On the question of numerical modeling of the flow stability of bodies

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Abstract. The paper deals with the flow of a thick slab from a material in a state of viscous plasticity. The stress-strain state of the plate is determined, taking into account the initial and boundary conditions. The problem is considered in the three-dimensional formulation in the Cartesian coordinate system. The stress state components depend on all three Cartesian coordinates. Currently, it is not advisable to consider problems in a one-dimensional or flat formulation due to the huge number of influencing parameters. The thick plate is loaded with forces parallel to the median plane. The task is considered in a statically definable setting, following A. Yu. Ishlinsky, a famous Soviet and Ukrainian scientist. The perturbation method modified for plastic flow is used by D.D. Ivlev to determine the stress-strain state. The effect of plate flow on material properties and boundary surface conditions is considered. The resulting solution is examined for stability. The limiting transition and special cases illustrating the non-uniqueness of the solution are considered, if we take it formally, without limiting generality. This term is not applicable to this. It is necessary choose the community very carefully.

1. Introduction
The theory of plasticity is a fairly young science [1], compared with the theory of elasticity. The question of the stability of the viscous-plastic flow of a strip belongs to A.A. Ilyushin. He built the solution in a Lagrangian production. Later A.Yu.Ishlinsky considered this and similar problems in the Euler formulation. He conducted a comparative analysis with the decision of Ilyushin. [2]. Later, using the method of small parameter of D.D. Ivlev [3], M.A. Artyomov solved the problem of the flow of the beam. Later L.A.Maximova solved the problem of stretching a thick plate [4-5].

One of the important tasks is to determine the stress-deformable state of a thick plate in a three-dimensional formulation. The dimension of the problem is of key importance in the theory of mechanics of a deformable solid. The transition from one dimension to another does not always go smoothly. Finding fundamental solutions is also not easy. In this work, all components of the stress state depend on all coordinates, as well as on initial and boundary conditions. The material has viscous properties. The viscosity of the material is taken into account according to Ishlinsky.

The uniqueness and stability of the solution obtained is of particular importance in the work. This question in the problems of mechanics and mathematics is always relevant. This indicates that the subject of this study is relevant not only for the considered task.

2. Analysis of literature data and problem statement
The paper [6] presents the fundamental results of the study on the flow of anisotropic bodies. It is shown that the above relations are in good agreement with the classical ideas of the theory of plasticity. But
such a material property as viscosity remained unaffected. This difficulty is not new, since it is very
difficult to cover all the parties in one model, one have to “sacrifice” something.

In the works \[7-9\], the study is conducted with an emphasis on those new properties that are indicated
in the names: plate dynamics under the action of an explosive type of load, dynamic bending, dynamic
deformation of the shells. Each of these tasks has its own formulation. When solving dynamic problems,
their own difficulties arise, and their solution can significantly differ from static deformation.

Researchers use certain classical representations when solving such complex problems. This is
sufficient for a one-dimensional or two-dimensional problem. Generalizing the problem to the three-
dimensional case, the dimension of the problem increases in several times. This kind of generalization
can have "pitfalls", some of which are investigated in this work. This problem is not new, but it is solved
in different ways for each specific class of problems.

The disturbed flow of a solid thick plate under the action of tensile loads is considered in the present
work. Elastic components are not considered. It is required to determine the stress
strain state of the
points of the thick plate, taking into account the existing changes on the surface of the plate. Analyze
the solution.

The aim of the work is to determine the stress-strain state of a thick viscoplastic plate.

The following tasks were set to achieve this goal: choose the defining relations, choose the solution
method, choose the type of the fundamental solution, solve the characteristic equation, solve the system
of equations, check the uniqueness of the solution.

3. The solution of the problem

3.1. Constitutive relation

Let us consider the viscoplastic flow of a thick rectangular plate, stretched in its plane, weakened by gentle
grooves. The axes of coordinates in the formulation of the problem are the main ones. The task is
considered in the formulation of A.Yu.Ishlinsky, who considered a viscous-plastic flow of a strip, a
round rod and a round plate.Unlike other representations of the plastic state, A.Yu.Ishlinsky generalizes
the well-known formulations in the case of a non-zero viscosity material. There was only one equation
that did not present much difficulty in the formulation of Ishlinsky. There were three of them in the
spatial formulation.

