Sound Probabilistic Inference via Guide Types

Technical Report

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Abstract

Probabilistic programming languages aim to describe and automate Bayesian modeling and inference. Modern languages support programmable inference, which allows users to customize inference algorithms by incorporating guide programs to improve inference performance. For Bayesian inference to be sound, guide programs must be compatible with model programs. One pervasive but challenging condition for model-guide compatibility is absolute continuity, which requires that the model and guide programs define probability distributions with the same support.

This paper presents a new probabilistic programming language that guarantees absolute continuity, and features general programming constructs, such as branching and recursion. Model and guide programs are implemented as coroutines that communicate with each other to synchronize the set of random variables they sample during their execution. Novel guide types describe and enforce communication protocols between coroutines. If the model and guide are well-typed using the same protocol, then they are guaranteed to enjoy absolute continuity. An efficient algorithm infers guide types from code so that users do not have to specify the types. The new programming language is evaluated with an implementation that includes the type-inference algorithm and a prototype compiler that targets Pyro. Experiments show that our language is capable of expressing a variety of probabilistic models with nontrivial control flow and recursion, and that the coroutine-based computation does not introduce significant overhead in actual Bayesian inference.

Keywords: Probabilistic programming, Bayesian inference, type systems, coroutines

1 Introduction

Probabilistic programming languages (PPLs) [1, 13, 23, 25, 26, 49, 52, 57, 62] provide a flexible way of describing statistical models and automatically performing Bayesian inference: a method for inferring the posterior of a statistical model from observed data. Bayesian inference accounts for uncertainty in latent variables that produce the observed data. It has applications in many fields, including artificial intelligence [22], cognitive science [28], and applied statistics [21].

Because there is not a single known inference algorithm that works well for all models [41], several PPLs have recently added support for programmable inference [8, 17, 20, 41, 46, 63]. This capability allows users to customize inference algorithms based on the characteristics of a particular model or dataset. Researchers have shown that programmable inference enables improved inference performance on a variety of modeling problems [8, 17, 19, 41].

Two important families of inference algorithms can be customized by incorporating guide programs, which are implemented by the user. The first family is Monte-Carlo methods, such as importance sampling and Markov-Chain Monte Carlo, where a guide program serves as a proposal, which generates random samples for latent variables. The second family is variational inference, where a guide program is a parameterized program that specifies a collection of approximating distributions on latent variables.

To ensure soundness of programmable inference, the guide programs have to be compatible with the implemented model program; incompatible guide programs could crash the inference process or lead to incorrect inference results [39, 40]. Recently, Lee et al. [39] developed a static analysis for finding bugs in model-guide pairs for variational inference in Pyro [8]. Lew et al. [40] proposed a type system that proves model-guide compatibility for multiple inference algorithms. However, neither approach handles general conditional statements that can influence the set of latent variables sampled by the model, and it is unclear how to extend them to analyze recursive programs precisely.

In this paper, we develop a new PPL that supports recursion and conditional statements, as well as guarantees absolute continuity, one of the most pervasive conditions for ensuring model-guide compatibility. Our PPL uses a new paradigm for writing inference code: users implement the model and guide programs as coroutines, which can communicate with each other during their execution. We develop a new type system, which we dub guide types, to describe the communication protocols between coroutines. These guide types can be automatically inferred and are proof certificates of absolute continuity for model-guide pairs. They apply to multiple kinds of Bayesian-inference algorithms.

In our development, we follow a common scheme of trace-based programmable inference that underlies Pyro [8], Venture [41], Gen [17], etc. These PPLs define the meaning of a probabilistic program by a probability distribution on sample
traces that record all the random samples that the program draws during its execution. A program $p$ is absolutely continuous with respect to a program $q$, if any set of sample traces with non-zero probability under the program $p$ must also have non-zero probability under the program $q$. In this paper, we reduce the problem of checking absolute continuity to the following verification task:

Given a model program $p$ and a guide program $q$, verify that they define probability distributions with the same support, i.e., they have the same set of possible sample traces.

The major challenge in our development is to reason about the sets of possible sample traces for the model and guide programs, when the two programs can diverge in their execution, as always with relational reasoning. Control-flow constructs make it difficult to keep track of sample sites precisely; for example, a conditional statement can sample different sets of random variables in its two branches. It is intractable to enumerate all possible execution paths in the two programs and compare the sample sites path-to-path, especially when the programs are recursive.

The first part of our solution is to think of the model and guide programs as coroutines that can exchange messages. Conceptually, we use coroutine-style communication to synchronize each pair of sample sites that represent the same random variable, as well as each branch selection that influences control flow. The communication between the two coroutines should then be conducted according to a protocol so that messages always occur in guidance pairs: when one partner sends, the other receives; and when one partner offers a selection, the other branches.

The second part of our solution is to develop guide types as guidance protocols between the model and guide coroutines. In our formalization, we structure the sequence of messages between two coroutines, rather than describe it as a collection of unrelated messages. To handle general recursion, we parameterize the guide type for each coroutine by a continuation type that describes the guidance protocol for the computation that continues after a recursive invocation. We also develop an efficient algorithm that infers guide types automatically from the code.

There have been several type systems for coroutines [2, 3, 29], but all of them require that all messages from a coroutine to another have the same type; thus, they are not sufficient to handle sample passing and branch selection in our coroutine-based paradigm. In our development of guide types, we took inspiration from type systems for communication protocols in concurrent systems, such as session types [31, 32]. Guide types have different semantics from and are simpler than session types, and use a parametrization technique to model recursive computation.

We then establish formal guarantees of our new PPL. First, we prove that guide types ensure safety of communication between coroutines, i.e., the coroutines send and receive messages in a consistent manner. Second, we prove that guide types serve as proof certificates of absolute continuity between the model and guide programs; consequently, we use guide types to justify soundness of importance sampling, Markov-Chain Monte Carlo, and variational inference. Note that for variational inference, the soundness guarantee is partial, because sound inference requires some additional conditions (e.g., differentiability), whereas this paper focuses just on absolute continuity.

We implemented a type-inference algorithm for guide types and a prototype compiler from our PPL to Pyro. We evaluated our PPL on a broad suite of probabilistic models, and our experimental results show that (i) our PPL is more expressive than a state-of-the-art PPL that ensures soundness of programmable inference [30], and (ii) type inference completes in several milliseconds, and the performance of Bayesian inference on the compiled code is similar to handwritten Pyro code, i.e., coroutine communication does not introduce significant overhead.

Contributions. We make four main contributions.

- We develop a new PPL with a coroutine-based paradigm for implementing model and guide programs.
- We propose guide types, which prescribe guidance protocols between the model and guide coroutines, and develop an efficient inference algorithm for guide types.
- We prove type safety of guide types, and show that guide types ensure key soundness conditions of model-guide pairs for multiple kinds of Bayesian-inference algorithms.
- We implemented our PPL and evaluated its effectiveness on a variety of probabilistic models.

2 Overview

In this section, we first review Bayesian inference and trace-based programmable inference (§2.1). We then demonstrate the coroutine-based paradigm for implementing inference code and the use of guide types to enforce guidance protocols between coroutines. (§2.2).

2.1 Bayesian Inference

Probabilistic programs specify generative models that sample random variables. The semantics of a probabilistic program can be defined as a probability distribution on the sample traces that record all the random values that a program draws during its execution [10, 38]. Consider the program $Model$ in Fig. 1; it specifies a probabilistic model on random variables introduced by commands $sample(\ell, d)$, where $\ell$ is a label that identifies a sample site in a program; and $d$ is a primitive distribution, such as Gamma distributions whose support is the positive real line $\mathbb{R}_+$. Normal distributions whose support is the real line $\mathbb{R}$, and Beta distributions whose support is the unit interval $\mathbb{R}_{(0,1)}$. Two possible sample traces in the program $Model$ are $[@x = 1; @z = -0.5]$ and $[@x = 3; @y = 0.9; @z = 0.7]$. More generally, the program specifies a distribution on sample traces whose support
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Figure 1. A program Model with a conditional statement.

\begin{verbatim}
1  proc Model() =
2       v ← sample[@x, Gamma(2; 1)];
3       if v < 2 then
4          _ ← sample[@z, Normal(−1; 1)];
5       return(v)
6   else
7       return(false)
8
proc Guide() =
1       v ← sample[@x, Gamma(1; 1)];
2       if v < 2 then
3          _ ← sample[@y, Beta(3; 1)];
4       return(v)
5   else
6       return(false)
\end{verbatim}

Figure 3. Sound and unsound guide programs for IS.

is

\[
\{ [[@x = a; @z = c] | 0 < a < 2] \cup \{ [@x = a; @y = b; @z = c] | a \geq 2, 0 < b < 1 \} \}
\]  \hfill (1)

Bayesian Inference amounts to conditioning a probabilistic model on observations and computing a posterior distribution on latent variables. For the program Model, we consider that @z is the single \( \mathbb{R} \)-valued observation, while both @x and @y are latent variables. Intuitively, latent variables encode knowledge about the “ground truth” that we cannot observe directly, and the model program specifies a prior distribution on the “ground truth.” Given a concrete value of the observation (e.g., \( @z = 0.8 \)), the objective of Bayesian inference is to approximate the posterior distribution of the latent variables (e.g., likely values of @x and @y under the condition that @z = 0.8). Fig. 2 plots the prior distribution of the random variable @x, and its posterior distribution under the observation @z = 0.8.

It is usually intractable to sample directly from or even derive posterior distributions. There have been two popular families of inference algorithms: Monte-Carlo methods and variational inference. These inference algorithms usually require some guide programs, which can have a substantial influence on the performance of the inference. Although many PPLs provide mechanisms for automatically generating those guide programs, the ability to allow users to customize them, has been shown to be helpful, and sometimes crucial, for effective inference [8, 17, 19, 41]. However, customizability introduces non-trivial challenges to ensuring soundness of Bayesian inference. We now illustrate some mistakes when programming guide programs for Monte-Carlo methods and for variational inference.

Monte-Carlo methods. A Monte-Carlo method generates iteratively random samples such that empirical distribution of the samples approximates the posterior distribution. Two popular Monte-Carlo methods are importance sampling (IS) and Markov-Chain Monte Carlo (MCMC). IS generates independent and identically distributed samples from a proposal distribution, and reweights the samples by their importance, which corrects the discrepancy between the posterior and proposal distributions. MCMC generates iteratively a new random sample from an old one; that is, it constructs a Markov chain whose stationary distribution is the posterior distribution.

We now illustrate a mistake when programming guide programs for IS. For IS to converge asymptotically to the posterior distribution, the posterior distribution must be absolutely continuous with respect to the proposal distribution, i.e., any set of samples with non-zero probability under the posterior distribution must also have non-zero probability under the proposal distribution. In §5, we will show that it suffices to verify if the model program conditioned with respect to a concrete observation and the guide program have the same set of possible sample traces. For example, for the program Model shown in Fig. 1, the support of a sound guide program could be

\[
\{ [@x = a] | 0 < a < 2 \}
\]

\[
\cup \{ [@x = a; @y = b] | a \geq 2, 0 < b < 1 \}, \hfill (2)
\]

which is obtained by factoring out the observation @z from the support of the unconditioned model shown in (1).

Fig. 3 presents two guide programs for performing IS from the program Model shown in Fig. 1, where the supports of the Pois and Unif distributions are natural numbers \( \mathbb{N} \) and the unit interval \( \mathbb{R}_{(0,1)} \), respectively. The support of the program Guide\( e_i \) is exactly the one shown in (2); thus, Guide\( e_i \) is a sound guide program; that is, Guide\( e_i \) samples the latent random variables @x, @y from the same space as Model does. On the other hand, the support of the program Guide\( e_i^\prime \) does not match (2), and it is actually an unsound guide program for two reasons:

- In the model program, the latent variable @x can be any positive real number, whereas the program Guide\( e_i^\prime \) only samples natural numbers for @x.
- In the model program, when the value of v (i.e., the latent variable @x) is greater than 2, the other latent variable @y should be present in the sample trace. However, when the value of v is greater than 10, the program Guide\( e_i^\prime \) will not produce a sample for @y.
Variational inference (VI). In contrast to Monte-Carlo methods, VI uses optimization (e.g., stochastic gradient descent) to find a candidate from an approximating family of distributions that minimizes the distance between the posterior distribution and the approximating distributions. In PPLs such as Pyro, users specify the approximating family by a parameterized probabilistic program called a guide; instantiating the parameters with a concrete valuation that produces a member of the approximating family. A widely used distance is the Kullback-Leibler (KL) divergence from the posterior distribution to the guide distribution. For the KL divergence to be well-defined, the guide distribution must be absolutely continuous with respect to the posterior distribution. In §5, we again reduce the verification of absolute continuity to checking a sufficient condition, namely, that the model conditioned with respect to a concrete observation and the guide have the same support. Note that VI requires several more conditions (such as differentiability) for inference to be sound [39]. In this paper, we focus on verification of absolute continuity.

Fig. 4 presents two guide programs for performing VI on the program Model shown in Fig. 1. The real-valued parameters of the guide programs are \( \theta_1, \ldots, \theta_4 \). The support of the program \( \text{Guide}_2 \) (instantiated with concrete parameters) is exactly the one shown in eq. (2). On the other hand, the program \( \text{Guide}_2 \) defines an unsound guide, because it samples \( \sigma \) from a normal distribution, whose support is the whole real line, whereas the program Model always samples a positive value for \( \sigma \).

2.2 Sound Bayesian Inference via Guide Types

Programs as coroutines. Our first contribution is a coroutine-based paradigm for implementing the model and guide programs for Bayesian inference. In an inference algorithm, the model program and its guide program have many connections. The two most significant patterns we can observe in common inference algorithms are as follows:

- The guide program needs to have similar control-flow structure to that of the model program. For example, if the model program has a conditional command whose two branches sample different sets of latent variables, the guide program should also have a conditional command with an equivalent branch condition.

- The guide program needs to have similar control-flow structure to that of the model program. For example, if the model program has a conditional command whose two branches sample different sets of latent variables, the guide program should also have a conditional command with an equivalent branch condition.

The first pattern illustrates a form of sample passing from the guide program to the model program, and the second pattern indicates that the model program should provide branch selection to the guide program. Such bidirectional guidance inspired us to treat the model and guide programs as coroutines that communicate with each other during their execution, rather than as totally independent programs. On the other hand, we do not want the coroutines to be tightly coupled: Bayesian practitioners usually maintain a separation between the model and the guide so that they can refine the guide iteratively to improve inference performance.

Therefore, we use message-passing communication to implement the coroutines; this formalism allows us to separate the model and the guide as individual programs, but connect them via channels over which coroutines exchange messages. Fig. 5 reimplements the model and guide programs in Fig. 1 and Fig. 3, respectively, by making the guidance communication explicit. The sample(·) commands and conditional commands are annotated with \( v \) (i.e., “receive”) or \( v \) (i.e., “send”) to indicate the direction of communication, and associated with a name of the channel on which the communication is carried out. In this example, we use two channels: latent for communication between the guide and the model, and obs for identifying observations in the model. Every channel has a unique provider and a unique consumer. Note that in this way we do not need to use labeled samples—as Pyro and some other PPLs do—because the sampling sites are synchronized through guidance communication.

Operationally, when a coroutine is executing a command associated with a channel \( c \), it resumes the other coroutine that accesses channel \( c \), until the other coroutine encounters a command that also communicates on channel \( c \). Then they perform synchronization; for example,

- When Model executes \( \text{sample}_c(\text{latent})(\text{Gamma}(2; 1)) \), it resumes the other end of the latent channel, i.e., the coroutine \( \text{Guide}_1 \), until \( \text{Guide}_1 \) reaches the command \( \text{sample}_d(\text{latent})(\text{Gamma}(1; 1)) \). Recall that the guide program is used in importance sampling; thus, the coroutine \( \text{Guide}_1 \) draws a sample from the distribution \( \text{Gamma}(1; 1) \), and then sends it to the coroutine Model, which uses the sample and the prior distribution \( \text{Gamma}(2; 1) \) to calculate the importance weight.

- When \( \text{Guide}_1 \) executes the conditional command on line 3 (where the \( * \) symbol indicates that the branch selection is received from the other coroutine), it resumes the other end of the latent channel, i.e., the coroutine Model, until Model reaches the conditional command on line 3. The coroutine Model is the sender of the branch selection;
When the synchronization is completed, either coroutine with a protocol of type $\text{latent}$ guides latent provider latent type for the channel $\mathcal{A}$ dually $\mathcal{A}$ the received selection. The type $\mathcal{A}$ guide types. The type $\mathcal{A}$ messages can be exchanged. The type $\mathcal{A}$ guide types. The type $\mathcal{A}$ messages on a channel, rather than describe it as a collection $\mathcal{A}$.

We sketch some type constructors in our development of guide types. We take inspiration from type systems for communication sessions types $\mathcal{A}$ [36]. The key idea is to structure the sequence of guidance messages on a channel, rather than describe it as a collection of unrelated messages.

We sketch some type constructors in our development of guide types. The type $\mathcal{A}$ types an ended channel, where no messages can be exchanged. The type $\mathcal{A} \& \mathcal{B}$ types a channel whose provider waits for a branch selection, and continues with a protocol of type $\mathcal{A}$ or a protocol of type $\mathcal{B}$ based on the received selection. The type $\mathcal{A} \& \mathcal{B}$ types a channel whose provider samples and sends a random value of type $\mathcal{A}$, and then continues with a type $\mathcal{B}$ protocol. The guide type for a channel is the same for the provider and the consumer of the channel, but the two ends of a channel interpret the guide type for the channel dually (e.g., sends as receives).

With these three type constructors, we can express the protocols for the $\text{latent}$ and $\text{obs}$ channels shown in Fig. 5 as

\begin{align*}
\text{latent} & : \mathbb{R}_+ \land (1 \& (\mathbb{R} \land \mathbb{R} \land 1)), \\
\text{obs} & : \mathbb{R} \land 1.
\end{align*}

(3) (4)

The provider and the consumer of the channel $\text{latent}$ are the coroutines $\text{Guide}_1$ and $\text{Model}$, respectively. From the provider $\text{Guide}_1$‘s perspective, the protocol shown as type (3) guides $\text{Guide}_1$ to draw a $\mathbb{R}_+$-valued sample and send it on $\text{latent}$, then wait for a branch selection, and finally end the communication on $\text{latent}$ if the received branch selection is then-branch, otherwise draw an $\mathbb{R}_+(0,1)$-valued sample before ending the communication. The coroutine $\text{Guide}_1$ implements this guidance protocol exactly. Meanwhile, from the consumer $\text{Model}$‘s perspective, the type constructors have dual semantics, i.e., $\text{send}$ becomes $\text{receive}$ and vice versa; thus, the protocol for $\text{latent}$ guides $\text{Model}$ to receive an $\mathbb{R}_+$-valued sample, and then send out a branch selection on channel $\text{latent}$; if $\text{Model}$ selects the else-branch, then it further receives an $\mathbb{R}_+(0,1)$-valued sample on channel $\text{latent}$.

