Deformation of semi-classical symmetries: Poisson-Hopf limit of quantum algebras

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Abstract. We present a method to construct the Poisson-Hopf structure associated to a quantum algebra $U_z(g)$. The procedure is based on the construction of a $\hbar$-family of quantum algebras $U_{z,\hbar}(g)$ and taking the “semiclassical” limit $\hbar \to 0$.

1. Introduction

Quantum groups appeared in relation with the quantization of classical integrable systems by the method of quantum inverse scattering. In this context, Poisson algebras of functions defined on Poisson manifolds that are Lie groups, $G$, are involved. These structures, called Poisson-Lie groups, are determined by certain compatibility conditions [1] between the Poisson manifold and the Lie group. The quantization of such (commutative) spaces of functions, $\text{Fun}(G)$, following the Moyal spirit [2] (i.e., by constructing a $*$-product) leads to the quantum groups $\text{Fun}_q(G)$. In these quantum structures there is a deformation parameter $q = e^z$, where $z$ is the constant that governs the noncommutativity of the algebra of functions

$$[f_1, f_2] := f_1 * f_2 - f_2 * f_1 = z\{f_1, f_2\} + \mathcal{O}(z^2),$$

being $\{f_1, f_2\}$ the Poisson bracket in $\text{Fun}(G)$ that has to be the “semiclassical limit” (or Poisson limit) of $[f_1, f_2]$, i.e.

$$\{f_1, f_2\} = \lim_{z \to 0} \frac{1}{z} [f_1, f_2].$$

In a physical context where the transition from classical to quantum physical models is studied, the deformation parameter $z$ has been interpreted as the Planck constant $\hbar$. But, in other contexts $z$ is a parameter whose meaning has to be elucidated for each particular case. Indeed from the beginning quantum groups and quantum algebras (the Hopf algebras dual of quantum groups [1,3–6]) has been also considered in an abstract way as Hopf algebras, that could describe new symmetries of physically relevant systems where the deformation parameter $z$ (or $q$) is completely independent with respect to the Planck constant. Thus, for instance, we can mention the $su_q(2)$ invariance of the Heisenberg spin XXZ chain [7, 8], where $z$ is the anisotropy of the chain. Also lattice systems related to deformations of inhomogeneous algebras, where $z$ is connected with the lattice length [9, 10], deformations of kinematical symmetries in which the
deformation parameter is a fundamental scale [11–15], effective models in nuclear physics [16,17], etc.

Thus, in this paper we consider a mathematical framework for quantum algebras in which two independent parameters, the deformation parameter \( z = \log q \) and the Planck constant \( \hbar \), are simultaneously considered. From a technical point of view we construct a one-parameter family \( \mathbb{C} \), \( \mathbb{C} \) independent parameters, the deformation parameter \( z \) etc.

is the Hopf algebra deformation of the universal enveloping algebra \( \mathfrak{u}_g \). It is worthy to note that the two sets of structure constants play symmetric roles: \((\text{integrable systems with a parameter } z)\) a quantum algebra symmetry. This approach can be interesting to construct new (classical) integrable systems with a parameter \( z \) that could control their dynamical behaviour. The method is illustrated by applying it to the standard quantum deformation of \( su(2) \). In this elementary case the \( q \)-Poisson-Hopf algebra obtained in the \( h \to 0 \) limit looks, in despite of the differences of the two structures, to be formally identical to the original quantum algebras, but this is not the case for more complex algebras like \( su(3) \) [18].

2. Poisson limit of quantum algebras

A Lie bialgebra [1,6] \((g, \delta)\) is a Lie algebra \( g \),

\[
[X_i, X_j] = f^k_{ij} X_k,
\]

(1)

equipped with a cocommutator map \( \delta : g \to g \otimes g \), i.e., a skew-symmetric linear map, given by

\[
\delta(X_i) = c_i^{jk} X_j \otimes X_k
\]

(2)

such that the structure constants \( c_i^{jk} \) fulfill the following compatibility conditions together with the structure constants of \( g, f^k_{ij} \),

\[
c_p^{qr} f^k_{st} = c_p^{qr} f^k_{ts} + c_s^{pr} f^q_{kt} + c_t^{pq} f^r_{st} + c_t^{pq} f^r_{st}.
\]

(3)

It is worthy to note that the two sets of structure constants play symmetric roles: \((g^*, \delta^*)\) is the Lie bialgebra dual of \((g, \delta)\) determined by

\[
[x^p, x^q] = c_p^{qr} x^r, \quad \delta^*(x^p) = f^p_{qr}, x^q \otimes x^r,
\]

(4)

where \( g^* \) is the dual space of \( g \) generated by the set of elements \( \{x^i\} \) such that \( x^i(X_j) = \delta^i_j \).

