Implications of a frame dependent dark energy for the spacetime metric, cosmography, and effective Hubble constant

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In earlier papers we showed that a frame dependent effective action motivated by the postulates of three-space general coordinate invariance and Weyl scaling invariance exactly mimics a cosmological constant in Friedmann-Robertson-Walker (FRW) spacetimes, but alters the linearized equations governing scalar perturbations around a spatially flat FRW background metric. Here we analyze the implications of a frame dependent dark energy for the spacetime cosmological metric within both a perturbative and a non-perturbative framework. Both methods of calculation give a one-parameter family of cosmologies which are in close correspondence to one another, and which contain the standard FRW cosmology as a special case. We discuss the application of this family of cosmologies to the standard cosmological distance measures and to the effective Hubble parameter, with special attention to the current tension between determinations of the Hubble constant at late time, and the Hubble value obtained through the cosmic microwave background (CMB) angular fluctuation analysis.

I. INTRODUCTION

The experimental observation of an accelerated expansion of the universe has been interpreted as evidence for a cosmological term in the gravitational action of the usual form

$$S_{\text{cosm}} = -\frac{\Lambda}{8\pi G} \int d^4x (g^{1/2}) ,$$

with $\Lambda = 3H_0^2\Omega_\Lambda$ in terms of the Hubble constant $H_0$ and the cosmological fraction $\Omega_\Lambda$. This functional form incorporates the usual assumption that gravitational physics is four-space general coordinate invariant, with no frame dependence in the fundamental action.

In a series of papers [1]-[4], motivated by the frame dependence of the CMB radiation, and ideas about scale invariance of an underlying pre-quantum theory, we have studied the implications of the assumption that there is an induced gravitational effective action that is three-space general coordinate and Weyl scaling invariant, but is not four-space general coordinate invariant. This
analysis leads to an alternative dark energy action given by

\[
S_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4x (g^{(4)})^{1/2} (g_{00})^{-2},
\]

which in Friedmann-Robertson-Walker (FRW) spacetimes where \(g_{00} = 1\) exactly mimics the cosmological constant effective action of Eq. (1).

To set up a phenomenology for testing for the difference between standard and frame-dependent dark energy actions, we made the ansatz that the observed cosmological constant arises from a linear combination of the two of the form

\[
S_\Lambda = (1 - f)S_\text{cosm} + f S_{\text{eff}} = -\frac{\Lambda}{8\pi G} \int d^4x (g^{(4)})^{1/2} [1 - f + f (g_{00})^{-2}],
\]

so that \(f = 0\) corresponds to only a standard cosmological constant, and \(f = 1\) corresponds to only an apparent cosmological constant arising from a frame dependent effective action. In [4] we gave detailed results for the scalar perturbation equations around a FRW background arising when dark energy is included through the action of Eq. (3). Our aim in this paper is to analyze the implications of the ansatz of Eq. (3) for the spacetime cosmological metric, with applications to standard cosmological distance measures and the effective Hubble parameter. We will carry out the calculations in two ways, first by using the linearized perturbation equations around a FRW background derived in [4], and then by a non-perturbative method, giving results that are in close correspondence. Both methods yield a one-parameter family of cosmologies, parameterized by the initial value at cosmic time \(t = 0\) of one of the metric components, with the standard FRW cosmology included as a special case.

This paper is organized as follows. In Sec. 2 we state our metric in non-perturbative and perturbative forms, and explain why when \(f \neq 0\) the model of Eq. (3) cannot be reduced to a standard cosmological constant by redefinition of the time variable. We also introduce some notations that will be used frequently later on. In Sec. 3 we solve for the metric dynamics implied by Eq. (3) using the perturbative formalism. In Sec. 4 we repeat this calculation by a non-perturbative method and show that the results closely correspond to those of Sec. 3. In Sec. 5 we give results for the standard cosmographic distance measures and the effective Hubble parameter, and apply our results to the recently much discussed “Hubble tension”. In the Appendix we give details of the perturbative derivation of the metric dynamical equation, which is also obtained as a by-product of the non-perturbative analysis.

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1 An earlier version has been merged into this one.
II. HOMOGENEOUS ISOTROPIC METRIC, WHY THE $f \neq 0$ CASE DOES NOT REDUCE TO $f = 0$, AND SOME NOTATIONS

Our starting point is the observation that the general form for the line element in a homogeneous, isotropic, zero spatial curvature universe in which physics is invariant under three-space general coordinate transformations, but not invariant under four-space general coordinate transformations, is

$$ds^2 = \alpha^2(t)dt^2 - \psi^2(t)d\vec{x}^2 = g_{00}dt^2 + g_{ij}dx^i dx^j$$

(4)

with metric components

$$g_{00} = \alpha^2(t) , \quad g_{ij} = -\delta_{ij}\psi^2(t) ,$$

$$g^{00} = 1/\alpha^2(t) , \quad g^{ij} = -\delta_{ij}/\psi^2(t) .$$

(5)

A. Why the $f \neq 0$ case cannot be reduced to $f = 0$ by redefining the time variable

An evident feature of Eq. (4) is that if we define a proper time $\tau$ by

$$d\tau = \alpha(t)dt , \quad \tau = \int_0^t du\alpha(u) ,$$

(6)

with inversion $t(\tau)$, and denote $\psi(t)$ as written in terms of $\tau$ by $\psi(t(\tau)) = \psi[\tau]$, then Eq. (4) takes the form

$$ds^2 = d\tau^2 - \psi^2[\tau]d\vec{x}^2 .$$

(7)

This has the same form as the standard FRW metric

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 ,$$

(8)

with $\tau$ replacing $t$ and with $\psi(t) = \psi[\tau]$ replacing the FRW expansion factor $a(t)$. However, this does not mean that Eq. (3) with $f \neq 0$ can be reduced to the $f = 0$ case. To see this, we rewrite Eq. (3) in terms of the metric components $\alpha(t)$ and $\psi(t)$, both assumed positive so that

$$(4^g)^{1/2} = \alpha(t)\psi^3(t) ,$$

giving

$$S_\Lambda = -\frac{\Lambda}{8\pi G} \int dt d^3x \alpha(t)\psi^3(t)[1 - f + f\alpha(t)^{-4}] = -\frac{\Lambda}{8\pi G} \int d\tau d^3x \psi^3[\tau][1 - f + f\alpha[\tau]^{-4}] .$$

(9)

We have chosen the arbitrary lower limit of the integral in Eq. (6) so that the proper time origin $\tau = 0$ coincides with the coordinate time origin $t = 0$. 

