Nonlinear Bogolyubov-Valatin transformations: 2 modes

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Abstract

Extending our earlier study of nonlinear Bogolyubov-Valatin transformations (canonical transformations for fermions) for one fermionic mode, in the present paper we perform a thorough study of general (nonlinear) canonical transformations for two fermionic modes. We find that the Bogolyubov-Valatin group for \( n = 2 \) fermionic modes which can be implemented by means of unitary \( SU(2^n = 4) \) transformations is isomorphic to \( SO(6; \mathbb{R})/\mathbb{Z}_2 \). The investigation touches on a number of subjects. As a novelty from a mathematical point of view, we study the structure of nonlinear basis transformations in a Clifford algebra [specifically, in the Clifford algebra \( C(0, 4) \)] entailing (supersymmetric) transformations among multivectors of different grades. A prominent algebraic role in this context is being played by biparavectors (linear combinations of products of Dirac matrices, quadriquaternions, sedenions) and spin bivectors (antisymmetric complex matrices). The studied biparavectors are equivalent to Eddington’s \( E \)-numbers and can be understood in terms of the tensor product of two commuting copies of the division algebra of quaternions \( \mathbb{H} \). From a physical point of view, we present a method to diagonalize any arbitrary two-fermion Hamiltonian. Relying on Jordan-Wigner transformations for two-spin-\( \frac{1}{2} \) and single-spin-\( \frac{3}{2} \) systems, we also study nonlinear spin transformations and the related problem of diagonalizing arbitrary two-spin-\( \frac{1}{2} \) and single-spin-\( \frac{3}{2} \) Hamiltonians. Finally, from a calculational point of view, we pay due attention to explicit parametrizations of \( SU(4) \) and \( SO(6; \mathbb{R}) \) matrices (of respective sizes \( 4 \times 4 \) and \( 6 \times 6 \)) and their mutual relation.

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1 Introduction

Like canonical transformations in classical mechanics, unitary transformations of quantum dynamical degrees of freedom often simplify the dynamical equations, or allow to introduce sensible approximation schemes. Such methods have wide-ranging applications, from the study of simple systems to many-body problems in solid-state or nuclear physics and quantum chemistry, up to the infinite-dimensional systems of quantum field theory [1–4]. Linear (unitary) canonical transformations (i.e., transformations preserving the canonical anticommutation relations (CAR)) for fermions have been introduced by Bogolyubov and Valatin (for two fermionic modes) in connection with the study of the mechanism of superconductivity [5, 10–13]. These (linear) Bogolyubov-Valatin transformations have been extended, initially by Bogolyubov and his collaborators [14, 15], [16], Appendix II, p. 123 ([17], p. 116, [18], p. 679), to involve $n$ fermionic modes [so-called generalized linear Bogolyubov-Valatin transformations, see, e.g., [19], часть III [chast’ III]/[part III], p. 247 ([20], p. 341, [21], p. 259, [22], p. 235)]. Such linear canonical transformations are important from a physical as well as from a mathematical point of view. Mathematically, they allow to relate quite arbitrary Hamiltonians quadratic in the fermion creation and annihilation operators to collections of Fermi oscillators whose mathematics is very well understood. From a physical point of view, canonical transformations implement the concept of quasiparticles in terms of which the physical processes taking place can be described and understood in an effective and transparent manner. To apply the powerful tool of canonical transformations to the physically interesting class of non-quadratic Hamiltonians, however, requires to go beyond linear Bogolyubov-Valatin transformations. Certain aspects of nonlinear Bogolyubov-Valatin transformations have received some attention over time [23–27], [28], Sec. 2–6, p. 52, [29–54] (We disregard here work done within the framework of the coupled-cluster method (CCM) [4] which is nonunitary.).\(^1\) However, a systematic analytic study of general (nonlinear) Bogolyubov-Valatin transformations had not been undertaken until the publication of our article [49].

In [49] we have initiated a systematic analytic study of general (nonlinear) Bogolyubov-Valatin transformations by studying in detail the prototypical case of one fermion mode. The mathematics of one fermion mode is closely related to the mathematics of the division algebra of quaternions $\mathbb{H}$. In the present paper, we now investigate the most general (nonlinear) Bogolyubov-Valatin transformations for two fermion modes. The original papers of Bogolyubov [5, 10] and Valatin [11] study the application of quasifermion operators constructed linearly (with certain real coefficients $u$ and $v$) from two fermionic modes in the most simple way. Our study can, therefore, be considered as the widest possible (take this with a grain of salt) generalization of the work of Bogolyubov and Valatin for two fermion modes. For example, we will encounter the generalization(s) of the condition ([5], p. 59, (p. 679))

\(^1\)Up to the referencing, this paragraph is identical to the introductory paragraph of our earlier article [49] on the one-mode case of nonlinear Bogolyubov-Valatin transformations.
42 of the English transl.), [10], p. 796)

\[ u^2 + v^2 = 1 \]  \hspace{1cm} (1)

which is the well-known condition for the original (linear) Bogolyubov-Valatin transformation to be canonical (We display this condition here symbolically without precisely defining its meaning.).

The study of nonlinear Bogolyubov-Valatin transformations requires the study of a new mathematical problem not considered previously – nonlinear basis transformations in a Clifford algebra. This involves transformations among multivectors of different grades which can be considered as a mathematical realization of the physical concept of a supersymmetry. Furthermore, nonlinear Bogolyubov-Valatin transformations of two fermion modes have connections to various concrete physical and mathematical problems. Via Jordan-Wigner transformations two-fermion systems are related to two-spin-\(\frac{1}{2}\) and single-spin-\(\frac{3}{2}\) systems which have been studied from various points of view in the past and have received broad attention in the context of quantum information and computation theory recently. Nonlinear Bogolyubov-Valatin transformations correspond to nonlinear spin transformations in these systems. As two-fermion systems are closely related to the Clifford algebra \(C(0, 4)\) (and their cousins of different signature), the study of nonlinear Bogolyubov-Valatin transformations is closely connected to Dirac matrices and the Dirac equation in general. Consequently, early studies of Dirac matrices and the Dirac equation performed by Eddington feature prominently in the list of references accompanying the article. And finally, via the relation of two-fermion systems to \(4 \times 4\) matrices the study of nonlinear Bogolyubov-Valatin transformations has close connections to the representation and parametrization of the group \(SU(4)\), and its related orthogonal group \(SO(6; \mathbb{R})\) to which it is the double cover.

As in our earlier paper [49], the main focus lies on structural and methodical aspects of the problem of nonlinear Bogolyubov-Valatin transformations. We start in Sec. 2 with a discussion of certain algebraic aspects of a (fixed) set of fermion creation and annihilation operators for two fermion modes. In particular, we make contact with Clifford algebras (Subsec. 2.1) and the concept of paravectors (Subsec. 2.2). In the main section 3 of our article we first relate the concept of nonlinear Bogolyubov-Valatin transformations to the mathematical concept of nonlinear basis transformations in a Clifford algebra (Subsec. 3.1) and then study those in mathematical detail for the Clifford algebra \(C(0, 4)\) (Subsec. 3.2). We find that nonlinear Bogolyubov-Valatin transformations have the structure of the group \(SO(6; \mathbb{R})/\mathbb{Z}_2\) [eq. (127)]. In section 4 we then apply the obtained insight to the problem of diagonalizing fermion (Subsec. 4.1) and spin (Subsec. 4.2) Hamiltonians. As the whole discussion in the sections 3 and 4 is closely linked to the intricate relationship between the groups \(SO(6; \mathbb{R})\) and \(SU(4)\) in section 5 we study explicitly and in detail the relationship between their corresponding \(6 \times 6\) and \(4 \times 4\) representation matrices. Discussions of various aspects of the present work and their relation with the work
of other authors are collected in Sec. 6. Short conclusions are presented in Sec. 7. To facilitate the readability of the article, a number of technical details are deferred to six Appendices. The article is accompanied by a comprehensive list of references which may serve as a guide to the relevant literature.

2 Algebraic preliminaries

2.1 Fermion creation and annihilation operators and the associated Clifford algebra

We consider two pairs of fermion creation and annihilation operators $\hat{a}_k^+, \hat{a}_k$ ($k = 1, 2$). We regard the creation operators $\hat{a}_k^+$ as the hermitian conjugates of the annihilation operators $\hat{a}_k$: $\hat{a}_k^+ = \hat{a}_k^\dagger$ (we will use the latter notation throughout). They are acting on the vacuum state $|0\rangle$ the following way

$$\hat{a}_k |0\rangle = 0 ,$$

$$\hat{a}_k^\dagger |0\rangle = |1\rangle_k .$$

The four-dimensional (complex) Hilbert space $\mathbb{C}_4$ (Sometimes, we will refer to it as spin space.) of the two-fermion system is spanned by the vacuum state $|0\rangle$, the two one-particle states $|1\rangle_k$, and the two-particle state $|2\rangle_{(2,1)}$ [be aware of the index convention in eq. (5)]

$$\hat{a}_k^\dagger \hat{a}_l^\dagger |0\rangle = |2\rangle_{(2,1)} \quad , \quad = -|2\rangle_{(1,2)} ,$$

$$\langle 0 |\hat{a}_1 \hat{a}_2 = \langle 2 |_{(2,1)} = -\langle 2 |_{(1,2)} .$$

The fermion creation and annihilation operators obey the canonical anticommutation relations (CAR), ($k, l = 1, 2$; $\mathbb{I}_4$ is the $4 \times 4$ unit matrix)$^2$

$$\{ \hat{a}_k^\dagger, \hat{a}_l^\dagger \} = \hat{a}_k^\dagger \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_k^\dagger = \delta_{kl} \mathbb{I}_4 ,$$

$$\{ \hat{a}_k, \hat{a}_l \} = \hat{a}_k \hat{a}_l + \hat{a}_l \hat{a}_k = 0 ,$$

the latter equation entailing

$$\{ \hat{a}_k^\dagger, \hat{a}_l^\dagger \} = \hat{a}_k^\dagger \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_k^\dagger = 0 .$$

It is now useful to consider the following pairs of anti-hermitian operators.

$$\hat{a}_k^{[1]} = -\hat{a}_k^{[1]*} = i \left( \hat{a}_k + \hat{a}_k^\dagger \right) ,$$

$$\hat{a}_k^{[2]} = -\hat{a}_k^{[2]*} = \hat{a}_k - \hat{a}_k^\dagger .$$

$^2$As we will emphasize in this paper on many occasions the concrete aspects of the problem under consideration we have decided to indicate the presence of unit operators explicitly.
As a consequence of the CAR, eqs. (6)-(8), these operators obey the equation \((p, q = 1, 2)\)

\[
\{ \hat{a}_k^{[p]}, \hat{a}_l^{[q]} \} = -2 \delta_{pq} \delta_{kl} \mathbb{1}_4. \tag{11}
\]

We introduce now the following useful notation:

\[
\hat{c}_{2k-1} = \hat{a}_k^{[1]}, \quad k = 1, \ldots, n (= 2), \tag{12}
\]

\[
\hat{c}_{2k} = \hat{a}_k^{[2]}, \quad k = 1, \ldots, n (= 2). \tag{12}
\]

The anti-Hermitian operators \(\hat{c}_k = -\hat{c}_k^\dagger\) obey the Clifford algebra relation \([k, l = 1, \ldots, 2n (= 4)]\)

\[
\{ \hat{c}_k, \hat{c}_l \} = 2 \hat{c}_k \cdot \hat{c}_l = 2 g_{kl} \mathbb{1}_4 = -2 \delta_{kl} \mathbb{1}_4, \tag{13}
\]

\[
g_{kl} = -\delta_{kl}, \tag{14}
\]

where \(\hat{c}_k \cdot \hat{c}_l\) denotes the inner product in the Clifford algebra. The operators \(\hat{c}_k\) generate the (real) Clifford algebra \(C(0, 4)\) which is isomorphic to the twofold tensor product of the algebra of quaternions \(\mathbb{H}\) (cf., e.g., [55], Chap. 15, p. 123, [56], Chap. 16, p. 205). To simplify the further calculations and to minimize sign errors, in the following we will raise and lower indices by means of \(\delta_{kl}\) and not by means of \(g_{kl}\).

### 2.2 Some useful algebraic structures in the Clifford algebra \(C(0, 4)\)

Having related the two-mode pairs of fermion creation and annihilation operators to the Clifford algebra \(C(0, 4)\) we will focus now our attention onto this algebraic structure. In this subsection, we will discuss certain algebraic objects in the Clifford algebra \(C(0, 4)\) that we will encounter in the course of the further investigation. Their significance will become clear in the further sections only. The somewhat unconventional index convention (including an index -1) to be used in the following has been chosen with an eye to possible future generalizations of it to a larger number of fermionic modes.

**Paravectors**

Define now

\[
\hat{c}_k = \hat{c}_k, \quad k = 1, \ldots, 4, \tag{15}
\]

\[
\hat{c}_{(-1)} = i\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4. \tag{16}
\]

These operators obey the equation \((k, l = -1, 1, \ldots, 4)\)

\[
\{ \hat{c}_k, \hat{c}_l \} = -2 \delta_{kl} \mathbb{1}_4. \tag{17}
\]
They are generators of the (non-universal) Clifford algebra \( C(0, 5) \). Going one step further we define
\[
\hat{c}_k = \hat{c}_k , \quad k = -1, 1, \ldots, 4 , \tag{18}
\]
\[
\hat{c}_0 = -\mathbb{I}_4 . \tag{19}
\]
These operators \( \hat{c}_k \) span the paravector space\(^3 \) \( V_6 \) associated with the Clifford algebra \( C(0, 5) \) the same way as the operators \( \hat{c}_k \) span the vector space that is associated with the Clifford algebra \( C(0, 4) \). It should be mentioned here that the choice in sign for \( \hat{c}_0 = -\mathbb{I}_4 \) is a matter of convenience. Our choice has been suggested by eq. (105) further below\(^4 \).

As a consequence of eq. (17) and the anti-Hermiticity of the operators \( \hat{c}_k \), the operators \( \hat{\hat{c}}_k \) [i.e., the basis elements of the paravector space associated with the Clifford algebra \( C(0, 5) \)] obey the relation \((k, l = -1, \ldots, 4)\)
\[
\hat{\hat{c}}_k \hat{\hat{c}}_l + \hat{\hat{c}}_l \hat{\hat{c}}_k = 2 \delta_{kl} \mathbb{I}_4 . \tag{20}
\]
Note, that eq. (20) has the form of the unitary Hurwitz-Radon matrix problem (\[65\], Subsec. 1.4, p. 25, also see [66], part II, p. 67 and [67]). One can convince oneself that the relation
\[
\hat{c}_m = -\frac{i}{5!} \epsilon_m^{npqrs} \hat{c}_n \hat{\hat{c}}_p \hat{\hat{c}}_q \hat{\hat{c}}_r \hat{\hat{c}}_s \tag{21}
\]
holds \((\epsilon_m^{npqrs} \text{ is the 6-dimensional totally antisymmetric tensor, } \epsilon_{(-1)01234} = 1; \text{ we have raised here the indices by means of } \delta_{kl} \text{ and not using } g_{kl})\), and consequently also \((n \neq p \neq q \neq r \neq s)\)
\[
\hat{\hat{c}}_n \hat{\hat{c}}_p \hat{\hat{c}}_q \hat{\hat{c}}_r \hat{\hat{c}}_s = -i \epsilon_m^{npqrs} \hat{\hat{c}}_m . \tag{22}
\]

**Biparavectors**

We define now the antisymmetric objects
\[
\hat{\hat{c}}_{m_1 m_2} = -\hat{\hat{c}}_{m_2 m_1} = \frac{1}{2} \left( \hat{\hat{c}}^{\dagger}_{m_1} \hat{\hat{c}}_{m_2} - \hat{\hat{c}}^{\dagger}_{m_2} \hat{\hat{c}}_{m_1} \right) = -\hat{\hat{c}}^{\dagger}_{m_1 m_2} , \tag{23}
\]
the decomposable/simple biparavectors of the paravector space \( V_6 \) associated with the Clifford algebra \( C(0, 5) \) (cf. [58], Sec. 3.4, p. 12). They span the 15-dimensional

\(^3\)For paravector space see, e.g., [57], Chap. 13, pp. 254-259, [55], Chap. 16, p. 140, [56], 1. and 2. ext. ed., Sec. 19.3, p. 247, [58]. The term paravector has been introduced in [59], p. 22. In a context related to our study, paravectors have been used earlier in [60], p. 14, eq. (5.3), (in some disguise) in [61], p. 340, eq. (1.2), and in [62], Sec. 2, p. 92; also note [63]. For an early, related discussion see [64], in particular, p. 3, eq. (5).

\(^4\)A different choice \( (\hat{c}_0 = \mathbb{I}_4) \) would have entailed to modify eq. (53) accordingly.
biparavector space $\bigwedge^2(V_6)$. To simplify the notation we will often write $\hat{c}_m$ instead of $\hat{c}_{m_1 m_2}$ [$M = (m_1, m_2)$; if capital letters, e.g., $M$, are used for summations, $m_1 < m_2$ is understood].

It seems worth noting here that the set of biparavectors $\hat{c}_m$ (supplemented by the unit operator $1_4$) is equivalent to the $E$-numbers introduced by Eddington [68]. It has been recognized from early on that the $E$-numbers of Eddington form an algebraic system which can be obtained by taking the tensor product of two (commuting) copies, $I_1, J_1, K_1 (I_1 J_1 = K_1$, etc.) and $I_2, J_2, K_2, (I_2 J_2 = K_2$, etc.), of the system of quaternions $\mathbb{H}$ (cf. also the comment at the end of Subsec. 2.1). To be more specific, one possible (symmetric) choice for the generators of the Clifford algebra $C(0, 4)$ is (for a different choice see Appendix D)

$$\hat{c}_1 = \hat{c}_1 = \frac{1}{\sqrt{2}} [1_4 - i K_1 K_2] \quad I_1 = \frac{1}{\sqrt{2}} [1_1 - i J_1 K_2], \quad (24)$$

$$\hat{c}_2 = \hat{c}_2 = \frac{1}{\sqrt{2}} [1_4 - i K_1 K_2] \quad J_1 = \frac{1}{\sqrt{2}} [J_1 + i I_1 K_2], \quad (25)$$

$$\hat{c}_3 = \hat{c}_3 = \frac{1}{\sqrt{2}} [1_4 + i K_1 K_2] \quad I_2 = \frac{1}{\sqrt{2}} [I_2 + i K_1 J_2], \quad (26)$$

$$\hat{c}_4 = \hat{c}_4 = \frac{1}{\sqrt{2}} [1_4 + i K_1 K_2] \quad J_2 = \frac{1}{\sqrt{2}} [J_2 - i K_1 I_2], \quad (27)$$

and, consequently,

$$\hat{c}_{(-1)} = i K_1 K_2. \quad (28)$$

Inversely, the two commuting copies of the system of quaternions can be given in terms of the generators of the Clifford algebra $C(0, 4)$ by means of the following equations.

$$I_1 = \frac{1}{\sqrt{2}} [1_4 + \hat{c}_{(-1)}] \quad \hat{c}_1 = -\frac{1}{\sqrt{2}} [\hat{c}_{01} + \hat{c}_{(-1)1}] = \frac{1}{\sqrt{2}} [\hat{c}_1 + i \hat{c}_2 \hat{c}_3 \hat{c}_4] \quad (29)$$

$$J_1 = \frac{1}{\sqrt{2}} [1_4 + \hat{c}_{(-1)}] \quad \hat{c}_2 = -\frac{1}{\sqrt{2}} [\hat{c}_{02} + \hat{c}_{(-1)2}] = \frac{1}{\sqrt{2}} [\hat{c}_2 - i \hat{c}_1 \hat{c}_3 \hat{c}_4] \quad (30)$$

\^5Also see [69-71]; a systematic presentation is given in [72], Part I, Chaps. 2-4, pp. 20-61, [73], Chaps. VI/VII, pp. 106-158, and a discussion of Eddington’s $E$-numbers from a modern perspective can be found in [74].

\^6[75-79], [72], §3.8, p. 47, [80], [73], §60, p. 120, [81, 82].

\^7One can convince oneself that the biparavector space $\bigwedge^2(V_6)$ supplemented by the unit operator $1_4$ spans the operator space of the Clifford algebra $C(0, 4)$. Initially, biparavectors (supplemented by the unit operator) have been named ‘sedenions’ ([75], p. 105, footnote 3) or ‘quadriquaternions’ [83]. Note, that these sedenions are different from the weird algebraic objects presently called ‘sedenions’ and obtained from the octonions in the following step of the Cayley-Dickson process.
\[ K_1 = \hat{c}_1 \hat{c}_2 = -\hat{c}_{12} = \hat{c}_1 \hat{c}_2 \] (31)
\[ I_2 = \frac{1}{\sqrt{2}} \left[ \mathbb{1}_4 - \hat{c}_{(-1)} \right] \hat{c}_3 = -\frac{1}{\sqrt{2}} \left[ \hat{c}_{03} - \hat{c}_{(-1)3} \right] = \frac{1}{\sqrt{2}} \left[ \hat{c}_3 - i \hat{c}_1 \hat{c}_2 \hat{c}_4 \right] \] (32)
\[ J_2 = \frac{1}{\sqrt{2}} \left[ \mathbb{1}_4 - \hat{c}_{(-1)} \right] \hat{c}_4 = -\frac{1}{\sqrt{2}} \left[ \hat{c}_{04} - \hat{c}_{(-1)4} \right] = \frac{1}{\sqrt{2}} \left[ \hat{c}_4 + i \hat{c}_1 \hat{c}_2 \hat{c}_3 \right] \] (33)
\[ K_2 = \hat{c}_3 \hat{c}_4 = -\hat{c}_{34} = \hat{c}_3 \hat{c}_4 \] (34)

As the system of quaternions \( \mathbf{I}, \mathbf{J}, \mathbf{K} \) is related to the group SU(2) and, therefore, to spin variables \( S^x, S^y, S^z \) \( (S^x = i \mathbb{1}/2, S^y = i \mathbf{J}/2, S^z = i \mathbf{K}/2) \), a quadiquaternion representation of the biparavectors \( \hat{c}_M \) is related to a two-spin-\( \frac{1}{2} \) system\(^8\). For example, this fact is relied on in the product operator formalism widely used in the field of nuclear magnetic resonance (NMR) \(^{85-87}\). Two-spin-\( \frac{1}{2} \) systems will be discussed in Subsection 4.2.1. Another approach relating quaternions to Eddington’s \( E \)-numbers has been put forward by Conway \(^{88}\) (also see \(^{89}\), Sec. 9, p. 48).

**Algebra of biparavectors**

Using the notation introduced in eq. (23), the eqs. (21), (22) can be transformed to read\(^9\)

\[ \hat{c}_M = \frac{i}{6} \epsilon_{M}^{\ PQ} \hat{c}_P \hat{c}_Q \] (35)

and

\[ \hat{c}_P \hat{c}_Q = -\delta_{PQ} \mathbb{1}_4 - \delta_{p_1 q_1} \hat{c}_{(p_2 q_2)} - \delta_{p_2 q_2} \hat{c}_{(p_1 q_1)} + \delta_{p_1 q_2} \delta_{p_2 q_1} \mathbb{1}_4 + \delta_{p_1 q_2} \hat{c}_{(p_2 q_1)} + \delta_{p_2 q_1} \hat{c}_{(p_1 q_2)} \]
\[ - i \epsilon_{PQ}^{\ N} \hat{c}_N \] (36)

In the last equation we have partially lifted the condition \( p_1 \neq p_2 \neq q_1 \neq q_2 \); it still applies \( p_1 \neq p_2, q_1 \neq q_2 \). Equation (36) can be found in a number of studies\(^{10}\). The

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\(^{8}\)In fact, the equations (24)-(34) are a modified version of a Jordan-Wigner transformation used in \([84]\), p. 13835, above of eq. (5).

\(^{9}\)Related equations can be found in:

- [90], p. 203, eq. (2),
- [91], p. 756, eq. (8),
- [92], p. 27, eq. (14),
- [94], p. 359, eq. (2.8),
- [95], p. 118, eq. (5), and
- [96], p. 205, eq. (9).

\(^{10}\) [97], p. 656, §8, eqs. (8.6), (8.7) (here, a five-dimensional formulation is still being used),

- [98], p. 32 (reprint # 1: p. 50, reprint # 2: p. 943), eq. (9),
- [99], p. 3121, eq. (F),
- [100], p. 136, eqs. (2.4), (2.10),
- [101], p. 127, eqs. (2), (3),
commutator \([\hat{c}_P, \hat{c}_Q]\) reads\(^{11}\):

\[
[\hat{c}_P, \hat{c}_Q] = \hat{c}_P \hat{c}_Q - \hat{c}_Q \hat{c}_P = -2 \delta_{pq1} \hat{c}_{(pq2)} - 2 \delta_{pq2} \hat{c}_{(pq1)} \\
+ 2 \delta_{pq11} \hat{c}_{(pq2)} + 2 \delta_{p2q11} \hat{c}_{(pq2)} ,
\]

(37)

while the anticommutator \(\{\hat{c}_P, \hat{c}_Q\}\) is given by\(^{12}\):

\[
\{\hat{c}_P, \hat{c}_Q\} = \hat{c}_P \hat{c}_Q + \hat{c}_Q \hat{c}_P \\
= (-2 \delta_{pq1} - 2 \delta_{pq2} \delta_{pq1}) \mathbb{1}_4 - 2i \epsilon_{pqN} \hat{c}_N .
\]

(38)

Equation (37) displays the structure of the Lie algebra of \(\mathfrak{su}(4) \cong \mathfrak{so}(6)\). Note that this Lie algebra (i.e., its generators) is realized here in terms of products of one to four fermion creation and annihilation operators. This representation of the Lie algebra \(\mathfrak{su}(4) \cong \mathfrak{so}(6)\) has been discussed earlier in \[112, 113\] (In general, for \(n\) fermionic modes the Lie algebra generated by all possible products of fermion creation and annihilation operators is \(\mathfrak{su}(2^n)\) \[114\], Sec. 5.2, p. 21, \[115\]). Most discussions, in the literature, of Lie algebras in terms of fermion creation and annihilation operators rely on their bilinear combinations only (see, e.g., \[116, 117\], Subsecs. 1.59, 1.60, pp. 88-93) or, at best, supplement these combinations by the fermion creation and annihilation operators themselves \[31, 35, 118, 119\], App. 1, p. 919 (p. 548 of the reprint), \[40, 121, 122\], p. 907, \[49\]. Cases that go beyond this linear/bilinear framework have been discussed in \[114\], Sec. 5.2, p. 21, \[112, 113, 115, 123, 124\]. Finally, we mention that certain graphical representations of the commutator/anticommutator relations (37), (38) have appeared in the literature \[125-130\]. To relate the works of Saniga and collaborators \[126-129\] and Rau \[130\] to the equations (37), (38) one must rely on the representation of the bivectors \(\hat{c}_M\) in terms of the (twofold) tensor product of quaternions \(\mathbb{H}\) discussed at

---

\(^{11}\)The commutator can be found in the literature in:

\[106\], p. 97, eq. (1a), \[91\], p. 756, eq. (7), \[92\], p. 27, eq. (12),

\[107\], p. 354, eq. (2), \[108\], p. B841, eq. (3.2), \[100\], p. 136, eq. (2.11),

\[109\], Subsec. 2d, p. 21, \[94\], p. 360, eq. (3.2), \[102\], p. 182, eq. (6),

\[110\], p. 368 (p. 1212 of the English transl.), below from eq. (7),

\[95\], p. 118, eq. (3), \[104\], p. 2244, eq. (III), \[96\], p. 204, eq. (3),

\[111\], Sec. 5.3, p. 168, \[105\], Appendix C, p. 105021-19, eq. (C.12).

\(^{12}\)The anticommutator can be found in the literature in:

\[98\], p. 33 (reprint # 1: p. 50, reprint # 2: p. 944), eq. (10a),

\[91\], p. 756, eq. (7), \[92\], p. 27, eq. (12), \[94\], p. 360, eq. (3.3),

\[102\], p. 182, eq. (7), \[95\], p. 122, eq. (40), \[61\], p. 341, eq. (1.11).
the end of Appendix D.

3 Nonlinear Bogolyubov-Valatin transformations

3.1 Clifford algebra formulation of canonical fermion transformations

3.1.1 The ansatz for nonlinear Bogolyubov-Valatin transformations

We proceed now to writing down an ansatz for the most general Bogolyubov-Valatin transformation for two fermionic modes. In view of the eqs. (6), (7), the new fermion annihilation and creation operators \( \hat{b}_k \), \( \hat{b}_k^\dagger \) read

\[
\hat{b}_k = B_k (\{\lambda\}; \{\hat{a}\}) = \hat{U} \hat{a}_k \hat{U}^\dagger
\]

\[
= \lambda_k^{(0)} \mathbb{1}_4 + \lambda_k^{(1)} \hat{a}_1 + \lambda_k^{(2)} \hat{a}_2 + \lambda_k^{(0,1)} \hat{a}_1 + \lambda_k^{(0,2)} \hat{a}_2
\]

\[
+ \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(1,1)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_1
\]

\[
+ \lambda_k^{(2)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(0,1,2)} \hat{a}_1 \hat{a}_2
\]

\[
+ \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_1 + \lambda_k^{(1,2)} \hat{a}_2 \hat{a}_2
\]

\[
+ \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_1
\]

\[
(39)
\]

\[
\hat{b}_k^\dagger = B_k (\{\lambda\}; \{\hat{a}\})^\dagger = \hat{U} \hat{a}_k^\dagger \hat{U}^\dagger
\]

\[
= \lambda_k^{(0)} \mathbb{1}_4 + \lambda_k^{(1)} \hat{a}_1^\dagger + \lambda_k^{(2)} \hat{a}_2^\dagger + \lambda_k^{(0,1)} \hat{a}_1^\dagger + \lambda_k^{(0,2)} \hat{a}_2^\dagger
\]

\[
- \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(1,1)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_1
\]

\[
+ \lambda_k^{(2)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(0,1,2)} \hat{a}_1 \hat{a}_2
\]

\[
- \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2 + \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2
\]

\[
- \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2
\]

\[
+ \lambda_k^{(1,2)} \hat{a}_1 \hat{a}_2
\]

\[
(40)
\]

We assume the coefficients to be complex numbers: \( \lambda^{(\cdot)} \in \mathbb{C} \). \( \{\lambda\} \) denotes the set of all 16 coefficients \( \lambda^{(\cdot)} \), and \( \{\hat{a}\} \) the set of the creation and annihilation operators.
\[ U = U(\{\lambda\}; \{\hat{a}\}) \] is an unitary operator belonging to the group \( SU(4) \) that implements the nonlinear Bogolyubov-Valatin transformation. The new annihilation operators \( \hat{b}_k \) act on the new vacuum state

\[ |0'\rangle = |0, \{\lambda\}\rangle = U(\{\lambda\}; \{\hat{a}\}) |0\rangle \tag{41} \]

the usual way

\[ \hat{b}_k |0'\rangle = 0 . \tag{42} \]

The new fermion creation and annihilation operators also must obey the canonical anticommutation relations [CAR, cf. eqs. (6), (7)], \((k,l = 1,2)\),

\[ \{\hat{b}_k^\dagger, \hat{b}_l\} = \hat{b}_k^\dagger \hat{b}_l + \hat{b}_k \hat{b}_l^\dagger = \delta_{kl} I_4 , \tag{43} \]

\[ \{\hat{b}_k, \hat{b}_l\} = \hat{b}_k \hat{b}_l + \hat{b}_l \hat{b}_k = 0 , \tag{44} \]

the latter equation entailing

\[ \{\hat{b}_k^\dagger, \hat{b}_l^\dagger\} = \hat{b}_k^\dagger \hat{b}_l^\dagger + \hat{b}_l^\dagger \hat{b}_k^\dagger = 0 . \tag{45} \]

Before proceeding further, it is useful to consider the expression for the trace of any annihilation (or creation) operator.

\[ \text{tr} \hat{b}_k \]

\[ = \langle 0'|\hat{b}_k|0'\rangle + \langle 1'|_1 \hat{b}_k|1'\rangle_1 + \langle 1'|_2 \hat{b}_k|1'\rangle_2 + \langle 2'|_{(1,2)} \hat{b}_k|2'\rangle_{(1,2)} = 0 \tag{46} \]

The trace has to vanish. From eq. (39), for eq. (46) it then follows:

\[ \text{tr} \hat{b}_k = 4\lambda_k^{(00)} + 2\lambda_k^{(11)} + 2\lambda_k^{(22)} - \lambda_k^{(1,2)} = 0 \tag{47} \]

(In the one-mode case studied earlier [49], the corresponding equation is eq. (13), p. 10247.)

To facilitate the further discussion, we now introduce the antihermitian operators

\[ \hat{b}_k^{(1)} = -\hat{b}_k^{(1)}\dagger = i \left( \hat{b}_k + \hat{b}_k^\dagger \right) \]

\[ = \text{Re} \kappa_k^{(1,0)} \hat{a}_1^{[1]} + \text{Re} \kappa_k^{(0,1)} \hat{a}_1^{[2]} + \text{Re} \kappa_k^{(2,0)} \hat{a}_1^{[1]} + \text{Re} \kappa_k^{(0,2)} \hat{a}_1^{[2]} + \text{Re} \kappa_k^{(1,1)} \hat{a}_1^{[1]} \hat{a}_2^{[1]} + \text{Re} \kappa_k^{(1,2)} \hat{a}_1^{[1]} \hat{a}_2^{[2]} + \text{Re} \kappa_k^{(2,1)} \hat{a}_1^{[2]} \hat{a}_2^{[1]} + \text{Re} \kappa_k^{(2,2)} \hat{a}_1^{[2]} \hat{a}_2^{[2]} + \text{Re} \kappa_k^{(3,0)} \hat{a}_1^{[3]} \hat{a}_2^{[0]} + \text{Re} \kappa_k^{(0,3)} \hat{a}_1^{[0]} \hat{a}_2^{[3]} + \text{Re} \kappa_k^{(1,3)} \hat{a}_1^{[1]} \hat{a}_2^{[3]} + \text{Re} \kappa_k^{(3,1)} \hat{a}_1^{[3]} \hat{a}_2^{[1]} + \text{Re} \kappa_k^{(2,3)} \hat{a}_1^{[2]} \hat{a}_2^{[3]} + \text{Re} \kappa_k^{(3,2)} \hat{a}_1^{[3]} \hat{a}_2^{[2]} + \text{Re} \kappa_k^{(3,3)} \hat{a}_1^{[3]} \hat{a}_2^{[3]} \]
and

\[ \hat{b}_k^{[2]} = -\hat{b}_k^{[2]†} = \hat{b}_k - \hat{b}_k^† \]

\[ = \text{Im} \kappa_k^{(1|0)} \hat{a}_1^{[1]} + \text{Im} \kappa_k^{(0|1)} \hat{a}_1^{[2]} + \text{Im} \kappa_k^{(2|0)} \hat{a}_2^{[1]} + \text{Im} \kappa_k^{(0|2)} \hat{a}_2^{[2]} \]

\[ + \text{Im} \kappa_k^{(1)1} \hat{a}_1^{[1]} \hat{a}_1^{[2]} + \text{Im} \kappa_k^{(1,2|0)} \hat{a}_1^{[2]} \hat{a}_2^{[1]} + \text{Im} \kappa_k^{(2|1)} \hat{a}_1^{[1]} \hat{a}_2^{[2]} \]

\[ + \text{Im} \kappa_k^{(1,2|1)1} \hat{a}_1^{[1]} \hat{a}_2^{[1]} \hat{a}_2^{[2]} + \text{Im} \kappa_k^{(1,2|1,2)} \hat{a}_1^{[1]} \hat{a}_2^{[1]} \hat{a}_2^{[2]} \]

\[ + \text{Im} \kappa_k^{(1,2|2)2} \hat{a}_1^{[1]} \hat{a}_1^{[2]} \hat{a}_2^{[2]} + \text{Im} \kappa_k^{(1,2|1,2)} \hat{a}_1^{[1]} \hat{a}_2^{[1]} \hat{a}_2^{[2]} \]

\[ + \text{Im} \kappa_k^{(1,2|1,2)} \hat{a}_1^{[1]} \hat{a}_2^{[1]} \hat{a}_2^{[2]} \]

\[ = -2 \delta_{pq} \delta_{kl} \mathbb{1}_4 . \quad (49) \]

With some hindsight, the primary ordering of the operators on the r.h.s. is being done in accordance with the mode number. In deriving the eqs. (48), (49) we have taken into account eq. (47). The explicit relation of the coefficients \(\kappa^{(\cdots)}\) in the above two equations to the coefficients in the eqs. (39), (40) is given in Appendix A. As a consequence of the CAR, eqs. (43)-(45), the operators \(\hat{b}_k^{[p]}\) must obey the equation:

\[ \left\{ \hat{b}_k^{[p]}, \hat{b}_l^{[q]} \right\} = -2 \delta_{pq} \delta_{kl} \mathbb{1}_4 . \quad (50) \]

### 3.1.2 Clifford algebra counterpart of nonlinear Bogolyubov-Valatin transformations

In analogy to eq. (12), we introduce the following useful notation:

\[ \hat{d}_{2k-1} = \hat{b}_k^{[1]} , \quad k = 1, \ldots, n (= 2) , \]

\[ \hat{d}_{2k} = \hat{b}_k^{[2]} , \quad k = 1, \ldots, n (= 2) . \quad (51) \]

The operators \(\hat{d}_k\) must obey the \(C(0, 2n = 4)\) Clifford algebra relation \([k, l = 1, \ldots, 2n = 4]\)

\[ \left\{ \hat{d}_k, \hat{d}_l \right\} = 2 \delta_{kl} \mathbb{1}_4 , \quad (52) \]

This relation is the Clifford algebra analogue of the canonical anticommutation relations, (43)-(45), for fermions. The study of nonlinear Bogolyubov-Valatin transformations can consequently be performed by studying nonlinear basis transformations in the Clifford algebra \(C(0, 4)\).

To prepare ourselves for the further investigation, the equations (48), (49) now can be compactly written as \([\text{The factor of } i \text{ in the last two terms can also be} \]

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The tensors \( \chi_k^{[m]} \), \( \chi_k^{[m,n]} \), \( \chi_k^{[m,n,p]} \), and \( \chi_k^{[m,n,p,q]} \) are real and the last three tensors are totally antisymmetric with respect to their upper indices. For the relation of the tensors \( \chi_k^{[l]} (\cdots) \) to the coefficients \( \kappa^{(\cdots)} \) see Appendix A.

In principle, already here one could go one step further and define the operators \( \hat{d}_k \) in terms of biparavectors (with some coefficients \( \chi_k^M \)):

\[
\hat{d}_k = - \hat{d}_k^\dagger = - \chi_k^M \hat{c}_M .
\]

However, this step can much better be understood in course of the further investigation and we postpone it, therefore, to Part II of Subsec. 3.2.3, eq. (105).

### 3.2 Nonlinear basis transformations in the Clifford algebra \( C(0, 4) \)

#### 3.2.1 Anticommutators – Conditions for canonical transformations

From the eqs. (53), (54), we find for the anticommutator, eq. (52),

\[
\left\{ \hat{d}_k, \hat{d}_l \right\} = \hat{d}_l \hat{d}_k + \hat{d}_k \hat{d}_l \overset{\text{def}}{=} 2 \hat{d}_k \cdot \hat{d}_l
\]
\[
\begin{align*}
&= -2 \left[ \chi_k^{[1]} (m) \chi_l^{[1]} (m) + \frac{1}{2!} \chi_k^{[2]} (m,n) \chi_l^{[2]} (m,n) \\
&\quad + \frac{1}{3!} \chi_k^{[3]} (m,n,p) \chi_l^{[3]} (m,n,p) + \frac{1}{4!} \chi_k^{[4]} (m,n,p,q) \chi_l^{[4]} (m,n,p,q) \right] \mathbb{I}_4 \\
&\quad - i \left[ \chi_k^{[2]} (m,n) \chi_l^{[3]} (m,n,q) + \chi_k^{[3]} (m,n,q) \chi_l^{[2]} (m,n) \right] \hat{c}_q \\
&\quad - i \left[ \chi_k^{[1]} (m) \chi_l^{[3]} (m,p,q) + \chi_k^{[3]} (m,p,q) \chi_l^{[1]} (m) \\
&\quad + \frac{1}{2} \chi_k^{[2]} (m,n) \chi_l^{[4]} (m,n,p,q) + \frac{1}{2} \chi_k^{[4]} (m,n,p,q) \chi_l^{[2]} (m,n) \right] \hat{c}_p \wedge \hat{c}_q \\
&\quad + \left[ \chi_k^{[1]} (n) \chi_l^{[2]} (p,q) + \chi_k^{[2]} (p,q) \chi_l^{[1]} (n) \right] \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q \\
&\quad + \frac{1}{2} \chi_k^{[2]} (m,n) \chi_l^{[2]} (p,q) \hat{c}_m \wedge \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q .
\end{align*}
\]

The above result has been derived by means of the eqs. (B.1)-(B.4) given in Appendix B [In the product \( \hat{d}_k \hat{d}_l \), use for the first factor the representation (53) and for the second factor (54)]. Now, imposing on the anticommutator (57) the Clifford algebra analogue of the canonical anticommutation relation (CAR)

\[
\left\{ \hat{d}_k, \hat{d}_l \right\} = 2 g_{kl} \mathbb{I}_4 = -2 \delta_{kl} \mathbb{I}_4 \tag{58}
\]

the equations (59)-(63) follow:

- **scalar part of eq. (58):**

\[
\begin{align*}
&\chi_k^{[1]} (m) \chi_l^{[1]} (m) + \frac{1}{2!} \chi_k^{[2]} (m,n) \chi_l^{[2]} (m,n) \\
&\quad + \frac{1}{3!} \chi_k^{[3]} (m,n,p) \chi_l^{[3]} (m,n,p) + \frac{1}{4!} \chi_k^{[4]} (m,n,p,q) \chi_l^{[4]} (m,n,p,q) \\
&= \left[ \chi_k^{[1]} (1) \chi_l^{[1]} (1) + \chi_k^{[2]} (1,2) \chi_l^{[2]} (1,2) + \chi_k^{[3]} (1,3) \chi_l^{[3]} (1,3) + \chi_k^{[4]} (1,4) \chi_l^{[4]} (1,4) \\
&\quad + \chi_k^{[2]} (1,2) \chi_l^{[1]} (2,1) + \chi_k^{[3]} (1,2,3) \chi_l^{[2]} (1,3) + \chi_k^{[2]} (1,3) \chi_l^{[1]} (2,1) + \chi_k^{[2]} (1,4) \chi_l^{[1]} (2,4) \\
&\quad + \chi_k^{[2]} (2,3) \chi_l^{[2]} (2,3) + \chi_k^{[2]} (2,4) \chi_l^{[2]} (2,4) + \chi_k^{[2]} (3,4) \chi_l^{[2]} (3,4) \\
&\quad + \chi_k^{[3]} (1,2,3) \chi_l^{[3]} (1,2,3) + \chi_k^{[3]} (1,2,4) \chi_l^{[3]} (1,2,4) + \chi_k^{[3]} (1,3,4) \chi_l^{[3]} (1,3,4) \\
&\quad + \chi_k^{[3]} (2,3,4) \chi_l^{[3]} (2,3,4) + \chi_k^{[4]} (1,2,3,4) \chi_l^{[4]} (1,2,3,4) \right] = \delta_{kl} \tag{59}
\end{align*}
\]
The above equation is the orthogonality/orthonormality condition for 4 \n- \text{vectors in a 15-dimensional Euclidean vector space.},

- vector part of eq. (58):
  \[
  \left[ \chi_k^{[2]} (m,n) \chi_l^{[3]} (m,n,q) + \chi_k^{[3]} (m,n,q) \chi_l^{[2]} (m,n) \right] \hat{c}_q = 0 , \quad (60)
  \]

- bivector part of eq. (58):
  \[
  \left[ \chi_k^{[1]} (m) \chi_l^{[3]} (m,p,q) + \chi_k^{[3]} (m,p,q) \chi_l^{[1]} (m) \right] \\
  + \frac{1}{2} \chi_k^{[2]} (m,n) \chi_l^{[4]} (m,n,p,q) + \frac{1}{2} \chi_k^{[4]} (m,n,p,q) \chi_l^{[2]} (m,n) \right] \hat{c}_p \wedge \hat{c}_q = 0 , \quad (61)
  \]

- trivector part of eq. (58):
  \[
  \left[ \chi_k^{[1]} (n) \chi_l^{[2]} (p,q) + \chi_k^{[2]} (p,q) \chi_l^{[1]} (n) \right] \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q = 0 , \quad (62)
  \]

- quadrivector part of eq. (58):
  \[
  \chi_k^{[2]} (m,n) \chi_l^{[2]} (p,q) \hat{c}_m \wedge \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q = 0 . \quad (63)
  \]

These equations have to be studied in order to investigate basis transformations in the Clifford algebra \( C(0,4) \) and, consequently, nonlinear Bogolyubov-Valatin transformations for two fermionic modes.

