Modelling and control of a spherical pendulum via a non–minimal state representation

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\textbf{ABSTRACT}
A spherical pendulum is a 2 degree-of-freedom mechanism consisting on a rod whose tip moves on the surface of a sphere. It is common to use two angular coordinates to describe such a system. This paper proposes the use of a non-minimal set of coordinates for modelling and controlling a fully-actuated torque-driven spherical pendulum. These coordinates are merely for the purpose of showing the application of unit quaternions as a useful tool for dealing with the orientation of rigid bodies. First, we recall the properties of unit quaternions, and explain how they can be employed for the definition of such non-minimal pendulum coordinates. Later, the control objective for orientation regulation is established and an inverse-dynamics controller, which uses joint displacement and velocity measurements but also some non-minimal states for the orientation error, is proposed. The stability analysis shows the fulfillment of the control objective and is validated through simulations.

\textbf{1. Introduction}

The so-called \textit{spherical pendulum} is a generalization of the simple planar pendulum (it is also known as 2–dof pendulum, due to the number of degrees of freedom (dof) it possesses). It consists on a thin rod mounted on a base through a universal joint (see Figure 1). The most common application of such mechanism is as an inverted pendulum, in which case the universal joint is not actuated and the base moves in an horizontal plane. Several papers can be found in the control literature about this kind of inverted pendulum. For example [1], gives a classification of 2-dof inverted pendulum systems, according to the way of controlling the base motion.

In this paper, however, we consider that the base of the pendulum is fixed and the universal joint is torque–driven and fully actuated, so that the pendulum’s tip can be driven to any point on the surface of a sphere. For such a system, modelling is far simple (see, e.g. [2]), and typical control formulations (as regulation and tracking of the angular displacements) are trivial when using the two coordinates of the pendulum’s universal joint (i.e., \(\phi\) and \(\theta\) in Figure 1) as the variables to be controlled. But, as will be shown later, the use of a non–minimal set of coordinates results in an interesting control problem.
even for the 2-dof pendulum. And that is, indeed, a contribution of this work.

*Euler parameters* are a set of four parameters with a unit norm constraint, so that they can be considered as points lying on the surface of the unit hypersphere of dimension 3, $S^3 \subset \mathbb{R}^4$. Therefore, Euler parameters are *unit quaternions*, a subset of the quaternion numbers, first described by Sir W. R. Hamilton in 1843 [3] as an extension to complex numbers, and with a well-defined algebra.

The importance of Euler parameters lies on the fact that they are a non-minimal parameterization of the three-dimensional orientation manifold (see, e.g. [4], or [5]). Other common parameterizations of the orientation are *Euler angles* (minimal) and *rotation matrices* (non-minimal). Reference [6] includes a listing of these and other parameterizations of orientation.

The use of Euler parameters in robotics has increased in the later years. They are a singularity-free alternative to rotation matrices for defining the kinematic relations between a robotic manipulator’s joint variables and its end-effector pose [7]. The advantages of using unit quaternions over rotation matrices appear in the computational aspects, mainly due to the reduction of floating-point operations, and thus, of processing time; in that same sense [8], shows how the use of unit quaternions is beneficial to the design and implementation of numerical simulation schemes for rigid-body dynamics.

![Figure 1. Configuration coordinates of a spherical pendulum.](image-url)
An additional benefit concerns to the definition of a more proper orientation error for control purposes. A reference for the use of Euler parameters in modelling, path planning and control of robot manipulators is [9].

It is worth mentioning that the aim of this paper is not simply to model and control a spherical pendulum, but rather to show the application of unit quaternions to obtain a non-minimal state description of such system, and then employ those non-minimal states to design an inverse-dynamics controller for regulation tasks. The motivation of this approach is reference [10], where unit complex numbers (a special case of unit quaternions) were employed for modelling and controlling a simple planar pendulum.

The remainder of this document is organized as follows. Section 2 recalls the definition, and interesting properties of the unit quaternion algebra. Section 3 introduces some concepts and theorems required for applying the Lyapunov stability theory to dynamical systems defined on manifolds. Section 4 explains how Euler parameters are employed to obtain the kinematics and dynamics models of the spherical pendulum using a non-minimal set of coordinates. Section 5 describes the controller which is proposed for regulation control and sketches the Lyapunov stability analysis of the closed-loop system. Simulations showing the performance of the proposed approach are discussed in Section 6, while concluding remarks are given in Section 7.

Throughout this paper we use the notation \( \lambda_m \{ A \} \) and \( \lambda_M \{ A \} \) to indicate the smallest and largest eigenvalues, respectively, of a symmetric matrix \( A(x) \in \mathbb{R}^{n \times n} \), for all \( x \in \mathbb{R}^n \). The Euclidean norm of vector \( x \in \mathbb{R}^n \) is defined as \( \| x \| = \sqrt{x^T x} \). Also, for a given symmetric matrix \( A(x) \), its spectral norm \( \| A(x) \| \) is defined as [11]:

\[
\| A(x) \| = \max_x \{ |\lambda_1(x)|, |\lambda_2(x)|, \ldots, |\lambda_n(x)| \}
\]

(1)

where \( \lambda_i(x) \) is the \( i \)-th eigenvalue of matrix \( A(x) \), for a given \( x \in \mathbb{R}^n \).

Now, we can state the following useful formulas with norms:

\[
\lambda_m \{ A \} \| y \|^2 \leq y^T A(x) y \leq \lambda_M \{ A \} \| y \|^2, \quad \forall y \in \mathbb{R}^n,
\]

(2)

which is known as the Rayleigh-Ritz theorem, and

\[-\| A(x) \| \| y \| \| z \| \leq y^T A(x) z \leq \| A(x) \| \| y \| \| z \|, \quad \forall y, z \in \mathbb{R}^n.
\]

(3)

In addition, \( \dot{x} \) denotes the time derivative of \( x \).

### 2. Euler parameters

For the purpose of this work, an \( n \)-dimensional manifold \( M^n \subseteq \mathbb{R}^m \), with \( n \leq m \), can be easily understood as a subset of the Euclidean space \( \mathbb{R}^m \) containing all the points whose coordinates \( x_1, x_2, \ldots, x_m \) satisfy \( r = m - n \) holonomic constraint equations of the form \( \gamma_i(x_1, x_2, \ldots, x_m) = 0 \) \( (i = 1, 2, \ldots, r) \). Thus, in the general case, we have

\[
M^n = \{ x \in \mathbb{R}^m : \gamma(x) = 0 \in \mathbb{R}^r \},
\]

(4)

where \( x = [x_1 \ x_2 \ \cdots \ x_m]^T \), and \( \gamma = [\gamma_1 \ \gamma_2 \ \cdots \ \gamma_r]^T \). It is worth noticing here that if the vector constraint \( \gamma(x) = 0 \) stands for all time \( t \geq t_0 \), then also the constraint \( \dot{\gamma}(x) = \frac{d}{dt} \gamma(x) = 0 \) is fulfilled. Conversely, \( \dot{\gamma}(x) = 0, \forall t \geq t_0 \), implies that the constraint \( \gamma(x) = k \), where \( k \in \mathbb{R}^r \) is a constant vector, is fulfilled for all \( t \geq t_0 \).
As a typical example of an \( n \)-dimensional differential manifold we have the generalized unit (hyper)sphere \( S^n \subset \mathbb{R}^{n+1} \), defined as
\[
S^n = \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \}.
\]
Other common example is the special orthogonal group of \( n \times n \) matrices, defined as
\[
SO(n) = \{ R \in \mathbb{R}^{n\times n} : R^T R = I, \det(R) = 1 \},
\]
where \( I \) is the identity matrix; it can be shown that the dimension of \( SO(n) \) is \( \frac{n(n-1)}{2} \).

It is well-known that the configuration space of a rigid body’s orientation is a three-dimensional manifold, \( M^3 \subset \mathbb{R}^m \), where \( m \) is the number of parameters employed to describe the orientation. Thus, we have minimal \( (m = 3) \) and non-minimal \( (m > 3) \) parameterizations of the orientation, being the most common the following:

a) Euler angles, \( M^3 \equiv \mathbb{R}^3 \).

b) Rotation matrices, \( M^3 \equiv SO(3) \subset \mathbb{R}^{3\times3} (\equiv \mathbb{R}^9) \).

c) Angle/axis pair, \( M^3 \equiv \mathbb{R} \times S^2 \subset \mathbb{R}^4 \).

d) Euler parameters, \( M^3 \equiv S^3 \subset \mathbb{R}^4 \).

A more detailed description of these parameterizations of orientation, and how they can be employed in robot control can be found in [12].

