Cosmic String in Scalar-Tensor Gravity

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Abstract

The gravitational properties of a local cosmic string in the framework of scalar-tensor gravity are examined. We find the metric in the weak-field approximation and we show that, contrary to the General Relativity case, the cosmic string in scalar-tensor gravitation exerces a force on non-relativistic, neutral test particle. This force is proportional to the derivative of the conformal factor $A(\phi)$ and it is always attractive. Moreover, this force could have played an important role at the Early Universe, although nowadays it can be negligible. It

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is also shown that the angular separation $\delta \varphi$ remains unaltered for scalar-tensor cosmic strings.

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\section{Introduction}

The scalar-tensor theories of gravity proposed by Bergmann \[1\], Wagoner \[2\] and Nordtverdt \[3\] - generalizing the original Brans-Dicke \[4\] theory - have been considerably revived in the last years. Indeed, the existence of a scalar field as a spin-0 component of the gravitational interaction seems to be a quite natural prediction of unification models such as supergravity or superstrings \[5\]. Apart from the fact that scalar-tensor theories may provide a solution for the problem of terminating inflation \[6, 7\], these theories by themselves have direct implications for cosmology and for experimental tests of the gravitational interaction: One expects that in the Early Universe the coupling to matter of the scalar component of the gravitational interaction would be of the same order of the coupling to matter of the long-range tensor component although in the present time the observable total coupling strength of scalars ($\alpha^2$) is generically small \[8\]. Besides, any gravitational phenomena will be affected by the variation of the gravitational “constant” $G_{\text{eff}} \sim \tilde{\phi}^{-1}$. So, it seems worthwhile to analyse the behaviour of matters in the framework of scalar-tensor theories, specially those which originated in the Early Universe such as topological defects, for exemple.

The aim of this paper is to study the modifications of the metric of a local cosmic string in the framework of the scalar-tensor gravity. These
modifications are induced by the coupling of a scalar field to the tensor field in the gravitational Lagrangean. For simplicity, we will consider a class of scalar-tensor theories where the potential $V(\tilde{\phi})$ (or in Wagoner’s notation $\lambda(\phi)$) is vanishing.
The action describing these theories is (in Jordan-Fierz frame)

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} \left[ \tilde{\phi} \tilde{R} - \frac{\omega(\tilde{\phi})}{\tilde{\phi}} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right] + S_m[\Psi_m, \tilde{g}_{\mu\nu}], \quad (1) $$

where $\tilde{g}_{\mu\nu}$ is the physical metric in this frame, $\tilde{R}$ is the curvature scalar associated to it and $S_m$ denotes the action of the general matter fields $\Psi_m$. These theories are metric, which means that matter couples minimally to $\tilde{g}_{\mu\nu}$ and not to $\tilde{\phi}$. For many reasons it is more convenient to work in the so-called Einstein (conformal) frame, in which the kinetic terms of tensor and scalar fields do not mix

$$ S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left[ R - 2g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] + S_m[\Psi_m, A^2(\phi)g_{\mu\nu}], \quad (2) $$

where $g_{\mu\nu}$ is the (unphysical) metric tensor in Einstein frame, $R$ is the curvature scalar associated to it and $A(\phi)$ is an arbitrary function of the scalar field. Action (2) is obtained from (1) by a conformal transformation in the physical metric

$$ \tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}, \quad (3) $$

and by a redefinition of the quantities

$$ GA^2(\phi) = \tilde{\phi}^{-1}, $$

$$ \alpha^2(\phi) \equiv \left( \frac{\partial \ln A(\phi)}{\partial \tilde{\phi}} \right)^2 = [2\omega(\tilde{\phi}) + 3]^{-1}. $$

It is important to remark that $\alpha(\phi)$ is the field-dependent coupling strength between matter and scalar fields. In the particular case of Brans-Dicke theory,
\( A(\phi) \) has the following dependence on \( \phi \): 
\[
A(\phi) = \exp[2\alpha \phi], \quad \text{with} \quad \alpha(\phi) = \alpha = \text{const.}
\]

In the Einstein frame, the field equations are

\[
R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi + 8\pi G(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T),
\]
\[
\square_g \phi = -4\pi G \alpha(\phi) T. \tag{4}
\]

