SPECTRAL GAP AND QUANTITATIVE STATISTICAL STABILITY FOR SYSTEMS WITH CONTRACTING FIBERS AND LORENZ-LIKE MAPS

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Abstract. We consider transformations preserving a contracting foliation, such that the associated quotient map satisfies a Lasota-Yorke inequality. We prove that the associated transfer operator, acting on suitable normed spaces, has a spectral gap (on which we have quantitative estimation).

As an application we consider Lorenz-like two dimensional maps (piecewise hyperbolic with unbounded contraction and expansion rate): we prove that those systems have a spectral gap and we show a quantitative estimate for their statistical stability. Under deterministic perturbations of the system of size $\delta$, the physical measure varies continuously, with a modulus of continuity $O(\delta \log \delta)$, which is asymptotically optimal for this kind of piecewise smooth maps.

1. Introduction. The study of the behaviour of the transfer operator restricted to a suitable functional space has proven to be a powerful tool for the understanding of the statistical properties of a dynamical system. This approach gave first results (see [25], [27] and [30]) in the study of the dynamics of piecewise expanding maps where the involved spaces are made of regular, absolutely continuous measures (see [6], [26], [10] and [17] for some introductory text). In recent years the approach was extended to piecewise hyperbolic systems by the use of suitable anisotropic norms (the expanding and contracting directions are managed differently), leading to suitable distribution spaces on which the transfer operator has good spectral properties (see e.g. [7], [8], [9], [14], [21] and [5],[13] for recent papers containing a survey of the topic). From these properties, several limit theorems or stability statements can be deduced. This approach has proven to be successful in non-trivial classes of systems like geodesic flows (see [26], [11]) or billiard maps (ess e.g. [15]

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where a relatively simple and unified approach to many limit and perturbative results is given for the Lorentz gas. In these approaches, usually some condition of boundedness of the derivatives or transversality between the map's singular set and the contracting directions is supposed.

In this work, we consider skew product maps preserving a uniformly contracting foliation. We show how it is possible, in a simple way, to define suitable spaces of signed measures (with an anisotropic norm) such that, under small regularity assumptions, the transfer operator associated to the dynamics has a spectral gap (in the sense given in Theorem 7.1). This shows an exponential convergence to 0 in a certain norm for the iteration of a large class of zero average measures by the transfer operator. In this approach the speed of this convergence can be quantitatively estimated, and depends on the rate of contraction of the stable foliation, the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of the induced quotient map (see Remark 4). We also remark that in our approach we can deal with piecewise continuous maps having piecewise $C^{1+\alpha}$ regularity, having unbounded derivatives, and where the discontinuity set is parallel to the contracting direction, as it happen in the Lorenz-like maps we consider in Section 8. These results allow to obtain in the second part of the paper a quantitative statistical stability estimate for deterministic perturbations of this kind of Lorenz-like systems. The result applies to deterministic perturbations of skew product maps with a piecewise expanding map on the base with $C^2$ branches and contracting behaviour on the fibers. Essentially the main theorem of the section states (see Theorem 9.2) that the physical measure of the system varies with a modulus of continuity of the type $\delta \log(\delta)$ under perturbations of size $\delta$ (see Section 9 for precise statements and definitions) in a strong topology determined by a certain anisotropic space of signed measures which will be described below. It is worth to remark that this bound is also asymptotically optimal (see Remark 7).

The function spaces we consider are defined by disintegrating signed measures on the phase space along the contracting foliation. The signed measure itself is then seen as a family of measures on the contracting leaves. We can then consider some notion of regularity for this family to define suitable spaces of more or less “regular” measures where to apply our transfer operator. To give an idea of these function spaces (see section 4), in the case of skew product maps of the unit square $I \times I$ to itself, the disintegration gives rise to a one dimensional family (a path) of measures defined on the contracting leaves, each leaf is isomorphic to the unit interval $I$, hence a measure on $I \times I$ is seen as a path of measures on $I$: a path in a metric space. The function spaces are defined by suitable notions of regularity for these paths. In the case $I \times I$ for example, the spaces which arise are included in $L^1(I, Lip(I))$ (the space of $L^1$ functions from the interval to the dual of the space of Lipschitz functions on the interval), imposing some kind of further regularity. This is a space of distribution valued functions. For simplicity we will only use normed vector spaces of signed measures in this paper, we do not need to consider the completion of the space of signed measure, which would lead to distribution spaces. Similar strong and weak function spaces have been used in [18] to investigate quantitatively the statistical stability of slowly mixing toral extensions (skew products with a non expanding preserved foliation).

**Plan of the paper.** The paper is structured as follows:

- in Section 2 we introduce the kind of systems we consider in the paper. Essentially, these are skew product maps, with a base map satisfying a Lasota-Yorke
inequality with respect to suitable spaces (piecewise expanding maps e.g.) and the fibers are contracted;
• in Section 3 we introduce the functional spaces used in the paper and discussed in the previous paragraphs;
• in Section 4 we show the basic properties of the transfer operator when applied to these spaces. In particular we see that there is an useful “Perron-Frobenius”-like formula (see Proposition 5).
• In Section 5 we see the basic properties of the iteration of the transfer operator on the spaces we consider. In particular we see Lasota-Yorke inequalities and a convergence to equilibrium statement (see Propositions 7 and 9).
• In Section 6 we use the convergence to equilibrium and the Lasota-Yorke inequalities to prove the spectral gap for the transfer operator associated to the system restricted to a suitable strong space (see Theorems 7.1 and 7.2).
• In Section 8 we present an application of our construction, showing a spectral gap for 2-dimensional Lorenz-like maps (piecewise $C^{1+\alpha}$ hyperbolic maps with unbounded expansion and contraction rates).
• In Section 9 we consider similar systems with some more regularity. We apply our construction to a class of piecewise $C^2$, two-dimensional Lorenz-like maps. We prove stronger (bounded variation like) regularity results for the iteration of probability measures on that systems, and use this to prove a quantitative statistical stability statement with respect to deterministic perturbations: we establish a modulus of continuity $\delta \log \delta$ for the stability of the physical measure in weak space $(L^1(I,\text{Lip}(I))$ after a “size $\delta$” perturbation (see Theorem 9.2). Qualitative statements, for classes of similar maps were shown in [1] and very recently in [4].

2. Contracting fiber maps. In this section we introduce the kind of systems we are considering in this paper and show some of its basic properties. Consider $\Sigma = N_1 \times N_2$, where $N_1$ and $N_2$ are compact and finite dimensional Riemannian manifolds such that diam($N_2$) = 1, where diam($N_2$) denotes the diameter of $N_2$ with respect to its Riemannian metric $d_2$. This is not restrictive but will avoid some multiplicative constants. Denote by $m_1$ and $m_2$ the Lebesgue measures on $N_1$ and $N_2$ respectively, generated by their corresponding Riemannian volumes, normalized so that $m_1(N_1) = m_2(N_2) = 1$ and $m = m_1 \times m_2$. Consider a map $F: (\Sigma, m) \to (\Sigma, m)$,

$$F(x, y) = (T(x), G(x, y)),$$

where $T: N_1 \to N_1$ and $G: \Sigma \to N_2$ are measurable maps. Suppose that these maps satisfy the following conditions

2.0.1. Properties of $G$.

$\textbf{G1:}$ Consider the $F$-invariant foliation

$$\mathcal{F}^s := \{\{x\} \times N_2\}_{x \in N_1}.$$ 

We suppose that $\mathcal{F}^s$ is contracted: there exists $0 < \alpha < 1$ such that for all $x \in N_1$ it holds

$$d_2(G(x, y_1), G(x, y_2)) \leq \alpha d_2(y_1, y_2), \text{ for all } y_1, y_2 \in N_2. \quad (1)$$
2.0.2. Properties of $T$ and of its associated transfer operator. Suppose that:

**T1:** $T$ is non-singular with respect to $m_1$ ($m_1(A) = 0$ $\Rightarrow$ $m_1(T^{-1}(A))) = 0$;

**T2:** There exists a disjoint collection of open sets $\mathcal{P} = \{P_1, \cdots, P_q\}$ of $N_1$, such that $m_1(\bigcup_{i=1}^q P_i) = 1$ and $T_i := T|_{P_i}$ is a diffeomorphism $T_i : P_i \rightarrow T_i(P_i) \subseteq N_1$, with $\det DT_i(x) \neq 0$ for all $x \in P_i$ and for all $i$, where $DT_i$ is the Jacobian matrix of $T_i$ with respect to the Riemannian metric of $N_1$;

**T3:** Let us consider the Perron-Frobenius Operator associated to $T$, $P_T$. We will now make some assumptions on the existence of a suitable functional analytic setting adapted to $P_T$. Let us hence denote the $L^1_{m_1}$ norm$^2$ by $|\cdot|_1$ and suppose that there exists a Banach space $(S_\beta, |\cdot|_s)$ such that

$$|P^n_T f|_s \leq \beta_0 |f|_s + C |f|_1;$$

**T3.1:** $S_\beta \subset L^1_{m_1}$ is $P_T$-invariant, $|\cdot|_1 \leq |\cdot|_s$ and $P_T : S_\beta \rightarrow S_\beta$ is bounded;

**T3.2:** The unit ball of $(S_\beta, |\cdot|_s)$ is relatively compact in $(L^1_{m_1}, |\cdot|_1)$;

**T3.3:** (Lasota-Yorke inequality) There exist $k \in \mathbb{N}$, $0 < \beta_0 < 1$ and $C > 0$ such that, for all $f \in S_\beta$, it holds

$$|P^n_T f|_s \leq \beta_0 |f|_s + C |f|_1.$$

**T3.4:** Suppose there is an unique $\psi_x \in S_\beta$ with $\psi_x \geq 0$ and $|\psi_x|_1 = 1$ such that $P_T(\psi_x) = \psi_x$, and if $\psi \in S_\beta$ is another density for a probability measure, then $P^n_T(\psi - \psi) \rightarrow 0$ as $n \rightarrow \infty$ in $S_\beta$.

It is known that in this case ([22], see also [10], [26]) the following holds.

**Theorem 2.1.** If $T$ satisfy T3.1, ..., T3.4 then there exist $0 < r < 1$ and $D > 0$ such that for all $f \in S_\beta$ with $\int f \, dm_1 = 0$ and for all $n \geq 0$, it holds

$$|P^n_T(f)|_s \leq Dr^n |f|_s. \tag{2}$$

In order to obtain spectral gap on $L^\infty$ like spaces, the following additional property on $|\cdot|_s$ will be supposed at some point in the paper.

**N1:** There is $H_N \geq 0$ such that $|\cdot|_\infty \leq H_N |\cdot|_s$ (where $|\cdot|_\infty$ is the usual $L^\infty_{m_1}$ norm on $N_1$).

The following is a standard consequence of item T3.3, allowing to estimate the behaviour of any given power of the transfer operator.

**Corollary 1.** There exist constants $B_3 > 0$, $C_2 > 0$ and $0 < \beta_2 < 1$, such that for all $f \in S_\beta$, and all $n \geq 1$, it holds

$$|P^n_T f|_s \leq B_3 \beta^n_2 |f|_s + C_2 |f|_1. \tag{3}$$

$^1$The unique operator $P_T : L^1_{m_1} \rightarrow L^1_{m_1}$ such that

$$\forall \phi \in L^1_{m_1} \text{ and } \forall \psi \in L^\infty_{m_1} \int \psi \cdot P_T(\phi) \, dm_1 = \int (\psi \circ T) \cdot \phi \, dm_1.$$

$^2$Notation. In the following we use $|\cdot|$ to indicate the usual absolute value or norms for signed measures on the basis space $N_1$. We will use $||\cdot||$ for norms defined for signed measures on $\Sigma$.

$^3$This assumption ensures that from our point of view the system is indecomposable. For piecewise expanding maps e.g., the assumption follows from topological mixing.
3. Weak and strong spaces.

3.1. $L^1$-like spaces. Through this section we construct some function spaces which are suitable for the systems defined in section 2. The idea is to define spaces of signed measures, where the norms are provided by disintegrating measures along the stable foliation. Thus, a signed measure will be seen as a family of measures on each leaf. For instance, a measure on the square with a vertical foliation will be seen as a one parameter family (a path) of measures on the interval (a stable leaf), where this identification will be done by means of the Rokhlin’s Disintegration Theorem. Finally, in the vertical direction (on the leaves), we will consider a norm which is the dual of the Lipschitz norm and in the “horizontal” direction we will consider essentially the $L^1_{m_1}$ norm.

Rokhlin’s disintegration theorem. Now we present a brief recall about disintegration of measures.

Consider a probability space $(\Sigma, B, \mu)$ and a partition $\Gamma$ of $\Sigma$ by measurable sets $\gamma \in B$. Denote by $\pi : \Sigma \to \Gamma$ the projection that associates to each point $x \in M$ the element $\gamma_x$ of $\Gamma$ which contains $x$, i.e., $\pi(x) = \gamma_x$. Let $\hat{\mathcal{B}}$ be the $\sigma$-algebra of $\Gamma$ provided by $\pi$. Precisely, a subset $Q \subset \Gamma$ is measurable if, and only if, $\pi^{-1}(Q) \in B$. We define the quotient measure $\mu_x$ on $\Gamma$ by $\mu_x(Q) = \mu(\pi^{-1}(Q))$.

The proof of the following theorem can be found in [29], Theorem 5.1.11.

**Theorem 3.1.** (Rokhlin’s Disintegration Theorem) Suppose that $\Sigma$ is a complete and separable metric space, $\Gamma$ is a measurable partition of $\Sigma$ and $\mu$ is a probability on $\Sigma$. Then, $\mu$ admits a disintegration relative to $\Gamma$, i.e. a family $\{\mu_\gamma\}_{\gamma \in \Gamma}$ of probabilities on $\Sigma$ and a quotient measure $\mu_x$ as above, such that:

(a) $\mu_\gamma(\gamma) = 1$ for $\mu_x$-a.e. $\gamma \in \Gamma$;

(b) for all measurable set $E \subset \Sigma$ the function $\Gamma \to \mathbb{R}$ defined by $\gamma \mapsto \mu_\gamma(E)$, is measurable;

(c) for all measurable set $E \subset \Sigma$, it holds $\mu(E) = \int \mu_\gamma(E) d\mu_x(\gamma)$.

The proof of the following lemma can be found in [29], proposition 5.1.7.

**Lemma 3.2.** Suppose the $\sigma$-algebra $B$, on $\Sigma$, has a countable generator. If $\{\mu_\gamma\}_{\gamma \in \Gamma}$ and $\{\mu'_\gamma\}_{\gamma \in \Gamma}$ are disintegrations of the measure $\mu$ relative to $\Gamma$, then $\mu_\gamma = \mu'_\gamma$, for $\mu_x$-almost every $\gamma \in \Gamma$.

3.1.1. The $L^1$ and $S^1$ spaces. Let $SB(\Sigma)$ be the space of Borel signed measures on $\Sigma$. Given $\mu \in SB(\Sigma)$ denote by $\mu^+$ and $\mu^-$ the positive and the negative parts of its Jordan decomposition, $\mu = \mu^+ - \mu^-$ (see remark 2). Let $\pi_x : \Sigma \to N_1$ be the projection defined by $\pi(x, y) = x$, denote by $\pi_x : SB(\Sigma) \to SB(N_1)$ the pushforward map associated to $\pi_x$. Denote by $AB$ the set of signed measures $\mu \in SB(\Sigma)$ such that its associated positive and negative marginal measures, $\pi_x(\mu)^+$ and $\pi_x(\mu)^-$, are absolutely continuous with respect to the volume measure $m_1$, i.e.,

$$AB = \{ \mu \in SB(\Sigma) : \pi_x(\mu)^+ << m_1 \text{ and } \pi_x(\mu)^- << m_1 \}.$$ 

Given a probability measure $\mu \in AB$ on $\Sigma$, theorem 4.1 describes a disintegration $(\{\mu_\gamma\}_{\gamma}, \mu_x)$ along $\mathcal{F}^x$ (see equation (2.0.1)) by a family $\{\mu_\gamma\}_{\gamma}$ of probability measures on the stable leaves$^4$ and, since $\mu \in AB$, $\mu_x$ can be identified with a

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$^4$In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with $\gamma$. 
non negative marginal density $\phi_x : N_1 \to \mathbb{R}$, defined almost everywhere, with $|\phi_x|_1 = 1$. For a general (non normalized) positive measure $\mu \in \mathcal{AB}$ we can define its disintegration in the same way. In this case $\mu_\gamma$ are still probability measures, $\phi_x$ is still defined and $|\phi_x|_1 = \mu(\Sigma)$.

**Definition 3.3.** Let $\pi_y : \Sigma \to N_2$ be the projection defined by $\pi_y(x, y) = y$. Let $\gamma \in \mathcal{F}^*$, let us consider $\pi_{\gamma,y} : \gamma \to N_2$, the restriction of the map $\pi_y : \Sigma \to N_2$ to the vertical leaf $\gamma$ and the associated pushforward map $\pi_{\gamma,y}$. Given a positive measure $\mu \in \mathcal{AB}$ and its disintegration along the stable leaves $\mathcal{F}^*$, $(\{\mu_\gamma\}_\gamma, \mu_x = \phi_x m_1)$, we define the **restriction of $\mu$ on $\gamma$** and denote it by $\mu_\gamma$, as the positive measure on $N_2$ (not on the leaf $\gamma$) defined, for all measurable set $A \subset N_2$, as

$$\mu_\gamma(A) = \pi_{\gamma,y}(\phi_x(\gamma)\mu_y)(A).$$

For a given signed measure $\mu \in \mathcal{AB}$ and its Jordan decomposition $\mu = \mu^+ - \mu^-$, define the **restriction of $\mu$ on $\gamma$** by

$$\mu_\gamma = \mu^+|_\gamma - \mu^-|_\gamma.$$

**Remark 1.** As we will prove in Corollary 4, the restriction $\mu_\gamma$ does not depend on the decomposition. Precisely, if $\mu = \mu_1 - \mu_2$, where $\mu_1$ and $\mu_2$ are any positive measures, then $\mu_\gamma = \mu_1|_\gamma - \mu_2|_\gamma$ m$_1$-a.e. $\gamma \in N_1$.

Let $(X, d)$ be a compact metric space, $g : X \to \mathbb{R}$ be a Lipschitz function and let $L(g)$ be its best Lipschitz constant, i.e.

$$L(g) = \sup_{x,y \in X, x \neq y} \left\{ \frac{|g(x) - g(y)|}{d(x, y)} \right\}. \quad (4)$$

**Definition 3.4.** Given two signed measures $\mu$ and $\nu$ on $X$, we define a Wasserstein-Kantorovich Like distance between $\mu$ and $\nu$ by

$$W^0_1(\mu, \nu) = \inf_{L(g) \leq 1, |g|_{\infty} \leq 1} \int g(\mu - \nu).$$

From now, we denote

$$||\mu||_W := W^0_1(0, \mu).$$

(5)

As a matter of fact, $|| \cdot ||_W$ defines a norm on the vector space of signed measures defined on a compact metric space. It is worth to remark that this norm is equivalent to the dual of the Lipschitz norm.

**Definition 3.5.** Let $\mathcal{L}^1 \subseteq \mathcal{AB}$ be defined as

$$\mathcal{L}^1 = \left\{ \mu \in \mathcal{AB} : \int_{N_1} W^0_1(\mu^+|_\gamma, \mu^-|_\gamma)dm_1(\gamma) < \infty \right\}$$

and define a norm on it, $|| \cdot ||_1 : \mathcal{L}^1 \to \mathbb{R}$, by

$$||\mu||_1 = \int_{N_1} W^0_1(\mu^+|_\gamma, \mu^-|_\gamma)dm_1(\gamma).$$

Here the measurability of the integrand follows by the measurability of the disintegration established at Item b) of Theorem 4.1.

Now, we define the following set of signed measures on $\Sigma$,

$$S^1 = \left\{ \mu \in \mathcal{L}^1 ; \phi_x \in S_x \right\}. \quad (6)$$
Consider the function $||\cdot||_{S^1} : S^1 \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{S^1} = |\phi_x|^s + ||\mu||_1,$$

where we denote $\phi_x = \phi^+_x - \phi^-_x$ with $\phi^\pm_x$ being the marginals of $\mu^\pm$ as explained before. Moreover, $\phi_x$ is the marginal density of the disintegration of $\mu$ and we remark that $\phi^+_x$ is not necessarily equal to the positive part of $\phi_x$.

The proof of the next proposition is straightforward. Details can be found in [28].

**Proposition 1.** $(L^1, ||\cdot||_1)$ and $(S^1, ||\cdot||_{S^1})$ are normed vector spaces.

In the following $(L^1, ||\cdot||_1)$ and $(S^1, ||\cdot||_{S^1})$ will play the role of a strong and weak space, for which we will prove a Lasota-Yorke inequality and deduce other important consequences, as the exponential convergence to equilibrium and spectral gap for the operator considered on the strong space.

### 3.2. $L^\infty$ like spaces

Stronger spaces which can be considered with the above approach can be defined easily, we show an example of a $L^\infty$ like space.

**Definition 3.6.** Let $L^\infty \subseteq AB(\Sigma)$ be defined as

$$L^\infty = \{\mu \in AB : \text{ess sup}(W^0_1(\mu^+|_\gamma, \mu^-|_\gamma)) < \infty\},$$

where the essential supremum is taken over $N_1$ with respect to $m_1$. Define the function $||\cdot||_{L^\infty} : L^\infty \longrightarrow \mathbb{R}$ by

$$||\mu||_{L^\infty} = \text{ess sup}(W^0_1(\mu^+|_\gamma, \mu^-|_\gamma)).$$

Finally, consider the following set of signed measures on $\Sigma$

$$S^\infty = \{\mu \in L^\infty ; \phi_x \in S\},$$

and the function, $||\cdot||_{S^\infty} : S^\infty \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{S^\infty} = |\phi_x|^s + ||\mu||_{L^\infty}.$$

The proof of the next proposition is straightforward and can be found in [28].

**Proposition 2.** $(L^\infty, ||\cdot||_{L^\infty})$ and $(S^\infty, ||\cdot||_{S^\infty})$ are normed vector spaces.

### 4. Weak and strong spaces

#### 4.1. $L^1$-like spaces

Through this section we construct some function spaces which are suitable for the systems defined in section 2. The idea is to define spaces of signed measures, where the norms are provided by disintegrating measures along the stable foliation. Thus, a signed measure will be seen as a family of measures on each leaf. For instance, a measure on the square with a vertical foliation will be seen as a one parameter family (a path) of measures on the interval (a stable leaf), where this identification will be done by means of the Rokhlin’s Disintegration Theorem. Finally, in the vertical direction (on the leaves), we will consider a norm which is the dual of the Lipschitz norm and in the “horizontal” direction we will consider essentially the $L^1_{m_1}$ norm.
Rokhlin’s disintegration theorem. Now we present a brief recall about disintegration of measures.

Consider a probability space \((\Sigma, \mathcal{B}, \mu)\) and a partition \(\Gamma\) of \(\Sigma\) by measurable sets \(\gamma \in \mathcal{B}\). Denote by \(\pi: \Sigma \rightarrow \Gamma\) the projection that associates to each point \(x \in M\) the element \(\gamma_x\) of \(\Gamma\) which contains \(x\), i.e. \(\pi(x) = \gamma_x\). Let \(\mathcal{B}\) be the \(\sigma\-)algebra of \(\Gamma\) provided by \(\pi\). Precisely, a subset \(\mathcal{Q} \subset \Gamma\) is measurable if, and only if, \(\pi^{-1}(\mathcal{Q}) \in \mathcal{B}\). We define the quotient measure \(\mu_x\) on \(\Gamma\) by \(\mu_x(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q}))\).

The proof of the following theorem can be found in [29], Theorem 5.1.11.

**Theorem 4.1.** (Rokhlin’s Disintegration Theorem) Suppose that \(\Sigma\) is a complete and separable metric space, \(\Gamma\) is a measurable partition of \(\Sigma\) and \(\mu\) is a probability on \(\Sigma\). Then, \(\mu\) admits a disintegration relative to \(\Gamma\), i.e. a family \(\{\mu_\gamma\}_{\gamma \in \Gamma}\) of probabilities on \(\Sigma\) and a quotient measure \(\mu_x\) as above, such that:

1. \(\mu_\gamma(\gamma) = 1\) for \(\mu_x\)-a.e. \(\gamma \in \Gamma\);
2. for all measurable set \(E \subset \Sigma\) the function \(\Gamma \rightarrow \mathbb{R}\) defined by \(\gamma \mapsto \mu_\gamma(E)\), is measurable;
3. for all measurable set \(E \subset \Sigma\), it holds \(\mu(E) = \int \mu_\gamma(E) d\mu_x(\gamma)\).

The proof of the following lemma can be found in [29], proposition 5.1.7.

**Lemma 4.2.** Suppose the \(\sigma\-)algebra \(\mathcal{B}\), on \(\Sigma\), has a countable generator. If

\[\{\mu_\gamma\}_{\gamma \in \Gamma}, \mu_x\]

and

\[\{\mu'_\gamma\}_{\gamma \in \Gamma}, \mu_x\]

are disintegrations of the measure \(\mu\) relative to \(\Gamma\), then \(\mu_\gamma = \mu'_\gamma\), for \(\mu_x\)-almost every \(\gamma \in \Gamma\).

