A New Method of 3-D Magnetic Field Reconstruction

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Abstract A method is described to model the magnetic field in the vicinity of three-dimensional constellations of satellites (at least four) using field and plasma current measurements. This quadratic model matches the measured values of the magnetic field and its curl (current) at each spacecraft, with V • B zero everywhere, and thus extends the linear curlometer method to second order. Near the spacecraft, it predicts the topology of magnetic structures, such as reconnecting regions or flux ropes, and allows a tracking of the motion of these structures relative to the spacecraft constellation. Comparisons to particle-in-cell simulations estimate the model accuracy. Reconstruction of two electron diffusion regions definitively confirms the expected field line structure. The model can be applied to other small-scale phenomena (e.g., bow shocks) and can also be modified to reconstruct the electric field, allowing tracing of particle trajectories.

1. Introduction

Measurements and models of the magnetic field are commonly studied in the extensive space physics literature. The magnetic field B is a predominant reservoir of energy available for acceleration of particles. Particle trajectories and energization processes are greatly influenced by the magnetic field and its topology. For these reasons, magnetometers are one of the most common instruments in ground observatories and space missions (see Kivelson & Russell, 1995). Over the past several decades, multiple-satellite mission designs (e.g., the International Sun-Earth Explorers, Cluster, and Magnetospheric Multiscale (MMS)) have been employed since they allow for approximate determination of the topologies of magnetic boundaries, which facilitates more complete analyses of related plasma phenomena.

The latest of these missions, MMS (Burch et al., 2015), has targeted magnetic reconnection in boundary regions of the magnetosphere. The magnetic topology plays a key role in energy conversion, fast flow acceleration, and energetic particle production, which are characteristic of reconnection. Reconstructions of B and its streamlines not only provide a more complete picture of fundamental features of reconnection regions but also allow simple recognition of where the spacecraft are located and how they are moving relative to magnetic structures.

There have been many approaches to reconstructing B: Grad-Shafranov techniques introduced by Sonnerup and Guo (1996), with the addition of flow parameters by Sonnerup and Teh (2008); using constraints imposed by magnetohydrodynamics (Sonnerup & Teh, 2009); and reconstructions under some simplifying topologies and the constraints of electron magnetohydrodynamics (Sonnerup et al., 2016). Using a linear approximation of the VB matrix (Dunlop et al., 1988), Shi et al. (2005) developed a method for estimating the motion of magnetic structures relative to spacecraft, and Denton et al. (2016) have refined this technique for the MMS tetrahedron configuration. However, an inaccuracy of this method is that the VB (matrix) derived from four separated points has no constraint that the trace be zero (i.e., V • B≠0) and the components corresponding to the curl are those of an average assumed for the barycenter of the spacecraft: that is, the current density within the spacecraft tetrahedron is assumed to be uniform.

One of the remarkable advances of the MMS mission is the very high fidelity of the current measurements using only particle data (Pollock et al., 2016 et al., 2014; Phan et al., 2016). The configuration of eight spectrometers per spacecraft that simultaneously measure flows in opposing directions for electrons every 30 ms and ions (150 ms) is a significant asset for this success. Furthermore, the magnetometers (Russell et al., 2014), assisted with independent measurements of the magnitude by the Electron Drift Instrument.
have provided one of the most accurate measurements of \( B \) ever acquired by high-altitude spacecraft, with an accuracy of \( \leq 0.1 \) nT. Using a “modified” curlometer, which employs both time and spatial variations of \( B \) to estimate current, Torbert et al. (2017), see Figure 1) showed that the particle data matched the magnetic variations at the highest cadence available on MMS within an electron diffusion region (EDR), where the current is far from uniform.

In this letter, we propose a new method where we use these accurate measurements at each spacecraft to produce a local, basically quadratic model of \( B \) that exactly matches the measurements of \( B \) and \( \nabla \times B \) and has zero divergence everywhere. This technique extends the linear curlometer method to second order and will allow better estimates of both local field line topology and the motion of magnetic structures by the spacecraft. The examples shown here will provide visual evidence that MMS has encountered reconnection electron diffusion regions.

