Acoustics of noise, vibration, and harshness

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Abstract. In this paper, we will go to a more fundamental level of treating acoustics. We introduced the gauge invariance into the acoustic wave equation of motion in 2007. This is an important milestone in the history of gauge invariance. The earliest gauge invariance is shown by Maxwell’s electromagnetic wave equations. The root of acoustics originated from fluid mechanics. We show that the usual form of acoustic wave equation for fluids is in fact the linearized form of the Navier Stokes equations. Hence there is in fact, no need to use acoustic analogy to derive the formula for the jet noise which was done by James Lighthill. In this paper we also treat finite amplitude sound wave and finite amplitude vibration using the deductive approach. This is a generalization of the linear theory to the nonlinear theory using relativistic and curvilinear spacetime treatment. This produces the linear case as a special case of the nonlinear case. The usual method is the reverse which starts from the linear theory and extend to the nonlinear theory. This will miss out some useful information. The treatment of vibration will need to incorporate the theory of elasticity. This paper is also the first introduction of singularities into vibration. Singularities is mathematically more sophisticated than resonance.

1. Introduction

We extend the gauge invariance property of the Maxwell’s equation for electromagnetic theory to the acoustic field equations. This is an important milestone in the history of gauge invariance. The earliest gauge invariance is used in Maxwell’s equations of electromagnetic wave equations. Gauge invariance which includes symmetries is a basic property of field theory which covers strong nuclear forces, electromagnetic force, and gravitational force. In extending gauge invariance approach to acoustic fields, it will be a more sophisticated approach than the vector theory of acoustic fields and will include also the characteristics of the vector theory. We will also show that vector theory is a subset of gauge theory. We address the symmetry properties of the acoustic field equations, and interpretation of the inhomogeneous wave equation in terms of gauge invariance. Gauge invariance has long been applied to electromagnetic wave theory. Due to the similarities between electromagnetic waves and acoustic waves, we feel that the interpretation of acoustic fields in terms of gauge invariance will provide more understanding of the acoustic fields and throw lights on new potential applications. This paper is the first application of gauge invariance to acoustic fields.

2. Gauge invariance formulation of the inhomogeneous acoustic wave equation

We will start with the derivation of the acoustic Lorentz gauge condition. First, we show that for electromagnetic waves, the Lorentz gauge condition gives rise to the equation of continuity. In electromagnetic theory, the Lorentz gauge condition is given as
\[ \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \]  
(1)

where $A = \text{vector potential}$ and $\phi = \text{scalar potential}$.

The potential theory is used here. By substituting in the retarded potential for $A$ and $\phi$, and with simplification, one can obtain:

\[ \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \]  
(2)

which is the equation of continuity.

On the other hand, the acoustic equation of continuity is:

\[ \nabla (\rho V) + \frac{\partial \rho}{\partial t} = 0 \]  
(3)

with just replacing $\vec{j}$ by $\rho V$ which is correct with charge density being analogous to the mass density.

So, one can use equation (1) as the acoustic Lorentz gauge condition where $c =$ sound velocity.

From the electromagnetic theory, the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are expressed in terms of potential functions as:

\[ \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \]  
(4)

and

\[ \vec{B} = \nabla \times \vec{A} \]  
(5)

These potentials are invariant under gauge transformations. Here one will use the analogy of $\vec{E}$ as $\vec{T}$ the acoustic stress field.

The usual acoustic wave equations are expressed in terms of the potentials $A$ and $\phi$. This gives only partial aspects of the acoustic fields $\vec{V}$ and $\vec{T}$.

From equation (4), one can write

\[ \vec{T} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \]  
(6)

From Maxwell’s equation

\[ \nabla \cdot \vec{E} = 4 \pi \rho \]  
(7)

For the acoustic field

\[ \nabla \cdot \vec{T} = 4 \pi \rho \]  
(8)

Substituting equation (6) into equation (8), one has

\[ \nabla \cdot (-\nabla \phi - \frac{\partial \vec{A}}{\partial t}) = 4 \pi \rho \]  
(9)

But Lorentz gauge condition, equation (1),

Thus,

\[ -\nabla^2 \phi + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 4 \pi \rho \]  
(10)

which is the inhomogeneous acoustic wave equation in $\phi$ and $\rho =$ field density.

