BUSEMANN–PETTY PROBLEMS FOR Lp MIXED INTERSECTION BODIES

XUEFU ZHANG, SHANHE WU AND YIBIN FENG

(Communicated by J. Pečarić)

Abstract. The notion of Lp mixed intersection bodies was introduced by Ma. In this paper, we consider the Busemann-Petty problems for the Lp mixed intersection bodies.

1. Introduction and main results

Let Sn denote the unit sphere in Euclidean space Rn. If K is a compact star-shaped (about the origin) set in Rn, then its radial function, \( \rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R} \), is defined by (see [9, 34])

\[
\rho(K, x) = \max \{ \lambda : \lambda x \in K \}.
\]

If \( \rho_K \) is positive and continuous, then K will be called a star body with respect to the origin. The set of all star bodies about the origin in \( \mathbb{R}^n \) is denoted by \( \mathcal{S}_o^n \), and the set of all origin-symmetric star bodies in \( \mathbb{R}^n \) will be denoted by \( \mathcal{S}^n \). Two star bodies K and L are said to be dilates of one another if \( \rho_K(u)/\rho_L(u) \) is independent of \( u \in S^{n-1} \).

Let \( \mathcal{K}^n \) denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean \( \mathbb{R}^n \). For \( u \in S^{n-1} \), \( u^\perp \) denotes the \((n-1)\)-dimensional subspace orthogonal to \( u \). We use \( V_k(M) \) to denote the \( k \)-dimensional volume of a \( k \)-dimensional compact convex set \( M \). Instead of \( V_n \) we usually write \( V \). For the standard unit ball \( B \) in \( \mathbb{R}^n \), we write \( \omega_n = V(B) \) for its volume. We also note that \( i \) denotes any real number in this article.

Busemann and Petty [3] posed a problem: Let K and L be origin-symmetric convex bodies in \( \mathbb{R}^n \). Is it true that for any \( u \in S^{n-1} \),

\[
V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp) \implies V(K) \leq V(L) ?
\]

A long list of authors contributed to the solution of this famous problem over a period of 40 years, see [1-2, 10-14, 17, 20-21, 24, 26, 33, 35, 42]. The question has
a negative answer for \( n \geq 5 \) and an affirmative answer for \( n = 3, 4 \). For a detailed account of the interesting history of the Busemann-Petty problems, see the books by Gardner [9, Chapter 8] and Koldobsky [22, Chapter 5].

The crucial idea solving the problem is to define a new convex body which is called the intersection body given by Lutwak [26]. For \( K \in \mathcal{S}_o^n \), the intersection body, \( IK \), of \( K \) is a star body whose radial function in the direction \( u \in S^{n-1} \) is equal to the \((n-1)\)-dimensional volume of the section of \( K \) by \( u^\perp \), i.e.,

\[
\rho(IK, u) = V_{n-1}(K \cap u^\perp).
\]

The intersection bodies have been intensively studied in recent years (see [15-16, 19, 23, 25, 31-32, 38-40] and the books [22, 36]). From (1.1) and the fact that \( K \subseteq L \) for \( K, L \in \mathcal{S}_o^n \) if and only if \( \rho(K, \cdot) \leq \rho(L, \cdot) \), we see that the Busemann-Petty problems can be rephrased in the following way:

For \( K, L \in \mathcal{S}_o^n \), is it true that

\[
IK \subseteq IL \implies V(K) \leq V(L)?
\]

Lutwak [26] showed that the problem has an affirmative answer if the body \( K \) restricted to the class of intersection bodies. In addition, Lutwak proved that if \( L \) is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body \( K \) such that \( IK \subseteq IL \) but \( V(K) > V(L) \).

