SELF-INJECTIVE COMMUTATIVE RINGS
HAVE NO NONTRIVIAL RIGID IDEALS

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Abstract. We establish a link between trace modules and rigidity in modules over Noetherian rings. Using the theory of trace ideals we make partial progress on a question of Dao, and on the Auslander-Reiten conjecture over Artinian Gorenstein rings.

1. Introduction

Let \( R \) be a ring and \( M, X \) \( R \)-modules. The \textit{trace module of} \( M \) \textit{in} \( X \), denoted \( \tau_M(X) \), is the \( R \)-module \( \sum \alpha(M) \) as \( \alpha \) ranges over \( \text{Hom}_R(M, X) \). Such a trace module is \textit{proper} provided \( \tau_M(X) \varsubsetneq X \). We identify a link between trace modules over \( R \) and the existence of self-extensions of \( R \)-modules. The main result is

\textbf{Theorem.} Let \( R \) be an local Artinian Gorenstein ring. If \( M \) is a syzygy of a proper trace module then \( \text{Ext}^1_R(M, M) \neq 0 \).

Every proper ideal in an Artinian Gorenstein ring is a proper trace module in \( R \); see Proposition 3.1. As a consequence, we obtain a positive answer for ideals to a question of Dao [4], and we settle the Auslander-Reiten conjecture for ideals and their syzygies over Artinian Gorenstein rings.

\textbf{Corollary.} Let \( R \) be an local Artinian Gorenstein ring. If an \( R \)-module \( M \) is rigid and appears as a positive or negative syzygy of an ideal then \( M \) is free. In particular, the Auslander-Reiten conjecture holds for all ideals \( I \subseteq R \).

Recall that we say a left \( R \)-module \( M \) is \textit{rigid} if \( \text{Ext}^1_R(M, M) = 0 \) and that a commutative Noetherian ring is Artinian Gorenstein if and only if it is self-injective.

In Section 2 we present the needed results concerning trace modules and establish a key lemma, Lemma 2.7. In Section 3 we discuss some cases in which rigidity implies projectivity and prove the main result, Theorem 3.9.

2. Trace Modules and Trace Ideals

Let \( R \) be a Noetherian ring. In this section \( R \) need not be commutative. We recall the properties of trace modules needed for subsequent sections.

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Given a left $R$-module $M$, we write $M^*$ for $\text{Hom}_R(M, R)$.

**Definition 2.1.** Let $M$ and $X$ be left $R$-modules. The trace module of $M$ in $X$ is

$$\tau_M(X) := \sum_{\alpha \in \text{Hom}_R(M, X)} \alpha(M).$$

We say an $R$-module $M$ is a trace module in $X$ provided $M = \tau_A(X) \subseteq X$ for some $R$-modules $A$ and $X$. We call such an $M$ a proper trace module when containment is strict and a trace ideal when $X = R$.

**Definition 2.2.** Let $M$ and $X$ be $R$-modules. We say $X$ is generated by $M$ if $\tau_M(X) = X$, that is, $X$ is the homomorphic image of a direct sum of copies of $M$.

**Lemma 2.3.** Consider $R$-modules $M \subseteq X$. The following are equivalent:

(i) $M$ is a trace module in $X$;

(ii) $M = \tau_M(X)$;

(iii) $\text{End}_R(M) = \text{Hom}_R(M, X)$, that is, every homomorphism from $M$ to $X$ has its image in $M$.

**Proof.** ($i \Rightarrow ii$): Evidently $M \subseteq \tau_M(X)$. If $M = \tau_A(X)$ for some $R$-module $A$, then $A$ generates $M$ and $M$ generates $\tau_M(X)$. It follows that $\tau_M(X) \subseteq \tau_A(X) = M$.

($ii \Rightarrow iii$): One has $\text{End}_R(M) \subseteq \text{Hom}_R(M, X)$. Now, given $\alpha$ in $\text{Hom}_R(M, X)$, by definition $\text{Im}(\alpha) \subseteq \tau_M(X) = M$. Therefore $\text{Hom}_R(M, X) \subseteq \text{End}_R(M)$.

($iii \Rightarrow i$): Given any $\alpha$ in $\text{Hom}_R(M, X)$, then $\text{End}_R(M) = \text{Hom}_R(M, X)$ implies that $\text{Im}(\alpha) \subseteq M$. It follows that $\tau_M(X) \subseteq M$ and therefore $\tau_M(X) = M$. \hfill $\square$

**Remark 2.4.** By Lemma 2.3, to say $M$ is a proper trace module in $X$ is to say $M \subset X$ and $M = \tau_M(X)$. Note, all modules are trivially trace modules since $M = \tau_M(M) = \tau_R(M)$. It is left to characterize proper trace modules and trace modules of modules that do not generate the entire category of $R$-modules.

