THE DAMPING TERM MAKES THE SMALE-HORSERHOE HETEROCLINIC CHAOTIC MOTION EASIER

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ABSTRACT. The nonlinear Rayleigh damping term that is introduced to the classical parametrically excited pendulum makes the parametrically excited pendulum more complex and interesting. The effect of the nonlinear damping term on the new excitable systems is investigated based on analytical techniques such as Melnikov theory. The threshold conditions for the occurrence of Smale-horseshoe chaos of this deterministic system are obtained. Compared with the existing conclusion, i.e. the smaller the damping term is, the easier the chaotic motions become when the damping term is linear, our analysis, however, finds that the smaller or the larger the damping term is, the easier the Smale-horseshoe heteroclinic chaotic motions become. Moreover, the bifurcation diagram and the patterns of attractors in Poincaré map are studied carefully. The results demonstrate the new system exhibits rich dynamical phenomena: periodic motions, quasi-periodic motions and even chaotic motions. Importantly, according to the property of transitive as well as the fractal layers for a chaotic attractor, we can verify whether a attractor is a quasi-periodic one or a chaotic one when the maximum Lyapunov exponent method is difficult to distinguish. Numerical simulations confirm the analytical predictions and show that the transition from regular to chaotic motion.

1. Introduction. Complex nonlinear phenomena that occur in a nonlinear driven oscillator are the interest of worldwide studies over the past decades [8, 9, 12, 11, 13, 1, 5, 2, 17, 16, 4, 15, 10, 14, 7, 3, 6]. A paradigmatic family of systems are assumed in the form of the second order nonautonomous differential equations

\[ \ddot{x} + \mu \dot{x} + dV(x)/dx + f(x, t) = 0, \]  

(1)
where \( x \) denotes a displacement from the equilibrium position, dots stand for differentiating with respect to time \( t \), \( \mu \dot{x} \) is a weak damping term, with parameter \( \mu \) denoting the damping intensity, \( f(x, t) \) represents periodic function of time with the period \( T = 2\pi/\omega \), and \( V(x) \) is the potential energy approximated by a finite Taylor series.

Note that the damping term \( \mu \dot{x} \) is linear with respect to the velocity \( \dot{x} \). However, linear damping is not always according with the reality [8]. By introducing a nonlinear damping instead of a linear damping term of Eq. (1), Eq. (1) is modified as follows

\[
\ddot{x} + h(x, \dot{x}) + dV(x)/dx + f(x, t) = 0, \tag{2}
\]

where \( h(x, \dot{x}) \) is a nonlinear damping function.

It can be easily verified that the Duffing oscillator, the Helmholtz oscillator, the parametrically excited pendulum, the periodically driven pendulum, with a nonlinear damping term are all special cases of Eq. (2). Several phenomenological models of a nonlinear damping are given in the literature [9, 12, 11, 13]. Especially, the effect of nonlinear damping in some nonlinear oscillators is discussed in detail in reference [12]. Therein \( h(x, \dot{x}) \) is chosen as \( \mu_p |\dot{x}|^{p-1} \), where \( p \geq 1 \) is the damping exponent, and \( \mu_p \) is the corresponding damping coefficient. The critical parameters in terms of damping coefficient and damping exponent is provided through rigorous mathematics calculation.

However, there exist still some unclear problems. For instance:

(i) The existing conclusion is that the smaller the damping term is, the easier the chaotic motions become when the damping term is linear. Does this conclusion remain valid when any nonlinear damping term is introduced?

(ii) Since a nonlinear damping makes a nonlinear driven oscillator system more complex, does it make the system have richer dynamics behaviors? What are the pattern of attractors in Poincaré map and bifurcations as the nonlinear damping coefficient changes?

(iii) When a bifurcation diagram is plotted, several phenomena can be observed: existence of a simple attractor with low period, coexistence, existence of a chaotic attractor (to be verified by the computation of Lyapunov exponents) and various bifurcations [10]. As is well known, negative largest Lyapunov exponent indicates an equilibrium point, zero indicates a periodic or quasi-periodic attractor, and a chaotic attractor has positive largest exponent. The calculation of the maximum Lyapunov exponents is approximate since the computer allows only a finite number, so the corresponding maximum Lyapunov exponent of a quasi-periodic attractor may not precisely equal to zero and may be a little bigger than 0. For instance when the maximum Lyapunov exponent is 0.05, what does this mean, a quasi-period attractor or a chaotic one?

