SEVERAL ANALYTIC INEQUALITIES IN SOME Q–SPACES

Pengtao Li and Zhichun Zhai*

Abstract. In this paper, we establish separate necessary and sufficient John-Nirenberg (JN) type inequalities for functions in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ which imply Gagliardo-Nirenberg (GN) type inequalities in $Q_{\alpha}(\mathbb{R}^n)$. Consequently, we obtain Trudinger-Moser type inequalities and Brezis-Gallouet-Wainger type inequalities in $Q_{\alpha}(\mathbb{R}^n)$.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

This paper studies several analytic inequalities in some $Q$ spaces. We first establish John-Nirenberg type inequalities in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ ($n \geq 2$). Then we get Gagliardo-Nirenberg, Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities in $Q_{\alpha}(\mathbb{R}^n)$. Here $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is the set of all measurable complex-valued functions $f$ on $\mathbb{R}^n$ satisfying

\begin{equation}
\|f\|_{Q^{\beta}_{\alpha}(\mathbb{R}^n)} = \sup_I \left( (l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}\,dx\,dy \right)^{1/2} < \infty
\end{equation}

for $\alpha \in (-\infty, \beta)$ and $\beta \in (1/2, 1]$, where the supremum is taken over all cubes $I$ with edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. Obviously, $Q^1_{\alpha}(\mathbb{R}^n) = Q_{\alpha}(\mathbb{R}^n)$ which was introduced by Essen, Janson, Peng and Xiao in [9]. It has been found that $Q_{\alpha}(\mathbb{R}^n)$ is a useful and interesting concept, see, for example, Dafni and Xiao [6, 7], Xiao [19], Cui and Yang [5]. As a generalization of $Q_{\alpha}(\mathbb{R}^n)$, $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is very useful in harmonic analysis and partial differential equations, see Yang and Yuan [20], Li and Zhai [14, 15] and Zhai [23] in which $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ was applied to study the well-posedness and regularity of mild solutions to fractional Navier-Stokes equations with fractional Laplacian $(-\Delta)^{\beta}$.

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  \item *Corresponding author.
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JN type inequality is classical in modern analysis and widely applied in theory of partial differential equations. In [10], John and Nirenberg proved the JN inequality for $\text{BMO}(\mathbb{R}^n)$. In this paper, we establish JN type inequalities in $Q_{\alpha}^{\beta}(\mathbb{R}^n)$ a special case of which implies Gagliardo-Nirenberg (GN) type inequalities meaning the continuous embeddings such as $L^r(\mathbb{R}^n) \cap Q_{\alpha}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for $-\infty < \alpha < 1$ and $1 \leq r \leq p < \infty$. Moreover, from GN type inequalities in $Q_{\alpha}(\mathbb{R}^n)$, we get Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities. See, for example, [1, 2, 8, 11, 12] for more information about Trudinger-Moser and Brezis-Gallouet-Wainger type inequalities.

To achieve our main goals, we need the characterization of $Q_{\alpha}^{\beta}(\mathbb{R}^n)$ in terms of the square mean oscillation over cubes.

We recall some facts about mean oscillation over cubes. For any cube $I$ and an integrable function $f$ on $I$, we define

$$ f(I) = \frac{1}{|I|} \int_{I} f(x) dx $$

the mean of $f$ on $I$, and for $1 \leq q < \infty$,

$$ \Phi_{f}^{q}(I) = \frac{1}{|I|} \int_{I} |f(x) - f(I)|^{q} dx $$

the $q$–mean oscillation of $f$ on $I$. Recall the well-known identities

$$ \frac{1}{|I|} \int_{I} |f(x) - a|^{2} dx = \Phi_{f}^{2}(I) + |f(I) - a|^{2} $$

for any complex number $a$, and

$$ \frac{1}{|I|^{2}} \int_{I} \int_{I} |f(x) - f(y)|^{2} dx dy = 2 \Phi_{f}^{2}(I). $$

Moreover, if $I \subset J$, then we have

$$ \Phi_{f}^{2}(I) \leq \frac{|J|}{|I|} \Phi_{f}^{2}(J) $$

and

$$ |f(I) - f(J)|^{2} \leq \frac{|J|}{|I|} \Phi_{f}^{2}(J). $$

Let $D_0 = D_0(\mathbb{R}^n)$ be the set of unit cubes whose vertices have integer coordinates, and let, for any integer $k \in \mathbb{Z}$, $D_k = D_k(\mathbb{R}^n) = \{2^{-k}I : I \in D_0\}$, then the cubes in $D = \cup_{k=0}^{\infty} D_k$ are called dyadic. Furthermore, if $I$ is any cube, $D_k(I)$, $k \geq 0$, denote the set of the $2^{kn}$ subcubes of edge length $2^{-kl}(I)$ obtained by $k$ successive bipartitions of each edge of $I$. Moreover, put $D(I) = \cup_{k=0}^{\infty} D_k(I)$. For any cube $I$ and a measurable function $f$ on $I$, we define
\[ \Psi_{f,\alpha,\beta}(I) = (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \Phi_{2k}^\beta(J) \]
\[ = (l(I))^{4\beta-4} \sum_{J \in D_k(I)} (\frac{f(J)}{l(I)})^{n-2(\alpha-\beta+1)} \Phi_{2k}^\beta(J). \tag{1.8} \]

We can prove the following proposition by a similar argument applied by Essen, Janson, Peng and Xiao for the case \( \beta = 1 \) in [9, Theorem 5.5]. The details are omitted here.

