Unimodal sequence generating functions arising from partition ranks

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Abstract
In this paper we study generating functions resembling the rank of strongly unimodal sequences. We give combinatorial interpretations, identities in terms of mock modular forms, asymptotics, and a parity result. Our functions imitate a relation between the rank of strongly unimodal sequences and the rank of integer partitions.

Keywords: Unimodal sequences, Strongly unimodal sequences, Partitions, Overpartitions, Unimodal ranks, Partition ranks, Dyson rank, $M_2$-rank, Asymptotics, Modular forms, Mock modular forms

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1 Introduction and statement of results
A sequence $\{a_j\}_{j=1}^n$ of positive integers is a unimodal sequence of size $n$ if it is of the form
$$a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_s \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n.$$The maximum value, $a_k$, is called the peak. If the inequalities are strict, the sequence is called strongly unimodal. Such sequences are related to integer partitions. Recall that a finite sequence $\{a_j\}_{j=1}^n$ of positive integers is a partition of size $n$ if it is of the form
$$a_1 \geq a_2 \geq \cdots \geq a_s \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n.$$Unimodal sequences, partitions, and similar objects appear throughout modern and classical literature on a variety of subjects including algebra, combinatorics, number theory, physics, and special functions [1,3,33,34]. To motivate our results, we discuss a few highlights in number theory. We focus on strongly unimodal sequences, rather than unimodal sequences. The enumeration function for strongly unimodal sequences seems to have first appeared as $X_p(n)$ in [1], along with a wealth of related functions; unimodal sequence type counting functions can also be found in older works such as [2,9,35].

We let $p(n)$ denote the number of partitions of size $n$ and let $u(n)$ denote the number of strongly unimodal sequences of size $n$. We note that the conventions for zero are somewhat inconsistent between strongly unimodal sequences and partitions, as we set $u(0) := 0$ but $p(0) := 1$. By standard counting techniques, the generating functions for partitions and strongly unimodal sequences are
$$P(q) := \sum_{n \geq 0} p(n) q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q)_{\infty}},$$
where we use the \( q \)-Pochhammer symbol, \((a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)\) for \( n \in \mathbb{N}_0 \cup \{\infty\}\). This extends to arbitrary \( n \) by setting \((a; q)_n := (a; q)_\infty (aq^n; q)_\infty\). Furthermore, we let \((a_1, a_2, \ldots, a_k; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n\). Throughout the article, \( q \) is a complex variable with \( 0 < |q| < 1 \).

Perhaps the three most famous results for the partition function are the following. There is the asymptotic formula of Hardy and Ramanujan [22, Eq. (1.41)]

\[
p(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi \sqrt{\frac{2n}{3}}}, \quad \text{as } n \to \infty.
\]

Moreover the function

\[
\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n) = q^{1/24} \frac{P(q)}{P(q')}, \quad (q := e^{2\pi i \tau} \text{ throughout}),
\]

is Dedekind’s eta function, which is a modular form of weight \( \frac{1}{2} \) (with multiplier). Lastly, there are Ramanujan’s congruences [31]

\[
p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7},
\]

\[
p(11n + 6) \equiv 0 \pmod{11}.
\]

By [32, Corollary 1.2], \( u(n) \) has a similar asymptotic behavior to (1.1)

\[
u(n) \sim \frac{1}{8 \cdot 6^{1/2} n^{3/2}} e^{\pi \sqrt{\frac{2n}{3}}}, \quad \text{as } n \to \infty.
\]

However, \( U(q) \) is essentially a mixed mock modular form instead of a modular form. A \textit{mock modular form} is the holomorphic part of a harmonic Maass form with nontrivial non-holomorphic part. A \textit{harmonic Maass form} is a function that transforms like a modular form, satisfies similar growth conditions, but needs to only be smooth and annihilated by the weighted hyperbolic Laplacian. A \textit{mixed mock modular form} is basically an element of the tensor space of modular forms and mock modular forms. These terms and their encompassing theory can be found in [17]. While \( u(n) \) does not satisfy congruences as elegant as those of \( p(n) \), it turns out that

\[
u \left( \ell^2 n + k\ell + \frac{1 - \ell^2}{24}\right) \equiv 0 \pmod{2},
\]

for any prime \( \ell \) satisfying \( \ell \not\equiv 3, 23 \pmod{24} \) and \( \ell \nmid k \) [18, Theorem 1.4].

Both partitions and strongly unimodal sequences have a statistic defined on them called the rank. The \textit{rank of a partition} is the largest part minus the number of parts. The \textit{rank of a strongly unimodal sequence} is the number of terms after the peak minus the number of terms before the peak. We note that the peak is unique for a strongly unimodal sequence, and so there is no ambiguity in this definition as we might have with ordinary unimodal sequences. We let \( N(m, n) \) denote the number of partitions of size \( n \) with rank \( m \) and let \( u(m, n) \) denote the number of strongly unimodal sequences of size \( n \) with rank \( m \). Again
by standard counting techniques we have that the relevant generating functions are given by

\[
R(\zeta; q) := \sum_{n \geq 0} N(m, n) \zeta^m q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(\zeta q, \zeta^{-1} q; q)_n},
\]

\[
U(\zeta; q) := \sum_{n \geq 1} u(m, n) \zeta^m q^n = \sum_{n \geq 1} (-\zeta q, -\zeta^{-1} q; q)_{n-1} q^n.
\]

The rank for strongly unimodal sequences is somewhat new and first appeared in [18]. However, the rank of partitions has a longer history. It was introduced by Dyson [19] in an attempt to provide a combinatorial refinement for the Ramanujan congruences modulo 5 and 7, which came to full fruition in [8]. Both \(R(\zeta; q)\) and \(U(\zeta; q)\) are of considerable interest because of their modular properties. All of Ramanujan’s third order mock theta functions are specializations of \(R(\zeta; q)\) with \(\zeta\) taken to be a root of unity multiplied by a fractional power of \(q\); that specializations of this type are always mock modular forms was established in [15]. Also, \(U(-1; q)\) is one of the most well known examples of a quantum modular form, which are explained at the end of Sect. 4.

A key link between the rank of partitions and the rank of strongly unimodal sequences is the relation between the summands in their generating functions. In particular,

\[
\frac{q^{-n^2}}{(\zeta q, \zeta^{-1} q; q)_n} = (\zeta, \zeta^{-1}; q)_n q^n.
\]

(1.2)

It is through this connection that one can easily explain the mock modular properties of \(U(\zeta; q)\). Specifically, using a certain \(\psi_2\) identity (see Eq. (3.2.2) and entry 3.4.7 in [5]), we have

\[
(1 + \zeta) (1 + \zeta^{-1}) U(\zeta; q) = -R(-\zeta; q) + \frac{1 + \zeta^{-1}}{(q; q)_\infty} \sum_{n \in \mathbb{Z}} \zeta^n q^{\frac{n(n+1)}{2}} 1 + \zeta^{-1} q^n.
\]

(1.3)

Due to work of Zwegers [37,38], the mock modular properties of the functions on the right hand-side of (1.3) are well understood.