The Lie bialgebras play a fundamental role in order to construct explicitly the deformation of a Lie algebra \( g \). Effectively, given an arbitrary Lie bialgebra \((g, \delta)\) the quantum algebra \((\mathfrak{u}_g, \Delta_z)\) is the Hopf algebra deformation of the universal enveloping algebra \( U(g) \), compatible with the deformed coproduct \( \Delta_z(X) \) whose leading order terms are

\[
\Delta_z(X) = \Delta_0(X) + z \delta(X) + \mathcal{O}(z^2),
\]

where \( \Delta_0 \) is the primitive coproduct (i.e. \( \Delta_0(X) = 1 \otimes X + X \otimes 1 \)). The deformed coproduct can be obtained using the analytic procedure described in [19,20].

Let \( M \) be a smooth manifold of (finite) dimension \( m \), and \( \mathcal{C}^\infty(M) \equiv \text{Fun}(M) \) the algebra of smooth real-valued functions on \( M \). A Poisson structure on \( M \) map is a \( \mathbb{R} \)-bilinear map (Poisson bracket),

\[
\{ , \} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)
\]

satisfying the following properties (\( \forall f_1, f_2, f_3 \in \mathcal{C}^\infty(M) \)):

(i) \( \{ f_1, f_2 \} = -\{ f_2, f_1 \} \) (skew-symmetry),
A map between Poisson manifolds preserving the Poisson brackets of both manifolds is called Poisson map.

A Poisson-Lie group is a Lie group and a Poisson manifold such that both structures are compatible. In more detail, a Poisson-Lie group is a Lie group \(G\) with a Poisson structure \(\{\ , \}\) (defined on \(C^\infty(G)\)) such that the multiplication on \(G, m : G \times G \rightarrow G\) (\(m(g_1, g_2) = g_1 \cdot g_2\)), is a Poisson map.

A homomorphism of Poisson-Lie groups will be a homomorphism of Lie groups and also a Poisson map.

A Poisson algebra over the field \(\mathbb{K}\) is a commutative algebra \(A\) over \(\mathbb{K}\) equipped with a Poisson bracket (i.e., a skew-symmetric \(\mathbb{K}\)-module) verifying the Jacobi identity and the Leibniz rule.

A homomorphism between two Poisson algebras \((A, \{\ , \}_A)\) and \((B, \{\ , \}_B)\) is a linear map, \(f : A \rightarrow B\), such that (\(\forall a_1, a_2 \in A\)):

\[
\begin{align*}
(i) \quad f(a_1a_2) &= f(a_1)f(a_2), \\
(ii) \quad f(\{a_1, a_2\}_A) &= \{f(a_1), f(a_2)\}_B.
\end{align*}
\]

A Poisson-Hopf algebra \(A\) is a Poisson algebra \((A, \{\ , \}_A)\) which is also a Hopf algebra \((A, \Delta_A)\) over \(\mathbb{K}\) if \(\Delta_A\) is a homomorphism of Poisson algebras between \((A, \{\ , \}_A)\) and \((A \otimes A, \{\ , \}_{A \otimes A})\).

