'Wave type' spectrum of the Gurtin-Pipkin equation of the second order

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Abstract
We study the complex part of the spectrum of the Gurtin-Pipkin integral-differential equation of the second order in time. We consider the model case when the kernel is a sum of exponentials \( a_k \exp(-b_k) \) with \( a_k = 1/k^\alpha \), \( b_k = k^\beta \). We show that there are two complex sequences of points of the spectrum asymptotically close to the spectrum points of the wave equation.

1 Introduction. Notations. Main theorem

1.1 Introduction
In several fields of physics such as heat transfer with finite propagation speed [2], systems with thermal memory [3], viscoelasticity problems [4], and acoustic waves in composite media [5], the following integro-differential equation arises

\[
\theta_{tt}(x, t) = a\theta_{xx} - \int_0^t k(t - s)\theta_{xx} ds, \quad x \in (0, \pi), \quad t > 0.
\]

(1)

with the Dirichlet boundary condition and with the initial data \( \theta(0, x) = \xi(x), \theta_x(0, x) = \eta(x) \). Regularity of this equation is studied in [6] and in [7].

First, apply the Fourier method: we set \( \varphi_n = \sqrt{\frac{2}{\pi}} \sin nx \) and expand the solution and the initial data in series in \( \varphi_n \)

\[
\theta(x, t) = \sum_{n=1}^{\infty} \theta_n(t) \varphi_n(x), \quad \xi(x) = \sum_{n=1}^{\infty} \xi_n \varphi_n(x), \quad \eta(x) = \sum_{n=1}^{\infty} \eta_n \varphi_n(x).
\]

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For the components we obtain

\[ \dot{\theta}_n(t) = -\alpha n^2 \theta_n(t) + n^2 \int_0^t k(t-s)\theta_n(s)ds, \quad t > 0, \quad \theta_n(0) = \xi_n, \quad \dot{\theta}_n(0) = \eta_n. \]  

(2)

We will denote the Laplace image by the capital characters. Applying the Laplace Transform to (2) we find

\[ z^2 \Theta_n(z) - z\xi_n - \eta_n = -an^2\Theta_n(z) + n^2K(z)\Theta_n(z) \]

or

\[ \Theta_n(z) = \frac{z\xi_n + \eta_n}{z^2 + an^2 - n^2K(z)}. \]

The set \( \Lambda \) of all zeros of the denominators \( z^2 + an^2 - n^2K(z), \ n = 1, 2, \ldots \) is called the spectrum of the equation (1).

1.2 The statement and the class of kernels

In application [1] the kernel \( k(t) \) is a sum of exponentials

\[ k(t) = \sum_{1}^{\infty} a_k e^{-b_k t}. \]

We consider the model case

\[ a = 1, \quad a_k = 1/k^\alpha, \quad b_k = k^\beta, \quad \alpha > 0. \]

The problem is to describe the zeros \( z_n \) with \( |\Re z_n| \gg 1 \) of the functions

\[ z^2/n^2 + 1 - K(z), \quad n \in \mathbb{N}, \]  

(3)

where \( K \) is the Laplace transform of \( k(t) \) in our case,

\[ K(z) = \sum_{1}^{\infty} a_k / z + b_k. \]

We are interested in the case

\[ \sum_{1}^{\infty} a_k = \infty, \quad \sum_{1}^{\infty} a_k / b_k < \infty. \]  

(4)

what gives

\[ \alpha \leq 1, \quad a + \beta > 1. \]

The conditions (4) means that \( k(t) \) has an integrable singularity at zero: \( k \notin L^\infty(0, \infty), \ k \in L^1(0, \infty). \)
1.3 The main result

Set

\[ r = \frac{\alpha + \beta - 1}{\beta}, \quad 0 < r \leq 1. \]

**Theorem 1** For any \( n \) there exist two zeros \( z^+_n, z^-_n = \overline{z^+_n} \) of (3) such that

(i) If \( r < 1 \),

\[ z^+_n = in + c_re^{-i(r+1)\pi/2}n^{1-r} + O(1) + O(n^{1-2r}), \quad c_r = \frac{\pi}{\beta \sin \pi r}. \]

(ii) If \( r = 1 \),

\[ z^+_n = \pm in - \frac{1}{2\beta} \log n + O(1). \]

2 The proof of the main theorem

We are going to find the zeros 'near' the positive imaginary axis and we set

\[ z_n = in + \tau_n n. \]

Then we obtain the equation

\[ \tau_n(\tau_n + 2i) = K(z_n). \]  

(5)

Consider the case \(|\tau_n| \ll 1.\)

**Remark 2** If we consider the case where the second factor in (5) is small: \(|\tau_n + 2i| \ll 1\) we obtain the complex conjugated roots. Indeed \( \bar{z}_n = -in + \bar{\tau}_n n \) is also the solution to (3).

