Abstract

We give various applications of essential circles (introduced in [16]) in a compact geodesic space $X$. Essential circles completely determine the homotopy critical spectrum of $X$, which we show is precisely $\frac{2}{3}$ the covering spectrum of Sormani-Wei. We use finite collections of essential circles to define “circle covers”, which extend and contain as special cases the $\delta$-covers of Sormani and Wei (equivalently the $\varepsilon$-covers of [16]); the constructions are metric adaptations of those utilized by Berestovskii-Plaut in the construction of entourage covers of uniform spaces. We show that, unlike $\delta$- and $\varepsilon$-covers, circle covers are in a sense closed with respect to Gromov-Hausdorff convergence, and we prove a finiteness theorem concerning their deck groups that does not hold for covering maps in general. This allows us to completely understand the structure of Gromov-Hausdorff limits of $\delta$-covers. Also, we use essential circles to strengthen a theorem of E. Cartan by finding a new (even for compact Riemannian manifolds) finite set of generators of the fundamental group of a semilocally simply connected compact geodesic space. We conjecture that there is always a generating set of this sort having minimal cardinality among all generating sets.

Keywords: Gromov-Hausdorff convergence, essential circles, covering maps, fundamental group, geodesic space, length spectrum
1 Introduction

Christina Sormani and Guofang Wei in [17], [18], [19], [20] and Valera Berestovskii and Conrad Plaut in [1], [2] studied covering space constructions that encode geometric information and stratify the topology of the underlying space. Sormani and Wei utilized a classical construction of Spanier ([21]) that provides a covering map \( \pi_\delta : \tilde{X}_\delta \to X \) corresponding to the open cover of a geodesic space \( X \) by open \( \delta \)-balls, which they called the \( \delta \)-cover of \( X \). Berestovskii-Plaut, developed a new construction for covers of uniform spaces – hence any metric space – that utilizes discrete chains and homotopies rather than traditional (i.e. continuous) paths and homotopies. Building on Berestovskii-Plaut, the special case of metric spaces was developed further by Plaut and Wilkins in [16] (see also [6] and [23]). For any connected metric space \( X \) and \( \varepsilon > 0 \), the Berestovskii-Plaut construction yields a covering map \( \phi_\varepsilon : X_\varepsilon \to X \), with deck group \( \pi_\varepsilon (X) \) that is a kind of coarse fundamental group at a given scale. In the geodesic case, which is the case of interest in this paper, we show that when \( \delta = \frac{3\varepsilon}{10} \), the covering maps \( \pi_\varepsilon^{3\varepsilon/2} \) and \( \phi_\varepsilon \) are naturally isometrically equivalent (Corollary 17). Both covering spaces have been used to obtain notable geometric and topological results (cf. [6], [16], [17], [18], [19], [20], [23]).

As Sormani-Wei observed in [17], one characteristic of the \( \delta \)-covers of a geodesic space – hence, the \( \varepsilon \)-covers, also – is that they are not “closed” with respect to Gromov-Hausdorff convergence. That is, if a sequence of compact geodesic spaces \( X_i \) converges to a compact space \( X \) in the Gromov-Hausdorff sense, the covers \( (X_i)_\varepsilon \) may not converge to \( X_\varepsilon \) in the pointed Gromov-Hausdorff sense. Even if the covers \( (X_i)_\varepsilon \) converge, their limit may not be any \( \varepsilon \)-cover of \( X \). Sormani and Wei did show that there is always a subsequence \( (X_i_k)_\varepsilon \) that pointed converges to a space \( Y \), that \( Y \) is covered by \( X_\varepsilon \), and that \( Y \) covers both \( X_\varepsilon \) and \( X_\varepsilon' \) for all \( \varepsilon' > \varepsilon \) (Theorem 3.6, [17] and Proposition 7.3, [18]). It follows that the \( \varepsilon \)-covers of a convergent sequence of compact geodesic spaces are precompact. In this paper we introduce the notion of “circle cover”, which extends the notion of \( \varepsilon \)-cover and is closed with respect to Gromov-Hausdorff convergence. As a consequence we can fully understand the limit space \( Y \) described by Sormani-Wei.

Discrete homotopy methods are quite amenable to questions that involve Gromov-Hausdorff convergence. For example, Gromov-Hausdorff convergence of compact metric spaces can be characterized by the existence of “almost isometries” that generally are not continuous, and therefore classical methods using continuous paths are not always easy to apply. However, for any function \( f : X \to Y \) between metric spaces, there is a naturally induced function \( f_\# : X_\varepsilon \to Y_\delta \) provided that \( f \) only satisfies a kind of coarse continuity: for any \( x, y \in X \), if \( d(x,y) < \varepsilon \) then \( d(f(x),f(y)) < \delta \) (see also Theorem 2 in [2]). In fact, we show that when \( f \) is a generalized almost isometry called a \( \sigma \)-isometry, \( f_\# \) is actually a quasi-isometry (see Remark 28 and Theorem 32) with distortion constants explicitly controlled by \( \sigma \). These tools play a significant role in the proof of our main theorem (Theorem 1), as these induced quasi-isometries allow
us to gain explicit control over the Gromov-Hausdorff distance between balls in the possibly non-compact spaces $X_\varepsilon$ and $Y_\delta$.

For simplicity, in what follows we will use the notation of [16] concerning $\varepsilon$-covers, although the results that apply to geodesic spaces can be equivalently stated for the $\delta$-covers of Sormani-Wei, due to Corollaries 17 and 18. In [16] we defined the notion of an essential $\varepsilon$-circle in a geodesic space (see Sections 2 and 3 for more details). Among their other applications, essential circles completely determine the Sormani-Wei Covering Spectrum (equivalently the Homotopy Critical Spectrum of [16]). Now suppose that $\varepsilon > 0$ and $T$ is any finite collection of essential $\delta$-circles such that $\delta \geq \varepsilon$. We define a natural normal subgroup $K_{\varepsilon}(T)$ of $\pi_{\varepsilon}(X)$ (which is the trivial group when $T = \emptyset$) that acts freely and properly discontinuously on $X_\varepsilon$ with quotient $X_{\varepsilon}^T$. Then there is a natural induced mapping $\phi_{T,\varepsilon}^*: X_{\varepsilon}^T \to X$ which is also a covering map with deck group naturally isomorphic to $\pi_T(X) = \pi_{\varepsilon}(X)/K_{\varepsilon}(T)$. We call $\phi_{T,\varepsilon}^*$ the $(T,\varepsilon)$-cover of $X$, and in general we will refer to these covers as circle covers (Definition 19). Note that when $T = \emptyset$, $\phi_{T,\varepsilon}^* = \phi_{\varepsilon}$, so circle covers extend the notion of $\varepsilon$-covers. In this paper, the arrow “$\to$” indicates Gromov-Hausdorff convergence (respectively, pointed Gromov-Hausdorff convergence) of the compact spaces $X_i$ (respectively, for the possibly non-compact covering spaces).

**Theorem 1** Suppose that $X_i \to X$, where each $X_i$ is compact geodesic, and for each $i$ there are an $\varepsilon_i > 0$ and a finite collection $T_i$ of essential $\tau$-circles in $X_i$ such that $\tau \geq \varepsilon_i$. If $\{\varepsilon_i\}$ has a positive lower bound then for any positive $\varepsilon \leq \liminf \varepsilon_i$ there exist a subsequence $\{X_{i_k}\}$ and a finite collection $T$ of essential $\tau$-circles in $X$ with $\tau \geq \varepsilon$, such that $(X_{i_k})_{T_{i_k}}^{T_{i_k}} \to X_{\varepsilon}^T$ and $\pi_{T_{i_k}}(X_{i_k})$ is isomorphic to $\pi_T(X)$ for all large $k$.

Note that without a positive lower bound on the size of the essential circles the situation is more complicated; in general, there may be no convergent subsequence of covers at all, and even if there is, it may not be a cover of the limiting space $X$ (see Example 12). Nevertheless, Theorem 1 is an explicit enhancement of some notable prior results. For instance, from Theorem 3.4 and Corollary 3.5 in [17], one can conclude that for any $0 < \varepsilon < \varepsilon'$, $\pi_{\varepsilon'}(X)$ is isomorphic to a quotient of $\pi_{\varepsilon}(X)$, for sufficiently large $i$. Theorem 1 on the other hand, not only gives a nice geometric picture of the explicit quotient, but it also handles the case where $\varepsilon = \varepsilon'$.

In Section 2 we recall some of the basics of discrete homotopy theory from [2] and [16], and present examples illustrating ideas outlined in the introduction. In Section 3 we establish some technical results regarding quotients of metric spaces, particularly in the case of subgroups of $\pi_{\varepsilon}(X)$ acting on $X_{\varepsilon}$. We establish in Theorem 15 that every cover of a compact geodesic space $X$ is naturally a quotient of $X_{\varepsilon}$ by a subgroup of $\pi_{\varepsilon}(X)$, for some sufficiently small $\varepsilon > 0$. In the process, we give an explicit description of the subgroup as well as necessary and sufficient conditions for the covering map to be regular; this strengthens Lemma 2.4 in [18]. As a byproduct of the results in Section 3, we use essential circles to create a new (even for Riemannian manifolds) set of generators of the
fundamental group of a compact geodesic space (Theorem 25), and we conjecture that one can always find such a generating set of minimal cardinality.

Section 4 begins by introducing the notion of \( \sigma \)-isometry and showing that pointed Gromov-Hausdorff convergence is implied by the existence of these maps. The induced maps \( f_\# \) mentioned above play a significant role in this section, and Proposition 30 provides the technical machinery necessary to control the distortion of these maps in the case of the \( \varepsilon \)-covers of a convergent sequence of compact spaces. The last step is the adaptation and translation of the preceding tools into an equivariant notion of pointed Gromov-Hausdorff convergence (Theorem 32 and Proposition 33); this facilitates the still rather technical proof of Theorem 1. We conclude the paper with some brief remarks and questions that naturally arise from these results.

## 2 Background and Examples

We begin with some background on discrete homotopy theory; proofs and further details regarding the results in this section may be found in [2] for the more general uniform case and in [10] for the special case of metric and geodesic spaces. In a metric space \( X \), and for \( \varepsilon > 0 \), an \( \varepsilon \)-chain is a finite sequence \( \{x_0, \ldots, x_n\} \) such that for all \( i \), \( d(x_i, x_{i+1}) < \varepsilon \). An \( \varepsilon \)-homotopy consists of a finite sequence \( \langle \gamma_0, \ldots, \gamma_n \rangle \) of \( \varepsilon \)-chains, where each \( \gamma_i \) differs from its predecessor by a "basic move": adding or removing a single point, always leaving the end-points fixed. The \( \varepsilon \)-homotopy equivalence class of an \( \varepsilon \)-chain \( \alpha \) will be denoted \( [\alpha]_\varepsilon \). Fixing a basepoint \( * \), \( X_\varepsilon \) is defined to be the set of all \( \varepsilon \)-homotopy equivalence classes of \( \varepsilon \)-chains starting at \( * \), and \( \phi_\varepsilon : X_\varepsilon \to X \) is the endpoint map. In a connected space, choice of basepoint is immaterial and \( \phi_\varepsilon \) is surjective, so we will not include the base point in our notation and will assume that all maps are base-point preserving. In particular, we always take the base point in \( X_\varepsilon \) to be the equivalence class \([*]_\varepsilon \) containing the trivial chain \( \{*\} \).

The group \( \pi_\varepsilon(X) \) is the subset of \( X_\varepsilon \) consisting of equivalence classes of \( \varepsilon \)-loops starting and ending at \( * \) with operation induced by concatenation, i.e., \([\alpha]_\varepsilon \cdot [\beta]_\varepsilon = [\alpha \cdot \beta]_\varepsilon \). We denote the reversal of a chain \( \alpha \) by \( \overline{\alpha} \). As expected, for \( [\alpha]_\varepsilon \in \pi_\varepsilon(X) \), \( ([\alpha]_\varepsilon)^{-1} = [\overline{\alpha}]_\varepsilon \), and the identity is \([*]_\varepsilon \).

