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ABSTRACT.

We consider correlation functions for the Wess-Zumino-Witten model on the torus with the insertion of a Cartan element; mathematically this means that we consider the function of the form

$$F = \text{Tr}(\Phi_1(z_1) \ldots \Phi_n(z_n) q^{-\partial} e^h)$$

where $\Phi_i$ are intertwiners between Verma modules and evaluation modules over an affine Lie algebra $\hat{\mathfrak{g}}$, $\partial$ is the grading operator in a Verma module and $h$ is in the Cartan subalgebra of $\mathfrak{g}$. We derive a system of differential equations satisfied by such a function. In particular, the calculation of $q \frac{\partial}{\partial q} F$ yields a parabolic second order PDE closely related to the heat equation on the compact Lie group corresponding to $\mathfrak{g}$ (cf. [Ber]). We consider in detail the case $n = 1$, $\mathfrak{g} = \mathfrak{sl}_2$. In this case we get the following differential equation ($q = e^{\pi i \tau}$):

$$\left(-2\pi i(K + 2) \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right) F = (m(m+1)\varphi(x + \frac{\tau}{2}) + c) F,$$

which for $K = -2$ (critical level) becomes Lamé equation. For the case $m \in \mathbb{Z}$ we derive integral formulas for $F$ and find their asymptotics as $K \to -2$, thus recovering classical Lamé functions.

Introduction.

We start with consideration of the Wess-Zumino-Witten model of conformal field theory on a torus. This is, we consider an affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to some simple finite-dimensional Lie algebra $\mathfrak{g}$. For technical reasons, it is more convenient to work with a twisted realization of $\hat{\mathfrak{g}}$. Next, we consider Verma modules $M_{\lambda,k}$ over $\hat{\mathfrak{g}}$. If $V$ is a representation of the finite-dimensional algebra $\mathfrak{g}$ then by definition a vertex operator $\Phi(z): M_{\lambda,k} \to M_{\mu,k} \otimes V$ is an operator valued formal Laurent series in $z$ satisfying the following commutation relations with the elements of $\hat{\mathfrak{g}}$:

$$\Phi(z)a \otimes t^m = ((a \otimes t^m) \otimes 1 + z^m 1 \otimes a) \Phi(z).$$

Let $M_{\lambda_i,k}, i = 0 \ldots n$ be a collection of Verma modules such that $\lambda_0 = \lambda_n$, and $\Phi_i(z_i): M_{\lambda_i,k} \to M_{\lambda_{i-1},k} \otimes V_i$ be vertex operators. Then we can consider the following “correlation function on the torus”:

$$F(z_1, \ldots, z_n) = \text{Tr}(\Phi_1(z_1) \ldots \Phi_n(z_n) q^{-\partial} e^h).$$
\[ F(z_1 \ldots z_n, q, h) = \text{Tr}|_{M_{\lambda, k}} (\Phi^1(z_1) \ldots \Phi^n(z_n) q^{-\partial} e^h) \]

where \( \partial \) is the grading operator in Verma modules\(^1\) and \( h \in h_R \). This function takes values in the module \( V = V_1 \otimes \ldots \otimes V_n \) and it is the main object of our study.

Our first goal is to derive differential equations for \( F \). We compute \( \frac{\partial}{\partial z_i} F \) using the same technique as for the usual Knizhnik-Zamolodchikov equations (see [TK], [FR]). However, this system of equations (Theorem 3.1) is not closed: it has the form

\[ z_i \frac{\partial}{\partial z_i} F = A_i(z_1 \ldots z_n) \Phi + \sum \pi_i(x_l) \frac{\partial}{\partial x_l} \Phi \]

where \( A_i \) are some operators in \( V \) and the sum is taken over an orthonormal basis \( x_l \) in \( h \). Since we do not have any information about \( \frac{\partial}{\partial x_l} F \), this system does not allow us to determine \( F \). This system of equations in another form appeared first in the papers of Bernard ([Ber]).

However, we can get additional information studying dependence on the modular parameter \( q \). This dependence takes a simpler form if we renormalize \( F \), considering \( F/F_0 \), where \( F_0 = \text{Tr}|_{M_{0, k}} (q^{-\partial} e^h) \). For this function we have the following equation (Theorem 4.1):

\[ (q \frac{\partial}{\partial q} - \frac{1}{2(k+1)} \Delta_h) \frac{F}{F_0} = \frac{1}{k+1} B \frac{F}{F_0}, \]

where \( B \) is some operator in \( V \) depending on \( z_i, q \).

This equation is especially interesting for \( n = 0 \) and \( n = 1 \), since in these cases there is no dependence on \( z \). For \( n = 0 \) we obtain that \( F/F_0 \) satisfies the heat equation on the maximal torus \( T \subset G \). This result was first obtained by Bernard; however, he worked in the untwisted setting which gave him heat equation on the group rather than on the torus.

We concentrate our attention on the case \( n = 1, g = \mathfrak{sl}_2 \) and take \( V \) to be \( 2m+1 \)-dimensional irreducible module over \( \mathfrak{sl}_2 \). In this case the intertwiner \( \Phi(z): M_{\lambda, k} \to M_{\lambda, k} \otimes V \) exists and is unique up to a constant for generic values of \( k, \lambda \). Therefore we can apply the results of the previous sections. It will be convenient to identify \( h_R \simeq \mathbb{R} \) and consider the following function of one real variable \( x \):

\[ F(x, q) = \frac{\text{Tr}|_{M_{\lambda, k}} (\Phi(z) q^{-\partial} e^{-\pi i z h})}{\text{Tr}|_{M_{0, k}} (q^{-\partial} e^{-\pi i z h})}, \]

where \( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2 \). Then the results of Section 4 show that thus defined \( F \) satisfies the following partial differential equation (eq. (5.1)):

\[ \left( -4\pi (k+1) \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2} \right) F = m(m+1) \left( \varphi \left( \frac{\tau}{2} + x \right) + c \right) F, \]

where \( q = e^{\pi i \tau} \).

\(^1\)We use the symbol \( \partial \) for the grading operator in twisted realization, reserving the standard notation \( d \) for the untwisted grading operator, see Section 1.
This equation has many nice properties and is worth studying in itself; it is a non-stationary Schrödinger equation. Equations of this type were considered and treated in [Kri] and references therein. What is particularly interesting is that in the limit \( k \to -1 \) this equation becomes the Lamé equation with precisely the same expression of the coefficient to \( \varphi \) as in the classical form (i.e., \( m(m+1) \), cf. [WW]). Note that the case \( k = -1 \) in the twisted setting corresponds to \( K = -2 \) in the untwisted one, i.e. to the critical level. Therefore we should expect that the asymptotics of the correlation functions on the torus at the critical level limit are closely related to the solutions of Lamé equation.

We make this relation more explicit. First we derive integral formulas for the function \( F \) defined above using the Wakimoto realization of Verma modules over \( \widehat{\mathfrak{sl}}_2 \). The answer is given by Theorem 5.1 and has the following form:

\[
F(x, q) = \int l(\zeta|\tau)^{1/\kappa} \phi(x, \zeta|\tau) d\zeta_1 \ldots d\zeta_m
\]

where both \( l(\zeta) \) and \( \phi(x, \zeta) \) are expressed in terms of theta functions and exponents and \( \kappa = 2(k+1) \).

Now we can find the asymptotics of these functions as \( \kappa \to 0 \). To do it we need the saddle-point method which says that under certain assumptions this integral has the following asymptotics:

\[
F(x, q) \sim C(\tau) \kappa^{m/2} l(\zeta_0|\tau)^{1/\kappa} \phi(x, \zeta_0|\tau),
\]

where \( \zeta_0 \) is a critical point for the function \( l(\zeta) \). Substituting this asymptotics in equation (5.1) we finally see that the function

\[
f(x) = \phi(x, \zeta_0)
\]

is a solution of the Lamé equation. Note that we cannot find the critical point \( \zeta_0 \) explicitly; we can only write a system of equations defining it; this system of equations will also depend on the highest weight \( \lambda \) of the Verma module.

It turns out that in this way we can get a generic eigenfunction of the Lamé operator; in particular, we can get all the doubly periodic eigenfunctions which are obtained for integer \( \lambda \). For example, for \( m = 1 \) we show how to get the classical solutions \( f(x) = \text{dn} \ 2Kx, \text{sn} \ 2Kx, \text{cn} \ 2Kx \), which correspond to even (the first solution) and odd (two latter solutions) values of \( \lambda \).

However, it is not very easy to check the applicability of the saddle-point method. For the case \( m = 1 \), we can check it in one special case (\( q \in \mathbb{R}, \lambda \) is small) and then use analytic continuation arguments to show that in fact every critical point gives rise to an eigenfunction of Lamé equation. In the general case one could repeat similar arguments to prove that again any critical point gives a solution of Lamé equation; however, we do not follow this way; instead, we say that it can be checked by direct calculation which is in fact carried out in the book by Whittaker and Watson [WW].

Remark. The asymptotics of solutions of the KZ equations at the critical level were computed by means of the saddle-point method by Reshetikhin and Varchenko [RV]. They found that the leading term of these asymptotics is a solution of the Bethe equations well known in physics.
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1. A twisted realization of affine Lie algebras.

Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) of rank \( r \). Denote by \( <,> \) the standard invariant form on \( \mathfrak{g} \) with respect to which the longest root has length \( \sqrt{2} \).

Let \( \mathfrak{h} \) denote a Cartan subalgebra of \( \mathfrak{g} \). The form \( <,> \) defines a natural identification \( \mathfrak{h}^* \to \mathfrak{h} : \lambda \mapsto h_\lambda \) for \( \lambda \in \mathfrak{h}^* \). We will use the notation \( <,> \) for the inner product in both \( \mathfrak{h} \) and \( \mathfrak{h}^* \).

Let \( \Delta^+ \) be the set of positive roots of \( \mathfrak{g} \). For \( \alpha \in \Delta^+ \), let \( e_\alpha \in \mathfrak{g}_\alpha \), \( f_\alpha \in \mathfrak{g}_{-\alpha} \) be such that \( < e_\alpha, f_\alpha > = 1 \). Then \( [e_\alpha, f_\alpha] = h_\alpha \). Also, let \( x_j, 1 \leq j \leq r \), be an orthonormal basis of \( \mathfrak{h} \) with respect to the standard invariant form. Then the elements \( e_\alpha, f_\alpha, x_i \) form a basis of \( \mathfrak{g} \).

Let \( g \) be the dual Coxeter number of \( \mathfrak{g} \). Let \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \). Then for \( \alpha \in \Delta^+, \ < \rho, \alpha > = |\alpha| \) is the number of summands in the decomposition of \( \alpha \) in the sum of simple positive roots.

Let \( \gamma = \text{Ad} e^{2\pi i \rho} \) be an inner automorphism of \( \mathfrak{g} \). This automorphism is of order \( g \).

The action of \( \gamma \) on root vectors is as follows: \( \gamma(e_\alpha) = e^{2\pi i |\alpha|} e_\alpha \), \( \gamma(f_\alpha) = e^{-2\pi i |\alpha|} f_\alpha \), where \( e = e^{2\pi i/g} \) is a primitive \( g \)-th root of unity. Note that since \( 0 < |\alpha| \leq g - 1 \), \( \gamma(a) = a \) if and only if \( a \in \mathfrak{h} \).

Let \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \) be the affine Lie algebra associated with \( \mathfrak{g} \). The commutation relations in this algebra are

\[
(1.1) \quad [a(t) + \lambda c, b(t) + \mu c] = [a(t), b(t)] + \frac{1}{2\pi i} \oint_{|t|=1} < a'(t)b(t) > t^{-1} dt \cdot c
\]

for any two \( \mathfrak{g} \)-valued Laurent polynomials \( a(t), b(t) \), and complex numbers \( \lambda, \mu \).