To introduce new restricted unimodal sequences, we take the relation in (1.2) as the guiding principle. We recall three additional well-known rank functions are defined by

\[
\overline{R}(\zeta; q) := \sum_{n \geq 0} (-1; q)_n q^{\frac{n(n+1)}{4}} \overline{R}(\zeta; q) := \sum_{n \geq 0} (-1; q)_n q^{\frac{n(n+1)}{4}},
\]

\[
\overline{R}(\zeta; q) := \sum_{n \geq 0} (-q; q^2)_n q^{n^2} \overline{R}(\zeta; q) := \sum_{n \geq 0} (-1; q^2)_n q^{n^2}.
\]

 Respectively, these are the generating functions of the Dyson rank of overpartitions [26], the \(M_2\)-rank of overpartitions [27], and the \(M_2\)-rank of partitions without repeated odd parts [13,28]. For completeness, an overpartition of size \(n\) is a partition of size \(n\) where the last appearance of each part may (or may not) be overlined. We replace \(n\) with \(-n\).
in the summands of the generating functions above and are led to the following three definitions:

$$\tilde{U}(\xi; q) = \sum_{n \geq 0 \atop m \in \mathbb{Z}} \pi(m, n) \xi^m q^n := \sum_{n \geq 1} \frac{(-\xi q^2 - \xi^{-1} q^2; q)_{n-1} q^n}{(-q; q)_n},$$

$$\tilde{U}^2(\xi; q) = \sum_{n \geq 0 \atop m \in \mathbb{Z}} \overline{u}(m, n) \xi^m (-1)^n q^n := \sum_{n \geq 1} \frac{(-\xi q^2 - \xi^{-1} q^2; q^2)_{n-1} q^{2n}}{(-q; q)_{2n}},$$

$$U(\xi; q) = \sum_{n \geq 0 \atop m \in \mathbb{Z}} u(m, n) \xi^m (-1)^n q^n := \sum_{n \geq 1} \frac{(-\xi q^2 - \xi^{-1} q^2; q^2)_{n-1} q^{2n}}{(-q; q^2)_n}. \quad (1.4)$$

The need for the factor $(-1)^n$ in the definitions of $\tilde{U}^2$ and $U(\xi)$ becomes apparent below when we give the combinatorial interpretations, which are given at the beginning of Sects. 3, 4, and 5. We also consider the $\xi = 1$ cases of these functions and set

$$\tilde{U}(q) := \tilde{U}(1; q) := \sum_{n \geq 0} \overline{u}(n) q^n, \quad \tilde{U}^2(q) := \tilde{U}^2(1; -q) := \sum_{n \geq 0} \overline{u}(n) q^n,$$

$$U(q) := U(1; -q) := \sum_{n \geq 0} u(n) q^n.$$

Unimodal sequence type ranks of a similar shape were introduced by Kim et al. [23]. Their functions are given by

$$V(\xi; q) := \sum_{n \geq 0} \frac{(-\xi q^2 - \xi^{-1} q^2; q)_{n} q^n}{(q; q^2)_n}, \quad W(\xi; q) := \sum_{n \geq 0} \frac{(\xi q^2 - \xi^{-1} q^2; q^2)_n q^{2n}}{(-q; q)_{2n+1}},$$

$$Z(\xi; q) := \sum_{n \geq 0} \frac{(-\xi q^2 - \xi^{-1} q^2; q; q^2)_n q^n}{(q; q)_{2n+1}}.$$

To recall one of the combinatorial interpretations, let

$$V(\xi; q) := \sum_{n \geq 0 \atop m \in \mathbb{Z}} v(m, n) \xi^m q^n.$$

Then $v(m, n)$ is the number of odd-balanced unimodal sequences of $2n + 2$ with rank $m$. A unimodal sequence being odd-balanced means that the peak is even, the subsequence of even parts is strongly unimodal, and each odd part appears to the left of the peak exactly as many times as it appears to the right of the peak. We note that since the odd parts appear identically on the left and right of the peak, the rank of an odd-balanced unimodal sequence is equal to the rank of the subsequence of even parts. Kim et al. investigated these functions in terms of their mock modular and quantum modular behavior, as well as giving some parity results. The functions $V(\xi; q)$ and $W(\xi; q)$ were further studied by Barnett et al. [12] for their mock and quantum modular properties. Our first result gives the mock modular properties of $\tilde{U}(\xi; q)$, $\tilde{U}^2(\xi; q)$, and $U(\xi; q)$.

**Theorem 1.1** The functions $\tilde{U}(\xi; q)$, $\tilde{U}^2(\xi; q)$, and $U(\xi; q)$, if $\xi$ is specialized to a root of unity times a fractional power of $q$, are essentially mixed mock modular forms.
Specifically, Theorem 1.1 follows from Corollaries 3.2, 4.2, and 5.2.

The next theorem gives the asymptotic behavior of $u2(n)$ and $u2(n)$ as $n \to \infty$.

**Theorem 1.2** We have, as $n \to \infty$,

$$u2(n) \sim \frac{1}{8(2n)^{\frac{3}{4}}} e^{\pi \sqrt{\frac{3}{2n}}}, \quad u2(n) \sim \frac{1}{4\sqrt{3}(6n)^{\frac{3}{4}}} e^{\pi \sqrt{\frac{1}{2n}}}. $$

Additionally, we fully determine the parity of $u2(n)$.

**Theorem 1.3** We have that $u2(n)$ is odd if and only if $8n - 1 = 3^k \ell^2 p^r$, where $p$ is a prime congruent to 5 or 23 modulo 24, $p \nmid \ell$, $b \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, and $c \equiv 1 \pmod{4}$.

We note that the generating function $U2(\zeta; q)$ was simultaneously and independently introduced by Barnett et al. [11]. There the relevant function is $N(z; \tau)$, which the authors study for its mock and quantum Jacobi properties. Furthermore, Theorem 1.3 given above was independently discovered and given as Conjecture 1.4 in [11], and Jeremy Lovejoy has given another proof in private communications.

