Thermodynamics of a New Phase: Classical (Cosmological) Time Crystal

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Mini-superspace version of a generalized gravity model in the spirit of $f(R_{\mu\nu})$ shows classical time crystal features. We study thermodynamic properties of a collection of such systems from classical mechanics perspective. We reveal possible existence of phase transition. Possible relevance of these results in context of multiverse needs to be explored.

Introduction: The classical and quantum versions of the Time Crystal (TC) have generated an enormous amount of interest just within a few years of its theoretical possibility, conceived by Shapere and Wilczek [1] and by Wilczek [2] respectively. A similar idea (leading to a spatially varying ground state condensate) in a different framework was also proposed by one of the present authors in [3]. (For a recent review and references, see [4].) After the critical assessment of [5] of the original quantum version given in [2], there has been major theoretical developments [6] of the quantum TC with remarkable experimental verifications [7]. On the other hand the classical version of TC is free from controversies [8]. However, the Classical TC (CTC) is comparatively less studied mainly due to lack of realistic models. In the cosmological context, a relativistic scalar field with a non-canonical kinetic term, in an expanding Friedmann-Robertson-Walker (FRW) universe, induces a TC behavior [9]. More recently, we have shown [10] that TC behavior in a noncommutative extended FRW model [11] where the scale factor, (being the only dynamical variable), exhibits periodic behavior indicating the possibility of a bouncing universe [12]. It is interesting specially because the bouncing behavior is achieved in a purely geometric setup, without any matter field. TC in $f(R)$ form of gravity has been proposed in [13]. The present work also deals with a specific form of $f(R, R^{\mu\nu} R_{\mu\nu})$ but our focus is on its thermodynamic aspects from a statistical mechanical perspective.

It is now established that (classical and quantum) TC is a novel and distinct phase. This immediately raises questions about its thermodynamic behavior. It is probably debatable whether conventional equilibrium statistical mechanics can be applied to TC that are curiously in the borderline between equilibrium and non-equilibrium system. Since we have an unambiguous mechanical (albeit non-canonical) particle model, we forge ahead to construct its partition function and come up with the remarkable result of a possible phase transition in the system. Higher derivative nature of the parent model leaves its signature in a slightly modified of the model, following the scheme proposed in [14] and requires a non-trivially modified measure [15] in the computation of the partition function.

Classical (generic) Time Crystal: The spontaneous symmetry breaking paradigm is pivotal in the general phenomenon of crystallization. In its conventional manifestation ground state (or minimum energy state) consists of the atoms arranging themselves into a definite periodic lattice throughout the configuration space, thereby breaking spontaneously the spatial translation symmetry. In [3] higher spatial derivative terms induced a spontaneous symmetry breaking in momentum space leading to the lifting of translational invariance. In [12] examples were provided whose ground state is endowed with a periodic motion leading to a Time Crystal that violates time translation symmetry. Note that Hamiltonian dynamics forbids the existence of CTC. For a Hamiltonian $H(p, \phi)$ with coordinate $\phi$ and conjugate momentum $p$, $H(p, \phi)$ minimizes at $\frac{\partial H}{\partial p} = \frac{\partial H}{\partial \phi} = 0$. But Hamilton’s equations of motion states $\dot{\phi} = \frac{\partial H}{\partial p}$. Put together we get for the minimum energy state $\dot{\phi} = \frac{\partial H}{\partial p} = 0$ indicating that $\phi$ should be a constant. Thus classical ground state should be static contrary to a CTC ground state. This negative conclusion can be bypassed if the structure of $H$ is such that the canonical momentum $p$ leads to a multivalued Hamiltonian as a function of $\phi$ with cusps at $\frac{\partial p}{\partial \phi} = 0$ where the Hamiltonian equations of motion are not valid. In the CTC models of [11][2][10] the system ground state has to adjust itself to the contrasting demands of a time invariant (constant position) and simultaneously time varying (constant velocity) state.

