THE MOTIVIC LAMBDA ALGEBRA
AND MOTIVIC HOPF INVARIANT ONE PROBLEM

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Abstract. We investigate forms of the Hopf invariant one problem in motivic homotopy theory over arbitrary base fields of characteristic not equal to 2. Maps of Hopf invariant one classically arise from unital products on spheres, and one consequence of our work is a classification of motivic spheres represented by smooth schemes admitting a unital product.

The classical Hopf invariant one problem was resolved by Adams, following his introduction of the Adams spectral sequence. We introduce the motivic lambda algebra as a tool to carry out systematic computations in the motivic Adams spectral sequence. Using this, we compute the $E_2$-page of the $R$-motivic Adams spectral sequence in filtrations $f \leq 3$. This universal case gives information over arbitrary base fields.

We then study the 1-line of the motivic Adams spectral sequence. We produce differentials $d_2(h_{a+1}) = (h_0 + \rho_1)h_2$ over arbitrary base fields, which are motivic analogues of Adams’ classical differentials. Unlike the classical case, the story does not end here, as the motivic 1-line is significantly richer than the classical 1-line. We determine all permanent cycles on the $R$-motivic 1-line, and explicitly compute differentials in the universal cases of the prime fields $F_q$ and $Q$, as well as $Q_p$ and $R$.

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1. Introduction

Motivic homotopy theory is a homotopy theory for algebraic varieties, developed by Morel and Voevodsky in the 1990s [MV99]. Since its conception and subsequent use by Voevodsky to resolve the Milnor [Voe03] and Bloch–Kato [Voe11] conjectures, an immense amount of work has gone into the theory, with applications to algebraic geometry, algebraic number theory, and algebraic topology.

Motivic stable homotopy theory is the home of $\mathbb{A}^1$-invariants on algebraic varieties, such as algebraic $K$-theory, motivic cohomology, and algebraic cobordism. The universal such invariants are motivic stable homotopy groups, and as such the internal structure of the motivic stable homotopy groups of spheres reflects the broad-scale structure of the motivic stable homotopy category. These motivic stable stems encode deep geometric and number-theoretic information; for example, Morel [Mor04] showed that the Milnor–Witt $K$-theory of a field appears in its stable stems, and Röndigs–Spitzweck–Østvær [RSO19, RSO21] have identified motivic stable stems in low Milnor-Witt stem in terms of variants of Milnor $K$-theory, Hermitian $K$-theory, and motivic cohomology.
Motivic homotopy theory was originally developed to apply ideas and tools from homotopy theory to problems in algebraic geometry and algebraic K-theory. Information now flows the other way as well. After $p$-completion, $\mathbb{C}$-motivic stable stems capture information about classical stable stems that is not seen using classical techniques. This has led to the highly successful program of Gheorghe–Isaksen–Wang–Xu [Isa19, IWX20, GWX21], yielding groundbreaking advances in computations of classical stable homotopy groups of spheres. A similar program using $\mathbb{R}$-motivic stable stems to capture information about $C_2$-equivariant stable stems has also developed [BHS20, BI20, DI16, DI17, GI20, BG21]. More recently, work of Bachmann–Kong–Wang–Xu [BKWX22] has related $F$-motivic stable homotopy theory over a general field $F$ to classical complex cobordism.

All of this has motivated a swath of explicit computations of motivic stable stems over particular base fields $F$. We refer the reader to [IO19] for a general survey, but mention the following $2$-primary computations:

- $F = \mathbb{C}$ Dugger–Isaksen [DI10] computed the $\mathbb{C}$-motivic stable stems through the 36-stem, and Isaksen [Isa19] and Isaksen–Wang–Xu [IWX20] pushed these computations out to the 90 stem.
- $F = \mathbb{R}$ Dugger–Isaksen [DI16] computed the first 4 Milnor–Witt stems over $\mathbb{R}$, and Belmont–Isaksen [BI20] expanded on this to compute the first 11 Milnor–Witt stems over $\mathbb{R}$.
- $F = \mathbb{F}_q$ Wilson [Wil16] and Wilson–Østvær [WO17] computed the motivic stable homotopy groups of finite fields in motivic weight zero through topological dimension 18.

There are still many mysteries contained in the motivic stable stems. All of the above computations were enabled by the motivic Adams spectral sequence, originally introduced by Morel [Mor99] and further developed by Dugger–Isaksen [DI10]. This is a motivic analogue of the classical Adams spectral sequence, which was developed by Adams [Ada58, Ada60] to resolve the Hopf invariant one problem. Adams used this spectral sequence to prove that the only elements of Hopf invariant one in the classical stable stems $\pi_1^s$ are the classical Hopf maps $\eta_1 \in \pi_1^s$, $\nu_1 \in \pi_3^s$, and $\sigma_1 \in \pi_7^s$. This theorem has a number of implications, including a classification of which spheres can be made into $\mathcal{H}$-spaces, which spheres are parallelisable, which 2-dimensional modules over the Steenrod algebra can be realized by cell complexes, which dimensions a finite-dimensional real division algebra can have, and more.

This paper is concerned with topics surrounding motivic analogues of the classical Hopf invariant one problem. There is an element $\eta$ in the motivic stable stems, represented by the canonical map $\eta: \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$, which refines the classical complex Hopf map $\eta_1$. Hopkins and Morel [Mor04] showed that $\eta$ is one of the generators of the Milnor–Witt $K$-theory of the base field. This motivic $\eta$ behaves quite differently from the classical Hopf map; most famously, $\eta$ is not nilpotent, and is generally not 2-torsion. Because $\eta$ is not nilpotent, one may consider the $\eta$-inverted stable stems $\pi_*, [\eta^{-1}]$. These are closely related to Witt $K$-theory [Bac20, BH20], and have been the subject of thorough investigation [AM17, GI15, GI16, OR20, Will18].

Using the theory of Cayley–Dickson algebras, Dugger–Isaksen [DI13] have shown that the classical quaternionic and octonionic Hopf maps $\nu_1$ and $\sigma_1$ also admit geometric refinements to motivic classes $\nu$ and $\sigma$. All of these motivic Hopf maps $\eta, \nu, \sigma$ are maps of Hopf invariant one, but unlike classically, they are not the only such maps. For example, the classical stable stems sit inside of the motivic stable stems, and $\eta_1, \nu_1, \sigma_1$ give rise to distinct examples of maps of Hopf invariant one in the motivic setting. If we reformulate the condition of a map $\alpha$ having nontrivial Hopf invariant as asking that the homology of the 2-cell complex with attaching map $\alpha$ is not split as a module over the motivic Steenrod algebra, then the situation becomes even richer: for example, $\sigma_1^2$ admits an $\mathbb{R}$-motivic refinement to a map of nontrivial Hopf invariant in this sense, closely related to the nonexistent Hopf map coming next in the sequence $\eta, \nu, \sigma$.

All of this motivates the present work, the purpose of which is three-fold:

1. To analyze the motivic Hopf invariant one problem and deduce geometric consequences;
2. To advance our understanding of motivic stable stems over general base fields;
3. To introduce the motivic lambda algebra, a new tool for motivic computations.
As mentioned above, Adams resolved the Hopf invariant one problem by introducing and studying the Adams spectral sequence. Morel and Dugger–Isaksen have already introduced the \textit{F-motivic Adams spectral sequence}, which takes the form

$$E_2^{*,*} = \text{Ext}^*_{A^F}(M^F, M^F) \Rightarrow \pi_{*,*}^F.$$ 

Here, $A^F$ is the \textit{F-motivic} Steenrod algebra \cite{Voe03} \cite{HKØ17}, which acts on $M^F$, the mod 2 motivic cohomology of $\text{Spec}(F)$. This spectral sequence converges to $\pi_{*,*}^F$, the homotopy groups of the \textit{2, $\eta$}-completed $F$-motivic sphere. Implicit is the assumption that 2 is invertible in $F$.

In this paper, we bring the motivic Adams spectral sequence back to its classical roots, using it to study the motivic Hopf invariant one problem. We do not follow Adams’ original approach. Instead, at least in broad outline, we follow J.S.P. Wang’s approach \cite{Wan67}, which proceeded by first gaining a good understanding of the $E_2$-page of the Adams spectral sequence. Importing this approach to motivic homotopy theory requires analyzing the $E_2$-page of the motivic Adams spectral sequence over general base fields in ranges beyond what is known by previous techniques.

To carry out this analysis, we bring another tool from classical stable homotopy theory into the motivic context: the \textit{lambda algebra}. The classical lambda algebra $\Lambda^{cl}$ is a certain differential graded algebra, originally constructed by Bousfield–Curtis–Kan–Quillen–Rector–Schlesinger \cite{6A66}, whose homology recovers the $E_2$-page of the Adams spectral sequence. The classical lambda algebra is now a standard member of the homotopy theorist’s toolbox, and we cannot hope to list all of its applications, but the following are a handful:

1. Wang’s computation of $E_2$-page of the Adams spectral sequence through the 3-line, and subsequent simplified resolution of the Hopf invariant one problem \cite{Wan67};
2. Some of the first automated computations of the $E_2$-page of the Adams spectral sequence, including products and Massey products \cite{Tan85, Tan93, Tan94, CGMM87};
3. The construction of Brown–Gitler spectra \cite{BJG73}, which played an important role in analyzing the \textit{bo}-resolution \cite{Mah81, Shi84}, the proof of the Immersion Conjecture \cite{Coh85}, and more \cite{Mah77, Goe99, HK99};
4. The algebraic Atiyah–Hirzebruch spectral sequence for $\mathbb{RP}^\infty$ \cite{WX16}, used as input to Wang–Xu’s proof of the nonexistence of exotic smooth structures on the 61-sphere \cite{WX17};
5. The only complete computations of the 4- and 5-lines of the Adams $E_2$-term \cite{Che11, Lin08}.

We expect that the motivic lambda algebra will likewise become a useful member of the motivic homotopy theorist’s toolbox. This paper focuses in particular on developing the lambda algebra and applying this to the motivic Hopf invariant one problem. We consider both the unstable problem, with applications to $H$-space structures on motivic spheres, and the stable problem, which is concerned with the 1-line of the motivic Adams spectral sequence. The motivic situation is substantially richer than the classical situation, and requires us to develop a number of new techniques for motivic computations across general base fields.

Adams’ resolution of the classical Hopf invariant one problem asserted the existence of differentials $d_2(h_{a+1}) = h_0 h_a^2$ in the Adams spectral sequence. There are classes $h_a$ in the \textit{F-motivic} Adams spectral sequence for any field $F$, corresponding to the motivic Hopf maps discussed above for $a \leq 3$. Using Betti realization, it is possible to lift Adams’ differentials to the \textit{\mathbb{C}-motivic} Adams spectral sequence. It follows that if $F$ admits a complex embedding, then $h_{a+1}$ must support a nontrivial differential for $a \geq 3$. However, this is insufficient to determine the precise target of the differential, as well as to determine what happens over other base fields, particularly fields of positive characteristic. The techniques we develop in this paper are geared towards resolving this sort of issue. We use these to obtain a number of new results; let us give the following here, as it is the most pleasant to state.

\textbf{Theorem A (Theorem 7.3.1).} For an arbitrary base field $F$ of characteristic not equal to 2, there are differentials of the form

$$d_2(h_{a+1}) = (h_0 + \rho h_1) h_a^2$$

in the \textit{F-motivic} Adams spectral sequence, which are nonzero for $a \geq 3$. <
It is worth making a couple remarks to distinguish this from the classical result.

Remark 1.0.1. Classically, there is at most one possible nontrivial target for a $d_2$-differential on $h_{a+1}$. As suggested by the target in Theorem A, the motivic situation is more complicated. For example, when $F = \mathbb{R}$, we show that if $a \geq 4$ then the group of potential values of $d_2(h_{a+1})$ is given by $\mathbb{F}_2(h_0^a h_1^a, \rho h_1 h_2^a)$. The general picture is similar, only where there may be additional interference coming from the mod 2 Milnor $K$-theory of $F$. This computation requires new techniques for computing the cohomology of the motivic Steenrod algebra, which is much richer than the analogous classical computation.

Remark 1.0.2. Even once we have carried out the algebraic work of identifying potential values of $d_2(h_{a+1})$, the classical proof does not directly generalize to yield Theorem A. In spirit, our proof follows Wang’s classical inductive proof [Wan67]. The base case of Wang’s induction is the differential $d_2(h_4) = h_0 h_3^2$, which follows easily from graded commutativity of stable stems. By contrast, our base case must include the differential $d_2(h_5) = (h_0 + \rho h_1) h_3^2$. Over $\mathbb{R}$, this differential may be deduced by combining complex and real Betti realization, but a completely different argument is required to obtain the differential for other fields. To obtain this differential over other base fields, we use a certain motivic Hasse principle to reduce to considering fields with simple mod 2 Milnor $K$-theory, then analyze how the classical Kervaire class $\theta_4$ appears in the motivic stable stems.

1.1. Brief overview. Now let us give a very brief overview of what we do in this paper, before giving a more thorough summary in Section 1.2 just below. This paper has three main parts. These parts are not independent, but none rely on the hardest aspects of the others.

The first part is purely algebraic, and is the most computationally intensive. In Section 2, we introduce the $F$-motivic lambda algebra (Theorem B), and in Section 4 we use the $\mathbb{R}$-motivic lambda algebra to compute $\text{Ext}_\mathbb{R}$ in filtrations $f \leq 3$ (Theorem C). The result is quite complicated, with 8 infinite families of multiplicative generators and numerous relations between these. As we explain in Section 7.1, this gives information about $\text{Ext}_F$ for any base field $F$ once the mod 2 Milnor $K$-theory of $F$ is known.

The second part is shorter, and does not rely on the above computation. In Section 6, after some preliminaries in Section 5, we consider the motivic analogue of the Hopf invariant one problem in its classical unstable formulation, concerning unstable 2-cell complexes with specified cup product, as well as concerning geometric applications such as to $H$-space structures on motivic spheres. Our analysis proceeds by a novel reduction to the classical case and other known results, by first formulating a certain motivic Lefschetz principle (Proposition 5.2.1), then using this to build unstable “Betti realization” functors over arbitrary algebraically closed fields (Proposition 5.3.2). One consequence of this analysis is a complete classification of motivic spheres which are represented by smooth schemes admitting a unital product (Theorem D).

The third part is our main homotopical contribution. In Section 7, we give a detailed study of the 1-line of the $F$-motivic Adams spectral sequence. This work has a direct geometric interpretation: permanent cycles on the 1-line of the motivic Adams spectral sequence classify how the motivic Steenrod algebra can act on the cohomology of a motivic 2-cell complex. This section does not rely on the full strength of our computation of $\text{Ext}_\mathbb{R}$, and should be understandable by the reader familiar with prior work on the $\mathbb{R}$-motivic Adams spectral sequence. The main theorems in this section are Theorem A above, together with much more detailed information about the 1-line of the $F$-motivic Adams spectral sequence for the particular fields $F = \mathbb{R}, F = \mathbb{F}_q$ with $q$ an odd prime-power, $F = \mathbb{Q}_p$ with $p$ any prime, and $F = \mathbb{Q}$ (Theorem E). As this includes all the prime fields, these computations give information that applies to an arbitrary base field. When $F = \mathbb{R}$, we completely determine all permanent cycles on the 1-line by comparison with a computation in Borel $C_2$-equivariant homotopy theory (Theorem F); both the equivariant computation and the method of comparison are of independent interest.
1.2. Summary of results. We now summarize our work in more detail. We begin with our introduction of the motivic lambda algebra. The nature of the classical lambda algebra $\Lambda^c$ [6A66] was greatly clarified by Priddy [Pri70], who introduced the notion of a Koszul algebra, and showed that $\Lambda^c$ is the Koszul complex of the classical Steenrod algebra. We carry out the motivic analogue of this, producing the following.

Theorem B (Section 2.4). There is a differential graded algebra $\Lambda^F$, the $F$-motivic lambda algebra, with the following properties.

1. $\Lambda^F$ may be described explicitly in terms of generators, relations, and monomial basis.
2. There is a surjective and multiplicative quasiisomorphism $C(\Lambda^F) \to \Lambda^F$ from the cobar complex of the $F$-motivic Steenrod algebra to $\Lambda^F$. In particular, there is an isomorphism $H_*\Lambda^F \cong \text{Ext}_F^*$
   
   (3) $\Lambda^F$ generalizes the classical lambda algebra, in the sense that if $F$ is algebraically closed then $\Lambda^F[\tau^{-1}] = \Lambda^c[\tau^{\pm 1}]$. In particular, it is considerably smaller than $C(\Lambda^F)$. 

Here, we have abbreviated $\text{Ext}_F^{*,*,*}(\mathbb{M}_F, \mathbb{M}_F)$ to $\text{Ext}_F^*$, where the single index refers to filtration, or homological degree, i.e. $\text{Ext}_F^f = H^f(\Lambda^F)$.

Remark 1.2.1. Several subtleties arise in the construction and identification of the motivic lambda algebra. We note two interesting points here:

1. Priddy’s notion of Koszul algebra in [Pri70] is not general enough for our situation: $\Lambda^F$ is generally not augmented as an $\mathbb{M}_F$-algebra, and $\mathbb{M}_F$ is generally not central in $\Lambda^F$. This forces us to consider a more general notion of a Koszul algebra, as well as to find new arguments to prove that $\Lambda^F$ is Koszul in this more general sense.
2. As readers familiar with the motivic Adem relations might suspect, the elements $\tau$ and $\rho$ of $\mathbb{M}_F$ appear in the relations defining the motivic lambda algebra, as well as in its differential and the endomorphism $\theta$ lifting $\text{Sq}^0$. Determining these formulas precisely is delicate and requires some careful arguments.

Remark 1.2.2. As indicated above, we construct the $F$-motivic lambda algebra as a certain Koszul complex for the $F$-motivic Steenrod algebra. The Koszul story produces other complexes as well: for any $\mathbb{A}_F$-modules $M$ and $N$ with $M$ projective over $\mathbb{M}_F$, there are complexes $\Lambda^F(M, N)$ serving as small models of the cobar complex computing $\text{Ext}_{\mathbb{A}_F}(M, N)$. An amusing special case of this produces a lambda algebra $\Lambda^{C_2}$ for the $C_2$-equivariant Steenrod algebra (Remark 2.3.5).

We use the motivic lambda algebra to study $\text{Ext}_F$ in low filtration. Before diving into our more extensive computations, we illustrate the structure of $\Lambda^F$ with some simple examples in Section 3.1, showing how it may be used to give easy rederivations of some well-known low-dimensional relations in $\text{Ext}_F$. We then carry out our main algebraic computation in Section 4, where we prove the following. Note that $\text{Ext}^0_F = \mathbb{F}_2[\rho]$.

Theorem C. The structure of $\text{Ext}_F$ in filtrations $f \leq 3$ is as described in Section 4; in particular, the $\mathbb{F}_2[\rho]$-module structure is described in Theorem 4.2.12, including a description of multiplicative generators and the action of $\text{Sq}^0$, and the majority of the multiplicative structure is described in Theorem 4.3.7.

Here, we are justified in focusing on $\text{Ext}_F$ as it is, in a certain precise sense, the universal case (see Remark 2.2.8). We explain in Section 7.1 how to pass from information about $\text{Ext}_F$ to information about $\text{Ext}_R$ for other base fields $F$.

Example 1.2.3 (Part (1) of Theorem 4.2.12). The computation of $\text{Ext}_F^{\leq 3}$ is much more involved than the corresponding classical computation, and the result is much richer. We refer the reader to
Section 4 for the full statements, but illustrate this here with the following sample. Classically, Ext_{cl}^{<3} is generated as an algebra by the classes $h_a$ and $c_a$ for $a \geq 0$. By contrast, a minimal multiplicative generating set of Ext_{cl}^{<3} as an $F_2[\rho]$-algebra is given by the classes in the following table:

| Multiplicative generator | $\rho$-torsion exponent |
|--------------------------|--------------------------|
| $h_{a+1}$                | $\infty$                |
| $c_{a+1}$                | $\infty$                |
| $\tau^{2^{a-1}(4n+1)}h_a$ | $2^a$                   |
| $\tau^{2^{2(4n+1)+1}}h_{a+2}$ | $2^{a+1} \cdot 3$ |
| $\tau^{2^{a-1}(2(16n+1)+1)}h_{a+3}$ | $2^a \cdot 13$ |
| $\tau^{2^{a-1}(4(4n+1)+1)}h_{a+3}$ | $2^a \cdot 7$ |
| $\tau^{2^{a-1}(16n+1)}c_a$ | $2^a \cdot 7$ |
| $\tau^{2^{a-1}(2(4n+1)+1)}c_a$ | $2^a \cdot 3$ |

Here, $a, n \geq 0$, we write $2^{a-1}(2n+1) = n$ for $a = 0$, and the $\rho$-torsion exponent of a class $\alpha$ is the minimal $r$ for which $\rho^r \alpha = 0$; the classes $h_{a+1}$ and $c_{a+1}$ are $\rho$-torsion-free. Note that all of the classes listed are named for their image in Ext_{cl}, and are not themselves products.

Example 1.2.4. Observe that the multiplicative generators $h_a$ and $c_a$ of Ext_{cl}^{<3} appear, with a shift, as $\rho$-torsion-free classes in Ext_{cl}. This is a general phenomenon: in [DI16, Theorem 4.1], Dugger–Isaksen produce an isomorphism Ext_{cl}[\rho^{-1}] \cong Ext_{cl}[\rho^{2+1}]; here, Ext_{cl} = Ext_{cl}, only given a motivic grading so that Ext_{cl}^{hf} = Ext_{cl}^{2+1};f,s+1}. As we discuss in Section 3.2, this in fact refines to a splitting Ext_{cl} \cong Ext_{cl}[\rho] \oplus Ext_{cl}^{\rho^{-1}}$, where Ext_{cl}^{\rho^{-1}} \subset Ext_{cl} is the subgroup of $\rho$-torsion; moreover, this splitting is modeled by a multiplicatively split inclusion $\tilde{\theta} : \Lambda^{cl} \to \Lambda^{cl}$. The general shape of Ext_{cl} forced by this may be illustrated by the following description of the 1-line:

$$
\operatorname{Ext}_{1}^{X, F} = F_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} F_2[\rho]/(\rho^{2^a})(\tau^{2^{a-1}(4n+1)}h_a : n \geq 0).
$$

As above, $\tau^{2^{a-1}(4n+1)}h_a = \tau^{2n}h_0$ for $a = 0$ by convention.

As Ext_{cl}^{<3} is entirely understood by Wang’s computation [Wan67], the hard work of Theorem C is in computing the $\rho$-torsion subgroup of Ext_{cl}^{<3}. This is the most computationally intensive part of the paper, and proceeds by a direct case analysis of monomials in $\Lambda^{cl}$ in low filtration. In the end, we find that Ext_{cl}^{<3} carries the multiplicative generators listed in Example 1.2.3, and that there are many exotic relations between these generators. Our computation describes all of this, with a few explicitly delineated unresolved products (see Conjecture 4.3.5 and Proposition 4.3.6).

With the algebraic computation of Theorem C in place, we turn to more homotopical topics, namely those surrounding the Hopf invariant one problem. There are (at least) two good motivic analogues of the Hopf invariant one problem: one which is unstable, concerning the construction of unstable 2-cell complexes with nontrivial cup product structure, and one which is stable, concerning the construction of stable 2-cell complexes with nontrivial $A^E$-module structure. As we recall in Section 6.2, understanding the latter question is equivalent to understanding the 1-line of the $F$-motivic Adams spectral sequence; we get to this in Section 7, which we will discuss further below.

It is the former unstable formulation which has more direct geometric applications. For example, following work of Dugger–Isaksen [DI13] on the Hopf construction in motivic homotopy theory, it is directly tied up with the question of which unstable motivic spheres $S^{a,b}$ admit $H$-space structures \footnote{Here, $S^{a,b}$ is the motivic sphere which is $\Lambda^1$-homotopy equivalent to $\Sigma^{a-b}C^b_m$.} (cf. Lemma 6.4.3). We discuss this unstable formulation in Section 6, which is independent of our other calculations. One pleasant consequence of this story is the following.
**Theorem D (Theorem 6.4.4).** The only motivic spheres which are represented by smooth $F$-schemes admitting a unital product are $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$.

The statement of Theorem D is directly analogous to the classical result that the only sphere admitting unital products are $S^0$, $S^1$, $S^3$, and $S^7$. Classically, the nonexistence of $H$-space structures on any other spheres may be reduced to the Hopf invariant one problem, which was then established by Adams. This reduction makes use of the instability condition that $Sq^0(x) = x^2$ whenever $x \in H^n(X)$ for some space $X$. There is an analogous instability condition for the motivic cohomology of a motivic space, but it holds only in a smaller range than we would need; as a consequence, some additional input is needed to analyze the unstable motivic Hopf invariant one problem (see Remark 6.3.2).

This additional input is interesting in itself. It follows from the formulation of the unstable motivic Hopf invariant one problem that, at least for nonexistence, one may reduce to the case where $F$ is algebraically closed. In Section 5.2, we explain how work of Wilson–Østvær [WØ17] implies a certain *Lefschetz principle* for suitable 2-primary categories of cellular motivic spectra. When combined with Mandell’s $p$-adic homotopy theory [Man01], this gives a 2-primary unstable “Betti realization” functor for any algebraically closed field $F$, which is well-behaved with respect to the mod 2 cohomology of motivic cell complexes; see Section 5.3. This gives a direct relation between motivic and classical homotopy theory, and we are then able to analyze the unstable motivic Hopf invariant one problem using a combination of classical results, work of Dugger–Isaksen [DI13] on the motivic Hopf construction, and work of Asok–Doran–Fasel on smooth models of motivic spheres [ADF17].

Finally, in Section 7, we turn to a study of the 1-line of the $F$-motivic Adams spectral sequence. After a few preliminaries, we begin by proving Theorem A, producing the differentials

$$d_2(h_{a+1}) = (h_0 + \rho h_1)h_n^2$$

valid for any $F$ (Theorem 7.3.1). As we mentioned above, the main content of this theorem is not the fact that the classes $h_{a+1}$ for $a \geq 3$ support nonzero differentials, but the exact value of the target of these differentials. We mention two interesting aspects of this computation here:

First, in order to get a more explicit handle on possible targets of $d_2(h_{a+1})$, we reduce to considering the case where $F$ is a prime field, i.e. $F = \mathbb{F}_p$ with $p$ odd or $F = \mathbb{Q}$. The latter case is then handled with the aid of a *Hasse principle*. We explain how work of Ormsby–Østvær on the structure of $\mathbb{M}_Q$ [OØ13] may be used to give a concrete description of $\text{Ext}_Q$ and of the Hasse map

$$\text{Ext}_Q \to \text{Ext}_R \times \prod_{p \text{ prime}} \text{Ext}_{Q_p},$$

in particular proving this map is injective (Proposition 7.1.3). In this way we reduce to computing the differentials $d_2(h_n)$ over the fields $\mathbb{F}_p$ with $p$ odd, $\mathbb{Q}_p$ with $p$ prime, and $\mathbb{R}$.

Second, the classical argument, using the fact that $2\sigma^2 = 0$, may be used to compute $d_2(h_4)$, but a new argument is required to produce the differential $d_2(h_5) = (h_0 + \rho h_1)h_4^2$ (Proposition 7.3.3). Once this differential is resolved, the remaining follow by an inductive argument analogous to Wang’s classical argument [Wan67]. After a further reduction when $F = \mathbb{R}$, the differential $d_2(h_5)$ may be resolved uniformly in the above choices of base field. In short, to resolve this differential, we lift the Hurewicz map $\pi_*^1 \to \pi_*^F$ to a map $\text{Ext}_{c_1}^* \to \text{Ext}_{c_0}^*$ of spectral sequences (Proposition 5.1.1), and by considering the effect of this on the Kervaire class $\theta_4$, deduce that $(h_0 + \rho h_1)h_4^2$ must be hit by $h_5$.

The story does not stop with the differentials $d_2(h_{a+1})$, as $\text{Ext}_F^1$ contains many more classes than these; recall for instance $\text{Ext}_R^1$ from Eq. (1). Having resolved these differentials, we move on to giving an explicit analysis of the 1-line of the $F$-motivic Adams spectral sequence for a number of base fields $F$. Our main results may be summarized in the following.

**Theorem E.** The following are carried out in Section 7.

1. In Theorem 7.4.9, we compute all $d_2$-differentials out of $\text{Ext}_R^1$, as well as all permanent cycles in $\text{Ext}_R^1$. 


(2) In Theorem 7.5.3, for $q$ a prime-power satisfying $q \equiv 1 \pmod{4}$, we compute all Adams differentials out of $\text{Ext}^1_{\rho}$, in particular giving all permanent cycles in $\text{Ext}^1_{\rho}$.

(3) In Theorem 7.5.6, for $q$ a prime-power satisfying $q \equiv 3 \pmod{4}$, we compute all $d_2$-differentials out of $\text{Ext}^1_{\rho}$, as well as all higher differentials in stems $s \leq 7$, in particular giving all permanent cycles in $\text{Ext}^1_{\rho}$ in stems $s \leq 7$.

(4) In Theorem 7.6.2, for $p$ an odd prime, we give as much information about $\text{Ext}^1_{\rho}$ as was given for $\text{Ext}^1_{\rho}$.

(5) In Theorem 7.6.6, we compute all $d_2$-differentials out of $\text{Ext}^1_{\rho}$, as well as all higher differentials in stems $s \leq 7$, in particular giving all permanent cycles in $\text{Ext}^1_{\rho}$ in stems $s \leq 7$.

(6) In Theorem 7.7.1, we give the same information for $\text{Ext}^1_{\rho}$ as was given for $\text{Ext}^1_{\rho}$.

Cases (2–6) of Theorem E proceed by a direct analysis, combining the Hopf differentials we produced in Theorem A with arithmetic differentials that may be obtained by comparison with the $F$-motivic Adams spectral sequence for integral motivic cohomology. The latter has been computed by Kylling [Ky15] for $F = \mathbb{F}_q$ with $q$ an odd prime-power, by Ormsby [Orm11] for $F = \mathbb{Q}_p$ with $p$ an odd prime, and by Ormsby–Østvær [OO13] for $F = \mathbb{Q}_2$ and $F = \mathbb{Q}$. Case (6), where $F = \mathbb{Q}$, may be read off the cases $F = \mathbb{R}$ and $F = \mathbb{Q}_p$, using our good understanding of the Hasse map Eq. (2). As with Ormsby and Østvær’s computations over $\mathbb{Q}$, the final description of the set of $d_2$-cycles in $\text{Ext}^1_{\rho}$ is quite intricate, but we feel that our techniques show that understanding the $\mathbb{Q}$-motivic Adams spectral sequence for $\pi^0_{\ast,\ast}$ is an accessible problem ripe for future investigation.

The $\mathbb{R}$-motivic computation requires more work. Recall the structure of $\text{Ext}^1_{\mathbb{R}}$ from Eq. (1). Theorem A describes what happens on the $\rho$-torsion-free summand of this, but says nothing about the large quantity of $\rho$-torsion classes. It is possible to use similar methods to compute all $d_2$-differentials supported on this $\rho$-torsion summand, and we do so in Proposition 7.4.8. However, this is insufficient to determine which classes in $\text{Ext}^1_{\mathbb{R}}$ are permanent cycles, as higher differentials may, and indeed must, occur.

We resolve this by comparison with Borel $C_2$-equivariant homotopy theory. In [BS20], Behrens–Shah formulate and prove an equivalence

$$(8\text{Sp}^e_{\mathbb{R}})_{(2,\rho)}[\tau^{-1}] \simeq \text{Fun}(BC_2, 8\text{Sp}^e_{\mathbb{R}})$$

between the $\tau$-periodic $(2,\rho)$-complete cellular $\mathbb{R}$-motivic category and the 2-complete Borel $C_2$-equivariant category. Define

$$\text{Ext}^s_{BC_2} = \text{Ext}^{s-w,f}_{BC_2}(\mathbb{F}_2, H^*P^\infty_w),$$

where $P^\infty_w$ is a stunted real projective space. These form the $E_2$-pages of the classical Adams spectral sequences for the stable cohomotopy groups of infinite stunted projective spaces. The equivalence of Behrens–Shah gives an effective method of computing these groups by “inverting $\tau$” in $\text{Ext}_{\mathbb{R}}$. The $\tau$-periodic behavior of $\text{Ext}_{\mathbb{R}}$ is plainly visible in our computation of $\text{Ext}^1_{\mathbb{R}}$, allowing us to directly read off the structure of $\text{Ext}^1_{BC_2}$ (Lemma 7.4.3). In particular,

$$\text{Ext}^1_{BC_2} = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2a})\{\tau^{2a-1}(4n+1)h_a : n \in \mathbb{Z}\}$$

(compare Eq. (1)). We warn the reader that this naming of classes is incompatible with viewing $\text{Ext}_{BC_2}$ as a collection of ordinary Adams spectral sequences; for example, $h_0$ does not detect 2, but instead the transfer $P^\infty_w \to S^0$. We may use the relatively simple structure of these 1-lines to verify that $\text{Ext}^1_{\mathbb{R}} \to \text{Ext}^1_{BC_2}$ reflects permanent cycles (Lemma 7.4.4), and this reduces the identification of permanent cycles in $\text{Ext}^1_{\rho}$ to the identification of permanent cycles in $\text{Ext}^1_{BC_2}$. The problem of $\rho$-torsion permanent cycles in $\text{Ext}^1_{BC_2}$ turns out to be equivalent to the vector fields on spheres problem (Lemma 7.4.5), which was resolved by Adams [Ada62]. Together with known information regarding the $\rho$-torsion-free classes, this leads to the following classification of maps $\Sigma^rP^\infty_w \to S^0$ detected in Adams filtration 1.
Theorem F (Theorem 7.4.7). For $a \geq 0$, write $a = c + 4d$ with $0 \leq c \leq 3$, and define $\psi(a) = 2^c + 8d$. Then the subgroup of permanent cycles in $\operatorname{Ext}^2_{BC_2}$ is given by

$$\mathbb{F}_2[\rho] \{h_1, h_2, h_3, ph_4\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{\psi(a)}) (\rho^{2^a-\psi(a)}r^{2^{a-1}(4n+1)}h_a : n \in \mathbb{Z}).$$

Moreover, one may characterize maps $S^\infty \to S^0$ detected by each of these classes.

