EXACT EFFECTIVE ACTION AND SPACETIME GEOMETRY IN GAUGED WZW MODELS †

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ABSTRACT

We present an effective quantum action for the gauged WZW model $G_{-k}/H_{-k}$. It is conjectured that it is valid to all orders of the central extension ($-k$) on the basis that it reproduces the exact spacetime geometry of the zero modes that was previously derived in the algebraic Hamiltonian formalism. Besides the metric and dilaton, the new results that follow from this approach include the exact axion field and the solution of the geodesics in the exact geometry. It is found that the axion field is generally non-zero at higher orders of $1/k$ even if it vanishes at large $k$. We work out the details in two specific coset models, one non-abelian, i.e. $SO(2,2)/SO(2,1)$ and one abelian, i.e $SL(2,\mathbb{R}) \otimes SO(1,1)^{d-2}/SO(1,1)$. The simplest case $SL(2,\mathbb{R})/\mathbb{R}$ corresponds to a limit.

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1. Introduction

A gauged WZW model can be rewritten in the form of a non-linear sigma model by choosing a unitary gauge that eliminates some of the degrees of freedom from the group element, and then integrating out the non-propagating gauge fields [1][2]. The remaining degrees of freedom are identified with string coordinates $X^\mu(\tau, \sigma)$. The resulting action exhibits a gravitational metric $G_{\mu\nu}(X)$ and an antisymmetric tensor $B_{\mu\nu}(X)$ at the classical level. At the one loop level there is also a dilaton $\Phi(X)$. These fields govern the spacetime geometry of the manifold on which the string propagates. Conformal invariance at one loop level demands that they satisfy coupled Einstein’s equations. Thanks to the exact conformal properties of the gauged WZW model these equations are automatically satisfied.

For a restricted list of non-compact gauged WZW models there is only one time coordinate [3][4], thus making them suitable for a string theory interpretation in curved spacetime. The list may be extended to supersymmetric heterotic models [5][6][7]. Then these models can be viewed as generating automatically a solution of these rather unyielding Einstein equations. One only needs to do some straightforward algebra based on group theory to extract the explicit forms of $G_{\mu\nu}, B_{\mu\nu}, \Phi$. Following the lead of [2] who interpreted the $SL(2, \mathbb{R})_{-k/4}$ case at $k = 9/4$ [1] as a string propagating in the background geometry of a black hole in two dimensions, several groups have worked out the geometry for all possible cases up to dimension four [8][9][5][10]. The resulting new geometries are generally non-isotropic and have singularities that are more intricate than a black hole, and may have physical interpretations in the early string universe. The global aspects of these higher geometries have been understood [11][12][7]. They all have very interesting duality properties that correspond to interchanges of patches of the global geometry. This duality may be viewed as inversions in group space [11] and are related to asymmetric left-right gauging that involves a twist on the right relative to the left of the group element [8].

Since these are singular geometries it is clearly desirable to go beyond the one loop expansion of the effective sigma model and consider the effect of the exact conformal invariance that underlies the gauged WZW model. It is the purpose of the present paper to accomplish this by considering the full quantum effective action. Of course, the full quantum action is of interest in its own right since the range of its applications goes far beyond the exact geometry of the model. However, at this point, rather than a complete
derivation we are able to present a conjecture on the form of the full quantum effective action. We will justify its form by deriving the exact geometry and comparing to our previous exact results obtained with algebraic Hamiltonian techniques. Therefore, let us first briefly review the status of conformally exact results.

In recent papers \[13\][14][12] we showed how to improve on the perturbative Lagrangian results by using algebraic Hamiltonian techniques to compute globally valid and conformally exact geometrical quantities such as the metric and dilaton (and, in principle, other fields) in gauged WZW models. We have applied the method to bosonic, heterotic and type-II supersymmetric 4D string models that use the non-compact cosets. The main idea is as follows. It is part of the folklore of string theory that \(L_0 + \bar{L}_0\) is the Laplacian, and that when applied to the tachyon \(T\) it takes the form

\[
(L_0 + \bar{L}_0)T = \frac{-1}{e^\Phi \sqrt{-G}} \partial_\mu (e^\Phi \sqrt{-GG}^{\mu \nu} \partial_\nu T).
\]

This equation follows from the general form of the low energy effective action of string theory which concentrates on the low lying spectrum. Eq. (1.1) was used in \[15\] where the \(SL(2, \mathbb{R})_{-k}/\mathbb{R}\) geometry to all orders in \(1/k\) was “conjectured” to arise from it. Indeed this simplest case has been checked to work up to four loops for the bosonic string and up to five loops for the type-II superstring \[16\]. In \[13\] we developed the general methods to use (1.1) to extract the global and conformally exact geometry for all \(G/H\) models, including the heterotic superstring case. This was based on the following proof of (1.1) which was implicit but was not stated explicitly in \[13\]: Evidently, \(L_0 + \bar{L}_0\), as constructed from currents in a \(G/H\) theory, is exact to all orders in \(1/k\). The tachyon is annihilated by all \(n \geq 1\) currents \(J^G_n\), so that only the zero mode currents \(J^G_0\) are relevant, as they appear in \(L_0 + \bar{L}_0\). We further made the reasonable assumption that the tachyon wavefunction depends only on the zero modes of the group parameters. Therefore, we only need to know how to construct the zero mode currents from the zero modes of the group parameters as differential operators. We have shown in \[13\] how to accomplish this, so that \(L_0 + \bar{L}_0\) becomes a second order differential operator. Then, after using the crucial observation that the tachyon \(T\) is constructed from certain gauge invariant combinations of group parameters, and then applying the chain rule as described in \[13\], \(L_0 + \bar{L}_0\) indeed takes the general form of the Laplacian in (1.1) with a non-trivial dilaton and metric. Since this Laplacian is exact to all orders in \(1/k\) the resulting metric and dilaton must be identified with the exact ones to all orders in \(1/k\).
The Hamiltonian approach has effectively concentrated on the zero modes. Therefore, in comparing the old results to the new exact quantum action of the present paper, we must take care that the exact geometry is in agreement first and foremost for the zero modes. We shall see that the geometry for the higher modes may be non-local on the world sheet. Our approach here will apply to abelian cosets such as $SL(2, \mathbb{R})/\mathbb{R}$, $SL(2, \mathbb{R}) \otimes SO(1, 1)^{d-2}/SO(1, 1)$ or $SL(2, \mathbb{R})_{-k'} \otimes SU(2)_k/\mathbb{R}$, as well as non-abelian ones such as $SO(2, 2)/SO(2, 1) \sim SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})/SL(2, \mathbb{R})$ or $SO(3, 2)/SO(3, 1)$. For related results for the abelian coset $SL(2, \mathbb{R})/\mathbb{R}$ see also a paper by A.A. Tseytlin [17] with whom the present investigations were initiated [18].

