Quantizing the discrete Painlevé VI equation: The Lax formalism

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Abstract

A discretization of Painlevé VI equation was obtained by Jimbo and Sakai in 1996. There are two ways to quantize it: 1) use the affine Weyl group symmetry (of $D_5^{(1)}$), 2) Lax formalism i.e. monodromy preserving point of view. It turns out that the second approach is also successful and gives the same quantization as in the first approach.

1 Introduction

The equation Painlevé VI is a well known nonlinear ordinary system with rich symmetry and structure. It can be treated as a non-autonomous Hamiltonian dynamical system and possesses an extended affine Weyl group symmetry of type $D_5^{(1)}$ [17].

The discrete Painlevé VI equation $(qPVI)$ found by Jimbo and Sakai is the following ordinary difference system: we take $t$ as the independent variable (time) of the system and $x(t), y(t)$ the dependent variables.

$$\begin{align*}
qPVI & \left\{ \begin{array}{l}
y(t)y(pt) = p^2t^{-2}x(t) + a_1^{-2}p^{-1}t^2x(t) + a_2^2p^{-1}t^2x(t) + a_0^2pt^{-1}, \\
x(t)x(p^{-1}t) = t^{-2} \frac{y(t) + a_4^{-2}t^2}{y(t) + a_5^{-2}t^1} \cdot \frac{y(t) + a_4^2t}{y(t) + a_5t^{-1}}.
\end{array} \right.
\end{align*}$$

We have five multiplicative parameters, $p = e^{δ}$ : step of time, and $a_i = e^{α_i}$ ($i = 0, 1, 4, 5$). The label of the parameters are consistently chosen according to the $W(D_5^{(1)})$-symmetry:

$$s_i(a_j) = a_i^{-C_{ij}}a_j = e^{s_i(α_j)},$$

$[C_{ij}] = \text{the Cartan matrix of type } D_5^{(1)}:$

\[
\begin{pmatrix}
0 & \downarrow & \downarrow & \downarrow & 5 \\
1 & \downarrow & 2 & 3 & 4
\end{pmatrix}
\]

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This is the discretized version of the symmetry in the original Painlevé VI system (PVI) investigated by Okamoto, where we adapted the convention used in Tsuda-Masuda [18]. The action of the subgroup \( \langle s_2 s_3 s_2 = s_3 s_2 s_3, s_i (i = 0, 1, 4, 5) \rangle \simeq W(D_4^{(1)}) \) commutes with the time evolution of \( q_{PVI} \), and in fact the time evolution itself is a translation by the lattice part element \( e_3 \) which is perpendicular to the root lattice \( D_4^{(1)} \) embedded in the \( D_5^{(1)} \) root lattice (see Appendix).

In our previous paper [5] we have succeeded in quantizing the affine Weyl group action and thereby construct the quantization of the \( q_{PVI} \) system. Here quantization means the noncommutativity of the dynamical variables \( x(t), y(t) \) in \( q_{PVI} \) and the resulting system \( \hat{q}_{PVI} \) (35) looks quite the same as to \( q_{PVI} \). Explicit formulae are gathered in the Appendix.

On the other hand, one can regard the isomonodromic deformation problem as the origin of the Painlevé VI equation. The aim of the present paper is to elucidate this point for the quantized discrete equation, that is, whether one can obtain \( \hat{q}_{PVI} \) as the quantization of the discretized isomonodromic deformation problem. Actually the answer is quite successful: we obtained the quantization of the Lax form or the Schlesinger equation for \( \hat{q}_{PVI} \) (Theorem 3).

For this aim, we employed the non-autonomous generalization of the quantized lattice system introduced by Faddeev-Volkov. The construction obeys deeply to the quantum group \( U_q(A_1^{(1)}) \) and its representation; we take the image of the universal R matrix as the Lax matrix or the discrete connection matrix. The non-autonomous feature comes from the term \( c \otimes d \) (where \( c \) denotes the canonical central element and \( d \) the scaling element, respectively) in the universal R matrix of type \( A_1^{(1)} \) and naturally enters in the pole structure of the Lax matrix, which comes from the Heisenberg part of the universal R matrix.

## 2 Review of the Lax formalism for discrete Painlevé VI equation

In this section we review how the isomonodromy deformation problem provides a natural origin of the Painlevé VI equation, and how one can discretize the problem to obtain qPVI.

Consider the \( 2 \times 2 \) regular- singular connection on the complex projective line \( \mathbb{P}^1 \),

\[
\nabla = L(z)dz = \sum_{j=1}^{n} \frac{L^{(j)}}{z-t_j}dz.
\]

We have \( n \) poles \( t_1, \ldots, t_n \) at finite points and one at the infinity, put \( L^{(\infty)} := \text{Res}_{\infty} \frac{L(z)}{dz} \). Let \( Y(z) \) be the fundamental solution of the linear problem \( \frac{dY}{dz} = L(z)Y(z) \). Then we have the monodromy matrix \( M_j \) along the contour \( C_j \in \pi_1(\mathbb{P}^1 - \{t_j\}, *) \) around
$t_j$, where $\ast$ stands for the fixed base point (which is different from the singularity):

$$C_{j\ast}(Y)(z) = Y(z) M_j.$$ 

The matrix $M_j$ is conjugate to $e^{2\pi i L(j)}$ and satisfy the relation $M_1 \cdots M_n M_\infty = 1$.

