Algebraic and topological structures on the set of mean functions and generalization of the AGM mean

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Abstract

In this paper, we present new structures and results on the set $M_\mathcal{D}$ of mean functions on a given symmetric domain $\mathcal{D}$ of $\mathbb{R}^2$. First, we construct on $M_\mathcal{D}$ a structure of abelian group in which the neutral element is simply the Arithmetic mean; then we study some symmetries in that group. Next, we construct on $M_\mathcal{D}$ a structure of metric space under which $M_\mathcal{D}$ is nothing else the closed ball with center the Arithmetic mean and radius $1/2$. We show in particular that the Geometric and Harmonic means lie in the border of $M_\mathcal{D}$. Finally, we give two important theorems generalizing the construction of the AGM mean. Roughly speaking, those theorems show that for any two given means $M_1$ and $M_2$, which satisfy some regular conditions, there exists a unique mean $M$ satisfying the functional equation: $M(M_1, M_2) = M$.

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1 Introduction

Let $\mathcal{D}$ be a nonempty symmetric domain of $\mathbb{R}^2$. A mean function (or simply a mean) on $\mathcal{D}$ is a function $M : \mathcal{D} \to \mathbb{R}$ satisfying the three following axioms:

i) $M$ is symmetric, that is for all $(x, y) \in \mathcal{D}$:

$$M(x, y) = M(y, x).$$
ii) For all \((x, y) \in D\), we have:
\[
\min(x, y) \leq M(x, y) \leq \max(x, y).
\]

iii) For all \((x, y) \in D\), we have:
\[
M(x, y) = x \implies x = y.
\]

Let us just remark that because of the axiom ii), the implication of the axiom iii) is actually even an equivalence.

Among the most known examples of mean functions, we cite:

- The arithmetic mean, noted \(A\) and defined on \(\mathbb{R}^2\) by:
  \[
  A(x, y) = \frac{x + y}{2}.
  \]

- The geometric mean, noted \(G\) and defined on \((0, +\infty)^2\) by:
  \[
  G(x, y) = \sqrt{xy}.
  \]

- The harmonic mean, noted \(H\) and defined on \((0, +\infty)^2\) by:
  \[
  H(x, y) = \frac{2xy}{x + y}.
  \]

- The Gauss arithmetic-geometric mean, noted \(AGM\) and defined on \((0, +\infty)^2\) by the following process:
  Given \(x, y\) positive real numbers, \(AGM(x, y)\) is the common limit of the two adjacent sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) defined by:
  \[
  \begin{align*}
  x_0 &= x, & y_0 &= y \\
  x_{n+1} &= \frac{x_n + y_n}{2} & (\forall n \in \mathbb{N}) \\
  y_{n+1} &= \sqrt{x_n y_n} & (\forall n \in \mathbb{N})
  \end{align*}
  \]

For a more profound survey on the mean functions, we refer to the chapter 8 of the book [1] in which the mean \(AGM\) takes the principal place. However, there are some differences between that reference and the present paper. Indeed, in [1], only the axiom ii) is considered for defining a mean function; the axiom iii) is added for obtaining the so called strict mean while the axiom of symmetry ii) is not taken into account. In this paper, we shall see that the three axioms i), ii) and iii) are both necessary and sufficient to define a good mean or a good set of mean functions on a given domain. In particular, the axiom ii), which is excluded in the book [1], is necessary for the foundation of...
our algebraic and topological structures announced in the title of this paper (see sections 2 and 3).

**Remark about the axiom iii).** The axiom iii) permits to avoid functions as 
\((x, y) \mapsto \min(x, y)\) and \((x, y) \mapsto \max(x, y)\) which are not *true* means although they satisfy the two axioms i) and ii). But beyond this simple constatation, the axiom iii) will play a vital role throughout this paper and especially in the generalization of the construction of the arithmetic-geometric mean established by Theorem 4.3.

Given a nonempty symmetric domain \(\mathcal{D}\) of \(\mathbb{R}^2\), we denote by \(\mathcal{M}_\mathcal{D}\) the set of mean functions on \(\mathcal{D}\). The purpose of this paper is on the one hand to establish important algebraic and topological structures on \(\mathcal{M}_\mathcal{D}\) and to study some of their properties and on the other hand to generalize in a natural way the arithmetic-geometric mean AGM. The article is composed of three sections:

In the first section, we define on \(\mathcal{M}_\mathcal{D}\) a structure of abelian group in which the neutral element is simply the arithmetic mean on \(\mathcal{D}\). The study of this group reveals us that the arithmetic, geometric and harmonic means lie in a very particular class of mean functions that we call *the normal mean functions*. We then study the symmetries on \(\mathcal{M}_\mathcal{D}\) and we discover that the symmetries with respect to one of the three means \(A\), \(G\) and \(H\) oddly coincides with another type of symmetry (with respect to the same means) we introduce and call *the functional symmetry*. The problem of describing the set of the all means realizing that curious coincidence is still open.

In the second section, we define on \(\mathcal{M}_\mathcal{D}\) a structure of metric space which turns out to be a closed ball with center \(A\) and radius \(1/2\). We then use the group structure (introduced in the first section) to calculate the distance between two arbitrary means on \(\mathcal{D}\); this permits us in particular to establish a simple characterization of the border of \(\mathcal{M}_\mathcal{D}\).

In the third section, we introduce the concept of *functional middle* of two mean functions on \(\mathcal{D}\) which generalize in a natural way the arithmetic-geometric mean, so that the latter is the functional middle of the arithmetic mean and the geometric mean. We establish two theorems, each provides a sufficiently condition for the existence and the uniqueness of the functional middle of two given means. The first one uses the topological structure of \(\mathcal{M}_\mathcal{D}\) by imposing on the two means in question the condition that they are not diametrically opposed. The second one (more important) imposes on the two means in question to be just continuous on \(\mathcal{D}\). In the proof of the latter one, the axiom iii) plays an extremely vital role.
2 An abelian group structure on $M_\mathcal{D}$

For the following, given $\mathcal{D}$ a nonempty symmetric domain of $\mathbb{R}^2$, we call $A_\mathcal{D}$ the set of asymmetric maps on $\mathcal{D}$; that is maps $f : \mathcal{D} \to \mathbb{R}$, satisfying:

$$f(x, y) = -f(y, x) \quad (\forall (x, y) \in \mathcal{D}).$$

It is clear that $(A_\mathcal{D}, +)$ (where $+$ is the usual addition law of the maps from $\mathcal{D}$ into $\mathbb{R}$) is an abelian group with neutral element the null map from $\mathcal{D}$ into $\mathbb{R}$.

Now, consider $\varphi : M_\mathcal{D} \to \mathbb{R}^\mathcal{D}$ the map defined by:

$$\forall M \in M_\mathcal{D}, \forall (x, y) \in \mathcal{D} : \quad \varphi(M)(x, y) := \begin{cases} \log \left( -\frac{M(x, y) - x}{M(x, y) - y} \right) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

The axioms i), ii) and iii) (verified by $M$, as a mean function) insure the well-definition of $\varphi$, that is they insure that the quantity $-\frac{M(x, y) - x}{M(x, y) - y} \,(\text{for } x \neq y)$ is well-defined and positive.

