A Hilbert space approach to fractional differential equations

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Abstract

We study fractional differential equations of Riemann-Liouville and Caputo type in Hilbert spaces. Using exponentially weighted spaces of functions defined on \( \mathbb{R} \), we define fractional operators by means of a functional calculus using the Fourier transform. Main tools are extrapolation- and interpolation spaces. Main results are the existence and uniqueness of solutions and the causality of solution operators for non-linear fractional differential equations.

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1 Introduction

The concept of a fractional derivative \( \partial^\alpha_0, \alpha \in [0, 1] \), which we utilize, will be based on inverting a suitable continuous extension of the Riemann-Liouville fractional integral of continuous functions \( f \in C_c(\mathbb{R}) \) with compact support given by

\[
t \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds
\]

as an apparently natural interpolation suggested by the iterated kernel formula for repeated integration. The choice of the lower limit as \( -\infty \) is determined by our wish to study dynamical processes, for which causality should play an important role. It is a pleasant fact that the classical definition of \( \partial^\alpha_0 \) in the sense

\[
t \mapsto \frac{1}{\sqrt{2\pi}} \chi_{[a, \infty[} (t) \int_{a}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds
\]

for \( a \in \mathbb{R} \), would lose time-shift invariance (a suggestive choice is \( a = 0 \)), which we consider undesirable. For our choice of the limit case \( a = -\infty \) it should be noted that the Riemann-Liouville and the Caputo fractional derivative essentially coincide.

¹Other frequent choices such as
Definition. Let $H$ be a Hilbert space, $p \in [1, \infty]$. For $f \in L^1_\text{loc}(\mathbb{R}; H)$ we denote $e^{-\varepsilon m} f := (\mathbb{R} \ni t \mapsto e^{-\varepsilon t} f(t))$. We define the normed spaces

$$L^p_\varepsilon(\mathbb{R}; H) := \left\{ f \in L^1_\text{loc}(\mathbb{R}; H); e^{-\varepsilon m} f \in L^p(\mathbb{R}; H) \right\},$$

with norm

$$\|f\|_{L^p_\varepsilon(\mathbb{R}; H)} := \|e^{-\varepsilon m} f\|_{L^p(\mathbb{R}; H)} = \left( \int_\mathbb{R} \|f(t)\|_H^p e^{-\varepsilon t} \, dt \right)^{1/p} \quad (p < \infty),$$

$$\|f\|_{L^\infty_\varepsilon(\mathbb{R}; H)} := \|e^{-\varepsilon m} f\|_{L^\infty(\mathbb{R}; H)} = \text{ess sup} \|e^{-\varepsilon m} f\|_H \quad (p = \infty).$$

Remark 2.1. The operator $e^{-\varepsilon m} : L^p_\varepsilon(\mathbb{R}; H) \to L^p(\mathbb{R}; H), f \mapsto e^{-\varepsilon m} f$ is an isometric isomorphism from $L^p_\varepsilon(\mathbb{R}; H)$ to $L^p(\mathbb{R}; H)$. Moreover $L^2_\varepsilon(\mathbb{R}; H)$ is a Hilbert space with scalar product

$$(f, g) \mapsto (f, g)_{L^2_\varepsilon(\mathbb{R}; H)} = \int_\mathbb{R} \langle f(t), g(t) \rangle_H e^{-2\varepsilon t} \, dt.$$

Next, we introduce the time derivative.

Definition. Let $H$ be a Hilbert space.

(a) Let $f, g \in L^1_\text{loc}(\mathbb{R}; H)$. We say that $f' = g$, if for all $\phi \in C_c^\infty(\mathbb{R})$

$$-\int_\mathbb{R} f \phi' = \int_\mathbb{R} g \phi.$$

2 Fractional derivative in a Hilbert space setting

In the present section, we introduce the necessary operators to be used in the following. We will formulate all results in the vector-valued, more specifically, in the Hilbert space-valued situation. On a first read, one may think of scalar-valued functions.

To begin with, we introduce an $L^2$-variant of the exponentially weighted space of continuous functions that proved useful in the proof of the Picard–Lindelöf Theorem and is attributed to Morgenstern, [4].

We denote by $L^p(\mathbb{R}; H)$ and $L^1_\text{loc}(\mathbb{R}; H)$ the space of $p$-Bochner integrable functions and the space of locally Bochner integrable functions on a Hilbert space $H$, respectively.

Definition. Let $H$ be a Hilbert space, $p \in [1, \infty]$. For $f \in L^1_\text{loc}(\mathbb{R}; H)$ we denote $e^{-\varepsilon m} f := (\mathbb{R} \ni t \mapsto e^{-\varepsilon t} f(t))$. We define the normed spaces

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$$\|f\|_{L^\infty_\varepsilon(\mathbb{R}; H)} := \|e^{-\varepsilon m} f\|_{L^\infty(\mathbb{R}; H)} = \text{ess sup} \|e^{-\varepsilon m} f\|_H \quad (p = \infty).$$

Remark 2.1. The operator $e^{-\varepsilon m} : L^p_\varepsilon(\mathbb{R}; H) \to L^p(\mathbb{R}; H), f \mapsto e^{-\varepsilon m} f$ is an isometric isomorphism from $L^p_\varepsilon(\mathbb{R}; H)$ to $L^p(\mathbb{R}; H)$. Moreover $L^2_\varepsilon(\mathbb{R}; H)$ is a Hilbert space with scalar product

$$(f, g) \mapsto (f, g)_{L^2_\varepsilon(\mathbb{R}; H)} = \int_\mathbb{R} \langle f(t), g(t) \rangle_H e^{-2\varepsilon t} \, dt.$$
Let \( g \in \mathbb{R} \). We define
\[
\partial_{0,g} : H^1_0(\mathbb{R}; H) \subseteq L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H)
\]
\[
f \mapsto f',
\]
where \( H^1_0(\mathbb{R}; H) := \{ f \in L^2(\mathbb{R}; H); f' \in L^2(\mathbb{R}; H) \} \).

The index 0 in \( \partial_{0,g} \) shall indicate that the derivative is with respect to time. We will introduce the fractional derivatives and fractional integrals by means of a functional calculus for \( \partial_{0,g} \). For this, we introduce the Fourier–Laplace transform.

**Definition.** Let \( H \) be a complex Hilbert space. Let \( g \in \mathbb{R} \).

(a) We define the Fourier transform of \( f \in L^1(\mathbb{R}; H) \) by
\[
\mathcal{F} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} \, dt, \quad \xi \in \mathbb{R}.
\]
(b) We define the Fourier–Laplace transform on \( L^1_0(\mathbb{R}; H) \) by \( \mathcal{L}_g := \mathcal{F} e^{-gm} \).
(c) We define the Fourier transform on \( L^2(\mathbb{R}; H) \) denoted \( \mathcal{F} : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H) \) to be the unitary extension of the operator \( \mathcal{F} : L^1(\mathbb{R}; H) \cap L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H) \).
(d) We define the Fourier–Laplace transform on \( L^2_0(\mathbb{R}; H) \) as the unitary mapping \( \mathcal{L}_g := \mathcal{F} e^{-gm} : L^2_0(\mathbb{R}; H) \to L^2(\mathbb{R}; H) \).

From now on, \( H \) denotes a complex Hilbert space. With the latter notion at hand, we provide the spectral representation of \( \partial_{0,g} \) as the multiplication-by-argument operator
\[
dom(m) := \{ f \in L^2(\mathbb{R}; H); (\exists g \mapsto f(\xi)) \in L^2(\mathbb{R}; H) \},
\]
\[
m : L^2(\mathbb{R}; H) \supseteq \dom(m) \to L^2(\mathbb{R}; H), \quad f \mapsto (\exists g \mapsto \xi f(\xi)).
\]

**Theorem 2.2.** Let \( g \in \mathbb{R} \). Then

(a) \( \partial_{0,0} = \mathcal{F}^* i m \mathcal{F} \),
(b) \( e^{-gm} \ast \partial_{0,0} e^{-gm} = \partial_{0,0} - g \),
(c) \( \partial_{0,0} = \mathcal{L}_g^*(i m + g) \mathcal{L}_g \).

**Proof.** For the proof of (a) we observe that the equality holds on the Schwartz space \( S(\mathbb{R}; H) \) of smooth, rapidly decaying functions. In fact, this is an easy application of integration by parts. The result thus follows from using that \( \mathcal{F} \) is a bijection on \( S(\mathbb{R}; H) \) and that \( S(\mathbb{R}; H) \) is an operator core, for both \( m \) and \( \partial_{0,0} \).

For the statement (b) let \( f \in H^1_0(\mathbb{R}; H) \) and \( \varphi \in C^\infty_c(\mathbb{R}) \). Then \( e^{-gm} \varphi \in C^\infty_c(\mathbb{R}) \) and
\[
\int_{\mathbb{R}} (e^{-gm} f) \varphi' = \int_{\mathbb{R}} f(e^{-gm} \varphi')
\]
\[
= \int_{\mathbb{R}} f((e^{-gm} \varphi)' + g e^{-gm} \varphi)
\]
\[
= - \int_{\mathbb{R}} \partial_{0,0} f e^{-gm} \varphi + \int_{\mathbb{R}} f g e^{-gm} \varphi
\]
\[ = - \int_{\mathbb{R}} (e^{-\alpha m} \partial_{\alpha,0} f - \alpha e^{-\alpha m} f) \varphi. \]

Hence \( e^{-\alpha m} f \in H^1(\mathbb{R}; H) \) and \( \partial_{\alpha,0} e^{-\alpha m} f = e^{-\alpha m} \partial_{\alpha,0} f - \alpha e^{-\alpha m} f \).

Next, we address [c]. By part (a) and (b) we compute

\[
\begin{align*}
\partial_{\alpha,0} &= (e^{-\alpha m})^* \partial_{\alpha,0} e^{-\alpha m} + \alpha \\
&= (e^{-\alpha m})^* F^* i F e^{-\alpha m} + \alpha \\
&= (e^{-\alpha m})^* F^* i F e^{-\alpha m} + (e^{-\alpha m})^* F^* \alpha F e^{-\alpha m} \\
&= \mathcal{L}_0^* (i \alpha + \alpha) \mathcal{L}_0. \tag*{\square}
\end{align*}
\]

Theorem 2.2 tells us that \( \partial_{\alpha,0} \) is unitarily equivalent to a multiplication operator with spectrum equal to \( i\mathbb{R} + \alpha = \{ z \in \mathbb{C}; \Re z = \alpha \} \). In particular, we are now in the position to define functions of \( \partial_{\alpha,0} \).

**Definition.** Let \( \alpha \in \mathbb{R} \) and \( F \colon \text{dom}(F) \subseteq \{it + \alpha; \ t \in \mathbb{R}\} \to \mathbb{C} \) be measurable such that \( \{t \in \mathbb{R}; it + \alpha \notin \text{dom}(F)\} \) has Lebesgue measure zero. We define

\[ F(\partial_{\alpha,0}) := \mathcal{L}_0^* F(i \alpha + \alpha) \mathcal{L}_0, \]

where

\[ F(i \alpha + \alpha) := \left( \mathbb{R} \ni \xi \mapsto F(i \xi + \alpha) f(\xi) \right) \]

in case \( f \in L^2(\mathbb{R}; H) \) is such that \( (\xi \mapsto F(i \xi + \alpha) f(\xi)) \in L^2(\mathbb{R}; H) \).

We record an elementary fact on multiplication operators.

**Proposition 2.3.** Let \( F \) be as in the previous definition. We denote \( \| F \|_{\infty, \mathcal{P}} := \text{ess sup}_{\xi \in \mathbb{R}} |F(i \xi + \alpha)| \in [0, \infty] \). The operator \( F(\partial_{\alpha,0}) \) is bounded, if and only if \( \| F \|_{\infty, \mathcal{P}} < \infty \). If \( F(\partial_{\alpha,0}) \) is bounded, then \( \| F(\partial_{\alpha,0}) \| = \| F \|_{\infty, \mathcal{P}} \).

