Dependence Measuring from Conditional Variances

DOI 10.1515/demo-2015-0007
Received February 28, 2015; accepted July 2, 2015

Abstract: A conditional variance is an indicator of the level of independence between two random variables. We exploit this intuitive relationship and define a measure $\nu$ which is almost a measure of mutual complete dependence. Unsurprisingly, the measure attains its minimum value for many pairs of non-independent random variables. Adjusting the measure so as to make it invariant under all Borel measurable injective transformations, we obtain a copula-based measure of dependence $\nu^*$ satisfying A. Rényi's postulates. Finally, we observe that every nontrivial convex combination of $\nu$ and $\nu^*$ is a measure of mutual complete dependence.

Keywords: conditional variances; measures of dependence; copulas; mutual complete dependence; shuffles of Min

MSC: 60A10, 62H20

1 Introduction

The problem of how to assign the level of dependence between two random variables in a consistent manner can never be solved completely by using only a single measure of dependence. There are many attributes to consider in choosing the “right” measure of dependence in a given situation. Among them are the nature of dependence (linear, monotone, or other types of dependence), a reference to the normal correlation coefficient and other specific purposes. Many measures of dependence have been proposed and studied since the beginning of the twentieth century. See [10, 14, 16, 18, 20, 22]. But it is not until the seminal paper of A. Rényi [16] that this problem attracted much wider attention. He proposed the following set of seven properties that should be valid for a generic measure of dependence $\delta$. To the best of our knowledge, the only measure satisfying all of these properties is the maximal correlation coefficient $R$: $R(X, Y) = \sup_{f,g} \gamma(f(X), g(Y))$, where the supremum is taken over all Borel measurable functions $f$ and $g$ such that the correlation coefficient $\gamma(f(X), g(Y))$ can be defined.

R0. $\delta(X, Y)$ is defined for all random variables $X$ and $Y$, neither of them being constant almost surely (a.s.).
R1. $\delta(X, Y) = \delta(Y, X)$.
R2. $0 \leq \delta(X, Y) \leq 1$.
R3. $\delta(X, Y) = 0$ if and only if $X$ and $Y$ are independent.
R4. $\delta(X, Y) = 1$ if $X$ and $Y$ are completely dependent, i.e. $X$ is almost surely a Borel measurable function of $Y$ or vice versa.
R5. $\delta(f(X), g(Y)) = \delta(X, Y)$ for all Borel measurable injective transformations $f$ and $g$.
R6. $\delta(X, Y) = |\rho|$ if $X$ and $Y$ are jointly normal with correlation coefficient $\rho$.

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Since the discovery of copulas and the famous Sklar’s theorem [13, 21], many measures of dependence defined via copulas - called copula-based measures of dependence - have been introduced. As a pair of random variables \(X\) and \(Y\) has a unique copula when they are continuous, a copula-based measure of dependence is guaranteed to be well-defined only for continuous random variables and hence \(R_0\) may not hold. For the copula-based measures of dependence, \(R_6\) is usually replaced by a weaker postulate that \(\delta(X, Y)\) is a strictly increasing function of \(|\rho|\). Two such measures are Schweizer and Wolff’s \(\sigma\) [18],

\[
\sigma(X, Y) = 12 \int \frac{|C - II|}{I} \, d\lambda_2,
\]

and Siburg and Stoimenov’s \(\omega\) [20],

\[
\omega(X, Y) = \left( 3 \int \frac{\left( (\partial_1 C)^2 + (\partial_2 C)^2 \right)}{I} \, d\lambda_2 - 2 \right)^{1/2},
\]

where \(C\) is the copula of \(X\) and \(Y\), denoted by \(C_{X,Y}\). Recall that \(II\), defined by \(II(x, y) = xy\), is the copula of independent continuous random variables; \(M(x, y) = \min(x, y)\) is the copula of comonotonic random variables; and \(W(x, y) = \max(x + y - 1, 0)\) is the copula of countermonotonic random variables. Both \(\sigma\) and \(\omega\) are defined for continuous random variables and satisfy \(R_1\)–\(R_3\). \(\sigma\) is called a measure of monotone dependence because its maximum value detects strict monotone dependence:

\[
\sigma_4. \quad \sigma(X, Y) = 1 \text{ if and only if } X \text{ and } Y \text{ are a.s. strictly monotonically dependent, i.e. } C_{X,Y} = M \text{ or } W.
\]

\[
\sigma_5. \quad \sigma(f(X), g(Y)) = \sigma(X, Y) \text{ for all a.s. strictly monotonic measurable transformations } f \text{ and } g.
\]

The measure \(\omega\) is called a measure of mutual complete dependence because its maximum value is attained exactly when the random variables are mutually completely dependent, i.e. they are completely dependent on each other:

\[
\omega_4. \quad \omega(X, Y) = 1 \text{ if and only if } X \text{ and } Y \text{ are mutually completely dependent.}
\]

\[
\omega_5. \quad \omega(f(X), g(Y)) = \omega(X, Y) \text{ for all a.s. strictly monotonic measurable transformations } f \text{ and } g.
\]

Observe that the properties \(R_4\)–\(R_5\) need to be adjusted according to which types of dependence the measures aim to detect. Historically, \(\sigma\) and \(\omega\) have their roots in the Spearman’s \(\rho\) and the (modified) Sobolev norm of copulas [13, 18–20].

The conditional variance of \(Y\) given \(X\) is an indicator of how weakly \(Y\) is dependent on \(X\). We make this relationship more explicit as follows. For the uniform \([0, 1]\) random variables \(X\) and \(Y\) with joint distribution function or copula \(C\), we observe that the \(L^1\)-norm of the conditional variance, called the total conditional variance,

\[
\sigma^2_{Y|X} = \int \frac{1}{0} \text{Var}(Y|X = x) \, dx
\]

is equal to the \(L^1\)-norm of the difference \((M - C^T \ast C)\). This suggests that the \(L^1\)-norms of \(C^T \ast C\) and \(C \ast C^T\) might possibly give rise to new measures of dependence with close ties to conditional variances. It turns out that the sum of these two \(L^1\)-norms give a “measure of mutual complete dependence” \(\nu\) that satisfies \(R_1\)–\(R_5\) except that in \(R_3\) it holds only that \(\nu(X, Y) = 0\) if \(X\) and \(Y\) are independent. Moreover, \(\nu(X, Y)\) is very close to \(|\rho|\) in the case when \(X\) and \(Y\) are jointly normal with correlation coefficient \(\rho\). All of the above are developed in section 3. In section 4, we overcome the inability of \(\nu\) in classifying the independence and define a new measure \(\nu^*\). It is proved to satisfy \(R_1\)–\(R_5\). We also show that the converse of \(R_4\) does not hold for \(\nu^*\). Finally, a class of measures of mutual complete dependence is given by all nontrivial convex combinations of \(\nu\) and \(\nu^*\). Note also that, by computation, both \(\nu\) and \(\nu^*\) are increasing functions of \(|\rho|\). Let us begin with a section summarizing all the necessary backgrounds on copulas including their properties and constructions.
2 Background on copulas

Denote $I = [0, 1]$, $\mathcal{B}(I)$ the Borel $\sigma$-algebra on $I$ and let $\lambda$ and $\lambda_2$ denote the Lebesgue measures on $I$ and $I^2$ respectively. The Lebesgue integral on $I$ is denoted simply by $\int_0^1 dx$. The symbol $\partial_1 C$ denotes the partial derivative of $C$ with respect to the $i$th variable.

