Gravitational radiation from a rotating magnetic dipole

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Abstract

The gravitational radiation emitted by a rotating magnetic dipole is calculated. Formulas for the polarization amplitudes and the radiated power are obtained in closed forms, considering both the near and radiation zones of the dipole. For a neutron star, a comparison is made with other sources of gravitational and electromagnetic radiation.

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1 Introduction

Gravitational radiation is an important source of energy in many astrophysical phenomena. A neutron star, for instance, radiates both electromagnetic [1, 2] and gravitational waves [3, 4, 5]; the main sources of radiation are the interior of the star (behaving as a magnetized fluid), the magnetic dipole field and the corotating magnetosphere.

In the present article, we study the gravitational waves (GWs) generated by one of these possible sources: a rotating magnetic dipole. The radiated
energy and the polarization amplitudes of the GWs are calculated, considering the electromagnetic field in either the near or the radiation zones of the dipole; it is shown that the contribution of the latter is negligible in all realistic cases. With a particular interest in neutron stars, the results are compared to other processes that can also produce an energy loss through gravitational or electromagnetic radiations.

The calculations are presented in Section 2, where the basic formulas of gravitational radiation are applied to the problem of a rotating magnetic dipole, taking into account the near zone of the electromagnetic field. Equivalent calculations for the radiation zone are given in section 3, showing that the corresponding gravitational radiation is entirely negligible. The results are discussed in Section 4 and compared with other sources of radiation. Some concluding remarks are presented in Section 5.

2 Near zone

Our starting point is the formula for the metric $h_{ij}^{TT}$ in the TT gauge [6],

$$h_{ij}^{TT} = \frac{1}{r} \frac{4G}{c^5} \Lambda_{ij,kl}(\hat{n}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \widetilde{T}_{kl}(\omega, \omega \hat{n}/c)e^{-i\omega(t-r/c)},$$  \hspace{1cm} (2.1)

where $\widetilde{T}_{kl}(\omega, k)$ is the Fourier transform of the energy-momentum tensor,

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2} P_{ij}P_{kl},$$  \hspace{1cm} (2.2)

and

$$P_{ij}(\hat{n}) = \delta_{ij} - \hat{n}_i \hat{n}_j$$  \hspace{1cm} (2.3)

is the projection tensor with respect to the unit vector $\hat{n}$. The energy radiated in the form of GWs in that same direction is

$$\frac{dE}{d\Omega} = \frac{r^2c^3}{32\pi G} \int_{-\infty}^{\infty} dt \ \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT},$$  \hspace{1cm} (2.4)

where dots represent derivation with respect to time $t$ (see Maggiore [6] for details).

Consider the field of a magnetic dipole of magnitude $m$. In the near zone, the electric field can be neglected and the magnetic field is

$$B_i = \frac{m}{r^3} \left[ -\dot{u}_i(t) + 3(\hat{n}(t) \cdot \dot{r})\dot{r}_i \right],$$  \hspace{1cm} (2.5)
where \( \mathbf{\hat{u}}(t) \) is the unit vector in the direction of the dipole, and \( \mathbf{r} \) is a unit radial vector. Further corrections to the electromagnetic field are of order \( \omega r/c \) with respect to \( B_i \) (where \( \omega \) is the rotation frequency of the dipole). For neutron stars of radius \( R \sim 10 \text{ km} \), the approximation is valid for \( \omega \ll c/R \sim 3 \times 10^4 \text{ s}^{-1} \).

The energy-momentum tensor is
\[
T_{ij} = \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} \delta_{ij} B_n B_n \right),
\]
with Fourier transform
\[
\tilde{T}_{ij}(\omega, \mathbf{k}) = c \int_{-\infty}^{\infty} dt \, d\mathbf{x} \, T_{ij}(t, \mathbf{x}) \, e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}},
\]
\[
= \frac{m^2}{4\pi} c \int_{-\infty}^{\infty} dt \left( I_{ij} - \frac{1}{2} \delta_{ij} I_{nn} \right) e^{i\omega t},
\]
where
\[
I_{ij}(t, \mathbf{k}) \equiv \int \frac{d^3x}{r^6} \left[ -\mathbf{\hat{u}}_i + 3(\mathbf{\hat{u}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}}_i \right] \left[ -\mathbf{\hat{u}}_j + 3(\mathbf{\hat{u}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}}_j \right] e^{-i\mathbf{k} \cdot \mathbf{x}}.
\]

