A boundary matrix for AdS/CFT $SU(1|1)$ spin chain

Qingyong Lin$^{1,2}$, Guangliang Li$^{1,2}$ and Yufei Huang$^2$

1 MOE Key Laboratory for Nonequilibrium Synthesis and Modulation of Condensed Matter, Xi’an Jiaotong University, Xi’an 710049, People’s Republic of China
2 Department of Applied Physics, Xi’an Jiaotong University, Xi’an 710049, People’s Republic of China
E-mail: linlqy@stu.xjtu.edu.cn, leegl@mail.xjtu.edu.cn and yfhuang@mailst.xjtu.edu.cn

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Abstract. By solving the right reflection equation proposed in the reference (Hofman and Maldacena, 2007 J. High Energy Phys. JHEP11(2007)063 [arXiv:0708.2272]) to describe the $Z = 0$ giant graviton branes, we obtain a boundary matrix with two free parameters for the AdS/CFT $SU(1|1)$ spin chain.

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1. Introduction

The discoveries of integrable structures in planar $N = 4$ SYM [1]–[3] and in superstring theory on $AdS_5 \times S^5$ [4]–[7] enable us to determine this system’s spectrum in the form of a set of nonlinear Bethe equations [8]–[10], which can be obtained by solving spin chain models with different boundaries in the framework of the quantum inverse scattering method (QISM) [11, 12].

In the past years, integrability has been extended to the open string/spin chain sector of AdS/CFT [13]–[15]. Hofman and Maldacena (HM) considered open strings attached to maximal giant gravitons in $AdS_5 \times S^5$ [16]. They proposed boundary $S$-matrices describing the reflection of world-sheet excitations (giant magnons) for two cases, namely the $Y = 0$ and $Z = 0$ giant graviton branes. For the $Y = 0$ case, the right boundary matrix $R^R$ satisfies the reflection equation [16]

$$S_{12} (p_1, p_2) R^R_1 (p_1) S_{21} (p_2, -p_1) R^R_2 (p_2) = R^R_2 (p_2) S_{12} (p_1, -p_2) R^R_1 (p_1) S_{21} (-p_2, -p_1),$$

(1)

where $S_{12}(p_1, p_2)$ is the $S$-matrix determined uniquely by $SU(2|2)$ symmetry [18], $R^R$ is a $4 \times 4$ matrix and has a diagonal solution [16, 17]. Murgan and Nepomechie extended Sklyanin’s construction of commuting open-chain transfer matrices to the $Y = 0$ case, where they proposed a left reflection equation and constructed a commuting transfer matrix [19]. Later the corresponding Bethe equations were obtained by Galleas [20] and Nepomechie [21] with the help of different Bethe methods.

For the $Z = 0$ case, the giant graviton brane has a boundary degree of freedom and full $SU(2|2)$ symmetry [16, 17]. The right boundary $S$-matrix $R^R$ satisfies the right boundary reflection equation (BYBE) [16]

$$S_{12} (p_1, p_2) R^R_{13} (p_1) S_{21} (p_2, -p_1) R^R_{23} (p_2) = R^R_{23} (p_2) S_{12} (p_1, -p_2) R^R_{13} (p_1) S_{21} (-p_2, -p_1),$$

(2)
where $R^R$ is a $16 \times 16$ matrix. Referring to the work of Nepomechie [19, 21], one can prove that if the left BYBE obeys the following equation:

$$S_{21}(p_2, p_1)^{t_1 t_2} R_{31}^L(p_1)^{t_1} C_1(-p_1) S_{21}(p_2, -p_1)^{t_2} C_1(-p_1)^{-1} R_{32}^L(p_2)^{t_2} = R_{32}^L(p_2)^{t_2} C_2(-p_2) S_{12}(p_1, -p_2)^{t_1} C_2(-p_2)^{-1} R_{31}^L(p_1)^{t_1} S_{12}(-p_1, -p_2)^{t_1 t_2},$$

the transfer matrix defined as

$$t(p; \{q_i\}) = tr_a R^L_{\alpha \alpha}(p) T_{a1\ldots L}(p; \{q_i\}) R^R_{\alpha L+1}(p) \tilde{T}_{a1\ldots L}(p; \{q_i\}),$$