Further, the Busemann-Petty problems have been considered in the context of \( L_p \) Brunn-Minkoski theory (see [4-8, 27-30, 37, 39]). In particular, Yuan and Cheung [41] generalized the intersection body to \( L_p \) analogue, and introduced the notion of \( L_p \) mixed intersection bodies. Let \( L \) be a star body and nonzero \( p < 1 \). The \( L_p \) intersection body \( IpL \), of \( L \), is the origin-symmetric star body whose radial function is defined by

\[
\rho(IpL, u)^p = \int_L |u \cdot x|^{-p} dx.
\]
for $u \in S^{n-1}$, where $\tilde{W}_{p,i}$ denotes the $L_p$ dual mixed quermassintegrals (see (2.3)).

Suppose that $f$ is a Borel function on $S^{n-1}$. The spherical Radon transform $Rf$ of $f$ was, in [18], defined by

$$(Rf)(u) = \int_{S^{n-1} \cap u^\perp} f(v) dS_{n-2}(v)$$

for $u \in S^{n-1}$. The spherical Radon transform is self-adjoint, i.e., if $f$ and $g$ are defined on $S^{n-1}$ bounded Borel function, then

$$\int_{S^{n-1}} f(u) Rg(u) dS(u) = \int_{S^{n-1}} Rf(u) g(u) dS(u).$$

(1.4)

Using the spherical Radon transform, the definition of $I_{p,i}K$ is rewritten by

$$\rho(I_{p,i}K, u) = \left( \frac{\tilde{W}_{p,i}(K, B \cap u^\perp)}{\tilde{W}_{p,i}(B, B \cap u^\perp)} \right)^{\frac{1}{p}} = \left( \frac{1}{(n-1) \omega_{n-1}} R(\rho^{n-p-i}_K(u)) \right)^{\frac{1}{p}}$$

$$= \left( \frac{1}{(n-1) \omega_{n-1}} \int_{S^{n-1} \cap u^\perp} \rho(K,v)^{n-p-i} dS_{n-2}(v) \right)^{\frac{1}{p}}$$

(1.5)

for $u \in S^{n-1}$.

Thus, such inequalities as the Busemann type inequality, monotonicity inequality and Brunn-Minkowski inequality were shown in [31]. The main aim of this paper is to study the Busemann-Petty problems for the $L_p$ mixed intersection bodies. We first solve the affirmative version for the problems. For convenience, let $\mathbb{I}_{p,i}$ denote the set of $L_p$ mixed intersection bodies.

**Theorem 1.1.** Let $p \geq 1$ and $K, L \in \mathcal{S}_0^n$. if $K \in \mathbb{I}_{p,i}$, then for $i < n - p$,

$$I_{p,i}K \subseteq I_{p,i}L$$

implies

$$\tilde{W}_i(K) \leq \tilde{W}_i(L),$$

with equality if and only if $K = L$; if $L \in \mathbb{I}_{p,i}$, then for $i > n$,

$$I_{p,i}K \subseteq I_{p,i}L$$

implies

$$\tilde{W}_i(K) \leq \tilde{W}_i(L),$$

with equality if and only if $K = L$.

Here, $\tilde{W}_i$ denotes the dual quermassintegrals (see (2.2)).

The following provides the negative version of the Busemann-Petty problems for the $L_p$ mixed intersection bodies.

**Theorem 1.2.** Let $p \geq 1$. If $K \notin \mathcal{S}_e^n$, then there exists $L \in \mathcal{S}_e^n$ such that when $i < n - p$,

$$I_{p,i}K \subset I_{p,i}L$$

has

$$\tilde{W}_i(K) > \tilde{W}_i(L).$$
2. Preliminaries

2.1. $L_p$ dual mixed quermassintegrals

For $K,L \in \mathcal{K}_o^n$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the $L_p$ radial combination, $\lambda \circ K_{+p} \mu \circ L \in \mathcal{K}_o^n$, of $K$ and $L$ is defined, cf. [34], by

$$\rho(\lambda \circ K_{+p} \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \tag{2.1}$$

