A Parallel Elicitation-Free Protocol for Allocating Indivisible Goods

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Abstract

We study the problem of allocating a set of indivisible goods to multiple agents. Recent work [Bouveret and Lang, 2011] focused on allocating goods in a sequential way, and studied what is the “best” sequence of agents to pick objects based on utilitarian or egalitarian criterion. In this paper, we propose a parallel elicitation-free protocol for allocating indivisible goods. In every round of the allocation process, some agents will be selected (according to some policy) to report their preferred objects among those that remain, and every reported object will be allocated randomly to an agent reporting it. Empirical comparison between the parallel protocol (applying a simple selection policy) and the sequential protocol (applying the optimal sequence) reveals that our proposed protocol is promising. We also address strategical issues.

1 Introduction

How to allocate resources among multiple agents in an efficient, effective, and fair way is one of the most important sustainability problems. Recently it has become an emerging research topic in AI. Many centralized approaches to allocating indivisible goods have been proposed (e.g., in [Cramton et al., 2006]). In these approaches, agents are required to fully reveal their preferences to some central authority (which computes the final allocation) and pay for the resources allocated to them at some prices. However, there are some drawbacks and limitations of these approaches:

- the elicitation process and the winner determination algorithm can be very expensive;
- agents have to reveal their full preferences, which they might be reluctant to do (sometimes an elicitation process is unwelcome);
- in many real world situations (e.g., assigning courses to students [Kalinowski et al., 2012]; Budish and Cantillon, 2012), and providing employment training opportunities to unemployed), resources must be allocated free and monetary side payments [Chevaleyre et al., 2010] are impossible or unwelcome.

So it is important to design a decentralized elicitation-free protocol for allocating indivisible goods. [Brams et al., 2012] adapted a cake-cutting protocol (a typical decentralized approach for the allocation of divisible goods [Chen et al., 2010]) to the allocation of indivisible goods. However, the protocol is typically designed for the cases when there are only two agents. [Bouveret and Lang, 2011] studied a sequential elicitation-free protocol. By applying this protocol, any number of objects can be allocated to any number of agents. The sequential protocol is parameterized by a sequential policy (i.e., a sequence of agents). Agents take turns to pick objects according to the sequence when the allocation process begins.

In this paper, we define and study a parallel elicitation-free protocol for allocating indivisible goods to multiple agents. According to this protocol, a parallel policy (i.e., an agent selection policy) has to be defined before the public allocation process can begin. At each stage of the allocation process, some agents will be selected (according to the parallel policy) to publicly report their preferred objects among those that remain, and every reported object will be allocated to an agent reporting it. If an object is reported by more than one agent, then the agents reporting it draw lots and the winner could get it. We give a general definition of parallel policies, which can consider the allocation history that had happened; and provide eight different criteria to measure the social welfare induced by parallel policies.

In fact, any sequential policy applied in the sequential protocol is in a specific class of parallel policies that are sensitive to identities. The social welfare criteria considered in [Bouveret and Lang, 2011] and [Kalinowski et al., 2012] are three of the eight criteria proposed in our paper. We introduce two simple parallel policies (i.e., \(\varpi_A\) and \(\varpi_L\)), which are insensitive to identities; and compare \(\varpi_A\) and the optimal sequential policies (for small numbers of objects and agents) with respect to the three social welfare criteria. The results show that the parallel protocol is promising because \(\varpi_A\) outperforms the optimal sequential policies in most cases.

We further consider strategical issues under \(\varpi_A\). We show that an agent who knows the preferences of other agents can find in polynomial time whether she has a strategy for getting a given set of objects regardless of uncertainty arising from lottery. We also show that if the scoring function of the manipulator is lexicographic, computing an optimal strategy in
the sense of pessimism is polynomial.

The remainder of this paper is structured as follows: Section 2 briefly reviews the basics of the sequential protocol. Section 3 presents the parallel protocol and introduces the two specific parallel policies (i.e., $\pi_A$ and $\pi_L$). Section 4 compares $\pi_A$ and sequential policies with respect to several social welfare criteria. Section 5 considers strategic issues under $\pi_A$. Section 6 summarizes the contributions of this work and discusses future work.

2 Preliminaries

A set of $m$ indivisible objects $\mathcal{O} = \{o_1, \ldots, o_m\}$ need to be allocated free to a set of $n$ agents $\mathcal{N} = \{1, 2, \ldots, n\}$. It is supposed that $m \geq n$ and all agents have strict preferences.

$\succ_i$ denotes agent $i$’s ordinal preference (which is a total strict order) over $\mathcal{O}$, and $\text{rank}_i(o) \in \{1, \ldots, m\}$ denotes the rank of object $o$ in $\succ_i$. A profile $R$ consists of a collection of rankings, one for each agent: $R = (\succ_1, \ldots, \succ_n)$; $\text{Prof}(\mathcal{O}, \mathcal{N})$ denotes the set of all possible profiles under $\mathcal{O}$ and $\mathcal{N}$. In the following discussion, if not specified, we only consider full independence case, where all preference orderings are equally probable (i.e., $Pr(R) = \frac{1}{\text{card}(\text{Prof}(\mathcal{O}, \mathcal{N}))}$).

