On the amplitudes in $\mathcal{N} = (1, 1)$ $D = 6$ SYM

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ABSTRACT: We consider the on-shell amplitudes in $\mathcal{N} = (1, 1)$ SYM in $D = 6$ dimensions within the spinor helicity and on-shell superspace formalism. This leads to an effective and straightforward technique reducing the calculation to a set of scalar master integrals. As an example, the simplest four point amplitude is calculated in one and two loops in the planar limit. All answers are UV and IR finite and expressed in terms of logs and Polylogs of transcendentality level 2 at one loop, and 4 and 3 at two loops. The all loop leading logarithmical asymptotics at high energy is obtained which exhibits the Regge type behaviour. The intercept is calculated in the planar limit and is equal to $\alpha(t) = 1 + \sqrt{2 \frac{\mu^2 \mathcal{N}_{|| T}^2}{32 \pi^2 t}}$.

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1 Introduction

In the last decade tremendous progress has been achieved in understanding the structure of the $S$-matrix of four dimensional gauge theories (for review see, for example, [1–6]). The most impressive results have been obtained in the theories with extended supersymmetry, the $\mathcal{N} = 4$ SYM is one of the important examples.

The $\mathcal{N} = 4$ SYM in addition to ordinary (super)conformal invariance has a new type of symmetry, dual (super)conformal invariance, i.e., the conformal symmetry in momentum space. Taking together, the algebras of these symmetries can be fused into an infinite dimensional Yangian algebra which in principle should completely define the $S$-matrix of the theory [7–9]. Also, the $\mathcal{N} = 4$ SYM possesses the unexpected dualities between the amplitudes and the Wilson loops, the amplitudes and the correlation functions and presumably between the form factors and the Wilson loops (see, for example, [10–16] for review). In addition to the above mentioned properties, the recent results suggest that in the description of the structure of the $S$-matrix of the $\mathcal{N} = 4$ SYM the language of algebraic geometry (the motive theory) might be useful [17]. All these intriguing properties of the $\mathcal{N} = 4$ SYM are intimately linked together and are not fully understood at the current moment.

It should be noted that the above mentioned results are almost impossible to obtain using the standard textbook computational methods. The new technique is intensively used: the spinor helicity and momentum twistors formalisms, different sets of recurrence relations for the tree level amplitudes, the unitarity based methods for loop amplitudes and different realizations of the on-shell superspace technique for theories with supersymmetry [1–6].
It is interesting to note that the spinor helicity formalism and the unitarity based methods can be generalised to space-time dimension greater than $D = 4$ [18, 19]. So the gauge theories in extra dimensions can also be studied by these methods. For example, the spinor helicity formalism was suggested for $D = 6$ in [20], so the $S$-matrix in $D = 6$ gauge theory can be calculated like its $D = 4$ counterpart.

In particular, the $D = 6$ gauge theories with maximal supersymmetry, namely, $(1,1)$ and $(2,0)$ supersymmetries, are of special interest [21–24]. At the tree level the amplitudes of the $(1,1)$ SYM theory can be interpreted as amplitudes of the $D = 4 \mathcal{N} = 4$ SYM theory with the Higgs regulator [20, 25]. Then, they can be used in the unitarity based computations of the loop amplitudes in the $D = 4 \mathcal{N} = 4$ massive SYM. They may also be useful in QCD computations of rational terms of the one loop amplitudes, and one can encounter other $D = 6$ objects as parts of the QCD multiloop computations [26–30]. In addition, the $(1,1)$ and $(2,0)$ theories can be considered as a special low energy limit (the effective actions on the 5-branes) of the string/M theory. After additional compactifications on two-torus both the theories reduce to the $D = 4 \mathcal{N} = 4$ SYM. One may wonder if the origin of dual (super)conformal symmetry and other "miracles" of the $\mathcal{N} = 4$ SYM lies in the properties of the string theory in 6 dimensions [17]. For these reasons it would be interesting to study the $S$-matrix of the $(1,1)$ and $(2,0)$ $D = 6$ gauge theories.

In this article we focus on the four point amplitudes in the $(1,1)$ $D = 6$ SYM theory. As was explained above, we use the spinor helicity formalism suggested in [20] and the on-shell momentum superspace formalism proposed in [31]. The tree level 3, 4 and 5 point amplitudes were obtained in [20]. In addition to it, the 4- and 5-point amplitudes were studied in [31, 36] at the one loop level. The symmetry properties of the amplitudes were discussed in [37]. Our main interest is the structure of the four point amplitude at the multiloop level and its comparison with the $D = 4 \mathcal{N} = 4$ SYM 4-point amplitude.

The article is organised as follows. In section 2, we discuss the spinor helicity formalism and the general structure of the on-shell momentum superspace in the $D = 6 (1,1)$ SYM. In section 3, we compute the 4-point amplitude in one and two loops in terms of scalar integrals by means of the iterated unitarity cuts. In section 4, we compute the scalar integrals and discuss the structure of the corresponding amplitude. We also investigate its asymptotic Regge behaviour of the 4-point amplitude in all loops and obtain the Regge asymptotics. In appendices, we discuss the computation of the $D = 6$ double box integral by means of the MB representation and give the derivation of the all-loop Regge asymptotics from the ladder diagrams.

2 The spinor helicity formalism and on-shell superspace in six dimensions

This section is based mostly on the papers [20] and [31]. We review first the $D = 6$ spinor helicity formalism used throughout the calculations.

Consider the massless $D = 6$ vector $p^\mu, p^2 = 0, \mu = 1, \ldots, 6$ which transforms under the vector representation of the $D = 6$ Lorentz group $SO(5,1)$. Using the $D = 6$ antisymmetric Pauli matrices $(\sigma^\mu)_{AB}$ and $(\bar{\sigma}_\mu)^{AB}$, where the indices $A, B = 1, \ldots, 4$ transform under the
fundamental representation of the Spin(SO(5, 1)) \simeq SU(4)^* (which is the covering group for SO(5, 1))\(^1\) we can rewrite \(p^\mu\) by analogy with the \(D = 4\) case as:

\[
p^{AB} = p^\mu (\sigma_\mu)^{AB}, \tag{2.1}
\]

or

\[
p_{AB} = p_\mu (\sigma^\mu)_{AB}. \tag{2.2}
\]

Note that one can lower and rise the indices \(A, B\) using absolutely antisymmetric objects \(\epsilon^{ABCD}\) and \(\epsilon_{ABCD}\) associated with \(SU(4)^*\):

\[
(\sigma^\mu)_{AB} = \frac{1}{2} \epsilon_{ABCD} (\sigma_\mu)^{CD}. \tag{2.3}
\]

The condition \(p^2 = 0\) in terms of the matrix \(p^{AB}\) is equivalent to \(\det(p) = 0\), so one can write \(p^{AB}\) as a product of two commuting \(SU(4)^*\) spinors:

\[
p^{AB} = \lambda^{Aa} \lambda^B_a, \quad p_{AB} = \tilde{\lambda}^{\dot{a}A} \tilde{\lambda}_{\dot{a}B}. \tag{2.4}
\]

Note that the spinors \(\lambda^{Aa}\) and \(\tilde{\lambda}^{\dot{a}A}\), in contrast to the \(D = 4\) case, in addition to the covering group index \(A\) also carry the little group \(SO(4)\) which is the covering group \(SO(5, 1)\) and \(SO(2)\) indices \(a = 1, 2\) and \(\dot{a} = \dot{1}, \dot{2}\). The little group for \(D\) dimensions is \(SO(D - 2)\), so for \(D = 4\) it is just \(SO(2) \simeq U(1)\) and the action of the \(D = 4\) little group on spinors is just the multiplication by a complex number \(z\), \(|z| = 1\). In the \(D = 6\) case the action of the little group is no longer trivial and helicity is no longer conserved in contrast to the \(D = 4\) case. Note also that one cannot raise and lower the \(SU(4)^*\) indices for spinors but the little group indices \(a\) and \(\dot{a}\) using the antisymmetric objects \(\epsilon_{ab}\) and \(\epsilon^{ab}\) associated with the \(SU(2)\) groups. Note also that there are no any constraints on the spinors \(\lambda^{Aa}\) and \(\tilde{\lambda}^{\dot{a}A}\) [31] as in the \(D = 4\) case.