When $f = 0$ the metric component $\alpha(t)$ is absorbed into the definition of the new time variable, as expected since the standard cosmological action is four-space general coordinate invariant. But when $f \neq 0$, the factor $\alpha(t)^{-4}$ cannot be similarly absorbed, reflecting the fact that the $f$ term is only three-space, but not four-space general coordinated invariant.\footnote{If one were to define $d\tau = dt\alpha(t)[1 - f + f/\alpha(t)]^4$, the Einstein-Hilbert action which is proportional to $\int dtd^3x\alpha(t)\psi^3(t)R$ would be left with a residual $\alpha(t)$ dependence. The same is true if one were to define $d\tau = \alpha(0)dt$.} So we anticipate that Eq. (9) with $f \neq 0$ will give a more general cosmology than the standard FRW cosmology. However, since when $\alpha(t) \equiv 1$ Eq. (9) reduces back to a standard cosmological term, we also anticipate that this more general dynamics will include the standard FRW cosmology as a special case.

### B. Some notations

We record the following notational conventions that will be used throughout.

- The metric components $\alpha(t)$ and $\psi(t)$ form the basis of our non-perturbative treatment. But in both the non-perturbative and perturbative discussions, we will also use the related components $\theta(t)$, $\Phi(t)$ and $\Psi(t)$ defined by

\begin{align*}
\alpha(t) &= 1 + \Phi(t) , \\
\psi(t) &= a(t)\theta(t) , \\
\theta(t) &= 1 - \Psi(t) ,
\end{align*}

with $a(t)$ the standard FRW expansion factor.

- Since our focus is on the matter-dominated era, we use the following convenient approximate formula \footnote{If one were to define $d\tau = dt\alpha(t)[1 - f + f/\alpha(t)]^4$, the Einstein-Hilbert action which is proportional to $\int dtd^3x\alpha(t)\psi^3(t)R$ would be left with a residual $\alpha(t)$ dependence. The same is true if one were to define $d\tau = \alpha(0)dt$.} for $a(t)$,

\begin{equation}
\begin{split}
a(t) &\simeq \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} (\sinh(x))^{2/3} , \\
\Omega_m &= 1 - \Omega_\Lambda ,
\end{split}
\end{equation}

with $x$ the dimensionless time variable

\begin{equation}
x = \frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t .
\end{equation}

At the present era $t = t_0$ this is

\begin{equation}
x_0 = \frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t_0 \simeq 1.169 ,
\end{equation}

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\end{equation}
where we have used the Planck 2018 values \( \Omega_\Lambda = 0.679, \Omega_m = 0.321, H_0 = 66.88 \pm 67 \text{km s}^{-1} \text{Mpc}^{-1}, H_0^{-1} = 14.62 \text{Gyr}, t_0 = 13.83 \text{Gyr} \). So to emphasize, by \( H_0 \) and \( a(t) \) we always mean the Hubble constant and FRW expansion factor as determined by the Planck CMB angular fluctuation analysis.\(^4\)

- The Hubble parameter \( H(t) \) defined by the standard FRW cosmology is

\[
H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\ddot{a}(t)}{a(t)} ,
\]

which in the matter-dominated era, using the approximate formula of Eq. (11), is

\[
H(t) = H_0 \sqrt{\Omega_\Lambda \coth(x)} = H_0 [\Omega_m (1 + z)^3 + \Omega_\Lambda]^{1/2} ,
\]

with \( \coth(x) = \cosh(x)/\sinh(x) \) and with the redshift \( z \) defined by \( 1 + z = 1/a(t) \). This is to be distinguished from the Hubble parameter \( H_{\text{eff}}(t) \) arising from the modified dynamics of Eq. (3), which is given by the proper time derivative

\[
H_{\text{eff}}[\tau] = \frac{d\psi[\tau]/d\tau}{\psi[\tau]} = H_{\text{eff}}(t) = \frac{\dot{\psi}(t)/dt}{\alpha(t)\psi(t)} .
\]

### III. METRIC PERTURBATION DERIVATION

#### A. Setting up the \( \Phi \) equation

In\(^4\) we gave detailed results for the scalar perturbation equations around a FRW background arising when dark energy is included through the action of Eq. (3). Writing

\[
g_{00} = 1 + E ,
\]

\[
g_{i0} = - a(t)(\partial_i F + \text{vector}) ,
\]

\[
g_{ij} = - a^2(t)\delta_{ij} + \partial_i \partial_j B + \text{vector + tensor} ,
\]

we gave formulas in \( B = 0 \) gauge for the linearized equations governing the scalar perturbations \( A, E, \) and \( F, \) for the general case in which these are functions of both space and time. For the analysis

\(^4\) Before the matter-dominated era, when the radiation content of the universe is significant, \( a(t) \) is no longer well approximated by Eq. (11). But this does not affect the numerical results given below.
of this paper, it is more convenient to express the $B = 0$ gauge linear perturbation equations in terms of $F$ and the functions $\Phi$ and $\Psi$ that are related to $E$ and $A$ by

\begin{align*}
E &= 2[\Phi - \partial_t(aF)] , \\
A &= -2[\Psi + \dot{a}F] ,
\end{align*}

(18)
as given in the Appendix. This form of the perturbation equations corresponds to writing the perturbed line element as

\begin{equation}
\begin{aligned}
ds^2 &= [1 + 2\Phi]dt^2 - a^2(t)[1 - 2\Psi]d\vec{x}^2 ,
\end{aligned}
\end{equation}

(19)
in agreement to linear order with Eqs. (4) and (10) above.

In the limit that the metric perturbations have no spatial dependence, the perturbation $F$, which appears in Eq. (17) acted on by spatial derivatives, can be assumed to have the $\vec{x}$-independent limit $F(t) = 0$ and so can be dropped from the metric perturbation equations.\(^5\) Also, in the matter-dominated era the anisotropic inertia $\pi^S = 0$, so assuming continuity in the limit of vanishing spatial dependence, the $Y = 0$ part of Eq. (A11) in the Appendix implies that $\Phi(t) = \Psi(t)$. With these simplifications, we show in the Appendix that the perturbation equations governing the time evolution of $\Phi(t)$ can be put into the form (with $\dot{\cdot} = d/dt$)

\begin{equation}
\begin{aligned}
\ddot{\Phi} + 4\frac{\dot{a}}{a}\dot{\Phi} + \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\Phi &= 4\pi G\delta p + 2\Lambda f \Phi ,
\end{aligned}
\end{equation}

(20)
with $\delta p$ the pressure perturbation. When the term proportional to $f$ is dropped, this agrees with Eq. (7.49) of Mukhanov\(^6\) and Eq. (23c) of Ma and Bertschinger\(^10\) when their conformal time derivatives are converted to time derivatives. In the matter-dominated era, $\delta p$ vanishes for adiabatic perturbations, so then we can drop the $\delta p$ term in Eq. (20), giving

\begin{equation}
\begin{aligned}
\ddot{\Phi} + 4\frac{\dot{a}}{a}\dot{\Phi} + \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\Phi &= 2\Lambda f \Phi .
\end{aligned}
\end{equation}

(21)