2. The effective quantum action

The effective quantum action for any field theory is derived by introducing sources and then applying a Legendre transform [19]. The effective action, which is then used as a classical field theory, incorporates all the higher loop effects. Based on a perturbative analysis in [20] [21] it has been argued [17] that for the ungauged WZW model $G_{-k}$ this procedure gives

$$S_{WZW}^{eff} = (-k + g)I_0(g),$$

$$I_0(g) = \frac{1}{8\pi} \int_M Tr(\partial_+ g^{-1} \partial_- g) + \frac{1}{24\pi} \int_B Tr(g^{-1} dg)^3, \quad (2.1)$$

Therefore the full quantum effective action differs from the classical one only by the overall renormalization that replaces $(-k)$ by $(-k + g)$, where $g$ is the Coxeter number for the group $G$, not to be confused with the group element $g(\sigma^+, \sigma^-)$ (we have also assumed a conformally critical theory with the Virasoro central charge at $c = 26$ that fixes the value of $k$). Instead of relying on the perturbative approach in [20] [21] [17] we can justify the result (2.1) by the following argument on the geometry: Before the quantum effects are taken into account the classical sigma model geometry of the WZW model is given by the group manifold metric and the antisymmetric tensor (the axion), both multiplied by $(-k)$. To derive the exact geometry by the algebraic Hamiltonian approach one must use the quantum exact stress tensor to construct $L_0 + \bar{L}_0$ as described in section-1. The conformally exact quantum stress tensor follows from the classical one by a well known renormalization that replaces $(-k)$ by $(-k + g)$. It follows from this that the exact geometry in the Hamiltonian approach is the same as the classical geometry except for the aforementioned
renormalization. To agree with this quantum result the exact effective action must be the same as the classical one except for the proportionality constant \((-k+g)\) as given in (2.1). Furthermore, \(g(\sigma^+, \sigma^-)\) is now treated as a classical field.

We now extend these arguments to the gauged WZW model (GWZW) for \(G_{-k}/H_{-k}\) which is defined by the classical action \([22][23]\)

\[
S_{GWZW} = -k I_0(g) - k I_1(g, A_+, A_-),
\]

\[
I_1(g, A_+, A_-) = \frac{1}{4\pi} \int_M Tr(A_- \partial_+ g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_+ A_-).
\]

(2.2)

Here \(g\) is a group element in \(G\) and \(A_{\pm}\) is valued in the Lie algebra for the subgroup \(H\). This action is invariant under the local gauge transformations that belong to the subgroup \(H\)

\[
g \rightarrow \Lambda^{-1} g \Lambda, \quad A_+ \rightarrow \Lambda^{-1} (A_+ - \partial_+) \Lambda, \quad A_- \rightarrow \partial_- \Lambda^{-1} h_{\pm}^{-1}.\]

(2.3)

It is useful to make a change of variables to group elements \(h_{\pm} \in H, A_+ = \partial_+ h_{\pm} h_{\pm}^{-1}, A_- = \partial_- h_{\pm} h_{\pm}^{-1}\). After picking up a determinant and an anomaly from the measure, the path integral is rewritten with a new form for the action \([24][23]\)

\[
S_{GWZW} = -k I_0(h_{\pm}^{-1} g h_{\pm}) + (k - 2h) I_0(h_{\pm}^{-1} h_{\pm})\]

(2.4)

which is manifestly gauge invariant under \(h_{\pm} \rightarrow \Lambda^{-1} h_{\pm}\). The new path integral measure is the Haar group measure \(Dg \ Dh_+ \ Dh_-\). We want to take advantage of the similarity of this action to the classical WZW action: the first term is appropriate for \(G\) with central extension \((-k)\) and the second term is appropriate for \(H\) with central extension \((k - 2h)\). Defining the new fields \(g' = h_{\pm}^{-1} g h_{\pm}, \ h' = h_{\pm}^{-1} h_{\pm}, \ h'' = h_{\pm}\) and taking advantage of the properties of the Haar measure, we can rewrite the measure and action in decoupled form \(Dg' \ Dh' \ Dh''\) and \(S = -k I_0(g') + (k - 2h) I_0(h')\). This decoupled form emphasizes the close connection to the WZW path integral, and gives us a clue for how to guess the effective quantum action.