Suppose that the median plane of the plate coincides with the plane \(xOy\), denote by \(2h\) the thickness
of the plate, so that \(|z| \leq h\)

The number of ratios increased three times. The condition of the viscoplastic state we write in the form

\[
(\sigma_x - \sigma - 2/3k - \mu\varepsilon_x)(\sigma_y - \sigma - 2/3k - \mu\varepsilon_y) = (\tau_{xy} - \mu\varepsilon_{xy})^2,
\]

\[
(\sigma_y - \sigma - 2/3k - \mu\varepsilon_y)(\sigma_z - \sigma - 2/3k - \mu\varepsilon_z) = (\tau_{yz} - \mu\varepsilon_{yz})^2,
\]

\[
(\sigma_z - \sigma - 2/3k - \mu\varepsilon_z)(\sigma_x - \sigma - 2/3k - \mu\varepsilon_x) = (\tau_{xz} - \mu\varepsilon_{xz})^2.
\]

Or in the form of

\[
(\sigma_x - \sigma - 2/3k - \mu\varepsilon_x)(\tau_{yz} - \mu\varepsilon_{yz}) = (\tau_{xy} - \mu\varepsilon_{xy})(\tau_{xz} - \mu\varepsilon_{xz}),
\]

\[
(\sigma_y - \sigma - 2/3k - \mu\varepsilon_y)(\tau_{xz} - \mu\varepsilon_{xz}) = (\tau_{xy} - \mu\varepsilon_{xy})(\tau_{yz} - \mu\varepsilon_{yz}),
\]

\[
(\sigma_z - \sigma - 2/3k - \mu\varepsilon_z)(\tau_{xy} - \mu\varepsilon_{xy}) = (\tau_{yz} - \mu\varepsilon_{yz})(\tau_{xz} - \mu\varepsilon_{xz}).
\]

Relations (2) are transformed from relations (1), but sometimes they are more acceptable.

3.2. Method of solution

We will determine the stress state by the small parameter method proposed by D.D. Ivlev. The initial
state will be denoted by and taken:

\[
\sigma_x^0 = k, \quad \sigma_y^0 = k, \quad \sigma_z^0 = k, \quad \tau_{xy}^0 = \tau_{xz}^0 = \tau_{yz}^0 = 0, \quad \varepsilon_x^0 + \varepsilon_y^0 + \varepsilon_z^0 = 0, \quad \varepsilon_{xy}^0 = \varepsilon_{yz}^0 = \varepsilon_{xz}^0 = 0.
\]
The components of the stress-strain state will be considered as the sum of the initial and perturbed states:

\[
\sigma_x = \sigma_x^0 + \delta\sigma_x, \quad \sigma_y = \sigma_y^0 + \delta\sigma_y, \quad \sigma_z = \sigma_z^0 + \delta\sigma_z, \\
\varepsilon_x = \varepsilon_x^0 + \delta\varepsilon_x, \quad \varepsilon_y = \varepsilon_y^0 + \delta\varepsilon_y, \quad \varepsilon_z = \varepsilon_z^0 + \delta\varepsilon_z.
\]

(4)

Where the mark at the top we assign to the components of the disturbed state, \(\delta\) a small parameter.

According to, (3) and [6], we introduce new viscoplastic constants depending on the initial state:

\[
\mu_1 = k l (\varepsilon_x^0 - \varepsilon_y^0), \quad \mu_2 = k l (\varepsilon_x^0 - \varepsilon_z^0).
\]

(5)

Linearizing the plasticity condition (1-2), we obtain expressions for the components of the stressed perturbed state

\[
\sigma_x' = \sigma_x' + \mu_1 \varepsilon_x', \quad \sigma_y' = \sigma_y' + \mu_1 \varepsilon_y', \quad \sigma_z' = \sigma_z' + \mu_1 \varepsilon_z', \quad \tau_{xy}' = \mu_2 (\varepsilon_x' + \varepsilon_y'), \quad \tau_{xz}' = \mu_2 (\varepsilon_x' + \varepsilon_z'), \quad \tau_{yz}' = \mu_2 (\varepsilon_y' + \varepsilon_z').
\]

