The Number of Different Effective Partitions and a Specific Goldbach Partition of Any Given Even Number Greater Than 6

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Abstract
Other than any odd prime whose factor is contained by the given even number $2N_m$ greater than 6, total $m$ ($m \geq 2$) odd primes within open interval $(1, 2N_m - 1)$ were defined as effective primes $p_{(i, m)}$ of $2N_m$, where $i = 1, 2, \ldots, m - 1, m$, and $p_{(1, m)}$ and $p_{(m, m)}$ is the minimum and maximum one among $p_{(i, m)}$, respectively.

Let Eq.s $\alpha_{(i, m)}$ & Eq.s $\beta_{(i, m)}$ represent $2m$ simultaneous equations $p_{(i, m)} + \alpha_{(i, m)} = 2N_m$ and $p_{(1, m)}p_{(i, m)} + \beta_{(i, m)} = 2N_m$, respectively.

By using two-part method starting from $m = 2$, it was verified that whatever $m$ is, the number of different effective partition(s) of $2N_m$ among $2N_m$ simultaneous Eq.s $\alpha_{(i, m)}$ & Eq.s $\beta_{(i, m)}$ is always less than $m$, and consequently any given even number $2N_m$ greater than 6 must be divided by $p_{(g, m)}$ into a Goldbach partition of $2N_m$, where $p_{(g, m)}$ represents the greatest one among all the effective primes of $2N_m$ whose factors are contained by $\alpha_{(i, m)}$.

A specific Goldbach partition of any given even number greater than 6 can be found definitely.

Text
Enlightened by Ben Green and Terence Tao\(^{(1)}\), this manuscript is to establish a way to definitely find a specific Goldbach partition of any given even number greater than 6.
§1. Basic concepts

I-1. Goldbach partition of even number $2N$ greater than 6
When a given even number $2N$ greater than 6 can be partitioned to two odd primes, such partition is called a Goldbach partition of $2N$.

I-2. Effective primes/products/integers/partitions of the given even number $2N_m$
Other than any odd prime whose factor is contained by the given even number $2N_m$ greater than 6, total $m$ odd primes within open interval $(1, 2N_m - 1)$ were defined as effective primes $p(i, m)$ of $2N_m$ ordered by magnitudes as $1 < p(1, m) < p(2, m) < \ldots < p(m - 1, m) < p(m, m) < 2N_m - 1$.

Within open interval $(1, 2N_m - 1)$, products which contain and only contain the effective prime factor(s) of $2N_m$ were defined as effective products of $2N_m$.

Effective primes and effective products of $2N_m$ were commonly named as effective integers of $2N_m$. It is obvious that every effective integer of $2N_m$ is within open interval $(1, 2N_m - 1)$.

The partition of $2N_m$ which consists of two effective integers of $2N_m$ was defined as an effective partition of $2N_m$.

I-3. The number of effective primes of any given even number $2N_m$ greater than 6 is always greater than 1
Any given even number $2N_m$ greater than 6 is corresponding to a definite number $m$ of effective primes of $2N_m$, and $m$ is always no less than 2 because effective integers $(N_m - k)$ and $(N_m + k)$ of $2N_m$ are relatively prime to each other, where $k = 1$ or 2 when $N_m$ is even or odd, respectively.

§2. Theorem 1

Theorem 1 states that:

Let

$$\begin{align*}
p(i, m) + \alpha_{(i, m)} &= 2N_m &\text{Eq.s}\ \alpha_{(i, m)} \\
p(1, m)p(i, m) + \beta_{(i, m)} &= 2N_m &\text{Eq.s}\ \beta_{(i, m)}
\end{align*}$$

where, $p(i, m)$ represent total $m$ effective primes of the given even number $2N_m$ greater than 6; $i = 1, 2, \ldots, m - 1, m$, respectively;
$m \geq 2$;
$p(1, m)$ and $p(m, m)$ is the minimum and maximum one among $p(i, m)$, respectively;
Each of $\alpha_{(i,m)}$ is an effective integer of $2N_m$, respectively; 
$\beta_{(i,m)}$ is an effective integer of $2N_m$ only when $p_{(i,m)} < 2N_m - 1$, respectively.

Whatever $m$ is, the number of different effective partition(s) of $2N_m$ among $2m$ simultaneous equations $Eq.s \alpha_{(i,m)}$ & $Eq.s \beta_{(i,m)}$ is always less than $m$.