It is convenient now to use the dimensionless time variable $x$ introduced above in Eq. (12), which when substituted into Eq. (21) gives finally the evolution equation for $\Phi$ in terms of $x$,

\begin{equation}
\begin{aligned}
\frac{d^2\Phi}{dx^2} + \frac{8}{3} \coth(x) \frac{d\Phi}{dx} &= \frac{4}{3}(2f - 1)\Phi .
\end{aligned}
\end{equation}

(22)

\(^5\) This still allows one to make three-space general coordinate transformations, but not four-space general coordinate transformations, which corresponds to the invariance properties of the action of Eq. (6).
Before proceeding to the numerical solution of Eq. (22), we examine analytically the large and small $x$ behavior of the solutions. For large $x$ the function $\coth(x)$ approaches unity, and Eq. (22) becomes an equation with constant coefficients solved by exponentials,

$$\Phi(x) = C_1 e^{\mu_+ x} + C_2 e^{\mu_- x},$$

$$\mu_{\pm} = -\frac{2}{3}[2 \pm (6f + 1)^{1/2}].$$

(23)

As suggested already by the factor $2f - 1$ in Eq. (22), there is a crossover in behavior at $f = 1/2$. For $f < 1/2$, both exponents in Eq. (23) are negative, and $\Phi$ decays to zero as $x \to \infty$. For $f = 1/2$, one exponent is negative, while the other is zero, and $\Phi$ approaches a constant as $x \to \infty$. For $f > 1/2$, one exponent remains negative, while one is positive, and $\Phi$ becomes infinite as $x \to \infty$. So for $f = 1$, the case of a scale invariant cosmological action, the metric perturbation $\Phi$ grows with time.

We examine next the small $x$ behavior, where the term proportional to $2f - 1$ becomes much less important than the terms on the left of Eq. (22). This equation is then approximated by

$$\frac{d^2 \Phi}{dx^2} + \frac{8}{3} \frac{d \Phi}{x \, dx} = 0,$$

(24)

with the general solution

$$\Phi(x) = C_3 + C_4 x^{-5/3}.$$  

(25)

This shows that sufficient boundary conditions for getting a unique solution are the requirements that the solution be regular at $x = 0$, together with specification of the value $\Phi(0)$.

With this analysis in mind, we rewrite Eq. (22) as an integral equation. Defining the normalized perturbation

$$\hat{\Phi}(x) = \Phi(x)/\Phi(0),$$

(26)

$\hat{\Phi}$ obeys the integral equation

$$\hat{\Phi}(x) = 1 + \frac{4}{3}(2f - 1) \int_0^x dw (\sinh(w))^{-8/3} \int_0^w du (\sinh(u))^{8/3} \hat{\Phi}(u),$$

(27)

which incorporates both the boundary conditions at $x = 0$ and the differential equation of Eq. (22). Starting from an initial assumption $\hat{\Phi}(x) = 1$, and then updating at each evaluation of the
right hand side of Eq. (27), the integral equation converges rapidly to an answer in 5 iterations on an 800 bin mesh\(^6\) taking integrand values at center-of-bin. The results for \(d\hat{\Phi}/dx\) with \(f = 1\) are given in Table I, for various values of \(x\) ranging from 0 to \(x_0\), showing that \(d\hat{\Phi}/dx|_{x=x_0}\) is positive. Consistent with the large time analysis given above, when we repeat the calculation with \(f = 0\) we find a negative value of \(d\hat{\Phi}/dx|_{x=x_0}\).

**TABLE I: Values of \(d\hat{\Phi}/dx\) and \(\hat{\Phi}\) versus \(x/x_0\) and redshift \(z\), calculated with \(f = 1\), that is, with all of the observed cosmological constant arising from a scale invariant but frame dependent action.**

| \(x/x_0\) | \(z\) | \(d\hat{\Phi}/dx\) | \(\hat{\Phi}\) |
|---|---|---|---|
| 1.0 | 0.00 | 0.409 | 1.244 |
| 0.9 | 0.10 | 0.369 | 1.198 |
| 0.8 | 0.22 | 0.331 | 1.158 |
| 0.7 | 0.36 | 0.290 | 1.121 |
| 0.6 | 0.54 | 0.251 | 1.089 |
| 0.5 | 0.77 | 0.209 | 1.062 |
| 0.4 | 1.08 | 0.169 | 1.040 |
| 0.3 | 1.55 | 0.126 | 1.023 |
| 0.2 | 2.36 | 0.0848 | 1.010 |
| 0.1 | 4.36 | 0.0416 | 1.0026 |
| 0.0341 | 10 | 0.0131 | 1.0003 |
| 0.00122 | 100 | 0.000018 | 1.0000 |

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**IV. NON-PERTURBATIVE DERIVATION**

**A. Einstein tensor and equations, and the dark energy and matter energy momentum tensors**

We return now to the general form for the line element given in Sec. 2,

\[
ds^2 = \alpha^2(t)dt^2 - \psi^2(t)d\vec{x}^2 .
\] (28)
Using a Mathematica notebook for general relativity [11], the nonzero Einstein tensor components for this metric are computed to be

\[ G_{00} = -3 \frac{\dot{\psi}^2(t)}{\psi^2(t)} , \]
\[ G_{ij} = \delta_{ij} \frac{2\psi(t)[\alpha(t) \ddot{\psi}(t) - \dot{\alpha}(t) \dot{\psi}(t)] + \alpha(t) \psi^2(t)}{\alpha^3(t)} \]  

(29)

where \( \dot{\psi}^2(t) = (d\psi(t)/dt)^2 \).

Writing the dark energy action of Eq. (3) as

\[ S_\Lambda = -\frac{\Lambda}{8\pi G} \int d^4x (g^{1/2} [1 - f + f/\alpha^4(t)]) , \]  

(30)

and varying this equation with respect to \( g_{ij} \), we get the \( ij \) component of the dark energy contribution to the energy momentum tensor,

\[ T^{ij}_\Lambda = \frac{\Lambda}{8\pi G} [(1 - f)g^{ij} + ft^{ij}] , \quad T_{\lambda ij} = \frac{\Lambda}{8\pi G} [(1 - f)g_{ij} + ft_{ij}] \]  

(31)

with

\[ t^{ij} = -\delta_{ij} \frac{1}{\alpha^4(t)\psi^2(t)} , \quad t_{ij} = -\delta_{ij} \frac{\psi^2(t)}{\alpha^4(t)} \]  

(32)

As detailed in [1]–[4], in order to use this as the source term in the Einstein equations, the remaining components of \( t_{\mu\nu} \) must be determined so that covariant conservation with respect to the metric is satisfied. The conserving completion of the \( g_{ij} \) part of Eq. (31) is just \( g_{\mu\nu} \). Since the metric is diagonal, the conserving completion of \( t_{ij} \) in Eq. (32) has \( t_{0i} = t_{i0} = 0 \), and \( t_{00} \) given by the solution of the covariant conservation equation, which is

\[ \dot{t}_{00}(t) + t_{00}(t) \left[ 3 \frac{\dot{\psi}(t)}{\psi(t)} - 2 \frac{\dot{\alpha}(t)}{\alpha(t)} \right] = 3 \frac{\dot{\psi}(t)}{\alpha^2(t)\psi(t)} \]  

(33)

This equation can be readily integrated to give

\[ t_{00}(t) = 3 \frac{\alpha^2(t)}{\psi^3(t)} \int_0^t du \frac{\dot{\psi}(u)\psi^2(u)}{\alpha^4(u)} , \]  

(34)

where we have arbitrarily taken the lower limit of integration as zero (more about this below).

