Local computation of influence propagation through Bayes linear belief networks

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Abstract

In recent years there has been interest in the theory of local computation over probabilistic Bayesian graphical models. In this paper, local computation over Bayes linear belief networks is shown to be amenable to a similar approach. However, the linear structure offers many simplifications and advantages relative to more complex models, and these are examined with reference to some illustrative examples.

Keywords: BAYES LINEAR METHODS; BELIEF PROPAGATION; DYNAMIC LINEAR MODELS; EXCHANGEABILITY; GRAPHICAL MODELS; LOCAL COMPUTATION.

1 Introduction

Conditional independence graphs are of vital importance in the structuring, understanding and computing of high dimensional complex statistical models. For a review of early work in this area, see [17], the references and the discussion, and also [2]. The above mentioned work is concerned with updating in discrete probability networks. For a discussion of updating in networks with continuous random variables, see [15], for example. For a general overview of the theory of graphical models, see [16].

Also relevant to this paper is the work on graphical Gaussian models. [16], [23] and [21] discuss the properties of such models. [18] examine data propagation through a graphical Gaussian network, and apply their results to a dynamic linear model (DLM). Here, the aim is to link the theory of local computation over graphical Gaussian networks to the Bayes linear framework for subjective statistical inference, and the many interpretive and diagnostic features associated with that methodology, in particular.

2 Bayes linear methods

2.1 Overview

In this paper, a Bayes linear approach is taken to subjective statistical inference, making expectation (rather than probability) primitive. An overview of the methodology is given in [4]. The foundations of the theory are quite general, and are outlined in the context of second-order exchangeability in [10], and discussed for more general situations in [11]. Bayes linear methods may be used in order to learn about any quantities of interest, provided only that a mean and variance specification is made for all relevant quantities, and a specification for the covariance between all pairs of quantities is made. No distributional assumptions are necessary. There are many interpretive and diagnostic features of the Bayes linear methodology. These are discussed with reference to [B/D] (the Bayes linear computer programming language) in [14].

2.2 Bayes linear conditional independence

Conventional graphical models are defined via strict probabilistic conditional independence [1]. However, as [13] demonstrates, all that is actually required is a tertiary operator ·Π· satisfying some simple properties. Any relation satisfying these properties is known as a generalised conditional independence relation. Bayes linear graphical models are based on what [13] refers to as weak conditional independence. In this paper, the

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relation will be referred to as *adjusted orthogonality*, in order to emphasise the linear structure underlying the relation.

Bayes linear graphical models based upon the concept of adjusted orthogonality are described in [8]. For completeness, and to introduce some notation useful in the context of local computation, the most important elements of the methodology are summarised here, and the precise form of the adjusted orthogonality relation is defined.

For vectors of random quantities, \( X \) and \( Y \), define

\[
\text{Cov} (X, Y) = E (XY^T) - E (X) E (Y)^T \\
\text{Var} (X) = \text{Cov} (X, X)
\]

Also, for any matrix, \( A \), \( A^\dagger \) represents the Moore-Penrose generalised inverse of \( A \).

**Definition 1** For all vectors of random quantities \( B \) and \( D \), define

\[
P_{\{D\}}^{\{B\}} = \text{Cov} (B, D) \text{Var} (D)^\dagger \\
T_{\{D\}}^{\{B\}} = P_{\{D\}}^{\{B\}} P_{\{B\}}^{\{D\}}
\]

These represent the fundamental operators of the Bayes linear methodology. \( P_{\{D\}}^{\{B\}} \) is the operator which updates the expectation vector for \( B \) based on the observation of \( D \), and \( T_{\{D\}}^{\{B\}} \) updates the variance matrix for \( B \) based on observation of \( D \). Local computation over Bayes linear graphical models is made possible by local computation of these operators.

**Definition 2** For all vectors of random quantities \( B, C \) and \( D \), define

\[
E_D (B) = E (B) + P_{\{D\}}^{\{B\}} [D - E (D)] \\
\text{Cov}_D (B, C) = \text{Cov} (B - E_D (B), C - E_D (C))
\]

\( E_D (B) \) is the expectation for \( B \) adjusted by \( D \). It represents the linear combination of a constant and the components of \( D \) closest to \( B \) in the sense of expected squared loss. It corresponds to \( E (B|D) \) when \( B \) and \( D \) are jointly multivariate normal. \( \text{Cov}_D (B, C) \) is the covariance between \( B \) and \( C \) adjusted by \( D \), and represents the covariance between \( B \) and \( C \) given observation of \( D \). It corresponds to \( \text{Cov} (B, C|D) \) when \( B, C \) and \( D \) are jointly multivariate normal.