The dual quermassintegrals of a body $K \in \mathcal{K}_o^n$ is

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i}dS(u). \tag{2.2}$$

Obviously,

$$\tilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n}dS(u) = V(K).$$

For $p \geq 1$, the $L_p$ dual mixed quermassintegrals, $\tilde{W}_{p,i}(K,L)$, of $K,L \in \mathcal{K}_o^n$ was defined, in [31], by

$$\frac{n-i}{p} \tilde{W}_{p,i}(K,L) = \lim_{\epsilon \to 0^+} \frac{\tilde{W}_i(K_{+p} \mu \circ L) - \tilde{W}_i(K)}{\epsilon}. \tag{2.3}$$

From definition (2.3), the following integral representation of $L_p$ dual mixed quermassintegrals was given in [31]: If $K,L \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$\tilde{W}_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-p-i} \rho(L,u)^p dS(u). \tag{2.4}$$

Apparently,

$$\tilde{W}_{p,i}(K,K) = \tilde{W}_i(K).$$

The Minkowski inequalities for the $L_p$ dual mixed quermassintegrals were established in [31]: If $K,L \in \mathcal{K}_o^n$ and $p \geq 1$, then for $i < n - p$,

$$\tilde{W}_{p,i}(K,L) \leq \tilde{W}_i(K) \frac{n-p-i}{n-1} \frac{\tilde{W}_i(L)}{\tilde{W}_i(L)}; \tag{2.5}$$

for $n - p < i < n$ or $i > n$,

$$\tilde{W}_{p,i}(K,L) \geq \tilde{W}_i(K) \frac{n-p-i}{n-1} \frac{\tilde{W}_i(L)}{\tilde{W}_i(L)}. \tag{2.6}$$

In every case, equality holds if and only if $K$ is a dilate of $L$.

2.2. $L_p$ dual mixed Blaschke body

For $K,L \in \mathcal{K}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the $L_p$ dual mixed Blaschke combination, $\lambda \ast K_{+p} \mu \ast L \in \mathcal{K}_o^n$, of $K$ and $L$ is defined by

$$\rho(\lambda \ast K_{+p} \mu \ast L, \cdot)^{n-p-i} = \lambda \rho(K, \cdot)^{n-p-i} + \mu \rho(L, \cdot)^{n-p-i}. \tag{2.7}$$
Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in (2.7), the $L_p$ dual mixed Blaschke body, $\mathbf{\check{W}}_{p,i}K$, of $K$ is defined by

$$\mathbf{\check{W}}_{p,i}K = \frac{1}{2} * K + \frac{1}{2} * (-K).$$

(2.8)

Obviously, the $L_p$ dual mixed Blaschke body is origin-symmetric.

3. Proofs of Theorems 1.1-1.2

The proof of Theorem 1.1 needs the following lemma.

**Lemma 3.1.** If $K, L \in \mathcal{S}_n^p$, then for $p \geq 1$,

$$\tilde{W}_{p,i}(K, I_{p,i}L) = \tilde{W}_{p,i}(L, I_{p,i}K).$$

**Proof.** From (1.4), (1.5) and (2.4), it follows that

$$\tilde{W}_{p,i}(K, I_{p,i}L) = \frac{1}{n} \int_{S_{n-1}} \varrho(K, u)^{n-p-i} \varrho(I_{p,i}L, u)^p dS(u)$$

$$= \frac{1}{n} \int_{S_{n-1}} \frac{1}{(n-1) \omega_{n-1}} \varrho(K, u)^{n-p-i} R(\varrho_L^{n-p-i})(u) dS(u)$$

$$= \frac{1}{n} \int_{S_{n-1}} \frac{1}{(n-1) \omega_{n-1}} R(\varrho_K^{n-p-i})(u) \varrho(L, u)^{n-p-i} dS(u)$$

$$= \frac{1}{n} \int_{S_{n-1}} \varrho(L, u)^{n-p-i} \varrho(I_{p,i}K, u)^p dS(u) = \tilde{W}_{p,i}(L, I_{p,i}K).$$