Classical (Cosmological) Time Crystal: In the present paper we study a generalized form of gravity, popularly known as $f(R_{\mu\nu})$ gravity with up to quadratic invariants made out of the Ricci tensor and scalar, given by

$$A = \frac{c^4}{16\pi G} \int \sqrt{-g} (R + CR^2 + DR^{\mu\nu}R_{\mu\nu} - 2\Lambda) \, d^4x$$ (1)
where $C, D$ are numerical constants. In the cosmological context, adhering to the minisuperspace formalism, the action (1) reduces to,

$$A = \frac{3c^4}{2G} \int dt \left[ -a\dot{a}^2 + ka - \frac{\Lambda}{3} a^3 \right] + p \left( \frac{\dot{a}^4}{a} + \frac{k^2}{a} \right) + q(\ddot{a}a) + p(a\ddot{a}) + 2pk\dot{a}^2$$

(2)

where $p = 6C + 2D$ and $q = 12C + 2D$. The $q$-term drops out being a total derivative. This action has a complication since it consists of higher (time) derivative terms and will be plagued by ghost problems. Recall that in [10] we had dropped higher derivative terms, (present in the noncommutative extended FRW model [11]), in an approximation scheme. Incidentally, $f(R)$ models constructed out of Ricci scalar $R$ only are free from the ghost menace. For the mechanical models, (as the present one), the ghost problem manifests itself via a non-positive definite Hamiltonian. We follow the scheme of [14] to develop a reduced model with a positive definite Hamiltonian. This method allows us to get rid of the higher time derivative term from and thereby yielding

$$L = \left( -a\dot{a}^2 + ka - \frac{\Lambda}{3} a^3 \right) + p \left( \frac{\dot{a}^4}{a} + \frac{k^2}{a} \right) - \frac{\sigma}{4} a^2 - 2pk\dot{a}^2.$$

(3)

From the Lagrangian (3), the equation of motion can be written as,

$$\ddot{a} \left[ -2a + 12p\frac{\dot{a}^2}{a} + 4p\frac{k}{a} - 3\sigma\frac{\dot{a}^2}{a} \right] - 3\dot{a}^2 \left( p - \frac{\sigma}{4} \right) - 2pk\dot{a}^2 + p\frac{k^2}{a^2} - k + \Lambda a^2 = 0.$$

(4)

We provide a plot in Fig. 1 depicting the evolution of the scale factor $a(t)$ with the cosmic time $t$ by solving the master equation eqn. (4) numerically assuming the spatial flatness of the universe (i.e., for $k = 0$). The figure shows that at a certain time $t$, the scale factor shows a single hump before it hits the singularity at around $t \sim 2$. This phenomenon can be interpreted as a cosmological bounce.

Defining the canonical momentum as

$$p = \frac{\partial L}{\partial \dot{a}} = -2a\dot{a} + 4p \left( \frac{\dot{a}^3}{a} + k \frac{\dot{a}}{a} \right) - \sigma \frac{\dot{a}^3}{a},$$

(5)

the Hamiltonian yields,

$$H = p\dot{a} - L = 3 \left( p - \frac{\sigma}{4} \right) \frac{\dot{a}^4}{a} + 2pk\frac{\dot{a}^2}{a} - a\dot{a}^2 - p\frac{k^2}{a} - ka + \frac{\Lambda}{3} a^3$$

$$= \frac{3(p - \frac{\sigma}{4})}{a} \left[ \ddot{a}^2 - \frac{(a^2 - 2pk)^2}{6(p - \frac{\sigma}{4})} \right] + V_{eff}$$

(6)

where the effective potential $V_{eff} = \frac{\Lambda}{3} a^3 - p\frac{k^2}{a} - ka - \frac{(a^2 - 2pk)^2}{12(p - \frac{\sigma}{4})}$.

Minimizing $H$ to show Time Crystal behavior: In our model $a$ being the scale factor is always positive. Hence
renaming \((p - \sigma/4) \sim p > 0\), notice that \(H\) can be minimized by separately minimizing the kinetic term, \(a_0^2 = \frac{(a_0^2 - 2pk)}{6p}\) with \(a_0\) determined from minimizing \(V_{eff}\), i.e., \(\partial V_{eff}/\partial a = 0\). Thus, as long as \(a_0 - 2pk \neq 0\), the system can behave as a CTC since the minimum energy state or ground state requires a non-zero velocity \((\dot{a}_0 \neq 0)\) as well as a non-zero coordinate \((a_0 \neq 0)\).

