Existence and uniqueness of a weak solution to fractional single-phase-lag heat equation

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Received: 14 October 2022 / Revised: 26 May 2023 / Accepted: 30 May 2023 / Published online: 19 June 2023

Abstract
In this article, we study the existence and uniqueness of a weak solution to the fractional single-phase lag heat equation. This model contains the terms $D_t^\alpha (\partial_t u)$ and $D_t^\alpha u$ (with $\alpha \in (0, 1)$), where $D_t^\alpha$ denotes the Caputo fractional differential operator (in time) of order $\alpha$. We consider homogeneous Dirichlet boundary data for the temperature. We rigorously show the existence of a unique weak solution under low regularity assumptions on the data. Our main strategy is to use the variational formulation and a semidiscretisation in time based on Rothe’s method. We obtain a priori estimates on the discrete solutions and show convergence of the Rothe functions to a weak solution. The variational approach is employed to show the uniqueness of this weak solution to the problem. We also consider the one-dimensional problem and derive a representation formula for the solution. We establish bounds on this explicit solution and its time derivative by extending properties of the multivariate Mittag-Leffler function.

Keywords Fractional heat equation · Single-phase-lag · Time discretisation · Existence · Uniqueness · Rothe’s method · Multivariate Mittag-Leffler function

Mathematics Subject Classification 35A01 · 35A02 · 35R11 · 65M12 · 33E12

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1 Introduction

1.1 Problem formulation

Consider a material contained in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ with a Lipschitz continuous boundary $\partial \Omega$. Set $Q_T := \Omega \times (0, T]$ and $\Sigma_T := \Gamma \times (0, T]$, where $T > 0$ denotes the final time. The function $u(x, t)$ represents the temperature at a material point $x \in \Omega$ at time $t$. The heat flux is denoted by $q(x, t)$. Let $\rho$ and $c$ be positive constants denoting the material’s density and specific heat, respectively. Moreover, let $k$ be the thermal conductivity, which may be space-dependent.

In [41], a modification of the classical Fourier law for heat conduction processes,

$$q(x, t + \tau_q) = -k(x) \nabla u(x, t + \tau_T), \quad (1.1)$$

was proposed to overcome the presence of infinite speed of heat propagation. This model links the heat flux $q$ to the temperature gradient $\nabla u$ at the spatial point $x$ at different times, allowing for a delay in the build-up of the heat flux or temperature gradient. Here $\tau_q$ and $\tau_T$ are the phase-lag parameters. These parameters are material properties and represent the relaxation and delay time, respectively. In case $\tau_q > 0$ and $\tau_T = 0$, the law is of single-phase-lag (SPL) type. If $\tau_q > 0$ and $\tau_T > 0$, the law is called to be of dual-phase-lag (DPL) type. The classical Fourier law is recovered if both lagging parameters equal zero.

In this contribution, we consider the model obtained by first-order Taylor approximation of (1.1) if $\tau_T = 0$, where we neglect the higher order terms and replace the time derivative by a fractional variant of Caputo type with order $\alpha \in (0, 1)$. The adapted form of (1.1) reads as

$$\left(1 + \tilde{\tau}_q^\alpha D_t^\alpha\right) q(x, t) = -k(x) \nabla u(x, t), \quad (1.2)$$

where $\tilde{\tau}_q^\alpha := \frac{\tau_q^\alpha}{\Gamma(1+\alpha)}$, see e.g. [27] and [1, Remark 2.4] for the fractional Taylor expansion formulas. In the above, the so-called Caputo differential operator of order $\alpha$, with $0 < \alpha < 1$ (see [6, Def. 3.2] and [1]) is considered, which is defined by

$$D_t^\alpha f(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} (f(s) - f(0)) \, ds, \quad t \in (0, T],$$

where $\Gamma$ denotes the Gamma function. We use the (non-local) Caputo derivative in (1.2) to allow for memory effects in a power-like fashion and because of the compatibility with the initial data, i.e. only integer order initial conditions are needed. For small $\tau_q$, the fractional Taylor expansion used to replace $Q(t + \tau_q)$ by $Q(t) + \tilde{\tau}_q^\alpha D_t^\alpha Q(t)$, yields a good approximation for functions of the form $Ct^\alpha$ since $D_t^\alpha t^\alpha = \Gamma(1+\alpha)$ for $\alpha \in (0, 1)$.

Moreover, we consider the classical energy conservation law given by

$$\rho c \partial_t u(x, t) + \nabla \cdot q(x, t) = G(x, t), \quad (1.3)$$
Existence and uniqueness...

where $G$ involves the heat sources and sinks. Following [14, 25, 36], we can write

$$
\left(1 + \tau_q^\alpha D_t^\alpha\right) G(x, t) = F(x, t) - a \left(1 + \tau_q^\alpha D_t^\alpha\right) u(x, t), \quad (1.4)
$$

where $a \geq 0$ is a constant and $F$ is a function representing the sources.

The form of (1.4) is attributed to applying the model in fractional bioheat equations describing heat transfer within skin tissue in thermal therapy [14, 36]. Herein, the heat source $G$ is decomposed into several components [14, p. 51]. These components involve the metabolic heat generation, the heat source due to blood circulation (modelled as a linear function of the temperature $u$) and the heat generated due to the absorption of electromagnetic radiation. This latter component only depends on the location of the tissue relative to the electromagnetic source.

Applying the operator $I + \tau_q^\alpha D_t^\alpha$ on (1.3), and employing (1.2) and (1.4), we obtain the following equation

$$
\rho c \tau_q^\alpha D_t^\alpha \partial_t u(x, t) + a \tau_q^\alpha D_t^\alpha u(x, t) + \rho c \partial_t u(x, t) + \mathcal{L} u(x, t) = F(x, t), \quad (1.5)
$$

for $(x, t) \in Q_T$, where the operator $\mathcal{L}$ is the following second-order linear differential operator

$$
\mathcal{L} u(x, t) := au(x, t) + \nabla \cdot (-k(x) \nabla u(x, t)).
$$

Equation (1.5) represents a fractional form of the Cattaneo-Vernotte type equation [2, Eq. (7.6) and (7.9)]. The presence of $\tau_q$ in (1.1) leads to the wave-like nature of the modified Fourier law (1.2), whilst the introduction of the fractional derivative allows for memory effects (time non-locality).

From now on, for convenience, we omit the tilde in $\tilde{\tau}^\alpha_q$ and simply write $\tau^\alpha_q$ for this parameter. The main target of this paper is to study the existence and uniqueness questions for the single-phase lag heat equation (1.5) subjected to given initial data and homogeneous Dirichlet boundary conditions. This SPL problem is given by

$$
\begin{align*}
\rho c \tau_q^\alpha D_t^\alpha \partial_t u(x, t) &+ a \tau_q^\alpha D_t^\alpha u(x, t) + \rho c \partial_t u(x, t) + \mathcal{L} u(x, t) = F(x, t), \quad (x, t) \in Q_T \\
\partial_t u(x, 0) &= U_0(x), \quad x \in \Omega \\
\partial_t u(x, 0) &= V_0(x), \quad x \in \Sigma_T \\
u(x, t) &= 0, \quad (x, t) \in \Sigma_T.
\end{align*}
$$

1.2 Literature overview, new aspects and outline

First, we provide an overview of the main results when considering classical derivatives. In [31], the stability of the corresponding problem was investigated. Models obtained by first-order approximation for $u$ and second-order approximation for $q$ in (1.1) yield a hyperbolic equation which is (exponentially) stable if $\tau_q < 2\tau_T$, whereas second-order approximation for both $u$ and $q$ yield stability if $\tau_T > (2 - \sqrt{3})\tau_q$ or $\tau_T \leq (2 - \sqrt{3})\tau_q$, and the (restrictive) condition [31, Eq. (3.10)] hold. For the
first-order approximation considered in this paper, their result shows no conditions on the phase-lag parameters for stability. The well-posedness of these problems on smoothly bounded domains and convex domains was studied in [20, 32] with the aid of the semigroup theory. In the case of first-order approximation for both \( u \) and \( q \), the well-posedness on a bounded domain was investigated in [50] for different boundary conditions and extended to higher dimensions in [49].

Fractional calculus has attracted many researchers because of the nonlocal property of the fractional derivatives and their use in the modelling part of complex systems, see, e.g. [40] and references therein. Fractional wave and diffusion equations (with constant order fractional derivatives) were studied in e.g. [11, 13, 22, 24, 28, 33] by means of eigenfunction expansions and in e.g. [43–46] by means of the Rothe method. For well-posedness results about equations of the form \( D^\alpha_t u + \partial_t u(x, t) + Lu = f \), we refer to e.g. [29, 37]. Similar results related to multi-term fractional equations can be found in [19, 23, 39, 51, 54], the fractional telegraph equation [5, 9, 15, 53, 56] and the recent works about variable-order operators [47, 48, 55]. To the best of our knowledge, there is no general well-posedness result known for the SPL problem (1.6) under low regularity conditions. The goal of this paper is to fill in this gap. Note that the inverse source problems considered in the work [25] assume the well-posedness of problem (1.6).

This paper is organised as follows. In Section 2, we fix the notations and assumptions, and give some valuable properties of the convolution kernel together with some technical results. The Fourier method will be used to obtain a representation formula for the solution in 1D in Section 3. Section 4 covers the existence of the solution. It includes the setup of the weak formulation and the time discretisation, as well as the a priori estimates and the Rothe functions. The uniqueness part is dealt with in Subsection 4.6.