The axiom of symmetry i) shows in addition that for all $M \in M_\mathcal{D}$, we have $\varphi(M) \in A_\mathcal{D}$. Indeed, for all $M \in M_\mathcal{D}$ and for all $(x, y) \in \mathcal{D}$ such that $x \neq y$, we have:

$$\varphi(M)(x, y) = \log \left( -\frac{M(x, y) - x}{M(x, y) - y} \right) = \log \left( -\frac{M(y, x) - x}{M(y, x) - y} \right) \quad (\text{according to the axiom i}))
$$

$$= -\log \left( -\frac{M(y, x) - y}{M(y, x) - x} \right)
$$

$$= -\varphi(M)(y, x).$$

Since we have in addition $\varphi(M)(x, x) = 0 \,(\forall (x, x) \in \mathcal{D})$, then it follows that $\varphi(M)$ is effectively an asymmetric map on $\mathcal{D}$, as claimed.

Conversely, if $f$ is an asymmetric map on $\mathcal{D}$, we easily verify that $M : \mathcal{D} \to \mathbb{R}$, defined by:

$$M(x, y) := \frac{x + ye^{f(x, y)}}{e^{f(x, y)} + 1} \quad (\forall (x, y) \in \mathcal{D})$$

is a mean function on $\mathcal{D}$ and that we have $\varphi(M) = f$.

So, it follows that the map $\varphi$ constitutes a bijection from $M_\mathcal{D}$ to $A_\mathcal{D}$ and that its inverse map $\varphi^{-1}$ is given by:

$$\forall f \in A_\mathcal{D}, \forall (x, y) \in \mathcal{D} : \quad \varphi^{-1}(f)(x, y) = \frac{x + ye^{f(x, y)}}{e^{f(x, y)} + 1}. \quad (1)$$
We thus can transport, by $\varphi$, the abelian group structure $(A_\mathcal{D}, +)$ on $M_\mathcal{D}$. This consists to define on $M_\mathcal{D}$ the following internal composition law $\ast$:

\[
\forall M_1, M_2 \in M_\mathcal{D} : \quad M_1 \ast M_2 = \varphi^{-1}(\varphi(M_1) + \varphi(M_2)).
\]

So $(M_\mathcal{D}, \ast)$ is an abelian group and $\varphi$ is a group isomorphism from $(M_\mathcal{D}, \ast)$ to $(A_\mathcal{D}, +)$.

Furthermore, since the null map on $\mathcal{D}$ is the neutral element of the group $(A_\mathcal{D}, +)$ and that $\varphi^{-1}(0) = A$, then we deduce that the arithmetic mean $A$ on $\mathcal{D}$ is the neutral element of the group $(M_\mathcal{D}, \ast)$.

By calculating explicitly $M_1 \ast M_2$ (for $M_1, M_2 \in M_\mathcal{D}$), we obtain the following theorem:

**Theorem 2.1** Let $\mathcal{D}$ be a nonempty symmetric domain of $\mathbb{R}^2$. Then, the law $\ast$ on $M_\mathcal{D}$ defined by:

\[
(M_1 \ast M_2)(x, y) := \begin{cases} \frac{x(M_1(x, y) - x)(M_2(x, y) - y) + y(M_1(x, y) - y)(M_2(x, y) - y)}{(M_1(x, y) - x)(M_2(x, y) - y) + (M_1(x, y) - y)(M_2(x, y) - y)} & \text{if } x \neq y \\ x & \text{if } x = y \end{cases}
\]

is an internal composition law on $M_\mathcal{D}$ and $(M_\mathcal{D}, \ast)$ is an abelian group with neutral element the arithmetic mean $A$ on $\mathcal{D}$.

In addition, the map $\varphi : M_\mathcal{D} \to A_\mathcal{D}$ defined by:

\[
\forall M \in M_\mathcal{D}, \forall (x, y) \in \mathcal{D} : \quad \varphi(M)(x, y) := \begin{cases} \log \left( -\frac{M(x, y) - x}{M(x, y) - y} \right) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}
\]

constitutes a group isomorphism from $(M_\mathcal{D}, \ast)$ to $(A_\mathcal{D}, +)$.

Now, let us calculate the images of the geometric and harmonic means by the isomorphism $\varphi$ (with $\mathcal{D} = (0, +\infty)^2$). For all $(x, y) \in \mathcal{D}, x \neq y$, we have:

\[
\varphi(G)(x, y) = \log \left( -\frac{G(x, y) - x}{G(x, y) - y} \right)
= \log \left( -\frac{\sqrt{xy} - x}{\sqrt{xy} - y} \right)
= \log \left( -\frac{\sqrt{x}(\sqrt{y} - \sqrt{x})}{\sqrt{y}(\sqrt{x} - \sqrt{y})} \right)
= \log \sqrt{x} - \log \sqrt{y}
= \frac{1}{2} \log x - \frac{1}{2} \log y.
\]
Since, in addition, \( \phi(G)(x, x) = 0 \) (\( \forall x \in (0, +\infty) \)), it follows that:

\[
\phi(G)(x, y) = \frac{1}{2}\log x - \frac{1}{2}\log y \quad (\forall (x, y) \in D). \tag{2}
\]

Similarly, for all \((x, y) \in D, x \neq y\), we have:

\[
\phi(H)(x, y) = \log \left( -\frac{H(x, y) - x}{H(x, y) - y} \right)
= \log \left( -\frac{\frac{2xy}{x+y} - x}{\frac{2xy}{x+y} - y} \right)
= \log \left( -\frac{x}{y} \right)
= \log x - \log y.
\]

Since we have in addition \( \phi(H)(x, x) = 0 \) (\( \forall x \in (0, +\infty) \)), it follows that:

\[
\phi(H)(x, y) = \log x - \log y \quad (\forall (x, y) \in D). \tag{3}
\]

From (2) and (3), we constat that \( \phi(G) \) and \( \phi(H) \) (with also \( \phi(A) \)) have a particular form of asymmetric maps: each of them can be written as \( h(x) - h(y) \), where \( h \) is a real function of one variable.

Conversely, given a nonempty subset \( I \) of \( \mathbb{R} \) and a map \( h : I \rightarrow \mathbb{R} \), it is clair that the map \( f : I^2 \rightarrow \mathbb{R} \), defined by:

\[
f(x, y) = h(x) - h(y) \quad (\forall x, y \in I)
\]

is an asymmetric map on \( I^2 \). Consequently, \( \varphi^{-1}(f) \) will give a mean on \( I^2 \).

Let us explicit the expression of that mean. For all \((x, y) \in I^2\), we have:

\[
\varphi^{-1}(f)(x, y) = \frac{x + ye^{f(x,y)}}{e^{f(x,y)} + 1} \quad (\text{according to (1)})
= \frac{x + ye^{h(x)-h(y)}}{e^{h(x)-h(y)} + 1}
= \frac{xe^{-h(x)} + ye^{-h(y)}}{e^{-h(x)} + e^{-h(y)}}
= \frac{XP(x) + yP(y)}{P(x) + P(y)},
\]

with \( P(t) = e^{-h(t)} \) (\( \forall t \in I \)). We remark that the only particularity of \( P \) is that it is a positive function on \( I \). This leads us to include the most known three means (arithmetic, geometric and harmonic means) in an important class of mean functions which we define in what follows:
Definition 2.2  Let $I$ be a nonempty subset of $\mathbb{R}$. We call *normal* mean function on $I^2$ any function $M : I^2 \to \mathbb{R}$, which can be written as:

$$M(x, y) = \frac{xP(x) + yP(y)}{P(x) + P(y)} \quad (\forall x, y \in I),$$

with $P : I \to \mathbb{R}$ is a positive function on $I$.

We call $P$ the *weight* function associated to $M$.

