Coset for Hopf fibration and squashing

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Abstract

We provide a simple derivation of metrics for fundamental geometrical deformations such as Hopf fibration, squashing and the $\mathbb{Z}_k$ quotient which play essential roles in recent studies on the AdS4/CFT3. A general metric formula of Hopf fibrations for complex and quaternion cosets is presented. Squashing is given by a similarity transformation which changes the metric preserving the isometric symmetry of the projective space. On the other hand, the $\mathbb{Z}_k$ quotient is given as a lens space which changes the topology preserving the ‘local’ metric.

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1. Introduction

In recent studies on AdS4/CFT3 \cite{1}, interesting subjects are explored by fundamental geometrical deformations such as Hopf fibration, squashing and the $\mathbb{Z}_k$ quotient \cite{2-4}. The AdS\textsubscript{4} \times $\mathbb{S}^7$ is a maximal supersymmetric solution of the M-theory, and a gauge theory on the three-dimensional membrane worldvolume including $N = 8$ supersymmetries is expected to be its CFT dual \cite{1, 5}. For the $N = 6$ superconformal symmetry, whose bosonic subgroup is $SU(4) \times U(1)$, the Chern–Simon-matter theory with level $k$ is conjectured to be CFT dual of the $M$ theory on the AdS\textsubscript{4} \times $\mathbb{S}^7/\mathbb{Z}_k$ \cite{1}. Dimensional reduction by increasing $k$ reduces it into the type IIA superstring theory on the AdS\textsubscript{4} \times $\mathbb{CP}^3$ \cite{6, 7}. The $N = 1$ superconformal theory with the bosonic subgroup $Sp(2) \times Sp(1)$, where $Sp(2) = SO(5)$ and $Sp(1) = SO(3)$, is conjectured to be dual of the $M$ theory on AdS\textsubscript{4} \times $\tilde{S}^7/\mathbb{Z}_k$ \cite{8} where $\tilde{S}^7$ is squashed $S^7$. There is also squashed $\mathbb{CP}^3$ solution \cite{9} and other squashing gravity duals in the AdS/CFT including $N = 2, 3$ superconformal cases have been examined \cite{10}.

Squashed solutions and $\mathbb{Z}_k$ quotient solutions were widely studied in the 11-dimensional supergravity theory \cite{3, 4, 11} and in the superstring theories \cite{12}. Recently intensive studies on such deformations have been explored in the 5-dimensional supergravity theory related to black
Table 1. Hopf fibration and coset.

| Field   | Hopf fibration | Coset  | Projective space |
|---------|----------------|--------|------------------|
| Real    | $S^N$ $\rightarrow$ RP$^N$ | $O(N + 1)$ $\rightarrow$ $O(N + 1)$ | |
| Complex | $S^{2N+1}$ $\rightarrow$ CP$^N$ | $U(N + 1)$ $\rightarrow$ $U(N + 1)$ | $U(N + 1)$ $\rightarrow$ $U(N) \times U(1)$ |
| Quaternion | $S^{4N+3}$ $\rightarrow$ HP$^N$ | $Sp(N + 1)$ $\rightarrow$ $Sp(N + 1)$ | $Sp(N + 1) \times Sp(1)$ |

hole solutions [13, 14]. They are also useful to explore 11-dimensional supergravity solutions from the point of view of analogy between 5-dimensional and 11-dimensional supergravity theories [15]. Squashing deforms a kind of 5-dimensional asymptotically flat black hole solution into an asymptotically locally flat Kaluza–Klein black hole solution with a compact extra dimension. The $Z_k$ quotient of a compact direction gives a Kaluza–Klein monopole solution with a magnetic charge $k$ [16]. Squashing preserves the global isometric symmetry but changes the local metric, while the $Z_k$ quotient changes the topological structure but does not change the local metric.

The Hopf fibration describes a three-dimensional sphere in terms of a one-dimensional fiber over a two-dimensional sphere base space whose metric is a projective space metric locally. Its generalization to real, complex, quaternonic and octonionic projective spaces exist. The corresponding cosets are known as listed [17] in table 1. It is denoted that $Sp(n)$ is a group of $n \times n$ matrices of quaternion numbers. In this paper, we present a simple description of these spaces by following the projective lightcone limit procedure proposed in [18]. The projective lightcone limit brings the AdS$_5$ space into the four-dimensional projective lightcone space where CFT lives in. It was generalized to a Hopf fibration, a map from a $(2N + 1)$-dimensional sphere to a $N$-dimensional complex projective space, as a simple derivation of the Fubini–Study metric [19]. There is a gauged sigma model derivation of $\mathbb{C}P^N$ metric [20] where constrained $N + 1$ complex coordinates are used. Starting from the $S^{2N+1}$ metric, a gauge field is introduced for the local $U(1)$ and then it is eliminated by the equation of motion to obtain the $\mathbb{C}P^N$ metric. Instead we use a whole $U(N + 1)$ matrix as coordinates which includes $U(1)$ field and $U(N)$ fields. In this formulation it is clear how coset, sigma model and geometry are related; for a coset $U(N + 1)/U(N)$ the kinetic term for $U(N)$ fields is absent resulting $S^{2N+1}$, and for a coset $U(N + 1)/(U(N) \times U(1))$ the kinetic term for $U(1)$ field in addition to $U(N)$ fields is absent resulting $\mathbb{C}P^N$.

In this paper, we extend this formulation to Hopf fibrations listed in table 1 and to deformations such as squashing and the $Z_k$ quotient. In the following section, a general formula of Hopf fibrations is derived where coordinates are embedded in a group matrix. Squashing is explained as a similarity transformation. After discussing a real coset case, a complex coset case is presented in section 4. The $Z_k$ quotient is presented as a lens space there. In section 5, a quaternion coset case is presented. Squashed $S^7$ in our formulation is also presented which is consistent to that obtained by Awada et al [3].