**Proposition 1.1.** Let \(-\infty < \alpha < \beta \) and \( \beta \in (1/2, 1] \). Then \( Q_{\alpha}^\beta(\mathbb{R}^n) \) equals the space of all measurable functions \( f \) on \( \mathbb{R}^n \) such that \( \sup_I \Psi_{f,\alpha,\beta}(I) \) is finite, where \( I \) ranges over all cubes in \( \mathbb{R}^n \). Moreover, the square root of this supremum is a norm on \( Q_{\alpha}^\beta(\mathbb{R}^n) \), equivalent to \( \| f \|_{Q_{\alpha}^\beta(\mathbb{R}^n)} \) as defined above.

Using this equivalent characterization of \( Q_{\alpha}^\beta(\mathbb{R}^n) \), we can establish the following JN type inequalities.

**Theorem 1.2.** Let \(-\infty < \alpha < \beta \), \( \beta \in (1/2, 1] \) and \( 0 \leq p < 2 \). If there exist positive constants \( B, C \) and \( c \), such that, for all cubes \( I \subset \mathbb{R}^n \), and any \( t > 0 \),
\[ (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in D_k(I)} \frac{m_J(t)}{|J|} \leq B \max \left\{ 1, \left( \frac{C}{t} \right)^n \right\} \exp(-ct), \tag{1.9} \]
then \( f \) is a function in \( Q_{\alpha}^\beta(\mathbb{R}^n) \). Here \( m_J(t) \) is the distribution function of \( f - f(I) \) on the cube \( I \):
\[ m_I(t) = |\{ x \in I : |f(x) - f(I)| > t \}|. \tag{1.10} \]

**Theorem 1.3.** Let \(-\infty < \alpha < \beta \), \( \beta \in (1/2, 1] \) and \( f \in Q_{\alpha}^\beta(\mathbb{R}^n) \). Then there exist positive constants \( B \) and \( b \), such that
\[ (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in D_k(I)} \frac{m_J(t)}{|J|} \leq B \max \left\{ 1, \left( \frac{\| f \|_{Q_{\alpha}^\beta(\mathbb{R}^n)}}{t} \right)^2 \right\} \exp \left( \frac{-bt}{\| f \|_{Q_{\alpha}^\beta(\mathbb{R}^n)}} \right), \tag{1.11} \]
holds for \( t \leq \| f \|_{Q_{\alpha}^\beta(\mathbb{R}^n)} \) and any cubes \( I \subset \mathbb{R}^n \), or for \( t > \| f \|_{Q_{\alpha}^\beta(\mathbb{R}^n)} \) and cubes \( I \subset \mathbb{R}^n \) with \( (l(I))^{4\beta-2} \geq 1 \). Moreover, there holds
\[ (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in D_k(I)} \frac{m_J(t)}{|J|} \leq B \tag{1.12} \]
for $t > \|f\|^{2}_{Q^{\beta}_{\alpha}(\mathbb{R}^{n})}$ and cubes $I \subset \mathbb{R}^{n}$ with $(l(I))^{2\beta - 2} < 1$.

For $\beta = 1$, the JN inequality in $Q_{\alpha}(\mathbb{R}^{n})$ was conjectured by Essen-Janson-Peng-Xiao in [9] and finally a modified version as in Theorems 1.2-1.3 was established by Yue-Dafni [21].

According to Essen, Janson, Peng and Xiao [9, Theorem 2.3] and Li and Zhai [14, Theorem 3.2], we know that if $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$, $Q^{\beta}_{\alpha}(\mathbb{R}^{n})$ is decreasing in $\alpha$ for a fixed $\beta$. Moreover, if $\alpha \in (-\infty, \beta - 1)$, then all $Q^{\beta}_{\alpha}(\mathbb{R}^{n})$ equal to $Q^{\beta}_{-\infty + \beta - 1}(\mathbb{R}^{n}) := BMO^{\beta}(\mathbb{R}^{n})$. Thus, when $k = 0$ and $\alpha = -\frac{n}{2} + \beta - 1$, (1.11) implies a special JN type inequality, that is, for $f \in L^{2}(\mathbb{R}^{n}) \cap BMO^{\beta}(\mathbb{R}^{n})$ and $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^{n})}$,

$$\left|\{x \in \mathbb{R}^{n} : |f(x)| > t\}\right| \leq \frac{B\|f\|^{2}_{L^{2}(\mathbb{R}^{n})}}{t^{2}} \exp\left(\frac{-bt}{\|f\|^{2}_{BMO^{\beta}(\mathbb{R}^{n})}}\right).$$

When $t > \|f\|_{BMO^{\beta}(\mathbb{R}^{n})}$, we get a weaker form of (1.13).

