Nonlinear dynamics of a cantilevered piezoelectric rectangular plate with voltage excitation

J H Yang¹,², W Zhang¹, M H Yao¹ and G Zhu²

¹College of Mechanical Engineering, Beijing University of Technology, Beijing, China
²Beijing Vocational College of Agriculture, Beijing, China

Abstract. The nonlinear dynamics of a cantilevered piezoelectric rectangular plate are studied, which are simultaneously forced by the base excitation and the excitation loaded by piezoelectric layers. The nonlinear partial differential governing equations of motion for the system are established by using the Hamilton’s principle. The Galerkin approach is used to discretize the partial differential equations to the ordinary differential equations with four-degree-of-freedom. Numerical simulations are presented to investigate the effects of the voltage excitation on the steady-state responses of the cantilevered piezoelectric plate. The bifurcation diagram of the system for \( W_1, W_2, W_3, W_4 \) via the voltage excitation amplitude is obtained. The motions of the system can be shown as follows: the chaotic motion \( \rightarrow \) the multiple periodic motion \( \rightarrow \) the periodic motion \( \rightarrow \) the chaotic motion. Based on the above bifurcation diagrams, the voltage excitation amplitude are changed to obtain the waveforms, two-dimensional phase portraits, three-dimensional phase portraits and Poincare maps. The results show that the amplitude of the system can reduce effectively and keep the stability by adjusting the voltage excitation.

1. Introduction
Piezoelectric materials, which include piezoelectric lead-zirconate-titanate (PZT) and piezoelectric polyvinylidene fluoride (PVDF) are new type functional materials and can be utilized as the actuators and sensors in engineering structures [1, 2]. Zhang et al. [3,4] investigated the bifurcations and chaotic dynamics of a simply support symmetric cross-ply composite laminated piezoelectric rectangular plate, which are simultaneously forced by the transverse and in-plane excitations. Correia et al. [5] developed a semi-analytical axisymmetric shell finite element model with embedded or surface bonded piezoelectric actuators or sensors to study active damping vibration control of structures.

2. Equations of motion
As shown in Figure 1, a cantilevered piezoelectric rectangular plate clamped at edge \( oa \). The upper and lower layers are made of the PVDF piezoelectric materials as actuators, while substrate layer is made of fiber-reinforced composite material. It is assumed that different layers are perfectly bonded to each other. A Cartesian coordinate \( Oxyz \) is located in the middle surface of cantilevered piezoelectric plate. The base excitation is represented by \( F \cos(\Omega t) \). The dynamic electrical loading is expressed as \( V \cos(\Omega t) \).
Figure 1 The mathematical model of a cantilevered piezoelectric plate structure

According to the von Karman type equations for the geometric nonlinearity and the Reddy’s classic deformation plate theory, we can write the displacement field of the cantilevered piezoelectric plate as followings [6]

\[ u_i = u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0}{\partial x}, u_2 = v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_0}{\partial y}, u_3 = w(x, y, z, t) = w_0(x, y, t) \] (1)

where \((u_1, u_2, u_3)\) are the displacement components along the \((x, y, z)\) directions, \((u_0, v_0, w_0)\) is the deflection of a point on the middle plane \((z = 0)\).

The nonlinear strain-displacement relations are given as follows

\[\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right), \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \right), \varepsilon_{zz} = \frac{\partial w}{\partial z}, \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \] (2)

The stress–strain relationship of the cantilevered piezoelectric plate is given

\[ \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \end{bmatrix} \]

(3)

where \(Q_{ij}\) is the elastic stiffness coefficient, \(e_{ij}\) is the piezoelectric constant, \(E\) is the electric field strength, shown as follows

\[ Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, Q_{12} = \frac{\nu_{12} E_1}{1 - \nu_{12} \nu_{21}}, Q_{21} = Q_{12}, Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}, Q_{44} = Q_{22}, Q_{55} = Q_{11}, Q_{66} = Q_{12} \] (4)

where \(E_1\) is the elastic modulus, \(G_{ij}\) is the shear modulus, \(\nu_{12}\) and \(\nu_{21}\) are Poisson’s ratio.

Using the Hamilton’s principle and the second kinds of piezoelectric constitutive equations, the nonlinear governing equations of motion for the cantilevered piezoelectric plate are obtained as

\[ \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_a \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x}, \frac{\partial N_{yy}}{\partial x} + \frac{\partial N_{xx}}{\partial y} = I_a \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial y}, \]

\[ \frac{\partial N_{xx}}{\partial x} \frac{\partial \dot{w}_0}{\partial x} + \frac{\partial N_{yy}}{\partial x} \frac{\partial \dot{w}_0}{\partial y} + \frac{\partial N_{xx}}{\partial y} \frac{\partial \dot{w}_0}{\partial x} + \frac{\partial N_{yy}}{\partial y} \frac{\partial \dot{w}_0}{\partial y} + F \cos(\Omega_1 t) + V \cos(\Omega_2 t) - c_3 \dot{w}_0 \]