2 Preliminaries

2.1 Notations and assumptions

We start by introducing the function spaces used throughout this work. The classical \( L^2 \)-inner product is denoted by \( \langle \cdot, \cdot \rangle \) and the corresponding norm by \( \| \cdot \| \). Let \( (X, \| \cdot \|_X) \) be a Banach space. Its dual space is denoted by \( X^* \). The duality pairing between \( H^1_0(\Omega)^* \) and \( H^1_0(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle_{H^1_0(\Omega)^* \times H^1_0(\Omega)} \) and is seen as a continuous extension of \( \langle \cdot, \cdot \rangle \). For \( p \geq 1 \) the space \( L^p((0, T), X) \) is the space of all measurable functions \( u : (0, T) \to X \) such that

\[
\| u \|_{L^p((0, T), X)} := \int_0^T \| u(t) \|_X^p \, dt < \infty.
\]

The space \( L^\infty((0, T), X) \) is the space of all measurable functions \( u : (0, T) \to X \) that are essentially bounded, i.e.

\[
\| u \|_{L^\infty((0, T), X)} := \text{ess sup}_{t \in [0, T]} \| u(t) \|_X < \infty.
\]
The space $C([0, T], X)$ consists of all continuous functions $u : [0, T] \rightarrow X$ such that

$$\|u\|_{C([0, T], X)} := \max_{t \in [0, T]} \|u(t)\|_X < \infty.$$ 

The space $H^k((0, T), X)$ consists of all functions $u : [0, T] \rightarrow X$ such that the weak derivative with respect to $t$ up to order $k$ exists and

$$\|u\|_{H^k((0, T), X)}^2 := \int_0^T \left( \sum_{i=0}^k \|u^{(i)}(t)\|_X^2 \right) dt < \infty.$$ 

Similar notations will be used for the case of vector-valued functions.

Additionally, the values of $C, \varepsilon$ and $C_\varepsilon$ are considered generic and positive constants. Their value can differ from place to place, but their meaning should be clear from the context. These constants are independent of the time discretisation parameter, where $\varepsilon$ is arbitrarily small and $C_\varepsilon$ arbitrarily large, i.e. $C_\varepsilon = C \left( 1 + \varepsilon + \frac{1}{\varepsilon} \right)$.

Now, we review some properties related to the operator $\mathcal{L}$. The bilinear form $\mathcal{L}$ associated to the operator $\mathcal{L}$ is given by

$$\mathcal{L}(u(t), \varphi) := \langle k \nabla u(t), \nabla \varphi \rangle + a(u(t), \varphi), \quad u(t), \varphi \in H^1_0(\Omega).$$

for $u, \varphi \in H^1_0(\Omega)$. Here $a \geq 0$ and the matrix $k$ is a symmetric matrix-valued function on $\Omega$ consisting of essentially bounded functions, i.e.

$$k = (k_{i,j}(x)) \in L^\infty(\Omega) := (L^\infty(\Omega))^{d \times d}, \quad k = k^\top.$$ 

We assume $k$ to be uniformly elliptic, that is there exists a constant $\tilde{k} > 0$ such that for all $x \in \Omega$ and all $\xi = (\xi_1, \ldots, \xi_d)^\top \in \mathbb{R}^d$ it holds that

$$\xi^\top \cdot k(x)\xi = \sum_{i,j=1}^d k_{i,j}(x)\xi_i\xi_j \geq \tilde{k} \|\xi\|^2.$$ 

These assumptions yield the following type of inequalities for $\mathcal{L}$:

$$\mathcal{L}(u, \varphi) \leq C \|u\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}$$

$$\mathcal{L}(\varphi, \varphi) \geq \tilde{k} \|\nabla \varphi\|^2,$$  

for $u, \varphi \in H^1_0(\Omega)$. It follows that $\mathcal{L}$ is $H^1_0(\Omega)$-elliptic by applying the Friedrichs’ inequality on (2.1).

Finally, we remember that the fractional order appearing in (1.6) satisfies $0 < \alpha < 1$ and that the fractional phase-lag parameters $\tau^{\alpha}_q$ is considered to be positive.
2.2 Properties of the kernel

We denote by \( g_{\alpha} \) the Riemann-Liouville kernel

\[
g_{\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0, \alpha \in (0, 1).
\]

The Caputo derivative operator \( D_t^{\alpha} \) can be rewritten as a convolution

\[
D_t^{\alpha} f(t) = \frac{d}{dt} \left( g_{\alpha} \ast (f - f(0)) \right)(t), \quad (2.2)
\]

where \( \ast \) denotes the Laplace convolution of two functions

\[
(k \ast z)(t) = \int_0^t k(t - s)z(s) \, ds.
\]

The ensuing lemma contains several properties of the kernel \( g_{\alpha} \), see e.g. [45, Section 2], [45, Corollary 3.1] and [13, Corollary 2].

Lemma 1 The function \( g_{\alpha}(t), t > 0, 0 < \alpha < 1 \) satisfies

(i) \( g_{\alpha} \in L^1(0, T) \);

(ii) \( g_{\alpha} \) is decreasing in time \( t \) and

\[
g_{\alpha}(t) \geq \frac{\min\{1, T^{-\alpha}\}}{\Gamma(1-\alpha)} > 0, \quad t \in (0, T];
\]

(iii) \( g'_{\alpha} \in L^1_{\text{loc}}(0, T) \);

(iv) \( g_{\alpha}(t) \geq 0, g'_{\alpha}(t) \leq 0, g''_{\alpha}(t) \geq 0 \) for all \( t > 0 \) and \( \partial_t g_{\alpha}(t) \neq 0 \), thus \( g_{\alpha} \) is strongly positive definite, i.e. for all \( v \in L^2_{\text{loc}}((0, \infty), L^2(\Omega)) \) it holds that

\[
\int_0^t \left( (g_{\alpha} \ast v)(s), v(s) \right) ds \geq 0, \quad t \geq 0;
\]

(v) For any \( v : [0, T] \rightarrow L^2(\Omega) \) satisfying \( v \in L^2((0, T), L^2(\Omega)) \) with \( g_{\alpha} \ast v \in H^1((0, T), L^2(\Omega)) \), it holds for all \( t \in [0, T] \) that

\[
\int_0^t \left( \partial_t (g_{\alpha} \ast v)(s), v(s) \right) ds \geq \frac{g_{\alpha}(T)}{2} \int_0^t \|v(s)\|^2_{L^2(\Omega)} \, ds. \quad (2.3)
\]

The Riemann-Liouville fractional integral

\[
I_{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds = g_{1-\alpha} \ast f(t), \quad \text{for } \alpha > 0, f \in L^1(0, T),
\]

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has the property [34, Eq. (2.21)]

\[
(I^{\alpha_1} (I^{\alpha_2} f)) (t) = (I^{\alpha_1 + \alpha_2} f) (t), \quad \forall t \in [0, T],
\]

(2.4)

if \( \alpha_1 > 0, \alpha_2 > 0 \) and \( f \in C([0, T]) \). This property holds true for a.e. \( t \in [0, T] \) if \( f \in L^1(0, T) \) and it leads to the following relation between the Riemann-Liouville kernel and the Caputo fractional derivative, see e.g. [34].

**Lemma 2** Let \( u \) be absolutely continuous on \( [0, T] \) and assume that \( \alpha \in (0, 1) \). Then for a.e. \( t \in [0, T] \) it holds that

\[
\partial_t (g_\alpha * u)(t) = g_\alpha(t)u(0) + (g_\alpha * \partial_t u)(t).
\]

### 3 Solution via Fourier method in 1D

In this part, we consider (1.6) with \( k(x) = \tilde{k} > 0 \) on the domain \( \Omega = (0, L), L > 0 \), and we suppose that the solution \( u \) can be written as \( u(x, t) = X(x)T(t) \). Employing the separation of variables technique leads to the eigenvalue problem

\[
\begin{aligned}
-\tilde{k}X''(x) + aX(x) &= \sigma X(x), \quad x \in (0, L) \\
X(0) &= X(L) = 0,
\end{aligned}
\]

where \( \sigma \) is the separation constant and \( a > 0 \). The normalised solutions to this problem are given by

\[
X_n(x) = \sqrt{\frac{2}{L}} \sin \left( \sqrt{\frac{\sigma_n - a}{\tilde{k}}} x \right) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi}{L} x \right),
\]

where \( \sigma_n = a + \tilde{k} \left( \frac{n\pi}{L} \right)^2 \) for \( n \in \mathbb{N} \). The functions \( T_n(t) \) satisfy a fractional differential equation of the form (assuming \( \partial_t u \) absolutely continuous in time)

\[
\rho c \tau_q^\alpha \partial_t^\alpha T' + \rho c T' + a \tau_q^\alpha \partial_t^\alpha T + \sigma_n T = 0
\]

where \( \partial_t^\alpha \) is the classical Caputo fractional derivative of order \( \alpha \) [4]. Adopting the approach of Luchko and Gorenflo in [21, Theorem 4.1], where an explicit formula for multi-term fractional differential equations is presented, to our setting, we recover the solution for \( T_n(t) \) in the following form

\[
T_n(t) = c_n T_n^1(t) + d_n T_n^2(t), \quad c_n, d_n \in \mathbb{R},
\]

where

\[
T_n^1(t) = E_{(\alpha+1, 1, \alpha)} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} \right)
\]
\[ + \frac{a}{\rho c} t E_{(\alpha+1,1,\alpha),2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha t^\alpha}, -\frac{1}{\rho c t^\alpha} \right) \]

\[ + \frac{t^\alpha}{\tau_q^\alpha} E_{(\alpha+1,1,\alpha),2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha t^\alpha}, -\frac{1}{\rho c t^\alpha} \right) \]

and

\[ T_n^2(t) = t E_{(\alpha+1,1,\alpha),2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha t^\alpha}, -\frac{1}{\rho c t^\alpha} \right). \]