A function $C: I^2 \to I$ is called a (bivariate) copula if for all $u, v \in I$,

1. $C(0, 0) = C(u, 0)$;
2. $C(1, v) = v, C(u, 0) = u$; and
3. $C(u', v) - C(u, v) - C(u, v') + C(u, v) \geq 0$ for all $[u, u'] \times [v, v'] \subseteq I^2$.

Every copula $C$ can be extended to a joint distribution function of uniform $[0, 1]$ random variables in a unique way. Let $X$ and $Y$ be any random variables whose distribution functions are $F$ and $G$, respectively. Sklar’s theorem states that every joint distribution function $H$ of $X$ and $Y$ can be written as

$$H(x, y) = C(F(x), G(y)), \quad (1)$$

for some copula $C$. If $F$ and $G$ are continuous, then $C$ is uniquely determined by (1) and called the copula of $X$ and $Y$. Conversely, putting an arbitrary copula $C$ into (1) always yields a joint distribution function $H$. A copula $C$ is said to be symmetric if its transpose $C^T$, given by $C^T(u, v) = C(v, u)$, is equal to $C$. For more details on the theory of copulas, see [13].

In a series of papers [3, 4, 15], Darsow, Nguyen and Olsen introduce a binary operation on the class of bivariate copulas, called the $*$-product, defined by

$$C * D(u, v) = \frac{1}{2} \int_0^1 \partial_2 C(u, t) \partial_1 D(t, v) dt. \quad (2)$$

$M$ is the identity ($M * C = C * M$) while $II$ is the zero ($II * C = II = C * II$). We say that $C$ is left invertible (right invertible) if $C^T * C = M$ ($C * C^T = M$). It was shown that a copula $C$ is left invertible (right invertible) if and only if $C$ is the copula of $X$ and $f(X)$ ($f(X)$ and $X$) for some continuous random variable $X$ and Borel measurable transformation $f$. Random variables $X$ and $Y$ are said to be completely dependent if $Y = f(X)$ a.s. or $X = f(Y)$ a.s. for some Borel measurable $f$. They are said to be mutually completely dependent if $Y = f(X)$ a.s. for some Borel measurable injection $f$. A mutual complete dependence copula is the copula of two continuous random variables which are mutually completely dependent. Note that the invertible copulas, whose class is denoted by $J$, are exactly the mutual complete dependence copulas.

Shuffles of Min are the copulas of random variables $X$ and $f(X)$ for which $f$ is a piecewise continuous injection. They are simple mutual complete dependence copulas in the sense that they can be constructed by cutting $I^2$ into a finite number of vertical stripes and shuffling the masses of $M(u, v) = \min(u, v)$ on the main diagonal with possible flipping of the stripes. See [12, 13] for more details on shuffles of Min. Note that the $*$-product of shuffles of Min is a shuffle of Min.

The ordinal sum of copulas $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$ with respect to a partition $\{[a_i, b_i]\}_{i=1}^n$ of $I$ is the copula $C$ given by

$$C(u, v) = \begin{cases} a_i + (b_i - a_i)C_i\left(\frac{u - a_i}{b_i - a_i}, \frac{v - a_i}{b_i - a_i}\right) & \text{for } (u, v) \in [a_i, b_i]^2, \\ M(u, v) & \text{otherwise}. \end{cases}$$

The mass of $C$ is spread in each square $[a_i, b_i]^2$ according to the copula $C_i$. So the ordinal sum of shuffles of Min is still a shuffle of Min. Recall from [3, Theorem 8.3] that:

**Lemma 2.1.** If $C$ and $D$ are the ordinal sums of $\{C_n\}$ and $\{D_n\}$, respectively, with respect to the same partition of $[0, 1]$, then $C * D$ is the ordinal sum of $\{C_n * D_n\}$ with respect to that partition.
3 Conditional variances and dependence measuring

Let $C$ be the copula of uniform $[0, 1]$ random variables $X$ and $Y$ on a common probability space and $x \in I$. Recall that the conditional distribution of $Y$ given $X = x$ satisfies $F_{Y|X=x}(y) = P(Y \leq y | X = x) = \partial_1 C(x, y)$ a.s. and so the conditional expectation of $Y$ given $X = x$ is given by

$$\mu_{Y|X}(x) = E(Y | X = x) = \int_0^1 y \partial_1 C(x, dy).$$

Denote the conditional variance of $Y$ given $X = x$ by

$$\sigma^2_{Y|X}(x) = \text{Var}(Y | X = x) = E((Y - \mu_{Y|X}(x))^2 | X = x) = E(Y^2 | X = x) - \mu^2_{Y|X}(x) \quad (2)$$

and the total conditional variance of $Y$ given $X$ by $\sigma^2_{Y|X} = \int_0^1 \sigma^2_{Y|X}(x) \, dx$.

**Proposition 3.1.** Suppose random variables $X$ and $Y$ are uniformly distributed on $[0, 1]$ with copula $C$. Then

$$\sigma^2_{Y|X} = \frac{1}{3} - \int_0^1 \int_0^1 C^T \ast C(x, y) \, dx \, dy.$$ 

Our proof will use some identities collected in the following Lemma.

