Maxwell-modified metric affine gravity

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Abstract We present a gauge formulation of the special affine algebra extended to include an antisymmetric tensorial generator belonging to the tensor representation of the special linear group. We then obtain a Maxwell modified metric affine gravity action with a cosmological constant term. We find the field equations of the theory and show that the theory reduces to an Einstein-like equation for metric affine gravity with the source added to the gravity equations with cosmological constant \( \mu \) contains linear contributions from the new gauge fields. The reduction of the Maxwell metric affine gravity to Riemann–Cartan one is discussed and the shear curvature tensor corresponding to the symmetric part of the special linear connection is identified with the dark energy. Furthermore, the new gauge fields are interpreted as geometrical inflaton vector fields which drive accelerated expansion.

1 Introduction

It is verified by the Solar System and cosmological tests that general relativity provides an elegant and powerful formulation of gravitation in terms of Riemannian geometry and forms our understanding of space-time [1]. Despite these successes, there are some reasons to believe that general relativity is unable to explain some gravity phenomena on both atomic and cosmological scales and should be either modified or replaced by a new theory of gravity. Recently many papers propose new types of dynamics to explain the dark energy phenomenon [2,3] as well as the dynamical role of the cosmological constant [4,5].

It is also known that the cosmological term, usually associated with the vacuum energy density, cannot be a valid theoretical explanation for the accelerated expansion of the universe [6]. A very different approach holds that cosmic acceleration is a manifestation of new gravitational physics rather than dark energy, i.e., that it involves a modification of the geometry as opposed to the stress–energy tensor side of the Einstein equations [4]. It is important to point out that one can accommodate a generalized cosmological term in the gravity theory using extended algebras. A way of introducing the generalized cosmological constant term using the Maxwell algebra was presented in [7] and even more, interestingly it has been argued that by making use of the gauged Maxwell algebra one can understand it as a source of an additional contribution to the cosmological term in Einstein gravity.

Maxwell symmetry was introduced around forty years ago [8,9], but it is only recently that has attracted more attention after the work of Soroka [10] in 2005. The Maxwell symmetry is the result of extending the Poincare symmetry by six additional tensorial Abelian symmetry generators that make the four-momenta non-commutative. Since then a variety of different Maxwell (super) symmetry algebras with interesting geometric and physical properties have been constructed and analyzed in the papers [11–19].

By gauging Maxwell symmetries, one can define modified gravitational theories that extend general relativity by including a generalized cosmological term [20–30]. Among these is the Maxwell extension of special affine symmetry and its gauging which will be the focus of our attention in this paper.

In 1974 Yang [31] put forward a gauge theory of gravity based on the affine group to construct a theory of (quantum) gravity in the high energy limit [32]. On the other hand, in nature, there is no conservation law corresponding to the (special) linear transformation and so the linear transformations must be dynamical, i.e., spontaneously broken [33]. Correspondingly, the papers [34–38] suggested that the renormalizability and unitarity problems in quantum gravity can be overcome by taking the affine group as the dynamical group in a gauge theory of gravity with the help of generalized linear connection [39]. There exists a series of papers...
in which an affine gauge gravitation theory is considered.

Our paper has the following structure. In Sect. 2, following [28,29], we briefly review the Maxwell extension of the special-affine group, \( \mathcal{MSA}(4,R) \). We also present the transformation rules for the generalized coordinates (coset parameters) and the corresponding differential realization of generators using the nonlinear realization technique. In Sect. 3, we gauge the Maxwell special linear algebra \([38,40–46]\) in which an affine gauge gravitation theory is invariant action. In Sect. 4, we introduced \( \mathcal{SL}(4,R) \) gauge covariant metric tensor in the affine space needed for the metric affine gravity (MAG). In Sect. 5, we propose an action for Maxwell metric affine gravity by using Euler or Gauss–Bonnet type topological action and derive the equations of motion of corresponding action. We present our conclusions in Sect. 6.