We additionally need the particulate matter energy momentum tensor, for which we take the relativistic perfect fluid form

\[ T_{\mu\nu}^m = (p + \rho)u_\mu u_\nu - pg_{\mu\nu} \]  

(35)
with \( p \) the pressure, \( \rho \) the energy density, and \( u_\mu = g_{\mu\nu}dx^\nu/ds \) the four velocity. For the metric of Eq. (4), \( u_0 \) is given by

\[
u_0 = \alpha^2(t)\frac{dx^0}{ds} = \alpha^2(t)\frac{dx^0}{\alpha(t)dt} = \alpha(t)\frac{dx^0}{dt} = \alpha(t),
\]

(36)

from which we find

\[
T_{00}^{pm} = \rho \alpha^2(t),
\]

\[
T_{ij}^{pm} = \delta_{ij}p \psi^2(t).
\]

(37)

Covariant conservation of \( T_{\mu\nu}^{pm} \) implies

\[
\frac{d(\rho(t)\psi^3(t))}{dt} = -3p(t)\dot{\psi}(t)\psi^2(t),
\]

(38)

which dividing by \( \alpha(t) \) on both sides is equivalent to

\[
\frac{d(\rho[\tau]\psi^3[\tau])}{d\tau} = -3p[\tau]\frac{d\psi[\tau]}{d\tau}\psi^2[\tau].
\]

(39)

B. Einstein equations

We now have all the ingredients needed to write down the Einstein equations. From the 00 component we get

\[
\frac{\ddot{\psi}(t)}{\alpha^2(t)\psi^2(t)} = \frac{\Lambda}{3} \left[ 1 - f + f \frac{3}{\psi^3(t)} \int_0^t \dot{\psi}(u)\psi^2(u) \frac{du}{\alpha^4(u)} \right] + \frac{8\pi G}{3} \rho(t),
\]

(40)

and from the spatial components we get

\[
\frac{2\psi(t)[\alpha(t)\dot{\psi}(t) - \dot{\alpha}(t)\dot{\psi}(t)] + \alpha(t)\dot{\psi}^2(t)}{\alpha^3(t)} = \psi^2(t) \left[ (1 - f)\Lambda + \frac{f\Lambda}{\alpha^4(t)} - 8\pi Gp(t) \right].
\]

(41)

The three equations Eq. (40), Eq. (41), and Eq. (38) are not independent, by virtue of the Bianchi identities for \( G_{\mu\nu} \) and covariant conservation of the energy momentum tensors. Multiplying Eq. (40) through by \( \psi^3(t) \), differentiating with respect to time, eliminating \((d/dt)(\psi^3(t)\rho(t))\) by Eq. (38), and dividing by \( \dot{\psi}(t) \), we recover Eq. (41). Hence effectively we have only one equation relating the metric components \( \alpha \) and \( \psi \), and a second equation will be needed, a topic which we address in the next section.

But first let us address several issues raised by the structure of the above equations, beginning with the arbitrary lower limit in the integral in Eq. (34). In the sequel we will be interested in
solving Eqs. (40) and (41) in the matter-dominated era, in which the pressure \( p = 0 \). Then Eq. (38) can be integrated to give \( \rho(t) = C/\psi^3(t) \), with \( C \) a constant of integration, and we see that a shift in the lower limit of the integral in Eqs. (34) and (40) can be absorbed into a shift in the integration constant \( C \). So there is no loss of generality in taking the lower limit of integration as zero.\(^7\)

A further property of Eqs. (40) and (41) is that when \( \alpha(t) \equiv 1 \), they reduce when \( \psi(0) = 0 \) to the usual FRW equations with \( \psi(t) \) playing the role of the FRW expansion factor \( a(t) \),

\[
\frac{\psi^2(t)}{\psi^2(t)} = \frac{\Lambda}{3} + \frac{8\pi G}{3} \rho(t) ,
\]

\[
2\psi(t)\dot{\psi}(t) + \ddot{\psi}^2(t) = \psi^2(t)[\Lambda - 8\pi G p(t)] .
\]

These equations have as the solution \( \psi(t) = a(t) \), a property that will be relevant for finding a second equation relating \( \alpha(t) \) and \( \psi(t) \).

C. Finding the second equation

To find a second equation relating \( \alpha(t) \) and \( \psi(t) \), we generalize to the case where these are also functions of the coordinate \( \vec{x} \), so that we have \( \alpha(t, \vec{x}) \), \( \psi(t, \vec{x}) \), and then assume a smooth limit as the coordinate dependence limits to zero. When \( \alpha \) and \( \psi \) are coordinate dependent, the Einstein tensor component \( G_{xy} \) is given by

\[
\psi^2(1+\Phi)_{\partial_x \partial_y} = -2\alpha_{\partial_x} \partial_y \psi + \alpha_{\partial_x} \partial_y \psi + \psi^2(1+\Phi)_{\partial_x} \partial_y \alpha - \psi(1+\Phi)_{\partial_x \partial_y} \psi - (1-\psi)(1+\Phi)_{\partial_x \partial_y} \psi .
\]

(43)

Since the energy momentum source tensors have no \( xy \) component, \( G_{xy} \) must be equated to zero.

To proceed further, with no loss of generality so far we write \( \alpha \) and \( \psi \) as

\[
\alpha(t, \vec{x}) = 1 + \Phi(t, \vec{x}) ,
\]

\[
\psi(t, \vec{x}) = a(t) \theta(t, \vec{x}) ,
\]

\[
\theta(t, \vec{x}) = 1 - \Psi(t, \vec{x}) ,
\]

(44)

corresponding to introducing \( \vec{x} \) dependence into Eq. (10) above. Substituting into Eq. (43), we get

\[
0 = -2(1+\Phi)_{\partial_x} \psi_{\partial_y} \psi - (1+\Phi)(1-\Psi)_{\partial_x} \partial_y \psi + (1-\Psi)^2(1-\Psi)(\partial_x \Phi)(\partial_y \psi + \partial_y \Phi \partial_x \psi) .
\]

(45)

\(^7\) When Eq. (10) is converted to Eq. (11) by multiplication by \( \psi^3(t) \) and differentiation, the constant \( C \) drops out, and is replaced by the additional constant of integration needed for a second order differential equation as opposed to a first order equation. This new constant will end up parameterizing the family of solutions of our model.
To leading linear order in $\Phi$ and $\Psi$, this reduces to

$$0 = \partial_x \partial_y [\Phi(t, \vec{x}) - \Psi(t, \vec{x})] \quad ,$$

which assuming a smooth limit to vanishing spatial dependence gives the leading order result

$$\Phi(t) = \Psi(t) \quad .$$

(47)