The article is organized as follows. In Sect. 2, we give the various definitions, identities, and general results required in our proofs. In Sect. 3, we discuss the function $U(\zeta; q)$, beginning with its combinatorial interpretation. As it turns out, this function is the one for which we can say the least, which is surprising as it comes from the simplest of the three ranks. The relevant statement of Theorem 1.1 is contained in Corollary 3.2. Sect. 4 is devoted to the investigation of the function $U2(\zeta; q)$. This includes the combinatorial interpretation, identities in terms mock modular objects for Theorem 1.1 in Corollary 4.2, and the asymptotic behavior given in Theorem 1.2 is proved toward the end of the section. We also give a brief note on the formal dual $U2(\zeta; q^{-1})$ and its quantum modularity. We study $U2(\zeta; q)$ in Sect. 5; again this includes the combinatorial interpretation, mock modular properties in Corollary 5.2 for Theorem 1.1, and the asymptotics of Theorem 1.2 are proved after Corollary 5.2. In this section we end with a proof of the parity classification in Theorem 1.3 for $u2(n)$, which is related to the arithmetic of $\mathbb{Q}(\sqrt{6})$. We conclude the article with a few remarks in Sect. 6.

**2 Preliminaries**

**2.1 Combinatorial results**

We require several known identities and transformation for $q$-series. We state these results in a series of lemmas. In the statements of these identities we give restrictions for convergence, however we make no mention of this in our proofs as the convergence conditions are clear and the resulting identities hold in greater generality due to analytic continuation.

The first lemma is Heine’s transformation.

**Lemma 2.1** [21, Eq. (III.1)] Suppose that $|t|, |b|, |q| < 1$. Then we have

$$\sum_{n \geq 0} \frac{(a, b; q)_n}{(c, q; q)_n} t^n (\ell; q)_n b^n = \frac{(b, at; q)_\infty}{(c, t; q)_\infty} \sum_{n \geq 0} \frac{(\ell; q)_n}{(at, q; q)_n} t^n b^n. $$

The following is known as Watson’s transformation.
Lemma 2.2 [21, Eq. (III.18)] Suppose that \(|aq| < |de|\). Then we have

\[
\sum_{n \geq 0} \left( \frac{aq}{de}, d, c; q \right)_n \left( \frac{aq}{de} \right)^n = \left( \frac{aq}{de}, q \right)_\infty \sum_{n \geq 0} \left( aq, \frac{aq}{de}, \frac{aq}{de}; q \right)_n (aq)^2 (-1)^n q^{\frac{n(n-1)}{2}} (aq)^n.
\]

The next lemma is often used with partial theta functions.

Lemma 2.3 [5, Theorem 6.2.1] Suppose that \(|b| < 1\) and \(|Abq| < |a|\). Then we have

\[
\sum_{n \geq 0} \left( \frac{Bq}{aq}, bq; q \right)_n q^n = \left( \frac{aq}{de}, q \right)_\infty \sum_{n \geq 0} \left( aq, \frac{aq}{de}, \frac{aq}{de}; q \right)_n (aq)^2 (-1)^n q^{\frac{n(n-1)}{2}} (aq)^n.
\]

Furthermore we use another identity related to Lemma 2.3.

Lemma 2.4 [5, entry 6.3.12] The following identity holds,

\[
\sum_{n \geq 0} \left( \frac{aq}{de}, b; q \right)_n q^n = \left( \frac{aq}{de}, q \right)_\infty \sum_{n \geq 0} \left( aq, \frac{aq}{de}, \frac{aq}{de}; q \right)_n (aq)^2 (-1)^n q^{\frac{n(n-1)}{2}} (aq)^n.
\]

We also make use of the Bailey pair machinery [3, Chap. 3]. Recall that a pair of sequences \((\alpha_n, \beta_n)\) is called a Bailey pair relative to \((a, q)\) if

\[ \beta_n = \sum_{0 \leq j \leq n} \frac{\alpha_j}{(q; q)_{n-j}(aq; q)_{n-j}}. \]

Bailey’s Lemma is as follows.

Lemma 2.5 [3, Theorem 3.4] If \((\alpha_n, \beta_n)\) is a Bailey pair relative to \((a, q)\), then, assuming convergence conditions,

\[
\sum_{n \geq 0} \left( \frac{aq}{de}, \varrho_1, \varrho_2; q \right)_n \beta_n = \left( \frac{aq}{de}, \varrho_1, \varrho_2; q \right)_\infty \sum_{n \geq 0} \left( \frac{aq}{de}, \varrho_1, \varrho_2; q \right)_n \alpha_n.
\]

The following theorem of Lovejoy gives a convenient formula for constructing Bailey pairs.
Lemma 2.6 [24, Theorem 8] The following is a Bailey pair relative to \((a, q)\):

\[
\alpha_n = \left( \frac{a, q, \eta; q}{b, c, q} \right)_n \left( 1 - aq^{2n} \right) \left( -bcdq \right)^m q^{\frac{1}{2}n(n-1)} \frac{1 - a(bq, cq, dq; q)_n}{a^n} \\
\beta_n = \left( \frac{adq}{bc; q} \right)_n.
\]

2.2 Analytic results

To recognize the functions of interest for this paper as mixed mock modular forms, we recall a few well-known functions. For \(z \in \mathbb{C}\), define the Jacobi theta function

\[
\vartheta(z; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{2\pi in(z + \frac{1}{2})} q^{\frac{n^2}{2}} = -iq^{\frac{1}{8}} \xi^{-\frac{1}{8}} (q, \xi, \xi^{-1}; q)_\infty, \quad (\xi := e^{2\pi iz}).
\]

Moreover we require Zwegers \(\mu\)-function for \(z_1, z_2 \in \mathbb{C}\),

\[
\mu(z_1, z_2; \tau) := e^{\pi iz_1} \vartheta(z_2; \tau) \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi iz_2} q^{\frac{mn}{2}}}{1 - e^{2\pi iz_1} q^n},
\]

and for \(\ell \in \mathbb{N}\) let the higher level Appel function be given by

\[
A_\ell(z_1, z_2; \tau) := e^{\pi iz_1} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi iz_2} q^{\frac{mn}{2}}}{1 - e^{2\pi iz_1} q^n}.
\]

The function \(\vartheta(z; \tau)\) is a holomorphic Jacobi form and the mock modular properties of \(\mu(z_1, z_2; \tau)\) and \(A_\ell(z_1, z_2; \tau)\) are described in [37,38].

To prove Theorem 1.2, we also require the following asymptotic behavior which follows directly from the modular transformation of the Dedekind \(\eta\)-function

\[
(e^{-w}; e^{-w})_\infty \sim \sqrt{\frac{2\pi}{w}} e^{-\frac{w^2}{2}} \quad \text{as } w \to 0,
\]

where the limit is taken in any region \(|\text{Arg}(w)| < \theta\), for fixed \(\theta < \frac{\pi}{2}\). Moreover we need the following Tauberian Theorem.

