Evolutionary Grad-Shafranov Equation for a Toroidal Plasma

Giovanni Montani, Matteo Del Prete, Nakia Carlevaro, and Francesco Cianfrani

1 ENEA, Fusion and Nuclear Safety Department, C. R. Frascati, Via E. Fermi 45, 00044 Frascati (Roma), Italy
2 Physics Department, “Sapienza” University of Rome, P.le Aldo Moro 5, 00185 Roma, Italy
3 Consorzio RFX, Corso Stati Uniti 4, 35127 Padova, Italy
4 PHM UMR7345, CNRS, Aix-Marseille University, Jardin du Pharo, 58 Boulevard Charles Livon, 13007 Marseille, France

We describe the adiabatic evolution of a plasma equilibrium having a toroidal topology in the presence of constant electric resistivity. After outlining the main analytical properties of the solution, we reproduce a realistic scenario for the upcoming Italian experiment Divertor Tokamak Test Facility, with a good degree of accuracy. Although the theoretical lifetime is of the order of $10^4$ s, we observe a macroscopic change in plasma volume on a timescale comparable to the predicted flat-top duration.

In the final part of the work, we compare our self-consistent solution to the more common Solov’ev configuration, and to a family of nonlinear configurations.

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I. INTRODUCTION

The theory underlying plasma equilibrium in axial symmetry, in particular in toroidal configurations like those ones of Tokamak devices [1], consists of the so-called Grad-Shafranov equation (GSE) [2–4]. Such equation is nothing more than the implementation of the basic magnetostatic equation (i.e., steady magnetohydrodynamics (MHD) in the absence of bulk plasma velocities) to the axial symmetry, once making explicit use of the magnetic flux function as fundamental variable [5] (see also Ref.[6] and, for a detailed review on this topic, see Ref.[7] and refs therein).

In the practice of Tokamak experiments, such stationary configurations have actually a finite lifetime, both for the intrinsic finiteness of the discharge duration and for the emergence of instabilities, able to grow and then to destroy such steady profiles. Nonetheless, the physical meaning of the GSE solutions is ensured by the different timescales of their existence and the growth rates of the most common instabilities, i.e., the perturbations have a so fast evolution to feel the background plasma, _de facto_, like constant in time [2]. The basic feature of Tokamak devices is their toroidal topology, characterized by a nearly constant toroidal magnetic field and a smaller poloidal component (the ratio of the latter to the former is typically taken of the order of the torus aspect ratio $\sim 1/3$ [1]). The presence of the poloidal magnetic component, mainly due to the induced current in the plasma, implies a certain rotation of the field lines around the torus axis, which improves the confinement properties of charged particles in such machines.

The role of resistive diffusion in non stationary axisymmetric tokamak plasma has been discussed originally in the seminal paper Ref.[8] (see also Refs.[9]–[14]). The theoretical analysis outlines that two transient processes are involved: the skin effect, _i.e._ the same mechanism responsible for the penetration of magnetic field lines in a solid conductor, and the non-linear diffusion of pressure across magnetic field lines. These two processes are, in general, nonlinearly coupled and they can be disentangled by considering the two limiting cases for the generalized Ohm’s law: no bulk velocity, resulting in $E = \sigma J$, and no electric field, resulting in $v \wedge B = \sigma J$ (here, and in the following, $E$ and $B$ denote the electric and magnetic fields, respectively, while $\sigma, J$ and $v$ are the plasma electric conductivity, current density and fluid velocity, respectively). Indeed, the characteristic time scales of both processes are generically longer than those of instabilities and of wave propagation. Over the past few years, the development of real time equilibrium solvers (e.g. Refs.[17]–[18]) has profited also from the strategy of alternating the solution of the GSE and the reconstruction of source functions from real data, imposed on the successive equilibrium computation in the form of a constraint.