Euler parameters are a set of four real parameters, \( \eta, \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R} \), subject to a unit norm constraint. Let \( \xi = [\eta \quad \epsilon^T]^T \in S^3 \), with \( \epsilon = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3]^T \), be the Euler parameters for a given orientation. Then, the unit norm constraint can be written as
\[
\xi^T \xi = \eta^2 + \epsilon^T \epsilon = 1. \tag{5}
\]

Euler parameters are then unit quaternions, belonging to the unit hypersphere \( S^3 \). The term \( \eta \) is said to be the scalar (or real) part of the quaternion, while \( \epsilon \) is the vector (or imaginary) part. Moreover, the conjugate of \( \xi = [\eta \quad \epsilon^T]^T \in S^3 \), and \( -\xi = [-\eta \quad -\epsilon^T]^T \in S^3 \) both represent the same orientation. This is due to the fact that \( S^3 \) is indeed a double cover of the orientation manifold \( M^3 \).

Now, given two unit quaternions \( [\eta_1 \quad \epsilon_1^T]^T, [\eta_2 \quad \epsilon_2^T]^T \in S^3 \), the quaternion multiplication, denoted here by \( \otimes \), is defined as
\[
\begin{bmatrix}
\eta_1 \\
\epsilon_1
\end{bmatrix} \otimes \begin{bmatrix}
\eta_2 \\
\epsilon_2
\end{bmatrix} = \begin{bmatrix}
\eta_1 \eta_2 - \epsilon_1^T \epsilon_2 \\
\eta_1 \epsilon_2 + \eta_2 \epsilon_1 + \text{Skew}(\epsilon_1) \epsilon_2
\end{bmatrix} \in S^3 \tag{6}
\]
where, for all \( x = [x_1 \quad x_2 \quad x_3]^T \in \mathbb{R}^3 \), the skew–symmetric matrix operator \( S(\cdot) \) gives:
\[
S(x) = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix},
\]
so that, given vectors \( x, y \in \mathbb{R}^3 \), the term \( S(x) y \) is equivalent to the vector cross product, usually denoted \( x \times y \). Skew–symmetric matrices satisfy several useful properties, such as the following (\( \forall x, y, z \in \mathbb{R}^3 \)):
\[
S(x)^T = S(-x) = -S(x), \tag{7}
\]
\[ S(x + y)z = S(x)y + S(x)z, \]  
\[ S(x)y = -S(y)x, \]  
\[ S(x)x = 0, \]  
\[ y^TS(x)y = 0, \]  
\[ x^TS(y)z = z^TS(x)y, \]  
\[ S(x)S(y) = yx^T - x^TyI. \]  

An important property of unit quaternions is that they form a multiplicative group using the quaternion multiplication defined in (6). This means that, for all \( \xi_1, \xi_2, \xi_3 \in S^3 \), the following group axioms are satisfied:

- Closure: \( \xi_1 \otimes \xi_2 \in S^3 \)
- Associativity: \( \xi_1 \otimes (\xi_2 \otimes \xi_3) = (\xi_1 \otimes \xi_2) \otimes \xi_3 \)
- Identity element: \( \exists \mathbf{1} \in S^3 : \mathbf{1} \otimes \xi_1 = \xi_1 \otimes \mathbf{1} = \xi_1 \)
- Inverse element: \( \exists \xi_1^{-1} \in S^3 : \xi_1 \otimes \xi_1^{-1} = \xi_1^{-1} \otimes \xi_1 = \mathbf{1} \)

The proof that the quaternion multiplication (6) is closed and associative relies in using the unit norm constraint (5) for each operand, and some of the above properties of the skew–symmetric operator. The identity element of the unit quaternion group is \( \mathbf{1} = \begin{bmatrix} 1 & 0^T \end{bmatrix}^T \in S^3 \), and the inverse element of \( \xi = \begin{bmatrix} \eta & \epsilon^T \end{bmatrix}^T \in S^3 \) is \( \xi^{-1} = \xi^* = \begin{bmatrix} \eta & -\epsilon^T \end{bmatrix}^T \in S^3 \), so that

\[ \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} \otimes \begin{bmatrix} \eta \\ -\epsilon \end{bmatrix} = \begin{bmatrix} \eta \\ -\epsilon \end{bmatrix} \otimes \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \]

is satisfied for all \( \begin{bmatrix} \eta & \epsilon^T \end{bmatrix}^T \in S^3 \).

A group which is also a differential manifold is called a Lie group. So unit quaternions form a 3–dimensional Lie group, which is a double cover of the orientation manifold. An implication of this fact is that quaternion multiplication is closely related to the composition of rotations in 3D space.

Suppose that a body has rotated an angle \( \theta \in \mathbb{R} \) around an axis defined by the unit vector \( \mathbf{u} \in S^2 \subset \mathbb{R}^3 \). The Euler parameters describing the final orientation of the body with respect to its initial orientation are given by

\[ \xi(\theta, \mathbf{u}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \mathbf{u} \end{bmatrix} \]  

The pair \( [\theta \quad \mathbf{u}^T]^T \in \mathbb{R} \times S^2 \subset \mathbb{R}^4 \) is the so–called angle/axis parameterization of orientation. Furthermore, the quaternion multiplication defined in (6) represents the composition of the two rotations given by the corresponding pairs \( [\theta_1 \quad \mathbf{u}_1^T]^T \in \mathbb{R} \times S^2 \) and \( [\theta_2 \quad \mathbf{u}_2^T]^T \in \mathbb{R} \times S^2 \) (which means first rotating an angle \( \theta_1 \) around \( \mathbf{u}_1 \), and then an angle \( \theta_2 \) around \( \mathbf{u}_2 \)). This fact was employed in [7] to propose a procedure based on
unit quaternions for obtaining the kinematic model of serial-chained robot manipulators.

In the case of a time–varying orientation, we need to establish a relation between the time derivatives of Euler parameters \( \dot{\xi} = [\eta \quad \dot{\xi}^T]^T \in \mathbb{R}^4 \) and the angular velocity of the rigid body \( \omega \in \mathbb{R}^3 \). That relation is given by the so–called quaternion propagation rule [13]:

\[
\dot{\xi} = \frac{1}{2} E(\xi) \omega.
\]

where

\[
E(\xi) = E(\eta, \varepsilon) = \left[ \begin{array}{c} -\varepsilon^T \\
\eta I - S(\varepsilon) \end{array} \right] \in \mathbb{R}^{4 \times 3}.
\]

By using properties (7), (13) and the unit norm constraint (5) it can be shown that \( E(\xi)^T E(\xi) = I \in \mathbb{R}^{3 \times 3} \), so that (15) can be solved for \( \omega \), resulting

\[
\omega = 2E(\xi)^T \dot{\xi}.
\]

3. Lyapunov stability on manifolds

In this section we present some basic concepts and theorems related to the Lyapunov stability theory when applied to dynamical systems defined on manifolds of the form given by (4). Such approach is employed in [14,15] and more recently in [16].

Let us consider the general system

\[
\dot{x}(t) = f(x(t), t), \quad x(t) \in M^n \subset \mathbb{R}^m, \quad \forall t \geq t_o
\]

where \( x(t) \) represents the system’s state at time \( t \), and the vector field \( f : M^n \times \mathbb{R} \rightarrow T_x M^n \), which is assumed to be continuously differentiable (or at least locally Lipschitz, so as to ensure a unique solution for all initial conditions) on \( M^n \), associates a vector \( \dot{x} \) of the tangent space \( T_x M^n \) to each point \( x \in M^n \) at every time \( t \). The initial conditions of (17) are \( x(t_0) = x_o \) and \( t = t_o \), where \( t_o \) is the initial time.

If function \( f \) does not explicitly depend on \( t \), that is \( f(x(t), t) = f(x(t)) \), then system (17) is said to be autonomous; otherwise it is a non-autonomous system.

If \( M^n = \mathbb{R}^m \) (with \( n = m \)) then \( T_x M^n = \mathbb{R}^m \), \( f : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m \), and (17) becomes

\[
\dot{x}(t) = f(x(t), t), \quad x(t) \in \mathbb{R}^m, \quad \forall t \geq t_o,
\]

which is the well–known formulation of a dynamical system defined on a Euclidean space. This way a system of the form (18) can be seen as a particular case of (17), when \( M^n = \mathbb{R}^m \).

A manifold \( M^n \subset \mathbb{R}^m \) is said to be an invariant manifold for system (17) if [17]:

\[
x_o = x(t_o) \in M^n \Rightarrow x(t) \in M^n, \quad \forall t \geq t_o.
\]

Also notice that a dynamical system of the form (17), which is defined on a manifold \( M^n \subset \mathbb{R}^m \), can be treated as a system of the form (18), defined on \( \mathbb{R}^m \), if the initial condition \( x_o \) belong to \( M^n \) and it is an invariant manifold for (18). In other words, if we
can prove that a manifold $M^n \subset \mathbb{R}^m$ is invariant for a given dynamical system \( \dot{x} = f(x, t) \), we can analyse the stability of such a system as if it were defined in $\mathbb{R}^m$ and subject to the corresponding $m - n$ holonomic constraint equations $\gamma(x) = 0$, for all $t \geq t_o$.