Theorem 2.7 [17, Theorem 14.4] Let \(f(q) = \sum_{n \geq 0} a(n) q^n\) be a power series with non-negative \(a(n)\) that are monotonically increasing and have radius of convergence equal to 1. Suppose that

\[
f(e^{-t}) \sim \lambda t^\alpha e^{\frac{A}{2}} \quad \text{as } t \to 0^+,
\]

\[
f(e^{-w}) \ll |w|^\alpha e^{\frac{A}{\alpha}} \quad \text{as } w \to 0 \text{ in each region } |\text{Arg}(w)| < \theta < \frac{\pi}{2},
\]

for \(A > 0, \lambda, \alpha \in \mathbb{R}\). Then we have

\[
a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^\frac{3}{2} + \frac{1}{2}}{n^{\frac{3}{2} + \frac{3}{2}}} e^{2\sqrt{A} n} \quad \text{as } n \to \infty.
\]

Remark 2.8 Theorem 2.7 is commonly stated without the additional boundedness condition. However, this seems to be in error and is discussed in detail in an upcoming article [14]. In the current article, we determine the asymptotic behavior of functions via modular transformations, which actually imply \(f(e^{-w}) \sim \lambda w^\alpha e^{\frac{A}{\alpha}} \) as \(w \to 0\) in each region.
\( |\text{Arg}(w)| < \theta < \frac{\pi}{2} \) (and as such, the required bound holds), so that this detail is not of major concern.

### 3 The function \( \overline{U}(\zeta; q) \)

Since the series expansions of the summands of (1.4) have both positive and negative coefficients, we interpret \( \overline{u}(n) \) and \( \overline{u}(m,n) \) both as the difference of two non-negative counts. A left-heavy overlined unimodal sequence of size \( n \) is a unimodal sequence of size \( n \) such that the parts up to and including all occurrences of the peak form an overpartition with largest part overlined, and the parts after the peak form an overpartition with all parts overlined. Then \( \overline{u}(n) \) is the number of left-heavy overlined unimodal sequences of size \( n \) with an even number of non-overlined parts minus those with an odd number of non-overlined parts. Furthermore, \( \overline{u}(m,n) \) is the same difference of counts as \( \overline{u}(n) \), but with the added restraint that the rank of the strongly unimodal sequence consisting of the overlined parts is \( m \).

**Example** The left-heavy overlined unimodal sequences of 3 are \((\overline{3}), (1, \overline{2}), (1, \overline{2}), (\overline{2}, \overline{1}), (1, 1, \overline{1})\). By accounting for the parity of the non-overlined parts, we find \( \overline{u}(3) = 3 \). The ranks of the strongly unimodal subsequences consisting of the overlined parts are, respectively, 0, 0, -1, 1, and 0.

The following lemma writes \( \overline{U}(\zeta; q) \) in terms of \( R(\zeta; q) \) and \( \overline{R}(\zeta; q) \).

**Lemma 3.1** We have

\[
(1 - \zeta)(1 - \zeta^{-1}) \overline{U}(\zeta; q) = \overline{R}(\zeta; q) - \frac{(-\zeta q, -\zeta^{-1}q; q)_{\infty}}{(-q; q)_{\infty}} R(\zeta; q).
\]

**Proof** Lemma 2.4 gives, shifting \( n \mapsto n + 1 \) in the definition of \( \overline{U}(\zeta; q) \),

\[
(1 + q) \overline{U}(\zeta; q) = \sum_{n \geq 1} \frac{(-\zeta q, -\zeta^{-1}q; q)_{n + 1}}{(-q^2; q)_{n}}
\]

\[
= \sum_{n \geq 1} \frac{(1 + q)(-1; q)_{n - 1} q^{n + 1}}{(\zeta, \zeta^{-1}; q)_{n}} - \frac{(-\zeta q, -\zeta^{-1}q; q)_{\infty}}{(-q^2; q)_{\infty}} \sum_{n \geq 1} \frac{q^{n - 1}}{(\zeta, \zeta^{-1}; q)_{n}}
\]

\[
= \frac{1 + q}{(1 - \zeta)(1 - \zeta^{-1})} \overline{R}(\zeta; q) - \frac{1 + q}{(1 - \zeta)(1 - \zeta^{-1})} \overline{R}(\zeta; q).
\]

shifting \( n \mapsto n + 1 \) in the definitions of \( \overline{R}(\zeta; q) \) and \( R(\zeta; q) \). This gives the claim. \( \square \)

It is not hard to conclude the following representation using (mock) modular objects, which are defined in Sect. 2.2.

**Corollary 3.2** We have

\[
\overline{U}(\zeta; q) = \frac{-2\zeta \eta(2\tau) A_2(z, \frac{1}{2}; \tau)}{(1 - \zeta^2) \eta(\tau)^2} - \frac{\theta(z + \frac{1}{2}; \tau) A_3(z, -\tau; \tau)}{(1 - \zeta^2) \eta(\tau) \eta(2\tau)} - \frac{\zeta}{1 - \zeta^2}.
\]
4 The function \( \overline{U_2}(\zeta; q) \)

Before we state the combinatorial interpretation of \( \overline{U_2}(n) \) and \( \overline{U_2}(m, n) \), note that the summands of \( \overline{U_2}(\zeta; -q) \) have non-negative coefficients since

\[
\frac{1}{(q; q)_{2n}} = \frac{(-q; q^2)_n}{(q^{2n+2}; q^3)_n}.
\]

A unimodal sequence is an \( M_2 \)-left-heavy overlined unimodal sequence if the peak is even and appears overlined exactly once (suppose it is \( \overline{2N} \)), the parts before and after \( \overline{2N} \) form an overpartition, all overlined odd parts appear to the left of \( \overline{2N} \), and all non-overlined parts are at least \( N + 1 \) and appear identically on the left and right of \( \overline{2N} \). Then \( \overline{U_2}(n) \) is the number of \( M_2 \)-left-heavy overlined unimodal sequences of size \( n \) and \( \overline{U_2}(m, n) \) is the number of \( M_2 \)-left-heavy overlined unimodal sequences of size \( n \) such that the strongly unimodal sequence consisting of the overlined even parts has rank \( m \).

Example We have \( \overline{U_2}(7) = 5 \) since the relevant sequences are: \((1, 6), (3, 4), (1, 2, 3, 4), (1, 4, 2), \) and \((1, 2, 2, 2)\). The residual ranks of these sequences are \(0, 0, -1, 1\), and \(0\), respectively.

The following proposition rewrites \( \overline{U_2}(\zeta; -q) \) in terms of generalized Lambert series.