2 Introducing the special-affine algebra and its maximal extension

We begin in this section by giving an overview of the Maxwell extension of the special affine group. For a more complete description of the details, the reader is referred to earlier works [28,29]. The special affine symmetry group \( \mathcal{SA}(4,R) \) is given by the semi-direct product of the special linear group \( \mathcal{SL}(4,R) \) and the translation group \( \mathcal{T}(4) \) and are generated by the fifteen special linear generators \( \hat{\mathcal{L}}^a_b \) and by the four affine translation generators \( P_a \), respectively. The commutators of the generators obey the following algebra,

\[
\begin{align*}
\left[ \hat{\mathcal{L}}^a_b, \hat{\mathcal{L}}^c_d \right] &= i \left( \delta^c_b \hat{\mathcal{L}}^a_d - \delta^a_b \hat{\mathcal{L}}^c_d \right), \\
\left[ \hat{\mathcal{L}}^a_b, P_c \right] &= -i \left( \delta^a_c P_b - \frac{1}{4} \delta_b^a P_c \right), \\
\left[ P_a, P_b \right] &= 0.
\end{align*}
\]

(1)

From this algebra, we can construct a group element by exponentiation,

\[
g(x, \omega) = e^{ix^a(x)P_a} e^{i\omega^a_b(x)\hat{\mathcal{L}}^a_b},
\]

(2)

where \( x^a(x) \) and \( \omega^a_b(x) \) are the real parameters. The Maurer–Cartan (MC) 1-forms is defined as \( \Omega = -ig^{-1}dg \), here \( g \) is the general element of the \( \mathcal{SA}(4,R) \) group and the structure equation is given by

\[
d\Omega + i \frac{1}{2} [\Omega, \Omega] = 0.
\]

(3)

Thus, one can show that the MC 1-forms satisfy following equations,

\[
\begin{align*}
0 &= d\Omega^a_p + \Omega^a_{Lb} \wedge \Omega^b_p - \frac{1}{4} \Omega^a_L \wedge \Omega^a_p, \\
0 &= d\Omega^a_{Lb} + \Omega^a_{Lc} \wedge \Omega^c_{Lb}.
\end{align*}
\]

(4)
\[ Z_{ab} = i \partial_{ab}, \]
\[ \tilde{L}_b^a = i \left( x^a \partial_b + 2 \theta^{ac} \partial_{be} - \frac{1}{4} \delta^a_b \left( x^c \partial_c + 2 \theta^{cd} \partial_{cd} \right) \right), \] (12)

where \( \theta \) is the derivative of the gauge parameter as defined by:
\[ \partial_{ab} \theta^{cd} = \frac{1}{2} \left( \delta^a_b \delta^d_c - \delta^d_b \delta^a_c \right). \]

It is an easy task to check that the generators satisfy the algebra Eqs. (5) and (6).

### 3 Gauging the Maxwell-special-affine algebra

Let us construct a gauge theory for the Maxwell special affine algebra \( \text{msa}(4, R) \). For this purpose, we follow the same methods given in [18, 22, 28]. The gauge field is a \( \text{msa}(4, R) \) valued one-form
\[ A = e^a P_a + B_{ab} Z_{ab} + \omega^b_a \tilde{L}_b^a. \] (13)

An infinitesimal gauge parameter is
\[ \zeta(x) = y^a(x) P_a + \psi_{ab}(x) Z_{ab} + \lambda^b_a(x) \tilde{L}_b^a, \] (14)

where \( y^a(x), \psi_{ab}(x), \) and \( \lambda^b_a(x) \) are the infinitesimal parameters corresponding to the affine translation, tensorial and special linear transformations respectively.

The gauge transformation are given by
\[ \delta A = -d \zeta - i [A, \zeta], \] (15)

evaluating (13), we get:
\[ \delta e^a = -dy^a - \omega^a_b y^b + \frac{1}{4} \omega y^a + \lambda^a_b e^b - \frac{1}{4} \lambda e^a, \] (16)
\[ \delta B_{ab} = -d\psi_{ab} - \psi_{ac} \psi^{cb} + \frac{1}{2} \omega \psi_{ab} + \lambda^{[a} \psi^{b]} + \frac{1}{2} \lambda \psi_{ab} + \frac{1}{2} e^{a[y} \psi^{b]}. \] (17)
\[ \delta \omega^a_b = -d \lambda^a_b - \omega^a_c \lambda^c_b - \omega^a_b \lambda^c_c + \omega^a_b \omega^c_c, \] (18)