(6)

Substituting these relations into the equilibrium equations, we turn to the components of the velocity of displacement using Cauchy formulas

\[
\varepsilon_{ij} = 1/2 (u_{i,j} + u_{j,i}),
\]

(7)

appending not compressibility condition

\[
\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0,
\]

(8)

we obtain four equations with four unknowns \(\sigma', u'.\)

\[
\frac{\partial^2 u'}{\partial x} + \frac{\partial^2 v'}{\partial y} + \frac{\partial^2 w'}{\partial z} = 0,
\]

\[
\frac{\partial \sigma'}{\partial x} + \frac{\mu_1}{2} \left( \frac{\partial^2 u'}{\partial z^2} + \frac{\partial^2 w'}{\partial x \partial z} \right) + \frac{\mu_2}{2} \left( 2 \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) = 0,
\]

\[
\frac{\partial \sigma'}{\partial y} + \frac{\mu_1}{2} \left( \frac{\partial^2 v'}{\partial z^2} + \frac{\partial^2 w'}{\partial y \partial z} \right) + \frac{\mu_2}{2} \left( 2 \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 v'}{\partial x^2} \right) = 0,
\]

\[
\frac{\partial \sigma'}{\partial z} + \frac{\mu_1}{2} \left( \frac{\partial^2 w'}{\partial x \partial z} + \frac{\partial^2 w'}{\partial z^2} \right) + \frac{\mu_2}{2} \left( 2 \frac{\partial^2 w'}{\partial y \partial z} + \frac{\partial^2 v'}{\partial z^2} + \frac{\partial^2 w'}{\partial z^2} \right) = 0.
\]

(9)

3.3. Fundamental decisions

The solution of the system of equations (9) will be sought in the form of trigonometric series, having the following form:

\[
\sigma' = A \cos mx \cos ny \cos \lambda z, \quad u' = B \sin mx \cos ny \cos \lambda z, \\
v' = C \cos mx \sin ny \cos \lambda z, \quad w' = D \cos mx \cos ny \sin \lambda z,
\]

(10)

where \(A, B, C, D, m, n, \lambda\) are constants. These functions satisfy the compatibility conditions. We omit the amount sign, if necessary, it will be possible to take several members of the series.

Substituting the functions (10) into the system of equations (9), we obtain

\[
\begin{bmatrix}
0 & m & n & \lambda \\
m \mu m^2 + \frac{\mu n^2 + \mu_1 \lambda^2}{2} & \frac{1}{2} \mu mn & \frac{1}{2} \mu_1 m \lambda \\
n \frac{1}{2} \mu mn & \mu n^2 + \frac{\mu_2 \lambda^2}{2} & \frac{1}{2} \mu_2 n \lambda \\
\lambda \frac{1}{2} \mu m \lambda & \frac{1}{2} \mu_1 n \lambda & \mu \lambda^2 + \frac{\mu_2 m^2 + \mu_2 n^2}{2}
\end{bmatrix}
= \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}
\]

(11)

Thus, we passed to the system of algebraic equations (11) from the system of differential equations (9).
3.4. Characteristic equation

In order for the solution of this system not to be trivial, we require that its determinant be zero. As a result, we obtain an equation for determining the relationship between \( m, n, \lambda \). Using various cases of these dependencies. Let's find the remaining variables.

A.Yu.Ishlinsky later considered other modifications of this problem. Many researchers were engaged in similar tasks.

The determinant of the resulting system of algebraic equations is as follows

\[
\mu \left( \mu \lambda^2 + \mu (m^2 + n^2) \right) n^2 + \left( \mu \lambda^2 + \mu (m^2 + n^2) \right) m^2 \left( 3 \lambda^2 - m^2 - n^2 \right) \left( \lambda^2 + m^2 + n^2 \right) + \\
+ \left( \mu \lambda^2 + \mu (m^2 + n^2) \right) \left( \mu \lambda^2 + \mu (m^2 + n^2) \right) \left( \lambda^2 - m^2 - n^2 \right)^2 = 0.
\]  

(12)

Analyzing this determinant, we obtained that the last factor can be equal to zero. It is possible to investigate the first factor in view of the set of variables. Thus, some connection between the frequency characteristics of voltage fluctuations has been established.

