Robustness, Scott Continuity, and Computability

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Abstract

Robustness is a property of system analyses, namely monotonic maps from the complete lattice of subsets of a (system’s state) space to the two-point lattice. The definition of robustness requires the space to be a metric space. Robust analyses cannot discriminate between a subset of the metric space and its closure, therefore one can restrict to the complete lattice of closed subsets. When the metric space is compact, the complete lattice of closed subsets ordered by reverse inclusion is $\omega$-continuous and robust analyses are exactly the Scott continuous maps. Thus, one can also ask whether a robust analysis is computable (with respect to a countable base). The main result of this paper establishes a relation between robustness and Scott continuity, when the metric space is not compact. The key idea is to replace the metric space with a compact Hausdorff space, and relate robustness and Scott continuity by an adjunction between the complete lattice of closed subsets of the metric space and the $\omega$-continuous lattice of closed subsets of the compact Hausdorff space. We demonstrate the applicability of this result with several examples involving Banach spaces.

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1 Introduction

The main contribution of this paper is relating robust analyses and Scott continuous maps (between \(\omega\)-continuous lattices). This contribution is relevant to the broader endeavor of developing software tools for system analysis based on mathematical models. Typically, the behavior of a controlled system is given a priori, but for most systems the open system approach is insufficient as the correctness of the controlling system depends on properties of the environment. This requires to model also the environment. Software tools (for system analysis) manipulate formal descriptions. The key point of formal descriptions is their mathematical exactness. However, exactness should not be confused with precision. In particular, mathematical descriptions should make explicit known unknowns and the amount of imprecision. There are two unavoidable sources of imprecision: errors in measurements (on physical systems) and representations of continuous quantities in software tools.

The key feature of robust analyses is the ability to cope with small amounts of imprecision. On the other hand, analyses can be implemented in software tools only if they are computable. A definition of computability for effectively given domains has been proposed in [20] Definition 3.1, where the domains considered include those of interest for us, namely \(\omega\)-continuous lattices. The key point of this, and similar proposals, is that computable maps (between effectively given domains) are necessarily Scott continuous.

For the benefit of readers, before outlining the main result of the paper, we review the context in which it is placed and recall related results, while keeping technicalities to a minimum.

**Systems.** First, one has to decide how to model systems. The simplest systems, i.e., discrete systems, can be modeled by a set \(S\) (of states) and a transition map \(t : S \to S\) describing the deterministic state change of the system in one step. The systems we consider are closed, i.e., they do not interact with the environment, or to put it differently, a model should account also for the environment. In this respect, it is important to non-determinism, namely, a state in \(S\) is replaced by a set of states and the transition map is replaced by a transition relation \(T \subseteq S \times S\). In theoretical computer science, the pair \((S, T)\) is called a transition system.

**Analyses.** We move from the category of sets and relations to the category of complete lattices and monotonic maps. More precisely, we replace \(S\) with the complete lattice \(\mathbb{P}(S)\) of subsets of \(S\), and a relation \(T' \subseteq S \times S'\) with the monotonic map \(T' : \mathbb{P}(S) \to \mathbb{P}(S')\) such that:

\[
T'_*(S) = \{ s' | \exists s : S.T(s, s') \}.
\]

We take as partial order \(\leq\) on \(\mathbb{P}(S)\) reverse inclusion, i.e., \(S' \subseteq S \iff S' \supseteq S\). The rational for this choice is that a smaller set (of states) provides more information on the actual state of the system. The transition relations on \(S\) form a complete lattice \(\mathbb{P}(S \times S)\), where \(T' \subseteq T\) means that \(T\) is more deterministic than \(T'\).

Several analyses correspond to monotonic maps between complete lattices. For instance, reachability analysis for transition systems on \(S\) corresponds to the map \(R : \mathbb{P}(S \times S) \times \mathbb{P}(S) \to \mathbb{P}(S)\) given by \(R(T, I) = T^*(I)\), i.e., the set of states reachable in a finite number of steps from (a state in) \(I\), while safety analysis corresponds to the map \(S : \mathbb{P}(S \times S) \times \mathbb{P}(S) \times \mathbb{P}(S) \to \Sigma\) in which \(\Sigma\) is the two-point lattice \(\bot < \top\) and \(S(T, I, E) = \top \iff T^*(I) \cap E = \emptyset\), i.e., no bad state in \(E\) is reachable from \(I\).

**Approximation.** The partial order on a complete lattice \(X\) allows (qualitative) comparisons, in particular we say that \(x'\) is an over-approximation of \(x\) when \(x' \leq x\). The category of complete lattices and monotonic maps is also the natural setting for abstract interpretation [6,7]. More precisely, given an interpretation \([-\cdot]\) of a (programming) language in a complete lattice \(X\), one can choose another complete lattice \(X_a\) related to \(X\) by an adjunction (also known as Galois connection) \(X_a \rightleftarrows X\), i.e., a pair of monotonic maps \(\alpha\) (called abstraction) and \(\gamma\) (called concretization) such that \(x' \leq a x \iff \gamma(x') \leq x\). In general, \(X_a\) is simpler than \(X\) (e.g., \(X_a\) can be finite) and allows interpretations \([-\cdot]_a\) of the language for computing over-approximations of \([\cdot]\), i.e., \(\gamma([p]_a) \leq [p]_a\) for every (program) \(p\) in the language.

The adjunction between \(X_a\) and \(X\) gives a systematic way of defining a \([-\cdot]_a\) from \([-\cdot]\). For instance, if the language is given by the BNF \(p ::= c | f(p)\), then an interpretation \([-\cdot]\) in \(X\) is uniquely determined by \([c]\) : \(X\) and a monotonic map \([f] : X \to X\), and the abstract interpretation in \(X_a\) (computing over-approximations) is determined by taking \([c]_a = \alpha([c]) : X_a\) and \([f]_a = \alpha [f] \circ \gamma : X_a \to X_a\). In static analysis, the choice of \(X_a\) and \([-\cdot]_a\) is a matter of trade-offs between the cost of computing \([p]_a\) and the information provided by \(\gamma([p]_a)\).

**Spaces.** To model more complex systems, e.g., continuous or hybrid [13], one may have to replace sets with more complex spaces. For instance, in [16], hybrid systems are modelled by triples \((S, F, G)\), where \(S\) is a Banach
space, and $F$ and $G$ are binary relations on $S$—called flow and jump relation, respectively—which constrain how the system may evolve continuously (in time) and discontinuously (instantaneously). As in transition systems, the relations $F$ and $G$ allow to model known unknowns. Despite the increased complexity of models, it is still possible and useful to move to the category of complete lattices and monotonic maps and use it for analyses and abstract interpretations of these more complex systems.

**Imprecision.** In defining reachability for hybrid systems, we realized the need to cope with imprecision (see also [10]). Thus, in [10], we introduced safe and robust reachability analysis. In [17], we use metric spaces to formalize the notions of imprecision and robust analysis. In a metric space $S$, a level of imprecision $\delta > 0$ means that one cannot distinguish two points $s$ and $s'$ when their distance $d(s, s')$ is less than $\delta$. If one considers subsets instead of points, and allows $\delta$ to become arbitrarily small, then one cannot distinguish two subsets that have the same closure, namely, $\forall \delta > 0. B(S, \delta) = B(\overline{S}, \delta)$, where $B(S, \delta)$ is the open subset $\{s' \mid \exists s : d(s, s') < \delta\}$ and the closure $\overline{S}$ is the smallest closed subset containing $S$. Therefore, one can replace $P(S)$ with the complete lattice $\mathcal{C}(S)$ of closed subsets, which is related to the former by the adjunction $\mathcal{C}(S) \xleftarrow{\alpha} P(S)$, with $\gamma$ an inclusion map and $\alpha$ a surjective map. This replacement is convenient, since the cardinality of $\mathcal{C}(S)$ can be smaller than that of $P(S)$, e.g., when $S$ is the real line $\mathbb{R}$.

**Robustness.** A monotonic map $A : P(S) \to P(S')$, with $S$ and $S'$ metric spaces, is robust when small input changes cause small output changes, i.e., $\forall s \in P(S). \forall \epsilon > 0. \exists \delta > 0. A(B(S, \epsilon)) \subseteq A(B(S, \delta))$.

When $A$ is robust, there is no loss of information in restricting to closed subsets, namely, there exists a unique monotonic map $A_c : \mathcal{C}(S) \to \mathcal{C}(S')$ such that $A_c \circ \alpha = \alpha \circ A$. Thus, the focus of [10] [17] is on analyses between complete lattices of closed subsets. [17] identifies sufficient (and almost necessary) conditions to ensure that every monotonic map $A : \mathcal{C}(S) \to \mathcal{C}(S')$ has a best robust approximation, i.e., the biggest robust map $\square_R(A) : \mathcal{C}(S) \to \mathcal{C}(S')$ such that $\square_R(A) \leq A$ in the lattice of monotonic maps with the point-wise order.

**Scott continuity.** We refer to [11] for the definitions of Scott continuous map, way-below relation $\ll$, and continuous lattice. Restricting to compact metric spaces is mathematically appealing, since in this case a monotonic map $A : \mathcal{C}(S) \to \mathcal{C}(S')$ is robust exactly when it is Scott continuous, and the complete lattices $\mathcal{C}(S)$ and $\mathcal{C}(S')$ are $\omega$-continuous [10].

A complete lattice $X$ is $\omega$-continuous when it has a countable base $B$, i.e., a countable subset of $X$ such for every $x \in X$, the subset $B_x := \{b \in B \mid b \ll x\}$ is directed and $x = \sup B_x$. Moreover, by fixing an enumeration $e$ of the base, one can define when an element $x : X$ is computable, namely, when the set $\{n \mid e(n) \in B_x\}$ is a recursively enumerable subset of $\mathbb{N}$. The notion of computable can be extended to Scott continuous maps between $\omega$-continuous lattices because these maps, under the point-wise order, form an $\omega$-continuous lattice.

Ideally, we would like to focus on computable analyses, but we settle for the broader class of Scott continuous analyses, since they are better behaved. For instance, every monotonic map $A : X \to X'$ between complete lattices has a best Scott continuous approximation $\sqcap_S(A) : X \to X'$, while there is no best computable approximation of a monotonic map between (effectively given) $\omega$-continuous lattices.

**Related results.** In [16], we define robustness as a property of analyses, i.e., monotonic maps $A : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$, where $S_i$ are metric spaces, and $\mathcal{C}(S_i)$ are the complete lattices of closed subsets of $S_i$, ordered by reverse inclusion. In the same paper, it was proven that:

- Robustness of $A$ amounts to continuity with respect to suitable $T_0$-topologies $\tau_R(S_i)$ on the carrier sets of $\mathcal{C}(S_i)$, called robust topologies (see Definition 2.20). In general, these topologies depend on the metric structures of $S_i$.
- When $S_i$ is compact, the topology $\tau_R(S_i)$ coincides with the Scott topology $\tau_S(S_i)$ on $\mathcal{C}(S_i)$. Thus, in this case, $\tau_R(S_i)$ depends only on the topology induced by the metric structure on $S_i$.

In particular, when both $S_1$ and $S_2$ are compact, robustness and Scott continuity are equivalent properties of $A$, and the complete lattices $\mathcal{C}(S_i)$ are $\omega$-continuous. In [17], we prove that every analysis $A : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ has a best robust approximation $\square_R(A)$, when $S_2$ is compact, with $\square_R(A)(C)$ given by $\bigcap\{A(C_\delta) \mid \delta > 0\}$, where the closed subset $C_\delta := \overline{B(C, \delta)}$ is called $\delta$-fattening of $C$. When $S_1$ is not compact, however, $\square_R(A)$ may fail to be Scott continuous, and $\mathcal{C}(S_1)$ may fail to be $\omega$-continuous.

**Motivating examples.** Examples of metric spaces that are not compact are Banach Spaces. In applications, one usually considers closed bounded subsets of Banach spaces. In finite-dimensional Banach spaces, all closed bounded subsets are compact, but this fails in the infinite-dimensional case. To motivate the need to go beyond
compact subsets, we present some examples of closed bounded subsets of infinite-dimensional Banach spaces that are not compact:

- Probability distributions for a system with a countable set of states form a closed bounded subset of $\ell_1$, i.e., the Banach space of sequences $(x_n | n : \omega)$ in $\mathbb{R}^\omega$ such that $\sum_{n=\omega}^\omega |x_n|$ is bounded.

More generally, probability distributions on a measurable space $(X, \Sigma)$ form a closed bounded subset of $ca(\Sigma)$, i.e., the Banach space of countably additive bounded signed measures on $X$. This subset is not compact when the cardinality of $\Sigma$ is infinite.

- Continuous maps from a compact Hausdorff space $X$ to a compact interval $[a, b]$ in $\mathbb{R}$ form a closed bounded subset of $C(X)$, i.e., the Banach space of continuous maps from $X$ to $\mathbb{R}$. For instance, these maps could represent the height as a function of the position.

- Closed bounded subsets of feature spaces arising from kernel methods in machine learning [14]. Usually feature spaces are Hilbert spaces, whose carrier sets consist of real-valued maps. All Hilbert spaces with a countable orthonormal base are isomorphic to $\ell_2$, i.e., the Hilbert space of sequences $(x_n | n : \omega)$ in $\mathbb{R}^\omega$ such that $\sum_{n=\omega}^\omega |x_n^2|$ is bounded.

- Closed bounded subsets of Sobolev spaces $W^{m,p}(\Omega)$, in which $\Omega \subseteq \mathbb{R}^n$ is an open set. These sets commonly appear in solution of partial differential equations [3].

**Contribution.** For simplicity, we consider analyses of the form $A : C(S) \rightarrow C(1)$, with 1 denoting the one-point metric space, although the results hold also when 1 is replaced by a compact metric space, e.g., a compact interval $[a, b]$ of the real line. $C(1)$ is (isomorphic to) the two-point lattice $\Sigma$ and the complete lattice $C(S) \rightarrow \Sigma$ is isomorphic to that of upward closed subsets of $C(S)$, ordered by inclusion.

This paper proposes a way to reconcile robustness and Scott continuity when $S$ is not compact. The general idea is to construct an $\omega$-continuous lattice $D$ related to $C(S)$ by an adjunction

$$C(S) \xrightarrow{\iota} D$$

such that the composite map $A' \circ \iota : C(S) \rightarrow \Sigma$ is robust whenever the map $A' : D \rightarrow \Sigma$ is Scott continuous. Therefore, given an analysis $A : C(S) \rightarrow \Sigma$, we can take the best Scott continuous approximation $A' = A \circ \iota^* : D \rightarrow \Sigma$—in fact, any Scott continuous approximation will do—and the composite map $A' \circ \iota^*$ is guaranteed to be a robust approximation of $A$.

The $\omega$-continuous lattice $D$ that we construct is of the form $C(\overline{S})$, where $\overline{S}$ is a compact Hausdorff space given by the limit of an $\omega^\theta$-chain of compact metric spaces related to $S$ (see Theorem 3.9), and the adjunction $\iota_\sigma \sim \iota^*$ is determined by a continuous map $\iota : S \rightarrow \overline{S}$. Thus, by moving from $S$ to $\overline{S}$, we gain compactness by giving up the metric structure.

In general, $\overline{S}$ is not uniquely determined by $S$, although Theorem 3.10 provides some criteria to choose the $\omega^\theta$-chain of compact metric spaces which determines $\overline{S}$.

**Summary**

The rest of the paper is organized as follows:

- Section 2 contains the mathematical preliminaries, where we fix notation and definitions. We will also present some basic results, usually without proofs, unless the results are not available in textbooks, in which case we provide proofs or pointers to other papers which include the relevant proofs. Most definitions are standard or taken from other papers. The only exception is the category $\text{Top}_A$ of topological analyses (Definition 2.4).

- The main theoretical results are in Section 3. These include properties of idempotents and their splittings in a generic category $A$ (e.g., Theorem 3.6) and the construction (Theorem 3.9) of a continuous map $\iota : S \rightarrow \overline{S}$ relating a metric space $S$ to a compact Hausdorff space $\overline{S}$.

- In Section 4 we apply the results of Section 3 to several examples of $S$, namely: finite-dimensional Banach spaces $\ell_m,p$, infinite dimensional Banach spaces $\ell_p$ (i.e., sequence spaces), and closed unit balls $B_p$ in sequence spaces.

- In Section 5 we investigate loss of precision when moving from the complete lattice $C(S)$ to the $\omega$-continuous lattice $C(\overline{S})$, when $S$ is a closed unit ball $B_p$. In Theorem 5.3 we characterize the closed subsets of $\ell_p$ (with $1 < p < \infty$) for which there is no loss of precisions as those that can be expressed as a non-empty intersections of finite unions of closed balls.

