Spectrum of some arrow-bordered circulant matrix

Wojciech Florek
Adam Mickiewicz University, Faculty of Physics, ul. Uniwersytetu Poznańskiego 2, 61-614 Poznań, Poland

Adam Marlewski
Poznań University of Technology, Institute of Mathematics, ul. Piotrowo 3A, 60-965 Poznań, Poland

(Dated: May 14, 2019)

Given a circulant matrix circ(c,a,0,0,...,0,a), a ≠ 0, of order n, we “border” it from left and from above by constant column and row, respectively, and we set the left top entry to be −nc. This way we get a particular title object, an example of what we call an abc matrix, or an arrow-bordered circulant (matrix). We find its eigenpairs and we discuss its spectrum with stress on extreme eigenvalues and their bounds. At last we notice its relation to a weighted wheel graph.

Keywords: Arrow matrix; Circulant matrix; Eigenvalue; Spectral graph theory; Wheel graph; MSC2010 15A18 & 05C50
I. INTRODUCTION AND MOTIVATION

By “gluing” a circulant matrix and an arrow(head) matrix we get an abc matrix ("abc" acronymizes "arrow-bordered circulant"). We notice that an abc matrix is the adjacency matrix of a wheel graph whose both vertices and edges have weights. By elementary methods we find eigenpairs of a regular abc matrix and we analyse its spectrum. The work with matrices we consider is motivated by problems such as that treated by Schmidt and Luban [1], where elements of spectral graph theory are applied to determine the lowest energy configuration(s) of finite classical Heisenberg spin systems. There exists a weighted adjacency matrix of a (simple) graph representing such system of interacting localized spin vectors (and this graph, $W_n$, is discussed in Section VII below), weights are assigned to both vertices and edges; in addition, this matrix is dressed with a “gauge” vector spread over the main diagonal (the notion “dressing with a vector” is explained in Section VII). The crucial point in the search of desired configurations is to find the minimum eigenvalue of the matrix at hand—the sought solution is determined by the maximum, determined with respect to the gauge, of the minimum eigenvalue. The other eigenvalues have no clear physical meaning, nevertheless properties of the whole spectrum are discussed below.

There is an extensive literature on spectral graph theory. Numerous relations between graphs and their 0-1 adjacency matrices are presented, among others, in monographs Biggs [2], Bondy and Murty [3], Cvetković [4], Haemers [5], and Nica [6], where, in particular, the Cauchy theorem on interlacing eigenvalues of a symmetric matrix and its submatrix, and Weyl upper bound for the eigenvalues of a sum of two symmetric matrices are recalled. Zhang and Zhang [7] discussed adjacency matrices of graphs obtained by attaching some pendant edges to complete graphs ($K_n$) and to cyclic graphs ($C_n$), whereas Das [8] dealt with eigenvalues of a friendship graph ($F_n$). Introduced in this work an abc matrix $M_n$ (see Section III) is the adjacency matrix of a regularly weighted wheel graph, the notion we define in Section VI. Graphs and their adjacency matrices are widely used among others in chemistry, physics and telecommunications [a survey is presented in 9], and mostly concerns unweighted graphs. Examples where edge- or vertex-weighted graphs are dealt with include Klavžar and Gutman [10], Latora and Marchiori [11], Ashrafi and Hamadanian [12], Mirchev [13], Jooya et al. [14], Cai et al. [15], and Antoš [16].

The paper is organized as follows: In Section II the most important properties of circulant and arrowhead matrices are recalled. The title object of this work is presented in Section III whereas its eigenpairs are determined in Section IV. Main theorems are included in three parts of Section VI. In Section VII it is shown that the abc matrices discussed in this paper can be treated as weighted adjacency matrices of wheel graphs. Finally, Section VII is devoted to some final remarks.

II. BASICS ON CIRCULANT MATRICES AND ARROW(HEAD) MATRICES

In linear algebra a matrix $[a_{j,k}]$ is called Toeplitz [17] or constant-diagonal, if its each descending diagonal from left to right is constant, so if $a_{j,k} = t_{j-k}$ (and we say that quantities $t_m$’s determine it). Such infinity matrices, $T = \{t_{j-k}\}_{j,k=1,2,3,...}$, were first investigated by Otto Toeplitz, he showed that the matrix $T$ defines a bounded linear operator, which transforms a sequence $x \in \ell^2$ in $Tx = y$, where $y_k = \sum_{j=0}^{\infty} t_{j-k}x_j$, iff $t_m$ are Fourier coefficients of an appropriate function. In fact, Toeplitz proved it in the special case of symmetric matrix ($t_{m-n} = t_m$), and several decades later his result was extended to the general case by Hartman and Wintner [18]. Below we deal with finite Toeplitz matrices. By definition, a $(m,n)$-Toeplitz matrix has $m$ rows and $n$ columns, its top/first row is $[t_0, t_1, t_2, \ldots, t_{n-2}, t_{n-1}]$, the second row is $[t_{-1}, t_0, t_1, \ldots, t_{n-3}, t_{n-2}]$, etc. In particular, a square Toeplitz matrix, $[a_{j,k}]_{j,k=1,\ldots,n}$, has its last/nth row $[t_{-(n-1)}, t_{-(n-2)}, t_{-(n-3)}, \ldots, t_{-1}, t_0]$. So, any Toeplitz matrix of order $n$ (and of $2n - 1$ degrees of freedom) is fully determined by the vector $t = [t_{-(n-1)}, t_{-(n-2)}, t_{-(n-3)}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{n-1}]$, and is denoted by $\text{toep}(t)$.

For $a_{j,k} = t_{j-k}$ we have a symmetric Toeplitz matrix of order $n$, and we denote it by the same symbol, $\text{toep}(t)$, where $t = [t_0, t_1, \ldots, t_{n-1}]$ is its first row. A review of results on eigenvalues of symmetric Toeplitz matrices is given in Delsarte and Genin [19], Reichel and Trefethen [20], Melman [21]; in Andrews [22], Cantoni and Butler [23], Liu [24], Abu-Jeib [25], Katona et al. [26], and Brualdi and Ma [27] there are mostly discussed, general or particular, centrosymmetric matrices.