Proposition 4.1 We have

\[
\overline{U_2}(\zeta; -q) = -\frac{\zeta q (\zeta q^2, \zeta^{-1} q^2, -q; q^2)_\infty}{(1 - \zeta)(q; q)_\infty (\zeta^{-1} q; q)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n q^{2n^2 + 3n} + \frac{\zeta^2 (q^2; q^4)_\infty}{2 (1 - \zeta^2) (q^4; q^4)_\infty} \sum_{n \in \mathbb{Z}} q^{n^2 + 3n + 1} \frac{\zeta q^{2n + 1}}{1 + \zeta q^{2n + 1}} - \frac{\zeta (q^2; q^4)_\infty}{2 (1 - \zeta^2) (q^4; q^4)_\infty} \sum_{n \in \mathbb{Z}} q^{n^2 + n} \frac{\zeta q^{2n + 1}}{1 + \zeta q^{2n + 1}}.
\]

Proof Applying Lemma 2.3 with \( q \mapsto q^2 \), \( a = -q \), \( b = q^2 \), \( \lambda = \zeta^{-1} q^{-2} \), and \( B = -\zeta q^2 \), we find that

\[
q^{-2}(1 - q) (1 + q^2) \overline{U_2}(\zeta; -q) = \sum_{n \geq 0} \frac{(-\zeta q^2, -\zeta^{-1} q^2; q)_n q^{2n}}{(q^3, -q^4; q^6)_n} = \frac{q^{-1} (-\zeta q^2, -\zeta^{-1} q^2; q^2)_\infty}{(1 + \zeta q) (q^3, -q^4; q^6)_\infty} \sum_{n \geq 0} \frac{(\zeta q^2; q^2)_n (-1)^n q^n \zeta^{-n}}{(-\zeta q^2; q^2)_n} - \frac{q^{-1} (1 - q) (1 + q^2)}{(1 + \zeta q) (1 + \zeta^{-1} q)} \sum_{n \geq 0} \frac{(q^2; q^4)_n (-1)^n q^{2n}}{(-\zeta q^3, -\zeta^{-1} q^3; q^4)_n}.
\]

We handle the two sums in (4.1) separately. For the first, we apply Lemma 2.1 with \( q \mapsto q^2 \), \( a = \zeta q^2 \), \( b = q^2 \), \( c = -\zeta q^2 \), and \( t = -\zeta^{-1} q \), which gives that

\[
\sum_{n \geq 0} \frac{(\zeta q^2, q^2)_n (-1)^n q^n \zeta^{-n}}{(-\zeta q^3; q^2)_n} = \frac{(q^2, -q^3; q^4)_\infty}{(-\zeta q^3; q^2)_\infty} \sum_{n \geq 0} \frac{(-\zeta q, -\zeta^{-1} q; q^2)_n q^{2n}}{(-q^3, q^2; q^2)_n}.
\]
Next we take \( q \mapsto q^2, a = q^2, b = -q, c \to \infty, d = -\zeta q, \) and \( e = -\zeta^{-1} q \) in Lemma 2.2 to find that

\[
\sum_{n \geq 0} \frac{(-\zeta q, -\zeta^{-1} q; q^2)_n q^{2n}}{(q^2, -q^3; q^2)_n} = -\zeta (1 - q)(1 - \zeta) (q^4, q^2; q^2)_{\infty} \sum_{n \in \mathbb{Z}} (1)^n q^{2n^2 + 3n} + q^{2n^2 + 3n} \cdot (4.3)
\]

Combining (4.2) and (4.3) gives the first summand in the proposition.

For the second series in (4.1) we apply Lemma 2.2 with \( q \mapsto q^2, a = q^2, b = -q, c = -\zeta^{-1} q, d = q, \) and \( e = -q. \) This gives that

\[
\sum_{n \geq 0} \frac{(-\zeta^3 q, -\zeta^{-1} q^2; q^2)_n}{(-\zeta^3 q^2, -\zeta^{-1} q^3; q^2)_n} = \frac{(1 + q)(1 + \zeta^{-1} q) (q^6; q^4)_{\infty}}{2 (1 - q^2, q^4; q^2)_{\infty}} \sum_{n \in \mathbb{Z}} (1 + \zeta^{-1} q^{2n+1})(1 + \zeta^{-1} q^{2n+1})
\]

Letting \( n \mapsto -n - 1 \) in the last sum and combining terms gives the claim. \( \square \)

We next rewrite \( \mathcal{U}_{\mathbb{Z}} \) in terms of (mock) modular objects.

**Corollary 4.2** We have

\[
\mathcal{U}_{\mathbb{Z}}(\zeta; -q) = -\frac{\zeta^{\frac{1}{2}} \eta(2\tau)^2 \vartheta(z + \frac{1}{2}; 2\tau)}{(1 - \zeta^2) \eta(\tau)^2 \eta(4\tau)^2} A_2 \left( z + \frac{1}{2} + \tau, \frac{1}{2} + \tau; 2\tau \right)
\]

\[
- i \zeta^{\frac{1}{2}} q^{-\frac{1}{4}} \frac{1}{1 - \zeta^2} \left( 2 \mu \left( z + \frac{1}{2} + \tau, \frac{1}{2}; 2\tau \right) + i \zeta^{\frac{1}{2}} q^{\frac{1}{4}} \right).
\]

The asymptotic formula for \( \mathcal{U}_{\mathbb{Z}}(n) \) in Theorem 1.2 follows from Theorem 2.7, once we establish that \( \mathcal{U}_{\mathbb{Z}}(n) \) is monotonic.

**Lemma 4.3** For \( n \in \mathbb{N}_0, \) we have \( \mathcal{U}_{\mathbb{Z}}(n + 1) \geq \mathcal{U}_{\mathbb{Z}}(n). \)

**Proof** To prove the claim, we show that \((1 - q) \mathcal{U}_{\mathbb{Z}}(1; -q) \) has non-negative coefficients. For this, we note that

\[
(1 - q) \mathcal{U}_{\mathbb{Z}}(1; -q) = \sum_{n \geq 1} F_n(q), \quad \text{where} \quad F_n(q) := \frac{(-q^2; q^2)^{n-1} q^{2n}}{(1 + q^{2n})(q^3; q^2)^{n-1}}.
\]

We first verify that \( F_n(q) \) has non-negative coefficients for \( n \geq 3. \) Given two power series \( A(q) \) and \( B(q), \) write \( A(q) \geq B(q) \) to indicate that \( A(q) - B(q) \) has non-negative coefficients. We see that

\[
G_n(q) = \sum_{m \geq 0} (g_{n,e}(m) - g_{n,e}(m)) q^m, \quad \text{where}
\]

\[
G_3(q) := \frac{q^6}{(1 - q^3)(1 + q^6)},
\]

\[
G_n(q) := \frac{q^{2n}}{(1 - q^3)(1 - q^{2n-3})(1 + q^{2n})} \quad \text{for} \quad n \geq 4,
\]
and $g_{n,o}(m)$ ($g_{n,e}(m)$, resp.) is the number of partitions of $m$ with largest part $2n$, where the only other allowed parts are 3 and $2n-3$, and $2n$ appears an odd (even, resp.) number of times. Taking an occurrence of $2n$ and replacing it by 3 and $2n-3$ gives an injection from the partitions counted by $g_{n,e}(m)$ to those of $g_{n,o}(m)$. Thus $G_n(q) \geq 0$, which implies $F_n(q) \geq 0$ for $n \geq 3$. However, $F_1(q)$ and $F_2(q)$ do have negative coefficients. By a careful grouping of $F_1(q) + F_2(q) + F_3(q) + F_4(q)$ as rational functions, we find $F_1(q) + F_2(q) + F_3(q) + F_4(q) \geq 0$. Thus \( \overline{w}(n+1) \geq \overline{w}(n) \) for all $n$. We carefully group $F_1(q) + F_3(q)$ and $F_2(q) + F_4(q)$ as follows,