Agent $i$’s value function $u_i : 2^\mathcal{O} \to \mathbb{R}$ specifies her valuation $u_i(B)$ on each bundle $B$ with $u_i(\emptyset) = 0$. When $B = \{o\}$, we also write $u_i(B)$ as $u_i(o)$. For any $i \in \mathcal{N}$, $B \subseteq 2^\mathcal{O}$, and $o \in \mathcal{O}$, it is assumed that:

- $u_i$ is additive, i.e., $u_i(B) = \sum_{o \in B} u_i(o)$; and
- $u_i(o) = g(\text{rank}_i(o))$, where $g$ is a non-increasing function from $\{1, \ldots, m\}$ to $\mathbb{R}^+$.

$g$ is called the scoring function. $g$ is convex if $g(x) - g(x + 1) \geq g(y) - g(y + 1)$ holds for any $x \leq y$. In this paper, we focus on two prototypical convex scoring functions (let $k \in \{1, \ldots, m\}$): (Borda) $g_B(k) = m-k+1$, and (lexicographic) $g_L(k) = 2^m - k$.

In the sequential protocol, agents take turns to pick objects according to a sequential policy $\pi \in \mathcal{P}^m$. $\pi(i)$ designates the $i^{th}$ agent designated by $\pi$. Given a profile $R = \langle \succ_1, \ldots, \succ_n \rangle$, if all the agents act truthfully, then the corresponding allocation history $h(k)^\pi$ is $(\pi(1), \alpha_1')$, $(\pi(m), \alpha_m')$ (i.e., agent $\pi(k)$ picks object $\alpha_k'$ at time $k$), where $\alpha_k' \succ_{\pi(k)} o$ for every $o \in \mathcal{O} \setminus \{\alpha'_1|1 \leq l \leq k\}$.

A scoring function $g$, agent $i$’s utility at $\pi$ and $R$ (i.e., $u_i(\pi, R)$) and $i$’s expected utility at $\pi$ (i.e., $u_i^*(\pi)$) are:

\[ u_i(\pi, R) = \sum_{o \in \mathcal{O}_i} g(\text{rank}_i(o)) \]

where $\mathcal{O}_i = \{\alpha'_k | 1 \leq k \leq m \text{ s.t. } \pi(k) = i\}$, and

\[ u_i^*(\pi) = \frac{\sum_{R \in \text{Prof}(\mathcal{O}, \mathcal{N})} u_i(\pi, R)}{m!} \]

Given an aggregation function $F$ (which is a symmetric, non-decreasing function from $[\mathbb{R}^+]^n$ to $\mathbb{R}^+$), the expected social welfare of a sequential policy $\pi$ is defined as:

\[ sw_F^\pi = F(u_1^*(\pi), \ldots, u_n^*(\pi)) \]

Sequential policy $\pi$ is optimal for $\langle \mathcal{O}, \mathcal{N}, g, F \rangle$ if $sw_F^\pi \geq sw_F^{\pi'}$ for every $\pi' \in \mathcal{P}^m$.

[Bouveret and Lang, 2011] considered two typical aggregation functions which correspond to the utilitarian criterion $F_u(u_1, \ldots, u_n) = \sum_i u_i$ and the Rawlsian egalitarian criterion $F_r(u_1, \ldots, u_n) = \min\{u_i | 1 \leq i \leq n\}$. They also showed that, strict alternation (i.e., $1, 2, 3$) is optimal for $\langle \mathcal{O}, \mathcal{N}, g_B, F_r \rangle$ when $m = 12$ and $n = 2$, and $m \leq 10$ and $n = 3$. But they did not know whether this is true for every $m$ and $n$.

The following example is modified from the one given in [Bouveret and Lang, 2011]. It illustrates the notions introduced in this section and will be used throughout the paper.

Example 1 Let $m = 5$, $n = 3$, and $\pi = 12332$. Then $\langle u_1(\pi), u_2(\pi), u_3(\pi) \rangle = (5, 7.2, 7.5)$ under $g_B$, and in (16, 17.8667, 17) under $g_L$. Consequently, $sw_{F_B}^\pi = 19.7$ under $g_B$, $sw_{F_L}^\pi = 16$ under $g_L$, etc.

Suppose $R = \langle \succ_1, \succ_2, \succ_3 \rangle$ s.t. $\succ_1 = o_1 \succ o_2 \succ o_3 \succ o_4 \succ o_5$, $\succ_2 = o_4 \succ o_2 \succ o_3 \succ o_5 \succ o_1$, and $\succ_3 = o_1 \succ o_3 \succ o_5 \succ o_4 \succ o_2$. Then $h_R^{\pi} = (1, o_1)\langle 2, o_4 \rangle\langle 3, o_3 \rangle\langle 3, o_5 \rangle\langle 2, o_2 \rangle\langle u_1(\pi, R), u_2(\pi, R), u_3(\pi, R) \rangle$ is $\langle 5, 9, 7 \rangle$ under $gL$, and in (16, 24, 12) under $g_L$.