Define the twisted affine algebra \( \hat{\mathfrak{g}}_\gamma \) as a subalgebra of \( \hat{\mathfrak{g}} \) consisting of all expressions \( a(t) + \lambda c \) with the property \( a(zt) = \gamma(a(t)) \). The basis in this subalgebra is given by elements \( e_\alpha \otimes t^{\alpha|m} \), \( f_\alpha \otimes t^{-\alpha|m} \), \( x_i \otimes t^m \), \( c \), for \( m \in \mathbb{Z}, \alpha \in \Delta^+ \). For brevity we write \( x[n] \) for \( x \otimes t^{n} \) and \( x \) for \( x \otimes t^{0} \).

The representation theory of \( \hat{\mathfrak{g}}_\gamma \) is quite parallel to that in the untwisted case. First of all, define the polarization of \( \hat{\mathfrak{g}}_\gamma \): \( \hat{\mathfrak{g}}_\gamma = \hat{\mathfrak{g}}^{++}_\gamma \oplus \hat{\mathfrak{g}}^{-}_\gamma \oplus h \oplus \mathbb{C}c \). Here \( \hat{\mathfrak{g}}^{++}_\gamma \) is the set of polynomials \( a(t) \) vanishing at 0, and \( \hat{\mathfrak{g}}^{-}_\gamma \) is the set of polynomials \( a(t) \) vanishing at infinity.

Next, define Verma modules over \( \hat{\mathfrak{g}}_\gamma \). This is done exactly in the same way as for the untwisted affine algebra. Let \( \lambda \in \mathfrak{h}^* \) be a weight, and let \( k \) be a complex

\footnote{Note that throughout the paper we denote the complex number \( i = \sqrt{-1} \) by a roman “i”, to distinguish it from the subscript \( i \), which is italic.}
number. Define $X_{\lambda,k}$ to be a one dimensional module over $\hat{g}^+_\gamma \oplus \mathfrak{h} \oplus \mathbb{C}c$ spanned by a vector $v$ such that $\hat{g}^+_\gamma$ annihilates $v$, and $cv = kv$, $hv = \lambda(h)v$, $h \in \mathfrak{h}$. Define the Verma module

\begin{equation}
M_{\lambda,k} = \text{Ind}_{\hat{g}^+_\gamma \oplus \mathfrak{h} \oplus \mathbb{C}c}^{\mathfrak{g}^+_\gamma} X_{\lambda,k}.
\end{equation}

Now define evaluation representations. Let $V$ be a module over $\mathfrak{g}$; we always assume that $V$ has a weight decomposition such that weight subspaces are finite-dimensional. Define the operator $C \in \text{End}(V)$ by $C = e^{2\pi i h_{\rho}}$; then $Caw = \gamma(a)Cw$ for any $w \in V$, $a \in \mathfrak{g}$, and $Cw_0 = w_0$, if $w_0$ is from the zero-weight subspace $V(0)$.

Let $V(z)$ denote the space of $V$-valued Laurent polynomials in $z$, and let $V_C(z)$ be the space of those polynomials which satisfy the equivariance condition $w(\varepsilon z) = Cw(z)$.

The natural (pointwise) action of $\hat{\mathfrak{g}}$ on $V(z)$ restricts to an action of $\hat{\mathfrak{g}}_{\gamma}$ on $V_C(z)$.

Note that the twisted affine algebra is isomorphic to untwisted one. More precisely, we have the following lemma.

**Lemma 1.1** (see [PS], p.36). The Lie algebras $\hat{\mathfrak{g}}_{\gamma}$ and $\hat{\mathfrak{g}}$ are isomorphic. Under this isomorphism, the Verma module $M_{\lambda,k}$ over $\hat{\mathfrak{g}}_{\gamma}$ is identified with the Verma module $\mathcal{M}_{\lambda+k\rho,\mathfrak{g}K}$ over $\hat{\mathfrak{g}}$, and evaluation representation $V_C(z)$ over $\hat{\mathfrak{g}}_{\gamma}$ is identified with the evaluation representation $V(z)$ over $\hat{\mathfrak{g}}$.

Therefore, all the results about the representations of $\hat{\mathfrak{g}}_{\gamma}$ can be as well obtained from the representations theory of $\hat{\mathfrak{g}}$. However, use of the twisted algebra is technically more convenient to us.

We can also extend the Lie algebra $\hat{\mathfrak{g}}_{\gamma}$, adding to it the degree operator $\partial$ which commutes with elements of $\hat{\mathfrak{g}}_{\gamma}$ by $[\partial, a(t)] = ta'(t), [\partial, c] = 0$. We denote this extended Lie algebra by $\hat{\mathfrak{g}}_{\gamma}$. Note that under the isomorphism of Lemma 1.1, $\partial \mapsto gd + h_{\rho}$, where $d$ is the standard grading operator for $\hat{\mathfrak{g}}$.

Since every Verma module has a natural $\mathbb{Z}$-gradation, we can uniquely extend the action of $\hat{\mathfrak{g}}_{\gamma}$ in $M_{\lambda,k}$ to the action of $\hat{\mathfrak{g}}_{\gamma}$ as soon as we define the action of $\partial$ on the highest weight vector. Let us define it by $\partial v_{\lambda,k} = -\frac{\langle \lambda, \lambda \rangle}{2(k+1)} v_{\lambda,k}$. Again, this agrees with the untwisted case: if we identify $M_{\lambda,k}$ with the Verma module $\mathcal{M}_{\Lambda,\mathfrak{g}K}$ over $\hat{\mathfrak{g}}$, where $\Lambda = \lambda + k\rho, K = gk$, then $\partial$ is identified with $gd + h_{\rho} - \frac{k^2\langle \rho, \rho \rangle}{2(k+1)}$, where the action of $d$ on Verma module $\mathcal{M}_{\Lambda,\mathfrak{g}K}$ is defined so that on the highest level, $d = -\frac{\langle \Lambda, \Lambda+2\rho \rangle}{2(k+g)}$. This gives us the following twisted analogue of Sugawara construction for $\partial$.

**Proposition 1.2** (see [E]). In every Verma module over $\hat{\mathfrak{g}}_{\gamma}$,

\begin{equation}
\partial = -\frac{1}{k+1} \sum_{m \in \mathbb{Z}} \left( \sum_{\alpha \in \Delta^+} :e_\alpha[|\alpha|+mg]f_\alpha[-|\alpha|-mg] : + \frac{1}{2} \sum_{j=1}^r :x_j[mg]x_j[-mg] : \right),
\end{equation}

where $::$ is the standard normal ordering:

\begin{equation}
:e_\alpha[n]f_\alpha[-n] := \begin{cases} e_\alpha[n]f_\alpha[-n] & n < 0 \\ f_\alpha[-n]e_\alpha[n] & n > 0 \end{cases}
\end{equation}

\begin{equation}
h[n]h[-n] := \begin{cases} h[n]h[-n] & n \leq 0 \\ h[-n]h[n] & n > 0 \end{cases}, \quad h \in \mathfrak{h}.
\end{equation}
We can also define the action of $\partial$ in the evaluation representation $V_C(z)$ by $\partial = z \frac{d}{dz}$. Thus $V_C(z)$ becomes a $\tilde{\mathfrak{g}}_\gamma$-module. Again, under the isomorphism of the Lemma 1.1, $\partial$ is identified with the operator $gd + h_\rho$ in the evaluation representation $V(z)$ over $\tilde{\mathfrak{g}}$, where we define the action of $d$ in $V(z)$ by $d = z \frac{d}{dz}$.3

Let us introduce the twisted version of currents. Set

$$J_{e_\alpha}(z) = \sum_{m \in \mathbb{Z}} e_\alpha[|\alpha| + mg] \cdot z^{-|\alpha| - mg - 1},$$

$$J_{f_\alpha}(z) = \sum_{m \in \mathbb{Z}} f_\alpha[-|\alpha| + mg] \cdot z^{|\alpha| - mg - 1},$$

$$J_h(z) = \sum_{m \in \mathbb{Z}} h[mg] \cdot z^{-mg - 1}, \quad h \in \mathfrak{h}.$$  

(1.5)

Thus by linearity we have defined $J_a(z)$ for any $a \in \mathfrak{g}$.

Define the polarization of currents:

$$J^+_a(z) = \sum_{m < 0} e_\alpha[|\alpha| + mg] \cdot z^{-|\alpha| - mg - 1},$$

$$J^+_f(z) = \sum_{m \leq 0} f_\alpha[-|\alpha| + mg] \cdot z^{|\alpha| - mg - 1},$$

$$J^+_h(z) = \frac{1}{2} h \cdot z^{-1} + \sum_{m < 0} h[mg] \cdot z^{-mg - 1}, \quad h \in \mathfrak{h}.$$  

(1.6)

This defines $J^+_a(z)$ for all $a \in \mathfrak{g}$. Now set

$$J^-_a(z) = J^+_a(z) - J_a(z).$$  

(1.7)

Note that this polarization is not quite the same as the standard polarization of currents for the untwisted $\tilde{\mathfrak{g}}$, i.e. the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}_\gamma$ does not match up these two polarizations.

One can easily write commutation relations between currents; however, we won’t need them. One thing we will need is the commutation relations with $\partial$. Namely, one can easily see that

$$J^\pm_a(z) q^{-\partial} = q^{-1} q^{-\partial} J^\pm_a(q^{-1}z)$$

(1.8)

2. Twisted intertwiners and correlation functions on a torus

We will be interested in $\tilde{\mathfrak{g}}$, intertwining operators $\Phi(z) : M_{\lambda,k} \to M_{\nu,k} \otimes z^\Delta V_C(z)$, where the highest weight of $V$ is $\mu$, $\otimes$ denotes the completed tensor product, and $\Delta$ is a complex number.

Let $z_0$ be a nonzero complex number. Evaluation of the operator $\Phi(z)$ at the point $z_0$ yields an operator $\Phi(z_0) : M_{\lambda,k} \to M_{\nu,k} \otimes V$, where $M$ denotes the completion of $M$ with respect to the grading.

3Note, however, that usually in the conformal field theory the action of $d$ on the evaluation representation is defined with some shift, namely: $d = z \frac{d}{dz} + \frac{C}{2(K + \gamma)}$, where $C$ is the Casimir operator. We will note use this convention here.
From now on the notation \( \Phi(z) \) will mean the operator \( \Phi \) evaluated at the point \( z \in \mathbb{C}^* \). This will give us an opportunity to regard the operator \( \Phi(z) \) as an analytic function of \( z \). This analytic function will be multivalued: \( \Phi(z) = z^\Delta \Phi^0(z) \), where \( \Phi^0 \) is a single-valued function in \( \mathbb{C}^* \), and \( \Delta = \frac{<\nu,\nu> - <\lambda,\lambda>}{2(k+1)} \).

Let \( u \) belong to the restricted dual module \( V^* \). Introduce the notation \( \Phi_u(z) = (1 \otimes u)(\Phi(z)) \). \( \Phi_u(z) \) is an operator: \( M_{\lambda,k} \rightarrow \hat{M}_{\nu,k} \).

The intertwining property for \( \Phi(z) \) can be written in the form

\[
(a[m], \Phi_u(z)) = z^m \Phi_{au}(z).
\]

It is convenient to write the intertwining relation in terms of currents.

**Lemma 2.2.**

\[
[J^\pm_{\mu}(\zeta), \Phi_u(z)] = \zeta^{-1} x^0(\zeta/z) \Phi_{hu}(z);
\]

\[
[J^\pm_{e\alpha}(\zeta), \Phi_u(z)] = \zeta^{-1} x^\alpha(\zeta/z) \Phi_{e\alpha u}(z), \quad \alpha \in \Delta_+;
\]

\[
[J^\pm_{f\alpha}(\zeta), \Phi_u(z)] = \zeta^{-1} x^{-\alpha}(\zeta/z) \Phi_{f\alpha u}(z), \quad \alpha \in \Delta_+,
\]

where

\[
x^0(t) = \frac{1 + t^g}{2(1 - t^g)},
\]

\[
x^\alpha(t) = t^{-|\alpha|} \frac{t^g}{1 - t^g},
\]

\[
x^{-\alpha}(t) = t^{|\alpha|} \frac{1}{1 - t^g}
\]

The identities marked with + make sense if \( |z| > |\zeta| \), and those marked with − make sense if \( |z| < |\zeta| \).

