The Lie algebra of the $sl(2, \mathbb{C})$-valued automorphic functions on a torus

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Abstract

It is shown that the Lie algebra of the automorphic, meromorphic $sl(2, \mathbb{C})$-valued functions on a torus is a geometric realization of a certain infinite-dimensional finitely generated Lie algebra. In the trigonometric limit, when the modular parameter of the torus goes to zero, the former Lie algebra goes over into the $sl(2, \mathbb{C})$-valued loop algebra, while the latter one - into the Lie algebra $(A_1^{(1)})'/\text{(centre)}$.

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1 Introduction

The Inverse Scattering Transform Method has been a source of many important algebraic constructions. The most spectacular of them are quantum groups associated with the quantum R-matrices, and their classical limits - the Lie bialgebras associated with the classical r-matrices. It is well-known, that the classical r-matrices can be classified into three categories: rational, trigonometric and elliptic r-matrices [2]. While the Lie bialgebras, as well as the quantum groups, associated with the rational (Yangians) and the trigonometric (quantized Kac-Moody affine Lie algebras) solutions of the Yang-Baxter equation are well understood by now [4,5], not so much is known about those associated with the elliptic R-matrices. Important exceptions are the Lie algebras of the automorphic, meromorphic sl(n, C)-valued functions on a torus introduced by Reyman and Semenov-Tyan-Shanskii [1]. These algebras are related to the elliptic solutions of the classical Yang-Baxter equation, the simplest of which (sl(2, C)-case) is being the r-matrix of the classical Landau-Lifshitz model [8].

The construction of Reyman and Semenov-Tyan-Shanskii is semi-geometric. Having in view the problem of quantization it is important to give a purely algebraic definition of their algebras in terms of finite number of generators and defining relations. The example of what we are looking for is provided by the trigonometric case, where the loop algebras can be considered as the geometric realizations of the appropriate affine Kac-Moody Lie algebras [7].

As it is shown in this letter, it indeed can be done, at least in the sl(2, C) case. We define an infinite-dimensional, finitely generated Lie algebra $E_{k,\nu^{\pm}}$, and show that the Lie algebra of automorphic, meromorphic sl(2, C)-valued functions on a torus provides a geometric realization of $E_{k,\nu^{\pm}}$. It will be also shown that in the trigonometric limit, when the modular parameter of the torus goes to zero, the Lie algebra $E_{k,\nu^{\pm}}$ goes over into the Lie algebra $L = (A^{(1)}_1)^{'}/(\text{centre})$. 
It turns out that the Lie (bi)algebra $\mathcal{E}_{k,\nu^\pm}$ can be quantized, the corresponding quantum group being related to the eight-vertex R-matrix. This will be the subject of the subsequent paper [9].

2 Definition of the Lie algebra $\mathcal{E}_{k,\nu^\pm}$ and the main Theorem.

Let $k, k' \in \mathbb{C}\setminus\{0, 1\}$, and $k^2 + k'^2 = 1$. Let $T = \mathbb{C}/(\mathbb{Z}4K + \mathbb{Z}4iK')$ be a torus with the periods defined by the complete elliptic integrals $K$ and $K'$ of the 1-st kind of the moduli $k$ and $k'$ correspondingly.

Introduce three meromorphic functions $\{w_i\}_{i=1,2,3}$ on $T$ defined in the following way [6]:

\[
\begin{align*}
  w_1(u) &= \frac{1}{\text{sn}(u)}, \quad w_2(u) = \frac{\text{dn}(u)}{\text{sn}(u)}, \quad w_3(u) = \frac{\text{cn}(u)}{\text{sn}(u)}; \quad u \in T
\end{align*}
\]

where the Jacobi elliptic functions are of the modulus $k$. These functions obey the following quadratic equations [6]:

\[
\begin{align*}
  w_i(u)^2 - w_j(u)^2 &= J_{ij}, \quad i, j \in \{1, 2, 3\}, \quad i \neq j.
\end{align*}
\]

where $J_{12} = k^2$, $J_{23} = k'^2$, $J_{31} = -1$.

Let $\nu^+, \nu^- \in T$ and $\nu^+ - \nu^- \neq n_1 2K + n_2 2iK'$; $(n_1, n_2) \in \mathbb{Z}_2^2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Definition** \(\mathcal{E}_{k,\nu^\pm}\) is a complex Lie algebra generated by six generators $\{x_i^\pm\}_{i=1,2,3}$ which obey the following defining relations:

\[
\begin{align*}
  [x_i^+, [x_j^+, x_k^\pm]] &= 0, \\
  [x_i^+, [x_j^+, x_k^\pm]] - [x_j^+, [x_j^+, x_k^\pm]] &= J_{ij}x_k^\pm, \\
  [x_i^+, x_i^-] &= 0, \\
  [x_i^+, x_j^-] &= \sqrt{-1}(w_i(\nu_i^+ - \nu_i^-)x_i^k - w_j(\nu_i^+ - \nu_i^-)x_j^k);
\end{align*}
\]
here, and throughout the rest of the letter, \(\{i, j, k\}\) is a cyclic permutation of \(\{1, 2, 3\}\).