1.3. Future directions. The classical lambda algebra has been applied broadly in stable homotopy theory. This suggests several natural directions for future work, and we list a few here.

1.3.1. Homological computations. The homology of the classical lambda algebra can be computed algorithmically as the Curtis algorithm. This procedure was refined and implemented by Tangora [Tan85] to compute the cohomology of the Steenrod algebra through internal degree 56, as well as to compute products and Massey products [Tan93, Tan94]; further computations of Curtis–Goerss–Mahowald–Milgram [CGMM87] pushed this out to describe the cohomology of the Steenrod algebra through stem 51. More recently, the Curtis algorithm was used by Wang–Xu to compute the algebraic Atiyah–Hirzebruch spectral sequence for $\mathbb{R} P^\infty$ [WX16], providing the data necessary for their proof of the uniqueness of the smooth structure on the 61-sphere [WX17].

Our method for computing $\operatorname{Ext}^\leq 3_{\mathbb{R}}$ is closely related to the homology algorithm of [Tan85], only modified to take into account the $\mathbb{F}_2[\rho]$-module structure of $\Lambda_{\mathbb{R}}$, as well to incorporate some additional flexibility in choosing representatives for the sake of a more digestible manual computation. By ignoring this additional flexibility, and incorporating the ideas of [Tan85, Section 3.4], one obtains a Curtis algorithm for computing the homology of the $\mathbb{R}$-motivic lambda algebra, as well as of other motivic lambda complexes. The effectiveness of these procedures in higher dimensions remains to be seen.

In addition to its use in computer-assisted computations, the classical lambda algebra has also been used by Lin [Lin107] and Chen [Che111] to completely compute the cohomology of the classical Steenrod algebra through filtration 5. In principle, there should be no obstruction to continuing our computation of $\operatorname{Ext}^\leq 3_{\mathbb{R}}$ to higher filtrations, other than the rather more involved calculations and bookkeeping that this would necessarily take.

1.3.2. Motivic Brown–Gitler spectra. Brown–Gitler spectra [BJG73] have many applications in classical algebraic topology, including Mahowald’s analysis of the $bo$-resolution [Mah81, Shi84], Cohen’s solution of the Immersion Conjecture [Coh85], and more [Mah77, HK99, Goe99]. The classical lambda algebra was essential for constructing and analyzing Brown–Gitler spectra; see [BJG73, Shi84] as above, as well as [GJM86]. In [CQ21], the last two authors introduced a motivic analog of the $bo$-resolution, the $kq$-resolution, and analyzed it over algebraically closed fields of characteristic zero. The analysis of the $kq$-resolution over more general base fields would be greatly simplified by the existence of motivic Brown–Gitler spectra.

1.3.3. Unstable motivic Adams spectral sequences. The classical lambda algebra $\Lambda^{cl}$ has certain subcomplexes $\Lambda^{cl}(n)$ which form the $E_1$-page of an unstable Adams spectral sequence:

$$E_1 \cong \Lambda^{cl}(n) \Rightarrow \pi_* S^n.$$ 

Moreover, James’s 2-local fiber sequence [Jan57]

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1},$$

which gives rise to the EHP sequence, is modeled by short exact sequences [Cur71, Section 11]

$$0 \to \Lambda^{cl}(n) \to \Lambda^{cl}(n + 1) \to \Sigma^n \Lambda^{cl}(2n + 1) \to 0,$$

which are useful both for understanding the unstable complexes $\Lambda^{cl}(n)$ and the stable complex $\Lambda^{cl}$. It is natural to ask whether there are analogous subcomplexes of $\Lambda^{F}$ related to a suitable motivic unstable Adams spectral sequence. The motivic situation seems to be much more delicate: it is...
not obvious how to define such subcomplexes of $\Lambda^F$, and the nature of the cohomology of motivic
Eilenberg–MacLane spaces suggests that a motivic unstable Adams spectral sequence may not be as
well-behaved. A better understanding of these topics would shed light both on the nature of $\Lambda^F$ and
on unstable $F$-motivic homotopy theory.

1.4. Conventions. We maintain the following conventions throughout the paper.

(1) We work solely at the prime 2.
(2) We write $F$ for a base field of characteristic not equal to 2.
(3) We write $\pi^F_{s,*}$ for the homotopy groups of the $(2, \eta)$-completed $F$-motivic sphere spectrum.
(4) Our homotopy and cohomology groups are bigraded by $(s, w)$, where $s$ is stem and $w$ is
weight.
(5) In particular, we write $S^{a,b}$ for the motivic sphere which is $A^1$-homotopy equivalent to
$\Sigma^{a-b} \mathbb{G}_m$.
(6) We write $H^{*,*}$ for reduced mod 2 $F$-motivic cohomology and $H^*$ for reduced ordinary mod
2 cohomology.
(7) We write, for instance, $H^{*,*}(X_\pm)$ for the unreduced mod 2 motivic cohomology of $X$.
(8) We will use homological grading even for cohomology classes, in the sense that if $x \in H^{a-b}(X)$
then we say $|x| = (-a, -b)$. This allows us to say, for instance, $|\tau| = (0, -1)$ and $|\rho| =
(-1, -1)$, regardless or whether we are working with homology or cohomology.
(9) We write $M^F = H^{*,*}(\text{Spec}(F)_+) \subset K$ for the unreduced mod 2 motivic cohomology of a point.
(10) We write $M^F_s$ for the portion of $M^F$ concentrated on the line $s = w$, so that $M^F = M^F_s[\tau]$. 
(The ring $M^F_0$ may be identified as the mod 2 Minov K-theory of $F$, by work of Voevodsky; see
[IØ19, Section 2.1] for an overview of the structure of $M^F$).
(11) We write $\text{Ext}_F$ for the cohomology of the $F$-motivic Steenrod algebra, employing the grading
conventions given in the following two points.
(12) We write $\text{Ext}_F^f$ for the filtration $f$ piece of $\text{Ext}_F$.
(13) We write $\text{Ext}^{s,f,w}_F \subset \text{Ext}_F$ for the subset of elements in filtration $f$ and topological stem $s$
and weight $w$.
(14) We use a subscript or superscript of $\text{cl}$ to denote classical objects; in particular, $\pi^\text{cl}$
are the classical 2-completed stable stems, $A^\text{cl}$ is the classical mod 2 Steenrod algebra, and $\text{Ext}_F^\text{cl}$ is
its cohomology.
(15) Given integers $a, n \geq 0$, we write $2^{n-a-1}(2n + 1)$ for the integer so represented for $a \geq 1$, and
to be $n$ for $a = 0$.
(16) We take the binomial coefficient $\binom{a}{b}$ to be $\frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, and to be zero otherwise.

1.5. Acknowledgements. The authors thank Uğur Yiğit for sharing a copy of his thesis which
proposes an alternative construction of a $C_2$-equivariant lambda algebra, as well as Eva Belmont
and Dan Isaksen for sharing data produced for their study of $\text{Ext}_F$. The first author thanks Nick
Kuhn for some enlightening conversations about real projective spaces. The second author would
like to thank the Max Planck Institute for providing a wonderful working environment and financial
support while this paper was being written. The third author thanks Eva Belmont, Dan Isaksen,
and Inna Zakharevich for helpful discussions.

2. The motivic lambda algebra

The classical lambda algebra was originally constructed by Bousfield–Curtis–Kan–Quillen–Rector–
Schlesinger [6A66] using methods from unstable homotopy theory. In [Pri70], Priddy defined the
notion of a Koszul algebra, and constructed (co)Koszul complexes, which are small models for the
cobar complex of a Koszul algebra. He then showed that the Steenrod algebra is Koszul, and that its
Koszul complex recovers the classical lambda algebra, thus giving a purely algebraic construction of
the classical lambda algebra.
Our approach to constructing the motivic lambda algebra proceeds along the same lines as Priddy’s. We cannot quite directly apply Priddy’s results, as the notion of Koszul algebra employed in [Pri70] is not sufficiently general to include to the motivic Steenrod algebra: among other things, the $M^F$-algebra $A^F$ is not generally augmented, nor does $M^F$ always live in the center of $A^F$. The notion of a Koszul algebra has been generalized in various ways; see for instance [PP05] for an account of some developments in this area. We will use the formulation given in [Bal21], as this gives a sufficiently general definition of Koszul algebra and explicit description of their associated Koszul complex; in our context, this functions as a generalization of Rezk’s formulation in [Rez12, Section 4] to the case of nonhomogeneous algebras. We review this theory in Section 2.1. The reader familiar with Koszul algebras will find no surprises in this material.

In Section 2.2, we review the structure of the $F$-motivic Steenrod algebra $A^F$. We show that $A^F$ is in fact a Koszul algebra in Section 2.3, ultimately by reducing to Priddy’s classical PBW criterion for Koszulity [Pri70, Section 5]. The $F$-motivic lambda algebra $Λ^F$ is then defined to be the Koszul complex of $A^F$. We compute the structure of $Λ^F$ explicitly, and introduce an endomorphism $θ$ of $Λ^F$ lifting the squaring operation $Sq^0$ on Ext$_F$. All of this structure is summarized in one place in Section 2.4.

Our work in this section applies somewhat more generally than to just the construction of the $F$-motivic lambda algebras $Λ^F$. This generality is in two directions. First, our proof that $A^F$ is Koszul applies to any algebra of a similar form to $A^F$, such as the $C_2$-equivariant Steenrod algebra; the exact notion of similarity is discussed in Remark 2.2.8. Second, from the fact that $A^F$ is Koszul, one recovers not just the Koszul complex $Λ^F$ modeling Ext$_A^F(M^F, M^F)$, but also Koszul complexes modeling any Ext$_A^F(M, N)$ with $M$ projective over $M^F$.

2.1. Review of Koszul algebras. This section summarizes some definitions and facts from [Bal21, Section 3] regarding Koszul algebras which we will use in constructing the motivic lambda algebra. We shall review this material in some detail, in order to specialize from the more general context considered there. Many of the results we need have appeared in varying levels of generality throughout the literature; in particular, the definition of Koszulity we use can be considered as a direct generalization of the homogeneous case considered by Rezk in [Rez12, Section 4].

We fix throughout this subsection an associative algebra $S$ to serve as our base ring, together with an associative algebra $A$ which is an $S$-algebra in the sense of being equipped with an algebra map $S → A$. Equivalently, $A$ is a monoid in the category of $S$-bimodules. We abbreviate $⊗ = ⊗_S$.

We are most interested in the case where $S = M^F$ and $A = A^F$, and so to avoid some subtle points regarding signs, we shall assume that $S$ is of characteristic 2. In addition, we suppose throughout that $A$ is projective as a left $S$-module.

Definition 2.1.1. Say that $A$ is a graded $S$-algebra if we have chosen a decomposition $A = ⊔_{n ≥ 0} A[n]$ of $S$-bimodules such that

1. $S ≃ A[0]$;
2. The product on $A$ restricts to $A[n] ⊗ A[m] → A[n + m]$.

Say that $A$ is a filtered $S$-algebra if we have chosen a filtration $A ≃ colim_{n → ∞} A ≤ n$ such that

1. $S ≃ A < 0$;
2. The product on $A$ restricts to $A ≤ n ⊗ A ≤ m → A ≤ n + m$.

Finally, say that the filtration on a filtered $S$-algebra $A$ is projective if (both $A$ and) the associated graded algebra

$$gr \ A := ⊔_{n ≥ 0} A[n], \quad A[n] := coker(A ≤ n - 1 → A ≤ n)$$

are projective as left $S$-modules.

Fix a left $A$-module $M$. Write $B^{un}(A, A, M)$ and $B(A, A, M)$ for the unreduced and reduced bar resolutions of $M$ relative to $S$; that is, for the unnormalized and normalized chain complexes associated to the standard monadic resolution of $M$ with respect to the adjunction LMod$_S ≃ LMod_A$. 

These are projective left $A$-module resolutions provided that $M$ is projective as a left $S$-module. If $A$ is a filtered algebra, then $B^m_n(A, A, M)$ is a filtered complex, with filtration defined by

\[ B^m_n(A, A, M)[\leq m] := \text{Im} \left( \bigoplus_{m_1 + \cdots + m_n = m} A \otimes A_{\leq m_1} \otimes \cdots \otimes A_{\leq m_n} \otimes M \to B^m_n(A, A, M) \right), \]

and this descends to a filtration of $B(A, A, M)$; compare for instance [Pri70, Section 10], [Rez12, Section 4], or [Bal21, Section 3.5]. If $A$ is augmented, then this augmentation makes $S$ into an $A$-bimodule, allowing us to form the bar complex $B(A) := S \otimes_A B(A, A, S)$ and consider the homology $H_*(A) := H_*(C(A))$. The filtration of Eq. (3) descends to a filtration on $B(A)$, and if $A$ is graded then this filtration is split, i.e. $B(A) \cong \bigoplus_{m \geq 0} B(A)[m]$. This then passes to a splitting $H_*(A) \cong \bigoplus_{m \geq 0} H_*(A)[m]$.

**Definition 2.1.2** ([Rez12, Definition 4.4], [Bal21, Definition 3.5.3]). We say that $A$ is a homogeneous Koszul $S$-algebra provided that

1. $A$ has been given the structure of a graded $S$-algebra;
2. $H_n(A)[m] = 0$ for $n \neq m$.

We say that $A$ is a Koszul $S$-algebra if

1. $A$ has been equipped with a projective filtration;
2. $gr A$ is a homogeneous Koszul $S$-algebra.

Suppose now that $A$ is projectively filtered, and fix a left $A$-module $M$ which is flat as a left $S$-module. The filtration of Eq. (3) on $B(A, A, M)$ induced by that on $A$ satisfies $gr B(A, A, M) \cong A \otimes B(gr A) \otimes M$, and so the convergent spectral sequence associated to this filtration is of signature

\[ E^1_{p,q} = A \otimes H_q(gr A)[p] \otimes M \Rightarrow H_qC(A, A, M), \quad d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q-1}. \]

**Definition 2.1.3.** Let $M$ be an $A$-module which is flat as a left $S$-module. The Koszul resolution of $M$ is the augmented chain complex

\[ M \leftarrow K(A, A, M) \]

defined by

\[ K_p(A, A, M) = E^1_{p,p} = A \otimes H_p(gr A)[p] \otimes M, \]

with differential given by the $d^1$ differential of the spectral sequence Eq. (4). When $M$ is projective as a left $S$-module, we define the Koszul complex\(^2\) $K_A(M, M')$ as the cochain complex

\[ K_A(M, M') := \text{Hom}_A(K(A, A, M), M') \cong \text{Hom}_S(H_*(gr A) \otimes M, M'), \]

with differential inherited from that on $K(A, A, M)$.

Observe that, by construction, $K(A, A, M)$ is a subcomplex of $B(A, A, M)$, and dually $K_A(M, M')$ is a quotient complex of the cobar complex $C_A(M, M') := \text{Hom}_A(B(A, A, M), M')$. When $A$ is Koszul, the spectral sequence of Eq. (4) collapses into the Koszul complex $K(A, A, M)$, proving the following.

**Theorem 2.1.4** (cf. [Pri70, Theorem 3.8], [Rez12, Proposition 4.8], [Bal21, Theorem 3.5.5]). Suppose that $A$ is a Koszul $S$-algebra, and fix left $A$-modules $M$ and $M'$.

1. If $M$ is flat over $S$, then there is an injective quasiisomorphism $K(A, A, M) \subset B(A, A, M)$;
2. If $M$ is projective over $S$, then there is a surjective quasiisomorphism $C_A(M, M') \to K_A(M, M')$.

In particular, if $M$ is projective over $S$, then the homology of $K_A(M, M')$ is isomorphic to $\text{Ext}_A(M, M')$.

\(^2\)Called a “co-Koszul complex” in [Pri70].
This allows us to define Koszul complexes in the generality we need. We now recall some facts from [Bal21, Sections 3.6–3.7] describing the structure of Koszul complexes; these are direct analogues of [Pri70, Theorem 4.6]. We begin by fixing some conventions.

**Definition 2.1.5.** Fix a left $S$-module $M$. Then the dual $M^\vee = \text{LMod}_S(M, S)$ carries the structure of a right $S$-module by

$$f \cdot s)(m) = f(s \cdot m).$$

If $M$ is in fact an $S$-bimodule, then $M^\vee$ also carries an $S$-bimodule structure, with left $S$-module structure

$$(s \cdot f)(m) = s \cdot (f(m)).$$

Now, if $M$ is a left $S$-module and $M'$ is an $S$-bimodule, then there is a comparison map

$$c: M^\vee \otimes M'^\vee \to (M' \otimes M)^\vee, \quad c(f \otimes f')(m' \otimes m) = f(m' f(m)).$$

If $M$ is finitely presented and projective as a left $S$-module, then this map is an isomorphism. In general, if $M''$ is another left $S$-module, then we write

$$M^\vee \otimes M'' := \text{LMod}_S(M, M''),$$

so that in particular

$$M^\vee \otimes M'^\vee \cong (M' \otimes M)^\vee;$$

in good cases, this may be realized as a topological tensor product, as the notation suggests. \hfill \blacktriangleleft

The theory of Koszul algebras is closely related to the theory of quadratic algebras; let us fix some notation for these.

**Definition 2.1.6.** Fix an $S$-bimodule $B$ and subbimodule $R \subset B \otimes B$. The *quadratic algebra* generated by the pair $(B, R)$ is the algebra

$$T(B, R) := \bigoplus_{n \geq 0} T_n(B, R),$$

$$T_n(B, R) := \ker \left( \sum_{i+j=n} B^\otimes i \otimes R \otimes B^\otimes j \to B^\otimes n \right),$$

with multiplication inherited from the tensor algebra $T(B)$. Similarly, given a subbimodule $R' \subset B^\vee \otimes B^\vee$,\footnote{To be precise, our definition of $\hat{T}(B^\vee, R')$ as written requires that $R'$ is dual to a quotient of $B \otimes B$; however this is the only case of interest.} we define the completed quadratic algebra

$$\hat{T}(B^\vee, R') := \prod_{n \geq 0} \hat{T}_n(B^\vee, R'),$$

$$\hat{T}_n(B^\vee, R') := \ker \left( \sum_{i+j=n} (B^\vee)^\otimes i \otimes R' \otimes (B^\vee)^\otimes j \to (B^\vee)^\otimes n \right).$$

Say that $(B, R)$ is a quadratic datum if $T(B, R)$ is projective. In this case, the *dual quadratic datum* to $(B, R)$ is the pair $(B^\vee, R^\perp)$, where $R^\perp = (T_2(B, R))^\vee$. \hfill \blacktriangleleft

The cohomology of a homogeneous Koszul algebra may be explicitly described as follows.

**Theorem 2.1.7** (cf. [Pri70, Theorem 2.5], [Rez12, Proposition 4.12], [Bal21, Theorem 3.6.4]).

1. Let $(B, R)$ be a quadratic datum. Then $H^1(T(B, R))[1] \cong B^\vee$, and the inclusion $B^\vee \subset H^*(T(B, R))$ extends to an isomorphism $\hat{T}(B^\vee, R^\perp) \cong \prod_{n \geq 0} H^n(T(B, R))[n]$.
2. Let $A = \bigoplus_{n \geq 0} A[n]$ be a homogeneous Koszul algebra, and let $R = \ker(A[1] \otimes A[1] \to A[2])$.
   
   Then $A \cong T(A[1], R)$ is quadratic, and $H^*(A) \cong \hat{T}(A[1]^\vee, R^\perp)$. \hfill \blacktriangleleft
Now fix a quadratic algebra $A = T(A[1], R)$ and left $A$-modules $M$ and $M'$, supposing that $M$ is projective as a left $S$-module. We may use Theorem 2.1.7 to describe the Koszul complex $K_A(M, M')$. Recall that

$$K^n_A(M, M') = \text{LMod}_A(A \otimes H_n(A)[u] \otimes M, M') \cong \text{LMod}_S(H_n(A)[u] \otimes M, M').$$

If we suppose that $H_n(A)$ is projective as a left $S$-module, as holds if $A$ is Koszul, then there is an isomorphism $(H_n(A))^\vee \cong H^n(A)$ of $S$-bimodules. In this case, we have

$$K^n_A(M, M') \cong \text{LMod}_S(M, H^n(\operatorname{gr} A)[u] \otimes M') \cong \text{LMod}_S(M, \tilde{T}_n(A[1]^\vee, R^\perp) \otimes M');$$

Thus $K^n_A(M, M')$ is completely described as a graded object by Theorem 2.1.7.

It remains to describe the differential on $K_A(M, M')$. Observe first that if $M''$ is an additional $A$-module, then there are pairings

$$i: K^n_A(M, M') \otimes_{\mathbb{Z}} K^n_A(M', M'') \to K^{n+n'}_A(M, M'').$$

This is a pairing of chain complexes compatible with analogous pairings on cobar complexes, and when $A$ is Koszul, it is a chain-level lift of the standard composition product in $\operatorname{Ext}_A$. In addition, it may be described in terms of the product structure on $\tilde{T}(A[1]^\vee, R^\perp)$ as follows (cf. [Bal21, Sections 3.2, 3.7]). Write $\mu$ for the multiplication on $\tilde{T}(A[1]^\vee, R^\perp)$. Then given $f: M \to \tilde{T}_n(A[1]^\vee, R^\perp) \otimes M'$ and $g: M' \to \tilde{T}_n(A[1]^\vee, R^\perp) \otimes M''$, we have

$$f \circ g = (\mu \otimes 1) \circ (1 \otimes g) \circ f.$$

In the special case where $M = M'$, these pairings give $K_A(M, M)$ the structure of a differential graded algebra, and give $K_A(M, M')$ the structure of a differential graded $K_A(M, M)$-$K_A(M', M')$-bimodule. Note that $K^n_A(M, M) = \text{LMod}_S(A[1] \otimes M, M)$. The $A$-module structure on $M$ restricts to an element $Q^M \in K^1_A(M, M)$, and we have the following.

**Theorem 2.1.8** ([Bal21, Theorem 3.7.1]). The differential on $K_A(M, M')$ is given by

$$\delta: K^n_A(M, M') \to K^{n+1}_A(M, M'), \quad \delta(f) = Q^M \circ f - f \circ Q^{M'}.$$

In particular, if $M = M'$, then $\delta(f)$ is the commutator $[Q^M, f]$. \hfill $\Box$

This theorem describes Koszul complexes for a homogeneous Koszul algebra. Suppose now that $A$ is an arbitrary Koszul $S$-algebra, and fix still left $A$-modules $M$ and $M'$ with $M$ projective as a left $S$-module. The additive and multiplicative structure of the Koszul complexes $K_A(M, M')$ depend only on the algebra $\operatorname{gr} A$ and left $S$-modules $M$ and $M'$, and so are still described by Theorem 2.1.7. In practice, the differential on $K_A(M, M')$ may be identified using the following.

Let $qR = \ker(A_{\leq 1} \otimes A_{\leq 1} \to A_{\leq 2})$, and observe that $(A_{\leq 1}, qR)$ is a quadratic datum. Let $A_{\text{big}}^\text{gr} = \bigoplus_{n \geq 0} A_{\leq n}$. This is a graded algebra, and the inclusion $A_{\leq 1} \subset A_{\text{big}}^\text{gr}$ extends to a map $T(A_{\leq 1}, qR) \to A_{\text{big}}^\text{gr}$ of graded algebras.

**Theorem 2.1.9** ([Bal21, Theorem 3.7.3]).

1. The map $T(A_{\leq 1}, qR) \to A_{\text{big}}^\text{gr}$ is an isomorphism of graded algebras;
2. $A_{\text{big}}^\text{gr}$ is a homogeneous Koszul algebra;
3. The surjection $A_{\text{big}}^\text{gr} \to A$ gives rise to short exact sequences

$$0 \to K^n_A(M, M') \to K^n_{A_{\text{big}}}(M, M') \to K^{n-1}_A(M, M') \to 0,$$

which are split if $A$ is augmented.

In particular, $K_A(M, M') \subset K_{A_{\text{big}}}(M, M')$ is a subcomplex with differential on the target described by Theorem 2.1.8. \hfill $\Box$
2.2. The motivic Steenrod algebra. We will construct the motivic lambda algebra by applying the theory recalled in Section 2.1 to the mod 2 motivic Steenrod algebra, whose structure we now recall. The conventions of Section 1.4 are in force throughout this section.

We note in particular that, following these conventions, we take the somewhat nonconventional approach of consistently using homological grading. Thus for example $\tau \in H^{n,1}(\text{Spec}(F)_+)$, but we shall write $|\tau| = (0, -1)$, as this is how it will appear in the lambda algebra.

We begin by recalling the general structure of the base ring $M_\alpha = H^{*,*}(\text{Spec}(F)_+)$. See also Section 7.1 for a discussion of Example 2.2.1. For any positive integers $a \geq 0$, these squares are generated by $\text{Sq}^{a+b}$ for $a \leq |\tau| = (0, -1)$, as this is how it will appear in the lambda algebra.

The following are some particular examples of the ring $M_\alpha$. We refer the reader to [IØ19, Section 2.1] for further details.

- For $F = \mathbb{F}_2$ algebraically closed, such as $F = \mathbb{C}$, we have
  
  $M_\alpha \cong \mathbb{F}_2$

- For $F = \mathbb{R}$ the real numbers, we have
  
  $M_\alpha \cong \mathbb{F}_2[\rho]$

  where $|\rho| = (-1, -1)$.

- For $F = \mathbb{F}_q$ a finite field of odd prime-power order $q$, we have
  
  $M_\alpha \cong \begin{cases} 
  \mathbb{F}_q[u]/u^2 & \text{if } q \equiv 1 \pmod{4}, \\
  \mathbb{F}_q[\rho]/\rho^2 & \text{if } q \equiv 3 \pmod{4},
  \end{cases}$

  where $|\rho| = |u| = (-1, -1)$.

- For $F = \mathbb{Q}_p$ the $p$-adic rationals, for $p$ an arbitrary prime, we have
  
  $M_\alpha \cong \begin{cases} 
  \mathbb{F}_2[\pi, u]/(\pi^2, u^2) & \text{if } q \equiv 1 \pmod{4}, \\
  \mathbb{F}_2[\pi, \rho, u]/(\rho^2, \rho \pi + \pi^2) & \text{if } q \equiv 3 \pmod{4}, \\
  \mathbb{F}_2[\pi, \rho, u]/(\rho^2, u^2, \pi^2, \rho u, \rho \pi, \rho^2 + u \pi) & \text{if } q = 2,
  \end{cases}$

  where $|\rho| = |u| = |\pi| = (-1, -1)$.

See also Section 7.1 for a discussion of $M_\alpha$. 

Voevodsky [Voe03] (with minor corrections by Riou [Rio12]) and Hoyois–Kelly–Østvær [HKØ17] have constructed Steenrod squares

$\text{Sq}^a : H^{m,n}(X) \to H^{m+a,n+[a/2]}(X)$

for $a \geq 0$ and shown that they generate the algebra $A^F$ of natural operations in mod 2 motivic cohomology. It is convenient to take the convention that $\text{Sq}^a = 0$ for $a < 0$. The relations between these squares are generated by $\text{Sq}^a = 1$ together with the Adem relations:

**Theorem 2.2.2** ([Voe03, Theorem 10.2], [Rio12, Théorème 4.5.1], [HKØ17, Theorem 5.1]). Fix positive integers $a$ and $b$ with $a < 2b$.

If $a$ is even and $b$ is odd, then

$$\text{Sq}^a \text{Sq}^b = \sum_{0 \leq j \leq [a/2]} \binom{b-1-j}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j + \sum_{1 \leq j \leq [a/2]} \binom{b-1-j}{a-2j} \rho \text{Sq}^{a+b-j-1} \text{Sq}^j.$$  

If $a$ and $b$ are odd, then

$$\text{Sq}^a \text{Sq}^b = \sum_{1 \leq j \leq [a/2]} \binom{b-1-j}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j.$$
If $a$ and $b$ are even, then
\[ \text{Sq}^a \text{Sq}^b = \sum_{0 \leq j \leq [a/2]} \tau^{j \mod 2} \binom{b-1-j}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j. \]

If $a$ is odd and $b$ is even, then
\[ \text{Sq}^a \text{Sq}^b = \sum_{0 \leq j \leq [a/2]} \binom{b-1-j}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j + \sum_{1 \leq j \leq [a/2]} \binom{b-1-j}{a-1-2j} \rho \text{Sq}^{a+b-j-1} \text{Sq}^j. \]

In all cases, the bounds on summation are not necessary, but give regions where the given binomial coefficients may be nonzero.

As with the classical Steenrod algebra, $A^F$ admits an admissible basis.

**Definition 2.2.3.** Given a sequence $I = (r_1, \ldots, r_k)$ with $r_i > 0$ for all $1 \leq i \leq k$, we abbreviate $\text{Sq}^I = \text{Sq}^{r_1} \ldots \text{Sq}^{r_k}$. Say that $\text{Sq}^I$ is admissible if $r_i \geq 2r_{i+1}$ for all $1 \leq i \leq k-1$.

**Proposition 2.2.4 ([Voe03, Section 11]).** $A^F$ is freely generated as a left $M^F$-module by the admissible squares $\text{Sq}^I$.

The mod 2 motivic cohomology $H^{*,*}(X_+)$ of any smooth scheme $X$ carries the structure of a left $A$-module. These actions satisfy the following Cartan formulas.

**Proposition 2.2.5 ([Voe03, Proposition 9.6], [Rio12, Proposition 4.4.2]).** Let $a \geq 0$ and $x, y \in H^{*,*}(X_+)$. Then
\[ \text{Sq}^{2a}(xy) = \sum_{r=0}^a \text{Sq}^{2r}(x) \text{Sq}^{2a-2r}(y) + \tau \sum_{s=0}^{a-1} \text{Sq}^{2s+1}(x) \text{Sq}^{2a-2s-1}(y), \]
\[ \text{Sq}^{2a+1}(xy) = \sum_{r=0}^a (\text{Sq}^{2r+1}(x) \text{Sq}^{2a-2r}(y) + \text{Sq}^{2r}(x) \text{Sq}^{2a-2r+1}(y)) + \rho \sum_{s=0}^{a-1} \text{Sq}^{2s+1}(x) \text{Sq}^{2a-2s-1}(y). \]

The action of $A^F$ on $M^F$ is determined by these Cartan formulas and the following.

**Proposition 2.2.6 ([Voe03]).** The action of $A^F$ on $M^F$ satisfies
\[ \text{Sq}^{2^1}(x) = 0, \quad x \in M^F_0 \]
\[ \text{Sq}^I(\tau) = \rho, \quad \text{Sq}^{2^2}(\tau) = 0. \]

As in the classical case, the Cartan formulas of Proposition 2.2.5 may be encoded in a coproduct on the algebra $A^F$. The resulting structure is not quite a Hopf algebra, but is dual to a Hopf algebroid structure on the dual Steenrod algebra $(A^F)^\vee$. This complication arises in part due to the following. The Steenrod algebra $A^F$ is an $M^F$-algebra, by way of the homomorphism $M^F \to A^F$ sending an element $x \in M^F$ to the stable operation given by left multiplication by $x$. However, $M^F$ does not land in the center of $A^F$; equivalently, $A^F$ has nontrivial $M^F$-bimodule structure. We may describe this structure explicitly as follows.

**Proposition 2.2.7.** The $M^F$-bimodule structure of $A^F$ is determined by the following:
\[ \text{Sq}^n x = x \text{Sq}^n, \quad x \in M^F_0, \]
\[ \text{Sq}^{2n} \tau = \tau \text{Sq}^{2n} + \rho \tau \text{Sq}^{2n-1}, \]
\[ \text{Sq}^{2n+1} \tau = \tau \text{Sq}^{2n+1} + \rho \text{Sq}^{2n} + \rho^2 \text{Sq}^{2n-1}. \]
Proof. It suffices to show both sides of each equality coincide when evaluated on an arbitrary cohomology class. For example, for any $X$ and $x \in H^{*,*}(X_+)$, we have

$$(\text{Sq}^{2n}\tau)(x) = \text{Sq}^{2n}(\tau x) = \sum_{i+j=n} (\text{Sq}^{2i}\tau)(\text{Sq}^{2j}x) + \tau \sum_{i+j=n-1} (\text{Sq}^{2i+1}\tau)(\text{Sq}^{2j+1}x) = \tau \text{Sq}^{2n}(x) + \rho \tau \text{Sq}^{2n-1}(x)$$

by Proposition 2.2.5. This proves the second equation, and the other cases are similar. \hfill \qed

Remark 2.2.8. Although we work in this section over an arbitrary base field $F$, there is a sense in which $F = \mathbb{R}$ represents the universal case: the class $\rho$ may be defined over any field $F$, making $M^F$ into an $M^\mathbb{R}$-module, and in all cases we have

$$A^F = M^F \otimes_{M^\mathbb{R}} A^\mathbb{R}.$$ 

In fact, the formulas of Proposition 2.2.6 describe an action of $A^\mathbb{R}$ on $M^F$ for which

$$\text{Ext}_{M^F}(M^\mathbb{R}, M^F),$$

and at least additively this depends only on the $\mathbb{R}_2[\rho]$-module structure of $M^F$.

