications of (W)}
\]

Then this proof belongs to \( C(\pi', \rho') \), and we can apply Lemma 5.2.

If our sequent is proved by the following proof

\[
\pi
\quad \frac{\Theta, A, \Gamma \vdash C}{\Theta, A \land B, \Gamma \vdash C}
\]

\[\land L\]
then, by the induction hypothesis, there is a W-normal proof

\[
\begin{array}{c}
\pi \\
\Lambda \vdash C \\
\vdots \\
\Theta, A, \Gamma \vdash C
\end{array}
\]

where \( \pi \) is tailless. Applications of (W) can be permuted with (\&L) as follows

\[
\begin{array}{ll}
\Theta, A, \Gamma \vdash C & \Theta, A, \Gamma \vdash C \\
\Theta', A, \Gamma' \vdash C & \Gamma, A \& B, \Gamma' \vdash C \\
\Theta', A \& B, \Gamma' \vdash C & \Theta, A \& B, \Gamma' \vdash C \\
\Theta, \Gamma \vdash A, \Gamma' \vdash C & \Theta, \Gamma \vdash C, \Gamma' \vdash A \\
\end{array}
\]

\[
\begin{array}{ll}
\Theta, A, \Gamma \vdash C & \Theta, A, \Gamma \vdash C \\
\Theta', A, \Gamma' \vdash C & \Gamma, A \& B, \Gamma' \vdash C \\
\Theta', A \& B, \Gamma' \vdash C & \Theta, A \& B, \Gamma' \vdash C \\
\Theta, \Gamma \vdash A \& B, \Gamma' \vdash C & \Theta, \Gamma \vdash C \& B, \Gamma' \vdash A \\
\end{array}
\]

And, by an induction analogous to that in the proof of Lemma 5.1, we show that there is a W-normal proof of \( \Theta, A \& B, \Gamma \vdash C \). We proceed analogously for the other (\&L) rule, involving \( B \).

If our sequent is proved by the following proof

\[
\begin{array}{c}
\pi \\
\Gamma \vdash A \\
\rho \\
\Gamma \vdash B \\
\Gamma \vdash A \& B
\end{array}
\]

then, by the induction hypothesis, there are W-normal proofs

\[
\begin{array}{l}
\pi' \\
\Gamma' \vdash A \\
\vdots \\
\Gamma \vdash A
\end{array}
\]

\[
\begin{array}{l}
\rho' \\
\Gamma'' \vdash B \\
\vdots \\
\Gamma \vdash B
\end{array}
\]

where \( \pi' \) and \( \rho' \) are tailless. Then we have the W-normal proof

\[
\begin{array}{c}
\pi' \\
\Gamma' \vdash A \\
\vdots \\
\Gamma'' \vdash A \\
\cdots \\
\Gamma \vdash A \& B \\
\vdots \\
\Gamma \vdash A \& B
\end{array}
\]

If our sequent is proved by the following proof

\[
\begin{array}{c}
\pi \\
\Theta, A, \Gamma \vdash C \\
\rho \\
\Theta, B, \Gamma \vdash C \\
\Theta, A \& B, \Gamma \vdash C
\end{array}
\]

\[
\text{\&L}
\]
we apply the induction hypothesis to \( \pi \) and \( \rho \), and next we apply Lemma 5.3.

If our sequent is proved by the following proof

\[
\begin{array}{c}
\pi \\
\Gamma \vdash A \\
\Gamma \vdash A \lor B \\
\end{array}
\]

we apply the induction hypothesis to \( \pi \), and we push applications of (W) below \( \lor R \) as follows

\[
\begin{array}{c}
\Theta \vdash A \\
\Theta' \vdash A \\
\Theta' \vdash A \lor B \\
\Theta \vdash A \lor B \\
\end{array}
\]

\[
\begin{array}{c}
\Theta' \vdash A \lor B \\
\Theta' \vdash A \lor B \\
\Theta \vdash A \lor B \\
\Theta \vdash A \lor B \\
\end{array}
\]

Of course, we proceed analogously with the other \( \lor R \) rule, involving \( B \).

