Poincaré series and deformations of Gorenstein local algebras with low socle degree *

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May 12, 2010

Abstract

Let $K$ be an algebraically closed field of characteristic 0, and let $A$ be an Artinian Gorenstein local commutative and Noetherian $K$–algebra, with maximal ideal $m$. In the present paper we prove a structure theorem describing such kind of $K$–algebras satisfying $m^4 = 0$. We use this result in order to prove that such a $K$–algebra $A$ has rational Poincaré series and it is always smoothable in any embedding dimension, if $\dim_K m^2/m^3 \leq 4$. We also prove that the generic Artinian Gorenstein local $K$–algebra with socle degree three has rational Poincaré series, in spite of the fact that such algebras are not necessarily smoothable.

1 Introduction

Throughout the whole paper by $K$–algebra we will mean a local commutative Noetherian $K$–algebra. When $A$ is a local ring we will usually denote by $m$ its unique maximal ideal. We always assume that $K$ is algebraically closed of characteristic zero. The socle degree of an Artinian algebra $A$ is the integer $s$ such that $m^s \neq 0$, but $m^{s+1} = 0$.

In the main result of this paper we present a structure theorem for the Artinian Gorenstein $K$–algebras $A$ of socle degree 3, Theorem 2.4. The key ingredients of the proof are some well–known facts about the classical Macaulay’s correspondence and

*2000 Mathematics Subject Classification. Primary 13D40; Secondary 13H10;
Key words and Phrases: Gorenstein, Artinian, Poincaré series, smoothable
†Partially supported by MTM2007-67493, Acción Integrada España-Italia 07-09
‡Partially supported by M.I.U.R.: PRIN 07-09, Azione Integrata Italia-Spagna 07-09
a recent result of J. Elias and M.E. Rossi (see [15]). From this result we get some applications to the rationality of the Poincaré series

\[ P_A(z) := \sum_{i \geq 0} \dim_K \text{Tor}_i^A(K, K) z^i \]

and the smoothability of \( \text{Spec}(A) \) in the Hilbert scheme \( \Hilb_d(\mathbb{A}^h) \) parameterizing punctual subschemes of multiplicity \( d \) of the affine space of dimension \( h \) over \( K \).

Actually the results on the Poincaré series of an Artinian Gorenstein local ring could be stated in any dimension \( r \). In fact, let \( (A, m) \) be a Gorenstein local ring of dimension \( r \) and let \( J \) be an ideal generated by a minimal reduction of \( m \), see [6]. Then by [30], Satz 1,

\[ P_A(z) = (1 + z)^r P_{A/J}(z) \]

where \( A/J \) is Artinian, still Gorenstein, whose length is the multiplicity of \( A \).

Notice that, when \( m^r = 0 \) for \( r = 1, 2 \), it is completely trivial to prove the rationality of \( P_A(z) \) and the smoothability in \( \Hilb_d(\mathbb{A}^h) \) since, in those cases, \( A \) has multiplicity either 1 or 2. When \( m^3 = 0 \) both such properties again hold true for Artinian Gorenstein \( K \)–algebras of arbitrary multiplicity \( d \) (see Proposition [2.1]).

J.P. Serre in [29] conjectured the rationality of \( P_A(z) \) and proved it when \( A \) is a regular local ring. The problem has motivations in commutative and non-commutative algebra and over the last 30 years has been a topic of much current interest. Despite many interesting results showing the rationality, J. Anick (see [1]) has given an example of an Artinian algebra \( A \) such that \( m^3 = 0 \) with transcendental Poincaré series. G. Sjödin proved that all Artinian Gorenstein rings with \( m^3 = 0 \) have a rational Poincaré series (see [31]), R. Bøgvad proved that there exist Artinian Gorenstein local rings with \( m^4 = 0 \) and transcendental Poincaré series (see [5]). Nevertheless, several results show that large classes of Gorenstein local rings have a good behaviour with respect to this problem. We mention the local rings which are complete intersections (see [32]), Gorenstein rings with \( \text{emdim}(A) - 4 \leq \text{depth}(A) \) (see [4] and [25]), stretched and almost stretched Gorenstein rings (see [28], [10], [17]), i.e. such that \( \dim_K m^2 / m^3 \leq 2 \), Gorenstein local rings such that \( \dim_K m^2 / m^3 = 3 \) and \( m^4 = 0 \) (see [10]).

