Limit theorems for out-of-sample extensions of the adjacency and Laplacian spectral embeddings

Keith Levin\textsuperscript{1}, Fred Roosta\textsuperscript{2,3}, Minh Tang\textsuperscript{4}, Michael W. Mahoney\textsuperscript{3,5} and Carey E. Priebe\textsuperscript{6}

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Abstract

Graph embeddings, a class of dimensionality reduction techniques designed for relational data, have proven useful in exploring and modeling network structure. Most dimensionality reduction methods allow out-of-sample extensions, by which an embedding can be applied to observations not present in the training set. Applied to graphs, the out-of-sample extension problem concerns how to compute the embedding of a vertex that is added to the graph after an embedding has already been computed. In this paper, we consider the out-of-sample extension problem for two graph embedding procedures: the adjacency spectral embedding and the Laplacian spectral embedding. In both cases, we prove that when the underlying graph is generated according to a latent space model called the random dot product graph, which includes the popular stochastic block model as a special case, an out-of-sample extension based on a least-squares objective obeys a central limit theorem about the true latent position of the out-of-sample vertex. In addition, we prove a concentration inequality for the out-of-sample extension of the adjacency spectral embedding based on a maximum-likelihood objective. Our results also yield a convenient framework in which to analyze trade-offs between estimation accuracy and computational expense, which we explore briefly.

1 Introduction

Graph embeddings are a class of dimensionality reduction techniques designed for network data, which have emerged as a popular tool for exploring and modeling network structure. Given a graph $G = (V, E)$ on vertex set $V = \{1, 2, \ldots, n\}$ with adjacency matrix $A \in \{0, 1\}^{n \times n}$, the graph embedding problem concerns how best to map $V$ to a $d$-dimensional vector space so that geometry in that vector space captures the topology of $G$. For example, we may ask that vertices that play similar structural roles in $G$ be mapped to nearby points. Two common approaches to graph embedding are the graph Laplacian embedding (Belkin and Niyogi 2003, Coifman and Lafon 2006) and the adjacency spectral embedding (ASE; Sussman et al. 2012), both of which are based on spectral decompositions of the adjacency matrix or a transformation thereof. In many settings, data collection or computational constraints may dictate that having computed an embedding of the graph $G$, a practitioner may wish to add vertices to $G$, and compute

\textsuperscript{1}Department of Statistics, University of Michigan
\textsuperscript{2}School of Mathematics and Physics, University of Queensland
\textsuperscript{3}International Computer Science Institute
\textsuperscript{4}Department of Statistics, North Carolina State University
\textsuperscript{5}Department of Statistics, University of California at Berkeley
\textsuperscript{6}Department of Applied Mathematics and Statistics, Johns Hopkins University
the corresponding embeddings of these new vertices. We call these new vertices out-of-sample vertices, in contrast to the in-sample vertices in $V$. Since constructing the in-sample embedding typically requires a comparatively expensive eigenvalue computation, it is preferable to compute this out-of-sample embedding without computing a new graph embedding from scratch. This problem is well-studied in the dimensionality reduction literature, where it is known as the out-of-sample extension problem. The focus of the present paper is to derive out-of-sample extensions for the ASE and a slight variant of Laplacian eigenmaps, and to establish their statistical properties under a particular natural choice of network model.

Latent space network models are a class of statistical models for graphs in which unobserved geometry drives network formation. Each vertex is assigned a latent position (Richardson and Domingos 2006), which concerns how to embed a less costly computation. This is the motivation for the out-of-sample (OOS) extension problem, which is well-studied in the dimensionality reduction literature, where it is known as the out-of-sample extension problem. The focus of the present paper is to derive out-of-sample extensions for the ASE and a slight variant of Laplacian eigenmaps, and to establish their statistical properties under a particular natural choice of network model.

1.1 Background and Notation

Most dimensionality reduction and embedding techniques begin with a collection of training data observations $D = \{z_1, z_2, \ldots, z_n\} \subseteq \mathcal{X}$, where $\mathcal{X}$ is the set of all possible observations (e.g., the set of all possible images, audio signals, etc.). $\mathcal{X}$ is endowed with a similarity measure $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \geq 0$, and most embedding procedures leverage the eigenstructure of the symmetric similarity matrix $M = [K(z_i, z_j)] \in \mathbb{R}^{n \times n}$. An embedding of the data $D$ assigns to each $z_i \in D$ a vector $x_i \in \mathbb{R}^d$, where $d$ is the embedding dimension, with the embeddings $\{x_1, x_2, \ldots, x_n\}$ chosen so as to preserve the structure of the sample $D$ as captured by the matrix $M$. This typically manifests as attempting to ensure that elements $z_i, z_j \in D$ for which $K(z_i, z_j)$ is large are mapped so that $\|x_i - x_j\|$ is small. Suppose that, having computed $x_1, x_2, \ldots, x_n$, we obtain a new out-of-sample observation $z \in \mathcal{X}$ (which may or may not appear in the training sample $D$), which we would like to embed along with the in-sample observations $D$. Letting $\tilde{D} = D \cup \{z\}$, a na"ive approach would simply construct a new embedding $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, \tilde{x}_{n+1}\}$ based on the sample $\tilde{D}$. This would involve computational complexity of the same order as that required to compute the initial embedding $\{x_1, x_2, \ldots, x_n\}$. Since computing the embedding $\{x_1, x_2, \ldots, x_n\}$ tends to involve expensive computations, most commonly eigendecompositions, it would be preferable to avoid paying this computational cost repeatedly, particularly if there exists a scheme whereby the embedding $\tilde{x}_{n+1}$ of out-of-sample observation $z$ can be well approximated by a less costly computation. This is the motivation for the out-of-sample (OOS) extension problem, which concerns how to embed $z$ into the same embedding space $\mathbb{R}^d$ based only on the existing in-sample embedding $\{x_1, x_2, \ldots, x_n\}$ and the similarity measurements $\{K(z, x_i) : i = 1, 2, \ldots, n\}$. That is, we wish to compute an embedding of $z$ without making recourse to the full similarity matrix $M \in \mathbb{R}^{n \times n}$.

As an illustrative example, consider the Laplacian eigenmaps embedding (Belkin and Niyogi 2003).
Recall that the *normalized Laplacian* of graph \( G = (V,E) \) with adjacency matrix \( A \in \mathbb{R}^{n \times n} \) is given by the matrix \( L = D^{-1/2}AD^{-1/2} \), where \( D \in \mathbb{R}^{n \times n} \) is the diagonal matrix of degrees, with \( D_{ii} = \sum_{j=1}^{n} A_{ij} \), and \( 0^{-1/2} = 0 \) by convention (Chung 1997, Luxenburg 2007, Vishnoi 2013). The \( d \)-dimensional normalized Laplacian eigenmaps embedding of \( G \) is then given by the rows of the matrix \( \tilde{U} \in \mathbb{R}^{n \times d} \), where the columns of \( \tilde{U} \) are the orthonormal eigenvectors corresponding to the top \( d \) eigenvectors of \( L \), excluding the trivial eigenvalue 1. Suppose now that we wish to add a vertex \( v \) to the graph, to form graph \( \tilde{G} \) with adjacency matrix

\[
\tilde{A} = \begin{bmatrix} A & \tilde{a} \\ \tilde{a}^T & 0 \end{bmatrix},
\]

where \( \tilde{a} \in \{0,1\}^n \) and has \( a_i = 1 \) if and only if \( v \) forms and edge with in-sample vertex \( i \in [n] \). Naïvely, one could simply apply the Laplacian eigenmaps embedding again to \( \tilde{A} \), at the cost of another eigendecomposition. Cheaper, however, would be an OOS extension, such as that given by Bengio et al. (2003) or Belkin et al. (2006), that only makes use of the embedding \( \tilde{U} \) and the vector of edges \( \tilde{a} \).

Out-of-sample extensions for multidimensional scaling (MDS; Torgerson 1952, Borg and Groenen 2005), spectral clustering (Weiss 1999, Ng et al. 2002), Laplacian eigenmaps (Belkin and Niyogi 2003) and ISOMAP (Tenenbaum et al. 2000) appear in Bengio et al. (2003). These extensions were obtained by formulating each of the dimensionality reduction techniques as a least-squares problem, which is possible owing to the fact that the in-sample embeddings are functions of the eigenvalues and eigenvectors of a similarity or distance matrix. Let matrix \( M = [K(x_i, x_j)]_{i,j=1}^{n} \) be the similarity matrix for some similarity function \( K \), and let \( \{(\lambda_i, u_i)\}_{i=1}^{n} \) be the eigenvalue-eigenvector pairs of \( M \). Bengio et al. (2003) derive the OOS extensions for a number of embeddings as solutions to the least-squares problem

\[
\min_{f(x) \in \mathbb{R}^d} \sum_{i=1}^{n} \left( K(x, x_i) - \frac{1}{n} \sum_{j=1}^{d} \lambda_j f_j(x_i) f_j(x) \right)^2,
\]

where \( D = \{x_1, x_2, \ldots, x_n\} \) are the in-sample observations and \( f_j(x_i) \) is the \( i \)-th component of \( u_j \). A different OOS extension for MDS was considered in Trosset and Priebe (2008). Instead of the least-squares framework of Bengio et al. (2003), Trosset and Priebe (2008) frame the MDS OOS extension problem as a modification of the optimization problem solved by the in-sample MDS embedding.

An approach to the Laplacian eigenmaps OOS extension, different from the one presented here, was pursued in Belkin et al. (2006), incorporating regularization in both the geometry of the training data and the geometry of the similarity function \( K \). Their approach can also be extended to regularized least squares, SVM and a variant of SVM in which a Laplacian penalty term is added to the SVM objective. The authors showed that all of these OOS extensions are the solutions to generalized eigenvalue problems. Levin et al. (2015) provides an illustrative example of the practical application of these OOS extensions, using the OOS extension of Belkin et al. (2006) to build an audio search system. More recent OOS extension techniques have attempted to avoid altogether the need to solve least squares or eigenvalue problems, instead training a neural net to learn the embedding, so that at out-of-sample embedding time one need only feed the out-of-sample observation as input to the neural net (see, for example, Quispe et al. 2016, Jansen et al. 2017).

As far as we are aware, the only work to date on the OOS extension for ASE appears in Tang et al. (2013a), in which the authors considered the OOS extension problem for certain latent space models of graphs (see, for example, Hoff et al. 2002). These are models in which each vertex has an associated latent vector in a Hilbert space, with edge probabilities determined by inner products between the latent vectors in this Hilbert space.
an OOS extension based on a least-squares objective and proved a result, analogous to our
Theorem 1, given the rate of growth of the error between this out-of-sample embedding and the
true out-of-sample latent position. Theorem 1 yields a simplification of the proof of the result
originally appearing in Tang et al. (2013a), specialized to the random dot product graph model
(see Definition 2 below). We note, however, that our results can be extended to more general
latent space network models under suitable conditions on the inner product.

Largely missing from the literature, but of particular importance to the assessment of OOS
extensions, is the comparison of the OOS estimate’s performance compared to its in-sample
counter-part. That is, for training sample $\mathcal{D}$ and out-of-sample observation $z \in \mathcal{X}$ (both drawn,
perhaps, from a probability distribution on $\mathcal{X}$), how closely does the out-of-sample embedding
approximate its in-sample counterpart computed based on $\mathcal{D} = \mathcal{D} \cup \{z\}$? In this work, we
address this question as it pertains to the adjacency spectral embedding (ASE) and the Laplacian
spectral embedding (LSE; an embedding closely related to the Laplacian eigenmaps embedding
but more amenable to analysis; see Section 2). In particular, we show the following:

- Two different approaches to the ASE OOS extension problem yield OOS extensions that
  recover the true out-of-sample latent position at a rate that matches the in-sample es-
timation error rate. The first (Theorem 1), based on a linear least squares objective,
  holds under essentially no conditions on the model. The second (Theorem 2), based on a
  maximum-likelihood objective, requires mild regularity conditions.
- An LSE OOS extension based on a linear least-squares objective that, similarly to the
  ASE OOS extensions, recovers the true out-of-sample latent position at the same rate as
  the in-sample embedding (Theorem 3).
- Both of the LLS-based OOS extensions obey central limit theorems (Theorems 4 and
  5), with each OOS extension asymptotically normally distributed about the true latent
  position (in the case of ASE) or a transformation thereof (in the case of LSE).

We believe that analogous central limit theorems can be obtained for other OOS extensions
such as those presented in Bengio et al. (2003) and for the maximum-likelihood ASE OOS
extension, but do not pursue this generalization here.

1.2 Notation

Before continuing, we pause to establish notation. For a matrix $M \in \mathbb{R}^{n_1 \times n_2}$, we denote by
$\sigma_i(M)$ the $i$-th singular value of $M$, so that $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_k(M) \geq 0$, where
$k = \min\{n_1, n_2\}$. For integer $k > 0$, we let $[k] = \{1, 2, \ldots, n\}$. Throughout the paper, $n$ will
denote the number of vertices in the observed graph $G$. For a vector $x$, the unadorned norm
$\|x\|$ will denote the Euclidean norm of $x$, while for all $p > 0$, $\|x\|_p$ will denote the $p$-norm of
$x$, where $\|x\|_\infty = \max_i |x_i|$. For a matrix $M$, $\|M\|_F$ will denote the Frobenius norm, $\|M\|$ will
denote the spectral norm
$$\|M\| = \sup_{x : \|x\| = 1} \|Mx\|,$$
and $\|M\|_{2, \infty}$ will denote the 2-to-$\infty$ norm,
$$\|M\|_{2, \infty} = \sup_{x : \|x\|=1} \|Mx\|_\infty.$$

Most of our results will concern the behavior of certain quantities as the number of vertices
$n$ increases to $\infty$. We will often, for ease of notation, suppress this dependence on $n$, but it
should be assumed throughout that all quantities are dependent on $n$, with the exception of the
distribution $F$ and the latent space dimension $d$. Thus, for example, we will in several places
refer to a “sequence of matrices” $Q \in \mathbb{R}^{d \times d}$, where we suppress what ought to be, say, a subscript
$n$. Throughout, $C > 0$ denotes a positive constant, not depending on $n$, whose value may change.
Suppose that with probability 1 there exists a graph Laplacian $L$. Given a collection of events $E_n$ indexed by $n$, suppose that with probability 1 there exists $n_0$ such that $E_n$ occurs whenever $n \geq n_0$. If this is the case, we say that $E_n$ occurs eventually or, by a slight abuse of terminology, say simply that $E_n$ occurs.

We make standard use of the big-$O$, big-$\Omega$ and big-$\Theta$ notation. Thus, for example, we write $f(n) = O(g(n))$ to denote the existence of a constant $C > 0$ such that for all suitably large $n$, $f(n) \leq C g(n)$. We write $f(n) = \tilde{O}(g(n))$ to mean that $f(n) = O(g(n))$ ignoring logarithmic factors. That is, if there exists a $c > 0$ such that $f(n) = O(g(n) \log^c n)$ (throughout the paper, $c$ is never larger than 2 or 3 and is typically 1). Our one slight abuse of this notation is in the case where, letting $\{Z_n\}$ be a sequence of random variables, we write $Z_n = O(g(n))$ to mean that there exists a constant $C > 0$ such that almost surely there exists $n_0$ such that $|Z_n| \leq C g(n)$ for all $n \geq n_0$, replacing the modulus with an appropriate norm when $Z_n$ is a vector or matrix. Most results in this paper are of this form. We note that throughout, we prove these results by showing first that $\Pr[|Z_n| \geq C g(n)] \leq C n^{-(1+c)}$ is summable for all suitably small $c > 0$. We then use the independence of $\{Z_n : n = 1, 2, \ldots\}$ to invoke the Borel-Cantelli lemma (Billingsley 1995) to conclude that $Z_n = O(g(n))$. Thus, though many of our results are stated as holding asymptotically, they all have finite-sample analogues obtained in the course of their proofs.

### 1.3 Roadmap
The remainder of this paper is structured as follows. In Section 2, we formalize the graph out-of-sample extension problem, and introduce a few methods for constructing such extensions. In Section 3, we present our main theoretical results, proving concentration and asymptotic distributions for these extensions. Section 4 gives an experimental investigation of the properties of these embeddings. We conclude in Section 5 with a brief discussion of directions for future work.

### 2 Out-of-sample Extension for ASE and LSE

Given a graph $G = ([n], E)$ with adjacency matrix $A \in \{0, 1\}^{n \times n}$, the adjacency spectral embedding (ASE; Sussman et al. 2012) and the Laplacian spectral embedding (LSE; Tang and Priebe 2018) each provide a mapping of the $n$ vertices of $G$ into $\mathbb{R}^d$. The ASE maps the vertices of $G$ to $d$-dimensional representations $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n \in \mathbb{R}^d$ given by the rows of the matrix

$$\hat{X} = \text{ASE}(A, d) = \tilde{U} \hat{S}^{1/2} \in \mathbb{R}^{n \times d},$$

where $\hat{S} \in \mathbb{R}^{d \times d}$ is the diagonal matrix with entries given by the top $d$ eigenvalues of $A$ and the columns of $\tilde{U} \in \mathbb{R}^{n \times d}$ are the corresponding orthonormal eigenvectors. The Laplacian spectral embedding (LSE; Tang and Priebe 2018) proceeds according to a similar eigenvalue truncation, applied to the normalized graph Laplacian,

$$L = \mathcal{L}(A) := D^{-1/2} A D^{-1/2},$$

where $D \in \mathbb{R}^{n \times n}$ is the diagonal degree matrix, with $D_{i,i} = \sum_{j=1}^n A_{i,j}$, with $0^{-1/2} = 0$ by convention. The LSE embeds the vertices of $G$ as $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n \in \mathbb{R}^d$ given by the rows of the matrix

$$\hat{X} = \text{LSE}(A, d) = \tilde{U} S^{1/2} \in \mathbb{R}^{n \times d},$$

where $S \in \mathbb{R}^{d \times d}$ is the diagonal matrix formed of the $d$ largest-magnitude eigenvalues of the graph Laplacian $L$ and $\tilde{U} \in \mathbb{R}^{n \times d}$ is the matrix formed of the $d$ corresponding orthonormal
eigenvectors. The well-known Laplacian eigenmaps embedding \cite{Belkin2003} corresponds to a rescaling of the LSE, in that the Laplacian eigenmaps embedding is given by the rows of $\tilde{U} \in \mathbb{R}^{n \times d}$. As such, results similar to those presented here for the LSE can be obtained for the Laplacian eigenmaps embedding as well.

We note that in both of the embeddings just described, there may be a concern that the $d$ largest-magnitude eigenvalues need not all be positive, and hence square roots $\sqrt{S}$ will be ill-defined. As a result, it may be preferable, in general, to consider instead the top-$d$ singular values of $\Sigma$ and $L$. We will not consider this issue in the present work, since under the model considered in this paper (see Definition 2 below), with probability 1 the $d$ largest-magnitude eigenvalues will be positive for all suitably large $n$.

**Remark 1** (Comparing ASE and LSE). Both the ASE and LSE yield low-dimensional representations of the vertices of $G$, and it is natural to ask which embedding is preferable. The answer, in general, is dependent on the precise model under consideration and the intended downstream task. For example, one can show that neither the ASE nor the Laplacian embedding strictly dominates in a vertex classification task. Section 4 of \cite{Tang2018} demonstrates that ASE performs better than the Laplacian embedding when applied to graphs with a core-periphery structure. Such structures are ubiquitous in real networks; see, for example, \cite{Leskovec2009} and \cite{Jeub2015}. We refer the interested reader to \cite{Cape2018} for a more thorough theoretical treatment of this point.

The two embeddings just discussed are especially well-suited to the random dot product graph (RDPG; \cite{Young2007} \cite{Athreya2018}, a model in which graph structure is driven by the geometry of latent positions associated to the vertices.

**Definition 1.** (Inner product distribution) A distribution $F$ on $\mathbb{R}^d$ is a $d$-dimensional inner product distribution if $0 \leq x^T y \leq 1$ whenever $x, y \in \text{supp} F$.

**Definition 2.** (Random Dot Product Graph) Let $F$ be a $d$-dimensional inner product distribution, and let $X_1, X_2, \ldots, X_n \sim F$ be collected in the rows of $X \in \mathbb{R}^{n \times d}$. Let $G$ be a random graph with adjacency matrix $A \in \{0, 1\}^{n \times n}$. We say that $G$ is a random dot product graph (RDPG) with latent positions $X_1, X_2, \ldots, X_n \in \mathbb{R}^d$, if the edges of $G$ are independent conditioned on $\{X_1, X_2, \ldots, X_n\}$, with

$$
\Pr[A|X] = \prod_{1 \leq i < j \leq n} (X_i^T X_j)^{A_{i,j}} (1 - X_i^T X_j)^{1-A_{i,j}}.
$$

We say that $X_i$ is the latent position associated to the $i$-th vertex in $G$, and write $(A, X) \sim \text{RDPG}(F, n)$ to mean that the rows of $X \in \mathbb{R}^{n \times d}$ are drawn i.i.d. from $F$ and that $A \in \{0, 1\}^{n \times n}$ is generated according to Equation 4 conditional on $X$.

