Fermion Determinant Calculus

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Abstract
The path-integral of the fermionic oscillator with a time-dependent frequency is analyzed. We give the exact relation between the boundary condition to define the domain in which the path-integral is performed and the transition amplitude that the path-integral calculates. According to this relation, the amplitude suppressed by a zero mode does not indicate any special dynamics, unlike the analogous situation in field theories. It simply says the path-integral picks up a combination of the amplitudes that vanishes. The zero mode that is often neglected in the reason of not being normalizable is necessary to obtain the correct answer for the propagator and to avoid an anomaly on the fermion number. We give a method to obtain the fermionic determinant by the determinant of a simple $2 \times 2$ matrix, which enables us to calculate it for a variety of boundary conditions.

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Fermionic determinant of the operator $\mathcal{D}$ is the one we first encounter in the analysis of the quantum physics in the path-integral formalism. It is the exact result of the Grassmann path-integral made from a bilinear Lagrangian $\bar{\psi} \mathcal{D} \psi$ over the fermionic degrees of freedom $\psi$ and $\bar{\psi}$. The determinant carries an important information about the time evolution of the fermions under the influence of bosonic background. Especially when $\mathcal{D}$ or its adjoint $\mathcal{D}^\dagger$ has a zero mode, the zero-frequency eigenmode of the operator, the determinant vanishes and corresponding transition is suppressed. A typical example of such situation happens with $\mathcal{D}$ the Dirac operator in SU(2) gauge theory. It possesses zero modes in the instanton background. The consequent suppression of the transition is interpreted to reflect the fermion number violation due to the anomaly on the fermion current $\bar{\psi}$. 
The physical significance of the zero modes of Dirac operators was first advocated in Ref. [2] and has been discussed in various places in physics. We still notice there seems to be a confusion on the treatment of zero modes in the case of fermionic oscillator, the simplest system containing only one fermionic degree of freedom. Its Lagrangian $\bar{\psi}D\psi$ in imaginary time formalism is given by a simple first order differential operator

$$D = \frac{d}{d\tau} + v(\tau)$$

(1)

with respect to the imaginary time $\tau$, where $v(\tau)$ is the time-dependent angular frequency induced by a coupling to a bosonic degree of freedom. We also define

$$D^\dagger = -\frac{d}{d\tau} + v(\tau).$$

(2)

Assume $v(\tau)$ changes its sign along the evolution in the time interval $[-T/2, T/2]$, for example, as $v(\tau) = \tanh(\tau)$. The solution of $D\phi = 0$ is then regarded as the zero mode since it is normalizable in the limit of $T \rightarrow \infty$. Gildener and Patrascioiu have argued by an explicit calculation that there is no zero mode available in the determinant calculus even in this simple example [3], while Salomonson and Van Holten have taken advantage of the zero mode in their calculation for the supersymmetry breaking [4].

In addition to the question of the existence of zero mode, we are concerned that the fermionic oscillator would have an anomaly if the numbers of the zero modes of $D$ and $D^\dagger$ are different. In the simple example mentioned above, the solution of $D^\dagger\phi = 0$ is not thought to be the relevant zero mode: for sufficiently large $T$, it becomes zero almost everywhere when normalized to one. Thus there appears to be the asymmetry in the numbers. According to the path-integral formulation of the anomaly [5], this asymmetry induces a phase in the path-integral measure under the global phase transformation, $\psi \rightarrow e^{i\theta} \psi$ and $\bar{\psi} \rightarrow e^{-i\theta} \bar{\psi}$, which indicates the non-conservation of the fermion number. This contradicts our naive intuition, that the fermionic oscillator should have no anomaly since we can calculate any amplitude with no regularization.
Motivated by this question of zero mode, we will clarify in this letter the exact calculation of the path-integral of the fermionic oscillator. In doing the path-integral, we first prepare an orthonormal complete set for all possible configurations. The eigenmodes of a certain self-adjoint operator are known to constitute such a complete set. The useful choice for the calculation is to use the differential expressions $D\dagger D$ and $DD\dagger$ as the candidate for the operator. As is known in mathematics, we need to impose the boundary condition to make the differential expressions to be self-adjoint. The important observation found in this letter is that the path-integral represents a different transition amplitude if the boundary condition is different. We will give the exact relation between the boundary condition and the transition amplitude, and verify it by comparing the path-integral results with those obtained by the operator formalism. The suppression for the zero mode will be understood within this boundary condition dependence.