For any \( \varepsilon \)-chain \( \alpha = \{x_0, \ldots, x_n\} \), we set \( \nu(\alpha) := n \) and define its length by

\[
L(\alpha) := \sum_{i=1}^n d(x_i, x_{i-1}).
\]

Defining \( |[\alpha]_\varepsilon| := \inf \{L(\gamma) : \gamma \in [\alpha]_\varepsilon \} \) leads to a metric on \( X_\varepsilon \) given by

\[
d([\alpha]_\varepsilon, [\beta]_\varepsilon) := |[\alpha \cdot \beta]_\varepsilon| = \inf \{L(\kappa) : \alpha \cdot \kappa \cdot \overline{\beta} \text{ is } \varepsilon\text{-null homotopic} \} \tag{1}
\]

This metric has a number of nice properties that we will need. For example, \( \pi_\varepsilon(X) \) acts on \( X_\varepsilon \) as isometries via the map induced by preconcatenation by an \( \varepsilon \)-loop. Additionally, the endpoint map \( \phi_\varepsilon : X_\varepsilon \to X \) (which is well-defined since \( \varepsilon \)-homotopies preserve endpoints), is a uniform local isometry and, provided
X is connected, a regular covering map with deck group naturally identified with \( \pi_e(X) \). When \( X \) happens to be a geodesic space (which will soon be our underlying assumption) then so is \( X_\varepsilon \), and in fact the above metric coincides with the traditional lifted geodesic metric on the covering space \( X_\varepsilon \) (Proposition 23, [16]). The definition using (1) is very useful for our purposes, but since we will need the lifted geodesic metric for arbitrary covering spaces, we will recall the definition now. Given a covering map \( \phi : X \rightarrow Y \), where \( Y \) is a geodesic space and \( X \) is connected, the lifted geodesic metric on \( X \) is defined by \( d(x,y) = \inf \{ L(\phi \circ c) : c \text{ is a path joining } x \text{ and } y \} \). As pointed out in [10], geodesic metrics are uniquely determined by their local values, and in particular the lifted geodesic metric is uniquely determined by the fact that \( \phi \) is a local isometry.

There is a mapping from fixed-endpoint homotopy classes of continuous paths to \( \varepsilon \)-homotopy classes of \( \varepsilon \)-chains defined as follows: For any continuous path \( c : [0,1] \rightarrow X \), choose \( 0 = t_0 < \cdots < t_n = 1 \) fine enough that every image \( c([t_i,t_{i+1}]) \) is contained in the open ball \( B(c(t_i),\varepsilon) \). Then the chain \( \{c(t_0),\ldots,c(t_n)\} \) is called a subdivision \( \varepsilon \)-chain of \( c \). Setting \( \Lambda([c]) := \{c(t_0),\ldots,c(t_n)\}_\varepsilon \) produces a well-defined function that is length non-increasing in the sense that \( |\Lambda([c])| \leq |[c]| := \inf \{L(d) : d \in [c] \} \). Restricting \( \Lambda \) to the fundamental group at any base point yields a homomorphism \( \pi_1(X) \rightarrow \pi_\varepsilon(X) \) that we will also refer to as \( \Lambda \). When \( X \) is geodesic, \( \Lambda \) is surjective since the successive points of an \( \varepsilon \)-loop \( \lambda \) may be joined by geodesics to obtain a path loop whose class goes to \([\alpha]_\varepsilon \). The kernel of \( \Lambda \) is precisely described by Corollary [18]. Variations of \( \Lambda \) and their applications to the fundamental group and universal covers are further examined by the second author in [23].

A partial inverse operation to \( \Lambda \) is given by the following notion: Let \( \alpha := \{x_0,\ldots,x_n\} \) be an \( \varepsilon \)-chain in a metric space \( X \), where \( \varepsilon > 0 \). A stringing of \( \alpha \) consists of a path \( \tilde{\alpha} \) formed by concatenating paths \( \gamma_i \) from \( x_i \) to \( x_{i+1} \) where each path \( \gamma_i \) lies entirely in \( B(x_i,\varepsilon) \). If each \( \gamma_i \) is a geodesic then we call \( \tilde{\alpha} \) a chording of \( \alpha \). Note that by “geodesic” in this paper we mean an arclength-parametrized path whose length is equal to the distance between its endpoints, and not a locally minimizing path as is the more common meaning in Riemannian geometry. We will need the following two basic results.

**Proposition 2** If \( \alpha \) is an \( \varepsilon \)-chain in a chain connected metric space \( X \) then the unique lift of any stringing \( \tilde{\alpha} \) starting at the basepoint \([*]_\varepsilon \) in \( X_\varepsilon \) has \([\alpha]_\varepsilon \) as its endpoint.

**Corollary 3** If \( \alpha \) and \( \beta \) are \( \varepsilon \)-chains in a chain connected metric space \( X \) such that there exist stringings \( \tilde{\alpha} \) and \( \tilde{\beta} \) that are path homotopic then \( \alpha \) and \( \beta \) are \( \varepsilon \)-homotopic.

We also need some basic technical results. The first of these quantifies the idea that “uniformly close” \( \varepsilon \)-chains are \( \varepsilon \)-homotopic. Of course “close” depends on \( \varepsilon \). Given \( \alpha = \{x_0,\ldots,x_n\} \) and \( \beta = \{y_0,\ldots,y_n\} \) with \( x_i, y_i \in X \), define \( \Delta(\alpha,\beta) := \max \{d(x_i,y_i)\} \). For any \( \varepsilon > 0 \), if \( \alpha \) is an \( \varepsilon \)-chain we define
A special case is a midpoint refinement is a call \( \alpha \) \( \beta \) pairs of points. Of course refinements always exist in geodesic spaces, but not with convergence questions, since the property of being an \( \epsilon \)-homotopic to its pointwise limit.

The reader will likely have noticed that the previous proposition requires that the chains in question have the same number of points. The next lemma shows that this is not really an issue. It is useful in many ways—for example to find “convergent subsequences” of classes of chains, much like a discrete version of Ascoli’s Theorem.

**Lemma 5** Let \( L, \epsilon > 0 \) and \( \alpha \) be an \( \epsilon \)-chain in a metric space \( X \) with \( L(\alpha) \leq L \). Then there is some \( \alpha' \in [\alpha]_\epsilon \) such that \( L(\alpha') \leq L(\alpha) \) and \( \nu(\alpha') = \left\lceil \frac{2L(\alpha)}{\epsilon} + 1 \right\rceil \).

For any \( \delta \geq \epsilon > 0 \), every \( \epsilon \)-chain (respectively \( \epsilon \)-homotopy) is also a \( \delta \)-chain (respectively \( \delta \)-homotopy) and there is a well-defined mapping \( \phi_{\epsilon \delta} : X_\epsilon \to X_\delta \) given by \( \phi_{\epsilon \delta}(\alpha)_\epsilon = [\alpha]_\delta \). When \( X \) is geodesic, this mapping is also a regular covering map and local isometry, though for non-geodesic metric spaces it may not be surjective. Restricting the map \( \phi_{\epsilon \delta} \) to the group \( \pi_\delta(X) \) induces a homomorphism \( \theta_{\epsilon \delta} : \pi_\epsilon(X) \to \pi_\delta(X) \), which is injective (respectively, surjective) if and only if \( \phi_{\epsilon \delta} \) is. Thus, for a geodesic space \( X \), one obtains parameterized collections of covering spaces \( \{X_\epsilon\}_{\epsilon > 0} \) and their corresponding deck groups \( \{\pi_\epsilon(X)\}_{\epsilon > 0} \), which actually form inverse systems (see [2]) via the surjective bonding maps/homomorphisms \( \phi_{\epsilon \delta} \) and \( \theta_{\epsilon \delta} \), respectively.

A number \( \epsilon > 0 \) is called a homotopy critical value for \( X \) if there is an \( \epsilon \)-loop \( \alpha \) based at \( * \) such that \( \alpha \) is not \( \epsilon \)-null (i.e. \( \epsilon \)-homotopic to the trivial chain) but is \( \delta \)-null for all \( \delta > \epsilon \). We have the following essential connection between homotopy critical values and \( \epsilon \)-covers:

**Lemma 6** If \( X \) is a geodesic space then the covering map \( \phi_{\epsilon \delta} : X_\delta \to X_\epsilon \) is injective if and only if there are no homotopy critical values \( \sigma \) with \( \delta \leq \sigma < \epsilon \).

**Corollary 7** If \( \lambda \) is an \( \epsilon \)-loop in a geodesic space \( X \) of length less than \( 3\epsilon \) then \( \lambda \) is \( \epsilon \)-null.

If \( \alpha = \{x_0, ..., x_n\} \) is a chain in a geodesic space \( X \) then a refinement of \( \alpha \) consists of a chain \( \beta \) formed by inserting between each \( x_i \) and \( x_{i+1} \) some subdivision chain of a geodesic joining \( x_i \) and \( x_{i+1} \). If \( \beta \) is an \( \epsilon \)-chain we will call \( \beta \) an \( \epsilon \)-refinement of \( \alpha \). Note that if \( \alpha \) is an \( \epsilon \)-chain, then any \( \epsilon \)-refinement of \( \alpha \) is \( \epsilon \)-homotopic to \( \alpha \), and, hence, any two \( \epsilon \)-refinements of \( \alpha \) are \( \epsilon \)-homotopic.

A special case is a midpoint refinement, which simply uses a midpoint between pairs of points. Of course refinements always exist in geodesic spaces, but not in general metric spaces. Refinements are an important tool when working with convergence questions, since the property of being an \( \epsilon \)-chain is an “open condition” and may not be preserved when passing from a sequence of chains to its pointwise limit.
Definition 8 If $X$ is a metric space and $\varepsilon > 0$, an $\varepsilon$-loop of the form $\lambda = \alpha * \tau * \pi$, where $\nu(\tau) = 3$, will be called $\varepsilon$-small. Note that this notation includes the case when $\alpha$ consists of a single point—i.e. $\lambda = \tau$.

The next proposition was established in [16] with the assumption that $\varepsilon < \delta$, but the same proof works for $\varepsilon = \delta$, and we will need that case in the present paper. In essence it “translates” a homotopy into a product of small loops.

Proposition 9 Let $X$ be a geodesic space and $0 < \varepsilon \leq \delta$. Suppose $\alpha, \beta$ are $\varepsilon$-chains and $(\alpha = \gamma_0, ..., \gamma_n = \beta)$ is a $\delta$-homotopy. Then $[\beta]_\varepsilon = [\lambda_1 * \cdots * \lambda_r * \varepsilon * \lambda_{r+1} * \cdots * \lambda_n]_\varepsilon$, where each $\lambda_i$ is an $\varepsilon$-refinement of a $\delta$-small loop.

An $\varepsilon$-triad in a geodesic space $X$ is a triple $T := \{x_0, x_1, x_2\}$ such that $d(x_i, x_j) = \varepsilon$ for all $i \neq j$; when $\varepsilon$ is not specified we will simply refer to a triad. We denote by $\alpha_T$ the loop $\{x_0, x_1, x_2, x_0\}$. We say that $T$ is essential if some (equivalently any) $\varepsilon$-refinement of $\alpha_T$ is not $\varepsilon$-null. The equivalence of “some” and “any” in the preceding definition—as well as the useful fact that the $\varepsilon$-refinement of $\alpha_T$ may always be taken to be a midpoint refinement—follow from Proposition 37 in [16]. Essential $\varepsilon$-triads $T_1$ and $T_2$ are defined to be equivalent if some (equivalently any) $\varepsilon$-refinement of $\alpha_{T_1}$ is freely $\varepsilon$-homotopic to an $\varepsilon$-refinement of either $\alpha_{T_2}$ or $\alpha_{T_2}^-$, and again it suffices to consider only midpoint refinements. See [16] for the definition of “free $\varepsilon$-homotopy”, which is analogous to the classical meaning for paths.

The image of a closed loop of length $3\varepsilon$ with the property that any $\varepsilon$-loop along this path loop is not $\varepsilon$-null is called an essential $\varepsilon$-circle. If one connects the points of an essential $\varepsilon$-triad with minimal geodesics to form a parameterized loop, the resulting loop forms an essential $\varepsilon$-circle. Conversely, if one subdivides an essential $\varepsilon$-circle into three equal segments, then the endpoints of those segments are an essential $\varepsilon$-triad (Proposition 37 and Corollary 41, [16]). Equivalence of the underlying triads is used to define equivalence of essential circles. Note that equivalent essential circles may not be freely path homotopic due to “small holes” that block traditional homotopies but not $\varepsilon$-homotopies. It should also be noted that while essential circles are necessarily the images of non-null, closed geodesics that are shortest in their respective homotopy classes, they have an even stronger property: an essential circle is metrically embedded in the sense that its metric as a subspace of $X$ is the same as the intrinsic metric of the circle (Theorem 39, [16]). There are examples of closed geodesics in compact geodesic spaces that are not essential circles (Example 44, [16]).