**Proof.** Since \( \mathcal{L}_0 \) is unitary we may prove that \( F(i \alpha + \alpha) \) is bounded on \( L^2(\mathbb{R}; H) \) if and only if \( \| F \|_{\infty, \mathcal{P}} < \infty \).

Suppose \( \| F \|_{\infty, \mathcal{P}} < \infty \). Then for \( f \in L^2(\mathbb{R}; H) \) we have \( \int_{\mathbb{R}} \| F(i \xi + \alpha) f(\xi) \|_H^2 \, d\xi \leq \| F \|_{\infty, \mathcal{P}}^2 \| f \|_{L^2}^2 \). Hence \( F(i \alpha + \alpha) \) is bounded with \( \| F(i \alpha + \alpha) f \| \leq \| F \|_{\infty, \mathcal{P}} \| f \|_{L^2} \). Let \( F(i \alpha + \alpha) \) be bounded. For \( f \in L^2(\mathbb{R}; H) \) with \( \| f \|_{L^2} = 1 \) we have

\[ \infty > \| F(i \alpha + \alpha) f \|^2 \geq \int_{\mathbb{R}} |F(i \xi + \alpha)|^2 \| f(\xi) \|^2_H \, d\xi. \]

There is a sequence \( (f_n)_{n \in \mathbb{N}} \in L^2(\mathbb{R}; H)^{\mathbb{N}} \) with \( \| f_n \|_{L^2} = 1 \) \( (n \in \mathbb{N}) \) and \( \int_{\mathbb{R}} |F(i \xi + \alpha)|^2 \| f_n(\xi) \|^2_H \, d\xi \to \text{ess sup} |F(i \cdot + \alpha)|^2 \) for \( n \to \infty \). This shows \( \infty > \| F(i \alpha + \alpha) \| \geq \| F \|_{\infty, \mathcal{P}} \). To construct \( (f_n)_{n \in \mathbb{N}} \) wlog. we may assume that \( \text{ess sup} |F(i \cdot + \alpha)| > 0 \). Let \( x \in H \) with \( \| x \|_H = 1 \) and let \( (c_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \) be a positive sequence with \( c_n \uparrow \text{ess sup} |F(i \cdot + \alpha)| \). Then for \( n \in \mathbb{N} \) set \( A_n := [-n, n] \cap \{ |F(i \cdot + \alpha)| > c_n \} \). By the definition of the essential supremum, we may assume that \( \lambda(A_n) > 0 \) and for \( n \in \mathbb{N} \) we set

\[ f_n := 1_{A_n} \frac{x}{\sqrt{\lambda(A_n)}}. \tag*{\square} \]

One important class of operators that can be rooted to be of the form just introduced are fractional derivatives and fractional integrals:
Example 2.4. Let $\alpha \in \mathbb{R}_{>0}$ and $\varrho \in \mathbb{R}$. Then the fractional derivative of order $\alpha$ is given by
\[
\partial_{0,\varrho}^\alpha = \mathcal{L}_\varrho^* (\text{i} \varrho + \varrho)^\alpha \mathcal{L}_\varrho
\]
and the fractional integral of order $\alpha$ is given by
\[
\partial_{0,\varrho}^{-\alpha} = \mathcal{L}_\varrho^* \left( \frac{1}{\text{i} \varrho + \varrho} \right)^\alpha \mathcal{L}_\varrho.
\]
Note that both expressions are well-defined in the sense of functions of $\partial_{0,\varrho}$ defined above and that $\partial_{0,\varrho}^{-\alpha}$ is bounded iff $\varrho \not= 0$. Moreover, $(\partial_{0,\varrho}^\alpha)^{-1} = \partial_{0,\varrho}^{-\alpha}$. We set $\partial_{0,\varrho}^0$ as the identity operator on $L_2^2(\mathbb{R}; H)$.

In order to provide the connections to the more commonly known integral representation formulas for the fractional integrals, we recall the multiplication theorem, that is,
\[
\sqrt{2\pi} \mathcal{F} f \cdot \mathcal{F} g = \mathcal{F}(f \ast g),
\]
for $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R}; H)$.

We recall the cut-off function
\[
\chi_{\mathbb{R}^+} = \begin{cases} 
1, & t > 0, \\
0, & t \leq 0.
\end{cases}
\]

Lemma 2.5. For all $\varrho, \alpha > 0$, and $\xi \in \mathbb{R}$, we have
\[
\sqrt{2\pi} \mathcal{L}_\varrho \left( t \mapsto \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \chi_{\mathbb{R}^+}(t) \right)(\xi) = \left( \frac{1}{\text{i} \xi + \varrho} \right)^\alpha.
\]

Proof. We start by defining the function
\[
f(\xi) := \int_0^\infty e^{-(\text{i} \xi + \varrho) s} s^{\alpha-1} ds
\]
for $\xi \in \mathbb{R}$. Then we have
\[
f'(\xi) = \int_0^\infty -\text{i} e^{-(\text{i} \xi + \varrho) s} s^\alpha ds
\]
\[
= -\frac{\alpha}{\text{i} \xi + \varrho} f(\xi),
\]
where we have used integration by parts. By separation of variables, it follows that
\[
f(\xi) = f(0) \frac{\varrho^\alpha}{(\text{i} \xi + \varrho)^\alpha}
\]
for $\xi \in \mathbb{R}$. Now, since
\[
f(0) = \int_0^\infty e^{-\varrho s} s^{\alpha-1} ds = \frac{1}{\varrho^\alpha} \Gamma(\alpha),
\]
we infer
\[
f(\xi) = \Gamma(\alpha) \frac{1}{(\text{i} \xi + \varrho)^\alpha}.
\]
Since the left hand side of (2.1) equals $\frac{1}{\Gamma(\alpha)} f(\xi)$, the assertion follows. \qed
Next, we draw the connection from our fractional integral to the one used in the literature.

**Theorem 2.6.** For all \( \varrho, \alpha > 0 \), \( f \in L^2_{\varrho}(\mathbb{R}; H) \), we have

\[
\partial^{-\alpha}_{0,\varrho} f(t) = \int_{-\infty}^{t} \frac{1}{(\alpha)} (t-s)^{\alpha-1} f(s) \, ds.
\]

**Proof.** For the proof we set \( g := (t \in \mathbb{R} \mapsto \frac{1}{(\alpha)} t^{\alpha-1} \chi_{\mathbb{R} > 0}(t)) \). Then \( g \in L^1_{\varrho}(\mathbb{R}) \). For \( f \in L^2_{\varrho}(\mathbb{R}; H) \) we have by Youngs convolution inequality

\[
(e^{-\text{em}} g) * (e^{-\text{em}} f) = e^{-\text{em}} (g * f) \in L^2(\mathbb{R}; H).
\]

Using the convolution property of the Fourier transform we obtain

\[
\sqrt{2\pi} \mathcal{L}_{\varrho} g \ast \mathcal{L}_{\varrho} f = \mathcal{L}_{\varrho} (g * f).
\]

Using Lemma 2.5 we compute

\[
\partial^{-\alpha}_{0,\varrho} f = \mathcal{L}_{\varrho}^* \left( \frac{1}{im + \varrho} \right) \mathcal{L}_{\varrho} f
= \mathcal{L}_{\varrho}^* (\sqrt{2\pi} \mathcal{L}_{\varrho} g \ast \mathcal{L}_{\varrho} f)
= \mathcal{L}_{\varrho}^* \mathcal{L}_{\varrho} (g * f)
= \int_{-\infty}^{t} \frac{1}{(\alpha)} (\cdot - s)^{\alpha-1} f(s) \, ds.
\]

\[\square\]

**Corollary 2.7.** Let \( \varrho, \alpha > 0 \). Then for all \( t \in \mathbb{R} \), we have for \( h \in H \)

\[
(\partial^{-\alpha}_{0,\varrho} \chi_{\mathbb{R} > 0} h)(t) = \begin{cases} \frac{1}{\Gamma(\alpha+1)} t^\alpha h, & t > 0, \\ 0, & t \leq 0. \end{cases}
\]

**Proof.** We use Theorem 2.6 and obtain for \( t \in \mathbb{R} \)

\[
(\partial^{-\alpha}_{0,\varrho} \chi_{\mathbb{R} > 0} h)(t) = \int_{-\infty}^{t} \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \chi_{\mathbb{R} > 0}(s) h \, ds
= \int_{\mathbb{R}} \chi_{\mathbb{R} > 0}(t-s) \chi_{\mathbb{R} > 0}(s) \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \, ds \, h.
\]

Thus, if \( t > 0 \), we obtain

\[
(\partial^{-\alpha}_{0,\varrho} \chi_{\mathbb{R} > 0} h)(t) = \int_{0}^{t} \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \, ds \, h = \frac{1}{\Gamma(\alpha)} \frac{1}{\alpha} t^\alpha \, h.
\]

For \( t \leq 0 \), we infer \( \chi_{\mathbb{R} > 0}(t-s) \chi_{\mathbb{R} > 0}(s) = 0 \) for \( s \in \mathbb{R} \), i.e. \( (\partial^{-\alpha}_{0,\varrho} \chi_{\mathbb{R} > 0} h)(t) = 0. \)

\[\square\]

**Remark 2.8.** It seems to be hard to determine analog formulas for the case \( \varrho < 0 \), although the operator \( \partial^{-\alpha}_{0,\varrho} \) for \( \varrho < 0, \alpha > 0 \) is bounded. The reason for this is that the corresponding multiplier \( (im + \varrho)^{-\alpha} \) is not defined in 0 and has a jump there. In particular, it cannot be extended to an analytic function on some right half plane of \( \mathbb{C} \). This, however, corresponds to the causality or anticausality of the operator \( \partial^{-\alpha}_{0,\varrho} \) by a Paley-Wiener result ([5] or [8, 19.2 Theorem]) and hence, we cannot expect to get a convolution formula as in the case \( \varrho > 0 \).
3 A reformulation of classical Riemann–Liouville and Caputo differential equations

There are two main concepts of fractional differentiation (or integration). In this section we shall start to identify both of these notions as being part of the same solution theory, related to the spectral representation construction above. We study Riemann–Liouville and Caputo differential equations and their respective integral equations. This section is only an introduction for the sections to come, where a comprehensive theory regarding well-posedness of fractional differential equations for a wide range of right-hand sides, is provided. In fact it will turn out that Caputo differential equations can be readily rephrased with the notions developed in Section 2. We shall see that for Riemann–Liouville differential equations some more theory has to be put in place.

To start off, we recall the Caputo differential equation. In [1], the author treated the following initial value problem of Caputo type for $1 \geq \alpha > 0$:

$$D_\alpha^* y(t) = f(t, y(t)) \quad (t > 0)$$

$$y(0) = y_0,$$

where a solution $y$ is continuous at zero and $y_0 \in C^n$ is a given initial value; $f: \mathbb{R}_{>0} \times \mathbb{C}^n \to \mathbb{C}^n$ is continuous, satisfying

$$|f(t, y_1) - f(t, y_2)| \leq c|y_1 - y_2|$$

for some $c \geq 0$ and all $y_1, y_2 \in \mathbb{C}^n$, $t > 0$. For definiteness, we shall also assume that

$$(t \mapsto f(t,0)) \in L^2_\varrho_0(\mathbb{R}_{>0}; \mathbb{C}^n)$$

for some $\varrho_0 \in \mathbb{R}$. In order to circumvent discussions of how to interpret the initial condition, we shall rather put [1, Equation (6)] into the perspective of the present exposition. In fact, this equation reads

$$y(t) = y_0 \chi_{\mathbb{R}_{>0}}(t) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \tilde{f}(s, y(s)) \, ds \quad (t > 0).$$

(3.3)

First of all, we remark that in contrast to the setting in the previous section, the differential equation just discussed ‘lives’ on $\mathbb{R}_{>0}$, only. To this end we put

$$\tilde{f}: \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n, \quad (t, y) \mapsto \chi_{\mathbb{R}_{>0}}(t) f(t, y),$$

with the apparent meaning that $\tilde{f}$ vanishes for negative times $t$. We note that by (3.1) and (3.2) it follows that

$$L^2_\varrho(\mathbb{R}) \ni y \mapsto (t \mapsto \tilde{f}(t,y(t))) \in L^2_\varrho(\mathbb{R})$$

is a well-defined Lipschitz continuous mapping for all $\varrho \geq \varrho_0$. Obviously, (3.3) is equivalent to

$$y(t) = y_0\chi_{\mathbb{R}_{>0}}(t) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \tilde{f}(s, y(s)) \, ds \quad (t > 0),$$

(3.4)

which in turn can be (trivially) stated for all $t \in \mathbb{R}$. Next, we present the desired reformulation of equation (3.3).