3 Parallel Protocol and Policies

Now we introduce a parallel protocol for allocating indivisible goods. At each stage $t$ of the allocating process, there is a designated set of agents $\mathcal{N}_t \subseteq \mathcal{N}$ s.t. each $i \in \mathcal{N}_t$ reports an object (her preferred object among those that remain). If object $o$ is reported by only one agent then it is allocated to the agent, otherwise the agents demanding $o$ draw lots for the right to get $o$.

The protocol is parameterized by a parallel policy. Formally, a parallel policy is a function $\pi : (2^\mathcal{N} \times 2^\mathcal{N})^* \to 2^\mathcal{N}$. Given a finite sequence $\sigma = \langle \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k, \mathcal{N}_{k+1} \rangle$ (where for every $1 \leq l \leq k$, $\mathcal{N}_l$ is the set of agents reporting at stage $l$, and $\mathcal{N}_k \subseteq \mathcal{N}$ is the set of agents losing some lottery at stage $l$), $\pi$ designates the set of agents reporting at stage $k + 1$. An allocation history induced by $\pi$ is in the form of $\langle O_1, D_1 \rangle\langle O_2, D_2 \rangle\ldots\langle O_p, D_p \rangle\langle O_{k} \rangle\langle O_{k+1} \rangle\langle \mathcal{O}_l \rangle\langle \mathcal{O}_{l+1} \rangle$ (where $1 \leq l \leq p$), and $1 < l < k$.

- $O_1 = \mathcal{O}, \mathcal{O}_l = \mathcal{O}(\mathcal{E})$;
- $D_k : \mathcal{N}_k \to \mathcal{O}_k, \mathcal{O}_k = \{o \in \mathcal{O}_k | \exists i \in \mathcal{N}_k, (D_k(i) = o)\}$, $\mathcal{N}_k \subseteq \mathcal{N}_k$ s.t. $\forall o \in \mathcal{O}_k \setminus \mathcal{O}_k' \langle i \in \mathcal{N}_k \setminus \mathcal{N}_k' | (D_k(i) = o) \rangle = 1$;
- $O_{l+1} = O_l \setminus O_l', O_{l+1} = \pi \langle O_1, \mathcal{N}_1 \rangle, \ldots, \langle O_l, \mathcal{N}_l \rangle)$;
- $O_k \neq \emptyset, \emptyset \subset \mathcal{N}_k \subseteq \mathcal{N}$, and $O_l = O_{l'}$.

Intuitively, at stage $k$, $O_k$ is the set of objects remaining, $O_k'$ is the set of objects reported by some $i \in \mathcal{N}_k$, and for every $i \in \mathcal{N}_k$, $D_k(i)$ is the object reported by $i$. $\langle O_k, D_k \rangle$ is called the demand situation at $k$, and $STOP$ is called the termination situation.

Given a parallel policy $\pi$ and a profile $R = \langle \succ_1, \ldots, \succ_n \rangle$, if all the agents act truthfully, the set of possible histories can be represented as an allocation structure $S^n_R = (\mathcal{V}, \mathcal{E})$ s.t. $\mathcal{V}$ and $\mathcal{E}$ are the minimal sets satisfying the following rules:

1. We suppose the lot is fair, i.e., if there are $k$ agents drawing lots then each one of these agents has $1/k$ chance of winning the lot.
2. “$\pi$” denotes the empty sequence.
• \((O, D : \varpi(\epsilon) \rightarrow O) \in \mathcal{V}\) s.t. \(\text{rank}_i(D(i)) = 1\) for every \(i \in \varpi(\epsilon)\);

• if there exists a history \(h = \ldots(O_k, D_k, N'_{k})\rangle_{k=1, k+1}, \ldots\) induced by \(\varpi\) such that:
  \- \((O_k, D_k) \in \mathcal{V}\), and
  \- \(\forall i \in N_{k+1} \forall o \in O_{k+1} \setminus \{D_{k+1}(i)\}, D_{k+1}(i) > o\),

then \(\langle(O_k, D_k), N'_{k} ; (O_{k+1}, D_{k+1}) \rangle \in \mathcal{E}\), and \(\langle(O_{k+1}, D_{k+1}) \in \mathcal{V}\);

• \(\text{STOP} \in \mathcal{V}\), and if there exists a history \(h = \ldots(O_k, D_k)\rangle_{k=1, k} \text{STOP}\) induced by \(\varpi\) s.t. \(\langle(O_k, D_k) \in \mathcal{V}\) then \(\langle(O_k, D_k), N'_{k}; \text{STOP}\rangle \in \mathcal{E}\).

It is easy to find that \(S^r_{\mathcal{R}}\) is acyclic, and \((O, D)\) is the root.