**Remark:** The defining relations which involve both \(x_i^+\) and \(x_i^-\) can be compactly written in the r-matrix form, namely, let:

\[
X^\pm = \begin{pmatrix}
x_3^\pm & x_1^\pm - ix_2^\pm \\
x_1^\pm + ix_2^\pm & -x_3^\pm
\end{pmatrix}
\]

\[
X_{i(2)}^\pm = X^\pm \otimes I(I \otimes X^\pm)
\]

where \(I\) is an identity in \(Mat_2\) and \(\{\sigma_n\}_{n=1,2,3}\) are the Pauli matrices; then (5,6) can be written as follows:

\[
[X_1^\pm, X_2^\mp]_{E_{k,\nu}} = [r_{12}(\nu^\mp - \nu^\pm), X_1^\pm + X_2^\mp]_{Mat_2}
\]

This representation turns out to be helpful when one is doing quantization of the Lie algebra \(E_{k,\nu}^{\pm}\) [9].

Now we formulate a theorem which describes the structure of the Lie algebra \(E_{k,\nu}^{\pm}\).

**Theorem**

1. \(E_{k,\nu}^{\pm} = E^+ \oplus E^- - E_{k,\nu}^{\pm}\) is, as a linear space, a direct sum of two Lie subalgebras \(E^+\) and \(E^-\) generated by \(\{x_i^+\}_{i=1,2,3}\) and \(\{x_i^-\}_{i=1,2,3}\) correspondingly.

2. \(E^\pm = \oplus_{n \in \mathbb{Z}_{\geq 0}} E_n^{\pm}\) where \(dim(E_n^{\pm}) = 3\). The elements \(\{g_i^{(n),\pm}\}_{i=1,2,3; n \in \mathbb{Z}_{\geq 0}}\) defined by the recurrent formula:

\[
g_i^{(0),\pm} = x_i^\pm,
\]

\[
g_i^{(n),\pm} = \frac{1}{\sqrt{-1}n} \sum_{m + l = n - 1} [g_i^{(m),\pm}, g_j^{(l),\pm}], \quad n \geq 1, \quad m, l \in \mathbb{Z}_{\geq 0},
\]

form a basis in \(E^\pm\). The elements \(\{g_i^{(n),\pm}\}_{i=1,2,3}\) form a basis in \(E_n^{\pm}\).

3. The commutation relations of \(E_{k,\nu}^{\pm}\) in the basis \(\{g_i^{(n),\pm}\}_{i=1,2,3; n \in \mathbb{Z}_{\geq 0}}\) are given by the formulae:

a).

\[
[g_i^{(m),\pm}, g_i^{(n),\pm}] = 0, \quad i \in \{1, 2, 3\},
\]
b).
\[
\frac{1}{\sqrt{-1}} [g_i^{(m),\pm}, g_j^{(n),\pm}] = g_k^{(m+n+1),\pm} + (-1)^m \sum_{r=m}^{m+n} \binom{r}{m} w^i_r g_k^{(m+n-r),\pm} - (-1)^n \sum_{r=n}^{m+n} (-1)^r \binom{r}{n} w^j_r g_k^{(m+n-r),\pm},
\]

(12)

c).
\[
[g_i^{(m),+}, g_i^{(n),-}] = 0, \quad i \in \{1, 2, 3\},
\]

(13)
d).
\[
\frac{1}{\sqrt{-1}} [g_i^{(m),\pm}, g_j^{(n),\mp}] = (-1)^m \sum_{r=m}^{m+n} \binom{r}{m} v^i_r (\pm (\nu^- - \nu^+)) g_k^{(m+n-r),\mp} - (-1)^n \sum_{r=n}^{m+n} (-1)^r \binom{r}{n} v^j_r (\pm (\nu^- - \nu^+)) g_k^{(m+n-r),\pm},
\]

(14)

\[m, n \in \mathbb{Z}_{\geq 0}, \text{ and the coefficients } w^i_r, v^i_r(\nu) \text{ are defined by the formulae:}
\]
\[
w_i(u) = \frac{1}{u} + \sum_{r=0}^{\infty} w^i_r u^r, \quad v^i_r(u) = \frac{1}{n!} \frac{d^n}{du^n} w_i(\nu).
\]

(15)

Proof of the theorem is given in the Appendix.