A square Toeplitz matrix whose $j$th row is the cyclic shift of the top/first row, $c = [c_0, c_1, \ldots, c_{n-1}]$ by $j - 1$ positions to the right, is called circulant (matrix) and denoted by $\text{circ}(c)$, or $\text{circ}(c_0, c_1, \ldots, c_{n-1})$. First circulant matrices were studied by Eugène Catalan in Recherches sur les déterminants (1846), William Spottiswoode in Elementary theorems relating to determinants (1856) and Alphonse Legoux in Application d’un déterminant (1883). Circulant matrices appear in numerous problems [see, e.g., 29, 31]. A classical book on circulant matrices is Davis [32], whereas more recent texts dedicated to them and their generalizations are Gray [33], Fuhrmann [34], Kra and Simanca [35], Bose et al. [36]. They all cite the following basic result on spectral properties of arbitrary circular matrix
Theorem 1 Given \( c := [c_0, c_1, \ldots, c_{n-1}] \), eigenpairs of \( \text{circ}(c) \) are \((\lambda_k, v_k)\), \( k = 0, 1, 2, \ldots, n - 1 \), where
\[
\begin{align*}
\lambda_k &= c_0 + c_1 \omega_n^k + c_2 \omega_n^{2k} + \cdots + c_{n-1} \omega_n^{(n-1)k}, \\
v_k &= [1, \omega_n^k, \omega_n^{2k}, \ldots, \omega_n^{(n-1)k}]^T, \\
\omega_n &= \exp(2\pi i/n);
\end{align*}
\]
\( \omega_n \) is the first prime root of degree \( n \) of the unity and, as usual, the superscript “\( T \)” stands for the transpose.

A square matrix containing zeros in all its entries except for the first row, first column, and main diagonal, is called (because of the pattern its nonzero entries form) an arrowhead, or (left up) arrow matrix [e.g., 37]. The notion “arrow matrix” embraces also all matrices similar to any arrowhead matrix via a symmetric permutation [but there are also in use more refined names, e.g., down-arrow matrix, see 58].

Diagonal matrices are arrow matrices, so examples of arrow matrices appeared when, in 1850 in Additions to the articles, “On a New Class of Theorems”, and “On Pascal’s Theorem”, Joseph James Sylvester coined the term “a matrix”. A good compendium on arrow matrices is O’Leary and Stewart [39]. Examples of centrosymmetrical arrow matrices include matrices \( A_n := A_n(h, b, d) \in \mathbb{R}^{(n+1) \times (n+1)}, n > 0 \), where \( h \) is the most left top entry (one can refer it to as a headpoint entry), all other bordering entries (i.e., elements in the top row and in the first column) are \( b \), and all other diagonal elements are \( d \). So, it has a form as follows:
\[
A_n(h, b, d) = \begin{bmatrix}
    h & b^T \\
    b & D
\end{bmatrix},
\]
where \( h \in \mathbb{R}, b = [b, b, \ldots, b]^T \in \mathbb{R}^n \) is a constant vector, \( D = \text{diag}(d, \ldots, d) \) is a diagonal matrix with \( d \in \mathbb{R} \). Such arrow matrices are referred to as regular ones. Their eigenvalues are given by

Theorem 2 Let \( A_n \in \mathbb{R}^{(n+1) \times (n+1)} \) be a regular arrowhead matrix of the form (1). Then, its eigenvalues are
\[
\begin{align*}
\lambda_+ &= \frac{(h + d + \sqrt{\Delta_n})}{2}, & \text{for} & \quad n \geq 1, \quad (2a) \\
\lambda_k &= d, & \text{for} & \quad n > 1 \quad \text{and} \quad k = 1, 2, \ldots, n - 1, \quad (2b)
\end{align*}
\]
where \( \Delta_n = (h - d)^2 + 4nb^2 \).

Proof 1 Let \( \alpha_n(\lambda) \) denote the characteristic polynomial of the matrix \( A_n \). Then
\[
\alpha_1(\lambda) = (\lambda - h)(\lambda - d) - b^2 = \lambda^2 - (h + d)\lambda + hd - b^2.
\]
For \( n > 1 \) we have
\[
\alpha_n(\lambda) = (\lambda - d)^{n-1}((\lambda - h)(\lambda - d) - nb^2).
\]
Indeed, applying the Laplace expansion we have inductively
\[
\alpha_{n+1}(\lambda) = (\lambda - d)\alpha_n(\lambda) - (\lambda - d)^n b^2 = (\lambda - d)^n ((\lambda - h)(\lambda - d) - (n+1)b^2).
\]
The first factor in Eq. (3) proves (2b), whereas the second part and Eq. (3) lead to a quadratic equation with the discriminant \( \Delta_n = (h - d)^2 + 4nb^2, n > 0 \), and, therefore, the other two eigenvalues are given by Eq. (2a), which completes the proof. □

For \( n > 1 \) this theorem can be proved applying Corollary 4 stated by Shen and Suter [37]. Note that \( \lambda_- < \lambda_+ \) if \( b \neq 0 \), despite values of \( h \) and \( d \). Immediately from Theorem 2 it follows

Corollary 3 For any real numbers \( h, d \) and \( b \neq 0 \) the spectrum \( \sigma(A_n(h, b, d)) \) is of cardinality
\[
|\sigma(A_n(h, b, d))| = \begin{cases} 
2, & \text{for} \quad n = 1, \\
3, & \text{otherwise}.
\end{cases}
\]
III. ARROW-BORDERED CIRCULANT MATRIX

Let us define an arrow-bordered (or arrowly bordered) circulant matrix, or an abc matrix for short, as a circulant matrix expanded on its left with a column, and on its top with a row. So an abc matrix

\[ m_n = \begin{bmatrix} h & r^T \\ b & t_n \end{bmatrix}, \]

where a scalar \( h \) is proposed to be referred to as a headpoint number (or a tip number), \( b \) and \( r^T \) are (bordering) vectors, and \( t_n \) is a circulant matrix of order \( n \). Symmetric arrow-bordered matrices (i.e., for \( b = r \) with a diagonal \( t_n = \text{diag}(t_1, t_2, \ldots, t_n) \), called headarrow matrices, are treated in O’Leary and Stewart \[39\], Pickmann et al. \[40\], Shen and Suter \[37\], and Jakovčević Stor et al. \[41\]. Below we deal with more general case, namely with \( t_n = \text{circ}(c, a, 0, \ldots, 0, a) \), where \( a \neq 0 \) in a general case.

