EIGENVALUES HOMOGENIZATION FOR THE FRACTIONAL
LAPLACIAN OPERATOR

ARIEL MARTIN SALORT

ABSTRACT. In this work we study the homogenization for eigenvalues of the fractional Laplace operator in a bounded domain. We obtain an explicit order of the convergence rates of the variational eigenvalues. Since the proof of our results do not use linear techniques, it is possible to prove the same homogenization results for the spectrum of the fractional $p-$laplacian operator.

1. Introduction

The purpose of this paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the eigenvalues of the following non-local problem

$$
\begin{cases}
(\Delta)^s u = \lambda_\varepsilon u & \text{in } \Omega \subset \mathbb{R}^n \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
$$

where for $\varepsilon > 0$ the weight functions $\rho_\varepsilon$ are positive and bonded away from zero and infinity and $\lambda_\varepsilon$ is the eigenvalue parameter. Here, for $s \in (0, 1)$ we denote by $(\Delta)^s$ the fractional Laplace operator which is defined as

$$(\Delta)^s u(x) = c(n, s) \text{ p.v. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

where the constant $c$ is given by (see, for instance [2])

$$c(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1}.$$  

The domain $\Omega$ is assumed to be a bounded and open set in $\mathbb{R}^n$, $n \geq 1$.

As $\varepsilon \to 0$, the following limit problem is obtained

$$
\begin{cases}
(\Delta)^s u = \lambda u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
$$

where $\rho(x)$ is the weak* limit in $L^\infty(\Omega)$ as $\varepsilon \to 0$ of the sequence $\{\rho_\varepsilon\}_\varepsilon$.

For each fixed value of $\varepsilon$ it is known that there exists a sequence of variational eigenvalues $\{\lambda_\varepsilon^k\}_{k \geq 1}$ of (1.1) such that $\lambda_\varepsilon^k \to \infty$ as $k \to \infty$. Analogously, for the limit problem (1.3), there exists a sequence of variational eigenvalues $\{\lambda_k\}_{k \geq 1}$ such that $\lambda_k \to \infty$ as $k \to \infty$.

We are interested in studying the behavior of the sequence $\{\lambda_\varepsilon^k\}_{k \geq 1}$ as $\varepsilon \to 0$.

When $s = 1$, (1.1) becomes the eigenvalue problem for the Laplacian operator with Dirichlet boundary conditions. This problem has been extensively studied and a complete description of the asymptotic behavior of its spectrum was obtained in

2010 Mathematics Subject Classification. 35B27, 35P15, 35P30, 34A08.

Key words and phrases. Eigenvalue homogenization, nonlinear eigenvalues, order of convergence, fractional laplacian.
the 70’s. Boccardo and Marcellini [3], and Kesavan [17] proved that for each fixed $k$,

$$\lim_{\varepsilon \to 0} \lambda_{\varepsilon}^k = \lambda_k.$$  

Later, in [4] and [11] this result was extended to $p-$Laplacian type operators.

One of the purposes of our paper is to extend this results to non-local eigenvalue problems. Our first result states the convergence of the $k-$th eigenvalue of problem (1.1) to the $k-$th eigenvalue of the limit problem (1.3).

**Theorem 1.1.** Let $\lambda_{\varepsilon}^k$ and $\lambda_k$ be the $k-$th (variational) eigenvalues of (1.1) and (1.3), respectively. Then

$$\lim_{\varepsilon \to 0} \lambda_{\varepsilon}^k = \lambda_k$$

for each $k \geq 1$ fixed.

Homogenization theory dates back to the late sixties with the works of Spagnolo and de Giorgi and it developed very rapidly during the last two decades. Homogenization theory tries to get a good approximation of the macroscopic behavior of the heterogeneous material by letting the parameter $\varepsilon \to 0$. A case of relevant importance is the study of periodic homogenization problems due to the many applications to physics and engineering. The main references for the homogenization theory of periodic structures are the books by Bensoussan-Lions-Papanicolaou [1], Sanchez–Palencia [23], Oleinik-Shamaev-Yosifian [20] among others.

