Height bounds for certain exceptional points in some variations of Hodge structures

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Abstract

We consider smooth projective morphisms $f : X \to S$ of $K$-varieties with $S$ an open curve and $K$ a number field. We establish upper bounds of the Weil height $h(s)$ by $[K(s) : K]$ at certain points $s \in S(K)$ that are "exceptional" with respect to the variation of Hodge structures $R^n(f^\text{an})_*(\mathbb{Q}_{X_{\text{an}}})$, where $n = \dim X - 1$. We work under the assumption that the generic special Mumford-Tate group of this variation is $\text{Sp}(\mu, \mathbb{Q})$, the variation degenerates in a strong fashion over some fixed point $s_0$ of a proper curve that contains $S$, the Hodge conjecture holds, and that what we define as a "good arithmetic model" exists for the morphism $f$ over the ring $\mathcal{O}_K$.

§1 Introduction

§1.1 History

In [And89] Y.André considers an abelian scheme $f : X \to S$, where $S = S' \setminus \{s_0\}$ with $S'$ a smooth connected curve defined over a number field $K$ and $s_0 \in S'(K)$. He then considers a Weil height $h$ on the curve $S'$. He assumes that the generic fiber $X_{\eta}$ is a simple abelian variety of odd dimension $g > 1$, and that the scheme has completely multiplicative reduction at the point $s_0$. He then shows that for any point in the set $\{s \in S(\mathbb{Q}) : \text{End}X_s \not\hookrightarrow M_g(\mathbb{Q})\}$ the height of the point is bounded from above by an effectively computable power of $[K(s) : K] + 1$.

Recently, in [DO20], C.Daw and M.Orr prove an analogous result for the case $g > 1$ under some stronger assumptions, see Theorem 9.1 in [DO20]. Using this height bound together with the Masser-Wustholz isogeny Theorem, in [DO20] and [DO19], they prove unconditionally a so called "Large Galois Orbit conjecture". The existence of large Galois orbits allows them
to prove unconditionally a significant part of the Zilber-Pink conjecture for curves such as $S$ embedded in $\mathcal{A}_2$, i.e. curves whose Zariski closure in the Baily-Borel compactification of $\mathcal{A}_2$ intersect the 0-dimensional stratum of the boundary of the compactification. Namely, they show that there are only finitely many points $s \in S(\overline{\mathbb{Q}}) \subset \mathcal{A}_2(\overline{\mathbb{Q}})$ for which the corresponding abelian surface has quaternionic multiplication or is of the form $E \times CM$. An abelian surface is said to be of the form $E \times CM$ if it is isogenous to $E_1 \times E_2$ where $E_1$ and $E_2$ are non-isogenous elliptic curves only one of which has complex multiplication. Most recently, in [DO21], Daw and Orr employ the same G-function method to establish some cases of the Zilber-Pink conjecture unconditionally for curves such as above in $\mathcal{A}_g$ for $g$ even.

Their method follows the general strategy set out by Pila and Zannier in the breakthrough paper [PZ08]. The Pila-Zannier method mainly rests on comparing two bounds. On the one hand, an upper bound for the number of rational points on a transcendental variety and on the other hand a lower bound for Galois orbits of certain points of interest. Establishing the existence of large Galois orbits seems, at least to the author, to be perhaps the biggest missing piece in solving several problems in the theory of unlikely intersections via the Pila-Zannier method.

With these thoughts in mind, motivated by the formulation of the Zilber-Pink conjecture in the setting of mixed variations of Hodge structures by B.Klingler in [Kli17], we prove height bounds for certain “exceptional” points on a curve $S$ with respect to a geometric variation of Hodge structures over $S^{an}$. The vague term “exceptional point” reflects that assigned to the points we study we have a Hodge structure that has smaller Mumford-Tate group than the generic Mumford-Tate of a variation of Hodge structures supported on the curve $S$. These results of ours depend on the validity of the Hodge conjecture and the existence of what we refer to as “good arithmetic models”.

§1.2 The main Theorem

**Our setting:** Let $K$ be a number field and let $S'$ be a smooth geometrically irreducible complete curve over $K$, $\Sigma_S \subset S'(K)$ a finite set of $K$-point of $S'$, and fix $s_0$ an element of $\Sigma_S$. Let us consider $S$ to be the curve $S' \setminus \Sigma_S$, $X$ a smooth variety over $K$, and let $f : X \to S$ be a smooth projective morphism that is also defined over $K$ and assume that the dimension of the fibers of $f$ is $n$.

For each $i \in \{0, \ldots, 2n\}$ the morphism $f$ defines variations of Hodge structures on the analytification $S^{an}$ of $S$, namely the variations given by $R^if_*^{an}\mathbb{Q}_{X^{an}} \otimes \mathcal{O}_{S^{an}}$. We focus on the variation with $i = n$ and set $\nabla := R^n f_*^{an}\mathbb{Q}_{X^{an}}$. We furthermore assume that there exists a smooth $K$-scheme
$X'$ and a projective morphism $f' : X' \to S'$ such that:

1. $f'$ is an extension of $f$, and

2. $Y = f^{-1}(s_0)$ is a union of transversally crossing smooth divisors $Y_i$ entering the fiber with multiplicity 1.

Let $\Delta \subset S'_{\mathbb{C}}$ be a small disk centered at $s_0$ such that $\Delta^* \subset S'_{\mathbb{C}}$. From work of Katz it is known that the residue at $s_0$ of the Gauss-Manin connection of the relative de Rham complex with logarithmic poles along $Y$ is nilpotent if we have (2) above. From this it follows, by [Ste76] Theorem 2.21, that the local monodromy around $s_0$ acts unipotently on the limit Hodge structure $H^n_{Q_{\text{lim}}}$. By the theory of the limit Hodge structure we then get the weight monodromy filtration $W_{\bullet}$. We let $h := \dim_{\mathbb{Q}} W_0$.

**Main result:** Our main result is the following theorem.

**Theorem 1.1.** Let $S'$, $s_0$, and $f : X \to S$ be as above and all defined over a number field $K$. We assume that the dimension $n$ of the fibers is odd, that the Hodge conjecture holds, and that a good arithmetic model, in the sense of §11, exists for the morphism $f$ over $\mathcal{O}_K$.

For the variation whose sheaf of flat sections is given by $\mathcal{V} := R^n f_{\text{an}}^* \mathcal{Q}_{X_{\text{an}}^n}$ we assume the following hold true:

1. the generic special Mumford-Tate group of the variation is $Sp(\mu, \mathbb{Q})$, where $\mu = \dim_{\mathbb{Q}} V_z$ for any $z \in S_{\mathbb{Z}}$, and

2. $h \geq 2$.

Let $\Sigma \subset S(\overline{\mathbb{Q}})$ be the set of points for which the decomposition $\mathcal{V}_s = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$ of $\mathcal{V}_s$ into simple polarized sub-$\mathbb{Q}$-HS and the associated algebra $D_s := M_{m_1}(D_1) \oplus \cdots \oplus M_{m_r}(D_r)$ of Hodge endomorphisms are such that:

1. $s$ satisfies condition $\star$ in §13, and either

2. $h > \frac{\dim_{\mathbb{Q}} V_j}{[\mathbb{Z}(D_j):\mathbb{Q}]}$ for some $j$, or

3. there exists at least one $D_i$ that is of type IV in Albert’s classification and $h \geq \min\left\{ \frac{\dim_{\mathbb{Q}} V_i}{[\mathbb{Z}(D_i):\mathbb{Q}]} : i \text{ such that } D_i = \text{End}_{\text{HS}}(V_i) \text{ is of type IV} \right\}$.

Then, there exist constants $C_1$, $C_2 > 0$ such that for all $s \in \Sigma$ we have

$$h(s) \leq C_1 [K(s) : K]^{C_2},$$

3
where $h$ is a Weil height on $S'$.

**Remark.** We note that CM-points of the variation will be in the set $\Sigma$ of this 1.1. We can also create concrete examples of possible algebras of Hodge endomorphisms for which the conditions that guarantee $s \in \Sigma$ above can be checked fairly easily, once we have information on the weight monodromy filtration defined by the local monodromy around the point of degeneration $s_0$. We return to this issue in §15.

### §1.3 Organization of the paper- A summary of the proof

We start by reviewing some aspects of the theory of G-functions in §2. The method that André uses to obtain his height bounds hinges on two results from the theory of G-functions. First is the fact that among the relative $n$-periods associated to the morphism $f : X \to S$ there are some that are G-functions. Namely they will be the ones that can be written as $\int_{\gamma} \omega$ where $\gamma \in Im((2\pi i N^*)^n)$ for $z \in \Delta^*$, where $\Delta^*$ and $N^*$ are the aforementioned punctured disc and nilpotent endomorphism. The second main result we will need is a result that can be described as a “Hasse principle” for the values of G-functions. This is what will ultimately allow us to extract height bounds.

We then move on in §3 where we review some standard facts about the structure of the algebra of Hodge endomorphisms of a Hodge structure. After this, in §4 we fix some general notation with the hope of making the exposition easier.

In §5 we address some technical issues that appear later on in our exposition. Namely we consider the isomorphism between algebraic de Rham and singular cohomology for a smooth projective variety $Y/k$, where $k$ is a subfield of $\overline{\mathbb{Q}}$ and $\mathbb{P}^n$. We then move on in §3 where we review some standard facts about the structure of the algebra of Hodge endomorphisms of a Hodge structure. After this, in §4 we fix some general notation with the hope of making the exposition easier.

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In §5 we address some technical issues that appear later on in our exposition. Namely we consider the isomorphism between algebraic de Rham and singular cohomology for a smooth projective variety $Y/k$, where $k$ is a subfield of $\overline{\mathbb{Q}}$.

$$ P^n : H_{DR}^n(Y/k) \otimes_k \mathbb{C} \to H^n(Y^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. $$

The singular cohomology is endowed with a Hodge structure and we consider its algebra of Hodge endomorphisms $D$. Later on we will want to create splittings of both de Rham and singular cohomology with respect to actions of $D$ on these vector spaces. To do that we will need to have an action of $D$ on $H_{DR}^n(Y/k)$, which a priori we do not. We show that assuming the absolute Hodge conjecture we may base change $Y$ by a finite extension $L$ of $k$ to obtain such an action that will be compatible with the action of $D$ on $H^n(Y^{an}, \mathbb{Q})$ via the isomorphism $P^n$. We also show that this field extension may be chosen so that its degree is bounded from above only in terms of the dimension $\dim_{\mathbb{Q}} H^n(Y^{an}, \mathbb{Q})$. We believe these results are known to experts.
in the field, however being unable to find a reference for these arguments we include them here for the sake of completeness.

Our next goal, realized in §7, is to describe the trivial relations among those relative $n$-periods associated to the morphism $f$ which are G-functions. This amounts to describing the polynomials defining the $\mathbb{Q}[x]$-Zariski closure of a certain $h \times \mu$ matrix, where $x$ here is a local parameter of $S'$ at the point $s_0$. This is achieved by a monodromy argument using André’s Theorem of the Fixed part.

The next part of our exposition, mainly §8, consists of creating relations among the values of the G-functions in question at certain exceptional points that are “non-trivial”. That means that these do not come from specializing the trivial relations we described earlier.

The last part of our exposition is dedicated to showing that the relations we created are “global”, see §2.1 for the term. To achieve this we need to assume the existence of certain good arithmetic models. We discuss these models in §11.

To achieve this we first study the relation between the algebra of Hodge endomorphisms $D_s = \text{End}_{HS}(H^n(X^m, \mathbb{Q}))$ and the algebra of inertia-invariant endomorphisms of the étale cohomology group $H^a_{\acute{e}t}(\tilde{X}_{s,v}, \mathbb{Q}_l)$. In §10.2 we prove that assuming the Hodge conjecture the former algebra naturally injects in the latter.

This forces an interplay between the algebra of Hodge endomorphisms and the endomorphisms of the graded quotients of the monodromy filtration of $H^a_{\acute{e}t}(\tilde{X}_{s,v}, \mathbb{Q}_l)$. Taking advantage of this interplay we establish conditions in §13 that guarantee the impossibility of the point $s$ being $v$-adically close to the degeneration $s_0$. Establishing this rests on the comparison between the local monodromy representation and the representation defined by inertia which follows from the theorem on the Purity of the branch locus. To employ this comparison we need to assume the existence of the arithmetic models of §11.

After this we put all the aforementioned ideas together in §14. In summary, the relations created in §8 are shown to be non-trivial and global, under the aforementioned conditions. Applying the “Hasse Principle” for the values of G-functions, we obtain the height bounds we want.

We finish with a section centered around examples of algebras where 1.1 applies. In particular we study the CM-points of variations satisfying the conditions of 1.1 and establish that these are in fact points of the set $\Sigma$.

We have also included an appendix on polarizations. The main result we need about polarizations in the text is a description of the relations they define among the $n$-periods. This description in the case where the weight of the Hodge structures is 1 already appears in [And89]. The description in
the case of arbitrary odd weight is not different at all. We include it in this appendix for the sake of completeness.

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Part I
Background material

§2 A short review of G-functions in Arithmetic Geometry

G-functions were first introduced by Siegel in [Sie14]. We start with a short review of G-functions and we list some of their main properties.

Definition. Let $K$ be a number field and let $y = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$. Then $y$ is called a **G-series at the origin** if the following are true:

1. $\forall v \in \Sigma_{K,\infty}$ we have that $i_v(y) \in C_v[[x]]$ defines an analytic function around 0,

2. there exists a sequence $(d_n)_{n \in \mathbb{N}}$ of natural numbers such that
   - $d_n a_m \in \mathcal{O}_K$ for all $m \leq n$,
   - there exists $C > 0$ such that $d_n \leq C^n$ for all $n \in \mathbb{N}$,

3. $y$ satisfies a linear homogeneous differential equation with coefficients in $K(x)$.

Examples of G-series at the origin are elements of $\bar{\mathbb{Q}}(x)$ without a pole at 0, the expansion of $\log(1 + x)$ at 0, and any element of $\widebar{\mathbb{Q}}[[x]]$ which is algebraic over $\bar{\mathbb{Q}}(x)$.

We note, see [DGS94], that we can naturally define “G-series at $\zeta$”, for any $\zeta \in \mathbb{C}$. We also remark that the number field $K$ can be replaced by $\bar{\mathbb{Q}}$ without problems thanks to the third condition, which implies that the $a_i$ are all in some finite extension of $\bar{\mathbb{Q}}$. Finally, we note that the set of G-series at $\zeta$ forms a ring.
Definition. A **G-function** is a multivalued locally analytic function $y$ on $\mathbb{C}\setminus S$, with $|S| < \infty$, such that for some $\zeta \in \mathbb{C}\setminus S$, $y$ can be represented by a $G$-series at $\zeta$.

Thanks to the Theorem of Bombieri-André and the Theorem of Chudnovsky we know that the global nature of a G-function is in fact very much dependent on the fact that it can be locally written as a G-series. That is why, essentially following [And89], we identify the two notions, especially since we will be only interested at power series centered at the origin.

For more on G-functions we direct the interested reader to the excellent introductory text [DGS94] and the more advanced [And89].

§2.1 A Hasse Principle for G-functions

The main tool we will need from the theory of G-functions is a theorem of André, that generalizes work of Bombieri in [Bom81], which plays the role of a “Hasse Principle” for G-functions. First we need some definitions. For the rest of this section consider $y_0, \ldots, y_{m-1}$ to be G-functions with coefficients in some number field $K$. We also define $Y := (y_0, \ldots, y_{m-1}) \in K[[x]]^m$, we fix some homogeneous polynomial $p \in K[t_1, \ldots, t_m]$, and a $\xi \in K$.

**Definition.** 1. We say that a relation $p(y_0(\xi), \ldots, y_{m-1}(\xi)) = 0$ holds $v$-adically for some place $v$ of $K$ if

$$i_v(p)(i_v(y_0)(i_v(\xi)), \ldots, i_v(y_{m-1})(i_v(\xi))) = 0.$$  

2. A relation like that is called **non-trivial** if it does not come by specialization at $\xi$ from a homogeneous relation of the same degree with coefficients in $K[x]$ among the $y_i$. Respectively, we call it **strongly non-trivial** if it does not occur as a factor of a specialization at $\xi$ of a homogeneous irreducible relation among the $y_i$ of possibly higher degree.

3. A relation $p(y_0(\xi), \ldots, y_{m-1}(\xi)) = 0$ is called **global** if it holds $v$-adically for all places $v$ of $K$ for which $|\xi|_v < \min\{1, R_v(Y)\}$.

**Theorem 2.1** (Hasse Principle for G-functions,[And89], Ch VII, §5.2). Assume that $Y \in \bar{\mathbb{Q}}[[x]]^m$ satisfies the differential system $\frac{d}{dx}Y = \Gamma Y$ where $\Gamma \in M_m(\bar{\mathbb{Q}}(x))$ and that $\sigma(Y) < \infty$. Let $\Pi_\delta(Y)$, resp. $\Pi'_\delta(Y)$, denote the set of ordinary points or apparent singularities $\xi \in \bar{\mathbb{Q}}^*$ where there is some non-trivial, resp. strongly non-trivial, and global homogeneous relation of degree $\delta$.

Then,
\[ h(\Pi_\delta(Y)) \leq c_1(Y)\delta^{3(m-1)}(\log \delta + 1), \text{ and} \]
\[ h(\Pi'_\delta(Y)) \leq c_2(Y)\delta^m(\log \delta + 1). \]

In particular any subset of \( \Pi_\delta(Y) \) with bounded degree over \( \mathbb{Q} \) is finite.

Remark. The quantity \( \sigma(Y) \) is called the size of \( Y \). G-functions have finite size. \(^1\)

\section{2.2 Periods and G-functions}

Our primary interest in the theory of G-functions stems from the connection between G-functions and relative periods. We give a brief review of the results in \cite{And89} that highlight this connection together with some basic facts and definitions that we will use later on.

Let \( T \) be a smooth connected curve over some number field \( k \subset \mathbb{C} \), \( S = T \setminus \{ s_0 \} \), where \( s_0 \in T(k) \) is some closed point, and let \( x \) be a local parameter of the curve \( T \) at \( s_0 \).

We also consider \( f : X \to S \) a proper smooth morphism and we let \( n = \dim X - 1 \). We then have the following isomorphism of \( \mathcal{O}_S \)-modules

\[ P^*_{X/S} : H^*_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^an} \to R^*f^*_an\mathbb{Q}_{X^an} \otimes_{\mathbb{Q}_{S^an}} \mathcal{O}_{S^an}. \]

In what follows we will be focusing on the isomorphism \( P^*_{X/S} \), which from now on we will simply denote by \( P_{X/S} \). We also let \( \mu = \dim_{\mathbb{Q}} H^n(X^an, \mathbb{Q}) \) where \( z \in S(\mathbb{C}) \).

This isomorphism is the relative version of Grothendieck’s isomorphism between algebraic de Rham and Betti cohomology and it can be locally represented by a matrix. Namely, if we choose a basis \( \omega_i \) of \( H^n_{DR}(X/S) \) over some affine open subset \( U \subset S \) and a frame \( \gamma_j \) of \( R^*f^*_an\mathbb{Q}_{X^an} \) over some open analytic subset \( V \) of the analytification \( U^an_C \), \( P_{X/S} \) is represented by a matrix with entries of the form \( \int_\gamma \omega_i \).

Definition. We define the relative \( n \)-period matrix (over \( V \)) to be the \( \mu \times \mu \) matrix

\[ \left( \frac{1}{(2\pi i)^m} \int_\gamma \omega_i \right). \]

Its entries will be called the relative \( n \)-periods.

\(^1\)For this fact and the definition of the notion of “size” of a power series see \cite{And89} Chapter I.
A result we will need in what follows guarantees the existence of G-functions among the relative $n$-periods under the hypothesis that the morphism $f$ extends over all of $T$. Namely, let us assume $f$ extends to a projective morphism $f_T: X_T \to T$ with $X_T$ a smooth $k$-scheme, such that $Y := f^{-1}(s_0)$ is a union of smooth transversally crossing divisors $Y_i$ entering the fiber with multiplicity $1$.

Under these assumptions we know, see [PS08] Corollary 11.19, that the local monodromy is unipotent. Let $\Delta$ be a small disk embedded in $T^{an}$ and centered around $s_0$. We let $2\pi i N^*$ be the logarithm of the local monodromy acting on the sheaf $R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*}$.

**Definition.** We denote the image of the map $(2\pi i N^*)^n$ by $M_0 R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*}$. We call $M_0$-period any relative $n$-period over a cycle $\gamma$ in $M_0 R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*}$.

By the formalism of the limit Hodge structure we have that for all $z \in \Delta^*$ the group $\pi_1(\Delta^*, z)$ acts unipotently on the fiber $(R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*})_z$. We also get that, letting $2\pi i N^*_z$ be the nilpotent logarithm of the image of a generator of $\pi_1(\Delta^*, z)$ via the monodromy representation, $(M_0 R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*})_z = \text{Im}((2\pi i N^*_z)^n)$.

**Theorem 2.2** ([And89], p.185). There exists a basis of sections $\omega_i$ of $H^0_{DR}(X/S)$ over some dense open subset of $S$, such that for any section $\gamma$ of $M_0 R_n(f^{an})_*(\mathbb{Q})|_{\Delta^*}$, the Taylor expansion in $x$ of the relative $M_0$-periods $\frac{1}{(2\pi i)^n} \int_\gamma \omega_i$ are globally bounded G-functions.

**Remark.** We may assume without loss of generality that the G-functions created have coefficients in $k$, when $k \subset \bar{\mathbb{Q}}$. For more on this see the proof of 9.1.

### §3 Endomorphism algebras of Hodge Structures

One of the central notions we will employ in what follows are the endomorphism algebras of polarized $\mathbb{Q}$-Hodge structures of pure weight. We present here a quick review of the main facts we will need later on about the structure of these algebras, as well as a few standard definitions and notation on Hodge-theoretic notions that we will use.

Given a $\mathbb{Q}$-Hodge structure, or $\mathbb{Q}$-HS for short, of pure weight we also get a group homomorphism $\hat{\varphi}: S \to GL(V)_\mathbb{R}$ of $\mathbb{R}$-algebraic groups, where $S$ is the Deligne torus. Let $U_1$ be the $\mathbb{R}$-subtorus of $S$ with $U_1(\mathbb{R}) = \{ z \in \mathbb{C}^*: |z| = 1 \}$ and let $\varphi := \hat{\varphi}|_{U_1}$.
Definition. Let $V$ be a pure weight $\mathbb{Q}$-HS and $\tilde{\varphi}$ and $\varphi$ be as above. The Mumford-Tate group of $V$, denoted by $G_{\text{mt}}(V)$, is defined as the $\mathbb{Q}$-Zariski closure of $\tilde{\varphi}(S(\mathbb{R}))$. The special Mumford-Tate group of $V$, denoted by $G_{\text{smt}}(V)$, is defined as the $\mathbb{Q}$-Zariski closure of $\varphi(U_1(\mathbb{R}))$.

Irreducible Hodge Structures: Albert’s Classification

It is well known that the category of polarizable $\mathbb{Q}$-Hodge structures is semi-simple. This implies that for a polarizable $\mathbb{Q}$-HS $V$, its endomorphism algebra $D := \text{End}(V)^{G_{\text{mt}}(V)}$ is a semi-simple $\mathbb{Q}$-algebra. If, furthermore, the polarizable $\mathbb{Q}$-HS $V$ that we consider is simple, then $D$ is a simple division $\mathbb{Q}$-algebra equipped with a positive involution, naturally constructed from the polarization. Such algebras are classified by Albert’s classification.

Theorem 3.1 (Albert’s Classification, [Mum08]). Let $D$ be a simple $\mathbb{Q}$-algebra with a positive (anti-)involution $\iota$, denoted $a \mapsto a^\dagger$. Let $F = \mathbb{Z}(D)$, be the center of $D$, $F_0 = \{ a \in F : a = a^\dagger \}$, $e_0 = [F_0 : \mathbb{Q}]$, $e = [F : \mathbb{Q}]$, and $d^2 = [D : F]$. Then $D$ is of one of the following four types:

**Type I:** $D = F = F_0$ is a totally real field, so that $e = e_0$, $d = 1$, and $\iota$ is the identity.

**Type II:** $D$ is a quaternion algebra over the totally real field $F = F_0$ that also splits at all archimedean places of $F$. If $a \mapsto a^* = tr_{D/F}(a) - a$ denotes the standard involution of this quaternion algebra, then there exists $x \in D$ with $x = -x^*$ such that $a^\dagger = xax^{-1}$ for all $a \in D$. Finally, in this case $e = e_0$ and $d = 2$.

**Type III:** $D$ is a totally definite\(^2\) quaternion algebra over the totally real field $F = F_0$. In this case $\iota$ is the standard involution of this quaternion algebra and as before $e = e_0$ and $d = 2$.

**Type IV:** $D$ is a division algebra of rank $d^2$ over the field $F$, which is a CM-field with totally real subfield $F_0$, i.e. $e = 2e_0$. Finally, the involution $\iota$ corresponds, under a suitable isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_d(\mathbb{C}) \times \ldots \times M_d(\mathbb{C})$, with the involution $(A_1, \ldots, A_{e_0}) \mapsto (\text{tr} A_1, \ldots, \text{tr} A_{e_0})$.

\(^2\)We remind the reader that a quaternion algebra $B$ over a number field $F$ is called totally definite if for all archimedean places $v \in \Sigma_{F, \infty}$ we have that the algebra $B$ is ramified at $v$. This requires that $F$ is totally real so that $B \otimes_F F_v \simeq \mathbb{H}$, with $\mathbb{H}$ the standard quaternion algebra over $\mathbb{R}$, for all $v \in \Sigma_{F, \infty}$.
Furthermore, in this case we have that for $\sigma$ a generator of $\text{Gal}(F/F_0)$ the following must hold:

1. if $v \in \Sigma_{F,f}$ is such that $\sigma(v) = v$ we have that $\text{inv}_v(D) = 0$, and
2. for all $v \in \Sigma_{F,f}$ we must have that $\text{inv}_v(D) + \text{inv}_{\sigma(v)}(D) = 0$.

The general case

Let $(V, \phi)$ be a polarized $\mathbb{Q}$-HS of weight $n$. Then, combining the semi-simplicity of the category of polarized $\mathbb{Q}$-HS and 3.1 we get a good description of the endomorphism algebra $D = \text{End}(V)^{G_{\text{sm}}(V)}$.

Indeed, we know that there exist simple polarized weight $n$ sub-$\mathbb{Q}$-Hodge structures $(V_i, \varphi_i)$ with $1 \leq i \leq r$, such that $V_i \not\cong V_j$ for all $i \neq j$ and we have a decomposition

$$V = V_1^{m_1} \oplus \ldots \oplus V_r^{m_r}. \tag{1}$$

Denoting by $D_i := \text{End}(V_i)^{G_{\text{sm}}(V_i)}$ the corresponding endomorphism algebras and by $F_i := Z(D_i)$ their respective centers, we then have a decomposition

$$D = M_{m_1}(D_1) \times \ldots \times M_{m_r}(D_r). \tag{2}$$

Finally, this implies that the center $F$ of $D$ is such that

$$F = F_1 \times \ldots \times F_r, \tag{3}$$

were each $F_i$ is diagonally embedded into $M_{m_i}(D_i)$, and the maximal commutative semi-simple sub-algebra $E$ of $D$ may be written as

$$E = F_1^{m_1} \times \ldots \times F_r^{m_r}. \tag{4}$$

For a proof of Albert’s classification see [Mum08], §21. For more on Mumford-Tate groups we direct the interested reader to our sources for this section, which are mainly [Moo99] and [GGK12].

§4 The main setting-notational conventions

Before delving into the technical parts of our argument we devote this section on describing the general setting that we will be working on in more detail. We give the definitions of the main objects and introduce the notation that we will, unless otherwise stated, keep uniform throughout our exposition.
Let \( S' \) be a smooth proper geometrically irreducible curve over some number field \( K \leq \bar{\mathbb{Q}} \), let \( \Sigma_S \subset S'(K) \) be a finite set of \( K \)-points and \( s_0 \in \Sigma_S \) be a fixed such point. We let \( S = S' \setminus \Sigma_S \) be the complement of \( \Sigma_S \) in \( S' \). We also fix \( x \) a local parameter of the curve \( S' \) at \( s_0 \) and \( \eta \) the generic point of \( S \).

Let us consider \( f : X \to S \) a smooth projective morphism and let \( n = \dim X - 1 \). Assume \( f \) extends to a projective morphism \( f' : X' \to S' \) with \( X' \) a smooth \( K \)-scheme and that \( Y = f^{-1}(s_0) \) is a simple normal crossings divisor.