In Section 3 we deal with the case of an Artinian Gorenstein local \( K \)–algebra \( A \) such that \( m^4 = 0 \) giving necessary and sufficient conditions for being \( P_A(z) \) rational. In particular we prove that \( A \) has rational Poincaré series if \( \dim_K m^2 / m^3 \leq 4 \) (see Corollary [3.2]).

In [26], [27] and [7] it is proved there that \( \Hilb_d(\mathbb{A}^h) \) is irreducible if either \( d \leq 7 \) or \( d = 8 \) and \( h \leq 3 \) and it has exactly two components when \( d = 8 \) and \( h \geq 4 \). In particular each Artinian \( K \)–algebra \( A \) of length \( d \leq 7 \) is smoothable i.e. can be flatly deformed to the trivial \( K \)–algebra \( K^d \). In [9] and [11] the attention is focused on the problem of smoothability of Gorenstein algebras in \( \Hilb_d(\mathbb{A}^h) \). The locus \( \Hilb^G_d(\mathbb{A}^h) \subseteq \Hilb_d(\mathbb{A}^h) \) of such \( K \)–algebras contains all the algebras isomorphic to the trivial \( K \)–algebra \( K^d \), thus it makes sense to ask when \( \Hilb^G_d(\mathbb{A}^h) \) is irreducible.
and what kind of algebras it contains. In the quoted papers it is proved that this is actually true if \( d \leq 10 \) whereas in [24] the authors prove the reducibility of \( \text{Hilb}_d(\mathbb{A}^h) \) when \( d \geq 14 \) asserting the existence of a non smoothable Artinian Gorenstein local \( K \)-algebra with Hilbert function \((1, 6, 6, 1)\) (see also [11] Section 4 for an explicit example).

In Section 4, starting from the structure theorem for Artinian Gorenstein \( K \)-algebras \( A \) with \( m^4 = 0 \), proved in Section 2, we achieve some results concerning the smoothability of \( \text{Spec}(A) \) in the Hilbert scheme \( \text{Hilb}_d(\mathbb{A}^h) \). This problem can be reduced to the Artinian Gorenstein graded algebras with Hilbert function \((1, h, h, 1)\) which have been well treated in the literature. In Corollary [4.3] we prove that \( A \) is smoothable if \( \dim_K m^2/m^3 \leq 4 \).

2 Structure theorem of Artin Gorenstein algebras with socle degree three

An Artinian Gorenstein algebra \( A \) with maximal ideal \( m \) over a field \( K \) is self dual, that is there exists an exact pairing from \( A \times A \) to \( K \) making \( A \) isomorphic as \( A \)-module to \( \text{Hom}_K(A, K) \).

The Hilbert function of \( A \) is by definition the Hilbert function of the associated graded algebra \( G = gr_m(A) := \bigoplus_{i \geq 0} m^i/m^{i+1} \) (by definition \( m^0 = A \)), i.e.

\[
HF_A(i) = \dim_K m^i/m^{i+1}.
\]

In the graded case, the Hilbert function of an Artinian Gorenstein algebra is symmetric. Little is known about the Hilbert function in the local case. The problem comes from the fact that the associated graded algebra \( G \) is in general no longer Gorenstein.