An interesting issue in the homogenization theory is to estimate the rates of convergence of the eigenvalues in (1.5), that is, to find bounds for the error $|\lambda_{\varepsilon}^k - \lambda_k|$. To this end we restrict our study to periodic weights because we need to know the explicit dependence on $\varepsilon$. We consider the family of weight functions $\rho_\varepsilon$ given in terms of a single-bounded $Q-$periodic function $\rho$ in the form $\rho_\varepsilon(x) := \rho(x/\varepsilon)$, $Q$ being the unit cube of $\mathbb{R}^n$. The weight $\rho$ is such that for some constants $\rho_-$ and $\rho_+$ it holds that

$$0 < \rho_- \leq \rho(x) \leq \rho_+ < \infty \quad x \in \Omega.$$  

In this case, the family of weight functions $\{\rho_\varepsilon\}_\varepsilon$ converges weakly* to $\bar{\rho}$ as $\varepsilon \to 0$, $\bar{\rho}$ being the average of $\rho$ on $Q$. Even the rates of that convergence can be computed. As consequence, it allows us to calculate the convergence rates of the eigenvalues.

Again, in the case $s = 1$, for the Laplace operator, the rates of convergence were studied in several papers. The authors in [20] proved some estimates by using tools from functional analysis in Hilbert spaces. Assuming that $\Omega$ is a Lipschitz domain they show that there exists a constant $C$ depending on $k$ and $\Omega$ such that

$$|\lambda_{\varepsilon}^k - \lambda_k| \leq C \varepsilon^{\frac{1}{2}}.$$  

Later on, under the same assumptions on $\Omega$ it was proved in [16] that

$$|\lambda_{\varepsilon}^k - \lambda_k| \leq C \varepsilon |\log \varepsilon|^{\frac{1}{2}+\gamma}$$  

for any $\gamma > 0$, $C$ depending on $k$ and $\gamma$. When the domain is more regular ($C^{1,1}$ is enough) in [19] explicit dependence of the constant $C$ on $k$ was obtained. It was proved that

$$|\lambda_{\varepsilon}^k - \lambda_k| \leq C_\varepsilon k^2 \varepsilon |\log \varepsilon|^{\frac{1}{2}+\gamma}$$  

for any $\gamma > 0$, $C$ depending on $\gamma$. In both cases, when the domain $\Omega$ is smooth, the logarithmic term can be removed.
Finally, in [11] the results were extended to the $p-$Laplace operator via non-linear techniques and the dependence on the constant was improved. The convergence of the spectrum as $\varepsilon \to 0$ was proved, obtaining the following bound for the Dirichlet eigenvalues of the Laplacian:

$$|\lambda^\varepsilon_k - \lambda_k| \leq C k^{s/2} \varepsilon$$

where $C$ is a constant depending only on $\Omega$ which can be explicitly computed.

Up to our knowledge, no investigation was made on the homogenization and convergence rates for the weighted fractional Laplacian eigenvalue problem. In contrast with the Laplacian operator, the non-local nature of (1.1) makes it more difficult to deal with the convergence rates. The main obstacle is how to manage the boundedness of fractional norms in order to obtain relations between the variational characterization of eigenvalues.

In our second result we obtain bounds for the convergence rates of the eigenvalues of (1.1) to those of (1.3) when periodic weights are considered.

Theorem 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$ and $\rho \in L^\infty(\mathbb{R}^n)$ be a $Q-$periodic function satisfying (1.6), $Q$ being the unit cube of $\mathbb{R}^n$. Let $\lambda^\varepsilon_k$ and $\lambda_k$ be the $k-$th variational eigenvalues of (1.1) and (1.3), respectively. Then

$$|\lambda^\varepsilon_k - \lambda_k| \leq C \varepsilon^s k^{s/2}.$$

The constant can be computed as

$$C = 4 c_1 (1 + c) n \omega_n \frac{\rho^+}{(\rho^-)^+} \|\rho - \rho\|_{L^\infty} (n \pi)^{2s} |\Omega|^{-1/2}$$

where $c_1$ is the fractional Poincaré constant in $L^2$ in the unit cube and $c$ is the Poincaré constant in $L^2$ in $\Omega$.

Remark 1.3. Since the proof of our results do not use linear techniques, it is possible to prove the same homogenization results for the spectrum of the fractional $p-$Laplacian operator. Given $p > 1$ we consider the following non-linear non-local eigenvalue problem

(1.7)

$$\begin{cases} (-\Delta)^s_p u = \lambda \varepsilon \rho \varepsilon |u|^{p-2} u & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where, up to a normalization constant $c(n, p, s)$ the fractional $p-$Laplacian operator is given by

$$(-\Delta)^s_p u(x) = p.v. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} \, dy$$

and $\rho \varepsilon$ satisfies (1.6). As $\varepsilon \to 0$, the following limit problem is obtained

(1.8)

$$\begin{cases} (-\Delta) u^\varepsilon \rho(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where $\rho(x)$ is the weak* limit in $L^\infty(\Omega)$ as $\varepsilon \to 0$ of the sequence $\{\rho \varepsilon\}_\varepsilon$.

