A Bang-Bang Principle of Time Optimal Internal Controls of the Heat Equation *

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Abstract. In this paper, we study a time optimal internal control problem governed by the heat equation in $\Omega \times [0, \infty)$. In the problem, the target set $S$ is nonempty in $L^2(\Omega)$, the control set $U$ is closed, bounded and nonempty in $L^2(\Omega)$ and control functions are taken from the set $U_{ad} = \{ u(\cdot, t) : [0, \infty) \to L^2(\Omega) \text{ measurable}; u(\cdot, t) \in U, \text{ a.e. in } t \}$. We first establish a certain null controllability for the heat equation in $\Omega \times [0, T]$, with controls restricted to a product set of an open nonempty subset in $\Omega$ and a subset of positive measure in the interval $[0, T]$. Based on this, we prove that each optimal control $u^*(\cdot, t)$ of the problem satisfies necessarily the bang-bang property: $u^*(\cdot, t) \in \partial U$ for almost all $t \in [0, T^*]$, where $\partial U$ denotes the boundary of the set $U$ and $T^*$ is the optimal time. We also obtain the uniqueness of the optimal control when the target set $S$ is convex and the control set $U$ is a closed ball.

Key words. Bang-bang principle, time optimal control, null-controllability, heat equation.

AMS subject classification. 93C35, 93C05.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n, n \geq 1$, with a $C^\infty$-smooth boundary. Let $\omega$ be an open subset of $\Omega$. Denote by $\chi_\omega$ the characteristic function of $\omega$. Consider the following

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controlled heat equation:

\[
\begin{aligned}
  y_t(x, t) - \Delta y(x, t) &= \chi_\omega(x)u(x, t) \quad \text{in } \Omega \times (0, \infty), \\
  y(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
  y(x, 0) &= y_0(x) \quad \text{in } \Omega,
\end{aligned}
\]  

(1.1)

where \( y_0(\cdot) \) is a function in \( L^2(\Omega) \) and \( u(x, t) \) is a control function taken from the set of functions as follows:

\[
U_{ad} = \{ v : [0, \infty) \to L^2(\Omega) \text{ measurable; } v(\cdot, t) \in U \text{ for almost all } t \geq 0 \}.
\]  

(1.2)

Here \( U \) is a closed, bounded and nonempty subset in \( L^2(\Omega) \). Notice that the control function \( u \) is acted internally (or locally) into the equation (1.1). If \( \omega = \Omega \), we say that the control is acted globally into the equation. We shall denote by \( y(x, t; u, y_0) \) or \( y(x, t) \) the solution of the equation (1.1) if there is no risk of causing confusion.

In this paper, we shall study the following time optimal control problem:

(P) \( \inf \{ \tilde{t}; \ y(\cdot, \tilde{t}; u, y_0) \in S, \ u \in U_{ad} \} \).

Where \( S \) is a nonempty subset in \( L^2(\Omega) \). We call the set \( S \) as the target set, the set \( U \) as the control set, the set \( U_{ad} \) as the control function set and \( y_0 \) as the initial state for the problem (P). For simplicity, we shall call a control function as a control. The number

\[
T^* \equiv \inf \{ \tilde{t}; \ y(x, \tilde{t}; u, y_0) \in S, \ u \in U_{ad} \}
\]

is called the optimal time for the problem (P), a control \( u^* \in U_{ad} \) having the property:

\[
y(x, T^*; u^*, y_0) \in S,
\]

is called an optimal control (or a time optimal control) for the problem (P), and a control \( u \in U_{ad} \) having the property:

\[
y(x, T; u, y_0) \in S \quad \text{for a certain positive number } T,
\]

is called an admissible control for the problem (P).

In this paper, we obtain that each optimal control \( u^* \) for the problem (P) satisfies the bang-bang property: \( u^*(\cdot, t) \in \partial U \) for almost all \( t \in [0, T^*] \). We further show that if the control set \( U \) is a closed ball \( B(0, R) \), centered at the origin of \( L^2(\Omega) \) and of positive radius \( R \), then each optimal control \( u^* \) for the problem (P) satisfies the property:

\[
\| \chi_\omega u^*(\cdot, t) \|_{L^2(\Omega)} = R \quad \text{for almost all } t \in [0, T^*].
\]

We also prove the uniqueness of the optimal control for the problem (P), when the target set \( S \) is convex and nonempty and the control set \( U \) is a closed ball. Combining these with the existence result of time optimal controls obtained in [17], (See also [14].) we derive that if the target set \( S \) is a closed, convex and nonempty subset, which contains the origin of \( L^2(\Omega) \), and if the
control set $U$ is the ball $B(0,R)$, then the problem \((P)\) has a unique optimal control $u^*$ satisfying the bang-bang property: $\|\chi_{\omega}u^*(\cdot,t)\|_{L^2(\Omega)} = R$ for almost all $t \in [0,T^*]$.

The bang-bang principle above can be explained physically as follows: If an outside force $u^*$, acted in an open subset $\omega$ of $\Omega$ and with the maximum norm bound: $\|u^*(\cdot,t)\|_{L^2(\Omega)} \leq R$ for almost all $t$, makes the temperature distribution in $\Omega$ change from an initial distribution $y_0(x)$ into the target set $S$ in the shortest time $T^*$, then $u^*$ takes necessarily the maximum norm for almost all $t$ in $[0,T^*]$, namely, $\|\chi_{\omega}u^*(\cdot,t)\|_{L^2(\Omega)} = R$ for almost all $t \in [0,T^*]$. This bang-bang principle is a weaker form if it is compared with the following stronger form: If $u^*$ is an optimal control of the problem \((P)\) where the control function set is

$$\{u(x,t) \in L^\infty(\Omega \times [0,\infty)); \ |u(x,t)| \leq R \text{ for almost all } (x,t)\},$$

then $|u^*(x,t)| = R$ for almost all $(x,t) \in \Omega \times [0,T^*]$.

In this work, we observe that the bang-bang principle for the problem \((P)\) is based on the following null controllability property for the heat equation:

\textbf{(C)} Let $T$ be a positive number and let $E$ be a subset of positive measure in the interval $[0,T]$. For each $\delta \geq 0$, we write $E_\delta$ for the set $\{t \in \mathbb{R}^1; t + \delta \in E\}$ and denote by $\chi_{E_\delta}$ the characteristic function of the set $E_\delta$. Then there exists a number $\delta_0$ with $0 < \delta_0 < T$ such that for each $\delta$ with $0 \leq \delta \leq \delta_0$ and for each element $y_0$ in $L^2(\Omega)$, there is a control $u_\delta$ in the space $L^\infty(0,T-\delta;L^2(\Omega))$ such that the solution $z^\delta$ to the following controlled heat equation:

$$\begin{cases}
z^\delta(x,t) - \Delta z^\delta(x,t) = \chi_{E_\delta}(t)\chi_\omega(x)u_\delta(x,t) & \text{in } \Omega \times (0,T-\delta), \\
z^\delta(x,t) = 0 & \text{on } \partial\Omega \times (0,T-\delta), \\
z^\delta(x,0) = y_0(x) & \text{in } \Omega,
\end{cases}$$

satisfies $z^\delta(x,T-\delta) = 0$ over $\Omega$. Moreover, the control $u_\delta$ satisfies the following estimate:

$$\|u_\delta\|^2_{L^\infty(0,T-\delta;L^2(\Omega))} \leq L\|y_0\|^2_{L^2(\Omega)},$$

where $L$ is a positive number independent of $\delta$ and $y_0$.

It is well known that the null controllability \((C)\) is equivalent to the following observability inequality:

\textbf{(O)} There exist positive numbers $L$ and $\delta_0$ with $\delta_0 < T$ such that

$$\left[\int_{\Omega} (p^\delta(x,0))^2 dx\right]^{\frac{1}{2}} \leq L \int_0^{T-\delta} \left\{ \int_{\Omega} [\chi_{E_\delta}(t)\chi_\omega(x)p^\delta(x,t)]^2 dx \right\}^{\frac{1}{2}} dt$$

for each number $\delta$ with $0 \leq \delta \leq \delta_0$ and each function $p^\delta_T(x) \in L^2(\Omega)$. Where $p^\delta(x,t)$ is the solution to the following adjoint equation:

$$\begin{cases}
p^\delta_t(x,t) + \Delta p^\delta(x,t) = 0 & \text{in } \Omega \times (0,T-\delta), \\
p^\delta(x,t) = 0 & \text{on } \partial\Omega \times (0,T-\delta), \\
p^\delta(x,T-\delta) = p^\delta_T(x) & \text{in } \Omega.
\end{cases}$$
However the inequality (O) is not a trivial consequence of the Carleman inequality for linear parabolic equation given in [6]. We establish the property (C) by applying an iterative argument stimulated by that in [7]. (See also [8] and [12].) Our iterative argument is based on a sharp observability estimate on the eigenfunctions of the Laplacian, due to G. Lebeau and E. Zuazua in [8] (See also [7].) and a special result in the measure theory given in [13].