**4.1.1. \(L^1\) and \(S^1\) spaces.** Let \(SB(\Sigma)\) be the space of Borel signed measures on \(\Sigma\). Given \(\mu \in SB(\Sigma)\) denote by \(\mu^+\) and \(\mu^-\) the positive and the negative parts of its Jordan decomposition, \(\mu = \mu^+ - \mu^-\) (see remark 2). Let \(\pi_x: \Sigma \rightarrow N_1\) be the projection defined by \(\pi(x, y) = x\), denote by \(\pi_{x\ast}: SB(\Sigma) \rightarrow SB(N_1)\) the pushforward map associated to \(\pi_x\). Denote by \(\mathcal{AB}\) the set of signed measures \(\mu \in SB(\Sigma)\) such that its associated positive and negative marginal measures, \(\pi_{x\ast}\mu^+\) and \(\pi_{x\ast}\mu^-\), are absolutely continuous with respect to the volume measure \(m_1\), i.e.

\[\mathcal{AB} = \{\mu \in SB(\Sigma) : \pi_{x\ast}\mu^+ << m_1 \text{ and } \pi_{x\ast}\mu^- << m_1\}\].

Given a probability measure \(\mu \in \mathcal{AB}\) on \(\Sigma\), theorem 4.1 describes a disintegration \((\{\mu_\gamma\}_{\gamma}, \mu_x)\) along \(\mathcal{F}^s\) (see equation (2.0.1)) by a family \(\{\mu_\gamma\}_{\gamma}\) of probability measures on the stable leaves and, since \(\mu \in \mathcal{AB}\), \(\mu_x\) can be identified with a non negative marginal density \(\phi_x: N_1 \rightarrow \mathbb{R}\), defined almost everywhere, with \(|\phi_x|_1 = 1\). For a general (non normalized) positive measure \(\mu \in \mathcal{AB}\) we can define its disintegration in the same way. In this case \(\mu_\gamma\) are still probability measures, \(\phi_x\) is still defined and \(|\phi_x|_1 = \mu(\Sigma)\).

**Definition 4.3.** Let \(\pi_\gamma: \Sigma \rightarrow N_2\) be the projection defined by \(\pi_\gamma(x, y) = y\). Let \(\gamma \in \mathcal{F}^s\), let us consider \(\pi_{\gamma:y} : \gamma \rightarrow N_2\), the restriction of the map \(\pi_\gamma: \Sigma \rightarrow N_2\) to the vertical leaf \(\gamma\) and the associated pushforward map \(\pi_{\gamma:y}\). Given a positive measure \(\mu \in \mathcal{AB}\) and its disintegration along the stable leaves \(\mathcal{F}^s\), \((\{\mu_\gamma\}_{\gamma}, \mu_x = \phi_x m_1)\),

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5In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with \(\gamma\).
we define the restriction of $\mu$ on $\gamma$ and denote it by $\mu|_\gamma$ as the positive measure on $N_2$ (not on the leaf $\gamma$) defined, for all measurable set $A \subset N_2$, as

$$\mu|_\gamma(A) = \pi_{\gamma,y,*}(\phi_x(\gamma)\mu_\gamma)(A).$$

For a given signed measure $\mu \in AB$ and its Jordan decomposition $\mu = \mu^+ - \mu^-$, define the restriction of $\mu$ on $\gamma$ by

$$\mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma.$$

**Remark 2.** As we will prove in Corollary 4, the restriction $\mu|_\gamma$ does not depend on the decomposition. Precisely, if $\mu = \mu_1 - \mu_2$, where $\mu_1$ and $\mu_2$ are any positive measures, then $\mu|_\gamma = \mu_1|_\gamma - \mu_2|_\gamma$ $m_1$-a.e. $\gamma \in N_1$.

Let $(X,d)$ be a compact metric space, $g : X \to \mathbb{R}$ be a Lipschitz function and let $L(g)$ be its best Lipschitz constant, i.e.

$$L(g) = \sup_{x,y \in X, x \neq y} \left\{ \frac{|g(x) - g(y)|}{d(x,y)} \right\}. \quad (8)$$

**Definition 4.4.** Given two signed measures $\mu$ and $\nu$ on $X$, we define a Wasserstein-Kantorovich Like distance between $\mu$ and $\nu$ by

$$W_1^0(\mu, \nu) = \sup_{L(g) \leq 1, |g|_\infty \leq 1} \left| \int g d\mu - \int g d\nu \right|.$$

From now, we denote

$$\|\mu\|_W := W_1^0(0, \mu). \quad (9)$$

As a matter of fact, $\| \cdot \|_W$ defines a norm on the vector space of signed measures defined on a compact metric space. It is worth to remark that this norm is equivalent to the dual of the Lipschitz norm.

**Definition 4.5.** Let $L^1 \subseteq AB$ be defined as

$$L^1 = \left\{ \mu \in AB : \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma) < \infty \right\}$$

and define a norm on it, $\| \cdot \|_1 : L^1 \to \mathbb{R}$, by

$$\|\mu\|_1 = \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma).$$

Here the measurability of the integrand follows by the measurability of the disintegration established at Item b) of Theorem 4.1.

Now, we define the following set of signed measures on $\Sigma$,

$$S^1 = \left\{ \mu \in L^1 ; \phi_x \in S_\gamma \right\}. \quad (10)$$

Consider the function $\| \cdot \|_{S^1} : S^1 \to \mathbb{R}$, defined by

$$\|\mu\|_{S^1} = |\phi_x|_x + \|\mu\|_1,$$

where we denote $\phi_x = \phi^+_x - \phi^-_x$ with $\phi^\pm_x$ being the marginals of $\mu^\pm$ as explained before. Moreover, $\phi_x$ is the marginal density of the disintegration of $\mu$ and we remark that $\phi^+_x$ is not necessarily equal to the positive part of $\phi_x$.

The proof of the next proposition is straightforward. Details can be found in [28].

**Proposition 3.** $(L^1, \| \cdot \|_1)$ and $(S^1, \| \cdot \|_{S^1})$ are normed vector spaces.
Proposition 4. Let \( L^\infty \) be defined as
\[
L^\infty = \{ \mu \in AB : \text{ess sup}(W_1^0(\mu^+|\gamma, \mu^-|\gamma)) < \infty \},
\]
where the essential supremum is taken over \( N_1 \) with respect to \( m_1 \). Define the function \( || \cdot || : L^\infty \to \mathbb{R} \) by
\[
||\mu||_\infty = \text{ess sup}(W_1^0(\mu^+|\gamma, \mu^-|\gamma)).
\]
Finally, consider the following set of signed measures on \( \Sigma \)
\[
S^\infty = \{ \mu \in L^\infty : \phi_x \in S \},
\]
and the function, \( || \cdot ||_{S^\infty} : S^\infty \to \mathbb{R} \), defined by
\[
||\mu||_{S^\infty} = |\phi_x|_s + ||\mu||_\infty.
\]

The proof of the next proposition is straightforward and can be found in [28].

Proposition 4. \( (L^\infty, || \cdot ||_\infty) \) and \( (S^\infty, || \cdot ||_{S^\infty}) \) are normed vector spaces.

4.2. \( L^\infty \) like spaces. Stronger spaces which can be considered with the above approach can be defined easily, we show an example of a \( L^\infty \) like space.

Definition 4.6. Let \( L^\infty \subseteq AB(\Sigma) \) be defined as
\[
L^\infty = \{ \mu \in AB : \text{ess sup}(W_1^0(\mu^+|\gamma, \mu^-|\gamma)) < \infty \},
\]
where the essential supremum is taken over \( N_1 \) with respect to \( m_1 \). Define the function \( || \cdot || : L^\infty \to \mathbb{R} \) by
\[
||\mu||_\infty = \text{ess sup}(W_1^0(\mu^+|\gamma, \mu^-|\gamma)).
\]

In this section we consider the transfer operator associated to skew product maps as defined in Section 2, acting on our disintegrated measures spaces defined in Section 3. For such transfer operators and measures we prove a kind of Perron-Frobenius formula, which is somewhat similar to the one used for one-dimensional maps.

Consider the pushforward map \( F_* \) associated with \( F \), defined by
\[
[F_* \mu](E) = \mu(F^{-1}(E)),
\]
for each signed measure \( \mu \in SB(\Sigma) \) and for each measurable set \( E \subset \Sigma \). When \( F_* \) is considered on the vector space \( SB(\Sigma) \) or on suitable vector subspaces of more regular measures, \( F_* \) is a linear map, beacuse of this we also call it ”transfer operator associated to \( F^\circ \).

Lemma 5.1. For all probability \( \mu \in AB \) disintegrated by \( \{ \mu_\gamma \}_\gamma, \phi_x \), the disintegration \( \{ (F_* \mu_\gamma, \gamma), (F_* \mu)x \} \) of the pushforward \( F_* \mu \) satisfies the following relations
\[
(F_* \mu)_x = P_T(\phi_x)m_1
\]
and
\[
(F_* \mu)_\gamma = \nu_\gamma := \frac{1}{P_T(\phi_x)(\gamma)} \sum_{i=1}^q \frac{\phi_x}{|\det DT_i|} \cdot T_i^{-1}(\gamma) \cdot \chi_T(P_i)(\gamma) \cdot F_* \mu_{T_i^{-1}(\gamma)}
\]
when \( P_T(\phi_x)(\gamma) \neq 0 \). Otherwise, if \( P_T(\phi_x)(\gamma) = 0 \), then \( \nu_\gamma \) is the Lebesgue measure on \( \gamma \) (the expression \( \frac{\phi_x}{|\det DT_i|} \cdot T_i^{-1}(\gamma) \cdot \chi_T(P_i)(\gamma) \cdot \frac{P_T(\phi_x)(\gamma)}{P_T(\phi_x)(\gamma)} \cdot F_* \mu_{T_i^{-1}(\gamma)} \) is understood to be zero outside \( T_i(P_i) \) for all \( i = 1, \cdots, q \). Here and above, \( \chi_A \) is the characteristic function of the set \( A \).
Proof. By the uniqueness of the disintegration (see Lemma 4.2) is enough to prove the following equation
\[ F_* \mu(E) = \int_{N_1} \nu_\gamma(E \cap \gamma) P_T(\phi_x)(\gamma) dm_1(\gamma), \]
for a measurable set \( E \subset \Sigma \). For this purpose, let us define the sets \( B_1 = \{ \gamma \in N_1; T^{-1}(\gamma) = \emptyset \} \), \( B_2 = \{ \gamma \in B_1^c; P_T(\phi_x)(\gamma) = 0 \} \) and \( B_3 = (B_1 \cup B_2)^c \). The following properties can be easily proven:

1. \( B_i \cap B_j = \emptyset \), \( T^{-1}(B_i) \cap T^{-1}(B_j) = \emptyset \), for all \( 1 \leq i, j \leq 3 \) such that \( i \neq j \) and \( 1 \leq i, j \leq 3 \) such that \( i \neq j \) and \( \bigcup_{i=1}^{3} B_i = \bigcup_{i=1}^{3} T^{-1}(B_i) = N_1; \)

2. \( m_1(T^{-1}(B_1)) = \phi_x m_1(T^{-1}(B_2)) = 0; \)

Using the change of variables \( \gamma = T_i(\beta) \) and the definition of \( \nu_\gamma \) (see (13)), we have

\[
\int_{N_1} \nu_\gamma(E \cap \gamma) P_T(\phi_x)(\gamma) dm_1(\gamma) = \int_{B_1} \sum_{i=1}^{q} \frac{\phi_x}{|\det DT_i|} \circ T^{-1}_i(\gamma) F_* \mu_{T^{-1}_i(\gamma)}(E) \chi_{T_i(P_i)}(\gamma) dm_1(\gamma)
\]

\[
= \sum_{i=1}^{q} \int_{T_i(P_i) \cap B_3} \frac{\phi_x}{|\det DT_i|} \circ T^{-1}_i(\gamma) F_* \mu_{T^{-1}_i(\gamma)}(E) dm_1(\gamma)
\]

\[
= \sum_{i=1}^{q} \int_{T^{-1}(B_3)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta)
\]

\[
= \int_{T^{-1}(B_1)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta)
\]

\[
= \int_{\bigcup_{i=1}^{3} T^{-1}(B_i)} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta)
\]

\[
= \int_{N_1} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta)
\]

\[
= \mu(F^{-1}(E))
\]

\[
= F_* \mu(E).
\]

And the proof is done. \( \square \)

As said in Remark 2, Corollary 4 yields that the restriction \( \mu|_\gamma \) does not depend on the decomposition. Thus, for each \( \mu \in \mathcal{L}^1 \), since \( F_* \mu \) can be decomposed as \( F_* \mu = F_*(\mu^+) - F_*(\mu^-) \), we can apply the above Lemma to \( F_*(\mu^+) \) and \( F_*(\mu^-) \) to get the following.

**Proposition 5.** Let \( \gamma \in F_* \) be a stable leaf. Let us define the map \( F_\gamma : N_2 \rightarrow N_2 \) by

\[ F_\gamma = \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}. \]  

(14)

Then, for each \( \mu \in \mathcal{L}^1 \) and for almost all \( \gamma \in N_1 \) (interpreted as the quotient space of leaves) it holds

\[
(F_* \mu)|_\gamma = \sum_{i=1}^{q} \frac{F_{T^{-1}_i(\gamma)*}}{|\det DT_i|} \circ T^{-1}_i(\gamma) \mu_{T^{-1}_i(\gamma)}(\gamma) \chi_{T_i(P_i)}(\gamma) m_1 - a.e. \ \gamma \in N_1 \]  

(15)

where \( F_{T^{-1}_i(\gamma)*} \) is the pushforward map associated to \( F_{T^{-1}_i(\gamma)} \).
6. Basic properties of the norms and convergence to equilibrium. In this section, we show important properties of the norms and their behaviour with respect to the transfer operator. In particular, we prove that the \( L^1 \) norm is weakly contracted. We prove Lasota-Yorke like inequalities for the strong norms and exponential convergence to equilibrium. All these properties will be used in next section to prove the spectral gap for the transfer operator associated to the system \( F : \Sigma \rightarrow \Sigma \).

**Proposition 6** (The weak norm is weakly contracted by \( F_* \)). If \( \mu \in L^1 \) then

\[
||F_* \mu||_1 \leq ||\mu||_1.
\]

In the proof of the proposition we will use the following lemma about the behaviour of the \( ||\cdot||_W \) norm (see equation (9)) which says that a contraction cannot increase the \( ||\cdot||_W \) norm.

**Lemma 6.1.** For every \( \mu \in AB \) and a stable leaf \( \gamma \in F^* \), it holds

\[
||F_{\gamma_*} \mu||_W \leq ||\mu||_W, \tag{16}
\]

where \( F_{\gamma} : N_2 \rightarrow N_2 \) is defined in Proposition 5 and \( F_{\gamma_*} \) is the associated push-forward map. Moreover, if \( \mu \) is a probability measure on \( N_2 \), it holds

\[
||F_{\gamma_*}^n \mu||_W = ||\mu||_W = 1, \quad \forall \ n \geq 1. \tag{17}
\]

**Proof.** (of Lemma 6.1) Indeed, since \( F_{\gamma} \) is an \( \alpha \)-contraction, if \( |g|_{\infty} \leq 1 \) and Lip\( (g) \leq 1 \) the same holds for \( g \circ F_{\gamma} \). Since

\[
\left| \int g \ dF_{\gamma_*} \mu_{|\gamma} \right| = \left| \int g(F_{\gamma}) \ d\mu_{|\gamma} \right|,
\]

taking the supremum over \( g \) such that \( |g|_{\infty} \leq 1 \) and Lip\( (g) \leq 1 \) we finish the proof of the inequality (16).

In order to prove equation (17), consider a probability measure \( \mu \) on \( N_2 \) and a Lipschitz function \( g : N_2 \rightarrow \mathbb{R} \), such that \( |g|_{\infty} \leq 1 \) we get immediately \( |\int g d\mu| \leq ||g||_{\infty} \leq 1 \), which yields \( ||\mu||_W \leq 1 \). Considering \( g \equiv 1 \) we get \( ||\mu||_W = 1 \). \( \square \)

**Proof.** (of Proposition 6)

In the following, we consider for all \( i \), the change of variable \( \gamma = T_i(\alpha) \). Thus, Lemma 6.1 and equation (15) yield

\[
||F_* \mu||_1 = \int_{N_1} ||(F_* \mu)_{|\gamma}||_W dm_1(\gamma)
\]

\[
\leq \sum_{i=1}^{q} \int_{T(p_i)} \left| \left| \frac{F_{T_i^{-1}(\gamma)_*} \mu|_{T_i^{-1}(\gamma)}}{||\det DT_i(T_i^{-1}(\gamma))||_W} \right| \right| dm_1(\gamma)
\]

\[
= \sum_{i=1}^{q} \int_{p_i} ||F_{\alpha_*} \mu_{|\alpha}||_W dm_1(\alpha)
\]

\[
= \sum_{i=1}^{q} \int_{p_i} ||\mu_{|\alpha}||_W dm_1(\alpha)
\]

\[
= ||\mu||_1.
\]

\( \square \)
The following proposition shows a regularizing action of the transfer operator with respect to the strong norm. Such inequalities are usually called Lasota-Yorke or Doeblin-Fortet inequalities.

**Proposition 7** (Lasota-Yorke inequality for $S^1$). Let $F : \Sigma \rightarrow \Sigma$ be a map satisfying $T1$, $T2$ and $T3$. Then, there exist $A$, $B_2 > 0$ and $\lambda < 1$ such that, for all $\mu \in S^1$, it holds

$$\| F^n \mu \|_{S^1} \leq A \lambda^n \| \mu \|_{S^1} + B_2 \| \mu \|_1, \quad \forall n \geq 1. \tag{18}$$

**Proof.** Firstly, we recall that $\phi_x$ is the marginal density of the disintegration of $\mu$. Precisely, $\phi_x = \phi_x^+ - \phi_x^-$, where $\phi_x^+ = \frac{d\pi_x^+}{dm_1}$ and $\phi_x^- = \frac{d\pi_x^-}{dm_1}$. By the definition of the Wasserstein norm it follows that for every $\gamma$ it holds $\| \mu \|_{W} \geq \int_1 d(\mu|_\gamma) = \phi_x(\gamma)$. Thus, $\| \phi_x \|_1 \leq \| \mu \|_1$. By this last remark, equation (3) and Proposition 6 we have

$$\| \Gamma_n^\mu \|_{S^1} = \| P^n_T \phi_x \|_1 + \| P_n^\mu \|_1$$

$$\leq B_3 \beta_2^2 \| \phi_x \|_1 + C_2 \| \mu \|_1$$

$$\leq B_3 \beta_2^2 \| \mu \|_{S^1} + (C_2 + 1) \| \mu \|_1.$$ 

We finish the proof by setting $\lambda = \beta_2$, $A = B_3$ and $B_2 = C_2 + 1$. \qed

6.1. **Convergence to equilibrium.** Let $X$ be a compact metric space. Consider the space $SB(X)$ of signed Borel measures on $X$. In the following we consider two further normed vectors spaces of signed Borel measures on $X$. The spaces $(B_*, || \cdot ||_s) \subseteq (B_*, || \cdot ||_w) \subseteq SB(X)$ with norms satisfying

$$\| \cdot \|_w \leq || \cdot ||_s.$$ 

We say that the a Markov operator $L : B_w \rightarrow B_w$ has convergence to equilibrium with speed at least $\Phi$ and with respect to the norms $|| \cdot ||_s$ and $|| \cdot ||_w$, if for each $\mu \in V_s$, where

$$V_s = \{ \mu \in B_s, \mu(X) = 0 \}$$

is the space of zero-average measures, it holds

$$\| L^n(\mu) \|_w \leq \Phi(n) \| \mu \|_s,$$

where $\Phi(n) \rightarrow 0$ as $n \rightarrow \infty$.

In this section, we prove that $F_*$ has exponential convergence to equilibrium. This is weaker with respect to the spectral gap. However, the spectral gap follows from the above Lasota-Yorke inequality and the convergence to equilibrium. Before the main statements we need some preliminary lemmata. The following is somewhat similar to Lemma 6.1 considering the behaviour of the $|| \cdot ||_W$ norm after a contraction. It gives a finer estimate for zero average measures. The following Lemma is useful to estimate the behaviour of our $W$ norms under contractions.

**Lemma 6.2.** For all signed measures $\mu$ on $N_2$ and for all $\gamma \in F^*$, it holds

$$\| F_{\gamma^*} \mu \|_W \leq \alpha \| \mu \|_W + \mu(N_2)$$

($\alpha$ is the rate of contraction of $G$, see (1)). In particular, if $\mu(N_2) = 0$ then

$$\| F_{\gamma^*} \mu \|_W \leq \alpha \| \mu \|_W.$$
Proof. If $\text{Lip}(g) \leq 1$ and $\|g\|_\infty \leq 1$, then $g \circ F_\gamma$ is $\alpha$-Lipschitz. Moreover, since $\|g\|_\infty \leq 1$, then $\|g \circ F_\gamma - \theta\|_\infty \leq \alpha$, for some $\theta$ such that $|\theta| \leq 1$. Indeed, let $z \in N_2$ be such that $|g \circ F_\gamma(z)| \leq 1$, set $\theta = g \circ F_\gamma(z)$ and let $d_2$ be the Riemannian metric of $N_2$. Since $\text{diam}(N_2) = 1$, we have

$$|g \circ F_\gamma(y) - \theta| \leq \alpha d_2(y, z) \leq \alpha$$

and consequently $\|g \circ F_\gamma - \theta\|_\infty \leq \alpha$.

This implies,

$$\left| \int_{N_2} g dF_\gamma \mu \right| = \left| \int_{N_2} g \circ F_\gamma d\mu \right| \leq \left| \int_{N_2} g \circ F_\gamma - \theta d\mu \right| + \left| \int_{N_2} \theta d\mu \right| = \alpha \left| \int_{N_2} \frac{g \circ F_\gamma - \theta}{\alpha} d\mu \right| + |\theta| |\mu(N_2)|.$$

And taking the supremum over $g$ such that $|g|_\infty \leq 1$ and $\text{Lip}(g) \leq 1$ we have $\|F_\gamma^* \mu\|_W \leq \alpha \|\mu\|_W + \mu(N_2)$. In particular, if $\mu(N_2) = 0$, we get the second part.

Now we are ready to show a key estimate regarding the behaviour of our weak $\| \cdot \|_1$ norm in Lorenz-like systems, as defined at beginning of Section 2.

**Proposition 8.** For all signed measure $\mu \in \mathcal{L}^1$, it holds

$$\|F_\gamma^* \mu\|_1 \leq \alpha \|\mu\|_1 + (\alpha + 1) |\phi_x|_1.$$  \hfill (20)

**Proof.** Consider a signed measure $\mu \in \mathcal{L}^1$ and its restriction on the leaf $\gamma$, $\mu|_\gamma = \pi_{\gamma, y^*}(\phi_x(\gamma)\mu|_y)$. Set

$$\overline{\mu}|_\gamma = \pi_{\gamma, y^*} \mu|_\gamma.$$  

If $\mu$ is a positive measure then $\overline{\mu}|_\gamma$ is a probability on $N_2$ and $\mu|_\gamma = \phi_x(\gamma)\overline{\mu}|_\gamma$. Then, the expression given by Proposition 5 yields

$$\|F_\gamma^* \mu\|_1 \leq \sum_{i=1}^{q} \int_{\mathcal{T}(F_i)} \left\| \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^+(T_\gamma^{-1}(\gamma)) - \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^-(T_\gamma^{-1}(\gamma)) \right\| \, dm_{\gamma}(\gamma).$$

where

$$I_1 = \sum_{i=1}^{q} \int_{\mathcal{T}(F_i)} \left| \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^+(T_\gamma^{-1}(\gamma)) - \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^-(T_\gamma^{-1}(\gamma)) \right| \, dm_{\gamma}(\gamma).$$

and

$$I_2 = \sum_{i=1}^{q} \int_{\mathcal{T}(F_i)} \left| \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^+(T_\gamma^{-1}(\gamma)) - \frac{F_{\gamma^{-1}(\gamma)} \mu}{|\det DT_i| \circ T_\gamma^{-1}(\gamma)} \phi_x^-(T_\gamma^{-1}(\gamma)) \right| \, dm_{\gamma}(\gamma).$$
In the following we estimate $I_1$ and $I_2$. By Lemma 6.1 and a change of variable we have

\[
I_1 = \sum_{i=1}^{q} \int_{T(P_i)} \left\| F_{T_i^{-1}(\gamma)} \cdot \mu^{2} \cdot |T_i^{-1}(\gamma)| \right\|_W \frac{\phi_+^i - \phi_-^i}{|\det DT_i|} \circ T_i^{-1}(\gamma)dm_1(\gamma)
\]

\[
\leq \int_{N_1} \left\| F_{\beta} \cdot \mu^{2} \cdot |\beta| \right\|_W \phi_+^i - \phi_-^i |\beta|dm_1(\beta)
\]

\[
= \int_{N_1} \phi_+^i - \phi_-^i |\beta|dm_1(\beta)
\]

\[
= \phi_1 |\phi_1|,
\]

and by Lemma 6.2 we have

\[
I_2 = \sum_{i=1}^{q} \int_{T(P_i)} \left\| F_{T_i^{-1}(\gamma)} \cdot \mu^{2} \cdot |T_i^{-1}(\gamma)| \right\|_W \frac{\phi_+^i - \phi_-^i}{|\det DT_i|} \circ T_i^{-1}(\gamma)dm_1(\gamma)
\]

\[
\leq \sum_{i=1}^{q} \int_{P_i} \left\| F_{\beta} \cdot \mu^{2} \cdot |\beta - \mu \cdot |\beta| \right\|_W \phi_+^i - \phi_-^i |\beta|dm_1(\beta)
\]

\[
\leq \alpha \int_{N_1} \left\| \mu^{2} \cdot |\beta - \mu \cdot |\beta| \right\|_W \phi_+^i - \phi_-^i |\beta|dm_1(\beta)
\]

\[
\leq \alpha \int_{N_1} \left\| \mu^{2} \cdot |\beta \cdot \phi_+^i - \beta \cdot |\beta| \phi_+^i |\beta| \right\|_W dm_1(\beta)
\]

\[
+ \alpha \int_{N_1} \left\| \mu^{2} \cdot |\beta \cdot \phi_-^i - \beta \cdot |\beta| \phi_-^i |\beta| \right\|_W dm_1(\beta)
\]

\[
= \alpha |\phi_1| + \alpha |\mu|.
\]

Summing the above estimates we finish the proof. \qed

Iterating (20) we get the following corollary.