One can also express the velocity field in terms of the scalar potential and vector potential as

\[ \vec{V} = \nabla \phi + \nabla \times \vec{A} \]  
(11)

The Christoffel equation for an isotropic medium with plane wave solutions and harmonic time variation can be written as
\[ c_{11} \nabla (\nabla \tilde{V}) - c_{44} \nabla \times \nabla \times \tilde{V} = \rho \frac{\partial^2}{\partial t^2} \tilde{V} \]  

(12)

Substitution of equation (11) into equation (12) gives

\[ \nabla (c_{11} \nabla^2 \phi - \rho \frac{\partial^2}{\partial t^2} \phi) \cdot \nabla \times (c_{44} \nabla \times \nabla \times \tilde{A} + \rho \frac{\partial^2}{\partial t^2} \tilde{A}) = 0 \]  

(13)

For the second term, the quantity in brackets is set equal to the gradient of an arbitrary function \( f \).
The vector potential \( \tilde{A} \) can thus be taken as a solution to the vector wave equation

\[ \nabla^2 \tilde{A} - \frac{1}{V_s^2} \frac{\partial^2}{\partial t^2} \tilde{A} = 0 \]  

(14)

The first term in equation (13) is made zero by simply by requiring that the scalar potential \( \phi \) satisfy the scalar equation

\[ \nabla^2 \phi - \frac{1}{V_t^2} \frac{\partial^2}{\partial t^2} \phi = 0 \]  

(15)

So, one has obtained homogeneous sound wave equation in \( A \) and \( \phi \).

This shows that the acoustic fields \( \tilde{T} \) and \( \tilde{V} \) can be expressed in terms of the solutions of the homogeneous acoustic wave equations \( A \) and \( \phi \). \( A \) and \( \phi \) in turn are only partial aspects of \( \tilde{T} \) and \( \tilde{V} \).

The Lorentz gauge condition has the advantage of introducing complete symmetry between the scalar and vector potentials, i.e., it makes both potentials satisfy the same wave equation as that obeyed by the fields. Equations (14) and (15) are a symmetrical set of equations.

By using the analogy of momentum density \( \tilde{P} \) as equivalent to \( \tilde{B} \), one has

\[ \tilde{T} = -\nabla \phi - \frac{\partial \tilde{A}}{\partial t} \]  

(16)

and

\[ \tilde{P} = \nabla \times \tilde{A} \]  

(17)

Equations (16) and (17) are unchanged by transformations of the type

\[ A' = A - \nabla \phi \]  

(18)

\[ \phi' = \phi + \frac{\partial \phi}{\partial t} \]  

(19)

These transformations are usually known as gauge transformations, and a physical law that is invariant under such a transformation is said to be gauge invariant.

Gauge invariance or gauge field theory is a more sophisticated treatment of acoustic fields than vector theory. It gives a more accurate solution for the velocity field and the stress field than those obtained from the inhomogeneous acoustic wave equations. It also gives the difference between the strain field and the velocity field which previously all treat them as wave functions. It provides better understanding of the stress field and the velocity field.

3. The root of acoustics is fluid mechanics - there is no need to use acoustic analogy

Acoustics for fluids originated from fluid mechanics. The acoustic wave equation for fluids is in fact the linearized form of the Navier Stokes equation. Hence there is no need to use the acoustic analogy to use the Navier Stokes equation to derive the formula for the jet noise as what James Lighthill [1] did.

The Navier Stokes equations consist of two equations:

1) The continuity equation. This equation is derived through the hypothesis of conservation of mass. In its typical form, it can be written down as
\[
\frac{\partial p}{\partial t} + \rho \nabla \cdot \mathbf{u} = 0 \tag{20}
\]

where \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \mathbf{u} \frac{\partial}{\partial x_i} \) known as the material derivative, and \( \rho \) the density of the fluid.