We have not found a model-independent way to go beyond leading order, so we proceed by introducing a specific model, from which we abstract a relation which we then use with no further reference to the model. Our model is to assume that $\Phi$ and $\Psi$ are dominated by a single plane wave so that

$$\Phi(t, \vec{x}) = \Phi(t) e^{i \vec{k} \cdot \vec{x}} \quad ,$$

$$\Psi(t, \vec{x}) = \Psi(t) e^{i \vec{k} \cdot \vec{x}} \quad .$$

(48)

With this assumption $\partial_x \equiv i k_x$, $\partial_y \equiv i k_y$, which can then be factored from all terms of Eq. (45). This leaves a purely algebraic relation between $\Phi(t)$ and $\Psi(t)$, which can be reduced to the simple form

$$\Phi(t) = \frac{\Psi(t)}{1 - 2\Psi(t)} \quad ,$$

(49)

which to first order reproduces Eq. (47). Reexpressing Eq. (49) in terms of $\psi(t)/a(t) = \theta(t)$ and $\alpha(t)$ we find

$$\alpha(t) = \frac{\psi(t)/a(t)}{2\psi(t)/a(t) - 1} = \frac{\theta(t)}{2\theta(t) - 1} \quad ,$$

$$\theta(t) = \psi(t)/a(t) = \frac{\alpha(t)}{2\alpha(t) - 1} \quad ,$$

$$2\theta(t)\alpha(t) = \theta(t) + \alpha(t) \quad .$$

(50)

These equations give the relation between $\alpha$ and $\psi$ that we will use in our analysis beyond leading order. It has the nice feature that $\alpha(t) \equiv 1$ implies that $\theta(t) \equiv 1$, $\psi(t) \equiv a(t)$, which is the same property found for Eqs. (40) and (41) above. Thus our model relating $\alpha$ and $\psi$ is structurally compatible with the Einstein equations derived in the preceding section.
D. Differential equation for $\theta$, reduction to the linearized case, and large and small time behavior

Since study of the linearized case in Sec. 3 shows that $\Psi(t) = 1 - \theta(t)$ is slowly varying in comparison to the FRW expansion factor $a(t)$, we turn Eq. (41) into an equation for $\theta$ by substituting $\psi(t) = a(t)\theta(t)$. Setting the pressure term $p$ to zero, using the zeroth order FRW equation $2\ddot{a}/a + (\dot{a}/a)^2 = \Lambda$, changing to the dimensionless variable $x$ of Eq. (12) and using $a'/a = (2/3)\coth(x)$ with $'$ denoting $d/dx$, we arrive at

$$\theta'' = F[\theta', \theta],$$

$$F[\theta', \theta] = \frac{2}{3}\alpha^2[1 - f + f/\alpha^4] \theta - \frac{(2\theta + 1) (\theta')^2}{2(2\theta - 1)} - \frac{4}{3} \coth(x) \frac{3\theta - 1}{2\theta - 1} \theta' - \frac{2}{3} \theta,$$  

(51)

with $\alpha$ in the first term of $F$ related to $\theta$ by $\alpha = \theta/(2\theta - 1)$. To check that this reduces to the linearized case, we set $\theta = 1 - \Psi$ and keep only terms of first order in $\Psi$, giving after a little algebra

$$\Psi'' + \frac{8}{3} \coth(x) \Psi' = \frac{4}{3}(2f - 1)\Psi,$$  

(52)

in agreement (using the perturbative relation $\Psi = \Phi$) with Eq. (22) above. We note that since $\theta = 1$ corresponds to $\alpha = 1$, Eq. (51) implies that $F(0, 1) = 0$, with the consequence that initial data $\theta'(0) = 0$ and $\theta(0) = 1$ propagate forward in time unchanged.

Since Eq. (51) is even in $x$ it admits $\theta$ to be an even function of $x$, and so at $x = 0$ there will be a regular solution with $\theta(0)$ an arbitrary constant, and $\theta'(0) = 0$. Making the ansatz that $\theta$ becomes infinite for large $x$, Eq. (51) reduces to an equation with constant coefficients. Substituting $\theta(x) = e^{\lambda x}$ we find the algebraic equation

$$\lambda^2 + \frac{4}{3} \lambda + \frac{4}{9} = \frac{1}{9}(15f + 1),$$  

(53)

with roots

$$\lambda = \frac{2}{3} \pm \frac{1}{3}(15f + 1)^{1/2},$$  

(54)

to be compared with

$$\lambda = -2/3 [2 \pm (6f + 1)^{1/2}],$$  

(55)

found from the linearized equation in Sec. 3. When $f = 1$ Eq. (54) gives an exponentially growing solution with $\lambda = 2/3 \approx 0.67$, somewhat larger than the exponent $(2/3)(\sqrt{7} - 2) \approx 0.43$ given for
$f = 1$ by Eq. (55). For the nonperturbative formula of Eq. (54) the exponentially growing solution appears for $f > 1/5$, whereas for the linearized result of Eq. (55) the exponentially growing solution first appears for the larger value $f > 1/2$.

E. Numerical solution

To solve for $\theta$ numerically, we employ stepwise forward integration of Eq. (51) starting from $x = 0$,

$$\theta'(i + 1) = \theta'(i) + F[\theta'(i), \theta(i)] \Delta x,$$
$$\theta(i + 1) = \theta(i) + \theta'(i) \Delta x,$$

with $\Delta x = 1/800$, and then as a check with $\Delta x = 1/1600$. These two calculations differ only in the 5th decimal place. In Table II we give results with $f = 1$ for $\theta$, and for the normalized ratio $\Psi(x)/\Psi(0) = [1 - \theta(x)]/[1 - \theta(0)]$, starting from $\theta(0) = 1.1$, or equivalently $\Psi(0) = -0.1$, and $\theta'(0) = 0$. The normalized ratio $\Psi(x)/\Psi(0)$ is directly comparable to $\hat{\Phi}$ calculated in Sec. 3 and given in Table I; in the final column of Table II we give $\hat{\Phi}$ recalculated by the stepwise forward integration method used here for solving the $\theta$ differential equation, which agrees to within one or two parts per thousand with the iterative integral equation method used in Sec. 3.\textsuperscript{8} We see that the final two columns of Table I agree to within a couple of percent, indicating that the linear perturbation results of Sec. 3 suffice at present for phenomenological applications.

We have also checked that with the iteration of Eq. (56), initial data $\theta(0) = 1$ and $\theta'(0) = 0$ propagate forward in time unchanged. For $\theta(0) > 1$, the numerical solution for $\theta(x)$ with $f = 1$ increases with increasing $x$, while for $\theta(0) < 1$ the numerical solution for $\theta(x)$ decreases with increasing $x$, corresponding respectively to a universe that expands more rapidly, or more slowly, than the standard FRW solution $a(t)$.