The Lorentz invariant products of spinors then can be given by:

\[
\lambda(i)^A \tilde{\lambda}(j)^\dot{a} \equiv \langle i_a | j_a \rangle = [j_a | i_a], \tag{2.5}
\]

where \(i\) and \(j\) are the labels of external momenta \(p_i^\mu\) and \(p_j^\mu\) associated with the spinors \(\lambda(i)^A\) and \(\tilde{\lambda}(j)^\dot{a}\). In addition, one has two Lorentz invariant combinations of spinors:

\[
\epsilon_{ABCD} \lambda(1)^A \lambda(2)^B \lambda(3)^C \lambda(4)^D \equiv \langle 1_a 2_{\dot{b}} 3_c 4_d \rangle, \tag{2.6}
\]

\[
\epsilon^{ABCD} \tilde{\lambda}(1)^\dot{a} \tilde{\lambda}(2)^\dot{b} \tilde{\lambda}(3)^\dot{c} \tilde{\lambda}(4)^\dot{d} \equiv [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}], \tag{2.7}
\]

One can also contract the momenta and spinors in the Lorentz invariant combinations:

\[
\lambda(i)^{A_1 a_1} p_{1, A_1 A_2} p_{2, A_2 A_3} \cdots p_{2n+1, A_{2n+1}} \lambda(j)^{A_{2n+1} b} \equiv \langle i^a | p_1 p_2 \cdots p_{2n+1} j^b \rangle \tag{2.8}
\]

\[
\lambda(i)^{A_1 a_1} p_{1, A_1 A_2} p_{2, A_2 A_3} \cdots p_{2n-1, A_{2n-1}} \tilde{\lambda}(j)^{\dot{a}} \equiv \langle i^a | p_1 p_2 \cdots p_{2n} j^\dot{a} \rangle \tag{2.9}
\]

\(^1\)\(SU(4)^*\) is a none compact generalization of \(SU(4)\). One also has \(Spin(SO(2, 1)) \simeq SL(2, \mathbb{R}), Spin(SO(3, 1)) \simeq SL(2, \mathbb{C}), Spin(SO(5, 1)) \simeq SL(2, \mathbb{Q})\), where \(\mathbb{R}, \mathbb{C}, \mathbb{Q}\) are sets of real and complex numbers and quaternions, respectively.
Using the spinor products one can write, for example, an explicit expression for the $D = 6$ gluon polarization vector with momentum $p^\mu$ as:

$$
\epsilon_{ab}^{AB}(p) = (\lambda(p)_a^A \lambda(q)_b^B - \lambda(p)_a^B \lambda(q)_b^A) \langle q_b|p|_a \rangle^{-1},
$$

(2.10)

or equivalently

$$
\epsilon_{a\dot{b}}^{A\dot{B}}(p) = \langle p^a|q_{\dot{c}} \rangle^{-1} \left( \tilde{\lambda}(q)_{\dot{a}}^A \tilde{\lambda}(p)_{\dot{b}}^\dot{B} - \tilde{\lambda}(q)_{\dot{b}}^\dot{B} \tilde{\lambda}(p)_{\dot{a}}^A \right),
$$

(2.11)

where it is implemented that the inverse matrices $\langle q_b|p|_a \rangle^{-1}$ and $\langle p^a|q_{\dot{c}} \rangle^{-1}$ are nondegenerate, which can be achieved by an appropriate choice of reference momenta $q$ and associated spinors $\lambda(q)_a^A$ and $\tilde{\lambda}(q)_{\dot{a}}^A$.

Consider now the essential parts of the $D = 6 \mathcal{N} = (1, 1)$ on-shell momentum super-space construction. Some aspects of $\mathcal{N} = (1, 1) D = 4, 6$ gauge theories in the standard coordinate off-shell superspace have been investigated in $[32-35]$. The on-shell $\mathcal{N} = (1, 1)$ superspace for $D = 6$ SYM was first formulated in $[31]$. It can be parameterized by the following set of coordinates

$$
\mathcal{N} = (1, 1) D=6 \text{ on-shell superspace} = \{ \lambda_a^A, \lambda_{\dot{a}}^A, \eta_{Ia}, \eta_{I'\dot{a}} \},
$$

(2.12)

where $\eta_{Ia}$ and $\eta_{I'\dot{a}}$ are the Grassmannian coordinates, $I = 1, 2$ and $I' = 1', 2'$ are the SU(2)$_R \times$ SU(2)$_R$ R-symmetry indices. Note that this superspace is not chiral. We have two types of supercharges $q^{AI}$ and $\overline{q}_{AI'}$ with the commutation relations

$$
\{q^{AI}, q^{BJ}\} = p^{AB} \epsilon^{IJ},
$$

$$
\{q^{AI'}, \overline{q}_{BJ'}\} = p_{AB} \epsilon_{I'J'},
$$

$$
\{q^{AI}, \overline{q}_{AI'}\} = 0.
$$

(2.13)

The $\mathcal{N} = (1, 1) D = 6$ SYM on-shell supermultiplet consists of the gluon $A_{a\dot{a}}$, two fermions $\Psi_I^a, \Psi_{I'}^{\dot{a}}$ and two complex scalars $\phi_{I'J'}$ (antisymmetric with respect to $I, I'$). It is CPT self-conjugated. However, to combine all the on-shell states in one superstate $|\Omega\rangle$ by analogy with the $\mathcal{N} = 4 D = 4$ SYM one has to perform a truncation of the full $\mathcal{N} = (1, 1)$ on-shell superspace $[31]$ in contrast to the former case. Indeed, if one expands any function $X$ (or $|\Omega\rangle$ superstate) defined on the full on-shell superspace in Grassmannian variables, one encounters terms like $\sim \eta_{Ia}^I \eta_{I'\dot{a}} A_I^{I'\dot{a}}$. Since there are no such bosonic states $A_I^{I'\dot{a}}$ in the $\mathcal{N} = (1, 1)$ SYM supermultiplet, one needs to eliminate these terms by imposing constraints on $X$, i.e., to truncate the full on-shell superspace. If one wishes to use the little group indices to label the on-shell states, the truncation has to be done with respect to R symmetry indices. This can be done by consistently using the harmonic superspace techniques $[31]$. Defining the harmonic variables $u_I^\pm$ and $\overline{u}_{I'}^\pm$ which parametrize the double coset space

$$
\frac{\text{SU}(2)_R}{\text{U}(1)} \times \frac{\text{SU}(2)_R}{\text{U}(1)}
$$