Nevertheless A. Iarrobino in [23] proved interesting results even in the local case. The information comes from a stratification of \( G \) by a descending sequence of ideals

\[
G = C(0) \supset C(1) \supset \ldots,
\]

whose successive quotient \( Q(a) = C(a)/C(a + 1) \) are reflexive \( G \)-modules. This reflexivity property imposes conditions on the Hilbert function \( HF_A \). If \( A \) has socle degree \( s \), then \( HF_A \) is a sum of symmetric functions \( HF_{Q(a)} \) with respect \( s - a \).

The first subquotient \( Q(0) \) of \( G \) is always a graded Gorenstein algebra and it is the unique (up to isomorphism) quotient of \( G \) with socle degree \( s \). If necessary, we will write \( Q_A(0) \). A. Iarrobino proved that if \( HF_A \) is symmetric, then \( G = Q_A(0) \) and it is Gorenstein. Hence (see [23] Proposition 1.7 and [19] Proposition 7) \( G \) is Gorenstein if and only if \( HF_A \) is symmetric, equivalently if \( G = Q_A(0) \).

For example, if \( A \) is a Gorenstein local \( K \)-algebra with socle degree three and embedding dimension \( h \) (= \( HF_A(1) \)), then we deduce that \( HF_A(2) = n \leq h \) and
clearly $HF_A(3) = 1$. In this case we will write that $A$ has Hilbert function $(1, h, n, 1)$; notice that $Q_A(0)$ has Hilbert function $(1, n, n, 1)$.

Macaulay's inverse system has an important role in the study of Artinian local $K$–algebras. The reader should refer to [19], [23] and [15] for an extended treatment. If $R$ is a local ring over a field $K$, we may regard the dual module $\text{Hom}_K(A, K)$ of a quotient $A = R/I$ as a submodule of the injective envelope $\text{Hom}_K(R, K)$.

If $R = K[x_1, \ldots x_h]$ is a power series ring with maximal ideal $n = (x_1, \cdot \cdot \cdot , x_h)$, then, being $K$ a characteristic zero field, $\text{Hom}_K(R, K)$ is a divided power ring $S = K[y_1, \ldots , y_h]$ and we get an explicit description of the duality.

Since $A$, being Artinian, is complete with respect to the $\mathfrak{m}$–adic topology, hence we may assume $A$ is a quotient of $R$.

It is known that $S$ has a structure of $R$–module by means the following action

$$\circ : R \times S \rightarrow S \quad (f, g) \rightarrow f \circ g = f(\partial_{y_1}, \ldots , \partial_{y_h})(g)$$

where $\partial_{y_i}$ denotes the partial derivative with respect to $y_i$.

J. Emsalem in [19], Section B, Proposition 2, A. Iarrobino in [23] Lemma 1.2, characterized Artinian local $K$–algebras in terms of suitable $R$–submodules of $S$ which are finitely generated.

A quotient local ring $A = R/I$ is an Artinian Gorenstein local $K$–algebra of socle degree $s$ if and only if its dual module is a cyclic $R$–submodule of $S$ generated by a polynomial $F \in S$ of degree $s$ (see also [23], Lemma 1.2.). If we consider the ideal of $R$

$$\text{Ann}_R(F) := \{g \in R \mid g \circ F = 0 \}$$

then $A = R/\text{Ann}_R(F)$. Hence each Artinian Gorenstein local $K$–algebra of socle $s$ will be equipped with a polynomial $F \in S$ of degree $s$. We will write $A = A_F$. The polynomial $F$ is not unique, but it is determined up an unit $u$ of $R$. In the general case, one can translate in terms of classification and deformation of polynomials, the respective problems of classification and deformation of the corresponding algebra. For example, when $A$ is an Artinian Gorenstein local $K$–algebra of socle $s = 2$ and embedding dimension $h$, we have $A \simeq A_F$ with $F = y_1^2 + \cdots + y_h^2 \in S$ due to the classification of quadrics up to projectivities. We have thus the following result.