The most general form of \( I_{ij} \) is
\[
I_{ij} = A\delta_{ij} + B(\mathbf{\hat{n}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}}_i + C(\mathbf{\hat{u}} \cdot \mathbf{\hat{n}}) \mathbf{\hat{u}}_i + D(\mathbf{\hat{u}} \cdot \mathbf{\hat{n}}) \mathbf{\hat{u}}_i,
\]
where \( \mathbf{k} = k \mathbf{\hat{n}} \) (\( k = \omega/c \)), and \( A, B, C, D \) are scalar functions of \( \mathbf{k} \) and \( t \). Since \( \Lambda_{ij,kl}\delta_{ij} = 0 \) and \( \Lambda_{ij,kl}\mathbf{n}_l = 0 \), it follows that
\[
\Lambda_{ij,kl}(\mathbf{\hat{n}})\tilde{T}_{kl}(\omega, \omega \mathbf{\hat{n}}/c) = \frac{m^2}{4\pi} \Lambda_{ij,kl}(\mathbf{\hat{n}}) \, c \int_{-\infty}^{\infty} dt \, \mathbf{\hat{u}}_k(t) \mathbf{\hat{u}}_l(t) D(t, \omega \mathbf{\hat{n}}) e^{i\omega t},
\]
and thus the function \( D \) is the only one to be calculated.

If we define \( \mathbf{\hat{n}} \cdot \mathbf{\hat{u}}(t) \equiv \cos \chi(t) \), then \( P_{ij} \mathbf{\hat{u}}_i \mathbf{\hat{u}}_j = \sin^2 \chi \), and it follows from Eq. \((2.9)\) that
\[
\frac{1}{2} D \sin^4 \chi = \Lambda_{ij,kl} \mathbf{\hat{u}}_i \mathbf{\hat{u}}_j I_{kl},
\]
\[
= (u_{\perp k} u_{\perp l} - \frac{1}{2} P_{kl} \sin^2 \chi) I_{kl},
\]
where \( u_{\perp i} \equiv P_{ij} \hat{u}_j \). The next step is to substitute \( I_{kl} \) in this last equation, and perform the volume integration in Eq. (2.8). For this purpose we set (provisionally) \( \hat{n} \) along the \( z \) axis, and use spherical coordinates such that

\[
\hat{r} = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta').
\]  

The \( \phi' \) integration can be performed with the general formula

\[
\int_0^{2\pi} (u_1 \cos \phi + u_2 \sin \phi)^n d\phi = C_n,
\]

where \( C_0 = 2\pi, C_2 = \pi \sin^2 \chi, C_4 = (3\pi/4) \sin^4 \chi, C_n = 0 \) if \( n \) is odd, and \( u_1^2 + u_2^2 = \sin^2 \chi \). It then follows with some straightforward algebra that

\[
D = \frac{\pi}{2} \int_R^\infty dr \ r^{-4} \int_0^\pi d\theta' \ \sin \theta' (1 - 3 \cos^2 \theta')^2 \ e^{-ikr \cos \theta'},
\]

where \( R \) is a cut-off: it is the minimum radius for the validity of the dipole approximation. For a neutron star, \( R \) can be interpreted as its radius.

Notice that the factor \( \sin^4 \chi(t) \) cancelled out in the calculations; thus \( D = D(k) \) does not depend on time or on \( \hat{n} \). The integral (2.13) can be performed analytically, but it is enough to note that for \( kR \ll 1 \)

\[
D = \frac{4\pi}{15 R^3} (1 + O([kR]^2)).
\]

The metric \( h_{ij}^{TT} \) can be calculated from (2.1) and (2.10). The corresponding integral is easily performed since it contains the Fourier transform of a delta-function. The final result is

\[
h_{ij}^{TT} = \alpha \ \Lambda_{ij,kl} \ \hat{u}_k(t') \ \hat{u}_l(t'),
\]

where

\[
\alpha = \frac{4Gm^2}{15c^4 R^3 r}
\]

and \( t' = t - r/c \).