Rewrite (5) as

\[ \tau = g_n(\tau), \quad g_n(\tau) = \frac{K(in + \tau n)}{\tau + 2i}. \]  

(6)

Show that for large \( n \) we have a contraction map and we can applied the fixed point theorem. We need

**Lemma 3** If for some \( \delta > 0 \)

\[ |\arg z| < \pi - \delta, \]  

(7)

then for \(|z| \to \infty\)

(i) \( r < 1 \)

\[ K(z) = c_r z^{-r} + O(|z|^{-1}). \]
(ii) $r = 1$.

\[ K(z) = \frac{1}{\beta} \frac{\log(1 + z)}{z} + O(1/|z|). \]

(iii)

\[ |zK'(z)| < \begin{cases} \frac{1}{\rho^r}, & r < 1 \\ \frac{\log \rho}{\rho}, & r = 1. \end{cases} \] \hspace{1cm} (8)

The proof is in Sec. 3. Using this lemma, we find the asymptotic of $\tau_n$. Let $|\tau| < 1/2$ (and $z = in + n\tau$)

\[ |g_n'(\tau)| = \left| \frac{K'(z) n}{\tau + 2i} - K(z) \frac{1}{(\tau + 2i)^2} \right| \]

\[ \leq |K'(z)| \frac{n}{|\tau + 2i|} + |K(z)| \frac{1}{|\tau + 2i|^2} < |K'(z)z| + |K(z)| \]

\[ < \begin{cases} \frac{1}{\rho^r}, & r < 1 \\ \frac{\log \rho}{\rho}, & r = 1. \end{cases} \]

as $|z| \to \infty$. Therefore we can find $R$ such that $|g_n'(\tau)| < \rho < 1$ for $|z| > R$ and $|\tau| < 1/2$. This implies that (6) has a zero $\tau_n$ with $|\tau_n| < 1/2$.

The iterations $\tau_n^{(0)} = 0$, $\tau_n^{(1)} = g_n(\tau_n^{(0)}) = g_n(0) = -\frac{1}{2} K(in)$, \ldots, $\tau_n^{(k+1)} = g_n(\tau_n^{(k+1)})$, \ldots, converge and we may write

\[ \tau_n = \tau_n^{(1)} + (\tau_n^{(2)} - \tau_n^{(1)}) + \cdots + (\tau_n^{(k+1)} - \tau_n^{(k)}) + \cdots = \tau_n^{(1)} + O(|\tau_n^{(2)} - \tau_n^{(1)}|). \]

Estimate $|\tau_n^{(2)} - \tau_n^{(1)}|$

\[ |\tau_n^{(2)} - \tau_n^{(1)}| = |g_n(\tau_n^{(1)}) - g_n(\tau_n^{(0)})| \leq \max_{|\tau| \leq |\tau_n^{(1)}|} |g_n(\tau)||\tau_n^{(1)}| \]

\[ < |zK'(z)| < \begin{cases} \frac{1}{\rho^{2r}}, & r < 1 \\ \frac{\log \rho}{\rho^2}, & r = 1. \end{cases} \]

Now Theorem follows from Lemma and from

\[ z_n^+ = in + n\tau_n = in - \frac{in}{2} K(in) + O(n|\tau_n^{(2)} - \tau_n^{(1)}|). \]
3 Asymptotic of $K(z)$

3.1 Integral instead series

Find the error when we replace the series $K(z)$ by the integral

$$h(z) = \int_{1}^{\infty} \frac{dx}{x^\alpha(z + x^\beta)}. $$

Lemma 4 Under (7) and $x \geq 1$, $b > 0$

$$|z + x^b|^2 \asymp |z|^2 + x^{2b}. $$

Proof. Set $z = \rho e^{i\varphi}$, $\rho = |z|$

$$|z + x^b|^2 = (\rho \cos \varphi + x^b)^2 + \rho^2 \sin^2 \varphi = \rho^2 + 2x^b \rho \cos \varphi + x^{2b}. $$

Evidently

$$\rho^2 + 2x^b \rho \cos \varphi + x^{2b} \leq (\rho + x^b)^2 \leq 2(\rho^2 + x^{2b})$$

and

$$\rho^2 + 2x^b \rho \cos \varphi + x^{2b} \geq \rho^2 - 2x^b \rho \cos \delta + x^{2b} \geq (\rho^2 + x^{2b}) - (\rho^2 + x^{2b}) \cos \delta$$

$$= (1 - \cos \delta)(\rho^2 + x^{2b}). $$

\[\square\]