**Lemma 3.2.** Let $C$ be a copula and $f$ be a nonnegative bounded measurable function on $[0, 1]$. Then for almost all $x \in I$,

$$\int_0^1 f(y) \partial_1 C(x, dy) = \frac{d}{dx} \left( \int_0^1 f(y) \partial_2 C(x, y) \, dy \right) \quad (3)$$

and

$$\int_0^1 \partial_1 C(x, y) \, dy = \frac{d}{dx} \int_0^1 C(x, y) \, dy. \quad (4)$$

**Proof.** (3) can be proved by repeating the arguments in the proof of Lemma 3.1 in [3].

To prove (4), let $x \in I$ be such that $\frac{d}{dx} \int_0^1 C(x, y) \, dy$ exists and $\partial_1 C(x, y)$ exists a.e. $y$. Note that almost every $x$ possesses these properties. Consider a sequence $\{x_n\}$ converging to $x$. By the Lipschitz condition of copula, $\left| \frac{C(x, y) - C(x_n, y)}{x - x_n} \right| \leq 1$ for all $y \in I$. So, by the dominated convergence theorem,

$$\frac{d}{dx} \int_0^1 C(x, y) \, dy = \int_0^1 \lim_{n \to \infty} \frac{C(x, y) - C(x_n, y)}{x - x_n} \, dy = \int_0^1 \partial_1 C(x, y) \, dy. \quad \square$$
Proof of Proposition 3.1. By (2), \( \sigma^2_{Y|X}(x) = \int_0^1 y^2 \partial_1 C(x, dy) - \mu^2_{Y|X}(x) \). So
\[
\sigma^2_{Y|X} = \int_0^1 \int_0^1 y^2 \partial_1 C(x, dy) \, dx - \int_0^1 \left[ \frac{1}{t} \partial_1 C(x, dt) \right]^2 \, dx 
= \int_0^1 \frac{d}{dx} \left( \frac{1}{t} \right) \left( \int_0^1 y^2 \partial_2 C(x, y) \, dy \right) \, dx - \int_0^1 \left[ \frac{d}{dx} \left( \int_0^1 \partial_2 C(x, t) \, dt \right) \right]^2 \, dx 
= \int_0^1 y^2 \left[ \partial_2 C(x, y) \right]_{y=0}^1 \, dy - \int_0^1 \left[ \frac{d}{dx} \left( \partial_2 C(x, t) \, dt \right) \right]^2 \, dx 
= \frac{1}{3} \left( 1 - \int_0^1 \partial_1 C(x, t) \, dt \right)^2 \, dx.
\]
We have used (3) twice in the second line, the first fundamental theorem of calculus and the method of integration by parts in the third line, and (4) in the last line. Applying Tonelli’s theorem, the second integral in the last line equals
\[
1 - 2 \int_0^1 \int_0^1 \partial_1 C(x, t) \, dx \, dt + \int_0^1 \int_0^1 \partial_2 C^T(s, x) \partial_1 C(x, t) \, dx \, ds \, dt
\]
which clearly equals \( \int_0^1 \int_0^1 C^T \cdot C(s, t) \, ds \, dt \) as desired. \( \square \)

The total conditional variance \( \sigma^2_{X|Y} \) of \( X \) given \( Y \) is defined similarly and can be proved to satisfy
\[
\sigma^2_{X|Y} = \frac{1}{3} - \int_0^1 \int_0^1 C \cdot C^T(x, y) \, dx \, dy.
\]

Motivated by this relationship, we define \([C]\) for every bivariate copula \( C \) by \([C] = [C]_1 + [C]_2\) where
\[
[C]_1 = \int_{\pi}^{\mathcal{I}} C^T \cdot C \, d\lambda_2 \quad \text{and} \quad [C]_2 = \int_{\pi}^{\mathcal{I}} C \cdot C^T \, d\lambda_2.
\]
Recall that if \( C \) is the copula of mutually completely dependent continuous random variables then \( C \) is invertible with inverse \( C^T \), i.e. \( C \cdot C^T = C^T \cdot C = M \). See [19, 20]. So
\[
[C] = \int_{\pi}^{\mathcal{I}} [C^T \cdot C + C^T \cdot C] \, d\lambda_2 = 2 \int_{\pi}^{\mathcal{I}} M \, d\lambda_2 = \frac{2}{3}.
\]
For the independence case, if \( C = \Pi \), then \([C] = 2 \int_{\pi}^{\mathcal{I}} uv \, du \, dv = \frac{1}{2} \). Let us note here that every idempotent copula \( C \) is symmetric [5, 23]. So \([C] = 2 \int_{\pi}^{\mathcal{I}} C \, d\lambda_2 \) and
\[
[C] = 2 \int_{\pi}^{\mathcal{I}} [(C - \Pi) \, d\lambda_2 + \frac{1}{2}] = \frac{1}{6} \rho(C) + \frac{1}{2}
\]
where \( \rho \) denotes the Spearman’s rho.

**Theorem 3.3.** Let \( C \) be a bivariate copula.

(i) \( \frac{1}{6} \leq [C]_i \leq \frac{1}{3} \) for \( i = 1, 2 \).

(ii) \([C]_1 = \frac{1}{3}\) if and only if \( C \) is left invertible.
(iii) $[C]_2 = \frac{1}{3}$ if and only if $C$ is right invertible.

(iv) $[C]_1 = \frac{1}{4}$ if and only if $\int_0^1 C(u, v) \, dv = \frac{u}{2}$ for all $u \in [0, 1]$.

(v) $[C]_2 = \frac{1}{4}$ if and only if $\int_0^1 C(u, v) \, du = \frac{v}{2}$ for all $v \in [0, 1]$.

Proof. Let $C$ be a bivariate copula. We only prove the statements for $[C]_1$.

(i): Since $C^T \otimes C \leq M$, $[C]_1 \leq \int_I M \, d\lambda_2 = \frac{1}{4}$. By Cauchy-Schwarz inequality,

$$\int_0^1 \int_0^1 \partial_1 C(t, u) \, du \, dt \leq \left( \int_0^1 \int_0^1 \partial_1 C(t, u) \, du \right)^2 \left( \int_0^1 1^2 \, dt \right)^{1/2}.$$ 

Therefore, by Tonelli’s theorem and the fundamental theorem of calculus,

$$\int_0^1 \int_0^1 C^T \otimes C(u, v) \, du \, dv = \int_0^1 \left( \int_0^1 \partial_1 C(t, u) \, du \right)^2 \, dt \leq \int_0^1 \int_0^1 C(t, u) \partial_1 C(t, v) \, dt \, du \, dv$$

$$= \int_0^1 \left( \int_0^1 \partial_1 C(t, u) \, du \right)^2 \, dt \geq \left( \int_0^1 \left( \int_0^1 \partial_1 C(t, u) \, du \right) \, dt \right)^2 = \left( \int_0^1 u \, du \right)^2 = \frac{1}{4}.$$ 

(ii): $C$ is left invertible, i.e. $C^T \otimes C = M$, if and only if $\int_I C^T \otimes C \, d\lambda_2 = \int_I M \, d\lambda_2 = \frac{1}{4}$.