Remark: Commutation relations in \(E_{k,\nu^\pm}\) can be compactly written down using the generating functions and the r-matrix representation. Let us define:
\[
G_i^\pm(\alpha) = \sum_{n \geq 0} \alpha^n g_i^{(n),\pm}, \quad \alpha \in \mathbb{T};
\]

(16)
\[
G^\pm = \begin{pmatrix} G_3^\pm & G_1^\pm - iG_2^\pm \\ G_1^\mp + iG_2^\mp & -G_3^\mp \end{pmatrix};
\]

(17)

then the commutation relations in the basis \(\{g_i^{(n),\pm}\}\) can be recovered from the following commutation relations for the generating functions:
\[
[G_1^\pm(\alpha), G_2^\mp(\beta)]_{E_{k,\nu^\pm}} = [r_{12}(\beta - \alpha), G_1^\mp(\alpha) + G_2^\pm(\beta)]_{Mat_2}.
\]

(18)
\[
[G_1^\pm(\alpha), G_2^\pm(\beta)]_{E_{k,\nu^\pm}} = [r_{12}(\beta - \alpha + (\nu^\mp - \nu^\pm)), G_1^\mp(\alpha) + G_2^\pm(\beta)]_{Mat_2}.
\]

(19)
3 Geometric realization of $\mathcal{E}_{k,\nu^\pm}$

In this section we describe a geometric realization of the Lie algebra $\mathcal{E}_{k,\nu^\pm}$ - an “elliptic analog” of the loop Lie algebras. This turns out to be the Lie algebra of the $sl(2,\mathbb{C})$-valued automorphic meromorphic functions on $T$ [1].

Consider a set $\tilde{F} = \{\varphi_i^{(n),\pm}\}_{i=1,2,3}, n \in \mathbb{Z}_{\geq 0}$ of meromorphic functions on $T$ defined as follows:

$$\varphi_i^{(n),\pm}(u) = \frac{(-1)^n}{n!} \frac{d^n}{du^n} w_i(u - \nu^\pm), \ u \in T$$  \hspace{1cm} (20)

Let $\mathcal{G} = sl(2,\mathbb{C})$. Fix a basis $\{s_i\}_{i=1,2,3}$ in $\mathcal{G}$ such that: $[s_i, s_j] = \sqrt{-1}s_k$. Let $F$ be a linear span of $\tilde{F}$ over $\mathbb{C}$. Consider a tensor product $\mathcal{A} = F \otimes_{\mathbb{C}} \mathcal{G}$. Single out a linear subspace of $\mathcal{A}$ spanned by the elements of the form $\varphi_i^{(n),\pm} \otimes s_i$. Denote this subspace by $\mathcal{E}_{k,\nu^\pm}$. The elements of this subspace satisfy the following automorphicity condition with respect to the action of the group $\mathbb{Z}_2^2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$\varphi_i^{(n),\pm}(u + n_1 2K + n_2 i 2K') \otimes s_i = \varphi_i^{(n),\pm}(u) \otimes (T^{(n_1,n_2)}s_i(T^{(n_1,n_2)})^{-1}) \ , \ (n_1, n_2) \in \mathbb{Z}_2^2$$  \hspace{1cm} (21)

where, in the $2 \times 2$ matrix realization of $\mathcal{G}$, one has $T^{(n_1,n_2)} = \sigma^1 \sigma^1$ $n_2$. Introduce a Lie bracket in $\mathcal{E}_{k,\nu^\pm}$ as follows: $[\varphi_i^{(n),\epsilon} \otimes s_i, \varphi_j^{(n'),\epsilon'} \otimes s_j] = \varphi_i^{(n),\epsilon} \varphi_j^{(n'),\epsilon'} \otimes [s_i, s_j]; \ i, j = 1, 2, 3; \ \epsilon, \epsilon' \in \{+, -\}$. Then the map $\gamma : \mathcal{E}_{k,\nu^\pm} \to \mathcal{E}_{k,\nu^\pm}^{-} : \ \gamma(g_i^{(n),\pm}) = \varphi_i^{(n),\pm} \otimes s_i$ is an isomorphism, as can be readily checked using the relations:

$$w_i(u)w_j(v) = w_k(u - v)w_i(v) - w_k(u - v)w_j(u) \ , \ u, v \in T$$  \hspace{1cm} (22)

Proposition 1 $\mathcal{E}_{k,\nu^\pm}$ is a Lie bialgebra [3,4], cocommutator map $\delta : \mathcal{E}_{k,\nu^\pm} \to \wedge^2 \mathcal{E}_{k,\nu^\pm}$ is being given by the following formulae:

$$\delta[g_i^{(n),\pm}] = \sqrt{-1} \sum_{l=0}^{n} g_i^{(n-l),\pm} \wedge g_j^{(l),\pm}$$  \hspace{1cm} (23)

or, in terms of the generating functions:

$$\delta[G_i^{\pm}(\cdot, \alpha)](u, v) = [G_i^{\pm}(u, \alpha) \otimes I + I \otimes G_i^{\pm}(v, \alpha), r(u - v)]_{U\mathcal{G} \otimes U\mathcal{G}}$$  \hspace{1cm} (24)
where \( G^\pm_i(u, \alpha) = \gamma(G^\pm_i(\alpha)) = w_i(u - \alpha - \nu^\pm) \otimes s_i, r(u) = 2 \sum_{k=1}^3 w_k(u)s_k \otimes s_k \in UG \otimes UG, \ u \in T, \ and \ I \ is \ an \ identity \ in \ UG\)-the universal enveloping algebra of \( G \).