**Lemma 1** For all vectors of random quantities \( B, C \) and \( D \)

\[
\text{Cov}_D (B, C) = \text{Cov} (B, C) - \text{Cov} (B, D) P_{\{D\}}^{\{C\}} T_{\{D\}}^{\{B\}} \\
\text{Var}_D (B) = (I - T_{\{D\}}^{\{B\}}) \text{Var} (B)
\]

**Proof**

Substituting \( \text{(3)} \) into \( \text{(4)} \) we get

\[
\text{Cov}_D (B, C) = \text{Cov} (B, C - E_D (C)) \\
= \text{Cov} \left( B, C - P_{\{D\}}^{\{C\}} D \right)
\]

which gives \( \text{(5)} \), and replacing \( C \) by \( B \) gives \( \text{(6)} \). \( \square \)

Note that \( \text{(6)} \) shows that \( T_{\{D\}}^{\{B\}} \) is responsible for the updating of variance matrices. Adjusted orthogonality is now defined.

**Definition 3** For random vectors \( B, C \) and \( D \)

\[
B \not\perp\!\!\!\!\perp C/D \iff \text{Cov}_D (B, C) = 0
\]

\( \text{(9)} \) shows that this relation does indeed define a generalised conditional independence property, and hence that all the usual properties of graphical models based upon such a relation hold.
2.3 Bayes linear graphical models

[8] defines a Bayes linear influence diagram based upon the adjusted orthogonality relation. [12] illustrate the use of Bayes linear influence diagrams in a multivariate forecasting problem. Relevant graph theoretic concepts can be found in the appendix of [3]. The terms moral graph and junction tree are explained in [22]. Briefly, an undirected moral graph is formed from a directed acyclic graph by marrying all pairs of parents of each node, by adding an arc between them, and then dropping arrows from all arcs. A junction tree is the tree of cliques of a triangulated moral graph. A tree is a graph without any cycles. A graph is triangulated if no cycle of length at least four is without a chord. A clique is a maximally connected subset of a triangulated graph.

In this paper, attention will focus on undirected graphs. An undirected graph consists of a collection of nodes \( B = \{ B_i | 1 \leq i \leq n \} \) for some \( n \), together with a collection of undirected arcs. Every pair of nodes, \( \{ B_i, B_j \} \) is joined by an undirected arc unless \( B_i \perp \!\!\!\!\perp B_j / B \{ B_i, B_j \} \). Here, the standard set theory notation, \( B \setminus A \) is used to mean the set of elements of \( B \) which are not in \( A \). An undirected graph may be obtained from a Bayes linear influence diagram by forming the moral graph of the influence diagram in the usual way.

In fact, local computation (the computation of global influences of particular nodes of the graph, using only information local to adjacent nodes) requires that the undirected graph representing the conditional independence structure is a tree. This tree may be formed as the junction tree of a triangulated moral graph, or better, by grouping together related variables “by hand” in order to get a tree structure for the graph. For the rest of this paper, it will be assumed that the model of interest is represented by an undirected tree defined via adjusted orthogonality.

3 Local computation on Bayes linear graphical models

3.1 Transforms for adjusted orthogonal belief structures

**Lemma 2** If \( B, C \) and \( D \) are random vectors such that \( B \perp \!\!\!\!\perp C / D \), then

\[
\text{Cov}(B, C) = \text{Cov}(B, D) P\{C\}^T_{\{D\}} \tag{10}
\]

This follows immediately from Definition 3 and (5).

**Lemma 3** If \( X, Y \) and \( Z \) are random vectors such that \( X \perp \!\!\!\!\perp Z / Y \), then

\[
\text{Cov}_X(Y, Z) = (I - T_{\{X\}}^{(Y)}) \text{Cov}(Y, Z) \tag{11}
\]

**Proof**

From (5)

\[
\text{Cov}_X(Y, Z) = \text{Cov}(Y, Z) - \text{Cov}(Y, X) \text{Var}(X)^\dagger \text{Cov}(X, Z) \tag{12}
\]

\[
= \text{Cov}(Y, Z) - P_{\{Y\}}^{(X)} \text{Cov}(X, Y) P_{\{X\}}^{(Z)} \tag{13}
\]

by Lemma 2

and the result follows.

