Weyl-semi symmetric special Para-Sasakian manifold

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Abstract
In this paper, we investigate the theory of Weyl-semi symmetric special Para-Sasakian. In section 1, we have defined special Para-Sasakian manifold and established a few properties thereof. Section 2 is devoted to the study of Weyl-pseudo symmetric and Weyl-semi symmetric special Para-Sasakian manifold. The results of this paper are believed to be new and unified in nature.

Keywords
Weyl-semi symmetric, Weyl-pseudo symmetric, Special Para-Sasakian manifold, Levi-Civita connection, Riemannian manifold.

AMS Subject Classification
53C25, 53Cxx, 53-XX.

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1. Introduction

Let $M$ be a connected $n$-dimensional Riemannian manifold of class $C^\infty$ with a positive definite metric $g$ which admits a unit 1-from $\eta$ satisfying

$$\nabla_\beta \eta_\alpha - \nabla_\alpha \eta_\beta = 0$$

and

$$\nabla_\gamma \nabla_\beta \eta_\alpha = -(g_{\gamma\beta} \eta_\alpha + g_{\gamma\alpha} \eta_\beta) + 2 \eta_\gamma \eta_\beta \eta_\alpha$$

wherein $\nabla$ denotes the covariant differentiation with regard to Levi-Civita connection.

If we take

$$\xi^\alpha = g^{\alpha\beta} \eta_\beta$$

$$\eta_\alpha = g_{\alpha\beta} \xi^\beta$$

we get

$$\phi^\alpha_\beta = \nabla_\beta \xi^\alpha$$

$$\phi_\alpha_\beta = g_{\alpha\gamma} \phi^\gamma_\beta$$

Consequently, we obtain

$$\eta_\alpha \xi^\alpha = 1$$

$$\phi_\alpha^\beta = \phi_\beta^\alpha$$

$$\phi_\alpha^\gamma \phi^\alpha_\gamma = \delta_\alpha^\beta - \eta_\beta \xi^\alpha$$

$$g_{\gamma\epsilon} \phi^\gamma_\beta \phi^\beta_\epsilon = g_{\alpha\beta} - \eta_\alpha \xi^\beta$$
and

\[ \text{rank}(\phi^a_\beta) = (n-1) \] \hspace{1cm} (1.13)

These relations shows that the manifold \( M \) is a special para contact Riemannian manifold with a structure \((\phi, \xi, \eta, g)\). Such a manifold is called a Para-Sasakian manifold \([1, 5]\).

If in a Para-Sasakian manifold \( M \) the unit 1-form \( \eta \)-satisfying the relation

\[ \nabla_\beta \eta_\alpha = \varepsilon (-g_\beta \alpha + \eta_\beta \eta_\alpha), \] \hspace{1cm} (1.14)

wherein \( \varepsilon = \pm 1 \), then the manifold \( M \) is termed as special Para-Sasakian manifold or briefly SP-Sasakian manifold \([4]\).

From \([2]\), we have

\[ S_{\alpha \beta} \xi_\beta = -(n-1) \eta_\alpha. \] \hspace{1cm} (1.15)

\[ \eta_\alpha R^\lambda_{\alpha \beta \gamma} = g_{\alpha \beta} \eta_\gamma - g_{\beta \gamma} \eta_\alpha. \] \hspace{1cm} (1.16)

\[ g^{\alpha \beta} S_{\alpha \beta} = \tau. \] \hspace{1cm} (1.17)

### 2. Weyl-Semi Symmetric Special Para-Sasakian Manifold

Let \( M \) be an \( n \)-dimensional \((n \geq 3)\) differentiable manifold of class \( C^\infty \) and \( \nabla \) denotes its Levi-Civita connection. Also let \( S \) is the Ricci tensor of \( n \)-dimensional differentiable manifold \( M \).