**Thermodynamics of Classical Time Crystal:** Thermodynamic properties of a generic CTC from a statistical mechanics perspective is the new element in our work where we exploit the CTC Hamiltonian. Thermodynamic features of this new phase of matter — CTC — have not been studied earlier and there are novelties both in computational procedure and in results. Very interestingly, we find indications of phase transition as \(\dot{a}\) increases from a small value to \(\dot{a}_0 = \pm \sqrt{(a_0^2 - 2pk)/6p}\) when the CTC phase sets in.

Although we have managed to replace higher time derivatives still the Lagrangian contains quartic term in \(\dot{a}\) that is in fact necessary for the TC behavior. However this makes the conjugate momentum \(p\) multiple valued resulting in different Hamiltonians. Conventionally one computes the partition function summing or integrating over phase space degrees of freedom, i.e., coordinate and momentum but for our model this is not convenient but coordinate \((a)\) and velocity \((\dot{a})\) prove to be the proper choice of variables, as advocated in [16]. However using \(a, \dot{a}\) instead of \(a, p\) is effectively a change of variables that brings in a Jacobian \(J(a, \dot{a})\) in the measure. This has been discussed with specific examples similar to our model in [15]. Hence, the partition function is (with the inverse temperature \(\beta = 1/T\))

\[
Z = \int_{\dot{a}=-\infty}^{\dot{a}=-\infty} d\dot{a} \int_{a=0}^{a=A} da \ J(a, \dot{a}) e^{-\beta H(a, \dot{a})}. \tag{7}
\]

Although the \(\dot{a}\)-integral is analytically doable but it is not possible to perform the \(a\)-integral analytically and thus it compels us to take recourse to numerical integration. Our aim is to consider a canonical ensemble and compute thermodynamic potential such as the Helmholtz free energy \(F\) and subsequently other thermodynamic observables such as average energy, entropy, and higher derivatives such as specific heat and compressibility. Our scheme is the following: we numerically calculate (7) for a set of different values of \(A, T\) (remember that “volume” is linear dimension \(a\) in our case) and thus generate a set of values for \(F\). Note that factor \(J\) in measure can become negative thus rendering \(Z\) unphysical. This forces us to restrict the upper limit of \(a_1\). In all the figures containing red and blue profiles, red line shows ideal gas behavior and blue line depicts the CTC behavior. In order to study rest of the thermodynamic quantities, analytic forms of \(F(T)\) for fixed \(a\) and \(F(a)\) for fixed \(T\) are obtained by curve fitting. Comparison with ideal one dimensional gas profile

\[
F_{\text{ideal}} = -T \left( \text{Constant} + \ln(A) + \frac{1}{2} \ln(T) \right),
\]

it is remarkably clear that the CTC has two distinct type of behaviors: in one sector it resembles the ideal gas whereas in the other sector it behaves differently. This requires two different analytic forms below and above the transitional

\footnote{Note that for a real gas such as e.g. van der Waal equation of state the volume can not be smaller that the total volume of gas particles.}
region. Our method is illustrated in Fig. 2 where the left panel of Fig. 2 we show the evolution of \( \ln Z \) as function of \( T \) for fixed \( a \) and in the right panel of Fig. 2 we show the evolution of \( \ln Z \) as a function of \( a \) for fixed \( T \).

Notice that the CTC curve becomes similar to the ideal gas curve in Fig. 2 after a critical value of \( T \) (or \( a \)) whereas the CTC curve profile is qualitatively different below the critical region. To faithfully represent the numerical results it seems natural to fit the CTC profiles similar to ideal gas for regions above the critical value and as arbitrary power law for region below the critical value. In a set of plots given in Fig. 3 we consider a prototype profile with a fixed and plot Free Energy \( F(T) = -T \ln Z \) (upper left panel of Fig. 3) where we assumed the Boltzmann constant \( k_B = 1 \), the entropy \( S(T) = - (\partial F)/ (\partial T) \) (upper right panel of Fig. 3), energy \( E(T) \) shown in the lower left panel of Fig. 3, and finally the specific heat \( C_a(T) = (\partial E)/(\partial T) = T(\partial F)/(\partial T) - F \) (analogous to \( C_V \)) in the lower right panel of Fig. 3.