In order to verify the invariance of a given manifold for a dynamical system, the following result is useful:

**Lemma 1.** Let us consider: (a) the dynamical system
\[
\dot{x}(t) = f(x(t), t)
\]
where $x(t) \in \mathbb{R}^m$, $f: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, for all $t \geq t_o$, and (b) the manifold $M^n \subset \mathbb{R}^m$, defined as
\[
M^n = \{ x \in \mathbb{R}^m : \gamma(x) = 0 \in \mathbb{R}^{m-n} \}.
\]
If
\[
\frac{\partial \gamma(x)}{\partial x} f(x, t) = 0 \in \mathbb{R}^{m-n}
\]
for all $x \in M^n$, and for all $t \geq t_o$, then
\[
x(t_o) \in M^n \Rightarrow x(t) \in M^n, \forall t \geq t_o.
\]

**Proof.**
\[
\frac{\partial \gamma(x)}{\partial x} f(t, x) = 0 \Rightarrow \frac{\partial \gamma(x)}{\partial x} \dot{x} = 0 \Rightarrow \frac{\partial y(x)}{\partial t} = 0.
\]
The fulfilment of constraint $\frac{d}{dt} \gamma(x) = 0$, for all $t \geq t_o$, implies that also the constraint $\gamma(x) = k$, where $k$ is a constant vector, is fulfilled for all $x = x(t)$; and as it is assumed that $x(t_o) \in M^n$, where $\gamma(x) = 0$, then $x(t) \in M^n$, for all $t \geq t_o$.

Among the basic concepts in Lyapunov theory that of the equilibrium of a dynamical system plays a central role.

**Definition 1.** Consider a system of the form (17). A constant vector $\bar{x} \in M^n$ is said to be an equilibrium point of such a system if $f(\bar{x}, t) = 0$, for all $t \geq t_o$.

The following two definitions, concerning the stability of equilibria, are based on similar ones found in [14]:

**Definition 2.** An equilibrium point $\bar{x}$ is stable (in the sense of Lyapunov) if, for every $\sigma > 0$ and every $t_o$, there exists a real number $\delta(t_o) > 0$ such that $\| x(t_o) - \bar{x} \| < \delta(t_o)$ implies $\| x(t) - \bar{x} \| < \sigma$, for all $t > t_o$. Moreover, the equilibrium point is uniformly stable if $\delta(t_o)$ can be chosen independent from $t_o$.

An equilibrium which is not stable is said to be an unstable equilibrium.

**Definition 3.** An equilibrium point $\bar{x}$ is (uniformly) asymptotically stable if it is (uniformly) stable and there exists a $\delta' > 0$ such that $\| x(t_o) - \bar{x} \| < \delta'$ implies
\[
\lim_{t \to \infty} x(t) = \bar{x}.
\] (19)

Moreover, the equilibrium is globally (uniformly) asymptotically stable if it is (uniformly) stable and (19) is satisfied for all \( x(t_0) \in M^n \).

The customary concept of stability in the literature on nonlinear control (see, e.g. [17], and [18]) is formulated for dynamical systems defined on Euclidean spaces, where the Euclidean norm (or Euclidean metric) is employed as a measure of the distance between any two vectors. Moreover, it is common to perform a change of variables to translate to the origin \( x = 0 \) the equilibrium to be analysed; that is possible since all points in a Euclidean space have the same properties, and usually simplifies the stability analysis.

The generalization of Lyapunov stability theory to systems defined on manifolds usually involves: (a) replacing the Euclidean metric with the corresponding metric for the manifold, and (b) avoiding the translation of the equilibrium to the origin, since it can cause an unnecessary increase in the complexity of the analysis (specially for symmetric manifolds such as origin–centred (hyper)spheres). To emphasize the stability properties of a particular equilibrium (not necessarily at the origin of the state–space), some authors employ phrases such as ‘locally stable in a neighbourhood of or around the equilibrium’.

In this paper, however, as we consider that the dynamical system under analysis is defined in \( \mathbb{R}^m \) (although belonging to an invariant manifold \( M^n \subset \mathbb{R}^m \)), we keep using the Euclidean norm as a metric, but we also allow the existence of equilibrium points outside the origin. Those aspects are taken into account in the following definition of Lyapunov functions for the analysis of equilibria on manifolds. Similar definitions can be found in [19].

**Definition 4.** A function \( V : M^n \times \mathbb{R} \to \mathbb{R} \) is said to be a locally positive definite function around an equilibrium \( \bar{x} \in M^n \) if it is continuous, \( V(\bar{x}, t) = 0 \), and there exists a constant \( r > 0 \) and a function \( W_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) of class \( K \) (i.e. continuous and strictly increasing; see details in [18]), such that \( \forall t \geq t_0 \), and \( \| x - \bar{x} \| < r \)

\[
V(x, t) \geq W_1(\| x - \bar{x} \|).
\]

\( V(x, t) \) is a locally negative definite function around \( \bar{x} \) if \( -V(x, t) \) is locally positive definite around \( \bar{x} \). Moreover, \( V \) is locally decrescent around \( \bar{x} \in M^n \) if there exist a constant \( r > 0 \) and a function \( W_2 \) of class \( K \) such that \( \forall t \geq t_0 \), and \( \| x - \bar{x} \| < r \)

\[
V(x, t) \leq W_2(\| x - \bar{x} \|).
\]

All the properties stated in Definition 4 are global if they are valid for every \( x \in M^n \). Moreover, if \( V(x, t) \) is independent of \( t \), i.e., if \( V(x, t) = V(x) \) then \( V \) is globally decrescent around \( x \).

Most of the Lyapunov–like theorems, including those for non–autonomous systems, may be formulated on general metric spaces (see, e.g [20]. and [21]). For example, the following theorem employs the definitions in this section.
Theorem 1. The equilibrium $\bar{x}$ of system (17) is locally/globally uniformly asymptotically stable if there exists a decrescent locally/globally positive definite function $V$ around $\bar{x}$ such that $\dot{V}$ is locally/globally negative definite around $\bar{x}$.

Now consider the following theorem for unstability, which will be useful for the purpose of this paper.

Theorem 2. The equilibrium $x$ of system (17) is unstable if there exists a function $V_e : M^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that:

I. $V_e(\bar{x}, t) = 0$, for all $t \geq t_o$.

II. $V_e(x, t)$ is decrescent around $\bar{x}$.

III. $V_e(x, t) \geq W(x)$, where $W(x)$ is a continuous function, such that $W(x_o) > 0$, for some $x_o \in M^n$ arbitrarily close to $\bar{x}$.

IV. $\dot{V}_e(x, t)$ is positive definite around $\bar{x}$, at least locally.

4. The spherical pendulum

A spherical pendulum is a 2-dof mechanism, so only two independent coordinates are required to describe its configuration. Several sets of coordinates can be used (see, e.g [22]). In this paper we chose the angles $\phi$ and $\theta$ shown in Figure 1, which usually characterize the motion of the universal joint. Now let us assume that the angle $\phi$ is bounded so that $0 \leq \phi \leq \phi_0$, where $\phi_0 \in [0, \pi) \subset \mathbb{R}$; this assumption is made for the stability analysis in Section 5.3 but it is easily attributable to the physical limitations of the universal joint at the base of the pendulum; $\theta$ is not bounded ($\theta \in \mathbb{R}$).

Also, let us consider the two coordinate frames in Figure 1. Frame $\Sigma_o(X_o, Y_o, Z_o)$, with origin at the base of the pendulum, is fixed, and thus considered an inertial frame. Frame $\Sigma_p(X_p, Y_p, Z_p)$ is attached to the pendulum, with origin at its tip, and rotates around the base. In addition, let us consider a frame $\Sigma_{p'}(X_{p'}, Y_{p'}, Z_{p'})$ which, for all time, has the same orientation as $\Sigma_p$ but its origin is placed at the base (i.e. it coincides with the origin of $\Sigma_o$).

The configuration of the pendulum is given by the pose of frame $\Sigma_p$ with respect to $\Sigma_o$, which can be defined in terms of $\phi$ and $\theta$ as follows:

- If $\phi = \theta = 0$ then frames $\Sigma_o$, $\Sigma_{p'}$, and $\Sigma_p$ have the same orientation (this is called the home configuration).
- Starting from the home configuration, the pendulum is rotated an angle $\phi$ around the $X_o$ ($= X_{p'}$) axis.
- After that first rotation, the pendulum is now rotated an angle $\theta$ around the (already rotated) $Y_{p'}$ axis.
- Finally, the position of frame $\Sigma_p$ is gotten simply by translating frame $\Sigma_{p'}$ along the $Z_{p'}$ axis a distance equal to the length of the pendulum; but as this length is constant, then it is not a configuration coordinate for the spherical pendulum.

It is worth noticing that $\phi$ and $\theta$, which describe the configuration of the spherical pendulum, can also be seen as Euler angles describing the orientation of the pendulum.
frame \((\Sigma_p)\) with respect to the base frame \((\Sigma_o)\). This is the key for obtaining a non-minimal parameterization of this system.