The map \( f \) defines a variation of polarized \( \mathbb{Q} \)-HS of weight \( n \) over \( S \) given by \( R^n f'^* \mathcal{Q}_{X'^{an}} \). We denote by \( G_{mt,p} \), respectively by \( G_{smt,p} \), the Mumford-Tate group, or respectively the special Mumford-Tate group, associated to the \( \mathbb{Q} \)-HS associated to the point \( p \in S(\mathbb{C}) \). We also let \( G_{mt,\eta} \), respectively \( G_{smt,\eta} \), be the generic Mumford-Tate group, or respectively the generic special Mumford-Tate group, of the variation. For each \( p \in S(\mathbb{C}) \) we also let \( V_p = H^n(X_p^{an}, \mathbb{Q}) \) be the fiber of the local system \( R^n f'^* \mathcal{Q}_{X'^{an}} \) and let \( \mu = \dim_\mathbb{Q} V_p \).

Consider \( z \in S(\mathbb{C}) \) to be a Hodge generic point for the above variation of \( \mathbb{Q} \)-HS. The main invariant of the variation we will be interested in is the \( \mathbb{Q} \)-algebra

\[
D := \text{End}(V_z)^{G_{smt,z}} = \text{End}(V_z)^{G_{smt,\eta}}.
\]

Similarly, for \( s \in S(\mathbb{C}) \) we let

\[
D_s := \text{End}(V_s)^{G_{smt,s}}.
\]

**Definition.** Let \( X, S, s \in S(\mathbb{C}) \), \( D_s \), and \( D \) be as above. We call \( D_s \) the **algebra of Hodge endomorphisms at** \( s \).

**Definition.** A variation of Hodge structures such as above, meaning a weight \( n \) geometric variation of \( \mathbb{Q} \)-HS parameterized by \( S \) whose degeneration at some \( s_0 \in \Sigma_S \subset S' \) is as above, with \( S = S' \setminus \{ \Sigma_S \} \), with all of the above defined over some number field \( K \), will be called **\( G \)-admissible**.

**Remark.** We remark that under these assumptions 2.2 applies by letting \( T = S' \setminus (\Sigma_S \setminus \{s_0\}) \) and \( f_T \) be the pullback of \( f' \) over \( T \). In particular we have the existence of \( G \)-functions among the entries of the relative period matrix as described in §2.2.

**Notation:** We fix some notation that appears throughout the text. By \( \Sigma_K, \Sigma_{K,f}, \Sigma_{K,\infty} \) we denote the set of all places of a number field \( K \), respectively finite or infinite places of \( K \). For \( v \in \Sigma_K \) we let \( i_v : K \to \mathbb{C}_v \) denote...
the inclusion of $K$ into $C_v$. For $y \in K[[x]]$ we let $i_v(y)$ denote the element of $C_v[[x]]$ given via $i_v$ acting coefficient-wise on $y$.

For a scheme $Y$ defined over a field $k$ we let $\bar{Y} := Y \times_{\text{Spec } k} \text{Spec } \bar{k}$ and $Y_L := Y \times_{\text{Spec } k} \text{Spec } L$ for any extension $L/k$.

§5 Hodge Endomorphisms and De Rham Cohomology

Let $K$ be a number field and $f : X \to S$ be a smooth projective $K$-morphism of $K$-varieties, with $S$ a curve as above. Let us also consider a point $s \in S(L)$ for some finite extension $L/K$ and set $Y := X_s$ which is a smooth projective variety defined over $L$.

In what follows we will need the existence of a natural action of the algebra of Hodge endomorphisms of $H^n(Y, \mathbb{Q})$ on both sides of the comparison isomorphism

$$P^n : H^n_{\text{DR}}(\bar{Y}/L) \otimes_L \mathbb{C} \to H^n(\bar{Y}^\text{an}_\mathbb{C}, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C},$$

such that these actions commute with this isomorphism.

In the case of abelian varieties this is automatic from the fact that the algebra of Hodge endomorphisms is naturally realized as the algebra of endomorphisms of the abelian variety. This in turn acts naturally on both sides of the comparison isomorphism and the actions commute with the isomorphism itself. In a general variety $Y$ we cannot hope for such a description without assuming the validity of the absolute Hodge Conjecture.

It is the author’s belief that the results in this section are known to experts in the field. Since we were not able to find an exact reference of the results we needed we have dedicated this section in providing proofs for these results.

§5.1 Existence of the action

For the rest of this subsection we fix a number field $L$ and a smooth projective $n$-dimensional variety $Y$ defined over $L$.

Proposition 5.1. Let $Y$ be a smooth projective variety over the number field $L$ of dimension $n$. Let $V := H^n(\bar{Y}^\text{an}_\mathbb{C}, \mathbb{Q})$ and $D := \text{End}_{HS}(V)$ be the algebra of Hodge endomorphisms. Then, assuming the absolute Hodge Conjecture, there exists a finite Galois extension $\hat{L}$ of $L$ such that there exists an injective homomorphism of algebras

$$i : D \hookrightarrow \text{End}_{\hat{L}}(H^n_{\text{DR}}(Y/L) \otimes_L \hat{L}).$$
Moreover, we have that $P^n(i(d)v) = d \cdot P^n(v)$ for all $d \in D$ and all $v \in H^n_{DR}(\bar{Y}/\bar{L}) \otimes_L \mathbb{C}$. In other words, the action of the algebra $D$, that is induced by $i$, on de Rham cohomology coincides with the usual action of $D$ on the Betti cohomology as endomorphisms of the Hodge structure under the comparison isomorphism $P^n$.

Proof. We start with some, well known, observations. First of all, the natural isomorphism $\alpha_0 : \text{End}_\mathbb{Q}(V) \cong V \otimes V^*$ is an isomorphism of $\mathbb{Q}$-HS. In particular, via $\alpha_0$ the elements of $D$ correspond to Hodge classes.

It is also known that the isomorphism $\alpha : H^n(\bar{Y}_C^\text{an}, \mathbb{Q})^* \to H^n(\bar{Y}_C^\text{an}, \mathbb{Q})(n)$, given by Poincaré duality, is an isomorphism of $\mathbb{Q}$-HS. As a consequence we get that the induced isomorphism

$$\alpha_1 : V \otimes_\mathbb{Q} V^* \cong (V \otimes_\mathbb{Q} V)(n)$$

is also an isomorphism of $\mathbb{Q}$-HS. Moreover, it is known that the injection $\alpha_2 : (H^n(\bar{Y}_C^\text{an}, \mathbb{Q}) \otimes_\mathbb{Q} H^n(\bar{Y}_C^\text{an}, \mathbb{Q}))(n) \hookrightarrow H^{2n}(\bar{Y}_C^\text{an} \times \bar{Y}_C^\text{an}, \mathbb{Q})(n)$, given by the Künneth formula is also an injective homomorphism of $\mathbb{Q}$-HS.

**Step 1: Reduction to the algebraic closure:** Let us start by fixing a basis $\beta$ of $D$ over $\mathbb{Q}$. By the above remarks, for each $d \in \beta$ we get a Hodge class $\phi_d := \alpha_2 \circ \alpha_1 \circ \alpha_0(d) \in H^{2n}(\bar{Y}_C^\text{an} \times \bar{Y}_C^\text{an}, \mathbb{Q})(n)$.

Now, assuming the absolute Hodge Conjecture, from Corollary 11.3.16 of [CS14] such a class $\phi_d$ has to be defined over the algebraically closed field $\bar{L}$, i.e. $\phi_d = P_{X,Y}^{2n}(\tilde{\phi}_d)$ where $\tilde{\phi}_d \in H^{2n}_{DR}(\bar{Y} \times L \bar{Y}/\bar{L})(n)$.

**Step 2: Reduction to finite extension:** Let us set $Z := Y \times_L Y$. We have an $L$-vector space $H^{2n}(Z/L)$ and we also have an isomorphism

$$H^{2n}_{DR}(Z/L) \otimes_L F \cong H^{2n}_{DR}(Z_F/F)$$

for every extension $F/L$. In particular (6) holds for $F = \bar{L}$.

If we consider $\delta := \{\delta_1, \ldots, \delta_m\}$ the image of an $L$-basis of $H^{2n}_{DR}(Z/L)$ in $H^{2n}_{DR}(Z/L)$ under the above isomorphism. For $\phi_d$ as above we may write

$$\tilde{\phi}_d = a_1(d)\delta_1 + \cdots + a_m(d)\delta_m.$$
Given these coefficients, we set $L_d$ to be the field $L(a_1(d), \cdots, a_m(d))$, which is a finite extension of $L$. Finally, we let $\hat{L}$ be the Galois closure of the compositum of the $L_d$ for all $d \in \beta$.

We observe that for any Galois extensions $F_1, F_2$ of the field $L$ with $L \subset F_1 \subset F_2$ the diagram

$$
\begin{array}{ccc}
H^n_{DR}(Z/L) \otimes L F_1 & \otimes F_1 F_2 & H^n_{DR}(Z_{F_1}/F_1) \otimes F_1 F_2 \\
\downarrow & & \downarrow \\
H^n_{DR}(Z/L) \otimes L F_2 & \otimes F_2 F_2 & H^n_{DR}(Z_{F_2}/F_2)
\end{array}
$$

is a commutative diagram of $\text{Gal}(F_2/L)$-modules. As a consequence of this, we may and do view from now on each $\tilde{\phi}_d$ as an element of $H^n_{DR}(\hat{L}/\hat{L})$.

**Step 3: Back to endomorphisms:** So far we have found classes $\tilde{\phi}_d \in H^n_{DR}(Z_L/\hat{L})$. We want to show that these naturally correspond to endomorphisms of $H^n_{DR}(Y_L/\hat{L})$ and that this correspondence behaves well with respect to the comparison isomorphism of Grothendieck.

To that end, we start by noting that Grothendieck’s comparison isomorphism between algebraic de Rham cohomology and Betti cohomology is compatible with the isomorphisms given by both Poincaré duality and the Künneth formula. We note that both of these, i.e. Poincaré duality and the Künneth formula, are defined for both cohomology theories in question, in fact for de Rham cohomology they are defined over $L$.

With that in mind we define $\alpha_{i,DR}$ mirroring the homomorphisms $\alpha_i$ we had earlier.

Therefore for every $d \in \beta$, viewing the class $\tilde{\phi}_d$ as an element of $H^n_{DR}(Z_{\hat{C}}/\hat{C})$, due to the aforementioned compatibility, we get an element $\tilde{d} \in \text{End}_C(H^n_{DR}(\hat{Y}_{\hat{C}}/\hat{C}))$ which is such that

1. it maps to $d$ via the comparison isomorphism, and
2. it maps to $\tilde{\phi}_d$ via the injective map $\alpha_{2,DR} \circ \alpha_{1,DR} \circ \alpha_{0,DR}$.

Property (1) above tells us that $P^n(\tilde{d}(v)) = d(P^n(v))$ for all $v \in H^n_{DR}(\hat{Y}_{\hat{C}}/\hat{C})$. Thus proving the “moreover” part of the proposition.

Since $Y$ is defined over the field $L$ the same is true for the $\alpha_{i,DR}$. In particular since their composition $\alpha_{DR} := \alpha_{2,DR} \circ \alpha_{1,DR} \circ \alpha_{0,DR}$ is an injective homomorphism

$$
\alpha_{DR} : \text{End}(H^n_{DR}(Y_L/\hat{L})) \hookrightarrow H^n_{DR}(Y_L \times_L Y_{\hat{L}}/\hat{L}),
$$

15
we get that in fact $\tilde{d} \in \text{End}(H^n_{DR}(Y_L/\hat{L}))$.

Since $d$ was a random element in a $\mathbb{Q}$-basis of $D$ we get an injective homomorphism

$$i : D \hookrightarrow \text{End}(H^n_{DR}(Y_L/\hat{L})) \cong \text{End}(H^n_{DR}(Y_L/L) \otimes_L \hat{L}).$$

(8)

Finally, because of the above comments, we also get that $i$ satisfies the “moreover” part of the proposition.

\[\square\]

§5.1.1 Bounds on the degree extension

Later on we want to have some control on the degree of the Galois extension $\hat{L}/L$ constructed in the proof of 5.1. In particular, we want an upper bound on the degree $[\hat{L} : L]$ that will be independent of the smooth projective variety $Y/L$ and the field $L$ itself. We want this bound to only depend on the dimension of $Y$ and its $n$-th Betti number. In making an analogy with the case of abelian varieties, we want upper bounds akin to those achieved in [Sil92].

**Proposition 5.2.** Assume the absolute Hodge Conjecture is true. Let $Y$ be a smooth $n$-dimensional projective variety defined over the number field $L$. Then the field extension $\hat{L}/L$ constructed in 5.1 may be chosen so that for its degree we have

$$[\hat{L} : L] \leq ((6.31)m^2)m^2,$$

where $m = \dim_{\mathbb{Q}} H^n(\hat{Y}_C^{an}, \mathbb{Q})$ is the $n$-th Betti number.

**Proof.** Let $\beta$ be a $\mathbb{Q}$-basis of $D$. From the proof of 5.1 we have an injective homomorphism of $\mathbb{Q}$-algebras $D \hookrightarrow \text{End}_L(H^n_{DR}(Y_L/\hat{L}))$, given in the basis elements by $d \rightarrow \tilde{d}$ in the notation of the proof of 5.1.

By base change we have a natural action of the finite Galois group $\text{Gal}(\hat{L}/L)$ on de Rham cohomology $H^n_{DR}(\hat{Y}_L/\hat{L})$, as an $L$-vector space. This induces a natural action of the same group on $\text{End}_L(H^n_{DR}(Y_L/\hat{L}))$, viewed as an $L$-vector space again. We start by proving the following claim.

**Claim:** The above action of the Galois group induces an action on the embedding of $D$ in $\text{End}_L(H^n_{DR}(Y_L/\hat{L}))$. In other words for all $\sigma \in \text{Gal}(\hat{L}/L)$ we have that $\sigma(D) = D$.

**Proof of the claim.** Assuming the absolute Hodge Conjecture, by our earlier construction, for every element $d$ of the basis $\beta$ we get an element $\tilde{d}$ =
$i(d) \in \text{End}_L(H_{DR}^n(Y_L/\hat{L}))$. By the previous proof, via Poincaré duality and the Künneth formula, we get classes $\hat{\phi}_d \in H_{DR}^{2n}(Y_L \times Y_L/\hat{L})$ that map to Hodge classes $\phi_d \in H^{2n}(Y^{an} \times Y^{an}, \mathbb{Q})$. As we did in our earlier proof we let $Z := Y \times_L Y$. In the above construction we implicitly consider a fixed embedding $\sigma_0 : \hat{L} \hookrightarrow \mathbb{C}$.

By our assumption that the absolute Hodge Conjecture holds true, we get that for all embeddings $\sigma : \hat{L} \hookrightarrow \mathbb{C}$ the class $\hat{\phi}_d \in H^{2n}(\sigma(Z_L)/\mathbb{C})$ is Hodge. Here $\sigma(Z_L)$ denotes the complex variety obtained from $Z_L$ when we base change via the embedding $\sigma$ to $\mathbb{C}$.

From the embedding $\sigma_0 : \hat{L} \hookrightarrow \mathbb{C}$ that we fixed earlier we get an embedding $\iota_0 : L \hookrightarrow \mathbb{C}$. Any embedding $\sigma : \hat{L} \hookrightarrow \mathbb{C}$ that is such that $\sigma|_L = \iota_0$ will correspond to an element of the Galois group $\text{Gal}(\hat{L}/L)$ via the bijective map $\text{Gal}(\hat{L}/L) \to \{\sigma : \hat{L} \hookrightarrow \mathbb{C} : \sigma|_L = \iota_0\}$ given by $\tau \mapsto \sigma_0 \circ \tau$. For notational brevity we suppress $\sigma_0$ from our notation from now on and identify $\tau \in \text{Gal}(\hat{L}/L)$ with $\sigma_0 \circ \tau$, in other words we identify the elements of $\text{Gal}(\hat{L}/L)$ with the corresponding embedding $\hat{L} \hookrightarrow \mathbb{C}$. With this notational convention we may and will write from now on $Y_\mathbb{C}$, or $Z_\mathbb{C}$ respectively, for the complex variety we would otherwise denote by $\sigma_0 Y_L$, or $\sigma_0 Z_L$ respectively.

For the above $\sigma$, since $Y$ and hence also $Z$ are defined over the field $L$, by the above remarks $H_{DR}^n(\sigma Z_L/\mathbb{C})$ may be identified with $H_{DR}^n(Z_\mathbb{C}/\mathbb{C})$. Via this identification $\hat{\phi}_d$ will get mapped to $\sigma^*(\hat{\phi}_d) \in H_{DR}^n(Z_L/\hat{L})$. Here $\sigma^* : H_{DR}^n(Z_L/\hat{L}) \to H_{DR}^n(Z_L/\hat{L})$ denotes the isomorphism of $L$-vector spaces induced by $\sigma \in \text{Gal}(\hat{L}/L)$ on cohomology.

Now, since $Y$ and $Z$ are both defined over the field $L$, both the Poincaré duality isomorphism and the Künneth formula on de Rham cohomology are defined over the field $L$ as well. These maps, by construction, commute with the isomorphisms $\sigma^*$ so we get that $\sigma^*(d)$ maps to $\sigma^*(\hat{\phi}_d) \in H^{2n}(Z_\mathbb{C}/\mathbb{C})$ via the map $\alpha_{DR}$ we had in the proof of 5.1.

Writing $P$ for Grothendieck’s comparison isomorphism we have that $P(\sigma^*(d)) \in D \subset \text{End}_\mathbb{Q} H^n(Y_\mathbb{C}^{an}, \mathbb{Q})$ is a Hodge endomorphism. Thus $\sigma^*(d) \in i(d)$ with the notation of 5.1 and the result follows.

By the claim therefore we get an action of $G := \text{Gal}(\hat{L}/L)$ on the $\mathbb{Q}$-vector space $D$, or more precisely its image in $\text{End}_L(H_{DR}^n(Y_L/\hat{L}))$. Let $\dim_\mathbb{Q} D = m_0$ and note that $m_0 \leq m^2$ trivially. We may and do assume, without loss of generality, that the field extension $\hat{L}/L$ constructed in the previous proof is minimal with the property that every cycle of the above basis $d$ is defined over $\hat{L}$. This implies that the corresponding group homomorphism $\text{Gal}(\hat{L}/L) \to \text{Aut}(D)$ is in fact injective.

Let $\Lambda_1$ be a lattice in $D$, and consider $\Lambda := \sum_{g \in G} g(\Lambda_0)$. This will be a
lattice that is also invariant by $G$. From the $G$-invariance of $\Lambda$ we get a group homomorphism

$$G \rightarrow \text{GL}(\Lambda).$$

This homomorphism will be injective as well by our earlier assumption about the minimality of the extension $\bar{L}/L$.

Let $N \geq 3$. Then, we know that the kernel of the surjective map $\text{GL}(\Lambda) \rightarrow \text{GL}(\bar{L}/N\Lambda)$ contains no element of finite order of the group $\text{GL}(\Lambda)$. As a result we get $G \hookrightarrow \text{GL}(\bar{L}/N\Lambda)$ which implies that $|G|$ divides $|\text{GL}(\bar{L}/N\Lambda)| = |\text{GL}_{m_0}(\mathbb{Z}/NZ)|$.

Following the notation of [Sil92] we let $g_r(N) := |\text{GL}_r(\mathbb{Z}/NZ)|$ and $G(r) := \text{gcd}\{g_r(N) : N \geq 3\}$. From Theorem 3.1 of [Sil92] we have that

$$G(r) < ((6.31)r)^r. \quad (9)$$

From the above argument we get that $|G|$ divides $G(m_0)$ and combining this with (9) and the fact that $m_0 \leq m^2$ we get that

$$|G| < ((6.31)m^2)^{m^2}. \quad (10)$$

$\square$

Part II

Determining the trivial relations

Given a $G$-admissible variation of Hodge structures we will show that for some exceptional points $s \in S(\bar{\mathbb{Q}})$ we get so called “non-trivial” relations among the values of the relative periods at the point $s$. To be able to say that these relations we will create are in fact non-trivial we need to know what the trivial ones are first!

We have devoted this part of the paper to determining these trivial relations in the case where the generic special Mumford-Tate group of our variation is a symplectic group.

§6 The action of the Local Monodromy

We start by reviewing a key property of the local monodromy that we will need during this process. This follows the ideas in Chapter X, Lemma 2.3 of
Let $\Delta$ be a small disc embedded in $S^n_{\mathbb{C}}$ centered at $s_0$ and such that $\Delta^* \subset S^n_{\mathbb{C}}$. We have already remarked in §2.2 that the logarithm of the local monodromy of $\Delta^* \subset S^n_{\mathbb{C}}$ acting on $R_n(f_{\mathbb{C}}^n)_*(\mathbb{Q})|_{\Delta^*}$ defines the local subsystem $\mathcal{M}_0 := M_0 R_n(f_{\mathbb{C}}^n)_*(\mathbb{Q})|_{\Delta^*}$. This is contained in the maximal constant subsystem of $R_n(f_{\mathbb{C}}^n)_*(\mathbb{Q})|_{\Delta^*}$, since $2\pi i N^*$, the nilpotent logarithm associated with the action of monodromy on the limit Hodge structure, has degree of nilpotency $\leq n + 1$.

We recall that, since the map $f : X \to S$ is smooth and projective, we have a bilinear form $\langle , \rangle$ on the local system $R_n f_{\mathbb{C}}^n \mathbb{Q}$ induced by the polarizing form.

**Lemma 6.1.** The local system $\mathcal{M}_0$ is a totally isotropic subsystem of $R_n f_{\mathbb{C}}^n \mathbb{Q}|_{\Delta^*}$ with respect to the polarizing form $\langle , \rangle$.

**Proof.** The skew-symmetric form $\langle , \rangle$ defines a morphism of local systems

$$R_n f_{\mathbb{C}}^n \mathbb{Q}|_{\Delta^*} \otimes R_n f_{\mathbb{C}}^n \mathbb{Q}|_{\Delta^*} \to \mathbb{Q}(n)|_{\Delta^*}.$$ 

Therefore it is invariant under the local monodromy and we conclude that for any $z \in \Delta^*$ and for all $v, w \in (R_n f_{\mathbb{C}}^n \mathbb{Q})_z$ we have

$$\langle N^*_z v, w \rangle + \langle v, N^*_z w \rangle = 0. \quad (11)$$

Now let $v, w$ be any two sections of $\mathcal{M}_0$. Then for any $z \in \Delta^*$ there exist $v_0, z, w_0, z \in (R_n f_{\mathbb{C}}^n \mathbb{Q})_z$ such that $v_z = (2\pi i N^*_z)^n(v_0, z)$ and $w_z = (2\pi i N^*_z)^n(w_0, z)$. Using (11) we thus get

$$\langle v_z, w_z \rangle = \langle (2\pi i N^*_z)^n(v_0, z), (2\pi i N^*_z)^n(w_0, z) \rangle = -\langle (2\pi i N^*_z)^{n-1}(v_0, z), (2\pi i N^*_z)^{n+1}(w_0, z) \rangle = 0,$$

where the last equality follows from the fact that $N^*_z$ has degree of nilpotency $\leq n + 1$.

Therefore we get that for all $v, w \in \mathcal{M}_0$ we have $\langle v, w \rangle = 0$. Hence $\mathcal{M}_0$ is a totally isotropic local subsystem.

### §7 Trivial relations

#### §7.1 Our setting and notations

Let $f : X \to S$ be a smooth projective morphism of $k$-varieties where $k$ is a subfield of $\overline{\mathbb{Q}}$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ so that we may consider $k$ as a subfield of $\mathbb{C}$. Assume that $S$ is a smooth irreducible curve, that the fibers
of $f$ are $n$-dimensional, and let $\mu := \dim_{\mathbb{Q}} H^n(X_{s}^{an}, \mathbb{Q})$ for some $s \in S(\mathbb{C})$. Throughout this section we assume that $n$ is odd and that $S = S' \setminus \{\Sigma_S\}$ for some finite subset $\Sigma_S$ and that there exists a $k$-point $s_0 \in \Sigma_S$ where our VHS has a non-isotrivial degeneration.

We consider
\[ P^n_{X/S} : H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{C}}^{an}} \rightarrow R^n f_*^{an} \mathbb{Q}_{X_{\mathbb{C}}^{an}} \otimes_{\mathbb{Q}_{S_{\mathbb{C}}^{an}}} \mathcal{O}_{S_{\mathbb{C}}^{an}}, \]

the relative period isomorphism.

### Choosing bases-The Riemann relations

Let $\omega_i$, $1 \leq i \leq \mu$, be a basis of $H^n_{DR}(X_\eta)$ over $k(S)$, where $\eta$ is the generic point of $S$. Then there exists some dense affine open subset $U$ of $S$ over which these $\omega_i$ are sections of the sheaf $H^n_{DR}(X/S)$. We also fix a trivialization $\gamma_i$ of $R_n f_*^{an} \mathbb{Q}_{X_{\mathbb{C}}^{an}}$, i.e. the relative homology, over an analytic open subset $V$ of $U_{\mathbb{C}}^{an}$. Since we are interested in describing the relations among the periods archimedeanly close to the point of degeneration $s_0$, we may and do assume that the set $V$ is simply connected and contained in a fixed small punctured disk $\Delta^*$ around $s_0$.

The matrix of $P^n_{X/S}$ with respect to this basis and trivialization will have entries in the ring $\mathcal{O}_V$. We multiply the matrix’s elements by $(2\pi i)^{-n}$ and, by abuse of notation, we denote the above $\mu \times \mu$ matrix of relative $n$-periods by

\[ P_{X/S} := ((2\pi i)^{-n} \int_{\gamma_j} \omega_i). \]

Since the morphism $f : X \rightarrow S$ is smooth, projective, and is also defined over $k$, it defines a polarization which will be defined over $k$ as a form on de Rham cohomology. In particular we get, since the weight $n$ of our variation is odd,

- a skew-symmetric form $\langle \cdot, \cdot \rangle_{DR}$ on $H^n_{DR}(X_\eta)$ with values in $k(S)$ and
- a skew-symmetric form $\langle \cdot, \cdot \rangle_B = (2\pi i)^n \langle \cdot, \cdot \rangle$ on $R_n f_*^{an} \mathbb{Q}$ with values in $\mathbb{Q}(n)$.

These two skew-symmetric forms are compatible with the isomorphism $P^n_{X/S}$, in the sense that the dual form of $\langle \cdot, \cdot \rangle_B$ coincides with the form induced by $\langle \cdot, \cdot \rangle_{DR}$ via the isomorphism $P^n_{X/S}$. The compatibility of the polarizing forms translates to relations among the periods. These relations can be described succinctly by the equality

\[ ^t P M^{-1}_{DR} P = (2\pi i)^{-n} M^{-1}_B, \tag{12} \]
where \( M_{DR} \) and \( M_B \) are the matrices of \( \langle \cdot, \cdot \rangle_{DR} \) and the dual of \( \langle \cdot, \cdot \rangle_B \) respectively with respect to some basis and trivialization.

For more on this see A. The relations given on the periods by (12) are practically a direct consequence of the well known Hodge-Riemann bilinear relations defining a polarization of a Hodge structure. For this reason from now on we shall refer to (12) as the Riemann relations for brevity.

With this in mind, we may and do select the above basis \( \omega_i \) and trivialization \( \gamma_j \) so that the following are satisfied:

1. the \( \omega_i \) are a symplectic basis of \( H^n_{DR}(X_\eta) \) so that \( \omega_1, \ldots, \omega_{\mu/2} \) constitute a basis of the maximal isotropic subspace \( F_{\mu+1}^{\perp n} H^n_{DR}(X_\eta) \) and the rest of the elements, i.e. \( \omega_{\mu/2+1}, \ldots, \omega_{\mu} \) are the basis of a transverse Lagrangian of \( F_{\mu+1}^{\perp n} H^n_{DR}(X_\eta) \), and

2. the \( \gamma_j \) is a symplectic trivialization of \( R_n f^a_{an} Q X_{an}|_V \), which is also such that \( \gamma_1, \ldots, \gamma_h \) are a frame of the space \( M_0|_V \) and the \( \gamma_1, \ldots, \gamma_{\mu/2} \) are a frame of a maximal totally isotropic subsystem that contains \( M_0 R_n f^a_{an} Q|_V \).

With these choices we may and do assume from now on that the matrices that correspond to the two aforementioned forms are \( M_{DR} = M_B = J_\mu = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \). With this (12) translates to

\[
^t P J_\mu P = (2\pi i)^{-n} J_\mu. \tag{13}
\]

The main result

Let \( y_{i,j} \) with \( 1 \leq i \leq \mu \) and \( 1 \leq j \leq h \) be the entries of the first \( h \) columns of the matrix \( P_{X/S} \). The aforementioned work of André, see 2.2, guarantees that these are G-functions. This is happening with respect to a local parameter \( x \) of \( S' \) at \( s_0 \), with respect to which the \( y_{i,j} \) can be written as power series.