Note that the RDPG has an inherent nonidentifiability, owing to the fact that the distribution of $A$ is unchanged by an orthogonal rotation of the latent positions: for latent position matrix $X \in \mathbb{R}^{n \times d}$ and orthogonal matrix $W \in \mathbb{R}^{d \times d}$, both $X \in \mathbb{R}^{n \times d}$ and $XW \in \mathbb{R}^{n \times d}$ give rise to the same distribution over adjacency matrices, in that $\mathbb{E}[A | X] = XX^T = XWX^T W^T$. Thus, we can only ever hope to recover the latent positions of the RDPG up to some orthogonal transformation. Throughout this work, we denote by $\Delta = \mathbb{E}X_1 X_1^T \in \mathbb{R}^{d \times d}$ the second moment matrix of the latent position distribution $F$. Our results require that $\Delta$ be of full rank, an assumption that we make without loss of generality owing to the fact that if $\Delta$ is of, say, rank $d' < d$, then we may equivalently think of $F$ as a $d'$-dimensional inner product distribution by restricting our attention to an appropriate $d'$-dimensional subspace of $\mathbb{R}^d$.

**Remark 2.** (Extension to other graph models) As alluded to above, the RDPG as defined here only captures graphs with positive semi-definite expected adjacency matrices. This limitation can be avoided by considering the generalized RDPG \cite{Rubin-Delanchy2017}. The results stated
in the present work can for the most part be extended to this model, at the expense of additional notational complexity, which we prefer to avoid here. Similarly, using standard concentration inequalities, most of the results presented here can be extended beyond binary edges to consider independent edges that are unbiased (\(\mathbb{E}A_{i,j} = X_i^TX_j\)) with sub-Gaussian or sub-gamma tails \cite{Boucheron2013, Tropp2013}.

Throughout this paper, we will assume that \((A, X) \sim \text{RDPG}(F, n)\) for some \(d\)-dimensional inner product distribution \(F\), and write \(P = \mathbb{E}[A \mid X] = XX^T\). Under this setting, it is clear that \(X = \text{ASE}(A, d)\) is a natural estimate of the matrix of true latent positions \(X\). Further, \(\hat{X} = \text{LSE}(A, d)\) is a natural estimate of \(\hat{X} = T^{-1/2}X\), where \(T \in \mathbb{R}^{n \times n}\) is a diagonal matrix with entries \(T_{i,i} = \sum_j X_j^TX_i\). The rows of \(\hat{X}\) can be thought of as the Laplacian spectral embeddings of the matrix \(P = XX^T\), in the sense that \(\hat{X}\hat{X}^T = \mathcal{L}(P)\). Indeed, it has been shown previously that the ASE consistently estimates the latent positions in the RDPG \cite{Sussman2012, Tang2013}, and successfully recovers community structure in the (positive semi-definite) stochastic block model \cite{Lyzinski2014}, which can be recovered as a special case of the RDPG by taking the distribution \(F\) to be a mixture of point masses. Similar results can be shown for the LSE \cite{Tang2018}.

**Lemma 1.** Let \((A, X) \sim \text{RDPG}(F, n)\) for some \(d\)-dimensional inner product distribution \(F\) and let \(X, \hat{X}, \tilde{X} \in \mathbb{R}^{n \times d}\) be as above. Then there exists a sequence of orthogonal matrices \(Q \in \mathbb{R}^{d \times d}\) such that

\[
\|\hat{X} - XQ\|_{2,\infty} = O\left(\frac{\log n}{\sqrt{n}}\right). \tag{5}
\]

Further, if there exists a constant \(\eta > 0\) such that \(\eta \leq x^Ty \leq 1 - \eta\) whenever \(x, y \in \text{supp}F\), then there exists a sequence of orthogonal matrices \(Q \in \mathbb{R}^{d \times d}\) such that

\[
\|\hat{X} - \tilde{X}Q\|_{2,\infty} = O\left(\frac{\log^{1/2} n}{n}\right). \tag{6}
\]

**Proof.** The bound in Equation (5) is Lemma 5 in \cite{Lyzinski2014}. A proof of Equation (6) can be found in Appendix A. \(\square\)

Suppose that graph \(G = ([n], E)\) with adjacency matrix \(A \in \mathbb{R}^{n \times n}\) is a random dot product graph, so that \((A, X) \sim \text{RDPG}(F, n)\), and we compute

\[
\hat{X} = \text{ASE}(A, d) = [\hat{X}_1 \hat{X}_2 \cdots \hat{X}_n]^T \in \mathbb{R}^{n \times d}\]

and

\[
\tilde{X} = \text{LSE}(A, d) = [\tilde{X}_1 \tilde{X}_2 \cdots \tilde{X}_n]^T \in \mathbb{R}^{n \times d},
\]

where \(\tilde{X}_i, \hat{X}_i \in \mathbb{R}^{d}\) are embeddings of the \(i\)-th vertex under ASE and LSE, respectively. Suppose now that a vertex \(v\) having latent position \(\tilde{w} \in \text{supp}F\) is added to the graph \(G\) to form \(\tilde{G} = ([n] \cup \{v\}, E \cup E_v)\), where \(E_v \subseteq \{(i, v) : i = 1, 2, \ldots, n\}\). The edges between the out-of-sample vertex \(v\) and the in-sample vertices \(\{1, 2, \ldots, n\}\) are specified by a vector \(\tilde{a} \in \{0, 1\}^n\) such that \(a_i = 1\) if \((i, v) \in E_v\) and \(a_i = 0\) otherwise. Thus, \(\tilde{G}\) has adjacency matrix \(\tilde{A}\) as in Equation (1) above. Having computed an embedding \(\hat{X}\) or \(\tilde{X}\), we would like to embed the vertex \(v\) to obtain an estimate of the true latent position \(\tilde{w}\) (in the case of ASE) or, in the case of LSE, its Laplacian spectral embedding \(\tilde{w} = \tilde{w}/\sqrt{n\mu^T}\tilde{w} \in \mathbb{R}^d\), where \(\mu = \mathbb{E}X_1\) is the mean of \(F\). In the case of ASE, the out-of-sample extension problem concerns how to compute an estimate of \(\tilde{w}\) based only on \(\hat{X}\) and \(\tilde{a}\). Similarly, in the case of LSE, the out-of-sample extension problem requires computing an estimate of \(\tilde{w}\) based only on the information in \(\hat{X}\), \(\tilde{a}\) and, for reasons that will become clear below, the vector of in-sample vertex degrees, \(\tilde{d} \in \mathbb{R}^n\).
2.1 Out-of-sample extension for ASE

Two natural approaches to the out-of-sample extension of ASE suggest themselves. The first, following Bengio et al. (2003), involves embedding the out-of-sample vertex $v$ as

$$\hat{w}_{LS} = \arg \min_{w \in \mathbb{R}^d} \sum_{i=1}^{n} \left( a_i - \hat{X}_i^T w \right)^2,$$

(7)

where $a_i$ is the $i$-th component of the vector $\hat{a} \in \mathbb{R}^n$ of edges between the out-of-sample vertex and the in-sample vertices. We refer to $\hat{w}_{LS}$ as the linear least squares out-of-sample (LLS OOS) extension of adjacency spectral embedding.

An alternative approach to the OOS extension problem, perhaps more appealing from a statistical perspective, but more computationally expensive, is to cast the OOS extension as a maximum-likelihood problem. Letting $X_1, X_2, \ldots, X_n \in \mathbb{R}^d$ be the true latent positions of the in-sample vertices and $\hat{w} \in \mathbb{R}^d$ be the true latent position of the out-of-sample vertex, the entries of $\hat{a}$ are independent Bernoulli random variables, with $a_i \sim \text{Bernoulli}(X_i^T \hat{w})$. Thus, the log likelihood (conditional on the in-sample latent positions) is

$$\ell(w) = \sum_{i=1}^{n} a_i \log X_i^T w + (1 - a_i) \log(1 - X_i^T w).$$

Of course, in practice we observe the latent positions only through their ASE estimates $\{\hat{X}_i\}_{i=1}^{n} \subseteq \mathbb{R}^d$. Thus, we define the maximum-likelihood out-of-sample extension for ASE as the maximizer of the plug-in likelihood, i.e., as the solution to

$$\max_{w \in \mathbb{R}^d} \sum_{i=1}^{n} a_i \log \hat{X}_i^T w + (1 - a_i) \log \left(1 - \hat{X}_i^T w\right).$$

(8)

Unfortunately, this objective need not achieve its optimum inside the support of $F$. Indeed, the objective need not even be bounded. Thus, we will settle for a slight reformulation of this objective, and define the maximum-likelihood out-of-sample (ML OOS) extension for ASE to be the solution to a constrained maximum-likelihood problem,

$$\hat{w}_{ML} = \arg \max_{w \in \mathcal{T}_\epsilon} \sum_{i=1}^{n} a_i \log \hat{X}_i^T w + (1 - a_i) \log \left(1 - \hat{X}_i^T w\right),$$

(9)

where $\mathcal{T}_\epsilon = \{ w \in \mathbb{R}^d : \epsilon \leq \hat{X}_i^T w \leq 1 - \epsilon, i \in [n] \}$, and $\epsilon > 0$ is some small constant. We note that we call this the maximum-likelihood OOS extension, though it is, strictly speaking, based on a plug-in approximation to the true likelihood given in Equation (8).

Note that, as required by the out-of-sample problem, both $\hat{w}_{LS}$ and $\hat{w}_{ML}$ are functions only of the in-sample embedding $\hat{X} \in \mathbb{R}^{n \times d}$ and the edges between the out-of-sample vertex $v$ and the in-sample vertices $[n]$, as encoded in the vector $\hat{a} \in \mathbb{R}^n$.

2.2 Out-of-sample extension for LSE

Recall that given the adjacency matrix $A$ of graph $G = ([n], E)$, we form the sample graph Laplacian $L = \mathcal{L}(A) = D^{-1/2}AD^{-1/2}$ and embed in-sample vertex $i \in [n]$ as $\hat{X}_i \in \mathbb{R}^d$, the $i$-th row of

$$\hat{X} = \hat{U}S^{1/2} \in \mathbb{R}^{n \times d},$$

where we remind the reader that $\hat{U} \in \mathbb{R}^{n \times d}$ denotes the matrix formed by the top $d$ orthonormal eigenvectors of $L$ with their corresponding eigenvalues collected in the diagonal matrix $\hat{S} \in \mathbb{R}^{d \times d}$. 8
Conditional on the latent positions $X_1, X_2, \ldots, X_n \sim^i \sim \mathcal{D} F$, we have $\mathbb{E}[A|X] = XX^T = P \in \mathbb{R}^{n \times n}$, and we view $L = \mathcal{L}(A)$ as an estimate of $L(P) = T^{-1/2} PT^{-1/2}$, where $T \in \mathbb{R}^{n \times n}$ is the matrix of (conditional) expected degrees, $T_{i,i} = \sum_{j=1}^{n} P_{i,j} = \sum_{j=1}^{n} X_i^T X_j$. Applying the LSE to $\mathcal{L}(P)$, we may think of the rows of

$$
\tilde{X} = \hat{U} \hat{S}^{1/2} \in \mathbb{R}^{n \times d}
$$

as the “true” Laplacian spectral embedding, and view $\hat{X}$ as an estimate of this quantity.

Given out-of-sample vertex $v$ with latent position $\tilde{w} \in \mathbb{R}^d$, the natural Laplacian embedding of $v$, in light of the definition of $\tilde{X}$, is given by $\tilde{w} = \tilde{w}/\sqrt{n \mu^T \tilde{w}}$, where $\mu = \mathbb{E}X_i \in \mathbb{R}^d$ is the mean of $F$. Of course, in practice we must compute the out-of-sample embedding of $v$ based on $\tilde{X} \in \mathbb{R}^{n \times d}$ and the vector of edges $\tilde{a} \in \mathbb{R}^n$ to obtain an estimate of $\tilde{w}$. In applying the least-squares approach suggested by Equation (7) and used in Bengio et al. (2003), it is most natural to consider the minimizer

$$
\tilde{w}_{LS} = \arg \min_{w \in \mathbb{R}^n} \sum_{i=1}^{n} \left( \frac{a_i}{\sqrt{d_i d_v}} - \tilde{X}_i^T w \right)^2,
$$

(10)

where $d_i = \sum_{j=1}^{n} A_{i,j}$ is the degree of the $i$-th in-sample vertex, and $d_v = \sum_i a_i$ is the degree of the out-of-sample vertex $v$. We refer to $\tilde{w}_{LS}$ as the LLS OOS extension of the Laplacian spectral embedding. We note that Equation (10) requires that we keep in-sample vertex degree information for use in the out-of-sample extension, which violates the typical requirement that we compute the out-of-sample extension using only $\tilde{X}$ and $\tilde{a}$. Nonetheless, it is reasonable to allow the use of the vector $\tilde{d}$, since typically the embedding dimension $d$ is of a smaller order than $n$ and thus the space required to store node degrees is of the same or smaller order as that required to store $\tilde{X} \in \mathbb{R}^{n \times d}$. We note that one could avoid this additional storage by replacing $d_i$ with $\sum_{j=1}^{n} \tilde{X}_j^T \tilde{X}_i$ and all our results below would go through (see Lemma 6), but this would come at the expense of notational inconvenience and longer proofs below. The motivation for the least-squares objective in Equation (10) becomes clear if we think of $d_v^{-1/2} d_i^{-1/2} a_i$ as an estimate of the normalized kernel

$$
\tilde{K}(i, v) = \frac{X_i^T \tilde{w}}{n \sqrt{X_i^T \mu \tilde{w}^T \mu}},
$$

where $\mu \in \mathbb{R}^d$ is again the mean of $F$.

3 Theoretical Results

The main results of this paper concern concentration inequalities and central limit theorems for the OOS extensions introduced in Section 2. We first present the concentration inequalities, which allow us to control the rate of convergence of the OOS extension to the parameter of interest, given by the true OOS latent position $\tilde{w}$ in the case of ASE, and by the transformed latent position $\tilde{w} = \tilde{w}/\sqrt{n \mu^T \tilde{w}}$ in the case of LSE.

3.1 Rates of convergence for OOS extensions

A first question surrounding the OOS extensions presented in the preceding section concerns their quality as estimators of their respective true parameters. Interestingly, all of the OOS extensions presented above recover their respective target parameters at asymptotic rates that match that of the full-graph embedding.

We begin by considering the ASE OOS extensions defined in Equations (7) and (9). Both of these estimates recover the true out-of-sample latent position $\tilde{w}$ at the same asymptotic rate.
(see Theorems 1 and 2 below), and this rate matches the one we would obtain if we were to compute the ASE of the augmented graph $G$ with adjacency matrix $A$, given in Lemma 1. We find that the estimation error between the least squares OOS extension for ASE $\hat{w}_{LS}$ and the true latent position $\bar{w}$ follows the same rate.

**Theorem 1.** Let $F$ be a $d$-dimensional inner-product distribution and suppose $(A, X) \sim \text{RDPG}(F, n)$. Let $v$ denote the out-of-sample vertex, and denote its latent position by $\bar{w} \in \text{supp} F$. Let $\hat{w}_{LS}$ denote the LS-based OOS extension for ASE based on $a = \text{ASE}(A, d)$ and the vector of edges $\vec{a} \in \mathbb{R}^n$ between $v$ and the in-sample vertices, as defined in Equation (7). There exists a sequence of orthogonal matrices $Q \in \mathbb{R}^{d \times d}$ such that

$$
\|Q\hat{w}_{LS} - \bar{w}\| = O(n^{-1/2} \log n),
$$

and this matrix $Q$ is the same one guaranteed by Lemma 1.

**Proof.** A standard result for solutions of perturbed linear systems allows us to show that with high probability, $\|Q\hat{w}_{LS} - w_{LS}\| \leq Cn^{-1/2} \log n$, where $Q \in \mathbb{R}^{d \times d}$ is the orthogonal matrix guaranteed by Lemma 1 above and $w_{LS}$ is the least-squares minimizer obtained if one uses the true latent positions $\{X_i\}$ rather than the ASE estimates $\{\hat{X}_i\}$ in Equation (7). Hoeffding’s inequality implies that $\|w_{LS} - \bar{w}\| = O(n^{-1/2} \log n)$. The result then follows by a triangle inequality applied to $\|Q\hat{w}_{LS} - \bar{w}\|$. A detailed proof can be found in Appendix B. 

In a similar vein, the ML-based OOS extension also recovers the true out-of-sample latent position at a rate that matches that of the in-sample embedding, given by Equation (5) in Lemma 1.

**Theorem 2.** Let $F$ be a $d$-dimensional inner-product distribution for which there exists a constant $\eta > 0$ such that $\eta < x^T y < 1 - \eta$ for all $x, y \in \text{supp} F$. Suppose that $(A, X) \sim \text{RDPG}(F, n)$ and let $v$ be an out-of-sample vertex with latent position $\bar{w} \in \text{supp} F$. Let $\hat{w}_{ML}$ be the out-of-sample embedding defined in Equation (9), with $\epsilon > 0$ chosen so that $\epsilon < \eta$. Then there exists a sequence of orthogonal matrices $Q \in \mathbb{R}^{d \times d}$ such that

$$
\|Q\hat{w}_{ML} - \bar{w}\| = O(n^{-1/2} \log n),
$$

and this matrix $Q$ is the same one guaranteed by Lemma 1.

**Proof.** Using the definition of $\hat{F}_n$ and a standard argument from convex optimization, one can show that with probability 1, it holds for all suitably large $n$ that

$$
\|Q\hat{w}_{ML} - \bar{w}\| \leq C\|\nabla \hat{F}(Q^T \bar{w})\| \frac{1}{n}.
$$

An application of the triangle inequality and standard concentration inequalities yields

$$
\|\nabla \hat{F}(Q^T \bar{w})\| = O(\sqrt{n} \log n).
$$

A detailed proof can be found in Appendix C.

In keeping with the above two results, the least-squares LSE OOS extension given in Equation (10) recovers the true out-of-sample Laplacian embedding $\hat{w}$ at a rate that matches that of the Laplacian spectral embedding $\hat{w}$ of the augmented graph $G$, given by Equation (6) in Lemma 1.

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Theorem 3. Let \( F \) be a \( d \)-dimensional inner-product distribution with mean \( \mu = \mathbb{E}X_1 \), and suppose that there exists a constant \( \eta > 0 \) such that \( \eta < x^T y < 1 - \eta \) for all \( x, y \in \text{supp} \, F \). Let \((A, X) \sim \text{RDPG}(F, n)\), let \( v \) be an out-of-sample vertex with latent position \( \bar{w} \in \text{supp} \, F \), and let \( \tilde{w} = \bar{w}/\sqrt{\mu^T \bar{w}} \) be the Laplacian spectral embedding of this latent position. Then there exists a sequence of orthogonal matrices \( Q \in \mathbb{R}^{d \times d} \) such that

\[
\| Q\tilde{w}_{LS} - \bar{w} \| \leq C n^{-1} \log^{1/2} n,
\]

and this matrix \( Q \) is the same one guaranteed by Lemma 1.

Proof. Letting \( \hat{w}_{LS} \) denote the LLS OOS solution if we had access to the true latent positions, the triangle inequality and unitary invariance of Euclidean norm bound

\[
\| Q\hat{w}_{LS} - \bar{w} \| \leq \| Q\hat{w}_{LS} - \hat{w}_{LS} \| + \| \hat{w}_{LS} - \bar{w} \|.
\]

Both of these terms can be bounded using standard concentration inequalities and properties of linear least-squares solutions. A detailed proof is given in Appendix D.

3.2 Central limit theorems for the OOS extensions

We now turn our attention to the question of the asymptotic distribution of the OOS extensions introduced in Section 2. Once again, we state the results for the case of Bernoulli edges, but similar results can be shown for a broader class of edge noise models, provided that noise model and the latent position distribution \( F \) obey suitable moment conditions.

Theorem 4. Let \( F \) be a \( d \)-dimensional inner-product distribution and suppose that \((A, X) \sim \text{RDPG}(F, n)\) and let \( v \) be the out-of-sample vertex with latent position \( \bar{w} \in \text{supp} \, F \). Let \( \hat{w}_{LS} \) be the least-squares OOS extension as defined in Equation (7). Then there exists a sequence of orthogonal \( d \)-by-\( d \) matrices \( Q \) such that

\[
\sqrt{n}(Q\hat{w}_{LS} - \bar{w}) \overset{\mathcal{D}}{\to} \mathcal{N}(0, \Sigma_{F,\bar{w}}),
\]

where for any \( w \in \text{supp} \, F \), we define

\[
\Sigma_{F, w} = \Delta^{-1} \mathbb{E} \left[ X_1^T w(1 - X_1^T w)X_1X_1^T \right] \Delta^{-1},
\]

and \( \Delta = \mathbb{E}X_1X_1^T \) is the second moment matrix of \( F \).