**Theorem 1** If \( X, Y \) and \( Z \) are random vectors such that \( X \perp \!\!\!\!\perp Z / Y \), then

\[
P_{\{Z\}}^{\{X\}} = P_{\{Z\}}^{\{Y\}} P_{\{X\}}^{\{Y\}} \tag{14}
\]

\[
T_{\{X\}}^{(Z)} = P_{\{Z\}}^{(Y)} T_{\{X\}}^{(Y)} P_{\{Y\}}^{\{Z\}} \tag{15}
\]

**Proof**

\[
P_{\{X\}}^{\{Z\}} = \text{Cov}(Z, X) \text{Var}(X)^\dagger \tag{16}
\]

\[
= \text{Cov}(Z, Y) P_{\{Y\}}^{\{X\}} \text{Var}(X)^\dagger \text{Cov}(X, Y) \tag{17}
\]

by Lemma 2
Theorem 1 contains the two key results which allow local computation over Bayes linear belief networks.

### 3.2 Local computation on trees

The implications of Theorem 1 to Bayes linear trees should be clear from examination of Figure 1. To examine the effect of observing node \( Z \) on the graph, since these operators contain all necessary information about the adjustment of \( Z \) by \( Y \). Note further that it is a property of the graph that\( X \perp Z \), which is shown in Figure 1. The direct predecessor of \( Z \) is denoted by \( Y \). This provides a recursive method for the calculation of the transforms, which leads to the algorithm for the propagation of transforms throughout the tree, which is described in the next section.

### 3.3 Algorithm for transform propagation

Consider a tree with nodes \( B = \{B_1, \ldots, B_n\} \) for some \( n \). Each node, \( B_i \), represents a vector of random quantities. It also has an edge set \( G \), where each \( g \in G \) is of the form \( g = \{B_k, B_l\} \) for some \( k, l \). The resulting tree should represent a conditional independence graph over the random variables in question. It is assumed that each node, \( B_i \), has an expectation vector \( E_{B(i)} = E(B_i) \) and variance matrix \( V_{B(i)} = Var(B_i) \) associated with it. It is further assumed that each edge, \( \{B_k, B_l\} \), has the covariance matrix, \( C_{B(k),B(l)} = Cov(B_k, B_l) \) associated with it. This is the only information required in order to carry out Bayes linear local computation over such structures.

Now consider the effect of adjustment by the vector \( X \), which consists of some or all of the components of node \( B_j \) for some \( j \). Then, starting with node \( B_j \), calculate and store \( T_{B(j)} = T_{[X]}^{B(j)} \) and \( P_{B(i)} = P_{[X]}^{B(i)} \). Then, for each node \( B_k \in b(B_j) \equiv \{B_l | \{B_i, B_j\} \in G\} \) calculate and store \( T_{B(k)} \) and \( P_{B(k)} \), then for each node \( B_l \in b(B_k)\backslash B_j \), do the same, using Theorem 1 to calculate

\[
T_{B(i)} = T_{B(j)}^{B(k)} T_{B(k)} \quad \text{and} \quad P_{B(i)} = P_{B(j)}^{B(k)} P_{B(k)}
\]

In this way, recursively step outward through the tree, at each stage computing and storing the transforms using the transforms from the predecessor and the variance and covariance information over and between the current node and its predecessor.

Once this process is completed, associated with every node, \( B_i \in B \), there are matrices \( T_{B(i)} = T_{[X]}^{B(i)} \) and \( P_{B(i)} = P_{[X]}^{B(i)} \). These operators represent all information about the adjustment of the structure by \( X \).
Note however, that $X$ has not yet been observed, and that expectations, variances and covariances associated with nodes and edges have not been updated.

It is a crucial part of the Bayes linear methodology that a priori analysis of the model takes place, and that the expected influence of potential observables is examined. Examination of the eigen structure of the belief transforms associated with nodes of particular interest is the key to understanding the structure of the model, and the benefits of observing particular nodes. It is important from a design perspective that such analyses can take place before any observations are made. See [7] for a more complete discussion of such issues, and [8] for a discussion of the technical issues it raises.