A symmetrical abc matrix with a constant vector \( b = [b, b, \ldots, b]^T = r \) is referred to as a regularly arrow-bordered matrix, or a regular abc matrix. In the next we consider traceless regular abc matrices, so matrices

\[ \tilde{M}_n := m_n(a, b, c), \]

where \( a, b, c \) are real numbers, \( t_n = \text{circ}(c, a, 0, \ldots, 0, a) \), and \( h = -nc \) (this choice makes that \( M_n \) is traceless).

For example (zero entries are marked by dots),

\[ M_6 = m_6(a, b, c) = \begin{bmatrix} -6c & b & b & b & b & b \\ b & c & a & \cdots & a \\ b & a & c & a & \cdots \\ b & \cdots & a & c & a \\ b & \cdots & a & c & a \\ b & a & \cdots & a & c \end{bmatrix}. \]

Remark 1 Some obvious but important remarks are in place.

1. With \( b = 0 \) we have a trivial situation, \( m_n(a, 0, c) = \text{diag}(-nc, t_n) \), so in this case the spectrum

\[ \sigma(m_n(a, 0, c)) = \{-nc\} \cup \sigma(t_n). \]

Therefore in the next we deal with \( b \neq 0 \).

2. Since \( m_n(a, b, c) = b \cdot m_n(a/b, 1, c/b) \), we can deal with abc matrices with fixed \( b = 1 \) or \( b = -1 \). Obviously, this does not restrict the generality of considerations, and we mainly discuss these cases, i.e., \( b = \pm 1 \).

3. The special case \( a = 0 \) gives \( m_n(0, b, c) = A_n(-nc, b, c) \) and we treated it in Theorem \[3\].

IV. EIGENPAIRS OF AN ABC MATRIX

All matrices that we treat in the next are regular abc matrices, and we pay our attention to traceless ones. Nevertheless, for reasons that appear clear later, we need to treat two kinds of abc matrices of order two and three \( (n = 1, 2, \text{respectively}) \), namely that defined by \[7\] and that defined by \[8\]; as we will see, formulas \[7\] perfectly match the general case \( (n > 2, a \neq 0) \), while formulas \[8\] do not (but, surprisingly, go well with Theorem \[5\]). In corresponding definitions we take different vectors \( t \).

We can take \( t = [c] \) and \( t = [c, a] \) and associate to them matrices \( \tilde{t}_1 = \text{circ}(c) \) and \( \tilde{t}_2 = \text{circ}(c, a) \), respectively. This way we get

\[ \tilde{M}_1 = \begin{bmatrix} -c & b \\ b & c \end{bmatrix} = A_1(-c, b, c) \quad \text{and} \quad \tilde{M}_2 = \begin{bmatrix} -2c & b & b \\ b & c & a \\ b & a & c \end{bmatrix}. \]

In the above definitions tilded symbols are used, since these matrices do not observe general formulas given below, in Theorem \[5\]. Note, that in this approach the matrix \( \tilde{M}_1 \) does not depend on the parameter \( a \). It is easy to see
Theorem 4 (Spectra of $\tilde{M}_1$ and $\tilde{M}_2$)

1. $\sigma(\tilde{M}_1) = \{\pm \sqrt{b^2 + c^2}\}$, so $|\sigma(\tilde{M}_1)| = 2$.

2. $\sigma(\tilde{M}_2) = \{\lambda_-, \lambda_+, \lambda_1\}$, where

$$\lambda_\pm := \frac{1}{2} \left(a - c \pm \sqrt{\Delta}\right)$$

$$\lambda_1 := c - a,$$

and $\Delta = (a + 3c)^2 + 8b^2$. Therefore,

$$|\sigma(\tilde{M}_2)| = \begin{cases} 2, & \text{if } a \neq 0 \text{ and } c = (a^2 - b^2)/(3a), \\ 3, & \text{otherwise}. \end{cases}$$

Let us pay attention that

1. For $c = (a^2 - b^2)/(3a)$, $a \neq 0$, there is $\sigma(\tilde{M}_2) = \{s, -s/2\}$, where $s := 2(2a^2 + b^2)/(3a)$ is the single eigenvalue.

2. The lines $\lambda = c + a$ and $\lambda = -2c$ are asymptotes to both curves $\lambda = \lambda_-$ and $\lambda = \lambda_+$.

3. Since $\Delta > 0$ for $b \neq 0$, then $\lambda_- < \lambda_+$.

The other possibility is to demand that a sum of entries of the vector $t$ equals $c + 2a$. Within this approach $t_1 := \text{circ}(c + 2a)$ and $t_2 := \text{circ}(c, 2a)$, so

$$M_1 = \begin{pmatrix} -c & b \\ b & c + 2a \end{pmatrix} = A_1(-c, b, c + 2a) \quad \text{and} \quad M_2 = \begin{pmatrix} -2c & b & b \\ b & c & 2a \\ b & 2a & c \end{pmatrix}.$$  \hfill (8)

The spectrum and its properties of the matrix $M_1$ can be determined from Theorem 2 For the matrix $M_2$ Theorem 4 may be applied replacing the parameter $a$ by $2a$. The notions introduced below in Eqs. (9) can be formally applied to the matrix $M_1$, but this matrix has a nonvanishing trace and is omitted in the further discussion in Sections V and VI. On the other hand, the matrix $M_2$ defined above obeys all assumptions and, therefore, can be included in general considerations, so it is assumed hereafter that the matrix $m_2(a, b, c)$ is constructed with $t_2 = \text{circ}(c, 2a)$. In Section VI where a relation of abc matrices to wheel graphs is presented, both matrices, $\tilde{M}_2$ and $M_2$, are taken into account.