Proof. This theorem follows by writing the ASE least-squares OOS extension as a sum of two vectors, one of which converges in probability to 0 using arguments similar to Theorem 1 and the other of which converges in distribution to a normal, and applying Slutsky’s lemma. A detailed proof can be found in Appendix E.

If the latent position \( \tilde{w} \) of the OOS vertex \( v \) is itself distributed according to \( F \), integrating \( \bar{w} \) above with respect to \( F \) yields the following corollary.

Corollary 1. Assume the same setup as Theorem 4 but suppose that the true latent position of the out-of-sample vertex \( v \) is given by \( \bar{w} \sim F \), independent of \((A, X)\). Then there exists a sequence of orthogonal matrices \( Q \in \mathbb{R}^{d \times d} \) such that

\[
\sqrt{n}Q\hat{w}_{LS} \overset{\mathcal{D}}{\to} \int \mathcal{N}(w, \Sigma_{F, w})dF(w),
\]

where \( \Sigma_{F, w} \) is as defined in Equation (11). That is, \( \sqrt{n}Q\hat{w}_{LS} \) converges in distribution to a mixture of normals with mixing distribution \( F \).
Turning our attention to the LSE, we can obtain a similar CLT result for the LSE OOS extension, once we adjust for the fact that the LSE does not estimate the latent position \( \bar{w} \) but instead estimates the vector \( \tilde{w} = \bar{w}/\sqrt{n \mu^T \bar{w}} \), where \( \mu \in \mathbb{R}^d \) is the mean of the inner-product distribution \( F \). We note that the scaling of \( \tilde{w} \) by the square root of the expected degree means that we must scale by \( n \) instead of the \( \sqrt{n} \) scaling in the ASE CLTs above.

**Theorem 5.** Let \( F \) be a \( d \)-dimensional inner-product distribution for which there exists a constant \( \eta > 0 \) such that \( \eta \leq x^T y \leq 1 - \eta \) whenever \( x, y \in \text{supp} F \). Let \((A, X) \sim \text{RDPG}(F, n)\) and let \( v \) be the out-of-sample vertex with latent position \( \bar{w} \in \text{supp} F \). Let \( \tilde{w}_{LS} \in \mathbb{R}^d \) denote the least-squares OOS extension of LSE as defined in Equation (10). Then there exists a sequence of orthogonal matrices \( \tilde{Q} \in \mathbb{R}^{d \times d} \) such that

\[
n(\tilde{Q}\tilde{w}_{LS} - \tilde{w}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tilde{\Sigma}_{F, \tilde{w}}),
\]

where for any \( w \in \text{supp} F \) we define

\[
\tilde{\Sigma}_{F, \tilde{w}} = \mathbb{E} \left[ X_J^T \tilde{w} (1 - X_J^T \tilde{w}) \mu^T \tilde{w} \left( \tilde{\Delta}^{-1} X_J \frac{\tilde{w}}{2 \mu^T \tilde{w}} - \frac{\tilde{w}}{2 \mu^T \tilde{w}} \left( \tilde{\Delta}^{-1} X_J \frac{\tilde{w}}{2 \mu^T \tilde{w}} - \frac{\tilde{w}}{2 \mu^T \tilde{w}} \right)^T \bigg) \right],
\]

with \( \tilde{\Delta} = \mathbb{E} X_1 X_1^T / \mu^T X_1 \).

**Proof.** The proof follows similarly to that of Theorem 4 though it requires a more careful analysis to control convergence of the degrees. Details are given in Appendix F.

### 4 Experiments

In this section, we briefly explore our results through simulations. We leave a more thorough experimental examination of our results, particularly as they apply to real-world data, for future work. We first give a brief exploration of how quickly the asymptotic distribution in Theorem 4 becomes a good approximation. Toward this end, let us consider a simple mixture of point masses, \( F = F_{\lambda, x_1, x_2} = \lambda \delta_{x_1} + (1 - \lambda) \delta_{x_2} \), where \( x_1, x_2 \in \mathbb{R}^2 \) and \( \lambda \in (0, 1) \). This corresponds to a two-block stochastic block model (Holland et al. 1983), in which the block probability matrix is given by

\[
\begin{bmatrix}
  x_1^T x_1 & x_1^T x_2 \\
  x_1^T x_2 & x_2^T x_2
\end{bmatrix}
\]

Corollary 1 implies that if all latent positions (including the OOS vertex) are drawn according to \( F \), then the OOS estimate should be distributed as a mixture of normals centered at \( x_1 \) and \( x_2 \), with respective mixing coefficients \( \lambda \) and \( 1 - \lambda \).

To assess how well the asymptotic distribution predicted by Theorem 4 and Corollary 1 holds, we generate RD\( \text{PGs} \) with latent positions drawn i.i.d. from distribution \( F = F_{\lambda, x_1, x_2} \) defined above, with

\[
\lambda = 0.4, \quad x_1 = (0.2, 0.7)^T \text{, and } x_2 = (0.65, 0.3)^T.
\]

For each trial, we draw \( n + 1 \) independent latent positions from \( F \), and generate a binary adjacency matrix from these latent positions. We let the \((n + 1)\)-th vertex be the OOS vertex. Retaining the subgraph induced by the first \( n \) vertices, we obtain an estimate \( \hat{X} \in \mathbb{R}^{n \times 2} \) via ASE, from which we obtain an estimate for the OOS vertex via the LS OOS extension as defined in (7). We remind the reader that for each RD\( \text{PG} \) draw, we initially recover the latent positions only up to a rotation. Thus, for each trial, we compute a Procrustes alignment (Gower and Dijksterhuis 2004) of the in-sample estimates \( \hat{X} \) to their true latent positions. This yields a
Figure 1: Observed distribution of the LLS OOS estimate for 100 independent trials for number of
vertices $n = 50$ (left), $n = 100$ (middle) and $n = 500$ (right). Each plot shows the positions of 100
independent OOS embeddings, indicated by crosses, and colored according to cluster membership.
Contours indicate two generalized standard deviations of the multivariate normal (i.e., 68% and
95% of the probability mass) about the true latent positions, which are indicated by solid circles.
We note that even with merely 100 vertices, the normal approximation is already quite reasonable.

rotation matrix $R$, which we apply to the OOS estimate. Thus, the OOS estimates are sensibly
comparable across trials. Figure 1 shows the empirical distribution of the OOS embeddings of
100 independent RDPG draws, for $n = 50$ (left), $n = 100$ (center) and $n = 500$ (right) in-sample
vertices. Each cross is the location of the OOS estimate for a single draw from the RDPG with
latent position distribution $F$, colored according to true latent position. OOS estimates with
true latent position $x_1$ are plotted as blue crosses, while OOS estimates with true latent position
$x_2$ are plotted as red crosses. The true latent positions $x_1$ and $x_2$ are plotted as solid circles,
colored accordingly. The plot includes contours for the two normals centered at $x_1$ and $x_2$
predicted by Theorem 4 and Corollary 1, with the ellipses indicating the isoclines corresponding
to one and two (generalized) standard deviations.

Examining Figure 1, we see that even with only 100 vertices, the mixture of normal distri-
butions predicted by Theorem 4 holds quite well, with the exception of a few gross outliers from
the blue cluster. With $n = 500$ vertices, the approximation is particularly good. Indeed, the
$n = 500$ case appears to be slightly under-dispersed, possibly due to the Procrustes alignment.
It is natural to wonder whether a similarly good fit is exhibited by the ML-based OOS extension.
We conjectured at the end of Section 3 that a CLT similar to that in Theorem 4 would also hold
for the ML-based OOS extension as defined in Equation (9). Figure 2 shows the empirical distri-
bution of 100 independent OOS estimates, under the same experimental setup as Figure 1, but
using the ML OOS extension rather than the linear least-squares extension. The plot supports
our conjecture that the ML-based OOS estimates are also approximately normally distributed
about the true latent positions. Broadly similar patterns hold for the same experiment applied
to the least-squares LSE OOS extension, as predicted by Theorem 5.

Figure 3 plots the same experiment as that performed in Figures 1 and 2, this time for the
linear least squares OOS extension of the Laplacian spectral embedding. Recall that Theorem 5
predicts that the out-of-sample extension should be asymptotically normally distributed about
the true (rescaled) latent position $\tilde{w} = \tilde{w}/\sqrt{n\tilde{w}^T\mu}$. Compared to the previous two experiments,
it is evident that the asymptotics are slightly slower to kick in, but modulo the same Procrustes-
induced underdispersion observed previously, the theorem appears to hold quite well with $n =
500$ vertices.

Figure 4 suggests that we may be confident in applying the large-sample approximation
suggested by Theorem 4 and Corollary 1. Applying this approximation allows us to investigate
the trade-offs between computational cost and classification accuracy, to which we now turn our
Distribution of ML OOS estimates as function of graph size

Figure 2: Observed distribution of the ML OOS estimate for 100 independent trials for number of vertices $n = 50$ (left), $n = 100$ (middle) and $n = 500$ (right). Each plot shows the positions of 100 independent OOS embeddings, indicated by crosses, and colored according to cluster membership. Contours indicate two generalized standard deviations of the multivariate normal about the true latent positions, which are indicated by solid circles. Once again, even with merely 100 vertices, the normal approximation is already quite reasonable, supporting our conjecture that the ML OOS estimates also distributed as a mixture of normals according to the latent position distribution $F$.

Distribution of LSE LLS OOS estimates as function of graph size

Figure 3: Observed distribution of the LSE OOS estimate for 100 independent trials for number of vertices $n = 50$ (left), $n = 100$ (middle) and $n = 500$ (right). Each plot shows the positions of 100 independent OOS embeddings, indicated by crosses, and colored according to cluster membership. Contours indicate two generalized standard deviations of the multivariate normal about the true latent positions, which are indicated by solid circles.

attention. The mixture distribution $F_{\lambda,x_1,x_2}$ above suggests a task in which, given an adjacency matrix $A$, we wish to classify the vertices according to which of two clusters or communities they belong. That is, we will view two vertices as belonging to the same community if their latent positions are the same (Holland et al. 1983, i.e., the latent positions specify an SBM). More generally, one may view the task of recovering vertex block memberships in a stochastic block model as a clustering problem. Lyzinski et al. (2014) showed that applying ASE to such a graph, followed by $k$-means clustering of the estimated latent positions, correctly recovers community memberships of all the vertices (i.e., correctly assigns all vertices to their true latent positions) with high probability.

For concreteness, let us consider a still simpler mixture model, $F = F_{\lambda,p,q} = \lambda\delta_p + (1 - \lambda)\delta_q$, where $0 < p < q < 1$, and draw an RDPG $(\tilde{A}, X) \sim \text{RDPG}(F, n+m)$, taking the first $n$ vertices
to be in-sample, with induced adjacency matrix $A \in \mathbb{R}^{n \times n}$. That is, we draw the full matrix

$$\tilde{A} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where $C \in \mathbb{R}^{n \times m}$ is the adjacency matrix of the subgraph induced by the $m$ OOS vertices and $B \in \mathbb{R}^{n \times m}$ encodes the edges between the in-sample vertices and the OOS vertices. The latent positions $p$ and $q$ encode a community structure in the graph $\tilde{A}$, and, as alluded to above, a common task in network statistics is to recover this community structure. Let $\tilde{w}^{(1)}, \tilde{w}^{(2)}, \ldots, \tilde{w}^{(m)} \in \{p, q\}$ denote the true latent positions of the $m$ OOS vertices, with respective least-squares OOS estimates $\hat{w}^{(1)}_{LS}, \hat{w}^{(2)}_{LS}, \ldots, \hat{w}^{(m)}_{LS}$, each obtained from the in-sample ASE $\tilde{X} \in \mathbb{R}^n$ of $A$. We note that one could devise a different OOS embedding procedure that makes use of the subgraph $C$ induced by these $m$ OOS vertices, but we leave the development of such a method to future work. Corollary 1 implies that each $\hat{w}^{(t)}_{LS}$ for $t \in [m]$ is marginally (approximately) distributed as

$$\hat{w}^{(t)}_{LS} \sim \lambda N(p, (n + 1)^{-1}\sigma_p^2) + (1 - \lambda)N(q, (n + 1)^{-1}\sigma_q^2),$$

where

$$\sigma_p^2 = \Delta^{-2} (\lambda p^2 (1 - p^2)p^2 + (1 - \lambda)pq(1 - pq)q^2),$$

$$\sigma_q^2 = \Delta^{-2} (\lambda pq(1 - pq)p^2 + (1 - \lambda)q^2(1 - q^2)q^2),$$

and $\Delta = \lambda p^2 + (1 - \lambda)q^2$.

Classifying the $t$-th OOS vertex based on $\hat{w}^{(t)}_{LS}$ via likelihood ratio thus has (approximate) probability of error

$$\eta_{n,p,q} = \lambda(1 - \Phi) \left( \frac{\sqrt{n} + 1(x_{n+1,p,q} - p)}{\sigma_p} \right) + (1 - \lambda)\Phi \left( \frac{\sqrt{n} + 1(x_{n+1,p,q} - q)}{\sigma_q} \right),$$

where $\Phi$ denotes the cdf of the standard normal and $x_{n,p,q}$ is the value of $x$ solving

$$\lambda \sigma_p^{-1} \exp\{n(x - p)^2/(2\sigma_p^2)\} = (1 - \lambda)\sigma_q^{-1} \exp\{n(x - q)^2/(2\sigma_q^2)\},$$

and hence our overall error rate when classifying the $m$ OOS vertices will grow as $m \eta_{n+1,p,q}$.

As discussed previously, the OOS extension allows us to avoid the expense of computing the ASE of the full matrix

$$\tilde{A} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

The LLS OOS extension is computationally inexpensive, requiring only the computation of the matrix-vector product $\tilde{S}^{-1/2}U^T\tilde{a}$, with a time complexity $O(d^2n)$ (assuming one does not precompute the product $\tilde{S}^{-1/2}U^T$). The eigenvalue computation required for embedding $\tilde{A}$ is far more expensive than the LLS OOS extension. Nonetheless, if one were intent on reducing the OOS classification error $\eta_{n+1,p,q}$, one might consider paying the computational expense of embedding $\tilde{A}$ to obtain estimates $\tilde{w}^{(1)}, \tilde{w}^{(2)}, \ldots, \tilde{w}^{(m)}$ of the $m$ OOS vertices. That is, we obtain estimates for the $m$ OOS vertices by making them in-sample vertices, at the expense of solving an eigenproblem on the $(m + n)$-by-$(m + n)$ adjacency matrix. Of course, the entire motivation of our approach is that the in-sample matrix $A$ may not be available. Nonetheless, a comparison against this baseline, in which all data is used to compute our embeddings, is instructive.

Theorem 1 in [Athreya et al. (2016)] implies that the $\hat{w}^{(t)}$ estimates based on embedding the full matrix $A$ are (approximately) marginally distributed as

$$\hat{w}^{(t)} \sim \lambda N(p, (n + m)^{-1}\sigma_p^2) + (1 - \lambda)N(q, (n + m)^{-1}\sigma_q^2),$$

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Figure 4: Ratio of the OOS classification error to the in-sample classification error as a function of the number of OOS vertices \( m \), for \( n = 100 \) vertices, \( n = 1000 \) vertices and \( n = 10000 \) vertices. We see that for \( m \leq 100 \), the expensive in-sample embedding does not improve appreciably on the OOS classification error. However, when many hundreds or thousands of OOS vertices are available simultaneously (i.e., \( m \geq 100 \)), we see that the in-sample embedding may improve upon the OOS estimate by a significant multiplicative factor.

with classification error

\[
\eta_{n+m,p,q} = \lambda \Phi \left( \frac{p - x_{n+m,p,q}}{\sigma_p} \right) + (1 - \lambda) \Phi \left( \frac{x_{n+m,p,q} - q}{\sigma_q} \right),
\]

where \( x_{n+m,p,q} \) is the value of \( x \) solving

\[
\lambda \sigma_p^{-1} \exp \left\{ (m + n)(x - p)^2/(2\sigma_p^2) \right\} = (1 - \lambda) \sigma_q^{-1} \exp \left\{ (m + n)(x - q)^2/(2\sigma_q^2) \right\},
\]

and it can be checked that \( \eta_{n+m,q,p} < \eta_{n,q,p} \) when \( m > 1 \). Thus, at the cost of computing the ASE of \( \tilde{A} \), we may obtain a better estimate. How much does this additional computation improve classification the OOS vertices? Figure 4 explores this question.

Figure 4 compares the error rates of the in-sample and OOS estimates as a function of \( m \) and \( n \) in the model just described, with \( \lambda = 0.4 \), \( p = 0.6 \) and \( q = 0.61 \). The plot depicts the ratio of the (approximate) in-sample classification error \( \eta_{n+m,p,q} \) to the (approximate) OOS classification error \( \eta_{n+1,p,q} \), as a function of the number of OOS vertices \( m \), for differently-sized in-sample graphs, \( n = 100, 1000, \) and 10000. We see that over several magnitudes of graph size, the in-sample embedding does not improve appreciably over the OOS embedding except when multiple hundreds of OOS vertices are available. When hundreds or thousands of OOS vertices are available simultaneously, we see in the right-hand side of Figure 4 that the in-sample embedding classification error may improve upon the OOS classification error by a large multiplicative factor. Whether or not this improvement is worth the additional computational expense will, depend upon the available resources and desired accuracy, but this suggests that the additional expense associated with performing a second ASE computation is only worthwhile in the event that hundreds or thousands of OOS vertices are available simultaneously. This surfeit of OOS vertices is rather divorced from the typical setting of OOS extension problems, where one typically wishes to embed at most a few previously unseen observations.

5 Discussion and Conclusion

We have presented theoretical results for out-of-sample extensions of graph embeddings, the adjacency spectral embedding and the Laplacian spectral embedding. In both cases, we have
shown that under the random dot product graph, a least squares-based OOS extension recovers the true latent position at the same rate as the more expensive in-sample embedding. Further, this linear least squares OOS extension obeys a CLT, whereby the OOS embedding is normally distributed about the true latent position. We have also presented results for an ASE OOS extension based on a maximum-likelihood objective function showing that this embedding recovers the true out-of-sample latent position at the same rate as the in-sample embedding. Experiments suggest that convergence to the predicted normal distribution is fairly fast, being a good approximation with only a few hundred vertices. Finally, we have briefly investigated how the approximation introduced by these OOS extensions might be traded off against the computational expense associated with computing the more expensive full graph embedding by investigating how the approximate classification error predicted by our CLT depends on the size of the size of the in-sample and the number of out-of-sample vertices.

The results in this work suggest a number of interesting directions for future work, a few of which we briefly enumerate here. Firstly, though all of the OOS extensions presented in this paper match the asymptotic estimation error rates of their respective in-sample embeddings, our results say little about the constants associated with those rates or about finite-sample behavior of those OOS extensions (aside from their obvious restatements as finite-sample results alluded to briefly in Section 1.2). A more thorough investigation of how these different OOS extensions behave for different sizes of the in-sample graph and for different latent position distributions $F$ would be of particular interest to practitioners faced with choosing between these different embeddings and OOS extensions as they apply to real data. Our discussion surrounding Figure 4 makes an initial step in this direction, but only suggests rules of thumb for when the speed/accuracy trade-off associated with out-of-sample extension is likely to be favorable.

A related line of questioning concerns how one should, when possible, select the in-sample vertices so as to yield optimal (as measured by, e.g., vertex classification or estimation accuracy of the latent positions) out-of-sample embeddings. Consider the setting where one has a graph $\tilde{G}$ of size $\tilde{n} = n + m$ that is far too large to be embedded via ASE or LSE. If $n$ is the largest number of vertices that can be feasibly embedded as a full in-sample graph, it is natural to choose $n$ vertices from $\tilde{G}$ to serve as the in-sample vertices, and embed the remaining $m$ vertices via one of the out-of-sample extensions discussed in this paper. In this setting, how should one choose these $n$ vertices from $\tilde{G}$? Problems of a similar nature have been considered elsewhere in the literature under the heading of anchor graphs or choosing anchor points (see, e.g., [Liu et al. 2010]), but we are not aware of any work in this area as it pertains to the ASE and LSE. This also suggests the problem of how best to embed $m$ out-of-sample vertices jointly, rather than applying an OOS extension to each of them in isolation, particularly in the setting where we have access to the subgraph induced by these $m$ out-of-sample vertices. Of most import here is the question, also explored by Figure 4 of how large the out-of-sample size $m$ must be before one should prefer the expense of the full-graph embedding, and whether an embedding that makes use of this out-of-sample induced graph might bridge the gap between these two extremes by providing an embedding which, while more expensive than performing $m$ OOS extensions in isolation, is still far less computationally intensive than embedding a graph of size $m+n$. A more thorough exploration of this trade-off from both a theoretical and empirical standpoint is the subject of on-going work.

A Technical Results for the Random Dot Product Graph

Here we collect a number of basic results that will be useful in our subsequent proofs of the main theorems. Most of the results in this section are adapted from existing results in Levin et al. (2017), Lyzinski et al. (2014) and Tang and Priebe (2018). We refer the interested reader to Athreya et al. (2018) for a more thorough overview of the RDPG and the statistical problems
that arise in relation to it.