**Corollary 2.** For all signed measure $\mu \in L^1$ it holds

\[
\| F^n_\mu \|_1 \leq \alpha^n \| \mu \|_1 + \alpha \| \phi \|_1,
\]

where $\alpha = \frac{1+\alpha}{1-\alpha}$.

Let us consider the set of zero average measures in $S^1$ defined by

\[
V_s = \{ \mu \in S^1 : \mu(\Sigma) = 0 \}.
\]

Note that, for all $\mu \in V_s$ we have $\pi_{s+1}(N_1) = 0$. Moreover, since $\pi_{s+1} = \phi_s m_1$ ($\phi_s = \phi_s^+ - \phi_s^-$), we have $\int_{N_1} \phi_s dm_1 = 0$. This allows us to apply Theorem 2.1 in the proof of the next proposition.

**Proposition 9** (Exponential convergence to equilibrium). There exist $D_2 \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that for every signed measure $\mu \in V_s$, it holds

\[
\| F^n_\mu \|_1 \leq D_2 \beta^n_1 \| \mu \|_{S^1},
\]

for all $n \geq 1$.

**Proof.** Given $\mu \in V_s$ and denoting $\phi_s = \phi_s^+ - \phi_s^-$, it holds that $\int \phi_s dm_1 = 0$. Moreover, Theorem 2.1 yields $|P^n_\mu(\phi_s)|_s \leq D^\mu_1|\phi_s|_s$ for all $n \geq 1$, then $|P^n_\mu(\phi_s)|_s \leq D^\mu_1|\mu|_{S^1}$ for all $n \geq 1$. 

Let \( l \) and \( 0 \leq d \leq 1 \) be the coefficients of the division of \( n \) by 2, i.e. \( n = 2l + d \). Thus, \( l = \frac{n-d}{2} \) (by Proposition 6, we have \( \| F^n_\ast \mu \|_1 \leq \| \mu \|_1 \) for all \( n \), and \( \| \mu \|_1 \leq \| \mu \|_{S^1} \)) and by Corollary 2, it holds (below, set \( \beta_1 = \max \{ \sqrt{r}, \sqrt{\alpha} \} \))

\[
\| F^n_\ast \mu \|_1 \leq \| F^{2l+d}_\ast \mu \|_1 \\
\leq \alpha^l \| F^{l+d}_\ast \mu \|_1 + \sqrt{d} \| \pi_{x_\ast} (F^{l+d}_\ast \mu) \|_{dm_1} \\
\leq \alpha^l \| \mu \|_1 + \beta_1 \beta_1 \| \mu \|_{S^1} \\
\leq D_2 \beta_1 \| \mu \|_{S^1},
\]

where \( D_2 = \frac{1 + \beta_1}{\beta_1} \).

**Remark 3.** The rate of convergence to equilibrium, \( \beta_1 \), for the map \( F \) found above, is directly related to the rate of contraction, \( \alpha \), of the stable foliation, and to the rate of convergence to equilibrium, \( r \), of the induced basis map \( T \) (see equation 2). More precisely, \( \beta_1 = \max \{ \sqrt{r}, \sqrt{\alpha} \} \). Similarly, we have an explicit estimate for the constant \( D_2 \), provided we have an estimate for \( D \) in the basis map\(^6\).

Now we show that under the assumptions taken, the system has a unique invariant measure \( \mu_0 \in S^1 \).

**Lemma 6.3.** A contracting fiber map \((N_1 \times N_2, F)\) satisfying assumptions G1, T1, ..., T3.4 has a unique invariant measure in \( S^1 \).

Before the proof of Lemma 6.3 we need a preliminary lemma.

**Lemma 6.4.** Let \( \mu_n \) be a sequence of probability measures which is a Cauchy sequence for the Wassertein like norm \( \| \cdot \|_W \) on a compact manifold \( N \). Then this sequence has a limit in the space of probability measures \( \mathcal{P}B(N) \) on \( N \). In other words \( \mathcal{P}B(N) \) is a complete metric space with the distance induced by \( \| \cdot \|_W \).

**Proof.** Consider \( \mathcal{P}B(N) \) with the weak* topology, i.e. the topology in which \( \mu_n \rightarrow \mu \) if and only if for each continuous \( f : N \rightarrow \mathbb{R} \) it holds \( \int f \, d\mu_n \rightarrow \int f \, d\mu \). This space is compact. Then \( \mu_n \) has subsequences \( \mu_{n_k} \) converging to some \( \mu_0 \in \mathcal{P}B(N) \) in this topology. Since \( N \) is compact we can approximate uniformly every continuous function \( f \) with Lipschitz functions \( g_i \). Given \( f \in C^0(N) \), \( \epsilon > 0 \) let us choose \( g_i \) such that \( \| f - g_i \|_\infty \leq \epsilon \) we have

\[
\left| \int f \, d(\mu_n - \mu_m) \right| \leq \left| \int (f - g_i) \, d(\mu_n - \mu_m) \right| + \left| \int g_i \, d(\mu_n - \mu_m) \right| \leq \epsilon + o(m, n)
\]

with \( o(m, n) \rightarrow 0 \) as \( \min(m, n) \rightarrow \infty \) hence \( \int f \, d(\mu_n - \mu_m) \leq 2\epsilon \) as \( \min(m, n) \rightarrow \infty \). Since \( \epsilon \) is arbitrary we get \( \int f \, d(\mu_n - \mu_m) \rightarrow 0 \) as \( \min(m, n) \rightarrow \infty \). This shows that \( \mu_n \) is a Cauchy sequence in the weak* topology, and then it converges to \( \mu_0 \) in that topology. Now conversely, suppose that this convergence was not in the \( \| \cdot \|_W \) norm, there is a subsequence \( \mu_{n_k} \) such that \( \forall k \| \mu_{n_k} - \mu_0 \|_W \geq \epsilon \), for some \( \epsilon < 0 \). Then it means there are uniformly bounded, 1-Lipschitz functions \( g_i \) such that for each \( i \), we have

\[
\int g_i \, d(\mu_{n_k} - \mu_0) \geq \epsilon \frac{\epsilon}{2}.
\]

\(^6\)It can be difficult to find a sharp estimate for \( D \). An approach allowing to find some useful upper estimates is shown in [19].
By Ascoli-Arzelà theorem however a subsequence $g_{i}$ converges uniformly to some continuous function $g$, for which $\int g_{i} \text{d}[\mu_{n_{i}} - \mu_{0}] \to 0$, contradicting (22). Then $||\mu_{n_{k}} - \mu_{0}||_{W} \to 0$, proving the statement. 

**Proof of Lemma 6.3.** By assumption T3.4, the base map $T$ has a unique invariant measure $\psi_{x}$ in $S_{x} \subseteq L^{1}$. Let us consider the following set of measures having $\psi_{x}$ as a marginal:

$$M_{\psi} = \{ \mu \in S^{1}, \pi_{x*}(\mu) = \psi_{x} \}.$$  

By Proposition 9 $F_{*}$ is a contraction on $M_{\psi}$, thus if we prove that there is a fixed point in $M_{\psi}$ this is unique. Let us consider the measure $\nu_{0} := \psi_{x} \times m_{2} \in S^{1}$ and let us iterate this by $F$. Every iterate $\nu_{n} := F_{*}^{n}(\nu)$ is a positive measure and because of Corollary 2 of $\nu_{n} \in S^{1}$. Furthermore, for each $n$, $\pi_{x*}(F_{*}^{n}(\nu)) = \psi_{x}$. By Proposition 9 $\nu_{n}$ is a Cauchy sequence in $M_{\psi}$, for the $|| ||_{1}$ norm. Let us consider the completion $\overline{M_{\psi}}$ of $M_{\psi}$. Being a contraction $F_{*}$ can be extended continuously to $\overline{M_{\psi}}$. Let $\mu_{0}$ be hence the limit of $\nu_{n}$ in $\overline{M_{\psi}}$. $\mu_{0}$ is then a fixed point of the contraction $F_{*}$. We now prove that $\mu_{0}$ is a Borel probability measure.

Let us consider the set of Borel probability measures $\mathcal{PB}(N_{1} \times N_{2})$ equipped with the Wassertein distance $d_{W}$ defined by $d_{W}(\mu, \nu) = \sup_{\text{Lip}(g) \leq 1} |\mu(g) - \nu(g)|$. $M_{\psi}$ is a closed subset of $\mathcal{PB}(N_{1} \times N_{2})$ for this topology. Indeed for each $\mu \in \mathcal{PB}(N_{1} \times N_{2})$, the projection $\pi_{x*}(\mu) \in \mathcal{PB}(N_{1})$ can be also characterized by its action on suitable Lipschitz observables: let $f \in \text{Lip}(N_{1})$, consider $\hat{f} \in \text{Lip}(N_{1} \times N_{2})$ be defined by $\hat{f}(x, y) = f(x)$. The projection $\pi_{x*}(\mu)$ can also be defined by the measure on $N_{1}$ for which

$$\int_{N_{1}} f \text{d} \pi_{x*}(\mu) = \int_{N_{1} \times N_{2}} \hat{f} \text{d} \mu.$$ 

If $\mu_{n} \to \mu$ in the $d_{W}$ topology and $\mu_{n} \in M_{\psi}$ for such a function $\hat{f}$ we have $\int_{N_{1} \times N_{2}} \hat{f} \text{d} \nu_{n} \to \int_{N_{1} \times N_{2}} \hat{f} \text{d} \mu$ this shows that $\pi_{x*}(\mu) = \psi_{x}$.

Furthermore, we have that if $\mu, \nu \in M_{\psi}$ it holds $d_{W}(\mu, \nu) \leq ||\mu - \nu||_{1}$. Indeed for every $g$ such that $\text{Lip}(g) \leq 1$, disintegrating the two measures on the stable foliation it holds

$$\int \text{d}[\mu - \nu] = \int_{N_{1}} \int_{N_{2}} g(\gamma, \cdot) \text{d}[\mu_{\gamma} - \nu_{\gamma}] \text{d} \psi_{x}.$$ 

For every $\gamma$ $g(\gamma, \cdot)$ is 1-Lipschitz on the stable leaf. Hence

$$\int \text{d}[\mu - \nu] \leq \int_{N_{1}} ||\mu_{\gamma} - \nu_{\gamma}||_{W} \text{d} \psi_{x} = ||\mu - \nu||_{1}.$$ 

By this a Cauchy sequence for the $|| ||_{1}$ norm is also a Cauchy sequence for $d_{W}(\mu, \nu)$. By Lemma 6.4 we have that $\nu_{n}$ has a limit in $\mathcal{PB}(N_{1} \times N_{2})$ in the $d_{W}$ topology. Since $M_{\psi}$ is closed in this topology, we get $\mu_{0} \in M_{\psi} \subseteq S^{1}$. Since this invariant measure is the fixed point of a contraction, it is unique. 

Another construction to show the existence of an invariant measure in the context of fiber contracting maps can be found in [3] (subsection 7.3.4.1). If the system satisfies the assumption $N1$ we can also prove a stronger statement

**Proposition 10.** If $N1$ is satisfied, $\mu_{0}$ is the unique $F$-invariant probability in $S^{\infty}$.

\footnote{See (11) for the definition of the space.}
Lemma 6.6. Under the assumptions where \( \psi_x \) is the unique \( T \)-invariant density (see T3.4) in \( S_- \). If N1 is satisfied, we have \(| x | \leq | x_s | \). Suppose that \( g : N_2 \rightarrow \mathbb{R} \) is a Lipschitz function such that \(| g(x) \leq 1 \) and \( L(g) \leq 1 \). Then, it holds \( | P g(\mu_0,\gamma) | \leq | g(x) \psi_x(\gamma) \leq | \psi_x(\gamma) | \leq | \psi_x | \). Hence, \( \mu_0 \in S^\infty \).

6.2. \( L^\infty \) norms. In this section we consider an \( L^\infty \) like anisotropic norm. We show how a Lasota Yorke inequality can be proved for this norm too.

Lemma 6.5. Under the assumptions G1, T1, ..., T3.3, for all signed measure \( \mu \in S^\infty \) with marginal density \( \phi_x \) it holds

\[
\| F \ast \mu \|_\infty \leq \alpha \| \mu \|_\infty + | P R \phi_x | \infty.
\]

Proof. Let \( T_i \) be the branches of \( T \), for all \( i = 1 \cdots q \). Applying Lemma 6.2 on the third line below, we have

\[
\| (F \ast \mu)(\gamma) \|_W = \left| \sum_{i=1}^{q} {\frac{F_{T_i^{-1}(\gamma)} \ast \mu_{T_i^{-1}(\gamma)}}{| \det DT_i(T_i^{-1}(\gamma)) |} \chi_T(P_i)(\gamma)} \right|_W
\]

\[
\leq \sum_{i=1}^{q} {\frac{| F_{T_i^{-1}(\gamma)} \ast \mu_{T_i^{-1}(\gamma)} |_W}{| \det DT_i(T_i^{-1}(\gamma)) |} \chi_T(P_i)(\gamma)}
\]

\[
\leq \sum_{i=1}^{q} \alpha | \mu_{T_i^{-1}(\gamma)} |_W + \phi_x(T_i^{-1}(\gamma))
\]

\[
\leq \alpha | \mu \|_\infty \sum_{i=1}^{q} {\frac{\chi_T(P_i)(\gamma)}{| \det DT_i(T_i^{-1}(\gamma)) |}} + \sum_{i=1}^{q} {\frac{\phi_x(T_i^{-1}(\gamma))}{| \det DT_i(T_i^{-1}(\gamma)) |} \chi_T(P_i)(\gamma)}.
\]

Hence, taking the supremum on \( \gamma \), we finish the proof of the statement. \( \square \)

Applying the last lemma to \( F^{\ast n} \) instead of \( F \) one obtains.

Lemma 6.6. Under the assumptions G1, T1, ..., T3.4, for all signed measure \( \mu \in S^\infty \) it holds

\[
\| F^{\ast n} \mu \|_\infty \leq \alpha^n | P \ast 1 \|_\infty | \mu \|_\infty + | P \ast \phi_x | \infty,
\]

where \( \phi_x \) is the marginal density of \( \mu \).

Proposition 11 (Lasota-Yorke inequality for \( S^\infty \)). Suppose \( F \) satisfies the assumptions G1, T1, ..., T3.4 and N1. Then, there are 0 < \( \alpha_1 < 1 \) and \( A_1, B_4 \in \mathbb{R} \) such that for all \( \mu \in S^\infty \), it holds

\[
\| F^{\ast n} \mu \|_S^\infty \leq A_1 \alpha^n | \mu \|_S^\infty + B_4 | \mu \|_1.
\]

Proof. By equation (3) and (N1) it follows \( | P \ast 1 \|_\infty \leq H_N(B_3 + C_2) \), for each \( n \). Then,

\[
\| F^{\ast n} \mu \|_S^\infty = | P \ast \phi_x |_S + \| F \ast \mu \|_\infty
\]

\[
\leq [B_3 \beta_2^n | \phi_x |_S + C_2 | \phi_x |_1] + [\alpha^n | P \ast 1 \|_\infty | \mu \|_\infty + | P \ast \phi_x | \infty]
\]

\[
\leq [B_3 \beta_2^n | \phi_x |_S + C_2 | \phi_x |_1] + [\alpha^n H_N(B_3 + C_2) | \mu \|_\infty + H_N(B_3 \beta_2^n | \phi_x |_S + C_2 | \phi_x |_1)]
\]

\[
\leq [\max(\alpha, \beta_2)^n | B_3 (1 + 2 H_N) + H_N C_2 | \mu \|_S^\infty + C_2 (1 + H_N) | \mu \|_1,
\]
where $|\phi_{x,1}| \leq ||\mu||_1$ and $|\phi_{x,\alpha}| \leq ||\mu||_{S^\alpha}$. We finish the proof, setting $\alpha_1 = \max(\alpha, \beta_2)$, $A_1 = [B_3(1 + 2H_N) + H_NC_2]$ and $B_4 = C_2(1 + H_N)$. \hfill \Box

7. Spectral gap. In this section, we prove a spectral gap statement for the transfer operator applied to our strong spaces. For this, we will directly use the properties proved in the previous section, and this will give a kind of constructive proof. We remark that, we cannot apply the traditional Hennion, or Ionescu-Tulcea and Marinescu’s approach to our function spaces because there is no compact immersion of the strong space into the weak one. This comes from the fact that we are considering the same “dual of Lipschitz”distance (see Definition 4.4) in the contracting direction for both spaces.

**Theorem 7.1** (Spectral gap on $S^1$). If $F$ satisfies $G1$, $T1$, ..., $T3.4$ given at beginning of section 2, then the operator $F_* : S^1 \rightarrow S^1$ (see (10)) can be written as

$$F_* = P + N,$$

where

1. $P$ is a projection i.e. $P^2 = P$ and $\dim Im(P) = 1$;
2. there are $0 < \xi < 1$ and $K > 0$ such that $\forall \mu \in S^1$

$$||N^n(\mu)||_{S^1} \leq ||\mu||_{S^1} \xi^n K;$$
3. $PN = NP = 0.$

**Proof.** First, let us show there exist $0 < \xi < 1$ and $K_1 > 0$ such that, for all $n \geq 1$, it holds

$$||F^n_*||_{\mathcal{V}_1 \rightarrow \mathcal{V}_s} \leq \xi^n K_1$$

where $\mathcal{V}_s$ is the zero average space defined in (21). Indeed, consider $\mu \in \mathcal{V}_s$ (see (21)) s.t. $||\mu||_{S^1} \leq 1$ and for a given $n \in \mathbb{N}$ let $m$ and $0 < d \leq 1$ be the coefficients of the division of $n$ by 2, i.e. $n = 2m + d$. Thus $m = \frac{n - d}{2}$. By the Lasota-Yorke inequality (Proposition 7) we have the uniform bound $||F^n_* \mu||_{S^1} \leq B_2 + A$ for all $n \geq 1$. Moreover, by Propositions 9 and 6 there is some $D_2$ such that it holds (below, let $\lambda_0$ be defined by $\lambda_0 = \max{\{\beta_1, \lambda\}}$)

$$||F^n_* \mu||_{S^1} \leq A\lambda^m ||F^m_* + \mu||_{S^1} + B_2 ||F^m_* + \mu||_1 \leq \lambda^m A(A + B_2) + B_2 ||F^m_* + \mu||_1 \leq \lambda^m A(A + B_2) + B_2 D_2 \beta_1^m \leq \lambda^m_0 [A(A + B_2) + B_2 D_2] \leq \lambda^m_0 \left(\sqrt{\lambda_0}\right)^m \frac{1}{\lambda_0} \frac{1}{2} [A(A + B_2) + B_2 D_2] \leq \xi^n K_1,$$

where $\xi = \sqrt{\lambda_0}$ and $K_1 = \left(\frac{1}{\lambda_0}\right)^{\frac{1}{2}} [A(A + B_2) + B_2 D_2]$. Thus, we arrive at

$$||(F_*|_{\mathcal{V}_s})^n||_{S^1 \rightarrow S^1} \leq \xi^n K_1.$$
Now, recall that \( F^*: S^1 \to S^1 \) has an unique fixed point \( \mu_0 \in S^1 \), which is a probability (see Proposition 10). Consider the operator \( P: S^1 \to [\mu_0] \) ([\( \mu_0 \)]) is the space spanned by \( \mu_0 \)), defined by \( P(\mu) = \mu(\Sigma)\mu_0 \). By definition, \( P \) is a projection and \( \text{dim} \text{Im}(P) = 1 \). Define the operator
\[
S: S^1 \to \mathcal{V}_s,
\]
by
\[
S(\mu) = \mu - P(\mu), \quad \forall \ \mu \in S^1.
\]
Thus, we set \( N = F^* \circ S \) and observe that, by definition, \( PN = NP = 0 \) and \( F^* = P + N \). Moreover, \( N^n(\mu) = F^* \circ S(\mu) \) for all \( n \geq 1 \). Since \( S \) is bounded and \( S(\mu) \in \mathcal{V}_s \), we get by (23), \( \|N^n(\mu)\|_{S^1} \leq \xi^n K\|\mu\|_{S^1} \), for all \( n \geq 1 \), where \( K = K_1\|S\|_{S^1 \to S^1} \).

In the same way, using the \( L^\infty \) Lasota-Yorke inequality of Proposition 11, and Lemma 6.6 it is possible to obtain exponential convergence to equilibrium (see the proof of Proposition 9) and spectral gap on the \( L^\infty \) like strong and weak spaces \((L^\infty, \|\cdot\|_{L^\infty}) \) and \((S^\infty, \|\cdot\|_{S^\infty}) \). We omit the proof which is essentially the same as above:

**Theorem 7.2** (Spectral gap on \( S^\infty \)). If \( F \) satisfies the assumptions \( G1, T1, ..., T3.4 \) and \( N1 \), then the operator \( F^*: S^\infty \to S^\infty \) can be written as
\[
F^* = P + N,
\]
where

a) \( P \) is a projection i.e. \( P^2 = P \) and \( \text{dim} \text{Im}(P) = 1 \);

b) there are \( 0 < \xi_1 < 1 \) and \( K_2 > 0 \) such that \( \|N^n(\mu)\|_{S^\infty} \leq \|\mu\|_{S^\infty} \xi_1^n K_2 \) \( \forall \ \mu \in S^\infty \);

c) \( PN = NP = 0 \).

**Remark 4.** The constant \( \xi \) for the map \( F \), found in Theorem 7.1, is directly related to the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of \( F \) found before (see Remark 3). More precisely, \( \xi = \max\{\sqrt{\lambda}, \sqrt{\beta_1}\} \).

We remark that, from the above proof we also have an explicit estimate for \( K \) in the exponential convergence, while many classical approaches are not suitable for this.

### 7.1. Exponential decay of correlations

In this section, we present one of the standard consequences of spectral gap. We will show how Theorems 7.1 and 7.2 implies an exponential rate of convergence for the limit
\[
\lim C_n(f, g) = 0,
\]
where
\[
C_n(f, g) := \left| \int (g \circ F^n) f d\mu_0 - \int g d\mu_0 \int f d\mu_0 \right|,
\]
g : \( \Sigma \to \mathbb{R} \) is a Lipschitz function and \( f \in \Theta_{\mu_0}^1 \) or \( f \in \Theta_{\mu_0}^\infty \). The sets \( \Theta_{\mu_0}^1 \) and \( \Theta_{\mu_0}^\infty \) are defined as
\[
\Theta_{\mu_0}^1 := \{ f : \Sigma \to \mathbb{R}; f\mu_0 \in S^1 \}
\]
and
\[
\Theta_{\mu_0}^\infty := \{ f : \Sigma \to \mathbb{R}; f\mu_0 \in S^\infty \},
\]
where the measure \( f\mu_0 \) is defined by \( f\mu_0(E) := \int_E f d\mu_0 \) for all measurable set \( E \).
Proposition 12. For all Lipschitz function \( g : \Sigma \rightarrow \mathbb{R} \) and all \( f \in \Theta_{\mu_0}^1 \), it holds
\[
\left| \int (g \circ F^n) f d\mu_0 - \int g d\mu_0 \int f d\mu_0 \right| \leq \|f\|_{S^1} K \|g\|_{\text{Lip}} \xi^n \quad \forall n \geq 1,
\]
where \( \xi \) and \( K \) are from Theorem 7.1 and \( \|g\|_{\text{Lip}} := \|g\|_{\infty} + L(g) \).

