Vertex operator algebras associated to modular invariant representations for \( A_1^{(1)} \) *

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Abstract

We investigate vertex operator algebras \( L(k,0) \) associated with modular-invariant representations for an affine Lie algebra \( A_1^{(1)} \), where \( k \) is 'admissible' rational number. We show that VOA \( L(k,0) \) is rational in the category \( \mathcal{O} \) and find all irreducible representations in the category of weight modules.

1 Introduction

Vertex operator algebras (VOA) are mathematical counterpart of conformal field theory (CFT). It is very interesting that some representations of affine Lie algebras carries the structure of VOA (or modules for VOA) [FLM], [FZ], [MP].

The new sight in the theory of representation of VOA was made by Frenkel and Zhu (see [FZ], [Z]) by introducing the associative algebra \( A(V) \) associated to VOA \( V \). So called \( A(V) \)-theory gave an (theoretical) elegant way for the classification of all irreducible representations of \( V \) and for calculating the 'fusion rules'. They also introduce the term of rational VOA which is a VOA with finite number of irreducible modules, such that every finitely generated module is completely reducible.

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In [FZ], [MP], [KWN] the irreducible representations of the VOA $L(k, 0), k \in \mathbb{N}$, associated to the irreducible highest weight representations for an affine Lie algebra, were classified. It seems that this case is much simpler because the associative algebra $A(L(k, 0))$ is finite dimensional (see [KWN]).

The main goal of this paper is a classification of the irreducible representations of the simple vertex operator algebra $L(k, 0)$ for $A_1^{(1)}$ on the admissible rational level $k$. Our main result is that irreducible $L(k, 0)$–modules from the category $\mathcal{O}$ are exactly modular invariant representations for $A_1^{(1)}$. To show this, we use $A(V)$–theory and identify $A(L(k, 0))$ with the certain quotient of $U(g)$. Here we used Malikov-Feigin-Fuchs formula for the singular vectors in the Verma modules. Then, by using classification of the irreducible representations in the category $\mathcal{O}$, we find all irreducible representations in the category of weight modules for $A_1^{(1)}$.

Feigin and Malikov in [FM] have the geometrical approach to the similar problem (see also [AY]). They calculated conformal blocks for three admissible modules associated to three different points on $\mathbb{CP}^1$. We interpret our result of the classification of irreducible modules in the terms of conformal blocks considered in [FM].

2 Preliminaries

2.1 Vertex operator algebras and modules

Definition 2.1 A vertex operator algebra is a $\mathbb{Z}$–graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ with a sequence of linear operators $\{a(n) \mid n \in \mathbb{Z}\} \subset \text{End } V$ associated to every $a \in V$, such that for fixed $a, b \in V$, $a(n)b = 0$ for $n$ sufficiently large. We call the generating series $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$, vertex operators associated to $a$, satisfy the following axioms:

(V1) $Y(a, z) = 0$ iff $a = 0$.

(V2) There is a vacuum vector, which we denote by $1$, such that

\[ Y(1, z) = I_V \]  ($I_V$ is the identity of $\text{End } V$).
(V3) There is a special element $\omega \in V$ (called the Virasoro element), whose vertex operator we write in the form

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega(n)z^{-n-1} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

such that

$$L_0 |_{V_n} = nI |_{V_n},$$

$$Y(L_{-1} a, z) = \frac{d}{dz} Y(a, z) \text{ for every } a \in V,$$  \hspace{1cm} (1)

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c,$$  \hspace{1cm} (2)

where $c$ is some constant in $\mathbb{C}$, which is called the rank of $V$.

(V4) The Jacobi identity holds, i.e.

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(a, z_1)Y(b, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) Y(b, z_2)Y(a, z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(a, z_0)b, z_2)$$  \hspace{1cm} (3)

for any $a, b \in V$.

The subspace $I$ of $V$ is called ideal if $Y(a, z)b \in I[[z, z^{-1}]]$ for every $a \in V, b \in I$. Given an ideal $I$ in $V$ such that $1 \notin I, \omega \notin I$, the quotient $V/I$ admits a natural VOA structure (see [FZ]).