**Lemma 2 (Levin et al. (2017), Observation 2).** Let \((A, X) \sim \text{RDPG}(F, n)\) for some \(d\)-dimensional inner product distribution \(F\). There exists constants \(0 < C_1 < C_2\), depending only on \(F\), such that with probability 1 it holds for all suitably large \(n\) that

\[
C_1 n \leq \lambda_d(P) \leq \lambda_1(P) \leq C_2 n
\]

and

\[
C_1 \sqrt{n} \leq \lambda_d(X) \leq \lambda_1(X) \leq C_2 \sqrt{n}.
\]

**Lemma 3 (Levin et al. (2017), Lemma 3).** With notation as above, let \(V_1 \Lambda V_2^T\) be the SVD of \(U^T \bar{U} \in \mathbb{R}^{d \times d}\), and define \(Q = V_1 V_2^T\). Then

\[
\|U^T \bar{U} - Q\|_F = O(n^{-1} \log n).
\]

**Lemma 4 (Tang and Priebe (2018), Proposition B.2).** With notation as above, let \(\hat{V}_1 \hat{\Lambda} \hat{V}_2^T\) be the SVD of \(\hat{U}^T \hat{U} \in \mathbb{R}^{d \times d}\) and define \(\hat{Q} = \hat{V}_1 \hat{V}_2^T\). Then

\[
\|\hat{U}^T \hat{U} - \hat{Q}\|_F = O(n^{-1}).
\]

**Lemma 5 (Lyzinski et al. (2017) Lemma 15; Tang and Priebe (2018) Lemma B.3).** With notation as above,

\[
\|\hat{U}^T \hat{U} S^{1/2} - \hat{S}^{1/2} \hat{U}^T \hat{U}\| = O(n^{-1}),
\]

\[
\|\hat{U}^T \hat{U} S^{-1/2} - \hat{S}^{-1/2} \hat{U}^T \hat{U}\|_F = O(n^{-3/2} \log n)
\]

and

\[
\|\hat{U}^T \hat{U} S^{1/2} - \hat{S}^{1/2} \hat{U}^T \hat{U}\|_F = O(n^{-1/2} \log n).
\]

**Lemma 6.** Let \(F\) be a \(d\)-dimensional inner-product distribution and let \((A, X) \sim \text{RDPG}(F, n)\), and let \(v\) be the out-of-sample vertex with latent position \(\bar{w} \in \text{supp} F\). For \(i \in [n]\), let \(d_i = \sum_j A_{i,j}\) denote the degree of vertex \(i\) and \(t_i = \sum_j X_j^T X_i = \mathbb{E}[d_i | X]\) denote its expectation conditional on the latent positions. Analogously, let \(d_v = \sum_j a_j\) denote the degree of the out-of-sample vertex and \(t_v = \sum_j X_j^T \bar{w}\) denote its expectation. Then

\[
\max \{ |d_i - t_i| : i \in [n] \cup \{v\} \} = O(\sqrt{n} \log^{1/2} n).
\]

Similarly, letting \(\mu = \mathbb{E} X_1 \in \mathbb{R}^d\) denote the mean of latent position distribution \(F\) and taking \(X_v = \bar{w}\),

\[
\max \{ |t_i - n \mu^T X_i| : i \in [n] \cup \{v\} \} = O(\sqrt{n} \log^{1/2} n).
\]

Further, uniformly over all \(i \in [n]\),

\[
|d_i^{-1/2} - t_i^{-1/2}| = O(n^{-1} \log^{1/2} n),
\]

\[
|d_i^{-1} - t_i^{-1}| = O(n^{-3/2} \log^{1/2} n),
\]

\[
t_i = \Theta(n)
\]

\[
(17)
\]

**Proof.** Fix some \(i \in [n] \cup \{v\}\). By definition, we have

\[
d_i - t_i = \begin{cases} \sum_{j \neq i} (A_{i,j} - P_{i,j}) & \text{if } i \in [n] \\
\sum_{j=1}^n a_j - X_j^T \bar{w} & \text{if } i = v,
\end{cases}
\]

a sum of independent random variables, each contained in \([-1, 1]\) and thus Hoeffding’s inequality immediately yields

\[
\Pr[|d_i - t_i| \geq s] \leq 2 \exp \left\{ \frac{-2s^2}{n} \right\}
\]

\[
(18)
\]
for any \( s \geq 0 \). Taking \( s = C\sqrt{n}\log^{1/2}n \) for suitably large constant \( C > 0 \), we have

\[
\Pr \left[ |d_i - t_i| \geq C\sqrt{n}\log^{1/2}n \right] \leq C'n^{-3}.
\]

Taking a union bound over all \( i \in [n] \cup \{v\} \), we conclude that

\[
\Pr \left[ \exists i : |d_i - t_i| \geq C\sqrt{n}\log^{1/2}n \right] \leq Cn^{-2},
\]

and an application of the Borel-Cantelli Lemma \cite{Billingsley1995} yields Equation (13).

Again by definition, we have for any \( i \in [n] \cup \{v\} \),

\[
t_i - nX_i^T\mu = X_i^T(X_i - \mu) + \sum_{j \neq i} X_i^T(X_j - \mu).
\]

The first term on the right-hand side is \( O(1) \), since \( X_i \sim F \) and \( \mu \) is constant. The sum over \( j \neq i \) is, conditioned on \( X_i \), a sum of independent unbiased random variables, which are bounded by the assumption that \( 0 \leq x^Ty \leq 1 \) whenever \( x, y \in \text{supp } F \). Thus, an application of Hoeffding’s inequality similar to that above yields that, conditioned on \( X_i = x_i \in \text{supp } F \),

\[
\sum_{j \neq i} x_i^T(X_j - \mu) \leq C\sqrt{n}\log^{1/2}n,
\]

where the constant \( C \) can be chosen independent of \( x_i \) again because \( \text{supp } F \) is bounded. Unconditioning establishes Equation (14), since \( X_i^T(X_i - \mu) = O(1) \). Equation (17) follows, since \( t_i = nX_i^T\mu + O(\sqrt{n}\log^{1/2}n) \). Writing

\[
\left| \frac{1}{\sqrt{d_i}} - \frac{1}{\sqrt{t_i}} \right| = \frac{|d_i - t_i|}{\sqrt{d_i}\sqrt{t_i}(\sqrt{d_i} + \sqrt{t_i})}
\]

and applying Equations (14) and (17) implies (15). A similar argument establishes (16). \( \square \)

**Lemma 7.** Let \( P = XX^T \in \mathbb{R}^{n \times n} \) with rows of \( X \) drawn i.i.d. from \( F \) as above. Then

\[
\lambda_d(\mathcal{L}(P)) = \Theta(1), \quad \lambda_1(\mathcal{L}(P)) = \Theta(1) \text{ and } \lambda_d(\tilde{X}) = \Theta(1).
\]

**Proof.** By definition, \( \mathcal{L}(P) = T^{-1/2}USU^TT^{-1/2} \), so that

\[
\lambda_d(\mathcal{L}(P)) \leq \lambda_1(\mathcal{L}(P)) \leq \|\mathcal{L}(P)\| \leq \|T^{-1/2}\|\|S\|\|T^{-1/2}\| \leq \frac{\|S\|}{\min_i t_i} \leq C,
\]

where the last inequality follows from Lemmas 2 and 6.

To show the corresponding lower-bound, we adapt an argument from the proof of Theorem 8.1.17 in \cite{GolubVanLoan2012} to write

\[
\lambda_2^2(T^{1/2})\lambda_d(\mathcal{L}(P)) \geq \lambda_d(P) \geq Cn,
\]

where the second lower-bound follows from Lemma 2. We conclude that

\[
\lambda_1(\mathcal{L}(P)) \geq \lambda_d(\mathcal{L}(P)) \geq \frac{Cn}{\lambda_1(T)} \geq C,
\]

since \( \lambda_2^2(T^{1/2}) = \lambda_1(T) \leq n \).

By definition of \( \tilde{X} \), \( \lambda_k(\tilde{X}) = \sqrt{\lambda_k(\mathcal{L}(P))} \) for all \( k \in [d] \), whence \( \lambda_d(\tilde{X}) = \Theta(1) \). \( \square \)
Lemma 8. Let $F$ be a $d$-dimensional inner-product distribution with mean $\mu$ and suppose that there exists a constant $\eta > 0$ such that $\eta \leq x^T y \leq 1 - \eta$ for all $x, y \in \text{supp } F$. Define $\Delta = \mathbb{E} X_1 X_1^T / X_1^T \mu$ where $X_1 \sim F$ and let $\tilde{S} = \tilde{X}^T \tilde{X}$ and $\hat{S} = \hat{X}^T \hat{X}$. Then

$$\|\tilde{Q}\tilde{S}\tilde{Q}^T - \hat{S}\| = O\left(\frac{1}{n}\right) \text{ and } \|\hat{S} - \tilde{S}\| = O\left(\frac{\log^{1/2} n}{\sqrt{n}}\right).$$

Proof. Adding and subtracting appropriate quantities and applying a triangle inequality followed by submultiplicativity, we have

$$\|\tilde{Q}\tilde{S}\tilde{Q}^T - \hat{S}\| = \left\|\tilde{Q}\tilde{S}^{1/2} \left(\tilde{S}^{1/2}\tilde{Q}^T - \tilde{Q}^T \tilde{S}^{1/2}\right) + \left(\tilde{Q}\tilde{S}^{1/2} - \tilde{S}^{1/2}\tilde{Q}\right) \tilde{Q}^T \tilde{S}^{1/2}\right\|$$

$$\leq \left(\|\tilde{Q}\tilde{S}^{1/2}\| + \|\tilde{Q}^T \tilde{S}^{1/2}\|\right) \|\tilde{Q}\tilde{S}^{1/2} - \tilde{S}^{1/2}\tilde{Q}\|,$$

where we have used the unitary invariance of the spectral norm to write

$$\|\tilde{Q}\tilde{S}\tilde{Q}^T - \hat{S}\| = \|\tilde{S}^{1/2}\tilde{Q}^T - \tilde{Q}^T \tilde{S}^{1/2}\|.$$ 

An additional application of the unitary invariance of the spectral norm yields

$$\|\tilde{Q}\tilde{S}\tilde{Q}^T - \hat{S}\| \leq \left(\|\tilde{S}^{1/2}\| + \|\tilde{S}^{1/2}\|\right) \|\tilde{Q}\tilde{S}^{1/2} - \tilde{S}^{1/2}\tilde{Q}\|.$$ \hspace{1cm} (19)

By definition of $\tilde{S}$ and $\hat{S}$ as the top $d$ eigenvalues of $\mathcal{L}(A)$ and $\mathcal{L}(P)$, respectively, we have

$$\|\tilde{S} - \hat{S}\| \leq \|\mathcal{L}(A) - \mathcal{L}(P)\|.$$

Theorem 3.1 in Oliveira (2010) implies that

$$\|\mathcal{L}(A) - \mathcal{L}(P)\| \leq C \left(\min_i t_i\right)^{-1/2} \log^{1/2} n,$$

and Lemma 6 implies that $\min_i t_i = \Omega(n)$, so that

$$\|\tilde{S} - \hat{S}\| = O(n^{-1/2} \log^{1/2} n),$$

and it follows that

$$\|\tilde{S}^{1/2}\| \leq \|\hat{S}^{1/2}\| (1 + o(1)).$$

Lemma 2 bounds the growth of $\|\tilde{S}\|$ as $O(1)$, whence $\|\tilde{S}^{1/2}\| = O(1)$ and we conclude that

$$\|\tilde{S}^{1/2}\| + \|\hat{S}^{1/2}\| = O(1).$$ \hspace{1cm} (20)

Once again adding and subtracting appropriate quantities, applying the triangle inequality followed by submultiplicativity,

$$\|\tilde{Q}\tilde{S}^{1/2} - \hat{S}^{1/2}\| \leq \|(\tilde{Q} - \tilde{U}^T \tilde{U}) \tilde{S}^{1/2}\| + \|\tilde{U}^T \tilde{U} \tilde{S}^{1/2} - \tilde{S}^{1/2}\| + \|\tilde{S}^{1/2}(\tilde{U}^T \tilde{U} - \tilde{Q})\|$$

$$\leq \left(\|\tilde{S}^{1/2}\| + \|\tilde{S}^{1/2}\|\right) \|\tilde{Q} - \tilde{U}^T \tilde{U}\| + \|\tilde{U}^T \tilde{U} \tilde{S}^{1/2} - \tilde{S}^{1/2}\| \|\tilde{U}^T \tilde{U}\|.$$ 

Equation (20) and Lemma 3 imply that

$$\left(\|\tilde{S}^{1/2}\| + \|\tilde{S}^{1/2}\|\right) \|\tilde{Q} - \tilde{U}^T \tilde{U}\| = O(n^{-1}),$$

and Lemma 5 implies that

$$\|\tilde{U}^T \tilde{U} \tilde{S}^{1/2} - \tilde{S}^{1/2}\| = O(n^{-1}).$$
Combining the above two displays, we conclude that
\[ \|\tilde{Q}\tilde{S}^{1/2} - \tilde{S}^{1/2}\tilde{Q}\| = O(n^{-1}). \]
Applying this and Equation (20) to Equation (19), we conclude that \( \|\tilde{Q}\tilde{S}\tilde{Q}^T - \tilde{S}\| = O(n^{-1}). \)
To bound \( \|\tilde{S} - \tilde{\Delta}\| \), note that
\[
\tilde{S} = \sum_{i=1}^{n} X_iX_i^T = \sum_{i=1}^{n} X_i X_i^T.
\]
Applying Lemma 6, \( \max_i |t_i^{-1} - (nX_i^T\mu)^{-1}| = O(n^{-3/2} \log^{1/2} n) \), and thus
\[
\tilde{S} = \frac{1}{n} \sum_{i=1}^{n} \frac{X_iX_i^T}{X_i^T\mu} + O(n^{-1/2} \log^{1/2} n).
\]

Hoeffding’s inequality applied to the sum implies \( \tilde{S} = \tilde{\Delta} + O(n^{-1/2} \log^{1/2} n) \), completing the proof.

\( \square \)

**Lemma 9.** Suppose that \( F \) is a d-dimensional inner-product distribution with \( X_1 \sim F \) for which \( \Delta = \mathbb{E}_F X_1X_1^T \in \mathbb{R}^{d \times d} \) is full rank. If \( X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F \), then with probability 1 there exists an \( n_0 \) such that \( X \in \mathbb{R}^{n \times d} \) has full column rank for all \( n \geq n_0 \).

**Proof.** Since the top \( d \) eigenvalues of \( P = XX^T \) are precisely the \( d \) eigenvalues of \( X^T X \), Lemma 2 implies that \( \lambda_d(X^T X) = \Omega(n) \). It follows that \( XX^T \in \mathbb{R}^{d \times d} \) is invertible for all suitably large \( n \).

\( \square \)

We now give a proof of the bound in Equation (9) in Lemma 1.

**Proof of Lemma 7.** Let \( \zeta_i \in \mathbb{R}^d \) denote the (transposed) \( i \)-th row of \( \tilde{X} - \tilde{X}\tilde{Q} \), where \( \tilde{Q} = \tilde{V}_1\tilde{V}_2^T \) as in Lemma 1 above. Define the event
\[
E_n = \left\{ \forall i \in [n] : \|\zeta_i\| \leq \frac{C \log^{1/2} n}{n} \right\}
\]
where \( C > 0 \) is a constant that we will specify below, depending on the latent position distribution \( F \) but not on \( n \). It will suffice for us to show that \( E_n \) holds eventually.

Fix some \( i \in [n] \) and define \( \mu = \mathbb{E}X_1 \in \mathbb{R}^d \) to be the mean of \( F \). Following the argument in Appendix B.1 of Tang and Priebe (2018), we have
\[
\zeta_i = \frac{(\tilde{X}^T \tilde{X})^{-1}}{n} \sqrt{n} \sum_{j \neq i} A_{i,j} - P_{i,j} \left( \frac{X_j}{X_j^T \mu} - \frac{\tilde{\Delta}X_j}{2X_i^T \mu} \right) + o(n^{-1}). \tag{21}
\]

For all \( j \in [n] \setminus \{i\} \), define
\[
Z_{j}^{(i)} = \frac{A_{i,j} - P_{i,j}}{\sqrt{n}} \left( \frac{X_j}{X_j^T \mu} - \frac{\tilde{\Delta}X_j}{2X_i^T \mu} \right).
\]

Condition on \( X_i = x_i \in \text{supp} \, F \) and fix \( k \in [d] \). Thanks to the assumption that \( 0 < \eta \leq x^T y \leq 1 - \eta \) whenever \( x, y \in \text{supp} \, F \), we have that \( \sum_{j \neq i} Z_{j,k}^{(i)} \) is a sum of independent 0-mean bounded random variables. Hoeffding’s inequality implies that
\[
\Pr\left[ \left| \sum_{j \neq i} Z_{j,k}^{(i)} \right| \geq s \mid X_i = x_i \right] \leq 2 \exp \left\{ \frac{-s^2}{2n^{-1} \sum_{j \neq i} V_{j,k}^2} \right\}, \tag{22}
\]

21
where

\[ V_{j,k} = \frac{X_{j,k}}{\mu_x} - \frac{(\tilde{\Delta} x_i)_k}{2x_i^\top \mu}. \]

Using the fact that \( X_j, x_i \in \text{supp } F \) and that \( X_j \) is independent of \( X_i \) for \( j \neq i \), we have

\[
\mathbb{E}\left[ V_{j,k}^2 \mid X_i = x_i \right] \leq C \left( \frac{\|\tilde{\Delta} x_i\|_2^2}{4(x_i^\top \mu)^2} + \mathbb{E}\left[ \frac{|X_{j,k}|^2}{(X_j^\top \mu)^2} \right] \right)^2 \leq C_F, \tag{23}
\]

where \( C_F \) depends on \( F \) but can be chosen independent of \( k \) and \( x_i \). By the law of large numbers (conditional on \( X_i = x_i \)),

\[
n^{-1} \sum_{j \neq i} V_{j,k}^2 \rightarrow \mathbb{E}[V_{j,k}^2 \mid X_i = x_i] \quad \text{almost surely.}
\]

Thus, applying Equation (23) and integrating out by \( X_i \),

\[
n^{-1} \sum_{j \neq i} V_{j,k}^2 \leq 2C_F \quad \text{eventually.}
\]

Integrating (22) with respect to \( F \) and using the above fact, we conclude that

\[
\text{Pr} \left[ \left\| \sum_{j \neq i} Z_{j,k}^{(i)} \right\| \geq C \log^{1/2} n \right] \leq 2n^{-3},
\]

for suitably large constant \( C > 0 \). A union bound over all \( k \in [d] \) yields

\[
\text{Pr} \left[ \left\| \sum_j Z_j^{(i)} \right\| \geq C \log^{1/2} n \right] \leq 2dn^{-3},
\]

and a further union bound over \( i \in [n] \) implies

\[
\max_{i \in [n]} \left\| \sum_{j \neq i} Z_{j,k}^{(i)} \right\| = O(\log^{1/2} n). \tag{24}
\]

Applying this result to Equation (21) and using the fact that \( \tilde{\Delta}^T \tilde{\Delta} \rightarrow \tilde{\Delta} \) almost surely and \( \sqrt{\mu_\nu t^{-1/2}} = O(1) \) by Lemmas 6 and 8 respectively, we have

\[
\max_{i \in [n]} \| \zeta_i \| \leq \frac{1}{n\| X^T X \|} \min_{i \in [n]} \sqrt{\nu} \max_{i \in [n]} \left\| \sum_{j \neq i} Z_j^{(i)} \right\| = O \left( \frac{\log^{1/2} n}{\sqrt{n}} \right), \tag{25}
\]

which completes the proof.

The following spectral norm bound will be useful at several points in our proofs.

**Theorem 6.** (Matrix Bernstein inequality, [Tropp 2015] ) Let \( \{Z_k\} \) be a finite collection of random matrices in \( \mathbb{R}^{d_1 \times d_2} \) with \( \mathbb{E} Z_k = 0 \) and \( \| Z_k \| \leq R \) for all \( k \), then

\[
\text{Pr} \left[ \left\| \sum_k Z_k \right\| \geq t \right] \leq (d_1 + d_2) \exp \left\{ -\frac{t^2}{\nu^2 + Rt/3} \right\},
\]

where

\[
\nu^2 = \max \left\{ \left\| \sum_k \mathbb{E} Z_k \right\|, \left\| \sum_k \mathbb{E} Z_k^T \right\| \right\}.
\]
B Proof of ASE LS-OOS Concentration Inequality

To prove Theorem 1, we must relate the least squares solution \( \hat{w}_{LS} \) of (7) to the true latent position \( \bar{w} \). We will proceed in two steps. First, we will show that \( \hat{w}_{LS} \) is close to a least squares solution based on the true latent positions \( \{ \bar{x}_i \}_{i=1}^n \) rather than on the estimates \( \{ \hat{x}_i \}_{i=1}^n \). That is, letting \( w_{LS} \) be the solution

\[
w_{LS} = \arg \min_{w \in \mathbb{R}^d} \|Xw - \bar{a}\|_F, \tag{26}
\]

we will bound the error introduced by the ASE, \( \|Q\hat{w}_{LS} - w_{LS}\| \), taking \( Q \in \mathbb{R}^{d \times d} \) to be as defined in Lemma 1. This is the content of Lemma 12. Second, we will show that \( w_{LS} \) is close to the true latent position \( \bar{w} \). That is, we will control the error introduced by the random in-sample latent positions and the network \( A \). This is done in Lemma 13. The triangle inequality will then yield Theorem 1.