- We conclude the paper with some remarks and suggestions for future work in Section 6.
Table 1 Categories of Spaces

| Ban | ← − | NVS | normed vector space |
| KMS | ← − | CMS | ← − | Met | (extended) metric space |
| KH | ← − | Haus | ← − | Top₀ | topological space |

The notation $\longrightarrow$ denotes a faithful forgetful functor, $\longleftarrow$ denotes the inclusion functor from a full subcategory, and $\top$ indicates the existence of a left adjoint to a functor. The left adjoints from top to bottom and left to right are: the Cauchy completion $\overline{X}$ (for $X$ in NVS or Met), the Stone-Cech compactification $\beta X$ (for $X$ in Haus), the Hausdorff reflection $HX$ (for $X$ in $\text{Top}_0$), the Discrete topology $DX$ (for $X$ in Set).

2 Mathematical Preliminaries

In this section, we present the basic technical background—including the notation—that will be used throughout the paper. We assume standard terminology for topological and metric spaces. At times, we may refer to a structure by its carrier set. For instance, for a metric space $(S, d)$, we may simply write the metric space $S$.

We use both $\sigma$ and $\tau$ to denote membership. A natural number is identified with the set of its predecessors, i.e., $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$, for any $n \geq 1$. We write $\mathbb{N}$ or $\omega$ for the set of natural numbers. When the order matters we denote the set of natural numbers ordered by inclusion, while $\mathbb{N}$ denotes the set of the natural numbers with the discrete order. We write $(x_n : n \in \omega)$ to denote a countable sequence, and when the indexing set is clear from the context, we just write $(x_n : n)$.

The power set of a set $X$ is denoted by $P(X)$, $\subseteq$ denotes subset inclusion, and $\subset$ denotes strict subset inclusion, i.e., $A \subset B \iff A \subseteq B \land A \neq B$. Similarly, the finite power set (i.e., the set of finite subsets) of $X$ is denoted by $P_f(X)$, and $A \subseteq_f B$ denotes that $A$ is a finite subset of $B$.

2.1 Categories of Spaces

The spaces of interest for this paper are (extended) metric spaces and Hausdorff spaces. However, in examples we restrict to Banach spaces, and some constructions extend to arbitrary topological spaces. Table 1 summarizes the relations among the following categories of spaces.

**Definition 2.1 (Categories of Spaces).**

- $\text{Top}_0$ is the category of $T_0$-topological spaces $(X, \tau)$ and continuous maps. $\text{Haus}$ and $\text{KH}$ are the full sub-categories consisting of Hausdorff spaces (aka $T_2$-spaces) and compact Hausdorff spaces, respectively.
- $\text{Met}$ is the category of extended metric spaces $(X, d)$, i.e., the distance $d$ can be $\infty$, and short maps, i.e., maps $f : X_1 \to X_2$ such that $d_2(f(x), f(x')) \leq d_1(x, x')$ for $x, x' : X_1$. There are other maps one can consider between (extended) metric spaces, in particular isometries, i.e., distance preserving maps. The forgetful functor $U : \text{Met} \longrightarrow \text{Haus}$ maps a distance $d$ on $X$ to the $T_2$-topology $\tau_d$ on $X$ generated by the open balls. $\text{CMS}$ and $\text{KMS}$ are the full sub-categories of $\text{Met}$ consisting of Cauchy complete extended metric spaces and compact extended metric spaces, respectively. The objects in $\text{KMS}$ are exactly the extended metric spaces whose underlying topological spaces are compact.
- $\text{NVS}$ is the category of normed vector spaces $(X, \cdot, +, \| - \|)$ and short linear maps. The forgetful functor $U : \text{NVS} \to \text{Met}$ maps a normed vector space to the metric space with (the same carrier and) distance $d(x', x) \triangleq \| x' - x \|$. $\text{Ban}$ is the full sub-category of $\text{NVS}$ consisting of Banach spaces. The objects in $\text{Ban}$ are exactly the normed vector spaces whose underlying metric spaces are complete.

The following theorems recall some properties of categories and functors in Table 1.

**Theorem 2.2.** The categories in the following diagram have finite limits and finite sums, and the functors preserve them:

\[ \text{KMS} \longrightarrow \text{CMS} \longrightarrow \text{Met} \longrightarrow \text{Haus} \]

1 We use extended metric spaces because they have better category-theoretic properties.
Table 2 Poset-enriched categories

| Poset-enriched categories |
|---------------------------|
| $\text{Top}_A$ | $\text{Top}_0$ | topologies |
| $\xrightarrow{A}$ | $\xrightarrow{U}$ |
| $\text{Po}_A$ | $\text{Po}$ | posets |

The notation $\rightarrow$ denotes a faithful forgetful functor, $\hookrightarrow$ denotes the inclusion functor from a full sub-category, and $\dashv$ indicates the existence of a left adjoint to a functor.

The categories $\text{CMS}$, $\text{Met}$ and $\text{Haus}$ have also small limits and small colimits.

For the existence of sums and infiniitary products it is essential to use extended metric spaces.

Theorem 2.3. The categories in the following diagram have small limits and finite sums, and the functors preserve them:

$$\text{KH} \hookrightarrow \text{Haus} \rightarrow \text{Top}_0$$

The categories have also small colimits.

2.2 Categories of Analyses

We define an analysis as a monotonic map between complete lattices. However, we need to consider further properties of analyses, that (with the exception of computability) can be defined as continuity with respect to suitable topologies on (the carrier set of) complete lattices. For this reason, we introduce the category $\text{Top}_A$ of topological analyses, which refines the category $\text{Po}_A$ of analyses (see Table 2).

Definition 2.4 (Category of Analyses).

- $\text{Po}$ is the category of posets and monotonic maps.
- The forgetful functor $U: \text{Top}_0 \rightarrow \text{Po}$ maps a $T_0$-topology $\tau$ on $X$ to the specialization order $\leq_\tau$ on $X$, i.e., $x \leq_\tau y$ is equivalent to $\forall O: \tau. x \in O \implies y \in O$.
- The inclusion functor $A: \text{Po} \rightarrow \text{Top}_0$ maps a poset $\leq$ on $X$ to the Alexandrov topology $\tau_\leq$ on $X$ consisting of the upward closed subsets, i.e., $O \in \tau_\leq$ if $\forall x, y: X. x \in O \land x \leq y \implies y \in O$.
- $\text{Po}_A$, the category of analyses, is the full sub-category of $\text{Po}$ consisting of complete lattices.
- $\text{Top}_A$, the category of topological analyses, is the full sub-category of $\text{Top}_0$ consisting of $T_0$-spaces whose specialization order is a complete lattice.

Theorem 2.5.

1. The categories $\text{Po}$ and $\text{Top}_0$ are $\text{Po}$-enriched and have small limits and small colimits.
2. The functors $U$ and $A$ are $\text{Po}$-enriched, and $A$ is left adjoint to $U$.
3. The functor $U$ preserves small limits and small sums.
4. The functor $A$ preserves finite limits and small colimits.

Proof. The $\text{Po}$-enrichment of $\text{Po}$ is given by its cartesian closed structure. Since $U$ is faithful, $\text{Top}_0(X, Y)$ is a subset of $\text{Po}(UX, UY)$ and inherits the $\text{Po}$-enrichment. It is easy to prove that $\text{Top}_0(AX, Y) = \text{Po}(X, UY)$. Thus, $A$ is left adjoint to $U$ also as $\text{Po}$-enriched functors.

Corollary 2.6.

1. The category $\text{Po}_A$ is $\text{Po}_A$-enriched and has small products.
2. The category $\text{Top}_A$ is $\text{Po}$-enriched and has small products.
3. The $\text{Po}$-enriched functors $U$ and $A$ restrict to functors between $\text{Top}_A$ and $\text{Po}_A$.

Proof. The $\text{Po}_A$-enrichment of $\text{Po}_A$ is given by its cartesian closed structure. The other claims are easy consequences of Theorem 2.5 and the definition of $\text{Top}_A$.  


Theorem 2.7 (Topologies on a poset). Given a partial order \( \leq \) on \( X \), the set of \( T_0 \)-topologies on \( X \) with specialization order \( \leq \) ordered by reverse inclusion forms a complete lattice \( \text{Top}(\leq) \), where:

- the least element \( \tau_1 \) is the Alexandrov topology \( \tau_\leq \).
- the top element \( \tau_\tau \) is the topology generated by the set \( \{ y \mid y : X \} \), where \( \{ y \mid x \not\leq y \} \).
- the nonempty sups are given by intersection.

Moreover, when \( X \) is finite \( \text{Top}(\leq) \) is trivial, i.e., \( \tau_1 = \tau_\tau \).

Proof. It is easy to show that \((X, \tau_\tau)\) and \((X, \tau_1)\) are \( T_0 \)-spaces with specialization order \( \leq \). Given a \( T_0 \)-topology \( \tau \) on \( X \) with specialization order \( \leq \) we have \( \tau_1 \subseteq \tau \subseteq \tau_\tau \), because:

- each \( O : \tau \) is upward closed, by definition of \( \leq_\tau \),
- each \( \{ y \mid y \not\leq \} \) is in \( \tau \), because \( \{ y \mid y \not\leq \} = \bigcup \{ O : \tau \mid y \not\in O \} \).

Therefore, \( \tau_\tau \) and \( \tau_1 \) are respectively the top and bottom element in \( \text{Top}(\leq) \). Since the topologies on a set \( X \) (ordered by reverse inclusion) form a complete lattice, with (nonempty) sups given by intersections, so do the topologies \( \tau \) on \( X \) such that \( \tau_1 \subseteq \tau \subseteq \tau_\tau \). Moreover, such topologies are \( T_0 \), because \( \tau_\tau \) is.

For every \( x : X \) we have \( \top x = \bigcap \{ \{ y \mid y : X \land x \not\leq y \} \) when \( X \) is finite, the rhs of the equality is a finite intersection of open sets in \( \tau_\tau \), thus \( \top x \in \tau_\tau \), and therefore \( \tau_1 \subseteq \tau_\tau \).

When \( \leq \) is a complete lattice (i.e., an object in \( \text{Po}_A \)), the topologies in \( \text{Top}(\leq) \) are objects in \( \text{Top}_A \).

2.3 Adjunctions and Best Approximations

A key property of categories of analyses is poset-enrichment, which provides a qualitative criterion for comparing analyses between two complete lattices, and allows to define adjunctions (aka Galois connections) between two complete lattices.

Definition 2.8 (Adjunction). An adjunction in a \( \text{Po} \)-enriched category \( \mathcal{A} \), notation \( f \dashv g \), is a pair of maps \( X \leftarrow f \rightarrow Y \) in \( \mathcal{A} \) such that \( f \circ g \leq \text{id}_Y \) and \( g \circ f \geq \text{id}_X \). The maps \( f \) and \( g \) are called left- and right adjoint, respectively. Moreover, any one of these two maps uniquely determines the other.

Remark 2.9. A characterization of adjunctions in \( \text{Po} \) is \( f \dashv g \) iff \( \forall x : X, y : Y. x \leq_X y \Leftrightarrow f(x) \leq_Y y \). This characterization implies that in \( \text{Po} \) left adjoints preserve sups and (dually) right adjoints preserve infs.

Theorem 2.10. The \( \text{Po} \)-enriched categories \( \text{Po}_A \) and \( \text{Top}_A \) have limits of \( \omega^A \)-chains of right adjoints.

Proof. First, we prove the property for \( \text{Po}_A \). Given an \( \omega^A \)-chain \( \{ p_n : D_{n+1} \rightarrow D_n \mid n \} \) of right adjoints in \( \text{Po}_A \), its limit in \( \text{Po} \) is the subset of \( \Pi_nD_n \) given by \( \{ d \mid \forall n.d_n = p_n(d_{n+1}) \} \) with the point-wise order \( \leq_D \). Since right adjoints preserve infs, infs in \( D \) exist and are computed point-wise. Thus, \( D : \text{Po}_A \) and the maps \( \pi_n : D \rightarrow D_n \) with \( \pi_n(d) = d_n \) form a limit cone (and preserve infs).

Given an \( \omega^A \)-chain \( \{ p_n : D_{n+1} \rightarrow D_n \mid n \} \) of right adjoints in \( \text{Top}_A \), its limit in \( \text{Top}_0 \) is the sub-space of \( \Pi_nD_n \) corresponding to the subset \( \{ d \mid \forall n.d_n = p_n(d_{n+1}) \} \), and the maps \( \pi_n : D \rightarrow D_n \) with \( \pi_n(d) = d_n \) form a limit cone. Since \( U : \text{Top}_0 \rightarrow \text{Po} \) is \( \text{Po} \)-enriched and preserves limits, we have that \( (Up_n \mid n) \) is an \( \omega^A \)-chain of right adjoints in \( \text{Po}_A \) and \( (U\pi_n \mid n) \) is a limit cone in \( \text{Po} \). By the result for \( \text{Po}_A \), we have \( UD : \text{Po}_A \). Hence, \( D : \text{Top}_A \).

A similar result holds if right adjoints are replaced by left adjoints (i.e., \( \text{Po}_A \) and \( \text{Top}_A \) have limits of \( \omega^A \)-chains of left adjoints). We recall further properties of adjunctions in \( \text{Po} \)-enriched categories (and in \( \text{Po} \)). Each of these properties has a dual, that we do not state explicitly.

Proposition 2.11. If \( f \dashv g \) in a \( \text{Po} \)-enriched category \( \mathcal{A} \) and \( f : X \rightarrow Y \) is mono, then \( g \circ f = \text{id}_X \).

Proof. Since \( f \dashv g \) one has \( f \circ g \circ f = f \). When \( f \) is mono, this implies \( g \circ f = \text{id}_X \).
\[ x \leq g(y) \implies \text{because } f \text{ preserves sups} \]
\[ f(x) \leq_Y f(g(y)) = \sup(f(x) \mid f(x) \leq_Y y) \leq_Y y. \]
The right-to-left implication follows from:
\[ f(x) \leq_Y y \implies \text{by definition of } g \]
\[ x \leq \sup\{z \mid f(z) \leq_Y y\} = g(y). \]

**Proposition 2.13.** If \( Y \) is a complete lattice and \( X \) is a sub-poset of \( Y \), then the inclusion \( f : X \to Y \) is a left adjoint in \( \text{Po} \) iff \( X \) is a complete lattice and sups in \( X \) are computed as in \( Y \) (i.e., \( f \) preserves sups).

**Proof.** The right-to-left implication follows from Proposition 2.12. For the other implication, consider the right adjoint \( g \) to \( f \). By Remark 2.12, \( f \) preserves sups and \( g \) preserves infs. Moreover, \( X \) has all infs (i.e., is a complete lattice), because \( g \circ f = \text{id}_X \) (by Proposition 2.11) and \( \inf D = g(\inf(D)) \) for any subset \( D \) of \( X \).

**Definition 2.14** (Best Approximation). Given a subset \( X \) of a poset \( Y \), \( x : X \) is the **best \( X \)-approximation** of \( y : Y \) iff \( \forall x' : X. x' \preceq x \iff x' \preceq y \).

A subset \( X \) of a poset \( Y \) can be identified with the sub-poset of \( Y \) with carrier \( X \) and the partial order inherited from \( Y \). Then, the inclusion \( f : X \to Y \) is a left adjoint in \( \text{Po} \) exactly when every \( y : Y \) has a best \( X \)-approximation, and the right adjoint to \( f \) maps \( y \) to its best \( X \)-approximation. When \( Y \) is a complete lattice, Proposition 2.13 characterizes the subsets \( X \) of \( Y \) for which every \( y : Y \) has a best \( X \)-approximation.

We are interested in best \( X \)-approximations of analyses in \( \text{Po}_A(\leq_1, \leq_2) \) for \( X \) of the form \( \text{Top}_A(\tau_1, \tau_2) \), where \( \leq_i \) is the specialization order of the topology \( \tau_i \), i.e., \( \leq_i = U(\tau_i) \). For existence of best \( X \)-approximations, the poset \( \text{Top}_A(\tau_1, \tau_2) \) must be a complete lattice with sups computed as in \( \text{Po}_A(\leq_1, \leq_2) \).