### 4.1. Kinematics model

Let us define \(q = [\phi \theta]^T \in D_q\), with \(D_q = [0, \phi_0] \times \mathbb{R} \subset \mathbb{R}^2\), as the vector of joint coordinates of the spherical pendulum. Moreover, let \(\xi(q) \in S^3\) be the Euler parameters describing the orientation of the pendulum frame with respect to the base frame, as a function of \(q\). Thus, according to (14) and the composition rule for rotations mentioned in the previous section, \(\xi(q)\) can be computed by as

\[
\xi(q) = \xi(\phi, \hat{\theta}) \otimes \xi(\theta, \hat{\phi}) = \begin{bmatrix} \cos(\frac{\phi}{2}) & \sin(\frac{\phi}{2}) & 0 & 0 \\ \sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) & 0 \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) & 0 \end{bmatrix}
\]

where \(\hat{\theta}\) and \(\hat{\phi}\) are the standard unit vectors in the direction of the \(X\) and \(Y\) axis of the \(\Sigma_p\) frame, respectively. Then, by using (6) we get

\[
\xi(q) = h(q) = \begin{bmatrix} \eta(q) \\ \varepsilon_1(q) \\ \varepsilon_2(q) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\phi}{2}) & \cos(\frac{\theta}{2}) \\ \sin(\frac{\phi}{2}) & \cos(\frac{\theta}{2}) \\ \cos(\frac{\phi}{2}) & \sin(\frac{\theta}{2}) \end{bmatrix}
\]

(20)

where \(h : D_q \rightarrow S^3\) corresponds to the direct kinematics function, which produces a non-minimal parameterization of the configuration space of the spherical pendulum. It satisfies (5) and the secondary holonomic constraint

\[
\eta \varepsilon_3 = \varepsilon_1 \varepsilon_2.
\]

Using double-angle trigonometric identities, we can obtain other useful relations between the minimal joint coordinates and the non-minimal quaternionic coordinates of the pendulum:

\[
\sin(\phi) = 2(\eta \varepsilon_1 + \varepsilon_2 \varepsilon_3)
\]

\[
\cos(\phi) = (\eta^2 + \varepsilon_2^2) - (\varepsilon_1^2 + \varepsilon_3^2)
\]

\[
\sin(\theta) = 2(\eta \varepsilon_2 + \varepsilon_1 \varepsilon_3)
\]

\[
\cos(\theta) = (\eta^2 + \varepsilon_1^2) - (\varepsilon_2^2 + \varepsilon_3^2)
\]

The inverse kinematics function of the spherical pendulum, which allows computing \(q\) from the corresponding Euler parameters \(\xi\), can thus be computed as

\[
q(\xi) = h^{-1}(\xi) = \begin{bmatrix} \phi(\xi) \\ \theta(\xi) \end{bmatrix} = \begin{bmatrix} 2 \arctan \left( \frac{2(\eta \varepsilon_1 + \varepsilon_2 \varepsilon_3)}{(\eta^2 + \varepsilon_2^2) - (\varepsilon_1^2 + \varepsilon_3^2)} \right) \\ 2 \arctan \left( \frac{2(\eta \varepsilon_2 + \varepsilon_1 \varepsilon_3)}{(\eta^2 + \varepsilon_1^2) - (\varepsilon_2^2 + \varepsilon_3^2)} \right) \end{bmatrix}
\]

Now, by taking the time derivative of (20), and using (16), we can show that the pendulum angular velocity \(\omega\) is given by
\[ \omega = J(\phi) \dot{q} \]  

(21)

where

\[ J(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & \cos(\phi) \\ 0 & \sin(\phi) \end{bmatrix} \in \mathbb{R}^{3 \times 2} \]  

(22)

It should be noticed that \( J(\phi)^T J(\phi) = I \), so that it is possible to write

\[ \dot{q} = J(\phi)^T \omega. \]

Furthermore, by combining (15) and (21) we get

\[ \dot{\xi} = \frac{1}{2} H(\xi) \dot{q} \]

where

\[ H(\xi) = E(\xi) J(\phi(\xi)) = \begin{bmatrix} -\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2 & \eta \\ -\varepsilon_3 & \eta \\ -\varepsilon_3 & \varepsilon_1 \end{bmatrix} \in \mathbb{R}^{4 \times 2} \]

and also

\[ \dot{q} = 2H(\xi)^T \dot{\xi}. \]

4.2. Dynamics model

Now we consider the application of the Lagrangian formulation for obtaining the dynamics model of the spherical pendulum. An important fact of this formulation is that, as it is based on the energy–conservation principle, it is independent of the coordinate system employed to describe the motion of the pendulum.

If we choose \( q = [\phi \ \theta]^T \) as the configuration coordinates, then the Lagrangian function is given by

\[ L(q, \dot{q}) = K(q, \dot{q}) - U(q) \]

where \( K(q, \dot{q}) \) is the kinetic energy and \( U(q) \) is the potential energy of the pendulum. The dynamics model can then be obtained by the Euler–Lagrange equations:

\[ \frac{d}{dt} \left( \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) - \frac{\partial L(q, \dot{q})}{\partial q} = \tau \]  

(23)

where \( \tau \) is the vector of joint torques applied to the pendulum. Adopting notation from robotics (see, e.g. [11]), Eq. (23) can be rewritten as the dynamics model:

\[ M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) = \tau \]  

(24)

where \( M(q) \in \mathbb{R}^{2 \times 2} \) is a symmetric positive definite matrix known as the inertia matrix, \( C(q, \dot{q}) \in \mathbb{R}^{2 \times 2} \) is the matrix of centripetal and Coriolis torques, and \( g(q) \in \mathbb{R}^2 \) is the vector of torques due to gravity.
Let us assume that the spherical pendulum has a mass \( m \) concentrated in its centre of mass point, at a distance \( l \) from the base, and it has a negligible moment of inertia (this is a common assumption which simplifies the analysis); then the kinetic and potential energy functions are

\[
K(q, \dot{q}) = \frac{1}{2} ml^2[\dot{\theta}^2 + \cos^2(\theta)\dot{\phi}^2],
\]

\[
U(q) = mgl \cos(\phi) \cos(\theta),
\]

with \( g = 9.81 \text{ [m/s}^2\text{]} \) the acceleration of gravity constant. In this case the matrices \( M(q), C(q, \dot{q}) \), and the vector \( g(q) \) of the dynamic model (24) are given by

\[
M(q) = ml^2 \begin{bmatrix} \cos^2(\theta) & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
C(q, \dot{q}) = ml^2 \begin{bmatrix} -\sin(\theta) \cos(\theta)\dot{\theta} & -\sin(\theta) \cos(\theta)\dot{\phi} \\ \sin(\theta) \cos(\theta)\dot{\phi} & 0 \end{bmatrix},
\]

\[
g(q) = -mlg[\cos(\theta) \sin(\phi) \quad \sin(\theta) \cos(\phi)]^T.
\]

5. Inverse–dynamics controller

5.1. Orientation error and control objective

As we mentioned before, it is assumed that the spherical pendulum is a torque–driven fully–actuated mechanism, and that we can measure angles \( \phi \) and \( \theta \) (i.e., \( q \)), which completely characterize the configuration of the 2–dof system.

To control the spherical pendulum, we could simply specify a constant desired configuration \( q_d = [\phi_d \quad \theta_d]^T \in D_q \) and use a joint–space controller to ensure that \( \lim_{t \to \infty} q(t) = q_d \). But in this paper we take a different approach, consisting in assuming that we want to control the orientation of the \( \Sigma_p \) frame attached to the pendulum’s tip (see Figure 1). It should be clear that such pendulum orientation depends on the values of \( \phi \) and \( \theta \) (which, indeed, can be seen as Euler angles describing such an orientation). But as we mentioned in Section 2 there are other parameterizations of orientation which are non–minimal.