(2.14)
we express the projected supercharges, the Grassmannian coordinates
\[ \bar{q}^\pm = u^I q^{A_I}, \quad \bar{q}_A^\pm = u^{I'} q_{A_I'}, \]
\[ q^\pm_A = u_I q^I, \quad q^\pm_A = u^{I'} q_{I'}^A, \]
and creation/annihilation operators of the on-shell states
\[ \phi^{--}, \quad \phi^{+-}, \quad \phi^{-+}, \quad \phi^{++}, \]
\[ \Psi^{-a}, \quad \Psi^{+a}, \quad \bar{\Psi}^{-a}, \quad \bar{\Psi}^{+a}, \]
\[ A^{a\dot{a}}. \]  
\[ (2.15) \]
\[ \text{in terms of the new harmonic variables.} \]

Now one has to consider only the objects \( X \) that depend on half of the Grassmannian coordinates \( \eta^-_a, \eta^+_a \), i.e., to impose the Grassmannian analyticity constraints on \( X \):
\[ D^+_A X = D^-_A X = 0, \]  
\[ (2.17) \]
where \( X \) is some function of the full on-shell superspace, and the projectors \( D^+_A \) and \( D^-_A \) are the super covariant derivatives with respect to the supercharges \( (2.13) \). This can be done in a self consistent way if the projectors obey the algebra \( [31] \):
\[ \{D^+_A, D^+_B\} = \{D^-_A, D^-_B\} = \{D^+_A, D^-_B\} = 0, \]
\[ (2.18) \]
which is indeed the case due to eq. \( (2.13) \). Therefore, in what follows we will consider only objects that depend on the set of variables which parametrize the subspace (“analytic superspace”) of the full \( \mathcal{N} = (1,1) \) on-shell superspace
\[ \{ \lambda^A_a, \tilde{\lambda}^{\dot{a}}_a, \eta^-_a, \eta^+_a \}. \]
\[ (2.19) \]
The projected supercharges acting on the analytic superspace for the one particle case can be explicitly written as:
\[ q^+_A = \lambda^A_a \eta^-_a, \quad \bar{q}^+_A = \tilde{\lambda}^{\dot{a}}_A \eta^+_a. \]
\[ (2.20) \]
Now one can combine all the on-shell state creation/annihilation operators \( (2.16) \) into one superstate \( |\Omega_i \rangle = \Omega_i |0 \rangle \) (here \( i \) labels the momenta carried by the state):
\[ |\Omega_i \rangle = \{ \phi^{--}_i + \phi^{+-}_i (\eta^- \eta^-)_i + \phi^{-+}_i (\eta^+ \eta^+_i) + \phi^{++}_i (\eta^- \eta^-)_i (\eta^+ \eta^+_i) + (\Psi^{-+}_i)_i + (\Psi^{-+}_i)_i (\bar{\Psi}^{++}_i)_i + (\Psi^{++}_i)_i (\bar{\Psi}^{++}_i)_i + (A \eta^- \eta^+_i)_i \} |0 \rangle, \]
\[ (2.21) \]
where \( (XY)_i = X^{a\dot{a}}_i Y_{a\dot{a}} \). Hereafter we will drop the \( \pm \) labels for simplicity.

Consider now the colour ordered n-particle superamplitude in the planar limit. The planar limit for the SU(\( N_c \)) gauge group is understood as usual as the limit when \( N_c \rightarrow \infty, g_Y^2 M \rightarrow 0 \) and \( \lambda = g_Y^2 M N_c \) is fixed. We want to note that strictly speaking \( g_Y M \) is
dimensional, so the real PT parameter would be $g_{YM} E$, where $E$ is some energy scale. We will see later the explicit form of $E$. We also use the "all ingoing notation" as usual. Then one has\footnote{We do not write the S-matrix operator in the definition of $A_n$ explicitly.} \[ A_n(\{\lambda^A_n, \lambda_\eta^A_n, \eta_i, \eta \}) = \langle 0 | \prod_{i=1}^n \Omega_i | 0 \rangle. \] (2.22)

The superamplitude should be translationally invariant, i.e., it should be invariant under the action of supercharges (2.20) and the total momenta, i.e., translationally invariant in "analytic superspace". This means that \[ q^n A_n = \eta A_n = p^{AB} A_n = 0, \] (2.23)
where for the n-particle case one has:
\[ q^n = \sum_i^n \lambda^A_i(n) \eta_i, \eta_A = \sum_i^n \lambda^A_i(n) \eta_{a,i}, p^{AB} = \sum_i^n \lambda^A_i(n) \lambda^B_i(n). \] (2.24)
From the later we conclude that the superamplitude should have the form:
\[ A_n(\{\lambda^A, \lambda^A_\eta, \eta_i, \eta \}) = \delta_6(p^{AB}) \delta^4(q^n) \delta^4(\eta_A) \mathcal{P}_n(\{\lambda^A, \lambda^A_\eta, \eta_i, \eta \}), \] (2.25)
where $\mathcal{P}_n$ is a polynomial with respect to $\eta$ and $\eta$ of degree of $2n - 8$. We will drop the momentum conservation delta function $\delta_6(p^{AB})$ from now on. Note that since there is no helicity as a conserved quantum number, there are no closed subsets of MHV, NMHV, etc. amplitudes, and this is all we can say about the general structure of the superamplitude from the supersymmetry alone. The Grassmannian delta functions $\delta^4(q^n)$ and $\delta^4(\eta_A)$ are defined in this case as:
\[ \delta^4(q^n) = \frac{1}{4!} \epsilon^{ABCD} \delta \left( \sum_i^n q_i^A \right) \delta \left( \sum_k^n q_k^A \right) \delta \left( \sum_p^n q_p^A \right) \delta \left( \sum_l^n q_l^A \right), \]
\[ \delta^4(\eta_A) = \frac{1}{4!} \epsilon^{ABCD} \delta \left( \sum_i^n \eta_{A,i}^j \right) \delta \left( \sum_k^n \eta_{A,k}^j \right) \delta \left( \sum_p^n \eta_{A,p}^j \right) \delta \left( \sum_l^n \eta_{A,l}^j \right). \] (2.26)

The integral over $\delta(X) \delta(\overline{X})$ is performed according to the rule:
\[ \int d\eta_i^a \int d\eta_j^a \delta(q^n) \delta(\overline{q}_B) = \lambda(i)^A \tilde{\lambda}_B(j)^b, \] (2.27)
so for the integral over the full superspace (2.19) is
\[ \int d^2 \eta_i^a \int d^2 \eta_j^a \int d^2 \eta_l^b \int d^2 \eta_{l_2}^b = \int d^4 \eta_{l_1 l_2}, \] (2.28)
and we obtain:\footnote{Note that we do not integrate over harmonics $u^{x'=i}$ and $\bar{v}^{x'=i}$, so in this formulation R symmetry is not explicit [31]. We choose different from [36] normalization of the supersum. This can be done by rescaling of $g_{YM}^2$ by 1/32.}
\[ \int d^4 \eta_{l_1 l_2} d^4 \eta_{l_2 l_3} \delta^4(q^n) \delta^4(\overline{q}_B) = 2(l_1, l_2)^2. \] (2.29)
This is an important relation because it allows one to compute sums over the states ("supersums") which appear in two particle cuts within the unitarity based computations. As we will see in the next section for the 4-point amplitude the two particle iterated cuts will be sufficient to construct the integrand up to two loops, just as in the $D = 4$ $\mathcal{N} = 4$ SYM case.

