Some results on \((p, k)\)-extension of the hypergeometric functions

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Abstract

In this study, we investigate a new natural extension of hypergeometric functions with the two parameters \(p\) and \(k\) which is so called \((p, k)\)-extended hypergeometric functions”. In particular, we introduce the \((p, k)\)-extended Gauss and Kummer (or confluent) hypergeometric functions. The basic properties of the \((p, k)\)-extended Gauss and Kummer hypergeometric functions, including convergence properties, integral and derivative formulas, contiguous function relations and differential equations. Since the latter functions contain many of the familiar special functions as sub-cases, this extension is enriches theory of \(k\)-special functions.

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1 Introduction

The motivation of the present work arises from:

(A) The special functions are very important tools in the different areas of mathematical physics, astronomy, applied statistics, applied sciences and engineering which have engaged many researchers, we refer the reader to [1–3].

(B) The hypergeometric function belongs to an important class of special functions of the mathematical physics and chemistry [3] with a large number of applications in different branches of the quantum mechanics, electromagnetic field theory, probability theory, analytic number theory, data analysis, etc (see, for instance, [1, 2, 4–6]). Indeed, various classes of extended special functions have been obtained as special cases for the hypergeometric functions.

Throughout this research, \(\mathbb{N} := \{1, 2, 3, \ldots\}\) denotes the set of natural numbers, \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\), \(\mathbb{Z}^- := \{-1, -2, -3, \ldots\}\) denotes the set of negative integers, \(\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}\), \(\mathbb{R}^+\) denotes the set of positive real numbers and \(\mathbb{C}\) denotes the set of complex numbers.

Traditionally, the hypergeometric function known as Gauss function is defined by

\[
W(v) = F(\delta_1, \delta_2, \delta_3; v) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n (\delta_2)_n}{(\delta_3)_n} \frac{v^n}{n!}, \quad v \in \mathbb{C},
\] (1.1)
which is absolutely and uniformly convergent if $|v| < 1$, divergent when $|v| > 1$, and is absolutely convergent when $|v| = 1$, if $\text{Re}(\delta_3 - \delta_1 - \delta_2) > 0$, where $\delta_1, \delta_2, \delta_3$ are complex parameters with $\delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and

$$(\delta_1)_n = \frac{\Gamma(\delta_1 + n)}{\Gamma(\delta_1)} = \begin{cases} 
\delta_1(\delta_1 + 1)\ldots(\delta_1 + n - 1), & n \in \mathbb{N}, \ \delta_1 \in \mathbb{C} \\
1, & n = 0; \ \delta_1 \in \mathbb{C} \setminus \{0\},
\end{cases} \quad (1.2)$$

is the Pochhammer symbol (or the shifted factorial) and $\Gamma(.)$ is gamma function. The function in (1.1) satisfy the following differential equation

$$v(1-v)W'' + [\delta_3 - (\delta_1 + \delta_2 + 1)v]W' - \delta_1 \delta_2 W = 0. \quad (1.3)$$

Nowadays, numerous investigations, for example, in recent works of Srivastava et al. [7, 8], Jana et al. [9, 10], Agarwal et al. [11, 12], Fuli et al. [13] and Abdalla [14, 15] to introduce extensions and generalizations of the hypergeometric functions, defined by Euler type integrals, associated with properties and applications.

In particular, Diaz and Pariguan [16] introduced the k-analogue of gamma, beta and hypergeometric functions and proved a number of their properties. Since that period, many different results concerning the k-hypergeometric function and related functions have been considered by many researchers, for instance, Agarwal et al. [17], Mubeen et al. [18–20], Rahman et al. [21], Chinra et al. [22], Korkmaz-Duzgun and Erkus-Duman [23], Nisar et al. [24], Li and Dong [25], Yilmaz et al. [26] and Yilmazer and Ali [27].

Motivated by some of these aforesaid studies of the k-hypergeometric functions and related functions, we introduce the $(p, k)$-extended Gauss and Kummer hypergeometric functions and their properties. Relevant connections of some of the discussed results here with those presented in earlier references are outlined.

The manuscript is organized as follows. In Section 2, we list some basic definitions and terminologies that are needed in the paper. In Section 3, we introduce the $(p, k)$-extended Gauss and Kummer (or confluent) hypergeometric functions and discuss their regions of convergence. In Section 4, we obtain integral and differentiation formulas of the $(p, k)$-extended Gauss and Kummer hypergeometric functions. In addition, contiguous function relations and differential equations connecting these functions are established in Section 4. Finally, we point out outlook and observations in Section 5.

