Critical Temperature for $\alpha$-Particle Condensation within a Momentum Projected Mean Field Approach

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Alpha-particle (quartet) condensation in homogeneous spin-isospin symmetric nuclear matter is investigated. The usual Thouless criterion for the critical temperature is extended to the quartet case. The in-medium four-body problem is strongly simplified by the use of a momentum projected mean field ansatz for the quartet. The self-consistent single particle wave functions are shown and discussed for various values of the density at the critical temperature.

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I. INTRODUCTION

Investigation of pairing in different Fermi systems is still on the forefront of active research. Examples are nuclear physics [1] and the physics of cold fermionic atoms [2]. The formation and condensation of heavier clusters in Fermi systems is much less studied.

In cold atom physics the recent advent of trapping three different species of fermions [2] has opened up the possibility of creating gases of heavier clusters. For the time being those may be trions (bound state of three different fermions) but in the future one also can think of quartets (bound state of four different fermions). The latter are specially interesting because of their bosonic nature and the possibility of Bose-Einstein condensation (BEC) of quartets. The description of quartet condensation to occur has been attempted with an extension of the so-called Cooper problem to the four body case in [1]. In [5] a variational procedure for the condensation of $(2s + 1)$-component fermion clusters, with $s$ the fermion spin, has been proposed. A quartet phase has been found in a one dimensional model with four different fermions [6].

On the other hand, nuclear physics, because it is a four-component fermion-system (proton/neutron spin up/down), all fermions attracting one another, leading to the very strongly bound $\alpha$-particle, is a proto-type system for quartetting. There, the formation of clusters has been an object of study almost since the beginning of nuclear physics [7]. Of course, pairing also exists in nuclei from where it is concluded that neutron stars are superfluid. Nuclei are very small quantum objects with only a (slowly) fluctuating phase (the conjugate variable to particle number $N$). Actually the number of Cooper pairs in nuclei generally does not exceed about a dozen (often much less) and yet clear signs of superfluidity are observed in nuclei (e.g., moments of inertia strongly reduced from their classical value), implying that no critical size exists from where signatures of superfluidity abruptly disappear. One, thus, can safely extrapolate from finite nuclei to superfluidity in neutron stars. On the other hand, as already mentioned, in nuclear physics the existence of quartets ($\alpha$-particles) as subclusters of nuclei is omnipresent. As well known, many lighter nuclei with equal proton and neutron numbers ($Z = N$) show, for instance in excited states, strong $\alpha$ clustering. The concept that these $\alpha$-particles may form a condensate in certain low density states of nuclei and that this may, in analogy to the pairing case, be a precursor sign of $\alpha$-particle condensation in infinite matter [8], has come up only recently [9]. Also heavy nuclei seem to have preformed $\alpha$-clusters in the surface because of their well known spontaneous $\alpha$ decay properties.

Symmetric nuclear matter does not exist in nature because of the too strong Coulomb repulsion. However, in collapsing stars, so-called proto-neutron stars, the fraction of protons is still high [10] and the formation of $\alpha$-particles and, at sufficiently low temperature, their condensation may eventually be possible. At any rate, it seems evident that nuclear matter at various degrees of asymmetry is unstable with respect to cluster formation in the low density regime. At zero temperature, the most stable nucleus is $^{56}$Fe but as a function of temperature, density, asymmetry, other cluster compositions of infinite baryonic matter may be formed. Several theoretical studies predict $\alpha$-phases to exist in certain temperature-density- asymmetry domains [11].

In view of the complexity of the task, the objective of the present work is quite modest. We want to study the critical temperature of $\alpha$-particle condensation as a function of density and temperature in symmetric nuclear matter. Still, even this task will not be carried out down to the BEC limit. We will study the critical temperature $T_c^\alpha$ for the onset of formation of $\alpha$-particles in a thermal
gas of nucleons. This shall be done with a theory analogous to the famous Thouless criterion for the onset of formation of Cooper pairs in a superconductor. On the microscopic level the problem is still very challenging, since it amounts to solve an in-medium four-body problem. In spite of that, solutions have already been worked out in the past, either solving the Faddeev-Yakubovsky equations [12] or with an approximate procedure [8].