3 The 4-point amplitude in $D=6$

Consider the simplest amplitudes with 4 legs. For $n = 4$ the degree of Grassmannian polynomial $\mathcal{P}_4$ is $2n - 8 = 0$, so $\mathcal{P}_4$ is a function of bosonic variables $\{\lambda^A_a, \tilde{\lambda}^\dot{A}_{\dot{a}}\}$ only

$$A_4(\{\lambda^A_a, \tilde{\lambda}^\dot{A}_{\dot{a}}, \eta_a, \tilde{\eta}_{\dot{a}}\}) = \delta^4(q^4)\delta^4(\overline{q}_A)\mathcal{P}_4(\{\lambda^A_a, \tilde{\lambda}^\dot{A}_{\dot{a}}\}). \quad (3.1)$$

At the tree level $\mathcal{P}_4$ can be found from the explicit answer in components for the 4 gluon amplitude $[20, 31]$ obtained by using the six dimensional version of the BCFW recurrence relation:

$$A_4^{(0)}(1a_2b_3c_4d_4) = -i g^2_{YM} \frac{\langle 1a_2b_3c_4d_4 \rangle \langle 1\dot{a}\dot{2}\dot{b}\dot{3}\dot{c}\dot{4}\dot{d}\rangle}{st}. \quad (3.2)$$

Comparing this expression with (3.1) and expanding (3.1) in powers of $\eta, \tilde{\eta}$ and then extracting the coefficient of $(\eta \eta)(\eta \eta)_2(\eta \tilde{\eta})_3(\eta \tilde{\eta})_4$ one concludes that:

$$\mathcal{P}_4^{(0)} = -i g^2_{YM}/st,$$

where $s$ and $t$ are the standard Mandelstam variables $s = (1 + 2)^2$, $t = (2 + 3)^2$. So, at the tree level the 4-point superamplitude can be written in a very compact form:

$$A_4^{(0)} = -i g^2_{YM} \frac{\delta^4(q^4)\delta^4(\overline{q}_A)}{st}. \quad (3.3)$$

Note that already at the tree level the 5-point amplitude is not so simple $[20, 31]$.

The $D = 6$ SYM theory has the dimensional coupling constant and it is not conformal already at the classical level. On other hand, the tree level amplitudes in this theory possess the dual conformal covariance $[37]$, i.e., they are covariant under inversions $I$ and special conformal transformations $K^\mu$. The explicit form of $I$, $K^\mu$ and the way how the dual coordinates are introduced can be found in $[37]$. Under these transformations of momenta the amplitudes transform as

$$I[\mathcal{P}_n^{(0)}] = \prod_{i=1}^{n} x_i^2 \mathcal{P}_n^{(0)},$$

$$K^\mu[\mathcal{P}_n^{(0)}] = \sum_{i=1}^{n} 2x_i^\mu \mathcal{P}_n^{(0)}.$$

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4We use the compact notation for scalar products of massless momenta $(p_i + p_j)^2 = (i + j)^2 = 2(ij)$. 

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where $x^\mu_i$ are the dual coordinates for momenta $p^\mu_i$. The combination of $\delta$-functions $\delta^6(p^{AB})\delta^4(q^A)\delta^4(p_A)$ transforms covariantly with the factor $(x^2_i)^2$; we will not write them hereafter. At the loop level the L-loop integrand $\text{Int}^P_n^{(L)}$ transforms under inversions $I$ as:

$$I[\text{Int}^P_n^{(L)}] = \prod_{i=1}^n x_{2i}^2 \prod_{k=1}^L (x_{4k}^2)^4 \text{Int}^P_n^{(L)}, \tag{3.5}$$

where $l_k$ is the loop momenta. So the loop amplitude is dually conformal covariant if the loop momentum integration $d^{D=6}l_i$ is restricted to the $d^{D=4}l_i$ subspace only.

It seems that despite the presence of the dual conformal invariance/covariance in the amplitudes of higher dimensional gauge theories its true power at the loop level manifests itself only in four dimensions. Intuitively one may think that it should work in the opposite way.

In the case of the $D = 4\,\mathcal{N} = 4$ SYM, the dual conformal invariance plays an extremely important role. For example, the whole existence of the exact BDS formulas for the 4- and 5-point MHV amplitudes can be understood as a consequence of the dual conformal invariance. In the case of $D = 6\,\mathcal{N} = (1, 1)$ SYM we do not expect similar all loop restrictions but in both theories the dual conformal invariance still restricts the form of the integrands.

Let us now make some short comments about the UV/IR structure of the $D = 6\,\mathcal{N} = (1, 1)$ SYM amplitudes. In [38, 39], the UV finiteness bound

$$D < 4 + \frac{6}{L}, \quad L \geq 2. \tag{3.6}$$

was suggested for the gauge theory amplitudes with maximal supersymmetry. The one loop level is exceptional and the first UV divergences may appear at $D=8$ and not at $D=10$ [40]. We see that for $\mathcal{N} = (1, 1)$ SYM at three loops $D = 6$ is a critical dimension in a sense that the first UV divergence may appear. This is consistent with the old estimates based on off-shell superspace considerations that suggest that the first UV divergence may appear after three loops, i.e. $\mathcal{N} = (1, 1)$ SYM is one and two loop finite [41]. As we will see in a moment this is just the case. In recent years, interest in the UV properties of the S-matrix of formally nonrenormalizable gauge theories with extended supersymmetry ($D = 4\,\mathcal{N} = 8$ SUGRA is a particular example) was reborn. One may hope that there are theories with the UV finite S-matrix which are formally nonrenormalizable [42]. The results obtained so far are in some sense controversial [46–48] but one may still hope for the UV finiteness of $D = 4\,\mathcal{N} = 8$ SUGRA. Regardless of these results we treat the $D = 6\,\mathcal{N} = (1, 1)$ SYM amplitudes in the following way. We compute the 4-point amplitude at the two loop approximation and study the high energy (Regge) asymptotics at all loops. In our considerations we do not encounter any UV divergences. If $D = 6\,\mathcal{N} = (1, 1)$ SYM at higher orders is not UV finite, still one can consider the high energy (Regge) asymptotic behaviour of the n-point amplitudes as computation of some particular limit of the corresponding string/M theory S-matrix. It is interesting to note that there are no IR divergences in $D = 6\,\mathcal{N} = (1, 1)$ SYM, so we obtain completely finite answers for the amplitudes, contrary to the $D = 4\,\mathcal{N} = 4$ case.
Consider now the structure of the four point amplitude at one and two loops. In the unitarity based approach this computation is essentially trivial. Since we expect no UV/IR divergences up to two loops, the amplitudes at this order of PT can be obtained by the unitary cut method without any regularization. The easiest way to obtain the answers in terms of scalar integrals is to use the super amplitude (3.3) and impose the two particle cuts. We use the notation
\[ q^A_L = q_1^A + q_2^A, \]
\[ q^A_R = q_3^A + q_4^A, \]
\[ q^A_{l_1 l_2} = q^A_{l_1} + q^A_{l_2}. \]  
(3.7)

and assume the momentum conservation conditions associated with the amplitudes on both sides of the cut: \( 1 + 2 + l_1 + l_2 = 0 \) and \( -l_1 - l_2 + 3 + 4 = 0 \) (see figure 1). The integrand for the s-channel two particle cut of the one loop amplitude takes the form
\[
\text{Int} A_4^{(1)} = \int d^4 \eta_{l_1 l_2} d^4 \eta_{l_1 l_2} A_4^{(0)}(1, 2, l_1, l_2) \times A_4^{(0)}(-l_1, -l_2, 3, 4) \]  
(3.8)