**Definition 2.2** Given an VOA $V$, a representation of $V$ (or $V$-module) is a $\mathbb{Z}_{+}$-graded vector space $M = \bigoplus_{n \in \mathbb{Z}_{+}} M_n$ and a linear map

$$V \longrightarrow (\text{End } M)[[z, z^{-1}]],$$

$$a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1},$$

satisfying
(M1) $a(n)M_m \subset M_{m+\deg a - n - 1}$ for every homogeneous element $a$.

(M2) $Y_M(1, z) = I_M$, and setting $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, we have

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c,$$

$$Y_M(L_{-1}a, z) = \frac{d}{dz} Y_M(a, z)$$

for every $a \in V$.

(M3) The Jacobi identity holds, i.e.

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(a, z_1) Y_M(b, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) Y_M(b, z_2) Y_M(a, z_1)$$

$$z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(a, z_0)b, z_2)$$

for any $a, b \in V$.

The submodules, quotient modules, irreducible modules, completely reducible modules are defined in the usual way ([FHL]).

2.2 Associative algebra $A(V)$

Let $V$ be a VOA. For any homogeneous element $a \in V$ and for any $b \in V$, following [Z] we define

$$a * b = \text{Res}_z \left( \frac{(1 + z)^{\text{wt} a}}{z} \right) Y(a, z)b. \quad (5)$$

Then extend this product bilinearly to the whole space $V$. Let $O(V)$ be the subspace of $V$ linearly spanned by the elements of type

$$\text{Res}_z \left( \frac{(1 + z)^{\text{wt} a}}{z^2} \right) Y(a, z)b \quad \text{for homogeneous elements } a, b \in V. \quad (6)$$

Set $A(V) = V/O(V)$. The multiplication $*$ induces the multiplication on the $A(V)$ and $A(V)$ becomes an associative algebra. The image of 1 in $A(V)$ becomes the identity element till the image of $\omega$ is in center of $A(V)$ (see [Z]). Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a $V$–module. For a homogeneous element $a \in V$ we define $o(a) = a(\deg a - 1)$. From the definition of $M$ follows that operator $o(a)$ preserves the graduation of $M$. 

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Theorem 2.1

(a) On $\text{End}(M_0)$ we have

$$o(a)o(b) = o(a \ast b)$$
$$o(x) = 0$$

for every $a, b \in V$, $x \in O(V)$. The top level $M_0$ is an $A(V)$–module.

(b) Let $U$ be an $A(V)$–module, there exists $V$–module $M$ such that $A(V)$–module $M_0$ and $U$ are isomorphic.

Thus, we have one-to-one correspondence between irreducible $V$–modules and irreducible $A(V)$–modules.

We have the following consequence of the definition of $A(V)$.

Proposition 2.1 Let $I$ be an ideal of $V$ Assume $1 \not\in I, \omega \not\in I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/[I]$, where $[I]$ is the image of $I$ in $A(V)$.

2.3 Vertex operator algebras associated to affine Lie algebras

Let $g$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. The affine Lie algebra $\hat{g}$ associated with $g$ is defined as $g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with the usual commutation relations. Let $g = n_- + h + n_+$ and $\hat{g} = \hat{n}_- + \hat{h} + \hat{n}_+$ be the usual triangular decompositions for $g$ and $\hat{g}$ and $P = \mathbb{C}[t] \otimes g \oplus \mathbb{C}c$ be upper parabolic subalgebra. Let $U$ be any $g$–module. Considering $U$ as a $P$–module, we have the induced module (so called generalized Verma module) $M(\ell, U) = U(\hat{g}) \otimes_{U(P)} U$, where the central element $c$ acts as multiplication with $l \in \mathbb{C}$.

For $\lambda \in h^*$ with $M(\lambda)$ we denote Verma module and with $V(\lambda)$ its irreducible quotient.