In this paper, we intend to provide a more rigorous mathematical treatment of the equilibrium of magnetized plasma in the presence of non ideal effects. We consider the case without convection and we demonstrate how the effect of resistive diffusion due to the skin effect can be treated analytically. This is due to a technical reason, already noted in Ref.[14]: the diffusion equations for the poloidal flux and for the toroidal current coincide in the limit of constant resistivity and no convection (see Eqs.[6] and [7]). This allows us to construct a consistent non-stationary equilibrium solution in which time-dependence is only within the poloidal magnetic flux function, dubbed $\psi$, whose dynamics are governed by the generalized Ohm’s law. In other words, we describe a diffusion process in which all of the relevant plasma quantities remain instantaneously frozen on a non-stationary magnetic field configuration. In this sense, we speak of a _non-stationary_ GSE. It is worth noting that our analytical setting differs from the Solov’ev-like configuration, which was originally studied in Ref.[19], and has formed
the basis of most analytical studies on tokamak plasma properties since then. In fact, we assume a linear relation for the pressure, like in the aforementioned work, and we assume the poloidal current to be linear too, differently from the Solov’ev case where it is proportional to the square root of the magnetic flux. This assumption on the form of the arbitrary functions was already considered in Ref. [20], even though here it is also motivated by the compatibility requirements of our dynamical approach. We also remark that a linear assumption on the poloidal current with respect to $\psi$ is of interest for machines different than tokamaks, where the plasma region includes the central axis of symmetry (e.g., see Ref. [21]), while Solov’ev-like profiles are not suitable in such geometries.

After defining the lifetime of the configuration, we outline the basic eigenvalue structure of the mathematical problem and solve the relevant equation. Then, we implement this model to a specific scenario for the Italian tokamak proposal named Divertor Tokamak Test (DTT) Facility [22, 23]. We show how our solution is able to reproduce the essential features of the 5 MA double-null scenario described in Ref. [22] with a good degree of accuracy. The reconstructed equilibrium is associated to a theoretical timescale of about $10^3$ s, while the predicted flat-top duration for the discharge is about $\approx 50$ s. However, we spot the emergence of an effective lifetime in our model, corresponding to the loss of confinement of the plasma configuration, which we observe on a timescale of $10^3$ s, which is comparable with the predicted discharge duration. It is important to remark how the obtained radial pressure profiles indicate that our model refers to low confinement states only and that the presence of a pedestal, typical of the H-mode [24, 25], could significantly increase the profile lifetime.

In our study, the GSE is self-consistently verified at all times along the plasma dynamics. A completely different approach, which is usually used in tokamak numerical simulations, corresponds to solve the GSE separately from the evolution problem, and to use this solution as initial condition for transport codes (e.g., see Ref. [27]). In this way, the dynamical constraint on the configuration is dropped during the evolution, until eventually a new equilibrium is recalculated according to the updated GSE. In the last part of this work, we address the question whether this is a viable strategy in the prediction of tokamak plasma evolution. Firstly, we consider a Solov’ev-like configuration, which we are able to solve analytically only after excluding one equation from the model (see Eq. [4]). Then, we use this novel solution to model the same double-null scenario previously considered, and find the two profiles to be in good accordance at all times up to deconfinement. Nevertheless, we question the generality of this result, considering that the linearity of the system is preserved in this setting. Hence, we also perform a numerical study on a family of nonlinear scenarios, which directly generalizes the compatible one. Taking the Solov’ev result as reference, we suggest that significantly larger errors could arise in nonlinear situations.

The paper is organized as follows. In Sec. II we describe the basic MHD equations which characterize the dynamics of the plasma configuration. In Sec. III the details of the considered dynamical scenario as due to resistivity are developed, outlining the analytical implications of this effect and the temporal decay of the profile. In Sec. IV we solve the resulting eigenvalue problem, and use our solution to model a DTT-like double-null plasma scenario. The lifetime of the configuration and its profile are outlined. In Sec. V we use the Solov’ev-like solution to model the same plasma scenario. We also provide estimates on the error associated to a class of nonlinear scenarios. Concluding remarks follow in Sec. VI.