For the remainder of this section we consider the above notation fixed. The rest of this section is dedicated to describing the generic, or “trivial”, relations among the G-functions \( y_{i,j} \). Indeed, we prove the following:

**Proposition 7.1.** With the above notation, assume that the generic special Mumford-Tate group of the variation of \( \mathbb{Q} \)-HS on \( S_{an}^{an} \) given by \( R^n f^a_{an} Q X_{an} \) is \( \text{Sp}(\mu, \mathbb{Q}) \).

Then, the Zariski closure of the \( \mu \times h \) matrix \( Y := (y_{i,j}) \) over \( \mathbb{Q}[x] \) in \( \mathbb{A}^{\mu \times h} \) is the variety whose ideal is given by the Riemann relations.
§7.2 Trivial relations over \( \mathbb{C} \) for the period matrix

Under the notations and assumptions of §7.1 and 7.1 we have the following:

**Lemma 7.1.** Let \( z \in V \subset \mathbb{U}^{an} \) be a Hodge generic point of the \( \mathbb{Q} \)-VHS given by \( R^n f_s^{an} \mathbb{Q}_{X^{an}_s} \). Then the monodromy group \( H_z \) at \( z \) is \( \text{Sp}(\mu, \mathbb{Q}) \).

**Proof.** Let \( \rho_H : \pi_1(S^{an}_s, z) \to GL(H^n(X^{an}_z, \mathbb{Q})) \) be the monodromy representation at \( z \). Then, by André’s Theorem of the fixed part [And92] we know that \( H_z \), which is the connected component of the \( \mathbb{Q} \)-algebraic group \( \rho_H(\pi_1(S^{an}_s, z))^{Q-Zar} \), is a normal subgroup of the derived subgroup of the Mumford-Tate group \( G_{mt,z} \) at \( z \). In other words

\[
H_z \leq \text{DG}_{mt,z}.
\]

On the other hand we have that \( \text{DG}_{mt,z} \leq G_{smt,z} \) and trivially that \( \text{DG}_{smt,z} \leq \text{DG}_{mt,z} \), where \( G_{smt,z} \) is the special Mumford-Tate group at \( z \). But, by assumption, we know that \( G_{smt,z} \simeq \text{Sp}(\mu, \mathbb{Q}) \), since \( z \) is Hodge generic for our variation. It is classical that \( \text{Sp}(\mu, \mathbb{Q}) \) satisfies \( D\text{Sp}(\mu, \mathbb{Q}) = \text{Sp}(\mu, \mathbb{Q}) \).

Hence we have \( \text{DG}_{mt,z} = \text{Sp}(\mu, \mathbb{Q}) \).

We thus get that \( H_z \leq \text{Sp}(\mu, \mathbb{Q}) \). Finally, \( \text{Sp}(\mu, \mathbb{Q}) \) is a simple \( \mathbb{Q} \)-algebraic group, therefore \( H_z = 1 \) or \( H_z = \text{Sp}(\mu, \mathbb{Q}) \). But, if we had \( H_z = 1 \), then the variation of \( \mathbb{Q} \)-HS in question would be isotrivial, and hence extend to \( T = S^{an} \cup \{ s_0 \} \). We get a contradiction since the local monodromy at \( s_0 \in S'(\mathbb{C}) \) is non-trivial by assumption. \( \square \)

From now on, by taking a finite étale cover of \( S \) if necessary, we may and do assume that \( \rho_H(\pi_1(S^{an}_s, z))^{Q-Zar} \) is connected, i.e. that for the Hodge generic points \( z \in V \) we have \( H_z = \rho_H(\pi_1(S^{an}_s, z))^{Q-Zar} \).

§7.2.1 The matrix of Periods and differential equations

Let us denote by \( M_\mu \) the variety of \( \mu \times \mu \) matrices over \( \mathbb{C} \), where \( \mu : = \dim_{\mathbb{Q}} H^n(X_s^{an}, \mathbb{C}, \mathbb{Q}) \) for any \( s \in S(\mathbb{C}) \).

The period matrix \( P_{X/S} \) defines a holomorphic map

\[
\phi : V \to M_\mu.
\]

We let \( Z \subset V \times M_\mu \) be the graph of this function. The first step in our process is determining the \( \mathbb{C} \)-Zariski closure of \( Z \).

**Lemma 7.2.** Let \( Z \) be as above then the \( \mathbb{C} \)-Zariski closure of \( Z \) is

\[
S_{\mathbb{C}} \times \{ M : 'M J_\mu M = (2\pi i)^{-n} J_\mu \}.
\]
In order to prove this we will employ the monodromy action in an essential way. For this purpose we will need to review some further properties of the isomorphism $P^n_{X/S}$.

To this end, let us consider

$$Q^n_{X/S} : R^n f^*_a \otimes_{\mathcal{O}_{S_{an}}} \mathcal{O}_{S_{an}} \cong H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{an}},$$

the inverse of $P^n_{X/S}$.

It is known, see [Kat72] Prop.4.1.2, that this isomorphism restricts to an isomorphism of local systems

$$Q : R^n f^*_a \mathcal{C}^{X_{an}} \cong \mathbb{R}^n f^*_a \Omega^\bullet \mathcal{C}^{X_{an}} \cong (H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{an}})^\nabla$$

where $(H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{an}})^\nabla \subset \mathbb{R}^n f^*_a \Omega^\bullet \mathcal{C}^{X_{an}}$ is the local system of horizontal sections with respect to the Gauss-Manin connection.

Note that we have an inclusion of local systems $R^n f^*_a \mathcal{Q} \hookrightarrow R^n f^*_a \mathcal{C}$ on $S_{an}$. This leads to a commutative diagram

$$\begin{array}{ccc}
\pi_1(S_{an}, z) & \xrightarrow{\rho_{H,C}} & \text{GL}(H^n(X^an, \mathbb{C})) \\
\rho & \downarrow & \\
& GL(H^n(X^an, \mathcal{Q}))
\end{array}$$

for any point $z \in S_{an}$.

In particular, we get, under our assumptions on the connectedness of the group $(\rho_{H}(\pi_1(S_{an}, z)))^{Q_{-Zar}}$, that the group $G_{mono,z} := (\rho_{H,C}(\pi_1(S_{an}, z)))^{C_{-Zar}}$, i.e. the $\mathbb{C}$-Zariski closure of the image of the fundamental group under $\rho_{H,C}$, is such that

$$G_{mono,z} = H_z \otimes_{\mathbb{Q}} \mathbb{C}. \quad (14)$$

Earlier we saw that we have an isomorphism $Q$ of local systems over $S_{an}$. By the equivalence of categories between local systems over $S_{an}$ and representations of the fundamental group $\pi_1(S_{an}, z)$ we thus have that the representations

$$\rho_{H,C} : \pi_1(S_{an}, z) \to \text{GL}(H^n(X^an, \mathbb{C})), \quad \text{and} \quad \rho_{DR} : \pi_1(S_{an}, z) \to \text{GL}((H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{an}})^\nabla),$$

are conjugate. In fact, keeping in mind that all actions are on the right, we have that $\rho_{DR}(\lambda) = Q(z)^{-1}\rho_{H,C}(\lambda)Q(z)$, for all $\lambda \in \pi_1(S_{an}, z)$, where $Q(z)$ is the fiber of $Q$ at $z$. From this we get that

$$G_{DR,z} := (\rho_{DR}(\pi_1(S_{an}, z)))^{C_{-Zar}} = Q(z)^{-1}G_{mono,z}Q(z). \quad (15)$$

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Let $B$ be the matrix of the isomorphism $Q|_V$ with respect to the frame \( \{ \gamma^*_j : 1 \leq j \leq \mu \} \) of the trivialization of $R^nf^\mu Q|_V \subset R^nf^\mu C|_V$, i.e. the dual of the frame given by the $\gamma_j$ on $R^nf^\mu Q|_V$, and the basis $\{ \omega_i : 1 \leq i \leq \mu \}$ chosen above. We then have that the rows $b_i$ of $B$, which will correspond to $Q|_V(\gamma^*_i)$ written in the basis $\omega_i$, will constitute a basis of the space $\Gamma(V, (H^0_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^n})^\nabla)$. In other words $B$ is a complete solution of the differential equation $\nabla(\omega) = 0$, defined by the Gauss-Manin connection. We note that in our setting the Gauss-Manin connection is known to be defined over the field $k$ by work of Katz and Oda. see [KO68] and [Kat70].

Let $\Gamma \in \text{M}_\mu(k(S))$ be the (local) matrix of $\nabla$ on $U$ with respect to the basis given by the $\omega_i$. Writing $\nabla(\omega) = d\omega + \omega \Gamma$, identifying the $\omega$ with the $1 \times n$ matrix given by the coefficients of $\omega$ in the basis given by the $\omega_i$, we may rewrite the above equation as $d\omega = -\omega \Gamma$. The corresponding matricial differential equation then becomes

$$X' = -X\Gamma.$$ \hfill (16)

The monodromy representation $\rho_{DR}$ defines analytic continuations of solutions at $z$ of the differential equation 16. So in considering the value at the point $z$ of the analytic continuation $B^\lambda$ of the matrix $B$ along the cycle $\lambda \in \pi_1(S^n, z)$, corresponding to a loop $\gamma$ passing through $z$, all we are doing is multiplying the matrix $B_z$ by $\rho_{DR}(\lambda)$. In other words for $\lambda \in \pi_1(S^n, z)$ we have that

$$(B^\lambda)_z = B_z \rho_{DR}(\lambda).$$ \hfill (17)

We apply the ideas presented in the above discussion to prove the following lemma.

**Lemma 7.3.** Consider $A$ to be the matrix of the isomorphism $P^n_{X/S}$ on the open analytic set $V$ with respect to the basis $\omega_i$ and frame $\gamma^*_j$ chosen above. Let $z \in V$ and let $\lambda \in \pi_1(S^n, z)$. Then the value at $z$ of the analytic continuation $A^\lambda$ of $A$ along the loop that corresponds to $\lambda$ is given by

$$(A^\lambda)_z = A_z \rho_{H,C}(\lambda)^{-1},$$

where $\rho_{H,C}$ is the above representation on Betti cohomology.

**Proof.** We have $A \cdot B = I_\mu$ hence $A^\lambda \cdot B^\lambda = I_\mu$. Using (17) we get that

$$(A^\lambda)_z = \rho_{DR}(\lambda)^{-1}B_z^{-1} = \rho_{DR}(\lambda)^{-1}A_z.$$

On the other hand, with the above notation we have that $\rho_{DR}(\lambda) = B_z^{-1}\rho_{H,C}(\lambda)B_z$. This combined with the above leads to the result. \hfill \Box

**Remark.** The same relation holds for the value $P_{X/S}(z)$ at $z$ of the matrix of relative periods $P_{X/S}$, since $P_{X/S} = (2\pi i)^{-n}A$.
We are now in the position to prove 7.2.

**Proof of 7.2.** Let \( Z \subset V \times M_\mu \subset S_\mathbb{C} \times M_\mu \) be the graph of the isomorphism \( P_{X/S}|_V \). Let \( \tilde{Z} \) be the union of the graphs of all possible analytic continuations of \( Z \). It is easy to see via analytic continuation that we have \( (\tilde{Z})^{\mathbb{C} - Zar} = Z^{\mathbb{C} - Zar} \). We also note that for all \( z \in V \) we have \( (\tilde{Z}_z)^{\mathbb{C} - Zar} \subset (Z^{\mathbb{C} - Zar})_z \) for trivial reasons.

We focus on the points \( z \in V \) that are Hodge generic for the variation of \( Q_{HS} \) given by \( R^n f_{an}^* \mathbb{Q}_{Xan}|_V \). We note that the set of such \( z \) in \( V \), which we denote by \( V_{Hgen} \), is uncountable.

By 7.3, and the fact that the rows of the matrix \( B \) above are a complete solution of the differential system (16), we know that \( \tilde{Z}_z = P_{X/S}(z)\rho_{H,\mathbb{C}}(\pi_1(S^{an}, z)) \).

From this we get that \( (\tilde{Z}_z)^{\mathbb{C} - Zar} = P_{X/S}(z)G_{mono,z} \).

From (14) we know that \( G_{mono,z} = H_z \otimes_{\mathbb{Q}} \mathbb{C} \) while from 7.1 we know that, since \( z \in V_{Hgen} \), we have \( H_z \simeq Sp(\mu, \mathbb{Q}) \), hence \( G_{mono,z} \simeq Sp(\mu, \mathbb{C}) \). Hence we have \( (\tilde{Z}_z)^{\mathbb{C} - Zar} = P_{X/S}(z)Sp(\mu, \mathbb{C}) \).

Using (13) together with the above we arrive through elementary reasoning to

\[
(\tilde{Z}_z)^{\mathbb{C} - Zar} = \{ M \in GL_\mu(\mathbb{C}) : {}^t M J_\mu M = (2\pi i)^{-n} J_\mu \}. \tag{18}
\]

Applying this to the fact that for all \( z \in V \) we have \( (\tilde{Z}_z)^{\mathbb{C} - Zar} \subset (Z^{\mathbb{C} - Zar})_z = (Z^{\mathbb{C} - Zar})_z \), we get that

\[
V_{Hgen} \times \{ M \in GL_\mu(\mathbb{C}) : {}^t M J_\mu M = (2\pi i)^{-n} J_\mu \} \subset Z^{\mathbb{C} - Zar}. \tag{19}
\]

Now, using the fact that \( V_{Hgen} \) is uncountable and taking Zariski closures in (19) we get that

\[
S_\mathbb{C} \times \{ M \in GL_\mu(\mathbb{C}) : {}^t M J_\mu M = (2\pi i)^{-n} J_\mu \} \subset Z^{\mathbb{C} - Zar}.
\]

On the other hand, once again from (13), we know that

\[
Z \subset S_\mathbb{C} \times \{ M \in GL_\mu(\mathbb{C}) : {}^t M J_\mu M = (2\pi i)^{-n} J_\mu \}
\]

which, by once again taking Zariski closures, gives the reverse inclusion. \( \square \)

**§7.3 Trivial relations over \( \mathbb{C} \) for the G-functions**

As we remarked earlier, the entries of the first \( h \) columns of our matrix \( P_{X/S} \) are G-functions, under our choice of basis and trivialization. Let us denote by \( y_{i,j} \) these entries and by \( Y \) the respective \( \mu \times h \) matrix they define. Consider the projection map \( \text{pr} : M_\mu \rightarrow \mathbb{A}^{\mu \times h} \) that maps a matrix \( (a_{i,j}) \in M_\mu \) to the \( \mu \times h \) matrix that consists of its first \( h \) columns. This maps \( P_{X/S} \) to \( Y \).
Lemma 7.4. Let $T$ be the subvariety of $\mathbb{A}^{\mu \times h}$ defined by the following set of polynomials

$$\{ t_{b_i} J_{b_j} : 1 \leq i, j \leq h \},$$

where $b_i$ denotes the $i$-th column of a matrix of indeterminates.

Then $Y^C(S) - \mathbb{Z}_S = T_C(S)$.

Proof. Let $Z_Y \subset V \times M_{\mu \times h}(\mathbb{C})$ denote the graph of $Y$ as a function $Y : V \to M_{\mu \times h}(\mathbb{C})$. It suffices to show that $Z_Y^{C-Zar} = S \times T$.

The inclusion $Z_Y^{C-Zar} \subset S \times T$ follows trivially from (13), which shows that $Z_Y \subset V \times T(\mathbb{C})$. On the other hand, we have that the map $\text{id}_S \times \text{pr} : S \times M_\mu \to S \times \mathbb{A}^{\mu \times h}$ is topologically closed, with respect to the Zariski topology. This implies that $Z_Y^{C-Zar} = (\text{id} \times \text{pr})(Z_{C-Zar}^{C-Zar})$.

By construction we have that the columns $c_i$ of any $\mu \times h$ matrix $C \in T(C)$ will be a basis that spans an isotropic subspace of dimension $h$ with respect to the symplectic form defined by $J_\mu$ on $C^\mu$. It is easy to see that we can extend this set of vectors to a basis $\{ c_j : 1 \leq j \leq \mu \}$ of $\mathbb{C}^\mu$ that satisfies

1. $t_{c_i} J_{c_j} = 0$ for all $i, j$ with $|i - j| \neq \mu/2$, and
2. $t_{c_i} J_{c_j} = (2\pi i)^{-n}$ for $i = j + \mu/2$.

In other words, this is a symplectic basis “twisted” by a factor $(2\pi i)^{-n/2}$. The $\mu \times \mu$ matrix with columns $c_i$ will then be such that $(s, M_C) \in Z_Y^{C-Zar}$ by 7.2 and by construction $\text{pr}(M_C) = C$.

Combining the above with the fact that $Z_Y^{C-Zar} = (\text{id} \times \text{pr})(Z_{C-Zar}^{C-Zar})$ we have that $S \times T \subset Z_Y^{C-Zar}$ and our result follows. □

§7.4 Trivial relations over $\bar{\mathbb{Q}}$

So far we have not used any arithmetic information about the $y_{i,j}$, namely the fact that they are G-functions.

Let $\xi \in \bar{\mathbb{Q}}$. Then a trivial polynomial relation with coefficients in $\bar{\mathbb{Q}}$ at the point $\xi$ among the $y_{i,j} \in \bar{\mathbb{Q}}[[x]]$ is a relation that satisfies the following:

1. there exists $p(x_{i,j}) \in \bar{\mathbb{Q}}[x_{i,j}]$ such that the relation we have is of the form $p(y_{i,j}(\xi)) = 0$,
2. the relation holds $v$-adically for some place $v$ of $\bar{\mathbb{Q}}$, i.e. letting $i_v : \mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_v$ we have that the $y_{i,j}$ converge at $i_v(\xi)$ and the above relation is an equality in $\bar{\mathbb{Q}}_v$. 26
3. there exists a polynomial \( q(x)(x_{i,j}) \in \overline{\mathbb{Q}}[x][x_{i,j} : 1 \leq i \leq \mu, 1 \leq j \leq h] \) such that it has the same degree as \( p \), with respect to the \( x_{i,j} \), and \( q(\xi)(x_{i,j}) = p(x_{i,j}) \)

Therefore, to describe the trivial relations among the values of our G-functions \( y_{i,j} \) at some \( \xi \in \overline{\mathbb{Q}} \), it is enough to determine the \( \overline{\mathbb{Q}}[x] \)-Zariski closure of the matrix \( Y \). We do this in the following lemma, which is practically a more detailed rephrasing of 7.1.

**Lemma 7.5.** Let \( Y \) be the \( \mu \times h \) we had above. Then the \( \overline{\mathbb{Q}}[x] \)-Zariski closure \( Y^{\overline{\mathbb{Q}}[x]-Zar} \) of \( Y \) is the subvariety of \( \mathbb{A}_ \overline{\mathbb{Q}[x]}^{\mu \times h} \) defined by the following set of polynomials

\[
\{ b_iJ_i b_j : 1 \leq i, j \leq h \},
\]

where \( b_i \) denotes the \( i \)-th column of a matrix of indeterminates.

**Proof.** We let \( \Sigma \) be the set of polynomials above and let \( I_R \) be the ideal generated by \( \Sigma \) in the ring \( R[x_{i,j}] \), where \( R \) will denote different fields in our proof.

In this case from 7.4 we know that \( Y^{C(S)-Zar} \) is equal to \( V(I_{\mathbb{C}(S)}) \). Note that the elements of \( \Sigma \) all have coefficients in \( \overline{\mathbb{Q}}[x] \), in fact they have coefficients in \( \overline{\mathbb{Q}} \). From this we get the result we wanted, i.e. \( Y^{\overline{\mathbb{Q}}[x]-Zar} = V(I_{\overline{\mathbb{Q}[x]}}) \).

**Remark.** Implicit in the previous proof is the fact that we have a polarization that is defined over \( k \subset \overline{\mathbb{Q}} \) as a cycle in some de Rham cohomology group.

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**Part III**

**Constructing non-trivial relations**

§8 Towards relations for exceptional points

Let \( f : X \to S \) be a G-admissible variation of \( \mathbb{Q}\text{-HS} \). We start with the following definition.

**Definition.** Let \( s \in S(\overline{\mathbb{Q}}) \). Assume that in the decomposition of \( V_s \) into irreducible \( \mathbb{Q}\)-Hodge structures, as in (20), there exists at least one irreducible factor \( V_i \) whose algebra of endomorphisms \( D_i \) is of type IV in Albert’s classification. We then say that the point \( s \), or equivalently the corresponding \( \mathbb{Q}\text{-HS} \), is pseudo-CM.
Remark. We note here that all CM-points $s \in S(\overline{Q})$ of the variation will satisfy the above definition. The term “pseudo-CM” reflects the fact that the center of a type IV algebra in Albert’s classification is a CM field. We note that the points considered here are far more general, at least in principle, than special points.

§8.1 Notational Conventions

Let $f : X \to S$ be a $G$-admissible variation as above and let $s \in S(L)$ with $L \subset \overline{Q}$.

First of all, note that from the semisimplicity of the category of polarized Hodge structures, we know that we may write
\[ V_s = V_{i_1}^{m_1} \oplus \ldots \oplus V_{i_r}^{m_r}, \]
with $(V_i, \varphi_i)$ irreducible polarized $\mathbb{Q}$-HS that are non-isomorphic to each other. Let $D_i := \text{End}(V_i)^{G_{m_1}(V_i)}$ be the respective endomorphism algebras so that
\[ D_s = M_{m_1}(D_1) \times \ldots \times M_{m_r}(D_r). \]

From 5.1 we know that, assuming the absolute Hodge conjecture, there exists a finite extension $\hat{L}$ of $L$ such that $D_s$ acts on $H^n_{DR}(X_s, \hat{L}/\hat{L})$ and that this action is compatible with the comparison isomorphism between algebraic de Rham and singular cohomology. Again assuming the absolute Hodge conjecture, we know from 5.2 that the degree $[\hat{L} : L]$ of the extension is bounded by a bound independent of the point $s$. We assume from now on that $\hat{L} = L$ and return to this issue in the proof of 1.1.

We let $F_i$ denote the center of the algebra $D_i$ for $1 \leq i \leq r$ and note that these are number fields due to Albert’s classification. We introduce the following notation
- $\hat{E}_s = F_1^{m_1} \times \ldots \times F_r^{m_r}$ the maximal commutative semi-simple algebra of $D_s$,
- $\hat{F}_i$ the Galois closure of the field $F_i$ in $\mathbb{C}$,
- $\hat{F}_s$ the compositum of the fields $\hat{F}_i$ together with the field $L$.

§8.2 Splittings in cohomology and homology

Let us assume $f : X \to S$ is a $G$-admissible variation as above and let $s \in S(L)$, where $L/K$ is a finite extension. We assume that $s$ is archimedeanly
close to the point $s_0$ on $S'$, with respect to a fixed inclusion $L \hookrightarrow \mathbb{C}$. In particular we assume that it is in the image of the inclusion of a punctured unit disc $\Delta^* \subset S_{\mathbb{C}}^{an}$ centered at $s_0$.

Under the above assumption, $L = \hat{L}$, we know that we have two splittings. Namely, on the one hand we get a splitting

$$H_n(X_{s,\mathbb{C}}^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \hat{F}_s = \bigoplus_{\sigma: \hat{E}_s \to \mathbb{C}} \hat{W}_\sigma, \quad (21)$$

induced from the splitting $\hat{E}_s \otimes_{\mathbb{Q}} \hat{F}_s = \bigoplus_{\sigma: \hat{E}_s \to \mathbb{C}} \hat{F}_s^\sigma$, where $\hat{F}_s^\sigma$ denotes the field $\hat{F}_s$ viewed as an $\hat{E}_s$-module with the action of $\hat{E}_s$ being multiplication by $\sigma$. We also note that on $\hat{W}_\sigma$ the algebra $\hat{E}_s$ acts again via multiplication with its character $\sigma$.

On the other hand, we have a splitting

$$H^n_{DR}(X_s/L) \otimes_L \hat{F}_s = \bigoplus_{\sigma: \hat{E}_s \to \mathbb{C}} \hat{W}_{\sigma}^{DR}, \quad (22)$$

which once again comes from the above splitting of $\hat{E}_s \otimes_{\mathbb{Q}} \hat{F}_s$. In particular, we note that the action of $\hat{E}_s$ on $\hat{W}_{\sigma}^{DR}$ comes once again via $\sigma$.

§8.2.1 Duality of the splittings

We start by highlighting how the two splittings interact with one another via the comparison isomorphism

$$P_{X_s}^n: H^n_{DR}(X_s/L) \otimes_L \mathbb{C} \to H^n(X_{s,\mathbb{C}}^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$ 

The following lemma is already noted as a property of the splittings by André, we include a short proof for the sake of completeness.

**Lemma 8.1.** For all $\sigma \neq \tau$ if $\omega \in W_{\sigma}^{DR}$ and $\gamma \in W_{\sigma}$ then

$$\int_{\gamma} \omega = 0.$$ 

**Proof.** Let us fix $\sigma \neq \tau$ as above and let $\omega \in W_{\sigma}^{DR}$ and $\gamma \in W_{\sigma}$.

For all $e \in \hat{E}_s$ we have that $P_{X_s}^n(d\omega) = P_{X_s}^n(\tau(d)\omega) = \tau(d)P_{X_s}^n(\omega) = d \cdot P_{X_s}^n(\omega)$, where the last equality follows from the moreover part of 5.1. The algebra $\hat{E}_s$, and in particular its group of invertible elements $\hat{E}_s^\times$, acts by
definition on $V_s$ as endomorphisms of the Hodge structure. The action of $\hat{E}_s$ on the dual space $V_s^*$ will thus be the dual of that of $V_s$.

In particular for any $\gamma \in \hat{W}_\sigma$, for any $e \in \hat{E}_s$, and for any $\delta \in V_s$, we get that $(e \cdot \gamma)(e \cdot \delta) = \gamma(\delta)$. Taking $\delta = P^n_{X_s}(\omega)$ we get that for all $\gamma \in \hat{W}_\sigma$ and for all $e \in \hat{E}_s$

$$\int_\gamma \omega = \gamma(P^n_{X_s}(\omega)) = (e \cdot \gamma)(e \cdot P^n_{X_s}(\omega)).$$

But we know that $(e \cdot \gamma)(e \cdot P^n_{X_s}(\omega)) = (\sigma(e^{-1})\gamma)(\tau(e)P^n_{X_s}(\omega))$, where we used the duality between the actions of $\hat{E}_s$ on $V_s$ and $V_s^*$. Putting everything together we get that for all $e \in \hat{E}_s$ we will have that

$$\int_\gamma \omega = \sigma(e)^{-1}\tau(e) \int_\gamma \omega.$$

Since $\sigma \neq \tau$ we can find such an $e$ with $\sigma(e) \neq \tau(e)$ and the lemma follows.

\[\square\]

§8.3 Involution and symplectic bases

In creating the relations we want we will need to construct symplectic bases with particular properties. To construct these we will need to review some facts about the involutions of the algebras of Hodge endomorphisms and see how they interact with the splittings we have.

For the weight $n$ $\mathbb{Q}$-HS given by $V_s$ we denote by $\langle , \rangle$ the symplectic form defined by the polarization on $V_s$. By duality we get a polarized $\mathbb{Q}$-HS of weight $-n$ on the dual space $V^*_s := H_n(\bar{X}^{an}_{s, \mathbb{C}}, \mathbb{Q})$, and we denote the symplectic form given by the polarization again by $\langle , \rangle$. We note that these two symplectic forms are dual.

The algebra $D_s$ comes equipped with an involution, which we denote by $d \mapsto d^*$, that is defined by the relation

$$\langle d \cdot v, w \rangle = \langle v, d^* \cdot w \rangle, \quad (23)$$

for all $d \in D_s$ and for all $v, w \in V_s^*$, or equivalently for all $v, w \in V_s$.

In the decomposition (20) of $V_s$, or its dual $V^*_s$, the polarization on each $V_i$, or $V^*_i$ respectively, is given by the restriction of the polarization of $V_s$, or its dual respectively. Therefore the involution $d \mapsto d^*$ of $D_s$ restricts to the positive involutions of the respective algebras $D_i$.