### 2.1 Rotating dipole

Let us now take a coordinate system in which the rotation axis of the dipole is in the \( z \) direction. Thus

\[
\hat{u}(t) = (u_\perp \cos(\omega t), u_\perp \sin(\omega t), u_\parallel),
\]
where $u_\parallel$ is the constant component of $\hat{u}(t)$ along the rotation axis, $u_\perp^2 = 1 - u_\parallel^2$, and $\omega$ is the angular velocity of the dipole. In this same system of coordinates we can define the three orthonormal vectors:

$$
\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\hat{\theta} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),
\hat{\phi} = (-\sin \phi, \cos \phi, 0),
$$

(2.18)

with the useful formulas

$$
\hat{\theta}_i \hat{\theta}_j \Lambda_{ij,kl} = \hat{\theta}_k \hat{\theta}_l - \frac{1}{2} P_{kl},
$$

$$
\hat{\phi}_i \hat{\phi}_j \Lambda_{ij,kl} = \hat{\phi}_k \hat{\phi}_l - \frac{1}{2} P_{kl},
$$

$$
\hat{\phi}_i \hat{\theta}_j \Lambda_{ij,kl} = \hat{\phi}_k \hat{\theta}_l.
$$

(2.19)

### 2.2 Metric

The metric potentials of the GW can be calculated from (2.15) and the formulas (2.19). The result is

$$
h_+ \equiv h_{ij}^{TT} \hat{\phi}_i \hat{\phi}_j = -h_{ij}^{TT} \hat{\theta}_i \hat{\theta}_j = \frac{1}{2} \alpha (u_\phi^2 - u_\theta^2)
$$

$$
h_\times \equiv -h_{ij}^{TT} \hat{\theta}_i \hat{\phi}_j = -\alpha u_\theta u_\phi,
$$

(2.20)

where $u_\theta = \hat{u} \cdot \hat{\theta}$ and $u_\phi = \hat{u} \cdot \hat{\phi}$, and also $(\hat{u} \cdot \hat{n})^2 + u_\theta^2 + u_\phi^2 = 1$. The above two formulas can be written as

$$
h_+ + ih_\times = \frac{1}{2} \alpha (u_\phi - iu_\theta)^2,
$$

(2.21)

and explicitly

$$
u_\phi = u_\perp \sin(\omega t' - \phi)
$$

$$
u_\theta = u_\perp \cos \theta \cos(\omega t' - \phi) - u_\parallel \sin \theta.
$$

(2.22)

Thus, in general, the spectrum of the GW has two lines, one at $\omega$, and one at $2\omega$, and its amplitude is $\alpha$ (a unique frequency $2\omega$ if the GW propagates along the rotation axis).
2.3 Radiated energy

In order to calculate the radiated power, notice that

\[ \dot{\mathbf{h}}_x + i \dot{\mathbf{h}}_\phi = \alpha (u_\phi - i u_\theta) (\dot{u}_\phi - i \dot{u}_\theta), \]

and thus

\[ \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} = \dot{h}_x^2 + \dot{h}_\phi^2 = \alpha^2 [1 - (\hat{\mathbf{u}} \cdot \hat{\mathbf{r}})^2] (\dot{u}_\phi^2 + \dot{u}_\theta^2) \]

\[ = \alpha^2 u_\perp^2 \omega^2 [1 - \left( u_\perp \sin \theta \cos (\omega t' - \phi) + u_\parallel \cos \theta \right)^2] \left[ 1 - \sin^2 \theta \sin^2 (\omega t' - \phi) \right]. \]

(2.24)

The power radiated in unit time follows from Eq. (2.4) performing the integration over one period \( T = 2\pi/\omega \) and dividing by \( T \). The result is

\[ \frac{dP}{d\Omega} = \frac{G}{3600\pi c^5} \frac{m^4}{R^6} u_\perp^2 \omega^2 \left[ 1 - \cos^4 \theta + \frac{1}{4} u_\perp^2 (5 \cos^4 \theta + 6 \cos^2 \theta - 3) \right]. \]

(2.25)

Finally, an integration over solid angles yields the total power radiated:

\[ P = \frac{G}{1125c^5} \frac{m^4}{R^6} u_\perp^2 \omega^2. \]

(2.26)

3 Radiation zone

Just for the sake of comparison we calculate the gravitational radiation from the radiation zone of the electromagnetic field. In the radiation zone, the electric and magnetic fields are

\[ E_i = \frac{m}{r} \epsilon_{ijk} \dot{r}_j \hat{u}_k''(t), \]

(3.27)

\[ B_i = \frac{m}{r} [-\ddot{u}_i''(t) + (\hat{\mathbf{u}}''(t) \cdot \hat{\mathbf{r}}) \dot{r}_i], \]

(3.28)

where the primes denote derivation with respect to \( ct \). The energy-momentum tensor, \( T_{ij} \equiv T_{ij}^B + T_{ij}^E \), has a magnetic component \( T_{ij}^B \), as given by Eq. (2.6), and an electric component:

\[ T_{ij}^E = \frac{1}{4\pi} \left( E_i E_j - \frac{1}{2} \delta_{ij} E_n E_n \right). \]