## 2 Preliminaries

In this section, we give some basic definitions and terminologies which are used further in this manuscript.

**Definition 2.1.** [16, 26] For $k \in \mathbb{R}^+$, the k-gamma function $\Gamma^k(u)$ is defined by

$$\Gamma^k(u) = \int_0^\infty y^{u-1}e^{-\frac{y^k}{k}} dy, \quad (2.1)$$
where \( u \in \mathbb{C} \setminus k\mathbb{Z}^- \). We note that \( \Gamma^k(u) \to \Gamma(u) \), for \( k \to 1 \), where \( \Gamma(u) \) is the classical Euler’s gamma function and \((u)_{m,k}\) is the k-Pochhammer symbol given in the form

\[
(u)_{m,k} = \frac{\Gamma^k(u + mk)}{\Gamma^k(u)} = \begin{cases} 
  u(u + k) \cdots (u + (m - 1)k), & m \in \mathbb{N}, \ u \in \mathbb{C} \\
  1, & m = 0, k \in \mathbb{R}^+, u \in \mathbb{C} \setminus \{0\},
\end{cases}
\]

the relation between the \( \Gamma^k(u) \) and the gamma function \( \Gamma(u) \) follows easily that

\[
\Gamma^k(u) = k^{\frac{1}{k} - 1} \Gamma\left(\frac{u}{k}\right).
\]

**Definition 2.2.** [16, 26] For \( u, v \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \), the k-beta function \( B_k(u, v) \) is defined by

\[
B_k(u, v) = \frac{1}{k} \int_0^1 y^{\frac{u}{k} - 1} (1 - y)^{\frac{v}{k} - 1} dy = \frac{\Gamma^k(u) \Gamma^k(v)}{\Gamma^k(u + v)},
\]

where \( \text{Re}(u) > 0 \) and \( \text{Re}(v) > 0 \).

Clearly, the case \( k = 1 \) in (2.3) reduces to the known beta function \( B(u, v) \), and the relation between the k-beta function \( B_k(u, v) \) and the original beta function \( B(u, v) \) is

\[
B_k(u, v) = \frac{1}{k} B\left(\frac{u}{k}, \frac{v}{k}\right).
\]

**Definition 2.3.** [16, 26, 27] Let \( k \in \mathbb{R}^+ \) and \( s_1, s_2, \eta \in \mathbb{C} \) and \( s_3 \in \mathbb{C} \setminus \mathbb{Z}_0^- \), then k-Gauss hypergeometric function is defined in

\[
U(\eta) = {}_2F_1^k\left[ s_1, s_2 \mid s_3 \mid \eta \right] = \sum_{m=0}^{\infty} \frac{(s_1)_{m,k} (s_2)_{m,k}}{(s_3)_{m,k} m!} \eta^m, \ |\eta| < \frac{1}{k},
\]

where \((s_1)_{m,k}\) is the k-Pochhammer symbol defined in (2.2). Obviously, if \( k = 1 \), equation (2.4) is reduced to (1.1).

**Proposition 2.1.** [16, 26] For any \( \delta \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \), the following identity holds

\[
_1F_0^k\left[ \delta \mid \eta \right] = \sum_{n=0}^{\infty} (\delta)_{n,k} \eta^n = (1 - k\eta)^{\frac{-\delta}{k}}, \ |\eta| < \frac{1}{k}.
\]

The k-hypergeometric differential equation of second order defined in [18, 25–27] by

\[
k\eta \ (1 - k\eta) U'' + [s_3 - (s_1 + s_2 + k)k\eta] U' - s_1 s_2 U = 0.
\]

Particular choices of the parameters \( s_1, s_2, s_3 \) and \( k \) in the linearly independent solutions of the differential equation (2.6) yield more than 24 special cases. Also, the k-hypergeometric function can be given an integral representation in the following result [20, 26]:

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**Theorem 2.1.** Assume that \( \eta, s_1, s_2, s_3 \in \mathbb{C} \) such that \( \text{Re}(s_3) > \text{Re}(s_2) > 0 \) and \( k \in \mathbb{R}^+ \), then the integral formula of the \( k \)-hypergeometric function is given by

\[
2F_1^k \left[ \begin{array}{c} s_1, s_2 \\ s_3 \end{array} ; \eta \right] = \frac{\Gamma^k(s_3)}{k \Gamma^k(s_2) \Gamma^k(s_3 - s_2)} \int_0^1 y^{s_2 - 1 \frac{k}{k}} (1 - y)^{s_3 - s_2 - 1 \frac{k}{k}} (1 - ky^{s_2})^{-\frac{s_2}{k}} \, dy. 
\tag{2.7}
\]