The set of homotopy critical values of a compact geodesic space make up what is called the homotopy critical spectrum of $X$, and these values indicate precisely when the equivalence type of the $\varepsilon$-covers changes as $\varepsilon$ decreases to 0. For example, given a standard geodesic (i.e. Riemannian) circle of circumference $a > 0$, the $\varepsilon$-covers are all isometries when $\varepsilon > \frac{a}{3}$, but are the standard universal cover when $\varepsilon \leq \frac{a}{3}$; that is, the homotopy critical spectrum of this circle is $\{\frac{a}{3}\}$. Equivalently, any $\frac{a}{3}$-loop that traverses the circle once in either direction is not $\frac{a}{3}$-null, but it is $\delta$-null homotopic for all $\delta > \frac{a}{3}$. In general one may imagine $\varepsilon$ as decreasing from the diameter of the space towards 0, as the $\varepsilon$-covers “unravel”
more of the topology of the space at each homotopy critical value. If the space is semilocally simply connected, then the process stabilizes with the universal cover, and the smallest homotopy critical value is $\frac{1}{3}$ of the 1-systole of the space (smallest non-null closed geodesic—Corollary 43, [16]). If the space is not semilocally simply connected, then the “unrolling” process may never end, although by using an inverse limit one will obtain what Berestovskii-Plaut call the uniform universal cover as defined in [2].

Sormani and Wei defined the covering spectrum of a compact geodesic space to be the set of all $\delta > 0$ such that $\tilde{X}_\delta \neq \tilde{X}_{\delta'}$ for all $\delta' > \delta$; hence, the covering spectrum also indicates where the $\delta$-covers change equivalence type. As was noted in the introduction, while the homotopy critical spectrum may be defined for more general metric spaces, it follows from the definitions and Corollary 17 below that the homotopy critical spectrum of a compact geodesic space differs from its covering spectrum only by a constant multiple of $\frac{2}{3}$. For compact geodesic spaces, the homotopy critical values are discrete and bounded above in $(0, \infty)$; in fact, if $X$ is a precompact collection of compact geodesic spaces, there is a uniform upper bound on the number of critical values in any positive interval $[a, b]$ for any $X \in \mathcal{X}$. This was first proved by Sormani-Wei when the spaces have universal covers (Corollary 7.7, [18]—though they prove discreteness directly without this assumption) and later without this assumption and with a different proof by Plaut-Wilkins (Theorem 11, [16]).

The connection between essential circles/triads and the homotopy critical spectrum was established by the authors in Theorem 6 of [16]: $\varepsilon$ is a homotopy critical value of a compact geodesic space $X$ if and only if $X$ contains an essential $\varepsilon$-circle. The multiplicity of a critical value $\varepsilon$ is the number of equivalence classes of essential $\varepsilon$-circles, which is always finite when $X$ is compact. Furthermore, Sormani and Wei showed that if compact geodesic spaces $X_i$ converge to the compact space $X$, then the covering (hence, homotopy critical) spectra of the spaces $X_i$ converge in the Hausdorff sense in $\mathbb{R}$ to the spectrum of $X$ (Theorem 8.4, [18]), which means that essential circles in the limit $X$ can only arise as “limits” of essential circles in the spaces $X_i$. With this in mind, one can see that the fact that $\varepsilon$-covers are not closed with respect to Gromov-Hausdorff convergence is related to the behavior of both the covering spaces, themselves, and the homotopy critical spectra, as the next example shows.

**Example 10** Suppose $X_i$ is a geodesic circle of circumference $a - \frac{1}{i}$, so $X_i \to X$, where $X$ is the circle of circumference $a$. The homotopy critical spectra of $X_i$ and $X$, respectively, are $\left\{ \frac{2}{3} - \frac{1}{3i} \right\}$ and $\{ \frac{2}{3} \}$. If we set $\varepsilon = \frac{2}{3}$, then $(X_i)_\varepsilon = X$, while $X_\varepsilon = \mathbb{R}$. Of course in this case $(X_i)_{\frac{2}{3}}$ does converge to an $\varepsilon$-cover of $X$, but it is not $X_{a/3}$. The covers $(X_i)_{a/3}$ converge to $X_\tau$ for any $\tau > \frac{2}{3}$. On the other hand, for any $\varepsilon \neq \frac{2}{3}$, it does hold that $(X_i)_\varepsilon \to X_\varepsilon$.

To see how it is possible for $(X_i)_\varepsilon$ to not converge to any $\varepsilon$-cover at all, it will be helpful to illustrate the notion of multiplicity of a critical value $\varepsilon$—which by definition is the number of equivalence classes of essential $\varepsilon$-circles. Suppose that $Y$ denotes the flat torus obtained by identifying the sides of a rectangle of
dimensions $0 < 3a \leq 3b$. When $a < b$, $a$ and $b$ are distinct homotopy critical values: $Y_\varepsilon = Y$ for $\varepsilon > b$, $Y_\varepsilon$ is a flat metric cylinder over a circle of length $3a$ for $a < \varepsilon \leq b$, and $Y_\varepsilon$ is the plane for $\varepsilon \leq a$. Each critical value $a, b$ has multiplicity 1 because there is one class of essential circles corresponding to each value. When $a = b$, however, the torus unrolls immediately into the plane at $\varepsilon = a$; that is, the $\varepsilon$-covers “skip” the cylinder. In this case, $a$ is a homotopy critical value of multiplicity 2, since both topological holes are of the same size and are detected by the $\varepsilon$-covers simultaneously.

**Example 11** Take a sequence of tori $T_i$ obtained from $(1 - \frac{1}{i}) \times (1 + \frac{1}{i})$-rectangles. Then each $T_i$ has homotopy critical values $\frac{1}{3} - \frac{1}{i}$ and $\frac{1}{3} + \frac{1}{i}$, each with single multiplicity, while the limiting torus has a single critical value $\frac{1}{3}$ with multiplicity 2. In fact, $(T_i)_{\frac{1}{3}}$ is a cylinder for all $i$, with limit $Y$ a cylinder of circumference 1, which as observed above is not an $\varepsilon$-cover of the limiting torus.

What happens in Example 11 is that distinct homotopy critical values merge in the limit to a single homotopy critical value with multiplicity greater than 1. Theorem 1 formally describes how this phenomenon occurs: by extending $\varepsilon$-covers to the notion of circle covers, we can tease apart the multiplicity to find the “missing” intermediate covers. We conclude this section with an example illustrating what can go wrong in Theorem 1 when there is no positive lower bound on the size of the essential circles.

**Example 12** Consider the torus $T_i = S_{\frac{1}{3}/i} \times S_1$ formed by circles of circumference $\frac{1}{3}$ and 1. Then $T_i \rightarrow S_1$. If we choose $\varepsilon_i$ so that $\varepsilon_i < \frac{1}{i}$ and $\varepsilon_i \rightarrow 0$, then each $(T_i)_{\varepsilon_i}$ for $i \geq 3$ is the universal cover $\mathbb{R}^2$, but of course $\mathbb{R}^2$ is not any kind of cover of $S_1$. This particular example satisfies the assumptions of Theorem 1.1, [10], in which Ennis and Wei showed the following: Suppose $X_i \rightarrow X$ are all compact geodesic spaces having (categorical, possibly not simply connected) universal covers. The latter assumption is equivalent to the homotopy critical spectra having positive lower bounds $\varepsilon_i, \varepsilon$, and in fact the universal covers are $(X_i)_{\varepsilon_i}$ and $X_\varepsilon$, respectively. Theorem 1.1, [10] says that if one additionally assumes that the spaces $X_i$ have dimension uniformly bounded above and the spaces $X_{\varepsilon_i}$ pointed Gromov-Hausdorff converge to a space $\bar{X}$, then there is a subgroup $H \subset \text{Iso}(\bar{X})$ such that $\bar{X}/H$ is the universal cover of $X$. Note that in the case of collapse, the subgroup $H$ need not be discrete.

On the other hand, if $H$ denotes the geodesic Hawaiian Earring, then as $\varepsilon \rightarrow 0$ the $\varepsilon$-covers $H_\varepsilon$ of $H$ contain graphs of higher and higher valency (see [2] for more details). Thus, if $\inf \{\varepsilon_i\} = 0$, then no subsequence of $(X_i)_{\varepsilon_i}$ can converge.

### 3 Covering Maps and Quotients

We begin by recalling some results concerning quotients of metric spaces. In this paper, all actions are by isometries and are discrete in the sense of [14]. Discreteness of an action is a uniform version of properly discontinuous, which
is implied by the following property for any \( G \) acting by isometries on a metric space \( Y \): There exists some \( \varepsilon > 0 \) such that for all \( y \in Y \) and non-trivial \( g \in G \), 
\[ d(y, g(y)) \geq \varepsilon. \]
Note that in the case where \( Y \) is the \( \varepsilon \)-cover of a metric space \( X \), 
\( \pi\varepsilon(X) \) acts discretely on \( X\varepsilon \), since the restriction of \( \phi_e \) to any \( B([a]_\varepsilon, \frac{\varepsilon}{2}) \subset X\varepsilon \)
is an isometry onto its image in \( X \) ([16]). When \( Y \) is geodesic, then the quotient metric on \( Y/G \) (cf. [14]) is the uniquely determined geodesic metric such that the quotient mapping is a local isometry ([16]). Combining this observation with Proposition 28 in [15] we obtain the following:

**Proposition 13** Suppose that \( X \) is a geodesic space, \( G \) acts discretely by isometries on \( X \), and \( H \) is a normal subgroup of \( G \). Then \( G/H \) acts discretely by isometries on \( X/H \) via \( gH \circ (Hx) = g(x)H \) and the mapping \( J : X \to (G/H)(Hx) \) an isometry from \( X/G \) to \( (X/H)/(G/H) \).

Among basic applications we have that for any geodesic space \( X \) and \( 0 < \delta < \varepsilon \), the covering map \( \phi_\varepsilon : X\varepsilon \to X \) is isometrically equivalent to the induced mapping \( \zeta : X\delta \circ \ker \theta_{\delta \varepsilon} \to X \). Here \( \ker \theta_{\delta \varepsilon} \) acts discretely and isometrically as a normal subgroup of \( \pi\delta(X) \) with \( X = X\delta / \pi\delta(X) \), and \( \zeta \) is the unique covering map such that \( \zeta \circ \pi = \phi_\delta \), while \( \pi : X\delta \to X\delta / \ker \theta_{\delta \varepsilon} \) is the quotient mapping.

We know already from the results of [2] that if \( X \) is a compact metric space, \( Y \) is connected, and \( f : Y \to X \) is a covering map then for small enough \( \varepsilon > 0 \) there is a covering map \( g : X\varepsilon \to Y \). The next proposition refines this statement when \( X \) is geodesic.

**Proposition 14** Let \( X \) be a compact geodesic space and suppose that \( f : Y \to X \) is a covering map, where \( Y \) is connected. Suppose that \( \varepsilon > 0 \) is at most \( \frac{\pi}{\delta} \) of a Lebesgue number for a covering of \( X \) by open sets evenly covered by \( f \). Then there is a covering map \( g : X\varepsilon \to Y \) such that \( \phi_\varepsilon = f \circ g \).

**Proof.** Choose a basepoint \( * \) in \( Y \) such that \( f(*) = * \) and define \( g([a]_\varepsilon) \) to be the endpoint lift of some stringing \( \hat{\alpha} \) starting at \( * \) in \( Y \). We need to check that \( g \) is well-defined. By iteration, it suffices to prove the following: If \( \alpha := \{x_0, ..., x_n\} \) and \( \alpha' \) is an \( \varepsilon \)-chain \( \{x_0, ..., x, x, x_{i+1}, ..., x_n\} \), then for any stringings \( \hat{\alpha} \) and \( \hat{\alpha}' \), the endpoints of the lifts of \( \hat{\alpha} \) and \( \hat{\alpha}' \) starting at \( * \) are the same. Let \( \{\gamma_i\} \) and \( \{\gamma'_i\} \) be geodesics joining \( x_i, x_{i+1} \) and \( \beta_1, \beta_2 \) be geodesics from \( x_i \) to \( x \) and \( x \) to \( x_{i+1} \), respectively. Note that each of the loops formed by each \( \gamma_i \) and the reversal of \( \gamma'_i \) has diameter smaller than \( \varepsilon \) and thus lifts as a loop in \( Y \). Moreover, the triangle formed by the geodesics \( \beta_1, \beta_2 \), and either \( \gamma_i \) or \( \gamma'_i \) has diameter less than \( \frac{\varepsilon}{4} \) and hence lifts as a loop. This proves that \( g \) is well-defined. It now follows from a standard result in topology that \( g \) is a covering map ([13]). (This result is stated with the additional assumption that all maps are continuous, but that assumption is superfluous because the third mapping - \( g \) in this case - is locally a homeomorphism.)