F. Correspondence of the non-perturbative and perturbative solutions

To conclude this section, we see that a nonperturbative treatment of the problem that we addressed by linearized perturbation theory in Sec. 3 leads again to a one-parameter family of

\textsuperscript{8} For example, interpolating into Table II we find for $z=0.54$ that $\hat{\Phi} = 0.6 \times 1.095 + 0.4 \times 1.080 = 1.089$, compared with 1.089 in Table I.
TABLE II: Values of $\theta(x)$ and $\Psi(x)/\Psi(0)$ calculated for $\theta(0) = 1.1$, and in the final column the corresponding linearized equation result $\hat{\Phi}(x)$, versus $x$ and redshift $z$, taking $f = 1$, that is, with all of the observed cosmological constant arising from a scale invariant but frame dependent action.

| $z$ | $x$ | $\theta(x)$ | $\Psi(x)/\Psi(0)$ | $\hat{\Phi}(x)$ |
|-----|-----|-------------|--------------------|------------------|
| 0   | 1.169 | 1.126       | 1.264              | 1.243            |
| 0.1 | 1.054 | 1.122       | 1.215              | 1.198            |
| 0.2 | 0.955 | 1.118       | 1.177              | 1.163            |
| 0.3 | 0.868 | 1.115       | 1.146              | 1.135            |
| 0.4 | 0.792 | 1.112       | 1.122              | 1.113            |
| 0.5 | 0.726 | 1.110       | 1.103              | 1.095            |
| 0.6 | 0.668 | 1.109       | 1.087              | 1.080            |
| 0.7 | 0.616 | 1.107       | 1.074              | 1.069            |
| 0.8 | 0.571 | 1.106       | 1.064              | 1.059            |
| 0.9 | 0.530 | 1.106       | 1.055              | 1.051            |
| 1.0 | 0.494 | 1.105       | 1.048              | 1.044            |

cosmologies, parameterized now by the initial value $\theta(0)$. In the perturbative treatment, the corresponding parameter was $\Phi(0)$, and this parameter is related to the one used in the nonperturbative analysis by $\Phi(0) \simeq \Psi(0) = 1 - \theta(0)$. For $\theta(0) = 1$, $\Phi(0) = 0$, the nonperturbative and perturbative evolutions both have as the solution the standard FRW expansion factor $a(t)$; that is, $\theta(t)$ remains equal to 1 (and $\Phi(t)$ remains 0) for all times $t$. For $\theta(0) > 1$, $\Phi(0) < 0$ the $f = 1$ nonperturbative and perturbative evolutions both give a universe that expands faster than the standard FRW solution $a(t)$, as discussed further below, and for $\theta(0) < 1$, $\Phi(0) > 0$ the $f = 1$ nonperturbative and perturbative evolutions both give a universe that expands more slowly than the standard FRW solution $a(t)$. There is thus a direct qualitative correspondence between the two methods of treating the model of Eq. (3), and for parameter values $\theta(0)$ close to unity, there is only a small quantitative difference. We have also seen that the nonperturbative treatment gives a relatively straightforward way of rederiving the perturbation equation of Eq. (22).

V. COSMOGRAPHIC EQUATIONS AND APPLICATION TO THE “HUBBLE TENSION”

To discuss cosmography we rewrite the metric in terms of proper time $\tau$ as in Eq. (7),

$$ds^2 = d\tau^2 - \psi^2[\tau]d\vec{x}^2,$$

(57)
which we recall takes the standard FRW form with $\tau$ replacing $t$ and with $\psi[\tau] \equiv \psi(t) = a(t)\theta(t)$ replacing $a(t)$. Thus, the nonperturbative generalizations of the standard cosmological distance measures are obtained by making these substitutions in the usual formulas, with the following results, where the subscript 0 denotes the present time, and 1 denotes the past time at which an object at coordinate distance $r_1$ emitted a signal.

- For waves of wavelength $\lambda$ and frequency $\nu$, we have

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = 1 + z_{\text{eff}} = \frac{\psi(t_0)}{\psi(t_1)} = \frac{a(t_0)\theta(t_0)}{a(t_1)\theta(t_1)} . \tag{58}$$

- The parallax distance distance $d_P$, luminosity distance $d_L$, angular diameter distance $d_A$, and proper motion or comoving angular diameter distance $d_M$, are given by

$$d_P = r_1\psi(t_0) ,$$
$$d_L = r_1\psi(t_0)(1 + z_{\text{eff}}) ,$$
$$d_A = r_1\psi(t_1) ,$$
$$d_M = r_1\psi(t_0) , \tag{59}$$

with ratios

$$\frac{d_A}{d_L} = \frac{1}{(1 + z_{\text{eff}})^2} ,$$
$$\frac{d_M}{d_L} = \frac{1}{1 + z_{\text{eff}}} ,$$
$$\frac{d_A}{d_P} = \frac{1}{1 + z_{\text{eff}}} . \tag{60}$$

- As noted in Sec. 2, the effective Hubble parameter is defined as the proper time derivative

$$H_{\text{eff}}(\tau) = \frac{d\psi[\tau]/d\tau}{\psi[\tau]} , \tag{61}$$

or in terms of $t$,

$$H_{\text{eff}}(t) = \frac{d\psi(t)/dt}{\alpha(t)\psi(t)} = \frac{\dot{a}(t) }{\alpha(t)} \left( \frac{\dot{a}(t)}{a(t)} + \frac{\dot{\theta}(t)}{\theta(t)} \right) = \frac{1}{\alpha(t)} \left( H(t) + \frac{\dot{\theta}(t)}{\theta(t)} \right) . \tag{62}$$

In the linearized perturbation limit this is (using $\alpha\theta \simeq 1 + \text{second order}$)

$$H_{\text{eff}}(t) \simeq [1 - \Phi(t)]H(t) - \dot{\Phi}(t) , \tag{63}$$
which at \( t = t_0 \) gives (recalling \( H_0 = H(t_0) \))

\[
\frac{H_{\text{eff}}(t_0)}{H_0} \simeq 1 - \Phi(0)[\dot{\Phi}(x_0) + \frac{3}{2}\sqrt{\Omega_\Lambda d\dot{\Phi}/dx|_{x=x_0}}] \simeq 1 - 1.75\Phi(0) , \tag{64}
\]

with \( \dot{\Phi}(x_0) \) contributing 1.244 and with \( \frac{3}{2}\sqrt{\Omega_\Lambda d\dot{\Phi}/dx|_{x=x_0}} \) contributing 0.5055 to the coefficient 1.75.