Theorem 3.1. Let $\varrho > \max\{0, \varrho_0\}$. Assume that $y \in L^2_\varrho(\mathbb{R})$. Then the following statements are equivalent:

(i) $y(t) = y_0\chi_{\mathbb{R}_{>0}}(t) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \tilde{f}(s, y(s)) \, ds$ for almost every $t \in \mathbb{R}$,
(ii) \( y = \partial_0^{-\alpha} \tilde{f}(\cdot, y(\cdot)) + y_0 \chi_{R>0}, \)

(iii) \( \partial_0^{-\alpha} (y - y_0 \chi_{R>0}) = \tilde{f}(\cdot, y(\cdot)). \)

**Proof.** The assertion follows trivially from Theorem 2.6. \( \square \)

**Remark 3.2.**

(a) For a real valued-function \( g : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}^n \) we may consider the Caputo differential equation with \( f : \mathbb{R}_{>0} \times \mathbb{C}^n \to \mathbb{C}^n, (t, z) \mapsto g(t, \text{Re}(z)). \)

(b) In particular, we have shown in Theorem 3.1 that the notions of so-called mild and strong solutions coincide.

Next we introduce Riemann–Liouville differential equations. Using the exposition in [9], we want to discuss the Riemann–Liouville fractional differential equation given by

\[
\frac{d^\alpha}{dx^\alpha} y(x) = f(x, y(x)), \\
\frac{d^{\alpha-1}}{dx^{\alpha-1}} y(x) \bigg|_{x=0^+} = y_0,
\]

where as before \( f \) satisfies (3.1) and (3.2) and \( y_0 \in \mathbb{R}, \) and \( \alpha \in (0, 1] \). Again, not hinging on too much of an interpretation of this equation, we shall rather reformulate the equivalent integral equation related to this initial value problem. According to [9, Chapter 42] this initial value problem can be formulated as

\[
y(t) = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \quad (t > 0).
\]

We abbreviate \( g_\beta(t) := \frac{1}{\Gamma(\beta+1)} t^\beta \chi_{R>0}(t) \) for \( t, \beta \in \mathbb{R}. \) For \( \alpha > 1/2 \) we have \( g_{\alpha-1} \in L^2_{\mathfrak{g}}(\mathbb{R}; H). \) Let us assume that \( \alpha > 1/2. \) Invoking the cut-off function \( \chi_{R>0} \) and defining \( \tilde{f} \) as before, we may provide a reformulation of the Riemann–Liouville equation on the space \( L^2_{\mathfrak{g}}(\mathbb{R}; H) \) by

\[
y = g_{\alpha-1} y_0 + \partial_0^{-\alpha} \tilde{f}(\cdot, y(\cdot)), \quad y \in L^2_{\mathfrak{g}}(\mathbb{R}; H).
\]

By a formal calculation and when applying Corollary 2.7, i.e. \( \partial_0^{-\alpha} \chi_{R>0} y_0 = g_{\alpha} y_0, \) we would obtain

\[
g_{\alpha-1} y_0 = \partial_0^{-\alpha} \chi_{R>0} y_0 = \partial_0^{-\alpha} \partial_0^{-\alpha} \chi_{R>0} y_0 = \partial_0^{-\alpha} \partial_0^{-\alpha} \chi_{R>0} y_0 = \partial_0^{-\alpha} \partial_0^{-\alpha} y_0 \delta_0,
\]

where \( \partial_0^{-\alpha} \chi_{R>0} y_0 \) is, when understood distributionally, the delta function \( y_0 \delta_0 \) and we could reformulate the Riemann–Liouville equation by

\[
\partial_0^{-\alpha} y = y_0 \delta_0 + \tilde{f}(\cdot, y(\cdot)). \quad (3.5)
\]

However, the calculation indicates that we have to extend the \( L^2_{\mathfrak{g}}(\mathbb{R}; H) \) calculus to understand Riemann–Liouville differential equations. This will be done in the coming sections.

## 4 Extra- and interpolation spaces

We begin to define extra- and interpolation spaces associated with the fractional derivative \( \partial_0^{-\alpha} \) for \( \varrho \neq 0, \alpha \in \mathbb{R}. \) Since by definition

\[
\partial_0^{-\alpha} = L^\varrho (im + \varrho)^\alpha L_\varrho,
\]

we will define the extra- and interpolation spaces in terms of the multiplication operators \( (im + \varrho)^\alpha \) on \( L^2(\mathbb{R}; H). \)
**Definition.** Let \( \varrho \neq 0 \). For each \( \alpha \in \mathbb{R} \) we define the space

\[
H^\alpha(\text{im} + \varrho) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}; H) : \int_{\mathbb{R}} \|(it + \varrho)^\alpha f(t)\|_H^2 \, dt < \infty \right\}
\]

and equip it with the natural inner product

\[
\langle f, g \rangle_{H^\alpha(\text{im} + \varrho)} := \int_{\mathbb{R}} \langle (it + \varrho)^\alpha f(t), (it + \varrho)^\alpha g(t) \rangle_H \, dt
\]

for each \( f, g \in H^\alpha(\text{im} + \varrho) \).

We shall use \( X \hookrightarrow Y \) to denote the mapping \( X \ni x \mapsto y \in Y \), if \( X \subseteq Y \) (under a canonical identification, which will always be obvious from the context).

**Lemma 4.1.** For \( \varrho \neq 0 \) and \( \alpha \in \mathbb{R} \) the space \( H^\alpha(\text{im} + \varrho) \) is a Hilbert space. Moreover, for \( \beta > \alpha \) we have

\[
j_{\beta \rightarrow \alpha} : H^\beta(\text{im} + \varrho) \hookrightarrow H^\alpha(\text{im} + \varrho)
\]

where the embedding is dense and continuous with \( \|j_{\beta \rightarrow \alpha}\| \leq |\varrho|^{\alpha - \beta} \).

**Proof.** Note that \( H^\alpha(\text{im} + \varrho) = L^2(\mu; H) \), where \( \mu \) is the Lebesgue measure on \( \mathbb{R} \) weighted with the function \( t \mapsto |it + \varrho|^{2\alpha} \). Thus, \( H^\alpha(\text{im} + \varrho) \) is a Hilbert space by the Fischer–Riesz theorem. Let now \( \beta > \alpha \) and \( f \in H^\beta(\text{im} + \varrho) \). Then

\[
\int_{\mathbb{R}} \|(it + \varrho)^\alpha f(t)\|_H^2 \, dt = \int_{\mathbb{R}} (t^2 + \varrho^2)^{\alpha - \beta} \|(it + \varrho)^\beta f(t)\|_H^2 \, dt \leq (\varrho^2)^{\alpha - \beta} \|f\|_{H^\beta(\text{im} + \varrho)}^2,
\]

which proves the continuity of the embedding \( j_{\beta \rightarrow \alpha} \) and the asserted norm estimate. The density follows, since \( C_c^\infty(\mathbb{R}; H) \) lies dense in \( H^\alpha(\text{im} + \varrho) \) for each \( \gamma \in \mathbb{R} \).

**Definition.** Let \( \varrho \neq 0 \) and \( \alpha \in \mathbb{R} \). We consider the space

\[
W^\alpha_{\text{loc}}(\mathbb{R}; H) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}; H) : \mathcal{L}_\varrho u \in H^\alpha(\text{im} + \varrho) \right\}
\]

equipped with the inner product

\[
\langle u, v \rangle_{\varrho, \alpha} := \langle \mathcal{L}_\varrho u, \mathcal{L}_\varrho v \rangle_{H^\alpha(\text{im} + \varrho)}
\]

and set \( H^\alpha_{\text{loc}}(\mathbb{R}; H) \) as its completion with respect to the norm induced by \( \langle \cdot, \cdot \rangle_{\varrho, \alpha} \).

**Lemma 4.2.** Let \( \varrho \neq 0 \).

(a) For \( \alpha \geq 0 \) we have that \( H^\alpha_{\text{loc}}(\mathbb{R}; H) = W^\alpha_{\text{loc}}(\mathbb{R}; H) = \text{dom}(\partial^\alpha_{\varrho, \varrho}) \).

(b) The operator

\[
\mathcal{L}_\varrho : W^\alpha_{\text{loc}}(\mathbb{R}; H) \subseteq H^\alpha_{\text{loc}}(\mathbb{R}; H) \rightarrow H^\alpha(\text{im} + \varrho)
\]

has a unique unitary extension, which will again be denoted by \( \mathcal{L}_\varrho \).

(c) For \( \alpha, \beta \in \mathbb{R} \) with \( \beta > \alpha \) we have that

\[
i_{\beta \rightarrow \alpha} : H^\beta_{\text{loc}}(\mathbb{R}; H) \hookrightarrow H^\alpha_{\text{loc}}(\mathbb{R}; H)
\]

is continuous and dense with \( \|i_{\beta \rightarrow \alpha}\| \leq |\varrho|^{\alpha - \beta} \).
(d) For each $\beta > 0$ and $\alpha \in \mathbb{R}$ the operator

$$\partial_{0,e}^{\beta} : H^{\beta + |\alpha|}_e(\mathbb{R}; H) \subseteq H^{\alpha}_e(\mathbb{R}; H) \to H^{-\beta}_e(\mathbb{R}; H)$$

has a unique unitary extension, which will again be denoted by $\partial_{0,e}^{\beta}$.

Proof.

(a) Let $\alpha \geq 0$. For $u \in H^\alpha(im + g)$, i.e. $u \in L^1_{loc}(\mathbb{R}; H)$ and $(im + g)^\alpha u \in L^2(\mathbb{R}; H)$, we infer that $u \in \text{dom}((im + g)^\alpha)$. Hence $H^\alpha(im + g) = \text{dom}((im + g)^\alpha)$. Moreover,

$$u \in W^\alpha_e(\mathbb{R}; H) \Leftrightarrow u \in L^2_e(\mathbb{R}; H) \wedge \mathcal{L}_e u \in H^\alpha(im + g)$$

$$\Leftrightarrow u \in L^2_e(\mathbb{R}; H) \wedge \mathcal{L}_e u \in \text{dom}((im + g)^\alpha)$$

$$\Leftrightarrow u \in \text{dom}(\partial_{0,e}^{\beta}),$$

by Example 2.4. Moreover, since

$$\mathcal{L}_e : W^\alpha_e(\mathbb{R}; H) \to H^\alpha(im + g)$$

is unitary, we infer that $W^\alpha_e(\mathbb{R}; H)$ is complete with respect to $\|\cdot\|_{e,\alpha} = \|\mathcal{L}_e \cdot\|_{H^\alpha(im + g)}$, and thus $H^\alpha_e(\mathbb{R}; H) = W^\alpha_e(\mathbb{R}; H)$.

(b) Obviously,

$$\mathcal{L}_e : W^\alpha_e(\mathbb{R}; H) \subseteq H^\alpha_e(\mathbb{R}; H) \to H^\alpha(im + g)$$

is isometric by the definition of the norm on $H^\alpha_e(\mathbb{R}; H)$. Moreover, its range is dense, since $\mathcal{L}_e^* \varphi \in W^\alpha_e(\mathbb{R}; H)$ for each $\varphi \in C^\infty_e(\mathbb{R}; H)$ and thus, $C^\infty_e(\mathbb{R}; H) \subseteq \text{dom } \mathcal{L}_e [W^\alpha_e(\mathbb{R}; H)].$ Hence, the continuous extension of $\mathcal{L}_e$ to $H^\alpha_e(\mathbb{R}; H)$ is onto and, thus, unitary.

(c) Since $\iota_{\beta \to \alpha} = \mathcal{L}_e^* \iota_{\beta \to \alpha} \mathcal{L}_e$, the assertion follows from Lemma 4.1.