Now we are ready to write down the twisted version of the operator Knizhnik-Zamolodchikov (KZ) equations.

**Theorem 2.3** (see[E]). The operator function \( \Phi_u(z) \) satisfies the differential equation

\[
(k + 1) \frac{d}{dz} \Phi_u(z) = \sum_{\alpha \in \Delta_+} (J^+_{e\alpha}(z) \Phi_{f\alpha u}(z) - \Phi_{f\alpha u}(z) J^-_{e\alpha}(z)) + \sum_{\alpha \in \Delta_+} (J^+_{f\alpha}(z) \Phi_{e\alpha u}(z) - \Phi_{e\alpha u}(z) J^-_{f\alpha}(z)) + \sum_{j=1}^r (J^+_{xj}(z) \Phi_{xj u}(z) - \Phi_{xj u}(z) J^-_{xj}(z)).
\]

In the paper [E] this proposition is proved in the case when \( V \) is a highest-weight module. However, one can check that the result is valid for any module with the weight decomposition.

**Remark.** Note that since we have the isomorphism of Lemma 1.1, we can identify the intertwiners for \( \tilde{g} \), with those for \( \mathfrak{g} \). Namely, if \( \tilde{\Phi}(\zeta): M_{\lambda,k} \rightarrow M_{N,K} \otimes V_{\mu} \) is an intertwiner for \( \tilde{g} \), then the results of Section 1 show that
\begin{equation}
\Phi(z) = z^{-h^\gamma} \hat{\Phi}(z^g)
\end{equation}

will be an intertwiner for $\tilde{\mathfrak{g}}_\gamma$.

The main object of our study will be the following twisted version of correlation functions on a torus. Let $V_1, \ldots, V_n$ be representations of $\mathfrak{g}$, and let $\Phi^i(z_i) : M_{\lambda_i,k} \rightarrow \hat{M}_{\lambda_{1-k}} \otimes V_i$, $1 \leq i \leq n$ be intertwining operators for $\tilde{\mathfrak{g}}_\gamma$ (sometimes for brevity we write $\Phi^i$ instead of $\Phi^i_{u_\gamma}(z_i)$). If $\lambda_0 = \lambda_n = \lambda$, then we can define the following correlation function on a torus:

\begin{equation}
\mathcal{F}_{u_1, \ldots, u_n}(z_1, \ldots, z_n; h) = \text{Tr}_{\lambda, 1} \left( \Phi^1_{u_1}(z_1) \cdots \Phi^n_{u_n}(z_n) q^{-\partial} e^h \right),
\end{equation}

where $q \in \mathbb{C}^\times$, $|q| < 1$, $h \in \mathfrak{h}_\mathbb{R}$ is an element from the compact form of $\mathfrak{h}$ (i.e., $h(\alpha) \in \mathbb{R}$ for all $\alpha \in \Delta$). This trace is a formal power series in $z_1 \ldots z_n$. However, it can be shown that it converges to an analytic function of $z_i$ in the region $|z_i| > \ldots > |z_n| > |qz_1|$. Therefore, we will consider $\mathcal{F}$ as a function of $q, z_1, \ldots, z_n, h$ with values in $V_1 \otimes \cdots \otimes V_n$ or, equivalently, as a function of $q, z_i$ with values in $V_1 \otimes \cdots \otimes V_n \otimes C^\infty(\mathfrak{h}_\mathbb{R})$. Our first goal is finding differential equations for this function.

3. Differential equations for correlation functions.

In this section we deduce the differential equations for the correlation function on the torus $\mathcal{F}$ defined by (2.6). The basic tool will be the application of operator KZ equation (2.3), which implies:

\begin{equation}
(k + 1) z_i \frac{\partial}{\partial z_i} \mathcal{F}_{u_1, \ldots, u_n}(z_1, \ldots, z_n) = \sum_{\alpha \in \Delta^+} z_i \text{Tr}(\Phi^1 \cdots (J_e^+ \Phi^j_{e_a u_i}(z_i) J_e^- \Phi^n_{e_a u_i}(z_i)) q^{-\partial} e^h) + \sum_{\alpha \in \Delta^+} z_i \text{Tr}(\Phi^1 \cdots (J^+_{e_a \alpha}(z_i) J^+_{e_a \alpha}(z_i)) q^{-\partial} e^h) + \sum_{l=1}^r z_i \text{Tr}(\Phi^1 \cdots (J^+_x(z_i) J^+_x(z_i)) q^{-\partial} e^h),
\end{equation}

Let us consider the first summand in (3.1). Using commutation relations of currents with intertwiners (2.2), we can move $J^+_{e_a}(z_i)$ to the utmost left, and $J^-_{e_a}(z_i)$ to the utmost right, which gives us the following expression for the first term in (3.1):

\begin{equation}
-\sum_{\alpha \in \Delta^+} \left( \sum_{j \neq i} x^\alpha(z_i/z_j) \text{Tr}(\Phi^1 \cdots \Phi^{j}_{e_a u_j}(z_i) \cdots \Phi^n_{e_a u_i}(z_i)) q^{-\partial} e^h \right)
\end{equation}

\begin{equation*}
- z_i \text{Tr}(J^+_{e_a}(z_i) \Phi^j_{e_a u_i}(z_i)) q^{-\partial} e^h
\end{equation*}

\begin{equation*}
+ z_i \text{Tr}(\Phi^1 \cdots \Phi^n_{e_a u_i}(z_i)) q^{-\partial} e^h).
\end{equation*}
Now, using the cyclic property of trace and the relation $q^{-\partial} e^h J^\pm_{e_\alpha}(z) = q e^{h(\alpha)} J^\pm_{e_\alpha}(qz) q^{-\partial} e^h$, we can rewrite (3.2) as follows:

\[- \sum_{\alpha \in \Delta^+} \left( \sum_{j \neq i} x^\alpha(z_i/z_j) \pi_i \otimes \pi_j(f_\alpha \otimes e_\alpha) \Phi \right. \\
\left. - e^{h(\alpha)} q z_i \text{Tr}(\Phi^1 \ldots \Phi^i_{x_\alpha} u_i(z_i) \ldots \Phi^n J^+_{e_\alpha}(qz_i) q^{-\partial} e^h) \\
+ e^{-h(\alpha)} q^{-1} z_i \text{Tr}(J^-_{e_\alpha}(q^{-1} z_i) \Phi^1 \ldots \Phi^i_{x_\alpha} u_i(z_i) \ldots \Phi^n q^{-\partial} e^h) \right)\]

Repeating this procedure and taking into account that $\lim_{m \to +\infty} q^{\pm m} z J^\pm_{e_\alpha}(q^{\pm m} z) = 0$, we finally get the following expression for the first summand in (3.1):

\[(3.3) - \sum_{\alpha \in \Delta^+} \sum_{m \in \mathbb{Z}, j \neq i} x^\alpha(q^m z_i/z_j) e^{mh(\alpha)} \pi_i \otimes \pi_j(f_\alpha \otimes e_\alpha) \Phi,\]

where $\sum'$ means that the sum is taken over all $m \in \mathbb{Z}, j = 1 \ldots n$ except $m = 0, j = i$. (We use the convention $\pi_i \otimes \pi_i(a \otimes b) = \pi_i(ba)$.) It is easy to see that the series $f_{\alpha, h} = \sum_{m \in \mathbb{Z}} x^\alpha(q^m z_i/z_j) e^{mh(\alpha)}$ converges; it can be expressed via theta-functions, though we won’t need it.

Similarly, one shows that the second term in (3.1) equals

\[(3.4) - \sum_{\alpha \in \Delta^+} \sum_{m \in \mathbb{Z}, j \neq i} x^{-\alpha}(q^m z_i/z_j) e^{-mh(\alpha)} \pi_i \otimes \pi_j(e_\alpha \otimes f_\alpha) \mathcal{F}.\]

As for the third summand, one must be more careful, since $J^\pm_{e_\alpha}(z)$ contains zero modes, and therefore we have: $\lim_{m \to +\infty} q^{\pm m} z J^\pm_{e_\alpha}(q^{\pm m} z) = \pm \frac{1}{2} h \otimes \tau^0$. This gives us the following expression for the third term in (3.1):

\[- \sum_{l=1}^r \text{V.P.} \sum_{m \in \mathbb{Z}, j \neq i} x^0(q^m z_i/z_j) \pi_i \otimes \pi_j(x_l \otimes x_l) \mathcal{F} \\
+ \sum_{l=1}^r \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^h x_l),\]

where we define the principal value of a series as $\text{V.P.} \sum_{m \in \mathbb{Z}} a_m = \lim_{M \to +\infty} \sum_{m = -M}^M a_m$.

Finally, we obtain the following theorem.

**Theorem 3.1.** The correlation function (2.6) satisfies the following system of differential equations:
\begin{align}
\left( k + 1 \right) z_i \frac{\partial}{\partial z_i} F &= - \sum_{\alpha \in \Delta^+} \sum_{j \neq i} f_{\alpha,h}(z_i/z_j) \pi_i \otimes \pi_j (f_{\alpha} \otimes e_{\alpha}) \Phi \\
&\quad - \sum_{\alpha \in \Delta^+} \sum_{j \neq i} f_{-\alpha,h}(z_i/z_j) \pi_i \otimes \pi_j (e_{\alpha} \otimes f_{\alpha}) \Phi \\
&\quad - \sum_{l=1}^{r} \sum_{j \neq i} f_0(z_i/z_j) \pi_i \otimes \pi_j (x_l \otimes x_l) \Phi \\
&\quad - \pi_i (\Omega_h) F + \sum_{l=1}^{r} \pi_i (x_l) \frac{\partial}{\partial x_l} F,
\end{align}

where the functions \( f_{\alpha,h} \) are defined as follows:

\begin{align}
\begin{split}
f_{\alpha,h}(t) &= \sum_{m \in \mathbb{Z}} x^\alpha (q^m t) e^{m h(\alpha)}, \quad \alpha \in \Delta^+ \\
\end{split} \\
\begin{split}
f_{-\alpha,h}(t) &= \sum_{m \in \mathbb{Z}} x^{-\alpha} (q^m t) e^{-m h(\alpha)}, \quad \alpha \in \Delta^+ \\
\end{split}
\end{align}

\begin{align}
f_0(t) = V.P. \sum_{m \in \mathbb{Z}} x^0 (q^m t),
\end{align}

and the operator \( \Omega_h \) equals

\begin{align}
\Omega_h = V.P. \sum_{m \neq 0} \left( \sum_{\alpha \in \Delta^+} (q^{-|\alpha|} e^{h(\alpha)})^m \frac{q^{mg}}{1 - q^{mg}} e_{\alpha} f_{\alpha} \\
+ \sum_{\alpha \in \Delta^+} (q^{|\alpha|} e^{-h(\alpha)})^m \frac{1}{1 - q^{mg}} f_{\alpha} e_{\alpha} \\
+ \sum_{l=1}^{r} \frac{1 + q^{mg}}{2(1 - q^{mg}) x_l^2} \right)
\end{align}

The functions \( f_{\alpha,h} \) can be easily written in terms of the theta-functions with characteristics. We do not give these expressions here since we are not going to use it in this paper.