Proof. Let \( g : \Sigma \rightarrow \mathbb{R} \) be a Lipschitz function and \( f \in \Theta_{\mu_0}^1 \). By Theorem 7.1, we have
\[
\left| \int (g \circ F^n) f d\mu_0 - \int g d\mu_0 \int f d\mu_0 \right| = \left| \int g dF^n(f\mu_0) - \int g dP(f\mu_0) \right| \leq \|F^n(f\mu_0) - P(f\mu_0)\|_W \max\{L(g), \|g\|_{\infty}\} \leq \|N^n(f\mu_0)\|_W \max\{L(g), \|g\|_{\infty}\} \leq \|f\|_{S^1} K \|g\|_{\text{Lip}} \xi^n.
\]

By the same argument as above and by Theorem 7.2 it holds the following.

Proposition 13. For all Lipschitz function \( g : \Sigma \rightarrow \mathbb{R} \) and all \( f \in \Theta_{\mu_0}^{\infty} \), it holds
\[
\left| \int (g \circ F^n) f d\mu_0 - \int g d\mu_0 \int f d\mu_0 \right| \leq \|f\|_{S^1} K \|g\|_{\text{Lip}} \xi^n \quad \forall n \geq 1,
\]
where \( \xi \) and \( K_2 \) are from Theorem 7.2.

In Proposition 20 we will see that under some further assumptions on the system, the sets \( \Theta_{\mu_0}^1 \) contains the set of Lipschitz functions on \( \Sigma \).

8. Application to Lorenz-like maps. In this section, we apply Theorems 7.1 and 7.2 to a large class of maps which are Poincaré maps for suitable sections of Lorenz-like flows. In these systems (see e.g. [3]), it can be proved that there is a two dimensional Poincaré section \( \Sigma \) which can be supposed to be a rectangle \( I^2 \), where \( I = [0,1] \), whose return map \( F_L : I^2 \rightarrow I^2 \), after a suitable change of coordinates, has the form \( F_L(x,y) = (T_L(x), G_L(x,y)) \), satisfying the properties, \( G1 \) and \( T1-T3 \), of section 2. The map \( T_L : I \rightarrow I \), in this case, can be supposed to be piecewise expanding with \( C^{1+\alpha} \) branches.

Hence, we consider a class of skew product maps \( F_L : I^2 \rightarrow I^2 \), where \( I = [0,1] \), satisfying \((G1),(T1),(T2)\), and the following properties on \( T_L \):

8.0.1. Properties of \( T_L \) in Lorenz-like systems.

\( (P'1) \quad \frac{1}{|T'_L|} \) is of universal bounded \( p \)-variation, i.e. for \( p \geq 1 \)
\[
\var_p \left( \frac{1}{|T'_L|} \right) := \sup_{0 \leq x_0 < \cdots < x_n \leq 1} \left( \sum_{i=0}^{n} \left| \frac{1}{T'_L(x_i)} - \frac{1}{T'_L(x_{i-1})} \right|^p \right)^{\frac{1}{p}} < \infty;
\]

\( (P'2) \inf |T''_L| \geq \lambda_1 > 1 \), for some \( n_0 \in \mathbb{N} \).
Proposition 14. It holds the following (see [23]).

A piecewise expanding map satisfying assumptions (P’1) and (P’2) has an in-

From properties P’1 and P’2, it follows (see [23]) that there exists a suitable

Definition 8.1. Let $m_1$ be the Lebesgue measure on $I = [0,1]$. For an arbitrary

Definition 8.2. Fix $A_1 > 0$ and denote by $\Phi$ the class of all isotonic maps $\phi : (0, A_1] \to [0, \infty]$, i.e. such that $x \leq y \implies \phi(x) \leq \phi(y)$ and $\phi(x) \to 0$ if $x \to 0$.

Set

- $R_1 = \{h : I \to \mathbb{C} ; \text{osc}_{1}(h, \cdot) \in \Phi\};$
- For $n \in \mathbb{N}$, define $R_{1,n,p} = \{h \in R_1 ; \text{osc}_{1}(h, \epsilon) \leq n \cdot \epsilon/p \ \forall \epsilon \in (0, A_1]\};$
- And set $S_{1,p} = \bigcup_{n \in \mathbb{N}} R_{1,n,p}.$

Definition 8.3. Let us consider the following spaces and semi-norms:

1. $BV_{1, p}$ is the space of $m_1$-equivalence classes of functions in $S_{1,p};$
2. Let $h : I \to \mathbb{C}$ be a measurable function. Set

$$\text{var}^{1, p}_{1}(h) = \sup_{0 \leq \epsilon \leq A_1} \left( \frac{1}{\epsilon^{p}} \text{osc}_{1}(h, \epsilon) \right).$$

Since $BV_{1, 1/p}$ was defined using a probability measure, $m_1$, then $\text{var}^{1, 1/p}_1(h) \leq 2^{1/p} \text{var}_p(h)$ (see [23], Lemma 2.7).

Let us consider $| \cdot |_{1, p} : BV_{1, p} \to \mathbb{R}$ defined by

$$|f|_{1, p} = \text{var}^{1, p}_1(f) + |f|_1,$$

it holds the following (see [23]).

Proposition 14. $(BV_{1, p}, | \cdot |_{1, p})$ is a Banach space.

In the above setting, G. Keller has shown (see [23]) that there is an $A_1 > 0$ (we

The universal bounded $p$-variation, $\text{var}_p$, is a generalization of the usual bounded variation. It is a weaker notion, allowing piecewise Holder functions. Indeed, for $p \geq 1$, a $1/p$-Holder function is of universal bounded $p$-variation. This definition is adapted to maps having $C^{1+\alpha}$ regularity.

From properties P’1 and P’2, it follows (see [23]) that there exists a suitable

strong space (the space $S_{-}$ in T3.1) for the Perron-Frobenius operator $P_T$ associated

to such a $T_L$, in a way that it satisfies the assumptions T1, ..., T3.3 and N1. In this

case, supposing a property like T3.4 then we can apply our results. Therefore, let

us introduce the space of generalized bounded variation functions with respect to

the Lebesgue measure: $BV_{1, p}$. The functions of universal bounded $p$-variation are

included in this space (for more details and results see [23], in particular Lemma

2.7 for a comparison of the two spaces).

A piecewise expanding map satisfying assumptions (P’1) and (P’2) has an in-
v

variant measure with density in

$BV_{1, p}$, moreover the transfer operator restricted to this space satisfies a Lasota-Yorke inequality and other interesting properties, as we will see in the following.

Definition 8.1. Let $m_1$ be the Lebesgue measure on $I = [0,1]$. For an arbitrary

function $h : I \to \mathbb{C}$ and $\epsilon > 0$ define $\text{osc}(h, B_{\epsilon}(x)) : I \to [0, \infty]$ by

$$\text{osc}(h, B_{\epsilon}(x)) = \text{ess sup}\{|h(y_1) - h(y_2)| ; y_1, y_2 \in B_{\epsilon}(x)\},$$

where $B_{\epsilon}(x)$ denotes the open ball of center $x$ and radius $\epsilon$ and the essential supremum is taken with respect to the product measure $m_1^2$ on $I^2$. Also define the real

function $\text{osc}_{1}(h, \epsilon)$, on the variable $\epsilon$, by

$$\text{osc}_{1}(h, \epsilon) = \int \text{osc}(h, B_{\epsilon}(x))dm(x).$$

Definition 8.2. Fix $A_1 > 0$ and denote by $\Phi$ the class of all isotonic maps $\phi : (0, A_1] \to [0, \infty]$, i.e. such that $x \leq y \implies \phi(x) \leq \phi(y)$ and $\phi(x) \to 0$ if $x \to 0$.

Set

- $R_1 = \{h : I \to \mathbb{C} ; \text{osc}_{1}(h, \cdot) \in \Phi\};$
- For $n \in \mathbb{N}$, define $R_{1,n,p} = \{h \in R_1 ; \text{osc}_{1}(h, \epsilon) \leq n \cdot \epsilon^{p} \ \forall \epsilon \in (0, A_1]\};$
- And set $S_{1,p} = \bigcup_{n \in \mathbb{N}} R_{1,n,p}.$

Definition 8.3. Let us consider the following spaces and semi-norms:

1. $BV_{1, p}$ is the space of $m_1$-equivalence classes of functions in $S_{1,p};$
2. Let $h : I \to \mathbb{C}$ be a measurable function. Set

$$\text{var}^{1, p}_{1}(h) = \sup_{0 \leq \epsilon \leq A_1} \left( \frac{1}{\epsilon^{p}} \text{osc}_{1}(h, \epsilon) \right).$$

Since $BV_{1, 1/p}$ was defined using a probability measure, $m_1$, then $\text{var}^{1, 1/p}_1(h) \leq 2^{1/p} \text{var}_p(h)$ (see [23], Lemma 2.7).

Let us consider $| \cdot |_{1, p} : BV_{1, p} \to \mathbb{R}$ defined by

$$|f|_{1, p} = \text{var}^{1, p}_1(f) + |f|_1,$$

it holds the following (see [23]).

Proposition 14. $(BV_{1, p}, | \cdot |_{1, p})$ is a Banach space.

In the above setting, G. Keller has shown (see [23]) that there is an $A_1 > 0$ (we

recall that definition 8.2 depends on $A_1$) such that:
Proposition 15

(a) \( \text{BV}_{1, \frac{1}{p}} \subset L^1 \) is \( \text{P}_T \)-invariant, \( \text{P}_T : \text{BV}_{1, \frac{1}{p}} \rightarrow \text{BV}_{1, \frac{1}{p}} \) is continuous and it holds
\[ |\cdot| \leq |\cdot|_{1, \frac{1}{p}}; \]
(b) The unit ball of \( (\text{BV}_{1, \frac{1}{p}}, |\cdot|_{1, \frac{1}{p}}) \) is relatively compact in \( (L^1, |\cdot|) \);
(c) There exists \( k \in \mathbb{N}, 0 < \beta_0 < 1 \) and \( C > 0 \) such that, for every signed measure \( \mu \in \text{BV}_{1, \frac{1}{p}} \),
\[
|P_T^n f|_{1, \frac{1}{p}} \leq \beta_0 |f|_{1, \frac{1}{p}} + C |f|_1.
\]

Analogously to the proof of inequality (3), we have
\[
|P_T^n f|_{1, \frac{1}{p}} \leq B_3 \beta_0^n |f|_{1, \frac{1}{p}} + C_2 |f|_1, \quad \forall n, \quad \forall f \in \text{BV}_{1, \frac{1}{p}},
\]
for \( B_3, C_2 > 0 \) and \( 0 < \beta_2 < 1 \).

Moreover, in [2] (Lemma 2), it was shown that
(d)
\[
|\cdot|_{\infty} \leq A_1^{\frac{1}{p}-1} |\cdot|_{1, \frac{1}{p}}. \tag{25}
\]

Therefore, the properties T1, T2, T3.1, ..., T3.3, N1 of section 2 are satisfied with \( S_- = \text{BV}_{1, \frac{1}{p}} \). If T3.4 is also satisfied, then we can apply our construction to such maps.

Thus, for \( 1 \leq p < \infty \), we set
\[
\text{BV}_{1, \frac{1}{p}} := \left\{ \mu \in L^1; \text{var}_{1, \frac{1}{p}}(\phi_x) < \infty, \text{ where } \phi_x = \frac{d\mu_x}{dm_1} \right\}
\]
and consider \( ||\cdot||_{1, \frac{1}{p}} : \text{BV}_{1, \frac{1}{p}} \rightarrow \mathbb{R} \), defined by
\[
||\mu||_{1, \frac{1}{p}} = |\phi_x|_{1, \frac{1}{p}} + ||\mu||_1.
\]

Clearly, \( \left( \text{BV}_{1, \frac{1}{p}}, ||\cdot||_{1, \frac{1}{p}} \right) \) is a normed space. If we suppose that the system, \( T_L : I \rightarrow I \), satisfies T3.4, then it has an unique absolutely continuous invariant probability with density \( \varphi_x \in \text{BV}_{1, \frac{1}{p}} \).

As defined in equation (21), for \( 1 \leq p < \infty \), consider the set of zero average measures in \( \text{BV}_{1, \frac{1}{p}} \),
\[
\mathcal{V}_s = \{ \mu \in \text{BV}_{1, \frac{1}{p}} : \mu(\Sigma) = 0 \}.
\]

Directly from the above settings, Proposition 9 and from Theorem 7.1, using \( \text{BV}_{1, \frac{1}{p}} \) as a strong space (playing the role of \( S^1 \) in Theorem 7.1) it follows convergence to equilibrium and spectral gap for these kind of maps.

**Proposition 15** (Exponential convergence to equilibrium). If \( F_L \) satisfies assumptions G1, T1, T2, T3.4, P’1 and P’2, then there exist \( D_2 > 0 \) and \( 0 < \beta_2 < 1 \) such that, for every signed measure \( \mu \in \mathcal{V}_s \subset \text{BV}_{1, \frac{1}{p}}, 1 \leq p < \infty \), it holds
\[
||F_L^n \mu||_1 \leq D_2 \beta_1^n ||\mu||_{1, \frac{1}{p}},
\]
for all \( n \geq 1 \).

**Theorem 8.4** (Spectral gap for \( \text{BV}_{1, \frac{1}{p}} \)). If \( F_L \) satisfies assumptions G1, T1, T2, T3.4, P’1 and P’2, then the operator \( F_{L*} : \text{BV}_{1, \frac{1}{p}} \rightarrow \text{BV}_{1, \frac{1}{p}} \) can be written as
\[
F_{L*} = P + N
\]
where
(a) \( P \) is a projection i.e. \( P^2 = P \) and \( \text{dim} \text{Im}(P) = 1; \)
b) there are $0 < \xi < 1$ and $K > 0$ such that for all $\mu \in BV_{1, \frac{1}{p}}$
\[ ||N^n(\mu)||_{BV_{1, \frac{1}{p}}} \leq \xi^n K ||\mu||_{BV_{1, \frac{1}{p}}}; \]

c) $PN = NP = 0$.

We can get the same kind of results for stronger $L^\infty$ like norms. Let us consider
\[ BV_{1, \frac{1}{p}} := \{ \mu \in L^\infty; \frac{d(\pi_x \mu)}{dm_1} \in BV_{1, \frac{1}{p}} \} \]
and the function, $||\cdot||_{1, \frac{1}{p}} : BV_{1, \frac{1}{p}} \rightarrow \mathbb{R}$, defined by
\[ ||\mu||_{1, \frac{1}{p}} = |\phi_x|_{1, \frac{1}{p}} + ||\mu||_{\infty}. \]
Applying Theorem 7.2 using $BV_{1, \frac{1}{p}}$ as a strong space (playing the role of $S^\infty$) we get

**Theorem 8.5** (Spectral gap for $BV_{1, \frac{1}{p}}$). If $F_L$ satisfies assumptions $G 1, T 1, T 2, T3.4, P'1$ and $P'2$, then the operator $F_{L^*} : BV_{1, \frac{1}{p}} \rightarrow BV_{1, \frac{1}{p}}$ can be written as
\[ F_{L^*} = P + N, \]
where
a) $P$ is a projection i.e. $P^2 = P$ and $\text{dim } \text{Im}(P) = 1$;
b) there are $0 < \xi_1 < 1$ and $K_2 > 0$ such that for all $\mu \in BV_{1, \frac{1}{p}}$
\[ ||N^n(\mu)||_{1, \frac{1}{p}} \leq \xi_1^n K_2 ||\mu||_{1, \frac{1}{p}}; \]

c) $PN = NP = 0$.

By Proposition 10 we immediately get

**Proposition 16.** If $F_L$ satisfies assumptions $G 1, T1, T2, T3.4, P'1$ and $P'2$, then the unique invariant probability for the system $F_L$ in $BV_{1, \frac{1}{p}}$ is $\mu_0$. Moreover, since $N1$ is satisfied (equation (25)), $\mu_0$ is the unique $F_L$-invariant probability in $BV_{1, \frac{1}{p}}$.

9. **Quantitative statistical stability.** Throughout this section, we consider small perturbations of the transfer operator of a particular system of the kind described in the previous sections and study the dependence of the physical invariant measure with respect to the perturbation. A classical tool that can be applied for this type of problems is the Keller-Liverani stability theorem [24]. Since in our setting the strong space is not compactly immersed in the weak one, we cannot directly apply it. We will use another approach giving us precise bounds on the statistical stability. In this section, this approach will be applied to a class of Lorenz-like maps with slightly stronger regularity assumptions than used in Section 8. We call such a system by $BV$ Lorenz-like map (see Definition 9.3) and precisely, we need the additional property stated in item (1) of Definition 9.3.
9.0.1. Uniform family of operators. In this subsection we present a general quantitative result relating the stability of the invariant measure of an uniform family of operators (Definition 9.1) and convergence to equilibrium.

In the following definition, for all \( \delta \in [0, 1) \), let \( L_\delta \) be a Markov operator acting on two vector subspaces of signed measures on \( X \), \( L_\delta : (B_s, ||\cdot||_s) \rightarrow (B_s, ||\cdot||_s) \) and \( L_\delta : (B_w, ||\cdot||_w) \rightarrow (B_w, ||\cdot||_w) \), endowed with two norms, the strong norm \( ||\cdot||_s \) on \( B_s \), and the weak norm \( ||\cdot||_w \) on \( B_w \), such that \( ||\cdot||_s \geq ||\cdot||_w \). Suppose that,

\[
B_s \subseteq B_w \subseteq SB(X),
\]

where \( SB(X) \) denotes the space of Borel signed measures on \( X \).

**Definition 9.1.** A one parameter family of transfer operators \( \{L_\delta\}_{\delta \in [0,1)} \) is said to be an uniform family of operators with respect to the weak space \((B_w, ||\cdot||_w)\) and the strong space \((B_s, ||\cdot||_s)\) if \( ||\cdot||_s \geq ||\cdot||_w \) and it satisfies

**UF1** Let \( \mu_\delta \in B_s \) be a probability measure fixed under the operator \( L_\delta \). Suppose there is \( M > 0 \) such that for all \( \delta \in [0, 1) \), it holds

\[
||\mu_\delta||_s \leq M;
\]

**UF2** \( L_\delta \) approximates \( L_0 \) when \( \delta \) is small in the following sense: there is \( C \in \mathbb{R}^+ \) such that:

\[
||(L_0 - L_\delta)\mu_\delta||_w \leq \delta C;
\]

**UF3** \( L_0 \) has exponential convergence to equilibrium with respect to the norms \( ||\cdot||_s \) and \( ||\cdot||_w \): there exists \( 0 < \rho_2 < 1 \) and \( C_2 > 0 \) such that

\[
\forall \mu \in V_s := \{\mu \in B_s : \mu(X) = 0\}
\]

it holds

\[
||L_0^n \mu||_w \leq \rho_2^n C_2 ||\mu||_s;
\]

**UF4** The iterates of the operators are uniformly bounded for the weak norm: there exists \( M_2 > 0 \) such that

\[
\forall \delta, n, \nu \in B_s \text{ it holds } ||L_\delta^n \nu||_w \leq M_2 ||\nu||_w.
\]

Under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when \( L_0 \) is perturbed to \( L_\delta \), for small values of \( \delta \). Moreover, the modulus of continuity can be estimated. We postpone the proof of Proposition 17 to the Appendix 3 (section 12).

**Proposition 17.** Suppose \( \{L_\delta\}_{\delta \in [0,1)} \) is a uniform family of operators as in Definition 9.1, where \( \mu_0 \) is the unique fixed point of \( L_0 \) in \( B_w \) and \( \mu_\delta \) is a fixed point of \( L_\delta \). Then, there exists \( \delta_0 \in (0, 1) \) such that for all \( \delta \in [0, \delta_0) \), it holds

\[
||\mu_\delta - \mu_0||_w = O(\delta \log \delta).
\]

9.1. Quantitative stability of Lorenz-like maps. In this subsection we apply the above general result on uniform family of operators (Proposition 17) to a suitable family of bounded variation Lorenz-like maps. We consider families of maps as defined in Section 8, with some further regularity assumptions defining uniform families of Bounded Variation Lorenz-like maps (see Definitions 9.3 and 9.6). For these families we prove that the invariant measures associated to a size \( \delta \) perturbation varies continuously as the map is perturbed, with modulus of continuity \( \delta \log \delta \). Precisely, the aim of this section is to prove the following theorem:
Theorem 9.2. [Quantitative stability for deterministic perturbations] Let \( \{F_\delta\}_{\delta \in [0,1)} \) be a Uniform BV Lorenz-like family (see definition 9.6). Denote by \( \mu_\delta \) the fixed probability measures of \( F_\delta \) in \( BV_{1,1} \) (also in \( BV_{1,1}^\infty \)), for all \( \delta \). Then, there exists \( \delta_0 \in (0,1) \) such that for all \( \delta \in [0, \delta_0) \), it holds
\[
||\mu_\delta - \mu_0||_1 = O(\delta \log \delta).
\]

The proof will be postponed to the end of the section.

Remark 5. We believe that using the techniques of [18] in which a sort of generalized bounded variation for disintegrated measures is considered in the spirit of the work [23] we could get a similar result removing the additional Bounded Variation regularity to the Lorenz-like family.

Remark 6. A straightforward computation (see the proof of Lemma 6.3) yields \( ||\cdot||_W \leq ||\cdot||_1 \). Then, by Theorem (9.2), it holds
\[
||\mu_\delta - \mu_0||_W \leq A\delta \log \delta,
\]
for some \( A > 0 \). Therefore, for all Lipschitz function \( g : [0,1]^2 \rightarrow \mathbb{R} \), the following estimate holds
\[
|\int gd\mu_\delta - \int gd\mu_0| \leq A||g||_{Lip}\delta \log \delta,
\]
where \( ||g||_{Lip} = ||g||_{\infty} + L(g) \) (see equation (8), for the definition of \( L(g) \)). Thus, for all Lipschitz functions, \( g : [0,1]^2 \rightarrow \mathbb{R} \), the limit \( \lim_{\delta \rightarrow 0} \int gd\mu_\delta = \int gd\mu_0 \) holds, with rate of convergence smaller than or equal to \( \delta \log \delta \).

Remark 7. It is well known (see [17] e.g) that the modulus of continuity \( \delta \log(\delta) \) is optimal for suitable deterministic perturbations of piecewise expanding maps (which are the basis maps of our piecewise hyperbolic system). Therefore, the estimate given in Theorem 9.2 is optimal too. To realize this, consider a sequence of piecewise expanding maps \( T_n \) with absolutely continuous invariant measures \( \mu_n \), realizing the modulus of continuity \( \delta \log(\delta) \). Consider \( F_n : I^2 \rightarrow I^2 \) given by \( F_n(x,y) = (T_n(x), \frac{y}{2}) \) (the second component contracts everything to \( \frac{1}{2} \)). The sequence \( F_n \) has a sequence of invariant measures \( \nu_n \) of the kind \( \nu_n = \mu_n \times \frac{1}{2} \) for which is easy to see that \( ||\nu_n - \nu_0||_1 \geq A\delta_n \log(\delta_n) \).

We now precise the definition of BV Lorenz-like map and BV Lorenz-like family considered in the Theorem 9.2.

Definition 9.3. A map \( F_L : [0,1]^2 \rightarrow [0,1]^2 \), \( F_L(x,y) = (T_L(x), G_L(x,y)) \), is said to be a BV Lorenz-like map if it satisfies

1. There are \( H \geq 0 \) and a partition \( P' = \{J_i := (b_{i-1}, b_i), i = 1, \ldots, d \} \) of \( I \) such that for all \( x_1, x_2 \in J_i \) and for all \( y \in I \) the following inequality holds
\[
|G_L(x_1, y) - G_L(x_2, y)| \leq H \cdot |x_1 - x_2|;
\]
2. \( F_L \) satisfy property G1 (hence is uniformly contracting on each leaf \( \gamma \) with rate of contraction \( \alpha \));
3. \( T_L : I \rightarrow I \) is a piecewise expanding map satisfying the assumptions given in the following definition 9.4.

The following definition characterizes a class of piecewise expanding maps of the interval with bounded variation derivative \( T_L : I \rightarrow I \) which is a subclass of the ones considered in section 8.0.1.
**Definition 9.4** (Piecewise expanding functions with bounded variation inverse of the derivative). Suppose there exists a partition $\mathcal{P} = \{P_i := (a_{i-1}, a_i), i = 1, \cdots, q\}$ of $I$ s.t. $T_L : I \to I$ satisfies the following conditions. For all $i$

1) $T_{L_i} = T_L|_{P_i}$ is of class $C^1$ and $g_i = \frac{1}{|T_{L_i}|}$ satisfies (P’1) of section 8, for $p = 1$.

2) $T_L$ satisfies (P’2) of section 8: $\inf |T_L^n|_g \geq \lambda_1 > 1$ for some $n_0 \in \mathbb{N}$.

3) $T_L$ satisfies T3.4.

In particular we assume that $T_{L_i}$ and $g_i$ admit a continuous extension to $\overline{T_i} = [a_{i-1}, a_i]$ for all $i = 1, \cdots, q$.