For the sake of completeness, here we also give the expression for the commutator of the \( \hat{d}_k \) (which, incidentally, also defines their wedge product).

\[
\left[ \hat{d}_k, \hat{d}_l \right] = \hat{d}_k \hat{d}_l - \hat{d}_l \hat{d}_k \stackrel{\text{def}}{=} 2 \hat{d}_l \wedge \hat{d}_k \\
= 2 \left[ \chi_k^{[2]} (m,q) \chi_l^{[1]} (m) - \chi_k^{[1]} (m) \chi_l^{[2]} (m,q) + \frac{1}{3!} \chi_k^{[4]} (m,n,p,q) \chi_l^{[3]} (m,n,p) \\
- \frac{1}{3!} \chi_k^{[3]} (m,n,p,q) \chi_l^{[4]} (m,n,p,q) \right] \hat{c}_q \\
+ 2 \left[ \chi_k^{[1]} (p) \chi_l^{[1]} (q) + \chi_k^{[2]} (p,m) \chi_l^{[2]} (m,q) \\
+ \frac{1}{2} \chi_k^{[3]} (m,n,p) \chi_l^{[3]} (m,n,q) \right] \hat{c}_p \wedge \hat{c}_q \\
+ i \left[ \frac{1}{3} \chi_k^{[4]} (m,n,p,q) \chi_l^{[1]} (m) - \frac{1}{3} \chi_k^{[1]} (m) \chi_l^{[4]} (m,n,p,q) \\
- \frac{1}{3} \chi_k^{[1]} (m) \chi_l^{[4]} (m,n,p,q) \\
- \frac{1}{3} \chi_k^{[4]} (m,n,p,q) \chi_l^{[1]} (m) \right]
\]
\[ + \chi_k^{[2]} (r,n) \delta_{rs} \chi_l^{[3]} (s,p,q) - \chi_k^{[3]} (s,p,q) \delta_{rs} \chi_l^{[2]} (r,n) \] \hat{c}_n \land \hat{c}_p \land \hat{c}_q

\[ + \frac{i}{3} \left[ \chi_k^{[1]} (m) \chi_l^{[3]} (n,p,q) - \chi_k^{[3]} (n,p,q) \chi_l^{[1]} (m) \right] \hat{c}_m \land \hat{c}_n \land \hat{c}_p \land \hat{c}_q \] (64)

At the end of this subsection, a conceptual comment is in order. The first two lines of eq. (57) define the inner product in the new set of vectors \( \hat{d}_k \). As the vectors \( \hat{d}_k \) are not linear combinations of the original vectors \( \hat{c}_k \), the vector space within the Clifford algebra \( C(0, 4) \) spanned by the vectors \( \hat{d}_k \) differs from the vector space spanned by the vectors \( \hat{c}_k \). Consequently, also the wedge products built over the respective vector spaces differ conceptually, as do concepts related to them such as the grade of a multivector. Consequently, the first line of eq. (64) defines the wedge product related to the vector space spanned by the vectors \( \hat{d}_k \) (We indicate this fact by means of the letter ‘d’ atop of the wedge symbol: \( \land \)). To see this explicitly compare eq. (64) with the following equation which has been evaluated using the linearity and associativity of the wedge product related to the vectors \( \hat{c}_k \).

\[
\hat{d}_k \land \hat{d}_l = \chi_k^{[1]} (p) \chi_l^{[1]} (q) \hat{c}_p \land \hat{c}_q \\
+ \frac{1}{2!} \left[ \chi_k^{[1]} (n) \chi_l^{[2]} (p,q) + \chi_k^{[2]} (p,q) \chi_l^{[1]} (n) \right] \hat{c}_n \land \hat{c}_p \land \hat{c}_q \\
+ \left[ \frac{i}{3!} \chi_k^{[1]} (m) \chi_l^{[3]} (n,p,q) - \frac{i}{3!} \chi_k^{[3]} (n,p,q) \chi_l^{[1]} (m) \right] \hat{c}_m \land \hat{c}_n \land \hat{c}_p \land \hat{c}_q \\
+ \frac{1}{4} \chi_k^{[2]} (m,n) \chi_l^{[2]} (p,q) \hat{c}_m \land \hat{c}_n \land \hat{c}_p \land \hat{c}_q \] (65)

Contrary to what naively one might expect, the product \( \hat{d}_k \land \hat{d}_l \) fails to be antisymmetric with respect to the indices \( k \) and \( l \) unless the condition \( \hat{d}_k \cdot \hat{d}_l = -\delta_{kl} \) \( 1 \) [eq. (58)] is imposed.

### 3.2.2 Solution of the Clifford algebra analogue of the CAR – Infinitesimal method

We are now going to solve the eqs. (59)-(63) by means of two different methods, an infinitesimal and a global one. We start with the infinitesimal method in the vicinity of the identical Bogolyubov-Valatin transformation. We apply the relations \( \chi_k^{[1]} (m) = \delta_{km} + \Delta \chi_k^{[1]} (m), \chi_k^{[2]} (m,n) = \Delta \chi_k^{[2]} (m,n) + \Delta \chi_k^{[2]} (m,n,p) + \Delta \chi_k^{[3]} (m,n,p), \chi_k^{[4]} (m,n,p,q) = \Delta \chi_k^{[4]} (m,n,p,q) \), where all quantities preceeded by \( \Delta \) are infinitesimally
small. Neglecting higher order terms the eqs. (59), (61), (62) yield the following conditions, respectively:

\[
\Delta \chi^{[1]}_{k \langle (l) \rangle} + \Delta \chi^{[1]}_{l \langle (k) \rangle} = 0 ,
\]

(66)

\[
\Delta \chi^{[3]}_{k \langle (m,n,l) \rangle} + \Delta \chi^{[3]}_{l \langle (m,n,k) \rangle} = 0 ,
\]

(67)

\[
\Delta \chi^{[2]*}_{l \langle (k,m) \rangle} + \Delta \chi^{[2]*}_{k \langle (l,m) \rangle} = 0 .
\]

(68)

Here, \(\Delta \chi^{[2]*}_{k \langle (l,m) \rangle} = \epsilon_{lmpq} \Delta \chi^{[2]}_{k \langle (p,q) \rangle} (\epsilon_{lmpq} \text{ is the completely antisymmetric tensor in four dimensions with } \epsilon_{1234} = 1.)\). To arrive at this form of the eq. (68) we have made use of the relation

\[
\epsilon_{mnpq} \epsilon_{rstq} = \delta_{mr} \delta_{ns} \delta_{pt} + \delta_{ms} \delta_{nt} \delta_{pr} - \delta_{mt} \delta_{nr} \delta_{ps} + \delta_{mt} \delta_{ns} \delta_{pr} .
\]

(69)

The eqs. (60) and (63) do not give rise to any condition for the infinitesimal parameters. From the eqs. (66)-(68) we can conclude that \(\Delta \chi^{[1]}_{k \langle (l) \rangle} ; \Delta \chi^{[2]*}_{k \langle (l,m) \rangle} ; \Delta \chi^{[3]}_{k \langle (m,n,l) \rangle}\) are objects which are completely antisymmetric in all of their lower indices. Consequently, we find that the infinitesimal (nonlinear) Bogolyubov-Valatin transformations depend on 15 parameters: 6 parameters given by \(\Delta \chi^{[1]}_{k \langle (l) \rangle}\), 4 parameters given by \(\Delta \chi^{[2]*}_{k \langle (l,m) \rangle}\), 1 parameter given by the totally antisymmetric object \(\Delta \chi^{[3]}_{k \langle (m,n,l) \rangle}\) and 4 more parameters given by \(\Delta \chi^{[4]}_{k \langle (m,n,p,q) \rangle}\) (The latter object does not receive any restrictions within our infinitesimal consideration.). The number of parameters for each object is in agreement with the global solution to be discussed further below [cf. eqs. (80), (74), (84), (89)] and the total number of 15 parameters stands in correspondence to insight coming from the implementation of the Bogolyubov-Valatin transformation by means of an unitary transformation \(U(\{\lambda\}; \hat{c}) [\text{cf. eq. (39)}]\)

\[
\hat{d}_{k} = B (\{\lambda\}; \hat{c}) = U (\{\lambda\}; \hat{c}) \hat{c}_{k} U (\{\lambda\}; \hat{c})^\dagger .
\]

(70)

\(U(\{\lambda\}; \hat{c})\) is for two \((n = 2)\) fermionic modes an element of \(SU(2^n = 4)\) which is a 15-parametric group and the double cover of the group \(SO(6; \mathbb{R})\).

3.2.3 Solution of the Clifford algebra analogue of the CAR – Global method

I. Structural analysis of the Clifford algebra analogue of the CAR

We turn our attention now to the global study of the eqs. (59)-(63). The best strategy appears to be to start with the consideration of the eq. (63), the quadrivector part of these equations. The reason for this lies in the fact that in difference to the other eqs. (59)-(62), eq. (63) depends on one type of coefficients \(\chi^{[2]}_{k \langle (m,n) \rangle}\), i.e., the bivector coefficients in eq. (53)] only.
(A) Quadrivector part

For the case $k = l$, eq. (63) reads

$$\hat{\chi}_k^{[2]} \wedge \hat{\chi}_k^{[2]} = 0 \quad (71)$$

where we have applied the notation $\hat{\chi}_k^{[2]} = \chi_k^{[2]} (m,n) \hat{c}_m \wedge \hat{c}_n$ for the bivector $\hat{\chi}_k^{[2]}$. Equivalently, this condition can be formulated as ([134], §7, p. 101, eq. (3) [p. 2250, eq. (7.3) of the English transl.])

$$\langle \hat{\chi}_k^{[2]*}, \hat{\chi}_k^{[2]} \rangle = \epsilon^{mnpq} \chi_k^{[2]} (m,n) \chi_k^{[2]} (p,q) = 0 . \quad (72)$$

Here, $\langle A, B \rangle$ denotes the scalar product of the bivectors $A$, $B$. For its definition see, e.g., [134], §1, p. 85, eq. (3) [eq. (1.3), p. 2240 of the English translation]. $\hat{\chi}_k^{[2]*}$ denotes the bivector that is related to the Hodge dual of $\chi_k^{[2]}$: $\chi_k^{[2]*} (m,n) = \epsilon^{mnpq} \chi_k^{[2]} (p,q)$. Finally, a third, equivalent formulation for the condition (71) is:

$$\text{Pf} \chi_k^{[2]} = 0 \quad (73)$$

where Pf denotes the Pfaffian of the matrix $\chi_k^{[2]}$. The eqs. (71)-(73) each are the condition for the decomposability of the bivector $\hat{\chi}_k^{[2]}$ 14. Consequently, we can write for the matrix $\chi_k^{[2]} (m,n)$ and the corresponding bivector [For the last part of eq. (76) use eq. (B.5).]

$$\chi_k^{[2]} (m,n) = \beta^{(m)} \mathcal{H}_k^{(n)} - \mathcal{H}_k^{(m)} \beta^{(n)} , \quad (74)$$

$$\chi_k^{[2]} = 2 \beta \wedge \mathcal{H}_k , \quad (75)$$

$$\hat{\chi}_k^{[2]} = 2 \hat{\beta} \wedge \hat{\mathcal{H}}_k = 2 \hat{\beta} \hat{\mathcal{H}}_k - 2 \mathcal{H}_k \beta^T 1_4 . \quad (76)$$

Here, $\hat{\mathcal{H}}_k = \mathcal{H}_k^{(m)} \hat{c}_m$. $\mathcal{H}_k$, $\beta$ denote row vectors ($1 \times 4$ matrices)15. The ansatz (74), (76) immediately also solves eq. (63) for $k \neq l$.

$$\hat{\chi}_k^{[2]} \wedge \hat{\chi}_l^{[2]} = 0$$

\[14\] [135], Vol. I, p. 26 (p. 19 of the English transl.),
[136], all eds., Chap. II, § 6, p. 27,
[137], p. 25, Exercise 5.1,
[138], Vol. 1, Chap. 3, end of § 5, p. 165,
[139], p. 69,
[140], p. 101 (p. 2250 of the English transl.),
[141], Chap. 8, end of § 4, Пример 3 [Primer 3]/[Example 3], p. 341, 2, and 3. ed. (p. 324 of the English transl.),
[142], Vol. 2 (Часть II: Линейная Алгебра [Chast’ II: Lineínaya Algebra]), гл. 6 [Chap. 6], § 5, pp. 293-295.
\[15\]Their elements will finally turn out to be real and can, for simplicity, immediately be chosen to be real in view of the fact that $\chi_k^{[2]} (m,n) \in \mathbb{R}$.
(B) Trivector part

Eq. (62) can be written as follows.

\[ \hat{\chi}^{[1]}_k \wedge \hat{\chi}^{[2]}_l + \hat{\chi}^{[2]}_k \wedge \hat{\chi}^{[1]}_l = 0 \quad (78) \]

Introducing the ansatz (76) into eq. (78), it can be transformed to read

\[ \hat{\beta} \wedge \left[ \hat{\chi}^{[1]}_k \wedge \hat{H}_l - \hat{H}_k \wedge \hat{\chi}^{[1]}_l \right] = 0 \quad (79) \]

and its solution can immediately be read off to be \((H_k^{(0)}, \beta^{(0)})\) are real numbers)

\[ \chi^{[1](m)}_k = \beta^{(0)} H^{(m)}_k - H^{(0)}_k \beta^{(m)} , \quad \hat{\chi}^{[2]}_k = \beta^{(0)} \hat{H}_k - H^{(0)}_k \hat{\beta} . \quad (80) \]

The significance of the chosen notation will become evident in a moment.

(C) Vector part

We can now also study eq. (60). Introducing the Hodge dual of \(\chi_k^{[3]}\) by writing \(\chi_k^{[3]}(m,n,p) = \epsilon^{mnpq} \chi_k^{[3]*} \) it reads

\[ \chi^{[2](m,n)}_k \epsilon^{mnpq} \chi^{[3]*}_l(q) + \chi^{[3]*}_k \epsilon^{mnpq} \chi^{[2](m,n)}_l = 0 . \quad (82) \]

Operating on it with the expression \(\epsilon_{rstp} \hat{c}^r \wedge \hat{c}^s \wedge \hat{c}^t\) and summing over repeated indices, eq. (82) can be brought to the form [taking into account eq. (69)]

\[ \hat{\chi}^{[2]}_k \wedge \chi^{[3]*}_l + \chi^{[3]*}_k \wedge \hat{\chi}^{[2]}_l = 0 . \quad (83) \]

This equation is analogous to eq. (78), and immediately its solution can be found to be \((H_k^{(-1)}, \beta^{(-1)})\) are real numbers, the sign is chosen with hindsight)

\[ \chi^{[3]*}(m) = - \left( \beta^{(-1)} H^{(m)}_k - H^{(-1)}_k \beta^{(m)} \right) , \quad \hat{\chi}^{[3]*} = - \left( \beta^{(-1)} \hat{H}_k - H^{(-1)}_k \hat{\beta} \right) , \quad (84) \]

or

\[ \chi^{[3](m,n,p)}_k = - \epsilon^{mnpq} \left( \beta^{(-1)} H_k(q) - H^{(-1)}_k \beta(q) \right) , \quad \hat{\chi}^{[3]}_k = - \left( \beta^{(-1)} \hat{H}_k^* - H^{(-1)}_k \hat{\beta}^* \right) . \quad (85) \]

where we have applied the notation \(\hat{\chi}^{[3]}_k = \chi^{[3]}_k(m,n,p) \hat{c}_m \wedge \hat{c}_n \wedge \hat{c}_p\) for the trivector \(\hat{\chi}^{[3]}_k\), and \(H_k^{*}(m,n,p) = \epsilon^{mnpq} H_k(q), \beta^{*}(m,n,p) = \epsilon^{mnpq} \beta(q)\). Again, the significance of the chosen notation will become evident in a moment.
We turn our attention now to eq. (61). Introducing the Hodge dual of $\chi^4$ by writing

$$\chi^4_k (m,n,p,q) = \epsilon^{mnpq} \chi^4_{k^*}$$

and making use of the eqs. (74), (80), (62) it reads:

$$\left[ \beta_{(m)} H_k (n) \left( \beta^{(0)} H_l^{(-1)} - \beta^{(-1)} H_l^{(0)} \right) \right.$$  

$$+ \left( \beta^{(0)} H_k^{(-1)} - \beta^{(-1)} H_k^{(0)} \right) \beta_{(m)} H_l (n)$$  

$$- \beta_{(m)} H_k (n) \chi^4_{l^*} - \chi^4_{k^*} \beta_{(m)} H_l (n) \right] \epsilon^{mnpq} c_p \wedge c_q = 0 \quad (88)$$

And the solution of this equation is found to be:

$$\chi^4_{k^*} = \beta^{(0)} H_k^{(-1)} - \beta^{(-1)} H_k^{(0)} \quad (89)$$

$$\chi^4_{k} (m,n,p,q) = \epsilon^{mnpq} \left( \beta^{(0)} H_k^{(-1)} - \beta^{(-1)} H_k^{(0)} \right) \quad (90)$$

The remaining task consists in studying eq. (59). Inserting the eqs. (80), (74), (86), (90) into it, it can be transformed to read:

$$\left[ \sum_{m=-1}^{4} (\beta^{(m)})^2 \right] \left[ \sum_{n=-1}^{4} H^{(n)}_k H^{(n)}_l \right]$$  

$$- \left[ \sum_{m=-1}^{4} H^{(m)}_k \beta^{(m)} \right] \left[ \sum_{n=-1}^{4} H^{(n)}_l \beta^{(n)} \right] = \delta_{kl} \quad (91)$$

This equation obviates the usefulness of the notation chosen earlier in the subparagaphs (A)-(D). From now on we denote by the symbol $H$ the $4 \times 6$ matrix $H$ with row number $k$ and column number $m$ matrix elements $H^{(m)}_k$ and by $\beta$ the row vector ($1 \times 6$ matrix) $\beta$. Eq. (91) can then compactly be written as

$$|\beta|^2 HH^T - (H\beta^T) (H\beta^T) = 1_4 \quad (92)$$

It is immediately obvious from eq. (92) that it is invariant under $(S)O(6;\mathbb{R})$ transformations $[H' = HL, \beta' = \beta L, L$ is a $6 \times 6$ matrix with $L \in (S)O(6;\mathbb{R})]$. Consequently, any two-mode nonlinear Bogolyubov-Valatin transformation can be parametrized by means of the $(S)O(6;\mathbb{R})$ transformed parameters of any particular (fixed) Bogolyubov-Valatin transformation. The most natural starting point is the identical Bogolyubov-Valatin transformation $\hat{d}_k = \hat{c}_k$, i.e., in view of eq. (80) we can choose

$$\beta = \beta_I = (0,1,0,0,0,0) \quad (93)$$

$$H^{(m)}_k = H^{(m)}_{I_k} = \delta_k^m, \quad k,m = 1,\ldots,4 \quad (94)$$

$$H^{(-1)}_k = H^{(0)}_k = 0 \quad (95)$$
We immediately find that always
\[ \mathcal{H} \beta^T = 0, \]  
\[ \beta \beta^T = |\beta|^2 = 1 \]  
(96)  
(97)

applies. Consequently, eq. (92) finally reads
\[ \mathcal{H}^T = \mathbb{I}_4. \]  
(98)

We can now go one step further in streamlining the notation by writing \( \beta = \mathcal{H}_0 \) (i.e., \( \beta^{(m)} = \mathcal{H}_0^{(m)} \)). From now on we denote by the symbol \( \mathcal{H} \) the 5 × 6 matrix \( \mathcal{H} \) with row number \( k = 0, \ldots, 4 \) and column number \( m = -1, \ldots, 4 \) matrix elements \( \mathcal{H}_k^{(m)} \). The equations (96)-(98) can then compactly be written in a single line as
\[ \mathcal{H}^T = \mathbb{I}_5. \]  
(99)

Any nonlinear Bogolyubov-Valatin transformation can conveniently be parametrized by means of a \((S)O(6; \mathbb{R})\) matrix \( L \) according to the equation
\[ \mathcal{H} = \mathcal{H}_I \mathcal{L}. \]  
(100)

The above structural analysis of the Clifford algebra analogue of the CAR has now paved the way for a systematic discussion of nonlinear basis transformations in the Clifford algebra \( C(0, 4) \) (and the nonlinear Bogolyubov-Valatin transformations related to them).

II. Biparavector structure of basis transformations in the Clifford algebra \( C(0, 4) \)

To structure the results obtained in the previous subsection, let us define the antisymmetric 6 × 6 matrix \( \chi_{0k}^{(m_1, m_2)} \) with matrix elements \( \chi_{0k}^{(m_1, m_2)} \) by means of the equation\([M = (m_1, m_2)]\)

\[ \chi_{0k}^{(m_1, m_2)} = \chi_{0k}^{M} = \mathcal{H}_{0}^{(m_1)} \mathcal{H}_{k}^{(m_2)} - \mathcal{H}_{0}^{(m_2)} \mathcal{H}_{k}^{(m_1)} = C_2(\mathcal{H})_{0k}^{m_1 m_2} \]  
(101)

or, symbolically,
\[ \chi_{0k} = \begin{pmatrix} 0 & -\chi_{k}^{[4]} & -\chi_{k}^{[3]} \\ \chi_{k}^{[4]} & 0 & \chi_{k}^{[1]} \\ \chi_{k}^{[3]} & -\chi_{k}^{[1]} & \chi_{k}^{[2]} \end{pmatrix}^T = -\chi_{0k}^T \]  
(102)

\[ = 2 \mathcal{H}_0 \wedge \mathcal{H}_k = 2 \left[ (\mathcal{H}_I \wedge \mathcal{H}_I) \mathcal{C}_2(\mathcal{L}) \right]_{0k}. \]  
(103)

Here, \( \chi_{k}^{[1]} \), \( \chi_{k}^{[3]} \) denote the row vectors (1 × 6 matrices) \( \chi_{k}^{[1]} \), \( \chi_{k}^{[3]} \) with components \( \chi_{k}^{[1]} (m) \), \( \chi_{k}^{[3]} (m) \), respectively, and \( \mathcal{C}_l(\mathcal{L}) \) denotes the order \( l \) compound matrix of the matrix \( \mathcal{L} \) (cf. Appendix C). Writing eq. (53) as
\[ \hat{d}_k = \chi_{k}^{[1]} (m) \hat{c}_m + \frac{1}{2} \chi_{k}^{[2]} (m,n) \hat{c}_m \hat{c}_n - \chi_{k}^{[3]} (q) \hat{c}_{(-1)} \hat{c}_q + \chi_{k}^{[4]} \hat{c}_{(-1)} \]  
(104)
one can convince oneself that the following representation applies [here, \( M = (m_1, m_2), m_1 < m_2 \)].

\[
\hat{d}_k = - \hat{d}_k^\dagger = - \frac{1}{2} \hat{\chi} \chi_{0k} \hat{\chi} = - \chi_{0k}^M \hat{\chi}_M 
\]

\[
= -2 (\mathcal{H}_0 \wedge \mathcal{H}_k)^M \hat{\chi}_M 
\]

(105)

We use the notation

\[
\hat{\chi} = \left( \begin{array}{c}
\hat{c}_{-1} \\
\hat{c}_0 \\
\hat{c}_1 \\
\hat{c}_2 \\
\hat{c}_3 \\
\hat{c}_4 \\
\end{array} \right) = \left( \begin{array}{c}
i\hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_4 \\
-\mathbb{I}_4 \\
\hat{c}_1 \\
\hat{c}_2 \\
\hat{c}_3 \\
\hat{c}_4 \\
\end{array} \right), 
\]

(107)

\[
\hat{\chi}^\dagger = \left( \begin{array}{c}
\hat{\chi}_{-1}^\dagger \\
\hat{\chi}_0^\dagger \\
\hat{\chi}_1^\dagger \\
\hat{\chi}_2^\dagger \\
\hat{\chi}_3^\dagger \\
\hat{\chi}_4^\dagger \\
\end{array} \right) = \left( \begin{array}{c}
-i\hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_4, -\mathbb{I}_4, -\hat{c}_1, -\hat{c}_2, -\hat{c}_3, -\hat{c}_4 \\
\end{array} \right). 
\]

(108)

\( \hat{\chi}_M \) used in eq. (105) is a biperavector [cf. eq. (23)]. Taking into account the eqs. (93)-(95), (103), eq. (106) can also be written as

\[
\hat{d}_k = - C_2(L)_{0k}^M \hat{\chi}_M. 
\]

(109)

This equation describes the structure of nonlinear basis transformations in the Clifford algebra \( C(0, 4) \). However, as these transformations must have a group theoretical structure the eqs. (106), (109) fail to reflect this structural, group theoretical, aspect because the new (transformed) generators \( \hat{d}_k \) of the Clifford algebra \( C(0, 4) \) are given in terms of other (however, related) objects – the biperavectors \( \hat{\chi}_M \).

We will now derive a more symmetric representation of the nonlinear basis transformations in the Clifford algebra \( C(0, 4) \). Taking into account eq. (23), we find (with some hindsight) for eq. (106)

\[
- \hat{d}_k = \hat{d}_{0k} = \frac{1}{2} \left( \hat{\chi}^\dagger \hat{\chi} - \hat{\chi}^\dagger \hat{\chi} \hat{d}_0 \hat{d}_0 \right) 
\]

\[
= 2 (\mathcal{H}_0 \wedge \mathcal{H}_k)^M \hat{\chi}_M = C_2(\mathcal{H})_{0k}^M \hat{\chi}_M 
\]

\[
= \frac{1}{2} \left( \hat{\mathcal{H}}_0 \mathcal{H}_k - \mathcal{H}_k \hat{\mathcal{H}}_0 \right). 
\]

(110)
Here, \( \hat{\mathcal{H}}_k = \mathcal{H}_k \hat{c} = \sum_{m=-1}^{4} \mathcal{H}^{(m)}_k \hat{c}_m \) is a paravector. From the eqs. (20), (99) we recognize that for the paravectors \( \mathcal{H}_k \) applies (\( k, l = 0, \ldots, 4 \))

\[
\hat{\mathcal{H}}_k \hat{\mathcal{H}}_l + \hat{\mathcal{H}}_l \hat{\mathcal{H}}_k = \hat{\mathcal{H}}_k \hat{\mathcal{H}}_l + \hat{\mathcal{H}}_l \hat{\mathcal{H}}_k = 2 \delta_{kl} \mathbb{I}_4 .
\]  

(111)

Consequently, we obtain

\[
- \hat{d}_k = \hat{d}_{0k} = \hat{d}_0 \hat{d}_k = - \hat{d}_k \hat{d}_0 = \mathcal{H}_0 \hat{\mathcal{H}}_k = - \hat{\mathcal{H}}_k \mathcal{H}_0 .
\]  

(112)

Using the relation [cf. eqs. (97), (111)]

\[
\hat{\mathcal{H}}_0 \hat{\mathcal{H}}_0 = \mathcal{H}_0 \mathcal{H}_0 = \mathcal{H}_0 \mathcal{H}_0^T \mathbb{I}_4 = \mathbb{I}_4
\]  

(113)

we find for \( \hat{d}_k \wedge \hat{d}_l, (k \neq l, k, l = 1, \ldots, 4) \)

\[
- \hat{d}_k \wedge \hat{d}_l = - \hat{d}_k \hat{d}_l = \hat{d}_{kl} = \frac{1}{2} \left( \hat{d}_k \hat{d}_l - \hat{d}_l \hat{d}_k \right)
\]

\[
= \frac{1}{2} \left( \hat{\mathcal{H}}_k \hat{\mathcal{H}}_l - \hat{\mathcal{H}}_l \hat{\mathcal{H}}_k \right) = - \hat{\mathcal{H}}_k \hat{\mathcal{H}}_l = \hat{\mathcal{H}}_l \hat{\mathcal{H}}_k
\]

\[
= 2 (\mathcal{H}_k \wedge \mathcal{H}_l)^M \hat{\mathcal{c}}_M = C_2(\mathcal{H})_{kl}^M \hat{\mathcal{c}}_M .
\]  

(114)

Finally, only studying objects with index \( k = -1 \) remains to be done. In view of eq. (21) it turns out to be useful to define and to study the object

\[
\hat{\mathcal{H}}_{(-1)} = -i \mathcal{H}_0 \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4 = -i \mathcal{H}_4 \mathcal{H}_3 \mathcal{H}_2 \mathcal{H}_1 \mathcal{H}_0 .
\]  

(115)

\( \mathcal{H}_{(-1)} \) implicitly defined by means of eq. (115) is a row vector (1 \( \times \) 6 matrix). For the identical Bogolyubov-Valatin transformation one quickly finds \( \mathcal{H}_{l(-1)} = (1, 0, 0, 0, 0, 0) \). From now on we denote by the symbol \( \mathcal{H} \) the (quadratic) 6 \( \times \) 6 matrix \( \mathcal{H} \) with row number \( k = -1, \ldots, 4 \) and column number \( m = -1, \ldots, 4 \) matrix elements \( \mathcal{H}^{(m)}_k \). Consequently, for the identical Bogolyubov-Valatin transformation we have

\[
\mathcal{H}_l = \mathbb{I}_6 .
\]  

(116)

For eq. (115), we can also write (\( \epsilon_{mpqrs} \) is the 6-dimensional totally antisymmetric tensor, \( \epsilon_{(-1)01234} = 1 \); we have raised here the indices by means of \( \delta_{kl} \) and not using \( g_{kl} \))

\[
\hat{\mathcal{H}}_{(-1)} = - \frac{i}{5!} \epsilon_{(-1)}^{mpqrs} \mathcal{H}_m \mathcal{H}_p \mathcal{H}_q \mathcal{H}_r \mathcal{H}_s .
\]  

(117)
Eq. (117) can be transformed [taking into account eq. (99)] to read

\[ \hat{\mathcal{H}}_{(-1)} = \frac{-i}{5!} \epsilon_{(-1)}^{pqrs} \times \sum_{n',p',q',r',s'=-1}^{4} \mathcal{H}^{(n')}_{n} \mathcal{H}^{(q')}_{p} \mathcal{H}^{(r')}_{q} \mathcal{H}^{(s')}_{r} \hat{c}_{n'} \hat{c}_{p'} \hat{c}_{q'} \hat{c}_{r'} \hat{c}_{s'} . \]  

(118)

Using eq. (22), taking into account eq. (121), and applying the Laplace expansion of a determinant [cf. eq. (C.13) in our Appendix C] we find

\[ \hat{\mathcal{H}}_{(-1)} = \frac{1}{5!} \epsilon_{(-1)}^{pqrs} \mathcal{H}^{(n')}_{n} \mathcal{H}^{(q')}_{p} \mathcal{H}^{(r')}_{q} \mathcal{H}^{(s')}_{r} \epsilon_{n'p'q'r's'}^{m} \hat{c}_{m} \]  

(119)

\[ = [C_5 (L^T)^\dagger]^{-1}_{(-1)} m \hat{c}_{m} = \frac{L_{(-1)}^{(m)}}{\det L} \hat{c}_{m} \]  

(120)

Consistency with eq. (116) now requires that \( \det L = 1 \), i.e., \( L \in SO(6; \mathbb{R}) \). This is closely related to the fact that the unitary group implementing the nonlinear Bogolyubov-Valatin transformations [i.e., \( SU(4) \)] is simply connected. From the eqs. (100), (120) we find in total

\[ \mathcal{H} = L , \quad LL^T = I_6 . \]  

(121)

One can convince oneself that eq. (111) also applies for the whole index range \( k, l = -1, \ldots, 4 \).

In analogy to eq. (110) we can write

\[ \hat{d}_{(-1)} = i \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 = i \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 \]  

\[ = \frac{1}{2} \left( \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 - \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 \right) \]

\[ = \frac{1}{2} \left( \mathcal{H}_{(-1)} \mathcal{H}_{0} - \mathcal{H}_{0} \mathcal{H}_{(-1)} \right) = i \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4 \]

\[ = 2 (\mathcal{H}_{(-1)} \wedge \mathcal{H}_{0})^M \hat{c}_M = C_2 (\mathcal{H}_{(-1)} \wedge M \hat{c}_M . \]  

(122)

Furthermore, we have

\[ -i \hat{d}_2 \hat{d}_3 \hat{d}_4 = -i \hat{d}_2 \hat{d}_3 \hat{d}_4 = - \hat{d}_{(-1)} \hat{d}_1 \]

\[ = \hat{d}_{(-1)} \hat{d}_1 = \frac{1}{2} \left( \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 - \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 \right) \]  

26
$$\begin{align*}
&= \frac{1}{2} \left( \hat{\mathcal{H}}_{(-1)} \hat{\mathcal{H}}_1 - \hat{\mathcal{H}}_1 \hat{\mathcal{H}}_{(-1)} \right) = -i \hat{\mathcal{H}}_0 \hat{\mathcal{H}}_2 \hat{\mathcal{H}}_3 \hat{\mathcal{H}}_4 \\
&= 2 \left( \mathcal{H}_{(-1)} \wedge \mathcal{H}_1 \right)^M \hat{c}_M = C_2(\mathcal{H})_{(-1)1}^M \hat{c}_M , \quad (123) \\
&= 2 \left( \mathcal{H}_{(-1)} \wedge \mathcal{H}_2 \right)^M \hat{c}_M = C_2(\mathcal{H})_{(-1)2}^M \hat{c}_M , \quad (124) \\
&= 2 \left( \mathcal{H}_{(-1)} \wedge \mathcal{H}_3 \right)^M \hat{c}_M = C_2(\mathcal{H})_{(-1)3}^M \hat{c}_M , \quad (125) \\
&= 2 \left( \mathcal{H}_{(-1)} \wedge \mathcal{H}_4 \right)^M \hat{c}_M = C_2(\mathcal{H})_{(-1)4}^M \hat{c}_M . \quad (126)
\end{align*}$$

**III. Final result**

Summarizing the above study of two-mode nonlinear Bogolyubov-Valatin transformations we can say that these can be studied in terms of nonlinear basis transformations of the Clifford algebra \(C(0, 4)\). The 15-dimensional space of nontrivial operators (the unit operator omitted) in the fermionic Fock space can by spanned in terms...
of biparavectors. In terms of biparavectors of the Clifford algebra $C(0,5)$, nonlinear Bogolyubov-Valatin transformations can be described by means of the equation $[L_m = \mathcal{H}_m$ are the row vectors with row number $m$ of the matrix $L = \mathcal{H} \in SO(6;\mathbb{R})].$

$$\hat{d}_M = U \hat{c}_M U^\dagger = \chi_M^N \hat{c}_N$$

$$= 2 (\mathcal{H}_{m1} \wedge \mathcal{H}_{m2})^N \hat{c}_N = 2 (L_{m1} \wedge L_{m2})^N \hat{c}_N = C_2(L)_M^N \hat{c}_N \quad (127)$$

In a more compact notation we can write

$$\hat{D} = U \hat{C} U^\dagger = \chi N \hat{C} \quad (128)$$

where $\hat{D}$ and $\hat{C}$ are column vectors with biparavector components (To achieve a more compact graphical display, we give here the Hermitian conjugate of $\hat{C}$.)

$$\hat{C}^\dagger = \left( \begin{array}{c} \hat{c}_{(-1)0}, \hat{c}_{(-1)1}, \hat{c}_{(-1)2}, \hat{c}_{(-1)3}, \hat{c}_{(-1)4}, \\
\hat{c}_{01}, \hat{c}_{02}, \hat{c}_{03}, \hat{c}_{04}, \hat{c}_{12}, \hat{c}_{13}, \hat{c}_{14}, \hat{c}_{23}, \hat{c}_{24}, \hat{c}_{34} \end{array} \right)$$

$$= (i\hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_4, i\hat{c}_1 \hat{c}_3 \hat{c}_4, -i\hat{c}_1 \hat{c}_3 \hat{c}_4, i\hat{c}_1 \hat{c}_3 \hat{c}_4, -i\hat{c}_1 \hat{c}_3 \hat{c}_4, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4, \hat{c}_1 \hat{c}_2 \hat{c}_3, \hat{c}_1 \hat{c}_2 \hat{c}_3, \hat{c}_2 \hat{c}_3, \hat{c}_2 \hat{c}_3, \hat{c}_3 \hat{c}_4) \quad (129)$$

and $\chi$ denotes the $15 \times 15$-matrix $\chi = C_2(L)$. In component notation, we have

$$\chi_{MN} = C_2(L)_{MN} = L_{m1n1} L_{m2n2} - L_{m1n2} L_{m2n1} \quad (130)$$

Finally, as a special case of eq. (127) we display again eq. (109):

$$\hat{d}_k = - \chi_{0k}^N \hat{c}_N = - C_2(L)_{0k}^N \hat{c}_N \quad (131)$$

expressing the new (transformed) generators of the Clifford algebra $C(0,4)$ in terms of the original (untransformed) biparavectors.

Eq. (127) can be reformulated in a different way. Consider now $\hat{D}$ and $\hat{C}$ not as column vectors in the 15-dimensional biparavector space $\Lambda^2(V_6)$ but rather as antisymmetric $6 \times 6$ matrices with biparavector entries for which we now write $\hat{D}$ and $\hat{C}$, respectively. For example, the antisymmetric matrix $\hat{D}$ has the matrix elements

$$\hat{d}_{kl} = \frac{1}{2} \left( \hat{d}_k^\dagger \hat{d}_l - \hat{d}_l^\dagger \hat{d}_k \right) \quad (132)$$
Eq. (127) then equivalently reads
\[ \hat{\mathcal{D}} = L \hat{\chi} L^T. \] (133)

Results corresponding to eq. (133) have been obtained earlier by Hristev, [142], ext.
French version in Rev. Roum. Math. Pures Appl., p. 637, eq. (15.7), [143], p. 176,
eq (4.1), ten Kate [102], p. 183, eq. (9), and Buchdahl [94], p. 363, Sec. 8.

As the matrices \( L \) and \( -L \) describe the same Bogolyubov-Valatin transforma-
tion (i.e., the same matrix \( \chi \)) the group of two-mode nonlinear Bogolyubov-Valatin
transformations is equivalent to the group \( SO(6; \mathbb{R})/\mathbb{Z}_2 \) (cf. also [102], specifically
Sec. II and Theorem 1 therein; for related considerations see [138], Vol. 2, Sec. 9.4,
p. 316, [104], [111], Sec. 5.3, p. 162).