**Remark.** The weight function associated to a normal mean function is defined up to a multiplicative positive constant. The weight functions associated to the three means $A$, $G$ and $H$ are respectively $P_A(x) = 1 \ (\forall x \in \mathbb{R})$, $P_G(x) = 1/\sqrt{x} \ (\forall x > 0)$ and $P_H(x) = 1/x \ (\forall x > 0)$.

Before continuing our study on the group structure defined above on $\mathcal{M}_G$, we would like to stress a quite interesting property concerning the comparison (in the sense of the usual order on $\mathbb{R}$) between two normal mean functions.

To compare between two normal mean functions, there is a simple and practical criterium which uses their associated weight functions. We have the following

**Proposition 2.3**  Let $I$ be a nonempty interval of $\mathbb{R}$ and $M_1$ and $M_2$ be two normal mean functions on $I^2$ with weight functions $P_1$ and $P_2$ respectively. Then, the two following properties are equivalent:

1) $\forall (x, y) \in I^2: M_1(x, y) \leq M_2(x, y)$.

2) The function $\frac{P_1}{P_2}$ is non-increasing on $I$.

The same holds for the following two properties:

3) $\forall (x, y) \in I^2, x \neq y: M_1(x, y) < M_2(x, y)$.

4) The function $\frac{P_1}{P_2}$ is decreasing on $I$.

**Proof.** We only prove the equivalence of the two properties 1) and 2). The prove of the equivalence of the two properties 3) and 4) is similar. For all $(x, y) \in I^2$, we have:

$$M_1(x, y) \leq M_2(x, y) \iff \frac{xP_1(x) + yP_1(y)}{P_1(x) + P_1(y)} \leq \frac{xP_2(x) + yP_2(y)}{P_2(x) + P_2(y)} \iff (xP_1(x) + yP_1(y))(P_2(x) + P_2(y)) \leq (xP_2(x) + yP_2(y))(P_1(x) + P_1(y)) \iff (x - y)(P_1(x)P_2(y) - P_1(y)P_2(x)) \leq 0 \iff (x - y) \left( \frac{P_1}{P_2}(x) - \frac{P_1}{P_2}(y) \right) \leq 0.$$
So, the property 1) of the proposition is equivalent to the property:
\[ \forall (x, y) \in I^2 : (x - y) \left( \frac{P_1}{P_2}(x) - \frac{P_1}{P_2}(y) \right) \leq 0, \]
which amounts to say that the function \( P_1/P_2 \) is non-increasing on \( I \). ■

From Proposition 2.3, we derive the following immediate corollary:

**Corollary 2.4** Let \( I \) be a nonempty interval of \( \mathbb{R} \) and \( M \) be a normal mean function on \( I^2 \), with weight function \( P \). Then

1) \( M \) is sub-arithmetic (that is \( M \) verifies: \( \forall (x, y) \in I^2 : M(x, y) \leq A(x, y) \)) if and only if \( P \) is non-increasing on \( I \).

2) \( M \) is strictly sub-arithmetic (that is \( M \) verifies: \( \forall (x, y) \in I^2, x \neq y : M(x, y) < A(x, y) \)) if and only if \( P \) is decreasing on \( I \).

3) \( M \) is super-arithmetic (that is \( M \) verifies: \( \forall (x, y) \in I^2 : M(x, y) \geq A(x, y) \)) if and only if \( P \) is non-decreasing on \( I \).

4) \( M \) is strictly super-arithmetic (that is \( M \) verifies: \( \forall (x, y) \in I^2, x \neq y : M(x, y) > A(x, y) \)) if and only if \( P \) is increasing on \( I \). ■

**An application of Proposition 2.3**. By remembering that the respective weight functions associated to the three normal means \( A \), \( G \) and \( H \) are \( 1 \), \( 1/\sqrt{x} \) and \( 1/x \), Proposition 2.3 immediately shows that for all \( x, y > 0, x \neq y \), we have:
\[ A(x, y) > G(x, y) > H(x, y). \]

**Study of some symmetries on the group \((\mathcal{M}_\mathcal{D}, \ast)\)**

Given a nonempty symmetric domain \( \mathcal{D} \) of \( \mathbb{R}^2 \), we are interested in what follows in the symmetric mean of a given mean \( M_1 \) with respect to another given mean \( M_0 \) via the group structure \((\mathcal{M}_\mathcal{D}, \ast)\). Denoting by \( S_{M_0} \) the symmetry with respect to a fixed mean \( M_0 \) in the group \((\mathcal{M}_\mathcal{D}, \ast)\), we have by definition:
\[ \forall M_1, M_2 \in \mathcal{M}_\mathcal{D} : S_{M_0}(M_1) = M_2 \iff M_1 \ast M_2 = M_0 \ast M_0. \]

The explicit expression of \( S_{M_0}(M_1) \) \((M_0, M_1 \in \mathcal{M}_\mathcal{D})\) is given by the following:
Proposition 2.5 Let $\mathcal{D}$ be a nonempty symmetric domain of $\mathbb{R}^2$ and $M_0$ and $M_1$ be two mean functions on $\mathcal{D}$. Then we have:

$$S_{M_0}(M_1) = \frac{x(M_1 - x)(M_0 - y)^2 - y(M_0 - x)^2(M_1 - y)}{(M_1 - x)(M_0 - y)^2 - (M_0 - x)^2(M_1 - y)},$$

where, for simplicity, we have noted $M_0$ for $M_0(x, y)$, $M_1$ for $M_1(x, y)$ and $S_{M_0}(M_1)$ for $S_{M_0}(M_1)(x, y)$.

Proof. Of course, we use the group isomorphism $\varphi$ introduced at the beginning of Section 2. Let $f_0 = \varphi(M_0)$, $f_1 = \varphi(M_1)$, $M_2 = S_{M_0}(M_1)$ and $f_2 = \varphi(M_2)$. The equality $M_2 = S_{M_0}(M_1)$ amounts to $M_1 * M_2 = M_0 * M_0$.

By applying $\varphi$ to the two sides of the latter, we get $f_1 + f_2 = 2f_0$, which gives $f_2 = 2f_0 - f_1$. It follows that for all $(x, y) \in \mathcal{D}$, we have:

$$M_2(x, y) = \varphi^{-1}(f_2)(x, y) = \frac{x + ye^{f_2(x,y)}}{e^{f_2(x,y)} + 1} = \frac{x + ye^{2f_0(x,y)-f_1(x,y)}}{e^{2f_0(x,y)-f_1(x,y)} + 1} = \frac{x e^{f_1(x,y)} + ye^{2f_0(x,y)}}{e^{2f_0(x,y)} + e^{f_1(x,y)}} \quad (4)$$

Further, since $f_0 = \varphi(M_0)$ and $f_1 = \varphi(M_1)$, we have:

$$e^{f_0(x,y)} = -\frac{M_0(x,y) - x}{M_0(x,y) - y} \quad \text{et} \quad e^{f_1(x,y)} = -\frac{M_1(x,y) - x}{M_1(x,y) - y}.$$

By inserting that two last equalities into (4) and after simplification and rearrangement, the identity of the proposition follows.

The following corollary gives us the expression of the symmetric mean of a given mean with respect to one of the three means $A$, $G$ and $H$. Remark that the obtained expressions for the symmetric means with respect to the arithmetic and geometric means supports the intuition that we can have about them. This curious fact shows the interest of the group structure defined on $\mathcal{M}_\mathcal{D}$.