2. Model Description

One of Helmholtz’s theorems states that a three-dimensional, continuously differential vector field is uniquely specified within an enclosed volume by (1) its curl, (2) its divergence, and (3) the normal component over the boundary (Arfken, 1985). The divergence of \( B \) is zero everywhere. Given that the displacement current is negligible for the time scales modeled here, the curl is the current (times \( \mu_0 \)) measured by the particle instrumentation on the four MMS satellites, but of course only at four vertices of the constellation tetrahedron, where the values of \( B \) are also determined. Thus, given that the curl and normal component are not fully specified, the complete vector field obviously cannot be modeled. However, if the spatial variation is restricted, a model can be reconstructed that is the simplest (most slowly varying) possible one that is consistent with the data. There are several approaches to this problem (interpolations along the boundary and within the volume, using Helmholtz constructions of the field), but the most straightforward is to Taylor expand the field at some convenient point within the MMS tetrahedron (here taken as the barycenter),
and truncate the series when there is a sufficient number of coefficients. The result will be nearly a quadratic expansion (by "nearly," see below) for each component (j):

$$B_j = B_{0j} + \sum_k x_k (\partial_k B_j)_0 + \frac{8}{3} \sum_{k \leq l} x_k x_l (\partial_k \partial_l B_j)_0$$

(1)

where each of the model coefficients $[B_{0j}, (\partial_k B_j)_0, (\partial_k \partial_l B_j)_0]$ are referenced to the expansion origin(0), and the $x_k$ are the components of the position of the field point referenced to that location. Without the quadratic term, such an expansion will replicate the normal curlometer method (Dunlop et al., 1988) when its 12 free parameters (three $B_{0j}$ and nine $(\partial_k B_j)_0$) are determined using the $3 \times 4$ measured components of $B$. Extending this expansion to the quadratic term allows us to model the magnetic field with our knowledge of the current at each spacecraft.

Given that the quadratic coefficient is symmetric in the k-l indices, there are 3(components) x (1 + 3 + 6 coefficients), or 30 unknowns in this expansion. However, the divergence-free requirement on $B$ implies that the trace of the linear term is zero and the gradient of the trace is also zero (constraining the quadratic coefficients), and therefore, four of these unknowns are determined, reducing the number to 26. We have four spacecraft observations of both $B$ and $J$, providing 24 elements of data. We thus need additional constraints on the expansion. We obtain these by using a minimum variance analysis (Sonnerup & Scheible, 1998) to produce a local LMN coordinate system (where M is the minimum variance direction and N is the maximum variance direction) and require that the three $\delta_M \delta_M B_j$ terms be zero, since there is little variation in this direction. Given that there are now 23 parameters for 24 measurements, the problem would seem to be overconstrained. However, since the model computes $J$ from the curl of $B$, it automatically delivers a divergence-free current, whereas the measured $J$ will not be so: not only because (1) there are errors in the current measurement itself, but also because (2) the current is measured at four separated points, and there is no requirement that the linear approximation of the gradient tensor of $J$, using these separated points, be traceless. Thus, if the data could be constrained so that the trace of $\mathbf{v} \mathbf{J}$ is zero, there would be 23 data values for 23 parameters, resulting in a unique solution.

It is tempting to resolve this issue, the nonzero divergence of $J$ data, by devising some method of adjusting that data itself, but analysis within regions of strong current usually shows a strong spatial and nonlinear variation such that, in fact, where the authors want to model the changing $B$ field (in reconnection diffusion regions), reason (2) above dominates: the errors are less than the variation over the tetrahedron, as is clearly seen in Figure 1. Another possibility is to use fewer expansion parameters and do a least squares fit to the data to determine a less varying model, which, however, does not match the measurements at each satellite (R. Denton, private communication). This approach may produce a better model extrapolated farther from the spacecraft, because some of the quadratic terms (especially cross-terms in the M-derivatives) will give spurious results at large distances. However, this letter describes the procedure to produce a higher-fidelity fit within, or very close to, the tetrahedron itself, for the future purposes of modeling particle trajectories within and near the tetrahedron, as described below.