Let us assume that the orientation of the pendulum is described by Euler parameters. The direct kinematics function (20) can be used for this. The desired orientation can be written in terms of Euler parameters by relations similar to (20), i.e.

\[
\xi_d(q_d) = h(q_d) = \begin{bmatrix} \eta_d(q_d) \\ \varepsilon_{1d}(q_d) \\ \varepsilon_{2d}(q_d) \\ \varepsilon_{3d}(q_d) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\phi_d}{2}) \cos(\frac{\theta_d}{2}) \\ \sin(\frac{\phi_d}{2}) \cos(\frac{\theta_d}{2}) \\ \cos(\frac{\phi_d}{2}) \sin(\frac{\theta_d}{2}) \\ \sin(\frac{\phi_d}{2}) \sin(\frac{\theta_d}{2}) \end{bmatrix} \in S^3 \subset \mathbb{R}^4. \tag{25}
\]

This \( \xi_d = [\eta_d \quad \varepsilon_d]^T \in S^3 \) gives the orientation of the pendulum frame at the desired configuration. Let \( \Sigma_{d} \) be such desired constant frame. Then the relative orientation of
frame $\Sigma_d$ with respect to the current pendulum frame $\Sigma_p$ (that is, the orientation error) in terms of Euler parameters is given by $\tilde{\zeta} = [\tilde{\eta} \; \tilde{\epsilon}^T]^T = [\tilde{\eta} \; \tilde{\epsilon}_1 \; \tilde{\epsilon}_2 \; \tilde{\epsilon}_3]^T \in S^3$, which can be computed as [23]:

$$\tilde{\zeta} = \xi_d \otimes \xi^* = \left[ \begin{array}{c} \eta_d \\ \epsilon_d \\ -\epsilon \\ \tilde{\eta} \end{array} \right] \times \left[ \begin{array}{c} \eta \\ -\epsilon \\ \tilde{\eta} \end{array} \right] = \left[ \begin{array}{c} \tilde{\eta} \\ \tilde{\epsilon} \end{array} \right]$$  \hspace{1cm} (26)

and applying (6) we get [13]:

$$\tilde{\zeta}(\xi, \xi_d) = \left[ \begin{array}{c} \tilde{\eta} \\ \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \tilde{\epsilon}_3 \end{array} \right] = \left[ \begin{array}{c} \eta \eta_d + \epsilon^T \epsilon_d \\ \eta \epsilon_d - \eta_d \epsilon + S(\epsilon) \epsilon_d \end{array} \right] \in S^3.$$  \hspace{1cm} (27)

And in terms of the minimal actual and desired coordinates, $q$ and $q_d$, respectively, the orientation error (27) becomes

$$\tilde{\zeta}(q, q_d) = \left[ \begin{array}{c} \tilde{\eta} \\ \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \tilde{\epsilon}_3 \end{array} \right] = \left[ \begin{array}{c} \epsilon \cos(\phi_d - \phi) \cos(\theta_d - \theta) \\ \epsilon \sin(\phi_d - \phi) \cos(\theta_d - \theta) \\ \epsilon \cos(\phi_d + \phi) \sin(\theta_d - \theta) \\ \epsilon \sin(\phi_d + \phi) \sin(\theta_d - \theta) \end{array} \right] \in S^3 \subset \mathbb{R}^4.$$  \hspace{1cm} (28)

Notice that when the end-effector orientation equals the desired orientation, i.e. $\xi = \xi_d$, the orientation error becomes $\tilde{\zeta} = [1 \; 0^T]^T \in S^3$, and that corresponds to the case when $\phi = \phi_d$, $\theta = \theta_d + 2\pi k$ ($k \in \mathbb{Z}$), and $k$ is even. Now, if $\xi = -\xi_d$, then $\tilde{\zeta} = [-1 \; 0^T]^T \in S^3$; this is the case when $k$ is odd, and clearly corresponds to exactly the same orientation as when $\xi = \xi_d$. This fact complies with the double-cover property of Euler parameters, mentioned in Section 2.

From the previous analysis we can establish the objective of orientation control in terms of the Euler parameters as

$$\lim_{t \to \infty} \left[ \begin{array}{c} \eta(t) \\ \epsilon(t) \end{array} \right] = \pm \left[ \begin{array}{c} \eta_d \\ \epsilon_d \end{array} \right] \in S^3 \subset \mathbb{R}^4,$n

or, in terms of the orientation error $\tilde{\zeta} = [\tilde{\eta} \; \tilde{\epsilon}^T]^T$ defined in (27), as

$$\lim_{t \to \infty} \left[ \begin{array}{c} \tilde{\eta}(t) \\ \tilde{\epsilon}(t) \end{array} \right] = \pm \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \in S^3 \subset \mathbb{R}^4.$$  \hspace{1cm} (28)

And in [24], Yuan showed that this control objective is fulfilled if and only if $\lim_{t \to \infty} \tilde{\epsilon} = 0$.

Now taking the time derivative of (27), considering (15), and properties (8), (9), and (13), it can be shown that:

$$\left[ \begin{array}{c} \dot{\tilde{\eta}} \\ \dot{\tilde{\epsilon}} \end{array} \right] = \frac{1}{2} \left[ \tilde{\epsilon}^T - [\tilde{\eta} I + S(\tilde{\epsilon})] \right] \omega.$$  \hspace{1cm} (29)

This expression will be useful to define the closed-loop dynamics in the following subsection.
5.2. Control law and closed–loop system

The inverse–dynamics control technique [25] assumes that the dynamics parameters of the system to control are completely known and compensates the whole dynamics so as to get a closed–loop system conveniently simple.

In order to obtain a nonminimal quaternion-based closed-loop equation for the spherical pendulum, we propose to use the following controller:

$$\tau = M(q)[k_p \tilde{\rho} - k_v \dot{\tilde{\rho}}] + C(q, \dot{q}) \dot{q} + g(q)$$

(30)

where $k_p > 0, k_v > 0 \in \mathbb{R}$ are the controller gains, and $\tilde{\rho} \in \mathbb{R}^2$ is defined in terms of $\tilde{\epsilon}$ (the vector part of the Euler parameters of the orientation error, given by (27)) as

$$\tilde{\rho} = \begin{bmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\epsilon}_1 \\ \sqrt{\tilde{\epsilon}_2^2 + \tilde{\epsilon}_3^2} \end{bmatrix}$$

(31)

in such a way that $\tilde{\rho} = 0 \iff \tilde{\epsilon} = 0$, and

$$\tilde{\eta}^2 + \tilde{\rho}^T \tilde{\rho} = \tilde{\eta}^2 + \tilde{\epsilon}^T \tilde{\epsilon} = 1.$$  

(32)

It should be noticed from (32), that $[\tilde{\eta} \quad \tilde{\rho}^T]^T$ belongs to the unit sphere $S^2 \subset \mathbb{R}^3$. Moreover, from (28), we have that

$$\tilde{\rho} = \begin{bmatrix} \sin\left(\frac{\phi}{2}\right) \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix},$$

where $\tilde{\phi} = \phi_d - \phi$ and $\tilde{\theta} = \theta_d - \theta$. Further, $\tilde{\epsilon}$ can be computed from $\tilde{\rho}$ as

$$\tilde{\epsilon} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cos\left(\frac{\phi_d + \phi}{2}\right) \\ \cos\left(\frac{\phi_d + \phi}{2}\right) & 0 \\ \cos\left(\frac{\phi_d + \phi}{2}\right) & \sin\left(\frac{\phi_d + \phi}{2}\right) \end{bmatrix} \tilde{\rho}$$

or, more conveniently, as

$$\tilde{\epsilon} = R(\tilde{\phi})J(\phi)\tilde{\rho}$$

(33)

with

$$R(\tilde{\phi}) = \begin{bmatrix} 1 \\ 0 & \cos\left(\frac{\phi}{2}\right) & -\sin\left(\frac{\phi}{2}\right) \\ 0 & \sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{bmatrix} \in SO(3).$$

(34)

and $J(\phi)$ defined in (22).

Also notice that $\tilde{\rho}$ can be solved from (33) to obtain

$$\tilde{\rho} = J(\phi)^T R(\tilde{\phi})^T \tilde{\epsilon}$$

(35)

and we can show that the time derivative of (35) is simply

$$\dot{\tilde{\rho}} = J(\phi)^T R(\tilde{\phi})^T \dot{\tilde{\epsilon}}.$$  

(36)
Figure 2 shows the schematic diagram of the closed-loop control system. Notice in that figure that $\tilde{\xi} = [\tilde{\eta} \quad \tilde{\rho}^T]^T$ is computed from $q$ and $\dot{q}_d$ using (20), (25), and (26), while for the computation of $\tilde{\rho}$ we employ (35), where $\tilde{\phi} = \phi_d - \phi$, and $\phi$ is fed back from the pendulum measurements.

In order to obtain the closed-loop system we replace the inverse-dynamics controller (30) in the spherical pendulum dynamics (24), resulting in

$$\ddot{q} = k_p\tilde{\rho} - k_v\dot{q}$$

and the dynamics of $\tilde{\rho}$ is given by (36), and (29).

Thus, employing (33) and (21), we can write the dynamics of the closed loop system in terms of the state vector $x = [\tilde{\eta} \quad \tilde{\rho}^T \quad \dot{q}^T]^T \in \mathbb{S}^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$ as

$$\frac{d}{dt} \begin{bmatrix} \tilde{\eta} \\ \tilde{\rho} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \tilde{\rho}^T J^T R^T J q \\ -\frac{1}{2} J^T R^T [\tilde{\eta} I + S(RJ\tilde{\rho})] J \dot{q} \\ k_p\tilde{\rho} - k_v\dot{q} \end{bmatrix} = f(x, t) \quad (37)$$

where we have omitted the arguments of matrices $J(\phi)$ and $R(\tilde{\phi})$. These matrices, which are full-rank, make system (37) non-autonomous.