It should be mentioned that the problem (P) may have no admissible control in many cases. For instance, if the target set $S$ is a closed ball $B(y_1, R)$ in $L^2(\Omega)$, centered at $y_1$ and of positive radius $R$ and if the control set $U$ is the closed ball $B(0, 1)$ in $L^2(\Omega)$, centered at the origin and of radius 1, then a necessary condition for the existence of an admissible control for the problem (P) is as follows: (See [14].)

$$\|y_1\|_{L^2(\Omega)} \leq (1 + \frac{1}{\lambda_1})(\|y_0\|_{L^2(\Omega)} + 1) + R,$$

where $\lambda_1$ is the first eigenvalue of the Laplacian. However, it was proved in [17] (See also [14].) that when the target set $S$ is the origin of $L^2(\Omega)$ and the control set is the ball $B(0, R)$ with $R > 0$, then the problem (P) has at least one time optimal control. From this, it follows that if the target set $S$ is a closed and convex subset, which contains the origin of $L^2(\Omega)$, and if the control set $U$ is the ball $B(0, R)$ with $R > 0$, then the problem (P) has at least one optimal control.

The time optimal control problems for parabolic equations have been extensively studied in the past years. Here, we mention the works [14], [17], [18] and [19], where the existence of time optimal controls for linear and some semi-linear parabolic equations was investigated. We mention the works [10] and [20], where both the existence and the maximum principle of time optimal controls governed by certain parabolic equations were studied. We mention the works [1] and [9], where the maximum principle for time optimal controls was derived. We mention the works [3], [4], [5] and [11], where the bang-bang principle (in the weaker form) for time optimal controls governed by linear parabolic and hyperbolic equations with the controls acted in the whole domain $\Omega$ or the whole boundary $\partial\Omega$ was established. We mention the work [16], where the bang-bang principle (in the stronger form) of time optimal controls for the heat equation where the control is restricted in the whole boundary was obtained. We mention the work [13], where the bang-bang principle (in the stronger form) for time optimal controls of the one-dimensional heat equation where the control is restricted in one ending point of the one-dimensional state space, was derived. Moreover, the authors in [13] observed that such a bang-bang principle is based on a certain exactly boundary null-controllability for the one-dimensional heat equation from arbitrary sets of positive measure in the time variable space. We also mention a more recent work [18], where the bang-bang principle (in the weaker form) of time optimal internal controls governed by the heat equation and with a ball centered at $0 \in L^2(\Omega)$ and of a positive radius as the target was obtained.
Moreover, in [18], the bang-bang principle was obtained by a certain unique continuation property for the heat equation involving a measurable set, and the maximum principle for the optimal controls.

This paper is organized as follows. In Section 2, we establish the null controllability (C). In Section 3, we give and prove the main results of the paper, namely, the bang-bang principle and the uniqueness of the optimal control for the problem (P).

## 2 The null controllability (C)

Let $T$ be a positive number and $E$ be a subset of positive measure in the interval $[0, T]$. We denote by $m(E)$ the Lebesgue measure of the set $E$ in $\mathbb{R}^1$. For each $\delta \geq 0$, we write $E_\delta$ for the set $\{t \in \mathbb{R}^1; t + \delta \in E\}$ and denote by $\chi_{E_\delta}$ the characteristic function of the set $E_\delta$. In what follows, we shall omit $(x, t)$ (or $t$) in functions of $(x, t)$ (or functions of $t$), if there is no risk of causing confusion. For each positive number $\delta$, we consider the following controlled equation:

\[
\begin{cases}
y_t(x, t) - \Delta y(x, t) = \chi_{E_\delta}(t) \chi_\omega(x) u(x, t) & \text{in } \Omega \times (0, T - \delta), \\
y(x, t) = 0 & \text{on } \partial \Omega \times (0, T - \delta), \\
y(x, 0) = y_0(x) & \text{in } \Omega,
\end{cases}
\]

where $y_0 \in L^2(\Omega)$ is a given function. The main result of this section is as follows:

**Theorem 2.1.** Let $T$ be a positive number and let $E$ be a subset of positive measure in the interval $[0, T]$. Then there exists a positive number $\delta_0$ with $\delta_0 < T$ such that for each number $\delta$ with $0 \leq \delta \leq \delta_0$ and for each element $y_0$ in the space $L^2(\Omega)$, there is a control $u_\delta$ in the space $L^\infty(0, T - \delta; L^2(\Omega))$ with the estimate

\[
\|u_\delta\|^2_{L^\infty(0, T - \delta; L^2(\Omega))} \leq L \|y_0\|^2_{L^2(\Omega)}
\]

for a certain positive constant $L$ independent of $\delta$ and $y_0$, such that the solution $y_\delta(x, t)$ to the equation (2.1) with $u$ being replaced by $u_\delta$ reaches zero value at time $T - \delta$, namely, $y_\delta(x, T - \delta) = 0$ over $\Omega$.

The proof of Theorem 2.1 is based on a sharp estimate on the eigenfunctions of the Laplacian due to G. Lebeau and E. Zuazua (See [8].) and a fundamental result in the measure theory, which will be given in the later. Let $\{\lambda_i\}_{i=1}^\infty$, $0 < \lambda_1 < \lambda_2 \leq \cdots$, be the eigenvalues of $-\Delta$ with the Dirichlet boundary condition and $\{X_i(x)\}_{i=1}^\infty$ be the corresponding eigenfunctions, which serve as an orthonormal basis of $L^2(\Omega)$. Then we have the following result. (See [8].)

**Theorem 2.2.** There exist two positive constants $C_1$, $C_2 > 0$ such that

\[
\sum_{\lambda_i \leq r} |a_i|^2 \leq C_1 e^{C_2 \sqrt{r}} \int_\omega \left| \sum_{\lambda_i \leq r} a_i X_i(x) \right|^2 dx
\]
for every finite $r > 0$ and every choice of the coefficients $\{a_i\}_{\lambda_i \leq r}$, with $a_i \in \mathbb{R}^1$.

Now, we shall first use Theorem 2.2 to derive a certain controllability result, which will help us in the proof of Theorem 2.1. For each $r > 0$, we set $X_r = \text{span} \{X_i(x)\}_{\lambda_i \leq r}$, and consider the following dual equation:

$$
\begin{cases}
\varphi_t(x,t) + \Delta \varphi(x,t) = 0 & \text{in } \Omega \times (0,T), \\
\varphi(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\
\varphi(x,T) \in X_r.
\end{cases} \tag{2.2}
$$

Here, each element $\varphi(x,T)$ in $X_r$ can be written as

$$
\varphi(x,T) = \sum_{\lambda_i \leq r} a_i X_i(x),
$$

for a certain sequence of real numbers $\{a_i\}_{\lambda_i \leq r}$. Then the solution $\varphi(x,t)$ to the equation (2.2) can be expressed by

$$
\varphi(x,t) = \sum_{\lambda_i \leq r} a_i e^{-\lambda_i(T-t)} X_i(x) \text{ for all } t \in [0,T].
$$

Set $b_i(t) = a_i e^{-\lambda_i(T-t)}$, $t \in [0,T]$. Then by Theorem 2.2, we have

$$
\sum_{\lambda_i \leq r} \int_{\omega} |b_i(t)|^2 \leq C_1 e^{C_2 \sqrt{r}} \int_{\omega} \sum_{\lambda_i \leq r} b_i(t)X_i(x) |^2 dx
$$

$$
= C_1 e^{C_2 \sqrt{r}} \int_{\omega} |\varphi(x,t)|^2 dx \text{ for all } t \in [0,T].
$$

On the other hand,

$$
\sum_{\lambda_i \leq r} |b_i(t)|^2 = \sum_{\lambda_i \leq r} a_i^2 e^{-2\lambda_i(T-t)} \geq \sum_{\lambda_i \leq r} a_i^2 e^{-2\lambda_i T}
$$

$$
= \int_{\Omega} \varphi^2(x,0) dx \text{ for all } t \in [0,T].
$$

Hence,

$$
\int_{\Omega} \varphi^2(x,0) dx \leq C_1 e^{C_2 \sqrt{r}} \int_{\omega} |\varphi(x,t)|^2 dx \text{ for all } t \in [0,T],
$$

or equivalently,

$$
\left[ \int_{\Omega} \varphi^2(x,0) dx \right]^2 \leq (C_1 e^{C_2 \sqrt{r}})^2 \left[ \int_{\omega} |\varphi(x,t)|^2 dx \right]^2 \text{ for all } t \in [0,T],
$$

from which, it follows that

$$
\int_E \left[ \int_{\Omega} \varphi^2(x,0) dx \right]^{\frac{3}{2}} dt \leq (C_1 e^{C_2 \sqrt{r}})^{\frac{3}{2}} \int_E \left[ \int_{\omega} |\varphi(x,t)|^2 dx \right]^{\frac{3}{2}} dt.
$$
Namely, we obtained that for each \( \varphi(\cdot, T) \in \mathbf{X}_r \),

\[
\int_{\Omega} \varphi^2(x,0)dx \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \left\{ \int_0^T \left[ \int_{\Omega} |\chi_E(t)\chi_\omega(x)\varphi(x,t)|^2dx \right]^{1/2}dt \right\}^2
\]

\[
= \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \|\chi_E\chi_\omega \varphi\|_{L^2(0,T;L^2(\Omega))}^2. \tag{2.3}
\]

Write \( P_r \) for the orthogonal projection from \( L^2(\Omega) \) to \( \mathbf{X}_r \). We next use (2.3) to obtain the following controllability result.