If our sequent is proved by the following proof

\[
\begin{array}{c}
\pi \\
\Delta \vdash A \\
\Theta, B, \Gamma \vdash D \\
\Theta, \Delta, A \rightarrow B, \Gamma \vdash D \\
\end{array}
\]

we apply the induction hypothesis to \( \pi \) and \( \rho \), and next we apply Lemma 5.4.

If our sequent is proved by the following proof

\[
\begin{array}{c}
\pi \\
\Delta \vdash A \\
\Theta, \Delta, A \rightarrow B, \Gamma \vdash D \\
\end{array}
\]

we apply the induction hypothesis to \( \pi \) to obtain the W-normal proof

\[
\begin{array}{c}
\pi' \\
\Phi \vdash B \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \text{applications of (W)} \\
A, \Gamma \vdash B \\
\end{array}
\]

We next push below \( \rightarrow R \) each of the applications of (W) not tied to \( A \) in \( A, \Gamma \vdash B \). \( \square \)

6 Maximal cuts

A cut in a proof of \( \mathcal{G} \) will be called maximal iff its rank is 2 and none of its premises is an axiom. A proof will be called maximalized iff all cuts in it are maximal. We can prove the following theorem, which covers the second phase of our cut-elimination procedure.

**Theorem 6.1** Every W-normal proof of a sequent in \( \mathcal{G} \) can be reduced to a maximalized W-normal proof, of the same or of a lower degree, of the same sequent.

**Proof:** It is enough to consider a W-normal proof of the form

\[
\begin{array}{c}
\pi \\
\Delta \vdash A \\
\Theta, \Delta, A \rightarrow C \\
\Theta, \Delta, \Gamma \vdash C \\
\end{array}
\]

\[
\begin{array}{c}
\rho \\
\Theta, A, \Gamma \vdash C \\
\end{array}
\]

\[
\text{cut}
\]
where the cut noted in this figure is not maximal and all cuts in \( \pi \) and \( \rho \) are maximal. The rank of such a proof is the rank of the nonmaximal cut. We show by induction on rank that this proof can be reduced to a maximized W-normal proof of the same degree of \( \Theta, \Delta, \Gamma \vdash C \).

Suppose the rank of our nonmaximal cut is 2. This means that one of its premises is an axiom. Then we eliminate this cut by standard reduction steps, as those in (1.1) - (1.3) of the proof of Theorem 4.1. At this point the degree of the proof may decrease.

Suppose now the rank of our nonmaximal cut is greater than 2. In order to decrease the rank of the proof we introduce a number of reduction steps that decrease first the left rank. When this rank is 1, we introduce other reduction steps that decrease the right rank. (This is opposite to Gentzen’s procedure, where the right rank is first reduced to 1. However, the matter is not essential, and we could proceed as Gentzen did. Gentzen need not have reduced rank to 1 on one side, before reducing the rank on the other side – he could as well have worked in a zig-zag manner, passing from one side to another before reaching 1. However, for us it is essential that the rank on one side has fallen to 1 before we attack the rank on the other side.)

Suppose now the left rank of the nonmaximal cut above is greater than 1. Then in addition to the standard reduction steps like those considered in (2) of the proof of Theorem 4.1 we have the following additional reduction steps.

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta_2 \vdash B & \quad \Delta_1, C, \Delta_3 \vdash A & \quad \rho \\
\Delta_1, \Delta_2, B \rightarrow C, \Delta_3 \vdash A & \rightarrow \text{L} & \Theta, A, \Gamma \vdash C \\
\Theta, \Delta_1, \Delta_2, B \rightarrow C, \Delta_3, \Gamma \vdash C & \text{cut}
\end{align*}
\]

is reduced to

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta_2 \vdash B & \quad \Delta_1, C, \Delta_3 \vdash A & \quad \Theta, A, \Gamma \vdash C \\
\Theta, \Delta_1, C, \Delta_3, \Gamma \vdash C & \text{cut}
\end{align*}
\]

\[
\Theta, \Delta_1, \Delta_2, B \rightarrow C, \Delta_3, \Gamma \vdash C \rightarrow \text{L}
\]

The cut in the lower figure has lower rank and we may apply the induction hypothesis to it.