Let us look at equations (3.6) more closely. This system looks very much like some (twisted) elliptic version of Knizhnik-Zamolodchikov (KZ) system. The only term which breaks this analogy is \( \sum \pi_i (x_l) \frac{\partial}{\partial x_l} F \). Due to this term, this system of equations is not closed, since we have no equations which would allow us to find the derivatives \( \frac{\partial}{\partial x_l} F \).

Let us move further. Note that our correlation function \( F \) depends not only on \( z_i \) but also on the modular parameter \( q \). Let us derive the equations for \( \frac{\partial}{\partial q} F \).

Obviously,

\begin{align}
q \frac{\partial}{\partial q} F &= \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^h(-\partial))
\end{align}
Now let us substitute into this equation the expression for $\partial$ given by the Sugawara construction (1.3). This yields:

\[(3.10) \quad (k + 1)q \frac{\partial}{\partial q} F = \sum_{m \in \mathbb{Z}} \left( \sum_{\alpha \in \Delta^+} a_{m,\alpha} + \frac{1}{2} \sum_{l=1}^{r} b_{m,l} \right), \]

where
\[
a_{m,\alpha} = \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^h : e_{\alpha}[|\alpha| + mg]|f_{\alpha}[-|\alpha| - mg] :)
\]
\[b_{m,l} = \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^h : x_l[|mg|]x_l[-mg] :)
\]

Let us calculate $a_{m,\alpha}$ and $b_{m,l}$. Consider first case $m \geq 0$. Then, using the defining commutation relation for intertwiners (2.1), we get

\[
a_{m,\alpha} = \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^h f_{\alpha}[|\alpha| - mg|e_{\alpha}[|\alpha| + mg] + \sum_j z_j^{|\alpha| + mg} \text{Tr}(\Phi^1 \ldots \Phi^j_{e_{\alpha}u_j}(z_j) \ldots \Phi^n q^{-\partial} e^h f_{\alpha}[|\alpha| - mg])
\]

\[+ \text{Tr}(\Phi^1 \ldots \Phi^n e_{\alpha}[|\alpha| + mg] |\alpha| - mg) + q^{|\alpha| + mg} e^{-h(\alpha)} a_{\alpha,\alpha} + \sum_j z_j^{|\alpha| + mg} \text{Tr}(\Phi^1 \ldots \Phi^j_{e_{\alpha}u_j}(z_j) \ldots \Phi^n q^{-\partial} e^h f_{\alpha}[|\alpha| - mg])
\]

\[+ q^{|\alpha| + mg} e^{-h(\alpha)} \left( k(|\alpha|) + \frac{\partial}{\partial h_{\alpha}} \right) F \]

Therefore, denoting $q^{|\alpha| + mg} e^{-h(\alpha)} = c$, we see that for $m \geq 0$

\[
a_{m,\alpha} = \frac{1}{1 - c} \left( \sum_j z_j^{|\alpha| + mg} \text{Tr}(\Phi^1 \ldots \Phi^j_{e_{\alpha}u_j}(z_j) \ldots \Phi^n q^{-\partial} e^h f_{\alpha}[|\alpha| - mg])
\]

\[+ c \left( k(|\alpha|) + \frac{\partial}{\partial h_{\alpha}} \right) F. \]

Similar arguments show that

\[
\text{Tr}(\Phi^1 \ldots \Phi^j_{e_{\alpha}u_j}(z_j) \ldots \Phi^n q^{-\partial} e^h f_{\alpha}[|\alpha| - mg]) = \frac{1}{1 - c - 1} \sum_i z_i^{-|\alpha| - mg} \pi_i \otimes \pi_j (f_{\alpha} \otimes e_{\alpha}) F.
\]

This gives us the final answer: for $m \geq 0$,

\[(3.11) \quad a_{m,\alpha} = \left( -\frac{c}{(1 - c)^2} \sum_{i,j} z_j/z_i \pi_i \otimes \pi_j (f_{\alpha} \otimes e_{\alpha}) + \frac{c}{1 - c} \left( k(|\alpha|) + \frac{\partial}{\partial h_{\alpha}} \right) \right) F, \]
where \( c = q^{|\alpha|+mg} e^{-h(\alpha)} \).

Similar considerations for \( m < 0 \) show that

\[
(3.12) \quad a_{m,\alpha} = \left( -\frac{c}{(1-c)^2} \sum_{i,j} (z_j/z_i)^{|\alpha|+mg} \pi_i \otimes \pi_j (f_\alpha \otimes e_\alpha) + \frac{1}{1-c} \left( k(|\alpha| + mg) + \frac{\partial}{\partial h_\alpha} \right) \right) F,
\]

where \( c \) is given by the expression above.

Now we change the notation: \( c = q^{mg} \). Then the expressions for \( b_{m,l} \) are obtained in a similar way:

for \( m > 0 \):

\[
(3.13) \quad b_{m,l} = \left( -\frac{c}{(1-c)^2} \sum_{i,j} (z_j/z_i)^{mg} \pi_i \otimes \pi_j (x_l \otimes x_l) + \frac{c}{1-c} kmg \right) F.
\]

for \( m < 0 \):

\[
(3.14) \quad b_{m,l} = \left( -\frac{c}{(1-c)^2} \sum_{i,j} (z_j/z_i)^{mg} \pi_i \otimes \pi_j (x_l \otimes x_l) + \frac{1}{1-c} kmg \right) F,
\]

for \( m = 0 \):

\[
(3.15) \quad b_{0,l} = \text{Tr}(\Phi^1 \ldots \Phi^n q^{-\partial} e^{2h_2 x_l^2}) = \frac{\partial^2}{\partial x_l^2} F.
\]

This completes the calculations. We can rewrite the result using the following identity:

\[
\sum_{m \in \mathbb{Z}} \frac{(qz)^{|\alpha|+mg} e^{-h(\alpha)}}{(1-q^{\alpha|+mg} e^{-h(\alpha)})^2} = e^{2\pi i \zeta |\alpha|/g} \phi \left( \frac{h(\alpha)}{2\pi i} - \frac{|\alpha|}{g} \tau, \zeta \right),
\]

where \( z = e^{2\pi i \zeta/g}, q = e^{2\pi i r/g} \) and the function \( \phi \) is given by

\[
\phi(x, \zeta) = \sum_{m \in \mathbb{Z}} e^{2\pi i m \tau} e^{-2\pi i x} e^{2\pi i m \zeta} \left( 1 - e^{2\pi i m \tau} e^{-2\pi i x} \right)^2.
\]

It is easy to check that for \( 0 < \text{Im} \zeta < \text{Im} \tau \) we have the following expression for \( \phi(x, \zeta) \):

\[
\phi(x, \zeta) = -\frac{1}{2\pi i} \frac{\partial}{\partial x} \sum_{m \in \mathbb{Z}} \frac{e^{2\pi i m \zeta}}{1 - e^{2\pi i m \tau} e^{-2\pi i x}} = -\frac{1}{4\pi} \frac{\partial}{\partial x} \left( \frac{\theta_1(\pi(x - \zeta)) \theta'_1(0)}{\theta_1(\pi x) \theta_1(\pi \zeta)} \right).
\]

This formula defines analytic continuation of \( \phi(x, \zeta) \) for all \( \zeta, x \neq m\tau + n \). Recall that by definition

\[
\theta_1(\zeta | \tau) = 2e^{\pi i r/4} \sin \zeta \prod_{n \geq 1} (1 - e^{2\pi i n \tau} e^{2i\zeta})(1 - e^{2\pi i n \tau} e^{-2i\zeta})(1 - e^{2\pi i n \tau})
\]

\[
= 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi i (n + \frac{1}{2})^2} \sin (2n + 1) \zeta.
\]
Similarly, one can show that
\[
\sum_{m \neq 0} \frac{(qz)^{mg}}{(1 - q^{mg})^2} = \lim_{x \to 0} (\phi(x, \zeta) + \frac{1}{4\sin^2\pi x}).
\]

Expanding the right-hand side of this equation in Laurent series around \(x = 0\), we get that
\[
\sum_{m \neq 0} \frac{(qz)^{mg}}{(1 - q^{mg})^2} = \frac{\theta''_1(\pi \zeta)}{8\theta_1(\pi \zeta)} - \frac{\theta'''_1(0)}{24\theta_1'(0)} + \frac{1}{12}.
\]

Finally, we can formulate the result in the following form:

**Theorem 3.2.** The correlation function \(F\) satisfies the following differential equation:

\[
(k + 1)q \frac{\partial}{\partial q} F = \sum_{\alpha \in \Delta^+} \left( -\sum_{i,j} e^{2\pi i|\alpha|} (\zeta_j - \zeta_i) \phi \left( \frac{h(\alpha)}{2\pi i} - \frac{|\alpha|}{g} \tau, \zeta_j - \zeta_i \right) \pi_i \otimes \pi_j (f_{\alpha} \otimes e_{\alpha}) \right.
\]
\[
+ u_{\alpha,h} \frac{\partial}{\partial h_{\alpha}} + v_{\alpha,h} k \bigg) F
\]
\[
+ \frac{1}{2} \sum_{l=1}^{r} \left( -\sum_{i,j} \phi_0(\zeta_j - \zeta_i) \pi_i \otimes \pi_j (x_l \otimes x_l) + v_0 k \right) F
\]
\[
+ \frac{1}{2} \Delta_{\theta} F,
\]

where \(z_i = e^{2\pi i \zeta_i/g}, q = e^{2\pi ir/g}\) and
\[
\phi(x, \zeta) = \sum_{m \in \mathbb{Z}} \frac{e^{2\pi i m \tau} e^{-2\pi i x} e^{2\pi i m \zeta}}{(1 - e^{2\pi i m r} e^{-2\pi i x})^2}
\]
\[
= -\frac{1}{4\pi} \frac{\partial}{\partial x} \left( \frac{\theta_1(\pi (x - \zeta)) \theta'_1(0)}{\theta_1(\pi x) \theta_1(\pi \zeta)} \right)
\]
\[
\phi_0(\zeta) = \sum_{m \neq 0} \frac{(qz)^{mg}}{(1 - q^{mg})^2} = \frac{\theta''_1(\pi \zeta)}{8\theta_1(\pi \zeta)} - \frac{\theta'''_1(0)}{24\theta_1'(0)} + \frac{1}{12}
\]

\[
u_{\alpha,h} = \sum_{m \geq 0} \frac{q^{m+|\alpha|+mg} e^{-h(\alpha)}}{1 - q^{m+|\alpha|+mg} e^{-h(\alpha)}} + \sum_{m < 0} \frac{1}{1 - q^{m+|\alpha|+mg} e^{-h(\alpha)}}
\]
\[
u_{0} = \sum_{m > 0} \frac{q^{mg} mg}{1 - q^{mg} mg} + \sum_{m < 0} \frac{1}{1 - q^{mg} mg}.
\]
Remark 1. We could as well write \( u_{\alpha,h}, v_{\alpha,h} \) in terms of theta-functions, but it is not necessary: in the next section we show that the terms containing \( u_{\alpha,h}, v_{\alpha,h} \) can be cancelled by a suitable renormalization of \( \mathcal{F} \).

Remark 2. Note that the function \( \frac{\theta_1(\pi(x-\zeta))\theta'_1(0)}{\theta_1(\pi\zeta)\theta_1(0)} \) plays a very special role in this whole story. The same function also appears later in the integral formulas for \( \mathcal{F} \) in the case \( g = \mathfrak{sl}_2 \) (see section 5 below); it also appears in the expressions for the functions \( f_{\alpha,h}(z) \) (cf. Theorem 3.1) for \( g = \mathfrak{sl}_2 \).