It is worth putting this observation in a slightly more general context. The Cartan formulas of Proposition 2.2.5 gives the category of left $A^\mathbb{R}$-modules a symmetric monoidal structure. If $R$ is a monoid in this category, then the tensor product $R \otimes_{M^\mathbb{R}} A^\mathbb{R}$ may be equipped with a product with the property that

$$\text{LMod}_{R \otimes_{M^\mathbb{R}} A^\mathbb{R}} \simeq \text{LMod}_{R}(\text{LMod}_{A^\mathbb{R}});$$

this is the semi-tensor product of $[MP65]$. Moreover, we have

$$\text{Ext}_{R \otimes_{M^\mathbb{R}} A^\mathbb{R}}(R, R) \cong \text{Ext}_{A^\mathbb{R}}(M^\mathbb{R}, R).$$

The algebras $A^F$ are obtained in the case where $R = M^F$. Another simple class of example is given by the algebras $A^\mathbb{R}/(\rho^n, \tau^m)$, where $n$ and $m$ are such that $\tau^m$ is central in $A^\mathbb{R}/(\rho^n)$. A more interesting example is the following: there is an isomorphism of algebras

$$A^{C_2} \cong M^{C_2} \otimes_{M^\mathbb{R}} A^\mathbb{R},$$

where $A^{C_2}$ is the $C_2$-equivariant Steenrod algebra, $M^{C_2}$ is the $C_2$-equivariant cohomology of a point, and $A^\mathbb{R}$ acts on $M^{C_2}$ as described, for instance, in [GHIR19, Section 2] (building on work of Hu–Kriz [HK01]). \hfill \triangleleft

2.3. The motivic lambda algebra. We now produce the motivic lambda algebra. For simplicity of notation, we consider the base field $F$ as fixed, and abbreviate

$$A = A^F, \quad M = M^F$$

throughout this subsection.

2.3.1. Koszulity of $A$. We begin by showing that $A$ is Koszul. The algebra $A$ is a projectively filtered $M$-algebra under the length filtration: $A_{\leq n} \subset A$ is the submodule generated by squares $\text{Sq}^I$ where $I$ is a sequence of length at most $n$. In particular,

$$A_{\leq 1} = M\{\text{Sq}^I : a \geq 0\}$$

as a left $M$-module, with the understanding that $\text{Sq}^0 = 1$ in $A$. By Definition 2.1.3, to show that $A$ is Koszul we must show that $\text{gr} A$ is homogeneous Koszul. To show that the classical Steenrod algebra is Koszul, Priddy developed a PBW criterion for Koszulity [Pri70, Theorem 5.3]. We cannot apply this criterion directly, in part due to the nontrivial $M$-bimodule structure of $\text{gr} A$. Our strategy is to filter this issue away, thereby reducing to Priddy’s criterion.

Theorem 2.3.1. $A$ is a Koszul $M$-algebra.
Proof. As \( A \) is a projectively filtered algebra, we must only show that \( \text{gr} A \) is a homogeneous Koszul algebra, i.e. that \( H_n(\text{gr} A)[m] = 0 \) for \( n \neq m \). To that end, we define a new filtration \( F_\bullet \text{gr} A \) on \( \text{gr} A \) by declaring \( F_{\leq m} \text{gr} A \subset \text{gr} A \) to be generated by elements of the form \( \text{Sq}^I \), where \( I = (r_1, \ldots, r_k) \) is a sequence satisfying \( r_1 + \cdots + r_k \leq m \). This filtration is multiplicative, and so we may form its associated graded algebra \( \text{gr} \text{gr} A \).

The same construction employed in Section 2.1 shows that the filtration \( F_\bullet \text{gr} A \) induces a filtration on the bar complex \( B(\text{gr} A) \) with associated graded \( B(\text{gr} A) \). This filtration is compatible with the decomposition

\[
B(\text{gr} A) \cong \bigoplus_{m \geq 0} B(\text{gr} A)[m],
\]

and so for each \( m \) there is a convergent spectral sequence

\[
E^1_n = H_n C(\text{gr} \text{gr} A)[m] \Rightarrow H_n(\text{gr} A)[m].
\]

It is thus sufficient to verify that \( \text{gr} \text{gr} A \) is a homogeneous Koszul algebra with respect to the grading \( \text{gr} \text{gr} A = \bigoplus_{m \geq 0} \text{gr}^m A \).\(^4\) By passing from \( \text{gr} A \) to \( \text{gr} \text{gr} A \), we have filtered away both the nontrivial \( \mathbb{M} \)-bimodule structure on \( \text{gr} A \) described in Proposition 2.2.7 and the parts of the Adem relations involving \( \rho \) which appear in Theorem 2.2.2, and in the end we may identify

\[
\text{gr} \text{gr} A \cong M^F \otimes_{F_2[r]} \text{gr} \mathcal{A}_C.
\]

From here, it is easily seen that the admissible basis of \( \text{gr} \text{gr} A \) satisfies Priddy's PBW criterion [Pri70, Section 5.1], and it thus follows from [Pri70, Proposition 5.3] that \( \text{gr} \text{gr} A \) is Koszul.\(^5\) \( \square \)

Remark 2.3.2. When \( F = \mathbb{R} \), the filtration \( F_\bullet \text{gr} A \) coincides with the \( \rho \)-adic filtration of \( \text{gr} A \). The use of \( F \) allows us to apply our argument uniformly to arbitrary base fields, but we could have also proved Theorem 2.3.1 in the \( \mathbb{R} \)-motivic case, and deduced the general case from this. Indeed, everything in Section 2.1 is compatible with base change (cf. [Bal21, Lemma 3.5.7]), so Koszulity of \( A^\mathbb{R} \) implies that any algebra obtained from the construction of Remark 2.2.8 is Koszul. As an example not explicitly covered by the statement of Theorem 2.3.1, \( A^C_2 \) is Koszul over \( M^C_2 \). \( \triangle \)

Definition 2.3.3. The \( F \)-motivic lambda algebra \( \Lambda^F \) is the Koszul complex \( K_{A^F}(M^F, M^F) \) associated to the Koszul \( M^F \)-algebra \( A^F \), as defined in Definition 2.1.3, where \( A^F \) acts on \( M^F \) as described in Proposition 2.2.6. \( \triangle \)

We shall abbreviate \( \Lambda = \Lambda^F \) throughout the rest of this subsection. Theorem 2.1.4 now implies the following:

Theorem 2.3.4. Let \( C(A) = C_A(M, M) \) denote the cobar complex of \( A \). Then there is a surjective multiplicative quasiisomorphism

\[
C(A) \to \Lambda.
\]

In particular,

\[
H_* \Lambda \cong \text{Ext}_{A^C}^*(M, M),
\]

and this isomorphism is compatible with all products and Massey products. \( \square \)

Remark 2.3.5. More generally, the theory recalled in Section 2.1 produces and describes Koszul complexes \( K_A(M, M') \) modeling the cobar complex \( C_A(M, M') \) for any left \( A \)-modules \( M \) and \( M' \) with \( M \) projective over \( M \). Classically, the case where \( M = H^*(\mathbb{R}P^\infty) \) and \( M' = F_2 \) is of particular importance. Another amusing example is given over \( F = \mathbb{R} \) with the observation that \( K_{A^\mathbb{R}}(M^\mathbb{R}, M^C_2) \cong K_{A^C_2}(M^C_2, M^C_2) = \Lambda C_2 \) (cf. Remark 2.2.8, Remark 2.3.2). \( \triangle \)

\(^4\)Note this is not the grading obtained by viewing \( \text{gr} \text{gr} A \) as the associated graded algebra of the filtration \( F_\bullet \text{gr} A \).

\(^5\)Although [Pri70, Proposition 5.3] assumes the base to be a field, the proof carries over unchanged to \( M^F \otimes_{F_2[r]} \text{gr} \mathcal{A}_C \).
2.3.2. The structure of the motivic lambda algebra. We will now apply the theory recalled in Section 2.1 to describe $\Lambda$ explicitly. First note that $\Lambda = \bigoplus_{m \geq 0} \Lambda[m]$ with $\Lambda[1] = (A[1])^\vee$, where $A[1] = \text{coker}(M \to A_{\leq 1})$. As a left $M$-module, we may identify

$$A[1] = M\{\text{Sq}^r : r \geq 1\}.$$  

Dualizing, we may identify

$$\Lambda[1] = \{\lambda_r : r \geq 0\}M$$  

as a right $M$-module, where $\lambda_r$ is dual to $\text{Sq}^{r+1}$ in the given basis. Considering internal algebraic degrees yields $|\lambda_r| = (r + 1, [(r + 1)/2])$; following our conventions (Section 1.4), we subtract off the filtration from the algebraic stem to obtain the topological stem, and so instead write $|\lambda_r| = (r, [r/2])$.

We now begin by describing the multiplicative structure of $\Lambda$.

**Proposition 2.3.6.** The left $M$-module structure on $\Lambda[1]$ is determined by

$$x\lambda_n = \lambda_n x, \quad x \in M_0,$$

$$\tau\lambda_{2n+1} = \lambda_{2n+1} \tau + \lambda_{2n+2}\rho,$$

$$\tau\lambda_{2n} = \lambda_{2n} \tau + \lambda_{2n+1}\rho + \lambda_{2n+2}\rho^2.$$  

**Proof.** This follows by dualizing Proposition 2.2.7. ☐

**Proposition 2.3.7.** If $a$ is odd or $b$ is even, then

$$\lambda_a\lambda_{2a+b+1} = \sum_{0 \leq r < b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r},$$

and if $a$ is even and $b$ is odd, then

$$\lambda_a\lambda_{2a+b+1} = \sum_{0 \leq r < b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r} \tau^{(r-1) \mod 2}$$

$$+ \sum_{0 \leq r \leq (b+1)/2} \lambda_{a+b+1-r}\lambda_{2a+1+r}\binom{b/2 - [r/2]}{[r/2]} \rho.$$  

**Proof.** By Theorem 2.1.7, the bimodule of relations defining $\Lambda$ as a quadratic algebra with generating bimodule $\Lambda[1]$ may be identified as $A[2]^\vee = \ker(A[1]^\vee \otimes A[1]^\vee \to R^\vee)$, where $R \subset A[1] \otimes A[1]$ is the projection of the subbimodule $qR \subset A_{\leq 1} \otimes A_{\leq 1}$ of Adem relations recalled in Theorem 2.2.2. It follows by direct computation that this kernel is generated by the indicated relations. ☐

**Remark 2.3.8.** Unless both $a$ and $b$ are even, the Adem relation expanding a product of the form $\lambda_a\lambda_b$ is exactly as in the classical lambda algebra. ☜

The additive structure of $\Lambda$ may be understood just as in the classical case.

**Definition 2.3.9.** Given a sequence $I = (r_1, \ldots, r_n)$, write $\lambda_I = \lambda_{r_1} \cdots \lambda_{r_n}$. Call the sequence $I$ coadmissible if $2n_i \geq r_{i+1}$ for all $1 \leq i \leq n-1$. ☜

**Proposition 2.3.10.** $\Lambda$ is freely generated as a right $M$-module by classes of the form $\lambda_I$ where $I$ is a coadmissible sequence.

**Proof.** The relations of Proposition 2.3.7 imply that the coadmissible classes $\lambda_I$ generate $\Lambda$ as a right $M$-module, and we must only verify that they do so freely. Following Remark 2.2.8 and Remark 2.3.2, there is an isomorphism

$$\Lambda \cong \Lambda[R] \otimes_{\Lambda[R]} M[1];$$

thus we may reduce to the case where $F = \mathbb{R}$. By construction, $\Lambda$ is free as a right $M$-module. Thus to show that the coadmissible classes $\lambda_I$ freely generate $\Lambda$ over $M$, it is sufficient to verify the same for $\Lambda/(\rho)[\tau^{-1}]$ over $M/(\rho)[\tau^{-1}]$. There is an isomorphism $\Lambda/(\rho)[\tau^{-1}] \cong \Lambda^{|\tau|} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau^\pm 1]$, so this follows from the classical case. ☐
Finally, we describe the differential on $\Lambda$ by applying Theorem 2.1.8.

**Proposition 2.3.11.** The differential on $\Lambda$ is determined by the Leibniz rule, together with

\[
\delta(x) = 0, \quad x \in M_0 \\
\delta(\tau) = \lambda_0 \rho, \\
\delta(\lambda_n) = \sum_{1 \leq r \leq n/2} \lambda_{n-r} \lambda_{r-1} \binom{n-r}{r}.
\]

**Proof.** Recall the construction $A^\text{big} = \bigoplus_{m \geq 0} A_{\leq m}$ used in the statement of Theorem 2.1.9. By inspection, we find that $A^\text{big}$ may be identified as the “big motivic Steenrod algebra”, defined with generators and relations the same as $A$ only without the stipulation that $Sq^0 = 1$. Let $\Lambda^\text{big} = K_{\text{mot}}(M, \mathcal{M})$, where $A^\text{big}$ acts on $\mathcal{M}$ through the quotient $A^\text{big} \to A$, i.e. with $Sq^0$ acting by the identity.

Theorem 2.1.9 tells us that $A^\text{big}$ is a homogeneous Koszul algebra, and that there is an inclusion $\Lambda \subset \Lambda^\text{big}$ of differential graded algebras. As $A^\text{big}$ is homogeneous Koszul, Theorem 2.1.7 applies to show that $\Lambda^\text{big}$ is generated by classes $\lambda_r$ for $r \geq -1$, subject to relations of the same form as described for $\Lambda$ in Proposition 2.3.6 and Proposition 2.3.7. The inclusion $\Lambda \subset \Lambda^\text{big}$ is the obvious one, identifying $\Lambda$ as the subalgebra of $\Lambda^\text{big}$ generated by the classes $\lambda_r$ for $r \geq 0$.

Theorem 2.1.8 describes the differential on $\Lambda^\text{big}$ as

\[
\delta(f) = [Q, f] = Q \cdot f - f \cdot Q,
\]

where $Q \in \Lambda^\text{big}[1] \cong (\Lambda^\text{big}[1])^\vee$ is the map $\Lambda^\text{big}[1] \cong A_{\leq 1} \otimes \mathcal{M} \to \mathcal{M}$ induced by the action of $A^\text{big}$ on $\mathcal{M}$. In the basis $A^\text{big}[1] = A_{\leq 1} = \mathcal{M}\{Sq^r : r \geq 0\}$, this map is the projection onto $Sq^0$, which by definition is the class $\lambda_{-1} \in \Lambda^\text{big}$. So the differential on $\Lambda^\text{big}$ is given by

\[
\delta(f) = [\lambda_{-1}, f] = \lambda_{-1} f - f \lambda_{-1},
\]

and $\Lambda \subset \Lambda^\text{big}$ is closed under this. The proposition follows upon expanding out this commutator using the relations defining the algebra $\Lambda^\text{big}$. \hfill $\Box$

**Remark 2.3.12.** The description of the differential on $\Lambda$ as the commutator $\delta(f) = [\lambda_{-1}, f]$ has appeared classically as well, see [Bru88, Page 83].

### 2.3.3. A closed formula for $\delta(\tau^n)$

**Proposition 2.3.11** gives a recursive process for computing $\delta(\tau^n)$. It is possible to solve this recursion, and we do so here. Recall that the pair $(\mathcal{M}, A^\vee)$ carries the structure of a Hopf algebroid. In particular, $A^\vee$ is a commutative ring, and $A_{\leq 1}^\vee$ is a quotient of this ring. Now, the differential $\delta: \Lambda[0] \to \Lambda[1]$ may be described as the composite

\[
\delta = \eta_R + \eta_L: \Lambda[0] = \mathcal{M} \to A^\vee \to A^\vee_{\leq 1} \to \text{coker}(\mathcal{M} \to A^\vee_{\leq 1}) = \Lambda[1],
\]

where $\eta_L, \eta_R: \mathcal{M} \to A^\vee$ are given by $\eta_R(m)(a) = \epsilon(ma)$ and $\eta_L(m)(a) = \epsilon(am)$, where $\epsilon: A = \Lambda \otimes_{\mathcal{M}} \mathcal{M} \to \mathcal{M}$ encodes the action of $\Lambda$ on $\mathcal{M}$.

We may use this interpretation to compute $\delta(\tau^n)$. The full structure of the Hopf algebroid $(\mathcal{M}, A^\vee)$ was determined by Voevodsky [Voe03]; however, we only need a small piece of this, which is easily computed by hand from the structure of $A$ recalled in Section 2.2. We record this piece in the following.

**Lemma 2.3.13.** There is an isomorphism of rings

\[
A_{\leq 1}^\vee = \mathcal{M}[\tau_0, \xi_1]/(\tau_0^2 + \xi_1 \tau_0 + \xi_1 \tau),
\]

where the quotient map

\[
A_{\leq 1}^\vee \to \Lambda[1]
\]

acts by

\[
\tau_0^e \xi_1^n \mapsto \lambda_{2n-1+e}
\]
for $\epsilon \in \{0, 1\}$ and $n \geq 0$, with the interpretation that $\lambda_{-1} = 0$. Moreover, the maps $\eta_L, \eta_R : \mathbb{M} \to A_{\leq 1}^\vee$ act by

$$
\eta_R(x) = x, \quad x \in \mathbb{M},
$$

$$
\eta_L(x) = x, \quad x \in \mathbb{M}_0,
$$

$$
\eta_L(\tau) = \tau + \tau_0 \rho.
$$

**Proof.** The structure of the ring $A_{\leq 1}^\vee$ may be read off the coproduct of $A$, as given in Proposition 2.2.5, and its relation with our basis of $\Lambda[1]$ then follows by construction. The behavior of the left and right units may be read off the $\mathbb{M}$-bimodule structure of $A_{\leq 1}$ as given in Proposition 2.3.11, together with knowledge of the counit map $\epsilon : A_{\leq 1} \to \mathbb{M}$ given in Proposition 2.2.6. □

The main input to our computation of $\delta(\tau^n)$ is the following elementary computation.

**Lemma 2.3.14.** In the ring $A_{\leq 1}^\vee$, we have

$$
\tau^n_0 = \sum_{0 \leq i \leq n} \tau_0^\epsilon_1 \left( \binom{i + \epsilon - 1}{n - i - 1} \right) \tau^{n-i-\epsilon} \rho^{2i-n+\epsilon}.
$$

These bounds on $i$ are not necessary, but give a region where the binomial coefficients may be nonzero.

**Proof.** We first compute $\tau^n$ in the quotient ring

$$
\mathbb{F}_2[\tau_0, \xi_1]/(\tau_0^2 + \xi_1 \tau_0 + \xi_1).
$$

of $A_{\leq 1}^\vee$ in which both $\tau$ and $\rho$ are set to 1. Clearly

$$
\tau^n_0 = \sum_{0 \leq i \leq n} (\xi_1^n c_{n,i} + \tau_0^\epsilon_1 d_{n,i})
$$

for some $c_{n,i}, d_{n,i} \in \mathbb{F}_2$. The relation

$$
\tau^n_0 = \xi_1(\tau_0^{n-1} + \tau_0^{n-2})
$$

gives rise to recurrence relations

$$
c_{n,i} = c_{n-1,i-1} + c_{n-2,i-1}
$$

$$
d_{n,i} = d_{n-1,i-1} + d_{n-2,i-1}.
$$

Set $c_{i,n}' = c_{n+i,i}$ and $d_{i,n}' = d_{n+i,i}$. Then these relations become

$$
c_{i,n}' = c_{i-1,n-1}' + c_{i-1,n}'
$$

$$
d_{i,n}' = d_{i-1,n-1}' + d_{i-1,n}'
$$

exactly as seen in Pascal’s triangle. Paired with the initial conditions

$$
c_{0,0}' = c_{0,1}' = d_{0,0}' = 0, \quad c_{1,1}' = 1 = d_{0,1}',
$$

we find that

$$
c_{i,n}' = \binom{i - 1}{n - 1}, \quad d_{i,n}' = \binom{i}{n - 1},
$$

and thus

$$
c_{n,i} = \binom{i - 1}{n - i - 1}, \quad d_{n,i} = \binom{i}{n - i - 1}.
$$

Plugging this back in, we find

$$
\tau^n_0 = \sum_{0 \leq i \leq n} \left( \xi_1 \left( \binom{i - 1}{n - i - 1} \right) + \tau_0^\epsilon_1 \left( \binom{i}{n - i - 1} \right) \right) = \sum_{\epsilon \in \{0, 1\}} \tau_0^\epsilon_1 \left( \binom{i + \epsilon - 1}{n - i - 1} \right).
$$
To compute $\tau_0^n$ in $A^\vee_{\leq 1}$ itself, recall that $|r| = (0, -1)$, $|\rho| = (-1, -1)$, $|\tau_0| = (1, 0)$, $|\xi_1| = (2, 1)$. Solving $|\tau_0^n| = |\tau_0^\varepsilon \xi_1^\varepsilon a^b|$
yields $a = n - i - \epsilon$, $b = 2i - n + \epsilon$.

It follows that

$$\tau_0^n = \sum_{\epsilon \in \{0, 1\}, 0 \leq i \leq n} \tau_0^\varepsilon \xi_1^\varepsilon {i + \epsilon - 1 \choose n - i - 1} \tau^{n-i-\epsilon} \rho^{2i-n+\epsilon}$$
in $A^\vee_{\leq 1}$. For this binomial coefficient to be nonzero we require

$$0 \leq i + \epsilon - 1, \quad 0 \leq n - i - 1, \quad n - i - 1 \leq i + \epsilon - 1,$$
giving the stated bounds on summation. \hfill \Box

**Proposition 2.3.15.** The differential $\delta$ satisfies

$$\delta(\tau^n) = \sum_{r \geq 0} \lambda_r \binom{n + \lfloor r/2 \rfloor}{r + 1} \tau^{n-\lfloor r/2 \rfloor-1} \rho^{r+1}.$$

**Proof.** Following Lemma 2.3.13, to compute $\delta(\tau^n)$ one may compute

$$\tau^n + (\tau + \tau_0 \rho)^n$$
in terms of the standard basis of $A^\vee_{\leq 1} = \mathbb{M}[\tau_0, \xi_1]/(\tau_0^2 + \xi_1 \tau_0 + \xi_1 \tau)$. Moreover, it is sufficient to work in the quotient of $A^\vee_{\leq 1}$ wherein $\tau$ and $\rho$ are set to 1, as the necessary quantity of $\tau$'s and $\rho$'s may be recovered by comparing degrees, just as in the proof of Lemma 2.3.14. Using Lemma 2.3.14, we find

$$1 + (1 + \tau_0)^n = \sum_{1 \leq k \leq n} {n \choose k} \tau_0^k = \sum_{1 \leq k \leq n} {n \choose k} \sum_{\epsilon \in \{0, 1\}, i \geq 0} {i + \epsilon - 1 \choose k - i - 1} \tau_0^\varepsilon \xi_1^\varepsilon;$$

here we are free to omit the bounds of summation on $i$, as they merely recorded when certain binomial coefficients were zero. The coefficient of $\tau_0^\varepsilon \xi_1^\varepsilon$ in this sum is

$$\sum_{1 \leq k \leq n} {n \choose k} {i + \epsilon - 1 \choose k - i - 1} = \sum_{1 \leq k \leq n} {n \choose k} {i + \epsilon - 1 \choose 2i + \epsilon - k} = {n + i + \epsilon - 1 \choose 2i + \epsilon};$$

here, the first equality uses the standard identity $\binom{a}{b} = \binom{a}{a-b}$, and the second uses Vandermonde’s identity. Adding in a sufficient number of $\rho$'s and $\tau$'s, and converting to $\Lambda[1]$, we learn

$$\delta(\tau^n) = \sum_{\epsilon \in \{0, 1\}, i \geq 0} \lambda_{2i+\epsilon-1} \binom{n + i + \epsilon - 1}{2i + \epsilon} \tau^{n-i-\epsilon} \rho^{2i+\epsilon}.$$ 

Set $r = 2i + \epsilon - 1$. Then $|r/2| = i + \epsilon - 1$, leading to the given description. \hfill \Box

2.3.4. Lift of $Sq^0$. The dual motivic Steenrod algebra $A^\vee$ is a commutative Hopf algebroid, and thus its cohomology, which agrees by definition with $\text{Ext}_{A}(\mathbb{M}, \mathbb{M})$, is equipped with algebraic Steenrod operations [BMMS86, Chapter 4]. The purpose of this section is to lift the operation $Sq^0$ to an endomorphism of $\Lambda$. Our approach essentially follows the proof of [Pal07, Proposition 1.4].

Let $C(A) = C_A(\mathbb{M}, \mathbb{M})$ denote the cobar complex of the algebra $A$; this is by definition the same as the cobar complex of the Hopf algebroid $A^\vee$. As $A^\vee$ is a commutative ring, $C(A)$ is the Moore complex of a cosimplicial commutative ring, and the levelwise Frobenius on this cosimplicial commutative ring induces a map

$$\sigma : C(A) \to C(A).$$

This is a map of differential graded algebras, and $Sq^0$ is the map induced by $\sigma$ in homology.
**Theorem 2.3.16.** The map $\sigma : C(A) \to C(A)$ induced by the levelwise Frobenius descends to a map $\theta : \Lambda \to \Lambda$ of differential graded algebras. This map is given on generators by

\[ \theta(x) = x^2, \quad x \in \mathcal{M}, \]
\[ \theta(\lambda_{2n-1}) = \lambda_{4n-1}, \]
\[ \theta(\lambda_{2n}) = \lambda_{4n+1}\tau + \lambda_{4n+2}\rho. \]

**Proof.** Recall $A^{\text{big}}$ and $\Lambda^{\text{big}}$ from the proof of Proposition 2.3.11. Let $C(A^{\text{big}})$ be the cobar complex for $A^{\text{big}}$ with respect to augmentation of $A^{\text{big}}$, so that $H_\ast C(A^{\text{big}}) = \Lambda^{\text{big}}$ as algebras. The levelwise Frobenius gives a map $\sigma : C(A^{\text{big}}) \to C(A^{\text{big}})$ of differential graded algebras, and by taking homology this induces a map $\theta' : \Lambda^{\text{big}} \to \Lambda^{\text{big}}$ of algebras. We claim that to produce $\theta$ it suffices to show that $\theta'$ restricts to an endomorphism of $\Lambda \subset \Lambda^{\text{big}}$ satisfying the given formulas. Indeed, there is an inclusion $C(A) \subset C(A^{\text{big}})$ of algebras, which does not respect differentials but does commute with the levelwise Frobenius $\sigma$. It would thus follow that the restriction $\theta$ of $\theta'$ to $\Lambda$ is induced by the levelwise Frobenius on $C(A)$. In particular, this would show that $\sigma : C(A) \to C(A)$ indeed descends to an algebra map $\theta : \Lambda \to \Lambda$. That $\theta$ moreover respects the differential is inherited from $\sigma$.

To understand $\theta'$, it suffices to understand its effect on the generators of $\Lambda^{\text{big}}$, i.e. to understand the map $\theta' : \Lambda^{\text{big}}[1] \to \Lambda^{\text{big}}[1]$. Recall that $\Lambda^{\text{big}}[1] = (A^{\text{big}}[1])^\vee = A^\vee_{\leq 1}$. This ring was described in Lemma 2.3.13, and $\theta'$ acts on it by the Frobenius. We find that $\theta'$ satisfies the same formulas as described for $\theta$, only with the addition that $\theta'(|-1) = \lambda_{-1}$. In particular $\theta'$ does restrict to $\Lambda$, and this restriction satisfies the stated formulas, proving the theorem. \(\square\)

**2.4. Summary.** For ease of reference, let us summarize what we have learned in one place. As always, $F$ is a base field of characteristic not equal to 2.

#### 2.4.1. Generators.** There is a differential graded algebra $\Lambda^F$, the $F$-motivic lambda algebra, together with a multiplicative quasiisomorphism $C(\Lambda^F) \to \Lambda^F$, where $C(\Lambda^F)$ is the reduced cobar complex of $\Lambda^F$. We write $\Lambda^F = \bigoplus_{m \geq 0} \Lambda^F[m]$, where the differential on $\Lambda^F$ is of the form $\delta : \Lambda^F[m] \to \Lambda^F[m+1]$.

The $F$-motivic lambda algebra $\Lambda^F$ is an $M^F$-bimodule algebra, generated by classes $\lambda_r \in \Lambda^F[1]$ for $r \geq 0$. In the trigrading (stem, filtration, weight), we have

\[ |\tau| = (0, 0, -1), \quad |\rho| = (-1, 0, -1), \quad |\lambda_a| = [(a, 1, [a/2])]. \]

A right $M^F$-module basis of $\Lambda^F$ is given by those $\lambda_{r_1} \cdots \lambda_{r_n}$ with $2r_i \geq r_{i+1}$ for $1 \leq i \leq n - 1$.

#### 2.4.2. Relations.** The $F$-motivic lambda algebra is a quadratic $M^F$-bimodule algebra. Recall that $M^F = M^F_0[\tau]$. The $M^F$-bimodule structure of $\Lambda^F$ is determined by

\[ x \lambda_n = \lambda_n x, \quad x \in M^F_0, \]
\[ \tau \lambda_{2n+1} = \lambda_{2n+1}\tau + \lambda_{2n+2}\rho, \]
\[ \tau \lambda_{2n} = \lambda_{2n}\tau + \lambda_{2n+1}\rho + \lambda_{2n+2}\rho^2, \]

and the quadratic relations are given as follows. Fix $a, b \geq 0$. If $a$ is odd or $b$ is even, then

\[ \lambda_a \lambda_{2a+b+1} = \sum_{0 \leq r < b/2} \lambda_{a+b-r} \lambda_{2a+1+r} \binom{b-r-1}{r}; \]
and if \( a \) is even and \( b \) is odd, then
\[
\lambda_a \lambda_{2a+b+1} = \sum_{0 \leq r < b/2} \lambda_{a+b-r} \lambda_{2a+1+r} \left( \frac{b-r-1}{r} \right) \tau^{(r-1) \mod 2} + \sum_{0 \leq r \leq \lfloor b/2 \rfloor} \lambda_{a+b+1-r} \lambda_{2a+1+r} \left( \frac{\lfloor b/2 \rfloor - \lfloor r/2 \rfloor}{\lfloor r/2 \rfloor} \right) \rho.
\]

2.4.3. Differentials. The differential on \( \Lambda \) is determined by the Leibniz rule, together with
\[
\delta(x) = 0, \quad x \in M^F_0,
\]
\[
\delta(\tau) = \lambda_0 \rho,
\]
\[
\delta(\lambda_n) = \sum_{1 \leq r \leq n/2} \lambda_{n-r} \lambda_{r-1} \left( \frac{n-r}{r} \right).
\]
Moreover, we have
\[
\delta(\tau^n) = \sum_{r \geq 0} \lambda_r \left( \frac{n+[r/2]}{r+1} \right) \tau^{n-[r/2]-1} \rho^{r+1}.
\]

2.4.4. The endomorphism \( \theta \). The squaring operation \( \text{Sq}^0 : \text{Ext}^{s,f,w}_F \to \text{Ext}^{2s+f,f,w+f}_F \) lifts to an endomorphism \( \theta : \Lambda^F \to \Lambda^F \) of differential graded algebras, determined by
\[
\theta(x) = x^2, \quad x \in M^F,
\]
\[
\theta(\lambda_{2n-1}) = \lambda_{4n-1},
\]
\[
\theta(\lambda_{2n}) = \lambda_{4n+1} \tau + \lambda_{4n+2} \rho.
\]

3. Some first examples, and the doubling map

3.1. First examples. Before continuing on, we give some basic examples illustrating the form of the motivic lambda algebra. In particular, we use \( \Lambda^F \) to define some classes in \( \text{Ext}_F \), and reprove some well-known low-dimensional relations. This material is meant only to familiarize the reader with \( \Lambda^F \); we give a more thorough and entirely self-contained investigation in Section 4.

Given a cycle \( z \in \Lambda^F \), in this section we write \([z] \in \text{Ext}_F\) for the corresponding cohomology class.

Lemma 3.1.1. \( \delta(\lambda_{2a-1}) = 0 \) for all \( a \geq 0 \).

Proof. The proof is identical to the proof of [Wan67, Proposition 2.2]. \( \square \)

This allows us to define the following Hopf elements.

Definition 3.1.2. \( h_a := [\lambda_{2a-1}] \). \( \triangleright \)

Lemma 3.1.3. If \( \rho = 0 \) in \( M^F \), such as if \( F \) is algebraically closed, then \( \delta(\tau^n) = 0 \) for all \( n \geq 0 \).

Proof. This is immediate from the differential \( \delta(\tau) = \lambda_0 \rho \). \( \square \)

In general, if \( \rho \) is nilpotent in \( M^F \), then various powers of \( \tau \) will be cycles in \( \Lambda^F \). We shall write \( \tau^n \) in place of \( [\tau^n] \) in this case. We begin by considering some examples in the case where \( F \) is algebraically closed.

Proposition 3.1.4. For \( F \) algebraically closed, there is a relation
\[
\tau \cdot h_4^1 = h_2^2 h_0^2.
\]

Proof. By definition, \( \tau \cdot h_4^1 = [\lambda_4^1 \tau] \) and \( h_2^2 h_0^2 = h_0^2 h_2 = [\lambda_0^2 \lambda_3] \). We have
\[
\lambda_0^2 \lambda_3 = \lambda_3^1 \tau,
\]
so these classes coincide in \( \text{Ext}_F \). \( \square \)
Proposition 3.1.5. For $F$ algebraically closed, there is a relation
\[ \tau \cdot h_1^4 = 0. \]

However, $h_1^n \neq 0$ for any $n$.