2. General formula

In this section, we present a simple derivation of metrics for a sphere and a projective space. A squashing is introduced as a similarity transformation.
Table 2. Embedding into orthogonal group of coset for sphere.

| Orthogonal group embedded G/H | Field | m  | n  | Condition | G/H \\ O(n+1) |
|------------------------------|-------|----|----|-----------|----------------|
| O(m+n+1) for n+1            | Real  | 1  | N - 1 | None      | O(N)           |
| O(n+1) for n+1              | Complex | 2  | 2N - 1 | g′Kg = K  | U(N)           |
|                              |        |    |      | g′Lg = L  | Sp(N)          |

We begin by a metric for a sphere whose corresponding coset G/H is in Table 1. For a general treatment we consider embedding these groups, G and H, into orthogonal group O(m+n+1) and O(n) where m and n take specific numbers as listed in Table 2. There are conditions on an element of U(n) and Sp(n) as in Table 2 with

\[ g'Kg = K, \quad K = \begin{pmatrix} \epsilon & 0 & \ldots & 0 \\ 0 & \epsilon & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \epsilon \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ -1 \\ 0 \end{pmatrix} \]  (2.1)

and

\[ g'Lg = L, \quad L = \begin{pmatrix} \epsilon_4 & 0 & \ldots & 0 \\ 0 & \epsilon_4 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \epsilon_4 \end{pmatrix}, \quad \epsilon_4 = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ -1_2 \\ 0 \end{pmatrix}. \]  (2.2)

For an element of the coset embedded in O(m+n+1) matrix \( z_A^B \) with \( A, B = 0, \ldots, m, \ldots, m+n \) it is convenient to denote \( z_A^0 = x_A \). The orthonormal condition, \( z'^t z = 1 \), leads to the metric for a (m+n)-dimensional sphere

\[ (z'z)^00 = \sum_{A=0}^{m+n} (x_A)^2 = 1, \quad \sum_{A=0}^{m+n} (dx_A)^2 = \sum_{A,B=0}^{m+n} \delta^{AB} (J_A^0)^t J_B^0, \]  (2.3)

where \( J_A^B = (z^{-1} dz)_A^B \) is the left invariant (LI) one form. This is invariant under the global O(m+n+1) transformation of \( x_a \), namely a round (m+n)-dimensional sphere.

On the other hand, the same isometric symmetry of the sphere, G, is realized by a projective space as listed in Table 2. An element of \( G \ni z \) is partitioned into four blocks

\[ z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]  (2.4)

where \( A, B, C, D \) are \( m \times m, m \times (n+1), (n+1) \times m, (n+1) \times (n+1) \) matrices, respectively. Diagonal blocks A and D are subgroup H of the coset for a projective space, while only D is included in H for a sphere coset. One of the two off-diagonal blocks, C, contains projective coordinates. In order to describe a projective space it is convenient to parametrize z as \[ z = \begin{pmatrix} 1 & 0 \\ (u) & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ (v) & 0 \end{pmatrix}. \]  (2.5)
The parameters in (2.4) and (2.5) are related as \(u = A\) and \(X = CA^{-1}\) showing that \(X\) is projective. Under the \(G\) transformation, block coordinates are transformed as

\[
z \to z' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G
\]

\[
\begin{align*}
X' &= (c + dX)(a + bX)^{-1} \\
u' &= (a + bX)u \\
v' &= (d - (c + dX)(a + bX)^{-1}b)v \\
Y' &= Y + u^{-1}(a + bX)^{-1}bv.
\end{align*}
\]

The \(X\) parameter is a projective coordinate representing \(G\) by a fractional linear transformation. The LI current is calculated as

\[
J = z^{-1} dz = \begin{pmatrix} J_u & J_y \\ J_x & J_v \end{pmatrix}
\]

\[
\begin{align*}
J_x &= v^{-1} dX u \\
J_u &= u^{-1} du - Yv^{-1} dX u \\
J_v &= v^{-1} dv + v^{-1} dX uY \\
J_y &= dY + u^{-1} du Y - Yv^{-1} dv - Yv^{-1} dX uY.
\end{align*}
\]

Each element is invariant under \(G\) transformation (2.6).

A sphere metric (2.3) is generalized to \(ds^2 = \|J_X\|^2 + \|J_u\|^2\) with a suitable norm. The contribution of \(\|J_Y\|^2\) equals \(\|J_X\|^2\) for the orthogonal group so we just use simpler expression \(\|J_X\|^2\) only. The norm \(\|w\|^2\) is defined in such a way that it equals the norm for a corresponding complex/quaternion number, \(|w|^2\). Complex conjugate ‘\(^*\)’ is taken care by transpose ‘\(^t\)’ when a complex matrix is embedded in an real \(O(n)\) matrix as shown in the following sections. For example, suppose that \(w\) is \(O(2)\) \(2 \times 2\) matrix written as \(aI_2 + ib\) with real numbers \(a, b\). The norm is defined as \(\|w\|^2 = \frac{1}{2} \text{tr} w^*w = a^2 + b^2\) which is equal to \(|w|^2 = w^*w = a^2 + b^2\) when it is recognized as a complex number, \(w = a + ib\). We denote ‘\(\text{tr} M^{AB}\)’ as \(\sum_{A=0}^{m-1} M^{AA}\), since we only need this trace for \(m \times m\) matrices, \((J_X^tJ_X)^{AB}\) and \((J_u^tJ_u)^{AB}\), with \(m = 2\) or for complex or quaternion cases, respectively. Both \(\|J_X\|^2\) and \(\|J_u\|^2\) are invariant under \(H\) transformation;

\[
z \to z' = z\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \in H
\]

(2.8)

and \(h'h = 1\) is used in the norm \(\|J_X\|^2\).