**Remark 8.** The definition 9.4 allows infinite derivative for $T_L$ at the extreme points of its regularity intervals.

**Definition 9.5.** Let $T_1$ and $T_2$ be two piecewise expanding maps of definition (9.4). Define the set $Int_n$, by

$$Int_n = \{A \subset [0,1], s.t. \ A = I_1 \cup \cdots \cup I_n, \text{ where } I_i \text{ are intervals}\}$$

the set of subsets of $[0,1]$ which is the union of at most $n$ intervals. Set

$$C(n, T_1, T_2) = \left\{ \epsilon : \exists A_1 \in Int_n \text{ and } \exists \sigma : I \to I \text{ a diffeomorphism s.t. } m_1(A_1) \geq 1 - \epsilon, \ T_1|_{A_1} = T_2 \circ \sigma|_{A_1} \text{ and } \forall x \in A_1, |\sigma(x) - x| \leq \epsilon, |\frac{1}{\sigma'(x)} - 1| \leq \epsilon \right\}$$

and define a distance from $T_1$ to $T_2$ as:

$$d_{S,n}(T_1, T_2) = \inf \{\epsilon | \epsilon \in C(n, T_1, T_2)\}. \quad (27)$$

If we denote by $d_S$ the classical notion of Skorokhod distance (see [10] e.g.), it is obvious that $\forall n \ d_{S,n} \geq d_S$. By [10], Lemma 11.2.1, it follows that $\forall n$:

$$|P_{T_1} - P_{T_2}|_{BV \to L^1} \leq 14d_{S,n}(T_1, T_2). \quad (28)$$

**Definition 9.6.** A family of maps $\{F_\delta\}_{\delta \in [0,1]}$ is said to be a *Uniform BV Lorenz-like family* if $F_\delta$ is a BV Lorenz-like map (see definition 9.3) for all $\delta \in [0,1)$ and $\{F_\delta\}_\delta$ satisfies the following assumptions:

(UBV1): there exist $0 < \lambda < 1$ and $D > 0$ s.t. for all $f \in BV_{1,1}$ and for all $\delta \in [0,1)$ it holds $|P_{T_\delta} f|_{1,1} \leq \lambda |f|_{1,1} + D |f|_1$ for all $n \geq 1$, where $P_{T_\delta}$ is the Perron-Frobenius operator of $T_\delta$.

When $\delta$ is small

(UBV2): $T_0$ and $T_\delta$ are close with the above Skorokhod-like distance. For some $n$ independent of $\delta$ it holds $\forall \delta$

$$d_{S,n}(T_0, T_\delta) \leq \delta.$$

(UBV3): For each $\delta$ there is a set $A_2$ (depending on $\delta$) such that $A_2 \in Int_{n_\delta}$ for some $n_\delta$ (depending on $\delta$) furthermore $m_1(A_2) \geq 1 - \delta$ and for all $x \in A_2, y \in I$:

$$|G_0(x, y) - G_\delta(x, y)| \leq \delta.$$

Let us furthermore suppose that the number of such intervals during the perturbation remains uniformly bounded: $\sup_\delta n_\delta < \infty$.

For all $\delta \in [0,1)$, let $n_0 = n_0(\delta) \in \mathbb{N}$ be the first integer such that there exists $\lambda_1(\delta) > 0$ satisfying $|T_{n_0}^\delta(x)| \geq \lambda_1(\delta) > 1$ for all $x \in P_{\delta,i}$ and for each $i = 1, \cdots, q(\delta)$, where $T_{n_0}^\delta := T_\delta|_{P_{\delta,i}}$. Also set $g_{i,\delta} = \frac{1}{|T_{n_0}^\delta|}$ and denote by $H_\delta > 0$
and $\mathcal{P}_\delta'$ the “Lipschitz” constant and the regularity partition associated to $G_\delta$, see item (1) of Definition 9.3 and Definition 9.4.

(UBV4): Suppose that:
(1) $\inf i, \lambda_1(\delta) > 1$, $\sup i, \lambda_1(\delta) < \infty$ and $\sup \delta \in [0,1) \{ n_0(\delta) \} < \infty$;
(2) there exists $C_4 > 0$ such that $\sup g_{\delta,i} \leq C_4$ and $\text{var} g_{\delta,i} \leq C_4$ for all $i = 1, \ldots, q(\delta)$ and all $\delta \in [0,1)$;
(3) $\inf \delta \in [0,1) \min i=1,\ldots,q(\delta) \{ m_1(P_\delta, i) \} > 0$;
(4) $\sup \delta \in [0,1) H_\delta < \infty$, $\sup \delta \in [0,1) \# \mathcal{P}_\delta < \infty$.

9.1.1. Measures with bounded variation. Here, we introduce a space of measures having bounded variation in some stronger sense, and prove that the invariant measure of a BV Lorenz-like map is in it. We use this fact in the proof of Proposition 23, where we prove that the family of transfer operators $\{ F_{\delta} \}_{\delta \in (0,1)}$ induced by a Uniform BV Lorenz-like family $\{ F_\delta \}_{\delta \in (0,1)}$ satisfies UF2.

We have seen that a positive measure on the square, $[0,1]^2$, can be disintegrated along the stable leaves $\mathcal{F}^s$ in a way that we can see it as a family of positive measures on the interval, $\{ \mu_t \}_{t \in \mathcal{F}^s}$. Since there is a one-to-one correspondence between $\mathcal{F}^s$ and $[0,1]$, this defines a path in the metric space of positive measures, $[0,1] \longrightarrow \mathcal{SB}(I)$, where $\mathcal{SB}(I)$ is endowed with the Wasserstein-Kantorovich like metric (see definition 4.4). It will be convenient to use a functional notation and denote such a path by $\Gamma_\mu : I \longrightarrow \mathcal{SB}(I)$ defined for all $\gamma \in I$ (where $\mu_x = \phi_x (m_1 \mu)$, $\mu_x(I) = 1$, and $\phi_x : I_\omega \longrightarrow \mathbb{R}$ is a non-negative marginal density. Denote by $\Gamma_\mu$, the class of equivalent paths associated to $\mu$.

Definition 9.7. Consider a positive Borel measure $\mu$ and a disintegration $\omega = (\{ \mu_\gamma \}_{\gamma \in I_\omega}, \phi_x)$, where $\{ \mu_\gamma \}_{\gamma \in I_\omega}$ is a family of probabilities on $\Sigma$ defined for all $\gamma \in I_\omega$ (where $\mu_x = \phi_x (m_1 \mu)$, $\mu_x(I) = 1$, and $\phi_x : I_\omega \longrightarrow \mathbb{R}$ is a non-negative marginal density. Denote by $\Gamma_\mu$, the class of equivalent paths associated to $\mu$

$$\Gamma_\mu = \{ \Gamma_\mu^\omega \}_{\omega}.$$ 

where $\omega$ ranges on all the possible disintegrations of $\mu$ on the stable foliation and $\Gamma_\mu : I_\omega \longrightarrow \mathcal{SB}(I)$ is the path associated to a given disintegration, $\omega$:

$$\Gamma_\mu^\omega(\gamma) = \mu|_\gamma = \pi_{\gamma,y_+} \phi_x(\gamma) \mu_\gamma.$$ 

Definition 9.8. Let $\mathcal{P} = \mathcal{P}(\Gamma_\mu^\omega)$ be a finite sequence $\mathcal{P} = \{ x_i \}_{i=1}^{n} \subset I_\omega$ and define the variation of $\Gamma_\mu^\omega$ with respect to $\mathcal{P}$ as (denote $\gamma_i := \gamma_{x_i}$)

$$\text{Var}(\Gamma_\mu^\omega, \mathcal{P}) = \sum_{j=1}^{n} ||\Gamma_\mu^\omega(\gamma_j) - \Gamma_\mu^\omega(\gamma_{j-1})||_W,$$

where we recall $|| \cdot ||_W$ is the Wasserstein-like norm defined by equation (9). Finally, we define the variation of $\Gamma_\mu^\omega$ by taking the supremum over the set of finite sequences of any length, as

$$\text{Var}(\Gamma_\mu^\omega) := \sup_{\mathcal{P}} \text{Var}(\Gamma_\mu^\omega, \mathcal{P}).$$

Remark 9. For an interval $\eta \subset I$, we define

$$\text{Var}_\eta(\Gamma_\mu^\omega) := \text{Var}(\Gamma_\mu^\omega|_\eta),$$
where $\bar{\eta}$ is the closure of $\eta$. We also remark that $\text{Var}_{\bar{\eta}}(\Gamma^\omega_{\mu}) = \text{Var}(\Gamma^\omega_{\mu} \cdot \chi_{\bar{\eta}})$, where $\chi_{\bar{\eta}}$ is the characteristic function of $\bar{\eta}$.

**Remark 10.** When no confusion can be done, to simplify the notation, we denote $\Gamma^\omega_{\mu}(\gamma)$ just by $\mu|_{\gamma}$.

**Definition 9.9.** Define the [variety of a positive measure $\mu$ by]

$$\text{Var}(\mu) = \inf_{\Gamma^\omega_{\mu} \in \Gamma_{\mu}} \{\text{Var}(\Gamma^\omega_{\mu})\}. \quad (29)$$

We remark that,

$$||\mu||_1 = \int W^0(0, \Gamma^\omega_{\mu}(\gamma)) d\mu(\gamma), \quad \text{for any } \Gamma^\omega_{\mu} \in \Gamma_{\mu}.$$

**Definition 9.10.** From the definition 9.8 we define the set of bounded variation positive measures $BV^+$ as

$$BV^+ = \{ \mu \in \mathcal{A}B : \mu \geq 0, \text{Var}(\mu) < \infty \}.$$

Now we are ready to state a proposition which will provide an estimative for the regularity of the iterates $F^n_{\mu}(m)$. Next inequality (31), is a Lasota-Yorke like inequality, where the variation, $\text{Var}(\mu)$, defined in 9.9, plays the role of the strong semi-norm. This is our main tool to estimate the regularity of the invariant measure of a BV Lorenz-like map (Proposition 19) and it is an immediate consequence of Theorem 10.2 and Remark 14 which are proved in Appendix 1.

**Proposition 18.** Let $F_L(x, y) = (T_L(x), G_L(x, y))$ be a BV Lorenz-like map. Then, there are $K_0$ and $0 < \lambda_0 < 1$ such that for all $\mu \in BV^+$, all disintegration $\omega$ of $\mu$ and all $n \geq 1$ it holds

$$\text{Var}(\Gamma^\omega_{F_*^n \mu}) \leq K_0 \lambda_0^n \text{Var}(\Gamma^\omega_{\mu}) + K_0 |\phi_x|_{1,1}. \quad (30)$$

**Remark 11.** Taking the infimum over all paths $\Gamma^\omega_{\mu} \in \Gamma_{\mu}$ on both sides of inequality (30), we get

$$\text{Var}(F_*^n \mu) \leq K_0 \lambda_0^n \text{Var}(\mu) + K_0 |\phi_x|_{1,1}. \quad (31)$$

A precise estimative for $K_0$ can be found in equation (45). Remember that, by Proposition 10, a Lorenz-like map has an invariant measure $\mu_0 \in S^\infty$.

**Remark 12.** Let $m$ be the Lebesgue measure on $\Sigma = I \times I$, i.e. $m = m_1 \times m_1$, where $m_1$ is the Lebesgue measure on $I = [0, 1]$. Besides that, consider its trivial disintegration $\omega_0 = \{(m_\gamma, \phi_x)\}$, given by $m_\gamma = \pi_{\gamma,1}^I m_1$, for all $\gamma$ and $\phi_x \equiv 1$. According to this definition, it holds that

$$m|_{\gamma} = m_1, \quad \forall \gamma.$$

In other words, the path $\Gamma^\omega_{\mu_0}$ is constant: $\Gamma^\omega_{\mu_0}(\gamma) = m_1$ for all $\gamma$. Moreover, for each $n \in \mathbb{N}$, let $\omega_n$ be the particular disintegration for the measure $F_*^n m$, defined from $\omega_0$ as an application of Lemma 5.1 and consider the path $\Gamma^\omega_{F_*^n m}$ associated with this disintegration. By Proposition 5 we have

$$\Gamma^\omega_{F_*^n m}(\gamma) = \sum_{i=1}^q \frac{F^n_{T_i^{-n}(\gamma)} \cdot m_1}{|\det DT_i^n \circ T_{-n}(\gamma)|} \chi_{P_i}(\gamma) \quad \forall \gamma \in \mathbb{N}_1, \quad (32)$$

where $P_i, i = 1, \ldots, q = q(n),$ ranges over the partition $\mathcal{P}(n)$ defined in the following way: for all $n \geq 1$, let $\mathcal{P}(n)$ be the partition of $I$ s.t. $\mathcal{P}(n)(x) = \mathcal{P}(n)(y)$ if and only if $\mathcal{P}(1)(T_j^i(x)) = \mathcal{P}(1)(T_j^i(y))$ for all $j = 0, \ldots, n - 1$, where $\mathcal{P}(1) = \mathcal{P}$ (see definition 9.4). This path will be used in the proof of the next proposition.
Proposition 19. Let $F_L(x,y) = (T_L(x),G_L(x,y))$ be BV Lorenz-like map and suppose that $F_L$ has an unique invariant probability measure $\mu_0 \in \text{BV}^\infty_1$. Then $\mu_0 \in \text{BV}^+$ and

$$\text{Var}(\mu_0) \leq 2K_0.$$ 

Proof. Consider the path $\Gamma_{\mu_0}^m$, defined in Remark 12, which represents the measure $F^m \mu_0$.

According to Proposition 16, let $\mu_0 \in \text{BV}^\infty_1$ be the unique $F_L$-invariant probability measure in $\text{BV}^\infty_1$. Consider the Lebesgue measure $m$ and the iterates $F^m(\mu_0)$. By Theorem 8.5, these iterates converge to $\mu_0$ in $\mathcal{L}^\infty$. It implies that the sequence $\{\Gamma_{\mu_0}^m\}_n$ converges $m$-a.e. to $\Gamma_{\mu_0}^\infty \in \Gamma_{\mu_0}$ (in $\text{SB}(I)$ with respect to the metric defined in definition 4.4), where $\Gamma_{\mu_0}^\infty$ is a path given by the Rokhlin Disintegration Theorem and $\{\Gamma_{\mu_0}^m\}_n$ is given by equation (32). It implies that $\{\Gamma_{\mu_0}^m\}_n$ converges pointwise to $\Gamma_{\mu_0}^\infty$ on a full measure set $\hat{I} \subset I$. Let us denote $\Gamma_n := \Gamma_{F^m}^\infty|_{\hat{I}}$ and $\Gamma := \Gamma_{\mu_0}^\infty|_{\hat{I}}$. Since $\{\Gamma_n\}_n$ converges pointwise to $\Gamma$ it holds $\text{Var}(\Gamma_n, \mathcal{P}) \rightarrow \text{Var}(\Gamma, \mathcal{P})$ as $n \rightarrow \infty$ for all finite sequences $\mathcal{P} \subset \hat{I}$. Indeed, let $\mathcal{P} = \{x_1, \cdots, x_k\} \subset \hat{I}$ be a finite sequence. Then,

$$\text{Var}(\Gamma_n, \mathcal{P}) = \sum_{j=1}^k ||\Gamma_n(x_j) - \Gamma_n(x_{j-1})||_W,$$

taking the limit, we get

$$\lim_{n \rightarrow \infty} \text{Var}(\Gamma_n, \mathcal{P}) = \lim_{n \rightarrow \infty} \sum_{j=1}^k ||\Gamma_n(x_j) - \Gamma_n(x_{j-1})||_W = \sum_{j=1}^k ||\Gamma(x_j) - \Gamma(x_{j-1})||_W = \text{Var}(\Gamma, \mathcal{P}).$$

On the other hand, $\text{Var}(\Gamma_n, \mathcal{P}) \leq \text{Var}(\Gamma_n) \leq 2K_0$ for all $n \geq 1$, where $K_0$ comes from Proposition 18. Then $\text{Var}(\Gamma_{\mu_0}^m, \mathcal{P}) \leq 2K_0$ for all partition $\mathcal{P}$. Thus, $\text{Var}(\Gamma_{\mu_0}^m) \leq 2K_0$ and hence $\text{Var}(\mu_0) \leq 2K_0$. \hfill \square

Remark 13. We remark that, Proposition 19 is an estimation of the regularity of the disintegration of $\mu_0$. Similar results are presented in [20] and [12].

In Section 7.1 we proved exponential decay of corretation for Lorenz like maps and observables in the set $f \in \Theta^1_{\mu_0}$. In this section we prove that for BV Lorenz like maps, the set $f \in \Theta^1_{\mu_0}$ contains the set of Lipschitz functions. Denote the space of the Lipschitz functions, $f : [0,1]^2 \rightarrow \mathbb{R}$ by $\text{Lip}(\Sigma)$. As a consequence of Proposition 19, next Proposition 20 yields $\text{Lip}(\Sigma) \subset \Theta^1_{\mu_0}$ (defined in subsection 7.1). In order to prove it, we need the next Lemma 9.11 on disintegration of absolutely continuous measures with respect to a measure $\mu_0 \in \mathcal{A}^B$, where its proof was postponed to the Appendix 4.

Lemma 9.11. Let $\{\mu_0, \gamma\}$ be the disintegration of $\mu_0$, along the partition $\mathcal{F}^\infty := \{\gamma\} \times N_2 : \gamma \in N_1\}$, and for a $\mu_0$ integrable function $f : N_1 \times N_2 \rightarrow \mathbb{R}$, denote by $\nu$ the measure $\nu := f\mu_0$ ($f\mu_0(E) := \int_E f d\mu_0$). If $\{\nu, \gamma\}$ is the
disintegration of \( \nu \), where \( \widehat{\nu} := \pi_{x_2} \nu \), then it holds \( \widehat{\nu} \ll m_1 \) and \( \nu_\gamma \ll \mu_{0,\gamma} \). Moreover, denoting \( \mathcal{F} := \frac{d\widehat{\nu}}{dm_1} \), it holds

\[
\mathcal{F}(\gamma) = \int_{N_1} f(\gamma, y)d(\mu_0|\gamma),
\]

and for \( \widehat{\nu}\text{-a.e.} \ \gamma \in N_1 \)

\[
\frac{d
u_{\gamma}}{d\mu_{0,\gamma}}(y) = \begin{cases} \frac{f(\gamma, y)}{\mathcal{F}(\gamma)}, & \text{if } \gamma \in B^c \\ 0, & \text{if } \gamma \in B, \end{cases}
\]

where \( B := \mathcal{F}^{-1}(0) \).

**Proposition 20.** Let \( F_L : [0,1]^2 \rightarrow [0,1]^2, F_L(x, y) = (T_L(x), G_L(x, y)) \), be a BV Lorenz-like map and \( \mu_0 \in BV_{1,1} \) the unique \( F_L \)-invariant measure in \( BV_{1,1} \). Then, \( Lip(\Sigma) \subset \Theta_{\mu_0} \)

**Proof.** Let \( \{ \mu_{0,\gamma} \} \) be the disintegration of \( \mu_0 \) and denote by \( \nu \) the measure \( \nu := f\mu_0 \) \( (f\mu_0(E) := \int_E f dm_0) \). If \( \{ \nu_{\gamma} \} \) is the disintegration of \( \nu \), then it holds \( \widehat{\nu} \ll m_1 \) and \( \nu_\gamma \ll \mu_{0,\gamma} \) (see appendix 4, section 13). Moreover, denoting \( \mathcal{F} := \frac{d\widehat{\nu}}{dm_1} \), it holds

\[
\mathcal{F}(\gamma) = \int_{[0,1]} f(\gamma, y)d(\mu_0|\gamma),
\]

and

\[
\frac{d\nu_{\gamma}}{d\mu_{0,\gamma}}(y) = \frac{f(\gamma, y)}{\mathcal{F}(\gamma)}, \quad \text{if } \mathcal{F}(\gamma) \neq 0.
\]

and

\[
\frac{d\nu_{\gamma}}{d\mu_{0,\gamma}}(y) \equiv 0, \quad \text{if } \mathcal{F}(\gamma) = 0.
\]

It is immediate that \( \nu \in \mathcal{L}^1 \). Let us check that \( \mathcal{F} \in BV_{1,1} \) by estimating the variation of \( \mathcal{F} \). For an arbitrary partition \( P = \{ 0 = \gamma_0, \gamma_1, \cdots, \gamma_n = 1 \} \) of \([0,1]\), we have

\[
|\mathcal{F}(\gamma_i) - \mathcal{F}(\gamma_{i-1})| \leq \left| \int_{[0,1]} f(\gamma_i, y)d(\mu_0|\gamma_i) - \int_{[0,1]} f(\gamma_{i-1}, y)d(\mu_0|\gamma_{i-1}) \right|
\]

\[
\leq \left| \int_{[0,1]} f(\gamma_i, y)d(\mu_0|\gamma_i) - \int_{[0,1]} f(\gamma_i, y)d(\mu_0|\gamma_{i-1}) \right|
\]

\[
+ \left| \int_{[0,1]} f(\gamma_i, y)d(\mu_0|\gamma_{i-1}) - \int_{[0,1]} f(\gamma_{i-1}, y)d(\mu_0|\gamma_{i-1}) \right|
\]

\[
\leq \left| \int_{[0,1]} f(\gamma_i, y)d(\mu_0|\gamma_i - \mu_0|\gamma_{i-1}) \right|
\]

\[
+ \left| \int_{[0,1]} f(\gamma_{i-1}, y)d(\mu_0|\gamma_{i-1}) \right|
\]

\[
\leq \|f\|_{Lip} ||\mu_0|\gamma_i - \mu_0|\gamma_{i-1}||_W + L(f)|\gamma_i - \gamma_{i-1}| \phi_\infty.
\]

Thus, \( \text{var } \mathcal{F} < \infty \) and \( \mathcal{F} \in BV_{1,1} \) (since \( \text{var}_{1,1} \mathcal{F} \leq 2 \text{var } \mathcal{F} \)).

The proof of the following proposition is postponed to the appendix.
Proposition 21. Let \( \{F_\delta\}_{\delta \in [0,1)} \) be a Uniform BV Lorenz-like family (definition (9.6)) and let \( f_\delta \) be the unique \( F_\delta \)-invariant probability in \( BV_{1,1} \) (also in \( BV_{1,1}^\infty \)). Then, there exists \( B_u > 0 \) such that
\[
\text{Var}(f_\delta) \leq 2B_u,
\]
for all \( \delta \in [0,1) \).

For the next proposition we will use the following notation. Given a probability measure \( f_\delta \) on \( I^2 \) and a measurable set \( E \subset I \), we define the measure \( 1_E f_\delta \) on \( I^2 \), by
\[
1_E f_\delta(A) := f_\delta(A \cap \pi_x^{-1}(E)) \quad \text{for all measurable set } A \subset I^2.
\]
We remark that, if \( \{f_\delta, \gamma\} \) is a disintegration of \( f_\delta \), then
\[
\{f_\delta, \gamma\}, \chi E \phi_{x, \delta},
\]
is a disintegration of \( 1_E f_\delta(A) \).