In this work, we will continue along those lines. The final objective is to reach the BEC regime in a treatment similar to the one of Nozières and Schmitt-Rink (NSR) theory [13], but for quartets. Needless to say that this only will be possible if the whole formalism can radically be simplified. Actually, as we will show in this work, such a procedure may well exist. In any case, it is not conceivable that one treats condensation of bosonic clusters built out of N fermions on the level of non-linear in-medium N-body equations for N > 2. On the other hand, it is well known, that nuclei can satisfactorily be described, in mean field approximation [14]. Projecting these mean field (Hartree-Fock) type of solutions on zero total momentum (K = 0) will then allow these mean field clusters to Bose condense. Actually it is well known among the nuclear physics community that even for such a small nucleus as the α-particle a momentum projected mean field approach yields a very reasonable description [15]. The reason for this stems, as already mentioned, from the presence of four different fermions, all attracting one another with about the same force.

In Fig. 1 we sketch the situation, indicating that the two protons and two neutrons occupy the lowest 0S level of the mean field potential of harmonic oscillator shape.

![FIG. 1: Sketch of α-particle configuration, indicating that the two protons and two neutrons occupy the lowest 0S level in the mean field potential of harmonic oscillator shape.](image)

In the present case of an in-medium quartet, the corresponding equation reads as follows [8]:

\[
(E - \varepsilon_{1234})\Psi_{1234} = \left(1 - f_1 - f_2\right) \sum_{1/2^+} v_{12,1'2'} \Psi_{1'2'34} + \left(1 - f_1 - f_3\right) \sum_{1/3'} v_{13,1'3'} \Psi_{1'2'3'4}
\]

+ permutations,

(1)

where \(f_i = f(\varepsilon_i) = \left[e^{(\varepsilon_i - \mu)/T} + 1\right]^{-1}\) with \(\varepsilon_i = \varepsilon(k_i) = k_i^2/(2m)\) is the Fermi-Dirac distribution and \(\varepsilon_{1234} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4\) \((\hbar = c = k_B = 1:\text{ natural units})\). The matrix element of the interaction is \(v_{12,1'2'}\) with the numbers 1, 2, 3, ... standing for all quantum numbers as momenta, spin, isospin, etc., as also in all other quantities in (1).

In Eq. (1), when \(E = 4\mu\), this signals quartet condensation in very much the same manner as in the two body equation

\[
(E - \varepsilon_{12})\Psi_{12} = (1 - f_1 - f_2) \sum_{1/2'} v_{12,1'2'} \Psi_{1'2'},
\]

(2)

where \(\varepsilon_{12} = \varepsilon_1 + \varepsilon_2\), the approach of \(T \rightarrow T_c\) such that \(E \rightarrow 2\mu\) signals the transition to a superconducting or superfluid state (the well known Thouless criterion [17]).

Of course, as already stated several times, the determination of \(T_c\) needs the heavy solution of the in-medium modified four particle equation (1).

Following the discussion in the introduction, we, therefore, make the following ‘projected’ mean field ansatz for the quartet wave function [11, 12, 18],

\[
\Psi_{1234} = (2\pi)^3\delta^{(3)}(k_1 + k_2 + k_3 + k_4) \prod_{i=1}^4 \varphi(k_i) \chi^{ST},
\]

(3)

where \(\chi^{ST}\) is the spin-isospin function which we suppose to be the one of a scalar \((S = T = 0)\). We will not further mention it from now on. We work in momentum space and \(\varphi(k)\) is the as-yet unknown single particle 0S wave function. In position space, this leads to the usual formula [14] \(\Psi_{1234} \rightarrow \int d^3R \prod_{i=1}^4 \hat{\varphi}(r_i - R)\), where \(\hat{\varphi}(r_i)\) is the Fourier transform of \(\varphi(k_i)\). If we take for \(\varphi(k)\) Gaussian shape, this gives: \(\Psi_{1234} \rightarrow \exp[-c \sum_{1 \leq i < k \leq 4} (r_i - r_k)^2]\) which is the translationally invariant ansatz often used to describe α-clusters in nuclei. For instance, it is also employed in the α-particle condensate wave function of Totsaki, Horiuichi, Schuck, Röpke (THSR) in [9].

Inserting the ansatz (3) into (1) and integrating over superfluous variables, or minimizing the energy, we arrive at the following non-linear, Hartree-Fock type of equation for the single particle 0S wave function \(\varphi(k) = \varphi(|k|)\)

\[
A(k)\varphi(k) + 3B(k) + 3C(k)\varphi(k) = 0,
\]