**Example 2.15.** Given a complete lattice \( \leq \) (on a set \( X \)), we can define two topologies in \( \text{Top}(\leq) \):

1. The Scott topology \( \tau_S(\leq) \) consists of upward closed subsets \( O \) of \( X \) such that \( \sup D \in O \implies \exists d : D.d \in O \) for any directed subset \( D \) of \( X \), equivalently, \( \sup S \in O \implies \exists S_0 \subseteq f.S. \sup S_0 \in O \) for any subset \( S \) of \( X \).
2. The \( \omega \)-topology \( \tau_\omega(\leq) \) consists of upward closed subsets \( O \) of \( X \) such that \( \sup D \in O \implies \exists d : D.d \in O \) for any directed \( \omega \)-chain \( D \) in \( X \), equivalently, \( \sup S \in O \implies \exists S_0 \subseteq f.S. \sup S_0 \in O \) for any countable subset \( S \) of \( X \).

Clearly, \( \tau_S(\leq) \subseteq \tau_\omega(\leq) \). Given a map \( f : \text{Po}_A(\leq_1, \leq_2) \), there is an order-theoretic characterization of continuity with respect to these two topologies, namely:

- **f** is Scott continuous, i.e., \( f : \text{Top}_A(\tau_S(\leq_1), \tau_S(\leq_2)) \), exactly when \( f \) preserves sups of directed sets.
- **f** is \( \omega \)-continuous, i.e., \( f : \text{Top}_A(\tau_\omega(\leq_1), \tau_\omega(\leq_2)) \), exactly when \( f \) preserves sups of \( \omega \)-chains.

These characterizations imply that \( \text{Top}_A(\tau_S(\leq_1), \tau_S(\leq_2)) \subseteq \text{Top}_A(\tau_\omega(\leq_1), \tau_\omega(\leq_2)) \subseteq \text{Po}_A(\leq_1, \leq_2) \) and that these subsets are closed with respect to sups computed in \( \text{Po}_A(\leq_1, \leq_2) \). Therefore, every analysis \( f : \text{Po}_A(\leq_1, \leq_2) \) has a best Scott-continuous approximation \( \sqcap_S(f) \), and a best \( \omega \)-continuous approximation \( \sqcap_\omega(f) \), with \( \sqcap_S(f) \leq \sqcap_\omega(f) \leq f \).

We introduce two sub-categories of \( \text{Po}_A \), related to the example above, which can be viewed also as full sub-categories of \( \text{Top}_A \), by mapping a complete lattice with order \( \leq \) to the Scott topology \( \tau_S(\tau) \).

**Definition 2.16** (Category of Continuous Lattices). A poset \( \leq \)

- **CL** is the category of continuous lattices, i.e., every element \( x \) in the lattice is the sup of the directed set formed by the elements way-below \( x \), and Scott continuous maps.
- **\omega CL** is the full sub-category of **CL** whose objects are \( \omega \)-continuous lattices, i.e., continuous lattices with a countable subset \( B \) (called a base) such that every element \( x \) in the lattice is the sup of the directed set formed by the elements in the base way-below \( x \).

**Proposition 2.17.** The \( \text{Po} \)-enriched categories in the following diagram have finite products and limits of \( \omega \)-chains of right adjoints and the functors preserve them:

\[
\omega \text{CL} \longrightarrow \text{CL} \longrightarrow \text{Top}_A \longrightarrow \text{Po}_A.
\]

Moreover, the categories \( \omega \text{CL} \) and \( \text{CL} \) have exponentials and the functor \( \omega \text{CL} \longrightarrow \text{CL} \) preserves them, and every \( \omega \)-continuous map between \( \omega \)-continuous lattices is necessarily Scott continuous.

**Proof.** For finite products in \( \omega \text{CL} \) and \( \text{CL} \), see [1 Proposition 3.2.4]. Exponentials in \( \text{CL} \) are discussed in [12 Section II-4]. Scott continuity of \( \omega \)-continuous maps between \( \omega \)-continuous lattices is proven in [1 Proposition 2.2.14].
2.4 From Spaces to Complete Lattices

Given a topological space $S$, the set of closed subsets of $S$, ordered by reverse inclusion, forms a complete lattice $C(S)$, with sups given by intersection. We introduce several topologies on these complete lattices, but first we give the main properties of $C$ as a functor from $\text{Haus}$ to $\text{Po}_A$.

**Definition 2.18.** The functor $C : \text{Haus} \longrightarrow \text{Po}_A$ is defined as follows:

- $C(S)$ is the complete lattice of closed subsets of $S$ under reverse inclusion.
- If $f : \text{Haus}(S, S')$, then the map $C(f) \triangle f_* : \text{Po}_A(C(S), C(S'))$ is given by $f_*(C) = \overline{f(C)}$, i.e., it maps $C$ to the closure of the image of $C$ along $f$.

There is also a contravariant version, whose action on maps $f^* : \text{Po}_A(C(S'), C(S))$ is given by $f^*(C') = f^{-1}(C')$, i.e., it maps $C'$ to the inverse image of $C'$ along $f$.

**Theorem 2.19.** If $S \rightarrow S'$ in $\text{Haus}$, then $f^* = f_*$ in $\text{Po}_A$.

**Proof.** See [16, Example 5.5].

**Definition 2.20.** (Topologies on $C(S)$ [16]).

1. Given a Hausdorff space $S$, we write $O(S)$ for the set of open subsets of $S$, $C(S)$ for the set of closed subsets of $S$, and $\uparrow S$ for the set of $C : C(S)$ such that $C \subseteq S$, where $S$ is a subset of $S$. The **Upper topology** $\tau_U(S)$ is the topology on $C(S)$ such that $U : \tau_U(S) \triangleleft \forall C : U.\exists O(S)C : \uparrow O \subseteq U$.

2. Given an extended metric space $S$, we write $B(x, \delta) : O(S)$ for the open ball with center $x : S$ and radius $\delta > 0$, and $B(S, \delta) : O(S)$ for the union of open balls $B(x, \delta)$ with $x : S$, where $S$ is a subset of $S$. The **Robust topology** $\tau_R(S)$ is the topology on $C(S)$ such that $U : \tau_R(S) \triangleleft \forall C : U.\exists \delta > 0.\exists C : B(C, \delta) \subseteq U$.

3. For uniformity we write $\tau_S(S)$ and $\tau_A(S)$ for the Scott and Alexandrov topologies on $C(S)$, and for conciseness we write $A_{XY}(S_1, S_2)$ for the poset $\text{Top}_A(\tau_X(S_1), \tau_Y(S_2))$.

We call Upper topology what is better known as upper Vietoris topology.

**Theorem 2.21.**

1. If $S : \text{Haus}$, then $\tau_U(S)$ is in $\text{Top}(\leq)$, where $\leq$ is the partial order on $C(S)$.

2. If $S : KH$, then $C(S) : \text{CL}$ and $\tau_U(S) = \tau_S(S)$.

3. If $S : \text{Met}$, then $\tau_S(S) \subseteq \tau_R(S) \subseteq \tau_U(S)$. Therefore, $\tau_S(S) = \tau_R(S) = \tau_U(S)$ when $S : \text{KMS}$.

4. If $S : \text{KMS}$ is finite, then $C(S)$ is finite and $\tau_S(S) = \tau_A(S)$.

**Proof.**

1. We prove that $\tau_U(S) \subseteq \tau_R(S) \subseteq \tau_A(S)$ (see Theorem 2.27). $\tau_U(S) \subseteq \tau_A(S)$, because the open subsets of the upper topology are upward closed with respect to reverse inclusion.

To prove $\tau_U(S) \subseteq \tau_A(S)$ we show that $C \triangle {C' \in C(S) | C \notin C'}$ is in $\tau_U(S)$ when $C : C(S)$. If $C \notin C'$, then there exists $x$ in $C - C'$. But every singleton is closed in $S$ (because $S : \text{Haus}$), thus the complement $O$ of the singleton $\{x\}$ is open and $C' \uparrow O \subseteq \overline{C}$.

2. Follows from [8, Proposition 3.3].

3. Follows from [16, Lemma A.3].

4. If $S$ is finite, then $C(S)$ is also finite. Hence, $\text{Top}(\leq)$ contains only one topology.

**Theorem 2.22.** If $f : \text{Haus}(S, S')$, then $f^* : A_{SS}(S, S')$ and $f_* : A_{UV}(S, S')$.

**Proof.** Since the map $f^*$ is a left adjoint in $\text{Po}_A$, it preserves all sups. Thus, it is Scott continuous. The map $f_*$ is Upper continuous, because for any $C : C(S)$ and $O' : O(S)$ we have $f(C) \subseteq O' \implies f(C) \subseteq O \implies C \subseteq f^{-1}(O')$ and $f^{-1}(O') : O(S)$ by continuity of $f$.

**Theorem 2.23.** The functor $C : \text{Haus} \longrightarrow \text{Po}_A$ restricted to $\text{KH}$ factors through $\text{CL}$, and when restricted to $\text{KMS}$ factors through $\omega \cdot \text{CL}$.
Proof. From Theorem 2.21 we know that for any $X : KH$, the lattice $C(X)$ is continuous. The fact that for any $X : KMS$, the lattice $C(X)$ is $\omega$-continuous is a straightforward consequence. It remains to show that, if $f : KH(X,Y)$, then $f_*$ is Scott continuous. But this also follows from item 2 of Theorem 2.21 and Theorem 2.22.

$$\square$$

Theorem 2.24. The functor $C : KH \to \text{Po}_A$ preserves limits of $\omega^p$-chains.

Proof. Given an $\omega^p$-chain $(p_n : X_{n+1} \to X_n | n)$ in KH, its limit in KH (and Haus) is the sub-space $X$ of $\prod_n X_n$ such that $|X| = \{x | \forall n.x_n = p_n(x_{n+1})\}$. By Theorem 2.24, the limit of $(C(p_n) | n)$ in Po$_A$ (and CL) is the sub-poset $D$ of $\prod_n D_n$ such that $|D| = \{d | \forall n.d_n = (p_n)(d_{n+1})\}$, where $D_n \triangleleft C(X_n)$. But for spaces in KH compact subsets and closed subsets coincide, thus $C(p_n)(C) = p_n(C)$, i.e., the image of $C$ along $p_n$.

By the universal property of $D$ there exists a unique map $\phi : C(X) \to D$ in Po$_A$ such that:

$$C(\pi_n : X \to X_n) = C(X) \to D \to \pi_n \to D_n,$$

namely, $\phi(C) = (\pi_n(C) | n)$ for every $C : C(X)$. Moreover, $\phi$ preserves infs, since the $C(\pi_n)$ are right adjoints and preserve infs. Therefore, $\phi$ has a left adjoint $\phi' : D \to C(X)$, namely $\phi'(d) = \bigcap_n \pi_n^*(d_n)$. We prove that $\phi'$ is the inverse of $\phi$. Because of the adjunction $\phi' \dashv \phi$ we have:

1. $\forall n. \forall d : D. \phi'(d)_n = \pi_n(\bigcap_n \pi_n^*(d_n)) \preceq d_n$, and
2. $\forall C : C(X). C \preceq \phi'(C) = \bigcap_n \pi_n^*(\pi_n(C))$.

Given $d : D$, we have (by the Axiom of Choice) $\forall n. \forall x : d_n. \exists y : d_{n+1}. x = p_n(y)$, which implies $\forall n. \forall x : d_n. \exists y : X.x = y_n \wedge (\forall i. y_i \in d_i)$. But $\{y : X | \forall n.y_n \in d_n\}$ is another way to denote $\bigcap_n \pi_n^*(d_n) : C(X)$, thus the first inclusion is an equality.

For the second inclusion, consider an $x \in \phi'(\phi(C))$, i.e., $\forall n. \forall x \in \pi_n(C)$, we prove that $x \in C$. Since $C$ is closed and a base for the topology on $X$ is given by the subsets $O_n = \{y : X | y_n \in O\}$ with $O : O(X)$, it suffices to prove that $\forall n. \forall O : O(X_n). x \in O_n \iff y_n \in O \iff y \in O$. But $x_n \in \pi_n(C)$ means that $x_n = y_n$ for some $y \in C$, thus $x \in O_n \iff x_n \in O \iff y_n \in O \iff y \in O$.

$$\square$$

3. Main Results

Ideally, given a metric space $S$ (or more generally, an extended metric space), we would like to find a compactification $\overline{S}$ of $S$ such that the complete lattice $C(\overline{S})$ is $\omega$-continuous.2 We establish a weaker result, namely, given an $\omega$-chain $(g_n | n)$ of short idempotents (with certain additional properties) on a metric space $S$, we define a compact Hausdorff space $\overline{S}$ with a countable base and a continuous map $\iota : S \to \overline{S}$ such that the monotonic map $C(\iota) : C(S) \to C(\overline{S})$ is continuous when $C(S)$ is equipped with the Robust topology and $C(\overline{S})$ is equipped with the Scott topology. In general, $\overline{S}$ is not a compactification of $S$, nor is it uniquely determined (up to isomorphism) by $S$, as it depends on the choice of $(g_n | n)$.

The result above follows from Theorem 3.10 which is applicable under more general assumptions than having an $\omega$-chain $(g_n | n)$ of short idempotents. Another result, Theorem 3.12, gives sufficient conditions to ensure that $\iota : S \to \overline{S}$ is both mono and epi, which is as close as we can get to have that $\overline{S}$ is a compactification of $S$.

3.1 Idempotents, Embeddings and Projections

In this section, we establish general properties of idempotents, split monos and split epis, that in this paper we call embeddings and projections, respectively.

Definition 3.1 (idempotents & co). In a category A:

1. $g$ is an idempotent on $X \triangleleft X \ominus g \cong X$ and $g \circ g = g$.

2. given two idempotents $g_1$ and $g_2$ on $X$ we write $g_1 \leq g_2 \triangleleft g_1 \circ g_2 = g_1 = g_2 \circ g_1$.

3. $(e,p)$ is an e-p pair from $X$ to $Y$, notation $(e,p) : X \Longrightarrow Y \triangleleft X \xrightarrow{\leftarrow P \ominus e} Y$ and $p \circ e = id_X$.

4. $(e,p)$ is a splitting of $g \Longleftrightarrow (e,p)$ is an e-p pair and $g = e \circ p$.

2For this, it suffices that the topology on $\overline{S}$ has a countable base.
5. **idempotents split** (in \(\mathcal{A}\)) \(\xrightarrow{\Delta}\) every idempotent (in \(\mathcal{A}\)) has a splitting.

Given an e-p pair \((e, p)\), we call e embedding and p projection. In general, e and p do not determine each other. \(A_{ep}\) denotes the category whose arrows are e-p pairs and composition \((e_2, p_2) \circ (e_1, p_1)\) is given by \((e_2 \circ e_1, p_1 \circ p_2)\). The forgetful functors \(E: A_{ep} \to \mathcal{A}\) and \(P: A_{ep} \to \mathcal{A}^{op}\) map the e-p pair \((e, p)\) to the embedding e and the projection p, respectively.

The notions above are absolute, i.e., they are preserved by functors, because they are defined only in terms of composition and identities. For instance, if \(F: \mathcal{A} \to \mathcal{A}'\) is a functor and g is an idempotent on X in \(\mathcal{A}\), then \(Fg\) is an idempotent on \(FX\) in \(\mathcal{A}'\). As is customary in Category Theory, a definition or result given for a generic category \(\mathcal{A}\) can be recast in the dual category \(\mathcal{A}^{op}\).

**Proposition 3.2** (Duality).

1. The definition of \(g_1 \leq g_2\) is self-dual, i.e., \(g_1 \leq g_2\) in \(\mathcal{A}\) \(\iff\) \(g_1 \leq g_2\) in \(\mathcal{A}^{op}\).

2. A pair \((e, p)\) is an e-p pair from X to Y in \(\mathcal{A}\) \(\iff\) \((p, e)\) is an e-p pair from X to Y in \(\mathcal{A}^{op}\). In particular, the swap functor \(S: \mathcal{A}_{ep} \to (\mathcal{A}^{op})_{ep}\), which maps \((e, p)\) to \((p, e)\), is an isomorphism of categories.

**Proposition 3.3** (Basic facts). In any category \(\mathcal{A}\), the following hold:

1. the relation \(\leq\) is a partial order (on idempotents), i.e., \(g_1 \leq g_2 \leq g_3\) \(\implies\) \(g_1 \leq g_3\).

2. if \((e, p)\) is an e-p pair, then \(g = e \circ p\) is an idempotent.

3. if \((g_i \mid i : I)\) is a family of idempotents on \(X\) which is jointly mono, i.e.,

\[
(\forall i.g_i \circ f = g_i \circ f') \implies f = f',
\]

then its sup is id\(X\). Similarly, when \((g_i \mid i : I)\) is jointly epi, i.e.,

\[
(\forall i.f \circ g_i = f' \circ g_i) \implies f = f',
\]

then its sup is id\(X\).