Lemma 5

$$|K(z) - h(z)| \ll 1/\rho. $$

(9)

Proof. Set

$$w(x) = \frac{1}{x^\alpha(z + x^\beta)}. $$

$$|K(z) - h(z)| = \left| \sum_{k=1}^{\infty} \int_{k}^{k+1} [w(k) - w(x)] \, dx \right| \leq \sum_{k=1}^{\infty} \int_{k}^{k+1} |w(k) - w(x)| \, dx$$

We have for $x \in [k, k+1]$

$$|w(x) - w(k)| \leq (x - k) \max |w'(x)|,$$

$$|w'(x)| = \frac{\alpha x^{\alpha - 1}z + (\alpha + \beta) x^{\alpha + \beta - 1}}{x^{2\alpha}(z + x^\beta)^2} \asymp \frac{\rho + x^\beta}{x^{\alpha + 1}(\rho^2 + x^{2b})}. $$
what gives

$$|w| < \frac{1}{k^{\alpha+1}(\rho + k^{\beta})}$$

For $\alpha > 0$ this gives

$$\rho|K(z) - h(z)| \leq \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+1}} = \text{Const} < \infty.$$  

Remark 6 The restriction $r \leq 1$ in the theorem connected with the estimate (9). If $\rho > 1$, then the error $1/\rho$ of replacing the series by the integral is more than the integral, which $h$ is $O(1/\rho^r)$.

3.2 Evaluating of the integral

Lemma 7 (i) For $r < 1$

$$h(z) = c_r z^{-r} + O(1/|z|)$$

(ii) For $r = 1$

$$h(z) = \frac{1}{\beta} \log(z + 1).$$

Proof.

(i)

First, reduce one parameter. Set $x^\beta = t$, $x = t^{1/\beta}$, $dx = \frac{1}{\beta} t^{1/\beta - 1}$. Then

$$h(z) = \frac{1}{\beta} \int_1^{\infty} \frac{dt}{t^r(z + t)}.$$ 

Replace this integral by the integral on $[0, \infty)$ with the error

$$\left| \int_0^1 \frac{dt}{t^r(z + t)} \right| \sim \frac{1}{|z|}.$$

Therefore

$$h(z) = \frac{1}{\beta} \int_0^{\infty} \frac{dt}{t^r(z + t)} + O \left( \frac{1}{|z|} \right) = \frac{1}{\beta} J(z) + O \left( \frac{1}{|z|} \right).$$
In \( J(z) \) we change the variable \( y = t/z \). We obtain the integral by the ray \( R(z) = \{ a e^{i \arg z} \mid a \geq 0 \} \).

\[
J(z) = z^{-r} \int_{R(z)} \frac{dy}{y^{r}(1+y)}.
\]

The integrand is an analytical function in the upper half plane and is \( o(1/M) \) on the circle \( |z| = M \). This implies that the integral does not depend on the ray. Now

\[
J(z) = z^{-r} \int_{0}^{\infty} \frac{dt}{t^{r}(1+t)} = \frac{\pi}{\sin \pi r} z^{-r},
\]

see [4, Probl. 28.22(7)] or [8, Probl. 878] (ii) This is direct evaluating.

Proof of Lemma 3 parts (i) and (ii) follows from Lemmas 3 and 5. Part (iii).

\[
|zR'(z)| \leq \sum_{k=1}^{\infty} \frac{|z|}{k^{\alpha}|z|^{2}} \quad \text{Lemma 4,} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}(\rho^{2} + k^{2\beta})}.
\]

Replace the series by integral:

\[
|zK'(z)| \leq \rho \frac{1}{\rho^{2} + 1} + \sum_{k=2}^{\infty} \frac{1}{k^{\alpha}(\rho^{2} + k^{2\beta})} < \frac{1}{\rho} + \int_{\rho}^{\infty} \frac{dx}{x^{\alpha}(\rho^{2} + x^{2\beta})}.
\]

Estimate the last integral setting \( x^{\beta} = \rho t \)

\[
J = \int_{\rho}^{\infty} \frac{dx}{x^{\alpha}(\rho^{2} + x^{2\beta})} < \rho^{1/\beta - \alpha/\beta - 2} \int_{\rho}^{\infty} \frac{dt}{y^{r}(1+t^{2})}.
\]

This gives

\[
J < \begin{cases} 
\frac{1}{\rho^{r}}, & r < 1 \\
\frac{1}{\rho} \log \rho, & r = 1.
\end{cases}
\]

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