Completed the discussion on $n \in \{1, 2\}$, we fix a natural number $n > 2$, take arbitrary real numbers $a, b \neq 0, c$, and denote

$$M_n := m_n(a, b, c),$$

$$\varphi_n := 2\pi/n, \quad \text{so} \quad \omega_n = \exp(\i \varphi_n),$$

$$\Delta_n := \Delta(a, b, c, n) := (2a + (n + 1)c)^2 + 4nb^2,$$

$$\beta_{n, \pm} := \beta_{\pm}(a, b, c, n) := -\left(2a + (n + 1)c \mp \sqrt{\Delta_n}\right)/(2b),$$

$$\lambda_{n, \pm} := \lambda_{\pm}(a, b, c, n) := b\beta_{n, \pm} + 2a + c = \left(2a - (n - 1)c \pm \sqrt{\Delta_n}\right)/2,$$

$$\lambda_{n,k} := c + 2a \cos(k\varphi_n), \quad k = 1, 2, \ldots, n - 1.$$  \hfill (9f)

For higher transparency, we omit the index $n$, so $M \equiv M_n$, $\beta_{n,-} \equiv \beta_-$ etc. With above denotations there holds true

Theorem 5 For arbitrary $n > 2$ the eigenpairs of the matrix $M$ are

$$\{\lambda_-, w_-, \lambda_+, w_+\} \quad \text{and} \quad \{\lambda_k, w_k\}, \quad k = 1, 2, \ldots, n - 1,$$  \hfill (10)

where

$$w_\pm = [\beta_\pm, 1, 1, 1, \ldots, 1, 1]^T,$$

$$w_k = [0, 1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-2)k}, \omega^{(n-1)k}]^T,$$  \hfill (11a)

$$i.e., (w_k)_0 = 0 \text{ and } (v_k)_j = \omega^{(j-1)k}, \text{ for } j = 1, 2, \ldots, n.$$  \hfill (11b)
Proof 2 The proof consists in demonstrating that there hold true the equalities \( Mw_k = \lambda_k w_k \) and \( Mw = \lambda w \), and that the collection \( \{w_-, w_+, w_1, w_2, \ldots, w_{n-1}\} \) is linearly independent. First we state that there exists a number \( \beta \) (and we specify it later) such that

\[
w := [\beta, 1, 1, \ldots, 1]^T
\]

is an eigenvector of \( M \). For every \( \beta \) there is

\[
Mw = [nb - nc\beta, \lambda, \lambda, \ldots, \lambda]^T,
\]

where \( \lambda = b\beta + 2a + c \), so \( \beta \) is the eigenvalue of \( M \) iff

\[
nb - nc\beta = \beta \lambda = \beta(b\beta + 2a + c).
\]

This condition is the quadratic equation in \( \beta \), with solutions \( \beta_- \) and \( \beta_+ \) (they are distinct because \( b \neq 0 \)). For them the eigenvalue \( \lambda \) assumes the value \( \lambda_- \) and \( \lambda_+ \), respectively, and the vector \( w \) is \( w_- \) and \( w_+ \). Since the discriminant \( \Delta \) of the equation at hand is positive for \( b \neq 0 \), so \( \lambda_- < \lambda_+ \). The eigenvectors corresponding to them have identical coordinates (up to a constant multiplier each coordinate is equal to 1) but the first one: up to the same multiplier this coordinate is \( \beta_- \) and \( \beta_+ \), respectively.

Now we go to show that \( M \) has \( n-1 \) eigenvectors such that their first coordinate is 0. We will see even more: these eigenvectors of \( M \) are of form

\[
w_k := [0, 1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}]^T,
\]

where \( k = 1, 2, \ldots, n-1 \). The index \( k \) fixed, we have \( w_k = [0, 1, z, z^2, \ldots, z^{n-1}]^T \), where \( z := \omega^k \). Then, by the properties of the roots of unity,

\[
Mw_k = \begin{bmatrix}
b(1 + z + z^2 + \cdots + z^{n-1}) \\
c + a(z + z^{n-1}) \\
\vdots \\
cz^{j-1} + a(z^{j-2} + z^j) \\
\vdots \\
cz^{n-1} + a(1 + z^{n-2})
\end{bmatrix}
= (c + a(z + z^{-1}))
\begin{bmatrix}
0 \\
1 \\
\vdots \\
z^{j-1} \\
\vdots \\
z^{n-1}
\end{bmatrix}
= \lambda_k w_k
\]

The above proves that eigenpairs of \( M \) are \((\lambda_-, w_-), (\lambda_+, w_+)\) and \((\lambda_k, w_k)\), where \( k = 1, 2, \ldots, n-1 \).

Eigenvectors \( w_k, k = 1, 2, \ldots, n-1 \), are linearly independent. Really, by neglecting their first coordinate (recall, it is 0) we turn them into vectors \( v_k := [1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}]^T \), \( k = 1, 2, \ldots, n-1 \). They and the vector \( v_0 := [1, 1, 1, \ldots, 1]^T \) form the set \( \{v_0, v_1, v_2, \ldots, v_{n-1}\} \). The matrix, whose columns are these \( n \) vectors, is the Vandermonde matrix \( V_n = [v_0|v_1|v_2|\ldots|v_{n-1}] = V_n(1, \omega, \omega^2, \ldots, \omega^{n-1}) \). Since \( V_n \) is nonsingular, its columns are linearly independent.

Recalling that \( v_0 \) is the reduced \( w_+ \) or \( w_- \) (obtained by throwing away its first coordinate \( \beta_+ \)), and that \( w_- \) and \( w_+ \) are linearly independent, we conclude that vectors \( w_- , w_+, w_1, w_2, \ldots, w_{n-1} \) are linearly independent. \( \square \)

Note that in produced formulas \( b \) appears only squared, so the sign of \( b \) has no importance. Plots in Fig. 1 show how eigenvalues of \( M_n \) change in \( c \), when \( b = 1 \) and the parameters \( a \) and \( n \) are fixed.

V. ANALYZING THE EIGENVALUES OF AN ABC MATRIX

In the whole section \( b = \pm 1 \) is assumed, so in all cases \(|b| = 1 \) and \( b^2 = 1 \). The results can be applied to the matrix \( M_2 \) introduced by formula \([3] \).

A. The spectrum

Direct examination proves

Lemma 6 For \( n > 2 \) and any fixed real numbers \( a, b \neq 0 \)
Lemma 8 For fixed $n > 2$, a real number $a \neq 0$ and $b = \pm 1$ there holds true:

1. For every $k = 1, 2, \ldots, n-1$ the line $\lambda = \lambda_{sep}(c)$ lies above (below) $\lambda_k(c)$ for $a > 0$ ($a < 0$, respectively).

2. There are $p_n \equiv p := [(n-1)/2]$ pairs of equal numbers: $\lambda_k = \lambda_{n-k}$, for $k = 1, 2, \ldots, p$.

3. The sequence $(\lambda_1, \lambda_2, \ldots, \lambda_q)$, $q_n \equiv q := [n/2]$, is strictly decreasing (increasing) for $a > 0$ ($a < 0$, respectively). They are bounded by $\lambda_{\text{lim}} := c - 2a$, which is equal to $\lambda_q = \lambda_{n/2}$ for an even number $n$.