- Much recent attention has been paid to a tension between the values of the Hubble constant at late cosmic times directly measured from redshifts \([12]\), and the value at early cosmic times inferred from the study of fluctuations in the CMB radiation \([7]\). This tension may require recalibration of the distance ladder used in direct redshift measurements \([13]\), or may indicate a need for new physics \([14]\). To interpret the Hubble tension in terms of new physics via a frame-dependent dark energy, we take \( f = 1 \) and use Eq. (64) to fit \( \frac{H_{\text{eff}}(t_0)}{H_0} \simeq 1.1 \), as needed to reconcile the CMB value \( H_0 \simeq 67 \text{km s}^{-1} \text{Mpc}^{-1} \) with the recent epoch \([12]\) value \( H_{\text{eff}} \simeq 74 \text{km s}^{-1} \text{Mpc}^{-1} \). This requires \( \Phi(0) \simeq -0.057 \) and thus \( \alpha(0) = 1 + \Phi(0) \simeq 0.943 \), corresponding to an early time proper time expansion rate of the universe that is faster than the rate expected from the standard FRW expansion factor \( a(t) \).

- We note from Table I that for redshifts \( z \) greater than 100, and therefore at and before decoupling, \( \Phi(t) \) is very accurately a constant \( \Phi(0) \), so the transformation from coordinate time \( t \) to proper time \( \tau \) is just a constant rescaling \( \tau = \alpha(0)t \). For \( \alpha(t) \simeq \alpha(0) \), Eqs. (40) and (41) simplify to

\[
\left(\frac{d\psi[\tau]}{d\tau}\right)^2 = \frac{\Lambda}{3} \left(1 - f + \frac{f}{\alpha^4(0)}\right) + \frac{8\pi G}{3}\rho[\tau] , \tag{65}
\]

and

\[
2\psi[\tau]\frac{d^2\psi[\tau]}{(d\tau)^2} + \left(\frac{d\psi[\tau]}{d\tau}\right)^2 = \psi^2[\tau] \left(1 - f + \frac{f}{\alpha^4(0)}\right) - 8\pi G\rho[\tau] , \tag{66}
\]

which have the same structure as Eqs. (42) apart from the use of proper time and the parameter redefinition \( \Lambda \to \Lambda(1 - f + f/\alpha^4(0)) \). Since these equations are homogeneous in \( \psi[\tau] \), they determine it only up to an overall constant factor, and so the solution will be a multiple of \( a[\tau] \), up to a rescaling of parameters. This can also be seen directly from the early time, first order formula

\[
\psi[\tau] = \theta(0)a(\tau/\alpha(0)) \simeq a(\tau/\alpha(0))/\alpha(0) , \tag{67}
\]
which shows that for $\alpha(0) < 1$, $\psi[\tau]$ evolves faster in $\tau$ than $a(t)$ evolves in $t$.

- For redshifts $z > 100$, the line element of Eq. (4) becomes to high accuracy

$$ds^2 = \alpha^2(0)dt^2 - \theta^2(0)a^2(t)d\vec{x}^2$$

(69)

corresponding to a rescaling of the units used to measure $dt$ and $d\vec{x}$ by effectively constant factors. We conjecture that this metric will give as good a fit to the CMB angular fluctuation analysis as the standard FRW cosmology, apart from a rescaling of dimensional parameters used in the fit. In particular, when the CMB analysis using our model is recast in terms of proper time, we expect the Hubble constant $H_0$, which has dimensions of inverse time, to be rescaled by a factor of $\alpha(0)^{-1}$, corresponding to the rescaling in Eq. (67). This will have to be checked by a repeat of the CMB analysis with $dt$ and $d\vec{x}$ rescaled by constant factors.

- When $\alpha(0) < 1$, the expansion rate in our model is enhanced over that in the standard FRW cosmology for all times. To see this, take the $\tau$ derivative of the logarithm of $\psi[\tau] = \theta(t(\tau))a(t(\tau))$, converting derivatives on the right to $t$ derivatives using $d/d\tau = (1/\alpha(t))d/dt$, to get

$$\frac{d}{d\tau} \log(\psi) = \frac{1}{\alpha(t)} \left[ \frac{d}{dt} \log(a) + \frac{d}{dt} \theta \right],$$

(70)

with $t$ on the right hand side understood to be $t(\tau)$. Since for $\alpha(0) < 1$ we have $\frac{1}{\alpha(t)} > 1$ and $d\theta(t)/dt > 0$, Eq. (71) implies that

$$\frac{d}{d\tau} \log(\psi) - \frac{d}{dt} \log(a) > 0;$$

(71)

that is, in our model with $f = 1$ and $\alpha(0) < 1$, $\theta(0) > 1$, the expansion rate of the universe at all times is greater than in standard FRW cosmology, because of the $\alpha(t)^{-1}$ factor arising from the proper time derivative. Thus, while potentially preserving the accuracy of the CMB angular analysis, our model is not exclusively a late time model that leaves the early time history of the universe unaltered. Hence it may not be subject to the criticism of late time models to explain the Hubble tension raised in the papers of Lemos et al.

9 The superficial mismatch between the $\alpha^4(0)$ factor in Eqs. (65) and (66), and the $\alpha(0)$ factor in Eq. (67), is only apparent, because to leading order

$$a(t) \simeq (3H_0\sqrt{\Omega_m/2})^{2/3}t^{2/3}[1 + O(t^2)]$$

(68)

only determines a single overall coefficient of $t^{2/3}$, not the full sinh($x$) functional form.

10 More generally, the dynamics of all physical quantities, such as stellar evolution rates, will be correspondingly more rapid than in the standard FRW cosmology.
Knox and Millea [16]. It will be of interest to analyze other Hubble determinations used in the analysis of [15], such as [17], using the modified cosmography of our model.

- Returning now to Eq. (70), introducing the leading order perturbation expansion this becomes

\[
H_{\text{eff}}[\tau] - H(t) = \frac{d\log(\psi)}{d\tau} - \frac{d\log(a)}{dt} = -\Phi(0) \left[ \dot{\Phi}(t)H(t) + \frac{d\Phi}{dt}(t) \right],
\]

(72)

which for any value of the parameter \(\Phi(0)\) and any \(t\) can be evaluated numerically from the formula for \(a(t)\) and the solution obtained above for \(\dot{\Phi}(t)\). Again, \(t\) in Eq. (72) is understood to be \(t(\tau)\).