Set $M(\ell, \lambda) = M(\ell, V(\lambda))$. Let $L(\ell, \lambda)$ denotes its irreducible quotient.
Theorem 2.2 ([FZ]) Every $M(\ell, 0)$ $\ell \neq -g$ (where $g$ denotes dual Coxeter number) has the structure of VOA. Let $U$ be any $g$–module. Then every $M(\ell, U)$ is a module for $M(\ell, 0)$. In particular $M(\ell, \lambda)$ is $M(\ell, 0)$–module.

Theorem 2.3 The associative algebra $A(M(\ell, 0))$ is canonically isomorphic to $U(g)$ and the isomorphism $F : A(M(\ell, 0)) \to U(g)$ is given by:

$$F \left[ a_1(-i_1 - 1) \cdots a_n(-i_n - 1) \right] = (-1)^{i_1+\cdots+i_n} a_n \cdots a_1. \quad (7)$$

for every $a_1, \cdots, a_n \in g$ and every $i_1, \cdots, i_n \in \mathbb{Z}_+$.

3 Irreducible modules for VOA $L(k, 0)$ in the category $\mathcal{O}$

3.1 Modular invariant representations for $A^{(1)}_1$

Let now $g = sl(2, \mathbb{C})$ with generators $e, f, h$ and relations: $[h, f] = -2f$, $[h, e] = 2e$, $[e, f] = h$. Let $\Lambda_0, \Lambda_1$ denote the fundamental weights for $g$, and $\omega$ the fundamental weight for $g$.

Definition 3.1 $k = p/q \in \mathbb{Q}$ is called admissible if $q \in \mathbb{N}$, $p \in \mathbb{Z}$, $(p, q) = 1$ and $2q + p - 2 \geq 0$.

In [KW] V.Kac and M.Wakimoto define modular invariant representations. They also define weights which have admissible level and satisfy some technical conditions (for definition see [KW]). They call them admissible weight.

The following proposition describes the admissible weights and modular invariant representations on level $k$:

Proposition 3.1 Let $k = p/q \in \mathbb{Q}$ be admissible. Set $t = k + 2$. Define :

$$P^k = \{(k-n+mt)\Lambda_0 + (n-mt)\Lambda_1, \ m, n \in \mathbb{Z}_+, \ n \leq 2q+p-2, \ m \leq q-1\}.$$
Let $M$ be any irreducible highest weight module, with the highest weight $\lambda$.

The following statements are equivalent:

1. $M$ is a modular-invariant.
2. $\lambda$ is an admissible weight.
3. $\lambda \in P^k$.

(For proof see [KW]).

We will need the following description of the set $P^k$.

**Lemma 3.1** Let $\lambda \in \hat{h}^*$. Then $\lambda \in P^k$ if and only if

$$\langle \lambda, c \rangle = k, \quad \langle \lambda, h \rangle = (N - it - j)$$

where $i \in \{0, \ldots, l\}, j \in \{1, \ldots, N\}$, $N = 2q + p - 1, l = q - 1$.

By using Cor. 2.1 in [KW] or Kac determinant formula we have

**Theorem 3.1** Let $k = p/q \in Q$ be admissible. Then

$$L(k, 0) = M(k, 0)/U(\hat{g})v_{sing},$$

where vector $v_{sing}$ is the unique singular vector of the weight $k\Lambda_0 - q(2q + p - 1)\delta + (2q + p - 1)\alpha$.

We also need the following theorem:

**Theorem 3.2** (Kac–Wakimoto) Let $M$ be a $\hat{g}$–module from the category $\mathcal{O}$ such that for any irreducible subquotient $L(\mu)$ the weight $\mu$ is admissible. Then $\hat{g}$–module $M$ is completely reducible.
3.2 Malikov-Feigin-Fuchs formula

Now, we recall the result of Malikov-Feigin-Fuchs which give us the singular vector in form with ”rational powers” (see [MFF]).

**Theorem 3.3** *(Malikov-Feigin-Fuchs)* The maximal submodule of \( M(k,0) \) is generated by the singular vector given by the \( v_{\text{sing}} = F(k) \) where

\[
F(k) = e^{-1}N+lt f(0)^{N+(l-1)t} \cdots f(0)^{N-(l-1)t} e^{-1}N-lt,
\]

for \( N = 2q + p - 1, l = q - 1 \) and \( t = p/q + 2 \).

