Anomalous Hydrodynamics of Fractional Quantum Hall States

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In this paper we propose a comprehensive framework for the quantum hydrodynamics of the Fractional Quantum Hall (FQH) states. We suggest that the electronic fluid in the FQH regime could be phenomenologically described by the quantized hydrodynamics of vortices in an incompressible rotating liquid. We demonstrate that such hydrodynamics captures all major features of FQH states including the subtle effect of Lorentz shear stress. We present a consistent quantization of hydrodynamics of an incompressible fluid providing a powerful framework to study FQHE and superfluid. We obtain the quantum hydrodynamics of the vortex flow by quantizing the Kirchhoff equations for vortex dynamics.
I. INTRODUCTION

Quantum systems with effectively strong interaction form liquids whose flows are coherent quantum collective motions. Among them there are interesting notable cases when such liquids allow a hydrodynamics description. That is when the long wave, slow flows can be effectively described solely in terms of a macroscopic, but quantum, pair of canonical fields of density $\rho(r, t)$ and velocity $v(r, t)$. Such quantum flows are the subject of quantum hydrodynamics.

In the classical case the principle of local equilibrium reduces the Boltzmann kinetic equation for the distribution function to the hydrodynamics equations for the density and the velocity (see, e.g., [1]). Local equilibrium occurs when the characteristic time of the flow exceeds the characteristic time of collisions, and the characteristic scale of the flow exceeds the mean free path of particles. A quantum analog of the principle of local equilibrium, is yet to be understood, but when it comes to effect, involves a long range coherent effects. Strong coherence emerges as a result of interactions. Noticeable examples of quantum hydrodynamics are superfluid helium, superconductors, trapped cooled atomic gases, and Luttinger liquids. A Fractional Quantum Hall (FQH) liquid is yet another case.

Electronic states confined within the lowest Landau level by quantizing magnetic field are holomorphic. The holomorphic nature of states make the hydrodynamic description possible.

A quest for the hydrodynamics of a FQH liquid originated in a seminal paper [2]. Earlier approaches to FQH states of Refs [3–5] are somewhat related to the hydrodynamics, as noted in Ref. [6]. Hydrodynamics of FQH states is in the focus of a renewed interest.

In hydrodynamics, a few basic principles, symmetries, and a few phenomenological parameters are sufficient to formulate the fundamental equations. In the case of FQHE we already possess sufficient characterizations of states. They can be used as a basis of the hydrodynamics approach. To this aim a microscopical Hamiltonian and a deeper understanding of underlying microscopic mechanisms of emergence of correlated liquid states are, in fact, not necessary.

In this paper we formulate a minimal number of principles sufficient to develop the hydrodynamics of FQH bulk states in a close analog of Feynman’s theory of superfluid helium [7], and the magneto-roton theory of collective excitations in FQH states of Ref. [2]. We discuss only the simplest Laughlin states. Elsewhere, we hope to be able to address the hydrodynamics of other, richer FQH states, possessing additional symmetries, such as the 5/2 state. They can be studied within the framework developed here.

We argue that states of the FQH liquid can be treated as

*Flows of quantized vortices in a quantum incompressible rotating inviscid liquid.*

On this basis we obtain the major features of FQHE including subtle effects such as Lorentz shear stress [15], missed by the previous approaches [3–6]. In particular, the Laughlin wave function

$$\psi_0(z_1, \ldots, z_N) = e^{\frac{-1}{4\ell^2} \sum_i |z_i|^2 \prod_{i>j}|z_i - z_j|^\beta}, \quad \beta = \frac{1}{\nu}. \tag{1}$$

emerges as the ground state of the vortex fluid. Here $\ell = \sqrt{\hbar c/eB}$ is the magnetic length.

To the author’s knowledge, a hydrodynamics of vortex flows has not been developed. It is an interesting subject of its own. Apart from FQHE it is also relevant to the theory of superfluids and classical hydrodynamics. In this paper we present a consistent quantum hydrodynamics of such a fluid.

Hydrodynamics of the vortex fluid differs from the Euler hydrodynamics. Its quantum version differs from the canonical quantum hydrodynamics of Landau [8]. The major difference is the anomalous terms. These terms represent the Lorentz shear force. Emergence of such forces in hydrodynamics, classical and quantum alike, is the major focus of this paper.

We start from the observation that the FQH states can be interpreted as the states of quantized Kirchhoff vortex matter, Sec. III, and then develop the hydrodynamics of the vortex matter. We summarize the main results in the Sec. V, followed by the details of derivations presented in Sec. VI–IX.

Some results presented below were obtained in collaboration with Alexander Abanov. This paper is an extended version of Ref. [9] written for the special issue of JETP dedicated to Anatoly Larkin. Discussions of hydrodynamics of quantum liquids with I. Rushkin, E. Bettelheim and T. Can and their help in understanding material presented below are acknowledged.

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II. FOUNDATIONAL PRINCIPLES OF HYDRODYNAMICS OF FQH LIQUID

A. Characterization of Fractional Quantum Hall states

Electrons in a quantizing magnetic field confined in 2D heterostructures in the regime dominated by the Coulomb interaction form an enigmatic quantum liquid. The most robust among the FQH states occurs at the filling fraction \( \nu = 1/3 \), that is the Laughlin state. Characterization of that liquid is:

- Flows are incompressible \(^{10}\), and almost dissipation-free \(^{11, 12}\);
- The spectrum of bulk excitations is gapped \(^{2, 12}\). The gap is less than the cyclotron energy \( \hbar \omega_c > \Delta_\nu \). Only edge states, excitations localized on the boundary, are soft \(^{13}\);
- The Hall conductance is fractionally quantized \(^{11}\);
- Elementary excitations in the bulk of the fluid are vortices. Vortices carry fractionally quantized negative electronic charge \(^{10}\).

More subtle features recently discussed in the literature are:

- Edge excitations consist of two branches of non-linear solitons: subsonic solitons with the fractional negative electronic charge and supersonic solitons with the unit electronic charge \(^{14}\);
- Quantized double layers of the density at boundaries and vortices \(^{14}\);
- The Lorentz shear stress and anomalous (or odd, or Hall) viscosity \(^{15}\).

From the listed properties we select a set of the foundational principles and attempt to obtain others as consequences. The set of basic principles is remarkably minimal. We only assume that electrons in the FQH regime form a quantum fluid, and that

The fluid is incompressible, and flows possess a macroscopic number of equally oriented vortices.

Since in the quantum fluid vorticity is quantized, a unit volume of the fluid contains the quantum of vorticity. We refer to such flow as a chiral flow. We want to demonstrate that the chiral flow captures all known physics of FQHE. We start with a general discussion of scales of FQH (bulk) states.

B. Scales, Holomorphic States, Incompressibility

There are two distinct energy scales: the cyclotron energy \( \hbar \omega_c = eB/(m_b c) \), which provides the distance between Landau levels, and the gap in the bulk excitation spectrum \( \Delta_\nu \). The former is determined by the band electronic mass \( m_b \) and by the magnetic field. The latter is the characteristic of the Coulomb energy. From the theoretical standpoint, the very existence of FQH states assumes that the cyclotron energy is larger than the gap \( \hbar \omega_c > \Delta_\nu \). If this limit holds \(^{29}\), the flows with energy \( E \) exceeding the gap may still be comprised of states on the lowest Landau level \( \Delta_\nu \ll E < \hbar \omega_c \). Such motion does not depend on the band electronic mass \( m_b \).