4. For $a > 0$ ($a < 0$) each $\lambda = \lambda_k(c)$ crosses $\lambda = \lambda_{-}(c)$ ($\lambda_{+}(c)$, respectively) at the unique point $C_k$ with the abscissa

$$
c_k = \frac{4a^2 \cos(k\varphi)(1 - \cos(k\varphi)) + n}{2(n + 1)a(\cos(k\varphi) - 1)}.
$$

When $k$ increases from 1 to $q$, these abscissas form an increasing (a decreasing) sequence for $a > 0$ ($a < 0$, respectively).
Proof 3 These four claims follow properties of the cosine function. For \(1 \leq k \leq q\) the angles \(k\varphi\) form the increasing sequence and \(0 < k\varphi \leq \pi\), so the sequence \((\cos(k\varphi))\) is decreasing, and this confirms (a). The point (b) is valid due to the upper bound \(\cos(k\varphi) < 1\) for \(k\varphi > 0\), whereas (b) is proved due to the parity of the cosine function, \(\cos((n-k)\varphi)\). The numbers \(c_k\) are unique solutions of the appropriate equations

\[
c + 2a \cos(k\varphi) = \left(2a - (n-1)c + \sqrt{\Delta}\right)/2.
\]

Due to the monotonic behavior of the sequence \((\lambda_k)\), the sequence \((c_k)\) is also monotonic and its character follows the properties of the curves \(\lambda = \lambda_\pm\) and the sign of the parameter \(a\). This completes the proof of (4), so of the whole lemma, as well.

Theorem 9 (Cardinality of the spectrum) With the same assumptions as in Lemma [8] we have

\[
|\sigma(M)| = \begin{cases} 
\lfloor n/2 \rfloor + 1, & \text{if } c = \{c_1, c_2, \ldots, c_q\}, \\
\lfloor n/2 \rfloor + 2, & \text{otherwise}.
\end{cases}
\]

Proof 4 This theorem follows immediately the claims in Lemma [8]

B. The special points

Since the function \(\lambda = \lambda_-\) is strictly concave in \(c\), then it may have the (local, so also global) maximum and, in fact, it has. Its uppermost point is \(U_n(a) := (c_{\text{upp}}, \lambda_{\text{upp}})\), where

\[
c_{\text{upp}}(a, n) := -\frac{(n-1) + 2a}{n + 1} = -1 - \frac{2(a-1)}{n+1}, \quad (14a)
\]

\[
\lambda_{\text{upp}}(a, n) := \frac{2n}{n+1}(a-1) = -n(c_{\text{upp}} + 1). \quad (14b)
\]

Similarly, the convex function \(\lambda = \lambda_+\) has the lowermost point (the global minimum) at \(L_n(a) := (c_{\text{low}}, \lambda_{\text{low}})\), with

\[
c_{\text{low}}(a, n) := \frac{(n-1) - 2a}{n + 1} = 1 - \frac{2(a+1)}{n+1}, \quad (15a)
\]

\[
\lambda_{\text{low}}(a, n) := \frac{2n}{n+1}(a+1) = -n(c_{\text{low}} - 1). \quad (15b)
\]

Considerations of the other “special point” we start with

Corollary 10 For fixed \(a \neq 0\) and any number \(n > 2\) the abscissas \(c_k\), \(k = 1, 2, \ldots, q\), belong to the interval \((-\infty, c_-]\), for \(a > 0\), and to the interval \([c_+, \infty)\), for \(a < 0\), where \(c_\pm\) denote the abscissa of the point at which \(\lambda_\pm(c) = \lambda_\text{lim}(c)\).

Proof 5 At first we note that for arbitrary fixed \(a \neq 0\), \(n > 2\), and \(k = 1\) Eq. (15) gives

\[
c_1 = -\frac{2a \cos \varphi}{n+1} + \frac{n}{n+1} \frac{1}{2a(\cos \varphi - 1)}.
\]

Obviously, the first summand tends to 0. Moreover, \(\cos \varphi < 1\) and \(\lim_{n \to \infty} (\cos \varphi - 1) = 0\), therefore \(\lim_{n \to \infty} c_1 = -\infty\) for \(a > 0\) and \(\lim_{n \to \infty} c_1 = \infty\), otherwise. The existence and properties of the points \(c_\pm\) follow from the points (3) and (7) in Lemma [5], so the proof is completed.
In the other domains, i.e. for \( c \in (c_-, \infty) \), for \( a > 0 \), and \( c \in (-\infty, c_+), \) for \( a < 0 \), the graphs \( \lambda = \lambda_k(c) \) and \( \lambda = \lambda_k(c) \) have no common points, so in these domains \( |\sigma(M)| = q + 2 \). The spectrum cardinality at \( c_\pm \) depends on the parity of \( n \), namely \( |\sigma(M)| = \left| (n + 3)/2 \right| \).

Taking into account Eq. (14) we see that both numbers \( c_\pm \) are determined by the same formula. We refer to the point, at which the one of the curves \( \lambda = \lambda_k(c) \) meets the limit line \( \lambda = \lambda_{\text{lim}} \), as a transition point (generated by \( a \neq 0 \) and \( n \)). Coordinates of this point, \( T_n(a) := (c_{\text{trans}}, \lambda_{\text{trans}}) \), can be determined substituting \( \cos(k\varphi) = -1 \) in Eq. (13) and then taking \( \lambda_{\text{trans}} = c_{\text{trans}} - 2a \). In this way we have

\[
\begin{align*}
c_{\text{trans}}(a, n) &= \frac{8a^2 - n}{4(n + 1)a}, \\
\lambda_{\text{trans}}(a, n) &= -\frac{n(8a^2 + 1)}{4(n + 1)a}.
\end{align*}
\]

The transition point \( T_n(a) \) sits on the curve \( \lambda = \lambda_-(c) \), for \( a > 0 \), and on the curve \( \lambda_+(c) \) for \( a < 0 \).