**Fact.** The monodromy matrices \{M_j\} are constant (isomonodromy) with respect to $t_j$’s if the following relations hold:

$$\frac{\partial Y}{\partial t_j} Y^{-1} = -\frac{L^{(j)}}{z - t_j} (=: B_j) \quad (j = 1, \ldots, n). \quad (3)$$

If this is the case, the compatibility of (3), called the **Schlesinger equation**, should be satisfied:

$$\left[ \frac{\partial}{\partial z} - L(z, t), \frac{\partial}{\partial t_j} - B_j(z, t) \right] = 0 \quad (i, j = 1, \ldots, n), \quad (4)$$

where $t := (t_1, \ldots, t_n)$ and $t$ dependence of $L$ and $B_j$ are explicitly written. See Jimbo-Miwa-Ueno [9] for details.

This is the Lax form of the isomonodromy problem. The case $n + 1 = 4$ reproduces the Painlevé VI equation: one can assume $(t_1, t_2, t_3, \infty) = (0, 1, t, \infty)$ and take the dependent variable $y(t)$ to be (roughly speaking) the off-diagonal element of $L(3)$.

According to Jimbo and Sakai [7], the difference equation case goes quite similarly. Let us consider the difference equation

$$\frac{Y(qz) - Y(z)}{qz - z} = L(z) Y(z) \quad (L(z) = \frac{L^{(1)}}{z} + \frac{L^{(2)}}{z - 1} + \frac{L^{(3)}}{z - t}, \text{ generic})$$

which can be rewritten as $Y(qz) = \{1 + (q - 1)zL(z)\} Y(z)$. There exists some function $\gamma$ such that $(z - 1)(z - t) = \gamma(qz) \gamma(z)^{-1}$. Put $\mathcal{Y} = \gamma Y$, then we have $\mathcal{Y}(qz) = \mathcal{L}(z) \mathcal{Y}(z)$, where $\mathcal{L}(z) := (z - 1)(z - t)\{1 + (q - 1)zL(z)\}$ is polynomial in $z$. Now singularities are 0 and $\infty$; 1 and $t$ can be detected as the zero of det $\mathcal{L}$.

There exists an solution at $z = 0$ of the form $\mathcal{Y}_0(z) = z^{\mathcal{L}(0)} \times \text{(power series in } z\text{), and similarly, } \mathcal{Y}_\infty(z) \text{ for } z = \infty$. The ‘connection’ matrix

$$\mathcal{M}(z) := \mathcal{Y}_0(z)^{-1} \mathcal{Y}_\infty(z), \quad \mathcal{M}(z) = \mathcal{M}(qz), \quad (5)$$

plays the role of the monodoromy matrix. Deformation preserving condition $\mathcal{M}(z) = \mathcal{M}(z, t)$ is then satisfied if we have some $\mathcal{B}(z, t)$ such that $\mathcal{Y}(z, qt) = \mathcal{B}(z, t) \mathcal{Y}(z, t)$.

The compatibility now reads as **discrete Schlesinger equation**,

$$\mathcal{L}(z, qt) \mathcal{B}(z, t) = \mathcal{B}(qz, t) \mathcal{L}(z, t) \quad (6)$$

from which Jimbo and Sakai derived the $qP_{VI}$ equation (1).

Our goal will be the quantization of (6) as well as to confirm that it reproduces the quantization $qP_{VI}$ (35) of $P_{VI}$ (1).
3 The quantized local Lax matrix

For our aim, we use non-autonomous modification of Faddeev-Volkov quantization of discrete sine-Gordon equation and its periodic reduction. In this section we will give the local Lax matrix, which can be said as nonautonomously modified Izergin-Korepin Lax matrix [6].

Let \( q \) be a complex number with \( 0 < |q| < 1 \). For \( \pm = + \) or \( - \) respectively, let \( U_q^{\pm} = U_q^{\pm}(A_1^{(1)}) \) be the upper/lower subalgebra of the quantum group \( U_q = U_q(A_1^{(1)}) \) generated by the upper/lower Chevalley generators \( e_i^{\pm} \) together with the Cartan part \( h_i(i = 0, 1), d \), where \( d \) is the scaling element. We write the canonical central element as \( c := h_0 + h_1 \).

Let \( c^{\pm} \in C \), and let \( \rho^{\pm} \) be the representation of \( U_q^{\pm} \) on the space \( V^{\pm} := C[e^{\pm\alpha_0}, e^{\pm\alpha_1}] \) defined by

\[
e_i^{\pm} \mapsto -(q - q^{-1})e^{\pm\alpha_i} =: E_i^{\pm}, \quad h_i \mapsto h_i \ (i = 0, 1), \quad c \mapsto c^{\pm} \in C
\]

(7)

respectively. By the definition \( h_i \) acts as the derivation satisfying \( [h_i, e^{\pm\alpha_j}] = \pm \alpha_j (h_i) e^{\pm\alpha_j} \).