(d) Since,

$$(im + g)^\beta : H^\alpha(im + g) \to H^{-\beta}(im + g)$$

$$f \mapsto (t \mapsto (it + g)^\beta f(t))$$

is obviously unitary, we infer that for $u \in H^{\beta + |\alpha|}_e(\mathbb{R}; H)$

$$\|\partial_{0,e}^{\beta} u\|_{e,\alpha - \beta} = \|\mathcal{L}_e \partial_{0,e}^{\beta} u\|_{H^{-\beta}(im + g)}$$

$$= \|\| (im + g)^\beta \mathcal{L}_e u \|_{H^{-\beta}(im + g)}$$

$$= \| \mathcal{L}_e u \|_{H^{-\beta}(im + g)}$$

$$= \| u \|_{e,\alpha},$$

which shows that $\partial_{0,e}^{\beta}$ is an isometry. Moreover, for $\varphi \in C^\infty_c(\mathbb{R}; H)$, we have that $(im + g)^\gamma \varphi \in C^\infty_c(\mathbb{R}; H)$ for all $\gamma \in \mathbb{R}$ and thus, in particular $\mathcal{L}_e^* (im + g)^{-\beta} \varphi \in \mathcal{L}_e^* [C^\infty_c(\mathbb{R}; H)] \subseteq H^\beta_e(\mathbb{R}; H)$. Next,

$$\partial_{0,e}^{\beta} \mathcal{L}_e^* (im + g)^{-\beta} \varphi = \mathcal{L}_e^* (im + g)^{\beta} \mathcal{L}_e \mathcal{L}_e^* (im + g)^{-\beta} \varphi = \mathcal{L}_e^* \varphi$$

and thus, $\mathcal{L}_e^* [C^\infty_c(\mathbb{R}; H)] \subseteq \partial_{0,e}^{\beta} [H^\beta_e(\mathbb{R}; H)]$. Since $C^\infty_c(\mathbb{R}; H)$ is dense in $H^\alpha(\mathbb{R}; H)$, we infer that $\mathcal{L}_e^* [C^\infty_c(\mathbb{R}; H)]$ is dense in $H^{-\beta}_e(\mathbb{R}; H)$ and thus, $\partial_{0,e}^{\beta}$ has dense range. This completes the proof. \(\square\)
We conclude this section by providing an alternative perspective to elements lying in \( H^\alpha_\varrho(\mathbb{R}; H) \) for some \( \alpha \in \mathbb{R} \) (with a particular focus on \( \alpha < 0 \)). In particular, we aim for a definition of a support for those elements which coincides with the usual support of \( L^2 \) functions in the case \( \alpha \geq 0 \).

**Lemma 4.3.** Let \( \varrho \neq 0 \) and \( \alpha \in \mathbb{R} \). Then

\[
\sigma_{-1} : W^\alpha_\varrho(\mathbb{R}; H) \subseteq H^\alpha_\varrho(\mathbb{R}; H) \to H^-_{\varrho}(\mathbb{R}; H)
\]

\[
f \mapsto (t \mapsto f(-t))
\]

extends to a unitary operator. Moreover, for \( f \in H^\alpha_\varrho(\mathbb{R}; H) \) we have

\[
\mathcal{L}_{-\varrho}\sigma_{-1}f = \sigma_{-1}\mathcal{L}_{\varrho}f \quad \text{and} \quad \mathcal{L}^*_{-\varrho}\sigma_{-1}f = \sigma_{-1}\mathcal{L}^*_{\varrho}f.
\]

**Proof.** For \( f \in W^\alpha_\varrho(\mathbb{R}; H) \) we have that

\[
\mathcal{L}_{-\varrho}\sigma_{-1}f = \sigma_{-1}\mathcal{L}_{\varrho}f
\]

and hence,

\[
\int_\mathbb{R} \| (it - \varrho)^\alpha (\mathcal{L}_{-\varrho}\sigma_{-1}f)(t) \|_H^2 \, dt = \int_\mathbb{R} (t^2 + \varrho^2)^\alpha \| (\mathcal{L}_{\varrho}f)(-t) \|_H^2 \, dt
\]

\[
= \int_\mathbb{R} (t^2 + \varrho^2)^\alpha \| (\mathcal{L}_{\varrho}f)(t) \|_H^2 \, dt = \| f \|_{H^\alpha_\varrho(\mathbb{R}; H)}^2,
\]

which proves the isometry of \( \sigma_{-1} \). Moreover, \( \sigma_{-1} \) has dense range, since \( \sigma_{-1}[W^\alpha_\varrho(\mathbb{R}; H)] = W^\alpha_{-\varrho}(\mathbb{R}; H) \). Hence, \( \sigma_{-1} \) extends to a unitary operator. The equality \( \mathcal{L}_{-\varrho}\sigma_{-1}f = \sigma_{-1}\mathcal{L}_{\varrho}f \) holds for \( f \in H^\alpha_\varrho(\mathbb{R}; H) \), since \( W^\alpha_{\varrho}(\mathbb{R}; H) \) is dense in its completion \( H^\alpha_\varrho(\mathbb{R}; H) \).

**Proposition 4.4.** Let \( \varrho \neq 0, \alpha \in \mathbb{R} \) and \( f \in H^\alpha_\varrho(\mathbb{R}; H) \). Then

\[
\langle f, \cdot \rangle : C_c^{\infty}(\mathbb{R}; H) \to \mathbb{C}
\]

given by

\[
\langle f, \varphi \rangle := \int_\mathbb{R} \langle \mathcal{L}_{\varrho}f(t), \mathcal{L}_{-\varrho}\varphi(t) \rangle_H \, dt
\]

defines a distribution. Moreover, for \( f \in H^\alpha_\varrho(\mathbb{R}; H) \) and \( \varphi \in C_c^{\infty}(\mathbb{R}; H) \) we have

\[
\langle f, \varphi \rangle = \langle f, \partial_{0, \varrho}^{-\alpha} e^{2\alpha \varrho} \sigma_{-1} \partial_{0, \varrho}^{-\alpha} \sigma_{-1} \varphi \rangle_{\varrho, \alpha}.
\]

In particular, for \( \alpha = 0 \)

\[
\langle f, \varphi \rangle = \int_\mathbb{R} \langle f(t), \varphi(t) \rangle_H \, dt.
\]

Note that the operator \( \partial_{0, \varrho}^{-\alpha} e^{2\alpha \varrho} \sigma_{-1} \partial_{0, \varrho}^{-\alpha} \sigma_{-1} \) maps \( H^{-\alpha}_{\varrho}(\mathbb{R}; H) \) to \( H^\alpha_{\varrho}(\mathbb{R}; H) \) unitarily.

**Proof.** Let \( f \in H^\alpha_\varrho(\mathbb{R}; H) \). We first prove that the expression \( \langle f, \cdot \rangle \) is indeed a distribution. Due to Lemma 4.2(c) it suffices to prove this for \( f \in H^{-k}_{\varrho}(\mathbb{R}; H) \) for some \( k \in \mathbb{N} \). Indeed, if \( f \in H^{-k}_{\varrho}(\mathbb{R}; H) \), then we know that

\[
(t \mapsto (it + \varrho)^{-k} (\mathcal{L}_{\varrho}f)(t)) \in L^2(\mathbb{R}; H)
\]
and hence, for \( \varphi \in C_c^\infty(\mathbb{R}; H) \) we obtain using Hölder’s inequality and the fact that \( \mathcal{L}_{-\varphi} \varphi^{(k)} = (i\alpha + \theta)^k \mathcal{L}_{-\varphi} \varphi \)

\[
|\langle f, \varphi \rangle| \leq \int_\mathbb{R} |(i\alpha + \theta)^k (\mathcal{L}_{\theta}f(t), (-i\alpha - \theta)^k (\mathcal{L}_{-\varphi} \varphi)(t))_H| \, dt \\
\leq \| \mathcal{L}_{\theta}f \|_{H^{-k}(i\alpha + \theta)} \| \mathcal{L}_{-\varphi} \varphi^{(k)} \|_{L^2(\mathbb{R}; H)} \\
\leq \| \mathcal{L}_{\theta}f \|_{H^{-k}(i\alpha + \theta)} \left( \int_{\text{spt} \varphi} e^{2\alpha t} \, dt \right)^{\frac{1}{2}} \| \varphi^{(k)} \|_\infty,
\]

which proves that \( \langle f, \cdot \rangle \) is indeed a distribution. Next, we prove the asserted formula. For this, we note the following elementary equality

\[
\sigma_{-1} \mathcal{L}_{\varphi} \varphi = \mathcal{L}_{\varphi} e^{2\alpha m} \sigma_{-1} \varphi
\]

for \( \varphi \in L^2(\mathbb{R}; H) \). Let \( f \in H^\alpha_c(\mathbb{R}; H) \) and compute

\[
\langle f, \varphi \rangle = \langle \mathcal{L}_{\theta}f, \mathcal{L}_{-\varphi} \varphi \rangle_{L^2(\mathbb{R}; H)} \\
= \langle \mathcal{L}_{\theta}f, \sigma_{-1} \mathcal{L}_{\theta} \sigma_{-1} \varphi \rangle_{L^2(\mathbb{R}; H)} \\
= \langle (i\alpha + \theta)^\alpha \mathcal{L}_{\theta}f, (im + \theta)^{-\alpha} \sigma_{-1} \mathcal{L}_{\theta} \sigma_{-1} \varphi \rangle_{L^2(\mathbb{R}; H)} \\
= \langle (i\alpha + \theta)^\alpha \mathcal{L}_{\theta}f, (im + \theta)^{-\alpha} \mathcal{L}_{\theta} \partial_{0,\alpha} e^{2\alpha m} \sigma_{-1} \varphi \rangle_{L^2(\mathbb{R}; H)} \\
= \langle (im + \theta)^\alpha \mathcal{L}_{\theta}f, (im + \theta)^{\alpha} \mathcal{L}_{\theta} \partial_{0,\alpha} e^{2\alpha m} \sigma_{-1} \varphi \rangle_{L^2(\mathbb{R}; H)} \\
= \langle f, \partial_{0,\alpha} e^{2\alpha m} \sigma_{-1} \partial_{0,\alpha} \sigma_{-1} \varphi \rangle_{\mathbb{R}, \alpha}
\]

for each \( \varphi \in C_c^\infty(\mathbb{R}; H) \). In particular, in the case \( \alpha = 0 \) we obtain

\[
\langle f, \varphi \rangle = \langle f, \varphi \rangle_{\mathbb{R}, 0} = \int_\mathbb{R} \langle f(t), \varphi(t) \rangle_H \, dt.
\]

**Remark 4.5.** The latter proposition shows that \( \bigcup_{\alpha \neq 0, \alpha \in \mathbb{R}} H^\alpha_c(\mathbb{R}; H) \subseteq D(\mathbb{R}; H)' \). In particular, the support of an element in \( H^\alpha_c(\mathbb{R}; H) \) is then well-defined by

\[
\bigcap_{U \subseteq \mathbb{R} \text{ open}, \forall \varphi \in C_c^\infty(U; H)} \{ f, \varphi = 0 \},
\]

and the second part of the latter proposition shows, that it coincides with the usual \( L^2 \)-support if \( \alpha \geq 0 \). Moreover, we can now compare elements in \( H^\alpha_c(\mathbb{R}; H) \) and \( H^\beta_c(\mathbb{R}; H) \) by saying that those elements are equal if they are equal as distributions. We shall further elaborate on this matter in Proposition 4.9. In particular, we shall show that \( f \mapsto \langle f, \cdot \rangle \) is injective. We shall also mention that the notation \( \langle f, \varphi \rangle \) is justified, as it does not depend on \( \varphi \) nor \( \alpha \).