§ 3. Some necessary preparative concepts for the verification of Theorem 1

3-1, Properties of $\alpha_{(i,m)}$ & $\beta_{(i,m)}$

Because of relatively prime theorem and the definitions of effective primes/integers of $2N_m$, there are 5 useful properties of $\alpha_{(i,m)}$ & $\beta_{(i,m)}$ for the verification of Theorem 1.

Prop. 1
Each of $\alpha_{(i,m)}$ is an effective integer without factor $p_{(i,m)}$ within open interval $(1, 2N_m - 1)$; 
$2N_m - 1 > \alpha_{(1,m)} > \alpha_{(2,m)} > \cdots > \alpha_{(m-1,m)} > \alpha_{(m,m)} \geq p_{(1,m)}$;

$\beta_{(i,m)}$ is an effective integer without factors $p_{(i,m)}$ and $p_{(1,m)}$ only when $p_{(1,m)p_{(i,m)}} < 2N_m - 1$; 
$2N_m - 1 > \beta_{(1,m)} > \beta_{(2,m)} > \cdots > \beta_{(m-1,m)} > \beta_{(m,m)}$;

$p_{(1,m)p_{(m,m)}} > p_{(1,m)p_{(m-1,m)}} > \cdots > p_{(1,m)p_{(2,m)}} > p_{(1,m)p_{(1,m)}}$.

Prop. 2
There is no any common divisor between $\alpha_{(i,m)}$ and $\beta_{(i,m)}$, respectively.

Prop. 3
At most one of $\alpha_{(i,m)}$ and $\beta_{(i,m)}$ may contain the maximum effective prime factor $p_{(m,m)}$ of $2N_m$, respectively.

Prop. 4
$\beta_{(m,m)}$ is not an effective integer of $2N_m$ as long as there is a Goldbach partition of $2N_m$.

Prop. 5
There are at most $(m - k)/2$ successive $\beta_{(j,m)}$ equals an effective prime of $2N_m$, respectively, where, $j = 1, 2, \ldots, (m - k)/2$, respectively; $k = 0$ or $1$ when $m$ is even or odd, respectively.

3-2, The number of different partition(s) of $2N_m$

There are totally $2m$ simultaneous equations $Eq.s \alpha_{(i,m)}$ & $Eq.s \beta_{(i,m)}$.

On the following 3 conditions only, the number $D_m$ of different effective partitions of $2N_m$ among these $2m$ simultaneous equations is reduced from $2m$:

Condition 1
When $\alpha_{(x, m)} = p_{(y, m)}$,

\[ p_{(x, m)} + p_{(y, m)} = 2N_m \quad \quad \text{Eq. } \alpha_{(x, m)} \]
\[ p_{(y, m)} + p_{(x, m)} = 2N_m \quad \quad \text{Eq. } \alpha_{(y, m)} \]

Eq. $\alpha_{(x, m)}$ and Eq. $\alpha_{(y, m)}$ are identical, and $D_m$ is one reduced from $2m$;

**Condition 2**

When $\beta_{(x, m)} = p_{(y, m)}$,

\[ p_{(1, m)}p_{(x, m)} + p_{(y, m)} = 2N_m \quad \quad \text{Eq. } \beta_{(x, m)} \]
\[ p_{(y, m)} + p_{(1, m)}p_{(x, m)} = 2N_m \quad \quad \text{Eq. } \beta_{(y, m)} \]

Eq. $\beta_{(x, m)}$ and Eq. $\alpha_{(y, m)}$ are identical, and $D_m$ is one reduced from $2m$;

**Condition 3**

When $p_{(1, m)}p_{(z, m)} \geq 2N_m - 1$,

because $1 \geq \beta_{(z, m)} > \beta_{(z+1, m)} > \cdots > \beta_{(m-1, m)} > \beta_{(m, m)}$, that is Eq. $\beta_{(z, m)}$, Eq. $\beta_{(z+1, m)}$, ..., Eq. $\beta_{(m-1, m)}$ and Eq. $\beta_{(m, m)}$ are not effective partitions of $2N_m$, and $D_m$ is $(m - z + 1)$ reduced from $2m$.