4 Trigonometric limit of \( \mathcal{E}_{k, \nu^\pm} \)

In this section it is shown that the trigonometric limit: \( k \to 0 \) of \( \mathcal{E}_{k, \nu^\pm} \), when \( \nu^+ = i\frac{3}{2}K', \nu^- = i\frac{1}{2}K' \) coincides with the \( sl(2) \)-loop algebra: \( \mathcal{L} = (\mathcal{G})'/(centre) \), where \( \mathcal{G} = A_1^{(1)} \).

Let us fix \( \nu^+ = i\frac{3}{2}K', \nu^- = i\frac{1}{2}K' \), let us also make a change of the generators of \( \mathcal{E}_{k, \nu^\pm} \): \( x_i^\pm \to x'_i^\pm \), where \( x_{1,2}^+ = \frac{1}{\sqrt{k}}x_{1,2}^\pm; \ x_3^\pm = x_3^\pm \). Then the defining relations of \( \mathcal{E}_{k, \nu^\pm} \) acquire a form (from now on we drop the prime):

\[
[x_1^+, x_2^-] = x_3^+, \quad [x_1^-, x_2^+] = -x_3^-,
\]

\[
[x_2^-, x_3^+] = -x_1^- + kx_1^+, \quad [x_3^-, x_1^+] = -x_2^- \quad [x_3^-, x_1^+] = x_2^-;
\]

\[
[x_1^+, [x_1^+, x_3^\pm]] - [x_2^+, [x_2^+, x_3^\pm]] = kx_3^\pm, \quad k[x_2^+, [x_3^\pm, x_1^+]] - [x_3^\pm, [x_3^\pm, x_1^+]] = k^2 x_1^+, \quad (25)
\]

\[
[x_3^\pm, [x_3^\pm, x_2^\pm]] - k[x_1^+, [x_1^+, x_2^\pm]] = -x_2^\pm, \quad [x_i^+, [x_j^+, x_k^\pm]] = 0, \quad [x_i^+, x_j^-] = 0, \quad i = 1, 2, 3. \quad (26)
\]

If \( k \neq 0 \), it follows from the above relations that

\[
[x_2^+, [x_1^+, [x_1^+, x_2^\pm]]] = 0, \quad \text{and} \quad [x_2^+, [x_2^+, [x_3^\pm, x_1^+]]) + [x_1^+, [x_1^+, [x_1^+, x_2^\pm]]] = k[x_1^+, x_2^\pm]. \quad (29)
\]

Let us add these new (non-independent only if \( k \neq 0 \)) relations to the relations (25-28) and then take a limit \( k \to 0 \). As a result \( \mathcal{E}_{k, \nu^\pm} \) goes over into the Lie algebra \( \hat{\mathcal{L}} \) generated by the elements \( \{y_i^\pm = lim_{k \to 0} x_i^\pm \}_{i=1,2,3} \). In the geometric realization this limit has the following explicit form:

\[
\frac{1}{\sqrt{k}} w_1(u - \nu^+) \otimes s_1 = \sqrt{k} sn(u \mp i\frac{1}{2}K') \otimes s_1 \to \mp ie^{\pm iu} \otimes s_1,
\]

\[
\frac{1}{\sqrt{k}} w_2(u - \nu^+) \otimes s_2 = i\sqrt{k} cn(u \mp i\frac{1}{2}K') \otimes s_2 \to ie^{\pm iu} \otimes s_2,
\]

\[
w_3(u - \nu^+) \otimes s_3 = idn(u \mp i\frac{1}{2}K') \otimes s_3 \to i \otimes s_3.
\]
\{y_i^\pm\} obey the following defining relations:

\[ [y_1^\pm, y_2^\pm] = \pm y_3^\pm, \quad [y_2^\pm, y_3^\pm] = \pm y_1^\pm, \quad [y_3^\pm, y_1^\pm] = \mp y_2^\pm, \]

(30)

\[ [y_1^\pm, [y_1^\pm, y_3^\pm]] - [y_2^\pm, [y_2^\pm, y_3^\pm]] = 0, \quad [y_3^\pm, [y_3^\pm, y_2^\pm]] = -y_2^\pm, \quad [y_3^\pm, [y_3^\pm, y_1^\pm]] = -y_1^\pm, \]