**Example 4.6.** Let \( f \in L^2(\mathbb{R}; H) \). Then, by definition, \( \partial_{0,\alpha} f \in H^{-1}_{\alpha}(\mathbb{R}; H) \). We shall compute the action of \( \partial_{0,\alpha} f \) as a distribution. For this let \( \varphi \in C_c^\infty(\mathbb{R}; H) \) and we compute with the formula outlined in Proposition 4.4 for \( \alpha = -1 \):

\[
\langle \partial_{0,\alpha} f, \varphi \rangle = \langle \partial_{0,\alpha} f, \partial_{0,\alpha} e^{2\alpha m} \sigma_{-1} \partial_{0,\alpha} \sigma_{-1} \varphi \rangle_{-1} \\
= \langle (im + \theta)^{-1} \mathcal{L}_{\theta} \partial_{0,\alpha} f, (im + \theta)^{-1} \mathcal{L}_{\theta} \partial_{0,\alpha} e^{2\alpha m} \sigma_{-1} \partial_{0,\alpha} \sigma_{-1} \varphi \rangle_{L^2(\mathbb{R}; H)}
\]

12
Indeed, by the product rule, the choice of

\[ \partial_{0, \varphi} L (b) : \]

Let \( n \in \mathbb{N} \) and \( \mu, \nu \neq 0 \). Let \( \psi \in C^\infty(\mathbb{R}; H) \cap H^\infty_\nu(\mathbb{R}; H) \). Then there is \( (\varphi_n)_{n \in \mathbb{N}} \in C^\infty_c(\mathbb{R}) \) s.t. \( \varphi_n \to \psi \) for \( n \to \infty \) in \( H^\nu_\mu(\mathbb{R}; H) \) and \( H^\nu_{\mu}(\mathbb{R}; H) \) and \( \text{spt}(\varphi_n) \subseteq \text{spt}(\psi) \) for \( n \in \mathbb{N} \).

**Proof.** (a) Let \( \alpha \in \mathbb{R} \). Let \( \mu, \varphi > 0 \). For \( \alpha > 0 \) it holds that \( \partial_{\varphi}^\alpha \varphi = \partial_{\varphi}^{\alpha-\left[\alpha\right]} \varphi \) and \( \varphi^{\left[\alpha\right]} = \partial_{\nu, \varphi}^\alpha \varphi \varphi \). Thus we may assume that \( \alpha < 0 \). By Theorem 2.6 we have \( \partial_{\varphi}^\alpha \varphi = \partial_{\nu, \varphi}^\alpha \varphi \) and \( \text{inf spt} \partial_{\nu, \varphi}^\alpha \varphi \) is invariant. It is well known that \( C \in \mathcal{D}(\mathbb{R}; H) \) such that \( \text{spt} \psi \subseteq [-n-1, n+1] \), \( \chi_n = 1 \) on \([-n, n]\) and

\[
\sup \left\{ \| \chi_n^{(j)} \|_\infty : j \in \{0, \ldots, k\}, n \in \mathbb{N} \right\} < \infty.
\]

Set \( \varphi_n := \chi_n \psi \in C^\infty(\mathbb{R}; H) \). Then \( \text{spt}(\varphi_n) \subseteq \text{spt}(\psi) \) for \( n \in \mathbb{N} \). Since \( H^\nu_{\mu}(\mathbb{R}; H) \) is dense in \( H^\nu_\mu(\mathbb{R}; H) \), it suffices to show that \( \varphi_n \to \psi \) for \( n \to \infty \) in \( H^\nu_\mu(\mathbb{R}; H) \) and \( H^\nu_{\mu}(\mathbb{R}; H) \).

Indeed, by the product rule, the choice of \( \chi_n \) and dominated convergence we obtain

\[
\varphi_n^{(k)} = \sum_{j=0}^{k} \chi_n^{(k-j)} \psi^{(j)} = \chi_n \psi^{(k)} + \sum_{j=0}^{k-1} \chi_n^{(k-j)} \psi^{(j)} \to \psi^{(k)}
\]

for \( n \to \infty \) in \( L^2(\mathbb{R}; H) \) and in \( L^2_\mu(\mathbb{R}; H) \).

**Lemma 4.7.** Let \( \varphi \in C^\infty(\mathbb{R}; H) \). For all \( \varphi > 0 \) we have \( \partial_{\varphi}^\alpha \varphi = \partial_{\nu, \varphi}^{\alpha-\left[\alpha\right]} \varphi \) and \( \varphi^{\left[\alpha\right]} = \partial_{\nu, \varphi}^\alpha \varphi \varphi \). Thus we may assume that \( \alpha < 0 \). By Theorem 2.6 we have \( \partial_{\varphi}^\alpha \varphi = \partial_{\nu, \varphi}^\alpha \varphi \) and inf \( \text{spt} \partial_{\nu, \varphi}^\alpha \varphi \) is invariant. It is well known that \( C \in \mathcal{D}(\mathbb{R}; H) \) such that \( \text{spt} \psi \subseteq [-n-1, n+1] \), \( \chi_n = 1 \) on \([-n, n]\) and

\[
\sup \left\{ \| \chi_n^{(j)} \|_\infty : j \in \{0, \ldots, k\}, n \in \mathbb{N} \right\} < \infty.
\]

**Lemma 4.8.** Let \( \varphi \neq 0 \) and \( \alpha \in \mathbb{R} \). Then \( C^\infty_c(\mathbb{R}; H) \) is dense in \( H^\alpha(\mathbb{R}; H) \).

**Proof.** We first note that it suffices to prove the assertion for \( \varphi > 0 \), since the operator \( \sigma_{-1} \) from Lemma 4.3 leaves \( C^\infty_c(\mathbb{R}; H) \) invariant. It is well known that \( C^\infty_c(\mathbb{R}; H) \) is dense in \( L^2(\mathbb{R}; H) \). We have \( \partial_{\varphi}^\alpha f \in L^2(\mathbb{R}; H) \). Let \( (\psi_n)_{n \in \mathbb{N}} \in C^\infty_c(\mathbb{R}; H) \) with \( \psi_n \to \partial_{\varphi}^\alpha f \) for \( n \to \infty \) in \( L^2(\mathbb{R}; H) \). By Lemma 4.7(a) we have \( \partial_{\varphi}^\alpha \psi_n \in C^\infty(\mathbb{R}; H) \cap H^\infty(\mathbb{R}; H) \) and by Lemma 4.7(b) we find \( (\varphi_n)_{n \in \mathbb{N}} \in C^\infty_c(\mathbb{R}; H) \) with

\[
\| \partial_{\varphi}^\alpha \psi_n - \varphi_n \|_{\varphi, \alpha} \to 0 \quad (n \to \infty).
\]

With this result at hand, we can characterize those distributions, which belong to \( H^\alpha(\mathbb{R}; H) \) for some \( \alpha \in \mathbb{R}, \varphi \neq 0 \), in the following way.

**Proposition 4.9.** Let \( \psi \in \mathcal{D}(\mathbb{R}; H) \) and \( \alpha \in \mathbb{R}, \varphi \neq 0 \). Then, there exists \( f \in H^\alpha(\mathbb{R}; H) \) such that

\[
\psi(\varphi) = \langle f, \varphi \rangle \quad (\varphi \in C^\infty_c(\mathbb{R}; H))
\]

13
in the sense of Propositions 4.4 if and only if there is $C \geq 0$ such that
\[ |\psi(\varphi)| \leq C \|\varphi\|_{p,-\alpha} \]
for each $\varphi \in C^\infty_c(\mathbb{R}; H)$.

**Proof.** Assume first that there is $f \in H^\alpha_0(\mathbb{R}; H)$ representing $\psi$. Then we estimate
\begin{align*}
|\psi(\varphi)| &= |(f, \varphi)| \\
&= \left| \int_\mathbb{R} (\mathcal{L}_\varphi f(t), \mathcal{L}_{-\varphi} \varphi(t))_H \, dt \right| \\
&= \left| \int_\mathbb{R} ((it + \varphi)^\alpha \mathcal{L}_\varphi f(t), (-it + \varphi)^{-\alpha} \mathcal{L}_{-\varphi} \varphi(t))_H \, dt \right| \\
&\leq \|\mathcal{L}_\varphi f\|_{H^\alpha_{i(m+\varphi)}} \|\mathcal{L}_{-\varphi} \varphi\|_{H^{-\alpha}_{i(m-\varphi)}} \\
&= \|f\|_{\varphi,\alpha} \|\varphi\|_{p,-\alpha} 
\end{align*}
for each $\varphi \in C^\infty_c(\mathbb{R}; H)$. Let $C \geq 0$ such that $\psi$ satisfies for $\varphi \in C^\infty_c(\mathbb{R}; H)$
\[ |\psi(\varphi)| \leq C \|\varphi\|_{p,-\alpha}. \]

The operator $A := \partial_0^{\alpha} e^{2\varphi t} \sigma_1 \partial_0^{\alpha} \sigma_1 : H^\alpha_0(\mathbb{R}; H) \to H^\alpha_0(\mathbb{R}; H)$ (cf. Proposition 4.4) is unitary. Thus for $\varphi \in C^\infty_c(\mathbb{R}; H)$
\[ |\psi(A^{-1} \varphi)| \leq C \|A^{-1} \varphi\|_{p,-\alpha} = C \|\varphi\|_{\varphi,\alpha}. \]
Moreover, $C^\infty_c(\mathbb{R}; H) \subseteq H^\alpha_0(\mathbb{R}; H)$ is dense. Thus $\psi(A^{-1})$ can be extended continuously to $H^\alpha_0(\mathbb{R}; H)$. By the Riesz representation theorem, there is a $f \in H^\alpha_0(\mathbb{R}; H)$ such that for $\varphi \in H^\alpha_0(\mathbb{R}; H)$
\[ \psi(A^{-1} \varphi) = (f, \varphi)_{\varphi,\alpha}. \]
By Theorem 4.4 we have for $\varphi \in C^\infty_c(\mathbb{R}; H)$
\[ \psi(\varphi) = \psi(A^{-1} A \varphi) = (f, A \varphi)_{\varphi,\alpha} = (f, \varphi). \]

In the next proposition, we shall also obtain the announced uniqueness statement, that is, the injectivity of the mapping $f \mapsto (f, \cdot)$. 

**Proposition 4.10.** Let $\alpha \in \mathbb{R}$ and $\mu, \varphi > 0$. Moreover, let $f \in H^\alpha_0(\mathbb{R}; H)$ and $g \in H^\alpha_0(\mathbb{R}; H)$. Then the following statements are equivalent:

(i) $f = g$ in the sense of distributions, i.e., for each $\varphi \in C^\infty_c(\mathbb{R}; H)$ we have that
\[ \int_\mathbb{R} \langle \mathcal{L}_\varphi f(t), \mathcal{L}_{-\varphi} \varphi(t) \rangle_H \, dt = \int_\mathbb{R} \langle \mathcal{L}_\varphi g(t), \mathcal{L}_{-\varphi} \varphi(t) \rangle_H \, dt. \]

(ii) $\partial_0^{\alpha} f = \partial_0^{\alpha} g$ as functions in $L^1_{\text{loc}}(\mathbb{R}; H)$.

(iii) There is a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C^\infty_c(\mathbb{R}; H)$ with $\varphi_n \to f$ in $H^\alpha_0(\mathbb{R}; H)$ and $\varphi_n \to g$ in $H^\alpha_0(\mathbb{R}; H)$ as $n \to \infty$. 