**Remark 3.1** In [MMF] were proved that this formula really make sense, because only with commuting we can transform formula (8) in the usually form in \( U(\hat{g}) \).

3.3 Fundamental lemma

First we define:

\[
\epsilon : \quad U(\hat{n}_-) \rightarrow U(g)
\]

\[
a_1(-i_1) \cdots a_s(-i_s) \mapsto a_1 \cdots a_s,
\]

for every \( a_1, \ldots, a_s \in g, \ s \in \mathbb{N} \).

In the same way as in [F] we have:

**Proposition 3.2**

\[
\epsilon(F(k)) = \prod_{i=1}^{l} \prod_{j=1}^{N} p_{i,j}(h) e^N,
\]

where \( p_{i,j}(h) = ef + (it + j - 1)h - (it + j)(it + j - 1) \).

We define the \( \mathbb{Z} \)-gradation on \( U(\hat{g}) \):

\[
\deg a_1(-i_1) \cdots a_k(-i_k) = i_1 + \cdots + i_k,
\]

for every \( a_1, \ldots, a_k \in g \).

In the following lemma we will use the ordinary transposing \( T \) in \( U(g) \) (see [Dix]).
Lemma 3.2 Let $g \in U(n_{-})$, such that $\deg g = n$. Then we have

$$\epsilon(g) \equiv (-1)^n(F[g.1])^T \mod U(g)n_{-}.$$ 

Proof. First notice that $n_{-}.1 = 0$. Since $\deg g = n$, one can write $g$ in a form

$$g = \sum_{i=0}^{r} g_i f(0)^i,$$

where

$$g_i = \sum a_{i_1}^{(i)} (-j_1 - 1) \cdots a_{i_t}^{(i)} (-j_t - 1),$$

$a_{i_1}^{(i)}, \ldots, a_{i_t}^{(i)} \in g$, $j_1, \ldots, j_t \in \mathbb{Z}_+$, $j_1 + \cdots + j_t + t = n$, $r \in \mathbb{Z}_+$, and get

$$\epsilon(g) \equiv g_0 \mod U(g)n_{-}.$$ 

Set $a_{ij} = a_{ij}^{(0)}$. Since

$$g.1 = g_0.1 = \sum a_{i_1} (-j_1 - 1) \cdots a_{i_t} (-j_t - 1),$$

we have that :

$$F([g.1])^T = \sum (-1)^{n-t}(a_{i_t} \cdots a_{i_1})^T$$

$$= (-1)^n \sum (a_{i_t} \cdots a_{i_1})$$

and lemma holds. $\square$

Set $Q = F([v_{sing}]) \in U(g)$. From proposition 3.2 and lemma 3.2 we have

Lemma 3.3

$$Q^T \equiv (-1)^{q(2q-p-1)} \prod_{i=1}^{l} \prod_{j=1}^{N} p_{i,j}(h)e^N \mod U(g)n_{-}$$

where polynomials $p_{i,j}$ are as in proposition 3.2.
3.4 Classification of representation

The vertex operator algebra $M(k,0)$ has the maximal ideal $M^1(k,0)$. It is generated by the vector $v_{\text{sing}}$. Let $L(k,0)$ be the quotient VOA. The proposition 2.1 and theorem 2.3 imply:

**Proposition 3.3** $A(L(k,0))$ is isomorphic to $U(\mathfrak{g})/I$ where $I$ is a two sided ideal generated by the vector $Q$.

Let $U$ be any $A(L(k,0))$–module. Then $U$ is a $\mathfrak{g}$–module. We have

**Proposition 3.4** Let $U$ be any $U(\mathfrak{g})$–module. Then the following statements are equivalent:

1. $U$ is a $A(L(k,0))$–module,

2. $Q.U = 0$.

Set $R = U(\mathfrak{g}).Q$ and $R^T = U(\mathfrak{g}).Q^T$. Clearly $R$ and $R^T$ are irreducible $\mathfrak{g}$–modules and $R \cong R^T \cong V(2N\omega) \cong V^*(2N\omega)$.