□

**Proof of Theorem 1.1.** For a star body $\overline{K}$ with $I_{p,i}\overline{K} = K$, it follows from Lemma 3.1 that

$$\tilde{W}_i(K) = \tilde{W}_{p,i}(K, K) = \tilde{W}_p(K, I_{p,i}\overline{K}) = \tilde{W}_{p,i}(\overline{K}, I_{p,i}K);$$

$$\tilde{W}_{p,i}(L, K) = \tilde{W}_{p,i}(L, I_{p,i}\overline{K}) = \tilde{W}_{p,i}(\overline{K}, I_{p,i}L).$$

Since

$$\tilde{W}_{p,i}(\overline{K}, I_{p,i}K) = \frac{1}{n} \int_{S_{n-1}} \varrho(\overline{K}, u)^{n-p-i} \left( \frac{\varrho(I_{p,i}K, u)}{\varrho(I_{p,i}L, u)} \right)^p \varrho(I_{p,i}L, u)^p dS(u)$$

$$\leq \max_{u \in S_{n-1}} \left( \frac{\varrho(I_{p,i}K, u)}{\varrho(I_{p,i}L, u)} \right)^p \tilde{W}_{p,i}(\overline{K}, I_{p,i}L),$$

we have

$$\frac{\tilde{W}_i(K)}{\tilde{W}_{p,i}(\overline{K}, I_{p,i}L)} \leq \max_{u \in S_{n-1}} \left( \frac{\varrho(I_{p,i}K, u)}{\varrho(I_{p,i}L, u)} \right)^p .$$

From $I_{p,i}K \subseteq I_{p,i}L$, we obtain that for $i < n - p$,

$$\tilde{W}_i(K) \leq \tilde{W}_{p,i}(\overline{K}, I_{p,i}L) = \tilde{W}_{p,i}(L, K) \leq \tilde{W}_i(L) \frac{n-p-i}{n-i} \tilde{W}_i(K) \frac{p}{n-i},$$
i.e.,
\[ \tilde{W}_i(K) \leq \tilde{W}_i(L). \]

From the equality condition of (2.5) and the condition \( I_{p,i}K \subseteq I_{p,i}L \), we know that equality holds if and only if \( K = L \).

Let \( I_{p,i}L = L \) for a star body \( L \). By Lemma 3.1, we have
\[
\tilde{W}_i(L) = \tilde{W}_{p,i}(L, L) = \tilde{W}_{p,i}(L, I_{p,i}L) = \tilde{W}_{p,i}(I_{p,i}L, I_{p,i}L); \\
\tilde{W}_{p,i}(K, L) = \tilde{W}_{p,i}(K, I_{p,i}L) = \tilde{W}_{p,i}(I_{p,i}L, I_{p,i}K).
\]

Thus
\[
\tilde{W}_{p,i}(I_{p,i}L, I_{p,i}K) = \frac{1}{n} \int_{S^{n-1}} \rho(\mathcal{L}, u)^{n-p-i} \left( \frac{\rho(I_{p,i}K, u)}{\rho(I_{p,i}L, u)} \right)^p \rho(I_{p,i}L, u)^p dS(u) \\
\leq \max_{u \in S^{n-1}} \left( \frac{\rho(I_{p,i}K, u)}{\rho(I_{p,i}L, u)} \right)^p \tilde{W}_{p,i}(I_{p,i}L, I_{p,i}K),
\]
i.e.
\[
\frac{\tilde{W}_{p,i}(K, L)}{\tilde{W}_i(L)} \leq \max_{u \in S^{n-1}} \left( \frac{\rho(I_{p,i}K, u)}{\rho(I_{p,i}L, u)} \right)^p.
\]

From \( I_{p,i}K \subseteq I_{p,i}L \), it follows that
\[ \tilde{W}_i(L) \geq \tilde{W}_{p,i}(K, L). \]

The above inequality implies that for \( i > n \),
\[ \tilde{W}_i(K) \leq \tilde{W}_i(L). \]