Based on these questions above, in this paper, we introduce the Rayleigh damping term to the parametrically excited pendulum. The system of Eq. (2) can be rewritten as follows

\[
\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} + \sin(x) + \gamma + F \sin(x) \cos(\omega t) = 0,
\]

or its equivalent form

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\sin x - \gamma + \mu(1 - y^2)y - F \sin x \cos(\omega t),
\end{align*}
\tag{3}
\]

where the angular displacement \( x \) is a one-degree-of-freedom state variable, and it is measured from the downward hanging position. The external periodic excitation
is presented by the term $F \sin x \cos(\omega t)$, where $F$ and $\omega$ are the amplitude and the frequency of the excitation, respectively. Let $f(x) = \sin x + \gamma$, and $f(x)$ denotes the nonlinear restoring force with a symmetric parameter $\gamma$. When $\gamma = 0$, Eq. (3) corresponds to a symmetric parametrically excited mathematical pendulum. When $\gamma \neq 0$, Eq. (3) is not invariant under the transformation of

$$x \rightarrow -x, \ t \rightarrow t + \pi/\omega.$$  

Eq. (3) is an asymmetric parametrically excited mathematically pendulum.

Nonlinear oscillators with parametric excitation do not have so extensive literature as the standard oscillators with external harmonic forcing. Discussion on the persistent chaos in a parametrically driven pendulum was opened by Bishop and Clifford in [2]. Recently, much work has been done by Zhou and Cao in [17] when the damping term is linear. The investigation of Eq. (3) including nonlinearities in the damping term has not received much attention. Our goal in this paper is to make a contribution in the study of the transition to chaos by using the Melnikov theory, and see what patterns of attractors are modified as the nonlinear damping term coefficient varied. In such a way, we wish to give a unified view of the nonlinear damping effects on behaviors of several oscillators.

The layout of this paper is organized as follows. In Section 2, we deal with the description, analysis of the system. In Section 3, the conditions of existence of Melnikov’s chaos under the Rayleigh damping term resulting from the heteroclinic and homoclinic bifurcation are performed. Section 4 discusses bifurcation diagrams and the patterns of attractors of the pendulum with Rayleigh damping term. Finally, some conclusions and comments are given in Section 5.

2. Description and analysis of the system. The unperturbed system of Eq. (3) is

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\sin x - \gamma,
\end{align*}
\]

with the potential function

$$V(x) = -\cos x + \gamma x,$$

and then the Hamiltonian function associated to Eq. (4) is

\[
H(x, y) = \frac{1}{2}y^2 - \cos x + \gamma x.
\]

From Eq. (4), we can compute the fixed points. There exist two types of fixed points, in which $x_{2k}(2k\pi - \arcsin \gamma, 0)$, $k = 0, \pm 1, \pm 2, \cdots$ are centers, and $x_{2k+1}((2k+1)\pi + \arcsin \gamma, 0)$, $k = 0, \pm 1, \pm 2, \cdots$ are hyperbolic saddle points.

Fig. 1 shows the corresponding curves of the potential function and the phase spaces portraits when $\gamma = 0$ and $\gamma = 0.05$, respectively. The plots of potential energy and the trajectories in the Cartesian plane, with the $x$ coordinate span from -4 to 10, is useful to illustrate several rotations of the pendulum. When $\gamma = 0$, the plot of the potential energy $V = V(x)$ legislates using the term ‘infinite’ with respect to the number of potential energy wells. $V_{\text{min}}$ occurs at $x_{2k}$ and $V_{\text{max}}$ occurs at $x_{2k+1}$, $k = 0, \pm 1, \pm 2, \cdots$ (Fig. 1a). When the system is viewed in cylindrical space, it becomes clear that in physical realization the system possesses single stable (hanging) and single unstable (inverted) equilibrium positions, related to $V_{\text{min}}$ and $V_{\text{max}}$, respectively. The closed trajectories that encircle the stable equilibria at $x_{2k}$ represent oscillation motion around the hanging position, while the
Figure 1. The corresponding curves of the potential function and the phase spaces portraits when $\gamma = 0$ and $\gamma = 0.05$. (a) The potential function $V(x)$ for $\gamma = 0$. (b) The phase spaces portraits when $\gamma = 0$. The closed trajectories represent oscillation motion around the hanging position, while the opened trajectories $R_1, R_2$ with unchanging sign of velocity visualize the rotating motion of the pendulum in the anti-clockwise or clockwise direction, respectively. (c) The potential function $V(x)$ for $\gamma = 0.05$. (d) The phase spaces portraits when $\gamma = 0.05$. Starting from each hyperbolic saddle, there exists an asymmetric homoclinic orbit located at the right-hand of each saddle, which means oscillation motions from one peak point can not reach to the next right peak point but turn downward.