### II. BASIC EQUATIONS

We study a plasma confined in a magnetic field $B$ and having negligible macroscopic motion, i.e., its fluid velocity $v$ identically vanishes. The plasma is also characterized by a finite electric conductivity $\sigma \simeq \text{const}$. The electric field, living in the plasma, is then provided, via the current density, according to the generalized Ohm’s law

$$E = \frac{1}{\sigma} J. \quad (1)$$

Expressing the electric field via the scalar and vector potentials ($\varphi$ and $A$, respectively), i.e.,

$$E = -\nabla \varphi - \partial_t A / c, \quad (2)$$

and observing that $B = \nabla \wedge A$, while in the Coulomb gauge (i.e., $\nabla \cdot A = 0$) $J = (e/4\pi) \Delta A$ ($e$ is the speed of light), we can rewrite Eq. (1) as follows

$$\partial_t A = -c\nabla \varphi + \frac{e^2}{4\pi \sigma} \Delta A. \quad (3)$$

We now consider an axial symmetry, associated to a toroidal topology, by the choice of cylindrical variables $r$, $\phi$ and $z$, having the following ranges of variation: $R_0 - a < r < R_0 + a$, $0 < \phi < 2\pi$. Here $R_0$ denotes the major radius of a standard Tokamak configuration, while $a$ is the minor radius (we also have $|z| \lesssim a$). The axial symmetry is implemented by requiring the independence of all the physical quantities on $\phi$.

We now fix the vector potential as follows

$$A = A_\rho + \frac{\psi}{2\pi r} \hat{e}_\phi, \quad (4)$$

where $\hat{e}_\phi$ is the toroidal versor and the poloidal (radial-axial) vector potential $A_\rho$ is fixed via the relations

$$\nabla \cdot A_\rho = 0, \quad \nabla \wedge A_\rho = B_\phi \equiv \frac{2 I}{c r}, \quad (5)$$
The functions $\psi = \psi(t, r, z)$ and $I = I(t, r, z)$ denote the flux function and the axial current (in the cross section $\pi r^2$), respectively, and they are the considered dynamical degrees of freedom.

Since the scalar electric potential gradient is poloidal in axial symmetry, we easily get the dynamics of $\psi$ from the toroidal component of Eq. (5), and the one of $I$ by taking the curl of the remaining poloidal components (so eliminating the gradient of $\phi$) and taking into account Eqs. (3) and (5). Thus, we arrive to the following two (identical) dynamical equations:

$$\partial_t \psi = \frac{c^2}{4\pi\sigma} \Delta^* \psi,$$

$$\partial_t I = \frac{c^2}{4\pi\sigma} \Delta^* I,$$

where we have defined $\Delta^*(...) \equiv r \partial_r (1/r \partial_r (...)) + \partial^2_z (...)$.

Now the toroidal component of the momentum conservation equation ($p$ denoting the plasma pressure), i.e.,

$$\nabla p = (\nabla \times B) \times B \times B / 4\pi,$$

reduces to the constraint $\partial_t \psi \partial_z I - \partial_z \psi \partial_z I = 0$, implying the basic restriction $I = I(\psi)$. Once we substitute this expression into Eq. (7), the compatibility with Eq. (6) leads to the condition

$$\frac{d^2 I}{d\psi^2} |\nabla \psi|^2 = 0,$$

which is either solved trivially by $\psi = \text{const}$, or by letting

$$\frac{d^2 I}{d\psi^2} = 0 \Rightarrow I = A_1 \psi + A_0,$$

where $A_{1,0}$ are two integration constants. Near the magnetic axis, this assumption would correspond to considering a first order expansion of $I(\psi)$, in a similar fashion as in Ref. [19], where analytical solutions of the GSE in the same regime are found by expanding the quantities $dp/d\psi$ and $IdI/d\psi$. In this context, however, the Solov’ev solution fails to guarantee the compatibility of the resistive system, de facto neglecting Eq. (7). In Sec. (V) we study this incompatible scenario in detail, providing a comparison with the formally correct solution, which is derived in the following Section.

The poloidal components of Eq. (8) reduce to the usual GSE,

$$\Delta^* \psi = -16\pi^3 r^2 \frac{dp}{d\psi} - \frac{16\pi^2}{c^2} I \frac{dI}{d\psi},$$

in which we also implement choice (10), i.e.,

$$\Delta^* \psi = -16\pi^3 r^2 \frac{dp}{d\psi} - \frac{16\pi^2}{c^2} (A_1^2 \psi + A_1 A_0).$$

Finally, the mass conservation equation ($\rho$ being the plasma mass density), i.e.,

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

becomes, in the present scenario, the simple relation $\rho \equiv \rho_0(r, z)$.