4. every arrow in \(A_{ep}\) is a mono. Furthermore, \((e, p)\) is an isomorphism in \(\mathcal{A}_{ep}\) \(\iff\) p is the inverse of e in \(\mathcal{A}\).

5. if \((e_i, p_i): X_1 \to Y\) is a splitting of \(g_i\) (for \(i = 1, 2\)), then \(g_1 \leq g_2\) \(\iff\) there exists (necessarily unique) \((e, p): X_1 \to X_2\) such that \((e_1, p_1) = (e_2, p_2) \circ (e, p)\).

**Proof.** The proofs for Items 1 and 2 are straightforward. For the other items, we have:

**Item 3** Consider an idempotent \(g\) of \(X\) which is an upper-bound of \((g_i \mid i)\) (i.e., \(\forall i.g_i \leq g\)). Then, \(\forall i.g_i \circ g = g_i\), which implies \(g = \text{id}_X\), because \((g_i \mid i)\) is jointly mono.

**Item 4** To prove that every arrow \((e, p): X \to Y\) in \(A_{ep}\) is a mono it suffices to observe that in \(\mathcal{A}\) every embedding e is mono (i.e., \(e \circ f_1 = e \circ f_2 \implies f_1 = f_2\)) and every projection p is epi (the dual of mono). If \((e', p')\) is the inverse of \((e, p)\) in \(A_{ep}\), then \(e'\) is the inverse of \(e\) in \(\mathcal{A}\), thus \(p = p \circ e \circ e' = e'\), because \(p \circ e = \text{id}_X\).

**Item 5** Uniqueness of \((e, p)\) is immediate, because \((e_2, p_2)\) is a mono in \(A_{ep}\). For existence, define \(e = p_2 \circ e_1\) and \(p = p_1 \circ e_2\). Then, from the assumption \(e_1 \circ p_1 = g_1 \leq g_2 = e_2 \circ p_2\) we derive:

1. \(p \circ e = \text{id}_{X_1}\) (i.e., \((e, p)\) is an e-p pair), because

\[
\begin{align*}
(p_1 \circ e_2) \circ (p_2 \circ e_1) &= \text{by definition of } g_2 \\
p_1 \circ g_2 \circ e_1 &= \text{by } (e_1, p_1) \text{ e-p pair} \\
p_1 \circ g_1 \circ e_1 &= \text{by definition of } g_1 \\
p_1 \circ g_1 \circ e_1 &= \text{by } g_1 \leq g_2 \\
p_1 \circ g_1 \circ e_1 &= \text{by definition of } g_1 \\
p_1 \circ (e_1 \circ p_1) \circ e_1 &= \text{by } (e_1, p_1) \text{ e-p pair} \\
\text{id}_{X_1} \circ \text{id}_{X_1} &= \text{id}_{X_1}.
\end{align*}
\]
Proposition 3.4 \((\omega\text{-}\text{colimits of embeddings})\). Given an \(\omega\text{-}\text{chain}: ((e_n,p_n): X_n \to X_{n+1} | n: \omega)\) in \(\mathcal{A}_{ep}\) and a colimit cone \((f_n: X_n \to X | n)\) in \(\mathcal{A}\) from the \(\omega\text{-}\text{chain}\) \((e_n:X_n \to X_{n+1} | n)\) to \(X\), there exists a unique cone \(((\overline{f_n},\overline{q_n}): X_n \rightarrow X | n)\) in \(\mathcal{A}_{ep}\) from the \(\omega\text{-}\text{chain}\) \(((e_n,p_n) \ | n)\) to \(X\). Moreover, if \(\overline{g_n}\) is the idempotent on \(X\) defined by the \(e\)-\(p\) pair \(((\overline{f_n},\overline{q_n})\)), then \(((\overline{g_n} | n)\) is a jointly epi \(\omega\text{-}\text{chain of idempotents on }X\).

\[\text{Proof.}\] First, we define the family \((h_{i,j}: X_i \to X_j | i,j: \omega)\) of maps in \(\mathcal{A}\) (by induction on \(|j-i|\))

- \(h_{i,i}\) is the identity on \(X_i\)
- \(h_{i,j} = h_{i+1,j} \circ e_i\) when \(i < j\), thus \(h_{i,i+1} = e_i\)
- \(h_{i,j} = p_j \circ h_{i,j+1}\) when \(i > j\), thus \(h_{i+1,i} = p_i\)

Second, we prove (by induction on \(|j-i|\)) the property \(\forall i,j, h_{i+1,j} \circ e_i = h_{i,j} = p_j \circ h_{i,j+1}\).

- case \(i = j\): immediate, since \(h_{i+1,i} = p_i\) and \(h_{i,i+1} = e_i\).
- case \(i < j\): \(h_{i+1,j} \circ e_i = h_{i,j}\) by definition of \(h_{i,j}\). For the other equality:
  1. \(h_{i,j} = \text{by definition of } h_{i,j}\)
  2. \(h_{i+1,j} \circ e_i = \text{by IH on } (i+1,j)\)
  3. \(p_j \circ h_{i+1,j+1} \circ e_i = \text{by definition of } h_{i,j+1}\)
  4. \(p_j \circ h_{i,j+1}\).
- case \(i > j\): \(h_{i,j} = p_j \circ h_{i,j+1}\) by definition of \(h_{i,j}\). For the other equality:
  1. \(h_{i+1,j} \circ e_i = \text{by definition of } h_{i+1,j}\)
  2. \(p_j \circ h_{i+1,j+1} \circ e_i = \text{by IH on } (i,j+1)\)
  3. \(p_j \circ h_{i,j+1}\)
  4. \(h_{i,j}\).

The property implies that \((h_{i,n} | i)\) is a cone in \(\mathcal{A}\) from \((e_i | i)\) to \(X_n\). Thus, there exists a unique \(\overline{g_n}: X \to X_n\) such that \(\forall i, \overline{g_n} \circ f_i = h_{i,n}\). In particular, \(\overline{g_n} \circ f_n = h_{n,n} = \text{id}_{X_n}\), i.e., \((f_n,\overline{g_n}): X \rightarrow X_n\). The property implies also that \(\overline{g_n} = p_n \circ \overline{q_n+1}\), because \(f_n\) is jointly epi. Thus, \(((f_n,\overline{g_n}) \ | n)\) is a cone in \(\mathcal{A}_{ep}\) from \(X\) to \(((e_n,p_n) \ | n)\).

To prove uniqueness of \((\overline{g_n} | n)\), we use again that \((f_n | n)\) is jointly epi and prove (by induction on \(|j-i|\)) that \(\forall i,j, q_{j}' \circ f_i = h_{i,j}\) when \((q_{j}' | n)\) is a cone from \(X\) to \((p_n | n)\) such that \(\forall n, q_{n}' \circ f_n = \text{id}_{X_n}\).

- case \(i = j\): immediate, since \(q_{j}' \circ f_i = \text{id}_{X_i} = h_{i,i}\) by assumption on \((q_{j}' | n)\) and definition of \(h_{i,i}\).
- case \(i < j\):
  1. \(q_j' \circ f_i = \text{by definition of } (f_n, | n)\)
  2. \(q_j' \circ f_{i+1} \circ e_i = \text{by IH on } (i+1,j)\)
  3. \(h_{i+1,j} \circ e_i = h_{i,j}\) by definition of \(h_{i,j}\).
- case \(i > j\):
  1. \(q_j' \circ f_i = \text{by assumption on } (q_j' | n)\)
  2. \(p_j \circ q_j' \circ f_i = \text{by IH on } (i,j+1)\)
3. \( p_j \circ h_{i,j+1} = h_{i,j} \) by definition of \( h_{i,j} \).

Consider the idempotents \( q_n = f_n \circ q_n \) on \( X \). From item 5 of Proposition 3.3, it follows that \( \forall n.q_n \leq q_{n+1} \). The family \((q_n | n)\) is jointly epi, because \((f_n | n)\) is jointly epi (as colimit cones are jointly epi) and each \( q_n \) is (split) epi.

The following is the dual of Proposition 3.4.

**Corollary 3.5.** Given an \( \omega \)-chain \((c_n, p_n) : X_n \to X_{n+1} | n) \) in \( \mathcal{K} \) and a limit cone \((\bar{q}_n : \bar{X} \to X_n | n) \) in \( \mathcal{K} \) from \( \bar{X} \) to the \( \omega \)-chain \((\bar{p}_n : \bar{X}_{n+1} \to \bar{X}_n | n) \), there exists a unique cone:

\[
((\bar{f}_n, \bar{q}_n) : X_n \to \bar{X} | n)
\]

in \( \mathcal{K}_{ep} \) from the \( \omega \)-chain \((c_n, p_n) | n) \) to \( \bar{X} \). Moreover, if \( \bar{q}_n \) is the idempotent on \( \bar{X} \) defined by the e-p pair \((\bar{f}_n, \bar{q}_n) \), then \((\bar{q}_n | n) \) is a jointly mono \( \omega \)-chain of idempotents on \( \bar{X} \).

The following result implies existence and uniqueness of a map \( \iota : \bar{X} \to X \) in \( \mathcal{K} \) such that \( \forall n.\iota \circ f_n = \bar{f}_n \) and \( \forall n.q_n = \bar{q}_n \circ \iota \), where \((f_n, q_n) : X_n \to X_{n+1} | n) \) and \((\bar{f}_n, \bar{q}_n) : X_n \to \bar{X} | n) \) are the cones in \( \mathcal{K}_{ep} \) given by Proposition 3.4 and Corollary 3.3. In general, there is no reason for \( \iota \) to be mono, epi or iso, e.g.: in \( \text{Set} \) the map \( \iota \) is always mono, but it may fail to be epi; in \( \text{Set}^{op} \) the converse holds; in \( \text{Haus} \) the map \( \iota \) is both mono and epi, but \( \bar{X} \) may fail to be a sub-space of \( \bar{X} \).

**Theorem 3.6.** If \((g_n | n : \omega) \) is an \( \omega \)-chain of idempotents on \( X \) in \( \mathcal{K} \) such that every \( g_n \) has a splitting, say \((f_n, q_n) : X_n \to X \), then there exists a unique \( \omega \)-chain \((c_n, p_n) : X_n \to X_{n+1} | n : \omega) \) in \( \mathcal{K}_{ep} \) such that \((f_n, q_n) : X_n \to X | n) \) is a cone in \( \mathcal{K}_{ep} \) from \((c_n, p_n) : X_n \to X_{n+1} | n) \) to \( X \). Moreover:

1. If \((f_n : X_n \to X | n) \) is a colimit cone in \( \mathcal{K} \) from the \( \omega \)-chain \((c_n : X_n \to X_{n+1} | n) \) to \( X \) and \( \iota : X \to X \) is the unique map such that \( \forall n.f_n = \iota \circ f_n \), then \( \forall n.q_n = \bar{q}_n \circ \iota \) (see Proposition 3.3 for \( \bar{q}_n \)).

2. If \((\bar{q}_n : \bar{X} \to X_n | n) \) is a limit cone in \( \mathcal{K} \) from \( \bar{X} \) to the \( \omega \)-chain \((\bar{p}_n : \bar{X}_{n+1} \to \bar{X}_n | n) \) and \( \iota : X \to \bar{X} \) is the unique map such that \( \forall n.q_n = \bar{q}_n \circ \iota \), then \( \forall n.q_n = \bar{q}_n \circ \iota \) (see Corollary 3.3 for \( \bar{f}_n \)).

**Proof.** By Item 3 of Proposition 3.3, we get that for each \( n : \omega \) there exists a unique e-p pair \((c_n, p_n) \) such that \((f_{n+1}, q_{n+1}) \circ (c_n, p_n) = (f_n, q_n) \), which implies that \((f_n, q_n) : X_n \to X | n) \) is a cone in \( \mathcal{K}_{ep} \) from the \( \omega \)-chain \((c_n, p_n) : X_n \to X_{n+1} | n) \) to \( X \).

**Item 1.** We rely on the proof of Proposition 3.3, where \( q_n \) is defined. Since \((f_{i+1}, q_{i+1}) \circ (e_i, p_i) = (f_i, q_i) \) in \( \mathcal{K}_{ep} \) follows from \( \forall i,j.q_j \circ f_j = g_i \circ f_j \), or equivalently, from \( \forall i,j.q_j \circ f_i = h_{i,j} \), which we prove by induction on \( |j-i| \):

- case \( i = j \): immediate, since \( q_i \circ f_i = id_{X_i} = h_{i,j} \), because \((f_i, q_i) \) is a map in \( \mathcal{K}_{ep} \) and by definition of \( h_{i,j} \).

- case \( i < j \):
  1. \( q_j \circ f_i = (f_{i+1}, q_{i+1}) \circ (e_i, p_i) = (f_i, q_i) \) in \( \mathcal{K}_{ep} \)
  2. \( q_j \circ f_i = h_{i+1,j} \circ e_i \) by IH on \((i+1, j)\)
  3. \( h_{i+1,j} \circ e_i = h_{i,j} \) by definition of \( h_{i,j} \).

- case \( i > j \):
  1. \( q_j \circ f_i = (f_{i+1}, q_{i+1}) \circ (e_i, p_i) = (f_i, q_i) \) in \( \mathcal{K}_{ep} \)
  2. \( p_j \circ q_{j+1} \circ f_i = \) by IH on \((i,j+1)\)
  3. \( p_j \circ h_{i,j+1} = h_{i,j} \) by definition of \( h_{i,j} \).
Proposition 3.7. Given two cones \((f_i : X_i \to X | i : I)\) and \((f'_i : X_i \to X' | i : I)\) and a map \(\iota : X \to X'\) such that \(\forall i. \iota \circ f_i = f'_i\), then \((f'_i | i)\) is jointly epi implies \(\iota\) is epi.

Proof. We have to prove that \(h \circ \iota = h' \circ \iota\) implies \(h = h'\).

1. \(h \circ \iota = h' \circ \iota\) implies
2. \(\forall i. h \circ f_i = h' \circ f_i\) implies, by \(\iota \circ f_i = f'_i\)
3. \(\forall i. h \circ f'_i = h' \circ f'_i\) implies, by \((f'_i | i)\) jointly epi
4. \(h = h'\)

The following result is the dual of Proposition 3.7.

Corollary 3.8. Given two cones \((q_i : X \to X_i | i : I)\) and \((q'_i : X' \to X_i | i : I)\) and a map \(\iota : X \to X'\) such that \(\forall i. q'_i \circ \iota = q_i\), then \((q_i | i)\) jointly mono implies \(\iota\) mono.

3.2 Extended Metric Spaces versus Compact Hausdorff Spaces

The following result requires to move between four categories (and we have added also \(\text{Set}\)) using four functors (where \(\longrightarrow\) denotes a faithful functor and \(\longleftarrow\) an inclusion of a full sub-category):

\[
\begin{array}{ccc}
\text{KMS} & \longrightarrow & \text{Met} \\
\downarrow & & \downarrow \\
\text{KH} & \longleftarrow & \text{Haus} \\
\end{array}
\]

All categories in the diagram have:

- finite limits and finite sums (Theorem 2.2);
- enough points, i.e., the global section functors \(\Gamma\) into \(\text{Set}\) are faithful;
- splittings of idempotents;

and all functors in the diagrams preserve finite limits and finite sums. Moreover, \(\text{Haus}\) has all small limits and small colimits (Theorem 2.3), where limits are computed as in \(\text{Top}\), and limits (computed in \(\text{Haus}\)) of diagrams in \(\text{KH}\) are in \(\text{KH}\). Also \(\text{Met}\) has all small limits and small colimits (Theorem 2.2), but the forgetful functor \(U\) from \(\text{Met}\) to \(\text{Haus}\) may not preserve all these (co)limits.

In applications, we start from a metric space \(S\), then identify an \(\omega\)-chain \((g_n | n)\) of idempotents on \(S\) in \(\text{Met}\), and by applying Theorem 3.6 in \(\text{Haus}\) we get a map \(\tau : S \to \overline{S}\) in \(\text{Haus}\). The theorems below provide sufficient conditions to ensure that \(\overline{S}\) is compact, \(\tau\) is mono and epi, and, above all, that the complete lattice \(C(\overline{S})\) is \(\omega\)-continuous and the monotonic map \(C(\tau)\) is in \(\mathcal{K}_{RS}(S, \overline{S})\). These properties can be proved for maps \(\iota : S \to \overline{S}\) that are not necessarily obtained through Theorem 3.6 and the theorems below capture this greater generality.

If we start from an \(\omega\)-chain \((g_n | n)\) of idempotents on \(S\) in \(\text{Met}\), then Theorem 3.6 provides candidates for \((p_n | n)\) and \((q_n | n)\) in the following theorem, because \(\text{Met}\) has splittings of idempotents.