where the \( \mathcal{SL}(4, R) \) valued exterior covariant derivative \( \mathcal{D} \) of a tensor density \( \Phi \) of affine weight \( w \) contains
\[ \left\{ w Tr \left( \omega^b_a \right) \Phi \right\}, \]
\[ (D \Phi)^a_b = \left[ \delta^a_b d + \omega^a_b + (\Phi) Tr (\omega^b_a) \right] \Phi. \] (20)

From transformation rules, we immediately infer that 1-forms \( e^a, B_{ab} \), and \( \omega^b_a \) have the following affine scaling weights \(-1/4, -1/2, \) and \( 0 \) respectively.

Now, acting the exterior covariant derivative on \( A \) we obtain the curvature \( F \) satisfying the structure equation and the Bianchi identity
\[ F = dA + \frac{i}{2} [A, A], \] (21)
\[ dF + i [A, F] = 0, \] (22)

where \( d \) is the exterior differential. Upon expressing the curvature form \( F \) as
\[ F = \mathcal{F}^a P_a + \mathcal{F}^{ab} Z_{ab} + \mathcal{R}^a_b \tilde{L}_b^a, \] (23)

the structure Eq. (21) becomes
\[ \mathcal{D} \mathcal{F}^a = d e^a + \hat{\omega}^a_b \wedge e^b - \frac{1}{4} \hat{\omega} \wedge e^a = \mathcal{D} e^a, \] (24)
\[ \mathcal{D} \mathcal{F}^{ab} = d B^{ab} + \hat{\omega}^{[a} \wedge B^{b]} - \frac{1}{2} \hat{\omega} \wedge B^{ab} - \frac{1}{2} e^{a[y} \wedge e^{b]}, \] (25)

\[ \mathcal{D} \mathcal{R}^a_b = d \hat{\omega}^a_b + \hat{\omega}^a_c \wedge \hat{\omega}^c_b = \mathcal{D} \hat{\omega}^a_b, \] (27)

Thus the curvature forms corresponding to the various generators of the algebra are \( (\mathcal{F}^a, \mathcal{F}^{ab}, \mathcal{R}^a_b) \), and they represent the torsion, the field strength associated with the \( B_{ab} \) field and the non-Riemannian affine curvature form, respectively. One concludes that the affine curvature \( \mathcal{R}^a_b \) and the torsion \( \mathcal{F}^a \) are given by the exterior covariant derivatives of the affine connection and vierbein respectively. On the other hand, the curvature 2-form \( \mathcal{F}^{ab} \) coming from Maxwell extension is not given by the exterior covariant derivative of the corresponding gauge field. The extra term in \( \mathcal{F}^{ab} \) represents the curvature of the local tensor space. This contribution is present because the commutator of two infinitesimal affine transformations equals to an element of the tensor space. Moreover, from the Bianchi identity Eq. (22), we get the following equations
\[ \mathcal{D} \mathcal{F}^{ab} = \mathcal{R}^{[a} \wedge B^{b]} - \frac{1}{2} \mathcal{R} \wedge B^{ab} - \frac{1}{2} \mathcal{F}^{[a} \wedge e^{b]}, \] (28)

Under infinitesimal gauge transformations with parameter \( \zeta \), the curvature 2-form \( F \) transform as
\[ \delta F = i \left[ \zeta, F \right], \] (29)

and hence one gets
\[ \delta \mathcal{F}^a = -\mathcal{R}^a_b \wedge \wedge e^b - \frac{1}{4} \mathcal{R} \wedge e^a - \lambda^{[a} \mathcal{F}^{b]} - \frac{1}{2} \lambda \mathcal{F}^a, \] (30)
\[ \delta \mathcal{F}^{ab} = -\mathcal{R}^{[a} \wedge \wedge \wedge e^{b]} + \frac{1}{2} \mathcal{R} \wedge e^{a} + \lambda^{[a} \mathcal{F}^{b]} - \frac{1}{2} \lambda \mathcal{F}^{ab} + \frac{1}{2} \mathcal{F}^{[a} \wedge e^{b]}, \] (31)
\[ \delta \mathcal{R}^a_b = \lambda^a_c \mathcal{R}^c_b - \lambda^b_c \mathcal{R}^a_c. \] (32)

Again from these transformation rules, one observes that curvature 2-form \( \mathcal{F}^a, \mathcal{F}^{ab} \), and \( \mathcal{R}^a_b \) have the following affine
scaling weights $-1/4$, $-1/2$, and 0 respectively and they will be useful for constructing invariant Lagrangian densities.