(4)

where \(A(k), B(k),\) and \(C(k)\) are given by:

\[
A(k_1) = \int \prod_{i=2}^4 \frac{d^3k_i}{(2\pi)^3} \left[ \sum_{i=1}^4 \frac{k_i^2}{2m} - 4\mu \right]
\]
\[ B(k_1) = \int \prod_{i=2}^{4} \frac{d^3 k_i}{(2\pi)^3} \delta^{(3)}(\sum_{i=1}^{4} k_i) \times |\varphi(k_2)|^2 |\varphi(k_3)|^2 |\varphi(k_4)|^2 (2\pi)^3 \delta^{(3)}(\sum_{i=1}^{4} k_i) \]  
\[ C(k_1) = \int \prod_{i=2}^{4} \frac{d^3 k_i}{(2\pi)^3} \delta^{(3)}(\sum_{i=1}^{4} k_i) \times |\varphi(k_2)|^2 |\varphi(k_3)|^2 |\varphi(k_4)|^2 \times (2\pi)^3 \delta^{(3)}(\sum_{i=1}^{4} k_i), \]  
where, in this pilot study, we neglect mean field shifts and effective mass contributions. 

From Eq. (10), we obtain the single particle wave function in momentum space as 
\[ \varphi(k) = \frac{-3B(k)}{A(k) + 3C(k)}. \]  
As seen in Eqs. (4), (6), and (7), since \( A(k), B(k), \) and \( C(k) \) depend on the wave function of \( \varphi(k) \), Eq. (8) is strongly non-linear. Its solution can be found by iteration. 

For a general two body force \( v_{k_1,k_2,k_3,k_4} \), the equation to be solved is still rather complicated. We, therefore, proceed to the last simplification and replace the two body force by a unique separable one, that is 
\[ v_{k_1,k_2,k_3,k_4} = \lambda e^{-k^2/k^2_0} e^{-k^2/k^2_0} (2\pi)^3 \delta^{(3)}(K - K'), \]  
where \( k = (k_1 - k_2)/2, k' = (k_1 - k_2)/2, K = k_1 + k_2, \) and \( K' = k'_1 + k'_2. \) This means that we take a spin-isospin averaged two body interaction and disregard that in principle the force may be somewhat different in the \( S, T = 0, 1 \) or \( 1, 0 \) channels. 

We are now ready to study the solution of (11) for the critical temperature \( T_{c}^{\alpha} \) when the eigenvalue hits \( 4\mu. \) For later comparison, the deuteron (pair) wave function at the critical temperature is also represented from Eqs. (2) and (10) to be 
\[ \phi(k) = -\frac{1 - 2f(\varepsilon)}{k^2/m - 2\mu} \lambda e^{-k^2/k^2_0} \int \frac{d^3 k'}{(2\pi)^3} e^{-k^2/k^2_0} \phi(k'), \]  
where \( \phi(k) \) is a relative wave function of two particles given by \( \Psi_{12} \to \phi(k_1 + k_2) \delta^{(3)}(k_1 + k_2), \) and \( \varepsilon = k^2/(2m). \) From Eq. (10), the critical temperature of pair condensation is obtained with the following equation: 
\[ 1 = -\lambda \int \frac{d^3 k}{(2\pi)^3} \frac{1 - 2f(\varepsilon)}{k^2/m - 2\mu} e^{-k^2/k^2_0}. \]  

III. RESULTS FOR THE CRITICAL TEMPERATURE \( T_{c}^{\alpha} \) 

In order to determine the critical temperature for \( \alpha \)-condensation as a function of density \( n \), we need to determine the chemical potential \( \mu \) via 
\[ n = 4 \int \frac{d^3 k}{(2\pi)^3} f(\varepsilon) \]  
and adjust the temperature so that the eigenvalue of (11) hits \( 4\mu. \) The two open constants \( \lambda \) and \( k_0 \) in Eq. (9) are determined so that binding energy (\( -28.3 \) MeV) and radius (\( 1.71 \) fm) of the free \( \alpha \)-particle come out right. The adjusted values are: \( \lambda = -992 \) MeV fm\(^3\), and \( b = 1.43 \) fm\(^{-1}\). The results of the calculation are shown in Fig. [2]. 

In Fig. [2] the maximum of critical temperature \( T_{c}^{\alpha,\text{max}} \) is at \( \mu = 5.5 \) MeV, and the \( \alpha \)-condensation can exist up to \( \mu_{\text{max}} = 11 \) MeV. It is very remarkable that the obtained results for \( T_{c}^{\alpha} \) well agree with a direct solution of (11). These results for \( T_{c}^{\alpha} \) are by about 25 percent higher than the ones of our earlier publication [8]. We, however, checked that the underlying radius of the \( \alpha \)-particle in that work is larger than the experimental value and that \( T_{c}^{\alpha} \) decreases with increasing radius of \( \alpha \)-particle. Furthermore a different variational wave function was used in [8]. 