**Remark 2** Note that Eqs. (16) for \( n = 2 \) give

\[
T_2(a) = \frac{1}{6a} (4a^2 - 1, -(8a^2 + 1))
\]

and this formulas correspond to the matrix \( M_2 \) given by formula (3). To obtain them for the matrix \( \tilde{M}_2 \), determined in by (7), we have to replace each \( 2a \) by \( a \), so (cf. Theorem 7)

\[
\tilde{T}_2(a) = \frac{1}{3a} (a^2 - 1, -(2a^2 + 1)).
\]

For fixed numbers \( n \) the set \( \{ T_n(a) \mid a \neq 0 \} \equiv \{(c_{\text{trans}}, \lambda_{\text{trans}})\} \subset \mathbb{R}^2 \) determines the graph \( T_n(a) \) hereafter referred to as the \( n \)-th transition curve. Note that this curve has two separate branches: one for \( a > 0 \) and the second for \( a < 0 \). These branches have asymptotes: \( \lambda = c \) and \( \lambda = -nc \). Sending \( n \) to infinity we get

\[
T_\infty(a) := \lim_{n \to \infty} T_n(a) = \frac{1}{4a} (1, 8a^2 + 1).
\]

This point referred to as a limit transition point (associated to \( a \)); the set (and the graph) \( T(a) = \{ T_\infty(a) \mid a \neq 0 \} \) is called a limit transition curve. This limit curve is explicitly described via the relation \( \lambda = c + 1/(2c) \), \( c \neq 0 \). For \( c < 0 \) (\( c > 0 \)), so also \( \lambda < 0 \) (\( \lambda > 0 \), respectively), this graph is the “limit curve” for the lower (the upper) branch of \( T_n(a) \), i.e., for \( a > 0 \) and \( a < 0 \), respectively. Since all the transition points satisfy the relation \( \lambda = \lambda_{\text{lim}}(c) = c - 2a \) and, moreover, it is also satisfied by points sitting on the graph \( T(a) \), then at each point \( T_\infty(a) \) the curve \( T(a) \) and the line \( \lambda_{\text{lim}} \) intersect each other. Fig. 2 shows some transition curves and the limit transition curve.

**C. The extreme eigenvalues**

The points \( T_n(a) \) are actual transitions points for an even number \( n \), so, hereafter, we restrict ourselves to this case. Solving equations (for \( a > 0 \))

\[
c_{\text{upp}} = c_{\text{trans}} \quad \text{and} \quad \lambda_{\text{upp}} = \lambda_{\text{trans}}
\]

we find the value \( a_{\text{crit}+} = 1/4 \) at which the transition point \( T_n(a) \) coincides with the uppermost point \( U_n(a) \). The same procedure gives \( a_{\text{crit}^-} = -1/4 \) for which \( T(a) = U(a) \) (\( a < 0 \)). Since \( \lambda_{\text{upp}} (\lambda_{\text{low}}) \) is the global maximum (minimum, respectively), then for \( a \neq a_{\text{crit} \pm} \) we always have \( \lambda_{\text{trans}} < \lambda_{\text{upp}} \), when \( a > 0 \), and \( \lambda_{\text{trans}} > \lambda_{\text{low}} \), when \( a < 0 \). Now, directly from Lemmas 6 and 8 it follows

**Corollary 11** The following inequalities are satisfied (for \( n > 2 \))

\[
\begin{align*}
c_{\text{trans}} &< c_{\text{low}}, \quad \text{for} \quad a < -1/4, \\
c_{\text{trans}} &> c_{\text{low}}, \quad \text{for} \quad -1/4 < a < 0, \\
c_{\text{trans}} &< c_{\text{upp}}, \quad \text{for} \quad 0 < a < 1/4, \\
c_{\text{trans}} &> c_{\text{upp}}, \quad \text{for} \quad a > 1/4.
\end{align*}
\]

The critical values \( a_{\text{crit} \pm} = \pm 1/4 \) do not depend on \( n \).
These relations are also satisfied for the matrix $M_2$. When the matrix $\tilde{M}_2$ is at hand, then $\tilde{a}_{\text{crit}} = \pm 1/2$.

Reassuring the above, we have

**Theorem 12 (Extreme eigenvalues)** For any natural number $n > 2$ the extreme eigenvalues $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ of the matrix $m_n(a,b = \pm 1,c)$ are given by the following formulas

**Case** $a < 0$::

$$\lambda_{\text{min}}(c) = \lambda_{-}(c), \quad \text{and} \quad \lambda_{\text{max}}(c) = \begin{cases} \lambda_{+}(c), & \text{for } c < c_{\text{trans}}, \\ \lambda_{n/2} = c - 2a, & \text{for } c \geq c_{\text{trans}}. \end{cases}$$ (20)

**Case** $a = 0$:: $\lambda_{\text{min}}(c) = \lambda_{-}(c)$ and $\lambda_{\text{max}}(c) = \lambda_{+}(c)$;

**Case** $a > 0$::

$$\lambda_{\text{min}}(c) = \begin{cases} \lambda_{n/2} = c - 2a, & \text{for } c \leq c_{\text{trans}}, \\ \lambda_{-}(c), & \text{for } c > c_{\text{trans}}. \end{cases}, \quad \text{and} \quad \lambda_{\text{max}}(c) = \lambda_{+}(c).$$ (21)

The preceding claims prove the final

**Corollary 13 (Extreme extrema)** The functions $\lambda = \lambda_{\text{min}}(c)$ and $\lambda = \lambda_{\text{max}}(c)$ have their extrema $\min_{c \in \mathbb{R}} \lambda_{\text{max}}(c)$ and $\max_{c \in \mathbb{R}} \lambda_{\text{min}}(c)$ at

$$c_{\text{min of max}}, \lambda_{\text{min of max}} = \begin{cases} (c_{\text{trans}}, \lambda_{\text{trans}}), & \text{for } a \leq -1/4 \\ (c_{\text{low}}, \lambda_{\text{low}}), & \text{for } a > -1/4; \end{cases}$$ (22a)

$$c_{\text{max of min}}, \lambda_{\text{max of min}} = \begin{cases} (c_{\text{trans}}, \lambda_{\text{trans}}), & \text{for } a \geq 1/4 \\ (c_{\text{upp}}, \lambda_{\text{upp}}), & \text{for } a < 1/4. \end{cases}$$ (22b)

Whenever such extremum coincides with the transition point, the corresponding eigenvalue (the minimum of $\lambda_{\text{max}}$ or the maximum of $\lambda_{\text{min}}$) is double, and it is single, otherwise.