III. DYNAMICAL IMPLICATIONS

The discussion above clarified how $\psi(t, r, z)$ is the only dynamical variable of the system, which has to obey Eqs. (10) and (12). They can be combined into a new equation:

$$\partial_t \psi = \frac{c^2}{4\pi\sigma} \left( -16\pi^3 r^2 \frac{dp}{d\psi} - \frac{16\pi^2}{c^2} (A_1^2 \psi + A_1 A_0) \right),$$

which remains coupled to Eq. (12).

In order to look for analytical solutions, having to deal with a $dp/d\psi$ term, we preserve the linearity of the system by assuming the following form for the pressure:

$$p(\psi) = C_2 \psi^2 / 2 + C_1 \psi + C_0,$$

where $C_{2,1,0}$ are generic real constants. A careful analysis of the compatibility of the system shows that $C_2$ must vanish, which implies that also the pressure, like the axial current function $I$, must be linear in the flux function. This kind of choice for the functions $I$ and $p$ has been previously studied in Ref. [20], although in our work we show how such a choice can be motivated by dynamical considerations.

It is now easy to check that the general solution to Eq. (13) takes the form

$$\psi(t, r, z) = \psi_0(r, z) e^{-\gamma t} + \delta(r),$$

where $\psi_0(r, z)$ is a generic function yet to be determined, while the quantities $\gamma$ and $\delta(r)$ explicitly read

$$\gamma \equiv \frac{4\pi A_1^2}{\sigma}, \quad \delta(r) \equiv -\frac{\pi e^2 C_1}{A_1^2} r^2 - \frac{A_0}{A_1}.$$

Now, by substituting the solution (10) into Eq. (12), we get an equation for $\psi_0(r, z)$, i.e.,

$$\Delta^* \psi_0 = -16\pi^2 A_0^2 \psi_0 / c^2.$$

Before studying the morphology of the plasma profile, we observe that the magnetic configuration is always damped in time, by a rate $\gamma$, towards an asymptotic constant field $B_\infty = (c^2 C_1 / A_1^2) \hat{e}_z$.

The present study has the merit of defining quantitatively a lifetime for a given plasma configuration, once resistive diffusion is consistently taken into account. In particular, we showed how the lifetime is very sensitive to $A_1$, the proportionality constant between $I$ and $\psi$. This approach in not intended as an alternative choice to standard transport studies on assigned equilibria. In fact, we simply clarify the influence of the considered correction to Ohm’s law on the adiabatic evolution of a plasma profile, which could play a significant role in the physics of future steady-state tokamak machines. Clearly, the impact of this study on real machine operations relies on the possibility to show that the adiabatic decay of the magnetic configuration possesses the same qualitative properties also when non-dramatic transport processes (i.e., in absence of disruptions or phase transitions) are taken into account.
IV. MAGNETIC PROFILE

In order to investigate the profile predicted by Eq. (18), we observe that its linearity allows to consider the following Fourier expansion:

\[ \psi_0(r, z) = \int_0^\infty dk \chi_k(r) e^{ikz} + c.c., \]  

where \( \chi_k(r) \) verifies the eigenvalue problem

\[ \frac{d^2 \chi_k}{dr^2} - \frac{1}{r} \frac{d \chi_k}{dr} = E_k \chi_k, \quad E_k \equiv k^2 - \frac{16 \pi^2 A^2}{c^2}. \]  

The equation for \( \chi_k \) admits an analytical solution in terms of Bessel functions. In particular, defining \( x \equiv r|E_k|^{1/2} \) and setting \( \chi_k \equiv r \varepsilon(k, x) \), Eq. (20) can be rewritten as

\[ x^2 \frac{d^2 \varepsilon}{dx^2} + x \frac{d \varepsilon}{dx} - (1 \pm x^2) \varepsilon = 0, \]  

where the sign \( \pm \) corresponds to \( E_k < 0 \), i.e., to \( k < k^* \), where \( k^* = 4 \pi A_1/c \), while the sign \( + \) to the case \( E_k > 0 \), i.e., to \( k > k^* \).