The multivariate Mittag-Leffler function, see [8, Eq. (3.16)], is given by

\[ E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1, \ldots, z_m) = \sum_{k=0}^{\infty} \sum_{\sum_{j=1}^m k_j = k} \left( \begin{array}{c} k \\ k_1, \ldots, k_m \end{array} \right) \frac{\prod_{j=1}^m z_j^{k_j}}{\Gamma(\beta + \sum_{j=1}^m \alpha_j k_j)}. \]

(3.1)

where \( \left( \begin{array}{c} \alpha_1, \ldots, \alpha_m \end{array} \right) \) is the multinomial coefficient. Let us remark that (3.1) is invariant under (the same) permutation on the parameters \( \alpha_j \) and the variables \( z_j \) with \( j = 1, \ldots, m \).

The unknown function \( u(x, t) \) can then be represented as the series

\[ u(x, t) = \sum_{n=1}^{\infty} X_n(x) \left[ (U_0, X_n) T_n^1(t) + (V_0, X_n) T_n^2(t) \right]. \]

(3.2)

Before considering the regularity of \( u \) we first list some useful results concerning the multivariate Mittag-Leffler function (3.1). The first one is a slight generalisation of [18, Lemma 3.3], which we recover by setting \( \gamma = \beta \).

**Lemma 3** Let \( \alpha_1, \ldots, \alpha_m \) be positive constants and \( q_1, \ldots, q_m, \beta \) be real. Then, for real \( \gamma \), we have that

\[ \frac{d}{dt} \left[ t^\gamma E_{(\alpha_1,\ldots,\alpha_m),\beta+1} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right) \right] \]

\[ = t^{\gamma-1} E_{(\alpha_1,\ldots,\alpha_m),\beta} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right) \]

\[ + (\gamma - \beta) t^{\gamma-1} E_{(\alpha_1,\ldots,\alpha_m),\beta+1} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right). \]

**Proof** First, we calculate

\[ \frac{d}{dt} \left[ t^\beta E_{(\alpha_1,\ldots,\alpha_m),\beta+1} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right) \right] \]

\[ = \frac{d}{dt} \left[ \sum_{k=0}^{\infty} \sum_{\sum_{j=1}^m k_j = k} \left( \begin{array}{c} k \\ k_1, \ldots, k_m \end{array} \right) m \prod_{j=1}^m q_j^{k_j} t^{\beta + \sum_{j=1}^m \alpha_j k_j} \right] \]

\[ \Gamma \left( \beta + 1 + \sum_{j=1}^m \alpha_j k_j \right) \]
\[
\sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1, \ldots, k_m} \prod_{j=1}^{m} q_j^{k_j} t^{\beta - 1 + \sum_{j=1}^{m} \alpha_j k_j} \frac{1}{\Gamma(\beta + \sum_{j=1}^{m} \alpha_j k_j)}
\]

Then, an application of the chain rule shows that
\[
\frac{d}{dt} \left[ t^{\gamma} E_{(\alpha_1, \ldots, \alpha_m), \beta} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right) \right] = t^{\beta - 1} E_{(\alpha_1, \ldots, \alpha_m), \beta} \left( q_1 t^{\alpha_1}, \ldots, q_m t^{\alpha_m} \right),
\]
which proves the result.

The next result is a generalization of the property \([7, \text{Eq. 4.2.3}]\),
\[
E_{\alpha, \beta}(z) = z E_{\alpha, \beta} + \frac{1}{\Gamma(\beta)} \]
for the two-parameter Mittag-Leffler function
\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0, \quad \beta \in \mathbb{R}.
\]
The proof can be performed as in \([18, \text{Lemma 3.1}]\). Let us remark that this result remains valid in the limit case as \(\lim_{\beta \to 0} 1/\Gamma(\beta) = 0\).

**Lemma 4** Let \(\alpha_1, \ldots, \alpha_m\) be positive constants, \(z_1, \ldots, z_m \in \mathbb{C}\) and \(\beta > 0\) be fixed. Then
\[
\sum_{j=1}^{m} z_j E_{(\alpha_1, \ldots, \alpha_m), \beta + \alpha_j} (z_1, \ldots, z_m) + \frac{1}{\Gamma(\beta)} = E_{(\alpha_1, \ldots, \alpha_m), \beta} (z_1, \ldots, z_m).
\]

Finally, we extend \([18, \text{Lemma 3.2}]\) so that it can be used in our problem setting (multivariate Mittag-Leffler function \((3.1)\) with \(\alpha_1 \in (1, 2)\)) as well. Hence, the next lemma provides a crucial bound in the analysis of multi-term time-fractional wave equations. A similar estimate for a special case of the multivariate Mittag-Leffler function has recently been obtained in \([35, \text{Lemma 3}]\) assuming \(0 < \beta < 1 + \alpha_1\). The result presented below is more general as we consider the multivariate Mittag-Leffler function in its general form \((3.1)\) and only need \(\beta > 0\).

**Lemma 5** Let \(\beta > 0\) and \(\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0\) be given with \(\alpha_1 \in (0, 2)\). Assume that \(\alpha_1 \pi/2 < \mu < \min(\alpha_1 \pi, \pi), \mu \leq |\arg(z)| \leq \pi\) and there exists \(K > 0\) such that \(-K \leq z_j < 0\) for \(j = 2, \ldots, m\). Then there exists a constant \(C > 0\) depending only on \(\mu, K, \alpha_j, j = 1, \ldots, m\) and \(\beta\) such that
\[
|E_{(\alpha_1, \ldots, \alpha_m), \beta} (z_1, \ldots, z_m)| \leq \frac{C}{1 + |z_1|}.
\]
Proof The proof goes along the same lines as that of [18, Lemma 3.2], therefore we point out some key steps and comment on the differences. The starting point is the contour representation of the reciprocal of the gamma function, [30, Eq. (1.52)]

\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma(R,\theta)} \exp\left(\frac{1}{\alpha_1} \right) \xi^{(1-z-\alpha_1)/\alpha_1} d\xi, \tag{3.3}
\]

which is applicable since \( \alpha_1 < 2 \) and \( \pi \alpha_1/2 < \mu < \min\{\alpha_1 \pi, \pi\} \). Here \( \gamma(R, \theta) \) is the contour consisting of an circular arc \( \{\xi \in \mathbb{C} : |\xi| = R, |\arg(\xi)| \leq \theta\} \) and two half lines \( \{\xi \in \mathbb{C} : |\xi| > R, |\arg(\xi)| = \pm \theta\} \). The radius \( R \) is to be fixed later and \( \theta \) is chosen such that \( \alpha_1 \pi/2 < \theta < \mu \).

By means of (3.3), we can rewrite

\[
E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m)
= \frac{1}{2\pi i} \int_{\gamma(R,\theta)} \exp\left(\frac{1}{\alpha_1} \right) \xi^{1-\beta/\alpha_1} - 1 \sum_{k=0}^{\infty} \sum_{k_1+\ldots+k_m=k} \left( \frac{k}{k_1,\ldots,k_m} \right) 
\times \prod_{j=1}^{m} \frac{z_j^k}{\xi^k} + \sum_{j=2}^{m} z_j \xi^{-\alpha_j/\alpha_1}
\]

by isolating the index \( j = 1 \) and applying the multinomial theorem. Note that \( 1 - \frac{\alpha_j}{\alpha_1} \in (0, 1) \) for \( j = 2, \ldots, m \). Therefore, taking \( R > |z_1| + K \sum_{j=2}^{m} R^{1-\frac{\alpha_j}{\alpha_1}} \) ensures convergence of the series. In case all \( |z_j| \leq K, j = 1, \ldots, m \) we can fix \( R \) as a constant depending only on \( K \) and \( \alpha_1, \ldots, \alpha_m \). In that case we deduce that

\[
E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) = \frac{1}{2\pi i} \int_{\gamma(R,\theta)} \frac{\exp\left(\frac{1}{\alpha_1} \right) \xi^{1-\beta/\alpha_1}}{\xi - z_1 - \sum_{j=2}^{m} z_j \xi^{-\alpha_j/\alpha_1}} d\xi.
\]

Repeating the arguments of [18] we obtain in case \( |z_1| > R \) and \( \mu \leq |\arg(z_1)| \leq \pi \) that

\[
|E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m)| \leq C \frac{1}{|z_1|},
\]

and in case \( |z_1| \leq R, \mu \leq |\arg(z_1)| \leq \pi \) that

\[
|E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m)| \leq C E_{\alpha_m,\beta}(R + (m - 1)K),
\]
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where \( C \geq \max\{1, \frac{\beta + \alpha_1}{\alpha_m}\} \). To proof this last inequality, we used the bound

\[
\frac{1}{\Gamma(\beta + \sum_{j=1}^{m} \alpha_j k_j)} \leq \max\{1, \frac{\beta + \alpha_1}{\alpha_m}\} \frac{\Gamma(\beta + \alpha_m k)}{\Gamma(\beta + \alpha_m k)}.
\] (3.4)