We first establish a bound on \( \|Q\hat{w}_{LS} - w_{LS}\| \), where \( \hat{w}_{LS} \) is the solution to Equation (7), \( w_{LS} \) is as defined by Equation (26), and \( Q \in \mathbb{R}^{d \times d} \) is the orthogonal matrix guaranteed to exist by Lemma 1. Our bound will depend upon a basic result for solutions of perturbed linear systems, which we adapt from Golub and Van Loan (2012). In essence, we wish to compare \( \hat{w}_{LS} \) against \( w_{LS} \).

Recall that for a matrix \( B \in \mathbb{R}^{n \times d} \) of full column rank, we define the condition number

\[
\kappa_2(B) = \frac{\sigma_1(B)}{\sigma_d(B)}.
\]

**Theorem 7** (Golub and Van Loan (2012), Theorem 5.3.1). Suppose that the quantities \( w_{LS}, \hat{w}_{LS} \in \mathbb{R}^d \) and \( r_{LS}, \hat{r}_{LS} \in \mathbb{R}^n \) satisfy

\[
\|Xw_{LS} - \bar{a}\| = \min_w \|Xw - \bar{a}\|, \quad r_{LS} = \bar{a} - Xw_{LS},
\]

\[
\|X\hat{w}_{LS} - \bar{a}\| = \min_w \|Xw - \bar{a}\|, \quad \hat{r}_{LS} = \bar{a} - \hat{X}\hat{w}_{LS},
\]

and that

\[
\|\hat{X} - XQ\| < \lambda_d(X). \tag{27}
\]

Assume \( \bar{a}, r_{LS} \) and \( w_{LS} \) are all non-zero and define \( \theta_{LS} \in (0, \pi/2) \) by \( \sin \theta_{LS} = \|r_{LS}\|/\|\bar{a}\| \).

Letting

\[
\nu_{LS} = \frac{\|Xw_{LS}\|}{\sigma_d(XQ)\|Q^Tw_{LS}\|},
\]

we have

\[
\frac{\|\hat{w}_{LS} - Q^Tw_{LS}\|}{\|Q^Tw_{LS}\|} \leq \frac{\nu_{LS}}{\|XQ\|} \left( \frac{\nu_{LS}}{\cos \theta_{LS}} + (1 + \nu_{LS} \tan \theta_{LS}) \kappa_2(XQ) \right) + O \left( \frac{\|\hat{X} - XQ\|^2}{\|XQ\|^2} \right). \tag{28}
\]

To apply Theorem 7, we will first need to show that the condition in Equation (27) and the non-zero conditions on \( \bar{a}, r_{LS} \) and \( w_{LS} \) all hold with high probability. This is done in Lemma 10. We will then show, using Lemma 10 and Lemma 11 that the right-hand side of Equation (28) is \( O(n^{-1/2} \log n) \).
Lemma 10. With notation as above, \( \vec{a} \), \( r_{LS} \) and \( w_{LS} \) are all nonzero eventually, and (27) holds eventually. That is, with probability 1, there exists a sequence of orthogonal matrices \( Q \in \mathbb{R}^{d \times d} \) such that
\[
\| \hat{X} - XQ \| < \lambda_d(X) \text{ eventually} \tag{29}
\]
Further,
\[
\frac{\| \hat{X} - XQ \|}{\| XQ \|} = O \left( \frac{\log n}{\sqrt{n}} \right). \tag{30}
\]

Proof. That \( \vec{a} \) is non-zero eventually is an immediate consequence of the model, and it follows that \( w_{LS} \) is non-zero eventually, from which it follows that the residual \( r_{LS} = \vec{a} - Xw_{LS} \) is also nonzero eventually. Let \( Q \in \mathbb{R}^{d \times d} \) be the orthogonal matrix guaranteed by Lemma 10. With notation as above, \( \parallel \vec{r} \parallel \) is non-zero eventually is an immediate consequence of the model, and it follows from the above display.

Lemma 11. With notation as in Theorem 7, there exists a constant \( 0 < \gamma < 1 \), not depending on \( n \), such that with probability 1, \( \cos \theta_{LS} \geq \gamma \) for all suitably large \( n \). That is, there exists a constant \( 0 < \gamma' \) such that
\[
\frac{\| XQw_{LS} - \vec{a} \|}{\| \vec{a} \|} \leq \gamma' \text{ eventually.} \tag{31}
\]

Proof. By definition of \( w_{LS} \), we have \( \| XQw_{LS} - \vec{a} \| \leq \| X\hat{w} - \vec{a} \| \). For ease of notation, set \( \vec{r} = \vec{a} - X\hat{w} \). It will suffice for us to show that for some constant \( \rho > 0 \), we have
\[
(1 - \rho)\| \vec{a} \|^2 - \| \vec{r} \|^2 \geq 0 \text{ eventually}, \tag{31}
\]
and then, after rearranging terms, \( \sin^2 \theta_{LS} \leq 1 - \rho \). To show (31), note that
\[
(1 - \rho)\| \vec{a} \|^2 - \| \vec{r} \|^2 = 2 \sum_{i=1}^{n} a_i X_i^\top \hat{w} - \sum_{i=1}^{n} (X_i^\top \hat{w})^2 - \rho \sum_{i=1}^{n} a_i^2 \geq \mathbb{E} \left[ (1 - \rho)\| \vec{a} \|^2 - \| \vec{r} \|^2 \right] + C \sqrt{n} \log^{1/2} n \text{ eventually,}
\]
where the inequality follows from an application of Hoeffding’s inequality to show that the sum concentrates about its expectation. We will have established (31) if we can show that \( \mathbb{E} \left[ (1 - \rho)\| \vec{a} \|^2 - \| \vec{r} \|^2 \right] \) grows faster than \( C \sqrt{n} \log^{1/2} n \). To establish this, let \( i \in [n] \) be arbitrary and write
\[
\mathbb{E} \left[ (1 - \rho)a_i^2 - r_i^2 \right] = \mathbb{E} \left[ (1 - \rho)a_i^2 - (a_i - X_i^\top \hat{w})^2 \right] = -\rho \mathbb{E} a_i^2 + 2\mathbb{E} a_i X_i^\top \hat{w} - \mathbb{E} (X_i^\top \hat{w})^2 \\
= -\rho \mathbb{E} a_i^2 - \mathbb{E} (a_i - X_i^\top \hat{w}) X_i^\top \hat{w} = \mathbb{E} a_i X_i^\top \hat{w} - \rho \mathbb{E} a_i^2.
\]
By our boundedness assumption on \( \supp F \), \( \mathbb{E} a_i X_i^\top \hat{w} = \mathbb{E} (X_i^\top \hat{w})^2 \) is bounded away from zero uniformly in \( i \in [n] \), and thus choosing \( \rho > 0 \) suitably small ensures that there exists a small constant \( \eta' > 0 \) such that \( \mathbb{E} \left[ (1 - \rho)a_i^2 - r_i^2 \right] \geq \eta' > 0 \). Summing over \( n \),
\[
\mathbb{E} \left[ (1 - \rho)\| \vec{a} \|^2 - \| \vec{r} \|^2 \right] = \sum_{i=1}^{n} \mathbb{E} \left[ (1 - \rho)a_i^2 r_i^2 \right] \geq n\eta' = \Omega(n),
\]
which proves the bound in (31), completing the proof. \( \square \)

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Lemma 12. With notation as in Theorem 7, there exists a sequence of orthogonal matrices $Q \in \mathbb{R}^{d \times d}$ such that

$$\|Q\hat{w}_{LS} - w_{LS}\| = O(n^{-1/2} \log n).$$

Proof. This is a direct result of Theorem 7 and the preceding Lemmas, once we establish bounds on $\kappa_2(XQ)$ and $\nu_{LS} = \frac{\|XQw_{LS}\|}{\lambda_1(XQ)\|w_{LS}\|}$.

By Lemma 2, we have $C_1 \sqrt{\nu} \geq \lambda_1(XQ) \geq \lambda_d(XQ) \geq C_2 \sqrt{\nu}$, and it follows immediately that $\kappa_2(XQ) \leq C$ eventually. Since $\|XQw_{LS}\|/\|w_{LS}\| \leq \|XQ\| \leq \sqrt{\nu}$, we also have $\nu_{LS} \leq C$ eventually.

By Lemma 10, we are assured that Theorem 7 applies eventually. Lemmas 10 and 11 ensure that the each of $(\cos \theta_{LS})^{-1}$ and $\tan \theta_{LS}$ are bounded by constants eventually. Thus, using Lemma 10 to bound $\|\hat{X} - XQ\|/\|XQ\|$, it follows that the right-hand side of Equation 28 is $O(n^{-1/2} \log n)$ and the result follows.

We now turn to showing that $w_{LS}$ is close to the true latent position $\hat{w}$. A combination of this result with Lemma 12 will yield Theorem 1.

Lemma 13. Let notation be as above and let $\hat{w} \in \text{supp} F$ be the (fixed) latent position of the out-of-sample vertex. Then for all but finitely many $n$,

$$\|w_{LS} - \hat{w}\| \leq C \frac{\log n}{\sqrt{n}}.$$  

Proof. Define $\hat{r} = \hat{a} - X\hat{w}$. As noted previously, by definition of $w_{LS}$, we have

$$\|Xw_{LS} - \hat{a}\|^2 \leq \|X\hat{w} - \hat{a}\|^2 = \|\hat{r}\|^2,$$

whence plugging in $\hat{a} = X\hat{w} + \hat{r}$ yields $\|Xw_{LS} - X\hat{w} - \hat{r}\|^2 \leq \|\hat{r}\|^2$. Thus,

$$\|Xw_{LS} - X\hat{w}\|^2 \leq 2r^T X(w_{LS} - \hat{w}).$$

By Lemma 9, $X$ has full column rank eventually, and thus also $\|X(w_{LS} - \hat{w})\| \geq \sigma_d(X)\|w_{LS} - \hat{w}\|$ eventually. Combining this fact with 32 and using the fact that $\sigma_d^2(X) = \sigma_d(P)$, we have

$$\|w_{LS} - \hat{w}\|^2 \leq \frac{\|X(w_{LS} - \hat{w})\|^2}{\sigma_d^2(X)} \leq \frac{2r^T X(w_{LS} - \hat{w})}{\sigma_d(P)}.$$

Applying the Cauchy-Schwartz inequality and dividing by $\|w_{LS} - \hat{w}\|,$

$$\|w_{LS} - \hat{w}\| \leq \frac{2\|X^T \hat{r}\|}{\sigma_d(P)}.$$

Thus, it remains for us to show that $\|X^T \hat{r}\|$ grows as at most $O(\sqrt{n} \log^2 n)$, from which Lemma 2 will yield our desired growth rate. Expanding, we have

$$\|X^T \hat{r}\|^2 = \sum_{k=1}^d \left( \sum_{i=1}^n (a_i - X_i^T \hat{w})X_{i,k} \right)^2.$$  

Fixing some $k \in [d]$, Hoeffding’s inequality implies that with probability at least $1 - O(n^{-2})$,

$$\sum_{i=1}^n (a_i - X_i^T \hat{w})X_{i,k} \leq 2\sqrt{\nu} \log n.$$

Since $d$ is assumed to be constant in $n$, a union bound over all $k \in [d]$ implies $\|X^T \hat{r}\|^2 \leq 4dn \log^2 n$ with probability at least $1 - O(n^{-2})$. Applying the Borel-Cantelli Theorem and taking square roots completes the proof. \qed
C Proof of ASE ML-OOS Concentration Inequality

To prove Theorem 2, we will apply a standard argument from convex optimization and use the properties of the set $\tilde{T}_\epsilon$ to show that

$$\|Q\hat{w}_{ML} - \bar{w}\| \leq \frac{\|\hat{\ell}'(Q^T \hat{w})\|}{Cn},$$

where $Q \in \mathbb{R}^{d \times d}$ is the orthogonal matrix guaranteed by Lemma 1. This is proven in Lemma 14. We then show in Lemma 15 that

$$\|\hat{\ell}'(Q^T \bar{w})\| = O(\sqrt{n} \log n),$$

which establishes Theorem 2 by the triangle inequality.

Recall the log-likelihood functions

$$\ell(w) = \sum_{i=1}^{n} a_i \log X_i^T w + (1 - a_i) \log(1 - X_i^T w)$$

$$\hat{\ell}(w) = \sum_{i=1}^{n} a_i \log \hat{X}_i^T w + (1 - a_i) \log(1 - \hat{X}_i^T w)$$

and observe that both are convex in their arguments.

**Lemma 14.** With notation as above, under the assumptions of Theorem 2, it holds almost surely that for all suitably large $n$, there exists an orthogonal matrix $Q \in \mathbb{R}^{d \times d}$ satisfying

$$\|Q\hat{w}_{ML} - \bar{w}\| \leq \frac{\|\hat{\ell}'(Q^T \bar{w})\|}{Cn}.$$

**Proof.** By a standard argument, we have

$$\left(\hat{\ell}'(Q^T \bar{w})\right)^T (Q^T \bar{w} - \hat{w}_{ML})$$

$$= \left(\hat{\ell}'(\hat{w}_{ML})\right)^T (Q^T \bar{w} - \hat{w}_{ML})$$

$$+ \int_0^1 (Q^T \bar{w} - \hat{w}_{ML})^T \nabla^2 \hat{\ell}(Q^T \bar{w} + t(Q^T \bar{w} - \hat{w}_{ML})) (Q^T \bar{w} - \hat{w}_{ML})dt$$

$$\geq \|\bar{w} - Q\hat{w}_{ML}\|^2 \min_{w \in \tilde{T}_\epsilon} \lambda_{\min} \left(\nabla^2 \hat{\ell}(w)\right).$$

Rearranging and applying the Cauchy-Schwarz inequality implies

$$\|\bar{w} - Q\hat{w}_{ML}\| \leq \frac{\|\hat{\ell}'(Q^T \bar{w})\|}{\left|\lambda_{\min} \left(\nabla^2 \hat{\ell}(w)\right)\right|}.$$  

The constraint that $w \in \tilde{T}_\epsilon$ implies that for suitably large $n$,

$$\min_{w \in \tilde{T}_\epsilon} \lambda_{\min} \left(\nabla^2 \hat{\ell}(w)\right) \geq Cn,$$

with $C > 0$ depending on $\epsilon$ and $F$ but not on $n$, where we have used Lemma 1 to ensure that $\{X_i\}_{i=1}^{n}$ are uniformly close to supp $F$. We conclude that eventually,

$$\|\bar{w} - Q\hat{w}_{ML}\| \leq \frac{\|\hat{\ell}'(Q^T \bar{w})\|}{Cn},$$

completing the proof. □
Lemma 15. With notation as above, under the assumptions of Theorem 2

\[ \| \nabla \ell(Q^T \bar{w}) \| = O(\sqrt{n} \log n) . \]

Proof. By the triangle inequality,

\[ \| \nabla \ell(Q^T \bar{w}) \| \leq \| \nabla \ell(\bar{w}) \| + \| \nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}) \| . \]

We will show that both terms on the right hand side of (35) are \( O(\sqrt{n} \log^{1/2} n) \).

Fix \( i \in [d] \). By our boundedness assumption on \( \text{supp} F \) and the fact that \( \bar{w}, X_1, X_2, \ldots, X_n \in \text{supp} F \),

\[ (\nabla \ell(\bar{w}))_k = \sum_{i=1}^{n} \left( \frac{a_i}{X_i^T \bar{w}} - \frac{1 - a_i}{1 - X_i^T \bar{w}} \right) X_{i,k} = \sum_{i=1}^{n} \frac{(a_i - X_i^T \bar{w})X_{i,k}}{X_i^T \bar{w}(1 - X_i^T \bar{w})} \]

is a sum of bounded zero-mean random variables. Applying Hoeffding’s inequality,

\[ \Pr \{ |(\nabla \ell(\bar{w}))_k| \geq t \} \leq 2 \exp \left( -\frac{2t^2}{Cn} \right) \]

for some constant \( C > 0 \) depending on \( F \) but not \( n \). Choosing \( t = \sqrt{Cn \log^{1/2} n} \), we have \( (\nabla \ell(\bar{w}))_k \geq \sqrt{Cn \log^{1/2} n} \) with probability at most \( O(n^{-2}) \). A union bound over all \( i \in [d] \), implies that with probability at least \( 1 - Cdn^{-2} \),

\[ \sum_{k=1}^{d} (\nabla \ell(\bar{w}))_k^2 \leq dCn \log n , \]

and the Borel-Cantelli Lemma implies \( \| \nabla \ell(\bar{w}) \| = O(\sqrt{n} \log^{1/2} n) \) after taking square roots.

Turning to the second term on the right hand side of (35), fixing \( i \in [d] \), we have

\[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k = \sum_{i=1}^{n} \frac{(a_i - X_i^T \bar{w})X_{i,k}}{X_i^T Q^T \bar{w}(1 - X_i^T \bar{w})} \sum_{i=1}^{n} \frac{(a_i - X_i^T \bar{w})X_{i,k}}{X_i^T \bar{w}(1 - X_i^T \bar{w})} \]

Taking expectation conditional on \( A \) and \( X \), the second sum has expectation 0, and

\[ \mathbb{E} \left[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k \bigg| A, X \right] = \sum_{i=1}^{n} \frac{(QX_i - X_i)^T \bar{w}}{(QX_i)^T \bar{w}(1 - (QX_i)^T \bar{w})} X_{i,k} . \]

By Lemma 1 and our boundedness assumptions on \( \text{supp} F \), the denominators of this sum are uniformly bounded away from zero over almost all sequences of \( (A, X) \). Lemma 1 also bounds the numerators in this sum uniformly by \( O(n^{-1/2} \log n) \), and it follows that

\[ \mathbb{E} \left[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k \bigg| A, X \right] = O(\sqrt{n} \log n) . \]

Our proof will be complete if we can show that

\[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k - \mathbb{E} \left[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k \bigg| A, X \right] \]

concentrates at the same rate. Toward this end, for ease of notation, for each \( i \in [n] \) define \( p_i = X_i^T \bar{w} \) and \( \hat{p}_i = \bar{X}_i^T \bar{w} \). Then

\[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k - \mathbb{E} \left[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k \bigg| A, X \right] \]

concentrates at the same rate. Toward this end, for ease of notation, for each \( i \in [n] \) define \( p_i = X_i^T \bar{w} \) and \( \hat{p}_i = \bar{X}_i^T \bar{w} \). Then

\[ (\nabla \ell(Q^T \bar{w}) - \nabla \ell(\bar{w}))_k = \sum_{i=1}^{n} \left[ \frac{(a_i - \hat{p}_i)X_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} - \frac{(a_i - p_i)X_{i,k}}{p_i(1 - p_i)} - \frac{(p_i - \hat{p}_i)X_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} \right] \]

\[ = \sum_{i=1}^{n} (a_i - p_i) \left( \frac{\hat{X}_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} - \frac{X_{i,k}}{p_i(1 - p_i)} \right) . \]
Conditional on \((A, X)\), this is a sum of \(n\) independent zero-mean random vectors, with the \(i\)-th summand bounded by

\[
\left| (a_i - p_i) \left( \frac{\hat{X}_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} - \frac{X_{i,k}}{p_i(1 - p_i)} \right) \right| \leq \left| \frac{\hat{X}_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} - \frac{X_{i,k}}{p_i(1 - p_i)} \right|
\]
since \(|a_i - p_i| \leq 1\). Let \(M_i\) denote this bound for each \(i \in [n]\). Let \(s > 0\) be a value which we will specify below, and let \(B_n\) denote the event that

\[
\left| \left( \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \right)_k \right| - \mathbb{E} \left[ \left( \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \right)_k \mid A, X \right] > s.
\]

Hoeffding’s inequality conditional on \(A, X\) implies that

\[
\Pr [B_n \mid A, X] \leq 2 \exp \left\{ -\frac{s^2}{2 \sum_{i=1}^n M_i^2} \right\}.
\]