Note that the Poisson bracket in \(A \otimes A\) is defined by

\[
\{a_1 \otimes a_1', a_2 \otimes a_2'\}_{A \otimes A} := \{a_1, a_2\}_A \otimes a_1'a_2' + a_1a_2 \otimes \{a_1', a_2'\}_A.
\]

Let us remember that quantization in physics tries to associate to a commutative algebra of physical observables, \(\text{Fun}(M)\), of a classical mechanical system (with phase space \(M\)) a non-commutative algebra of linear operators, \(\text{Op}(\mathcal{H})\), on the suitable complex Hilbert space, \(\mathcal{H}\), of the quantum system. More precisely, the idea of quantization is to find a linear bijective map \(Q : \text{Fun}(M) \rightarrow \text{Op}(\mathcal{H})\) such that for all \(f \in \text{Fun}(M)\)

\[
Q(\{f_1, f_2\}) = \frac{i}{\hbar} [Q(f_1), Q(f_2)].
\] (5)

This condition comes from the fact that if \(f_2\) is the classical Hamiltonian of the system and \(Q(f_2)\) its quantum counterpart then \(\{f_1, f_2\}\) and \([Q(f_1), Q(f_2)]\) give the time evolution of the physical observable \(f_1\) in the classical and in the quantum case, respectively. However such a map is not known in general, even for very simple cases.

Let us consider the case of \(M = \mathbb{R}^{2n}\) and \(\mathcal{H} = L^2(\mathbb{R}^n)\). The fundamental Poisson brackets associated to the canonical variables \((q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)\) are

\[
\{q_i, p_j\} = \delta_{ij}.
\] (6)

The canonical quantization [21] allows us to replace the Poisson brackets (6) by the commutators

\[
[Q_i, P_j] = \hbar \delta_{ij} I,
\] (7)

where \(Q_i\) and \(P_j\) are the linear operators associated to \(q_i\) and \(p_j\) respectively (i.e. \(Q_i\) and \(P_j\) are the quantum counterparts of the classical observables \(q_i\) and \(p_j\): the operator \(Q_i\) corresponds to multiplication by \(q_i\) and \(P_j\) to \(-i\hbar \frac{\partial}{\partial q_j}\)). In both cases (6) and (7) reflect that the Heisenberg algebra \(\mathfrak{h}\) is involved. Now the condition (5) is verified by the canonical coordinates. This observation leads to the idea, pointed out also by Dirac, that the correspondence between
classical and quantum theories has to be based on a certain analogy among their mathematical structures, in other words, a homomorphism must exist between the Lie algebras of the classical and quantum observables.

When there is a set of quantum observables $X_i$ that close a Lie algebra

$$\{ Q^{-1}(X_i), Q^{-1}(X_j) \} := \lim_{\hbar \to 0} \frac{1}{\hbar} [X_i, X_j]. \quad (8)$$

On the other hand, the dual space of $g, g^*$, can be equipped with a Poisson structure (Kostant-Kirillov-Souriau structure) whose fundamental brackets are

$$\{ x^i, x^j \} = f^{ij}_k x^k, \quad (9)$$

where the $x^i$ can be considered as local coordinates on $g^*$ and $f^{ij}_k \equiv f^{ik}_j$. For any two functions on $\text{Fun}(g^*)$ their Poisson bracket will be given by

$$\{ f_1, f_2 \} = \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j} f^{ij}_k x^k.$$  

Also we have a new bialgebra $(g^*, \delta)$ determined by (9) and (2).

Keeping in mind all these ideas we see that easily we can quantize the Kostant-Kirillov-Souriau structure as follows

$$\{ x^i, x^j \} = f^{ij}_k x^k \quad \Rightarrow \quad [X_i, X_j] = \hbar f^{ij}_k X_k. \quad (10)$$

So, we have obtained a “new” Lie algebra $g_\hbar$ isomorphic to $g^*$ and also to $g$. In this way we have built up a one-parameter family of equivalent Lie bialgebras $(g_\hbar, \delta)$ determined by

$$[X_i, X_j] = \hbar f^{ij}_k X_k \quad (10)$$

and (2). Obviously, the mathematical structure of the original bialgebra $(g, \delta)$ is recovered when $\hbar = 1$.

We can quantize this one-parameter family of Lie bialgebras $(g_\hbar, \delta)$ by using the method of the analytical deformation described in [20], and a two-parameter quantum algebra $(U_{z, \hbar}(g), \Delta_{z, \hbar})$, depending explicitly on $\hbar$, is thus obtained.

The Poisson limit (i.e., $\hbar \to 0$) of $(U_{z, \hbar}(g), \Delta_{z, \hbar})$ gives, in a unique way, the Poisson-Hopf algebra $(\text{Fun}(g_z), \Delta_z^P)$ which is just a Poisson-Lie structure on the group $g_z$ whose Lie algebra is characterized by $\delta^z (4)$, i.e. by the structure tensor $f^{jk}_i$.

