Mirror Fermions in Noncommutative Geometry

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Abstract

In a recent paper we pointed out the presence of extra fermionic degrees of freedom in a chiral gauge theory based on Connes Noncommutative Geometry. Here we propose a mechanism which provides a high mass to these mirror states, so that they decouple from low energy physics.

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Recently in Ref. [1] we have pointed out a doubling of the fermionic degrees of freedom in the Connes [2, 3] approach to chiral gauge theories based on Noncommutative Geometry [4]. In this brief report we suggest a mechanism which solves the problems posed by these extra degrees of freedom, by giving them a high mass. The mechanism on which the solution is based is loosely modelled in analogy with lattice gauge theories where a similar phenomenon occurs. The origin of the two phenomena are however quite distinct, at least with the present level of understanding.

As is known the essential ingredients of Noncommutative Geometry are an algebra $\mathcal{A}$, which encodes the topology of space–time (or its noncommutative generalization), an Hilbert space $\mathcal{H}$ which represents the fermionic mass degrees of freedom, and an operator $D$ which generalizes the Dirac operator and which encodes the metric structure of the space. In addition to these three components of the so called spectral triple, there are other two essential elements: the real structure $J$, which represents charge conjugation, and a grading $\gamma$ which generalizes the usual $\gamma_5$.

At present the noncommutative geometric structure of chiral gauge theories is understood as the product of continuous geometry representing the usual (commutative) space–time times an internal geometry. Thus the algebra is chosen as $\mathcal{A} = C^\infty(\mathbb{R}^4, \mathbb{C}) \otimes \mathcal{A}_F$, where $C^\infty(\mathbb{R}^4, \mathbb{C})$ is the algebra of smooth complex valued functions on $\mathbb{R}^4$, and $\mathcal{A}_F$ a matrix algebra whose unimodular group is the gauge group. Analogously $\mathcal{H} = L^2(S_{\mathbb{R}^4}) \otimes \mathcal{H}_F$, with $L^2(S_{\mathbb{R}^4})$ the space of spinors and $\mathcal{H}_F$ an internal Hilbert space which comprises all fermionic degrees of freedom. The other ingredients are obtained in a similar way, for details we refer to the literature on the subject in its various versions ([3, 5]–[9] and references therein). Other versions of gauge theory based on Noncommutative Geometry [10] have some of the basic ingredients of the construction which differ in essential ways from the ones treated here, and in general the considerations about mirror fermions will not apply.

It is evident that the full power of noncommutative geometry is still used in a very limited way. The theory is some sort of Kaluza–Klein in which there is a continuous commutative space–time, still made of usual points with the usual Hausdorff topology. At each point then there is a noncommutative space of the simplest kind possible, the one represented by finite dimensional matrix algebras. Despite the promising phenomenological features of the model [4, 11, 12], this simple choice of the space as a product creates some problems. The main one arises in $\mathcal{H}$. For the consistency of the model it must be the tensor product of spinors times all fermionic degrees of freedom, and therefore some degrees of freedom will appear more than once. Moreover the chirality assignments of the extra degrees of freedom are incorrect. We will be more detailed in the following.

The problem could be solved by projecting out the unwanted degrees of freedom, but as we showed in Ref. [1] this procedure is ambiguous and can only be made in a highly ad hoc fashion.

In this paper we would like to explore another possibility, namely that the mirror fermionic degrees of freedom are actually real ones, but that the mass they have is
too large to be detected, or to have any effect at present energies. There might be
some consequences for the early universe, and we will comment on this later.

In what follows we will work in Minkowski space. It has been already stressed
in Ref.[1] that the appearance of mirror fermions is independent of the choice of the
signature. However, as will be clear in the following, this aspect is crucial for the
solution of the problem we propose here. For the bosonic terms in the action, the
choice of the scalar product is not so important, since the euclidean and lorentzian
theories can be related by Wick rotation. This is not true for fermionic terms. In
this case, in fact, since the involved representations are complex and the invariants
are written in terms of a hermitian form rather than a scalar product, there are no
transformations which can relate the positive definite hermitian form of euclidean
theories, with the ones, with no definite sign, of lorentzian models. In other words,
as far as the bosons are concerned, the invariants under \( SO(3,1) \) build up in terms of
scalar product of the fields become, via Wick rotation, the corresponding invariants
under \( SO(4) \), whereas this does not occur for spinors. Note that, the
euclidean theories introduced as a way to regularize functional integration are just the Wick
rotation of the lorentzian models, and thus not invariant, in general, under \( SO(4) \).