Furthermore, the \( k \)-Kummer (confluent) hypergeometric function \( 1F_1^k \) defined in [24] in the form

\[
1F_1^k \left[ \begin{array}{c} s_1 \\ s_2 \end{array} ; \eta \right] = \sum_{m=0}^{\infty} \frac{(s_1)_{m,k}}{(s_2)_{m,k}} \eta^m. 
\tag{2.8}
\]

**3 The \((p, k)\)-extended hypergeometric functions**

In this section, we introduce and discuss the \((p, k)\)-extended Gauss hypergeometric function \( \mathcal{W}(p, k; \xi) \) and \((p, k)\)-extended Kummer (or confluent) hypergeometric function \( \mathcal{Y}(p, k; \xi) \) as follows, respectively

\[
\mathcal{W}(p, k; \xi) = 2F_1^{(p, k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k} (\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \xi^m \frac{1}{(pm)!},
\tag{3.1}
\]

\[
\mathcal{Y}(p, k; \xi) = 1F_1^{(p, k)} \left[ \begin{array}{c} \zeta_1 \\ \zeta_3 \end{array} ; \xi \right] = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k}}{(\zeta_3)_{m,k}} \xi^m \frac{1}{(pm)!},
\tag{3.2}
\]

where \( k \in \mathbb{R}^+ \) and \( \zeta_1, \zeta_2, \xi \in \mathbb{C} \) and \( \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^- \), \( p \) is appositive integer and \((\zeta)_{m,k}\) is the \( k \)-Pochhammer symbol defined in (2.2).

**Remark 3.1.** Some important special cases of the \( \mathcal{W}(p, k; \xi) \) and the \( \mathcal{Y}(p, k; \xi) \) for some particular choice of the parameters \( p \) and \( k \) are enumerated below:

1. Putting \( p = 1 \), we produce the \( k \)-analogue of Gauss and Kummer hypergeometric functions are given in (2.4) and (2.8), respectively.

2. Setting \( k = 1 \), we obtain a \( p \)-extension of the Gauss and Kummer hypergeometric functions in the following forms, respectively

\[
\mathcal{W}(p; \xi) = 2F_1^{(p, 1)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k} (\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \xi^m \frac{1}{(pm)!},
\tag{3.3}
\]

\[
\mathcal{Y}(p; \xi) = 1F_1^{(p, 1)} \left[ \begin{array}{c} \zeta_1 \\ \zeta_3 \end{array} ; \xi \right] = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k}}{(\zeta_3)_{m,k}} \xi^m \frac{1}{(pm)!}.
\tag{3.4}
\]
3. Taking $k = 1$ and $p = 1$ in (3.1), we produce the standard Gauss hypergeometric function in (1.1).

4. When $k = 1$ and $p = 1$, (3.2) yields the following special case in (see, e.g., [1, 2]).

The following theorem shows that the convergence property of the series (3.1).

**Theorem 3.1.** For all $k \in \mathbb{R}^+$ and $p > 1$, then the $(p, k)$-extended Gauss hypergeometric function $\mathcal{W}(p, k; \xi)$ given by (3.1) is an entire function.

**Proof.** For this prove, we relabel and write (3.1) as

$$2F_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] = \sum_{m=0}^{\infty} U_m(\xi),$$

where

$$U_m(\xi) = \frac{(\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \frac{\xi^m}{(pm)!}.$$ (3.5)

By using the ratio test and according to the identity $(\zeta)_{m+1,k} = (\zeta + mk)(\zeta)_{m,k}$, we see that

$$\lim_{m \to \infty} \left| \frac{U_{m+1}(\xi)}{U_m(\xi)} \right| = \lim_{m \to \infty} \frac{(\zeta_1)_{m+1,k}(\zeta_2)_{m+1,k} \xi^{m+1}}{(p(m+1))!(\zeta_3)_{m+1,k}} \frac{(pm)!(\zeta_3)_{m,k}}{(\zeta_1)_{m,k}(\zeta_2)_{m,k} \xi^m} = \lim_{m \to \infty} \frac{(\zeta_1 + mk)(\zeta_2 + mk)(pm)! \xi}{(\zeta_3 + mk)(pm + p)(pm + p - 1)...(pm + 1)(pm)!} \frac{(\zeta_3 + mk)(pm + p)(pm + p - 1)...(pm + 1)}{(\zeta_1 + mk)(\zeta_2 + mk)} |\xi| = \lim_{m \to \infty} \frac{m^2 (\frac{\zeta_1}{m} + k)(\frac{\zeta_2}{m} + k)}{m^{p+1}(\frac{\zeta_3}{m} + k)(p + \frac{p}{m})(p + \frac{p-1}{m})...(p + \frac{1}{m})} |\xi| = 0.