Proof. Observe that $\lambda_0 \lambda_1 = 0$, and thus $h_1 h_0 = 0$. Combined with Proposition 3.1.4, we find
\[ \tau \cdot h_1^4 = \tau h_1^3 \cdot h_1 = h_2 h_0^2 \cdot h_1 = 0. \]

Alternately, $\tau h_1^4 = [\lambda_1^4 \tau]$, and there is a differential
\[ \delta(\lambda_2^3 \lambda_1) = \lambda_1^4 \tau. \]

On the other hand, for $h_1^n$ to vanish, the class $\lambda_2^3$ must be nullhomotopic, i.e. $\delta(x) = \lambda_1^4$ for some $x \in \Lambda$. The class $x$ must live in stem $n + 1$, weight $n$, and filtration $n - 1$, and in this degree $\Lambda$ is generated by the cycle $\lambda_3 \lambda_1^{n-2}$. So no such $x$ exists. \hfill \Box

Next we consider some examples relevant to base fields $F$ over which $\rho$ does not vanish. We begin by defining some classes. Note that the differential
\[ \delta(\tau) = \lambda_0 \rho \]
implies that $\delta(\tau^m) \equiv 0 \pmod{\rho^2}$. This allows for the following definition.

Definition 3.1.6. If $F = \mathbb{R}$, then
\[ \tau^{2^a-1} h_a := \left[ \frac{1}{\rho^2} \delta(\tau^{2^a}) \right] \]
for $a \geq 1$. In general, $\tau^{2^a-1} h_a \in \text{Ext}_F$ is defined by pushing these classes forward along the map $\Lambda^R \to \Lambda^F$ induced by $\mathbb{M}^R \to \mathbb{M}^F$ (cf. Remark 2.2.8). \hfill \Box

Remark 3.1.7. Following our convention that $\Lambda^F$ is considered primarily as a right $\mathbb{M}^F$-module, it would be more natural to write $h_a \tau^{2^a-1}$ for the classes introduced above. We have chosen instead to work with naming conventions compatible with those in [BI20], as no confusion should arise. \hfill \Box

Remark 3.1.8. If $\tau^{2^a-1}$ is a cycle in $\text{Ext}_F$, then $\tau^{2^a-1} h_a = \tau^{2^a-1} \cdot h_a$. \hfill \Box

Example 3.1.9. We have
\[ \tau h_1 = [\lambda_1 \tau + \lambda_2 \rho], \]
\[ \tau^2 h_2 = [\lambda_3 \tau^2 + \lambda_5 \tau \rho^2 + \lambda_6 \rho^3], \]
\[ \tau^4 h_3 = [\lambda_7 \tau^4 + \lambda_{11} \tau^2 \rho^4 + \lambda_{13} \tau \rho^6 + \lambda_{14} \rho^7]. \]

In fact, we may identify $\tau^{2^a-1} h_a = \left[ \tau^{2^a} \lambda_1^{2^{a-1}} \right]$ for all $a \geq 1$. \hfill \Box

The following relation was proved over $\mathbb{R}$ by Dugger–Isaksen [DI16, Proof of Lemma 6.2] using Massey products and May’s Convergence Theorem. We may use the lambda algebra to provide an explicit direct proof.

Proposition 3.1.10. There is a relation
\[ (h_0 + \rho h_1) \cdot \tau h_1 = 0. \]

Proof. By definition,
\[ h_0 \cdot \tau h_1 = [\lambda_0 (\lambda_1 \tau + \lambda_2 \rho)], \]
\[ \rho h_1 \cdot \tau h_1 = [\rho \lambda_1 (\lambda_1 \tau + \lambda_2 \rho)]. \]

Expanding, we have
\[ \lambda_0 (\lambda_1 \tau + \lambda_2 \rho) = \lambda_2^2 \tau \rho + \lambda_1 \lambda_2 \rho^2 + \lambda_2 \lambda_1 \rho^2, \]
\[ \rho h_1 (\lambda_1 \tau + \lambda_2 \rho) = \lambda_2^2 \tau \rho + \lambda_1 \lambda_2 \rho^2. \]
But
\[ \delta(\lambda_3 \tau \rho + \lambda_4 \rho^2) = \lambda_2 \lambda_1 \rho^2, \]
so \( h_0 \cdot \tau h_1 = \rho h_1 \cdot \tau h_1 \). The proposition follows. \( \square \)

The fact that \( \delta(\tau^n) \equiv 0 \pmod{\rho} \) allows for the following definition.

**Definition 3.1.11.** If \( F = \mathbb{R} \), then
\[ \tau^{2n} h_0 := \left[ \frac{1}{\rho} \delta(\tau^{2n+1}) \right]. \]
In general, \( \tau^{2n} h_0 \in \text{Ext}_F \) is defined by pushing these classes forward along the map \( \Lambda^R \to \Lambda^F \) induced by \( M^R \to M^F \) (see Remark 2.2.8).

\( \triangleright \)

**Example 3.1.12.** We have
\[ h_0 = [\lambda_0], \]
\[ \tau^2 h_0 = [\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho + \lambda_3 \tau \rho^3 + \lambda_4 \rho^4], \]
\[ \tau^4 h_0 = [\lambda_0 \tau^4 + \lambda_3 \tau^3 \rho^3 + \lambda_4 \tau^3 \rho^4 + \lambda_5 \tau^2 \rho^5 + \lambda_7 \tau \rho^7 + \lambda_8 \rho^8]. \]
\( \triangleright \)

The following proposition was originally proved over \( \mathbb{R} \) by Dugger–Isaksen [DI16, Proof of Lemma 6.2] using Massey products, May’s Convergence Theorem, and analysis of the \( \rho \)-Bockstein spectral sequence. Using the lambda algebra, the proof amounts to checking that the products of cycle representatives are equal.

**Proposition 3.1.13.** There is a relation
\[ \tau^2 h_0 \cdot h_1 = \rho(\tau h_1)^2. \]
**Proof.** We may directly compute
\[ \tau^2 h_0 \cdot h_1 = [(\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho + \lambda_3 \tau \rho^3 + \lambda_4 \rho^4) \lambda_1] = [\lambda_2 \tau^2 \rho + \lambda_2 \lambda_1 \tau \rho^2 + \lambda_2 \lambda_3 \rho^3 + \lambda_2 \lambda_4 \rho^4] = [\rho(\lambda_1 + \lambda_2 \rho) \lambda_1] = \rho(\tau h_1)^2. \]
\( \square \)

**3.2. The doubling map.** In [DI16, Theorem 4.1], Dugger–Isaksen produce an isomorphism
\[ \text{Ext}_{\mathbb{R}}[\rho^\pm 1] \cong \text{Ext}_F[\rho^{-1}], \]
which doubles internal degrees. We can lift this isomorphism to a quasiisomorphism of lambda algebras.

**Proposition 3.2.1.** Let \( \Lambda^{\text{cl}} \) denote the classic lambda algebra, only given a motivic grading where \( |\lambda_n| \) has stem \( 2n + 1 \) and weight \( n + 1 \). For any \( F \), there is a retraction
\[ \Lambda^{\text{cl}} \xrightarrow{\hat{\theta}} \Lambda^F \xrightarrow{q} \Lambda^{\text{cl}} \]
with the following properties:
\begin{itemize}
  \item[(1)] All maps shown are maps of differential graded algebras respecting \( \theta \);
  \item[(2)] \( \hat{\theta} \) is given on generators by \( \hat{\theta}(\lambda_n) = \lambda_{2n+1} \);
  \item[(3)] \( q \) is given on generators by \( q(\tau) = 0, q(\lambda_{2n}) = 0, \) and \( q(\lambda_{2n+1}) = \lambda_n \).
\end{itemize}
Now say \( F = \mathbb{R} \), and write \( \text{Ext}_{\mathbb{R}}^{\rho\text{-tors}} \subset \text{Ext}_{\mathbb{R}} \) for the \( \rho \)-torsion subgroup of \( \text{Ext}_{\mathbb{R}} \).
\begin{itemize}
  \item[(4)] The map \( \text{Ext}_{\mathbb{R}}[\rho] \oplus \text{Ext}_{\mathbb{R}}^{\rho\text{-tors}} \to \text{Ext}_{\mathbb{R}} \) induced by \( \hat{\theta} \) and the inclusion of \( \rho \)-torsion is an isomorphism;
  \item[(5)] In particular, \( \hat{\theta} \) extends to a quasiisomorphism \( \Lambda^{\text{cl}} \otimes_{\mathbb{Z}_2} F_2[\rho^\pm 1] \to \Lambda^R[\rho^{-1}] \).
Proof. The assignments given in (2) and (3) are easily seen to extend to maps of differential graded algebras, proving (1), and that the resulting sequence is a retraction is clear. Evidently (4) implies (5), so we are left with proving (4).

It is equivalent to verify that the composite \( \text{Ext}_{\text{cl}}[\rho] \to \text{Ext}_R \to \text{Ext}_R / \text{Ext}_R^{\text{tors}} \) is an isomorphism. This is a split inclusion of free \( \mathbb{F}_2[\rho] \)-modules, so for it to be an isomorphism it is sufficient to verify that it is an isomorphism after inverting \( \rho \), and for this it is sufficient for the injection \( \text{Ext}_{\text{cl}}[\rho^{\pm 1}] \to \text{Ext}_R[\rho^{-1}] \) to be an isomorphism. By Dugger–Isaksen’s isomorphism \( \text{Ext}_R[\rho^{-1}] \cong \text{Ext}_{\text{cl}}[\rho^{\pm 1}] \) [DI16, Theorem 4.1], we find that our map \( \text{Ext}_{\text{cl}}[\rho^{\pm 1}] \to \text{Ext}_R[\rho^{-1}] \) is an injection between vector spaces of equal finite rank in each degree, and is thus an isomorphism. \( \square \)

Remark 3.2.2. We point out the following amusing corollary of Proposition 3.2.1: there is a multiplicative injection

\[
Q: \ker(Sq^0: \text{Ext}_{\text{cl}} \to \text{Ext}_{\text{cl}}) \to \text{Ext}_{\text{cl}}^{\tau-\text{tors}},
\]

acting in degrees as \( Sq^0 \) would. For example, as \( \theta \lambda_2^n = \lambda_1^n \), we find that \( Q(h_1^n) = h_1^n \). This provides another explanation of the fact that \( h_1 \) is not nilpotent in \( \text{Ext}_C \). It is natural to ask whether \( Q \) accounts for all indecomposable \( \tau \)-torsion classes in \( \text{Ext}_C \), but a counterexample is given by the class \( B_5 \) in stem 55 and filtration 7, as \( \text{Ext}_{\text{cl}}^{24,7} = 0 \). \( \triangle \)

4. \( \text{Ext}_R \) in filtrations \( f \leq 3 \)

In this section, we use the \( \mathbb{R} \)-motivic lambda algebra to compute \( \text{Ext}_R^f \) for \( f \leq 3 \). Throughout this section, we shall abbreviate

\[
\Lambda = \Lambda^\mathbb{R}.
\]

4.1. Preliminaries. We begin by describing our strategy for computing \( \text{Ext}_R \). We rely on the following device, which uses ideas from Tangora’s work on the classic lambda algebra [Tan85] to produce something like a chain-level lift of the \( \rho \)-Bockstein spectral sequence [Hi11]. We require some preliminary definitions.

Definition 4.1.1. Let \( V = \mathbb{F}_2 \{ x_s : s \in S \} \) be a (locally) finite \( \mathbb{F}_2 \)-vector space with ordered basis.

1. The leading term of a class \( x \in V \) is the largest term appearing when \( x \) is written as a sum of basis elements.
2. We write \( x < x' \) when the leading term of \( x \) is less than that of \( x' \).
3. Given another vector space \( U = \mathbb{F}_2 \{ x_s : s \in T \} \) with ordered basis, map \( \phi: V \to U \), and \( s \in S, t \in T \), we write

\[
\phi(x_s + <) = y_t + <
\]

for the relation that there exist some classes \( u < x_s \) and \( v < y_t \) for which \( \phi(x_s + u) = y_t + v \). \( \triangle \)

The main technical lemma we need is the following. The reader is invited to skip this lemma on first reading; the details are not necessary to understand our computation, and we rephrase what we need in the context of \( \Lambda \) in Theorem 4.1.4.

Lemma 4.1.2. Let \( (C,d) \) be a chain complex of locally finite and free \( \mathbb{F}_2[\rho] \)-modules, and suppose (for simplicity) that \( H_mC[\rho^{-1}] = 0 \). Choose an ordered basis \( \mathbb{F}_2 \{ x_s : s \in T \} \) for \( C/(\rho) \), and extend this to a basis \( \mathbb{F}_2 \{ \rho^n x_s : (s,n) \in T \times \mathbb{N} \} \) for \( C \), itself ordered by \( \rho^n x_s < \rho^m x_t \) whenever \( n > m \), or else \( n = m \) and \( s < t \). Let \( \{ \alpha_s : s \in B \} \) be a basis for \( H_s(C/(\rho)) \), indexed by a subset \( B \subset T \) with the property that for each \( \alpha_s \) there is some \( z_s \in C \) with leading term \( x_s \) which projects to a cycle representative of \( \alpha_s \). Let \( B_1 \subset B \) be the subset of those \( s \) for which \( x_s \) is the leading term of some cycle in \( C \), and let \( B_0 = B \setminus B_1 \).

There is then a unique injection \( t: B_0 \to B \) such that

\[
d(x_s + <) = \rho^{t(s)} x_{t(s)} + <
\]
Adding these together, we find
\[ H_\ast C = \bigoplus_{s \in B_0} \mathbb{F}_2[\rho]/(\rho^{r(s)}), \]
where we may take the summand indexed by \( s \) to be generated by any class of the form \( \rho^{-r(s)} \cdot d(x_s + c) \) with leading term \( x_{t(s)} \).

Proof. We begin by defining a function \( t^{-1} : B_1 \to B \). Fix \( b \in B_1 \); we claim that there exists some \( s \in B \) such that \( d(x_s + <) = x_b + <. \) The function \( t^{-1} \) will then be defined by declaring \( t^{-1}(b) \) to be the minimal \( s \) for which \( d(x_s + <) = x_b + <. \)

Indeed, let \( z_b \) be a cycle with leading term \( x_1 \), which projects to a cycle representative for \( \alpha_0 \). As \( H_\ast C[\rho^{-1}] = 0 \), necessarily \( \rho^r z_b \) is nullhomologous for some minimal \( r \geq 1 \). That is, there is some \( y \in C \) not divisible by \( \rho \) such that \( d(y) = \rho^r z_b \). If \( y = x_s + < \) with \( s \in B \), then we are done. Otherwise, as \( y \) is a cycle in \( C/(\rho) \), necessarily \( y \) is homologous to some \( x_u + \) with \( u < x_s \) and \( s \in B \), in which case there exists some \( v \) with \( d(v) = x_s + u + y \). We find that
\[ d(x_s + <) = d(x_s + u) = d(x_s + u + d(v)) = d(y) = \rho^r z_b = \rho^r x_b + < \]
as claimed. Thus we have produced the function \( t^{-1} \).

Next we claim that \( t^{-1} \) restricts to a function \( t^{-1} : B_1 \to B_0 \). Indeed, suppose towards contradiction that there were some \( b \in B_1 \) such that \( x_{t^{-1}(b)} \) is the leading term of some cycle. That is to say, there are some \( u,v < x_{t^{-1}(b)} \) such that
\[ d(x_{t^{-1}(b)} + u) = x_b + <, \quad d(x_{t^{-1}(b)} + v) = 0. \]
Adding these together, we find
\[ d(u + v) = x_b + <. \]
As \( u + v < x_{t^{-1}(b)} \), this contradicts minimality of \( t^{-1}(b) \). Thus we have a function \( t^{-1} : B_1 \to B_0 \).

Next we claim that \( t^{-1} \) is a bijection. It is a function between locally finite sets, and the assumption that \( H_\ast C[\rho^{-1}] = 0 \) implies that these sets have the same cardinality in each degree. So it is sufficient to verify that \( t^{-1} \) is an injection. Indeed, suppose towards contradiction that there were some \( b < c \) in \( B_1 \) for which \( t^{-1}(b) = s = t^{-1}(c) \). Thus there are \( u,v < x_s \) such that
\[ d(x_s + u) = x_b + <, \quad d(x_s + v) = x_c + <. \]
Adding these together, we find
\[ d(u + v) = x_c + <. \]
As \( u + v < x_s \), this contradicts minimality of \( t^{-1}(c) \).

By taking the inverse of \( t^{-1} : B_1 \to B_0 \), we have thus proved the existence of a bijection \( t : B_0 \to B_1 \) with the property that \( d(x_s + <) = x_{t(s)} + < \) for all \( s \in B_0 \). With this \( t \), the given description of \( H_\ast C \) is clear; in effect, we have described how to choose a basis for \( C \) for which \( d \) is upper triangular, where if a diagonal entry is divisible by \( \rho^r \) so too are all entries above it. Compare the notion of a tag from [Tan85].

It remains to verify uniqueness. Suppose towards contradiction that we have found some other injection \( t' : B_0 \to B \) such that \( d(x_a + <) = x_{t'(a)} + < \) for all \( s \in B_0 \). The condition that \( t' \neq t \) means that there exists some \( s \in B_0 \) for which \( d(x_s + <) = x_{t'(s)} + < \), but \( s \) is not minimal among possible \( a \in B_0 \) with \( d(x_a + <) = x_{t'(s)} + < \). Choose such \( s \) with \( t'(s) \) maximal, and let \( a = t^{-1}(t'(s)) \) be the minimal \( a \in B_0 \) with \( d(x_a + <) = x_{t'(s)} + < \). So there are \( u,v < x_a \) for which
\[ d(x_a + u) = x_{t'(s)} + <, \quad d(x_a + v) = x_{t'(a)} + <. \]
Adding these together, we find that
\[ d(u + v) = x_{t'(s)} + x_{t'(a)} + <, \]
where \( u + v < x_a \). If \( t'(a) < t'(s) \), then this reduces to
\[
d(u + v) = x_{t'(s)} + <,
\]
contradicting minimality of \( a \). If \( t'(s) < t'(a) \), then this reduces to
\[
d(u + v) = x_{t'(a)} + <,
\]
contradicting maximality of \( t'(s) \). So there is no such \( t' \), proving that \( t \) is the unique injection satisfying the required property. \( \square \)

We now specialize to the computation of \( \text{Ext}_R \). Observe that by Proposition 3.2.1, we may reduce to considering only the \( \rho \)-torsion subgroup of \( \text{Ext}_R \). In terms of \( \Lambda \), this amounts to ignoring monomials of the form \( \lambda_I \) where \( I \) is a sequence of odd numbers. We will apply Lemma 4.1.2 to compute this \( \rho \)-torsion subgroup as follows.

We take as basis of \( \Lambda/\langle \rho \rangle \) the standard basis \( \lambda_I \) where \( I \) is coadmissible (Definition 2.3.9) and \( n \geq 0 \). We also need to order this basis; in the region where we will compute, our choice of order makes no difference, \(^6\) but for concreteness let us say \( \lambda_I < \lambda_J \) if \( n > m \), or else \( n = m \) and \( I < J \) lexicographically. \(^7\)

We must fix some further notation. Let \( \{ \alpha_s : s \in S \} \) be a basis for \( \text{Ext}_C \), and write \( \alpha_s \in \text{Ext}_C \) for the image of \( \alpha_s \) under the map induced by \( \bar{\theta} : \Lambda^{cl} \to \Lambda^C \) (see Proposition 3.2.1). Extend this to a minimal generating set \( \{ \alpha_s : s \in S \} \) for \( \text{Ext}_C \) as an \( \mathbb{F}_2[\tau] \)-module. For \( s \in S \), let \( n_s \) denote the \( \tau \)-torsion exponent of \( \alpha_s \), so that \( \{ \alpha_s \tau^n : s \in S, n < n_s \} \) is an \( \mathbb{F}_2 \)-basis for \( \text{Ext}_C \). For each \( s \in S \), choose a distinct coadmissible monomial \( \lambda_{I(s)} \) which is the leading term of a cycle representative for \( \alpha_s \) in \( \Lambda_C \), making this choice so that if \( s \in S_0 \) then \( \lambda_{I(s)} \) is in the image of \( \bar{\theta} \). See the discussion following Proposition 4.2.1 for the particular choices we will take in our computation.

Let \( B' = \{(s,n) : s \in S, n < n_s \} \). Given \( b = (s,n) \in B' \), write \( x_b = \lambda_{I(s)} \tau^n \in \Lambda^S \). Let \( B \subset B' \) be the subset of pairs not of the form \( (s,0) \) with \( s \in S_0 \). Let \( B_1 \subset B \) be the subset of those \( b \) such that \( x_b \) is the leading term of some cycle, and let \( B_0 = B \setminus B_1 \). Let \( B[f] \subset B \) be the subset of those \( b \) for which \( x_b \) is in filtration \( f \), and extend this notation to all the indexing sets under consideration.

For our computation, we will produce for every \( b \in B_0[f] \) with \( f \leq 2 \), some \( t(b) \in B \) such that
\[
\delta(x_b + <) = \rho^{r(b)} x_{t(b)} + <,
\]
making this choice so that \( t : B_0 \to B \) is injective. Here, \( r(b) \geq 1 \) is some integer which may be determined by comparing the stems of \( x_b \) and \( x_{t(b)} \).

**Definition 4.1.3.** In the above situation, we shall write \( x_b \to x_{t(b)} \rho^{r(b)} \). \( \diamond \)

**Theorem 4.1.4.** Fix notation as above. Then

1. \( t \) is uniquely determined (given our choice of ordered basis);
2. \( t \) restricts to bijections \( t : B_0[f] \cong B_1[f + 1] \);
3. The \( \rho \)-torsion subgroup of \( \text{Ext}_C^{d+1} \) is isomorphic to
\[
\bigoplus_{b \in B_0[f]} \mathbb{F}_2[\rho]/(\rho^{r(b)}),
\]
where the summand corresponding to \( b \in B_0[f] \) is generated by any class of the form
\[
\delta(x_b + <) / \rho^{r(b)}
\]
with leading term \( x_{t(b)} \).

**Proof.** This follows by specializing Lemma 4.1.2 to the complementary summand of \( \bar{\theta} : \Lambda^{cl} \subset \Lambda \). \( \square \)

---

\(^6\)All \"error terms\" appearing in \"+ <\" will be divisible by \( \rho \).

\(^7\)That is, if \( I = (i_1, \ldots, i_f) \) and \( J = (j_1, \ldots, j_f) \), then \( i_1 < j_1 \), or else \( i_1 = j_1 \) and \( i_2 < j_2 \), and so forth.
Most notably, the $\rho$-torsion in $\text{Ext}_f^{f+1}$ is obtained by understanding differentials out of $\Lambda[f]$; this is significantly easier than finding cycles in $\Lambda[f+1]$ directly.

We end with two remarks, which could have been made in the more general context of Lemma 4.1.2.\par

**Remark 4.1.5.** More, generally, $H^\ast(A^R/\langle \rho^m \rangle^\ast) = H_\ast(A/\langle \rho^m \rangle)$ (denoted $\text{Ext}(m)$ in Section 7) may be read off our computation as follows. For each $b \in B_0$, choose $u_b \in \Lambda$ such that $u_b < x_b$ and $\delta(x_b + u_b) = \rho^{-r(b)}x_b + <$, and let $z_b = \rho^{-r(b)} \cdot \delta(x_b + u_b)$. Then $H_\ast(A/\langle \rho^m \rangle)$ is given as follows.

1. For each $x_b \in S_0$, there is a summand of the form $\mathbb{F}_2[\rho]/\langle \rho^m \rangle$, generated by the image of $\alpha_s$.
2. For each $x_b \in \rho^r(b)x_b(b)$, there is a summand of the form $\mathbb{F}_2[\rho]/\langle \rho^\min(m, r(b)) \rangle$, generated by the class with cycle representative $z_s$.
3. For each $x_b \in \rho^{-r(b)}x_b(b)$, there is a summand of the form $\mathbb{F}_2[\rho]/\langle \rho^{\max(0, m-r(b))} \rangle(x_b + u_b)$.

**Remark 4.1.6.** Our approach to computing $\text{Ext}_R$ via $\Lambda$ is closely related to the computation of $\text{Ext}_R$ via the $\rho$-Bockstein spectral sequence $\text{Ext}_R[\rho] \Rightarrow \text{Ext}_R$ [Hil11]. The precise relation is as follows. For $b = (s, n) \in B$, let $\alpha_b = \alpha_s \tau^n$, so that $\{\alpha_b : b \in B\}$ is a basis of $\text{Ext}_C$. Our ordering on $\Lambda$ and choice of classes $x_b$ gives $B$ an order, thus making this into an ordered basis of $\text{Ext}_C$. Now, $x_b \sim \rho^r(b)x_b$ if and only if $d_r(b)(\alpha_b + <) = \rho^r(b)\alpha_t(b) + <$ in the $\rho$-Bockstein spectral sequence. \hfill \small \triangledown

The above discussion describes how we will compute $\text{Ext}_{R}^{S^3}$ as an $\mathbb{F}_2[\rho]$-module. The computation gives more, as it produces explicit cocycle representatives for our generators of $\text{Ext}_{R}^{S^3}$. We will use this in Section 4.3 to compute products in $\text{Ext}_{R}^{S^3}$.

### 4.2. $\text{Ext}_R^f$ for $f \leq 3$.

We now proceed to the computation. We begin by understanding $\Lambda_R^f/\langle \rho \rangle \cong \Lambda^C$.

**Proposition 4.2.1.** $\text{Ext}_C^{S^3}$ is generated as a commutative $\mathbb{F}_2[\tau]$-algebra by classes $h_a$ for $a \geq 0$, represented in $\Lambda^C$ by $\lambda_{2^a}$, and $c_a$ for $a \geq 0$, represented in $\Lambda^C$ by $\lambda_{2^a+1}$.

A full set of relations is given by

\[
\begin{align*}
h_{a+1} + h_a &= 0, & h_{a+2} + h_a &= 0, & h_2 h_0 &= \tau h_1, & h_{a+3} h_{a+1} &= h_{a+2},
\end{align*}
\]

for all $a \geq 0$. This is free over $\mathbb{F}_2[\tau]$, with basis given by the classes in the following table.

| Class | Constraints |
|-------|-------------|
| 1     |             |
| $h_a$ | $a \geq 0$  |
| $h_a \cdot h_b$ | $a \geq b \geq 0$ and $a \neq b + 1$ |
| $h_a \cdot h_b \cdot h_c$ | $a \geq b \geq c \geq 0$, where $a \neq b + 1$ and $b \neq c + 1$, and where if $b = c$ or $a = b$ then $a \neq c + 2$ |
| $c_a$ | $a \geq 0$. |

The only such classes not in the image of $\bar{\theta}: \text{Ext}_{dcl} \to \text{Ext}_C$ are those in which either $h_0$ or $c_0$ appears.

**Proof.** This is essentially well-known, owing to work of Isaksen on the cohomology of the C-motivic Steenrod algebra [Isa19]. Alternately, one may compute $H_{S^3}(\Lambda^C/\langle \tau \rangle)$ following Wang’s approach [Wan67], and run the $\tau$-Bockstein spectral sequence to recover $\text{Ext}_{S^3}$. One finds that $H_{S^3}(\Lambda^C/\langle \tau \rangle)$ agrees with $\text{Ext}_{S^3}$, with two exceptions:

1. Instead of $h_2 \cdot h_2 = h_4$, one has $h_2 \cdot h_2 = 0$;  
2. There is a new cycle $\alpha$ represented by $\lambda_2 \lambda_1$.

There is a $\tau$-Bockstein differential $d_1(\alpha) = \tau h_1$, after which we recover the claimed $\mathbb{F}_2[\tau]$-module basis of $\text{Ext}_{S^3}$. The hidden extension $h_3^2 \cdot h_2 = \tau h_3$ was shown in Proposition 3.1.1; alternately, it is the only relation compatible with $\text{Sq}^0(h_3^2 \cdot h_2) = \tau^2 h_3^2 = \tau^2 h_3^2 = \text{Sq}^0(\tau h_3^2)$. \hfill \square
Proposition 4.2.1 describes a basis for $\text{Ext}^3_{=3}C$, thus giving our set $S[\leq 3]$. We must also choose lambda algebra representatives of these classes. We shall choose $c_n$ to be represented by $\lambda_2n_1\lambda_2n_2\cdots \lambda_2n_k$ and a product $h_{n_1}\cdots h_{n_k}$ with $n_1 \geq \cdots \geq n_k$ to be represented by $\lambda_2n_1\cdots \lambda_2n_k$. We warn that these representatives are not minimal; for example, we have chosen $\lambda_3\lambda_0$ as our representative for $h_2h_0$, rather than the minimal representative which is $\lambda_2\lambda_1$. However, they are easily defined, and convenient enough for our computation.

To better consolidate various cases in our computation, we will take the following convention.

**Definition 4.2.2.** Given integers $a, n \geq 0$, we define
\[
2^{a-1}(2n+1)
\]
to be the integer so represented for $a \geq 1$, and to be $n$ for $a = 0$. □

The purpose of this definition is the following, immediately seen from the description of $\theta$ given in Theorem 2.3.16.

**Lemma 4.2.3.** We have
\[
\theta^a(\lambda_0\tau^n) = \lambda_2^{a-1}\tau^{2^{a-1}(2n+1)} + O(\rho^{2^{a-1}}).
\]
for all $n \geq 0$. □

We now produce the relation "→" described in Definition 4.1.3, proceeding filtration by filtration. To start, observe that $B_0[0] = \{\tau^n : n \geq 1\}$.

**Proposition 4.2.4.** We have
\[
\delta(\tau^{2a}(2m+1)) = \lambda_2^{2a-1}\tau^{2^{a-1}(4m+1)}\rho^{2^a} + O(\rho^{2^{a+2}})
\]
for all $a, m \geq 0$. In particular,
\[
\tau^{2^{a}(2m+1)} \to \lambda_2^{2a-1}\tau^{2^{a-1}(4m+1)}\rho^{2^a}.
\]

**Proof.** By Lemma 4.2.3, we may reduce to the case where $a = 0$. Here, as $\tau^2$ is a cycle mod $\rho^2$, we may compute
\[
\delta(\tau^{2m+1}) = \delta(\tau)\tau^{2m} + O(\rho^2) = \lambda_0\tau^{2m}\rho + O(\rho^2),
\]
as claimed. □

**Corollary 4.2.5.** $B_0[1]$ consists of those $\lambda_2\tau^n$ such that $n$ is not of the form $2^{a-1}(4m+1)$ for any $m$. □

We have located the following indecomposable classes.

**Definition 4.2.6.** For $a, n \geq 0$, we declare
\[
\tau^{2^{a-1}(4n+1)}h_a
\]
to be the class represented by
\[
\rho^{-2^n} \cdot \delta(\tau^{2^n}(2n+1)).
\]

We now compute out of $B_0[1]$.

**Proposition 4.2.7.** $\lambda_2^{a-b}\tau^{2^{a}(2m+1)} \to$ the following monomial.

1. If $a < b - 1$ or $a = b$, then
\[
\lambda_2^{a-b}\lambda_2^{b-1}\tau^{2^{a-1}(4m+1)}\rho^{2^a}.
\]

2. If $a > b + 1$ and $b \neq 0$, then
\[
\lambda_2^{a-1}\lambda_2^{b-1}\tau^{2^{a-1}(4m+1)}\rho^{2^a}.
\]
(3) If \( a = b - 1 \) and \( m = 2n + 1 \) is odd, then
\[
\lambda_{2^b-1}^2 \tau^{2^b(4n+1)} \rho^2.
\]
(4) If \( a = b + 1 \) and \( b \neq 0 \), then
\[
\lambda_{2^{b+1}+1}^2 \tau^{2^b(8m+1)} \rho^{2^b}.
\]
Moreover, these cases are mutually exclusive, and altogether exhaust \( B_0[1] \).

**Proof.** That these cases are mutually exclusive and altogether exhaust \( B_0[1] \) is seen by direct inspection. As the monomials arising as targets are \( \rho \)-multiples of distinct elements of \( B[2] \), it suffices to only verify that for each claim of \( x \rightarrow y \) we have \( \delta(x + <) = y + < \).