Corollary 2.6 Let $\mathcal{D}$ be a nonempty symmetric domain of $\mathbb{R}^2$ and $M$ a mean function on $\mathcal{D}$. Then we have:

1) $S_A(M) = x + y - M = 2A - M$.

2) $S_G(M) = \frac{xy}{M} = \frac{G^2}{M} \quad (\text{by supposing } \mathcal{D} \subset (0, +\infty)^2)$.

3) $S_H(M) = \frac{xy}{(x+y)(M-xy)} = \frac{HM}{2M-H} \quad (\text{by supposing } \mathcal{D} \subset (0, +\infty)^2)$.

4) $S_H = S_G \circ S_A \circ S_G$. 

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Proof. To obtain the formulas of the items 1), 2) and 3), it suffices to apply the formula of Proposition 2.5 respectively for $M_0 = A$, $M_0 = G$ and $M_0 = H$. The formula of the item 4) is derived from those of the previous one.

Now, we are going to define another symmetry on the set $\mathcal{M}_\mathcal{D}$ (having a certain form) which is completely independent of the group structure $(\mathcal{M}_\mathcal{D}, *)$. This new symmetry is defined by solving a functional equation but it curiously coincides, in many cases, with the symmetry defined above which is rather related to the law of the group $(\mathcal{M}_\mathcal{D}, *)$.

Definition 2.7 Let $I$ be a nonempty interval of $\mathbb{R}$, $\mathcal{D} = I^2$ and $M_0$, $M_1$ and $M_2$ be three mean functions on $\mathcal{D}$ such that $M_1$ and $M_2$ take their values in $I$. We say that $M_2$ is the functional symmetric mean of $M_1$ with respect to $M_0$ if the following functional equation is satisfied:

$$M_0(M_1(x, y), M_2(x, y)) = M_0(x, y) \quad (\forall (x, y) \in \mathcal{D}).$$

Equivalently, we also say that $M_0$ is the functional middle of $M_1$ and $M_2$.

According to the axiom iii) verified by the mean functions, it is immediate that if the functional symmetric of a mean with respect to another mean exists then it is unique. This justifies the following notation:

Notation. Given $M_0$ and $M_1$ two mean functions on $\mathcal{D} = I^2$ with values in $I$ (where $I$ is an interval of $\mathbb{R}$), we denote by $\sigma_{M_0}(M_1)$ the functional symmetric (if it exists) of $M_1$ with respect to $M_0$.

The following proposition gives us a sufficient condition for the existence of the functional symmetric mean.

Proposition 2.8 Let $I$ be a nonempty interval of $\mathbb{R}$, $\mathcal{D} = I^2$ and $M_0$ be a mean function on $\mathcal{D}$. For all $x \in I$, set:

$$I_x := \{M_0(x, t) \mid t \in I\}$$

and suppose that we have:

$$\forall x, y \in I : \quad x \leq y \implies I_x \subset I_y. \quad (5)$$

Suppose also that $M_0$ is increasing, that is $M_0$ is increasing with respect to one (so to each) of its two variables. Then, any mean function on $\mathcal{D}$ has a functional symmetric with respect to $M_0$. 10
Proof. Let us fix a mean function \( M_1 \) on \( \mathcal{D} \). For all \((x, y) \in \mathcal{D}\), we are going to show the existence of a unique real number, which we denote by \( M_2(x, y) \), such that
\[
M_0(M_1(x, y), M_2(x, y)) = M_0(x, y).
\]
then, we will show that the obtained function \( M_2 \) (on \( \mathcal{D} \)) is actually a mean on \( \mathcal{D} \).

Let \( x, y \in I \), fixed. consider the map:
\[
i : I \rightarrow \mathbb{R} \\
\quad t \mapsto M_0(M_1(x, y), t).
\]
According to the hypothesis that \( M_0 \) is increasing, the map \( i \) increases on \( I \).

It follows that \( i \) is a bijection from \( I \) into \( i(I) = I_{M_1(x,y)} \).
Now, since \( M_0(x, y) \in I_{\min(x,y)} \) (because \( M_0(x, y) \in I_x \) and \( M_0(x, y) \in I_y \)) and \( I_{\min(x,y)} \subset I_{M_1(x,y)} \) (according to the hypothesis \( (5) \)) then \( M_0(x, y) \in I_{M_1(x,y)} \).

The bijectivity of \( i \) from \( I \) into \( I_{M_1(x,y)} \) thus implies that there exists a unique \( t_* \in I \) satisfying \( i(t_*) = M_0(x, y) \), so satisfying
\[
M_0(M_1(x, y), t_*) = M_0(x, y).
\]
It suffices to set \( M_2(x, y) = t_* \) to establish \( (6) \) for the couple \((x, y)\).

So the existence and the uniqueness of \( M_2 \) as a function on \( \mathcal{D} \) satisfying the functional equation \( (6) \) are confirmed and it just remains to show that \( M_2 \) is a mean on \( \mathcal{D} \).

\[\bullet\] Let us show that \( M_2 \) is symmetric on \( \mathcal{D} \):
Given \((x, y) \in \mathcal{D} \), \( M_2(y, x) \) is (by definition) the unique real number of \( I \) satisfying
\[
M_0(M_1(y, x), M_2(y, x)) = M_0(y, x).
\]
But because \( M_0 \) and \( M_1 \) are symmetric on \( \mathcal{D} \) (as mean functions on \( \mathcal{D} \)), the last relation amounts to
\[
M_0(M_1(x, y), M_2(y, x)) = M_0(x, y),
\]
which implies (according to the definition of \( M_2(x, y) \)) that:
\[
M_2(x, y) = M_2(y, x).
\]
So the function \( M_2 \) is symmetric on \( \mathcal{D} \) as we claimed it to be.

\[\bullet\] Let us show that \( M_2 \) satisfies:
\[
\forall (x, y) \in \mathcal{D} : \quad \min(x, y) \leq M_2(x, y) \leq \max(x, y).
\]
We argue by contradiction. Assume that there exists a couple \((x, y) \in D\) for which \((7)\) is not valid. So, we have:

Either \(M_2(x, y) < \min(x, y)\):

In this case, since \(M_1(x, y) \leq \max(x, y)\) (because \(M_1\) is a mean on \(D\)), then according to the hypothesis that \(M_0\) is increasing, we have:

\[ M_0(M_1(x, y), M_2(x, y)) < M_0(\max(x, y), \min(x, y)) = M_0(x, y), \]

which contradicts the relation \((6)\) satisfied by \(M_2\). This first case is thus impossible.

Or \(M_2(x, y) > \max(x, y)\):

In this case, since \(M_1(x, y) \geq \min(x, y)\) (because \(M_1\) is a mean on \(D\)), then according to the hypothesis that \(M_0\) is increasing, we have:

\[ M_0(M_1(x, y), M_2(x, y)) > M_0(\min(x, y), \max(x, y)) = M_0(x, y), \]

which again contradicts the relation \((6)\) satisfied by \(M_2\). This second case is thus also impossible.

The function \(M_2\) thus satisfies the relation \((6)\) satisfied by \(M_2\). This second case is thus also impossible.

The function \(M_2\) thus satisfies the relation \((7)\).

• Let us finally show that \(M_2\) satisfies the third axiom of mean functions, that is:

\[ \forall (x, y) \in D : \quad M_2(x, y) = x \implies x = y. \]

For all \((x, y) \in D\), we have:

\[
M_2(x, y) = x \implies M_0(M_1(x, y), M_2(x, y)) = M_0(M_1(x, y), x) \\
\implies M_0(x, y) = M_0(x, M_1(x, y)) \\
\implies M_1(x, y) = y \\
\implies x = y,
\]

where in the second implication, we have used the relation \((6)\) together with the symmetry of \(M_0\); in the third implication, we have used the increasing of \(M_0\) and in the forth implication, we have used the third axiom of mean functions for the mean \(M_1\).