**Example 2.5.** Consider the ring $R = k[x, x^2, x^3]$, where $k$ is a field. Note that $R$ is a one-dimensional Cohen-Macualay domain which is not Gorenstein and the maximal ideal of $R$ is $\mathfrak{m} = (x^3, x^4, x^5)$. The set of trace ideals in $R$ is $\{0, \mathfrak{m}, R\}$; see [3, Example 30]

**Remark 2.6.** Auslander and Green [2] identify several classes of proper trace modules. Let $\Lambda$ be an Artin algebra. If $X$ is a $\Lambda$-module, then the socle of $X$ is a trace module in $X$; see the proof of [2, Proposition 2.5]. If $C$ is a non-rigid $\Lambda$-module such that $\text{End}_\Lambda(C)$ is a division ring, then $C$ is a proper trace module in some $\Lambda$-module $X$; see [2, Proposition 2.3]. Also, if $M \subseteq X$ is a waist in $X$ then $M$ is a trace module in $X$, that is, if for any $N \subseteq X$ either $N \subseteq M$ or $M \subseteq N$; see [2, Proposition 2.1].

See Proposition 3.1 for further examples of trace ideals.

**Lemma 2.7.** Suppose $M$ is a proper submodule of $X$. If $\text{Hom}_R(M, X/M) = 0$ then $M$ is a trace module in $X$. The converse holds when $M$ is also rigid.
Proof. Applying \( \text{Hom}_R(M, -) \) to the exact sequence

\[
0 \to M \to X \to X/M \to 0.
\]

yields an exact sequence

\[
0 \to \text{End}_R(M) \to \text{Hom}_R(M, X) \to \text{Hom}_R(M, X/M) \to \text{Ext}^1_R(M, M).
\]

If \( \text{Hom}_R(M, X/M) = 0 \) then \( \text{End}_R(M) = \text{Hom}_R(M, X) \) and \( M \) is a trace module by Lemma 2.8. On the other hand, if \( M \) is a rigid trace module in \( X \), then \( \text{Hom}_R(M, X/M) = 0 \) by the exactness of the sequence. \( \square \)

Remark 2.8. One may use Lemma 2.7 to show that certain proper trace modules cannot be rigid, and also to show that certain rigid trace modules \( M \) in \( X \) cannot be proper, that is, \( M = X \); see, for example, Theorem 3.9 and Proposition 3.5 respectively. Note, the converse to the statement in Lemma 2.7 does not hold when \( M \) is not rigid. Section 3 focuses on trace ideals such that \( \text{Hom}_R(I, R/I) \neq 0 \).

The remainder of this paper considers the consequences of Lemma 2.7 over commutative Noetherian rings.

Remark 2.9. Suppose \( R \) is commutative. By Hom-Tensor adjunction the hypothesis \( \text{Hom}_R(I, R/I) = 0 \) is equivalent to \( (I/I^2)^* = 0 \), where the dual is taken with respect to \( R/I \). Lemma 2.7 proves that \( (I/I^2)^* = 0 \) implies \( I \) is a trace ideal; see Example 2.11 (b).

The \( R/I \)-module \( I/I^2 \) is called the conormal module of \( I \). It is well-known that the conormal module of \( I \) can detect if \( R/I \) is a complete intersection; see [11]. Here we see that the conormal module can also detect when \( I \) is a trace ideal.

Remark 2.10. Consider the relationship between these three conditions:

(i) \( M \) is a proper trace module in \( X \),

(ii) \( \text{Hom}_R(M, X/M) \neq 0 \), and

(iii) \( M \) is rigid.

Lemma 2.7 shows that a pair of modules \( (M, X) \) cannot have all three. However, Example 2.11 shows that pairs \( (M, X) \) may have any two of these three properties.

Example 2.11.

(a) Any proper free ideal, \( I \), is a rigid ideal and \( \text{Hom}_R(I, R/I) \neq 0 \). For any such \( I \) one has \( \tau_I(R) = R \).

(b) Let \( R = k[[x, y]]/(xy) \). Then \( I = (y) \) is a rigid trace ideal which is not free. Here \( \text{Hom}_R(I, R/I) = 0 \) because \( \text{Ann}_R I = (x) \) consist of nonzerodivisors on \( R/I \cong k[[x]] \); see Lemma 3.6 and [5, Example 1.2].