**Lemma 2.3.** For each \( r > 0 \), there exists a control \( u_r \) in the space \( L^\infty(0,T;L^2(\Omega)) \) with the estimate

\[
\|u_r\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2}\|y_0\|_{L^2(\Omega)}^2, \tag{2.4}
\]

such that \( P_r(y(\cdot, T)) = 0 \), where \( y(x,t) \) is the solution of the equation (2.1) with \( \delta = 0 \) and \( u = u_r \), and where \( C_1 \) and \( C_2 \) are the positive constants given in Theorem 2.2.

**Proof:** Let \( y(x,t) \) be the solution of the equation (2.1) with \( \delta = 0 \) and let \( \varphi(x,t) \) be a solution of the equation (2.2). Then

\[
<y(\cdot, T), \varphi(\cdot, T)> - <y_0(\cdot), \varphi(\cdot, 0)> = \int_0^T \int_{\Omega} \chi_E(t)\chi_\omega(x)u_r(x,t)\varphi(x,t)dxdt.
\]

Here and in what follows, \( <\cdot, \cdot> \) denotes the inner product in \( L^2(\Omega) \). If we can show that \( <y(\cdot, T), \varphi(\cdot, T)>, 0 \) for all \( \varphi(x, T) \in \mathbf{X}_r \), then \( P_r(y(\cdot, T)) = 0 \). Thus, it suffices to prove that there exists a control \( u_r \in L^\infty(0,T;L^2(\Omega)) \) with the estimate (2.4) such that

\[
-y(\cdot, T), \varphi(\cdot, 0)> = \int_0^T \int_{\Omega} \chi_E(t)\chi_\omega(x)u_r(x,t)\varphi(x,t)dxdt \quad \text{for all } \varphi(\cdot, T) \in \mathbf{X}_r.
\]

Now, we set

\[
\mathbf{Y}_r = \{\chi_E(t)\chi_\omega(x)\varphi(x,t); \varphi(x,t) \text{ is the solution to the equation (2.2) with } \varphi(\cdot, T) \in \mathbf{X}_r\}.
\]

It is clear that \( \mathbf{Y}_r \) is a linear subspace of \( L^1(0,T;L^2(\Omega)) \). We define a linear functional \( F_r : \mathbf{Y}_r \to \mathbf{R}^1 \) by \( F_r(\chi_E\chi_\omega \varphi) = -<y_0(\cdot), \varphi(\cdot, 0)> \). By the inequality (2.3), we see that

\[
|F_r(\chi_E\chi_\omega \varphi)|^2 \leq \|y_0\|_{L^2(\Omega)}^2 \cdot \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \|y_0\|_{L^2(\Omega)}^2 \cdot \|\chi_E\chi_\omega \varphi\|_{L^1(0,T;L^2(\Omega))}^2.
\]

Namely,

\[
\|F_r\|^2 \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \|y_0\|_{L^2(\Omega)}^2,
\]

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where $\|F_r\|$ denotes the operator norm of $F_r$. Thus, $F_r$ is a bounded linear functional on $Y_r$. By the Hahn-Banach Theorem, there is a bounded linear functional

$$G_r : L^1(0, T; L^2(\Omega)) \to \mathbb{R}^1$$

such that

$$G_r = F_r \text{ on } Y_r,$$

and such that

$$\|G_r\|^2 = \|F_r\|^2 \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \|y_0\|^2_{L^2(\Omega)}.$$

Then, by making use of the Riesz Representation Theorem in [2], (See p.61, [2].) there exists a function $u_r$ in the space $L^\infty(0, T; L^2(\Omega))$ such that

$$G_r(f) = \int_0^T \int_\Omega f u_r dx dt \quad \text{for all } f \in L^1(0, T; L^2(\Omega)),$$

and such that

$$\|u_r\|^2_{L^\infty(0, T; L^2(\Omega))} = \|G_r\|^2 \leq \frac{C_1 e^{C_2 \sqrt{T}}}{(m(E))^2} \|y_0\|^2_{L^2(\Omega)}.$$

In particular,

$$F_r(\chi_E \chi_\omega \varphi) = \int_0^T \int_\Omega \chi_E \chi_\omega \varphi u_r dx dt \quad \text{for all } \chi_E \chi_\omega \varphi \in Y_r.$$

Namely,

$$- <y_0(\cdot), \varphi(\cdot, 0) >= \int_0^T \int_\Omega \chi_E \chi_\omega \varphi u_r dx dt \quad \text{for all } \varphi(\cdot, T) \in X_r.$$

This completes the proof.\

The following lemma from the measure theory will be used in our later discussion, whose proof can be found in [11]. (See p. 256-257, [11].)

**Lemma 2.4.** For almost all $\bar{t}$ in the set $E$, there exists a sequence of numbers $\{t_i\}_{i=1}^\infty$ in the interval $[0, T]$ such that

$$t_1 < \cdots < t_i < t_{i+1} < \cdots < \bar{t}, \quad t_i \to \bar{t} \text{ as } i \to \infty, \quad (2.5)$$

$$m(E \cap [t_i, t_{i+1}]) \geq \rho(t_{i+1} - t_i), \quad i = 1, 2, \cdots, \quad (2.6)$$

and

$$\frac{t_{i+1} - t_i}{t_{i+2} - t_{i+1}} \leq C_0, \quad i = 1, 2, \cdots, \quad (2.7)$$

where $\rho$ and $C_0$ are two positive constants.
Now we are going to prove Theorem 2.1. Before proceeding the proof, we introduce briefly our main strategy. By applying Lemma 2.4, there exist a number \( \tilde{t} \) and a sequence \( \{t_N\}_{N=1}^{\infty} \) in the interval \((0, T)\) such that (2.5)-(2.7) hold. The main part of the proof is to show that for each \( \tilde{y}_0 \) in \( L^2(\Omega) \), there exists a control \( \tilde{u} \) in the space \( L^\infty(t_1, \tilde{t}; L^2(\Omega)) \) with the estimate \( \|\tilde{u}\|^2_{L^\infty(t_1, \tilde{t}; L^2(\Omega))} \leq L\|\tilde{y}_0\|^2_{L^2(\Omega)} \) for a certain positive constant \( L \) independent of \( \tilde{y}_0 \), such that the solution \( \tilde{y}(x, t) \) to the equation:

\[
\begin{align*}
\tilde{y}_t(x, t) - \Delta \tilde{y}(x, t) &= \lambda(x) \chi(x) \tilde{u}(x, t) & \text{in } \Omega, (t_1, \tilde{t}), \\
\tilde{y}(x, t) &= 0 & \text{on } \partial \Omega, (t_1, \tilde{t}), \\
\tilde{y}(x, t_1) &= \tilde{y}_0(x) & \text{in } \Omega,
\end{align*}
\]

has zero value at time \( \tilde{t} \), namely, \( \tilde{y}(x, \tilde{t}) = 0 \) over \( \Omega \). To this end, we write

\[
[t_1, \tilde{t}] = \bigcup_{N=1}^{\infty} (I_N \cup J_N),
\]

where \( I_N = [t_{2N-1}, t_{2N}] \) and \( J_N = [t_{2N}, t_{2N+1}] \), \( N = 1, 2, \ldots \). Then we choose a suitable sequence of positive numbers \( \{r_N\}_{N=1}^{\infty} \) having the following properties:

(a) \( r_1 < r_2 < \cdots < r_N < \cdots \),

(b) \( r_N \to \infty \) as \( N \to \infty \).