(5)

where the dot represents partial differentiation with respect to time \(t\); \(c_3\) is the damping coefficient.

\[ N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dz (\alpha = x, y; \beta = x, y), I_i = \int_{-h/2}^{h/2} \rho(z) dz, (i = 0, 1, 2) \]

(6)

Considering the variation of electric field in thickness, piezoelectric force are given as follows
\[ N_{xx}^p = \int_{\Omega_1} e_{31} Edv, \quad N_{yy}^p = \int_{\Omega_2} e_{32} Edv, \quad N_{xy}^p = N_{yx}^p = 0 \] (7)

The stress-strain relations are given as follows

\[
\begin{align*}
N_{xx}^p &= \left[ A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{12} \frac{\partial^2 v_0}{\partial y^2} + (A_{12} + A_{20}) \frac{\partial^2 u_0}{\partial y^2} + A_{11} \frac{\partial v_0}{\partial x} \frac{\partial^2 v_0}{\partial y^2} + A_{20} \frac{\partial^2 w_0}{\partial x^2} + (A_{21} + A_{20}) \frac{\partial^2 w_0}{\partial y^2} \right] + (A_{21} + A_{20}) \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} = I_0 \frac{\partial^2 u_0}{\partial x^2} - I_1 \frac{\partial^2 w_0}{\partial x^2} \quad \text{(9a)} \\
A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{20} \frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{20}) \frac{\partial^2 u_0}{\partial y^2} + A_{22} \frac{\partial v_0}{\partial x} \frac{\partial^2 v_0}{\partial y^2} + A_{20} \frac{\partial^2 w_0}{\partial x^2} + (A_{21} + A_{20}) \frac{\partial^2 w_0}{\partial y^2} + I_0 \frac{\partial^2 v_0}{\partial y^2} - I_1 \frac{\partial^2 w_0}{\partial y^2} \quad \text{(9b)} \\
-\frac{D_{11}}{\partial x^2} + (-D_{22} - 4D_{30} - D_{12}) \frac{\partial^4 w_0}{\partial x^4} + (\frac{3A_{11}}{2} \frac{\partial u_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + \frac{3A_{22}}{2} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + (\frac{1}{2} A_{12} + A_{20}) \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2} A_{21} + A_{20} \frac{\partial^2 w_0}{\partial y^2} + A_{11} \frac{\partial w_0}{\partial x} + A_{22} \frac{\partial w_0}{\partial y} + A_{22} \frac{\partial^2 w_0}{\partial x^2} + A_{22} \frac{\partial^2 w_0}{\partial y^2} + A_{22} \frac{\partial^4 w_0}{\partial x^4} - N_{xx}^p \cos(\Omega t) \frac{\partial^2 w_0}{\partial x^2} - N_{yy}^p \cos(\Omega t) \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial N_{xx}^p}{\partial x} \frac{\partial w_0}{\partial x} - \frac{\partial N_{yy}^p}{\partial y} \frac{\partial w_0}{\partial y} \quad \text{(9c)}
\end{align*}
\]

where \((A_{ij}, D_{ij})\) respectively are the stiffness elements of the cantilevered plate, which are denoted as

\[ (A_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}^k (1, z, z^2) dz, (i, j = 1, 2, 6) \] (10)

We consider nonlinear dynamics of the cantilevered piezoelectric plate in the first mode of \(u_0, v_0\) and the first four modes of \(w_0\). We write \(u_0, v_0, w_0, F\) in the following forms [7]:

\[ u_0(x, y, t) = u_1(t) \sin(\frac{\pi x}{2a}) \cos(\frac{\pi y}{2b}), \quad v_0(x, y, t) = v_1(t) \sin(\frac{\pi x}{2a}) \cos(\frac{\pi y}{2b}), \]

\[ w_0(x, y, t) = w_1(t) \alpha_1 \beta_1 + w_2(t) \alpha_2 \beta_2 + w_3(t) \alpha_3 \beta_3 + w_4(t) \alpha_4 \beta_4, \]

\[ \alpha_i = \cos(\frac{k_i x}{a} - \cos(\frac{k_i x}{a} - \frac{\cos(k_i x)}{a}), \quad \alpha_i = \cos(\frac{k_i x}{a} - \cos(\frac{k_i x}{a} - \frac{\cos(k_i x)}{a}), \]

\[ \cos k_i \cos k_j + 1 = 0, k_i^4 = -\frac{2A}{EJ} (i = 1, 2), \quad \phi_i = \frac{\sin k_i - \sin k_j}{\cos k_i + \cos k_j} (i = 1, 2, \beta_1 = 1, \beta_2 = \sqrt{3}(1 - \frac{2}{y}) \]