The orthonormal condition of \(z\) of (2.5) is given by

\[
z'z = 1 \Rightarrow \begin{align*}
u u' &= [1 + X'X]^{-1} \\
u u' &= [1 - X(1 + X'X)^{-1}X']^{-1} \\
Y &= -u'X'v.
\end{align*}
\]

(2.9)

Inserting this relation into (2.7), the square of \(J_X\) is given by

\[
\|J_X\|^2 = \frac{1}{m} \text{tr}(J_X)^tJ_X = \frac{1}{m} \sum_{B=0}^{m-1} \sum_{A=m}^{m+n} (J_{X^{A^{B}}})^t J_{X^{A^{B}}}
\]

\[
= \frac{1}{m} \text{tr}(1 + X'X)^{-1}[dX'dX - dX'X(1 + X'X)^{-1}X' dX]
\]

\[
= \frac{\|dX\|^2}{1 + \|X\|^2} - \frac{\|X' dX\|^2}{(1 + \|X\|^2)^2} = ds^2_{\text{Fubini–Study}}.
\]

(2.10)
This is a Fubini–Study metric for a projective space [19]. From the orthonormal condition of the first line of (2.9) u can be parametrized as

$$u = \frac{1}{\sqrt{1 + \|X\|^2}} U, \quad U' U = 1_m$$

(2.11)

where $1_m$ is a $m$-dimensional unit matrix. The square of $J_u$ is given by

$$\|J_u\|^2 = \frac{1}{m} \text{tr} J_u^t J_u = \frac{1}{m} \text{tr}[u^{-1} du + u' X' dX u][u^{-1} du + u' X' dX u]$$

$$= \left\| dU U^{-1} + \frac{X' dX - dX' X}{2(1 + |X|^2)} \right\|^2 = d^2_{\text{Hopf-fiber}}$$

(2.12)

It turns out that this is a metric for the Hopf-fiber.

Now let us perform a similarity transformation with a parameter $\lambda$ as

$$z \rightarrow z_\lambda = \Lambda z \Lambda^{-1}, \quad \Lambda = \left( \begin{array}{cc} \lambda 1_m & 0 \\ 0 & I_{n+1} \end{array} \right).$$

(2.13)

Although this similarity transformation does not change the algebra, it scales a part of LI 1-forms as

$$J \rightarrow \left( \begin{array}{cc} J_u & \lambda J_Y \\ \lambda J_X & J_v \end{array} \right).$$

(2.14)

Taking into account the rescaling $ds^2 \rightarrow ds^2/\lambda^2$ for a normalization, the metric for a sphere becomes

$$ds^2_{\text{round}} = ds^2_{\text{Fubini-Study}} + \lambda^2 ds^2_{\text{Hopf-fiber}}$$

$$= \left\{ \begin{array}{ll}
\frac{|dX|^2}{1 + |X|^2} & \text{Fubini-Study} \\
\frac{|X^* dX|^2}{(1 + |X|^2)^2} & \text{Hopf-fiber}
\end{array} \right.$$

$$ds^2_{\text{Hopf-fiber}} = \left\| dU U^{-1} + \frac{X^* dX - dX^* X}{2(1 + |X|^2)} \right\|^2$$

(2.15)

where $X$ and $U$ variables are complex or quaternion numbers and the norm $\|w\|^2$ is replaced by the usual norm $|w|^2$; for a complex case $2N \times 2$ rectangular matrix $X$ is recognized as a complex $N$ vector, and for a quaternion case $4N \times 4$ rectangular matrix is recognized as a quaternion $N$ vector. The parameter $\lambda$ in (2.15) is ‘squashing’ parameter which is in general a function of coordinates [13, 14]. For such cases the similarity transformation (2.13) gives extra term in (2.14), which however is not added for preserving the $G$ symmetry. This transformation is different from the Weyl transformation so the Weyl tensor becomes non-vanishing in many cases which causes supersymmetry breaking. The metric (2.15) has $G$ invariance independently on the value of $\lambda$ since each term is invariant under $G$ transformations (2.6). In addition to $G$ symmetry $\|J_u\|^2$ term has another symmetry which is a shift of the Hopf fiber coordinate, independently on the value of $\lambda$.

This deformation is recognized as the symmetry breaking of the orthonormal metric. A round sphere metric (2.3) can be written in terms of the orthonormal frame as

$$ds^2_{\text{round}} = \frac{1}{m} \sum_{C=0}^{m-1} \sum_{\Delta, \beta = 0}^{m+n} \delta \Delta_{\Delta'} \left( J_{\Delta'}^C \right)^{J_{\Delta}}_{C}$$

$$\delta \Delta_{\Delta'} = \left( \begin{array}{cc} 1_m & 0 \\ 0 & I_{n+1} \end{array} \right)$$

(2.16)
where $O(m + n + 1)$ symmetry is manifest. However the squashed
sphere metric (2.15) gives the following expression:

$$\text{d}s^2_{\text{squashed}} = \lambda^2 \| J_u \|^2 + \| J_X \|^2$$

$$= \frac{1}{m} \sum_{c=0}^{m-1} \left( \lambda^2 \sum_{A,B=0}^{m-1} \delta^{AB} (J_A^C)^t J_B^C + \sum_{A,B=m}^{m+n} \delta^{AB} (J_A^C)^t J_B^C \right)$$

$$\delta^{AB} = \begin{pmatrix} \lambda^2 I_m & 0 \\ 0 & I_{n+1} \end{pmatrix},$$

(2.17)

where the symmetry of the orthonormal metric is broken to $O(m) \times O(n + 1)$. Despite of this smaller vacuum symmetry the projective coordinates can realize a larger symmetry $G$ by the fractional linear transformation. Only for $\lambda^2 = 1$ the $O(m + n + 1)$ invariance of the orthonormal metric is recovered in addition to $G$ symmetry. In the $\lambda \to 0$ limit a sphere becomes a projective space with the Fubini–Study metric (2.10) where the space dimension is reduced.