Let \( R \in U_q^+ \otimes U_q^- \) be the universal \( R \) matrix of \( U_q \) and \( \square \) be the two dimensional evaluation representation of \( U \). Write \( k := q^h, \Delta^{\pm} = E_0^{\pm} E_1^{\pm} \). We have:

\[
k E_1^{\pm} k^{-1} = q^{\mp2} E_1, \quad k E_0^{\pm} k^{-1} = q^{\mp2} E_0^{\mp}, \quad [E_1^{\pm}, E_1^{-}] = (k - k^{-1})(q - q^{-1})
\]

and also

\[
q^d \Delta^{\pm} q^{-d} = q^{d+1} \Delta^{\pm}.
\]

(8)

Other than (8), \( \Delta^{\pm} \) commutes with the generators \( e_i^{\pm}, h_i \ (i = 0, 1) \) of \( U_q^{\pm} \), i.e. \( \Delta^{\pm} \in \mathcal{Z}(U_q^{\pm}) \), the center of the derived algebra of \( U_q^{\pm} \). Put

\[
L_z^{+}(\Delta^{+}) := (\rho^{+} \otimes \square)(\mathcal{R}), \quad L_z^{-}(\Delta^{-}) := (\square \otimes \rho^{-})(\mathcal{R}).
\]

(9)

These are the local Lax matrices for our aim.

**Proposition 1** We have

\[
L_z^{+}(\Delta^{+}) = \frac{(q^4 z^{-1} \Delta^{+}, q^4)_{\infty}}{(q^2 z^{-1} \Delta^{+}, q^4)_{\infty}} \left[ 1 \ E_0^{\pm} \ E_1^{\pm} \right] \begin{bmatrix} k^{-\frac{1}{z}} & 0 \\ 0 & k^{\frac{1}{z}} \end{bmatrix} q^{-c_d},
\]

(10)

\[
L_z^{-}(\Delta^{-}) = \frac{(q^4 z \Delta^{-}, q^4)_{\infty}}{(q^2 z \Delta^{-}, q^4)_{\infty}} \left[ 1 \ E_0^{-} \ E_1^{-} \right] \begin{bmatrix} k^{-\frac{1}{z}} & 0 \\ 0 & k^{\frac{1}{z}} \end{bmatrix} q^{-c_d},
\]

(11)

where we used the standard notation for the infinite product : \( (x,Q)_{\infty} := \prod_{n=0}^{\infty}(1 - xQ^n) \).

This is derived from the formula for the universal \( R \) matrix (see e.g. [2]) and the above definition of the corresponding representations. The case \( c^{\pm} = 0 \) is essentially the one used in [6] and later reproduced by [3]. The infinite product factors come from the Heisenberg part (contribution from the null root vectors) in the product formula of \( \mathcal{R} \), and therefore we have infinite poles here. The pole location will move according to (8) during the dynamics defined in the next section; this is the non-autonomous nature of our Lax matrix. In the \( q \to 1 \) limit, we have

\[
\lim_{q \to 1} \left( \frac{(q^4 z^{-1} \Delta^{+}, q^4)_{\infty}}{(q^2 z^{-1} \Delta^{+}, q^4)_{\infty}} \right)^{-2} = 1 - \frac{\Delta^{+}}{z}
\]

(12)
which is equal to the determinant of the matrix
\[
\begin{bmatrix}
1 & \frac{1}{z}E_0^+ \\
E_1^+ & 1
\end{bmatrix}.
\] (13)

Similar relation holds for \(L^-\), showing the \(SL(2)\) nature of these local Lax matrices.

Note also
\[q^d L^\pm_z(\Delta^\pm)q^{-d} = L^\pm_{zq^\pm 1}(\Delta^\pm),\] (14)
which can be easily seen from (10) and (11).

Let \(R(\Delta^+, \Delta^-) := (\rho^+ \otimes \rho^-)(R)\). According to the multiplicative formula of the universal R matrix, it is explicitly written in terms of the quantum dilogarithm [12],
\[R(\Delta^+, \Delta^-) = (qE_1^+ \otimes E_1^- , q^2)^{-1}(q^2 \Delta^+ \otimes \Delta^- , q^4)^{-1}(qE_0^+ \otimes E_0^- , q^2)^{-1}q^{-T},\] (15)
where \(T := \frac{1}{2}h_1 \otimes h_1 + c^+ \otimes d + d \otimes c^-\).