(iv): By Cauchy-Schwarz inequality,

$$\left( \int_0^1 \left( \int_0^1 \partial_1 C(t, u) \, du \right) \, dt \right)^2 = \int_0^1 \int_0^1 \partial_1 C(t, u) \, du \, dt$$

only if $\int_0^1 \partial_1 C(t, u) \, du = K$, a constant function of $t$. Then

$$K = \int_0^1 \int_0^1 \partial_1 C(t, u) \, dt \, du = \int_0^1 u \, du = \frac{1}{2}$$

and hence $\int_0^1 C(t, u) \, du = \frac{1}{2}$ for all $t \in [0, 1]$. The converse is clear. \hfill \Box

Therefore, $[C]$ takes value in the range $\left[ \frac{1}{2}, \frac{1}{3} \right]$ where the maximum is attained if and only if $C$ is invertible. However, the minimum value of $[C]$ cannot identify independence of random variables because $\left[ \frac{M + W}{2} \right] = \frac{1}{2} = \{I\}$. In fact, the minimum value of $[C]$ is attained for every copula of continuous random variables which are jointly symmetric about $(0.5, 0.5)$. Recall [13] that the **jointly symmetric copulas** are precisely the copulas whose associated doubly stochastic measures are symmetric with respect to the line $x = 0.5$ and the line $y = 0.5$.

**Proposition 3.4.** For every jointly symmetric copula $C$, $[C] = \frac{1}{2}$.

Proof. A copula $C$ which is symmetric with respect to the line $x = 0.5$ and the line $y = 0.5$ satisfies $C(x, y) = y - C(1 - x, y)$ and $C(x, y) = x - C(x, 1 - y)$, for all $x, y \in I$. So

$$\int_0^1 C(x, y) \, dx = y - \int_0^1 C(x, y) \, dx \quad \text{and} \quad \int_0^1 C(x, y) \, dy = x - \int_0^1 C(x, y) \, dy.$$ 

Consequently, $\int_0^1 C(x, y) \, dx = \frac{y}{2}$ and $\int_0^1 C(x, y) \, dy = \frac{x}{2}$ for all $x, y \in I$. By Theorem 3.3 (iv) and (v), $[C]_2 = \frac{1}{4} = [C]_1$. Thus, $[C] = \frac{1}{2}$. \hfill \Box
However, there are some non-jointly-symmetric copulas $C$ for which $[C] = \frac{1}{2}$ as demonstrated in Example 3.6. The following lemma will be useful in computing $[C]$.

**Lemma 3.5.** Let $F_1$ and $F_2$ denote the uniform distributions on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. If $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ are copulas for which $\int_{F_i} A_{ij} d\lambda_2 = \frac{1}{4}$ for all $i, j = 1, 2$ then the function $A : I^2 \rightarrow I$ defined by

$$A(u, v) = \frac{1}{4} \sum_{i,j=1} A_{ij}(F_i(u), F_j(v))$$

is a copula satisfying $\int_{I^2} A d\lambda_2 = \frac{1}{4}$.

**Proof.** By considering all pertinent cases, it can be verified straightforwardly that $A$ is a copula. In fact, $A$ is called the uniform $\{A_{ij}\}_{i,j=1}^{2}$-patched copula. See [2, 24]. Next, let us denote $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. By appropriate linear changes of variables, the integrals of $A$ over each of the four squares $I_i \times I_j$, $i, j = 1, 2$, can be written in terms of the integrals of $A_{ij}$’s over $I^2$ as follows:

$$\int_{I_1 \times I_1} A d\lambda_2 = \frac{1}{16} \int_{I^2} A_{11} d\lambda_2, \quad \int_{I_1 \times I_2} A d\lambda_2 = \frac{1}{16} \int_{I^2} A_{12} d\lambda_2,$$

$$\int_{I_1 \times I_2} A d\lambda_2 = \frac{1}{32} + \frac{1}{16} \int_{I^2} A_{12} d\lambda_2, \quad \int_{I_2 \times I_2} A d\lambda_2 = \frac{1}{16} \int_{I^2} A_{22} d\lambda_2.$$

Summing the four integrals gives $\int_{I^2} A d\lambda_2 = \frac{3}{16} + \int_{I^2} A_{ij} d\lambda_2 = \frac{5}{16}$.

**Example 3.6.** Let $C$ be the copula whose mass is spread uniformly on the line segments shown in Figure 1. It follows that $C \times C^T$ and $C^T \times C$ are the uniform $\{A_{ij}\}_{i,j=1}^{2}$-patched copula where $A_{11} = A_{22} = E_0 \equiv \frac{M + W}{2}$ and $A_{12} = A_{21} = E_1$, the uniform $\{E_0\}_{i,j=1}^{2}$-patched copula. See Figure 1. The integral of each $A_{ij}$ is $\frac{1}{8}$. By Lemma 3.5, $\int_{I^2} C \times C^T d\lambda_2 = \int_{I^2} C^T \times C d\lambda_2 = \frac{4}{8}$ and hence $[C] = \frac{1}{2}$.

Observe that the uniform patched copula of four copies of the same copula with minimum $[\cdot]$ still has minimum $[\cdot]$. As an example, starting from $E_0 = \frac{M + W}{2}$, the iterative uniform patching gives a sequence $\{E_n\}$ for which $E_n$ is the uniform patched copula of four copies of $E_{n-1}$ and $[E_n] = \frac{1}{2}$.

**Proposition 3.7.** Recall that $F_1$ and $F_2$ denote the uniform distributions on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. If $D_0$ is an idempotent copula with $[D_0] = \frac{1}{2}$ then the uniform $\{D_0\}_{i,j=1}^{2}$-patched copula $D_1$ given by

$$D_1(u, v) = \frac{1}{4} \sum_{i,j=1} D_0(F_i(u), F_j(v))$$

Figure 1: The supports of $C$, $C^T$ and $C \times C^T$
is idempotent with \([D_1] = \frac{1}{2}\). Note that \(D_1\) may not be jointly symmetric if \(D_0\) is not.

**Proof.** By a straightforward but tedious computation, we obtain

\[
D_1 \ast D_1(u, v) = \frac{1}{4} \sum_{i,j=1}^{2} D_0 \ast D_0(F_i(u), F_j(v)) = \frac{1}{4} \sum_{i,j=1}^{2} D_0(F_i(u), F_j(v)) = D_1(u, v).
\]

So \(D_1\) is idempotent. By Lemma 3.5, \([D_1] = 2 \int_0^1 D_1 \, d\lambda_2 = \frac{1}{2}\).