However, \(g', h'\) are not really decoupled, since we must consider sources coupled to the original fields. Indeed, to derive the quantum effective action one must introduce source

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1 The more general left-right asymmetric gauging of \([8]\) may also be discussed in a straightforward fashion (for an application see \([12]\) ).
terms and perform a Legendre transform. Since these coupled $g', h', h''$ integrations are not easy to perform, we will guess \[18\] the answer based on the remarks above and then try to justify it. By analogy to \(2.1\) we suggest that the quantum effective action is given by simply shifting \((-k)\) to \((-k + g)\) and \((k - 2h)\) to \((k - 2h) + h = k - h\).

\[
S_{GW ZW}^{\text{eff}} = (-k + g)I_0(h_-^{-1}gh_+) - (-k + h)I_0(h_-^{-1}h_+) .
\]  

(2.5)

This may now be rewritten back in terms of classical fields $g, A_+, A_-$ by using the definitions given before. We obtain

\[
S_{GW ZW}^{\text{eff}} = (-k + g)[I_0(g) + I_1(g, A_+, A_-) + \frac{g - h}{-k + g}I_2(A_+, A_-)],
\]

\[
I_2(A_+, A_-) = I_3(A_+) + I_3(A_-) + \frac{1}{4\pi} \int d^2\sigma \ Tr(A_+A_-) .
\]  

(2.6)

where we have defined $I_3(A_+) \equiv I_0(h_+), I_3(A_-) \equiv I_0(h_-^{-1})$ and used the Polyakov-Wiegman formula \[24\] to rewrite $I_0(h_-^{-1}h_+) \equiv I_2(A_+, A_-)$ in the form above. Note that $I_2(A_+, A_-)$ is gauge invariant. Our proposed effective action differs from the purely classical action \(2.2\) by the overall renormalization \((-k + g)\) and by the additional term proportional to \((g - h)\). In the large $k$ limit (which is equivalent to small $\bar{h}$) the effective quantum action reduces to the classical action, as it should.

This is not yet the end of the story, because what we are really interested in is the effective action for the sigma model after the gauge fields are integrated out (and a unitary gauge fixed for $g$). In other words, sources are not introduced for the original $A_\pm$, but only for $g$. The effect of this is that the path integral over the above $A_\pm$ (or $h_\pm$) still needs to be performed. At the outset, with the classical action, the path integral over $A_\pm$ was purely gaussian, and therefore it could be performed by simply substituting the classical solutions for $A_\pm = A_\pm(g)$ back into the action. This integration also introduces an anomaly which can be computed exactly as a one loop effect. The anomaly gives the dilaton piece to be added to the effective action

\[
S_{\text{dil}} \sim \int d^2\sigma \sqrt{\gamma} \ R^{(2)}(\gamma) \ \Phi(g) ,
\]  

(2.7)

where $\gamma_{ab}, \gamma, R^{(2)}$ are the metric, its determinant and curvature on the worldsheet for any genus. In order to obtain the exact dilaton we need to perform the $A_\pm$ integrals with the effective action, not the classical one. However, in \(2.6\) the parts $I_3(A_\pm)$ are non-local in the $A_\pm$ (although they are local in $h_\pm$). The reason is that $I_3(A_+) = I_0(h_+) \sim
\[ \int Tr(A_+ \partial_- h_+ h_-^{-1}) + \cdots, \] and we cannot write \( \partial_- h_+ h_-^{-1} \) as a local function of \( A_+ \). Furthermore, in the non-abelian case \( I_3(A_\pm) \) have additional non-linear terms. So, if we believe that the quantum effective action is indeed (2.6), then the effective sigma model action we are seeking seems to be generally non-local even in the abelian case (see also [17]). We will therefore concentrate on just the zero modes. As shown below, we have managed to obtain exactly the zero mode sector of the sigma model and proven that the geometry does indeed reproduce correctly the exact geometry derived before in the Hamiltonian formalism [13] [14]. This is our justification for (2.6).

3. The zero mode sector

To restrict ourselves to the zero mode sector we do dimensional reduction by taking all the fields as functions of only \( \tau \) (i.e. worldline rather than worldsheet). This extracts the low energy point particle content of the string. This technique proved to be very useful in the analysis of the GWZW model at the classical limit [11] and we now use it for the conformally exact action. The derivatives \( \partial_\pm \) get replaced by \( \partial_\tau \) and \( A_\pm \) get replaced by \( a_\pm = \partial_\tau h_\pm h_\pm^{-1} \). Then all non-local and non-linear terms drop out and we obtain the effective action in the zero mode sector

\[
S_{\text{eff}} = \frac{-k + g}{4\pi} \int d\tau \ Tr\left( \frac{1}{2} \partial_\tau g^{-1} \partial_\tau g + a_- \partial_\tau gg^{-1} - a_+ g^{-1} \partial_\tau g + a_- ga + g^{-1} - a_+ a_- \right) \\
- \frac{g - h}{8\pi} \int d\tau \ Tr(a_+ - a_-)^2,
\]

(3.1)

This action is gauge invariant for \( \tau \)-dependent gauge transformations \( \Lambda(\tau) \). Most notably the path integral over \( a_\pm \) is now Gaussian, and this permits the elimination of \( a_\pm \) through the classical equations of motion

\[
(D_+ gg^{-1})_H = \frac{g - h}{k - g} (a_+ - a_-), \quad (g^{-1} D_- g)_H = \frac{g - h}{k - g} (a_+ - a_-),
\]

(3.2)

where we have defined the covariant derivatives \( D_\pm \) on the worldline \( D_\pm g = \partial_\tau g - [a_\pm, g] \) and the subscript \( H \) indicates a projection to the Lie algebra of the subgroup \( H \). The system of equations (3.2) is linear and algebraic in \( a_\pm \) and therefore it can be easily solved. To do that and for further convenience it is useful to introduce a set of matrices \( \{ t_A \} \) in the Lie algebra of \( G \) which obey \( Tr(t_A t_B) = \eta_{AB} \), where the Killing metric \( \eta_{AB} \) is diagonal and
normalized to have ±1 eigenvalues. The subset of matrices belonging to the Lie algebra of the subgroup \( H \) will be denoted by \( \{ t_a \} \) with lower case subscripts or superscripts. Then we define the following quantities

\[
L^H = (g^{-1} \partial_\tau g)_H, \quad L^A_\mu \partial_\tau X^\mu = Tr(g^{-1} \partial_\tau g t^A)
\]
\[
R^H = (-\partial_\tau g g^{-1})_H, \quad R^A_\mu \partial_\tau X^\mu = -Tr(\partial_\tau g g^{-1} t^A)
\]
\[
M_{ab} = Tr(t_a g t_b g^{-1} - t_a t_b)
\]