(3.29)

The Fourier transform is

\[ \tilde{T}_{ij}(\omega, \mathbf{k}) = \frac{m^2}{4\pi} c \int_{-\infty}^{\infty} dt \left( J_{ij} - \frac{1}{2} \delta_{ij} J_{nn} + K_{ij} - \frac{1}{2} \delta_{ij} K_{nn} \right) e^{i\omega t}, \]

(3.30)
where
\[ J_{ij}(t, \mathbf{k}) \equiv \int \frac{d^3x}{r^2} \left[ \hat{u}_i'' + (\hat{\mathbf{u}}'' \cdot \hat{\mathbf{r}}_i)\hat{r}_i \right] \left[ \hat{u}_j'' + (\hat{\mathbf{u}}'' \cdot \hat{\mathbf{r}}_j)\hat{r}_j \right] e^{-ik \cdot x}, \] (3.31)
and
\[ K_{ij}(t, \mathbf{k}) \equiv \epsilon_{imn} \epsilon_{jpq} \int \frac{d^3x}{r^2} \hat{r}_m \hat{u}_n'' \hat{r}_p \hat{u}_q'' e^{-ik \cdot x}. \] (3.32)

For the magnetic part, we can follow exactly the same algebra as in the previous section with only minor changes of coefficients. We have, instead of (2.10),
\[ \Lambda_{ij,kl}(\hat{n}) \tilde{T}_B^{kl}(\omega, \omega \hat{n}/c) = \frac{m^2}{4\pi} \Lambda_{ij,kl}(\hat{n}) c \int_{-\infty}^{\infty} dt \ \hat{u}_k''(t)\hat{u}_l''(t) \tilde{D}(t, \omega \hat{n}) e^{i\omega t}, \] (3.33)
where now
\[ \frac{1}{2} k^8 u_{\perp}^4 \sin^4 \bar{\chi} = \Lambda_{ij,kl} \hat{u}_i'' \hat{u}_j'', \] (3.34)
and \( \hat{n} \cdot \hat{u}''(t) = k^2 u_{\perp} \cos \bar{\chi}(t). \) Thus
\[ \Lambda_{ij,kl} \hat{u}_i'' \hat{u}_j'' = u_{\perp}'' (u_{\perp}'' - \frac{1}{2} k^4 u_{\perp}^2 P_{ij} \sin^2 \bar{\chi}), \] (3.35)
and \( u_{\perp}'' \equiv P_{ij} \hat{u}_j''. \)

The same algebra as for the near zone leads to the result
\[ \tilde{D} = \frac{\pi}{2} \int_0^\infty dr \int_0^\pi d\theta' \ \sin \theta' (1 + \cos^2 \theta')^2 e^{-ikr \cos \theta'} = \frac{\pi^2}{2k}, \] (3.36)
in place of (2.13) and (2.14). Notice that the integral is finite and thus no cut-off is necessary.

For the electric part of the energy-momentum tensor we use the formula
\[ \int \frac{d^3x}{r^2} \hat{r}_i \hat{r}_j e^{-ik \cdot x} = \frac{\pi^2}{k} P_{ij}, \] (3.37)
from where
\[ K_{ij} = \frac{\pi^2}{k} \left[ |\hat{\mathbf{u}}''|^2 \delta_{ij} - \hat{u}_i'' \hat{u}_j'' - (\hat{\mathbf{u}}'')_i (\hat{\mathbf{u}}'' \times \hat{n})_j \right]. \] (3.38)
It follows with some straightforward algebra, using formulas (2.19), that
\[ \Lambda_{ij,kl} K_{kl} \hat{\phi}_i \hat{\phi}_j = \Lambda_{ij,kl} K_{kl} \hat{\theta}_i \hat{\theta}_j = \Lambda_{ij,kl} K_{kl} \hat{\phi}_i \hat{\theta}_j = 0. \]
Therefore the electric part of the energy-momentum tensor makes no contribution to the potential functions $h_{ik}^{TT}$.

Following exactly the same procedure leading to Eq. (2.15), and taking into account that $(\mathbf{u}' \cdot \hat{n})^2 + (u_0''^2) + (u_\phi'')^2 = u_\perp^2 k^4$, we find

$$(h_+ + i h_\times) = \frac{1}{2} \beta (u_\phi'' - i u_\theta'')^2,$$  \hspace{1cm} (3.39)

where now

$$\beta = \pi G m^2/(2c^4 kr).$$

The power radiated from the radiation zone turns out to be

$$P_{rz} = \frac{G \pi^2}{80 c^4} m^4 u_\perp^4 \omega^8. \hspace{1cm} (3.40)$$

Compared with the power emitted from the near zone, as given by Eq. (2.26), we find

$$\frac{P_{rz}}{P} = \frac{225}{16} \pi u_\perp^2 (\frac{\omega R}{c})^6. \hspace{1cm} (3.41)$$

Since $\omega R \ll c$, the contribution of the radiation zone is entirely negligible with respect to that of the near zone.