The algebra homomorphisms $\sigma : \hat{E}_s \to \mathbb{C}$ have a convenient description. Writing

$$\hat{E}_s = F_1^{m_1} \times \ldots \times F_r^{m_r},$$

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we let \( pr_{j,l} : \hat{E}_s \rightarrow F_j \), where \( 1 \leq j \leq r \) and \( 1 \leq l \leq m_j \), denote the projection of \( \hat{E}_s \) onto the \( l \)-th factor of \( F_j^{m_j} \), which will act respectively on the \( l \)-th factor of \( V_j^{m_j} \) that appears in the decomposition. Then any algebra homomorphism \( \sigma : \hat{E}_s \rightarrow \mathbb{C} \) can be written as
\[
\sigma = \hat{\sigma} \circ pr_{j,l}
\]
for some \( j \) and \( l \) as above and some \( \hat{\sigma} : F_j \hookrightarrow \mathbb{C} \). For convenience, from now on we define the notation
\[
\tilde{\sigma}_{j,l} := \hat{\sigma} \circ pr_{j,l}.
\]

**Lemma 8.2.** Consider the splitting (21). Then for the subspaces \( \hat{W}_\sigma \) the following hold:

1. If \( \sigma = \hat{\sigma}_{j,l} \) then \( \hat{W}_\sigma \) is contained in the \( l \)-th factor of \( (V_j^*)^{m_j} \),
2. Let \( \sigma = \hat{\sigma}_{j,l} \) and let \( \tau \) be some non-zero algebra homomorphism \( \hat{E}_s \rightarrow \mathbb{C} \). Consider non-zero vectors \( v \in \hat{W}_\sigma \) and \( w \in \hat{W}_\tau \). If we assume that \( \langle v, w \rangle \neq 0 \) then one of the following cases holds
   (a) \( \sigma = \tau \) and the algebra \( D_j \) of Hodge endomorphisms is of Type I, II or III in Albert’s classification, or
   (b) \( \sigma = \bar{\tau} \), where \( \bar{\cdot} \) denotes complex conjugation, and \( D_j \) is of Type IV in Albert’s classification.

**Proof.** The first part of the lemma is trivial.

For the second part let \( v \in \hat{W}_\sigma \) and \( w \in \hat{W}_\tau \) be non-zero vectors as above with \( \langle v, w \rangle \neq 0 \). From the preceding discussion there exists a pair \((j',l')\) for \( \tau \) such that \( \tau = \bar{\tau} \circ pr_{j',l'} \), where \( \bar{\tau} : F_{j'} \hookrightarrow \mathbb{C} \). From the first part of this lemma we also know that \( \hat{W}_\tau \) is contained in the \( l' \)-th factor of \( (V_{j'}^*)^{m_{j'}} \).

The subspaces \( V_s^* \) of \( V_s^* \) are symplectic, with their symplectic inner product being the restriction of that of \( V_s^* \). This immediately implies that \( (j,l) = (j',l') \).

For any \( d \in \hat{E}_s \) we have that \( \langle d \cdot v, d \cdot w \rangle = \langle \sigma(d)v, \tau(d)w \rangle = \sigma(d)\tau(d)\langle v, w \rangle \).

On the other hand using the defining property of the involution we get
\[
\langle d \cdot v, d \cdot w \rangle = \langle v, (d^d) \cdot w \rangle = \tau(d^d)\langle v, w \rangle.
\]

Since, by assumption \( \langle v, w \rangle \neq 0 \) the above relations imply that for all \( d \in \hat{E}_s \) we have
\[
\sigma(d)\tau(d) = \tau(d^d)\tau(d).
\]
Let $F_j$ be the center of the algebra $D_j$. Then (27) implies that for all $d \in F_j$
\[ \hat{\sigma}(d)\hat{\tau}(d) = \hat{\tau}(d^\dagger)\hat{\tau}(d). \] (28)
In particular, this implies that for all $d \in F_j$ we have that
\[ \hat{\tau}(d^\dagger) = \hat{\sigma}(d). \] (29)

If $D_j$ is of Type I in Albert’s classification then the involution restricts to the identity and we get trivially that $\hat{\tau} = \hat{\sigma}$, and hence also $\sigma = \tau$. So our result follows in this case.

If $D_j$ is of Type II then we have that $F_j$ is a totally real field, $D_j$ is a quaternion algebra over $F_j$ and there exists $a \in D_j$ such that the involution is given by $d^\dagger = ad^*a^{-1}$ on $D_j$, where $d^* = \text{tr}_{D_j/F_j}(d) - d$. Note that for $d \in F_j = Z(D_j)$ we have that $\text{tr}_{D_j/F_j}(d) = 2d$, so that $d^\dagger = d$ for all $d \in F_j$. Combining these observations with (29) we get that $\tau = \sigma$.

The same argument we just used for the case of Type II algebras works for the case of Type III algebras, though we do not need to introduce any element $a$ as above since the involution in this case is equal to the canonical involution.

Finally, let us assume that $D_j$ is of type IV in Albert’s classification. In this case $F_j$ is a CM-field. In this case the involution is known to restrict to complex conjugation on the field $F_j$. In other words $d^\dagger = \bar{d}$ for $d \in F_j$. This, together with (29), implies that $\hat{\sigma}(d) = \hat{\tau}(\bar{d})$. Since $F_j$ is a CM-field this implies that $\hat{\sigma} = \bar{\hat{\tau}}$ and by extension $\sigma = \tau$.

**Remark.** The above lemma shows that the splitting (21) of $V_s^*$ is comprised of two types of mutually skew-orthogonal symplectic subspaces. On the one hand, we have the symplectic subspaces $W_\sigma$ that are contained in some $V_j$ that is of Type I-III, and on the other hand we have the symplectic subspaces of the form $W_\tau \oplus W_\bar{\tau}$, where $W_\tau$ is contained in some $V_j$ that is of Type IV. For the second type, note that we also have that $W_\tau$ and $W_\bar{\tau}$ are transverse Lagrangians of these symplectic subspaces.

### §8.4 Constructing relations

We return to our original $G$-admissible variation of $\mathbb{Q}$-HS, restricted to $\Delta^*$. We let $s \in S(L)$ be a fixed point which we assume is in $\Delta^*$. We then have the totally isotropic local subsystem of rank $h$ over the ring $\mathcal{O}_{\mathbb{S}^n}|_{\Delta^*}$
\[ \mathcal{M}_0 := M_0R_n(f_{\mathbb{C}}^n)_*(\mathbb{Q})|_{\Delta^*}. \]
of the local system $R_n(f^\text{an})_*(\mathbb{Q})|_{\Delta^*}$, which has rank $\mu := \dim_\mathbb{Q} V_\ast$.

We fix a basis $\{\omega_i : 1 \leq i \leq \mu\}$ of $H^n_{\text{DR}}(X/S)$ over some dense open subset $U \subset S$ and a trivialization $\{\gamma_j : 1 \leq j \leq \mu\}$ of $R_n f^\text{an}_* \mathbb{Q}|_V$ where $V$ is some open analytic subset of $U^\text{an}$ with $s \in V \subset \Delta^*$. We may and do choose these so that the following conditions are satisfied:

1. the matrices of the skew-symmetric forms on $H^n_{\text{DR}}(X/S)$ and $R_n f^\text{an}_* \mathbb{Q}$ induced by the polarization written with respect to the basis $\{\omega_i\}$ and trivialization $\{\gamma_j\}$ respectively are both equal to $J_\mu$;

2. $\gamma_1, \ldots, \gamma_h \in M_0|_V$ and $\gamma_1, \ldots, \gamma_{\mu/2} \in M^+$, where $M^+$ is a maximal totally isotropic local subsystem of $R_n(f^\text{an})_*(\mathbb{Q})|_V$ that contains $M_0|_V$.

Let us now consider the relative comparison isomorphism

$$P^n_{X/S} : H^n_{\text{DR}}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^\ast} \to R^n f^\text{an}_* \mathbb{Q}_{X^\ast} \otimes_{\mathcal{O}_{S^\ast}} \mathcal{O}_{S^\ast},$$

and restrict it over the set $V$. With respect to the above choices we let $P_{X/S} = \frac{1}{(2\pi i)^n}(\int_{\gamma_j} \omega_i)$ for the matrix of periods of $f$.

Let us write $P_{X/S} = \begin{pmatrix} \Omega_1 & \Omega_2 \\ N_1 & N_2 \end{pmatrix}$. From 2.2, we know that the first $h$ columns of this matrix have entries that are G-functions. It is among their values at $\xi = x(s)$ that we want to find some relation that reflects the action of $\hat{E}_s$.

**Lemma 8.3.** Assume that $h \geq 2$ and that for the point $s \in S(L)$ one of the following is true

1. there exists $\tau : \hat{E}_s \to \mathbb{C}$ such that $h > \dim_{\hat{E}_s} \hat{W}_\tau$, or

2. $s$ is a pseudo-CM point and

$$h \geq \min\{\dim \hat{W}_\tau : \hat{W}_\tau \subset V_i(\tau), \text{ with } V_i(\tau) \text{ of type IV}\}.$$

Then, there exists an algebraic relation among the values at $\xi = x(s)$ of the entries of the first $h$ columns of the matrices $\Omega_1$ and $N_1$. Moreover, this relation corresponds to some homogeneous polynomial with coefficients in $\hat{F}_s$ and degree $\leq 2$.

**Proof.** We take cases depending on the interplay between $\mathcal{M}_{0,s} \otimes \hat{F}_s$ and the splitting (21). We also assume that $V_s = H^n(\tilde{X}_s,\mathbb{C}; \mathbb{Q})$ has a decomposition as in (20).
**Case 1:** Assume there exists some \( \tau : \hat{E}_s \to \mathbb{C} \) such that the following holds

\[
\left( \bigoplus_{\sigma : \hat{E}_s \to \mathbb{C}} \hat{W}_\sigma \right) \cap (\mathcal{M}_s \otimes \hat{F}_s) \neq 0, \tag{30}
\]

then we get, at least one relation of degree 1.

Note that for dimension reasons (30) is satisfied for the \( \tau \) of the first condition above.

Indeed, let \( \gamma \in \Gamma(V, R_n(f_{\mathbb{Q}}^m)_s(\mathbb{Q})) \) be a section such that \( \gamma(s) \) belong to the non-zero space of (30). From 8.1 we get that for all \( \omega \in \hat{W}^{DR}_s \) we have

\[
\frac{1}{(2\pi i)^n} \int_{\gamma(s)} \omega = 0. \tag{31}
\]

Writing \( \gamma \) as an \( \hat{F}_s \)-linear combination of the \( \gamma_j \) with \( 1 \leq j \leq h \) and \( \omega \) as an \( \hat{F}_s \)-linear combination of the \( \omega_i \) with \( 1 \leq i \leq \mu \), we have that (31) leads to a linear equation among the values of the G-functions in question at \( \xi \).

**Case 2:** Assume that for all \( \tau : \hat{E}_s \to \mathbb{C} \) we have

\[
\left( \bigoplus_{\sigma : \hat{E}_s \to \mathbb{C}} \hat{W}_\sigma \right) \cap (\mathcal{M}_s \otimes \hat{F}_s) = 0, \tag{32}
\]

then we want to show that we can create a relation of degree 2.

First of all, we may assume, which we do from now on, that \( \dim_{\hat{F}_s} \hat{W}_\sigma \geq h \) for all \( \sigma \), otherwise we are in case 1, for dimension reasons.

The first step in creating the relations we want is defining symplectic bases with particular properties, which we do in the following claims.

**Claim 1:** There exists a symplectic basis \( e_1, \ldots, e_{\mu/2}, f_1, \ldots, f_{\mu/2} \) of the symplectic vector space \( V_s^* \otimes_{\mathbb{Q}} \hat{F} : = H_n(\bar{X}_s, \mathbb{C}) \otimes \hat{F}_s \) that satisfies the following properties

1. \( \langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0 \) and \( \langle e_i, f_j \rangle = \delta_{i,j} \) for all \( i, j \).
2. \( e_j = \gamma_j(s) \) for \( 1 \leq j \leq h \).
3. There exists \( \tau : \hat{E}_s \to \mathbb{C} \) such that
we know that choosing any basis of local sections satisfies the last condition of our claim.

Proof of Claim 1. From 6.1 we know that any basis of local sections $\gamma_j(s)$ of $\mathcal{M}_{0,s}$, its vectors will satisfy $\langle \gamma_i(s), \gamma_j(s) \rangle = 0$ for all $i, j$. Assume that we have fixed one such basis as above and fix an indexing of the set $\{\sigma : \sigma : \hat{E}_s \to \mathbb{C}\} = \{\sigma_i : 1 \leq i \leq m(s)\}$. We can then write uniquely

$$\gamma_j(s) = w_{j,1} + \ldots + w_{j,m(s)}$$

where $1 \leq j \leq h$ and $w_{j,i} \in \hat{W}_{\sigma_i}$.

By assumption the $\mathbb{Q}$-HS $V_s$ is pseudo-CM, therefore there exists $\tau$ such that $W_\tau$ is as we want in the claim and the same holds for $\hat{W}_\tau$. Without loss of generality assume that $\bar{s} = \sigma_1$. Since we are in Case 2, we also know that $\dim_{\hat{F_s}} W_\tau \geq h$ and that (32) holds. From (32) we get that the vectors $w_{j,1} \in \hat{W}_\tau$ are in fact linearly independent.

By 8.2 we know that $\hat{W}_\tau \oplus \hat{W}_\tau$ is a symplectic vector space with $\hat{W}_\tau$ and $\hat{W}_\tau$ being transverse Lagrangians.

Let $v_j$ with $1 \leq j \leq \dim_{\hat{F_s}} \hat{W}_\tau$ be a basis of $\hat{W}_\tau$ with $v_j = w_{j,1}$ for $1 \leq j \leq h$. We complete this to a symplectic basis $v_i, f_j$, with $1 \leq j \leq \dim_{\hat{F_s}} \hat{W}_\tau$ of $\hat{W}_\tau \oplus \hat{W}_\tau$ such that the $f_j$ are a basis of $\hat{W}_\tau$. Then we have, by construction and by 8.2, that

$$\langle \gamma_i(s), f_j \rangle = \delta_{i,j}$$

for all $1 \leq i, j \leq h$.

Therefore, setting $e_i := \gamma_i(s)$ for $1 \leq i \leq h$ the result follows by extending the set of vectors $\{e_i, f_i : 1 \leq i \leq h\}$ to a symplectic basis of $V_s^* \otimes \hat{F_s}$. Finally, note that the $\tau$ was arbitrary with $\hat{W}_\tau$ being contained in a type IV sub-Hodge structure of $V_s^*$. Therefore, by the assumption that $h \geq \min\{\dim W_\tau : W_\tau \subset V_{s(t)}^*, \text{with } V_{s(t)}^* \text{ of type IV}\}$ in our lemma and the assumption in this second case that $\dim W_\sigma \geq h$ for all $\sigma$ we get that we may find such a $\tau$ that also satisfies the last condition of our claim.

From now on we fix the $\tau$ we found in Claim 1. Having created a symplectic basis for $H_n(X_{s, C}, \mathbb{Q}) \otimes \hat{F_s}$ we want to construct a symplectic basis of $H^n_{DR}(X_s) \otimes L \hat{F_s}$ in a way that lets us take advantage of 8.1.

Claim 2: There exists a symplectic basis $e^1_{DR}, \ldots, e^1_{\mu/2}, f^1_{DR}, \ldots, f^{\mu/2}_{DR}$ of $H^n_{DR}(X_s) \otimes L \hat{F_s}$ such that the following holds
1. \( \forall j \) we have that \( e_j^{DR} \in \hat{W}^*_\sigma \) for some \( \sigma \neq \tau \),

2. for \( 1 \leq j \leq h \) we have that \( f_j^{DR} \in \hat{W}^*_\tau \),

3. for \( h + 1 \leq h \leq \mu / 2 \) we have that \( f_j^{DR} \in \hat{W}^*_\sigma \) for some \( \sigma \neq \tau \).

**Proof of Claim 2.** We start by noting that the results of 8.2 apply easily via duality to the splitting (22) via \( P^n_{\mathcal{X}_s} \), due to our assumption that \( L = \hat{L} \). In particular, via duality we get that for \( \sigma : \hat{E}_s \to \mathbb{C} \) the subspaces \( \hat{W}_{\sigma}^s \) are once again divided into two categories

- \( \hat{W}_{\sigma}^s \) that are symplectic subspaces, corresponding to \( \hat{W}_{\sigma}^s \) that are contained in simple sub-Hodge structures of \( V^*_s \), after these are tensored with \( \hat{F}_s \), that are of Type I, II or III, and

- \( \hat{W}_{\sigma}^s \) that are isotropic subspaces appearing in pairs such that \( \sigma \) and \( \bar{\sigma} \) are both algebra homomorphisms \( \hat{E}_s \to \mathbb{C} \) and \( \hat{W}_{\sigma}^s \oplus \hat{W}_{\bar{\sigma}}^s \) is a symplectic subspace. These correspond via duality to the \( \hat{W}_{\sigma}^s \) that are contained in simple sub-Hodge structures of \( V^*_s \), again after these are tensored with \( \hat{F}_s \), that are of Type IV.

With that in mind, for each \( \sigma \) we pick vectors \( e_{\sigma i} \) so that

- the \( e_{\sigma i} \) are the basis of a Lagrangian subspace of \( \hat{W}_{\sigma}^s \) if we are in the first case above, so that in this case \( 1 \leq i \leq \frac{\dim \hat{F}_s \hat{W}_{\sigma}^s}{2} \),

- the \( e_{\bar{\sigma} i} \) are a basis of \( \hat{W}_{\bar{\sigma}}^s \) of our fixed \( \bar{\tau} \), and

- in the second case above for each \( \sigma \neq \tau, \bar{\tau} \) we pick one \( \sigma \) for each pair \((\sigma, \bar{\sigma})\) and we let \( e_{\sigma i} \) be a basis of \( \hat{W}_{\sigma}^s \).

Let \( e_j^{DR} \), with \( 1 \leq j \leq \mu \), be any indexing of the set of all the \( e_{\sigma i} \) above. The spanning set of these defines a Lagrangian subspace of \( H^*_{DR}(X_s) \otimes_L \hat{F}_s \). In a similar manner, by the above remarks derived from 8.2, we can construct a basis of a transverse Lagrangian to the Lagrangian spanned by the \( e_j^{DR} \) with \( f_j^{DR} \) also elements of the various \( \hat{W}_{\sigma}^s \). It is also straightforward from the above that we may pick \( f_1^{DR}, \ldots, f_h^{DR} \in \hat{W}_\tau^s \).

**Step 1: Changing bases.** We note that the bases \( \beta_2 := \{e_i, f_i : 1 \leq i \leq \mu / 2\} \) and \( \beta_2^{DR} := \{e_i^{DR}, f_i^{DR} : 1 \leq i \leq \mu / 2\} \) that were created above are \( \hat{F}_s \)-linear combinations of the bases \( \beta_1 := \{\gamma_j(s) : 1 \leq j \leq \mu\} \) and \( \beta_1^{DR} := \{\omega_j(s) : 1 \leq j \leq \mu\} \) respectively. Since all bases are by construction symplectic the base change matrices are all symplectic matrices. Note that for the change of base matrix \([I_{\mu}]_{\beta_1}^{\beta_2}\) we will have by construction of \( \beta_2 \) have that its first \( h \) columns will be

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\[
\begin{pmatrix}
I_h \\
0
\end{pmatrix}.
\]

Let us consider the isomorphism \( P^n_{X_s} : H^n_{DR}(\bar{X}_s, \mathbb{Q}) \otimes \mathbb{C} \to H^n(\bar{X}^an_{\mathcal{C}}, \mathbb{Q}) \otimes \mathbb{C} \). Let \( \tilde{P}_j \) for \( j = 1, 2 \) be the matrix\(^4\) of this isomorphism with respect the basis \( \beta_j^{DR} \) and the dual of the basis \( \beta_j \). We are interested in the matrices \( P_j := (2\pi i)^n \tilde{P}_j \), note that \( P_1 \) is the value of the relative period matrix at \( \xi = x(s) \). For the matrices \( P_j \) we have
\[
P_2 = [I_{2\mu}]^{\beta_j^{DR}} P_1 [I_{2\mu}]^{\beta_i^j},
\]
and all these matrices are symplectic, while the two change of base matrices will have coefficients in the field \( \hat{F}_s \).

**Step 2: Relations on \( P_2 \).** Let us examine the matrix \( P_2 \) in more detail. Write
\[
P_2 = \begin{pmatrix}
\Gamma_1 & \Gamma_2 \\
\Delta_1 & \Delta_2
\end{pmatrix},
\]
where \( \Gamma_i \) and \( \Delta_i \) are \( \mu/2 \times \mu/2 \) matrices. For convenience we also let \( \tilde{\Gamma}_i \) and \( \tilde{\Delta}_i \) for the \( \mu/2 \times h \) matrices defined by the first \( h \) first columns of the matrices \( \Gamma_i \) and \( \Delta_i \) for \( i = 1, 2 \) respectively.

From the fact that \((2\pi i)^n P_2 \) is symplectic we have the relations\(^5\)
\[
\begin{align*}
^t \Delta_j \Gamma_j &= ^t \Gamma_j \Delta_j, \\
^t \Delta_2 \Gamma_1 - ^t \Gamma_2 \Delta_1 &= \frac{I_{\mu/2}}{(2\pi i)^n}
\end{align*}
\]

By construction of the bases in the two claims above and 8.1 we immediately get that \( \tilde{\Gamma}_2 = 0 \).

Let us set \( \Delta_{2,h} \) to be the \( h \times h \) matrix given by \((\int \gamma_i f^i_{DR})_{1 \leq i,j \leq h}\). Let us also set \( \Delta_{1,h} \) be the \( h \times h \) matrix given by \((\int \gamma_i f^i_{DR})_{1 \leq i,j \leq h}\). In other words, \( \Delta_{i,h} \) is the submatrix of \( \Delta_i \) that is comprised of the entries in the first \( h \) columns and first \( h \) rows of \( \Delta_i \).

**Claim 3:** There exists an \( h \times h \) matrix \( T \in \text{GL}_h(\hat{F}_s) \) which is such that
\[
\Delta_{1,h} = \Delta_{2,h} ^t T.
\]

\(^4\)Note that in keeping with our earlier notation the matrix acts via multiplication on the right, i.e. \( P^n(x) = [x]^{\beta_j^{DR}} \tilde{P}_j \).

\(^5\)This follows from the Riemann relations. See the A for more details.

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Proof of Claim 3. We have a pairing

\[ \hat{W}_{DR}^\tau \times H_n(\bar{X}_{s,C}, \mathbb{Q}) \otimes \hat{F}_s \to \mathbb{C}, \quad (40) \]

defined by \((\omega, \gamma) \mapsto \int_\gamma \omega\).

By 8.1 we have that this induces a perfect pairing

\[ \hat{W}_{DR}^\tau \times \left( \left( H_n(\bar{X}_{s,C}, \mathbb{Q}) \otimes \hat{F}_s \right) / \left( \bigoplus_{\sigma:E_s\to C, \sigma \neq \tau} \hat{W}_\sigma \right) \right) \to \mathbb{C}, \]

On the one hand, we know that \(\{f_1, \ldots, f_h\}\), the basis of \(\hat{W}_\tau\), maps to a basis in the quotient \(\left( H_n(\bar{X}_{s,C}, \mathbb{Q}) \otimes \hat{F}_s \right) / \left( \bigoplus_{\sigma:E_s\to C, \sigma \neq \tau} \hat{W}_\sigma \right)\). On the other hand, from the assumption (32), we get that the basis \(\{\gamma_1, \ldots, \gamma_h\}\) also maps to a basis of the same quotient.

Let \(T\) be the transpose of the change of basis matrix from the basis induced by the \(\gamma_j\) on \(\left( H_n(\bar{X}_{s,C}, \mathbb{Q}) \otimes \hat{F}_s \right) / \left( \bigoplus_{\sigma:E_s\to C, \sigma \neq \tau} \hat{W}_\sigma \right)\), to that induced on the same space by the \(f_j\). Then \(T \in \text{GL}_h(\hat{F}_s)\) and (40) holds.

Let \(\Gamma_{1,h}\) be, once again, the submatrix of \(\Gamma_1\) that is comprised of the entries in the first \(h\) columns and first \(h\) rows of \(\Gamma_1\). We have already seen that \(\tilde{\Gamma}_2 = 0\) and, by the construction of Claims 1 and 2 and 8.1, that

\[ \tilde{\Delta}_2 = \begin{pmatrix} \Delta_{2,h} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \]

Using these facts we derive from (38) that

\[ t^{\dagger} \Delta_{2,h} \Gamma_{1,h} = \frac{1}{(2\pi i)^n} I_h, \quad (41) \]

Let us now multiply both sides of (41) by \(T\). Then, using (39), we get

\[ t^{\dagger} \Delta_{1,h} \Gamma_{1,h} = \frac{T}{(2\pi i)^n}. \quad (42) \]

Step 3: Relations on \(P_1\). The relation (42) we created on the first \(h\) columns of the matrix \(P_2\) will translate to relations among the coefficients of the first \(h\) columns of the matrix \(P_1\). Since these are the values of the
G-functions we are interested in, this will finish the proof of this lemma.

We start with introducing some notation let \( I_{\mu}^{\beta_1} = (A_1 B_1) \) and \( I_{\mu}^{\beta_2} = (A_2 B_2) \), were the \( A_i, B_i, C_i, \) and \( D_i \in M_{\mu/2}(\hat{F}_s) \). In keeping the same notation as above, for any matrix \( A \in M_{\mu/2}(C) \) we define \( \tilde{A} \) to be the \( \mu/2 \times h \) matrix defined by the first \( h \) columns of \( A \). Note that by our construction in Claim 1 we know that \( \tilde{C}_1 = 0 \) and \( \tilde{A}_1 = \begin{pmatrix} I_h \\ 0 \end{pmatrix} \).

With this notation (35) becomes

\[
\begin{pmatrix}
\Gamma_1 & \Gamma_2 \\
\Delta_1 & \Delta_2
\end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} \Omega_1(s) & \Omega_2(s) \\ N_1(s) & N_2(s) \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}. \tag{43}
\]

From this we get the following two relations

\[
\Gamma_1 = A_2 \Omega_1(s)A_1 + A_2 \Omega_2(s)C_1 + B_2 N_1(s)A_1 + B_2 N_2(s)C_1, \tag{44}
\]

and

\[
\Delta_1 = C_2 \Omega_1(s)A_1 + C_2 \Omega_2(s)C_1 + D_2 N_1(s)A_1 + D_2 N_2(s)C_1. \tag{45}
\]

Now we notice that for any matrices \( A, B \in M_\mu(C) \) we have that \( \tilde{A} \tilde{B} = A \tilde{B} \) and that \( A \cdot \begin{pmatrix} I_h \\ 0 \end{pmatrix} = \tilde{A} \). Using these observations on (44) and (45) we get

\[
\tilde{\Gamma}_1 = A_2 \tilde{\Omega}_1(s) + B_2 \tilde{N}_1(s), \tag{46}
\]

and

\[
\tilde{\Delta}_1 = C_2 \tilde{\Omega}_1(s) + D_2 \tilde{N}_1(s). \tag{47}
\]

Substituting these in (42) we get

\[
^t((C_2 \tilde{\Omega}_1(s) + D_2 \tilde{N}_1(s))_h)(A_2 \tilde{\Omega}_1(s) + B_2 \tilde{N}_1(s))_h = \frac{T}{(2\pi i)^n}, \tag{48}
\]

where, using the same notation as earlier, the subscript \( h \) signifies that we are considering the \( h \times h \) submatrices that are comprised by the first \( h \) rows of these \( \mu/2 \times h \) matrices.

Since we are assuming that \( h \geq 2 \), equation (48) provides relations among the values of the G-functions we want at \( \xi = x(s) \) that, upon getting rid of the factor \( (2\pi i)^n \), correspond to homogeneous polynomials with coefficients in \( \hat{F}_s \) and degree \( \leq 2 \). \( \Box \)
§8.4.1 Some cleaning up

The technical conditions

\[ \exists \tau : \hat{E}_s \to \mathbb{C} \text{ such that } h > \dim_{\hat{E}_s} \hat{W}_\tau \]  \hspace{1cm} (49)

\[ h \geq \min \{ \dim \hat{W}_\tau : \hat{W}_\tau \subset V_{i(\tau)}, \text{ with } V_{i(\tau)} \text{ of type IV } \}. \]  \hspace{1cm} (50)

that appear in 8.3 are by no means aesthetically pleasing! We have dedicated this short section to remedy this fact. In fact we prove the following lemma.

**Lemma 8.4.** Condition (50) is equivalent to the condition

\[ h > \frac{\dim Q V_i}{[Z(D_i) : \mathbb{Q}]} \text{ for some } j, \]

and condition (50) is equivalent to the condition

\[ h \geq \min \{ \dim Q V_i : i \text{ such that } D_i = \text{End}_{HS} (V_i) \text{ is of type IV } \}. \]

To prove this we work in greater generality with modules of semisimple algebras over \( \mathbb{Q} \). The material in this section is definitely not new but we include it for the sake of completeness of our exposition.