The Ricci operator \( S \) is defined as

\[ S_{\alpha \beta} \eta^\beta = S_\alpha^\gamma \] \hspace{1cm} (2.1)

and the covariant tensor of rank two \((S^2)\) is defined as

\[ (S^2)_{\alpha \beta} = (S.S)_{\alpha \beta} = S_{\alpha \alpha} S_\beta^\alpha. \] \hspace{1cm} (2.2)

The Weyl conformal curvature operator is defined as

\[ C^\alpha_\beta = R^\alpha_\beta - \frac{1}{(n-2)} \left[ \delta^\alpha_\alpha \delta_\beta^\gamma + S_\alpha^\gamma \delta_\beta^\gamma - k \frac{1}{(n-1)} \delta_\beta^\gamma \right] \] \hspace{1cm} (2.3)

and the Weyl conformal curvature tensor is defined as

\[ C_{\alpha \beta \gamma} = g_{\gamma \epsilon} C_{\alpha \beta}. \] \hspace{1cm} (2.4)

wherein \( k \) is the scalar curvature of \( n \)-dimensional differentiable manifold \( M \).

**Definition 2.1.** If the tensor \( R.C \) and \( Q(g, C) \) are linearly dependent then the manifold \( M \) is termed as Weyl-Pseudo symmetric special Para-Sasakian manifold \([2, 3]\).

This is equivalent to

\[ R.C = L_C Q(g, C). \] \hspace{1cm} (2.5)

**Definition 2.2.** A special Para-Sasakian manifold \( M \) with the properties

\[ C.S = 0 \] \hspace{1cm} (2.6)

is termed as Weyl semi-symmetric special Para-Sasakian manifold.

**Remark 2.3.** It is noteworthy that a conformally symmetric special Para-Sasakian manifold is Weyl semi-symmetric.

Next, we define the tensor \( C.S \) on \((M, g)\) as follows

\[ C^\alpha_\beta S_\epsilon^\gamma = -(S_\beta \epsilon C^\alpha_\gamma + S^\alpha_\gamma C_{\beta \epsilon}). \] \hspace{1cm} (2.7)

Equation (2.7) can be written as

\[ S_{\alpha \gamma} C^\gamma_\beta + S_{\alpha \epsilon} C^\epsilon_\beta = 0. \] \hspace{1cm} (2.8)

Contracting equation (2.8) by \( \xi^\alpha \) and using equation (1.15) yields

\[ \eta_\alpha C^\gamma_\beta + \eta_\epsilon C^\epsilon_\beta = 0. \] \hspace{1cm} (2.9)

By virtue of equations (1.15), (1.16), (2.2) and (2.3), we obtain

\[ \eta_{\beta} S_{\alpha \gamma} + \eta_{\beta} S_{\alpha \beta} - (1-n)(\eta_{\beta} g_{\alpha \beta} + \eta_{\beta} g_{\alpha \gamma}) + \frac{1}{(n-2)} \left[ (S.S)_{\alpha \gamma} + \eta_{\beta} (S.S)_{\alpha \beta} - (1-n)^2 (\eta_{\beta} g_{\alpha \gamma}) + \eta_{\gamma} g_{\alpha \beta} \right] + \frac{k}{(n-1)(n-2)} \left[ (1-n)(\eta_{\beta} g_{\alpha \gamma}) + \eta_{\gamma} g_{\alpha \beta} - \eta_{\beta} S_{\alpha \gamma} - \eta_{\beta} S_{\alpha \beta} \right] = 0. \] \hspace{1cm} (2.10)

Contracting equation (2.10) by \( \xi_\gamma \) and using equations (1.15), (2.2), we get

\[ (S.S)_{\alpha \beta} = \frac{k}{(n-1)(n-2)} S_{\alpha \beta} + (k+n-1) g_{\alpha \beta}. \] \hspace{1cm} (2.11)

In view of above discussion, we observe the following theorem:

**Theorem 2.4.** If \( n \)-dimensional special Para-Sasakian manifold is Weyl-semi symmetric then the following condition (2.11) holds good.

Let us consider an \( \eta \)-Einstein special Para-Sasakian manifold, then we can write \([2]\):

\[ S_{\alpha \beta} = a g_{\alpha \beta} + b \eta_\alpha \eta_\beta, \] \hspace{1cm} (2.12)

wherein \( a \) and \( b \) are smooth functions on \( M \).

Contracting equation (2.12) with \( g^{\alpha \beta} \) and using equation (1.17), we get

\[ na + b = \tau. \] \hspace{1cm} (2.13)

Further, contracting equation (2.12) with \( \xi_\beta \) and using equations (1.7), (1.15) yields

\[ a + b = (1-n). \] \hspace{1cm} (2.14)
Subtracting equation (2.14) from equation (2.13), we get

\[ a = 1 - \frac{\tau}{(1-n)}. \quad (2.15) \]

Inserting this value of \( a \) in equation (2.14), we obtain

\[ b = \frac{\tau}{1-n} - n. \quad (2.16) \]

Consequently, we have a theorem:

**Theorem 2.5.** If \( \eta \)-Einstein special Para-Sasakian manifold is Weyl-semi-symmetric admits a vector field \( \xi^a \) characterised by the relation (2.12) then the smooth functions are connected by the relations (2.15) and (2.16).