1. We have
\[
\delta(\lambda_{2^b-1} \tau^{2^b(2m+1)}) = \lambda_{2^b-1} \lambda_{2^{2m+1}} \tau^{2^{2m+1} - (4m+1)} \rho^{2^b} + O(\rho^{2^b+2^{2m+1}}).
\]

2. Note that
\[
\tau^{2^b(2m+1)} \lambda_{2^b-1} = \lambda_{2^b-1} \tau^{2^b(2m+1)} + <,
\]
as \( \tau \) is central mod \( \rho \). Now we have
\[
\delta(\tau^{2^b(2m+1)} \lambda_{2^b-1}) = (\lambda_{2^b-1} \tau^{2^{2m+1} - (4m+1)} \rho^{2^b} + O(\rho^{2^b+2^{2m+1}})) \lambda_{2^b-1}
\]
\[
= \lambda_{2^b-1} \lambda_{2^{2m+1}} \tau^{2^{2m+1} - (4m+1)} \rho^{2^b} + O(\rho^{2^b+1}).
\]

3. Note that
\[
\delta(\lambda_{2^b-1} \tau^{2^b(2m+1)}) = \lambda_{2^b-1} \tau^{2^b(2m+1)} + O(\rho).
\]

Now we have
\[
\delta(\delta(\lambda_{2^b-1} \tau^{2^b(2m+1)})) = \delta(\lambda_{2^b-1} \tau^{2^b(2m+1)} + O(\rho)) = \lambda_{2^b-1} \tau^{2^b(4m+1)} \rho^{2^b} + O(\rho^{2^b+1}).
\]

4. We have
\[
\delta(\lambda_{2^b-1} \tau^{2^b+1(2m+1)}) = \delta(\lambda_{2^b-1} \tau^{2^b+1(2m+1)}) = \delta(\lambda_{2^b-1} \tau^{2^b+1(2m+1)} + O(\rho^7)) = \lambda_{2^b+1}^2 \tau^{2^b+1(8m+1)} \rho^{2^b} + O(\rho^{2^b+2^{2m+1}}).
\]

Here, the third equality uses the Adem relations \( \lambda_1 \lambda_3 = 0 \) and \( \lambda_1 \lambda_5 = \lambda_3 \lambda_3 \) to determine the leading term of \( \lambda_1 \delta(\tau^4) \). \( \square \)

**Corollary 4.2.8.** \( B_0[2] \) consists of those \( \lambda_{2^b-1} \lambda_{2^{2m+1}} \tau^n \) where \( b = c \) or \( b \geq c + 2 \), and where moreover

1. \( n \neq 2^{b-1}(4m+1) \) and \( n \neq 2^{c-1}(4m+1) \) for any \( m \);
2. If \( b = c = 0 \), then \( n \) is odd;
3. If \( b = c \geq 1 \), then \( n \neq 2^b(4m+1) \) for any \( m \);
4. If \( b = c \geq 2 \), then \( n \neq 2^{b-2}(8m+1) \) for any \( m \). \( \square \)

We have located the following indecomposable classes.

**Definition 4.2.9.** For \( a, n \geq 0 \), we declare
\[
\tau^{2^n(8n+1)} \rho^{a+2}
\]
to be the class represented by
\[
\rho^{-2^{a+1}+3} \cdot \delta(\lambda_{2^{a+1}+1} \tau^{2^n(2m+1)})
\]

We now compute out of \( B_0[2] \).

**Proposition 4.2.10.** For \( b = c \) or \( b \geq c + 2 \), we have \( \lambda_{2^b-1} \lambda_{2^{2m+1}} \tau^{2^n(2m+1)} \rightarrow \) the following.

1. If \( b = c = 0, a = -1, m = 2n + 1 \), then
\[
\lambda_{0}^3 \tau^{2n} \rho.
\]
(2) If $b = c \geq 1$, $a = b - 1$, $m = 2n + 1$, then
\[ \lambda_{2b-1}^3 \tau^{2(2n+1)} \rho^2. \]

(3) If $b = c \geq 0$, $a = c$, $m = 2n + 1$, then
\[ \lambda_{2b-1}^3 \tau^{2b-1} (4(2n+1)+1). \]

(4) If $b = c \geq 1$, $a = b + 1$, then
\[ \lambda_{2b+3-1} \lambda_{2b+5-1} \tau^{2b-2} (16m+1) \rho^{2b-1}. \]

(5) If $b = c \geq 1$, $a = b + 2$, then
\[ \lambda_{2b+2-1} \lambda_{2b+4-1} \tau^{2b-2} (2(4n+1)+1) \rho^{2b-2}. \]

(6) If $b = c \geq 1$, $a \geq b + 3$, then
\[ \lambda_{2b-1}^2 \lambda_{2b-1}^2 \tau^{2a-1}(4m+1) \rho^a. \]

(7) If $b = c \geq 2$, $a = b - 2$, $m = 4n + 2$, then
\[ \lambda_{2b-1}^3 \tau^{2b-2} (2(4n+1)+1) \rho^{2b}. \]

(8) If $b = c \geq 2$, $a = b - 2$, $m = 2n + 1$, then
\[ \lambda_{2b-3-1} \lambda_{2b-1}^2 \tau^{2b-3} (2(4n+1)+1) \rho^{2b-2}. \]

(9) If $b = c \geq 3$, $a \leq b - 3$, then
\[ \lambda_{2b-1} \lambda_{2b-1}^2 \tau^{2a-1}(4m+1) \rho^a. \]

(10) If $b - 2 \geq c = 0$, $a = 0$, $m = 2n + 1$, then
\[ \lambda_{2b-1} \lambda_{2b-1}^2 \tau^{2n} \rho. \]

(11) If $b - 2 = c \geq 1$, $a = b$, then
\[ \tau^{2(8n+1)} \lambda_{2b-3-1} \lambda_{2b+2-1} \rho^{2b-1}. \]

(12) If $b - 2 = c \geq 1$, $a \geq b + 2$, then
\[ \lambda_{2b-1}^3 \lambda_{2b-1} \lambda_{2b-1}^2 \tau^{2a-1}(4m+1) \rho^a. \]

(13) If $b - 3 \geq c \geq 1$, $a \geq b$, $a \neq b + 1$, then
\[ \lambda_{2b-1} \lambda_{2b-1} \lambda_{2b-1}^2 \tau^{2a-1}(4m+1) \rho^a. \]

(14) If $b - 2 \geq c \geq 1$, $c \leq a < b$, $a \notin \{ c + 1, b - 1 \}$, then
\[ \lambda_{2b-1}^2 \lambda_{2b-1} \lambda_{2b-1} \tau^{2a-1}(4m+1) \rho^a. \]

(15) If $b - 2 = c \geq 1$, $a = c - 1$, $m = 2n + 1$, then
\[ \lambda_{2b-1}^3 \tau^{2(2n+1)} \rho^2. \]

(16) If $b - 3 \geq c \geq 1$, $a = c - 1$, $m = 2n + 1$, then
\[ \lambda_{2b-1}^2 \lambda_{2b-1}^2 \tau^{2(2n+1)} \rho^2. \]

(17) If $b - 2 = c \geq 1$, $a = c + 1$, $m = 2n + 1$, then
\[ \lambda_{2b-1}^3 \tau^{2b+3} (4(4n+1)+1) \rho^{2b+2}. \]

(18) If $b - 3 = c \geq 1$, $a = c + 1$, then
\[ \lambda_{2b-1}^3 \lambda_{2b+3-1} \tau^{2b+3} (8m+1) \rho^{2b+3}. \]

(19) If $b - 4 \geq c \geq 1$, $a = c + 1$, then
\[ \lambda_{2b-1}^3 \lambda_{2b+3-1} \tau^{2b+3} (8m+1) \rho^{2b+3}. \]
(20) If $b - 3 \geq c \geq 1$, $a = b - 1$, $m = 2n + 1$, then
$$\lambda^2_{2b-1} \lambda^2_{2c-1} \tau^{2b(2n+1)} \rho^{2b}.$$  
(21) If $b - 2 \geq c \geq 1$, $a = b - 1$, then
$$\lambda^2_{2b+1} \lambda^2_{2c-1} \tau^{2b(8m+1)} \rho^{2b3}.$$  
(22) If $b - 2 \geq c \geq 2$, $a \leq c - 2$, then
$$\lambda^2_{2b-1} \lambda^2_{2a-1} \tau^{2b(4m+1)} \rho^{2a}.$$  

Moreover, these cases are mutually exclusive, and altogether exhaust $B_0[2]$.

**Proof.** That these cases are mutually exclusive and altogether exhaust $B_0[2]$ is seen by direct inspection. As the monomials arising as targets are $\rho$-multiples of distinct elements of $B[3]$, it suffices to only verify that for each claim of $x \rightarrow y$ we have $\delta(x + \langle \rangle) = y + \langle \rangle$.

Each case represents a collection of families of monomials whose leading terms are connected by $\theta$. Thus we may always reduce to the smallest possible $c$, with the exception of (9) and (22), where doing so would place extra constraints on $a$. In addition, by working modulo the smallest possible $\rho$ in which the proposed target does not vanish, we may always reduce to the smallest possible $m$.

We may further divide the list of cases provided into three types: those which require no calculations beyond those carried out in Proposition 4.2.7, cases (15) and (18), and the more interesting cases which do require additional calculation, producing new indecomposable classes in $\text{Ext}^3_{\overline{\mathbb{R}}}$. Here, cases (15) and (18) are not really exceptional; they could be consolidated into cases (16) and (19), only this would require slightly modifying the setup of Section 4.1, and it is easier to just separate them out. The more interesting cases are (4), (5), (8), (11), and (17). The remaining less interesting cases may all be handled exactly the same way as the first two cases of Proposition 4.2.7 were handled. Thus we shall not handle them individually, and instead only illustrate this point with a verification of (21). With these reductions in place, the proposition is proved by the following calculations.

(4) Here, we are claiming $\delta(\lambda^2_1 \tau^4 + \langle \rangle) = \lambda^2_2 \lambda^2_3 \rho^7 + \langle \rangle$. In fact $\delta(\lambda^2_1 \tau^4) = \lambda^2_2 \lambda^2_3 \tau^7$ on the nose.

(5) Here, we are claiming $\delta(\lambda^2_1 \tau^8 + \langle \rangle) = \lambda^2_2 \lambda^2_0 \tau^{13} + \langle \rangle$. Observe that $\delta(\lambda^2_1 \tau^8) = \lambda^2_3 \tau^4 \rho^8 + O(\rho^{12})$, but $\lambda_3 \tau^4 \rho^8$ is already seen as a target in case (1). Thus some additional correction term must be added to $\lambda^2_1 \tau^8$ to get down to $\lambda^2_2 \lambda^2_0 \tau^{13}$. Such a correction term is given by

$$u = \lambda^2_2 \tau^6 \rho^4 + \lambda^2_3 \lambda^2_5 \tau^5 \rho^6 + \lambda^2_3 \lambda^2_6 \tau^4 \rho^7 + \lambda^2_5 \lambda^2_7 \tau^3 \rho^{10} + (\lambda^2_5 \lambda^2_8 + \lambda^2_6 \lambda^2_7) \tau^2 \rho^{11} + (\lambda^2_{11} \lambda^2_3 + \lambda^2_5 \lambda^2_9) \tau^2 \rho^{12} + (\lambda^2_8 \lambda^2_7 + \lambda^2_7 \lambda^2_8 + \lambda^2_6 \lambda^2_9) \tau \rho^{13};$$  

with this choice of $u$, we have $\delta(\lambda^2_1 \tau^8 + u) = \lambda^2_2 \lambda^2_0 \tau^{13} + O(\rho^{14}).$

(8) Here, we are claiming $\delta(\lambda^2_1 \tau^3 + \langle \rangle) = \lambda^2_2 \lambda^2_3 \tau^3 + \langle \rangle$. Indeed, let

$$u = (\lambda^2_3 \lambda^4 + \lambda^2_4 \lambda^2_3) \tau^2 \rho + \lambda^2_3 \lambda^2_5 \tau^2 \rho^2 + \lambda^2_4 \lambda^2_5 \tau^3;$$  

then we have $\delta(\lambda^2_1 \tau^3 + u) = \lambda^2_2 \lambda^2_3 \tau^3 + O(\rho^4).$

(11) Here, we are claiming $\delta(\lambda^2_1 \tau^8 + \langle \rangle) = \lambda^2_5 \lambda^2_7 \tau^2 \rho^{12}$. Indeed, let

$$u = \lambda^2_9 \lambda^2_7 \tau^4 \rho^8 + \lambda^2_9 \lambda^2_1 \tau^2 \rho^{12};$$  

then we have $\delta(\lambda^2_1 \tau^8 + u) = \lambda^2_3 \lambda^2_5 \tau^2 \rho^{12} + O(\rho^{14}).$

(15) Here, we are claiming $\delta(\lambda^2_1 \tau^{13} + \langle \rangle) = \lambda^2_3 \tau^2 \rho^2 + \langle \rangle$. Indeed, let

$$u = \lambda^2_7 \lambda^2_2 \tau^2 \rho + (\lambda^2_9 \lambda^2_1 + \lambda^2_5 \lambda^2_3) \tau^2 \rho^2;$$  

then we have $\delta(\lambda^2_1 \tau^{13} + \langle \rangle) = \lambda^2_3 \tau^2 \rho^2 + O(\rho^3).$

(17) Here, we are claiming $\delta(\lambda^2_1 \tau^{12} + \langle \rangle) = \lambda^2_3 \tau^5 \rho^{14} + \langle \rangle$. Indeed, let

$$u = \lambda^2_3 \lambda^2_9 \tau^6 + (\lambda^2_1 \lambda^2_4 + \lambda^2_4 \lambda^2_3 + \lambda^2_7 \lambda^2_8 + \lambda^2_8 \lambda^2_7) \tau^8 \rho^7 + \lambda^2_2 \tau^9 \rho^6 + \lambda^2_9 \lambda^2_7 \tau^8;$$  

then we have $\delta(\lambda^2_1 \tau^{12} + u) = \lambda^2_3 \tau^5 \rho^{14} + O(\rho^{15}).
(18) Here, we are claiming \( \delta(\lambda_1^2 \lambda_2^4 + \langle \rangle) = \lambda_2^3 \tau \rho^6 + \langle \rangle \). Indeed, let
\[
\tau = (\lambda_1^2 \lambda_3 + \lambda_{11} \lambda_{11}) \tau \rho^6;
\]
then we have \( \delta(\lambda_1^2 \lambda_2^4 + \langle \rangle) = \lambda_2^3 \tau \rho^6 + O(\rho^7) \).
(21) Here, we are claiming \( \delta(\lambda_2^2 \lambda_2^4 \tau^{2k+1}(2m+1) + \langle \rangle) = \lambda_2^2 \lambda_{2+1} \tau^{2k-1}(8m+1) \rho^2 \lambda_{2} + \langle \rangle \), at least provided \( b - 2 \geq c \geq 1 \). This case is intended to illustrate all the remaining cases, and is identical in form to case (2) of Proposition 4.2.7. Recall from Proposition 4.2.7 that
\[
\delta(\lambda_2^2 \lambda_2^4 \tau^{2k+1}(2m+1) + O(\rho)) = \lambda_2^2 \lambda_{2+1} \tau^{2k-1}(8m+1) \rho^2 \lambda_{2} + O(\rho^2)\lambda_{2-1}.
\]
As
\[
\lambda_2^2 \lambda_2^4 \tau^{2k+1}(2m+1) \equiv \lambda_2^2 \lambda_{2+1} \tau^{2k+1}(2m+1) \lambda_{2-1} \pmod{\rho},
\]
it follows that
\[
\delta(\lambda_2^2 \lambda_2^4 \tau^{2k+1}(2m+1) + O(\rho)) = \delta(\lambda_2^2 \lambda_2^4 \tau^{2k-1}(2m+1) \lambda_{2-1} + O(\rho))
\]
\[
= (\lambda_2^2 \lambda_{2+1} \tau^{2k-1}(8m+1) \rho^2 \lambda_{2-1} + O(\rho^2))\lambda_{2-1}
\]
\[
= \lambda_2^2 \lambda_{2+1} \lambda_{2-1} \tau^{2k-1}(8m+1) \rho^2 \lambda_{2-1} + O(\rho^2)\lambda_{2-1},
\]
which gives the desired relation. The remaining cases are either identical in form to this, or simpler in that they do not require one to first move \( \tau \) around to reduce to a case already considered in Proposition 4.2.7.

This produces the indecomposable classes
\[
\tau^{2a-1}(2(16a+1)+1)h_{a+3}^3, \quad \tau^{2a-1}(4(4a+1)+1)h_{a+3}^3,
\]
\[
\tau^{2a-1}(16a+1)c_{a}, \quad \tau^{2a-1}(8a+1)c_{a+1}, \quad \tau^{2a-1}(2(4a+1)+1)c_{a},
\]
for \( a, n \geq 0 \), following the same recipe as employed in Definition 4.2.6 and Definition 4.2.9, only where one must employ \( \theta \)-iterates of \( \tau \)-multiples of the correction terms \( u \) given in Proposition 4.2.10.

Proposition 4.2.10 concludes the work necessary for our computation of the \( \mathbb{F}_2[\rho] \)-module structure of \( \text{Ext}_R^2 \). Let us now summarize in one theorem what we have learned. We wish to give a minimal generating set of \( \text{Ext}_R^2 \) whose elements are products of the indecomposable classes we have found. Before doing so, let us treat the following subtlety.

By way of example, let \( x = \frac{1}{\rho^2} \delta(\lambda_2^2 \lambda_{2} \tau^{2k-1}(4m+1)) \), and let \( \alpha \in \text{Ext}_R \) be the class represented by \( x \). Our computation in Proposition 4.2.7 combined with the result of Theorem 4.1.4 would yield \( \alpha \) as an element of a minimal generating set for \( \text{Ext}_R \). Observe that \( x \) has leading term \( \lambda_2^2 \lambda_{2} \tau^{2k-1}(4m+1) \).

It follows quickly from this that \( x \) has the same leading term as the cocycle representative of \( (\tau^{2k-1}(4m+1))h_{b}^2 \) given by the product of those cocycle representatives for \( \tau^{2k-1}(4m+1)h_{b} \) given in Definition 4.2.6. However, this does not prove that \( \alpha = (\tau^{2k-1}(4m+1))h_{b}^2 \); we have not ruled out the possibility that \( \alpha + \beta = (\tau^{2k-1}(4m+1))h_{b}^2 \) for some nonzero \( \beta \) represented by a cycle \( y < \lambda_2^2 \lambda_{2} \tau^{2k-1}(4m+1) \).

This is still sufficient to deduce that we may, if necessary, replace \( \alpha \) with \( \alpha + \beta \) in our minimal generating set in order to obtain a minimal generating set built as products of indecomposables. It turns out that no such correction is necessary.

Lemma 4.2.11. Write \( \phi: \text{Ext}_R \to \text{Ext}_C \) for the quotient. Fix classes \( \alpha, \beta \in \text{Ext}_R^1 \) or \( \text{Ext}_R^2 \), at least one of which is \( \rho \)-torsion, and not both in \( \text{Ext}_R^2 \). Let \( r \) be minimal for which \( \rho^r \alpha = 0 \) or \( \rho^r \beta = 0 \). Fix \( \gamma \in \text{Ext}_R^2 \) not divisible by \( \rho \) and such that \( \rho^r \gamma = 0 \), and suppose \( \phi(\alpha) \cdot \phi(\beta) = \phi(\gamma) \). Then \( \alpha \cdot \beta = \gamma \).

Proof. Under the given conditions, there is in fact a unique class in the degree of \( \alpha \cdot \beta \) which is not divisible by \( \rho \) and is killed by \( \rho^r \). This may be seen by direct inspection of the propositions preceding this.

We may now state the main theorem of this section.
Theorem 4.2.12. (1) A minimal multiplicative generating set for $\text{Ext}_{R}^{3}$ as an $\mathbb{F}_2[\rho]$-algebra is given by the given by the classes in the following table:

| Multiplicative generator | $\rho$-torsion exponent |
|--------------------------|-------------------------|
| $h_{a+1}$                | $\infty$               |
| $c_{a+1}$                | $\infty$               |
| $\tau^{2a-1}(4n+1)h_{a}$ | $2^a$                  |
| $\tau^{2a}(8n+1)h_{a+2}$ | $2a+1 \cdot 3$        |
| $\tau^{2a-1}(2(16n+1)+1)h_{a+3}^2h_a$ | $2^a \cdot 7$ |
| $\tau^{2a-1}(4(4n+1)+1)h_{a+3}^3$ | $2^a \cdot 7$ |
| $\tau^{2a-1}(16n+1)c_a$ | $2^a \cdot 3$.         |
| $\tau^{2a-1}(2(4n+1)+1)c_a$ | $2^a \cdot 3$.         |

Here, $a, n \geq 0$, we write $2^{a-1}(4n+1) = 2n$ for $a = 0$, and the $\rho$-torsion exponent of a class $\alpha$ is the minimal $r$ for which $\rho^r \alpha = 0$; the classes $h_{a+1}$ and $c_{a+1}$ are $\rho$-torsion-free.

(2) The operation $\text{Sq}^1$ acts on these classes by incrementing $a$ in each row.

(3) The image of these classes under $\text{Ext}_{R}^{3} \to \text{Ext}_{C}^{3}$ is as their name suggests.

(4) A minimal $\mathbb{F}_2[\rho]$-module generating set for $\text{Ext}_{R}^{3}$ is given in the following table. In all cases, the $\rho$-torsion exponent of a given class is the minimal $\rho$-torsion exponent of the multiplicative generators it is written as a product of.

| $\mathbb{F}_2[\rho]$-module generator | Constraints |
|-----------------------------------------|-------------|
| $h_a$                                   | $a \geq 1$  |
| $\tau^{2a-1}(4n+1)h_a$                  | $a, n \geq 0$ |
| $h_a \cdot h_b$                         | $a \geq b \geq 1$ and $a \neq b + 1$ |
| $h_a \cdot \tau^{2a-1}(4n+1)h_b$       | $a \geq 1$ and $b, n \geq 0$, and $a \neq b \pm 1$ |
| $\tau^{2a-1}h_a \cdot \tau^{2a-1}(4n+1)h_a$ | $a, n \geq 0$ |
| $\tau^{2a}(8n+1)h_{a+2}$               | $a, n \geq 0$ |
| $h_a \cdot h_b \cdot h_c$              | $a \geq b \geq c \geq 1$, where $a \neq b + 1$ and $b \neq c + 1$, and where if $b = c$ or $a = b$ then $a \neq c + 2$ |
| $h_a \cdot h_b \cdot \tau^{2a-1}(4n+1)h_c$ | $a \geq b \geq 1$ and $c, n \geq 0$, with $a \neq b + 1$ and $c \notin \{a \pm 1, b \pm 1\}$, and if $a = b$ then $c \notin \{a - 2, a, a + 2\}$, and if $a = b + 2$ then $c \neq a$ |
| $h_a \cdot \tau^{2a-1}h_b \cdot \tau^{2a-1}(2n+1)h_a$ | $a \geq 1$ and $b, n \geq 0$, and $a \notin \{b - 2, b - 1, b + 1\}$ |
| $h_a \cdot \tau^{2a-1}h_b \cdot h_0$    | $a \geq 1$ and $b, n \geq 0$, and $a \notin \{b - 2, b - 1, b + 1\}$ |
| $h_a \cdot \tau^{2a-1}h_b \cdot h_0$    | $a \geq 1$ and $b, n \geq 0$, and $a \notin \{b - 2, b - 1, b + 1\}$ |
| $h_a \cdot \tau^{2a}(8n+1)h_{a+2}$     | $a \geq 1$ and $b, n \geq 0$, and either $a \leq b - 1$ or $a \geq b + 4$ |
| $\tau^{2a-1}(2(16n+1)+1)h_{a+3}^2h_a$ | $a, n \geq 0$ |
| $\tau^{2a-1}(4(4n+1)+1)h_{a+3}^3$     | $a, n \geq 0$ |
| $c_a$                                   | $a \geq 1$ |
| $c_a$                                   | $a \geq 1$ |
| $c_a$                                   | $a \geq 1$ |
| $c_a$                                   | $a \geq 1$ |
| $c_a$                                   | $a \geq 1$ |

Proof. All of this may be read off the preceding computations, using Lemma 4.2.11 with Proposition 4.2.1 if necessary to write a given class as a product of classes in the given generating set. 

We point out the following corollary.
Theorem 7.4.9. Corollary 4.2.13.

Remark 4.2.14. As indicated in Remark 4.1.6, one may also read off our computation a description of all differentials in the $\rho$-Bockstein spectral sequence $\text{Ext}_C[\rho] \Rightarrow \text{Ext}_R$ emanating out of filtration at most 2. We leave this to the interested reader.

4.3. Multiplicative structure. We now compute the multiplicative structure of $\text{Ext}^{\leq 3}_R$. This material is mostly not needed for our study of the 1-line of the motivic Adams spectral sequence in Section 7; the exception is that we will use the relation Proposition 4.3.4(4) in the proof of Theorem 7.4.9.

Already Lemma 4.2.11 produces a large number of relations. For example, it implies that we may always shift powers of $\tau$ around in products that do not vanish in $\text{Ext}_C$, provided it makes sense to do so, yielding relations such as

$$\tau^{2a-1}(4n+1)h_a \cdot \tau^{2b-1}(4m+1)h_b = h_a \cdot \tau^{2a-1}(4m+2n+1-4(n+1)+1)h_b$$

for $a \geq b + 2$. These were implicitly used in the proof of Theorem 4.2.12. The condition that the product does not vanish in $\text{Ext}_C$ is necessary; see Example 4.3.3 below.

We are left only with relations that would be realized as hidden extensions in the $\rho$-Bockstein spectral sequence. These arise from the possible failure of the relations $h_{a+1}h_a = 0$ and $h_{a+2}^2h_a = 0$ to lift through $\text{Ext}_R \Rightarrow \text{Ext}_C$.

Remark 4.3.1. The following computations will involve some explicit calculations with cocycle representatives. For ease of reference, we list some important cocycle representatives here.

1. $h_0, h_1, h_2, h_3$ are represented by $\lambda_0, \lambda_1, \lambda_3, \lambda_7$;
2. $c_0$ and $c_1$ are represented by $\lambda_2\lambda_3^2$ and $\lambda_5\lambda_4^2$;
3. $\tau^{2a-1}h_a$ is represented by $\rho^2 \cdot \delta(\tau^{2a}) = \delta(\rho^2(\lambda_0)) = \tau^{2a-1}h_{a-1} = \lambda_2\lambda_{a-1}^2 + O(\rho^{2a-1})$;
4. $\tau^2h_4$ is represented by $\lambda_0\tau^2 + \lambda_3\tau\rho + \lambda_7\rho^3 + \lambda_4\rho^4$;
5. $\tau^2h_5$ is represented by $\lambda_2\tau + (\lambda_3\lambda_4 + \lambda_4\lambda_3)\rho = \tau\lambda_3^2$;
6. $\tau^2h_5$ is represented by $\lambda_2^2\tau + (\lambda_3\lambda_4 + \lambda_4\lambda_3)\rho = \tau\lambda_3^2$;
7. $\tau^2h_5$ is represented by $\lambda_2^2\tau + (\lambda_3\lambda_4 + \lambda_4\lambda_3)\rho = \tau\lambda_3^2$.

We will use these without further comment.

We begin with some products in $\text{Ext}^{\leq 3}_R$ which lift the relation $h_{a+1}h_a = 0$.

Proposition 4.3.2. The following hold.

1. $h_{a+1} \cdot \tau^{2n-1}(4(2n+1)+1)h_a = \rho^{2a} \cdot \tau^{2n}h_{a+1} \cdot \tau^{2a-1}(4n+1)h_a$;
2. $h_{a+1} \cdot \tau^{2n-1}(8n+1)h_a = 0$;
3. $\tau^{2n}(4n+1)h_{a+2} \cdot h_{a+1} = \rho^{2a+1} \cdot \tau^{2n}(8n+1)h_{a+2}^2$;
4. $\tau^{2n}(8n+1)h_{a+2} \cdot h_{a+1} = \rho^{2a} \cdot \tau^{2n-1}(8n+1)h_a$;
5. $\tau^{2n}(8n+1)h_{a+2} \cdot \tau^{2n}(4m+1)h_{a+1} = \rho^{2a} \cdot \tau^{2n-1}(2(4m+2n)+1)h_a$;
6. $h_{a+3} \cdot \tau^{2n}(16n+1)h_{a+2} = 0$;
7. $h_{a+3} \cdot \tau^{2n}(8(2n+1)+1)h_{a+2}^2 = \rho^{2a+3} \cdot \tau^{2n}(4(4n+1)+1)h_{a+2}^2$.

Proof. In each of these, we may use $\text{Sq}^0$ to reduce to the case $a = 0$. In all cases where the product does not vanish, the claimed value of the product is the unique nonzero class in its degree which is both $\rho$-torsion and divisible by $\rho$, so it suffices to verify the product working modulo the smallest power of $\rho$ in which the claimed value does not vanish. In doing so, we may in each case reduce to $n = m = 0$. With these reductions in place, the proposition is proved by the following computations.

1. Here, we are claiming $h_1 \cdot \tau h_0 = \rho \cdot \tau h_1 \cdot \tau h_1$. Indeed, we may compute

$$\rho^2(\lambda_0\tau^2 + \lambda_3\tau\rho^3 + \lambda_4\rho^4) \cdot \lambda_1 = \lambda_1^2\tau^2 \rho + \lambda_3\lambda_3\tau\rho^2 + \lambda_2\lambda_2\rho^3 + \lambda_3\lambda_3\rho^4 = \rho^2(\lambda_1\tau + \lambda_2\rho^2) = \rho^2(\lambda_0)^2,$$
which represents $\rho \cdot \tau h_1 \cdot \tau h_1$.

(2) There are no nonzero $\rho$-torsion classes in this degree, so the product must vanish.

(3) Here, we are claiming $h_1 \cdot \tau^2 h_2 = \rho^2 \cdot \tau h_2$. Indeed, we may compute

$$\lambda_1 \cdot \tau^2 \lambda_3 = \rho^2 \cdot \tau \lambda_3^2$$
on the nose.

(4) Here, we are claiming $h_1 \cdot \tau h_2^2 = \rho \cdot c_0$. Indeed, we may compute

$$\lambda_1 \cdot \tau \lambda_3^2 = \lambda_2 \lambda_3^2 \rho$$
on the nose.

(5) Here, we are claiming $\tau h_1 \cdot \tau h_2 = \rho \cdot \tau c_0$. For this, it suffices to work mod $\rho^2$. Here we may compute

$$\tau \lambda_1 \cdot \tau \lambda_2^2 = \rho \cdot \lambda_2 \lambda_3^2 \tau + O(\rho^2),$$

and the claim follows.

(6) Here, we have reduced to $a = 0$ but not yet to $n = 0$. The only nonzero $\rho$-torsion class in this degree is $\rho^6 \tau^{16n+1} c_1$, so it suffices to work mod $\rho^7$. In doing so, we may now reduce to $n = 0$. Indeed, we have

$$\tau \lambda_3^2 \cdot \lambda_7 = 0,$$

and the claim follows.

(7) Here, we are claiming $h_3 \cdot \tau^3 h_2^2 = \rho^8 \cdot \tau^5 h_3^3$. For this, it suffices to work mod $\rho^9$. Here, we may compute

$$(\lambda_3^2 \tau^9 + (\lambda_3 + \lambda_3 \lambda_3) \tau^8 \rho + \lambda_5 \lambda_3 \tau^8 \rho^2) \cdot \lambda_7 = \lambda_3^2 \tau^5 \rho^8 + O(\rho^9),$$

yielding the claim. $\square$

**Example 4.3.3.** We have

$$\tau^2 h_2 \cdot h_2^2 = \rho^3 c_0, \quad h_2 \cdot (\tau h_1)^2 = 0.$$  

This serves as a warning that one cannot in general freely shift around powers of $\tau$ in products. $\triangleleft$

We now give some products that lift the relation $h_{a+2}^2 \cdot h_a = 0$.

**Proposition 4.3.4.** The following hold.

1. $h_{a+2}^2 \cdot \tau^{2^a-1}(16n+1) h_a = 0$;
2. $h_{a+2}^2 \cdot \tau^{2^a-1}(2(4n+1)+1) h_a = \rho^{2^{a+1}} \cdot \tau^{2^a-1}(2(4n+1)+1) c_a$;
3. $h_{a+2}^2 \cdot \tau^{2^a-1}(8(2n+1)+1) h_a = \rho^{2^{a+2}} \cdot \tau^{2^a+1} h_{a+2} \cdot \tau^{2^a}(8n+1) h_{a+2}^2$;
4. $\tau^{2^{a+1}} h_{a+3} \cdot \tau^{2^{a+1}(4n+1)} h_{a+3} \cdot h_{a+3} = \rho^{2^{a+1}} \cdot \tau^{2^{a+1}(4n+1)+1} h_{a+3}$;
5. $h_{a+1} \cdot h_{a+3} \cdot \tau^{2^{a+2}} (4n+1) h_{a+3} = \rho^{2^{a+2}} \cdot \tau^{2^{a+1} (8n+1)} c_{a+1}$;
6. $\tau^{2^{a+1} (8+1)} h_{a+3}^2 \cdot h_{a+3} = 0$.

**Proof.** As in the proof of Proposition 4.3.2, we may use $\text{Sq}^0$ to reduce to the case $a = 0$, and in all cases where the product does not vanish may reduce to $n = 0$. With these reductions in place, the proposition is proved by the following computations.

1. There are no nonzero $\rho$-torsion classes in this degree, so the product must vanish.
2. Here, we are claiming $h_2^2 \cdot \tau^2 h_0 = \rho^2 \cdot \tau c_0$. For this, it suffices to work mod $\rho^3$. Recall that $\tau^2 h_0$ is represented by $\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho + O(\rho^3)$. We may compute

$$\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho \cdot \lambda_2^2 = \rho^2 \cdot \lambda_2 \lambda_3^2 \tau + O(\rho^3),$$

and the claim follows.
3. Here, we are claiming $h_2^2 \cdot \tau^4 h_0 = \rho^3 \cdot \tau^2 h_2 \cdot \tau h_2^2$. For this, it suffices to work mod $\rho^4$. Observe that

$$h_2 \cdot h_2 \cdot \tau^4 h_0 = h_2 \cdot \tau^2 h_2 \cdot \tau^2 h_0 = \tau^2 h_2 \cdot h_2 \cdot \tau^2 h_0 = \tau^2 h_2 \cdot \tau^2 h_2 \cdot h_0$$
The second term vanishes, as we may compute
\[ \lambda_0 \cdot \tau^2 \lambda_3 \cdot \tau^2 \lambda_3 = \rho^3 \cdot \lambda_3^3 \tau^3 + O(\rho^4), \]
yielding the claim.