3.4 Updating of expectation and covariance structures after observation

After observation of $X = x$, updating of the expectation, variance and covariance structure over the tree is required. Start at node $B_j$ and calculate

$$E'_{B(j)} = E_{B(j)} + P_{B(j)} (x - E(X))$$

$$V'_{B(j)} = (I - T_{B(j)}) V_{B(j)}$$

(23) (using 5). Replace $E_{B(j)}$ by $E'_{B(j)}$ and $V_{B(j)}$ by $V'_{B(j)}$. Then for each $B_k \in b(B_j)$ do the same, and also update the arc between $B_j$ and $B_k$ by calculating

$$C'_{B(j),B(k)} = (I - T_{B(j)}) C_{B(j),B(k)}$$

(24) (using Lemma 8), and replacing $C_{B(j),B(k)}$ by $C'_{B(j),B(k)}$. Again, step outwards through the tree, updating nodes and edges using the transforms previously calculated.

3.5 Pruning the tree

Once the expectations, variances and covariances over the structure have been updated, the tree should be pruned. If the adjusting node was completely observed (i.e. $X = B_j$), then $B_j$ should be removed from $B$, and $G$ should have any arcs involving $B_j$ removed. Further, leaf nodes and their edges may always be dropped without affecting the conditional independence structure of the graph. This is important if a leaf node is partially observed and the remaining variables are unobservable and of little diagnostic interest, since it means that the whole node may be dropped after observation of its observable components.

If a non-leaf node is partially observed, or a leaf node is observed, but its remaining components are observable or of interest, then the graph itself should remain unaffected, but the expectation, variance and covariance matrices associated with the node and its arcs should have the observed (and hence redundant) rows and columns removed (for reasons of efficiency — use of the Moore-Penrose generalised inverse ensures that no problems will arise if observed variables are left in the system).

3.6 Sequential adjustment

As data becomes available on various nodes, it should be incorporated into the tree one node at a time. For each node with observations, the transforms should be computed, and then the beliefs updated in a sequential fashion. The fact that such sequential updating provides a coherent method of adjustment is demonstrated in [3].

3.7 Local computation of diagnostics

Diagnostics for Bayes linear adjustments are a crucial part of the methodology, and are discussed in [8]. It follows that for local computation over Bayes linear networks to be of practical value, methodology must be developed for the local computation of Bayes linear diagnostics such as the size, expected size and bearing of an adjustment. The bearing represents the magnitude and direction of changes in belief. The magnitude of the bearing, which indicates the magnitude of changes in belief, is known as the size of the adjustment.

Consider the observation of data, $X = x$, and the partial bearing of the adjustment it induces on some node, $B_p$. Before observation of $X$, record $E = E_{B(p)}$ and $V = V_{B(p)}$. Also calculate the Cholesky factor, $A$ of $V$, so that $A$ is lower triangular, and $V = AA^T$. Once the observed value $X = x$ is known, propagate the
revised expectations, variances and covariances through the Bayes linear tree. The new value of $E_{B(p)}$ will be denoted $E'_{B(p)}$. Now the quantity

$$E' = A^{\dagger}(E'_{B(p)} - E)$$

represents the adjusted expectation for an orthonormal basis, $F = A^{\dagger}(B_p - E)$ for $B_p$ with respect to the a priori beliefs, $E$ and $V$. Therefore, $E'$ gives the coordinates of the bearing of the adjustment with respect to that basis.

The size of the partial adjustment is given by

$$\text{Size}_X (B_p) = ||E'||^2$$

where $|| \cdot ||$ represents the Euclidean norm. The expected size is given by

$$E(\text{Size}_X (B_p)) = \text{Tr}(T_{B(p)})$$

and so the size ratio for the adjustment (often of most immediate interest) is given by

$$\text{Sr}_X (B_p) = \frac{||E'||^2}{\text{Tr}(T_{B(p)})}$$

A size ratio close to one indicates changes in belief close to what would be expected. A size ratio smaller than one indicates changes in belief of smaller magnitude than would have been anticipated a priori, and a size ratio bigger than one indicates changes in belief of larger magnitude than would have been expected a priori. Informally, a size ratio bigger than 3 is often taken to indicate a diagnostic warning of possible conflict between a priori belief specifications and the observed data.