The Yang-Baxter equation for the universal R matrix
\[R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in U_q^+ \otimes U_q \otimes U_q^-\]
immediately implies the following (although it can be checked directly):

**Proposition 2**
\[L^+ z(\Delta^+) R(\Delta^+, \Delta^-) L^- z(\Delta^-) = L^- z(\Delta^-) R(\Delta^+, \Delta^-) L^+ z(\Delta^+).\] (16)

**Figure 1.** The Yang-Baxter equation (16).

We can rewrite the above Yang-Baxter relation (Figure 1) among \(L^\pm\) and \(R(\Delta^+, \Delta^-)\) as follows (exchange dynamics of \(L^+\) and \(L^-\), Figure 2):

**Figure 2.** The exchange dynamics (17).

\[R^{-1} [(L^-)^{-1} L^+ u]_{ij} R = [L^+_z (L^-)^{-1}]_{ij} \in (\rho^+ \otimes \rho^-)(U_q^+ \otimes U_q^-)\] (17)
where \([\cdot]_{ij}\) stands for the matrix element. If we work with \(U_q(A_n^{(1)})\), one can see that this gives the quantized periodic Toda lattice equation (cf. [8]).
4 The discrete time lattice dynamics and its reduction

Let us define the one dimensional lattice system. Let us write the even/odd lattice points of $\mathbb{Z}$ as $n^+, n^-$, rather than $2n, 2n+1$. Then we attach the representation

$\rho^\pm = \rho^{n^\pm}: U_q^\pm \to \text{End}(V^{n^\pm})$, $V^{n^\pm} \simeq \mathbb{C}[e^{\pm a_0}, e^{\pm a_1}]$

specified by the parameters $c^{n^\pm} := \rho^{n^\pm}(c), \Delta^{n^\pm} := \rho^{n^\pm}(\Delta^\pm)$, for each of these points $n^\pm$. Let

$\mathcal{V} := \otimes_{n^\pm} V^{n^\pm}, \quad \mathcal{A} := \otimes'_{n^\pm} \text{End}(V^{n^\pm})$, \hspace{1cm} (18)

where $\otimes'$ stands for the restricted tensor product with respect to 1. $\mathcal{A}$ is the algebra finitely generated by the local operators on $V^{n^\pm}$ (i.e. elements of $\text{End}(V^{n^\pm})$).

We write the $R$ matrix $\rho^m^{n^\pm} \otimes \rho^{n^-} (\mathcal{R})$ as $R^{m+n^-}$ for short and define the Hamiltonian of our dynamics to be $\mathcal{H} = \mathcal{H}_0 \mathcal{H}_1$, where (cf. [3])

$\mathcal{H}_0 := \cdots R^{1+1} - R^{2+2} \cdots, \quad \mathcal{H}_1 := \cdots R^{2+1} - R^{3+2} \cdots$. \hspace{1cm} (19)

Note that the operators $\{R^{n+n^-}\}$ (resp. $\{R^{n+(n-1)^-}\}$) are commuting among themselves here; we may similarly define $\mathcal{H}_m := \cdots R^{m+0} - R^{m+1-1} \cdots$. Then the discrete dynamics

$\mathcal{T} := \text{Ad}(\mathcal{H}^{-1}) : \mathcal{O} \mapsto \mathcal{H}^{-1} \mathcal{O} \mathcal{H} \quad \mathcal{O} \in \mathcal{A}$

on $\mathcal{A}$ is well-defined since any $\mathcal{O} \in \mathcal{A}$ is locally supported (i.e. of the form $\cdots 1 \otimes a \otimes 1 \cdots$). This dynamics is explicitly described in terms of matrix elements of (finite products of) local Lax matrices as we will see shortly (Theorem 1).

Consider successive products of the local Lax matrices and express them as e.g.

$\mathcal{L}_z(1^{-1} 2^+ 2^-) := L_z^-(\Delta^1)^{-1} L_z^+(\Delta^2)^{-1} L_z^+(\Delta^2)^{-1} L_z^-(\Delta^1)^{-1} L_z^-(\Delta^2)^{-1} L_z^+(\Delta^2)^{-1}$

(four points case) and so on. If we exploit some more graphics, the dynamics $\mathcal{T}$ applied to (the matrix elements of) $\mathcal{L}$ can be depicted as follows (Fig. 3):

*Figure 3. The time evolution $\mathcal{T}$ (19).*

\[\mathcal{L}(\cdots 1^{-1} 2^+ 2^- \cdots) = \quad \mathcal{T}(\mathcal{L}) = \]

\[\mathcal{L}_0 \quad \mathcal{H}_0 \quad \mathcal{H}_0^{-1} \quad \mathcal{H}_1 \quad \mathcal{H}_1^{-1} \]

\[\mathcal{H}_1 := \cdots R^{1+1} - R^{2+2} \cdots \quad \mathcal{H}_0 := \cdots R^{0+1} - R^{2+2} \cdots \]
To describe the dynamics more explicitly, put $\Delta(m^+n^-) := \Delta^{m+} \otimes \Delta^{n-}$ and

$$w_i(m^+n^-) := \begin{cases} (E_i^+)^{m+} \otimes (E_i^-)^{n-} & (m \equiv n \mod 2), \\ (E_i^+)^{m+} \otimes (k_i^{-1} E_i^-)^{n-} & \text{(otherwise)}. \end{cases}$$   \hspace{1cm} (20)$$

Then $T$ is locally determined by neighbouring $w$’s and $\Delta$’s as follows.