- Corresponding to the fact that expansion rate of the universe is altered in our model for \(\Phi(0) \neq 0\), the age of the universe is changed. This can be calculated, to leading order in perturbations, as follows. We first note that the age \(t_0\) of the FRW universe is fixed by the requirement \(a(t_0) = 1\), which from Eq. (11) becomes \(\sinh(x_0) = (\Omega_\Lambda/\Omega_m)^{1/2}\). When solved for \(x_0\) this gives \(x_0 = 1.169\), and combining this with the CMB value of \(H_0\) gives the CMB value of \(t_0\). In similar fashion, in our model the proper time age of the universe \(\tau_0\) is fixed\(^{11}\) by the condition \(\psi[\tau_0] = 1\). Writing \(t(\tau_0) = t_0 + \Delta t\) we get from \(\psi = \theta a\) the equation

\[
1 = \psi[\tau_0] = \theta(a(t(\tau_0))a(t(\tau_0)))
\]

\[
= [1 - \Phi(0)\dot{\Phi}(t_0 + \Delta t)]a(t_0 + \Delta t)
\]

\[
\approx 1 - \Phi(0)\dot{\Phi}(t_0) + \Delta t \frac{da}{dt}(t_0),
\]

(73)

which can be solved to give \(\Delta t\),

\[
\Delta t = \frac{\Phi(0)\dot{\Phi}(t_0)}{\frac{da}{dt}(t_0)} = \frac{\Phi(0)\dot{\Phi}(t_0)}{H_0}.
\]

(74)

\(^{11}\) To elaborate, the age of the universe is the elapsed proper time between redshift zero and redshift infinity. When \(\psi = 1\), the effective redshift \(z_{\text{eff}}\) vanishes, and at \(\tau = 0\) the effective redshift is infinite since \(\psi[0] = \theta(0)a(0) = 0\).
From the definition of the proper time in Eq. (6) we get

\[
\tau_0 = \int_0^{t(\tau_0)} du \alpha(u) = \int_0^{t_0 + \Delta t} du [1 + \Phi(0) \dot{\Phi}(u)]
= t_0 + \Delta t + \Phi(0) \int_0^{t_0} du \dot{\Phi}(u),
\]

(75)

which gives

\[
\tau_0 - t_0 = \Delta t + \Phi(0) \int_0^{t_0} du \dot{\Phi}(u) = \frac{\Phi(0)}{H_0} \left[ \dot{\Phi}(t_0) + \frac{2}{3\sqrt{\Omega_\Lambda}} \int_0^{x_0} dx \dot{\Phi}(x) \right].
\]

(76)

Numerical evaluation of this from Table I,\textsuperscript{12} for \(\Phi(0) = -0.057\) as needed to fit the Hubble tension, gives \(\tau_0 - t_0 = -1.77\) Gyr, \(\tau_0 = 12.06\) Gyr.

To summarize, a scale invariant but frame dependent dark energy can potentially enhance the late time Hubble constant value without spoiling the excellent CMB angular fits, with a universe that is expanding faster and hence is younger than is suggested by the standard FRW cosmology. We look forward to future experiments to give an enlarged and improved data set against which to test our model.

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Appendix A: Perturbation equations rewritten in terms of \(\Phi\) and \(\Psi\)

In [4] we gave the metric perturbation equations for \(f \neq 0\) in \(B = 0\) gauge in terms of \(A, E,\) and \(F,\) taken as functions of both \(t\) and \(\vec{x}.\) Here we give the same equations when rewritten in terms of \(\Psi, \Phi,\) and \(F,\) obtained by substituting Eq. (18) and algebraic simplification.

\textsuperscript{12} Trapezoidal integration from Table I gives \(\int_0^{x_0} dx \dot{\Phi}(x) = 1.083.\)
The $ij$ equation takes the form

$$0 = \delta_{ij} X + \partial_i \partial_j Y ,$$

$$X = 2[a\ddot{a} + 2(a^2\dot{a}^2)\Phi + a\dot{a} \dot{\Phi} + 6a\dot{a} \dot{\Psi} + a^2 \ddot{\Psi} - \nabla^2 \Psi$$

$$+ 4\pi G a^2(\delta \rho - \delta p - \nabla^2 \pi S) - 4\pi G[a a^2 (p + \rho) + a^3 (\dot{\rho}/3 + \dot{\rho})]F$$

$$+ \Lambda f a^2(t_{00}/2 - E) ,$$

$$Y = 8\pi G a^2 \pi S + \Phi - \Psi .$$

(A1)

The $i0$ equation becomes

$$0 = -2(\dot{a}/a)\partial_i \Phi - 2\partial_i \dot{\Psi} - 8\pi G(p + \rho)(\partial_i \delta u - a \partial_i F) + \Lambda f t_{0i} ,$$

(A2)

and the $00$ equation becomes

$$0 = Z = -(1/a^2)\nabla^2 \Phi - 3(\dot{a}/a)\dot{\Phi} - 6(\dot{a}/a)\dot{\Psi} - 6(\dot{a}/a)\ddot{\Psi}$$

$$+ 4\pi G[\delta \rho + 3 \delta p + \nabla^2 \pi S + a(\dot{\rho} + 3\dot{\rho})F]$$

$$+ \Lambda f (t_{00}/2 + 3E) .$$

(A3)

Taking the linear combination $(1/4)a^2 Z + (3/4)X + (1/4)\nabla^2 Y$ gives

$$0 = -\nabla^2 \Psi + 3a\dot{a} \dot{\Psi} + 3(\dot{a}/a)\dot{\Phi} + a^2 t_{00}/2$$

$$+ 4\pi G a^2[\delta \rho - 3\dot{a}(p + \rho)F] ,$$

(A4)

and using this to eliminate $-\nabla^2 \Psi$ from $X$ we get

$$a^2 \ddot{\Psi} + [2a\ddot{a} + (\dot{a})^2]\Phi + a\dot{a} \dot{\Phi} + 3a\dot{a} \dot{\Psi} = 4\pi G a^2(\delta \rho + \nabla^2 \pi S) + 2\Lambda f a^2[\Phi - \dot{a}F - aF]$$

$$+ 4\pi G a^3 \rho F .$$

(A5)

\[13\] Under the gauge changes $\Delta_\rho(a^2 A) = 2a\dot{\epsilon}_0, \Delta_\varphi E = 2\epsilon_0, \Delta_a(a F) = -\epsilon_0, \Delta_\rho \Delta_\varphi = \dot{\rho}\epsilon_0, \Delta_\rho \Delta_\varphi = \dot{\rho}\epsilon_0, \Delta_\rho \pi S = 0, \Delta_\rho \delta u = -\epsilon_0, \Delta_\rho \Phi = \Delta_\rho \Psi = 0$, all terms of Eqs. (13) and (A1) - (A3) are invariant except the $\Lambda f$ terms, reflecting the fact that the effective action of Eq. (2) breaks four-space general coordinate invariance. When only three-space general coordinate transformations are considered (which is the most general invariance of Eq. (3)), one can take $\epsilon_0 = 0$, which is consistent with our then setting $F(t) = 0$. 
In the limit that the metric perturbations are functions only of $t$, using $F(t) = 0$ and $\nabla^2 \pi S(t) = 0$, this simplifies, after division by $a^2$, to

$$\ddot{\Psi} + \left[2\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]\Phi + \frac{3}{a} \frac{\dot{a}}{a} \dot{\Psi} = 4\pi G \delta \rho + 2\Lambda f \Phi \quad \text{(A6)}$$

When $\Psi = \Phi$, this gives Eq. (20) of the text. The point of the manipulations leading to Eq. (A6) is to eliminate both energy densities $\delta \rho$ and $t_{00}$ from the equation used to solve for $\Phi$. 



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