The number of different partition(s) of $2N_m$ among total $2m$ simultaneous equations Eq.s $\alpha_{(i, m)}$ & Eq.s $\beta_{(i, m)}$ could be expressed as the following Eq.(3, 1):

\[ D_m = 2m - a_m/2 - b_m - c_m \quad \quad \text{Eq.(3, 1)} \]

where, $D_m$ represents the number of different effective partition(s) of $2N_m$ among total $2m$ simultaneous equations Eq.s $\alpha_{(i, m)}$ & Eq.s $\beta_{(i, m)}$;

$a_m$ among $m$ $\alpha_{(i, m)}$ equals an effective prime of $2N_m$, respectively;

$a_m$ is a non-odd integer;

$b_m$ among $m$ $\beta_{(i, m)}$ equals an effective prime of $2N_m$, respectively;

$b_m \geq 0$;

$c_m$ among $m$ $\beta_{(i, m)}$ is not an effective integer of $2N_m$, respectively;

$c_m \geq 0$;
\[ m \geq b_m + c_m. \]

3-3 \( D_m, D'_m \) and \( d_m \)
On other hand, \( D_m \) could also be expressed as the following Eq.(3, 2):

\[ D_m = D'_m + d_m \quad \text{Eq.}(3, 2) \]

where, \( D_m \) represents the number of different effective partition(s) of \( 2N_m \) among total \( 2m \) simultaneous equations \( Eq.s \alpha_{(i, m)} \) & \( Eq.s \beta_{(i, m)} \), where \( i = 1, 2, \ldots, m - 1, m \), respectively;

\( D'_m \) represents the number of different effective partition(s) of \( 2N_m \) in which the maximum effective prime factor \( p_{(m, m)} \) of \( 2N_m \) not take part among total \( 2(m-1) \) simultaneous equations \( Eq.s \alpha_{(i, m)} \) & \( Eq.s \beta_{(i, m)} \), where \( i = 1, 2, \ldots, m - 2, m - 1 \), respectively;

\( d_m \) represents the number of different effective partition(s) of \( 2N_m \) in which the maximum effective prime factor \( p_{(m, m)} \) of \( 2N_m \) takes part among total \( 2m \) equations \( Eq.s \alpha_{(i, m)} \) & \( Eq.s \beta_{(i, m)} \), where \( i = 1, 2, \ldots, m - 1, m \), respectively;

\[ m \geq 2. \]

3-4 Relationship between \( D_{m-1} \) and \( D'_m \)
According to the similarity between the definitions of \( D_{m-1} \) and \( D'_m \), the following In-eq.(3, 3) holds:

\[ D'_m \leq w \quad \text{when} \quad D_{m-1} \leq w \quad \text{In-eq.}(3, 3) \]

where, \( w \) represents an integer no less than \( l \).

\[ m \geq 2. \]

§4, Verification of Theorem 1
Based on these preparative concepts mentioned above and by using two-part method starting from \( m = 2 \), Theorem 1 was verified as shown below.

4-1, Theorem 1 holds when \( m = 2 \)
There are total two effective primes, \( p_{(1, 2)} \) and \( p_{(2, 2)} \), of the given even number \( 2N_2 \) when \( m = 2 \).

4 simultaneous equations \( Eq.s \alpha_{(i, 2)} \) & \( Eq.s \beta_{(i, 2)} \) were listed below:
According to Propositions 1 and 2, \( a_{(1, 2)} \) and \( a_{(2, 2)} \) has to equal an exact power of prime factor \( p_{(2, 2)} \) and \( p_{(1, 2)} \), respectively, and both \( \beta_{(1, 2)} \) and \( \beta_{(2, 2)} \) contains neither factor \( p_{(1, 2)} \) nor factor \( p_{(2, 2)} \).

According to the definition of effective integers of \( 2N_2 \), both \( \beta_{(1, 2)} \) and \( \beta_{(2, 2)} \) are not effective integers of \( 2N_2 \), and the following In-eq.(4, 1) holds:

\[ I \geq \beta_{(1, 2)} > \beta_{(2, 2)} \quad \text{In-eq.(4, 1)} \]

Because \( p_{(1, 2)}p_{(2, 2)} \geq p_{(1, 2)}p_{(2, 2)} \geq 2N_2 - 1 \) when In-eq.(4, 1) holds, \( a_{(1, 2)} \) and \( a_{(2, 2)} \) has to equal prime \( p_{(2, 2)} \) and \( p_{(1, 2)} \), respectively.