**Theorem 1** We have

$$T(w_0(1^+1^-)) = \frac{w_0(1^+0^-) - q \Delta(1^+0^-)}{w_0(1^+0^-) - q} \cdot \frac{w_0(2^+1^-) - q \Delta(2^+1^-)}{w_0(2^+1^-) - q} \cdot \frac{w_0(2^+0^-) - q \Delta(2^+0^-)}{w_0(2^+0^-) - q},$$

$$T^{-1}(w_0(2^+1^-)) = \frac{w_0(1^+1^-) - q \Delta(1^+1^-)}{w_0(1^+1^-) - q} \cdot \frac{w_0(2^+2^-) - q \Delta(2^+2^-)}{w_0(2^+2^-) - q} \cdot \frac{w_0(1^+2^-) - q \Delta(1^+2^-)}{w_0(1^+2^-) - q}. \hspace{1cm} (21)$$

As for $w_0(m^+, m^-)$ or $w_0((m+1)^+, m^-)$, we read $0^\pm, 1^\pm, 2^\pm$ above as $(m-1)^\pm, m^\pm, (m+1)^\pm$. Since (recall (8))

$$T(\Delta^{m\pm}) = q^{\mp c^{m\pm}} \Delta^{m\pm},$$

$$T(w_1(m^+, m^-)), T(w_1(m^+, m-1^-))$$

are determined by these formulae.

**Remark** From (17), we see that $T$ induces the exchange dynamics on the lattice:

$$H_0^{-1} \mathcal{L}(1^-1^+2^-2^+ \cdots n^-n^+)H_0 = \mathcal{L}(1^+1^-2^+2^- \cdots n^+n^-)$$

$$H_1^{-1} \mathcal{L}(1^-2^+2^-3^+ \cdots n^-n+1^+)H_1 = \mathcal{L}(2^+1^-3^+2^- \cdots (n+1)^+n^-) \hspace{1cm} (24)$$

Unfortunately, the result $H_1^{-1} \mathcal{L}(1^-1^+ \cdots n^-n^-)$ or $H_0^{-1} \mathcal{L}(2^+1^- \cdots (n+1)^+n^-)$ are not simple enough so that $T(\mathcal{L})$ can be said as “exchange dynamics” if we take the Lax matrix $\mathcal{L}$ as a representative of the conjugacy class of $L \sim L^{-1} \mathcal{L} L$, together with the following periodic condition.

Now, let us assume $c^{m\pm} = c^{(m+2)\pm}$ for $\pm = +,-$ and $m = 0, 1$ in what follows. Then we have the trivial $U_q^\pm$-isomorphisms $\iota^{m\pm}: V^{m\pm} \to V^{(m+2)\pm}$ ($1 \mapsto 1$); $\text{End} V^{m\pm} \to \text{End} V^{m\pm}$ and therefore

$$S := \otimes \iota^{m\pm} : \mathcal{V} \tilde{\to} \mathcal{V}; \hspace{1cm} \mathcal{A} \tilde{\to} \mathcal{A}.$$  

The isomorphism $S$ is nothing but the dilation in the space direction. It is obvious that $H_0$ and $H_1$ do not change with respect to this dilation and hence

**Lemma 1** (*Periodic reduction*) We have

$$[H_0, S] = [H_1, S] = 0$$

so that the dynamics $T$ descends to the quotient

$$\mathcal{A} := \mathcal{A}/\text{Im}(S - 1).$$
That is, under the assumption $c_m^\pm = c_{m+2}^\pm$, the dynamics $\mathcal{T}$ preserves the conditions
\[
w_i(m^+n^-) = w_i((m + 2)^+n^-) = w_i(m^+(n + 2)^-),
\]
\[
\Delta(m^+n^-) = \Delta((m + 2)^+n^-) = \Delta(m^+(n + 2)^-).
\]

Under this periodic reduction, comparison of the obtained formulae (21), (22) with the ones (35) in the Appendix via the Weyl group approach, we can identify the resulting system with the quantum discrete Painlevé system $\hat{q}P_{VI}$ with the following identification of the parameters and the dynamical variables. Write $w_i^{mn} := w_i(m^+n^-)$ and $\Delta^{mn} := \Delta(m^+n^-)$ for short. It turns out that we should identify as follows,
\[
-F = \left( w_0^{11} \right) \left( w_1^{21} \right) \left( w_1^{11} \right), \quad -G = \left( w_0^{10} \right) \left( w_0^{21} \right) \left( w_1^{01} \right),
\]
and
\[
a_0 = \frac{w_0^{21}}{w_0^{01}}, \quad a_1 = \frac{w_1^{21}}{w_1^{01}}, \quad a_2 = \frac{1}{\Delta_{21}}, \quad a_3 = \Delta_{11}, \quad a_4 = \frac{w_1^{22}}{w_1^{11}}, \quad a_5 = \frac{w_0^{22}}{w_0^{10}}.
\]
It is easy to see that $a_i$ for $i = 0, 1, 4, 5$ and $p = a_0 a_1 a_2^2 a_3^2 a_4 a_5$ are central elements among the algebra of observables, so that they are constants with respect to our dynamics $\mathcal{T}$. Moreover,
\[
t := \Delta(0^+0^-) \Delta(1^+1^-)
\]
satisfies $\mathcal{T}(t) = q^{2c} t$, where $c = c^0^- + c^1^- - c^0^+ - c^1^+$ (cf. (23)), meaning that $t$ can be regarded as the time parameter of the dynamics.