The Poisson bracket on $\text{Fun}(g_z)$ is given by

$$\{ X_i, X_j \} := \lim_{\hbar \to 0} \frac{[X_i, X_j]}{\hbar}, \quad (11)$$

and the coproduct map by

$$\Delta_z^P(X_i) := \lim_{\hbar \to 0} \Delta_{z, \hbar}(X_i), \quad (12)$$

where $X_i = Q^{-1}(X_i)$ can be seen as the local coordinates on the manifold $g_z$, so the Poisson brackets (11) are the so-called fundamental brackets. The coproduct (12) is a Poisson algebra homomorphism between $\text{Fun}(g_z)$ and $\text{Fun}(g_z) \otimes \text{Fun}(g_z)$. Effectively, for non-deformed Lie
algebras, its coproduct is the primitive one and the commutation rules are linear: thus the Poisson limits (11) and (12) lead to the same formal structure where commutation rules have been just replaced by Poisson brackets and the coproduct remains primitive. However, in quantum algebras nonlinear functions of the generators appear at the level of the commutation rules and of the coproduct. This implies that the \( q \)-Poisson structure given by the limit \( \hbar \to 0 \) can be formally different from the original quantum algebra structure. In fact, the Poisson limit allows us to remove contributions in the deformation that arise as reordering terms.

In the next section we will illustrate this general approach constructing the standard \( q \)-Poisson algebra associated to \( su_q(2) \). In [18] the reader can find, to best of our knowledge, the first example of a Hopf algebra deformation of the Poisson \( su(3) \) algebra.

3. The basic example: \( q \)-Poisson-Hopf algebra of \( su(2) \)

In the \( su(2) \) case, as it is well-known, two quantum deformations that do exist: the standard one [22,23], that we consider here, and the non-standard (or Jordanian) deformation [24].

The \( su(2) \) algebra is generated by \( (J_3, J_+, J_-) \) with commutation rules

\[
[J_+, J_-] = 2 J_3, \quad [J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-.
\] (13)

The standard \( su(2) \) Lie bialgebra is characterized by the cocommutator

\[
\delta(J_3) = 0, \quad \delta(J_+) = J_3 \wedge J_+, \quad \delta(J_-) = J_3 \wedge J_-.
\] (14)

The quantum algebra deformation of (13) and (14) is given, respectively, by

\[
[J_+, J_-] = \frac{\sinh(2zJ_3)}{z}, \quad [J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-
\] (15)

and

\[
\Delta_z(J_3) = J_3 \otimes 1 + 1 \otimes J_3,
\]
\[
\Delta_z(J_+) = e^{zJ_3} \otimes J_+ + J_+ \otimes e^{-zJ_3},
\]
\[
\Delta_z(J_-) = e^{zJ_3} \otimes J_- + J_- \otimes e^{-zJ_3}.
\] (16)

Let us introduce the one-parameter \( \hbar \)-family of Lie algebras \( (su(2)_\hbar, \delta) \) (10) determined by the commutators

\[
[J_+, J_-] = 2 \hbar J_3, \quad [J_3, J_+] = \hbar J_+, \quad [J_3, J_-] = -\hbar J_-
\] (17)

combined with the cocommutator (14). Since the cocommutator of \( su(2)_\hbar \) is the same as of \( su(2) \) we would obtain a quantization in which the coproduct (16) does not formally change, however we get

\[
[J_+, J_-] = \hbar \frac{\sinh(2zJ_3)}{z}, \quad [J_3, J_+] = \hbar J_+, \quad [J_3, J_-] = -\hbar J_-.
\] (18)

The \( (\hbar, z) \)-deformed Casimir operator is now

\[
C_{z, \hbar} = \frac{\sinh(zJ_3) \sinh z(J_3 - \hbar)}{z} + J_+ J_-.
\] (19)

It is worthy noticing that the following substitution

\[
J_3 \to J_3/\hbar, \quad J_+ \to J_+/\hbar, \quad J_- \to J_-/\hbar, \quad z \to \hbar z,
\] (20)
allows us to reabsorb the parameter $\hbar$ and, hence, to recover the standard expressions (15) and (16) depending only on the parameter $z$. In other words, $U_z(\mathfrak{su}(2))$ depends on only one essential parameter instead on two. But, in the limit $\hbar \to 0$ this mapping (20) is singular, so the parameter $\hbar$ is essential in order to define properly the $q$-Poisson algebra.