Substituting the values of \( a \) and \( b \) in equation (2.12), we get

\[ S_{\alpha \beta} = (1 - \frac{\tau}{(1-n)})g_{\alpha \beta} + (\frac{\tau}{(1-n)} - n)\eta_{\alpha} \eta_{\beta}. \quad (2.17) \]

Consequently, we have a theorem:

**Theorem 2.6.** If an \( \eta \)-Einstein special Para-Sasakian manifold admits \( C.S. = 0 \), and a vector field \( \xi^a \) characterised by the relation (2.12) then the Ricci tensor holds the relation (2.17).

In this regard, we have a theorem:

**Theorem 2.7.** For an \( \eta \)-Einstein special Para-Sasakian manifold with the condition \( C.S. = 0 \), the following relation \( S_{\alpha \beta} \phi^\beta_\gamma = (1 - \frac{\tau}{(1-n)})\phi_{\alpha \gamma} \) holds good.

**Proof.** Contracting equation (2.17) with \( \phi^\beta_\gamma \) and using equations (1.6), (1.10) yields

\[ S_{\alpha \beta} \phi^\beta_\gamma = (1 - \frac{\tau}{(1-n)})\phi_{\alpha \gamma}. \quad (2.18) \]

From equations (1.12) and (2.17), we get

\[ S_{\alpha \beta} = (1-n)g_{\alpha \beta} - (\frac{\tau}{1-n} - n)g_{\gamma \epsilon} \phi^\gamma_\epsilon \phi^\alpha_\beta. \quad (2.19) \]

As a consequence of equations (1.6) and (2.19), we obtain

\[ S_{\alpha \beta} = (1-n)g_{\alpha \beta} - (\frac{\tau}{1-n} - n)\phi_{\epsilon \alpha} \phi^\epsilon_\beta. \quad (2.20) \]

By virtue of equations (1.5) and (2.20), we observe that

\[ S_{\alpha \beta} = (1-n)g_{\alpha \beta} - (\frac{\tau}{1-n} - n)(\nabla \epsilon \eta_\alpha)(\nabla \beta \xi^\epsilon). \quad (2.21) \]

Contracting equation (2.20) with \( \xi^\beta \) and using equation (1.9) yields

\[ S_{\alpha \beta} \xi^\beta = -(n-1)\eta_\alpha. \quad (2.22) \]

This expression obtained above is similar to the expression (1.15) given by Mileva Prvanovic [2].

In view of above, we have the following theorems:

**Theorem 2.8.** For \( \eta \)-Einstein special Para-Sasakian manifold, the relation \( \tau = -(n-1) \) holds good.

**Proof.** Contracting equation (2.22) with \( \eta_\beta \) and using equation (1.7), we obtain

\[ S_{\alpha \beta} = -(n-1)\eta_\alpha \eta_\beta. \quad (2.23) \]

Again contracting equation (2.23) with \( g^{\alpha \beta} \) and using equations (1.3), (1.7) yields

\[ g^{\alpha \beta} S_{\alpha \beta} = -(n-1). \quad (2.24) \]

From equations (1.17) and (2.24), we get

\[ \tau = -(n-1) \quad (2.25) \]

**Theorem 2.9.** If \( \eta \)-Einstein special Para-Sasakian manifold admits \( C.S. = 0 \), then the following relation \( (S.S)_{\alpha \beta} \phi^\beta_\gamma = (k+n-1)\phi_{\alpha \gamma} \) holds good.

**Proof.** Contracting equation (2.23) with \( \phi^\beta_\gamma \) and using the equation (1.10) yields

\[ S_{\alpha \beta} \phi^\beta_\gamma = 0. \quad (2.26) \]

Contracting equation (2.11) with \( \phi^\beta_\gamma \) and using equations (2.26), we get

\[ (S.S)_{\alpha \beta} \phi^\beta_\gamma = (k+n-1)\phi_{\alpha \gamma}. \quad (2.27) \]

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