We first carry out the calculation in the operator formalism. The Hamiltonian of the oscillator is

$$H(\tau) = v(\tau) \frac{1}{2} \left( \Psi\dagger \Psi - \Psi\Psi\dagger \right),$$

(3)

where $\Psi$ and $\Psi\dagger$ are the annihilation and creation operators in the two-dimensional space spanned by vacant $|0\rangle$ and occupied $|1\rangle$ states,

$$\Psi|0\rangle = \Psi\dagger|1\rangle = 0, \quad \Psi|1\rangle = |0\rangle, \quad \Psi\dagger|0\rangle = |1\rangle.$$  

(4)

From this Hamiltonian, the evolution operator from the initial time $T_i$ to the final $T_f$ is obtained by

$$U(T_f, T_i) = \mathcal{T} \exp \left[ -\int_{T_i}^{T_f} d\tau' H(\tau') \right],$$

(5)

where $\mathcal{T}$ represents the time-ordered product. The Hamiltonian commutes with the fermion number operator $\Psi\dagger\Psi$, and it is obvious that the matrix elements of $U$ are written as

$$\langle 0|U(T_f, T_i)|0\rangle = \exp \left[ \frac{1}{2} \int_{T_i}^{T_f} d\tau \, v(\tau) \right], \quad \langle 1|U(T_f, T_i)|1\rangle = \exp \left[ -\frac{1}{2} \int_{T_i}^{T_f} d\tau \, v(\tau) \right],$$

(6)

and two off-diagonal ones equal to zero. These are all we have to know to obtain any transition amplitude.
We turn to the path-integral formalism and consider

\[ I = \int [d\bar{\psi}d\psi] \exp \left[ -\int_{-T/2}^{T/2} d\tau \bar{\psi}D\psi \right], \tag{7} \]

where the functional measure is defined by discretizing imaginary-time [6]. We will come back this point later. Eq. (7) is related to the matrix elements of \( U(T/2, -T/2) \). Before seeing it, let us proceed the calculation of Eq. (7) usually adopted in literatures. The most decent way of the calculation for non-Hermitian \( D \) such as the one in the present case is to choose the domains \( D_{\varphi} \) and \( D_{\phi} \) of the square-integrable functions in the interval \([-T/2, T/2]\), in which \( D_{\varphi}^{\dagger}D_{\varphi} \) and \( DD_{\phi}^{\dagger} \) are self-adjoint and positive semi-definite, and the non-zero eigenmodes have the one-to-one correspondence (see for example [6]). The normalized eigenmodes \( \varphi^{(n)}(\in D_{\varphi}) \) and \( \phi^{(n)}(\in D_{\phi}) \) \( (n = 1, 2, 3, \ldots) \),

\[ D_{\varphi}^{\dagger}D_{\varphi}^{(n)} = \lambda_{n}\varphi^{(n)}, \quad DD_{\phi}^{\dagger} = \lambda_{n}\phi^{(n)}, \tag{8} \]

constitute a complete orthonormal set and they are related by

\[ \frac{1}{\sqrt{\lambda_{n}}}D_{\varphi}^{(n)} = \phi^{(n)}, \quad \frac{1}{\sqrt{\lambda_{n}}}D_{\phi}^{\dagger} = \varphi^{(n)}, \tag{9} \]

except zero modes. The eigenvalues so obtained are non negative. Using the expansions with these eigenmodes,

\[ \psi(\tau) = \sum_{n} a_{n}\varphi^{(n)}(\tau), \quad \bar{\psi}(\tau) = \sum_{n} \bar{a}_{n}\phi^{(n)*}(\tau), \tag{10} \]