2) **Momentum equation.** The fluid must also conserve momentum. We ensure this by requiring that

\[
\frac{d}{dt} \mathbf{u}_j = \frac{\partial T_{ij}}{\partial x_j} \rho g_j \tag{21}
\]

with

\[
T_{ij} = -p \delta_{ij} + 2\mu \varepsilon_{ij} + \lambda \varepsilon_{mm} \delta_{ij} \tag{22}
\]

where \( \mathbf{u} \) is the fluid velocity, \( T_{ij} \) is the stress tensor, \( P \) is the pressure, \( g_j \) is the constant gravitational acceleration, \( \delta_{ij} \) is the unity matrix, \( \varepsilon_{mm} \) is the strain tensor, with \( \mu \) and \( \lambda \) are fluid dependent scalars.

If \( \mathbf{u} \) is incompressible, \( (\nabla \cdot \mathbf{u} = 0) \) these equations reduce to the Navier-Stokes Equations. On the other hand, usually the acoustic wave equation is derived as follows [2] as what is done by B. A. Auld [2]:

These equations are used. The first is the equation of motion:

\[
\nabla \mathbf{T} = \rho \frac{\partial}{\partial t} \mathbf{u} - \mathbf{F} \tag{23}
\]

where \( \mathbf{F} = \) body force and the second equation are the strain-displacement relation:

\[
\mathbf{S} = \nabla \mathbf{u} \mathbf{L} \tag{24}
\]

where \( \mathbf{S} = \) strain, and \( \mathbf{L} = \) displacement.

One will realize that equation (21) is equivalent to equation (23) and equation (22) and equivalent to equation (24). The acoustic wave equation of motion is obtained by the combination of equation (23) and (24). This shows that the Navier Stokes equation and the acoustic wave equation of motion are equivalent. And they have the same origin.

4. **Nonlinear acoustics on the curvilinear spacetime- a deductive approach**

Curvilinear spacetime is a coordinates system that satisfy the general covariance requirements of all physics equations. All physics equations must remain the same form on whatever type of coordinate system being used. The curvilinear coordinate system was used by Albert Einstein in his general theory of relativity [3]. He introduced the curvilinear coordinates to analyse the general theory of relativity because besides fulfilling the general covariance requirement, it also describes accurately the curvilinear path of the gravitational force. Curvilinear spacetime is useful for explaining intense sound fields due to their curvilinear paths and hence will be suitable for the treatment of nonlinear acoustics. This is a more accurate approach and appropriate for the design of extremely sensitive equipment.

The use of curvilinear spacetime will enable a deductive approach to a nonlinear acoustics problem. The usual treatment of nonlinear acoustics is the proceeding from the to the general; from the special, linear case to the more general, non-linear case. This will tend to miss out some useful information’s. But it has become clear in recent years that in many cases it is advantageous to reverse this procedure. That is, to proceed from the general to the particular. Take an example, to regard the case of a material with linear constitutive relations as a special case of a more general class of materials.

5. **Treatment of vibration and introduction of singularities into vibration problems**

To understand vibration, one needs to incorporate in the theory of elasticity. For example, for the treatments of vibration in structures, in shells, in plates. This is because one has to deal with the stress-strain relationships.
Resonance is a very common phenomenon in vibration problems and the usual treatment on resonance is to deal with the solution of the free vibration and forced vibration partial differential equations. Singularities is not a shortcoming of theories but is evident as the pattern of nature. Singularities play an important role in several branches of physics such as the Big Bang and the origin of the universe, singularities in second order phase transition, singularities in the Yang Mills theory of Standard Model of particle physics and so on.

Mathematically, singularities are more sophisticated than resonance. Usually it is only mentioned that the amplitude reaches a peak during resonance but not mentioned that in fact the amplitude tends to move towards infinity and causes instability during resonance. The figure below shows the amplitude of vibration tends towards infinity at the resonance frequency of a driven damped simple harmonic oscillator. Here the term disastrous resonance is used instead of the term singularities.

![Resonance Transmissibility](image)

**Figure 1.** The amplitude of vibration tends towards infinity at the resonance frequency of a driven damped simple harmonic oscillator.

### 6. References

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3. Einstein, A 1916 The Foundation of the General Relativity (Annalen der Physik) 354 (7) 790