The next theorem shows that any covering space of a compact geodesic space can be obtained in a particularly natural way as a quotient of the space \( X\varepsilon \) obtained in Proposition 14.
Theorem 15 Let $X$ be a compact geodesic space and suppose that $f:Y\to X$ is a covering map, where $Y$ is connected and has the lifted geodesic metric from $X$. Let $\varepsilon > 0$ be such that there is a covering map $g:X_\varepsilon \to Y$ with $\phi_\varepsilon = f \circ g$. Define a subgroup $K$ of $\pi_\varepsilon(X)$ by

$$K := \{ [\lambda]_\varepsilon : \exists \kappa \in [\lambda]_\varepsilon \text{ such that some stringing } \hat{\kappa} \text{ of } \kappa \text{ lifts as a loop in } Y \}.$$  

Then

1. $K = \{ [\lambda]_\varepsilon : \text{for all } \lambda \in [\lambda]_\varepsilon, \text{ every stringing } \hat{\kappa} \text{ of } \kappa \text{ lifts as a loop to } Y \}$. 

2. There is a covering equivalence $\phi:Y\to X_\varepsilon/K$ such that $\pi = \phi \circ g$, where $\pi:X_\varepsilon \to X_\varepsilon/K$ is the quotient map. 

3. $K$ is a normal subgroup if and only if $f$ is a regular covering map.

Proof. The first part follows from Proposition [2]. Define $\phi(y) := \pi(x)$, where $x \in g^{-1}(y)$. To see why $\phi$ is well-defined, suppose that $g([\alpha]_\varepsilon) = g([\beta]_\varepsilon) = y$. Since the lifts $\hat{\alpha}$ and $\hat{\beta}$ of $\alpha$ and $\beta$ to the basepoint in $X_\varepsilon$ have $[\alpha]_\varepsilon$ and $[\beta]_\varepsilon$ as their endpoints (Proposition [2]), $g \circ \hat{\alpha}$ and $g \circ \hat{\beta}$ both end in $y$. But these two curves are the unique lifts of $\alpha$ and $\beta$ to $Y$, so they both end in $y$. In other words, if $\lambda := \alpha \cdot \beta$, then $[\lambda]_\varepsilon \in K$. Moreover, $[\alpha]_\varepsilon := [\lambda]_\varepsilon * [\beta]_\varepsilon$, so $[\alpha]_\varepsilon$ and $[\beta]_\varepsilon$ lie in the same orbit of $K$, and $\pi([\alpha]_\varepsilon) = \pi([\beta]_\varepsilon)$. This shows that $\phi$ is well-defined, and clearly $\pi = \phi \circ g$.

For surjectivity, let $K[\alpha]_\varepsilon \in X_\varepsilon/K$ and define $y := g([\alpha]_\varepsilon)$. We have

$$\phi(y) = \phi(g([\alpha]_\varepsilon)) = \pi([\alpha]_\varepsilon) = K[\alpha]_\varepsilon.$$  

For injectivity, suppose that $y,z \in Y$ satisfy $\phi(y) = \phi(z)$. Then for $[\alpha]_\varepsilon, [\beta]_\varepsilon$ such that $g([\alpha]_\varepsilon) = y$ and $g([\beta]_\varepsilon) = z$ we have that $[\alpha]_\varepsilon, [\beta]_\varepsilon$ lie in the same orbit, so $[\alpha]_\varepsilon * [\beta]_\varepsilon = [\lambda]_\varepsilon \in K$. Consequently, the lifts of stringings $\hat{\alpha}$ and $\hat{\beta}$ to $Y$ at the basepoint must have the same endpoint. But one lift ends in $y$ and the other ends in $z$, so $y = z$. This shows that $\phi$ is a homeomorphism. The natural covering map $\xi:X_\varepsilon/K \to X$ is defined by $\xi(K[\alpha]_\varepsilon) = \phi_*([\alpha]_\varepsilon)$ and therefore $\xi \circ \phi = f$, showing that $\phi$ is a covering equivalence.

If $K$ is normal then by Lemma 39 in [15], the covering map $\xi:X_\varepsilon/K \to X$ is a topological quotient map with respect to a well-defined induced action of $\pi_\varepsilon(X)/K$, defined by $[\lambda]_\varepsilon K([\alpha]_\varepsilon K) = ([\lambda]_\varepsilon ([\alpha]_\varepsilon))K$. Since $\xi$ is also a covering map, it follows from standard results in topology that $\xi$ is regular. Since $\phi$ is a covering equivalence, $f$ is also regular. Conversely, suppose that $\xi$ is regular and let $H$ denote its group of covering transformations. Define a function $h:\pi_\varepsilon(X) \to H$ as follows: Given $[\alpha]_\varepsilon \in \pi_\varepsilon(X)$, let $y$ be the endpoint of the lift of some chording $\hat{\alpha}$ to $Y$ starting at the basepoint. Since $\phi_\varepsilon = g \circ f$, by uniqueness $y$ must also be the endpoint $f \circ c$, where $c$ is the lift of $\alpha$ to $X_\varepsilon$. By Proposition [2] the endpoint of $c$ is just $[\alpha]_\varepsilon$, and hence $y = f([\alpha]_\varepsilon)$ depends only on $[\alpha]_\varepsilon$. That is, if we let $h([\alpha]_\varepsilon)$ be the (unique) $\mu \in H$ such that $\mu(+) = y$, then $h$ is well-defined. Sorting through the definition shows that $h$ is a homomorphism with kernel $K$, since $y$ is the basepoint exactly when $\hat{\alpha}$ lifts as a loop.
Corollary 16 Let $X$ be compact geodesic and $\varepsilon > 0$. Then an $\varepsilon$-loop $\alpha$ is $\varepsilon$-null if and only if some (equivalently any) stringing of $\alpha$ lifts to a loop in $X_{\varepsilon}$.

Proof. We only note that $\alpha$ is $\varepsilon$-null if and only if $\beta * \alpha * \overline{\beta}$ is $\varepsilon$-null for any $\varepsilon$-chain from the basepoint to the starting point of $\alpha$, and the analogous statement also holds for null-homotopies of any stringing of $\alpha$. ■

The $\delta$-covering map defined by Sormani-Wei is obtained using a construction of Spanier that provides for any open cover $\mathcal{U}$ of a connected, locally path connected topological space $Z$ a covering map $\phi_{\mathcal{U}} : W \to Z$. The covering map is characterized by the fact that a path loop $c$ at the basepoint in $Z$ lifts as a loop to $W$ if and only if its homotopy equivalence class $[c]$ lies in the subgroup $S_{\mathcal{U}}$ of $\pi_1(Z)$, which we will call the Spanier Group of $\mathcal{U}$, generated by all loops of the following form: $c * L \ast \overline{c}$, where $c$ is a path loop starting at the basepoint and $L$ is a path loop lying entirely in some set in the open cover $\mathcal{U}$. For their construction, Sormani-Wei took $\mathcal{U}$ to be the cover of $X$ by open $\delta$-balls. We will denote the corresponding Spanier Group by $S_{\delta}$.

Corollary 17 For any geodesic space $X$ and $\delta = \frac{n}{3} > 0$, there is an equivalence of the covering maps (hence an isometry) $h : \tilde{X}^\delta \to X_{\varepsilon}$.

Proof. An immediate consequence of the definition of $\tilde{X}^\delta$ is that all open $\delta$-balls are evenly covered by $\pi^3$, and hence we may take $\delta$ for the Lebesgue number in Proposition 14. That proposition gives a covering map $\psi : X_{\varepsilon} \to \tilde{X}^\delta$, and Theorem 15 will finish the proof if we can show that the group $K$ for $Y := \tilde{X}^\delta$ is trivial. If $[\lambda]_{\varepsilon} \in K$ then by definition some chording $\tilde{\lambda}$ lifts as a loop in $\tilde{X}^\delta$. In other words, $\tilde{\lambda}$ is homotopic to a concatenation of paths of the form $c_i \ast L_i \ast \overline{c}_i$ where $L_i$ is a path loop that lies entirely in an open $\delta$-ball. According to Corollary 8, we need only show that any refinement $\varepsilon$-chain of any such path is $\varepsilon$-null. In other words, it is enough to show the following: Any $\varepsilon$-refinement of a rectifiable loop $f$ in an open ball $B(x, \delta)$ is $\varepsilon$-null. Given such a loop $f$, the distance from points on $f$ to $x$ has a maximum $D < \delta$. Now subdivide $f$ into segments $\sigma_i$ whose endpoints $x_i, x_{i+1}$ satisfy $d(x_i, x_{i+1}) < \delta - D$. Then each of the $\varepsilon$-loops $\kappa_i := \{x, x_i, x_{i+1}, x\}$ (where $x_n = x_0$ for the highest index $n$) has length less than $D + D + \delta - D = D + \delta < 2\delta = 3\varepsilon$. But then each $\kappa_i$, and hence $f$, is $\varepsilon$-null by Corollary 16. ■

Corollary 18 If $X$ is a geodesic space, $\varepsilon > 0$, and $\Lambda : \pi_1(X) \to \pi_{\varepsilon}(X)$ is the natural homomorphism defined in the second section, then $\ker \Lambda = S_{0\varepsilon}$. ■

Proof. We have $[c] \in \ker \Lambda$ if and only if some subdivision $\varepsilon$-chain $\alpha$ of $c$ is $\varepsilon$-null. Equivalently, by Corollary 16 any stringing of $\alpha$ lifts to a loop in $X_{\varepsilon}$. But since $X_{\varepsilon} = \tilde{X}^\delta$ by Corollary 17 this is equivalent to $[c] \in S_{0\varepsilon}$. ■

Definition 19 Suppose $\mathcal{T}$ is a finite collection of essential triads in a compact geodesic space and $\varepsilon > 0$ is such that each $T \in \mathcal{T}$ is a $\delta$-triad for some $\delta \geq \varepsilon$. Define $K_{\varepsilon}(\mathcal{T})$ to be the subgroup of $\pi_{\varepsilon}(X)$ generated by the collection $\Gamma_\varepsilon(\mathcal{T})$ of
all \([\alpha \ast T' \ast \overline{\alpha}]_\varepsilon\), where \(\alpha\) is an \(\varepsilon\)-chain starting at \(*\) and \(T'\) is an \(\varepsilon\)-refinement of \(T \in \mathcal{T}\). Finally, we define \(\Phi^\varepsilon_T : X / K_T(T) \to X\) by \(\Phi^\varepsilon_T([\alpha]_\varepsilon) := \phi^\varepsilon([\alpha]).\) We will call a covering map equivalent to some \(\Phi^\varepsilon_T\) a circle covering map (including the case when \(T\) is empty, in which case we take \(K_T(T)\) to be the trivial group, so \(\Phi^\varepsilon_T = \phi^\varepsilon\)).

Remark 20 First, note that \(K_T(T)\) is normal in \(\pi_T(X)\), since any conjugate of \([\alpha \ast T' \ast \overline{\alpha}]_\varepsilon\) has the same form. If \(\mathcal{T} = \{T_i\}_{i=1}^n\), it is also easy to check that for any fixed choice of \(\varepsilon\)-chains \(\alpha_i\) from \(*\) to \(T_i\), \(K_T(T)\) is the smallest normal subgroup containing the finite set \([\alpha_i \ast T_i' \ast \overline{\alpha_i}]_{\varepsilon}\}_{i=1}^n.\) However, we do not know in general whether \(K_T(T)\) is finitely generated. Finally, a word of caution. While \(\varepsilon\)-refinements of an essential \(\delta\)-triad \(T\) are all \(\varepsilon\)-homotopic, different \(\varepsilon\)-refinements of \(T\) need not be \(\varepsilon\)-homotopic when \(\delta\) is larger than \(\varepsilon\). In fact, as simple geodesic graphs show, the vertices of \(T\) may be joined by different geodesics that together form loops that are always \(\varepsilon\)-null but are not \(\varepsilon\)-null. That is, in general, replacing a \(\delta\)-triad \(T \in \mathcal{T}\) with a \(\delta\)-equivalent \(\delta\)-triad may change the group \(K_T(T)\). This further emphasizes the essential dependence of \(K_T(T)\) on not just the collection \(\mathcal{T}\) but on the value of \(\varepsilon\).

Notation 21 When convenient we will denote \(X\) by \(X_\infty\); this makes sense since any chain can be considered as an \(\infty\)-chain and every sequence of chains in which one point is removed or added to get from one chain to the next is an \(\infty\)-homotopy. Then every chain is \(\infty\)-homotopy equivalent to the chain \(\{x, y\}\), where \(x\) and \(y\) are its endpoints, and hence the mapping \(\phi_\varepsilon : X \to X\) is naturally identified with \(\phi_\infty : X_\varepsilon \to X_\infty\). This saves us from having to consider the mapping \(\phi_\varepsilon\) as a special case in the statements that follow.