**Lemma 3.2.** If \( K, L \in \mathcal{K}^n_o \), \( p \geq 1 \) and \( \lambda, \mu \geq 0 \) (not both zero), then for \( i < n - p \),
\[
\tilde{W}_i(\lambda \ast K, \mu \ast L)^{\frac{n-p-i}{n-i}} \leq \lambda \tilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \tilde{W}_i(L)^{\frac{n-p-i}{n-i}}; \tag{3.1}
\]
for \( n - p < i < n \) or \( i > n \),
\[
\tilde{W}_i(\lambda \ast K, \mu \ast L)^{\frac{n-p-i}{n-i}} \geq \lambda \tilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \tilde{W}_i(L)^{\frac{n-p-i}{n-i}}. \tag{3.2}
\]
In every inequality, with equality if and only if \( K \) and \( L \) are dilates.

**Proof.** From (2.4), (2.5) and (2.7), we have for any \( Q \in \mathcal{K}^n_o \) and \( i < n - p \)
\[
\tilde{W}_{p,i}(\lambda \ast K, \mu \ast L, Q) = \lambda \tilde{W}_{p,i}(K, Q) + \mu \tilde{W}_{p,i}(L, Q) \\
\leq \left[ \lambda \tilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \tilde{W}_i(L)^{\frac{n-p-i}{n-i}} \right] \tilde{W}_i(Q)^{\frac{p}{n-i}}. \tag{3.3}
\]
Let \( Q = \lambda \ast K, \mu \ast L \) in (3.3). Thus we have
\[
\tilde{W}_i(\lambda \ast K, \mu \ast L)^{\frac{n-p-i}{n-i}} \leq \lambda \tilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \tilde{W}_i(L)^{\frac{n-p-i}{n-i}}.
\]
For $n - p < i < n$ or $i > n$, similar to the above method, we have

$$\tilde{W}_i(\lambda \ast K_\ast_{p,i} \mu \ast L)^{\frac{n-p-i}{n-i}} \geq \lambda \tilde{W}_i(K)^{\frac{n-p-i}{n-i}} + \mu \tilde{W}_i(L)^{\frac{n-p-i}{n-i}}.$$  

Together with the equality conditions of (2.5) and (2.6), we see that equality in every inequality holds if and only if $K$ and $L$ are dilates. □

Let $\lambda = \mu = \frac{1}{3}$, $L = -K$ in (3.1) and (3.2). Then the following is an immediate result of Lemma 3.2.

**Corollary 3.1.** If $K \in \mathcal{S}_o^n$ and $p \geq 1$, then for $i < n - p$ or $n - p < i < n$,

$$\tilde{W}_i(\tilde{\nabla}_{p,i} K) \leq \tilde{W}_i(K); \quad (3.4)$$

for $i > n$,

$$\tilde{W}_i(\tilde{\nabla}_{p,i} K) \geq \tilde{W}_i(K). \quad (3.5)$$

In every inequality, with equality if and only if $K$ is origin-symmetric.

**Lemma 3.3.** If $K \in \mathcal{S}_o^n$, then for $p \geq 1$,

$$I_{p,i}(\tilde{\nabla}_{p,i} K) = I_{p,i} K.$$  

**Proof.** From (1.5), (2.7) and (2.8), we have

$$\rho(I_{p,i}(\tilde{\nabla}_{p,i} K), u)^p = \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1} \cap u^+} \rho \left(\frac{1}{2} \ast K_{\ast_{p,i}} \frac{1}{2} \ast (-K), v\right)^{n-p-i} dS_{n-2}(v)$$

$$= \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1} \cap u^+} \left[\frac{1}{2} \rho(K, v)^{n-p-i} + \frac{1}{2} \rho(-K, v)^{n-p-i}\right] dS_{n-2}(v)$$

$$= \frac{1}{2} \rho(I_{p,i} K, u)^p + \frac{1}{2} \rho(I_{p,i}(-K), u)^p. \quad (3.6)$$