**VI. AN ABC MATRIX AND ITS CORRESPONDING WHEEL GRAPH**

Just introduced abc matrix $m_n(a,b,c)$ can be interpreted as the weighted adjacency matrix of a regularly weighted wheel graph, the notion we define below and we mostly follow the nomenclature presented by Bondy and Murty [3] and Brandstädt et al. [43] [see also [44]. To unify the nomenclature, we say that an $N$-vertex graph, $N > 3$, in
In this paper the special case has been investigated: regular \(M\) matrices (see Section III). They inherit some matrix properties of their “parents”, arrowhead and circulant matrices. We have determined eigenpairs of the \(abc\) matrices in each row. Such object can be considered as weighted adjacency matrices of wheel graphs. For \(n = 2\) (b) a weighted cyclic graph \(C_3\), with the adjacency matrix \(M_2\), or (c) a multigraph, with the adjacency matrix \(M_2\), can be considered. In all cases black full circles denote vertices of weight \(c\), whereas gray (and larger) ones denote those of weight \(-nc\).

An example, the \(W_7\) graph, is presented in Fig. 3(a).

When to each of \(2n\) edges and to each vertex \(j = 0, 1, \ldots, n\) of \(W_N\) there are assigned some real numbers (referred to as weights), we have a \textit{weighted wheel graph}, also denoted by \(W_N\). It is said to be \textit{regular} if: (a) every edge \((j, j + 1)\), \(n + 1 \equiv 1\), in \(C_n\) is of weight \(w(j, j + 1) = a\); (b) every spoke \((0, j)\) is of weight \(w(0, j) = b\); (c) every tire vertex is of the same weight \(w(j) = c\); (d) the weight of the central vertex is \(w(0) = -nc\). Here \(b\) can be seen as a scale, and without the loss of generality we can take into account the sign of \(b\) only, i.e., we restrict discussion to the cases \(b = \pm 1\) (cf. Remark 1). Clearly, \(W_N\) is of dihedral symmetry \(D_n\), although this group does not act transitively on all vertices. In a more general case, it can be assumed \(w(0) = c' \neq -nc\), which preserves the dihedral symmetry. However, in physical problems the matrices \(M_n\) are applied to, one needs traceless matrices (see further in the text for details), so we demand \(w(0) = -nc\) in a regular weighted wheel graph \(W_N\). Such graphs, and corresponding weighted adjacency matrices, are considered in some papers representing various fields of science, see, e.g., [15, 47–49]. Note that for \(a = 0\) we have no wheel graph \(W_N\) any longer (since the edges with zero weights are removed), but we have a star graph \(S_N\) ([7] applies accordingly), which is connected. This is why we have chosen the weight of spokes not of the tire (cyclic) edges to be a scale in problems discussed here.

In the main text there are considered matrices \(\tilde{M}_2\) and \(M_2\). Formally, they are \textit{not} weighted adjacency matrices of wheel graphs, since usually it is assumed that a wheel graph \(W_N\) has at least four vertices. By formula (7) it is clear that the matrix \(\tilde{M}_2\) is a weighted adjacency matrix of the cyclic graph \(C_3\) with one edge of different weight \(a\), whereas the other two edges are of equal weights \(b\) [see Fig. 3(b)]. So this graph (and the corresponding matrix) has the \(D_2\) (the Klein group) symmetry. A doubled weight \(2a\) in the matrix \(M_2\) appears when we sketch the cyclic graph \(C_3\), with two vertices and two edges, i.e., a digon (two-gon): due to two undirected edges (1.2) it is a multigraph [see Fig. 3(c)]. However, it has, again, the \(D_2\) symmetry and connecting its vertices to the hub (labeled by 0) we obtain “a centered digon”. The tire edges have the same weight \(a\), so (at least in some applications) this graph (its adjacency matrix, in fact) has the same properties as the cyclic graph \(C_3\) with one distinguished edge of weight \(2a\). This shows that the matrix \(M_2\) can be included in the discussion presented, whereas the matrix \(\tilde{M}_2\) constitutes the special case.

VII. FINAL REMARKS

In this paper we introduced arrow-bordered circulant (“abc” for short) matrices \(m_n(a, b, c)\) for \(n \geq 2\) and real parameters \(a, b,\) and \(c\). For \(a = 0\) and \(bc \neq 0\) the arrowhead matrices are revealed, whereas assuming \(b = 0\) and \(a \neq 0\) one obtains two blocks: a trivial one-dimensional matrix and a circulant matrix with at most three nonzero elements in each row. Such object can be considered as weighted adjacency matrices of wheel graphs \(W_N\) (or star graphs \(S_N\) for \(b = 0\)). In this paper the special case has been investigated: regular abc matrices (see Section III). They inherit some properties of their “parents”, arrowhead and circulant matrices. We have determined eigenpairs of the abc matrices.

FIG. 3. (Color online) (a) The wheel graph \(W_7 \equiv W_{n+1}\) (the hub or the central vertex is labeled by 0, and tire vertices are labeled by 1, 2, \ldots, \(n\)) with its edge-weights (the tire edges of weight \(a\), solid lines, and the spokes of weight \(b\), dashed lines). For \(n = 2\) (b) a weighted cyclic graph \(C_3\), with the adjacency matrix \(M_2\), or (c) a multigraph, with the adjacency matrix \(M_2\), can be considered. In all cases black full circles denote vertices of weight \(c\), whereas gray (and larger) ones denote those of weight \(-nc\).
and discussed their eigenvalues. We have put stress on their bounds, asymptotic behavior, and extrema. Since such matrices are widely used in different fields of science, then this work provides results of some interest. It is desirable to have results in more general cases, e.g., for irregular abc matrices, when the vector \( b \) is not constant [cf., 57], or arrow-bordered alternating circulant matrices [50]. In the latter case signs of consecutive rows are alternated, but in more general cases rows with alternated values of nonzero matrix elements should also be included.