Since \([C]\) is defined in terms of \(C \ast C \ast C \ast C\) and \(C \ast C \ast T\), let us investigate further some properties of the self-map \(\Psi\) on the class of copulas \(C\) defined by \(\Psi(C) = C \ast C \ast T\). The mapping \(C \mapsto C \ast C \ast T\) has the analogous properties.

**Proposition 3.8.** Let \(C\) be a copula.

1. \(\Psi\) is neither one-to-one nor onto.
2. \(\Psi(C)\) is symmetric and hence \(\|\Psi(C)\|_1 = \|\Psi(C)\|_2\).
3. \(\|\Psi(C)\|_1 = \frac{1}{4}\) if and only if \([C]_1 = \frac{1}{4}\).
4. \(\Psi\) is a continuous function from \(C\) endowed with the Sobolev norm into itself. That is, if a sequence of copulas \(\{C_n\}\) converges to a copula \(C\) in the Sobolev norm, then \(\{\Psi(C_n)\}\) converges to \(\Psi(C)\) in the Sobolev norm.

**Proof.** 2: This is clear as \((C \ast C \ast C) \ast T = C \ast T = C \ast C \ast T\).

1: Since all the left invertible copulas map to \(M\), \(\Psi\) is not one-to-one. \(\Psi\) is not onto because all the left invertible copulas \(S \neq M\) are not in the range of \(\Psi\). For if a left invertible copula \(S \neq M\) were of the form \(C \ast C\) for some copula \(C\), then \(C\) could not be left invertible but

\[
M = S \ast S = C \ast C \ast C \ast C,
\]

which means that \(C\) is left invertible, a contradiction.

3: \((\Leftarrow)\) Since \([C]_1 = \frac{1}{4}\), it follows from Theorem 3.3 (3.3) that \(\int_0^1 C(t, v) \, dv = \frac{1}{2}\) for all \(t \in I\). Then for all \(u \in I\), by (4),

\[
\int_0^1 C \ast C(u, v) \, dv = \int_0^1 \int_0^1 \partial_2 C(u, t) \partial_1 C(t, v) \, dt \, dv
\]

\[
= \int_0^1 \partial_2 C(u, t) \frac{d}{dt} \left( \int_0^1 C(t, v) \, dv \right) \, dt
\]

\[
= \int_0^1 \partial_2 C(u, t) \frac{d}{dt} \left( \frac{t}{2} \right) \, dt
\]

\[
= \frac{1}{2} \int_0^1 \partial_2 C(u, t) \, dt = \frac{1}{2} C(u, 1) = \frac{u}{2}.
\]

Again, by Theorem 3.3 (iv), \(\|\Psi(C)\|_1 = \left\|\frac{1}{4} C \ast C\right\|_1 = \frac{1}{8}\).

\((\Rightarrow)\) If \([\Psi(C)]_1 = \frac{1}{4}\), then by Theorem 3.3 (iv), \(\int_0^1 C \ast C \ast C(u, v) \, dv = \frac{2}{3}\) for all \(u\). So \([C]_1 = \int_0^1 \frac{u}{2} \, du = \frac{1}{4}\).

4: This follows from the fact that the \(\ast\)-product is jointly continuous with respect to the Sobolev norm. See Theorem 4.2 in [4].

We are now ready to define the first candidate for a measure of dependence \(v\). For all continuous random variables \(X\) and \(Y\), let

\[
v(X, Y) = \sqrt{6 \left[\|C_{X,Y}\| - 3\right]}.
\]
Theorem 3.9. The measure \( \nu \) satisfies the following properties:

\begin{itemize}
\item[v1.] \( \nu(X, Y) = \nu(Y, X) \).
\item[v2.] \( 0 \leq \nu(X, Y) \leq 1 \).
\item[v3.] \( \nu(X, Y) = 0 \) if \( X \) and \( Y \) are independent.
\item[v4.] \( \nu(X, Y) = 1 \) if and only if \( X \) and \( Y \) are mutually completely dependent.
\item[v5.] \( \nu(f(X), g(Y)) = \nu(X, Y) \) for all strictly monotonic transformations \( f \) and \( g \).
\end{itemize}

Proof. v1-v4 follow directly from the definitions of \( \cdot \) and \( \nu \) and Theorem 3.3. To prove v5, let \( C_{X,Y} \) be the copula of \( X \) and \( Y \) and consider the following four cases. If \( f \) and \( g \) are strictly increasing, then \( C_{f(X),g(Y)} = C_{X,Y} \) [13, Theorem 2.4.3]. So \( \nu(f(X), g(Y)) = \nu(X, Y) \). If \( f \) is strictly increasing and \( g \) is strictly decreasing, then \( C_{f(X),g(Y)}(x, y) = x - C_{X,Y}(x, 1 - y) = C_{X,Y} \ast W(x, y) \) [13, Theorem 2.4.4 and Example 6.7]. Consequently,

\[
C_{f(X),g(Y)} \ast C_{f(X),g(Y)} = C_{X,Y} \ast W \ast W^\top \ast C_{X,Y} = C_{X,Y} \ast C_{X,Y}^T \text{ and so } ||C_{f(X),g(Y)}||_2 = ||C_{X,Y}||_2.
\]

We also have

\[
||C_{f(X),g(Y)}||_1 = \int_0^1 \int_0^1 (C_{X,Y} \ast W)^T \ast (C_{X,Y} \ast W)(x, y) \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 W \ast C_{X,Y}^T \ast C_{X,Y} \ast W(x, y) \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 \left( x + y - 1 + C_{X,Y}^T \ast C_{X,Y}(1 - x, 1 - y) \right) \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 C_{X,Y}^T \ast C_{X,Y}(x, y) \, dx \, dy = ||C_{X,Y}||_1.
\]

Thus, \( ||C_{f(X),g(Y)}||_1 = ||C_{X,Y}||_1 \), i.e. \( \nu(f(X), g(Y)) = \nu(X, Y) \). The case where \( f \) is strictly decreasing and \( g \) is strictly increasing follows from the symmetry of \( \nu \). The last case when \( f \) and \( g \) are strictly decreasing can be proved using the fact that \( C_{f(X),g(Y)} = W \ast C_{X,Y} \ast W \).

\( \square \)

Note that the property v5 in Theorem 3.9 is not valid for all Borel measurable injections where we utilize the same counterexample as the one in page 109.