(c) Let \( R = k[[x, y]]/(xy) \). Then \( J = (x, y) \) is a trace ideal with \( \text{Hom}_R(J, R/J) \neq 0 \), that is not rigid and not free. Whenever \( R \) is a local ring which is not a DVR, its maximal ideal will be an example of this kind.
3. Rigid Trace Modules over Gorenstein Rings

In this section $R$ is a commutative Noetherian ring. We use Lemma 2.7 to establish
theorems concerning rigid modules. To that end, we first identify examples of
trace ideals and pairs of $R$-modules $M \subseteq X$ for which $\text{Hom}_R(M, X/M) \neq 0$.

The proof of the result below, is an application of the proof of Example 2.4 in [7].

**Proposition 3.1.** If $\text{Ext}^1_R(R/I, R) = 0$ then $I$ is a trace ideal.

**Proof.** Applying $\text{Hom}_R(-, R)$ to the sequence

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0,$$

one gets the exact sequence

$$0 \rightarrow \text{Hom}_R(R/I, R) \rightarrow \text{Hom}_R(R, R) \rightarrow \text{Hom}_R(I, R) \rightarrow \text{Ext}^1_R(R/I, R) \rightarrow \cdots.$$

Because $\text{Ext}^1_R(R/I, R) = 0$, the map $i^*$ is surjective and each $R$-homomorphism
from $I$ to $R$ is given by multiplication by an element of $R$. Therefore $\text{Hom}_R(I, R) = \text{End}_R(I)$ and $I$ is a trace ideal by Lemma 2.3. 

**Remark 3.2.** When $R$ is a commutative Noetherian ring

$$\text{grade } I = \min \{i \mid \text{Ext}^i_R(R/I, R) \neq 0\},$$

by [3, Theorem 1.2.5]. Also, $R$ is Artinian Gorenstein if and only if it is self-injective.
It follows from Proposition 3.1 that $I$ is a trace ideal if

(i) $\text{grade } I \geq 2$ or

(ii) $R$ is an Artinian Gorenstein ring.

In fact, a Noetherian ring $R$ is Artinian Gorenstein if and only if every ideal is a
trace ideal; see [3]. However, not every grade zero ideal in a Gorenstein ring is a
trace ideal; see Example 3.3. Nevertheless, if $\text{Min}(I) \cap \text{Supp}(I) \neq \emptyset$, where $\text{Min}(I)$

is the set of prime ideals minimally containing $I$, then a standard reduction to the
Artinian case still precludes rigidity in proper grade zero ideals in Gorenstein rings;
see Proposition 3.5.

**Example 3.3.** Let $R = \mathbb{Q}[x, y]/(x^2y^2)$. For the grade zero ideal $I = (x^5, xy^7)$,
$\tau_R(I) = (x^2, xy^2)$. To see this, recall that $\tau_R(I)$ is the ideal generated by the
entries of the left kernel of the presentation matrix of $I$; see [12, Remark 3.3].

**Definition 3.4.** We say $R$ is generically Gorenstein provided $R_p$ is Gorenstein for
each $p$ minimal in $\text{Spec}(R)$.

**Proposition 3.5.** Let $R$ be a local ring that is generically Gorenstein and let $I \subseteq R$
be an ideal. Suppose $\text{grade } I = 0$ and $\text{Min}(I) \cap \text{Supp}(I) \neq \emptyset$ or $\text{grade } I > 1$. Then
$I$ is not rigid.

**Proof.** Assume that $\text{grade } I = 0$. Pick $p \in \text{Min}(I) \cap \text{Supp}(I)$, then $R_p$ is Artinian
Gorenstein.
$I_p$ is a trace ideal in $R_p$; see Proposition 3.1. The ideal $I_p$ is $pR_p$-primary therefore $\text{Hom}_{R_p}(I_p, R_p/I_p) \neq 0$ by Remark 3.7 and

$$0 = \text{Ext}^1_{R_p}(I_p, I_p) = \text{Ext}^1_{R_p}(I_p, I_p).$$

By Lemma 2.7, $I_p = R_p$. This contradicts the containment $I \subseteq p$. It follows that grade $I \neq 0$.

Now assume grade $I \geq 2$. If $I \neq R$, $I$ is a rigid proper trace ideal such that $\text{Hom}_{R_p}(I, R/I) \neq 0$; see Remark 3.2, Lemma 3.6 and Remark 3.7. This contradicts Lemma 2.7, hence $I = R$. □

The following result is well-known; see, for example, [6, Lemma 8.1].

**Lemma 3.6.** Let $R$ be a commutative Noetherian ring and let $M$ and $N$ be finitely generated $R$-modules, such that $M \otimes_R N \neq 0$. Then $\text{Hom}_R(M, N) = 0$ if and only if $\text{Ann}_R M$ contains an $N$-regular element.