The channel $\text{obs}$, whose provider is the coroutine $\text{Model}$, is used to identify observations in the probabilistic model. The coroutine $\text{Model}$ accesses $\text{obs}$ on lines 4 and 8, each of which lies in a branch of the conditional command on line 3. Because the conditional command is associated with $\text{latent}$, it should not bother with the communication on channel $\text{obs}$; thus, we require that the two branches of the conditional command have the same guidance protocol for $\text{obs}$. The protocol shown as type (4) specifies that the coroutine $\text{Model}$ produces a single $\mathbb{R}$-valued observation, and $\text{Model}$ implements this protocol exactly.

**Recursion.** Probabilistic programs can use recursion to express complex generative models, such as a probabilistic context-free grammar (PCFG), which is a popular model for constructing languages [36]. Fig. 6 shows a recursive model that generates a random expression tree with two constructors: $\text{Const}(\cdot)$ for leaf nodes and $\text{Add}(\cdot; \cdot)$ for internal nodes.

To support recursion in probabilistic programs, we add a standard recursive-type constructor to guide types. However, composition of the guide types from multiple procedure calls in a non-tail-recursive program remains a challenge. One straightforward approach is to add a sequencing type $\mathcal{A} \uparrow \mathcal{B}$ that types a channel whose provider starts with a type $\mathcal{A}$ protocol and then continues with a type $\mathcal{B}$ protocol, but such sequencing types will complicate the type system, because they allow a guidance protocol to be described by different types. For example, both $(\mathbb{R} \land \mathbb{R} \land 1)$ and $(\mathbb{R} \land 1) \uparrow (\mathbb{R} \land 1)$ types the communication on $\text{latent}$.
1]) describe a channel whose provider sends two \( \mathbb{R} \)-valued random samples.

To sidestep the need for a nontrivial equivalence check in the type system, we adapt the idea of type-level polymorphism, and parameterize the guide type for a recursive coroutine by a continuation type that describes the communication after a procedure call to this coroutine returns. For example, consider the following parametric type \( R[\cdot] \).

\[
R[X] \triangleq R(\{1\}) \land ((R \land X) \otimes R[R[X]]).
\]

It specifies a guidance protocol by prepending messages to the continuation protocol defined by the type parameter \( X \). The type \( R[X] \) precisely describes the behavior of the \textit{PcfgGen} coroutine shown in Fig. 6: the coroutine first receives an \( R(\{1\}) \)-valued random sample (line 6); evaluates and sends out a branch selection (line 7); and then based on the branch selection, the coroutine either receives an \( R \)-valued random sample (line 8) and then returns (i.e., continues with the continuation protocol \( X \)), or makes two recursive procedure calls (lines 11 and 12). The guide type of the else-branch can be justified by backward reasoning: at line 13, the coroutine returns (i.e., continues with the continuation protocol \( X \)); at line 12, because the guide type after the procedure call is \( X \), we obtain the guide type before the procedure call by instantiating \( R \) with \( X \); and at line 11, because the guide type after the procedure call is \( R[X] \), we again instantiate \( R \), but with \( R[X] \), to derive the guide type of the else-branch. Finally, for the coroutine \textit{Pcfg} shown in Fig. 6, we derive \( R(\{0,1\}) \land R[1] \) as the guidance protocol for channel \textit{latent}.

Control-flow divergence. In Fig. 5, the model program \textit{Model} and the guide program \textit{Guide}_1 have very similar control-flow. In general, our type system permits the guide’s control-flow structure to diverge from the model’s, as long as the two programs communicate with each other in a consistent way, i.e., the two programs follow the same guidance protocol for the channel over which they communicate. For example, the program below implements a part of a Bayesian linear-regression model with outliers [17], where the latent variable \textit{prob_outlier} describes how likely a data point does not conform to the linear relationship, and \textit{is_outlier} is a Boolean-valued latent variable that indicates if a data point is an outlier.

1  \textit{prob_outlier} \leftarrow \text{sample}_{\text{nd}}(\text{latent})(\text{Unif});
2  \textit{is_outlier} \leftarrow \text{sample}_{\text{nd}}(\text{latent})(\text{Ber}(\text{prob_outlier}));
3  \text{return}()

For MCMC algorithms, the guide program generates a new random sample from an old one; thus, for better inference performance, an MCMC guide usually behaves differently for different old samples. The following program implements a part of a guide that branches on \textit{is_outlier} from the old sample [17]. Intuitively, this guide proposes the negation (with a small amount of noise) of the old \textit{is_outlier}, which is bound to a program variable \textit{old_is_outlier}; i.e., if the old \textit{is_outlier} is true (resp., false), then the guide is likely to propose false (resp., true).

\[
\tau := 1 | 2 | R(\{0,1\}) | R_+ | R | N_0 | N | \tau_1 \to \tau_2 | \text{dist}(r)
\]
\[
e := x | \text{triv} | \text{true} | \text{false} | \text{if}(e_1; e_2) | \text{f} | \# \mid \text{op}_{\phi}(e_1; e_2)
\]
\[
\lambda(x.e) \mid \text{app}(e_1; e_2) \mid \text{let}(e_1; e_2)
\]
\[
\mid \text{Ber}(e) \mid \text{Unif} \mid \text{Beta}(e_1; e_2) \mid \text{Gamma}(e_1; e_2)
\]
\[
\mid \text{Normal}(e_1; e_2) \mid \text{Cat}(e_1, \ldots, e_n) \mid \text{Geo}(e) \mid \text{Pois}(e)
\]
\[
v := \text{triv} | \text{true} | \text{false} | \text{f} | \# \mid \text{clo}(V, \lambda(x.e))
\]
\[
\mid \text{Ber}(v) \mid \text{Unif} \mid \text{Beta}(\{0,1\}; v_2) \mid \text{Gamma}(\{0,1\}; v_2)
\]
\[
\mid \text{Normal}(\{0,1\}; v_2) \mid \text{Cat}(e_1, \ldots, e_n) \mid \text{Geo}(v) \mid \text{Pois}(v)
\]
\[
m := \text{ret}(e) \mid \text{bdn}(m_1; x.m_2) \mid \text{call}(f; e)
\]
\[
\mid \text{sample}_{\text{nd}}(a)(e) \mid \text{sample}_{\text{ad}}(a)(e)
\]
\[
\mid \text{cond}_{\text{vd}}(a)(m_1; m_2) \mid \text{cond}_{\text{ad}}(a)(e; m_1; m_2)
\]
\[
\mathcal{D} := \text{fix}(a; b)(f.x.m)
\]

\textbf{Figure 7. Syntax of the core calculus.}

1  \textit{prob_outlier} \leftarrow \text{sample}_{\text{nd}}(\text{latent})(\text{Beta}(2; 5));
2  \textbf{if} \ \textit{old_is_outlier} \ \textbf{then}
3  \textit{is_outlier} \leftarrow \text{sample}_{\text{nd}}(\text{latent})(\text{Ber}(0.1));
4  \textbf{return}()
5  \textbf{else}
6  \textit{is_outlier} \leftarrow \text{sample}_{\text{ad}}(\text{latent})(\text{Ber}(0.9));
7  \textbf{return}()

Although the model and the guide have divergent control-flow structures, in our type system, we can express the guidance protocol for channel \textit{latent} as \( R(\{0,1\}) \land 2 \land 1 \); that is, both programs sample an \( R(\{0,1\}) \)-valued random variable and then sample a Boolean-valued one.

\textbf{Type inference.} Guide types can be automatically inferred from code; in practice, they can still be used as specifications of the programs for better understanding. Our implementation can infer guide types for the examples mentioned so far, including the recursive one shown in Fig. 6.

3 A Coroutine-Based PPL

In this section, we formulate a core monadic calculus for coroutine-based probabilistic programming.

\textbf{Syntax.} Fig. 7 presents the grammar of basic types \( \tau \), expressions \( e \), values \( v \), commands \( m \), and programs \( \mathcal{D} \) in the core calculus via abstract binding trees [29]. There is a modal distinction in the core language: expressions describe purely deterministic computations, while commands describe probabilistic computations. Intuitively, we treat randomness as a kind of monadic effect [45].

The purely deterministic fragment is a simply-typed lambda calculus augmented with \textit{scalar} types (i.e., nullary products \( \mathbb{1} \), Booleans \( \mathbb{2} \), unit interval \( \mathbb{R}_{(0,1)} \), positive real numbers \( \mathbb{R}_+ \), real numbers \( \mathbb{R} \), integer rings \( \mathbb{Z}_n \), and natural numbers \( \mathbb{N} \)), as well as a \textit{distribution} type \text{dist}(\tau). The syntactic form \text{op}_\phi(e_1; e_2) represents expressions that perform built-in binary operations \( \diamond \) on scalar values. Inhabitants of
dist(r) are the primitive distributions from which probabilistic programs can draw a random value of type r; for example, Bernoulli distributions Ber(·) have type dist((0,1)), the uniform distribution on unit interval Unif has type dist([0,1]), and geometric distributions Geo(·) have type dist(N). For each primitive distribution d, we assume that it admits two fields: d.support and d.density are the support and the density function of the distribution, respectively. In the core calculus, the type of a primitive distribution characterizes the support of the distribution precisely: for a distribution d of type dist(r) and a value v, it holds that v ∈ d.support if and only if v is an inhabitant of type r. Primitive distributions can be generalized to density-carrying expressions [5, 6] to further improve language expressibility.

The probabilistic fragment is a monadic calculus augmented with probabilistic constructs and communication primitives for coroutine-based programming. The sampling commands sample_a(a)(e) and sample_v(a)(e) first evaluate the expression e to a primitive distribution d. Then the send version sample_a(a)(d) draws a value from d and sends it on channel a, whereas the receive version sample_v(a)(d) receives a value from channel a and treats it as a sample from d. The random samples can influence the likelihoods of computations; thus, randomness can be seen as a source of side effects. The branching commands also have a send version cond_a(a)(e; m_1; m_2), which evaluates e to a Boolean value and sends it as the branch selection on channel a; and a receive version cond_v(a)(m_1; m_2), which receives a branch selection from channel a. The syntactic form call(f; e) represents a procedure call, where f is a procedure name and e is the argument.

A probabilistic program D is a collection of (mutually recursive) procedures, each of which has the form fix(a; b)(f . x. m), where f is the procedure name, x is the parameter, m is a command that represents the procedure body, a is the name of the channel consumed by f, and b is the name of the channel provided by f. Note that both a and b are optional; that is, the procedure f might not consume any channel, and it might not provide any channel.

**Semantics.** We develop a big-step operational semantics for the core calculus. Details of the semantics are included in appendix B. The evaluation judgments for expressions have the form V ⊢ e ▼ v, where V is an environment that maps program variables to values. The evaluation rules for expressions are skipped here because they are standard.

We adopt a trace-based approach [10, 38] in our semantics of probabilistic computations. A guidance trace e is a finite sequence of guidance messages exchanged on a channel; each guidance message has the form val_f(o) (resp. dir_f(o)) for a sample value o (resp., a branch selection o) from the provider to the consumer, the form val_f(o) (resp. dir_f(o)) for a sample value o (resp., a branch selection o) from the consumer to the provider, or a procedure-call indicator fold. The evaluation judgments for commands have the form V ⊢ c ▼ v, where V is an environment, c is a command that consumes channel a and provides channel b, σ_a and σ_b are guidance traces on the channels, v is the evaluation result, and w ≥ 0 is a weight that expresses how likely the guidance traces are. Intuitively, a probabilistic program specifies a probability distribution on guidance traces, and the weights represent probability densities with respect to the distribution.

**Fig. 8.** Selected evaluation rules for commands.

1The fold message is only useful in the theoretical development; it can be seen as the introduction form for guidance traces whose type is a type-operator instantiation (see §4).
the guidance traces. The (EM::Cond::Send::L) rule evaluates the branch predicate to obtain a Boolean value, and ensure that the branch selection from the guidance trace of the consumed channel must be the same as the predicate’s value; if the guidance trace sets the branch selection to a different value, we simply set the weight of this trace to zero. The (EM::Call) rule requires the guidance traces start with a fold message, and proceeds by evaluating the body of the callee.

Example 3.1. Consider the command
\[
m_1 \overset{\text{def}}{=} \text{bdn( sample}_a\{a\}(\text{NORMAL}(0;1)); \ x. \\
\text{bdn( sample}_a\{b\}(\text{NORMAL}(x;1)); \ y. \\
\text{ret(op}_a(x; y)) )
\]
which consumes a channel \(a\) and provides a channel \(b\). Let \(\varphi \overset{\text{def}}{=} \lambda x. \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}\) be the probability density function of the standard normal distribution \(\text{NORMAL}(0;1)\). Let \(\sigma_a \overset{\text{def}}{=} [\text{val}^P(1)]\) and \(\sigma_b \overset{\text{def}}{=} [\text{val}^P(2)]\). Then we can derive the evaluation judgment
\[
\emptyset \vdash (a: \sigma_a); (b: \sigma_b) + m_1 \parallel \varphi(1) \varphi(1) \hat{3},
\]
for the command \(m_1\) and the guidance traces \(\sigma_a, \sigma_b\).

Communication. There are a lot of formalisms for communication in (concurrent) programming systems, such as CCS [42], Theoretical CSP [30], and \(\pi\)-calculus [43, 44]. In this paper, we use a lightweight approach to handling communication; that is, in the semantics, we assume we have all the messages exchanged on all the communication channels. We use this formalism because (i) our focus is to reason about soundness of Bayesian inference, rather than concurrency-related properties (e.g., deadlock freedom); and (ii) the inference algorithms we study in \S3 involve only two coroutines—one for the model and the other for the guide—so the communication in our system is much simpler than that in general concurrent systems.

Example 3.2. Consider the command
\[
m_2 \overset{\text{def}}{=} \text{bdn( sample}_a\{a\}(\text{NORMAL}(3;1)); \ a. \\
\text{bdn( sample}_a\{b\}(\text{NORMAL}(x;1)); \ b. \\
\text{ret(triv) )}
\]
which provides a channel \(a\) that is consumed by the command \(m_1\) in Ex. 3.1. To model the communication between \(m_2\) and \(m_1\), we simply use the guidance trace \(\sigma_a = [\text{val}^P(1)]\) as the sequence of messages exchanged on channel \(a\) in the semantics, and derive evaluation judgments for \(m_2\) and \(m_1\) separately. We showed the judgment for \(m_1\) in Ex. 3.1; here, we can derive the judgment
\[
\emptyset \vdash \emptyset; (a: \sigma_a) + m^* \parallel \varphi(2) \text{triv},
\]
for command \(m_2\) and guidance trace \(\sigma_a\). We use the \(\emptyset\) symbol to indicate that \(m_2\) does not consume any channel.

4 Guide Types

Type formation. We take inspiration from a structuring principle in session types [31, 32], and develop guide types to enforce protocols for guidance traces. The grammar shown below formulates the syntax of guide types. We write \(A, B\) for guide types, \(X\) for type variables, \(T\) for unary type operators, and \(F\) for procedure signatures.

\[
A, B ::= X \mid 1 \mid T[A] \mid \tau \land A \mid \tau \lor A \mid A \oplus B \mid A \& B
\]

\[
F ::= t_1 \rightarrow t_2 \mid (a: T_1); (b: T_2)
\]

\[
T ::= \text{typedef}(T,X.A)
\]
The type \(1\) indicates an ended channel, where the guidance trace is empty. The type \(T[A]\) instantiates a unary type operator \(T\) with a guide type \(A\). For sample passing and branch selection, each type constructor has a dual version that reverses the role of the provider and the consumer. The type \(\tau \land A\) types a channel whose provider samples a random value, sends it on the channel, and then continues with a type \(A\) guidance protocol; dually, the type \(\tau \lor A\) types a channel whose consumer samples and sends a random value. Similarly, the type \(A \oplus B\) types a channel whose provider evaluates a branch predicate, sends a branch selection on the channel, and then continues with a type \(A\) guidance protocol or a type \(B\) protocol based on the branch selection; dually, the type \(A \& B\) types a channel whose consumer evaluates and sends a branch selection.

Remark 4.1. In the rest of this paper, we will not use the dual types \(\tau \lor A\) and \(A \oplus B\). We introduce these types here for theoretical completeness, and they may be used in some future development.

Type operators prescribe guidance protocols for procedures by parameterizing with a continuation type that describes the guidance protocol after a procedure call. A procedure signature \(t_1 \rightarrow t_2 \mid (a: T_a); (b: T_b)\) types a procedure that takes a parameter of type \(t_1\), returns a result of type \(t_2\), consumes a channel \(a\), and provides a channel \(b\), such that if the guidance protocols for \(a\) and \(b\) after a procedure call are \(A\) and \(B\), respectively, then the guidance protocols for \(a\) and \(b\) before the procedure call are \(T_a[A]\) and \(T_b[B]\), respectively.

A type definition typedef\((T,X.A)\) declares a unary type operator \(T\) that takes a type parameter \(X\) and produces a guide type \(A\), which can reference \(X\). Because type operators are used to prescribe procedure signatures, we assume that a probabilistic program is always accompanied by a collection \(T\) of (mutually recursive) type definitions.

Example 4.2. We can formally declare the type operator Recur for the PcfgGen procedure shown in Fig. 6 as typedef\((R, X, R[0,1]) \land ((R \land X) \& R[R[X]])\).

Typing rules. The typing judgments for expressions have the form \(\Gamma \vdash e : \tau\), where \(\Gamma\) is a typing context that maps program variables to basic types (defined in Fig. 7). A full list of typing rules is included in appendix B. The typing rules for expressions are skipped here because they are standard.

The typing judgments for commands have the form
\[
\Gamma \vdash (a: A); (b: B) \tau m \vdash \tau \mid (a: A'); (b: B'),
\]
where 𝜎 maps procedure identifiers to procedure signatures. The intuitive meaning of the typing judgment is that if the channels a and b are of the guidance protocols A and B, respectively, then we can evaluate the command m to a value of type τ, and after the evaluation, the channels a and b are of the guidance protocols A’ and B’, respectively.