**Proposition 1.4.** Let $\beta \in (1/2, 1]$. If $f \in BMO^{\beta}(\mathbb{R}^{n}) \cap L^{2}(\mathbb{R}^{n})$, then

(i) (1.13) holds for all $t \leq \|f\|_{BMO^{\beta}(\mathbb{R}^{n})}$;

(ii) (1.14) $$\left|\{x \in \mathbb{R}^{n} : f(x) > t\}\right| \leq \frac{B\|f\|^{2}_{L^{2}(\mathbb{R}^{2})}}{\|f\|^{2}_{BMO^{\beta}(\mathbb{R}^{n})}}$$

holds for all $t > \|f\|_{BMO^{\beta}(\mathbb{R}^{n})}$.

When $\beta = 1$ and $t > \|f\|_{BMO(\mathbb{R}^{n})}$, (1.13) also holds and implies the following GN type inequalities in $Q_{\alpha}(\mathbb{R}^{n})$ which can also be deduced from [4, Theorem 2] and [9, Theorem 2.3]: for $-\infty < \alpha < 1$ and $1 \leq r < p < \infty$,

$$\|f\|_{L^{r}(\mathbb{R}^{n})} \leq C_{n,p}\|f\|^{r/p}_{L^{r}(\mathbb{R}^{n})}\|f\|_{Q_{\alpha}(\mathbb{R}^{n})}^{1-r/p},$$

for $f \in L^{r}(\mathbb{R}^{n}) \cap Q_{\alpha}(\mathbb{R}^{n})$. Here, $C_{k,\ldots,k}$ denotes a constant which depends only on the quantities appearing in the subscript indexes.

As an application of (1.15), we establish the Trudinger-Moser type inequality which implies a generalized JN type inequality.

**Theorem 1.5.**

(i) There exists a positive constant $\gamma_{n}$ such that for every $0 < \zeta < \gamma_{n}$

$$\int_{\mathbb{R}^{n}} \Phi_{p}\left(\zeta \left(\frac{|f(x)|}{\|f\|_{Q_{\alpha}(\mathbb{R}^{n})}}\right)\right) dx \leq C_{n,\zeta} \left(\frac{\|f\|_{L^{p}(\mathbb{R}^{n})}}{\|f\|_{Q_{\alpha}(\mathbb{R}^{n})}}\right)^{p},$$

where $\Phi_{p}(s) = \frac{1}{s^{p}} - 1$.
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holds for all
\[ f \in L^p(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \text{with} \quad 1 < p < \infty \quad \text{and} \quad -\infty < \alpha < 1. \]

Here \( \Phi_p \) is the function defined by
\[ \Phi_p(t) = e^t - \sum_{j < p, j \in \mathbb{N} \cup \{0\}} \frac{t^j}{j!}, \quad t \in \mathbb{R}. \]

(ii) There exists a positive constant \( \gamma_n \) such that
\[
(1.17) \quad \{|x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)} \left( \exp \left( \frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) - 1 \right) - \frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}}}
\]
holds for all \( t > 0 \) and
\[ f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \text{with} \quad -\infty < \alpha < 1. \]

In particular, we have
\[
(1.18) \quad \{|x \in \mathbb{R}^n : |f| > t\}| \leq C_n \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{\|f\|^2_{Q_\alpha(\mathbb{R}^n)} \left( \exp \left( -\frac{t\gamma_n}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) \right)}
\]
holds for all \( t > \|f\|_{Q_\alpha(\mathbb{R}^n)} \) and
\[ f \in L^2(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \text{with} \quad -\infty < \alpha < 1. \]

We can also get the following Brezis-Gallouet-Wainger type inequalities.

**Proposition 1.6.** For every \( 1 < q < \infty \) and \( n/q < s < \infty \), we have
\[
\|f\|_{L^\infty(\mathbb{R}^n)} \leq C_{n,p,q,s} \left( 1 + (\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha(\mathbb{R}^n)}) \log(e + \|(-\triangle)^{s/2}f\|_{L^q(\mathbb{R}^n)}) \right)
\]
holds for all \( (-\triangle)^{s/2}f \in L^q(\mathbb{R}^n) \) satisfying
\[ f \in L^p(\mathbb{R}^n) \cap Q_\alpha(\mathbb{R}^n) \quad \text{when} \quad 1 \leq p < \infty \quad \text{and} \quad -\infty < \alpha < 1. \]