\textbf{Theorem 2} The quantum Painlevé VI system $\hat{q}P_{VI}$ (35) is reproduced by the above construction:
\[
\mathcal{T}(F) = \frac{p^2}{q^2 t^2} \cdot \frac{G + p^{-1} a_1^{-2} t}{G + p a_0^2 t^{-1}} \cdot \frac{G + p a_0^2 t^{-1} F^{-1}}{G + p a_0^2 t^{-1} F^{-1}},
\]
\[
\mathcal{T}^{-1}(G) = \frac{1}{q^2 t^2} \cdot \frac{F + a_4^2 t}{F + a_5^2 t^{-1}} \cdot \frac{F + a_5^2 t^{-1} G^{-1}}{F + a_5^2 t^{-1} G^{-1}}.
\]

\textbf{Remark.} In (28) we should employ the fourth root so as to getting the same formula as $\hat{q}P_{IV}$. In fact we can find the $W(D_5^{(1)})$ action without these fourth root and allows us to recover $\hat{q}P_{IV}$ as in the manner in the Appendix. As in the Faddeev-Volkov system, $w_0(1^+1^-), w_0(1^+2^-)$ (or $F$ and $G$) together with $a_i^4$ ($i = 0, \ldots, 5$) generates the diagonal-gauge invariants:
\[
\langle w_0(1^+1^-), w_0(1^+2^-), a_0^4, \ldots, a_5^4 \rangle = \langle \{ \mathcal{L}(1^-1^+2^-2^+)_{ij} \mid i, j = 1, 2 \} \rangle^{AdH}
\]
where $H = \left\{ \left[ \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \mid a, b \in \mathbb{C}; a, b \neq 0 \right\}$.

We also remark that the quantized lattice Liouville equation [4, 11] appears as a limit of our equation. That is, if we assume $\mathcal{T}^N \Delta(1^+1^-), \mathcal{T}^N \Delta(2^+1^-), \mathcal{T}^N \Delta(1^+1^-), \mathcal{T}^N \Delta(2^+2^-) \rightarrow 0$ as $N \rightarrow \infty$ (i.e. $\text{Re}(c^m^+ - c^n^-) > 0$ for all $m, n$), then from (21), (22) we respectively have
\[
\mathcal{T}(w_0(1^+1^-))w_0(2^+0^-) = \frac{w_0(1^+0^-)}{w_0(1^+0^-) - q} \frac{w_0(2^+1^-) - q}{w_0(2^+1^-)},
\]
\[
\mathcal{T}^{-1}(w(2^+1^-))w(1^+2^-) = \frac{w_0(1^+1^-) - q}{w_0(1^+1^-)} \frac{w_0(2^+2^-) - q}{w_0(2^+2^-) - q}.
\]
or
\[
\begin{aligned}
& w_0(2^{+0^{-}})^{-1} \mathcal{T}(w_0(1^{+1^{-}})^{-1}) = (1 - qw_0(1^{+0^{-}})^{-1})(1 - qw_0(2^{+1^{-}})^{-1}), \\
& w_0(1^{+2^{-}})^{-1} \mathcal{T}^{-1}(w(2^{+1^{-}})^{-1}) = (1 - qw_0(1^{+1^{-}})^{-1})(1 - qw_0(2^{+2^{-}})^{-1}).
\end{aligned}
\]  

(29)

5 Quantized discrete Schlesinger equation

Theorem 3 Let \( B(z) := \mathcal{H}_0 \mathcal{L}_z(2^{-}) \mathcal{H}_1 \). We have
\[
\mathcal{L}(1^{+1^{+}2^{-}2^{+}}) B(z) = B(z) \mathcal{L}(1^{+2^{-}2^{+}1^{-}}).
\]

(30)

Let us write \( \mathcal{L}(1^{+1^{+}2^{-}2^{+}}) = q^D L(1^{+1^{+}2^{-}2^{+}}) \), where \( q^D \) stands for the difference operator part,
\[
D = \sum_{i=1,2; \pm} \mp c_i^\pm d = cd.
\]

Then the above relation (30) is equivalent to (cf. (14))
\[
L(1^{+1^{+}2^{-}2^{+}}) B(z) = B(zq^{-c}) \mathcal{T}(L(1^{+1^{+}2^{-}2^{+}})),
\]

(31)

which can be recognized as the quantization of (6).

Proof
\[
\text{LHS} = \mathcal{L}(1^{+1^{+}2^{-}2^{+}}) \mathcal{H}_0 L(2^{-}) \mathcal{H}_1
\]
\[
= \mathcal{H}_0 \mathcal{L}(1^{+1^{+}2^{-}2^{+}}) L(2^{-}) \mathcal{H}_1
\]
\[
= \mathcal{H}_0 L(2^{-}) \mathcal{L}(2^{+1^{+}1^{-}2^{+}}) \mathcal{H}_1
\]
\[
= \mathcal{H}_0 L(2^{-}) \mathcal{H}_1 \mathcal{L}(1^{+2^{-}2^{+}1^{-}}) = \text{RHS}.
\]

The above proof uses the Yang-Baxter equation under the periodicity condition, which can be depicted as Fig. 4.