Since the allocation process from some demand situation \(v \in \mathcal{V}\) is nondeterministic in general, each rational agent \(i\) is often concerned with her expected utility \(u_i(v)\) and the minimal utility \(\underline{u}_i(v)\) that she can get regardless of uncertainty. Formally, given a scoring function \(g, \hat{u}_i(v) = \underline{u}_i(v) = 0\) if \(v = \text{STOP}\); otherwise (suppose \(v = (O', D' : N' \rightarrow O')\)):

\[
\hat{u}_i(v) = v + \frac{\sum_{v' \in E} \#E_{v'} \cdot \hat{u}_i(v')}{\#out_x},
\]

\(\underline{u}_i(v) = \min\{u_i(N'_i), v(N', v') \in \mathcal{E}\},\)

- \(w = \frac{g(\text{rank}_i(D'(i)))}{\# \{N'_i \in N \mid \text{rank}_i(D'(i)) = (\text{.rank}_i(D'(i)))\}}\) if \(i \in N'_i\), \(w = 0\) otherwise;
- \(#E_{v'} = |\{N'_i \in N \mid \langle v(N', v') \in \mathcal{E}\}\};
- \(#out_x = |\{\langle N'_i, v' \in 2N \setminus \forall \langle v(N', v') \in \mathcal{E}\};
- \(u_i(N', v') = u_i(v')\) if \(i \in N'_i \cup (N \setminus N'_i), u_i(N', v') = \frac{\hat{u}_i(v') + g(\text{rank}_i(D'(i)))}{\#out_x}\) otherwise.

\(\hat{u}_i(v, v) = \hat{u}_i(v)\) and \(\hat{u}_i(v, v) = u_i(v)\) are called agent \(i\)'s expected utility and minimum utility at \(\varpi\) and \(R\), respectively, where \(v\) is the root of \(S^r_{\mathcal{R}}\).

Each agent \(i \in N\) can evaluate a given parallel policy \(\varpi\) according to 4 values, i.e., \(v_i(y, z, \varpi)\) where:

- \(y, z \in \{u, e\}\),
- \(v_i(u) = \min\{u_i(z, \varpi, R) \mid R \in \text{Prof}(O, N)\}\),
- \(v_i(e) = \min\{u_i(z, \varpi, R) \mid R \in \text{Prof}(O, N)\}\),
- \(u_i(u, \varpi) = u_i(u, \varpi, R)\), and \(u_i(e, \varpi) = u_i(e, \varpi, R)\).

The social welfare induced by \(\varpi\) (i.e., \(sw(x, y, z, \varpi)\)) can be measured by the most possible orderings over 3 elements taken from \(\{e, u\}\). Formally, let \(x, y, z \in \{u, e\}\),

\[
sw(u, y, z, \varpi) = \sum_{i=1}^{n} v_i(y, z, \varpi) = \min\{v_i(y, z, \varpi) \mid 1 \leq i \leq n\}.
\]

Any sequential policy \(\pi\) can be seen as a parallel policy \(\varpi_{\pi}\) s.t. \(\varpi_{\pi}(\epsilon) = \pi(1)\) and \(\varpi_{\pi}(\sigma_k) = \pi(k + 1)\) for every \(1 \leq k < n\), since \(\pi(k)\). In particular, for every profile \(R\), there is only one possible history in \(S^r_{\mathcal{R}}\). So \(u_i(w, \varpi, R) = u_i(w, \varpi, R) = u_i(w, \varpi, R) = u_i(w, \varpi, R)\), and \(sw(x, u, u, \varpi, \pi) = sw_{\varpi}(\pi)\).

In this paper, we introduce two specific parallel policies: all-reporting \(\varpi_A\), where all the agents report at every stage, and loser-reporting \(\varpi_L\), where all the agents losing some lot at the current stage report at the next stage. Formally, \(\varpi_A(\sigma) = N\) for any sequence \(\sigma\), \(\varpi_L(\epsilon) = N\), and \(\varpi_L(\ldots, (N_k, N'_{k}) = \left\{ \begin{array}{ll} N_k' & \text{if } N_k' \neq \emptyset \\ N_k & \text{otherwise} \end{array} \right.\)

\(\varpi_L\) guarantees that every agent can get \(\varpi\) objects at least. So in the eyes of pessimists, it may be a better choice than \(\varpi_A\).

Note that neither \(\varpi_A\) nor \(\varpi_L\) mentions identities of agents.

We called this kind of parallel policies are insensitive to identities. We can get Lemma 1 directly.

Lemma 1 Let parallel policy \(\varpi\) be insensitive to identities. Then for every \(y, z \in \{u, e\}\), and \(i, j \in N\), we have \(v_i(y, z, \varpi) = v_j(y, z, \varpi)\), and \(sw(u, y, z, \varpi) = n \cdot v_i(y, z, \varpi) = n \cdot sw(e, y, z, \varpi)\).