Let us summarize the results obtained so far. We consider \( \mathcal{F} \) as a function of \( z_1 \ldots z_n, q \) with values in \( V = V_1 \otimes \ldots \otimes V_n \otimes C^\infty(\mathfrak{h}_R) \). Then theorems 3.1 and 3.2 give us differential equations for \( \mathcal{F} \) of the following form:

\[
\frac{\partial}{\partial z_i} \mathcal{F} = A_i(z_1 \ldots z_n, q) \mathcal{F}
\]

\[
q \frac{\partial}{\partial q} \mathcal{F} = B(z_1 \ldots z_n, q) \mathcal{F},
\]

where \( A_i, B \) are some operators in \( V \) which are defined for \( |z_1| > \ldots > |z_n| > |qz_1| \).

Note also that we can easily find asymptotics of \( \mathcal{F} \) as \( q \to 0 \). Indeed, it is easy to show that as \( q \to 0 \),

\[
\mathcal{F} \sim < v_{\lambda,k}, \Phi^1 \ldots \Phi^n q^{-\partial} e^h v_{\lambda,k} > \]

\[
= e^{h(\lambda)} q^{< \lambda,\lambda >} \Psi
\]

where \( \Psi = < v_{\lambda,k}, \Phi^1(z_1) \ldots \Phi^n(z_n)v_{\lambda,k} > \) is the “correlation function on the sphere” with values in \( V_1 \otimes \ldots \otimes V_n \). For the case when \( V_i \) are highest or lowest weight representations, which is our main example, this function is very well studied; in particular, it is known that it is well defined for \( |z_1| > \ldots > |z_n| \) and satisfies trigonometric KZ equation in this region (see [FR] and references therein); there are integral formulas for this function (see [SV],[Ch]). So we consider \( \Psi \) as a well-known function.

Now standard arguments from the theory of ordinary differential equations show that the equation (3.18) together with boundary condition (3.19) uniquely determine \( \mathcal{F} \); in fact, we only need the equation for \( q \frac{\partial}{\partial q} \mathcal{F} \).

Remark. Note that we never used the fact that \( M_{\lambda,k} \) are Verma modules. In fact, all the statements of this section are still valid if we consider any modules from the category \( \mathcal{O} \) instead of Verma modules – provided that the intertwiners exist.

4. Connection with the heat equation

In this section we will show, following the ideas of Bernard [Ber] that the equation (3.16) is closely related to the heat equation on \( \mathfrak{h}_R \). This is of special interest for \( n = 0 \) and \( n = 1 \). In this cases \( \mathcal{F} \) does not depend on \( z_i \), so (3.16) is the only non-trivial equation for \( \mathcal{F} \).

Let us denote \( \mathcal{F}_0 = \text{Tr}|_{M_{0,k}}(q^{-\partial} e^h) \). It can be easily calculated explicitly: since the character of \( M_{0,k} \) coincides with the character of \( U \hat{\mathfrak{g}}_{-} \), we see that
\begin{equation} \mathcal{F}_0 = \frac{1}{\prod_{m > 0} (1 - q^m g^r) \prod_{\alpha, m} (1 - e^{-h(\alpha)} q^{|\alpha| + mg})}, \end{equation}

where the second product is taken over all \( \alpha \in \Delta, m \in \mathbb{Z} \) such that \(|\alpha| + mg > 0\). This can be rewritten as follows:

\begin{equation} \frac{1}{\mathcal{F}_0} = \prod_{m > 0} (1 - q^m g^r) \prod_{\alpha \in \Delta^+} \left( \prod_{m \geq 0} (1 - e^{-h(\alpha)} q^{|\alpha| + mg}) \prod_{m < 0} (1 - e^{h(\alpha)} q^{-|\alpha| - mg}) \right). \end{equation}

Explicit calculation shows that

\begin{equation} q \frac{\partial}{\partial q} \frac{1}{\mathcal{F}_0} = - \sum_{\alpha \in \Delta^+} \left( \sum_{m \geq 0} \frac{q^{|\alpha| + mg} e^{-h(\alpha)}}{1 - q^{|\alpha| + mg} e^{-h(\alpha)}} (|\alpha| + mg) + \sum_{m < 0} \frac{1}{1 - q^{|\alpha| + mg} e^{-h(\alpha)}} (|\alpha| + mg) \right) \frac{1}{\mathcal{F}_0} \end{equation}

\begin{equation} - r \sum_{m > 0} \frac{q^m g}{1 - q^m g} \frac{1}{\mathcal{F}_0} \end{equation}

\begin{equation} = \left( - \sum_{\alpha \in \Delta^+} v_{\alpha, h} - \frac{r}{2} v_0 \right) \frac{1}{\mathcal{F}_0}, \end{equation}

where \( v_{\alpha, h}, v_0 \) are defined in Theorem 3.2.

Similarly, if \( x \in \mathfrak{h} \),

\begin{equation} \frac{\partial}{\partial x} \frac{1}{\mathcal{F}_0} = \left( \sum_{\alpha \in \Delta^+} x(\alpha) u_{\alpha, h} \right) \frac{1}{\mathcal{F}_0} \end{equation}

where \( u_{\alpha, h} \) is defined in Theorem 3.2.

Now let us come back to Theorem 3.2, which we rewrite in the following form

\begin{equation} q \frac{\partial}{\partial q} \mathcal{F} = \frac{1}{k + 1} A \mathcal{F} + \sum_{\alpha \in \Delta^+} \left( \frac{1}{k + 1} u_{\alpha, h} \frac{\partial}{\partial h_{\alpha}} + \frac{k}{k + 1} v_{\alpha, h} \right) \mathcal{F} \end{equation}

\begin{equation} + \frac{k}{k + 1} \frac{r}{2} v_0 \mathcal{F} \end{equation}

\begin{equation} + \frac{1}{2(k + 1)} \Delta_{\mathfrak{h}} \mathcal{F}, \end{equation}

where we for brevity denoted

\begin{equation} A = - \sum_{\alpha \in \Delta^+} \sum_{i, j} e^{2\pi i |\alpha| / g} \phi \left( \frac{h(\alpha)}{2\pi i} - \frac{|\alpha|}{g} \tau, \zeta_j - \zeta_i \right) \pi_i \otimes \pi_j (f_{\alpha} \otimes e_{\alpha}) \end{equation}

\begin{equation} - \frac{1}{2} \sum_{l=1}^r \sum_{i, j} \phi_0 (\zeta_j - \zeta_i) \pi_i \otimes \pi_j (x_l \otimes x_l), \end{equation}

where \( \phi(x, \zeta), \phi_0 (\zeta) \) are defined in Theorem 3.2. Comparing (4.5) with (4.4) and (4.3), we get the following statement.
Proposition 4.1.

(4.7) \[ q \frac{\partial}{\partial q} \left( \frac{F}{F_0} \right) = \frac{1}{k+1} \left( (AF) \frac{1}{F_0} + (q \frac{\partial}{\partial q} F_0) F + \sum_{l=1}^{r} \frac{\partial F}{\partial x_l} \frac{\partial (1/F_0)}{\partial x_l} + \frac{1}{2} (\Delta_b F) \frac{1}{F_0} \right) \]

Proof. The only not immediately obvious step in the proof is dealing with the terms containing \( u_{\alpha,h} \). As for them,

\[
\left( \sum_{\alpha \in \Delta^+} u_{\alpha,h} \frac{\partial F}{\partial H_{\alpha}} \right) \frac{1}{F_0} = \left( \sum_{\alpha \in \Delta^+} u_{\alpha,h} < h_{\alpha}, \text{grad } F > \right) \frac{1}{F_0}.
\]

Comparing it with (4.4), we see that this equals to

\[ < \text{grad } F, \text{grad } \frac{1}{F_0} > = \sum_l \frac{\partial F}{\partial x_l} \frac{\partial (1/F_0)}{\partial x_l}. \]

\[ \square \]

Corollary.

(4.8) \[ \left( q \frac{\partial}{\partial q} - \frac{1}{2} \Delta_b \right) \frac{1}{F_0} = 0. \]

Proof. Rewrite (4.7) in the form

\[
\left( q \frac{\partial}{\partial q} - \frac{1}{2} \Delta_b \right) \left( \frac{F}{F_0} \right) = \frac{1}{k+1} (AF) \frac{1}{F_0} + \frac{1}{k+1} F \left( q \frac{\partial}{\partial q} - \frac{1}{2} \Delta_b \right) \frac{1}{F_0}
\]

and substitute there \( F = F_0 = \text{Tr}|_{\mathcal{M}_{0,k}} (q^{-\partial} e^h) \).

So, finally we have the following theorem:

Theorem 4.1.

(4.9) \[
\left( q \frac{\partial}{\partial q} - \frac{1}{2} \Delta_b \right) \frac{F}{F_0} = \]

\[
- \frac{1}{k+1} \sum_{i,j} \left( \sum_{\alpha \in \Delta^+} e^{2\pi i |\alpha|/g} \phi \left( \frac{h(\alpha)}{2\pi i} - \frac{|\alpha|}{g} \tau, \zeta_j - \zeta_i \right) \pi_i \otimes \pi_j (f_\alpha \otimes e_\alpha) \right) \]

\[
+ \frac{1}{2} \sum_{l=1}^{r} \phi_0 (\zeta_j - \zeta_i) \pi_i \otimes \pi_j (x_l \otimes x_l) \left( \frac{F}{F_0} \right),
\]

where \( \phi(x, \zeta), \phi_0(\zeta) \) are defined in Theorem 3.2.
Corollary. Let $n = 0$, i.e. $\mathcal{F} = \text{Tr}(q^{-\partial}e^h)$. Then $\mathcal{F}/\mathcal{F}_0$ satisfies the heat equation:

\begin{equation}
\left(q \frac{\partial}{\partial q} - \frac{1}{2(k+1)} \Delta_h\right) \frac{\mathcal{F}}{\mathcal{F}_0} = 0.
\end{equation}

This equation (for untwisted algebra) was first derived by Bernard ([Ber]).

Let us consider more interesting example $n = 1$, $\mathcal{F} = \text{Tr}|_{\mathcal{M}_\lambda,k}(\Phi_1(z)q^{-\partial}e^h)$. In this case $\mathcal{F}$ takes values in some module $V$ over $\mathfrak{g}$. From the weight considerations it is clear that in fact $\mathcal{F}$ takes values in the zero-weight subspace $V(0)$. Therefore, the equation (4.9) reduces to

\begin{equation}
\left(q \frac{\partial}{\partial q} - \frac{1}{2(k+1)} \Delta_h\right) \frac{\mathcal{F}}{\mathcal{F}_0} = -\frac{1}{k+1} \sum_{\alpha \in \Delta^+} \phi \left(\frac{h(\alpha)}{2\pi i} - \frac{|\alpha|}{g}, 0\right) f_\alpha e_\alpha
\end{equation}

Note that $\phi(x,0)$ is an elliptic function of a very simple form. Indeed, from the expression

$\phi(x,0) = \sum_{m \in \mathbb{Z}} \frac{e^{2\pi im\tau} e^{-2\pi ix}}{(1 - e^{2\pi im\tau} e^{-2\pi ix})^2}$

it is obvious that $\phi(x,0)$ is an even elliptic function with periods $1, \tau$ and poles of order 2 at the points $m\tau + n, m, n \in \mathbb{Z}$. Finding the leading coefficient at $u = 0$, we see that

\begin{equation}
\phi(x,0) = -\frac{1}{(2\pi)^2} (\varphi(x) + c),
\end{equation}

for some constant $c = c(q)$ (not to be confused with $c$ of section 3); we won’t need an explicit expression for it. We will, however, use two properties of $c(q)$: as $q \to 0$, $c(q) \to 0$ and if $q \in \mathbb{R}$ then $c(q) \in \mathbb{R}$.