In this case, to produce a model where the variation in $J$ is not completely linear, an additional cubic term is required to produce a fit, given the 24 independent measurements of $B$ and $J$. If we impose requirements that (1) no terms may have more than a linear dependence in the M-direction, consistent with the approach above; (2) only a single cubic term be added, to make the simplest and most slowly varying addition to the expansion that results in a unique solution; and (3) the divergence of the field be constant (in the case of $B$, namely, zero), then, given the symmetry in partial derivatives, careful examination of all the combinations shows that there are only eight possibilities to add a single additional cubic term, $(\partial_i \partial_j \partial_k B_j)_0$, to equation (1) above: namely, [ikl] = [1132], [1123], [3321], [1332], [1113], [3331], [3332], or [1112], where 123 = NML in the above coordinate system. In principle, any one of these will give an exact fit to the data. However, since our objective is to find the smoothest (least varying) model fit over the tetrahedron, the coefficients with the largest scaling length (favoring the smallest cubic coefficient) are preferred. In practice, a solution is obtained for each of the eight cubic terms, and a weighted average of all of the solutions with scale lengths within a factor of 4 of the maximum scale length is computed as the final model. The final result usually involves two to three of the possible solutions and allows a continuous time evolution of the field. In principle, a solution can be obtained for every time step where there is a reliable current measurement.
(on MMS, every 30 ms, usually). Since the result is a linear combination of exact solutions, the final model also has $\nabla \cdot \mathbf{B} = 0$ everywhere; it matches the observed $\mathbf{B}$ and $\mathbf{J}$ at each spacecraft, and varies spatially as slowly as possible, and very nearly quadratically throughout the tetrahedron. We call this solution the “25-parameter” fit: 23 from equation (1), one cubic term, plus a 25th parameter which is a constant divergence. For $\mathbf{B}$, this parameter is identically zero. However, as described below, there are mathematical reasons to retain this quantity as a constant parameter. The matrix that results, and which must be inverted to obtain the 25 parameters, is given in S1.

Since the model is basically only quadratic, it is critical, before solving for coefficients, to average the data to the appropriate time scale, corresponding to an appropriate spatial scale with an assumption about the average speed of structures past the spacecraft. Clearly waves, for example, with wavelengths much smaller than the spacecraft separation, cannot be replicated in a model that uses data at four separated points. In the examples below, timing methods were used to estimate structure velocity, and then the data were averaged on a time scale corresponding to a spatial scale of about half the tetrahedron size, providing an approximate three-point fit to a quadratic.

3. Comparisons With Model Simulations

As an initial test of the procedure, in the absence of a known field throughout a real tetrahedron, we have compared our model results with those obtained in particle-in-cell (PIC) simulations of reconnection. Data on the magnetic field and currents from the simulation of Nakamura et al. (2018) were obtained...
along four tracks, with separation corresponding to those of MMS on 11 July 2017 as reported by Torbert et al. (2018). The data were averaged over time as described above. This simulation was 2.5D; thus, the PIC data are constant in the M direction. However, the reconstruction algorithm was not informed or adjusted for this. Nevertheless, the resulting coefficients all showed negligible values for M-derivatives.

Plots of the comparison between simulation data and reconstruction in Figure 2 show excellent agreement in both the magnitude of the field (Figure 2b) and the direction (Figure 2d) within a volume about twice that of the tetrahedron. Figure 2c, showing the BN component, clearly shows the effect of quadratic terms when farther out from the center of the spacecraft constellation.

4. Two Example EDRs

Two of many cases that have been reconstructed with this method are the dayside asymmetric reconnection event of 16 October 2015 (Burch et al., 2016) and the magnetotail symmetric reconnection event of 11 July 2017 used above (Torbert et al., 2018). Figure 3a shows the results of the three-dimensional (3-D) reconstruction when MMS was very near an EDR on 16 October 2015 at the magnetopause. The field vectors are computed every 30 ms on a cubic grid with spacing of 2 km, and the view is through this cubic lattice along the M-direction of an LMN coordinate system given in Burch et al. (2016). The field lines are 3D but projected into the L-N plane. The four MMS spacecraft locations are color coded by the pattern given in top right. (b, top right) Approximately 4 s of data taken during this encounter on the four MMS spacecraft, from the top: Bx, Ey, EN, E_parallel-M (the reconnecting current component), J_x, J_E, and J_M, indicating that most of the energy is converted from the perpendicular components; the Bx change marks the approach of the EDR at 13:07:02.2 s. (bottom right) Electron phase space density (V_{par}, parallel to B, horizontal axis; V_{perp} in the E_perp direction) from MMS4 and MMS3 at the indicated times, showing electrons streaming left (southward) on MMS4 and right (northward) on MMS3 consistent with their locations in the reconstruction.
The advantage of the quadratic reconstruction is apparent in Figure 3a: even though all four measurements of $B$ at this time are in the same direction (see top panel of Figure 3b), the measured current ($\nabla \times B$) demands that the field reverse just beyond the tetrahedron (at larger N position) and that an X-line lies near MMS4. The reconstruction of this and other EDRs gives a definitive visual confirmation of the topological changes implied by reconnection.