Theorem 3.9. If \((p_n : S_{n+1} \to S_n | n)\) is an \(\omega^\omega\)-chain in \(\text{KMS}\), \((q_n | n)\) is a cone from \(S\) to \((p_n | n)\) in \(\text{Met}\), \((\tau_n : S \to \overline{S} | n)\) is a limit cone from \(S\) to \((p_n | n)\) in \(\text{Haus}\), and \(\iota : S \to \overline{S}\) is the unique map in \(\text{Haus}\) such that \(q_n = \tau_n \circ \iota\), then:

1. \(\overline{S}\) is compact and has a countable base, thus \(C(\overline{S})\) is \(\omega\)-continuous.
2. The monotonic map \(C(\iota)\) is in \(\mathcal{K}_{RS}(S, \overline{S})\).

\[\text{Given a category } \mathcal{K} \text{ with a terminal object } 1, \text{ the global section functor from } \mathcal{K} \text{ to } \text{Set} \text{ is given by } \mathcal{K}(1, -).\]
The Hausdorff space $\mathbb{S}$ can be identified with the set $\{s : \prod_n S_n | \forall n, s_n = p_n(s_{n+1})\}$, equipped with the coarsest topology $O(\mathbb{S})$ making the maps $\tau_n(s) = s_n$ continuous, i.e., the topology generated by the sub-base $[O]_n = \{s : \mathbb{S} | \tau_n(s) \in O\}$, where $O$ is an open set in $S_n$.

By Theorem 2.23, the limit $\mathbb{S}$ is in $\text{KH}$, because it is the limit (in $\text{Haus}$) of a diagram in $\text{Met}$. The topology on $S_n$ has a countable base $\tau^b_n$ because $S_n$ is in $\text{KMS}$. Thus, the topology on $\mathbb{S}$ has a countable sub-base too, namely the set of open subsets of the form $[B]_n$, with $B \in \tau^b_n$.

By Theorem 2.21, the complete lattice $C(X)$ is in $\text{CL}$ and the topologies $\tau_S(X)$ and $\tau_U(X)$ coincide, when $X \neq \text{KH}$, as in the case of $S_n$ and $\mathbb{S}$. Therefore, $\tau_S(X)$ is generated by $\{K : C(X) | K \subseteq O\}$ with $O : C(X)$, and the way-below relation is given by $K_1 \ll K_2 \iff K_2 \subseteq K_1^\circ$, where $K_1^\circ$ is the interior of $K_1$.

By Theorem 2.22, the continuous lattice $C(\mathbb{S})$ is (isomorphic to) the limit of the $\omega^\alpha$-chain of right adjoints $(C(p_n) | n)$ in $\text{CL}$ (and in $\text{Po}_A$). To be more precise, the isomorphism is $K \mapsto (\tau_n(K) | n)$ and $(K_n | n) \mapsto \bigcap_n \tau_n(K_n)$.

The sub-base of $O(\mathbb{S})$ given above, i.e., the set of $[O]_n = \{s : \mathbb{S} | \tau_n(s) \in O\}$, where $O$ is an open set in $S_n$, is actually a base, because $[O]_n = [p_n'(O)]_{n+1}$. Therefore, every $O$ of $\mathbb{S}$ is of the form $\bigcup_{i \in I} [O_i]_n$, with $O_i : O(S_n)$ for $i \in I$. Since $\mathbb{S}$ is compact, also $K : C(\mathbb{S}$) is compact, and $K \subseteq O$ implies $K \subseteq \bigcup_{i \in I} [O_i]_n$ for some $J \subseteq I$. In particular, $O \supseteq \bigcup_{i \in I} [O_i]_n \supseteq [O]_n$, where $n_J = \sup_{i \in J} n_i$ and $O_J : O(S_{n_J})$ is the union for $i : J$ of the $O_i$ moved from $S_{n_i}$ to $S_{n_J}$, therefore

$$\forall K : C(\mathbb{S}), \forall O : O(\mathbb{S}), K \subseteq O \implies \exists n_0 : O(\mathbb{S}), \tau_n(K) \subseteq O_n \wedge [O]_n \subseteq O. \quad (1)$$

Using the above property, we have: 

- $C(i) : A_{RS}(S, \mathbb{S})$ means $\forall C : C(S), \forall O : O(\mathbb{S}), \tau(C) \subseteq O \implies \exists \delta : \delta(C) \subseteq O$.

- By property 1 this is implied by $\forall C : C(S), C(\mathbb{S}), \forall O : O(\mathbb{S}), \tau(C) \subseteq [O]_n \implies \exists \delta : \delta(C) \subseteq [O]_n$.

- By $q_n = \tau_n \circ i$ and the definition of $[O]_n$, this is equivalent to:

$$\forall C : C(S), C(\mathbb{S}), \forall O : O(\mathbb{S}), q_n(C) \subseteq O \implies \exists \delta : \delta(q_n(C)) \subseteq O.$$

- In general, $q : \text{Met}(S, S') \implies \delta \circ O : C(S), q(C) \subseteq \delta(C)$, and $S' : \text{KMS}$ implies $\forall K : C(S'), \forall O : O(S'), K \subseteq O \implies \exists \delta : \delta(K) \subseteq O$.

- Since $q_n : \text{Met}(S, S_n)$ and $S_n : \text{KMS}$, for any $C : C(S)$ and $O : O(S_n)$

$$q_n(C) \subseteq O \implies \exists \delta : \delta(q_n(C)) \subseteq O \implies \exists \delta : \delta(q_n(C)) \subseteq O.$$

If we start from an $\omega$-chain $(g_n | n)$ of idempotents on $S$ in $\text{Met}$, rather than in $\text{Haus}$, then $((f_n, q_n) | n)$ and $((e_n, p_n) | n)$ in the following theorem consist of short maps. Moreover, families of maps that are jointly mono in $\text{Met}$ are also jointly mono in $\text{Haus}$, because these categories have enough points. Finally, if an $\omega$-chain $(g_n | n)$ of idempotents on $S$ (in the category $A$) is jointly mono (in $A$), then the identity on $S$ is the sup of the $\omega$-chain in the poset of idempotents on $S$.

Theorem 3.10. Given an $\omega$-chain $(g_n | n)$ of idempotents on $S$ in $\text{Haus}$, which is jointly mono, consider:

- a splitting $S \twoheadrightarrow q_n \twoheadrightarrow S_n \twoheadrightarrow f_n \hookrightarrow S$ of $g_n$;

- the unique $\omega$-chain $((e_n, p_n) : S_n \rightarrow S_{n+1} | n)$ in $\text{Haus}_{ep}$ such that $((f_n, q_n) : S_n \rightarrow S | n)$ is a cone in $\text{Haus}_{ep}$ from $((e_n, p_n) : S_n \rightarrow S_{n+1} | n)$ to $\mathbb{S}$ (see Theorem 3.6);

- the limit cone $(\tau_n : S \rightarrow S | n)$ in $\text{Haus}$ from $\mathbb{S}$ to the $\omega^\alpha$-chain $(p_n : S_{n+1} \rightarrow S_n | n)$ (see Corollary 3.8);

- the unique map $i : S \rightarrow \mathbb{S}$ in $\text{Haus}$ such that $\forall n, q_n = \tau_n \circ i$.

Then, $i$ is both mono and epi in $\text{Haus}$.

Proof. If $(g_n | n)$ is jointly mono, then also $(q_n | n)$ is jointly mono. Therefore, $i$ is mono by Corollary 3.8.

If $((f_n, q_n) : S_n \rightarrow \mathbb{S} | n)$ is the unique cone in $\text{Haus}_{ep}$ given by Corollary 3.8 then $\tau_n \circ i \circ f_n = \tau_n$, by Theorem 3.6. Therefore, to prove that $i$ is epi it suffices, by Proposition 3.7, to prove that $(\tau_n : n)$ is jointly epi in $\text{Haus}$. This amounts to proving that the union of the images of the maps in $(\tau_n | n)$ is dense in $\mathbb{S}$. To do this we use the base of $O(\mathbb{S})$ as in the proof of Theorem 3.9. For every $s : S \rightarrow [O]_n$, in the base (i.e., $O : O(S_n)$) such that $\forall n, i. e., q_n(s) : O$, we have to give an $s_n : S_n$ such that $\tau_n(s_n) : [O]_n$. It suffice to take $s_n = q_n(s)$, since $s_n = q_n(\tau_n(s_n))$.
4 Examples

In this section, we consider examples of Banach spaces $S$, demonstrating how to apply the results of Section 3 to define a specific $S$ in $KH$, and in which cases $S$ is a compactification of $S$ (a summary is given at the end of this section, see Table 3). In Section 3 we will study loss of precision when going from $S$ to $S$. All examples considered in this section are sequence spaces. Hence, we recall some general definitions and fix notation.

Definition 4.1 (Uniform notation). We write $R$ for the standard Banach space on the reals, and also for the underlying vector space, metric space, topological space, and set.

- Given a set $I$, we write $R^I$ for the product of $I$ copies of the set $R$, which is also the carrier of the product of $I$ copies of $R$ in the categories of the vector spaces, extended metric spaces and Hausdorff spaces.
- Given a real number $p$ in the interval $[1,\infty)$, we write $\|x\|_p$ for the map from $R^I$ to $[0,\infty]$ given by $\|x\|_p^p = (\sum_{i \in I} |x_i|^p)^{1/p}$, and it is extended to $p = \infty$ by defining $\|x\|_\infty = \sup_{i \in I} |x_i|$. Since $I$ is determined by $x$, we drop the subscript $I$ and write $\|x\|_p$.
- We write $\ell_{I,p}$ for the Banach space with carrier the sub-space $\{x : R^I \mid \|x\|_p < \infty\}$ of $(\text{the vector space}) R^I$ and norm $\|x\|_p$. We write $B_{I,p}$ for the closed unit ball in $\ell_{I,p}$, whose elements are those $x$ such that $\|x\|_p \leq 1$. The subset $B_{I,p}$ inherits from $\ell_{I,p}$ the metric space structure.
- If $I \subseteq J$, then $\ell_{I,p}$ is isomorphic (in the category of Banach spaces and short linear maps) to the sub-space of $\ell_{J,p}$ with carrier $\{x \mid \exists j : I \ni j, x_j = 0\}$, and $B_{I,p}$ is a sub-space of $B_{J,p}$ (modulo the isomorphism).

We consider only countable $I$, specifically, either $\omega$ or a natural number $m$. We write $\ell_p$ for $\ell_{\omega,p}$ and $\ell_{*,p}$ for the (normed vector) sub-space of $\ell_p$ with carrier $\{x \mid \exists n \in \mathbb{N}, \forall i, x_i = 0\}$. Usually $\ell_{*,p}$ is denoted $c_{00}$. We write $B_p$ for $B_{\omega,p}$, and $B_{*,p}$ for $B_{\omega,p} \cap \ell_{*,p}$. Note that $\ell_{0,p}$ is trivial and $\ell_{1,p} = R$ for every $p$.

In the sequel, we use the following characterization of limits in $\text{Top}$ and general properties of limits and colimits (valid in any category).

Proposition 4.2. Given a small diagram $D : I \to \text{Top}$, a limit cone $(\pi_i : (X, \tau) \to D_i \mid i : I)$ in $\text{Top}$ is obtained by taking a limit cone $(\pi_i : X \to U(D_i) \mid i : I)$ of $U \circ D : I \to \text{Set}$ in $\text{Set}$, and by defining $\tau$ as the coarsest topology on $X$ making the maps $\pi_i : (X, \tau) \to D_i$ continuous.

Proposition 4.3 (Limits commute with Limits). Given an $I \times J$-diagram $D : I \times J \to \mathbb{A}$ in a category $\mathbb{A}$ (with the relevant limits), if for each $i : I$, $(p_{i,j} : X_i \to D_{i,j} \mid j : J)$ is a limit cone for the $J$-diagram $D(i, -) : J \to \mathbb{A}$, then the family $(X_i \mid i : I)$ extends canonically to an $I$-diagram $X : I \to \mathbb{A}$, namely, for $f : i \to i'$ in $I$, the map $X_f : X_i \to X_{i'}$ is the unique map in $\mathbb{A}$ such that for all $j : J$, the following diagram commutes:

Moreover, if $(p_i : x \to X_i \mid i : I)$ is a limit cone for $X : I \to \mathbb{A}$, then $(p_{i,j} \circ p_i : x \to D_{i,j} \mid i : I, j : J)$ is a limit cone for $D$. Since one can exchange the role of $I$ and $J$, there are two alternative ways of computing limits of $I \times J$-diagrams, which necessarily produce canonically isomorphic results.

Proposition 4.4 (Colimits of cofinal diagrams). Given an $\omega$-diagram $D : \omega \to \mathbb{A}$ in a category $\mathbb{A}$ (with the relevant colimits), if $(f_n : D_n \to X \mid n : \omega)$ is a colimit cone for $D$ and $h : \omega \to \omega$ is a strictly increasing map, then $(f_{h(n)} : D_{h(n)} \to X \mid n : \omega)$ is a colimit cone for the $\omega$-diagram $D \circ h : \omega \to \mathbb{A}$.

4.1 Banach space $R$

Consider the metric space $R$ with distance $d(x,y) = |x-y|$ and the $\omega$-chain $(r_n \mid n : \omega)$ such that

$$r_n(x) \triangleq \begin{cases} n, & \text{if } n < x, \\ x, & \text{if } |x| \leq n, \\ -n, & \text{if } x < -n. \end{cases} \quad (2)$$

Each $r_n$ is idempotent and short, because:

$$d(r_n(x), r_n(y)) = \begin{cases} x < -n & \text{if } n < x, \\ -n \leq x \leq n & n < x, \\ n < x & y < -n, \\ n + x & 0 < n + x, \\ 2n & \text{if } 2n - x \leq n \leq 2n - y, \\ n - x & n - y \leq y \leq n, \\ 0 & n < y. \end{cases}$$

\[d(x,y).\]
The image of \( r_n \) is the compact sub-space \( \mathbb{R}_n \triangleq [-n, n] \), and the union \( S_n \) of the sub-spaces \( \mathbb{R}_n \) is \( \mathbb{R} \).

Let \( \mathbb{R} \to q_n \to \mathbb{R}_n \to f_n \to \mathbb{R} \) be the splitting of \( r_n \) through \( \mathbb{R}_n \) and \( p_n = q_n \circ f_{n+1} : \mathbb{R}_{n+1} \to \mathbb{R}_n \). Let \( (q_n \mid n) \) be the limit cone from \( \mathbb{S} \) to the \( \omega\)-\( p \)-chain \( (p_n \mid n) \) in \( \text{Haus} \). Then, by Theorem 3.10, \( \mathbb{S} \) is compact, and by Theorem 3.10 the map \( i : \mathbb{R} \to \mathbb{S} \) is both epi and mono in \( \text{Haus} \).

We show that \( (q_n \mid n) \) is isomorphic to the cone \( (\hat{q}_n \mid n) \) from \( \mathbb{R} = [-\infty, +\infty] \) (the two-point compactification of \( \mathbb{R} \)) to \( (p_n \mid n) \), where \( \hat{q}_n \) is the extension of \( q_n \) to \( \mathbb{R} \) mapping \( -\infty \) to \( -n \) and \( +\infty \) to \( +n \). Let \( \phi : \mathbb{R} \to \mathbb{S} \) be the unique map such that \( \forall n. \hat{q}_n = \overline{\varnothing} \circ \phi \), namely \( \phi(x) \triangleq (\hat{q}_n(x) \mid n) \). The map \( \phi \) is a bijection (in \( \text{Set} \)), since the elements \( (s_n \mid n) \in \mathbb{S} \) satisfy one of the following disjoint properties:

- \( \forall n. s_n = -n \), i.e., \( (s_n \mid n) = \phi(-\infty) \);
- \( (s_n \mid n) \) is eventually constant. This happens when \( |s_m| < m \) for some \( m : \omega \). In this case \( (s_n \mid n) = \phi(s_m) \);
- \( \forall n. s_n = +n \), i.e., \( (s_n \mid n) = \phi(+\infty) \).

Therefore, \( (\hat{q}_n \mid n) \) is a limit cone from \( \mathbb{R} \) to \( (p_n \mid n) \) in \( \text{Set} \). To prove that \( \phi \) is an isomorphism in \( \text{Top} \), it suffices to show that the topology on \( \mathbb{R} \) is the coarsest topology making the maps \( \hat{q}_n \) continuous (Proposition 4.2). Since a base for the topology on \( \mathbb{R} \) consists of the subsets of the form \([-\infty, x \rangle \), \( (x, y) \) and \( (y, +\infty] \) for \( x, y : \mathbb{R} \), it suffices to show that every element in the base is of the form \( \hat{q}_n \cdot (O) \) for some \( n : \omega \) and open subset \( O : \mathcal{O}(\mathbb{R}_n) \). This is immediate by taking \( n \) such that \( |x|, |y| < n \), and taking \( O \) of the form \([-n, x \rangle \), \( (x, y) \) and \( (y, n] \), respectively.