(4) Here, we are claiming \( \tau^4 h_3 \cdot \tau^4 h_3 \cdot h_1 = \rho^6 \cdot \tau^5 h_3^3 \). For this, it suffices to work mod \( \rho^7 \). Here, we may compute
\[ \lambda_1 \cdot \tau^4 \lambda_7 \cdot \tau^4 \lambda_7 = \rho^6 \cdot \lambda_3^2 \tau^5 + O(\rho^7), \]
yielding the claim.

(5) Here, we are claiming \( h_1 \cdot h_3 \cdot \tau^4 h_3 = \rho^4 \cdot \tau^2 c_1 \). For this, it suffices to work mod \( \rho^5 \). Here, we may compute
\[ \lambda_1 \cdot \tau^4 \lambda_7 \cdot \lambda_7 = \rho^4 \cdot \lambda_3 \lambda_7^2 + O(\rho^5), \]
yielding the claim.

(6) There are no nonzero \( \rho \)-torsion classes in this degree, so the product must vanish. \( \square \)

The preceding propositions leave open three families of products. We have managed to only partially pin these down; a full computation would be resolved by a positive solution to the following conjecture.

**Conjecture 4.3.5.** There are relations
\[
\tau^{4m+1} h_1 \cdot \tau^{2l} h_0 = \tau^2 h_1 \cdot \tau^{2(2m+l)} h_0 \\
\tau^{4(4m+1)} h_3 \cdot \tau^{8l+1} h_2^2 = \tau^4 h_3 \cdot \tau^{8(2m+1)+1} h_2^2 \\
\tau^{8m+1} h_2^2 \cdot \tau^{4l} h_0 = \tau h_2^2 \cdot \tau^{2(2m+l)} h_0.
\]

From here, we have the following.

**Proposition 4.3.6.** Write \( 2m + l + 1 = 2^k(2n + 1) \). Assuming Conjecture 4.3.5, the following hold.

1. \( \tau^{2(4m+1)} h_{a+1} \cdot \tau^{2-1(4l+1)} h_a = \rho^{2^k(2^{k+1}-1)} \cdot h_{a+1} \cdot \tau^{2^{k+2}(4n+1)} h_{a+k+1} \);
2. \( \tau^{2^{k+2}(4m+1)} h_{a+3} \cdot \tau^{2^{k+8}(4l+1)} h_{a+2} = \rho^{2^{k+1}(2^{k+2}-3)} \cdot h_{a+1} \cdot h_{a+3} \cdot \tau^{2^{k+2}(4n+1)} h_{a+k+3} \);
3. \( \tau^{2^{k+8}(4m+1)} h_{a+2} \cdot \tau^{2^{k+1}(4l+1)} h_a = \rho^{2^{k+1}(2^{k+1}-1)} \cdot h_{a+2} \cdot \tau^{2^{k+2}(4n+1)} h_{a+k+1} \).

These hold unconditionally when \( m = 0 \) or when \( l \) is even.

**Proof.** In each of these, we may use \( \text{Sq}^0 \) to reduce to the case \( a = 0 \). By working modulo the smallest power of \( \rho \) in which the claimed product does not vanish, we may reduce to the case \( n = 0 \). If \( l \) is even, then this also allows us to reduce to the case \( m = 0 \); otherwise, Conjecture 4.3.5 allows us to reduce to the case \( m = 0 \). The proposition is now proved by the following computations.

1. Here, we are claiming \( \tau h_1 \cdot \tau^{2(k+1)-1} h_0 = \rho^{2^{k+1}-1} \cdot h_1 \cdot \tau^{2} h_{k+1} \). Recall that \( \tau^{2(k+1)-1} h_0 \) is represented by \( \rho^{-1} \delta(\tau^{2(k+1)-1}) \). Now, the Leibniz rule implies
\[ \rho^{-1} \delta(\tau^{2(k+1)-1}) \cdot \tau \lambda_3 = \rho^{-1} \delta(\tau^{2(k+1)-1}) \cdot \lambda_1 + \rho^{-1} \tau^{2(k+1)-1} \cdot \delta(\tau) \cdot \lambda_1. \]
The second summand vanishes, as \( \delta(\tau) \cdot \lambda_1 = \rho \lambda_0 \cdot \lambda_1 = 0 \). The first summand represents \( \rho^{2^{k+1}-1} \cdot \tau^{2} h_{k+1} \cdot h_1 \), yielding the claimed relation.

2. Here, we are claiming \( \tau h_3 \cdot \tau^{8(k+1)+1} h_3^2 = \rho^{2^{k+2}-3} \cdot h_1 \cdot h_3 \cdot \tau^{2^{k+2}} h_{k+3} \). Recall that \( \tau^{8(k+1)+1} h_3^2 \) is represented by \( \rho^{-6} \delta(\lambda_1 \tau^{8(k+3)-1}) \). Now, the Leibniz rule implies
\[ \rho^{-6} \delta(\lambda_1 \tau^{8(k+3)-1}) \cdot \tau \lambda_7 = \rho^{-6} \lambda_1 \cdot \delta(\tau^{k+3}) \cdot \tau \lambda_7 + \rho^{-6} \lambda_1 \cdot \tau^{8(k+3)-1} \cdot \delta(\tau) \cdot \lambda_7. \]
The second term vanishes, as \( \delta(\tau) \cdot \lambda_3 = \tau^2 \lambda_3 \cdot \lambda_7 = 0 \). The first term represents \( \rho^{2^{k+2}-3} \cdot h_1 \cdot h_3 \cdot \tau^{2^{k+2}} h_{k+3} \), yielding the claimed relation.
(3) Here, we are claiming \( \tau h_2^2 \cdot \tau^{2(2^k - 1)} h_0 = \rho^{2k+1-1} \cdot h_2^2 \cdot \tau^2 h_{k+1} \). Recall that \( \tau^{2(2^k - 1)} h_0 \) is represented by \( \rho^{-1} \delta(\tau^{2(2^k - 1)+1}) \). Now, the Leibniz rule implies
\[
\rho^{-1} \delta(\tau^{2(2^k - 1)+1}) \cdot \lambda^2_3 = \rho^{-1} \delta(\tau^{2k+1}) \cdot \lambda^2_3 + \rho^{-1} \tau^{2(2^k - 1)+1} \cdot \delta(\tau) \cdot \lambda^2_3.
\]
The second term vanishes, as \( \delta(\tau) \cdot \lambda^2_3 = \rho \lambda_0 \cdot \lambda^2_3 = 0 \). The first summand represents \( \rho^{2k+1-1} h_{k+1} \cdot h^2_2 \), yielding the claimed relation. \( \Box \)

The relations above suffice to write any product in \( \text{Ext}^{\leq 3}_F \) in terms of the basis given in Theorem 4.2.12.\(^8\) Thus we have the following.

**Theorem 4.3.7.** Modulo Conjecture 4.3.5, a full set of relations for \( \text{Ext}^{\leq 3}_F \) is given by those visible relations which may be deduced from Lemma 4.2.11, together with the products listed in Proposition 4.3.2, Proposition 4.3.4, and Proposition 4.3.6. \( \Box \)

5. Some homotopical preliminaries

With the algebraic computation of Section 4 out of the way, we now proceed to more homotopical considerations. This brief section collects a couple constructions that will be used in the following sections. Explicitly, Section 5.1 will be used in our computation of \( d_2(h_0) \) in Section 7, and Section 5.3 will be used in our discussion of the unstable Hopf invariant one problem in Section 6.

5.1. The Hurewicz map. The constant functor \( c : \text{Sp}^{cl} \to \text{Sp}^F \) has a lax symmetric monoidal right adjoint \( c^* \), described by
\[
c^*(X) = \text{Sp}^F(S^{0,0}, X).
\]
In particular, the unit of \( c^*(S^{0,0}) \) gives a ring map
\[
S^0 \to c^*(S^{0,0}),
\]
and on homotopy groups this yields a Hurewicz map
\[
c : \pi_*^{cl} \to \pi_*^F.
\]

**Proposition 5.1.1.** For any \( F \), there is map
\[
c : \text{Ext}^{*,f}_{cl} \to \text{Ext}^{*,f,0}_F
\]
of multiplicative spectral sequences, converging to the Hurewicz map
\[
c : \pi_*^{cl} \to \pi_*^F.
\]
Moreover, \( c \) commutes with \( \text{Sq}^0 \) and satisfies \( c(h_0) = h_0 + \rho h_1 \).

**Proof.** Write \( HF_2 \) for the ordinary mod 2 Eilenberg-MacLane spectrum and \( HF^F_2 \) for the motivic spectrum representing mod 2 motivic cohomology. Then \( c^*(HF^F_2) = HF_2 \), thereby giving maps
\[
HF^F_2 \otimes_n \simeq c^*(HF^F_2) \otimes_n \to c^*((HF^F_2) \otimes_n).
\]
Thus there is a map from the canonical Adams resolution of the sphere to the restriction along \( c^* \) of the canonical Adams resolution of the \( F \)-motivic sphere. On homotopy groups, this gives a map from the cobar complex of \( A^{cl} \) to the weight 0 portion of the cobar complex of \( A^F \), and passing to homology we obtain a map
\[
\text{Ext}^{*,f}_{cl} \to \text{Ext}^{*,f,0}_F
\]
which is multiplicative and commutes with \( \text{Sq}^0 \), and by construction this is a map of spectral sequences converging to the Hurewicz map. That \( c(h_0) = h_0 + \rho h_1 \) follows as these are the classes detecting 2 (see for instance [IO19, Remark 6.3]). \( \Box \)

\(^8\)Not directly; for example, case (2) of Proposition 4.3.6 with \( k = 0 \) must be combined with case (5) of Proposition 4.3.4.
5.2. **The Lefschetz principle.** The Lefschetz principle asserts, informally, that “everything” which is true over \( \mathbb{C} \) is true over any algebraically closed field. In this subsection, we note how one may read off a certain motivic Lefschetz principle from work of Wilson–Østvær [WØ17].

In this paper, we have primarily been concerned with \( F \)-motivic homotopy theory for \( F \) a field of characteristic not equal to 2. For this subsection, we extend our notation to apply also when \( F \) is some ring in which 2 is invertible. We shall write \( S^0_F \) for the \( HF^2_F \)-nilpotent completion of the \( F \)-motivic sphere spectrum. When \( F \) is a field, this is the \((2, \eta)\)-completion of the \( F \)-motivic sphere spectrum, and when \( F \) is an algebraically closed field, this reduces to a 2-completion [HKO11a, KW19]. Let \( S^F_{cl,cell} \subset S^F_2 \) denote the cellular subcategory, i.e. the category generated by the spheres \( S^{a,b} \) under colimits.

**Proposition 5.2.1.** Let \( F \) be an algebraically closed field. Then there is an equivalence
\[
S^F_{cl,cell} \simeq S^C_{cl,cell}.
\]
Moreover, this is compatible on Adams spectral sequences with the isomorphism \( \text{Ext}_F \cong \text{Ext}_C \).

**Proof.** First suppose that \( F \) is of odd characteristic \( p \). We follow the methods of [WØ17, Section 6].

Let \( W(F) \) be the ring of Witt vectors on \( F \), and choose an algebraically closed field \( L \) of characteristic 0 together with embeddings
\[
\mathbb{C} \rightarrow L \leftarrow W(F) \rightarrow F.
\]

This gives rise to base change functors
\[
S^F \rightarrow S^L \leftarrow S^{W(F)} \rightarrow S^F,
\]
and in particular maps
\[
\pi^{C}_{*,*} \rightarrow \pi^{L}_{*,*} \leftarrow \pi^{W(F)}_{*,*} \rightarrow \pi^{F}_{*,*}.
\]
(5)

Although \( W(F) \) is not a field, Wilson–Østvær show that its Steenrod algebra and Adams spectral sequence are still well-behaved, and [WØ17, Corollary 6.3] shows that the above maps are modeled on motivic Adams spectral sequences by a zigzag of isomorphisms
\[
\text{Ext}_C \rightarrow \text{Ext}_L \leftarrow \text{Ext}^{W(F)} \rightarrow \text{Ext}_F.
\]

It follows that Eq. (5) is a zigzag of isomorphisms.\(^9\) In particular, consider the zigzag
\[
S^C_{cl,cell} \rightarrow S^L_{cl,cell} \leftarrow S^{W(F)}_{cl,cell} \rightarrow S^F_{cl,cell}.
\]

This is a zigzag of colimit-preserving functors of compactly generated stable categories which are equivalences on subcategories of compact generators, and is thus a zigzag of equivalences. This yields the canonical equivalence \( S^C_{cl,cell} \cong S^F_{cl,cell} \).

If \( F \) is of characteristic zero, then we may apply the same argument instead to a zigzag of the form
\[
\mathbb{C} \rightarrow L \leftarrow F
\]
with \( L \) algebraically closed. \( \square \)

5.3. **Betti realization.** If \( X \) is a smooth scheme over \( \mathbb{C} \), then the space of complex points of \( X \) is a complex manifold. This refines to give Betti realization functors [MV99] from \( \mathbb{C} \)-motivic spaces to ordinary spaces, and from \( \mathbb{C} \)-motivic spectra to ordinary spectra, with a number of nice properties. We may use the Lefschetz principle of **Proposition 5.2.1** to obtain an analogue for an arbitrary algebraically closed field \( F \).

Let \( S^0 \) denote the 2-completed sphere spectrum, and \( S^F_2 \) the category of modules thereover.

\(^9\)The restriction on degrees in [WØ17, Theorem 6.6] concerns when \( \pi^{F}_{*,*} \) agrees with the algebraic 2-completion of the homotopy groups of the non-completed \( F \)-motivic sphere, and is not relevant here.
Proposition 5.3.1. Let $F$ be an algebraically closed field. Then there is a symmetric monoidal “Betti realization” functor
\[
\text{Be}: \text{Sp}_2^{\text{cell}} \to \text{Sp}_2^{\text{cl}},
\]
factoring through an equivalence from the category of modules over $S^0.0[\tau^{-1}]$ in $\text{Sp}_2^{\text{cell}}$ to $\text{Sp}_2^{\text{cl}}$, with the following properties.

1. $\text{Be}(\tau) = 1$. In particular, $\text{Be}(S^{a,b}) = S^a$, so that $\text{Be}$ induces a map $\pi_{s,w}^F \to \pi_{s,w}^{\text{cl}}$, and these patch together to an isomorphism $\pi_{s,w}^F[\tau^{-1}] \cong \pi_{s,w}^F[\tau^{\pm 1}]$.
2. The above isomorphism is modeled on Adams spectral sequences by the map $\text{Ext}_F \to \text{Ext}_F[\tau^{-1}] \cong \text{Ext}_G[\tau^{\pm 1}]$.
3. The composite $\text{Be} \circ c: \text{Sp}_2^{\text{cl}} \to \text{Sp}_2^{\text{cell}} \to \text{Sp}_2^{\text{cl}}$ is an equivalence. In particular, the map $c: \text{Ext}_F \to \text{Ext}_F$ of Proposition 5.1.1 extends to an equivalence $\text{Ext}_{\tau^{\pm 1}}[\tau^{-1}] \to \text{Ext}_F[\tau^{-1}]$.

Proof. These facts are known of the Betti realization functor for $\mathbb{A}^1$-algebraically closed fields. Then there is a symmetric monoidal $\text{Be}: \text{Sp}_2^{\text{cell}} \to \text{Sp}_2^{\text{cl}}$, as can be seen by inspection of homotopy groups. Let $\text{Spd}_\infty^F$ be the category of $F$-motivic spaces and $\text{Spd}_\infty^2$ be the category of 2-complete spaces.

Proposition 5.3.2. Let $F$ be an algebraically closed field, and define
\[
\text{Be}: \text{Spd}_\infty^F \to \text{Spd}_\infty^2, \quad \text{Be}(X) = \text{CAlg}_{H_{F_2}}(\text{Be}((H_{F_2}^F)^{X^+}), \mathbb{F}_2).
\]
Then $\text{Be}(S^{a,b}) = (S^a)_{2}^\wedge$, and, at least when restricted to the full subcategory of $\text{Spd}_\infty^F$ consisting of simply connected finite motivic cell complexes, the functor $\text{Be}$ preserves finite colimits and satisfies
\[
H_{F_2}^{\text{Be}(X)^+} \simeq \text{Be}((H_{F_2}^F)^{X^+}).
\]

Proof. We begin by recalling two facts from Mandell’s work on $p$-adic homotopy theory [Man01]. First, if $Y$ is a connected nilpotent space with locally finite mod 2 cohomology, then the unit map $Y \to \text{CAlg}_{H_{F_2}}((H_{F_2}^F)^{Y^+}, H_{F_2})$ realizes the target as the 2-completion of $Y$. Second, the functor
\[
\text{CAlg}_{H_{F_2}}^{op} \to \text{Spd}_\infty, \quad R \mapsto \text{CAlg}_{H_{F_2}}(R, H_{F_2})
\]
lands in $\text{Spd}_\infty^2$ and preserves finite colimits when restricted to the full subcategory of $E_\infty$ algebras $R$ over $F_2$ such that $R_*$ is locally finite-dimensional, $R_0 = F_2$, $R_1 = 0$, and the Dyer–Lashof operation $Q^0$ acts by the identity on $R_*$. We now apply this to our situation. The stable Betti realization functor is symmetric monoidal, and thus $\text{Be}((H_{F_2}^F)^{X^+})$ is indeed an $E_\infty$ ring over $F_2$. Moreover, as $\text{Sq}^0$ acts by the identity on $H^{*,*}(X)$, the Dyer–Lashof operation $Q^0$ acts by the identity on $\pi_* \text{Be}((H_{F_2}^F)^{X^+})$. In particular, $\text{Be}((H_{F_2}^F)^{X^+}) \simeq H_{F_2}^{X^+}$, and so the proposition follows by applying Mandell’s theory.

Remark 5.3.3. We have focused in this section on 2-primary motivic homotopy theory over a field $F$ of characteristic not 2. However, we note that our discussion applies in general to $p$-primary motivic homotopy theory over a field $F$ of characteristic not $p$. \(\diamond\)
6. The motivic Hopf invariant one problem

In this section, we formulate and discuss motivic analogues of the Hopf invariant one problem. The material in this section is not needed for Section 7.

Recall the following conventions from Section 1.4. First, $H^*$ and $H^{*,*}$ always refer to reduced ordinary and motivic mod 2 cohomology. Second, we write e.g. $H^*(X_+)$ for the unreduced cohomology of a space $X$. Third, we use homological grading even for cohomology classes, in the sense that if $x \in H^a(X)$ then we say $|x| = -a$.

6.1. The unstable Hopf invariant one problem. Classically, Adams’ determination of the permanent cycles in $\text{Ext}^3$ resolved the Hopf invariant one problem. The Hopf invariant one problem may be formulated motivically using the following.

**Definition 6.1.1.** Let $f: S^{2a-1,2b} \to S^{a,b}$ be an unstable map between motivic spheres; in particular, $a \geq b \geq 0$ and $a \geq 1$. Write $C(f)$ for the cofiber of $f$. The map $f$ vanishes in mod 2 motivic cohomology for degree reasons, and thus

$$H^{*,*}(C(f)_+) \cong M^F\{1, x, y\}$$

as $M^F$-modules, where $|x| = (-a, -b)$ and $|y| = (-2a, -2b)$. Say that $f$ has **Hopf invariant one** if $x^2 = y$, i.e. if $H^{*,*}(C(f)_+) \cong M^F/(x^3)$; otherwise $x^2 = 0$ and $f$ has Hopf invariant zero.

The unstable motivic Hopf invariant one problem is then the following question.

**Question 6.1.2.** For which $(a, b)$ does there exist a map $f: S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant 1?  

This turns out to mostly reduce to the classical case, by way of the following.

**Lemma 6.1.3.** Let $f: S^{2a-1,2b} \to S^{a,b}$ be an unstable $F$-motivic map. Then $f$ has Hopf invariant one if and only if its base change to an algebraic closure of $F$ is of Hopf invariant one.

**Proof.** This is immediate from the definitions. \qed

**Proposition 6.1.4.** Suppose that $F$ is algebraically closed, and fix an unstable $F$-motivic map $f: S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one. Then the Betti realization (see Proposition 5.3.2) of $f$ is an odd multiple of $\eta, \nu$, or $\sigma$. In particular, $a \in \{2, 4, 8\}$.

**Proof.** By Lemma 6.1.3, we may as well suppose that $F$ is algebraically closed. Let $C(f)$ denote the cofiber of $f$ and $C(\text{Be}(f))$ the cofiber of $\text{Be}(f)$. Then $\text{Be}(C(f)) = C(\text{Be}(f))$ by Proposition 5.3.2, and thus $H^*(C(\text{Be}(f))_+) = H^*(\text{Be}(C(f))_+) = H^*(\text{Be}(f))_+ = F_2[x]/(x^3)$ with $|x| = -a$. In other words, the map between 2-completed spheres $\text{Be}(f): S^{2a-1} \to S^a$ has Hopf invariant one. The proposition now follows from Adams’ resolution of the Hopf invariant one problem [Ada60]. \qed

Proposition 6.1.4 is not a complete answer to Question 6.1.2, as we have not given any bounds on $b$, nor have we discussed the existence of maps of Hopf invariant one. Although we will not end up with a complete answer in general, there is more we can say. Before this, we recall what information is encoded in the 1-line of the $F$-motivic Adams spectral sequence.

6.2. The stable Hopf invariant one problem. Question 6.1.2 can be rephrased as asking when there exists an unstable 2-cell complex, with cells in dimension $(a, b)$ and $(2a, 2b)$, such that in cohomology the bottom cell squares to the top cell. In the stable category, one no longer has cup-squares; instead, one has Steenrod operations. Thus we may consider the stable motivic Hopf invariant one problem to be the following question.

**Question 6.2.1.** What $A^F$-modules arise as the cohomology of 2-cell complexes? In particular, for which $(a, b)$ does there exist a 2-cell complex, with cells in dimensions $(0, 0)$ and $(a, b)$ and attaching map vanishing in mod 2 motivic cohomology, such that $H^{*,*}X = M^F\{x, y\}$ is not split as an $A^F$-module?  


This is a particular case of the realization problem for $A^F$-modules, and is exactly what the 1-line of the $F$-motivic Adams spectral sequence encodes. By definition, Ext$_F$ classifies $A^F$-module extensions of $M^F$ by $M^F$. In particular, a class $\epsilon \in \text{Ext}_{F}^{a-1,b}$ classifies an extension of the form $0 \to M^F \{y\} \to E \to M^F \{x\} \to 0$ 
with $|x| = (0,0)$ and $|y| = (-a,-b)$. Such an extension always splits as an $M^F$-module, i.e. $E \cong M^F \{x,y\}$ as $M^F$-modules. It splits as an $A^F$-module if and only if $\epsilon = 0$. The relation between Question 6.2.1 and the 1-line of the $F$-motivic Adams spectral sequence may now be summarized in the following standard proposition.

**Proposition 6.2.2.** Fix a class $\epsilon \in \text{Ext}_{F}^{a-1,b}$, corresponding to an extension $E$ as above. Then the following are equivalent:

1. There is a stable 2-cell complex $C$ with cells in dimensions $(0,0)$ and $(a,b)$ such that $H^{*,*}C \cong E$.
2. The class $\epsilon$ is a permanent cycle in the $F$-motivic Adams spectral sequence, and thus detects a stable class $\alpha \in \pi_{a-1,b}^F$.

Explicitly, if $\epsilon \in \text{Ext}_{F}^{a-1,b}$ detects $\alpha \in \pi_{a-1,b}^F$, then the cofiber $C(\alpha)$ satisfies $H^{*,*}C(\alpha) \cong E$; and if $C$ is a stable 2-cell complex with $H^{*,*}C = E$, then the fiber of the inclusion $S^0 \to C$ is a map $\alpha : S^{a-1,b} \to S^{0,0}$ detected by $\epsilon \in \text{Ext}_{F}^{a-1,b}$.

As we will see in Section 7, the 1-line of the $F$-motivic Adams spectral sequence is already quite rich, and strongly depends on the base field $F$. Thus in considering the stable Hopf invariant one problem, one may not reduce to the case where $F$ is algebraically closed, unlike in the unstable case.

### 6.3. Relation between the unstable and stable motivic Hopf invariant one problems

We may now relate the unstable and stable questions, Question 6.1.2 and Question 6.2.1.

**Proposition 6.3.1.** Let $f : S^{2a-1,2b} \to S^{a,b}$ be a map of Hopf invariant one. Then the associated stable class $\alpha \in \pi_{a-1,b}^F$ is detected by a permanent cycle in $\text{Ext}_{F}^{a-1,b}$, which, after base change to the algebraic closure of $F$, is one of the following:

- $h_1, \tau h_1, h_2, \tau h_2, h_3, \tau h_3, \tau^2 h_3, \tau^3 h_3, \tau^4 h_3$.

In particular, if $\text{Ext}_{F}^{a-1,b}$ does not contain any such permanent cycle, then there is no map $f : S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one.

**Proof.** By Lemma 6.1.3, we may suppose that $F$ itself is algebraically closed. By stabilizing Proposition 6.1.4, we find that $B_2(\alpha)$ is detected by $h_1, h_2, h_3$ in Ext$_{F}^{1}$. Recall from Proposition 5.3.1 that Betti realization is modeled on Adams spectral sequences by the map

$$\text{Ext}_{F} \to \text{Ext}_{F}[^{-1}] \cong \text{Ext}_{F}(\tau^{\pm 1})$$

In particular, the structure of Ext$_{F}$ (see Proposition 4.2.1) implies that $\alpha$ must be detected by a permanent cycle in Ext$_F$ of the form $\tau^n h_1, \tau^n h_2, \tau^n h_3$ for some $n \geq 0$. As $f$ is an unstable map, this class must have nonnegative weight, reducing to the listed classes. \hfill \Box

**Remark 6.3.2.** Our method of relating the unstable motivic Hopf invariant one problem to the stable motivic Hopf invariant one problem, going through the “Betti realization” functors of Section 5.3, may seem somewhat roundabout. This route was taken for the following reason. Recall the classical story. If $f : S^{2a-1} \to S^a$ is a map of Hopf invariant 1, then $H^{*,*}(C(f)_+) = F_2[\tau(x)/(x^3)]$ with $|x| = -a$. That $H^{*,*}(C(f))$ is nonsplit as a $A^1$-module, and thus the associated stable class $\alpha \in \pi_{a-1}^F$ is detected in Ext$_{F}^{1}$, follows from the instability condition $Sq^a(x) = x^2$.

Motivically, the analogous instability condition asserts that if $X$ is a motivic space and $x \in H^{2a-1}(X_+)$, then $Sq^a(x) = x^2$ [Rio12, Proposition 4.4.4]. Now suppose that $f : S^{2a-1,2b} \to S^{a,b}$ is an unstable map of Hopf invariant one, and write $H^{*,*}(C(f)_+) = M^F[\tau(x)/(x^3)]$ with $|x| = (-a,-b)$. If $a$ is even and $b \leq a/2$, then one may set $c = a/2 - b$ and deduce $Sq^c(\tau^c x) = \tau^{2c} x^2$, so that $H^{*,*}(C(f))$ is not split as an $A^F$-module. If $a$ is odd, then one may argue by appealing to an integral motivic Hopf
invariant and graded commutativity, as in the classical case. Thus it is to rule out the possibility of a map \( f : S^{2a-1,2b} \to S^{a,b} \) of Hopf invariant one with \( b > a/2 \) that we have taken our approach. \( \blacklozenge \)

Our computations in Section 7 show, for a variety of base fields \( F \), when \( \text{Ext}_{F}^{1} \) contains a permanent cycle whose image over the algebraic closure is one of the classes listed in Proposition 6.3.1, yielding various nonexistence results. To obtain existence results, we must recall how maps of Hopf invariant one arise.

6.4. **Geometric applications.** Adams’ resolution of the classical Hopf invariant one problem had geometric consequences; notably, it implied that the only spheres which admit \( H \)-space structures are the spheres \( S^0, S^1, S^3, \) and \( S^7 \). It makes sense to ask for the motivic analogue of this, i.e. to ask which spheres \( S^{a,b} \) admit \( H \)-space structures.

This question is in some sense geometric, but we can also ask for something even more concrete. The spheres \( S^{a,b} \) are certain sheaves on the Nisnevich site of smooth \( F \)-schemes, and so it is reasonable to ask when \( S^{a,b} \) is in fact represented by a smooth \( F \)-scheme. This question was raised and studied by Asok–Doran–Fasel in [ADF17]; in particular, they produce explicit smooth affine schemes representing \( S^{a,[a/2]} \), as well as prove that \( S^{a,b} \) is not represented by a smooth scheme for \( a > 2b \). Motivated by this, we are led to ask the following question.

**Question 6.4.1.** For what pairs \((a,b)\) is \( S^{a,b} \) a motivic \( H \)-space? Of these, when is it represented by a smooth \( F \)-scheme which admits a unital product? \( \blacklozenge \)

Classically, the connection between the \( H \)-space structures and the Hopf invariant one problem is via the *Hopf construction*. This construction may also be carried out in the motivic category, and has been studied in this context by Dugger–Isaksen [DI13]. We recall the key points.

**Definition 6.4.2.** [DI13, Definition C.1] Let \( X, Y, \) and \( Z \) be pointed spaces, and let \( h : X \times Y \to Z \) be a pointed map. The *Hopf construction* of \( h \) is the map \( H(h) : X \star Y \to \Sigma Z \) obtained by taking homotopy colimits of the rows of the diagram

\[
\begin{array}{ccc}
X & \leftarrow & X \times Y \rightarrow & Y \\
\downarrow & & \downarrow & \\
* & \leftarrow & Z \rightarrow & * \\
\end{array}
\]

Here, \( \star \) is the join. Note that \( S^{a,b} \star S^{c,d} \simeq S^{a+c+1,b+d} \); thus the Hopf construction may be used to construct maps between motivic spheres. Using the theory of Cayley–Dickson algebras, Dugger–Isaksen [DI13, Section 4] used this to define *motivic Hopf maps* \( \eta \in \pi_{1,1}, \nu \in \pi_{3,2} \), and \( \sigma \in \pi_{5,4} \). As noted in [DI13, Remark 4.14], these motivic Hopf maps have Hopf invariant one. This is a general property of the Hopf construction, which we may summarize in the following.

**Lemma 6.4.3.** If \( \mu : S^{a-1,b} \times S^{a-1,b} \to S^{a-1,b} \) is an \( H \)-space product, then its Hopf construction \( H(\mu) : S^{2a-1,2b} \to S^{a,b} \) has Hopf invariant one.

**Proof.** The proof of the analogous fact for topological spaces [Ste62, Section I.5] extends to motivic spaces. We summarize the key points.

A pointed map \( f : S^{a-1,b} \times S^{a-1,b} \to S^{a-1,b} \) of motivic spaces is said to have *degree* \((\alpha, \beta)\) if \( f|_{S^{a-1,b} \times \{p_2\}} \) has degree \( \alpha \) and \( f|_{\{p_1\} \times S^{a-1,b}} \) has degree \( \beta \). Since \( \mu \) is an \( H \)-space product, its restrictions to \( S^{a-1,b} \times \{p_2\} \) and \( \{p_1\} \times S^{a-1,b} \) are homotopic to the identity, so \( \mu \) has degree \((1,1)\). The lemma follows by showing that, more generally, the integral Hopf invariant, defined in the evident way, of the Hopf construction of a map of degree \((\alpha, \beta)\) is \( \alpha \cdot \beta \).

Steenrod–Epstein’s proof of [Ste62, Lemma 5.3] carries over to the motivic setting to complete the proof. The main point is that Steenrod–Epstein work with particular models of the cone, join,
homotopy cofiber, and suspension in their proof, but any model would work, as all of their statements only depend on the homotopy types of the relevant spaces and homotopy classes of the relevant maps.

More precisely, with notation as in their proof, one may replace $E_1$, $E_2$, $E_+$, and $E_-$ by the cones on $S_1$, $S_2$, $S$, and $S$, respectively, to avoid any potential point-set issues. In particular, one regards $E_1$, $E_2$, $E_+$, and $E_-$ as suspension data in the sense of [DI13, Remark 2.9] for the various suspensions appearing in the Hopf construction. In this language, the identifications of various pushouts in the proof of [Ste62, Lemma 5.3] are examples of induced orientations [DI13, Remark 2.10]. The proof carries through unchanged with these new choices of $E_1$, $E_2$, $E_+$, and $E_-$. □

We can now summarize what is known in the following.

**Theorem 6.4.4.** A motivic sphere is represented by a smooth $F$-scheme admitting a unital product if and only if it is one of the following:

$$S^{0,0}, S^{1,1}, S^{3,2}, S^{7,4}. $$

In addition to the motivic spheres listed above, the following motivic spheres admit $H$-space structures:

$$S^{1,0}, S^{3,0}, S^{7,0}. $$

The only other motivic spheres that could possibly admit $H$-space structures are the following:

$$S^{3,1}, S^{7,3}, S^{7,2}, S^{7,1},$$

moreover, an $H$-space structure on such a sphere produces a permanent cycle in $\text{Ext}_F$ whose image over the algebraic closure is $\tau h_2$, $\tau h_3$, $\tau^3 h_3$, or $\tau^3 h_3$, respectively.

**Proof.** That the spheres $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$ are represented by smooth $F$-schemes admitting a unital product is given by work of Dugger–Isaksen [DI13]. The spheres $S^{1,0}$, $S^{3,0}$, and $S^{7,0}$ are the images of $S^1$, $S^3$, and $S^7$, respectively, under the unstable constant functor from spaces to motivic spaces, and so inherit $H$-space structures from their classical structures. That all the spheres listed are the only spheres which may admit $H$-space structures follows from Lemma 6.4.3 and Proposition 6.3.1, as does the final claim concerning the $F$-motivic Adams spectral sequence. Finally, Asok–Doran–Fasel prove in [ADF17, Proposition 2.3.1] that if $S^{a-1,b}$ is represented by a smooth $F$-sphere, then necessarily $2b \geq a - 1$, and the only possible $H$-spaces satisfying this are $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$ as listed. □

We note the following special case.