The third axiom of mean functions is thus satisfied by \(M_2\).

In conclusion, \(M_2\) is a mean function on \(D\). This completes the proof of the proposition. \(\blacksquare\)

We can see easily that the three means Arithmetic (on \(D = \mathbb{R}^2\)), Geometric and Harmonic (on \(D = (0, +\infty)^2\)) satisfy the hypothesis of Proposition 2.8. So, we can establish for any given mean (on a suitable domain \(D\)) its symmetric mean (in the functional sense) with respect to one of the three means A, G and H. Precisely, We have the following
Proposition 2.9 Let $M$ be a mean function on a suitable symmetric domain $\mathcal{D}$ of $\mathbb{R}^2$. Then the functional symmetric means of $M$ with respect to the three means Arithmetic, Geometric and Harmonic are respectively given by:

$$
\sigma_A(M) = x + y - M \\
\sigma_G(M) = \frac{xy}{M} \quad \text{(for } \mathcal{D} \subset (0, +\infty)^2) \\
\sigma_H(M) = \frac{xyM}{(x+y)M - xy} \quad \text{(for } \mathcal{D} \subset (0, +\infty)^2).
$$

So, those functional symmetric means coincide with the symmetric means in the sense of the group law defined above on $\mathcal{M}_\mathcal{D}$.

Proof. Given $M$ a mean function on $\mathbb{R}^2$, its functional symmetric mean with respect to the arithmetic mean $A$ is defined by the functional equation:

$$
A(M, \sigma_A(M)) = A,
$$

which amounts to:

$$
\frac{M + \sigma_A(M)}{2} = \frac{x + y}{2}.
$$

Hence:

$$
\sigma_A(M) = x + y - M,
$$
as claimed.

Similarly, given $M$ a mean function on $(0, +\infty)^2$, its functional symmetric means with respect to the two means $G$ and $H$ are respectively defined by:

$$
G(M, \sigma_G(M)) = G \quad \text{and} \quad H(M, \sigma_H(M)) = H,
$$

that is:

$$
\sqrt{M \sigma_G(M)} = \sqrt{xy} \quad \text{and} \quad \frac{2M \sigma_H(M)}{M + \sigma_H(M)} = \frac{2xy}{x + y}.
$$

This gives:

$$
\sigma_G(M) = \frac{xy}{M} \quad \text{and} \quad \sigma_H(M) = \frac{xyM}{(x+y)M - xy},
$$
as claimed. The proposition is proved.

The remarkable phenomenon of the coincidence of the two symmetries defined on $\mathcal{M}_\mathcal{D}$ in the three particular cases corresponding to the most known means $A$, $G$ and $H$ leads us to formulate the following important open question:
Open question. What are all the mean functions $M$ on $D = (0, +\infty)^2$ for which the two symmetries with respect to $M$ (in the sense of the group law introduced on $\mathcal{M}_\mathcal{D}$ and in the functional sense) coincide?

Mathematically speaking, we ask about the description of the set

$$\left\{ M \in \mathcal{M}_{(0, +\infty)^2} : S_M = \sigma_M \right\}$$

which contains (as proved above) at least the three means A, G and H.

We end this section by giving an important example of functional symmetry. We have the following

**Proposition 2.10** The two means A and G are symmetric in the functional sense with respect to the AGM mean.

**Proof.** Given $x, y > 0$, each of the two real numbers $\text{AGM}(x, y)$ and $\text{AGM}(\frac{x+y}{2}, \sqrt{xy})$ is defined as the common limit of two adjacent sequences. But, it is easy to see that the two adjacent sequences defining $\text{AGM}(\frac{x+y}{2}, \sqrt{xy})$ are the shifted by one term of the two adjacent sequences defining $\text{AGM}(x, y)$. Consequently, we have:

$$\text{AGM}\left(\frac{x+y}{2}, \sqrt{xy}\right) = \text{AGM}(x, y),$$

that is:

$$\text{AGM}(A(x, y), G(x, y)) = \text{AGM}(x, y).$$

This shows that the means A and G are symmetric, in the functional sense, with respect to the AGM mean. The proposition is proved.

\[\blacksquare\]

3 A metric space structure on $\mathcal{M}_\mathcal{D}$

Throughout this section, we fix a nonempty symmetric domain $\mathcal{D}$ of $\mathbb{R}^2$. In the case where all the points of $\mathcal{D}$ have the form $(x, x)$ ($x \in \mathbb{R}$), the set $\mathcal{M}_\mathcal{D}$ is reduced to a unique element and consequently there is a unique topology on $\mathcal{M}_\mathcal{D}$ which is trivial. So suppose that $\mathcal{D}$ contains at least a point $(x_0, y_0)$ of $\mathbb{R}^2$ such that $x_0 \neq y_0$. For all couple $(M_1, M_2)$ of mean functions on $\mathcal{D}$, define:

$$d(M_1, M_2) := \sup_{(x, y) \in \mathcal{D}, x \neq y} \left| \frac{M_1(x, y) - M_2(x, y)}{x - y} \right|.$$

We have the following:
Proposition 3.1 \textit{The map $d$ of $\mathcal{M}_D^2$ into $[0, +\infty]$ is a distance on $\mathcal{M}_D$. In addition, the metric space $(\mathcal{M}_D, d)$ is identic to the closed ball with center $A$ (the arithmetic mean) and radius $\frac{1}{2}$.}

\textbf{Proof.} First let us show that for all couple $(M_1, M_2)$ of $\mathcal{M}_D^2$, the nonnegative quantity $d(M_1, M_2)$ is finite. Given $M_1, M_2 \in \mathcal{M}_D$, for all $(x, y) \in D$, $x \neq y$, the two real numbers $M_1(x, y)$ and $M_2(x, y)$ lie in the same interval $[\min(x, y), \max(x, y)]$, so we have:

\[ |M_1(x, y) - M_2(x, y)| \leq \max(x, y) - \min(x, y) = |x - y|. \]

Hence

\[ \sup_{(x, y) \in D, x \neq y} \left| \frac{M_1(x, y) - M_2(x, y)}{x - y} \right| \leq 1, \]

that is $d(M_1, M_2) \leq 1$. This shows that $d$ is actually a map from $\mathcal{M}_D^2$ into $[0, 1]$. Further, since the three properties

- $\forall M_1, M_2 \in \mathcal{M}_D$: $d(M_1, M_2) = d(M_2, M_1)$
- $\forall M_1, M_2 \in D$: $d(M_1, M_2) = 0 \iff M_1 = M_2$
- $\forall M_1, M_2, M_3 \in \mathcal{M}_D$: $d(M_1, M_3) \leq d(M_1, M_2) + d(M_2, M_3)$

are trivially satisfied by $d$, then $d$ is a distance on $\mathcal{M}_D$.

Now, given $M \in \mathcal{M}_D$, let us show that $d(M, A) \leq \frac{1}{2}$. For all couple $(x, y) \in D$, $x \neq y$, the real number $M(x, y)$ lies in the closed interval limited by $x$ and $y$, so we have:

\[ |M(x, y) - A(x, y)| \leq \max(x - A(x, y), y - A(x, y)) \]
\[ = \max \left( x - \frac{x + y}{2}, y - \frac{x + y}{2} \right) \]
\[ = \max \left( \frac{x - y}{2}, \frac{y - x}{2} \right) \]
\[ = \frac{1}{2} |x - y|. \]

It follows that:

\[ \sup_{(x, y) \in D, x \neq y} \left| \frac{M(x, y) - A(x, y)}{x - y} \right| \leq \frac{1}{2}, \]

that is $d(M, A) \leq \frac{1}{2}$, as required.