Fig. 9 presents the typing rules for commands. We assume a fixed global 𝜎 that we omit from the rules. Intuitively, the rules formulate a backward-reasoning system: we start with continuation types A’ and B’ for the channels a and b, respectively, and then preprend the guidance messages sent or received by the command m to A’ and B’, to obtain the guide protocols A and B for the channels a and b before the evaluation of m, respectively. For sample passing and branch selection, each guide type has two derivation rules: one for the consumed channel, and the other for the provided channel b. For example, the type τ ∧ A represents a channel whose provider sends a sample of type τ; thus, if the consumed channel a has such a type, the rule (TM:Sample:Recv:L) receives a sample from the provider of a, and if the provided channel b has such a type, the rule (TM:Sample:Send:R) sends a sample to the consumer of b.

The rule (TM:Call) handles procedure calls. For a procedure call call(f; e), the rule fetches from the procedure f’s signature τ₁ ⊁ τ₂ | (a: Tₛ; b: T_b), and then instantiates the type operators Tₛ, T_b with continuation types A, B, respectively, to obtain the guide types Tₐ[A] and Tₐ[B] for the channels a and b before the procedure call, respectively.

**Example 4.3.** Consider the command

\[ m_3 \overset{def}{=} \text{bind}(\text{call}(f; k); \_); \text{bind}(\text{sample}_v\{\text{Normal}(0; 1)\}(a); \_); \text{bind}(\text{call}(f; k); \_); \text{ret(triv) }); \]

where the variable k has type Rₐ(0; 1) and the procedure f has signature Rₐ(0; 1) ⊁ 1 | (a: T); ⊙, i.e., the procedure f consumes channel a but does not provide any channel, and channel a is associated with a type operator T. Now we show that we can derive a typing judgment for m₃ by backward reasoning. First, by (TM:Ret), we have

\[ k : Rₐ(0; 1) \mid (a: 1); \odot \triangledown m₃ \text{ ret(triv) } \overset{\_}{\overset{\_}{\_}} 1 \mid (a: 1); \odot. \]

Then by (TM:Call), we derive

\[ k : Rₐ(0; 1) \mid (a: T[1]); \odot \triangledown m₃ \text{ call(f; k) } \overset{\_}{\overset{\_}{\_}} 1 \mid (a: 1); \odot. \]

Define \( m₄ \overset{def}{=} \text{bind}(\text{call}(f; k); \_); \text{ret(triv)} \). Thus, by (TM:Bnd),

\[ k : Rₐ(0; 1) \mid (a: T[1]); \odot \triangledown m₄ \overset{\_}{\overset{\_}{\_}} 1 \mid (a: 1); \odot. \]

Define \( m₅ \overset{def}{=} \text{bind}(\text{sample}_v\{\text{Normal}(0; 1)\}; \_); m₄ \). By (TM:Sample:Recv:L) and (TM:Bnd), we have

\[ k : Rₐ(0; 1) \mid (a: R \wedge T[1]); \odot \triangledown m₅ \overset{\_}{\overset{\_}{\_}} 1 \mid (a: 1); \odot. \]

Finally, we again apply (TM:Call) and (TM:Bnd) to derive

\[ k : Rₐ(0; 1) \mid (a: T[R \wedge T[1]]); \odot \triangledown m₅ \overset{\_}{\overset{\_}{\_}} 1 \mid (a: 1); \odot. \]

**Type safety.** We present some theoretical results about type safety of guide types. Proofs are included in appendix B.

We first formulate two judgments for well-formedness of values and guidance traces. The judgment \( \nu : \tau \) means that value \( \nu \) has type \( \tau \). The judgment \( \sigma : A \) means that the guidance trace is a sequence of messages that satisfies protocol A. Rules for these judgments are straightforward; we omit them here but include them in appendix B.

The theorem below states that if \( m \) is a well-typed closed command, and it evaluates to a value \( \nu \) under guidance traces \( \sigma_a, \sigma_b \), then \( \nu \) is a well-typed value, and \( \sigma_a, \sigma_b \) are well-typed guidance traces.

**Theorem 4.4.** If \( \| (a: A); (b: B) \triangledown m \overset{\_}{\overset{\_}{\_}} \tau \mid (a: 1); (b: 1) \) and \( \emptyset \mid (a: \sigma_a); (b: \sigma_b) \overset{\_}{\overset{\_}{\_}} m \|\| \nu, \text{ then } \sigma_a : A, \sigma_b : B, \text{ and } \nu : \tau. \)

Furthermore, we can show some normalization properties of guide types. The theorem below states that if \( m \) is a well-typed closed command, and \( \sigma_a, \sigma_b \) are well-typed guidance traces, then \( m \) can evaluate to some well-typed \( \nu \) under \( \sigma_a, \sigma_b \).

**Theorem 4.5.** If \( \| (a: A); (b: B) \triangledown m \overset{\_}{\overset{\_}{\_}} \tau \mid (a: 1); (b: 1), \sigma_a : A, \text{ and } \sigma_b : B, \text{ then there exist } w, \nu \text{ such that } \emptyset \mid (a: \sigma_a); (b: \sigma_b) \overset{\_}{\overset{\_}{\_}} m \|\| \nu \text{ and } \nu : \tau. \)
We can strengthen the normalization property when a command will not send out any branch selections. The theorem below states that if a well-typed command \( m \) consumes a channel \( a \) with a type \( A \) that does not contain \( \&, \) and provides a channel \( b \) with a type \( B \) that does not contain \( \oplus \), and \( \sigma_a, \sigma_b \) are well-typed guidance traces, then \( m \) can evaluate to some well-typed value \( v \) under \( \sigma_a, \sigma_b \) with a strictly positive weight \( w \).

**Theorem 4.6.** If \( \vdash (a: A); (b: B) \mapsto_\Sigma m \vdash \tau \mid (a: 1); (b: 1) \), \( A \) is \( \& \)-free, \( B \) is \( \oplus \)-free, \( \sigma_a : A \), and \( \sigma_b : B \), then there exist \( w, v \) such that \( \emptyset \vdash (a: \sigma_a); (b: \sigma_b) \vdash m \Downarrow^w v, v : \tau \), and \( w > 0 \).

**Type-inference algorithm.** We now sketch a type-inference algorithm that derives guide types automatically from the implementation. In the algorithm, we assume we have information about basic types—such as the parameter and result types for procedures and the typing contexts that map program variables to basic types—because without guide types, our core language is a simply-typed lambda calculus, for which type inference is decidable.

First, for each procedure \( \text{fix}\{a;b\}(f.x.m) \) in the program, we create two fresh type operators \( T_a \) and \( T_b \) for the channels \( a \) and \( b \), respectively, and obtains \( \tau_1 \leadsto \tau_2 \mid (a: T_a); (b: T_b) \) as the signature of this procedure. Then we collect signatures of all the procedures in the program to obtain the map \( \Sigma \).

Now the task is to derive definitions of the type operators. We observe that the rules in Fig. 9 are syntax directed, and they can be turned into an algorithmic system by interpreting

\[
\Gamma \vdash (a: A); (b: B) \mapsto_\Sigma m \vdash \tau \mid (a: A'); (b: B')
\]

as a function from \( \Sigma, \Gamma, m, \tau, a, b, A', B' \) to \( A, B \); i.e., we assume we know all the basic types, and we perform backward reasoning to infer guide types. Therefore, for each procedure \( \text{fix}\{a;b\}(f.x.m) \) with signature \( \tau_1 \leadsto \tau_2 \mid (a: T_a); (b: T_b) \), we create two fresh type variables \( X_a \) and \( X_b \), derive two guide types \( A \) and \( B \) through

\[
x : \tau_1 \mid (a: A); (b: B) \mapsto_\Sigma m \vdash \tau_2 \mid (a: X_a); (b: X_b),
\]

and then add type definitions `typedef(T_a,X_a,A)` and `typedef(T_b,X_b,B).

5 Soundness of Bayesian Inference

In this section, we use guide types to reason about Bayesian inference. We first present a measure-theoretic formulation of Bayesian inference in the coroutine-based PPL, and prove that guide types are certificates of absolute continuity (§5.1). We then sketch how guide types ensure key soundness conditions for multiple Bayesian-inference algorithms (§5.2). Appendix C includes the details (e.g., formalizations and proofs) of this section.

5.1 Verification of Absolute Continuity

We use the following notions from measure theory: \( \sigma \)-algebras, measurable spaces, measurable functions, measures, and Lebesgue integration. Appendix A provides a review of these notions.

**Semantic domains.** For each scalar type \( \tau \), we equip it with a standard Borel space \( [\tau] \) on the inhabitants of \( \tau \), i.e., \( [\tau] \) is a measurable space isomorphic to a countable set or the real line. We then equip each type \( \tau \) with a stock measure \( \lambda^{[\tau]} \): if \( [\tau] \) is a countable set, we define \( \lambda^{[\tau]} \) to be the counting measure; otherwise, \( [\tau] \) is a subset of the real line, so we define \( \lambda^{[\tau]} \) to be the Lebesgue measure.

Because guidance traces are finite sequences of messages that contain values of scalar types, we can define \( [A] \) as a standard Borel space on guidance traces that satisfy protocol \( A \). We then construct the stock measure \( \lambda_{[A]} \) for \( A \) by decomposing \( A \) to products and/or sums of scalar types, and then combining the stock measures for scalar types via product and/or coproduct measures.

**Denotation of commands.** For a well-typed closed command \( m \), i.e., \( \vdash (a: A); (b: B) \mapsto_\Sigma m \vdash \tau \mid (a: 1); (b: 1) \), we define the density function of \( m \) as

\[
P_m(\sigma_a, \sigma_b) \triangleq \begin{cases} 
 w & \text{if } \emptyset \vdash (a: \sigma_a); (b: \sigma_b) \vdash m \Downarrow^w v, v : \tau \text{, and } w > 0. \\
 0 & \text{otherwise}
\end{cases}
\]

We can prove that \( P_m \) is a measurable function from \( [A] \otimes [B] \) is the product measurable space of \( [A] \) and \( [B] \) to non-negative real numbers. Thus, we construct a measure denotation \( [m] \) for \( m \), by integrating \( P_m \) with respect to the stock measure on the product space \( [A] \otimes [B] \), i.e.,

\[
[m](S_{a,b}) \triangleq \int_{S_{a,b}} P_m(\sigma_a, \sigma_b) \lambda_{[A] \otimes [B]}(d(\sigma_a, \sigma_b)),
\]

where \( S_{a,b} \) is a measurable set in \( [A] \otimes [B] \).

**Bayesian inference.** Let us fix a well-typed model program \( m_m \) that consumes latent random variables on a channel \( latent \) and provides observations on a channel \( obs \), i.e., \( \vdash (latent: A); (obs: B) \mapsto_\Sigma m_m \vdash \tau \mid (latent: 1); (obs: 1) \).

Usually, the program \( m_m \) does not receive any branch selections, i.e., \( A \) is \( \& \)-free and \( B \) is \( \oplus \)-free. Given a concrete observation \( \sigma_O : B \) such that \( \int P_m(\sigma, \sigma_O) \lambda_{[A]}(d\sigma) > 0 \), Bayesian inference is the problem of approximating the posterior \( [m_m]_{\sigma_O} \), a measure conditioned with respect to \( \sigma_O \), defined by

\[
[m_m]_{\sigma_O}(S_{\ell}) \triangleq \frac{\int_{S_{\ell}} P_m(\sigma, \sigma_O) \lambda_{[A]}(d\sigma)}{\int P_m(\sigma, \sigma_O) \lambda_{[A]}(d\sigma)},
\]

where \( S_{\ell} \) is a measurable set in \( [A] \), i.e., a set of guidance traces of type \( A \). Note that if we fix the observation \( \sigma_O \), then the denominator of eq. (5) is a constant independent of \( S_{\ell} \). Thus, it is sufficient for an inference algorithm to ignore the denominator and approximate the measure \( S_{\ell} \mapsto \int_{S_{\ell}} P_m(\sigma, \sigma_O) \lambda_{[A]}(d\sigma) \).

**Guide programs.** Bayesian-inference algorithms usually require some guide programs, such as proposals for importance
sampling and approximating families for variational inference. These guide programs specify measures on latent random variables; in our system, we implement a guide program \( m_g \) as a coroutine that works with the model program \( m_m \), and provides the latent channel with guide type \( A \) that \( m_m \) consumes, i.e.,

\[
\cdot | \; \emptyset; (\text{latent}: A) \vdash \Sigma \; m_g \downarrow \tau_g | \; \emptyset; (\text{latent}: 1),
\]

\[
\cdot | (\text{latent}: A); (\text{obs}: B) \vdash \Sigma \; m_m \downarrow \tau_m | \; (\text{latent}: 1); (\text{obs}: 1).
\]

The guide and model have the same guide type \( A \) on channel \( \text{latent} \). Because the guide provides the channel and the model consumes the channel, the two programs interpret the guide type \( A \) dually; thus, their communication is compatible.

The coroutine-based paradigm folds the model and guide programs into a single entity; thus, during the inference, both the model and guide coroutines execute. To model possible combinations of traces for a model-guide system, we introduce a reduction relation \( V \vdash (a: \sigma_a); (b: \sigma_b) \vdash m \downarrow \nu \), where \( V \) is an environment, \( m \) is a command, \( \sigma_a \) and \( \sigma_b \) are guidance traces on channel \( a \) and channel \( b \), respectively, and \( \nu \) is the reduction result. The reduction relation is essentially the same as the evaluation relation for the operational semantics, except that reduction does not account for probabilities.

Below are two example rules.

\[
\frac{}
\frac{V \vdash e \downarrow d \quad \nu \in d\text{-support}}{V \vdash (e: \nu) \vdash m \downarrow \nu}
\]

\[
\frac{V \vdash \nu_e}{V \vdash (\text{cond}: \nu_e) \vdash m \downarrow \nu}
\]

We prove that for a model-guide pair, guide types serve as certificates for absolute continuity.

**Theorem 5.2.** Suppose that \( A \) is \( \oplus \)-free, \( B \) is \( \otimes \)-free,

\[
\cdot | \; \emptyset; (\text{latent}: A) \vdash \Sigma \; m_g \downarrow \tau_g | \; \emptyset; (\text{latent}: 1),
\]

\[
\cdot | (\text{latent}: A); (\text{obs}: B) \vdash \Sigma \; m_m \downarrow \tau_m | \; (\text{latent}: 1); (\text{obs}: 1),
\]

and \( \sigma : B \) such that \( \int P_{m_m}(\sigma_t, \sigma); \lambda[A](d\sigma_t) > 0 \). Then the measure \( [m_m]_{\sigma} \) is absolutely continuous with respect to the measure \( [m_g]_{\sigma} \), and vice versa.

### 5.2 Soundness of Inference Algorithms

We now describe how guide types can help us reason about inference algorithms.

**Importance sampling (IS).** IS approximates the posterior distribution by drawing latent variables using the guide program, and then reweights the samples by their importance.

The operational rule below formulates a single step in the algorithm: given a model program \( m_m \), a guide program \( m_g \), and a concrete observation \( \sigma_o \), IS performs joint execution of the two programs to draw a sample \( \sigma_t \) with density \( w_g \) and compute \( \frac{[m_g]_{\sigma_t}}{[m_m]_{\sigma_o}} \) as the importance of \( \sigma_t \).

\[
\frac{0 | \; \emptyset; (\text{latent}: \sigma_o) \vdash m_m \downarrow w_o}{0 | (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \vdash m_m \downarrow \frac{w_t}{w_o}}
\]

By Thm. 5.2, if the model and guide programs are well-typed, then the posterior \( [m_m]_{\sigma_o} \) is absolutely continuous with respect to \( [m_g]_{\sigma_t} \); thus, IS is able to sample any possible latent variables \( \sigma_t \) in the posterior. With the importance ratios, IS can be seen as generating \( \sigma_t \) with density \( w_g \cdot \frac{[m_g]_{\sigma_t}}{[m_m]_{\sigma_o}} = w_m \). Thus, IS generates a measure proportional to \( [m_m]_{\sigma_o} \).

**Markov-Chain Monte Carlo (MCMC).** MCMC uses a transition kernel to generate iteratively a new random sample from an old one. A popular MCMC algorithm is Metropolis-Hastings (MH), which constructs the transition kernel from a proposal subroutine. To implement proposal subroutines in our system, we extend the core calculus such that guidance traces can be used as first-class data. Then we implement the proposal subroutine as a procedure \( g \) whose argument is a guidance trace on the channel for latent random variables. The operational rule below formulates a single step in the MH algorithm; given a proposal procedure \( g \), a model \( m_m \), an observation \( \sigma_o \), and the current latent trace \( \sigma_t \), MH first performs joint execution of call \( (g; \sigma_t) \) and \( m_m \) to generate a new latent trace \( \sigma'_t \) with density \( w_{\text{bwd}} \cdot \frac{[m_g]_{\sigma'_t}}{[m_m]_{\sigma_o}} \), and then uses the new \( \sigma'_t \) and the old \( \sigma_t \) to calculate a backward density \( w_{\text{bwd}} \cdot \frac{[m_m]_{\sigma'_t}}{[m_m]_{\sigma_o}} \).
and accepts the new sample $\sigma'_j$ with probability $\alpha$.

\[
\begin{align*}
\emptyset \mid \sigma; \text{latent: } \sigma'_j \Rightarrow & \text{call}(g; \sigma)_j \parallel_{\text{w}} \emptyset \\
\emptyset \mid \text{latent: } \sigma'_j; (\text{obs: } \sigma_0)_j \Rightarrow & m_j \parallel_{\text{v}} \emptyset \\
\emptyset \mid \text{latent: } \sigma'_j; \text{call}(g; \sigma)_j \parallel_{\text{w}} \emptyset \\
\emptyset \mid \text{latent: } \sigma'_j; (\text{obs: } \sigma_0)_j \Rightarrow m_j \parallel_{\text{v}} \emptyset \\
\end{align*}
\]

Similar to IS, MH requires that the command call($g; \sigma_j$) be able to sample any possible latent variables $\sigma'_j$ in the posterior. We prove the soundness of MH by a variant of Thm. 5.2, where the programs do not need to be closed so that they can reference data in the environment (e.g., the old samples).

**Variational inference (VI).** VI uses optimization to find a candidate from an approximating family of guide programs that minimizes the distance from the posterior distribution to the guide distribution. We focus on verifying if the distance is well-defined, whereas VI requires extra conditions for the optimization problem to be well-formed. Here, we parameterize the guide $m_{\theta,0}$ by a vector $\theta \in \Theta$ of parameters, and use KL divergence as the distance, which is defined by

\[
\text{KL}(\mu \parallel v) \overset{\text{def}}{=} \int p_{\mu}(\sigma) (\log p_{\mu}(\sigma) - \log p_v(\sigma)) d\lambda[A] (d\sigma),
\]

where $\mu$ and $v$ are measures on $[A]$ with densities $p_{\mu}$ and $p_v$, respectively, and $\mu$ is absolutely continuous with respect to $v$.