On the subinterval \( I_N \), we control the heat equation with a control \( u_N \) restricted on the subdomain \( \omega \times (I_N \cap E) \) such that \( P_{r_N}(\tilde{y}_N(\cdot, t_{2N})) = 0 \), where \( P_{r_N} \) denotes the orthogonal projection from \( L^2(\Omega) \) onto span \( \{X_i(x)\}_{i=1}^{r_N} \). On the subinterval \( J_N \), we let the heat equation to evolve freely. We start with the initial data for the equation on \( I_1 \) to be \( y_0 \). For the initial data on \( I_N, N = 2, 3, \ldots \), we define it to be the ending value of the solution for the equation on \( J_{N-1} \). The initial data of the equation on \( J_N, N = 1, 2, \ldots \), is given by the ending value of the solution for the equation on \( I_N \). Moreover, by making use of Lemma 2.3 and Lemma 2.4, we will show that there is a sequence \( \{r_N\}_{N=1}^{\infty} \), having the properties (a) and (b) as above, such that the \( L^\infty(I_N; L^2(\Omega)) \)-norm of the control \( u_N \) is bounded by \( L^2(\Omega) \|\tilde{y}_0\|_{L^2(\Omega)} \) for a certain positive constant \( L \) independent of \( N \) and \( \tilde{y}_0 \). Then, we construct a control \( \tilde{u} \) by setting

\[
\tilde{u}(x, t) = \begin{cases} 
  u_N(x, t), & x \in \Omega, \ t \in I_N, \ N = 1, 2, \ldots, \\
  0, & x \in \Omega, \ t \in J_N, \ N = 1, 2, \ldots.
\end{cases}
\]

We can show that this control \( \tilde{u} \) makes the corresponding trajectory \( \tilde{y} \) of the equation (2.8) have zero value at time \( \tilde{t} \).

Now, we set

\[
u(x, t) = \begin{cases} 
  \tilde{u}(x, t), & x \in \Omega \times (t_1, \tilde{t}), \\
  0, & x \in \Omega \times ((0, T) \setminus (t_1, \tilde{t}))
\end{cases}
\]

and take \( \tilde{y}_0 \) to be \( \psi(x, t_1) \), where \( \psi(x, t) \) is the solution of the heat equation on \( \Omega \times (0, t_1) \) with the initial data \( y_0 \). Then it is clear that this control \( u \) makes the trajectory \( y(x, t) \)
of the equation (2.1) with \( \delta = 0 \) have zero value at time \( T \). Moreover, \( \| u \|^2_{L^\infty(0,T;L^2(\Omega))} \leq L\| y_0 \|^2_{L^2(\Omega)} \).

We next replace the sequence \( \{ t_N \}_{N=1}^\infty \) and the number \( \tilde{t} \) by the sequence \( \{ t_N - \delta \}_{N=1}^\infty \) and the number \( (\tilde{t} - \delta) \) respectively, where the number \( \delta \) is such that \( 0 \leq \delta \leq t_1 \). Then by making use of the same argument as above, we obtain that for each number \( \tilde{t} \) with \( 0 \leq \delta \leq t_1 \), there exists a control \( u_\delta \) in the space \( L^\infty(0,T-\delta;L^2(\Omega)) \) with the estimate \( \| u_\delta \|^2_{L^\infty(0,T-\delta;L^2(\Omega))} \leq L_\delta \| y_0 \|^2_{L^2(\Omega)} \) for a certain positive number \( L_\delta \) independent of \( y_0 \), such that the corresponding solution \( y_\delta \) to the equation (2.1) reaches zero value at time \( T-\delta \), namely, \( y_\delta(x,T-\delta) = 0 \) over \( \Omega \). We finally prove that \( L_\delta = L \) is independent of \( \delta \).

Now we turn to prove Theorem 2.1.

**Proof of Theorem 2.1.** Without loss of generality, we can assume that \( C_1 \geq 1 \), where \( C_1 \) is the positive constant given in Theorem 2.2. By making use of Lemma 2.4, we can take a number \( \tilde{t} \) in the set \( E \) with \( \tilde{t} \leq T \) and a sequence \( \{ t_N \}_{N=1}^\infty \) in the open interval \( (0,T) \) such that (2.5)-(2.7) hold for certain positive numbers \( \rho \) and \( C_0 \) and such that

\[
\tilde{t} - t_1 \leq \min\{ \lambda_1, 1 \}.
\]

We shall first prove that for each \( \tilde{y}_0 \) in \( L^2(\Omega) \), there exists a control \( \tilde{u} \) in the space \( L^\infty(t_1, \tilde{t};\Omega) \) with the estimate \( \| \tilde{u} \|^2_{L^\infty(t_1, \tilde{t};\Omega)} \leq L\| \tilde{y}_0 \|^2_{L^2(\Omega)} \) for a certain positive constant \( L \) independent of \( \tilde{y}_0 \), such that the solution \( \tilde{y} \) to the equation (2.8) reaches zero value at time \( \tilde{t} \), namely, \( \tilde{y}(x,\tilde{t}) = 0 \) over \( \Omega \).

To this end, we shall use the strategy presented above. We set \( I_N = [t_{2N-1}, t_{2N}] \) for \( N = 1, 2, \ldots \). Then

\[
[t_1, \tilde{t}) = \bigcup_{N=1}^\infty (I_N \cup J_N).
\]

Notice that for each \( N \geq 1 \), it holds that \( m(E \cap I_N) > 0 \).

Now, on the interval \( I_1 \equiv [t_1, t_2] \), we consider the following controlled heat equation:

\[
\begin{cases}
y_1'(x,t) - \Delta y_1(x,t) = \chi_E(t)\chi_\omega(x)u_1(x,t) \quad &\text{in } \Omega \times (t_1, t_2), \\
y_1(x,t) = 0 \quad &\text{on } \partial\Omega \times (t_1, t_2), \\
y_1(x,t_1) = \tilde{y}_0(x) \quad &\text{in } \Omega.
\end{cases}
\]

By Lemma 2.3, for any \( r_1 > 0 \), there exists a control \( u_1 \) in the space \( L^\infty(t_1, t_2;L^2(\Omega)) \) with the estimate:

\[
\| u_1 \|^2_{L^\infty(t_1, t_2;L^2(\Omega))} \leq \frac{C_1 e^{C_2 \sqrt{r_1}}}{(m(E \cap [t_1, t_2]))^2} \| \tilde{y}_0 \|^2_{L^2(\Omega)},
\]

such that \( P_{r_1}(y_1(\cdot,t_2)) = 0 \). Then, by (2.6) and (2.7) in Lemma 2.4, we see that

\[
\| u_1 \|^2_{L^\infty(t_1, t_2;L^2(\Omega))} \leq \frac{C_1 e^{C_2 \sqrt{r_1}}}{\rho^2(t_2 - t_1)^2} \| \tilde{y}_0 \|^2_{L^2(\Omega)}
\]

\[
= \frac{C_1}{\rho^2(t_2 - t_1)^2} \cdot \alpha_1 \| \tilde{y}_0 \|^2_{L^2(\Omega)},
\]

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where }\alpha_1 = e^{C_2\sqrt{r_1}}\text{. Moreover, we have }\begin{align*}
\|y_1(\cdot, t_2)\|^2_{L^2(\Omega)} &\leq \|y_1(\cdot, t_1)\|^2_{L^2(\Omega)} + \frac{1}{\lambda_1} \int_{t_1}^{t_2} \|u_1(\cdot, s)\|^2_{L^2(\Omega)} ds \\
&
\leq \|\tilde{y}_0\|^2_{L^2(\Omega)} + \frac{(t_2-t_1)}{\lambda_1} \|u_1\|^2_{L^\infty(t_1,t_2;L^2(\Omega))} \\
&\leq 2 \frac{C_1}{\rho^2(t_2-t_1)^2} \alpha_1 \|\tilde{y}_0\|^2_{L^2(\Omega)}.
\end{align*}

Here we have used the facts that }\langle t_2 - t_1 \rangle \leq \min(\lambda_1, 1), \rho < 1 \text{ and } C_1 > 1.