**Proposition 2.1.** An Artinian Gorenstein local $K$–algebra $A$ of embedding dimension $h > 1$ has socle $s = 2$ if and only if $A \simeq R/I$ where

$$I = (x_i x_j, x_i^2 - x_1^2)_{1 \leq i < j \leq h, u=2,..,h}.$$ 

In this case $P_A(z)$ is rational and $A$ is smoothable.

**Proof.** The first part is an immediate consequence of the classification of quadrics up to projectivities (see also [28]). For the rationality of $P_A(z)$ see e.g. [31] or [13], Proposition 2.12. For the smoothability see [8], Section 3. \qed
It is interesting to point out that the Gorenstein assumption on $A$ is necessary because Anick’s example has socle degree $s = 2$, nevertheless $P_A(z)$ is not rational. It is thus a natural question to ask what happens when $s > 2$.

The $G$–module $Q(0)$ will play a crucial role in this investigation. It can be computed in terms of the corresponding polynomial in the inverse system. Let $F \in S$ be a polynomial of degree $s$ such that $A = A_F$ and denote by $F_s$ the form of highest degree in $F$, that is $F = F_s +$ terms of lower degree, then (see [19] Proposition 7 and [23] Lemma 1.10)

$$Q(0) \cong R/\text{Ann}_R(F_s).$$

We say that a homogenous form $F \in S$ of degree $d$ is non-degenerate if the $K$–vector space of the derivatives of order $d−1$ has maximal dimension, that is $h = \dim_K[S]_1$.

Now we focus our attention on the case $s = 3$, i.e. the case of Artinian Gorenstein local algebras $A$ with Hilbert function $(1, h, n, 1)$ with $h \geq n$. Notice that the Hilbert function of $Q(0) \cong R/\text{Ann}_R(F_3)$ is $(1, n, n, 1)$.

From now on we let $R_j = K[[x_1, \ldots, x_j]]$ and $S_j = K[y_1, \ldots, y_j]$ for every positive integer $j \leq h$. Hence $R_h = R$ and $S_h = S$ and in this case we will write $R$ and $S$. In the following $h, n$ will denote positive integers such that $n \leq h$. We assume $A = R/I$ of embedding dimension $h$, that is $I \subseteq n^2$.

J. Elias and M.E. Rossi proved the following result.

**Theorem 2.2** ([15], Theorem 4.1). Let $A$ be an Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$, then $A \cong R/\text{Ann}_R(F)$ where

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S$$

with $F_3$ a non-degenerate degree three form in $S_n$ ($F = F_3$ if $n = h$).

Starting from the above result we can prove a structure theorem for Artinian Gorenstein local $K$–algebra $A = R/I$ with maximal ideal $m$ and socle degree three.

**Lemma 2.3.** Let $h, n$ be positive integers such that $h > n$ and let

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S$$

where $F_3$ a non-degenerate degree three form in $S_n$. Then

$$\text{Ann}_R(F) = \text{Ann}_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j, n+1 \leq j \leq h}$$

where $\sigma \in R_n$ is any form of degree 3 such that $\sigma \circ F_3 = 1$. 

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Proof. It is easy to check that
\[ I := \text{Ann}_R(F) \supseteq J := \text{Ann}_R(F_3)R + (x_ix_j, x^2_j - 2\sigma(x_1, \cdots, x_n))_{i<j, n+1 \leq j \leq h} \]
Since \( R/I \) and \( R/J \) are finitely generated \( K \)-vector spaces (\( n^4 \subseteq I, J \)) and there is a surjection between \( R/J \) and \( R/I \), the equality \( I = J \) follows if they have the same colength. In particular we prove that \( HF_{R/I}(i) = HF_{R/J}(i) \) for every \( i \geq 0 \). If we denote by \((\ ))^*\) the homogeneous ideal generated by the initial forms of an ideal of \( R \), we have
\[ L = (\text{Ann}_R(F_3))^* + (x_ix_j, x^2_j)_{i<j, n+1 \leq j \leq h} \subseteq J^* \subseteq I^* \]
thus
\[ HF_{R/L}(i) \geq HF_{R/J}(i) \geq HF_{R/I}(i) \]
for every \( i \geq 0 \). In order to prove the assertion we prove that the Hilbert function of \( R/L \) is \((1, h, n, 1)\), i.e. the Hilbert function of \( R/I \) (see Theorem 2.2). The equality easily follows since \( R_3/\text{Ann}_R(F_3) \) has Hilbert function \((1, n, n, 1)\), being \( F_3 \) a non-degenerate degree three form in \( S_n \) and \( L_2 = [\text{Ann}_R(F_3)]^* \).