The second example (Figure 4) uses data from the 11 July 2017 encounter with an EDR in the magnetotail (Torbert et al., 2018). The authors there showed that the MMS constellation traversed an EDR in the earthward and (meandering) northward directions while remaining near the neutral plane ($B_L \approx 0$). Figure 4a shows six seconds of magnetic, electric, and current field data consistent with that interpretation. In Figure 4b, the constellation is seen earthward and southward of the X-line, not yet fully within the earthward electron exhaust at the neutral plane, indicated by the colored arrows which are the L-N projection of the electron bulk flow. There are flow arrows in (b), but so small as to be unnoticeable. Some of the field lines terminate as they exit the reconstruction cubic lattice in the M direction before they reach the LN boundary.
the outflow jet and creates the “Hall” magnetic field (Sonnerup, 1979). The reconstruction software developed for this method allows 3-D visualizations of these field lines where such effects are more readily seen. In a movie of the entire six seconds of Figure 4, the motion, sometimes back-and-forth, of the field line structure can be seen. In this movie, sometimes quite convoluted topologies are seen, but it should be emphasized that each solution is the simplest one consistent with the data: they could be, and probably are, often even more complicated!

5. Conclusions and Future Work

The reconstructions described here are done over time intervals of typically 0.1 s, consistent with the averaging procedure above, but can in principle be done on at the fastest cadence for the current: on MMS, that is 7.5 ms. The averaging, however, reduces the statistical fluctuations in the current measurement. Besides the comparison with simulations above, we have assessed the sensitivity of the generated topology to current errors (see S3): errors of 5–10% in the current have only a small effect owing to the fact that the field itself is fixed at each spacecraft and the errors in $\mathbf{B}$ are very small for MMS (~0.1 nT). As is seen before and after the EDR in Movie S1 (where there is very little current and this percentage error is not unexpected), the field lines are very regular and this percentage error has little effect. Where the currents are large, within EDRs such as seen between 12:59:14.1 and 14.2 s in Figure 1, this error is well within the capabilities of the FPI instrument. Work is continuing to analyze how robust the model is to these errors, particularly to assess whether spurious features appear if the differences in spacecraft currents are less than the errors. Future work by R. Denton (private communication) is also exploring the use of least squares fitting to this model, but with fewer parameters which may result in models of the topology being more valid at larger distances from the tetrahedron.

The reconstructions provide important confirmation of the presence of EDRs near the MMS spacecraft. Besides magnetic topological contributions, there were two other motivations for this model. The first follows from another Helmholtz theorem that states that the $\mathbf{VB}$ matrix at any point can be decomposed into three parts: the divergence (trace), the curl (antisymmetric part), and a third traceless symmetric part. This last part encodes the values of the normal component of a surrounding volume (consistent with the previously cited theorem). We know the first part, and measure directly the second part at each spacecraft; the model provides an estimate of the third consistent with the observations of $\mathbf{B}$ and $\mathbf{J}$ at the other three spacecraft. Now with a matrix valid at each spacecraft, a modification of the Shi et al. (2005) method may produce a more reliable estimate of the motion of structures.

The second additional motivation was to model particle trajectories in EDRs. The gyroradius of the electrons of relevant energies are of the order of, or usually larger, than the spatial variations, and, of course, the flow is not frozen-in, so that the field lines by themselves do not indicate plasma motion. However, MMS has successfully flown an accurate 3-D electric field ($\mathbf{E}$) measurement, again calibrated with Electron Drift Instrument (Torbert et al., 2015; Lindqvist et al., 2016; Ergun et al., 2016). For this field, we know the values of $\mathbf{E}$ and its curl ($\nabla \times \mathbf{E}$) and the measurements of $\mathbf{E}$ are sufficiently accurate to estimate the divergence using the linear curlometer technique. This divergence is certainly not constant, but comparison to simulations shows that it is varying over a scale not much smaller than the spacecraft separation in some EDRs.

Using the same 25-parameter fit with these $\mathbf{E}$ data, the model technique then produces a self-consistent solution of quasi-static Maxwell’s equations (no displacement current) for $\mathbf{B}$ and $\mathbf{E}$ around the tetrahedron. Although the assumption of constant divergence limits the spatial applicability of the $\mathbf{E}$ solution, initial work shows that the model has promise for understanding the acceleration of electrons around the EDR and also results in a robust and self-consistent calculation of the terms of Poynting’s theorem (electromagnetic energy flux and $\mathbf{J} \cdot \mathbf{E}$) for studies in the vicinity of the tetrahedron.

Use of these many aspects of this quadratic model for fields around the MMS tetrahedron promises to guide the interpretation of the motion of structures past the spacecraft constellation and further our understanding of acceleration in the very complicated dynamics of reconnection. In addition, the model can be used in the same manner for many other space physics phenomena where the scale sizes are appropriate for the expansion, such as bow shock encounters, and with plasma waves whose wavelengths are comparable to the spacecraft separation.
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