IV. Further analysis
On the basis of the equation [cf. the eqs. (105), (131)]
\[ \hat{d}_k = - \chi_{0k}^M \hat{\chi}_M \] (134)
and taking into account the eq. (36), we can quickly rederive and reformulate the
expression for the anticommutator (57). We find
\[ \hat{d}_k \hat{d}_l = - \chi_{0k}^M \chi_{0l}^M \mathbb{1}_4 - \sum_{m=-1}^4 \chi_{0k}^{mn1} \chi_{0l}^{mn2} \hat{\chi}_{n1n2} \]
\[ - i \chi_{0k}^M \chi_{0l}^N \epsilon_{MP} \hat{\chi}_P \] (135)
and, consequently,
\[ \{ \hat{d}_k, \hat{d}_l \} = \hat{d}_k \hat{d}_l + \hat{d}_l \hat{d}_k = 2 g_{kl} \mathbb{1}_4 = - 2 \delta_{kl} \mathbb{1}_4 \]
\[ = - 2 \chi_{0k}^M \chi_{0l}^M \mathbb{1}_4 - 2 i \chi_{0k}^M \chi_{0l}^N \epsilon_{MP} \hat{\chi}_P. \] (136)
The eqs. (59)-(63) can be written compactly as
\[ \chi_{0k}^M \chi_{0l}^M = \delta_{kl}, \] (137)
\[ \chi_{0k}^M \chi_{0l}^N \epsilon_{MP} = 0. \] (138)
Eq. (135) now reads
\[ \hat{d}_k \hat{d}_l = - \sum_{m=-1}^4 \chi_{0k}^{mn1} \chi_{0l}^{mn2} \hat{\chi}_{n1n2} - \delta_{kl} \mathbb{1}_4 \] (139)
\[ = -\chi_{kl}^N \hat{c}_N - \delta_{kl} \mathbb{I}_4 \tag{140} \]

\[ \chi_{kl}^N = \sum_{m=-1}^{4} (\chi_{0k}^{mn_1} \chi_{0l}^{mn_2} - \chi_{0k}^{mn_2} \chi_{0l}^{mn_1}) \tag{141} \]

Introducing the complex vectors \( (e'_k)^T = \left( \kappa_k^{(1|0)}, \kappa_k^{(0|1)}, \kappa_k^{(2|0)}, \kappa_k^{(0|2)}, \kappa_k^{(1|1)}, \kappa_k^{(1,2|0)}, \kappa_k^{(1|2)}, \kappa_k^{(2|1)}, \kappa_k^{(0|1,2)}, \kappa_k^{(1,2|1)}, \kappa_k^{(1,2|2)}, \kappa_k^{(2|2)}, \kappa_k^{(2|1,2)}, \kappa_k^{(1,2|1,2)} \right) \) \( (k = 1, 2) \), i.e.,

\[
\text{Re } e'_k M = \chi_{0(2k-1)} M, \text{ Im } e'_k M = \chi_{0(2k)} M \]

the eqs. (137), (138) can be written as

\[ (e')^T_k \overline{e'_l} = 2 \delta_{kl} \tag{142} \]

\[ (e')^T_k e'_l = 0 \tag{143} \]

\[ e'_k M \overline{e'_l}^N \epsilon_{MNP} = 0 \tag{144} \]

\[ e'_k M e'_l^N \epsilon_{MNP} = 0 \tag{145} \]

Consequently, the 15-component complex vectors \( e'_1, e'_2 \) should be two orthogonal isotropic vectors of length \( \sqrt{2} \) fulfilling the additional conditions (144), (145).

Eq. (137) which is the generalization of the condition (1) characteristic for the original (linear) Bogolyubov-Valatin transformations can be understood as an orthonormality condition in the 15-dimensional biparavector space \( \bigwedge^2(V_6) \) and eq. (138) is (for \( k = l \)) a decomposability condition. Consequently, any set of four operators \( \hat{d}_k \) fulfilling the Clifford algebra analogue of the canonical anticommutation relations should be a set of four decomposable orthonormal biparavectors. A related result has been obtained earlier (in a somewhat more general form, i.e., for pentades) by Haantjes, [144], p. 51, stelling 5 [proposition 5], and a corresponding comment can also be found in ref. [94], §14, p. 269, above of eq. (14.8). For a further discussion of these aspects see Subsec. 6.3.

The eqs. (137), (138) can be further generalized. Inserting the equation [cf. eq. (127)]

\[ \hat{d}_M = \chi_M^N \hat{c}_N \tag{146} \]

into the anticommutator (38) we find in generalization of the eq. (137)

\[ \chi_{p_1 p_2}^M \chi_{q_1 q_2} M = \delta_{p_1 q_1} \delta_{p_2 q_2} - \delta_{p_1 q_2} \delta_{p_2 q_1} \tag{147} \]

and also

\[ \chi_{p_1 p_2}^M \chi_{M q_1 q_2} = \delta_{p_1 q_1} \delta_{p_2 q_2} - \delta_{p_1 q_2} \delta_{p_2 q_1} \tag{148} \]
The generalizations of eq. (138) read

\[ \chi^M_J \chi^N_K \chi^P_L \epsilon_{MNP} = \epsilon_{JKL}, \tag{149} \]

\[ \epsilon_{MNP} \chi^M_J \chi^N_K \chi^P_L = \epsilon_{JKL}. \tag{150} \]

If all the six indices \( j_1, j_2, k_1, k_2, l_1, \) and \( l_2 \) are chosen pairwise differently, the eqs. (149), (150) can be transformed to read

\[ \det \chi = 1. \tag{151} \]

This sharpens the condition \( \det \chi = \pm 1 \) that can be derived from the eqs. (147), (148). If two of the six indices \( j_1, j_2, k_1, k_2, l_1, \) and \( l_2 \) are equal, say \( k_1 = l_1 = j \) (The cases \( r_1 = r_2 \) are trivial and need not to be discussed.), the eqs. (149), (150) can be transformed to read [using the orthogonality condition (147), (148)]

\[ \chi^M_M \chi^N_N \epsilon_{MNP} = 0. \tag{152} \]

Finally, we study the connection between the conditions (147), (148), and (149), (150). Using the known expression for \( \chi \) in terms of \( L \) [eq. (130)], the eq. (147) can be written as

\[ C_2 (L) C_2 \left( L^T \right) = \mathbb{I}_{15}. \tag{153} \]

Taking into account the compound matrix relation (Laplace expansion of a determinant; for the notation see Appendix C)

\[ C_2 (L) C_4 (L)^* = \mathbb{I}_{15} \tag{154} \]

we can immediately conclude that

\[ C_2 (L) = C_4 \left( L^T \right)^*. \tag{155} \]

This equation can also be written in the following form

\[ \chi_{PQ} = \frac{1}{3!} \epsilon^M_P \epsilon^L_Q \epsilon^N_L \chi_{KL} \chi_{MN} \tag{156} \]

which can also be derived from the eqs. (149), (150) using the eqs. (147), (148).

4 Diagonalizing Hamiltonians

Relying on the insight obtained in the previous section into the structure of nonlinear Bogolyubov-Valatin transformations, in this section we will study the diagonalization of certain fermion and spin Hamiltonians. In principle, the methods of this section can also be applied, with certain modifications, to other quantities of physical interest, for example, the density matrix which is widely being investigated in quantum information theory.
4.1 Diagonalizing two-fermion Hamiltonians

Let us now look at an arbitrary two-fermion Hamiltonian $H'$.

$$
H' = h^{(0|0)} \mathbb{1}_4 + h^{(1|0)} \hat{a}_1^\dagger + h^{(2|0)} \hat{a}_2^\dagger + h^{(0|1)} \hat{a}_1 + h^{(0|2)} \hat{a}_2
$$

$$
+ h^{(1,2|0)} \hat{a}_1^\dagger \hat{a}_2^\dagger + h^{(1|1)} \hat{a}_1^\dagger \hat{a}_1 + h^{(1|2)} \hat{a}_2^\dagger \hat{a}_2
$$

$$
+ h^{(2|1)} \hat{a}_2^\dagger \hat{a}_1 + h^{(2|2)} \hat{a}_2^\dagger \hat{a}_2 + h^{(0|1,2)} \hat{a}_1 \hat{a}_2
$$

$$
+ h^{(1,2|1)} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 + h^{(1,2|2)} \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2
$$

$$
+ h^{(1|1,2)} \hat{a}_1^\dagger \hat{a}_1 + h^{(1|1,2)} \hat{a}_2^\dagger \hat{a}_2 + h^{(1,2|1,2)} \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1^\dagger \hat{a}_2
$$

(157)

We assume it to be hermitian ($H' = H'\dagger$). From the hermiticity condition the following 10 relations derive.

$$
h^{(0|0)} = \overline{h^{(0|0)}}
$$

(158)

$$
h^{(1|0)} = \overline{h^{(0|1)}}
$$

(159)

$$
h^{(2|0)} = \overline{h^{(0|2)}}
$$

(160)

$$
h^{(1,2|0)} = -\overline{h^{(0|1,2)}}
$$

(161)

$$
h^{(1|1)} = \overline{h^{(1|1)}}
$$

(162)

$$
h^{(1|2)} = \overline{h^{(2|1)}}
$$

(163)

$$
h^{(2|2)} = \overline{h^{(2|2)}}
$$

(164)

$$
h^{(1,2|1)} = -\overline{h^{(1|1,2)}}
$$

(165)

$$
h^{(1,2|2)} = -\overline{h^{(2|1,2)}}
$$

(166)

$$
h^{(1,2|1,2)} = \overline{h^{(1,2|1,2)}}
$$

(167)

Taking into account the equations (158)-(167) (in particular, that $h^{(0|0)}, h^{(1|1)}, h^{(2|2)}, h^{(1,2|1,2)}$ are real), eq. (157) reads

$$
H' = \left( h^{(0|0)} + \frac{1}{2} h^{(1|1)} + \frac{1}{2} h^{(2|2)} - \frac{1}{4} h^{(1,2|1,2)} \right) \mathbb{1}_4
$$

$$
- i \text{ Re} \left( h^{(1|0)} + \frac{1}{2} h^{(1,2|2)} \right) \hat{c}_1 - i \text{ Im} \left( h^{(1|0)} + \frac{1}{2} h^{(1,2|2)} \right) \hat{c}_2
$$

$$
- i \text{ Re} \left( h^{(2|0)} - \frac{1}{2} h^{(1,2|1)} \right) \hat{c}_3 - i \text{ Im} \left( h^{(2|0)} - \frac{1}{2} h^{(1,2|1)} \right) \hat{c}_4
$$
\[-\frac{i}{2} h^{(1|1)} \hat{c}_1 \hat{c}_2 - \frac{i}{2} \text{Im} \left( h^{(1,2|0)} + h^{(1|2)} \right) \hat{c}_1 \hat{c}_3
\]

\[+ \frac{i}{2} \text{Re} \left( h^{(1,2|0)} - h^{(1|2)} \right) \hat{c}_1 \hat{c}_4 + \frac{i}{2} \text{Re} \left( h^{(1,2|0)} + h^{(1|2)} \right) \hat{c}_2 \hat{c}_3
\]

\[+ \frac{i}{2} \text{Im} \left( h^{(1,2|0)} - h^{(1|2)} \right) \hat{c}_2 \hat{c}_4 - \frac{i}{2} h^{(2|2)} \hat{c}_3 \hat{c}_4
\]

\[-\frac{i}{2} \text{Re} h^{(1,2|1)} i \hat{c}_1 \hat{c}_2 \hat{c}_3 - \frac{i}{2} \text{Im} h^{(1,2|1)} i \hat{c}_1 \hat{c}_2 \hat{c}_4
\]

\[+ \frac{i}{2} \text{Re} h^{(1,2|2)} i \hat{c}_1 \hat{c}_3 \hat{c}_4 + \frac{i}{2} \text{Im} h^{(1,2|2)} i \hat{c}_2 \hat{c}_3 \hat{c}_4
\]

\[-\frac{i}{4} h^{(1,2|1,2)} i \hat{c}_1 \hat{c}_2 \hat{c}_3 \hat{c}_4
\]

\[= \left( h^{(0|0)} + \frac{1}{2} h^{(1|1)} + \frac{1}{2} h^{(2|2)} - \frac{1}{4} h^{(1,2|1,2)} \right) \mathbb{1}_4
\]

\[-\frac{i}{4} \text{tr}_V \left( Y \hat{C} \right).
\]

Here, the subscript $V_6$ indicates that the trace operation is carried out with respect to the six-dimensional paravector space $V_6$ and the antisymmetric matrix $Y$ has the explicit form (To simplify the display we have omitted the lower triangle matrix elements.)
\[ Y = -Y^T = \]

\[
\begin{pmatrix}
0 & -\frac{h^{(1,2|1,2)}}{2} & -\text{Im } h^{(1,2|2)} & \text{Re } h^{(1,2|2)} & \text{Im } h^{(1,2|1)} & -\text{Re } h^{(1,2|1)} \\
. & 0 & \text{Re } (2h^{(1|0)} + h^{(1,2|2)}) & \text{Im } (2h^{(1|0)} + h^{(1,2|2)}) & \text{Re } (2h^{(2|0)} - h^{(1,2|1)}) & \text{Im } (2h^{(2|0)} - h^{(1,2|1)}) \\
. & . & 0 & \text{Re } h^{(1|1)} & \text{Im } (h^{(1,2|0)} + h^{(1|2)}) & -\text{Re } (h^{(1,2|0)} - h^{(1|2)}) \\
. & . & . & 0 & -\text{Re } (h^{(1,2|0)} + h^{(1|2)}) & -\text{Im } (h^{(1,2|0)} - h^{(1|2)}) \\
. & . & . & . & 0 & h^{(2|2)} \\
. & . & . & . & . & 0
\end{pmatrix}
\]  

(170)
Eq. (169) expresses the Hamiltonian (157) in terms of a biparavector \( \text{tr}_{V_0} \left( \hat{Y} \hat{C} \right) \); plus some constant. This biparavector stands in an one-to-one correspondence to an antisymmetric matrix (exactly as this is the case for ordinary bivectors). The further analysis of the Hamiltonian \( H' \) will be based on this correspondence. It should be mentioned here that recently Uskov and Rau [145], Appendix B, p. 022331-8 [see the first equation below from eq. (B.1)] have given a representation of the Hamiltonian analogous to eq. (168). To explicitly see that both formulations are equivalent one must rely on our equations (D.7)-(D.12) given in the Appendix D.

The matrix \( Y \) can be brought to the standard block diagonal form given by the matrix \( Z \)

\[
Z = - Z^T = \begin{pmatrix}
0 & \nu_{(-1)0} & 0 & 0 & 0 & 0 \\
-\nu_{(-1)0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu_{12} & 0 & 0 \\
0 & 0 & -\nu_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu_{34} & 0 \\
0 & 0 & 0 & 0 & -\nu_{34} & 0
\end{pmatrix}
\]  

(171)

by means of an orthogonal transformation \( L \): \( Z = L Y L^T \). Eq. (169) then reads

\[
H' = \left( h^{(0)(0)} + \frac{1}{2} h^{(1)(1)} + \frac{1}{2} h^{(2)(2)} - \frac{1}{4} h^{(1,2)(1,2)} \right) \mathbb{1}_4 \\
- \frac{i}{4} \text{tr}_{V_0} \left( \hat{Z} \hat{D} \right)
\]  

(172)

\[
= \left( h^{(0)(0)} + \frac{1}{2} h^{(1)(1)} + \frac{1}{2} h^{(2)(2)} - \frac{1}{4} h^{(1,2)(1,2)} \right) \mathbb{1}_4 \\
+ \frac{i}{2} \nu_{(-1)0} \hat{d}_{(-1)0} + \frac{i}{2} \nu_{12} \hat{d}_{12} + \frac{i}{2} \nu_{34} \hat{d}_{34}
\]  

(173)

\[
= \left( h^{(0)(0)} + \frac{1}{2} h^{(1)(1)} + \frac{1}{2} h^{(2)(2)} - \frac{1}{4} h^{(1,2)(1,2)} \right) \mathbb{1}_4 \\
+ \frac{i}{2} \nu_{(-1)0} i\hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4 - \frac{i}{2} \nu_{12} \hat{d}_1 \hat{d}_2 - \frac{i}{2} \nu_{34} \hat{d}_3 \hat{d}_4
\]  

(174)

\[
= \left[ h^{(0)(0)} + \frac{1}{2} \left( h^{(1)(1)} - \nu_{12} \right) + \frac{1}{2} \left( h^{(2)(2)} - \nu_{34} \right) \\
- \frac{1}{4} \left( h^{(1,2)(1,2)} + 2 \nu_{(-1)0} \right) \right] \mathbb{1}_4
\]
with \( \hat{D} = L \hat{C} L^T \) [cf. eq. (133)]. The Hamiltonian is given here in terms of three (commuting) Cartan elements: \( \hat{b}_1 \hat{b}_2, \hat{b}_2 \hat{b}_1, \hat{b}_2 \hat{b}_1 \hat{b}_2 \hat{b}_1 \) [This is the maximal number for the group \( SO(6; \mathbb{R}) \simeq SU(4) \).]. For a related consideration see ref. [146]. In a general situation, the number of nonvanishing pairs of eigenvalues \( \pm iv_K \) \( [K = (1)0, 12, 34] \) of the (similar) matrices \( Y, Z \) is called the \textit{length} ([147], vol. 2, Sec. 4.1, Exercise 26, pp. 52-53), \textit{rank} ([148], Vol. II, Sec. 9.3.5, p. 331), or \textit{mass} ([139], p. 67) of the related b(para)vectors. We will use the term rank to diminish the risk of any misunderstanding in any physics-related context. We should point out here that due to the one-to-one relation between antisymmetric matrices and bivectors on one hand and the one-to-one relation between antisymmetric matrices and biparavectors on the other hand results available in the literature concerning bivectors carry over to biparavectors (we are concerned with) with little change. The standard orthogonal decomposition of bivectors\(^{16}\) relies on the fact that any antisymmetric matrix can be brought to block diagonal form [the diagonal blocks are proportional to \( (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \)]; see, for example, [151], Chap. XI, §4, various pp. in the different Russian editions (Vol. 2, pp. 12-18 of the English transl.). For the general case of three mutually different pairs of eigenvalues \( \pm iv_{(-1)0}, \pm iv_{12}, \pm iv_{34}, \) of the (similar) matrices \( Y, Z \) a geometric algebra formalism to calculate these eigenvalues and the orthogonal transformation \( L \) to transform the matrix \( Y \) into the matrix \( Z \) exists (We will not review it here. Cf. [152], Chap. 3, Sec. 4, p. 78, [153]). This general situation, for example, will apply for Hamiltonians of (quasi)fermions which have been obtained from Jordan-Wigner transformations of two-spin-\( \frac{1}{2} \) systems (Those will be discussed in some detail further below.). However, in the present subsection primarily we have in mind the interpretation of \( H' \) as the Hamiltonian of two (equivalent) fermionic modes. Consequently, we expect for a physical Hamiltonian that two of the eigenvalue pairs [with an eye to eq. (175) chosen to be \( \pm iv_{12}, \pm iv_{34} \) should be equal \( iv_{12} = iv_{34} \) [i.e., the Hamiltonian should not change under exchange of the fermion indices 1 and 2 - this entails further identities for the coefficients in eq. (157)]. Consequently, eq. (175) exhibits a residual \( O(2) \otimes O(2) \) symmetry related to the twofold degenerate eigenspaces of the matrices \( Y, Z \) related to the eigenvalues \( iv_{12} = iv_{34} \) and \( -iv_{12} = -iv_{34} \), and a certain arbitrariness exists in constructing the orthogonal transformation \( L \) (and the related canonical transformation) [56], Sec. 17.3, p. 222 (In this reference, this fact is expressed in terms of a bivector expansion.).

The eigenvalues \( \lambda = \pm iv_K \) \( [K = (1)0, 12, 34] \) of the matrix \( Y \) can be calculated by means of the characteristic equation (For the trace operation we omit here the subscript \( V_6 \) because no misunderstanding may occur.)

\[
0 = \det (Y - \lambda \mathbb{I}_6)
\]

\(^{16}\)[136], all eds., Chap. II, §11, pp. 35-36, [149], [148], Vol. II, Sec. 9.3.5, p. 331, [139], [150], Sec. 1, p. 184, [134], p. 103 (p. 2251 of the English transl.).
\[ \lambda^6 - \frac{1}{2} (\text{tr} Y^2) \lambda^4 + \frac{1}{8} \left[ (\text{tr} Y^2)^2 - 2 \text{tr} Y^4 \right] \lambda^2 + \det Y \quad (176) \]

which is a cubic equation in terms of \( \lambda^2 \). The invariants of the matrices \( Y, Z \) entering the above equation are

\[
\begin{align*}
\text{tr} Y^2 &= \text{tr} Z^2 = -2 \left( \nu_{(-1)0}^2 + \nu_{12}^2 + \nu_{34}^2 \right), \\
\text{tr} Y^4 &= \text{tr} Z^4 = 2 \left( \nu_{(-1)0}^4 + \nu_{12}^4 + \nu_{34}^4 \right), \\
\det Y &= \det Z = \left( \nu_{(-1)0} \nu_{12} \nu_{34} \right)^2.
\end{align*}
\]

By calculating the discriminant \( \Delta \) of the equation \((176)\) (We do not display here the explicit expression of it, see any standard reference on cubic equations.) one can check if indeed two of the eigenvalue pairs \( \lambda = \pm i \nu \) agree as expected. If this is the case the eqs. \((177)-(179)\) read

\[
\begin{align*}
\text{tr} Y^2 &= \text{tr} Z^2 = -2 \left( \nu_{(-1)0}^2 + 2 \nu^2 \right), \\
\text{tr} Y^4 &= \text{tr} Z^4 = 2 \left( \nu_{(-1)0}^4 + 2 \nu^4 \right), \\
\det Y &= \det Z = \left( \nu_{(-1)0} \nu^2 \right)^2.
\end{align*}
\]

and one can quickly find from these equations the eigenvalues \( \lambda = \pm i \nu_{(-1)0}, \lambda = \pm \nu = \pm i \nu_{12} = \pm i \nu_{34} \) without the need to resort to the standard machinery of solving a cubic equation in general. From the eqs. \((180), (181)\) one finds for the squared eigenvalues \( \lambda^2 \)

\[
\begin{align*}
\nu^2 &= \frac{1}{6} \left( -\text{tr} Y^2 \pm \sqrt{3 \text{tr} Y^4 - \frac{1}{2} (\text{tr} Y^2)^2} \right), \\
\nu_{(-1)0}^2 &= \frac{1}{6} \left( -\text{tr} Y^2 \mp 2 \sqrt{3 \text{tr} Y^4 - \frac{1}{2} (\text{tr} Y^2)^2} \right).
\end{align*}
\]

Note, that \( 3 \text{tr} Y^4 - \frac{1}{2} (\text{tr} Y^2)^2 = 4 \left( \nu^2 - \nu_{(-1)0}^2 \right)^2. \) Consequently, in the eqs. \((183), (184)\) on the r.h.s. the upper sign applies to weak (quasi)fermion coupling \( (\nu^2 > \nu_{(-1)0}^2) \) and the lower sign to strong (quasi)fermion coupling \( (\nu^2 < \nu_{(-1)0}^2) \). Which sign applies can be determined by calculating the determinant (or the Pfaffian) of the matrix \( Y \).

One may wonder why the analysis of the 4-level Hamiltonian \( H' \) [eq. \((157)\)] which can be represented by means of a \( 4 \times 4 \) matrix leads to a cubic equation and not to a quartic one. The answer is as follows: Writing the Hamiltonian \( H' \) in terms of a biparavector plus some constant \([eq. \((168)\)]\) amounts to splitting the Hamiltonian \( H' \) into a traceless part plus some diagonal term (proportional to \( \mathbb{1}_4 \)). Consequently, as
this way one degree of freedom for the eigenvalues of the Hamiltonian $H'$ has been separated out, the remaining characteristic equation for the traceless part of $H'$ is reduced in order by one degree to a cubic equation.

Finally, let us dwell on certain general considerations. Qualitatively, the class of physical Hamiltonians $H'$ (with $\nu_{12} = \nu_{34} = \nu$) can have rank 2 ($\nu_{(1)0} = 0$) or 3 ($\nu_{(1)0} \neq 0$) - interpreted in terms of the biparavector(s) related to it. For the generic case $\nu_{(1)0} \neq 0$ (i.e., rank 3) it is clear that it is impossible to write the Hamiltonian $H'$ as the sum of two noninteracting quasiparticle fermion oscillators, i.e., it is impossible to define canonical transformations of the fermion creation ($\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$) and annihilation ($\hat{a}_1$, $\hat{a}_2$) operators such a way that the Hamiltonian $H'$ can be written in terms of noninteracting quasiparticles [(quasi)fermions] ($\hat{b}_1$, $\hat{b}_2$). However, if

$$\det Y = (\text{Pf } Y)^2 = \det Z = (\text{Pf } Z)^2 = (\nu_{(1)0} \nu^2)^2 = 0$$

(i.e., $\nu_{(1)0} = 0$) this is possible.

We conclude this subsection with some comments on the related literature. The Hamiltonian (175) with $\nu_{12} = \nu_{34} = \nu$ has been studied for fermion systems in [48, 50, 52, 154–158], while in the references [159–161] the somewhat more general situation where not necessarily $\nu_{12} = \nu_{34}$ is being considered. Particular versions of the general Hamiltonian (157) have been studied in [154, 155, 162]. However, the case studied in [162] can be treated by means of linear Bogolyubov-Valatin transformations.

### 4.2 Nonlinear spin transformations

#### Diagonalizing spin Hamiltonians

##### 4.2.1 Two-spin-$\frac{1}{2}$ systems

It is well-known that fermion systems can be related to spin systems by means of Jordan-Wigner transformations [163]. This, of course, also applies to the two-fermion system under consideration. From a group-theoretical point of view, each elementary spin ($\frac{1}{2}$) corresponds to some $SU(2)$ subgroup of some larger group. Inasmuch as spin operators (Pauli operators) of different spins commute among each other a system of $n$ spins corresponds to $n$ pairwise commuting $SU(2)$ subgroups of the larger group related to the system under consideration. In our setting, the transition from fermion operators to spin (Pauli) operators is related to the choice of two commuting $su(2)$ Lie subalgebras within the $su(4)$ Lie algebra of the two-fermion system. By means of this choice a two-fermion system is mapped to a two-spin-$\frac{1}{2}$ system. Such systems have been studied from various points of view in the past [164–172], [2], Chap. 2, Sec. 13, pp. 52-56, Table 8, pp. 294-299, [173], [174], Chap. 5, p. 120, [84, 175–186]. We leave here aside the broad range of studies performed
in recent years concerning the problem of entanglement in two-spin-\(\frac{1}{2}\) systems – the ”harmonic oscillator of quantum information theory”. It is clear that there are many possible Jordan-Wigner transformations (This question seems not to have been studied systematically in the literature so far.). Here, we will follow [84] with some modifications arising from certain esthetic considerations related to the equations (24)-(27). We want to choose a version of the Jordan-Wigner transformation that is fairly symmetric with respect to the mode number indices involved. The authors of ref. [84] define (p. 13835, above of eq. (5); \([S^x_k, S^y_k] = iS^z_k, S^\pm_k = S^x_k \pm iS^y_k\))

\[
S^+_1 = \hat{a}_1^\dagger \left(\mathbb{1}_4 - (1 + i) \hat{a}_2^\dagger \hat{a}_2\right),
\]

\[
S^-_1 = \hat{a}_1 \left(\mathbb{1}_4 - (1 - i) \hat{a}_2^\dagger \hat{a}_2\right),
\]

\[
S^z_1 = \hat{a}_1^\dagger \hat{a}_1 - \frac{1}{2} \mathbb{1}_4,
\]

\[
S^+_2 = \hat{a}_2^\dagger \left(\mathbb{1}_4 - (1 - i) \hat{a}_1^\dagger \hat{a}_1\right),
\]

\[
S^-_2 = \hat{a}_2 \left(\mathbb{1}_4 - (1 + i) \hat{a}_1^\dagger \hat{a}_1\right),
\]

\[
S^z_2 = \hat{a}_2^\dagger \hat{a}_2 - \frac{1}{2} \mathbb{1}_4.
\]

One can convince oneself without any difficulty that the inverse Jordan-Wigner transformation from the spin-\(\frac{1}{2}\) (Pauli) operators to fermion operators is given by the following equations.

\[
\hat{a}_1 = -i S^-_1 \left[\frac{1}{2} (1 + i) \mathbb{1}_4 + (1 - i) S^z_2\right]
\]

\[
\hat{a}_1^\dagger = i S^+_1 \left[\frac{1}{2} (1 - i) \mathbb{1}_4 + (1 + i) S^z_2\right]
\]

\[
\hat{a}_2 = i S^-_2 \left[\frac{1}{2} (1 - i) \mathbb{1}_4 + (1 + i) S^z_1\right]
\]

\[
\hat{a}_2^\dagger = -i S^+_2 \left[\frac{1}{2} (1 + i) \mathbb{1}_4 + (1 - i) S^z_1\right]
\]

In difference to ref. [84], we define spin-\(\frac{1}{2}\) operators by means of the following equations [Our choice is related to the choice for the eqs. (24)-(27)].

\[
S^+_1 = \frac{i}{\sqrt{2}} \hat{a}_1 \left[(1 + i) \mathbb{1}_4 - 2 \hat{a}_2^\dagger \hat{a}_2\right]
\]

\[
S^-_1 = -\frac{i}{\sqrt{2}} \hat{a}_1^\dagger \left[(1 - i) \mathbb{1}_4 - 2 \hat{a}_2 \hat{a}_2\right]
\]
And the inverse transformation is given by

\[
\hat{a}_1 = -\frac{1}{\sqrt{2}} S_1^+ \left[ \mathbb{1}_4 - 2i S_2^z \right], \quad (202)
\]

\[
\hat{a}_1^\dagger = -\frac{1}{\sqrt{2}} S_1^- \left[ \mathbb{1}_4 + 2i S_2^z \right], \quad (203)
\]

\[
\hat{a}_2 = -\frac{1}{\sqrt{2}} S_2^+ \left[ \mathbb{1}_4 + 2i S_1^z \right], \quad (204)
\]

\[
\hat{a}_2^\dagger = -\frac{1}{\sqrt{2}} S_2^- \left[ \mathbb{1}_4 - 2i S_1^z \right]. \quad (205)
\]

It is clear that any two-spin-$\frac{1}{2}$ Hamiltonian can be transformed by means of the eqs. (186)-(191) to the general form (157) of the two-fermion Hamiltonian. It is furthermore possible, as discussed above, to bring any two-fermion Hamiltonian to the special form (175). Of course, also the quasifermion operators $\hat{b}_1, \hat{b}_2$ can be related to quasi-spin-$\frac{1}{2}$ operators $T_k^+, T_k^-, T_k^z (k = 1, 2)$ by means of a Jordan-Wigner transformation. In analogy to the above equations we can write

\[
\hat{b}_1 = -\frac{1}{\sqrt{2}} T_1^+ \left[ \mathbb{1}_4 - 2i T_2^z \right], \quad (206)
\]

\[
\hat{b}_1^\dagger = -\frac{1}{\sqrt{2}} T_1^- \left[ \mathbb{1}_4 + 2i T_2^z \right], \quad (207)
\]

\[
\hat{b}_2 = -\frac{1}{\sqrt{2}} T_2^+ \left[ \mathbb{1}_4 + 2i T_1^z \right], \quad (208)
\]

\[
\hat{b}_2^\dagger = -\frac{1}{\sqrt{2}} T_2^- \left[ \mathbb{1}_4 - 2i T_1^z \right]. \quad (209)
\]

Then, the quasifermion Hamiltonian $H'$ [cf. eq. (175)] can be written as a two-quasi-spin-$\frac{1}{2}$ Hamiltonian. It reads

\[
H' = \left( h^{(00)} + \frac{1}{2} h^{(11)} + \frac{1}{2} h^{(22)} - \frac{1}{4} h^{(1,2|1,2)} \right) \mathbb{1}_4
\]
\[ \left( \nu_{(-1)0} + \nu_{12} \right) T^z_1 - \left( \nu_{(-1)0} + \nu_{34} \right) T^z_2 + 2 \nu_{(-1)0} T^z_1 T^z_2. \]  
\hspace{1cm} (210)

Consequently, any two-spin-\( \frac{1}{2} \) Hamiltonian can be brought to the above form. A related result has been found (for a somewhat restricted class of two-spin-\( \frac{1}{2} \) Hamiltonians) in [2], Chap. 2, Sec. 13, pp. 52-56. The unitary transformations reducing any two-spin-\( \frac{1}{2} \) Hamiltonian to the form (210) correspond, in general, to nonlinear spin transformations. Such transformations have been considered in a somewhat different context in [187], see p. 1186, eq. (6a), [188–193].

4.2.2 Single-spin-\( \frac{3}{2} \) systems

Two-fermion Hamiltonians are not only related to two-spin-\( \frac{1}{2} \) systems but can also be related to a single-spin-\( \frac{3}{2} \) system (A concrete choice for the latter relation can be found in [194]). Consequently, any system which is given in terms of one of these representations can also equivalently be formulated in terms of the other two. The study of single spin-\( \frac{3}{2} \) systems has a long history (For certain theoretical aspects see [195–198] and references therein.) and they have found recent attention in the field of quantum computation [199–202]. To make the relation of any two-fermion system to a single-spin-\( \frac{3}{2} \) system explicit we will follow here [194]. We will apply the same line of reasoning as for a two-spin-\( \frac{1}{2} \) system applied above. The three (radial) spin-\( \frac{3}{2} \) spin operators \( I^+, I^-, I^z \) can be given in terms of the fermion creation and annihilation operators the following way ([194], p. L506, eqs. (19), (20)).

\[ I^+ = \sqrt{3} \hat{a}_2 + 2 \hat{a}_2^\dagger \hat{a}_1, \]  
\hspace{1cm} (211)

\[ I^- = \sqrt{3} \hat{a}_2^\dagger + 2 \hat{a}_1^\dagger \hat{a}_2, \]  
\hspace{1cm} (212)

\[ I^z = \frac{1}{2} \left[ I^+, I^- \right] = 2 \hat{a}_2^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{a}_1 - \frac{3}{2} \mathbb{I}_4, \]  
\hspace{1cm} (213)

The inverse transformation reads ([194], p. L506, eq. (21)):

\[ \hat{a}_1^\dagger = -\frac{1}{\sqrt{3}} I^+ I^z I^+, \]  
\hspace{1cm} (214)

\[ \hat{a}_1 = -\frac{1}{\sqrt{3}} I^- I^z I^-, \]  
\hspace{1cm} (215)

\[ \hat{a}_2^\dagger = \frac{1}{\sqrt{3}} I^+ \left( \frac{1}{2} \mathbb{I}_4 + I^z \right)^2, \]  
\hspace{1cm} (216)

\[ \hat{a}_2 = \frac{1}{\sqrt{3}} \left( \frac{1}{2} \mathbb{I}_4 + I^z \right)^2 I^-. \]  
\hspace{1cm} (217)

Again, any single-spin-\( \frac{3}{2} \) Hamiltonian can be transformed by means of the eqs. (211)-(213) to the general form (157) of the two-fermion Hamiltonian. It is furthermore
possible, as discussed above, to bring any two-fermion Hamiltonian to the special form (175). Then, also the quasifermion operators \( \hat{b}_1, \hat{b}_2 \) can be related to quasi-spin-\( \frac{3}{2} \) operators \( J^+, J^-, J^z \) by means of a Jordan-Wigner transformation. In analogy to the above equations we can write

\[
\begin{align*}
J^+ &= \sqrt{3} \hat{b}_2 + 2 \hat{b}_2^\dagger \hat{b}_1, \\
J^- &= \sqrt{3} \hat{b}_2^\dagger + 2 \hat{b}_1^\dagger \hat{b}_2, \\
J^z &= \frac{1}{2} [J^+, J^-] = 2 \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 - \frac{3}{2} \mathbb{1}_4,
\end{align*}
\]

and

\[
\begin{align*}
\hat{b}_1^\dagger &= -\frac{1}{\sqrt{3}} J^+ J^z J^+, \\
\hat{b}_1 &= -\frac{1}{\sqrt{3}} J^- J^z J^-, \\
\hat{b}_2^\dagger &= \frac{1}{\sqrt{3}} J^+ \left( \frac{1}{2} \mathbb{1}_4 + J^z \right)^2, \\
\hat{b}_2 &= \frac{1}{\sqrt{3}} \left( \frac{1}{2} \mathbb{1}_4 + J^z \right)^2 J^-.
\end{align*}
\]

The quasifermion Hamiltonian \( H' \) [cf. eq. (175)] can be written as a single-quasi-spin-\( \frac{3}{2} \) Hamiltonian. It then reads

\[
H' = \left( h^{(0)(0)} + \frac{1}{2} h^{(1)[1]} + \frac{1}{2} h^{(2)[2]} - \frac{1}{4} h^{(1,2)[1,2]} + \frac{13}{8} \nu_{(-1)0} \right) \mathbb{1}_4
\]
\[
+ \frac{1}{12} \left( 17 \nu_{(-1)0} - 5 \nu_{12} + 22 \nu_{34} \right) J^z + \frac{1}{2} \nu_{(-1)0} (J^z)^2
\]
\[
+ \frac{1}{3} \left( \nu_{(-1)0} - \nu_{12} + 2 \nu_{34} \right) (J^z)^3.
\]

Finally, we would like to mention that on the basis of the two above (generalized) Jordan-Wigner transformations for two-spin-\( \frac{1}{2} \) and single-spin-\( \frac{3}{2} \) systems relations can be established between spin-\( \frac{1}{2} \) and spin-\( \frac{3}{2} \) operators\(^{17}\). Combined with the full range of nonlinear Bogolyubov-Valatin transformations discussed we thus obtain a large manifold of expressing single-spin-\( \frac{3}{2} \) operators in terms of two-spin-\( \frac{1}{2} \) operators and vice versa.

\(^{17}\)There seems to have been done some related research in the past [203].
5  $SU(4), SO(6; \mathbb{R}), SO(6; \mathbb{R})/\mathbb{Z}_2$ transformations and their parametric relations

In the previous sections unitary $SU(4)$ transformations $U$, orthogonal $SO(6; \mathbb{R})$ transformations $L$, and $SO(6; \mathbb{R})/\mathbb{Z}_2$ transformations $\chi$ have played an important role. So far, we have not discussed (except for the expression of $\chi$ in terms of $L$) their concrete mutual relationship which is of importance for any explicit calculation. In this section we finally will consider this technical problem. To set the frame for this discussion we will first specify which sort of parametrization for the matrices $U$ and $L$ we are going to use.

Let us write for the unitary $4 \times 4$ matrix $U = U(\{\lambda\}; \{\hat{c}\})$ implementing a given nonlinear Bogolyubov-Valatin transformation ($T_0$ is a complex number while $T_{m_1m_2}$ are the matrix elements of a complex antisymmetric $6 \times 6$ matrix $T$.)

$$U = T_0 \mathbb{I}_4 + T^M \hat{c}_M = T_0 \mathbb{I}_4 + \frac{1}{2} T^{m_1m_2} \hat{c}_{m_1m_2}.$$  \hspace{2cm} (226)

From the unitarity condition $UU^\dagger = \mathbb{I}_4$ follow the equations

$$|T_0|^2 + T^M \overline{T}_M = 1,$$  \hspace{2cm} (227)

$$T_0 \overline{T}_P + T_0 T_P - T_{p_1} m T_{m_2} + T_{p_1} m T_{m_2} + i T^M \overline{T}_N \epsilon_{MNP} = 0.$$  \hspace{2cm} (228)

These equations have been given earlier (in some less general notation) in [204], p. 243, eq. (35). For another version of these equations see eq. (332). For any given unitary matrix $U$ the coefficients $T_0, T_M$ can be calculated by means of the equations [cf. eq. (36)]

$$T_0 = \frac{1}{4} \text{tr} U,$$  \hspace{2cm} (229)

$$T_M = -\frac{1}{4} \text{tr} (\hat{c}_M U).$$  \hspace{2cm} (230)

For the orthogonal $6 \times 6$ matrix $L$ we rely on the Cayley representation

$$L = \frac{\mathbb{I}_6 + A}{\mathbb{I}_6 - A} = \mathbb{I}_6 + \frac{2A}{\mathbb{I}_6 - A} = -\mathbb{I}_6 + \frac{2}{\mathbb{I}_6 - A},$$  \hspace{2cm} (231)

in terms of a real antisymmetric matrix $A$ (We disregard here all problems of the Cayley representation related to any eigenvalues $= -1$ of the orthogonal matrix $L$.) The matrix $L$ can be expressed as a sum over a finite number of powers of the matrix $A$. For an explicit expression see eq. (274) further down. In turn, the antisymmetric matrix $A$ can be expressed in terms of $L$ as

$$A = \frac{L - \mathbb{I}_6}{L + \mathbb{I}_6} = \frac{L(-)}{\mathbb{I}_6 + L(+)},$$  \hspace{2cm} (232)
\[ L^{(+)} = \frac{1}{2} \left( L + L^T \right), \]  
\[ L^{(-)} = \frac{1}{2} \left( L - L^T \right). \]  

The antisymmetric matrix \( A \) can also be expressed as a sum over a finite number of powers of the matrix \( L \), however, already for the present case these expressions are quite involved and, therefore, we will not display them here (for details see [205]).

To simplify navigation through the present section we now give in Figure 1 a schematic overview of its content. Each arrow in Figure 1 stands for an equation expressing the quantity at the end point of the arrow by another one at its starting point.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Schematic overview over the content of Sec. 5.}
\end{figure}
5.1 From $U \in SU(4)$ to $L \in SO(6; \mathbb{R})$

In this and the following subsection, we will study the explicit relation between the $SO(6; \mathbb{R})$ transformations $L$ and the corresponding unitary $SU(4)$ transformations $U(\{\lambda\}; \{\hat{c}\})$. In the present subsection, we will assume that $U = U(\{\lambda\}; \{\hat{c}\})$ is known and derive from it an expression for $L$. We will perform this task by relying on the discussion presented in [206, 207] (for follow-up work see [208]).