opened trajectories $R_1, R_2$ with unchanging sign of velocity visualize the rotating motion of the pendulum in the anti-clockwise or clockwise direction, respectively (Fig. 1b). In the rotating motion, the system escapes the potential well and crosses the potential barrier $V = V_{max}$ multiple, or infinite number of times, as $t \rightarrow \infty$. The heteroclinic orbits denote oscillation motion from one peak point of potential function to another. When $\gamma = 0.05$, the peak points of potential function will not be on the same level with the increasing of $x$ (see Fig. 1c). $V_{max} \rightarrow +\infty$ as $x \rightarrow +\infty$ and $V_{max} \rightarrow -\infty$ as $x \rightarrow -\infty$. From Fig. 1d the previous heteroclinic orbits...
have broken. Starting from each hyperbolic saddle, there exists an asymmetric homoclinic orbit located at the right-hand of each saddle, which means oscillation motions from one peak point can not reach to the next right peak point but turn downward. The closed trajectories that encircle the stable equilibria at $x_{2k}$ also represent oscillation motions around the hanging position, and there exists a special kind of motion whose portraits extend to the left direction infinitely.

For a clear illustration it is useful to present computational results in the plane with $x$ coordinate confined to the region $[\pi + \arcsin \gamma, \pi + \arcsin \gamma]$.

3. **Prediction for the existence of chaotic behaviors.** In this section, the generalized Melnikov method developed by Wiggins [15] will be used to predict the appearance of chaos. The unperturbed system for system Eq. (3) has two heteroclinic orbits ($\gamma = 0$) and one homoclinic orbit ($\gamma \neq 0$) at region $[\pi + \arcsin \gamma, \pi + \arcsin \gamma]$. A transformation of $\mu \rightarrow \varepsilon \mu, F \rightarrow \varepsilon F$ is done in order to apply the first-order $\varepsilon$ perturbation scheme of Melnikov theory. Hence, the system of Eq. (3) may be written as

$$
\begin{align*}
\dot{x} & = y, \\
\dot{y} & = -\sin x - \gamma + \varepsilon [\mu(1 - y^2)y - F \sin x \cos(\omega t)].
\end{align*}
$$

(6)

Some notations are introduced as follows

$$f = (f_1, f_2) = (y, -\sin x - \gamma),$$

and

$$g = (g_1, g_2) = (0, \mu(1 - y^2)y - F \sin x \cos(\omega t)).$$

![Figure 2](image)

**Figure 2.** (a) Two heteroclinic orbits, two hyperbolic saddles $x_1, x_{-1}$ and a turning point $x_{TP}$. (b) A homoclinic orbit, a hyperbolic saddle $x_{-1}$ and a turning point $x_{TP}$.

**Case 1: Heteroclinic bifurcation**

Let us first consider the case of heteroclinic orbit. Suppose that the upper heteroclinic orbit is presented as $\Gamma_1 : q_{-1,1}(t) = (x_{-1,1}(t), y_{-1,1}(t))$, and the lower heteroclinic orbit is presented as $\Gamma_2 : q_{1,1}(t) = (x_{1,1}(t), y_{1,1}(t))$, associated with the hyperbolic fixed points $x_{-1}$ and $x_1$ as shown in Fig. 2a, respectively. Then the corresponding Melnikov function $M_{-1,1}$ for $q_{-1,1}(t)$ is calculated as follows
From Eq. (8), we obtain

\[
M_{-1,1}(t_0) = \int_{-\infty}^{\infty} y_{-1,1}(t)[\mu(1-y_{-1,1}^2(t))]y_{-1,1}(t) - F\sin(x_{-1,1}(t))\cos(\omega(t+t_0))dt
\]

\[
= \mu \int_{-\infty}^{\infty} y_{-1,1}^2(t)dt - \mu \int_{-\infty}^{\infty} y_{-1,1}^4(t)dt + F\sin\omega t_0 \int_{-\infty}^{\infty} y_{-1,1}(t)\sin(x_{-1,1}(t))\sin(\omega t)dt
\]

\[
= \mu I_0 - \mu I_1 + F\sin(\omega t_0)I_2,
\]

where \( t_0 \) is the cross-section time of the Poincaré map and \( t_0 \) can be interpreted as the initial time of the forcing term, and

\[
I_0 = \int_{-\infty}^{\infty} y_{-1,1}^2(t)dt,
\]

\[
I_1 = \int_{-\infty}^{\infty} y_{-1,1}^4(t)dt,
\]

\[
I_2 = \int_{-\infty}^{\infty} y_{-1,1}(t)\sin(x_{-1,1}(t))\sin(\omega t)dt.
\]