Therefore, Eq.s \( a_{(i, 2)} \) & Eq.s \( \beta_{(i, 2)} \) could be revised as the following:

\[
\begin{align*}
p_{(1, 2)} + p_{(2, 2)} &= 2N_2 & \text{Eq. } a_{(1, 2)} \\
p_{(2, 2)} + p_{(1, 2)} &= 2N_2 & \text{Eq. } a_{(2, 2)} \\
p_{(1, 2)} + p_{(2, 2)} + \beta_{(1, 2)} &= 2N_2 & \text{Eq. } \beta_{(1, 2)} \\
p_{(1, 2)}p_{(2, 2)} + \beta_{(2, 2)} &= 2N_2 & \text{Eq. } \beta_{(2, 2)}
\end{align*}
\]

where, both \( \beta_{(1, 2)} \) and \( \beta_{(2, 2)} \) are not effective integers of \( 2N_2 \).

Observing the revised Eq.s \( a_{(i, 2)} \) & Eq.s \( \beta_{(i, 2)} \) gave that \( a_2 = 2, b_2 = 0 \) and \( c_2 = 2 \). According to Eq.(3, 1), \( D_2 = 1 \) when \( m = 2 \), and consequently Theorem 1 holds when \( m = 2 \).

On other hand, observing the revised Eq.s \( a_{(i, 2)} \) & Eq.s \( \beta_{(i, 2)} \) gave that \( D’_2 = 0 \) and \( d_2 = 1 \). According to Eq.(3, 2), \( D_2 = 1 \) when \( m = 2 \), and consequently Theorem 1 holds when \( m = 2 \).

Therefore, Theorem 1 holds when \( m = 2 \).

4-2 Theorem 1 holds when \( m = M + 1 \) if Theorem 1 holds when \( m = M \geq 2 \)

Eq.s \( a_{(i, M+1)} \) & Eq.s \( \beta_{(i, M+1)} \) when \( m = M + 1 \) were listed below:

\[
\begin{align*}
p_{(i, M+1)} + a_{(i, M+1)} &= 2N_{M+1} & \text{Eq.s } a_{(i, M+1)} \\
p_{(i, M+1)}p_{(i, M+1)} + \beta_{(i, M+1)} &= 2N_{M+1} & \text{Eq.s } \beta_{(i, M+1)}
\end{align*}
\]
where, \( p(i, M+1) \) represents total \( M+1 \) effective primes of the given even number \( 2N_{M+1} \); 
\( i = 1, 2, \ldots, M, M+1 \), respectively; 
\( M \geq 2 \); 
\( p(1, M+1) \) and \( p(M+1, M+1) \) is the minimum and maximum one among \( p(i, M+1) \), respectively; 
Each of \( a(i, M+1) \) is an effective integer of \( 2N_{M+1} \), respectively; 
\( \beta(i, M+1) \) is an effective integer of \( 2N_{M+1} \) only when \( p(1, M+1)p(i, M+1) < 2N_{M+1} - 1 \).

It was verified that \( D_{M+1} \leq (M+1) - 1 \) if \( D_M \leq M - 1 \) as shown below:

4-2-1 \( D'_{M+1} \leq M - 1 \) if Theorem 1 holds when \( m = M \)
Assuming Theorem 1 holds when \( m = M \) implies that the following In-eq.(4, 2) holds:

\[
D_M \leq M - 1 \quad \text{In-eq.(4, 2)}
\]

where, \( D_M \) represents the number of different effective partitions of \( 2N_M \) among \( 2M \) equations 
\( \text{Eq.s } a(i, M) \) & \( \text{Eq.s } \beta(i, M); \)
\( i = 1, 2, \ldots, M - 1, M \), respectively.

According to In-eq.(3, 3) and In-eq.(4, 2), the following In-eq.(4, 3) holds, too:

\[
D'_{M+1} \leq M - 1 \text{ when } D_M \leq M - 1 \quad \text{In-eq.(4, 3)}
\]

where, \( D'_{M+1} \) represents the number of different effective partitions of \( 2N_{M+1} \) in which only factors \( p(i, M+1) \) take part among \( 2M \) simultaneous equations \( \text{Eq.s } a(i, M+1) \) & \( \text{Eq.s } \beta(i, M+1); \)
\( D_M \) represents the number of different effective partitions of \( 2N_M \) in which only factors \( p(i, M) \) take part among \( 2M \) simultaneous equations \( \text{Eq.s } a(i, M) \) & \( \text{Eq.s } \beta(i, M); \)
\( i = 1, 2, \ldots, M - 1, M \), respectively.