This implies

Proposition 4.2. Let $n = 1$. Then the correlation function $\mathcal{F}$ satisfies the following equation:

\begin{equation}
\left(q \frac{\partial}{\partial q} - \frac{1}{2(k+1)} \Delta_h\right) \frac{\mathcal{F}}{\mathcal{F}_0} = \frac{1}{(2\pi)^2(k+1)} \sum_{\alpha \in \Delta^+} \left(\varphi \left(\frac{\tau|\alpha|}{g} - \frac{h(\alpha)}{2\pi i}\right) + c\right) f_\alpha e_\alpha \frac{\mathcal{F}}{\mathcal{F}_0}.
\end{equation}

5. Example: $\mathfrak{g} = \mathfrak{sl}_2, n = 1$.

Let us consider the simplest possible case $\mathfrak{g} = \mathfrak{sl}_2, n = 1$. Let $e, f, h$ be the standard basis of $\mathfrak{sl}_2$. We can identify $\mathbb{R} \simeq h_{\mathbb{R}}$ by $x \mapsto -\pi ixh$. Under this identification, the Killing form becomes $-2\pi^2x^2$, and the Laplace operator: $\Delta_h = -\frac{d^2}{2\pi^2 d x^2}$.

Let $V_\mu, \mu \in \mathbb{C}$ be an irreducible highest-weight module over $\mathfrak{sl}_2$ with the highest weight $\frac{\mu}{2}\alpha$, where $\alpha$ is the positive root of $\mathfrak{sl}_2$. In other words, the action of $h_{\mathbb{R}}$ on the highest weight vector of $V_\mu$ is given by $hv_\mu = \mu \nu_\mu$.

Note that the zero-weight subspace $V_\mu(0)$ is not empty iff $\mu = 2m, \mu \in \mathbb{Z}_+$. For this reason, from now on we assume that $V_\mu = V_{2m}$ is a finite-dimensional module.
of dimension $2m+1$. Then $V_{2m}(0)$ is one-dimensional and $f_{\alpha e_\alpha}|_{V_{2m}(0)} = m(m+1)$. Also, it is well-known that in this case the intertwining operator $\Phi: M_{\lambda,k} \to M_{\lambda,k} \otimes V_\mu$ exists and is unique up to a constant factor for generic values of $\lambda, k$.

So we can rewrite (4.12) as follows:

$$\left(-4\pi i(k+1)\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right) F = m(m+1) \left( \varphi \left( \frac{\tau}{2} + x \right) + c \right) F,$$

where $F = \mathcal{F}/\mathcal{F}_0$ is a $\mathbb{C}$-valued function of $x \in \mathbb{R}, \tau \in \mathcal{H}$, $\mathcal{H}$ being the upper half-plane.

Note that this equation does not depend on the highest weight $\lambda$ of the Verma module we consider: the only dependence is the boundary conditions (3.19). Note, however, that $F$ has the following periodicity property: $F(x+1) = e^{-\pi i \lambda} F(x)$; in particular, if $\lambda \in 2\mathbb{Z}$ then $F(x)$ is periodic: $F(x+1) = F(x)$.

Note that in the limit $k \to -1$ the equation (5.1) becomes the Lamé equation (see [WW]):

$$\frac{d^2}{dx^2} F = m(m+1) \left( \varphi \left( \frac{\tau}{2} + x \right) + c \right) F.$$

We discuss this relation in the next section.

To find an explicit expression for $F$, we use formula (2.5), which allows us to rewrite $F$ in terms of untwisted intertwiners $\hat{\Phi}(\zeta)$ as follows:

$$F(x, q) = \frac{\text{Tr}|_{\mathcal{M}_{\Lambda, K}} \left( \hat{\Phi}[0] q^{2d - \frac{h}{2} e^{-\pi i x h}} \right)}{\text{Tr}|_{\mathcal{M}_{\mu, K}} \left( q^{2d - \frac{h}{2} e^{-\pi i x h}} \right)},$$

where $K = 2k, \Lambda = \lambda + k$.

This enables us to use well-known integral formulas for $\hat{\Phi}(\zeta)$. For this purpose we recall the Wakimoto realization of $\widehat{\mathfrak{sl}}_2$ ([Wa]), following [BF]. Let us introduce the algebra $A$, generated by $\alpha_n, \beta_n, \gamma_n, n \in \mathbb{Z}$ with the relations:

$$[\alpha_n, \alpha_m] = 2n \delta_{n+m,0}$$
$$[\beta_n, \gamma_m] = \delta_{n+m,0}$$

and all the other commutators vanish.

Next, we can define the module $H_{\Lambda, \Lambda} \in \mathbb{C}$ over $A$ as a module generated by a vacuum vector $v_{\Lambda}$ with the properties:

$$\alpha_n v_{\Lambda} = 0, \quad n > 0$$
$$\beta_n v_{\Lambda} = \gamma_{n+1} v_{\Lambda} = 0, \quad n \geq 0$$
$$\alpha_0 v_{\Lambda} = \frac{\Lambda}{\sqrt{K}} v_{\Lambda},$$

where $\kappa = K + 2 \neq 0$.

Define the normal ordering in $A$ by:
\[\alpha_n \alpha_m := \begin{cases} \alpha_n \alpha_m, & m > 0 \\ \alpha_m \alpha_n, & m \leq 0 \end{cases}\]
\[\beta_n \gamma_m := \begin{cases} \beta_n \gamma_m, & m > 0 \\ \gamma_m \beta_n, & m \leq 0 \end{cases}\]

Finally, introduce the free bosonic fields as follows:

\[\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}\]
\[\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}\]
\[\gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n}\]

Then we have the following proposition, due to Wakimoto [Wa].

**Proposition.** cf. [BF] The following formulas give an action of \(\widehat{sl}_2\) in \(H_\Lambda\). Moreover, \(H_\Lambda\) endowed with this action is isomorphic to Verma module \(M_{\Lambda,K}\) for generic \(K, \Lambda\).

\[J_\epsilon(z) = \beta(z)\]
\[J_h(z) = -2 : \beta(z) \gamma(z) : + \sqrt{\kappa} \alpha(z)\]
\[J_f(z) = - : \gamma^2(z) \beta(z) : + \sqrt{\kappa} \alpha(z) \gamma(z) + K \gamma'(z)\]

We also need explicit formulas for the intertwiners in Wakimoto realization. Let us introduce the operator \(e^{c_{\alpha}} : H_\Lambda \to H_{\Lambda+2c_+\sqrt{\kappa}}\) which maps \(v_\Lambda\) to \(v_{\Lambda+2c_+\sqrt{\kappa}}\) and commutes with all the generators of \(A\) except \(\alpha_0\). Then we can introduce the vertex operators \(X(\mu), \mu \in \mathfrak{h}^*\) by:

\[X(c_{\alpha}, z) =: \exp \left( c \int \alpha(z) \, dz \right) =: \exp \left( c \sum_{n<0} \frac{\alpha_n}{-n} z^{-n} \right) \exp \left( c \sum_{n>0} \frac{\alpha_n}{-n} z^{-n} \right) e^{c_{\alpha}} z^{c_{\alpha}}\]

and the screening operators \(U(t)\):

\[U(t) = \beta(t) X(-\frac{\alpha}{\sqrt{\kappa}}, t)\]

Then the intertwining operator \(\hat{\Phi}(z) : M_{\Lambda,K} \to M_{\mu,K} \otimes V_\mu\) can be written as follows (we assume that \(\mu \in \mathbb{Z}\), so \(V_\mu\) is finite-dimensional, and \(\lambda + \mu - \nu = 2m\))

\[\hat{\Phi}(z)v = \sum_{n \geq 0} \left( \int_{\Delta} X\left( \frac{\mu}{2\sqrt{\kappa}} \alpha \right)(-\gamma(z))^n U(t_1) \cdots U(t_m) dt_1 \cdots dt_m \right) v \otimes \frac{e^n}{n!} v_{-\mu},\]
where \(v_{-\mu}\) is the lowest-weight vector in \(V_\mu\) and the cycle of integration \(\Delta \subset \mathbb{C}^m \setminus \{t_i = 0, z, t_i = t_j\}\) is chosen so that one can define single-valued branches of \(\log t_i, \log (t_i - t_j), \log (t_i - z)\) on \(\Delta\). General results of the homology theory of local systems (see [SV]) show that such cycle is essentially unique.

In particular, this shows that the zero-mode component \(\hat{\Phi}[0] : \mathcal{M}_{\Lambda,K} \rightarrow \mathcal{M}_{\Lambda,K} \otimes V_\mu(0)\) is given by:

\[
(5.9) \quad \hat{\Phi}[0] = \int_{\Delta} X\left(\frac{m}{\sqrt{\kappa}} \alpha, z\right)(-\gamma(z))^m U(t_1) \ldots U(t_m) dt_1 \ldots dt_m
\]

To find \(F\), we must calculate \(\text{Tr}|_{\mathcal{M}_{\Lambda,K}} \left( \hat{\Phi}[0] q^{-2d} e^{-\pi i x h} \right)\). Traces of this type can be calculated in the same way as we did in section 3; we borrow the answer from [BF, formula 3.14], which can be rewritten as follows:

\[
(5.10) \quad \text{Tr}|_{H_\Lambda} \left( (-\gamma(t_0))^m \prod_{i=1}^m \beta(t_i) \prod_{i=0}^m X\left(\frac{l_i}{\sqrt{\kappa}} \alpha, t_i\right) q^{-2d} e^{-\pi i x h} \right) = 
\]

\[
\text{Ch}_{H_\Lambda} \times \prod_{i=0}^m t_i^{(-l_i^2 + l_i \Lambda)/\kappa} \prod_{0 \leq i < j \leq m} \left( \frac{\theta_1(\pi(\zeta_i - \zeta_j)|\tau)}{\theta_1'(0|\tau)} \right)^{2l_i l_j/\kappa} \times m! \prod G(t_0, t_i|\tau, x)
\]

where

\[
q = e^{\pi i r} \quad t_i = e^{2\pi i \zeta_i} \quad \text{Ch}_{H_\Lambda} = \text{Tr}|_{H_\Lambda} \left( q^{-2d} e^{-\pi i x h} \right) \quad G(t_0, t_i|\tau, x) = \sum_{r \in \mathbb{Z}} \frac{e^{2\pi i x}}{1 - q^{2r} e^{-2\pi i x}}
\]

Applying this formula in our case, we get

\[
\text{Tr}|_{\mathcal{M}_{\Lambda,K}} \left( \hat{\Phi}[0] q^{-2d} e^{-\pi i x h} \right) = 
\]

\[
\int_{\Delta} \text{Tr}|_{H_\Lambda} \left( (-\gamma(z))^m \prod_{i=1}^m \beta(t_i) \prod_{i=0}^m X\left(\frac{-\alpha}{\sqrt{\kappa}}, t_i\right) q^{-2d} e^{-\pi h(x + \frac{r}{2})} \right) dt_1 \ldots dt_m = 
\]

\[
\text{Tr}|_{\mathcal{M}_{\Lambda,K}} \left( q^{-2d} e^{-\pi i x h} \right) z^{m(m+\Lambda)/\kappa} \int_{\Delta} \left( \prod_{i=1}^m t_i^{(1-\Lambda)/\kappa} \right)
\]

\[
\prod_{i=1}^m \left( \frac{\theta_1(\pi(\zeta_0 - \zeta_i)|\tau)}{\theta_1'(0|\tau)} \right)^{-m/\kappa} \prod_{1 \leq i < j \leq m} \left( \frac{\theta_1(\pi(\zeta_i - \zeta_j)|\tau)}{\theta_1'(0|\tau)} \right)^{2/\kappa} m! \prod G(z, t_i|\tau, x + \frac{\tau}{2})
\]