3. Real coset

At first we consider a real coset $O(N + 1)/O(N)$ as $S^0(\pm 1)$ points fibration over $\mathbb{R}P^N$. An element of $G = O(N + 1) \ni z$ is decomposed into four blocks as (2.4). In the parametrization of (2.5), $u$ and $X$ are $1 \times 1$ and $N \times 1$ matrices, respectively. For example $N = 7$ case, $S^7$ is written as $S^0(\pm 1)$ fibration over $\mathbb{R}P^7$. A round 7-sphere is easily seen in the parametrization of $z \in O(8)$ as

$$z = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

(3.1)

with $x_0 = u$ and $x_A = X_A^0 u$ for $A = 1, \ldots, 7$, while the Hopf fibration is easily seen in the $\mathbb{R}P^7$ coordinates $X_A^0$ with $A = 1, \ldots, 7$ and the fiber coordinate $u$.

The fiber coordinate ‘$u$’ can be chosen as

$$u = \frac{1}{\sqrt{1 + X^2}} U, \quad X^2 = \sum_{A=1}^{n} X_A^0 X_A^0, \quad U = \pm 1.$$  

(3.2)

The LI one form responsible for the fiber is zero, $J_u = 0$. The metric (2.15) becomes the one for $\mathbb{R}P^N$ calculated as

$$\text{d}s^2_{\mathbb{R}P^N} = \frac{\text{d}X^2}{1 + X^2} = \frac{(X \cdot \text{d}X)^2}{(1 + X^2)^2}$$

$$= \frac{\text{d}r^2}{(1 + r^2)^2} + \frac{r^2 \text{d}\Omega^2_{(N-1)}}{1 + r^2}$$

$$= \text{d}\theta^2 + \sin^2 \theta \text{d}\Omega^2_{(N-1)}$$

(3.3)

which is the metric for a ‘locally’ round $S^N$. In the second line from the bottom $X^2 = r^2$ and $\text{d}X^2 = \text{d}r^2 + r^2 \text{d}\Omega^2_{(N-1)}$ are used and $r = \tan \theta$ is used for the last line. There is no room to introduce $\lambda$ for the $S^0$ fiber.
4. Complex coset

Next we consider a complex coset $U(N+1)/U(N)$ which corresponds to $S^1$ fibration over $\mathbb{CP}^N$. When an element of $U(N + 1)$ matrix is embedded in $(2N+2) \times (2N+2)$ orthogonal matrix, $z$, it preserves a Kähler form metric $z\bar{z}K = K$ in (2.1) as well as $z\bar{z} = 1$. Since $e^{i\epsilon} = -1$ and $e^{i\epsilon} = -e$, $\epsilon$ is an imaginary base ‘i’ and ‘transpose’ is replaced with complex conjugate ‘*’. Then $z$ is recognized as $(N + 1)$ matrix of complex numbers by recognizing a $2 \times 2$ matrix $w = aI_2 + b\epsilon$ as a complex number $w = a + ib$ with real numbers $a$ and $b$.

Let us decompose a coordinate $z \in U(N + 1)$ into four blocks as (2.4). $u$ and $X$ in the parametrization (2.5) are $2 \times 2$ and $2N \times 2$ matrices, respectively. $S^7$ is given by $N = 3$ which is $S^1$ fibration over $\mathbb{CP}^3$. A round 7-sphere and a Hopf fibration are easily seen in the following parametrizations of $z \in U(4)$ respectively as

$$z = \begin{pmatrix} x_0 & -x_1 \\
-1 & x_0 \\
x_2 & -x_3 \\
x_3 & x_2 \\
x_4 & -x_5 \\
x_5 & x_4 \\
x_6 & -x_7 \\
x_7 & x_6 \end{pmatrix} = \begin{pmatrix} u \\ X_1u \\ X_2u \\ X_3u \end{pmatrix}. \quad (4.1)$$

The fiber coordinate in ‘$u$’ is parametrized as

$$u = \frac{1}{\sqrt{1 + \sum_{I=1}^{N} |X_I|^2}} e^{i\phi}. \quad (4.2)$$

From the metric formula (2.15) the metric for a $S^{2N+1}$ as $S^1$ fibration over $\mathbb{CP}^N$ including a squashed parameter $\lambda$ is given as

$$\begin{aligned}
ds^2_{S^{2N+1}} &= ds^2_{\mathbb{CP}^N} + ds^2_{S^1} \\
ds^2_{\mathbb{CP}^N} &= \frac{\sum_{I=1}^{N} |dX_I|^2}{1 + \sum_{I=1}^{N} |X_I|^2} - \frac{\sum_{I=1}^{N} dX_I^* X_I}{(1 + \sum_{I=1}^{N} |X_I|^2)^2}, \\
ds^2_{S^1} &= \lambda^2(d\phi + A)^2, \\
A &= -\frac{\sum_{I=1}^{N} \bar{X}_I^* dX_I - d\bar{X}_I^* X_I}{2(1 + \sum_{I=1}^{N} |X_I|^2)}. \quad (4.3)
\end{aligned}$$

d$s^2_{S^{2N+1}}$ is invariant under the $G = U(N + 1)$ symmetry with the transformation rule (2.6) independently from the value of $\lambda$, where the $\mathbb{CP}^N$ coordinates are transformed as a linear fractional transformation. In addition to $U(N + 1)$ there exists another $U(1)$ symmetry; $\phi \to \phi + c$ with a constant number $c$. Combining this $U(1)$ and $U(1)$ in $U(N + 1)$, the whole symmetry of the $(2N + 1)$-dimensional sphere metric $ds^2_{S^{2N+1}}$ in (4.3) is $SU(N + 1) \times U(1)$.