At this point it is important to verify that the manifold $\mathbb{S}^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$, defined as

$$\{ x = [\tilde{\eta} \quad \tilde{\rho}^T \quad \dot{q}^T]^T \in \mathbb{R}^5 : \gamma(x) = \tilde{\eta}^2 + \tilde{\rho}^T \tilde{\rho} - 1 = 0 \in \mathbb{R} \} \quad (38)$$

is invariant for system (37). In order to do so, we compute

$$\frac{\partial \gamma(x)}{\partial x} f(x) = \begin{bmatrix} 2\tilde{\eta} & 2\tilde{\rho}^T & 0^T \end{bmatrix} \begin{bmatrix} \frac{1}{2} \tilde{\rho}^T J^T R^T J \dot{q} \\ -\frac{1}{2} J^T R^T [\tilde{\eta} I + S(RJ\tilde{\rho})] J \dot{q} \\ k_p\tilde{\rho} - k_v\dot{q} \end{bmatrix}$$

$$= \tilde{\eta} \tilde{\rho}^T J^T R^T J \dot{q} - \tilde{\rho}^T J^T R^T [\tilde{\eta} I + S(RJ\tilde{\rho})] J \dot{q}$$

$$= \tilde{\eta} \tilde{\rho}^T J^T R^T J \dot{q} - \tilde{\rho}^T J^T R^T J \dot{q} = 0$$

where we have applied properties (11) and (12) of the operator $S$ to eliminate the term $\tilde{\rho}^T J^T R^T S(RJ\tilde{\rho}) J \dot{q}$. Thus by invoking Lemma 1, we conclude that the manifold (38) is invariant for system (37).

![Figure 2. Schematic diagram of the closed-loop control system.](image-url)
With respect to the equilibria of system (37) we have that, according to Definition 1, those are the constant values of \( \begin{bmatrix} \tilde{\eta} \\ \tilde{\rho} \\ \tilde{\varphi} \\ \tilde{q} \end{bmatrix} \in S^2 \subset \mathbb{R}^3 \) and \( \tilde{q} \in \mathbb{R}^2 \), which are solution to the following system of equations:

\[
\tilde{\rho}^T J^T R^T J \tilde{q} = 0
\]

(39)

\[
J^T R^T [\eta I + S(RJ \tilde{\rho})] J \tilde{q} = 0
\]

(40)

\[
k_p \tilde{\rho} - k_v \tilde{q} = 0
\]

(41)

subject to the constraint \( \tilde{\eta}^2 + \tilde{\rho}^T \tilde{\rho} = 1 \). And after replacing \( J = J(\varphi) \) and \( R = R(\tilde{\varphi}) \), from (22) and (34), respectively, we can verify that equations (39)–(41) become

\[
\tilde{\rho}_1 \dot{\tilde{q}}_1 + \cos\left(\frac{\varphi}{2}\right) \tilde{\rho}_2 \dot{\tilde{q}}_2 = 0
\]

\[
\tilde{\eta} \dot{\tilde{q}}_1 - \sin\left(\frac{\varphi}{2}\right) \tilde{\rho}_2 \dot{\tilde{q}}_2 = 0
\]

\[
\left[ \cos\left(\frac{\varphi}{2}\right) \tilde{\eta} + \sin\left(\frac{\varphi}{2}\right) \tilde{\rho}_1 \right] \dot{\tilde{q}}_2 = 0
\]

\[
k_p \tilde{\rho}_1 - k_v \tilde{q}_1 = 0
\]

\[
k_p \tilde{\rho}_2 - k_v \tilde{q}_2 = 0
\]

and the constraint \( \tilde{\eta}^2 + \tilde{\rho}_1^2 + \tilde{\rho}_2^2 - 1 = 0 \).

Now, solving this system of six homogeneous equations in a symbolic mathematical software, considering \( \varphi \) as a constant, we can verify that the closed-loop system (37) has only two equilibria:

\[
E_1 = \begin{bmatrix} \tilde{\eta} \\ \tilde{\rho} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5, \quad \text{and} \quad E_2 = \begin{bmatrix} \tilde{\eta} \\ \tilde{\rho} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \in S^2 \times \mathbb{R}^2
\]

(42)

The following subsection is devoted to the stability analysis of these equilibria.

### 5.3. Stability analysis

As stated before, \( \varphi, \varphi_d \in [0, \varphi_0] \subset \mathbb{R} \), with \( \varphi_0 \in [0, \pi) \subset \mathbb{R} \); thus we have that \( \tilde{\varphi} \in [-\varphi_0, \varphi_0] \subset \mathbb{R} \), and for the analysis let us define the \( \delta \) parameter so that

\[
0 < \delta = \cos\left(\frac{\varphi_d}{2}\right) \leq \cos\left(\frac{\varphi}{2}\right) \leq 1,
\]

(43)

and it is worth noticing that \( \delta = 1 \) implies that \( \varphi_d = 0 \), meaning that the pendulum has no possible motion. But, on the other hand, as will be shown below, very small values of \( \delta \) make it impossible to prove the stability or unstability of the two equilibria, \( E_1 \) and \( E_2 \). Thus, for the analysis, let us consider that the minimum value of \( \delta \) can be expressed in
terms of the control gains \( k_p, k_v \), and a parameter \( \gamma > 0 \) to be defined later, according to the following expression:

\[
\delta > \frac{2\sqrt{2(2 + 3\mu) - (3 + 2\mu)}}{7 - 2\mu} = \delta_1(\mu) \tag{44}
\]

where

\[
\mu = \frac{y k_p}{k_v^2}.
\]

It should be noticed in (44) that \( \delta_1(\mu) \) is a strictly increasing function, well defined for all \( \mu \geq 0 \), except at \( \mu = \frac{7}{2} \), where it has a removable discontinuity \( (\lim_{\mu \to \frac{7}{2}} \delta_1(\mu) = \frac{\gamma}{2}) \); moreover, \( \delta_1(0) = \frac{1}{7} \) and \( \lim_{\mu \to \infty} \delta_1(\mu) = 1 \). This ensures that, for all \( k_p > 0 \) and \( k_v > 0 \), we can find a \( \delta \), such that \( \frac{1}{7} < \delta_1(\mu) < \delta < 1 \).

### 5.3.1. Equilibrium \( E_1 \)

In order to test the stability properties of the equilibrium \( E_1 \) in (42) let us consider the function

\[
V(\tilde{\eta}, \tilde{\rho}, \dot{\tilde{q}}) = a(\tilde{\eta} - 1)^2 + a\tilde{\rho}^T \tilde{\rho} + \frac{c}{2k_p} \dot{\tilde{q}}^T \dot{\tilde{q}} - \tilde{\rho}^T \dot{\tilde{q}}, \tag{45}
\]

with

\[
a = \frac{(1 + 5\delta)k_v}{(1 - \delta)^2}, \quad \text{and} \quad c = \frac{(1 + 5\delta)(1 + \delta)k_v}{2(1 - \delta)^2}.
\tag{46}
\]

Notice that \( V(\tilde{\eta}, \tilde{\rho}, \dot{\tilde{q}}) \) can be lower bounded as

\[
V(\tilde{\eta}, \tilde{\rho}, \dot{\tilde{q}}) \geq (\tilde{\eta} - 1)^2 + \left[ \left\| \tilde{\rho} \right\| \right]^T P \left[ \left\| \tilde{\rho} \right\| \right]
\]

with

\[
P = \begin{bmatrix}
a & -1/2 \\
-1/2 & \frac{c}{2k_p}
\end{bmatrix},
\]

which is a positive definite matrix if

\[
2ac > k_p. \tag{47}
\]

Now, taking the time derivative of (45) along the trajectories of system (37) we get:

\[
\dot{V}(\tilde{\eta}, \tilde{\rho}, \dot{\tilde{q}}) = 2a(\tilde{\eta} - 1)\ddot{\tilde{\eta}} + 2a\tilde{\rho}^T \dot{\tilde{\rho}} + \frac{c}{k_p} \dot{\tilde{q}}^T \dot{\tilde{q}} - \tilde{\rho}^T \ddot{\tilde{q}} - \dot{\tilde{\rho}}^T \dot{\tilde{q}}
\]

\[
= -k_p\tilde{\rho}^T \tilde{\rho} - \dot{\tilde{q}}^T \left[ \frac{ck_v}{k_p} I - \frac{1}{2} (\eta \Lambda + \Gamma) \right] \dot{\tilde{q}} + \tilde{\rho}^T [k_v I + (c I - a\Lambda)] \dot{\tilde{q}},
\tag{48}
\]

where
\[
\Lambda = J^T R^T J = \begin{bmatrix}
1 & 0 \\
0 & \cos\left(\frac{\phi}{2}\right)
\end{bmatrix},
\]  
(49)
and
\[
\Gamma = J^T R^T S(R J \bar{\rho}) J = \begin{bmatrix}
0 & -\sin\left(\frac{\phi}{2}\right) \sin\left(\frac{\theta}{2}\right) \\
0 & \sin^2\left(\frac{\phi}{2}\right) \cos\left(\frac{\theta}{2}\right)
\end{bmatrix}.
\]  
(50)

In order to find a bound for \( \hat{V}(\bar{\eta}, \bar{\rho}, \hat{q}) \) in (48), we need to first find a bound for matrices \( \bar{\eta}\Lambda + \Gamma \) and \( cI - a\Lambda \). We can take the symmetric part of matrix of \( \Gamma, \Gamma_\varepsilon \), and apply the Rayleigh-Ritz theorem (2) to show that

\[
\frac{1}{2} \hat{q}^T [\bar{\eta}\Lambda + \Gamma] \hat{q} = \frac{1}{2} q^T [\bar{\eta}\Lambda + \Gamma] q \leq \lambda_M \{\bar{\eta}\Lambda + \Gamma_\varepsilon\} \| \hat{q} \|^2 = \gamma \| \hat{q} \|^2.
\]

Thus we have that \( \gamma = \lambda_M \{\bar{\eta}\Lambda + \Gamma_\varepsilon\} \) and we can verify, from (49) and (50), that for the domain \( \bar{\phi} \in [-\phi_0, \phi_0] \) and \( \bar{\theta} \in \mathbb{R} \), we get \( \gamma = 2 \).