Cumulative sizes and bearings may be calculated in exactly the same way, simply by updating several times before computing $E'$. However, to calculate the expected size of the adjustment, in order to compute the size ratio, the cumulative belief transform must be recorded and updated at each stage, using the fact that

$$T_{[X+Y]}^{(B)} = I - (I - T_{[Y]}^{(B/)})(I - T_{[X]}^{(B)})$$

where $T_{[Y]}^{(B/)}$ represents the partial transform for $B$ by $Y$, with respect to the structure already adjusted by $X$. In other words, I minus the transforms at each stage multiply together to give I minus the cumulative transform. See [9] for a more complete discussion of the multiplicative properties of belief transforms.

### 3.8 Efficient computation for evidence from multiple nodes

To adjust the tree given data at multiple nodes, it would be inefficient to adjust the entire tree sequentially by each node in turn, if the nodes in question are “close together”. Here again, ideas may be borrowed from theory for the updating of probabilistic expert systems. It is possible to propagate transforms and projections from each node for which adjustment is required, to a strong root, and then propagate transforms from the strong root out to the rest of the tree. A strong root is a node which separates the nodes for which there is information, from as much as possible of the rest of the tree. In practice, there are many ways in which one can use the strong root in order to control information flow through the tree. An example of its use is given in Section 5.

### 3.9 Geometric interpretation, and infinite collections

In this paper, attention has focussed exclusively on finite collections of quantities, and matrix representations for Bayes linear operators. All of the theory has been developed from the perspective of pushing matrix representations of linear operators around a network. However, the Bayes linear methodology may be formulated and developed from a purely geometric viewpoint, involving linear operators on a (possibly infinite dimensional) Hilbert space. This is not relevant to practical computer implementations of the theory and algorithms – hence the focus on matrix formulations in this paper. However, from a conceptual viewpoint, it is very important, since one sometimes has to deal, in principle, with infinite collections of quantities, or probability measures over an infinite partition. In fact, all of the theory for local computation over Bayes linear belief networks developed in this paper is valid for the local computation of Bayes linear operators on an arbitrary Hilbert space. Consequently, the results may be interpreted geometrically, as providing a method of pushing linear operators around a Bayes linear Hilbert space network. A geometric form of Theorem 1 is derived and utilised in [13].
4 Example: A dynamic linear model

Figure 2 shows a Bayes linear graphical tree model for the first four time points of a dynamic linear model. Local computation will be illustrated using the example model, beliefs and data from [24]. Here, \( \theta_t \) represents the vector \((M_t, N_t)^T\) from that paper. The model takes the form

\[
X_t = (1, 0) \theta_t + \nu_t \tag{31}
\]

\[
\theta_t = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \theta_{t-1} + \omega_t \tag{32}
\]

where \( \text{Var}(\theta_1) = \text{diag}(400, 9) \), \( E(\theta_1) = (20, 0)^T \), \( E(\nu_t) = 0 \), \( E(\omega_t) = 0 \), \( \text{Var}(\nu_t) = 171 \), \( \text{Var}(\omega_t) = \text{diag}(4.75, 0.36) \) \( \forall t \) and the \( \nu_t \) and \( \omega_t \) are uncorrelated.

First, the nodes and arcs shown in Figure 2 are defined. Then the expectation and variance of each node is calculated and associated with each node, and the covariances between pairs of nodes joined by an arc is also computed, and associated with the arc. All of the expectations variances and covariances are determined by the model. For example, node \( X_1 \) has expectation vector \((20, 0)^T\) associated with it. Node \( \theta_1 \) has expectation vector \((20, 0)^T\) and variance matrix \( \text{diag}(400, 9) \) associated with it. The arc between \( X_1 \) and \( \theta_1 \) has associated with it the covariance matrix \((400, 9)\).