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta_2 \vdash B & \quad \Delta_1, B, \Delta_3 \vdash A \quad \text{cut} & \quad \rho \\
\Delta_1, \Delta_2, \Delta_3 \vdash A & \text{cut} & \Theta, A, \Gamma \vdash C \\
\Theta, \Delta_1, \Delta_2, \Delta_3, \Gamma \vdash C & \text{cut}
\end{align*}
\]

is reduced to

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
\Delta_2 \vdash B & \quad \Delta_1, B, \Delta_3 \vdash A \quad \text{cut} & \quad \rho \\
\Theta, \Delta_1, B, \Delta_3, \Gamma \vdash C & \text{cut} \\
\Theta, \Delta_1, \Delta_2, \Delta_3, \Gamma \vdash C & \text{cut}
\end{align*}
\]

By the induction hypothesis, the subproof of the reduced proof ending with the right premise of the lower cut can be reduced to a maximized W-normal proof, of the same or of a lower degree, of the same sequent. The first step of this reduction, which is one of the reduction steps (2.1)-(2.3), makes the lower cut maximal, and subsequent steps leave it so. We must apply (2.1)-(2.3) because the left rank of the upper cut in the lower figure is greater than 1 (the proof \( \pi_2 \) cannot be an axiom, and the right rank of the upper cut in the first figure is 1), and, moreover, \( \pi_2 \) cannot end with a cut. Note that in the reduction step (2.1) the rule R cannot be (W).

Suppose now the left rank of our cut is 1 and the right rank is greater than 1. Then in addition to the standard reduction steps like those considered in (3) of the proof of Theorem 4.1 (except
for (3.1) with \( R \) being (W), and (3.4), which we don’t have because of W-normality), we have the following additional cases.

\[
\begin{align*}
\pi & \quad \rho_1 \quad \rho_2 \\
\Delta \vdash A & \quad \Theta_2, A, \Gamma_1 \vdash B & \quad \Theta_1, D, \Gamma_2 \vdash C \\
\Theta_1, \Theta_2, A, \Gamma_1, B \rightarrow D, \Gamma_2 \vdash C & \quad \rightarrow L \\
\Theta_1, \Theta_2, \Delta, \Gamma_1, B \rightarrow D, \Gamma_2 \vdash C & \quad \text{cut}
\end{align*}
\]

is reduced to

\[
\begin{align*}
\pi & \quad \rho_1 \\
\Delta \vdash A & \quad \Theta_2, A, \Gamma_1 \vdash B \\
\Theta_2, \Delta, \Gamma_1 \vdash B & \quad \text{cut} \\
\Theta_1, \Theta_2, \Delta, \Gamma_1, B \rightarrow D, \Gamma_2 \vdash C & \quad \rightarrow L \\
\Theta_1, D, \Gamma_2 \vdash C & \quad \text{cut}
\end{align*}
\]

We have analogous reduction steps when \( A \) in the initial proof is in \( \Theta_1 \) or \( \Gamma_2 \).

\[
\begin{align*}
\rho & \quad \\
C_1, \ldots, C_1, \Gamma_1, A, \Gamma_2 \vdash C_2 & \quad \text{applications of (W)} \\
\pi & \quad \\
\Delta \vdash A & \quad \text{cut} \\
\Gamma_1, A, \Gamma_2 \vdash C_1 \rightarrow C_2 & \quad \rightarrow R \\
\Gamma_1, \Delta, \Gamma_2 \vdash C_1 \rightarrow C_2 & \quad \text{cut}
\end{align*}
\]

provided \( \rho \) is tailless, is reduced to

\[
\begin{align*}
\pi & \quad \rho \\
\Delta \vdash A & \quad C_1, \ldots, C_1, \Gamma_1, A, \Gamma_2 \vdash C_2 \\
C_1, \ldots, C_1, \Gamma_1, \Delta, \Gamma_2 \vdash C_2 & \quad \text{cut} \\
\ldots & \quad \text{applications of (W)} \\
C_1, \Gamma_1, \Delta, \Gamma_2 \vdash C_2 & \quad \Gamma_1, \Delta, \Gamma_2 \vdash C_1 \rightarrow C_2 \\
& \quad \rightarrow R \\
\Gamma_1, \Delta, \Gamma_2 \vdash C_1 \rightarrow C_2 & \quad \text{cut}
\end{align*}
\]