Thus, the power series (3.1) is convergent for all $|\xi| < \infty$, under the hypothesis $p > 1$, $k \in \mathbb{R}^+$ and $\zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0$. Thus, yields our desired result.

The following result can be verified in a similar way.

**Theorem 3.2.** For all $k \in \mathbb{R}^+$ and $p > 1$, then the $(p, k)$-extended Kummer hypergeometric function $\mathcal{Y}(p, k; \xi)$ given by (3.2) is an entire function.
**Corollary 3.1.** For all $p > 1$, then the power series (3.3) and (3.4) are an entire function.

**Remark 3.2.** For $p = 1$ in Theorem 3.1, and Theorem 3.2, we get the convergence property of the $k$-Gauss hypergeometric function $\mathcal{W}(1, k; \xi)$ and the $k$-Kummer hypergeometric function $\mathcal{Y}(1, k; \xi)$, provided that $k \in \mathbb{R}^+$ and $\zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$ (see [16]).

**Remark 3.3.** For $p = 1$ in Corollary 3.1, we obtain the convergence property of the usual Gauss and Kummer hypergeometric series (see [1, 2]).

### 4 Integral representations and derivative formulae

#### 4.1 Integral representations

Following, we establish the following theorems in terms of the $k$-integral representations of the $(p, k)$-extended Gauss and Kummer hypergeometric functions.

**Theorem 4.1.** The following integral representation for $2\mathcal{G}_{1}^{(p,k)}$ in (3.1) holds true:

$$2\mathcal{G}_{1}^{(p,k)} \left[ \frac{\zeta_1; \zeta_2}{\zeta_3}; \xi \right] = \frac{\Gamma^k(\zeta_3)}{k \Gamma^k(\zeta_2) \Gamma^k(\zeta_3 - \zeta_2)} \int_0^1 t^{\frac{\zeta_3 - \zeta_2}{k} - 1} (1 - t)^{\frac{\zeta_1 - \zeta_2}{k} - 1} \frac{1}{\Gamma^0(\zeta_1 - t \xi)} dt. \quad (4.1)$$

$(\zeta_1, \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \text{Re}(\zeta_3) > \text{Re}(\zeta_2) > 0, \text{Re}(\zeta_1) > 0, k \in \mathbb{R}^+ \text{ and } p > 1).$

**Proof.** Considering the following elementary identity involving the $k$-Beta function $B_k(u, v)$:

$$\frac{(\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} = \frac{\Gamma_k(\zeta + mk)}{\Gamma_k(\zeta) \Gamma_k(\zeta + mk)} B_k(\zeta + mk, \zeta_3 - \zeta_2) = \frac{1}{\Gamma_k(\zeta_2) \Gamma_k(\zeta_3 - \zeta_2)} \frac{1}{k} \int_0^1 t^{\frac{\zeta_2 + m - 1}{k}} (1 - t)^{\frac{\zeta_3 - \zeta_2}{k} - 1} dt, \quad (\text{Re}(\zeta_3) > \text{Re}(\zeta_2))$$

in (3.1) and using the relation (3.4), we get the required integral formula (4.1). □

**Theorem 4.2.** The following integral representation for $2\mathcal{G}_{1}^{(p,k)}$ in (3.1) holds true:

$$2\mathcal{G}_{1}^{(p,k)} \left[ \frac{\zeta_1; \zeta_2}{\zeta_3}; \xi \right] = \frac{1}{\Gamma^k(\zeta_1)} \int_0^\infty t^{\zeta_1 - 1} e^{-\frac{t}{k}} 1\mathcal{G}_{1}^{(p,k)} \left[ \frac{\zeta_2}{\zeta_3}; t \xi \right] dt. \quad (4.2)$$

$(\zeta_1, \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \text{Re}(\zeta_3) > 0, p > 1, k \in \mathbb{R}^+ \text{ when } p = 1).$

**Proof.** Inserting the $k$-Pochhammer symbol $(a_1)_{n,k}$ from (2.2) in the definition (3.1) by its integral form given by (2.1) and from the relation (3.2), we thus obtain the desired result (4.2). □