The Einstein condition allows only $\lambda^2 = 1$ which corresponds to a round $S^{2N+1}$ metric with $O(2N + 2)$ symmetry. Reducing dimensions allows another Einstein solution with $\lambda = 0$ corresponding to the Fubini–Study metric for $\mathbb{CP}^N$. At $\lambda^2 = 1$ the $O(2N + 2)$ symmetry is recovered from $U(N + 1)$ which is reflection of recovery of the vacuum symmetry as shown in (2.17). It is also easily seen by introducing the coordinates $\tilde{X}_I = X_I e^{i\phi}$, then the metric is...
rewritten as
\[ ds^2_{\mathbb{S}^{2N+1}} = \frac{d\phi^2 + \sum_{I=1}^{N} |d\hat{X}_I|^2}{1 + \sum_{I=1}^{N} |\hat{X}_I|^2} - \frac{\left( \frac{1}{2} \sum_{I=1}^{N} |\hat{X}_I|^2 \right)^2}{\left( 1 + \sum_{I=1}^{N} |\hat{X}_I|^2 \right)^2} + (\lambda^2 - 1)^2 \frac{(d\phi + \tilde{a})^2}{\left( 1 + \sum_{I=1}^{N} |\hat{X}_I|^2 \right)^2} \] (4.4)

\[ \tilde{a} = -\frac{1}{2} \sum_{l=1}^{N} (\hat{X}_l \cdot d\hat{X}_l - d\hat{X}_l^* \hat{X}_l). \]

At \( \lambda^2 = 1 \) with changing variables, \( \sum_{l} |\hat{X}_l|^2 = r^2, \sum_{l} |d\hat{X}_l|^2 = dr^2 + r^2 d\Omega_{(2N-1)} \) and \( r = \tan^2 \theta \), the metric becomes
\[ ds^2_{\mathbb{S}^{2N+1}} = d\theta^2 + \sin^2 \theta d\Omega^2_{(2N-1)} + \cos^2 \theta d\phi^2 \] (4.5)
describing a round \( \mathbb{S}^{2N+1} \). The first two terms in (4.4) contain real symmetric combinations, \( |V|^2 \) for a complex vector \( V \), whose symmetry is not only \( U(N+1) \) but enlarged to \( O(2N+2) \). On the other hand, the third term in (4.4) contains ‘imaginary’ skew combinations \( V^*W - W^*V \) for another complex vector \( W \) whose invariance is \( Sp(2N+2;\mathbb{R}) \). Requiring both invariances at \( \lambda^2 \neq 1 \) reduces to \( U(N+1) \) invariance.

Now let us consider the \( \mathbb{Z}_k \) quotient of a sphere which is important deformation to change the topology \([1, 14]\). Lens spaces, \( \mathbb{S}^3/\mathbb{Z}_k \), were considered for the geometrical interpretation of D-branes \([21]\). We present a general description of lens spaces in our coset formalism. Let \( \omega \) be a primitive \( k \)-th root of unity, \( \omega^k = 1 \) and \( q_0, q_1, q_2, \ldots \) be coprime to \( k \). A lens space \( L(k; q_0, q_1, q_2, \ldots) \) is a quotient of the \( \omega \) action for complex coordinates \( \mathbb{Z}_k \) as \( (Z_0, Z_1, Z_2, \ldots) \rightarrow (\omega^{q_0} Z_0, \omega^{q_1} Z_1, \omega^{q_2} Z_2, \ldots). \) This is realized as the ‘left’ action in our formalism as
\[ z = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ \vdots \end{pmatrix} \rightarrow z_{\omega} = \begin{pmatrix} \omega^{q_0} \\ \omega^{q_1} \\ \omega^{q_2} \\ \vdots \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ \vdots \end{pmatrix}. \] (4.6)

Therefore it does not change the LI 1-form, \( z_{\omega}^{-1} dz_{\omega} = z^{-1} dz \), and the ‘local’ metric. But it changes periods of the fiber coordinates, so it changes a topology in general. A lens space \( L(2;1, \ldots, 1) \) is \( \mathbb{RP}^{2N+1} \), and lens spaces \( L(k; q_0, q_1, q_2, \ldots) \) are \( \mathbb{S}^{2N+1}/\mathbb{Z}_k \). We consider a lens space \( L(k; 1, \ldots, 1) \) as \( \mathbb{S}^{2N+1}/\mathbb{Z}_k \) \([1]\). Contrast to the squashing action (2.13), the \( \mathbb{Z}_k \) quotient is obtained by the left action of \( \omega = e^{2\pi i/k} \) operation:
\[ z = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ \vdots \end{pmatrix} \rightarrow z_{\omega} = \begin{pmatrix} \omega \\ \omega \\ \omega \\ \vdots \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \omega & 0 \\ \hat{X}_1 & 1 & 0 & \omega \\ \hat{X}_2 & 0 & 1 & \omega \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ \omega u \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} Y \\ 1 \\ 1 \\ 1 \end{pmatrix} \] (4.7)

In the second line, it is shown that this action reduces into the action only on \( u \). The \( \mathbb{Z}_k \) quotient \( z \sim z_{\omega} \) reduces to \( u \sim \omega u \) which changes the boundary condition of \( \phi \) introduced in (4.2),
\[ u \sim \omega u \Leftrightarrow \phi \sim \phi + \frac{2\pi}{k}. \] (4.8)
In general, boundary conditions in lense space produce more general topological deformations. We can examine the effect of the $Z_k$ quotient at $\lambda^2 = 1$ for example, since the $Z_k$ quotient is independent procedure from squashing. For a round sphere metric (4.5) there is $d\phi^2$ term where $\phi$ has period $2\pi/k$. If we rewrite it in terms of a $2\pi$ period coordinate $\varphi$, then it becomes $d\phi^2 \rightarrow d\varphi^2/k^2$ so the orthonormal metric is distorted as $\delta_{\varphi}^\varphi = 1 \rightarrow \delta_{\varphi}^\varphi = 1/k^2$.

In the last, we consider a quaternion coset $5$. Quaternion coset condition in 2

Let us decompose a coset element $z \in Sp(N+1)$ into four blocks as (2.4). $u$ and $X$ in the parametrization (2.5) are $4 \times 4$ and $4N \times 4$ matrices, respectively. $S^7$ is given by $N = 1$ case, which is $S^3$ fibration over $\mathbb{HP}$. An element $z \in Sp(2)$ is parametrized as follows:

$$
\begin{pmatrix}
0 & -x_1 & -x_2 & -x_3 \\
x_1 & 0 & -x_3 & x_2 \\
x_2 & x_3 & 0 & -x_1 \\
x_3 & -x_2 & x_1 & 0 \\
x_4 & -x_5 & x_6 & x_7 \\
x_5 & x_4 & x_7 & -x_6 \\
x_6 & x_7 & -x_6 & x_5 \\
x_7 & -x_6 & x_5 & x_4
\end{pmatrix}
= 
\begin{pmatrix}
u \\
Xu
\end{pmatrix}
$$

(5.2)
The first parametrization is convenient for a round 7-sphere and the second parametrization is convenient for the Hopf fibration.