This last result (3.4) can be shown in the following way. For \( k = 0 \) there is nothing to show, so assume \( k \geq 1 \). For \( \beta \geq 1 \), we have \( \Gamma(\beta + \sum_{j=1}^{m} \alpha_j k_j) \leq \Gamma(\beta + \alpha_m k) \), since \( \Gamma(z) \) is monotonically increasing for \( z \geq 1 \) and \( \alpha_1 > \alpha_2 > \cdots > \alpha_m > 0 \). For \( 0 < \beta < 1 \), we use that \( \Gamma(z + 1) = z \Gamma(z) \) to write

\[
\frac{\Gamma(\beta + \alpha_m k)}{\Gamma(\beta + \sum_{j=1}^{m} \alpha_j k_j)} \leq \frac{\beta + \alpha_m k}{\beta + \sum_{j=1}^{m} \alpha_j k_j} \leq \frac{\beta + \alpha_1}{\alpha_m k} + \frac{\alpha_1}{\alpha_m}.
\]

which is a constant that can be brought out of the summations. □

Using Lemma 4 or scrutinising the above proof, we see that Lemma 5 also stays valid in the limit case \( \beta \searrow 0 \).

Now, we continue to investigate the regularity of the solution (3.2) of the one-dimensional problem considered in this section. From Lemma 4, we have that

\[
E_{(\alpha+1,1,\alpha),1} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} t^{\alpha} \right) = 1 - \frac{\sigma_n}{\rho c \tau_q^\alpha} t^{\alpha+1} E_{(\alpha+1,1,\alpha),\alpha+2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} t^{\alpha} \right) - \frac{a}{\rho c} t E_{(\alpha+1,1,\alpha),2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} t^{\alpha} \right) - \frac{1}{\tau_q^\alpha} t^{\alpha} E_{(\alpha+1,1,\alpha),\alpha+1} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} t^{\alpha} \right).
\]

Hence, \( T_n^1 \) simplifies to

\[
T_n^1(t) = 1 - \frac{\sigma_n}{\rho c \tau_q^\alpha} t^{\alpha+1} E_{(\alpha+1,1,\alpha),\alpha+2} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha}, \frac{-a}{\rho c}, \frac{-1}{\tau_q^\alpha} t^{\alpha} \right).
\]
Moreover, from Lemma 5, it follows that

\[
\max_{t \in [0, T]} \left| T_n^i(t) \right| \leq T_i, \quad \forall n \in \mathbb{N}, \quad i = 1, 2.
\]

Turning back to (3.2), we have for all \( t \in [0, T] \) that

\[
\| u(t) \|_{L^2(0, L)}^2 = \sum_{n=1}^{\infty} \left| (U_0, X_n) T_n^1(t) + (V_0, X_n) T_n^2(t) \right|^2 \\
\leq 2 \max\{ T_1^2, T_2^2 \} \sum_{n=1}^{\infty} \left[ |(U_0, X_n)|^2 + |(V_0, X_n)|^2 \right] \\
\leq 2 \max\{ T_1^2, T_2^2 \} \left( \| U_0 \|_{L^2(0, L)}^2 + \| V_0 \|_{L^2(0, L)}^2 \right).
\]

Using Lemma 3, we obtain that

\[
\left( T_n^1 \right)'(t) = \frac{-\sigma_n}{\rho c \tau_q^\alpha} t^{\alpha+1} E_{\alpha+1, 1, \alpha} \left( \frac{\sigma_n}{\rho c \tau_q^\alpha}, \frac{\alpha+1}{\rho c t}, \frac{-1}{\tau_q^\alpha} \right)
\]

and

\[
\left( T_n^2 \right)'(t) = E_{\alpha+1, 1, \alpha} \left( \frac{-\sigma_n}{\rho c \tau_q^\alpha} t^{\alpha+1}, \frac{-1}{\tau_q^\alpha} \right).
\]

Therefore, from Lemma 5, it follows that for all \( n \in \mathbb{N} \)

\[
\left| \left( T_n^1 \right)'(t) \right| \leq \left( \frac{\sigma_n}{\rho c \tau_q^\alpha} \right)^{\frac{1}{2}} \left( \frac{\sigma_n}{\rho c \tau_q^\alpha} \right)^{\frac{1}{2}} t^{\alpha} C \\
\leq \sigma_n \frac{1}{2} \tilde{M} C := \sigma_n \frac{1}{2} \tilde{T}_1
\]

and

\[
\max_{t \in [0, T]} \left| \left( T_n^2 \right)'(t) \right| \leq \tilde{T}_2,
\]

where

\[
\tilde{M} := \max_{(r, t) \in [0, \infty] \times [0, T]} \frac{r^{\frac{1}{2}} t^{\alpha}}{1 + rt^{\alpha+1}} < +\infty.
\]

Using these bounds, we have for all \( t \in (0, T] \) that

\[
\| \partial_t u(t) \|_{L^2(0, L)}^2 = \sum_{n=1}^{\infty} \left| (U_0, X_n) \left( T_n^1 \right)'(t) + (V_0, X_n) \left( T_n^2 \right)'(t) \right|^2
\]
\[ \leq 2\hat{T}_1^2 \sum_{n=1}^{\infty} \sigma_n |(U_0, X_n)|^2 + 2\hat{T}_2^2 \sum_{n=1}^{\infty} |(V_0, X_n)|^2 \]
\[ \leq 2 \max\{\hat{T}_1^2, \hat{T}_2^2\} \left( \|U_0\|_{L^2_0(0,L)}^2 + \|V_0\|_{L^2_0(0,L)}^2 \right) . \]

Summarising the calculations above, we can conclude that
\[ \max_{t \in [0,T]} \|u(t)\|_{L^2_0(0,L)}^2 + \max_{t \in [0,T]} \|\partial_t u(t)\|_{L^2_0(0,L)}^2 \leq C \]
if \( U_0 \in H^1_0(0,L) \) and \( V_0 \in L^2(0,L) \).

### 4 The single-phase-lag problem

This section is devoted to constructing a weak solution to (1.6) through Rothe’s method [10]. This section is structured as follows. First, the variational formulation and its discrete analogue are posed. Next, the existence and uniqueness of a solution \( u_i \) to the discrete variational problem at each time slice \( i \) is deduced. Given those solutions, we prove some a priori estimates in appropriate norms. Finally, we construct the Rothe functions and show that they possess a converging subsequence whose limit constitutes a weak solution of (1.6). In the following section, the uniqueness of a weak solution will be established, implying that the whole Rothe sequence converges to this weak solution.

#### 4.1 Weak formulation

Note that the convolution map \( f \mapsto g_\alpha * f \) is a bounded operator from \( L^2((0,T) \times \Omega) \) to itself with norm bounded by \( \|g_\alpha\|_{L^1(0,T)} \), as can be seen from Young’s inequality for convolutions. Therefore, if \( \partial_t u \in L^2((0,T) \times \Omega) \), by Lemma 2, we have that \( g_\alpha * \partial_t u = \partial_t (g_\alpha * (u - U_0)) \) is an element of \( L^2((0,T) \times \Omega) \). Now, the weak formulation can be stated as follows.

**Definition 1** (Weak formulation (SPL)) Find \( u \in L^\infty((0,T), H^1_0(\Omega)) \cap C([0,T], L^2(\Omega)) \) with \( \partial_t u \in L^2((0,T), L^2(\Omega)) \) and \( \partial_t (g_\alpha * (\partial_t u - V_0)) \) in \( L^2((0,T), H^1_0(\Omega)^*) \) such that for a.a. \( t \in (0,T) \) it holds that

\[
\rho c \tau^\alpha_q (\partial_t (g_\alpha * (\partial_t u - V_0)))(t), \varphi)_{H^1_0(\Omega)^* \times H^1_0(\Omega)} + a \tau^\alpha_q (\partial_t (g_\alpha * (u - U_0)), \varphi) + \rho c (\partial_t u(t), \varphi) + L (u(t), \varphi) = (F(t), \varphi) ,
\]

for all \( \varphi \in H^1_0(\Omega) \).

Remark that under the assumption \( \partial_t u \in L^2((0,T), L^2(\Omega)) \) we have that

\[ \|\partial_t u\|_{L^1((0,T),L^2(\Omega))} \leq \sqrt{T} \|\partial_t u\|_{L^2((0,T),L^2(\Omega))} < \infty . \]
hence $\partial_t u \in L^1((0, T), L^2(\Omega))$ and we find that $u$ is absolutely continuous. Lemma 2 shows that $\partial_t (g_\alpha \ast (u - U_0)) = g_\alpha \ast \partial_t u$. For simplicity, we work with the latter expression in the analysis below.