In Fig. [2] we also show the critical temperature for
deuteron condensation derived from Eq. (11). In this case, we take \( \lambda = -1305 \text{ MeV fm}^3 \) and \( k_0 = 1.46 \text{ fm}^{-1} \) to get experimental energy \((-2.2 \text{ MeV})\) and radius \((1.95 \text{ fm})\) of the deuteron. It is seen that at higher densities deuteron condensation wins over the one of \( \alpha \)-particle. The latter breaks down rather abruptly at a critical positive value of the chemical potential. Roughly speaking, this corresponds to the point where the \( \alpha \)-particles start to overlap. This behavior stems from the fact that Fermi-Dirac distributions in the four body case, see (13), can never become step-like, as in the two body case, even not at zero temperature, since the pairs in an \( \alpha \)-particle are always in motion. As a consequence, \( \alpha \)-condensation generally only exists as a BEC phase and the weak coupling regime is absent.

Fig. 3 shows the normalized self-consistent solution of the wave function in momentum space derived from Eq. (3) and the wave function in position space defined by its Fourier transform \( \tilde{\varphi}(r) \). This corresponds to the point where the \( \alpha \)-particles start to overlap. This behavior stems from the fact that Fermi-Dirac distributions in the four body case, see (13), can never become step-like, as in the two body case, even not at zero temperature, since the pairs in an \( \alpha \)-particle are always in motion. As a consequence, \( \alpha \)-condensation generally only exists as a BEC phase and the weak coupling regime is absent.

IV. DISCUSSION AND CONCLUSIONS

In this work we again took up the study of the critical temperature of \( \alpha \)-particle (quartet) condensation in homogeneous symmetric nuclear matter. We essentially confirm the behavior of two previous studies [8, 12]. The objective of the paper was to show that practically same results as before can be obtained with a strongly simplifying ansatz for the four particle wave function. Namely, this time, we used a momentum projected mean field variational wave function. This is based on the fact that the four different fermions of the quartet can occupy the same single particle 0S-wave function in the mean field. The latter is to be determined from a self-consistent non linear HF-type of equation as a function of chemical potential or density. The relation between the chemical potential and density is taken from the free Fermi gas relation, Eq. (12). However, the total nucleon density of the system must be calculated from the mean single nucleon state occupation number taking into account correlations, so that the contribution of bound states to the total nucleon density is taken into account, see Ref. [20]. To calculate the critical temperature not as function of the free nucleon density, see Fig. 2b, but of the total nucleon density, a generalization à la NSR 13 must be performed, that is we have at least to incorporate the contribution of the \( \alpha \)-particle density including the condensate to the single particle occupation numbers. This shall be investigated in future work.
Besides, in this work, we used the isospin-independent separable potential, Eq. (9), for the two-body interaction as a simplification. Comparison with a realistic two-body interaction is interesting. This study also shall be done in the future.

The self-consistent wave function has been studied in momentum and position space. For negative chemical potential the single particle wave function behaves like a Gaussian. However, once the chemical potential turns positive, then the single particle wave function in $r$-space starts to oscillate. This is a well known feature from ordinary pairing.

We, therefore, have demonstrated that a very simplifying momentum projected mean field ansatz suffices to account for the salient features of quartet condensation. This is very helpful for the next step which is more complicated, i.e. the incorporation of quartet condensation self-consistently into the Equation of State (EOS).

We should, however, be aware of the fact that our projected mean field ansatz for the quartet wave function can only be a valid approximation so long as well defined quartets exist. In the break down region seen on Fig. 2, this is certainly no longer the case. How the quartet phase evolves into a superfluid phase of pairs is an open question. A possibility to study this very interesting problem could be to write down the in-medium four body equation directly in the BCS formalism, i.e. with the corresponding BCS coherence factors. It may be foreseen that the latter only catch on close to the transition region. Another interesting problem for the future is how the present results are modified in the asymmetric case, that is in the case of neutron excess.

The success of our study to employ a very simplifying ansatz of the mean field type for the quartet wave function, may open wide perspectives. Besides to push the description of quartet condensation much further, there might exist the possibility that even for the case of a gas of trions such a projected mean field ansatz is a quite valid approach. In the case of three colors, like quarks in the constituent quark model for nucleons, a harmonic confining potential is frequently assumed and the three quarks can occupy the lowest $0S$ state, analogously to the case of quartets treated in the present paper. Of course, trions are composite fermions and cannot be treated in the same way as bosonic composites, since they shall form a new Fermi gas with their own new Fermi level. How this situation can eventually be treated has recently been outlined in [21].

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