Let us fix some notation. We consider a \( \mathbb{Q} \)-HS \( V \) with \( \mu := \dim_{\mathbb{Q}} V \) that decomposes as \( V = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r} \). We write \( D = M_{m_1}(D_1) \oplus \cdots \oplus M_{m_r}(D_r) \) for the algebra of Hodge endomorphisms of \( V \), where \( D_i \) is the algebra of Hodge endomorphisms of \( V_i \). For each \( i \) we let \( F_i := Z(D_i) \) be the center of \( D_i \) and \( f_i := [F_i : \mathbb{Q}] \). Finally, we let \( \hat{F} \) be the Galois closure of the compositum of the fields \( F_i \) and \( \hat{E} := F_1^{m_1} \oplus \cdots \oplus F_r^{m_r} \) be the maximal commutative semisimple algebra of \( D \).

For the non-trivial homomorphisms of algebras \( \sigma : \hat{E} \to \hat{F} \) we write \( \sigma = \tilde{\sigma}_{j,l} \) as we did earlier. The above result then follows from the following lemma.

**Lemma 8.5.** The \( \hat{E} \otimes_{\mathbb{Q}} \hat{F} \)-module \( V \otimes_{\mathbb{Q}} \hat{F} \) has a decomposition as an \( \hat{E} \otimes_{\mathbb{Q}} \hat{F} \)-module as

\[ V \otimes_{\mathbb{Q}} \hat{F} = \bigoplus_{\sigma : \hat{E} \to \hat{F}} \hat{W}_\sigma, \]

where \( \hat{W}_\sigma \) are \( \hat{F} \)-subspaces of \( V \otimes_{\mathbb{Q}} \hat{F} \) on which \( \hat{E} \otimes_{\mathbb{Q}} \hat{F} \) acts via multiplication by \( \sigma \). Moreover, \( \dim_{\hat{E}} W_\sigma = \frac{\dim_{F_i} V_{i(\sigma)}}{f_i(\sigma)} \) where \( i(\sigma) \in \{1, \ldots, r\} \) is such that \( \sigma = \tilde{\sigma}_{i(\sigma),l} \) for some \( l \) and \( \tilde{\sigma} \) with our previous notation.
Proof. First of all, note that \( \forall i \) we have \( F_i \hookrightarrow \text{End}_Q V_i \) trivially. Therefore \( V_i \) is isomorphic, as an \( F_i \)-module, to \( V_i \cong F_i^{t_i} \) for some \( t_i \). Counting dimensions of these as \( Q \)-vector spaces we get that \( t_i = \dim_Q V_i \).

Tensoring both sides of (51) by \( \otimes Q \hat{F} \) we get that \( V_i \otimes Q \hat{F} \cong (F_i \otimes Q \hat{F})^{t_i} \) as \( F_i \)-modules. Now note that since \( \hat{F} \) is a Galois extension that contains \( F_i \) we have that \( F_i \otimes Q \hat{F} \cong \bigoplus_{\bar{\sigma}:F_i \hookrightarrow \hat{F}} F_i^{t_i} \bar{\sigma} \), where \( \hat{F} \bar{\sigma} \) is just \( \hat{F} \) viewed as an \( F_i \)-module via the action of the embedding \( \bar{\sigma}:F_i \hookrightarrow \hat{F} \). Combining the above we get

\[
V_i \otimes Q \hat{F} \cong \bigoplus_{\bar{\sigma}:F_i \hookrightarrow \hat{F}} (\hat{F} \bar{\sigma})^{t_i}.
\]

The result now follows trivially. \( \square \)

§9 Non-trivial relations

The relations created in 8.3 were created after fixing a place \( v \in \Sigma_{L,\infty} \), corresponding to an inclusion \( i_v: L \to \mathbb{C} \). This is because we assumed that \( s \) is archimedeanly close to \( s_0 \), with respect to this fixed embedding \( L \hookrightarrow \mathbb{C} \).

Definition. Let \( s \in S(L) \) for some \( L/K \) and set \( \xi = x(s) \in L \). For a place \( v \in \Sigma_L \) we say that \( s \) is \( v \)-adically close to \( s_0 \) if \( |\xi|_v < \min\{1, R_v(\bar{y}) := R_v(y_1, \ldots, y_{h_\mu}) \} \), where \( y_j \) are the G-functions we had earlier.

We want to create relations among the values of the G-functions \( y_i \in K[[x]] \) at \( \xi = x(s) \) for all places \( v \in \Sigma_{L,\infty} \). In order to be able to create these we will need the following technical lemma, following the exposition in Ch.X, §3.1 of [And89]. We fix a priori, the matrix

\[
G = \begin{pmatrix}
y_1 & \cdots & y_h \\
\vdots & \ddots & \vdots \\
y_{h\mu-h+1} & \cdots & y_{h\mu}
\end{pmatrix} \in M_{\mu \times h}(K[[x]])
\] (52)

Let us now consider \( \iota: K \hookrightarrow \mathbb{C} \) to be a random complex embedding of \( K \). We then have the complex Taylor series \( \iota(y_i) \). We also let \( G_\iota \) be the matrix defined analogously to \( G \) with the \( y_i \) replaced by the \( \iota(y_i) \).
Lemma 9.1. For any \( i \) as above the matrix \( G_i \) is again the matrix that consists of the entries in the first \( h \) columns of a period matrix with respect to the same basis of local sections of \( H^n_{DR}(X/S) \) and to some local frame of the local system \( R_n(f^n_{\ast,C})_\ast(Q_{X^n}) \).

Remark. Here by \( f^n_{\ast,C} \) we denote the analytification of the morphism \( f_\ast \), where \( f \) is the morphism induced from \( f : X \to S \) via the base change given by \( i : \text{Spec} \mathbb{C} \to \text{Spec} K \).

Proof. This follows essentially from the proof of Theorem 2 in Ch.X, §4.1 of [And89], which constitutes §4.4 of the same chapter. We review the main parts we need. We let \( S_1 = S \cup \{ s_0 \} \) and write \( f_1 : X_1 \to S_1 \) for the pullback of \( f' : X' \to S' \) via \( S_1 \to S' \). We note that the pair \( (S_1, X_1) \) will be the pair \( "(S', X')" \) in the notation of loc. cit.

Step 1: A short review of the construction. For each point \( Q \in Y^{[n]} \) we can find an affine open subset \( U^Q \) of \( X_1 \) admitting algebraic coordinates \( x_{Q,1}, \ldots, x_{Q,n+1} \) such that \( Y_i \cap U^Q = Z(x_{Q,i}) \) and the local parameter \( x \) of \( S_1 \) at \( s_0 \) lifts to \( x = x_{Q,1} \cdots x_{Q,n+1} \). To ease our notation we write simply \( x_i \) for \( x_{Q,i} \). We also fix the inclusion \( i_Q : U^Q \to X_1 \). Then loc.cit. describes a horizontal map \( T_Q : H^n_{DR}(X_1/S_1(\log Y)) \to K[[x]] \).

This map \( T_Q \) also has an analytic description. To define it one needs some cycles \( i_{Q} \gamma_Q \). We briefly review the definition of these cycles. For each \( z \in \Delta \) we have the cycle \( \gamma_{Q,z} \in H^n_n((U^Q)^{an}, \mathbb{Z}) \) defined by the relations \( |x_2| = \cdots = |x_{n+1}| = \epsilon \) and \( x_1x_2 \cdots x_{n+1} = x(z) \), where \( \epsilon > 0 \) is small. These cycles glue together to define a section \( \gamma_Q \in H^n_0(\Delta, R_n((f_1)_\ast i_Q)_\ast(Q)) \) which we can push-forward to a cycle \( i_{Q} \gamma_Q \in H^n_0(\Delta, R_n((f_1)_\ast)_\ast(Q)) \).

From this it follows that \( (i_{Q} \gamma_Q)_z \in H_n(X^{an}_z, \mathbb{Q}) \) is also invariant by the action of \( \pi_1(\Delta^*, z) \). In fact from the exposition in loc. cit. we know that the cycles \( (i_{Q} \gamma_Q)_z \) span the fiber \( M_0 R_n(f^n_{\ast,C})_\ast(Q)_z \) for \( z \in \Delta^* \).

From the analytic description of \( T_Q \) we get that for \( 1 \leq i \leq h_\mu \) there exists a point \( Q \in Y^{[n]}(\mathbb{Q}) \) such that the entry \( y_i|_\Delta \) is equal to

\[
T_Q(\omega) = \int_{i_{Q} \gamma_Q} \omega
\]

for some \( \omega \in H^n_{DR}(X_1/S_1(\log Y)) \), where \( \Delta \) is a unit disk centered at \( s_0 \).

To be able to work over \( K \), instead of \( \mathbb{Q} \) as loc. cit. does, we assume without loss of generality that all of the above, i.e. the points \( Q \), the algebraic coordinates, and the coefficients of the \( y_i \) are actually defined over our original field \( K \). To achieve this we might have to a priori base change everything, i.e. \( f : X \to S \) and \( f' : X' \to S' \), by some fixed finite extension.
\(K\) of our original field \(K\). This does not affect our results so we may and do assume it.

**Step 2: Changing embeddings.** Implicit in the definition of the cycles \(i_{Q,*}\gamma_Q\) is the fixed embedding \(K \hookrightarrow \mathbb{C}\). Shifting our point of view to the embedding \(\iota : K \hookrightarrow \mathbb{C}\) we get a similar picture. Given a \(K\)-variety \(Z\) we define \(Z_\iota := Z \times_{\text{Spec}K} \text{Spec} \mathbb{C}\) the base change of \(Z\) via \(\iota : \text{Spec} \mathbb{C} \to \text{Spec} K\) and similarly for the base change of a morphism \(\phi : Z_1 \to Z_2\) between \(K\)-varieties. In other words we suppress reference to the original embedding \(K \hookrightarrow \mathbb{C}\) but keep track of the new embeddings.

The algebraic coordinates \(x_1,\ldots,x_{n+1}\) on \(U^Q\) pullback to algebraic coordinates \(\iota^* x_1,\ldots,\iota^* x_{n+1}\) on \(U_i^Q\). We write \(x_{i,t}\) for \(\iota^* x_i\) and also consider a unit disk \(\Delta_t \subset (S_t)_{\iota_0}^\infty\) centered at \(s_0\).

Once again we have that \(x_{i_1,t}\cdots x_{i_{n+1},t} = x_t\). We define the cycles \(\gamma_{Q,t}\) similarly:

\[
\text{for } z \in \Delta_i \text{ we let } \gamma_{Q,t,z} \in H_n((U_i^Q)_{\iota_0}^\infty, \mathbb{Z}) \text{ be defined by } |x_{2,i}| = \ldots = |x_{n+1,i}| = \epsilon \text{ and } x_{1,i} x_{2,i} \cdots x_{n+1,i} = x_i(z). \text{ Once again these glue together to give cycles } i_{Q,*}\gamma_{Q,t} \in H^0(\Delta_i, R_n((f_{i})_{\iota,\Delta_i})_{\iota_0}^\infty \mathbb{Q}).
\]

The cycles \((i_{Q,*}\gamma_{Q,t})_z\), for \(Q\) varying in the set \(Y_0[n]\), will span the fiber of the local system \(M_0 R_n(f_{i}^\infty)_*(\mathbb{Q}_{X_{i,t}^\infty})_z\) for \(z \in \Delta_i^\infty\). This follows from the exposition in loc.cit. since the proof does not depend on the embedding \(K \hookrightarrow \mathbb{C}\).

Among these we may choose a frame of \(M_0 R_n(f_{i}^\infty)_*(\mathbb{Q}_{X_{i,t}^\infty})_V\) and then extend that to a frame of \(R_n(f_{i}^\infty)_*(\mathbb{Q}_{X_{i,t}^\infty})_V\), where \(V \subset \Delta_i^\infty\) is some open subset of \(\Delta_i^\infty\). We thus get a relative period matrix \(P_i\) of the morphism \(f\).

Finally, Remark 1 page 21 of [And89] together with the exposition in the aforementioned proof show that in fact \(G_1 = G_i\), where \(G_i\) is the matrix that consists of the first \(h\) columns of \(P_i\), and the result follows.

\[\square\]

§9.0.1 Construction of the actual relations

Let \(s \in S(L)\) be a point of the variation satisfying either of the conditions of 8.3. We assume that \(s\) is \(v_0\)-adically close for some fixed \(v_0 \in \Sigma_{L,\infty}\). Considering the embedding \(i_{v_0} : L \to \mathbb{C}\), which we drop from notation from now on writing just \(L \to \mathbb{C}\), the construction of 8.3 goes through.

We consider now \(G\), as in (52) above, to be the matrix of \(G\)-functions created with respect to that embedding. For any other place \(v \in \Sigma_{L,\infty}\) such that \(s\) is \(v\)-adically close to \(s_0\) we repeat the process of 8.3, this time we replace \(K\) by \(i_v(K)\), \(L\) by \(i_v(L)\), and \(X_s\) by \(X_s \times_L i_v(L)\). Thanks to
we may choose trivializations so that the corresponding $\mu \times h$ matrix of $G$-functions we are interested in is $G_{\nu}$. As a result for any such archimedean place $v$ we get a polynomial $q_v$ with coefficients in $L$ such that $i_v(q_v)(i_v(y_1)(\xi), \ldots, i_v(y_h)(i_v(\xi))) = 0$. We let

$$q = \prod_{v \in \Sigma_{L,\infty}} q_v.$$  \hfill (53)

The relation we were looking for is the one coming from this polynomial. This relation holds $v$-adically for all archimedean places for which $|\xi|_v < \min\{1, R_v(\vec{y})\}$, by construction.

Later on, we describe conditions that guarantee that $s$ cannot be $v$-adically close to $s_0$ for $v \in \Sigma_{L,f}$. This will, effectively, make (53), or to be more precise the relation it induces at $\xi$, global in that case.

Leaving the proof of globality for later we note that the relation induced from (53) satisfies the other key property we want. Namely we have the following lemma.

**Lemma 9.2.** The relations created above are non-trivial, assuming that the generic special Mumford-Tate group of our variation is $Sp(\mu, \mathbb{Q})$.

**Proof.** This follows by comparing the relations in (31) and (48) with the polynomials described in 7.5. For Case 2 above, it is easier to see the non-triviality of the relations in question by looking at (42) instead of (48). \hfill $\square$

**Part IV**

**Towards globality**

**§10 Review on the action of the inertia group**

Here we review some notions about the action of the inertia group on étale cohomology of varieties over local fields. We then apply these results to our case of interest that of $G$-admissible variations of $\mathbb{Q}$-HS.

**§10.1 Grothendieck’s monodromy Theorem**

Let $X/K$ be a smooth projective variety with $K$ a local field whose residue field has characteristic $p$. We let $G_K := \text{Gal}(K_s/K)$ be the absolute Galois group of the field $K$, $I_K \leq G_K$ be its inertia subgroup, and we also let $X_{K_s} = X \times_K K_s$. We have a natural action of these groups, which we denote by
\[ \rho_l : G_K \to Aut(H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l)), \]
on the étale cohomology groups of \( X_{K_s} \) for \( l \neq p \). This action for the inertia group is described by the following classical theorem of Grothendieck.

**Theorem 10.1.** [ST68] Let \( X \) be a smooth projective variety over \( K \). Then the inertia group \( I \) acts quasi-unipotently on the étale cohomology group \( H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \).

From this we get that there exists a finite field extension \( L/K \) for which the inertia group \( I_L \) acts unipotently on \( H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \). The unipotency of this action can be described more explicitly and provides an ascending filtration, called the monodromy filtration, \( M_* \) of \( H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \). We present a short review of these facts here.

Let us choose a uniformizer \( \varpi \in \mathcal{O}_L \) and consider, for each \( n \geq 0 \) the map

\[ t_{L,n} : I_L \to \mu_n, \]

which is defined by \( \sigma(\varpi^n) = t_{L,n}(\sigma)\varpi^n. \) We then define the map

\[ t_l : I_L \to \mathbb{Z}_l(1), \]
as the inverse limit of the maps \( t_{L,n} \).

Then there exists a nilpotent map, called the monodromy operator, \( N : H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l)(1) \to H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \) such that for all \( \sigma \in I_L \) we have \( \rho_l(\sigma) = \exp(Nt_l(\sigma)) \).

**The monodromy filtration**

From the above operator \( N \) we can construct the monodromy filtration on \( H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \) written as

\[ 0 = M_{-i-1} \subset M_{-i} \subset \ldots \subset M_{i-1} \subset M_i = H^i_{\text{et}}(X_{K_s}, \mathbb{Q}_l) \]

we also define the \( j \)-th graded quotient of the filtration to be \( \text{Gr}^M_j := M_j/M_{j-1} \).

We record the main properties of the monodromy filtration in the following lemma.

**Lemma 10.1.** Let \( M_* \) be the above monodromy filtration. Then the following hold

1. \( NM_j(1) \subset M_{j-2} \) for all \( j \),
2. the map $N^j$ induces an isomorphism $Gr^M_j(j) \xrightarrow{\cong} Gr^M_j$ for all $j$,

3. the monodromy filtration is the unique ascending filtration satisfying the above two properties, and

4. the inertia group $I_L$ acts trivially on $Gr^M_j$ and $M^I \subset M_0$.

Proof. This follows from [Del80], Prop. 1.6.1 and the above discussion. □

§10.1.1 Filtrations and endomorphisms

In what follows we will need the following lemma.

Lemma 10.2. Let $k$ be a field of characteristic 0 and let $A = A_1 \oplus \ldots \oplus A_r$ be a semisimple algebra over $k$, where $A_i$ are its simple summands, and assume that $A \hookrightarrow (\text{End}(V))^N$, where $N$ is a nilpotent endomorphism of the finite dimensional $k$-vector space $V$.

Let $W_\bullet$ be the ascending filtration of $V$ defined by $N$ and let $h_i := \dim_k Gr^W_i$. Then, for each $1 \leq i \leq r$ we have that there exists $j(i)$ with

$$A_i \hookrightarrow \text{End}(Gr^W_{j(i)}) \cong M_{h_{j(i)}}(k).$$

Proof. We assume without loss of generality that $i = 1$ and proceed by induction on the degree $n+1$ of nilpotency of $N$, and hence the length $2n$ of the filtration $W_\bullet$,

$$0 \subset W_{-n} \subset W_{-n+1} \subset \ldots \subset W_{n-1} \subset W_n = V.$$

Let $\psi_1 : \text{End}(V)^N \rightarrow \text{End}(W_{-n})$ be the homomorphism of $k$-algebras defined by $F \mapsto F|_{W_{-n}}$. Since the algebra $A_i$ is simple, either one of two things will happen

1. $\ker \psi_1 \cap A_1 = \{0\}$, in which case we are done, or

2. $A_1 \subset \ker \psi_1$.

From now on we assume that we are in the second case.

Let $N_1$ be the nilpotent endomorphism induced by $N$ on the quotient $W_{n-1}/W_{-n}$. We note that the degree of nilpotency of $N_1$ is $n$, i.e. it has dropped by 1. We than make the following

Claim. There is a natural embedding of $k$-algebras

$$A_1 \hookrightarrow \text{End}(W_{n-1}/W_{-n})^{N_1}.$$

Proof of the Claim. First of all, note that we have the map
\[ \phi_1 : \ker \psi_1 \to \text{End}(W_{n-1}/W_{-n})^{N_1}, \]
given by \( F \mapsto F|_{W_{n-1}} \mod W_{-n}. \)

Since \( A_1 \subset \ker \psi_1 \) is simple we get that, once again, either \( A_1 \subset \ker \phi_1 \) or \( A_1 \cap \ker \phi_1 = \{0\} \). We clearly cannot have the first case since all elements of \( \ker \phi_1 \) are nilpotent, in fact \( F^3 = 0 \) for all such \( F \). Therefore, \( A_1 \cap \ker \phi_1 = \{0\} \) and the algebra homomorphism \( \phi_1 \) restricts to an embedding of \( A_1 \) into \( \text{End}(W_{n-1}/W_{-n})^{N_1} \) as we wanted.

By induction our result follows, noting that the case \( n = 1 \) is trivially dealt with by the above argument.

\section{Inertia and endomorphisms}

We believe the results in this subsection are broadly known to experts. Unable to find a reference for these we include them here for the sake of completeness.

Let us consider \( f : X \to S \) a G-admissible variation of \( \mathbb{Q}\text{-HS} \) defined over the fixed number field \( K \) as usual. We also fix a point \( s \in S(L) \) that is \( v \)-adically close to \( s_0 \) for some \( v \in \Sigma_{L_f} \). Our main goal in this section is a brief study of the relation between the algebra of Hodge endomorphisms at \( s \in S(L) \) and its relation with the endomorphisms of \( \text{H}^n_{\text{et}}(\bar{X}_{s,v}, \mathbb{Q}_l) \) where \( l \neq p(v) \) and \( \bar{X}_{s,v} = (X_s \times_L \bar{L}_v) \times_{\bar{L}_v} \bar{L}_v. \)

This relation is captured by the following proposition.

**Proposition 10.1.** Assume the Hodge conjecture is true for \( X_s \) and that the action of the inertia group \( I_{L_v} \) on \( \text{H}^n_{\text{et}}(\bar{X}_{s,v}, \mathbb{Q}_l) \) is unipotent. Then

\[
D_s \otimes \mathbb{Q}_l := (\text{End}(\text{H}^n(\bar{X}_{s,v}^{an}, \mathbb{Q})))^{M_s} \otimes_{\mathbb{Q}_l} \mathbb{Q}_l \hookrightarrow (\text{End}(\text{H}^n_{\text{et}}(\bar{X}_{s,v}, \mathbb{Q}_l)))^{I_{L_v}}.
\]

In order to prove this we follow the same strategy of proof as that of Theorem 1 of \cite{Rib75}. We first start with the following lemma.

**Lemma 10.3.** There is a natural endomorphism of \( \mathbb{Q}_l \) algebras

\[
(\text{End}(\text{H}^n(\bar{X}_{s,v}^{an}, \mathbb{Q})))^{G_{\text{mt},s}} \otimes_{\mathbb{Q}_l} \mathbb{Q}_l \hookrightarrow \text{End}(\text{H}^n_{\text{et}}(\bar{X}_{s,v}, \mathbb{Q}_l)).
\]

**Proof.** As a corollary\(^6\) of the Smooth base change Theorem for lisse \( l \)-adic sheaves we have that

\[
\text{H}^n_{\text{et}}(\bar{X}_{s,v}, \mathbb{Q}_l) \cong \text{H}^n_{\text{et}}(\bar{X}_s, \mathbb{Q}_l) \cong \text{H}^n_{\text{et}}(\bar{X}_{s,C}, \mathbb{Q}_l), \tag{54}
\]

\(^6\)See Corollary 4.3, Ch. VI of \cite{Mil80} applied to the field extension \( \bar{L}_v/L \) and \( C/L. \)
where $\bar{X}_s = X_s \times_L \bar{L}$ and $\bar{X}_{s,\mathbb{C}} = X_s \times_{\bar{L}} \mathbb{C}$.

Applying Artin’s comparison theorem for lisse $l$-adic sheaves, we get

$$H^n_{\text{ét}}(\bar{X}_{s,\mathbb{C}}, \mathbb{C}, \mathbb{Q}_l) \cong H^n(\bar{X}^\text{an}_{s,\mathbb{C}}, \mathbb{Q}_l).$$

Combining (54) with (55) we get

$$H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l) \cong H^n(\bar{X}^\text{an}_{s,\mathbb{C}}, \mathbb{Q}_l),$$

and the inclusion map we want follows.

Proof of 10.1. It suffices to show that given $\sigma \in I_{L,v}$ and $f \in (\text{End}(H^n(\bar{X}^\text{an}_{s,\mathbb{C}}, \mathbb{Q}_l)))^{G_{\text{mt},s}}$ the corresponding elements of $\text{End}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l))$ commute with each other.

Since $f \in (\text{End}(H^n(\bar{X}^\text{an}_{s,\mathbb{C}}, \mathbb{Q}_l)))^{G_{\text{mt},s}}$ the Hodge conjecture implies that $f$ is defined over some finite extension $\bar{L}/L$. Therefore, by compatibility of the cycle class maps in étale and singular cohomology with Artin comparison, $f$ commutes with $\sigma^k$ for some $k \geq 1$. The result then follows from Lemma 1.2 of [Rib75] since the action in question is unipotent.

§11 Good models and $v$-adic proximity

We return from now on to the notation of §4. So let us fix for the remainder a $G$-admissible variation of $\mathbb{Q}$-HS given by the map $f : X \to S$ with all the relevant notions of §4 defined over some fixed number field $K$. Let us also fix a local parameter $x \in K(S')$ at the point $s_0$.

We start by considering a regular proper model $\bar{S}$ of the curve $S'$ over the ring of integers $\mathcal{O}_K$. We also consider a point $s \in S(L)$, where $L/K$ is some finite extension. By the valuative criterion of properness $s_0$ and $s$ give sections, which we denote by $\bar{s}_0$ and $\bar{s}$, of the arithmetic pencil

$$\bar{S} \times_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_L.$$

Lemma 11.1. The basis $\omega_i$ of $H^n_{\text{DR}}(X/S)$ in 2.2 may be chosen so that the following property holds:

For any $L/K$ and any $s \in S(L)$ we have the if $s$ is $v$-adically close to $s_0$ for some $v \in \Sigma_{L,f}$ then $\bar{s}$ and $\bar{s}_0$ have the same image in $\bar{S}(\mathbb{F}_{q(v)})$.

Proof. This follows from the discussion on page 209 of [And89]. In essence we might need to multiply a given basis of sections of $H^n_{\text{DR}}(X/S)$ by a factor of the form $\frac{\zeta}{\zeta - x}$, for an appropriately chosen $\zeta \in K^\times$. This amounts to multiplying the G-functions by the same factor.

We still get G-functions but these will have possibly smaller local radii for a finite set of finite places to ensure the property above. 

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§11.1 Good arithmetic models

We aim to apply the results of §4 of [PST⁺21]. To apply these results we need to assume the existence of good arithmetic models over \( \mathcal{O}_K \) both for the curve \( S \) and the morphism \( f \). We consider an indexing of the elements of \( \Sigma_S = S' \setminus S = \{ s_0, \ldots, s_g \} \).

**Definition.** We say that \( S \) has a good arithmetic model over \( \mathcal{O}_K \) if for the triple \((S', S, \Sigma_S)\) there exists a model \((\tilde{S}, \mathcal{C}, D)\) over \( \mathcal{O}_K \) such that \( \tilde{S} \) is smooth and proper and \( D \) is a normal crossings divisor.

Furthermore, we assume that for all \( i \neq 0 \) and all primes \( q \) we have that \((\tilde{s}_i)_\overline{\mathbb{F}}_q \neq (\tilde{s}_0)_\overline{\mathbb{F}}_q\), where \( \tilde{s}_i \) are the sections \( \mathcal{O}_K \to \tilde{S} \) defined by these points as above.

The smooth proper \( K \)-morphism \( f : X \to S \) provides us, as we have seen, with a weight \( n = (\dim X - 1) \) variation of Hodge structures on the \( \mathbb{C} \)-manifold \( S_{an} \). We are also provided with a \( \mathbb{Z} \)-local system \( \mathbb{V} := R^n f_{an}^* \mathbb{Z}_{X_{an}} \), contained in the local system of flat sections of the variation of Hodge structures we study.

On the other hand, we know that for any prime \( l \) the morphism \( f \) defines an associated \( l \)-adic étale local system (lisse \( l \)-adic sheaf) over \( S \) which we denote by \( \mathbb{V}_l := \lim_{\leftarrow} R^n f_*(\mathbb{Z}/l^m \mathbb{Z}) \). We note that the analytification of \( \mathbb{V}_l \) is nothing but the \( l \)-adic completion of the local system \( \mathbb{V} \).

From the proper base change theorem in étale cohomology and Artin’s comparison theorem we know that for each \( s \in S(L) \) we have an isomorphism

\[
H^n_{\text{ét}}(\overline{X}_s, \mathbb{Z}_l) = (\mathbb{V}_l)_s \cong (\mathbb{V}_l)^{an}_s = H^n(X_{an}^s, \mathbb{Z}_l). \quad (56)
\]

§11.1.1 Extending the étale local system

Let us fix some notation. For a finite place \( v \) of an extension \( L/K \), let \( p = p(v) \) be the characteristic of the residue field \( \mathcal{O}_{L_v}/m_{L_v} \), and \( l \neq p(v) \) a prime. We also let \( M := L_v^{nr} \) and \( \mathcal{O}_M \) be its ring of integers. We consider \((\tilde{S}, \mathcal{C}, D)\) to be as in §11.1, and let \( \mathbb{V} \) and \( \mathbb{V}_l \) be as in the previous section.