To begin with, let us start by repeating some elements of the very lucid account given in [207] (We will follow here also the notation used in this article with slight modifications.). Consider a set of six linearly independent antisymmetric (complex) $4 \times 4$ matrices\(^\text{18}\): $\Gamma_k = -\Gamma_k^T$, $k = -1, \ldots, 4$. These matrices represent spin (space) bivectors ($\in \bigwedge^2(\mathbb{C}_4)$, where $\mathbb{C}_4$ is the spin space) and can be chosen to obey $(\epsilon^{abcd},$\(^\text{18}\)For an early discussion of the role of antisymmetric matrices in the present context see [209], related discussions can be found in [210, 211], [212], Chap. IV, Secs. 5-9, pp. 36-40, [213] (There are many other papers by Schouten and collaborators containing related but less focused material.), [144, 214] (very clear mathematical accounts), [215], § 11, p. 49-52, [217], [94] (also note [103]), [104], [218], \textit{Jekl}. [Lekts.] 13, pp. 258-299 (pages 247-285 of the English transl.), and in [219–222]. A somewhat related discussion for the case $SL(2; \mathbb{C})$, $SO(1, 3; \mathbb{R})$ is given in [223]. A different approach of dealing with the homomorphism between the groups $SU(4)$ and $SO(6; \mathbb{R})$ related to the physical problem of the 3-particle problem is discussed in [224], Secs. VI.A, VI.B, pp. 557-559.
\[ \epsilon_{abcd}, \epsilon^{1234} = \epsilon_{1234} = 1 \] is the completely antisymmetric tensor operating in spin space \( C_4 \)

\[ \frac{1}{8} \epsilon^{abcd} \left( \bar{\Gamma}_k \right)_{ab} \left( \bar{\Gamma}_l \right)_{cd} = \delta_{kl} \quad (235) \]

([207], p. 10, eq. (1), an example for these matrices can be found in [206], p. 814, eq. (12); also see our Appendix D). One can then define antisymmetric 4 \( \times \) 4 matrices \( \bar{\Gamma}_k = -\bar{\Gamma}_k^T \) (the spin space Hodge duals of \( \Gamma_k \))\(^{19}\) by writing

\[ \left( \bar{\Gamma}_k \right)_{ab} = -\frac{1}{2} \epsilon^{abcd} \left( \Gamma_k \right)_{cd}. \quad (236) \]

Then, eq. (235) can be written compactly as

\[ \frac{1}{4} \text{tr} \left( \bar{\Gamma}_k \bar{\Gamma}_l \right) = \delta_{kl}. \quad (237) \]

From this equation one recognizes\(^{20}\) that the matrices \( \bar{\Gamma}_k, \bar{\Gamma}_k \) obey the equation\(^{21}\)

\[ \bar{\Gamma}_k \bar{\Gamma}_l + \bar{\Gamma}_l \bar{\Gamma}_k = \bar{\Gamma}_k \bar{\Gamma}_l + \bar{\Gamma}_l \bar{\Gamma}_k = 2 \delta_{kl} \mathbb{I}_4. \quad (238) \]

One can also derive the following useful relations\(^{22}\)

\[ \left( \bar{\Gamma}_k \right)_{ab} \left( \bar{\Gamma}_k \right)_{cd} = 2 \delta_a^d \delta_b^c - 2 \delta_a^c \delta_b^d, \quad (239) \]

\[ \left( \bar{\Gamma}_k \right)_{ab} \left( \bar{\Gamma}_k \right)_{cd} = 2 \epsilon^{abcd}, \quad (240) \]

\[ \left( \bar{\Gamma}_k \right)_{ab} \left( \bar{\Gamma}_k \right)_{cd} = 2 \epsilon_{abcd}. \quad (241) \]

\(^{19}\)Incidentally, the minus sign on top of the \( \Gamma \) symbol should not be confused with the sign for complex conjugation – a longer bar.

\(^{20}\)Cf. [210], p. 410, eq. (2.14), [212], p. 36 (p. 216 of the whole volume), bottom of the page,

[213], p. 184, eq. (60), [144], p. 48, eq. (2), [214], p. 139, eq. (1.5),

[94], p. 368, eq. (13.8), [207], pp. 10-11, [221], Subsec. 2.1.1, p. 19, eq. (49),

[222], p. 11 (p. 9 of the English transl.), eq. (2), [225], p. 24, Appendix A, eqs. (A2).

\(^{21}\)Incidentally, such a relation has also emerged in [226], p. 1066, eq. (6). For a somewhat related discussion also see [64], in particular, p. 3, eq. (5).

\(^{22}\)Cf. [211], p. 510, eq. (4.10), [213], p. 177, eq. (7), [144], p. 48, eq. (6), [214], p. 138, eqs. (1.2),

[215], p. 50, eq. (33) (incidentally, in the preceding eq. (32) \( \eta^{25}_4 \) and \( \eta^{25}_3 \) should correctly read \( \eta^{23}_4 \) and \( \eta^{23}_3 \), respectively), [216], p. 159, eq. (21), [94], p. 367, eq. (13.6),

[62], p. 93, eq. (2.10), [207], p. 11, eqs. (11), (12), [220], p. 44, eq. (2),

[221], Subsec. 2.1.1, pp. 17-20, [222], p. 11 (p. 9 of the English transl.), eq. (1),

[225], p. 24, Appendix A, eqs. (A4)-(A6).

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It is now possible to represent the paravectors $\hat{c}_k$ in terms of the antisymmetric matrices $\tilde{\Gamma}_k$, $\hat{\Gamma}_k$. One finds\(^{23}\)

$$\hat{c}_k = - \tilde{\Gamma}_0 \hat{\Gamma}_k .$$

(242)

The choice of $\tilde{\Gamma}_0$ singles out a certain spin bivector and in the spin bivector space $\bigwedge^2(\mathbb{C}_4)$ a certain direction (vector, line). Haantjes calls this direction (line) the axis of the pentade $\hat{c}_k$, $k = -1, 1 \ldots , 4$ ([144], p. 51, stelling 4 [proposition 4]). Furthermore, it is possible to choose the matrices $\tilde{\Gamma}_k$, $\hat{\Gamma}_k$ such a way that

$$\hat{\Gamma}_k^\dagger = \tilde{\Gamma}_k$$

(243)

applies. The condition (243) determines in the (complex six-dimensional) spin bivector space $\bigwedge^2(\mathbb{C}_4)$ a (real six-dimensional) subspace which we denote in view of eq. (237) by $\bigwedge^2_{\mathbb{R}_0}(\mathbb{C}_4) \simeq \mathbb{R}_6$ [also see the comment below from eq. (247)]. As a technical comment, we would like to mention here that $\hat{\Gamma}_k^\dagger$ on the l.h.s. of eq. (243) is an object with upper (spin) indices related by hermitian conjugation to the object $\tilde{\Gamma}_k$ with lower (spin) indices. To cut a long story short, this is related to the fact that for bivectors in $\bigwedge^2_{\mathbb{R}_0}(\mathbb{C}_4) \simeq \mathbb{R}_6$, we can raise and lower indices in spin space by means of $s_{ab} = \delta_{ab}$, $s^{ab} = \delta^{ab}$ supplemented by an operation of hermitian conjugation [cf. [221], Subsec. 2.2.3, p. 40, table 1, and Subsec. 2.2.2, p. 31/32, Teorem [Teorema]/[Theorem] 2, eq. (121); [222], p. 15 (p. 12 of the English transl.), table, and p. 13 (p. 11 of the English transl.), Teorem [Teorema]/[Theorem] 2, eq. (10)]\(^{24}\).

By virtue of eq. (236), eq. (243) leads to the condition (--- denotes complex conjugation)

$$\left(\hat{\Gamma}_k\right)_\text{ba} = \left(\Gamma_k\right)^{ab} = - \frac{1}{2} \epsilon^{abcd} \left(\hat{\Gamma}_k\right)^{cd}$$

(244)

which entails that the real part of the matrix $\hat{\Gamma}_k^\dagger$ should be a selfdual matrix while the imaginary part should be an anti-selfdual matrix. We give in Appendix D an explicit example for the set of matrices $\hat{\Gamma}_k$ that obey eq. (243) [Up to the choice of signs this set of matrices agrees with that given in [206], p. 814, eq. (12).]. Taking into account eq. (243) we find

$$\hat{c}_k^\dagger = - \tilde{\Gamma}_k \hat{\Gamma}_0 .$$

(245)

---

\(^{23}\)Also cf. [210], p. 410, eq. (2.13), [212], p. 37 (p. 217 of the whole volume), [144], p. 49, eq. (7). The analogous equation for an arbitrary biparavector $\hat{c}_M$ [see eq. (336), Subsec. 6.3] is given in [94], p. 367, eq. (13.7), [104], p. 2244, eq. (9), [207], p. 11, eq. (18), [220], p. 44, eq. (5a), [111], Sec. 5.3, p. 167, eq. (5.21), [221], Subsec. 6.1.1, p. 97, eq. (472), [222], p. 12 (p. 10 of the English transl.), the first eq. on the page.

\(^{24}\)We are indebted to K. V. Andreev for pointing out to us these subtleties and refer the interested reader to his publications [220–222] for further details.
Equations analogous to the eqs. (242), (245) also hold for \( \hat{d}_k \), \( \hat{d}_k^\dagger \) with the matrices \( \hat{\Gamma}_k \) replaced by some other matrices \( \hat{\Gamma}'_k \). These two sets of matrices are related by the equation

\[
\hat{\Gamma}'_k = L_{kl} \hat{\Gamma}_l = U \hat{\Gamma}_k U^T,
\]

(246)

\[
\hat{\Gamma}'_k = L_{kl} \bar{\Gamma}_l = \bar{U} \hat{\Gamma}_k U^\dagger.
\]

(247)

In general, here the matrix \( L \) may be an element of the group \( SO(6; \mathbb{C}) \), however, by virtue of the condition (243) the matrix \( L \) must be real in our case and, therefore, belongs to the group \( SO(6; \mathbb{R}) \). It seems worth mentioning that eq. (244) is invariant under the transformations (246), (247) (To see this one has to rely on the Laplace expansion of the determinant of the \( SU(4) \) matrix \( U \).). The eqs. (246), (247) provide us with additional motivation to denote the subspace of the spin bivector space \( \wedge^2 (\mathbb{C}_4) \) we are using by the symbol \( \mathbb{R}_6 \) because its invariance group is \( SO(6; \mathbb{R}) \).

The antisymmetric matrices \( \hat{\Gamma}_k \) can be understood as the orthonormal basis in the spin bivector space \( \wedge^2 (\mathbb{C}_4) \) [cf. eq. (237)].

From eq. (247) one finds the relation

\[
L_{kl} = \frac{1}{4} \text{tr} \left( \hat{\Gamma}_k' \bar{\Gamma}_l \right) = \frac{1}{4} \text{tr} \left( \hat{\Gamma}_k U^\dagger \bar{\Gamma}_l U \right)
\]

(248)

\[
= \frac{1}{4} \text{tr} \left( \hat{\Gamma}_k' \bar{\Gamma}_l \right) = \frac{1}{4} \text{tr} \left( \hat{\Gamma}_k U^T \bar{\Gamma}_l U \right)
\]

(249)

Inserting now eq. (226) into it and calculating the occurring traces by means of the eqs. (E.1)-(E.6) we obtain [cf. [206], p. 822, eq. (53)]

\[
L = \left( T_0^2 + T_M^M T_M \right) \mathbb{1}_6 - 2 T_0 T + 2 T^2 - 2i P c^2 (T)
\]

(250)

\[
= \left( T_0^2 + T_M^M T_M \right) \mathbb{1}_6 - 2 T_0 T + 2 T^2 + 2i P c^2 (T).
\]

Here, we have introduced the matrix \( P c^2 (T) \) for the antisymmetric matrix \( T \) by defining

\[
P c^2 (T)_{kl} = \frac{\partial}{\partial T_{lk}} \text{Pf} T = -\frac{1}{2} T^M T^N \epsilon_{M N k l}.
\]

(251)

We call the matrix \( P c^2 (T) \) the (second) supplementary Pfaffian compound matrix. We have defined it in analogy to the definition of supplementary compound matrices

\[25\text{[206], p. 815, eq. (10), p. 821, eq. (47), [207], p. 12, below from eq. (23), p. 13, eq. (30). Note that Stepanovskii has studied the more general case } SL(4; \mathbb{C}) / \mathbb{Z}_2 \simeq SO(6; \mathbb{C}) \text{ and that we have interchanged } U \text{ and } U^\dagger \text{ compared with the use in [206, 207].}
\]

\[26\text{Cf. [206], p. 822, eq. (52), also see [227], p. 766, the equation below from eq. (2.4), [60], p. 13, eq. (4.9), [104], p. 2244, eq. (6), [221], Subsec. 2.2.1, p. 22, eq. (62), and [222], p. 12 (p. 10 of the English transl.), eq. (4). A related result can be found in [100], p. 139, eq. (3.13).}
\]
from a determinant [cf. Appendix C, eq. (C.9)]. The matrix $P_c^2(T)$ obeys the following equations.

\[
P_c^2(T) T = T P_c^2(T) = Pf T \mathbb{1}_6
\]  

\[
[P_c^2(T)]^2 = -\frac{1}{4} \left[ \frac{1}{2} \left( \text{tr} \, T^2 \right)^2 - \text{tr} \, T^4 \right] \mathbb{1}_6 + \frac{1}{2} T^2 \, \text{tr} \, T^2 - T^4
\]  

Under the assumption $Pf T \neq 0$ we can derive from the above two equations the relation [by multiplying eq. (253) by $T$]

\[
P_c^2(T) = -\frac{T}{Pf T} \left\{ \frac{1}{4} \left[ \frac{1}{2} \left( \text{tr} \, T^2 \right)^2 - \text{tr} \, T^4 \right] \mathbb{1}_6 - \frac{1}{2} T^2 \, \text{tr} \, T^2 + T^4 \right\}. \]  

Eq. (252) can be viewed as a special form of the Pfaffian analogue of the Laplace expansion of a determinant\(^{27}\).

It is clear that the imaginary part of the expressions on the r.h.s. of eq. (250) must vanish, consequently the following relations apply.

\[
\text{Im} \left( T_0^2 + T^M T_M \right) \, \mathbb{1}_6 + 2 \, \text{Im} \left( T^2 \right) = 0
\]  

\[
\text{Im} \left( T_0 \, T \right) - \text{Re} \left( P_c^2(T) \right) = 0
\]  

Eq. (255) originates from the symmetric part of eq. (250) while eq. (256) is derived from its antisymmetric part. From eq. (255) we can obtain the equation

\[
\text{Im} \left( T_0^2 \right) = \text{Im} \left( \text{tr} \, T^2 \right).
\]  

5.2 From $SO(6; \mathbb{R})$ back to its double cover $SU(4)$

Having obtained eq. (250), we will now reverse reasoning and study the relation between the $SO(6; \mathbb{R})$ transformations $L$ and the corresponding unitary $SU(4)$ transformations $U (\{\lambda\}; \{\hat{c}\})$ in the opposite direction. We will now assume that $L$ is known and derive from it an expression for $U = U (\{\lambda\}; \{\hat{c}\})$. In the literature, one finds two approaches to this task. One is due to Fedorov and collaborators [235, 236] and another one has been given by Klotz [104] (relying on earlier work by Macfarlane [100, 237]). In the first part of the following discussion (Subsec. 5.2.1) we will rely to a large extent on the former while in a second part (Subsec. 5.2.2) we will describe the approach by Klotz.

\(^{27}\)Cf. [228], Part 1, Chap. 1, Sec. 3, p. 6, eq. (1.5). This expansion is originally due to Tanner [229], has also been noted later by Baker [230], and proved in [231–233]. For a recent discussion see [234].
5.2.1 Plane orthogonal transformations – The approach by Fedorov and collaborators

To begin with, it turns out to be useful to recall that any orthogonal transformation $SO(6; \mathbb{R})$ can be represented in terms of 3 commuting orthogonal transformations within mutually orthogonal planes \cite{238, 239, 135}, Act. Sci. Ind. 643, §45, pp. 46/47 (p. 36 of the English transl.). In the Cayley representation of an orthogonal transformation $L$ [eq. (231)] each of these plane orthogonal transformations can be given in terms of a decomposable (real antisymmetric) matrix $A$. For such a plane orthogonal transformation [obeying $A^2 = \frac{1}{2} \left( \text{tr} A^2 \right)$] the orthogonal matrix $L$ can be written as ($a \neq 1$; \cite{235}, p. 1034, p. 1350 of the English transl.)

$$L = \mathbb{I}_6 + \frac{2A}{1-a} \left( \mathbb{I}_6 + A \right), \quad a = \frac{1}{2} \text{tr} A^2. \quad (258)$$

Now, comparing the antisymmetric parts of the equations (250) and (258) we immediately find the relation [The last term on the r.h.s. of eq. (250) vanishes for a plane orthogonal transformation.]

$$\frac{A}{1-a} = -T_0 T \quad (259)$$

where

$$T_0 = \pm \left( 1 + \frac{1}{2} \text{tr} T^2 \right)^{\frac{1}{2}} = \pm (1-a)^{-\frac{1}{2}} = \pm \left( 1 - \frac{1}{2} \text{tr} A^2 \right)^{-\frac{1}{2}} \quad (260)$$

in view of eq. (227) [$T_0$ and $T$ are real for a plane orthogonal transformation.]. For a plane orthogonal transformation parametrized by the antisymmetric matrix $A$ it holds

$$1 - \frac{1}{2} \text{tr} A^2 = \text{det} (\mathbb{I}_6 - A). \quad (261)$$

Consequently, eq. (259) can be written as

$$T = \mp \left( 1 - \frac{1}{2} \text{tr} A^2 \right)^{-\frac{1}{2}} A = \mp \frac{A}{\sqrt{\text{det} (\mathbb{I}_6 - A)}}, \quad (262)$$

$$A = -\frac{T}{T_0} = -\text{sign} T_0 \left( 1 + \frac{1}{2} \text{tr} T^2 \right)^{-\frac{1}{2}} T. \quad (263)$$

Finally, any unitary transformation $U$ standing in correspondence to a plane orthogonal transformation $L = L(A)$ can be written as

$$U = U(A) = \pm [\text{det} (\mathbb{I}_6 - A)]^{-\frac{1}{2}} \left( \mathbb{I}_4 - A^N \hat{c}_N \right). \quad (264)$$
Having obtained this representation we can now go over to the general case. Any real antisymmetric matrix $A$ representing a general orthogonal transformation $SO(6; \mathbb{R})$ can be written as the sum of three decomposable real antisymmetric matrices $A_k$ that represent three (commuting) plane orthogonal transformations in mutually orthogonal planes:

$$A = A_1 + A_2 + A_3, \quad A_k A_l = 0 \text{ for } k \neq l.$$  \hfill (265)

We have

$$L = L(A) = L(A_1) L(A_2) L(A_3)$$  \hfill (266)

and, consequently,

$$U = U(A) = U(A_1) U(A_2) U(A_3).$$  \hfill (267)

Inserting eq. (264) into eq. (267) and performing the algebra [by means of eq. (36) and taking into account the decomposability of the matrices $A_k$ (no summation over $k$ here): $A_k^M A_k^N \epsilon_{MNP} = 0$ ] we find

$$U = U(A) = T_0 \mathbb{1}_4 + T^P \hat{c}_P,$$

$$= \pm [\det (\mathbb{1}_6 - A)]^{-\frac{1}{2}}$$

$$\times \left[ (1 - i \text{Pf} A) \mathbb{1}_4 - (A^P - i \text{Pc}^2(A)^P) \hat{c}_P \right],$$  \hfill (268)

$$T_0 = \pm [\det (\mathbb{1}_6 - A)]^{-\frac{1}{2}} (1 - i \text{Pf} A),$$  \hfill (269)

$$T_P = \mp [\det (\mathbb{1}_6 - A)]^{-\frac{1}{2}} (A_P - i \text{Pc}^2(A)^P).$$  \hfill (270)

Eq. (268) agrees with eq. (27) of [236] (p. 988, p. 258 of the English transl.). For a related result see [104], p. 2245, eq. (13) [or its discussion in our Subsec. 5.2.2, eq. (281)]. It also yields an alternative to eq. (250) by relying on the Cayley representation [eq. (231)] of the matrix $L$ [cf. eq. (26) of [236] (p. 988, p. 258 of the English transl.)]

$$A = - \frac{\text{Re} T}{\text{Re} T_0}.$$  \hfill (271)

The eqs. (269), (270) can now be inserted into eq. (250) to obtain the following further version of eq. (231).

$$L = \mathbb{1}_6 + \frac{2A(\mathbb{1}_6 + A)}{\det (\mathbb{1}_6 - A)} \left[ \left( 1 - \frac{1}{2} \text{tr} A^2 \right) \mathbb{1}_6 + A^2 - \left[ \text{Pc}^2(A) \right]^2 \right]$$  \hfill (272)

Besides more elementary ones we have made use of the following result in deriving eq. (272).

$$\epsilon_{KMN} \text{Re} T^M \text{Im} T^N = \frac{1}{\det (\mathbb{1}_6 - A)} \left[ -\frac{1}{2} A_K \text{tr} A^2 + (A^3)_K \right]$$  \hfill (273)
Eq. (272) can be further transformed to read

\[ L = \mathbb{I}_6 + 2A - \frac{2(\mathbb{I}_6 + A)}{\det(\mathbb{I}_6 - A)} \left[ (\det A) \mathbb{I}_6 - A^2 \left( \left( 1 - \frac{1}{2} \text{tr} A^2 \right) \mathbb{I}_6 + A^2 \right) \right]. \]  

(274)

This form agrees with the general eq. (3) in ref. [235].

5.2.2 The approach by Klotz

The approach by Klotz [104] (adapted to our case) for relating \( SO(6; \mathbb{R}) \) transformations \( L \) and the corresponding unitary \( SU(4) \) transformations \( \hat{U}(\{\lambda\}; \{\hat{c}\}) \) starts by considering the relation

\[ \left( \hat{\epsilon}_N \right)_a^b \left( \hat{\epsilon}_N \right)_c^d = \delta_a^b \delta_c^d - 4 \delta_a^d \delta_c^b \]  

(275)

which goes back to Pauli. Here, \( \left( \hat{\epsilon}_N \right)_a^b \) denotes the matrix elements of the operator \( 4 \times 4 \) matrix \( \hat{\epsilon}_N \). Eq. (275) can easily be derived by means of the eqs. (23), (242), (245), (239)-(241). Multiplying eq. (275) by \( \hat{U} \) and \( U^\dagger \) [i.e., certain matrix elements of the unitary operator \( \hat{U}(\{\lambda\}; \{\hat{c}\}) \in SU(4) \)] and performing the sum over the indices \( a, b, c \), one can derive the following equation (in operator notation).

\[ U \text{tr} \hat{U} = \frac{1}{4} \left[ \mathbb{I}_4 - U \hat{\epsilon}_N U^\dagger \hat{\epsilon}_N \right] \]  

(276)

Recognizing, that \( U \hat{\epsilon}_N U^\dagger \) can be written as \( \chi_N M \hat{\epsilon}_M \) and taking into account the relation \( \text{tr} \hat{U} = 4 T_0 \) [cf. eq. (229)] we find the expression

\[ U = \frac{T_0}{16 |T_0|^2} \left[ \mathbb{I}_4 - \chi_{NM} \hat{\epsilon}_M \hat{\epsilon}_N \right]. \]  

(277)

By taking the trace of eq. (277) one can derive the relation (\( \text{tr} \chi = \chi_{M}^M \))

\[ |T_0|^2 = \frac{1}{16} \left[ 1 + \text{tr} \chi \right] = \frac{1}{16} \left[ 1 + \frac{1}{2} (\text{tr} L)^2 - \frac{1}{2} \text{tr} L^2 \right] \]  

(278)

Note, that an analogous approach has been used earlier by Macfarlane (leaving aside here the problem of the signature in the spaces used) in the cases of \( SO(4; \mathbb{R}) \) [237] and \( SO(5; \mathbb{R}) \) [100], p. 145.\[240\], p. 32 (p. 725 of Vol. 2 of the reprint [241]), eq. (I), [242], p. 118 (p. 762 of Vol. 2 of the reprint [241]), eq. (II) Also cf. e.g., [243], p. 210, eq. (233), [92], p. 35, eq. (11), [100], p. 137, eq. (2.22), [94], p. 361, eq. (4.2), [104], p. 2245, Sec. IV, [96], p. 208, eq. (50), [111], Sec. 5.3, p. 167, eq. (5.22), [220], p. 44, eq. (5a), [221], Subsec. 2.1.1, p. 18, eq. (47), [222], p. 12 (p. 10 of the English transl.), eq. (2').
which, however, still leaves the (complex) phase of $T_0$ undetermined. One can convince oneself that this result agrees with the corresponding expression derived from eq. (269). To see this explicitly one has to rely on the relation

$$\frac{1}{6} (\text{tr } L)^4 - (\text{tr } L)^2 \text{tr } L^2 + \frac{4}{3} \text{tr } L \text{tr } L^3$$

$$+ \frac{1}{2} (\text{tr } L^2)^2 - \text{tr } L^4 = 2 (\text{tr } L)^2 - 2 \text{tr } L^2$$  (279)

which can be derived from eq. (155) by means of taking appropriate trace operations on both sides. Further progress in discussing eq. (277) can now be made by relying on eq. (36). Finally, eq. (277) is found to read $[\text{tr } L = \text{tr } L^{(+)}$, $Pc^2(L) = Pc^2(L^{(-)})]$

$$U = T_0 \left[ \mathbb{1}_4 - \frac{1}{16 |T_0|^2} (\chi_{nk}^{nl} - i \chi^{PQ} \epsilon_{PQkl}) \hat{c}^{kl} \right]$$  (280)

$$= T_0 \left[ \mathbb{1}_4 - \frac{1}{8 |T_0|^2} \{L^{(-)} [(\text{tr } L) \mathbb{1}_6 - 2 L^{(+)}] - 2i Pc^2(L)\}_M \hat{c}^M \right].$$  (281)

This entails

$$T_M = -\frac{2 T_0}{1 + \frac{1}{2} (\text{tr } L)^2 - \frac{1}{2} \text{tr } L^2}$$

$$\times \left\{L^{(-)} [(\text{tr } L) \mathbb{1}_6 - 2 L^{(+)}] - 2i Pc^2(L)\}_M \right\}$$  (282)

which stands in correspondence to eq. (270)\(^{30}\).

It now remains to calculate the (complex) phase of $T_0$. Klotz [104] incorrectly states on p. 2245 [above of eq. (13)] that the (complex) phase of $T_0$ can assume the (fixed!) values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ only. This statement does not follow (as Klotz suggests) from the condition $\det U = 1$ and is also in contradiction to eq. (269). Following the reasoning of Klotz [104], Laufer [111] has also discussed (Sec. 5.3, p. 162-179) the homomorphism between $SU(2, 2)$ and $SO(2, 4; \mathbb{R})$ (The difference in signature to our case $SU(4)$, $SO(6; \mathbb{R})$, can be disregarded for the present purpose.) and makes the same erroneous statement\(^{31}\). As a matter of fact, $T_0$ can be calculated from the condition $\det U = 1$ as some fourth (complex!) root and any root will differ from the other three by one of the phases $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (mod $2\pi$) but can be complex itself, in

\(^{30}\)Infinitesimally $L^{(-)} \approx 2A$, and for this domain one immediately recognizes that eq. (282) agrees with eq. (270). However, we have not attempted to explicitly check the equivalence of the eqs. (270) and (282) in general.

\(^{31}\)Bottom of p. 169; incidentally, the problem of the complex phase does not occur for smaller groups and, therefore, does not show up in the work of Macfarlane [100].
The determinant can be calculated by means of the formula

\[ \det U = T_0^4 - T_0^2 \text{tr} T^2 - \frac{1}{4} (\text{tr} T^2)^2 + \text{tr} T^4 + 8i T_0 \text{Pf} T . \]  

(283)

The sign of the last term is determined by our choice of the paravector space (specifically, our choice for the sign of the component \( \hat{c}_0 = -1 \)). The explicit calculation of \( T_0 \) in terms of the matrix \( L \) by means of eq. (283) turns out to be quite tedious. In view of the fact, that eq. (269) provides us with an explicit expression of \( T_0 \) in terms of the matrix \( A \) we will not further pursue this subject here. The algebraic complexity of the calculation of \( T_0 \) in terms of the matrix \( L \) seems to be related to the fact mentioned below from eq. (234) that the expression relating the antisymmetric matrix \( A \) to a sum of a finite number of powers of the matrix \( L \) is algebraically involved.

The alert reader will have noticed that the method of Klotz \[104] delivers twice as many results for the matrix \( U \) than the method of Fedorov and collaborators \[235, 236\] (Of course, the difference lies in the number of allowed values for the total phase of \( U \) only.) and than is required for the group \( SU(4) \) as the double cover of \( SO(6; \mathbb{R}) \). The explanation lies in the respective starting points of the two methods. In our version of the approach, the starting point of Klotz is invariant under the center \( Z_4 \) of \( SU(4) \).

5.3 From \( SO(6; \mathbb{R})/Z_2 \) back to its double cover \( SO(6; \mathbb{R}) \)

We will now invert eq. (130), i.e., we will express the elements of the matrix \( L \) in terms of the elements of the (compound) matrix \( \chi = C_2(L) \). This task occurs if one wants to find from the coefficients \( \chi_{0k}^M \) of a given Bogolyubov-Valatin transformation the \( SO(6; \mathbb{R}) \) matrix \( L \) generating it. We would like to discuss two possible lines of attack to this problem which, however, both have their deficiencies.

The first method relies on eq. (277) [or eq. (280)] which can be inserted into eq. (248) to obtain the matrix \( L \) directly. As a first step, this method requires the (tedious) calculation of the complete matrix \( \chi_{MN} \) from the coefficients \( \chi_{0k}^M \) [eq. (141) and its generalizations]. Furthermore, \( T_0 \) is known from eq. (278) up to its phase only and its calculation entails the difficulties discussed at the end of Subsec. 5.2.2.

The second method to calculate the matrix \( L \) from the coefficients \( \chi_{0k}^M \) takes a different route. Consider the following matrix elements (We also allow \( k, l = 0 \) for

\[ \text{Cf. [72], Sec. 3.6, p. 42, eq. (3.62), [244], [98], p. 37 (reprint # 1: p. 55, reprint # 2: p. 948), eq. (44), [73], Sec. 65, p. 129, eq. (65.1), [245], Part II, §11, pp. 160/161, [206], Sec. 3, p. 819, eq. (40). Be aware of misprints and errors in the various versions of the equation as displayed in the references cited.} \]
which $\chi_{0k\,0l}$ vanishes individually.)

$$\chi_{0k\,0l} = \beta_I^m \beta_J^n \chi_{mk\,nl}$$

$$= (\beta_I L \beta_I^T) L_{kl} - (\beta_I L^T) L_{00} = L_{00} L_{kl} - L_{k0} L_{0l} . \quad (284)$$

Of course, the choice of $\beta = \beta_I$ [eq. (93)] involves a certain element of arbitrariness in expressing the matrix $L$ in terms of $\chi$ and is dictated here by physics considerations and convenience. We can now formally write (assuming $L_{00} \neq 0$)

$$L_{kl} = \frac{\chi_{0k\,0l} + (L \beta_I^T) L_{0l}}{\beta_I L \beta_I^T} . \quad (285)$$

Note that $\chi_{0(-1)\,0l}$ is given by

$$- \chi_{0(-1)\,p_1p_2} = \chi_{(-1)0\,p_1p_2}$$

$$= \frac{1}{4!} \epsilon_{0k_1k_2l_1l_2} \epsilon_{p_1p_2q_1q_2s_1s_2} \chi_{0k_1\,mq_1} \chi_{0k_2\,mq_2} \chi_{0l_1\,ns_1} \chi_{0l_2\,ns_2} . \quad (287)$$

which is obtained by relying on the eqs. (128), (105) and $\hat{d}_{(-1)} = \hat{d}_{(-1)0} = i\hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4$. The other components of $\chi_{0k\,0l}$ can be read off from eq. (102). We now have to express $L_{k0}$, $L_{0l}$ in terms of $\chi_{0k\,0l}$ yet. Let us start with $L_{00}$. Taking the determinant on both sides of eq. (285) we find

$$\det L = \frac{\det \chi_{00}}{L_{00}^4} . \quad (288)$$

where $\chi_{00}$ denotes the $5 \times 5$ matrix with the matrix elements $\chi_{0k\,0l}$ ($k, l \neq 0$). Taking into account the orthogonality of the matrix $L$ (det $L = 1$) we finally obtain the relation

$$|L_{00}| = (\det \chi_{00})^\frac{1}{4} . \quad (289)$$

The non-diagonal elements $L_{k0}$, $L_{0l}$ can be found by means of the following consideration. Taking into account the orthogonality relation $LL^T = \mathbb{1}_6$ we find the following equation.

$$\chi_{0k\,0m} \chi_{0l\,0m} = (L_{k0} - L_{00} \delta_{k0})(L_{l0} - L_{00} \delta_{l0}) + L_{00}^2 (\delta_{kl} - \delta_{k0} \delta_{l0}) . \quad (290)$$

This equation can be transformed to read

$$(L_{k0} - L_{00} \delta_{k0})(L_{l0} - L_{00} \delta_{l0}) = \chi_{0k\,0m} \chi_{0l\,0m} - L_{00}^2 (\delta_{kl} - \delta_{k0} \delta_{l0}) . \quad (291)$$
Setting \( k = l \ (\neq 0) \) we immediately find

\[
|L_{k0}| = \sqrt{\chi_{0k}^{0m} \chi_{0k}^{0m} - L_{00}^2} = \sqrt{\chi_{0k}^{0m} \chi_{0k}^{0m} - (\det \chi_{00})^{\frac{1}{2}}}. \tag{292}
\]

Considering \( \chi_{0k}^{0m} \chi_{0m}^{0l} \) we obtain in an analogous way

\[
|L_{0k}| = \sqrt{\chi_{0k}^{0m} \chi_{0k}^{0m} - L_{00}^2} = \sqrt{\chi_{0k}^{0m} \chi_{0k}^{0m} - (\det \chi_{00})^{\frac{1}{2}}}. \tag{293}
\]

As an aside, we also find the following expressions for \( k \neq l \ (\neq 0) \)

\[
L_{k0}L_{l0} = \chi_{0k}^{0m} \chi_{0l}^{0m}, \tag{294}
\]

\[
L_{0k}L_{0l} = \chi_{0k}^{0m} \chi_{0m}^{0l}, \tag{295}
\]

which entail the following relations for the matrix \( \chi_{00} \) (no summation over \( k, l \) on the r.h.s.)

\[
\chi_{0k}^{0m} \chi_{0l}^{0m} - \delta_{kl} (\det \chi_{00})^{\frac{1}{2}}
\]

\[
= \sqrt{\left( \chi_{0k}^{0m} \chi_{0k}^{0m} - (\det \chi_{00})^{\frac{1}{2}} \right)
\left( \chi_{0l}^{0m} \chi_{0l}^{0m} - (\det \chi_{00})^{\frac{1}{2}} \right)}, \tag{296}
\]

\[
\chi_{0m}^{0k} \chi_{0m}^{0l} - \delta_{kl} (\det \chi_{00})^{\frac{1}{2}}
\]

\[
= \sqrt{\left( \chi_{0m}^{0k} \chi_{0m}^{0k} - (\det \chi_{00})^{\frac{1}{2}} \right)
\left( \chi_{0m}^{0l} \chi_{0m}^{0l} - (\det \chi_{00})^{\frac{1}{2}} \right)}. \tag{297}
\]

Collecting all results and inserting them into eq. (286), our final expression for the matrix \( L \) in terms of the matrix \( \chi \) (more specifically, in terms of its submatrix \( \chi_{00} \)) reads

\[
|L_{kl}| = (\det \chi_{00})^{\frac{1}{4}} \left| \chi_{0k}^{0l} \right|
\]

\[
\pm \sqrt{\left( \chi_{0k}^{0m} \chi_{0k}^{0m} - (\det \chi_{00})^{\frac{1}{2}} \right)
\left( \chi_{0l}^{0m} \chi_{0l}^{0m} - (\det \chi_{00})^{\frac{1}{2}} \right)} \right|. \tag{298}
\]

Already from eq. (130) it is clear that the matrix elements of the matrix \( L \) can be determined from the matrix \( \chi \) up to a minus sign only. The \( \pm \)-sign in (298) reflects an uncertainty and, therefore, weakness of the applied method to reconstruct the matrix \( L \) from its second compound matrix \( \chi = C_2(L) \).
6 Discussion

In this section we collect the discussion of various aspects of the study performed in the preceding parts of the article. We have postponed these considerations until now in order to streamline the presentation in the other sections.

6.1 Bogolyubov-Valatin transformations

To prepare ourselves for the following discussion we first have a look at the problem how the 15-dimensional irreducible representation of the Bogolyubov-Valatin group $SO(6; \mathbb{R})/\mathbb{Z}_2$ operating in the biparavector space $\bigwedge^2(V_6)$ decomposes into representations of smaller dimension under the subgroup chain $SO(6; \mathbb{R}) \supset O(5; \mathbb{R}) \supset O(4; \mathbb{R}) \supset O(3; \mathbb{R}) \supset O(2; \mathbb{R})$. This decomposition can be described by the graphical representation given in Figure 2.

![Diagram](image)

**Figure 2:** Decomposition of the 15-dimensional biparavector space $\bigwedge^2(V_6)$ under the subgroup chain of $SO(6; \mathbb{R})$. 

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6.1.1 Linear Bogolyubov-Valatin transformations

Now we will examine the well-known class of linear \( O(2n = 4; \mathbb{R}) \subset SO(6; \mathbb{R}) \) Bogolyubov-Valatin transformations. (Then, the only nonvanishing coefficients in the eq. (39) are \( \lambda_k^{(1)} \), \( \lambda_k^{(0)} \), \( \lambda_k^{(2)} \), \( \lambda_k^{(02)} \) ) and see how the decomposition of the bivector space \( \wedge^2(V_6) \) is structured under this class of canonical transformations. The 6 \( \times \) 6 matrix \( L \in SO(6; \mathbb{R}) \) then has the form (somewhat symbolically written)

\[
L = \begin{pmatrix}
\det A & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & A
\end{pmatrix}
\]  

(299)

where the 4 \( \times \) 4 matrix \( A \in O(4; \mathbb{R}) \). The matrix \( A = A(\{\lambda\}) \) reads [cf. the eqs. (48), (49), and (A.30)]:

\[
A = A(\{\lambda\}) = \begin{pmatrix}
\text{Re} \lambda_1^{(1)} & \text{Re} \lambda_1^{(0)} & \text{Re} \lambda_1^{(2)} & \text{Re} \lambda_1^{(02)} \\
\text{Im} \lambda_1^{(1)} & \text{Im} \lambda_1^{(0)} & \text{Im} \lambda_1^{(2)} & \text{Im} \lambda_1^{(02)} \\
\text{Re} \lambda_2^{(1)} & \text{Re} \lambda_2^{(0)} & \text{Re} \lambda_2^{(2)} & \text{Re} \lambda_2^{(02)} \\
\text{Im} \lambda_2^{(1)} & \text{Im} \lambda_2^{(0)} & \text{Im} \lambda_2^{(2)} & \text{Im} \lambda_2^{(02)}
\end{pmatrix} = \chi^{[1]} \cdot (300)
\]

Here, \( \chi^{[1]} \) denotes the matrix with row number \( k \) and column number \( m \) matrix elements \( \chi_k^{[1]}(m) \). Eq. (299) can be found by relying on Subsec. 5.3 and on eq. (127).

Consider now the linear space \( W \) (of operators in Fock space) which is the direct sum of the space spanned by the identity operator in spin space \( \mathbb{C}_4 \) (by the way, an invariant subspace) and the space of bivectors \( \wedge^2(V_6) \). It is spanned by the basis \( ^3^{34} \)

\[
a^\dagger = \begin{pmatrix}
\mathbb{I}_4, -i\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]} \wedge \hat{a}_2^{[1]} \wedge \hat{a}_2^{[2]}, \\
\hat{a}_1^{[2]} \wedge \hat{a}_2^{[1]} \wedge \hat{a}_2^{[2]}, -i\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]} \wedge \hat{a}_2^{[1]} \wedge \hat{a}_2^{[2]}, i\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]} \wedge \hat{a}_2^{[1]} \wedge \hat{a}_2^{[2]}, -i\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]} \wedge \hat{a}_2^{[1]} \wedge \hat{a}_2^{[2]}, \\
-\hat{a}_1^{[1]}, -\hat{a}_1^{[2]}, -\hat{a}_1^{[1]}, -\hat{a}_1^{[2]}, \\
-\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]}, -\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]}, -\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]} \\
-\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]}, -\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]}, -\hat{a}_1^{[1]} \wedge \hat{a}_1^{[2]}
\end{pmatrix}
\]

(301)

\[
= \begin{pmatrix}
\mathbb{I}_4, -i\hat{c}_1 \wedge \hat{c}_2 \wedge \hat{c}_3 \wedge \hat{c}_4, \\
i\hat{c}_2 \wedge \hat{c}_3 \wedge \hat{c}_4, -i\hat{c}_1 \wedge \hat{c}_3 \wedge \hat{c}_4, i\hat{c}_1 \wedge \hat{c}_2 \wedge \hat{c}_4, -i\hat{c}_1 \wedge \hat{c}_2 \wedge \hat{c}_3
\end{pmatrix}
\]

\( ^3^{34} \)Cf. e.g., [1], Sec. 2.2, p. 38. For further references concerning linear Bogolyubov-Valatin transformations see [49], p. 10247, below from eq. (16).

\( ^{34} \)To achieve a more compact graphical display, we give here the Hermitian conjugates \( a^\dagger \). Be aware of the minus signs and note that in the last line for simplicity we have used a somewhat imprecise notation [cf. eq. (129)].