### 4.2 Time discretisation

Let $n \in \mathbb{N}$ be given. We discretise the time interval $(0, T)$ in $n$ subintervals according to the nodes $t_i = i \tau$ for $i = 0, 1, \ldots, n$ where $\tau = T/n$ is the time step. We define $[t]_\tau = t_i$ for $t \in (t_{i-1}, t_i]$. For a function $z(t)$ we will write $z_i = z(t_i)$ for its evaluation at the time steps. The backward Euler method is used for approximating the first and second order derivatives at each $t_i$, i.e.

$$
\delta z_i = \frac{z_i - z_{i-1}}{\tau} \quad \text{and} \quad \delta^2 z_i = \frac{\delta z_i - \delta z_{i-1}}{\tau} = \frac{z_i - z_{i-1}}{\tau^2} - \frac{\delta z_{i-1}}{\tau}. \quad (4.2)
$$

We set $u_0 = U_0$ and $\delta u_0 = V_0$ according to the initial conditions.

Given a kernel $\kappa : (0, T] \to \mathbb{R}$ and a function $z : [0, T] \to \mathbb{R}$, we define the following discrete convolution approximations of $(\kappa \ast z)(t_i)$:

$$
(\kappa \ast z)^c_i := \sum_{\ell=1}^i \kappa_{i+1-\ell} z_{\ell} \tau. \quad (4.3)
$$

If $z_0 = 0$, then we can properly define $(\kappa \ast z)^c_0 := 0$.

We record the following two lemmas which will be needed and crucial in the estimates below. It is a discrete version of the inequality (2.3). For the proof, we refer to [38, Lemma 3.2].

**Lemma 6** Let $\tau = T/n$ be the time step, where $n \in \mathbb{N}$ is the number of time discretisation intervals. Let $\{z_i\}_{i \in \mathbb{N}}$ and $\{\kappa_i\}_{i \in \mathbb{N}}$ be two sequences of real numbers. Assume that $\kappa_{i+1} \leq \kappa_i$ for all $i \in \mathbb{N}$ and $(\kappa \ast z)^c_0 = 0$. Then

$$
2\delta(\kappa \ast z)^c_i z_i \geq \delta \left( (\kappa \ast z^2)^c_i \right) + \kappa_i z_i^2
$$

As a consequence, we get the following lemma.

**Lemma 7** Let the assumptions of Lemma 6 be fulfilled. Then, it holds that

$$
2 \sum_{i=1}^j \delta(\kappa \ast z)^c_i z_i \geq \left( (\kappa \ast z^2)^c_j \right) + \sum_{i=1}^j \kappa_i z_i^2
$$

assuming $(\kappa \|z\|^2)^c_0 = 0$.

For convenience, we state the summation by parts formula for bilinear mappings.
Lemma 8 Let $b: V \times V \to \mathbb{R}$ be a bilinear form on a vector space $V$ and let \( \{z_i\}_{i \in \mathbb{N}}, \{w_i\}_{i \in \mathbb{N}} \) be two sequences in $V$. Then

$$
\sum_{i=1}^{j} b(z_i, w_i - w_{i-1}) = b(z_j, w_j) - b(z_0, w_0) - \sum_{i=1}^{j} b(\delta z_i, w_{i-1}) \tau.
$$

4.3 Existence at each time step

The discrete variational formulation is obtained by approximating (4.1) at time $t = t_i$ by means of (4.2) and (4.3).

Definition 2 (Discrete weak formulation (SPL)) Find $u_i \in H^1_0(\Omega), i = 1, \ldots, n$, such that

$$
\rho c \tau_q^\alpha \left( (g_\alpha * (\delta u - V_0))_i^c, \varphi \right) + a \tau_q^\alpha \left( ((g_\alpha * \delta u)_i^c, \varphi \right)
$$

$$
+ \rho c (\delta u_i, \varphi) + \mathcal{L}(u_i, \varphi) = (F_i, \varphi),
$$

for all $\varphi \in H^1_0(\Omega)$.

The existence of a solution on a single time step is established in the following lemma.

Lemma 9 Suppose that $F \in L^2([0, T], L^2(\Omega))$, $U_0 \in L^2(\Omega)$ and $V_0 \in L^2(\Omega)$. Then for any $i = 1, \ldots, n$ there exists a unique solution $u_i \in H^1_0(\Omega)$ to (4.4).

Proof We have for $i \geq 1$ that

$$
\delta (g_\alpha * (\delta u - V_0))_i^c = (g_\alpha * \delta (\delta u - V_0))_i^c = (g_\alpha * \delta^2 u)_i^c.
$$

Hence, using (4.2) and (4.3), the discrete problem (4.4) can be rewritten into the form

$$
A(u_i, \varphi) = \mathcal{F}_i(\varphi), \quad \text{for all } \varphi \in H^1_0(\Omega),
$$

where $A$ is the $H^1_0(\Omega)$-elliptic and continuous bilinear form given by

$$
A(u_i, \varphi) := \left( \frac{\rho c \tau_q^\alpha}{\tau} g_\alpha(\tau) + a \tau_q^\alpha g_\alpha(\tau) + \frac{\rho c}{\tau} \right) (u_i, \varphi) + \mathcal{L}(u_i, \varphi),
$$

and

$$
\mathcal{F}_i(\varphi) := (F_i, \varphi) + \rho c \tau_q^\alpha g_\alpha(\tau) \left( \frac{u_{i-1}}{\tau} + \delta u_{i-1}, \varphi \right) + \frac{\rho c}{\tau} (u_{i-1}, \varphi)
$$

$$
- \rho c \tau_q^\alpha \sum_{k=1}^{i-1} g_\alpha(t_{i+1-k}) \left( \delta^2 u_k, \varphi \right) \tau + a \tau_q^\alpha g_\alpha(\tau) (u_{i-1}, \varphi) + a \tau_q^\alpha \sum_{k=1}^{i-1} g_\alpha(t_{i+1-k}) (\delta u_k, \varphi) \tau,
$$

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where in case $i = 1$ the summations yield no contribution. The boundedness of $\mathcal{F}_i$ follows inductively under the assumptions $F \in L^2 \left( (0, T), L^2(\Omega) \right)$ and $U_0 \in H^1_0(\Omega), V_0 \in L^2(\Omega)$. An application of the Lax-Milgram lemma [52, Theorem 18.E] consecutively gives the existence and uniqueness of $u_i \in H^1_0(\Omega), i = 1, \ldots, n$ to the discrete problem (4.4). □

4.4 A priori estimates

Our next goal is to derive estimates on the discrete solutions $u_i$ and various related objects. These estimates will be crucial in the convergence study in the following subsection.

Lemma 10 Suppose that $F \in L^2 \left( [0, T], L^2(\Omega) \right)$, $U_0 \in H^1_0(\Omega)$ and $V_0 \in L^2(\Omega)$. Then, there exist a positive constant $C$ such that for every $j = 1, \ldots, n$ the following estimate holds

$$
\left( g \alpha \star \| \delta u - V_0 \|^2 \right)_j^c + \sum_{i=1}^j \| \delta u_i \|^2 \tau
+
\| u_j \|^2_{H^1(\Omega)} + \sum_{i=1}^j \| u_i - u_{i-1} \|^2_{H^1(\Omega)} \leq C.
$$

Proof Take $\phi = \delta u_j \tau$ in (4.4) and sum the resulting equations for $i = 1, \ldots, j$ with $1 \leq j \leq n$, we obtain that

$$
\rho c \tau_q \sum_{i=1}^j \left( \delta \left( g \alpha \star \delta (u - V_0) \right)_i^c, \delta u_i \right) + a \tau_q \sum_{i=1}^j \left( (g \alpha \star \delta u)_i^c, \delta u_i \right) + \rho c \sum_{i=1}^j \| \delta u_i \|^2 \tau + \sum_{i=1}^j \mathcal{L} (u_i, \delta u_i) \tau = \sum_{i=1}^j (F_i, \delta u_i) \tau.
$$

The proof now proceeds by handling each term of the above expression. The first convolution term on the left-hand side of (4.5) can be rewritten as

$$
\sum_{i=1}^j \left( \delta \left( g \alpha \star \delta (u - V_0) \right)_i^c, \delta u_i \right) \tau = \sum_{i=1}^j \left( \delta \left( g \alpha \star \delta (u - V_0) \right)_i^c, \delta u_i - V_0 \right) \tau + \left( (g \alpha \star \delta u - V_0)_j^c, V_0 \right).
$$

The first term on the right-hand side of (4.6) is bounded from below by

$$
\sum_{i=1}^j \left( \delta \left( g \alpha \star \delta (u - V_0) \right)_i^c, \delta u_i - V_0 \right) \tau \geq \frac{1}{2} \left( g \alpha \star \| \delta u - V_0 \|^2 \right)_j^c.
$$
as an application of Lemma 7 shows. The last term of (4.6) is moved to the right-hand side of (4.5), where it is bounded by the Cauchy Schwarz and \( \varepsilon \)-Young inequality as follows

\[
\left| \left( g_\alpha * (\delta u - V_0) \right)^c_j, V_0 \right| \leq C_{\varepsilon_1} \| V_0 \|^2 + \varepsilon_1 \left\| (g_\alpha * (\delta u - V_0))^c_j \right\|^2.
\]

Since

\[
\sum_{l=1}^{j} (g_\alpha)_{j-l+1} \tau = \sum_{l=1}^{j} g_\alpha(t_j - t_{l-1}) \tau \leq \int_0^{t_j} g_\alpha(t_j - s) \, ds \leq \| g_\alpha \|_{L^1(0,T)},
\]

we obtain that

\[
\left| \left( g_\alpha * (\delta u - V_0) \right)^c_j, V_0 \right| \leq C_{\varepsilon_1} + \varepsilon_1 \left( g_\alpha * \| \delta u - V_0 \|^2 \right)^c_j,
\]

where the constant \( C_{\varepsilon_1} \) only depends on \( \| V_0 \| \). By [37, Eq. (3.2)], we have that

\[
\sum_{i=1}^{j} \left( (g_\alpha * \delta u)^c_i, \delta u_i \right) \tau \geq 0.
\]