The sequence of nonlinear eigenvalue problems (1.7) was studied in [19] and [12] and a variational characterization of the eigenvalues is given in [14]. If we denote
\( \lambda_\varepsilon^k \) and \( \lambda_k \) the \( k \)-th (variational) eigenvalues of (1.7) and (1.8), respectively, in an analogous way to Theorem 1.1 can be proved that

\[
\lim_{\varepsilon \to 0} \lambda_\varepsilon^k = \lambda_k
\]

for each \( k \geq 1 \) fixed. Again, when the weights \( \rho_\varepsilon \) are assumed to be \( Q \)-periodic functions, by using the same arguments of Theorem 1.2 it is possible to prove that

\[
|\lambda_\varepsilon^k - \lambda_k| \leq C\varepsilon k \max\{\lambda_k, \lambda_\varepsilon^k\}^p
\]

for some constant \( C \) independent on \( k \) and \( \varepsilon \).

Moreover, using the estimates on the growth of the fractional \( p \)-eigenvalues given in [14], the last inequality becomes

\[
|\lambda_\varepsilon^k - \lambda_k| \leq C\varepsilon k \frac{\varepsilon^{2-s} k^s}{n}
\]

for some constant \( C \) independent on \( k \) and \( \varepsilon \).

As the authors pointed out in [14], their estimates are non-optimal. They suspect that a sharper Weil law for the eigenvalues holds: \( \max\{\lambda_k, \lambda_\varepsilon^k\} \leq Ck \frac{\varepsilon^{2-s} k^s}{n} \), which would imply that

\[
|\lambda_\varepsilon^k - \lambda_k| \leq C\varepsilon k \frac{\varepsilon^{2-s} k^s}{n},
\]

the natural generalization of Theorem 1.2.

This paper is organized as follows: in Section 2 we introduce some definitions and properties of the eigenvalues of problem (1.1), and in Section 3 we prove the results stated.

2. EIGENVALUES OF THE FRACTIONAL LAPLACIAN

In this section we present some well-known results for the (variational) eigenvalues of the weighted fractional Laplacian as well as a minimax characterization. In order to do that, we first introduce the fractional Sobolev spaces, the natural spaces to work with. For more detailed information we refer to the reader to the Hitchhiker's Guide to the Fractional Sobolev Spaces [9].

Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^n \), \( n \geq 1 \). For any \( s \in (0, 1) \) and \( p \geq 1 \) we denote \( W^{s,p}(\Omega) \) the fractional Sobolev space defined as follows

\[
W^{s,p}(\Omega) := \{ u \in L^p(\Omega) : \frac{u(x) - u(y)}{|x - y|^{n+sp}} \in L^p(\Omega \times \Omega) \}
\]

endowed with the natural norm

\[
||u||_{W^{s,p}(\Omega)} := (||u||_{L^p(\Omega)}^p + ||u||_{W^{s,p}(\Omega)}^p)^{\frac{1}{p}}
\]

where \( ||u||_{W^{s,p}(\Omega)} \) is the so-called Gagliardo semi-norm of \( u \) defined as

\[
||u||_{W^{s,p}(\Omega)}^p = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy.
\]

We denote \( W^{s,p}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in the norm \( || \cdot ||_{W^{s,p}(\Omega)} \). A function \( u \in W^{s,p}_0(\Omega) \) is always assumed to be defined in the whole \( \mathbb{R}^n \) by extending it by zero. Also, for the sake of simplicity we define \( H^s(\Omega) := W^{s,2}(\Omega) \) and \( H_0^s(\Omega) := W^{s,2}_0(\Omega) \).

An useful space inclusion we will use is the following.
Lemma 2.1 (Proposition 2.1 in [9]). Let $0 < s \leq s' < 1$ and $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$, then the inclusion $H^{s'}(\Omega) \subset H^s(\Omega)$ holds. Moreover, the following inequality is true

$$|u|^2_{H^{s'}(\Omega)} \leq c_2 \|u\|^2_{H^{s'}(\Omega)}$$

where $c_2 = 4 \int_{|x| \geq 1} |x|^{-(n+2s)} \, dx$.