Example 2 Consider the situation depicted in Example 1. Figure 1 shows the allocation structures of \(\varpi_A\) and \(\varpi_L\), where (let \(1 \leq p \leq 3, 1 \leq q \leq 5, \text{ and } ud\) denote the undefined value):

- \(v_i = (O, D : N \rightarrow O)\), \(v_i' = (O_i', D_i' : N_i' \rightarrow O_i')\);\)
- \(O = O_i = O_2 = O_3 = O_i = \{2, 3, 5\}, O_3 = \{5\}, O_3 = \{2, 5\};\)
- \(N_i = N_i = N_i = N_i = \{1\}, \text{ and } N_i = \{3\};\)
- \(d_i = (O_i, \{u, e\}^N) s.t. d_i[1] = D_i(i) \text{ for every } i \in N, \text{ i.e., } d_1 = \{1, 4, 1\}, d_2 = \{2, 2, 3\}, \text{ and } d_4 = \{5, 5, 5\};\)
- \(d_i = (O_i, \{u, e\}^N) s.t. d_i = D_i(i) \text{ if } i \in N, \text{ otherwise } d_i = ud\);\)

Under \(g\), \(u_i = (u_i(\varpi_A, R), u_i(\varpi_A, R), u_i(\varpi_A, R) = (4, 3, 3), 8, 7, 5)\), and \(sw(u, u, u, \varpi_A) = 20.382, \text{ etc.} \text{ Under } g,\ (u_i(\varpi_A, R), u_i(\varpi_A, R)) = (4, 3, 3), 8, 7, 5)\), and \(sw(u, u, u, \varpi_A) = 20.382, \text{ etc.} \text{ Under } g,\ (u_i(\varpi_A, R), u_i(\varpi_A, R)) = (4, 3, 3), 8, 7, 5)\).

4 Comparison

Bouwer and Lang, 2011} studied what are the sequential policies maximizing social welfare. They considered a utilitarian principle and an egalitarian principle, in which the social welfare induced by a sequential policy \(\pi\) is measured.
Table 1: $\pi^*$, $sw(u,u,u,\pi^*)$, and $sw(u,u,u,\pi_A)$ under $g_B$

|   | $\pi^*$ | $sw(u,u,u,\pi^*)$ | $\pi_A$ | $sw(u,u,u,\pi_A)$ | $\pi^*$ | $sw(u,u,u,\pi^*)$ | $\pi_A$ | $sw(u,u,u,\pi_A)$ |
|---|---|---|---|---|---|---|---|---|
| n = 2 | 4 | 12.292 | 31.051 | 4 | 13.983 | 32.797 | 4 | 12.783 | 32.885 |
| n = 3 | 121 | 18.625 | 45.360 | 121 | 20.033 | 42.382 | 121 | 20.800 | 41.304 |
| n = 4 | 444 | 22.106 | 59.309 | 444 | 20.622 | 62.840 | 444 | 21.000 | 60.341 |

Table 2: $\pi^*$, $sw(u,u,u,\pi^*)$, and $sw(u,u,u,\pi_A)$ under $g_L$

|   | $\pi^*$ | $sw(u,u,u,\pi^*)$ | $\pi_A$ | $sw(u,u,u,\pi_A)$ | $\pi^*$ | $sw(u,u,u,\pi^*)$ | $\pi_A$ | $sw(u,u,u,\pi_A)$ |
|---|---|---|---|---|---|---|---|---|
| n = 2 | 4 | 20.468 | 28.448 | 4 | 31.051 | 35.189 | 4 | 22.417 | 29.458 |
| n = 3 | 121 | 44.274 | 74.625 | 121 | 56.933 | 61.029 | 121 | 59.350 | 58.477 |
| n = 4 | 444 | 95.371 | 135.311 | 444 | 114.281 | 138.617 | 444 | 129.200 | 132.890 |
| n = 5 | 1222 | 190.409 | 299.110 | 1222 | 224.554 | 249.125 | 1222 | 249.200 | 249.125 |
| n = 6 | 4444 | 412.91 | 612.91 | 4444 | 516.69 | 520.79 | 4444 | 520.79 | 520.79 |

Table 3: $\pi^*$, $sw(e,u,u,\pi^*)$, and $sw(e,u,u,\pi_A)$ under $g_B$

|   | $\pi^*$ | $sw(e,u,u,\pi^*)$ | $\pi_A$ | $sw(e,u,u,\pi_A)$ | $\pi^*$ | $sw(e,u,u,\pi^*)$ | $\pi_A$ | $sw(e,u,u,\pi_A)$ |
|---|---|---|---|---|---|---|---|---|
| n = 2 | 4 | 6.000 | 6.146 | 4 | 6.790 | 4.432 | 4 | 6.560 | 5.438 |
| n = 3 | 122 | 9.000 | 9.314 | 122 | 9.000 | 6.794 | 122 | 9.000 | 6.794 |
| n = 4 | 444 | 14.229 | 14.195 | 444 | 17.000 | 9.613 | 444 | 17.000 | 9.613 |

Conjecture 1 Under any convex scoring function $g$ and for any number $n \geq 1$ of objects, $sw(u,u,u,\pi^*) = sw(u,u,u,\pi_A)$ when $n = 2$, and $sw(u,u,u,\pi^*) < sw(u,u,u,\pi_A)$ when $n > 2$.