For our argument to work, we need to have the analogue of Lemma 4.3 in [PST⁺21]. In other words we would like to be able to extend the \( l \)-adic étale local system \( \mathbb{V}_l \) to some \( l \)-adic étale local system \( \overline{\mathbb{V}}_l \) on \( \mathcal{C}_{\mathcal{O}_M} := \mathcal{C} \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_M) \).

The argument in [PST⁺21] assumes that there is at least one CM point that is integral. This is achieved by a standard spreading out argument.
Since we want to be able to deal with all of the finite places \( v \) of the field \( L \), we cannot employ a similar tactic.

With that in mind we start with the following definition.

**Definition (Arithmetic models for \( f \)).** Let \( f : X \to S \) be projective smooth morphism of \( K \)-varieties with \( S \) a curve, as above. We say \( f \) has a **good arithmetic model over** \( \mathcal{O}_K \) if there exists a good arithmetic model for the triple \((S', S, \Sigma_S)\) over \( \mathcal{O}_K \), such that for each \( L/K \) finite and each \( v \in \Sigma_{L,f} \) we have that

1. there exists an \( \mathcal{O}_M \)-scheme \( \mathcal{X}_v \) such that \( (\mathcal{X}_v)_{L^ur} = X_{L^ur} \), and
2. there exists a smooth proper morphism \( \tilde{f}_v : \mathcal{X}_v \to \mathcal{C}_{\mathcal{O}_M} \) of \( \mathcal{O}_M \)-schemes whose generic fiber is the morphism \( f_v \) (see below).

Let us fix a finite extension \( L/K \) and a place \( v \in \Sigma_{L,f} \). Assume the existence of such a pair \((\mathcal{X}_v, \tilde{f}_v)\) and let \( f_v \) be the base change of \( f \) via the morphism \( \text{Spec } L^ur \to \text{Spec } K \). We then define \( \tilde{V}_l \) to be the \( l \)-adic étale sheaf on \( \mathcal{C}_{\mathcal{O}_M} \) given by

\[
R^n(\tilde{f}_v)_*(\mathbb{Z}_l) = \lim_{\leftarrow} R^n(\tilde{f}_v)_*(\mathbb{Z}/l^m\mathbb{Z}).
\]

**Lemma 11.2.** Let \( f : X \to S \) over \( K \) be as above. Assume that there exists a good arithmetic model \((\mathcal{X}, \tilde{f})\) for the morphism \( f \) over \( \mathcal{O}_K \). Then the \( l \)-adic étale sheaf \( \tilde{V}_l := R^n(\tilde{f}_v)_*(\mathbb{Z}_l) \) on \( \mathcal{C}_{\mathcal{O}_M} \) is an \( l \)-adic étale local system that extends the \( l \)-adic étale local system \( R^n(f_v)_*(\mathbb{Z}_l) \) on \( S_M \).

**Proof.** From the proper base change theorem for lisse \( l \)-adic sheaves we have that \( R^n(\tilde{f}_v)_*(\mathbb{Z}_l) \) is an extension of the sheaf \( R^n(f_v)_*(\mathbb{Z}_l) \).

The fact that \( \tilde{V}_l \) is also an \( l \)-adic étale local system on \( \mathcal{C}_{\mathcal{O}_M} \) follows from the smooth proper base change theorem for lisse \( l \)-adic sheaves. \( \Box \)

**§12 A comparison of monodromy operators**

The crux of our argument rests on a comparison between the action of inertia and that of the local monodromy around the degeneration \( s_0 \). To be able to establish this connection we need to assume the existence of the aforementioned good arithmetic models.

**§12.1 Review on the monodromy weight filtration**

We present a short review of the two monodromy operators we want to compare.
The local monodromy operator

Let $f : X \to S$ be a G-admissible variation defined over the number field $K$ and, as usual, $\Delta^* \subset S_C^m$ a small disk, relative to a fixed embedding $K \hookrightarrow \mathbb{C}$, centered at the point of degeneration $s_0 \in S'(K)$.

We have by our assumptions that the local monodromy around $s_0$ on $V_z := H^n(X^m, \mathbb{Q})$ is in fact unipotent for any $z \in \Delta^*$. We denote by $N_z : V_z \to V_z$ the corresponding nilpotent endomorphism that is the logarithm of the unipotent endomorphism that defines this action.

We know that $N_z$ has degree of nilpotency $\leq n+1$ and hence, by (6.4) of [Sch73], we get an ascending filtration

$$0 \subset W^B_0 \subset W^B_1 \subset \ldots \subset W^B_{2n-1} \subset W^B_{2n} = V_z. \quad (57)$$

The filtration $W^B_\bullet$ is called the weight monodromy filtration. By Lemma (6.4) of [Sch73], we know that this is the unique ascending filtration characterized by the properties

1. $N_z(W^B_i) \subset W^B_{i-2}$ for all $i$, and

2. for all $1 \leq i \leq n$ the endomorphism $N^i_z$ defines an isomorphism $\text{Gr}^{W^B}_{i+1} \to \text{Gr}^{W^B}_{n-i}$.

We define from now on $h^B_i := \dim_{\mathbb{Q}} \text{Gr}^{W^B}_i$ and let $h^B_{\text{max}} := \max\{h^B_j : 0 \leq j \leq 2n\}$. We note that the number $h$, the dimension of the local system $M_0 R_n(f^m \circ Q_{X_C^m})_{|\Delta^*}$, is equal to $h^B_0$.

We note that while the matrix $N_z$ depends on the point $z$ the numbers $h^B_i$ do not.

The monodromy operator from inertia

Let $s \in S(L)$ with $L/K$ finite, be a point of the G-admissible variation of $\mathbb{Q}$-HS given by the morphism $f : X \to S$.

Let $v \in \Sigma_{L,f}$ be a finite place of $L$. We then have the nilpotent endomorphism $N_v$, given by the action of some subgroup of finite index of the inertia group $I_w$, acting on the étale cohomology group $H^m_{\text{et}}(\overline{X}_s, \mathbb{Q}_l)$, where $l \neq p(v)$.

Let $W^\text{et}_\bullet$ be the monodromy filtration defined on $H^m_{\text{et}}(\overline{X}_s, \mathbb{Q}_l)$ via the action of the above $N_v$. We also let $h^\text{et}_i := \dim_{\mathbb{Q}_l} \text{Gr}^{W^\text{et}}_i$.
§12.2 $v$-adic proximity and comparison of operators

Consider $f : X \to S$ some $G$-admissible variation of Hodge structures. Throughout this subsection we consider $s \in S(L)$ a fixed point, where $L/K$ is some finite extension, and $v \in \Sigma_{L,f}$ some fixed finite place of $L$. We let $p = p(v)$ be the characteristic of the finite field $\mathcal{O}_{L,v}/m_v$ and we fix some $l \neq p$.

We assume that a good arithmetic model, in the sense of §11, $(\breve{S}, \mathcal{C}, D)$ over $\mathcal{O}_K$ exists for $S$ and that a good arithmetic model for the morphism $f$ exists over $\mathcal{O}_K$ with respect to this triple.

Motivated by the exposition in §4 of [PST$^+21$] we prove the following lemma.

**Lemma 12.1.** Under the above assumptions, we have that if $s$ is $v$-adically close to $s_0$ we have $h_i^{et} = h_i^{B} + \eta_i$ for all $i$, where $h_i^{B}$ and $h_i^{et}$ are as in §12.1.

**Proof.** We start with a bit of notation following the exposition in §4 of [PST$^+21$]. First of all, for any group $G$ we will denote by $G(\breve{\mathbb{Z}}_p)$ its maximal pro-$l$ quotient. As always we fix a punctured unit disk $\Delta^* \subset S^{an}$ centered at $s_0$.

We let $y$ and $y_0$ be the sections of the arithmetic pencil $\breve{S} \times_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_L$ whose generic fibers are the points $s$ and $s_0$ respectively. By 11.1 we may assume without loss of generality that $y_{\overline{F}_p} = y_{0,\overline{F}_p} \in D(\overline{\mathbb{F}}_p)$. We also let $t$ be such that it cuts out $\breve{s}_0 \subset D$ in $\breve{S}$. In particular, for this choice we get, by our assumption that for all $s_i \in \Sigma_S$ and all primes $q$ we have that $(\breve{s}_i)_{\overline{F}_q} \neq (\breve{s}_0)_{\overline{F}_q}$ when $i \neq 0$, that

$$\hat{\mathcal{O}}_{\breve{S}_{y_0,\overline{F}_p}} \cong \mathcal{O}_{\mathbb{F}_v}[\![t]\!] .$$

We have, by our assumptions, that the local monodromy $\pi_1(\Delta^*, z)$ acts unipotently on the fiber $(\mathbb{R}^n f^{an} \mathbb{Z})_z$ for all $z \in \Delta^*$. Letting $\gamma_0 \in \pi_1(\Delta^*, z)$ be a generator, we denote by $U_0 \in \text{GL}(H^n(X^{an}_z, \mathbb{Z}))$ the unipotent endomorphism it maps to via the local monodromy representation. We define $N_z$ to be the nilpotent logarithm of $U_0$.

By Grothendieck’s quasi-unipotent action theorem we know that there exists a finite extension $F/L_v$ such that the Galois representation $\rho_i : G_{L_v} \to \text{GL}(H^n_{\text{et}}(X_{s,v}, \mathbb{Z}))$ restricts to a unipotent representation of the inertia group $I_F$. In other words $\rho_i|_{I_F}$ is given by $\sigma \mapsto \exp(N_v I_1(\sigma))$ with $N_v$ nilpotent with degree of nilpotency $\leq n + 1$.

We let $M := F^{ur}$ and $\mathcal{O}_M$ be its ring of integers. We consider the rings $R_1 := \mathcal{O}_M[[\tau]]\left[\left[\frac{1}{\tau}\right]\right]$, $R_2 := M[[\tau]]\left[\left[\frac{1}{\tau}\right]\right]$, and $R_3 := \mathbb{C}[[\tau]]\left[\left[\frac{1}{\tau}\right]\right]$. We note that, after fixing an inclusion $\mathcal{O}_M \hookrightarrow \mathbb{C}$, these define the following commutative diagram...
with $g_1$, $g_2$ and $g_3$ being étale. In fact, note that the Spec $R_i$ are étale neighborhoods of the geometric point $\bar{y}_M = \bar{s}$. Originally, Spec $R_1$ is an étale neighborhood of $\bar{y}_M$ in $\tilde{S}_M$ but it will not intersect the divisor $D$, by construction and our assumption that for all $s_j$, $s_i \in \Sigma_S$ and all primes $q$ we have that $(\bar{s}_i)_{\bar{F}_q} \neq (\bar{s}_0)_{\bar{F}_q}$ when $i \neq 0$.

Since $y\bar{s}_p = y_0\bar{s}_p$, we get that $t$ pulls back to an element $t_M$ of the maximal ideal of $\mathcal{O}_M$ via $\text{Spec}(\mathcal{O}_M) \xrightarrow{y_{\Sigma}} \tilde{S}_M$. From this we get a morphism $f_1 : \mathcal{O}_M[[t]] \llbracket \frac{1}{t} \rrbracket \to M$, since $v_M(t_M) > 0$.

By Lemma 4.2 of [PST+21] we get an induced map

$$\phi_1 : G^{(p)} \to \pi_1^{\text{ét}}(\text{Spec } R_1, \bar{y}_M)^{(p)},$$

between the prime-to-$p$ Galois groups, whose image is completely determined by $v_M(t_M) > 0$. In particular we get a map

$$\phi_{1,M} : (G_{\mathcal{M}})_{(l)} \to \pi_1^{\text{ét}}(\text{Spec } R_1, \bar{y}_M)_{(l)}.$$

With respect to the above embedding $\mathcal{O}_M \hookrightarrow \mathbb{C}$ we get a map $f_2 : R_1 \to R_3$ which induces an isomorphism of prime-to-$p$ Galois groups, and hence of their maximal pro-$l$ quotients, which we denote by

$$\phi_{2,M} : \pi_1^{\text{ét}}(\text{Spec } R_3, \bar{y}_M)_{(l)} \to \pi_1^{\text{ét}}(\text{Spec } R_1, \bar{y}_M)_{(l)}.$$

We also consider the composition $\phi_1 := \phi_{2,M} \circ \phi_{1,M} : (G_{\mathcal{M}})_{(l)} \to \pi_1^{\text{ét}}(\text{Spec } R_3, \bar{y}_M)_{(l)}$.

Finally, we let $F : R_3 \to R_3$ be the map defined by $t \mapsto t^{v_M(t_M)}$, by abuse of notation we also let $F : \text{Spec}(R_3) \to \text{Spec}(R_3)$ be the étale cover induced from $F$. Letting $\tilde{y}_1$ be any geometric point in the fiber of $\bar{y}_M = \bar{s}$ over $F$, we then get from $F$ an induced morphism

$$\psi_1 : \pi_1^{\text{ét}}(\text{Spec } R_3, \tilde{y}_1)_{(l)} \to \pi_1^{\text{ét}}(\text{Spec } R_3, \tilde{y}_1)_{(l)}.$$

We also note that by construction we have that $\psi_1$ has the same image as $\phi_1$.

On $S_M$ we have the lisse $l$-adic sheaf $\mathcal{V}_l := (\lim_i R^n(f_v)_*(\mathbb{Z}/l^n\mathbb{Z}))$. Its analytification $\mathcal{V}_l^{an}$ is nothing but the local system $R^n f_{an}^* \mathbb{Z}_\mathbb{C}^{an} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l$ on $S_{\mathbb{C}}^{an}$, which is the $l$-adic completion of the $\mathbb{Z}$-local system that underlies the variation of $\mathbb{Z}$-HS we are studying. The assumption that good arithmetic models for the morphism $f$ exist over $\mathcal{O}_K$ then implies, see 11.2, that we have that $\mathcal{V}_l$ extends to a lisse $l$-adic sheaf $\bar{\mathcal{V}}_l$ on $\mathcal{C}_{\mathcal{O}_M}$. We let $\bar{\mathcal{V}}_{l,1}$ be the pullback of this.
sheaf via \( g_1 \). Note that the pullback of \( V \) via the morphism \( \mathrm{Spec}(R_2) \to S_M \), which we will denote by \( V_{l,1} \), is nothing but the generic fiber of the lisse \( l \)-adic sheaf \( \tilde{V}_{l,1} \). By abuse of notation we also denote by \( V_{l,2} \) the pullback of this last sheaf via the morphism \( \mathrm{Spec}(R_3) \to \mathrm{Spec}(R_2) \) above. Finally, we denote by \( V_{l,3} \) the pullback of \( V_{l,1} \) via the map \( F \).

For any \( z \in \Delta^* \) we have the local monodromy representation given by \( \rho : \pi_1(\Delta^*, z) \to \mathrm{GL}(\mathcal{V}_{l,1}^{an}) \). By our assumptions this action is unipotent given by \( \gamma_0 \mapsto U_0 \), where \( \gamma_0 \) a generator of \( \pi_1(\Delta^*, z) \). From the tower of étale covers \( \mathrm{Spec}(R_3)^{an} \to \mathrm{Spec}(R_3)^{an} \to \mathcal{S}_C^{an} \) we get an inclusion \( \pi_1(\mathrm{Spec}(R_3)^{an}, z_2) \to \pi_1(\mathrm{Spec}(R_3)^{an}, z_1) \) where \( z_1 \in (g_3^{an})^{-1}(z) \) and \( z_2 \in (F^{an})^{-1}(z_1) \). Letting \( \gamma_i \in \pi_1(\mathrm{Spec}(R_3)^{an}, z_i) \) for \( i = 1, 2, \) be generators of these groups we can identify these with \( \gamma_i a_i \), where \( a_i \in \mathbb{Z} \) and \( a_1 | a_2 \). Without loss of generality we may and do assume that \( a_i > 0 \).

By construction we then have the following commutative diagrams for \( i = 1, 2 \)

\[
\begin{array}{ccc}
\pi_1(\mathrm{Spec}(R_3)^{an}, z_i) & \xrightarrow{\rho_i} & \mathrm{GL}(\mathcal{V}_{l,1}^{an}) \\
\downarrow & & \downarrow \\
\pi_1(\Delta^*, z) & \xrightarrow{\rho} & \mathrm{GL}(\mathcal{V}_{l,1}^{an})
\end{array}
\]

from which we see that \( \gamma_i \) maps to \( U_{i,z} := U_0^{a_i} \). In particular, the nilpotent logarithm \( N_i, z \) of \( U_{i,z} \) is \( a_i N_z \).

Now, let \( \gamma_3 \) be a generator of the group \( \pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) \simeq \mathbb{Z} \). Considering the representation \( \pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) \to \mathrm{GL}(\mathcal{V}_{l,1}^{an}) \) with \( \gamma_3 \to U_3 \). We get that \( U_3 \) is conjugate to \( U_{1,z} \), i.e. there exists some \( P \) with \( U_3 = PU_{1,z}P^{-1} \). So that \( U_3 \) is also unipotent and we may write \( N_3 := PN_{1,z}P^{-1} \) for its nilpotent logarithm.

Putting everything together we get that

\[
N_3 = a_1 PN_z P^{-1}.
\] (58)

We also have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) & \xrightarrow{\rho_1} & \mathrm{GL}(\mathcal{V}_{l,1}^{an}) \\
\downarrow & & \downarrow \\
\pi^\text{ét}_1(\mathrm{Spec}(R_3), \bar{y}_M) & \xrightarrow{\rho^\text{ét}_1} & \mathrm{GL}(\mathcal{V}_{l,1}^{an})
\end{array}
\]

We note that \( \pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) \simeq \mathbb{Z} \) and that it is dense in the étale fundamental group \( \pi^\text{ét}_1(\mathrm{Spec}(R_3), \bar{y}_M) \). Thus, the continuity of \( \rho^\text{ét}_1 \) implies that
the image of \( \rho_{1}^\ell \) is \( P_{1} U_{3}^{Z_{l}} P_{1}^{-1} := \{ P_{1} \exp(\alpha N_{3}) P_{1}^{-1} : \alpha \in Z_{l} \} \). This is a pro-\( l \)
group and we therefore get that \( \rho_{1}^\ell \) factors through \( \pi_{1}^\ell (\text{Spec}(R_{3}), \bar{y}_{M})_{(l)} \).

We know turn our attention to the action of inertia. We know that the following diagram is commutative

\[
\begin{array}{c}
G_{M} \longrightarrow \pi_{1}^\ell (\text{Spec} R_{3}, \bar{y}_{M}) \longrightarrow \text{GL}(H^{n}_{\text{ét}}(\bar{X}_{s,v}, Z_{l}))
\end{array}
\]

where \( c \) is the homomorphism induced by the composition of the identification of \( (\mathbb{V}_{l,1})_{\bar{y}_{M}} \simeq H^{n}_{\text{ét}}(\bar{X}_{s,v}, Z_{l}) \) and the action of \( \pi_{1}^\ell (\text{Spec} R_{3}, \bar{y}_{M}) \) on \( (\mathbb{V}_{l,1})_{\bar{y}_{M}} \).

By construction, the image of \( \rho_{l} \) is a unipotent group. This implies that \( \text{Im}(\rho_{l}) \) is in fact a pro-\( l \) group. In more detail, we have \( \text{Im}(\rho_{l}) = \{ \exp(\alpha N_{v}) : \alpha \in Z_{l} \} \) which is a pro-\( l \) group.

From our earlier comments we get that the image of \( c \) is a pro-\( l \) group as well. We thus have that the above diagram induces a commutative diagram

\[
\begin{array}{c}
(G_{M})_{(l)} \longrightarrow \pi_{1}^\ell (\text{Spec} R_{3}, \bar{y}_{M})_{(l)} \longrightarrow \text{GL}(H^{n}_{\text{ét}}(\bar{X}_{s,v}, Z_{l}))
\end{array}
\]

where \( \phi_{l} \) is as above.

In particular we get that \( \exp(N_{v}) \in \text{Im}(c) \). In more detail this implies that \( \exp(N_{v}) = Q \rho_{1} P_{1} \exp(\alpha N_{3}) P_{1}^{-1} Q^{-1} \) for some \( \alpha \in Z_{l} \). Taking logarithms and combining this with (58) this translates to

\[
N_{v} = \alpha a_{1} P_{0} N_{z} P_{0}^{-1},
\]

where \( P_{0} \in \text{GL}_{\mu}(Z_{l}) \), upon considering \( Z_{l} \)-bases of the fibers of the various lisse \( l \)-adic sheaves.

Note that \( \alpha a_{1} \neq 0 \) since \( \text{Im}(\phi_{l}) = \text{Im}(\psi_{l}) \). Indeed, in this case we would have that \( \text{Im}(\psi_{l}) = 0 \) and, hence, that the monodromy representation \( \rho_{2} : \pi_{1}(\text{Spec}(R_{3}), z_{2}) \rightarrow GL((\mathbb{V}_{l,2})_{z_{2}}) \) is trivial. This would imply that \( N_{2,z} = 0 \) and hence that \( N_{z} = 0 \) which is impossible, since we have a non-isotrivial singularity at \( s_{0} \) by assumption of having a \( G \)-admissible variation.

Combining this with 12.2, our result follows.

**Lemma 12.2.** Let \( k \) be a field with \( \text{char}(k) = 0 \) and let \( N_{1} \) and \( N_{2} \) be two nilpotent elements of \( \text{of End}(V) \), where \( V \) is some \( \mu \)-dimensional \( k \)-vector space. Assume that \( N_{1} \) is conjugate to \( aN_{2} \) for some \( a \in k^{\times} \). Let \( W_{1} \) and
$W_i^2$ be the ascending filtrations of $V$ associated to $N_1$ and $N_2$ respectively, and let $h_i^j := \dim_k \text{Gr}_i W^j$ for $j = 1, 2$. Then $h_i^1 = h_i^2$ for all $i$.

Proof. The easiest way to see this is via the explicit formulas for the filtration $W_\bullet$ associated to a nilpotent endomorphism $N$, see Remark (2.3) of [SZ85]. The fact that $N_1 = PaN_1P^{-1}$ for some $P \in \text{GL}(V)$ and some $a \in k^\times$ defines, using the aforementioned formulas, isomorphisms $\text{Gr}_i W_1^j \cong \text{Gr}_i W_2^j$ for all $i$ and the result follows.

§13 Conditions for $v$-adic proximity

Motivated by 10.2 we make the following definition.

Definition. Let $f : X \to S$ be a $G$-admissible variation defined over the number field $K$. Let $s \in S(\bar{\mathbb{Q}})$ be a point of the variation whose associated algebra of Hodge endomorphism is $D_s = M_{m_1}(D_1) \oplus \cdots \oplus M_{m_r}(D_r)$. Let us also define $F_i := Z(D_i)$, $d_i^2 := [D_i : F_i]$, and $f_i := [F_i : \mathbb{Q}]$.

We say that the point $s$ satisfies condition $\star$ if it satisfies any of the following

$\star_1$ if we have that
\[
\sum_{i=1}^r m_if_i > \mu - \dim_{\mathbb{Q}} \text{Im}(N^B).
\]
(\textit{\textbf{\textit{\textbf{(\star_1)}}}})

$\star_2$ if there exists $i$ such that for the set
\[
\Pi_{D_i} := \{l \in \Sigma_{Q,f} : \exists w \in \Sigma_{F_i,f}, \text{ with } w|l \text{ and } [F_i,w : \mathbb{Q}] > \frac{h_B}{m_i}\}
\]
we have that
\[
|\Pi_{D_i}| \geq 2.
\]
(\textit{\textbf{\textit{\textbf{(\star_2)}}}})

$\star_3$ if we have that $\exists i \in \{1, \ldots, r\}$ such that
\begin{enumerate}
  \item $d_im_i \geq h_B^\text{max}$, and
  \item for the sets $P_{D_i} := \{l \in \Sigma_{Q,f} : l \text{ is totally split in } F_i\}$ and $Q_{D_i} := \{l \in \Sigma_{Q,f} : \exists w \in \Sigma_{F_i,f}, w|l \text{ and } \text{inv}_w(D_i) \notin \mathbb{Z}\}$ we have that $|P_{D_i} \cap Q_{D_i}| \geq 2$.
\end{enumerate}

$\star_4$ if for some $i$ we have that $D_i$ is a quaternion algebra, and letting $R_{D_i} := \{l \in \Sigma_{Q,f} : \exists w \in \Sigma_{F_i,f}, w|l, \text{ inv}_w(D_i) \notin \mathbb{Z}, \text{ and } m_i[F_i,w,\mathbb{Q}] \not| h_B^j \forall j\}$, we have that $|R_{D_i}| \geq 2$.
⋆5 if for some $i$ we have that $D_i$ is a quaternion algebra, and letting $S_{D_i} := \{ l \in \Sigma_{Q,f} : \exists w \in \Sigma_{F_i,f}, w|l, \ inv_w(D_i) \in \mathbb{Z}, and m_i 2[F_{i,w}, \mathbb{Q}] / h_B^j \forall j \}$, we have that $|S_{D_i}| \geq 2$.

⋆6 if for some $i$ we have that $D_i$ is a quaternion algebra and for the above sets $|R_{D_i} \cup S_{D_i}| \geq 2$.

⋆7 if for some $i$ we have that $D_i$ is of Type IV with $d_i^2 = \dim_{\mathbb{Q}}[D_i : F_i]$ and for the set

$$T^i_{D_i} := \{ l \in \Sigma_{Q,f} : \exists w \in \Sigma_{F_i,f}, w|l and m_i d_i[F_i,F_i,F_i,w, \mathbb{Q}] / h_B^j \forall j \},$$

we have that $|T^i_{D_i}| \geq 2$.

**Remark.**

1. We remark that all conditions above only depend on data coming from the original $G$-admissible variation $f : X \to S$. These should be thought of as demanding that we have “a lot of endomorphisms”. This “a lot” here should be contrasted with the fact that the generic algebra of Hodge endomorphisms is just $\mathbb{Q}$.

2. The necessity for demanding that several of the sets of primes in these conditions have at least two elements arises from 12.1. As we will see in the proof of 13.1, once we fix the place $v$, and by extension the prime $p \in \mathbb{Q}$ with $v|p$, we need some condition as those in ⋆2–⋆7 to hold for some prime $l \neq p$ to be able to apply the aforementioned result.

3. We note that the sets in conditions ⋆2, ⋆5, ⋆6, and ⋆7 will potentially have infinitely many elements, while the sets in ⋆3 and ⋆4 will have at most finitely many elements. This rests on the fact that $\inv_w(D) \notin \mathbb{Z}$ only for finitely many primes.

4. We note that, assuming that the extension $F_i/\mathbb{Q}$ is cyclic, condition ⋆5 is equivalent to the existence of some $i$ for which $m_i 2[F_i : \mathbb{Q}] / h_B^j$ for all $j$.

5. The different treatment needed for general type IV algebras, addressed in ⋆7, stems from the fact that even if we know that the restriction of the division algebra at some place $w$ is not of the form $M_t(F_{i,w})$ for some it might still be of the form $M_t(D')$ with $D'$ a central division algebra $D'$ with $[D' : F_{i,w}] > 1$ and $t > 1$. This issue does not appear in the case of quaternion algebras since the only two possibilities are $D_i \otimes_{F_i,F_i,F_i} \simeq M_2(F_{i,w})$ or $D_i \otimes_{F_i,F_i,F_i} \simeq M_2(F_{i,w})$ is a quaternion algebra over $F_{i,w}$. 

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§13.1 Some linear algebraic lemmas

To prove the result we want, first we will need some elementary lemmas from linear algebra and the theory of linear algebraic groups.

**Lemma 13.1.** Let $N \in M_n(k)$, where $k$ is a field with $\text{char}(k) = 0$, be a non-zero nilpotent upper-triangular matrix with $\dim_k \text{Im}(N) = r$. Then, there exist unipotent upper triangular matrices $Q_L, Q_R \in \text{GL}_n(k)$ such that the matrix $N_{\text{red}} := Q_L N Q_R$ is strictly upper triangular with at most one non-zero entry in each row and column.

**Proof.** We employ row and column reduction together with induction.

**Remark.** Another way of phrasing the above lemma is that $N_{\text{red}} = (\epsilon_{i,j})$ such that there exist exactly $r$ entries $\epsilon_{i,j} \neq 0$, which we can take without loss of generality to be equal to 1. These entries are all in distinct rows and columns, i.e. if $\epsilon_{i,j}, \epsilon_{i',j'} \neq 0$ then $i \neq i'$ and $j \neq j'$.

**Lemma 13.2.** Let $V$ be an $n$-dimensional vector space over an algebraically closed field $k$ of characteristic 0 and let $N \in \text{End}(V)$ be a nilpotent linear operator with $r := \dim_k \text{Im}(N) > 0$. Let $T$ be a sub-torus of the algebraic group $\text{GL}(V)_N$ of automorphisms of $V$ commuting with $N$. Then $\dim_k T \leq n - r$.