On the interval } J_1 \equiv [t_2, t_3], \text{ we consider the following heat equation without control:

\begin{align*}
&\begin{cases}
    z_1'(x, t) - \Delta z_1(x, t) = 0 & \text{in } \Omega \times (t_2, t_3), \\
    z_1(x, t) = 0 & \text{on } \partial \Omega \times (t_2, t_3), \\
    z_1(x, t_2) = y_1(x, t_2) & \text{in } \Omega.
\end{cases}
\end{align*}

Since } P_{r_1}(y_1(\cdot, t_2)) = 0, \text{ we have }
\begin{align*}
\|z_1(\cdot, t_3)\|^2_{L^2(\Omega)} &\leq \exp (-2r_1(t_3 - t_2)) \cdot \|y_1(\cdot, t_2)\|^2_{L^2(\Omega)} \\
&\leq 2 \frac{C_1}{\rho^2(t_2-t_1)^2} \alpha_1 \cdot \exp (-2r_1(t_3 - t_2)) \cdot \|\tilde{y}_0\|^2_{L^2(\Omega)}.
\end{align*}

On the interval } J_2 \equiv [t_3, t_4], \text{ we consider the controlled heat equation as follows:

\begin{align*}
&\begin{cases}
    y_2'(x, t) - \Delta y_2(x, t) = \chi_E(t)\chi_\omega(x)u_2(x, t) & \text{in } \Omega \times (t_3, t_4), \\
    y_2(x, t) = 0 & \text{on } \partial \Omega \times (t_3, t_4), \\
    y_2(x, t_3) = z_1(x, t_3) & \text{in } \Omega.
\end{cases}
\end{align*}

Then by Lemma 2.3, for any } r_2 > 0, \text{ there exists a control } u_2 \text{ in the space } L^\infty(t_3, t_4; L^2(\Omega)) \text{ with the estimate:

\begin{align*}
\|u_2\|^2_{L^\infty(t_3,t_4;L^2(\Omega))} \leq \frac{C_1 e^{C_2\sqrt{r_2}}}{m(E \cap [t_3, t_4])^2} \cdot \|z_1(\cdot, t_3)\|^2_{L^2(\Omega)},
\end{align*}

such that } P_{r_2}(y_2(\cdot, t_4)) = 0. \text{ By (2.6) and (2.7) in Lemma 2.4, we get }
\begin{align*}
\|u_2\|^2_{L^\infty(t_3,t_4;L^2(\Omega))} \leq 2 \left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4 \cdot \alpha_1 \cdot \alpha_2 \cdot \|\tilde{y}_0\|^2_{L^2(\Omega)}
\end{align*}

where } \alpha_2 = \exp (C_2\sqrt{r_2})\exp (-2r_1(t_3 - t_2)). \text{ Moreover, it holds that }
\begin{align*}
\|y_2(\cdot, t_4)\|^2_{L^2(\Omega)} &\leq \|z_1(\cdot, t_3)\|^2_{L^2(\Omega)} + \frac{1}{\lambda_1} (t_4 - t_3) \|u_2\|^2_{L^\infty(t_3,t_4;L^2(\Omega))} \\
&\leq 2 \left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4 \cdot \alpha_1 \cdot \alpha_2 \cdot \|\tilde{y}_0\|^2_{L^2(\Omega)}.
\end{align*}
On the interval $J_2 \equiv [t_4, t_5]$, we consider the following heat equation without control:

\[
\begin{cases}
    z'(x, t) - \Delta z(x, t) = 0 & \text{in } \Omega \times (t_4, t_5), \\
    z(x, t) = 0 & \text{on } \partial \Omega \times (t_4, t_5), \\
    z(x, t_4) = y_2(x, t_4) & \text{in } \Omega.
\end{cases}
\]

Since $P_{r_2}(y_2(\cdot, t_4)) = 0$, we have

\[
\|z_2(\cdot, t_5)\|_{L^2(\Omega)}^2 \leq \exp(-2r_2(t_5 - t_4))\|y_2(\cdot, t_4)\|_{L^2(\Omega)}^2 \\
\leq 2^{2\left(\frac{C_1}{\rho^2(t_2 - t_1)^2}\right)^2}C^4_0 \cdot C^{4\cdot 2}_0 \cdot \|y_0\|_{L^2(\Omega)} \cdot \exp(-2r_2(t_5 - t_4)).
\]

On the interval $I_3 \equiv [t_5, t_6]$, we consider the following controlled heat equation:

\[
\begin{cases}
    y'(x, t) - \Delta y(x, t) = \chi_E(t)\chi_\omega(x)u_3(x, t) & \text{in } \Omega \times (t_5, t_6), \\
    y(x, t) = 0 & \text{on } \partial \Omega \times (t_5, t_6), \\
    y(x, t_5) = z_2(x, t_5) & \text{in } \Omega.
\end{cases}
\]

Then by Lemma 2.3, for any $r_3 > 0$, there exists a control $u_3$ in the space $L^\infty(t_5, t_6; L^2(\Omega))$ with the estimate:

\[
\|u_3\|^2_{L^\infty(t_5, t_6; L^2(\Omega))} \leq \frac{C_1 e^{C_2\sqrt{r_3}}}{(m(E \cap [t_5, t_6]))^2}\|z_2(\cdot, t_5)\|^2_{L^2(\Omega)},
\]

such that $P_{r_3}(y_3(\cdot, t_6)) = 0$. By making use of (2.6) and (2.7) again, we get

\[
\|u_3\|^2_{L^\infty(t_5, t_6; L^2(\Omega))} \leq 2^{2\left(\frac{C_1}{\rho^2(t_2 - t_1)^2}\right)^2}C^4_0 \cdot C^{4\cdot 2}_0 \cdot \|y_0\|_{L^2(\Omega)},
\]

where $\alpha_3 = \exp(C_2\sqrt{r_3}) \exp(-2r_2(t_3 - t_2)C_0^{-2})$.

Generally, on the interval $I_N$, we consider the controlled heat equation:

\[
\begin{cases}
    y'(x, t) - \Delta y(x, t) = \chi_E(t)\chi_\omega(x)u_N(x, t) & \text{in } \Omega \times (t_2N-1, t_2N), \\
    y(x, t) = 0 & \text{on } \partial \Omega \times (t_2N-1, t_2N), \\
    y(x, t_2N-1) = z_{N-1}(x, t_2N-1) & \text{in } \Omega.
\end{cases}
\]

On the interval $J_N$, we consider the following heat equation without control:

\[
\begin{cases}
    z'_N(x, t) - \Delta z_N(x, t) = 0 & \text{in } \Omega \times (t_2N, t_2N+1), \\
    z_N(x, t) = 0 & \text{on } \partial \Omega \times (t_2N, t_2N+1), \\
    z_N(x, t_2N) = y_N(x, t_2N) & \text{in } \Omega.
\end{cases}
\]

Then by making use of induction argument, we can obtain the following: For each $r_N > 0$, there exists a control $u_N$ in the space $L^\infty(I_N; L^2(\Omega))$ with the following estimate:

\[
\|u_N\|^2_{L^\infty(I_N; L^2(\Omega))} \leq 2^{N-1}\left(\frac{C_1}{\rho^2(t_2 - t_1)^2}\right)^NC^4_0 \cdot C^{4\cdot 2}_0 \cdots C^{4(N-1)}_0 \cdot \alpha_1 \cdot \alpha_2 \cdots \alpha_N \cdot \|y_0\|_{L^2(\Omega)},
\]

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where
\[
\alpha_N = \begin{cases} 
\exp \left( C_2 \sqrt{r_1} \right), & N = 1, \\
\exp \left( C_2 \sqrt{r_N} \right) \exp \left( -2r_{N-1}(t_3 - t_2)C_0^{-2(N-2)} \right), & N \geq 2,
\end{cases}
\]

such that \( P_{r_N}(y_N(\cdot, t_{2N})) = 0 \). It is easily seen that for each \( N \geq 1 \),
\[
\| u_N \|_{L^\infty(I_N; L^2(\Omega))} \leq \left( \tilde{C} \right)^{(N-1)} \alpha_1 \cdots \alpha_N \cdot \| \tilde{y}_0 \|_{L^2(\Omega)},
\]
where
\[
\tilde{C} = \frac{2C_1}{\rho^2(t_2 - t_1)^2} \cdot C_0^2.
\]

Now, we set
\[
r_N = \left[ \frac{2}{(t_3 - t_2)} \tilde{C}^{N-1} \right]^4 \equiv \left[ A \cdot \tilde{C}^{N-1} \right]^4, \quad N \geq 1.
\]

Because we have \( \tilde{C} > C_0^2 > 1 \) and \( t_3 - t_2 < 1 \), it holds that
\[
2^4 < r_1 < r_2 < \cdots < r_N < r_{N+1} < \cdots, \quad \text{and} \quad r_N \to \infty \quad \text{as} \quad N \to \infty.
\]

Moreover, we have
\[
r_{N-1}^{\frac{1}{2}}(t_3 - t_2)C_0^{-2(N-2)} \geq 2 \quad \text{for each} \quad N \geq 2.
\]

Then we get
\[
\exp \left\{ -2r_{N-1}(t_3 - t_2)C_0^{-2(N-2)} \right\} \leq \exp \left( -4r_{N-1}^{\frac{3}{2}} \right) \quad \text{for each} \quad N \geq 2.
\]