Conjecture 2 Under any convex scoring function $g$, for any number $n$ of agents and any number $m \geq n$ of objects, $sw(e,u,u,\pi^*) < sw(e,u,u,\pi_A)$. 

Note that in Tables 1 to 4, $\pi^*$, $sw(u,u,u,\pi^*)$, and $sw(u,u,u,\pi_A)$ denote the sequence 12...n, the social welfare induced by $\pi^*$ and $\pi_A$,
policy $\pi$ is measured by the value of $sw(u, e, u, \varpi_\pi)$. They also computed the optimal sequential policies (denoted by $\pi^*$) under $g_B$ when $n = 2$ and $p \leq 8$. We compute the values of $sw(u, e, u, \varpi_{\pi^*})$ and $sw(u, e, u, \varpi_A)$ by use of an exhaustive method. The result is shown in Table 5. Again, $\varpi_A$ outperforms sequential policies in all the test cases.

## 5 Strategic Issues under $\varpi_A$

In this section, we will discuss strategic issues under all-reporting policy $\varpi_A$, which is one of the simplest parallel policies that are insensitive to identities. As most collective decision mechanisms, $\varpi_A$ is not strategy-proof. See Example 2. If all the agents play sincerely, i.e., report their preferred object at each stage, then $u_1(\varpi_A, R) = \frac{1}{2}g(1) + \frac{1}{2}g(2) + \frac{1}{4}g(5)$ and $u_1(\varpi_A, R) = 0$. Suppose 1 is a pessimist and believes that she cannot win any lottery. Then she is concerned only with the utility she can get regardless of uncertainty. If I knows other agents’ preferences and plays strategically, then she reports $o_2$ first and she can get $g(2)$ units of utility at least, which is better than $0 = u_1(\varpi_A, R)$.

Someone may want to study the impact of strategic behavior on the complete-information extensive-form game of such parallel allocation procedures (In). However, it is supposed that the environment matches decentralized elicitation-free protocols’ application motivation. That is to say, we suppose that it is hard to learn self-interested agents’ preferences in advance (In). So we accept the assumption made in [Bouveret and Lang, 2011], i.e., all agents but the only one manipulator act truly. In the following discussion, without loss of generality, let 1 be the manipulator that knows the rankings of the other agents (i.e., $\langle \succ_2, \ldots, \succ_n \rangle$), and $o_1 \succ_1 o_2 \succ_1 \ldots \succ_1 o_m$.

Under $\varpi_A$, a strategy for agent 1 is a sequence of objects $\tau = o_1', \ldots, o_T'$ s.t. $\forall t, t' \in \{1, \ldots, T\}$ ($o_1' \in O$ and $o_t' = o_t'$ iff $t = t'$) holds. That is to say, $\tau$ specifies which object 1 should report at any stage $1 \leq t \leq T$. Some strategies may fail because some object that 1 intends to report has already been allocated. We say strategy $\tau$ is well-defined with respect to ($\succ_2, \ldots, \succ_n$) if at any stage $t \in \{1, \ldots, T\}$, object $o_t'$ is still available, and there is no object available after stage $T$.

A manipulation problem $M$ (for agent 1) consists of ($\succ_2, \ldots, \succ_n$), and a target set of objects $S \subseteq O$. A well-defined strategy $\tau$ is successful for $M$ if, assuming the agents 2 to n act sincerely, $\tau$ ensures that agent 1 gets all objects in $S$. Solving $M$ consists in determining whether there exists a successful strategy. Below we show that the manipulation problem $M$ can be solved in polynomial time. First, we define some notions: for every $i \in N^*$, $A, B \subseteq O$ s.t. $A \cap B = \emptyset$, $\text{Better}_i(A, B) = \{o \in A\mid (\forall o' \in B) o \succ_i o'\}$, and $\text{Best}_i(A) = \{o \in A\mid o \succ_i o' \text{ for every } o' \in A \setminus \{o\}\}$. We can get Lemma 2 directly.

**Lemma 2** Let $A \subseteq C \subseteq O$, $B \subseteq D \subseteq O$, and $C \cap D = \emptyset$. Then for any $i \in N^*$, $\text{Better}_i(A, D) \subseteq \text{Better}_i(C, D) \subseteq \text{Better}_i(C, B)$, and if $\text{Best}_i(C) \in A$ then $\text{Best}_i(C) = \text{Best}_i(A)$, otherwise $\text{Best}_i(C) \succ_i \text{Best}_i(A)$.

Second, for a target set $S \subseteq O$, we construct a sequence $\rho_S = \langle \langle O_1', S_1, O_1 \rangle, \langle O_2', S_2, O_2 \rangle, \ldots \rangle$ as follows:

- $O'_1 = O, N' = \{2, \ldots, n\}$.
- $S_k = \bigcup_{j \in N'} \text{Better}(O'_k \cap S, O'_k \setminus S)$.
- $O_k = \{o \in O'_k \setminus S\mid (\exists i \in N') o \in \text{Best}_i(O'_k \setminus S)\}$.
- $O_{k+1} = O'_k \setminus (S_k \cup O_k)$.