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\[ -\hat{c}_1, -\hat{c}_2, -\hat{c}_3, -\hat{c}_4, \\
-\hat{c}_1 \land \hat{c}_2, -\hat{c}_1 \land \hat{\hat{c}}_3, -\hat{c}_1 \land \hat{c}_4, -\hat{c}_2 \land \hat{c}_3, -\hat{c}_2 \land \hat{\hat{c}}_4, -\hat{c}_3 \land \hat{\hat{c}}_4, \]
\[ = \left( I_4, \hat{\hat{\hat{c}}}_0, \hat{\hat{\hat{c}}}_1, \hat{\hat{\hat{c}}}_2, \hat{\hat{\hat{c}}}_3, \hat{\hat{\hat{c}}}_4 \right) \]
\[ = \left( I_4, \hat{\hat{C}}^\dagger \right). \quad (302) \]

Also defining \(^{35}\)
\[ \hat{b}^\dagger = \left( I_4, -i\hat{\hat{b}}_1[1] \land \hat{\hat{\hat{b}}}_1[2] \land \hat{\hat{b}}_2[1] \land \hat{\hat{\hat{b}}}_2[2], \\
i\hat{\hat{b}}_1[2] \land \hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{b}}_2[1] \land \hat{\hat{\hat{b}}}_2[2], -i\hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{b}}_2[1] \land \hat{\hat{b}}_2[2], i\hat{\hat{b}}_1[2] \land \hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{b}}_2[2], -i\hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{\hat{b}}}_1[2] \land \hat{\hat{b}}_2[1], \\
-\hat{\hat{b}}_1[1], -\hat{\hat{\hat{b}}}_2[1], -\hat{\hat{\hat{b}}}_1[2], -\hat{\hat{\hat{b}}}_2[2], \\
-\hat{\hat{b}}_1[1] \land \hat{\hat{\hat{b}}}_1[2], -\hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{\hat{b}}}_1[2], -\hat{\hat{\hat{b}}}_1[1] \land \hat{\hat{\hat{b}}}_1[2], \\
-\hat{\hat{\hat{b}}}_2[2] \land \hat{\hat{\hat{b}}}_2[1], -\hat{\hat{\hat{b}}}_2[2] \land \hat{\hat{\hat{b}}}_2[1], -\hat{\hat{\hat{b}}}_2[2] \land \hat{\hat{\hat{b}}}_2[1] \right) \quad (303) \]
\[ = \left( I_4, -id_1 \land \hat{d}_2 \land \hat{d}_3 \land \hat{d}_4, \\
id_2 \land \hat{d}_3 \land \hat{d}_4, -id_1 \land \hat{d}_3 \land \hat{d}_4, id_1 \land \hat{d}_2 \land \hat{d}_4, -id_1 \land \hat{d}_2 \land \hat{d}_3, \\
-\hat{d}_1, -\hat{d}_2, -\hat{d}_3, -\hat{d}_4, \\
-\hat{d}_1 \land \hat{d}_2, -\hat{d}_1 \land \hat{d}_3, -\hat{d}_1 \land \hat{d}_4, -\hat{d}_2 \land \hat{d}_3, -\hat{d}_2 \land \hat{d}_4, -\hat{d}_3 \land \hat{d}_4 \right) \]
\[ = \left( I_4, \hat{\hat{d}}^\dagger \right). \quad (304) \]

\(^{35}\)Below we have not indicated that the wedge product relates to a different vector space than in eq. (301), i.e.: \(\land = \hat{\land} \land \hat{\land}\).
we can write any two-mode Bogolyubov-Valatin transformation as a basis transfor-
mation in the linear space \( W \).

\[
\begin{align*}
b &= A(\{\lambda\}) a 
\end{align*}
\]  

(305)

For a linear Bogolyubov-Valatin transformation, the matrix \( A \) assumes a block di-
agonal form. It reads [cf. eq. (127)]:

\[
A(\{\lambda\}) = \text{diag}[C_0(A), C_4(A), C_3(A^T)^*, C_1(A), C_2(A)]
\]  

(306)

Here, \( C_i(A) \) denotes compound matrices of the matrix \( A \) (cf. Appendix C). Let us

denote the subspace in which the compound matrix \( C_l(A) \) operates by \( W_l \)
\( (W = W_0 \oplus W_4 \oplus W_3 \oplus W_1 \oplus W_2) \). With the usual relations for compound matrices and
the orthogonality condition for the matrix \( A \) we can write eq. (306) as:

\[
A(\{\lambda\}) = \text{diag} [1, \det A, (\det A)A, A, C_2(A)]
\]  

(307)

For the indices of the compound matrices we have applied here the usual convention
of the lexicographical order of the indices. Note that the above representation of \( A \)
in terms of compound matrices requires (and, therefore, induces) a certain order in
the sequence of the basis elements of the linear subspaces \( W_2, W_3 \) invariant under
the \( O(4; \mathbb{R}) \) transformation [cf. eqs. (301), (303)] once the sequence in the subspace
\( W_1 \) (spanned by \( \hat{a}_1[1] = \hat{c}_1, \hat{a}_1[2] = \hat{c}_2, \hat{a}_1[1] = \hat{c}_3, \hat{a}_1[2] = \hat{c}_4 \) ) has been chosen.

6.1.2 Nonlinear Bogolyubov-Valatin transformations for one fermion

mode

To make further contact with previously known results, we consider now the case of
a full nonlinear \( SO(3; \mathbb{R}) \) Bogolyubov-Valatin transformation of one
fermion mode (say, with mode number 1) considered in [49].

\[
\hat{b}_1 = B_1(\{\lambda\}; \{\hat{a}\}) = \lambda_1^{(01)} \hat{a}_1 + \lambda_1^{(10)} \hat{a}_1^\dagger + \lambda_1^{(11)} \left( \hat{a}_1^\dagger \hat{a}_1 - \frac{1}{2} \mathbb{1}_4 \right)
\]  

(308)

The coefficients \( \lambda_1^{(01)}, \lambda_1^{(10)}, \lambda_1^{(11)} \) obey the equations (cf. [49], p. 10247)

\[
\begin{align*}
|\lambda_1^{(01)}|^2 + |\lambda_1^{(10)}|^2 + \frac{1}{2} |\lambda_1^{(11)}|^2 &= 1, \\
4 \lambda_1^{(10)} \lambda_1^{(11)} + \left( \frac{\lambda_1^{(11)}}{\lambda_1^{(10)}} \right)^2 &= 0.
\end{align*}
\]  

(309)-(310)

For this case, the \( 6 \times 6 \) matrix \( L \in SO(6; \mathbb{R}) \) has the form [cf. eq. (127)]

\[
L = \begin{pmatrix}
P_{(-1)(-1)} & 0 & 0 & 0 & P_{(-1)3} & P_{(-1)4} \\
0 & Q_{00} & Q_{01} & Q_{02} & 0 & 0 \\
0 & Q_{10} & Q_{11} & Q_{12} & 0 & 0 \\
0 & Q_{20} & Q_{21} & Q_{22} & 0 & 0 \\
P_{3(-1)} & 0 & 0 & 0 & P_{33} & P_{34} \\
P_{4(-1)} & 0 & 0 & 0 & P_{43} & P_{44}
\end{pmatrix}
\]  

(311)
where the $3 \times 3$ matrices $P$, $Q \in SO(3; \mathbb{R})$. The orthogonal matrix $Q$ is related to the orthogonal matrix $A(\{\lambda\})$ used in eq. (23), p. 10248 in [49] the following way

$$A(\{\lambda\}) = \begin{pmatrix}
    \text{Re} \kappa^{(1|0)} & \text{Re} \kappa^{(0|1)} & \text{Re} \kappa^{(1|1)} \\
    \text{Im} \kappa^{(1|0)} & \text{Im} \kappa^{(0|1)} & \text{Im} \kappa^{(1|1)} \\
    \text{Im} \left( \kappa^{(0|1)} \kappa^{(1|1)} \right) & \text{Im} \left( \kappa^{(1|1)} \kappa^{(1|0)} \right) & \text{Im} \left( \kappa^{(1|0)} \kappa^{(0|1)} \right)
\end{pmatrix}
\quad = C_2(Q) .$$

(312)

Here,

$$\kappa^{(1|0)} = \lambda^{(0|1)} + \lambda^{(1|0)},$$

(313)

$$\kappa^{(0|1)} = i \left( \lambda^{(0|1)} - \lambda^{(1|0)} \right),$$

(314)

$$\kappa^{(1|1)} = \lambda^{(1|1)} .$$

(315)

Relying on the equation (for the notation applied cf. Appendix C)

$$A^{-1} = A^T = C_1(Q)^* = Q^* = \varepsilon Q^T \varepsilon$$

(316)

where

$$\varepsilon = \begin{pmatrix}
    0 & 0 & 1 \\
    0 & -1 & 0 \\
    1 & 0 & 0
\end{pmatrix}$$

(317)

we quickly find

$$Q = \varepsilon A(\{\lambda\}) \varepsilon$$

$$= \begin{pmatrix}
    \text{Im} \left( \kappa^{(0|1)} \kappa^{(1|1)} \right) & -\text{Im} \kappa^{(1|1)} & \text{Im} \kappa^{(1|0)} \\
    -\text{Im} \kappa^{(1|1)} & \text{Im} \kappa^{(0|1)} & -\text{Im} \kappa^{(1|0)} \\
    \frac{\text{Re} \kappa^{(1|1)}}{\text{Im} \kappa^{(1|1)}} & \frac{\text{Re} \kappa^{(0|1)}}{\text{Im} \kappa^{(0|1)}} & \frac{\text{Re} \kappa^{(1|0)}}{\text{Im} \kappa^{(1|0)}}
\end{pmatrix} .$$

(318)

The orthogonal $3 \times 3$ matrix $P$ must be a representation of the special orthogonal transformation given by the matrix $Q$ (in view of the group properties of general Bogolyubov-Valatin transformations). The simplest choice consists in setting $P = \mathbb{1}_3$, however, also putting $P = Q$ (or setting it equal to some equivalent representation matrix) is possible. At this point we are mainly interested in the question of whether a full nonlinear Bogolyubov-Valatin transformation of the fermion mode with mode number 1 necessarily induces a related (nonidentical) transformation of the fermion mode with mode number 2 in such a way that the combined transformation is canonical. One can now convince oneself of the fact that no admissible choice of the matrix $P$ leaves the fermion mode with mode number 2 unchanged.
For the “minimal” choice $P = \mathbb{I}_3$ we find

$$
\hat{b}_2 = B_2 (\{\lambda\}; \{\hat{a}\}) \\
= \left[ \left( |\lambda_1^{(0)1}|^2 - |\lambda_1^{(1)0}|^2 \right) \mathbb{I}_4 + \left( \lambda_1^{(11)} \overline{\lambda_1^{(0)1}} - \lambda_1^{(11)} \lambda_1^{(1)0} \right) \hat{a}_1 \\
+ \left( \lambda_1^{(11)} \overline{\lambda_1^{(1)0}} - \lambda_1^{(11)} \lambda_1^{(0)1} \right) \hat{a}_1^\dagger \right] \hat{a}_2 .
$$

Of course, for a linear Bogolyubov-Valatin transformation of mode 1 (i.e., $\lambda_1^{(11)} = 0$) the two fermion modes decouple.

This consideration demonstrates the truly nonlinear character of general Bogolyubov-Valatin transformations. A nonlinear Bogolyubov-Valatin transformation for one mode necessarily requires us to perform corresponding changes for the other mode(s) in order to maintain the correct CAR. The picture of independent quasi-particles characteristic for linear Bogolyubov-Valatin transformations no longer applies in general.

### 6.1.3 Nonlinear Bogolyubov-Valatin transformations – Some comment on the ansatz of Fukutome and collaborators

One of the earlier studies of nonlinear Bogolyubov-Valatin transformations is contained in an article by Fukutome and collaborators [31]. Our study of these transformations allows us to see the ansatz used in ref. [31] from a different perspective. For the interested reader (who, for easy reference, is advised to have a copy of the article [31] close at hand), we would like to mention that our eq. (105) bears some resemblance to eq. (2.11) on p. 1557 in [31] specialized to the case of two fermion modes. However, while in part it corresponds to our more general equation (105) the eq. (2.11) in [31] does not adequately reflect in its display the general group theoretical structure of nonlinear Bogolyubov-Valatin transformations. To see this note that the case considered in ref. [31] corresponds to the choice $\chi_{k}^{[4]} = 0, \chi_{k}^{[3]} = 0$ in our eq. (102). Consequently, under the reduction $SO(5; \mathbb{R}) \subset SO(6; \mathbb{R})$ considered in [31] the 15-dimensional irreducible representation of the nonlinear Bogolyubov-Valatin transformation [equivalent to $SO(6; \mathbb{R})/\mathbb{Z}_2$] operating in biparavector space $\wedge^2(V_6)$ splits into two irreducible representations of $SO(5; \mathbb{R})$ - a 5-dimensional representation [operating in the space of biparavectors with indices $k = (-1)$ and $l \neq (-1)$] and a 10-dimensional representation [operating in the space of biparavectors with indices $k, l \neq (-1)$]; cf. Figure 2. The 5-dimensional linear space (of operators in Fock space) is spanned by products of three or four different fermion creation and annihilation operators while the 10-dimensional space (of operators in Fock space) is spanned by the fermion creation and annihilation operators themselves and all possible products of two (different) of them. The unit operator in Fock space does not transform under nonlinear Bogolyubov-Valatin transformations and it spans an
invariant subspace of the 16-dimensional space of linear operators in Fock space. Consequently, eq. (2.11) on p. 1557 in [31] must be the above characterized 5-dimensional representation in disguise and not a transformation in paravector space [associated with the Clifford algebra C(0, 4)] as its display suggests.

6.1.4 The group of nonlinear Bogolyubov-Valatin transformations as the adjoint representation of the group SU(2^n = 4)

Recalling eq. (127)

\[ U \hat{c}_M U^\dagger = \chi_M^N \hat{c}_N \]  

(320)

we recognize ([275], Vol. II, Chap. 11, Sec. 5, Theorem III, p. 417) that the matrix \( \chi^T \) (i.e., the transposed of \( \chi \)) is the representation matrix of the group element \( U \) of the Lie group \( SU(2^n = 4) \) taken in the adjoint representation, i.e., we can write

\[ \chi^T = [\chi(U)]^T = \text{Ad}(U). \]  

(321)

Consequently, we find [cf. eq. (127)]

\[ \text{Ad}(U) = C_2(L^T) \]  

(322)

where the matrix \( L = L(U) \in SO(6; \mathbb{R}) \) is given by [cf. eq. (249)]

\[ L_{kl} = \frac{1}{4} \text{tr} \left( \Gamma_k^+ U^T \Gamma_l U \right). \]  

(323)

Clearly, the adjoint representation of the Lie group \( SU(2^n = 4) \) is not faithful because \( SU(2^n = 4) \) posesses a non-trivial center (i.e., \( \mathbb{Z}_4 \)). Let us finally note that the adjoint representation of the group \( SU(3) \) has been discussed in [246].

6.2 Biparavectors and parametrizations of \( SU(4) \) and \( SO(6; \mathbb{R}) \) transformations

Low-dimensional unitary and orthogonal matrices play an essential role in many investigations in theoretical physics and beyond. The choice of representation for them can significantly facilitate or inhibit their use. In the mathematical and physical literature many different parametrizations can be found. Concerning the unitary matrix \( U \in SU(4) \) we are of course interested in representations in terms of the biparavectors \( \hat{c}_M \), i.e., in terms of fermion creation and annihilation operators (For a related discussion see [247, 248],). We have discussed in Sec. 5 such a representation of \( SU(4) \) matrices in terms of biparavectors [cf. eq. (226)]. In this subsection, we would like to supplement this discussion by some further comments.
6.2.1 The representation of SU(4) transformations by Östlund and Mele

An interesting representation of the unitary operator \( U \) related to eq. (226) (i.e., in terms of biparavectors) has been considered by Östlund and Mele [43]. Define the following states (We have changed somewhat the numbering in comparison with ref. [43]):

\[
\begin{align*}
\Psi_{-1} &= |2\rangle_{(2,1)} = \hat{a}_2\hat{a}_1|0\rangle, \\
\Psi_0 &= |0\rangle, \\
\Psi_1 &= |1\rangle_1 = \hat{a}_1^\dagger|0\rangle, \\
\Psi_2 &= |1\rangle_2 = \hat{a}_2^\dagger|0\rangle.
\end{align*}
\]

Then, one defines the elements of the matrix \( \hat{m} \) (\( \equiv m \), in the notation used in [43]) by means of the relation

\[
\hat{m}_{ij} = \Psi_i\Psi_j^\dagger. 
\]

It is then useful to recall the relation

\[
|0\rangle\langle 0| = \Psi_0\Psi_0^\dagger = (\mathbb{1}_4 - \hat{a}_1\hat{a}_1^\dagger)(\mathbb{1}_4 - \hat{a}_2\hat{a}_2^\dagger)
\]

to explicitly construct the matrix \( \hat{m} \equiv m \) in terms of the creation and annihilation operators \( \hat{a}_k^\dagger, \hat{a}_k \) (cf. eq. (1) of ref. [43]). Consequently, the matrix elements \( \hat{m}_{ij} \) of the matrix \( \hat{m} \) are linear combinations of the operators \( \hat{c}_M \) (and transform under \( SO(6;\mathbb{R})/\mathbb{Z}_2 \) transformations correspondingly). For the explicit relations see Appendix F.

One can now write \((X_{ij} \in \mathbb{C})\)

\[
U = U(X) = \sum_{i,j=-1}^2 X_{ji} \hat{m}_{ij}.
\]

A remarkable property of the representation of the unitary operator \( U \) in terms of the matrix \( \hat{m} \) is represented by the following equation where \( XY \) denotes the matrix multiplication of the two \( 4 \times 4 \) matrices \( X \) and \( Y \).

\[
U(X)U(Y) = U(XY)
\]

Taking into account the relation \( [U(X)]^\dagger = U(X^\dagger) \) (eq. (3) of [43]), the unitarity condition for the unitary operator \( U \) is then found to read [This is another version of the eqs. (227), (228)].

\[
XX^\dagger = \mathbb{1}_4.
\]

Consequently, eq. (330) provides us with a representation of the unitary matrix \( U \) by means of some, in general other, unitary matrix \( X \).
6.2.2 Exponential representation

Next we want to have a short look at some other version of the parametrizations of the unitary \([U \in SU(4)]\) (and orthogonal \([L \in SO(6; \mathbb{R})]\)) transformations we have applied. Besides the representation we are using [see eq. (226)] of most interest appears to be an exponential representation of the unitary matrix \(U\)

\[
U = T_0 \mathbb{1}_4 + T^M \hat{c}_M = T_0 \mathbb{1}_4 + \frac{1}{2} T^{m_1 m_2} \hat{c}_{m_1 m_2}
\]

\[
= \exp \left( V^M \hat{c}_M \right) = \exp \left( \frac{1}{2} V^{m_1 m_2} \hat{c}_{m_1 m_2} \right)
\]

(333)

in terms of a real antisymmetric matrix \(V\) (remember: \(\hat{c}^\dagger_M = -\hat{c}_M\)). Formally,

\[
V^M \hat{c}_M = \frac{1}{2} V^{m_1 m_2} \hat{c}_{m_1 m_2} = \ln U.
\]

(334)

There is a comprehensive mathematical literature on the subject of the logarithm of a matrix and its subtleties. Restricting our attention to the main sheet of the logarithmic function, a calculation of the matrix \(V\) in terms of \(T_0\) and the matrix \(T\) could rely on the approach presented in [249], in particular Sec. 2.C.3, p. 021108-5 (also note [250]). However, we will not pursue this calculation here.

Concerning some of the subtleties of the matrix logarithm it is useful to recall that any \(n \times n\) unitary matrix operating in a complex space of \(n\) dimensions can be represented as a \(2n \times 2n\) orthogonal matrix operating in a real space of dimension \(2n\). Consequently, the problem of finding the exponential representation of the unitary matrix \(U\) is closely related to the problem of the exponential representation of the orthogonal matrix \(L\) in terms of an antisymmetric matrix \(B\)

\[
L = e^B.
\]

(335)

For our case of the \(6 \times 6\) matrix \(L\), by virtue of the Cayley-Hamilton theorem the antisymmetric \((6 \times 6)\) matrix \(B\) must be a linear combination of the matrices \(A^k\), \(k = 1, 3, 5\), in the vicinity of the unity transformation \(L = \mathbb{1}_6\). For the subject of the logarithm of orthogonal matrices we refer the reader to [251–253]. Finally, we want to mention that for bi(para) vectors two different notions of the exponential function exist (see [254], [56], Sec. 17.3, p. 221, both editions). But, we will not further discuss this subject here.

6.2.3 On Cayley-Klein parameters for \(SO(6; \mathbb{R})\) transformations

In the light of the equations (246), (247) (Subsec. 5.1) reflecting the relation between \(SO(6; \mathbb{R})\) and \(SU(4)\) transformations it seems to be interesting to add a comment on
the introduction of generalized Cayley-Klein parameters for the orthogonal transformations $L$ in the Euclidean space $\mathbb{R}_6$. For a good introduction to Cayley-Klein parameters in $\mathbb{R}_3$ we refer the reader to [255], §4.5, 1. ed. p. 109, 2. ed. p. 148 (Note, that in the third edition [256] this material is no longer included.). Leaving aside the problem of the signature of the (pseudo-) Euclidean space on which the (pseudo-) orthogonal transformations operate, Cayley-Klein parameters for (pseudo-) orthogonal transformations in a 4-dimensional space have been studied by Cayley already [257]. They have later been discussed in [258–262], [263], also see [264], Mathematische Zusätze und Ergänzungen, Sec. 17, pp. 806-813, [89, 265–269]. Corresponding studies of Cayley-Klein parameters for (pseudo-) orthogonal transformations in five dimensions can be found in [270–272]. A five-dimensional space seems to be the space of largest dimension to which the standard Cayley-Klein parameters can be generalized. From our perspective, the reason is that for orthogonal transformations $L \in SO(5; \mathbb{R})$ the 15-dimensional biparavector space $\bigwedge^2(V_6)$ decomposes into two invariant subspaces. One subspace is a 5-dimensional space in which the five-dimensional transformation $L$ acts linearly. The second invariant subspace is a 10-dimensional space in which the 5-dimensional transformation $L$ acts via its second compound matrix exactly in the same manner as an orthogonal transformation $L \in SO(6; \mathbb{R})$ is acting in the full 15-dimensional biparavector space $\bigwedge^2(V_6)$ [cf. eq. (127) and Fig. 2]. Consequently, generalized Cayley-Klein parameters for orthogonal transformations in $\mathbb{R}_6$ must be based on the spin bivector space $\bigwedge^2(\mathbb{C}_4)$ according to the eqs. (246), (247), rather than on eq. (127) related to biparavector space $\bigwedge^2(V_6)$.

6.3 Biparavectors as bi(para)vectors of the spin bivector space $\bigwedge^2(\mathbb{C}_4)$

Here, we would like to extend somewhat the discussion performed in paragraph IV of Subsec. 3.2.3. By means of the eqs. (242), (245) (Subsec. 5.1) one finds for the decomposable/simple biparavectors [cf. eq. (23); $m_1 \neq m_2$]

$$\hat{c}_{m_1m_2} = \frac{1}{2} \left( \hat{c}^\dagger_{m_1} \hat{c} m_2 - \hat{c}^\dagger_{m_2} \hat{c} m_1 \right) = \hat{c}^\dagger_{m_1} \hat{c} m_2$$

$$= \frac{1}{2} \left( \hat{\Gamma}^+_m \hat{\Gamma}_m m_2 - \hat{\Gamma}^+_m \hat{\Gamma}_m m_1 \right) = \hat{\Gamma}^+_m \hat{\Gamma}_m m_2$$

$$= \frac{1}{2} \left( \hat{\Gamma}^+_m \hat{\Gamma}_m m_2 - \hat{\Gamma}^+_m \hat{\Gamma}_m m_1 \right) = \hat{\Gamma}^+_m \hat{\Gamma}_m m_2$$

$$= \frac{1}{2} \left( \hat{\Gamma}^+_m \hat{\Gamma}_m m_2 - \hat{\Gamma}^+_m \hat{\Gamma}_m m_1 \right) = \hat{\Gamma}^+_m \hat{\Gamma}_m m_2$$

$$= \frac{1}{2} \left( \hat{\Gamma}^+_m \hat{\Gamma}_m m_2 - \hat{\Gamma}^+_m \hat{\Gamma}_m m_1 \right) = \hat{\Gamma}^+_m \hat{\Gamma}_m m_2$$

(336)

For the related references see footnote 23 to eq. (242) in Subsec. 5.1.
In view of eq. (336), we can write

\[ \bigwedge^2 (V_6) = \bigwedge^2 \left( \bigwedge^2_{\mathbb{R}_6} (\mathbb{C}_4) \right) = \bigwedge^2 (\mathbb{R}_6) , \]  

but clearly \( V_6 \neq \mathbb{R}_6 \) (The vectors of the spin bivector space \( \mathbb{R}_6 \) are antisymmetric matrices while one direction in the paravector space \( V_6 \) is given by the unit matrix \( \mathbb{1}_4 \)).

Eq. (336) tells us that the decomposable/simple biparavectors \( \hat{c}_M \) of the paravector space \( V_6 \) can also be understood as decomposable/simple bi(para)vectors of the spin bivector space \( \mathbb{R}_6 \). In other and more plain words, any \((4 \times 4) \) Dirac matrix and any of their (multiple) products can be given as the product of two antisymmetric \((4 \times 4) \) matrices that exhibit a certain generalized anticommutation relation. This insight into the structure of decomposable/simple biparavectors \( \hat{c}_M \) has in part been spelled out by Haantjes, [144], p. 51, stelling 5 [proposition 5], and a corresponding comment can also be found in ref. [94], §14, p. 269, above of eq. (14.8). While we have concentrated our study on the structure of \( SO(6; \mathbb{R}) \) transformations in paravector space \( \bigwedge^2 (V_6) \) Haantjes [144] and Buchdahl [94] reflect the decomposability property from out the construction of the paravectors/biparavectors in terms of the product of spin bivectors [cf. eq. (242)], i.e., their representation in terms of products of antisymmetric matrices. For the special choice discussed in Appendix D this product representation of the biparavectors corresponds to a product representation in terms of two commuting sets of quaternions \( \mathbb{H} \). Another product representation (in terms of matrices called \( S_\alpha \) and \( D_\alpha \)) has been discussed by Eddington [68], [72], Chap. III, Secs. 3.1, 3.2, pp. 34-36. Nikol’ski˘ı [273] has shown that this product representation is related to transformations of tetrahedral coordinates (For a related discussion concerning the Dirac equation also see [274]).

6.4 Miscellaneous

Finally, we would like to add some comments related to the choice of the metric (in particular its signature) in spin bivector space. According to the eqs. (246), (247) (Subsec. 5.1), unitary transformations \([\in SU(4)]\) in the spin space \( \mathbb{C}_4 \) correspond to orthogonal transformations \([\in SO(6; \mathbb{R})] \) in the spin bivector space \( \mathbb{R}_6 \). On the other hand, it has been known for a long time that quadriquaternions (or, equivalently, the biparavectors \( \hat{c}_M \)) are related to conformal transformations [83]. To arrive at these one must change from the spin bivector space \( \mathbb{R}_6 \) to the related spin bivector space \( \mathbb{R}_{5,1} \). This change in metric can be achieved, for example, by means of the transition \( \Gamma_{-1} \rightarrow i \Gamma_{-1} \) [entailing, e.g., \( \left( \Gamma_{-1} \right)^2 = -\mathbb{1}_4 \)]. The \( SO(5,1; \mathbb{R}) \) transformations in

\[ ^{37} \text{We tend to use the term 'biparavectors' of spin bivector space here because their definition involves a Hermitian conjugation in one of the two factors in analogy to the biparavectors defined in eq. (23).} \]
\( \mathbb{R}_{5,1} \) then stand in correspondence to \( SU^*(4) \) transformations in spin space\(^{38}\). It is then possible to represent the pseudo-orthogonal \( SO(5,1;\mathbb{R}) \) transformations in \( \mathbb{R}_{5,1} \) by means of conformal transformations in a subspace \( \mathbb{R}_4 \subset \mathbb{R}_{5,1} \). However, the corresponding transformations in spin space then are no longer unitary and do not correspond to canonical transformations in which we are interested. From a particle physics perspective, transformations not in Euclidean space \( \mathbb{R}_4 \) but in Minkowski space \( \mathbb{R}_{3,1} \) are interesting. The Minkowski space is then to be embedded in a six-dimensional pseudo-Euclidean space \( \mathbb{R}_{4,2} \) and pseudo-orthogonal \( SO(4,2;\mathbb{R}) \) transformations in \( \mathbb{R}_{4,2} \) then correspond to pseudo-unitary \( SU(2,2) \) transformations in spin space. We will not further dwell on this subject here but confine ourselves to some comments on the related literature: For a good, general mathematical account of conformal transformations see [277], Chap. 11, p. 480. Ref. [278] covers the subject from a physical perspective (including many references, also see [279]). For articles dealing with various aspects of the above relationship that are interesting within the present context see [95, 107, 142, 143, 280, 281]. This subject has also been discussed, in part recently, from a twistor perspective (see, e.g., [104, 282, 283]). Leaving aside special conformal transformations and dilatations in the pseudo-Euclidean space \( \mathbb{R}_{3,1} \), the relevance of the restricted class of \( SO(4,1;\mathbb{R}) \) transformations in \( \mathbb{R}_{4,2} \) for massive (4-component) fermions (Dirac particles) has been discussed in the physics literature (Dirac equation in \( 4 + 1 \) dimensions) [270, 284–287]. Also, (generalized) Foldy-Wouthuysen transformations in spin space fit into the latter framework [288, 289]. A discussion of the algebra of Dirac matrices related to the group \( SO(3,2;\mathbb{R}) \) can be found in [290–292]. Finally, for the sake of completeness, we also want to mention the papers [293] and [294–298] which have attempted to address certain problems related to the general spin space transformations we have studied.

7 Conclusions

Having reached the end of the present study of nonlinear Bogolyubov-Valatin transformations for two fermion modes, it seems fair to say that even this apparently simple and basic case has revealed its complexity and broad significance. Among the more general aspects, we would like to point out that unlike linear Bogolyubov-Valatin transformations which can be performed for each mode of quasifermions independently and separately, nonlinear Bogolyubov-Valatin transformations exhibit the truly nonlinear nature of the canonical fermion anticommutation relations because any nonlinear Bogolyubov-Valatin transformation – even within one fermion mode – immediately impacts the other mode(s) and, in general, the (linear) picture of independent quasifermions no longer applies.

\(^{38}\)For the notation see, e.g., [275], Vol. 2, p. 392, Table 10.1. \( SU^*(4) \) is the noncompact version of \( SU(4) \) related to it by the “unitary trick” of Weyl [276].
From a mathematical point of view, nonlinear Bogolyubov-Valatin transformations are accompanied by transformations of multivectors of different grades and, therefore, can be viewed as concrete realizations of supersymmetric transformations. The preservation of the canonical anticommutation relations under nonlinear Bogolyubov-Valatin transformation is closely connected to the decomposability of the biparavectors generating the Clifford algebra $C(0, 4)$. This is the Clifford algebra that is associated with the creation and annihilation operators of the two fermion modes. From a somewhat different perspective, the preservation of the canonical anticommutation relations under nonlinear Bogolyubov-Valatin transformation can be described in other words as being closely connected to the fact that Dirac matrices can always be written as the product of two antisymmetric matrices. This is a fact that was first recognized in the 1930’s already by Dutch mathematicians (Schouten, Struik, Haantjes).

Looking to future generalizations of the present work to more than two fermion modes it should be said that one should expect to meet mathematical structures which have not let their presence in the study of one and two fermion modes be known. The present investigation of nonlinear Bogolyubov-Valatin transformations for $n = 2$ fermion modes exhibits a number of features that are related to fact that the group $SU(2^n = 4)$ is the double cover of the group $SO(6; \mathbb{R})$ [For one fermion mode $SU(2)$ is related to $SO(3; \mathbb{R})$.]. For $n > 2$ fermion modes no such relation of the group $SU(2^n)$ to any of the orthogonal groups exists. This fact necessarily will raise its head in any future study of nonlinear Bogolyubov-Valatin transformation for more than two fermion modes. This will lead to a further increase in the complexity of the problem to be studied but certainly also to further interesting mathematical and physical insight into the nature of fermionic systems.

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Appendix A

We define the coefficients [cf. Sec. 2, eqs. (39), (40)]

\[
k^{(0)0}_k = \lambda^{(0)0}_k, \quad (A.1)
\]

\[
k^{(1)0}_k = \lambda^{(1)0}_k + \frac{1}{2} \lambda^{(1,2|2)}_k - \frac{1}{2} \lambda^{(2|1,2)}_k, \quad (A.2)
\]

\[
k^{(0)1}_k = -i \left( \lambda^{(0)1}_k - \frac{1}{2} \lambda^{(0,2|2)}_k + \frac{1}{2} \lambda^{(2|1,2)}_k \right), \quad (A.3)
\]

\[
k^{(2)0}_k = \lambda^{(2)0}_k + \lambda^{(0)2}_k - \frac{1}{2} \lambda^{(1,2|1)}_k + \frac{1}{2} \lambda^{(1|1,2)}_k, \quad (A.4)
\]

\[
k^{(0)2}_k = -i \left( \lambda^{(0)2}_k - \frac{1}{2} \lambda^{(0,1|2)}_k - \frac{1}{2} \lambda^{(2|1,2)}_k \right), \quad (A.5)
\]

\[
k^{(1)1}_k = \lambda^{(1)1}_k, \quad (A.6)
\]

\[
k^{(1,2)0}_k = -\frac{i}{2} \left( \lambda^{(1,2|0)}_k + \lambda^{(0|1,2)}_k + \lambda^{(1|2)}_k - \lambda^{(2|1)}_k \right), \quad (A.7)
\]

\[
k^{(1)2}_k = \frac{1}{2} \left( -\lambda^{(1,2|0)}_k + \lambda^{(0|1,2)}_k + \lambda^{(1|2)}_k + \lambda^{(2|1)}_k \right), \quad (A.8)
\]

\[
k^{(2)1}_k = \frac{1}{2} \left( -\lambda^{(1,2|0)}_k + \lambda^{(0|1,2)}_k - \lambda^{(1|2)}_k - \lambda^{(2|1)}_k \right), \quad (A.9)
\]

\[
k^{(0|1,2)}_k = \frac{i}{2} \left( \lambda^{(1,2|0)}_k + \lambda^{(0|1,2)}_k - \lambda^{(1|2)}_k + \lambda^{(2|1)}_k \right), \quad (A.10)
\]

\[
k^{(2|2)}_k = \lambda^{(2|2)}_k, \quad (A.11)
\]

\[
k^{(1,2|1)}_k = \frac{1}{2} \left( -\lambda^{(1|1,2)}_k + \lambda^{(1,2|1)}_k \right), \quad (A.12)
\]

\[
k^{(1|1,2)}_k = -\frac{i}{2} \left( \lambda^{(1|1,2)}_k + \lambda^{(1,2|1)}_k \right), \quad (A.13)
\]

\[
k^{(1,2|2)}_k = \frac{1}{2} \left( \lambda^{(2|1,2)}_k - \lambda^{(1,2|2)}_k \right), \quad (A.14)
\]

\[
k^{(2|1,2)}_k = \frac{i}{2} \left( \lambda^{(2|1,2)}_k + \lambda^{(1,2|2)}_k \right), \quad (A.15)
\]

\[
k^{(1|2|1,2)}_k = \frac{1}{2} \lambda^{(1|2|1,2)}_k. \quad (A.16)
\]
This in turn implies:

\[ \lambda_{k}^{(1|0)} = \frac{1}{2} \left( \kappa_{k}^{(1|0)} + i\kappa_{k}^{(0|1)} + \kappa_{k}^{(1,2|2)} + i\kappa_{k}^{(2|1,2)} \right), \]  
(A.17)

\[ \lambda_{k}^{(0|1)} = \frac{1}{2} \left( \kappa_{k}^{(1|0)} - i\kappa_{k}^{(0|1)} + \kappa_{k}^{(1,2|2)} - i\kappa_{k}^{(2|1,2)} \right), \]  
(A.18)

\[ \lambda_{k}^{(2|0)} = \frac{1}{2} \left( \kappa_{k}^{(2|0)} + i\kappa_{k}^{(0|2)} + \kappa_{k}^{(1,2|1)} + i\kappa_{k}^{(1|1,2)} \right), \]  
(A.19)

\[ \lambda_{k}^{(0|2)} = \frac{1}{2} \left( \kappa_{k}^{(2|0)} - i\kappa_{k}^{(0|2)} + \kappa_{k}^{(1,2|1)} - i\kappa_{k}^{(1|1,2)} \right), \]  
(A.20)

\[ \lambda_{k}^{(1,2|0)} = \frac{1}{2} \left( -\kappa_{k}^{(2|1)} - \kappa_{k}^{(1|2)} - i\kappa_{k}^{(1,2|0)} + i\kappa_{k}^{(0|1,2)} \right), \]  
(A.21)

\[ \lambda_{k}^{(1|2)} = \frac{1}{2} \left( -\kappa_{k}^{(2|1)} + \kappa_{k}^{(1|2)} - i\kappa_{k}^{(1,2|0)} - i\kappa_{k}^{(0|1,2)} \right), \]  
(A.22)

\[ \lambda_{k}^{(2|1)} = \frac{1}{2} \left( -\kappa_{k}^{(2|1)} + \kappa_{k}^{(1|2)} + i\kappa_{k}^{(1,2|0)} + i\kappa_{k}^{(0|1,2)} \right), \]  
(A.23)

\[ \lambda_{k}^{(0|1,2)} = \frac{1}{2} \left( \kappa_{k}^{(2|1)} + \kappa_{k}^{(1|2)} - i\kappa_{k}^{(1,2|0)} - i\kappa_{k}^{(0|1,2)} \right), \]  
(A.24)

\[ \lambda_{k}^{(1,2|1)} = \kappa_{k}^{(1,2|1)} + i\kappa_{k}^{(1|1,2)}, \]  
(A.25)

\[ \lambda_{k}^{(1,1,2)} = -\kappa_{k}^{(1,2|1)} + i\kappa_{k}^{(1|1,2)}, \]  
(A.26)

\[ \lambda_{k}^{(1,2|2)} = -\kappa_{k}^{(1,2|2)} - i\kappa_{k}^{(2|1,2)}, \]  
(A.27)

\[ \lambda_{k}^{(2,1,2)} = \kappa_{k}^{(1,2|2)} - i\kappa_{k}^{(2|1,2)}, \]  
(A.28)

\[ \lambda_{k}^{(1,2,1,2)} = 2 \kappa_{k}^{(1,2|1,2)}. \]  
(A.29)

Furthermore, the following relations apply [cf. eqs. (48), (49) and (55); \( k = 1, 2 \)]

\[ \chi_{2k-1}^{[1]} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(1|0)}, \quad \chi_{2k}^{[1]} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(1|0)}, \]  

\[ \chi_{2k-1}^{[1]} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(0|1)}, \quad \chi_{2k}^{[1]} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(0|1)}, \]  

\[ \chi_{2k-1}^{[1]} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(0|0)}, \quad \chi_{2k}^{[1]} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(0|0)}, \]  

\[ \chi_{2k-1}^{[1]} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(0|2)}, \quad \chi_{2k}^{[1]} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(0|2)}, \]  

\[ \chi_{2k-1}^{[2]} \begin{bmatrix} 1,2 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(1|1)}, \quad \chi_{2k}^{[2]} \begin{bmatrix} 1,2 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(1|1)}, \]  

\[ \chi_{2k-1}^{[2]} \begin{bmatrix} 1,3 \\ 1 \end{bmatrix} = \text{Re} \kappa_{k}^{(1,2|0)}, \quad \chi_{2k}^{[2]} \begin{bmatrix} 1,3 \\ 1 \end{bmatrix} = \text{Im} \kappa_{k}^{(1,2|0)}, \]  

[55]
\begin{align}
&\chi_{2k-1}^{[2]}(1,4) = \text{Re} \, \kappa_k^{(1|2)}), \quad \chi_{2k}^{[2]}(1,4) = \text{Im} \, \kappa_k^{(1|2)}, \\
&\chi_{2k-1}^{[2]}(2,3) = \text{Re} \, \kappa_k^{(2|1)}, \quad \chi_{2k}^{[2]}(2,3) = \text{Im} \, \kappa_k^{(2|1)}, \\
&\chi_{2k-1}^{[2]}(2,4) = \text{Re} \, \kappa_k^{(0|1,2)}, \quad \chi_{2k}^{[2]}(2,4) = \text{Im} \, \kappa_k^{(0|1,2)}, \\
&\chi_{2k-1}^{[2]}(3,4) = \text{Re} \, \kappa_k^{(2|2)}, \quad \chi_{2k}^{[2]}(3,4) = \text{Im} \, \kappa_k^{(2|2)}, \\
&\chi_{2k-1}^{[3]}(1,2,3) = \text{Re} \, \kappa_k^{(1,2|1)}, \quad \chi_{2k}^{[3]}(1,2,3) = \text{Im} \, \kappa_k^{(1,2|1)}, \\
&\chi_{2k-1}^{[3]}(1,2,4) = \text{Re} \, \kappa_k^{(1|1,2)}, \quad \chi_{2k}^{[3]}(1,2,4) = \text{Im} \, \kappa_k^{(1|1,2)}, \\
&\chi_{2k-1}^{[3]}(1,3,4) = \text{Re} \, \kappa_k^{(1,2|2)}, \quad \chi_{2k}^{[3]}(1,3,4) = \text{Im} \, \kappa_k^{(1,2|2)}, \\
&\chi_{2k-1}^{[3]}(2,3,4) = \text{Re} \, \kappa_k^{(2|1,2)}, \quad \chi_{2k}^{[3]}(2,3,4) = \text{Im} \, \kappa_k^{(2|1,2)}, \\
&\chi_{2k-1}^{[4]}(1,2,3,4) = \text{Re} \, \kappa_k^{(1,2|1,2)}, \quad \chi_{2k}^{[4]}(1,2,3,4) = \text{Im} \, \kappa_k^{(1,2|1,2)}. \quad (A.30)
\end{align}

Appendix B

To work out the anticommutator (and commutator) of the operators \( \hat{d}_k \) (see Subsec. 3.2.1) it turns out to be useful to calculate the following products first (On the r.h.s. terms with more than four factors in the wedge product have been omitted as they vanish in the case under consideration.).

\begin{align}
\hat{c}_q \hat{d}_l &= - \chi_i^{[1]}(q) \mathbb{1}_4 - \chi_i^{[2]}(q,s) \hat{c}^s + \chi_i^{[1]}(r) \hat{c}_q \wedge \hat{c}_r - \frac{1}{2!} \chi_i^{[3]}(q,s,t) i \hat{c}^s \wedge \hat{c}^t \\
&\quad + \frac{1}{2!} \chi_i^{[2]}(r,s) \hat{c}_q \wedge \hat{c}_r \wedge \hat{c}_s - \frac{1}{3!} \chi_i^{[4]}(q,s,t,u) i \hat{c}^s \wedge \hat{c}^t \wedge \hat{c}^u \\
&\quad + \frac{1}{3!} \chi_i^{[3]}(r,s,t) i \hat{c}_q \wedge \hat{c}_r \wedge \hat{c}_s \wedge \hat{c}_t \quad (B.1)
\end{align}