The effect of the observation of \( X_1 \) on the tree structure will be examined, and the effect on the node \( \theta_4 \) in particular, which has a priori variance matrix \( \begin{pmatrix} 500.3 & 29.2 \\ 29.2 & 10.1 \end{pmatrix} \) associated with it. Before actual observation of \( X_1 \), the belief transforms for the adjustment, may be computed across the tree structure. The transforms are computed recursively, in the following order.

\[
T_{X(1)} = \begin{pmatrix} 1 \end{pmatrix} \quad T_{\theta(1)} = \begin{pmatrix} 0.7 & 0 \\ 0 & 0 \end{pmatrix} \quad T_{\theta(2)} = \begin{pmatrix} 0.692 & -0.692 \\ 0 & 0 \end{pmatrix} \tag{33}
\]

\[
T_{\theta(3)} = \begin{pmatrix} 0.684 & -1.342 \\ 0 & 0 \end{pmatrix} \quad T_{\theta(4)} = \begin{pmatrix} 0.674 & -1.949 \\ 0 & 0 \end{pmatrix} \quad T_{X(4)} = \begin{pmatrix} 0.417 \end{pmatrix} \tag{34}
\]

The \( P \) matrices are calculated similarly. In particular, \( P_{\theta(4)} = (0.7, 0)^T \). A priori analysis of the belief transforms is possible. For example, \( \text{Tr}(T_{\theta(4)}) = 0.674 \), indicating that observation of \( X_1 \) is expected to reduce overall uncertainty about \( \theta_4 \) by a factor of 0.674. This is also the expected size of the bearing for the adjustment of \( \theta_4 \) by \( X_1 \).
Now, $X_1$ is observed to be 17, and so the expectations, variances and covariances may be updated across the structure. For example, beliefs about node $\theta_4$ were updated so that $E_{\theta(4)} = (17.9, 0)^T$ and $V_{\theta(4)} = \begin{pmatrix} 220.0 & 29.2 \\ 29.2 & 10.1 \end{pmatrix}$ after propagation. Also, calculating the bearing for the adjustment of $\theta_4$ by $X_1 = 17$, using \( \text{[26]} \) gives $E' = (-0.094, 0.042)^T$. Consequently, the size of the adjustment is 0.011 and the size ratio is 0.016.

Once evidence from the observed value of $X_1$ has been taken into account, the $X_1$ node, and the arc between $X_1$ and $\theta_1$ may be dropped from the graph. Note also that $\theta_1$ then becomes an unobservable leaf node, which may be of little interest, and so if desired, the $\theta_1$ node, and the arc between $\theta_1$ and $\theta_2$ may also be dropped. Observation of $X_2$ may now be considered. Using the updated, pruned tree, projections and transforms for the adjustment by $X_2$ may be calculated and propagated through the tree. For example, the (partial) belief transform for the adjustment of $\theta_4$ by $X_2$ is $\begin{pmatrix} 0.46 & -0.87 \\ 0.03 & -0.05 \end{pmatrix}$. If cumulative diagnostics are of interest, then it is necessary to calculate the cumulative belief transform, $\begin{pmatrix} 0.82 & -1.92 \\ 0.01 & 0.00 \end{pmatrix}$. This has trace 0.82, and so the resolution for the combined adjustment of $\theta_4$ by $X_1$ and $X_2$ is 0.82. Similarly, the expected size of the cumulative bearing is 0.82. $X_2$ is observed to be 22. The new expectations, variances and covariances may then be propagated through the tree. For example, the new expectation vector for $\theta_4$ is $(19.95, 0.13)^T$. The size of the cumulative bearing is 0.002, giving a size ratio of approximately 0.002. Again, the tree may be pruned, and the whole process may continue.
5 Example: n-step exchangeable adjustments

An ordered collection of random quantities, \( \{X_1, X_2, \ldots \} \) is said to be (second-order) n-step exchangeable if (second-order) beliefs about the collection remain invariant under an arbitrary translation or reflection of the collection, and if the covariance between any two members of the collection is fixed, provided only that they are a distance of at least \( n \) apart. Such quantities arise naturally in the context of differenced time series \( [23] \). n-step exchangeable quantities may be written in the form

\[
X_i = M + R_i, \quad \forall i
\]  

where the \( R_i \) are a mean zero n-step exchangeable collection such that the covariance between them is zero provided they are a distance of at least \( n \) apart. \( M \) represents the underlying mean for the collection, and \( R_i \) represents the residual uncertainty which would be left if the underlying mean became known. Introduction of a mean quantity helps to simplify a graphical model for an n-step exchangeable collection. For example, Figure 3 (top) shows an undirected graphical model for a 3-step exchangeable collection. Note that without the introduction of the mean quantity, \( M \), all nodes on the graph would be joined, not just those a distance of one and two apart. Figure 3 (bottom) shows a conditional independence graph for the same collection of quantities, duplicated and grouped together so as to make the resulting graph a tree. Note that each node contains 3 quantities, and that there is one less node than observables.