14
Proof. (i)⇒(ii): Let \( \psi \in C_\infty^\infty(\mathbb{R}; H) \) and \( \tilde{\psi} := \sigma_1 \partial_{\psi}^\alpha \sigma_{-1} \psi = \sigma_1 \partial_{\psi}^\alpha \sigma_{-1} \psi \). Then by Lemma 4.4(a), \( \tilde{\psi} \in C_\infty^\infty(\mathbb{R}; H) \cap H_\psi^\infty(\mathbb{R}; H) \cap H_\psi^\infty(\mathbb{R}; H) \). By Lemma 4.4(b) there's \( (\varphi_n)_{n \in \mathbb{N}} \in C_\infty^\infty(\mathbb{R}; H)^H \) with \( \varphi_n \to \psi \) \((n \to \infty)\) in \( H_\psi^{-\alpha}(\mathbb{R}; H) \) and in \( H_\psi^{-\alpha}(\mathbb{R}; H) \). Thus

\[
\int_\mathbb{R} \langle \partial_{\psi}^\alpha f(t), \psi(t) \rangle_H \, dt = \langle \partial_{\psi}^\alpha f, e^{2\mu \sigma} \psi \rangle_{H,0}
\]

\[
= \langle f, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} (\sigma_1 \partial_{\psi}^\alpha \sigma_{-1} \psi) \rangle_{H,0}
\]

\[
= \lim_{n \to \infty} \langle f, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} \varphi_n \rangle_{H,0}
\]

\[
= \lim_{n \to \infty} \langle f, \varphi_n \rangle
\]

\[
= \lim_{n \to \infty} \langle g, \varphi_n \rangle
\]

\[
= \lim_{n \to \infty} \langle f, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} \varphi_n \rangle_{H,\mu,0}
\]

\[
= \langle \tilde{f}_n(t), \psi(t) \rangle_H \, dt.
\]

(ii) ⇒ (iii): Define \( \tilde{f}_n := \chi_{[-n,n]} \cdot \partial_{\psi}^\alpha f = \chi_{[-n,n]} \cdot \partial_{\psi}^\alpha g \) for \( n \in \mathbb{N} \). Without loss of generality let \( g < \mu \).

Take a function \( \psi_n \in C_\infty^\infty(\mathbb{R}; H) \) with \( \psi_n \subseteq [-n,n] \) such that

\[
\| \tilde{f}_n - \psi_n \|_{\psi,0} \leq \frac{1}{n} e^{(e-\mu)n}.
\]

Then, we estimate

\[
\| \tilde{f}_n - \psi_n \|^2_{H,\mu,0} = \int_{-n}^{n} \| \tilde{f}_n(t) - \psi_n(t) \|^2 e^{-2\mu t} \, dt = \int_{-n}^{n} \| \tilde{f}_n(t) - \psi_n(t) \|^2 e^{-2\mu t} e^{2(e-\mu)t} \, dt
\]

\[
\leq \| \tilde{f}_n - \psi_n \|^2_{\psi,0} e^{2(e-\mu)n} \leq \frac{1}{n^2}.
\]

Hence \( \psi_n \to \partial_{\psi}^\alpha f = \partial_{\psi}^\alpha g \) in \( L_2^{\infty}(\mathbb{R}; H) \) and in \( L_2^{\infty}(\mathbb{R}; H) \) by the triangle inequality and dominated convergence. We set \( \tilde{\varphi}_n := \partial_{\psi}^\alpha \varphi_n = \partial_{\psi}^\alpha \psi_n \in C_\infty^\infty(\mathbb{R}; H) \cap H_\psi^\infty(\mathbb{R}; H) \cap H_\psi^\infty(\mathbb{R}; H) \). Then \( \tilde{\varphi}_n \to f \) and \( \tilde{\varphi}_n \to g \) in \( H_\psi^\infty(\mathbb{R}; H) \) and in \( H_\psi^\infty(\mathbb{R}; H) \) respectively. We use Lemma 4.4(b) and choose a sequence \( (\varphi_n)_{n \in \mathbb{N}} \in C_\infty^\infty(\mathbb{R}; H)^H \) with \( \| \tilde{\varphi}_n - \varphi_n \|_{\psi,\alpha} \to 0 \). Then

\[
\| f - \varphi_n \|_{\psi,\alpha} \leq \| f - \tilde{\varphi}_n \|_{\psi,\alpha} + \| \tilde{\varphi}_n - \varphi_n \|_{\psi,\alpha} \to 0 \quad (n \to \infty),
\]

\[
\| g - \varphi_n \|_{\mu,\alpha} \leq \| g - \tilde{\varphi}_n \|_{\mu,\alpha} + \| \tilde{\varphi}_n - \varphi_n \|_{\mu,\alpha} \to 0 \quad (n \to \infty).
\]

(iii) ⇒ (i): Let \( (\varphi_n)_{n \in \mathbb{N}} \) be a sequence in \( C_\infty^\infty(\mathbb{R}; H) \) such that \( \varphi_n \to f \) and \( \varphi_n \to g \) in \( H_\psi^\infty(\mathbb{R}; H) \) and \( H_\psi^\infty(\mathbb{R}; H) \) respectively. Let \( \varphi \in C_\infty^\infty(\mathbb{R}; H) \). Then we have according to Proposition 4.4

\[
\int_\mathbb{R} \langle L_\psi f(t), L_\psi \varphi(t) \rangle_H \, dt = \langle f, \varphi \rangle
\]

\[
= \langle f, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} \varphi \rangle_{H,\alpha,0}
\]

\[
= \lim_{n \to \infty} \langle \varphi_n, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} \varphi \rangle_{H,\alpha,0}
\]

\[
= \lim_{n \to \infty} \langle \varphi_n, \varphi \rangle
\]

\[
= \lim_{n \to \infty} \langle \varphi_n, \partial_{\psi}^\alpha e^{2\mu \sigma} \sigma_{-1} \partial_{\psi}^\alpha \sigma_{-1} \varphi \rangle_{H,\mu,\alpha}
\]

15
We shall need the following elementary observation later on. Let \( \alpha \) continuous, with convergence in \( L^2 \) with convergence in \( H^1 \).

Indeed, by Proposition 4.10 there exists a sequence \((\varphi_n)\) satisfying sup \( \varphi_n \leq 0 \) such that for each \( \varphi \geq \varphi_n \) the function \( \varphi \) has a Lipschitz continuous extension

\[
F_\varphi : H^2_0(\mathbb{R}; H) \to H^1_0(\mathbb{R}; H)
\]

satisfying sup \( \varphi \geq 0 \) \( |F_\varphi|_{\text{Lip}} < \infty \). Moreover, we call \( F \) eventually \((\beta, \gamma)-\text{contracting}, \) if \( F \) is eventually \((\beta, \gamma)-\text{Lipschitz continuous and} \limsup_{\varphi \to \infty} |F_\varphi|_{\text{Lip}} < 1 \). Here, we denote by \( |\cdot|_{\text{Lip}} \) the smallest Lipschitz constant of a Lipschitz continuous function:

\[
|F_\varphi|_{\text{Lip}} := \sup_{f, g \in H^2_0(\mathbb{R}; H), f \neq g} \frac{|F_\varphi(f) - F_\varphi(g)|_{\beta, \gamma}}{\|f - g\|_{\beta, \gamma}}.
\]

Note that by Lemma 4.3 any eventually Lipschitz continuous function is densely defined. Thus, the Lipschitz continuous extension \( F_\varphi \) is unique.

**Remark 5.1.** (a) If \( f \in H^2_0(\mathbb{R}; H) \) and \( g \in H^2_0(\mathbb{R}; H) \) generate the same distribution, we have that

\[
F_\varphi(f) = F_\mu(g).
\]

Indeed, by Proposition 4.10 there exists a sequence \((\varphi_n)\) in \( C^\infty_c(\mathbb{R}; H) \) with \( \varphi_n \to f \) and \( \varphi_n \to g \) in \( H^2_0(\mathbb{R}; H) \) and \( H^2_0(\mathbb{R}; H) \), respectively. We infer that

\[
F_\varphi(f) = \lim_{n \to \infty} F(\varphi_n) \quad \text{and} \quad F_\mu(g) = \lim_{n \to \infty} F(\varphi_n)
\]

with convergence in \( H^2_0(\mathbb{R}; H) \) and \( H^2_\mu(\mathbb{R}; H) \) respectively. Consequently

\[
\partial_{0, \varphi} F_\varphi(f) \leftarrow \partial_{0, \varphi} F(\varphi_n) = \partial_{0, \mu} F(\varphi_n) \to \partial_{0, \mu} F_\mu(g)
\]

with convergence in \( L^2_0(\mathbb{R}; H) \) and hence almost everywhere for a suitable subsequence of \((\varphi_n)\). The assertion follows from Proposition 4.11.

(b) We shall need the following elementary observation later on. Let \( F \) be eventually \((\beta, \gamma)-\text{Lipschitz continuous}, \alpha \in \mathbb{R} \). Let \( \varphi \geq \varphi_0 \). Then

\[
\tilde{F} : C^\infty_c(\mathbb{R}; H) \ni \varphi \mapsto F_\varphi(\alpha \varphi)
\]

which completes the proof. \( \Box \)

5 A unified solution theory — well-posedness and causality of fractional differential equations

We are now able to study abstract fractional differential equations of the form

\[
\partial_{0, \varphi}^\alpha u = F(u).
\]

In order to obtain well-posedness of the latter problem, we need to restrict the class of admissible right-hand sides \( F \) in the latter equation.

**Definition.** Let \( \varphi_0 > 0 \) and \( \beta, \gamma \in \mathbb{R} \). We call a function \( F : \text{dom}(F) \subseteq \bigcap_{\varphi \geq \varphi_0} H^2_0(\mathbb{R}; H) \to \bigcap_{\varphi \geq \varphi_0} H^2_0(\mathbb{R}; H) \) eventually \((\beta, \gamma)-\text{Lipschitz continuous}, \) if \( \text{dom}(F) \supseteq C^\infty_c(\mathbb{R}; H) \) and there exists \( \nu \geq \varphi_0 \) such that for each \( \varphi \geq \nu \) the function \( F \) has a Lipschitz continuous extension

\[
F_\varphi : H^2_0(\mathbb{R}; H) \to H^1_0(\mathbb{R}; H)
\]

which completes the proof. \( \Box \)
is eventually \((\beta + \alpha, \gamma)\)-Lipschitz continuous. Indeed, the assertion follows from part (a) and

\[
\left\| \tilde{F}(f) - \tilde{F}(g) \right\|_{\mu, \gamma} \leq |F_\mu|_{Lip} \left\| \partial_0^\alpha f - \partial_0^\alpha g \right\|_{\mu, \beta} = |F_\mu|_{Lip} \left\| f - g \right\|_{\mu, \alpha + \beta},
\]

for \(\mu \geq \nu, f, g \in C^\infty_c(\mathbb{R}; H)\).

**Theorem 5.2.** Let \(\alpha > 0, \beta, \gamma \in \mathbb{R}, \gamma_0 > 0\) and \(F : \text{dom}(F) \subseteq \bigcap_{t \geq \gamma_0} H^{\beta}_\phi(\mathbb{R}; H) \to \bigcap_{t \geq \gamma_0} H^{\beta - \gamma}_0(\mathbb{R}; H)\) be eventually \((\beta, \beta - \alpha)\)-contracting. Then there exists \(\nu \geq \gamma_0\) such that for each \(\varphi \geq \nu\) there is a unique \(u_\varphi \in H^{\beta}_\phi(\mathbb{R}; H)\) satisfying

\[
\partial_0^\alpha u_\varphi = F_\varphi(u_\varphi). \tag{5.1}
\]

**Proof.** This is a simple consequence of the contraction mapping theorem. Indeed, choosing \(\nu \geq \gamma_0\) large enough, such that \(|F_\varphi|_{Lip} < 1\) for each \(\varphi \geq \nu\), we obtain that

\[
\partial_0^\alpha F_\varphi : H^{\beta}_\phi(\mathbb{R}; H) \to H^{\beta}_\phi(\mathbb{R}; H)
\]

is a strict contraction, since \(\partial_0^\alpha : H^{\beta - \gamma}_0(\mathbb{R}; H) \to H^{\beta}_0(\mathbb{R}; H)\) is unitary by Lemma 4.2. Hence, the mapping \(\partial_0^\alpha F_\varphi\) admits a unique fixed point \(u_\varphi \in H^{\beta}_\phi(\mathbb{R}; H)\), which is equivalent to \(u_\varphi\) being a solution of (5.1). \(\square\)

**Corollary 5.3.** Let \(\alpha > 0, \beta, \gamma \in \mathbb{R}, \gamma_0 > 0\) and \(F : \text{dom}(F) \subseteq \bigcap_{t \geq \gamma_0} H^{\beta}_\phi(\mathbb{R}; H) \to \bigcap_{t \geq \gamma_0} H^{\beta - \gamma}_0(\mathbb{R}; H)\) for some \(\gamma \in [0, \alpha]\) be eventually \((\beta, \beta - \gamma)\)-Lipschitz continuous. Then there exists \(\nu \geq \gamma_0\) such that for each \(\varphi \geq \nu\) there is a unique \(u_\varphi \in H^{\beta}_\phi(\mathbb{R}; H)\) satisfying

\[
\partial_0^\alpha u_\varphi = F_\varphi(u_\varphi).
\]

**Proof.** It suffices to prove that \(i_{\beta - \gamma \to \beta - \alpha} \circ F\) is eventually \((\beta, \beta - \alpha)\)-contracting by Theorem 5.2. Let \(\nu \geq \gamma\), s.t. for \(\varphi \geq \nu\), \(F_\varphi\) exists. Then for \(\varphi \geq \nu\)

\[
|\beta - \gamma \to \beta - \alpha| \circ F_\varphi|_{Lip} \leq \left\| i_{\beta - \gamma \to \beta - \alpha} \right\| |F_\varphi|_{Lip} \leq \gamma^{-\alpha} |F_\varphi|_{Lip}
\]

by Lemma 4.2. Since \(|F_\varphi|_{Lip}\) is bounded in \(\varphi\) on \([\nu, \infty]\) by assumption, we infer

\[
\limsup_{\varphi \to \infty} |\beta - \gamma \to \beta - \alpha| \circ F_\varphi|_{Lip} = 0 < 1.
\]

Next, we want to show that the solution \(u_\varphi\) of (5.1) is actually independent of the particular choice of \(\varphi\). For doing so, we need to rule out the possibility of causality, which will be addressed in the next propositions.