Proposition 22 Let \(\varepsilon > 0\) be a homotopy critical value for a compact geodesic space \(X\). Then there is some \(\delta > \varepsilon\) such that if \(\{x_0, x_1, x_2, x_0\}\) is \(\delta\)-small with a midpoint refinement \(\alpha\) that is not \(\varepsilon\)-null then \(\alpha\) is \(\varepsilon\)-homotopic to a midpoint refinement of an essential \(\varepsilon\)-triad.

Proof. If the statement were not true then there would exist \((\varepsilon + \tfrac{1}{\varepsilon})\)-small loops \(\{x_i, y_i, z_i, x_i\}\) having midpoint subdivision chains \(\mu_i = \{x_i, m_i, y_i, n_i, z_i, p_i, x_i\}\) that are not \(\varepsilon\)-null but are not \(\varepsilon\)-homotopic to a midpoint refinement of an essential \(\varepsilon\)-triad. By taking subsequences if necessary, we may suppose that \(\{x_i, n_i, m_i, p_i, y_i, z_i, x_i\} \to \{x, m, y, n, z, p, x\}\), where \(\{x, y, z\}\) is an \(\varepsilon\)-small loop which is not \(\varepsilon\)-null. Hence \(\{x, y, z\}\) must be an essential \(\varepsilon\)-triad. Moreover, for large \(i\), \(\{x_i, m_i, y_i, n_i, z_i, p_i, x_i\}\) is \(\varepsilon\)-homotopic to a midpoint subdivision of \(\{x, y, z\}\), a contradiction. \(\blacksquare\)

Definition 23 In a geodesic space \(X\), we will call a path (resp. \(\varepsilon\)-chain) of the form \(k \ast c \ast \overline{k}\), where \(k\) is a path (resp. \(\varepsilon\)-chain), \(\overline{k}\) is its reversal, and \(c\) is an arclength parameterization of an essential circle (resp. a midpoint refinement of an essential \(\varepsilon\)-triad), a lollipop (resp. \(\varepsilon\)-lollipop). If the path \(k\) in the lollipop is locally length minimizing (possibly not minimal) then we call the lollipop a geodesic lollipop.
Note that for fixed \( \varepsilon \), the homomorphism \( \Lambda : \pi_1(X) \to \pi_\varepsilon(X) \) maps classes of lollipops determined by essential \( \varepsilon \)-circles to \( \varepsilon \)-lollichains. In fact, if \( c \) is an essential \( \varepsilon \)-circle determined by an essential \( \varepsilon \)-triad \( T \) then we can choose a midpoint refinement \( \beta \) of \( \alpha_T \) such that each point of \( \beta \) lies on \( c \). By definition, \([c]\) is mapped via \( \Lambda \) to \([\beta]\) \( \varepsilon \). Conversely, if \( \beta \) is a midpoint refinement of an essential \( \varepsilon \)-triad \( T \), we can define an essential \( \varepsilon \)-circle \( c \) containing \( \beta \) by joining the points of \( \beta \) by geodesics. Then \( \Lambda \) will map \([c]\) to \([\beta]\). Anchoring the essential circles to the base point by adjoining paths \( k \) and choosing corresponding \( \varepsilon \)-chains \( \kappa \) so that \([\kappa]\) \( = \Lambda([k]) \) yields the full conclusion.

**Theorem 24** Let \( X \) be a compact geodesic space, \( 0 < \varepsilon < \delta \leq \infty \), and \( T \) any (possibly empty!) collection that contains a representative for every essential \( \tau \)-triad with \( \varepsilon \leq \tau < \delta \). Then \( \ker \theta_{\delta \varepsilon} = K_\varepsilon(T) \). Consequently, \( \pi_\delta(X) = \pi_\varepsilon(X)/K_\varepsilon(T) \), and the covering map \( \phi_{\delta \varepsilon} : X_\varepsilon \to X_\delta \) is equivalent to the quotient covering map \( \pi : X_\varepsilon \to X_\delta/K_\varepsilon(T) \).

\( \textbf{Proof.} \) First of all, note that the inequality \( \tau < \delta \) shows that each element of \( K_\varepsilon(T) \) is \( \delta \)-null and hence \( K_\varepsilon(T) \subset \ker \theta_{\delta \varepsilon} \). For the opposite inclusion, let \([\lambda]\) \( \in \ker \theta_{\delta \varepsilon} \), meaning that \( \lambda \) is \( \delta \)-null. We will start with the case when \( \varepsilon \) is a homotopy critical value of \( X \) and \( \delta > \varepsilon \) is close enough to \( \varepsilon \) that \( \delta < 2\varepsilon \) and Proposition \( \ref{prop:critical} \) is valid: whenever \( \{x_0,x_1,x_2,x_0\} \) is \( \delta \)-small with a midpoint refinement \( \alpha \) that is not \( \varepsilon \)-null then \( \alpha \) is \( \varepsilon \)-homotopic to a midpoint refinement of an essential \( \varepsilon \)-triad. By Proposition \( \ref{prop:homotopy} \) \( \lambda \) is \( \varepsilon \)-homotopic to a product of midpoint refinements \( \lambda_i \) of \( \delta \)-small loops. Since Proposition \( \ref{prop:critical} \) holds, each \( \lambda_i \) is either \( \varepsilon \)-null or \( \varepsilon \)-homotopic to a non-null \( \varepsilon \)-lollichain. That is, \([\lambda]\) \( \in K_\varepsilon(T) \).

Next, observe that if there are no homotopy critical values \( \tau \) with \( \varepsilon \leq \tau < \delta \), then on the one hand \( T \) must be empty, and on the other hand, \( \theta_{\delta \varepsilon} \) is an isomorphism so its kernel is trivial, and we are finished. Suppose now that there is a single critical value \( \tau \) between \( \varepsilon \) and \( \delta \), which, by the previous case, may be assumed to satisfy \( \varepsilon \leq \tau < \delta \). We may choose \( \delta_1 \) with \( \tau < \delta_1 < \delta \) satisfying the requirement of the special case proved in the first paragraph to obtain that \( \ker \theta_{\delta_1 \tau} = K_\varepsilon(T) \). Now both \( \theta_{\tau \varepsilon} \) and \( \theta_{\delta_1 \tau} \) are isomorphisms, and \( \theta_{\delta \varepsilon} = \theta_{\delta \delta_1} \circ \theta_{\delta_1 \tau} \circ \theta_{\tau \varepsilon} \) by definition. Therefore,

\[ \ker \theta_{\delta \varepsilon} = \theta_{\tau \varepsilon}^{-1}(\ker \theta_{\delta_1 \tau}) = \theta_{\tau \varepsilon}^{-1}(K_\varepsilon(T)) = K_\varepsilon(T). \]

For the general case we have \( \varepsilon \leq \varepsilon_i < \cdots < \varepsilon_j < \delta := \varepsilon_{j+1} \) where \{\varepsilon_i,...,\varepsilon_j\} is the set of all homotopy critical values between \( \varepsilon \) and \( \delta \), which has at least two elements. For \( i \leq k \leq j \), let \( T_k \) be the set of all \( \varepsilon_k \)-triads in \( T \), so that \( T = \cup T_k \). By the previous case we have for all \( k \) that \( \ker \theta_{\varepsilon_{k+1} \varepsilon_k} = K_{\varepsilon_k}(T_k) \).

If \([\lambda]\) \( \in \ker \theta_{\delta \varepsilon} \), then \( x := \theta_{\varepsilon_j}(\lambda) \) \( \in \ker \theta_{\varepsilon_{j+1} \varepsilon_j} = K_{\varepsilon_j}(T_j) \), so we may write \( x \) as a finite product \( \Pi_{\varepsilon_j}[\beta_{\varepsilon_j}] \) with each \( \beta_{\varepsilon_j} \) is an \( \varepsilon_j \)-lollichain made from an \( \varepsilon_j \)-refinement of some element of \( T_j \). Now let \( \lambda^1_{\varepsilon_j} \) be any \( \varepsilon \)-refinement of \( \beta_{\varepsilon_j} \). By definition, \([\lambda^1_{\varepsilon_j}]\) \( \in K(T) \) for all \( r \), and \( \theta_{\varepsilon_j}(\lambda^1_{\varepsilon_j}) = [\beta_{\varepsilon_j}] \).

Therefore we may write \( [\lambda]\) \( = [\lambda^1_{\varepsilon_j}] \) \( (\Pi_{\varepsilon_j}[\lambda^1_{\varepsilon_j}] \) for some \( [\lambda^1_{\varepsilon_j}] \) \( \in \ker \theta_{\varepsilon_j} \). Since \( [\lambda^1_{\varepsilon_j}] \) \( \in \ker \theta_{\varepsilon_j} \), we may repeat the same argument to write \( [\lambda^2_{\varepsilon_j}] \) as a product of some \( [\lambda^2_{\varepsilon_j}] \) \( \in \ker \theta_{\varepsilon_j-1} \), and a finite product of elements of \( K(T) \) (consisting
of $\varepsilon$-lollichains formed using $\varepsilon$-refinements of elements of $T_{j-1}$). After finitely many iterations of this argument we obtain that $[\lambda]_{\varepsilon} \in K(T)$.

It is a straightforward extension of a theorem of E. Cartan that in a compact semi-locally simply connected geodesic space, every path contains a shortest path in its fixed-endpoint homotopy class, and that path is a locally minimizing geodesic. Likewise, every path loop contains a shortest element in its free homotopy class, and this curve is a closed geodesic. (This is not true in general without semilocal simple connectivity.) Consequently, in such spaces the fundamental group is generated by homotopy classes of loops of the form $\alpha \ast c \ast \overline{\alpha}$, where $c$ is a closed geodesic that is shortest in its homotopy class, and $\alpha$ is a locally minimizing geodesic. Now in the case $\delta = \infty$ in Theorem 24, since we know from [16] that $\pi_{\varepsilon}(X)$ is finitely generated, we obtain that $\pi_{\varepsilon}(X)$ is generated by finitely many $\varepsilon$-lollichains. If $X$ is semilocally simply connected then for $\varepsilon$ small enough, $\pi_{\varepsilon}(X)$ is isomorphic to $\pi_1(X)$ via the map $\Lambda : \pi_1(X) \to \pi_{\varepsilon}(X)$ (c.f. [16] or [23]). By the discussion following Definition 23, applying $\Lambda^{-1}$ to a basis of $\varepsilon$-lollichains give a basis of lollipops for $\pi_1(X)$, and we may replace any lollipop by a geodesic lollipop, up to homotopy equivalence. We thus have shown:

**Theorem 25** Let $X$ be a compact geodesic space. Then $\pi_{\varepsilon}(X)$ is either trivial or is generated by a finite collection $[\lambda_1]_{\varepsilon}, \ldots, [\lambda_n]_{\varepsilon}$, where each $\lambda_i$ is an $\varepsilon$-refinement of a $\delta$-lollichain with $\delta \geq \varepsilon$. In particular, if $X$ is semilocally simply connected and not simply connected then $\pi_1(X)$ is generated by a finite collection of equivalence classes of geodesic lollipops.

Recalling that the notion of essential circle is stronger than just being a non-null, closed geodesic that is shortest in its homotopy class, we see that Theorem 25 is stronger than Cartan’s result even in the Riemannian case. In fact, Example 44 of [16] shows that such closed geodesics need not be essential circles even when they are shortest in their homotopy class in a Riemannian manifold.

We conjecture that there is always a collection of $\delta$-lollichains that gives a generating set of $\pi_{\varepsilon}(X)$ having minimal cardinality.

**Proposition 26** Let $X$ be a compact geodesic space, $0 < \delta < \varepsilon$, and $T$ be a collection of essential $\tau$-triads such that $\tau \geq \varepsilon$ for each element of $T$. Let $S = T \cup T'$, where $T'$ consists of one representative of each essential $\tau$-triad with $\delta \leq \tau < \varepsilon$. Then

1. The covering map $\phi_T^X : X_T^X \to X$ is isometrically equivalent to $\phi_S^X : X_S^X \to X$ and

2. $\pi_{\varepsilon}(X)$ is isomorphic to $\pi_S^X(X)$.