Since
\[
\tilde{C}^{N(N-1)} \exp \left( -r_{N-1}^{\frac{3}{2}} \right) = \frac{\tilde{C}^{N(N-1)}}{(\exp(r_{N-1}^{\frac{3}{2}}))^{r_{N-1}^{\frac{3}{2}}}} \leq \frac{\tilde{C}^{N(N-1)}}{(\exp(2\tilde{C}^{N-1}))^{r_{N-1}^{\frac{3}{2}}}} \leq \frac{\tilde{C}^{N(N-1)}}{\tilde{C}^{(N-1)2(r_{N-1}^{\frac{3}{2}})}}
\]
for each \( N \geq 2 \), we derive from (2.12) that there exists a natural number \( N_1 \) with \( N_1 \geq 2 \) such that for each \( N \geq N_1 \),
\[
\tilde{C}^{N(N-1)} \exp \left( -r_{N-1}^{\frac{3}{2}} \right) \leq 1.
\]

By making use of (2.12) again, we obtain that for each \( N \geq 2 \),
\[
\exp \left( C_2 \sqrt{r_N} \right) \exp \left( -r_{N-1}^{\frac{3}{2}} \right) = \exp \left( C_2 A^2 \tilde{C}^{2(N-1)} \right) \exp \left( -A^3 \tilde{C}^{3(N-2)} \right).
\]

Thus, there exists a natural number \( N_2 \) with \( N_2 \geq 2 \) such that for each \( N \geq N_2 \),
\[
\exp \left( C_2 \sqrt{r_N} \right) \exp \left( -r_{N-1}^{\frac{3}{2}} \right) \leq 1.
\]
Now we set
\[ N_0 = \max \{ N_1, N_2 \}. \] (2.16)
Then by (2.13), (2.14) and (2.15), we see that for all \( N \geq N_0, \)
\[
\tilde{C}^{(N-1)} \alpha_N
\]
\[
= \tilde{C}^{(N-1)} \exp (C_2 \sqrt{r_N}) \exp (-2r_{N-1} (t_3 - t_2) C_0^{-2(N-2)})
\]
\[
\leq \tilde{C}^{(N-1)} \exp (C_2 \sqrt{r_N}) \exp (-4r_{N-1} \frac{t_3}{t_2})
\]
\[
\leq \exp (-2r_{N-1} \frac{t_3}{t_2}).
\] (2.17)
Moreover, it is obvious that
\[ \alpha_N \leq 1 \text{ for all } N \geq N_0. \] (2.18)
Now, we set
\[ L = \max \{ (\tilde{C})^{(N-1)} \alpha_1 \cdots \alpha_N, \ 1 \leq N \leq N_0 \}. \] (2.19)
It follows from (2.10), (2.17), (2.18) and (2.19) that for all \( N \geq 1, \)
\[
\|u_N\|_{L^\infty(I_N;L^2(\Omega))}^2 \leq L \|y_0\|_{L^2(\Omega)}^2.
\] (2.20)
Then we construct a control \( \tilde{u} \) by setting
\[
\tilde{u}(x, t) = \left\{ \begin{array}{ll}
u_N(x, t), & x \in \Omega, \ t \in I_N, \ N \geq 1, \\0, & x \in \Omega, \ t \in J_N, \ N \geq 1, \end{array} \right.
\] (2.21)
from which and by (2.20), we easily see that the control \( \tilde{u} \) is in the space \( L^\infty(t_1, \tilde{t}; L^2(\Omega)) \)
and satisfies the estimate:
\[
\|\tilde{u}\|_{L^\infty(t_1, \tilde{t}; L^2(\Omega))}^2 \leq L \|y_0\|_{L^2(\Omega)}^2.
\]
Let \( \tilde{y} \) be the solution of the equation (2.8) corresponding to the control \( \tilde{u} \) constructed in (2.21). Then on the interval \( I_N, \ \tilde{y}(\cdot, t) = y_N(\cdot, t). \) Since \( P_{r_N}(y_N(\cdot, t_{2N})) = 0 \) for all \( N \geq 1 \)
and \( r_1 < r_2 < \cdots < r_N < \cdots, \) by making use of (2.21) again, we see that
\[
P_{r_N}(\tilde{y}(\cdot, t_{2M})) = 0 \text{ for all } M \geq N.
\] (2.22)
On the other hand, since \( t_{2M} \to \tilde{t} \) as \( M \to \infty, \) we obtain that
\[
\tilde{y}(\cdot, t_{2M}) \to \tilde{y}(\cdot, \tilde{t}) \text{ strongly in } L^2(\Omega), \ \text{as } M \to \infty.
\]
This, together with (2.22), implies that \( P_{r_N}(\tilde{y}(\cdot, \tilde{t})) = 0 \) for all \( N \geq 1. \) Since \( r_N \to \infty \)
when \( N \to \infty, \) it holds that \( \tilde{y}(\cdot, \tilde{t}) = 0. \) Thus, we have proved that for each \( \tilde{y}_0 \in L^2(\Omega), \)
there exists a control \( \tilde{u} \in L^\infty(t_1, \tilde{t}; L^2(\Omega)) \) with the estimate
\[
\|\tilde{u}\|_{L^\infty(t_1, \tilde{t}; L^2(\Omega))}^2 \leq L \|\tilde{y}_0\|_{L^2(\Omega)}^2,
\]
where the constant \( L \) is given by (2.19), such that the solution \( \tilde{y} \) to the equation (2.8)
reaches zero value at time \( \tilde{t}, \) namely, \( \tilde{y}(x, \tilde{t}) = 0 \) over \( \Omega. \)
Now, we take $\tilde{y}_0(x)$ to be $\psi(x,t_1)$, where $\psi(x,t)$ is the solution to the following equation:

$$
\begin{align*}
\psi_t(x,t) - \Delta \psi(x,t) &= 0 & \text{in } \Omega \times (0, t_1), \\
\psi(x,t) &= 0 & \text{on } \partial \Omega \times (0, t_1), \\
\psi(x,0) &= y_0(x) & \text{in } \Omega
\end{align*}
$$

and construct a control $u$ by setting

$$
u(x,t) = \begin{cases} 
0 & \text{in } \Omega \times (0, t_1), \\
\bar{u}(x,t) & \text{in } \Omega \times (t_1, \bar{t}), \\
0 & \text{in } \Omega \times (\bar{t}, T).
\end{cases}
$$

(2.23)

It is clear that this control $u$ is in the space $L^\infty(0, T; L^2(\Omega))$ and that the corresponding solution $y$ of the equation (2.1) with $\delta = 0$ reaches zero value at time $T$, namely, $y(x,T) = 0$ over $\Omega$. Moreover, the control $u$ constructed in (2.23) satisfies the following estimate:

$$
\|u\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq L \|y_0\|_{L^2(\Omega)}^2,
$$

where $L$ is given by (2.19).

Next, we take $\delta_0$ to be the number $t_1$ given above. For each $\delta$ with $0 \leq \delta \leq \delta_0$, we set

$$
\bar{t}_\delta = \bar{t} - \delta \quad \text{and} \quad t_{N,\delta} = t_N - \delta \quad \text{for all } N = 1, 2, \cdots.
$$

Then it holds that

$$
0 \leq t_{1,\delta} < t_{2,\delta} < \cdots < t_{N,\delta} \to \bar{t}_\delta < T - \delta.
$$

Moreover, we have for each $N \geq 1$,

$$
m(E_{\delta} \cap [t_{N,\delta}, t_{N+1,\delta}]) = m(E \cap [t_N, t_{N+1}]) \geq \rho(t_{N+1} - t_N),
$$

and

$$
\frac{t_{N+1,\delta} - t_{N,\delta}}{t_{N+2,\delta} - t_{N+1,\delta}} = \frac{t_{N+1} - t_N}{t_{N+2} - t_{N+1}} \leq C_0,
$$

where $C_0$ and $\rho$ are the positive constants as above.