Proposition 22 (to obtain UF2). Let \( \{F_\delta\}_{\delta \in [0,1)} \) be a family of BV Lorenz-like maps which satisfies UBV2, UBV3 and UBV4 of definition 9.6. Denote by \( F_{\delta*} \) their transfer operators and by \( f_\delta \) their fixed points (probabilities) in \( BV_{1,1} \) (also in \( BV_{1,1}^\infty \)). Suppose that \( f_\delta \) has uniformly bounded variation,
\[
\text{Var}(f_\delta) \leq M_2, \quad \forall \delta.
\]
Then, there is a constant \( C_1 \) such that for \( \delta \) small enough, it holds
\[
|||\{F_{\delta*} - F_{\delta*}\}f_\delta|||_1 \leq C_1 \delta(M_2 + 1).
\]

Proof. Set \( A = A_1 \cap A_2 \) where \( A_1 \) comes from definition of \( d_{S,n} \) (see equation (27)) and \( A_2 \) is from (UBV3) (see definition 9.6). Remark that these sets depend on \( \delta \). Let us estimate
\[
|||\{F_{\delta*} - F_{\delta*}\}f_\delta|||_1 \leq \int_I |||F_{\delta*}(1_A f_\delta)|_{\gamma} - F_{\delta*}(1_A f_\delta)|_{\gamma}|||W dm_1(\gamma)
+ \int_I |||F_{\delta*}(1_{A_c} f_\delta)|_{\gamma} - F_{\delta*}(1_{A_c} f_\delta)|_{\gamma}|||W dm_1(\gamma).
\]
By the assumptions, for a.e. \( \gamma \), \( |||f_\delta|||_W \leq (M_2 + 1) \) and \( ||1_{A_c} f_\delta||_1 \leq 2\delta(M_2 + 1) \).
Indeed, since \( \text{Var}(f_\delta) \leq M_2, \quad \forall \delta \), we have (below, we denote \( \phi_{x, \delta} = \frac{d\pi_x(f_\delta)}{dm_1} \))
\[
|||f_\delta|||_W \leq ||f_\delta|_{\gamma} - f_\delta|_{\gamma_2}||_W + ||f_\delta|_{\gamma_2}||_W
= ||f_\delta|_{\gamma} - f_\delta|_{\gamma_2}||_W + \phi_{x, \delta}(\gamma_2).
\]
Integrating with respect to \( \gamma_2 \) we get
\[
||f_\delta|_{\gamma}||_W \leq (M_2 + 1). \number{(36)}
\]
To prove the inequality \( ||1_{A_c} f_\delta||_1 \leq 2\delta(M_2 + 1) \) we use the previous equation, \( m_1(A^c) \leq 2\delta \) and the fact that (see equation (35))
\[
||1_{A_c} f_\delta||_1 = \int_{A_c} ||f_\delta|_{\gamma}||_W dm_1.
\]
Since \( F_{\delta*} \) is a contraction for the weak norm, we have
\[
\int_I |||F_{\delta*}(1_{A_c} f_\delta)|_{\gamma} - F_{\delta*}(1_{A_c} f_\delta)|_{\gamma}|||W dm_1(\gamma) \leq 4\delta(M_2 + 1).
\]
Now, let us estimate the first summand of (36) by estimating the integral

$$
\int \left\| (F_{0*}\mu - F_{\delta*}\mu) \right\|_W dm_1(\gamma),
$$

where $\mu = 1_A f_\delta$. Denote by $T_{0,i}$, with $0 \leq i \leq q$, the branches of $T_0$ defined in the sets $P_i \in \mathcal{P}$ and set $T_{\delta,i} = T_\delta|_{P_i \cap A}$. These functions will play the role of the branches for $T_\delta$. Since in $A$, $T_0 = T_\delta \circ \sigma_\delta$ (where $\sigma_\delta$ is the diffeomorphism in the definition of the Skorokhod distance), then $T_{\delta,i}$ are invertible. Then

$$
(F_{0*}\mu - F_{\delta*}\mu)|_\gamma = \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast [T_{0,i}'(\gamma)] \chi_{T_0(P_i \cap A)}}{|T_{0,i}'(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)} \ast [T_{\delta,i}'(\gamma)] \chi_{T_\delta(P_i \cap A)}}{|T_{\delta,i}'(T_{\delta,i}^{-1}(\gamma))|} \mu_x - a.e. \gamma \in I.
$$

Let us now consider $T_0(P_i \cap A), T_\delta(P_i \cap A)$ and remark that $T_0(P_i \cap A) = \sigma_\delta(T_\delta(P_i \cap A))$ where $\sigma_\delta$ is a diffeomorphism near to the identity. Let us denote $B_i = T_0(P_i \cap A) \cap T_\delta(P_i \cap A)$ and $C_i = T_0(P_i \cap A) \triangle T_\delta(P_i \cap A)$. Then, we have

$$
\int_f \left\| (F_{0*}\mu - F_{\delta*}\mu) \right\|_W dm_1(\gamma) \leq O_1 + O_2,
$$

where

$$
O_1 = \int_f \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast [T_{0,i}'(\gamma)] \chi_{B_i}}{|T_{0,i}'(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)} \ast [T_{\delta,i}'(\gamma)] \chi_{B_i}}{|T_{\delta,i}'(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm_1,
$$

and

$$
O_2 = \int_f \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast [T_{0,i}'(\gamma)] \chi_{C_i}}{|T_{0,i}'(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)} \ast [T_{\delta,i}'(\gamma)] \chi_{C_i}}{|T_{\delta,i}'(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm_1.
$$

And since $m_1(C_i) = O(\delta)$, we get that there is $K_1 \geq 0$ such that $O_2 \leq qK_1(M_2 + 1)\delta$. In order to estimate $O_1$, we note that

$$
O_1 = \int_f \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast [T_{0,i}'(\gamma)] \chi_{B_i}}{|T_{0,i}'(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)} \ast [T_{\delta,i}'(\gamma)] \chi_{B_i}}{|T_{\delta,i}'(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm_1
$$

$$
\leq \int_f \left\| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast [T_{0,i}'(\gamma)] \chi_{B_i}}{|T_{0,i}'(T_{0,i}^{-1}(\gamma))|} \right\|_W dm_1
$$

$$
+ \int_f \left\| \sum_{i=1}^q \frac{F_{\delta,T_{\delta,i}^{-1}(\gamma)} \ast [T_{\delta,i}'(\gamma)] \chi_{B_i}}{|T_{\delta,i}'(T_{\delta,i}^{-1}(\gamma))|} \right\|_W dm_1
$$

$$
= \int_I \left( \chi_{\operatorname{dim}_1(\gamma)} \right) dm_1(\gamma) + \int_I \left( \chi_{\operatorname{dim}_1(\gamma)} \right) dm_1(\gamma).
$$

Remark that $m_1(T_\delta(P_i \cap A) \triangle T_\delta(P_i \cap A)) = O(\delta)$ because $T_\delta(P_i \cap A) = \sigma(T_\delta(P_i \cap A))$ where $\sigma$ is a diffeomorphism near to the identity as in the definition of the Skorokhod distance and $P_i \cap A$ is a finite union of intervals whose number is uniformly bounded with respect to $\delta$. 


The two summands will be treated separately. Let us denote \( \bar{\mu}_\gamma = \pi_{\gamma,y} \ast \mu_\gamma \) (note that \( \mu_\gamma = \phi_\mu(\gamma)\bar{\mu}_\gamma \) and \( \bar{\mu}_\gamma \) is a probability measure).

\[
I(\gamma) = \left| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W} - \left| \sum_{i=1}^q \frac{F_{\delta,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W}
\]

\[
\leq \left| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W} - \left| \sum_{i=1}^q \frac{F_{\delta,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W}
\]

\[
+ \left| \sum_{i=1}^q \frac{F_{\delta,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W} - \left| \sum_{i=1}^q \frac{F_{\delta,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W}
\]

\[
= I_a(\gamma) + I_b(\gamma).
\]

Since \( f_\delta \) is a probability measure it holds, posing \( \beta = T_{0,i}^{-1}(\gamma) \)

\[
\int I_a(\gamma) \, dm_{1} = \int \left| \sum_{i=1}^q \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W} \, dm_{1}(\gamma)
\]

\[
\leq \int \sum_{i=1}^q \left| \frac{F_{0,T_{0,i}^{-1}(\gamma)} \ast \mu_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T_{0,i}^{-1}(\gamma)|} \right|_{W} \, dm_{1}(\gamma)
\]

We remark \( T_{0,i}^{-1}(B_i) \subset P_i \cap A \) and \( T_{0,i}^{-1}(T_{0,i}^{-1}(B_i)) \subset P_i \cap A \). Moreover, since \( |T_{\delta,i}(\beta) - T_{0,i}(\beta)| \leq \delta \) and \( T_{0,i}^{-1} \) is a contraction, then

\[
|T_{0,i}^{-1} \circ T_{\delta,i}(\beta) - \beta| \leq \delta.
\]

Therefore

\[
\left| F_{0,\beta} \ast \mu_\beta - F_{\delta,T_{\delta,i}(T_{0,i}(\beta))} \ast \mu_\beta \right|_{W} \leq \left| F_{0,\beta} \ast \mu_\beta - F_{\delta,T_{0,i}(\beta)} \ast \mu_\beta \right|_{W}
\]

By (UBV3) and equation (36),

\[
\left| F_{0,\beta} \ast \mu_\beta - F_{\delta,T_{0,i}(\beta)} \ast \mu_\beta \right|_{W} \leq \delta(M_2 + 1).
\]

Then, by (37), we have

\[
\left| F_{\delta,\beta} \ast \mu_\beta - F_{\delta,T_{0,i}(\beta)} \ast \mu_\beta \right|_{W} \leq H_\delta(M_2 + 1)
\]

when \( d(\beta, \cup \partial J_i) \geq \delta \). For the other values of \( \beta \) we remark that the set of points \( \{x \text{ s.t. } d(x, \cup \partial J_i) \leq \delta \} \) is of measure bounded by \( \delta \# P_i \), thus

\[
\int I_b \, dm_1 = O(\delta).
\]
To estimate $I_b(\gamma)$, we have

$$I_b(\gamma) = \left\| \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} \right\|_W,$$

and

$$\int I_b(\gamma) \, dm_1(\gamma) \leq |(P_{T_0} - P_{T_\delta})(1)| (M_2 + 1).$$

By [10], Lemma 11.2.1,

$$\int_{A_1} I_b(\gamma) \, dm_1(\gamma) \leq 14(M_2 + 1)\delta.$$

Now, let us estimate the integral of the second summand

$$II(\gamma) = \left\| \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} \right\|_W.$$

Let us make the change of variable $\gamma = T_{\delta,i}(\beta)$.

$$\int I(\gamma) \, dm_1(\gamma) = \int \left\| \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \mu_{T_\gamma^{-1}(\gamma)} \cdot \chi_{B_i}}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} \right\|_W \, dm_1(\gamma)$$

$$\leq \sum_{i=1}^{q} \int_{B_i} \left\| \frac{1}{|T_{\delta,i}(T_{\gamma,i}^{-1}(\gamma))|} \right\| \left\| F_{\delta,T_\gamma^{-1}(\gamma)} \cdot \left( \mu_{T_\gamma^{-1}(\gamma)} - \mu_{T_{\delta,i}^{-1}(\gamma)} \right) \right\|_W \, dm_1(\gamma)$$

$$\leq \sum_{i=1}^{q} \int_{B_i} \left\| \mu_{T_{\delta,i}^{-1}(\gamma)} \right\|_{W_{\delta,i}(\gamma)} \, dm_1(\gamma)$$

Hence, by (37)

$$\int I(\gamma) \, dm_1(\gamma) \leq \int \sup_{x,y \in B(\beta)} \left( \|\mu_x - \mu_y\|_W \right) \, dm_1(\beta)$$

and then

$$\int I(\gamma) \, dm_1(\gamma) \leq 2\delta(M_2 + 1).$$

Summing all, the statement is proved. 

9.1.2. Proof of Theorem 9.2. Before to establish Theorem 9.2, we need to prove the following proposition.

**Proposition 23.** Let $\{F_\delta\}_{\delta \in [0,1)}$ be a Uniform BV Lorenz-like family and let $\{F_{\delta,*}\}_{\delta \in [0,1)}$ be the induced family of transfer operators. Then, $\{F_{\delta,*}\}_{\delta \in [0,1)}$ is a uniform family of operators with weak space ($L^1, \|\cdot\|_1$, and strong space ($BV_{1,1}, \|\cdot\|_{1,1}$).

**Proof.** To prove UF1, note that, by (UBV1) there exist $0 < \alpha_1 < 1$ and $\overline{D} > 0$ s.t. for all $\mu \in BV_{1,1}$ and for all $\delta$ it holds $\|F_{\delta,*} \mu\|_{1,1} \leq \overline{D}\alpha_1^2\|\mu\|_{1,1} + \overline{D}\|\mu\|_1$, for all $n \geq 1$. Indeed, by Lemma 6 we have

$$\|F_{\delta,*} \mu\|_{1,1} = \|P_{T_{\delta}} \phi_{x,1,1} + \|F_{\delta,*} \mu\|_1 \leq \overline{D}\alpha_1^n\|\mu\|_{1,1} \leq \overline{D}\alpha_1^n\|\mu\|_{1,1} + (D + 1)\|\mu\|_1.$$
Therefore, if \( f_\delta \) is a fixed probability measure for the operator \( F_\delta \), by the above inequality we get UF1 with \( M = D + 1 \).

Proposition 22 and Proposition 21 immediately give UF2. The items UF3 and UF4 follow, respectively, from Proposition 15 and Lemma 6 applied to each \( F_\delta \).

Once this is done, we apply the above result together with Proposition 17 and the proof of Theorem 9.2 is established.

10. Appendix 1: Proof of propositions 18 and 21. In this section, we obtain Proposition 18 as a particular case of Theorem 10.2. We also prove Proposition 21.

Note that, for all \( \mu \in BV^+ \) it holds
\[
\|\mu\|_1 = |\phi_x|_1 \quad \text{and} \quad \|\mu\|_\infty = |\phi_x|_\infty,
\]
where \( \phi_x = \frac{d\pi_x \mu}{dm} \). We also remark, for each \( \mu \in BV^+ \) we have \( \phi_x \in BV_{1,1} \).

For a measurable map \( F : [0,1]^2 \rightarrow [0,1]^2 \), of the type \( F(x,y) = (T(x),G(x,y)) \), and a given \( \gamma \in F^s(\gamma = \{x\} \times [0,1]) \), consider the function \( F_\gamma : [0,1] \rightarrow [0,1] \), defined by equation (14).

Definition 10.1. Consider a function \( f : [0,1]^2 \rightarrow \mathbb{R} \) and let \( x_1 \leq \cdots \leq x_n \) and \( y_1 \leq \cdots \leq y_n \) be such that \((x_i)_{i=1}^n \subset I\) and \((y_i)_{i=1}^n \subset I\). We define
\[
\text{var}^\diamond(f,(x_i)_{i=1}^n,(y_i)_{i=1}^n) := \sum_{i=1}^{n-1} |f(x_{i+1},y_i) - f(x_i,y_i)|,
\]
and
\[
\text{var}^\diamond(f) := \sup_{(x_i)_{i=1}^n,(y_i)_{i=1}^n} \text{var}^\diamond(f,(x_i)_{i=1}^n,(y_i)_{i=1}^n).
\]
If \( \eta \subset I \) is an interval, we define \( \text{var}^\diamond_\eta(f) = \text{var}^\diamond(f|\eta \times I) \), where \( \overline{\eta} \) is the closure of \( \eta \).

Since preliminaries results are necessary, we postponed the proof of the next theorem to the end of the section.

Theorem 10.2. Let \( F(x,y) = (T(x),G(x,y)) \) be a measurable transformation such that

1. \( \text{var}^\diamond(G) < \infty \)
2. \( F \) satisfy property G1 (hence is uniformly contracting on each leaf \( \gamma \) with rate of contraction \( \alpha \));
3. \( T : [0,1] \rightarrow [0,1] \) is a piecewise expanding map satisfying the assumptions given in the definition 9.4.

Then, there are \( K_0 \) and \( 0 < \lambda_0 < 1 \) such that for all path \( \Gamma_\mu \), where \( \mu \in BV^+ \), and all \( n \geq 1 \) it holds
\[
\text{Var}(\Gamma_{F^n\mu}) \leq K_0 \lambda_0^n \text{Var}(\Gamma_\mu) + K_0 |\phi_x|_{1,1}.
\]

Remark 14. If \( F_L \) is a BV Lorenz-like map (definition 9.3), a straightforward computation yields
\[
\text{var}^\diamond(G_L) \leq H,
\]
where \( H \) comes from equation (26). This shows that Proposition 18 is a direct consequence of Theorem 10.2.
Lasota-yorke inequality for positive measures. Henceforth, we fix a positive measure \( \mu \in \mathcal{B}^{+} \subset \mathcal{A} \) and a path, \( \Gamma_{\mu}^{\ast} \), which represents \( \mu \) (i.e. a pair \( (\{\mu_{\gamma}\}, \phi_{x}) \) s.t. \( \Gamma_{\mu}^{\ast}(\gamma) := \mu_{\gamma} \)). To simplify, we will denote the path \( \Gamma_{\mu}^{\ast} \in \Gamma_{\mu} \), just by \( \Gamma_{\mu} \).

Remark 15. Consider \( T : [0, 1] \rightarrow [0, 1] \) a piecewise expanding map from definition 9.4 and \( g_{i} = \frac{1}{|T_{i}|} \). For all \( n \geq 1 \), let \( \mathcal{P}^{(n)} \) be the partition of \( I \) s.t. \( \mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y) \) if and only if \( \mathcal{P}^{(1)}(T_{j}(x)) = \mathcal{P}^{(1)}(T_{j}(y)) \) for all \( j = 0, \cdots, n - 1 \), where \( \mathcal{P}^{(1)} = \mathcal{P} \) (see definition 9.4). Given \( P \in \mathcal{P}^{(n)} \), define \( g_{P}^{(n)} = \frac{1}{|T_{n}^{(P)}|} \). Item 2) implies that there exists \( C_{1} > 0 \) and \( 0 < \theta < 1 \) s.t.

\[
\sup\{g_{P}^{(n)}\} \leq C_{1}\theta^{n}, \quad \text{for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1.
\]

(38)

Moreover, equation (38) and some basic properties of real valued \( BV \) functions imply (see [31], page 41, equation (3.1)) there exists \( \lambda_{2} \in (\theta, 1) \) and \( C_{2} > 0 \) such that

\[
\text{var}(g_{P}^{(n)}) \leq C_{2}\lambda_{2}^{n}, \quad \text{for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1.
\]

Then, there is an iterate of \( F \), \( \tilde{F} := F^{k} \), such that \( T^{k} \) satisfies

\[
\beta_{k} := \text{var}(g_{P}^{(k)}) + 3\sup g_{P}^{(k)} < 1, \quad \forall P \in \mathcal{P}^{(k)}.
\]

(39)

We also remark that \( G^{k} := \pi_{y} \circ F^{k} \) also satisfies

\[
\text{var}(G^{k}) < \infty.
\]

(40)

Next lemma provides equation (40) and its proof can be found in [2].

Lemma 10.3. If \( F \) satisfy definition 9.3, then for all \( n \geq 1 \) and all \( f : [0, 1]^{2} \rightarrow \mathbb{R} \) such that

\[
\sup_{x, y, y \in [0, 1]} \frac{|f(x, y) - f(x, y_{1})|}{|y_{2} - y_{1}|} < \infty
\]

and

\[
|f|_{\infty} < \infty,
\]

it holds

\[
\text{var}(f \circ F^{n}) \leq q^{n}\text{var}(f) + \sum_{i=1}^{n-1} q^{i}(\text{var}(G)|f|_{lip} + 2q|f|_{\infty}),
\]

where \( q \) is the number of branches of \( T \) \( (q := \#\mathcal{P}) \).

Recalling equation (43), set

\[
\Gamma_{\mu_{\gamma}}(\gamma) := F_{\gamma} \ast \Gamma_{\mu}(\gamma).
\]

(41)

With the above notation and following the strategy of the proof of Lemma 5.1, the path \( \Gamma_{F, \mu} \), defined on a full measure set by

\[
\Gamma_{F, \mu}(\gamma) = \sum_{i=1}^{q} (g_{i} \ast \Gamma_{\mu}) \circ T_{L_{i}}^{-1}(\gamma) \cdot \chi_{T_{L_{i}}(P_{i})}(\gamma), \quad \text{where } g_{i} = \frac{1}{|T_{L_{i}}|},
\]

represents the measure \( F_{\ast} \mu \).

By equations (16) and (41), it holds

\[
||\Gamma_{\mu_{\gamma}}(\gamma)||_{W} \leq ||\Gamma_{\mu}(\gamma)||_{W},
\]

(10)
Lemma 10.4. Let $\gamma_1$ and $\gamma_2$ be two leaves such that $G(\gamma_i, \cdot) : I \to I$ is a contraction, $i = 1, 2$. Then for every path $\Gamma_\mu$, where $\mu \in AB$, it holds

$$||\Gamma_{\mu_1} - \Gamma_{\mu_2}||_W \leq ||\Gamma_{\mu_1} - \Gamma_{\mu_2}||_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)||\phi_2|_\infty,$$

for some $y_0 \in I$.

Proof. Consider $g$ such that $|g|_\infty \leq 1$ and $\text{Lip}(g) \leq 1$, and observe that since $G_{\gamma_1} - G_{\gamma_2} : I \to I$ is continuous, it holds

$$\sup_{\gamma} |G(\gamma_1, y) - G(\gamma_2, y)| = |G(\gamma_1, y_0) - G(\gamma_2, y_0)|,$$

for some $y_0 \in I$. Moreover, by equations (16) and (41), we have

$$\left| \int g d\Gamma_{\mu_1}(\gamma_1) - \int g d\Gamma_{\mu_2}(\gamma_2) \right| = \left| \int g d\Gamma_{\mu_1}(\gamma_1) - \int g d\Gamma_{\mu_2}(\gamma_2) \right| \leq \left| \int g d\Gamma_{\mu_1}(\gamma_1) - \int g d\Gamma_{\mu_2}(\gamma_2) \right| \leq ||\Gamma_{\mu_1} - \Gamma_{\mu_2}||_W + \sup_{\gamma} |G(\gamma_1, y) - G(\gamma_2, y)| \int |d\mu|_{\gamma_2(y)} \leq ||\Gamma_{\mu_1} - \Gamma_{\mu_2}||_W + \sup_{\gamma} |G(\gamma_1, y) - G(\gamma_2, y)| \int |d\mu|_{\gamma_2(y)} = \left| \int \Gamma_{\mu_1}(\gamma_1) - \Gamma_{\mu_2}(\gamma_2) \right|_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)| |\phi_2|_\infty.$$

Taking the supremum over $g$, we finish the proof. \qed

The proofs of the next three lemmas are straightforward and analogous to the one dimensional $BV$ functions. So, we omit them (details can be found in [28]).

Lemma 10.5. Given paths $\Gamma_{\mu_0}, \Gamma_{\mu_1}$ and $\Gamma_{\mu_2}$ (where $\mu_i(\gamma) = \mu_i|_{\gamma}$) representing the positive measures $\mu_0, \mu_1, \mu_2 \in BV^+$ respectively, a function $\varphi : I \to \mathbb{R}$, an homeomorphism $h : \eta \subset I \to h(\eta) \subset I$ and a subinterval $\eta \subset I$, then the following properties hold

P1) If $\mathcal{P}$ is a partition of $I$ by intervals $\eta$, then

$$\text{Var}(\Gamma_{\mu_0}) = \sum_{\eta} \text{Var}_{\mathcal{P}}(\Gamma_{\mu_0});$$

P2) $\text{Var}_{\mathcal{P}}(\Gamma_{\mu_1} + \Gamma_{\mu_2}) \leq \text{Var}_{\mathcal{P}}(\Gamma_{\mu_1}) + \text{Var}_{\mathcal{P}}(\Gamma_{\mu_2})$

P3) $\text{Var}_{\mathcal{P}}(\varphi \Gamma_{\mu_0}) \leq \left( \sup_{\gamma} |\varphi| \right) \left( \text{Var}_{\mathcal{P}}(\Gamma_{\mu_0}) \right) + \left( \sup_{\gamma \in \mathcal{P}} ||\Gamma_{\mu_0}(\gamma)||_W \text{var}_{\mathcal{P}}(\varphi) \right)$

P4) $\text{Var}_{\mathcal{P}}(\Gamma_{\mu_0} \circ h) = \text{Var}_{\mathcal{P}}(\Gamma_{\mu_0}).$
Lemma 10.6. For every path $\Gamma_\mu$, $\mu \in AB$ and an interval $\eta \subset I$, it holds
\[
\sup \|\Gamma_\mu(\gamma)\|_W \leq \operatorname{Var}_{\tau}(\Gamma_\mu) + \frac{1}{m(\eta)} \int_{\eta} \|\Gamma_\mu(\gamma)\|_W \, dm_1(\gamma),
\]
where $\eta$ is the closure of $\eta$.

A straightforward application of Lemma 10.4 yields the following.

Lemma 10.7. For all $\Gamma_\mu$, where $\mu \in BV^+$, and all $P \in \mathcal{P}$ it holds
\[
\operatorname{Var}_{\tau}(\Gamma_{\mu\nu}) \leq \operatorname{Var}_{\tau}(\Gamma_\mu) + \varphi_{\tau}(G)\phi_{x,\infty}.
\]

Lemma 10.8. For all path $\Gamma_\mu$, where $\mu \in BV^+$, it holds
\[
\operatorname{Var}(\Gamma_{F, \mu}) \leq \sum_{i=1}^{q} \left[ \operatorname{var}_{\tau}(g_i) + 2 \sup_{P_i} g_i \right] \cdot \sup_{\gamma \in P_i} \|\Gamma_\mu(\gamma)\|_W + \sup_{\gamma \in P_i} g_i \cdot \operatorname{Var}_{\tau}(\Gamma_{\mu\nu}),
\]
where $\Gamma_{\mu\nu}$ is defined by equation (41).

Proof. Using the properties P2, P3, P4, $\sup_{\gamma \in P_i} \|\Gamma_{\mu\nu}(\gamma)\|_W \leq \sup_{\gamma \in P_i} \|\Gamma_\mu(\gamma)\|_W$ and
\[
\sup_{\gamma \in P_i} g_i = \sup_{\gamma \in P_i} g_i,
\]
we have
\[
\operatorname{Var}(\Gamma_{F, \mu}) \leq \sum_{i=1}^{q} \operatorname{var}_{\tau}(g_i) \cdot \operatorname{var}_{\tau}(\Gamma_{F, \mu}) \leq \sum_{i=1}^{q} \operatorname{var}_{\tau}(g_i) \cdot \operatorname{var}_{\tau}(\Gamma_{F, \mu}) + 2 \sup_{T_i(P_i)} g_i \cdot \operatorname{var}_{\tau}(\Gamma_{F, \mu}) \cdot \varphi_{x,\infty}.
\]

Lemma 10.9. For all path $\Gamma_\mu$, where $\mu \in BV^+$, it holds
\[
\operatorname{Var}(\Gamma_{F, \mu}) \leq \beta \operatorname{Var}(\Gamma_\mu) + K_3|\phi_x|_{1,1}.
\]
Remark 16. Remember that, the coefficients of inequality (42) are given by the formulas

\[ \beta := \max_{i=1, \ldots, q} \left\{ \text{var}_{\mathcal{T}}(g_i) + 3 \sup_{\mathcal{T}_i} g_i \right\} \]

and

\[ K_3 = \max_{i=1, \ldots, q} \left\{ \sup_{\mathcal{T}_i} g_i \right\} \text{var}^\circ(G) + \max_{i=1, \ldots, q} \left\{ \frac{\text{var}_{\mathcal{T}}(g_i) + 2 \sup_{\mathcal{T}_i} g_i}{m(\mathcal{T}_i)} \right\} \].