Theorem 2.4. Let \( A \) be an Artinian local \( K \)-algebra of embedding dimension \( h \) and let \( n = HF_A(2) \).

\( A \) is Gorenstein of socle degree three if and only if \( n \leq h \) and there exists a non-degenerate cubic form \( F_3 \in S_n \) such that \( A \cong R/I \) where
\[ I = \begin{cases} \text{Ann}_R(F_3)R + (x_ix_j, x^2_j - 2\sigma(x_1, \cdots, x_n))_{i<j, n+1 \leq j \leq h} & \text{if } n < h \\ \text{Ann}_R(F_3) & \text{if } n = h \end{cases} \]
being \( \sigma \in R_n \) any form of degree 3 such that \( \sigma \circ F_3 = 1 \).

Proof. Let \( A \) be an Artinian Gorenstein local \( K \)-algebra \( A \) with socle degree three. The Hilbert function of \( A \) is of the form \((1, h, n, 1)\). Due to the decomposition theorem proved in [23], then \( h \geq n \), hence Theorem 2.2 yields that \( A \cong R/\text{Ann}_R(F) \) where
\[ F = F_3 + y^2_{n+1} + \cdots + y^2_h \in S \]
if \( h > n \) and \( F = F_3 \) if \( h = n \). Then the result follows now by Lemma 2.3.
Conversely, the result follows by Macaulay’s correspondence and again by Lemma 2.3.

\[ \square \]


3 Poincaré series of Gorenstein local algebras with socle degree three

In this section we will reduce the computation of the Poincaré series of Gorenstein local $K$–algebras with socle degree three to the Poincaré series of graded Gorenstein algebras with the same socle degree. By [30], Satz 1, we may assume that $A$ is an Artinian Gorenstein algebra.

We recall that if $x \in m \setminus m^2$ is an element in the socle $(0 :_A m)$ of $A$, then, by [22],

$$P_A(z) = \frac{P_{A/xA}(z)}{1 - z P_{A/xA}(z)}.$$  

Moreover if $A$ is an Artinian Gorenstein local ring, then, by [2],

$$P_A(z) = \frac{P_{A/(0: m)}(z)}{1 + z^2 P_{A/(0: m)}(z)}$$ (2)

If $A = A_F$ with $F = F_3 + \ldots$ we will write the Poincaré series of $A$ in terms of those of the Artinian Gorenstein graded $K$–algebra $Q_A(0) \simeq R/Ann_R(F_3)$. Notice that if $A$ has Hilbert function $(1, h, n, 1)$, then $n \leq h$ and $Q_A(0)$ is a Gorenstein graded algebra with Hilbert function $(1, n, n, 1)$.

**Theorem 3.1.** Let $A$ be an Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$. Then

$$P_A(z) = \frac{P_{Q_A(0)}(z)}{1 - (h - n) P_{Q_A(0)}(z)}.$$  

In particular $P_A(z)$ is rational if and only if $P_{Q_A(0)}(z)$ is rational.