**Corollary 6.4.5.** Suppose there is an $R$-motivic map $f: S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one. Then $(a, b)$ is one of

$$(1, 0), (2, 1), (4, 2), (8, 4), (2, 0), (4, 0), (8, 0).$$

Moreover, all of these are realized, and in fact

$$S^{0,0}, S^{1,1}, S^{3,2}, S^{7,4}, S^{1,0}, S^{3,0}, S^{7,0}$$

are all the $R$-motivic spheres admitting $H$-space structures.

**Proof.** This is immediate from Theorem 6.4.4, either appealing to the fact that $\text{Ext}_R$ vanishes in the degrees detecting the remaining possibilities, or else noting that the real points of $S^{a,b}$ are $S^{a-b}$, so that if $S^{a,b}$ is an $H$-space then $a - b \in \{0, 1, 3, 7\}$. □

7. The 1-line of the motivic Adams spectral sequence

We now analyze the 1-line of the $F$-motivic Adams spectral sequence. We begin in Section 7.1 by explaining how to read off the structure of $\text{Ext}_F$ for various fields $F$ from our computation of $\text{Ext}_R$. After some additional preliminaries in Section 7.2, we give a direct motivic analogue of the classical differentials in Section 7.3, proving $d_2(h_n + 1) = (h_0 + \rho h_1)h_n^2$ for $a \geq 3$ over arbitrary base fields. We then proceed to give more detailed information about the 1-line for the particular fields $F$ of the form $R, F_q$ with $q$ an odd prime-power, $\mathbb{Q}_p$, with $p$ any prime, and $\mathbb{Q}$. 
7.1. **Computing** $\text{Ext}_F$. As a general rule, $\text{Ext}_F$ is largely understood once $M^F$ and $\text{Ext}_R$ are both understood. Rather than formulate a precise statement, let us just describe $\text{Ext}_F$ for the various particular fields $F$ we shall encounter, namely those described in Example 2.2.1 as well as $F = \mathbb{Q}$.

Recall from Remark 2.3.2 that for any field $F$, we may view $M^F$ as an $A^R$-module, and there is an isomorphism

$$\text{Ext}_F \cong \text{Ext}_{A^R}(M^R, M^F).$$

Thus, the main point is to understand $M^F$ as an $A^R$-module, and this is in fact determined by $M^F_0$ as an $\mathbb{F}_2[\rho]$-module. For the examples of interest, we have the following. Abbreviate

$$M = \mathbb{F}_2[\tau, \rho], \quad M_{(r)} = M/(\rho^r).$$

**Lemma 7.1.1.** As $A^R$-modules, we have the following.

1. If $F = \mathbb{R}$, then $M^F = M$.
2. If $F = \mathbb{F}_p$ is algebraically closed, then $M^F = M_{(1)}$.
3. If $F = \mathbb{F}_p$ with $q \equiv 1 \pmod{4}$, then $M^F = M_{(1)}\{1, u\}$.
4. If $F = \mathbb{F}_p$ with $q \equiv 3 \pmod{4}$, then $M^F = M_{(2)}$.
5. If $F = \mathbb{Q}_p$ with $p \equiv 1 \pmod{4}$, then $M^F = M_{(1)}\{1, \pi, u, \pi u\}$.
6. If $F = \mathbb{Q}_p$ with $p \equiv 3 \pmod{4}$, then $M^F = M_{(2)}\{1, \pi\}$.
7. If $F = \mathbb{Q}_2$, then $M^F = M_{(3)}\{1\} \oplus M_{(1)}\{u, \pi\}$.
8. If $F = \mathbb{Q}$, then $M^\mathbb{Q} = M\{1\} \oplus M_{(1)}\{[2]\} \oplus M_{(1)}\{[p], a_p : p \equiv 1 \pmod{4}\} \oplus M_{(2)}\{u_p : p \equiv 3 \pmod{4}\}$.

**Proof.** All but the case $F = \mathbb{Q}$ may be read off the examples listed in Example 2.2.1. When $F = \mathbb{Q}$, the ring $M^\mathbb{Q}$ is described by Ormsby–Østvær in [OØ13, Propositions 5.3 and 5.4], following work of Milnor [Mil70]. Our description may be read off this upon setting $u_p = [p] + \rho$ for $p \equiv 3 \pmod{4}$.

For $r \geq 0$, define

$$\text{Ext}_{(r)} = \text{Ext}_{A^R}(M, M_{(r)}) = H_*(A^R/(\rho^r)).$$

The $\mathbb{F}_2[\rho]$-module structure of $\text{Ext}_{(r)}$ may be easily computed from $\text{Ext}_R$ via the long exact sequence associated to the cofiber of $\rho^r$. Even less work is necessary when $\text{Ext}_R$ has been computed by some method compatible with the $\rho$-Bockstein spectral sequence such as ours; see in particular Remark 4.1.5. Thus Theorem 4.2.12 allows us to read off $\text{Ext}^f_{(r)}$ for $f \leq 2$, as well as the image of $\text{Ext}_{(r)} \rightarrow \text{Ext}^3_{(r)}$. This does not give the entirety of $\text{Ext}^3_{(r)}$; however, we at least know that whatever remains is generated by classes which appear in the $\rho$-Bockstein spectral sequence as $\rho^k \alpha$ with $\alpha \in \text{Ext}^1_{(4)}$ and $k < r$, and this is enough information for our purposes.

**Lemma 7.1.1** describes for various $F$ how $\text{Ext}_F$ may be written as a direct sum of copies of various $\text{Ext}_{(r)}$. For example, $\text{Ext}_{\mathbb{Q}_2} = \text{Ext}_{(3)}\{1\} \oplus \text{Ext}_{(1)}\{u, \pi\}$. We may use this to prove a Hasse principle for $\text{Ext}_{\mathbb{Q}}$.

**Lemma 7.1.2.** The map

$$M^\mathbb{Q} \rightarrow M^{\mathbb{Q}_0}$$

satisfies

$$[p] \mapsto \pi, \quad a_p \mapsto u\pi, \quad u_p \mapsto \pi + \rho.$$ 

Here, the first is relevant for $p = 2$ or $p \equiv 1 \pmod{4}$, the second for $p \equiv 1 \pmod{4}$, and the third for $p \equiv 3 \pmod{4}$.

**Proof.** The behavior of these maps is described by Ormsby–Østvær in [OØ13, Proposition 5.3], following work of Milnor [Mil70]. Our description follows immediately; note we have defined $u_p = [p] + \rho$ for $p \equiv 3 \pmod{4}$.

**Proposition 7.1.3.** The Hasse map

$$\text{Ext}_{\mathbb{Q}} \rightarrow \text{Ext}_R \times \prod_p \text{Ext}_{\mathbb{Q}_p}$$

is injective.
Theorem 7.3.1. For an arbitrary base field \( F \) will follow a similar pattern to Wang’s computation of the corresponding classical Adams differentials. The summand \( \text{Ext} \) of these builds on the case where \( F \) is a permanent cycle by Lemma 7.2.1. In [DI13], Dugger–Isaksen construct the \( F \)-motivic Adams differentials \( d_2(h_{a+1}) \) will follow a similar pattern to Wang’s computation of the corresponding classical Adams differentials [Wan67] (differentials which were first computed by Adams [Ada60]). This is an inductive argument, beginning with information about the Hopf elements which are known to exist. We record some of this information in this subsection.

Write \( \epsilon \in \pi^F_{0,0} \) for the class represented by the twist map \( S^{1,1} \otimes S^{1,1} \to S^{1,1} \otimes S^{1,1} \).

Lemma 7.1.4. Ext\(_{(1)} \) is given in stems \( s \leq 6 \) by the following module:

\[
\mathbb{F}_2[\tau] \otimes (\mathbb{F}_2\{h_n^a : n \geq 0\} \oplus \mathbb{F}_2\{h_1^1, h_1^2, h_2^1, h_2, h_0 h_1, h_2^2\}) \oplus \mathbb{F}_2[\tau]/(\{h_1^1, h_1^5, h_1^6\}).
\]

Proof. These groups have been computed by Dugger–Isaksen [DI10].

7.2. Existence of Hopf elements. Our computation of the \( F \)-motivic Adams differentials \( d_2(h_{a+1}) \) will follow a similar pattern to Wang’s computation of the corresponding classical Adams differentials [Wan67] (differentials which were first computed by Adams [Ada60]). This is an inductive argument, beginning with information about the Hopf elements which are known to exist. We record some of this information in this subsection.

Write \( \epsilon \in \pi^F_{0,0} \) for the class represented by the twist map \( S^{1,1} \otimes S^{1,1} \to S^{1,1} \otimes S^{1,1} \).

Lemma 7.2.1. Fix \( \alpha \in \pi^F_{a,b} \) and \( \beta \in \pi^F_{c,d} \). Then there is an identity

\[
\alpha \cdot \beta = (-1)^{\langle a-b \rangle(n-d)} \cdot \beta \cdot \alpha.
\]

Moreover, \( 1 - \epsilon \) is detected in \( Ext_F \) by \( h_0 \) and 2 by \( h_0 + \rho h_1 \).

Proof. The claimed graded commutativity is given in [Mor04, Corollary 6.1.2]; see also [IO19, Section 6.1] for a discussion. That \( 1 - \epsilon \) is detected by \( h_0 \) and 2 by \( h_0 + \rho h_1 \) is noted in [IO19, Remark 6.3].

Lemma 7.2.2. For any field \( F \), the class \( h_a \) is a permanent cycle for \( a \in \{0, 1, 2, 3\} \).

Proof. The class \( h_0 \) is a permanent cycle by Lemma 7.2.1. In [DI13], Dugger–Isaksen construct the motivic Hopf elements \( \eta, \nu, \) and \( \sigma \), and in [DI13, Remark 4.14] they indicate that these are detected by \( h_1, h_2, \) and \( h_3, \) respectively; see also our discussion in Section 6.4. Thus these classes must be permanent cycles.

7.3. Nonexistence of Hopf elements. The purpose of this subsection is to prove the following.

Theorem 7.3.1. For an arbitrary base field \( F \) of characteristic not equal to 2, there are differentials of the form

\[
d_2(h_{a+1}) = (h_0 + \rho h_1) h_a^2
\]

in the \( F \)-motivic Adams spectral sequence, which are nonzero for \( a \geq 3 \).

By naturality, it suffices to produce these differentials in the case where \( F \) is a prime field, i.e. \( F = \mathbb{F}_p \) or \( F = \mathbb{Q} \), and when \( F \) is algebraically closed. Moreover, by the Hasse principle given in Proposition 7.1.3, the case \( F = \mathbb{Q} \) may be deduced from the cases \( F = \mathbb{Q}_p \) and \( F = \mathbb{R} \) combined. All of these build on the case where \( F \) is algebraically closed, which may be treated as follows.
Proposition 7.3.2. If $F = \mathbb{F}$ is algebraically closed, then

$$d_2(h_{a+1}) = h_0 h_a^2.$$ 

This is nonzero for $a \geq 3$.

Proof. The corresponding classical differentials are known due to work of Adams [Ada60]. The proposition could be reduced to this by appealing to Proposition 5.3.1; however, we shall instead proceed as follows.

In [Wan67, Section 3], Wang gives another proof of the classical differentials, combining only a minimal amount of homotopical input with a good understanding of $\text{Ext}_q$. His argument may be applied essentially verbatim to produce the claimed $\mathbb{F}$-motivic differentials. It is this argument that may be adapted to work for other base fields, so to motivate our later computations let us recall this argument in full.

The proof proceeds by induction on $a$, where only the base case requires any homotopical input.

Consider the base case $a = 3$. The class $h_3$ is a permanent cycle, detecting the Hopf element $\sigma$; see Lemma 7.2.2. By Lemma 7.2.1, we find that $2\sigma^2 = 0$. As $2$ is detected by $h_0$ over algebraically closed fields, it follows that $h_0 h_3^2$ cannot survive the Adams spectral sequence. The structure of $\text{Ext}_F$ implies that $d_2(h_4) = h_0 h_3^2$ is the only way for $h_0 h_3^2$ to die.

Now, suppose we have produced the differential $d_2(h_a) = h_0 h_a^{2-1}$ for some $n \geq 4$. The relation $h_{a+1} h_a = 0$ together with the Leibniz rule implies

$$0 = d_2(h_{a+1} h_a) = d_2(h_{a+1}) \cdot h_a + h_{a+1} \cdot d_2(h_a).$$

Applying our inductive hypothesis and the relation $h_{a+1} \cdot h_{a-1}^2 = h_a^3$, this reduces to

$$(d_2(h_{a+1}) + h_0 h_a^2) \cdot h_a = 0.$$ 

The algebraic structure of $\text{Ext}_F^3$ implies that $d_2(h_{a+1}) \in \mathbb{F}_2 \{h_0 h_a^2\}$, so it suffices to verify that $h_0 h_a^2 \neq 0$ for $a \geq 4$. This follows from Wang’s computation [Wan67, Proposition 3.4] by comparison along the map $\text{Ext}_F \to \text{Ext}_F[\tau^{-1}] \cong \text{Ext}_q[\tau^{-1}]$, and this concludes the proof. \hfill $\Box$

The base step for the inductive argument given in Proposition 7.3.2 works for arbitrary base fields, but the inductive step fails apart. This inductive step relies on the algebraic fact that, when working over an algebraically closed field, multiplication by $h_a$ is injective on the degree of $d_2(h_{a+1})$ for $a \geq 4$. Over other base fields, this fails for $a = 4$: this degree may contain elements of the form $\omega h_3 h_a^2$ where $\omega \in \text{Ext}_F^{1,0,-1}$ is a sum of elements such as $\rho$, $\pi$, and $u$, and

$$\omega h_3 h_a^2 \cdot h_4 = \omega h_1 \cdot h_3^2 = \omega h_1 \cdot h_3^2 \cdot h_5 = 0.$$ 

Luckily, the inductive step fails only for $a = 4$; once we have resolved $d_2(h_5)$, the remaining differentials will follow via the same argument. To resolve this differential, we proceed as follows.

Proposition 7.3.3. Let $F$ be a field of the form $\mathbb{F}_q$ for $q$ odd, $\mathbb{Q}_p$ for any $p$, or $\mathbb{R}$. Then there is a differential

$$d_2(h_5) = (h_0 + \rho h_1) h_4^2$$

in the $F$-motivic Adams spectral sequence.

Proof. When $F = \mathbb{R}$, we first make the following reduction. Observe that $\text{Ext}_\mathbb{R}$ in the degree of $d_2(h_5)$ is given by $\mathbb{F}_2 \{h_0 h_4^2, \rho h_1 h_4^2\}$, and that neither of these classes are divisible by $\rho^2$. Thus it is sufficient to verify this differential in the Adams spectral sequence for the cofiber of $\rho^2$. By [BS20, Lemma 7.8], this cofiber is a ring spectrum, and so its Adams spectral sequence is multiplicative. Having made this reduction, the remainder of the argument is uniform in the given choices of $F$. For brevity of notation, in the following we shall write $\text{Ext}_F$ for the object so named when $F = \mathbb{F}_q$ or $F = \mathbb{Q}_p$, and write the same for $\text{Ext}_{(2)}$ when $F = \mathbb{R}$.\hfill $\Box$
First, observe that as $\tau^4 \in \text{Ext}_F^0$, the class $\tau^{16}$ is a square and thus a $d_2$-cycle. As $\tau^{16}$ acts injectively on $\text{Ext}_F^f$ for $f \leq 3$, it suffices to show

$$d_2(\tau^{16}h_5) = (h_0 + \rho h_1)\tau^{16}h_4^2.$$ 

Consider the Hurewicz map $c: \pi_* \to \pi_*^{E_0}$. Let $\theta_4 \in \pi_{30}S^0$ be the Kervaire class, detected by $h_4^2$ and satisfying $2\theta_4 = 0$. By Proposition 5.1.1, we find that $c(\theta_4)$ is detected by $\tau^{16}h_4^2$. As $2 \cdot c(\theta_4) = 0$, the class $(h_0 + \rho h_1)\tau^{16}h_4^2$ cannot survive. The only possibility is that $d_2(\tau^{16}h_4) = (h_0 + \rho h_1)\tau^{16}h_4^2$, yielding the desired differential.

**Remark 7.3.4.** When $F = \mathbb{R}$, the differential $d_2(h_5)$, and in fact all the differentials $d_2(h_{a+1})$, may also be produced as follows. By comparison with $\mathbb{C}$, one finds $d_2(h_5) \in h_0h_4^2 + \mathbb{F}_2(\rho h_1h_4^2)$. Thus it suffices to verify that $d_2(h_5)$ is not $\rho$-torsion. This is a consequence of the fact that the isomorphism $\text{Ext}_{\mathbb{R}}[p^{-1}] \cong \text{Ext}_{\mathbb{C}l}[p^{+1}]$ commutes with Adams differentials. □

We need just one more algebraic fact for the proof of Theorem 7.3.1.

**Lemma 7.3.5.** Let $\omega \in \text{Ext}_F^0$ be nonzero. Then $\omega h_1h_4^3 \neq 0$ for all $a \geq 5$.

**Proof.** The class $h_0h_4^2h_3^{1-a}$ is nonzero in $\text{Ext}_F^1$ for $a \geq 5$ by [Wan67, Proposition 3.4]. Proposition 3.2.1 gives an injection $\text{Ext}_F^1 \to \text{Ext}_F^1$, which extends by linearity to an injection $\text{Ext}_F^0 \otimes_{\mathbb{F}_2} \text{Ext}_F^1 \to \text{Ext}_F^1$. The class $\omega h_1h_4^3$ is the image of $\omega \otimes h_0h_4^2h_3^{1-a}$ under this map, yielding the claim. □

We may now give the following.

**Proof of Theorem 7.3.1.** As discussed, it suffices to consider only the cases where $F$ is of the form $\mathbb{F}_q$ for some $q$ odd, $\mathbb{Q}_q$ for some $q$, or $\mathbb{R}$. So let $F$ be one of these. We now induct on $a$, with base cases $a = 3$ and $a = 4$.

First, consider the case $a = 3$. By Lemma 7.2.2, the class $h_3$ is a permanent cycle detecting the class $\sigma$. By Lemma 7.2.1, $2\sigma^2 = 0$, and so $(h_0 + \rho h_1)h_3^2$ must be the target of a differential. The only possibility is that $d_2(h_4) = (h_0 + \rho h_1)h_3^2$.

The case $a = 4$ was handled in Proposition 7.3.3.

Now, suppose inductively that we have produced the differential $d_2(h_a) = (h_0 + \rho h_1)h_3^{2-a}$ for some $a \geq 5$. Combining the Leibniz rule with the relation $h_{a+1}h_a = 0$, we find

$$0 = d_2(h_{a+1}h_a) = d_2(h_{a+1})h_a + h_{a+1}d_2(h_a).$$

Applying our inductive hypothesis and the relation $h_{a+1}h_3^{2-a} = h_4^a$, we find

$$(d_2(h_{a+1}) + (h_0 + \rho h_1)h_3^{2-a})h_a = 0. $$

It follows that $d_2(h_{a+1}) = (h_0 + \rho h_1)h_3^{2-a} + x$ where $x$ is some class killed by $h_a$. The only classes in this degree are $h_0h_4^2$ and those of the form $\omega h_1h_3^2$ where $\omega \in \text{Ext}_F^0$. By comparison with $\mathbb{F}$, we find that $x$ must be zero or a nonzero class of the form $\omega h_1h_4^2$ with $\omega \in \text{Ext}_F^{-1,0,-1}$. As $a \geq 5$, Lemma 7.3.5 implies that none of the latter are killed by $h_a$. Thus $x = 0$, yielding the desired differential. □

This concludes our uniform analysis of differentials out of $\text{Ext}_F^1$. The rest of this section is dedicated to studying the 1-line in more detail for particular fields $F$.

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11When $F = \mathbb{R}$, instead $c(\theta_4)$ is detected by $(\tau^8h_4)^2 + \rho^3h_4^2$

12To be explicit, we note that $(h_0 + \rho h_1)h_3^2 = h_0h_4^2$, although this is not relevant to the argument.
7.4. The real numbers. We now study the case $F = \mathbb{R}$ in more detail. Recall from Theorem 4.2.12 that
\[
\text{Ext}^1_\mathbb{R} = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2a})(\tau^{2a-1}(4n+1)h_a : n \geq 0). \]
Here, recall that $2^{a-1}(4n+1) = 2n$ for $a = 0$. Theorem 7.3.1 allows one to understand the fate of the classes in the $\rho$-torsion-free summand, so we turn our attention to the $\rho$-torsion subgroup. We shall first pin down which of these $\rho$-torsion classes are permanent cycles, and then by separate methods compute all $d_2$-differentials on these $\rho$-torsion classes. A comparison reveals that there must be numerous higher differentials, but determining these is outside the scope of our computation. The first point of order is the following.

Definition 7.4.1. For $a \geq 0$, write $a = c + 4d$ with $0 \leq c \leq 3$, and define $\psi(a) = 2^c + 8d$.\(^{13}\)

Proposition 7.4.2. The class $\rho^r \tau^{2a-1}(4n+1)h_a$ is a permanent cycle if and only if $r \geq 2^n - \psi(a)$. \(^{13}\)

The proof of Proposition 7.4.2 requires some preliminaries. We proceed by comparison with Borel $C_2$-equivariant stable homotopy theory. Let $\text{Ext}_{BC_2}$ denote the $E_2$-page of the Borel $C_2$-equivariant Adams spectral sequence. Explicitly,
\[
\text{Ext}^{s,f,w}_{BC_2} = \text{Ext}^{s-f,w}_{A^{cl}}(\mathbb{F}_2, H^*P_w^\infty);\]
this is just a combination of the ordinary Adams spectral sequences for the stable cohomotopy groups of infinite stunted projective space. By Lin’s positive resolution of the Segal conjecture [Lin80], this spectral sequence converges to $\pi_*^{Cl}$, the homotopy groups of the 2-completion of the $C_2$-equivariant sphere spectrum.

Betti realization follows by Borel completion yields a functor from the stable $\mathbb{R}$-motivic category to the Borel $C_2$-equivariant stable category $\text{Fun}(BC_2, \mathbb{S}_p)$, and Behrens-Shah [BS20, Section 8] show that this may be understood as completing at $\rho$ and inverting $\tau$. Applying this to an Adams resolution, we find that
\[
\text{Ext}_{BC_2} = \lim_{n \to \infty} \text{Ext}_{(2^n)[\tau^{-2n}]}.\]
The simple form of $\text{Ext}^\leq_3$ allows us to immediately read off $\text{Ext}^\leq_3_{BC_2}$.

Lemma 7.4.3. $\text{Ext}^\leq_3_{BC_2}$ is exactly as $\text{Ext}^\leq_3_{R}$ is described in Theorem 4.2.12, only where $n$ is allowed to be negative, and in place of the map $\text{Ext}_{R} \to \text{Ext}_{C}$ is a map $\text{Ext}_{BC_2} \to \text{Ext}_{C}[\tau^{-1}] \cong \text{Ext}_{cl}[\tau^\pm 1]$. \(\square\)

In particular,
\[
\text{Ext}_{BC_2} = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2a})(\tau^{2a-1}(4n+1)h_a : n \in \mathbb{Z}).
\]
We have introduced $\text{Ext}_{BC_2}$ in order to make the following reduction.

Lemma 7.4.4. Write $h : \text{Ext}_{R} \to \text{Ext}_{BC_2}$ for the canonical map of spectral sequences. Fix a $\rho$-torsion class $x \in \text{Ext}^1_{BC_2}$. Then $x$ is a permanent cycle if and only if $h(x)$ is a permanent cycle.\(^{14}\)

Proof. Clearly, if $x$ is a permanent cycle, then the same must be true of $h(x)$. Conversely, suppose that $h(x)$ is a nontrivial permanent cycle; we claim that $x$ is a permanent cycle.

Write $\text{Ext}_{C_2}$ for the $E_2$-page of the $C_2$-equivariant Adams spectral sequence [HK01, Section 6], converging to the same target as $\text{Ext}_{BC_2}$. This splits additively as $\text{Ext}_{C_2} = \text{Ext}_R \oplus \text{Ext}_{NC}$ for a certain summand $\text{Ext}_{NC}$ (see [GHR19, Section 2]), and $h$ factors as $h = g \circ f : \text{Ext}_{R} \to \text{Ext}_{C_2} \to \text{Ext}_{BC_2}$, the first map being the obvious inclusion and the second map killing the summand $\text{Ext}_{NC}$.

As $h(x)$ is a nontrivial permanent cycle, it detects a class $\alpha$ in Borel Adams filtration 1. The class $\alpha$ must then be detected in $\text{Ext}^{\leq 1}_{C_2}$. By [BGI21], the map $\text{Ext}_{R} \to \text{Ext}_{C_2}$ is an isomorphism in the degrees under consideration, so the same must be true for $\text{Ext}_{C_2} \to \text{Ext}_{BC_2}$. As there is at

\(^{13}\)OEIS sequence A003485.

\(^{14}\)The $\rho$-torsion assumption may be removed by combining this with our understanding of the remaining classes.
most one nonzero \( \rho \)-torsion class in these degrees, the only possibility is that \( \alpha \) is detected by \( f(x) \) in \( \text{Ext}_{C_2}^1 \), implying that \( f(x) \) is a permanent cycle. As \( \text{Ext}_R \rightarrow \text{Ext}_{C_2} \) is the inclusion of a summand, this implies that \( x \) is a permanent cycle, as claimed. \( \square \)

Thus it suffices to understand permanent cycles in \( \text{Ext}_{BC_2}^1 \). The main point is the following.

**Lemma 7.4.5.** There exists a nonzero \( \rho \)-torsion class \( \alpha \in \pi_{s,w}^{BC_2} \) detected in Borel Adams filtration 1 if and only if the inclusion of the bottom cell of \( P_{w-s-1}^{w-1} \) is split.

**Proof.** First, suppose given such a map \( \alpha \). The structure of \( \text{Ext}_{BC_2}^1 \) implies that \( \alpha \) must have \( \rho \)-torsion exponent \( s + 1 \), and so there is a lift \( \overline{\alpha} \) in the diagram

\[
\begin{array}{ccc}
\Sigma^{s-w+1} P_{w-s-1}^{w-1} & \xrightarrow{\partial} & S^0 \\
\uparrow \scriptstyle{\overline{\alpha}} & & \downarrow \scriptstyle{\alpha} \\
\Sigma^{s-w} P_{w-s-1}^\infty & \xrightarrow{} & S^0 \\
\end{array}
\]

As \( \alpha \) and \( \partial \) have Adams filtration 1, necessarily \( \overline{\alpha} \) has Adams filtration 0. It follows that precomposing \( \overline{\alpha} \) with the inclusion of the bottom cell \( S^0 \rightarrow \Sigma^{s-w+1} P_{w-s-1}^{w-1} \) gives a map \( S^0 \rightarrow S^0 \) which is nonzero in mod 2 cohomology, and must therefore be an equivalence. In other words, \( \overline{\alpha} \) splits off the bottom cell of \( P_{w-s-1}^{w-1} \).

Conversely, if the inclusion of the bottom cell of \( P_{w-s-1}^{w-1} \) is split, then its splitting gives a nonzero map \( \overline{\alpha} \) as above in Adams filtration 0. Let \( \alpha = \overline{\alpha} \circ \partial \); we claim that \( \alpha \) is a nonzero class detected in Adams filtration 1. Indeed, the cofibering \( P_{w-s-1}^{w-1} \rightarrow P_{w-s-1}^\infty \rightarrow P_{w-s-1}^\infty \) gives an exact sequence

\[
\text{Ext}^0(\mathbb{F}_2, H^* P_{w-s-1}^\infty) \rightarrow \text{Ext}^0(\mathbb{F}_2, H^* P_{w-s-1}^{w-1}) \rightarrow \text{Ext}^0(\mathbb{F}_2, H^* P_{w-s-1}^{w-1}) \xrightarrow{\partial'} \text{Ext}^1(\mathbb{F}_2, H^* P_{w-s-1}^\infty),
\]

where \( \partial' \) models restriction along \( \partial \) in the previous diagram. The first map is exactly

\[
\rho^{s+1}: \text{Ext}_{BC_2}^{s,0,w} \rightarrow \text{Ext}_{BC_2}^{s,0,w-s-1}.
\]

As \( \text{Ext}_{BC_2}^0 = \mathbb{F}_2[\rho] \), we find that the kernel of \( \partial' \) consists of only that class represented by the inclusion \( \mathbb{F}_2 \rightarrow H^0 P_{w-s-1}^{w-1} \). So \( \partial' \) is injective in the relevant degrees, implying that \( \alpha \) is nonzero and of Adams filtration 1 as claimed. \( \square \)

We may now give the following.

**Proof of Proposition 7.4.2.** By Lemma 7.4.4, it suffices to show that a class \( \rho^r \tau^{2^{n+1}(4n+1)} h_a \in \text{Ext}_{BC_2}^1 \) is a permanent cycle if and only if \( r \geq 2^n - \psi(a) \). By sparseness of \( \text{Ext}_{BC_2}^1 \), the class \( \rho^r \tau^{2^{n+1}(4n+1)} h_a \) is a permanent cycle if and only if there is some \( \rho \)-torsion class \( \alpha \in \pi_{C_2}^{2^{n+1}(4n+1)} \) detected in Borel Adams filtration 1. By Lemma 7.4.5, this holds if and only if inclusion of the bottom cell of \( P_{2^{n+1}(4n+1)}^{2^{n+1}(4n+1)} \) is split. By James periodicity [Jam58, Jam59], this holds if and only if the inclusion of the bottom cell of \( P_{2^{N+1}(4n+1)}^{2^{N+1}(4n+1)} \) is split for some sufficiently large \( N \gg 0 \); that is, we may assume ourselves to be working with suspension spectra of honest real projective spaces. When this happens was resolved by Adams’ solution of the vector fields on spheres problem [Ada62, Theorem 1.2], yielding the condition claimed. \( \square \)

**Corollary 7.4.6.** The classes \( \tau^{2^{n+1}(4n+1)} h_a \) are permanent cycles for \( a \leq 3 \).
Corollary 7.4.6 could also be proved more directly, applying the technique used in the proof of Theorem 7.3.1 or Proposition 7.4.8 below to reduce to the region considered by Belmont–Isaksen.

It is worth summarizing what we have learned from the proof of Proposition 7.4.2 about the stable cohomotopy groups of projective spaces.

**Theorem 7.4.7.** The subgroup of permanent cycles in \( \text{Ext}^1_{BC_2} \) is given by
\[
\mathbb{F}_2[\rho] \{ h_1, h_2, h_3, \rho h_4 \} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{\psi(a)}) \{ \rho^{2^a-\psi(a)} \tau^{2^a-1}(4n+1) h_a : n \in \mathbb{Z} \}.
\]

A choice of maps \( \Sigma^r \mathbb{P}^\infty \to S^0 \) detected by these permanent cycles is given by the following.

1. For all \( r \geq 0 \), there are maps
\[
\begin{align*}
P_{1-r}^\infty & \to P^\infty_1 \to S^0 \\
\Sigma P_{2-r}^\infty & \to \Sigma P^\infty_2 \to S^0 \\
\Sigma^3 P_{4-r}^\infty & \to \Sigma^3 P^\infty_3 \to S^0
\end{align*}
\]
where \( \Sigma \) are equivariant refinements of the Hopf maps with the same names. The above composites are detected by \( \rho^r h_1, \rho^r h_2, \rho^r h_3 \), respectively.

2. For all \( r \geq 0 \), there is a map
\[
\Sigma^7 P_{7-r}^\infty \to \Sigma^7 P^\infty_7 \overset{\text{Sq}(\sigma)}\to S^0,
\]
where \( \text{Sq}(\sigma) \) is the symmetric square of \( \sigma : S^7 \to S^0 \). The above composite is detected by \( \rho^{1+r} h_4 \).

3. For all \( a \geq 0, n \in \mathbb{Z} \), and \( 1 \leq r \leq \psi(a) \), there are maps
\[
\Sigma^{2^a(2n+1)-1} P_{-2^a(2n+1)+r}^\infty \to \Sigma^{2^a(2n+1)} P_{-2^a(2n+1)+r-1}^\infty \to S^0.
\]
Here, \( \partial \) is the cofiber of \( \Sigma^{2^a(2n+1)-1} P_{-2^a(2n+1)}^\infty \to \Sigma^{2^a(2n+1)-1} P_{-2^a(2n+1)+r}^\infty \), and \( s \) is any map splitting off the bottom cell of \( P_{-2^a(2n+1)+r-1}^\infty \). The above composite is detected by \( \rho^{2^a-r \tau^{2^a-1}(4n+1) h_a} \).