(31)

\[ [y_i^\pm, [y_j^\pm, y_k^\pm]] = 0, \quad [y_i^\pm, y_i^-] = 0, \quad [y_2^\pm, [y_1^\pm, [y_1^\pm, y_2^\pm]]] = 0, \]

(32)

\[ [y_2^\pm, [y_2^\pm, [y_2^\pm, y_1^\pm]]] + [y_1^\pm, [y_1^\pm, [y_1^\pm, y_2^\pm]]] = 0. \]

(33)

**Proposition 2** \( \tilde{\mathcal{L}} \) contains an ideal \( \mathfrak{i} \) generated by the elements:

\[ [y_2^\pm, y_3^\pm] \mp y_1^\pm, \quad [y_3^\pm, y_1^\pm] \mp y_2^\pm, \quad y_3^\pm - y_3^- \]  

(34)

**Proposition 3** Lie algebra \( \mathcal{L} = \tilde{\mathcal{L}}/\mathfrak{i} \) is isomorphic to the \( sl(2, \mathbb{C}) \)-loop Lie algebra \( (\hat{\mathcal{G}})'/(\text{centre}) \), \( \hat{\mathcal{G}} = A_1^{(1)} \).

Remark: Non-independent relations (29) in \( E_{k,\nu^\pm} \) go over, as \( k \to 0 \) into the Serre relations in \( \mathcal{L} \).

### 5 Conclusion

The fact, that the Lie algebra of the automorphic, meromorphic \( sl(2, \mathbb{C}) \)-valued functions on a torus admits a purely algebraic description as the finitely generated infinite dimensional Lie algebra, suggests that it is possible to develop a theory of the algebras of such a type which would be parallel to the theory of the Kac-Moody affine Lie algebras [7]. Such a development would be clearly important taking into account an abundance of physical and mathematical constructions related to the latter. Among the more immediate ramifications of the result given in this letter, one can point out the problem of finding the quantum groups related to the elliptic quantum R-matrices [9].
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Appendix

Proof of the theorem. From the defining relations (3-6) and the Jacobi identity it follows that $\mathcal{E}_{k,\nu}^{\pm} = \mathcal{E}^+ \cup \mathcal{E}^-$, where $\mathcal{E}^+$ and $\mathcal{E}^-$ are Lie subalgebras generated by $\{x_i^+\}_{i=1,2,3}$ and $\{x_i^-\}_{i=1,2,3}$ correspondingly.

Consider $\mathcal{E}^+$. $\mathcal{E}^+ = \bigcup_{\nu \geq 0} \mathcal{E}^+_\nu$, where $\mathcal{E}^+_\nu$ is a linear subspace of $\mathcal{E}^+$ spanned by all the elements of the form $[x_i^+, \ldots, x_i^n, x_i^{n+1}]$, $i, j \in \{1, 2, 3\}$ if $n \geq 1$, and $\mathcal{E}^+_0$ is a linear span of $\{x_i^+\}_{i=1,2,3}$.

In what follows $g_i^{(n)} \overset{\text{def}}{=} g_i^{(n),+}$, $i \in \{1, 2, 3\}$, $n \geq 0$. We remind, that the indices $\{i, j, k\}$ denote any cyclic permutation of $\{1, 2, 3\}$. We will prove the statements 2 and 3 a), b) of the theorem using induction.

Fix $n \in \mathbb{Z}$, $n \geq 2$ and assume, that:

$A_n$. The elements $\{g_i^{(s)}\}_{i=1,2,3}$, $0 \leq s \leq n$ form a basis in $\bigcup_{s=0}^n \mathcal{E}^+_s$. Denote by $\mathcal{E}^+_s$ a linear subspace of $\bigcup_{s=0}^n \mathcal{E}^+_s$ spanned by the elements $\{g_i^{(s)}\}_{i=1,2,3}$. We have $\bigcup_{m=0}^n \mathcal{E}^+_m = \bigoplus_{m=0}^n \mathcal{E}^+_m$.

$B_n$. $[g_i^{(l)}, g_i^{(m)}] = 0$, if $0 \leq l + m \leq n - 1$.

$C_n$. There exist the coefficients $A_{k,r}^{(l,m)}$ such, that

$$\frac{1}{\sqrt{-1}}[g_i^{(l)}, g_j^{(m)}] = g_k^{(l+m+1)} + \sum_{r=0}^{l+m-1} A_{k,r}^{(l,m)} g_k^{(r)},$$

if $0 \leq l + m \leq n - 1$.

The fact that these assumptions are true for $n = 2$, and that $A_{k,0}^{(0,1)} = \frac{1}{2} J_{ij} = w_1^i - w_1^j$, $A_{k,0}^{(1,0)} = -\frac{1}{2} J_{ij}$, follows immediately from the defining relations (3,4). We will show that under the above assumptions the statements $A_{n+1}$, $B_{n+1}$, $C_{n+1}$ hold true.