A fundamental difference between the fractional Laplacian and the usual Laplacian is the behavior of the boundary value problem. Due to the nonlocal character of the operator, in order to obtain a well-posed Dirichlet eigenvalue problem, the boundary condition is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial \Omega$. We consider the following equation

$$\begin{cases}
(-\Delta)^s u = \mu u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases} \tag{2.1}$$

For a fixed value of $s$, problem (2.1) has a sequence of positive eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$ tending to $+\infty$. In 1959, Blumenthal and Getoor [2] proved a Weyl’s asymptotic formula for the eigenvalues of $s-$stable symmetric processes, whose generators are the fractional Laplacians, by using Karamata’s Tauberian Theorem. More precisely, they proved that the eigenvalues of (2.1) satisfy

$$\mu_k \sim (4\pi)^s \left( \frac{k\Gamma(1 + \frac{1}{s})}{|\Omega|} \right)^\frac{1}{s}, \quad k \to +\infty. \tag{2.2}$$

It should be noticed that when $s = 1/2$, we have the Weyl’s formula for the Klein-Gordon operator (see [13]). In contrast, when $s = 1$, we obtain Weyl’s formula for the Dirichlet Laplacian eigenvalue problem.

Also, in [6] the following relation for the eigenvalues of the spherically symmetric $s-$stable process $X^\varphi$ killed upon leaving $\Omega$ (which are equivalent to the eigenvalues of (2.1)) was proved

$$c \Lambda_k \leq \mu_k \leq \Lambda_k, \quad k \geq 1 \tag{2.3}$$

where $\Lambda_k$ is the $k-$th eigenvalue of the Laplacian with Dirichlet boundary conditions on $\Omega$. When $\Omega$ is convex the constant $c$ above can be taken as $1/2$. In particular, when $n = 1$ and $\Omega = (0, \ell)$, (2.3) states that

$$\frac{1}{2} \left( \frac{k\pi}{\ell} \right)^{2s} \leq \mu_k \leq \left( \frac{k\pi}{\ell} \right)^{2s}, \quad k \geq 1. \tag{2.4}$$

Now we focus on the following eigenvalue problem involving a weight function, i.e.,

$$\begin{cases}
(-\Delta)^s u = \lambda \rho u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases} \tag{2.5}$$

where $\rho$ is a positive function bounded away from zero and infinity.

This problem has a variational structure. We say that $u \in H^s_0(\Omega)$ is a weak solution of (2.5) if

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} = \lambda \int_{\Omega} \rho(x)uv$$

for every $v \in H^s_0(\Omega)$. 

As we pointed in the introduction, there exists a sequence \( \{ \lambda_k \}_{k \geq 1} \) of variational eigenvalues of (2.5) going to \( +\infty \). By the min-max principle for eigenvalues (see for example [5], [6], [22]), they can be characterized as

\[
\lambda_k = \min_{u \perp W_k - 1} \frac{|u|_{H^s(\mathbb{R}^n)}^2}{\| \mu^\frac{1}{2} u \|_{L^2(\Omega)}^2} = \min_{c \in D_k} \max_{u \in c} \frac{|u|_{H^s(\mathbb{R}^n)}^2}{\| \mu^\frac{1}{2} u \|_{L^2(\Omega)}^2}
\]

where \( W_k = \langle u_1, \ldots, u_k \rangle \) and \( D_k = \{ W \subset H^s_0(\Omega) : \dim W = k \} \).

The variational characterization of eigenvalues plays a fundamental role in our analysis and the proof of our results since it allows to reduce the eigenvalues convergence to the study of oscillating integrals.

\section*{3. Proof of the results}

In the previous work [11] we obtained estimates for the eigenvalue convergence rates in problems involving rapidly oscillating weights for the \( p \)-Laplacian operator. We proved that these estimates can be reduced to the study of oscillating integrals. In this section, adapting the arguments used in [11, 20] we obtain results concerning oscillating integrals in fractional Sobolev spaces. First, we prove the results in the periodic weight case.

A useful tool to be used is the following fractional Poincaré inequality:

\begin{lemma}
\text{(Bourgain-Brezis-Mironescu, [21]).} Let \( Q \) be the unit cube in \( \mathbb{R}^n \), \( n \geq 1 \). Then for each \( q \geq 1 \) and \( s \in (0, 1) \) there exists a constant \( c \) depending on \( s, n \) and \( q \) such that
\[
\| u - (u)_Q \|_{L^q(Q)} \leq c \| u \|_{W^{s,q}(Q)},
\]
for every \( u \in L^q(Q) \),
\end{lemma}

where \( (u)_Q \) is the average of \( u \) in \( Q \).