In the next section, we prove our main results. We verify Theorem 1.2-1.3 for \( \beta \in (1/2, 1] \) by applying similar arguments in the proof of Yue and Dafni [21, Theorems 1-2] for \( \beta = 1 \). We deduce Proposition 1.4 from a special case of Theorem 1.3. Finally, we demonstrate Theorem 1.5 and Proposition 1.6 by applying (1.15) and the \( L^p - L^q \) estimates for \( e^{-t(-\triangle)^{s/2}} \).
2. PROOFS OF MAIN RESULTS

2.1. Proof of Theorem 1.2

According to Proposition 1.1, it suffices to prove that \( \Psi_{f,\alpha,\beta}(I) \) is bounded independent of \( I \). More specially, we will prove for any \( p < q \), we have

\[
(2.1) \quad \Psi_{f,\alpha,\beta}^q(I) := (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta+1)-n-k} \sum_{J \in D_k(I)} \Phi_J^q(J) \leq BK_{C,C,q,p},
\]

where \( B, C, c \) are the constants appearing in (1.9), and \( K_{C,C,q,p} \) is a constant depending only on \( C, c, p, \) and \( q \). When \( q = 2 \), \( \Psi_{f,\alpha,\beta}^q(I) = \Psi_{f,\alpha,\beta}(I) \), so this implies the theorem.

For a fixed cube \( I \), and any \( J \in D_k(I) \), let \( \int_J |f(x) - f(J)|^q dx = q \int_0^\infty t^{q-1} m_J(t) dt \).

Using the Monotone Convergence Theorem and the inequality (1.9), we have

\[
\Psi_{f,\alpha,\beta}^q(I) = (l(I))^{4\beta-4} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta+1)-n-k} \sum_{J \in D_k(I)} \frac{q}{|J|} \int_0^\infty t^{q-1} m_J(t) dt \\
\leq q \int_0^\infty t^{q-1} B(1 + \left( \frac{C}{t} \right)^p ) e^{-ct} dt \\
= qB \left( c^{-q} \int_0^\infty u^{q-1} e^{-u} du + C^p c^{-(q-p)} \int_0^\infty u^{q-p-1} e^{-u} du \right) \\
= qB(c^{-q} \Gamma(q) + C^p c^{-(q-p)} \Gamma(q-p))
\]

where \( \Gamma(y) = \int_0^\infty u^{y-1} e^{-u} du \). Since \( 0 \leq p < q \), \( \Gamma(q) \) and \( \Gamma(q-p) \) are finite. Thus, we can get the desired inequality by taking \( K_{C,C,p,q} = q(c^{-q} \Gamma(q) + C^p c^{-(q-p)} \Gamma(q-p)) \).

2.2. Proof of Theorem 1.3

Assume that \( f \) is a nontrivial element of \( Q^\beta_c(\mathbb{R}^n) \). Then \( \gamma = \sup_I (\Psi_{f,\alpha,\beta}(I))^{1/2} < \infty \). For all cubes \( I \) we have

\[
(2.2) \quad (l(I))^{2\beta-2} \frac{1}{|I|} \int_I |f(x) - f(I)| dx \\
\leq ((l(I))^{4\beta-4} \Phi_J^2(I))^{1/2} \leq (\Psi_{f,\alpha,\beta}(I))^{1/2} \leq \gamma.
\]

For a cube \( I \) and each \( J \in D_k(I) \), we have by the Chebyshev inequality, for \( t > 0 \),

\[
m_J(t) \leq t^{-2} \int_J |f(x) - f(J)|^2 dx.
\]
Thus we get
\begin{equation}
(l(I))^{4\beta+4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in D_k(I)} \frac{m_J(t)}{|J|} \leq t^{-2} \Psi_{f,\alpha,\beta}(I) \leq t^{-2} \gamma^2.
\end{equation}

Thus, if \( t \leq \gamma \), then (1.11) holds with \( B = e \) and \( b = 1 \).

To consider the case of \( t > \gamma \), we need the Calderón-Zygmund decomposition, see Calderón and Zygmund [3], and Neri [17].

**Lemma 2.1.** Assume that \( f \) is a nonnegative function in \( L^1(\mathbb{R}^n) \) and \( \xi \) is a positive constant. There is a decomposition \( \mathbb{R}^n = P \cup \Omega, \ P \cap \Omega = \emptyset, \) such that
\begin{enumerate}[(a)]
  \item \( \Omega = \bigcup_{k=1}^{\infty} I_k \), where \( I_k \) is a collection of cubes whose interiors are disjoint;
  \item \( f(x) \leq \xi \) for a.e. \( x \in P \);
  \item \( \xi < \frac{1}{|I|} \int_I f(x) \, dx \leq 2^n \xi \), for all \( I \) in the collection \( \{I_k\} \).
  \item \( \xi |\triangle| \leq \int_{\triangle} f(x) \, dx \leq 2^n \xi |\triangle| \), if \( \triangle \) is any union of cubes \( I \) from \( \{I_k\} \).
\end{enumerate}

In the following we fix a cube \( I \). For \( \xi = t(l(I))^{2-2\beta} \) with any \( t > 0 \), we apply the Calderón-Zygmund decomposition to \( |f(x) - f(J)| \) on a subcube \( J \in D_k(I) \). Set \( \Omega = \Omega_J(t), \ P = J \setminus \Omega_J(t) \).