From $A_n$ it follows that $\mathcal{E}^+_{n+1}$ is a linear span of $\{[g_i^{(m)}, g_i^{(n-m)}]\}_{i,i' \in \{1,2,3\}, 0 \leq m \leq n}$. Let us
show that among the elements \( \{[g_i^{(m)}: g_j^{(n-m)}]\}_{0 \leq m \leq n} \) only three are linearly independent \( \mod(\oplus_{s=0}^{n} \mathcal{E}_s^+) \). This fact follows from the:

**Lemma 1.** Under the assumptions \( A_n, B_n, C_n \):

1. \( [g_i^{(m+1)}, g_j^{(n-m-1)}] = [g_i^{(m)}, g_j^{(n-m)}] + \eta_k^{(n,m)}, \ 0 \leq m \leq n - 1, \) where
   \[
   \eta_k^{(n,m)} = \sqrt{-1} \sum_{s=0}^{n-1} B_{k,s}^{(n,m)} g_k^{(s)} \in \oplus_{s=0}^{n-1} \mathcal{E}_s^+ . \tag{36}
   \]

2. The coefficients \( B_{k,s}^{(n,m)} \) are expressed in terms of the coefficients \( \{A_{k,s}^{(l,m)}\}_{0 \leq l+m \leq n-1} \) by the formulae:
   \[
   B_{k,s}^{(n,m)} = \theta(s - m - 1) A_{i,s-n+m-1}^{(n-2m-2, m+1)} - \theta(s - m - 2) A_{i,s-n+m-2}^{(n-2m-2, m+1)} + \\
   \sum_{r=0}^{n-m-2} \theta(m + r - 1 - s) A_{j,r}^{(n-2m-2, m+1)} A_{i,s}^{(m-r)} - \sum_{r=0}^{n-m-3} \theta(m + r - s) A_{j,r}^{(n-2m-2, m+1)} A_{k,l}^{(m-r, l)}, \]
   if \( 0 \leq m \leq p - 1, \ n = 2p \) or \( n = 2p + 1 \); (37)

   \[
   B_{k,s}^{(n,m)} = \theta(s - n + m - 1) A_{i,n-m}^{(n-1, 2m-n)} - \theta(s - n + m) A_{i,n-m}^{(n-1, 2m-n)} + \\
   \sum_{r=0}^{m-2} \theta(r + n - m - 1 - s) A_{i,r}^{(n-1, 2m-n)} A_{k,s}^{(r,n-m)} - \\
   \sum_{r=0}^{m-1} \theta(r + n - m - 2 - s) A_{i,r}^{(n-1, 2m-n)} A_{k,s}^{(r,n-m-1)}, \]
   if \( p \leq m \leq n = 2p, \) or \( p + 1 \leq m \leq n = 2p + 1 \); (38)

   \[
   B_{k,s}^{(2p+1, p)} = \theta(s - p - 2) A_{i,s-p-2}^{(p-1, 0)} - \theta(s - p) A_{i,s-p}^{(p+1, 0)} + \sum_{r=0}^{p-2} \theta(r + p - s) A_{i,r}^{(p-1, 0)} A_{k,s}^{(r, p+1)} - \\
   \sum_{r=0}^{p} \theta(r + p - 2 - s) A_{i,r}^{(p+1, 0)} A_{k,s}^{(r, p-1)} - B_{k,s}^{(2p+1, p+1)}, \text{ where } \theta(x \geq (<) 0) = 1(0). \tag{39}
   \]

**Proof.** Let \( n = 2p \) or \( n = 2p + 1 \).

Let \( 0 \leq m \leq p - 1 \). According to the assumption \( C_n \):

\[
[g_i^{(m)}, g_j^{(n-m)}] = \sqrt{-1} [g_i^{(m+1)}, g_j^{(n-m-1)}] + \tilde{\eta}_k^{(n,m)}, \tag{40}
\]

\[
-\tilde{\eta}_k^{(n,m)} = \sqrt{-1} \left( \sum_{r=0}^{n-m-1} A_{j,r}^{(n-2m-2, m+1)} g_k^{(m+r+1)} + \sum_{r=0}^{n-m-2} A_{j,r}^{(n-2m-2, m+1)} \sum_{s=0}^{m+r-1} A_{k,s}^{(m,r)} g_k^{(s)} \right). \tag{41}
\]
It is clear, that $\vartheta_{k}^{(n,m)} \in \oplus_{r=0}^{n-1} \mathcal{E}_{r}^{+}$.