In the following Lemma we compute the Poincaré constant on the cube of side \( \varepsilon \) in terms of the Poincaré constant of the unit cube.

\begin{lemma}
Let \( Q \) be the unit cube in \( \mathbb{R}^n \), \( n \geq 1 \) and let \( c_1 \) be the constant in \( L^q \) in the unit cube given by \( [5] \). Then, for every \( u \in W^{s,q}(Q_\varepsilon) \) we have
\[
\| u - (u)_{Q_\varepsilon} \|_{L^q(Q_\varepsilon)} \leq c_1 \varepsilon^n \| u \|_{W^{s,q}(Q_\varepsilon)},
\]
where \( Q_\varepsilon = \varepsilon Q \).
\end{lemma}

\begin{proof}
Let \( u \in W^{s,q}(Q_\varepsilon) \). We may assume that \( (u)_{Q_\varepsilon} = 0 \). Now, if we denote \( u_\varepsilon(t) = u(\varepsilon t) \), we find that \( u_\varepsilon \in W^{s,q}(Q) \) and by the change of variables formula, we get
\[
\int_{Q_\varepsilon} |u|^q = \int_Q |u_\varepsilon|^q \varepsilon^n \int_{Q \times Q} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x - y|^{n+sq}} \, dx \, dy
\]
\[
= c_1 \varepsilon^n \varepsilon^{-2n} \varepsilon^{n+sq} \int_{Q_\varepsilon \times Q_\varepsilon} |u(x) - u(y)|^q \frac{|x - y|^{n+sq}}{|x - y|^{n+sq}} \, dx \, dy
\]
\[
= c_1 \varepsilon^{sq} \| u \|_{W^{s,q}(Q_\varepsilon)}^q.
\]
\end{proof}

The proof is now complete.

The following Lemma allows us to bound oscillating integrals in which a function is multiplying.
Lemma 3.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and denote by $\mathcal{Q}$ the unit cube in $\mathbb{R}^n$. Let $g \in L^\infty(\mathbb{R}^n)$ be a $\mathcal{Q}$-periodic function such that $\bar{g} = 0$. Then the inequality
\[
\left| \int \Omega g(\frac{x}{\varepsilon})v \right| \leq c_1 \|g\|_{L^\infty(\mathbb{R}^n)} \varepsilon^s |v|_{W^{s,1}(\Omega)}
\]
holds for every $v \in W^{s,1}_0(\Omega)$ with $s \in (0,1)$, where $c_1$ is the constant given in Lemma 3.2.

Proof. Denote by $I^\varepsilon$ the set of all $z \in \mathbb{Z}^n$ such that $Q_{z,\varepsilon} \cap \Omega \neq \emptyset$, $Q_{z,\varepsilon} := \varepsilon(z + Q)$. Given $v \in W^{s,1}_0(\Omega)$ we consider the function $\bar{v}_\varepsilon$ given by the formula
\[
\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{Q_{z,\varepsilon}} v(y)dy
\]
for $x \in Q_{z,\varepsilon}$. We denote by $\Omega_1 = \cup_{z \in I^\varepsilon} Q_{z,\varepsilon} \supset \Omega$. Thus, we can write
\[
\int \Omega g \varepsilon v = \int_{\Omega_1} g_\varepsilon(v - \bar{v}_\varepsilon) + \int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon.
\]

Now, by Lemma 3.2 we get
\[
\|v - \bar{v}_\varepsilon\|_{L^1(\Omega_1)} = \sum_{z \in I^\varepsilon} \int_{Q_{z,\varepsilon}} |v - \bar{v}_\varepsilon|dx \leq c_1 \varepsilon^s \sum_{z \in I^\varepsilon} |v|_{W^{s,1}(Q_{z,\varepsilon})} \leq c_1 \varepsilon^s |v|_{W^{s,1}(\Omega)}.
\]

Finally, since $\bar{g} = 0$ and since $g$ is $Q$-periodic, we get
\[
\int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon = \sum_{z \in I^\varepsilon} \bar{v}_\varepsilon |Q_{z,\varepsilon} \int_{Q_{z,\varepsilon}} g_\varepsilon = 0.
\]

Now, combining $\text{(1.2)}$ and $\text{(1.3)}$, we can bound $\text{(1.1)}$ by
\[
\left| \int \Omega g_\varepsilon v \right| \leq c_1 \|g\|_{L^\infty(\mathbb{R}^n)} \varepsilon^s |v|_{W^{s,1}(\Omega)}.
\]

This finishes the proof. $\square$

The next Lemma is essential to estimate the convergence rate of eigenvalues since it allows us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube.