Proof. By lemma 10.7, remark 10.6, lemma 10.8, P1, equation (39) of remark 15 and by \( \sum_{i=1}^q \text{var}_{\mathcal{T}_i}^\circ G = \text{var}^\circ(G) \), we get

\[ \text{Var}(\Gamma_{F_* \mu}) \leq \sum_{i=1}^q \left[ \text{var}_{\mathcal{T}_i}(g_i) + 2 \sup_{\mathcal{T}_i} g_i \right] \sup_{\mathcal{T}} \|\mu|\|w + \sup_{\mathcal{T}} \text{Var}_{\mathcal{T}}(\Gamma_{\mu F}) \]

\[ \leq \sum_{i=1}^q \left[ \text{var}_{\mathcal{T}_i}(g_i) + 2 \sup_{\mathcal{T}_i} g_i \right] \left( \text{Var}_{\mathcal{T}}(\Gamma_{\mu}) + \frac{1}{m_1(\mathcal{T}_i)} \int_{\mathcal{T}_i} \|\mu|\|w \, dm_1(\gamma) \right) \]

\[ + \sum_{i=1}^q \sup_{\mathcal{T}_i} g_i \left( \text{Var}_{\mathcal{T}_i}(\Gamma_{\mu}) + \text{var}^\circ_{\mathcal{T}_i}(G) \right) |\phi_x|_{\infty} \]

\[ \leq \sum_{i=1}^q \left[ \text{var}_{\mathcal{T}_i}(g_i) + 3 \sup_{\mathcal{T}_i} g_i \right] \text{Var}_{\mathcal{T}_i}(\Gamma_{\mu}) \]

\[ + \max_{i=1, \ldots, q} \left\{ \frac{\text{var}_{\mathcal{T}_i}(g_i) + 2 \sup_{\mathcal{T}_i} g_i}{m_1(\mathcal{T}_i)} \right\} |\phi_x|_1 \]

\[ + |\phi_x|_{\infty} \max_{i=1, \ldots, q} \left\{ \sup_{\mathcal{T}_i} g_i \right\} \text{var}^\circ(G) \]

\[ \leq \beta \text{Var}(\Gamma_{\mu}) + K_3 |\phi_x|_{\infty} \]

\[ \leq \beta \text{Var}(\Gamma_{\mu}) + K_3 |\phi_x|_{1,1}. \]

Remark 16. Remember that, the coefficients of inequality (42) are given by the formulas

\[ \beta := \max_{i=1, \ldots, q} \left\{ \text{var}_{\mathcal{T}_i}(g_i) + 3 \sup_{\mathcal{T}_i} g_i \right\} \]

and

\[ K_3 = \max_{i=1, \ldots, q} \left\{ \sup_{\mathcal{T}_i} g_i \right\} \text{var}^\circ(G) + \max_{i=1, \ldots, q} \left\{ \frac{\text{var}_{\mathcal{T}_i}(g_i) + 2 \sup_{\mathcal{T}_i} g_i}{m_1(\mathcal{T}_i)} \right\} \].

We will use these expressions in the next result and later on.

From Lemma 10.9 and taking the infimum over the paths \( \Gamma_{\mu} \) we have the following.

Corollary 3. If \( F : [0, 1]^2 \to [0, 1]^2 \) satisfies all the hypothesis of Theorem 10.2. Then, for all \( \mu \in BV^+ \), it holds

\[ \text{Var}(F_* \mu) \leq \beta \text{Var}(\Gamma_{\mu}) + K_3 |\phi_x|_{1,1}. \]
where $\beta$ and $K_3$ were given by Remark 16.

**Proposition 24.** If $F : [0, 1]^2 \to [0, 1]^2$ satisfies all the hypothesis of Theorem 10.2. Then, there exist $k \in \mathbb{N}$, $0 < \beta_k < 1$ and $C_k > 0$ such that for all path $\Gamma_{\mu}$, where $\mu \in BV^+$, it holds

$$\text{Var}(\Gamma_{F, \mu}) \leq \beta_k \text{Var}(\Gamma_{\mu}) + C_k |\phi_x|_{1,1}.$$  

**Proof.** The proof is a straightforward consequence of the above Remark 16 and Remark 15, where $\beta_k$ was defined by equation (39). \hfill \Box

**Proposition 25.** If $F : [0, 1]^2 \to [0, 1]^2$ satisfies all the hypothesis of Theorem 10.2. Then, there exist $k \in \mathbb{N}$, $C_0$ and $0 < \beta_k < 1$ such that for all path $\Gamma_{\mu}$, where $\mu \in BV^+$, and all $n \geq 1$ it holds

$$\text{Var}(\Gamma_{F, \mu}) \leq C_0 \beta_k^n \text{Var}(\Gamma_{\mu}) + C_0 |\phi_x|_{1,1}.$$  

**Proof.** Inequality (24) gives us

$$|P^T f|_{1,1} \leq B_3 \beta_k^n |f|_{1,1} + C_2 f \|1, \forall f \in BV_{1,1},$$

for $B_3, C_2 > 0$ and $0 < \beta_2 < 1$. Then, since $|f|_1 \leq |f|_{1,1}$, it holds

$$|P^T f|_{1,1} \leq K_2 |f|_{1,1}, \forall f \in BV_{1,1}, \quad (43)$$

where

$$K_2 = B_3 + C_2.$$  

In particular, inequality (43) holds if we replace $f$ by $\phi_x = \frac{d(\pi_x, \mu)}{dm_1}$ for each $\mu \in BV^+$.

By inequality (43), Proposition 24 and a straightforward induction we have

$$\text{Var}(\Gamma_{F, \mu}) \leq \beta_k^n \text{Var}(\Gamma_{\mu}) + C_k \max\{K_2, 1\} \sum_{i=0}^{n-1} \beta_k^i |\phi_x|_{1,1}, \forall n \geq 0.$$  

We finish the proof by setting

$$C_0 := \max \left\{1, \frac{C_k \max\{K_2, 1\}}{1 - \beta_k} \right\}.$$  

\hfill \Box

**Proof.** (of Theorem 10.2) Let $k \in \mathbb{N}$ be from Proposition 25. For a given $n$, we set $n = kq_n + r_n$, where $0 \leq r_n < k$. Applying Proposition 10.9 and iterating $r_n$ times the inequality (42) we have

$$\text{Var}(\Gamma_{F, \mu}) \leq \max_{i=0, \ldots, k} \{\beta^i\} \text{Var}(\Gamma_{\mu}) + K_2 \sum_{j=0}^{k} \beta^j |\phi_x|_{1,1}, \quad (44)$$

where $K_2$ was defined in equation (43). Thus, by Proposition 25 and the above inequality (44), we have

$$\text{Var}(\Gamma_{F, \mu}) = \text{Var}(\Gamma_{F, kq_n + r_n \mu}) \leq C_0 \beta_k^{q_n} \text{Var}(\Gamma_{F, \mu}) + C_0 |\phi_x|_{1,1} \leq C_0 \max_{i=0, \ldots, k} \{\beta^i\} \beta_k^{q_n} \text{Var}(\Gamma_{\mu}) + \left[C_0 \beta_k^{q_n} K_2 \sum_{j=0}^{k} \beta^j + C_0\right] |\phi_x|_{1,1}$$
Proposition 26. If \( \{F_\delta\}_{\delta \in [0,1]} \) is a Uniform BV Lorenz-like family. Then, there exist uniform constants \( \beta_u > 0 \) and \( K_u > 0 \) such that for every \( \mu \in BV^+ \), it holds

\[
\text{Var}(F_\delta, \mu) \leq \beta_u \text{Var}(\mu) + K_u |\phi_x|_{1,1}, \quad \forall \delta \in [0,1).
\]  

Proof. Since \( \text{var}^\omega(G_\delta) \leq H_\delta \), we can apply Corollary 3 to each \( F_\delta \) to get (see Remark 16)

\[
\text{Var}(F_\delta, \mu) \leq \beta_\delta \text{Var}(\mu) + K_{3, \delta} |\phi_x|_{1,1}, \quad \forall \delta \in [0,1),
\]

where

\[
\beta_\delta = \max_{i=1, \ldots, q} \left\{ \text{var}_{P_i}(g_{i, \delta}) + 3 \sup_{P_i} g_{i, \delta} \right\}
\]

and

\[
K_{3, \delta} = \max_i \left\{ \sup_{P_i} g_{i, \delta} \right\} \text{Var}^\omega(G_\delta) + \max_i \left\{ \frac{\text{var}_{P_i}(g_{i, \delta}) + 2 \sup_{P_i} g_{i, \delta}}{m(P_i)} \right\}.
\]

Since \( \text{var}^\omega(G_\delta) \leq H_\delta \), UBV4 ((2), (3), (4)) yields the existence of uniform constants

\[
\beta_u := \sup_{\delta \in [0,1]} \beta_\delta < \infty \quad \text{and} \quad K_u := \sup_{\delta \in [0,1]} K_{3, \delta} < \infty.
\]

Note that, we do not necessarily have \( \beta_u < 1 \). In what follows, we will prove that there exists a uniform \( k \in \mathbb{N} \) such that this property is satisfied for the map \( F_\delta^k \), for all \( \delta \in [0,1) \). We also remark that, if \( \{F_\delta\}_{\delta \in [0,1]} \) is a BV Lorenz-like family, then \( F_\delta^n \) also satisfies the hypothesis of Theorem 10.2, for all \( n \geq 1 \) and all \( \delta \), in a way that we can apply Lemma 10.9 to \( F_\delta^n \), for all \( n \geq 1 \).

Lemma 10.10. Let \( \{T_\delta\}_{\delta \in [0,1]} \) be a family of piecewise expanding maps satisfying Definition 9.4, item (1), item (2), item (3) and item (4) of UBV4 (see Definition 9.6). Then, there is \( k \) (which does not depends on \( \delta \)) such that

\[
\sup_{\delta \in [0,1)} \max_i \text{var}_{P_i}(g_{i, \delta}^{(k)}) + 3 \sup_{P_i} g_{i, \delta}^{(k)} < 1.
\]

Proof. (of the Lemma) First of all, consider a piecewise expanding map, \( T : [0,1] \rightarrow [0,1] \) satisfying Definition 9.4. For all \( n \geq 1 \), let \( P^{(n)} \) be the partition of I s.t. \( P^{(n)}(x) = P^{(n)}(y) \) if and only if \( P^{(1)}(T^j(x)) = P^{(1)}(T^j(y)) \) for all \( j = 0, \ldots, n-1 \), where \( P^{(1)} = P \). For each \( n \) define \( T^n_\delta T^n P_i \) and \( g^{(n)}_i = \frac{1}{|T^n_i|} \), for all \( P_i \in P^{(n)} \).
Let us consider \( n_0 \) and \( \lambda_1 \) from item 2) of Definition 9.4: \( \inf |T^{n_0}_n| \geq \lambda_1 > 1 \). For a given \( n \geq 1 \), we write \( n = n_0 q_n + r_n \), where \( 0 \leq r_n < n_0 \). Thus, for all \( x \in P_i \in \mathcal{P}(n) = \{ P_1, \cdots, P_{q(n)} \} \), we have

\[
|T^{n}_i(x)| = |(T^{n_0}_i)^(r_n)(x)| \\
\geq (\lambda_1)^{r_n} |(T^{n_0}_i)'(x)|.
\]

Then,

\[
g_i^{(n)}(x) \leq \left( \frac{1}{\lambda_1} \right)^{q_n} \frac{1}{|T^{n_0}_i|'}(x) \leq \left( \frac{1}{\lambda_1} \right)^{\frac{n_0}{n}} \sup_{0 \leq j \leq n_0} \{ g_i \}_j \leq \lambda_4^n C_5,
\]

where \( \lambda_4 = \frac{1}{\sqrt{\lambda_1}} < 1 \) and \( C_5 = \lambda_1 \max_{0 \leq j \leq q} \{ \max_{0 \leq j \leq n_0} \sup_{i} \{ g_i \}_j \} \). Therefore,

\[
\sup_{i} \{ g_i^{(n)} \} \leq \lambda_4^n C_5,
\]

for all \( n \geq 1 \) and all \( i \).

Now, set \( C_6 := \max \{ C_5, \max_{i} \{ \text{var}(g_i) \} \} \). Thus, for all \( n \geq 1 \) it holds (see [31], page 41, equation (3.1))

\[
\text{var} g_i^{(n)} \leq \frac{n C_6^3}{\lambda_4^n} \lambda_1^4 \forall \delta \in [0, 1) \text{ and } \forall i = 1, \cdots q.
\]

Then,

\[
\text{var} g_i^{(n)} \leq C_7 \lambda_5^n, \forall n \geq 1, \forall i,
\]

where \( \lambda_5 \in (\lambda_4, 1) \) and \( C_7 := \sup_{n \geq 1} \left\{ \frac{C_6^3}{\lambda_4^n} \left( \frac{\lambda_4}{\lambda_5} \right)^{\frac{1}{n}} \right\} \).

Now, let us consider a family of piecewise expanding maps, \( \{ T_\delta \}_{\delta \in [0, 1)} \), satisfying Definition 9.4, item (1), item (2), item (3) and item (4) of UBV4 (see Definition 9.6). Applying the above equations to \( T_\delta \) we get, for all \( i \) and all \( \delta \)

\[
\sup_{i} \{ g_i^{(n, \delta)} \} \leq \lambda_4^n C_{5, \delta},
\]

where \( \lambda_{4, \delta} = \frac{1}{\sqrt{\lambda_1(\delta)}} \) and \( C_{5, \delta} = \lambda_1(\delta) \max_{i} \{ \max_{0 \leq j \leq n_0(\delta)} \sup_{i} \{ g_i \}_j \} \). By item (1) of UBV4, we get

\[
\lambda_{4, u} := \sup_{\delta \in [0, 1)} \{ \lambda_{4, \delta} \} = \sup_{\delta \in [0, 1)} \left( \frac{1}{n_0(\delta)} \right) < 1
\]

and by items (1) and (2) of UBV4 it holds

\[
C_{5, u} := \sup_{\delta \in [0, 1)} C_{5, \delta} < \infty.
\]

Then, we get the uniform estimate

\[
\sup_{i} \{ g_i^{(n, \delta)} \} \leq \lambda_{4, u} C_{5, u},
\]

for all \( \delta \), all \( i \) and all \( n \geq 1 \).
By item (2) of UBV4, set $C_{6,u} := \max\{C_{5,u}, \sup_\delta \max_i \{\text{var}(g_{i,\delta})\}\}$. Thus, for all $n \geq 1$ it holds
\[
\text{var} g_{i,\delta}^{(n)} \leq \frac{nC_{6,u}^3}{\lambda_{4,u}} \lambda_{4,u} \forall i \text{ and } \forall \delta \in [0, 1].
\]
Then,
\[
\text{var} g_{i,\delta}^{(n)} \leq C_{7,u} \lambda_{5,u} \lambda_{4,u} \forall i, \forall \delta
\]
where $\lambda_{5,u} \in (\lambda_{4,u}, 1)$ and $C_{7,u} := \sup_{n \geq 1} \left\{\frac{C_{6,u}^3}{\lambda_{4,u}} \left(\frac{\lambda_{4,u}}{\lambda_{5,u}}\right)^n\right\}$. \hfill \Box

**Proposition 27.** If $\{F_\delta\}_{\delta \in [0,1]}$ is a BV Lorenz-like family. Then, there exist uniform constants $0 < \lambda_u < 1$, $C_u > 0$ and $k \in \mathbb{N}$ such that for every $\mu \in BV^+$, it holds
\[
\text{Var}(F_\delta \ast k \mu) \leq \lambda_u \text{Var}(\mu) + C_u |\phi_x|_{1,1}, \forall \delta \in [0, 1).
\]
(48)

**Proof.** Consider the iterate $F_\delta^k$, where $k \in \mathbb{N}$ is from Lemma 10.10. Applying Corollary 3, we get
\[
\text{Var}(F_\delta \ast k \mu) \leq \beta_\delta \text{Var}(\mu) + K_{3,\delta} |\phi_x|_{1,1}
\]
where
\[
\beta_\delta := \max_i \{\text{var} g_{i,\delta}^{(k)} + 3 \sup_{i} g_{i,\delta}^{(k)}\}
\]
and
\[
K_{3,\delta} := \max_i \{\sup_{i} g_{i,\delta}^{(k)}\} \text{var}^\circ (G_\delta^k) + \max_i \left\{\frac{\text{var}^\circ (g_{i,\delta}^{(k)}) + 2 \sup_{i} g_{i,\delta}^{(k)}}{m_i(P_i)}\right\}.
\]
By Lemma 10.3, replacing $f$ by $\pi_y$, we have
\[
\text{var}^\circ (G_\delta^k) \leq q^k \sum_{j=1}^{k} q^j \{2 \text{var}^\circ (G_\delta) + 2q\}
\]
\[
\leq q^k \sum_{j=1}^{k} q^j \{2H_\delta + 2q\}.
\]
Since by item (4) of UBV4 we have $\sup_{\delta \in [0,1]} H_\delta < \infty$, we get $\sup_{\delta \in [0,1]} \text{var}^\circ (G_\delta^k) < \infty$. By the previous comments, item (2) and item (3) of UBV4, we define
\[
C_u := \sup_{\delta \in [0,1]} \{K_{3,\delta}\} < \infty.
\]
We also set
\[
\lambda_u := \sup_{\delta \in [0,1]} \{\beta_\delta\},
\]
where, by Lemma 10.10, it holds $\lambda_u < 1$. With these definitions we arrive at inequality (48). \hfill \Box

**Proposition 28.** If $\{F_\delta\}_{\delta \in [0,1]}$ is a BV Lorenz-like family. Then, there exist uniform constants $0 < \xi_u < 1$, $B_u > 0$ such that for every $\mu \in BV^+$, all $\delta \in [0, 1)$ and all $n \geq 1$, it holds
\[
\text{Var}(F_\delta \ast n \mu) \leq \xi_u^n B_u \text{Var}(\mu) + B_u |\phi_x|_{1,1}.
\]
Proof. By UBV1 we have gives us
\[
|P^n f|_{1,1} \leq D^n|f|_{1,1} + D|f|_1, \quad \forall n, \quad \forall f \in BV_{1,1},
\]
where \( D > 0 \) and \( 0 < \lambda < 1 \). Then, since \( |f|_1 \leq |f|_{1,1} \), it holds
\[
|P^n f|_{1,1} \leq 2D|f|_{1,1}, \quad \forall n, \quad \forall f \in BV_{1,1},
\]
where \( 2D \geq 1 \). In particular, (49) holds if we replace \( f \) by \( \phi_x = \frac{d(\pi_x \mu)}{dm_1} \) for each \( \mu \in BV^+ \).

By Proposition 27 and a straightforward induction we have
\[
\Var(F_{\delta_n}^{nk} \mu) \leq \lambda_u^n \Var(\mu) + 2DC_u \sum_{i=0}^{n-1} \lambda_u^i |\phi_x|_{1,1}, \quad \forall n \geq 0.
\]

Then,
\[
\Var(F_{\delta_n}^{k \mu}) \leq \lambda_u^n \Var(\mu) + 2DC_u \frac{1}{1-\lambda_u} |\phi_x|_{1,1}, \quad \forall n \geq 0.
\]

Consider \( D (2D \geq 1) \) from equation (49) and set \( n = kq_n + r_n \), where \( 0 \leq r_n < k \).

Applying Proposition 26 iterating \( r_n \) times the inequality (47) we get
\[
\Var(F_{\delta_n}^{r_n \mu}) \leq \max_{i=0,\ldots,k} \{ \beta_u^i \} \Var(\mu) + 2DK_u \sum_{j=0}^{k} \beta_u^j |\phi_x|_{1,1}.
\]

Thus,
\[
\Var(F_{\delta_n}^{\mu}) \leq \lambda_u^n \Var(F_{\delta_n}^{r_n \mu}) + 2DC_u \frac{1}{1-\lambda_u} |P^n f_{\delta_n}^{(\phi_x)}|_{1,1}
\]
\[
\leq \lambda_u^n \max_{i=0,\ldots,k} \{ \beta_u^i \} \Var(\mu) + 2DC_u \sum_{j=0}^{k} \beta_u^j |\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u} |\phi_x|_{1,1}
\]
\[
\leq \lambda_u^n \max_{i=0,\ldots,k} \{ \beta_u^i \} \Var(\mu) + 2DK_u \sum_{j=0}^{k} \beta_u^j |\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u} |\phi_x|_{1,1}
\]
\[
\leq \left( \frac{\lambda_u}{\sqrt{\lambda_u}} \right)^n \max_{i=0,\ldots,k} \{ \beta_u^i \} \Var(\mu) + 2DK_u \sum_{j=0}^{k} \beta_u^j |\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u} |\phi_x|_{1,1}
\]
\[
\leq \xi_u B_u \Var(\mu) + B_u |\phi_x|_{1,1},
\]
where \( B_u := \max \left\{ \frac{\max_{i=0,\ldots,k} \{ \beta_u^i \}}{\lambda_u}, 2DK_u \sum_{j=0}^{k} \beta_u^j |\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u} \right\} \) and \( \xi_u := \sqrt{\lambda_u} \).

With all results established in this section, the proof of Proposition 21 is analogous to the Proposition 19, where \( B_u \) comes from Proposition 28.

11. Appendix 2: Linearity of the restriction. Let us consider the measurable spaces \((N_1, \mathcal{N}_1)\) and \((N_2, \mathcal{N}_2)\), where \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are the Borel’s \( \sigma \)-algebras of \( N_1 \) and \( N_2 \) respectively. Let \( \mu \in \mathcal{A} \mathcal{B} \) be a positive measure on the measurable space \((\Sigma, \mathcal{B})\), where \( \Sigma = N_1 \times N_2 \) and \( \mathcal{B} = \mathcal{N}_1 \times \mathcal{N}_2 \) and consider its disintegration \((\{ \mu_\gamma \}, \{ \mu_x \})\) along \( \mathcal{F}^\gamma \), where \( \mu_x = \pi_x \mu \) and \( d(\pi_x \mu) = \phi_x dm_1 \), for some \( \phi_x \in L^1(N_1, m_1) \). We will suppose that the \( \sigma \)-algebra \( \mathcal{A} \mathcal{B} \) has a countable generator.
Proposition 29. Suppose that $\mathcal{B}$ has a countable generator, $\Gamma$. If $\{\mu_\gamma\}_\gamma$ and $\{\mu'_\gamma\}_\gamma$ are disintegrations of a positive measure $\mu$ relative to $\mathcal{F}_x$, then $\phi_x(\gamma)\mu_\gamma = \phi_x(\gamma)\mu'_\gamma$ $m_1$-a.e. $\gamma \in N_1$.

Proof. Let $\mathcal{A}$ be the algebra generated by $\Gamma$. $\mathcal{A}$ is countable and $\mathcal{A}$ generates $\mathcal{B}$. For each $A \in \mathcal{A}$ define the sets

$$G_A = \{\gamma \in N_1 | \phi_x(\gamma)\mu_\gamma(A) < \phi_x(\gamma)\mu'_\gamma(A)\}$$

and

$$R_A = \{\gamma \in N_1 | \phi_x(\gamma)\mu_\gamma(A) > \phi_x(\gamma)\mu'_\gamma(A)\}.$$

If $\gamma \in G_A$ then $\gamma \not\in \pi^{-1}_x(G_A)$ and $\mu_\gamma(A) = \mu'_\gamma(A \cap \pi^{-1}_x(G_A))$. Otherwise, if $\gamma \notin G_A$ then $\gamma \cap \pi^{-1}_x(G_A) = \emptyset$ and $\mu_\gamma(A \cap \pi^{-1}_x(G_A)) = 0$. The same holds for $\mu'_\gamma$. Then, it holds

$$\mu(A \cap \pi^{-1}_x(G_A)) = \begin{cases} \int \mu_\gamma(A \cap \pi^{-1}_x(Q_A)) \phi_x(\gamma)dm_1 = \int Q_A \mu_\gamma(A)\phi_x(\gamma)dm_1 & \text{if } \phi_x(\gamma) \neq 0 \\ \int \mu'_\gamma(A \cap \pi^{-1}_x(Q_A)) \phi_x(\gamma)dm_1 = \int Q_A \mu'_\gamma(A)\phi_x(\gamma)dm_1 & \text{if } \phi_x(\gamma) = 0. \end{cases}$$

Since $\phi_x(\gamma)\mu_\gamma(A) < \mu'_\gamma(\gamma)\phi_x(\gamma)$ for all $\gamma \in G_A$, we get $m_1(G_A) = 0$. The same holds for $R_A$. Thus

$$m_1\left(\bigcup_{A \in \mathcal{A}} R_A \cup G_A\right) = 0.$$

It means that, $m_1$-a.e. $\gamma \in N_1$ the positive measures $\phi_x(\gamma)\mu_\gamma$ and $\mu'_\gamma\phi_x(\gamma)$ coincides for all measurable set $A$ of an algebra which generates $\mathcal{B}$. Therefore $\phi_x(\gamma)\mu_\gamma = \mu'_\gamma\phi_x(\gamma)$ for $m_1$-a.e. $\gamma \in N_1$. \square

Proposition 30. Let $\mu_1, \mu_2 \in AB$ be two positive measures and denote their marginal densities by $d(\mu_1)_x = \phi_x dm_1$ and $d(\mu_2)_x = \psi_x dm_1$, where $\phi_x, \psi_x \in L^1(m_1)$ respectively. Then $(\mu_1 + \mu_2)|_\gamma = \mu_1|_\gamma + \mu_2|_\gamma$ $m_1$-a.e. $\gamma \in N_1$.