Figure 4. The discrete Schlesinger equation (30).

In general, without the periodicity, we have (cf. (24) (25))
\[
\mathcal{L}(1^{+1^{+} \cdots n^{+}n^{-}}) B_n(z) = B_0(z) \mathcal{L}(1^{+0^{-}2^{+1^{-}} \cdots n^{+}(n-1)^{-})
\]

(32)
on $\mathcal{Y}$, where $\mathcal{B}_k(z) := \mathcal{H}_0 L(k^-) \mathcal{H}_1$ for $k = 0, n$. If we assume the $n$-periodicity $c^{m\pm} = c^{(m+n)\pm}$, $V^{m\pm} := V^{(m+n)\pm}$, then $L(1+0\cdot 2+1\cdot \ldots + n\cdot (n-1)) = L(1+n-2+1\cdot \ldots + n\cdot (n-1))$ in the right-hand side is conjugate to $\mathcal{T}(L(1+1\cdot \ldots + n\cdot n))$, so that the equation (32) or (31) in $n = 2$) can be considered as the compatibility condition of the linear problem

$$\mathcal{Y} = \mathcal{L} \mathcal{Y}, \quad \mathcal{L} = L(1+1\cdot \ldots + n\cdot n),$$

or, equivalently (write $\mathcal{L} = q^D L(z)$ as in (31) )

$$q^{-D} \mathcal{Y}(z) = \mathcal{Y}(q^{-c} z) = L(z) \mathcal{Y}(z),$$

(where $\mathcal{Y}$ should be regarded as $V \otimes \mathbb{C}^2$-valued) and the time evolution $\mathcal{T}$.

Thus we have succeeded in quantizing the isomonodromy problem or the Lax form for the quantum discrete Painlevé system $q^D \mathcal{Y}$ explicitly.

It is interesting to note that two fundamental solutions of (34) can be at least formally obtained as

$$(\mathcal{Y}_0 L(z q^c) L(z q^{2c}) \cdots)$$

and

$$(\mathcal{Y}_0 (z q^{-c})^{-1} L(z q^{-2c})^{-1} \cdots)$$

and then the quantization of (3) can be simply written as

$$\mathcal{M}(z) := \mathcal{Y}_0^{-1} \mathcal{Y}_0 = \cdots L(z q^{-2c}) L(z q^{-c}) L(z) L(z q^c) L(z q^{2c}) \cdots.$$

6 The Weyl group action

Further comparison with the formula in the Appendix give us the formula for the $\mathcal{W}(D_5^{(1)}) = \langle W, \sigma \rangle$-action. We have

$$\sigma : w_0^{11} \mapsto q^{-2} w_1^{11}, \quad w_0^{21} \mapsto q^{-2} w_1^{21},$$

and

$$s_0 : w_0^{11} \mapsto w_0^{11}, \quad w_0^{21} \mapsto w_0^{10},$$

$$s_2 : w_0^{11} \mapsto w_0^{11} \frac{w_0^{21} - 1}{w_0^{20} - \Delta_1}, \quad w_0^{21} \mapsto \frac{1}{w_0^{11}},$$

$$s_1 : w_0^{11} \mapsto w_0^{11}, \quad w_0^{21} \mapsto w_0^{21},$$

$$s_3 : w_0^{11} \mapsto \frac{1}{w_0^{11}}, \quad w_0^{21} \mapsto w_0^{11} - 1,$$

$$s_4 : w_0^{11} \mapsto w_0^{11}, \quad w_0^{21} \mapsto w_0^{21}. $$

There should be a Lax matrix point of view elucidation of these symmetry: it is quite plausible that these symmetry arise from the choice of multiplicative decompositions of our Lax matrix (cf. [1]), and is related to the tesseract of the projective line with four points so that the time evolution $\mathcal{T}$ can be regarded as the Dehn twist ([10]). We would like to report this point seperately in a near future.

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7 Appendix. Review of the Weyl group approach to $\hat{q}P_{VI}$ [5]

Here we recite the results of [5]. We use the $D_5^{(1)}$ root system. Let $\{e_j\}_{1 \leq j \leq 5}$ be the orthonormal basis $\subset R^6 = R^5 \perp R\delta$, where $\delta$ is identified with the null root, then the symple roots will be realized as follows.

\[
\begin{array}{cccc}
\alpha_0 & \alpha_5 & \delta - e_1 - e_2 & e_4 + e_5 \\
\alpha_2 - \alpha_3 & \alpha_4 & e_2 - e_3 & e_3 - e_4 \\
\alpha_1 & & e_1 - e_2 & e_4 - e_5 \\
\end{array}
\]