By virtue of the same theorems, we give the following theorem:
**Theorem 4.3.** The following integral representations for \( \mathcal{D}_1^{(p,k)} \) in (3.2) hold true:

\[
\mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] = \frac{\Gamma^k(\zeta_3)}{k \Gamma^k(\zeta_3 - \zeta_2)} \int_0^1 t^{\frac{\zeta_2}{k} - 1} (1 - t)^{\frac{\zeta_3 - \zeta_2}{k} - 1} \mathcal{D}_0^{(p,k)} \left[ \begin{array}{c} - \\ - \end{array} ; t \xi \right] dt. 
\]

(\( \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \text{Re}(\zeta_3) > \text{Re}(\zeta_2) > 0, p > 1, k \in \mathbb{R}^+ \) when \( p = 1 \)).

\[
\mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] = \frac{1}{\Gamma^k(\zeta_2)} \int_0^\infty t^{\zeta_2 - 1} e^{-t^k} \mathcal{D}_0^{(p,k)} \left[ \begin{array}{c} - \\ - \end{array} ; t \zeta_3 \right] dt. 
\]

(\( \zeta_2 \in \mathbb{C}, \zeta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \text{Re}(\zeta_1) > 0, p > 1, k \in \mathbb{R}^+ \) when \( p = 1 \)).

**Remark 4.1.** The substitution \( p = 1 \) in (3.6) – (3.9) leads to the integral representations of the k-analogue of Gauss and Kummer hypergeometric functions (see [20, 24, 26]).

**Remark 4.2.** The substitution \( p = 1 \) in (4.1) – (4.4) leads to the integral formulas of the k-analogue of Gauss and Kummer hypergeometric functions (see [20, 24, 26]).

Also, the special cases of (3.6) – (3.9) when \( k = 1 \) and \( p = 1 \) are seen to yield the classical integral representations of the Gauss and Kummer hypergeometric functions (see, e.g., [1, 2, 6]).

### 4.2 Derivative formulae

**Theorem 4.4.** The following derivative formulas hold true:

\[
\frac{d^n}{d\xi^n} \left\{ 2\mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] \right\} = \frac{(\zeta_1)_{n,k}(\zeta_2)_{n,k}}{p^n (\zeta_3)_{n,k}} 2\mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 + nk, \zeta_2 + nk \\ \zeta_3 + nk \end{array} ; \xi \right] 
\]

and

\[
\frac{d^n}{d\xi^n} \left\{ \mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 \\ \zeta_3 \end{array} ; \xi \right] \right\} = \frac{(\zeta_1)_{n,k}}{p^n (\zeta_3)_{n,k}} \mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1 + nk \\ \zeta_3 + nk \end{array} ; \xi \right].
\]

\((k \in \mathbb{R}^+, p > 1, n \in \mathbb{N}_0)\).

**Proof.** The result (4.5) is obviously valid in the trivial case when \( n = 0 \). For \( n = 1 \), by the power series representation (3.1) of \( 2\mathcal{D}_1^{(p,k)} \), we see from (4.5) that

\[
\frac{d}{d\xi} \left\{ 2\mathcal{D}_1^{(p,k)} \left[ \begin{array}{c} \zeta_1, \zeta_2 \\ \zeta_3 \end{array} ; \xi \right] \right\} = \sum_{m=1}^\infty \frac{(\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3)_{m,k} p(pm - 1)!} \xi^{m-1}.
\]
Replacing the k-pochhammer symbols \((\zeta_1 + k)_{m,k}\) by the relation (2.2), we arrive at

\[
\frac{d}{d\xi} \left\{ 2\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1, \zeta_2}{\zeta_3}; \xi \right] \right\} = \frac{1}{p} \frac{\zeta_1 \zeta_2}{\zeta_3} 2\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1 + k, \zeta_2 + k}{\zeta_3 + k}; \xi \right].
\]

Therefore, the general result (4.5) can now be easily derived by using the principle of mathematical induction on \(n \in \mathbb{N}_0\).

A similar procedure yields the desired representation (4.6).

**Theorem 4.5.** The following derivative formulas hold true:

\[
\frac{d^n}{d\xi^n} \left\{ \xi^{\zeta_3-1} \, 2\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1, \zeta_2}{\zeta_3}; \xi \right] \right\} = \frac{\xi^{\zeta_3-n-1} \Gamma^k(\zeta_3)}{\Gamma^k(\zeta_3 - n)} \, 2\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1, \zeta_2}{\zeta_3 - n}; \xi \right].
\]

(4.7)

and

\[
\frac{d^n}{d\xi^n} \left\{ \xi^{\zeta_1-1} \, 1\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1}{\zeta_3}; \xi \right] \right\} = \frac{\xi^{\zeta_3-n-1} \Gamma^k(\zeta_3)}{\Gamma^k(\zeta_3 - n)} \, 1\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1}{\zeta_3 - n}; \xi \right].
\]

(4.8)

\((k \in \mathbb{R}^+, p > 1, n \in \mathbb{N}_0)\).