The $q$-Poisson-Hopf algebra $su^P_q(2) \equiv (\text{Fun}(g_z), \Delta^P_z)$ is obtained as the commutative algebra of functions defined on $g_z$ endowed with the Poisson bracket and coproduct map obtained by performing, respectively, the limits (11) and (12):

$$\{J_+, J_-\} = \frac{\sinh(2z J_3)}{z}, \quad \{J_3, J_+\} = J_+, \quad \{J_3, J_-\} = -J_-;$$

$$\Delta^P_z(J_3) = J_3 \otimes 1 + 1 \otimes J_3,$$
$$\Delta^P_z(J_+) = e^{zJ_3} \otimes J_+ + J_+ \otimes e^{-zJ_3},$$
$$\Delta^P_z(J_-) = e^{zJ_3} \otimes J_- + J_- \otimes e^{-zJ_3}.$$  

The $q$-deformed Casimir function obtained from (19) performing the limit $\hbar \to 0$ is:

$$C_z^P = \lim_{\hbar \to 0} C_{q, \hbar} = \frac{1}{2z^2} \sinh^2(z J_3) + J_+ J_-.$$

The non-standard $q$-Poisson algebra $su^P_{q,n}(2)$ and the $q$-Poisson $su^P_q(3)$ algebra have been also introduced in [18], where the quantum algebras $U^{n_q}(su(n))$ and $U_{\hbar, q}(su(3))$ were obtained by making use of the quantization approach described in [20]. In this case $su^P_q(3)$ and $U_{\hbar, q}(su(3))$ are quite different unlike the $su(2)$ cases, where the corresponding $q$-Poisson-Hopf algebras are formally equivalent to the quantum algebras from which they have been obtained as one can see on simple inspection of the corresponding expressions of the commutators and the coproduct.

4. General framework of $\hbar$-quantization and $z$-deformation

In the canonical quantization when the phase space is $M = \mathbb{R}^{2n}$ and $\mathcal{H} = L^2(\mathbb{R}^n)$ it is well-known that appears the Heisenberg algebra $\mathfrak{h}$ (6) or $\mathfrak{h}_h$ (7). In some sense, the idea of this quantization would be to find a map between the universal enveloping algebra of $\mathfrak{h}$, $U(\mathfrak{h})$, and $\text{Fun}(\mathfrak{h})$ (in reality, power series in the canonical variables). This map is found constructing a relation between the classical observables $(x^i, p^j)$ and their quantum counterparts $(X_i, P_j)$. We can summarized this quantization in the following diagram

$$\langle X_i, P_j \rangle \equiv \mathfrak{h}_h \xrightarrow{\hbar \to 0} U(\mathfrak{h}) \xrightarrow{\hbar \to 0} \langle x^i, p^j \rangle \equiv \mathfrak{h} \xrightarrow{\hbar \to 0} \text{Fun}(\mathfrak{h})$$

(21)

We can add the procedure of $q$-deformation to the previous diagram obtaining the following one including the $q$-deformation together with $\hbar$-quantization

$$\langle X_i, P_j \rangle_z \equiv \mathfrak{h}_z \xrightarrow{z \to 0} U_z(\mathfrak{h}_z) \xrightarrow{z \to 0} \langle x^i, p^j \rangle \equiv \mathfrak{h} \xrightarrow{\hbar \to 0} \text{Fun}(\mathfrak{h})$$
The Heisenberg algebra $\mathfrak{h}_h$ is $q$-deformed obtaining the $q$-algebra $\mathfrak{H}_z$, [19, 20], that is related with the usual $q$-algebra $U_z(\mathfrak{h}_h)$, which is the $q$-deformation of $U(\mathfrak{h}_h)$, making use of a generalized version of the well-known Friedrichs theorem for Lie algebras [25].