4-2-2, \( d_{M+1} = 1 \) if Theorem 1 holds when \( m = M \)
The following In-eq.(4, 4) was derived from Eq.(3, 1), Eq.(3, 2) and In-eq.(4, 3)

\[
a_{M+1} + b_{M+1} + c_{M+1} + d_{M+1} \geq M + 3 \text{ when } D_M \leq M - 1 \quad \text{In-eq.(4, 4)}
\]
where, \(a_{M+1}\) among \(M+1\) \(a_{(i, M+1)}\) equals an effective prime of \(2N_{M+1}\), respectively; \(a_{M+1}\) is a non-odd integer;
\(b_{M+1}\) among \(M+1\) \(\beta_{(i, M+1)}\) equals an effective prime of \(2N_{M+1}\), respectively; \(b_{M+1} \geq 0\);
\(c_{M+1}\) among \(M+1\) \(\beta_{(i, M+1)}\) is not an effective integer of \(2N_{M+1}\), respectively; \(c_{M+1} \geq 0\);
\(d_{M+1}\) represents the number of different effective partition(s) of \(2N_{M+1}\) in which the factor \(p_{(M+1, M+1)}\) of \(2N_{M+1}\) takes part among total \(2(M+1)\) simultaneous equations \(Eq.s \ a_{(i, M+1)}\) & \(Eq.s \ \beta_{(i, M+1)}\);
\(M+1 \geq b_{M+1} + c_{M+1}\).

\(i = 1, 2, \ldots, M, M+1\), respectively.

According to Prop. 3, in terms of how \(a_{(i, M+1)}\) and \(\beta_{(i, M+1)}\) contain factor \(p_{(M+1, M+1)}\), there are totally 4 sub-conditions in principle when \(m = M+1\).

It was verified that the following In-eq.(4, 5) always holds on the each logical sub-conditions:

\[d_{M+1} = 1\] when \(D_M \leq M - 1\) \hspace{1cm} \text{In-eq.(4, 5)}

**Sub-condition 1. None of \(a_{(i, M+1)}\) and \(\beta_{(i, M+1)}\) contained factor \(p_{(M+1, M+1)}\)**

\(d_{M+1} \leq 2\), because factor \(p_{(M+1, M+1)}\) only takes part in the following 2 simultaneous equations:

\[
p_{(M+1, M+1)} + a_{(M+1, M+1)} = 2N_{M+1} \hspace{1cm} Eq. \ a_{(M+1, M+1)} \\
p_{(1, M+1)}p_{(M+1, M+1)} + \beta_{(M+1, M+1)} = 2N_{M+1} \hspace{1cm} Eq. \ \beta_{(M+1, M+1)}
\]

1. When \(p_{(1, M+1)}p_{(M+1, M+1)} \geq 2N_{M+1} - 1\)

\(Eq. \ \beta_{(M+1, M+1)}\) is not an effective partition of \(2N_{M+1}\), and only \(Eq. \ a_{(M+1, M+1)}\) is an effective partition of \(2N_{M+1}\), \(d_{M+1} = 1\);

2. If \(p_{(1, M+1)}p_{(M+1, M+1)} < 2N_{M+1} - 1\)

According to Prop. 4, there would be no any Goldbach partition of \(2N_{M+1}\), \(a_{M+1} = 0\);

Each of \(\beta_{(i, M+1)}\) would be an effective integer of \(2N_{M+1}\), respectively, \(c_{M+1} = 0\);

Because \(a_{M+1} = 0\), \(c_{M+1} = 0\), and \(d_{M+1} \leq 2\), observing In-eq.(4, 4) would gave In-eq.(4, 6):
According to Prop. 5, this is illogical.

Therefore, because \( p(1, M+1)p(M+1, M+1) \geq 2N_{M+1} - 1 \) always holds, \( d_{M+1} = 1 \) on Sub-condition 1.

