ON THE REMOVABLE SINGULARITIES FOR MEROMORPHIC MAPPINGS

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Abstract. If \( E \) is a nonempty closed subset of the locally finite Hausdorff \((2n-2)\)-measure on an \( n \)-dimensional complex manifold \( \Omega \) and all points of \( E \) are nonremovable for a meromorphic mapping of \( \Omega \setminus E \) into a compact Kähler manifold, then \( E \) is a pure \((n-1)\)-dimensional complex analytic subset of \( \Omega \).

1. This paper was inspired by the following question of E.L. Stout [7]: Let \( E \) be a closed subset of the complex projective space \( \mathbb{P}^n \) \((n \geq 2)\) such that the Hausdorff \((2n-2)\)-measure of \( E \) (with respect to the Fubini-Study metric) is less than that of any complex hyperplane of \( \mathbb{P}^n \). If it is true that \( E \) is then a removable singularity for meromorphic functions? Using natural projections of \( \mathbb{P}^n \) onto hyperplanes G. Lupacciolu [5] has shown the removability of \( E \) under additional conditions on the sizes of \( E \) and a maximal ball in the complement. (The projection of \( \mathbb{P}^n \) onto a hyperplane does not decrease Hausdorff measures, as it take place in the Euclidean space.)

Using an Oka–Nishino theorem [6] on pseudoconcave sets we prove here the following.

Theorem. Let \( E \) be a closed subset of the locally finite Hausdorff \((2n-2)\)-measure on an \( n \)-dimensional complex manifold \( \Omega \) and let \( f \) be a meromorphic mapping of \( \Omega \setminus E \) into a complex manifold \( X \). If \( X \) has the meromorphic extension property and \( E \) does not contain any \((n-1)\)-dimensional closed analytic subset of \( \Omega \) then \( f \) is continued to a meromorphic mapping of \( \Omega \) into \( X \).

Here we say that \( X \) has the meromorphic extension property, if any meromorphic map \( \varphi : T \to X \) of the "squeezed polydisc"

\[
T = (z, w) \in \mathbb{C}^{n-1}_z \times \mathbb{C}_w : |z| < r, |w| < 1 \text{ or } |z| < 1, 1 - r < |w| < 1,
\]

\( 0 < r < 1, \, n \geq 2 \), extends to a meromorphic map \( \tilde{\varphi} : U \to X \) of the unit polydisc \( U : |z| < 1, |w| < 1 \). By a recent result of S.M. Ivashkovich [3] every compact Kähler manifold \( X \) has the meromorphic extension property, so we have a lot of nice examples of such \( X \). The case of meromorphic functions \((X = \mathbb{P}^1)\) is almost trivial in the consideration: every meromorphic function in a squeezed polydisc is represented as a ratio of two holomorphic functions (see [4]) and thus it is meromorphically continued into \( U \). So the answer on the question of Stout is..
positive because the mentioned $E$ cannot contain any complex analytic subset of $\mathbb{P}^n$ (by a Chow theorem such a set is algebraic, and the Hausdorff $(2n - 2)$-measure of it is not less then the measure of a complex hyperplane (see, e.g. [C]). As pointed me E.L.Stout, this question was solved already by his student Mark Lawrence by a similar method.

2. A closed subset $\Sigma$ of a complex manifold is called locally pseudoconcave if for every point $a \in \Sigma$ there exists a Stein neighbourhood $V$ such that $V \setminus \Sigma$ is Stein.

Let $E'$ be the set of points $a \in E$ such that $f$ meromorphically extends into a neighbourhood of $a$. Then $S := E \setminus E'$ is closed. As the complement to $E$ is locally connected in $\Omega$, these local meromorphic continuations of $f$ in points of $E'$ glue together into the unique meromorphic map $f : \Omega \setminus S \to X$ (we preserve the notation $f$).

The proof of the Theorem is accomplished now in two steps. Firstly we prove that $S$ is locally pseudoconcave (Lemma 1), and secondly we prove that $S$ is complex analytic (Lemma 2).

**Lemma 1.** $S$ is locally pseudoconvex in $\Omega$.

**Proof.** Let $a \in S$, $V \ni a$ is biholomorphic to a ball in $\mathbb{C}^n$ and $\varphi : T \to V \setminus S$ is a holomorphic embedding. Then (V is Stein) $\varphi$ extends to a holomorphic embedding $\tilde{\varphi} : U \to V$ (see [2]). As $X$ has the meromorphic extension property, the meromorphic map $f \circ \varphi : T \to X$ extends to a meromorphic map of $U$ into $X$ and thus $f$ meromorphically continues into the domain $\tilde{\varphi}(U) \subset V$. By the definition of $S$, $\tilde{\varphi}(U)$ does not intersect $S$, and thus we have proved that $V \setminus S$ satisfies the condition "$p^r$-convexity" of Docquier–Grauert [2]. It follows that $V \setminus S$ is Stein.

**Lemma 2.** Let $S$ be a nonempty locally pseudoconcave subset of the finite Hausdorff $(2n - 2)$-measure on an $n$-dimensional complex manifold $\Omega$. Then $S$ is a pure $(n - 1)$-dimensional complex analytic subset of $\Omega$.

**Proof.** The statement is local, so we can assume that $\Omega$ is a domain in $\mathbb{C}^n$. Let $a \in S$ and coordinates $(z, w)$, $z = (z_1, \ldots, z_{n-1})$ in $\mathbb{C}^n$ are chosen in such a way that $a = 0$ and the set $S \cap \{z = 0\}$ is finite or countable (it can be done obviously). Then there exists a neighbourhood $V = V' \times V_n \ni 0$ such that the projection of $S \cap V$ into $V'$ is proper. It follows that fibres $S \cap V \cap \{z = c\}$ are finite for almost every $c \in V'$. By an Oka–Nishino theorem [6] $S \cap V$ is then a complex analytic set of pure dimension $n - 1$.

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**References**

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