we employ the integration measure \([d\bar{a}_{n}da_{n}]\) instead of \([d\bar{\psi}d\psi]\) and obtain

\[ I = N \int [d\bar{a}da]e^{-\sum_{n} \sqrt{\lambda_{n}}a_{n}\bar{a}_{n}} = N \left[ \text{det}(D_{\varphi}^{\dagger}D) \right]^{1/2}, \tag{11} \]

where \( N \) is the Jacobian between the measures, \( \text{det}(D_{\varphi}^{\dagger}D) \) is the infinite product of the eigenvalues. The latter is well-defined in the combination with \( N \). A different background \( v(\tau) \) results in a different complete set and different amplitudes in Eq. (10), say \( b_{n} \) and \( \bar{b}_{n} \). The Jacobian between the measures \([d\bar{a}da]\) and \([d\bar{b}db]\) is the determinant of the matrix that represents the
linear transformation between the eigenmodes in the different sets. Since each set is complete and orthonormal anyway, the matrix is unitary and the Jacobian is a phase. Thus background dependence of $\mathcal{N}$ in (11) can be at most a phase factor. In the explicit examples adopted in later calculations, however, we can choose the eigenmodes as real function of $\tau$. This is because either the differential equations (8) or boundary conditions (see Eqs. (26) and (27)) to solve the eigenvalue problem does not have any imaginary variable or constant. Thus the phase factor is 1 or $-1$. The Jacobian $\mathcal{N}$ then becomes background-independent except a possibility that it changes the sign at some special background.

Although the differential expression $\mathcal{D}^\dagger \mathcal{D}$ or $\mathcal{D} \mathcal{D}^\dagger$ seems self-adjoint by its form, it is not immediately so. One needs to define the domains $D_\varphi$ and $D_\phi$ properly in the functional space by imposing the boundary condition. We notice that any $\varphi_1$ and $\varphi_2$ in $D_\varphi$ should obey

$$\varphi^*_2(-T/2)\mathcal{D}\varphi_1(-T/2) - \varphi^*_2(T/2)\mathcal{D}\varphi_1(T/2) = 0.$$ (12)

Under the usual definition of the inner product,

$$(\varphi_2, \varphi_1) \equiv \int_{-T/2}^{T/2} d\tau \varphi^*_2(\tau)\varphi_1(\tau),$$ (13)

Eq. (12) means $(\varphi_2, \mathcal{D}^\dagger \mathcal{D}\varphi_1) = (\mathcal{D}\varphi_2, \mathcal{D}\varphi_1)$ which guarantees that $\mathcal{D}^\dagger \mathcal{D}$ is non-negative, and leads to $(\varphi_2, \mathcal{D}^\dagger \mathcal{D}\varphi_1) = (\mathcal{D}^\dagger \mathcal{D}\varphi_2, \varphi_1)$ which holds if it is self-adjoint. Note also that the equation

$$\int_{-T/2}^{T/2} d\tau \bar{\psi}\mathcal{D}\psi = \int_{-T/2}^{T/2} d\tau (\mathcal{D}^\dagger \bar{\psi})\psi.$$ (14)

holds as far as we expand $\psi$ and $\bar{\psi}$ as in (10) with $\varphi$ and $\phi$ that obey (8). This equation means that any $\varphi \in D_\varphi$ and any $\phi \in D_\phi$ satisfy

$$\phi^*(-T/2)\varphi(-T/2) - \phi^*(T/2)\varphi(T/2) = 0.$$ (15)

Since $\mathcal{D}\varphi \in D_\phi$ if $\varphi \in D_\varphi$, Eq. (12) and (13) are equivalent.