Let us consider a small modulation of electronic density \( \rho(r) \) and ignore for the moment the electrostatic interaction of a nonuniform charged fluid \(^{30}\). Such flow has a momentum flux \( P(r) \) and propagates with velocity \( v(r) \). We assume that at small modulations the momentum flux is equal to \( P = m_* \rho v \), where \( m_* \) is the inertia of the flow. It seems naturally to assume that the inertia is set by the scale provided by the gap \( \Delta_\nu \sim \hbar^2/(m_* \ell^2) \). The mass \( m_* \) exceeds that of a band electron \( m_* / m_b \sim (h \omega_c) / \Delta_\nu > 1 \).

Waves propagating through the bulk of the FQH liquid are essentially non-linear. Linear waves in the bulk are possible as a response of the ground state to electric and magnetic fields and to spatial curvature. That sector of states often called topological.

Wave functions of states with energy less than the the cyclotron energy (the lowest Landau level) are holomorphic. It is customary to describe the set of states on the lowest Landau level as the Bargmann space \(^{21}\). Coherent states of the Bargmann space are labeled by symmetric polynomials of the holomorphic coordinates of particles \( z_i = x_i + iy_i \) and the holomorphic momenta \( \partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i \partial_{y_i}) \). Let \( Q \) be such a polynomial, and \( Q^\dagger \) the Hermitian conjugated polynomial.
which depends on anti-holomorphic coordinates $\bar{z}_i = x_i - iy_i$, and anti-holomorphic momenta $\partial^T_{\bar{z}_i} = -\frac{i}{2}(\partial^T_{x_i} + i\partial^T_{y_i})$. The symbol $^T$ is the transposition. Then in the notations of the Bargmann space the "bra" and "ket" states are

$$\langle Q | = \prod_{i>j}(\bar{z}_i - \bar{z}_j)^\beta \cdot Q^1 e^{-\frac{1}{2\pi} \sum |z_i|^2}, \quad Q \cdot \prod_{i>j}(z_i - z_j)^\beta = |Q \rangle$$

(2)

Flows within the first Landau level are incompressible. Sometimes the term "incompressible flow" is attributed to the gapped spectrum. Rather, the incompressibility reflects the holomorphic nature of FQH-states. This is seen from the following argument. For simplicity, let us consider a coherent state characterized by a polynomial $Q$ which depends only on coordinates $z_i$. The phase of the wave function of such state differs from the phase of the ground state by the phase of the holomorphic polynomial $\text{Im} \log Q$. Since the velocity is a gradient of the phase, the phase is the hydrodynamic potential. The phase is harmonic everywhere except points where the wave function vanishes. Since the wave function is single valued it vanishes as an integer power of holomorphic coordinates. Therefore the allowed singularities of the phase correspond to quantized vortices. There are no sources so the gradient of the phase is divergence-free

$$\omega_c \to \infty : \quad \nabla \cdot v = 0.$$  

(3)

There are two immediate consequences of incompressibility. One is that the material derivative of the density vanishes

$$D_t \rho \equiv \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \rho = 0.$$  

(4)

the second is that that flows in homogeneous 2D incompressible liquids do not possess linear waves. Only available bulk flows are non-linear flows of vorticity [24]. The flow can be viewed as a motion of a neutral gas of quasi-holes and quasi-particles.

In the next section we identify the the FQH states with vortices in a quantum incompressible rotating fluid. Then develop the hydrodynamics of the quantum vortex matter.

### III. KIRCHHOFF EQUATIONS

We start by recalling classical Kirchhoff equations for the rotating incompressible inviscid Euler flows with a constant density (see e.g.,[25]), and then proceed with the quantization.

#### A. Classical Kirchhoff Equations for an incompressible fluid

In two dimensions an incompressible fluid with the constant density is fully characterized by its vorticity. The curl of the Euler equation for the incompressible fluid with a constant density

$$D_t u \equiv (\partial_t + u \cdot \nabla) u = -\nabla p.$$  

(5)

yields the single (pseudo) scalar equation for the vorticity.

$$D_t (\nabla \times u) = 0.$$  

(6)

In this form the Euler equation has a simple geometrical meaning: the material derivative of the vorticity vanishes. Vorticity is transported along divergence-free velocity field $u$.

Helmholtz, and later Kirchhoff realized that there is a class of solutions of the vorticity equation (6) which consists of a finite number of point-like vortices. In this case the complex velocity of the fluid $u = u_x - iu_y$ is the meromorphic function

$$u(z, t) = -i\Omega \bar{z} + i \sum_{i=1}^N \Gamma_i \frac{z - z_i(t)}{z - z_i(t)}.$$  

(7)

where $\Omega$ is an angular velocity if the fluid is rotated, $N$ is the number of vortices, $\Gamma_i$ and $z_i(t)$ are circulations and positions of vortices.
A substitution of this "pole Ansatz" into the Euler equation (6) yields that the number of vortices $N$ and the circulations $\Gamma_i$ do not change in time, while the moving positions of vortices $z_i(t)$ obey the Kirchhoff equations:

$$\dot{z}_i = -i\Omega \bar{z}_i + i \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}.$$  \hspace{1cm} (8)

The Kirchhoff equations replace the non-linear PDE (6) by a dynamical system. They can be used for different purposes. Equations describe chaotic motions of a finite number of vortices if $N > 3$. Alternatively they can be used to approximate virtually any flow.

B. Chiral flow

The flows relevant for FQHE are such that a large number of vortices largely compensates rotation. We refer such flows as the chiral flow.

Bearing in mind the quantum case we assume that vortices have the same (minimal) circulation $\Gamma_i = \Gamma$. Then Kirchhoff equations read

$$v_i \equiv \dot{z}_i = -i\Omega \bar{z}_i + i \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}.$$  \hspace{1cm} (9)

We want to study the vortex system in the limit of large number of vortices distributed with the mean density $\bar{\rho}$

$$N \rightarrow \infty : \quad \bar{\rho} = \frac{\Omega}{\pi \Gamma}. \hspace{1cm} (10)$$

The chiral flow is a very special flow in fluid mechanics. There we distinguish two types of motion: fast motion of the fluid around vortex cores, and a slow motion of vortices. In this respect vortices themselves may be considered as a (secondary) fluid. In the ground state of the vortex fluid vortices do not move, but the fluid does.

Circulation of vortices in units of the Plank constant has dimension inverse to the mass unit. We set the dimensionless parameter

$$\nu = \frac{\hbar}{m^* \Gamma}.$$  \hspace{1cm} (11)

Later we show that the quantized chiral flow models FQHE with a filling fraction $\nu$. We denote $\beta = \nu^{-1}$.