Lemma 3.4. Let $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$, and $g \in L^\infty(\mathbb{R}^n)$ be a $Q$-periodic function such that $0 < g^- \leq g \leq g^+ < \infty$, $Q$ being the unit cube in $\mathbb{R}^n$. Then,
\[
\left| \int \Omega (g_\varepsilon(x) - \bar{g})u^2 \right| \leq c_3 \|g - \bar{g}\|_{L^\infty(\mathbb{R}^n)} \varepsilon^s |u|_{H^s(\mathbb{R}^n)}^2.
\]

for every $u \in H^s_0(\Omega)$ with $0 < s < 1$. Then constant $c_3$ is given by $4n\omega_n c_1(1 + c)$ being $c_1$ the Poincaré constant given in Lemma 3.2 and $c$ the Poincaré constant given in Lemma 3.1.
\textbf{Proof.} Let $\varepsilon > 0$ be fixed. Now, denote by $h = g - \tilde{g}$ and so, by Lemma 3.3 we obtain that
\begin{equation}
\left| \int_{\Omega} h \varepsilon |u|^2 \right| \leq c_1 \|h\|_{L^\infty(\mathbb{R}^n)} \varepsilon^s \|u\|^2_{W^{s,1}(\Omega)}.
\end{equation}

First, observe that
\begin{equation}
\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))u(x)}{|x - y|^{n+s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{u(x)u(x) - u(y)u(x)}{|x - y|^{n+s}} \, dx \, dy
\leq \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x)u(x) + u(y)u(y) - 2u(y)u(x)|}{|x - y|^{n+s}} \, dx \, dy
= \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} \, dx \, dy,
\end{equation}
and analogously
\begin{equation}
\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))u(y)}{|x - y|^{n+s}} \, dx \, dy \leq \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} \, dx \, dy.
\end{equation}

Now, from inequalities (3.5) and (3.6) we get
\begin{equation}
\|u\|^2_{W^{s,1}(\Omega)} = \int_{\Omega} \int_{\Omega} \frac{u(x)^2 - u(y)^2}{|x - y|^{n+s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(u(x) + u(y))}{|x - y|^{n+s}} \, dx \, dy
\leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s-\delta}} \, dx \, dy
\leq \|u\|_{H^{s-1}(\Omega)} \|u\|_{H^s(\Omega)}
\end{equation}
for any $0 < \delta < s$.

By using Lemma 2.1 with $\delta = s/2$ together with the fractional Poincaré inequality it follows that
\begin{equation}
\begin{aligned}
\|u\|^2_{W^{s,1}(\Omega)} &\leq \|u\|_{H^{s-1/2}(\Omega)} \|u\|_{H^{s+1/2}(\Omega)} \\
&\leq c_2 \|u\|^2_{H^s(\Omega)} = c_2(\|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)}) \\
&\leq c_2(1 + c)|u|^2_{H^s(\Omega)} \\
&\leq c_2(1 + c)|u|^2_{H^s(\Omega)}
\end{aligned}
\end{equation}
with $c_2 = 4 \int_{|x| \geq 1} |x|^{-(n+s)} \, dx$, which with a straightforward calculation derives in $4n\omega_n$, being $\omega_n$ the volume of the unit ball in $\mathbb{R}^n$. \hfill \Box

Now we are ready to prove the main result.

\textbf{Theorem 3.5.} Let $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$ and $\rho \in L^\infty(\mathbb{R}^n)$ be a $Q$–periodic function satisfying $1 \leq |Q|$, $Q$ being the unit cube of $\mathbb{R}^n$. Let $\lambda_1^Q$ and $\lambda_k$ be the $k$–th variational eigenvalues of $11$ and $18$, respectively. Then
\begin{equation}
|\lambda_k^Q - \lambda_k| \leq C\varepsilon^s \max\{\lambda_k, \lambda_k^Q\}^2
\end{equation}
where $C = 4n\omega_n c_1 (1 + c)^\delta \|\rho - \overline{\rho}\|_\infty$, being $c_1$ the fractional Poincaré constant of the unit cube in $L^2$ given in Lemma 3.2 and $c$ the Poincaré constant given in Lemma 3.4.
EIGENVALUES HOMOGENIZATION FOR THE FRACTIONAL LAPLACIAN OPERATOR

Proof. Let $C_k \subset H_s^0(\Omega)$ be a set of dimension $k$ such that

$$\lambda_k = \min_{C \in \mathcal{D}_k} \max_{u \in C} \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2} = \max_{u \in C_k} \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2}$$

where $\mathcal{D}_k = \{W \in H_s^0(\Omega) : \text{dim } W = k\}$.