For the term involving the bilinear form \( \mathcal{L} \), we use that \( k \) is uniformly elliptic and symmetric. Lemma 8 shows that

\[
\sum_{i=1}^{j} (k \nabla u_i, \delta \nabla u_i) \tau = \frac{1}{2} (k \nabla u_j, \nabla u_j) - \frac{1}{2} (k \nabla U_0, \nabla U_0) + \frac{1}{2} \sum_{i=1}^{j} (k (\nabla u_i - \nabla u_{i-1}), \nabla u_i - \nabla u_{i-1}) \geq \frac{\tilde{\kappa}}{2} \| \nabla u_j \|^2 - C + \frac{\tilde{\kappa}}{2} \sum_{i \leq j} \| \nabla u_i - \nabla u_{i-1} \|^2
\]

and (as \( a \geq 0 \))

\[
\sum_{i=1}^{j} (u_i, \delta u_i) \tau = a \left( \| u_j \|^2 - \| U_0 \|^2 + \sum_{i=1}^{j} \| u_i - u_{i-1} \|^2 \right) \geq -\frac{a}{2} \| U_0 \|^2,
\]

where \( C \) depends on \( \| k \|_{L^\infty(\Omega)} \) and \( \| U_0 \|_{H^1(\Omega)} \).

The right-hand side of (4.5) is estimated by use of the \( \varepsilon \)-Young inequality as

\[
\sum_{i=1}^{j} (F_i, \delta u_i) \tau \leq C_{\varepsilon_2} + \varepsilon_2 \sum_{i=1}^{j} \| \delta u_i \|^2 \tau.
\]
where \( C_{\epsilon_2} \) only depends on \( \| F \|_{L^2([0,T],L^2(\Omega))} \).

Summarising the above results, we obtain the estimate

\[
\rho \tau \frac{\alpha}{2} \left( \frac{1}{2} - \epsilon_1 \right) \left( g_{\alpha} * \| \delta u - V_0 \| \right)^c_j + (\rho \epsilon - \epsilon_2) \sum_{i=1}^{j} \| \delta u_i \| \tau \\
+ \frac{\tilde{k}}{2} \| \nabla u_j \| \tau + \frac{\tilde{k}}{2} \sum_{i=1}^{j} \| \nabla u_i - \nabla u_{i-1} \| \leq C + C_{\epsilon_1} + C_{\epsilon_2}.
\]

Fixing \( \epsilon_1 \) and \( \epsilon_2 \) sufficiently small and applying the Friedrichs’ inequality yield the stated result.

\( \square \)

**Lemma 11** Let the assumptions of Lemma 10 be fulfilled. Then, there exists a positive constant \( C \) such that

\[
\sum_{j=1}^{n} \left\| (g_{\alpha} * \delta u)^c_j \right\|^2 \tau \leq C.
\]

**Proof** Note that by the discrete Young inequality, we have that

\[
\sum_{j=1}^{n} \left\| (g_{\alpha} * \delta u)^c_j \right\|^2 \tau = \int_{\Omega} \sum_{j=1}^{n} \left( g_{\alpha} * \delta u_j \right)^c_j (\mathbf{x}) \left\| \tau \right. dx \\
\leq \left( \sum_{j=0}^{n-1} (g_{\alpha})_j \tau \right)^2 \sum_{j=1}^{n} \left\| \delta u_j \right\|^2 \tau \\
\leq C \| g_{\alpha} \|_{L^1(0,T)}
\]

by means of Lemma 10.

\( \square \)

**Lemma 12** Let the assumptions of Lemma 10 be satisfied. Then, there exists a positive constant \( C \) such that

\[
\sum_{j=1}^{n} \left\| \delta (g_{\alpha} * (\delta u - V_0))^c_j \right\|_{H^1_0(\Omega)}^2 \tau \leq C.
\]

**Proof** Note that

\[
\left\| \delta (g_{\alpha} * (\delta u - V_0))^c_j \right\|_{H^1_0(\Omega)}^2 = \sup_{\| \varphi \|_{H^1_0(\Omega)} = 1} \left| \left\langle \delta (g_{\alpha} * (\delta u - V_0))^c_j, \varphi \right\rangle_{H^1_0(\Omega) \times H^1_0(\Omega)} \right| \\
= \frac{1}{\rho \tau c_{\alpha}} \sup_{\| \varphi \|_{H^1_0(\Omega)} = 1} \left( F_j, \varphi \right) - \mathcal{L} (u_j, \varphi)
\]

\( \square \)
- $\rho c (\delta u_j, \varphi) - a \tau (g* \delta u)_j, \varphi)
\leq C \left( \|F_j\| + \|u_j\|_{H^1(\Omega)} + \|\delta u_j\| + \|(g* \delta u)_j\|_c \right).

Applying the rule $(a + b)^2 \leq 2a^2 + 2b^2$ yields

$$\sum_{j=1}^n \| \delta (g* (\delta u - V_0))_j^c \|^2 \tau$$

$$\leq C \left( \sum_{j=1}^n \|F_j\|^2 \tau + \sum_{j=1}^n \|u_j\|^2_{H^1(\Omega)} \tau + \sum_{j=1}^n \|\delta u_j\|^2 \tau + \sum_{j=1}^n \|(g* \delta u)_j^c\|^2 \tau \right),$$

which is uniformly bounded by Lemma 10 and 11. \square

### 4.5 Existence

The existence of a weak solution is shown in this subsection. To achieve this goal, we introduce the Rothe functions $[0, T] \rightarrow L^2(\Omega)$, which are built from the solutions $u_i, i = 0, \ldots, n$ at each time slice as follows,

$$v_n: t \mapsto \begin{cases} U_0, & t = 0 \\ u_{i-1} + (t - t_{i-1}) \delta u_i, & t \in (t_{i-1}, t_i], \ 1 \leq i \leq n, \end{cases}$$

$$\overline{w}_n: t \mapsto \begin{cases} U_0, & t = 0 \\ u_i, & t \in (t_{i-1}, t_i], \ 1 \leq i \leq n, \end{cases}$$

$$\overline{w}_n: t \mapsto \begin{cases} V_0, & t = 0 \\ \delta u_i, & t \in (t_{i-1}, t_i], \ 1 \leq i \leq n. \end{cases}$$

Note that $\partial_t v_n = \overline{w}_n$. Similar notations are used for $\overline{F}_n$ and $\overline{g}_\gamma$. Additionally, we define for two functions $\kappa$ and $z$ [43]:

$$C_{\kappa, z}^n: t \mapsto \begin{cases} 0, & t = 0 \\ (\kappa * z)_{i-1} + (t - t_{i-1}) \delta (\kappa * z)_i^c, & t \in (t_{i-1}, t_i], \end{cases}$$

$$\overline{C}_{\kappa, z}^n: t \mapsto \begin{cases} 0, & t = 0 \\ (\kappa * z)_i^c, & t \in (t_{i-1}, t_i]. \end{cases}$$

Note that for $t \in (t_{i-1}, t_i]$, we have that

$$\partial_t \left( C_{\kappa, z}^n \overline{w}_n - V_0 \right) (t) = \delta \left( \overline{g}_\gamma (\overline{w}_n - V_0)_i^c \right) = \delta (g* (\delta u - V_0))_i^c.$$
as \( \overline{w}_n(t_k) = \delta u_k \). Moreover, we have that

\[
\left( \overline{g}_\alpha \ast \overline{w}_n \right)([t]_\tau) = \int_0^{t_i} \overline{g}_\alpha(t_i - s) \overline{w}_n(s) \, ds = \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \overline{g}_\alpha(t_i - s) \delta u_j \, ds
\]

\[
= \sum_{j=1}^i g_\alpha(t_i - s) \delta u_j \tau
\]

as for \( s \in (t_{j-1}, t_j] \), we have \( t_i - s \in [t_{j-1}, t_{j}+1) \) and so \( \overline{g}_\alpha(t_i - s) = g_\alpha(t_i - s) \).

Now, we extend (4.4) to the whole time frame. In terms of the Rothe functions, the discrete variational formulation now reads as

\[
\rho c \tau^\alpha \left( \partial_t \left( C_{\overline{g}_\alpha \ast \overline{w}_n - V_0}^\alpha(t), \varphi \right) + a \tau^\alpha \left( \overline{u}_n (\tau], \varphi \right) 
+ \rho c (\partial_t v_n(t), \varphi) + L (\overline{v}_n(t), \varphi) = (\overline{F}_n(t), \varphi), \quad \forall \varphi \in H^1_0(\Omega). \quad (4.7)
\]

In the remainder of this subsection, we show that a subsequence of the Rothe functions converges to a weak solution. The following subsection is then devoted to proving the uniqueness of a solution.