\[ \left[ g_{i}^{(m+1)}, g_{j}^{(n-m-1)} \right] = \sqrt{-1} \left[ g_{i}^{(m+1)}, g_{k}^{(n-2m-2)} \right] + \nu_{k}^{(n,m+1)}, \quad \text{(42)} \]

\[ -\nu_{k}^{(n,m+1)} = \sqrt{-1} \left( \sum_{r=0}^{n-3} A_{j,r}^{(n-2m-2,m)} g_{k}^{(m+r+2)} + \sum_{r=0}^{n-3} A_{j,r}^{(n-2m-2,m)} \sum_{s=0}^{m+r} A_{k,s}^{(m+1,r)} g_{k}^{(s)} \right) \in \oplus_{r=0}^{n-1} \mathcal{E}_{r}^{+}, \quad \text{(43)} \]

Using the Jacobi identity and the assumption $B_{n}$ we get

\[ \left[ g_{i}^{(m)}, g_{j}^{(m+1)}, g_{j}^{(n-2m-2)} \right] = \left[ g_{i}^{(m)}, g_{j}^{(m)}, g_{j}^{(n-2m-2)} \right]. \quad \text{(44)} \]

Thus $\left[ g_{i}^{(m+1)}, g_{i}^{(n-m-1)} \right] = \left[ g_{i}^{(m)}, g_{j}^{(n-m)} \right] + \eta_{k}^{(n,m)}, \quad \eta_{k}^{(n,m)} = \nu_{k}^{(n,m+1)} - \nu_{k}^{(n,m)} \in \oplus_{r=0}^{n-1} \mathcal{E}_{r}^{+}.$

The statement 1. is proven in the case $0 \leq m \leq p - 1$. The statement 2. in this case follows from the above formulae for $\eta_{k}^{(n,m)}$. The cases $p \leq m \leq n = 2p$, $p+1 \leq m \leq n = 2p+1$ and $m = p$, $n = 2p + 1$ are proven analogously $\square$.

From the Lemma 1. it follows that $\left[ g_{i}^{(m)}, g_{j}^{(n-m)} \right] = \left[ g_{i}^{(0)}, g_{j}^{(n)} \right] + \sum_{r=0}^{n-1} \eta_{k}^{(n,r)}, \quad 1 \leq m \leq n.$ By the definition of $g_{k}^{(n+1)}$ we also have: $\sqrt{-1} (n+1) g_{k}^{(n+1)} = \sum_{m=0}^{n} [g_{i}^{(m)}, g_{j}^{(n-m)}].$ Hence we obtain the following expressions for the commutators $\left[ g_{i}^{(m)}, g_{j}^{(n-m)} \right]$:

\[ \left[ g_{i}^{(0)}, g_{j}^{(n)} \right] = \sqrt{-1} g_{k}^{(n+1)} - \frac{1}{n+1} \sum_{s=1}^{n} \sum_{r=0}^{s-1} \eta_{k}^{(n,r)}, \quad \text{(45)} \]

\[ \left[ g_{i}^{(m)}, g_{j}^{(n-m)} \right] = \sqrt{-1} g_{k}^{(n+1)} + \sum_{r=0}^{m-1} \eta_{k}^{(n,r)} - \frac{1}{n+1} \sum_{s=1}^{n} \sum_{r=0}^{s-1} \eta_{k}^{(n,r)} , \quad 1 \leq m \leq n. \quad \text{(46)} \]

From these expressions and the fact, that $\left[ g_{i}^{(m)}, g_{i}^{(n-m)} \right] = 0$, $0 \leq m \leq n; \ i \in \{1, 2, 3\}$, which will be proven in the Lemma 2., it follows, that $\left[ g_{i}^{(n+1)} \right]_{i=1,2,3}$ form a basis in $\mathcal{E}_{n+1}^{+} \cup_{0 \leq r \leq n} \mathcal{E}_{r}^{+}$. Thus we have proven that $A_{n+1}$ holds. From (45,46) and the second statement of the Lemma 1. it also follows, that $C_{n+1}$ is true. The coefficients $A_{k,r}^{(m,n-m)}$ are expressed in terms of the coefficients $\{ A_{k,s}^{(l,m)} \}_{0 \leq l+m \leq n-1}$ by the following formulae:

\[ A_{k,s}^{(m,n-m)} = (1 - \delta_{0m}) \sum_{r=0}^{m-1} B_{k,s}^{(n,r)} - \frac{1}{n+1} \sum_{r=0}^{n-1} (n-r) B_{k,s}^{(n,r)} , \quad 0 \leq m \leq n , \quad 0 \leq s \leq n - 1. \quad \text{(47)} \]

Recall, that the coefficients $B_{k,s}^{(n,r)}$ are expressed in terms of $\{ A_{i,s}^{(l,m)} \}_{i=1,2,3,0 \leq l+m \leq n-1}$ as given by the second statement of the Lemma 1. Combining the formulae (37-39) and (47) we obtain
recurrent relations for $A_{k,s}^{(l,m)}$. From these relations it is evident, that all the coefficients $A_{k,s}^{(l,m)}$ are completely and uniquely determined by $A_{k,0}^{(0,1)}$ and $A_{k,0}^{(1,0)}$ which enter into the defining relations (3,4).