We use now the set $C_k$, which is admissible in the variational characterization of the $k$th–eigenvalue of (1.3), in order to find a bound for it as follows,

$$\lambda_{\varepsilon k} \leq \max_{u \in C_k} \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2} = \max_{u \in C_k} \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2} \bar{\rho} \int_\Omega |u|^2.$$

To bound $\lambda_{\varepsilon k}$ we look for bounds of the two quotients in (3.7). For every function $u \in C_k$ we have that

$$\frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2} \leq \max_{v \in C_k} \frac{|v|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |v|^2} = \lambda_k.$$

Since $u \in C_k \subset H_0^s(\Omega)$, by Lemma 3.4 we obtain that

$$\frac{\bar{\rho} \int_\Omega |u|^2}{\bar{\rho} \int_\Omega |u|^2} \leq 1 + c_3 \varepsilon^s \rho - \overline{\rho} ||\rho - \overline{\rho}||_{\infty} \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2}$$

$$\leq 1 + c_3 \varepsilon^s \rho - \overline{\rho} ||\rho - \overline{\rho}||_{\infty} \rho \frac{|u|^2_{H^s(\mathbb{R}^n)}}{\bar{\rho} \int_\Omega |u|^2}$$

$$\leq 1 + C \varepsilon^s \lambda_k$$

where

$$C = 4n\omega_n c_1 (1 + c) \rho^+ ||\rho - \overline{\rho}||_{\infty}.$$

Then combining (3.7), (3.8) and (3.9) we find that

$$\lambda_{\varepsilon k} \leq \lambda_k (1 + C \varepsilon^s \lambda_k),$$

from where follows that

$$\lambda_{\varepsilon k} - \lambda_k \leq C \varepsilon^s \lambda_k^2.$$

In a similar way, interchanging the roles of $\lambda_k$ and $\lambda_{\varepsilon k}$, we obtain that

$$\lambda_k - \lambda_{\varepsilon k} \leq C \varepsilon^s (\lambda_{\varepsilon k})^2.$$

Hence, from (3.11) and (3.12), we arrive at

$$|\lambda_{\varepsilon k} - \lambda_k| \leq C \varepsilon^s \max\{\lambda_k, \lambda_{\varepsilon k}\}^2,$$

and so the proof is complete. □

By combining all these facts, we immediately can prove Theorem 1.2.

Proof of Theorem 1.2 Since $\rho$ satisfies (1.6) from the variational characterization of the eigenvalues we obtain the following relation

$$\frac{1}{\rho^+} \mu_k \leq \lambda_k, \lambda_{\varepsilon k} \leq \frac{1}{\rho^-} \mu_k$$

where $\mu_k$ is the $k$–th eigenvalue of (2.1).

From (3.13) and (2.3) it follows that

$$\lambda_k, \lambda_{\varepsilon k} \leq \frac{1}{\rho^-} \Lambda_k^+.$$
where $\lambda_k$ is the $k$–th eigenvalue of the Laplacian with Dirichlet boundary conditions on $\partial \Omega$. Finally, in [3] it is proved that

$$\lambda_k \leq n \pi \left( \frac{k}{|\Omega|} \right)^{\frac{2}{n}}.$$  

From the last inequalities, we get that

$$(3.14) \quad \max \{\lambda_\varepsilon^2 \lambda_k^2 \} \leq \left( \frac{1}{\rho^2(n\pi)^s|\Omega|^{-\frac{2s}{n}}} \right)^2 k^\frac{4}{n}$$  

and the result follows from Theorem 3.5.

**Remark 3.6.** The constant $C$ in Theorem 1.2 can be computed. From Lemma 3.4, (3.14) and (3.10) it follows that the constant $C$ is given by

$$C = 4c_1 (1 + c) n \omega_n \left( \frac{\rho^2}{\rho^2 - |\rho - \rho|_\infty(n\pi)^{2s}|\Omega|^{-\frac{2s}{n}}} \right)$$

where $c_1$ is the fractional Poincaré constant in $L^2$ in the unit cube given in Lemma 3.2 and $c$ is the Poincaré constant given in Lemma 3.1.

In order to prove the convergence of the full spectrum of (1.1) in the general weight case (without periodic assumption) we use the same proof of Theorem 1.2 by changing the use of Lemma 3.4 by the following one.

**Theorem 3.7.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $\{g_\varepsilon\}_{\varepsilon>0}$ be a set functions such that $0 < g^- \leq g_\varepsilon \leq g^+ < +\infty$ for $g^\pm$ constants and $g_\varepsilon \rightharpoonup g$ weakly* in $L^\infty(\Omega)$. Then

$$\lim_{\varepsilon \to 0} \int_\Omega (g_\varepsilon - g)|u|^2 = 0$$

for every $u \in H^s(\Omega)$, $0 < s < 1$.