**Theorem 1** (Existence (SPL)) Suppose that \( F \in L^2 ([0, T], L^2(\Omega)) \), \( U_0 \in H^1_0(\Omega) \) and \( V_0 \in L^2(\Omega) \). Then, there exists a weak solution \( u \) to (4.1) with \( u \in L^\infty ((0, T), H^1_0(\Omega)) \cap C ([0, T], L^2(\Omega)) \) for which \( \partial_t u \in L^2 ((0, T), L^2(\Omega)) \) and \( \partial_t (g_\alpha \ast (\partial_t u - V_0)) \in L^2 ((0, T), H^1_0(\Omega)^*). \)

**Proof** The Rellich-Kondrachov theorem [26, Theorem 6.3] gives us the compact embedding of \( H^1_0(\Omega) \) in \( L^2(\Omega) \). The estimates from Lemma 10 show in particular that

\[
\| \overline{v}_n(t) \|_{H^1_0(\Omega)} \leq C, \quad \forall t \in [0, T],
\]

and

\[
\int_0^T \| \partial_t v_n(t) \|_{L^2(\Omega)}^2 \, dt = \sum_{i=1}^n \| \delta u_i \|_{L^2(\Omega)}^2 \tau \leq C.
\]

Hence, by the Aubin-Lions lemma [10, Lemma 1.3.13], we find a function \( v \in C ([0, T], L^2(\Omega)) \cap L^\infty ((0, T), H^1_0(\Omega)) \) with \( \partial_t v \in L^2 ((0, T), L^2(\Omega)) \) and a corresponding subsequence \( \{v_{n_k}\}_{k \in \mathbb{N}} \) of \( \{v_n\}_{n \in \mathbb{N}} \) having the following properties, as \( k \to \infty \),

\[
v_{n_k} \to v \text{ in } C ([0, T], L^2(\Omega)) \]
\[
v_{n_k}(t) \to v(t) \text{ in } H^1_0(\Omega), \text{ for all } t \in [0, T] \]
\[
\overline{v}_{n_k}(t) \to \overline{v}(t) \text{ in } H^1_0(\Omega), \text{ for all } t \in [0, T] \]
\[
\partial_t v_{n_k} = \overline{w}_{n_k} - \overline{v}_{n_k} \text{ in } L^2 ((0, T), L^2(\Omega)).
\]
Note that
\[
\int_0^T \| \nabla v_{nk}(t) - \nabla \overline{v}_{nk}(t) \|^2 \, dt = \sum_{i=1}^{n_k} \int_{t_{i-1}}^{t_i} \| (t - t_i) \delta \nabla u_i \|^2 \, dt
\]
\[
= \frac{\tau_{nk}}{3} \sum_{i=1}^{n_k} \| \nabla \delta u_i \|^2 \tau_{nk}^2
\]
\[
= \frac{\tau_{nk}}{3} \sum_{i=1}^{n_k} \| \nabla u_i - \nabla u_{i-1} \|^2 \leq C \tau_{nk},
\]
where we have set \( \tau_{nk} = T/n_k \). As a result, \( \{v_{nk}\}_{k \in \mathbb{N}} \) and \( \{\overline{v}_{nk}\}_{k \in \mathbb{N}} \) have the same limit
\( v \in L^2((0, T), H^1_0(\Omega)) \).

Next, we integrate (4.7) over the interval \((0, \eta) \subset (0, T)\) to arrive at
\[
\rho \tau \alpha q \left( C^{\alpha}_{g_{nk}(t) \overline{w}_{nk} - V_0(t)} \right) + a \tau \alpha q \int_{0}^{\eta} \left( (g_{nk} \overline{w}_{nk}) ([t]_\tau), \varphi \right) \, dt
\]
\[
+ \rho c \int_{0}^{\eta} (\partial_t v_{nk}(t), \varphi) \, dt + \int_{0}^{\eta} L (\overline{v}_{nk}(t), \varphi) \, dt = \int_{0}^{\eta} (\overline{F}_{nk}(t), \varphi) \, dt. \tag{4.8}
\]
Now, we will first discuss some convergence results. Using Lemma 12, we can estimate
\[
\left| \int_{0}^{T} \left( C^{\alpha}_{g_{nk}(t) \overline{w}_{nk} - V_0(t)} - C^{\alpha}_{g_{nk}(t) \overline{w}_{nk} - V_0(t)} \right) \, dt \right|
\]
\[
= \sum_{i=1}^{n_k} \int_{t_{i-1}}^{t_i} \left| (t - t_i) \delta (g_{nk} \overline{w}_{nk} - V_0) \right| \, dt
\]
\[
\leq \sum_{i=1}^{n_k} \tau_{nk} \left\| \delta (g_{nk} \overline{w}_{nk} - V_0) \right\|_{H^1(\Omega)} \tau_{nk}
\]
\[
\leq C \tau_{nk} \sum_{i=1}^{n_k} \left\| \delta (g_{nk} \overline{w}_{nk} - V_0) \right\|_{H^1(\Omega)} \tau_{nk}
\]
\[
\leq C \tau_{nk} \to 0 \quad \text{as } k \to \infty.
\]

We note that
\[
C^{\alpha}_{g_{nk}(t) \overline{w}_{nk} - V_0(t)} = \sum_{j=1}^{i} g_{nk}(t_{i+1} - j) (\delta u_j - V_0) \tau_{nk} = (g_{nk} \overline{w}_{nk} - V_0) ([t]_\tau).
\]

The same arguments as in [45, p. 22–25] yield the limit transition
\[
\left| \int_{0}^{\xi} \left( C^{\alpha}_{g_{nk}(t) \overline{w}_{nk} - V_0(t)} - (g_{nk} \overline{w}_{nk} - V_0(t)) \right) \, d\eta \right| \to 0, \quad \text{as } k \to \infty.
\]
for any $\xi \in (0, T]$. It is here used that $\frac{g_{\alpha n_k}}{\delta u_0} \to g_\alpha$ pointwise in $(0, T)$ and that the estimate 
\[
\left( g_\alpha \ast \|\delta u - V_0 \|_2^2 \right) + \sum_{i=1}^{J} \|\delta u_i \|_2^2 \tau_{n_k} \leq C
\]
is available (see Lemma 10). The operator
\[
w \mapsto \int_0^\eta \left( (g_\gamma \ast w)(t), \varphi \right) dt,
\]
with $\gamma \in (0, 1)$ and $\eta \in (0, T)$ fixed, is a bounded linear functional on the space $L^2 ((0, T), L^2(\Omega))$ since by Young’s inequality for convolutions it holds that
\[
\left| \int_0^\eta \left( (g_\gamma \ast w)(t), \varphi \right) dt \right| \leq \sqrt{T} \|g_\gamma\|_{L^1(0,T)} \|\varphi\| \|w\|_{L^2((0,T),L^2(\Omega))}.
\]
Therefore, using $\overline{w}_{n_k} \to \partial_t v$ in $L^2 ((0, T), L^2(\Omega))$ it finally holds that
\[
\left| \int_0^\xi \left( \overline{C_{n_k}} \overline{\alpha_{\alpha n_k}} \overline{w}_{n_k} - V_0(\eta) - (g_\alpha \ast (\partial_t v - V_0))(\eta), \varphi \right) d\eta \right| \to 0, \quad \text{as } k \to \infty,
\]
for any $\xi \in (0, T)$. Next, we note that
\[
\overline{C_{n_k}} \overline{\alpha_{\alpha n_k}} \overline{w}_{n_k}(t) = \left( \overline{g_{\alpha n_k}} \ast \overline{w}_{n_k} \right) ([t] \tau_{n_k}).
\]
Hence, we immediately see that
\[
\left| \int_0^\xi \left( \overline{C_{n_k}} \overline{\alpha_{\alpha n_k}} \overline{w}_{n_k}(t) - (g_\alpha \ast \partial_t v)(t), \varphi \right) dt \right| \to 0 \quad \text{as } k \to \infty.
\]
Furthermore, we have that
\[
\left| \int_0^\eta \left( \overline{\Delta n_k}(t), \varphi \right) dt - \int_0^\eta \left( \Delta n(t), \varphi \right) dt \right| \to 0 \quad \text{as } k \to \infty,
\]
\[
\left| \int_0^\eta \left( \overline{F_{n_k}}(t), \varphi \right) dt - \int_0^\eta \left( F(t), \varphi \right) dt \right| \to 0 \quad \text{as } k \to \infty.
\]
Before making the limit transition, we need to integrate (4.8) in time over $\eta \in (0, \xi) \subset (0, T)$. We obtain that
\[
\rho c r^a q \int_0^\xi \left( \overline{C_{n_k}} \overline{\alpha_{\alpha n_k}} \overline{w}_{n_k} - V_0(\eta), \varphi \right) d\eta + a r^a q \int_0^\xi \int_0^\eta \left( \overline{g_{\alpha n_k}} \ast \overline{w}_{n_k} \right) ([t] \tau), \varphi \right) dt d\eta
\]
\[
\quad + \rho c \int_0^\xi \int_0^\eta \left( \partial_t v_{n_k}(t), \varphi \right) dt d\eta + \int_0^\xi \int_0^\eta \left( \overline{\Delta n_k}(t), \varphi \right) dt d\eta
\]
\[
= \int_0^\xi \int_0^\eta \left( \overline{F_{n_k}}(t), \varphi \right) dt d\eta.
\]
Using the previously obtained results, we can make the limit transition in (4.9) to receive that
\[
ρcτ_q^α \int_0^ξ ((gα * (∂_t v - V_0)) (η), ϕ) dη + aτ_q^α \int_0^ξ ((gα * ∂_t v) (t), ϕ) dt dη \\
+ ρc ∫_0^ξ ∫_0^η (∂_t v(t), ϕ) dt dη + ∫_0^ξ ∫_0^η L(v(t), ϕ) dt dη \\
= ∫_0^ξ ∫_0^η (F(t), ϕ) dt dη.
\]
(4.10)