From the Hamiltonian function Eq. (5), and \( \dot{x} = y \), we have

\[
y = \frac{dx}{dt} = \sqrt{2(\mu + \cos(x))}.
\]

From Eq. (8), we obtain

\[
t = \int_{-\infty}^{x_1} \frac{1}{\sqrt{2(\mu + \cos(\xi))}} d\xi.
\]

Here a numerical method is carried out to compute the Melnikov integrals,

\[
I_0 = \int_{-\infty}^{\infty} y_{-1,1}^2(t)dt = 2 \int_{x_{TP}}^{x_1} \sqrt{2(\mu + \cos(x))} dx,
\]

\[
I_1 = \int_{-\infty}^{\infty} y_{-1,1}^4(t)dt = 4 \int_{x_{TP}}^{x_1} \sqrt{2(\mu + \cos(x))}^3 dx,
\]

\[
I_2 = \int_{-\infty}^{\infty} y_{-1,1}(t)\sin(x_{-1,1}(t))\sin(\omega t)dt
\]

\[
= 2 \int_{x_{TP}}^{x_1} \sin x \sin(\omega(\int_{x_{TP}}^{x_1} \frac{1}{\sqrt{2(\mu + \cos(\xi))}} d\xi)) dx.
\]

Similarly, the Melnikov function \( M_{1,-1}(t_0) \) for \( q_{1,-1}(t) \) becomes

\[
M_{1,-1}(t_0) = \mu I_0 - \mu I_1 - F\sin(\omega t_0)I_2,
\]

where we have used the following symmetry equations

\[
x_{-1,1}(t) = -x_{-1,1}(-t), \quad y_{-1,1}(t) = -y_{-1,1}(-t).
\]

It follows from the Melnikov theory that the conditions for \( M_{-1,1}(t_0) \) and \( M_{1,-1}(t_0) \) change sign when

\[
\mu_{M_{-1,1}(t_0)} \leq \mu_{cr} = F[I_2/I_1 - I_0], \tag{9}
\]

\[
\mu_{M_{1,-1}(t_0)} \geq \mu_{cr} = F[I_2/I_1 - I_0], \tag{10}
\]

respectively, where \( \mu_{cr} \) is the heteroclinic bifurcation critical value. From Eqs. (9) and (10), the Melnikov function \( M_{-1,1}(t_0) \) and \( M_{1,-1}(t_0) \) have simple zeros, then
stable manifolds and unstable manifolds intersect transversely, signifying the possibility of chaotic behavior. Interestingly, the critical values of \( \mu \) for \( M_{-1,1}(t_0) \) and \( M_{1,-1}(t_0) \) are the same, but regions of chaos will occur are different. Furthermore, the two possible chaotic areas disjoint each other. In other words, there are two possible chaotic regions as \( \mu \) varying from 0 to 1.

![Figure 3](image)

**Figure 3.** The critical heteroclinic bifurcation curves are plotted when \( \gamma = 0 \). (a) \( F = 2 \). The region (I) below the heteroclinic bifurcation curve represents \( M_{-1,1}(t) \) has simple zeros, which means the upper heteroclinic orbit will break and then the chaotic behavior may appear. Similarly, the region (II) over the heteroclinic bifurcation curve represents \( M_{1,-1}(t) \) has simple zeros, which means the lower heteroclinic orbit will break and then the chaotic behavior may appear. (b) \( F = 2 \) and \( F = 3 \). The larger the exciting term coefficient \( F \) is, the easier the chaotic motions become.

With \( F \) constant, we study the chaotic threshold as a function of the damping intensity \( \mu \). A typical plot of \( \mu \) against \( \omega \) is shown in Fig. 3, in which the critical heteroclinic bifurcation curves are plotted versus the frequency parameter \( \omega \). Noting from Fig. 3a, first of all, let \( \mu = \mu_{cr} \), with the decreasing or increasing of \( \mu \), a heteroclinic bifurcation takes place when \( \mu \) crosses its critical value, so that a hyperbolic Cantor set appears in a neighborhood of the saddles. The region (I) below the heteroclinic bifurcation curve represents \( M_{-1,1}(t) \) has simple zeros, which means the upper heteroclinic orbit will break and then the chaotic behavior may appear. Similarly, the region (II) over the heteroclinic bifurcation curve represents \( M_{1,-1}(t) \) has simple zeros, which means the lower heteroclinic orbit will break and then the chaotic behavior may appear. Therefore, we predict that the smaller or the larger the damping term is, the easier the heteroclinic chaotic motions become. It also means that the possible chaotic motions will be getting popular in system with a nonlinear damping term. Later on we will use numerical simulations to verify this conclusion in Sec. 4. Second, seen from Fig. 3b, the larger the exciting term coefficient \( F \) is, the easier the chaotic motions become, which can also be verified by the same method as described in Case 1 of Sec. 4.