**Proof.** By using the series (3.1) in (4.7) and differentiating term by term under the sign of summation, we observe that

\[
\frac{d^n}{d\xi^n} \left\{ \xi^{\zeta_3-1} \, 2\, \psi^{(p,k)}_1 \left[ \frac{\zeta_1, \zeta_2}{\zeta_3}; \xi \right] \right\} = \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k} (\zeta_2)_{m,k}}{(\zeta_3)_{m,k}} \frac{d^n}{d\xi^n} \left\{ \xi^{kn+\zeta_3-1} \right\}
\]

\[
\quad = \Gamma^k(\zeta_3) \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k} (\zeta_2)_{m,k}}{\Gamma^k(\zeta_3 + mk - n) (pm)!} \xi^{km+\zeta_3-1} - \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k} (\zeta_2)_{m,k}}{\Gamma^k(\zeta_3 + nk - n) (pm)!} \xi^{km},
\]

which, in view of the series (3.1), yields the coveted formula (4.7).

Similarly, we can derive the derivative formula (4.8).
Remark 4.3. The special cases of (4.7) and (4.8) when $p = 1$ are easily seen to reduce to the known derivative formulas of the $k$-Gauss and Kummer hypergeometric functions

Remark 4.4. If we take $p = 1$ and $k = 1$ in the above mentioned theorems, we obtain the corresponding results for the classical hypergeometric functions $\,_{2}F_{1}^{(1,1)}$ and $\,_{1}F_{1}^{(1,1)}$ (cf. [6]).

5 Contiguous function relations and differential equations

The $k$-analogue of theta operator $k\Theta$ as given in [18, 19, 25], takes the form $k\Theta = k\frac{\xi}{d\xi}$. This operator has the particularly pleasant property that $k\Theta \xi^{m} = km\xi^{m}$, which makes it handy to be used on power series. In this section, relying on definition 2.1, we present some results concerning contiguous function relations and differential equations for the $(p, k)$-extended Gauss hypergeometric function $\,_{2}F_{1}^{(p,k)}$ and $(p, k)$-extended Kummer hypergeometric function $\,_{1}F_{1}^{(p,k)}$.

To realize that, we increase or decrease one and more of the parameters of the $(p, k)$-extended Gauss hypergeometric function, $W = W(p, k; \xi) = \,_{2}F_{1}^{(p,k)}\left[\frac{\zeta_{1}, \zeta_{2}}{\zeta_{3}}; \xi\right], \quad k \in \mathbb{R}^{+}, \quad p > 1,$

by $\pm k$, then the resultant function is said to be contiguous to $W(k; \xi)$. For simplicity, we use the following notations

$W(p, k; \zeta_{1} \pm) = \,_{2}F_{1}^{(p,k)}\left[\frac{\zeta_{1} \pm k, \zeta_{2}}{\zeta_{3}}; \xi\right].$

Now consider

$W(p, k; \zeta_{1}+) = \sum_{m=0}^{\infty} \frac{(\zeta_{1} + m)_{m,k}}{(\zeta_{3})_{m,k}} \frac{\xi^{m}}{(pm)!}$

$= \sum_{m=0}^{\infty} \frac{(\zeta_{1} + mk)}{\zeta_{1}} U_{m,k}, \quad k \in \mathbb{R}^{+}, \quad p > 1,$

where $\alpha_{1}(\alpha_{1} + k)_{n,k} = (\alpha_{1} + nk)(\alpha_{1})_{n,k}$ and $U_{m,k}(\xi)$ is defined in (3.5).

Similarly, we can write $W(p, k; \zeta_{1} -)$ as

$W(p, k; \zeta_{1} -) = \sum_{n=0}^{\infty} \frac{(\zeta_{1} - k)}{(\zeta_{1}(m - 1)k)} U_{m,k},$

where $(\zeta_{1}(m - 1)k)(\zeta_{1} - k)_{m,k} = (\zeta_{1} - k)(\zeta_{1})_{m,k}$. Similarly, for $W(p, k; \zeta_{2} \pm)$, and $W(p, k; \zeta_{2} \pm)$.