Example 3.10. Consider jointly normal random variables \( X \) and \( Y \) with correlation coefficient \( \rho \). Then \( \nu(X, Y) \) is a strictly increasing function of \( |\rho| \). Its graph obtained from a Matlab implementation is shown in Figure 2. Note the small difference between them whose graph is shown in Figure 3.

![Figure 2: \( \nu(X, Y) \) as a function of \( \rho \) for jointly normal \( X, Y \)](image)
we define where the supremum is taken over all Borel measurable injective transformations \( f \) and \( g \), respectively. More generally, \( C_{f(X),X} \ast C_{X,Y} = C_{f(X),Y} \) for any Borel measurable injection \( f \) and \( g \). See [17, Theorem 4.1]. Such a copula \( C_{f(X),X} \) is invertible with inverse \( C_{f(X),X}^{-1} = C_{X,f(X)}^{-1} \). In light of this observation, given continuous random variables \( X \) and \( Y \) with copula \( C_{X,Y} \), we define

\[
\left[ C_{X,Y} \right]_* = \sup_{f,g} \left[ C_{f(X),g(Y)} \right]_1 \quad \text{and} \quad \nu_*(X,Y) = \sqrt{6 \left[ C_{X,Y} \right]_* - 3},
\]

where the supremum is taken over all Borel measurable injective transformations \( f \) and \( g \). Using the facts that \( C_{f(X),g(Y)} = C_{f(X),X} \ast C_{X,Y} \ast C_{Y,g(Y)} \) (see [17, Corollary 4.6]) and that \( C_{f(X),X} \) and \( C_{Y,g(Y)} \) are invertible, we obtain

\[
\left[ C_{f(X),g(Y)} \right]_1 = \int_{p_2} C_{f(X),X}^T \ast C_{X,Y} \ast C_{f(X),X} \ast C_{X,Y} \ast C_{Y,g(Y)} \, d\lambda_2
\]

\[
= \int_{p_2} C_{Y,g(Y)}^T \ast C_{X,Y} \ast C_{X,Y} \ast C_{Y,g(Y)} \, d\lambda_2 = \left[ C_{X,Y} \ast C_{Y,g(Y)} \right]_1.
\]

Likewise, we can show that \( \left[ C_{f(X),g(Y)} \right]_* = \left[ C_{f(X),X} \ast C_{X,Y} \right]_* \). Therefore,

\[
\left[ C \right]_* = \sup_{S \in J} \left[ C \ast S \right]_1 + \sup_{S \in J} \left[ S \ast C \right]_2.
\]

The following theorem shows that \( \nu_* \) is a copula-based measure of dependence in the sense of A. Rényi [16], i.e. all R1-R5 are satisfied.

**Theorem 4.1.** The measure \( \nu_* \) satisfies the following properties:
v.1. \( \nu (X, Y) = \nu (Y, X) \).

v.2. \( 0 \leq \nu (X, Y) \leq 1 \).

v.3. \( \nu (X, Y) = 0 \) if and only if \( X \) and \( Y \) are independent.

v.4. \( \nu (X, Y) = 1 \) if \( X \) and \( Y \) are completely dependent.

v.5. \( \nu (f(X), g(Y)) = \nu (X, Y) \) for all Borel measurable injective transformations \( f \) and \( g \).

**Proof.** Properties v.1 and v.5 follow immediately from the definitions of \([\cdot] \) and \([\cdot, \cdot] \). The bounds of \([\cdot] \) in Theorem 3.3 give v.2.

In order to prove v.4, we will show only that \( \nu (X, Y) = 1 \) if \( Y \) is a Borel measurable function of \( X \) as the other case is similar. This is equivalent to proving that \([C]_* = \frac{1}{2} \) when \( C \) is left invertible. Suppose a copula \( C \) is left invertible. Then \( C^T \ast C = M \) and \( [C \ast S]_1 = \int_{\mathbb{R}} S^T \ast C \ast S \, d\lambda_2 = \int_{\mathbb{R}} M \, d\lambda_2 = \frac{1}{2} \) for every invertible copula \( S \). For \( \sup_{S \in \mathbb{S}} [S \ast C]_{2} \), let us start from the fact, see [5], that \( E = C \ast C^T \) is an idempotent copula whose invariant sets form a nonatomic \( \sigma \)-algebra \( \mathcal{E} \subseteq \mathcal{B}(I) \). As a consequence, by Corollary 1.12.10 in [1], for every integer \( n = 1, 2, \ldots \), there exists an invariant partition consisting of sets \( P_1, \ldots, P_n \) in \( \mathcal{E} \) in the sense that

\[
\lambda \left( \bigcup_{i=1}^{n} P_i \right) = 1, \quad \lambda(P_i) = \frac{1}{n} \text{ for all } i \text{ and } \lambda(P_i \cap P_j) = 0 \text{ for } i \neq j.
\]

Since each \( P_i \) is an invariant set of \( E \), we have

\[
\mu_{E}([a, b] \times P_i) = \int_{a}^{b} T_{E} x_{P_{i}} \, d\lambda = \int_{a}^{b} \chi_{P_{i}} \, d\lambda = \lambda([a, b] \cap P_i),
\]

where the first equality follows from a standard measure theoretic argument starting from \( P_i \) being intervals. By the symmetry of the idempotent \( E \), we get \( \mu_{E}(P_i \times [a, b]) = \lambda([a, b] \cap P_i) \). So \( E \) has zero mass in \( (I \setminus P_i) \times P_i \) or \( P_i \times (I \setminus P_i) \) for \( i = 1, \ldots, n \).

We will then use \( \{P_i\} \) to construct an invertible copula \( S_n \) for which \( S_n \ast E \ast S_n^T \) is supported in \( \bigcup_{i=1}^{n} I_i^2 \), where \( I_i = \left[ \frac{i-1}{n}, \frac{i}{n} \right] \). In fact, by Lemma 4,2 in [5], there exist measure preserving Borel functions \( h_i : P_i \rightarrow I_i \) and measure preserving Borel functions \( g_i : I_i \rightarrow P_i \) such that \( h_i \circ g_i(x) = x \) and \( g_i \circ h_i(x) = x \) for almost every \( x \). The functions \( g_i \) and \( h_i \) are called essential inverses of each other. Now, define \( R_n, T_n : I \rightarrow I \) by

\[
R_n(x) = g_i(x) \quad \text{if } x \in I_i \quad \text{and} \quad T_n(x) = h_i(x) \quad \text{if } x \in P_i.
\]