Sub-condition 2, Only \( \alpha_{(a, M+1)} \) contains factor \( p(M+1, M+1) \)
\( d_{M+1} \leq 3 \), because factor \( p(M+1, M+1) \) only takes part in the following 3 simultaneous equations:

\[
\begin{align*}
p(a, M+1) + Q_a p(M+1, M+1) f(a) &= 2N_{M+1} \\
p(M+1, M+1) + \alpha_{(M+1, M+1)} &= 2N_{M+1} \\
p(1, M+1)p(M+1, M+1) + \beta_{(M+1, M+1)} &= 2N_{M+1}
\end{align*}
\]

where, \( p(a, M+1) \) is an effective prime of \( 2N_{M+1} \);
\( a \neq M + 1 \).
\( Q_a \) is an effective integer of \( 2N_{M+1} \) or 1;
Exponent \( f(a) \) is an integer no less than 1.

1. When \( p(1, M+1)p(M+1, M+1) \geq 2N_{M+1} - 1 \)
Eq. \( \beta_{(M+1, M+1)} \) is not an effective partition of \( 2N_{M+1} \);

According to Prop. 1, effective integer \( \alpha_{(a, M+1)} \) of \( 2N_{M+1} \) has to equal effective prime \( p(M+1, M+1) \) of \( 2N_{M+1} \), and therefore Eq. \( \alpha_{(a, M+1)} \) and Eq. \( \alpha_{(M+1, M+1)} \) are identical.

There is only one different effective partition of \( 2N_{M+1} \) among Eq. \( \alpha_{(a, M+1)} \), Eq. \( \alpha_{(M+1, M+1)} \) and Eq. \( \beta_{(M+1, M+1)} \).

\( d_{M+1} = 1 \).

2. If \( p(1, M+1)p(M+1, M+1) < 2N_{M+1} - 1 \)
According to Prop. 4, there would be no any Goldbach partition of \( 2N_{M+1} \), \( d_{M+1} = 0 \);

Each of \( \beta_{(i, M+1)} \) would be an effective integer of \( 2N_{M+1} \), respectively, \( c_{M+1} = 0 \);

Because \( a_{M+1} = 0 \), \( c_{M+1} = 0 \), and \( d_{M+1} \leq 3 \), observing In-eq.(4, 4) would gave In-eq.(4, 7):

\[
b_{M+1} \geq M \text{ when } D_M \leq M - 1 \quad \text{In-eq.(4, 7)}
\]
According to Prop. 5, this is illogical.

Therefore, because \( p(1, M+1)p(M+1, M+1) \geq 2N_{M+1} - 1 \) always holds, \( d_{M+1} = 1 \) on **Sub-condition 2**.

**Sub-condition 3. Only \( \beta(b, M+1) \) contains factor \( p(M+1, M+1) \)**

\( d_{M+1} \leq 3 \), because factor \( p(M+1, M+1) \) only takes part in the following 3 simultaneous equations:

\[
\begin{align*}
    p(1, M+1)p(b, M+1) + Q_b p(M+1, M+1)^{f(b)} &= 2N_{M+1} & & \text{Eq. } \beta(b, M+1) \\
p(M+1, M+1) + a(M+1, M+1) &= 2N_{M+1} & & \text{Eq. } \alpha(M+1, M+1) \\
p(1, M+1)p(M+1, M+1) + \beta(M+1, M+1) &= 2N_{M+1} & & \text{Eq. } \beta(M+1, M+1)
\end{align*}
\]

where, \( p(b, M+1) \) is an effective prime of \( 2N_{M+1} \); \( b \neq M + 1 \).

\( Q_b \) is an effective integer of \( 2N_{M+1} \) or \( 1 \);

Exponent \( f(b) \) is an integer no less than \( 1 \).

1. When \( p(1, M+1)p(M+1, M+1) \geq 2N_{M+1} - 1 \)

Eq. \( \beta(M+1, M+1) \) is not an effective partition of \( 2N_{M+1} \);

Eq. \( \beta(b, M+1) \) is not an effective partition of \( 2N_{M+1} \) unless \( Q_b p(M+1, M+1)^{f(b)} = p(M+1, M+1) \);

When \( Q_b p(M+1, M+1)^{f(b)} = p(M+1, M+1) \), Eq. \( \beta(b, M+1) \) and Eq. \( \alpha(M+1, M+1) \) are identical.