From Cauchy-Schwarz inequality and (d) of Lemma 2.1, we get
\begin{equation}
(t(l(I))^{2-2\beta})^2 |\triangle| \leq \int_{\triangle} |f(x) - f(J)|^2 \, dx
\end{equation}
for any union \( \triangle \) of the cubes \( K \) in the decomposition of \( \Omega_J(t) \). Inequality (2.4) with \( \triangle = \Omega_J(t) \) gives us a variant of inequality (2.3):
\begin{equation}
(l(I))^{4\beta+4} \sum_{k=0}^{\infty} 2^{(2(\alpha-\beta+1)-n)k} \sum_{J \in D_k(I)} \frac{|\Omega_J(t)|}{|J|} \leq \frac{\Psi_{f,\alpha,\beta}(I)}{(t(l(I))^{2-2\beta})^2} \leq \left( \frac{\gamma}{(t(l(I))^{2-2\beta})^2} \right)^2
\end{equation}
for all \( t > 0 \).

When \( t \geq \gamma \), we can strengthen the estimate (c) in Lemma 2.1 as follows:
\begin{equation}
t(l(I))^{2-2\beta} \leq \frac{1}{|K|} \int_K |f(x) - f(J)| \, dx \leq (2^n \gamma + t)(l(I))^{2-2\beta}
\end{equation}
for all cubes \( K \) in the decomposition of \( \Omega_J(t) \). In fact, note that \( K \) is such a cube, then \( K \neq J \). Otherwise, (2.2) implies
\begin{equation}
\frac{1}{|J|} \int_J |f(x) - f(J)| \, dx \leq \gamma (l(I))^{2-2\beta} \leq t(l(I))^{2-2\beta}.
\end{equation}
This contradicts (c). It follows from the proof of the Calderón-Zygmund decomposition (see, Stein [18]) that $K$ must have a "parent" cube $K^* \subset J$ satisfying $K \in \mathcal{D}_1(K^*)$, $l(K^*) = 2l(K)$ and

$$|f(K^*) - f(J)| \leq |K^*|^{-1} \int_{K^*} |f(x) - f(J)| \, dx \leq t(l(I))^{2-2\beta}.$$ 

Then (2.2) implies

$$t(l(I))^{2-2\beta} \leq \frac{1}{|K|} \int_K |f(x) - f(J)| \, dx$$

$$\leq \frac{1}{|K|} \int_K |f(x) - f(K^*)| \, dx + |f(K^*) - f(J)|$$

$$\leq \frac{2^n}{|K^*|} \int_{K^*} |f(x) - f(K^*)| \, dx + t(l(I))^{2-2\beta}$$

$$\leq (2^n \gamma + t)(l(I))^{2-2\beta}.$$ 

There holds $\Omega_J(t') \subset \Omega_J(t)$ for $0 < t < t'$. In fact, for any cube $K \in \Omega_{J(t')} \setminus \Omega_J(t)$, we get $K \subset J \setminus \Omega_J(t)$. So, property (b) tells us

$$t(l(I))^{2-2\beta} \geq \frac{1}{|K|} \int_K |f(x) - f(J)| \, dx > t'(l(I))^{2-2\beta}.$$ 

This is a contradiction.

Letting $t' = t + 2^{n+1} \gamma$ for $t \geq \gamma$, we claim that

(2.7) $|\Omega_J(t')| \leq 2^{-n}|\Omega_J(t)|$. 

To prove this, take a cube $K$ in the decomposition for $\Omega_J(t)$. Then (2.6) implies that

$$\frac{1}{|K|} \int_K |f(x) - f(J)| \, dx \leq (2^n \gamma + t)(l(I))^{2-2\beta} < t'(l(I))^{2-2\beta}.$$ 

Thus, $K$ is not a cube in the decomposition of $\Omega_J(t')$, and was further subdivided. Set $\Delta' = K \cap \Omega_J(t')$. If $\Delta' \neq \emptyset$, it must be a union of cubes from the decomposition of $\Omega_J(t')$. Thus, according to (d) of Lemma 2.1, (2.2) and (2.6),

$$t'(l(I))^{2-2\beta} \leq |\Delta'|^{-1} \int_{\Delta'} |f(x) - f(J)| \, dx$$

$$\leq |\Delta'|^{-1} \int_{\Delta'} |f(x) - f(K)| \, dx + |f(K) - f(J)|$$

$$\leq |\Delta'|^{-1} |K| \frac{1}{|K|} \int_{\Delta'} |f(x) - f(K)| \, dx + \frac{1}{|K|} \int_K |f(x) - f(J)| \, dx$$