Obviously, for every $o \in O$, there exists one and only one $k$ s.t. $o \in S_k \cup O_k$. We denote by $\text{app}_o(o)$ the number $k$.

**Lemma 3** Let $S' \subseteq S \subseteq O$, $\rho_{S'} = \langle \langle O_1', S_1, O_1 \rangle, \langle O_2', S_2, O_2 \rangle, \ldots \rangle$ and $\rho_S = \langle \langle O_1', S_1', O_1' \rangle, \langle O_2', S_2', O_2' \rangle, \ldots \rangle$. Then for every $k \geq 1$ we have $\bigcup_{i \geq 1} S_i \subseteq \bigcup_{i \geq 1} S'_i$ and $\bigcup_{i \geq 1} (S_i \cup O_i) \subseteq \bigcup_{i \geq 1} (S'_i \cup O'_i)$ for any $k < p$. Then $O'_p \supseteq O_p$.

Let $N' = \{2, \ldots, n\}$.

1. According to Lemma 2, $S_p = \bigcup_{j \in N'} \text{Better}(O'_p \cap S', O'_p \setminus S') \subseteq \bigcup_{j \in N'} \text{Better}(O'_p \cap S, O'_p \setminus S)$. Pick an object $o$ from $S_p$. If $o \in O'_p \cap S$ then there must be $i \in N'$ s.t. $o \in \text{Better}(O'_p \cap S, O'_p \setminus S)$, i.e., $o \in S'$. Otherwise $o \in (O'_p \cap O'_p \setminus S)$, i.e., there must be some $q \leq p$ s.t. $o \in S_q$. So according to the assumption, we have $\bigcup_{i \geq 1} S_i \subseteq \bigcup_{i \geq 1} S'_i$.

2. Pick an object $o'$ from $S_p$. Then there exists $i \in N'$ s.t. $o' \in \text{Best}_i(O'_p \cap S)$, i.e., $o' \in O'_p \cap S'$. So $o' \in (O'_p \cap S)$ then $o' \in \text{Best}_i(O'_p \setminus S)$, i.e., $o' \in O'_p \setminus S$. Otherwise, $o' \in O'_p \setminus S'$, then there exists some $q \leq p$ s.t. $o \in S_q$. According to items 1 and 2, we have $\bigcup_{i \geq 1} S_i \subseteq \bigcup_{i \geq 1} S'_i$ and $\bigcup_{i \geq 1} (S_i \cup O_i) \subseteq \bigcup_{i \geq 1} (S'_i \cup O'_i)$.

Now we can give a simple characterization of a successful strategies in manipulation problems.

**Theorem 1** Let $M = \langle \langle \succ_2, \ldots, \succ_n \rangle, S \rangle$ be a manipulation problem, and $\rho_S = \langle \langle O_1, S_1, O_1 \rangle, \langle O_2, S_2, O_2 \rangle, \ldots \rangle$. There exists a successful strategy for $M$ iff for any $k \geq 1$ we have $k > \bigcup_{i \leq k} S_i'$. Moreover, in this case any strategy $\tau$ starting by reporting the objects in $S$, and reporting $o$ before...
stage \( \text{app}_o(o) \) for every \( o \in S \), (and completed so as to be well-defined) is successful.

**Proof.** (Sketch) We prove the statement by induction on the size of the target set \( S \). In the case when \( S \) is a singleton \( \{o\} \), it is easy to find that \( |\bigcup_{1 \leq i \leq k} S_i'| \leq |\{o\}| = 1 \) for every \( k \geq 1 \). So \( S_1' = \emptyset \) (i.e., \( \text{app}_o(o) > 1 \)) if \( k > |\bigcup_{1 \leq i \leq k} S_i'| \) for any \( k \geq 1 \). If \( S_1' = \emptyset \) (i.e., no agent in \( \{2, \ldots, n\} \) reports \( o \) at stage 1) then 1 can get \( o \) by reporting \( o \) first. If \( S_1' \neq \emptyset \) (i.e., there must be some agent in \( \{2, \ldots, n\} \) reporting \( o \) at stage 1) then there exists no successful strategy for \( M \).

Now assume that the statement holds for any target set whose size is no more than \( p - 1 \). Consider a target set \( S = \{o_1', o_2'\} \) s.t. \( \text{app}_S(o_1') \leq \ldots \leq \text{app}_S(o_2') \). Then \( k > |\bigcup_{1 \leq i \leq k} S_i'| \) for any \( k \geq 1 \) if \( k > |\bigcup_{1 \leq i \leq k} S_i'| \) for any \( p \geq k \geq 1 \). Let \( S' = S \setminus \{o_p'\} \) and \( p_{SB} = \langle O_1', S_1, O_1 \rangle, \langle O_2', S_2, O_2 \rangle, \ldots \).