From formula (1.5) we easily see $I_{p,i}(-K) = I_{p,i}(K)$. Thus, we have

$$\rho(I_{p,i}(\tilde{\nabla}_{p,i} K), u)^p = \rho(I_{p,i} K, u)^p,$$

i.e.,

$$I_{p,i}(\tilde{\nabla}_{p,i} K) = I_{p,i} K. \quad \Box$$

**Proof of Theorem 1.2.** Since $K \notin \mathcal{S}_e^n$, (3.4) implies that for $i < n - p$,

$$\tilde{W}_i(\tilde{\nabla}_{p,i} K) < \tilde{W}_i(K).$$

Let $\varepsilon > 0$ such that $\tilde{W}_i((1+\varepsilon)\tilde{\nabla}_{p,i} K) < \tilde{W}_i(K)$. Taking $L = (1+\varepsilon)\tilde{\nabla}_{p,i} K$ we have

$$\tilde{W}_i(K) > \tilde{W}_i(L).$$

However, from formula (1.5) and Lemma 3.3 we get

$$I_{p,i} L = I_{p,i}((1+\varepsilon)\tilde{\nabla}_{p,i} K) = (1+\varepsilon)^{\frac{n-p-i}{p}} I_{p,i}(\tilde{\nabla}_{p,i} K) = (1+\varepsilon)^{\frac{n-p-i}{p}} I_{p,i} K \ni I_{p,i} K. \quad \Box$$
Acknowledgement. The authors would like to thank the anonymous referee for encouraging comments and helpful suggestions which improved greatly the quality of this paper. This work was supported by the Scientific Planning of Education of Gansu (GS[2017]GHBZ051) and Introduction and Use of Open Online Courses of Gansu (2016-47).

REFERENCES

[1] K. M. BALL, Some remarks on the geometry of convex sets, in: J. Lindenstrauss, V. D. Milman (Eds.), Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1317, Springer, Berlin, 1988, pp. 224–231.
[2] J. BOURGAIN, On the Busemann-Petty problem for perturbations of the ball, Geom. Funct. Anal. 1(1991), 1–13.
[3] H. BUSEMANN AND C. M. PETTY, Problems on convex bodies, Math. Scand. 4(1956), 88–94.
[4] W. CHEN, LP Minkowski problem with not necessarily positive data, Adv. Math. 201(2006), 77–89.
[5] K. CHOU AND X. J. WANG, The LP Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205(2006), 33–83.
[6] Y. B. FENG AND W. D. WANG, The Shephard type problems and monotonicity for LP mixed centroid body, Indian J. Pure Appl. Math. 45(2014), 265–283.
[7] Y. B. FENG AND W. D. WANG, Blaschke-Minkowski homomorphisms and affine surface area, Publ. Math. Debrecen 85(2014), 297–308.
[8] Y. B. FENG AND W. D. WANG, Shephard type problems for LP centroid body, Math. Inequal. Appl. 17(2014), 865–877.
[9] R. J. GARDNER, Geometric Tomography, Second ed., Cambridge Univ. Press, New York, 2006.
[10] R. J. GARDNER, Intersection bodies and the Busemann-Petty problem, Trans. Amer. Math. Soc. 342(1994), 435–445.
[11] R. J. GARDNER, A positive answer to the Busemann-Petty problem in three dimensions, Ann. of Math. 140(1994), 435–447.
[12] R. J. GARDNER, A. KOLDOSKY AND T. SCHLUMPRECHT, An analytic solution to the Busemann-Petty problem on sections of convex bodies, Ann. of Math. 149(1999), 691–703.
[13] A. GIANNOPoulos, A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies, Mathematika 37(1990), 239–244.
[14] M. GIERTZ, A note on a problem of Busemann, Math. Scand. 25(1969), 145–148.
[15] P. GOODEY, E. LUTWAK AND W. WEIL, Functional analytic characterizations of classes of convex bodies, Math. Z. 222(1996), 363–381.
[16] E. GRINBERG AND G. ZHANG, Convolutions, transforms, and convex bodies, Proc. London Math. Soc. 78(1999), 77–115.
[17] H. HADWIGER, Radialpotenzintegrale zentralsymmetrischer Rotationskörper und Ungleichheitsaussagen Busemannscher Art, Math. Scand. 23(1968), 193–200.
[18] S. HELGASON, The Radon transform, Second edition, Progress in Mathematics, Vol. 5, Birkhäuser, Boston, 1999.
[19] N. J. KALTON AND A. KOLDOSKY, Intersection bodies and LP spaces, Adv. Math. 196(2005), 257–275.
[20] A. KOLDOSKY, Intersection bodies in R4, Adv. Math. 136(1998), 1–14.
[21] A. KOLDOSKY, Intersection bodies, positive definite distributions, and the Busemann-Petty problem, Amer. J. Math. 120(1998), 827–840.
[22] A. KOLDOSKY, Fourier Analysis in Convex Geometry, Amer. Math. Soc., Providence, RI, 2005.
[23] A. KOLDOSKY, A functional analytic approach to intersection bodies, Geom. Funct. Anal. 10(2000), 1507–1526.
[24] D. G. LARMAN AND C. A. ROGERS, The existence of a centrally symmetric convex body with central cross-sections that are unexpectedly small, Mathematika 22(1975), 164–175.
[25] M. LUDWIG, Intersection bodies and valuations, Amer. J. Math. 128(2006), 1409–1428.
[26] E. LUTWAK, Intersection bodies and dual mixed volumes, Adv. Math. 71(1988), 232–261.
[27] E. LUTWAK, The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem, J. Differential Geom. 38(1993), 131–150.