From the orthonormal condition $u$ can be parametrized as (2.11)

$$u = \frac{1}{\sqrt{1 + \sum_{i=1}^{N} |X_i|^2}} U, \quad U \in Sp(1).$$

From the metric formula (2.15) the metric for a $S^{4N+3}$ which is a $S^7$ fibration over $\text{HP}^N$ is obtained as

$$ds^2_{S^{4N+3}} = ds^2_{\text{HP}^N} + ds^2_{S^7},$$

$$ds^2_{\text{HP}^N} = \frac{\sum_{j=1}^{N} |dX_j|^2}{1 + \sum_{j=1}^{N} |X_j|^2} - \frac{N}{(1 + \sum_{j=1}^{N} |X_j|^2)^2} \sum_{j=1}^{N} |dX_j|^2,$$

$$ds^2_{S^7} = \lambda^2 \sum_{i=1,2,3} (\nu_i + A_i)^2, \quad \nu_i = \text{Re} (dU U^{-1} \xi_i^{-1})$$

where $\text{Re}(w)$ denotes a real part of $w$. This metric has $Sp(N+1) \times Sp(1)$ symmetry independently from the value of $\lambda$; the $\text{HP}^N$ coordinates, $X_i$, are transformed as a linear fractional transformation given by (2.6) realizing $Sp(N+1)$ symmetry. The $ds^2_S$ has additional $Sp(1)$ symmetry under which $X_i$'s are inert.

As in the general argument (2.16) it becomes a round $(4N+3)$-dimensional sphere at $\lambda^2 = 1$ and the global symmetry is enhanced to $O(4N+4)$. At $\lambda^2 = 1$ the metric (5.4) becomes the one for a round $S^{4N+3}$ which is easily seen in the following coordinates $\tilde{X}_i = X_i U$ as

$$ds^2_{S^{4N+3}} = \sum_{i=1,2,3} \nu_i^2 + \sum_{j=1}^{N} |d\tilde{X}_j|^2 - \frac{N}{(1 + \sum_{j=1}^{N} |\tilde{X}_j|^2)^2} \sum_{j=1}^{N} |d\tilde{X}_j|^2,$$

$$= \frac{d\theta^2 + \sin^2 \theta d\Omega^2_{(4N-1)} + \cos^2 \theta d\Omega^2_{(4)} \sum_{j=1}^{N} |\tilde{X}_j|^2}{1 + \sum_{j=1}^{N} |\tilde{X}_j|^2},$$

where $\sum_{j=1}^{N} |\tilde{X}_j|^2 = (\tan \theta)^2$ and $\sum_{j=1}^{N} |d\tilde{X}_j|^2 = \theta d\theta^2 + \sin \theta d\Omega^2_{(4N-1)}$ are used.

Next let us focus on $N = 1$ case to examine a non-trivial squashed $S^7$ solution. In order to correspond a round $S^7$ coordinates and a squashed $S^7$ coordinates, $Sp(1) \ni U$ and $\text{HP}^1$ coordinate $X$ are expressed in terms of $x_4, A = 1, \ldots, 7$ with $|X|^2 = x_4^2$ and $X = x_4 V$ as

$$U = \frac{1 + \sum_{i=1,2,3} x_i \xi_i}{\sqrt{1 + \sum_{i=1,2,3} x_i^2}}, \quad V = \frac{1 + \sum_{i=1,2,3} x_4 x_i \xi_i}{\sqrt{1 + \sum_{i=1,2,3} x_4^2 x_i^2}},$$

This is an embedding of an instanton solution on $S^4$ into $SU(2)$ fiber. The metric for $S^7$ as a $S^3$ fibration over $\text{HP}^1$ is given by

$$ds^2_{S^7} = \frac{1}{4} d\theta^2 + \frac{\sin \theta^2}{4} \tilde{v}_i^2 + \lambda^2 \sum_{i=1,2,3} (\nu_i + A_i)^2$$

$$A_i = \frac{1 - \cos \theta}{2} \tilde{v}_i,$$

$$\nu_i = \text{Re} (dU U^{-1} \xi_i^{-1}) = \frac{1}{1 + \sum_{i=1,2,3} x_i^2} (dx_i + \sum_{i=1,2,3} x_i^2 dx_i),$$

$$\tilde{v}_i = \text{Re} (V^{-1} dV \xi_i^{-1}) = \frac{1}{1 + \sum_{i=1,2,3} x_4 x_i^2} (dx_4 + \sum_{i=1,2,3} x_4 x_i dx_i),$$

$$\text{E} = \frac{1}{2} \sum_{i=1,2,3} |\tilde{v}_i|^2 + \frac{\sin \theta^2}{4} \tilde{v}_i^2.$$
\[ |X|^2 = (\tan \frac{\theta}{2})^2 \]

which is different from \( \theta \) in (5.5). Two \( Sp(1) \) currents, \( \nu_i \) and \( \tilde{\nu}_i \), satisfy

\[ \sum \nu_i^2 = \sum \tilde{\nu}_i^2 = d \Omega(3)^2, \quad d\nu_i = \epsilon_{ijk} v_j \wedge v_k \quad \text{and} \quad d\tilde{\nu}_i = -\epsilon_{ijk} \tilde{v}_j \wedge \tilde{v}_k. \]