Let us now prove $B_{n+1}$.

**Lemma 2.** Under the assumptions $A_n, B_n, C_n$:

$$[g_i^{(m)}, g_i^{(n-m)}] = 0 ; \; i \in \{1, 2, 3\}, \; 0 \leq m \leq n.$$  

**Proof.** First, let us prove, that $[g_i^{(0)}, g_i^{(n)}] = 0$. From the assumptions $B_n$ and $C_n$ we have:

$$\sqrt{-1}[g_i^{(0)}, g_i^{(n)}] = [g_i^{(0)}, [g_j^{(0)}, g_k^{(n-1)}]] = [g_i^{(0)}, [g_j^{(1)}, g_k^{(n-2)}]] = \ldots = [g_i^{(0)}, [g_j^{(n-1)}, g_k^{(0)}]]. \quad (48)$$

By the Jacobi identity and the assumptions $B_n$, $C_n$ we also have:

$$[g_i^{(0)}, g_i^{(n)}] = -[g_j^{(m)}, g_j^{(n-m)}] - [g_k^{(n-m-1)}, g_k^{(m+1)}]. \quad (49)$$

From (48) and (49) we get:

$$n[g_i^{(0)}, g_i^{(n)}] = -\sum_{m=0}^{n-1} ([g_j^{(m)}, g_j^{(n-m)}] + [g_k^{(n-m-1)}, g_k^{(m+1)}]) = -[g_j^{(0)}, g_j^{(n)}] - [g_k^{(0)}, g_k^{(n)}], \quad (50)$$

and

$$(n - 1)([g_i^{(0)}, g_i^{(n)}] - [g_j^{(0)}, g_j^{(n)}]) = 0. \quad (51)$$

Recall, that we consider only $n \geq 2$, so : $[g_i^{(0)}, g_i^{(n)}] = [g_j^{(0)}, g_j^{(n)}] = \xi$. From (50) we have $(n + 2)\xi = 0$. Thus $\xi = 0$.

Now let us prove, that $[g_i^{(l)}, g_i^{(m)}] = 0$, $l + m = n$, $l \geq 1$.

From the Jacobi identity and the assumptions $B_n$ and $C_n$ we have:

$$\sqrt{-1}[g_i^{(m)}, g_i^{(n-m)}] = [g_i^{(m)}, [g_j^{(0)}, g_k^{(n-1-m)}]] = -\sqrt{-1}[g_j^{(0)}, g_j^{(0)}] - \sqrt{-1}[g_k^{(n-1-m)}, g_k^{(m+1)}], \quad (52)$$

$$0 \leq m \leq n - 1.$$ 

hence $[g_i^{(m)}, g_i^{(n-m)}] = [g_k^{(m+1)}, g_k^{(n-m-1)}], 0 \leq m \leq n - 1$. Taking into account that, as has been proven, $[g_i^{(0)}, g_i^{(n)}] = 0$, we obtain the statement of the lemma $\square$. 

11
Thus it is shown, that the statements $A_{n+1}$, $B_{n+1}$, $C_{n+1}$ are true if the $A_n$, $B_n$, $C_n$ are true. From this the statements 2. and 3. a). of the theorem follow for the subalgebra $\mathcal{E}^+$. For the subalgebra $\mathcal{E}^-$ which is isomorphic to $\mathcal{E}^+$ a proof is identical.

By a direct calculation using the relations (22) one can verify that the structure constants appearing in the RHS of the statement 3. b). satisfy the recurrent relations (47),(37-39) with the initial conditions $A^{(0,1)}_{k,0} = \frac{1}{2}J_{ij} = w^i_1 - w^j_1$, $A^{(1,0)}_{k,0} = -\frac{1}{2}J_{ij}$. This proves the statement 3. b).\)

The statement 1. follows from the explicit form of the bases in $\mathcal{E}^+$ and $\mathcal{E}^-$.

The proof of the statements 3 c).,d). is very similar to that one of the statements 3 a)., b)., for this reason we describe it only schematically. Using induction one can derive recurrent relations for the structure constants appearing in the decompositions of the commutators $[g_i^{(l)}\pm, g_i^{(l')\mp}]$ in the basis $\{g_i^{(n)\pm}\}$. Then, we uniquely resolve these - the initial conditions being provided by the defining relations (3-6). The relation (22) again plays the crucial role in this solution.

To finish the proof of the theorem we need to check that the Lie bracket in $\mathcal{E}_{k,\nu\pm}$ as given by the formulas (11-14) satisfies the Jacobi identity. This follows from the r-matrix representation (18,19) of the commutation relations (11-14), and the well-known fact that $r(u)$ (see (7)) satisfies the Classical Yang-Baxter equation $\Box$.

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12
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