C. Quantum Kirchhoff Equations

Kirchhoff himself wrote equations (9) in the Hamiltonian form identifying holomorphic and anti-holomorphic coordinates of vortices as canonical variables. In the chiral case the canonical variables are $m^* \Omega \bar{z}_i$, and $z_i$. The Hamiltonian reads

$$\mathcal{H} = m^* \Omega \left( \sum_i \Omega |z_i|^2 - \Gamma \sum_{j \neq i} \log |z_i - z_j|^2 \right), \quad (m^* \Omega) \{\bar{z}_i, z_j\}_{P.B.} = -i\delta_{ij}.$$  \hspace{1cm} (12)

The parameter $m^*$ we put in the Hamiltonian and the Poisson brackets sets the scale of energy. It is a phenomenological parameter which does not appear in the Kirchhoff equations.

We emphasize that the Kirchhoff Hamiltonian is only a part of the energy of the fluid. This part of energy is transported by vortices. Another part of the energy is related to the vortices at rest. It diverges at vortex cores. This part is omitted in (12).

The Kirchhoff vortex system is readily canonically quantized. We replace the Poisson brackets by the commutators

$$i\hbar \{\bar{z}_i, z_j\}_{P.B.} \rightarrow [\bar{z}_i, z_j] = 2\ell^2 \delta_{ij}.$$  \hspace{1cm} (13)
The parameter $2\ell^2 = \hbar/(\Omega m_*)$ has dimension of the area. It is a phenomenological parameter arising in the quantization. We measure it in units of area per particle $2\ell^2 = \nu/(\pi \bar{\rho})$. Dimensionless number $\nu$ (11) is a semiclassical parameter. We will see that $\nu$ is identified with the filling fraction and $\ell$ with the magnetic length.

The next step is the choice of states. We assume that states are holomorphic polynomials of $z_i$. Then operators $\bar{z}_i$ are canonical momenta

$$\bar{z}_i = 2\ell^2 \partial_{z_i}. \tag{14}$$

Finally, we have to specify the inner product. We impose the chiral condition: operators $\bar{z}_i$ and $z_i$ are Hermitian conjugated

$$chiral \ condition: \quad \bar{z}_i = z_i^\dagger. \tag{15}$$

This condition combined with the representation (14) identifies the space of states with the Bargmann space (21) (see also 2). That is the Hilbert space of analytic polynomials $\psi(z_1, \ldots, z_N)$ with the inner product

$$\langle \psi' | \psi \rangle = \int d\mu \bar{\psi}' \psi, \quad d\mu = \prod_i e^{\frac{-|z_i|^2}{2\ell^2}} d^2 z_i. \tag{16}$$

Eqs. (9,14) help to write down quantum velocity operators of vortices

$$m_* v_i = -i2\hbar \partial_{z_i} + i\hbar \sum_{i \neq j} \frac{\beta}{z_i - z_j}, \quad \dot{z}_i = v_i, \quad \beta = \nu^{-1}, \tag{17}$$

where we set $h = 1$. It should be noted a subtlety in quantizing velocities. Velocities are not the linear operators. They act on the phase of the w.f. rather than on the w.f. itself $v_i [\exp(i \text{Arg} \psi)] = |\psi|^{-1}(\partial_{z_i} + i\hbar \sum_{j \neq i} \frac{\beta}{z_i - z_j}) \psi$. The linear operators are the momenta

$$p_i = -i\hbar \left( \partial_{z_i} - \sum_{j \neq i} \frac{\beta}{z_i - z_j} \right). \tag{18}$$

Eqs. (9-18) are quantum chiral Kirchhoff equations. They are not difficult to be generalized to a sphere, or a torus.

**IV. QUANTUM CHIRAL KIRCHHOFF EQUATIONS AND FQHE**

The quantum chiral Kirchhoff equations are readily to identify with FQHE.

The ground state of the vortex liquid is the state where the vortices are at rest. We repeat that this state is highly excited state of the fluid. Rather it is a state of the fluid at a very high angular moment. When vortices are at the ground state, the fluid moves with very high energy.

The ground state is the analytic function whose phase is nulled by all momenta operators. The common solution of the set of 1st order PDEs $p_i \psi_0 = 0$ in the class of holomorphic polynomials is the Laughlin w.f. in the Bargmann representation

$$\psi_0(z_1, \ldots, z_N) = \prod_{i > j} (z_i - z_j)^\beta, \quad \beta = 1/\nu. \tag{19}$$

The wave function is single valued if $\beta$ is integer, antisymmetric if $\beta$ is an odd integer, symmetric if $\beta$ is even integer. We see that vortices could be fermions or bosons depending of the choice of $\beta$. In particular, if $\beta = 2$ the same hydrodynamic equations describe the rotating trapped Bose. We notice how naturally the Laughlin wave function emerges in quantum hydrodynamics, both for fermionic and bosonic FQH states.

The correspondence will be completed if we assign the electronic charge to vortices and to identify the angular velocity with the effective cyclotron frequency

$$\Omega = \frac{eB}{m_e c} = \frac{m_b}{m_e} \omega_c. \tag{20}$$

The hydrodynamic interpretation of the FQHE is subtly different from Laughlin’s original interpretation. There, the coordinates entering the Laughlin w.f. are interpreted as bare band electrons. The fluid itself is absent in
the Laughlin picture. The hydrodynamic interpretation suggests that electrons (and their charge) are localized on topological excitations (vortices) of a neutral incompressible fluid. The neutral fluid is real. It serves as the agent of the interaction between electrons.

In the hydrodynamic interpretation, the quasi-hole is a hole in the uniform background of vortices. It corresponds to a state characterized by the polynomial with simple zeros located at a given point

$$Q(z_1, \ldots, z_N) = \prod_{i} (z - z_i).$$

(20)

The momentum of this state is $p_i|Q\rangle = i\nu \Gamma z_i - z_j |Q\rangle$. It shows that the Magnus force between vortices and a quasi-hole is the opposite to the fraction of the Magnus forces between vortices. Thus in the hydrodynamic interpretation the quasi-hole appears as a vortex with a fractional negative circulation $-\nu$.

Identifying vortices and electric charges, we must assume that the external fields (potential well, gradients of temperature etc.) are coupled to the vortices, not to the fluid. Let us examine how vortices move in the external potential well $U(r)$. The potential adds the term $\sum_i U(r_i)$ to the Hamiltonian, where $r_i$ are coordinates of vortices, and the force $-i[U, \bar{z}_i] = i2F^2\partial_{z_i}U$ to the Kirchhoff equations

$$p_i = -i\hbar \partial_{z_i} + i\hbar \sum_{i \neq j} \frac{\beta}{z_i - z_j} + im_x eE(z_i),$$

(21)

where $eE = -\nabla U$ is the electric field.

Fractionally quantized Hall conductance follows from Kirchhoff equations easily. Let us assume for now that the electric field is uniform. Then the center of mass of the fluid stays at the origin $\sum_i \partial_{z_i} = 0$. Summing (21) over all vortices we obtain the Hall current per particle $N^{-1} \sum_i e v_i = i e^2 F^2 E$ and the current per volume $i e^2 F^2 \bar{\rho} E$. We read that the Hall conductance equals to the fraction of $e^2 / h$.

$$\sigma_{xy} = \nu e^2 / h.$$  

(22)

Our next step is to develop the hydrodynamics of a system of quantum vortices described by the Kirchhoff equations. To the best of our knowledge this has not been done even for the classical fluids. We start by the summary of main results. The derivation and details follow.