The first of the above equations can also be written in terms of the Einstein tensor \( G_{\mu\nu} \)

\[
G_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 8\pi G T_{\mu\nu}. \tag{5}
\]

The energy-momentum tensor is defined as usual

\[
T_{\mu\nu} \equiv \frac{2}{\sqrt{g}} \delta S_m[A^2(\phi) g_{\mu\nu}],
\]

and in the Einstein frame it is no longer conserved

\[
\nabla_\nu T^\nu_\mu = \alpha(\phi) T \nabla_\mu \phi. \tag{5}
\]

It is clear from transformation (3), that we can relate quantities from both frames such that \( \tilde{T}^{\mu\nu} = A^{-6} T^{\mu\nu} \) and \( \tilde{T}^\mu_\nu = A^{-4} T^\mu_\nu \).

In what follows, we will search for a regular solution of an isolated static straight cosmic string in the scalar-tensor gravity described above. Hence, the cosmic string arises from the action of the Abelian-Higgs model where a charged scalar Higgs field \( \Phi \) minimally couples to the \( U(1) \) gauge field \( A_\mu \)

\[
S_m = \int dx^4 \sqrt{\tilde{g}} \left[ \frac{1}{2} D_\mu \Phi D^\mu \Phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(|\Phi|) \right], \tag{6}
\]
with $D_\mu \equiv \partial_\mu + ieA_\mu$, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Higgs potential

$$V(|\Phi|) = \lambda(|\Phi|^2 - \eta^2)^2.$$  

$e, \lambda$ and $\eta$ are positive constants, $\eta$ being the characteristic energy scale of the symmetry breaking (for typical grand unified theories (GUT), $\eta \sim 10^{16} \text{GeV}$).

We confine our attention to the static configurations of vortex type about the $z$-axis. In cylindrical coordinate system $(t, z, \rho, \phi)$ with $\rho \geq 0$ and $0 \leq \phi < 2\pi$, we impose the following form for the scalar $\Phi$ and the gauge $A_\mu$ fields:

$$\Phi \equiv R(\rho)e^{i\phi} \quad \text{and} \quad A_\mu \equiv \frac{1}{e}[P(\rho) - 1]\delta_\mu^\phi. \tag{7}$$

where $R, P$ are functions of $\rho$ only. Moreover, we require that these functions are regular and finite everywhere and that they satisfy the usual boundary conditions for vortex solutions:

$$R(0) = 0 \quad \text{and} \quad P(0) = 1$$

$$\lim_{\rho \to \infty} R(\rho) = \eta \quad \text{and} \quad \lim_{\rho \to \infty} P(\rho) = 0. \tag{8}$$

In General Relativity, a metric for the cosmic string described by the action (6) above has been already found in the asymptotic limit by Garfinkle [10] and exactly by Linet [11] provided the particular relation $e^2 = 8\lambda^2$ between the constants $e, \lambda$ is satisfied. The question is whether one can find a solution for the cosmic string in the context of the scalar-tensor gravity. We anticipate that an exact solution for both the matter fields and Einstein equations is impossible to be determined analytically. So that, we will

\footnote{For simplicity, we set the winding number $n = 1$.}
consider the weak-field approximation for this solution in the same way as Vilenkin did in the framework of General Relativity [14]. In fact, the weak-field approximation breaks down at large distances from the cosmic string. Therefore, we assume that, at large distances, the $\phi$ dependence on the right-hand side of the first of Einstein eqs. (4) must dominate over the $T^\mu_\nu$ term. So that, we can neglect the energy-momentum tensor in Einstein eqs. (4) and find the vacuum solution as an asymptotic behaviour of $g_{\mu\nu}$ and $\phi$ far from the cosmic string and, then, match it with the metric in the weak-field approximation.