There is only one different effective partition of \( 2N_{M+1} \) among Eq. \( \beta(b, M+1) \), Eq. \( \alpha(M+1, M+1) \) and Eq. \( \beta(M+1, M+1) \).

\( d_{M+1} = 1 \).

2. If \( p(1, M+1)p(M+1, M+1) < 2N_{M+1} - 1 \)

According to Prop. 4, there would be no any Goldbach partition of \( 2N_{M+1} \), \( a_{M+1} = 0 \);

Each of \( \beta(i, M+1) \) would be an effective integer of \( 2N_{M+1} \), respectively, \( c_{M+1} = 0 \);

Because \( a_{M+1} = 0 \), \( c_{M+1} = 0 \), and \( d_{M+1} \leq 3 \), observing In-eq.(4, 4) would gave In-eq.(4, 8):

\( b_{M+1} \geq M \) when \( D_{M+1} \leq M - 1 \)  \[ \text{In-eq.(4, 8)} \]
According to Prop. 5, this is illogical.

Therefore, because \( p_{(1, M+1)} p_{(M+1, M+1)} \geq 2N_{M+1} - 1 \) always holds, \( d_{M+1} = 1 \) on **Sub-condition 3**.

**Sub-condition 4.** Only \( \alpha \) \((a, M+1)\) and \( \beta \) \((b, M+1)\) contain factor \( p_{(M+1, M+1)} \)

\( d_{M+1} \leq 4 \), because factor \( p_{(M+1, M+1)} \) only takes part in the following 4 simultaneous equations:

\[
\begin{align*}
    p_{(a, M+1)} + Q_a p_{(M+1, M+1)} f(a) &= 2N_{M+1} & \text{Eq. } \alpha_{(a, M+1)} \\
    p_{(1, M+1)} p_{(b, M+1)} + Q_b p_{(M+1, M+1)} f(b) &= 2N_{M+1} & \text{Eq. } \beta_{(b, M+1)} \\
    p_{(M+1, M+1)} a_{(M+1, M+1)} &= 2N_{M+1} & \text{Eq. } \alpha_{(M+1, M+1)} \\
    p_{(1, M+1)} p_{(b, M+1)} + \beta_{(M+1, M+1)} &= 2N_{M+1} & \text{Eq. } \beta_{(M+1, M+1)} 
\end{align*}
\]

where, \( p_{(a, M+1)} \) is an effective prime of \( 2N_{M+1} \);

\( a \neq M + 1 \).

\( Q_a \) is an effective integer of \( 2N_{M+1} \) or \( 1 \);

Exponent \( f(a) \) is an integer no less than \( 1 \).

\( p_{(b, M+1)} \) is an effective prime of \( 2N_{M+1} \);

\( b \neq M + 1 \).

\( Q_b \) is an effective integer of \( 2N_{M+1} \) or \( 1 \);

Exponent \( f(b) \) is an integer no less than \( 1 \).

Observing Eq. \( \alpha_{(a, M+1)} \) and Eq. \( \beta_{(b, M+1)} \) gave that if this sub-condition existed, the following Eq.(4, 9) and Eq.(4, 10) would have to hold:

\[
\begin{align*}
    Q_a p_{(M+1, M+1)} f(a) &= p_{(1, M+1)} p_{(M+1, M+1)} & \text{Eq.(4, 9)} \\
    Q_b p_{(M+1, M+1)} f(b) &= p_{(M+1, M+1)} & \text{Eq.(4, 10)}
\end{align*}
\]

The 4 simultaneous equations in which factor \( p_{(M+1, M+1)} \) takes part were revised below:

\[
\begin{align*}
    p_{(a, M+1)} + p_{(1, M+1)} p_{(M+1, M+1)} &= 2N_{M+1} & \text{Eq. } \alpha_{(a, M+1)} \\
    p_{(1, M+1)} p_{(b, M+1)} + p_{(M+1, M+1)} &= 2N_{M+1} & \text{Eq. } \beta_{(b, M+1)} \\
    p_{(M+1, M+1)} + p_{(1, M+1)} p_{(b, M+1)} &= 2N_{M+1} & \text{Eq. } \alpha_{(M+1, M+1)} \\
    p_{(1, M+1)} p_{(M+1, M+1)} + p_{(a, M+1)} &= 2N_{M+1} & \text{Eq. } \beta_{(M+1, M+1)}
\end{align*}
\]

Observing these revised equations gave that Eq. \( \alpha_{(a, M+1)} \) and Eq. \( \beta_{(M+1, M+1)} \) would be identical;
Observing these revised equations gave that Eq. α(M+1, M+1) and Eq. β(b, M+1) would be identical;

This implies that \( d_{M+1} \leq 2 \), and Sub-condition 4 would be similar to Sub-condition 1.