$$\leq |\Delta'|^{-1} |K| |\gamma| (l(K))^{2-2\beta} + (2^n \gamma + t)(l(I))^{2-2\beta}$$

$$\leq |\Delta'|^{-1} |K| |\gamma| (l(I))^{2-2\beta} + (2^n \gamma + t)(l(I))^{2-2\beta}$$
since \(2 - 2\beta > 0\) and \(K \subset I\). Replacing \(t'\) by \(t + 2^{n+1} \gamma\), dividing by \((l(I))^{2 - 2\beta}\), subtracting \(t\) and dividing by \(\gamma\), we have

\[
(2^{n+1} - 2^n) \leq |\Delta'|^{-1}|K| \quad \text{and} \quad |K \cap \Omega_J(t')| = |\Delta'| \leq 2^{-n}|K|
\]

for any cube \(K\) in the decomposition of \(\Omega_J(t)\). Summing over all such \(K\), and noting that \(\Omega_J(t') = \Omega_J(t) \cap \Omega_J(t')\), we prove (2.7).

For each \(J \in \mathcal{D}_k(I)\), property (b) of the decomposition for \(|f - f(J)|\) implies that

\[
(2.8) \quad m_J(t(l(I))^{2 - 2\beta}) = |\{x \in J : |f(x) - f(J)| > t(l(I))^{2 - 2\beta}\}| \leq |\Omega_J(t)|.
\]

For \(t > \gamma\), let \(j\) be the integer part of \(\frac{k - \gamma}{2^n}\) and \(s = (1 + j2^{n+1})\gamma\). Then \(\gamma \leq s \leq t\). Thus one obtains from (2.8) that

\[
(l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{m_J(t)}{|J|}
\]

\[
= (l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{m_J((l(I))^{2 - 2\beta}l(l(I))^{2\beta - 2})}{|J|}
\]

\[
\leq (l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{m_J(l(I))^{2 - 2\beta}l(l(I))^{2\beta - 2}}{|J|}
\]

\[
\leq (l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{\Omega_J((1 + j2^{n+1})\gamma(l(I))^{2\beta - 2})}{|J|}
\]

\[
\leq (l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{\Omega_J(\gamma(l(I))^{2\beta - 2} + j2^{n+1}\gamma)}{|J|}
\]

\[
\leq 2^{-n}(l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{\Omega_J(\gamma(l(I))^{2\beta - 2} + (j - 1)2^{n+1}\gamma)}{|J|}
\]

if \((l(I))^{2\beta - 2} \geq 1\), by using (2.7) for

\[
t = ((l(I))^{2\beta - 2} + (j - 1)2^{n+1})\gamma \quad \text{and} \quad t' = ((l(I))^{2\beta - 2} + j2^{n+1})\gamma.
\]

Iterating the previous estimate \(j\) times and using (2.5) with \(t = \gamma(l(I))^{2\beta - 2}\), one has

\[
(l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{m_J(t)}{|J|}
\]

\[
\leq 2^{-nj}(l(I))^{4\beta - 4} \sum_{k=0}^{\infty} 2^{2(\alpha - \beta + 1) - n-k} \sum_{J \in \mathcal{D}_k(I)} \frac{\Omega_J(\gamma(l(I))^{2\beta - 2})}{|J|}
\]
\[ \leq 2^{-nj} \gamma^{2} \gamma^{-2} \]
\[ \leq 2^{-n \left( \frac{t - \gamma}{2^{n+1} \gamma} - 1 \right)} \]
\[ = 2^{-n \frac{n}{2^{n+1} \gamma}} \frac{n}{2^{n+1} \gamma} + n. \]

Taking \( B = \frac{2^n}{2^{n+1} \gamma} + n \) and \( b = \frac{n}{2^{n+1} \gamma} \ln 2 \), we get (1.11) when \((l(I))^{2\beta-2} \geq 1\).

If \((l(I))^{2\beta-2} < 1\), using (2.8) and (2.4), one has
\[ \begin{align*}
(l(I))^{2\beta-4} & \sum_{k=0}^{\infty} 2^{2(\alpha-\beta+1) - n} k \sum_{J \in D_k(I)} \frac{m_J(t)}{|J|} \\
& \leq (l(I))^{2\beta-4} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta+1) - n} k \sum_{J \in D_k(I)} \frac{|\Omega_J(t(l(I))^{2\beta-2})|}{|J|} \\
& \leq \gamma^2 t^{-2} \leq 1
\end{align*} \]

which yields (1.12).