Let $q = e^h \in C^x, |q| < 1$. Let $a_j := e^{h\alpha_j}, p := e^{h\delta} = a_0a_1a_2a_3a_4a_5$ be elements of the group algebra of the $D_5^{(1)}$ root lattice. We have the $W = W(D_5^{(1)})$ action given by

\[
s_i(a_j) = a_i^{G_{ij}}a_j \quad (s_i(p) = p, \forall i).
\]

We also need diagram automorphisms. They are

\[
\tau : a_j \leftrightarrow a_{5-j}^{-1}(j = 0, \cdots, 5),
\]

and

\[
\sigma : a_0 \leftrightarrow a_1^{-1}, a_4 \leftrightarrow a_5^{-1}, a_j \leftrightarrow a_j^{-1}(j = 2, 3).
\]

We have defined the action of the extended affine Weyl group $\hat{W} = \langle W, \tau, \sigma \rangle$ on the group algebra of the root lattice $Q$. Let further $K := C(a_0, \cdots, a_5)\langle F, G \rangle$ where $F$ and $G$ are noncommutative letters; we let $FG = q^2GF$ later.

**Theorem** (1) We have $\hat{W}(D_5^{(1)})$-action

\[
\langle W, \sigma \rangle \xrightarrow{\text{hom}} \text{Aut}(K)
\]

given by

\[
\sigma : F \leftrightarrow q^{-2}F^{-1}, G \leftrightarrow q^{-2}G^{-1},
\]

\[
s_2(F) := F\frac{a_0a_1^{-1}G + a_2^2}{a_0a_1^{-1}a_2^2G + 1}, \quad s_j(F) := F \quad (j \neq 2)
\]

\[
s_3(G) := \frac{a_2^2a_4a_5^{-1}F + 1}{a_4a_5^{-1}F + a_3^2G}, \quad s_j(G) := G \quad (j \neq 3)
\]

(2) If $FG = q^2GF$, this action is Hamiltonian: namely we have $\Sigma, S_j$ such that

\[
\sigma_0\sigma_45(\phi) = \Sigma \phi \Sigma^{-1}, \quad s_j(\phi) = S_j\phi S_j^{-1}(j = 0, \cdots, 5)
\]

for any $\phi \in K$. Recall $(x)_\infty = (x, q) = \prod_{m=0}^\infty (1 - xq^m)$ and put $\Psi(a, x) := \frac{(qx)^{(x^{-1})}_\infty}{(aqx)^{(ax^{-1})}_\infty}$. Then

\[
\Sigma := (FG)_\infty (qG^{-1}F^{-1})_\infty (G^{-1}F)_\infty (qF^{-1}G)_\infty (F)_{\infty}^2 (qF^{-1})_\infty^2 (G)_{\infty}^2 (qG^{-1})_\infty^2,
\]
and
\[
S_2 := \Psi(a_2, a_0 a_1^{-1} G)e^{\frac{\pi i}{2} \alpha_2 \partial_2}, \quad S_3 := \Psi(a_3, a_5 a_4^{-1} G)e^{\frac{\pi i}{2} \alpha_3 \partial_3}, \quad S_j := e^{\frac{\pi i}{2} \alpha_j \partial_j (j \neq 2, 3)},
\]
where the derication \(\partial_j\) is defined by \(\partial_j(\alpha_k) := C_{j,k}\) (the Cartan matrix).

Now consider the lattice element \(T_3 := s_2 s_1 s_0 s_2 s_3 s_4 s_5 s_0 \sigma_{01} \sigma_{45}: e_j \mapsto e_j - \delta_{j,3}\delta, \delta \mapsto \delta\). In \(q = 1\) case, \(T_3\) reproduces the \(qPVI\) of Jimbo-Sakai [18]. Put \(t = q^{2e_3} = a_3^2 a_4 a_5\).

**Theorem/Definition.** The \(T_3\) action is given as follows (the quantized difference Painlevé VI system \(\hat{qPVI}\)), which commutes with \(W(D_4^{(1)}) \simeq \langle s_0, s_1, s_2 s_3 s_2, s_4, s_5 \rangle\).

\[
T_3(a_0, a_1, t, a_4, a_5) = (a_0, a_1, t/p, a_4, a_5),
\]

\[
\hat{qPVI}:
\begin{align*}
T_3(F) &= \frac{p^2}{q^{2t^2}} \cdot \frac{G + p^{-1} a_1^{-2} t}{G + p a_0^{-2} t^{-1}} \cdot \frac{G + p^{-1} a_3^{-2} t}{G + p a_2^{-2} t^{-1}} F^{-1}, \\
T_3^{-1}(G) &= \frac{1}{q^{2t^2}} \cdot \frac{F + a_1^2 t}{F + a_3^2 t^{-1}} \cdot \frac{F + a_4^2 t}{F + a_5^2 t^{-1}} G^{-1}.
\end{align*}
\]

If \(FG = q^2 GF\), then by construction, \(\hat{qPVI}(= T_3\) action\) has the Hamiltonian,

\[
T_3 = Ad(\mathcal{H}), \quad \mathcal{H} := S_2 S_1 S_0 S_2 S_3 S_4 S_5 S_3 \Sigma.
\]
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