Because \( p^{(1, M+1)}p^{(M+1, M+1)} \geq 2N_{M+1} - 1 \) always holds if \( d_{M+1} \leq 2 \) as verified on Sub-condition 1, Sub-condition 4 is illogical.

Summarizing above gave that \( d_{M+1} = 1 \) holds when \( m = M + 1 \) if Theorem 1 holds when \( m = M \).

Because \( d_{M+1} = 1 \) if Theorem 1 holds when \( m = M \), the following In-eq.(4, 11) was derived from Eq.(3, 2) and In-eq.(4, 3):

\[
D_{M+1} \leq (M+1) - 1 \text{ when } D_M \leq M - 1 \quad \text{In-eq.}(4, 11)
\]

Because Theorem 1 holds when \( m = 2 \) as verified in 4-1, according to In-eq.(4, 11), whatever \( m \) is, Theorem 1 always holds.

§5. Theorem 2 and its verification

5-1. Theorem 2

Theorem 2 states that:

Any given even number \( 2N_m \) greater than 6 must be divided by its effective prime \( p^{(g, m)} \) into a Goldbach partition of \( 2N_m \), where \( p^{(g, m)} \) represents the greatest one among all the effective primes of \( 2N_m \) whose factors are contained by \( a^{(i, m)} \).

5-2. Verification of Theorem 2

According to Theorem 1, there are at most \( (m - 1) \) different effective partitions of \( 2N_m \) among \( m \) simultaneous equations Eq.s \( a^{(i, m)} \) \( (i = 1, 2, \ldots, m - 1, m, \text{ respectively}) \). Therefore, there is at least one Goldbach partition of \( 2N_m \):

\[
p^{(a, m)} + p^{(a', m)} = 2N_m \quad \text{Eq.(5, 1)}
\]

where, \( p^{(a, m)} \) and \( p^{(a', m)} \) are two effective primes of \( 2N_m \).
The following *In-eq. (5, 2), In-eq. (5, 3) and In-eq. (5, 4) always holds:*

\[
p(m, m) \geq p(g, m) > p(\alpha, m) > N_m > p(\alpha', m) \geq p(1, m) \quad \text{In-eq. (5, 2)}
\]
\[
p(1, m) \geq 3 \quad \text{In-eq. (5, 3)}
\]
\[
p(1, m)p(g, m) > 2N_m - 1 \quad \text{In-eq. (5, 4)}
\]

where, \(p(1, m)\) and \(p(m, m)\) is the minimum and maximum effective prime of \(2N_m\), respectively;
\(p(g, m)\) is the greatest one among all the effective primes of \(2N_m\) whose factors take part in \(\alpha(i, m)\);
\(i = 1, 2, \ldots, m-1, m\), respectively.

Let \(\alpha(g', m)\) contains factor \(p(g, m)\):

\[
p(g', m) + Q_{g'} p(g', m)^{f(g')} = 2N_m \quad \text{Eq. } \alpha(g', m)
\]

where, \(Q_{g'}\) is an effective integer of \(2N_m\) or 1;
Exponent \(f(g')\) is an integer no less than 1.

Because of *In-eq. (5, 4)*, there is no any effective product containing factor \(p(g, m)\) within open interval \((1, 2N_m - 1)\), and *Eq. \(\alpha(g', m)\)* can be revised as the following:

\[
p(g', m) + p(g, m) = 2N_m \quad \text{Eq. } \alpha(g', m)
\]

where, both \(p(g', m)\) and \(p(g, m)\) are effective primes of \(2N_m\).

Therefore, Theorem 2 always holds.

**§ 6, Conclusion**
A specific Goldbach partition of any given even number greater than 6 can be found definitely.
References

[1] Ben Green and Terence Tao, arXiv:math/0606088

The End