4 Construction of the metric for the affine space

Using the definition of the local metric,

$$g^{ab}(x) = e^a \otimes e^b,$$  \hspace{1cm} (33)

one deduces $\mathcal{S} \mathcal{L}(4, R)$ gauge variation of the metric tensor with the help of Eq. (16) by omitting diffeomorphism part

$$\delta_b g^{ab} = \left( \Lambda^a_c e^c - \frac{1}{4} \lambda^a e^a \right) \otimes e^b + e^a \otimes \left( \lambda^b_c e^c - \frac{1}{4} \lambda^b e^b \right) = \Lambda^a_c g^{cb} - \frac{1}{2} \lambda^a g^{ab},$$  \hspace{1cm} (34)

where round brackets denote symmetrization. Similarly from the definition of the Kronecker delta tensor

$$\delta^a_b = e^a \otimes e_b,$$  \hspace{1cm} (35)

one can obtain the $\mathcal{S} \mathcal{L}(4, R)$ gauge variation of $e_a$ as

$$\delta_x e_a = -\Lambda^a_b e_b + \frac{1}{4} \lambda^a e_a.$$  \hspace{1cm} (36)

and the last equation implies

$$\delta_x g_{ab} = -\Lambda^c_{(a} g_{cb)} + \frac{1}{2} \lambda^a g_{ab}.$$  \hspace{1cm} (37)

With the use of vierbein and local metric, the $\mathcal{S} \mathcal{L}(4, R)$ gauge variation of the coordinate metric becomes

$$\delta_x g_{\mu\nu}(x) = \left( -\Lambda^c_{(a} g_{cb)} + \frac{1}{2} \lambda^a g_{ab} \right) e^a_{\mu} e^b_{\nu} + g_{ab} \left( \lambda^c_{(a} e^c_{\mu} - \frac{1}{4} \lambda^c e^c_{\mu} \right) e^b_{\nu} + g_{ab} e^c_{\mu} \left( \lambda^b_c e^c_{\nu} - \frac{1}{4} \lambda^b e^b_{\nu} \right) = 0.$$  \hspace{1cm} (38)

Moreover, the gauge variation of determinant of the vierbein is

$$\delta_x e = \frac{1}{2} e^{\mu\nu} \delta_x g_{\mu\nu} = 0.$$  \hspace{1cm} (39)

Defining the fully antisymmetric tensor $\eta_{abcd}$ by

$$\eta_{abcd} = e_{abcd},$$  \hspace{1cm} (40)

where $e_{abcd}$ is the Levi-Civita symbol, its variation under local $\mathcal{S} \mathcal{L}(4, R)$ transformation becomes

$$\delta_x \eta_{abcd} = -\Lambda^e_a \eta_{ebcd} - \Lambda^e_b \eta_{aecd} - \Lambda^e_c \eta_{abed} - \Lambda^e_d \eta_{abce} + \lambda^e \eta_{abcd} = 0,$$ \hspace{1cm} (41)

and has affine scaling weight 1.