**Remark 3.7.** When $R$ is local and $M$ and $N$ are nonzero finitely generated $R$-modules $M \otimes_R N \neq 0$. Therefore, over a local ring, the hypothesis $\text{Hom}_R(M, X/M) \neq 0$ in Lemma 2.7 is a mild condition equivalent to

$$\text{Supp}(M) \cap \text{Ass}(X/M) = \text{Ass}(\text{Hom}_R(M, X/M)) \neq \emptyset.$$ 

Among other pairs of modules, it is held by

(i) $(I, R)$ for proper $m$-primary ideals $I \subseteq R$;

(ii) $(I, R)$ for all proper ideals $I \subseteq R$ of positive grade;

(iii) The $R$-modules $(M, X)$ whenever $X/M$ is nonzero and has finite length.

**Remark 3.8.** Over Gorenstein rings, rigidity passes from an MCM module to its (negative and positive) syzygies; see [10] Proposition 7.3 (1)(i) and [11] Proposition 7]. This allows us to provide a partial answer to Question 9.1.4 in [4]:

**Question.** Let $R$ be a commutative Artinian, Gorenstein local ring and $M$ be a finitely generated $R$-module. If $M$ is rigid is $M$ free?

**Theorem 3.9.** Let $R$ be an local Artinian Gorenstein ring. If an $R$-module $M$ is rigid and appears as a positive or negative syzygy of an ideal, then $M$ is free. In particular, the Auslander-Reiten conjecture holds for all ideals $I \subseteq R$.

**Proof.** Suppose $M$ is a syzygy of a proper trace module $N$ in $X$. Since $R$ is Artinian, the module $X/N$ has finite length. It follows that $\text{Hom}_R(N, X/N) \neq 0$; see Remark 3.7. By Lemma 2.7, $N$ is not rigid. Since rigidity passes to syzygies, it follows that $M$ is not rigid. □

**Conjecture** (Auslander-Reiten conjecture). Let $A$ be an Artin algebra and $M$ a finitely generated $A$-module. If $\text{Ext}_A^i(M, M \oplus A) = 0$ for all $i > 0$ then $M$ is projective.

**Corollary 3.10.** Let $R$ be an local Artinian Gorenstein ring. If an $R$-module $M$ is rigid and appears as a positive or negative syzygy of an ideal, then $M$ is free. In particular, the Auslander-Reiten conjecture holds for all ideals $I \subseteq R$. 
Proof. Every ideal $I$ in an Artinian Gorenstein ring is a trace ideal; see Proposition 3.1. By Theorem 3.9, a rigid $I$ cannot be a proper trace ideal. It follows that $I = R$ and that its syzygies are free $R$-modules. □

Remark 3.11. It is left to determine the full set of modules that can be realized as the syzygies of proper trace modules over Artinian Gorenstein rings.

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References

[1] Tokuji Araya, The Auslander-Reiten conjecture for Gorenstein rings, Proc. Amer. Math. Soc. 137 (2009), no. 6, 1941–1944. MR2480274
[2] Maurice Auslander and E. L. Green, Trace quotient modules, Illinois J. Math. 32 (1988), no. 3, 534–556. MR947045
[3] Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)
[4] Hailong Dao, Some homological properties of modules over a complete intersection, with applications, Commutative algebra, 2013, pp. 335–371. MR3051378
[5] Craig Huneke and Roger Wiegand, Tensor products of modules and the rigidity of Tor, Math. Ann. 299 (1994), no. 3, 449–476. MR1282227 (95m:13008)
[6] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, Twenty-four hours of local cohomology, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR2355715
[7] Haydee Lindo, Trace ideals and centers of endomorphism rings of modules over commutative rings, J. Algebra 482 (2017), 102–130. MR3646286
[8] Haydee Lindo and Nina Pande, Trace ideals and the Gorenstein property, 2017. In preparation.
[9] Thomas G. Lucas, The radical trace property and primary ideals, J. Algebra 184 (1996), no. 3, 1093–1112. MR1407887
[10] Ryo Takahashi, Remarks on modules approximated by $G$-projective modules, J. Algebra 301 (2006), no. 2, 748–780. MR2236766
[11] Wolmer V. Vasconcelos, Ideals generated by $R$-sequences, J. Algebra 6 (1967), 309–316. MR0213435
[12] Wolmer V. Vasconcelos, Computing the integral closure of an affine domain, Proc. Amer. Math. Soc. 113 (1991), no. 3, 633–638. MR1055780 (92b:13013)

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