To explain the problem and the solution we propose, the subtleties of the full
construction are unnecessary. We will therefore deal with a simplified model, in
which a single generation contains only one spinor. The generalization is absolutely
straightforward.

As usual we start defining the various elements of the Connes construction. For
the example under consideration of a spontaneously broken \( U(1) \) theory, we start with
the algebra \( \mathcal{A} = C^\infty(\mathbb{R}^4, \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C}) \). The unimodularity condition will reduce the
gauge group \( U(1)_L \otimes U(1)_R \), the unitary elements of the algebra, to \( U(1)_A \). The Hilbert
space has the usual tensor product structure \( \mathcal{H} = L^2(S_{\mathbb{R}^4}) \otimes \mathcal{H}_F \), where \( \mathcal{H}_F = \mathbb{C}^4 \).
A generic element of \( \mathcal{H} \), can be expressed as a linear combination of elements of the form
\( \Psi = \psi \otimes h_F \), with \( \psi \) a Dirac spinor in \( L^2(S_{\mathbb{R}^4}) \) and \( h_F = (h_L, h_R, h_L^c, h_R^c) \in \mathcal{H}_F \).
On \( \mathcal{H} \), an element \( \alpha \) of \( \mathcal{A} \) is represented as follows
\[
\rho(\alpha) = \begin{pmatrix}
a_L(x) & a_R(x) \\
a_L^\ast(x) & a_R^\ast(x)
\end{pmatrix}, \tag{1}
\]
where \( a_L(x) \) and \( a_R(x) \) belong to \( C^\infty(\mathbb{R}^4, \mathbb{C}) \). By observing that for each \( x \in \mathbb{R}^4 \) a
generic spinor \( \psi \) can be decomposed as \( \psi(x) = \psi_L + \psi_R + \psi_R^c + \psi_L^c \), among the 16
possible combinations in \( \Psi = \psi \otimes h_F \), the following have a chirality mismatch
\[
(\psi_L + \psi_R^c) \otimes (h_R + h_L^c) + (\psi_R + \psi_L^c) \otimes (h_L + h_R^c). \tag{2}
\]
These are the spurious degrees of freedom which behave as the mirror fermions on
lattice gauge theories and should be eliminated from the theory.
As Dirac operator we consider the following generalization of the customary one
\[
D = i\partial \otimes I + I \otimes \mathcal{M} - \gamma_5 \otimes \gamma_F \mathcal{M}',
\] (3)
where \(\gamma_F\) is the grading in the finite space \(\mathcal{H}_F\)
\[
\gamma_F = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}.
\] (4)

The grading in \(\mathcal{H}\), denoted with \(\gamma = \gamma_5 \otimes \gamma_F\) satisfies the following relations: \(\gamma = \gamma^\dagger\), \(\gamma^2 = I\), and \(\{\gamma, D\} = 0\).

It is worth observing that \(\gamma\) has eigenvalues +1 and -1 on the physical and unphysical fermionic states, respectively, and therefore the combination \(P = (I + \gamma)/2\) is the projection operator on the subspace of physical states. Interestingly one of the natural structure in the algebraic construction of gauge theories in Noncommutative Geometry, namely the grading \(\gamma\), distinguishes the real physical fermionic states in the underlying Hilbert space from their mirror partners.

At this point one could consider that the solution of the problem is to project out the unwanted degrees of freedom with the natural operator \(P\). However, we showed in Ref.[1] that this projection must be done only in the fermionic part of the action, since for the bosonic part it would eliminate the self-dual or the anti self-dual part of the gauge tensor fields. One can still be satisfied with treating the two terms of the action in a different way, but, apart from issues of naturality and aesthetics, this leaves open the problem of definition of the actual Hilbert space of the theory.