**Proof.** We may assume that the sub-torus $T$ is maximal in $G_N := \text{GL}(V)_N$. The torus $T$ will be contained in a maximal sub-torus $T_m$ of the group $\text{GL}(V)$. From the theory of linear algebraic groups\(^7\) all maximal tori of $\text{GL}(V)$ are conjugate. Hence, there exists $P \in \text{GL}(V)(k)$ such that $PT_m P^{-1}$ is the torus of diagonal matrices in $\text{GL}(V)$, with respect to a fixed basis $\{ \vec{e}_i : 1 \leq i \leq n \}$.

We notice that, setting $N_P := PNP^{-1}$, we have that $P \text{GL}(V)_N P^{-1} = \text{GL}(V)_{N_P}$, and the sub-torus $T$ will be isomorphic to the sub-torus $PTP^{-1}$. For the element $N_P$ we will have that it is nilpotent and that $r = \dim_k \text{Im}(N) = \dim_k \text{Im}(N_P)$. We may thus assume from now on, replacing $T$ by $PTP^{-1}$ and $N$ by $N_P$, that $T$ is comprised of diagonal matrices and is contained in the subgroup of diagonal matrices $\mathbb{G}_m^n \subset \text{GL}_n$. Let $A \in T(k)$ be generic and write $A = \text{diag}(a_1, \ldots, a_n)$. Since $A$ and $N$ commute they are simultaneously upper triangularizable, so there exists $P_1 \in \text{GL}_n(k)$ such that $A_1 = P_1 A P_1^{-1}$ and $N_1 := P_1 N P_1^{-1}$ are both upper triangular and $A_1 N_1 = N_1 A_1$.

We note that we can write

$$A_1 = \text{diag}(a_1', \ldots, a_n') + A' \quad (60)$$

\(^7\)The results we used from the theory of Linear algebraic groups can be found in Chapter 6 of [Spr98].
where \( A' \) is some strictly upper triangular matrix. We note that \((a'_1, \ldots, a'_n) = (a_{(r_1)}, \ldots, a_{(r_n)}) \) where \( i \mapsto r_i \) defines a permutation of the set \( \{1, \ldots, n\} \).

We apply 13.1 to the matrix \( N_1 \) and let \( Q_L, Q_R \) be unipotent matrices as in the lemma. We then have

\[
(Q_LN_1Q_R)(Q_R^{-1}A_1Q_R) = (Q_LA_1Q_L^{-1})(Q_LN_1Q_R). \tag{61}
\]

Let \( B = (b_{i,j}) := (Q_LN_1Q_R)(Q_R^{-1}A_1Q_R) \) and \( QLN_1QR = (\epsilon_{i,j}) = N_{\text{red}} \).

Note that, since the matrices \( Q_L \) and \( Q_R \) are unipotent and upper triangular, the matrices \( (Q_R^{-1}A_1Q_R), (Q_L^{-1}A_1Q_L), \) and \( A_1 \) are all upper triangular matrices that have the same diagonal.

Let \( I := \{(i_t, j_t) : 1 \leq t \leq r\} \) denote the set of indices with \( \epsilon_{i,j} = 1 \), note that the rest of the entries of the matrix \( N_{\text{red}} \) are zero. For each pair \((i_t, j_t) \in I\) we compute via the left hand side of (61)

\[
b_{i_t,j_t} = a'_{j_t}, \tag{62}
\]

while computing these using the right hand side of (61) we have

\[
b_{i_t,j_t} = a'_{i_t}, \tag{63}
\]

Combining the above we find that for \( 1 \leq t \leq r \) we have

\[
a'_{i_t} = a'_{j_t}. \tag{64}
\]

These imply that the Zariski closure of \( \text{diag}(a'_1, \ldots, a'_n) \) in \( \mathbb{G}_m^n \) is contained in the subvariety defined by the ideal

\[
J_T := \langle X_{i_t} - X_{j_t} : (i_t, j_t) \in I \rangle \tag{65}
\]

of \( k[X_1, \ldots, X_n] \).

Let \( f_t = X_{i_t} - X_{j_t} \) be the above polynomials generating the ideal \( J_T \). Note that each indeterminant \( X_i \) appears at most twice among the polynomials \( f_t \), at most once as one of the \( X_{i_t} \) and at most once as one of the \( X_{j_t} \). We also note that by construction we have \( i_t < j_t \) for all \( t \).

We start by proving the following claim:

**Claim:** The polynomials \( f_t \) are linearly independent over \( k \).

**Proof of Claim.** We proceed by induction on \( r \). The cases \( r = 1, 2 \), are trivial. So let us assume that \( r = 3 \).

Assume, without loss of generality, that

\[
f_3 = \lambda_1 f_1 + \lambda f_2. \tag{66}
\]
Since $X_{i_3}$ appears on the left it must appear on the right, so either $i_3 = j_1$ or $i_3 = j_2$, due to the above remarks. Assuming without loss of generality that $i_3 = j_1$ we get that $\lambda_1 = -1$. Similarly we get that $j_3 = i_1$ or $j_3 = i_2$. If $j_3 = i_1$ it would contradict the fact that $i_t < j_t$ for all $t$, so $j_3 = i_2$.

This implies $\lambda_2 = -1$ and (66) becomes
\[
X_{i_3} - X_{j_3} = -(X_{i_1} - X_{j_1}) - (X_{i_2} - X_{j_2}).
\]

Applying the above we get $X_{j_2} - X_{i_1} = 0$ which implies $i_1 = j_2$.

Putting everything together we get
\[
i_3 < j_3 = i_2 < j_2 = i_1 < j_1 = i_3,
\]
which is obviously a contradiction.

Now assume that $r \geq 4$. To reach a contradiction we assume that there exist $\lambda_i \in k$ such that
\[
X_{i_r} - X_{j_r} = \lambda_1 f_1 + \ldots + \lambda_{r-1} f_{r-1}.
\]
Again, since $j_r$ appears on the left there exists a unique $i_t$ such that $i_t = j_r$. We assume, without loss of generality that $t = r - 1$ so that $\lambda_{r-1} = -1$. Similarly without loss of generality we may assume that $i_r = j_{r-2}$ and $\lambda_{r-2} = -1$.

Using the above with (69) we get
\[
X_{i_r} - X_{j_r} = -(X_{j_{r-2}} - X_{i_{r-2}}) - (X_{i_{r-3}} - X_{j_{r-3}}) + \lambda_1 (X_{i_1} - X_{j_1}) + \ldots + \lambda_{r-3} (X_{i_{r-3}} - X_{j_{r-3}}),
\]
which, after canceling, becomes
\[
X_{i_{r-2}} - X_{j_{r-1}} = \lambda_1 (X_{i_1} - X_{j_1}) + \ldots + \lambda_{r-3} (X_{i_{r-3}} - X_{j_{r-3}}).
\]
This gives a contradiction by the inductive step as follows:

First, note that $i_{r-2} < j_{r-2} = i_r < j_r = i_{r-1} < j_{r-1}$. Now, consider the following set of pairs of indices
\[
I' = \{(i_1, j_1), \ldots, (i_{r-3}, j_{r-3}), (i_{r-2}, j_{r-1})\}.
\]
Notice that this satisfies all the properties of the original set of pairs of indices, namely $i_t < j_t$ for all $t$, where we can “rename” $j_{r-2}$ as $j_{r-1}$, each $i_t$ appears at most once in the first coordinate of the pairs and the same hold for $j_t$ in the second coordinate of the pairs.

Since the $f_t$ are linearly independent linear polynomials it follows that $Z(J_T) \subset A^n_k$ has dimension $n - r$ and the result follows.
§13.2 Ruling out \( v \)-adic proximity

With an eye towards “globality” of relations among the values of G-functions at a certain point, we establish the following proposition.

**Proposition 13.1.** Let \( f : X \to S \) be a G-admissible variation of \( \mathbb{Q} \)-HS defined over some number field \( K \). Let \( s \in S(L) \), where \( L/K \) is some finite extension, and let \( v \in \Sigma_{L,f} \) be a finite place of \( L \).

If \( s \) satisfies condition \( \star \) then, assuming the Hodge conjecture holds and that a good arithmetic model exists for the morphism \( f \), the point \( s \) cannot be \( v \)-adically close to the degeneration \( s_0 \).

*Proof. Step 1:* Assume that \( s \) is \( v \)-adically close to \( s_0 \). We then get by 12.1 that \( h^{\text{ét}}_i = h_B^{i+n} \), assuming the existence of a good arithmetic model for \( f \) over \( \mathcal{O}_K \).

From 10.1, assuming the validity of the Hodge conjecture, we get that
\[
D_s \otimes_{\mathbb{Q}} \mathbb{Q}_l \hookrightarrow \text{End}(H^{n}_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l))^{N_v},
\]
(72)
where \( l \neq p(v) \). We are thus in a position to use 10.2 for the algebra \( D_s \otimes_{\mathbb{Q}} \mathbb{Q}_l \) and the nilpotent endomorphism \( N_v \) on the space \( H^{n}_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l) \).

From this we get that we must have that for all \( 1 \leq i \leq r \) and all \( w \in \Sigma_{F_i,f} \) with \( w \mid l \) there exists \( j(i,w) \) such that
\[
M_{m_i}(D_i \otimes_{F_i} F_{i,w}) \hookrightarrow M_{h^B_{j(i,w)}}(\mathbb{Q}_l),
\]
(73)

*Step 2:* We start by ruling out points that satisfy \( \star_1 \). This follows from 13.2. Indeed the dimension of the maximal subtorus of \( (D_s \otimes_{\mathbb{Q}} \mathbb{Q}_l)^{\times} \) is \( \sum_{i=1}^{r} m_i f_i \).

On the other hand, the maximal subtorus of \( \text{GL}(H^{n}_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l))^{N_v} \) has dimension \( \leq \mu - \dim_{\mathbb{Q}_l} N_v \). The result then follows from 12.1, which implies that \( \dim_{\mathbb{Q}_l} \text{Im}(N_v) = \dim_{\mathbb{Q}} \text{Im}(N_B) > 0 \).

Assume now that \( s \) satisfies condition \( \star_2 \). We then get that there exists \( i \), a prime \( l \neq p(v) \), and a place \( w \in \Sigma_{F_i,f} \) with \( w \mid l \) for which \( [F_{i,w} : \mathbb{Q}_l]m_i > h^B_{\text{max}} \).

This contradicts the validity of (73). Indeed the maximal commutative semisimple algebra of \( M_{m_i}(D_j \otimes_{F_i} F_{i,w}) \) has dimension \( \geq [F_{i,w} : \mathbb{Q}_l]m_i \) over \( \mathbb{Q}_l \), while that of \( M_{h^B_{\text{max}}}(\mathbb{Q}_l) \) has dimension \( h^B_{\text{max}} \) over \( \mathbb{Q}_l \).

Assume that \( s \) satisfies condition \( \star_3 \) and choose \( i \) as in \( \star_3 \). Then we have that there exists \( l \in \Sigma_{\mathbb{Q},f} \) that is totally split in \( F_i \) with \( l \neq p(v) \). Therefore we have that \( F_{i,w} = \mathbb{Q}_l \) for all \( w \in \Sigma_{F_i,f} \) with \( w \mid l \).
Also by \( \star_3 \) we know that we can find \( w \in \Sigma_{F_{i,f}} \) with \( w|l \) such that \( \text{inv}_v(D_i) \notin \mathbb{Z} \). Since once again by assumption we have \( (m_i d_i) \geq h_{\text{max}}^B \) we get that (73) is an isomorphism in this case, with \( h_{j(i,w)}^B \). This would imply that \( \text{inv}_w(D_i) = 0 \in \mathbb{Q}/\mathbb{Z} \) contradicting the above assumption.

Assume that \( s \) satisfies condition \( \star_4 \). Then by assumption there exists a prime \( l \neq p(v) \) such that there exists \( w \in \Sigma_{F_{i,f}} \) for which \( D_{i,w} = D_i \otimes_{F_i} F_{i,w} \) is a quaternion algebra over \( F_{i,w} \). If a \( j(i,w) \) satisfying (73) existed for the simple summand \( M_{m_i}(D_{i,w}) \) of \( D_i \otimes \mathbb{Q} \mathbb{Q}_l \), we would have \( m_i \text{dim}_{\mathbb{Q}_l} D_{i,w} | h_{j(i,w)}^B \) which contradicts our assumptions. Indeed, such an embedding would imply an isomorphism of \( M_{m_i}(D_{i,w}) \)-modules \( \mathbb{Q}_l \) with \( h_{j(i,w)}^B \). Comparing \( \mathbb{Q}_l \)-dimensions the contradiction follows.

The argument for \( \star_5 \) and \( \star_6 \) is practically identical to that of condition \( \star_4 \).

Finally, the proof in the case where \( s \) satisfies condition \( \star_7 \) is practically identical to that of \( \star_4 \) but has a small twist. Let us assume that \( |T_{D_i}| \geq 2 \) and let \( l \in T_{D_i} \) and \( w|l \) be such that \( l \neq p(v) \) and \( m_i d_i [F_{i,w} : \mathbb{Q}_l] \) for all \( j \).

If \( \text{inv}_w(D) \neq 0 \) we have that \( D_i \otimes_{F_i} F_{i,w} \simeq M_r(D') \) with \( D' \) a division algebra with center \( F_{i,w} \). Let \( d' := \sqrt{|D' : F_{i,w}|} \). If an index \( j(i,w) \) satisfying (73) existed, by the same argument as above, we would have that \( m_i d' \text{dim}_{\mathbb{Q}_l}(D') | h_{j(i,w)}^B \). Since \( r d' = d_i \) and \( \text{dim}_{\mathbb{Q}_l}(D') = d'^2 [F_{i,w} : \mathbb{Q}_l] \) we get the contradiction we wanted.

The case \( \text{inv}_w(D) = 0 \) follows from the same argument, though we do not have to introduce a new division algebras \( D' \) since \( D_i \otimes_{F_i} F_{i,w} \simeq M_{d_i}(F_{i,w}) \).

**Remark.** The proof in the case where condition \( \star_7 \) is satisfied shows that \( \star_7 \) is weaker than the strongest condition we can actually impose on these algebras.

In fact the proof shows that we would need to guarantee the impossibility of the occurrence of

\[
rd'^2 \cdot [F_{i,w} : \mathbb{Q}_l] h_j^B \text{ for some } j,
\]

where \( r \) and \( d' \) are as in the proof of \( \star_7 \).

**§14 Proof of 1.1**

Finally, we combine all parts of our exposition to prove 1.1.
Proof of 1.1: Let \( s \in S(L) \) be a point satisfying the conditions in 1.1 for the variation \( V = R^n f_s^* \mathbb{Q}_{X_s^n} \) where \( L/K \) is some finite extension. We let \( \xi := x(s) \), where \( x \) is the local parameter of \( S' \) at \( s_0 \) with respect to which the \( y_i \) are written as power series.

By 5.1 there exists a finite extension \( \hat{\mathbb{L}}/\mathbb{L} \) such that \( D_s \) acts on \( H^n_{DR}(X_s \times_{\hat{\mathbb{L}}} \mathbb{L}/\hat{\mathbb{L}}) \). From 5.2 we also know that \( \hat{\mathbb{L}} \) may be chosen so that \( [\hat{\mathbb{L}} : \mathbb{L}] \) is bounded only in terms of \( \mu := \dim_{\mathbb{Q}} H^n(X_{s}^{an}, \mathbb{Q}) \).

Let \( y_1, \ldots, y_{h\mu} \) be the G-functions that comprise the first \( h \) columns of the relative \( n \)-period matrix associated to the morphism \( f \). We then have the polynomials (53) with coefficients in \( \hat{F}_s \) and degree \( \leq [\hat{\mathbb{L}} : \mathbb{Q}] \). By 9.2 we know that these polynomials define relations among the values of the G-functions in question at \( \xi \) that are non-trivial, as long as there exists at least one archimedean place \( v \in \Sigma_{\hat{\mathbb{L}}, \infty} \) for which \( s \) is \( v \)-adically close to \( s_0 \).

Consider now \( v \in \Sigma_{\mathbb{L},f} \) to be any finite place of \( \hat{\mathbb{L}} \). By 13.1 we know that \( s \) cannot be \( v \)-adically close to \( s_0 \), in other words that \( |\xi|_v \geq \min\{1, R_v(\bar{y})\} \), where \( R_v(\bar{y}) \) is the local radius of convergence of the \( y_i \).

Now we split into two cases.

Case 1: For all archimedean places \( v \in \Sigma_{\mathbb{L},\infty} \) we have that \( |\xi|_v \geq \min\{1, R_v(\bar{y})\} \).

Combining this assumption with the above result we get that

\[
h(\xi^{-1}) \leq \rho(\bar{y}),
\]

where \( \rho(\bar{y}) \) is the global radius of the collection of power series \( y_i \). Combining Lemma 2 of I.§2.2 of [And89] with the Corollary of VI.§5 of loc.cit., we get that \( h(s) = h(\xi) = h(\xi^{-1}) \leq \rho(\bar{y}) < \infty \).

This concludes this case.

Case 2: There exists at least one archimedean place \( v \in \Sigma_{\mathbb{L},\infty} \) for which \( s \) is \( v \)-adically close to \( s_0 \).

In this case the relation defined by the polynomials in (53) among the values at \( \xi \) of the \( y_i \) is by construction global, since there are no other places \( v \in \Sigma_{\mathbb{L},f} \) for which \( s \) is \( v \)-adically close to \( s_0 \).

Since we know that the relation (53) is both non-trivial and global we get from 2.1 that

\[
h(\xi) \leq c_1(\bar{y})\delta^{3n-1}(\log \delta + 1),
\]

(74)

where \( \delta \) is the degree of the polynomial (53) in \( \mathbb{Q}[x_1, \ldots, x_{h\mu}] \).

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By construction of (53) we know that \( \delta \leq [\hat{L} : \mathbb{Q}] = [\hat{L} : L] \cdot [L : \mathbb{Q}] \). On the other hand 5.2 gives the bound \( [\hat{L} : L] \leq ((6.31)^2)^{\mu^2} \). Combining these remarks with (74) we get that there exist positive constants \( C_1, C_2 \), independent of the point \( s \), such that

\[
h(\xi) \leq C_1([L : \mathbb{Q}] + 1)^{C_2}, \tag{75}
\]

as we wanted.

The result follows by combining the two above cases, or simply by replacing \( C_1 \) in (75) by \( \max\{C_1, \rho(\vec{y})\} \).

\[\square\]

\section{15 Complex Multiplication and other examples}

We construct examples of possible algebras of endomorphisms of Hodge structures where the conditions for points \( s \) to be in the set \( \Sigma \) of 1.1 are easy to check. We start with the case of CM-Hodge structures and then construct infinite families of possible such algebras of all possible types, i.e. type I-IV in Albert’s classification 3.1.

\subsection{15.1 Hodge Structures with complex multiplication}

Hodge structures with complex multiplication, or simply CM Hodge structures, play the same role for Hodge structures with weight \( \geq 2 \) that abelian varieties with complex multiplication play for the weight 1 Hodge structures. For an introduction to these we point the interested reader to \([GGK12]\) and \([Moo99]\).

The main ingredient we will need about CM Hodge structures is the following lemma that describes their algebra of Hodge endomorphisms. Following \([GGK12]\) we write “CMpHS” as an abbreviation of the term “polarized Hodge structure with complex multiplication”.

\textbf{Lemma 15.1.} Let \( V \) be a CMpHS then there is a unique decomposition of \( V \) into simple Strongly CMpHS’s \( V = V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r} \). In particular for the algebra \( D \) of Hodge endomorphisms of \( V \) we have that

\[
D \cong \bigoplus_{i=1}^{r} M_{m_i}(K_i),
\]

where \( K_i \cong D_i \), the algebra of Hodge endomorphisms of \( V_i \), is a CM field with \( [K_i : \mathbb{Q}] = \dim_{\mathbb{Q}} V_i \).
Proof. See [GGK12] Ch.V and especially the facts on page 195. For a proof see [Moo99] where the necessary machinery is developed.

Motivated by this we make the following definition.

**Definition.** Let $\mathcal{V}$ be a polarized variation of $\mathbb{Q}$-HS over some base $T$. We say that the point $t \in T(\mathbb{C})$ is a **CM point of the variation**, or a **special point of the variation**, if the Hodge structure $\mathcal{V}_t$ is a CMpHS.

**Remarks.**

1. Consider a CMpHS $(V, \phi)$ as in 15.1. Let $E$ be the maximal commutative semi-simple algebra of the algebra $D$. We have by 15.1 that

$$E = K_1^{m_1} \times \cdots \times K_r^{m_r}. \quad (76)$$

We also have that $\dim_{\mathbb{Q}} E = \sum_{j=1}^r m_j[K_j : \mathbb{Q}]$. Noting that $\dim_{\mathbb{Q}} V = \sum_{j=1}^r m_j \dim_{\mathbb{Q}} V_j$ and that $\dim_{\mathbb{Q}} V_j = [K_j : \mathbb{Q}]$ we get that

$$\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} E. \quad (77)$$

2. The above property guarantees that CM-points satisfy the conditions of 8.3. Indeed, from the above and 8.4 we have that CM-points satisfy (49) and so we get non-trivial relations among the values of the $G$-functions we study at $\xi = x(s)$.

The globality of these relations follows from the fact that CM-points satisfy condition $\star_2$.

§15.1.1 **CM-points have potentially good reduction**

Given a CM abelian variety $A$ defined over a number field $K$ it is well known, see [ST68], that $A$ will have potentially good reduction at each finite place of $K$. The linear algebraic lemma 13.2 has an interesting consequence. Namely, it shows that a similar picture holds true for smooth projective varieties with CM polarized Hodge structure defined over a number field $K$. Even though the term “good reduction” or “potentially good reduction” cannot be expected to hold in general, as in the existence of Néron models, the term still makes sense from the point of view of Galois representations.

**Proposition 15.1.** Let $f : X \rightarrow S$ be a $G$-admissible variation of $\mathbb{Q}$-HS defined over the number field $K$. Let $L/K$ be a finite extension, let $s \in S(L)$ be a CM point for this variation, and let $v \in \Sigma_{L,f}$ be a finite place of $L$.

Assume the Hodge conjecture holds. Then the $l$-adic Galois representation of $G_{K_v}$ on $H^n_{\text{ét}}(\overline{X}_{s,v}, \mathbb{Q}_l)$ has potentially good reduction for all $l \neq p(v)$.
Proof. Let $D_s$ be the algebra of Hodge endomorphisms at $s$. We set $V_s := H^n(X^a_{s,\overline{Q}}, \mathbb{Q})$ to be the $\mathbb{Q}$-HS corresponding to $s$, with $\dim_{\mathbb{Q}} V_s = 2\mu$.

From 15.1 we know that $V_s$ decomposes as $V_s = \bigoplus_{i=1}^r V_i^{\oplus m_i}$, so that $D_s \cong \bigoplus_{i=1}^r M_{m_i}(K_i)$, where $V_i$ are irreducible CM polarized $\mathbb{Q}$-HS with algebras of Hodge endomorphisms $K_i$ being CM fields with $n_i := [K_i : \mathbb{Q}] = \dim_{\mathbb{Q}} V_i$. We also note that, trivially from the above, we have that $\mu = \sum_{i=1}^r m_i n_i$, where $\mu := \dim_{\mathbb{Q}} H^n(X^a_{s,\overline{Q}}, \mathbb{Q})$.

Let $p = p(v)$ be the characteristic of the residue field $\kappa(v) := \mathcal{O}_{L_v}/m_{L_v}$. We fix $l \in \mathbb{N}$ a prime with $l \neq p$. We then notice that the $\overline{\mathbb{Q}}_l$-algebra $\overline{D}_{s,l} := D_s \times_{\mathbb{Q}} \overline{\mathbb{Q}}_l \cong (D_s \times_{\mathbb{Q}} \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$ is such that $\overline{D}_{s,l}$ contains all of the closed $\overline{\mathbb{Q}}_l$-points of a $\mu$-dimensional torus.

From 10.1 we know that the inertia group $I_{L_v}$ acts quasi-unipotently on $H^\dagger_{\dagger}(X_{s,v}, \mathbb{Q}_l)$. Therefore up to a finite extension of $L_v$ we may and do assume that it in fact acts unipotently. In this case we get an associated nilpotent endomorphism $N_v$, as we saw in our earlier discussion, whose exponential determines the action of the inertia group.

In this case, from 10.1, we have that $\overline{D}_{s,l}^\times \hookrightarrow \text{GL}(H^\dagger_{\dagger}(X_{s,v}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l)^{N_v}$, where $N_v$ is the above nilpotent matrix associated to the action of the inertia group. The result now follows from 13.2. Indeed, if $N_v \neq 0$ we get that $\dim_{\overline{\mathbb{Q}}_l} \overline{D}_{s,l}^\times \leq \mu - \text{rank}(N_v) < \mu$ which contradicts the above.

§15.2 Some more complicated examples

We start with highlighting some examples where 1.1 applies. We fix for the remainder some general notation as in the main part of our exposition.

We consider $f : X \to S$ a $G$-admissible variation of Hodge structures with fibers of odd dimension $n$ and generic special Mumford-Tate group $Sp(\mu, \mathbb{Q})$, where $\mu := \dim_{\mathbb{Q}} H^n(X^a_z, \mathbb{Q})$ for any $z \in S^a$ and let $h := \dim_{\mathbb{Q}} \text{Im}((N^*)^n)$.

As before, for a point $s \in S(\overline{\mathbb{Q}})$ we let $V_s := H^n(X^a_{s,\overline{Q}}, \mathbb{Q})$ and assume that the decomposition of $V_s$ into simple polarized sub-$\mathbb{Q}$-HS is given by $V_1^{m_1} \oplus \ldots \oplus V_r^{m_r}$. We also let $D_s = M_{m_1}(D_1) \oplus \ldots \oplus M_{m_r}(D_r)$ be its algebra of Hodge endomorphisms.
For our process to kick in we need to have that

\[ h > \frac{\dim_{\mathbb{Q}} V_j}{[Z(D_j) : \mathbb{Q}]} \text{ for some } j, \text{ or that } \]

\[ h \geq \min \{\frac{\dim_{\mathbb{Q}} V_i}{[Z(D_i) : \mathbb{Q}]} : i \text{ such that } D_i = \text{End}_{HS}(V_i) \text{ is of type IV} \} . \]

Some examples of CM-fields and totally real fields

The fields \( F_{0,\beta} \) and \( F_\beta \): Let \( \beta \in \mathbb{N} \) and let \( p \) be a prime with \( p \equiv 1 \mod 2\beta \).

It is well known that the field \( \mathbb{Q}(\zeta_p) \) is a cyclic extension of \( \mathbb{Q} \) with totally real subfield \( \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \). We can therefore find a subfield \( F_{0,\beta} \) of \( \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \) with \( [F_{0,\beta} : \mathbb{Q}] = \beta \). Note that by construction this extension is also cyclic and totally real.

By Frobenius density it follows\(^8\) that the set

\[ \text{In}(F_{0,\beta}) := \{ l \in \Sigma_{\mathbb{Q},f} : l \text{ is inert in } F_{0,\beta} \} \]

is infinite. Fix \( l_1, l_2 \in \text{In}(F_{0,\beta}) \) with \( l_1 \neq l_2 \). Then from quadratic reciprocity we can find a prime \( q = q(l_1, l_2) \in \mathbb{Q} \) such that in the extension \( F_\beta := F_{0,\beta}(i\sqrt{q}) \) the prime ideals \( l_j \mathcal{O}_{F_{0,\beta}}, j = 1, 2 \), split in \( F_\beta \).

In conclusion we get that for the field \( F_\beta \) constructed above there are distinct primes \( l_1, l_2 \in \mathbb{Q} \) whose splitting in \( F_\beta \) is given by

\[ l_j \mathcal{O}_{F_\beta} = w_{j,1} : w_{j,2}. \]

In particular we get that \([ (F_\beta)_{w_{j,i}} : \mathbb{Q}_{l_j}] = n \) for all \( i, j \) and that, trivially by construction, if \( \sigma \) denotes complex conjugation in \( F_\beta/\mathbb{Q} \), we have \( \sigma w_{j,1} = w_{j,2} \).

Note that the family of CM-fields constructed above is infinite, for fixed \( \beta \), by varying either the infinitely many pairs \( (l_1, l_2) \) or the infinite choices \( q(l_1, l_2) \).