**Case 2: Homoclinic bifurcation**

Now consider the case of homoclinic orbit. Suppose that the homoclinic orbit is presented as \( \Gamma : q_{-1}(t) = (x_{-1}(t), y_{-1}(t)) \), associated with the hyperbolic fixed points \( x_{-1} \). \( x_{TP} \) denotes turning point as shown in Fig. 2(b), the corresponding
Melnikov function $M_{-1}(t)$ for $q_{-1}(t)$ is calculated as follows

\[
M_{-1}(t_0) = \int_{-\infty}^{\infty} y_{-1}(t)[\mu(1 - y_{-1}^2(t))y_{-1}(t) - F\sin(x_{-1}(t))\cos(\omega(t + t_0))]dt
\]

\[
= \mu \int_{-\infty}^{\infty} y_{-1}^2(t)dt - \mu \int_{-\infty}^{\infty} y_{-1}^4(t)dt + F\sin\omega t_0 \int_{-\infty}^{\infty} y_{-1}(t)\sin(x_{-1}(t))\sin(\omega t)dt
\]

\[
= \mu I_0 - \mu I_1 + F\sin(\omega t_0)I_2,
\]

where $t_0$ is the cross-section time of the Poincaré map and $t_0$ can be interpreted as the initial time of the forcing term, and

\[
I_0 = \int_{-\infty}^{\infty} y_{-1}^2(t)dt,
\]

\[
I_1 = \int_{-\infty}^{\infty} y_{-1}^4(t)dt,
\]

\[
I_2 = \int_{-\infty}^{\infty} y_{-1}(t)\sin(x_{-1}(t))\sin(\omega t)dt.
\]

From the Hamiltonian function Eq. (5), and $\dot{x} = y$, we have

\[
y = \frac{dx}{dt} = \pm \sqrt{2(H + \cos(x) - \gamma x)},
\]

where ‘+’ denotes the upper homoclinic orbit and ‘−’ denotes the lower homoclinic orbit. From Eq. (11), we obtain

\[
t = \mp \int_{x_{TP}}^{x_1} \frac{1}{\sqrt{2(H + \cos \xi - \gamma \xi)}}d\xi.
\]

Here a numerical method is also carried out to compute the Melnikov integrals.

\[
I_0 = \int_{-\infty}^{\infty} y_{-1}^2(t)dt = \mp 2 \int_{x_{TP}}^{x_1} \sqrt{2(H + \cos(x) - \gamma x)}dx,
\]

\[
I_1 = \int_{-\infty}^{\infty} y_{-1}^4(t)dt = 2 \int_{0}^{x_1} y_{-1}^4(t)dt = \mp 4 \int_{x_{TP}}^{x_1} \sqrt{2(H + \cos(x) - \gamma x)^3}dx,
\]

\[
I_2 = \int_{-\infty}^{\infty} y_{-1}(t)\sin(x_{-1}(t))\sin(\omega t)dt
\]

\[
= 2 \int_{x_{TP}}^{x_1} \sin x\sin(\mp \int_{x_{TP}}^{x_1} \frac{1}{\sqrt{2(H + \cos \xi - \gamma \xi)}}d\xi)dx.
\]

It follows from the Melnikov theory that the condition for $M_{-1}(t_0)$ changes sign

\[
\mu|I_1 - I_0| \leq F|I_2|,
\]

i.e.

\[
\mu \leq \mu_{cr} = F|I_2/I_1 - I_0|,
\]

where $\mu_{cr}$ is the homoclinic bifurcation critical value. From Eq. (12), the Melnikov function $M_{-1}(t_0)$ has simple zeros, then stable manifolds and unstable manifolds intersect transversely, signifying the possibility of chaotic behavior.

With $F$ constant, the corresponding homoclinic bifurcation curve is plotted in the parameter plane $(\omega, \mu)$. Note from Fig. 4, we can derive the following conclusions: First of all, the threshold of the chaotic motion increases with the decreasing of the damping coefficient $\mu$. Second, the region where leads to chaos will increase when $F$ increase. That is to say the possible chaotic motions will be getting popular in
system with decrease of $\mu$ and the increase of $F$. It is also found that for a fixed value of $F$, there are three critical values under which homoclinic bifurcation may occur.