(3.3)

where \( X^\mu, \mu = 0, 1, \ldots, d - 1 \) are the \( d = \text{dim}(G/H) \) parameters in \( g \) that are left over after going to a unitary gauge for \( g \). Then the solution of (3.2) for \( a_\pm \) is

\[
a_+ = (M^T M - \lambda (M + M^T))^{-1} (M^T R^H - \lambda (L^H + R^H))
\]
\[
a_- = (M M^T - \lambda (M + M^T))^{-1} (M L^H - \lambda (L^H + R^H))
\]

(3.4)

where \( \lambda = \frac{g-h}{k-g} \). Substitution of these expressions back into (3.1) gives

\[
S_{\text{point}}^{\text{eff}} = \frac{k-g}{\pi} \int d\tau G^{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu,
\]

(3.5)

where the metric \( G^{\mu\nu} \) is defined as follows

\[
G^{\mu\nu} = g^{\mu\nu} + \frac{1}{8} ([M^T M - \lambda (M + M^T)]^{-1} (M^T - \lambda I))_{ab} L^a_\mu R^b_\nu
\]
\[
- \frac{1}{8} \lambda (M^T M - \lambda (M + M^T))_{ab} L^a_\mu L^b_\nu
\]
\[
- \frac{1}{8} \lambda (M M^T - \lambda (M + M^T))_{ab} R^a_\mu R^b_\nu,
\]

(3.6)

with \( g^{\mu\nu} \) being the part of the metric due to the kinetic first term in \( I_0(g) \)

\[
g^{\mu\nu} = L^A_\mu L^B_\nu \eta_{AB} = R^A_\mu R^B_\nu \eta_{AB},
\]

(3.7)

and where the curly brackets denote symmetrization with respect to the appropriate indices.

We will illustrate applications of the above general result for several abelian and non-abelian cosets. The simplest case is \( SL(2, \mathbb{R})/\mathbb{R} \), but since this can be presented as a limit of more complicated cases, we will give the results for it after discussing others. This provides a check of our methods.

Let us specialize to the three dimensional non-abelian coset \( SO(2, 2)/SO(2, 1) \) whose exact metric and dilaton was found in [13] with the Hamiltonian approach. We will find
the metric in a patch of the manifold corresponding, in the notation of [11], [13] to \( b = \cosh 2r \), \( u = \sin^2 \theta (\cosh 2t - 1) \), \( v = \cosh 2t + 1 \), where \( \{ b, u, v \} \) are the global coordinates which cover the entire manifold. The set of three matrices \( \{ t_a \} \) in the subgroup \( H = SO(2,1) \) is

\[
\begin{align*}
t_{01} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t_{02} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
t_{12} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{align*}
\tag{3.8}
\]

The columns of the matrices \( L_{\mu}^a, R_{\mu}^a \) may be given as vectors \( L_{\mu}, R_{\mu}, \mu = t, \theta, r \)

\[
\begin{align*}
L_t &= \sqrt{2} \begin{pmatrix} 2c_r - 2s^2_\theta (c_r - 1) \\ 2s_\theta c_\theta (c_r - 1) \\ 0 \end{pmatrix}, \\
R_t &= \sqrt{2} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}, \\
L_\theta &= \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ 1 - c_r \end{pmatrix}, \\
R_\theta &= \sqrt{2} \begin{pmatrix} 0 \\ s_t (1 - c_r) \\ c_t (1 - c_r) \end{pmatrix}, \\
L_r &= 0, \\
R_r &= 0
\end{align*}
\tag{3.9}
\]

and similarly the matrix \( M_{ab} \) is

\[
(M_{ab}) = \begin{pmatrix} c^2_r (c_r - 1) & s_\theta c_\theta (c_r - 1) & 0 \\ c_t s_\theta c_\theta (c_r - 1) & c_t (c^2_r + s^2_\theta) - 1 & s_t c_r \\ -s_t s_\theta c_\theta (c_r - 1) & -s_t (c^2_\theta + s^2_\theta c_r) & 1 - c_t c_r \end{pmatrix},
\tag{3.10}
\]

where \( c_r = \cosh 2r \), \( c_\theta = \cos \theta \), \( c_t = \cosh 2t \) and \( s_r = \sinh 2r \), \( s_\theta = \sin \theta \), \( s_t = \sinh 2t \). The non-zero components of the matrix \( g_{\mu\nu} \) are \( g_{tt} = g_{rr} = 1 \), \( g_{\theta\theta} = (c_r - 1)/2 \).

Using \( (3.6)-(3.10) \) the non-zero components of the metric \( (3.6) \) take the following form

\[
\begin{align*}
G_{rr} &= 1 \\
G_{tt} &= \beta (\tanh^2 r \coth^2 t \tan^2 \theta - \coth^2 r \frac{1}{\cos^2 \theta} - \frac{1}{k - 1} \frac{1}{\cos^2 \theta \sinh^2 t}) \\
G_{\theta\theta} &= \beta (\tanh^2 r - \frac{1}{k - 1} \frac{1}{\cos^2 \theta}) \\
G_{t\theta} &= \beta (\tanh^2 r \coth t \tan \theta),
\end{align*}
\tag{3.11}
\]

where the function \( \beta(r,t,\theta) \) is defined as follows

\[
\beta^{-1} = 1 - \frac{1}{k - 1} (\coth^2 r - \tanh^2 r (\frac{1}{\sinh^2 t} + \coth^2 t \tan^2 \theta)) - \frac{1}{(k - 1)^2} \frac{1}{\cos^2 \theta \sinh^2 t}.
\tag{3.12}
\]
It is not hard to check that the expression for the metric (3.11) is the same as the one we found in [13] with the Hamiltonian approach.