**Proof.** We will use Proposition 13. First observe that $K_\delta(T')$, as a normal subgroup of $\pi_1(X)$, is a normal subgroup of $K_\delta(S)$. By Theorem 24, $\ker \theta_{\varepsilon, \delta} = K_\delta(T')$ and therefore we may identify the action of $\pi_{\varepsilon}(X)$ on $X_{\varepsilon}$ with the action of $K_\delta(T)/K_\delta(T')$ on $X_{\delta}/K_\delta(T') = X_{\varepsilon}$. In order to apply Proposition 13 and
finish the proof of the first part, we need to show that $\theta_{\varepsilon\delta}(K_\delta(S)) = K_\varepsilon(T)$. Since $T \subset S$, $K_\varepsilon(T) = \theta_{\varepsilon\delta}(K_\delta(T)) \subset \theta_{\varepsilon\delta}(K_\delta(S))$. On the other hand, let $[\lambda]_\varepsilon \in \theta_{\varepsilon\delta}(K_\delta(S))$. By definition, $[\lambda]_\varepsilon = [\lambda_1]_\varepsilon \cdots [\lambda_k]_\varepsilon$, where each $\lambda_i = \alpha_i \ast \beta_i \ast \lambda_i$, $\alpha_i$ is a $\delta$-chain and each $\beta_i$ is either in $T$ or $T'$. But if $\beta_i \in T'$ then $\beta_i$ is $\varepsilon$-null, so $[\lambda_i]_\varepsilon$ is trivial and we may therefore eliminate $[\lambda_i]_\varepsilon$ from the product. The remaining terms are all in $K_\varepsilon(T)$.

For the second part, note that we have shown both $\ker \theta_{\varepsilon\delta} = K_\delta(T') \subset K_\delta(S)$ and $\ker \theta_{\varepsilon\delta}(K_\delta(S)) = K_\varepsilon(T)$. Therefore, from a basic theorem in algebra we may conclude that $\pi^T_\varepsilon(X) = \pi_\varepsilon(X)/K_\varepsilon(T)$ is isomorphic to $\pi_\varepsilon(X)/K_\delta(S) = \pi^S_\varepsilon(X)$.

4 Gromov-Hausdorff Convergence

**Definition 27** Suppose $f : X \to Y$ is a function between metric spaces and $\sigma$ is a first degree polynomial with non-negative coefficients. We say that $f$ is a $\sigma$-isometry if for all $x, y \in X$, $z \in Y$,

1. $|d(x, y) - d(f(x), f(y))| \leq \sigma(d(x, y))$ and
2. $d(z, f(w)) \leq \sigma(0)$ for some $w \in X$.

We refer to the first condition as “distortion at most $\sigma$”.

If $\sigma = 0$ then a $\sigma$-isometry is an isometry, and if $\sigma = \varepsilon > 0$ is constant then Definition 27 agrees with the notion of an $\varepsilon$-isometry given in [4]. If $X$ and $Y$ are compact of diameter at most $R$, then any $\sigma$-isometry is a $\sigma(R)$-isometry. In fact, if $X$ and $Y$ are compact, $\sigma$ is constant, and $d$ denotes the Gromov-Hausdorff distance, then $d(X, Y) < 2\sigma$ if there is a $\sigma$-isometry $f : X \to Y$, and such a $\sigma$-isometry exists if $d(X, Y) < \frac{\varepsilon}{2}$ (Corollary 7.3.28, [4]). In other words, for purposes involving convergence of compact spaces we might as well use constant functions $\sigma$. However, our extended definition is needed to study the induced mapping $f_\# : X_\delta \to Y_\varepsilon$ since $X_\sigma$ and $Y_\varepsilon$ are not generally compact even when $X$ and $Y$ are.

**Remark 28** Recall that a quasi-isometry (c.f. [4]) is a map $f : X \to Y$ such that $f(X)$ is a $D$-net in $Y$ for some $D > 0$ (i.e. for every $y \in Y$ we have $d(y, f(x)) < D$ for some $x \in X$) and $\frac{1}{D}d(x, y) - C \leq d(f(x), f(y)) \leq \lambda d(x, y) + C$ for all $x, y$ and some constants $\lambda \geq 1$, $C > 0$. If $\sigma(x) = mx + b$ then a $\sigma$-isometry is a quasi-isometry with $\lambda := \frac{1+m}{1-m}$ and $C = b$, with $\lambda \to 1$ as $m \to 0$. However, it is simpler for our purposes to use Definition 27.

**Proposition 29** Suppose that $(X_i, x_i)$, $(X, x)$ are proper geodesic spaces and $f_i : X_i \to X$ is a basepoint preserving $\sigma_i$-isometry for all $i$, where $\sigma_i$ is a first degree polynomial with $\sigma_i \to 0$ pointwise. Then $(X_i, x_i)$ is pointed Gromov-Hausdorff convergent to $(X, x)$.
Let $f : X \to Y$ be a function between metric spaces. We will extend the notion of a $\sigma$-isometry $f$ in $X$. The statement is not the most general possible for example it is possible to consider non-constant $\sigma$ and the first part of the statement is not the most general possible because the properties of a $\sigma$-isometry $f$ in $X$ are well-defined and satisfies the properties of a $\sigma$-isometry $f$ in $X$. Moreover, whenever the first concatenation is defined and $\sigma < \epsilon$, let $x = \delta(\sigma)$. Then, we can use the distortion of $\sigma$, to this ball to improve this estimate and complete the proof.

\[ f_{\#}(a \ast \beta) = f_{\#}(a) \ast f_{\#}(\beta) \]

\[ f((a \ast \beta) - x) = f((a) - x) \ast f((\beta) - x) \]

[17]

Proof. It suffices to show that for any constants $0 < \sigma < 1$, there is an $\alpha$-isometry $g : (B_r(x) \to B_r(y))$ for all large $r$. For large $r$, $\alpha(R) < \frac{\sigma}{4}$ and the restriction of $f$ to $B_r(x)$, $B_r(y)$ has distortion less than $\frac{\sigma}{4}$. Otherwise, since $X$ is geodesic, there is some $w$ in $B_r(x)$ such that $d(f(w), w) < \frac{\sigma}{4}$. In particular, if $y \in B_r(x)$, then $d(f(y), y) < \frac{\sigma}{4}$. By the triangle inequality, the distortion of $g$, $g'(w) = f(w)$, is at most $\frac{\sigma}{4}$. To finish, we need only check condition 3 of the definition of $\alpha$-isometry for the constant $\sigma$. If $g'(w) = f(w) + \delta(w)$, then by our choice of $\delta$, we have $m_n, b < \frac{\sigma}{4}$. Let $z \in B_r(x)$ such that $|z - w| < \frac{\sigma}{4}$. Since $g'(w) = f(w)$, we may find $w' \in X$ such that $d(w', w) < \frac{\sigma}{4}$. By the triangle inequality, $d(f(w), w') < \frac{\sigma}{4}$. Again, by the triangle inequality, $d(f(w'), w) < \frac{\sigma}{4}$.

\[ (w, f(w')) = \frac{\sigma}{4} \leq \frac{\sigma}{4} \leq \frac{\sigma}{4} \]

Then, $d(w, f(w')) < \frac{\sigma}{4}$.
Proposition 30 Let $X,Y$ be metric spaces, $0 < \frac{\omega}{2} < \omega < \delta < \varepsilon$. Suppose that $f : X \to Y$ is basepoint preserving $\sigma$-isometry for some constant $\sigma$ with $0 \leq \sigma < \min \{ \varepsilon, \frac{\delta - \omega}{2} \}$. Then the induced map $f_\# : X_\varepsilon \to Y_\varepsilon$ is defined, and the following hold.

1. For any $\delta$-chain $\alpha$ in $X$,
   \[ |f_\#([\alpha]_\delta)| \leq ||\alpha||_\delta + \sigma \left( \frac{4||\alpha||_\delta}{\varepsilon} + 1 \right). \]

2. For any $\omega$-chain $\alpha$ in $X$,
   \[ ||f(\alpha)||_{\omega + \sigma} \geq ||\alpha||_\omega - \sigma \left( \frac{4}{\varepsilon} + \frac{16\sigma}{\varepsilon^2} \right) |||\omega|| - \sigma \left( \frac{4\sigma}{\varepsilon} + 1 \right). \]

3. If $\beta$ is an $(\omega - 3\sigma)$-chain in $Y$ starting at $*$ then there exists some $\omega$-chain $\alpha$ in $X$ starting at $*$ such $d(f_\#([\alpha]_\delta), [\beta]_\varepsilon) < \sigma$ in $Y_\varepsilon$.

**Proof.** That the induced map is defined follows from $\delta + \sigma < \varepsilon$. According to Lemma 5 up to $\delta$-homotopy and without increasing the length of $\alpha$, we may assume that $\alpha = \{ x_0, \ldots, x_n \}$ where $n = \left\lceil \frac{2L(\alpha)}{\delta} + 1 \right\rceil$. We have

\[ L(f(\alpha)) \leq L(\alpha) + n\sigma = L(\alpha) + \left( \frac{2L(\alpha)}{\delta} + 1 \right) \sigma. \]

By definition, $f(\alpha) \in f_\#([\alpha]_\delta)$, therefore $|f_\#([\alpha]_\delta)| \leq L(f(\alpha))$. The proof of part 1 follows by taking the infimum of the right side and using $\delta > \frac{\omega}{2}$.

For the second part, fix $\tau > 0$ and let $\alpha' := \{ x_0, \ldots, x_m \}$ be an $\omega$-chain such that $[\alpha']_\omega = [\alpha]_\omega$ and $L(\alpha') \leq ||\alpha||_\omega + \tau$. By Lemma 5 we may suppose that $m = \left\lceil \frac{2L(\alpha')}{\delta} + 1 \right\rceil$. Then $\beta := f(\alpha') = \{ f(x_0), \ldots, f(x_m) \}$ is an $(\omega + \sigma)$-chain of length at most $K = L(\alpha') + m\sigma$. Let $\eta = \langle \eta_0, \ldots, \eta_n = \beta' \rangle$ be an $(\omega + \sigma)$-homotopy, where $L(\beta') \leq L(\beta)$; again by Lemma 5 we may assume that $\nu(\beta') = m' := \left\lceil \frac{2K}{\omega + \sigma} + 1 \right\rceil$. Denote by $y_{ij}$ the $i^{th}$ point of $\eta_j$; note that the points $y_{ij}$ are not necessarily distinct! Nonetheless, we may iteratively choose for each $y_{ij}$ a point $x_{ij} \in X$ such that $d(f(x_{ij}), y_{ij}) < \sigma$ and the following are true: $x_{i0} = x_i$ (which is possible since $\eta_0 = \beta = f(\alpha)$), and if $y_{ij} = y_{ab}$ then $x_{ij} = x_{ab}$. For any $x_{ij}, x_{ab}$,

\[ d(x_{ij}, x_{ab}) \leq d(f(x_{ij}), f(x_{ab})) + \sigma \leq d(y_{ij}, y_{ab}) + 3\sigma \]

and therefore if we let $\eta_i'$ denote the chain in $X$ having $x_{ij}$ as its $i^{th}$ point, $\eta' := \langle \eta'_0, \ldots, \eta'_n \rangle$ is an $(\omega + 4\sigma)$-homotopy between $\alpha'$ and a chain $\alpha''$ of length at most $L(\beta') + m'\sigma$. Since $\omega + 4\sigma < \delta$, $\eta'$ is in fact a $\delta$-homotopy. That is,

\[ ||\alpha||_\delta \leq L(\beta') + \sigma \left( \frac{2K}{\omega + \sigma} + 1 \right) < L(\beta') + \sigma \left( \frac{4K}{\varepsilon} + 1 \right) \]
non-negative coefficients such that 

\[ p(x) = \sum_{i=0}^{n} a_i x^i \]

and since \( \Delta(\beta, \tau) \leq \sigma + |\beta| + |\tau| \), the following property holds. Let \( X, Y \) be geodesic spaces such that \( \beta \) is an injection and \( \omega \) is a \( \sigma \)-isometry for some positive constant \( \omega \). Taking the infimum over all \( \omega \) gives us

\[ L(\beta') \geq L(\alpha) - \sigma \left( \frac{4}{\varepsilon} + 1 \right) \left( |\beta| + |\alpha| \right) + \sigma \left( \frac{4}{\varepsilon} + 1 \right) \].