This work has been motivated by some physical problems, in particular these related to magnetic (finite) systems. Classical counterpart of the Heisenberg model describes a set of localized spin vectors \( s_j \) with interactions determined by real numbers \( J_{ij} \), where “locations” of vectors \( s_j \) are labeled by \( i, j = 1, 2, \ldots, N \); this numbers are interpreted as graph vertices (to each vertex \( j \) of a given graph \( G \) a spin vector \( s_j \) is assigned). Within this approach couplings \( J_{ij} \) are nondiagonal elements of a weighted adjacency matrix (with vanishing diagonal) of the graph \( G \). The total energy of this system equals \( E = \sum_{(i,j)} J_{ij} s_i \cdot s_j \), where the standard inner product is denoted by dot “\( \cdot \)”. Schmidt and Luban [1] [see also a series of papers 51–54] introduced the so-called gauge vector \( c \in \mathbb{R}^N \) and then they “dressed” the matrix \( J \) assuming \( J_{jj} = c_j \), so each number \( c_j \) can be considered as weight of the vertex \( j \). They showed that for traceless dressed matrix (i.e., for \( \sum_j c_j = 0 \)) some physical quantities are “gauge independent”. For fixed parameters of the model the eigenvalues of the dressed matrix, including its minimum eigenvalue \( \lambda_{\min}(c) \), depend on the gauge vector \( c \). They proved that there exists \( \max_{c \in \mathbb{R}^N} \lambda_{\min}(c) \) and it equals, up to a constant factor, the minimum of the system energy \( E \). The considerations performed in this paper give us a general solution of this problem for spins placed at vertices and the center of a regular \( n \)-gon with an even number \( n \) [see, e.g., 52–58]. The further physical analysis take into account also eigenvectors of the dressed matrix \( J \), but this problem is out of scope here. It has to be emphasized that the degeneracy of the determined maximum of the minimum eigenvalue \( \lambda_{\min}(c) \) says whether the lowest energy configuration of spin vectors is collinear, coplanar or spatial—it happens for single, double and triple \( \lambda_{\min}(c) \) (cf. remarks at the end of Section V A). Note that higher degeneracy is not excluded and Schmidt [5] have provided an example of a classical spin system, when the lowest energy configuration can be realized in nonphysical four-dimensional space \( \mathbb{R}^4 \). For small systems some results have been obtained with simple calculus [55, 59], but more general considerations need strictly proved properties, i.e., the results of this paper [see 60]. It is worth noting that for actually synthesized magnetic molecules [see, e.g., 61–63] there is needed analysis of some more general matrices (not regularly weighted graphs), e.g., alternating circulant matrices or arrow-bordered alternating circulant ones.

1. H.-J. Schmidt and M. Luban, J. Phys. A: Math. Gen. 36, 6351 (2003)
2. N. L. Biggs, Algebraic Graph Theory (Cambridge University Press, Cambridge, 1974).
3. J. A. Bondy and U. S. R. Murty, Graph Theory with Applications (North-Holland, New York, 1976).
4. D. M. Cvetković, M. Doob, I. Gutman, and A. Torgašev, Recent Results in the Theory of Graph Spectra (North-Holland, New York, 1988).
5. A. E. Brouwer and W. H. Haemers, Spectra of Graphs (Springer, 2012).
6. B. Nica, A Brief Introduction to Spectral Graph Theory (EMS Publishing House, Zuerich, 2018).
7. X. Zhang and H. Zhang, Lin. Alg. Appl. 431, 1443 (2009)
8. K. C. Das, Discrete Math. 313, 19 (2013).
9. D. M. Cvetković and I. Gutman, eds., Selected topics on applications of graph spectra, Zbornik Radova (Eng. Collection of Works), Vol. 14(22) (Math. Inst. Serbian Academy of Sciences and Arts (SANU), 2011).
10. S. Klavžar and I. Gutman, Discrete Appl. Math. 80, 73 (1997)
11. V. Latora and M. Marchiori, Eur. Phys. J. B 32, 249 (2003).
12. A. R. Ashrafi and M. Hamadanian, Croatica Chemica Acta 78, 159 (2005)
13. M. J. Mirchev, in Proc. XXIII TELECOM edited by S. Mirchev and S. Patchedjiev (Sofia, 2015) pp. 141–148.
14. H. Z. Jooya, K. Reihani, and C. Shih-I, Sci. Rep. 6, 37544 (2016).
15. H. Cai, F. L. Lewis, G. Hu, and J. Huang, Automatica 75, 299 (2017)
16. K. Antoš, Periodica Polytechnica Translational Engineering (2018), 10.3311/PPtr.11171 ; Onlinefirst issue, 6 pages.
17. In 1911, in the paper Über allgemeine lineare Mittelbildungen (published in annals Prace Matematyczno-Fizyczne; they appeared in Warsaw, its 48 volumes were published in years 1888–1952), Otto Toepplitz (1881–1940), when discussing the summability of series, treated specific matrices, in particular triangular ones. Two years later his result was generalized (and it is now called the Silverman-Toeplitz theorem), and infinite matrices satisfying some conditions became to be referred to as Toepplitz; note that these Toepplitz matrices are objects distinct from that considered in linear algebra.
18. P. Hartman and A. Wintner, American Journal of Mathematics 76, 867 (1954).
19. P. Delmay and Y. Genin, in Mathematical Theory of Networks and Systems, edited by P. A. Fuhrmann (Springer, Berlin, Heidelberg, 1984) pp. 194–213.
20. L. Reichel and L. N. Trefethen, Lin. Alg. Appl. 162-164, 153 (1992)
21. A. Melman, Math. Comp. 70, 649 (2001)
A. L. Andrew, [Lin. Alg. Appl. 7, 151 (1973)].
A. Cantoni and P. Butler, [Lin. Alg. Appl. 13, 275 (1976)].
Z.-Y. Liu, [Appl. Math. Comput. 141, 297 (2003)].
I. T. Abu-Setb, New Zealand J. Math. 33, 105 (2004).
G. Y. Katona, M. Faghani, and A. R. Ashrafi, Discussiones Mathematicae Graph Theory 34, 751 (2014).
R. A. Brualdi and S.-M. Ma, in Mathematics across contemporary sciences, Springer Proceedings in Mathematics & Statistics, Vol. 190, edited by T. Abualrub, A. S. Jarrah, S. Kallel, and H. Sulieman (Springer, Berlin, 2017) pp. 17–31.
A matrix $a = [a_{j,k}]_{j=1,2,...,n}$ is said to be centrosymmetric, or cross-symmetric, if it is symmetric about its center, $a_{j,k} = a_{n+1-j,n+1-k}$. A necessary and sufficient condition a matrix $[a_{j,k}]$ to be centrosymmetric its commutation with so-called exchange matrix $J$ (by definition, all entries of $J$ are 0, but $J_{j,n+1-j} = 1$ for all $j = 1, 2, \ldots, n$). Thus every symmetric Toeplitz matrix is centrosymmetric. The centrosymmetry is one of patterns of symmetry [see, e.g., 63–66].