In correspondence to the sign \( \mp \), the solutions of the equation above read

\[ \varepsilon_-(k, x) = \varepsilon_1(k) J_1(x) + \varepsilon_2(k) Y_1(x), \quad \]  
\[ \varepsilon_+(k, x) = \varepsilon_3(k) J_1(x) + \varepsilon_4(k) K_1(x), \]  

where \( J_1, Y_1, K_1 \) denote ordinary (modified) Bessel functions of index 1, while the coefficients \( \varepsilon_j(k) \) (with \( j = 1, 2, 3, 4 \)) have to be assigned via the initial condition \( \psi(0, r, z) = \psi_0(r, z) + \delta(r) \). In this scheme, taking into account that the \( r \) variable is bounded and that \( J_1(\infty) \) is divergent, the flux function \( \psi(t, r, z) \) admits the following representation:

\[ \psi = -A_0/A_1 + \Lambda r^2 + \]
\[ + e^{-\gamma t} \int_{k_0}^{k} dk \left[ r \varepsilon_1(k) J_1(r \sqrt{k^2 - k^{*2}}) + r \varepsilon_2(k) Y_1(r \sqrt{k^2 - k^{*2}}) e^{ikz} + c.c. \right] + \]
\[ + e^{-\gamma t} \int_{k_*}^{\infty} dk \left[ r \varepsilon_3(k) K_1(r \sqrt{k^2 - k^{*2}}) e^{ikz} + c.c. \right]. \]  

Here, \( \Lambda = -\pi c^2 C_1/A^2_1 \) and, to exclude wave-lengths greater than the machine diameter, we have introduced a minimum wavenumber \( k_0 = \pi/a \), i.e., \( k \geq k_0 \). We also remark that, by suitably choosing the constant \( C_0 \) in Eq. (15) for the plasma pressure, the basic requirement \( p \geq 0 \) can be easily implemented in our confined plasma region.

A. Realistic scenario

In order to investigate the morphology of the plasma configuration, we analyze the level surfaces of \( \psi(r, z, t) \) at given times, together with the surface \( p = 0 \) (representing the plasma boundary layer). The general solution for \( \psi \) as in Eq. (24) can be adapted to a realistic scenario by imposing specific initial conditions. In this respect, for the sake of simplicity, we assign to the functions \( \varepsilon_j(k) \) a set of sufficiently narrow Gaussians, centered around arbitrarily given wave vectors \( k_{j,i} \) and weighted by amplitudes \( \varepsilon_j(k) \). Then, a given set of points \((r_1, z_1)\) lying along the boundary curve of the addressed realistic scenario generates an associated set of algebraic equations of the form \( \psi(r_1, z_1, 0) = \psi_B \), where \( \psi_B \) is the value of the magnetic flux at the plasma boundary. Since we require \( p = 0 \) on the same surface, recalling Eq. (15) and that \( C_2 = 0 \), we set \( C_0 = -C_1 \psi_B \). The rest of the constants have to be determined according to the relevant plasma parameters.

As an illustrative example, let us assume the parameters characterizing a Tokamak equilibrium specified for the DTT facility, as in Ref. [22]. In particular, the main machine parameters are major radius \( R_0 = 2.11 \) m, minor radius \( a = 0.64 \) m, averaged electron density \( n_e = 1.8 \times 10^{20} \) m\(^{-3} \) and electron temperature \( T_e = 6.1 \) keV. The resulting plasma frequency is \( \omega_p = 7.57 \times 10^{11} \) s\(^{-1} \), and, considering the Spitzer electric conductivity for a hydrogen plasma (with the Coulomb logarithm of \( O(10) \)), we obtain \( \sigma = 5.34 \times 10^8 \) \( \Omega^{-1} \) m\(^{-1} \). We implement an initial condition matching the double-null scenario, with main plasma parameters as reported in Table I. In the same Table, we also report the fitted values associated to our analytic solution, which is able to correctly reproduce most of the parameters. In this scheme, we obtain a configuration characteristic time of \( \gamma^{-1} = 1.1 \times 10^4 \) s. As a stability check, the safety factor \( q \) meets the Kruskal-Shafranov condition for stability \( q > 1 \) over the whole plasma region, with an average value of 2.6.

In Fig. 1 we plot the level surfaces of the flux function \( \psi(t, r, z) \) for different instants of time, where the initial condition at \( t = 0 \) is shown in the first panel. At later stages, the allowed domain for the plasma configuration decreases, and the central pressure is correspondingly suppressed (cf. Eq. (15)). Since the area inside the separatrix is decreasing in time, the axial symmetry implies that the confined plasma volume is also diminishing, but keeping a constant plasma density \( \rho \equiv \rho_0 \). As a consequence, the plasma evolution has to be associated with an adiabatic loss of particles throughout the boundary layer of the toroidal plasma profile. The behaviour of this outgoing flux of matter must be described in a different physical setting, having to deal with the behaviour of non confined plasma in the scrape-off layer.