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1The procedure described here has a greater advantage in this property than the direct use of the eigenmodes of non self-adjoint $\mathcal{D}$ as was done in Ref. [3].
In fact the mathematical theory of the self-adjoint extension of the differential expression tells us that we need to specify two linearly independent boundary conditions on the values, \( \varphi(-T/2) \) and \( \varphi(T/2) \), and the first derivatives, \( \varphi(-T/2) \) and \( \varphi(T/2) \) in the present case of the second order differential expression. We refer the readers to the mathematical textbook [7] about the details. The problem is that the boundary condition so obtained is not unique. Then there naturally occurs a question about the boundary condition dependence of the path-integral.

To see the boundary condition dependence, we review the derivation of Eq. (7) based on Ref. [6]. We prepare the states

\[
\langle \bar{\theta} \rangle \equiv \langle 0 \rangle + \langle 1 \rangle \theta, \quad \langle \bar{\theta} \rangle \equiv \langle 0 \rangle + \bar{\theta} \langle 1 \rangle
\]

making use of Grassmann numbers \( \theta \) and \( \bar{\theta} \). They satisfy the completeness relation,

\[
\int d\bar{\theta} d\theta (1 - \bar{\theta} \theta) \langle \theta \rangle \langle \bar{\theta} \rangle = \langle 0 \rangle \langle 0 \rangle + \langle 1 \rangle \langle 1 \rangle.
\]

The path-integral (11) calculates the evolution from \( |\theta_0\rangle \) to \( |\bar{\theta}_N\rangle \), \( \langle \bar{\theta}_N | U | \theta_0 \rangle \), where we have abbreviated \( U(T/2, -T/2) \) to \( U \). To evaluate this, we discretize the time interval into \( N \) segments, each of which has the length \( \epsilon = T/N \) and write \( U \) as

\[
U = \lim_{N \to \infty} T \left( \prod_{n=1}^{N} (1 - \epsilon H(\tau_n)) \right),
\]

where \( \tau_n = n\epsilon - T/2 \). We insert \((N - 1)\) pairs of Grassmann integrals of \( (\bar{\theta}_n, \theta_n) \)(\( n = 1, ..., N - 1 \)) in the form of the completeness relation (17) as the \((N - 1)\) junctions of the \( N \) factors \( (1 - \epsilon H(\tau_n))(n = 1, ..., N) \). Using the explicit form (3) and (4), we obtain

\[
\langle \bar{\theta}_N | U | \theta_0 \rangle = \lim_{N \to \infty} \int d\bar{\theta}_{N-1} d\theta_{N-1} ... d\bar{\theta}_1 d\theta_1 \\
\times \left[ 1 + \epsilon \frac{v(\tau_N)}{2} + \left( 1 - \epsilon \frac{v(\tau_N)}{2} \right) \bar{\theta}_N \theta_{N-1} \right] \\
\times \left[ 1 + \epsilon \frac{v(\tau_{N-1})}{2} - \left( 1 + \epsilon \frac{v(\tau_{N-1})}{2} \right) \bar{\theta}_{N-1} \theta_{N-1} + \left( 1 - \epsilon \frac{v(\tau_{N-1})}{2} \right) \bar{\theta}_{N-1} \theta_{N-2} \right] \\
\times ... \\
\times \left[ 1 + \epsilon \frac{v(\tau_1)}{2} - \left( 1 + \epsilon \frac{v(\tau_1)}{2} \right) \bar{\theta}_1 \theta_1 + \left( 1 - \epsilon \frac{v(\tau_1)}{2} \right) \bar{\theta}_1 \theta_0 \right].
\]
All the bracket factors except the first one have the form

\[
\left[ 1 + e^{v(\tau_i) \frac{1}{2}} - \left( 1 + e^{v(\tau_i) \frac{1}{2}} \right) \bar{\theta}_i \theta_i + \left( 1 - e^{v(\tau_i) \frac{1}{2}} \right) \bar{\theta}_i \theta_{i-1} \right], \quad (i = 1, ..., N - 1)
\]

and can be safely replaced with

\[
\exp \left\{ -\bar{\theta}_i \left[ \left( \frac{\theta_i - \theta_{i-1}}{\epsilon} \right) + v(\tau_i) \left( \frac{\theta_i + \theta_{i-1}}{2} \right) \right] \right\}
\]