\[
\begin{align*}
F_1(q) + F_3(q) &= \frac{(1 + q^2 + q^6) q^2}{1 - q^{12}} + \frac{(q^4 + q^7 + 2q^8 + 2q^{11} + q^{15} + q^{19}) q^2}{1 - q^{12}} - q^4, \\
F_2(q) + F_4(q) &= \frac{(1 + q^2) q^4}{1 - q^4} + \frac{(1 + q^2) (1 + q^4) (1 + q^6) q^8}{1 - q^4} \\
&\geq \frac{(1 + q^2) q^4}{(1 - q^4)(1 + q^4)} + \frac{(1 + q^2) (1 + q^4) (1 + q^6) q^8}{(1 - q^4)(1 + q^4)} \\
&= \frac{(1 + 2q + q^2 + q^4 + q^7 + q^8 + q^{10}) q^{13}}{(1 - q^4)(1 - q^{16})} \\
&+ \frac{(1 + q^2 + 2q^2 + q^{10} + 2q^{13} + q^{19}) q^4}{(1 - q^4)(1 - q^{16})},
\end{align*}
\]

where we make use of the fact that $G_4(q) \geq 0$. We then find $F_1(q) + F_2(q) + F_3(q) + F_4(q) \geq 0$, and thus $\overline{w}(n+1) \geq \overline{w}(n)$ for all $n$. \qed

The following calculation gives the asymptotics of $\overline{w}(n)$.

**Proof of the asymptotics for $\overline{w}(n)$ in Theorem 1.2.** We begin with the representation in (4.1),

\[
\overline{U}_2(1; -q) = \frac{q}{(q; q^2)_\infty} \sum_{n \geq 0} \frac{(q^2; q^2)_n (-1)^n q^n}{(-q; q^2)_n} - q \sum_{n \geq 0} \frac{(q^2; q^2)_n (-1)^n q^{2n}}{(-q; q^2)_n^2}.
\]

If $q = e^{-w}$ and $w \to 0$ in a region where $|\text{Arg}(w)| < \theta$, we have $q \to 1$ and the sums become $\frac{1}{2}$ and $\frac{1}{4}$ respectively. Thus

\[
\overline{U}_2(1; -e^{-w}) \sim \frac{(e^{-2w}; e^{-2w})_\infty}{2(e^{-w}; e^{-2w})_\infty} = \frac{1}{4} \frac{(e^{-4w}; e^{-4w})_\infty}{2(e^{-w}; e^{-w})_\infty} \quad (\text{as } w \to 0).
\]

Using (2.1) gives that $\overline{U}_2(1; -e^{-w}) \sim \frac{1}{4} e^{\frac{3w^2}{8}}$ as $w \to 0$. We now use Theorem 2.7 with $\lambda = \frac{1}{4}, \alpha = 0$, and $A = \frac{\pi^2}{8}$ to obtain the claim.

For the reader concerned about taking the limit $w \to 0$ inside the sums, one can instead apply (mock) modular transformations to the representation in Corollary 4.2 to obtain the same results. However, these calculations are considerably longer. \qed

While $\overline{U}_2(\zeta; -q)$ does not appear to possess any quantum modular properties, the formal dual $\overline{U}_2(\zeta; -q^{-1})$ does. Recall that a function $f : \mathbb{Q} \to \mathbb{C}$, $(\mathbb{Q} \subset \mathbb{Q})$ is a quantum modular form, of weight $k$ with respect to $\Gamma$, if the obstruction to modularity,

\[
f(\tau) - \chi(M)^{-1} (c\tau + d)^{-k}f \left( \frac{a\tau + b}{c\tau + d} \right), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,
\]
can be extended to an analytic function on an open subset of \( \mathbb{R} \). Quantum modular forms were introduced by Zagier in [36]. Noting that \((w; q^{-1})_n = (w^{-1}; q)_n(-1)^n w^n q^{a(n-1)}\), we obtain

\[
\mathcal{U}_2 (\zeta; q^{-1}) = \sum_{n \geq 1} \frac{(-\zeta q^2, -\zeta^{-1} q^2; q^2)_n (-1)^n q^n}{(q, -q^2; q^2)_n} = \frac{\zeta}{1 - \zeta^2} \sum_{n \geq 1} (-1)^n q^n (\zeta^n - \zeta^n),
\]

where the second equality is Theorem 15 of [6] with \( q \mapsto -q \). In particular,

\[
\mathcal{U}_2 (1; q^{-1}) = \sum_{n \geq 1} n(-1)^n q^n. \tag{4.4}
\]

Using the now standard techniques for false theta functions [16], one can show that (4.4) is a quantum modular form, where the values of the function on the rationals are given by taking radial limits. However, since neither the series for \( \mathcal{U}_2 (\zeta; q) \) nor \( \mathcal{U}_2 (\zeta; q^{-1}) \) truncates if \( q \) is a root of unity, the behavior of one function as \( q \) approaches a root of unity says nothing about the other.

5 The function \( \mathcal{U}_2 (\zeta; q) \)

A unimodal sequence is \( M_2 \)-left heavy if the largest part is even, all odd parts appear to the left of the peak, and the subsequence consisting of the even parts is strongly unimodal. Then \( u2(n) \) is the number of \( M_2 \)-left heavy unimodal sequences of size \( n \) and \( u2(m, n) \) is the number of such sequences where the strongly unimodal sequence consisting of the even parts has rank \( m \).

Example We have \( u2(6) = 5 \) since the relevant sequences are: (6), (2, 4), (4, 2), (1, 1, 4), and (1, 1, 1, 2). The residual ranks of these sequences are 0, −1, 1, 0, and 0, respectively.

The following proposition rewrites \( \mathcal{U}_2 (\zeta; q) \) in terms of generalized Lambert series. We note that a similar expression for \( \mathcal{U}_2 (\zeta; q) \) can be found in [30, Eq. (4.29)].