The plan of this work is as follows. In section 2, we find the exact metric for the vacuum Einstein equations. In section 3, we find the metric of the cosmic string in the weak-field approximation of the scalar-tensor gravity and analyse under which conditions it can be matched to the vacuum metric of section 2. In passing, we show that, contrary to the General Relativity case [14], the cosmic string in scalar-tensor gravity exerces a force on a test-particle. This force is always attractive and proportional to $A'(\phi) \sim \alpha(\phi)$. However, the angular separation $\delta \phi$ remains unaltered ($\delta \phi \sim 10^{-5}\text{rad}$) for scalar-tensor cosmic strings. Finally, in section 4 we add some concluding remarks.

\[ As shown by Laguna-Castillo and Matzner [12], the $T^\mu_\nu$ components vanish far from the cosmic string in General Relativity. Gundlach and Ortiz [13] showed that the same occurs if one considers cosmic string in Brans-Dicke theory. One must expect that this result remains valid in general scalar-tensor theories. \]
2 The Vacuum Metric in Scalar-Tensor Gravity

In this section we will show that there exists an exact static vacuum metric which is solution to the Einstein equations in scalar-tensor theories. Since this solution is supposed to match the (weak-field) solution of the cosmic string, it seems natural to impose that the vacuum spacetime has the same symmetries than the string. So that, we write the following metric

$$ds^2 = g_1(\rho)dt^2 - g_2(\rho)dz^2 - d\rho^2 - g_3(\rho)d\varphi^2,$$

where $g_1, g_2, g_3$ are functions of $\rho$ only, and $(t, z, \rho, \varphi)$ are cylindrical coordinates with $\rho \geq 0$ and $0 \leq \varphi < 2\pi$. Defining $u \equiv (g_1 g_2 g_3)^{1/2}$, the Einstein equations (4) in vacuum are

$$R^i_i = \frac{1}{2u} |u g_i'| = 0, \quad (i = t, z, \varphi)$$

and

$$G^\rho_\rho = -\frac{1}{4} \left[ \frac{g_1' g_2'}{g_1 g_2} + \frac{g_1' g_3'}{g_1 g_3} + \frac{g_2' g_3'}{g_2 g_3} \right] = - (\varphi')^2,$$

where the prime means derivative with respect to $\rho$. Moreover, since $T^\mu_\mu = 0$, the following expression is also valid

$$\Sigma_i R^i_i = \frac{u''}{u} = 0, \quad (i = t, z, \varphi).$$

From (13), it follows that $u$ is a linear function of $\rho$ ($u \sim B\rho$). This result enables us to solve eqs. (10) and (12), respectively
$g_i = k_i^{(0)} \left( \frac{\rho}{\rho_0} \right)^{k_i}$ and, \hspace{1cm} (14)

\[ \phi = \phi_0 + \kappa \ln \frac{\rho}{\rho_0}, \] \hspace{1cm} (15)

where $B, k_i^{(0)}, k_i$ and $\kappa$ are constants to be determined later. Combining the solution for $u$ and (14) and (15) in the differential eq. (11), we find the following relations between the constants

\[ (k_1^{(0)} k_2^{(0)} k_3^{(0)})^{1/2} = B, \]

\[ k_1 + k_2 + k_3 = 2 \quad \text{and}, \]

\[ k_1 k_2 + k_1 k_3 + k_2 k_3 = 4 \kappa^2. \]

Moreover, if we suppose that the Lorentz invariance along the $z$-axis is still valid ($g_1 = g_2$), we finally get

\[ k_1^{(0)} (k_3^{(0)})^{1/2} = B, \] \hspace{1cm} (16)

\[ k_3 = 2 - 2k_1 \quad \text{and}, \] \hspace{1cm} (17)

\[ \kappa^2 = k_1 (1 - \frac{3}{4} k_1). \] \hspace{1cm} (18)

Since the constant $k_1^{(0)}$ can always be absorbed by a redefinition of $t$ and $z$, we get the final form for the vacuum metric

\[ ds^2 = \left( \frac{\rho}{\rho_0} \right)^{k_1} (dt^2 - dz^2) - d\rho^2 - \left( \frac{\rho}{\rho_0} \right)^{2-2k_1} B^2 d\phi^2, \] \hspace{1cm} (19)

in which $k_1$ (and consequently $k_3$), $B$ and $\kappa$ will be fully determined after the introduction of matter fields. It is worthwhile to mention that in the
particular case of Brans-Dicke theory (i.e., substituting \( A^2(\phi) = \exp[2\alpha\phi] \) in expression (19)), we get the same result as Gundlach and Ortiz in ref. [13] if we correct the expression \((r - r_0)\) to \(r/r_0\) in metric (14) in their paper. Note that (19) can also be written in Taub-Kasner form [15] after a suitable transformation of coordinates.