because \( \bar{\theta}_i^2 = 0 \) and \( \int d\bar{\theta}_i = 0 \). Then we arrive at a very close expression to Eq. (7) when we regard \( \theta_n \) as \( \psi(\tau_n) \) and \( \bar{\theta}_n \) as \( \bar{\psi}(\tau_n) \). Note, however, a difference of the first bracket factor in Eq. (19) from the others. We also wonder where \( \bar{\theta}_N \) and \( \theta_0 \) have gone when we got the result (11).

These questions are solved naturally by assuming that the exact definition of the measure \( [d\bar{\psi}d\psi] \) in Eq. (7), corresponding to the way we have integrated it out, does include the integral over \( \bar{\theta}_N \) and \( \theta_0 \) as

\[
I \equiv \int d\bar{\theta}_N d\theta_N (1 - \bar{\theta}_N \theta_N) \langle \bar{\theta}_N | U | \theta_0 \rangle
\]

where \( \theta_0 \) and \( \theta_N \) are related by the boundary condition to make \( D^\dagger D \) and \( DD^\dagger \) self-adjoint. The relevant terms in the integrand in (22) is those quadratic in the Grassmann variables,

\[
(1 - \bar{\theta}_N \theta_N) \langle \bar{\theta}_N | U | \theta_0 \rangle = ... - \bar{\theta}_N \theta_N \langle 0 | U | 0 \rangle + \bar{\theta}_N \theta_0 \langle 1 | U | 1 \rangle + ...
\]

This shows

\[
I = \langle 0 | U | 0 \rangle + \beta \langle 1 | U | 1 \rangle
\]

if the boundary conditions imposed to define \( D_\varphi \) is \( \theta_0 + \beta \theta_N = 0 \), or equivalently

\[
\varphi(-T/2) + \beta \varphi(T/2) = 0.
\]

Eqs. (24) and (23) are consistent with the fact that the fermionic path-integral presents the trace of \( U \) when carried out in the anti-periodic configurations (\( \beta = 1 \)).
We verify Eq. (24) by explicitly calculating the determinant det($D^\dagger D$) at various boundary conditions. The boundary conditions we are interested in is written as

$$\varphi(-T/2) + \beta \varphi(T/2) = 0, \quad \beta \varphi(-T/2) + D \varphi(T/2) = 0,$$

(26)

and

$$\beta \phi(-T/2) + \phi(T/2) = 0, \quad D^\dagger \phi(-T/2) + \beta D^\dagger \phi(T/2) = 0,$$

(27)

with a real parameter $\beta$. This condition includes the periodic one at $\beta = -1$ as well as the anti-periodic at 1. It is also available for the domain in which the zero mode lives. Once we put the boundary condition on $\varphi$, Eq. (12) and the one-one correspondence between $\varphi$ and $\phi$ determine the conditions on $D \varphi$, $\phi$, and $D^\dagger \phi$. The operators $D^\dagger D$ and $DD^\dagger$ are proved to be self-adjoint under these boundary conditions [7]. To calculate det($D^\dagger D$), the $2 \times 2$ matrix

$$M(z) \equiv \begin{pmatrix} u_1(z; -T/2) + \beta u_1(z; T/2) & u_2(z; -T/2) + \beta u_2(z; T/2) \\ \beta D u_1(z; -T/2) + D u_1(z; T/2) & \beta D u_2(z; -T/2) + D u_2(z; T/2) \end{pmatrix}$$

(28)

plays the central role, where $u_i(z; \tau) (i = 1, 2)$ are the linearly independent solutions of the equation

$$D^\dagger D u_i(z; \tau) = zu_i(z; \tau)$$

(29)

and the parameter $z$ is complex in general. We fix the normalization of these solutions by

$$u_1(z; -T/2) = 1, \quad u_2(z; -T/2) = 0,$$

$$u_1(z; -T/2) = 0, \quad u_2(z; -T/2) = 1.$$ 

(30)