The Friedrichs theorem allows us to obtain a Lie algebra basic set of elements $\{X_i\}$ with linear commutation relations and primitive coproducts that generate the pair $(g,\Delta_0)$ from an arbitrary basic set of $U(g)$. Its generalized version [20] can be applied to obtain similarly $(g_z,\Delta)$ from an arbitrary basic set of $U_z(g)$.

$$
(g,\Delta_0) \xrightarrow{\text{Friedrichs theorem}} U(g) \\
(g_z,\Delta) \xrightarrow{\text{Generalized Friedrichs th.}} U_z(g)
$$

In this paper we have considered the $\hbar$-quantization for $q$-deformed objects. The following diagram, similar to that corresponding to the canonical quantization (21), summarizes these ideas:

$$
\langle X_i \rangle_z \equiv g_{z,\hbar} \xrightarrow{\hbar \to 0} U_{z,\hbar}(g) \\
\langle \chi^i \rangle_z \equiv g_z \xrightarrow{\hbar \to 0} \text{Fun}(g_z)
$$

Here, we have put in relation $U_{z,\hbar}(g)$ and $\text{Fun}(g_z)$ via the semiclassical limit (11).

The results of this paper show the existence of a hierarchy of complexity of Hopf algebras summarized in the following commutative diagram:

$$
(U(g),\Delta_0) \xrightarrow{\hbar = 1} (U_{\hbar}(g),\Delta_0) \xrightarrow{\hbar \to 0} (\text{Fun}(g),\Delta^P_0) \\
(U_z(g),\Delta_z) \xrightarrow{\hbar = 1} (U_{z,\hbar}(g),\Delta_z,\hbar) \xrightarrow{\hbar \to 0} (\text{Fun}(g_z),\Delta_z^P)
$$

Thus, we see that quantum algebras $U_{z,\hbar}(g)$ are the richest and most complex structures (they are non-commutative because $\hbar \neq 0$ and non-cocommutative since $z \neq 0$), the $q$-Poisson algebras $\text{Fun}(g_z)$ are intermediate objects (being commutative and non-cocommutative) and, finally, the Lie algebras (and the universal enveloping algebras $U(g)$) are, in some sense, the simplest ones (non-commutative and cocommutative).

The procedure to construct a $q$-Poisson algebra from a given quantum algebra is thus perfectly defined. First, one goes from $U_z(g)$ to $U_{z,\hbar}(g)$ by constructing analytically the deformation from the very beginning, i.e. starting from the bialgebra $(g_\hbar,\delta)$, and by taking into account explicitly both parameters $(\hbar, z)$. This two-parameter deformation seems to be irrelevant since for any finite value of $\hbar$, by using the Lie algebra automorphism $X \to X/\hbar$ and by rescaling the deformation parameter as $z \to \hbar z$, the quantum algebra $(U_{z,\hbar}(g),\Delta_z,\hbar)$ can be converted into $(U_z(g),\Delta_z)$. But the situation changes when the Poisson limit ($\hbar \to 0$) is performed since it does not commute with the rescaling automorphism and the result thus obtained is not trivial.

We have presented only the Poisson limit for the commutators and the coproduct, however the extension to counity and antipode is trivial. In both cases the Poisson limit would be similar to the coproduct limit (12).

Commutation rules instead $q$-commutators have been used in order to make explicit the quantum algebra structure since, in general, the Poisson limit of the $q$-commutators is not well-defined. This is the reason because $q$-Poisson structures for algebras of rank greater than one have not been constructed so far. However, the analytical bases approach presented in [19, 20]
provides a quantization framework based on pure commutators, thus leading to a well-defined Poisson limit for arbitrary quantum algebras. Hence, analytical bases looks also to have a privileged connection with the semiclassical limit.

In [18] we have shown, moreover, that it is possible to consider the corresponding Poisson limit and to obtain, what could be called, the $q$-Poisson-Serre relations. This result is based on the fact that the $q$-Serre relations can be rewritten in terms of commutators [26,27].

A potential application for this kind of $q$-Poisson-Hopf algebras could be their use for constructing integrable deformations of integrable systems. In particular $su_q^P(3)$ could be related with higher rank classical Gaudin models [28–31] by means of the approach introduced in [32], or to study the semiclassical limit of the $U_x(su(3))$ dynamics from the viewpoint of [33].

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