Next, let \( \beta'' := \{ z_0, \ldots, z_{n-1}, z_n, y_n \} \) and \( \beta''' := \{ z_0, \ldots, y_n, y_n \} \); note that \( |\beta'''| = |\beta| \). Since \( \beta''' \) is an \( (\omega - 3\sigma) \)-chain, \( E(\beta''') > \sigma - \omega - 3\sigma > 2\sigma \), and since \( \Delta(\beta''', \beta'') < \sigma \), we may apply Proposition 4 to conclude that \( |\beta'''| = |\beta'\| = |\beta| \). Finally:

\[ d([\beta']_\varepsilon, [\beta''']_\varepsilon) = d([\beta']_\varepsilon, [\beta''']_\varepsilon) = |[\beta']_\varepsilon, [\beta''']_\varepsilon| = |z_n, y_n| \leq d(z_n, y_n) < \sigma \]

\[ \blacksquare \]

**Theorem 31** For every \( \varepsilon, \sigma > 0 \) there is a first degree polynomial \( p(\sigma, \varepsilon) \) with non-negative coefficients such that \( p \to 0 \) as \( \sigma \to 0 \) (with \( \varepsilon \) fixed) and the following property holds. Let \( X, Y \) be geodesic spaces such that \( \phi_{\varepsilon\omega} : Y_{\omega} \to Y_{\varepsilon} \) is an injection and \( \omega < \delta < \varepsilon \). If \( f : X \to Y \) is a basepoint preserving \( \sigma \)-isometry for some positive constant \( \sigma < \min \{ \varepsilon - \delta, \frac{\delta}{\omega} \} \) then \( f_{\#} : X_{\delta} \to Y_{\varepsilon} \) is a \( p(\sigma, \varepsilon) \)-isometry.
\textbf{Proof.} Since $4\sigma < \delta - \omega_0$, we can choose $\omega$ such that $\omega_0 < \omega < \delta - 4\sigma$. Now we have the remaining two conditions, $\omega < \delta$ and $\sigma < \frac{\delta - \omega}{4}$, that are needed to apply Proposition 30. Note that since $Y$ is geodesic and hence all maps $\phi_{ab}$ are surjective, $\phi_{c\omega_0}$ is an isometry. But then $\phi_{c\omega}$ is also an isometry. Since $X$ is geodesic, we may always refine a $\delta$-chain to an $\omega$-chain of the same length, which means that on the right side of the inequality in the second part of Proposition 30 we may replace $|\alpha|_\omega$ by $|\alpha|_\delta$. Also since $\phi_{c\omega}$ is an isometry, $|f(\alpha)|_{\omega + \sigma} = |f(\alpha)|_\epsilon = |f_\#([\alpha]_\delta)|$. That is, we have

$$|\alpha|_\delta - |f_\#([\alpha]_\delta)| \leq \sigma \left( \frac{4}{\epsilon} + \frac{16\sigma}{\epsilon^2} \right) (|\alpha|_\delta) + \sigma \left( \frac{4\sigma}{\epsilon} + 1 \right).$$

The first part of Proposition 30 gives us:

$$|f_\#([\alpha]_\delta)| - |\alpha|_\delta \leq \frac{4\sigma}{\epsilon} |\alpha|_\delta + \sigma.$$

Let $m(\sigma, \epsilon) := \sigma \left( \frac{4}{\epsilon} + \frac{16\sigma}{\epsilon^2} \right) > \frac{4\sigma}{\epsilon}$, $b(\sigma, \epsilon) := \sigma \left( \frac{4\sigma}{\epsilon} + 1 \right) > \sigma$ and $p(\sigma, \epsilon)(t) = m(\sigma, \epsilon)t + b(\sigma, \epsilon)$. The coefficients of $p$ then have the desired property. Since

$$d(f_\#([\alpha]_\delta), f_\#([\beta]_\delta)) = d([f(\alpha)]_\epsilon, [f(\beta)]_\epsilon) = \left| \left[ f(\alpha) \right] \ast f(\beta) \right|_\epsilon = \left| \left[ f(\alpha) \ast f(\beta) \right] \right|_\epsilon,$$

and $d([\alpha]_\delta, [\beta]_\delta) = |[\alpha]_\delta|$ we see that $f_\#$ has distortion at most $p(\sigma, \epsilon)$.

Finally, let $[\beta]_\epsilon \in Y$; since $Y$ is geodesic (and all maps $\phi_{ab}$ are surjective) there is some $(\omega - 3\sigma)$-chain $\beta'$ such that $[\beta]_\epsilon = [\beta']_\epsilon$. The proof is now finished by the third part of Proposition 30. \hfill \blacksquare

We need an equivariant version of pointed Gromov-Hausdorff convergence. Origins of such an idea may be found in [11], [12], and something like this was used, for example, in [10]. Let $f : X \to Y$ be a function between metric spaces and suppose there are groups $H$ and $K$ of isometries on $X$ and $Y$, respectively, with a homomorphism $\psi : H \to K$. As usual, we say $f$ is equivariant (with respect to $\psi$) if for all $h \in H$ and $x \in X$, $f(h(x)) = \psi(h)(f(x))$. Then there is a well-defined induced mapping $f_\#: X/H \to Y/K$ defined by $f_\#(Hx) = Kf(x)$, where $Hx := \{h(x) : h \in H\}$ is the orbit of $x$. We will take the quotient pseudo-metric on $X/H$ and $Y/K$:

$$d(Hx, Hy) = \inf \{ d(k(x), h(y)) : h, k \in H \} = \inf \{ d(x, h(y)) : h \in H \}.$$

When the orbits of the action are closed sets, $d$ is a \textit{bona fide} metric.

In the next theorem note that some $\omega_0 > \epsilon$ to satisfy the hypothesis always exists since the homotopy critical values are discrete. The requirement that $\frac{\epsilon}{2} < \omega_0$ isn’t firm, but it simplifies the calculation. All that really matters is that the statement is true for every $\delta < \epsilon$ that is sufficiently close to $\epsilon$.

\textbf{Theorem 32} Let $\{X_i\}$ be a collection of compact geodesic spaces such that for all $i$ there is a basepoint preserving $\sigma_i$-isometry $f_i : X_i \to X$ for some sequence of constants $\sigma_i \to 0$. Let $\epsilon > 0$, suppose that $\phi_{c\omega_0} : X_{\omega_0} \to X_\epsilon$ is injective and let $\frac{\epsilon}{2} < \omega_0 < \delta < \epsilon$. Then for all large $i$, the following hold.
1. \((f_i)_\# : (X_i)_\delta \rightarrow X_\varepsilon\) is a \(p(\sigma_i, \varepsilon)\)-isometry, and in particular, \((X_i)_\delta\) is Gromov-Hausdorff pointed convergent to \(X_\varepsilon\).

2. The restriction of \((f_i)_\#\) to \(\pi_\delta(X_i)\) - denoted hereafter by \((f_i)_\#\) - is an isomorphism onto \(\pi_\varepsilon(X)\).

3. \((f_i)_\#\) is equivariant with respect to \((f_i)_\#\).

**Proof.** The first part is an immediate consequence of Theorem 31. One need only observe that \(\sigma_i < \min \{\varepsilon - \delta, \frac{\varepsilon - \delta}{m_i}\}\) for all large \(i\). For the next two parts, note that since \(f_i\) is basepoint preserving, \((f_i)_\#\) does map into \(\pi_\varepsilon(X)\). That \((f_i)_\#\) is a homomorphism, and that \((f_i)_\#\) is equivariant, both follow from Equation (2), and it remains to be shown that \((f_i)_\#\) is an isomorphism for large \(i\). To do this we will add a few more conditions that are satisfied for all large \(i\). We first require \(\sigma_i < \frac{\varepsilon}{6}\). Next note that if we let \(p(\sigma_i, \varepsilon)(t) := m_i t + b_i\), then \(m_i, b_i \rightarrow 0\).

If \(d((f_i)_\#([\alpha]_\delta), (f_i)_\#([\beta]_\delta)) = D\) then \(d([\alpha]_\delta, [\beta]_\delta) \leq \frac{D}{1 - m_i}\). In particular, we may conclude the following for all large \(i\):

\[
\text{If } d((f_i)_\#([\alpha]_\delta), (f_i)_\#([\beta]_\delta)) < \frac{\varepsilon}{3}, \text{ then } d([\alpha]_\delta, [\beta]_\delta) < \frac{\varepsilon}{2}. \tag{3}
\]

For large enough \(i\) the following also hold:

\[
\text{If } d([\beta]_\varepsilon \in X_\varepsilon, \text{ then there is some } [\alpha]_\delta \text{ such that } d((f_i)_\#([\alpha]_\delta), [\beta]_\varepsilon) < \frac{\varepsilon}{6}. \tag{4}
\]

\[
\text{If } d([\alpha]_\delta, [\beta]_\delta) < \frac{\varepsilon}{3}, \text{ then } d((f_i)_\#([\alpha]_\delta), (f_i)_\#([\beta]_\delta)) < \frac{\varepsilon}{2}. \tag{5}
\]

Suppose that \([\lambda]_\delta \in \ker(f_i)_\#.\) Then \(d((f_i)_\#([\lambda]_\delta), [*[\varepsilon]) = 0 < \frac{\varepsilon}{5}\) and by (4), \(d([\lambda]_\delta, [*[\varepsilon]) < \frac{\varepsilon}{5}\). But \(\phi_\delta\) is injective on \(B([*], \delta)\) and since \(\lambda\) is a loop, \([\lambda]_\delta = [*[\varepsilon].\)

Let \([\lambda]_\varepsilon \in \pi_\varepsilon(X).\) By (4) there is \([\alpha]_\delta\) such that \(d((f_i)_\#([\alpha]_\delta), [\lambda]_\varepsilon) < \frac{\varepsilon}{5}.\)

Letting \(\alpha := \{*[x_0, \ldots, x_n]\}\) we have

\[
d(x_n, *[x_0, \ldots, x_n]) \leq d(f(x_n), *[x_0, \ldots, x_n]) + \sigma_i \leq d((f_i)_\#([\alpha]_\delta), [\lambda]_\varepsilon) + \sigma_i < \frac{\varepsilon}{3} < \delta.
\]

Then \(\lambda' := \{*[x_0, \ldots, x_n, x]\}\) is a \(\delta\)-loop with \(d([\lambda']_\delta, [\alpha]_\delta) \leq d(x_n, *[x_0, \ldots, x_n]) \leq \frac{\varepsilon}{5}.\)

Therefore by (4) \(d((f_i)_\#([\alpha]_\delta), (f_i)_\#([\lambda']_\delta)) < \frac{\varepsilon}{5}.\) The triangle inequality now shows \(d([\lambda]_\varepsilon, (f_i)_\#([\lambda']_\delta)) < \frac{\varepsilon}{5} < \varepsilon.\) But once again, the injectivity of \(\phi_\varepsilon\) on open \(\varepsilon\)-balls implies that \([\lambda]_\varepsilon = (f_i)_\#([\lambda']_\delta) = (f_i)_\#([\lambda]_\delta)\), finishing the proof of surjectivity. \(\blacksquare\)

**Proposition 33** Suppose that \(X\) and \(Y\) are metric spaces with groups \(H, K\) acting on \(X, Y\), respectively, by isometries with closed orbits and \(\phi : H \rightarrow K\) is an epimorphism. If \(f : X \rightarrow Y\) is a \(\sigma\)-isometry for some first degree polynomial \(\sigma\) and equivariant with respect to \(\phi\), then \(f_\pi : X/H \rightarrow Y/K\) is a \(\sigma\)-isometry.
Proof. For any \( x, y \in X \), we have \( D := d(f_\pi(Hx), f_\pi(Hy)) = d(Kf(x), Kf(y)) \), from which we obtain
\[
D = \inf\{d(f(x), h(f(y))) : h \in K\} \\
= \inf\{d(f(x), \phi(g)(f(y))) : g \in H\} \\
= \inf\{d(f(x), f(g(y))) : g \in H\}
\]
(The first equality follows because \( \phi \) is surjective.) Now for any \( g \in H \),
\[
d(x, g(y)) - \sigma(d(x, g(y))) \leq d(f(x), f(g(y))) \leq d(x, g(y)) + \sigma(d(x, g(y))).
\]
Letting \( D' := d(Hx, Hy) = \inf\{d(x, g(y))\} \), the infimum of the right side is \( D' + \sigma(D') \). For the left side, for arbitrary \( \varepsilon > 0 \) we may suppose that
\[
D' \leq d(x, g(y)) < D' + \varepsilon
\]
which gives us
\[
d(x, g(y)) - \sigma(d(x, g(y))) > D' - \sigma(D' + \varepsilon)
\]
and therefore the infimum of the left side is \( D' - \sigma(D') \). That is, the distortion of \( f_\pi \) is at most \( \sigma \).

Finally, for any \( Ky \in Y/K \), there is some \( f(x) = z \in Y \) such that \( d(z, y) \leq \sigma(0) \). But then by definition \( Kz = f(Kx) \) and \( d(Kz, Ky) \leq d(z, y) \leq \sigma(0) \).

Next, we prove a special instance of Theorem 1 in the case where \( \varepsilon \) is not a critical value of the limit space. This result can be extracted from a combination of results established by Sormani and Wei in [17]. We include it here as a proposition in part because it has not yet been stated elsewhere in this specific form, and also because we provide an alternative proof using our discrete methods.