We also plot the stable and unstable manifolds of fixed points of Eq. (3), which can verify the Figs. 3 and 4. In Figs. 5 and 6, the stable and unstable manifolds of fixed points are plotted. In Fig. 5(a), the parameters are $\gamma = 0, F = 2, \omega = 1.3$ and $\mu = 0.12$. In the upper left window, the fixed points are marked by crosses. In the upper right window, a part of stable manifold of the fixed point, approximately $(3.067, -0.622)$, is superimposed. In the lower left window, a part of unstable manifold of the fixed point, approximately $(3.067, -0.622)$, is superimposed. The lower right window contains a superposition of the pictures of the upper left, upper right, lower left windows. The intersection of the stable manifolds and unstable manifolds implies the existence of chaos. In Fig. 5(b), the parameters are $\gamma = 0, F = 2, \omega = 1.3$ and $\mu = 0.83$. A part of stable and unstable manifold of fixed point, approximately $(3.063, -0.021)$. The intersection of stable manifold and unstable manifold also implies the existence of chaos. In Fig. 6(a), the parameters are $\gamma = 0.05, F = 3, \omega = 2.4$ and $\mu = 0.23$. In the upper left window, the fixed points are marked by crosses. In the upper right window, a part of stable manifold of the fixed point, approximately $(-3.052, -0.018)$, is superimposed. In the lower left window, a part of unstable manifold of the fixed point, approximately $(-3.052, -0.018)$, is superimposed. The lower right window contains a superposition of the pictures of the upper left, upper right, lower left windows. The intersection of the stable manifolds and unstable manifolds implies the existence of chaos. In Fig. 6(b), the parameters are $\gamma = 0.05, F = 3, \omega = 2.4$ and $\mu = 0.72$. A part of stable and unstable manifold of fixed point, approximately $(-3.064, -0.032)$. The separation of the stable and unstable manifolds means that chaos does not exist.
Figure 5. The stable and unstable manifolds of the fixed points of equation $\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} + \sin x + \gamma + F\sin x\cos(\omega t) = 0$ (with $\gamma = 0, F = 2, \omega = 1.3, \mu = 0.12$ in (a) and $\mu = 0.83$ in (b)). (a) Fixed points, stable and unstable manifolds of the fixed point (approximately $(3.067, -0.622)$). In the upper left window, the fixed points are marked by crosses. In the upper right window, a part of stable manifold (blue online) of the fixed point (approximately $(3.067, -0.622)$) is superimposed. In the lower left window, a part of unstable manifold (red online) of the fixed point (approximately $(3.067, -0.622)$) is superimposed. The lower right window contains a superposition of the pictures of the upper left, upper right, lower left windows. The intersection of the stable and unstable manifolds implies the existence of chaos. (b) A part of the stable (blue online) and unstable manifold (red online) of the fixed point (approximately $(3.063, -0.021)$). The intersection of the stable and unstable manifolds implies the existence of chaos.

4. Bifurcations and attractors. In this section, we focus on studying the bifurcation diagrams, the maximal Lyapunov exponents and the type of attractors plotted by the software package Dynamics [12].

Now we study the behavior of the system given by Eq. (3) as a function of the damping parameter. The bifurcation diagram and the maximal Lyapunov exponents have been represented for the variables $x$ and $L_{\text{max}}$, respectively (see Fig. 7). Numerical simulations have been carried out for $\gamma = 0, F = 2, \omega = 1.3$ and $\gamma = 0.05, F = 3, \omega = 2.4$ in Fig. 7, respectively.

Case 1: The Melnikov method gives a necessary condition when the damping coefficient $\mu$ is larger or smaller than the critical heteroclinic values. According to the previous calculation (see Fig. 3a), the $\mu_{\text{cr}} = 0.4242$. Seen from Figs. 7a and 7b, now provided $\mu = \mu_{\text{cr}} = 0.4242$, with the decrease or increase of $\mu$, there exist period-double bifurcations, and then there exist quasi-periodic windows. When $\mu \in [0, 0.24] \cup [0.82, 1]$, the corresponding maximum Lyapunov exponents are obviously larger than 0, which means there exist the heteroclinic chaotic motions.

Case 2: Similarly, the Melnikov method gives a necessary condition when the damping coefficient $\mu$ is smaller than the critical homoclinic values. According to the previous calculation (see Fig. 4), the $\mu_{\text{cr}} = 0.3600$. So the possible homoclinic
The damping term makes the Smale-Horseshoe chaotic region is $\mu \in [0, 0.36]$, which can be confirmed by Fig. 7c and 7d. Now provided $\mu = 1$, with the decrease of $\mu$, a period 2 orbit bifurcate into a period 4 orbit, followed rapidly by an infinite sequence, period 8, 16 etc, and then quasi-period orbits occur, finally chaos achieves. When $\mu \in [0.16, 0.28] \subset [0, 0.36]$, the corresponding maximum Lyapunov exponents are obviously larger than 0, which signifying the appearance of homoclinic chaos motions. When $\mu \in [0.36, 1]$, the corresponding maximum Lyapunov exponents are less than or equal to or a little larger than 0, which means there exist periodic motions or quasi-periodic motions.