Since
\[ \text{Tr}_{\mathcal{M}_{\Lambda,K}} \left( q^{-2d-\frac{b}{2}e^{-\pi i h x}} \right) = e^{-\pi i (\Lambda - k)x} q^{\frac{\lambda^2}{4(k+1)}} \text{Tr}_{\mathcal{M}_{k,K}} \left( q^{-2d-\frac{b}{2}e^{-\pi i h x}} \right) \]

we finally get

\[ F(x, q) = e^{-\pi i x \lambda + \pi i \tau \frac{\lambda^2}{2\kappa}} z^{m(\Lambda - m)/\kappa} \int_{\Delta} \prod_{i=1}^{m} \left( \frac{t_i^{(-1-\Lambda)/\kappa}}{\theta_1(\pi(x - \zeta_i)|\tau)} \theta_1(0|\tau) \right)^{-2m/\kappa} \prod_{1 \leq i < j \leq m} \left( \frac{\theta_1(\pi(\zeta_i - \zeta_j)|\tau)}{\theta_1'(0|\tau)} \right)^{2/\kappa} m! \prod_{i=1}^{m} G(z, t_i|\tau, x + \frac{\tau}{2}) \, dt_1 \ldots dt_m \]

(5.11)

Since we know that this function in fact does not depend on \( z \) (though it is not immediately obvious from the formula above), we can put \( z = 1 \). In this case we can use the following formula:

\[ G(1, e^{2\pi i \zeta}|\tau, x) = \frac{i}{2} e^{-2\pi i \zeta} \frac{\theta_1(\pi(x - \zeta))\theta_1'(0)}{\theta_1(\pi x)\theta_1(\pi \zeta)} , \]

(5.12)

So we have proved the following theorem:

**Theorem 5.1.** For any \( \lambda, \kappa \in \mathbb{C}, \kappa \neq 0, m \in \mathbb{Z}_+ \), the function \( F_{\lambda,\kappa}(x, q) \) given by

\[ F_{\lambda,\kappa}(x, q) = e^{-\pi i x \lambda + \pi i \tau \frac{\lambda^2}{2\kappa}} \int_{\Delta} \prod_{i=1}^{m} e^{2\pi i \zeta_i(-1-\Lambda)} l(\zeta_i|\tau)^{-2m/\kappa} \prod_{1 \leq i < j \leq m} \left( \frac{\theta_1(\pi(\zeta_i - \zeta_j)|\tau)}{\theta_1'(0|\tau)} \right)^{2/\kappa} m! \prod_{i=1}^{m} G(z, t_i|\tau, x + \frac{\tau}{2}) \, dt_1 \ldots dt_m , \]

(5.13)

where

\[ l(\zeta|\tau) = \frac{\theta_1(\pi \zeta|\tau)}{\theta_1'(0|\tau)} \]

satisfies the partial differential equation

\[ \left( -2\pi ik \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2} \right) F_{\lambda,\kappa} = m(m + 1) \left( \varphi \left( \frac{\tau}{2} + x \right) + c \right) F_{\lambda,\kappa} , \]

(5.14)
and the following boundary conditions when \( q \to 0 \),

\[
F_{\lambda, \kappa}(x, q) \sim C(\lambda, \kappa) e^{-\pi i x \lambda + \pi i \tau \lambda^2 / 2\kappa},
\]

where

\[
C(\lambda, \kappa) = \int_{\Delta} \left( \prod_{i=1}^{m} e^{-2\pi i \zeta \lambda / \kappa (\sin \pi \zeta)(-2m / \kappa)} \prod_{1 \leq i < j \leq m} (\sin \pi (\zeta_i - \zeta_j))^2 / \kappa \prod_{i=1}^{m} \left(1 - e^{2\pi i \zeta_i} \right) \right) d\zeta_1 \ldots d\zeta_m.
\]

**Remark.** The function \( F(x, q) \) in the theorem above differs from the function \( F(x, q) \) given by (5.11) by a constant factor.

Note that in the limit \( q \to 0 \) the right-hand side of equation (5.14) tends to zero and the equation itself tends to the heat equation:

\[
(-2\pi i \kappa \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}) f(x, \tau) = 0,
\]

and the asymptotics (5.15) of the functions \( F_{\lambda, \kappa} \) given by Theorem 5.1 form a basis in the space of solutions of the heat equation. This allows us to find all the solutions of equation (5.14) having good asymptotics as \( \tau \to i\infty \). Let us consider the case then \( \tau = it, t \in \mathbb{R}_+ \) and \( \kappa \in i\mathbb{R} \). In this case equation (5.14) can be considered as non-stationary Schrödinger equation:

\[
\frac{\partial f}{\partial t} = \frac{1}{2\pi \kappa} H f,
\]

with the Hamiltonian

\[
H = \frac{d^2}{dx^2} - m(m + 1) \left( \varphi(x + \frac{\tau}{2}) + c(\tau) \right).
\]

Note that this Hamiltonian is self-adjoint with respect to the \( L^2 \) norm.

In the limit \( q \to 0 \), (5.18) tends to

\[
\left( \frac{\partial}{\partial t} - \frac{1}{2\pi \kappa} \frac{\partial^2}{\partial x^2} \right) f(x, \tau) = 0
\]
Theorem 5.2. Let us fix $\kappa \in \mathbb{R}$ and assume that $\tau = it, t \in \mathbb{R}$. Let $f(x, \tau)$ be a solution of (5.18) such that

$$f(x, \tau) = f_0(x, \tau)(1 + g(x, \tau))$$

where $f_0(x, \tau)$ is a solution of (5.20) and $\sup_{x \in \mathbb{R}} |g(x, \tau)| \to 0$ as $q \to 0(t \to +\infty)$. Also, let us assume that for some $\tau_0 f_0(x, \tau_0)$ is a rapidly decreasing function of $x$.

Then $f(x, \tau)$ can be presented in the following form:

$$(5.21) \quad f(x, \tau) = \int_{\mathbb{R}} \frac{F_{\lambda, \kappa}(x, \tau)}{C(\lambda, \kappa)} \rho(\lambda) \, d\lambda$$

with the function $\rho(\lambda)$ given by

$$\rho(\lambda) = \frac{1}{2} \int_{\mathbb{R}} e^{\pi i \lambda x} f_0(x, 0) \, dx$$

Proof. The fact that Hamiltonian (5.19) is self-adjoint with respect to the $L^2$ norm implies that if $f(x, \tau)$ is a solution of Schrödinger equation (5.18) then $|f(x, \tau)|_{L^2(\mathbb{R})}$ does not depend on $\tau$. The same applies to $f_0$. Next, $f_0$ – as any solution of (5.20) – can be written in the form

$$f_0(x, \tau) = \int_{\mathbb{R}} e^{-\pi i \lambda x + \pi i \tau \lambda^2 / 2\kappa} \rho(\lambda) \, d\lambda$$

for some density function $\rho$. Letting $\tau \to 0$ and applying inverse Fourier transform, we see that

$$\rho(\lambda) = \frac{1}{2} \int_{\mathbb{R}} e^{\pi i \lambda x} f_0(x, 0) \, dx$$

Now let us define the function $\tilde{f}$ by

$$\tilde{f}(x, \tau) = \int_{\mathbb{R}} \frac{F_{\lambda, \kappa}(x, \tau)}{C(\lambda, \kappa)} \rho(\lambda) \, d\lambda.$$  

Then Theorem (5.1) implies that $\tilde{f}(x, \tau)$ satisfies Schrödinger equation (5.18). Moreover, it is easy to check that convergence of $F_{\lambda, k}$ to asymptotics (5.15) is in fact uniform in $x$. Therefore, as $q \to 0 \tilde{f}(x, \tau)$ can be written in the form

$$\tilde{f}(x, \tau) = f_0(x, \tau)(1 + \tilde{g}(x, \tau))$$

with the same function $f_0$ as above and $\tilde{g}$ satisfying $\sup_{x \in \mathbb{R}} |g(x, \tau)| \to 0$ as $q \to 0$. Comparing this with the assumption of the theorem and taking into account that the $L^2$ norms $|f(x, \tau) - \tilde{f}(x, \tau)|, |f_0(x, \tau)|$ do not depend on $\tau$, we see that $f(x, \tau) = \tilde{f}(x, \tau)$, which completes the proof. □
6. Lamé functions.

In this section we use the theorem proved in the previous section to find explicit formulas for Lamé functions. The basic idea is to let \( k \to -1 \); obviously, in this limit the equation (5.1) becomes Lamé equation. Unfortunately, we can not just repeat all the arguments of the previous sections for \( k = -1 \), since in this case (critical level) the whole theory fails; we can not even claim the existence of the intertwiners between Verma modules and evaluation representations. Instead, we can use the integral formula (5.13) for \( F \) and then find the asymptotics when \( k \to -1 \).

We illustrate this general idea on the simplest example \( m = 1 \), when all the calculations can be done absolutely explicitly. In this case the expression for \( F \) given at the end of previous section reduces to:

\[
F(x, q) = e^{-\pi i \lambda x + \pi i \lambda^2 / 2k} \int_{\Delta} e^{2\pi i \zeta (-1 - \frac{1}{2})} \left( \frac{\theta_1(\pi \xi | \tau)}{\theta_1'(0 | \tau)} \right)^{-2/\kappa} \frac{\theta_1(\pi (x - \zeta + \frac{\xi}{2}) \theta_1'(0)}{\theta_1(\pi (x + \frac{\xi}{2}) \theta_1(\pi \zeta)} d\zeta
\]

In this case one can take \( \Delta \) to be the Pochhammer loop in the \( t \)-plane, i.e. the element \( a_1 \circ a_0 \circ a_1^{-1} \circ a_0^{-1} \) of \( \pi_1(\mathbb{C} \setminus \{0, 1\}) \), where \( a_{0,1} : [0,1] \to \mathbb{C} \) is the loop going around 0,1 in the anticlockwise/clockwise direction. This cycle is the same as the cycle \( a_0 \circ a_1^{-1} \) in the \( \zeta \)-plane.

Now, let us assume that \( k \) approaches \(-1\) along the real line from below: \( k \in \mathbb{R}, k < -1 \). Then \( \kappa < 0 \) and therefore \(-2/\kappa > 0\). In this case we can use the saddle-point method for finding asymptotics of the integral (6.1) (see, for example, E.T. Copson, “Asymptotic expansions”, Cambridge Univ. Press, 1965).

**Proposition.** Let \( f(\nu) \) be defined by the integral

\[
f(\nu) = \int_{\Delta} e^{\nu w(z)} \phi(z) dz,
\]

where \( \Delta \) is a contour in \( \mathbb{C} \), \( w, \phi \) are analytic functions on \( \Delta \). Let us assume that there exists \( z_0 \in \Delta \) such that:

1. \( w'(z_0) = 0, w''(z_0) \neq 0 \)
2. \( \phi(z_0) \neq 0 \)
3. For any \( z \in \Delta, z \neq z_0 \) we have \( |e^{w(z)}| < |e^{w(z_0)}| \).

Then \( f(\nu) \) has the following asymptotics as \( \nu \to +\infty \):

\[
f(\nu) \sim \phi(z_0) e^{\nu w(z_0)} \left( \frac{-2\pi}{\nu w''(z_0)} \right)^{\frac{1}{2}}.
\]

Moreover, if \( \phi, w \) also depend analytically on some parameters \( x_i \) then the asymptotics (6.3) can be differentiated to get asymptotics for \( \frac{\partial}{\partial x_i} f(\nu) \).

Therefore, to find the asymptotics of the integral (6.1) we must find critical points of \( e^{\pi i \lambda \zeta} \theta_1(\pi \zeta) \), i.e. points \( \zeta_0 \) satisfying

\[
\lambda = \frac{\theta_1'(\pi \zeta_0)}{\theta_1(\pi \zeta_0)}
\]

(6.4)
Of course, it is hopeless to find explicit expression for $\zeta_0$ as a function of $\lambda$; however, for some special values of $\lambda$ – for example, $\lambda \in \mathbb{Z}$ – it can be done; we return to this later. Therefore, it is better to consider $\zeta_0$ as given and define $\lambda$ by (6.4).