**Proof.** We recall that $n \leq h$. By Theorem [2.2], we may assume $n < h$ and $A = R/Ann_R(F)$ where, as usual, $R = K[x_1, \ldots, x_h]$ and $F = F_3 + y_{n+1}^2 + \cdots + y_h^2$, with $F_3$ a non-degenerate degree three form in $S_n$. By Lemma [2.3] we know that

$$Ann_R(F) = Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i < j, n+1 \leq j \leq h}$$

where $\sigma \in S_n$ is a cubic form such that $\sigma \circ F_3 = 1$. So the coset of $\sigma$ in $A$ is a generator of the socle $(0 :_A m)$ of $A$, $m$ being the maximal ideal of $A$. Hence by (2) we get that

$$P_A(z) = \frac{P_C(z)}{1 + z^2 P_C(z)}$$

where

$$C := \frac{A}{(0 :_A m)} \cong \frac{R}{Ann_{R_n}(F_3)R + (\sigma) + (x_i x_j, x_j^2)_{i < j, n+1 \leq j \leq h}}.$$  

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Since $x_h \in \socle(C)$, by (11), we get that

$$PC(z) = \frac{PC/(x_h)(z)}{1 - z \cdot PC/(x_h)(z)}.$$ 

Iterating the process we deduce that

$$PC(z) = \frac{PD(z)}{1 - (h - n)z \cdot PD(z)},$$

where

$$D = \frac{R}{Ann_{R_n}(F_3)R + (\sigma) + (x_{n+1}, \ldots, x_h)}.$$ 

Since $F_3, \sigma \in S_n$ we get that

$$D \cong \frac{R_n}{Ann_{R_n}(F_3) + (\sigma)}.$$ 

Notice that again the coset of $\sigma$ in $B := R_n/Ann_{R_n}(F_3) \cong R/Ann(R_3) = Q_A(0)$ is a generator of its socle, so

$$D \cong \frac{R_n}{Ann_{R_n}(F_3) + (\sigma)} \cong \frac{B}{\socle(B)}.$$ 

Hence from (2) we deduce

$$PD(z) = \frac{PB}{1 - z^2 \cdot PB}.$$ 

From the above information, summing up, we get

$$PA(z) = \frac{PB}{1 - (h - n) \cdot PB}.$$ 

Since $PA$ is a rational function of $PB$ it follows that $PA$ is a rational if and only if the same is true for $PB$. 

The above result reduces the problem of the rationality of $PA(z)$ to the rationality of the Poincaré series of a graded Gorenstein $K$–algebra with socle degree three. This situation has been studied by a great number of researchers. By taking advantage of this we present the following corollaries.

**Corollary 3.2.** Let $A$ be an Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$. If $n \leq 4$, then $PA(z)$ is rational.

**Proof.** By Theorem 3.1 we may reduce the problem to graded Gorenstein $K$–algebras with embedding dimension $n \leq 4$. Hence the result follows by [4] and [25].

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We recall that a Koszul graded algebra $C$ has rational Poincaré series, in particular $P_C(z)H_C(-z) = 1$ where $H_C(z)$ is the Hilbert series of $G$, i.e. $H_C(z) = \sum_{j \geq 0} HF_C(j)z^j$. The following result specializes, for Gorenstein local $K$–algebras with socle degree three, a result of Fröberg in [21].

**Corollary 3.3.** Let $A$ be an Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$. If $Q_A(0)$ is Koszul, then

$$P_A(z) = \frac{1}{n - h + 1 - nz + nz^2 - z^3}$$

**Proof.** By Theorem 3.1 we may reduce the problem to graded Gorenstein $K$–algebras which are Koszul. Since $Q_A(0)$ has Hilbert series $H(z) = 1 + nz + nz^2 + z^3$, then $P_{Q_A(0)}(z) = 1$ and the result follows from an easy computation. \qed

In [14] Koszul filtrations were introduced for studying the Koszulness of a standard graded algebra $C$. It has been proved in [14], Proposition 1.2 that if $C$ has a Koszul filtration then all the ideals of the filtration have a linear $C$–free resolution, hence $C$ is Koszul. For Gorenstein graded algebras with Hilbert function $(1, n, n, 1)$, the property of having a Koszul filtration can be detected directly on the cubic form (see [13] and [12] Theorem 3.2).