**Proof.** Recall that
\[
\text{Ext}^1_{BC_2} = \mathbb{F}_2[\rho] \{ h_a : a \geq 1 \} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a}) \{ \tau^{2^a-1}(4n+1) h_a : n \in \mathbb{Z} \}.
\]
We have just analyzed which classes in the \( \rho \)-torsion summand are permanent cycles, leading to exactly the claimed \( \rho \)-torsion permanent cycles with representatives as described in (3). Lemma 7.2.2 implies that \( h_1, h_2, \) and \( h_3 \) are permanent cycles, and these detect the maps described in (1). Theorem 7.3.1 shows that \( \rho^h h_a \) supports a \( d_2 \)-differential for \( a \geq 5 \) and \( n \geq 0 \), and that \( h_4 \) supports a \( d_2 \)-differential but \( \rho h_4 \) does not. We are left with verifying that \( \rho h_4 \) is a permanent cycle detecting the map \( \text{Sq}(\sigma) \). Indeed, taking geometric fixed points yields an isomorphism \( \pi^\ast \mathbb{P}^\infty_* \cong \pi^\ast \mathbb{P}^\infty^{\text{cl}} \) which sends \( \text{Sq}(\alpha) \to \alpha \) for any \( \alpha \in \pi^\ast \mathbb{P}^\infty \). This isomorphism is modeled on Adams spectral sequences by \( \text{Ext}^1_{\mathbb{P}^\infty} \cong \text{Ext}^1_{\mathbb{P}^\infty^{\text{cl}}} \cong \text{Ext}^1_{\mathbb{P}^\infty^{\text{cl}}} \). As \( \rho h_4 \) is the only class in its degree lifting \( h_3 \in \text{Ext}^1_{\mathbb{P}^\infty^{\text{cl}}} \), it must be that \( \rho h_4 \) detects \( \text{Sq}(\sigma) \).

**Proposition 7.4.2** implies that the classes \( \tau^{2^a-1}(4n+1) h_a \) must support Adams differentials for \( a \geq 4 \). Although we do not compute all these differentials, we do give the following.

**Proposition 7.4.8.** For all \( n \geq 0 \) and \( a \geq 3 \), there is a differential
\[
d_2(\tau^{2^a(4n+1)} h_{a+1}) = (h_0 + \rho h_1)(\tau^{2^a-1}(4n+1) h_a)^2.
\]
Theorem 7.4.9. The class \( \tau^{4(4n+1)}h_3 \) is a permanent cycle by Corollary 7.4.6, detecting a class which we might call \( \tau^{4(4n+1)}\sigma \). By Lemma 7.2.1, \( 2 \cdot (\tau^{4(4n+1)}\sigma)^2 = 0 \), and so \( (h_0 + \rho h_1) \cdot (\tau^{4(4n+1)}h_3)^2 \) must die. This class is not divisible by \( \rho \), and the only non-\( \rho \)-divisible classes that may hit it are \( \tau^8h_4 \) and \( \tau^8h_4 + \rho^3h_5 \). By Theorem 7.3.1, if \( d_2(\tau^8h_4 + \rho^3h_5) = (h_0 + \rho h_1) \cdot (\tau^{4(4n+1)}h_3)^2 \), then \( d_2(\tau^8h_4) = (h_0 + \rho h_1) \cdot (\tau^{4(4n+1)}h_3 + h_4)^2 \). This is not possible, as \( \tau^8h_4 \) is \( \rho \)-torsion and this target is not. Thus in fact \( d_2(\tau^8h_4) = (h_0 + \rho h_1) \cdot (\tau^{4(4n+1)}h_3)^2 \) as claimed.

Next consider the case \( a > 3 \). The \( \rho \)-torsion subgroup of \( \text{Ext}_R \) in the degree of \( d_2(\tau^{2a+1}(4n+1)h_{a+1}) \) is given by \( \mathbb{F}_2[h_0, \rho h_1] \otimes \mathbb{F}_2((\tau^{2a+1}(4n+1)h_a)^2) \). These classes are not divisible by \( \rho^2 \), and so it suffices to verify the differential in the Adams spectral sequence for the cofiber of \( \rho^2 \). By [BS20, Lemma 7.8], this cofiber is a ring spectrum, so its Adams spectral sequence is multiplicative. As \( \tau^2 \) is a cycle, \( \tau^4 \) is a \( d_2 \)-cycle, so we reduce to showing \( d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2 \). This was shown in Theorem 7.3.1. □

We may summarize what we have learned as follows.

**Theorem 7.4.9.** The nontrivial \( d_2 \)-differentials out of the 1-line of the \( \mathbb{R} \)-motivic Adams spectral sequence are exactly those given in the following table.

| Source | Target | Constraints |
|--------|--------|-------------|
| \( h_4 \) | \( h_0h_3^2 \) | \( a \geq 5, r \geq 0 \) |
| \( \rho \cdot h_a \) | \( \rho^3(h_0 + \rho h_1)h_a^2 \) | \( a \geq 4, 0 \leq r \leq 2^{a-1} - 1 \) |
| \( \rho^r \cdot h_a \) | \( \rho^r(h_0 + \rho h_1)(\tau^{2a-2}(4n+1)h_{a-1})^2 \) | \( a \geq 4, 0 \leq r \leq 2^{a-1} - 1 \) |

The 1-line of the \( E_3 \)-page of the \( \mathbb{R} \)-motivic Adams spectral sequence has a basis given by the elements in the following table.

| \( \mathbb{F}_2[\rho] \)-module generator | Constraints | \( \rho \)-torsion exponent |
|---------------------------------------|-------------|--------------------------|
| \( h_a \) | \( a \in \{0, 1, 2, 3\} \) | \( \infty \) |
| \( \rho h_1 \) | \( \infty \) | \( \infty \) |
| \( \tau^{2a+1}(4n+1)h_a \) | \( n \geq 0 \) and \( a \in \{0, 1, 2, 3\} \) | \( 2^a \) |
| \( \rho^{a-1} \cdot \tau^{2a-1}(4n+1)h_a \) | \( n \geq 0 \) and \( a \geq 4 \) | \( 2^{a-1} + 1 \) |

Those classes in \( \text{Ext}_R^1 \) which are permanent cycles are given in the following table.

| \( \mathbb{F}_2[\rho] \)-module generator | Constraints | \( \rho \)-torsion exponent | Stem |
|---------------------------------------|-------------|--------------------------|-----|
| \( h_a \) | \( a \in \{0, 1, 2, 3\} \) | \( \infty \) | \( 2^a - 1 \) |
| \( \rho h_1 \) | \( \infty \) | 14 | |
| \( \tau^{2a+1}(4n+1)h_a \) | \( n \geq 0 \) and \( a \in \{0, 1, 2, 3\} \) | \( 2^a \) | \( 2^a - 1 \) |
| \( \rho^{a-1} \cdot \tau^{2a-1}(4n+1)h_a \) | \( n \geq 0, a \geq 4 \) | \( \psi(a) \) | \( \psi(a) - 1 \) |

**Proof.** All of this is immediate from Theorem 7.3.1, Proposition 7.4.2, Proposition 7.4.8, Theorem 7.4.7, and the \( \rho \)-torsion exponent of generators of \( \text{Ext}_R^3 \) given in Theorem 4.2.12, with the following exception: Proposition 7.4.8 produces differentials \( d_2(\tau^{8(4n+1)}h_4) = (h_0 + \rho h_1)(\tau^{4(4n+1)}h_3)^2 \), and one must use Proposition 4.3.4(4) to check that this target has \( \rho \)-torsion exponent 7. □

7.5. Finite fields. We now study the case where \( F \) is a finite field. For the most part, this case follows by combining Theorem 7.3.1 with differentials out of \( \text{Ext}_F^0 \) that may be deduced from work of Kylling [Kyl15]. By naturality, our discussion in this subsection gives information for \( F \) an arbitrary field of odd characteristic.

We will need the following definition.
**Definition 7.5.1.** For an integer $q$, let $\nu_2(q)$ denote the 2-adic valuation of $q$, i.e.

$$q = 2^{\nu_2(q)}(2n + 1)$$

for some integer $n$, and let

$$\varepsilon(q) = \nu_2(q - 1), \quad \lambda(q) = \nu_2(q^2 - 1).$$

We now split into cases based on congruence of the order of the field mod 4.

7.5.1. $q \equiv 1 \pmod{4}$. Fix a prime-power $q$ such that $q \equiv 1 \pmod{4}$. We work over $F = \mathbb{F}_q$. Recall that $\text{Ext}_q = \text{Ext}_{(1)}\{1, u\}$. In particular,

$$\text{Ext}^1_q = \mathbb{F}_2[\tau]\{1, u\} \otimes \mathbb{F}_2\{h_a : a \geq 0\}.$$ 

The class $u$ is a permanent cycle for degree reasons, and we have already computed the differential on all the classes $h_a$. However the story does not stop there; instead, we have the following.

**Lemma 7.5.2.** There are differentials

$$d_{\varepsilon(q)+s}(\tau^{2^s}) = u\tau^{2^s-1}h_0^{\varepsilon(q)+s}$$

for all $s \geq 0$.

**Proof.** In [Kyl15, Lemma 4.2.1], Kylling produces identical differentials in the $\mathbb{F}_q$-motivic Adams spectral sequence for $H\mathbb{Z}$. Recall that the claimed differentials follow by naturality. \hfill \Box

This may be combined with Theorem 7.3.1 to easily compute all differentials out of the 1-line.

**Theorem 7.5.3.** For $q \equiv 1 \pmod{4}$, the 1-line of the $\mathbb{F}_q$-motivic Adams spectral sequence supports only the nontrivial differentials given in the following table.

| Source | $d_r$ | Target | Constraints |
|--------|-------|--------|-------------|
| $\tau^nh_0$ | $d_{\varepsilon(q)+\nu_2(n)}$ | $\tau^{n-1}h_0^{\varepsilon(q)+\nu_2(n)+1}$ | $n \geq 1$ |
| $\tau^{2n+1}h_2$ | $d_2$ | $u\tau^{2n}h_2h_0^2$ | $n \geq 0$, $\varepsilon(q) = 2$ |
| $\tau^{2n+1}h_3$ | $d_3$ | $u\tau^{2n}h_3h_0^2$ | $n \geq 0$, $\varepsilon(q) = 2$ |
| $\tau^{2n+1}h_4$ | $d_3$ | $u\tau^{2n+1}h_3h_0^3$ | $n \geq 0$, $\varepsilon(q) = 3$ |
| $\tau^{2n+2}h_3$ | $d_3$ | $u\tau^{2n+1}h_3h_0^3$ | $n \geq 0$, $\varepsilon(q) = 2$ |
| $\tau^{n}h_b$ | $d_2$ | $\tau^n h_0 h_{b-1}^2 + d_2(\tau^n)h_b$ | $n \geq 0$, $b \geq 4$ |
| $u\tau^n h_b$ | $d_2$ | $u\tau^n h_0 h_{b-1}^2$ | $n \geq 0$, $b \geq 4$ |

After these have been run, the 1-line of the $E\infty$-page of the $\mathbb{F}_q$-motivic Adams spectral sequence has a basis given by the elements in the following table.

| Class | Constraints |
|-------|-------------|
| $h_0$ | $n \geq 0$ |
| $\tau^n h_1$ | $n \geq 0$, where if $\varepsilon(q) = 2$ then $n \equiv 0 \pmod{2}$ |
| $\tau^n h_2$ | $n \geq 0$, where if $\varepsilon(q) = 2$ then $n \equiv 0 \pmod{4}$, and if $\varepsilon(q) = 3$ then $n \equiv 0 \pmod{2}$ |
| $u\tau^n h_b$ | $n \geq 0$, $b \in \{0, 1, 2, 3\}$ |

**Proof.** The first four families of differentials follow immediately from Lemma 7.5.2 and Lemma 7.2.2, and the remaining two by combining Lemma 7.5.2 with Theorem 7.3.1. Note in particular that $d_2(\tau^n) \equiv 0 \pmod{u}$, and thus $d_2(\tau^n h_b) \neq 0$ for $b \geq 4$. The second table may be easily read off the first, provided we verify that we have not missed any differentials, i.e. that the classes listed...
Theorem 7.5.6. For degree reasons, the only possible nontrivial differentials on the classes $\tau^n h_b$ with $b \in \{1, 2, 3\}$ would be of the form

1. $d_\tau(\tau^n h_1) \equiv \tau^{n-1} h_0^{-1}$;
2. $d_2(\tau^n h_2) \equiv \tau^{n-1} h_0^2 h_2$;
3. $d_2(\tau^n h_3) \equiv \tau^{n-1} h_0^2 h_3$;
4. $d_3(\tau^n h_3) \equiv \tau^{n-1} h_0^2 h_3$;

with $n \geq 1$. The first is impossible for $n = 1$ as $h_0$ detects 2 and thus no power of $h_0$ may be killed, and is impossible for $n \geq 2$ as the class $\tau^{n-1} h_0^{-1}$ must support the differential given the first row of the first table. The remaining three differentials may occur, and when they occur is accounted for in the given tables. □

7.5.2. $q \equiv 3 \pmod{4}$. Now fix a prime-power $q$ such that $q \equiv 3 \pmod{4}$. We work over $F = \mathbb{F}_q$. Recall that $\text{Ext}_{\mathbb{F}_q} = \text{Ext}_{(2)}$.

**Lemma 7.5.4.** We may identify

\[
\text{Ext}^2_{\mathbb{F}_q} = \mathbb{F}_2[\tau^2, \rho, \tau\rho]/(\rho^2 = \rho \cdot (\tau\rho) = (\tau\rho)^2 = 0),
\]

and $\text{Ext}^1_{\mathbb{F}_q}$ is the tensor product of $\mathbb{F}_2[\tau^2]$ with the following:

\[
\mathbb{F}_2\{h_0, \rho \cdot h_0\} \oplus \mathbb{F}_2\{h_1, \rho \cdot h_1, \rho \tau \cdot h_1, \tau h_1\} \oplus \mathbb{F}_2\{h_b, \rho \cdot h_b, \rho \tau \cdot h_b : b \geq 2\}.
\]

**Proof.** This follows quickly from our computation of $\text{Ext}_{\mathbb{F}_q}$, following the recipe of Remark 4.1.5. Alternately, one may compute the $\rho$-Bockstein spectral sequence

\[
\text{Ext}^1(\rho)/(\rho^2) \Rightarrow \text{Ext}^2_{(2)}
\]
directly (cf. [W017]); the only relevant differential is $d_1(\tau) = \rho h_0$. □

As in the previous case, powers of $\tau$ support arbitrarily long differentials.

**Lemma 7.5.5.** There are differentials

\[
d_{\lambda(q)+s}(\tau^{2^s+1}) = \rho \tau^{2^s+1} h_0^{\lambda(q)+s}
\]

for all $s \geq 0$. On the other hand, $\rho \tau$ is a permanent cycle.

**Proof.** The class $\rho \tau$ is a permanent cycle for degree reasons. In [Kyl15, Lemma 4.2.2], Kylling produces identical differentials in the $\mathbb{F}_q$-motivic Adams spectral sequence for $\mathbb{F}_q$-motivic $\mathbb{H}Z$. The claimed differentials follow by naturality. □

**Theorem 7.5.6.** For $q \equiv 3 \pmod{4}$, the 1-line of the $\mathbb{F}_q$-motivic Adams spectral sequence supports the differentials given in the following table.

| Source | $d_r$ | Target | Constraints |
|--------|--------|--------|-------------|
| $\tau^{2n} h_0$ | $d_{\lambda(q)+\nu_2(n)}$ | $\rho \tau^{2n-1} h_0^{\lambda(q)+\nu_2(n)+1}$ | $n \geq 1$ |
| $\tau^{4n+2} h_3$ | $d_4$ | $\rho \tau^{4n+1} h_0^2 h_3$ | $n \geq 0, \lambda(q) = 3$ |
| $\tau^{2n} h_b$ | $d_2$ | $\tau^{2n}(h_0 + \rho h_1) h_b^{2-1}$ | $n \geq 1, b \geq 4$ |
| $\rho \tau^{2n+1} h_b$ | $d_2$ | $\rho \tau^{2n+1} h_0^2 h_b^{2-1}$ | $n \geq 0, b \geq 4$ |

After the $d_2$-differentials have been run, the 1-line of the $E_3$-page of the $\mathbb{F}_q$-motivic Adams spectral sequence has a basis given by the classes in the following table.

| Class | Constraints |
|-------|-------------|
| $h_0$ | $n \geq 0, \epsilon \in \{0, 1\}, b \in \{1, 2, 3\}$ |
Of these, all the classes in the first region are permanent cycles, with the exception that \( \tau^{4n+2} h_3 \)
supports a \( d_3 \)-differential if \( \lambda(q) = 3 \). The classes \( \tau^{2n} h_0 \) for \( n \geq 1 \) are not permanent cycles, and we leave open the fate of the classes \( \rho \tau^{2n} h_b \) for \( n \geq 1 \) and \( b \geq 4 \).

**Proof.** The given differentials follow quickly by combining Theorem 7.3.1 with Lemma 7.5.5, and this accounts for all \( d_2 \)-differentials. Note in particular that \( \tau^2 \) is a \( d_2 \)-cycle as \( \lambda(q) \geq 3 \) whenever \( q \equiv 3 \) (mod 4). Thus the given \( E_3 \)-page may be produced by linearly propagating the differentials of Theorem 7.3.1. Note also that \( d_3(\rho \tau^{2n} h_b) = \rho \tau^{2n}(h_0 + ph_1)h_b^{2n-1} = 0 \) for all \( n \geq 0 \) and \( b \geq 4 \), yielding the classes in final row of the second table.

It remains only to verify that the permanent cycles provided are indeed permanent cycles. As \( \rho \) and \( \rho \tau \) are permanent cycles for degree reasons, we may reduce to considering only the classes \( \tau^{2n} h_b \), \( \rho \tau^{4n+1} h_0 \), and \( \tau^{4n+1} h_1 \) for \( b \in \{1, 2, 3\} \) and \( n \geq 0 \). For degree reasons, the only possible nontrivial differentials supported by these classes would be of the form

1. \( d_2(\tau^{2n} h_b) \equiv \rho \tau^{2n-1} h_b^2 h_b \) for \( b \in \{2, 3\} \);\n2. \( d_2(\tau^{2n} h_3) \equiv \rho \tau^{2n-1} h_3^2 h_1 \);\nwith \( n \geq 1 \). The first does not hold, as \( \tau^2 \) and \( h_b \) are \( d_2 \)-cycles. The second holds only when \( \lambda(q) = 3 \), and this is accounted for in the theorem statement. \( \square \)

### 7.6. The \( p \)-adic rationals

We now work over \( F = \mathbb{Q}_p \), the \( p \)-adic rationals. This is very similar to the case where \( F = \mathbb{F}_q \), only where the additional input necessary to understand differentials out of \( \text{Ext}^1_{\mathbb{Q}_p} \) comes from work of Ormsby [Orm11] for \( p \) odd and Ormsby–Østvær [OØ13] for \( p = 2 \). The case where \( p \) is odd turns out to entirely reduce to what we have already done.

**Lemma 7.6.1.** There are the following differentials in the \( \mathbb{Q}_p \)-motivic Adams spectral sequence.

1. If \( p \equiv 1 \) (mod 4), then \( d_{a(q)+s}(\tau^{2^r}) = ur^{2^r-1}h_0^{a(q)+s} \);\n2. If \( p \equiv 3 \) (mod 4), then \( d_{a(q)+s}(\tau^{2^r+1}) = \rho r^{2^r+1-1}h_0^{a(q)+s} \).

**Proof.** In [Orm11, Theorem 5.2], Ormsby produces identical differentials in the \( \mathbb{Q}_p \)-motivic Adams spectral sequence for \( BP(0) \). The claimed differentials follow by naturality. \( \square \)

We may summarize the situation as follows.

**Theorem 7.6.2.** Fix an odd prime \( p \), and consider the facts outlined about the \( \mathbb{F}_p \)-motivic Adams spectral sequence in Theorem 7.5.3 and Theorem 7.5.6. The same facts hold for the \( \mathbb{Q}_p \)-motivic Adams spectral sequence upon tensoring with \( \mathbb{F}_p(1, \pi) \).

**Proof.** The class \( \pi \) is a permanent cycle for degree reasons, and the differentials given in Lemma 7.6.1 agree with those given in Lemma 7.5.2 and Lemma 7.5.5. All of the work carried out over \( \mathbb{F}_p \) then goes through verbatim, only where everything in sight has a twin copy indexed by \( \pi \). \( \square \)

**Remark 7.6.3.** The somewhat awkward phrasing of Theorem 7.6.2 is necessary as we did not wish to repeat two verbatim copies of both Theorem 7.5.3 and Theorem 7.5.6, but we have not shown that the 1-line of the \( \mathbb{Q}_p \)-motivic Adams spectral sequence is a direct sum of two copies of the 1-line of the \( \mathbb{F}_p \)-motivic Adams spectral sequence. The possible failure of this arises from the fact that when \( p \equiv 3 \) (mod 4), the classes \( \rho \tau^{2n} h_b \) for \( b \geq 4 \) could support different higher differentials over \( \mathbb{F}_p \) and \( \mathbb{Q}_p \). <
The case where \( p = 2 \) requires a separate analysis. Recall that
\[
\text{Ext}_{\mathbb{Q}_2} = \text{Ext}_{(3)} \{1\} \oplus \text{Ext}_{(1)} \{u, \pi\}.
\]

**Lemma 7.6.4.** We may identify
\[
\text{Ext}_{(3)}^0 = F_2(\tau^4, \rho^2\tau, \rho^2\tau^3, \rho) \subset F_2[\tau, \rho]/(\rho^3),
\]
and \( \text{Ext}_{(3)}^1 \) is the tensor product of \( F_2(\tau^4) \) with the direct sum of the following modules:
\[
F_2 \left\{ h_0, \tau^2 h_0, \rho^2 \tau h_0, \rho^2 \tau^3 h_0 \right\} \\
F_2 \{1, \rho\} \otimes F_2 \{\tau h_1\} \oplus F_2 \{\rho \tau^3 h_1\} \oplus F_2 \{1, \rho, \rho^2, \rho^2 \tau, \rho^2 \tau^2, \rho^2 \tau^3\} \otimes F_2 \{h_1\} \\
F_2 \{1, \rho, \rho^2, \rho^2 \tau, \rho^2 \tau^2, \rho^2 \tau^3\} \otimes F_2 \{h_b : b \geq 2\}.
\]

**Proof.** As with Lemma 7.5.4, this follows from our computation of \( \text{Ext}_R \) via the recipe in Remark 4.1.5, or via the \( \rho \)-Bockstein spectral sequence; here, the relevant \( \rho \)-Bockstein differentials are \( d_1(\tau) = \rho h_0 \) and \( d_2(\tau^2) = \rho^2 \tau h_1 \).

**Lemma 7.6.5.** The classes
\[
\tau^{4n+1} \rho^2, \quad \tau^{2n} \rho, \quad \tau^{4n+3} \rho^2, \quad \pi \tau^n, \quad u, \quad u \tau^{2n+1}
\]
are permanent cycles. There are differentials
\[
d_{4+r}(\tau^{2r+2}) = \pi \tau^{2r+2-1} h_0^{4+r}, \quad d_{3+r}(u \tau^{2r+1}) = \rho^2 \tau^{2r+1-1} h_0^{3+r}, \quad d_{3+r}(\tau^{2r+1} h_0) = \pi \tau^{2r+1-1} h_0^{4+r}
\]
for all \( r \geq 0 \).

**Proof.** In [OO13, Lemma 5.7], Ormsby–Østvær compute differentials in the \( \mathbb{Q}_2 \)-motivic Adams spectral sequence for \( BP(0) \).\(^{15}\) The claimed facts follow by comparison.

**Theorem 7.6.6.** The 1-line of the \( \mathbb{Q}_2 \)-motivic Adams spectral sequence supports the following nontrivial differentials.

| Source | \( d_r \) | Target | Constraints |
|--------|-----------|--------|-------------|
| \( \tau^{2n} h_0 \) | \( d_{3+v_2(n)} \) | \( \pi \tau^{2n-1} h_0^{4+v_2(n)} \) | \( n \geq 1 \) |
| \( \tau^{4n} h_b \) | \( d_2 \) | \( \tau^{4n}(h_0 + \rho h_1) h_0^{3-1} \) | \( n \geq 0, b \geq 4 \) |
| \( \rho \tau^{2n} h_b \) | \( d_2 \) | \( \rho^2 \tau^{2n} h_0^{3-1} \) | \( n \geq 0, b \geq 5 \) |
| \( u \tau^{2n} h_b \) | \( d_2 \) | \( u \tau^n h_0 h_0^{2-1} \) | \( n \geq 0, b \geq 4 \) |
| \( \pi \tau^{2n} h_b \) | \( d_2 \) | \( \pi \tau^n h_0 h_b^{2-1} \) | \( n \geq 0, b \geq 4 \) |
| \( u \tau^{4n+2} h_3 \) | \( d_3 \) | \( \rho^2 \tau^{4n+1} h_0^{3} h_3 \) | \( n \geq 0 \) |

After all the \( d_2 \)-differentials have been run, the 1-line of the \( E_3 \)-page of the \( \mathbb{Q}_2 \)-motivic Adams spectral sequence has a basis given by the classes in the following table.

| Class | Constraints |
|-------|-------------|
| \( h_0 \) | \( n \geq 0, \delta \in \{0, 1, 2\}, b \in \{1, 2, 3\} \) |
| \( \rho^2 \tau^{2n+1} h_0 \) | \( n \geq 0 \) |
| \( \rho^2 \tau^{2n+1} h_1 \) | \( n \geq 0, e \in \{0, 1\} \) |
| \( \rho^{1+e} \tau^{2n+1} h_1 \) | \( n \geq 0, e \in \{0, 1\} \) |
| \( u h_0 \) | \( n \geq 0 \) |
| \( u \tau^{2n+1} h_0 \) | \( n \geq 0 \) |
| \( u \tau^n h_b \) | \( n \geq 0, b \in \{1, 2\} \) |

\(^{15}\)Note that \( \tau^2 \) lives on the \( E_2 \)-page in their computation.
We must verify that all the classes we give as permanent cycles are indeed permanent cycles.

**Definition 7.5.1.**

Theorem 7.7.1. The 1-line of the $E_3$-page of the $\mathbb{Q}$-motivic Adams spectral sequence is given by a direct sum of that for the $\mathbb{R}$-motivic Adams spectral sequence with the classes in the following table, where $p$ ranges through all primes.

| Class            | Constraints |
|------------------|-------------|
| $\tau^2n_0$     | $n \geq 0$  |
| $\tau^4n_0$     |             |
| $\tau^n h_0$    | $n \geq 0$, $b \in \{0, 1, 2, 3\}$ |
| $\rho^2 \tau^n h_0$ | $n \geq 1$, $\epsilon \in \{0, 1\}$ |
| $\rho^2 \tau^{n+2} h_0$ | $n \geq 0$, $\epsilon \in \{0, 1\}$ |
| $\rho^2 \tau^{n+1} h_0$ | $n \geq 0$ |
| $\rho^2 \tau^{n+1} h_3$ | $n \geq 0$, $b \geq 5$ |

Of these, the classes in the first region are permanent cycles, the classes $u^2 \tau^{2n+1} h_3$ with $n \geq 1$ and $\epsilon \in \{0, 1\}$, as well as $u^2 \tau^{4n+2} h_3$ with $n \geq 0$, support higher differentials, and we leave open the fate of the classes $\rho^1 \tau^{-4n} h_4$ and $\rho^2 \tau^{-4n} h_4$ for $n \geq 0$, $\epsilon \in \{0, 1\}$, and $b \geq 5$.

**Proof.** The given differentials follow by combining Theorem 7.3.1 with Lemma 7.6.5. For example,

$$d_2(\rho \tau^{2n} h_3) = \rho \tau^{2n} \cdot d_2(h_3) = \rho \tau^{2n} \cdot (h_0 + \rho h_1)h_{b-1}^2 = \rho^2 \tau^{2n} h_1 h_{b-1}^2$$

for $b \geq 4$, which is nonzero precisely when $b \geq 5$; as another example,

$$d_3(u \tau^{4n+2} h_3) = d_3(u \tau^2) \cdot \tau^{4n} h_3 = \rho^2 \tau \cdot \tau^{4n} h_3 = \rho^2 \tau^{4n+1} h_3.$$  

We must verify that all $d_2$-differentials are accounted for in this table; the claimed description of the $E_3$-page follows quickly. We must also verify that the classes we give as permanent cycles are indeed permanent cycles. It suffices to verify the latter.

We may cut down the number of classes to consider by taking into account the classes which are products of the permanent cycles given in Lemma 7.6.5 with some other class. After this reduction, degree considerations rule out all differentials except for possibly

1. $d_2(\tau^{4n+1} h_3) \Rightarrow \tau^{4n} h_{0}^{r+1};$
2. $d_2(\rho \tau^{4n+3} h_1) \in \mathbb{F}_2 \{u, \tau\} \otimes \mathbb{F}_2 \{\tau^{4n+2} h_{0,1}^{r+1}\};$
3. $d_2(\rho \tau^{4n+2} h_1) \Rightarrow \rho \tau^{2n+1} h_{0,1}^{r+1};$
4. $d_2(u \tau^{2n} h_1) \Rightarrow \rho \tau^{2n-1} h_{0,1}^{r+1};$

with $n \geq 0$, and in the fourth case $n \geq 1$. In all cases, the possible nonzero targets are present and not boundaries in Ormsby–Østvær’s computation of the Adams spectral sequence for $\mathbb{Q}_2$-motivic $BP(0)$ [0013], so by naturality they cannot be boundaries in the Adams spectral sequence for the sphere. Thus these possible nonzero differentials are in fact not possible, yielding the theorem. □

### 7.7. The rational numbers

We end by considering the case $F = \mathbb{Q}$. By naturality, this gives information over arbitrary fields of characteristic zero. Recall the functions $\varepsilon$ and $\lambda$ defined in Definition 7.5.1.

**Theorem 7.7.1.** The 1-line of the $E_3$-page of the $\mathbb{Q}$-motivic Adams spectral sequence is given by a direct sum of that for the $\mathbb{R}$-motivic Adams spectral sequence with the classes in the following table, where $p$ ranges through all primes.

| Class            | Constraints |
|------------------|-------------|
| $\tau^n h_0[2]$  | $n \geq 0$, $b \in \{0, 1, 2, 3\}$ |
| $h_0[p]$         | $p \equiv 1 (4)$ |
| $\tau^n h_1[p]$  | $p \equiv 1 (4)$, $n \geq 0$ |
| $\tau^{2n} h_2[p]$ | $p \equiv 1 (4)$, $n \geq 0$ |
| $\tau^{2n+1} h_2[p]$ | $p \equiv 1 (4)$, $n \geq 0$, $\varepsilon(p) \geq 3$ |
| $\tau^{4n} h_3[p]$ | $p \equiv 1 (4)$, $n \geq 0$ |
| $\tau^{4n+2} h_3[p]$ | $p \equiv 1 (4)$, $n \geq 0$, $\varepsilon(p) \geq 3$ |
| $\tau^{2n+1} h_3[p]$ | $p \equiv 1 (4)$, $n \geq 0$, $\varepsilon(p) \geq 4$ |
Moreover, we have the following information about higher differentials. The classes in the first region of this table are permanent cycles, as are the classes $h_a$ and $\tau^{2n-1}(4n+1)h_a$ for $a \leq 3$. The classes in the second region of this table support higher differentials, as do the classes in $\text{Ext}^3_k$ which must support higher differentials by Theorem 7.4.9. We leave open the fate of the classes in the third region of this table, as well as the possibility of exotic higher differentials on the classes $\rho h_4$ and $\rho^{2n-\psi(a)}\tau^{2n-1}(4n+1)h_a$ for $a \geq 4$.

Proof. Recall the splitting

$$\text{Ext}_q = \text{Ext}_R \oplus \text{Ext}_{(1)}\{[2]\} \oplus \text{Ext}_{(1)}\{[p]\}, a_p : p \equiv 1 \pmod{4}\} \oplus \text{Ext}_{(2)}\{u_p : p \equiv 3 \pmod{4}\}$$

implied by Lemma 7.1.1. As in the proof of Proposition 7.1.3, each of these summands is itself either $\text{Ext}^3_k$ or an identifiable summand of some corresponding $\text{Ext}^3_{q_p}$; for $p$ odd, this summand looks like $\text{Ext}^3_{q_p}$. We may thus read the given table off the information given in Theorem 7.4.9, Theorem 7.6.2 (with Theorem 7.5.3 and Theorem 7.5.6), and Theorem 7.6.6, provided we verify the following claim: if $\alpha[p] \in \text{Ext}^1_q$ is a class in stem $s \leq 6$, then $\alpha[p]$ or $\alpha u_p$ is a $d_s$-cycle if and only if it projects to a $d_s$-cycle in the $Q_p$-motivic Adams spectral sequence; and likewise if $a \in \text{Ext}^3_k$ is a class in stem $s \leq 7$, then $a$ is a $d_s$-cycle in the $Q$-motivic Adams spectral sequence if and only if it projects to a $d_s$-cycle in the $R$-motivic Adams spectral sequence.

As in the proofs of Theorem 7.5.3, Theorem 7.5.6, and Theorem 7.6.6, differentials on the classes $\alpha[p]$ and $\alpha u_p$ in stems $s \leq 6$ are completely determined by the structure of differentials on the classes $[p] \tau^{2n}$ and $u_p \tau^{2n}$ in the $Q$-motivic Adams spectral sequence for $BP(0)$, together with the fact that $h_0$, $h_1$, $h_2$, and $h_3$ are permanent cycles. The $Q$-motivic Adams spectral sequence for $BP(0)$ was computed by Ormsby–Östvær in [OO13, Theorem 5.8]. We find that differentials on the classes $[p] \tau^{2n}$ and $u_p \tau^{2n}$ in the $Q$-motivic Adams spectral sequence for $BP(0)$ are entirely detected over $Q_p$, and our first claim follows. That the classes $h_a \in \text{Ext}^3_k$ for $a \leq 3$ are permanent cycles was seen in Lemma 7.2.2, and the classes $\tau^{2n-1}(4n+1)h_a \in \text{Ext}^3_k$ must be permanent cycles for $a \leq 3$ as there is no room for exotic higher differentials. □

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