Differentiation of (4.10) with respect to ξ yields
\[
ρcτ_q^α ((gα * (∂_t v - V_0)) (ξ), ϕ) + aτ_q^α ∫_0^ξ ((gα * ∂_t v) (t), ϕ) dt \\
+ ρc ∫_0^ξ (∂_t v(t), ϕ) dt + ∫_0^ξ L(v(t), ϕ) dt = ∫_0^ξ (F(t), ϕ) dt.
\]
(4.11)

From (4.11), it follows that \(\lim_{ξ \searrow 0} ((gα * (∂_t v - V_0)) (ξ), ϕ) = 0\). Hence, differentiating (4.11) again w.r.t. ξ we obtain that
\[
ρcτ_q^α (∂_t (gα * (∂_t v - V_0)) (ξ), ϕ)_H^1_0(Ω)^* × H^1_0(Ω) + aτ_q^α ((gα * ∂_t v) (ξ), ϕ)
\]
\[+ ρc (∂_t v(ξ), ϕ) + L(v(ξ), ϕ) = (F(ξ), ϕ). \]

Therefore, v satisfies equation (4.1).

4.6 Uniqueness

In this subsection, we show the uniqueness of the solution to problem (1.6) under the additional assumption that ∂_t u ∈ L^∞((0, T), L^2(Ω)). This assumption is reasonable considering the regularity of the solution to the SPL problem in one dimension, see Section 3.

**Theorem 2** (Uniqueness SPL) The weak solution to problem (4.1) satisfying \(u ∈ L^∞((0, T), H^1_0(Ω)) \cap C([0, T], L^2(Ω)), ∂_t u ∈ L^∞((0, T), L^2(Ω))\) and ∂_t (gα * (∂_t u − V_0)) ∈ L^2((0, T), H^1_0(Ω)^*) is unique.

**Proof** Let \(u_1\) and \(u_2\) be solutions to the problem (4.1) and consider their difference \(u = u_1 - u_2\). Then \(u\) is a solution to the homogeneous problem (i.e. \(F = 0\) and \(U_0 = 0 = V_0\)) with vanishing Dirichlet boundary conditions. By Lemma 2, the function \(u\) satisfies
\[
ρcτ_q^α (∂_t (gα * ∂_t u) (t), ϕ)_{H^1_0(Ω)^* × H^1_0(Ω)} + aτ_q^α (∂_t (gα * u) (t), ϕ)
\]
\[+ ρc (∂_t u(t), ϕ) + L(u(t), ϕ) = 0, \]
(4.12)
for all $\varphi \in H^1_0(\Omega)$. Note that we cannot choose $\varphi = \partial_t u(t)$ as $\partial_t u(t) \notin H^1_0(\Omega)$. We use a method due to Ladyzhenskaya (see [16] and [48, p.202]). Fix $s \in (0, T)$ and let $\varphi = v(t)$ where $v(t)$ is given by

$$v(t) = \begin{cases} \int_t^s u(\tau) \, d\tau & \text{if } 0 \leq t \leq s \\ 0 & \text{otherwise.} \end{cases}$$

The crucial property of $v$ is that $v'(t) = -u(t)$ if $t \leq s$. We substitute $\varphi = v(t)$ in (4.12) and integrate $t$ over $(0, s)$ to obtain that

$$\rho c \alpha^\alpha \int_0^s \left( \partial_t (g_\alpha * \partial_t u)(t), \int_t^s u(\tau) \, d\tau \right) \, d\tau + a \alpha^\alpha \int_0^s \left( \partial_t (g_\alpha * u)(t), \int_t^s u(\tau) \, d\tau \right) \, dt + \rho c \int_0^s \left( \partial_t u(t), \int_t^s u(\tau) \, d\tau \right) \, dt + \int_0^s \mathcal{L} (u(t), \int_t^s u(\tau) \, d\tau) = 0.$$

The additional assumption on $\partial_t u$ implies that

$$\lim_{t \to 0} \| (g_\alpha * \partial_t u)(t) \| \leq \lim_{t \to 0} (g_\alpha * \| \partial_t u \|)(t) \leq C \lim_{t \to 0} \int_0^t (t - s)^{-\alpha} = 0.$$

Hence, partial integration in the first fractional term yields

$$\int_0^s \left( \partial_t (g_\alpha * \partial_t u)(t), \int_t^s u(\tau) \, d\tau \right) \, d\tau = \left. \left( (g_\alpha * \partial_t u)(t), \int_t^s u(\tau) \, d\tau \right) \right|_{t=0}^{t=s} + \int_0^s ((g_\alpha * \partial_t u)(t), u(t)) \, dt$$

$$= \int_0^s (\partial_t (g_\alpha * u)(t), u(t)) \, dt$$

$$\geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \int_0^s \| u(t) \|^2 \, dt,$$

where the last inequality follows from Lemma 1(v). Analogously, for the second fractional term, partial integration shows that

$$\int_0^s \left( \partial_t (g_\alpha * u)(t), \int_t^s u(\tau) \, d\tau \right) \, dt = \int_0^s ((g_\alpha * u)(t), u(t)) \, dt \geq 0,$$

which is positive by Lemma 1(iv). Since

$$\frac{d}{dt} \left( u(t), \int_t^s u(\tau) \, d\tau \right) = \left( \partial_t u(t), \int_t^s u(\tau) \, d\tau \right) + \left( u(t), \partial_t \int_t^s u(\tau) \, d\tau \right)$$
we find that

\[
\int_0^s \left( \frac{\partial_t u(t)}{t} , \int_t^s u(\tau) \, d\tau \right) \, dt = \int_0^s \frac{d}{dt} \left( u(t), \int_t^s u(\tau) \, d\tau \right) \, dt + \int_0^s \|u(t)\|^2 \, dt
\]

\[
= - \left( u(0), \int_0^s u(\tau) \, d\tau \right) + \int_0^s \|u(t)\|^2 \, dt,
\]

and therefore

\[
\rho c \int_0^s \left( \frac{\partial_t u(t)}{t} , \int_t^s u(\tau) \, d\tau \right) \, dt = \rho c \int_0^s \|u(t)\|^2 \, dt.
\]

Next, we have that

\[
a \int_0^s \left( u(t), \int_t^s u(\tau) \, d\tau \right) \, dt = -a \int_0^s \left( -u(t), \int_t^s u(\tau) \, d\tau \right) \, dt
\]

\[
= -\frac{a}{2} \int_0^s \frac{\partial_t}{\partial_t} \left( \int_t^s u(\tau) \, d\tau \right)^2 \, dt
\]

\[
= \frac{a}{2} \left\| \int_0^s u(t) \, dt \right\|^2 \geq 0.
\]

Similarly, by the symmetry of \( k \), we obtain that

\[
\int_0^s \left( k \nabla u(t), \int_t^s \nabla u(\tau) \, d\tau \right) \, dt = -\int_0^s \left( -k \nabla u(t), \int_t^s \nabla u(\tau) \, d\tau \right) \, dt
\]

\[
= -\frac{1}{2} \int_0^s \partial_t \left( k \int_t^s \nabla u(\tau) \, d\tau, \int_t^s \nabla u(\tau) \, d\tau \right)
\]

\[
\geq \tilde{k} \frac{1}{2} \left\| \int_0^s \nabla u(t) \right\|^2 \, dt.
\]

Summarising, we obtain the inequality

\[
\rho c \left( 1 + \tau_q^{\alpha} \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \right) \int_0^s \|u(t)\|^2 \, dt + \frac{\tilde{k}}{2} \left\| \int_0^s \nabla u(t) \, dt \right\|^2 \leq 0.
\]

Therefore, \( u = 0 \) a.e. in \( QT \), which shows that \( u_1 = u_2 \) a.e. in \( QT \). \( \square \)

**Remark 1** From theoretical viewpoint, Theorem 2 stays valid when the condition on \( \|\partial_t u(t)\| \) is replaced by

\[
\|\partial_t u(t)\| \leq C \left( 1 + t^{-\beta} \right), \quad 0 < \beta < \min \left\{ \frac{1}{2}, 1 - \alpha \right\}.
\]
5 Conclusion

In this article, we have investigated the existence and uniqueness of a weak solution to the fractional single-phase-lag heat equation. Rothe’s method has been employed to show the existence of a solution \( u \in L^\infty \left( (0, T), H^1_0(\Omega) \right) \cap C \left( [0, T], L^2(\Omega) \right) \) satisfying \( \partial_t u \in L^2 \left( (0, T), L^2(\Omega) \right) \). Under the additional assumption \( \partial_t u \in L^\infty \left( (0, T), L^2(\Omega) \right) \), the uniqueness of a solution has been shown by contradiction. Moreover, we have derived an explicit solution to the one-dimensional problem. Bounds on the solution and its time derivative have been obtained by generalising the uniform boundedness property of the multivariate Mittag-Leffler function. Future work will concern the existence of a unique solution to the DPL model corresponding with (1.1).

Acknowledgements Van Bockstal is supported by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021).

Author Contributions The authors contributed equally to this work.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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