The metric in vielbein form is given as

\[ ds_S^2 = e^{a_5} \delta_{ab} e^{b}, \quad a = (1, 2, 3, \theta, 5, 6, 7) \]

\[ e^{a_5} = \lambda \left( v_i + \frac{1 - \cos \theta}{2} \tilde{v}_i \right) \]

\[ e^{\theta} = \frac{1}{2} d\theta \]

\[ e^{a_4j} = \frac{\sin \theta}{2} \tilde{v}_j. \]

The Ricci tensor with local Lorentz indices is

\[ R_{ij}^L = \left( \frac{2}{\lambda^2} + 4\lambda^2 \right) \delta_{ij}^L, \quad R_{4+i+4j}^L = 6(2 - \lambda^2) \delta_{ij}^L, \quad R_{ij}^\theta = 6(2 - \lambda^2). \]  

The Einstein metric condition, \( R_{ab} = c \delta_{ab} \Leftrightarrow R_{mn} = c g_{mn} \), equates these coefficients to be equal. There are two solutions

\[ 6(2 - \lambda^2) = \frac{2}{\lambda^2} + 4\lambda^2 \Rightarrow \lambda^2 = \frac{1}{5}. \]  

Detailed of the computation is in the appendix. The \( \lambda^2 = 1 \) solution corresponds to an \( O(8) \) invariant round 7-sphere solution, while the \( \lambda^2 = 1/5 \) solution corresponds to a squashed 7-sphere which has only \( Sp(2) \times Sp(1) = SO(5) \times SO(3) \) invariance [3, 4]. Our coordinate system in (5.6) is slightly different from that of the Awada et al., but it gives the same result which is the same ratio of sizes of the fiber \( S^3 \) and the base \( HP^1 = S^4 \). For \( \lambda^2 = 0 \) it becomes the Fubini–Study metric of \( HP^1 \).

6. Conclusion and discussions

We have presented a simple derivation of metrics for Hopf fibrations by a coset formulation. The coordinate is a group matrix and it is decomposed into four blocks. Fiber coordinates and projective coordinates are treated differently by embedding into different blocks. The former and the latter are embedded in an upper-left diagonal block and a lower-left off-diagonal block, respectively. The remaining diagonal block corresponds to the stability group. Projective coordinates realize the isometric symmetry manifestly by a fractional linear transformation. Squashing is introduced as a similarity transformation which preserves the isometric symmetry of the projective space. It changes the metric and curvature tensors. Squashed \( S^7 \) is also obtained in this formulation which is consistent with the one [3]. The \( \mathbb{Z}_k \) quotient is introduced as a lens space which also preserves the isometric symmetry of the projective space. It does not change the ‘local’ metric but changes the topology. It may be interesting to examine general lens space solutions \( S^{2N+1}/\mathbb{Z}_k = L(k; q_0, q_1, \ldots, q_N) \) where \( q_0, q_1, \ldots, q_N \) are coprime to \( k \).

Key to our simple description is the four blocks partition of a coordinate matrix (2.4) and (2.5) where the Hopf fiber coordinate is embedded in the upper-left diagonal block. This is not the case for the supersymmetrization of AdS\(_4 \times S^7 \) which is described by the supergroup \( OSP(8|4) \). The Hopf fibration breaks \( OSP(8|4) \) into \( OSP(6|4) \) where \( OSP(6|4) \) is embedded in a diagonal block, then the Hop fiber \( U(1) \) is not embedded in a diagonal block of this \( SO(8) \) spinor representation. In the supergroup matrix there is another \( U(1) \) under which \( OSP(6|4) \) is singlet, so this is embedded in the diagonal block. There are several important \( U(1) \)'s in
OSp(8|4) clarified by Gomis et al [7]. The Hopf fiber \(U(1)\) and \(U(1)\) in SU(4) which is SU(3) invariant are given by

\[
T_{\text{Hopf-fiber}} = M_{12}
\]

\[
T_{SU(3)\text{-inv}} = -3M_{12} + M_{34} + M_{56} + M_{78} \Rightarrow T_1
\]

where \(M_{ab} a = 1, \ldots, 8\) are SO(8) generators in the vector representation. On the other hand, the spinor representation is obtained by multiplying gamma matrices \(\Gamma_a\). Spinor states are classified by the chirality operator \(\Gamma_1 \cdots \Gamma_8\) and \(U(1)\) charge, \(T_2\),

\[
T_2 = M_{12} + M_{34} + M_{56} + M_{78}
\]

\[
\sim \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3
\]

(6.2)

\[
T_1 = T_{SU(3)\text{-inv}} = -3M_{12} + M_{34} + M_{56} + M_{78}
\]

with \(M_{12} = M_{12}\Gamma_{12}\). The \(U(1)\) generated by \(T_2\) is embedded in a diagonal block. \(T_2\) corresponds to \(K_{ab}\Gamma_{ab}\) in (2.1). The positive chirality states are \(|\uparrow\uparrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\downarrow\rangle, |\uparrow\uparrow\downarrow\downarrow\rangle, \ldots\). \(T\) charge \((-2, 0, 0, 0, 0, 0)\) respectively, which is \(1_2 + 1_{-2} + 6_0\). The negative chirality states are \(|\uparrow\uparrow\downarrow\downarrow\rangle, |\downarrow\downarrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle, \ldots\) with \(T\) charge \((1, 1, 1, -1, -1, -1, -1)\), respectively, which is \(4_1 + 4_{-1}\). Survived \(T_2\) invariant supergroup is OSp(6|4). The remaining spinor states with \(T_2 = \pm 2\) in the positive chirality sector must make a closed supergroup OSp(2|4) where this \(O(2|1)\) generated by \(T_2\). Therefore the Hopf-fiber \(U(1)\) and another \(U(1)\) are given by

\[
T_{\text{Hopf-fiber}} = \frac{1}{2}(T_2 - T_1) = M_{12}
\]

\[
T' = \frac{1}{2}(T_1 + T_2) = -M_{12} + M_{34} + M_{56} + M_{78}.
\]

(6.3)

In the spinor representation, \(T_{\text{Hopf-fiber}} cannot be embedded in a block among four, because \(|M_{12}| = 1_6\). The bosonic part of the coset is SU(4) \(\times U(1)/SU(3) \times U(1)'\), where \(U(1)\) in the numerator is generated by \(T_2\) and \(U(1)'\) in the denominator is generated by \(T'\). It will be useful to have a simple treatment of the supersymmetric Hopf fibration and deformations. We put this problem forward as a future problem including integrability analysis of the AdS_4/CFT_3 correspondence.