### 4 Comparisons

For the dipole field, we can set $m = B_0 R^3$, where $B_0$ is the average strength of the magnetic field at the surface of the star. If $B_0 \sim 10^{12}$ G and $\omega \sim 1$ s$^{-1}$, the power radiated in the form of gravitational radiation is

$$P \approx 2 \times 10^{21} \left( \frac{B_0}{10^{12} \text{G}} \right)^4 \left( \frac{R}{10 \text{ km}} \right)^6 (\omega \text{ s})^2 u_\perp^2 \text{ ergs/s}$$

according to formula (2.26). In comparison, the power emitted in the form of electromagnetic waves is [1] [7]

$$P_{em} = \frac{2m^2}{3c^3} u_\perp^2 \omega^4 \approx 2 \times 10^{28} \left( \frac{B_0}{10^{12} \text{G}} \right)^2 \left( \frac{R}{10 \text{ km}} \right)^6 (\omega \text{ s})^4 u_\perp^2 \text{ ergs/s}.$$ 

Thus the ratio of the gravitational power $P$ to $P_{em}$ depends only on $B_0/\omega$ and not on the radius of the star:

$$\frac{P}{P_{em}} \approx 10^{-7} \left( \frac{B_0}{10^{12} \text{G}} \right)^2 (\omega \text{ s})^{-2}.$$
One can also calculate the characteristic decay time, $\tau = E/P$, of a rotating dipole due to the emission of gravitational radiation; the rotational energy is $E = \frac{1}{2}I\omega^2$ and $I$ is the moment of inertia. For, say, $I \sim 10^{45}$ g cm$^2$ we have

$$\tau^{-1} \sim 10^{-24}\left(\frac{B_0}{10^{12} \text{G}}\right)^4 u_\perp^2 \text{s}^{-1},$$

which is independent of $\omega$. Compare this with the equivalent formula for the emission of GWs by an interior toroidal magnetic field of order $\sim 10^{15}$ G, as given by Eq. (2.9) of Cutler [4]: $\sim 5 \times 10^{-25}(\omega \text{s})^4 \text{s}^{-1}$. Clearly, this effect dominates for frequencies higher than $\omega \sim 10$ s$^{-1}$, while the dipole radiation dominates for lower frequencies.

It is also instructive to compare the above results with those obtained by Bonazzola and Gourgoulhon [3] for the GW emission of a neutron star due to its rotation as a distorted fluid:

$$|h_+ + ih_x| = \frac{1}{r} \frac{4G}{c^4} I \epsilon \omega^2,$$

(4.42)

where $\epsilon$ is the ellipticity of the star. Typical values of the parameters are $\epsilon \sim 10^{-6}$, and again $I \sim 10^{45}$ g cm$^2$. Accordingly, the GW produced by the rotating magnetic dipole is much larger, in general, than that due to the rotation of the star as a fluid, the ratio of the amplitudes being

$$10^{2}(B_0\omega^{-1}/10^{12} \text{G s}^{-1})^2.$$

5 Concluding remarks

As shown in Section 2.2, the GW is basically a superposition of two waves with frequencies $\omega$ and $2\omega$. The power of the corresponding gravitational radiation is proportional to the square of the rotational frequency, as implied by Eq. (2.20), while other processes produce a radiation proportional to the fourth-power of the frequency. Consequently, the gravitational radiation emitted by a magnetic dipole is a dominant source of energy for relatively slow rotations, but not for very rapidly rotating stars such as microsecond pulsars.

In general, we have shown that the energies emitted as gravitational and electromagnetic radiations are comparable if $B_0/\omega \sim 10^{16} \text{ G s}$, that is, for relatively high magnetic fields combined with relatively slow rotation.
As already mentioned, there are other important sources of gravitational radiation besides the magnetic dipole of a pulsar. Hence our results must be taken as a lower limit to the radiation of gravitational waves from neutron stars. Nevertheless, the presence of two lines in the spectrum of the GW, at $\omega$ and $2\omega$, would reveal a rotating magnetic dipole as its origin.

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