V. SUMMARY OF THE MAIN RESULTS AND DISCUSSION

Quantum hydrodynamics of the chiral vortex flow consists of three sets of data: the operator content and their algebra, the chiral constituency relation between operators and the dynamic equation. We summarize them below, but first we comment on notations.

A. Notations

We will use holomorphic coordinates $z = x + iy$, $\partial = \frac{1}{2}(\nabla_x - i\nabla_y)$. We use the roman script for complex velocity vectors $v = (v_x, v_y)$, $\bar{v} = v_x - iv_y$, and holomorphic components of symmetric flux tensors $\Pi_{ab} : \Pi = \Pi_{xx} - \Pi_{yy} - 2i\Pi_{xy}$, $\Pi_{zz} = \Pi_{xx} + \Pi_{yy}$.

We emphasize the difference between Hermitian conjugation $v^\dagger$ and complex conjugation $\bar{v}$, but still may use the classical notation for the divergence and the curl of the velocity. In particular, the divergence and the curl we abbreviate as $\nabla \cdot v = 0$ actually means $\nabla \cdot v = \partial v + \partial \bar{v}$, $\nabla \times v = i(\partial v - \partial \bar{v})$. Similarly, the term $v \cdot \nabla \rho$ in (4) is understood as $v^\dagger \cdot \partial \rho + \partial \bar{\rho} \cdot v$.

The divergence-free velocity of the incompressible liquid is expressed through the stream function operator

$$v = -2i\partial \Psi.$$  

(23)
We denote the momentum flux of the flow as
\[ P = m* \rho v. \quad (24) \]
and for reasons which will be clear later call it the vortex flux. The vortex flux operators null the ground state
\[ P|0\rangle = \langle 0|P^\dagger = 0. \quad (25) \]
Throughout the paper we set \( m* = 1 \) measuring the momentum per particle in units of units of velocity, or equivalently, treating the particle density as a mass density, as it customary is hydrodynamics. We emphasize that \( m* \) is not related to the band electronic mass.

**B. Commutation relation**

Commutation relations of the vortex flux operators differ from the canonical commutation relations of quantum hydrodynamics of Landau \[8\] by the anomalous terms
\[
\hbar^{-1}[P(r), P^\dagger(r')] = -\frac{1}{2}(P \times \nabla)\delta(r-r') + \frac{\hbar}{2\nu} \left( 2\pi \rho^2 \delta(r-r') + \frac{1}{4} \nabla [\rho \cdot \nabla \delta(r-r')] \right). \quad (26)
\]
The commutation relation between the flux and the density is canonical
\[ [P(r), \rho(r')] = -i\hbar \rho \partial \delta(r-r'). \quad (27) \]
The vortex flux operator can be conveniently represented in terms of canonical fields \( u \) and \( u^\dagger \)
\[ [u(r), u^\dagger(r')] = \pi \frac{\hbar^2}{\nu} \delta(r-r'), \quad [u(r), \rho(r')] = -i\hbar \partial \delta(r-r') \quad (28) \]
We introduce the axillary operators
\[ J = \rho u \quad (29) \]
which we call vorticity flux. The hydrodynamic interpretation of this operator will be clarified in the text. It has a canonical commutation relation with itself and with the density, but does not annihilate the vacuum. The vortex flux \( P \) does.

We show that vortex flux and vorticity flux are differed by the anomalous term
\[ P = J + i\frac{\hbar}{2\nu} \partial \rho. \quad (30) \]
The anomalous term adds to the diamagnetic energy of the flow in the background e.m. field
\[ \frac{e}{c} \int (A \cdot P) d^2r = \frac{e}{c} \int (A \cdot J) d^2r + \frac{\hbar e}{4\nu c} \int B \rho d^2r, \]
effectively reducing the orbital moment of particles. Similarly, the anomalous term contributes to the angular momentum of the flow
\[ N^{-1} \int (r \times P) d^2r = N^{-1} \int (r \times J) d^2r + \frac{\hbar}{4\nu} \]
The meaning of the anomalous term is seen directly from the monodromy of FQH states \[2\]. Monodromy with respect to a closed path is the phase acquired by the wave function when a particle is moved along that path. That is a circulation of the particle. It equals to the number of zeros of the wave function with respect to each coordinate. This number is \( (n-1)/\nu \), where \( n \) is the number of particles enclosed by the path. It is less by \( \nu^{-1} \) from the number of magnetic flux quanta piercing the system, simply because the vortex does not interfere with itself. The anomalous term accounts for that difference. The anomalous term can be seen as a local version of the global relation between the monodromy of states and the number of particles. The difference, often called the shift \( 2s \), has been emphasized in Ref. \[22\]. For the Laughlin states \( 2s = \nu^{-1} \).
C. Anomalous term in the chiral constituency relation

Unlike in a regular fluid mechanics where the density $\rho$ and velocity $v$ are independent fields in the chiral flow they are related by the chiral constituency relation. This means that the set of states on the lowest Landau level is restricted in such manner that the velocity operator acts as a certain functional of the density operator.

In a very rough approximation the chiral relation states that the vorticity per particle is the inverse filling factor in units of the Planck constant as suggested in [6]. This view refers to a popular picture of the FQH states as electronic states with an additional amount of flux attached to each particle. The actual relation between the vorticity and the density is more complicated. It involves the anomalous term

$$\nabla \times v = \frac{\hbar}{\nu} [\rho - \bar{\rho} + \frac{1}{4\pi} (\frac{1}{2} - \nu) \Delta \log \rho],$$

where $\bar{\rho} = \nu (2\pi \ell^2)^{-1} = \nu \frac{e}{hc} B$ is the mean density of electrons, and $h = 2\pi \hbar$.

An accurate reading of this relation is: the action of the operators in the r.h.s and the l.h.s of (31) on the Bargmann "bra" state $\langle Q |$ are equal. They are not equal if the "bra"-state is not in the Bargmann state.

In particular, a quasi-hole, a source for vorticity localized at $r_0$, corresponds to the polynomial $Q = \prod_i (z_0 - z_i)$. It deforms the density outside of the core $r = r_0$ according to the equation [31]

$$-\nu \delta (r - r_0) = \rho - \bar{\rho} + \frac{1}{4\pi} (\frac{1}{2} - \nu) \Delta \log \rho.$$

An equivalent form of the chiral relation connects the stream function and the density

$$v_a = -\epsilon_{ab} \nabla_b \Psi, \quad \Psi = \frac{\hbar}{2\nu} [\varphi - (\frac{1}{2} - \nu) \log \rho],$$

where the "regular part" of the stream function $\varphi$ is a solution of the Poisson equation

$$\Delta \varphi = -4\pi (\rho - \bar{\rho}).$$

We comment that the chiral relation readily to extend to the case of the inhomogeneous magnetic field. In this case the mean density $\bar{\rho} = \nu \frac{e}{hc} B$ in [31] and [33] is the function of coordinates. There is no other changes. In particular, in the ground state, where the velocity vanishes, the density in a non-uniform magnetic field obeys the "Liouville equation with the background". That is [31] with zero in the l.h.s. In the leading order in gradients, the ground state density acquires a universal correction

$$\rho = \nu \frac{e}{hc} B - \frac{1}{4\pi} (\frac{1}{2} - \nu) \Delta \log B + \ldots$$

The integrated form of [31] is the sum rule connecting the angular momentum $L = (\hbar \ell^2)^{-1} \int (r \times P) d^2r$ (per particle in units of $\hbar$) to the gyration per particle $N^{-1} G = \int r^2 (\rho - \bar{\rho}) d^2r$ of the flow

$$\ell^2 [L - (\frac{1}{\nu} - 2)] = G.$$

The ground state version of this formula is the familiar sum rule of the Laughlin w.f.

$$\nu \langle 0 | \sum_i |z_i|^2 |0 \rangle = \ell^2 N (N - 1 + 2\nu).$$

D. Anomalous term in the Euler Equation: Lorentz shear stress

The constituency relation [24], the chiral condition [31], the continuity equation [34], and the operator algebra [25,27] is the full set of hydrodynamics equations for the chiral incompressible quantum fluid.