From this facts and proposition 3.4 one can obtain:

**Lemma 3.4** Let $V(\mu)$ be the irreducible highest weight $\mathfrak{g}$–module with the highest weight vector $v_\mu$. The following statements are equivalent:

1. $V(\mu)$ is a $A(L(k,0))$–module ,

2. $RV(\mu) = R^T V(\mu)^* = 0$,

3. $R_0 v_\mu = R^T_0 v_\mu^* = 0$,

where $R_0$ ( $R^T_0$ ) denotes the zero-weight subspace of $R$ ( $R^T$ ).

For $p \in S(\mathfrak{h})$ and $\mu \in \mathfrak{h}^*$ define $p(\mu) \in \mathbb{C}$ with $p(h).v_\mu = p(\mu)v_\mu$.

Let $u_1 \in R_0$ and $u_2 \in R^T_0$. Clearly there exists unique polynomials $p_1, p_2 \in S(\mathfrak{h})$ such that

$$u_1 \equiv p_1(h) \mod U(\mathfrak{g})n_+ \quad u_2 \equiv p_2(h) \mod U(\mathfrak{g})n_-.$$

Then $u_1.v_\mu = p_1(\mu)v_\mu$ and $u_2.v_\mu^* = p_2(-\mu)v_\mu^*$.

We have:
Lemma 3.5 There is one-to-one correspondence between each two of the following three sets:

1. $\mu \in h^*$ such that $V(\mu)$ is $A(L(k,0))$–module,
2. $\mu \in h^*$ such that $p_1(\mu) = 0$,
3. $\mu \in h^*$ such that $p_2(-\mu) = 0$.

3.5 The main theorem
The following lemma is obtained by direct calculation:

Lemma 3.6

$$[f^N, ef + (it + j - 1)(h - (it + j))] = (-N - 1 + it + j)(h - it - j + N)f^N$$

Proposition 3.5 All irreducible $A(L(k,0))$–modules from the category $\mathcal{O}$ are: $V(r\omega)$, $r \in S$, where

$$S = \{N - it - j : i = 0, ..., l; j = 1, ..., N\}.$$  \hspace{1cm} (10)

Proof. Let $u \in R^T_0$. Then $u = (ad f)^NQ^T \equiv f^NQ^T \mod U(g)n_-$. By using lemma 3.6 we have

$$u \equiv c_1 \prod_{i=1}^{l} \prod_{j=1}^{N} q_{i,j}(h)f^N e^N \mod U(g)n_-,$$

where $q_{i,j}(h) = h - it - j + N$, $c_1 \in C$. Since $f^N e^N \equiv c_2 h(h + 1) \cdots (h + n - 1) \mod U(g)n_-$, for some $c_2 \in C$, we conclude that polynomial $p_2$ from lemma 3.3 is proportional to

$$\prod_{i=0}^{l} \prod_{j=1}^{N} (h - it - j + N).$$

Now, proposition follows from lemma 3.5. $\blacksquare$

We can obtain the main theorem:
Theorem 3.4 The set \{L(k, r_\omega) : r \in S\} provides a complete list of irreducible \(L(k, 0)\)-modules from the category \(\mathcal{O}\). Moreover, the irreducible \(L(k, 0)\)-modules from the category \(\mathcal{O}\) are exactly irreducible highest weight representations with admissible highest weights.

Proof. Proposition \ref{prop:3.5} and theorem \ref{thm:2.1} imply that \(L(k, r_\omega)\), for \(r \in S\), are all irreducible \(L(k, 0)\)-modules from the category \(\mathcal{O}\). The second statement follows from lemma \ref{lem:3.1}. \(\square\)

Theorem 3.5 Let \(M\) be a \(L(k, 0)\)-module from the category \(\mathcal{O}\). Then \(M\) is completely reducible \(L(k, 0)\)-module.