Proposition 5.1 We have

\[
(1 + \zeta) (1 + \xi^{-1}) \mathcal{U}_2 (\zeta; q) = -R_2 (-\xi; q) - \frac{\xi^{-1} (-\xi, -\xi^{-1}; q^2)_\infty}{(1 + \zeta q) (q; q^2)_\infty} R (-\xi q; q^2) + \frac{(q; q)_\infty (q; q^2)_\infty^2}{(-\zeta q, -\zeta^{-1} q; q)_\infty} + \frac{\xi^{-1} (-\zeta, -\zeta^{-1}; q^2)_\infty}{(q; q^2)_\infty}.
\]

Proof We use Lemma 2.4 with \( q \mapsto q^a, a = \zeta, b = \zeta^{-1}, \) and \( c = -q \), which gives that

\[
\mathcal{U}_2 (\zeta; q) = \sum_{n \geq 1} \frac{(q; q^2)_n (-1)^n q^n}{(-\zeta q, -\zeta^{-1} q; q^2)_n} + \frac{(-\zeta q^2, -\zeta^{-1} q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{n \geq 1} \frac{q^{2(n-1)+1}}{(-\zeta q, -\zeta^{-1} q; q^2)_n}. 
\]
By entry 12.3.2 of [4], we obtain that
\[
\sum_{n \geq 1} \frac{(-1)^n (q; q^2)_n}{(-\xi q, -\xi^{-1} q; q^2)_n} = - \frac{R2(-\xi; -q)}{(1 + \xi)(1 + \xi^{-1})} + \frac{(q; q^2)_{\infty}}{q R(-\xi; -q)}.
\]

Using Eq. (12.2.5) of [4] and Lemma 7.9 of [20] yields
\[
\sum_{n \geq 1} \frac{q^{n(n-1)+1}}{(-\xi q, -\xi^{-1} q; q^2)_n} = \xi^{-1} - \frac{1}{1 + \xi q} R(-\xi; q^2),
\]
giving the claim. \(\Box\)

In the following corollary, we rewrite \(U/2(\xi; -q)\) in terms of known (mock) modular objects.

**Corollary 5.2** We have
\[
U/2(\xi; -q) = \frac{q^{1/2} \eta(\tau) A_2(z + \frac{1}{2}, -\tau; 2\tau)}{(1 + \xi) \eta(2\tau)^2} + \frac{i \xi^{-2} q^{-3} \bar{\theta} (z + \frac{1}{2}; 2\tau) A_3(z + \tau + \frac{1}{2}, -2\tau; 2\tau)}{(1 + \xi) \eta(2\tau)} - \frac{\xi^{1/2} q^{1/2} \eta(\tau)^4}{(1 + \xi) \eta(2\tau)^2 \theta(z + \frac{1}{2}, \tau)} - \frac{\xi^{-1/2} q^{1/2} \eta(\tau)^4}{(1 + \xi) \eta(2\tau)}.
\]

Next we prove the asymptotic formula for \(u_2(n)\). We note that \(u_2(n+1) \geq u_2(n)\) follows trivially by taking a sequence counted by \(u_2(n)\) and adding a single 1 to the left of the peak. Furthermore, this also shows that \(u_2(m, n + 1) \geq u_2(m, n)\) for fixed \(m\).

**Proof of the asymptotics for \(u_2(n)\) in Theorem 1.2.** By using Proposition 5.1, we have
\[
U/2(1; -q) = -\frac{1}{4} R2(-1; -q) - \frac{(1; q^2)^2}{4(1 + q)(q; q^2)^2} R(-q; q^2) + \frac{(q; q)_{\infty}}{4(-q; q^2)_{\infty}^2} + \frac{(1; q^2)^2}{4(q; q^2)_{\infty}^2}.
\]

With \(q = e^{-w}, w \to 0\), and \(|\text{Arg}(w)| < \theta\), we find that
\[
\frac{(1; q^2)^2}{(q; q^2)_{\infty}^2} = \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty} (q^2; q^2)_{\infty}^2} \sim \sqrt{2} e^{-w/2},
\]
\[
\frac{(q; q)_{\infty}}{(-q; q^2)_{\infty}^2} = \frac{(q; q)^5}{(q^2; q^2)_{\infty}^4} \sim 4 \sqrt{\frac{2\pi}{w}} e^{-w/2}.
\]

Moreover,
\[
\lim_{w \to 0} R2(-1; -e^{-w}) = \lim_{q \to 1} \sum_{n \geq 0} \frac{(q; q^2)_n}{(-q^2; q^2)_n} = 1,
\]
\[
\lim_{w \to 0} R(-q; q^2) = \lim_{q \to 1} \sum_{n \geq 0} \frac{q^{2n^2}}{(-q^3, -q; q^2)_n} = \sum_{n \geq 0} \left(\frac{1}{4}\right)^n = \frac{4}{3}.
\]
Thus
\[ U2(1; -q) \sim -\frac{1}{4} + \frac{(-1; q^2)^2}{4(q; q^2)_\infty} \left(-\frac{1}{2} R(-q; q^2) + 1\right) \sim \frac{1}{6\sqrt{2}} e^\frac{\pi}{16}. \]

We now use Theorem 2.7 with \( \lambda = \frac{1}{6\sqrt{2}}, \alpha = 0, \) and \( A = \frac{\pi^2}{6} \) to obtain the claim.

Again for the reader concerned with taking limits inside sums, one may instead use modular transformations with the representation in Corollary 5.2. While one can save some effort by noting \( R(-q; q^2) = 1 + q - q(1 + q)\omega(-q), \) where \( \omega(q) \) is a third order mock theta function, these calculations are still somewhat lengthy.

A first step to prove Theorem 1.3 is to rewrite \( U2(1; q) \) modulo 2.

**Proposition 5.3** We have that
\[
U2(1; -q) \equiv \sum_{n \geq 0 \atop 0 \leq j \leq n} \left(1 + q^{2j+1}\right) q^{3n^2 + 6n - 2j^2 - 3j + 2} \pmod{2}.
\]

**Proof** By taking \( q \mapsto q^2, a = q^4, b = q, d = c^2, \) and then letting \( c \to 0 \) in Lemma 2.6 we have the following Bailey pair relative to \((q^4, q^2),\)
\[
\alpha_n = \frac{(-1)^n (1 - q^{4n+4}) q^{3n^2 + 4n(1 - q)} (1 - q^2)}{(1 - q^4)(1 - q^4)} \sum_{0 \leq j \leq n} (1 + q^{2j+1}) q^{-2j^2 - 3j},
\]
\[
\beta_n = \frac{1}{(q^3; q^2)_n^2}.
\]

We then apply Lemma 2.5, with \( \varrho_1 = \varrho_2 = q^2, \) to this Bailey pair to obtain
\[
\sum_{n \geq 0} \frac{(q^2; q^2)_n^2 q^{2n}}{(q^3; q^2)_n^2} = (1 - q) \sum_{n \geq 0} \sum_{0 \leq j \leq n} (-1)^n \left(1 + q^{2n+2}\right) (1 + q^{2j+1}) q^{3n^2 + 6n - 2j^2 - 3j} 1 - q^{2n+2}.
\]

Thus, changing \( n \mapsto n + 1 \) in the definition of \( U2, \) and then using (5.1), we obtain the claim.