Concerning the lifetime of the configuration, it is important to stress that we observe the opening of all magnetic lines, determining the loss of confinement, at \( t = 99 \) s. This timescale is two orders of magnitude shorter than \( \gamma^{-1} \), and is comparable with the predicted flat-top duration of the discharge of about \( \approx 50 \) s. This behaviour can be understood if we consider that the plasma region can also be defined as the points satis-
fying $\psi \geq \psi_B$. Indicating the initial peak value of the magnetic flux as $\psi_A$, in correspondence to the magnetic axis, it is clear that after an overall decrease in $\psi$ of the order $\Delta \psi \equiv |\psi_A - \psi_B|$ the whole profile will lie below the $\psi_B$ threshold, i.e., all magnetic lines will be open. Then, it is natural to define an effective lifetime according to the condition $\bar{\psi}_0 (1 - e^{-\gamma t^*}) = \Delta \psi$ (where $\bar{\psi}_0$ is the order of magnitude of the function $\psi_0(r,z)$), which provides the expression

$$t^* = -\gamma^{-1} \ln \left(1 - \Delta \psi / \bar{\psi}_0 \right).$$

(25)

In the case under study, $\Delta \psi \approx 5.5$ Vs and $\bar{\psi}_0 \approx 600$ Vs, so that $t^* = 99$ s, as observed.

As a final remark, we remind that the solution outside the boundary layer takes a different character, being described by a vacuum problem. In such an outer region, the current density must be set to zero, according to the pressure profile. Therefore, outside the region $p = 0$, we must require that $A_1 = A_0 = C_1 = C_0 = 0$ and also that the toroidal current $J_\phi$ vanishes. This last condition leads to the equation

$$\Delta^* \psi(t,r,z) = 0,$$

(26)

which is the only surviving equation for the vacuum configuration. Clearly, the time dependence of the magnetic flux function in vacuum is ensured by the matching conditions on the boundary layer.

V. COMPARISON WITH ALTERNATIVE APPROACHES

In order to compare our self-consistent approach to other standard methods, we now study an alternative analytical solution in correspondence to the well known Solov'ev-like configuration [19]. In this sense, we go back to the original GSE, Eq. (11), and assume its right-hand side to be independent of $\psi$:

$$\Delta^* \psi = -16\pi^3 C_{1s} r^2 - \frac{16\pi^2}{c^2} A_{1s},$$

(27)

where $C_{1s}$ and $A_{1s}$ are constants (the subscript $s$ indicates quantities relative to the Solov'ev-like scenario). In our formulation, the corresponding choice for $p(\psi)$ and $I(\psi)$ is

$$p_s(\psi) = C_{1s} \psi + C_{0s}, \quad I_s(\psi) = \sqrt{2A_{1s} \psi + A_{0s}},$$

(28)

so that the pressure is of the same kind as previously considered, while the axial current has a different functional form. It is important to remark that substituting the latter expression into Eq. (9), we get

$$\frac{d^2I_s}{d\psi^2} |\nabla \psi|^2 = -\frac{A_{1s}^2}{(2A_{1s} \psi + A_{0s})^{3/2}} |\nabla \psi|^2 = 0$$

(29)
which admits only the trivial solutions $A_{1s} = 0$ or $\psi = \text{const}$. However, if this equation is excluded from the model, Eqs. (6) and (27) lead to the expression

$$\psi_s(r, z, t) = -a(r) t + b(r) + \psi_0(r, z),$$

with

$$a(r) = \frac{4\pi^2 e^2}{\sigma} \left( C_{1s} r^2 + \frac{A_{1s}}{\pi c^2} \right),$$

$$b(r) = 2\pi^3 r^2 \left[ -C_{1s} r^2 + \frac{2A_{1s}}{\pi c^2} (1 - 2 \log r) \right].$$

Here, $\psi_0(r, z)$ is formally equivalent to the solution already considered in Sec. IV, since it must satisfy Eqs. (19) and (20), with the only difference that now $E_k = k^-$. The most striking feature of $\psi_s(r, z, t)$ is the linear time dependence, which differs radically from the exponential decay of the consistent solution. To test this discrepancy, we fit the new expression to the same double-null scenario of Sec. IV A. The agreement is surprisingly very good, as can be noted from the fitted values in Table I. Moreover, the time evolutions of the two profiles follow the same dynamics up to the loss of confinement, which, in this case, takes place after 98 s. Considering that the process develops on a timescale much shorter than the characteristic time $\gamma^{-1}$, the exponential decay in Eq. (24) can be linearly expanded around $t=0$, recovering the linearity and explaining this behaviour.