By definition of \(M_i\), we have

\[
M_i = \left| \frac{\hat{X}_{i,k}}{\hat{p}_i(1 - \hat{p}_i)} - \frac{X_{i,k}}{p_i(1 - p_i)} \right| \\
\leq \left| \frac{\hat{X}_{i,k} - X_{i,k}}{p_i(1 - p_i)} \right| + \left| \frac{1}{\hat{p}_i(1 - \hat{p}_i)} - \frac{1}{p_i(1 - p_i)} \right| \left| X_{i,k} \right| \\
\leq \frac{O(n^{-1/2} \log n)}{p_i(1 - p_i)} + \frac{||p_i - \hat{p}_i||(1 - p_i) + p_i||p_i - \hat{p}_i||}{p_i(1 - p_i)\hat{p}_i(1 - \hat{p}_i)},
\]

where the first inequality follows from the triangle inequality, and the second inequality follows from Lemma 1 and the fact that \(\|X_i\| \leq 1\) by definition of \(F\) being an inner product distribution. Lemma 1 implies that \(\|p_i - \hat{p}_i\| = O(n^{-1/2} \log n)\), since \(\|\bar{w}\| \leq 1\). Our boundedness assumptions on the support of \(F\), along with yet another application of Lemma 1 imply that both denominators are bounded away from 0 eventually. Thus, uniformly over all \(i \in [n]\), \(M_i = O(n^{-1/2} \log n)\), so that \(\sum_{i=1}^n M_i^2 = O(\log^2 n)\), and integrating with respect to \((A, X)\) implies that

\[
\Pr [B_n \mid A, X] \leq 2 \exp \left\{ -\frac{Cs^2}{\log^2 n} \right\}.
\]

Taking \(s = C \log^{3/2} n\) for suitably large constant \(C\) and applying the Borel-Cantelli Lemma ensures that \(B_n\) occurs eventually, and we have that

\[
\left( \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \right)_k - \mathbb{E} \left[ \left( \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \right)_k \mid A, X \right] = O(\log^{3/2} n).
\]

Combining this with Equation (36), we conclude that

\[
\left( \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \right)_k = O(\sqrt{n} \log n).
\]

Since \(d\) is assumed constant, this rate holds uniformly over all \(k \in [d]\), and we conclude that

\[
\| \nabla \hat{\ell}(Q^T \bar{w}) - \nabla \ell(\bar{w}) \| = O(\sqrt{n} \log n),
\]

completing the proof. \(\square\)
D Proof of LSE LS-OOS Concentration Inequality

Here we provide a proof of Theorem 3. The argument proceeds similarly to the proof of Theorem 1 in Appendix B above. Recall that \( \hat{w}_{LS} \in \mathbb{R}^d \) denotes the least-squares OOS extension, given by the solution to
\[
\min_{w \in \mathbb{R}^d} \frac{1}{2} \sum_{i=1}^n \left( \frac{a_i^2}{d_i^2} - \frac{\hat{X}_T w}{\sqrt{t_i}} \right)^2,
\]
where \( \hat{X}_i \in \mathbb{R}^d \) is the LSE estimate of the Laplacian spectral embedding of the true latent position of the \( i \)-th vertex and \( d_i \) denotes the degree of vertex \( i \) for \( i \in [n] \cup \{v\} \). We define \( \tilde{w}_{LS} \in \mathbb{R}^d \) to be the least-squares OOS extension if we had access to the true latent positions. That is, \( \tilde{w}_{LS} \) is the solution to the least-squares problem
\[
\min_{w \in \mathbb{R}^d} \frac{1}{2} \sum_{i=1}^n \left( \frac{a_i^2}{d_i^2} - \frac{\tilde{X}_T w}{\sqrt{t_i}} \right)^2.
\]
Letting \( \tilde{Q} \in \mathbb{R}^{d \times d} \) denote the orthogonal matrix guaranteed by Lemma 1, our proof of Theorem 3 will proceed by showing that both \( \| \tilde{w}_{LS} - \tilde{w} \| \) and \( \| \hat{w}_{LS} - \tilde{Q}^T \tilde{w}_{LS} \| \) are \( O(n^{-1/2} \log^{1/2} n) \), after which the triangle inequality will yield our desired result.

**Lemma 16.** With notation as above,
\[
\| \hat{w}_{LS} - \tilde{w} \| = O(n^{-1/2} \log^{1/2} n).
\]

**Proof.** Recall that \( D \in \mathbb{R}^{n \times n} \) is the diagonal matrix of in-sample vertex degrees and \( d_v = \sum_{i=1}^n a_i \) denotes the degree of the out-of-sample vertex \( v \). Define \( \tilde{b} = d_v^{-1/2} D^{-1/2} \tilde{d} \), and let \( \tilde{z} = \tilde{b} - \hat{X} \hat{w} \). By definition of \( \hat{w}_{LS} \) as a least squares solution, we have
\[
\| \hat{X} \hat{w}_{LS} - \tilde{b} \| \leq \| \tilde{z} \|.
\]
Substituting \( \tilde{b} = \tilde{z} + \hat{X} \hat{w} \), expanding the squares of both sizes and rearranging,
\[
\| \hat{X} (\hat{w}_{LS} - \hat{w}) \|^2 \leq 2 \tilde{z}^T \hat{X} (\hat{w}_{LS} - \hat{w}) \tag{37}
\]
By Lemma 10, \( \hat{X} \) is full rank eventually, and therefore
\[
|\hat{X} (\hat{w}_{LS} - \hat{w})| \geq \sigma_d(\hat{X}) |\hat{w}_{LS} - \hat{w}| \text{ eventually.}
\]
Combining this with (37) and making use of the Cauchy-Schwarz inequality,
\[
|\hat{w}_{LS} - \hat{w}| \leq \frac{2 \| \hat{X}^T \tilde{z} \|}{\sigma_d^2(\hat{X})} \text{ eventually.}
\]
Lemma \( \Box \) implies that \( \sigma_d^2(\hat{X}) = \Theta(1) \), so our proof will be complete if we can bound the growth of \( \| \hat{X}^T \tilde{z} \| \). We have
\[
\| \hat{X}^T \tilde{z} \|^2 = \sum_{k=1}^d \left( \sum_{i=1}^n z_i \hat{X}_{i,k} \right)^2 = \sum_{k=1}^d Y_k^2,
\]
where \( Y_k = \sum_{i=1}^n z_i \hat{X}_{i,k} \). Fixing some \( k \in [d] \),
\[
Y_k = \sum_{i=1}^n \left( \frac{X_i^T \tilde{w}}{\sqrt{t_i} \sqrt{n \mu^T \tilde{w}}} - \frac{a_i}{\sqrt{d_i \sqrt{d_v}}} \right) \frac{X_i}{\sqrt{t_i}}.
\]
Adding and subtracting appropriate quantities,
\[
Y_k = \sum_{i=1}^{n} \frac{(X_i^T \bar{w} - a_i)}{t_i \sqrt{n \mu_i^T \bar{w}}} X_{i,k} + \sum_{i=1}^{n} \frac{a_i X_{i,k}}{\sqrt{t_i}} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right).
\] (38)

Conditional on \(X\), the first term is a sum of independent mean-0 random variables, with
\[
\frac{(X_i^T \bar{w} - a_i) X_{i,k}}{t_i \sqrt{n \mu_i^T \bar{w}}} \in \left[ -\frac{1}{t_i \sqrt{n \mu_i^T \bar{w}}}, \frac{1}{t_i \sqrt{n \mu_i^T \bar{w}}} \right]
\] almost surely for each \(i \in [n]\). Let \(G_n\) denote the event that \(\min_i t_i \geq Cn\) for some suitably-chosen constant \(C > 0\). Lemma 6 ensures that \(\Pr[G_n^c] = O(n^{-2})\), and integrating with respect to \(X \in \mathbb{R}^{n \times d}\) yields
\[
\Pr[G_n] \leq \Pr[G_n \mid B_n] + \Pr[B_n^c] \leq 2 \exp \left\{ -\frac{n \mu_i^T \bar{w} s^2}{\sum_{i=1}^{n} t_i^{-2}} \right\} + O(n^{-2}).
\]

Taking \(s = Cn^{-1} \log^{1/2} n\) for \(C > 0\) suitably large ensures that both terms on the right-hand side are \(O(n^{-2})\), and we have
\[
\sum_{i=1}^{n} \left| \frac{(X_i^T \bar{w} - a_i) X_{i,k}}{t_i \sqrt{n \mu_i^T \bar{w}}} \right| = O(n^{-1} \log^{1/2} n). \tag{39}
\]

Lemma 6 similarly bounds the second sum in (38):
\[
\sum_{i=1}^{n} \frac{a_i X_{i,k}}{\sqrt{t_i}} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right) \leq C \sqrt{n} \sum_{i=1}^{n} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right) a_i X_{i,k}. \tag{40}
\]

Adding and subtracting appropriate quantities, the sum becomes
\[
\sum_{i=1}^{n} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right) a_i X_{i,k} = \sum_{i=1}^{n} \frac{a_i X_{i,k}}{\sqrt{t_i}} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_v}} \right) + \sum_{i=1}^{n} \frac{a_i X_{i,k}}{\sqrt{d_v}} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right),
\]
and several applications of Lemma 6 yields that
\[
\sum_{i=1}^{n} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right) a_i X_{i,k} = O(n^{-1/2} \log^{1/2} n),
\]
whence, applying this to Equation (40), we have
\[
\sum_{i=1}^{n} \frac{a_i X_{i,k}}{\sqrt{t_i}} \left( \frac{1}{\sqrt{t_i \sqrt{n \mu_i^T \bar{w}}}} - \frac{1}{\sqrt{d_i \sqrt{d_v}}} \right) = O(n^{-1} \log^{1/2} n).
\]

Applying this and (39) to the right-hand side of (38), \(|Y_k| = O(n^{-1} \log^{1/2} n)\) and a union bound over \(k \in [d]\) completes the proof. \(\square\)
Lemma 17. With notation as above, there exists a sequence of orthogonal matrices \( \tilde{Q} \in \mathbb{R}^{d \times d} \) such that

\[
\| \tilde{Q} \tilde{w}_{LS} - \tilde{w}_{LS} \| = O(n^{-1/2} \log^{1/2} n).
\]

Proof. Recall from above our definition \( \tilde{b} = d_c^{-1/2} D^{-1/2} \tilde{a} \), where \( d_c \) is the degree of the out-of-sample vertex and \( D \in \mathbb{R}^{n \times n} \) is the diagonal matrix of in-sample vertex degrees, and note that \( \tilde{w}_{LS} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \tilde{b} \). Our main tool, as in Section B, is Theorem 5.3.1 from Golub and Van Loan [2012], quoted above as Theorem [1]. Applying that theorem, we have that so long as \( \tilde{b}, \tilde{b} - \tilde{X} \tilde{w}_{LS} \) and \( \tilde{w}_{LS} \) are all non-zero,

\[
\frac{\| \tilde{w}_{LS} - \tilde{Q}^T \tilde{w}_{LS} \|}{\| \tilde{Q}^T \tilde{w}_{LS} \|} \leq \frac{\| \tilde{X} - \tilde{X} \hat{Q} \|}{\| \tilde{X} \hat{Q} \|} \left( \frac{\nu_{LS}}{\cos \theta_{LS}} + (1 + \nu_{LS} \tan \theta_{LS}) \kappa_2(\tilde{X} \hat{Q}) \right) + C \frac{\| \tilde{X} - \tilde{X} \hat{Q} \|^2}{\| \tilde{X} \hat{Q} \|^2}.
\]

where \( \theta_{LS} \in (0, \pi/2) \) with

\[
\sin \theta_{LS} = \frac{\| \tilde{r}_{LS} \|}{\| \tilde{r} \|}, \quad \text{and} \quad \nu_{LS} = \frac{\| \tilde{X} \tilde{w}_{LS} \|}{\sigma_d(\tilde{X}) \| \tilde{Q}^T \tilde{w}_{LS} \|}.
\]

In order to apply Theorem [7] we must first show that eventually

1. \( \| \tilde{X} - \tilde{X} \hat{Q} \| < \sigma_d(\tilde{X}) \) and

2. the quantities \( \tilde{b}, \tilde{b} - \tilde{X} \tilde{w}_{LS} \), and \( \tilde{w}_{LS} \) are all non-zero.

The first condition holds eventually by Lemma [7] and the fact that, using the relations between the spectral, Frobenius and \( (2, \infty) \)-norms,

\[
\| \tilde{X} - \tilde{X} \hat{Q} \|^2 \leq \| \tilde{X} - \tilde{X} \hat{Q} \|_{2, \infty}^2 \leq \frac{C \log n}{n}, \quad (41)
\]

where the last inequality holds eventually by Lemma [1]. As in the proof of Lemma [10] it is immediate from the model that condition 2 holds eventually.

Equation (41), along with another application of Lemma [7] to control \( \lambda_d(\mathcal{L}(P)) \) implies that

\[
\frac{\| \tilde{X} - \tilde{X} \hat{Q} \|}{\| \tilde{X} \hat{Q} \|} \leq \frac{C \log^{1/2} n}{\sqrt{n} \sigma_d(\mathcal{L}(P))} \leq \frac{C \log^{1/2} n}{\sqrt{n}} \quad \text{eventually} \quad (42)
\]

Thus, applying Theorem [7] we have

\[
\| \tilde{w}_{LS} - \tilde{Q}^T \tilde{w}_{LS} \| \leq \frac{C \| \tilde{w}_{LS} \| \log^{1/2} n}{\sqrt{n}} \left( \frac{\nu_{LS}}{\cos \theta_{LS}} + (1 + \nu_{LS} \tan \theta_{LS}) \kappa_2(\tilde{X} \hat{Q}) \right) + C \frac{\log^2 n}{n^2}. \quad (43)
\]

Lemma [7] bounds the condition number \( \kappa_2(\tilde{X} \hat{Q}) = \kappa_2(\tilde{X}) \leq C \), whence

\[
\nu_{LS} = \frac{\| \tilde{X} \tilde{w}_{LS} \|}{\sigma_d(\tilde{X}) \| \tilde{Q}^T \tilde{w}_{LS} \|} = \frac{\| \tilde{X} \tilde{w}_{LS} \|}{\sigma_d(\tilde{X}) \| \tilde{w}_{LS} \|} \leq \frac{\| \tilde{X} \|}{\sigma_d(\tilde{X})} = \kappa_2(\tilde{X}) \leq C \quad \text{eventually}.
\]

By the triangle inequality, the definition of \( \tilde{w} \) and using Lemma [16] to bound \( \| \tilde{w}_{LS} - \tilde{w} \| \),

\[
\| \tilde{w}_{LS} \| = \frac{\| \tilde{w} \|}{\sqrt{n} \mu^T \tilde{w}} + O(n^{-1} \log^{1/2} n) = O(n^{-1/2}) + O(n^{-1} \log^{1/2} n),
\]

whence Equation (43) becomes

\[
\| \tilde{Q} \tilde{w}_{LS} - \tilde{w}_{LS} \| \leq \frac{C \log^{1/2} n}{n} \left( 1 + \frac{1 + \sin \theta_{LS}}{\cos \theta_{LS}} \right) + C \frac{\log^2 n}{n^2} \quad \text{eventually}.
\]
Thus, to complete the proof, it will suffice to bound \( \cos \theta_{LS} \) away from 0. To do this, we will show by an argument similar to that in Lemma 11 that there exists a constant \( \rho \in (0, 1) \) such that \( \sin \theta_{LS} \leq 1 - \rho \) eventually.

Toward this end, define \( \tilde{b} = t_v^{-1/2}T^{-1/2} \tilde{a} \), where we remind the reader that \( t_v = \sum_{i=1}^n X_i^T \bar{w} \) is the expected degree of the out-of-sample vertex conditioned on the latent positions, and \( T \in \mathbb{R}^{n \times n} \) is the diagonal matrix of in-sample vertex expected degrees, i.e., \( T_{i,i} = \sum_{j=1}^n X_j^T X_i \).

Letting \( \tilde{X}^\dagger = (X^T T^{-1} X)^{-1} X^T T^{-1/2} \) denote the pseudoinverse of \( \tilde{X} \), (with the inverse existing eventually by Lemma 9), we have

\[
\sin \theta_{LS} = \frac{\| \tilde{b} - \tilde{X} \hat{w}_{LS} \|}{\| \tilde{b} \|} = \frac{\| (I - \tilde{X} \tilde{X}^\dagger) \tilde{b} \|}{\| \tilde{b} \|} = \frac{\| (I - \tilde{X} \tilde{X}^\dagger) \tilde{b} \|}{\| \tilde{b} \|} \leq \frac{\| (I - \tilde{X} \tilde{X}^\dagger) \tilde{b} \|}{\| \tilde{b} \|} + \frac{\| (I - \tilde{X} \tilde{X}^\dagger) \tilde{b} \|}{\| \tilde{b} \|},
\]

where the inequality follows from the triangle inequality and submultiplicativity. By definition of \( \tilde{b} \) and \( \hat{b} \), we have

\[
\frac{\| \tilde{b} - \hat{b} \|}{\| \hat{b} \|} = \frac{\left\| d_v^{-1/2}D^{-1/2} - t_v^{-1/2}T^{-1/2} \tilde{a} \right\|}{\| t_v^{-1/2}T^{-1/2} \tilde{a} \|} \leq \frac{\left\| d_v^{-1/2}D^{-1/2} - t_v^{-1/2}T^{-1/2} \tilde{a} \right\|}{t_v^{-1/2} / \max_i \sqrt{t_i}},
\]

where we have used submultiplicativity to upper bound the numerator, \( \| T^{-1/2} \tilde{a} \| \geq \| \tilde{a} \| / \max_i \sqrt{t_i} \) to lower-bound the denominator, and cancelled the resulting factor of \( \| \tilde{a} \| \). Cancelling factors of \( t_v^{-1/2} \), we have

\[
\frac{\| \tilde{b} - \hat{b} \|}{\| \hat{b} \|} \leq \| t_v^{-1/2} d_v^{-1/2} D^{-1/2} - T^{-1/2} \| \max_i \sqrt{t_i}.
\]

Lemma 9 implies \( \max_i \sqrt{t_i} = O(\sqrt{n}) \), and a second application of Lemma 9 implies that \( \| t_v^{-1/2} d_v^{-1/2} D^{-1/2} - T^{-1/2} \| = O(n^{-1} \log^{1/2} n) \), from which

\[
\frac{\| \tilde{b} - \hat{b} \|}{\| \hat{b} \|} = O(n^{-1/2} \log^{1/2} n),
\]

and it follows from the triangle inequality that

\[
\frac{\| \tilde{b} \|}{\| \hat{b} \|} \leq \frac{\| \tilde{b} \| + \| \tilde{b} - \hat{b} \|}{\| \hat{b} \|} = 1 + O(n^{-1/2} \log^{1/2} n) = O(1).
\]

Applying Equations (45) and (46) to Equation (44) and using the bound \( \| I - \tilde{X} \tilde{X}^\dagger \| \leq 1 \),

\[
\sin \theta_{LS} \leq O \left( \frac{\log^{1/2} n}{\sqrt{n}} \right) + C \frac{\| (I - \tilde{X} \tilde{X}^\dagger) \hat{b} \|}{\| \hat{b} \|}.
\]

Letting \( P_{\tilde{X}}^\perp = (I - \tilde{X} \tilde{X}^\dagger) \) denote the orthogonal projection onto the orthogonal complement of the column space of \( \tilde{X} = T^{-1/2} X \), we have, canceling factors of \( t_v^{-1/2} \) in the numerator and denominator,

\[
\frac{\| (I - \tilde{X} \tilde{X}^\dagger) \hat{b} \|}{\| \hat{b} \|} = \frac{\| (I - \tilde{X} \tilde{X}^\dagger)T^{-1/2} \tilde{a} \|}{\| T^{-1/2} \tilde{a} \|} = \frac{\| P_{\tilde{X}}^\perp T^{-1/2} \tilde{a} \|}{\| T^{-1/2} \tilde{a} \|} = \frac{\| P_{\tilde{X}}^\perp T^{-1/2} (\tilde{a} - \bar{w}) \|}{\| T^{-1/2} \tilde{a} \|},
\]

32
where we have used the fact that \( P_{\tilde{X}}^T T^{-1/2} X \bar{w} = 0 \), since \( T^{-1/2} X \bar{w} = \hat{X} \bar{w} \) is in the column space of \( \hat{X} \). Thus, defining \( \bar{r} = \bar{a} - X \bar{w} \), we have
\[
\frac{\|(I - \hat{X} \hat{X}^T) \bar{b}\|}{\|\bar{b}\|} = \frac{\|P_{\tilde{X}}^T T^{-1/2} \bar{r}\|}{\|T^{-1/2} \bar{a}\|} \leq \frac{\|T^{-1/2}\|\|\bar{r}\|}{\|\bar{a}\|/ \max_i \sqrt{t_i}},
\]
where the last inequality follows from the fact that the expected degrees \( \{t_i\}_{i=1}^n \) are all of the same order by Lemma 9. The same argument as that given in the proof of Lemma 11 lets us bound \( \|\bar{r}\|/\|\bar{a}\| \) by a constant \( \rho > 0 \) smaller than 1/(2C). Applying this to (47), we obtain
\[
\sin \theta_{LS} \leq 1 - \rho + O(n^{-1/2} \log^{1/2} n)
\]
It follows that
\[
\sin \theta_{LS} \leq 1 - \frac{\rho}{2}
\]
eventually, i.e., \( \sin \theta_{LS} \) is bounded away from 1, completing the proof. \( \square \)

**E Proof of ASE linear least squares out-of-sample CLT**

In this section, we prove Theorem 4 which shows that taking \( \{Q_n\}_{n=1}^\infty \) to be the sequence of orthogonal \( d \)-by-\( d \) matrices guaranteed to exist by Lemma 1, the quantity \( \sqrt{n} (\hat{w}_{LS} - Q^T \bar{w}) \) is asymptotically multivariate normal. We begin by recalling that
\[
\hat{w}_{LS} = (\hat{X}^T \hat{X})^{-1} \hat{X}^T \bar{a} = \hat{S}^{-1/2} \hat{U}^T \bar{a}.
\]
Our proof will consist of writing \( \sqrt{n} (\hat{w}_{LS} - Q^T \bar{w}) \) as a sum of two random vectors,
\[
\sqrt{n} (\hat{w}_{LS} - Q^T \bar{w}) = \sqrt{n} \hat{g} + \sqrt{n} \hat{h},
\]
and showing that \( \sqrt{n} \hat{g} \) converges in law to a normal, while \( \sqrt{n} \hat{h} \) converges in probability to 0. The multivariate version of Slutsky’s Theorem will then yield the desired result. We begin by showing that \( \hat{g} = \sqrt{n} S^{-1/2} U^T (\bar{a} - X \bar{w}) \) will suffice. We remind the reader that \( \Delta = \mathbb{E} X_1 X_1^T \in \mathbb{R}^{d \times d} \) is the second moment matrix of the latent position distribution \( F \).