4. The exact axion

To obtain the axion $B_{\mu\nu}$ we need to retain the $\partial_\pm$ on the worldsheet and then read off the coefficient of $\frac{1}{2}(\partial_\pm X^\mu \partial_\pm X^\nu - \partial_\pm X^\nu \partial_\pm X^\mu)B_{\mu\nu}(X)$. As already explained above we cannot do this fully because of the non-local terms and non-abelian non-linearities, but we can still obtain the axion as follows. We formally replace the $R^H, L^H$ in the expressions for $a_\pm$ and elsewhere by $R^H_\pm, L^H_\pm$, where $R^H_\pm = (-\partial_\pm gg^{-1})_H$ and $L^H_\pm = (g^{-1}\partial_\pm g)_H$. We justify this step by the conformal transformation properties for left and right movers. We then substitute these forms of $A_\pm$ back into the action (2.6) and extract the desired axion from the quadratic part (which is local and a partner of the metric). The expression we find for the axion $B_{\mu\nu}(X)$ is

$$B_{\mu\nu} = b_{\mu\nu} + \frac{1}{8}([M^T M - \lambda (M + M^T)]^{-1}(M^T - \lambda I))_{ab} L^a_{[\mu} R^b_{\nu]} ,$$

(4.1)

where $b_{\mu\nu}$ is the part of the axion due to the Wess-Zumino term in $I_0(g)$ and the brackets denote symmetrization with respect to the appropriate indices.

In the particular case of the $SO(2,2)/SO(2,1)$ coset model we have found [8] that for the semiclassical geometry ($k \to \infty$) the axion field vanishes. However, when $k$ is finite we obtain a non-vanishing result, which is given by the following expression

$$B_{t\theta} = \frac{\beta}{2(k-1)} \tan^2 r \coth t \tan \theta ,$$

(4.2)

with the rest of the components being zero. In terms of the global coordinates $\{b, u, v\}$ the corresponding expression is

$$B_{vu} = \frac{\beta}{8(k-1)} \frac{b-1}{b+1} \frac{1}{(v-2)(v-u-2)} .$$

(4.3)

In section 7 we will obtain the exact axion for the three dimensional black string model discussed in the semiclassical limit $k \to \infty$ in (2nd ref. in [9]) and for any $k$ in [14].

2 To compare one should change variables from $(b, u, v) \to (t, \theta, r)$ according to the prescription above.
5. The exact dilaton

To obtain the exact dilaton we must compute the anomaly in the integration over $A_{\pm}$. However, as it was the case with the metric and the axion, the local part of the dilaton can be obtained by going to the point particle limit. The effective action (3.1) contains a quadratic part in the gauge fields which can be rewritten as follows

$$\frac{-k + g}{4\pi} \int d\tau \text{Tr} \left(a_- (M - \lambda I)a_+ + \frac{\lambda}{2} (a_-^2 + a_+^2)\right). \tag{5.1}$$

Integrating out the gauge fields $a_{\pm}$ gives a determinant that produces the exact dilaton by identifying, determinant $= e^{\Phi}$, that is

$$\Phi(X) = \ln \left(\det(M) \sqrt{\det[1 - \lambda(M^{-1} + (MT)^{-1})]}\right) + \text{const.}. \tag{5.2}$$

As an example, for the non-abelian coset $SO(2,2)/SO(2,1)$ this gives

$$\Phi = \ln \left(\frac{\sinh^2 2r \sinh^2 t \cos^2 \theta}{\sqrt{\beta}}\right) + \text{const.}, \tag{5.3}$$

or in terms of the global coordinates $\{b, u, v\}$

$$\Phi = \ln \left(\frac{(b^2 - 1)(v - u - 2)}{\sqrt{\beta}}\right) + \text{const.}, \tag{5.4}$$

which is exactly the expression found in [13] with the Hamiltonian approach.

We can use the general expressions for the exact metric (3.6) and dilaton (5.2) to check a theorem which we suggested before [13]. We noticed sometime ago [8] that the combination $e^{\Phi} \sqrt{G}$ that appears in the Laplacian (1.1) is actually independent of $k$. We had first conjectured this by noting that, in the large $k$ limit, we could write this quantity as the product of the Haar measure for $g$ times the Faddeev-Popov determinant for fixing any $G/H$ gauged WZW model

$$e^{\Phi} \sqrt{G} = (\text{Haar}) \times (\text{Faddeev - Popov}). \tag{5.5}$$

\footnote{For a related statement for $SL(2, \mathbb{R})/\mathbb{R}$ see also [25], and for clarifications see [17]. In our previous work [3][8] we erroneously stated that the path integral for the GWZW model requires an extra gauge invariant factor $F(g)$ in the measure. Our error was due to the omission of an anomaly factor. The correct measure at the outset is the Haar measure for $g$ times the naive measure for the gauge fields $A_{\pm}$, and $F = 1$. This correction does not alter our theorem. We thank E. Kiritsis and A.A. Tseytlin for comments on this point.}
Both $G_{\mu\nu}$ and $\Phi$ receive $1/k$ corrections. But, by noting that the right hand side is purely group theoretical we first conjectured that the combination $e^\Phi \sqrt{-G}$ must remain $k$-independent. In our later work for several non-abelian cases \[13\] we verified that this conjecture is indeed true. Therefore, we stated the following theorem

$$e^\Phi \sqrt{-G} \text{ (any } k) = e^\Phi \sqrt{-G} \text{ (at } k = \infty).$$ \hspace{1cm} (5.6)