The chiral condition helps to write the continuity equation [34] as a non-linear equation of the density alone

$$\partial_t \rho = \frac{\hbar}{2\nu} \nabla \times \nabla \rho = 0, \quad \Delta \varphi = -4\pi (\rho - \bar{\rho})$$

(36)
The equation is identical to the Euler equation for the vorticity in the incompressible fluid. Naturally the anomalous term disappears from this equation. It appears in the boundary conditions, in the response to external fields and, also, determines forces acting in the fluid.

Forces are rendered by the momentum flux tensor \( \Pi_{ab} \), entering the Euler equation, written in the form of the conservation law

\[
\partial_t P_a + \nabla_b \Pi_{ab} = \rho F_a.
\]

Here \( F = eE - \frac{e}{c} B \times v \) is the Lorentz force. The gradient of the flux tensor is the negative of the force exerted on an element of the liquid.

The anomalous viscous stress emerges in the momentum stress tensor. A general fluid momentum flux tensor of incompressible fluid consists of the kinetic part, the stress, and the traceless viscous stress \( \sigma'_{ab} \). In the incompressible fluid the stress is expressed through the velocity. We write \( \Pi_{ab} = \pi_{ab} - \sigma'_{ab} \), where \( \pi_{ab} \) accounts for the kinetic part and the stress. At a fixed density \( \pi_{ab} \) is symmetric with respect to a change of the direction of the velocity \( v \to -v \). The viscous term is linear in gradients of the velocity. It changes the sign under this transformation. With the exception of the diamagnetic term, the viscous term has a lesser degree of velocity among terms of the flux tensor. This is the only term enters the linear response theory.

Our fluid is dissipation-free. Therefore the anomalous viscous stress produces no work. This is possible if the viscous stress represents forces acting normal to a shear. Such stress could only be a traceless pseudo-tensor. It changes sign under the spatial reflection. In the vortex flow the anomalous viscous stress reads

\[
\sigma'_{ab} = -\frac{\hbar}{2\nu} \hat{\rho}(\nabla_a \nabla_b - \frac{1}{2} \delta_{ab} \Delta)\Psi.
\]

It is often called the Lorentz shear stress. There is a noticeable difference with the dissipative shear viscous stress. That stress is given by the same formula where the stream function is replaced by the hydrodynamic potential.

Components of the anomalous viscous stress tensor are

\[
\sigma'_{xx} = -\sigma'_{yy} = -\frac{\hbar}{4\nu} \hat{\rho}(\nabla_x v_y + \nabla_y v_x), \quad \sigma'_{xy} = \sigma'_{yx} = \frac{\hbar}{4\nu} \hat{\rho}(\nabla_x v_x - \nabla_y v_y)
\]

They represent the force exerted by the flow on a line element of the liquid. The divergence of the Lorentz shear stress is the Lorentz shear force

\[
\nabla_b \sigma'_{ab} = \frac{\hbar}{4\nu} \hat{\rho} \nabla_a (\nabla \times v)
\]

exerted by the flow on the volume element of the liquid. It is proportional to the gradient of the vorticity.

A noticeable feature of the anomalous stress is that the kinetic coefficient \( \frac{1}{4\nu} \) (in units of \( \hbar \)) is universal and has a geometrical origin. The anomalous conservative viscosity is referred as odd-viscosity, or Hall viscosity. It has been introduced in Ref. \[16\] for the Integer Hall Effect as a linear response to a shear. Its notion has been extended to the FQHE in \[17, 18\], see \[15\] for further incomplete list of references. In this paper we show how the anomalous viscosity appears in the hydrodynamics of the chiral flow.

The anomalous term \( \frac{\hbar}{4\nu} \hat{\rho} \) represents the force acting normal to the shear (in contrast, the shear viscous force acts in the direction parallel and opposite to the shear). The stress is also referred as the Lorentz stress, and the force is referred as the Lorentz shear force \[17\].

Emergence of the Lorentz shear stress can be interpreted in terms of semiclassical motion of electrons. A motion of electrons consists of a fast motion along small orbits and a slow motion of orbits. A shear flow strains orbits elongating them normal to the shear, boundaries and vortices. Elongation yields an additional Lorentz shear stress.

E. Topological sector

The topological sector consists of flows driven by slow long-wave external fields, such as curvature of the space, a non-uniform electric and magnetic fields, etc., which does not produce excitations over the gap. The fluid may be driven, but remains in the ground state. Hall current is the most familiar example. The flow in the topological sector is steady.

The topological sector can be singled out in the limit \( m_\ast \to \infty \). In this limit the momentum flux tensor reduces to the anomalous viscous stress modified by the quantum corrections. Then the dynamics reduces to the balance between the Lorentz shear force and the Lorentz force.
In the linear approximation the stationary Euler equation reads

$$\left( \frac{1}{4\nu} - \frac{1}{2} \right) \nabla(\nabla \times v) = eE - \frac{e}{c} B \times v. \quad (40)$$

Solution of the is equation in the leading gradient approximation yields the universal correction to the Hall conductance \[19\]

$$\sigma_{xy}(k) = \frac{\nu e^2}{\hbar} \left( 1 + \left( \frac{1}{4\nu} - \frac{1}{2} \right)(k\ell)^2 \right). \quad (41)$$

The Hall current increases with the wave-vector.

The factor $1/2$ in these equations represents the diamagnetic energy. This term does not appear explicitly in the quantum equation (37). Rather it is hidden in the normal ordering of the kinetic part of the vortex flux tensor. If in addition, particles possess an orbital moment $M$ which is intrinsically related to the band, the term $\frac{m^*}{m_s}M$ is added to the factor $-1/2$ in both equations. Apart from this effect the correction to the Hall conductance is universal.

The universal correction to the Hall conductance can be also seen directly from the Eq. (34) describing the density in a non-uniform magnetic field. The Hall conductance connects the density and magnetic field in the linear approximations (the Streda formula): $e\rho_k = \frac{1}{2}\sigma_{xy}(k)B_k$.