On the other hand, from (3), we have that

\[
\bar{\rho}^T [cI - a\Lambda] \hat{q} \leq || cI - a\Lambda || || \bar{\rho} || || \hat{q} || = \alpha || \bar{\rho} || || \hat{q} ||,
\]
and as \( cI - a\Lambda \) is a symmetric matrix, given by

\[
cI - a\Lambda = \begin{bmatrix}
c - a & 0 \\
0 & c - a \cos\left(\frac{\phi}{2}\right)
\end{bmatrix}
\]
then, considering (1), we have

\[
\alpha = || cI - a\Lambda || = \max\{ |c - a|, |c - a\delta| \},
\]

where the \( \delta \) parameter was defined in (43). It is worth noticing here that \( \alpha \) can also be written as

\[
\alpha = \begin{cases}
a - c, & \text{if } c \leq \frac{a(1+\delta)}{2}, \\
c - a\delta, & \text{if } c \geq \frac{a(1+\delta)}{2},
\end{cases}
\]

and that for any \( a \), the minimum value that \( \alpha \) can take is

\[
\alpha = \frac{a(1-\delta)}{2},
\]  
(51)
and that value is obtained when

\[
c = \frac{a(1+\delta)}{2}.
\]  
(52)

Notice that (52) represents a line in the \( a-c \) plane; and that is why it is called here the line of minimum \( \alpha \).

Once parameters \( \gamma \) and \( \alpha \) are defined, \( \hat{V}(\bar{\eta}, \bar{\rho}, \hat{q}) \) can be upper bounded as

\[
\hat{V}(\bar{\eta}, \bar{\rho}, \hat{q}) \leq -\frac{k_\varepsilon}{4} (1 - \bar{\eta}^2) - \frac{\| \bar{\rho} \|^T}{\| \hat{q} \|^T} Q \frac{\| \bar{\rho} \|}{\| \hat{q} \|},
\]
where

\[ Q = \begin{bmatrix}
    -\frac{3}{2}k_p & -\frac{1}{2}(k_v + \alpha) \\
    -\frac{1}{2}(k_v + \alpha) & \frac{\delta^2}{k_p} - \frac{v^2}{2}
\end{bmatrix}, \quad (53)
\]

and we have used (32), taking a part of the quadratic factor for \( \rho \), to form the term with \( 1 - \tilde{\eta}^2 \). Moreover, it is worth noticing that if \( \alpha = 0 \), it would be easy to find a condition in terms of \( c \) for the determinant of \( Q \) to be positive. And that suggests that a good option to ensure that \( P > 0 \) is to select the minimum alpha possible.

In order to do so, let us replace in (53) \( a \) and \( c \), from (51) and (52), respectively, so that the condition for \( Q \) to be a positive definite matrix is:

\[
(1 - \delta)^2 a^2 - 2(1 + 5\delta)k_v a + 4k_v^2 + 6yk_p < 0. \quad (54)
\]

Notice that as the left-hand side of (54) is a quadratic polynomial in terms of \( a \), then the values of \( a \) that satisfy this inequality are those complying with

\[
a_1 = a_0 - \frac{\sqrt{a_r}}{(1 - \delta)^2} < a < a_0 + \frac{\sqrt{a_r}}{(1 - \delta)^2} = a_2 \quad (55)
\]

with

\[
a_0 = \frac{(1 + 5\delta)k_v}{(1 - \delta)^2}, \quad (56)
\]

and

\[
a_r = (1 + 5\delta)^2k_v^2 - 2(1 - \delta)^2 \left[ 2k_v^2 + 3yk_p \right] \quad (57)
\]

as long as \( a_r > 0 \) or, equivalently, as

\[
\mu_1(\delta) = \frac{(1 + 5\delta)^2 - 4(1 - \delta)^2}{6(1 - \delta)^2} > \frac{yk_p}{k_v^2} = \mu; \quad (58)
\]

and, solving inequality (58) for \( \delta \), it can be shown that it is satisfied whenever (44) is, at least for \( \frac{1}{7} < \delta < 1 \). Indeed, it can be shown that provided (44) is satisfied, there is a region \( D_0 \) in the \( a-c \) plane for which any point in \( D_0 \) makes the determinant of \( Q \) to be positive. Thus, (44) is the condition that ensures the existence of a point \( P(a, c) \) which makes \( Q \) a positive definite matrix.

Moreover, the line of minimum \( \alpha \) cuts the boundary of \( D \) at points \( P_1(a_1, c_1) \) and \( P_2(a_2, c_2) \), and the midpoint of segment \( P_1P_2 \) is \( P_0(a_0, c_0) \). The abscissa coordinate of these points can be computed employing (55)-(57), and for the ordinate, we can apply mapping (52) with \( a \in \{a_0, a_1, a_2\} \); that is the reason why expressions (46), which correspond to \( (a, c) = (a_0, c_0) \) are suggested to be used in (45).

It only remains to verify that the condition for the positivity of matrix \( P \), (47), is still met by this choice of \( a \) and \( c \). Replacing in (47) these parameters by the expressions in (46), and multiplying by \( \gamma \) in both sides, we get

\[
\mu_2(\delta) = \frac{(1 + 5\delta)^2(1 + \delta)\gamma}{(1 - \delta)^4} > \frac{yk_p}{k_v^2} = \mu. \quad (59)
\]
Notice that the fulfilment of condition (59) ensures that matrix $P$ is positive definite, for any $\delta \in (\frac{1}{2}, 1) \subset \mathbb{R}$, but in the same interval we can verify that

$$\mu_1(\delta) < \mu_2(\delta).$$

Thus, in order to ensure that matrices $P$ and $Q$ are both positive definite we only have to check condition (58), and that is done by taking a $\delta$ such that (44) is met.

Notice that $P > 0$ ensures that $V(\tilde{\eta}, \tilde{\rho}, \tilde{q})$ is globally positive definite around the equilibrium $E_1 \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$; that is to say that $V(1, 0, 0) = 0$ and $V(\tilde{\eta}, \tilde{\rho}, \tilde{q}) > 0$, for all $[\tilde{\eta} \tilde{\rho}^T \tilde{q}^T]^T \neq [1 \ 0^T \ 0^T]^T \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$. Also notice that, as function $V$ is independent of time, then it is a globally decrescent function around $E_1$. In addition, $Q > 0$ implies that $\dot{V}(\tilde{\eta}, \tilde{\rho}, \tilde{q})$ is (locally) negative definite around the equilibrium $E_1 \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$; that is to say that $\dot{V}(1, 0, 0) = 0$ and there is a neighbourhood $B_r \subset S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$ around that equilibrium such that $\dot{V}(\tilde{\eta}, \tilde{\rho}, \tilde{q}) < 0$, for all $[\tilde{\eta} \tilde{\rho}^T \tilde{q}^T]^T \neq [1 \ 0^T \ 0^T]^T \in B_r$. But it is worth noticing that the neighbourhood $B_r$ is the whole manifold $S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$, excluding both the equilibria $E_1$ and $E_2$.

Thus, according to Theorem 1, we can conclude that the equilibrium $[1 \ 0^T \ 0^T]^T$ is asymptotically stable, and its domain of attraction is given by $B_r = S^2 \times \mathbb{R}^2 - \{E_1, E_2\} \subset \mathbb{R}^5$. This implies that, starting from an initial condition in $B_r$, the stabilization of the spherical pendulum to a constant desired orientation is fulfilled.