If \( p \neq q \), the r.h.s. is antisymmetric in \( p \) and \( q \):

\begin{align}
\hat{c}_p \hat{c}_q \hat{d}_l &= - \chi_i^{[2]}(p,q) \mathbb{1}_4 - \chi_i^{[1]}(q) \hat{c}_p + \chi_i^{[1]}(p) \hat{c}_q - \chi_i^{[3]}(p,q,t) i \hat{c}^t \\
&\quad - \chi_i^{[2]}(q,s) \hat{c}_p \wedge \hat{c}^s + \chi_i^{[2]}(p,s) \hat{c}_q \wedge \hat{c}^s \\
&\quad - \frac{1}{2!} \chi_i^{[4]}(p,q,t,u) i \hat{c}^t \wedge \hat{c}^u + \chi_i^{[1]}(r) \hat{c}_p \wedge \hat{c}_q \wedge \hat{c}_r \\
&\quad - \frac{1}{2!} \chi_i^{[3]}(q,s,t) i \hat{c}_p \wedge \hat{c}^s \wedge \hat{c}_t + \frac{1}{2!} \chi_i^{[3]}(p,s,t) i \hat{c}_q \wedge \hat{c}^s \wedge \hat{c}_t \\
&\quad + \frac{1}{2!} \chi_i^{[2]}(r,s) \hat{c}_p \wedge \hat{c}_q \wedge \hat{c}_r \wedge \hat{c}_t. \quad (B.2)
\end{align}
\[ n \neq p \neq q, \text{ the r.h.s. is completely antisymmetric in } n, p \text{ and } q: \]
\[
\hat{c}_n \hat{c}_p \hat{c}_q \hat{d}_l = \chi_{l}^{[3]}(n,p,q) \hat{c}_n \hat{d}_l - \chi_{l}^{[2]}(p,q) \hat{c}_n \hat{c}_p - \chi_{l}^{[1]}(n,q) \hat{c}_n \hat{c}_q - \chi_{l}^{[1]}(p,q) \hat{c}_p \hat{c}_q \\
+ \chi_{l}^{[4]}(n,p,q,u) \hat{c}_n \hat{c}_p \hat{c}_q \hat{c}_u - \chi_{l}^{[1]}(q) \hat{c}_n \hat{c}_p + \chi_{l}^{[1]}(p) \hat{c}_n \hat{c}_q - \chi_{l}^{[3]}(p,q,t) \hat{c}_n \hat{c}_p \hat{c}_q - \chi_{l}^{[3]}(p,q,t) \hat{c}_n \hat{c}_p \hat{c}_q \\
+ \chi_{l}^{[2]}(n,q) \hat{c}_n \hat{c}_p \hat{c}_q \hat{s} + \chi_{l}^{[2]}(p,s) \hat{c}_n \hat{c}_p \hat{c}_q \hat{s} - \chi_{l}^{[2]}(n,s) \hat{c}_n \hat{c}_p \hat{c}_q \hat{s} - \chi_{l}^{[1]}(r) \hat{c}_n \hat{c}_p \hat{c}_q \hat{r} . \quad (B.3)
\]

\[ m \neq n \neq p \neq q, \text{ the r.h.s. is completely antisymmetric in } m, n, p \text{ and } q: \]
\[
\hat{c}_m \hat{c}_n \hat{c}_p \hat{c}_q \hat{d}_l = \chi_{l}^{[4]}(m,n,p,q) \hat{c}_m \hat{d}_l + \chi_{l}^{[3]}(n,p,q) \hat{c}_m \hat{c}_n - \chi_{l}^{[3]}(m,n,p,q) \hat{c}_m \hat{c}_n + \chi_{l}^{[3]}(m,n,q) \hat{c}_m \hat{c}_n \\
+ \chi_{l}^{[3]}(m,n,p,u) \hat{c}_m \hat{c}_n \hat{c}_p - \chi_{l}^{[2]}(p,q) \hat{c}_m \hat{c}_n \hat{c}_p - \chi_{l}^{[2]}(n,q) \hat{c}_m \hat{c}_n \hat{c}_p - \chi_{l}^{[1]}(q) \hat{c}_m \hat{c}_n \hat{c}_p - \chi_{l}^{[1]}(p) \hat{c}_m \hat{c}_n \hat{c}_p \\
+ \chi_{l}^{[2]}(m,q) \hat{c}_m \hat{c}_n \hat{c}_p \hat{s} + \chi_{l}^{[2]}(m,p) \hat{c}_m \hat{c}_n \hat{c}_p \hat{s} - \chi_{l}^{[2]}(m,n) \hat{c}_m \hat{c}_n \hat{c}_p \hat{s} + \chi_{l}^{[2]}(m,n) \hat{c}_m \hat{c}_n \hat{c}_p \hat{s} \\
+ \chi_{l}^{[1]}(r) \hat{c}_m \hat{c}_n \hat{c}_p \hat{r} + \chi_{l}^{[1]}(m) \hat{c}_m \hat{c}_n \hat{c}_p \hat{r} . \quad (B.4)
\]

In deriving the above relations we have used the Clifford algebra relations (\( \mathbf{x}, \mathbf{y} \) are vectors and \( A \) is an arbitrary multivector).
\[
\mathbf{x} A = \mathbf{x} | A + \mathbf{x} \wedge A \quad \text{(B.5)}
\]
and (\( \hat{\mathbf{y}}_k \) denotes a vector to be omitted.)
\[
\mathbf{x} | (\mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_p) = \sum_{k=1}^{p} (-1)^{k+1} (\mathbf{x} \cdot \mathbf{y}_k) (\mathbf{y}_1 \wedge \ldots \wedge \hat{\mathbf{y}}_k \wedge \ldots \wedge \mathbf{y}_p) \quad \text{(B.6)}
\]
(Cf., e.g., ref. [152], Chap. 1, p. 9, eq. (1.33). Note, that Hestenes and Sobczyk use a somewhat different notation for the operation of left contraction \( | \).).
We introduce \( \binom{n}{k} \times \binom{n}{k} \) matrices \( A^{(2k)} \) \((k = 1, \ldots, n)\) by writing (choose \( l_1 < l_2 < \ldots < l_k, m_1 < m_2 < \ldots < m_k \))

\[
A^{(2k)}_{LM} = A^{(2k)}_{l_1 \ldots l_k, m_1 \ldots m_k}
\]

(We identify the indices \( L, M \) with the ordered strings \( l_1 \ldots l_k, m_1 \ldots m_k \)) or, more generally (not requesting \( l_1 < l_2 < \ldots < l_k, m_1 < m_2 < \ldots < m_k \))

\[
A^{(2k)}_{LM} = \text{sgn} [\sigma_a(l_1, \ldots, l_k)] \text{sgn} [\sigma_b(m_1, \ldots, m_k)] A^{(2k)}_{l_1 \ldots l_k, m_1 \ldots m_k}
\]

(C.2)

The indices \( L, M \) label the equivalence classes of all permutations of the indices \( l_1, \ldots, l_k \) and \( m_1, \ldots, m_k \), respectively, and \( \sigma_a, \sigma_b \) are the permutations which bring the indices \( l_i, m_i \) \((i = 1, \ldots, k)\) into order with respect to the \(<\) relation (i.e., \( \sigma_a(l_1) < \sigma_a(l_2) < \ldots < \sigma_a(l_k) \), \( \sigma_b(m_1) < \sigma_b(m_2) < \ldots < \sigma_b(m_k) \)). The matrix elements of the matrix \( A^{(2k)} \) are arranged according to the lexicographical order of the row and column indices \( L, M \). We also define a set of (dual) \( \binom{n}{k} \times \binom{n}{k} \) matrices \( A^{(2k)*} \) \((k = 1, \ldots, n)\) by writing

\[
A^{(2k)*} = \mathcal{E}^{(k)} A^{(2k)T} \mathcal{E}^{(k)T}
\]

(C.3)

where the \( \binom{n}{k} \times \binom{n}{k} \) matrix \( \mathcal{E}^{(k)} \) is defined by

\[
\mathcal{E}^{(k)}_{LM} = \epsilon_{l_1 \ldots l_k, m_1 \ldots m_k}
\]

(C.4)

consequently,

\[
\mathcal{E}^{(k)T} = (-1)^{(n-k)k} \mathcal{E}^{(n-k)}
\]

(C.5)

(Quite generally, for any \( \binom{n}{k} \times \binom{n}{k} \) matrix \( B \) we define \( B^* \) by \( B^* = \mathcal{E}^{(k)} B^T \mathcal{E}^{(k)T} \)). It holds (\( \mathbb{1}_r \) is the \( r \times r \) unit matrix)

\[
\mathcal{E}^{(k)} \mathcal{E}^{(k)T} = \mathbb{1}_{\binom{n}{k}}
\]

(C.6)

\[
\mathcal{E}^{(k)T} \mathcal{E}^{(k)} = \mathbb{1}_{\binom{n}{k}}
\]

(C.7)

We can now give some formulas for compound matrices\(^{39}\). Let \( B, D \) be \( n \times n \) matrices. The compound matrix \( C_k(B) \), \( 0 \leq k \leq n \), is a \( \binom{n}{k} \times \binom{n}{k} \) matrix of all order \( k \) minors of the matrix \( B \). The indices of the compound matrix entries are given by ordered strings of length \( k \). These strings are composed from the row and column indices of the matrix elements of the matrix \( B \) the given minor of the

\(^{39}\)In the context of projective geometry, these matrices and their elements often are referred to as Plücker-Grassmann coordinates.
matrix $B$ is composed of. Typically, the entries of a compound matrix are ordered lexicographically with respect to the compound matrix indices (We also apply this convention.). The *supplementary* (or *adjugate*) compound matrix $C^{n-k}(B)$ (sometimes also referred to as the *matrix of the $k$th cofactors*) of the matrix $B$ is defined by the equation (cf. eq. (C.3))

$$C^{n-k}(B) = C_{n-k}(B)^*.$$  

(C.8)

The components of the supplementary compound matrix $C^{n-k}(B)$ can also be defined by means of the following formula (here, $l_1 < l_2 < \ldots < l_k, m_1 < m_2 < \ldots < m_k$; [300], Chap. IV, §89, p. 75, [301], Chap. 3, p. 18)\footnote{Note, that in the eqs. (A.2), p. 5443, (31)-(33), p. 5421 in [299] the indices $l$ and $m$ on the r.h.s. should be interchanged to correct the display of these equations.}

$$C^{n-k}(B)_{LM} = \frac{\partial}{\partial B_{m_1l_1}} \ldots \frac{\partial}{\partial B_{m_kl_k}} \text{det } B$$

(C.9)

This comparatively little known definition of (matrices of) cofactors (supplementary compound matrices) is essentially due to Jacobi\footnote{Note, that in the eqs. (A.2), p. 5443, (31)-(33), p. 5421 in [299] the indices $l$ and $m$ on the r.h.s. should be interchanged to correct the display of these equations.}, §10, p. 301, p. 273 of the ‘Gesammelte Werke’, p. 25 of the German transl. (also see the corresponding comment by Muir in [303], Part I, Chap. IX, pp. 253-272, in particular pp. 262/263).

For compound matrices holds ($I_r$ is the $r \times r$ unit matrix, $\alpha$ some constant)

$$C_k(\alpha I_n) = \alpha^k I_k.$$  

(C.10)

Important relations are given by the *Binet-Cauchy formula*

$$C_k(B)C_k(D) = C_k(BD)$$

(C.11)

from which immediately follows

$$C_k(B^{-1}) = C_k(B)^{-1},$$

(C.12)

the *Laplace expansion*

$$C_k(B)C^{n-k}(B) = C^{n-k}(B)C_k(B) =$$

$$C_k(B)C_{n-k}(B)^* = C_{n-k}(B)^*C_k(B) = \text{det } B I_k,$$

(C.13)

*Jacobi’s theorem* (a consequence of the eqs. (C.13) and (C.12))

$$C_k(B^{-1}) = \frac{1}{\text{det } B} C^{n-k}(B) = \frac{1}{\text{det } B} C_{n-k}(B)^*,$$

(C.14)

and the *Sylvester-Franke theorem*

$$\text{det } C_k(B) = (\text{det } B)^{n-k}.$$  

(C.15)
Compound matrices are treated in a number of references. A comprehensive discussion of compound matrices can be found in [304], Chap. V, pp. 63-87, [305], Chap. V, pp. 90-110, and, in a modern treatment, in [306], Chap. 6, pp. 142-155 of the English translation. More algebraically oriented modern treatments can be found in [147], Part I, Chap. 2, Sect. 2.4, pp. 116-159, Part II, Chap. 4, pp. 1-164 (very thorough), [307], Chap. 7, Sect. 7.2, pp. 411-420, and [308], Vol. 3, Chap. 2, Sect. 2.4, pp. 58-68. Concise reviews of the properties of compound matrices are given in [309, 310]. Also note [311].

Appendix D

In this Appendix, we first give a concrete example for the complex antisymmetric matrices introduced in Subsec. 5.1:

\[
\Gamma_{(-1)}^{\dagger} = \Gamma_{(-1)}^{-\dagger} = -\Gamma_{(-1)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (D.1)
\]

\[
\Gamma_0^{\dagger} = \Gamma_0^{-\dagger} = \Gamma_0 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (D.2)
\]

\[
\Gamma_1^{\dagger} = \Gamma_1^{-\dagger} = \Gamma_1 = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (D.3)
\]

\[
\Gamma_2^{\dagger} = \Gamma_2^{-\dagger} = \Gamma_2 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (D.4)
\]

\[
\Gamma_3^{\dagger} = \Gamma_3^{-\dagger} = -\Gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (D.5)
\]

\[
\Gamma_4^{\dagger} = \Gamma_4^{-\dagger} = -\Gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (D.6)
\]
They represent spin space bivectors ($\in \bigwedge^2(\mathbb{C}_4)$). For $k = 0, 1, 2$, the relation $\hat{\Gamma}_k = \Gamma_k$ applies, consequently, these three matrices span the eigenspace of the Hodge operator to the eigenvalue $-1$ [cf. eq. (236)]. For $k = -1, 3, 4$, the relation $\hat{\Gamma}_k = -\Gamma_k$ applies and, consequently, these three matrices span the eigenspace of the Hodge operator to the eigenvalue $1$ (For a related discussion see [312]). The antisymmetric matrices $\hat{\Gamma}_k$ with indices $k = 0, 1, 2$ and $k = -1, 3, 4$ are related to a matrix representation of two commuting copies, $I'_1$, $J'_1$, $K'_1$ ($I'_0J'_1 = K'_1$, etc.) and $I'_2$, $J'_2$, $K'_2$, ($I'_2J'_2 = K'_2$, etc.), of the system of quaternions ([262], p. 329, eq. (5)). We attach here a prime to the quaternionic units in order to distinguish them from the different choice made in the eqs. (29)-(34) in Subsec. 2.2. With some hindsight we set

\begin{align}
I'_1 &= i \hat{\Gamma}_2 = i \hat{\Gamma}_2 , \\
J'_1 &= -i \hat{\Gamma}_1 = -i \hat{\Gamma}_1 , \\
K'_1 &= i \hat{\Gamma}_0 = i \hat{\Gamma}_0 , \\
I'_2 &= \hat{\Gamma}_3 = -\hat{\Gamma}_3 , \\
J'_2 &= \hat{\Gamma}_4 = -\hat{\Gamma}_4 , \\
K'_2 &= \hat{\Gamma}_{(-1)} = -\hat{\Gamma}_{(-1)} .
\end{align}

Note, that these relations are valid only if the matrices $\hat{\Gamma}_k$ with the indices $k = 0, 1, 2$ and $k = -1, 3, 4$, respectively, span the two eigenspaces of the Hodge operator to the eigenvalues $-1$ and $1$. This will not be the case in general. Taking into account the above identifications we find the following relations of the quaternionic units to the elements of the biparavector space$^{41}$.

\begin{align}
I'_1 &= J'_1K'_1 = -\hat{\Gamma}_0\hat{\Gamma}_1 = -\hat{c}_{01} = \hat{c}_1 , \\
J'_1 &= K'_1I'_1 = -\hat{\Gamma}_0\hat{\Gamma}_2 = -\hat{c}_{02} = \hat{c}_2 , \\
K'_1 &= I'_1J'_1 = -\hat{\Gamma}_1\hat{\Gamma}_2 = -\hat{c}_{12} = \hat{c}_1\hat{c}_2 , \\
I'_2 &= J'_2K'_2 = \hat{\Gamma}_{(-1)}\hat{\Gamma}_4 = \hat{c}_{(-1)4} = i\hat{c}_1\hat{c}_2\hat{c}_3 , \\
J'_2 &= K'_2I'_2 = -\hat{\Gamma}_{(-1)}\hat{\Gamma}_3 = -\hat{c}_{(-1)3} = i\hat{c}_1\hat{c}_2\hat{c}_4 .
\end{align}

$^{41}$A related approach has been used in [180], Sec. 4.1, pp. 99-102, however, without relating the quaternionic units to the bivectors of spin space. The approach in [180] has rather some similarities with the one used by Milner [81]. Incidentally, a paper by Rau [176], p. 4, eq. (14), contains some related considerations.
\[ K_2' = \Gamma_2 J_2' = - \hat{\Gamma}_0 \hat{\Gamma}_2 \quad = - \hat{c}_3 \hat{c}_4 = - \hat{c}_3 \hat{c}_4 \tag{D.18} \]

For the generators of the Clifford algebra \( C(0, 4) \) we obtain\(^{42}\)

\[ \hat{c}_1 = \hat{c}_1 = I_1, \tag{D.19} \]
\[ \hat{c}_2 = \hat{c}_2 = J_1, \tag{D.20} \]
\[ \hat{c}_3 = \hat{c}_3 = i K_1' I_2', \tag{D.21} \]
\[ \hat{c}_4 = \hat{c}_4 = i K_1' J_2', \tag{D.22} \]

and, consequently,

\[ \hat{c}_{(-1)} = i K_1' K_2'. \tag{D.23} \]

The above identifications are up to minor detail the same as given by Schouten \(^{75}\) p. 106, eqs. (5), (6), and Lemaître \(^{79}\) p. 170 (For related discussions also see \(^{72}\) Sec. 2.3, p. 23 and Sec. 3.8, p. 47, \(^{73}\) Chap. VI, § 53, p. 108, \(^{81}\)).

### Appendix E

Here we calculate traces of products of the operators \( \hat{c}_k \) [For related results see \(^{100}\) p. 136, eqs. (2.12)-(2.21)]. These expressions are necessary for the discussion in Subsec. 5.1 [see eqs. (248)-(250)]. We find

\[ - \quad \text{tr} \quad \hat{c}_l = \text{tr} \left( \hat{\Gamma}_0 \hat{\Gamma}_l \right) = 4 \delta_{0l} \tag{E.1} \]

which is a special case of

\[ \text{tr} \left( \hat{c}_k \hat{c}_l \right) = \text{tr} \left( \hat{\Gamma}_k \hat{\Gamma}_l \right) = 4 \delta_{kl}. \tag{E.2} \]

Furthermore, we have

\[ - \quad \text{tr} \left( \hat{c}_l \hat{c}_m \hat{c}_n \right) = \text{tr} \left( \hat{\Gamma}_0 \hat{\Gamma}_l \hat{\Gamma}_m \hat{\Gamma}_n \right) = 4 (\delta_{0l} \delta_{mn} - \delta_{lm} \delta_{0n} + \delta_{0m} \delta_{ln}) \tag{E.3} \]

which is a special case of

\[ \text{tr} \left( \hat{c}_k \hat{c}_l \hat{c}_m \hat{c}_n \right) = \text{tr} \left( \hat{\Gamma}_k \hat{\Gamma}_l \hat{\Gamma}_m \hat{\Gamma}_n \right) = 4 (\delta_{kl} \delta_{mn} - \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}). \tag{E.4} \]

\(^{42}\)This choice corresponds to the Cartan extension of the Clifford algebra \( C(0, 2) \) generated by \( \hat{c}_1, \hat{c}_2 \), cf. \(^{313}\), Sec. 6.3, p. 80, eq. (6.26).
and
\[
- \text{tr} \left( \hat{c}_l \hat{c}_m \hat{c}_n \hat{c}_p \hat{c}_q \right) = \text{tr} \left( \Gamma_0 \Gamma_l \Gamma_m \Gamma_n \Gamma_p \Gamma_q \right)
\]
\[= 4 \left[ i \epsilon_{0lmnpq} + \delta_{kl} (\delta_{mn} \delta_{pq} - \delta_{mp} \delta_{nq} + \delta_{mq} \delta_{np}) - \delta_{km} (\delta_{ln} \delta_{pq} - \delta_{lp} \delta_{nq} + \delta_{lq} \delta_{np}) + \delta_{kn} (\delta_{lm} \delta_{pq} - \delta_{lp} \delta_{mq} + \delta_{lq} \delta_{mp}) - \delta_{kp} (\delta_{lm} \delta_{nq} - \delta_{ln} \delta_{mq} + \delta_{lp} \delta_{mn}) + \delta_{kq} (\delta_{lm} \delta_{np} - \delta_{ln} \delta_{mp} + \delta_{lp} \delta_{mn}) \right] (E.5)
\]

which is a special case of
\[
\text{tr} \left( \hat{c}_k \hat{c}_l \hat{c}_m \hat{c}_n \hat{c}_p \hat{c}_q \right) = \text{tr} \left( \Gamma_k \Gamma_l \Gamma_m \Gamma_n \Gamma_p \Gamma_q \right)
\]
\[= 4 \left[ i \epsilon_{klmnpq} + \delta_{kl} \delta_{mn} \delta_{pq} - \delta_{kl} \delta_{mp} \delta_{nq} + \delta_{kl} \delta_{mq} \delta_{np} - \delta_{km} \delta_{ln} \delta_{pq} + \delta_{km} \delta_{lp} \delta_{nq} - \delta_{km} \delta_{lq} \delta_{np} + \delta_{kn} \delta_{lm} \delta_{pq} - \delta_{kn} \delta_{lp} \delta_{mq} + \delta_{kn} \delta_{lq} \delta_{mp} - \delta_{kp} \delta_{lm} \delta_{nq} + \delta_{kp} \delta_{ln} \delta_{mq} - \delta_{kp} \delta_{lq} \delta_{mn} + \delta_{kq} \delta_{lm} \delta_{np} - \delta_{kq} \delta_{ln} \delta_{mp} + \delta_{kq} \delta_{lp} \delta_{mn} \right] . (E.6)
\]

Further traces can easily be obtained from the above results by means of the relation
[This is a special version of eq. (238) in disguise.]
\[
\hat{c}_k \hat{c}_l = -2 \delta_{k0} \mathbb{1}_4 . \quad (E.7)
\]

Appendix F

In this Appendix we give the explicit relation between the representation of SU(4) matrices discussed by Östlund and Mele [43] (cf. Subsec. 6.2.1) and our parametrization (226). The coefficients of the two parametrizations transform into each other in 4 groups of 4 coefficients.
Group 1 reads:

\[ X_{(-1)(-1)} = \frac{1}{4} \left( T_0 - i T_{(-1)0} - i T_{12} - i T_{34} \right), \]  
(F.1)

\[ X_{00} = \frac{1}{4} \left( T_0 - i T_{(-1)0} + i T_{12} + i T_{34} \right), \]  
(F.2)

\[ X_{11} = \frac{1}{4} \left( T_0 + i T_{(-1)0} - i T_{12} + i T_{34} \right), \]  
(F.3)

\[ X_{22} = \frac{1}{4} \left( T_0 + i T_{(-1)0} + i T_{12} - i T_{34} \right), \]  
(F.4)

and the inverse relations are

\[ T_0 = \frac{1}{4} \left( X_{(-1)(-1)} + X_{00} + X_{11} + X_{22} \right) = \frac{1}{4} \text{tr} X, \]  
(F.5)

\[ T_{(-1)0} = \frac{i}{4} \left( X_{(-1)(-1)} + X_{00} - X_{11} - X_{22} \right), \]  
(F.6)

\[ T_{12} = \frac{i}{4} \left( X_{(-1)(-1)} - X_{00} + X_{11} - X_{22} \right), \]  
(F.7)

\[ T_{34} = \frac{i}{4} \left( X_{(-1)(-1)} - X_{00} - X_{11} + X_{22} \right). \]  
(F.8)

Group 2 reads:

\[ X_{(-1)0} = \frac{1}{4} \left( -T_{13} - i T_{14} - i T_{23} + T_{24} \right), \]  
(F.9)

\[ X_{0(-1)} = \frac{1}{4} \left( T_{13} - i T_{14} - i T_{23} - T_{24} \right), \]  
(F.10)

\[ X_{12} = \frac{1}{4} \left( T_{13} - i T_{14} + i T_{23} + T_{24} \right), \]  
(F.11)

\[ X_{21} = \frac{1}{4} \left( -T_{13} - i T_{14} + i T_{23} - T_{24} \right), \]  
(F.12)

and the inverse relations are

\[ T_{13} = \frac{1}{4} \left( -X_{(-1)0} + X_{0(-1)} + X_{12} - X_{21} \right), \]  
(F.13)

\[ T_{14} = \frac{i}{4} \left( X_{(-1)0} + X_{0(-1)} + X_{12} + X_{21} \right), \]  
(F.14)

\[ T_{23} = \frac{i}{4} \left( X_{(-1)0} + X_{0(-1)} - X_{12} - X_{21} \right), \]  
(F.15)

\[ T_{24} = \frac{1}{4} \left( X_{(-1)0} - X_{0(-1)} + X_{12} - X_{21} \right). \]  
(F.16)
Group 3 reads:

\[ X_{(-1)1} = \frac{1}{4} \left( -T_{(-1)3} - i \ T_{(-1)4} - i \ T_{03} + T_{04} \right), \quad (F.17) \]

\[ X_{02} = \frac{1}{4} \left( T_{(-1)3} - i \ T_{(-1)4} - i \ T_{03} - T_{04} \right), \quad (F.18) \]

\[ X_{1(-1)} = \frac{1}{4} \left( -T_{(-1)3} + i \ T_{(-1)4} - i \ T_{03} - T_{04} \right), \quad (F.19) \]

\[ X_{20} = \frac{1}{4} \left( T_{(-1)3} + i \ T_{(-1)4} - i \ T_{03} + T_{04} \right), \quad (F.20) \]

and the inverse relations are

\[ T_{(-1)3} = \frac{1}{4} \left( -X_{(-1)1} - X_{02} + X_{1(-1)} + X_{20} \right), \quad (F.21) \]

\[ T_{(-1)4} = \frac{i}{4} \left( X_{(-1)1} - X_{02} + X_{1(-1)} - X_{20} \right), \quad (F.22) \]

\[ T_{03} = \frac{i}{4} \left( X_{(-1)1} + X_{02} + X_{1(-1)} + X_{20} \right), \quad (F.23) \]

\[ T_{04} = \frac{1}{4} \left( X_{(-1)1} - X_{02} - X_{1(-1)} + X_{20} \right). \quad (F.24) \]

Group 4 reads:

\[ X_{(-1)2} = \frac{1}{4} \left( T_{(-1)1} + i \ T_{(-1)2} + i \ T_{01} - T_{02} \right), \quad (F.25) \]

\[ X_{01} = \frac{1}{4} \left( -T_{(-1)1} + i \ T_{(-1)2} - i \ T_{01} - T_{02} \right), \quad (F.26) \]

\[ X_{10} = \frac{1}{4} \left( T_{(-1)1} + i \ T_{(-1)2} - i \ T_{01} + T_{02} \right), \quad (F.27) \]

\[ X_{2(-1)} = \frac{1}{4} \left( -T_{(-1)1} + i \ T_{(-1)2} + i \ T_{01} + T_{02} \right), \quad (F.28) \]

and the inverse relations are

\[ T_{(-1)1} = \frac{1}{4} \left( X_{(-1)2} - X_{01} + X_{10} - X_{2(-1)} \right), \quad (F.29) \]

\[ T_{(-1)2} = \frac{i}{4} \left( -X_{(-1)2} - X_{01} - X_{10} - X_{2(-1)} \right), \quad (F.30) \]

\[ T_{01} = \frac{i}{4} \left( -X_{(-1)2} + X_{01} + X_{10} - X_{2(-1)} \right), \quad (F.31) \]

\[ T_{02} = \frac{i}{4} \left( -X_{(-1)2} - X_{01} + X_{10} + X_{2(-1)} \right). \quad (F.32) \]
References

[For references in Cyrillic letters, we apply the (new) Mathematical Reviews transliteration (transcription) scheme (to be found at the end of index issues of Mathematical Reviews).]

[1] J.-P. Blaizot, G. Ripka: Quantum Theory of Finite Systems. The MIT Press, Cambridge, MA, London, 1986. 4, 58

[2] M. Wagner: Unitary Transformations in Solid State Physics. Modern Problems in Condensed Matter Sciences, Vol. 15. North-Holland, Amsterdam, 1986. 4, 38, 41

[3] P. Ring, P. Schuck: The Nuclear Many-Body Problem. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1. ed. 1980, 2. ed. 2000. 4

[4] R. F. Bishop: An overview of coupled cluster theory and its applications in physics. Theoretica Chimica Acta 80(1991)95-148 (DOI: 10.1007/BF01119617). 4

[5] Н. Н. Боголюбов [N. N. Bogolyubov]: О новом методе в теории сверхпроводимости. I. [O novom metode v teorii sverkhprovodimosti. I.] Журнал Экспериментальной и Теоретической Физики [Zhurnal Ekperimental'noi i Teoreticheskoi Fiziki] 34(1958)58-65 (preprint version: Joint Institute of Nuclear Research Preprint R-94, Dubna, 1957). English translation: N. N. Bogoliubov: A new method in the theory of superconductivity. I. Soviet Physics – JETP 7(1958)41-46. The Russian original is reprinted in 1. [6], Vol. 3, pp. 29-38, 2. [7], pp. 132-142, 3. [8], item 6, pp. 177-188, and in an extended version in 4. [9], Tr. Mat. Inst. im. V. A. Steklova, pp. 3-16 (The latter article is freely available online at the Mathnet site: http://mi.mathnet.ru/eng/tm1558.). The English translation is reprinted in [12], pp. 399-404, an English translation of the reprint # 4: [9], Tr. Mat. Inst. im. V. A. Steklova, pp. 3-16, is printed in [9], Proc. Steklov Math. Inst., pp. 1-14: N. N. Bogolyubov: On a new method in the theory of superconductivity. 4

[6] Н. Н. Боголюбов [N. N. Bogolyubov]: Избранные Труды в Трех Томах /Izbrannye Trudy v Trekh Tomakh/. Наукова Думка [Naukova Dumka], Kiev, Vol. 1: 1969, Vol. 2: 1970, Vol. 3: 1971. 82, 84

[7] Н. Н. Боголюбов [N. N. Bogolyubov]: Избранные Труды по Статистической Физике /Izbrannye Trudy po Statisticheskoi Fizike/. Издательство Московского Университета [Izdatel'ство Moskovskogo Universiteta], Moscow, 1979 (The book is freely available online at the EqWorld site: http://eqworld.ipmnet.ru/library/books/Bogolyubov1979ru.djvu.). 82, 84

[8] Н. Н. Боголюбов [N. N. Bogolyubov]: Собрание Научных Трудов в Двенаадцати Томах /Sobranie Nauchnykh Trudov v Dvenadtsati Tomakh = Collection of Scientific Works in Twelve Volumes. Классики Науки [Klassiki Nauki].
Statistical Mechanics: [in 4 vols.]. Vol. 8: Theory of Nonideal Bose Gas, Superfluidity and Superconductivity, 1946-1992. N. M. Plakida, A. D. Sukhanov (Eds.). [Nauka], Moscow, 2007.

82, 84
principle in the many-body problem. Soviet Physics – Doklady 3(1958)292-294. The Russian original is reprinted in: 1. [6], Vol. 3, pp. 48-50, 2. [7], pp. 190-192, 3. [8], item 19, pp. 411-418. 4

[15] Н. Н. Боголюбо́в [N. N. Bogolyubov], В. Г. Соловьёв [V. G. Solov’ev]: Об одном вариационном принципе в проблеме многих тел [Ob одном вариационном принципе в проблеме многих тел]. Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR] 124 (1959)1011-1014 (preprint version: Joint Institute of Nuclear Research Preprint R-262, Dubna, 1958). English translation: N. N. Bogolyubov, B. G. Solov’ev [correctly: V. G. Solov’ev]: On a variational principle in the many-body problem. Soviet Physics – Doklady 4(1959)143-146. The Russian original is reprinted in: 1. [6], Vol. 3, pp. 51-55, 2. [8], item 20, pp. 419-423. 4

[16] Н. Н. Боголюбо́в [N. N. Bogolyubov], В. В. Толмацкий [V. V. Tolmachev], Д. В. Ширков [D. V. Shirkov]: Новый Метод в Теории Сверхпроводимости [Novyi Metod v Teorii Sverkhprovodimosti]. Издательство Академии Наук СССР [Izdatel’stvo Akademii Nauk SSSR], Moscow, 1958 (preprint version: Joint Institute of Nuclear Research Preprint R-139, Dubna, 1958) (The book is freely available online at the EqWorld site: http://eqworld.ipmnet.ru/ru/library/books/BogolubovTolmachevShirkov1958ru.djvu). English translations: 1. [17], 2. (shortened version) [18]. A partial reprint [§5, §6 only] of the Russian book can be found in: [8], pp. 530-565. 4

[17] N. N. Bogoliubov, V. V. Tolmachev, D. V. Shirkov: A New Method in the Theory of Superconductivity. Consultants Bureau, Inc., New York; Chapman & Hall, Ltd., London, 1959. 4, 84

[18] N. N. Bogolyubov, V. V. Tolmachev, D. V. Shirkov: A new method in the theory of superconductivity. Fortschritte der Physik 6(1958)605-682 (DOI: 10.1002/prop.19580061102). Reprinted in: [13], pp. 278-355. 4, 84

[19] Н. Н. Боголюбов [N. N. Bogolyubov], Н. Н. Боголюбов (мл.) [N. N. Bogolyubov (ml.)]: Введение в Квантовую Статистическую Механику [Vvedenie v Kvantovuyu Statisticheskuyu Mekhaniku]. Наука [Nauka], Moscow, 1984. English translations: 1. [20], 2. [21], 2. ext. ed. only. Reprint of the Russian book: [22], pp. 11-374. 4

[20] N. N. Bogolubov, N. N. Bogolubov Jnr: An Introduction to Quantum Statistical Mechanics. Gordon and Breach Science Publishers, Switzerland, 1994. 4, 84

[21] N. N. Bogolubov, N. N. Bogolubov Jr: Introduction to Quantum Statistical Mechanics. World Scientific, Singapore, 1. ed.: 1982, 2. ext. ed.: 2009. 4, 84
[22] N. N. Bogolyubov [N. N. Bogolyubov]: Собрание Научных Трудов в Двенадцати Томах [Sobranie Nauchnykh Trudov v Dvenadtsati Tomakh] = Collection of Scientific Works in Twelve Volumes. Классики Науки [Klassiki Nauki]. Наука [Nauka], Moscow, 2005-2009. Vol. 7: Статистическая Механика: [в 4 т.]. T. 7: Н. Н. Боголюбов, Н. Н. Боголюбов (мл.): Введение в квантовую статистическую механику. Аспекты теории полярона [Statisticheskaya Mekhanika: [v 4 t.]. T. 7: N. N. Bogolyubov, N. N. Bogolyubov (ml.): Vvedenie v kvantovuyu statisticheskuyu mekhaniku. Aspekty teorii polyarona] = Statistical Mechanics: [in 4 vols.]. Vol. 7: N. N. Bogoliubov, N. N. Bogoliubov (Jr.): Introduction to Quantum Statistical Mechanics. Some Aspects of the Polaron Theory. N. N. Боголюбов (мл.), А. Д. Суханов (ред.) [N. N. Bogolyubov (ml.), A. D. Sukhanov (Eds.)]. Наука [Nauka], Moscow, 2007.

[23] V. G. Soloviev: Effect of quadruple correlations in light nuclei. Nuclear Physics 18(1960)161-172 (DOI: 10.1016/0029-5582(60)90396-5).

[24] В. Г. Соловьев [V. G. Solov’ev]: О четверных корреляциях в легких ядрах [O chetvernykh korrelyatsiyakh v legkih yadrah]. Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR] 131(1960)286-289. English translation: B. G. Solob’ev (correctly: V. G. Solov’ev): Parity correlations in light nuclei (correctly: Quadruple correlations in light nuclei). Soviet Physics – Doklady 5(1961)298-300.

[25] D. ter Haar, W. E. Perry: On determining excitation energies from the poles of Green functions. Physics Letters 1(1962)145-146 (DOI: 10.1016/0031-9163(62)90328-1).

[26] B. Brémond, J. G. Valatin: A method to describe pairing correlations of protons and neutrons. Nuclear Physics 41(1963)640-659 (DOI: 10.1016/0029-5582(63)90543-1).

[27] M. Ichimura: A generalization of Brémond-Valatin method for four-body correlation and its application to a charge independent pairing interaction. Progress in Theoretical Physics 31(1964)575-594 (DOI: 10.1143/PTP.31.575).

[28] J. R. Schrieffer: Theory of Superconductivity. Frontiers in Physics, Vol. 20. W. A. Benjamin, Inc., New York, 1964 and various later editions. Revised reprint: Advanced Book Classics. Perseus Books, Reading, MA, 1999.

[29] B. B. Varga, J. J. Sienicki, S. Ø. Aks: The Bogoliubov-Valatin quasiparticles in a finite superconductor. Journal of Low Temperature Physics 5(1971)495-498 (DOI: 10.1007/BF00629540).
[30] A. L. Kuzemsky, A. Pawlikowski: Note on the diagonalization of a quadratic linear form defined on the set of second quantization fermion operators. Reports on Mathematical Physics 3(1972)201-207 (DOI: 10.1016/0034-4877(72)90004-3).

[31] H. Fukutome, M. Yamamura, S. Nishiyama: A new fermion many-body theory based on the SO(2N+1) Lie algebra of the fermion operators. Progress of Theoretical Physics 57(1977)1554-1571 (DOI: 10.1143/PTP.57.1554).

[32] H. Fukutome: On the SO(2N+1) regular representation of operators and wave functions of fermion many-body systems. Progress of Theoretical Physics 58(1977)1692-1708 (DOI: 10.1143/PTP.58.1692).

[33] H. Fukutome: A new Tamm-Dancoff method based on the SO(2N+1) regular representation of fermion many-body systems. Progress of Theoretical Physics 60(1978)1624-1639 (DOI: 10.1143/PTP.60.1624).

[34] J. H. P. Colpa: Diagonalisation of the quadratic fermion Hamiltonian with a linear part. Journal of Physics A: Mathematical and General 12(1979)469-488 (DOI: 10.1088/0305-4470/12/4/008).

[35] H. Fukutome: The group theoretical structure of fermion many-body systems arising from the canonical anticommutation relations. I. – Lie algebras of fermion operators and exact generator coordinate representations of state vectors. Progress of Theoretical Physics 65(1981)809-827 (DOI: 10.1143/PTP.65.809).

[36] S. Nishiyama: Note on the new type of the SO(2N+1) time-dependent Hartree-Bogoliubov equation. Progress of Theoretical Physics 68(1982)680-683 (DOI: 10.1143/PTP.68.680).

[37] S. Nishiyama: An equation for the quasi-particle RPA based on the SO(2N+1) Lie algebra of the fermion operators. Progress of Theoretical Physics 69(1983)1811-1814 (DOI: 10.1143/PTP.69.1811).

[38] O. B. Zaslavskii, V. M. Tsukernik: Relaxation of one-dimensional spin systems in a transverse variable magnetic field. Soviet Journal of Low Temperature Physics 9(1983)33-37.