The rule below formulates the computation of KL divergence for a specific $\theta$, via joint execution of the two programs.

\[
\begin{align*}
\emptyset \mid \sigma; \text{latent: } \sigma_j \Rightarrow & m_{\theta,0} \parallel_{w} \emptyset \\
\emptyset \mid \text{latent: } \sigma_j; (\text{obs: } \sigma_0)_j \Rightarrow m_j \parallel_{v} \emptyset \\
\emptyset \text{latent: } \sigma_j \parallel_{w} m_{\theta,0} \parallel_{v} (\sigma_0, \log w_m - \log w_k)
\end{align*}
\]

The rule can be seen as defining a map $\sigma_j \mapsto w_k \cdot (\log w_m - \log w_k)$, which is the integrand of the KL divergence $\text{KL}(m_{\theta,0} \parallel m_{\theta,0})$. By Thm. 5.2, if the model and guide programs are well-typed, then $m_{\theta,0} \parallel m_{\theta,0}$ is absolutely continuous with respect to $m_{\theta,0} \parallel m_{\theta,0}$; thus, the KL divergence used in VI is well-defined.

### 6 Experimental Evaluation

**Implementation.** We implemented the coroutine-based PPL in OCaml. Our implementation consists of about 2,000 LOC; it contains a parser, a type checker with automatic inference of guide types, and a prototype compiler from our PPL to Pyro [8]. Our implementation extends the core calculus with tensors (i.e., multi-dimensional matrices) and primitive iteration operators for them. The prototype compiler supports code generation for importance sampling and variational inference. We use the Python package greenlet [60] to support coroutines in the compiled code.

**Evaluation setup.** We evaluated our implementation to answer the following two research questions:

1. How expressive is the coroutine-based PPL, compared to a state-of-the-art probabilistic programming language that ensures soundness of programmable inference [40]?

2. How efficient is our implementation, in terms of the time for type inference, and the performance of Bayesian inference on the compiled code?

For the first question, we obtained 23 benchmarks from prior work [40] and collected 6 new benchmarks. The 29 benchmark programs consist of (i) example models from Anglican [62], Turing [20], and Pyro [8], as well as (ii) PCFG models, including a Gaussian-process domain-specific language (DSL) [50] and synthetic models (such as examples shown in this paper). Compared to prior work [40], a larger subset of benchmark models are expressible and type-checked in our PPL. Particularly, our PPL is capable of expressing models with recursion and general conditional branches, whereas prior work [40] is not.

For the second question, we ran Bayesian inference on the compiled code, and compared the performance with non-coroutine-based, but equivalent, Pyro code. We obtained guide programs from where we obtained the benchmark models, and then reimplemented them in our PPL; for example, we implemented the encoder component of a variational autoencoder as the guide program [8]. For those benchmark models without guides, we first invoked our PPL to type-check the model program and infer a guide type for the model, and then implemented a guide program whose type was the guide type. The compiled model and guide use Pyro’s primitives (such as pyro.sample) to sample random data and condition on given data, as well as exchange messages and switch control with each other using the concurrent-programming package greenlet. We leveraged Pyro’s inference engines to carry out importance sampling or variational inference. Type inference is very fast in practice; our implementation completed the type-inference phase in several milliseconds on all of the benchmarks. Our experiments showed that coroutines (implemented via messaging passing) do not introduce significant overhead in actual Bayesian inference.

The experiments were performed on a machine with an Intel Core i7 3.6GHz processor and 16GB of RAM under macOS Catalina 10.15.7.

**Results.** Tab. 1 gives an overview of selected benchmark models. Our benchmarks cover a wide range of Bayesian models, such as linear regression, Gaussian mixtures, hidden Markov models, Bayesian networks, and variational autoencoders. Our benchmarks also include the classic Marsaglia algorithm (which generates a normal distribution from a uniform distribution), a Poisson-trace algorithm (shown in Fig. 10, which generates a Poisson distribution from a uniform distribution), and a Gaussian-process DSL (which uses a PCFG to generate the kernel function of a Gaussian process).

As shown in Tab. 1, our coroutine-based PPL is capable of expressing most of the benchmarks, except those involving stochastic memoization [25], such as the program dp. The programs branching, marsaglia, ptrace, and ex-1 have non-trivial branching, and the programs marsaglia, ptrace, ex-2,
Figure 10. An algorithm to generate Poisson-distributed numbers given by Knuth [37].

Table 1. Selected benchmark descriptions. T? = is type-checked in our PPL; LOC = #lines of code in the model in our PPL; TP? = is type-checked by prior work [40].

| Program | Description                  | T? | LOC | TP? |
|---------|------------------------------|----|-----|-----|
| tr      | Bayesian Linear Regression   | ✓  | 16  | ✓   |
| gmm     | Gaussian Mixture Model       | ✓  | 44  | ✓   |
| kalman  | Kalman Smoother              | ✓  | 32  | ✓   |
| sprinkler | Bayesian Network           | ✓  | 22  | ✓   |
| hmm     | Hidden Markov Model          | ✓  | 31  | ✓   |
| branching | Random Control Flow        | ✓  | 19  | ✗   |
| marsaglia | Marsaglia Algorithm      | ✓  | 22  | ✗   |
| dp      | Dirichlet Process            | ✗  | N/A | ✗   |
| ptrace  | Poisson Trace                | ✓  | 11  | ✗   |
| aircraft | Aircraft Detection           | ✓  | 32  | ✓   |
| weight  | Unreliable Weigh            | ✓  | 8   | ✓   |
| vae     | Variational Autoencoder     | ✓  | 26  | ✓   |
| ex-1    | Fig. 5                      | ✓  | 13  | ✓   |
| ex-2    | Fig. 6                      | ✓  | 21  | ✗   |
| gp-dsl  | Gaussian Process DSL        | ✓  | 58  | ✓   |

and gp-dsl define recursive models; our implementation successfully inferred guide types for these programs, whereas prior work [40] could not express them. Our implementation derived guide types for 25 of the 29 benchmarks, whereas prior work was able to express only 18 of them.

For all the benchmarks, we assume that each guide program samples random variables in the same order as its corresponding model program does. However, this assumption can sometimes be too restrictive: it has been shown that the ability to allow the model and the guide to sample random variables in different orders is desirable for inference amortization methods [59]. Prior work [40] allows different sampling orders in the model and the guide, whereas our system cannot handle such scenarios.

Tab. 2 presents performance statistics of selected benchmark programs. We evaluated our PPL’s performance under two criteria: (i) the time for type inference and code generation, and (ii) the time for Bayesian inference compared to handwritten inference code under the same set of hyperparameters (e.g., iteration rounds, optimization algorithms, and initial values of parameters). Our experiments showed that our implementation usually completes type inference and code generation in several milliseconds, and the compiled code, although using coroutines, has similar performance to handwritten inference code.

Table 2. Selected performance statistics. BI = Bayesian-inference algorithm (IS or VI); CG (ms) = time for type inference and code generation in milliseconds; GLOC = #lines of code in compiled code (model + guide); GI (s) = time for Bayesian inference on compiled code in seconds; HLOC = #lines of code in handwritten code (model + guide); HI (s) = time for Bayesian inference on handwritten code in seconds.

| Program | BI   | CG (ms) | GLOC | GI (s) | HLOC | HI (s) |
|---------|------|---------|------|-------|------|-------|
| ex-1    | IS   | 0.75    | 57   | 5.44  | 16   | 5.27  |
| branching | IS   | 1.74    | 58   | 8.49  | 16   | 7.48  |
| gmm     | IS   | 8.03    | 185  | 64.13 | 38   | 56.00 |
| weight  | VI   | 0.66    | 35   | 2.76  | 7    | 2.66  |
| vae     | VI   | 10.36   | 72   | 34.96 | 26   | 32.69 |

7 Related Work

**Sound Bayesian inference.** Most closely related to our work are techniques for reasoning about soundness of trace-based programmable inference. Lee et al. [39] developed a static analysis of stochastic variational inference with guide programs, which describe custom approximating families in Pyro. Their analysis supports nontrivial features of Pyro, such as tensor manipulation and plates, i.e., vectors of conditionally independent samples. Their approach aims at proving that the model and guide programs have the same support and satisfy differentiability-related conditions. Their static analysis does not handle the case when a conditional statement determines the set of random samples. Lew et al. [40] proposed trace types as precise signatures for sampling traces of probabilistic programs, and then used the type system to prove absolute continuity in multiple kinds of inference algorithms. Trace types can be seen as a type-and-effect system, where a trace type records the precise set of samples drawn by a single program. Trace types support higher-order functions, stochastic branches that can influence the set of random samples, as well as three forms of loops, including stochastic while-loops with an unbounded number of iterations, but not general recursion. Because the value of a conditional predicate cannot be determined in general at static-analysis time, trace types do not support general conditional statements that can influence the set of random samples. Both Lee et al. [39] and Lew et al. [40]’s approach allow the model and the guide to sample random variables in different orders. In this paper, we propose a new PPL that guarantees absolute continuity between a model-guide pair, and features general programming constructs, including recursion and branching. A key innovation of our work is the coroutine-based paradigm of writing inference code; this paradigm makes the relational reasoning of the
support-match property explicit, and in particular enables precise analysis of complex control flow. However, compared to prior work, our system only supports scenarios where the model and the guide sample random variables in the same order.

There has been a line of work on validating Monte-Carlo inference algorithms. Scibior et al. [53] developed a semantic framework to verify the soundness of Monte-Carlo inference algorithms with generic proposal distributions. Atkinson et al. [4] presented a type system for verifying hand-coded Monte-Carlo algorithms that explicitly manipulate densities, rather than use proposal distributions. For MCMC methods, Borgström et al. [10] and Hur et al. [35] developed provably correct MH algorithms. Castellan and Paquet [14] proposed an intensional semantics, which captures execution traces of programs, to validate an incremental MH algorithm. Several systems [4, 9, 34, 40] studied sound combiners for kernels used by MCMC. In contrast to the aforementioned work, our PPL is based on trace-based programmable inference. It would be interesting to develop programmable versions of those sound inference algorithms in our PPL.

Narayanan et al. [47] and Zinkov and Shan [63] validated the soundness of program transformations in Hakaru, which contains a programmable MH algorithm. The development of Hakaru is not centered around sample traces, and it uses symbolic disintegration [15, 54] to calculate the marginal densities for computing the acceptance ratio in an MH step. In this paper, we focus on a trace-based scheme for programmable inference. Establishing the relationship among different schemes of programmable inference is an interesting future research direction.

Session types. Honda et al. [31, 32] introduced session types to prescribe binary communication protocols for message-passing processes. Session types can be interpreted either classically [58], or intuitionistically [11, 12]. To enable non-binary communication, researchers proposed multiparty session types [16, 33, 51]. The tail-recursive structure of standard session types imposes communication protocols that can be described by a regular language. Recently, several systems have been developed to go beyond tail-recursive protocols, such as context-free [55], label-dependent [56], and nested [18] session types.

In our development of guide types, we took inspiration from the structuring principle of session types. Compared to session types, guide types have different semantics (i.e., sending and receiving random samples drawn from probability distributions), have simpler forms (i.e., no process spawning or higher-order channels), and enjoy an efficient type-inference algorithm, which can also analyze non-tail-recursive communication protocols. Developing a truly concurrent probabilistic programming system, and concurrent Bayesian inference algorithms with general session types, would be interesting future work.

8 Conclusion

We have presented a new probabilistic programming language that supports programmable Bayesian inference, and guarantees model-guide absolute continuity, thereby ensuring key soundness properties of multiple kinds of inference algorithms. Our language implements the model and guide programs as coroutines, and we develop guide types to prescribe the communication protocols between coroutines. We have proved that well-typed model and guide coroutines execute safely, and they are guaranteed to enjoy absolute continuity. We have also developed an efficient type-inference algorithm that reconstructs guide types directly from the code. Finally, we have implemented our language with a prototype compiler to Pyro, and evaluated our implementation on a suite of diverse probabilistic models.

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A Preliminaries on Measure Theory

Interested readers can refer to textbooks and notes in the literature [7, 61] for more details.

A measurable space is a pair \((S, \mathcal{S})\), where \(S\) is a nonempty set, and \(\mathcal{S}\) is a \(\sigma\)-algebra on \(S\), i.e., a family of subsets of \(S\) that contains \(\emptyset\) and is closed under complement and countable unions. The smallest \(\sigma\)-algebra that contains a family \(\mathcal{A}\) of subsets of \(S\) is said to be generated by \(\mathcal{A}\), denoted by \(\sigma(\mathcal{A})\). Every topological space \((S, \tau)\) admits a Borel \(\sigma\)-algebra, given by \(\sigma(\tau)\). This gives canonical \(\sigma\)-algebras on \(\mathbb{R},\mathbb{Q},\mathbb{N}\), etc. A measurable space \((S, \mathcal{S})\) is said to be a standard Borel space, if \(S\) is a Borel \(\sigma\)-algebra generated by a complete metric space on \(S\). A measurable space \((S, \mathcal{S})\) is a standard Borel space if and only if it is isomorphic to \(\mathbb{R}\) or a subset of \(\mathbb{N}\). A function \(f : S \to T\), where \((S, \mathcal{S})\) and \((T, \mathcal{T})\) are measurable spaces, is said to be \((S, \mathcal{T})\)-measurable, if \(f^{-1}(B) \in \mathcal{S}\) for each \(B \in \mathcal{T}\). If \(T = \mathbb{R}\), we tacitly assume that the Borel \(\sigma\)-algebra is defined on \(T\), and we simply call \(f\) measurable, or a random variable. Measurable functions form a vector space, and products, maxima, and limiting operations preserve measurability.

A measure \(\mu\) on a measurable space \((S, \mathcal{S})\) is a mapping from \(S\) to \([0, \infty]\) such that (i) \(\mu(\emptyset) = 0\), and (ii) for all pairwise-disjoint \(\{A_n\}_{n \in \mathbb{Z}}\)-in \(\mathcal{S}\), it holds that \(\mu\left(\bigcup_{n \in \mathbb{Z}} A_n\right) = \sum_{n \in \mathbb{Z}} \mu(A_n)\). The triple \((S, \mathcal{S}, \mu)\) is called a measure space. A measure \(\mu\) is called a probability measure, if \(\mu(S) = 1\). A measure \(\mu\) is called a sub-probability measure, if \(\mu(S) \leq 1\). A measure \(\mu\) is called \(\sigma\)-finite, if \(S\) is the countable union of measurable sets with finite measure. We denote the collection of probability measures on \((S, \mathcal{S})\) by \(\mathcal{D}(S, \mathcal{S})\). For each \(x \in S\), the Dirac measure \(\delta(x)\) is defined as \(\lambda_A[x \in A]\). For measures \(\mu\) and \(\nu\), we write \(\mu \preceq \nu\) for the measure \(\lambda_{A\mu}(A) + \nu(A)\). For measure \(\mu\) and scalar \(c \geq 0\), we write \(c \cdot \mu\) for the measure \(\lambda_{A\cdot c} \cdot c\). The integral of a measurable function \(f \in S\) with respect to a measure \(\mu\) on \((S, \mathcal{S})\) is defined following Lebesgue’s theory and is denoted by \(\mu(f; A)\). \(\int_A f \, d\mu\), or \(\int_A f(x) \, d\mu(x)\). If \(A = S\), we tacitly omit \(A\) from the notations. For each \(A \in \mathcal{S}\), it holds that \(\mu(f; A) = \mu(f|_A)\), where \(f|_A\) is the indicator function for \(A\).

Let \(f\) be a nonnegative measurable function on \((S, \mathcal{S})\). We can transform a measure \(\mu\) on \((S, \mathcal{S})\) through \(f\) by integration:

\[ f \mu \stackrel{\text{def}}{=} \lambda_{A\mu}(f; A) \text{ if } \nu \text{ denotes the measure } f \mu, \text{ say that } \nu \text{ has density } f \text{ relative to } \mu, \text{ and express this by } \frac{d\nu}{d\mu} = f. \]

In this case, we have for \(A \in \mathcal{S}\), \(\mu(A) = 0\) implies that \(\nu(A) = 0\), i.e., \(\nu\) is absolutely continuous with respect to \(\mu\).

A kernel from a measurable space \((S, \mathcal{S})\) to another \((T, \mathcal{T})\) is a mapping from \(S \times T\) to \([0, \infty]\) such that: (i) for each \(x \in S\), the function \(\lambda_{B\mu}(x, B)\) is a measure on \((T, \mathcal{T})\), and (ii) for each \(B \in \mathcal{T}\), the function \(\lambda_{x\mu}(x, B)\) is measurable. We write \(k : (S, \mathcal{S}) \otimes (T, \mathcal{T}) \to \lambda_{B\mu}(x, B) = \lambda_{x\mu}(x, B)\) is measurable. We write \(k : (S, \mathcal{S}) \otimes (T, \mathcal{T}) \to (T, \mathcal{T})\) to declare that \(k\) is a kernel from \((S, \mathcal{S})\) to \((T, \mathcal{T})\). Intuitively, kernels describe measure transformers from one measurable space to another. A kernel \(k\) is called a probability kernel, if \(\lambda_{x\mu}(x, T) = 1\) for all \(x \in S\). We denote the collection of probability kernels from \((S, \mathcal{S})\) to \((T, \mathcal{T})\) by \(\mathcal{K}(S, \mathcal{S}, T, \mathcal{T})\). If the two measurable spaces coincide, we simply write \(\mathcal{K}(S, \mathcal{S})\). We can “push-forward” a measure \(\mu\) on \((S, \mathcal{S})\) to a measure on \((T, \mathcal{T})\) through a kernel \(k : (S, \mathcal{S}) \otimes (T, \mathcal{T})\) by integration:

\[ \mu \gg k \stackrel{\text{def}}{=} \lambda_{B\mu}(x, B) \mu(dx). \]

The product of two measurable spaces \((S, \mathcal{S})\) and \((T, \mathcal{T})\) is defined as \((S, \mathcal{S}) \otimes (T, \mathcal{T}) \stackrel{\text{def}}{=} (S \times T, \mathcal{S} \otimes \mathcal{T})\), where \(S \otimes \mathcal{T}\) is the smallest \(\sigma\)-algebra that makes coordinate maps measurable, i.e., \(\sigmaapest{(\pi_1^{-1}(A) \mid A \in \mathcal{S}) \cup (\pi_2^{-1}(B) \mid B \in \mathcal{T})}\), where \(\pi_i\) is the \(i\)-the coordinate map. If \(\mu_1\) and \(\mu_2\) are two measures on \((S, \mathcal{S})\) and \((T, \mathcal{T})\), respectively, then there exists a measure on \((S, \mathcal{S}) \otimes (T, \mathcal{T})\), called the product measure and written \(\mu_1 \otimes \mu_2\), such that \((\mu_1 \otimes \mu_2)(A \times B) = \mu_1(A)\mu_2(B)\). When \(\mu_1\) and \(\mu_2\) are \(\sigma\)-finite, the product measure is uniquely defined and also \(\sigma\)-finite.