The metric space $(\mathcal{M}_D, d)$ is thus identic to the closed ball with center $A$ and radius $\frac{1}{2}$. This completes the proof of the proposition. \hfill \blacksquare
Remark 3.2 Given $M_1, M_2 \in \mathcal{M}_\mathcal{D}$, since the map $(x, y) \mapsto \frac{M_1(x, y) - M_2(x, y)}{x - y}$ is obviously asymmetric (on the set $\{(x, y) \in \mathcal{D} : x \neq y\}$), we have also

$$d(M_1, M_2) = \sup_{(x, y) \in \mathcal{D}, x \neq y} \frac{M_1(x, y) - M_2(x, y)}{x - y}.$$ 

By using the group isomorphism $\varphi$ from $\mathcal{M}_\mathcal{D}$ into $\mathcal{A}_D$, defined at Section 2, we will establish in what follows a practice formula to calculate the distance between two mean functions on $\mathcal{D}$.

Proposition 3.3 Let $M_1$ and $M_2$ two mean functions on $\mathcal{D}$. Set $f_1 = \varphi(M_1)$ and $f_2 = \varphi(M_2)$. Then we have:

$$d(M_1, M_2) = \sup_{(x, y) \in \mathcal{D}} \frac{e^{f_1} - e^{f_2}}{(e^{f_1} + 1)(e^{f_2} + 1)} = \sup_{(x, y) \in \mathcal{D}} \left(\frac{1}{e^{f_1} + 1} - \frac{1}{e^{f_2} + 1}\right).$$

Proof. According to the relation (1) of Section 2, we have for all $(x, y) \in \mathcal{D}$: $M_1(x, y) = \varphi^{-1}(f_1)(x, y) = \frac{x + ye^{f_1(x, y)}}{e^{f_1(x, y)} + 1}$ and $M_2(x, y) = \varphi^{-1}(f_2)(x, y) = \frac{x + ye^{f_2(x, y)}}{e^{f_2(x, y)} + 1}$. It follows that for all $(x, y) \in \mathcal{D}$, we have:

$$M_1(x, y) - M_2(x, y) = \frac{x + ye^{f_1(x, y)}}{e^{f_1(x, y)} + 1} - \frac{x + ye^{f_2(x, y)}}{e^{f_2(x, y)} + 1} = (x - y)\left(\frac{e^{f_1(x, y)} - e^{f_2(x, y)}}{(e^{f_1(x, y)} + 1)(e^{f_2(x, y)} + 1)}\right).$$

Then, according to Remark 3.2 and the fact that the functions $f_1$ and $f_2$ are zero at the points $(x, x)$ of $\mathcal{D}$ (because they are asymmetric on $\mathcal{D}$), we have:

$$d(M_1, M_2) = \sup_{(x, y) \in \mathcal{D}, x \neq y} \frac{M_1(x, y) - M_2(x, y)}{x - y} = \sup_{(x, y) \in \mathcal{D}} \frac{e^{f_1} - e^{f_2}}{(e^{f_1} + 1)(e^{f_2} + 1)}.$$ 

The proposition is proved. \hfill \blacksquare

Now, using Proposition 3.3 we will establish in what follows a practice criterium which permits to locate easily any mean function on $\mathcal{D}$ in the metric space $\mathcal{M}_\mathcal{D}$, seen as the closed ball with center $A$ and radius $\frac{1}{2}$.

Corollary 3.4 Let $M$ be a mean function on $\mathcal{D}$ and $f := \varphi(M)$, where $\varphi$ is the group isomorphism defined at Section 2. Then, setting $s := \sup_{\mathcal{D}} f \in [0, +\infty]$, we have:

$$d(M, A) = \frac{1}{2} \cdot \frac{e^s - 1}{e^s + 1}.$$ 

(We naturally suppose that $\frac{e^s + 1}{e^s - 1} = 1$ when $s = +\infty$).

In particular, the mean $M$ lies in the border of $\mathcal{M}_\mathcal{D}$ (that is on the circle with center $A$ and radius $\frac{1}{2}$) if and only if $\sup_{\mathcal{D}} f = +\infty$. 

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Proof. Since the asymmetric function associated to the arithmetic mean by the isomorphism $\varphi$ is the zero function (i.e., $\varphi(A) \equiv 0$), then according to Proposition 3.3 we have:

$$d(M, A) = \sup_{\varphi} \frac{e^f - 1}{2(e^f + 1)} = \frac{1}{2} \sup_{\varphi} \frac{e^f - 1}{e^f + 1}. $$

Next, since the function $x \mapsto \frac{e^x - 1}{e^x + 1}$ is increasing on $\mathbb{R}$, then we have $\sup_{\varphi} \frac{e^f - 1}{e^f + 1} = e^s - 1$. The corollary follows. 

Applications.

1) The two means Geometric and Harmonic (on $\mathcal{D} = (0, +\infty)^2$) lie on the border of the metric space $\mathcal{M}_{\mathcal{G}}$. Indeed, the asymmetric functions associated to $G$ and $H$ by the isomorphism $\varphi$ are respectively:

$$\varphi(G)(x, y) = \frac{1}{2}(\log x - \log y) \quad \text{and} \quad \varphi(H)(x, y) = \log x - \log y \quad \text{(see §2)}. $$

Since we have clearly $\sup_{\varphi} \varphi(G) = \sup_{\varphi} \varphi(H) = +\infty$, the second statement of Corollary 3.4 insures that $G$ and $H$ lie on the border of $\mathcal{M}_{\mathcal{G}}$ as we claimed it to be.

2) The distance between the two means Geometric and Harmonic (on $\mathcal{D} = (0, +\infty)^2$) is somewhat more difficult to calculate. Using Proposition 3.3 we can show that:

$$d(G, H) = \sup_{t > 0} \frac{t^2 - t}{(t + 1)(t^2 + 1)} \simeq 0.15. $$

Actually, we can show that this distance is an algebraic number with degree 4; it is a root of the equation $x^4 + 10x^3 + 3x^2 - 14x + 2 = 0$.