[39] H. Fukutome, S. Nishiyama: Time dependent SO(2N+1) theory for unified description of Bose and Fermi type collective excitations. Progress of Theoretical Physics 72(1984)239-251 (DOI: 10.1143/PTP.72.239).
[40] A. Bulgac: *New Type of Quasiparticles for a Many-Fermion System*. Unpublished manuscript, 9 pp.. Philadelphia, (around) 1987. 4, 11, 69

[41] J. M. F. Gunn, M. W. Long: Correlations near the atomic limit of the Anderson lattice. *Journal of Physics C: Solid State Physics* **21**(1988)4567-4589 (DOI: 10.1088/0022-3719/21/25/006). 4

[42] T. Suzuki: Theory of many-fermion system on unitary-transformation method. *Progress of Theoretical Physics* **79**(1988)330-342 (DOI:10.1143/PTP.79.330), Erratum *ibid.* **79**(1988)1249 (DOI: 10.1143/PTP.79.1249). 4

[43] S. Östlund, E. Mele: Local canonical transformations of fermions. *Physical Review B* **44**(1991)12413-12416 (DOI: 10.1103/PhysRevB.44.12413). 4, 64, 79

[44] J.-W. van Holten: *Grassmann Algebras and Spin in Quantum Dynamics*. Lecture notes, Dutch Summerschool of Mathematical Physics, University of Twente, unpublished, 66 pp.. Amsterdam, 1992. 4

[45] S. Nishiyama: Time dependent Hartree-Bogoliubov equation on the coset space SO(2N+2)/U(N+1) and quasi anti-commutation relation approximation. *International Journal of Modern Physics E* **7**(1998)677-707 (DOI: 10.1142/S0218301398000397). 4

[46] M. Abe, K. Kawamura: Nonlinear transformation group of CAR fermion algebra. *Letters in Mathematical Physics* **60**(2002)101-107 [arXiv:math-ph/0110004, Research Institute for Mathematical Sciences Kyoto preprint RIMS-1334] (DOI: 10.1023/A:1016114322568). 4, 18

[47] J. Katriel: A nonlinear Bogoliubov transformation. *Physics Letters A* **307**(2003)1-7 (DOI: 10.1016/S0375-9601(02)01671-7 ). 4

[48] P. Caban, K. Podlaski, J. Rembieliński, K. A. Smoliński, Z. Walczak: Entanglement and tensor product decomposition for two fermions. *Journal of Physics A: Mathematical and General* **38**(2005)L79-L86 [arXiv:quant-ph/0405108] (DOI: 10.1088/0305-4470/38/6/L02). 4, 38

[49] J.-W. van Holten, K. Scharnhorst: Nonlinear Bogolyubov-Valatin transformations and quaternions. *Journal of Physics A: Mathematical and General* **38**(2005)10245-10252 [arXiv:quant-ph/0411170, Preprint NIKHEF 04-024] (DOI: 10.1088/0305-4470/38/47/012). 4, 5, 11, 13, 18, 58, 60, 61

[50] P. Caban, K. Podlaski, J. Rembieliński, K. A. Smoliński, Z. Walczak: Tensor product decomposition, entanglement, and Bogoliubov transformations for two fermion system. *Open Systems and Information Dynamics* **12**(2005)179-188 (DOI: 10.1007/s11080-005-5729-8). 4, 38

87
[51] N. Ilieva, H. Narnhofer, W. Thirring: Supersymmetric models for fermions on a lattice. *Fortschritte der Physik* **54**(2006)124-138 [arXiv:quant-ph/0502100], CERN preprint CERN-PH-TH/2005-025, Erwin Schrödinger International Institute for Mathematical Physics, Vienna, preprint ESI 1607] (DOI: 10.1002/prop.200510261). 4

[52] P. Caban, K. Podlaski, J. Rembieliński, K. A. Smoliński, Z. Walczak: Bogoliubov transformations and entanglement of two fermions. *The Old and New Concepts of Physics* **4**(2007)389-398 (DOI: 10.2478/v10005-007-0017-8). 4, 38

[53] S. Östlund: Strong coupling Kondo lattice model as a Fermi gas. *Physical Review B* **76**(2007)153101, 4 pp. [arXiv:cond-mat/0703768] (DOI: 10.1103/PhysRevB.76.153101). 4

[54] S. Nishiyama, J. da Provindência, C. Provindência, F. Cordeiro: Extended supersymmetric $\sigma$-model based on the $SO(2N+1)$ Lie algebra of the fermion operators. *Nuclear Physics B* **802**(2008)121-145 [arXiv:0712.4208] (DOI: 10.1016/j.nuclphysb.2008.05.008). 4

[55] I. R. Porteous: *Clifford Algebras and the Classical Groups*. Cambridge Studies in Advanced Mathematics, Vol. 50. Cambridge University Press, Cambridge, 1995. 7, 8

[56] P. Lounesto: *Clifford Algebras and Spinors*. London Mathematical Society Lecture Notes Series, Vol. 239 (1. ed.), Vol. 286 (2. ext. ed.). Cambridge University Press, Cambridge, 1. ed. 1997, 2. ed. 2001. 7, 8, 36, 65

[57] I. R. Porteous: *Topological Geometry*. 1. ed.: The New University Mathematical Series. Van Nostrand Reinhold Company, London, 1969; 2. ed.: Cambridge University Press, Cambridge, 1981. 8

[58] W. E. Baylis: Multiparavector subspaces of $Cl_n$: Theorems and Applications. In: R. Abłamowicz, B. Fauser (Eds.): *Clifford Algebras and Their Applications in Mathematical Physics. Volume 1: Algebra and Physics*. Progress in Physics, Vol. 18. Birkhäuser, Boston, 2000, pp. 3-20. 8

[59] J. G. Maks: Modulo $(1,1)$ Periodicity of Clifford Algebras and Generalized (Anti)Möbius Transformations. Proefschrift (Ph. D. thesis), Technical University Delft, Delft, 1989 (The thesis is freely available online at the TU Delft site: http://repository.tudelft.nl/file/673039/374496). 8

[60] A. O. Barut: Introduction to de Sitter and conformal groups and their physical applications. In: A. O. Barut, W. E. Brittin (Eds.): *De Sitter and Conformal Groups and Their Applications*. Lectures in Theoretical Physics, Vol. XIII. Colorado Associated University Press, Boulder, CO, 1971, pp. 3-25. 8, 48

88
[61] E. A. Lord: Six-dimensional formulation of meson equations. *International Journal of Theoretical Physics* 5(1972)339-348 (DOI: 10.1007/BF00678224). 8, 11

[62] E. A. Lord: Generalised quaternion methods in conformal geometry. *International Journal of Theoretical Physics* 13(1975)89-102 (DOI: 10.1007/BF01806830). 8, 46

[63] E. A. Lord: Clifford algebras and representations of complex orthogonal groups. *Journal of Mathematical Analysis and Applications* 40(1972)509-539 (DOI: 10.1016/0022-247X(72)90066-2). 8

[64] S. R. Milner: On wave matrices, and some properties of the wave equation. *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences* 157(1936)1-27 (DOI: 10.1098/rspa.1936.0177, stable JSTOR URL: http://www.jstor.org/stable/96727). 8, 46

[65] B. Eckmann: Hurwitz-Radon matrices revisited: From effective solution of the Hurwitz matrix equations to Bott periodicity. In: G. Mislin (Ed.): *The Hilton Symposium 1993 – Topics in Topology and Group Theory*. CRM Proceedings & Lecture Notes, Vol. 6. American Mathematical Society, Providence, RI, 1994, pp. 23-35. Reprinted in: B. Eckmann: *Mathematical Survey Lectures 1943-2004*. Springer, Berlin, 2006, pp. 141-153 (DOI: 10.1007/978-3-540-33791-1_12). 8

[66] Y. C. Wong: *Isoclinic n-Planes in Euclidean 2n-Space, Clifford Parallels in Elliptic (2n − 1)-Space, and the Hurwitz Matrix Equations*. Memoirs of the American Mathematical Society, Vol. 41. American Mathematical Society, Providence, RI, 1961, 2. printing 1971. 8

[67] Y. C. Wong, K. P. Mok: Normally related $n$-planes and isoclinic $n$-planes in $R^{2n}$ and strongly linearly independent matrices of order $n$. *Linear Algebra and Its Applications* 139(1990)31-52 (DOI: 10.1016/0024-3795(90)90386-Q). 8

[68] A. S. Eddington: A symmetrical treatment of the wave equation. *Proceedings of the Royal Society of London, Series A, Containing Papers of a Mathematical and Physical Character* 121(1928)524-542 (DOI: 10.1098/rspa.1928.0217, stable JSTOR URL: http://www.jstor.org/stable/95117). 9, 67

[69] A. S. Eddington: The properties of wave tensors. *Proceedings of the Royal Society of London, Series A, Containing Papers of a Mathematical and Physical Character* 133(1931)311-324 (DOI: 10.1098/rspa.1931.0150, stable JSTOR URL: http://www.jstor.org/stable/95659). 9

[70] A. S. Eddington: On sets of anticommuting matrices. *Journal of the London Mathematical Society* 7(1932)58-68 (DOI: 10.1112/jlms/s1-7.1.58). 9
[71] A. S. Eddington: On sets of anticommuting matrices. Part II: The factorization of $E$-numbers. *Journal of the London Mathematical Society* 8(1933)142-152 (DOI: 10.1112/jlms/s1-8.2.142). 9

[72] A. Eddington: *Relativity Theory of Protons and Electrons.* Cambridge University Press, Cambridge, 1936 (The book is freely available online at the Internet Archive site: [http://www.archive.org/details/relativitytheory031459mbp](http://www.archive.org/details/relativitytheory031459mbp)). 9, 54, 67, 78

[73] A. S. Eddington: *Fundamental Theory.* Cambridge University Press, Cambridge, 1946, reprinted 1949, 1953. 9, 54, 78

[74] N. Salingaros: Some remarks on the algebra of Eddington’s $E$ numbers. *Foundations of Physics* 15(1985)683-691 (DOI: 10.1007/BF00738296). 9

[75] J. A. Schouten: Ueber die in der Wellengleichung verwendeten hyperkomplexen Zahlen [On the hypercomplex numbers used in the wave equation]. *Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam* 32(1929)105-108. [in German] Dutch abstract: J. A. Schouten: Over de hypercomplexe getallen, die in de golfvergelijking worden gebruikt. *Verslag van de Gewone Vergadering der Afdeeling Natuurkunde, Koninklijke Akademie van Wetenschappen te Amsterdam* 38(1929)22. 9, 78

[76] D. Iwanenko, K. Nikolsky: ¨Uber den Zusammenhang zwischen den Cauchy-Riemannschen und Diracschen Differentialgleichungen [On the relation between the Cauchy-Riemann and Dirac differential equations]. *Zeitschrift für Physik* 63(1930)129-137 (DOI: 10.1007/BF01336772). [in German] 9

[77] D. E. Littlewood: Note on the anticommuting matrices of Eddington. *Journal of the London Mathematical Society* 9(1934)41-50 (DOI: 10.1112/jlms/s1-9.1.41). 9

[78] B. Kwal: La théorie des équations de Maxwell et le calcul des opérateurs matriciels [The theory of the Maxwell equations and matrix operator calculus]. *Le Journal de Physique et le Radium, 7.* Series, 5(1934)445-448 (DOI: 10.1051/jphysrad:0193400508044500, the article is freely available online at the HAL site: [http://hal.archives-ouvertes.fr/jpa-00233258/fr/](http://hal.archives-ouvertes.fr/jpa-00233258/fr/). [in French] 9

[79] G. Lemaître: Sur l’interprétation d’Eddington de l’équation Dirac [On the interpretation of the Dirac equation by Eddington]. *Annales de la Société Scientifique de Bruxelles, Série I, Sciences Mathématiques et Physiques* 57(1937)165-140. [in French] 9, 78

[80] W. H. McCrea: Quaternion analogy of wave-tensor calculus. *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, 7.* Series, 30(1940)261-281 (DOI: 10.1080/14786444008520717). 9
[81] S. R. Milner: The relation of Eddington’s E-numbers to the tensor calculus. I. The matrix form of E-numbers. Proceedings of the Royal Society of London, Series A, Containing Papers of a Mathematical and Physical Character 214 (1952)292-311 (DOI: 10.1098/rspa.1952.0170, stable JSTOR URL: http://www.jstor.org/stable/99030). 9, 77, 78

[82] A. Deprit: A. S. Eddington’s E-numbers. Annales de la Société Scientifique de Bruxelles, Série I, Sciences Mathématiques, Astronomiques et Physiques 69 (1955)50-78. 9

[83] G. Combebiac: Sur un système numérique complexe représentant le groupe des transformations conformes de l’espace [On a complex number system representing the group of conformal transformations of a space]. Bulletin de la Société Mathématique de France 30 (1902)1-12 (The article is freely available online at the Numdam site: http://www.numdam.org/numdam-bin/fitem?id=BSMF_1902__30__1_0). [in French] 9, 67

[84] C. Sire, C. M. Varma, H. R. Krishnamurthy: Theory of non-Fermi-liquid transition point in the two-impurity Kondo model. Physical Review B 48 (1993)13833-13839 (DOI: 10.1103/PhysRevB.48.13833). 10, 38, 39

[85] O. W. Sørensen, G. W. Eich, M. H. Levitt, G. Bodenhausen, R. R. Ernst: Product operator formalism for the description of NMR pulse experiments. Progress in Nuclear Magnetic Resonance Spectroscopy 16 (1983)163-192 (DOI: 10.1016/0079-6565(84)80005-9). See also: R. R. Ernst, G. Bodenhausen, A. Wokaun: Principles of Nuclear Magnetic Resonance in One and Two Dimensions. The International Series of Monographs on Chemistry, Vol. 14. Clarendon Press, Oxford, 1987, Chap. 2, p. 9. 10

[86] F. J. M. van de Ven, C. W. Hilbers: A simple formalism for the description of multiple-pulse experiments. Application to a weakly coupled two-spin (I = 1/2) system. Journal of Magnetic Resonance 54 (1983)512-520 (DOI: 10.1016/0022-2364(83)90331-1). 10

[87] K. J. Packer, K. M. Wright: The use of single-spin operator basis sets in the N.M.R. spectroscopy of scalar-coupled spin systems. Molecular Physics 50 (1983)797-813 (DOI: 10.1080/0026897830102691). 10

[88] W. A. Conway: Quaternion treatment of the relativistic wave equation. Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences 162 (1937)145-154 (DOI: 10.1098/rspla.1937.0173, stable JSTOR URL: http://www.jstor.org/stable/96941). Reprinted in: J. McConnell (Ed.): Selected Papers of Arthur William Conway. Instituíúd Árd-Léinn Bhaile Átha Cliath/Dublin Institute for Advanced Studies, Dublin, 1953, item XVIII, pp. 179-188. 10
[89] J. L. Synge: Quaternions, Lorentz transformations, and the Conway-Dirac-Eddington matrices. Sraith A = Communications of the Dublin Institute for Advanced Studies, Series A, 21(1972)1-67. 10, 66

[90] A. P. Hristev: Deducerea covariantă a identităților matriciale și tensoriale Dirac [Covariant derivation of Dirac matrix and tensor identities]. In: Lucrările Confațuirii de Geometrie și Topologie. Iași, 2-5 iunie 1958. Editura Academiei Republicii Populare Romîne, București, 1962, pp. 203-204. [in Romanian] 10, 92

[91] A. P. Hristev: Deducerea covariantă a identităților matriciale Dirac în $E_6$ [Covariant derivation of Dirac matrix identities in $E_6$]. Comunicările Academiei Republicii Populare Romîne 11(1961)755-759. [in Romanian] For an abstract version of this article see [90]. 10, 11, 92

[92] A. P. Hristev: Les identités algébriques matricielles et tensorielles covariantes dans $C_6$ de type Dirac [The covariant algebraic Dirac matrix and tensor identities in $C_6$]. Bulletin Mathématique de la Société des Sciences Mathématiques et Physiques de la République Populaire Roumaine 5(53):1-2(1961)23-55. [in French] (Note, that the title page of the article displays the erroneous volume number 4(52).) The article is an extended version of [91, 93]. 10, 11, 52

[93] A. P. Hristev: Identitățile tensoriale Dirac în $C_6$ și deducerea lor covariantă [Tensorial Dirac identities in $C_6$ and their covariant derivation]. Comunicările Academiei Republicii Populare Romîne 11(1961)761-766. [in Romanian] For an abstract version of this article see [90]. 92

[94] H. A. Buchdahl: On the calculus of four-spinors. Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences 303(1968)355-379 (DOI: 10.1098/rspa.1968.0055, stable JSTOR URL: http://www.jstor.org/stable/2415737). 10, 11, 29, 30, 45, 46, 47, 52, 67

[95] A. P. Hristev: Conformal charge conjugation. Journal of Mathematical Physics 12(1971)118-124 (DOI: 10.1063/1.1665469). 10, 11, 68

[96] P. L. Nash: Identities satisfied by the generators of the Dirac algebra. Journal of Mathematical Physics 25(1984)204-209 (DOI: 10.1063/1.526141). 10, 11, 52

[97] J. A. Schouten, D. van Dantzig: Generelle Feldtheorie [General field theory]. Zeitschrift für Physik 78(1932)639-667 (DOI: 10.1007/BF01351689). [in German] 10

[98] Harish-Chandra: Algebra of the Dirac-matrices. Proceedings of the Indian Academy of Sciences A 22(1945)30-41 (The article is freely available online at the journal site: http://www.ias.ac.in/j_archive/proca/22/1/30-41/
A. Popovici, A. Hristev, J. Popovici: Théorie conforme dans $C_6$ des particules de spin maximal 1 et 1/2 [Conformal theory of particles with maximal spin 1 and 1/2 in $C_6$]. *Comptes Rendus Hebdomadaires des Séances de l’Académie des Sciences (Paris)* 257 (1963) 3120-3122 (The article is freely available online at the Gallica site of the Bibliothèque Nationale de France: [http://gallica.bnf.fr](http://gallica.bnf.fr)). [in French] 10

A. J. Macfarlane: Dirac matrices and the Dirac matrix description of Lorentz transformations. *Communications in Mathematical Physics* 2 (1966) 133-146 (DOI: 10.1007/BF01773348, the article is freely available online at the Project Euclid site: [http://projecteuclid.org/euclid.cmp/1103815014](http://projecteuclid.org/euclid.cmp/1103815014)). 10, 11, 48, 49, 52, 53, 78

F. I. Fedorov: Рекуррентная формула для произведений матриц Дирака [Recurrence formula for products of Dirac matrices]. *Vestsi Akademii Navuk Belaruskai SSR, Seryya Fizika-Matematychnykh Navuk* (1967) No. 1, 127-128. [in Russian] 10

A. ten Kate: Dirac algebra and the six-dimensional Lorentz group. *Journal of Mathematical Physics* 9 (1968) 181-185 (DOI: 10.1063/1.1664566). 11, 29

H. A. Buchdahl: On certain tensors and spinors associated with the Dirac algebra. *Tensor*, New Series, 25 (1972) 137-147. 11, 45

F. S. Klotz: Twistors and the conformal group. *Journal of Mathematical Physics* 15 (1974) 2242-2247 (DOI: 10.1063/1.1666606). 11, 29, 45, 47, 48, 49, 51, 52, 53, 54, 68

V. I. Tselyaev: Majorana spinors and extended Lorentz symmetry in four-dimensional theory. *Classical and Quantum Gravity* 25 (2008) 105021, 22 pp. [arXiv:0710.1890] (DOI: 10.1088/0264-9381/25/10/105021). 11

J. G. Valatin: Le couplage des variables de spineur d’un système d’électrons de Dirac [The coupling of spinor variables of a system of Dirac electrons]. *Le Journal de Physique et le Radium* 11 (1950) 97-101 (DOI: 10.1051/jphysrad:0195000110209700, the article is freely available online at the HAL site: [http://hal.archives-ouvertes.fr/jpa-00234224/fr/](http://hal.archives-ouvertes.fr/jpa-00234224/fr/)). [in French] 11
[107] W. A. Hepner: The inhomogeneous Lorentz group and the conformal group. *Il Nuovo Cimento*, 10. Series, 26(1962)351-368 (DOI: 10.1007/BF02787046). 11, 68

[108] A. O. Barut: Dynamical symmetry group based on Dirac equation and its generalization to elementary particles. *Physical Review* 135(1964)B839-B842 (DOI: 10.1103/PhysRev.135.B839). 11

[109] J. A. de Wet: *The Relationship Between Sedenion Algebra and the Spin Representations of the Rotation Group with an Application to Elementary Particle Theory*. M. Sc. thesis, University of South Africa, Pretoria, 1966. An abstract of the thesis is printed in: Die Publikasiekomitee/The Publication Committee [F. W. Blignaut (Red./Ed.): *Opsommings van Tesise aanvaar deur die Universiteit van Suid-Afrika in 1967 = Summaries of Theses Accepted by the University of South Africa in 1967*, University of South Africa, Pretoria, 1969, pp. 7-9. This edition is part of: *Mededelings van die Universiteit van Suid-Afrika. Reeks D, Bibliografiee = Communications of the University of South-Africa. Series D, Bibliography* 5(1969). 11, 69

[110] Б. Г. Конопельченко [B. G. Konopel’chenko]: Дискретные преобразования в шестимерной модели пространства Минковского [Diskretnye preobrazovaniya v shestimernoi modeli prostranstva Minkovskogo]. *Teoreticheskaya i Matematicheskaya Fizika* 5(1970)366-371 (The article is freely available online at the Mathnet site: [http://mi.mathnet.ru/eng/tmf4215](http://mi.mathnet.ru/eng/tmf4215)). English translation: B. G. Konopel’chenko: Discrete transformations in a six-dimensional model of Minkowski space. *Theoretical and Mathematical Physics* 5(1970)1211-1215 (DOI: 10.1007/BF01035252). 11

[111] A. Laufer: *The Exponential Map, Clifford Algebras, Spin Representations and Spin Gauge Theory of U*(1)×T4. Konstanzer Dissertationen, Vol. 547. Hartung-Gorre Verlag, Konstanz, 1997. 11, 29, 47, 52, 53

[112] Ч. Палев [Ch. Palev]: Максимальная простая алгебра Ли, построенная из двух пар Ферми операторов [Maksimal’naya prostaya algebra Li, postroen-naya iz dvukh par Fermi operatorov]// [Maximal simple Lie algebra constructed from two pairs of Fermi operators]. *Izvestiya na Fizicheskiya Institut s ANEB* [Izvestiya na Fizicheskiya Institut s ANEB] (Bulletin de l’Institut de Physique et de Recherche Atomique) 22(1972)129-136 (preprint version: Joint Institute of Nuclear Research Preprint R2-5267, Dubna, 1970). [in Russian] 11

[113] B. Gruber, M. Ramek: Boson and fermion operator realisations of su(4) and its semisimple subalgebras. In: B. J. Gruber, T. Otsuka (Eds.): *Symmetries in Science VII, Spectrum-Generating Algebras and Dynamic Symmetries in Physics*. Plenum Press, New York, London, 1993, pp. 223-243. 11
[114] B. R. Judd: Topics in atomic theory. In: B. R. Judd, J. P. Elliott: *Topics in Atomic & Nuclear Theory*. University of Canterbury Publications, Vol. 12. University of Canterbury, New Zealand, Christchurch, 1970, pp. ix-xii, 1-60. 11

[115] Ч. Д. Палев [Ch. D. Palev]: Максимальная простая алгебра Ли, построенная из заданного числа Ферми-операторов [Maksimal’naia prostaya algebra Li, postroennaya iz zadannogo chisla Fermi-operatorov] / [Maximal simple Lie algebra constructed from a given number of Fermi operators]. Joint Institute of Nuclear Research Preprint R2-5303, Dubna, 1970. [in Russian, English abstract] 11

[116] C. Doran, D. Hestenes, F. Sommen, N. Van Acker: Lie groups as spin groups. *Journal of Mathematical Physics* **34**(1993)3642-3669 (DOI: 10.1063/1.530050). 11

[117] L. Frappat, A. Sciarrino, P. Sorba: *Dictionary on Lie Algebras and Superalgebras*. Academic Press, San Diego, CA, 2000. 11

[118] B. G. Wyborne: Lie algebras in quantum chemistry: Symmetrized orbitals. *International Journal of Quantum Chemistry* **7**(1973)1117-1137 (DOI: 10.1002/qua.560070608). 11

[119] S. K. Bose: Dynamical algebra of spin waves in localised-spin models. *Journal of Physics A: Mathematical and General* **18**(1985)903-922 (DOI: 10.1088/0305-4470/18/6/014). Reprinted in: [120], Vol. 1, pp. 532-551. 11

[120] A. Bohm, Y. Ne’eman, A. O. Barut: *Dynamical Groups and Spectrum Generating Algebras*, 2 Vols.. World Scientific, Singapore, 1988. 95, 98

[121] C. Buzano, M.G. Rasetti, M.L. Rastello: Dynamical superalgebras of the “dressed” Jaynes-Cummings model. *Physical Review Letters* **62**(1989)137-139 (DOI: 10.1103/PhysRevLett.62.137). 11

[122] Wei-Min Zhang, Da Hsuan Feng, R. Gilmore: Coherent states: Theory and some applications. *Reviews of Modern Physics* **62**(1990)867-927 (DOI: 10.1103/RevModPhys.62.867). 11

[123] J. Keller, S. Rodríguez-Romo: Multivectorial representation of Lie groups. *International Journal of Theoretical Physics* **30**(1991)185-196 (DOI: 10.1007/BF00670711). 11

[124] Fu-Lin Zhang, Jing-Ling Chen: Non-standard Schwinger fermionic representation of unitary group. *International Journal of Theoretical Physics* **48**(2009)414-421 [arXiv:0801.1359] (DOI: 10.1007/s10773-008-9816-9). 11
[125] D. M. Goodmanson: A graphical representation of the Dirac algebra. *American Journal of Physics* **64** (1996) 870-880 (DOI: 10.1119/1.18113).

[126] M. Saniga [M. Saniga], M. Planat [M. Planat], P. Prachna [M. Prachna]: Проективные кривые над кольцом, включающие в себя два-кубиты [Проективные кривые над кольцом, включающие в себя два-кубиты]. *Теоретическая и математическая физика* [Teoreticheskaya i Matematicheskaya Fizika] **155** (2008) 463-473. English translation: M. Saniga, M. Planat, P. Pracna: Projective ring line encompassing two-qubits. *Theoretical and Mathematical Physics* **155** (2008) 905-913 [arXiv:quant-ph/0611063] (DOI: 10.1007/s11232-008-0076-x).

[127] M. Planat, M. Saniga: On the Pauli graph of N-qudits. *Quantum Information and Computation* **8** (2008) 0127-0146 [arXiv:quant-ph/0701211].

[128] M. Planat, M. Saniga: Pauli graph and finite projective lines/geoemtries. *Proceedings of SPIE* **6583** (2007) 65830W, 12 pp. [arXiv:quant-ph/0703154] (DOI: 10.1117/12.721687). The article is part of: I. Prochazka, A. L. Migdall, A. Pauchard, M. Dusek, M. S. Hillery, W. P. Schleich (Eds.): Photon Counting Applications, Quantum Optics, and Quantum Cryptography: 18-19 April 2007, Prague, Czech Republic. *Proceedings of SPIE* **6583** (2007) 658301-658311.

[129] M. Saniga, M. Planat, P. Pracna, H. Havlicek: The Veldkamp space of two-qubits. *SIGMA – Symmetry, Integrability and Geometry: Methods and Applications* **3** (2007) 075, 7 pp. [arXiv:0704.0495] (DOI: 10.3842/SIGMA.2007.075, the article is freely available online at the journal site: http://www.emis.de/journals/SIGMA/2007/075).

[130] A. R. P. Rau: Mapping two-qubit operators onto projective geometries. *Physical Review A* **79** (2009) 042323, 6 pp. [arXiv:0808.0598] (DOI: 10.1103/PhysRevA.79.042323).

[131] G. Bergdolt: Orthonormal basis sets in Clifford algebras. In: E. Ablamowicz, P. Lounesto, J. M. Parra (Eds.): *Clifford algebras with Numeric and Symbolic Computations*. Birkhäuser, Boston, 1996, pp. 269-284.

[132] G. Bergdolt: Isomorphism groups in Clifford algebras. *Advances in Applied Clifford Algebras* **9** (1999) 95-101 (The article is freely available online at the journal site: http://www-clifford-algebras.org/v9/v91/bergdol91.pdf).

[133] R. A. Mosna, D. Miralles, J. Vaz Jr.: Multivector Dirac equation and $\mathbb{Z}_2$-gradings of Clifford algebras. *International Journal of Theoretical Physics* **41** (2002) 1651-1671 (DOI: 10.1023/A:1021003016189).
[134] S. E. Kozlov [S. E. Kozlov]: Geometry of real Grassmann manifolds. Parts I, II. Journal of Mathematical Sciences (New York) 100 (1997) 2239-2253 (DOI: 10.1007/s10958-000-0008-2). This article is part of: B. A. Zalgaller [V. A. Zalgaller] (Ed.): Geometriya i Topologiya. 2 [Geometriya i Topologiya. 2]. Zapiski Nauchnykh Seminarov POMI [Zapiski Nauchnykh Seminarov POMI] 246 (1997) 1-195. English translation: Journal of Mathematical Sciences (New York) 100:3 (1997) 2189-2309. 20, 36

[135] É. Cartan: Leçons sur la Théorie des Spineurs, 2 Vols.. Vol. I: Les Spineurs de l’Espace a Trois Dimensions. Actualités Scientifiques et Industrielles, Vol. 643, Exposés de Géometrie, Vol. 9. Vol. II: Les Spineurs de l’Espace a n > 3 dimensions. Les Spineurs en Géométrie Riemannienne. Actualités Scientifiques et Industrielles, Vol. 701, Exposés de Géometrie, Vol. 11. Hermann, Paris, 1938. [in French] English translation: The Theory of Spinors. Hermann, Paris, 1966. Reprinted: Dover Publications, Inc., New York, 1981. 20, 50

[136] J. A. Schouten: Tensor Analysis for Physicists. Clarendon Press, Oxford, 1. ed. 1951, 2. rev. ed. 1954. Reprint of the second edition: Dover Publications, Inc., New York, 1989. 20, 36

[137] S. Sternberg: Lectures on Differential Geometry. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964. 20

[138] R. Penrose, W. Rindler: Spinors and Space-Time. Vol. 1: Two-Spinor Calculus and Relativistic Fields. Vol. 2: Spinor and Twistor Methods in Space-Time Geometry. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, Vol. 1: 1984, Vol. 2: 1986. 20, 29

[139] С. Е. Козлов [S. E. Kozlov]: Ортогонально совместимые бивекторы [Ortogonal’no sovmestimye bivektory] / [Orthogonally compatible bivectors]. Український Геометричний Сбірник [Ukrains’kii Geometricheskii Sbornik] 27 (1984) 68-75. [in Russian] 20, 36

[140] Э. Б. Винберг [È. B. Vinberg]: Курс Алгебры [Kurs Algebry]. Series Университетский Учебник [Universitet-skii Uchebnik] Факториал Пресс [Faktorial Press], Moscow, 1. ed. 1999, 2. rev. ext. ed. 2001, 3. rev. ext. ed. 2002. [in Russian] English translation of the second Russian ed.: E. B. Vinberg: A Course in Algebra. Graduate Studies in Mathematics, Vol. 56. American Mathematical Society, Providence, RI, 2003. 20
[141] A. I. Kostrikin [A. I. Kostrikin]: Введение в Алгебру, [Vvedenie v Algebru], 3 Vols., Физико-математическая литература [Fiziko-matematicheskaya literatura], Moscow, 2000. [in Russian] 20

[142] A. P. Hristev: Teoria covariantă a relaţiilor algebrice spinoriale ale cîmpurilor de spin maxim 1 şi 1/2 [Covariant theory of algebraic spinor identities of particles with maximal spin 1 and 1/2]. Studii şi Cercetări Matematice 16(1964)849-860. [in Romanian] Extended French version: Théorie covariante des identités spinorielles algébriques des champs de spin maximal 1 et 1/2. Revue Roumaine de Mathématiques Pures et Appliquées 9(1964)619-642. 29, 68

[143] A. P. Hristev: Asupra covariantăei conforme [On conformal invariance]. Analele Universităţii Bucureşti, Seria Știinţele Naturii, Matematică – Mecanică 13(1964)167-188. [in Romanian, Russian and French abstracts]. 29, 68

[144] J. Haantjes: Over de door de pentadensystemen gevormde X₉ in de E₁₆ der sedenionen [On the X₉ formed by the pentade systems in the E₁₆ of the sedenions]. Nieuw Archief voor Wiskunde, 2. Series, 18:4(1936)46-58. [in Dutch] 30, 45, 46, 47, 67

[145] D. Uskov, A. R. P. Rau: Geometric phases and Bloch-sphere construction for SU(N) groups with a complete description of the SU(4) group. Physical Review A 78(2008)022331, 10 pp. [arXiv:0801.2091] (DOI: 10.1103/PhysRevA.78.022331). 35

[146] J. L. Birman, A. L. Solomon: Dynamical group SO(6) and coexistence: superconductivity and charge-density-waves. Physical Review Letters 49(1982)230-233 (DOI: 10.1103/PhysRevLett.49.230), erratum ibid. 49(1982)421 (DOI: 10.1103/PhysRevLett.49.421.2). Reprinted in: [120], Vol. 1, pp. 528-531. 36

[147] M. Marcus: Finite Dimensional Multilinear Algebra, 2 Parts. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 23. Marcel Dekker, New York, Part 1: 1973, Part 2: 1975. 36, 76

[148] R. Shaw: Linear Algebra and Group Representations, 2 Vols.. Volume I: Linear Algebra and Group Representations. Volume II: Multilinear Algebra and Group Representations. Academic Press, London, Vol. 1: 1982, Vol. 2: 1983. 36

[149] R. Westwick: A representation theorem for real or complex bivectors. Linear and Multilinear Algebra 10(1981)89-91 (DOI: 10.1080/03081088108817398). 36

[150] A. McDaniel, S. M. Zoltek: Curvature operators which preserve normal spaces. Linear and Multilinear Algebra 30(1991)183-194 (DOI: 10.1080/030810889108818101). 36
[151] Ф. Р. Гантмахер [F. R. Gantmakher]: *Теория Матриц* [Teoriya Matrits]. 1. ed.: Гостехиздат [Gostekhizdat], Moscow, 1953; Наука [Nauka], Moscow, 2. rev. and ext. ed. 1966, 3. ed. 1967, 4. ed. 1988; Физматлит [Fizmatlit], Moscow, 5. ed. 2004 (The 2. edition of the book is freely available online at the EqWorld site: http://eqworld.ipmnet.ru/ru/library/books/Gantmaxer_matric_1966ru.djvu.). English translation of the 1. Russian edition: F. R. Gantmacher: *The Theory of Matrices*, 2 Vols.. Chelsea Publishing Company, New York, NY, 1959. Reprinted: AMS Chelsea Publishing, Vols. 131, 133. American Mathematical Society, Providence, RI, 2000. 36

[152] D. Hestenes, G. Sobczyk: *Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics*. Fundamental Theories of Physics, Vol. 5. D. Reidel Publishing Company, Dordrecht, 1984, corrected reprint 1992. 36, 73

[153] E. M. S. Hitzer: Antisymmetric matrices are real bivectors. *Memoirs of the Faculty of Engineering, Fuku University* 49(2001)283-298. 36

[154] R. Delbourgo, L. M. Jones, M. White: Anharmonic Grassmann oscillator. *Physical Review* D 40(1989)2716-2719 (DOI: 10.1103/PhysRevD.40.2716). 38

[155] R. Delbourgo, L. M. Jones, M. White: Anharmonic Grassmann oscillator. II. *Physical Review* D 41(1990)679-681 (DOI: 10.1103/PhysRevD.41.679). 38

[156] M. T. Thomaz, A. F. R. de Toledo Piza: Equivalence of classical spins and Hartree-Fock-Bogoliubov approximation of the fermionic anharmonic oscillator. *Physica A* 218(1995)237-244 (DOI: 10.1016/0378-4371(95)00100-L). 38

[157] M. C. D. Barrozo, M. T. Thomaz, A. F. R. de Toledo Piza: Dynamical evolution of a fermionic anharmonic oscillator. *American Journal of Physics* 63(1995)463-467 (DOI: 10.1119/1.17913). 38

[158] M. T. Thomaz: Fermionic four-level system in the presence of a time-dependent magnetic field. *Modern Physics Letters* B 10(1996)643-651 (DOI: 10.1142/S0217984996000717). 38

[159] M. D. Girardeau: Fock-Tani representation for composite particles in a soluble model. *Journal of Mathematical Physics* 21(1980)2365-2375 (DOI: 10.1063/1.524693). 38

[160] S. M. de Souza, M. T. Thomaz: Grassmann algebra and fermions in the lattice: A simple example. *Journal of Mathematical Physics* 32(1991)3455-3462 (DOI: 10.1063/1.529459). 38

[161] S. M. de Souza, M. T. Thomaz, S. Moss de Oliveira: Thermodynamic properties of an anharmonic fermionic oscillator. *American Journal of Physics* 60(1992)1122-1126 (DOI: 10.1119/1.16958). 38
[162] A. C. Aguiar Pinto, R. Bruno, M. T. Thomaz: Dynamics of one-site fermionic model in the presence of precessing magnetic field. *Physica A* 241(1997)535-548 (DOI: 10.1016/S0378-4371(97)00172-6). 38

[163] P. Jordan, E. P. Wigner: Über das Paulische Äquivalenzverbot [On Pauli’s exclusion principle]. *Zeitschrift für Physik* 47(1928)631-651 (DOI: 10.1007/BF01331938). [in German] Reprinted in: 1. A. S. Wightman (Ed.): *The Collected Works of Eugene Paul Wigner. Part A, The Scientific Papers*, Vol. 1. Springer-Verlag, Berlin, 1993, pp. 109-129. 2. J. Schwinger (Ed.): *Selected Papers on Quantum Electrodynamics*. Dover Publications, Inc., New York, 1958, paper 4, p. 41. 38

[164] C. N. Banwell, H. Primas: On the analysis of high-resolution nuclear magnetic resonance spectra. I. Methods of calculating N.M.R. spectra. *Molecular Physics* 6(1963)225-256 (DOI: 10.1080/00268976300100281). 38

[165] P. Erdös: Dynamical behavior of exchange-coupled spins. *Helvetica Physica Acta* 39(1966)155-163. 38

[166] P. Erdös: Theory of ion pairs coupled by exchange interaction. *Journal of Physics and Chemistry of Solids* 27(1966)1705-1720 (DOI: 10.1016/0022-3697(66)90100-4). 38

[167] J. D. Patterson, W. H. Southwell: Green’s function theory of ferromagnetism. *American Journal of Physics* 36(1968)343-350 (DOI: 10.1119/1.1974520). 38

[168] G. L. Lucas: Green function theory of the two-spin system. *American Journal of Physics* 36(1968)942-943 (DOI: 10.1119/1.1974359). 38

[169] L. K. Kuiper, J. J. Adels: Green’s function treatment of two-spin Heisenberg ferromagnet. *American Journal of Physics* 39(1971)253-256 (DOI: 10.1119/1.1986118). 38

[170] Л. А. Максимов [L. A. Maksimov], А. Л. Куземский [A. L. Kuzemskii]: К теории ферромагнитного кристалла с двумя спинами в узле [K teorii ferromagnitnogo kristalla s dvumya spinami v uzle]. *Физика Металлов и Металловедение [Fizika Metallov i Metallovedenie]* 31(1971)5-12. English translation: L. A. Maksimov, A. L. Kuzemskiy: A contribution to the theory of a ferromagnetic crystal with two spins per site. *The Physics of Metals and Metallography* 31(1971)1-8. 38

[171] J. O. Lawson, S. J. Brient: On obtaining the exact Green’s function solution for the two-spin-1/2 Heisenberg ferromagnet. *American Journal of Physics* 40(1972)1643-1646 (DOI: 10.1119/1.1987003). 38

[172] O. Kühler, H. D. Zeh: Dynamics of quantum correlations. *Annals of Physics (New York)* 76(1973)405-418 (DOI: 10.1016/0003-4916(73)90040-7). 38
[173] J. L. Cado, W. Figueiredo: Biquadratic exchange in a two-spin system. *physica status solidi (b)* **153**(1989)K73-K77 (DOI: 10.1002/pssb.2221530157). 38

[174] J. S. Townsend: *A Modern Approach to Quantum Mechanics*. International Series in Pure and Applied Physics. McGraw-Hill, New York, 1992; University Science Books, Sausalito, 2000. 38

[175] G. J. Bowden, M. J. Prandolini: Complete solution for two coupled spin-1/2 nuclei evolving under chemical shift and dipolar interaction. *Journal of Mathematical Chemistry* **16**(1994)1-7 (DOI: 10.1007/BF01169190). 38

[176] A. R. P. Rau: Manipulating two-spin coherences and qubit pairs. *Physical Review A* **61**(2000)032301, 5 pp. (DOI: 10.1103/PhysRevA.61.032301). 38, 77

[177] J. Vala, K. B. Whaley: Encoded universality for generalized anisotropic exchange Hamiltonians. *Physical Review A* **66**(2002)022304, 10 pp. [arXiv:quant-ph/0204016] (DOI: 10.1103/PhysRevA.66.022304). 38

[178] D. V. Efremov, R. A. Klemm: Heisenberg dimer single molecule magnets in a strong magnetic field. *Physical Review B* **66**(2002)174427, 12 pp. [arXiv:cond-mat/0206406] (DOI: 10.1103/PhysRevB.66.174427). 38

[179] J. Zhang, J. Vala, S. Sastry, K. B. Whaley: Geometric theory of nonlocal two-qubit operations. *Physical Review A* **67**(2003)042313, 18 pp. [arXiv:quant-ph/0209120] (DOI: 10.1103/PhysRevA.67.042313). 38

[180] T. F. Havel, C. J. L. Doran: A Bloch-sphere-type model for two qubits in the geometric algebra of a 6-D Euclidean vector space. *Proceedings of SPIE* **5436**(2004)93-106 [arXiv:quant-ph/0403136] (DOI: 10.1117/12.540929). The article is part of: E. Donkor, A. R. Pirich, H. E. Brandt (Eds.): Quantum Information and Computation II: 12-14 April 2004, Orlando, Florida, USA. *Proceedings of SPIE* **5436**(2004)1-400. 38, 77

[181] A. R. P. Rau, G. Selvaraj, D. Uskov: Four-level and two-qubit systems, subalgebras, and unitary integration. *Physical Review A* **71**(2005)062316, 8 pp. [arXiv:quant-ph/0501048] (DOI: 10.1103/PhysRevA.71.062316). 38

[182] V. G. Bagrov, M. C. Baldiotti, D. M. Gitman, A. D. Levin: Two interacting spins in external fields. Four-level systems. *Annalen der Physik (Leipzig)*, 8. Series, **16**(2007)274-285 [arXiv:quant-ph/0608036 Universidade de São Paulo, Instituto de Física Report IF-1622/2006] (DOI: 10.1002/andp.200610231). 38

[183] T. F. Jordan, A. Shaji, E. C. G. Sudarshan: One qubit almost completely reveals the dynamics of two. *Physical Review A* **76**(2007)012101, 6 pp. [arXiv:quant-ph/0611141] (DOI: 10.1103/PhysRevA.76.012101). 38
[184] Ю. С. Волков [Yu. S. Volkov], Д. О. Синицын [D. O. Sinitsyn]: Резонансная динамика двух спиновой системы с накачкой [Rezonansnaya dinamika dvuhspinovoi sistemy s nakachkoi]. Журнал Экспериментальной и Теоретической Физики [Zhurnal Èxperimental'noi i Teoreticheskoi Fiziki] 132(2007)1296-1301 (The article is freely available online at the journal site: http://www.jetp.ac.ru/cgi-bin/dn/r_132_1296.pdf). English translation: Yu. S. Volkov, D. O. Sinitsyn: Dynamics of a resonantly driven two-spin system. Journal of Experimental and Theoretical Physics 105(2007)1136-1140 (DOI: 10.1134/S1063776107120059). 38

[185] M. C. Baldiotti, D. M. Gitman: Four-level systems and a universal quantum gate. Annalen der Physik (Leipzig), 8. Series, 17(2008)450-459 [arXiv:0710.1112, Universidade de São Paulo, Instituto de Física Report IF-1643/2007] (DOI: 10.1002/andp.200810303). 38

[186] M. Sebawe Abdalla, E. Lashin, G. Sadiek: Entropy and variance squeezing for time-dependent two-coupled atoms in an external magnetic field. Journal of Physics B: Atomic, Molecular and Optical Physics 41(2008)015502, 13 pp. (DOI: 10.1088/0953-4075/41/1/015502). 38