Since \( \{I_i\} \) and \( \{P_i\} \) are essential partitions of \( I \), the self-maps \( R_n \) and \( T_n \) are measure preserving. Moreover, \( R_n \circ T_n(x) = x \) and \( T_n \circ R_n(x) = x \) for almost all \( x \in I \). Finally, let \( S_n \) be the copula of \( U \) and \( R_n(U) \) where \( U \) is a uniform \([0, 1]\) random variable. Equivalently, \( S_n = C_{e,R_n} = C_{T_n,e} \) where \( e \) denotes the identity map and \( C_{k,e}(x, y) = \lambda(k^{-1}([0, x]) \cap \mathcal{E}^{-1}([0, y])) \) for \( x, y \in I \). Hence, \( S_n \) is invertible. Recall that the left multiplication of \( E \) by \( S_n \) amounts to shuffling (moving) masses of \( E \) on the vertical stripes \( P_i \times I \) to \( I_i \times I \) for \( i = 1, \ldots, n \). The right multiplication by \( S_n^T \) shuffles the horizontal stripes. From the fact shown above that \( E \) has no mass in \( \bigcup_{i=1}^{n} I_i^2 \) or \( \bigcup_{i=1}^{n} P_i \times (I \setminus P_i) \), \( S_n \ast E \ast S_n^T \) is supported in \( \bigcup_{i=1}^{n} I_i^2 \). So \( S_n \ast E \ast S_n^T \) converges to \( M \) pointwise and thus \( \sup_{S \in \mathbb{S}} [S \ast C]_{2} = \frac{1}{2} \).

Clearly, \( \mathcal{I}_n = \mathcal{I}_2 = \frac{1}{2} \). To prove the opposite direction of v.3, or equivalently that \( [\cdot]_* = \frac{1}{2} \) implies that \( C = \mathcal{I} \), we define \( S_{\alpha,\beta} \), for \( 0 \leq \alpha \leq \beta \leq 1 \), as the shuffle of \( \mathcal{M} \) whose support consists of at most three lines

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1 A measurable set \( S \) is called an invariant set of a copula \( A \) if the characteristic function \( \chi_{S} \) is a fixed point of the Markov operator \( T_A : L^{\infty}(I) \rightarrow L^{\infty}(I) \) defined by \( T_A \psi(x) = \frac{d}{dx} \int_{I} \frac{dA}{dt}(x, t) \psi(t) \, dt \). It also holds that \( A(x, y) = \int_{0}^{y} T_A(x, t) \, d\lambda \). A \( \sigma \)-algebra \( \mathcal{E} \) is said to be nonatomic if for every \( S \in \mathcal{E} \) there exists a subset \( S' \) of \( S \) in \( \mathcal{E} \) such that \( 0 < \lambda(S') < \lambda(S) \).
segments of slope 1 shown in Figure 4, i.e.

\[
S_{a,\beta}(x, y) = \begin{cases} 
0 & \text{if } 0 \leq x \leq a, 0 \leq y \leq \beta - a, \\
\min(x, y - (\beta - a)) & \text{if } 0 \leq x \leq a, \beta - a < y \leq \beta, \\
\min(x - a, y) & \text{if } a < x \leq \beta, 0 \leq y \leq \beta - a, \\
x + y - \beta & \text{if } a < x \leq \beta, \beta - a < y \leq \beta, \\
\min(x, y) & \text{otherwise.}
\end{cases}
\]

For every copula \(C\) and \(0 \leq a \leq \beta \leq 1\), a direct computation gives

\[
S_{a,\beta} \ast C(x, y) = \begin{cases} 
C(x + \beta - a, y) - C(\beta - a, y) & \text{if } 0 \leq x \leq a, \\
C(x - a, y) + C(\beta, y) - C(\beta - a, y) & \text{if } a < x \leq \beta, \\
C(x, y) & \text{if } \beta < x \leq 1.
\end{cases}
\]

Integrating \(S_{a,\beta} \ast C\) with respect to \(x\) and making suitable changes of variables yield

\[
\int_0^1 S_{a,\beta} \ast C(x, y) \, dx = \int_0^1 C(x, y) \, dx + (\beta - a) C(\beta, y) - \beta C(\beta - a, y) = \frac{\beta - a}{2} C(\beta, y) - \beta C(\beta - a, y). \tag{7}
\]

Then \([C]\). = 0 implies that both \(\sup_{S \in \mathcal{I}} [C \ast S]_1\) and \(\sup_{S \in \mathcal{I}} [S \ast C]_2\) attain the minimum value \(\frac{1}{4}\). So \([C \ast S]_1 = \frac{1}{4} = [S \ast C]_2\) for all \(S \in \mathcal{I}\). Let \(n \in \mathbb{N}, i \in \{0, 1, 2, \ldots, 2^n - 1\}\) and \(y \in [0, 1]\). Applying Theorem 3.3 (v) to \(C\) and \(S_{\frac{i}{2^n}, \frac{i+1}{2^n}} \ast C\) yields

\[
\int_0^1 C(x, y) \, dx = \frac{y}{2} \quad \text{and} \quad \int_0^1 S_{\frac{i}{2^n}, \frac{i+1}{2^n}} \ast C(x, y) \, dx = \frac{y}{2}.
\]

By (7), \(\frac{i+1}{2^n} C \left(\frac{i}{2^n}, y\right) = \frac{i}{2^n} C \left(\frac{i+1}{2^n}, y\right)\). Since \(i\) is arbitrary, repeated use of this equation gives

\[
C \left(\frac{i}{2^n}, y\right) = \frac{i}{i+1} C \left(\frac{i+1}{2^n}, y\right) = \cdots = \frac{i}{2^n} C (1, y) = \frac{i}{2^n} y.
\]

By the continuity of \(C\) and the denseness of the dyadic rationals in \([0, 1]\), we have \(C = II\) as desired. \(\square\)

Remark. We then give an example to demonstrate that the converse of \(v.4\) is not true. By the proof of \(v.4\) above, any nonatomic idempotent copula gives maximum \([\cdot]\). However, in order to illustrate the shuffling in the proof, let us consider \(E_0 = \frac{M + W}{2}\) which is neither left nor right invertible. Equivalently, if \(E_0\) is the copula...
of $X$ and $Y$ then they cannot be completely dependent. We will show that $\{E_0\}_* = \frac{2}{3}$ and so $\nu(X, Y) = 1$. Since $E_0$ is idempotent and hence symmetric, for every $S \in \mathcal{J}$

$$\left[ E_0 \ast S^T \right]_1 = [S \ast E_0]_2 = \int_{\mu} (S \ast E_0 \ast E_0 \ast S^T) \, d\lambda_2 = \int_{\mu} (S \ast E_0 \ast S^T) \, d\lambda_2.$$ 

So it suffices to show that $\sup_{S \in \mathcal{J}} \int_{\mu} (S \ast E_0 \ast S^T) \, d\lambda_2 = \frac{1}{4}$. We shall acquire this by constructing a sequence $\{S_1, S_2, \ldots\}$ of shuffles of Min such that $\int_{\mu} (S_n \ast E_0 \ast S_n^T) \, d\lambda_2$ converges to $\frac{1}{4}$.