**Lemma 5.4.** Let \(\varphi > 0, \alpha \in \mathbb{R}\) and \(a \in \mathbb{R}\). Let \(f \in H^{\alpha}_\phi(\mathbb{R}; H)\) with \(\text{spt} f \subseteq \mathbb{R}_{\geq a}\). Then there is a sequence \((\varphi_n)_{n \in \mathbb{N}} \subseteq C^\infty_c(\mathbb{R}; H)^N\) with \(\text{spt} \varphi_n \subseteq \mathbb{R}_{\geq a}\) for \(n \in \mathbb{N}\) and \(\varphi_n \to f\) in \(H^{\alpha}_\phi(\mathbb{R}; H)\) as \(n \to \infty\).

**Proof.** Let \((\tilde{\psi}_n)_{n \in \mathbb{N}} \subseteq C^\infty_c(\mathbb{R}; H)^N\) be such that \(\tilde{\psi}_n \to \partial_0^\alpha f\) in \(H^{\alpha}_\phi(\mathbb{R}; H)\) as \(n \to \infty\). We may assume that \(\text{spt} \tilde{\psi}_n \subseteq \mathbb{R}_{\geq a}\). We set \(\psi_n := \partial_0^\alpha \tilde{\psi}_n\) for \(n \in \mathbb{N}\). Then \(\psi_n \to f\) as \(n \to \infty\) in \(H^{\alpha}_\phi(\mathbb{R}; H)\) and \(\text{spt} \psi_n \to a\) by Lemma 4.2(a). We use Lemma 4.2(b) and pick a sequence \((\varphi_n)_{n \in \mathbb{N}} \subseteq C^\infty_c(\mathbb{R}; H)^N\) with \(\text{spt}(\varphi_n) \subseteq \text{spt}(\psi_n)\) for \(n \in \mathbb{N}\) and \(\varphi_n - \psi_n \to 0\) in \(H^{\alpha}_\phi(\mathbb{R}; H)\) when \(n \to \infty\). Then

\[
\|\varphi_n - f\|_{\phi, \alpha} \leq \|\varphi_n - \psi_n\|_{\phi, \alpha} + \|\psi_n - f\|_{\phi, \alpha} \to 0 \quad (n \to \infty).
\]

**Proposition 5.5.** Let \(f \in H^{\alpha}_\phi(\mathbb{R}; H)\) for some \(\alpha \in \mathbb{R}, \varphi > 0\). Assume that \(\text{spt} f \subseteq \mathbb{R}_{\geq a}\) for some \(a \in \mathbb{R}\). Then

\[
\text{spt} \partial_0^\beta f \subseteq \mathbb{R}_{\geq a}
\]

for all \(\beta \in \mathbb{R}\).
On the other hand, according to Theorem 5.2, the mapping \( \partial_{0,e}^\beta \) where we have used that \( \partial \).

According to Proposition 5.5 we have that \( \text{spt} \).

First of all, we shall show the result for \( \partial_{0,e}^\beta \) - \( \partial_{0,e}^\beta \).\( \partial_{0,e}^\beta \)

The proof of the following theorem outlining causality of \( \partial_{0,e}^\beta \), is in spirit similar to the approach in [3, Theorem 4.5]. However, one has to adopt the distributional setting and the (different) definition of eventually Lipschitz continuity here accordingly.

**Theorem 5.6.** Let the assumptions of Theorem 5.2 be satisfied. Then, for each \( g \geq \nu \), where \( \nu \) is chosen according to Theorem 5.2, the mapping

\[
\partial_{0,e}^\beta F_\nu : H_\nu^\alpha(\mathbb{R}; H) \to H_\nu^\alpha(\mathbb{R}; H)
\]

is causal, that is, for each \( u, v \in H_\nu^\alpha(\mathbb{R}; H) \) satisfying \( \text{spt}(u - v) \subseteq \mathbb{R}_{\geq a} \) for some \( a \in \mathbb{R} \), it holds that \( \text{spt}(\partial_{0,e}^\beta F_\nu(u) - \partial_{0,e}^\beta F_\nu(v)) \subseteq \mathbb{R}_{\geq a} \). Here, the support is meant in the sense of distributions.

**Proof.** First of all, we shall show the result for \( u, v \in C_c^\infty(\mathbb{R}; H) \). So, let \( u, v \in C_c^\infty(\mathbb{R}; H) \) with \( \text{spt}(u - v) \subseteq \mathbb{R}_{\geq a} \). Take \( \varphi \in C_c^\infty(\mathbb{R}; H) \) with \( \text{spt} \varphi \subseteq \mathbb{R}_{<a} \). Let \( \mu \geq g \). Then \( F_\nu(u) = F_\mu(u) \) and

\[
\langle \partial_{0,e}^\alpha (F_\nu(u) - F_\nu(v)), \varphi \rangle = \langle \partial_{0,e}^\alpha (F_\mu(u) - F_\mu(v)), \varphi \rangle
\]

\[
= \langle \partial_{0,\mu}^\alpha (F_\mu(u) - F_\mu(v)), \partial_{0,\mu} \partial_{0,e}^\beta \varphi \rangle_{\mu,\beta}
\]

\[
= \langle F_\mu(u) - F_\mu(v), \partial_{0,\mu} \partial_{0,e}^\beta \varphi \rangle_{\mu,\beta - \alpha}
\]

\[
\leq \| F_\mu(u) - F_\mu(v) \|_{\mu,\beta - \alpha} \left\| \partial_{0,\mu} \partial_{0,e}^\beta \varphi \right\|_{\mu,\beta - \alpha}
\]

\[
\leq \| u - v \|_{\mu,\beta} \left\| \partial_{0,\mu} \partial_{0,e}^\beta \varphi \right\|_{\mu,0}
\]

where we have used that \( \partial_{0,\mu}^{(\beta-\alpha)} e^{2\mu \sigma_{-1}} \) : \( H_\mu^\beta(\mathbb{R}; H) \to H_\mu^{\beta-\alpha}(\mathbb{R}; H) \) is unitary and \( \varphi \in H_{-\mu}^{\beta}(\mathbb{R}; H) \).

According to Proposition 5.5 we have that \( \text{spt} \partial_{0,\mu}^{(\beta-\alpha)} \subseteq \mathbb{R}_{\geq a} \) and hence, we compute

\[
\left\| \partial_{0,\mu}^{\beta} \varphi \right\|_{\mu,0}^2 = \int_{-a} \left\| \partial_{0,\mu}^{\beta} \varphi \right\|_H^2 e^{-2\mu t} dt = \int_{0}^\infty \left\| \partial_{0,\mu}^{\beta} \varphi \right\|_H^2 e^{-2\mu t} dt e^{2\mu a}
\]

On the other hand

\[
\| u - v \|_{\mu,\beta} = \| \partial_{0,\mu} (u - v) \|_{\mu,0}^2
\]
Finally, let $C$ be given by

$$C = \{ \tau \in w \mid \text{spt}(\tau) \subseteq w \}$$

and consequently,

$$\|F_\mu\|_{\text{Lip}} \leq \|v\|_{\mu,0} \|\partial_{0,\mu}^{-1} \varphi\|_{\mu,0}$$

by dominated convergence. Summarizing, we have shown that spt($\partial_{0,\mu}^{-1} \varphi$) $\subseteq \mathbb{R}_{\geq 0}$ for $u, v \in C_\infty^c(\mathbb{R}; H)$ satisfying spt($u - v$) $\subseteq \mathbb{R}_{\geq 0}$.

Before we conclude the proof, we show that $(w_n)_{n \in \mathbb{N}}$ is a convergent sequence in $H_\phi^\beta(\mathbb{R}; H)$ with spt $w_n \subseteq \mathbb{R}_{\geq 0}$ for each $n \in \mathbb{N}$, then its limit $w$ also satisfies spt $w \subseteq \mathbb{R}_{\geq 0}$. For doing so, let $\varphi \in C_\infty^c(\mathbb{R}; H)$ with spt $\varphi \subseteq \mathbb{R}_{<0}$. Then

$$(w, \varphi) = \langle w, \partial_{0,\mu}^{-1} e^{2\alpha \sigma_\mu} \partial_{0,\mu}^{-1} \partial_{0,\mu}^{-1} \varphi \rangle_{\alpha,0} = \lim_{n \to \infty} \langle w_n, \partial_{0,\mu}^{-1} e^{2\alpha \sigma_\mu} \partial_{0,\mu}^{-1} \partial_{0,\mu}^{-1} \varphi \rangle_{\alpha,0} = \lim_{n \to \infty} (w_n, \varphi) = 0.$$ 

Finally, we prove that our solution is independent of the particular choice of the parameter $\beta > \nu$ in Theorem 5.2. The precise statement is as follows.

**Proposition 5.7.** Let the assumptions of Theorem 5.2 be satisfied and $\nu$ be chosen according to Theorem 5. Let $\mu, \mu > 0$ and $u_\mu \in H_\phi^\beta(\mathbb{R}; H)$, $u_\mu \in H_\phi^\beta(\mathbb{R}; H)$ satisfying

$$\partial_{0,\mu}^{-1} u_\mu = F_\mu(u_\mu) \text{ and } \partial_{0,\mu}^{-1} u_\mu = F_\mu(u_\mu).$$

Then $u_\mu = u_\mu$ as distributions in the sense of Proposition 4.4.