We can reinforce this result by making additional observations. First the path integral reasoning that allowed us to observe (5.5) is equally valid when the effective action (2.6) is used in place of the classical action (2.2). Since the right hand side of (5.3) is purely group theoretical, (5.3) should be valid both for the exact and classical $G_{\mu\nu}$ and $\Phi$. Since we have already computed the exact metric and dilaton one is now in principle in a position to check the relation (5.6) in general. However, the algebra required to compute $\sqrt{-G}$ is hard. Instead, the result for all cases relevant to strings in four dimensional curved spacetime has already been computed explicitly in our previous papers and indeed for abelian and non-abelian cases the theorem (6.3) is true.

To include the effects of the dilaton we must add one more piece to the effective sigma model action

$$S_{\text{eff total}}^\prime = S_{\text{eff sigma}} + S_{\text{dil}}^\text{eff},$$ \hspace{1cm} (5.7)

where $S_{\text{dil}}^\text{eff}$ has the same form as (2.7) but with the exact dilaton replacing the perturbative one. Here we have discussed mainly the zero mode part of the total effective string action. The effective action for the higher modes that follows from (2.5) and (2.6) is generally non-local.

### 6. Geodesics in the exact geometry

In \[11\] the string (or particle) coordinates were defined as certain gauge invariant combinations of the group parameters in $g$. In a specific unitary gauge these invariants are related to the gauge fixed form of $g$ that defines the string coordinates. Using this formalism a group theoretical method for obtaining the solution to the geodesic equation was found and used to obtain the geodesics in the classical geometry. It was shown that the solution to the geodesic equation, which generally are complicated non-linear differential equations for the string coordinates and hard to solve directly, could be obtained by first solving the
equations of motion of the original variables $g(\tau), a_{\pm}(\tau)$ (which is easy) and then forming
the gauge invariant combinations from the solutions for the group parameters in $g(\tau)$. We
now apply the same method to solve the geodesic equations in the exact geometry. So, we
seek a solution to the classical equations of motion given by (3.2) and

$$D_-(D_+gg^{-1}) = \partial_\tau(a_- - a_+) + [a_-, a_+] ,$$

(6.1)

which follows from varying $g$, and where $D_\pm$ have the same meaning as in (3.2). The
method for solving these equations is identical to [11] and the solution as a function of
proper time $\tau$ is

$$g(\tau) = \exp\left(\frac{k - g}{k - h} \alpha \tau\right) g_0 \exp\left((P - \alpha) \tau\right) , \quad \left[g_0(P - \alpha)g_0^{-1}\right]_H + \alpha = 0 ,$$

(6.2)

where $\alpha, P$ are constant matrices in the Lie algebra of $H$ and $G/H$ respectively, and $g_0$ is
a constant group element. These matrices, which are constrained by the second equation
in (6.2) define the initial conditions for any geodesic at $\tau = 0$. The line element evaluated
at this general solution becomes

$$\left(\frac{ds}{d\tau}\right)^2 = \frac{k - g}{8\pi} Tr\left(P^2 + \frac{g - h}{k - g} \alpha^2\right) .$$

(6.3)

The sign of this quantity determines whether the geodesic is timelike, spacelike or lightlike,
and it can be chosen \textit{a priori} as an initial condition. The large $k$ analysis of (6.2) was
given in [11]. With the new $k$-dependence, and using the same methods as [11], we have
checked in a few specific cases that the geodesic equations for the exact metric are indeed
solved with this group theoretical technique.

7. Axial gauging and the $SL(2, \mathbb{R}) \otimes SO(1, 1)^{d-2}/SO(1, 1)$ models

So far we have concentrated on the vector gauging of WZW models. For the axial
gauging the subgroup $H$ should be abelian with zero Coxeter number. The action is given
by (2.2) but with $I_1(g, A_+, A_-)$ replaced by

$$I_1^{\text{axial}}(g, A_+, A_-) = \frac{1}{4\pi} \int_M Tr(A_- \partial_+ gg^{-1} + A_+ g^{-1} \partial_- g - A_- g A_+ g^{-1} - A_+ A_-) .$$

(7.1)
Then if $A_\pm = -\partial_\pm h_\pm h_\pm^{-1}$ the analog of (2.6) is
\[
S_{GWZW}^{\text{eff,axial}} = (-k + g) [I_0(g) + I_1^{\text{axial}}(g, A_+, A_-) + \frac{g}{-k + g} I_0(h_\pm^{-1} h_\pm)] .
\] (7.2)

Let us specialize to the $SL(2, \mathbb{R}) \otimes SO(1, 1)^{d-2}/SO(1, 1)$ coset models. For $d = 3$ the semiclassical aspects of the model were worked out in the 2nd ref. in [9], for $d = 4$ in the 5th ref. in [9] and for general $d$ in [4]. The conformally exact geometry was found in [14] with the Hamiltonian approach. It is convenient to parametrize the group element of $G = SL(2, \mathbb{R}) \otimes SO(1, 1)^{d-2}$ as follows
\[
g = \begin{pmatrix}
g_0 & 0 & \cdots & 0 \\
0 & g_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & g_{d-2}
\end{pmatrix},
\] (7.3)

where
\[
g_0 = \begin{pmatrix}
a & u \\
-u & b
\end{pmatrix}, \quad ab + uv = 1
\] (7.4)

and
\[
g_i = \begin{pmatrix}
\cosh 2r_i & \sinh 2r_i \\
\sinh 2r_i & \cosh 2r_i
\end{pmatrix}, \quad i = 1, 2, \ldots, d - 2 .
\] (7.5)