4 Construction of a functional middle of two mean functions (generalization of the AGM mean)

Let $I$ be a nonempty subset of $\mathbb{R}$ and let $\mathcal{D} = \mathbb{R}^2$. The aim of this section is to prove, under some regular conditions, the existence and the uniqueness of the functional middle of two given means $M_1$ and $M_2$ on $\mathcal{D}$; that is the existence and the uniqueness of a new mean $M$ on $\mathcal{D}$ satisfying the functional equation:

$$M(M_1, M_2) = M.$$
In this context, we obtain two results which only differ in the imposed condition on the two means $M_1$ and $M_2$. The first one imposes to $M_1$ and $M_2$ the condition $d(M_1, M_2) \neq 1$ (where $d$ is the distance on $M_D$ defined at Section 3) while the second one simply imposes to $M_1$ and $M_2$ to be continuous on $\mathcal{D}$ (by taking $I$ an interval of $\mathbb{R}$). Notice further that our way of establishing the existence of the functional middle is constructive and generalizes the idea of the AGM mean. Our first result is the following:

**Theorem 4.1** Let $M_1$ and $M_2$ be two mean functions on $\mathcal{D} = I^2$, with values in $I$ and such that $d(M_1, M_2) < 1$. Then there exists a unique mean function $M$ on $\mathcal{D}$ satisfying the functional equation:

$$M(M_1, M_2) = M.$$ 

Besides, for all $(x, y) \in \mathcal{D}$, $M(x, y)$ is the common limit of the two real sequences $(x_n)_n$ and $(y_n)_n$ defined as follows:

$$\begin{cases} x_0 = x, & y_0 = y \\ x_{n+1} = M_1(x_n, y_n) & (\forall n \in \mathbb{N}) \\ y_{n+1} = M_2(x_n, y_n) & (\forall n \in \mathbb{N}) \end{cases}$$

**Proof.** Let $k := d(M_1, M_2)$. By hypothesis, we have $k < 1$. Let $(x_n)_n$ and $(y_n)_n$ be the two real sequences introduced at the second statement of the theorem and let $(u_n)_n$ and $(v_n)_n$ be the two real sequences (with values in $I$) defined by:

$$u_n := \min(x_n, y_n) \quad \text{and} \quad v_n := \max(x_n, y_n) \quad (\forall n \in \mathbb{N}).$$

Let us show that $(u_n)_n$ and $(v_n)_n$ are adjacent sequences. For all $n \in \mathbb{N}$, we have:

$$u_{n+1} = \min(x_{n+1}, y_{n+1}) = \min(M_1(x_n, y_n), M_2(x_n, y_n)) \geq \min(x_n, y_n) = u_n$$

(because $M_1(x_n, y_n) \geq \min(x_n, y_n)$ and $M_2(x_n, y_n) \geq \min(x_n, y_n)$, since $M_1$ and $M_2$ are mean functions). This shows that $(u_n)_n$ is a non-decreasing sequence.

Similarly, for all $n \in \mathbb{N}$, we have:

$$v_{n+1} = \max(x_{n+1}, y_{n+1}) = \max(M_1(x_n, y_n), M_2(x_n, y_n)) \leq \max(x_n, y_n) = v_n$$

(because $M_1(x_n, y_n) \leq \max(x_n, y_n)$ and $M_2(x_n, y_n) \leq \max(x_n, y_n)$, since $M_1$ et $M_2$ are mean functions). The sequence $(v_n)_n$ is then non-increasing.
Next, for all \( n \in \mathbb{N} \), we have:

\[
|v_{n+1} - u_{n+1}| = |\max(x_{n+1}, y_{n+1}) - \min(x_{n+1}, y_{n+1})| \\
= |x_{n+1} - y_{n+1}| \\
= |M_1(x_n, y_n) - M_2(x_n, y_n)| \\
\leq k|x_n - y_n| \quad \text{(by definition of } k) \\
= k|v_n - u_n|.
\]

By induction on \( n \), we get:

\[
|v_n - u_n| \leq k^n|v_0 - u_0| \quad (\forall n \in \mathbb{N}.
\]

It follows (since \( k \in [0, 1) \)) that \((v_n - u_n)\) tends to 0 as \( n \) tends to infinity.

The two sequences \((u_n)_n\) and \((v_n)_n\) are thus adjacent, as claimed. Consequently, \((u_n)_n\) and \((v_n)_n\) are convergent and have the same limit. In addition, since we have clearly:

\[
u_n \leq x_n \leq v_n \quad \text{and} \quad u_n \leq y_n \leq v_n \quad (\forall n \in \mathbb{N}),
\]

then \((x_n)_n\) and \((y_n)_n\) are also convergent and have the same limit which coincides with the common limit of the two sequences \((u_n)_n\) and \((v_n)_n\). In the sequel, let \( M(x, y) \) denote this limit.

Now let us show that the map \( M : \mathcal{D} \rightarrow \mathbb{R} \), just defined, is a mean function on \( \mathcal{D} \) and satisfies \( M(M_1, M_2) = M \). First let us show that \( M \) satisfies the three axioms of a mean function.

i) Given \((x, y) \in \mathcal{D}\), by changing \((x, y)\) by \((y, x)\) in the definition of the sequences \((x_n)_n\) and \((y_n)_n\), those lasts still unchanged except their first terms (since \( M_1 \) and \( M_2 \) are symmetric). So, we have:

\[
M(x, y) = M(y, x) \quad (\forall (x, y) \in \mathcal{D}).
\]

In other words, \( M \) is symmetric on \( \mathcal{D} \).

ii) Given \((x, y) \in \mathcal{D}\), since the corresponding sequences \((u_n)_n\) and \((v_n)_n\) are respectively non-decreasing and non-increasing and since \( M(x, y) \) is the common limit of \((u_n)_n\) and \((v_n)_n\) then we have:

\[
u_0 \leq M(x, y) \leq v_0,
\]

that is:

\[
\min(x, y) \leq M(x, y) \leq \max(x, y)
\]

iii) Let \((x, y) \in \mathcal{D}\), fixed. Suppose that \( M(x, y) = x \) and show that \( x = y \). Let us argue by contradiction; so assume that \( x \neq y \). Since \( M_1 \) and \( M_2 \)
are mean functions on $D$, we have (according to the third axiom of mean functions):

$$M_1(x, y) \neq x \quad \text{and} \quad M_2(x, y) \neq x \quad (8)$$

We distinguish the following two cases:

1st case: (if $x < y$)
In this case, we have $M(x, y) = x = \min(x, y) = u_0$. So the sequence $(u_n)_n$ is non-decreasing and converges to $u_0$. It follows that $(u_n)_n$ is necessarily constant and we have in particular $u_1 = u_0$, that is:

$$\min(M_1(x, y), M_2(x, y)) = x,$$

which contradicts (8).

2nd case: (if $x > y$)
In this case, we have $M(x, y) = x = \max(x, y) = v_0$. So the sequence $(v_n)_n$ is non-increasing and converges to $v_0$. It follows that $(v_n)_n$ is necessarily constant and we have in particular $v_1 = v_0$, that is:

$$\max(M_1(x, y), M_2(x, y)) = x,$$

which again contradicts (8).

Thus, we have $x = y$, as required.

From i), ii) and iii), we conclude that $M$ is effectively a mean function on $D$.

Next, let us show that $M$ satisfies the functional equation $M(M_1, M_2) = M$. To do so, we constat that the fact to change in the definition of the sequences $(x_n)_n$ and $(y_n)_n$ a couple $(x, y)$ of $D$ by the couple $(M_1(x, y), M_2(x, y))$ just amounts to shift by one term these sequences (namely we obtain $(x_{n+1})_n$ instead of $(x_n)_n$ and $(y_{n+1})_n$ instead of $(y_n)_n$). Consequently, this changing conserves the common limit of the two sequences $(x_n)_n$ and $(y_n)_n$ (which is $M(x, y)$); that is:

$$M(M_1(x, y), M_2(x, y)) = M(x, y).$$

Since this last equation holds for all $(x, y) \in D$, we have $M(M_1, M_2) = M$, as required.