### F. Trace and mixed anomaly

The meaning of the Lorentz shear stress is best illustrated when the fluid is placed into a curved space. In this case the energy receives an addition $H' = -\frac{1}{2} \int g^{ab} \sigma'_{ab} \sqrt{g} d^2\xi$ from the viscous tensor, where $g_{ab}$ is the spatial metric. Integrating by parts this term appears in the suggestive form $H' = \frac{\hbar}{16\pi\nu} \int R \psi \sqrt{g} d^2\xi$, where $R$ is the spatial curvature. This addition yields the **trace anomaly**: the flux tensor acquires a trace proportional to the curvature

$$-\sigma'_{aa} = \bar{\rho} \frac{\hbar^2}{16\pi\nu} R. \quad (42)$$

It is traceless if the space is flat.

The trace anomaly yields to a uniform force acting toward the region with the access curvature. This force squeeze particles toward the curved regions (**mixed anomaly**)

$$\delta\rho = \frac{1}{8\pi} R \sqrt{g}. \quad (43)$$

Accumulation of charges at curved parts of space has been suggested in [22] and further discussed in [19].

These formulas represent the effect of the anomalous terms valid in a semiclassical approximation at large $\nu^{-1}$. They experience quantum corrections, effectively replacing $\nu^{-1}$ in the formulas by $\nu^{-1} - 2$.

### G. Dispersion of density modulation

The anomalous term in the commutation relations [20] yields a universal correction to the kinetic energy of small density modulations $|k\rangle = \sum_i e^{ik\cdot r_i} |0\rangle$ [30]

$$\Delta_\nu(k) = \frac{1}{m^*\ell^2} \langle k | P(r) P^\dagger(r) | k \rangle. \quad (44)$$

We will show that at small wave vectors the dispersion is negative

$$\Delta_\nu(k) = \Delta_\nu(0) \left( 1 - \frac{1}{2\nu} \left( \frac{1}{2\nu} - 1 \right)(k\ell)^2 \right), \quad \Delta_\nu(0) = \frac{\hbar^2}{2m_s\ell^2}. \quad (45)$$

Such behavior signals the magneto-roton minimum discussed in Ref. [2], similar to the roton minimum known in superfluid helium. The dispersion of the excitation has been measured in the recent work [28]. There the excitation spectrum has been probed through the resonant absorption in the regime where surface acoustic waves propagate across the sample.
H. Boundary double layer and dispersion of edge modes

A striking manifestation of the anomalous terms is seen on the boundary. The Lorentz shear force squeezes flow lines with different velocities. As a result the charge there is an accumulation of density on the edge. The density at the edge \( r = R \) forms the double layer

\[
\rho(r) = \bar{\rho} + \frac{1 - \nu}{4\pi} \nabla_n \delta(r - R). \tag{46}
\]

Here the derivative is taken in the direction normal to the boundary.

A consequence of the double layer is the correction to the spectrum of edge modes

\[
\omega(k) = c_0 k + \frac{1}{2} \Delta \nu \left( \frac{1}{\nu} - 1 \right) \text{sign}(k)(k\ell)^2, \quad c_0 = \frac{E}{B}. \tag{47}
\]

These results were obtained in \[9\].

In the rest of the paper we obtain these (and some other) properties starting from quantized chiral fluid. It happens that many calculations are merely identical in the classical and the quantum cases. To simplify the matter, we first derive the hydrodynamics of the vortex fluid in the classical case, and then consider the quantum case.

VI. RELATION BETWEEN VELOCITY OF THE VORTEX FLOW AND VELOCITY OF THE FLUID.

Eulerian hydrodynamics of the vortex flow describes the flow in terms of the density and velocity field \( v(r) \) of vortices. We construct the velocity field starting from velocities of individual vortices. The calculations are merely identical in the classical and the quantum cases. We proceed with the classical calculations.

The density of vortices is the vorticity of the fluid \( \nabla \times u \) modulo the vorticity of the solid rotation. For the purpose of the hydrodynamics of the vortex flow we denote it as

\[
\rho(r) = \sum_i \delta(r - r_i) = \bar{\rho} + \frac{1}{2\pi \Gamma}(\nabla \times u). \tag{48}
\]

Having the density one finds the velocity, or the stream function of the fluid. The stream function of the fluid is the potential \( \varphi \)

\[
u = -2i\Gamma \partial \varphi, \quad \Delta \varphi = -4\pi(\rho - \bar{\rho}). \tag{49}
\]

The object of interest is the vortex flux

\[
P(r) = \sum_i \delta(r - r_i) v_i. \tag{50}
\]

Having the flux, we define the velocity field of the vortex fluid as \( P = \rho v \). We want to compute the velocity of the vortex flow \( v(r) \), and to compare it with the velocity of the original fluid \( u(r) \). Obviously, they are different. The former describes the slow motion of vortices, the latter the fast motion of the fluid around vortices and the drift together with vortices. Nevertheless there is a simple relation between the two.

We compute the vortex flux \( P \) and compare it with the vorticity flux \( J = \rho u \), where the velocity of the fluid \( u \) is given by \( \varphi \).

Using \( \bar{\partial} \) and the \( \bar{\partial} \)-formula \( \pi \delta = \bar{\partial}(\frac{1}{z}) \) we write

\[
P(r) = \sum_i \delta(r - r_i)[-i\Omega \bar{z}_i + i \sum_{i,j \neq j}^N \Gamma z_i - z_j]\]

\[
= -i\Omega \bar{z} \rho(r) + i \frac{\Gamma}{\pi \bar{\partial}} \sum_{i \neq j}^N \frac{1}{z - z_i} \frac{1}{z_i - z_j}. \tag{51}
\]
Then use the identity
\[
2 \sum_{i \neq j} \frac{1}{z - z_i} \frac{1}{z - z_j} = \left( \sum_{i} \frac{1}{z - z_i} \right)^2 - \sum_{i} \left( \frac{1}{z - z_i} \right)^2 = \left( \sum_{i} \frac{1}{z - z_i} \right)^2 + \partial \sum_{i} \frac{1}{z - z_i}
\]  
(52)

and apply \( \bar{\partial} \)
\[
\rho v = -i \Omega \bar{\rho} + i \sum_{i} \delta(r - r_i) \sum_{j} \frac{\Gamma}{z - z_j} + \frac{\Gamma}{2} \bar{\partial} \sum \delta(r - r_i).
\]  
(53)

We obtain the relations
\[
P = \rho u + \frac{\Gamma}{2} i \partial \rho, \quad v = u + \frac{\Gamma}{2} \rho^{-1} i \partial \rho.
\]  
(54)

The difference between the velocity of the vortex fluid and the velocity of the fluid has a simple meaning. Velocity of the fluid \( u \) diverges at the core of an isolated vortex (as it is seen in (7)). However, velocities of vortices are finite. The anomalous term removes that singularity.

The anomalous term changes only the transverse part of the velocity, so that the flow of vortices is incompressible like the fluid itself \( \nabla \cdot v = \nabla \cdot u = 0 \). Also, the anomalous term does not change the divergency of the flux \( \nabla \cdot P = v \cdot \nabla \rho = \nabla \cdot (\rho u) = u \cdot \nabla \rho \).