The coproduct (i.e., disjoint union) of two measurable spaces \((S, \mathcal{S})\) and \((T, \mathcal{T})\) is defined as \((S, \mathcal{S}) \amalg (T, \mathcal{T}) \stackrel{\text{def}}{=} (S \cup T, \mathcal{S} \amalg \mathcal{T})\), where \(S \amalg T \eqdef (\{i_1(x) \mid x \in S\} \cup \{i_1(y) \mid y \in T\}, i_1 \eqdef \lambda_{x\mu}(1, x), i_2 \eqdef \lambda_{y\mu}(2, y), \text{ and } S \amalg T\) is the smallest \(\sigma\)-algebra that makes injection maps measurable, i.e., \(\sigmaapest{(\pi_1^{-1}(A) \mid A \in \mathcal{S}) \cup (\pi_2^{-1}(B) \mid B \in \mathcal{T})}\), where \(\pi_1\) and \(\pi_2\) are two measures on \((S, \mathcal{S})\) and \((T, \mathcal{T})\), respectively, then we can define their coproduct measure, written \(\mu_1 \amalg \mu_2\), as \((\mu_1 \amalg \mu_2)(\pi_1^{-1}(A) \cup \pi_2^{-1}(B)) \eqdef \mu_1(A) + \mu_2(B)\), for any \(A \in \mathcal{S}, B \in \mathcal{T}\). We can easily extend the binary coproducts to arbitrary coproducts. Particularly, the countable coproduct of \(\sigma\)-finite measures is still \(\sigma\)-finite.

Standard Borel spaces are closed under countable products and coproducts. We will use this property in our construction of semantic domains in appendix C.

B Full Development of Guide Types

Fig. 11 presents a complete list of evaluation rules for expressions and commands. Fig. 12 presents a complete list of typing rules for expressions, commands, and programs. Fig. 13 presents typing rules for values, environments, and guidance traces. In the rest of this section, we prove type safety of guide types.

Proposition B.1. If \(\text{d} : \text{dist}(\tau)\) and \(\nu\) is a value, then \(\nu : \tau\) if and only if \(\nu \in \text{d}.\text{support}\) (i.e., \(\text{d}.\text{density}(\nu) > 0\)).

\footnote{We use a monad bind notation \(\gg\) here. Indeed, the category of measurable spaces admits a monad with sub-probability measures [24, 48].}
Proof. Appeal to mathematical properties of primitive distributions.

Proposition B.2.
- If $\Gamma \vdash e : \tau$, then $V : \Gamma$, and $V : \tau$, then $\alpha : \tau$.
- If $\Gamma \vdash e : \tau$, then there exists a value $\nu$ such that $V : \tau$.

Proof. Appeal to type soundness and strong normalization of the simply-typed lambda calculus.
Figure 12. Typing rules for expressions, commands, and programs.
Figure 13. Type rules for values, environments, and guidance traces.

Lemma B.3 (Substitution). If \( \Gamma \vdash (a : A); (b : B) \vdash m \vdash \tau \vdash (a : A'); (b : B') \), then for any \( X_a, X_b, A_o, B_o \), it holds that

\[ \Gamma \vdash (a : [A_o/X_a]A); (b : [B_o/X_b]B) \vdash m \vdash \tau \vdash (a : [A_o/X_a]A'); (b : [B_o/X_b]B') \].

Proof. By induction on the derivation of \( \Gamma \vdash (a : A); (b : B) \vdash m \vdash \tau \vdash (a : A'); (b : B') \). We show several nontrivial cases; others are similar to one of these cases.

Case:

\[ \text{(TM:Ret)} \]

\[ \Gamma \vdash e : \tau \]

\[ \Gamma \vdash (a : A); (b : B) \vdash \text{ret}(e) \vdash \tau \vdash (a : A); (b : B) \]

\[ \Gamma \vdash (a : [A_o/X_a]A); (b : [B_o/X_b]B) \vdash \text{ret}(e) \vdash \tau \vdash (a : [A_o/X_a]A); (b : [B_o/X_b]B) \]

\[ \text{(TM:Ret)} \]

Case:

\[ \text{(TM:Bnd)} \]

\[ \Gamma \vdash (a : A); (b : B) \vdash m_1 \vdash \tau_1 \vdash (a : A'); (b : B') \]

\[ \Gamma, x : \tau_1 \vdash (a : [A_o/X_a]A'); (b : [B_o/X_b]B') \vdash m_2 \vdash \tau \vdash (a : A'); (b : B') \]

\[ \Gamma \vdash (a : [A_o/X_a]A); (b : [B_o/X_b]B) \vdash \text{bnd}(m_1; x; m_2) \vdash \tau \vdash (a : [A_o/X_a]A'); (b : [B_o/X_b]B') \]

\[ \text{(I.H.)} \]

Case:

\[ \text{(TM:Call)} \]

\[ \Sigma(f) = \tau_1 \vdash \tau \vdash (a : T_a); (b : T_b) \quad \Gamma \vdash e : \tau_1 \]

\[ \Gamma \vdash (a : T_a[A']); (b : T_b[B']) \vdash \text{call}(f; e) \vdash \tau \vdash (a : A'); (b : B') \]

\[ \Gamma \vdash (a : T_a[A'/A']); (b : T_b[B'/B']) \vdash \text{call}(f; e) \vdash \tau \vdash (a : [A_o/X_a]A'); (b : [B_o/X_b]B') \]

\[ T_a[A_o/X_a']A' = [A_o/X_a](T_a[A']) \]

\[ T_b[B_o/X_b']B' = [B_o/X_b](T_b[B']) \]

\[ A = T_a[A'], B = T_b[B'] \]

\[ \text{(assumption)} \]

\[ \text{(TM:Call)} \]

\[ \text{Theorem B.4 (Well-typed programs evaluate to well-typed values). If } \Gamma \vdash (a : A); (b : B) \vdash m \vdash \tau \vdash (a : A'); (b : B') \text{, } V \vdash (a : a_o); (b : a_b) \vdash m \Downarrow^w v \text{, and } V : \Gamma, \text{ then } v : \tau. \]

Proof. By induction on the derivation of \( V \vdash (a : a_o); (b : a_b) \vdash m \Downarrow^w v \), followed by inversion on \( \Gamma \vdash (a : A); (b : B) \vdash m \vdash \tau \vdash (a : A'); (b : B') \). We show several nontrivial cases; others are similar to one of these cases.

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Case: (EM:Ret)
\[ V \vdash e \parallel v \]
\[ V \parallel (a:\{\}); (b:\{\}) \vdash \text{ret}(e) \vdash v \]
\[ \Gamma \vdash e : \tau \quad (\text{assumption}) \]
\[ V \vdash e \parallel v \quad (\text{Prop. B.2}) \]

Case: (EM:Bnd)
\[ V \vdash (a:\sigma_{a_1}); (b:\sigma_{b_1}) \vdash m_1 \parallel_{\nu_1} v_1 \]
\[ V[x \mapsto v_1] \vdash (a:\sigma_{a_2}); (b:\sigma_{b_2}) \vdash m_2 \parallel_{\nu_2} v \]
\[ V \vdash (a:\sigma_{a_1} + \sigma_{a_2}); (b:\sigma_{b_1} + \sigma_{b_2}) \vdash \text{bnd}(m_1; x;m_2) \parallel_{\nu_1+\nu_2} v \]
\[ \Gamma \vdash (a:A); (b:B) \vdash m_1 \parallel_{\nu_1} v_1, V \parallel (a:A''); (b:B''), V \vdash (a:a_1); (b:b_1) \vdash m_1 \parallel_{\nu_1} v_1, V : \Gamma \quad (\text{I.H.}) \]
\[ v_1 : \tau_1 \quad (\text{Prop. B.2}) \]
\[ V[x \mapsto v_1] : (\Gamma, x : \tau_1) \]
\[ \Gamma, x : \tau_1 \parallel (a:A''); (b:B'') \vdash m_2 \parallel_{\tau} \parallel (a:A'); (b:B') \]
\[ V[x \mapsto v_1] \parallel (a:a_2); (b:b_2) \vdash m_2 \parallel_{\nu_2} v \quad (\text{I.H.}) \]

Case: (EM:CALL)
\[ V \vdash e \parallel v_1 \]
\[ D(f) = \text{fix}(a:b)(f.xy.mf) \]
\[ \emptyset [x \mapsto v_1] \parallel (a:\sigma_{a_1}); (b:\sigma_{b_1}) \vdash mf \parallel_{\nu} v \]
\[ V \parallel (a:\{\text{fold}\} + \sigma_{a_1}); (b:\{\text{fold}\} + \sigma_{b_1}) \vdash \text{call}(f;e) \vdash_{\nu} v \quad (\text{I.H.}) \]

Case: The reasoning below also works for (TM:Sample:Recv:R) and (TM:Sample:Send:R).
\[ V \vdash e \parallel d \quad v \in d.\text{support} \]
\[ \text{w} = d.\text{density}(v) \]
\[ V \parallel (a:\{\text{val}P(v)\})(b:\{\}) \parallel \text{sample}_{\tau_1}(a)(e) \vdash_{\nu} v \]
\[ \Gamma \vdash e : \text{dist}(\tau) \quad (\text{assumption}) \]
\[ d : \text{dist}(\tau) \]
\[ V \vdash e \parallel d, V : \Gamma \quad (\text{assumption}) \]
\[ v : \tau \quad (\text{Prop. B.2}) \]

Case: The reasoning below also works for (TM:Cond:Recv:R).
\[ i = \text{ite}(a_0,1,2) \]
\[ V \parallel (a:\sigma_{a_1}); (b:\sigma_b) \vdash m_1 \parallel_{\nu} v \]
\[ V \parallel (a:\{\text{dir}P(a_0)\} + \sigma_{a_1}); (b:\sigma_b) \vdash \text{cond}_{\tau_1}(a)(m_1;m_2) \parallel_{\nu} v \]
\[ \Gamma \vdash (a:A_1); (b:B) \vdash m_1 \parallel_{\tau} \parallel (a:A'); (b:B') \quad (\text{I.H.}) \]
\[ \Gamma \vdash (a:A_2); (b:B) \vdash m_2 \parallel_{\tau} \parallel (a:A'); (b:B') \]
\[ V \parallel (a:a_1); (b:b_1) \vdash m_1 \parallel_{\nu} v, \Gamma \vdash (a:A_1); (b:B) \vdash m_1 \parallel_{\tau} \parallel (a:A'); (b:B') \quad (\text{assumption}) \]
\[ v : \tau \quad (\text{Prop. B.1}) \]
Case: The reasoning below also works for (TM:COND:SEND::R).

\[(\text{EM:COND:L})\]
\[V \vdash e \Downarrow v_e \quad i = \text{ite}(v_a, 1, 2) \quad V \mid (a: \sigma_{a1}); (b: \sigma_b) \Downarrow m \Downarrow^w v \]
\[V \mid (a: [\text{dir}^L(v_a)] + \sigma_{a1}); (b: \sigma_b) + \text{cond}_{ad}(a)(e;m_1;m_2) \Downarrow^w [u_a = v_a] \]

\[(\text{TM:COND:L})\]
\[\Gamma \vdash e : 2 \quad \Gamma \mid (a: A_1); (b: B) \Downarrow m \Downarrow^w v \quad \Gamma \mid (a: A_2); (b: B) \Downarrow m \Downarrow^w v \]
\[\Gamma \mid (a: A_1 \& A_2); (b: B) \Downarrow \text{cond}_{ad}(a)(e;m_1;m_2) \Downarrow^w (a: A'_1); (b: B') \]

\[V \mid (a: \sigma_{a1}); (b: \sigma_b) \Downarrow m \Downarrow^w v, \Gamma \mid (a: A_1); (b: B) \Downarrow m \Downarrow^w v \quad (a: A'_1); (b: B') \]
\[(\text{assumption})
\]

(I.H.)

**Theorem B.5** (Well-typed programs produce well-typed traces). If \(\Gamma \mid (a: A_1); (b: B) \Downarrow m \Downarrow^w v \), followed by inversion on \(\Gamma \mid (a: A); (b: B) \Downarrow m \Downarrow^w v \), then \((a: A); (b: B) \Downarrow m \Downarrow^w v \)

Proof: By induction on the derivation of \(V \mid (a: \sigma_a); (b: \sigma_b) \Downarrow m \Downarrow^w v \), followed by inversion on \(\Gamma \mid (a: A); (b: B) \Downarrow m \Downarrow^w v \). We show several nontrivial cases; others are similar to one of these.

Case:

\[(\text{EM:RET})\]
\[V \vdash e \Downarrow v \quad V \mid (a: [\text{ret}(e)]) = \Downarrow^w v \]
\[
\begin{align*}
\sigma_a &= [\cdot], \sigma_b = [\cdot] \\
(\sigma_a + \sigma'_a) &= \sigma'_a, (\sigma_b + \sigma'_b) &= \sigma'_b \\
\sigma'_a : A', \sigma'_b : B', A = A', B = B' \\
(\sigma_a + \sigma'_a) : A, (\sigma_b + \sigma'_b) : B
\end{align*}
\]

(assumption)

Case:

\[(\text{EM:BND})\]
\[V \mid (a: \sigma_{a1}); (b: \sigma_{b2}) \Downarrow m \Downarrow^w v_i \]
\[V[x \mapsto v_1] \mid (a: \sigma_{a2}); (b: \sigma_{b2}) \Downarrow m \Downarrow^w v, \Gamma, x : r_1 \mid (a: A'''); (b: B''') \Downarrow m \Downarrow^w v \]
\[V \mid (a: \sigma_{a2} + \sigma'_a); (b: \sigma_{b2} + \sigma'_b) : A', (\sigma_b + \sigma'_b) : B' \]
\[V \mid (a: \sigma_{a2} + \sigma'_a); (b: \sigma_{b2} + \sigma'_b) : A', \sigma_b : B' \]
\[V \mid (a: \sigma_{a2} + \sigma'_a); (b: \sigma_{b2} + \sigma'_b) : B' \]

\[(\text{TM:BND})\]
\[\Gamma \mid (a: A'); \sigma_b : B' \]
\[\Gamma, x : r_1 \mid (a: A'''); (b: B''') \Downarrow m \Downarrow^w v \]
\[\Gamma \mid (a: A); (b: B) \Downarrow m \Downarrow^w v \quad (a: A'); (b: B') \]

(Thm. B.4)

(assumption)

(TC:EXTEND)

(assumption)

(assumption)

(I.H.)

(I.H.)

(assumption)

Case:

\[(\text{EM:CALL})\]
\[V \vdash e \Downarrow v_1 \quad D(f) = \text{fix}(a; b)(f.x.f,m_f) \quad \emptyset[x_f \mapsto v_1] \mid (a: \sigma_{a1}); (b: \sigma_{b1}) \Downarrow m_f \Downarrow^w v \]
\[V \mid (a: [\text{fold}] + \sigma_{a1}); (b: [\text{fold}] + \sigma_{b1}) \Downarrow \text{call}(f; e) \Downarrow^w v \]

\[(\text{TM:CALL})\]
\[\Sigma(f) = r_1 \Downarrow r_1 \quad (a: T_a); (b: T_b) \]
\[\Gamma \vdash e : r_1 \quad \Gamma \mid (a: T_a); (b: T_b) \Downarrow \text{call}(f; e) \Downarrow r_1 \quad (a: A'); (b: B') \]
\[\Gamma \vdash e : r_1, V \Downarrow v_1, V \Downarrow \Gamma \]
\[\emptyset[x_f \mapsto v_1] \mid (x_f : r_1) \]
\[x_f : r_1 \mid (a: A''); (b: B''') \Downarrow m_f \Downarrow^w v \quad (a: X_a); (b: X_b) \]
\[x_f : r_1 \mid (a: A'''); (b: B''') \Downarrow m_f \Downarrow^w v \quad (a: A'''); (b: B''') \]
\[\sigma'_a : A', \sigma'_b : B', \emptyset[x_f \mapsto v_1] \mid (a: \sigma_{a1}); (b: \sigma_{b1}) \Downarrow m_f \Downarrow^w v \]
\[(\sigma_{a1} + \sigma'_a) : [A'/X_a]A'', (\sigma_{b1} + \sigma'_b) : [B'/X_b]B''
\]

(TP:DEC)

\[\text{typedef}(T_a,X_a:A''), \text{typedef}(T_b,X_b:B''') \in \Gamma \]
\[\text{fix}(a; b)(f.x.f.m_f) : r_1 \Downarrow r_1 \quad (a: T_a); (b: T_b) \]

(assumption)

(Prop. B.2)

(TC:EXTEND)

(assumption)

(Lem. B.3)

(assumption)

(I.H.)
\begin{proof}

\textbf{Case:} The reasoning below also works for (TM:Sample:Recv:R) and (TM:Sample:Send:*).

\begin{array}{c}
\begin{prooftree}
\mbox{(EM:Sample:Recv:L)}
V \vdash e \Downarrow d \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v) \quad (b: [] \vdash \text{sample}_\nu(a)(v) \Downarrow^w v)}{V \vdash (a: (\texttt{val}^p(v)) ; (b: []) \vdash \text{sample}_\nu(a)(v) \Downarrow^w v)}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(TM:Sample:Recv:L)}
G \vdash e : \text{dist}(\tau) \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v) \quad (\exists \sigma' : \tau \land A', \exists \sigma'' : B') \\ 
A = \tau \land A', B = B'}{V \vdash (a: \tau \land A'); (b: B') \vdash \text{sample}_\nu(a)(v) \Downarrow^\tau \quad (a: A'); (b: B')}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(Thm. B.4)}
\end{prooftree}
\end{array}
\end{proof}

\begin{proof}

\textbf{Case:} The reasoning below also works for (TM:Cond:Recv:R).