Note the differential equation (29) does not have the first derivative term. Thus the Wronskian $(u_1 \dot{u}_2 - \dot{u}_1 u_2)$ conserves and is equal to 1 at any $\tau$. The zero points of det $M$ coincide with the eigenvalues of $D^\dagger D$: if $\lambda$ is one of the eigenvalues, there exists a non-trivial linear combination $\gamma_1 u_1(\lambda; \tau) + \gamma_2 u_2(\lambda; \tau)$ that satisfies Eq. (26); the equation for $\gamma_i$ turns out to be

$$M(\lambda)_{ij} \gamma_j = 0 (i, j = 1, 2)$$

and thus det $M(z)$ is zero at the eigenvalue $\lambda$; one can reverse this argument in the opposite direction.
We then see the ratio $[\det(D^\dagger D - z) / \det M(z)]$ is independent from the background $v(\tau)$. The proof is essentially the same as the one given in Ref. [8] in the calculation of the other type of determinants. Let us consider two different operators $D_1^\dagger D_1$ and $D_2^\dagger D_2$ containing different backgrounds $v_1(\tau)$ and $v_2(\tau)$, and denote their $n$-th eigenvalue by $\lambda_{1;n}$ and $\lambda_{2;n}$, respectively. Correspondingly let $M_1$ and $M_2$ denote the matrix made by (28) and (29) with $D_1^\dagger D_1$ and $D_2^\dagger D_2$, respectively. The ratio defined by

$$\frac{\det(D_1^\dagger D_1 - z)}{\det(D_2^\dagger D_2 - z)} \equiv \prod_{n=1}^{\infty} \left( \frac{\lambda_{1;n} - z}{\lambda_{2;n} - z} \right)$$

(31)

is a meromorphic function of $z$, and it has a simple zero at each $\lambda_{1;n}$ and a simple pole at each $\lambda_{2;n}$. It goes to one as $z$ goes to infinity in any direction except along the real positive axis. The ratio $[\det M_1(z) / \det M_2(z)]$ is also a meromorphic function that has poles and zeros at exactly the same $z$. Note further we obtain

$$\det M(z) = Du_2(z; T/2) + \beta^2 [u_1(z; T/2) - v(-T/2)u_2(z; T/2)] + 2\beta$$

(32)

using the condition (30) and the conservation of the Wronskian. For sufficiently large $|z|$, $\sqrt{|z|} \gg |v(\tau)^2 - \dot{v}(\tau)|T$, the frequency $\nu(\tau)$ in (29) becomes negligible. The solutions $u_i(z; \tau)$ is then well-approximated by their free solutions, and $\dot{u}_2(T/2) \simeq u_1(T/2) \simeq e^{\sqrt{-z}T}/2$. The determinant of $M$ then grows as $(1 + \beta^2)e^{\sqrt{-z}T}/2$ at sufficiently large $|z|$ (except along the real positive axis) independently from the backgrounds. The ratio $[\det M_1(z) / \det M_2(z)]$ also goes to one in the same limit. Thus

$$\frac{\det(D_1^\dagger D_1 - z)}{\det(D_2^\dagger D_2 - z)} = \frac{\det M_1(z)}{\det M_2(z)}.$$  

(33)

Eq. (33) establishes that $[\det(D^\dagger D - z) / \det M(z)]$ is background independent.

We can now write

$$\mathcal{N}[\det(D^\dagger D)]^{1/2} = \mathcal{N}' [\det M(0)]^{1/2}.$$  

(34)

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2Similar formulae that relate the determinant of differential operators with that of a matrix have been found in condensed matter physics [7].
The factor $\mathcal{N}'$ is background independent as is $\mathcal{N}$. The calculation of $\det M(0)$ is elementary.