### 4.2 Banach spaces \( \ell_{m,\infty} \) for \( 1 < m \)

Fix a natural number \( m > 1 \), consider the metric space \( \mathbb{S} = \ell_{m,\infty} \) with distance \( d_\infty(x, y) = \max_{i : m} d(x_i, y_i) \), which coincides with the finite product \( \mathbb{R}_m \) in \( \text{Met} \), and the \( \omega \)-chain \( (g_n \mid n : \omega) \) defined by:

\[
\forall x : \mathbb{R}_m. g_n(x) \triangleq (r_n(x_i) \mid i : n),
\]

where \( r_n \) is as defined in 4.9. Since \( g_n \) is defined pointwise, it is idempotent and short by inheritance, since \( d_\infty(g_n(x), g_n(y)) = \max_{i : m} d(r_n(x_i), r_n(y_i)) \leq \max_{i : m} d(x_i, y_i) = d_\infty(x, y) \). The image of \( g_n \) is the compact sub-space \( \mathbb{S}_n \triangleq \mathbb{R}_m \) and, once again, the union \( S_n \) of the sub-spaces \( \mathbb{S}_n \) is \( \mathbb{S} \).

From Section 4.1 and Proposition 4.3 we have that \( \mathbb{S} \) is isomorphic to \( \mathbb{R}_m \) in \( \text{KH} \). In fact, \( \text{KH} \) has all small limits. Thus, we can take \( I = m \) and \( J = \omega^\omega \), and consider the \( I \times J \)-diagram \( D : I \times J \to \text{KH} \) such that \( D(i, n) = \mathbb{R}_n \) and \( D(i, n + 1 \to n) \) is the map \( p_n : \mathbb{R}_{n+1} \to \mathbb{R}_n \) defined in Section 4.1. The limit \( \mathbb{S} \) is obtained by first computing the limits \( \mathbb{R}_m \) of \( I \)-diagrams \( D(-, n) \) and then the \( J \)-limit, while \( \mathbb{R}_m \) is obtained by first computing the limits \( \mathbb{R} \) of the \( J \)-diagrams \( D(i, -) \) and then the \( I \)-limit.

### 4.3 Banach spaces \( \ell_{m,p} \) for \( 1 < m \) and \( 1 \leq p < \infty \)

This is a modification of Section 4.2 where we consider the metric space \( \mathbb{S} = \ell_{m,p} \) that has the carrier of \( \ell_{m,\infty} = \mathbb{R}_m \), but with distance

\[
d_p(x, y) \triangleq \left( \sum_{i : m} d(x_i, y_i)^p \right)^{1/p} \geq d_\infty(x, y).
\]

We take the same \( g_n \) used for \( \ell_{m,\infty} \), as defined in 4.9. Clearly \( g_n \) is idempotent, since this property does not depend on the distance, and is short also with respect to \( d_p \) (again by inheritance), since

\[
d_p(g_n(x), g_n(y)) = \left( \sum_{i : m} d(r_n(x_i), r_n(y_i))^p \right)^{1/p} \leq \left( \sum_{i : m} d(x_i, y_i)^p \right)^{1/p} = d_p(x, y).
\]

Therefore, the metric space \( S_n \) has the same carrier of \( \mathbb{R}_m \) and distance \( d_p \). Since \( d_p \) and \( d_\infty \) induce the same topology on \( \mathbb{R}_m \), we have that \( \mathbb{S} \) for \( \ell_{m,p} \) and for \( \ell_{m,\infty} \) are equal and isomorphic to \( \mathbb{R}_m \) in \( \text{KH} \).

### 4.4 Banach space \( \ell_\infty \)

Consider the metric space \( \mathbb{S} = \ell_\infty \) with distance \( d_\infty(x, y) \triangleq \sup_{i : \omega} d(x_i, y_i) \), and the \( \omega \)-chain \( (g_n \mid n : \omega) \) of maps on \( \ell_\infty \), defined by

\[
\forall x : \ell_\infty. g_n(x) \triangleq (r_n(x_i) \mid i : n) \cdot 0^\omega.
\]

Since \( g_n \) is defined pointwise, it is idempotent and short by inheritance, e.g.,

\[
d_\infty(g_n(x), g_n(y)) = \sup_{i : \omega} d(r_n(x_i), r_n(y_i)) \leq \sup_{i : \omega} d(x_i, y_i) \leq d_\infty(x, y).
\]
The image of $g_n$ is the compact sub-space $S_n = \ell_{\infty,n} \triangleleft \mathbb{R}_+^n \times [0]^\omega$, which is isomorphic to the finite product $\mathbb{R}_+^n$ in $\text{Met}$, and the union $S_\omega$ of the sub-spaces $S_n$ is the sub-space $\ell_{*,\infty}$ of $\omega$-sequences eventually equal to 0, which is not dense in $\ell_\infty$, and its closure is the sub-space $c_0$ of $\omega$-sequences converging to 0.

From Section 4.4 Proposition 4.3 and the dual of Proposition 4.4, we have that $\mathfrak{F}$ for $\ell_\infty$, which we denote with $\overline{\ell}_\infty$, is isomorphic to $\mathbf{R}^\omega$ in $\text{KH}$. In fact, take $I = \mathbb{N}$, $J = \omega^{op}$, and consider the $I \times J$-diagram $D : I \times J \rightarrow \text{KH}$ such that $D(i,n) = \mathbb{R}_n$ if $i < n$, else $[0]$, and $D(i,n+1 \rightarrow n) \triangleq p_n : \mathbb{R}_{n+1} \rightarrow \mathbb{R}_n$ if $i < n$, else the unique map from $D(i,n+1)$ to $[0]$. The limit $\mathfrak{F}$ is obtained by first computing the limits of $I$-diagrams $D(\cdot,n)$, which are isomorphic to $\mathfrak{F}_n$, and then the $J$-limit, while $\mathbf{R}^\omega$ is obtained by first computing the limits of $\mathbf{R}$-diagrams $D(i,\cdot)$ and then the $I$-limit.

Note that the map $\iota : \ell_\infty \rightarrow \overline{\ell}_\infty$ given by Theorem 3.10 is mono and epi in $\text{Haus}$ (the subset $\ell_{*,\infty}$ is not dense in $\ell_\infty$, but it is dense in $\ell_\infty$).

### 4.5 Banach spaces $\ell_p$ for $1 \leq p < \infty$

Consider the metric spaces $\mathbb{S} = \ell_p$ with distance $d_p(x,y) \triangleq (\sum_i d(x_i,y_i)^p)^{1/p} \geq d_\infty(x,y)$. The carrier $|\ell_p|$ of $\ell_p$ satisfies the following strict inclusions

$$|\ell_{*,\infty}| \subset |\ell_p| \subset |c_0| \subset |\ell_\infty|.$$

We can consider the restrictions to $\ell_p$ of the idempotents $g_n$ defined in (3). It is straightforward to prove that $g_n$ is short and idempotent on $\ell_p$, its image $S_n = \ell_{p,n}$ is isomorphic to the metric space with carrier $\mathbb{R}_+^n$ and distance $d_n$, and the union $S_\omega$ of the sub-spaces $S_n$ is the dense sub-space $\ell_{*,p}$ of $\ell_p$.

By analogy with Section 4.4, we have that the map $\iota : \mathbb{S} \rightarrow \mathfrak{F}$ is mono and epi in $\text{Haus}$, and $\mathfrak{F}$ for $\ell_p$ is isomorphic to $\mathbf{R}^\omega$ in $\text{KH}$. In particular, $\mathfrak{F}$ is independent of $p$. In summary, the relations between $\ell_p$ and $\ell_q$, for $1 \leq p < q \leq \infty$, are:

- The carrier of $\ell_p$ is a proper subset of the carrier of $\ell_q$, and the inclusion of $\ell_p$ into $\ell_q$ is a short map in $\text{Met}$, since $\forall x,y : \ell_p,d_q(x,y) \leq d_p(x,y)$.
- The compact metric spaces $\ell_{p,n}$ and $\ell_{q,n}$ have the same carrier, but different distances, and the inclusion of $\ell_{p,n}$ into $\ell_{q,n}$ is a short map in $\text{KMS}$.
- As topological spaces, $\ell_{p,n}$ and $\ell_{q,n}$ are equal. Thus, $\overline{\ell}_p = \overline{\ell}_q$.

### 4.6 Unit ball $B_\infty$

Let us now consider the unit ball $B_\infty$ in the metric space $\ell_\infty$. As a metric space, $B_\infty$ coincides with the infinite product $\prod \mathbb{R}_1^\omega$ in $\text{Met}$, and the idempotents $g_n$ defined is Section 4.4 restrict to idempotents on $B_\infty$. The image of $g_n$ (restricted to $B_\infty$) is the compact sub-space $S_n = B_{\infty,n} \triangleleft \mathbb{R}_1^\omega \times [0]^\omega$, which is isomorphic to the finite product $\mathbb{R}_1^n$ in $\text{Met}$, and coincides with closed unit ball $B_{\infty,n}$ in $\ell_{\infty,n} \triangleleft \mathbb{R}_n^\omega$. The union $S_\omega$ of the sub-spaces $S_n$ is the sub-space $B_{\infty,\omega}$, and the closure of $S_\omega$ is the sub-space $B_\infty \cap c_0$.

Similar to Section 4.4, we can prove that $\mathfrak{F}$ for $B_\infty$, which we denote with $\overline{B}_\infty$, is isomorphic to product $\mathbb{R}_1^\omega$ in $\text{KH}$. More precisely, take $I = \mathbb{N}$, $J = \omega^{op}$, and consider the $I \times J$-diagram $D : I \times J \rightarrow \text{KH}$ such that $D(i,n) = \mathbb{R}_1$ if $i < n$ else $[0]$, and $D(i,n+1 \rightarrow n)$ is the identity on $\mathbb{R}_1$ if $i < n$, else the unique map from $D(i,n+1)$ to $[0]$.

As in the case of $\ell_\infty$, the map $\iota : B_\infty \rightarrow \overline{B}_\infty$ given by Theorem 3.10 is mono and epi in $\text{Haus}$. Moreover, as a set-theoretic map, $\iota$ is a bijection. In fact, the metric space $B_\infty$ is (equal to) the product $\mathbb{R}_1^\omega$ in $\text{Met}$ and the topological space $\overline{B}_\infty$ is (isomorphic to) the product $\mathbb{R}_1^\omega$ in $\text{KH}$. As such, without loss of generality, we assume that the map $\iota$ is an identity map.

### 4.7 Unit ball $B_p$ for $1 \leq p < \infty$

Let us now consider the unit ball $B_p$ in the metric space $\ell_p$. One can proceed in analogy with Section 4.6. In particular, $B_{p,n}$ is isomorphic to the closed unit ball $B_{p,n}$ in $\ell_{p,n}$, the map $\iota : B_p \rightarrow \overline{B}_p$ given by Theorem 3.10 is mono and epi in $\text{Haus}$.

Moreover, as a set-theoretic map, $\iota$ is a bijection. The map $\iota$ is clearly injective. So, it suffices to prove that it is surjective. Let us consider an element $(x_n | n)$ in $\overline{B}_p$. Each $x_n$ may differ from $x_{n+1}$ only in the $n$-th component, namely $x_{n,n} = 0$, while $x_{n,n+1} : \mathbb{R}_1$. Consider $y : \mathbb{R}_1^\omega$ defined by $y_n \triangleq x_{n+1,n}$ for $n : \omega$. We have $\|y\|_p = \lim_{n \rightarrow \infty} \|x_n\|_p \leq 1$, which entails that $y : B_p$. Furthermore,

$$\forall i,n : \omega, \quad x_{n,i} = \begin{cases} y_i, & \text{if } i < n, \\ 0, & \text{otherwise.} \end{cases}$$
Table 3 Summary of Examples of Section 4
Notation: \(\rightarrow\) (sub-space) and \(\twoheadrightarrow\) (sub-object) in \textbf{Met} (left) and \textbf{Haus} (right); \(X^I\) product of \(I\) copies of \(X\) (in \textbf{Met} or \textbf{Haus}), the forgetful functor from \textbf{Met} to \textbf{Haus} preserves only finite products.

\[
\begin{array}{ccccccccc}
B_\infty \equiv \mathbb{R}^\infty & \twoheadrightarrow & \ell_\infty & \twoheadrightarrow & \ell_{m,\infty} \equiv \mathbb{R}^m \\
B_\infty & \rightarrow & \ell_\infty & \rightleftharpoons & \ell_{m,\infty} \equiv \mathbb{R}^m \\
B_p & \rightarrow & \ell_p & \rightleftharpoons & \ell_{m,p}
\end{array}
\]

In the following table, for each complete metric space \(S\) considered in Section 4 (first column), we give:

- the compact metric sub-space \(S_n\) (second column), i.e., its \(n\)th-approximant;
- the metric sub-space \(S_*\) (third column), i.e., the union of its approximants;
- the Hausdorff space corresponding to \(S\) (fourth column);
- the compact Hausdorff sub-space corresponding to \(S_n\) (fifth column);
- the compact Hausdorff space \(\Sigma\) given by our construction (sixth column);
- the property of the map \(\iota: S \rightarrow \Sigma\) in \textbf{Haus} (seventh column);
- the section where the example is explained in details (eighth column).

| Met | Haus | see |
|-----|------|-----|
| \(S\) | \(S_n\) | \(S_*\) | \(S\) | \(S_n\) | \(\Sigma\) | \(\iota: S \rightarrow \Sigma\) | Section |
| finite-dimensional Banach spaces |
| \(\mathbb{R}\) | \(\mathbb{R}_n\) | \(\mathbb{R}\) | \(\mathbb{R}_n\) | \(\mathbb{R}\) | sub-space | 1.1 |
| \(\ell_{m,\infty}\) \(\rightarrow\) \(\mathbb{R}^m\) | \(\mathbb{R}^m_n\) \(\rightarrow\) \(\mathbb{R}^m\) | \(\ell_{m,\infty}\) \(\rightarrow\) \(\mathbb{R}^m\) | \(\mathbb{R}^m\) | \(\mathbb{R}^m_n\) | sub-space | 1.2 |
| \(\ell_{m,p}\) \(\rightarrow\) \(\mathbb{R}^m\) | \(\langle \mathbb{R}^m_n, d_p \rangle\) | \(\ell_{m,p}\) | \(\mathbb{R}^m_n\) | \(\mathbb{R}^m\) | sub-space | 1.3 |
| infinite-dimensional Banach spaces |
| \(\ell_\infty\) | \(\mathbb{R}^n\) | \(\ell_{*,\infty}\) | \(\ell_\infty\) | \(\mathbb{R}^n\) | \(\mathbb{R}^n\) | | sub-object | 1.4 |
| \(\ell_p\) \(\rightarrow\) \(\ell_\infty\) | \(\langle \mathbb{R}^n, d_p \rangle\) | \(\ell_{*,p}\) | \(\ell_p\) | \(\mathbb{R}^n\) | \(\mathbb{R}^n\) | | sub-object | 1.5 |
| closed bounded convex subsets of infinite-dimensional Banach spaces |
| \(B_\infty\) \(\twoheadrightarrow\) \(\mathbb{R}^\infty\) | \(B_{\infty,n}\) = \(\mathbb{R}^n\) | \(B_{*,\infty}\) = \(\mathbb{R}^\infty\) | \(B_\infty\) \(\twoheadrightarrow\) \(\mathbb{R}^\infty\) | \(B_1\) | \(B_{\infty,n}\) | \(B_p\) | \(B_{\infty,n}\) | sub-object | 1.6 |
| \(B_p\) \(\twoheadrightarrow\) \(\mathbb{R}^\infty\) | \(B_{n,p}\) | \(B_{*,p}\) | \(B_p\) | \(B_{n,p}\) | \(\mathbb{R}^\infty\) | \(\mathbb{R}^\infty\) | \(\mathbb{R}^\infty\) | sub-object | 1.7 |

Hence, \(\iota(y) = (x_n \mid n)\), and \(\iota\) is surjective.