Having defined the local metric for the affine space-time, the metricity is obtained by taking the covariant derivative of the local metric, i.e., $Q^{ab} = \nabla g^{ab}$ and its explicit form follows

$$Q^{ab} = \mathcal{D} g^{ab} = d g^{ab} + \delta^e_{(a} g^{eb)} - \frac{1}{2} \lambda^e g^{ab},$$  \hspace{1cm} (42)

$$Q_{ab} = \mathcal{D} g_{ab} = d g_{ab} - \delta^e_{(a} g_{eb)} + \frac{1}{2} \lambda^e g_{ab}.$$  \hspace{1cm} (43)

This in turn leads to the covariant derivative of the metricity

$$\mathcal{D} Q^{ab} = \mathcal{R}^{(a} g^{eb)} - \frac{1}{2} \mathcal{R} g^{ab}.$$  \hspace{1cm} (44)

Likewise,

$$\mathcal{D} Q_{ab} = -\mathcal{R}^{(a} g_{eb)} + \frac{1}{2} \mathcal{R} g_{ab}.$$  \hspace{1cm} (45)

5 Maxwell-modified mag field equations

One way of constructing the action is to begin from the covariant quantities with manifest geometric meanings. To prescribe the dynamics of the gauge fields, we have to introduce an action, invariant under local $\mathcal{S} \mathcal{L}(4, R)$ transformation. We need then curvatures $\mathcal{R}^a_b, \mathcal{F}^{ab}$ and the metric $g^{ab}$ obtained in the last section. We start with following topological action,

$$S = \frac{1}{2\chi} \int \mathcal{J} \wedge \ast \mathcal{J} = \frac{1}{4\chi} \int \eta_{abcd} \mathcal{J}^{ab} \wedge \mathcal{J}^{cd},$$  \hspace{1cm} (46)

known as Euler or Gauss–Bonnet type action, where $\chi = 8\pi G/c^4$ is the Einstein’s constant, $\ast$ is the Hodge dual and $\eta_{abcd}$ is defined by Eq. (40). Contracting $\mathcal{R}^a_b$ with $g^{bc}$, we can form curvature 2-form $\mathcal{R}^{ab} = \mathcal{R}^{a} g^{bc}$ and it’s gauge transformation is given by

$$\delta \mathcal{R}^{ab} = \lambda^a \mathcal{R}^{cb} + \lambda^b \mathcal{R}^{ac} - \frac{1}{2} \lambda \mathcal{R}^{ab}.$$  \hspace{1cm} (47)

It has the same form as Eq. (31) when the diffeomorphism part omitted. So, one can introduce a shifted curvature 2-form,

$$\mathcal{J}^{ab} = \mathcal{R}^{ab} - \mu \mathcal{F}^{ab},$$  \hspace{1cm} (48)

where $\mu$ is a dimensionful constant. Its gauge transformation becomes

$$\delta \mathcal{J}^{ab} = \lambda^a \mathcal{J}^{cb} + \lambda^b \mathcal{J}^{ac} - \frac{1}{2} \lambda \mathcal{J}^{ab}.$$  \hspace{1cm} (49)

The gauge transformation of $\ast \mathcal{J}^{ab} = \frac{1}{2} \eta_{abcd} \mathcal{J}^{cd}$ has the following form

$$\delta \ast \mathcal{J}^{ab} = \frac{1}{2} \left( \lambda^c \eta_{abcd} + \lambda^e \eta_{abce} - \frac{1}{2} \eta_{abcd} \right) \mathcal{J}^{cd},$$  \hspace{1cm} (50)

the term in the parentheses can be written another form after re-indexing the labels as

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\[ \lambda_a \eta_{abcd} + \lambda_b \eta_{abcd} - \frac{1}{2} \lambda_\eta_{abcd} = -\frac{1}{2} \lambda_a \eta_{abcd} - \lambda_b \eta_{aecd} + \frac{1}{2} \lambda_\eta_{abcd}, \]  

then variation of the Hodge dual of \( J \) becomes

\[ \delta \ast J = -\lambda_a \ast J_{eb} - \lambda_b \ast J_{ae} + \frac{1}{2} \lambda \ast J_{ab}. \]  