Notice that if $F_3$ is a generic cubic, then the corresponding Gorenstein algebra $R/\text{Ann}_R(F_3) = Q_A(0)$ has a Koszul filtration (see [13], Theorem 6.3), hence $P_A(z)$ is rational.

We say that $A_F$ is a generic Artinian Gorenstein local $K$–algebra if $F$ is a generic polynomial of $S$. By the previous remark we get the following result.

**Corollary 3.4.** The generic Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$ has rational Poincaré series.

**Remark 3.5.** J. Elias and G. Valla proved that if the multiplicity of $A$ is either $d \leq 9$ or $d \leq h + 4$, then $P_A(z)$ is rational (see [18]). If $d \leq h + 6$ then by Corollary 3.2 and [18] we get that $P_A(z)$ is rational but the case that $A$ has Hilbert function $(1, h, 3, 1, 1)$ is still open.

### 4 Deformations of Gorenstein local algebras with socle degree three

We start this section by recalling some facts about families of $K$–algebras of fixed degree. Let $d \geq 2$ be an integer and let $A$ be an Artinian $K$–algebra of length $d$, hence $A \cong K^d$ as $K$–vector spaces. As explained in the introduction it is interesting to understand weather such an $A$ is smoothable in the following sense.
Definition 4.1. An Artinian $K$–algebra $A$ of length $d$ is smoothable if the scheme $\text{Spec}(A) \in \text{Hilb}_d(\mathbb{A}^d)$ is in the closure $\text{Hilb}_d^{\text{gen}}(\mathbb{A}^d)$ inside $\text{Hilb}_d(\mathbb{A}^d)$ of the locus of points representing reduced schemes.

We focus here our attention to Artinian Gorenstein local $K$–algebras $A$ with Hilbert function $(1, h, n, 1)$. In this case we are able to say something interesting using Lemma 2.3.

Theorem 4.2. Let $A$ be an Artinian Gorenstein local $K$–algebra with Hilbert function $(1, h, n, 1)$. Then $A$ is smoothable if $Q_A(0)$ is smoothable.

Proof. We will extend the methods used in [8] and [9], proving the statement by induction on $h$.

We will assume the statement true for $h - 1$ and we will prove it for $h$. Recall that $A \cong R/\text{Ann}_R(F)$ where (see Theorem 2.2)

$$ F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S = K[y_1, \ldots, y_h] $$

with $F_3$ a non-degenerate degree three form in $K[y_1, \ldots, y_h]$. If $h = n$, then the ring $A \cong R/\text{Ann}_R(F_3)$ is graded Artinian Gorenstein and local with socle in degree 3 and Hilbert function $(1, n, n, 1)$. Thus $A \cong G \cong Q_A(0)$ and the statement is trivial in this case.

We assume the statement is true for $h - 1$: we will prove it for $h > n$. Recall that we have (see Theorem 3.1)

$$ \text{Ann}_R(F) = \text{Ann}_R(F_3)R + (x_ix_j, x_i^2 - 2\sigma(x_1, \cdots, x_n))_{i \prec j, n+1 \leq j \leq h} $$

where $s$ is a degree three form in $S_n$ such that $\sigma \circ F_3 = 1$. For the techniques used in this proof, it will be useful working in the polynomial ring. Notice that in our setting we may assume $R = K[x_1, \ldots, x_h]$ because in $R/\text{Ann}_R(F)$ we have $m^4 = 0$.

For each $b \in \mathbb{A}^1$, let us consider the ideal in $R$

$$ J_b := \text{Ann}_R(F_3)R + (x_ix_j, x_i^2 - 2\sigma(x_1, \cdots, x_n), x_h - bx_h - 2\sigma(x_1, \cdots, x_n))_{i \prec j, n+1 \leq j \leq h, n+1 \leq k < h} $$

Notice that

$$ J_0 = \text{Ann}_R(F) = \text{Ann}_R(F_3)R + (x_ix_j, x_i^2 - 2\sigma(x_1, \cdots, x_n))_{i \prec j, n+1 \leq j \leq h}, $$

hence $A \cong R/J_0$.