\[
\begin{align*}
\pi & \quad \rho_1 \quad \rho_2 \\
\Delta \vdash A & \quad \Theta_2, A, \Gamma_1 \vdash B & \quad \Theta_1, B, \Gamma_2 \vdash C \\
\Theta_1, \Theta_2, A, \Gamma_1, \Gamma_2 \vdash C & \quad \text{cut} \\
\Theta_1, \Theta_2, \Delta, \Gamma_1, \Gamma_2 \vdash C & \quad \text{cut}
\end{align*}
\]

is reduced to

\[
\begin{align*}
\pi & \quad \rho_1 \\
\Delta \vdash A & \quad \Theta_2, A, \Gamma_1 \vdash B \\
\Theta_2, \Delta, \Gamma_1 \vdash B & \quad \text{cut} \\
\Theta_1, \Theta_2, \Delta, \Gamma_1, \Gamma_2 \vdash C & \quad \text{cut}
\end{align*}
\]

By the induction hypothesis, the subproof of the reduced proof ending with the left premise of the lower cut can be reduced to a maximized W-normal proof, of the same or of a lower degree, of
the same sequent. The first step of this reduction, which is one of the reduction steps (3.1)-(3.6), makes the lower cut maximal, and subsequent steps leave it so. We must apply (3.1)-(3.6) because the left rank of the upper cut in the lower figure is equal to 1 and the right rank is greater than 1 (the proof \( \rho_1 \) cannot be an axiom, and the left rank of the upper cut in the first figure is 1), and, moreover, \( \rho_1 \) cannot end with a cut.

\[
\begin{array}{c}
\pi \\
\Delta \vdash A \\
\hline
\Theta_3 \vdash B & \Theta_1, A, \Theta_2, B, \Gamma \vdash C \\
\hline
\Theta_1, \Delta, \Theta_2, \Theta_3, \Gamma \vdash C
\end{array}
\]

is reduced to

\[
\begin{array}{c}
\pi \\
\rho_1 \\
\Theta_3 \vdash B \\
\hline
\Delta \vdash A & \Theta_1, A, \Theta_2, B, \Gamma \vdash C \\
\hline
\Theta_1, \Delta, \Theta_2, B, \Gamma \vdash C
\end{array}
\]

and we reason as for (3.7), by applying the induction hypothesis to the subproof of the reduced proof ending with the right premise of the lower cut. We have an analogous reduction step when \( A \) in the initial proof is in \( \Gamma \).

In terms of categories, the reduction steps (2.4) and (3.7) in the proof above correspond to associativity of composition, whereas (3.8) corresponds to bifunctoriality equalities.

We can now finally go into the third phase of our cut-elimination procedure, which is covered by the following theorem.

**Theorem 6.2** Every maximalized proof of degree greater than 0 of a sequent of \( \mathcal{G} \) can be reduced to a proof of lower degree of the same sequent.

**Proof:** Take a maximalized proof of \( \mathcal{G} \) of degree greater than 0, and starting from the top of the proof apply to every maximal cut of the initial proof either the standard reduction steps like those of (1.5) and (1.6) of the proof of Theorem 4.1, or the standard reduction step that consists in replacing

\[
\begin{array}{c}
A, \Gamma \vdash B \\
\Gamma \vdash A \rightarrow B \\
\hline
\Delta \vdash A & \Theta, B, \Xi \vdash C \\
\Theta, A \rightarrow B, \Xi \vdash C \\
\hline
\Theta, \Delta, \Xi \vdash C
\end{array}
\]

by

\[
\begin{array}{c}
\Delta \vdash A \\
\Theta, \Delta, \Gamma, \Xi \vdash C
\end{array}
\]

and

\[
\begin{array}{c}
A, \Gamma \vdash B \\
\Delta, \Gamma \vdash B \\
\hline
\Theta, B, \Xi \vdash C
\end{array}
\]

The result is a proof whose degree has decreased. \( \square \)