The relations established in Section 4 imply the following relations between \(B_p\) and \(B_q\) for \(1 \leq p < q \leq \infty\):

- The carrier of \(B_p\) is a proper subset of the carrier of \(B_q\), and the inclusion of \(B_p\) into \(B_q\) is a short map in \textbf{Met}.
- The carrier of \(B_{p,n}\) is a proper subset of the carrier of \(B_{q,n}\), and the inclusion of \(B_{p,n}\) into \(B_{q,n}\) is a short map in \textbf{KMS}.
- As a topological space, \(B_{p,n}\) is a closed sub-space of \(B_{q,n}\) in \textbf{KH}, since the distances \(d_p\) and \(d_q\) induce the same topology on \(\mathbb{R}^n\). Thus, \(B_p\) is a closed sub-space of \(B_q\) in \textbf{KH}.

The last point implies that the carrier of \(\overline{B}_p\) depends on \(p\), but the topology does not, namely, it is the topology induced by that on \(\overline{B}_\infty\).

5 Precision

Table 3 gives a summary of the examples in Section 4. We observe the following:
1. In the finite-dimensional cases, \( S \) is a dense sub-space of \( \overline{S} \) in Haus, and \( \overline{S} \) is a compactification of \( S \). Therefore, every closed subset \( C \) of \( S \) is the intersection \( C' \cap S \) of some closed subset \( C' \) of \( \overline{S} \). Since the distances \( d_p \) and \( d_{\infty} \) induce the same topology on \( \mathbb{R}^m \)—the carrier set of both \( \ell_{m,p} \) and \( \ell_{m,\infty} \)—we have \( C(\ell_{m,p}) = C(\ell_{m,\infty}) \) for each \( p : [1, \infty] \). Furthermore, \( S \) does not depend on \( p \). Hence, it suffices to consider the cases \( S = \ell_{m,\infty} \). The map \( i : \ell_{m,\infty} \to \ell_{m,\infty} \) is a sub-space inclusion and \( i' \circ i = \text{id}_{C(\ell_{m,\infty})} \).

2. In the infinite-dimensional cases—as we will demonstrate—\( S \) is not a sub-space of \( \overline{S} \). More precisely, \( \iota : S \to \overline{S} \) is mono and epi, thus the image of \( \iota \) is dense in \( \overline{S} \). However, \( \iota \) is not a sub-space inclusion, thus there are closed subsets \( C \) of \( S \) which cannot be written as \( C' \cap S \) for some closed subset \( C' \) of \( \overline{S} \), and \( \iota' \circ \iota \) is not the identity on \( C(S) \).

In what follows, we focus mainly on the case of the unit ball \( B_p \). As discussed in Sections 4.3 and 4.4, without loss of generality, we assume that the bijective map \( \iota : B_p \to \overline{B}_p \) is an identity. As a result, by going from \( B_p \) to \( \overline{B}_p \), the carrier set does not change, and we have to compare only the topologies, or equivalently \( C(\overline{B}_p) \subset C(B_p) \). Since the left adjoint \( \iota'^* \) is the inclusion map of \( C(\overline{B}_p) \) into \( C(B_p) \), we have \( \iota'^* (\iota_*(C)) = \iota_*(C) \).

Thus, the loss of precision is measured by how bigger is \( \iota \). Since the left adjoint \( \iota'^* \) is mono and epi, thus the image of \( \iota \) is dense in \( \overline{S} \). Since \( \iota \) is continuous, we have \( \iota'^* (\tau_2) \subseteq \tau_1 \). Therefore, if \( K \) is \( \tau_1 \)-compact and \( U \) is an open cover of \( K \) in \( \iota'^* (\tau_2) \), then \( U \) is also an open cover of \( K \) in \( \tau_1 \). Hence, it has a finite sub-cover \( U_0 \).

**Remark 5.1.** In applications, it is reasonable to restrict to closed bounded subsets of Banach spaces, i.e., closed subsets included in a ball of finite radius. The claims in this section are true for closed balls in \( \ell_p \) with center \( x \) and radius \( r > 0 \), but we state them for the paradigmatic case \( x = 0 \) and \( r = 1 \), to avoid extra parameters in notation. Note that, compact subsets of a Banach space are always closed and bounded, and in the finite-dimensional case the converse also holds.

To start, we present a positive result where there is no loss of precision.

**Proposition 5.2 (Compact sets).** If \( S_1 \xrightarrow{\iota} S_2 \) in Haus, then compact subsets of \( S_1 \) are compact in \( S_2 \).

**Proof.** Write \( \tau_1 \) for the topology on \( S_1 \) and \( \iota \) for the mono \( S_1 \xhookrightarrow{\iota} S_2 \) which we can assume to be an inclusion between the carriers. Let \( \iota'^* (\tau_2) \) be the topology on the carrier of \( S_1 \) induced by \( \tau_2 \). Since \( \iota \) is continuous, we have \( \iota'^* (\tau_2) \subseteq \tau_1 \). Therefore, if \( K \) is \( \tau_1 \)-compact and \( U \) is an open cover of \( K \) in \( \iota'^* (\tau_2) \), then \( U \) is also an open cover of \( K \) in \( \tau_1 \). Hence, it has a finite sub-cover \( U_0 \).

**Corollary 5.3.** Assume that \( S : \text{Haus} \), and \( \iota : S \to \overline{S} \) (in Haus) is as in Theorem 5.10. If \( C : C(S) \) is compact, then \( C : C(\overline{S}) \), and therefore \( \iota_*(C) = C \).

**Proof.** As \( S \xrightarrow{\iota} \overline{S} \), by Proposition 5.2 every compact \( C : C(S) \) is also compact in \( \overline{S} \). The result now follows from the fact that in Hausdorff spaces, compact subsets are closed.

Compact subsets of infinite-dimensional Banach spaces are not so relevant in applications (e.g., they always have empty interior). For \( 1 < p < \infty \), however, we have a characterization of the closed bounded subsets of \( \ell_p \) for which there is no loss of precision, i.e., \( C = \iota'^* (\iota_*(C)) \).

**Theorem 5.4 (No loss of Precision).** For any \( 1 < p < \infty \) and \( C : C(\ell_p) \), the following are equivalent:

1. \( C \) is bounded and \( C = \iota'^* (\iota_*(C)) \), i.e., there is no loss of precision over \( C \).
2. \( C \) is a non-empty intersection of finite unions of closed balls in \( \ell_p \).
3. \( C \) is a non-empty intersection of finite unions of bounded-closed-convex subsets of \( \ell_p \).

**Proof.** See the proof on page 22.

To prove Theorem 5.4 we relate the topology on \( \overline{B}_p \) (indeed on any closed ball in \( \ell_p \)) to the weak-* topology, one of the fundamental topologies studied in functional analysis.

### 5.1 Weak-* topology on \( |B_p| \) for \( 1 \leq p \leq \infty \)

Let us first present a quick reminder of weak and weak-* topologies for the case of normed vector spaces over the field of real numbers. The reader may refer to any standard book on functional analysis, e.g., [19] [4], for the more general treatment of these topologies.

**Definition 5.5 (dual).** Given a normed vector space \( X : \text{NVS} \) with norm \( \| \cdot \|_X \), its continuous dual \( X' \) is the Banach space of linear continuous functions from \( X \) to \( \mathbb{R} \) with norm \( \| f \|_{X'} = \sup \{ |f(x)| : x \in X \land \| x \|_X \leq 1 \} \).

**Proposition 5.6.** If \( Y : \text{Ban} \) is the Cauchy completion of \( X : \text{NVS} \), then \( Y' \) and \( X' \) are isomorphic.
Table 4 Some duals and double duals (up to iso).

| X   | X'  | X'' | Note |
|-----|-----|-----|------|
| ℓ₀  | ℓ₁  | ℓ∞  |      |
| ℓₚ  | ℓ₀ᵣ | ℓₚ  | 1 < p < ∞ |

Proof. See, e.g., [18 Page 270].

Definition 5.7 (reflexive, weak, weak-*) For \( X : \text{NVS} \), the map \( \eta_X : X \to X'' \) (a linear isometry) is defined as \( \eta_X(x)(f) \overset{\Delta}{=} f(x) \) for \( x \in X \) and \( f : X' \). When the map \( \eta_X \) is an iso, \( X \) is called reflexive.

1. The weak topology \( W_X \) on \( X \) is the coarsest topology making each \( f \in X' \) continuous.
2. The weak-* topology \( W_X^* \) on \( X' \) is the coarsest topology on \( X' \) making \( \eta_X(x) \) continuous for each \( x : X \).

Proposition 5.8. Let \( p' \) denote the conjugate of \( p : [1, \infty] \), i.e., the unique \( q : [1, \infty] \) such that \( 1/p + 1/q = 1 \).

1. For every \( p : [1, \infty] \), the Cauchy completion of \( \ell_p \) is the Banach space \( \ell_p \).
2. The Cauchy completion of \( \ell_{p, \infty} \) is the Banach sub-space \( \ell_0 \).
3. For every \( p : [1, \infty] \), the map \( \xi : \ell_{p'} \to (\ell_{p'})' \), given by \( \xi(x')(x) \overset{\Delta}{=} \sum x_i \cdot x_1 \), is an isomorphism.

Proof. Proofs of (1) and (2) are straightforward. To prove (3) by Proposition 5.6 and by (1) and (2) we may regard \( \xi \) as a function from \( \ell_p \) to \( (\ell_{p'})' \), when \( p : [1, \infty] \), or to \( (\ell_0)' \), when \( p = \infty \). The proof that \( \xi \) is an isomorphism may now be found in:

- [4] Appendix B, for \( p : [1, \infty] \).
- [2] Page 50, for \( p = \infty \).

The duals and double duals of relevance in this section are summarized in Table 4.

Definition 5.9 (Topologies \( \tau_p, \tau_{p}', \tau_{p''} \)). We define the following topologies on the carrier of \( \ell_p \):

1. \( \tau_p \) is the original (or, norm) topology on \( \ell_p \), i.e., the topology induced by the norm \( \| \cdot \|_p \).
2. \( \tau_p' \) denotes the weak-* topology on \( \ell_p \) as the continuous dual of \( \ell_{p'} \).
3. \( \tau_p'' \) is the topology on \( \ell_p \) as a subset of the compact Hausdorff space \( \mathbb{R}^\infty \).

We use the same notation for the topologies when restricted to a subset of \( \ell_p \), such as \( B_p \).

According to Table 4 by using the notation of Definition 5.9 we have that \( \bar{\mathbb{S}} = (|B_p|, \tau_p) \) when \( \mathbb{S} = (|B_p|, \tau_p) \).

Therefore, we obtain:

Proposition 5.10. For every \( C : \mathbb{C}(\ell_p) \), the set \( \iota^*(\iota_*(C)) \) is the \( \tau_p \) closure of \( C \).

In Theorem 5.12 we show that the topological spaces \( (|B_p|, \tau_p) \) and \( (|B_p|, \tau_p') \) coincide for any \( 1 \leq p \leq \infty \).

First, we recall the lemma on the ‘rigidity’ of compact Hausdorff topologies in [19 Section 3.8].

Lemma 5.11. If \( \tau_1 \subseteq \tau_2 \) are topologies on a set \( X \), with \( \tau_1 \) Hausdorff and \( \tau_2 \) compact, then \( \tau_1 = \tau_2 \).

Proof. Let \( F \subseteq X \) be \( \tau_2 \)-closed. Since \( X \) is \( \tau_2 \)-compact, so is \( F \). Since \( \tau_1 \subseteq \tau_2 \), it follows that \( F \) is \( \tau_1 \)-compact. Since \( \tau_1 \) is a Hausdorff topology, it follows that \( F \) is \( \tau_1 \)-closed.

Theorem 5.12. For each \( 1 \leq p \leq \infty \), \( \overline{B_p} \) is (isomorphic to) the topological space \( (|B_p|, \tau_p^*) \).

Proof. The topology \( \tau_{p} \) is Hausdorff. On the other hand, by the Banach-Alaoglu theorem (see, e.g., [19 Section 3.8]) the closed unit ball \( |B_p| \) is weak-* compact, i.e., \( \tau_{p}^* \)-compact. Therefore, by Lemma 5.11 it suffices to prove that \( \tau_{p} \subseteq \tau_{p}^* \).

For \( i : \omega \), consider the projections \( \pi_i : \ell_p \to \mathbb{R} \) defined by

\[
\forall \ x : \ell_p : \pi_i(x) \overset{\Delta}{=} x_i. \tag{5}
\]
The set $\mathcal{Y} = \{ |B_B| \cap \pi_i^{-1}(O) | i : \omega, O \subseteq \mathbb{R} \text{ Euclidean open} \}$ is a sub-base for $\mathfrak{T}_p$. For each $i : \omega$, consider the sequence $e_i$ defined by

$$\forall n : \omega. \ e_i(n) \triangleq \begin{cases} 0, & \text{if } i \neq n, \\ 1, & \text{if } i = n. \end{cases}$$

(6)

For all $i : \omega$, the sequence $e_i$ is in all the relevant pre-duals as specified in Definition 5.10 above. Furthermore

$$\forall i : \omega. \forall x : \ell_p : \pi_i(x) = x(e_i).$$

Thus, each $\pi_i$ is weak-* continuous and $\mathcal{Y} \subseteq \tau^*_p$, which entails that $\mathfrak{T}_p \subseteq \tau^*_p$. □

**Remark 5.13.** As pointed out in Remark 5.1, although we present results for closed unit balls, they hold for arbitrary closed balls. In particular, Theorem 5.12 holds for closed balls in $\ell_p$, because they are all $\tau^*_p$-compact.

**Corollary 5.14.** For each $1 \leq p \leq \infty$ and $C : \mathcal{C}(B_p)$, $\mathfrak{I}(C) = C$ iff $C$ is a $\tau^*_p$-closed subset of $|B_p|$.

**Proof.** Follows from Proposition 5.10 and Theorem 5.12. □

**Corollary 5.15.** For each $1 < p < \infty$ and $C : \mathcal{C}(B_p)$, $\mathfrak{I}(C) = C$ is a weakly-closed subset of $|B_p|$.

**Proof.** Since the space $\ell_p$ for $1 < p < \infty$ is reflexive, i.e., the weak and weak-* topologies on $\ell_p$ coincide. The result follows from Corollary 5.14. □

We have established all the preliminaries for presenting the proof of Theorem 5.3.

**Proof.** (Theorem 5.4)

1. $\Rightarrow$ 2. As $C$ is assumed to be bounded, then it must be a subset of a closed ball $B$ of finite radius. If $\mathfrak{I}(\mathfrak{I}(C)) = C$, then, by Corollary 5.15, $C$ must be weakly closed. Thus, $C$ is a weakly closed subset of (the weakly compact set) $B$. As a result, it is weakly compact.

According to [3], a subset of a separable reflexive space is weakly compact if and only if it is the non-empty intersection of finite unions of closed balls. Each space $\ell_p$ for $1 < p < \infty$ is separable and reflexive. Hence, the result follows.

2. $\Rightarrow$ 3. This is straightforward as every closed ball is a bounded-closed-convex subset.

3. $\Rightarrow$ 1. Assume that $C$ is a non-empty intersection of finite unions of bounded-closed-convex subsets of $\ell_p$. To be precise, $C = \bigcap_{i \in I} \bigcup_{j \in J} C_{i,j}$ with $k : \omega^I$ and $C_{i,j}$ bounded-closed-convex subset of $\ell_p$.

As a consequence of Hahn-Banach separation theorem, every closed and convex subset of a Banach space is weakly closed (see, e.g., [13, Theorem 3.12]). This entails that each $C_{i,j}$ is weakly closed. As the set of closed subsets (under any topology) are closed under finite unions and arbitrary intersections, then $C$ itself is also weakly closed. Clearly, $C$ is also bounded. The result now follows from Corollary 5.14.

Item 3 of Theorem 5.4 provides examples of practical importance where no loss of precision is incurred.

**Example 5.16** (Sequence intervals). For each pair $s, t : \ell_p$, we define the sequence interval $[s, t]$ by:

$$[s, t] \triangleq \{ u : \ell_p | \forall i : \omega. \ s_i \leq u_i \leq t_i \}.$$ 

In general, sequence intervals are not norm-compact in $\ell_p$. They are, however, bounded, norm-closed, and convex. Hence, when $1 < p < \infty$, there is no loss of precision over sequence intervals, or indeed, over any subset $C$ of $\ell_p$ which may be written as an intersection of finite unions of sequence intervals.

### 5.1.1 Metrizability of the weak-* topology over closed balls

Consider the map $d_s : \ell_\infty \times \ell_\infty \to \mathbb{R}$ defined as follows:

$$\forall x, y : \ell_\infty. \ d_s(x, y) \triangleq \sum_{n \in \omega} \frac{d(x_n, y_n)}{2^{n+1}}.$$ 

(7)

We prove that $d_s$ is a metric on the carrier of $\ell_\infty$, which induces the topology on $\mathfrak{T}_\infty$.