To relate the parity of \( u2(n) \) to norms of ideals in \( \mathbb{Q}(\sqrt{6}), \) we rewrite the sum in Proposition 5.3.

**Proposition 5.4** We have that
\[
\sum_{n \geq 0 \atop 0 \leq j \leq n} \left(1 + q^{2j+1}\right) q^{3n^2 + 6n - 2j^2 - 3j + 2} = \frac{1}{2} \sum_{N \geq 2} \sum_{\frac{N}{2} \equiv 3 \pmod{4}} \sum_{\frac{N}{2} \equiv 1 \pmod{2}} \sum_{\frac{N}{16} \equiv 1 \pmod{2}, \frac{N}{16} \equiv 1 \pmod{2}} q^{N^2 - 6\ell^2 + \frac{1}{8}}.
\]

**Proof** Letting \( n \mapsto n + j, \) and then swapping \( n \) and \( j, \) we rewrite the left-hand side as
\[
\sum_{n,j \geq 0} q^{n^2 + 3n + 3j^2 + 6j + 6nj + 2} + \sum_{n,j \geq 0} q^{n^2 + 5n + 3j^2 + 6j + 6nj + 3} = \sum_{n \geq 1} q^{n^2 + n - 6j^2 - 3j}.
\]
For the last step, we let \( n \mapsto n - 1 \) in the first double sum and in the second we let \( n \mapsto n + 1 \).

To finish the claim, we have to prove that

\[
2 \sum_{n \geq 1} q^{(4n+2)^2-6(4j+1)^2} = \sum_{N \equiv 2 \pmod{4}} \sum_{-\frac{N}{2} < J \leq \frac{N}{2}} q^{N^2-6J^2}. \tag{5.2}
\]

Substituting \( N = 4n + 2 \) and \( J = 4j \pm 1 \), we find that the right-hand side equals

\[
\sum_{n \geq 1} q^{(4n+2)^2-6(4j+1)^2} + \sum_{n \geq 1} q^{(4n+2)^2-6(4j-1)^2}
\]

\[
= 2 \sum_{n \geq 1} \left( \sum_{-\frac{1}{2} \leq J \leq \frac{1}{2}} + \sum_{-\frac{1}{2} < J < -\frac{1}{2}} - \sum_{\frac{1}{2} < J < \frac{3}{2}} \right) q^{(4n+2)^2-6(4j+1)^2},
\]

letting \( j \mapsto -j \) in the second sum of the left-hand side. From this it is not hard to prove (5.2).

For the proof of Theorem 1.3, the proof requires a small amount of standard algebraic number theory. For the reader not familiar with the definitions, we offer [10, Chap. 11] and [29] as two references. By Propositions 5.3 and 5.4,

\[
q^{-2} U2 \left( 1; -q^{16} \right) = \frac{1}{2} \sum_{N \equiv 2 \pmod{4}} \sum_{-\frac{N}{2} < J \leq \frac{N}{2}} q^{N^2-6J^2}, \quad \pmod{2}.
\]

We note that \( N \equiv 2 \pmod{4} \) and \( J \equiv 1 \pmod{2} \) if and only if \( N^2 - 6J^2 \equiv 6 \pmod{8} \).

Let \( D = 6 \) of [7, Lemma 3] states that each equivalence class of solutions to \( u^2 - 6v^2 = m \), with \( m \) positive, contains a unique \((u, v)\) such that \( u > 0 \) and \(-\frac{4}{3} < v \leq \frac{4}{3}\). Recall that two solutions \((u, v)\) and \((u', v')\) are equivalent if \( u' + v' \sqrt{6} = (5 + 2\sqrt{6})^j (u + v\sqrt{6})\) with \( r \in \mathbb{Z}\). As such, two solutions \((u, v)\) and \((u', v')\) are equivalent exactly if \( u + v\sqrt{6} \) and \( u' + v'\sqrt{6} \) generate the same ideal in \( \mathcal{O}_K \), where \( K := \mathbb{Q}(\sqrt{6}) \). Since \( \mathcal{O}_K \) is a principal ideal domain, we see that

\[
\sum_{N \equiv 2 \pmod{4}} \sum_{-\frac{N}{2} < J \leq \frac{N}{2}} q^{N^2-6J^2} = \sum_{a \in \mathcal{O}_K} q^{N(a)} \quad \text{if } N(a) \equiv 0 \pmod{8}, \quad \sum_{a \in \mathcal{O}_K} q^{N(a)} \quad \text{if } N(a) \equiv 6 \pmod{8}.
\]

Let \( a(m) \) denote the number of ideals of \( \mathcal{O}_K \) of norm \( m \). A formula for \( a(m) \) can be determined by standard methods and for our choice of \( K \) it is given in the proof of Theorem 1.3 of [25]. In particular, suppose that \( m \) is positive and \( m = 2^{a_3} 3^{b_3} \cdots p_1^{e_1} r_1^{f_1} \cdots r_k^{e_k} s_1^{g_1} \cdots s_\ell^{g_\ell} \), where the \( p_i, r_i, \) and \( s_i \) are distinct primes with \( p_i \equiv \pm 7, \pm 11 \pmod{24} \), \( r_i \equiv 1, 19 \pmod{24} \), and \( s_i \equiv 5, 23 \pmod{24} \). Then

\[
a(m) = \begin{cases} 0 & \text{if any } e_t \text{ or } a + \sum_{1 \leq t \leq \ell} g_t \text{ is odd}, \\ (f_1 + 1) \cdots (f_k + 1)(g_1 + 1) \cdots (g_\ell + 1) & \text{otherwise}. \end{cases}
\]
It is not hard to see that the parity of $u_2(n)$ is as claimed, since
$$u_2(n) \equiv \frac{1}{2} Q(16n - 2) \pmod{2}. \qedhere$$

6 Concluding remarks

This paper introduces and proves various properties of the functions $U(\zeta; q)$, $\overline{U}(\zeta; q)$, and $U_2(\zeta; q)$. It is not difficult to see additional results remain and so we briefly mention a few of these. The interpretation of $\overline{\pi}(n)$ and $\pi(m, n)$ is as the difference of non-negative counts, but it appears that $\pi(m, n)$ is always non-negative. We leave it as an open problem to prove that $\pi(m, n)$ is non-negative and to give a combinatorial interpretation that clearly demonstrates this. Theorem 1.2 gives the asymptotics of $\overline{u_2}(n)$ and $u_2(n)$. One could also ask for the asymptotics of $\overline{\pi}(m, n)$, $\overline{u_2}(m, n)$, and $u_2(m, n)$, as well as how these ranks are asymptotically distributed as $n \to \infty$. Also one could introduce and study the moments of these rank functions.

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