Henceforth, for a copula $A$, let us denote by $\text{ord}_n(A)$ the ordinal sum of $n$ copies of $A$ with respect to the partition $\{[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, 1]\}$ of $[0, 1]$. Observe that $\text{ord}_m(\text{ord}_n(A)) = \text{ord}_{mn}(A)$ for any $m, n \in \mathbb{N}$.

![Figure 5: The supports of $E_0$, $S_1 \ast E_0$ and $E_1 \equiv S_1 \ast E_0 \ast S_1^T$ respectively](image)

Denote $S_1 \equiv S_{\frac{1}{2},1}$ which is the shuffle of Min supported on the main diagonals of the squares $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ as shown in Figure 4. Our proof hinges on an observation that the product $S_1 \ast E_0$ is the result of (horizontally) shuffling the mass of $E_0$ on the rectangles $[0, \frac{1}{2}] \times I$ and $[\frac{1}{2}, 1] \times I$ that $S_1 \ast E_0 \ast S_1^T$ can be obtained by (vertically) shuffling the mass of $S_1 \ast E_0$ on the rectangles $I \times [0, \frac{1}{2}]$ and $I \times [\frac{1}{2}, 1]$. Therefore, $E_1 \equiv S_1 \ast E_0 \ast S_1^T$ is equal to the ordinal sum of $(E_0, E_0)$ with respect to $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, i.e. $E_1 = \text{ord}_2(E_0)$. Their supports are shown in Figure 5. For more details on shuffles of copulas, see [7, 11], specifically [17] for their relations with the $*$-product used here. This observation can be made more rigorous but probably less transparent by using the conditional probability to decompose $E_0$ according to the partition $\{[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, 1]\}$ of $[0, 1]$ on both axes. Both horizontal and vertical shuffles of this so-called patched decomposition, introduced and studied in [2, 24], can then be conveniently manipulated. For details, see [9].

![Figure 6: The supports of $E_0$, $E_1$, $E_2$ and $E_3$ respectively](image)

For each $n \geq 2$, we put $E_n = S_n \ast E_0 \ast S_n^T$ where

$$S_n = \text{ord}_{2^{n-1}}(S_1) \ast \text{ord}_{2^{n-2}}(S_1) \ast \cdots \ast \text{ord}_2(S_1) \ast S_1.$$
It follows that $S_n$ is a shuffle of Min, $S_n = \text{ord}_{2^n-1}(S_1) * S_{n-1}$ and

$$E_n = \text{ord}_{2^n-1}(S_1) * E_{n-1} * \text{ord}_{2^n-1}(S_1)^T.$$  \hfill (8)

Using the recursive relation (8), it can be shown by induction on $n$ that $E_n = \text{ord}_{2^n}(E_0)$. The first few $E_n$’s are illustrated in Figure 6. Since the value of this ordinal sum agrees with $M$ except possibly on the union $\bigcup_{i=1}^{2^n}[\frac{i-\frac{1}{2}}{2^n}, \frac{i+\frac{1}{2}}{2^n}]$ whose area is $\frac{1}{2^n} \to 0$, $\int_{I_n} S_n * E_0 * S_n^T d\lambda_2$ converges to $\int_{I_n} M d\lambda_2 = \frac{1}{2}$.

Owing to one of the anonymous referees, we are pleased to propose a class of measures of mutual complete dependence $\nu_a$ defined as a nontrivial convex combination $\nu_a = a\nu + (1-a)\nu_*$ for $0 < a < 1$.

**Theorem 4.2.** The measure $\nu_a$ satisfies the following properties:

1. $\nu_a(X, Y) = \nu_a(Y, X)$.
2. $0 \leq \nu_a(X, Y) \leq 1$.
3. $\nu_a(X, Y) = 0$ if and only if $X$ and $Y$ are independent.
4. $\nu_a(Y, X) = 1$ if and only if $X$ and $Y$ are mutually completely dependent.
5. $\nu_a(f(X), g(Y)) = \nu_a(X, Y)$ for all strictly monotonic transformations $f$ and $g$.

**Proof.** The proof uses corresponding properties of $\nu$ and $\nu_*$ in Theorems 3.9 and 4.1. $\nu_a 1$ and $\nu_a 2$ clearly follow from the same properties of $\nu$ and $\nu_*$.

If $X$ and $Y$ are independent, then $\nu(X, Y) = \nu_*(X, Y) = 0$ and hence $\nu_a(X, Y) = 0$. Conversely, if $\nu_a(X, Y) = 0$ then $\nu_*(X, Y)$ must be zero and so $X, Y$ are independent.

If $X, Y$ are mutually completely dependent, then $\nu(X, Y) = \nu_*(X, Y) = 1$ and hence $\nu_a(X, Y) = 1$. Conversely, if $\nu_a(X, Y) = 1$ then $\nu(X, Y)$ must be one and so $X, Y$ are mutually completely dependent.

$\nu_a 5$ is a result of $\nu 5$ and $\nu_* 5$. \hfill $\Box$

## 5 Conclusion

We show that $\|C^T \ast C + C \ast C^T\|_1$ gives rise to a [0, 1]-valued function $\nu$ of continuous random variables which is almost a measure of mutual complete dependence as it cannot identify independence. We then prove that the measure $\nu_*$, modified from $\nu$ in such a way that it is invariant under all one-to-one transformations, satisfies the five essential properties in Rényi’s postulates for measures of dependence. Finally, every nontrivial convex combination of $\nu$ and $\nu_*$ is a measure of mutual complete dependence.

**Acknowledgement:** We sincerely thank Wayne Lawton for his insight and expertise as the initial idea of this research occurred in a discussion with him in 2012 and Pongpol Ruankong for several insightful suggestions. The first author would like to express his gratitude toward the Development and Promotion of Science and Technology Talents Project for the support during his undergraduate and Master’s study. The last author is partially supported by the Commission on Higher Education and the Thailand Research Fund for the support through grant no. RSA5680037. Lastly, the authors extend our sincere thanks to two anonymous referees whose comments and suggestions improved the manuscript considerably.

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