Now, we can use exactly the same argument as above to get for each $\delta$ with $0 \leq \delta \leq \delta_0$, the existence of a control $u_\delta(t)$ in the space $L^\infty(0, T-\delta; L^2(\Omega))$ such that the corresponding solution $y^\delta$ to the equation (2.1) reaches zero value at time $T-\delta$, namely, $y^\delta(x, T-\delta) = 0$ over $\Omega$. Moreover, this control $u_\delta$ satisfies the following estimate: (See (2.9)-(2.12) and (2.19).)

$$
\|u_\delta\|_{L^\infty(0, T-\delta; L^2(\Omega))}^2 \leq L_\delta \cdot \|y_0\|_{L^2(\Omega)}^2.
$$

The constant $L_\delta$ is given by

$$
L_\delta = \max \{(\bar{C}_\delta)^{N(N-1)}\alpha_{1,\delta} \cdots \alpha_{N,\delta}, \ 1 \leq N \leq N_0\},
$$
where
\[ \tilde{C}_\delta = \frac{2C_1}{\rho^2(t_{2,\delta} - t_{1,\delta})^2} \cdot C_0^2 \]
and
\[ \alpha_{N,\delta} = \begin{cases} \exp \left( C_2 \sqrt{t_{1,\delta}} \right), & N = 1, \\ \exp \left( C_2 \sqrt{t_{N,\delta}} \right) \exp \left( -2r_{N,\delta}(t_{3,\delta} - t_{2,\delta})C_0^{-2(N-2)} \right), & N \geq 2, \end{cases} \]
with
\[ r_{N,\delta} = \frac{2}{(t_{3,\delta} - t_{2,\delta})} \tilde{C}_\delta^{N-1}, \quad N = 1, 2, \ldots, \]
and where the natural number \( N_0 \) is given by (2.16). Since
\[ t_{N+1,\delta} - t_{N,\delta} = t_{N+1} - t_N, \quad \text{for all } N = 1, 2, \ldots, \]
we see easily that \( \tilde{C}_\delta = \tilde{C} \) and \( \alpha_{N,\delta} = \alpha_N \) for all \( N \geq 1 \). Then it holds that \( L_\delta = L \) for all \( \delta \) with \( 0 \leq \delta \leq \delta_0 \). This completes the proof.

3 The bang-bang principle for time optimal control

In this section, we shall prove the main result of the paper, namely, each optimal control for the problem (P) satisfies the bang-bang principle in the weaker form. Moreover, we shall show the uniqueness of the optimal control for the problem (P), when the target set \( S \) is convex and the control set is a closed ball. Throughout this section, we shall denote by \( y(t; u, y_0) \) the solution of the equation (1.1) corresponding to the control \( u \) and the initial data \( y_0 \), and write \( \{ G(t) \}_{t \geq 0} \) for the semigroup generated by \( \Delta \) with the Dirichlet boundary condition.

**Theorem 3.1.** Suppose that the control set \( U \) is closed, bounded and nonempty in \( L^2(\Omega) \) and the target set \( S \) is nonempty in \( L^2(\Omega) \). Let \( T^* \) be the optimal time and \( u^* \) be an optimal control for the problem (P). Then it holds that \( u^*(t) \in \partial U \) for almost all \( t \in [0, T^*] \). If we further assume that \( \chi_\omega U \subset U \), then it holds that \( \chi_\omega u^*(t) \in \partial U \) for almost all \( t \in [0, T^*] \).

**Proof of Theorem 3.1.** Seeking a contradiction, we suppose that there exist a subset \( E \) of positive measure in the interval \([0, T^*]\) and a positive number \( \varepsilon \) such that the following holds:
\[ u^*(t) \in U \quad \text{and} \quad d(u^*(t), \partial U) \geq \varepsilon \quad \text{for each } t \text{ in the set } E, \]
where \( d(u^*(t), \partial U) \) denotes the distance of the point \( u^*(t) \) to the set \( \partial U \) in \( L^2(\Omega) \). Then we would get
\[ B(u^*(t), \frac{\varepsilon}{2}) \subset U \quad \text{for each } t \text{ in the set } E. \]
We shall obtain from (3.1) that there exist a positive number $\delta$ with $\delta < T^*$ and a control $v_\delta$ in the set $U_{ad}$ such that the following holds:

$$y(T^* - \delta; v_\delta, y_0) = y(T^*; u^*, y_0).$$

(3.2)

Thus, $T^*$ could not be the optimal time for the problem (P), which leads to a contradiction.

We first observe that

$$y(T^*; u^*, y_0) = G(T^*)y_0 + \int_0^{T^*} G(T^* - \sigma)\chi_\omega u^*(\sigma)d\sigma.$$  

$$y(T^* - \delta; v_\delta, y_0) = G(T^* - \delta)y_0 + \int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega v_\delta(\sigma)d\sigma.$$  

Hence, (3.2) is equivalent to the following: There exist a positive number $\delta$ with $\delta < T^*$ and a control $v_\delta$ in the set $U_{ad}$ such that the following holds:

$$\int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega v_\delta(\sigma)d\sigma = [G(T^*) - G(T^* - \delta)]y_0 + \int_0^{T^*} G(T^* - \sigma)\chi_\omega u^*(\sigma)d\sigma.$$  

(3.3)

Notice that for any positive number $\delta$ with $\delta < T^*$, we have

$$\int_0^{T^*} G(T^* - \sigma)\chi_\omega u^*(\sigma)d\sigma$$

$$= \int_0^{\delta} G(T^* - \sigma)\chi_\omega u^*(\sigma)d\sigma + \int_{\delta}^{T^*} G(T^* - \sigma)\chi_\omega u^*(\sigma)d\sigma$$

$$= G(T^* - \delta)\int_0^{\delta} G(\delta - \sigma)\chi_\omega u^*(\sigma)d\sigma + \int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega u^*(\delta + \sigma)d\sigma$$

and

$$[G(T^*) - G(T^* - \delta)]y_0 = G(T^* - \delta)\{(G(\delta) - I)y_0\}.$$  

Therefore, (3.3) is equivalent to the following: There exist a positive number $\delta$ with $\delta < T^*$ and a control $v_\delta$ in the set $U_{ad}$ such that the following holds:

$$\int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega v_\delta(\sigma)d\sigma$$

$$= G(T^* - \delta)\int_0^{\delta} G(\delta - \sigma)\chi_\omega u^*(\sigma)d\sigma + (G(\delta) - I)y_0$$

$$+ \int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega u^*(\sigma + \delta)d\sigma$$

$$\equiv G(T^* - \delta)h_\delta + \int_0^{T^* - \delta} G(T^* - \delta - \sigma)\chi_\omega u^*(\sigma + \delta)d\sigma,$$

where

$$h_\delta = \int_0^{\delta} G(\delta - \sigma)\chi_\omega u^*(\sigma)d\sigma + (G(\delta) - I)y_0.$$  

(3.4)

(3.5)
For each positive number $\delta$, we write $E_\delta$ for the set \( \{ t; t + \delta \in E \} \) and denote by $\chi_{E_\delta}$ the characteristic function of the set $E_\delta$. We first claim the following: For each positive number $\delta$ sufficiently small, there exists a control $u_\delta$ in the space $L^\infty(0, \infty; L^2(\Omega))$ such that
\[
\|u_\delta(t)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} \text{ for almost all } t \geq 0,
\] (3.6)
and such that
\[
y(T^* - \delta; \chi_{E_\delta} u_\delta, 0) = G(T^* - \delta) h_\delta.
\] (3.7)
Recall that $y(t; \chi_{E_\delta} u_\delta, 0)$ is the solution of the controlled heat equation (1.1) with $u$ and $y_0$ being replaced by $\chi_{E_\delta} u_\delta$ and 0 respectively, and that $\varphi(t) \equiv G(t) h_\delta$ is the solution of the equation (1.1) with $u$ and $y_0$ being replaced by 0 and $h_\delta$ respectively. Then, what we claimed above is obviously equivalent to the following: For each positive number $\delta$ sufficiently small, there exists a control $u_\delta$ with the estimate:
\[
\|u_\delta(t)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} \text{ for almost all } t \geq 0,
\]
such that the following holds:
\[
z^\delta(T^* - \delta) = 0,
\]
where $z^\delta(t)$ is the solution to the following controlled heat equation:
\[
\begin{cases}
z^\delta_t(t) - \Delta z^\delta(t) = \chi_\omega \chi_{E_\delta}(t) u_\delta(t) & \text{in } (0, T^* - \delta), \\
z^\delta(0) = -h_\delta.
\end{cases}
\] (3.8)
However, by Theorem 2.1, there exist positive numbers $\delta_0$ and $L$ such that for each $\delta$ with $0 < \delta \leq \delta_0$, there is a control $u_\delta$ in the space $L^\infty(0, T^* - \delta; L^2(\Omega))$ with the estimate:
\[
\|u_\delta\|^2_{L^\infty(0, T^* - \delta; L^2(\Omega))} \leq L\|h_\delta\|^2_{L^2(\Omega)},
\] (3.9)
such that the following holds:
\[
z^\delta(T^* - \delta) = 0.
\] (3.10)
On the other hand, by (3.5), we can get a positive number $\bar{\delta}$ such that for each positive number $\delta$ with $\delta \leq \bar{\delta}$, the following holds:
\[
\|h_\delta\|^2_{L^2(\Omega)} \leq \left(\frac{\varepsilon}{2}\right)^2 / L.
\]
This, together with (3.9), implies that for each positive number $\delta$ with $\delta \leq \min\{\delta_0, \bar{\delta}\}$, there is a control $u_\delta$ with the estimate:
\[
\|u_\delta\|_{L^\infty(0, T^* - \delta; L^2(\Omega))} \leq \frac{\varepsilon}{2},
\] (3.11)
such that the corresponding solution $z^\delta$ to the equation (3.8) satisfies (3.10).
Next, we fix such a positive number \( \delta \) and the corresponding control \( u_\delta \) that \((3.10)\) and \((3.11)\) hold. Then we extend the control \( u_\delta(\cdot) \) by setting it to be zero on the interval \((T^* - \delta, \infty)\), and still denote the extension by \( u_\delta(\cdot) \). Clearly, this extended control \( u_\delta \) is in the space \( L^\infty(0, \infty; L^2(\Omega)) \) and makes \((3.6)\) and \((3.7)\) hold. Thus, we have proved the above mentioned claim.