3.5. The study of the system of equations

Let's consider the system of four equations to determine the amplitudes of the stress functions (11). This system is homogeneous and it is necessary to define basic and free variables.

From the condition of existence of the solution, it is necessary that the determinants of the system of equations for finding the perturbation amplitudes are equal to zero at the same time \( \Delta_1=0, \Delta_2=0, \Delta_3=0, \Delta_4=0 \). When deriving these conditions, it is also possible to get mismatched solutions.

In this case, the rank of the system can be two or three.

If the rank of the system is two, then the system can be degenerate and the choice of a set of basic variables cannot be any. This study is symmetrical. Due to the specificity of the matrix of the system of equations, the constant \( A \) cannot be free. One can take any other than the parameter \( A \) for a free variable. We take the constant \( C \) without limiting the generality. as a free variable. The same system, theoretically, can be solved by the Kramer method, but for this it is necessary to reduce the determinants included in the formulas, which is quite difficult. We present two numerical solutions of similar systems for clarity.

3.6. Numerical solution 1

Take the values \( m = 1, n = 2, \lambda = 3 \). According to these values, select the remaining coefficients so that the determinant is zero.

For example, the determinant:

\[
\begin{vmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 5 & 7 \\
3 & 4 & 7 & 10
\end{vmatrix}
\]  

(13)

Accordingly, these coefficients satisfy the condition for solving the system (11). We take the following value as a free variable \( C = -1 \).

We find the basic solutions by substituting the values from the above determinant into the system (11).

It is necessary to delete the unnecessary line to solve such a system. The choice of the line depends on the coefficients for unknowns. It is impossible to select the desired row in the source system. Using a numerical example, we consider all possible options and draw conclusions. Remove the first line.

Obtained solution:

\[
A = C_1 (-2D + 1); \quad B = C_1 (-D + 1); \quad D = C_1 D.
\]  

(14)

It follows that the rank of the system is two, since there are two free variables.

1) Crossing out the second line, instead of the first one, we get another system of equations.

Got the following values:

\[
A = C_2 * 0; \quad B = C_2 / 2; \quad D = C_2 / 2.
\]  

(15)
2) Now, similarly, delete the third line and get the system. Obtained solution:
\[ A = C_1 \times 0; B = C_1 / 2; D = C_1 / 2. \] (16)

3) Next, we get the following system deleting the fourth line. Got the basic solution:
\[ A = C_1 \times 0; D = C_1 / 2; B = C_1 / 2. \] (17)

When solving a system of equations, we get the same solutions when we remove the second, third, and fourth rows, where the rank of the matrix is three. In the case where we delete the first row, the rank of the matrix is two and, accordingly, we get another solution.

3.7. Numerical solution 2

Take the determinant with the following values:

\[
\Delta = \begin{vmatrix}
0 & 1 & 2 & 3 \\
1 & 6 & 10 & 15 \\
2 & 10 & 17 & 19 \\
3 & 15 & 19 & 61 \\
\end{vmatrix}
\] (18)

It is necessary that the determinant be equal to zero. This is not difficult to check it. Accordingly, these coefficients satisfy the condition for solving the system (11). We take the following value as a free variable \( C = -1 \).

We find the basic solutions by substituting the values from the above determinant into the system (11).

1) Cross out the first line and get the system. Got values \( A, B, D \)
\[ A = C_1 \times (-13/5); B = C_1 \times 13/5; D = C_1 \times (-1/5). \] (19)

2) Cross out the second line and get the system. Got
\[ A = C_1 \times (-13/5); D = C_1 \times 13/5; D = C_1 \times (-1/5). \] (20)

3) Delete the third line and get the system. Obtained value:
\[ A = C_1 \times (-13/5); B = C_1 \times 13/5; D = C_1 \times (-1/5). \] (21)

4) Remove the fourth line and get the system. Got
\[ A = C_1 \times (-13/5); B = C_1 \times 13/5; D = C_1 \times (-1/5). \] (22)

In comparison with the solutions obtained in the previous case, in solving this system of equations, we obtained that we get the same solutions by removing each of the rows, since the rank of the matrix is three.