Using (3.3) and (2.29) and momentum conservation conditions one gets: (the common factor \( g_Y^2 N_c \) is omitted)
\[
\text{Int} A_4^{(1)} = -\int d^4 \eta_{l_1 l_2} d^4 \eta_{l_1 l_2} \frac{\delta^4(q^A_R + q^A_{l_1 l_2}) \delta^4(q^A_L - q^A_{l_1 l_2}) \delta^4(\eta_{A,R} + \eta_{A,l_1 l_2}) \delta^4(\eta_{A,L} - \eta_{A,l_1 l_2})}{s^2(2 + l_1)^2(4 + l_2)^2} \\
= -\delta^4(q^A_R + q^A_L) \delta^4(\eta_{A,R} + \eta_{A,L}) \frac{2(l_1 l_2)^2}{s^2(2 + l_1)^2(4 + l_2)^2} \\
= A_4^{(0)} \frac{st}{2} \frac{-i}{(2 + l_1)^2(4 + l_2)^2}, \]  
(3.9)

which is consistent with the following ansatz for part of the amplitude associated with the s-channel cut
\[
-A_4^{(0)} \frac{st}{2} B(s, t), \]  
(3.10)

where \( B(s, t) \) is the \( D = 6 \) scalar box function. The t-channel cut gives the same result, so we conclude that the full one loop level amplitude has the form:
\[
A_4^{(1)} = -A_4^{(0)} \frac{g_Y^2 N_c}{2} \frac{st}{2} B(s, t). \]  
(3.11)
Consider now the two loops. Applying the two particle cut for the s-channel in a similar way as at the one loop level one gets (see figure 2)

\[
\begin{align*}
\text{Int}A_4^{(2)} &= \int d^4\eta_{l_1l_2}d^4\eta_{l_3l_4} A_4^{(0)}(1, 2, l_1, l_2) \times A_4^{(1)}(-l_1, -l_2, 3, 4) \\
&= \int d^4\eta_{l_1l_2}d^4\eta_{l_3l_4} \frac{\delta^4(q_R^A + q_{l_1l_2}^A)\delta^4(q_L^A - q_{l_1l_2}^A)}{2s} \\
&\times \frac{\delta^4(\bar{q}_{AR} + \bar{q}_{ARl_1l_2})\delta^4(\bar{q}_{AL} - \bar{q}_{ALl_1l_2})B(s, (4 + l_2)^2)}{(2 + l_1)^2} \\
&= \delta^4(q_R^A + q_L^A)\delta^4(\bar{q}_{AR} + \bar{q}_{ARl_1l_2})\frac{(l_1l_2)^2}{s(2 + l_1)^2}B(s, (4 + l_2)^2) \\
&= A_4^{(0)} \frac{s^2t}{4} \frac{i}{(2 + l_1)^2}B(s, (4 + l_2)^2),
\end{align*}
\]

which is consistent with the following ansatz for part of the amplitude associated with the s-channel cut

\[
A_4^{(2)}|_s = A_4^{(0)} \frac{s^2t}{4} DB(s, t),
\]

where \(DB(t, s)\) is the \(D = 6\) scalar double box function. The t-channel two particle cut gives a similar result:

\[
A_4^{(2)}|_t = A_4^{(0)} \frac{st^2}{4} DB(t, s),
\]

so combining the two contributions together we obtain:

\[
A_4^{(2)} = A_4^{(0)} \left(\frac{g^2 N_c}{4}\right) \left( s^2 t \; DB(s, t) + st^2 \; DB(t, s) \right).
\]
The six dimensional boxes and the high energy limit

The $D = 6$ scalar boxes $B(s, t)$ and $DB(s, t)$ are completely finite functions of the Mandelstam variables and can be computed in terms of logarithms, Polylogarithms and harmonic sums by means of either the Feynman parametrization or the MB representation technique (see appendix).

The evaluation of the single box $B(s, t)$ is straightforward and gives

$$B(s, t) = \frac{\pi^3}{(2\pi)^6} \frac{b_2(x)}{s + t}, \quad b_2(x) = \frac{L^2(x) + \pi^2}{2}, \quad x = \frac{t}{s},$$

(4.1)

where $L(x) \equiv \log(x)$.

The double box $DB(s, t)$ can be evaluated by means of the MB representation. We reproduce this derivation in appendix A. One has [49]:

$$DB(s, t) = \left( \frac{\pi^3}{(2\pi)^6} \right)^2 \left( \frac{b_4(x)}{t} + \frac{b_3(x)}{s + t} \right),$$

(4.2)

where

$$b_4(x) = \left( 2\zeta_3 - 2Li_3(-x) - \frac{\pi^2}{3} L(x) \right) L(1 + x) + \left( \frac{1}{2} L^2(x) + \frac{\pi^2}{2} \right) L^2(1 + x)$$

$$+ \left( 2L(x)L(1 + x) - \frac{\pi^2}{3} \right) Li_2(-x) + 2L(x)S_{1,2}(-x) - 2S_{2,2}(-x),$$

(4.3)

$$b_3(x) = -2\zeta_3 + \frac{\pi^2}{3} L(x) - \left( L^2(x) + \pi^2 \right) L(1 + x) - 2L(x)Li_2(-x) + 2Li_3(-x)$$

(4.4)

and $S_{1,2}$ and $S_{2,2}$ are the harmonic polynomials, their explicit form can be found in [58]. Note that both expressions for $B(s, t)$ and $DB(s, t)$ are real when $s > 0, t > 0$ (euclidian region), while in the physical region $t < 0$ the imaginary part appears.

The expression for the double box (4.2)–(4.4) contains all typical transcendental structures and does not reduce to logarithms contrary to the 4-point function in the $D = 4 \mathcal{N} = 4$ case. This does not happen for the full answer (3.15) as well, the Polylog functions remain. Note that even for the full amplitude in contrast to the $D = 4 \mathcal{N} = 4$ SYM case the maximal transcendentality principle no longer holds. While both $b_2(x)$ and $b_4(x)$ are
uniform and obey the maximal transcendentality criteria,\(^5\) \(b_3(x)\) is also uniform but has a lower transcendentality level.

In the case of \(D = 4\) \(\mathcal{N} = 4\) SYM there is a conjecture that the maximal transcendentality principle can be explained by using recurrence relations for the integrands written in terms of momentum twistor variables [17]. One may speculate that some generalization of the maximal transcendentality principle still holds for the \(D = 6\) \(\mathcal{N} = (1, 1)\) SYM. Indeed, in the \(D = 6\) case, there exists a supertwistor formalism based on \(OSp^*(8|2)\) superconformal group [31]. Also, the \(D = 6\) \(\mathcal{N} = (1, 1)\) SYM amplitudes at the tree level can be understood as the Higgs regulated \(D = 4\) \(\mathcal{N} = 4\) SYM ones and the construction of the integrand is based only on properties of the tree level amplitudes.

Let us now consider the high energy behaviour of our four point amplitude. In this regime it is usually possible to obtain simple expressions in each order of PT so that the all loop summation in terms of known functions becomes possible. One can also think of the high energy behaviour of the field theory amplitudes as a special limit of the corresponding string theory S-matrix.

Let’s consider the so-called Regge limit for the four point amplitude and investigate the leading logarithmic behaviour in this limit. In the Regge limit when \(s \to +\infty\) and \(t < 0\) is fixed the main logarithmical contribution comes from the vertical ladder diagrams. At the one loop order one has:

\[
B(s, t)|_{s \to \infty} \simeq \frac{1}{2} \frac{L^2(x)}{s} + \ldots \tag{4.5}
\]

At two loops the main contribution comes from the vertical double box \(DB(t, s)\) which is equal to

\[
DB(t, s)|_{s \to \infty} \simeq \frac{1}{12} \frac{L^4(x)}{s} + \ldots \tag{4.6}
\]

We neglect here all the terms \(\sim L^k(x), \) with \(k < 2n\) and the constants.