**Proposition 5.17.** For any closed ball $B$ of finite radius in $\ell_\infty$, the weak-* topology on $B$ is induced by $d_s$. □
Corollary 5.18. For each topology on $B$ centered at $x$, the metric $(\ell_p, x, y) = \ell_\infty(x, y) < \delta$. Furthermore, each $\pi_i$ is weak-* continuous. Thus, it suffices to take $f_n = \pi_n/2$, and the claim follows from [19, Section 3.8(c), page 63].

The metric $d_\ast$ on the carrier of $\ell_\infty$ satisfies the following property

$$\forall p : [1, \infty]. \forall x, y : \ell_\infty, d_\ast(x, y) = \sum_{n=0}^\infty \frac{d(x_n, y_n)}{2^{n+1}} \leq \sum_{n=0}^\infty \frac{d_\infty(x, y)}{2^{n+1}} = d_\infty(x, y) \leq d_\ast(x, y). (8)$$

Corollary 5.18. For each $p : [1, \infty]$, the weak-* topology on a closed ball of finite radius in $\ell_p$ is induced by $d_\ast$.

Proof. Let $B$ be the closed ball in $\ell_p$ centered at $x_0$ with radius $r > 0$. Let $B'$ denote the closed ball in $\ell_\infty$ centered at $y_0$ with radius $r > 0$. The weak-* topology on $B$ is the relative topology induced from the weak-* topology on $B'$. The claim now follows from Proposition 5.17.

By going from $B_p$ to $\mathcal{B}_p$, robustness with respect to the metric $d_p$ is replaced by robustness with respect to the metric $d_\ast$. Indeed, inequality $[3]$ shows that for any subset $S$ of $|B_p|$ its $\delta$-neighborhood $B(S, \delta)$ under $d_p$ is included its $\delta$-neighborhood under $d_\ast$.

Proposition 5.19. For every $C : \mathbb{C}(B_p)$:
1. The set $\iota_\ast(C)$ is the closure of $C$ under the $d_\ast$ metric.
2. For every $x : B_p$, $x \in \iota_\ast(C)$ if and only if:

$$\forall n : \omega, \forall \delta > 0.3y : C. \exists i : n. d(x_i, y_i) < \delta. (9)$$

Proof.
1. Follows from Corollary 5.18.
2. From [1] we deduce that $x \in \iota_\ast(C)$ if and only if:

$$\forall \epsilon > 0.3y : C. d_\ast(x, y) < \epsilon. (10)$$

Thus, it suffices to prove that $[9]$ and $[10]$ are equivalent.

$[9] \Rightarrow [10]$: For any given $\epsilon > 0$, choose $n$ large enough such that $\epsilon 2^{n-2} > 1$ and $\epsilon = \epsilon/2$. By [9], there exists a $y : C$ such that $\forall i : n. d(x_i, y_i) < \delta$. For this $y$ we have:

$$d_\ast(x, y) \leq \sum_{i=0}^n \frac{d(x_i, y_i)}{2^{i+1}} + \sum_{i=n+1} \frac{d(x_i, y_i)}{2^{i+1}} \leq \sum_{i=0}^n \frac{\delta}{2^{i+1}} + \sum_{i=n+1} \frac{2}{2^{i+1}} \leq \delta + 2 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.

[10] \Rightarrow [9]$: For any given $n : \omega$ and $\delta > 0$, choose $\epsilon = \delta/2^{n+1}$. By [10], there exists a $y : C$ such that $d_\ast(x, y) < \epsilon$, which implies that:

$$\sum_{i=0}^\infty \frac{d(x_i, y_i)}{2^{i+1}} < \epsilon = \frac{\delta}{2^{n+1}}. (11)$$

From $[11]$, we obtain $\forall i : \omega. d(x_i, y_i) < \epsilon \ast 2^{i-n}$. In particular, for every $i : n$, we have $d(x_i, y_i) < \delta$. 

$\square$
5.2 Loss of precision

In this subsection, we discuss loss of precision through a few examples.

Example 5.20 (bounded, closed, non-convex, discrete). Take the sequences \( (e_i \mid i : \omega) \) as defined in \([4]\), i.e.:

\[
\forall n : \omega. \quad e_i(n) \triangleq \begin{cases} 
0, & \text{if } i \neq n, \\
1, & \text{if } i = n, 
\end{cases}
\]

and consider the set \( C \triangleq \{ e_i \mid i : \omega \} \), which is a discrete, bounded, and closed subset of \( B_p \) for \( p : [1, \infty] \). Using the metric \( d_\ast \) of \([4]\), we show that \( \iota_\ast(C) = \bar{C} \cup \{0\} \).

It is indeed clear that 0 is in the weak-* closure of \( C \) as:

\[
\lim_{i \to \infty} d_\ast(e_i, 0) = \lim_{i \to \infty} 2^{-(i+2)} = 0.
\]

Furthermore, \( \forall i, j : \omega. \quad d_\ast(e_i, e_j) \leq 2^{-(i+2)} + 2^{-(j+2)} \). This entails that \( \lim_{i, j \to \infty} d_\ast(e_i, e_j) = 0 \). Hence, the sequence \( (e_i \mid i : \omega) \) is a Cauchy sequence under \( d_\ast \) and it can have only one limit point, i.e., zero.

Example 5.21 (bounded, closed, non-convex, connected). Consider the unit sphere \( S_p \triangleq \{ s : B_p \mid \| s \|_p = 1 \} \), which is norm-closed and non-convex. The weak-* closure of the unit sphere in \( |B_p| \) is the entire unit ball \( |B_p| \). In this case, the loss of precision is quite noticeable.

In Theorem 5.3, the assumption \( p \notin \{1, \infty\} \) is crucial, as demonstrated in Example 5.22 and Proposition 5.25.

Example 5.22 (bounded, closed, convex, \( p = \infty \)). Consider the set \( C \triangleq c_0 \cap B_\infty \), which is a bounded, closed, and convex subset of \( B_\infty \). In Section 4.6 we proved that \( \iota_\ast(C) = B_\infty \).

Theorem 5.3 relies on the fact that norm-closed and convex subsets of reflexive Banach spaces are weakly closed. On the other hand, norm-closed and convex subsets of non-reflexive Banach spaces are not, in general, weak-* closed. Recall that the map \( \eta_X \) in Definition 5.7 is a linear isometry which embeds \( X \) into \( X'' \). When \( X \) is a non-reflexive Banach space, we have \( X'' \nsubseteq X \neq \emptyset \).

Proposition 5.23. Assume that \( X \) is a non-reflexive Banach space. Then, for any element \( \theta \in X'' \setminus X \), the kernel \( \theta^{-1}(0) \) of \( \theta \) is a closed linear sub-space of \( X' \) which is not weak-* closed.

Proof. This is a consequence of \([4]\) Theorem 3.1, page 108 and \([19]\) Theorem 3.10, page 64].

Using the following result, for any given non-reflexive Banach space, one may construct many examples of sets for which there is a loss of precision, provided the space is the dual of another Banach space, i.e., has a pre-dual. This rules out spaces such as \( c_0 \) which have no pre-duals \([2]\) Theorem 6.3.7).

Corollary 5.24 (bounded, closed, convex, non-reflexive with pre-dual). Consider a non-reflexive Banach space \( X \) and let \( B \) denote the closed unit ball of \( X' \). For any \( \theta \in X'' \setminus X \), define \( C \triangleq B \cap \theta^{-1}(0) \). Then, the set \( C \) is closed and convex—hence, weakly closed—but not weak-* closed.

Proof. As a consequence of Proposition 5.23, the set \( C \) is closed and convex, hence, weakly closed. By \([4]\) Corollary 12.6, page 160], however, \( C \) cannot be weak-* closed.

In general, an exact description of the weak-* closures of sets \( C \) obtained in Corollary 5.24 is not known. There are, however, inner and outer approximations available in the literature. For instance, according to \([15]\) Proposition 2], the weak-* closure of \( C \) must contain a closed ball centered at the origin. Proposition 5.25 gives an instance of this result, which can be proved directly.

Proposition 5.25 (bounded, closed, convex, \( p = 1 \)). Take the set \( C \triangleq \{ x : B_1 \mid \sum_{n \omega} x_n = 0 \} \) and the closed ball \( D \) in \( \ell_1 \) centered at 0 with radius \( 1/2 \). Furthermore, let \( \overline{C} \) denote the \( \tau_{\ell_1}^* \)-closure of \( C \) in \( B_1 \). Then:

1. \( C \) is a closed and convex—hence, also weakly closed—subset of \( B_1 \). But, \( C \) is not weak-* closed.
2. \( D \cap C = \emptyset \).
3. \( D \subset \overline{C} \subset B_1 \), where both inclusions are strict.

Proof. Take \( X = c_0 \) (which implies that \( X' = \ell_1 \) and \( X'' = \ell_\infty \)) and let \( \theta = [1]^\omega : \ell_\infty \setminus c_0 \). We have \( C = B_1 \cap \theta^{-1}(0) \).

1. Follows from Corollary 5.24.
2. \( \{ \frac{1}{2^n} : n : \omega \} \in D \setminus C \).
3. Assume that $x = (x_n \mid n : \omega) : D$. For any $n : \omega$ and $\delta > 0$, consider $\hat{x} = (\hat{x}_i \mid i : \omega)$ defined as follows:

$$\forall i : \omega. \hat{x}_i \triangleq \begin{cases} x_i, & \text{if } i < n, \\ -x_{i-n}, & \text{if } n \leq i < 2n, \\ 0, & \text{otherwise}. \end{cases}$$

We have $\theta(\hat{x}) = \sum_{i : \omega} \hat{x}_i = 0$. Furthermore, $\| \hat{x} \|_1 \leq 2\| x \|_1 \leq 1$. Hence, $\hat{x} \in \theta^{-1}(0) \cap B_1 = C$. Clearly, for all $i : n$, we have $d(x_i, \hat{x}_i) = 0 \leq \delta$. Therefore, equation (10) of Proposition 5.19 holds, and we have $D \subseteq C$.

Take the sequence $y = (y_n : n : \omega)$ defined as follows:

$$\forall n : \omega. y_n \triangleq \begin{cases} 0.5, & \text{if } n = 0, \\ -0.5, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $y \in C \setminus D \subseteq \overline{C} \setminus D$. Thus, we have proved that $D \subset \overline{C}$.

It remains to prove that $B_1 \setminus \overline{C} \neq \emptyset$. Take the sequence $z = (z_n : n : \omega)$ defined as follows:

$$\forall n : \omega. z_n \triangleq \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, $z \in B_1$. We use Proposition 5.19 to prove that $z \notin \overline{C}$. In (10), choose $n = 1$ and $\delta = 1/2$. For any given $\hat{z} : C$, we have $\sum_{n : \omega} \hat{z}_n = 0$. Thus, $\sum_{n=1}^{\infty} \hat{z}_n = -\hat{z}_0$, which implies that:

$$\left| \sum_{n=1}^{\infty} \hat{z}_n \right| = | \hat{z}_0 | .$$

(12)

On the other hand, as $\hat{z} : C$, we must have $\| \hat{z} \|_1 \leq 1$. As a result:

$$1 \geq \| \hat{z} \|_1 = \sum_{n : \omega} | \hat{z}_n | \geq | \hat{z}_0 | + \sum_{n=1}^{\infty} | \hat{z}_n | .$$

(13)

By combining (12) and (13), we obtain $| \hat{z}_0 | \leq 1/2$, which implies $1 - \hat{z}_0 \geq 1/2$, equivalently $d(z_0, \hat{z}_0) \geq 1/2 = \delta$.

\[\square\]

6 Concluding Remarks

The results in this paper are part of an overall study of robust maps. We have chosen the theory of $\omega$-continuous lattices, within which computability can be studied using the framework of effectively given domains \[20\], and robustness can be analyzed by using the Robust topology (Definition \[2.20\]) over the lattice of closed subsets of the state space.

In a related work, Edalat \[8\] has considered locally compact Hausdorff spaces instead of metric spaces, and has worked with the domain of compact subsets (ordered by reverse inclusion) instead of the complete lattice of closed subsets. Furthermore, he has investigated the relationship between Scott topology and the upper Vietoris topology, but has not studied robustness. The Robust topology lies in between Scott and the upper Vietoris topologies (Theorem \[2.21\]).

The case of compact metric spaces has been studied in \[16\]. This suffices to deal with the input space of typical machine learning systems, and the state space of common hybrid systems. In this paper, the focus has been non-compact metric spaces, for which a novel approach has been presented based on approximation of the space via a (growing) sequence of compact metric sub-spaces. Non-compact spaces are relevant when dealing with perturbations of the model parameters of a system, e.g., perturbations of the activation function(s) of a neural network, or the flow $F$ and jump $G$ relations of a hybrid system $(S, F, G)$.

We presented a detailed account of some examples, including (closed bounded subsets of) infinite-dimensional Banach spaces, and analyzed the important issue of precision, when it is retained, and when precision is lost. In particular, we have obtained a complete characterization of the closed subsets of reflexive spaces $F_p$ (i.e., those with $1 < p < \infty$), for which there is no loss of precision (Theorem \[5.1\]).

All examples studied in this paper are sequence spaces. As such, studying other relevant spaces provides an immediate direction for future work. For instance, let $\Omega \subseteq \mathbb{R}^n$ be an open set. Lebesgue spaces $L^p(\Omega)$ are examples for future work, which are relevant in the study of partial differential equations \[3\].
Other cases for future study include infinite-dimensional feature spaces arising in machine learning and spaces of bounded measures. In particular, by applying our results to (closed subsets of) probability measures, we obtain a framework for computation of probability measures using finitary approximations. It will be interesting to compare the finitary approximations obtained in this way, with those obtained by Edalat \[9\] for computation of probability measures over separable metric spaces.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, Oxford, 1994.

[2] Fernando Albiac and Nigel J. Kalton. *Topics in Banach Space Theory*. Springer, 2006.

[3] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[4] John B. Conway. *A Course in Functional Analysis*. Springer, 2nd edition edition, 1990.

[5] H. H. Corson and J. Lindenstrauss. On weakly compact subsets of Banach spaces. *Proceedings of the American Mathematical Society*, 17(2):407–412, 1966.

[6] Patrick Cousot. Abstract interpretation. *ACM Computing Surveys (CSUR)*, 28(2):324–328, 1996.

[7] Patrick Cousot and Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Proceedings of the 4th ACM SIGACT-SIGPLAN symposium on Principles of programming languages*, pages 238–252, 1977.

[8] Abbas Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, July 1995.

[9] Abbas Edalat. When Scott is weak on the top. *Mathematical Structures in Computer Science*, 7(5):401–417, 1997.

[10] Martin Fränzle. Analysis of hybrid systems: An ounce of realism can save an infinity of states. In *Computer Science Logic*, pages 126–139. Springer, 1999.

[11] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encycloedia of Mathematics and its Applications*. Cambridge University Press, 2003.

[12] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott. *A Compendium of Continuous Lattices*. Springer, 1980.

[13] Rafał Goebel, Ricardo G Sanfelice, and A Teel. Hybrid dynamical systems. *Control Systems, IEEE*, 29(2):28–93, 2009.

[14] Thomas Hofmann, Bernhard Schölkopf, and Alexander J. Smola. Kernel methods in machine learning. *The Annals of Statistics*, 36(3):1171–1220, 2008.

[15] G. J. O. Jameson. The weak-star closure of the unit ball in a subspace. *Proceedings of the Edinburgh Mathematical Society*, 25(1):87–95, 1982.

[16] Eugenio Moggi, Amin Farjudian, Adam Duracz, and Walid Taha. Safe & robust reachability analysis of hybrid systems. *Theoretical Computer Science*, 747:75–99, 2018.

[17] Eugenio Moggi, Amin Farjudian, and Walid Taha. System analysis and robustness. In Alessandra Cherubini, Nicoletta Sabadini, and Simone Tini, editors, *Proceedings of the 20th Italian Conference on Theoretical Computer Science, ICTCS 2019, Como, Italy, September 9-11, 2019*, volume 2504 of *CEUR Workshop Proceedings*, pages 1–7. CEUR-WS.org, 2019.

[18] Lawrence Narici and Edward Beckenstein. *Topological Vector Spaces*. Chapman and Hall/CRC, second edition, 2011.

[19] Walter Rudin. *Functional Analysis*. McGraw-Hill, 2 edition, 1991.

[20] Michael B. Smyth. Effectively given domains. *Theoretical Computer Science*, 5(3):257–274, 1977.