By the help of differential operator $k\Theta = k\frac{\eta}{d\eta}$, we get the following relations

$(k\Theta + \zeta_{1})W = \zeta_{1} W(p, k; \zeta_{1} +)$

$(k\Theta + \zeta_{2})W = \zeta_{2} W(p, k; \zeta_{2} +)$

$(k\Theta + \zeta_{3} - k)W = (\zeta_{1} - k)W(p, k; \zeta_{3} -)$. 


From the above relations, we can easily obtain the following results

\[(\zeta_1 - \zeta_2)\mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1+) - \zeta_2 \mathcal{W}(p, k; \zeta_2+),\]  \hspace{1cm} (5.1)

\[(\zeta_1 - \zeta_2)\mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1+, \zeta_2+) - \zeta_2 \mathcal{W}(p, k; \zeta_2+),\]  \hspace{1cm} (5.2)

\[(\zeta_1 - \zeta_3 + k)\mathcal{W}(p, k; \xi) = \zeta_1 \mathcal{W}(p, k; \zeta_1+) - (\zeta_3 - k) \mathcal{W}(p, k; \zeta_1 + 2k),\]  \hspace{1cm} (5.3)

\[(\zeta_2 - \zeta_3 + k)\mathcal{W}(p, k; \xi) = \zeta_2 \mathcal{W}(p, k; \zeta_2+) - (\zeta_3 - k) \mathcal{W}(p, k; \zeta_2 + 2k),\]  \hspace{1cm} (5.4)

**Remark 5.1.** Other contiguous function relations for the k-Gauss hypergeometric function may be derived from the relations in (5.1) to (5.4) and the same manner, other results can also be obtained.

**Remark 5.2.** We can easily obtain many known results in [19, 22] by setting the parameters in our main findings. Therefore, the obtained results here extend to that results.

**Remark 5.3.** It is easy to see that in (5.1) to (5.4), if we take \(k = 1\), we get hypergeometric contiguous function relations (see [6]).

Furthermore, the operator \(k\Theta = \frac{\partial}{\partial \xi}\), that is used in the derivation of the contiguous function relations, is also in deriving the differential equations satisfied by \(\mathcal{W}(p, k; \xi)\) and \(\mathcal{Y}(p, k; \xi)\) as follows:

\[
\left[\Theta\left(\Theta - \frac{1}{p}\right)\left(\Theta - \frac{2}{p}\right)\ldots\left(\Theta - \frac{p-1}{p}\right)\right] \mathcal{W}(p, k; \xi)
= \sum_{m=1}^{\infty} \left(\frac{m(m-\frac{1}{p})(m-\frac{2}{p})\ldots(m-\frac{p-1}{p})}{(\zeta_3)m,k (pm)!}\right) (\zeta_1)_{m,k}(\zeta_2)_{m,k} \xi^m
= \frac{\xi}{p^p} \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m+1,k}(\zeta_2)_{m+1,k}}{(pm)!} (\zeta_3)_{m+1,k} \xi^m,
\]

using the following identity \((a)_{n+1,k} = (a)_{n,k}(a + nk)\), we find that

\[
\left[\Theta\left(\Theta - \frac{1}{p}\right)\left(\Theta - \frac{2}{p}\right)\ldots\left(\Theta - \frac{p-1}{p}\right)\right] \mathcal{W}(p, k; \xi)
= \frac{\xi}{p^p} \sum_{m=0}^{\infty} \frac{(\zeta_1 + mk)(\zeta_2 + mk)}{(\zeta_3 + mk)} U_{m,k}(\xi)
= \frac{\xi}{p^p} \sum_{m=0}^{\infty} \left[ mk + (\zeta_1 + \zeta_2 - \zeta_3) + \frac{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)}{(\zeta_3 + mk)} \right] U_{m,k}(\xi)
= \frac{\xi}{p^p} \left[ k\Theta \mathcal{W}(p, k; \xi) + (\zeta_1 + \zeta_2 - \zeta_3) \mathcal{W}(p, k; \xi) + \frac{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)}{\zeta_3} \mathcal{W}(p, k; \zeta_3+)ight].
\]
Thus, we get the following differential equation

\[
\left\{ \Theta (\Theta - \frac{1}{p} ) (\Theta - \frac{2}{p} ) ... (\Theta - \frac{p-1}{p} ) \right\} \mathcal{W}(p, k; \xi) \\
- \frac{\xi}{p^p} \left\{ k \Theta \mathcal{W}(p, k; \xi) + (\zeta_1 + \zeta_2 - \zeta_3) \mathcal{W}(p, k; \xi) + \frac{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)}{\zeta_3} \mathcal{W}(p, k; \zeta_3+) \right\} = 0.
\]