**Lemma 18.** Let \( F \) be a \( d \)-dimensional inner product distribution, with \( (A, X) \sim \text{RDPG}(F, n) \) and let \( \bar{w} \in \text{supp} F \) be the fixed latent position of the out-of-sample vertex. Then
\[
\sqrt{n} S^{-1/2} U^T (\bar{a} - X \bar{w}) \overset{\mathcal{L}}{\to} \mathcal{N}(0, \Sigma_{F, \bar{w}}),
\]
where \( \Sigma_{F, \bar{w}} = \Delta^{-1} \mathbb{E} \left[ X_1^T \bar{w}(1 - X_1^T \bar{w}) X_1 X_1^T \right] \Delta^{-1}. \)

**Proof.** We begin by observing that since \( \bar{w} \in \mathbb{R}^d \) is fixed,
\[
n^{-1/2} X^T (\bar{a} - X \bar{w}) = n^{-1/2} \sum_{i=1}^n (\bar{a}_i - X_i^T \bar{w}) X_i
\]
is a scaled sum of \( n \) independent 0-mean \( d \)-dimensional random vectors, each with covariance matrix
\[
V_{\bar{w}} = \mathbb{E} X_1^T \bar{w}(1 - X_1^T \bar{w}) X_1 X_1^T \in \mathbb{R}^{d \times d}.
\]
The multivariate central limit theorem implies that
\[
n^{-1/2} X^T (\bar{a} - X \bar{w}) X_i \overset{\mathcal{L}}{\to} \mathcal{N}(0, V_{\bar{w}}). \]
We have $\sqrt{n}S^{-1/2}UT(\bar{a} - X\bar{w}) = nS^{-1}n^{-1/2}X^{T}(\bar{a} - X\bar{w})$. By the WLLN, $S/n \xrightarrow{P} \Delta$, and hence by the continuous mapping theorem, $nS^{-1} \xrightarrow{P} \Delta^{-1}$. Thus, the multivariate version of Slutsky’s Theorem implies that

$$\sqrt{n}S^{-1/2}UT(\bar{a} - X\bar{w}) \xrightarrow{L} \mathcal{N}(0, \Delta^{-1}V\Delta^{-1}),$$

as we set out to show. \hfill \Box

The following technical lemma will be crucial for proving one of the convergence results required by our main theorem. Its comparative complexity merits stating it here rather than including it in the proof of Theorem \textit{4} below. We remind the reader that $\hat{S}, S \in \mathbb{R}^{d \times d}$ are the diagonal matrices formed by the top $d$ eigenvalues of $A$ and $P$, respectively, and $\hat{U}, U \in \mathbb{R}^{n \times d}$ are the matrices whose columns are the corresponding unit eigenvectors.

**Lemma 19.** With notation as above,

$$\sqrt{n}\hat{S}^{-1/2}(\hat{U}^{T} - \hat{U}^{T}UU^{T})(\bar{a} - X\bar{w}) \xrightarrow{P} 0.$$

**Proof.** For ease of notation, define the vector

$$\bar{z} = (\hat{U}^{T} - \hat{U}^{T}UU^{T})(\bar{a} - X\bar{w}).$$

Let $\epsilon > 0$ be a constant, and note that for suitably large $n$,

$$\Pr\left[\sqrt{n}\|\hat{S}^{-1/2}\bar{z}\| > \epsilon\right] \leq \Pr\left[\sqrt{n}\|\hat{S}^{-1/2}\bar{z}\| > C_0n^{-1/4}\right],$$

where $C_0 > 0$ is a constant that we are free to choose. Define the events

$$E_{1,n} = \{\|\hat{S}^{-1/2}\| \leq C_1n^{-1/2}\},$$

and

$$E_{2,n} = \{\sqrt{n}\|\bar{z}\| \leq C_2n^{1/4}\},$$

and note that $\Pr\left[\sqrt{n}\|\hat{S}^{-1/2}\bar{z}\| > C_0n^{-1/4}\right] \leq \Pr\left[\|E_{1,n} \cap E_{2,n}\|\right]$ so long as $C_1C_2 \leq C_0$. Thus, it will suffice for us to show that $\lim_{n \to \infty} \Pr\left[\|E_{1,n} \cap E_{2,n}\|\right] \to 0$. The proof of Lemma \textit{2} implies that $\lim_{n \to \infty} \Pr[E_{1,n}] = 0$, so our proof will be complete once we show that $\lim_{n \to \infty} \Pr[E_{2,n}] = 0$.

Toward this end, define the matrix

$$W = e_n^T \otimes \bar{w} = \begin{bmatrix} \bar{w} & \bar{w} & \ldots & \bar{w} \end{bmatrix} \in \mathbb{R}^{d \times n}$$

and let $B \in \mathbb{R}^{n \times n}$ be a random matrix with independent binary entries with $\mathbb{E}B_{i,j} = (XW)_{i,j} = X^T\bar{w}$. Define the event

$$E_{3,n} = \{\|((\hat{U}^{T} - \hat{U}^{T}UU^{T})(B - XW))\|_{F}^2 \leq C\log^2 n\}.$$

Since $\Pr[E_{3,n}^c] \leq \Pr[E_{3,n}^c|E_{3,n}] + \Pr[E_{3,n}], \text{ it will suffice to show that}$

1. $\lim_{n \to \infty} \Pr[E_{3,n}] = 0$, and
2. $\lim_{n \to \infty} \Pr[E_{3,n}^c|E_{3,n}] = 0$.

By submultiplicativity, we have

$$\|(\hat{U}^{T} - \hat{U}^{T}UU^{T})(B - XW))\|_{F}^2 \leq \|(\hat{U}^{T} - \hat{U}^{T}UU^{T})^2\|_{F}\|B - XW\|^2.$$

(48)

Theorem \textit{5} applied to $B - XW$ implies that with probability $1 - O(n^{-2})$,

$$\|B - XW\| \leq Cn^{1/2}\log^{1/2} n.$$

(49)
Theorem 2 in [Yu et al. (2015)] guarantees an orthogonal $R^* \in \mathbb{R}^{d \times d}$ such that

$$\| \hat{U} - U R^* \|_F \leq \frac{C \| A - P \|}{\lambda_d(P)} = O \left( \frac{\log^{1/2} n}{\sqrt{n}} \right),$$

(50)

where we have used Lemma 2 to lower-bound $\lambda_d(P)$ and bounded $\| A - P \| = O(n^{1/2} \log^{1/2} n)$ by a result in Oliveira (2010). Since $R = U^T U$ solves the minimization

$$\min_{R \in \mathbb{R}^{d \times d}} \| \hat{U}^T R - \hat{U}^T U U^T \|_F,$$

Equation (50) implies

$$\| \hat{U}^T - \hat{U}^T U U^T \|_F \leq \| \hat{U}^T - R^* U^T \|_F = O(n^{-1/2} \log^{1/2} n).$$

Plugging this and (49) back into (48), we have that with probability $1 - O(n^{-2})$,

$$\| (\hat{U}^T - \hat{U}^T U U^T)(B - X W) \|_F^2 \leq C \log^2 n$$

(51)

which is to say, $\Pr[E_{3,n}] = O(n^{-2})$.

It remains to show that $\Pr[E_{2,n} \mid E_{3,n}] = 0$. By construction, the columns of the matrix $(\hat{U}^T - \hat{U}^T U U^T)(B - X W)$ are $n$ independent copies of $\tilde{\varepsilon}$. Using this fact and the conditional Markov inequality, we have

$$\Pr[E_{2,n} \mid E_{3,n}] = \Pr[\sqrt{n} \| \tilde{\varepsilon} \| > C_2 n^{1/4} \mid E_{3,n}] \leq \frac{n \mathbb{E}[\| \tilde{\varepsilon} \|^2 \mid E_{3,n}]}{C_2^2 n^{1/2}} \leq \frac{C \log^2 n}{n^{1/2}},$$

where the last inequality follows from the definition of event $E_{3,n}$. This quantity goes to zero in $n$, thus completing the proof.

The following technical lemma will prove useful in our proof of Theorem 4 below. We state it here rather than proving it in-line for the sake of clarity.

**Lemma 20.** With notation as above,

$$\| U^T (\vec{a} - X \vec{w}) \| = O(n^{1/2} \log^{1/2} n).$$

**Proof.** For $k \in [d]$ and $i \in [n]$, observe that

$$(U^T (\vec{a} - X \vec{w}))_{k,i} = \sum_{j=1}^{n} (U)_{j,k} (a_j - X^T j \vec{w})$$

is a sum of independent 0-mean random variables, and Hoeffding’s inequality yields

$$\Pr[\| U^T (\vec{a} - X \vec{w}) \|_{k,i} \geq t] \leq 2 \exp \left\{ \frac{-t^2}{2 \sum_{j=1}^{n} (U)_{k,j}^2} \right\} = 2 \exp \left\{ \frac{-t^2}{2} \right\}.$$

Taking $t = C \log^{1/2} n$ for suitably large constant $C > 0$, a union bound over all $k \in [d]$ and $i \in [n]$ followed by the Borel-Cantelli Lemma yields the result.

We are now ready to present the proof of Theorem 4.
Lemma 19 shows that the second term in (57) also goes to zero in probability, and Equation (54)

\[
\|\sqrt{n}Q\hat{w}_{LS} - \bar{w}\| = \sqrt{n}\left( \hat{S}^{-1/2}\hat{U}^T\hat{a} - Q^T\bar{w} \right) = \sqrt{n}\hat{S}^{-1/2}U^T(\hat{a} - X\bar{w}) + \sqrt{n}\hat{Q}\hat{S}^{-1/2}(\hat{U}^T - Q^TU^T)(\hat{a} - X\bar{w}) \tag{52}
\]

By Lemma 18, the first of these terms converges in law:

\[
\sqrt{n}\hat{S}^{-1/2}U^T(\hat{a} - X\bar{w}) \xrightarrow{D} N(0, \Sigma_{F,\hat{w}}), \tag{53}
\]

where \(\Sigma_{F,\hat{w}}\) is as defined in Lemma 18. Thus, by Slutsky’s Theorem, our proof will be complete once we show that the remaining terms in Equation (52) go to zero in probability.

Since \(Q\) is orthogonal, it suffices to prove that

\[
\sqrt{n}\hat{Q}\hat{S}^{-1/2}(\hat{U}^T - Q^TU^T)(\hat{a} - X\bar{w}) \xrightarrow{P} 0, \tag{54}
\]

\[
\sqrt{n}(\hat{S}^{-1/2}\hat{U}^TX - QT)\bar{w} \xrightarrow{P} 0, \tag{55}
\]

and

\[
\sqrt{n}(\hat{S}^{-1/2}Q^T - Q^TS^{-1/2})U^T(\hat{a} - X\bar{w}) \xrightarrow{P} 0. \tag{56}
\]

We will address each of these three convergences in order.

To see the convergence in (54), adding and subtracting appropriate quantities gives

\[
\sqrt{n}\hat{S}^{-1/2}(\hat{U}^T - Q^TU^T)(\hat{a} - X\bar{w}) = \sqrt{n}\hat{S}^{-1/2}(\hat{U}^TU^T - Q^TU^T)(\hat{a} - X\bar{w}) + \sqrt{n}\hat{S}^{-1/2}(\hat{U}^T - \hat{U}U^T)(\hat{a} - X\bar{w}) \tag{57}
\]

To bound the first of these two summands, Lemmas 2, 20 and 3 imply

\[
\|\sqrt{n}\hat{S}^{-1/2}(\hat{U}^TU^T - Q^TU^T)(\hat{a} - X\bar{w})\| \leq \sqrt{n}\|\hat{S}^{-1/2}\|\|U^TU - Q^TU\|\|U^T(\hat{a} - X\bar{w})\|_F \leq O(n^{-1/2}\log^{3/2} n).
\]

Lemma 19 shows that the second term in (57) also goes to zero in probability, and Equation (54) follows.

To see (55), note that

\[
\sqrt{n}(\hat{S}^{-1/2}\hat{U}^T\hat{a} - Q^T\bar{w}) = \sqrt{n}\left( \hat{S}^{-1/2}\hat{U}^TUS^{1/2} - Q^T \right)\bar{w} = \sqrt{n}\hat{S}^{-1/2}(\hat{U}^TU - Q^T)S^{1/2}\bar{w} + \sqrt{n}\hat{S}^{-1/2}(Q^TS^{1/2} - \hat{S}^{1/2}Q^T)\bar{w}. \tag{58}
\]

Submultiplicativity of matrix norms combined with Lemmas 2 and 3 and the fact that \(\|\bar{w}\| \leq 1\) imply

\[
\|\sqrt{n}\hat{S}^{-1/2}(\hat{U}^TU - Q^T)\|S^{1/2}\bar{w}\| \leq C\sqrt{n}\|\hat{S}^{-1/2}\|\|\hat{U}^TU - Q^T\|_F\|S^{1/2}\|\|\bar{w}\| \leq O(n^{-1/2}\log n). \tag{59}
\]

Applying Lemma 2 again and taking the Frobenius norm as a trivial upper bound on the spectral norm, Lemma 4 implies

\[
\|\sqrt{n}\hat{S}^{-1/2}(Q^TS^{1/2} - \hat{S}^{1/2}Q^T)\bar{w}\| \leq C\sqrt{n}\|\hat{S}^{-1/2}\|\|Q^TS^{1/2} - \hat{S}^{1/2}Q^T\|\|\bar{w}\| \leq C\|QS^{1/2} - \hat{S}^{1/2}Q\|, \tag{60}
\]

Proof of Theorem 4.
where we have used the fact that the spectral norm is preserved by matrix transposition. Adding and subtracting appropriate quantities,
\[ QS^{1/2} - \tilde{S}^{1/2}Q = (Q - \hat{U}^T U)S^{1/2} + \tilde{S}^{1/2}(\hat{U}^T U - Q) + \hat{U}^T U S^{1/2} - \tilde{S}^{1/2}\hat{U}^T U. \]

By the triangle inequality and submultiplicativity,
\[ \|QS^{1/2} - \tilde{S}^{1/2}Q\| \leq \|S^{1/2}\| + \|\tilde{S}^{1/2}\| \|\hat{U}^T U - Q\| + \|\hat{U}^T U S^{1/2} - \tilde{S}^{1/2}\hat{U}^T U\|. \quad (61) \]

Lemmas 2 and 3 bound the first term as \( O(n^{-1/2}\log n) \), and the second term is bounded by Lemma 5 and thus Equation (60) is bounded as
\[ \|\sqrt{n}\tilde{S}^{-1/2}(Q^T S^{1/2} - \tilde{S}^{1/2}Q^T)\tilde{w}\| = O(n^{-1/2}\log n). \]

Applying this and Equation (59) to Equation (58) proves (55) by the triangle inequality. Finally, to prove (56), note that
\[ \|\sqrt{n}(\tilde{S}^{-1/2}Q^T - Q^T S^{-1/2})U^T(\bar{a} - X\tilde{w})\| \leq \sqrt{n}\|\tilde{S}^{-1/2}Q^T - Q^T S^{-1/2}\||U^T(\bar{a} - X\tilde{w})|_F, \]

and Lemmas 5 and 20 along with an argument similar to the bound in Equation (61) imply that
\[ \|\sqrt{n}(\tilde{S}^{-1/2}Q^T - Q^T S^{-1/2})U^T(\bar{a} - X\tilde{w})\| = O(n^{-1/2}\log^{3/2} n), \]

which completes the proof. \( \square \)

F Proof of LSE linear least squares out-of-sample CLT

In this section, we prove Theorem 3 which shows that the least-squares out-of-sample extension for the Laplacian spectral embedding is, in the large-\( n \) limit, normally distributed about the true embedding \( \tilde{w} = \hat{w}/\sqrt{n\mu^T \hat{w}} \), after appropriate rescaling. We remind the reader that \( \bar{a} \in \mathbb{R}^n \) denotes the vector of edges between the out-of-sample vertex \( v \) and the in-sample vertices \( V = [n] \) and \( D \in \mathbb{R}^n \) is the diagonal matrix of in-sample node degrees, so that \( D_{i,i} = d_i = \sum_{j=1}^n A_{i,j} \).

Below, we will also need to define the matrix
\[ T = \text{diag}(t_1, t_2, \ldots, t_n) \in \mathbb{R}^{n \times n}, \quad t_i = \sum_{j=1}^n X_j^T X_i, \]

the matrix of in-sample expected degrees conditioned on the latent positions. Analogously, we denote the out-of-sample vertex degree \( d_v = \sum_{j=1}^n a_j \), and its expectation \( t_v = \sum_{j=1}^n X_j^T \tilde{w} \).