- If \( k > |\bigcup_{1 \leq i \leq k} S_i'| \) for any \( k \geq 1 \) then:
  1. \( \text{app}_S(o_p') > |\bigcup_{1 \leq i \leq k} \text{app}_S(o_p') S_i'| = p \). So \( o_p' \notin \bigcup_{p+1}^{p_k} S_i' \cup O_i \). From Lemma 3, we have \( o_p' \notin \bigcup_{p+1}^{p_k} S_i' \cup O_i \) \( \bigcup_{p+1}^{p_k} S_i' \cup O_i \) and \( k > |\bigcup_{1 \leq i \leq k} S_i'| \geq |\bigcup_{1 \leq i \leq k} S_i'| \) for any \( p \geq k \geq 1 \).
  2. According to item 1 and the assumption, there exists a successful strategy \( \tau' \) for \( \langle \tau_2', \ldots, \tau_n' \rangle, S' \rangle \) starting by reporting the objects in \( S' \).
  3. According to items 1 and 2, if at each stage \( k < p \), 1 reports the object specified by \( \tau' \), then \( o_p' \) is available and not reported by any \( i \in \{2, \ldots, n\} \) at stage \( p \). Let \( \tau \) be a well-defined strategy reporting the object specified by \( \tau' \) at any stage \( k < p \), and reporting \( o_p' \) at stage \( p \). It is easy to find that \( \tau \) is successful for \( M \).

- If there exists some \( p \geq k \geq 1 \) s.t. \( k \leq |\bigcup_{1 \leq i \leq k} S_i'| \), then there must be some \( i \in \{2, \ldots, n\} \) reporting some \( o \in \bigcup_{1 \leq i \leq k} S_i' \) at some stage \( k' \leq k \). In this case, there is no successful strategy for \( M \).

So the statement holds for any target set whose size is \( p \).

We develop Algorithm 1 (in which the set of objects \( O = \{o_1, \ldots, o_n\} \) and the set of agents \( N = \{1, \ldots, n\} \) are supposed to be global variables) to find successful strategies. The soundness and completeness of the algorithm is from the proof of Theorem 1. It is not hard to find that Algorithm 1 always terminates and is polynomial in \( m \) and \( n \).

We say a well-defined strategy \( \tau \) is optimal (in the sense of pessimism) if it maximizes 1’s benefit under the assumption that 1 can not win any lottery. In fact, it is not hard to find that if agent 1’s scoring function is \( g_1 \), then she can find an optimal strategy (in the sense of pessimism) in polynomial time by applying Algorithm 2.

**Algorithm 2:** Finding an optimal strategy

**Input:** a profile \( R = \langle \tau_1, \ldots, \tau_n \rangle \)

**Output:** an optimal strategy \( \tau \) in the sense of pessimism

1. \( O' \leftarrow O, S \leftarrow \emptyset; \quad \tau \leftarrow \tau_1 \uparrow \phi \)
2. \( \tau \leftarrow \text{Algorithm 1}(\langle \tau_2, \ldots, \tau_n \rangle, \emptyset); \)
3. while \( O' \neq \emptyset \)
4. \( o \leftarrow \text{BEST}_1(O'), O' \leftarrow O' \setminus \{o\}; \)
5. \( \tau \leftarrow \text{Algorithm 1}(\langle \tau_2, \ldots, \tau_n \rangle, S \cup \{o\}); \)
6. if \( \tau \neq \phi \) failure
7. \( \tau \leftarrow \tau \cup \tau; S \leftarrow \tau \cup \{o\}; \)
8. return \( \tau \)

Let’s run Algorithm 2 on the profile \( R \) given in Example 1. Then \( \{o_2\} \), i.e., the best set of objects that 1 can manage to get is found, and a successful strategy for the set (i.e., \( o_2, o_3 \) or \( o_2, o_5 \)) is returned. We conjecture that under the Borda scoring function \( g_B \), the problem of finding an optimal strategy is NP-hard, but we do not have a proof.

**6 Conclusion**

We have defined and studied a parallel elicitation-free protocol for allocating indivisible goods. The protocol is parameterized by a parallel policy (i.e., an agent selection policy), which can consider the allocation history that had happened. We have compared a special parallel policy (i.e., \( \omega_A \)) with sequential policies for small numbers of objects and agents with respect to the three social welfare criteria considered in [Bouveret and Lang, 2011] and [Kalinowski et al., 2012]. The results show that \( \omega_A \) outperforms the optimal sequential policies in most cases. We have also proved that an agent who knows the preferences of other agents can find in polynomial time whether she has a successful strategy for a target set; and that if the scoring function of the manipulator is \( g_L \), she could compute an optimal strategy (in the sense of pessimism) in polynomial time.

There are several directions for future work. One direction would be to prove the conjectures about the social welfare induced by \( \omega_A \), and to design other parallel policies that can outperform \( \omega_A \) in some social welfare criterion. Another direction would be to find the missing complexity results for manipulation under \( g_B \), and to consider strategical issues under the assumption that the manipulator believes any lottery is fair. Furthermore, we plan to design an elicitation-free protocol for allocating sharable goods [Ariaud and Endriss, 2010].

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