Substituting eqs. (4.5) and (4.6) into eq. (3.15) one gets

\[
A_4 \simeq A^{(0)}_4 \left[ 1 + \left( \frac{g^2_{YM}\mathcal{N}_s|t|}{128\pi^3} \right) \frac{L^2(x)}{2} + \left( \frac{g^2_{YM}\mathcal{N}_s|t|}{128\pi^3} \right)^2 \frac{L^4(x)}{12} + \ldots \right]. \tag{4.7}
\]

Note that the dimensional coupling \(g^2_{YM}\) is always multiplied by \(t\) forming the dimensionless expansion parameter \(g^2_{YM}\mathcal{N}_s|t|\).

In higher orders of PT the main leading logarithmic contribution in this limit also comes from the vertical multiple boxes, the so-called ladder diagrams. Their asymptotics are well known in \(D = 4\) and can be similarly evaluated in \(D = 6\). We consider this

\(^5\)If we attach to each logarithm and \(\pi\) the level of transcendentality equal to 1 and to Polylogarithms \(Li_n(x)\) and \(\zeta_n\), the level of transcendentality equal to \(n\), then at the given order of perturbation theory the coefficient for the \(n\)-th pole \(1/\epsilon^n\) has the overall transcendentality equal to \(2l - n\), where \(l\) is the number of loops; \(n = 0\) corresponds to a finite part. For a product of several factors it is given by the sum of transcendentals of each factor.
derivation in appendix B. The result for the Regge limit of the vertical n-loop ladder diagram is UV and IR finite and takes the form

\[ \frac{1}{s} \frac{L^{2n}(x)}{n!(n+1)!}. \]  
(4.8)

This is similar to the case of SU(N赁) matrix gφ3 theory in D = 6 in large N赁 limit. The ladder diagrams gives leading contribution in the former case, but coefficients before ladder diagrams are different in comparison to the \( \mathcal{N} = (1, 1) \) SYM. One also has to take into account the running of the coupling g due to the presence of UV divergences in gφ3 theory in D = 6. After resummation this leads to different Regge behaviour than in the \( \mathcal{N} = (1, 1) \) SYM. The details can be found in [50, 51].

Combined with the coefficients \( s^n/2^n \) contributions from vertical ladder diagrams in the case of D = 6 \( \mathcal{N} = (1, 1) \) SYM leads to the series

\[ A_4 \approx A_4^{(0)} + \sum_{n=0}^{\infty} \frac{\lambda^n L^{2n}(x)}{n!(n+1)!} \lambda \equiv \frac{g_{YM}^2 N_c|t|}{128\pi^3}. \]  
(4.9)

This series can be summed and represents the Bessel function of the imaginary argument

\[ A_4 \approx A_4^{(0)} \frac{I_1(2y)}{y}, \quad y \equiv \sqrt{\lambda} L(x). \]  
(4.10)

In the Regge limit when \( y \to \infty \) \( I_1(2y) \to \exp(2y)/(2\sqrt{\pi y}) \) and one gets the Regge type behaviour\(^6\)

\[ \frac{A_4}{A_4^{(0)}} \sim \left( \frac{s}{t} \right)^{\alpha(t)-1} \]  
(4.11)

with

\[ \alpha(t) = 1 + 2\sqrt{\lambda} = 1 + \sqrt{\frac{g_{YM}^2 N_c|t|}{32\pi^3}}. \]  
(4.12)

We want to stress once again that all contributions from the terms \( \lambda^n L^x(x) \) with \( k < 2n \) are omitted. One can see that as expected for the gauge theory \( \alpha(0) = 1 \). Note that because there are no UV/IR divergences in the Regge limit in the \( D = 6 \mathcal{N} = (1, 1) \) SYM our result for the amplitude is completely independent of any kind of regulator. Notice also that the limit \( y \to \infty \) can be achieved in two regimes:

\[ L(x) \gg 1, \quad \lambda < 1 \quad \text{or} \quad \lambda > L(x) \gg 1. \]  
(4.13)

The first one is the weak coupling regime while the second one resembles the strong coupling limit.

It is interesting to compare this result to the Regge behaviour of the \( D = 4 \) gauge and gravity theories with maximal supersymmetry: \( \mathcal{N} = 4 \) SYM and \( \mathcal{N} = 8 \) SUGRA. For the

\(^6\)Note that the tree amplitude \( A_4^{(0)} \sim s/t \) just like in four dimensions, this can be seen from (3.1), noting that \( \langle 1_a 2_b 3_c 4_d \rangle \sim [1_a 2_b 3_c 4_d] \sim s. \)
In the Regge limit in dimensional regularization the BDS ansatz reduces to [52, 53] (see also [55] for the recent discussion):

\[ \frac{A_4}{A_4^{(0)}} \sim \left( \frac{s}{t} \right)^{\alpha(t)-1}, \]  

(4.14)

with (we assume that \( t \gg \mu^2 \), where \( \mu^2 \) is the dimensional parameter of the dimensional regularization parameter, and \( \lambda_4 = g_4^2 N_c \), where \( g_4 \) is the dimensionless coupling constant of the \( D = 4 \) SYM theory)

\[ \alpha(t) = 1 - \frac{f(\lambda_4)}{4} L \left( \frac{t}{\mu^2} \right), \]  

(4.15)

where \( f(\lambda_4) \) is the cusp anomalous dimension. In the weak/strong coupling regimes one has:

\[ \alpha(t) = 1 - \frac{\lambda_4^2}{16 \pi^2} L \left( \frac{t}{\mu^2} \right) + \ldots, \quad \lambda_4 \ll 1 \]

(4.16)

\[ \alpha(t) = 1 - \left( \frac{\sqrt{\lambda_4}}{\pi} \right) L \left( \frac{t}{\mu^2} \right) + \ldots, \quad \lambda_4 \gg 1. \]

It is remarkable that in \( D = 6 \mathcal{N} = (1, 1) \) the dependence of \( \alpha(t) \) on the effective coupling \( \lambda \) is similar to that in the \( D = 4 \mathcal{N} = 4 \) SYM in the strong coupling regime. Note also that the result of summation of the leading logarithms (4.10) is similar to the exact result for the vacuum expectation of a circular Wilson loop in the \( D = 4 \mathcal{N} = 4 \) SYM [56].

For the \( \mathcal{N} = 8 \) SUGRA one has [53, 57]:

\[ \frac{A_4}{A_4^{(0)}} \sim \left( \frac{s}{t} \right)^{\alpha(t)-2}, \]  

(4.17)

with (\( k \) is the dimensional \( D = 4 \) gravitational coupling constant)

\[ \alpha(t) = 2 - \frac{k^2 t}{2} L \left( \frac{t}{\mu^2} \right) + \ldots. \]

(4.18)

The effective coupling constant here is \( k^2 t \) like in the \( D = 6 \mathcal{N} = (1, 1) \) SYM.

5 Conclusion

In this article we discussed the structure of the four point amplitude in the \( D = 6 \mathcal{N} = (1, 1) \) SYM at one and two loop orders in the planar limit and studied the high energy asymptotics in the Regge limit.

The reduction of the one and two loop amplitudes to the scalar integrals is essentially trivial when the \( D = 6 \) spinor helicity and the on-shell momentum superspace formalisms are used. Up to two loops all the scalar integrals can be written in terms of the box and double box functions in \( D = 6 \) which can be evaluated by the MB representation method. These functions are IR and UV finite in agreement with the UV finiteness bounds. The
three loop computations are also possible; however, they are more involved since the Barnes lemmas are no more sufficient to compute the $D = 6$ three loop boxes. The answers for the one and two loop $D = 6$ boxes can be written in terms of logarithms, Polylogarithms and harmonic polynomials (harmonic Polylogarithms) of transcendentality 2 and 4 or 3 at one and two loops, respectively.