**Proof.** We note that it suffices to show $v_\mu := \partial_{0,\mu}^{-1} u_\mu = \partial_{0,\mu}^{-1} u_\mu = v_\mu$ as $L_{\text{loc}}(\mathbb{R}; H)$ functions by Proposition 4.10. We consider the function

$$\tilde{F} : \text{dom}(\tilde{F}) \subseteq \bigcap_{\theta \geq 0} H_\phi^\theta(\mathbb{R}; H) \to \bigcap_{\theta \geq 0} H_\phi^{\beta - \alpha}(\mathbb{R}; H)$$

given by

$$\tilde{F}(v) := F(\partial_{0,\mu}^{-1} v) \quad (v \in \text{dom}(\tilde{F}))$$

with maximal domain dom($\tilde{F}$) $= \{ v \in \bigcap_{\theta \geq 0} H_\phi^\theta(\mathbb{R}; H) : \forall \theta \geq \theta_0 : \partial_{0,\mu}^{-\theta} w \in \text{dom}(F) \}$. Note that the expression on the right hand side of (5.2) does not depend on the particular choice of $\theta \geq \theta_0$ by Proposition 4.10. Clearly, $\tilde{F}$ is eventually $(0, \beta - \alpha)$-contracting (see also Remark 5.1 b)) and

$$\tilde{F}_\phi = F_\phi(\partial_{0,\mu}^{-\beta}(\cdot)) \quad (\phi \geq \phi_0).$$
In particular,
\[ \partial_{0,\mu}^{\alpha-\beta} v_\mu = \partial_{0,\mu}^{\alpha} u_\mu = F_\mu(u_\mu) = \tilde{F}_\mu(v_\mu) \]
and analogously
\[ \partial_{0,\mu}^{\alpha-\beta} v_\mu = \tilde{F}_\mu(v_\mu). \]

Let now \( a \in \mathbb{R} \) and assume without loss of generality that \( \mu < \tilde{\mu} \). We note that \( \text{spt}(v_\tilde{\mu} - \chi_{\mathbb{R} \leq a} v_\tilde{\mu}) \subseteq \chi_{\mathbb{R} \geq a} \). We obtain, applying Theorem 5.6 that
\[ \chi_{\mathbb{R} \leq a} v_\tilde{\mu} = \chi_{\mathbb{R} \leq a} \partial_{0,\tilde{\mu}}^{\beta-\alpha} \tilde{F}_\mu(v_\tilde{\mu}) = \chi_{\mathbb{R} \leq a} \partial_{0,\tilde{\mu}}^{\beta-\alpha} \tilde{F}_\mu(\chi_{\mathbb{R} \leq a} v_\tilde{\mu}). \]

Now, since \( \chi_{\mathbb{R} \leq a} v_\tilde{\mu} \in L^2_\mu(\mathbb{R}; H) \cap L^2_\mu(\mathbb{R}; H) \), we infer that
\[ \chi_{\mathbb{R} \leq a} v_\tilde{\mu} = \chi_{\mathbb{R} \leq a} \partial_{0,\tilde{\mu}}^{\beta-\alpha} \tilde{F}_\mu(\chi_{\mathbb{R} \leq a} v_\tilde{\mu}) = \chi_{\mathbb{R} \leq a} \partial_{0,\tilde{\mu}}^{\beta-\alpha} \tilde{F}_\mu(\chi_{\mathbb{R} \leq a} v_\tilde{\mu}), \]
i.e. \( \chi_{\mathbb{R} \leq a} v_\tilde{\mu} \) is a fixed point of \( \chi_{\mathbb{R} \leq a} \partial_{0,\tilde{\mu}}^{\beta-\alpha} \tilde{F}_\mu \). However, since \( \chi_{\mathbb{R} \leq a} v_\mu \) is also a fixed point of this mapping, which is strictly contractive, we derive
\[ \chi_{\mathbb{R} \leq a} v_\tilde{\mu} = \chi_{\mathbb{R} \leq a} v_\mu \]
and since \( a \in \mathbb{R} \) was arbitrary, the assertion follows. \( \Box \)

6 Riemann–Liouville and Caputo differential equations revisited

In this section, we shall consider the differential equations introduced in Section 5 and prove their well-posedness and causality. First of all, we gather some results ensuring the Lipschitz continuity property needed to apply either of the well-posedness theorems presented in the previous section. As in Section 5 we fix \( \alpha \in (0,1] \).

**Proposition 6.1.** Let \( \varrho_0 > 0 \), \( n \in \mathbb{N} \), \( y_0 \in \mathbb{R}^n \), \( f : \mathbb{R}_{>0} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) continuous. Assume there exists \( c \geq 0 \) such that for all \( y_1, y_2 \in \mathbb{R}^n \), \( t > 0 \) we have
\[ |f(t,y_1) - f(t,y_2)| \leq c|y_1 - y_2|. \]
Moreover, we assume that
\[ (t \mapsto f(t,0)) \in L^2_{\varrho_0}(\mathbb{R}_{>0}; \mathbb{C}^n). \]
Define \( \tilde{f} : \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) by
\[ \tilde{f}(t,y) := \begin{cases} f(t,y) & \text{if } t > 0, \\ 0 & \text{else}. \end{cases} \]
Then the mapping \( F : C_\infty(\mathbb{R}; \mathbb{C}^n) \rightarrow C(\mathbb{R}; \mathbb{C}^n) \) given by
\[ F(\varphi)(t) := \tilde{f}(t,\varphi(t) + y_0) \quad (\varphi \in C_\infty(\mathbb{R}; \mathbb{C}^n), t \in \mathbb{R}) \]
is eventually \((0,0)\)-Lipschitz continuous.

**Proof.** Let \( \varrho \geq \varrho_0 \). In order to prove that \( F \) attains values in \( L^2_{\varrho}(\mathbb{R}; \mathbb{C}^n) \), we shall show \( F(0) \in L^2_{\varrho}(\mathbb{R}; \mathbb{C}^n) \) first. For this we compute
\[ \int_{\mathbb{R}} |F(0)(t)|^2 e^{-2\varrho t} \, dt = \int_{\mathbb{R}_{>0}} |f(t,y_0)|^2 e^{-2\varrho t} \, dt \]

20
of functions for which the limit at 0 exists, then the mentioned conditions imply equation (3.5). To this end, we note that

\[ spt (y - y_0 \chi_{\mathbb{R}_{>0}}) = \tilde{f} (\cdot, y(\cdot)). \]

Moreover, \( spt y \subseteq \mathbb{R}_{\geq 0} \).

\textbf{Proof.} With \( F \) as defined in Proposition 6.1 we may apply Corollary 5.3 with \( \beta = \gamma = 0 \) to obtain unique existence of \( z \in H_\varrho^\alpha (\mathbb{R}; \mathbb{C}^n) \) such that

\[ \partial_{0+, \varrho}^\alpha z = F_\varrho (z). \]

Setting \( y := z + y_0 \chi_{\mathbb{R}_{>0}} \), we obtain in turn unique existence of a solution of the desired equation. Since \( spt F_\varrho (z) \subseteq \mathbb{R}_{\geq 0} \), we obtain with Proposition 6.1 that \( spt z = spt \partial_{0+, \varrho}^\alpha F_\varrho (z) \subseteq \mathbb{R}_{\geq 0} \). Thus, \( spt y \subseteq \mathbb{R}_{\geq 0} \).

We remark here that the condition \( spt y \subseteq \mathbb{R}_{\geq 0} \) together with \( y - y_0 \chi_{\mathbb{R}_{>0}} \in H_\varrho^\alpha (\mathbb{R}; \mathbb{C}^n) \) describes how the initial value \( y_0 \) is attained. Indeed, if \( \alpha \) is large enough (e.g. \( \alpha > 1/2 \)) so that \( H_\varrho^\alpha (\mathbb{R}; \mathbb{C}^n) \) is a subset of functions for which the limit at 0 exists, then the mentioned conditions imply

\[ 0 = (y - y_0 \chi_{\mathbb{R}_{>0}})(0-) = (y - y_0 \chi_{\mathbb{R}_{>0}})(0+) = y(0+) - y_0, \]

that is, the initial value is attained.

We conclude this section by having a look at the case of the Riemann–Liouville fractional differential equation 5.5. To this end, we note that \( \chi_{\mathbb{R}_{>0}} y_0 \in H_\varrho^\alpha (\mathbb{R}; H) \) for \( \varrho > 0 \) and by Example 4.6 we have

\[ \partial_{0+, \varrho} \chi_{\mathbb{R}_{>0}} y_0 = \delta_0 y_0 \in H_\varrho^{-1}(\mathbb{R}; H). \]

We also recall the notation \( g_\beta(t) := \frac{1}{\Gamma (\beta + 1)} t^\beta \chi_{\mathbb{R}_{>0}} \) for \( \beta, t \in \mathbb{R} \).
Proposition 6.3. Let \( y_0 \in \mathbb{C}^n \). Assume that \( C_c^\infty(\mathbb{R}; H) \ni \varphi \mapsto \tilde{f}(\cdot, \varphi(\cdot)) \) is eventually \((\alpha - 1, \alpha - 1)\)-Lipschitz continuous and denote with \( H^{\alpha-1}_\varphi(\mathbb{R}; H) \) its Lipschitz-continuous extension for some \( \varphi > 0 \). There is \( \varrho_1 > 0 \) such that for \( \varrho \geq \varrho_1 \) we have a unique solution \( y \in H^{\alpha-1}_\varphi(\mathbb{R}; H) \) of the equation

\[
\partial_{0, \varphi}^\alpha y = y_0 \partial_0 + \tilde{f}(\cdot, \varphi(\cdot)), \quad y \in H^{\alpha-1}_\varphi(\mathbb{R}; H),
\]

with \( \partial_{0, \varphi}^{\alpha-1} y - y_0 \chi_{\mathbb{R}^+} \in H^0_\varphi(\mathbb{R}; H) \) and \( \text{spt}(y) \subseteq \mathbb{R}_{\geq 0} \).

Proof. The mapping \( G \) defined by

\[
G(\varphi)(t) := \tilde{f}(t, \partial_{0, \varphi}^{\alpha-1} \varphi(t) + g_{\alpha-1}(t)y_0), \quad \varphi \in C_c^\infty(\mathbb{R}; H), \quad t \in \mathbb{R}
\]

is eventually \((0, \alpha - 1)\)-Lipschitz continuous. Indeed, this fact follows from \( g_{\alpha-1}y_0 \in H^{\alpha-1}_\varphi(\mathbb{R}; H) \) and the unitarity of \( \partial_{0, \varphi}^{\alpha-1} : H^{\alpha-1}_\varphi(\mathbb{R}; H) \to H^0_\varphi(\mathbb{R}; H) \). Let \( \varrho_1 > 0 \) be such that \( \tilde{f}_\varphi \) and therefore \( G_\varphi \) exist for \( \varrho \geq \varrho_1 \). Let \( \varrho \geq \varrho_1 \). The Riemann–Liouville equation is equivalent to

\[
\partial_{0, \varphi}^{\alpha-1} y - \chi_{\mathbb{R}^+} y_0 = \partial_{0, \varphi}^{\alpha-1} \tilde{f}(\cdot, \varphi(\cdot)), \quad y \in H^0_\varphi(\mathbb{R}; H).
\]

With the transformation \( z = \partial_{0, \varphi}^{\alpha-1} y - \chi_{\mathbb{R}^+} y_0 \) and using \( \partial_{0, \varphi}^{\alpha-1} \chi_{\mathbb{R}^+} y_0 = \partial_{0, \varphi}^{\alpha-1} g_{\alpha-1} y_0 = g_{\alpha-1} y_0 \) (cf. Corollary 2.7), this equation is equivalent to

\[
\partial_{0, \varphi} z = G_\varphi(z), \quad z \in H^0_\varphi(\mathbb{R}; H).
\]

By Corollary 5.3 (with \( \gamma = \alpha - 1 \)) we find a unique solution \( z \in H^0_\varphi(\mathbb{R}; H) \). We have \( \text{spt}(\partial_{0, \varphi}^{-1} G_\varphi(\cdot)) \subseteq \mathbb{R}_{\geq 0} \). By Proposition 5.5 \( \text{spt} z \subseteq \mathbb{R}_{\geq 0} \). Hence we have a unique solution \( y = \partial_{0, \varphi}^{\alpha-1} (z - \chi_{\mathbb{R}^+} y_0) \in H^{\alpha-1}_\varphi(\mathbb{R}; H) \) of the Riemann-Liouville equation with \( \text{spt} y \subseteq \mathbb{R}_{\geq 0} \) and \( \partial_{0, \varphi}^{\alpha-1} y - \chi_{\mathbb{R}^+} y_0 = z \in H^0_\varphi(\mathbb{R}; H) \).

Remark 6.4. The space \( H^0_\varphi(\mathbb{R}; H) \) is continuously embedded into \( H^{\alpha-1}_\varphi(\mathbb{R}; H) \). Thus, the assumption that \( C_c^\infty(\mathbb{R}; H) \ni \varphi \mapsto \tilde{f}(\cdot, \varphi(\cdot)) \) is eventually \((\alpha - 1, \alpha - 1)\)-Lipschitz continuous, can be replaced by the stronger assumption that \( C_c^\infty(\mathbb{R}; H) \ni \varphi \mapsto \tilde{f}(\cdot, \varphi(\cdot)) \) is eventually \((\alpha - 1, 0)\)-Lipschitz continuous, which might be easier to compute.

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