\begin{array}{c}
\begin{prooftree}
\mbox{(EM:Cond:Recv:L)}
i = \text{ite}(v_a, 1, 2) \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v_a) \quad (\exists \sigma' : \tau \land A', \exists \sigma'' : B') \\ 
A_1 = \text{ite}(v_a, A_1, A_2)}{V \vdash (a: \tau \land A'); (b: B') \vdash \text{cond}_v(a)(v_a) \Downarrow^w v}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(TM:Cond:Recv:L)}
G \vdash e : 2 \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v_a) \quad (\exists \sigma' : \tau \land A', \exists \sigma'' : B') \\ 
A_1 = \text{ite}(v_a, A_1, A_2)}{V \vdash (a: \tau \land A'); (b: B') \vdash \text{cond}_d(a)(v_a) \Downarrow^w v}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(Thm. B.4)}
\end{prooftree}
\end{array}
\end{proof}

\begin{proof}

\textbf{Case:} The reasoning below also works for (TM:Cond:Send:R).

\begin{array}{c}
\begin{prooftree}
\mbox{(EM:Cond:Send:L)}
V \vdash e \Downarrow v_e \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v_a) \quad (\exists \sigma' : \tau \land A', \exists \sigma'' : B') \\ 
A = A_1 \land A_2)}{V \vdash (a: \tau \land A'); (b: B') \vdash \text{cond}_d(a)(v_e) \Downarrow^w v}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(TM:Cond:Send:L)}
G \vdash e : 2 \\
\frac{V \vdash \tau \quad V \vdash \text{\texttt{val}}^p(v_a) \quad (\exists \sigma' : \tau \land A', \exists \sigma'' : B') \\ 
A = A_1 \land A_2)}{V \vdash (a: \tau \land A'); (b: B') \vdash \text{cond}_d(a)(v_e) \Downarrow^w v}
\end{prooftree}
\end{array}

\begin{array}{c}
\begin{prooftree}
\mbox{(Thm. B.4)}
\end{prooftree}
\end{array}
\end{proof}

\textbf{Corollary (Thm. 4.4).} If $\vdash (a: A); (b: B) \vdash \gamma m \vdash \tau \quad (a: 1); (b: 1)$ and $\emptyset \vdash (a: \sigma_a); (b: \sigma_b) \vdash m \Downarrow^w v$, then $\sigma_a : A, \sigma_b : B$, and $v : \tau$.

\begin{proof}

Appeal to Thms. B.4 and B.5.
\end{proof}

\textbf{Theorem B.6 (Normalization, part I).} If $G \vdash (a: A); (b: B) \vdash m \vdash \tau \quad (a: A'); (b: B')$, $V : G, \sigma_a : A, \text{ and } \sigma_b : B$, then there exist $w, v, \sigma_a', \sigma_b', \sigma_a'' \sigma_b''$ such that $V \vdash (a: \sigma_a''); (b: \sigma_b'') \vdash m \Downarrow^w v, \sigma_a = \sigma_a'' + \sigma_a'$, $\sigma_b = \sigma_b'' + \sigma_b'$, $A'$, and $B'$.

\begin{proof}

By nested induction on the derivation of $\sigma_a : A, \sigma_b : B$, and $G \vdash (a: A); (b: B) \vdash m \vdash \tau \quad (a: A'); (b: B')$. We show several nontrivial cases; others are similar to one of these cases.
\end{proof}
Case:

\[(\text{TM:Ret})\]

\[Γ \vdash e : τ\]

\[Γ \mid (a : A); (b : B) \vdash \text{ret}(e) \not\vdash τ \mid (a : A); (b : B)\]  

\(V : Γ, Γ \vdash e : τ\)  

\(V \vdash e \nmid v\) for some \(v\)  

\(V \mid (a : []); (b : []) \vdash \text{ret}(e) \nmid^1 v\)

\(σ_a : A, σ_b : B\)

\(\sigma_v : A, \sigma_v : B\)

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\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)

\(\sigma_v : A, \sigma_v : B\)
Case: The reasoning below also works for (TM:COND;SEND;R).

\[(\text{TM:COND;SEND;L})\]
\[
\begin{align*}
\Gamma \vdash e : 2 & \quad \Gamma \mid (a : A_1); (b : B) \vdash m_1 \vdash \tau \mid (a : A'); (b : B') \\
\Gamma \mid (a : A_2); (b : B) \vdash m_2 \vdash \tau \mid (a : A'); (b : B') \\
\Gamma \mid (a : A_1 \land A_2); (b : B) \vdash \text{cond}_\mathbf{d}(a)(e;m_1;m_2) \vdash \tau \mid (a : A'); (b : B')
\end{align*}
\]
\[V : \Gamma, \Gamma \vdash e : 2\] (assumption)
\[V \vdash e \downarrow v_c \text{ for some } v_c \text{ s.t. } v_c : 2\] (Prop. B.2)
\[\sigma_a = A_1, A_2\]
\[\sigma_a = [\text{dir}^\mathbf{d}(v_a)] \circ \sigma_{a1}, \sigma_{a1} : \text{ite}(v_a, A_1, A_2)\]

Subcase: \(\sigma_a = \text{true}, \sigma_{a1} : A_1\)

\[V \mid (a : a_1'); (b : \sigma_b') \vdash m_1 \downarrow^w v \text{ s.t. } \sigma_{a1} = \sigma_{a1}' \circ \sigma_a, \sigma_b = \sigma_b'' \circ \sigma_b', \sigma_a : A', \sigma_b : B'\] (I.H.)

Let \(\sigma_a' \overset{\text{def}}{=} \text{dir}^\mathbf{d}(\text{true}) \circ \sigma_{a1}'\)

\[V \mid (a : a_1'); (b : \sigma_b'') \vdash \text{cond}_\mathbf{d}(a)(m_1;m_2) \downarrow^w [v_a = \text{true}] \downarrow v\] (EM:COND;SEND;L)

Subcase: \(\sigma_a = \text{false}, \sigma_{a1} : A_2\)

\[V \mid (a : a_1'); (b : \sigma_b') \vdash m_2 \downarrow^w v \text{ s.t. } \sigma_{a1} = \sigma_{a1}' \circ \sigma_a, \sigma_b = \sigma_b'' \circ \sigma_b', \sigma_a : A', \sigma_b : B'\] (I.H.)

Let \(\sigma_a' \overset{\text{def}}{=} \text{dir}^\mathbf{d}(\text{false}) \circ \sigma_{a1}'\)

\[V \mid (a : a_1'); (b : \sigma_b'') \vdash \text{cond}_\mathbf{d}(a)(m_1;m_2) \downarrow^w [v_a = \text{false}] \downarrow v\] (EM:COND:RECV:L)

Case: The reasoning below also works for (TM:SAMPLE;RECV;R) and (TM:SAMPLE;SEND;').

\[(\text{TM:SAMPLE;RECV;L})\]
\[
\begin{align*}
\Gamma \vdash e : \text{dist}(\tau) & \quad \Gamma \mid (a : \tau \land A'); (b : B) \vdash \text{sample}_\mathbf{r}(a)(e) \downarrow^w \tau \mid (a : A'); (b : B')
\end{align*}
\]
\[V : \Gamma, \Gamma \vdash e : \text{dist}(\tau)\] (assumption)
\[V \vdash e \downarrow d \text{ for some } d \text{ s.t. } d : \text{dist}(\tau)\] (Prop. B.2)
\[\sigma_a : \tau \land A'\]
\[\sigma_a = [\text{val}^\mathbf{d}(v)] \circ \sigma_a, v : \tau, \sigma_a : A'\] (inversion)
\[v \in d.\text{support}\]

Let \(w \overset{\text{def}}{=} d.\text{density}(v)\)

\[V \mid (a : [\text{val}^\mathbf{d}(v)]); (b : []) \vdash \text{sample}_\mathbf{r}(a)(e) \downarrow^w \downarrow v\] (EM:SAMPLE:RECV:L)
\[\sigma_a' : A', \sigma_b : B\] (assumption)

\[\Box\]

Corollary (Thm. 4.5). If \(\vdash (a : A); (b : B) \vdash m \vdash \tau \mid (a : 1); (b : 1), \sigma_a : A, \text{ and } \sigma_b : B, \text{ then there exist } w, v \text{ such that } 0 \mid (a : \sigma_a); (b : \sigma_b) \vdash m \downarrow^w v \text{ and } v : \tau.\]

Proof. Appeal to Thms. B.4 and B.6. \(\Box\)

Theorem B.7 (Normalization, part II). If \(\Gamma \mid (a : A); (b : B) \vdash m \vdash \tau \mid (a : A'); (b : B'), V : \Gamma, A \text{ is } \\& -\text{free}, B \text{ is } \oplus -\text{free}, \sigma_a : A, \text{ and } \sigma_b : B, \text{ then } A' \text{ is } \\& -\text{free}, B' \text{ is } \oplus -\text{free}, \text{ and there exist } w, v, \sigma_a', \sigma_a'', \sigma_a', \sigma_a'' \text{ such that }\)

\[V \mid (a : \sigma_a''); (b : \sigma_b''') \vdash m \downarrow^w u, v \rangle^> 0, \sigma_a = \sigma_a'' \circ \sigma_a', \sigma_a = \sigma_a'' \circ \sigma_a', \sigma_a : A', \text{ and } \sigma_a' : B'.\]

Proof. By nested induction on the derivation of \(\sigma_a : A, \sigma_b : B, \text{ and } \Gamma \mid (a : A); (b : B) \vdash m \vdash \tau \mid (a : A'); (b : B'). \text{ We show several nontrivial cases; others are similar to one of these cases.}\)

Case: \(\text{(TM:Ret)}\)

\[\begin{align*}
\Gamma \vdash e : \tau \\
\Gamma \mid (a : A); (b : B) \vdash \text{ret}(e) \downarrow^w \tau \mid (a : A); (b : B)
\end{align*}\]
\[V : \Gamma, \Gamma \vdash e : \tau\] (assumption)
\[V \vdash e \downarrow v \text{ for some } v\] (Prop. B.2)
\[V \mid (a : []); (b : []) \vdash \text{ret}(e) \downarrow^1 v\] (EM:Ret)
\[A \text{ is } \\& -\text{free}, B \text{ is } \oplus -\text{free}, \sigma_a : A, \sigma_b : B\] (assumption)

Case: \(\text{(TM:Bsm)}\)

\[\begin{align*}
\Gamma \mid (a : A); (b : B) \vdash m_1 \vdash \tau_1 \mid (a : A'''); (b : B''') & \quad \Gamma, x : \tau_1 \mid (a : A''''); (b : B''') \vdash m_2 \vdash \tau \mid (a : A''); (b : B'') \\
\Gamma \mid (a : A); (b : B) \vdash \text{bnd}(m_1;x,m_2) \vdash \tau \mid (a : A'); (b : B')
\end{align*}\]
Case:

\[ A'' \text{ is } \&\text{-free, } B'' \text{ is } \oplus\text{-free} \quad \text{(I.H.)} \]

\[ V \vdash (a : \sigma_a); (b : \sigma_b) \vdash m_1 \uplus^v w_1 \text{ s.t. } w_1 > 0, \sigma_a = \sigma_{a,1} \oplus \sigma_{a,2}, \sigma_b = \sigma_{b,1} \oplus \sigma_{b,2}, \sigma_{a,2} : A'', \sigma_{b,2} : B'', \text{ and} \]

\[ A'' \text{ is } \&\text{-free, } B'' \text{ is } \oplus\text{-free} \quad \text{(I.H.)} \]

\[ v_1 : \tau_1 \quad \text{(Thm. B.4)} \]

\[ V[x \mapsto v_1] \mid (\Gamma, x : \tau) \]

\[ V[\tau \mapsto \mathcal{R} w_2 > 0, \sigma_a = \sigma_{a,1} \oplus \sigma_{a,2}, \sigma_b = \sigma_{b,1} \oplus \sigma_{b,2}, \sigma_{a,2} : A', \sigma_{b,2} : B', \text{ and} \]

\[ A' \text{ is } \&\text{-free, } B' \text{ is } \oplus\text{-free} \quad \text{(I.H.)} \]

Let \( \sigma''_a \equiv \sigma_{a,1} \oplus \sigma_{a,2}, \sigma''_b \equiv \sigma_{b,1} \oplus \sigma_{b,2} \)

\[ V \mid (a : \sigma''_a); (b : \sigma''_b) \vdash \text{bd}(m_1 : x.m_2) \uplus^v w \]

\[ w_1 > 0, w_2 > 0 \text{ thus } w_1 : w_2 > 0 \quad \text{(EM:Bnd)} \]

Case:

\[ \Gamma \vdash e : \tau_1 \]

\[ \text{(TC:Extend)} \]

\[ \Gamma \vdash e : \tau_1 \quad \text{(Prop. B.2)} \]

\[ V \vdash e \uplus^v v_1 \text{ for some } v_1 \text{ s.t. } v_1 : \tau_1 \]

\[ \text{(EM:Call)} \]

\[ v_1 : \tau_1 \quad \text{(Thm. B.4)} \]

\[ \text{(TC:Extend)} \]

\[ \text{(Lemma B.3)} \]

\[ V \vdash (a : \sigma''_a); (b : \sigma''_b) \vdash \text{call}(f ; e) \uplus^v w \]

\[ w_1 > 0, w_2 > 0 \quad \text{thus } w_1 : w_2 > 0 \quad \text{(EM:Bnd)} \]

Case: The reasoning below also works for (TM:COND:RECV:R).

\[ \Gamma \vdash (a : A_1); (b : B') \vdash m_1 \uplus^v w \quad \text{(assumption)} \]

\[ \text{(TM:COND:RECV:L)} \]

\[ \text{(assumption)} \]

\[ (A_1 \oplus A_2) \&\text{-free implies } A_1 \&\text{-free, } A_2 \&\text{-free} \]

\[ (A_1 \oplus A_2) \quad \text{(inv.)} \]

\[ \sigma_a = \text{dir}^P(v_a) \oplus \sigma_{a,1}, \sigma_{a,1} : \text{it}(v_a, A_1, A_2) \quad \text{(inversion)} \]

Subcase: \( u_a = \text{true} \), \( \sigma_{a,1} : A_1 \)

\[ A' \text{ is } \&\text{-free, } B' \text{ is } \oplus\text{-free} \quad \text{(I.H.)} \]

\[ V \vdash (a : \sigma''_{a,1}); (b : \sigma''_{b,1}) \vdash m_1 \uplus^v w \text{ s.t. } w > 0, \text{ and } \sigma_{a,1} = \sigma'_{a,1} \oplus \sigma'_{a,2}, \sigma_b = \sigma'_{b,1} \oplus \sigma'_{b,2}, \sigma'_a : A', \sigma'_b : B' \quad \text{(I.H.)} \]

Let \( \sigma''_{a,1} \equiv \text{dir}^P(\text{true}) \oplus \sigma'_{a,1} \)

\[ V \vdash (a : \sigma''_a); (b : \sigma''_b) \vdash \text{cond}(a) (m_1 : m_2) \uplus^v w \quad \text{(EM:COND:RECV:L)} \]

Subcase: \( u_a = \text{false} \), \( \sigma_{a,1} : A_2 \)

\[ A' \text{ is } \&\text{-free, } B' \text{ is } \oplus\text{-free} \quad \text{(I.H.)} \]

\[ V \vdash (a : \sigma''_{a,1}); (b : \sigma''_{b,1}) \vdash m_2 \uplus^v w \text{ s.t. } w > 0, \text{ and } \sigma_{a,1} = \sigma'_{a,1} \oplus \sigma'_{a,2}, \sigma_b = \sigma'_{b,1} \oplus \sigma'_{b,2}, \sigma'_a : A', \sigma'_b : B' \quad \text{(I.H.)} \]

Let \( \sigma''_{a,1} \equiv \text{dir}^P(\text{false}) \oplus \sigma'_{a,1} \)

\[ V \vdash (a : \sigma''_a); (b : \sigma''_b) \vdash \text{cond}(a) (m_1 : m_2) \uplus^v w \quad \text{(EM:COND:RECV:L)} \]

Case: The reasoning below also works for (TM:SAMPLE:RECV:R) and (TM:SAMPLE:SEND:*).

\[ \text{(TM:SAMPLE:RECV:L)} \]

\[ \Gamma \vdash e : \text{dist}(\tau) \quad \text{(assumption)} \]

\[ V \vdash \Gamma, \Gamma \vdash e : \text{dist}(\tau) \]
Theorem B.8. By induction on the derivation of the evaluation judgment, followed by inversion on \( \Gamma \vdash (a : A); (b : B) \vdash m \vdash \tau \mid (a : A') \). We show several nontrivial cases; others are similar to one of these cases.