So far, the following questions immediately arise.

(i) How do we verify that the chaos attractor existence besides based on the maximum Lyapunov exponent theory?

(ii) For Case 2 how do we prove there exist a quasi-periodic attractor not a chaotic attractor when $\mu = 0.41$?

As we know, chaotic attractors demonstration randomly transitional phenomena usually exhibit the following three outstanding properties [14]: (i) Sensitive dependence on initial conditions; (ii) Fractal structure of alpha-branches of principal saddles; (iii) Transitive property, i.e., one trajectory or sequence of images can come arbitrarily close to every point in the attractor. Among the three properties...

Figure 6. The stable and unstable manifolds of the fixed points of equation $\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} + \sin x + \gamma + F \sin x \cos(\omega t) = 0$ (with $\gamma = 0.05, F = 3, \omega = 2.4, \mu = 0.23$ in (a) and $\mu = 0.72$ in (b)). (a) Fixed points, stable and unstable manifolds of the fixed point (approximately $(-3.052, -0.018)$). In the upper left window, the fixed points are marked by crosses. In the upper right window, a part of stable manifold (blue online) of the fixed point (approximately $(-3.052, -0.018)$) is superimposed. In the lower left window, a part of unstable manifold (red online) of the fixed point (approximately $(-3.052, -0.018)$) is superimposed. The lower right window contains a superposition of the pictures of the upper left, upper right, lower left windows. The intersection of the stable and unstable manifolds implies the existence of chaos. (b) A part of stable (blue online) and unstable (red online) manifolds of the fixed point (approximately $(-3.064, -0.032)$). The separation of the stable and unstable manifolds means that chaos does not exist.
Figure 7. Bifurcation diagrams and the corresponding maximal Lyapunov exponents of equation $\ddot{x} - \mu (1 - \dot{x}^2) \dot{x} + \sin x + \gamma + F \sin x \cos(\omega t) = 0$, (a) and (b) with $\gamma = 0, F = 2, \omega = 1.3$, (c) and (d) with $\gamma = 0.05, F = 3, \omega = 2.4$, respectively, where $\mu$ represents the bifurcation parameter and $L_{max}$ denotes the maximum Lyapunov exponents.

Given above, the second and third properties are always observed in simulation experiments. However, rigorous mathematical proofs have not been provided yet.

Chaotic attractor is shown in Fig. 8 with magnification sequences of the $(x, y)$ plane. These will show the transitive property as well as the fractal layers, which are common to chaotic attractors. But in Fig. 9, we can not see the fractal layers. Therefore, this is a quasi-periodic attractor not a chaotic attractor when $\mu = 0.41$.

Corresponding to the parameter values of $\mu = 0.8, \gamma = 0, F = 2$ and $\omega = 1.3$, there are two stable fixed point attractors $(-1.5964, 1.7136)$ and $(1.5964, -1.7136)$ as shown in the left window of Fig. 10. The right window of Fig. 10 shows the basins of attraction of the two attractors represented by the black (pink online) and light grey (green online), respectively. Corresponding to the parameter values of $\mu = 0.52$, $\gamma = 0.05, F = 3$ and $\omega = 2.4$, there is a period-8 attractor, which constitutes a period-8 orbit: $(1.7426, -0.5614) \rightarrow (-2.0013, 0.4297) \rightarrow (0.7126, -0.1422) \rightarrow (-1.8694, 0.5246) \rightarrow (0.9541, -0.4504) \rightarrow (-2.0245, 0.4132) \rightarrow (0.6680, -0.0750) \rightarrow (-1.7964, 0.5659) \rightarrow (1.7426, -0.5614)$ (see Fig. 11).
THE DAMPING TERM MAKES THE SMALE-HORSHOE

Figure 8. A chaotic attractor and consecutive blow-ups in \((x, y)\) plane. (a) shows a chaotic attractor with the parameter values \(\mu = 0.1, \gamma = 0, F = 2, \omega = 1.3\). The coordinates of the (b) are the coordinates of the rectangle in (a). Similarly, the coordinates of (c) are the coordinates of the rectangle in (b).