Proof. Note that $d(\mu_1 + \mu_2) = (\phi_x + \psi_x) dm_1$. Moreover, consider the disintegration of $\mu_1 + \mu_2$ given by

$$\{(\mu_1 + \mu_2)|_\gamma, (\phi_x + \psi_x)_\gamma m_1\},$$

where

$$(\mu_1 + \mu_2)|_\gamma = \begin{cases} \frac{\phi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_1|_\gamma + \frac{\psi_x(\gamma)}{\phi_x(\gamma) + \psi_x(\gamma)} \mu_2|_\gamma, & \text{if } \phi_x(\gamma) + \psi_x(\gamma) \neq 0 \\ 0, & \text{if } \phi_x(\gamma) + \psi_x(\gamma) = 0. \end{cases}$$

Then, by Proposition 29 for $m_1$-a.e. $\gamma \in N_1$, it holds

$$(\phi_x + \psi_x)(\gamma) (\mu_1 + \mu_2)|_\gamma = \phi_x(\gamma)\mu_1|_\gamma + \psi_x(\gamma)\mu_2|_\gamma.$$

Therefore, $(\mu_1 + \mu_2)|_\gamma = \mu_1|_\gamma + \mu_2|_\gamma$ $m_1$-a.e. $\gamma \in N_1$. \square

Definition 11.1. We say that a positive measure $\lambda_1$ is disjoint from a positive measure $\lambda_2$ if $(\lambda_1 - \lambda_2)^+ = \lambda_1$ and $(\lambda_1 - \lambda_2)^- = \lambda_2$.

Remark 17. A straightforward computations yields that if $\lambda_1 + \lambda_2$ is disjoint from $\lambda_3$, then both $\lambda_1$ and $\lambda_2$ are disjoint from $\lambda_3$, where $\lambda_1, \lambda_2$ and $\lambda_3$ are all positive measures.

Lemma 11.2. Suppose that $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are the Jordan decompositions of the signed measures $\mu$ and $\nu$. Then, there exist positive measures $\mu_1, \mu_2, \mu^{++}, \mu^{--}, \nu^{++}$ and $\nu^{--}$ such that $\mu^+ = \mu^{++} + \mu_1$ $\mu^- = \mu^{--} + \mu_2$ and $\nu^+ = \nu^{++} + \mu_2$, $\nu^- = \nu^{--} + \mu_1$. 

Proof. Suppose $\mu = \nu_1 - \nu_2$ with $\nu_1$ and $\nu_2$ positive measures. Let $\mu^+$ and $\mu^-$ be the Jordan decomposition of $\mu$. Let $\mu' = \nu_1 - \mu^+$, then $\nu_1 = \mu^+ - \mu'$. Indeed $\mu^+ - \mu^- = \nu_1 - \nu_2$ which implies that $\mu^+ - \nu_1 = \mu^- - \nu_2$. Thus if $\nu_1, \nu_2$ is a decomposition of $\mu$, then $\nu_1 = \mu^+ + \mu'$ and $\nu_2 = \mu^- + \mu'$ for some positive measure $\mu'$. Now, consider $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$. Since the pairs of positive measures $\mu^+, \nu^-$ and $(\mu^+ - \nu^-)^+$, $(\mu^+ - \nu^-)^-$ are both decompositions of $\mu^+ - \nu^-$, by the above comments, we get that $\mu^+ = (\mu^+ - \nu^-)^+ + \mu_1$ and $\nu^- = (\mu^+ - \nu^-)^- + \mu_1$, for some positive measure $\mu_1$. Analogously, since the pairs of positive measures $\mu^-, \nu^+$ and $(\nu^+ - \mu^-)^+$, $(\nu^+ - \mu^-)^-$ are both decompositions of $\nu^+ - \mu^-$, by the above comments, we get that $\nu^+ = (\nu^+ - \mu^-)^+ + \mu_2$ and $\mu^- = (\nu^+ - \mu^-)^- + \mu_2$, for some positive measure $\mu_2$. By definition $11.1$, $\mu^+$ and $\mu^-$ are disjoint, and so are $(\mu^+ - \nu^-)^+$ and $(\nu^+ - \mu^-)^-$. Analogously, $\nu^+$ and $\nu^-$ are disjoint, and so are $(\mu^+ - \nu^-)^-$ and $(\nu^+ - \mu^-)^+$. Moreover, since $(\mu^+ - \nu^-)^+$ and $(\nu^+ - \mu^-)^-$ are disjoint, so are $(\nu^+ - \mu^-)^+$ and $(\nu^+ - \mu^-)^-$. This gives that, the pair $(\mu^+ - \nu^-)^+ + (\nu^+ - \mu^-)^+$, $(\nu^+ - \mu^-)^- + (\nu^+ - \mu^-)^-$ is a Jordan decomposition of $\mu + \nu$ and we are done. $\square$

Proposition 31. Let $\mu, \nu \in A\mathcal{B}$ be two signed measures. Then $(\mu + \nu)|_{\gamma} = \mu|_{\gamma} + \nu|_{\gamma}$ $m_1$-a.e. $\gamma \in N_1$.

Proof. Suppose that $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are the Jordan decompositions of $\mu$ and $\nu$ respectively. By definition, $\mu|_{\gamma} = \mu^+|_{\gamma} - \mu^-|_{\gamma}$, $\nu|_{\gamma} = \nu^+|_{\gamma} - \nu^-|_{\gamma}$.

By Lemma 11.2, suppose that $\mu^+ = \mu^{++} + \mu_1$, $\mu^- = \mu^{--} + \mu_2$ and $\nu^+ = \nu^{++} + \mu_2$, $\nu^- = \nu^{--} + \mu_1$. In a way that $(\mu + \nu)^+ = \mu^{++} + \nu^{++} + \mu_1$ and $(\mu + \nu)^- = \mu^{--} + \nu^{--}$. By Proposition 30, it holds $\mu^+|_{\gamma} = \mu^{++}|_{\gamma} + \mu_1|_{\gamma}$, $\mu^-|_{\gamma} = \mu^{--}|_{\gamma} + \mu_2|_{\gamma}$, $\nu^+|_{\gamma} = \nu^{++}|_{\gamma} + \mu_1|_{\gamma}$ and $\nu^-|_{\gamma} = \nu^{--}|_{\gamma} + \mu_2|_{\gamma}$.

Moreover,

$(\mu + \nu)^+|_{\gamma} = (\mu + \nu)^+|_{\gamma} = \mu^{++}|_{\gamma} + \nu^{++}|_{\gamma}$

$(\mu + \nu)^-|_{\gamma} = \mu^{--}|_{\gamma} + \nu^{--}|_{\gamma}$

Putting all together, we get:

$(\mu + \nu)|_{\gamma} = (\mu + \nu)^+|_{\gamma} - (\mu + \nu)^-|_{\gamma}$

$= \mu^{++}|_{\gamma} + \nu^{++}|_{\gamma} - (\mu^{--}|_{\gamma} + \nu^{--}|_{\gamma})$

$= \mu^{++}|_{\gamma} + \mu_1|_{\gamma} + \nu^{++}|_{\gamma} + \mu_2|_{\gamma} - (\mu^{--}|_{\gamma} + \mu_2|_{\gamma} + \nu^{--}|_{\gamma} + \mu_1|_{\gamma})$

$= (\mu^+|_{\gamma} + \mu^-|_{\gamma} + \nu^+|_{\gamma} + \nu^-|_{\gamma})$

$= \mu|_{\gamma} + \nu|_{\gamma}$. $\square$

We immediately arrive at the following

Corollary 4. Let $\mu \in A\mathcal{B}$ be a signed measure and $\mu = \mu^+ - \mu^-$ its Jordan decomposition. If $\mu_1$ and $\mu_2$ are positive measures such that $\mu = \mu_1 - \mu_2$, then $\mu|_{\gamma} = \mu_1|_{\gamma} - \mu_2|_{\gamma}$. It means that, the restriction does not depends on the decomposition of $\mu$.

12. Appendix 3: Uniform family of operators. In this section, we prove the main results on uniform families of operators stated in Section 9.0.1. We state a general lemma on the stability of fixed points satisfying certain assumptions. Consider two operators $L_0$ and $L_\delta$ preserving a normed space of signed measures $\mathcal{B} \subseteq \mathcal{S}(X)$ with norm $|| \cdot ||_\mathcal{B}$. Suppose that $f_0, f_\delta \in \mathcal{B}$ are fixed points of $L_0$ and $L_\delta$, respectively.
Lemma 12.1. Suppose that:

\( a) \ |\|L_\delta f_\delta - L_0 f_\delta||_B < \infty; \)

\( b) \) For all \( i \geq 1, \ L_0^i \) is continuous on \( B \): for each \( i \geq 1, \ \exists C_i \) s.t. \( \forall g \in B, \ |\|L_0^i g||_B \leq C_i |\|g||_B. \)

Then, for each \( N \geq 1, \) it holds

\[
|\|f_\delta - f_0||_B| \leq |\|L_0^N (f_\delta - f_0)||_B| + |\|L_\delta f_\delta - L_0 f_\delta||_B| \sum_{i \in [0, N-1]} C_i.
\]

Proof. The proof is a direct computation. First note that,

\[
|\|f_\delta - f_0||_B| \leq |\|L_\delta^N f_\delta - L_0^N f_0||_B|
\]

\[
\leq |\|L_0^N f_0 - L_0^N f_\delta||_B| + |\|L_\delta^N f_\delta - L_\delta^N f_\delta||_B|\]

\[
\leq |\|L_\delta^N (f_0 - f_\delta)||_B| + |\|L_0^N f_\delta - L_\delta^N f_\delta||_B|.
\]

Moreover,

\[
L_0^N - L_\delta^N = \sum_{k=1}^{N} L_0^{N-k} (L_0 - L_\delta) L_\delta^{(k-1)}
\]

hence

\[
(L_0^N - L_\delta^N) f_\delta = \sum_{k=1}^{N} L_0^{N-k} (L_0 - L_\delta) L_\delta^{(k-1)} f_\delta
\]

\[
= \sum_{k=1}^{N} L_0^{N-k} (L_0 - L_\delta) L_\delta f_\delta
\]

by item \( b) \), we have

\[
|\|(L_0^N - L_\delta^N) f_\delta||_B| \leq \sum_{k=1}^{N} C_{N-k} |\|(L_0 - L_\delta) f_\delta||_B|
\]

\[
\leq |\|(L_0 - L_\delta) f_\delta||_B| \sum_{i \in [0, N-1]} C_i
\]

and then

\[
|\|f_\delta - f_0||_B| \leq |\|L_0^N (f_\delta - f_\delta)||_B| + |\|(L_0 - L_\delta) f_\delta||_B| \sum_{i \in [0, N-1]} C_i.
\]

Now, let us apply the statement to our family of operators satisfying assumptions UF1–UF4, supposing \( B_w = B \). We have the following

Proposition 32. Suppose \( \{L_\delta\}_{\delta \in [0, 1)} \) is a uniform family of operators as in Definition 9.1, where \( f_0 \) is the unique fixed point of \( L_0 \) in \( B_w \) and \( f_\delta \) is a fixed point of \( L_\delta \). Then, there is a \( \delta_0 \in (0, 1) \) such that for all \( \delta \in (0, \delta_0] \) it holds

\[
|\|f_\delta - f_0||_w = O(\delta \log \delta).
\]

Proof. First note that, if \( \delta \geq 0 \) is small enough, then \( \delta \leq -\delta \log \delta. \) Moreover, \( x - 1 \leq |x| \), for all \( x \in \mathbb{R}. \)

By UF2,

\[
|\|L_\delta f_\delta - L_0 f_\delta||_w| \leq \delta C
\]

(see Lemma 12.1, item a) ) and UF4 yields \( C_i \leq M_2. \)
Hence, by Lemma 12.1 we have
\[ \| f_\delta - f_0 \|_w \leq \delta CM_2 N + \| L_\delta \delta (f_\delta - f_0) \|_w. \]

By the exponential convergence to equilibrium of \( L_\delta \) (UF3), there exists \( 0 < \rho_2 < 1 \) and \( C_2 > 0 \) such that (recalling that by UF1 \( \| (f_\delta - f_0) \|_s \leq 2M \))
\[ \| L_\delta \delta (f_\delta - f_0) \|_w \leq C_2 \rho_2^N \| (f_\delta - f_0) \|_s \]
\[ \leq 2C_2 \rho_2^N M \]

hence
\[ \| f_\delta - f_0 \|_w \leq \delta CM_2 N + 2C_2 \rho_2^N M. \]

Choosing \( N = \left\lceil \frac{\log \delta}{\log \rho_2} \right\rceil \), we have
\[ \| f_\delta - f_0 \|_w \leq \delta \log \delta CM_2 \left\lceil \frac{\log \delta}{\log \rho_2} \right\rceil + 2C_2 \rho_2 \left\lceil \frac{\log \delta}{\log \rho_2} \right\rceil M \]
\[ \leq \delta \log \delta CM_2 \frac{\log \delta}{\log \rho_2} + 2C_2 \rho_2 \frac{\log \delta}{\log \rho_2} M \]
\[ \leq \delta \log \delta CM_2 \frac{1}{\log \rho_2} - \frac{2C_2 \rho_2}{\log \rho_2} M \]
\[ \leq \delta \log \delta CM_2 \frac{1}{\log \rho_2} + \frac{2C_2 \rho_2}{\log \rho_2} M \]
\[ \leq \delta \log \delta \left( \frac{CM_2}{\log \rho_2} - \frac{2C_2 M}{\rho_2} \right). \]

\[ \square \]

13. **Appendix 4: On disintegration of measures.** In this section, we prove some results on disintegration of absolutely continuous measures with respect to a measure \( \mu_0 \in \mathcal{A}\mathcal{B} \). Precisely, we are going to prove Lemma 9.11.

Let us fix some notations. Denote by \((N_1, m_1)\) and \((N_2, m_2)\) the spaces defined in section 2. For a \( \mu_0 \)-integrable function \( f : N_1 \times N_2 \to \mathbb{R} \) and a pair \((\gamma, y) \in N_1 \times N_2\) \((\gamma \in N_1 \text{ and } y \in N_2)\) we denote by \( f_{\gamma} : N_2 \to \mathbb{R} \), the function defined by \( f_{\gamma}(y) = f(\gamma, y) \) and \( f|_{\gamma} \) the restriction of \( f \) on the set \( \{\gamma\} \times N_2 \). Then \( f_{\gamma} = f \circ \pi_{y,\gamma} \) and \( f_{\gamma} \circ \pi_{x,\gamma} = f_{\gamma} \), where \( \pi_{y,\gamma} \) is restriction of the projection \( \pi_y(\gamma, y) := y \) on the set \( \{\gamma\} \times N_2 \). When no confusion can be done, we will denote the leaf \( \{\gamma\} \times N_2 \) just by \( \gamma \).

From now and ahead, for a given positive measure \( \mu \in \mathcal{A}\mathcal{B} \), on \( N_1 \times N_2 \), \( \tilde{\mu} \) stands for the measure \( \pi_{x,\mu} \). Where \( \pi_x \) is the projection on the first coordinate, \( \pi_x(x, y) = x \).

For a given measurable set \( A \subset N_1 \), define \( g : N_1 \to \mathbb{R} \), by
\[ g(\gamma) = \phi_x(\gamma) \int \chi_{\pi_{x_{\gamma}}(A)}(y) f|_{\gamma}(y) d\mu_{0,\gamma}(y) \]
and note that
\[ g(\gamma) = \begin{cases} 
\phi_x(\gamma) \int f|_{\gamma}(y) d\mu_{0,\gamma}, & \text{if } \gamma \in A \\
0, & \text{if } x \notin A.
\end{cases} \]
Then, it holds
\[ g(\gamma) = \chi_A(\gamma) \phi_x(\gamma) \int f|_{\gamma}(y) d\mu_{0,\gamma}. \]

Proof. (of Lemma 9.11)
For each measurable set \( A \subset N_1 \), we have
\[
\int_A \frac{\pi_x^*(f\mu_0)}{dm_1} dm_1 = \int \chi_A \circ \pi_x d(f\mu_0) = \int \chi_{\pi^{-1}(A)} f d\mu_0 = \int \left[ \int \chi_{\pi^{-1}(A)}|_{\gamma}(y) f|_{\gamma}(y) d\mu_{0,\gamma}(y) \right] d(\phi_x m_1)(\gamma) = \int \left[ \phi_x(\gamma) \int \chi_{\pi^{-1}(A)}|_{\gamma}(y) f|_{\gamma}(y) d\mu_{0,\gamma}(y) \right] d(m_1)(\gamma) = \int g(\gamma) d(m_1)(\gamma) = \int A \left[ \int f|_{\gamma}(y) d\mu_{0,\gamma}(y) \right] d(m_1)(\gamma).
\]

Thus, it holds
\[ \frac{\pi_x^*(f\mu_0)}{dm_1}(\gamma) = \int f|_{\gamma}(y) d\mu_{0,\gamma}, \text{ for } m_1 - \text{a.e. } \gamma \in N_1. \]

And by a straightforward computation
\[ \frac{\pi_x^*(f\mu_0)}{dm_1}(\gamma) = \phi_x(\gamma) \int f|_{\gamma}(y) d\mu_{0,\gamma}, \text{ for } m_1 - \text{a.e. } \gamma \in N_1. \] (50)

Thus, equation (33) is established.

Remark 18. Setting,
\[ \overline{f} := \frac{\pi_x^*(f\mu_0)}{dm_1}, \] (51)

we get, by equation (50), \( \overline{f}(\gamma) = 0 \) iff \( \phi_x(\gamma) = 0 \) or \( \int f|_{\gamma}(y) d\mu_{0,\gamma}(y) = 0 \), for \( m_1 \)-a.e. \( \gamma \in N_1 \).

Now, let us see that, by the \( \overline{\nu} \)-uniqueness of the disintegration, equation (34) holds. To do it, define, for \( m_1 \)-a.e. \( \gamma \in N_1 \), de function \( h_\gamma : N_2 \rightarrow \mathbb{R} \), in a way that
\[ h_\gamma(y) = \begin{cases} \frac{f|_{\gamma}(y)}{\int f|_{\gamma}(y) d\mu_{0,\gamma}(y)}, & \text{if } \gamma \in B^c \\ 0, & \text{if } \gamma \in B. \end{cases} \] (52)

Let us prove equation (34) by showing that, for all measurable set \( E \subset N_1 \times N_2 \), it holds
\[ f\mu_0(E) = \int_{N_1} \int_{E \cap \gamma} h_\gamma(y) d\mu_{0,\gamma}(y) d(\pi_x^*(f\mu_0))(\gamma). \]
In fact, by equations (50), (51), (52) and remark 18, we get

\[
f_{\mu_0}(E) = \int_E f d\mu_0 = \int_{N_1} \int_{E \cap \gamma} f |_{\gamma} d\mu_{0,\gamma} d(\phi_x m_1)(\gamma) = \int_{B^c} \int_{E \cap \gamma} f |_{\gamma} d\mu_{0,\gamma} d(\phi_x m_1)(\gamma) = \int_{B^c} \int_{E \cap \gamma} f |_{\gamma}(y) d\mu_{0,\gamma}(y) \int_{E \cap \gamma} f |_{\gamma} d\mu_{0,\gamma} \] d\mu_1(\gamma)

\[
= \int_{B^c} T(\gamma) \left[ \int_{E \cap \gamma} f |_{\gamma}(y) d\mu_{0,\gamma}(y) \int_{E \cap \gamma} f |_{\gamma} d\mu_{0,\gamma} \right] d\mu_1(\gamma)

= \int_{B^c} \int_{E \cap \gamma} h_{\gamma}(y) d\mu_{0,\gamma}(y) d(\pi_x * (f \mu_0))(\gamma)

= \int_{N_1} \int_{E \cap \gamma} h_{\gamma}(y) d\mu_{0,\gamma}(y) d(\pi_x * (f \mu_0))(\gamma).
\]

And we are done. \(\square\)

REFERENCES

[1] J. F. Alves and M. Soufi, Statistical stability of geometric Lorenz attractors, Fund. Math., 224 (2014), 219–231.

[2] V. Araújo, S. Galatolo and M. Pacifico, Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors, Math. Z., 276 (2014), 1001–1048.

[3] V. Araújo and M. Pacifico, Three-Dimensional Flows, A Series of Modern Surveys in Mathematics, 53, Springer, Heidelberg, 2010.

[4] W. Bahsoun and M. Ružiboev, On the statistical stability of Lorenz attractors with a \(C^{1+\alpha}\) stable foliation, Ergodic Theory Dynam. Systems, 39 (2019), 3169–3184.

[5] V. Baladi, The quest for the ultimate anisotropic Banach space, J. Stat. Phys., 166 (2017), 525–557.

[6] V. Baladi, Positive Transfer Operators and Decay of Correlations, Advanced Series in Nonlinear Dynamics, 16, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[7] V. Baladi and M. Tsujii, Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms, Ann. Inst. Fourier (Grenoble), 57 (2007), 127–154.

[8] V. Baladi and S. Gouëzel, Banach spaces for piecewise cone-hyperbolic maps, J. Mod. Dyn., 4 (2010), 91–137.

[9] V. Baladi and S. Gouëzel, Good Banach spaces for piecewise hyperbolic maps via interpolation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 1453–1481.

[10] A. Boyarsky and P. Gora, Laws of Chaos. Invariant Measures and Dynamical Systems in One Dimension, Probability and its Applications, Birkhäuser, Boston, MA, 1997.

[11] O. Butterley and C. Liverani, Smooth Anosov flows: Correlation spectra and stability, J. Mod. Dyn., 1 (2007), 301–322.

[12] O. Butterley and I. Melbourne, Disintegration of invariant measures for hyperbolic skew products, Israel J. Math., 219 (2017), 171–188.

[13] M. Demers, A gentle introduction to anisotropic Banach spaces, Chaos Solitons Fractals, 116 (2018), 29–42.

[14] M. Demers and C. Liverani, Stability of statistical properties in two-dimensional piecewise hyperbolic maps, Trans. Amer. Math. Soc., 360 (2008), 4777–4814.

[15] M. Demers and H. Z. Zhang, Spectral analysis of the transfer operator for the Lorentz gas, J. Mod. Dyn., 5 (2011), 665–709.
[16] M. Demers and H. Z. Zhang, A functional analytic approach to perturbations of the Lorentz gas, Comm. Math. Phys., 324 (2013), 767–830.
[17] S. Galatolo, Statistical properties of dynamics. Introduction to the functional analytic approach, preprint, arXiv:math/1510.02615.
[18] S. Galatolo, Quantitative statistical stability, speed of convergence to equilibrium and partially hyperbolic skew products, J. Éc. polytech. Math., 5 (2018), 377–405.
[19] S. Galatolo, I. Nisoli and B. Saussol, An elementary way to rigorously estimate convergence to equilibrium and escape rates, J. Comput. Dyn., 2 (2015), 51–64.
[20] S. Galatolo and M. J. Pacifico, Lorenz-like flows: Exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence, Ergodic Theory Dynam. Systems, 30 (2010), 1703–1737.
[21] S. Gouezel and C. Liverani, Banach spaces adapted to Anosov systems, Ergodic Theory Dynam. Systems, 26 (2006), 189–217.
[22] C. Ionescu-Tulcea and G. Marinescu, Théorie ergodique pour des classes d’opérateurs non complètement continues, Ann. of Math. (2), 52 (1950), 140–147.
[23] G. Keller, Generalized bounded variation and applications to piecewise monotonic transformations, Z. Wahrsch. Verw. Gebiete, 69 (1985), 461–478.
[24] G. Keller and C. Liverani, Stability of the spectrum for transfer operators, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 28 (1999), 141–152.
[25] A. Lasota and J. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc., 186 (1973), 481–488.
[26] C. Liverani, Invariant Measures and Their Properties. A Functional Analytic Point of View, Dynamical Systems, Part II, Scuola Norm. Sup., Pisa, 2003, 185–237.
[27] C. Liverani, Decay of correlations, Ann. of Math. (2), 142 (1995), 239–301.
[28] R. Lucena, Spectral Gap for Contracting Fiber Systems and Applications, Ph.D thesis, Universidade Federal do Rio de Janeiro in Brazil, 2015.
[29] K. Oliveira and M. Viana, Fundamentos da Teoria Ergódica, Coleção Fronteiras da Matemática - SBM, Brazil, 2014.
[30] J. Rousseau-Egele, Un théorème de la limite locale pour une classe de transformations dilatantes et monotones par morceaux, Ann. Probab., 11 (1983), 772–788.
[31] M. Viana, Stochastic dynamics of deterministic systems, Brazilian Math. Colloquium, IMPA, 1997, IMPA. Available from: http://w3.impa.br/~viana/out/odds.pdf.

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