In equation (3) \(\mathcal{M}\) and \(\mathcal{M}'\) are the mass matrices, defined as usual by
\[
\mathcal{M} = \begin{pmatrix} 0 & m & 0 & 0 \\ m^* & 0 & 0 & 0 \\ 0 & 0 & 0 & m^* \\ 0 & 0 & m & 0 \end{pmatrix},
\] (5)
and similarly for \(\mathcal{M}'\). In this simple example \(m\) and \(m'\) are just numbers, while in more complex cases, like the Standard Model, they would be matrices.

Finally, the real structure \(J = J_D \otimes J_F \mathcal{C}\), \(\mathcal{C}\) being complex conjugation, satisfies the relations
\[
[J, D] = 0 , \quad J\gamma = \gamma J , \quad [\alpha, J^\dagger \alpha' J] = [[D, \alpha], J^\dagger \alpha' J] = 0 , \quad \alpha, \alpha' \in \mathcal{A}.
\] (6)
Using these conditions one easily find that \(J_D = \gamma_2\) and
\[
J_F = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\] (7)

All relations uniquely fix \(J\) up to a phase and are compatible with the generalized Dirac operator introduced in (3) with a new mass matrix \(\mathcal{M}'\).
It is important to notice at this point that this is not true anymore if we try to implement the same structure in euclidean space. In fact, in this case, starting with the same algebra, grading and Hilbert space, the euclidean Dirac operator has the form

$$D = \partial \otimes 1 + 1 \otimes \mathcal{M} - \gamma_5 \otimes \gamma_F \mathcal{M}'$$ \hspace{1cm} (8)

where, for example, we choose all $\gamma^\mu$ to be antihermitean.

The condition $[D, J] = 0$ in particular requires the following conditions, as it is easy to verify

$$J_D \gamma_\mu^* = \gamma_\mu J_D \hspace{1cm} [J_F \mathcal{C}, \mathcal{M}] = 0 \hspace{1cm} [J, \gamma_5 \otimes \gamma_F \mathcal{M}'] = 0$$ \hspace{1cm} (9)

The first two give, up to a phase, $J_D = \gamma_0 \gamma_2$, and $J_F$ as in Eq.(8). Thus the third relation of Eq.(11) implies $m'\{\gamma_5, J_D\} = 0$ which is satisfied only if $\mathcal{M}'$ identically vanishes.

This is also an example which shows that working in euclidean space, while having advantages for the definitions of the mathematical objects involved in the theory, can have some non-trivial consequences from the physical point of view.

Going back to the Minkowski case, in order to construct the lagrangian density for the abelian model we are considering, we first need to construct the gauge connection one-forms, as the elements $\rho = \alpha[D, \alpha']$. Removing junk forms we get

$$\rho = \left( \begin{array}{c} A_{\rho} \\ \phi - \phi_0 - \gamma_5 (\phi' - \phi'_0) \end{array} \right) \hspace{1cm} (10)$$

where the Yang–Mills connection (one–form) $A_{L,R} \equiv \sum_i a^i_{L,R} i \partial a^i_{L,R}$, with the condition $A_{L,R}^\mu = A^\mu_{L,R}$ and with $\phi - \phi_0 \equiv \sum_i a^i_{L}(M a^i_{R} - a^i_{L} \mathcal{M})$. A similar expression holds for $\phi' - \phi'_0$. Unimodularity condition $Tr(\rho) = 0$ reduces, as already mentioned, the gauge group to the axial term only $U(1)_A$, tracing out the vector part

$$\rho = \left( \begin{array}{c} A \\ \phi - \phi_0 - \gamma_5 (\phi' - \phi'_0) \end{array} \right) \hspace{1cm} (11)$$

The two Higgs fields $\phi$ and $\phi'$ represent the connection fields in the discrete direction and are related to the terms proportional to $\mathcal{M}$ and $\mathcal{M}'$, respectively. Under a gauge transformation, represented by the unitary elements $u$ of the algebra $\mathcal{A}$, with the condition $u_{L} = u^*_{R}$, $\rho$ transforms as $\rho \rightarrow u[D, u^*] + u\rho u^*$.