VII. CLASSICAL HYDRODYNAMICS OF THE CHIRAL FLOW

A. Anomalous vortex flux tensor: Lorentz shear stress

Global symmetries of space and time, such as translation and rotation yield familiar conservation laws of the flux, energy and angular momentum. In addition, the 2D incompressible flows with a constant density possess conservation laws which are not directly related to global symmetries. One conservation law is familiar. This is the conservation of vorticity. With the help of (54) the Euler equation in the form of (6) appears as the continuity equation for the mass density of the vortex fluid
\[
D_t \rho \equiv \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \rho = 0.
\]  
(55)

In addition to conservation of vorticity, the vorticity flux \( J \) and the vortex flux \( P \) are also conserved
\[
J = \rho u, \quad P = \rho v.
\]  
(56)

Conservations of the vorticity and the vortex flux are obvious in the Kirchhoff picture. They are conservation of mass and mass flux of the vortex system. The vorticity flux is conserved due to the relation (54). In continuum fluid mechanics conservation of the vorticity and vortex fluxes are less obvious. They follow from the observation that the vorticity flux is the flux of the fluid plus a divergency of the symmetric and traceless tensor
\[
J_{a} = \bar{\rho} u_{a} + \frac{1}{2 \pi \Gamma} \epsilon_{ab} \partial_{c} t_{bc}, \quad t_{bc} = u_{b} u_{c} - \frac{1}{2} \delta_{bc} u^{2}.
\]  
(57)

The conservation law for the vorticity flux introduces the vorticity flux tensor
\[
\partial_{t} J_{a} + \nabla_{b} \pi_{ab} = \frac{-c}{\epsilon} \rho (B \times u)_{a}
\]  
(58)

The r.h.s. of this equation is the Lorentz force.

The vorticity flux tensor can be locally and explicitly expressed through the velocity and the pressure. Expression is cumbersome and we do not need it for the purpose of this paper. In the leading approximation in the density gradients the second term in (57) could be dropped. Then the vorticity flux tensor is identical to the flux tensor of the incompressible fluid with the constant density \( \pi_{ab} \equiv \rho u_{a} u_{b} + \rho \delta_{ab} \).

The next step is to determine the vortex flux tensor \( \Pi_{ab} \). It enters into the conservation law for the vortex flux
\[
\partial_{a} P_{a} + \nabla_{b} \Pi_{ab} = \frac{-c}{\epsilon} (B \times v)_{a}.
\]  
(59)
We see it as a transformation of the vorticity flux tensor induced by the transformation of the velocity \[ u \rightarrow v, \quad J \rightarrow P, \quad \pi \rightarrow \Pi. \] (60)

Under the shift (54) we have \( \hat{\Pi}_a = \hat{\Pi}_a + \Gamma \epsilon_{ab} \nabla_b \hat{\rho}. \) With the help of the continuity equation (55), we obtain the transformation

\[ \pi_{ab} \rightarrow \Pi_{ab} = \pi_{ab} + \Gamma \epsilon_{ac} \nabla^c (\rho v_b) + \epsilon_{bc} \nabla^c (\rho v_a). \]

In the leading approximation in gradients we replace the the density in the last equation by its mean \( \bar{\rho}. \) We observe that the stress tensor acquires the anomalous viscous term

\[ \Pi_{ab} \approx \pi_{ab} - \sigma'_{ab}, \quad \sigma'_{ab} = -\frac{\Gamma}{4} \bar{\rho} (\epsilon_{ac} \nabla^c v_b + \epsilon_{bc} \nabla^c v_a). \] (61)

This is the Lorentz shear stress \[ 15. \]

We see that the Lorentz shear stress naturally appears in the vortex liquid. Chiral flows consist of a fast motion along small orbits around vortex cores and a slow drift of centers of these orbits. A shear flow strains orbits elongating them normal to the shear, boundaries and vortices. Elongation yields to the Lorentz shear stress.

**VIII. QUANTUM HYDRODYNAMICS OF THE VORTEX MATTER**

We start by quantizing the incompressible chiral 2D fluid and then proceed with the quantization of the vortex flow.

### A. Quantum hydrodynamics of incompressible liquid

The canonical fields in hydrodynamics are density and velocity. In the chiral fluid with a constant fluid density the canonical hydrodynamics variables are velocity \( u \) and vorticity \( \rho, \) or rather, holomorphic and anti-holomorphic components of the velocity \( u \) and \( u^\dagger. \)

It should be noted a subtlety of quantizing hydrodynamics in the Bargmann space. The density (48) is real and consists of holomorphic and anti-holomorphic variables. We "decompose" it into the holomorphic and anti-holomorphic parts using the \( \bar{\partial} \) formula \( \pi \delta(r) = \bar{\partial} (\frac{1}{z}) = \partial (\frac{1}{z}) \) as \( \rho(r) = \rho_+ + \rho_- = \frac{1}{2\pi} \bar{\partial} \sum_i \frac{1}{z - z_i} + \frac{1}{2\pi} \partial \sum_i \frac{1}{z - z_i}. \) In the Bargmann space the action of the holomorphic operator \( 2\partial_{z_i} \) acting on the density is not just a differentiation over coordinates \( \partial_{z_i} - i \partial_{\bar{z}_i} \), as it may appear from the notation. It acts only on the holomorphic part \( \rho_+ \). Hence \( 2\partial_{z_i} \rho = -\partial \delta(r - r_i) \) is half the regular derivative. We already encountered this subtlety in Sec. III C discussing the action of the velocity in the "first quantized" formalism.

Bearing this nuance in mind the quantization of the fluid velocity amounts to the replacement of the term \( -i\Omega \bar{z} \) in (7) by \( \bar{\partial} \pi_{\rho} \), where \( \pi_{\rho} = -i\hbar \frac{\partial}{\partial \rho} \) is the canonical momentum the density. Also we replace the sum in (7) by the integral \( \sum_i \frac{\Gamma}{z - z_i} \rightarrow \Gamma \int \frac{\rho(\xi)}{z - \xi} d^2 \xi = -i\hbar \partial \varphi(\rho + \pi \bar{\rho}|z|^2). \)

We obtain the velocity of the quantum chiral fluid

\[ u = \partial \left( \pi_{\rho} - i\frac{\hbar}{\nu} (\varphi + \pi \bar{\rho}|z|^2) \right). \] (62)

This formula yields the canonical commutation relation between vorticity and velocity and between components of velocity

\[ [u(r), \rho(r')] = -i\hbar \partial \delta(r - r'), \quad \nabla \times u = i(\bar{\partial} u - \partial u^\dagger) = \frac{\hbar}{\nu} (\rho - \bar{\rho}). \] (63)

The commutation relations between components of velocity is the canonical Heisenberg algebra as it is known to be in a quantizing magnetic field

\[ [u(r), u^\dagger(r')] = \frac{\hbar^2}{\pi \nu} \delta(r - r'), \quad [u(r), u(r')] = 0. \] (64)

The algebra is completed by the equal point commutator

\[ [u(r), \rho(r)] = -i\hbar \partial \rho(r). \] (65)
The remaining element of the quantization is the chiral condition. The holomorphic derivative \( \partial z_i \) acting to the left on the anti-holomorphic "bra" states of the Bargmann space \( \pi \) differentiates only the factor \( \exp(-\sum_i |z_i|^2) \) of the measure \( \langle Q| (2\ell^2 \partial^T + \bar{z}_i) = 0 \). Similarly the operator \( \partial \bar{\pi} \), acting on the left acts only on the factor \( \exp(-1/2\ell^2 \int \rho d^2r) \).

\[
\langle Q| \left( \partial \pi + i\hbar \nu \partial \varphi \right) |Q' \rangle = 0.
\]

Therefore, when the holomorphic velocity operator acts on the ant-holomorphic "bra" state the first two terms in (62) cancel. We return to the classical formula (49)

\[
\langle Q| u + i\hbar \nu \partial \varphi |Q' \rangle = 0.
\]

We emphasize that this relation does not hold if the operator is not sandwiched between anti-holomorphic and holomorphic states.