Now, we take an element \( u_0 \) from the control set \( U \) and construct a control \( v_\delta \) by setting

\[
v_\delta(t) = \begin{cases} u^*(t + \delta) + \chi_{E_\delta}(t)u_\delta(t), & \text{if } t \in [0, T^* - \delta], \\ u_0, & \text{if } t > T^* - \delta. \end{cases}
\]

It is clear that \( v_\delta(\cdot) : [0, \infty) \to L^2(\Omega) \) is measurable. We shall prove \( v_\delta(t) \in U \) for almost all \( t \geq 0 \). Here is the argument: When \( t \) is in the set \([0, T^* - \delta] \cap E_\delta \), we have \( t + \delta \in E \). Then by \((3.1)\), we get \( B(u^*(t + \delta), \frac{\delta}{2}) \in U \). Since \( \|u_\delta(t)\|_{L^2(\Omega)} \leq \frac{\delta}{2} \) for almost all \( t \geq 0 \), we have

\[
\|v_\delta(t) - u^*(t + \delta)\|_{L^2(\Omega)} = \|u_\delta(t)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} \quad \text{for almost all } t \in [0, T^* - \delta] \cap E_\delta,
\]

namely, \( v_\delta(t) \in B(u^*(t + \delta), \frac{\delta}{2}) \) for almost all \( t \) in the set \([0, T^* - \delta] \cap E_\delta \). Hence, \( v_\delta(t) \in U \) for almost all \( t \) in the set \([0, T^* - \delta] \cap E_\delta \). On the other hand, for almost all \( t \in [0, T^* - \delta] \cap (E_\delta)^c \), we have \( v_\delta(t) = u^*(t + \delta) \in U \). Therefore, we have proved \( v_\delta \in U_{ad} \).

Then, by \((3.7)\) and \((3.12)\), we see easily that this control \( v_\delta \) makes the equality \((3.4)\) hold, which leads to a contradiction to the optimality of \( T^* \) for the problem \((P)\). Thus we have proved \( u^*(t) \in \partial U \) for almost all \( t \in [0, T^*] \).

Finally, if the control set \( U \) has the additional property: \( \chi_\omega U \subset U \), then we have \( \chi_\omega u^* \in U_{ad} \). It is clear that \( y(T^*; \chi_\omega u^*, y_0) = y(T^*; u^*, y_0) \). Thus, \( \chi_\omega u^* \) is also an optimal control for the problem \((P)\). Hence, it holds that \( \chi_\omega u^*(t) \in \partial U \) for almost all \( t \in [0, T^*] \). This completes the proof. \( \blacksquare \)

By Theorem 3.1, we immediately get the following consequence.

**Corollary 3.2.** Suppose that the control set \( U \) is the ball \( B(0, R) \) with \( R > 0 \) and the target set \( S \) is nonempty in \( L^2(\Omega) \). Let \( T^* \) be the optimal time and \( u^* \) be an optimal control for the problem \((P)\). Then it holds that \( \|\chi_\omega u^*(\cdot, t)\|_{L^2(\Omega)} = R \) for almost all \( t \in [0, T^*] \).

**Remark 3.3.** From the proof of Theorem 3.1, we see that if an admissible control \( u(\cdot, t) \) does not take its value on the boundary of the control set \( U \) in a subset of positive measure in the interval \([0, T]\), where the number \( T \) is such that \( y(T; u, y_0) \in S \), then there exists a ”room” for us to construct another admissible control \( v \) such that the corresponding trajectory \( y(t; v, y_0) \) reaches \( y(T; u, y_0) \) before the time \( T \). Hence, such an admissible control \( u \) can not be optimal. This idea has been used in \([4], [11], [13] \) and \([16]\). The key point is how to use this ”room” to construct such an admissible control \( v \). In this work, the null controllability property \((C)\) ( Theorem 2.1) leads us to such a way. It was already observed in \([13]\) that the null controllability of the boundary controlled one-dimensional heat equation in \((0,1) \times (0,T)\), with controls restricted on an arbitrary
subset \( E \subset [0, T] \) of positive measure leads to a bang-bang principle of time optimal boundary controls for the one-dimensional heat equation.

Next, we shall use Theorem 2.1 to derive the uniqueness of the optimal control for the problem (P) with certain target sets and control sets.

**Theorem 3.4.** Suppose that the target set \( S \) is convex and nonempty and the control set \( U \) is a closed ball. Then the optimal control of the problem (P) is unique.

**Proof.** Let \( U \) be the closed ball \( B(v_0, R) \) in \( L^2(\Omega) \), centered at \( v_0 \) and of positive radius \( R \). Let \( T^* \) be the optimal time for the problem (P). Seeking a contradiction, we suppose that there exist two different optimal controls \( u^* \) and \( v^* \) for the problem (P). Then there would exist a subset \( E_1 \) of positive measure in the interval \([0, T^*]\), such that \( u^*(t) \neq v^*(t) \) for every \( t \in E_1 \). We first observe that

\[
y(T^*; u^*, y_0), y(T^*; v^*, y_0) \in S.
\]

Then we construct a control \( w^*(t) \) by setting

\[
w^*(t) = \frac{u^*(t) + v^*(t)}{2} \quad \text{for almost all } t \in [0, \infty).
\]

It is clear that \( w^* \in U_{ad} \). Moreover, since \( S \) is convex, we have

\[
y(T^*; w^*, y_0) = \frac{y(T^*; u^*, y_0) + y(T^*; v^*, y_0)}{2} \in S.
\]

On the other hand, we see that for almost all \( t \in E_1 \),

\[
\|w^*(t) - v_0\|_{L^2(\Omega)}^2 = 2(\|\frac{u^*(t) - v_0}{2}\|_{L^2(\Omega)}^2 + \|\frac{v^*(t) - v_0}{2}\|_{L^2(\Omega)}^2) - \|\frac{u^*(t) - v_0}{2}\|_{L^2(\Omega)}^2 - \|\frac{v^*(t) - v_0}{2}\|_{L^2(\Omega)}^2
\]

\[
= R^2 - \frac{1}{4}\|u^*(t) - v^*(t)\|_{L^2(\Omega)}^2 < R^2.
\]

Thus, there exist a positive number \( \varepsilon \) and a subset \( E \) of positive measure in the set \( E_1 \) such that for each \( t \in E \), \( d(w^*(t), \partial B(v_0, R)) \geq \varepsilon \). Then, we can use the same argument as that in the proof of Theorem 3.1 to derive a contradiction to the optimality of \( T^* \). This completes the proof.

With regard to the existence of the time optimal controls for the problem (P), we recall (See [17].) that if the target set \( S \) is closed and convex in \( L^2(\Omega) \), which contains the origin in \( L^2(\Omega) \), and if the control set \( U \) is the ball \( B(0, R) \) with \( R > 0 \), then the problem (P) with any initial data \( y_0 \in L^2(\Omega) \) has an optimal control. ( See also [14].) Thus, by combining Corollary 3.2, Theorem 3.4 and the existence result mentioned above, we have the following consequence.

**Theorem 3.5.** Suppose that the target set \( S \) is a closed, convex and nonempty subset, which contains the origin of \( L^2(\Omega) \), and the control set \( U \) is the ball \( B(0, R) \) with \( R >
0. Then the problem \((\textbf{P})\) has a unique optimal control \(u^*\) which satisfies the bang-bang property: \(\|x_0 u^*(t)\|_{L^2(\Omega)} = R\) for almost all \(t \in [0, T^*]\), where \(T^*\) is the optimal time for the problem \((\textbf{P})\).

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