For the cantilevered thin plate, the transverse displacement is more obvious than other directions, so all inertia terms in equation (9a) (9b) and equation(9c) about \(u_0, v_0\) are neglected. By means of the Galerkin method, substituting equation (11) into equation (9), the expressions of \(w_1, w_2, w_3, w_4\) are obtained as follows
\[ \ddot{w}_i + \mu_i \dot{w}_i + a_i^2 w_i + a_{1i} w_{i-1} + a_{2i} w_{i+1} + a_{3i} w_{i-2} + a_{4i} w_{i+2} + a_{5i} w_{i-3} + a_{6i} w_{i+3} + a_{7i} w_{i-4} + a_{8i} w_{i+4} = \sum_{j=1}^{8} \frac{a_{ij} \dot{w}_j + a_{2ij} \dot{w}_{j-1} + a_{3ij} \dot{w}_{j+1} + a_{4ij} \dot{w}_{j-2} + a_{5ij} \dot{w}_{j+2} + a_{6ij} \dot{w}_{j-3} + a_{7ij} \dot{w}_{j+3} + a_{8ij} \dot{w}_{j+4}}{\frac{1}{2} + \frac{3}{2}} \]  

\[ \sum_{j=1}^{8} \frac{a_{ij} \dot{w}_j + a_{2ij} \dot{w}_{j-1} + a_{3ij} \dot{w}_{j+1} + a_{4ij} \dot{w}_{j-2} + a_{5ij} \dot{w}_{j+2} + a_{6ij} \dot{w}_{j-3} + a_{7ij} \dot{w}_{j+3} + a_{8ij} \dot{w}_{j+4}}{\frac{1}{2} + \frac{3}{2}} \]  

(12)

3. Numerical simulation

We employ the fourth-order Runge-Kutta algorithm to analyze numerically the nonlinear dynamic responses of the system. We chose the physical parameters respectively as Table 1, where \( a \) denotes the length of plate, \( b \) denotes the width of plate, \( h \) denotes the thickness of the plate, the subscript \( s \) denotes the substrate layer, the subscript \( p \) denotes the piezoelectric layer.

| Physical quantity | value | Physical quantity | value | Physical quantity | value |
|-------------------|-------|-------------------|-------|-------------------|-------|
| \( a \)           | 1.5m  | \( G_{12s} \)     | 4.0Gpa| \( G_{12p} \)     | 1.0Gpa|
| \( b \)           | 0.8m  | \( v_{12s} \)     | 0.33  | \( v_{12p} \)     | 0.35  |
| \( h_s \)         | 0.005m| \( \rho_s \)      | 1570kg/m\(^3\)| \( \rho_p \)      | 2500  |
| \( h_p \)         | 0.001m| \( c_3 \)        | 2.83\times10\(^{-5}\)| \( e_{31} \) | 17\( \epsilon_0 \)/m\(^2\)|
| \( E_{1s} \)      | 125Gpa| \( E_{1p} \)     | 2.5Gpa| \( e_{32} \)      | 6\( \epsilon_0 \)/m\(^2\)|
| \( E_{2s} \)      | 7.2Gpa| \( E_{2p} \)     | 2.5Gpa|                   |       |

When the voltage excitation amplitude \( V \) is located in the interval 0V–2000V and the base excitation amplitude \( F = 1000N \), substituting the physical parameters to the equations (12), the bifurcation diagrams for \( w_1, w_2, w_3, w_4 \) via the voltage excitation \( V \) are shown in Figure 5. Figure 5 shows that the cantilevered piezoelectric plate change from the chaotic motion to the periodic motion, and then to the chaotic motion with the increase of the voltage excitation amplitude \( V \).

![Figure 2 The bifurcation diagram of the first four modes of \( w_0 \) for the voltage excitation \( V \)](image)
We change the voltage excitation amplitude $V$ to obtain the waveforms (a) (b), the two-dimensional phase portraits (c) (d), Poincare maps (e) (f) and three-dimensional phase portraits (g). Figure 3 illustrates that the chaotic motion of the cantilevered piezoelectric plate exists when the voltage excitation changes to $V=10V$.

![Waveforms and phase portraits]

Figure 3 The chaotic motion of the system when $V=10V$

When the voltage excitation changes to $V=800V$, Figure 4 illustrates that the periodic motion of the composite laminated cantilevered piezoelectric plate exists.

![Waveforms and phase portraits]

Figure 4 The periodic motion of the system when $V=800V$

4. Conclusions
We investigate the nonlinear dynamics responses of a cantilevered piezoelectric plate subjected to the base and voltage excitations. The governing equations of motion for the cantilevered piezoelectric plate are established. Analytical study is given by using the fourth-order Runge-Kutta algorithm.

The bifurcation diagram of the cantilevered piezoelectric plate for $w_1, w_2, w_3, w_4$ via the voltage excitation $V$ is obtained. The bifurcation diagram shows that voltage excitation can significantly reduce the vibration amplitude of the cantilevered piezoelectric plate. By changing the voltage excitation amplitude $V$, we can control the nonlinear dynamic responses of the cantilevered piezoelectric plate from the chaotic motion to the periodic motion.

Acknowledgments
This work was partially supported by National Natural Science Foundation of China (No.11072008) and Research Project from Beijing Vocational College of Agriculture (No. XY-YF-14-42).
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