In Sec. 3, as we have said, there are three critical values under which homoclinic bifurcation may occur. Let’s check the existence of chaotic attractor when \(\omega \in [0.47, 6.00]\). Take \(\omega = 5.1\) as an example, we present the bifurcation diagram and the corresponding maximal Lyapunov exponents in Fig. 12, respectively. Almost all of the maximum Lyapunov exponents are equal to or approximately equal to 0 except two small regions \([0.04, 0.08] \cup [0.85, 0.89]\), which means there exist only period points and quasi-periodic points and no chaotic attractors. This phenomenon can be explained by the fact that in intersections of stable and unstable manifolds are the necessary conditions for the existence of chaos. In fact, there are 5 different attractors, which can be divided into three types: (i) A limit cycle (as shown in the upper left window of the left window in Fig. 13a); (ii) Period-2 points (as shown in the upper right window of the left window in Fig. 13a). There are 4 period-2 points which constitute two distinct orbits. The one orbit is \((0.2831, -2.5612) \to (-3.0273, 2.5313)\) (indicated by the smaller crosses), the other one is \((2.8137, 2.5026) \to (-0.7421, 2.5653)\) (indicated by the larger crosses).
Figure 9. (a) A quasi-periodic attractor with the parameter values \( \mu = 0.41, \gamma = 0.05, F = 3, \omega = 2.4 \). (b) A blow-up of (a). The coordinates of the (b) are the coordinates of the rectangle in (a).

Figure 10. (a) Two attractors of fixed points \((-1.5964, 1.7136)\) and \((1.5964, -1.7136)\) in \((x, y)\) plane with the parameters are \( \mu = 0.8, \gamma = 0, F = 2 \) and \( \omega = 1.3 \). (b) Basins of attraction for the two fixed points in \((x, y)\) plane.

Period-6 points (as shown in the lower right window of the left window in Fig. 13a). There are 12 period-6 points which constitute two distinct orbits. The one orbit (indicated by the smaller crosses) is
\[
\begin{align*}
(-1.8780, 0.0679) &\rightarrow (-1.1092, 1.1587) \rightarrow (0.9517, 1.1114) \\
(1.6421, -0.0854) &\rightarrow (0.7362, -1.2071) \rightarrow (-1.2831, -1.0470),
\end{align*}
\]
the other one (indicated by the larger crosses) is
\[
\begin{align*}
(2.5810, -1.2856) &\rightarrow (0.5767, -1.9969) \rightarrow (-2.2906, -1.8048) \\
(2.2799, -1.8541) &\rightarrow (-0.5668, -1.9273) \rightarrow (-2.5284, -1.2487).
\end{align*}
\]
Attractive basins of the 5 different attractors are represented in the right window of Fig. 13b. The green color region are the attractive basins of limit cycle. The yellow and the blue
regions are the attractive basins of period-2 points in two distinct orbits, respectively. The pink and grey regions are the the attractive basins of period-6 points in two distinct orbits, respectively.

5. **Conclusion.** We have studied the dynamical behaviors by introducing the Rayleigh damping term on the classical parameterically excited pendulum in this paper.

At first, we give a brief analysis of unperturbed system for $\gamma = 0$ and $\gamma = 0.05$, respectively. When $\gamma = 0$, there exist two heteroclinic orbits connecting two adjacent saddles. While $\gamma \neq 0$, starting from each hyperbolic saddle, there exists an asymmetric homoclinic orbit located at the right-hand of each saddle. In the
Figure 13. (a) Five different attractors when $\mu = 0.01, \gamma = 0.05, F = 3, \omega = 5.1$. Limit cycle (as shown in the upper left window); Period-2 points (as shown in the upper right window); Period-6 points (as shown in the lower left and right windows). (b) The corresponding attractive basins of the five attractors.

perturbed system, the heteroclinic orbits and the homoclinic orbits will break. So the discussion of existence of chaotic motions is necessary.

Second, we predict the existence of Smale-horse shoe chaos in two different cases: (i) heteroclinic bifurcation; (ii) homoclinic bifurcation. Melnikov method is used to obtain the threshold condition for the occurrence of Smale-horseshoe chaos. The results show that, the smaller or the larger the damping term is, the easier the heteroclinic chaotic motions become, which is different from the existing conclusion. While in homoclinic bifurcation case, we obtain that the smaller the damping term is, the easier the homoclinic chaotic motions become, which is in line with the existing conclusion. Numerical simulations confirm these analytical predictions.

Finally, the bifurcation diagrams, the maximal Lyapunov exponents and the types of attractors are carefully investigated by the software package Dynamics. We have verified the transitive property as well as the fractal layers of chaotic attractors. Moreover, we also have showed an attractor is not a chaotic one but a quasi-periodic one, since it doesn’t satisfy the property of fractal layers for a chaotic attractor. All analyses above demonstrate the new system exhibits rich dynamical phenomena.

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