Proof. Let \(M\) be a \(L(k, 0)\)-module from the category \(\mathcal{O}\) and let \(N\) be an irreducible subquotients of \(M\). Then \(N\) is irreducible \(L(k, 0)\)-module. From the theorem \ref{thm:3.4} follows that \(N\) is a irreducible highest weight module with admissible highest weight. Now theorem \ref{thm:3.2} implies that \(M\) is completely reducible \(\hat{g}\)-module and so completely reducible \(L(k, 0)\)-module. \(\square\)

Remark 3.2 Vertex operator algebra is by definition rational if it has only finitely many irreducible modules and if every finitely generated module is a direct sum of irreducible ones. We have showed that VOA \(L(k, 0)\) has finitely many irreducible modules in the category \(\mathcal{O}\) and every module from the category \(\mathcal{O}\) is completely reducible. By using this arguments we say that the vertex operator algebra \(L(k, 0)\), for \(k \in \mathbb{Q}\) admissible, is rational in the category \(\mathcal{O}\).

Remark 3.3 In \[A\] were considered some modular invariant representations for \(C_1^{(1)}\) and proved that the VOA \(L(n - \frac{3}{2}, 0)\), \(n \in \mathbb{N}\), is rational in the category \(\mathcal{O}\).

We have :

Conjecture 3.1 Let \(g\) be any simple finite-dimensional Lie algebra and \(L(k, 0)\) associated vertex operator algebra such that the highest weight of \(L(k, 0)\) is admissible. Then \(L(k, 0)\) is rational in the category \(\mathcal{O}\).
4 Irreducible modules for $L(k,0)$ in the category of weight modules

Let $M$ be any irreducible $L(k,0)$–module. From [FHL] we have that the contragradient $L(k,0)$–module $M^*$ is also irreducible. Moreover, $M^{**}$ and $M$ are isomorphic $L(k,0)$–modules. One can easily see that for $M = L(k, \lambda)$ $M_0^*$ is isomorphic to $V(\lambda)^*$. We have:

**Proposition 4.1** $V(r\omega)^*, r \in S$, are all irreducible lowest weight $A(L(k,0))$–modules.

Set $E_{r,\mu} = t^\mu C[t, t^{-1}]$ where $r, \mu \in C$ and $E_i = t^{\mu+i}$. We define $U(g)$ action on $E_{r,\mu}$ by the following formulas:

$$
e_i.E_i = -(\mu+i)E_{i-1}, \quad h.E_i = (-2\mu-2i+r), \quad f.E_i = (\mu+i-r)E_{i+1}. \quad (11)$$

We will find all pairs $(r, \mu)$, such that $E_{r,\mu}$ is irreducible $A(L(k,0))$–module.

**Theorem 4.1** Set $T = \{ (r, \mu) : r \in S - Z_+, \mu \notin Z, r - \mu \notin Z \}$. Then $E_{r,\mu}$ is irreducible $A(L(k,0))$–module if and only if $(r, \mu) \in T$.

**Proof.** First, we notice that $E_{r,\mu}$ is an irreducible $U(g)$–module iff $\mu \notin Z$ and $r - \mu \notin Z$.

By using (11) we have:

$$Q.E_i = (p_0(r) + p_1(r)(i+\mu) + \cdots + p_N(r)(i+\mu)^N)E_{i-N}, \quad (12)$$

for some polynomials $p_0, p_1 \cdots p_N$ and $\mu \in C$.

**Step 1.** Let $E_{r,\mu}$ be an irreducible $A(L(k,0))$–module, then $r \in S - Z_+$.

From the proposition 3.4 follows $Q.E_i = 0$ for all $i \in Z$. From this fact and from (12) we have that:

$$p_0(r) = p_1(r) = \cdots = p_N(r) = 0.$$ 

If $\mu = 0$ we have that $C[t]$ is a submodule of $C[t, t^{-1}]$ isomorphic to $M(r\omega)$. From (12) and proposition 3.4 follows that $M(r\omega)$ is $A(L(k,0))$–module. Then the theorem 3.4 implies that $r \in S - Z_+$ (in this case $V(r\omega) = M(r\omega)$).