In general, for a collection of \( N \), n-step exchangeable quantities, the variables can be grouped together to obtain a simple chain graph, in the obvious way, so that there are \( N - n + 2 \) nodes, each containing \( n \) quantities. The resulting graph for 5-step exchangeable quantities is shown in Figure 4 (with the first four nodes missing).

In [23], 3-, 4-, and 5-step exchangeable collections, \( \{X_3^{(1)}, X_4^{(1)}, \ldots\}, \{X_4^{(2)}, X_5^{(2)}, \ldots\}, \) and \( \{X_5^{(3)}, X_6^{(3)}, \ldots\} \) are used in order to learn about the quantities, \( V_1, V_2 \) and \( V_3 \), representing the variances underlying the DLM discussed in the previous section. Since the observables sequences are 3-, 4-, and 5-step exchangeable, they may all be regarded as 5-step exchangeable, and so Figure 4 represents a graphical model for the variables, where \( V = (V_1, V_2, V_3)^T \), and \( \forall i, \ Z_i = (X_i^{(1)}, X_i^{(2)}, X_i^{(3)})^T \). Note that \( V \) represents (a known linear function of) the mean of the 5-step exchangeable vectors, \( Z_i \). Each node of the graph actually contains 15 quantities. For example, the first node shown contains \( \{V_1, V_2, V_3, X_3^{(1)}, X_4^{(1)}, X_5^{(2)}, X_5^{(3)}, X_6^{(1)}, X_6^{(2)}, X_6^{(3)}, X_7^{(1)}, X_7^{(2)}, X_7^{(3)}, X_8^{(1)}, X_8^{(2)}, X_8^{(3)}\} \). Note that the fact that quantities are duplicated in other nodes does not affect the analysis in any way. Observation of a particular quantity in one node will reduce to zero the variance of that quantity in any other node, as they will have a correlation of unity. Here, the fact that the Moore-Penrose generalised inverse is used in the definition of the projections and transforms becomes important.

For this model may be locally computed over this tree structure in the usual way.

Suppose now that information for quantities \( Z_5 \) to \( Z_9 \) is to become available simultaneously. This corresponds to information on the first two nodes (and others, but this will be conveyed automatically). The second node acts as a strong root for information from the first two nodes. The transform for the first node may be calculated using information on the first node, thus allowing computation of the transform for the second node given information on the first. Once the information from the first node has been incorporated into the first two nodes, the transform for the second node given information from the first two nodes may be calculated, and the resulting transform for the second node given information on the first two may be used in order to propagate information to the rest of the tree.

6 Implementation considerations

A test system for model building and computation over Bayes linear belief networks has been developed by the author using the MuPAD computer algebra system, described in [24] and [25]. MuPAD is a very high level object-oriented mathematical programming language, with symbolic computing capabilities, ideal for the rapid prototyping of mathematical software and algorithms. The test system allows definition of nodes and arcs of a tree, and attachment of relevant beliefs to nodes and arcs. Recursive algorithms allow computation of belief transforms for node adjustment, and propagation of updated means, variances and covariances through the tree. Note that whilst propagating outwards through the tree, updating of the different branches of the tree may proceed in parallel. MuPAD provides a "parallel for" construct which allows simple exploitation of this fact on appropriate hardware. Simple functions to allow computation of diagnostics for particular nodes also exist.
7 Conclusions

The algorithms described in this paper are very simple and easy to implement, and very fast compared to many other algorithms for updating in Bayesian belief networks. Further, by linking the theory with the machinery of the Bayes linear methodology, full \textit{a priori} and diagnostic analysis may also take place. \textit{A priori} analysis is particularly important in large sparse networks, where it is often not clear whether or not it is worth observing particular nodes, which may be “far” from nodes of interest. Similarly, diagnostic analysis is crucial, both for diagnosing misspecified node and arc beliefs, and for diagnosing an incorrectly structured model.

For those who already appreciate the benefits of working within the Bayes linear paradigm, the methodology described in this paper provides a mechanism for the tackling of much larger structured problems than previously possible, using local computation of belief transforms, adjustments and diagnostics.

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