It is interesting to note that with our fit for \( \ln Z \) as function of \( T \), \( F = -T \ln Z \) is continuous across the critical value of \( T \) with existence of metastable states along extension along both sides of the critical value of \( T \) in the upper left panel of Fig. 3. However, discontinuities show up in the first and second derivative of \( F \) at a critical \( T \). The jump in \( E(T) \) (upper right panel of Fig. 3) and in \( S(T) \) (lower left panel of Fig. 3) indicate possibility of a structural change with a latent heat given by \( L = T \Delta S \). This is further corroborated in the finite gap in \( C_a \) profile against \( T \) displayed in the lower right panel of Fig. 3.

In the next set of graphs, summarized in Fig. 4 we plot the Free Energy as a function of \( a \) for fixed \( T \), pressure \( P = (\partial F)/(\partial a) \) and compressibility (or the inverse of bulk modulus) \( \kappa = \{(\partial^2 F)/(\partial a^2)\}^{-1} \) respectively. Once again we find that \( F(a) \) is continuous across the critical \( a \)-value, see the upper left panel of Fig. 4. There is a discontinuity in the pressure \( P \) against \( a \) graph indicating the possibility of a liquid-gas like transition where below critical \( a \) pressure rises much sharply with lowering volume in comparison to the zone above critical \( a \) (see the upper right panel of Fig. 4). Consistent with this feature a jump is observed in the profile of compressibility \( \kappa \) versus \( a \) (see the lower panel of Fig. 4).

Lastly in Figure 5 we show the behavior of another useful parameter, the compressibility factor \( z_c = (Pa)/T \) against...
FIG. 4: Upper left panel: Free Energy $F$ vs. length dimension $a$ for fixed $T$. Upper right panel: Pressure $P$ vs. length dimension $a$ for fixed $T$ showing the jump. Lower panel: Compressibility $\kappa$ vs. length dimension $a$ for fixed $T$ showing jump. In all the plots the red and blue curves respectively represent the ideal gas and CTC profiles.

FIG. 5: Plot of the compressibility factor $z_c$ vs. pressure $P$ for the CTC profile (blue curve) showing a deviation from the ideal gas nature (represented by the red curve).

pressure $P$ where $z_c,\text{ideal} = 1$ for ideal gas and any deviation of $z_c$ from unity indicates the non-ideal behavior. Note that $z_c,\text{CTC}$ peaks to a value close to 1 for a specific pressure and falls for larger $P$.

**Thermodynamics of multiverse?** Let us return to cosmology and try to interpret our model of a collection of mini-superspaces as a classical multiverse. Generally multiverse is considered in a quantum framework (see [17] for a recent perspective, also see [18]) and its thermodynamic aspects are studied from entanglement entropic point of view [19]. In the multiverse picture each of our “gas” particles represents a post inflation bubble universe. As suggested by Tegmark [20], the multiverse contains completely disconnected universes, governed by different physical
laws or mathematical structures. According to Linde [21] our world (or multiverse) might be a collection of infinitely many exponentially large parts and in each of them different low-energy laws of physics might prevail. Being very large, each such part evolves effectively independent of other parts. In our model we have considered a simplification where the laws of physics are identical in all member universes. The inflationary scenario is believed to provide such a multiverse structure where it is argued that our observable domain is just a tiny part of a single bubble universe that underwent an extra-fast accelerating phase at some early time due to the effect of a scalar field. According to Vilenkin [22] and Linde [23], the concept of “eternal” inflation may also lead to the multiverse structure where the universe is continually self-reproducing, so that an infinite number of bubbles are extending in both space and time. Proposals for observational signatures for multiverse related ideas are in progress [24].

In the light of above ideas the present work can be tentatively thought to be a study of classical statistical mechanics of multiverse consisting of a collection of identically behaving mini-superspaces. The roles of different thermodynamic observables along with their discontinuous behavior, leading to the possibility of a phase transition, need to be explored.

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