We obtain

$$u_1(0; \tau) = x_1(\tau) + v(-T/2)x_2(\tau), \quad u_2(0; \tau) = x_2(\tau),$$

where

$$x_1(\tau) = \exp \left[ -\int_{-T/2}^{\tau} d\tau' v(\tau') \right], \quad y_1(\tau) = \exp \left[ \int_{-T/2}^{\tau} d\tau' v(\tau') \right], \quad x_2(\tau) = x_1(\tau) \int_{-T/2}^{\tau} d\tau' (y_1(\tau'))^2.$$ (36)

Putting these solutions into (32), we find

$$\det M(0) = \left( [y_1(T/2)]^{1/2} + \beta[x_1(T/2)]^{1/2} \right)^2.$$ (37)

Note $y_1 = (x_1)^{-1}$. Recalling Eqs. (11), (34), and (37), we obtain

$$I = \mathcal{N}' \left\{ [y_1(T/2)]^{1/2} + \beta[x_1(T/2)]^{1/2} \right\}.$$ (38)

This is exactly Eq. (24), where the matrix elements are calculated explicitly by Eqs. (3) and (36). The factor $\mathcal{N}'$ turns out to be 1. We like to make a comment here. $I$ in Eq. (38) becomes negative, for example, when $\beta = -1$ and $x_1(T/2) > y_1(T/2)$. It means that we have chosen the negative solution in taking the square-root of $\det M$ in this case. This choice is justified by considering that $I$ should be analytic with respect to the functional variation of $v(\tau)$. The flow of $\sqrt{\lambda_n}$ that appears in (9) and (11) must be smooth when the background changes continuously.

We are prepared to answer the question of zero mode. Note that the zero mode candidate is $x_1$ or $y_1$ in (36). Let us start with assuming that the normalizable one in the usual sense is $x_1$. The domain in which $x_1$ resides is given by the boundary condition (26) with $\beta = -y_1(T/2)$. Now that zero is an eigenvalue, the path-integral is zero. This is readily verified by putting the value of $\beta$ into (38). The reason of the vanishing amplitude is, however, quite different from our interpretation of the same occurrence in the SU(2) gauge theory. There is no dynamical reason; we have just chosen a vanishing combination accidentally by the boundary conditions.

We also notice the other zero solution $y_1$ cannot be neglected. It satisfies the boundary condition Eq. (27) with the same value of $\beta$. Since we can normalize it any way as long as
the time interval $T$ is finite (no matter how long it is), we cannot find any legitimate reason to abandon it. We can confirm the necessity of $y_1$ in the calculation of the propagator. Let us define

$$F(\tau, \tau') \equiv \theta(\tau - \tau')U(T/2, \tau)\Psi U(\tau, \tau')\Psi^\dagger U(\tau', -T/2)$$

$$-\theta(\tau' - \tau)U(T/2, \tau')\Psi^\dagger U(\tau', \tau)\Psi U(\tau, -T/2),$$

and consider its path-integral representation

$$G(\tau, \tau') = \int [d\bar{\psi}d\psi] \exp \left[ -\int_{-T/2}^{T/2} d\tau \bar{\psi} D\psi \right] \psi(\tau)\bar{\psi}(\tau').$$

The exact relation of $G$ integrated in the domain defined by Eqs. (26) and (27) to the corresponding matrix element of $F$ is obtained by applying the same argument that leads to Eq. (24). It is

$$G(\tau, \tau') = \theta(\tau - \tau') \langle 0|U(T/2, \tau)\Psi U(\tau, \tau')\Psi^\dagger U(\tau', -T/2)|0 \rangle$$

$$+ \theta(\tau' - \tau) y_1(T/2)\langle 1|U(T/2, \tau')\Psi^\dagger U(\tau', \tau)\Psi U(\tau, -T/2)|1 \rangle.$$

In the operator formalism, Eqs. (3) and (4) yield

$$\langle 0|U(T/2, \tau)\Psi U(\tau, \tau')\Psi^\dagger U(\tau', -T/2)|0 \rangle = y_1(T/2)^{1/2}x_1(\tau)y_1(\tau'),$$

$$\langle 1|U(T/2, \tau)\Psi^\dagger U(\tau', \tau)\Psi U(\tau, -T/2)|1 \rangle = x_1(T/2)^{1/2}x_1(\tau)y_1(\tau'),$$

and, thus, using Eq. (41) we obtain

$$G(\tau, \tau') = y_1(T/2)^{1/2}x_1(\tau)y_1(\tau').$$

Interestingly the final result does not have any remnant of the time-ordered procedure in (41).