If $b \neq 0$, we claim that

$$ J_b = (x_1, \ldots, x_h - b) \cap (J_b + (x_h^2)) \subseteq R. $$

Since $J_b \subseteq (x_1, \ldots, x_h - b)$, by the modular law (see [3], Chapter I), it is enough to prove

$$ (x_1, \ldots, x_h - b) \cap (x_h^2) = (x_1x_h^2, \ldots, x_{h-1}x_h^2, (x_h - b)x_h^2) \subseteq J_b. $$
This is trivial because \( x_i x_h \in J_b \) for every \( i = 1, \ldots, h-1 \) and \( x_h^2 (x_h - b) \equiv 2x_h \sigma(x_1, \ldots, x_n) \equiv 0 \mod J_b \).

If \( b \neq 0 \), then \( x_h \in J_b + (x_h^2) \), thus the ideals \( (x_1, \ldots, x_h-1, x_h - b) \) and \( J_b + (x_h^2) \) are coprime, whence

\[
A_b := R/J_b \cong R/(x_1, \ldots, x_{h-1}, x_h - b) \oplus R/(J_b + (x_h^2)).
\]

Since \( x_h \in J_b + (x_h^2) \) we also have

\[
R/(J_b + (x_h^2)) \cong A' := R_{h-1}/(\Ann_{R_n}(F_3) R + (x_i x_j, x_j^2 - 2\sigma(x_1, \ldots, x_n))_{i < j, n+1 \leq j \leq h-1}).
\]

By induction hypothesis \( A' \) is smoothable. It follows easily that \( A_b \cong K \oplus A' \) turns out to be smoothable for \( b \neq 0 \) by induction hypothesis (for reader’s benefit see e.g. Lemma 4.2 of \[7\]).

Let \( \mathcal{R} := K[b] \otimes_K R \cong K[b, x_1, \ldots, x_h] \) and consider the family \( \mathcal{A} \cong \mathcal{R}/J \to \mathbb{A}^1 \cong \text{Spec}(k[b]) \) where

\[
J := \Ann_{R_n}(F_3) \mathcal{R} + (x_i x_j, x_j^2 - 2\sigma(x_1, \ldots, x_n), x_h^2 - bx_h - 2\sigma(x_1, \ldots, x_n))_{i < j, n+1 \leq j \leq h, n+1 \leq k < h}.
\]

Due to the discussion above all the fibres of \( \mathcal{A} \to \mathbb{A}^1 \) are Artinian \( K \)-algebras of degree \( d \), thus the family is flat. The universal property of Hilbert scheme guarantees the existence of a curve inside \( \text{Hilb}_d(\mathbb{A}^d) \) whose general point is in \( \text{Hilb}_d^{\text{gen}}(\mathbb{A}^d) \), thus \( \text{Spec}(A) \in \text{Hilb}_d^{\text{gen}}(\mathbb{A}^d) \).

As already noticed above Theorem \[4.2\] reduces the smoothability of \( A \) to the smoothability of a graded Artinian Gorenstein local \( K \)-algebra with socle degree three (see e.g. \[20\]). Our last result is the following corollary

**Corollary 4.3.** Let \( A \) be an Artinian Gorenstein local \( K \)-algebra with Hilbert function \( (1, h, n, 1) \). If \( n \leq 4 \), then \( A \) is smoothable.

**Proof.** It follows by Theorem \[4.2\] and \[11\].

We remark that the previous result cannot be generalized for \( n \geq 6 \). Indeed Iarrobino found an example of non smoothable local \( K \)-algebra \( A \) with Hilbert function \( (1, 6, 6, 1) \) (see \[11\], Section 4). The case \( n = 5 \) is still open.

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