Case:

\[
\Gamma \vdash e : \tau
\]

\[
\Gamma \vdash \text{ret}(e) \vdash \tau \mid (a : A); (b : B)
\]
• The “if” direction:

\[(\text{EM:Ret})\]

\[
\Gamma \vdash e \Downarrow v
\]

\[
\frac{V \vdash \phi \Downarrow \nu}{V \mid (a: [\phi]); (b: [\phi]) \vdash \text{ret}(e) \Downarrow \nu}
\]

(assumption)

\[
V \vdash e \Downarrow \nu
\]

\[
V \mid (a: [\phi]); (b: [\phi]) \vdash \text{ret}(e) \Downarrow \nu
\]

(RM:Ret)

• The “only if” direction:

\[(\text{RM:Ret})\]

\[
\Gamma \vdash e \Downarrow \nu
\]

\[
\frac{V \vdash \phi \Downarrow \nu}{V \mid (a: [\phi]); (b: [\phi]) \vdash \text{ret}(e) \Downarrow \nu}
\]

(assumption)

\[
V \vdash e \Downarrow \nu
\]

\[
V \mid (a: [\phi]); (b: [\phi]) \vdash \text{ret}(e) \Downarrow \nu \text{ with } 1 > 0
\]

Case:

\[(\text{TM:Bnd})\]

\[
\begin{align*}
\Gamma & \mid (a: A); (b: B) \vdash m_1 \Downarrow \tau_1 \mid (a: A''); (b: B'') \\
\Gamma, x : \tau_1 & \mid (a: A'''); (b: B''') \vdash m_2 \Downarrow \tau \mid (a: A'); (b: B') \\
\Gamma & \mid (a: A); (b: B) \vdash \text{bnd}(m_1; x; m_2) \Downarrow \tau \mid (a: A'); (b: B')
\end{align*}
\]

(I.H.)

\[(\text{Thm. B.4})\]

(TC:Extend)

• The “if” direction:

\[(\text{EM:Bnd})\]

\[
\begin{align*}
V & \mid (a: \sigma_{a1}); (b: \sigma_{b1}) \vdash m_1 \Downarrow \nu_1 \nu_1, V \mid x \mapsto v_1 \mid (a: \sigma_{a2}); (b: \sigma_{b2}) \vdash m_2 \Downarrow \nu_2 \nu_2 \\
V & \mid (a: \sigma_{a1} + \sigma_{a2}); (b: \sigma_{b1} + \sigma_{b2}) \vdash \text{bnd}(m_1; x; m_2) \Downarrow \nu_1 \nu_2 \nu_1 \nu_2
\end{align*}
\]

(w_1 > 0 and w_2 > 0)

(assumption)

Case:

\[(\text{TM:Call})\]

\[
\begin{align*}
\Sigma(f) = \tau_1 & \Downarrow \tau \mid (a: T_a); (b: T_b) \\
\Gamma & \vdash e : \tau_1
\end{align*}
\]

\[
\Gamma \mid (a: T_a[A]); (b: T_b[B']) \vdash \text{call}(f; e) \Downarrow \tau \mid (a: A'); (b: B')
\]

(TP:Dec)

\[
\begin{align*}
\text{typedef}(T_a; X_a; A'''); \text{typedef}(T_b; X_b; B''') & \in T \\
x_f & : \tau_1 \mid (a: A'''); (b: B''') \vdash \exists m_f \Downarrow \tau \mid (a: X_a); (b: X_b)
\end{align*}
\]

• The “if” direction:

\[(\text{EM:Call})\]

\[
\begin{align*}
\begin{align*}
D(f) & = \text{fix}(a; b)(f; x_f; m_f) \\
0[x_f & \mapsto v_1] \mid (a: \sigma_{a1}); (b: \sigma_{b1}) \vdash m_f \Downarrow \nu \nu \nu
\end{align*}
\end{align*}
\]

(V | (a: [fold] + \sigma_{a1}); (b: [fold] + \sigma_{b1}) \vdash \text{call}(f; e) \Downarrow \nu \nu \nu
\]

(assumption)
The "if" direction:

\[ \Gamma \vdash e : \tau, \Gamma \vdash e \downarrow v, V : \Gamma \]
\[ \vdash e : \text{dist}(\tau) \]
\[ \Gamma \mid (a : \tau \land A') ; (b : B') \vdash \text{sample}_v(a)(e) \downarrow \tau \mid (a : A') ; (b : B') \]

- The "if" direction:

\[ \Gamma \mid (a : A_1) ; (b : B_1) \vdash m_1 \land \tau \mid (a : A') ; (b : B') \]
\[ \Gamma \mid (a : A_2) ; (b : B) \vdash m_2 \land \tau \mid (a : A') ; (b : B') \]
\[ \Gamma \mid (a : A_1 \oplus A_2) ; (b : B) \vdash \text{cond}_v(a)(m_1, m_2) \downarrow \tau \mid (a : A') ; (b : B') \]

The "only if" direction:

\[ \vdash e : \text{dist}(\tau), \Gamma \vdash e : \downarrow v, V : \Gamma \]
\[ d : \text{dist}(\tau) \]
\[ v \in \text{d.support} \]
\[ v \vdash \tau \text{ and d.density}(v) > 0 \]
\[ V \mid (a : \text{val}_v(l))(b : []) \vdash \text{sample}_v(a)(e) \downarrow d \land \text{density}(v) v \]

Case: The reasoning below also works for (TM:COND:RECV:R).

\[ \vdash e : \text{it}(v_a, 1, 2) \]
\[ V \mid (a : \sigma_{a_1}) ; (b : \sigma_b) \vdash m_1 \vdash v \]
\[ V \mid (a : \sigma_{a_1}) ; (b : \sigma_b) + \text{cond}_v(a)(m_1, m_2) \downarrow \tau \mid (a : A') ; (b : B') \]

The "if" direction:

\[ V \mid (a : \sigma_{a_1}) ; (b : \sigma_b) \vdash m_1 \downarrow v, V \mid (a : A_1) ; (b : B) \vdash m_1 \land \tau \mid (a : A') ; (b : B'), V : \Gamma \]
\[ \vdash e : \text{cond}_v(a)(m_1, m_2) \downarrow \tau \mid (a : A') ; (b : B') \]

- The "only if" direction:

\[ V \mid (a : \sigma_{a_1}) ; (b : \sigma_b) \vdash m_1 \downarrow v \]
\[ V \mid (a : \sigma_{a_1}) ; (b : \sigma_b) + \text{cond}_v(a)(m_1, m_2) \downarrow \tau \mid (a : A') ; (b : B') \]

EM:COND:RECV:L
• The “only if” direction:

\begin{align*}
V & \vdash (a: \sigma_a); (b: \sigma_b) \vdash_{\text{red}} m_i \downarrow v \\
V & \vdash (a: \text{dir}^P(a_u)) + \ sigma_1); (b: \sigma_b) \vdash_{\text{red}} \text{cond}_{\text{rv}} (a)(m_1; m_2) \downarrow v \\
V & \vdash (a: \sigma_a); (b: \sigma_b) \vdash_{\text{red}} m_i \downarrow v, \Gamma \vdash (a: A_1); (b: B) \vdash_{\text{dir}^P(a_u)} + \ sigma_1); (b: \sigma_b) \vdash_{\text{red}} \text{cond}_{\text{rv}} (a)(m_1; m_2) \downarrow v \\
\end{align*}

\textbf{Case:} The reasoning also below works for (TM:COND:SEND:L).

(TM:COND:SEND:L)

\begin{align*}
\Gamma & \vdash e : 2 \\
\Gamma & \vdash (a: A_1); (b: B) \vdash_{\text{dir}^P(a_u)} + \ sigma_1); (b: \sigma_b) \vdash_{\text{red}} m_i \downarrow v \\
\end{align*}

• The “if” direction:

\begin{align*}
V & \vdash e \downarrow v_e \\
V & \vdash (a: \sigma_a); (b: \sigma_b) \vdash_{\text{red}} m_i \downarrow v, \Gamma \vdash (a: A_1); (b: B) \vdash_{\text{dir}^P(a_u)} + \ sigma_1); (b: \sigma_b) \vdash_{\text{red}} m_i \downarrow v \\
\end{align*}

\textbf{Corollary B.9.} Suppose that \( \vdash (a: A); (b: B) \vdash m \uparrow \tau \vdash (a: 1); (b: 1) \). Then for any \( \sigma_a, \sigma_b, v \), we have \( \emptyset \vdash (a: \sigma_a); (b: \sigma_b) \vdash_{\text{red}} m \downarrow v \) if and only if \( \emptyset \vdash (a: \sigma_a); (b: \sigma_b) \vdash_{\text{red}} m \downarrow v \) for some \( w > 0 \).

\textbf{Proof.} Appeal to Thm. B.8.

\section{Full Development of Sound Bayesian Inference}

\textbf{Semantic domains.} For each scalar type \( \tau \), we equip it with a standard Borel space \( \llbracket \tau \rrbracket \) on inhabitants of \( \tau \):

\begin{align*}
\llbracket [1] \rrbracket & \overset{\text{def}}{=} \{ \text{triv}, \varphi(\text{triv}) \} \\
\llbracket [2] \rrbracket & \overset{\text{def}}{=} \{ \text{true}, \text{false}, \varphi(\{\text{true}, \text{false}\}) \} \\
\llbracket \mathbb{R}_{(0,1)} \rrbracket & \overset{\text{def}}{=} \{ (0, 1), \mathcal{B}(0, 1) \} \\
\llbracket \mathbb{R} \rrbracket & \overset{\text{def}}{=} \{ (0, \infty), \mathcal{B}(0, \infty) \} \\
\llbracket \mathbb{N}_+ \rrbracket & \overset{\text{def}}{=} \{ (0, 1, \ldots, n - 1), \varphi(\{0, 1, \ldots, n - 1\}) \} \\
\llbracket \mathbb{N} \rrbracket & \overset{\text{def}}{=} \{ (0, 1, \ldots), \varphi(\{0, 1, \ldots\}) \}
\end{align*}
For each guide type $A$, we construct a standard Borel space $\llbracket A \rrbracket$ on guidance traces of type $A$. We first present a parameterized construction $\llbracket A \rrbracket_{(C,C)}$, where $(C,C)$ is a standard Borel space, for guide types excluding type-level applications:
\[
\llbracket 1 \rrbracket_{(C,C)} \overset{\text{def}}{=} (C,C)
\]
\[
\llbracket \tau \land A \rrbracket_{(C,C)} \overset{\text{def}}{=} \text{val}^p(\llbracket \tau \rrbracket) \otimes \llbracket A \rrbracket_{(C,C)}
\]
\[
\llbracket \tau \lor A \rrbracket_{(C,C)} \overset{\text{def}}{=} \text{val}^c(\llbracket \tau \rrbracket) \otimes \llbracket A \rrbracket_{(C,C)}
\]
\[
\llbracket A \oplus B \rrbracket_{(C,C)} \overset{\text{def}}{=} (\text{dir}^p(\text{true}) \otimes \llbracket A \rrbracket_{(C,C)}) \parallel (\text{dir}^p(\text{false}) \otimes \llbracket B \rrbracket_{(C,C)})
\]
\[
\llbracket A \& B \rrbracket_{(C,C)} \overset{\text{def}}{=} (\text{dir}^c(\text{true}) \otimes \llbracket A \rrbracket_{(C,C)}) \parallel (\text{dir}^c(\text{false}) \otimes \llbracket B \rrbracket_{(C,C)})
\]

We use product measurable spaces to construct trace spaces by treating a trace as a pair of its head and tail, and coproduct measurable spaces to join trace spaces from different branches. Then, for every type definition `typedef(T.X.A)`, we define a function $f_T$ that maps a standard Borel space to another one:
\[
f_T \overset{\text{def}}{=} \lambda(C,C) \cdot \text{fold} \otimes \llbracket A \rrbracket_{(C,C)}.
\]

We can now construct a map $F_T$ that computes fixed points:
\[
F_T \overset{\text{def}}{=} \lambda(C,C). \lim_{n \to \infty} f_T^n(C,C).
\]

Because standard Borel spaces are closed under countable coproducts, we know that $F_T$ is well-defined. We can then add the construction for type-level applications:
\[
\llbracket T[B] \rrbracket_{(C,C)} \overset{\text{def}}{=} F_T(\llbracket B \rrbracket_{(C,C)}).
\]

For closed programs, we usually set the continuation space $(C,C)$ to $(\{ \}, \varnothing((\{ \})))$. Thus, we obtain the following definitions:
\[
\llbracket 1 \rrbracket \overset{\text{def}}{=} (\{ \}, \varnothing((\{ \})))
\]
\[
\llbracket \tau \land A \rrbracket \overset{\text{def}}{=} \text{val}^p(\llbracket \tau \rrbracket) \otimes \llbracket A \rrbracket
\]
\[
\llbracket \tau \lor A \rrbracket \overset{\text{def}}{=} \text{val}^c(\llbracket \tau \rrbracket) \otimes \llbracket A \rrbracket
\]
\[
\llbracket A \oplus B \rrbracket \overset{\text{def}}{=} (\text{dir}^p(\text{true}) \otimes \llbracket A \rrbracket) \parallel (\text{dir}^p(\text{false}) \otimes \llbracket B \rrbracket)
\]
\[
\llbracket A \& B \rrbracket \overset{\text{def}}{=} (\text{dir}^c(\text{true}) \otimes \llbracket A \rrbracket) \parallel (\text{dir}^c(\text{false}) \otimes \llbracket B \rrbracket)
\]
\[
\llbracket T[B] \rrbracket \overset{\text{def}}{=} F_T(\llbracket B \rrbracket)
\]

**Stock measures.** For each scalar type $\tau$, we equip it with a stock measure $\lambda_{\llbracket \tau \rrbracket}$ on its semantic domain $\llbracket \tau \rrbracket$. For nullary products $\llbracket 1 \rrbracket$, Booleans $\llbracket \text{true} \rrbracket$, positive real line $\llbracket \mathbb{R}^+ \rrbracket$, and real line $\llbracket \mathbb{R} \rrbracket$, we define $\lambda_{\llbracket \text{true} \rrbracket}$ to be the counting measure, i.e., $\lambda_{\llbracket \text{true} \rrbracket}(S) = |S|$. For unit interval $\llbracket 0,1 \rrbracket$, positive real line $\llbracket \mathbb{R}^+ \rrbracket$, and real line $\llbracket \mathbb{R} \rrbracket$, we define $\lambda_{\llbracket \text{true} \rrbracket}$ to be the Lebesgue measure $\text{Leb}$, i.e., the unique measure that satisfies $\text{Leb}([a,b]) = b - a$ for any interval $[a,b]$. All these measures are $\sigma$-finite.

For each guide type $A$, we construct a stock measure $\lambda_{\llbracket A \rrbracket}$ on its semantic domain $\llbracket A \rrbracket$. Similar to the construction of semantic domains, we first present a parameterized construction $\lambda_{\llbracket A \rrbracket}[\mu_C]$, where $\mu_C$ is a measure on a standard Borel space $(C,C)$, for guide types excluding type-level applications:
\[
\lambda_{\llbracket 1 \rrbracket}[\mu_C] \overset{\text{def}}{=} \mu_C
\]
\[
\lambda_{\llbracket \tau \land A \rrbracket}[\mu_C] \overset{\text{def}}{=} \text{val}^p(\lambda_{\llbracket \tau \rrbracket}) \otimes \lambda_{\llbracket A \rrbracket}[\mu_C]
\]
\[
\lambda_{\llbracket \tau \lor A \rrbracket}[\mu_C] \overset{\text{def}}{=} \text{val}^c(\lambda_{\llbracket \tau \rrbracket}) \otimes \lambda_{\llbracket A \rrbracket}[\mu_C]
\]
\[
\lambda_{\llbracket A \oplus B \rrbracket}[\mu_C] \overset{\text{def}}{=} (\text{dir}^p(\text{true}) \otimes \lambda_{\llbracket A \rrbracket}[\mu_C]) \parallel (\text{dir}^p(\text{false}) \otimes \lambda_{\llbracket B \rrbracket}[\mu_C])
\]
\[
\lambda_{\llbracket A \& B \rrbracket}[\mu_C] \overset{\text{def}}{=} (\text{dir}^c(\text{true}) \otimes \lambda_{\llbracket A \rrbracket}[\mu_C]) \parallel (\text{dir}^c(\text{false}) \otimes \lambda_{\llbracket B \rrbracket}[\mu_C])
\]

We use product measures to construct sequencing trace spaces, and coproduct measures to join measures for trace spaces from different branches. Then, for every type definition `typedef(T.X.A)`, we define a function $f_T$ that maps a measure to another one:
\[
f_T \overset{\text{def}}{=} \lambda(\mu_C) \cdot \text{fold} \otimes \lambda_{\llbracket A \rrbracket}[\mu_C].
\]
We can now construct a map $\mathcal{F}_T$ that computes fixed points:

$$\mathcal{F}_T \overset{\text{def}}{=} \lambda (\mu_C). \prod_{n=0}^{\infty} \mathcal{P}^n_T(\mu_C).$$

Because arbitrary coproduct of measures is well-defined, we know that $\mathcal{F}_T$ is also well-defined. We can then add the construction for type-level applications:

$$\lambda_{[T[\mathcal{B}]]}^{\mu_C} \overset{\text{def}}{=} \mathcal{F}_T(\lambda_{[\mathcal{B}]}^{\mu_C}).$$

Finally, we drop the $\mu_C$ parameter to obtain the following constructions:

$$\lambda_{[\mathcal{A}]} \overset{\text{def}}{=} \text{counting measure on } [\mathcal{1}]$$

$$\lambda_{[\tau \cdot \mathcal{A}]} \overset{\text{def}}{=} \text{val}^P(\lambda_{[\tau]} \otimes \lambda_{[\mathcal{A}]})$$

$$\lambda_{[\tau \cdot \Sigma \mathcal{A}]} \overset{\text{def}}{=} \text{val}^C(\lambda_{[\tau]} \otimes \lambda_{[\mathcal{A}]})$$

$$\lambda_{[\mathcal{A} \otimes \mathcal{B}]} \overset{\text{def}}{=} (\text{dir}^P(\text{true}) \otimes \lambda_{[\mathcal{A}]}) \mathbb{I} (\text{dir}^P(\text{false}) \otimes \lambda_{[\mathcal{B}]})$$

$$\lambda_{[\mathcal{A} \cdot \mathcal{B}]} \overset{\text{def}}{=} (\text{dir}^C(\text{true}) \otimes \lambda_{[\mathcal{A}]}) \mathbb{I} (\text{dir}^C(\text{false}) \otimes \lambda_{[\mathcal{B}]})$$

In addition, $\lambda_{[\mathcal{A}]}$ is a $\sigma$-finite measure for any guide type $\mathcal{A}$, because (i) the stock measure $\lambda_{[\tau]}$ for any scalar type $\tau$ is $\sigma$-finite, (ii) binary product of $\sigma$-finite measures is still $\sigma$-finite, and (iii) countable coproduct of $\sigma$-finite measures still $\sigma$-finite.

**Denotation of commands.** For a well-typed closed command $m$, i.e., $\cdot | (a: A); (b: B) \vdash m \vdash \tau_{\mathcal{A}} | (a: \mathcal{1}); (b: \mathcal{1})$, we define the density function of $m$ as

$$P_m(\sigma_a, \sigma_b) \overset{\text{def}}{=} \begin{cases} \omega & \text{if } \emptyset | (a: \sigma_a); (b: \sigma_b) \vdash m \vdash \omega \nu \\ 0 & \text{otherwise} \end{cases} .$$

**Proposition C.1.** $P_m$ is measurable.

**Proof.** We follow the proof strategy of Borgström et al. [10], where they proved measurability of density functions in an untyped probabilistic lambda calculus. \qed

Then, we construct a measure denotation $\llbracket m \rrbracket$ for $m$, by integrating $P_m$ with respect to the stock measure on the product space $\llbracket A \rrbracket \otimes \llbracket B \rrbracket$, i.e.,

$$\llbracket m \rrbracket (S_{\sigma_a, \sigma_b}) \overset{\text{def}}{=} \int_{S_{\sigma_a, \sigma_b}} P_m(\sigma_a, \sigma_b) \lambda_{[A]}^{\mathcal{1}} \mathbb{I} (d(\sigma_a, \sigma_b)),$$

where $S_{\sigma_a, \sigma_b}$ is a measurable set in $\llbracket A \rrbracket \otimes \llbracket B \rrbracket$.