Although this result suggests that Eq. (9) can be safely disregarded in the DTT plasma scenario, this is far from being a general proof. In fact, the Solov’ev case has the good property of preserving the linearity of the system, which is broken in the context of numerical equilibrium solvers, such as EFIT [28], where nonlinear forms of $\frac{dp}{d\psi}$ and $\frac{IdI}{d\psi}$ are usually assumed. In such scenarios, Eqs. (11) cannot be solved through simple analytic means, so an exact comparison lies outside the scope of the present work. We propose an effective estimate of the incompatibility of the nonlinear case, by considering the following generalization of Eq. (10):

$$I_n(\psi) = A_{1n} \psi^n + A_{0n},$$

where coefficients $A_{1n}$ and $A_{0n}$ are determined according to the relevant plasma parameters of Table I. Assuming $|\nabla \psi|^2$ to be of the same order of magnitude in all configurations (i.e., $\sim \Delta \psi / a$), the error committed in Eq. (6) is quantified by the second derivative of $I_n$ with respect to $\psi$:

$$\frac{d^2 I_n}{d\psi^2} = n(n-1) A_{1n} \psi^{n-2}. $$

The same quantity, calculated for the Solov’ev configuration, is taken as reference, so we study the function

$$\varepsilon(\psi, n) \equiv \log_{10} \left| \frac{d^2 I_n}{d\psi^2} \right| = A_{1n} \frac{n(n-1) \psi^{n-2}}{(2A_{1n} \psi + A_{0n})^3/2},$$

defined as a logarithm for convenience.

In Fig. 2 it clearly emerges how the magnitude of $\varepsilon$ grows quickly for $n$ different than 0 and 1 (the only two analytically correct values). In particular, the whole region below $n = 0$ takes up values larger than 3, i.e., the left hand side of Eq. (9) is at least three times larger in these cases than in the Solov’ev case. A fiducial interval can be defined around $n = 1$, in which the discrepancy is less than one order of magnitude. According to this estimate, more detailed studies should be performed on the viability of the coupling of evolutive codes with nonlinear GSE configurations.

**VI. CONCLUDING REMARKS**

We analyzed an adiabatically varying equilibrium, in which the magnetic field profile is damped by resistive effects. In such a dynamical scheme, the GSE is coupled with an evolutionary equation for the magnetic flux function dynamics, i.e., the induction equation.

The main result has been the determination of a lifetime for the plasma confinement, here discussed in the particular case of an initial condition corresponding to a 5 MA double-null scenario predicted for the DTT tokamak proposal, as in Ref [22]. A secondary, effective lifetime also arises from the observation of the loss of magnetic confinement on a time scale much shorter than expected, and comparable with the flat-top duration of the discharge. This result suggests that, for incoming machines approaching a steady-state regime, the effect of resistivity on the magnetic profile could significantly influence the equilibrium properties. It is worth noting that our capability to analytically reproduce the predicted
DTT scenario is affected by a certain degree of approximation, since our solution lacks the sufficient number of parameters to constrain all the relevant plasma quantities. In this respect, the amount of parameters that can be fixed is naturally related to the linear prescription $I \propto \psi$, which is a remarkable conceptual implication of including resistivity into the magnetic flux function dynamics. We also studied two cases where this prescription is not respected, \textit{de facto} disregarding Eq.[9]. In the Solov’ev configuration, which keeps the system linear, no dramatic changes are observed on the profile, while in nonlinear cases we obtain numerical evidence of a larger discrepancy. We conclude that before saying a definitive word on the relevance of resistive diffusion in the equilibrium properties, a more systematic study on commonly used nonlinear plasma codes is required.

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