The infinitesimal generators for $SL(2, \mathbb{R})$ are
\[
\begin{align*}
\hat{j}_0 &= -\frac{g_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\hat{j}_+ &= q_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
\hat{j}_- &= q_0 \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\end{align*}
\] (7.6)

and those for the $SO(1, 1)$’s
\[
\hat{j}_i = q_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i = 1, 2, \ldots, d - 2 .
\] (7.7)

The coefficients $q_i$ parametrize the embedding of $H = SO(1, 1)$ into the factored $SO(1, 1)$’s in $G$ and are normalized to $\sum_{i=0}^{d-2} q_i^2 = 1$. The subgroup elements $h_\pm$ are parametrized in terms of two variables $\phi_\pm$ as follows
\[
h_\pm = e^{-\frac{1}{q_0} \hat{j}_0 u(1) \phi_\pm} ,
\] (7.8)
where

$$J_{U(1)} = \begin{pmatrix} j_0 & 0 & \cdots & 0 \\ 0 & j_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & j_{d-2} \end{pmatrix}.$$  \tag{7.9}

If we define two new variables $\phi = \phi_- - \phi_+$, $\tilde{\phi} = \phi_- + \phi_+$ then in the gauge $b = \pm a$ the action (7.2) takes the following form

$$S_{GWZ}^{eff, axial} = \frac{k'}{16\pi} \int \frac{1}{uv - 1} \left( \partial_+(uv) \partial_-(uv) - 2(uv - 1)(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \right)$$

$$+ 4 \frac{k}{k'} \sum_{i=1}^{d-2} \kappa_i \partial_+ r_i \partial_- r_i$$

$$+ (u \partial_+ v - v \partial_+ u - 2 \frac{k}{k'} \sum_{i=1}^{d-2} \kappa_i \eta_i \partial_+ r_i) (\partial_- \phi + \partial_- \tilde{\phi})$$

$$+ (u \partial_- v - v \partial_- u + 2 \frac{k}{k'} \sum_{i=1}^{d-2} \kappa_i \eta_i \partial_- r_i) (\partial_+ \phi - \partial_+ \tilde{\phi})$$

$$+ (uv - 1 - \frac{k}{k'} \rho^2 - 2 \frac{k}{k'} \partial_+ \phi \partial_- \phi$$

$$+ (1 - uv + \frac{k}{k'} \rho^2)(\partial_+ \tilde{\phi} \partial_- \phi + \partial_- \phi \partial_+ \tilde{\phi} - \partial_+ \phi \partial_- \tilde{\phi})$$ \tag{7.10}

where $k' = k - 2$ is the renormalized value for the central extension $k$ and $\eta_i \equiv q_i / q_0$, $\kappa_i \equiv k_i / k$, $\rho^2 \equiv \sum_{i=1}^{d-2} \eta_i^2 \kappa_i$.

To extract the effective string model we now need to integrate out $\phi$ and $\tilde{\phi}$, which is equivalent to integrating out $A_{\pm}$. As discussed before this gives non-local contributions. Therefore, we may again concentrate on the zero modes by dimensional reduction. Furthermore, as discussed in section 4, we may restore formally $\partial_\tau \rightarrow \partial_{\pm}$ in order to compute the axion. In some sense this procedure extracts the local part of the effective action and preserves gauge invariance with respect to $\tau$-dependent gauge transformations $\Lambda(\tau)$. In fact, the local part of the effective action is an ambiguous notion and the principle of $\tau$-dependent gauge invariance resolves this ambiguity. The upshot of these steps boils

\[ \text{Our gauge invariant results differ in general from the local part discussed in [17] which is not gauge invariant with respect to $\Lambda(\tau)$. Our form is required to produce the correct geometry that agrees with the algebraic results. However, for the special case $SL(2, \mathbb{R})/\mathbb{R}$ the results for the metric and dilaton agree accidentally with [17].} \]
down to keeping the local part of the solution of the classical equations for the gauge fields \( \phi \) and \( \tilde{\phi} \)

\[
\begin{align*}
\partial_{\pm} \phi |_{\text{local}} &= \frac{v \partial_{\pm} u - u \partial_{\pm} v}{uv - 1 - \frac{1}{k'} \rho^2 - \frac{1}{k'}} \\
\partial_{\pm} \tilde{\phi} |_{\text{local}} &= -\frac{k}{k'} \sum_{i=1}^{d-2} \kappa_i \eta_i \partial_{\pm} r_i .
\end{align*}
\] (7.11)

Substitution of the above expressions into the action (7.10) gives the following expression for the local part of the effective action

\[
S_{\text{local}}^{\text{eff}} = \frac{k'}{4\pi} \int \frac{1}{k'/k (uv - 1) - \rho^2 - 2/k'} \left[ -\rho^2 + 2/k \right. \left( uv + \partial_{\pm} (uv) \partial_{\pm} (uv) \right. + \frac{1}{2} \left( \partial_{\pm} u \partial_{\pm} v + \partial_{\pm} u \partial_{\pm} v \right) + \kappa_i (\delta_{ij} + \frac{\eta_i \eta_j \kappa_j}{k'/(uv - 1) - \rho^2}) \partial_{\pm} r_i \partial_{\pm} r_j \\
&+ \frac{1}{k'/(1 - uv)} + \rho^2 \sum_{i=1}^{d-2} ((u \partial_{\pm} v - v \partial_{\pm} u) \eta_i \partial_{\pm} r_i - (u \partial_{\pm} v - v \partial_{\pm} u) \eta_i \partial_{\pm} r_i) .
\] (7.12)

The first two lines in the above expression define a metric which is precisely that found in [14] with the Hamiltonian approach. The third line defines an antisymmetric tensor (axion). As in ref. [14] it is useful to diagonalize the metric. Since the procedure is exactly the same we are not going to repeat it here. The answer is that, only a three dimensional part of the metric is non-trivial, and the rest corresponds to flat directions.