3.8. Definition of constant \( C \)

All the properties of the material are taken into account to determine the stress-strain state of a viscoplastic plate, in particular, the viscosity of the body \( \mu \).

It is well known that when solving spatial problems, initial and boundary conditions are of great importance. The above takes into account the influence of the initial conditions (constants \( \mu_1, \mu_2 \)).

The equation of the surface of the plate will be given in the form \( z = \pm (h - \delta f (x, y)) \).

Further we set
\[ f (x, y) = K \cos m x \cos n y, \quad K = \text{const}. \] (23)

Further assume that the surface of the plate is free from stresses, from (6), we find
\[ \sigma'_z = 0, \tau'_z + k \frac{\partial f}{\partial x} = 0, \tau'_z + k \frac{\partial f}{\partial y} = 0, z = \pm h. \] (24)

From (10) it follows
\[
\sigma'_z = (A + \mu D\lambda) \cos mx \cos ny \cos \lambda z, \varepsilon'_y = (B\lambda + Dm) / 2 \sin mx \cos ny \sin \lambda z, \\
\varepsilon'_{yz} = (C\lambda + Dn) / 2 \cos mx \cos ny \sin \lambda z.
\]

From (24), (25) we obtain
\[
\cos \lambda h = 0, \lambda h = \pi / 2 + \pi k, k = 0, \pm 1, \ldots
\]

\[
\frac{\partial \varepsilon_y}{\partial x} = -\frac{\mu_1}{k} \varepsilon'_y = -Km \sin mx \cos ny, \frac{\partial \varepsilon'_y}{\partial y} = -\frac{\mu_2}{k} \varepsilon'_{yz} = -Kn \sin ny \cos mx.
\]

The following is as follows
\[
K = \frac{\mu_1}{2mk} (B\lambda + Dm) = \frac{\mu_2}{2mk} (C\lambda + Dn).
\]

The constants \(B, D\) and \(A\), according to the formulas, are expressed in terms of one constant \(C\), which is determined from the last formula through a given amplitude of the perturbation \(K\).

4. Discussion of the results of the determination of the state of the plate for stability

The advantage of the presented research is fundamental and analytic. All methods used in solving are classical methods of mechanics and mathematics. These studies can be applied to other problems of the viscoplastic flow of various bodies and structures. They are a continuation of the research of the classics of mechanics. It is assumed to continue the study with anisotropic and composite materials.

The results obtained are natural with a full study of a complex equation (characteristic) and systems of equations (obtained to determine the parameters of the components of the stress state). The possibility of obtaining an erroneous solution, and, in particular, inexplicable paradoxes with this approach is excluded.

The limitations of this study are defined by the statement: thick rectangular plate, stretchable in its plane. In another formulation, some relationships will change and the study can be continued.

With an increase in additional restrictions, for example, fracture (cracks), the system of equations may not have a solution, then it is necessary to change any initial hypotheses. For example, the fulfillment of conditions at the border, not exactly, but on average.

5. Conclusions

As a result of the research:

The stress-deformable viscoplastic state of a thick plate in a three-dimensional formulation, which is a natural continuation of the development of the fundamental mathematical theory of the mechanics of a deformable solid, is determined, continuing the studies of Ilyushin and Ishlinsky.

The existence of a solution in this formulation, its uniqueness and stability is shown.

Examples are given in which it is possible to get a wrong solution and it is shown how to avoid it.

The proposed solution for determining the stress state of a thick viscoplastic plate exists and is the only one consistent with the main hypotheses of the mechanics of a deformable solid. Numerical examples have shown that non-uniqueness of the solution may be the wrong choice of basic variables. The amplitudes and frequencies of oscillations of the components of fundamental solutions are interconnected and depend on all the parameters of the problem.

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