**Step 2.** If $(r, \mu) \in T$ then $E_{r,\mu}$ is the irreducible $A(L(k,0))$–module.
Since \( r \in S - \mathbb{Z}_+ \), we have that \( M(r) = V(r) \cong \mathbb{C}[t] \) and \( Q.C[t] = 0 \). By using (11) for \( \mu = 0 \), we conclude that \( p_0(r) = p_1(r) = \cdots = p_N(r) = 0 \). This fact implies that
\[
Q.E_i = 0 \quad \text{for all} \quad i \in \mathbb{Z},
\]
and we obtain that for \((r, \mu) \in T, E_{r,\mu} \) is \( A(L(k,0)) \) –module. ☐

Recall that \( U(g) \)–module \( U \) is called weight module if \( h \) acts semisimple on \( U \) and all weight subspaces are finite-dimensional. We know that irreducible weight modules are : highest weight, lowest weight and modules \( E_{r,\mu} \) defined by (11).

We have obtained:

**Corollary 4.1** Let \( U \) be an irreducible \( A(L(k,0)) \)–weight module. Then \( U \) is one of the following modules :

1. \( V(r\omega), r \in S, \)
2. \( V(r\omega^*), r \in S \) or
3. \( E_{r,\mu}, (r, \mu) \in T. \)

**Theorem 4.2** Let \( M \) be an irreducible \( L(k,0) \)– module such that \( M_0 \) is a weight module. Then \( M \) is one of the following modules :

1. \( L(k, V(r\omega)), r \in S, \)
2. \( L(k, V(r\omega^*)), r \in S \) or
3. \( L(k, E_{r,\mu}), (r, \mu) \in T. \)

**5 Connection with geometrical approach**

Let \( E \) be any finite subset of \( \mathbb{CP}^1 \) and let \( g(E) \) denotes the Lie algebra of meromorphic functions on \( \mathbb{CP}^1 \) holomorphic outside \( E \) with values in \( g \). For every \( z \in \mathbb{CP}^1 \) and \( \lambda \in h^* \) we can define irreducible highest weight \( g(z) \)–module \( L(k, \lambda, z) \) attached to \( z \) (for definition see [FM]).
Let $z_1, z_2, z_3$ be three different points on $\mathbb{CP}^1$ and $\lambda_1, \lambda_2, \lambda_3 \in h^*$. We consider $g(z_1, z_2, z_3)$-module $L(k, \lambda_1, z_1) \otimes L(k, \lambda_2, z_2) \otimes L(k, \lambda_3, z_3)$ and the space of coinvariants

$$H^\circ(g(z_1, z_2, z_3), L(k, \lambda_1, z_1) \otimes L(k, \lambda_2, z_2) \otimes L(k, \lambda_3, z_3)).$$

In previous section we showed that the irreducible $L(k, 0)$-modules (in the category $\mathcal{O}$) are exactly $L(k, r\omega)$, $r \in S$. Those modules were considered in [FM]. They calculated dimension of the space

$$H^\circ(g(0, 1, \infty), L(k, r_1\omega, 0) \otimes L(k, r_2\omega, 1) \otimes L(k, r_3\omega, \infty))$$

(this space is also called conformal block) for all triples $r_1, r_2, r_3 \in S$ and obtained "fusion algebra".

When $r_3 = 0$ their result implies that

$$\dim H^\circ(g(0, 1, \infty), L(k, r_1\omega, 0) \otimes L(k, r_2\omega, 1) \otimes L(k, 0, \infty)) = \begin{cases} 1 & \text{if } r_1 = r_2 \in S \\ 0 & \text{otherwise} \end{cases}$$

We have the following characterisation of $L(k, 0)$-modules:

**Theorem 5.1** $L(k, \lambda)$ is a $L(k, 0)$-module if and only if

$$\dim H^\circ(g(0, 1, \infty), L(k, \lambda, 0) \otimes L(k, \lambda, 1) \otimes L(k, 0, \infty)) = 1.$$

As in [FHL], for three modules we can define so called fusion rules (dimension of the space of intertwining operators). From the previous theorem follows that when one of the modules is $L(k, 0)$ then fusion rules and dimension of corresponding conformal block are equal. It seems that this is thrue for any three modules.

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