Examples of algebras of Type IV

Consider \( F_\beta \) a field as those constructed above. Let \( D \) be a division algebra over \( F_\beta \) and let \( d^2 = [D : F_\beta] \). For such an algebra to be of type IV in Albert’s classification it needs to satisfy the following conditions:

1. for any finite place \( v \in \Sigma_{F_\beta,f} \) we have that \( \text{inv}_v(D) + \text{inv}_{\sigma(v)}(D) = 0 \), and

---

\(^8\)See [Jan73] Ch. IV, Corollary 5.4.
2. inv_v(D) = 0 for all such places that satisfy \( \sigma(v) = v \).

By construction of our fields \( F_\beta \) we get places \( w_{j,i}, 1 \leq i, j \leq 2 \), that come in pairs with \( \sigma(w_{j,1}) = w_{j,2} \) for \( j = 1, 2 \). We want, in view of the conditions \(*\) introduced in §13, to work with algebras \( D \) that ramify at these finite places. For that reason we consider the following set of central division algebras, or CDA for short, with center \( F_\beta \):

\[
D_{IV}(\beta, d) := \{ D : CDA/F_\beta, \ [D : F_\beta] = d^2, \ \text{inv}_{w_{j,i}}(D) \not\in \mathbb{Z} \}.
\]

We note that the set \( D_{IV}(\beta, d) \) actually depends on the numbers \( \beta, l_j, q(l_1, l_2) \), and \( d \) so it would be perhaps more accurate to denote this by \( D_{IV}(d, \beta, l_1, l_2, q(l_1, l_2)) \), though we avoid this for notational brevity.

We note that for all choices of \( \beta, l_j, \) and \( q(l_1, l_2) \) as above the set \( D_{IV}(\beta, 2) \), i.e. the set of quaternion algebras satisfying the above conditions is non-empty. This follows from the classification theorem of quaternion algebras over global fields, see [Vig80] Ch. III, Théorème 3.1.

Let \( f : X \to S \) be as above and let \( s \in S(\overline{\mathbb{Q}}) \) and assume that one of the algebras \( D_k \) that appear in the decomposition of the algebra of Hodge endomorphisms \( D_s \) at the point \( s \) is an element of \( D_{IV}(\beta, d) \) for some choice of \( (\beta, d, l_1, l_2, q(l_1, l_2)) \). Then we can find simple conditions to check whether \( s \) is in the set \( \Sigma \) of 1.1.

Indeed, assuming that

\[
2\beta \geq \frac{\dim_{\mathbb{Q}} V_k}{h}
\]

is enough to guarantee the validity of (79). After this we just need to check the validity of at least one of the conditions in §13. In our case, by construction of the fields \( F_\beta \), conditions \(*_4\) and \(*_7\) translate into easy to check conditions, mainly thanks to the fact that \( \left[ (F_\beta)_{w_{j,i}} : \mathbb{Q}_{l_i} \right] = \beta \).

In more detail, by our construction in the case \( d = 2 \) condition \(*_4\) follows from

\[
4\beta m_k \mid h_j^B \text{ for all } j,
\]

where \( h_j^B \) are as in §12.1. Condition \(*_7\), only applicable in our case when \( d = d_k \geq 3 \), follows from

\[
m_k d \beta \mid h_j^B \text{ for all } j.
\]

Finally, for the case \( d = 1 \) so that \( D_k = F_\beta \), or in other words the case where \( V_k \) is a CM-HS while the other \( V_l \) are arbitrary polarized sub-HS of \( V_s \), it
is easy to create a condition analogous to the conditions \( \star \) created in §13. Indeed, it is easy to check, using arguments as in 13.1, that the the condition

\[
m_k \beta \not| h_j^B \quad \text{for all } j, \tag{83}
\]

is enough to guarantee the impossibility of \( v \)-adic proximity for all finite places \( v \in \Sigma_L \), where \( L = K(s) \) with our usual notation.

With these observations in mind we define \( \Sigma_{IV} \subset S(\overline{\mathbb{Q}}) \) to be the set that consists of the points \( s \in S(\overline{\mathbb{Q}}) \) whose corresponding algebra of Hodge endomorphisms satisfies the above hypothesis, i.e. for some \( k \) we have that \( D_k \in \mathcal{D}_{IV}(\beta, d) \) for some choice of \((\beta, d, l_1, l_2, q(l_1, l_2))\), condition (80) holds, and condition (81) holds if \( d_k = 2 \), or (82) holds if \( d_k \geq 3 \), or (83) holds if \( d_k = 1 \).

For the points in \( \Sigma_{IV} \) we get that they satisfy the conditions needed so that they are in the set \( \Sigma \) of 1.1. We thus have the following corollary.

**Corollary 15.1.** Let \( f : X \to S \) be a morphism over \( K \) defining a \( G \)-admissible variation of \( \mathbb{Q} \)-HS satisfying the conditions of 1.1. Let \( \Sigma_{IV} \) be the above set of points.

Then, there exist constants \( C_1, C_2 > 0 \) such that for all \( s \in \Sigma_{IV} \) we have

\[
h(s) \leq C_1[K(s) : K]^{C_2},
\]

where \( h \) is a Weil height on \( S' \).

**Example.** The simplest example of this nature that we could find is the following:

Assume that \( \mu = \dim_{\mathbb{Q}} V_s = 16 \) and that \( h = h_0^B = 4 \) for a \( G \)-admissible variation with \( n = 3 \) satisfying the conditions of 1.1.

In this case we let \( \beta = 2 \) and \( d = 2 \) above. In other words we can consider points for which \( D_s \) is a quaternion algebra over a CM-field constructed as above. Note that (80) is satisfied, leaving us with checking \( 8 \not| h_j^B \) for all \( j \).

We have \( h_0^B = h_0^B = 4 \) so all we need to check the validity of (81) is to make sure that \( 8 \not| h_j^B \) for \( j = 1, 2, \) and 3. To establish this we note that \( W_3^B \neq W_6^B \) and \( W_3^B \neq W_5^B \) in the notation of (57), which can be shown for example using the description of the \( W_i^B \) in [SZ85] remark (2.3). Since in our case \( \dim_{\mathbb{Q}} W_3^B = 12 \) and \( \dim_{\mathbb{Q}} W_0^B = 4 \) the result follows.

**Algebras of types I-III**

The totally real fields \( F_{0,\beta} \) with \([ F_{0,\beta} : \mathbb{Q} ] = \beta \) that we created above, help us construct convenient examples to check the conditions in §13, mainly since
they are cyclic. In fact, every such field constitutes a type I algebra in Albert’s classification.

By the aforementioned classification of quaternion algebras over number fields, see [Vig80] or [Voi21], we have a bijection between the set

\[ \{ \text{Quatnion algebras over } F_{0,\beta} \text{ up to isomorphism} \} \]

and the set\(^9\)

\[ \{ P \subset \Sigma_F : |P| \equiv 0 \mod 2 \} \]

given by \( B \mapsto \text{Ram}(B) \), with Ram(B) the set of places over which the quaternion algebra \( B \) ramifies.

With this in mind we define, in parallel to the type IV case above, the following sets of quaternion algebras \( D/F_{0,\beta} \):

\[ \mathcal{D}_{II}(\beta) := \{ D : \text{Ram}(D) \cap \Sigma_{F,\infty} = \emptyset, w_{j,i} \in \text{Ram}(D) \text{ for all } j, i \} \]

\[ \mathcal{D}_{III}(\beta) := \{ D : \Sigma_{F,\infty} \subset \text{Ram}(D), w_{j,i} \in \text{Ram}(D) \text{ for all } j, i \} \].

**Remark.** We note that by the aforementioned classification theorem it follows that these sets are in fact infinite, for fixed \( \beta \) and pair of primes \((l_1, l_2)\).

Note that for these algebras we will have by construction the following:

1. \( [(F_{0,\beta})_{w_{j,i}} : \mathbb{Q}_{l_j}] = [F_{0,\beta} : \mathbb{Q}] = \beta \) for all \( 1 \leq i, j \leq 2, \) and

2. for \( D \in \mathcal{D}_{III}(\beta) \cup \mathcal{D}_{II}(\beta) \) we will have that \( \text{inv}_{w_{j,i}}(D) \neq 0 \).

Given these it is much easier to check for the validity of the conditions for a point \( s \in S(\overline{\mathbb{Q}}) \) to be in the set \( \Sigma \) of 1.1, assuming that one of the algebras \( D_k \) appearing in the decomposition of the algebra of Hodge endomorphisms \( D_s \) is an element of \( \mathcal{D}_{III}(\beta) \cup \mathcal{D}_{II}(\beta) \), or even that \( D_k = F_{0,\beta} \) for some \( \beta \).

Indeed, assume that for some \( k \) we have \( D_k \in \mathcal{D}_{III}(\beta) \cup \mathcal{D}_{II}(\beta) \). Then, condition (78) translates to simply checking

\[ \beta > \frac{\dim_{\mathbb{Q}} V_k}{h}. \quad (84) \]

On the other hand, checking \( \star_4 \), which is the strongest out of the conditions in §13, becomes straightforward. Indeed, letting \( h_{ij}^B \) be the dimensions

\(^9\)Normally we would have to make sure that the the subsets \( P \) in question do not have any complex archimedean places. Since our fields are totally real this condition is simply vacuous.
of the quotients resulting from the weight monodromy filtration as in §12.1, \( \star_4 \) in this case follows from

\[
4m_k \beta \| h_j^B \text{ for all } j. \tag{85}
\]

In the case where \( D_k = F_{0, \beta} \) it is easy to check, with the same arguments as in the proof of 13.1, that the condition

\[
m_k \beta \| h_j^B \text{ for all } j, \tag{86}
\]

is enough to guarantee the impossibility of \( v \)-adic proximity for all finite places \( v \in \Sigma_L \), where \( L = K(s) \) with our usual notation.

With the above in mind, for our fixed morphism \( f : X \to S \), we consider the set \( \Sigma_{I-II} \subset S(\bar{\mathbb{Q}}) \) that consists of the points \( s \in S(\bar{\mathbb{Q}}) \) that are such that, if the corresponding algebra of Hodge endomorphisms is given by \( D_s = M_{m_1}(D_1) \oplus \ldots \oplus M_{m_r}(D_r) \), we have that there exists \( 1 \leq k \leq r \) and \( \beta, l_1, l_2 \in \mathbb{Q} \) as above such that \( (84) \) holds and either \( D_k \in D_{III}(\beta) \cup D_{II}(\beta) \) and \( (85) \) holds, or \( D_k = F_{0, \beta} \) and \( (86) \) holds.

In this case, 1.1 applies to such points and we have the following corollary.

**Corollary 15.2.** Let \( f : X \to S \) be a morphism over \( K \) defining a \( G \)-admissible variation of \( \mathbb{Q} \)-HS satisfying the conditions of 1.1. Let \( \Sigma_{I-II} \) be the above set of points.

Then, there exist constants \( C_1, C_2 > 0 \) such that for all \( s \in \Sigma_{I-II} \) we have

\[
h(s) \leq C_1[K(s) : K]^{C_2},
\]

where \( h \) is a Weil height on \( S' \).

**Remarks.** 1. We note that in both 15.1 and 15.2 the conditions imposed on the points \( s \) revolve around only one of the division algebras \( D_k \) that appear in the decomposition of the algebra of Hodge endomorphisms \( D_s \). The rest of the algebras \( D_t \) with \( t \neq k \) could have arbitrary properties.

2. We could simplify the situation by considering points \( s \) for which the algebra \( D_s \) is equal to one of the algebras constructed in the above examples, i.e., we are in the case where \( r = m_1 = 1 \) and \( D_s \) is a central simple algebra itself.
Appendix

A Some notes on polarizations

A.1 The non-relative case

Notation: Let $X/k$ be a smooth projective variety over a subfield $k$ of $\mathbb{C}$ and let $n = \dim_k X$.

Short review on polarizing forms

For all $d \in \mathbb{N}$ there exist non-degenerate bilinear polarizing forms

$$\langle , \rangle_{DR} : H^d_{DR}(X/k) \otimes_k H^d_{DR}(X/k) \to k,$$

$$\langle , \rangle_B : H^d(X^a_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^d(X^a_{\mathbb{C}}, \mathbb{Q}) \to (2\pi i)^{-d} \mathbb{Q} = \mathbb{Q}(-d),$$

on de Rham cohomology and Betti cohomology respectively. We also write

$$\langle , \rangle_B = (2\pi i)^{-d} \langle , \rangle,$$

where $\langle , \rangle$ has values in $\mathbb{Q}$ and is of the same type, i.e. symmetric or skew-symmetric, as $\langle , \rangle_B$.

These two are the polarizing forms of the corresponding cohomology group. Their existence follows from the fact that $X$ is projective and smooth and they are constructed via a very ample line bundle $[\text{Del71}]$.

We also have that, via the two embeddings $k \hookrightarrow \mathbb{C}$ and $(2\pi i)^{-d} \mathbb{Q} \hookrightarrow \mathbb{C}$, and the comparison isomorphism

$$P^d_X : H^d(X/k) \otimes_k \mathbb{C} \to H^d(X^a_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

the two bilinear forms $\langle , \rangle_{DR}$ and $\langle , \rangle_B$ are compatible under $P^d_X$, meaning that

$$\langle v, w \rangle_{DR} = \langle P^d_X(v), P^d_X(w) \rangle_B, \forall v, w \in H^d_{DR}(X/k) \otimes_k \mathbb{C}. \quad (87)$$

Relations on periods-Notation

From now on we assume that $d = n = \dim_k X$. We can and do consider from now on the above polarizing forms $\langle , \rangle_{DR}$, $\langle , \rangle_B$, and the form $\langle , \rangle$ as vectors in the spaces $H^n_{DR}(X/k)^* \otimes_k H^n_{DR}(X/k)^*$, $(H^n(X^a_{\mathbb{C}}, \mathbb{Q})^* \otimes_{\mathbb{Q}} H^n(X^a_{\mathbb{C}}, \mathbb{Q})^*)(-n)$, and $H^n(X^a_{\mathbb{C}}, \mathbb{Q})^* \otimes_{\mathbb{Q}} H^n(X^a_{\mathbb{C}}, \mathbb{Q})^*$ respectively.

In this case, i.e. $d = n$, via Poincaré duality, these forms will correspond to elements $t_{DR} \in H^n_{DR}(X/k) \otimes_k H^n_{DR}(X/k)$, $t_B \in (H^n(X^a_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X^a_{\mathbb{C}}, \mathbb{Q}))(-n)$, and $t \in H^n(X^a_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X^a_{\mathbb{C}}, \mathbb{Q})$, respectively.
The compatibility of the comparison isomorphism $P^n_X$ with Poincaré duality implies that $P^n_X \otimes P^n_X(t_{DR}) = t_B = (2\pi i)^n t$. In particular $t_{DR}$ is a Hodge class defined over the field $k$. For cycles such as this it is known$^{10}$ that they impose polynomial relations among the $n$-periods with coefficients in the field $k((2\pi i)^n)$.

In what follows we show that the relations constructed by $t_{DR}$ are in fact the Riemann-relations, i.e. they are the equations imposed on the $n$-periods by (87). This is used without proof by André in, essentially, the case were $n = 1$. The author is sure that this part is known to experts in the field and includes it only for the sake of completeness of the exposition.

**Notation:** We consider from now on a fixed basis $\{\gamma^*_i : 1 \leq i \leq \mu := \dim QH^n(X_{an}, \mathbb{Q})\}$ of $H^n(X_{an}, \mathbb{Q})$ and we let $\gamma^*_i$ be the elements of its dual basis, which constitutes a basis of $H^n(X_{an}, \mathbb{Q})$. We also consider $\omega_i, 1 \leq i \leq \mu$, a fixed $k$-basis of $H^n_{DR}(X/k)$.

With respect to these choices the isomorphism $P^n_X$ corresponds to the matrix $(\int \gamma_j^* \omega_i)$. We denote this matrix also by $P^n_X$ so that the isomorphism is nothing but $P^n_X(v) = t v P^n_X$, were on the right we have the matrix acting on the right. Vectors in the various spaces will be considered as column vectors in the various bases. Finally, we denote the matrix of the $n$-periods by $P := (2\pi i)^{-n} P^n_X$.

With the above notation fixed we let $\langle \omega_i, \omega_j \rangle_{DR} = d_{i,j}$ and let $M_{DR} = (d_{i,j}) \in \text{GL}_\mu(k)$, which will be the matrix corresponding to the form $\langle , \rangle_{DR}$, i.e.

$$\langle v, w \rangle_{DR} = t v M_{DR} w.$$  

Considering, alternatively as above, $\langle , \rangle_{DR}$ as an element of the space $H^n_{DR}(X/k)^* \otimes_k H^n_{DR}(X/k)$, the above are equivalent to

$$\langle , \rangle_{DR} = \sum_{i,j=1}^\mu d_{i,j} \omega^*_i \otimes \omega^*_j.$$  

Similarly we let $q_{i,j} = \langle \gamma^*_i, \gamma^*_j \rangle \in \mathbb{Q}$ and set $M_B = (q_{i,j}) \in \text{GL}_\mu(\mathbb{Q})$. This implies that $\langle \gamma^*_i, \gamma^*_j \rangle = (2\pi i)^{-n} q_{i,j}$. Same as above these relations can be rewritten as

$$\langle v, w \rangle = t v M_B w \text{ and } \langle v, w \rangle_B = t v ((2\pi i)^{-n} M_B) w,$$

$^{10}$See page 169 of [And89].
for all \( v, w \in H^n(X^m_C, \mathbb{C}) \). Alternatively, if we were to consider \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_B \) as elements of \( H^n(X^m_C, \mathbb{C})^* \otimes \mathbb{C} H^n(X^m_C, \mathbb{C})^* \) we can write these as \( \langle \cdot, \cdot \rangle = \sum_{i,j=1}^{\mu} q_{ij} \gamma_i \otimes \gamma_j \), and \( \langle \cdot, \cdot \rangle_B = \sum_{i,j=1}^{\mu} (2\pi i)^{-n} q_{ij} \gamma_i \otimes \gamma_j \) respectively.

We now consider the Poincaré duality isomorphisms \( \Pi_{DR} : H^n_{DR}(X/k) \to H^n_{DR}(X/k)^* \) and \( \Pi_B : H^n(X^m_C, \mathbb{Q}) \to H^n(X^m_C, \mathbb{Q}) \), which we have since \( \text{dim}_k X = n \). With respect to the bases \( \{ \omega_i \} \) and \( \{ \omega_i^* \} \) the isomorphism \( \Pi_{DR} \) corresponds to an invertible matrix which we denote by \( A_{DR} \in \text{GL}_\mu(k) \).

Similarly, with respect to the bases \( \{ \gamma_i \} \) and \( \{ \gamma_i^* \} \) we get the invertible matrix \( A_B \) corresponding to \( \Pi_B \).

Finally, let us write \( t_{DR} = \sum_{i,j=1}^{\mu} \lambda_{i,j} \omega_i \otimes \omega_j \), \( t = \sum_{i,j=1}^{\mu} \tau_{i,j} \gamma_i \otimes \gamma_j \), and \( t_B = \sum_{i,j=1}^{\mu} (2\pi i)^n \tau_{i,j} \gamma_i^* \otimes \gamma_j^* \), where \( \lambda_{i,j} \in k \) and \( \tau_{i,j} \in \mathbb{Q} \). We also define \( A_{DR} = (\lambda_{i,j}) \) and \( A_Q = (\tau_{i,j}) \).

**Classes and forms**

With the above notation fixed from now on we turn to describing the relation between the classes \( t_{DR} \), and \( t \) and the respective forms.

By definition we have \( \Pi_{DR}^{\otimes 2}(t_{DR}) = \langle \cdot, \cdot \rangle_{DR} \). This implies that

\[
\sum_{i,j=1}^{\mu} \lambda_{i,j} \Pi_{DR}(\omega_i) \otimes \Pi_{DR}(\omega_j) = \sum_{i,j=1}^{\mu} d_{i,j} \omega_i^* \otimes \omega_j^*. \tag{88}
\]

We know that \( \Pi_{DR}(\omega_i) = \Sigma a_{i,j} \omega_j^* \), with \( A_{DR} = (a_{i,j}) \). Applying this to (88) it is easy to see, with a few trivial computations, that \( t^* A_{DR} A_{DR} = M_{DR} \), or equivalently we get the equality

\[
A_{DR} = t^* A_{DR}^{-1} M_{DR} A_{DR}^{-1}. \tag{89}
\]

Similarly for the pair \( t \) and \( \langle \cdot, \cdot \rangle \) we find that

\[
A_Q = t^* A_B^{-1} M_B A_B^{-1}, \tag{90}
\]

coming from the equality \( \Pi_B^{\otimes 2}(t) = \langle \cdot, \cdot \rangle \).

**The relation given by \( t_{DR} \).**

We review how a relation on the \( n \)-periods is constructed from \( t_{DR} \). We start from the equality \( (2\pi i)^{-n}(P^n_X)^{\otimes 2}(t_{DR}) = t \). This in turn implies that for all \( l, m \), with the notation as above, we have

\[
\sum_{i,j=1}^{\mu} \lambda_{i,j} ((2\pi i)^{-n} \int_{\tau_l} \omega_i) ((2\pi i)^{-n} \int_{\tau_m} \omega_j) = (2\pi i)^{-n} \tau_{l,m}. \tag{91}
\]
These equations are the relations between $n$-periods that we eluded to earlier. Putting them altogether the previous relation is equivalent to the equality

$$^tP \Lambda_{DR} P = (2\pi i)^{-n} \Lambda_Q.$$  \hfill (92)

Comparing the matrices $A_B$ and $A_{DR}$.

Earlier on we had the matrices $A_{DR}$ and $A_B$ corresponding to the respective Poincaré duality isomorphisms. We saw in (89) and (90) how these matrices relate the “Λ-matrices” and “M-matrices”. We would like to replace the “Λ-matrices in (92) by the corresponding “M-matrices”, showing thus that the relations created are nothing but the Riemann-relations$^{11}$. The first step is to describe how the matrices $A_{DR}$ and $A_B$ relate to one another.

We had the isomorphisms $\Pi_{DR}$ and $\Pi_B$ and the matrices $A_{DR}$ and $A_B$ that represented these with respect to the bases we have chosen. We know that the comparison isomorphism $P_X^n$ respects Poincaré duality, meaning that the following diagram commutes:

$$
\begin{array}{ccc}
H^n_{DR}(X/k) \otimes_k \mathbb{C} & \xrightarrow{P_X^n} & H^n(X^an_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\
P_{DR}\otimes_k \mathbb{C} \downarrow & & \downarrow \Pi_B \otimes \mathbb{Q} \\
H^n_{DR}(X/k)^* \otimes_k \mathbb{C} & \xrightarrow{(P_X^n)^*} & H^n(X^an_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\
\end{array}
$$

where $(P_X^n)^*(f) = f \circ P_X^n$ for all $f \in H^n(X^an_{\mathbb{C}}, \mathbb{Q})^* \otimes_{\mathbb{Q}} \mathbb{C}$.

Looking at what the relation of the above diagram, i.e. $\Pi_{DR} \otimes_k \mathbb{C} = (P_X^n)^* \circ (\Pi_B \otimes_\mathbb{Q} \mathbb{C}) \circ P_X^n$, does to the basis $\{\omega_i\}$, and using the fact that with respect to the bases $\{\gamma_j\}$ and $\{\omega^*_i\}$ the matrix representing $(P_X^n)^*$ will be the matrix $^t(\int\gamma_j \omega_i)$, i.e. the transpose of $P_X^n$, we conclude that

$$A_{DR} = (\int\gamma_j \omega_i) \cdot A_B \cdot (^t(\int\gamma_j \omega_i)).$$  \hfill (93)

Conclusions

Combining (92) with (89) and (90) we get

$$^tP(^tA_{DR}^{-1}M_{DR}A_{DR}^{-1})P = (2\pi i)^{-n}(^tA_B^{-1}M_BA_B^{-1}).$$  \hfill (94)

$^{11}$See §7.1 for a definition and the reason of why we needed these.
From (93) we get
\[
{^t}P{^t}A_{DR}^{-1} = \frac{1}{(2\pi i)^n}{^t}A_B^{-1}P, \quad \text{and} \quad (95)
\]
\[
A_{DR}^{-1}P = \frac{1}{(2\pi i)^n}{^t}P^{-1}A_B^{-1}. \quad (96)
\]

Using (95) and (96) together with (94) we get
\[
PM_B{^t}P = (2\pi i)^{-n}M_{DR}. \quad (97)
\]

But, this is the relation we get between the above matrices by looking at the equation (87) and translating it in terms of matrices. Indeed, (87) translates to
\[
{^t}vP^n((2\pi i)^{-n}M_B)\{^t}wP^n = {^t}vM_{DR}w \quad \text{for all} \quad v, w \in H^n_{DR}(X/k) \otimes_k \mathbb{C}.
\]

From this we recover (97) by multiplying on both sides by $(2\pi i)^{-n}$ and noting that $P = (2\pi i)^{-n}P^n$.

What is actually of use to us is not exactly (97) but rather the same relation for the transpose of $P$. To obtain this, first from (97) we get trivially
\[
(2\pi i)^nM_B = P^{-1}M_{DR}{^t}P^{-1},
\]
then taking inverses on both sides we get
\[
{^t}PM_B^{-1}P = (2\pi i)^{-n}M_B^{-1}. \quad (98)
\]

A.2 The relative case

Setting: We consider $f : X \rightarrow S$ a smooth projective morphism of $k$-varieties itself defined over the same subfield $k$ of $\mathbb{C}$. We also assume that $S$ is a smooth connected curve which is not necessarily complete over $k$ and the dimension of the fibers of $f$ is $n$.

We then have, for all $d \in \mathbb{N}$, the relative version of the comparison isomorphism between the algebraic de Rham and the Betti cohomology
\[
P^d_{X/S} : H^d_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^n} \rightarrow R^df^*_{\ast}Q_{X_{\overline{S}^n}} \otimes_{\mathcal{O}_{\overline{S}^n}} \mathcal{O}_{\overline{S}^n}. \quad (99)
\]

Once again we let, in parallel to the non-relative case we studied earlier, $\mu$ denote the rank of these sheaves.

We once again have the same picture, as far as polarizing forms are concerned, as in the non-relative case. In other words we have $(\cdot)_{DR}$ a polarizing form of the de Rham cohomology sheaves $H^d_{DR}(X/S)$ which is defined over
the field $k$, and a polarizing form $(\cdot, \cdot)_B = (2\pi i)^{-n} \langle \cdot, \cdot \rangle$ of the sheaves on the right of (99). These two forms will be compatible with the relative isomorphism (99), meaning that we have

$$\langle P^d_{X/S}(v), P^d_{X/S}(w) \rangle_B = \langle v, w \rangle_{DR},$$

holds for all sections $v, w$ of the sheaf on the right of (99).

From now on we focus on the case $d = n$. We choose $U \subset S$ a non-empty affine open subset. Then the form $(\cdot, \cdot)_B|_U$ will map, via the relative version of the Poincaré duality isomorphism, to a class $t_{DR} \in H^n_{DR}(X/S) \otimes \mathcal{O}_S H^n_{DR}(X/S)|_U$.

Similarly we repeat this process for the forms $(\cdot, \cdot)_B$ and $(\cdot, \cdot)_{DR}$ over the analytification $U_\text{an}^n$, to get elements $t \in (R^n f_*\mathbb{Q} \otimes_{\mathbb{Q}_{S^an}} R^n f_*\mathbb{Q})|_{U_\text{an}^n}$ and $t_B \in (R^n f_*\mathbb{Q} \otimes_{\mathbb{Q}_{S^an}} R^n f_*\mathbb{Q})(n)|_{U_\text{an}^n}$ with $t_B = (2\pi i)^n t$.

Compatibility of Poincaré duality with the relative comparison isomorphism shows that $P^n_{X/S} \otimes P^n_{X/S}|_U(t_{DR}) = t_B$. In other words the class $t_{DR}$ is a relative Hodge class thus defining polynomial relations among the relative $n$-periods.

Now we can repeat the arguments we made in the non-relative case. First, we choose $\{\omega_i\}$ a basis of section of $H^n_{DR}(X/S)$ over the affine open subset $U \subset S$ and $\{\gamma_j\}$ a frame of $R^n f_{*an}^\text{an}Q_{X_{\text{an}}}|_V$, or equivalently a frame $\{\gamma_j\}$ of the relative homology $R_n f_{*an}^\text{an}Q_{X_{\text{an}}}|_V$, where $V \subset U_\text{an}^n$ is some open analytic subset. We get that the matrix $P^i_{X/S} \otimes P^i_{X/S}|_U(t_{DR}) = t_B$. In other words the class $t_{DR}$ is a relative Hodge class thus defining polynomial relations among the relative $n$-periods.

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The same process as before shows us that (101) is equivalent to the validity of the polynomial relations on the relative $n$-periods defined by the relative Hodge class $t_{DR}$. Finally, the same elementary argument as before shows the validity of the relative analogue of relation (98), i.e. the Riemann relations that we use in §7.1.

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