[187] D. C. Mattis, S. B. Nam: Exactly soluble model of interacting electrons. Journal of Mathematical Physics 13(1972)1185-1189 (DOI: 10.1063/1.1666120). 41

[188] P.-A. Lindgård, J. F. Cooke: Theory of spin waves in strongly anisotropic magnets. Physical Review B 14(1976)5056-5059 (DOI: 10.1103/PhysRevB.14.5056). 41

[189] J. F. Cooke, P.-A. Lindgård: Canonical transform method for treating strongly anisotropic magnets. Physical Review B 16(1977)408-418 (DOI: 10.1103/PhysRevB.16.408). 41

[190] J. F. Cooke, P.-A. Lindgård: Canonical transform theory of spin-waves in strongly anisotropic magnets. Journal of Applied Physics 49(1978)2136-2138. This article is part of: J. J. Becker, G. H. Lander (Eds.): Proceedings of the Twenty-Third Annual Conference on Magnetism and Magnetic Materials, 8-11 November 1977, Minneapolis, MN, USA. Journal of Applied Physics 49(1978)1299-2215 (DOI: 10.1063/1.324763). 41

[191] D. A. Garanin, V. S. Lutovinov: High temperature spin wave dynamics of the uniaxial antiferromagnets. Solid State Communications 44(1982)1359-1362 (DOI: 10.1016/0038-1098(82)90893-6). 41

[192] В. Л. Сафонов [V. L. Saфонов]: К теории параметрического возбуждения волн в спиновых системах [K teorii parametricheskogo vozbuždeniya voln v spinovykh sistemakh]/[On the theory of the parametric excitation of waves in spin systems]. Preprint Институт Атомной Энергии им. И. В. Курчатова
[193] V. L. Safonov: Method of spin Hamiltonian diagonalization. *Physics Letters A* 97(1983)164-167 (DOI: 10.1016/0375-9601(83)90206-2). 41

[194] S. V. Dobrov: On the spin-fermion connection. *Journal of Physics A: Mathematical and General* 36(2003)L503-L508 [arXiv:cond-mat/0306495, Institute of Electrophysics, Ekaterinburg, Russia, Preprint IEP-03-06] (DOI: 10.1088/0305-4470/36/39/101). 41

[195] R. B. Creel: Analytic solution of fourth degree secular equations: $I = 3/2$ Zeeman-quadrupole interactions and $I = 7/2$ pure quadrupole interaction. *Journal of Magnetic Resonance* 52(1983)515-517 (DOI: 10.1016/0022-2364(83)90179-8). 41

[196] G. M. Muha: Exact solution of the eigenvalue problem for a spin $I = 3/2$ system in the presence of a magnetic field. *Journal of Magnetic Resonance* 53(1983)85-102 (DOI: 10.1016/0022-2364(83)90073-2). 41

[197] A. D. Bain: Exact calculation, using angular momentum, of combined Zeeman and quadrupolar interactions in NMR. *Molecular Physics* 101(2003)3163-3175 (DOI: 10.1080/00268970310001626298). 41

[198] A. D. Bain, M. Khasawneh: From NQR to NMR: The complete range of quadrupole interactions. *Concepts in Magnetic Resonance Part A* 22A(2004)69-78 (DOI: 10.1002/cmra.20013). 41

[199] A. R. Kessel’, V. L. Ermakov: Multikubitny˘ı spin [Mnogokubitn˘ı spin]. Письма в Журнал Экспериментальной и Теоретической Физики [Pisma v Zhurnal Èkperimental'noï i Teoreticheskoi Fiziki] 70(1999)59-63 (The article is freely available online at the journal site: http://www.jetpletters.ac.ru/ps/911/article_13968.shtml). English translation: A. R. Kessel’, V. L. Ermakov: Multiquantum spin. *JETP Letters* 70(1999)61-65 [arXiv:quant-ph/9912047] (DOI: 10.1134/1.568130). 41

[200] A. R. Kessel’ [correctly: Kessel’], V. L. Ermakov: Virtualnye kubity – mnogourovnevnost’ vместо многочастичности [Virtual’nye kubity – mnogourovnenovost’ vmensto mnogochastichnosti]. Журнал Экспериментальной и Теоретической Физики [Zhurnal Èkperimental’noï i Teoreticheskoi Fiziki] 117(2000)517-525 (The article is freely available online at the journal site: http://www.jetp.ac.ru/cgi-bin/dn/r_117_517.pdf). English translation: A. R. Kesel’ [correctly: Kessel’], V. L. Ermakov: Virtual qubits – multilevels instead of multiparticles. *Journal of Experimental and Theoretical Physics* 90(2000)452-459 (DOI: 10.1134/1.559125). 41
[201] A. R. Kessel, N. M. Yakovleva: Implementation schemes in NMR of quantum processors and the Deutsch-Josza algorithm by using virtual spin representation. Physical Review A 66(2002)062322, 7 pp. (DOI: 10.1103/PhysRevA.66.062322). 41

[202] A. R. Kessel: Variants of qubit encoding in quantum information science. Laser Physics 12(2002)581-585 (The article is freely available online at the journal site: http://www.maik.ru/full/lasphys/02/3/lasphys3_02p581full.pdf). 41

[203] G. J. Grube, J. Y. Park: Pauli spinor representations for spin 3/2 operators. (abstract only) In: Proceedings of the International Conference on Nucleus-Nucleus Collisions, 26 September-1 October 1982, East Lansing, MI. National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, MI, 1982, p. 141. 42, 69

[204] Р. Зайков [R. Zaikov]: Върху спинорните трансформации [V’rkhu spinor-nite transformatmsii]/[On spinor transformations]. Известия на Българската Академия на Науките, Отделение за Физико-Математически и Технически Науки, Серия Физическа [Izvestiya na B”lgarskata Akademiya na Naukite, Otdelenie za Fiziko-Matematicheski i Tekhnickieski Nauki, Seriya Fizicheska], journal title on cover: Известия на Физическия Институт с АНЕБ [Izvestiya na Fizicheskiya Institut s ANEB] (Bulletin de l’Institut de Physique a. CA) 7(1959)239-259 (note the accompanying list of misprints/corrections of the volume). [in Bulgarian] 43

[205] А. А. Богуш [A. A. Bogush]: О матрицах конечных унитарных преобразований [O matritsakh konechnykh unitarnyh preobrazovaniй]/[On the matrices of finite unitary transformations]. Весті Академії Наук Білоруськай ССР, Серія Фізика-Математичних Навук [Vesti Akademiї Navuk Belaruskai SSR, Seryya Fizika-Matematychnyk Navuk] (1973) No. 5, 105-112. [in Russian] 44, 45

[206] Ю. П. Степановский [Yu. P. Stepanovskii]: Алгебра матриц Дираха у шестимерному вигляді [Algebra matrits’ Diraka u shestimërnому viglyadi]/[Six-dimensional form of Dirac matrix algebra]. Український Фізичний Журнал [Ukrains’kii Fizichni Zhurnal] 11(1966)813-824 (The article is freely available online at the Google Scholar site: http://scholar.google.com/scholar_host?q=info:ZzRNyT_Q0ewJ:scholar.google.com/&output=viewport&pg=1). [in Ukrainian, English abstract] 45, 46, 47, 48, 54

[207] Ю. П. Степановский [Yu. P. Stepanovskii]: Спиноры шестимерного пространства и их применение к описанию поляризованных частиц со спином 1/2 [Spinory shestimernoho prostranstva i ikh primenienie k opisaniyu polarizirovannykh chastits so spinom 1/2]/[Spinors of a six-dimensional space and their application to the description of polarized particles with spin]
1/2. Проблемы Ядерной Физики и Космических Лучей [Problemy Yadernoĭ Fiziki i Kosmicheskikh Luchey] 4(1976)9-21. [in Russian] 45, 46, 47, 48

[208] M. V. Lyubchenko, Yu. P. Stepanovsky: On six-dimensional Dirac equation and some relations between polarisations of the spin $\frac{3}{2}$ particle. Вопросы Атомной Науки и Техники [Voprosy Atomnoi Nauki i Tekhniki]/ Питання Атомної Науки і Техніки [Pitannya Atomnoi Nauki i Tekhniki]/ Problems of Atomic Science and Technology (2007) No. 3, 47-50 (The article is freely available online at the journal site: http://vant.kipt.kharkov.ua/ARTICLE/VANT_2007_3/article_2007_3_47.pdf). 45

[209] G. Lemaître: Sur l’interprétation d’Eddington de l’équation Dirac [On the interpretation of the Dirac equation by Eddington]. Annales de la Société Scientifique de Bruxelles, Série B, Sciences Physiques et Naturelles 51(1931)83-93. [in French] 45

[210] J. A. Schouten: Zur generellen Feldtheorie. Raumzeit und Spinraum. (G. F. V) [On general field theory. Space-time and spin space. (G. F. V)]. Zeitschrift für Physik 81(1933)405-417 (DOI: 10.1007/BF01344556). [in German] 45, 46, 47

[211] O. Veblen: Geometry of four-component spinors. Proceedings of the National Academy of Sciences of the United States of America 19(1933)503-517 (The article is freely available online at the journal site: http://www.pnas.org/content/19/5/503.full.pdf+html?ck=nck). 45, 46

[212] D. J. Struik: Theory of Linear Connections. Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. 3, No. 2, pp. I-IX, 1-68 (= pp. 173-248 of the whole volume). Verlag von Julius Springer, Berlin, 1934. 45, 46, 47

[213] J. A. Schouten, J. Haantjes: Konforme Feldtheorie II; $R_6$ und Spinraum [Conformal field theory II; $R_6$ and spin space]. Annali della R. Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, 2. Series, 4(1935)175-189 (The article is freely available online at the Numdam site: http://www.numdam.org/item?id=ASNSP_1935_2_4_2_175_0). [in German] 45, 46

[214] H.-C. Lee: On the projective theory of spinors. Compositio Mathematica 6(1939)136-152 (The article is freely available online at the Numdam site: http://www.numdam.org/numdam-bin/fitem?id=CM_1939__6__2_136_0). 45, 46

[215] А. П. Норден [A. P. Norden]: О комплексном представлении тензоров бипланарного пространства [O kompleksnom predstavlenii tensorov biplanarnogo prostranstva]/[On a complex representation of the tensors of biplanar space]. Учёные Записки Казанского Государственного Университета им. В. И. Ульянова-Ленина [Uchënye Zapiski Kazanskogo Gosudarstvennogo
The article is freely available online at the Mathnet site:
http://mi.mathnet.ru/eng/uzku343.

[216] A. P. Норден [A. P. Norden]: О комплексном представлении тензоров пространства Лоренца
[O kompleksnom predstavlenii tenzorov prostranstva Lorentsa]/[On a complex representation of the tensors of Lorentz space].
Известия Высших Учебных Заведений, Математика [Izvestiya Vysshikh Uchebnikh Zavedenii, Matematika] (1959) No. 1 (No. 8), 156-163
(The article is freely available online at the Mathnet site:
http://mi.mathnet.ru/eng/ivm2415./ [in Russian] 46, 106

[217] A. Esteve, P. G. Sona: Conformal group in Minkowsky space. Unitary irreducible representations. Il Nuovo Cimento, 10. Series, 32 (1964)473-485
(.DOI: 10.1007/BF02733974). 45

[218] М. М. Постников [M. M. Postnikov]: Группы и Альгебры Ли [Grupy i Algebry Li]. Лекции по Геометрии, Семестр V [Lektsii po Geometrii, Semestr V].
Наука [Nauka], Moscow, 1982. Supplemented English translation: M. Postnikov: Lie Groups and Lie Algebras. Lectures in Geometry, Semester 5. Mir,
Moscow, 1986; URSS Publishing, Moscow, 1994. 45

[219] К. В. Андреев [K. V. Andreev]: К вопросу о тензоре кривизны и бивекторах 6-мерных римановых пространств
[K voprosu o tenzore krivizny i bivektoraakh 6-mernykh rimanovykh prostranstv]/[On the curvature tensor and bivectors of 6-dimensional Riemannian spaces]. In: И. Ф. Красичков-Терновский, [I. F. Krasichkov-Ternovskii (Ed.): Комплексный Анализ, Дифференциальные Уравнения, Численные Методы и Приложения. VI. Численные Методы [Kompleksnyi Analiz, Differentsial'nye Uravneniya, Chislennye Metody i Prilozheniya. VI. Chislennye Metody]. Институт Математики с Вычислительным Центром Российской Академии Наук [Institut Matematiki s Vychislitel'nym Tsentrrom Rossiiskoi Akademii Nauk], Ufa, 1996, pp. 134-139.
[in Russian] 45, 69

[220] К. В. Андреев [K. V. Andreev]: О структуре тензора кривизны 6-мерных римановых пространств
[O strukture tensora krivizny 6-mernykh rimanovykh prostranstv]/[On the structure of the curvature tensor of 6-dimensional Riemannian spaces]. Вестник Башкирского Университета
[Vestnik Bashkirskogo Universiteta] (1996) No. 2, 44-47. [in Russian] 45, 46, 47, 52, 69
[221] K. V. Андреев [K. V. Andreev]: Спинорный Формализм и Геометрия Шестимерных Римановых Пространств [Spinornyi Formalizm i Geometriya Shestimernykh Rimanovkh Prostranstv] /[Spinor Formalism and the Geometry of Six-Dimensional Riemannian Spaces]. Кандидатская Диссертация [Kandidat-skaya Dissertatsiya] /[Ph. D. thesis], Башкирский Государственный Университет [Bashkirskiî Gosudarstvennyî Universitet] /[Bashkir State University], Ufa, 1997. [in Russian] 45, 46, 47, 48, 52, 69

[222] K. V. Андреев [K. V. Andreev]: О спинорном формализме при $n = 6$ [O spinornom formalizme pri $n = 6$]. Известия Высших Учебных Заведений, Математика [Izvestiya Vysshikh Uchebnikh Zavedenii, Matematika] (2001) No. 1 (No. 464), 11-23 (The article is freely available online at the Mathnet site: http://mi.mathnet.ru/eng/ivm838, the two figures of the article are missing in the electronic version, however.). English translation: K. V. Andreyev: On spinor formalism for $n = 6$. Russian Mathematics (Iz. VUZ) 45:1(2001)9-20. 45, 46, 47, 48, 52, 69

[223] J. R. Ellis: A spinor approach to quaternion methods in relativity. Proceedings of the Royal Irish Academy, Section A, Mathematical, Astronomical, and Physical Science 64(1966)127-142 [= 64(1966) No. 9] (stable JSTOR URL: http://www.jstor.org/stable/20488640). 45

[224] J. D. Louck, H. W. Galbraith: Application of orthogonal and unitary group methods and the N-body problem. Reviews of Modern Physics 44(1972)540-601 (DOI: 10.1103/RevModPhys.44.540). 45

[225] C. Cheung, D. O’Connell: Amplitudes and spinor-helicity in six dimensions. Journal of High Energy Physics (JHEP) (2009) No. 07, 075, 22 pp. [arXiv:0902.0981] (DOI: 10.1088/1126-6708/2009/07/075). 46

[226] R. L. Ingraham: On the classification of the new particles. Il Nuovo Cimento, 10. Series, 10(1958)1060-1070 (DOI: 10.1007/BF02859568). 46

[227] E. A. Lord: The Dirac spinor in six dimensions. Proceedings of the Cambridge Philosophical Society 64(1968)765-778 (DOI: 10.1017/S0305004100043474). 48

[228] E. R. Caianiello: Combinatorics and Renormalization in Quantum Field Theory. Frontiers in Physics, Vol. 38. W. A. Benjamin Inc., Reading, MA, 1973. 49

[229] H. W. L. Tanner: A theorem relating to Pfaffians. The Messenger of Mathematics 8(1878/1879)56-60 (The article is freely available online at the Internet Archive site: http://www.archive.org/details/messengermathem04glaigoog). 49

[230] H. F. Baker: Note on a property of Pfaffians. Proceedings of the London Mathematical Society 29(1898)141-142 (DOI: 10.1112/plms/s1-29.1.141). 49
[231] W. Zajackowski (correctly: Zajączkowski): A theorem relating to Pfaffians. *The Messenger of Mathematics* **10**(1880/1881)36-37 (The article is freely available online at the Internet Archive site: http://www.archive.org/details/messengermathem00glaigoog). 49

[232] W. Zajączkowski: O pewnej własności pfaffianu [On a property of the Pfaffian]. *Rozprawy i Sprawozdania z Posiedzeń Wydziału Matematyczno-Przyrodniczego Akademii Umiejętności* **7**(1880)67-74. [in Polish] 49

[233] J. Brill: Note on the algebraic properties of Pfaffians. *Proceedings of the London Mathematical Society* **34**(1901)143-151 (DOI: 10.1112/plms/s1-34.1.143). 49

[234] D. E. Knuth: Overlapping pfaffians. *The Electronic Journal of Combinatorics* **3**:2(1996)#R5 (The article is freely available online at the journal site: http://www.emis.de/journals/EJC/Volume_3/Abstracts/v3i2r5.html). The article is part of: J. Désarménien, A. Kerber, V. Strehl (Guest Eds.): The Foata Festschrift. A special issue of The Electronic Journal of Combinatorics dedicated to Dominique Foata on the occasion of his 60th birthday. *The Electronic Journal of Combinatorics* **3**:2(1996)#R1-#R27. 49

[235] A. A. Bogush [A. A. Bogush], F. I. Fedorov [F. I. Fedorov]: О плоских ортогональных преобразованиях [O ploskich ortogonal’nykh preobrazovaniyakh]. *Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR]* **206**(1972)1033-1036, erratum *ibid.* **208**(1973)viii. English translation (erratum incorporated): A. A. Boguš, F. I. Fedorov: On plane orthogonal transformations. *Soviet Mathematics Doklady* **13**(1972)1349-1353. 49, 50, 52, 54

[236] A. A. Bogush [A. A. Bogush], F. I. Fedorov [F. I. Fedorov], A. M. Fedorovyh [A. M. Fedorovykh]: О конечных преобразованиях группы SO(n,R) и ее представления [O konechnykh preobrazovaniyakh gruppy SO(n,R) i ee predstavleniya]. *Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR]* **214**(1974)985-988. English translation: A. A. Boguš, F. I. Fedorov, A. M. Fedorovyh: On finite transformations of the group SO(n,R) and its representations. *Soviet Mathematics Doklady* **15**(1974)255-259. 49, 51, 54

[237] A. J. Macfarlane: On the restricted Lorentz group and groups homomorphically related to it. *Journal of Mathematical Physics* **3**(1962)1116-1129 (DOI: 10.1063/1.1703854). 49, 52

[238] P. H. Schoute: Le déplacement le plus général dans l’espace a n dimensions [The most general transformation in a space of n dimensions]. *Annales de l’École Polytechnique de Delft* **7**(1891)139-158. [in French] 50

[239] G. Vitali: Sulle sostituzioni lineari ortogonalli [On linear orthogonal substitutions]. *Bollettino della Unione Matematica Italiana* **7**(1928)1-7. [in Italian] 50

108
[240] W. Pauli: Beiträge zur mathematischen Theorie der Dirac’schen Matrizen [Contributions to the mathematical theory of Dirac’s matrices]. In: Pieter Zeeman, 1965 – 25 Mei – 1935: Verhandelingen op 25 Mei 1935 Aangeboden aan Prof. Dr. P. Zeeman. Martinus Nijhoff, The Hague, 1935, pp. 31-43. [in German] Reprinted in: [241], Vol. 2, pp. 724-736. 52

[241] R. Kronig, V. F. Weisskopf (Eds.): Collected Scientific Papers by Wolfgang Pauli. 2 Vols.. Interscience, New York, 1964. 52, 109

[242] W. Pauli: Contributions mathématiques à la théorie des matrices de Dirac [Mathematical contributions to the theory of Dirac matrices]. Annales de la Institute Henri Poincaré 6(1936)109-136 (The article is freely available online at the Numdam site: http://www.numdam.org/numdam-bin/fitem?id=AIHP_1936__6_2_109_0). [in French] Reprinted in: [241], Vol. 2, pp. 753-780. 52

[243] R. H. Good Jr.: Properties of the Dirac matrices. Reviews of Modern Physics 27(1955)187-211 (DOI: 10.1103/RevModPhys.27.187). 52

[244] W. H. McCrea: A theorem concerning Eddington’s E-numbers. Journal of the London Mathematical Society 13(1938)283-288 (DOI: 10.1112/jlms/s1-13.4.283). 54

[245] H. Takeno: Contributions to the theory of sedenions I. Tensor, New Series, 7(1957)143-159. H. Takeno: Contributions to the theory of sedenions II. Tensor, New Series, 7(1957)160-172. H. Takeno: Contributions to the theory of sedenions III. Tensor, New Series, 8(1958)21-37. 54

[246] A. J. Macfarlane: Description of the symmetry group SU 3/Z 3 of the octet model. Communications in Mathematical Physics 11(1968)91-98 (DOI: 10.1007/BF01645898, the article is freely available online at the Project Euclid site: http://projecteuclid.org/euclid.cmp/1103841196). 63

[247] A. A. Bogush, V. M. Red’kov: On unique parametrization of the linear group GL(4,C) and its subgroups by using the Dirac algebra basis. Nonlinear Phenomena in Complex Systems 11(2008)1-24 [arXiv:hep-th/0607054]. 63

[248] V. M. Red’kov, A. A. Bogush, N. G. Tokarevskaya: On parametrization of the linear GL(4,C) and unitary SU(4) groups in terms of Dirac matrices. SIGMA – Symmetry, Integrability and Geometry: Methods and Applications 4(2008)021, 46 pp. [arXiv:0802.2634] (DOI: 10.3842/SIGMA.2008.021, the article is freely available online at the journal site: http://www.emis.de/journals/SIGMA/2008/021). The article is part of: Proceedings of the Seventh International Conference “Symmetry in Nonlinear Mathematical Physics” (June 24-30, 2007, Kyiv, Ukraine). SIGMA – Symmetry, Integrability and Geometry: Methods and Applications 4(2008) (the Proceedings are
freely available online at the journal site: http://www.emis.de/journals/SIGMA/symmetry2007.html. 63

[249] T. S. Untidt, N. Ch. Nielsen: Closed solution to the Baker-Campbell-Hausdorff problem: Exact effective Hamiltonian theory for analysis of nuclear-magnetic-resonance experiments. Physical Review E 65(2002)021108, 17 pp. (DOI: 10.1103/PhysRevE.65.021108). 65

[250] D. Siminovitch, T. Untidt, N. Ch. Nielsen: Exact Hamiltonian theory. II. Polynomial expansion of matrix functions and entangled unitary exponential operators. Journal of Chemical Physics 120(2004)51-66 (DOI: 10.1063/1.1628216). 65

[251] G. Bachman, M. J. Hellman: On the parametrization of the proper orthogonal group. Archiv der Mathematik (Basel) 10(1959)93-100 (DOI: 10.1007/BF01240768). 65

[252] R. Shaw, G. Bowtell: The bivector logarithm of a Lorentz transformation. The Quarterly Journal of Mathematics, Oxford Second Series 20(1969)496-503 (DOI: 10.1093/qmath/20.1.497). 65

[253] R. Kurth: Logarithms of real orthogonal matrices. Tamkang Journal of Mathematics 10(1979)237-239 (preprint version: Southern Illinois University at Edwardsville, Preprints in Mathematics and Mathematical Sciences, no. 62). 65

[254] P. Lounesto: Cayley transform, outer exponential and spinor norm. Supplemento di Rendiconti del Circolo Matematico di Palermo, II. Series, 16(1987)191-198. The article is part of: Proceedings of the Winter School on Geometry and Physics, Srní, 10-17 January, 1987. Supplemento di Rendiconti del Circolo Matematico di Palermo, II. Series, 16(1987). 65

[255] H. Goldstein: Classical Mechanics. Addison-Wesley, 1. ed. Cambridge, MA, 1950, 2. ed. Reading, MA, 1980. 66

[256] H. Goldstein, Ch. Poole, J. Safko: Classical Mechanics, 3. Ed.. Addison-Wesley, San Francisco, CA, 2002. 66

[257] A. Cayley: Recherches ultérieures sur les déterminants gauches [Further studies on skew determinants]. Journal für die Reine und Angewandte Mathematik 50(1855)299-313 (The article is freely available online at the DigiZeitschriften site: http://www.digizeitschriften.de/index.php?id=resolveppn&PPN=GDZPPN002149087 and at the GDZ Document Server http://gdz.sub.uni-goettingen.de/index.php?id=resolveppn&PPN=GDZPPN002149087). [in French] Reprinted in: The Collected Mathematical Papers of Arthur Cayley, Vol. II. Cambridge University Press, Cambridge, 1889, item 137, pp. 202-215. (The book is freely available
online at the Internet Archive site:  
http://www.archive.org/details/collmathpapers02caylrich).  

[258] L. Silberstein: Quaternionic form of relativity. *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 6. Series, 23(1912)790-809 (DOI: 10.1080/14786440508637276).  

[259] N. Rosen: Note on the general Lorentz transformation. *Journal of Mathematics and Physics* 9(1929-30)181-187.  

[260] K. Nikolsky: Bemerkungen über quantenmechanische Matrixgeometrie [Remarks on quantum-mechanical matrix geometry]. *Zeitschrift für Physik* 65 (1930)273-279 (DOI: 10.1007/BF01397039). [in German]  

[261] K. Nikolsky: Zur Theorie der Spinoren [On the theory of spinors]. *Zeitschrift für Physik* 83(1933)284-290 (DOI: 10.1007/BF01342086). [in German]  

[262] B. Kwal: Sur la représentation matricielle des quaternions [On the matrix representation of quaternions]. *Bulletin des Sciences Mathématiques*, 2. Series, 59 (vol. 70 of the series altogether)(1935)328-332. [in French]  

[263] A. Sommerfeld: Über die Klein’schen Parameter α, β, γ, δ und ihre Bedeutung für die Dirac-Theorie [On the Klein parameters α, β, γ, δ and their significance for the Dirac theory]. *Sitzungsberichte, Abteilung IIa: Mathematik, Astronomie, Physik, Meteorologie und Technik, Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse* 145(1936)639-650. [in German] Reprinted in: A. Sommerfeld: *Gesammelte Schriften*, Vol. IV. Edited by F. Sauter. Friedrich Vieweg & Sohn, Braunschweig, 1968, pp. 45-56.  

[264] A. Sommerfeld: *Atombau und Spektrallinien*, II. Band. Friedrich Vieweg, Braunschweig, 1944. [in German] Reprinted: Verlag Harri Deutsch, Thun, Frankfurt/M., 1978.  

[265] O. F. Fischer: Lorentz transformation and Hamilton’s quaternions. *The London, Edinburgh & Dublin Philosophical Magazine and Journal of Science*, 7. Series, 30(1940)135-150 (DOI: 10.1080/14786444008520703).  

[266] F. B. Estabrook: Nonclassical transformation in special relativity. *Physical Review* 103(1956)1579-1580 (DOI: 10.1103/PhysRev.103.1579).  

[267] W. Greub: Über die Drehungen des vierdimensionalen Raumes, die man durch das Quaternionenprodukt erhält [On the rotations of four-dimensional space obtained by means of the quaternion product]. *Mathematisch-Physikalische Semesterberichte* 8(1962)30-46. [in German]  

[268] P. Rastall: Quaternions in relativity. *Reviews in Modern Physics* 36(1964)820-832 (DOI: 10.1103/RevModPhys.36.820).  

111
[269] A. Zaddach: Fix-Bivektoren bei 4-dimensionalen Drehungen [Fix-bivectors for 4-dimensional rotations]. *Mathematische Semesterberichte* **36**(1989)205-226. [in German] 66

[270] H. A. S. Eriksson: Spinor representation of rotations and Dirac’s equations in five-dimensional space. *Arkiv för Matematik, Astronomi och Fysik* **29** A:14(1943)1-9. 66, 68

[271] J. Kociński: Dirac equation and de Sitter groups SO(4,1) and SO(3,2). Published in: A. E. Chubykalo, V. V. Dvoeglazov, D. J. Ernst, V. G. Kadyshevsky, Y. S. Kim (Eds.): *Proceedings of the International Workshop Lorentz Group, CPT and Neutrinos*, Zacatecas, Mexico, 23-26 June 1999. World Scientific Publishing, Singapore, 2000, pp. 19-26. 66

[272] J. Kociński: De Sitter quasigroups. *International Journal of Theoretical Physics* **41**(2002)231-250 ([DOI: 10.1023/A:1014054705801]). 66

[273] К. В. Никольский [K. V. Nikol’skii]: Геометрия матриц Дирака [Geometriya matrizzes Diraka]/[The geometry of Dirac matrices]. *Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR], (Series) A (Comptes Rendus des Sciences de l’Académie des Sciences de l’URSS, A)* (1930)667-673. [in Russian] 67

[274] К. В. Никольский [K. V. Nikol’skii]: О геометрии уравнения Дирака [O geometrii uravneniya Diraka]/[On the geometry of the Dirac equation]. *Доклады Академии Наук СССР [Doklady Akademii Nauk SSSR], (Series) A (Comptes Rendus de l’Académie des Sciences de l’URSS, A)* (1930)701-708. [in Russian] 67

[275] J. F. Cornwell: *Group Theory in Physics*, 3 Vols.. Techniques of Physics, Vols. 7, 10. Academic Press, London, Vols. 1, 2: 1984, Vol. 3: 1989. 63, 68

[276] R. Gilmore: *Lie Groups, Lie Algebras, and Some of Their Applications*. John Wiley & Sons, New York, 1974. 68

[277] Б. А. Розенфельд [B. A. Rozenfel’d]: *Многомерные Пространства* [Mnyogomernye Prostranstva]. Nauka, Moscow, 1966. [in Russian] 68

[278] H. A. Kastrup: Zur physikalischen Deutung und darstellungstheoretischen Analyse der konformen Transformationen von Raum und Zeit [On the physical interpretation and representation-theoretic analysis of the conformal transformations of space and time]. *Annalen der Physik*, 7. Series, **9**(1962)388-428 ([DOI: 10.1002/andp.19624640706]). [in German] 68

[279] H. A. Kastrup: On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics. *Annalen der Physik*, 8. Series, **17**(2008)631-690 ([arXiv:0808.2730], Preprint DESY 08-107) ([DOI: 10.1002/andp.200810324]). The article is part of the Special Topic

112
Issue: The Minkowski Spacetime of Special Relativity Theory – 100 Years after its Discovery –. *Annalen der Physik*, 8. Series, 17:9-10(2008)613-851. 68

[280] A. P. Hristev: Conformal matrix identities. *Revue Roumaine de Mathématiques Pures et Appliquées* 15(1970)1171-1179. 68

[281] R. F. Sigal: Conformal invariance and the six-dimensional formalism. *International Journal of Theoretical Physics* 11(1974)45-68 (DOI: 10.1007/BF01807936). 68

[282] R. da Rocha, J. Vaz Jr.: Conformal structures and twistors in the paravec tor model of spacetime. *International Journal of Geometric Methods in Modern Physics* 4(2007)547-576 [arXiv:math-ph/0412074v2] (this is a combined version of [arXiv:math-ph/0412074v1], [arXiv:math-ph/0412075], [arXiv:math-ph/0412076]) (DOI: 10.1142/S0219887807002193). 68

[283] E. Arcaute, A. Lasenby, C. Doran: A Representation of Twistors Within Geometric (Clifford) Algebra [arXiv:math-ph/0603037v2]. 68

[284] H. A. S. Eriksson: On a generalization of Dirac’s equation. *Arkiv för Matematik, Astronomi och Fysik* 33 B:6(1946)1-7. 68

[285] H. A. S. Eriksson: Space reflection, time reversal and charge conjugation of spinor fields. *Arkiv för Fysik* 6(1953)349-358. 68

[286] A. J. Bracken, H. A. Cohen: On canonical SO(4,1) transformations of the Dirac equation. *Journal of Mathematical Physics* 10(1969)2024-2032 (DOI: 10.1063/1.1664798). 68

[287] J. Kociński: A five-dimensional form of the Dirac equation. *Journal of Physics A: Mathematical and General* 32(1999)4257-4277 (DOI: 10.1088/0305-4470/32/23/306). 68

[288] E. de Vries: Foldy-Wouthuysen transformations and related problems. *Fortschritte der Physik* 18(1970)149-182 (DOI: 10.1002/prop.19700180402). 68

[289] R.-K. R. Loide: Преобразование Foldy-Wouthuysen для уравнений, связанных с группой de Sittera [Preobrazovanie Foldy-Vautkhoizena dlya uravnenii, svyazannykh s gruppoi de Sittera]. *Teoreticheskaya i Matematicheskaya Fizika* 23(1975)42-50 (The article is freely available online at the Mathnet site: http://mi.mathnet.ru/eng/tmf3742.). English translation: R.-K. R. Loide: Foldy-Wouthuysen transformation for equations connected to the de Sitter group. *Theoretical and Mathematical Physics* 23(1975)336-342 (DOI: 10.1007/BF01038217). 68
[290] P. A. M. Dirac: A remarkable representation of the 3 + 2 de Sitter group. *Journal of Mathematical Physics* 4(1963)901-909 (DOI: 10.1063/1.1704016).

[291] H. A. Buchdahl: On a calculus which reflects $SO(3, 2) \approx Sp(2, R)$. *Tensor*, New Series, 27(1973)329-336.

[292] Dae-Gyu Lee: The Dirac gamma matrices as “relics” of a hidden symmetry?: As fundamental representations of the algebra $sp(4, R)$. *Journal of Mathematical Physics* 36(1995)524-530 (DOI: 10.1063/1.531320).

[293] N. D. Sen Gupta: On the invariance properties of the Dirac equation. *Il Nuovo Cimento*, 10. Series, 36(1965)1181-1216 (DOI: 10.1007/BF02750699).

[294] C. A. Linhares, J. A. Mignaco: SU(4) for the Dirac equation. *Physics Letters* 153B(1985)82-86 (DOI: 10.1016/0370-2693(85)91446-7).

[295] C. A. Linhares, J. A. Mignaco: SU(4) properties of the Dirac equation. *Notas de Física*, CBPF-NF-057/88. Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, 1988 (The report is freely available online at the CBPF site: ftp://ftp2.biblioteca.cbpf.br/pub/apub/1988/nf/nf_zip/nf05788.pdf).

[296] C. A. Linhares, J. A. Mignaco: New symmetries for the Dirac equation. In: H. Falomir, R. E. Gamboa Saraví, P. Leal Ferreira, F. A. Schaposnik (Eds.): *J. J. Giambiagi Festschrift*. World Scientific, Singapore, 1990, pp. 281-296 (preprint version: *Notas de Física*, CBPF-NF-003/90. Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, 1990. The report is freely available online at the CBPF site: ftp://ftp2.biblioteca.cbpf.br/pub/apub/1990/nf/nf_zip/nf00390.pdf).

[297] C. A. Linhares, J. A. Mignaco: SU(4) properties of the Dirac-Kähler equation. *Notas de Física*, CBPF-NF-008/91. Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, 1991 (The report is freely available online at the CBPF site: ftp://ftp2.biblioteca.cbpf.br/pub/apub/1991/nf/nf_zip/nf00891.pdf).

[298] J. A. Mignaco, C. A. Linhares: Lie algebras for the Dirac-Clifford ring. *Journal of Mathematical Physics* 34(1993)2066-2074 (DOI: 10.1063/1.530156, preprint version: *Notas de Física*, CBPF-NF-037/92. Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, 1992. The report is freely available online at the CBPF site: ftp://ftp2.biblioteca.cbpf.br/pub/apub/1992/nf/nf_zip/nf03792.pdf).

[299] K. Scharnhorst: A Grassmann integral equation. *Journal of Mathematical Physics* 44(2003)5415-5449 [arXiv:math-ph/0206006] (DOI: 10.1063/1.1612896).
[300] T. Muir: *A Treatise on the Theory of Determinants*, revised and enlarged by W. H. Metzler. Privately published, Albany, New York, 1930. Longman’s, Green and Co., New York, 1933. Reprints: Dover Books on Advanced Mathematics, Vol. 670; Dover Publications, Inc., New York, 1960. Dover Phoenix Editions; Dover Publications, Inc., New York, 2003. 75

[301] R. Vein, P. Dale: *Determinants and Their Applications in Mathematical Physics*. Applied Mathematical Sciences, Vol. 134. Springer, New York, 1999. 75

[302] C. G. J. Jacobi: De formatione et proprietatibus determinantium [On the formation and the properties of determinants]. *Journal f"ur die Reine und Angewandte Mathematik* 22(1841)285-318 (The article is freely available online at the DigiZeitschriften site: [http://www.digizeitschriften.de/index.php?id=resolveppn&PPN=GDZPPN002142716](http://www.digizeitschriften.de/index.php?id=resolveppn&PPN=GDZPPN002142716) and at the GDZ Document Server [http://gdz.sub.uni-goettingen.de/index.php?id=resolveppn&PPN=GDZPPN002142716](http://gdz.sub.uni-goettingen.de/index.php?id=resolveppn&PPN=GDZPPN002142716). [in Latin] Reprinted in: K. Weierstrass (Weierstraß) (Ed.): *C. G. J. Jacobi’s Gesammelte Werke*, Vol. 3. Georg Reimer, Berlin, 1884, pp. 354-392. Reprint: Chelsea Publishing/American Mathematical Society, New York/Providence, 1969 (The book is freely available online at the Internet Archive site: [http://www.archive.org/details/gesammeltewerke01weiegoog](http://www.archive.org/details/gesammeltewerke01weiegoog) and at the Gallica site of the Bibliothèque Nationale de France: [http://gallica.bnf.fr](http://gallica.bnf.fr)). A German translation of this article is available: C. G. J. Jacobi: *Ueber die Bildung und die Eigenschaften der Determinanten (De formatione et proprietatibus Determinantium)*, edited by P. Stäckel. Ostwald’s Klassiker der exakten Wissenschaften, Vol. 77. Verlag von Wilhelm Engelmann, Leipzig, 1896, pp. 3-49, comments by P. Stäckel on pp. 66-73 (The book is freely available online at the Internet Archive site: [http://www.archive.org/details/ueberdiebildungu00jacouoft](http://www.archive.org/details/ueberdiebildungu00jacouoft)). 75

[303] T. Muir: *The Theory of Determinants in the Historical Order of Development*, 2. ed., Vol. 1. Macmillan and Co., New York, 1906. Reprint (together with the first edition of vol. 2, 1911): Dover Publications, Inc., New York, 1960. (The book is freely available online at the Internet Archive site: [http://www.archive.org/details/theoryofdetermin01muiruoft](http://www.archive.org/details/theoryofdetermin01muiruoft)). 75

[304] J. H. M. Wedderburn: *Lectures on Matrices*. American Mathematical Society, Colloquium Publications, Vol. 17. American Mathematical Society, New York, 1934. 76

[305] A. C. Aitken: *Determinants and Matrices*, 9. rev. ed., University Mathematical Texts. Oliver and Boyd, Edinburgh, 1956 (1. ed. 1939). 76

[306] M. Fiedler: *Speciální Matice a Jejich Použití v Numerické Matematice*. Teoretická Knihnice Inženýra. SNTL - Nakladatelství Technické Literatury, Prague, 1981. [in Czech] English translation: *Special Matrices and Their Applications*
in Numerical Mathematics. Martinus Nijhoff Publishers, Dordrecht, 1986.
Reprint: Dover Books on Mathematics. Dover Publications, Inc., New York, 2008. 76

[307] N. Jacobson: Basic Algebra I. 2. ed.. W. H. Freeman and Company, New York, 1985. 76

[308] P. M. Cohn: Algebra, 3 Vols., 2. ed.. John Wiley & Sons, Chichester, Vol. 1: 1982, Vol. 2: 1989, Vol. 3: 1991. 76

[309] D. L. Boutin, R. F. Gleeson, R. M. Williams: Wedge Theory/Compound Matrices: Properties and Applications. Report NAWCADPAX–96-220-TR, Naval Air Warfare Center Aircraft Division, Patuxent River, MD, USA, 1996. Available from the U.S. National Technical Information Service, 5285 Port Royal Road, Springfield, VA 22161, USA (http://www.ntis.gov). This report is freely available online at the Defense Technical Information Center (DTIC) site: http://oai.dtic.mil/oai/oai?&verb=getRecord&metadataPrefix=html&identifier=ADA320264. 76

[310] U. Prells, M. I. Friswell: Compound matrices and Pfaffians: A representation of geometric algebra; in: L. Dorst, C. Doran, J. Lasenby (Eds.): Applications of Geometric Algebra in Computer Science and Engineering. Birkhäuser, Boston, 2002, pp. 109-118. 76

[311] M. Barnabei, A. Brini, G.-C. Rota: On the exterior calculus of invariant theory. Journal of Algebra 96(1985)120-160 (DOI: 10.1016/0021-8693(85)90043-2). 76

[312] С. Е. Козлов [S. E. Kozlov], М. Ю. Никанорова [M. Yu. Nikanorova]: Геометрия алгебры Лин ортогональной группы $O(\mathbb{R}^4)$ [Geometriya algbery Li ortogonal’noi gruppy $O(\mathbb{R}^4)$]. Записки Научных Семинаров ПОМИ [Zapiski Nauchnych Seminarov POMI] 261(1999)119-124 (The article is freely available online at the Mathnet site: http://mi.mathnet.ru/eng/znsl1092). [in Russian] English translation: S. E. Kozlov, M. Yu. Nikanorova: The geometry of the Lie algebra of the orthogonal group $O(\mathbb{R}^4)$. Journal of Mathematical Sciences (New York) 110(1999)2820-2823 (DOI: 10.1023/A:1015354329606). This article is part of: В. А. Залгаллер [V. A. Zalgaller], Н. Ю. Нетветаев [N. Yu. Netsvetaev] (Eds.): Геометрия и Топология. 4 [Geometriya i Topologiya. 4]. Записки Научных Семинаров ПОМИ [Zapiski Nauchnych Seminarov POMI] 261(1999)1-265. English translation: Journal of Mathematical Sciences (New York) 110:4(1999)2755-2906. 77

[313] P. Budinich, A. Trautman: The Spinorial Chessboard. Trieste Notes in Physics. Springer-Verlag, Berlin, 1988. 78