The chiral condition projects all operators onto the lowest Landau level. The projected velocity is manifestly divergence-free. Projection onto the lowest Landau level is summarized by the condition \( \Delta \pi = -4\pi \bar{\rho} \).

The Heisenberg algebra of velocities (64), the continuity equation for the vorticity \( D_t \rho = 0 \) (55), and the chiral condition (67) summarize the quantization of hydrodynamics of incompressible chiral flow.

Finally we are ready to proceed with the quantization of the vortex fluid.

### B. Quantization of the vortex fluid

The classical formula for the flux (50) must be treated as an ordered product of operators

\[
P(r) = \sum_i \delta(r - r_i)p_i = \sum_i (p_i + i\hbar \partial z_i)\delta(r - r_i),
\]

where the momenta \( p_i \) are given by (21). The relation between the velocity (54) holds on the quantum level

\[
P = \rho u + i\hbar \nu \partial \rho.
\]

The chiral condition is obtained by placing \( u \) to the left. Using (65), or equivalently (68), pull \( u \) to the left and reduce it to its classical value (49). This yields the chiral conditions of Sec. V C

\[
P = -i\hbar \nu \partial \varphi + i\hbar \left( \frac{1}{2\nu} - 1 \right) \partial \rho.
\]

The commutation relations of flux components presented in Sec. VII B by Eqs. (26, 27) follow.

Computation of the quantum vortex flux tensor is not much different from the classical version of the Sec. VII A. All the formulas remain the same provided that normal ordering of operators is respected. However, when the velocity is pulled to the left the coefficient in front of the Lorentz force acquires the quantum correction \( \frac{1}{2\nu} \rightarrow \frac{1}{2\nu} - 1 \).

### IX. APPLICATIONS

#### A. Structure function

Anomalous commutation relations help to compute the structure function. This is \( s_k = N^{-1} \langle 0| \rho_k \rho_{-k} |0 \rangle \), where \( \rho_k = \sum_i^{N} e^{i k \cdot r_i} \) is the Fourier mode of a small density modulation with the wave vector \( k \).

The chiral condition connects modes of density and flux. Let us evaluate it in the linear approximation in density modes. Using \( k^2 \varphi_k = 4\pi \rho_k \) in (70) we write the Fourier mode of the flux in terms of density modes

\[
P_k = \frac{\hbar k}{(\ell k)^2} \left( 1 - \frac{1}{2\nu} \right) \left( 1 - \frac{1}{2\nu} \right) \rho_k, \quad k = k_x - ik_y.
\]

---

The text continues with further details on quantization and applications, including discussions on structure functions and other aspects of hydrodynamics in the context of chiral flow.
On the other hand the commutation relation \( [P_k, \rho_{-k}] = \frac{1}{2} Nh \hbar \).

Since \( P_k \) annihilates the ground state \( \langle 0| [P_k, \rho_{-k}] |0 \rangle = 0 \), we obtain the relation

\[
\langle 0| P_k \rho_{-k} |0 \rangle = \left( \frac{\hbar k}{(\ell k)^2} \left( 1 - \frac{1}{2} \left( 1 - \frac{1}{2\nu} - 1 \right) (k\ell)^2 \right) \right) \langle 0| \rho_{k\rho_{-k}} |0 \rangle = \frac{1}{2} \hbar k N.
\]

The known result \( [2] \) for the spectral function follows

\[
s_k = \langle 0| \rho_{k\rho_{-k}} |0 \rangle \approx (k\ell)^2 \left( 1 + \frac{1}{2} \left( 1 - \frac{1}{2\nu} - 1 \right) (k\ell)^2 \right) + \ldots.
\]

We see that the anomalous term accounts for the universal correction to the Hall conductance (41). Comparing the expressions of the spectral function \( s_k \), we obtain the relation

\[
\Delta_\nu(k) = \frac{\hbar^2 k^2}{2m_s s_k}.
\]

These formulas make sense in the leading order in \((k\ell)^2\).

We observe that the excitation spectrum is gapped and has a negative dispersion. The energy starts to increase at a larger \((k\ell)^2\). It oscillates at intermediate wave-length. To avoid possible confusion, we emphasize that we evaluated the kinetic energy over the states \( |k\rangle = \sum e^{ik\cdot r}|0\rangle \). These states are different from the "projected waves" of Ref. \( [2] \). Projected plane waves are created by the normal ordered wave operator \( \sum e^{-ik^2 \partial_i e^{-i\frac{\mathbf{B}}{c} \cdot \mathbf{r}}}|0\rangle \). Operators expanded in this basis is an interesting question of its own. We will address it elsewhere. Here we comment that it is arguable that acoustic waves used in the experiment \( [28] \) are projected plane waves. Rather they are regular waves \( |k\rangle = \sum e^{ik\cdot r}|0\rangle \).

C. Hall conductance in a non-uniform background

The formulas of the previous section are readily adapted to study transport in the topological sector, say a transport in a non-uniform electric field.

An electric field acts only on vortices. Thus we add the term \( e\rho E \) in the r.h.s. of the conservation law \( [59] \). In the topological sector \((m_s \to \infty)\) the flow is steady, and the only part of the flux tensor survives in the limit is the Lorentz shear force. It balances the Lorentz force

\[
-\nabla_b \sigma'_{ab} = \rho (eE - \frac{e}{c} B \times v)_a
\]

Pulling velocity to the left and in the leading approximation in gradient we obtain Eq. (40) of the Sec. V.E. That equation yields the universal correction to the Hall conductance \( [11] \). Comparing the expressions of the spectral...
function \[74\] and the Hall conductance we observe a simple relation between the two objects. It can be obtained from a general theory of the linear response.

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[29] In experiments the cyclotron energy is only few times larger than the gap.
[30] In FQH liquids the Coulomb forces essentially block propagating waves in the bulk. In this paper we neglect Coulomb forces in order to unmask laws of quantum hydrodynamics.
[31] Incidentally a similar equation exists inside the vortex core. There the quantum corrections changes the last term to \(-\frac{1}{4\pi}j\Delta \log \rho\). Accidentally a similar equation followed from the effective action of Refs. [3,4] erroneously featuring the term \(-\frac{1}{4\pi}j\Delta \log \rho\) inside and outside of the vortex.
[32] See Feynman’s book [7] p. 319 for an interesting critique of Landau’s quantum hydrodynamics approach to the superfluid helium "Proof that superfluidity is not explained by quantum hydrodynamics".