By using the matrix representation for the algebra and Eq.(11), we see in particular that both $\phi$ and $\phi'$ have equal non vanishing charge with respect to the $U(1)_A$ gauge group. Their expectation values on the vacuum state would therefore break axial gauge symmetry.

The bosonic contribution to the action of the model can be obtained by evaluating the square of the curvature $\theta = d\rho + \rho^2$, traced over the entire fermion Hilbert space.

$$S_B = Tr \theta^2 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)(D^\mu \phi)^* + (D_\mu \phi')(D^\mu \phi')^* - V(\phi, \phi') \right] \hspace{1cm} (12)$$
where $F^{\mu\nu}$ is the tensor field for the axial gauge potential $A^\mu$, and $D_\mu = i \partial_\mu + 2 A_\mu$. The Higgs potential $V(\phi, \phi')$ takes the form

$$V(\phi, \phi') \propto (|\phi|^2 + |\phi'|^2 - \mu^2)^2 + (\phi^* \phi' - \lambda^2)^2,$$

(13)

where $\mu^2 = |\phi_0|^2 + |\phi'_0|^2$ and $\lambda^2 = \phi_0 \phi'^*_0 + \phi'^*_0 \phi_0$.

Finally, the fermionic action is expressed in terms of the invariant scalar product

$$S_F = \langle \Psi, (D + \rho + J^\dagger \rho J) \Psi \rangle,$$

(14)

which gives, together with kinetic and interaction terms with gauge potential and Higgs fields, the following mass terms

$$S_F\text{(mass terms)} = \int d^4x (\bar{\psi} \psi) \left( h_F^\dagger \mathcal{M} h_F - (\bar{\psi} \gamma_5 \psi) \left( h_F^\dagger \gamma_F \mathcal{M}' h_F \right) \right)$$

$$= \int d^4x \bar{\Psi} \left( \left( \frac{1 + \gamma}{2} \right) \left( \mathcal{M} - \mathcal{M}' \right) + \left( \frac{1 - \gamma}{2} \right) \left( \mathcal{M} + \mathcal{M}' \right) \right) \Psi,$$

(15)

where $\psi \in L^2(S_M)$ and $h_F \in \mathcal{H}_F$. Decomposing $\psi$ and $h_F$ as shown before it follows that all fermions belonging to the physical subspace of $\mathcal{H}$ acquire mass equal to $m - m'$, while their mirror partner get instead $m + m'$. In particular, if both $m$ and $m'$ are very large, namely if the breaking of $U(1)_A$ occurs at a very high scale, all mirror states completely decouple from the low energy theory. If all physical states should remain instead massless or take a very small mass term, one has to impose $m - m' << m, m'$. In this scheme this fine-tuning condition seems to be unavoidable.

This mechanism which gives a high mass to the mirror fermions via a spontaneous breaking at a high scale, may provide a scalar dynamics which would drive chaotic inflation. We have already discussed the appearance of inflation in noncommutative geometry models in Ref. [13].

There is only a drawback in this solution of the doubling problem. To obtain masses which are very high for mirror states only, and practically vanishing for the observed fermions, one has a serious problem of fine tuning. Actually, fine tuning problems are not new to Noncommutative Geometry [7].

Despite this feature, the mechanism we propose here is a dynamical way of eliminating the problem. Actually, we think that the main point is the fact that we are too naive in considering the geometry as the product of a continuous commutative space, times the space of finite dimensional matrix algebra. The real structure of space time is probably a more complicated one, and Noncommutative Geometry seems the ideal tool to study its structure. In this respect, it is worth pointing out, to conclude, that the solution of mirror fermion problem we have discussed is suggesting that Planck mass or some other high mass scale should be the natural scale where noncommutative structure of space time should manifest itself. Shadows of these effects, as for example a particular tensor product form for the fermion Hilbert space, or the choice for the algebra, could well be present for low energy theories as the Standard Model.
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