The three dimensional non-trivial part of the metric, which describes a black string, has the following form [14]

\[
ds_{3d}^2 = -(1 - \frac{r_+}{r}) \, dt^2 + (1 - \frac{r_- - r_q}{r - r_q}) \, dx^2 + \frac{k'}{8r^2} \left( 1 - \frac{r_+}{r} \right)^{-1} \left( 1 - \frac{r_-}{r} \right)^{-1} \, dr^2 ,
\] (7.13)

where [14] \( r_+ = \sqrt{2/k'} (\rho^2 + 1) \, C \), \( r_- = \sqrt{2/k'} (\rho^2 + 2/k) \, C \) and \( r_q = 2/k \sqrt{2/k'} \, C \) (for \( C \) see below the expression for the dilaton). For the axion and its field strength we obtain a new result: \( B_{tr} = B_{xr} = 0 \), and

\[
\begin{align*}
B_{tx} &= \sqrt{\frac{r_+ - r_q}{r_+}} \frac{r - r_+}{r - r_q} \\
H_{rtx} &= \partial_r B_{tx} = \sqrt{\frac{r_+ - r_q}{r_+}} \frac{r_+ - r_q}{(r - r_q)^2} .
\end{align*}
\] (7.14)

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To obtain the dilaton one has to integrate out $\phi$ and $\tilde{\phi}$ in (7.10). Then one gets for the conformally exact dilaton the expression which was found in [14]

\[ C e^\Phi = (1 - uv) \sqrt{1 + \rho^2 + (\rho^2 + 2/k) \frac{uv}{1 - uv}} \left[ 1 + \rho^2 - 2/k + \rho^2 \frac{uv}{1 - uv} \right], \]

(7.15)

where $C$ is an arbitrary constant. In the variables which diagonalize the metric, the dilaton takes the following form [14]

\[ \Phi = \frac{1}{2} \ln(r - r_q) + \frac{1}{2} \ln k'. \]

(7.16)

Therefore an additional piece $S_{\text{dil}}^{\text{eff}}(\Phi)$ must added to the action in (7.12). The expressions for the metric, the dilaton, the axion and its field strength tend to their semiclassical values (see 2nd ref. in [9]) in the $k \to \infty$ limit, because then $r_q \to 0$.

It would be interesting to check that the expressions we found for the metric, the axion and dilaton in this simple abelian model indeed satisfy the perturbative equations for conformal invariance beyond the 1-loop approximation. For large $k$ the backgrounds of the $(2d$ black hole$) \otimes \mathbb{R}$ and the $3d$ black string are related by a duality transformation as it was shown in [26]. Knowing the exact backgrounds (any $k$) for both geometries, may shed some light into the form of the duality transformation beyond the leading order in $\alpha' \sim 1/k$.

8. The $SL(2, \mathbb{R})/\mathbb{R}$ model

Since the simplest case $SL(2, \mathbb{R})/\mathbb{R}$ is just a limit of the previous case we will briefly derive in this section all the well known results. In order to specialize the action (7.12) to the case of the $SL(2, \mathbb{R})_{-k}/\mathbb{R}$ model one should take $k_i = 0$. It follows that $\kappa_i = \rho^2 = 0$ and the action (7.12) and dilaton (7.13) take the following form

\[ S_{\text{local}}^{\text{eff}} = \frac{k}{8\pi} \int \frac{1}{uv - 1 - 2/k} \left( -\frac{1/k}{uv - 1} \partial_{+}(uv)\partial_{-}(uv) + (\partial_{+}u\partial_{-}v + \partial_{-}u\partial_{+}v) \right) + S_{2d}^{\text{dil}}(\Phi), \]

(8.1)

and

\[ C' e^\Phi = (1 - uv) \sqrt{1 - \frac{2}{k} \frac{uv}{uv - 1}}, \]

(8.2)
where $C'$ is a constant related to $C$ in (7.13). In the region where $uv > 1$ we change variables from $(u, v) \to (t, r)$ as follows

$$u = \cosh r \ e^t, \quad v = \cosh r \ e^{-t}. \quad (8.3)$$

Then the action (8.1) and the dilaton (8.2) can be written as

$$S_{2d}^{\text{local}} = \frac{k'}{4\pi} \int \partial_+ r \partial_- r - f(r) \partial_+ t \partial_- t + S_{2d}^{\text{dil}}(\Phi)$$

$$\Phi = \ln(\sinh 2r/f(r)) + \text{const.} \quad , \quad (8.4)$$

where $f(r) = 1/(\tanh^2 r - 2/k)$, thus reproducing the exact expressions for the metric and dilaton of the 2$d$ black hole as they were computed in [15] [13]. One could also use the effective action appropriate for vectorial gauging (2.6) to obtain all of the results in this section.

9. Conclusion

We have suggested the form of the effective quantum action for the general Abelian or non-Abelian GWZW model, and verified that it works, at least in the zero mode sector. Furthermore, we have obtained new general results for the conformally exact axion field and geodesics. The zero mode sector determines the point particle behavior of the underlying string theory and is the only part relevant for the low energy physics. Therefore, although our methods have yielded incomplete results for the full string theory, they are adequate to extract the most relevant physical information on the curved spacetime geometry. Based on the agreement with the algebraic Hamiltonian approach in the zero mode sector, we conjecture that, before the integration over $A_\pm$, the forms (2.5) (2.6) may be trusted for all the higher modes as well.

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