Proposition 34 Suppose that \( X_i \to X \), where each \( X_i \) is compact geodesic and let \( \varepsilon > 0 \). Then for any \( \delta < \varepsilon \) sufficiently close to \( \varepsilon \), \( (X_i)_\delta \to (X)_\varepsilon \) and \( \pi_\delta(X_i) \) is isomorphic to \( \pi_\varepsilon(X) \) for all large \( i \). In particular, if \( \varepsilon \) is not a homotopy critical value of \( X \) then \( (X_i)_\varepsilon \to X_\varepsilon \) and \( \pi_\varepsilon(X_i) \) is eventually isomorphic to \( \pi_\varepsilon(X) \).

Proof. Let \( f_i : X_i \to X \) be basepoint-preserving \( \sigma_i \)-isometries with constants \( \sigma_i \to 0 \). Since the homotopy critical values of \( X \) are discrete, we may choose \( \omega_0 \), and hence \( \delta \) with \( \frac{\varepsilon}{2} < \delta < \varepsilon \), so that the assumptions of Theorem 1 are satisfied. Eliminating finitely many terms if needed, we obtain the following properties for all \( i \) that we will use now and below: (1) \( (f_i)_\# : (X_i)_\delta \to X_\varepsilon \) is a \( p(\sigma_i, \varepsilon) \)-isometry. (2) The restriction \( (f_i)_\# \) of \( (f_i)_\# \) to \( \pi_\delta(X_i) \) is an isomorphism onto \( \pi_\varepsilon(X) \). (3) \( (f_i)_\# \) is equivariant with respect to \( (f_i)_\# \). The first statement of Theorem 1 is an immediate consequence of (1) and (2). If \( \varepsilon \) is not a homotopy critical value of \( X \) then there are some \( \varepsilon' > \varepsilon > \omega_0 \) such that \( \phi_{x',\omega_0} \) is an isometry. We may now apply the first part of the theorem using \( \varepsilon' \) to see
that \((X_i)_\varepsilon \to (X)_\varepsilon\) and \(\pi_\varepsilon(X_i)\) is eventually isomorphic to \(\pi_\varepsilon(X)\). But since 
\(\phi_{\varepsilon,\varepsilon} : X_\varepsilon \to X_{\varepsilon'}\) is an isometry, \(\theta_{\varepsilon,\varepsilon} : \pi_\varepsilon(X) \to \pi_{\varepsilon'}(X)\) is an isomorphism. This completes the proof.

**Proof of Theorem 1** For the proof of Theorem 1 we will continue with the same notation and \(\delta\) as chosen in the proof of Proposition 22. By eliminating terms if needed we may assume that \(\varepsilon_i > \delta\) for all \(i\). By Proposition 20 the covering space \(X^T_{\varepsilon_i}\) is isometrically equivalent to \(X^S_{\delta_i}\), and \(\pi^S_{\delta_i}(X)\) is isomorphic to \(\pi^S_{\delta_i}(X)\), where \(\delta_i\) is obtained by adding to \(T_i\) one representative for each essential \(t\)-circle with \(\varepsilon_i > t \geq \varepsilon\). Therefore we need only show that there is some collection \(T\) as in the statement of the theorem, and a subsequence such that \((X_{\delta_k})_{\delta_k} \to X^T_{\varepsilon}\) and \(\pi^S_{\delta_k}(X_{\delta_k})\) is eventually isomorphic to \(\pi^S_{\varepsilon}(X)\).

Let \(g_i : (X_{\delta_k})_{\delta_k} \to X_{\delta}\) denote \(\phi_{\varepsilon,\varepsilon}^{-1} \circ (f_i)\), which is a \(p(\varepsilon_i,\varepsilon)\)-isometry, and let \(h_i\) be the restriction of \(g_i\) to \(\pi_\varepsilon(X_{\delta})\). By Conditions (2) and (3) above, \(h_i\) is an isomorphism onto \(\pi_\delta(X)\) for all \(i\), and the maps \(g_i\) are invariant with respect to the isomorphisms \(h_i\). According to Theorem 11 of [16], the number of homotopy critical values \(\geq \varepsilon\) (counted with multiplicity) in the Gromov-Hausdorff precompact collection \(\{X_i\}\) has a uniform upper bound. Therefore by removing equivalent essential triads and taking a subsequence if necessary, we may assume that for some \(n\), \(\delta_i = \{T_{i1}, \ldots, T_{in}\}\) \((n \text{ could be } 0, \text{ in which case the following statements about } T_i \text{ are true for the empty set})\). Suppose that \(T_{ij} = \{x^0_{ij}, x^1_{ij}, x^2_{ij}\}\) is a \(\delta_{ij}\)-triad and \(T'_{ij}\) is an \(\varepsilon\)-refinement of \(\alpha_{T_{ij}}\). Since the diameters of the spaces \(X_i\) have a uniform upper bound, the number of points needed to refine each \(\alpha_{T_{ij}}\) has a uniform upper bound; by adding points if necessary we may assume that for some fixed \(w\), \(T'_{ij} = \{z^0_{ij} = x^0_{ij}, \ldots, z^w_{ij} = x^w_{ij}\}\) for all \(i,j\). The uniform upper bound on diameters also implies that for some fixed \(m\) we may find \(\varepsilon\)-chains \(\alpha_{ij} := \{*= y^0_{ij}, \ldots, y^m_{ij} = x^0_{ij}\}\) for all \(i,j\) (i.e. subdivide geodesics).

By choosing a subsequence yet again we may suppose that for all \(j,k\), \(f_i(z^j_{ij}) \to z^j_{ij}, f_i(x^k_{ij}) \to x^k_{ij}\) and \(f_i(y^k_{ij}) \to y^k_{ij}\). Let \(\alpha_{ij} := \{y^0_{ij}, \ldots, y^m_{ij}\}, \ T_j := \{x^0_{ij}, x^1_{ij}, x^2_{ij}\}, \ T'_j := \{z^0_{ij}, \ldots, z^w_{ij}\}, \ \text{and} \ \lambda_{ij} := \alpha_{ij} \ast T'_j \ast \overline{\alpha_{ij}}\). Since all the limiting chains have the property that each point is of distance at most \(\frac{\varepsilon}{2}\) from its successor, Proposition 13 implies that there is some \(N\) such that if \(i \geq N\), then

\[
[f_i(\lambda_{ij})]_\rho = [\lambda_{ij}]_\rho \text{ for all } j \text{ and any } \rho \geq \frac{\varepsilon}{2} > \frac{\varepsilon}{3},
\]

We assume \(i \geq N\) in what follows. One immediate consequence of (6) is that

\[
(f_i)_\theta(\lambda_{ij}) = (f_i)_\theta(\lambda_{ij}) = (f_i)_\theta(\lambda_{ij}) = [f_i(\lambda_{ij})]_\varepsilon = [\lambda_{ij}]_\varepsilon
\]

where \((f_i)_\theta\) is the restriction of \((f_i)_\#\) to \(\pi_\theta(X_i)\).

We can now argue that \(T_j\) is an essential \(\delta_{ij}\)-triad where \(\delta_{ij} := \lim \delta_{ij} \geq \varepsilon\). In fact, by continuity of the distance function, \(\delta_j = \lim \delta_{ij} \geq \varepsilon\) exists and \(T_j\) is a \(\delta_{ij}\)-triad. If \(T'_j\) were \(\varepsilon\)-null then \(f_i(\lambda_{ij})\) would be also \(\varepsilon\)-null, which by (7) means \([\lambda_{ij}]_\delta \in \ker (f_i)_\theta\). Since \((f_i)_\theta\) is an isomorphism, \(\lambda_{ij}\), and hence \(T'_j\), is \(\varepsilon\)-null. This contradicts that \(T_{ij}\) is \(\delta_{ij}\)-essential with \(\delta_{ij} > \varepsilon\). So \(T'_j\) is not \(\varepsilon\)-null, and \(\delta_j \geq \varepsilon\) by Corollary 7. If \(T_j\) were not essential then \(T'_j\) would be \(\delta_j\)-null. Then
for some $\delta' < \delta_j$ and close enough to $\delta_j$, $T'$ would also be $\delta'$-null. But then for large enough $i$, $\delta_{ij} > \delta'$ and (6) implies $T'_{ij}$ is $\delta'$-null, a contradiction to the fact that $T'_{ij}$ is not $\delta_{ij}$-null.

The next consequence of (7), and the characterization of $K_\delta(T)$ in Remark 24 is that $h_i(K_\delta(S_i)) = K_\delta(T)$, where $T := \{T_1, \ldots, T_n\}$. At this point we may assume the following, having chosen subsequences several times (but avoiding double subscripts for simplicity): the functions $g_i : (X_i)_\delta \to X_\delta$ are $p(\sigma, \varepsilon)$-isometries and the restrictions $k_i$ of $h_i$ to $K_\delta(S_i)$ are isometries onto $K_\delta(T)$ that are equivariant with respect to $g_i$. By Propositions 29 and 33, $(X_i)_\delta^S = (X_i)_\delta/K_\delta(S_i) \to X_\delta/K_\delta(T) = X_T^\varepsilon$. By the choice of $\delta$, $\phi_\delta$ is an isometry, so $X_\delta/K_\delta(T)$ is isometric to $X_\varepsilon/K_\varepsilon(T) = X_T^\varepsilon$ by Proposition 26.

Finally, recall that $h_i : \pi_\delta(X_i) \to \pi_\delta(X)$ is an isomorphism that takes $K_\delta(S_i)$ to $K_\delta(T)$. Combining this with the first part of Theorem 24 gives us that $\pi_\varepsilon^S(X_i) = \pi_\delta(X_i)/K_\delta(S_i)$ is isomorphic to $\pi_\delta(X)/K_\delta(T)$. But $\theta_{\varepsilon\delta}$ is an isomorphism from $\pi_\delta(X)$ to $\pi_\varepsilon(X)$ taking $K_\delta(T)$ to $K_\varepsilon(T)$, completing the proof of the theorem. ■

5 Some Open Questions and Problems

There are some questions that naturally arise from these results and might make interesting motivations for future work. For example, is it possible to characterize circle covers among all covers (this extends a question from [16] about characterizing $\varepsilon$-covers)? We know that not all covers of a compact geodesic space are circle covers. For example, the standard geodesic circle has only two circle covers: the trivial cover and the universal cover; other non-equivalent covers like the double cover cannot be circle covers. But at this point we are only able to identify when a cover is a circle cover in the following ways: (1) by exclusion when we know all the circle covers in a particular example, (2) if the cover is explicitly defined as a circle cover, or (3) if it is known to be so by Theorem 1. In this connection we note that the natural analog of Theorem 1 for covering maps in general is not true. In fact, for a circle cover $\pi$ of a compact geodesic space $X$, a lower bound $\varepsilon > 0$ on the size of the circles is equivalent to being covered by the $\varepsilon$-cover of $X$. Now let $\psi_k : C_k \to C_1$ be the $k$-fold cover of the geodesic circle $C_1$ by the geodesic circle of length $k$, which as we have mentioned is not a circle cover for $k > 1$. Each of these covers is covered by the universal covering space of $C_1$, which is the $\frac{1}{k}$-cover of $C_1$. It is also true that $C_k \to \mathbb{R}$ in the pointed Gromov-Hausdorff sense (in fact it is not hard to show in general precompactness of covering spaces covered by an $\varepsilon$-cover). However, the deck groups of these covering maps are $\mathbb{Z}_k$, which of course are all distinct.

Another question of interest is related to the fairly old question concerning the degree to which various spectra (Laplacian, length, covering) determine geometric properties in a compact geodesic space, including whether they must be isometric. Note that, up to a multiplied constant, the covering and homotopy critical spectra are contained in the length spectrum. While this was already observed by Sormani-Wei in [15], this is an immediate consequence of the pre-
viously mentioned fact that $X$ contains an essential $\varepsilon$-circle if and only if $\varepsilon$ is a homotopy critical value. The relationship between the length and the Laplace spectra was first considered in [3], [5], [9]. Already de Smit, Gornet, and Sutton have shown that the covering spectrum is not a spectral invariant (7, 8) by extending Sunada’s method [22] to determine when two manifolds have the same Laplace spectrum. However, essential circles allow one to enhance the notion of covering/homotopy critical spectrum in the following way. Given a compact geodesic space $X$, each circle covering of $X$ corresponds to a subgroup of $\pi_1(X)$, which we will call a circle group. Specifically, a circle group is the kernel of the natural map $\Lambda : \pi_1(X) \to \pi_\varepsilon(X)$ mentioned in Section 2, composed with the quotient map from $\pi_\varepsilon(X)$ to $\pi_\varepsilon(X)/K_\varepsilon(\mathcal{T})$ described above. The collection of all circle groups, partially ordered by inclusion, provides an algebraic refinement of the homotopy critical spectrum which in principle should say more about how similar two spaces are. That is, what can be said about compact geodesic spaces that not only share the same homotopy critical spectra, including multiplicity, but also share the same partially ordered collection of circle groups up to isomorphism?

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