Similarly, we can derive the following result of the \((p, k)\)-extended Kummer hypergeometric function \(\mathcal{Y}(p, k; \xi)\)

\[
\left\{ \Theta (\Theta - \frac{1}{p} ) (\Theta - \frac{2}{p} ) ... (\Theta - \frac{p-1}{p} ) \right\} \mathcal{Y}(p, k; \xi) - \frac{\xi(\zeta_1 - \zeta_3)}{p^p} \mathcal{Y}(k; \zeta_3+) \right] = 0.
\]

In addition, we consider

\[
\left\{ \Theta (\Theta - \frac{1}{p} ) (\Theta - \frac{2}{p} ) ... (\Theta - \frac{p-1}{p} ) \right\} \mathcal{W}(p, k; \xi) \\
= \frac{1}{p^p} \sum_{m=1}^{\infty} \frac{(km - \zeta_3 - k) \ (\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(pm - p)! \ (\zeta_3)_{m,k}} \xi^m
\]

Replacing \(m\) by \(m + 1\) and according to the identity \((\zeta_1)_{m+1,k} = (\zeta_1 + mk)(\zeta_1)_{m,k}\), we have

\[
\left\{ \Theta (\Theta - \frac{1}{p} ) (\Theta - \frac{2}{p} ) ... (\Theta - \frac{p-1}{p} ) \right\} \mathcal{W}(p, k; \xi) \\
= \frac{1}{p^p} \sum_{m=0}^{\infty} \frac{(km - \zeta_3) \ (\zeta_1)_{m+1,k}(\zeta_2)_{m+1,k}}{(mp)! \ (\zeta_3)_{m+1,k}} \xi^{m+1} \\
= \frac{\xi}{p^p} \sum_{m=0}^{\infty} \frac{(\zeta_1)_{m,k}(\zeta_1 + mk) \ (\zeta_2)_{m,k}(\zeta_2 + mk)}{(mp)! \ (\zeta_3)_{m,k}} \xi^m \\
= \left[ \frac{\xi}{p^p} \ (\zeta_1 + mk) \ (\zeta_2 + mk) \right] \mathcal{W}(p, k; \xi).
\]

We thus get the following differential equation

\[
\left[ \Theta (\Theta - \frac{1}{p} ) (\Theta - \frac{2}{p} ) ... (\Theta - \frac{p-1}{p} ) \left( k \Theta + \zeta_3 - k \right) \\
- \left\{ \frac{\xi}{p^p} \ (\zeta_1 + mk) \ (\zeta_2 + mk) \right\} \right] \mathcal{W}(p, k; \xi) = 0.
\]
Remark 5.4. For $p = 1$ in (5.5), it obviously reduces to the usual differential equation of the $k$-Gauss hypergeometric function in (2.6).

A similar procedure yields differential equation of the $(p, k)$-extended Gauss hypergeometric function $W(p, k, \xi)$, by using the operator $k \Theta$ in (3.2). We thus obtain differential equation of the $(p, k)$-extended Kummer hypergeometric function $Y(p, k, \xi)$ in the form

$$
\left[ \Theta \left( \Theta - \frac{1}{p} \right) \left( \Theta - \frac{2}{p} \right) \ldots \left( \Theta - \frac{p-1}{p} \right) \left( k \Theta + \frac{\zeta_3}{3} - k \right) - \left\{ \frac{\xi}{p^p} \left( \zeta_1 + mk \right) \right\} \right] Y(p, k; \xi) = 0. \quad (5.6)
$$

Remark 5.5. The special cases of (5.5) and (5.6) when $k = 1$ and $p = 1$ are seen to yield the classical differential equations of Gauss and kummer hypergeometric functions (see, for details, [6]).

6 Concluding remarks

Recently, many studies and extensions of the well-known special functions have been considered by various researchers.

In this paper, we obtained a new extension of the Gauss and Kummer hypergeometric functions, so-called $(p, k)$-extended Gauss and Kummer hypergeometric functions. Also, we given some of their main properties, namely the convergence properties, integral representations, differential formulas, contiguous function relations and differential equations.

We have spotted that by setting $p = 1$, the various outcomes presented in this article will reduce to some the corresponding outcomes derived earlier in [16, 18, 19, 24, 25]. Further, if we let $k = 1$, then we obtain several interesting new outcomes for the $p$-extended Gauss and Kummer hypergeometric functions. Finally, we have spotted that, if $p = 1$ and $k = 1$, then we obtain some known results for the usual Gauss and Kummer hypergeometric functions defined and established in [1, 2, 6]. Additional research and application on this topic is now under preparation and will be presented in forthcoming articles.

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