### 2.3. Proof of Proposition 1.4

Taking \( k = 0 \) and \( \alpha = -\frac{n}{2} + \beta - 1 \) in (1.11), we get that
\[ (l(I))^{4\beta-4} \frac{m_I(t)}{|I|} \leq B \frac{\|f\|_{BMO^\beta(\mathbb{R}^n)}}{t^2} \exp \left( \frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}} \right) \]
holds for \( t \leq \|f\|_{BMO^\beta(\mathbb{R}^n)} \) and any cube \( I \). Thus for \( t \leq \|f\|_{BMO^\beta(\mathbb{R}^n)} \) and any cube \( I \), we have

\[ \begin{align*}
(l(I))^{4\beta-4} \frac{m_I(t)}{|I|} & \int_I |f(x) - f(I)|^2 \, dx \\
& \leq B \frac{\|f\|_{BMO^\beta(\mathbb{R}^n)}^2}{t^2} \exp \left( \frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}} \right) \int_I |f(x) - f(I)|^2 \, dx \\
& \leq B \frac{\|f\|_{BMO^\beta(\mathbb{R}^n)}^2}{t^2} \exp \left( \frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}} \right) \int_I |f(x)|^2 \, dx \\
& \leq B \frac{\|f\|_{BMO^\beta(\mathbb{R}^n)}^2}{t^2} \exp \left( \frac{-bt}{\|f\|_{BMO^\beta(\mathbb{R}^n)}} \right) \int_{\mathbb{R}^n} |f(x)|^2 \, dx.
\end{align*} \]
Similarly, we can prove (1.14) since
\[ (2.9) \]
This tells us
\[ (2.10) \]
We obtain
\[ 2.4. \text{ Proof of Theorem 1.5} \]
(i) According to (1.15), we have
\[ \int_{\mathbb{R}^n} \Phi_p \left( \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right) dx = \int_{\mathbb{R}^n} \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \left( \frac{|f(x)|}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right)^j dx \]
\[ \leq \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \frac{\|f\|_{L_1(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \]
\[ \leq \sum_{j \geq p, j \in \mathbb{N}} \frac{\zeta^j}{j!} \left( C_n j \|f\|_{L_1^1(\mathbb{R}^n)} \|f\|_{Q_\alpha(\mathbb{R}^n)} \right)^j \]
\[ \leq \sum_{j \geq p, j \in \mathbb{N}} a_j (\zeta C_n)^j \left( \frac{\|f\|_{L^p(\mathbb{R}^n)}}{\|f\|_{Q_\alpha(\mathbb{R}^n)}} \right)^p \]
with $a_j = \frac{j^2}{j!}$. Since $\lim_{j \to \infty} \frac{a_j}{a_{j+1}} = e^{-1}$, the power series of the above right hand side converges provided $\zeta C_n < e^{-1}$ i.e. $\zeta < \gamma_n := (C_n e)^{-1}$.

(ii) According to (i) with $p = 2$, we have

$$\int_{\mathbb{R}^n} \left( \exp \left( \gamma_n \frac{|f(x)|}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) - 1 - \gamma_n \frac{|f(x)|}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) \, dx \leq C_n \frac{\|f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{Q_n(\mathbb{R}^n)}}.$$  

On the other hand, since the distribution function $m(t) = |\{ x \in \mathbb{R}^n : |f(x)| > t \}|$ is non-increasing, we have

$$\int_{\mathbb{R}^n} \left( \exp \left( \gamma_n \frac{|f(x)|}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) - 1 - \gamma_n \frac{|f(x)|}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) \, dx$$

$$= \sum_{j=2}^\infty \frac{\gamma_j^j}{j!} \frac{\|f\|_{Q_n(\mathbb{R}^n)}^j}{\|f\|_{Q_n(\mathbb{R}^n)}^{j}} \int_0^\infty m(s) s^{j-1} \, ds$$

$$\geq m(t) \sum_{j=2}^\infty \frac{\gamma_j^j}{j!} \frac{1}{\|f\|_{Q_n(\mathbb{R}^n)}} \int_0^t s^{j-1} \, ds$$

$$= m(t) \sum_{j=2}^\infty \frac{1}{j!} \left( \frac{\gamma_n t}{\|f\|_{Q_n(\mathbb{R}^n)}} \right)^j$$

$$= m(t) \left( \exp \left( \frac{\gamma_n t}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) - 1 - \frac{\gamma_n t}{\|f\|_{Q_n(\mathbb{R}^n)}} \right)$$

for all $t > 0$. Thus, we have

$$m(t) \leq C_n \frac{\|f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{Q_n(\mathbb{R}^n)}} \frac{1}{\exp \left( \frac{\gamma_n t}{\|f\|_{Q_n(\mathbb{R}^n)}} \right) - 1 - \frac{\gamma_n t}{\|f\|_{Q_n(\mathbb{R}^n)}}}.$$  

2.5. Proof of Proposition 1.6

We will use some facts about the fractional heat equations

$$\partial_t v(t, x) + (-\Delta)^{s/2} v(t, x) = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^n$$

with initial data $v(0, x) = g(x)$ for $x \in \mathbb{R}^n$. The fractional heat equations have been studied by Miao-Yuan-Zhang [16], Zhai [22, 24] and references therein. Here

$$\mathcal{F}((-\Delta)^{s/2} v(t, x))(\xi) = |\xi|^s \mathcal{F} v(t, \xi)$$
Thus $v_g(t, x) = e^{-t(\triangle)^{s/2}} g(x) = K_t^s(x) * g(x)$ with $K_t^s(\cdot) = \mathcal{F}^{-1}(e^{-t|\cdot|^s})$ where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and its inverse. We need the $L^p \to L^q$ estimates for the semigroup $\{e^{-t(\triangle)^{s/2}}\}_{t \geq 0}$. For the proof, see, for example, Kozono-Wadade [13, Lemma 3.4] or Miao-Yuan-Zhang [16, Lemma 3.1].