The Ricci tensor in the vacuum spacetime described by (19) is regular (vanishes everywhere) if and only if \( \phi = \phi_0 = \text{constant} \) (i.e., \( \kappa = 0 \)). This should not be surprising in views of the structure of Einstein eqs. (10-12), in particular eq. (11). Moreover, \( \kappa = 0 \) implies that the only allowed values for \( k_1 \) are \( k_1 = 4/3 \) and \( k_1 = 0 \). This latter value corresponds to the conical metric and only in this case \( B^2 = k_3^{(0)} \) can be interpreted as the deficit angle. As we will see in the next section, these values (\( k_1 = 0 \) and \( k_1 = 4/3 \)) are precisely the needed conditions for matching metric (19) to the metric of the cosmic string in the weak-field approximation.

Finally, let us remind that the physical vacuum metric is obtained by multiplying (19) by the conformal factor \( A^2(\phi) \), with \( \phi = \phi_0 + \kappa \ln \rho/\rho_0 \) and \( \kappa^2 = k_1(1 - \frac{2}{3}k_1) \).

### 3 Cosmic String Solution in Scalar-Tensor Gravity: the weak-field approximation

The full Einstein equations are
\[ R^t_t = R^z_z = \frac{1}{2u}[u \frac{g'_1}{g_1}]' = 8\pi G(T^t_t - \frac{1}{2}T), \quad (20) \]

\[ R^\rho_\rho = \frac{1}{2}[2(\frac{g''_1}{g_1}) - (\frac{g'_1}{g_1})^2 + (\frac{g''_3}{g_3}) - \frac{1}{2}(\frac{g'_3}{g_3})^2] \]
\[ = -2(\phi')^2 + 8\pi G(T^\rho_\rho - \frac{1}{2}T), \quad (21) \]

\[ R^\varphi_\varphi = \frac{1}{2u}[u \frac{g'_3}{g_3}]' = 8\pi G(T^\varphi_\varphi - \frac{1}{2}T), \quad (22) \]

\[ \frac{1}{u}[u\phi']' = 4\pi G\alpha(\phi)T, \quad (23) \]

with the matter fields equations (where form (7) for the material fields were imposed)

\[ R''' + R'\left(\frac{g'_1}{g_1} + \frac{1}{2}\frac{g'_3}{g_3} - 2\alpha(\phi)\phi'\right) - R(g_3^{-1}P^2 + 4\lambda A^2(R^2 - \eta^2)) = 0, \quad (24) \]

\[ P''' + P'\left(\frac{g'_1}{g_1} + \frac{1}{2}\frac{g'_3}{g_3} - 4\alpha(\phi)\phi'\right) - e^2 A^2 R^2 P^2 = 0. \quad (25) \]

The non-vanishing components of \( T^\nu_\mu \) are

\[ T^t_t = T^z_z = \frac{1}{2}A^2(\phi) \left[(R')^2 + g_3^{-1}R^2P^2 + \frac{1}{e^2g_3^{-1}}A^{-2}(P')^2 + 2\lambda A^2(R^2 - \eta^2)^2\right], \]

\[ T^\rho_\rho = -\frac{1}{2}A^2(\phi) \left[(R')^2 - g_3^{-1}R^2P^2 + \frac{1}{e^2g_3^{-1}}A^{-2}(P')^2 - 2\lambda A^2(R^2 - \eta^2)^2\right], \]

\[ T^\varphi_\varphi = \frac{1}{2}A^2(\phi) \left[(R')^2 - g_3^{-1}R^2P^2 - \frac{1}{e^2g_3^{-1}}A^{-2}(P')^2 + 2\lambda A^2(R^2 - \eta^2)^2\right]. \quad (26) \]

In views of the impossibility to find an exact solution for eqs. (20-25), we will consider the metric of a cosmic string in scalar-tensor gravity in the
weak-field approximation. In fact, this approximation can be justified only if we consider the scalar field $\phi$ as a small perturbation on the gravitational field of the cosmic string. So that, it may be expanded in terms of a small parameter $\epsilon$ about the values $\phi = \phi_0$ and $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\phi = \phi_0 + \epsilon\phi_{\!\!(1)} + \ldots$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + \ldots$$