Let us assume the following:

1. $\theta''_1(\pi \zeta_0) \neq 0$

2. The contour $\Delta$, described above, can be deformed so that it passes through $\zeta_0$ precisely twice, and $|e^{\pi i \lambda \zeta_0} \theta_1(\pi \zeta)|$ attains its absolute maximum on $\Delta$ at $\zeta_0$.

In this situation we can apply the saddle-point method, which gives us the following asymptotics for $F$ as $k \to -1$:

\begin{equation}
F(x, q) \sim (1 - e^{4 \pi i / \kappa})(A(q))^{-2 / \kappa} \sqrt{-\kappa / 2} \times B(x, q),
\end{equation}

where

\begin{align*}
A(q) &= e^{-\pi i (\lambda + \tau \lambda^2 / 4)} \frac{\theta_1(\pi \zeta_0)}{\theta'_1(0)} \\
B(x, q) &= e^{-2 \pi i \zeta_0 - \pi i \lambda x} \frac{\theta_1(\pi (x - \zeta_0 + \frac{\tau}{2}))}{\theta_1(\pi (x + \frac{\tau}{2}))} \frac{1}{\sqrt{\pi \theta''_1(0) / \theta_1(\pi \zeta_0)}} \\
&= e^{-2 \pi i \zeta_0 - \pi i \lambda x} \frac{\theta_1(\pi (x - \zeta_0 + \frac{\tau}{2}))}{\theta_1(\pi (x + \frac{\tau}{2}))} \frac{1}{\sqrt{\pi \lambda^2 / \theta_1(\pi \zeta_0)}} \\
&= e^{-2 \pi i \zeta_0 - \pi i \lambda x} \frac{\theta_1(\pi (x - \zeta_0 + \frac{\tau}{2}))}{\theta_1(\pi (x + \frac{\tau}{2}))} \frac{1}{\sqrt{\pi \lambda^2}}.
\end{align*}

The factor $1 - e^{4 \pi i / \kappa}$ in front of the integral appears due to the fact the cycle of integration passes through $\zeta_0$ twice.

Substituting this asymptotics in the equation (5.1), we get:

\begin{equation}
4 \pi i \frac{\partial}{\partial \tau} A(q) \frac{\partial}{\partial \tau} B(x, q) - 2 \pi i \kappa \frac{\partial}{\partial \tau} B(x, q) + \frac{\partial^2}{\partial x^2} B(x, q) = 2(\varphi(x + \frac{\tau}{2}) + c)B(x, q),
\end{equation}

or – letting $\kappa \to 0$ –

\begin{equation}
\frac{\partial^2}{\partial x^2} B(x, q) = 2 \left( \varphi(x + \frac{\tau}{2}) + c - 2 \pi i \frac{\partial}{\partial \tau} A(q) \right) B(x, q).
\end{equation}

This shows that the function $B(x, q)$, given by (6.6) is a solution of Lamé equation with $m = 1$. Therefore, we have proved the following

**Proposition 6.1.** Under the assumptions 1 and 2 above, the function

\begin{equation}
f(x) = e^{-\pi i \lambda x} \frac{\theta_1(\pi (x - \zeta_0 + \frac{\tau}{2}))}{\theta_1(\pi (x + \frac{\tau}{2}))},
\end{equation}

where $\lambda$ and $\zeta_0$ are related by (6.4), is a solution of Lamé equation of degree $m = 1$.

Let us return to these assumptions. Consider first the case when $q \in \mathbb{R}$, $\zeta_0 = 1/2$. One can easily check that in this case $\lambda = 0$, and that the function $|\theta_1(\pi \zeta_0)|$ reaches at $\zeta = 1/2$ its absolute maximum on the real line. Therefore, in this case the assumptions 1 and 2 are satisfied. Therefore, in this case the function $f(x)$ is
indeed an asymptotics for the function $F(x, q)$ and therefore is a solution of Lamé equation. In general these assumptions are not true. However, the result is still valid: the above defined function $f(x)$ is a solution of Lamé equation. To see it, note that the assumptions 1, 2 are satisfied for $q, \zeta_0$ in some neighborhood of $q_0 \in \mathbb{R}, \zeta_0 = 1/2$. Therefore, for all these $q, \zeta_0$ the function $f(x)$ defined above is a solution of Lamé equation. Since $f$ analytically depends on $q, \zeta_0$ it implies that in fact $f(x)$ is a solution of Lamé equation for all $q, \zeta_0$ for which it is well-defined (though in general we cannot claim that it is obtained as an asymptotics of the correlation function $F(x, q)$). So we have obtained the following result (which is of course well-know, see [WW]):

**Proposition 6.2.** For any $\lambda, \zeta_0$ satisfying (6.4), the function

$$f(x) = e^{-\pi i \lambda x} \frac{\theta_1(\pi(x - \zeta_0 + \frac{\tau}{2}))}{\theta_1(\pi(x + \frac{\tau}{2}))},$$

is a solution of Lamé equation of degree $m = 1$.

It is an instructive exercise to check this by direct computation.

Let us consider some examples. If $\lambda = 0$, it is easy to see that the critical point is $\zeta_0 = 1/2$, and $f(x) = G(1, -1|\tau, x + \frac{\tau}{2}) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} = \frac{K}{\pi} \text{dn} (2Kx)$, where the periods $K, K'$ of Jacobi elliptic functions are related with $q$ by $q = e^{-\pi \frac{K}{K'}}$. More generally, if $\lambda \in 2\mathbb{Z}$ then the critical point is $\zeta_0 = \frac{1}{2} + \frac{\lambda}{2}\tau$, and the corresponding function $f(x)$ is again $\text{dn} (2Kx)$.

Similarly, for $\lambda = 1$ the critical points are $\zeta_0 = \frac{\tau}{2}, \zeta_0 = \frac{1}{2} + \frac{\tau}{2}$, and the function $f(x) = \text{sn} (2Kx), f(x) = \text{cn} (2Kx)$; the same functions are obtained for every odd integer $\lambda$. Note that we have two different solutions corresponding to the same value of $\lambda$; in the best case only one of them is an asymptotics of the correlation function on the torus; so the interpretation of these solutions in terms of representation theory is still unclear.

We have shown that $\lambda \in \mathbb{Z}$ gives us the classical periodic solutions of Lamé equation of level $m = 1$: $\text{dn} (2Kx), \text{sn} (2Kx), \text{cn} (2Kx)$ (which are of course well known). The general theory (see [WW]) says that there are no other periodic solutions. Therefore, the asymptotics of the integral representation of the function $F$ for $m = 1$ gave us all the periodic solutions of Lamé equation of degree 1.

The same technique can be applied for arbitrary $m$. Though it is not easy to see what kind of contour we must take and check the necessary conditions for the applicability of the multi-dimensional saddle point method, Proposition 6.2 suggests the following formula:

**Proposition 6.3.** For any $m \in \mathbb{Z}_+, \zeta = \zeta_1 \ldots \zeta_m \in \mathbb{C}^m, \lambda \in \mathbb{C}$ satisfying

$$i\lambda + m \frac{\theta'_1(\pi \zeta_i)}{\theta_1(\pi \zeta_i)} - \sum_{j \neq i} \frac{\theta'_1(\pi(\zeta_i - \zeta_j))}{\theta_1(\pi(\zeta_i - \zeta_j))} = 0 \text{ for all } i = 1 \ldots m,$$

the function $f(x)$, defined by

$$f(x) = e^{-\pi i \lambda x} \prod_{j=1}^{m} \frac{\theta_1(\pi(x - \zeta_j + \frac{\tau}{2}))}{\theta_1(\pi(x + \frac{\tau}{2}))}$$
is a solution of the Lamé equation of degree $m$.

This proposition can be proved by a direct calculation, which shows that a function of the form (6.11) is a solution of Lamé equation if and only if $\lambda$ and $\zeta_i$ are related by a system of equations (6.10) (see [WW, §23.71]). It is also known that almost any solution — not necessarily periodic — can be written in this form. The only exception is the case when the constant $c$ in the Lamé equation (5.2) is an eigenvalue, i.e. when this equation has a doubly-periodic solution. In this case the other solution is not periodic and cannot be expressed in terms of theta functions. Therefore we see that our approach gave us almost all solutions of Lamé equation.

This general formula has been known for a long time (in [WW] it is given with a reference to Hermite’s note of 1872), though, of course, it was never obtained as an asymptotics of an integral in the way we do.

Conclusion

We’ve considered in detail the case $\mathfrak{g} = \mathfrak{sl}_2$. As for the case of arbitrary $\mathfrak{g}$, it is known there always exists an analogue of Wakimoto realization (see [FF]) and therefore, integral formulas for intertwiners similar to (5.8) (see [ATY]). This in turn implies the existence of integral formulas for the traces $\mathcal{F}$ of the form (2.6). Therefore, it is natural to suggest the following conjecture (for the case $n = 1$, i.e. trace with insertion of one intertwiner):

Conjecture. If $V$ is a finite-dimensional representation then the correlation function $\mathcal{F}$ defined by (2.6) has the following asymptotics as $\kappa \to 0$ (with all the derivatives):

$$\mathcal{F} \sim C_1 \kappa^{N/2} C_2^{1/\kappa} \phi(x),$$

where $C_1, C_2$ do not depend on $x, \kappa$ (but may depend on $q, \lambda$).

This would immediately imply that the function $\phi/\mathcal{F}_0$ satisfies the equation

$$\Delta_6 \frac{\phi}{\mathcal{F}_0} = -\frac{1}{2\pi^2} \sum_{\alpha \in \Delta^+} \left( \varphi \left( \frac{\tau |\alpha|}{g} - \frac{h(\alpha)}{2\pi i} \right) + c \right) f_\alpha e_\alpha \frac{\phi}{\mathcal{F}_0},$$

which is obtained from (6.12) by letting $\kappa \to 0$. This equation is a natural multidimensional analogue of the Lamé equation. Moreover, if the highest weight $\lambda$ of the Verma module is integral then the solutions we get must possess some remarkable periodicity properties and should be considered as natural analogues of Lamé functions. Note that in general $V_\mu(0)$ is not one-dimensional, so $\phi$ is a vector function; however, for $\mathfrak{g} = \mathfrak{sl}_n$ there exist finite-dimensional modules with one-dimensional zero-weight component. In this case we obtain the elliptic analogue of Calogero-Sutherland operator for the root system $A_n$ (cf. [OP]). This will be discussed in detail in our future paper.

It is an interesting problem to consider the correlation functions in the case where $V$ is not a finite-dimensional representation. For example, for $\mathfrak{g} = \mathfrak{sl}_2$ one can define the module $W_\mu = \{(x_1 x_2)^\mu f(x_1, x_2) | f$ is a Laurent polynomial in $x_1, x_2$ of total degree zero$\}$ with the action of $\mathfrak{sl}_2$ given by

$$e = x_1 \frac{\partial}{\partial x_2}, \quad h = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \quad f = x_2 \frac{\partial}{\partial x_1}.$$
In this form this module has an obvious generalization to the case $g = sl_n$.

It is easy to see that $W_\mu(0)$ is one-dimensional and $ef|_{W_\mu(0)} = \mu(\mu+1)$. Also note that if $\mu \in \mathbb{Z}_+$ then this module has a finite dimensional submodule of dimension $2\mu + 1$.

Therefore, the traces of the form (2.6) for the intertwiners $\Phi(z): M_{\lambda,k} \rightarrow M_{\lambda,k} \otimes W_\mu$ satisfy the parabolic PDE (5.14) with the constant $m = \mu - \text{an arbitrary complex number}$. It is an interesting open question if these correlation functions have asymptotics of the form

$$\mathcal{F} \sim f(q, \lambda, \kappa)\phi(q, \lambda, x).$$

Then the function $\phi(q, \lambda, x)$ would be a solution of the Lamé equation. This would provide a method of constructing Lamé functions with arbitrary complex value of the constant $m$ from representation theory of $\widehat{sl}_2$.

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