**Proof.** The weak* convergence of $g_\varepsilon$ in $L^\infty(\Omega)$ says that $\int_\Omega g_\varepsilon \varphi \to \int_\Omega g \varphi$ for all $\varphi \in L^1(\Omega)$. In particular, by using the same arguments that in the proof of Lemma 3.4 we obtain that $|u|^2 \in W^{s,1}(\Omega)$, whence $|u|^2 \in L^1(\Omega)$ and the result is proved. 

**Proof of Theorem 1.1** It follows immediately by following the same arguments that in Theorem 3.5 and by using Lemma 3.7 instead of Lemma 3.4.

**References**

1. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011, Corrected reprint of the 1978 original [MR0503330]. MR 2839402
2. R. M. Blumenthal and R. K. Getoor, *The asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math. 9 (1959), 399–408. MR 0107298 (21 #6023)
3. Lucio Boccardo and Paolo Marcellini, *Sulla convergenza delle soluzioni di disequazioni variazionali*, Ann. Mat. Pura Appl. (4) 110 (1976), 137–159. MR 0425344 (54 #13300)
4. Thierry Champion and Luigi De Pascale, *Asymptotic behaviour of nonlinear eigenvalue problems involving $p$-Laplacian-type operators*, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), no. 6, 1179–1195. MR 2376876 (2009b:35315)
5. Isaac Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. MR 768584 (86g:58140)
6. Zhen-Qing Chen and Renming Song, *Two-sided eigenvalue estimates for subordinate processes in domains*, J. Funct. Anal. 226 (2005), no. 1, 90–113. MR 2158176 (2006d:60116)
7. Christ, F. M. and Weinstein, M. I., Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal., Journal of Functional Analysis, 1991. MR 1124294
8. R. Courant and D. Hilbert, Methods of mathematical physics. Vol. I, Interscience Publishers, Inc., New York, N.Y., 1953. MR 0065391 (16.426a)
9. Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), no. 5, 521–573. MR 2944369
10. Bartłomiej Dyda and Rupert L. Frank, Fractional Hardy-Sobolev-Maz’ya inequality for domains, Studia Math. 208 (2012), no. 2, 151–166. MR 2910884
11. Julián Fernández Bonder, Juan Pablo Pinasco, and Ariel M. Salort, Convergence rate for quasilinear eigenvalue homogenization, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 5, 1901–1925. MR 3082248
12. G. Franzina, G. Palatucci, Fractional p-eigenvalues, Riv. Mat. Univ. Parma, to appear. 2, 9
13. Evans M. Harrell, II and Selma Yıldırım Yolcu, Eigenvalue inequalities for Klein-Gordon operators, J. Funct. Anal. 256 (2009), no. 12, 3977–3995. MR 2521917 (2010c:35198)
14. A. Iannizzotto and M. Squassina Weyl-type laws for fractional p–eigenvalue problems
15. Carlos Kenig, Fanghua Lin, and Zhongwei Shen, Estimates of eigenvalues and eigenfunctions in periodic homogenization, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 5, 1901–1925. MR 3082248
16. Carlos E. Kenig, Fanghua Lin, and Zhongwei Shen, Convergence rates in $L^2$ for elliptic homogenization problems, Arch. Ration. Mech. Anal. 203 (2012), no. 3, 1009–1036. MR 2928140
17. Srinivasan Kesavan, Homogenization of elliptic eigenvalue problems. II, Appl. Math. Optim. 5 (1979), no. 3, 197–216. MR 546068 (80j:65110)
18. Elliott H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. (2) 118 (1983), no. 2, 349–374. MR 717827 (86i:42010)
19. E. Lindgren, P. Lindqvist, Fractional eigenvalues, Calc. Var., to appear. 2
20. O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian, Mathematical problems in elasticity and homogenization, Studies in Mathematics and its Applications, vol. 26, North-Holland Publishing Co., Amsterdam, 1992. MR 1195131 (93k:35025)
21. Augusto C. Ponce, An estimate in the spirit of Poincaré’s inequality, J. Eur. Math. Soc. (JEMS) 6 (2004), no. 1, 1–15. MR 2041005 (2005a:26033)
22. Michael Reed and Barry Simon, Methods of modern mathematical physics. IV. Analysis of operators, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. MR 0493421 (58 #12429c)
23. E. Sánchez-Palencia, Équations aux dérivées partielles dans un type de milieux hétérogènes, C. R. Acad. Sci. Paris Sér. A-B

DEPARTAMENTO DE MATEMÁTICA
FCEN - UNIVERSIDAD DE BUENOS AIRES AND
IMAS - CONICET.
CIUDAD UNIVERSITARIA, PABELLÓN I
(1428) AV. CANTILLO S/N.
BUENOS AIRES, ARGENTINA.

E-mail address, A.M. Salort: asalort@dm.uba.ar