$$A(\phi) = A(\phi_0) + \epsilon A'(\phi_0)\phi_{\!\!(1)} + \ldots$$

$$T^\mu_\nu = T^\mu_{(0)\nu} + \epsilon T^\mu_{\!(1)\nu} + \ldots$$

The term $(\phi')^2$ is neglected in the process of linearisation of the Einstein eqs. (20-23). Moreover, in this approximation, the $T^\mu_{(0)\nu}$ term is obtained from (26) by a limit process $\lambda \to \infty$ [11]. Therefore, it tends to the Dirac distribution on the hypersurface $t = \text{and } z=\text{constant}$. In this way, the linearized equations reduce to those of General Relativity [14] (except that in our case $T^\mu_{(0)\nu}$ and $h_{\mu\nu}$ carry the conformal factor $A^2(\phi)$)

$$\nabla^2 h_{\mu\nu} = 16\pi G(T^\mu_{(0)\nu} - \frac{1}{2}\eta_{\mu\nu}T_{(0)}).$$

(27)

Besides, the linearized equation for the scalar field is

$$\nabla^2 \phi_{\!\!(1)} = 4\pi G\alpha(\phi_0)T_{(0)}.$$  

(28)

The solution for eq. (28) is

$$\phi_{\!\!(1)} = 4\mu GA^2(\phi_0)\alpha(\phi_0)\ln(\rho/\rho_0).$$

(29)
This solution matches with the vacuum metric if and only if

\[ \kappa_{\text{lin}} = 4G\mu\alpha(\phi_0)A^2(\phi_0). \quad (30) \]

From (18) we see that the only allowed values for \( k_1 \) are \( k_1 = 0 + O(G^2\mu^2) \) and \( k_1 = 4/3 + O(G^2\mu^2) \). The only physical-meaning result is \( k_1 = 0 \) plus negligible corrections of order \( O(G^2\mu^2) \); the other value corresponding to nonphysical metric.

The solutions for eq. (27) differs from those found by Vilenkin [14] by a conformal factor \( A^2(\phi) \). However, the procedure is the same as in his paper. We obtain

\[
\begin{align*}
\frac{ds^2}{(31)} &= A^2(\phi_0)(1 + 8\mu GA^2(\phi_0)\alpha^2(\phi_0) \ln(\rho/\rho_0))\{dt^2 - dz^2 - d\rho^2 \\
&\quad - [1 - 8\mu GA^2(\phi_0)](\rho/\rho_0)^2d\phi^2 \}.
\end{align*}
\]

Therefore, metric (31) represents an isolated scalar-tensor cosmic string in the weak-field approximation. It is important to remark that the cosmic string in this theory exerces a force on a neutral test-particle given by

\[
\begin{align*}
f^\rho &= -\frac{1}{2}\frac{dh_{00}}{d\rho} \\
&= -4\mu GA^2(\phi_0)\alpha^2(\phi_0)\frac{1}{\rho}.
\end{align*}
\]

and it is always attractive. Moreover, although this force seems to be negligible nowadays, in the Early Universe it was not so in views of the factor \( GA^2(\phi) \) in (32). This result differs from Brans-Dicke [13, 16] because
the parameters in this theory are constants. The light deflection \( \delta \varphi \) is
\[
\delta \varphi = 4\pi GA^2(\phi_0) = 4\pi \tilde{G}_0 \mu,
\]
where \( \tilde{G}_0 \) is the effective Newtonian constant. Therefore, the value \( \delta \varphi \sim 10^{-5}\text{rad} \) for GUT strings remains valid in scalar-tensor gravity as well as in General Relativity. It means that if one believes that the effect of double image of some astrophysical objects is due to the presence of a cosmic string between them and the observer, the fact that the angular separation \( \delta \varphi \sim 10^{-5}\text{rad} \) remains unaltered reveals that it would be impossible to distinguished, at least by this observation, scalar-tensor cosmic strings from General Relativity ones.

## 4 Conclusion

In this work we found the metric of a cosmic string in the weak field approximation of scalar-tensor gravity. It is shown that this metric is conformal to the metric found by Vilenkin in the framework of General Relativity. We showed that, in contrast with the General Relativity case, the cosmic string in scalar-tensor gravity exerces a force on a test particle and that this force is always attractive. However, the light deflection remains the same, at least in the linear approximation. It is worthwhile to point out that in this paper we generalized the works made by Gundlach and Ortiz \cite{Gundlach} and recently by Barros and Romero \cite{Barros}, both in the framework of Brans-Dicke theory of gravity.
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