Invariance of the action under gauge transformation can be checked easily with the help of Eqs. (47) and (50). By construction, the action is automatically invariant under diffeomorphism and has affine scaling weight zero. Introducing dynamics, we still have to discuss the nature of the gauge and general coordinate transformations (diffeomorphism). From a gauge theory perspective, infinitesimal displacements are a "local translation". Under a local translation, the action is automatically invariant under diffeomorphism. This has resemblance to the usual Einstein’s field equation. However, the curvature tensor \( R_{\mu \nu} \) and \( F_{\mu \nu} \) may not necessarily be symmetric. \( F_{\mu \nu} \) acts as sources in the field equation of gravity. This equation can be written in a more familiar form by going from differential form to space-time tensors as

\[ \frac{1}{2} F_{\mu \nu} \, dx^{\rho} \wedge dx^{\sigma}. \]  

so we get explicit form of \( \mathcal{F}^{\mu \nu \rho \sigma} \).

\[ \mathcal{F}^{\mu \nu \rho \sigma} = e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \nu} - \frac{1}{2} \delta^{\mu \nu}_{\rho \sigma} + \frac{1}{2} \delta^{\mu \rho}_{\nu \sigma}, \]  

then \( F_{\mu} \) and \( F \) can be extracted respectively as

\[ F_{\mu} = \mathcal{F}^{\mu \nu} = e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \nu} - \frac{3}{2} \delta^{\mu}_{\nu}, \]  

Thanks to the last three equations, we can re-expressed the right hand side of Eq. (61),

\[ F_{\mu} = \frac{1}{2} \delta^{\mu}_{\nu} \mathcal{F} = e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \rho} - \frac{1}{2} \delta^{\mu a}_{\nu b} \mathcal{D}_{[a} B_{b] \rho} + \frac{3}{2} \delta^{\mu}_{\nu}, \]  

so the Eq. (61) takes the following form,

\[ \mathcal{R}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \mathcal{F} = e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \rho} - \frac{1}{2} \delta^{\mu a}_{\nu b} \mathcal{D}_{[a} B_{b] \rho} + \frac{3}{2} \delta^{\mu}_{\nu}, \]  

where \( \frac{1}{2} \delta^{\mu}_{\nu} e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \rho} = \delta^{\mu}_{\nu} e_\mu^a \epsilon_b \mathcal{D}_{[a} B_{b] \rho} \).
We see that the source added to the gravity equations with cosmological constant $\mu$ contains linear contributions from the new gauge fields. The second term on the right-hand side of (67) provides a field-dependent modification of the cosmological constant at the left-hand side of the equation [30].

To the decomposition above there corresponds a splitting of the connection 1-form into its Riemannian and non-Riemannian parts $\omega^a_{\,b}$ and $\nu^a_{\,b}$, respectively, as

$$
\omega^a_{\,b} = \omega^a_{\,b} + \nu^a_{\,b},
$$

(68)

where $\omega^a_{\,b}$ is antisymmetric Lorentz connection and $\nu^a_{\,b}$ is symmetric shear connection. In terms of these forms Eq. (67) becomes,

$$
\mathcal{R}^\mu_v - \frac{1}{2} \delta^\mu_v \mathcal{R} - \frac{3}{2} \mu \delta^\mu_v
= \mu e^\alpha_d e^\beta_b \left( D_{[v B^{ab}_\rho]} + v^{[a}_{\rho} \wedge B^{c b}_{\rho]} - \frac{1}{2} \mu v B^{ab}_\rho \right)
- \frac{1}{2} \rho^a_{\,e} \rho^b_{\,e} \left( D_{\mu} B^{ab}_\rho + v^{[a}_{\rho} \wedge B^{c b}_{\rho]} - \frac{1}{2} \mu v B^{ab}_\rho \right),
$$

(69)

where $D$ is the Lorentz exterior covariant derivative. We see that this is simply Einstein’s equation for metric affine gravity with a cosmological constant term. It is then sensible to identify the expression in the curly bracket as the source of the gravitational field. Note also that, if the affine curvature tensor is decomposed into the Riemannian and shear strength parts, it can end up in a Riemann–Cartan theory with additional degrees of freedom represent uniform gauge field strengths in (super)space which leads to uniform constant energy density [13]. Also, it is known that such an additional term may be related to dark energy [4, 5]. Moreover, the Maxwell symmetry provides a geometric background to define vector inflaton in cosmological models [30] and the additional terms may be interpreted as geometrical inflation vector fields which drive accelerated expansion. To sum up, these results show the importance and potential of the Maxwell symmetry.

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