Recall that the LSE least-squares out-of-sample extension is given by
\[ \hat{\tilde{w}}_{LS} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T D^{-1/2} \frac{\bar{a}}{\sqrt{\bar{d}_v}}. \]

Our aim is to prove that for a suitably-chosen sequence of orthogonal matrices \( \hat{Q} \in \mathbb{R}^{d \times d} \),
\[ n(\hat{Q}\hat{\tilde{w}}_{LS} - \tilde{w}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \hat{\Sigma}_{F,\tilde{w}}), \]

where \( \hat{\Sigma}_{F,\tilde{w}} \) depends only on the latent position distribution \( F \) and the true out-of-sample latent position \( \tilde{w} \in \text{supp} \ F \), and is given by
\[ \hat{\Sigma}_{F,\tilde{w}} = \mathbb{E} \left[ X_j^T \tilde{w}(1 - X_j^T \tilde{w}) \frac{\tilde{X}_j^T \tilde{w}}{\mu^T \tilde{w}} \left( \tilde{X}_j^T \mu - \frac{\hat{\tilde{w}}}{2\mu^T \hat{\tilde{w}}} \right) \left( \tilde{X}_j^T \mu - \frac{\hat{\tilde{w}}}{2\mu^T \hat{\tilde{w}}} \right)^T \right] \in \mathbb{R}^{d \times d}, \]

where \( \hat{\Delta} = \mathbb{E} X_1 X_1^T / (X_1^T \mu) \) with \( \mu = \mathbb{E} X_1 \) is the mean of \( F \).
Proof of Theorem 5. Take $\tilde{Q} \in \mathbb{R}^{d \times d}$ to be the matrix guaranteed by Lemma 1. Similarly to the proof of Theorem 4, our proof will proceed by writing $n(\tilde{w}_{LS} - \tilde{Q}\tilde{w})$ as a sum,

$$ n(\tilde{Q}\tilde{w}_{LS} - \tilde{w}) = n\tilde{g} + n\tilde{h}, $$

where $n\tilde{h} \xrightarrow{P} 0$ and $n\tilde{g}$ converges in law to our desired normal distribution, whence Slutsky’s Theorem will yield the result. We begin by writing

$$ n(\tilde{w}_{LS} - \tilde{Q}^T\tilde{w}) = n(X^T\tilde{X})^{-1} \tilde{X}^T D^{-1/2} \tilde{a} \sqrt{d_v} - n\tilde{U}^T\tilde{U}\tilde{w} - n(\tilde{Q}^T - \tilde{U}^T\tilde{U})\tilde{w}. $$  \((62)\)

By submultiplicativity of the spectral norm, Lemma 4 and the definition of $\tilde{w} = \tilde{w}/\sqrt{n\mu^T\tilde{w}}$,

$$ \| (\tilde{Q}^T - \tilde{U}^T\tilde{U})\tilde{w} \| \leq \|\tilde{Q}^T - \tilde{U}^T\tilde{U}\|\|\tilde{w}\| \leq \frac{C\|\tilde{w}\|}{n^{3/2}}. $$

Applying this to Equation \((62)\) and using the fact that $\|\tilde{w}\|$ is bounded, we have

$$ n(\tilde{w}_{LS} - \tilde{Q}^T\tilde{w}) = n(X^T\tilde{X})^{-1} \tilde{X}^T D^{-1/2} \tilde{a} \sqrt{d_v} - n\tilde{U}^T\tilde{U}\tilde{w} + O(n^{-1/2}). $$  \((63)\)

Adding and subtracting quantities,

$$ \tilde{U}^T\tilde{U}\tilde{w} = S^{-1/2}\tilde{U}^T\tilde{U}\tilde{S}^{1/2}\tilde{w} - (\tilde{S}^{-1/2}\tilde{U}^T\tilde{U}\tilde{S}^{1/2} - \tilde{U}^T\tilde{U})\tilde{w}. $$  \((64)\)

By Lemma 5,

$$ \| \tilde{U}^T\tilde{S}^{1/2} - S^{1/2}\tilde{U}^T\tilde{U} \| = O(n^{-1}), $$

so that, applying submultiplicativity followed by Lemmas 7 and 5,

$$ \| (\tilde{S}^{-1/2}\tilde{U}^T\tilde{U}\tilde{S}^{1/2} - \tilde{U}^T\tilde{U})\tilde{w} \| \leq \|\tilde{S}^{-1/2}\|\|\tilde{U}^T\tilde{S}^{1/2} - \tilde{S}^{1/2}\tilde{U}^T\tilde{U}\|\|\tilde{w}\| = O(n^{-3/2}). $$

Plugging this into Equation \((64)\) we have shown that

$$ n\tilde{U}^T\tilde{U}\tilde{w} = n\tilde{S}^{-1/2}\tilde{U}^T\tilde{U}\tilde{S}^{1/2}\tilde{w} + O(n^{-1/2}), $$

and plugging this, in turn, into Equation \((63)\), we have

$$ n(\tilde{w}_{LS} - \tilde{Q}^T\tilde{w}) = n(X^T\tilde{X})^{-1} \tilde{X}^T D^{-1/2} \tilde{a} \sqrt{d_v} - n\tilde{S}^{-1/2}\tilde{U}^T\tilde{U}\tilde{S}^{1/2}\tilde{w} + O(n^{-1/2}) $$

$$ = n(X^T\tilde{X})^{-1} \tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \tilde{X}\tilde{w} \right) + O(n^{-1/2}), $$

where the second equality follows from the definitions of $\tilde{X}$ and $\tilde{X}$ and $\tilde{X}^T\tilde{X} = \tilde{S}$. Again adding and subtracting quantities, we have

$$ n(\tilde{w}_{LS} - \tilde{Q}^T\tilde{w}) = n(X^T\tilde{X})^{-1} \tilde{Q}^T\tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \tilde{X}\tilde{w} \right) $$

$$ + n(X^T\tilde{X})^{-1}(\tilde{X} - \tilde{X}\tilde{Q})^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \tilde{X}\tilde{w} \right) + O(n^{-1/2}). $$  \((65)\)

Expanding the second term on the right-hand side,

$$ (\tilde{X} - \tilde{X}\tilde{Q})^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \tilde{X}\tilde{w} \right) = \sum_{j=1}^{n} \left( \frac{a_j}{\sqrt{d_j d_v}} - \frac{X_j^T \tilde{w}}{\sqrt{t_j n\mu^T \tilde{w}}} \right) (\tilde{X}_j - \tilde{Q}^T\tilde{X}_j) $$

$$ = \sum_{j=1}^{n} \frac{a_j - X_j^T \tilde{w}}{\sqrt{d_j d_v}} (\tilde{X}_j - \tilde{Q}^T\tilde{X}_j) + \sum_{j=1}^{n} \left( \frac{1}{\sqrt{t_j n\mu^T \tilde{w}}} - \frac{1}{\sqrt{d_j d_v}} \right) X_j^T \tilde{w} (\tilde{X}_j - \tilde{Q}^T\tilde{X}_j). $$
Recalling that $\tilde{a}$ is independent of $A$ conditioned on $X$ and that $E[a_j \mid X_j] = X_j^T \tilde{w}$, the first of these two summations is a sum of independent zero-mean random variables, and an application of Hoeffding’s inequality along with Lemmas 1 and 6 yields

$$
\sum_{j=1}^{n} \frac{a_j - X_j^T \tilde{w}}{\sqrt{d_j d_v}} (\tilde{X}_j - \tilde{Q}^T \tilde{X}_j) = O(n^{-3/2} \log n).
$$

Again applying Lemmas 1 and 6

$$
\sum_{j=1}^{n} \left( \frac{1}{\sqrt{t_j n \mu^2}} - \frac{1}{\sqrt{d_j d_v}} \right) X_j^T \tilde{w} (\tilde{X}_j - \tilde{Q}^T \tilde{X}_j) \\
= \sum_{j=1}^{n} \left( \frac{1}{\sqrt{n \mu^2 \tilde{w}}} - \frac{1}{\sqrt{d_v \tilde{t}_j}} \right) X_j^T \tilde{w} (\tilde{X}_j - \tilde{Q}^T \tilde{X}_j) + \sum_{j=1}^{n} \left( \frac{1}{\sqrt{d_j}} - \frac{1}{\sqrt{d_v \tilde{t}_j}} \right) X_j^T \tilde{w} (\tilde{X}_j - \tilde{Q}^T \tilde{X}_j) \\
= O(n^{-3/2} \log n)
$$

Thus, the above two displays imply that

$$
(X - \tilde{X} \hat{Q})^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \hat{X} \tilde{w} \right) = O(n^{-3/2} \log n).
$$

Recalling that $\hat{S} = \tilde{X}^T \tilde{X}$, Lemmas 8 and 9 imply that $\hat{S}$ is invertible eventually, and $\| (\tilde{X}^T \tilde{X})^{-1} \| = O(1)$. Equation (65) thus becomes

$$
n (\tilde{w}_{LS} - \tilde{Q}^T \tilde{w}) = n \hat{S}^{-1} \tilde{Q}^T \tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \hat{X} \tilde{w} \right) + O(n^{-1/2}),
$$

and multiplying through by $\tilde{Q}$ yields

$$
n (\tilde{Q} \tilde{w}_{LS} - \tilde{w}) = n \hat{Q} \hat{S}^{-1} \tilde{Q}^T \tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \hat{X} \tilde{w} \right) + O(n^{-1/2}).
$$

Lemma 8 and the continuity of the inverse imply that

$$
\hat{Q} \hat{S}^{-1} \tilde{Q}^T \xrightarrow{P} \Delta^{-1}.
$$

An application of Slutsky’s Theorem will thus yield our result, provided we can show that

$$
n \tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \hat{X} \tilde{w} \right) \xrightarrow{L} \mathcal{N}(0, \Sigma_{F, w}),
$$

where

$$
\Sigma_{F, w} = \mathbb{E} \left[ \frac{X_j^T \tilde{w} (1 - X_j^T \tilde{w})}{\mu^T \tilde{w}} \left( \frac{X_j}{X_j^T \mu} - \frac{\Delta \tilde{w}}{2 \mu^T \tilde{w}} \right) \left( \frac{X_j}{X_j^T \mu} - \frac{\Delta \tilde{w}}{2 \mu^T \tilde{w}} \right)^T \right].
$$

To establish (66), we recall $t_v = \sum_{j=1}^{n} X_j^T \tilde{w} = \hat{E} d_v$ and note that

$$
n \tilde{X}^T \left( \frac{D^{-1/2} \tilde{a}}{\sqrt{d_v}} - \hat{X} \tilde{w} \right) = n \tilde{X}^T \tilde{T}^{-1/2} (\tilde{a} - X \tilde{w}) + n \tilde{X}^T \left( \frac{D^{-1/2}}{\sqrt{d_v}} - \frac{T^{-1/2}}{\sqrt{t_v}} \right) X \tilde{w} \\
+ n \tilde{X}^T \left( \frac{D^{-1/2}}{\sqrt{d_v}} - \frac{T^{-1/2}}{\sqrt{t_v}} \right) (\tilde{a} - X \tilde{w}).
$$
The last of these terms is \( O(n^{-1/2} \log n) \) by a Hoeffding inequality followed by an application of Lemma 6 so that

\[
n \hat{X}^T \left( \frac{D^{-1/2} \vec{a}}{\sqrt{d_v}} - \hat{X} \hat{w} \right) = n \hat{X}^T T^{-1/2} \left( \frac{\vec{a} - X \hat{w}}{\sqrt{T_v}} \right) + n \hat{X}^T \left( \frac{D^{-1/2}}{\sqrt{d_v}} - \frac{T^{-1/2}}{\sqrt{T_v}} \right) X \hat{w} + O(n^{-1/2} \log n).
\]

Multiplying numerator and denominator and applying Lemma 6, it holds for all \( i \in [n] \)

\[
\frac{1}{\sqrt{d_i}} - \frac{1}{\sqrt{T_i}} = \frac{t_i - d_i}{(\sqrt{d_i} + \sqrt{T_i}) \sqrt{d_i T_i}} = \frac{t_i - d_i}{2t_i^3/2} + (t_i - d_i) \frac{t_i (\sqrt{T_i} - \sqrt{d_i}) + (t_i - d_i) \sqrt{T_i}}{2t_i^3/2 (d_i \sqrt{T_i} + t_i \sqrt{d_i})}
\]

\[
= \frac{t_i - d_i}{2t_i^3/2} + O(n^{-3/2} \log n),
\]

and a similar result holds for the out-of-sample vertex, in that

\[
\frac{1}{\sqrt{d_i}} - \frac{1}{\sqrt{T_i}} = \frac{t_v - d_v}{2t_v^3/2} + O(n^{-3/2} \log n).
\]

Thus,

\[
\hat{X}^T \left( \frac{D^{-1/2}}{\sqrt{d_v}} - \frac{T^{-1/2}}{\sqrt{T_v}} \right) X \hat{w}
= \hat{X}^T T^{-1/2} \left( \frac{1}{\sqrt{d_v}} - \frac{1}{\sqrt{T_v}} \right) X \hat{w} + \hat{X}^T \left( \frac{D^{-1/2} - T^{-1/2}}{\sqrt{d_v}} \right) X \hat{w}
= \hat{X}^T T^{-1/2} \frac{t_v - d_v}{2t_v^3/2} X \hat{w} + \frac{\hat{X}^T T^{-3/2} (T - D) X \hat{w}}{2 \sqrt{d_v}} + \sum_{j=1}^{n} \xi_j X_j^T \hat{w} \left( \frac{1}{\sqrt{T_j}} + \frac{1}{\sqrt{d_j}} \right) \frac{X_j}{\sqrt{T_j}}
\]

where \( \xi_j \in \mathbb{R}, j = 1, 2, \ldots, n \) satisfy \( \xi_j = O(n^{-3/2} \log n) \). Using Lemma 6 this last sum is itself \( O(n^{-3/2} \log n) \), so that

\[
n \hat{X}^T \left( \frac{D^{-1/2}}{\sqrt{d_v}} - \frac{T^{-1/2}}{\sqrt{T_v}} \right) X \hat{w} = n \hat{X}^T T^{-1/2} \frac{t_v - d_v}{2t_v^3/2} X \hat{w}
+ n \hat{X}^T \frac{T^{-3/2} (T - D) X \hat{w}}{2 \sqrt{d_v}} + O(n^{-1/2} \log n).
\]

Plugging this into Equation \( 67 \),

\[
n \hat{X}^T \left( \frac{D^{-1/2} \vec{a}}{\sqrt{d_v}} - \hat{X} \hat{w} \right) = n \hat{X}^T T^{-1/2} \left( \frac{\vec{a} - X \hat{w}}{\sqrt{T_v}} \right) + n \hat{X}^T T^{-1/2} \frac{t_v - d_v}{2t_v^3/2} X \hat{w}
+ n \hat{X}^T \frac{T^{-3/2} (T - D) X \hat{w}}{2 \sqrt{d_v}} + O(n^{-1/2} \log n).
\]

To complete our proof, it will suffice to show the following two facts:

\[
n \hat{X}^T T^{-1/2} \left( \frac{\vec{a} - X \hat{w}}{\sqrt{T_v}} \right) \xi \sim \mathcal{N}(0, \Sigma_{F, \hat{w}})
\]

\[
n \hat{X}^T \frac{T^{-3/2} (T - D) X \hat{w}}{2 \sqrt{d_v}} \xrightarrow{p} 0
\]

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To see the latter of these two points, observe that by our definitions of \(d_i = \sum_{j=1}^{n} A_{i,j}\) and 
\[ t_i = \sum_{j=1}^{n} X_j^T X_i, \]

\[
n \hat{X}^T \frac{T^{-3/2}(T - D)X \bar{w}}{2\sqrt{d_v}} = \frac{n}{2\sqrt{d_v}} \sum_{i=1}^{n} \frac{(t_i - d_i)}{t_i^2} X_i^T \bar{w}X_i
\]

\[
= \frac{n}{2\sqrt{d_v}} \sum_{i=1}^{n} \frac{X_i^T X_i}{t_i^2} X_i^T \bar{w}X_i + \frac{n}{2\sqrt{d_v}} \sum_{1 \leq i < j \leq n} (X_j^T X_i - A_{i,j}) \left( \frac{X_j^T \bar{w}}{t_j^2} X_j + \frac{X_i^T \bar{w}}{t_i^2} X_i \right).
\]

The former of these two sums is \(O(n^{-1/2})\) by an application of Lemma 6 and using the fact that \(X_i \in \text{supp } F\) are bounded. The latter of these two sums is, conditioned on \(\{X_i\}_{i=1}^{n}\), a sum of independent 0-mean random variables, with \(\|t_j^{-2}(X_j^T X_i - A_{i,j})X_j^T \bar{w}X_j\| \in [-Ct_j^{-2}, Ct_j^{-2}]\) for all \(j \in [n]\). Thus,

\[
\Pr \left[ \left| \sum_{1 \leq i < j \leq n} t_j^{-2}(X_j^T X_i - A_{i,j})X_j^T \bar{w}X_j \right| \geq s \right] \leq 2 \exp \left\{ -Cs^2 \sum_{1 \leq i < j \leq n} t_j^{-4} \right\}.
\]

Let \(E_n = \{t_j \geq Cn : j = 1, 2, \ldots, n\}\) denote the high-probability event of Lemma 6 for which we have \(\Pr[E_n^c] \leq Cn^{-2}\) for all suitably large \(n\). Taking \(s = Cn^{-1} \log^{1/2} n\) for suitably large \(C > 0\), letting \(\mathbb{P}_{E_n}\) denote conditional probability \(\Pr[\cdot | E_n]\),

\[
\mathbb{P}_{E_n} \left[ \left| \sum_{1 \leq i < j \leq n} t_j^{-2}(X_j^T X_i - A_{i,j})X_j^T \bar{w}X_j \right| \geq Cn^{-1} \log^{1/2} n \right] \leq Cn^{-2}.
\]

Thus,

\[
\Pr \left[ \left| \sum_{1 \leq i < j \leq n} t_j^{-2}(X_j^T X_i - A_{i,j})X_j^T \bar{w}X_j \right| \geq Cn^{-1} \log^{1/2} n \right]
\leq \mathbb{P}_{E_n} \left[ \left| \sum_{1 \leq i < j \leq n} t_j^{-2}(X_j^T X_i - A_{i,j})X_j^T \bar{w}X_j \right| \geq Cn^{-1} \log^{1/2} n \right] + \Pr[E_n^c]
\leq Cn^{-2},
\]

and we conclude that, bounding \(d_v^{-1/2} = O(n^{-1/2} \log^{1/2} n)\) by Lemma 6,

\[
n \hat{X}^T \frac{T^{-3/2}(T - D)X \bar{w}}{2\sqrt{d_v}} = O(n^{-1/2} \log^{1/2} n),
\]

which establishes \(69\).

It remains only to prove Equation 68. Let \(m_i = nX_i^T \mu\) for \(i \in [n]\) and define the diagonal matrix

\[ M = \text{diag}(m_1, m_2, \ldots, m_n) \in \mathbb{R}^{n \times n}. \]

The argument in Lemma 6 allows us to bound \(\|t_v^{-1/2} - (n\mu^T \bar{w})^{-1/2}\|\), so an argument similar to that above wherein we apply Hoeffding’s inequality followed by Lemma 6 implies

\[
n \left( \frac{1}{\sqrt{t_v}} - \frac{1}{\sqrt{n\mu^T \bar{w}}} \right) \hat{X}^T T^{-1/2} (\bar{a} - X \bar{w}) = O(n^{-1/2} \log n).
\]
Lemma 6 also bounds \( \max_i |t_i^{-1/2} - m_i^{-1/2}| \), whence
\[
\frac{n \hat{X}^T (T^{-1/2} - M^{-1/2})(\bar{a} - \bar{X} \bar{w})}{\sqrt{n \mu T \bar{w}}} = O(n^{-1/2} \log n).
\]

The same Hoeffding-style argument once again yields, recalling that \( \hat{X} = T^{-1/2} X \),
\[
\frac{n X^T (T^{-1/2} - M^{-1/2}) M^{-1/2} (\bar{a} - X \bar{w})}{\sqrt{n \mu T \bar{w}}} = O(n^{-1/2} \log n).
\]

Combining the above three displays, the first term in the quantity of interest in Equation 68 is
\[
\frac{n \hat{X}^T T^{-1/2}(\bar{a} - X \bar{w})}{\sqrt{\nu_T}} = \frac{n X^T M^{-1}(\bar{a} - X \bar{w})}{\sqrt{n \mu T \bar{w}}} + O(n^{-1/2}).
\]

Turning to the second term on the left-hand side of Equation 68, rearranging terms and recalling the definition of \( \tilde{\Delta} = E X_1^T (X_1^T \mu) \),
\[
\frac{n \hat{X}^T T^{-1/2}(t_v - d_v) X \bar{w}}{2t_v^{3/2}} = \frac{n(t_v - d_v) \hat{X}^T \hat{X} \bar{w}}{2(n \mu T \bar{w})^{3/2}} + O(n^{-1/2}) = \frac{n(t_v - d_v) \hat{\Delta} \bar{w}}{2(n \mu T \bar{w})^{3/2}} + O(n^{-1/2}),
\]
where the first equality follows from Lemma 6 and the second equality follows from using (multivariate) Hoeffding’s inequality to bound
\[
\| \hat{X}^T \hat{X} - \hat{\Delta} \| = \left\| \sum_{i=1}^n \frac{X_i X_i^T}{X_i^T \mu} - \hat{\Delta} \right\| = O(n^{-1/2} \log^{1/2} n).
\]

Thus, combining with Equation 70, the quantity on the left-hand side of Equation 68 is
\[
n \hat{X}^T T^{-1/2} \left( \frac{\bar{a} - X \bar{w}}{\sqrt{\nu_T}} + \frac{t_v - d_v}{2t_v^{3/2}} X \bar{w} \right) = \frac{n X^T M^{-1}(\bar{a} - X \bar{w})}{\sqrt{n \mu T \bar{w}}} + \frac{n(t_v - d_v) \hat{\Delta} \bar{w}}{2(n \mu T \bar{w})^{3/2}} + O(n^{-1/2} \log^{1/2} n).
\]

Rearranging, and recalling \( m_j = n X_j^T \mu, t_v = \sum_{j=1}^n X_j \bar{w} \) and \( d_v = \sum_{j=1}^n a_j \),
\[
n \hat{X}^T T^{-1/2} \left( \frac{\bar{a} - X \bar{w}}{\sqrt{\nu_T}} + \frac{t_v - d_v}{2t_v^{3/2}} X \bar{w} \right) = n \sum_{j=1}^n \frac{a_j - X_j \bar{w}}{\sqrt{\mu_j T \bar{w}}} \left( \frac{X_j}{n X_j^T \mu} - \frac{\hat{\Delta} \bar{w}}{2n \mu T \bar{w}} \right) + O(n^{-1/2} \log^{1/2} n)
\]
\[
= \sum_{j=1}^n \frac{(a_j - X_j \bar{w})}{\sqrt{\mu_j T \bar{w}}} \left( \frac{X_j}{n X_j^T \mu} - \frac{\hat{\Delta} \bar{w}}{2n \mu T \bar{w}} \right) + O(n^{-1/2} \log^{1/2} n).
\]

Observe that this is a sum of \( n \) independent mean-zero random variables, so that by the multivariate CLT and Slutsky’s Theorem,
\[
n \hat{X}^T T^{-1/2} \left( \frac{\bar{a} - X \bar{w}}{\sqrt{\nu_T}} + \frac{t_v - d_v}{2t_v^{3/2}} X \bar{w} \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{F, \bar{w}}),
\]
where
\[
\Sigma_{F, \bar{w}} = E \left[ \frac{X_j \bar{w} (1 - X_j^T \bar{w})}{\mu_j T \bar{w}} \left( \frac{X_j}{X_j^T \mu} - \frac{\hat{\Delta} \bar{w}}{2n \mu T \bar{w}} \right) \left( \frac{X_j}{X_j^T \mu} - \frac{\hat{\Delta} \bar{w}}{2n \mu T \bar{w}} \right)^T \right],
\]
completing the proof. \( \square \)
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