By applying successively the first phase, the second phase and the third phase of our procedure, i.e. Theorems 5.5, 6.1 and 6.2, and then again the first phase, the second phase etc., we must obtain after one second or third phase a proof of degree 0. If this phase was a second phase, then there are no cuts in this proof, whereas if this phase was a third phase, then there are cuts in the proof and all of them have atomic cut formulae. By applying in the latter case once more the first and second phase we will end up with a cut-free proof, because there are no maximal cuts of degree 0.
7 Concluding comments

It is instructive to compare Gentzen’s cut-elimination procedure with ours at the place where Gentzen has critical mixes (see the end of Section 2). These critical mixes correspond to the maximal cuts whose reduction we postpone until the third phase of our procedure. Gentzen’s separation of a critical mix out of a mix, and leaving it below, corresponds to something achieved in the first and second phase of our procedure. When in the first phase a cut is pushed above a contraction and is replaced by two cuts, the second phase will ensure that the maximal cut that corresponds to the critical mix will be at its proper place below other cuts.

To work in the presence of the lattice connectives $\land$ and $\lor$ our procedure presupposes the presence of thinning (see the proofs of Lemma 5.3 and Theorem 5.5, case with $(\land R)$). So this procedure as it is formulated here cannot be transferred to relevant logic, which has contraction but lacks thinning, except if in this logic we omit the “additive”, i.e. lattice, connectives and restrict ourselves to “multiplicative” connectives.

The problem with the lattice connectives $\land$ and $\lor$ is that in the rules $(\land R)$ and $(\lor L)$ there are implicit contractions: in terms of a multiplicative rule, $(\land R)$ could be reconstructed as

$$
\begin{array}{c}
\Gamma \vdash A \\
\Gamma \vdash B
\end{array}
\quad
\begin{array}{c}
\Gamma, \Gamma \vdash A \land B
\end{array}
$$

while for $(\lor L)$ there is no such simple reconstruction, but similar contractions are involved. The W-normalization of the first phase of our procedure does not take care of these implicit contractions; i.e., these are not pushed below other rules as far as they can go. Because of that we can say that when in the second phase of that procedure, in cases (2.2) and (3.2) of the proof of Theorem 6.1 (which are taken over from the proof of Theorem 4.1), there is an increase in size in the transformed proof, this increase is again due to contraction. Contraction is, of course, to blame for the increase in size that occurs in the first phase of the procedure.

All the steps of our cut-elimination procedure are covered by equalities of bicartesian closed categories, which is not the case for all the steps of Gentzen’s procedure. The categorically unjustified steps of [Gentzen 1935] are like the following step, licenced by 3.121.1 in which

$$
\begin{array}{c}
A \vdash A \\
A, A \rightarrow A \vdash A
\end{array}
\quad
\begin{array}{c}
A \vdash A \\
A, A \rightarrow A \vdash A
\end{array}
\rightarrow L
$$

is replaced by

$$
\begin{array}{c}
A \vdash A \\
A, A \rightarrow A \vdash A
\end{array}
\quad
\begin{array}{c}
A \vdash A \\
A, A \rightarrow A \vdash A
\end{array}
\rightarrow L
$$

thinning and interchange

Another problem is that Gentzen’s mix

$$
\begin{array}{c}
\Gamma \vdash A \\
\Delta + C
\end{array}
\quad
\begin{array}{c}
\Gamma, \Delta^* \vdash C
\end{array}
$$

is strict in the sense that in $\Delta^*$ we must omit all the occurrences of $A$, whereas a “liberal” mix where in $\Delta^*$ we must omit some, but not necessarily all, occurrences of $A$ is better justified categorically. In terms of Gentzen’s strict mix the following cut

$$
\begin{array}{c}
\Gamma \vdash A \\
A, A \vdash C
\end{array}
\quad
\begin{array}{c}
\Gamma, A \vdash C
\end{array}
$$

cut
is reconstructed as

\[
\Gamma \vdash A, A \vdash C \\
\text{mix} \\
\Gamma \vdash C \\
\text{thinning and interchanges} \\
\Gamma, A \vdash C
\]

which is not always justified. However, it is possible to mend Gentzen’s mix-elimination procedure so that all of its steps are justified by equalities of bicartesian closed categories.

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