**Lemma 2.2.** For every $0 < s < \infty$, there exists a constant $C_{n,s}$ depending only on $n$ and $s$ such that

$$\|e^{-t(-\triangle)^{s/2}} g\|_{L^q(\mathbb{R}^n)} \leq C_{n,s} t^{-\frac{s}{2}(\frac{1}{p} - \frac{1}{q})} \|g\|_{L^p(\mathbb{R}^n)}.$$

holds for all $g \in L^p(\mathbb{R}^n)$, $t > 0$ and $1 \leq p \leq q < \infty$.

For any $g(x)$ in the Schwartz class of rapidly decreasing functions $S(\mathbb{R}^n)$, define $v_g(t, x) = e^{-t(-\triangle)^{s/2}} g(x)$ as the solution of fractional heat equation

$$\partial_t v(t, x) + (-\triangle)^{s/2} v(t, x) = 0$$

with initial data $g$. Fix $f \in L^2(\mathbb{R}^n) \cap Q^s_0(\mathbb{R}^n)$ with $(-\triangle)^{s/2} f \in L^q$. Then

$$\int_0^t \langle -(-\triangle)^{s/2} f(x), v(s, x) \rangle ds = \int_0^t \langle f(x), -(-\triangle)^{s/2} v(s, x) \rangle ds$$

$$= \int_0^t \langle f(x), \partial_s v(s, x) \rangle dt$$

$$= \langle f(x), v(t, x) \rangle - \langle f(x), g(x) \rangle.$$ 

Thus

$$\|\langle f, g \rangle\| \leq \|\langle f(x), v(t, x) \rangle\| + \int_0^t \|\langle -(-\triangle)^{s/2} f(x), v(s, x) \rangle\| ds = I_1 + I_2$$

for all $t > 0$. Here $\langle \cdot, \cdot \rangle$ denote the inner-product in $L^2$. Thus Holder inequality, Lemma 2.2 and (1.15) imply that

$$I_1 \leq \|f\|_{L^{q_1}(\mathbb{R}^n)} \|v(t, \cdot)\|_{L^{q_1'}(\mathbb{R}^n)} = \|f\|_{L^{q_1}(\mathbb{R}^n)} \|e^{-t(-\triangle)^{s/2}} g\|_{L^{q_1'}(\mathbb{R}^n)}$$

$$\leq C_{n,s} q_1 t^{-\frac{s}{p}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q^s_0(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}$$

for all $t > 0$ and $p \leq q_1 < \infty$. Similarly, we have

$$I_2 \leq \int_0^t \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|v(s, \cdot)\|_{L^{q_1'}(\mathbb{R}^n)} ds$$

$$= \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \int_0^t \|e^{-t(-\triangle)^{s/2}} g\|_{L^{q_1'}(\mathbb{R}^n)} ds$$

$$\leq C_{n,s} q \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \int_0^t s^{-\frac{1}{q_1}} ds$$

$$\leq C_{n,s} q t^{1-\frac{1}{q_1}} \|(-\triangle)^{s/2} f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}$$
for all $t > 0$. Combing the duality argument and these two estimates, we have

$$
\|f\|_{L^\infty(\mathbb{R}^n)} = \sup_{\|g\|_{L^1(\mathbb{R}^n)} \leq 1, g \in \mathcal{S}} |\langle f, g \rangle| \\
\leq C_{n,s,q} \left( q_1 t^{-\frac{n}{q_1}} \left( \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{Q_\alpha(\mathbb{R}^n)} \right) + t^{1 - \frac{2s}{q}} \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} \right)
$$

for all $t > 0$ and $p \leq q_1 < \infty$. Take

$$
q_1 = \log(1/t), \quad t = \left( e^p + \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} \right)^{1 - \frac{n}{q}}.
$$

Then $t^{-n/(sq_1)} = (t^{1/\log t})^{n/s} = e^{n/s}$ and

$$
t^{1 - \frac{2s}{q}} \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} = \left( e^p + \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} \right)^{1 - \frac{n}{q}} \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} \leq 1.
$$

Since we can find constant $C_{n,s,p,q}$ such that $q_1 \leq C_{n,s,p,q} \log \left( e + \|(-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)} \right)$, (1.19) holds.

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