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Appendix. Ricci tensor of squashed S^7

The metric (5.6) is written as

\[
dS^7_{12}^2 = \frac{1}{4}(d\theta^2 + \sin \theta^2 \tilde{\nu}_i \tilde{\nu}_i + \lambda^2 \sum_{j=1,2,3} (\nu_j + \frac{1 - \cos \theta}{2} \tilde{\nu}_j)^2).
\]

(A.1)

In the vielbein is determined as \(g_{mn} = \epsilon_{mnab} X^b\)

\[
\epsilon_{mnab} = \begin{pmatrix}
\lambda \hat{\gamma}_I^L & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
\frac{1}{\lambda} \hat{\gamma}_I^L & 0 & \frac{1}{\lambda} \hat{\gamma}_I^L
\end{pmatrix},
\]

\(m = (\{i, \theta, \hat{l}\} = (1, 2, 3, \theta, 5, 6, 7)\)

\(a = (\{i, \theta, \hat{l}\} = (1, 2, 3, \theta, 5, 6, 7)\)

(A.2)
with \( \hat{\lambda} = \frac{1 - \cos \theta}{2} \) and \( \hat{s} = \frac{\sin \theta}{2} \) and

\[
\Upsilon_{ij} = \frac{\delta_{ij} + \epsilon_{ijk} x^k}{1 + \sum_{l=1,2,3} x_{lj}^2}, \quad \hat{\Upsilon}_{ij} = \frac{\delta_{ij} - \epsilon_{ijk} x^{4+k}}{1 + \sum_{l=1,2,3} x_{4+l}^2}
\]

\( \Upsilon_{-1}^{ij} = \delta_{ij} - \epsilon_{ijk} x^k + x_j x_j, \quad \hat{\Upsilon}_{-1}^{ij} = \delta_{ij} + \epsilon_{ijk} x^{4+k} + x_{4+i} x_{4+j}. \)  

(A.3)

The inverse vielbein is

\[
e_{a}^{m} = \left( \begin{array}{ccc}
1 & \frac{1}{\hat{\lambda}} \Upsilon_{-1}^{i,j} & 0 \\
0 & 2 & 0 \\
-\frac{\hat{s}}{\hat{\lambda}} & \frac{1}{\hat{\lambda}} \Upsilon_{-1}^{j,i} & 0
\end{array} \right). \]

(A.4)

Derivative operators in the local Lorentz frame indices are closed as

\[
e_{a}^{m} = e_{a}^{m} \partial_{m}, \quad [e_{a}^{m}, e_{b}^{n}] = c_{ab}^{n} e_{c}^{n}, \quad c_{abc}^{n} = -e_{a}^{m} e_{b}^{k} \delta_{mn} e_{c}^{x}. \]

(A.5)

Covariant derivative operators

\[
\nabla_{a} = e_{a}^{m} \partial_{m} + \frac{1}{2} \omega_{abc}^{mn} M_{abc}^{m} \]

(A.6)

satisfy

\[
[V_{a}, V_{b}] = T_{abc} e_{c}^{a} + \frac{1}{2} R_{abc}^{ad} M_{abc}^{d}
\]

with Lorentz generator \( M_{abc} \). If the torsion is zero, \( T_{abc}^{m} = 0 \), curvature is written in terms of the structure constant

\[
\omega_{abc}^{mn} = \frac{1}{2} (c_{abc} - c_{bac} + c_{bca}), \quad R_{abc}^{ad} = e_{a}^{m} \omega_{b}^{n} e_{c}^{d} - c_{ab}^{n} \omega_{c}^{d} e_{b}^{m} + \omega_{a}^{n} \omega_{b}^{d} e_{c}^{m}. \]

(A.8)

The covariant derivative operators for the \( Sp(2) \times Sp(1) \) space \( (A.1) \), which is torsionless, are computed as

\[
\nabla_{a} = \frac{1}{\hat{\lambda}} \Upsilon_{-1}^{i,j} \partial_{j} + \frac{1}{2} \left( \frac{1}{\hat{\lambda}} \epsilon_{ijk} M_{jk} + \lambda \epsilon_{ijk} M_{jk} - \hat{\lambda} \delta_{jk} M_{ij} \right)
\]

\[
\nabla_{a} = 2 \partial_{a}
\]

\[
\nabla_{a} = \frac{1 - \cos \theta}{\sin \theta} \frac{\epsilon_{ijk}}{\lambda} \Upsilon_{-1}^{i,k} \partial_{k} + 2 \frac{\sin \theta}{\sin \theta} \hat{\Upsilon}_{-1}^{i,k} \partial_{k} + \frac{1}{2} \left( \frac{2(1 - \cos \theta)}{\sin \theta} \epsilon_{ijk} M_{jk} - \lambda \epsilon_{ijk} M_{jk} - \frac{2 \cos \theta}{\sin \theta} \lambda \delta_{jk} M_{ij} \right)
\]

(A.9)

The Ricci tensor with local Lorentz indices, \( R_{a}^{b} = R_{a}^{b} \)

\[
R_{a}^{b} = \left( \frac{2 + 4 \lambda^2}{\lambda^2} \right) \lambda \delta_{a}^{b}, \quad R_{a}^{4+b} = 6(2 - \lambda^2) \delta_{a}^{b}, \quad R_{a}^{d} = 6(2 - \lambda^2)
\]

(A.10)

others = 0.

The Ricci tensor with curved indices is \( R_{mn} = e_{m}^{a} e_{n}^{b} R_{ab} \).

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