We do the corresponding path-integral (40). The normalized zero modes are

$$\varphi^{(1)}(\tau) = \frac{x_1(\tau)}{\sqrt{\int_{-T/2}^{T/2} d\tau' x_1(\tau')^2}},$$

$$\phi^{(1)}(\tau) = \frac{y_1(\tau)}{\sqrt{\int_{-T/2}^{T/2} d\tau' y_1(\tau')^2}}.$$
The Grassmann variables \( a_1 \) and \( \bar{a}_1 \) in the expansion (10), the coefficients of \( \varphi^{(1)} \) and \( \phi^{(1)} \), do not appear in the action. Only the \( \psi \) and \( \bar{\psi} \) in the integrand in (10) can supply them, and one gets

\[
G(\tau, \tau') = \mathcal{N} \left[ \det \left( D^\dagger D \right) \right]^{1/2} \varphi^{(1)}(\tau) \phi^{(1)}(\tau'),
\]

where \( \det \left( D^\dagger D \right) \) is the product of the eigenvalues except zero. It is evaluated by

\[
\mathcal{N}^2 \det \left( D^\dagger D \right) = \lim_{z \to 0} \frac{\det M(z)}{(-z)}.
\]

For the calculation of the right-hand side in Eq. (46), we use the fact that the solutions \( u_i(z; \tau)(i = 1, 2) \) have an expansion around \( z = 0 \),

\[
u_i(z; \tau) = u_i(0; \tau) + z \delta u_i(\tau) + ...
\]

and the first order term is given by

\[
\delta u_i(\tau) = \int_{-T/2}^{T/2} d\tau' \left[ u_1(0; \tau) u_2(0; \tau') - u_2(0; \tau) u_1(0; \tau') \right] u_i(0; \tau').
\]

Since \( \delta u_i(-T/2) = \delta u_i(-T/2) = 0 \), the expansion (17) satisfies the initial conditions (30). Putting Eqs. (47) and (48) into Eq. (32) and using Eqs. (35) and (36), we obtain

\[
\lim_{z \to 0} \frac{\det M(z)}{(-z)} = y_1(T/2) \left[ \int_{-T/2}^{T/2} d\tau y_1(\tau)^2 \right] \left[ \int_{-T/2}^{T/2} d\tau x_1(\tau)^2 \right].
\]

Eqs. (14), (15), (16) and (49) give the correct result (13). We would have a wrong answer without \( \phi^{(1)} \). The numbers of the zero mode belonging to \( D_\varphi \) and \( D_\phi \) are the same independently of the specific boundary condition to define them. This is consistent with the absence of the anomaly in the fermionic oscillator.

In summary, we have revealed and confirmed that the path-integral in different boundary conditions calculates different matrix elements of the time evolution operator. The exact relation between the boundary condition and the matrix element tells us the reason of vanishing path-integral by the zero mode. It does not have a dynamical reason such as the fermion
number violation, but it occurs just because the zero mode takes the vanishing combination of the matrix elements. One may wonder whether the boundary condition dependence vanishes in the limit of $T \to \infty$. It does not vanish. For example, the path-integral with the periodic boundary condition can be either positive or negative depending on the background, while that with the anti-periodic one is always positive no matter what the background is. It is natural to expect the existence of the similar boundary condition dependence in field theories. Then any conclusion based on a path-integral calculation should be spelled out with the boundary condition, with respect to the imaginary time, explicitly specified. The conclusion might change for different boundary conditions. As far as we know there is little discussion on this dependence. A careful investigation of field theories taking consciously boundary conditions into account may reveal their new aspects.

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