### 5.3.2. Equilibrium $E_2$

It is assumed that equilibrium $E_2$ is unstable. In order to show that, let us consider the following function

$$V_c(\tilde{\eta}, \tilde{\rho}, \tilde{q}) = a(\tilde{\eta} + 1)^2 + a\tilde{\rho}^T \tilde{\rho} - \frac{c}{2k_p} \tilde{q}^T \tilde{q} + \tilde{\rho}^T \tilde{q},$$

with $a$ and $c$ chosen as in (46). Notice the similarity between functions (45) and (60).

In order to apply Theorem 2 we have to verify that all the conditions on $V_c$ are satisfied. Notice that as $V_c$ does not depend explicitly on $t$, then the condition II is trivially satisfied. Moreover, $V_c(-1, 0, 0) = 0$ (condition I) and $V_c(\tilde{\eta}, \tilde{\rho}, \tilde{q}) > 0$, for $\tilde{q} = 0 \in \mathbb{R}^2$ and $[\tilde{\eta} \ \tilde{\rho}]^T \in S^2$ arbitrarily close to the equilibrium $E_2 \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$ (condition III). Thus, the application of Theorem 2 goes to prove that the time derivative of $V_c(\tilde{\eta}, \tilde{\rho}, \tilde{q})$ is positive definite around the equilibrium $E_2 \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$.

Now, taking the time derivative of (60) along the trajectories of system (37) we get:

$$\dot{V}(\tilde{\eta}, \tilde{\rho}, \tilde{q}) = 2a(\tilde{\eta} + 1) \dot{\tilde{\eta}} + 2a\tilde{\rho}^T \tilde{\rho} - \frac{c}{k_p} \tilde{q}^T \tilde{q} + \tilde{\rho}^T \tilde{q} + \tilde{\rho}^T \dot{\tilde{q}}$$

$$= k_p \dot{\tilde{\rho}}^T \tilde{\rho} + \tilde{\dot{\eta}}^T \left[ \frac{c_k}{k_p} I - \frac{1}{2} (\tilde{\eta} \Lambda + \Gamma) \right] \tilde{q} - \tilde{\rho}^T [k_r I + (cI - a\Lambda)] \tilde{q},$$

where $\Lambda$ and $\Gamma$ have been already defined in (49) and (50).

It is worth noticing that $V_c(\tilde{\eta}, \tilde{\rho}, \tilde{q}) = -\dot{V}(\tilde{\eta}, \tilde{\rho}, \tilde{q})$, which was obtained in the previous section. This should be no surprise, if we notice that, as $\tilde{\eta}^2 + \tilde{\rho}^T \tilde{\rho} = 1$, then:
(\tilde{\eta} - 1)^2 + \tilde{\rho}^T \tilde{\rho} = 2(1 - \tilde{\eta}) \quad \text{and} \quad (\tilde{\eta} + 1)^2 + \tilde{\rho}^T \tilde{\rho} = 2(1 + \tilde{\eta}).

This is an interesting and useful result. And according to the analysis in Section 5.3.1, we can conclude that \( \dot{V}_c(\tilde{\eta}, \tilde{\rho}, \tilde{q}) \) is a (locally) positive definite function around equilibrium \( E_2 \). Thus, as all the conditions of Theorem 2 are fulfilled, the conclusion of instability of the equilibrium \( E_2 \) arises.

### 5.3.3. Discussion

From the previous analysis it is clear that the spherical pendulum system can be stabilized by the inverse–dynamics controller (30), which uses a non–minimal description of the orientation given by the Euler parameters. It is worth noticing that, due to the properties of Euler parameters, equilibria \( E_1 \) and \( E_2 \) represent the same orientation (given when the pendulum orientation equals the constant desired orientation), so we can say that the convergence to the desired orientation is ensured globally.

### 6. Simulation results

In order to validate the performance of the proposed approach, we carried out some simulations in Matlab/Simulink. Figure 2 shows how the controller was implemented. Two scenarios were considered:

1. A step reference, given by \( \phi_d \) and \( \theta_d \), starting from the home (vertical) configuration, where \( \phi = \theta = 0 \).
2. A series of step references, where the last configuration is the initial condition for the next step.

For the simulations, in both scenarios, we use \( m = 0.2 \) [kg], and \( l = 0.3 \) [m] as the pendulum parameters. The control gains were chosen as \( k_p = 180 \) [rad/s²] and \( k_v = 18 \) [1/s], and as \( \gamma = 2 \), then \( \mu = \frac{l}{m} \), and we can verify that if we take \( \phi_0 = \frac{4\pi}{5} \) (or 144°), so that \( \delta = 0.3090 \), then (44) is satisfied; and by using (46) we have that with \( a = 95.94 \), \( c = 62.79 \), then \( \det(P) = 16.48 \), and \( \det(Q) = 58.58 \), so the asymptotic stability of equilibrium \( E_1 \) is guaranteed.

### 6.1. Simulation scenario 1

The simulation starts at the home configuration, i.e. \( \phi(0) = \theta(0) = 0 \), so the initial orientation is given by \( \xi(0) = [1 \ 0 \ 0 \ 0]^T \in S^3 \). The desired orientation is expressed in terms of the Euler parameters as \( [0.8536 \ 0.3536 \ 0.3536 \ 0.1464]^T \in S^3 \), which corresponds to the angles \( \phi_d = \theta_d = \frac{\pi}{4} \) [rad]. The initial values of the closed–loop states are then computed using (27) and (31); we get

\[
[\tilde{\eta}(0) \ \tilde{\rho}(0)^T \ \tilde{q}(0)^T] = [0.8536 \ 0.3536 \ 0.3825 \ 0 \ 0]^T \in S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5.
\]

Figure 3 shows the time evolution of the three closed–loop non–minimal states. Graphs show the convergence to the desired values without overshooting. And as \( \tilde{\rho} = 0 \Leftrightarrow \tilde{\epsilon} = 0 \), the convergence to the desired orientation is also ensured. Figure 4
Figure 3. Simulation results for scenario 1: Time evolution of (a) $\tilde{\eta}$; (b) the norm of $\rho$; (c) the norm of $q$.

Figure 3. (Continued).
Figure 3. (Continued).

Figure 4. Simulation results for scenario 1: Trace of the pendulum’s tip.
shows the trace of the pendulum’s tip on the surface of the sphere; notice that a frame representing the change of orientation is included along the trace.

### 6.2. Simulation scenario 2

For this scenario we considered four desired orientation references, each applied during a time interval of different duration. The simulation also starts at the home configuration. The first desired orientation is set at \( t = 0 \) [s], and it is expressed in terms of the Euler parameters as \([0.3827 \quad 0.9239 \quad 0 \quad 0]^T \in S^3\), which corresponds to the angles \( \phi_d = \frac{3\pi}{4} \) (\( < \phi_0 \)) and \( \theta_d = 0 \) [rad]. The desired orientation is changed at \( t = 1 \) [s], and now is given by \([0.8924 \quad 0.3696 \quad 0.2391 \quad 0.0990]^T \in S^3\), which corresponds to \( \phi_d = \frac{\pi}{4} \) and \( \theta_d = \frac{\pi}{6} \) [rad]. At \( t = 2.5 \) [s] the desired orientation changes again, and is defined by \([0.6871 \quad 0.3967 \quad 0.5272 \quad 0.3044]^T \in S^3\), corresponding to the angles \( \phi_d = \frac{\pi}{4} \) and \( \theta_d = \frac{\pi}{6} \) [rad]. Finally, the desired orientation reference is set to \([0.9239 \quad 0 \quad 0.3827 \quad 0]^T \in S^3\), or equivalently \( \phi_d = 0 \) and \( \theta_d = \frac{\pi}{4} \) [rad], at \( t = 5 \) [s].

Figure 5 shows the time evolution of the three non–minimal states of the closed–loop system. Graphs show again the convergence to the desired values, where \( \tilde{\eta} = 0 \) and \( \tilde{\rho} = 0 \).

The trace of the pendulum’s tip for scenario 2 is shown in Figure 6. It is worth noticing that the paths traced on the surface of the sphere are not the ones with minimal length. The reason of this is that, as the proposed controller works only for the regulation (or point-to-point) case, it is only guaranteed that the final orientation of each interval is reached, but the intermediate path cannot be determined a priori.

![Figure 5](image-url) Simulation results for scenario 2: Time evolution of (a) \( \tilde{\eta} \); (b) the norm of \( \tilde{\rho} \); (c) the norm of \( \dot{q} \).
Figure 5. (Continued).
The most general case, where a motion controller for the spherical pendulum using the proposed non-minimal representation is to be designed for tracking a time-varying continuous desired–orientation trajectory, is still an open problem.

7. Conclusion

An spherical pendulum is a 2-dof mechanism, thus requiring only two independent coordinates to describe its configuration. In this paper, however, we have shown how to use unit quaternions to define a non–minimal set of configuration coordinates for modelling and controlling a fully–actuated torque–driven spherical pendulum.

A Lyapunov–based stability analysis is given, which allows us to conclude that the pendulum configuration asymptotically converges to the desired constant orientation. Simulations corroborated this fact. Thus, it has been shown that this approach is useful for orientation control of mechanisms.

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