[28] E. LUTWAK, The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas, Adv. Math. 118(1996), 244–294.

[29] E. LUTWAK, D. YANG AND G. ZHANG, $L_p$ affine isoperimetric inequalities, J. Differential Geom. 56(2000), 111–132.

[30] E. LUTWAK, D. YANG AND G. ZHANG, Sharp affine $L_p$ Sobolev inequalities, J. Differential Geom. 62(2002), 17–38.

[31] T. Y. MA, $L_p$ mixed intersection body, J. Math. Inequal. 8(2014), 559–579.

[32] V. D. MILMAN AND A. PAJOR, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in: Geometric Aspects of Functional Analysis, 1987-1988, in: Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 64–104.

[33] M. PAPADIMITRAKIS, On the Busemann-Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$, Mathematika 39(1992), 258–266.

[34] R. SCHNEIDER, Convex Bodies: The Brunn-Minkowski Theory, Second ed., Cambridge Univ. Press, New York, 2014.

[35] F. E. SCHUSTER, Valuations and Busemann-Petty type Problems, Adv. Math. 219(2008), 344–368.

[36] A. C. THOMPSON, Minkowski Geometry, Cambridge Univ. Press, New York, 1996.

[37] W. D. WANG AND Y. N. LI, Busemann-Petty problems for general $L_p$-intersection bodies, Acta Math. Sin. (Engl. Ser.) 31(2015), 777–786.

[38] W. D. WANG AND Y. N. LI, General $L_p$ intersection bodies, Taiwanese J. Math. 19(2015), 1247–1259.

[39] J. Y. WANG AND W. D. WANG, $L_p$ dual affine surface area forms of Busemann-Petty type problems, Proc. Indian Acad. Sci. (Math. Sci.) 125(2015), 71–77.

[40] W. Y. YU, D. H. WU AND G. S. LENG, Quasi $L_p$ intersection bodies, Acta Math. Sinica 11(2007), 1937–1948.

[41] J. YUAN AND W. CHEUNG, $L_p$ intersection bodies, J. Math. Anal. Appl. 339(2008), 1431–1439.

[42] G. ZHANG, A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$, Ann. of Math. 149(1999), 535–543.

(Received May 13, 2018)