**Bayesian inference.** Let us fix a well-typed model program $m_m$ that consumes latent random variables on a channel latent and provides observations on a channel obs, i.e.,

$$\cdot | (\text{latent}: A); (\text{obs}: B) \vdash m_m \vdash \tau_m | (\text{latent}: \mathcal{1}); (\text{obs}: \mathcal{1}).$$

Usually, the program $m_m$ does not receive any branch selections, i.e., $A$ is $\otimes$-free and $B$ is $\otimes$-free. Given a concrete observation $\sigma_o : B$, Bayesian inference is the problem of approximating the posterior $\llbracket m_m \rrbracket_{\sigma_o}$, a measure conditioned with respect to $\sigma_o$, defined by

$$\llbracket m_m \rrbracket_{\sigma_o} (S_{\tau}) \overset{\text{def}}{=} \int_{S_{\tau}} P_m(\sigma_{\tau}, \sigma_o) \lambda_{[A]}^{\mathcal{1}} (d\sigma_{\tau})$$

where $S_{\tau}$ is a measurable set in $\llbracket A \rrbracket$, i.e., a set of guidance traces of type $A$.

**Guide programs.** In our system, we implement a guide program $m_g$ as a coroutine that works with the model program $m_m$ and provides the latent channel with guide type $A$ that $m_m$ consumes, i.e.,

$$\cdot | \emptyset; (\text{latent}: A) \vdash m_g \vdash \tau_g | \emptyset; (\text{latent}: \mathcal{1}) ,$$

$$\cdot | (\text{latent}: A); (\text{obs}: B) \vdash m_m \vdash \tau_m | (\text{latent}: \mathcal{1}); (\text{obs}: \mathcal{1}).$$

The coroutine-based paradigm folds the model and guide programs into a single entity; thus, during the inference, both the model and guide coroutines execute. However, to distinguish the two measures defined by the model and guide, respectively, we define a denotation for the guide $m_g$, accompanied by the model $m_m$ and conditioned on a concrete observation $\sigma_o : B$, as a
measure defined by
\[
\| m_g \|_{\sigma_o}^{m_m}(S_t) \overset{\text{def}}{=} \int_{S_t} [P_{m_m}(\sigma_t, \sigma_o) \neq 0] \cdot P_{m_g}(\sigma_t) \lambda_A(d\sigma_t),
\]
where \( S_t \) is a measurable set in \([A]\), i.e., a set of guidance traces of type \( A \).

To justify the inclusion of \( [P_{m_m}(\sigma_t, \sigma_o) \neq 0] \) in the denotation of the guide program, we consider possible traces for a model-guide system. A combination of traces \((\sigma_t, \sigma_o)\) is said to be possible for the model program \( m_m \) and the guide program \( m_g \), if \( \emptyset \mid (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \triangleright m_m \downarrow v_m \) and \( \emptyset \mid \emptyset; (\text{latent}: \sigma_t) \triangleright m_g \downarrow v_g \) for some values \( v_m \) and \( v_g \).

**Lemma (Lem. 5.1).** Suppose that \( A \) is \( \oplus \)-free, \( B \) is \( \otimes \)-free, and
\[
\emptyset \mid \emptyset; (\text{latent}: A) \triangleright m_B \downarrow \tau_B \mid \emptyset; (\text{latent}: 1),
\]
\[
\emptyset \mid (\text{latent}: A); (\text{obs}: B) \triangleright m_m \downarrow \tau_m \mid (\text{latent}: 1); (\text{obs}: 1).
\]
Then a combination of traces \((\sigma_t, \sigma_o)\) is possible for the model \( m_m \) and the guide \( m_g \) if and only if \( P_{m_m}(\sigma_t, \sigma_o) \neq 0 \).

**Proof:**
- The “if” direction:
  \[
  P_{m_m}(\sigma_t, \sigma_o) \neq 0
  \]
  \[
  \emptyset \mid (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \triangleright m_m \downarrow^w v \text{ for some } v, w \text{ such that } w > 0
  \]
  \[
  A \text{ is } \oplus \text{-free}
  \]
  \[
  \emptyset \mid \emptyset; (\text{latent}: \sigma_t) \triangleright m_g \downarrow^w v' \text{ for some } v', w' \text{ such that } w' > 0
  \]
  \[
  \emptyset \mid \emptyset; (\text{latent}: \sigma_t) \triangleright m_g \downarrow v'
  \]
- The “only if” direction:
  \[
  \emptyset \mid (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \triangleright m_m \downarrow v \text{ for some } v
  \]
  \[
  \emptyset \mid (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \triangleright m_m \downarrow^w v \text{ for some } w > 0
  \]
  \[
  \sigma_t : A, \sigma_o : B
  \]
  \[
  P_{m_m}(\sigma_t, \sigma_o) = w > 0
  \]

**Absolute continuity.** Recall that a measure \( \mu \) is said to be absolutely continuous with respect to a measure \( \nu \), if \( \mu \) and \( \nu \) are defined on the same measurable space, and \( \nu(S) \neq 0 \) for every measurable set \( S \) for which \( \mu(S) \neq 0 \).

We prove that for a model-guide pair, guide types serve as certificates for absolute continuity.

**Theorem (Thm. 5.2).** Suppose that
\[
\emptyset \mid \emptyset; (\text{latent}: A) \triangleright m_B \downarrow \tau_B \mid \emptyset; (\text{latent}: 1),
\]
\[
\emptyset \mid (\text{latent}: A); (\text{obs}: B) \triangleright m_m \downarrow \tau_m \mid (\text{latent}: 1); (\text{obs}: 1),
\]
\( A \) is \( \oplus \)-free, \( B \) is \( \otimes \)-free, and \( \sigma_o : B \) such that \( \int P_{m_m}(\sigma_t, \sigma_o) \lambda_A(d\sigma_t) > 0 \). Then the measure \( \| m_m \|_{\sigma_o} \) is absolutely continuous with respect to the measure \( \| m_g \|_{\sigma_o} \), and vice versa.

**Proof:** We first claim that for any \( \sigma_t \in \| A \| \), it holds that \( P_{m_m}(\sigma_t, \sigma_o) \neq 0 \) if and only if \( [P_{m_m}(\sigma_t, \sigma_o) \neq 0] \cdot P_{m_g}(\sigma_t) \neq 0 \).

( \( \rightarrow \) ) Assume \( P_{m_m}(\sigma_t, \sigma_o) \neq 0 \). Thus, there exists \( v_1 \) such that \( \emptyset \mid (\text{latent}: \sigma_t); (\text{obs}: \sigma_o) \triangleright m_m \downarrow^w v_1 \), where \( v_1 \overset{\text{def}}{=} P_{m_m}(\sigma_t, \sigma_o) \). Because \( A \) is \( \oplus \)-free, we can apply Thm. 4.6 to program \( m_g \). Thus, there exist \( w_2, v_2 \) such that \( \emptyset \mid \emptyset; (\text{latent}: \sigma_t) \triangleright m_g \downarrow^w v_2 \) and \( w_2 > 0 \). In other words, we have \( P_{m_g}(\sigma_t) > 0 \). Then, we conclude that \( [P_{m_m}(\sigma_t, \sigma_o) \neq 0] \cdot P_{m_g}(\sigma_t) \neq 0 \).

( \( \leftarrow \) ) Assume \( [P_{m_m}(\sigma_t, \sigma_o) \neq 0] \cdot P_{m_g}(\sigma_t) \neq 0 \). Thus, we have both \( P_{m_m}(\sigma_t, \sigma_o) \neq 0 \) and \( P_{m_g}(\sigma_t) \neq 0 \). Then, we conclude that \( P_{m_m}(\sigma_t, \sigma_o) \neq 0 \) directly.
Fix a measurable set \( S_{\ell} \) in \([A]\). It suffices to show that \( \| m_m \|_{\sigma_o}(S_{\ell}) = 0 \) if and only if \( \| m_g \|_{\sigma_o}^{m_m}(S_{\ell}) = 0 \). We conclude by the following reasoning:

\[
\| m_m \|_{\sigma_o}(S_{\ell}) = 0 \iff \int_{S_{\ell}} \mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \lambda_{[A]}(d\sigma_{\ell}) = 0 \\
\iff \lambda_{[A]}(\mathbb{P}_{m_m}(\cdot, \sigma_o) \cdot 1_{S_{\ell}}) = 0 \\
\iff \lambda_{[A]}((\mathbb{P}_{m_m}(\cdot, \sigma_o) \cdot 1_{\bar{S}_{\ell}} > 0)) = 0 \\
\iff \lambda_{[A]}((\mathbb{P}_{m_m}(\cdot, \sigma_o) > 0 \land 1_{\bar{S}_{\ell}} > 0)) = 0 \\
\iff \lambda_{[A]}((\{ \mathbb{P}_{m_m}(\cdot, \sigma_o) = 0 \} \cdot \mathbb{P}_{m_g}(\cdot) > 0 \land 1_{\bar{S}_{\ell}} > 0)) = 0 \\
\iff \lambda_{[A]}((\{ \mathbb{P}_{m_m}(\cdot, \sigma_o) = 0 \} \cdot \mathbb{P}_{m_g}(\cdot) \cdot 1_{S_{\ell}}) = 0 \\
\iff \int_{S_{\ell}} [\mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \neq 0] \cdot \mathbb{P}_{m_g}(\sigma_{\ell}) \lambda_{[A]}(d\sigma_{\ell}) = 0 \\
\iff \| m_g \|_{\sigma_o}^{m_m}(S_{\ell}) = 0.
\]

**Importance sampling (IS).** Recall the operational rule below for a single step in the IS algorithm: given a model program \( m_m \), a guide program \( m_g \), and a concrete observation \( \sigma_o \), IS performs joint execution of the two programs to draw a sample \( \sigma_{\ell} \) with density \( w_g \) and compute \( \frac{w_m}{w_g} \) as the importance of \( \sigma_{\ell} \).

\[
0 \mid \emptyset ; (\text{latent: } \sigma_{\ell}) ; m_m \Downarrow w_m - 0 \mid (\text{latent: } \sigma_{\ell}) ; (\text{obs: } \sigma_o) ; m_m \Downarrow w_m - m_g ; m_m ; \sigma_o \vdash_{\text{is}} (\sigma_{\ell}, w_m/w_g)
\]

Define a density function \( \text{is}(\sigma_{\ell}) \) as \( w_g \cdot \frac{w_m}{w_g} = w_m \) on the space \([A]\) of guide traces for the computation of IS. Then, the measure defined by IS can be defined as \( \mu_{\text{is}}(S_{\ell}) \triangleq \int_{S_{\ell}} \text{is}(\sigma_{\ell}) \lambda_{[A]}(d\sigma_{\ell}) \).

**Lemma C.2.** Suppose that

\[
\emptyset \mid \emptyset ; (\text{latent: } A) \vdash_{A} m_g \Downarrow r_g \mid \emptyset ; (\text{latent: } 1),
\]

\[
\emptyset \mid (\text{latent: } A) ; (\text{obs: } B) \vdash_{A} m_m \Downarrow r_m \mid (\text{latent: } 1) ; (\text{obs: } 1),
\]

\( A \) is \( \oplus \)-free, \( B \) is \( \otimes \)-free, and \( \sigma_o : B \) such that \( \int \mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \lambda_{[A]}(d\sigma_{\ell}) > 0 \). Then \( \mu_{\text{is}} \propto \| m_m \|_{\sigma_o} \).

**Proof.** By Thm. 5.2, we know that the posterior \( \| m_m \|_{\sigma_o} \) is absolutely continuous with respect to \( \| m_g \|_{\sigma_o}^{m_m} \). Thus, \( \| m_m \|_{\sigma_o} \) is also absolutely continuous with respect to \( \mu_{\text{is}} \). In other words, the density function \( \text{is}(\cdot) \) is positive at all possible latent variables \( \sigma_{\ell} \) in the posterior. Then, for any \( S_{\ell} \) such that \( \| m_m \|_{\sigma_o}(S_{\ell}) \neq 0 \), we have

\[
\| m_m \|_{\sigma_o}(S_{\ell}) = \int_{S_{\ell}} \mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \lambda_{[A]}(d\sigma_{\ell}) = \int_{S_{\ell}} \text{is}(\sigma_{\ell}) \lambda_{[A]}(d\sigma_{\ell}) = \int \mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \lambda_{[A]}(d\sigma_{\ell}) = \int \mathbb{P}_{m_m}(\sigma_{\ell}, \sigma_o) \lambda_{[A]}(d\sigma_{\ell}),
\]

and we conclude by the fact that the denominator is a constant. □

**Variational inference (VI).** Recall that we parameterize the guide program \( m_{g,\theta} \) by a vector \( \theta \in \Theta \) of parameters, and use KL divergence as the distance metric, which is defined by

\[
\KL(\mu \parallel \nu) \triangleq \int p_\mu(\sigma_{\ell})(\log p_\nu(\sigma_{\ell}) - \log p_\nu(\sigma_{\ell})) \lambda_{[A]}(d\sigma_{\ell}),
\]

where \( \mu \) and \( \nu \) are measures on a space \([A]\) of guide traces of type \( A \) with densities \( p_\mu \) and \( p_\nu \), respectively, and \( \mu \) is absolutely continuous with respect to \( \nu \). The rule below formulates the computation of KL divergence for a specific \( \theta \), via joint execution of the two programs.

\[
0 \mid \emptyset ; (\text{latent: } \sigma_{\ell}) ; m_{g,\theta} \Downarrow w_m - 0 \mid (\text{latent: } \sigma_{\ell}) ; (\text{obs: } \sigma_o) ; m_m \Downarrow w_m - m_{g,\theta} ; m_m ; \sigma_o \vdash_{\text{VI}} (\sigma_{\ell}, \log w_m - \log w_g)
\]

The rule can be seen as defining a map \( \sigma_{\ell} \mapsto w_g \cdot (\log w_m - \log w_g) \), which is the integrand of the divergence \( \KL(\| m_{g,\theta} \|_{\sigma_o}^{m_m} \parallel \| m_m \|_{\sigma_o}) \).
Lemma C.3. Suppose that
\[ \cdot | \emptyset; (\text{latent: } A) \vdash \tau_B \div \tau_B | \emptyset; (\text{latent: } 1), \]
\[ \cdot | (\text{latent: } A); (\text{obs: } B) \vdash m_B \div m_B | (\text{latent: } 1); (\text{obs: } 1), \]
A is \( \oplus \)-free, B is \&-free, and \( \sigma_o : B \) such that \( \int P_{m_m}(\sigma_I, \sigma_o) \lambda_{\mathcal{A}}(d\sigma_I) > 0 \). Then, KL(\( [m_{\sigma_B}]_{\sigma_o} \div [m_B]_{\sigma_o} \)) is well-defined.

Proof. By Thm. 5.2, we know that \( [m_{\sigma_B}]_{\sigma_o} \div [m_B]_{\sigma_o} \) is absolutely continuous with respect to \( [m_B]_{\sigma_o} \). Thus, the KL divergence used in VI is well-defined. \( \square \)

Markov-Chain Monte Carlo (MCMC). We focus on Metropolis-Hastings (MH), which constructs the transition kernel from a proposal subroutine, and a probabilistic decision subroutine that either accepts the proposed random sample, or rejects it and keeps the old one. To implement proposal subroutines in our system, we extend the core calculus such that guidance traces can be used as first-class data. Then we implement the proposal subroutine as a procedure \( g \) whose argument is a guidance trace on the channel for latent random variables. The operational rule below formulates a single step in the MH algorithm; given a proposal procedure \( g \), a model program \( m_m \), a concrete observation \( \sigma_o \), and the current latent trace \( \sigma_I \), MH first performs joint execution of call(\( g; \sigma_I \)) and \( m_m \) to generate a new latent trace \( \sigma_I' \) with density \( \omega_{\text{fwd}} \), and then uses the new \( \sigma_I' \) and the old \( \sigma_I \) to calculate a backward density \( \omega_{\text{bwd}} \). MH then uses these densities to compute an acceptance ratio \( \alpha \), and then accepts the new sample \( \sigma_I' \) with probability \( \alpha \).

\[
\begin{align*}
\emptyset | \emptyset; (\text{latent: } \sigma_I') & \vdash \text{call}(g; \sigma_I') \Downarrow \omega_{\text{fwd}} - 0 | (\text{latent: } \sigma_I'); (\text{obs: } \sigma_o) \vdash m_B \Downarrow \omega_{m_B} - \\
\emptyset | \emptyset; (\text{latent: } \sigma_I) & \vdash \text{call}(g; \sigma_I') \Downarrow \omega_{\text{bwd}} - 0 | (\text{latent: } \sigma_I); (\text{obs: } \sigma_o) \vdash m_B \Downarrow \omega_{m_B} - \\
\text{g}; m_m; \sigma_o \vdash_{\text{MH}} \sigma_I ' \quad &\overset{\omega_{\text{bwd}} \circ \sigma_I'}{\underset{\omega_{\text{fwd}} \circ \sigma_I}{\longrightarrow}} \sigma_I''
\end{align*}
\]

The MH algorithm specifies a transition density \( f_1(\sigma_I, \sigma_I') \overset{\text{def}}{=} \omega_{\text{fwd}} \cdot \alpha \) and an unchanged density \( f_2(\sigma_I, \sigma_I') \overset{\text{def}}{=} \omega_{\text{bwd}} \cdot (1 - \alpha) \). Then we can use the two density functions to construct the MH kernel \( \kappa_{\text{MH}}(\sigma, S) \overset{\text{def}}{=} \int_S (f_1(\sigma, \sigma') + f_2(\sigma, \sigma') \cdot [\sigma \in S] \lambda_{\mathcal{A}}(d\sigma')). \)

Lemma C.4. Suppose that
\[
\text{old} : |A| | \emptyset; (\text{latent: } A) \vdash \tau_B \div \tau_B | \emptyset; (\text{latent: } 1), \]
\[
\cdot | (\text{latent: } A); (\text{obs: } B) \vdash m_B \div m_B | (\text{latent: } 1); (\text{obs: } 1), \]
mg is the procedure body of the proposal program \( g \), which has a single parameter \( \text{old} \) with type \( |A| \) that describes first-class guidance traces, A is \( \oplus \)-free, B is \&-free, and \( \sigma_o : B \) such that \( \int P_{m_m}(\sigma_I, \sigma_o) \lambda_{\mathcal{A}}(d\sigma_I) > 0 \). Then the posterior \( [m_B]_{\sigma_o} \div [m_B]_{\sigma_o} \) is stationary for the kernel \( \kappa_{\text{MH}} \).

Proof. We can extend Thm. 5.2 by allowing the environments to contain values and then using Thm. B.7 instead of Thm. 4.6 in the proof. Then, we derive that for any latent variables \( \sigma \), the posterior \( [m_B]_{\sigma_o} \div [m_B]_{\sigma_o} \) is absolutely continuous with respect to the measure \( \kappa_{\text{MH}}(\sigma, \cdot) \); that is, the guide program is able to sample any latent variables in the posterior, no matter what the current sample is. We then conclude by applying the Metropolis-Hastings-Green theorem [27]. \( \square \)