We see that for the full amplitude the contributions with transcendentality 3 do not cancel, so the maximal transcendentality principle no longer holds. Still one may wonder whether some generalization of the maximal transcendentality principle may be formulated. Indeed, the integrands of $D = 6 \mathcal{N} = (1,1)$ SYM can be interpreted as integrands of $D = 4 \mathcal{N} = 4$ SYM on a Coulomb branch (Higgs regulated). Also, for $D = 6$ there exists a twistor formalism based on the $OSp^*(8|2)$ superconformal group, which may be useful in explicit computations as our experience with the $D = 4 \mathcal{N} = 4$ SYM tells us.

The leading logarithmic behaviour in high energy limit ($s \gg 1$) of the four point amplitude is determined by the contributions of the vertical $D = 6$ L-rung boxes, whose leading asymptotics can be evaluated. The all order summation gives the Bessel function from which the Regge behaviour of the amplitude with $\alpha(0) = 1$ can be obtained as expected. It is interesting to note a similar dependence of $\alpha(t)$ on the effective coupling $\lambda$, as in the strong coupling limit of the $D = 4 \mathcal{N} = 4$ SYM.

In our analysis we completely ignored a possible nonperturbative contribution from the classical field configurations. They might be interesting by themselves. The instantons from $D = 4$ when uplifted to $D = 6$ become instantonic strings, the one dimensional objects with their own nontrivial dynamics. It would be interesting to study how such contributions might affect the scattering amplitudes.

Another interesting question is whether there is some form of geometrical realization of symmetries of $D = 6 \mathcal{N} = (1,1)$ SYM, i.e., some analog of the Wilson loop/amplitude duality for the $D = 6 \mathcal{N} = (1,1)$ SYM [18]. We hope that the results obtained here might be useful in this quest.

### A Calculation of massless $D = 6$ double box integral by means of MB representation

Here we present the evaluation of the box and double box integrals. The box integral is defined as

$$Box(s,t) = \frac{1}{i} \int \frac{d^6k}{(2\pi)^6} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2}. \quad (A.1)$$

This integral can be easily evaluated by Feynman parametrization. The result is given by (4.1).

The double box integral is defined as

$$DBox(s,t) = \frac{1}{i^2} \int \frac{d^6k}{(2\pi)^6} \frac{d^6l}{(2\pi)^6} \frac{1}{k^2l^2(k+p_1)^2(k+p_1+p_2)^2} \times \frac{1}{(l+p_1+p_2)^2(l+p_1+p_2+p_4)^2(k-l)^2}. \quad (A.2)$$
This integral is evaluated with the help of the Mellin-Barnes representation method. We use the MB expression of the horizontal double box integral from V. Smirnov’s book [58]

\[
DBox_6(s, t) = \frac{-\pi^6}{s} \int_{-i\infty}^{i\infty} \frac{dz_1 \ldots dz_4}{(2\pi i)^4} x^{z_1} \times
\frac{\Gamma(1 + z_1)\Gamma(-z_1 - z_2)\Gamma(-z_1 - z_3)\Gamma(-z_2 - z_3 - z_4)}{(z_2 + z_4)(z_3 + z_4)(2 + z_1 - z_4)(1 + z_4)}
\times \Gamma(-z_1)\Gamma(1 + z_1 + z_2 + z_3 + z_4)\Gamma(1 + z_1 - z_3)\Gamma(1 + z_2)\Gamma(1 + z_3)\Gamma(z_4),
\]

(A.3)

where \( x = t/s \). The Mellin-Barnes integrals can be evaluated by using the Barnes lemmas (see [58], Ch.D). To control the correctness of the choice of the integration contour, we check each step numerically. For this purpose one has to choose first the real parts of the integration variables \( z_i \) in such a way that all the arguments of the \( \Gamma \) functions in (A.3) are positive. One of the possible choices is \( z_1 = -1/4, z_2 = -9/32, z_3 = -27/64, z_4 = 9/16 \). The result does not depend on a particular choice.

The integral over \( z_2 \) is straightforward with the help of the first lemma and the second lemmas

\[
\int_{-i\infty}^{i\infty} \frac{dz_2}{2\pi i} \frac{\Gamma(1 + z_2)\Gamma(-z_1 - z_2)\Gamma(-z_2 - z_3 - z_4)\Gamma(1 + z_1 + z_2 + z_3 + z_4)}{(z_2 + z_4)} = \Gamma(1 - z_1)\Gamma(z_3 + z_4)\Gamma(z_1)\Gamma(1 - z_3 - z_4) \left(1 - \frac{\Gamma(-z_1 + z_4)\Gamma(-z_3)}{\Gamma(z_4)\Gamma(-z_1 - z_3)} \right).
\]

(A.4)

The integral over \( z_3 \) is already tricky due to the degeneracy of the arguments of the \( \Gamma \) functions, and one has to modify the contour to keep all the poles to the right. The result is

\[
- \int_{-i\infty}^{i\infty} \frac{dz_3}{2\pi i} \Gamma(1 + z_3)\Gamma(-z_1 - z_3)\Gamma(-z_3 - z_4)\Gamma(z_3 + z_4) \left(1 - \frac{\Gamma(-z_1 + z_4)\Gamma(-z_3)}{\Gamma(z_4)\Gamma(-z_1 - z_3)} \right)
= \Gamma(1 - z_1)\Gamma(-z_1 + z_4) \left[\psi(z_4) - \psi(-z_1 + z_4) + \psi(1 - z_1) - \psi(1) \right].
\]

(A.5)

The integral over \( z_4 \) can be composed into 2 integrals which again can be evaluated with the help of derivative of the first lemma

\[
\int_{-i\infty}^{i\infty} \frac{dz_4}{2\pi i} \frac{\Gamma(1 - z_4)\Gamma(-z_1 + z_4)\Gamma(1 + z_1 - z_4)\Gamma(z_4)}{(2 + z_1 - z_4)(1 + z_1 - z_4)} \times
\times [\psi(z_4) - \psi(-z_1 + z_4) + \psi(1 - z_1) - \psi(1)]
= - \int_{-i\infty}^{i\infty} \frac{dz_4}{2\pi i} \Gamma(1 - z_4)\Gamma(-1 - z_1 + z_4)\Gamma(1 + z_1 - z_4)\Gamma(z_4) \times
\times [\psi(z_4) - \psi(-z_1 + z_4) + \psi(1 - z_1) - \psi(1)]
- \int_{-i\infty}^{i\infty} \frac{dz_4}{2\pi i} \Gamma(1 - z_4)\Gamma(-2 - z_1 + z_4)\Gamma(2 + z_1 - z_4)\Gamma(z_4) \times
\times [\psi(z_4) - \psi(-z_1 + z_4) + \psi(1 - z_1) - \psi(1)].
\]