It finally remains to show that $M$ is the unique mean on $D$ satisfying the functional equation $M(M_1, M_2) = M$. Let $M'$ be a mean function on $D$, satisfying $M'(M_1, M_2) = M'$ and let us show that $M'$ coincides with $M$. So, let $(x, y) \in D$, fixed and let us show that $M'(x, y) = M(x, y)$. We associate to $(x, y)$ the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ of couples of $D$, defined like in the second statement of the theorem. Using the relation $M'(M_1, M_2) = M'$, we have:

$$M'(x, y) = M'(x_1, y_1) = M'(x_2, y_2) = \cdots = M'(x_n, y_n) = \cdots$$

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But since $M'$ is a mean on $\mathcal{D}$, it follows that for all $n \in \mathbb{N}$, we have:

$$\min(x_n, y_n) \leq M'(x, y) \leq \max(x_n, y_n).$$

Finally, since (according to the first part of this proof) $x_n$ and $y_n$ tend to $M(x, y)$ as $n$ tends to infinity, then by tending $n$ to infinity in the three hand-sides of the previous double inequality, we get:

$$M'(x, y) = M(x, y),$$

as required. This confirms the uniqueness of $M$ as a mean function on $\mathcal{D}$ satisfying $M(M_1, M_2) = M$. The proof is complete. ■

From Theorem 4.1 we derive the following important corollary:

**Corollary 4.2** Let $\mathcal{M}$ be a mean function on $\mathcal{D} = I^2$, with values in $I$. Then, there exists a unique mean function on $\mathcal{D}$ satisfying the functional equation:

$$M\left(\frac{x+y}{2}, \mathcal{M}(x, y)\right) = M(x, y) \quad (\forall (x, y) \in \mathcal{D}).$$

In addition, for all $(x, y) \in \mathcal{D}$, $M(x, y)$ is the common limit of the two real sequences $(x_n)_n$ and $(y_n)_n$ defined by:

$$\begin{align*}
  x_0 &= x, \quad y_0 = y \\
  x_{n+1} &= \frac{x_n + y_n}{2} \quad (\forall n \in \mathbb{N}) \\
  y_{n+1} &= \mathcal{M}(x_n, y_n) \quad (\forall n \in \mathbb{N})
\end{align*}$$

**Proof.** Since the metric space $(\mathcal{M}, d)$ is the closed ball with center $A$ and radius $1/2$ (see Proposition 4.1), we have $d(\mathcal{M}, A) \leq 1/2 < 1$. The corollary then follows from Theorem 4.1 applied to the two means $A$ and $\mathcal{M}$ on $\mathcal{D}$. The proof is finished. ■

**Definition.** Given $\mathcal{M}$ a mean on $\mathcal{D}$, we suggest to call the mean $M$ given by Corollary 4.2 the mean $\mathcal{M}$-arithmetic. So the mean G-arithmetic is nothing else than the AGM mean.

**Remark.** Since the metric space $(\mathcal{M}, d)$ is a closed ball with radius $1/2$ then the distance between any two means $M_1$ and $M_2$ on $\mathcal{D}$ is at most equal to 1. It follows that the unique case for which Theorem 4.1 cannot apply is the extremal case where $d(M_1, M_2) = 1$ (remark also that in such case the means $M_1$ and $M_2$ are obligatory on the border of $\mathcal{M}$). However, the functional middle between two means $M_1$ and $M_2$ on $\mathcal{D}$ may exist and be unique even if the condition $d(M_1, M_2) = 1$ is fulfilled. Indeed, taking $\mathcal{D} = (0, +\infty)^2$, 21
$M_1 = G$ and $M_2(x, y) = x + y - \sqrt{x y} \quad (\forall (x, y) \in \mathcal{D})$, we easily verify that $d(M_1, M_2) = 1$ although the functional middle of $M_1$ and $M_2$ exists and it is unique (it is simply the arithmetic mean $A$ on $\mathcal{D}$).

In the following theorem, we establish the existence and the uniqueness of the functional middle of two mean functions $M_1$ and $M_2$ on $\mathcal{D}$ by changing the condition “$d(M_1, M_2) < 1$” (required by Theorem 4.1) by the condition with different nature which simply requires to $M_1$ and $M_2$ to be continuous on $\mathcal{D}$.

**Theorem 4.3** Suppose that $I$ is an interval of $\mathbb{R}$ and let $M_1$ and $M_2$ be two mean functions on $\mathcal{D} = I^2$ with values in $I$. We suppose that $M_1$ and $M_2$ are continuous on $\mathcal{D}$. Then, there exists a unique mean function $M$ on $\mathcal{D}$ satisfying the functional equation:

$$M(M_1, M_2) = M.$$ 

In addition, for all $(x, y) \in \mathcal{D}$, $M(x, y)$ is the common limit of the two real sequences $(x_n)_n$ and $(y_n)_n$ defined by:

$$\begin{cases} x_0 = x, \ y_0 = y \\
 x_{n+1} = M_1(x_n, y_n) \quad (\forall n \in \mathbb{N}) \\
 y_{n+1} = M_2(x_n, y_n) \quad (\forall n \in \mathbb{N}) \end{cases}$$

**Proof.** Let $(x, y) \in \mathcal{D}$ fixed and let $(x_n)_n$ and $(y_n)_n$ be the two real sequences related to $(x, y)$ which are introduced in the second part of the theorem. Let also $(u_n)_n$ and $(v_n)_n$ be the two real sequences defined by:

$$u_n := \min(x_n, y_n) \quad \text{and} \quad v_n := \max(x_n, y_n) \quad (\forall n \in \mathbb{N}).$$

It has been already shown during the proof of Theorem 4.1 that $(u_n)_n$ is non-decreasing and that $(v_n)_n$ is non-increasing. Since we have in addition $u_n \leq v_n \ (\forall n \in \mathbb{N})$ then $(u_n)_n$ bounded from above by $u_0$ and $(v_n)_n$ is bounded from below by $v_0$. It follows that $(u_n)_n$ and $(v_n)_n$ are convergent. Let $u = u(x, y)$ and $v = v(x, y)$ denote the respective limits of $(u_n)_n$ and $(v_n)_n$ (so $u$ and $v$ lie in $[u_0, v_0] = [\min(x, y), \max(x, y)] \ subset I$).

Now, since $M_1$ and $M_2$ are symmetric on $\mathcal{D}$ (as mean functions on $\mathcal{D}$), we have for all $n \in \mathbb{N}$:

$$x_{n+1} = M_1(u_n, v_n) \quad \text{and} \quad y_{n+1} = M_2(u_n, v_n).$$

This implies (according to the hypothesis of continuity of $M_1$ and $M_2$ on $\mathcal{D}$) that the two sequences $(x_n)_n$ and $(y_n)_n$ are also convergent and that their respective limits are $M_1(u, v)$ and $M_2(u, v)$.
Then, by tending \( n \) to infinity in the two hand-sides of the relation \( x_{n+1} = M_1(x_n, y_n) \), we obtain (according to the continuity of \( M_1 \)) that

\[
M_1(u, v) = M_1(M_1(u, v), M_2(u, v)),
\]

which implies (according to the third axiom of mean functions) that

\[
M_1(u, v) = M_2(u, v).
\]

The two sequences \((x_n)_{n}\) and \((y_n)_{n}\) thus converge to a same limit. Denoting (for all \((x, y) \in \mathcal{D}\) \(M(x, y)\) the common limit of \((x_n)_{n}\) and \((y_n)_{n}\), we show in the same way as in the proof of Theorem 4.1 that \(M\) is a mean function on \(\mathcal{D}\) and that it is the unique mean on \(\mathcal{D}\) which satisfies the functional equation \(M(M_1, M_2) = M\). The proof is achieved.

References

[1] J. M. Borwein, and P. B. Borwein, *Pi and the AGM, (A study in Analytic Number Theory and Computational Complexity)*, John Wiley & Sons Inc., New York, 1987.