(A.6)
After a careful choice of the integration contours one has for each term separately:

\[ I_{11} = -\Gamma(1 + z_1)\Gamma(-z_1) \left[ \psi^2(1 + z_1) - \psi(1 + z_1)\psi(1) + \psi'(1 + z_1) - \psi'(1) \right], \]

\[ I_{21} = -\Gamma(2 + z_1)\Gamma(-1 - z_1) \left[ \psi^2(2 + z_1) - \psi(2 + z_1)\psi(1) + \psi'(2 + z_1) - \psi'(1) \right], \]

\[ I_{12} = \frac{1}{2} \Gamma(1 + z_1)\Gamma(-z_1) \left[ 2\psi(1 + z_1)\psi(1) - 2\psi(-z_1)\psi(1) + \psi^2(-z_1) - \psi^2(1) + \frac{\pi^2}{2} - \psi'(1) \right], \]

\[ I_{22} = \frac{1}{2} \Gamma(2 + z_1)\Gamma(-1 - z_1) \left[ 2\psi(2 + z_1)\psi(1) - 2\psi(-1 - z_1)\psi(1) + \psi^2(-1 - z_1) - \psi^2(1) + \frac{1}{1 + z_1} + 2\psi(1 + z_1) - 2\psi(-z_1) - 2\psi(1) \right], \]

\[ I_{31} = -\Gamma(1 + z_1)\Gamma(-z_1) \left[ \psi(1 + z_1) - \psi(1) \right] \left( \psi(1 - z_1) - \psi(1) \right), \]

\[ I_{32} = -\Gamma(2 + z_1)\Gamma(-1 - z_1) \left[ \psi(2 + z_1) - \psi(1) - 1 \right] \left( \psi(1 - z_1) - \psi(1) \right). \]

Summing up one finds the result for the integral over \( z_1 \)

\[ -2\Gamma(1 + z_1)\Gamma(-z_1) \frac{z_1}{1 + z_1} (\psi(1 + z_1) - \psi(1)). \quad (A.6) \]

Eventually, one gets the remaining integral over \( z_1 \)

\[ DBox_6(s, t) = \left( \frac{\pi^3}{(2\pi)^6} \right)^2 \frac{2}{s} \int_{i\infty}^{i\infty} \frac{dz_1}{2\pi i} x^{z_1} \left[ \Gamma(1 + z_1)\Gamma(-z_1) \right]^3 \left[ \psi(1 + z_1) - \psi(1) \right] \times \left( 1 - \frac{1}{1 + z_1} \right), \quad (A.7) \]

which can be calculated taking the residues at \( z_1 = 0, 1, \ldots \) and evaluating the sum. The last step can be performed with the help of the formulae from [58], Ch.C. The result is

\[ DBox_6(s, t) = \left( \frac{\pi^3}{(2\pi)^6} \right)^2 \left( \frac{b_4(x)}{t} + \frac{b_3(x)}{s + t} \right), \quad (A.8) \]

where the functions \( b_i(x) \) are given above and coincide with the ones obtained in [49] by using differential equations method.

**B Regge limit of the vertical \( l \)-loop ladder diagram**

Consider the \( D = 6 \) box type scalar integral with \( l \)-rungs shown in figure 4. It is UV/IR finite in all orders of PT. We are interested in its asymptotics in the Regge limit when
Figure 4. The Box type scalar integral with l-rungs $B_l(t,s)$.

$s \to +\infty$, $t < 0$ and fixed. In what follows we use the evaluation method suggested by E. Kuraev.

First consider the one loop box $B_{l=1}(t,s)$. It is given by the integral:

$$B_{l=1}(t,s) = \frac{1}{i} \int \frac{d^6k}{(2\pi)^6} \frac{1}{k^2(k - p_2)^2(k + p_1)^2(k + p_1 + p_4)^2}.$$  \hspace{1cm} \text{(B.1)}$$

Note that $B_{l=1}(t,s) = B_{l=1}(s,t)$. Using the Sudakov variables to parametrize the loop momentum

$$k = \alpha p_2 + \beta p_1 + k_\perp,$$  \hspace{1cm} \text{(B.2)}$$

one gets

$$d^6k = \frac{s}{2} \, d\alpha \, d\beta \, d^3k_\perp, \ \ d^4k_\perp = k^2_\perp \, dk^2_\perp \, d\Omega_4,$$  \hspace{1cm} \text{(B.3)}$$

where as usual

$$s = (p_1 + p_2)^2, \ \ t = (p_1 + p_4)^2, \ \ s > 0, t < 0.$$  \hspace{1cm} \text{(B.4)}$$

In the limit of $s \gg 1$ $(k + p_1)^2$ and $(k - p_2)^2$ can be estimated as

$$(k + p_1)^2 \simeq s\alpha, \ (k - p_2)^2 \simeq -s\beta,$$  \hspace{1cm} \text{(B.5)}$$

so we can rewrite $B_{l=1}$ as (hereafter we will omit common $(-\pi^3/(2\pi)^6)^l$ factor)

$$B_{l=1}(t,s) \simeq \frac{1}{s} \int_{t/s}^1 \frac{d\alpha \, d\beta}{\alpha \beta} \int \frac{d^4k_\perp}{k^2(k + p_1 + p_4)^2}.$$  \hspace{1cm} \text{(B.6)}$$

The leading asymptotic of the bubble type integral can be estimated as

$$\int \frac{d^4k_\perp}{k^2(k + p_1 + p_4)^2} \simeq \theta(s\alpha\beta - t),$$  \hspace{1cm} \text{(B.7)}$$
So for the box integral one gets (remind that $x = s/t$, $L(x) \doteq \log(x)$)

\[
B_{l=1}(t, s) \simeq \frac{1}{s} \int_{t/s}^{1} \frac{d\alpha d\beta}{\alpha \beta} \theta(s\alpha \beta - t) = \int_{0}^{L(x)} da \int_{0}^{-L(x)} db \theta(a + b + L(x))
\]

\[
= \frac{L^2(x)}{s} \int_{0}^{1} da \ db \ \theta(a + b - 1) = \frac{1}{2} \frac{L^2(x)}{s}, \quad (B.8)
\]

which is consistent with the explicit result (4.1).

For the double box $B_{l=2}(t, s)$ ($B_{l=2}(t, s) = DB(t, s)$) using the same approximations we obtain ($\alpha_i, \beta_i$ correspond to the $d_k^6$ loop momenta):

\[
B_{l=2}(t, s) \simeq \frac{1}{s} \int_{t/s}^{1} \frac{d\alpha_1 d\beta_2}{\alpha_1 \beta_2} \frac{d\alpha_2 d\beta_1}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)} \theta(s\alpha_1 \beta_1 - t)\theta(s\alpha_2 \beta_2 - t)
\]

\[
\simeq \frac{L^4(x)}{s^2} \int_{0}^{1} \prod_{i=1}^{2} da_i d b_1 \theta(a_1 + b_1 - 1)\theta(a_2 + b_2 - 1)\theta(a_1 - a_2)\theta(b_2 - b_1)
\]

\[
= \frac{1}{12} \frac{L^4(x)}{s}, \quad (B.9)
\]

which once again is consistent with the explicit result (4.2).

For the l-rung box integral one can get along the same lines

\[
B_l(t, s) \simeq \frac{L^{2l}(x)}{s} I_l, \quad l \geq 2,
\]

with

\[
I_l = \int_{0}^{1} \prod_{i=1}^{l} da_i d b_i \prod_{k=1}^{l} \theta(a_k + b_k - 1) \prod_{p=1}^{l-1} \theta(a_p - a_{p+1}) \prod_{m=1}^{l-1} \theta(b_{m+1} - b_m). \quad (B.11)
\]

The easiest way to treat this integral is to evaluate it numerically for several values of $l$. The result coincides with the analytical formula

\[
I_l = \frac{1}{l!(l + 1)!}. \quad (B.12)
\]

So, finally, we have the following result for the leading logarithmic asymptotics of the $D = 6$ l-rung box function:

\[
B_l(t, s) \simeq \frac{1}{l!(l + 1)!} \frac{L^{2l}(x)}{s}. \quad (B.13)
\]

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