can consist of a commuting family, where $C(p)$ is the charge conjugation matrix [22], $\bar{p}$ denotes the antiparticle momentum and

$$T_{a1\ldots L}(p; \{q_i\}) = S_{aL}(p, q_L) \ldots S_{a1}(p, q_1),$$

$$\tilde{T}_{a1\ldots L}(p; \{q_i\}) = S_{1a}(q_1, -p) \ldots S_{1a}(q_L, -p).$$

As discussed in reference [20], the Hamiltonian for equation (4) can be defined by

$$H = \frac{d \ln t(p; \{q_i = \pi\})}{dp} \bigg|_{p=\pi}.$$  

Much work has been done on the solution to the reflection equation like equation (1) [23–25]. However, there is a little work on the solution to the reflection equation like equation (2) [16, 17, 26]. For the AdS/CFT $SU(1|1)$ spin chain [27], a boundary $S$-matrix satisfying equation (1) was found in [19], while the boundary $S$-matrix satisfying equation (2) is not obtained so far as we know. In this paper we will solve equation (2) to find the boundary matrix for the $SU(1|1)$ spin chain.

The outline of this paper is organized as follows. In section 2, we introduce the $SU(1|1)$ spin chain. In section 3, we solve equation (2) and present a boundary matrix with two free parameters. Some discussions are given in section 4.

2. Bulk $S$-matrix of $SU(1|1)$ spin chain

The graded $SU(1|1)$ bulk $S$-matrix takes the form [27, 19]

$$S^g(p_1, p_2) = \begin{pmatrix}
 x_1^+ - x_2^- & 0 & (x_1^+ - x_1^-) \frac{\omega_1}{\omega} & 0 \\
 0 & x_1^+ - x_2^- & 0 & 0 \\
 (x_2^+ - x_2^-) \frac{\omega_1}{\omega_2} & 0 & x_1^+ - x_2^- & 0 \\
 0 & 0 & 0 & x_1^+ - x_2^+
\end{pmatrix},$$

where $x_1^\pm, \omega_i$ denote $x^\pm(p_i), \omega(p_i)$. This $S$-matrix owns unitarity property and crossing-like property [28].

Here, for the sake of simplicity, we use the non-graded $SU(1|1)$ bulk $S$-matrix:

$$S(p_1, p_2) = \begin{pmatrix}
 x_1^+ - x_2^- & 0 & (x_1^+ - x_1^-) \frac{\omega_1}{\omega} & 0 \\
 0 & x_1^+ - x_2^- & 0 & 0 \\
 (x_2^+ - x_2^-) \frac{\omega_1}{\omega_2} & 0 & x_1^+ - x_2^- & 0 \\
 0 & 0 & 0 & -(x_1^+ - x_2^+)
\end{pmatrix}.$$
and let \( x^\pm(p) \) satisfy the following constraints:

\[
x^+(p) + \frac{1}{x^+(p)} - x^-(p) - \frac{1}{x^-(p)} = \frac{i}{g}, \quad \frac{x^+(p)}{x^-(p)} = e^{i\varphi}.
\] (10)

At the same time, we choose \( w(p) = 1 \) due to that \( w(p) \) can be gauged away by performing a gauge transformation [28].

The relation between \( S \) and \( S^\natural \) is [19]

\[
S(p_1, p_2) = \mathcal{P} \mathcal{P}^\natural S^\natural(p_1, p_2),
\] (11)

where \( \mathcal{P}^\natural \) and \( \mathcal{P} \) are the graded and non-graded permutation matrix, respectively. \( \mathcal{P}^\natural_{kl} = (-1)^{p(i)p(j)}\delta_{ik}\delta_{jk} \), \( p(1) = 0, p(2) = 1 \) and \( \mathcal{P}_{kl} = \delta_{ik}\delta_{jk} \).

3. Boundary \( S \)-matrix of \( SU(1|1) \) spin chain

We now write the right BYBE equation (2) in the matrix element form:

\[
S(p_1, p_2)_{ij, j_1} \mathcal{R}^R (p_1)_{i_1 k_1} S(p_2, -p_1)_{j_2 j_2} \mathcal{R}^R (p_2)_{j_2 k_2}
= \mathcal{R}^R (p_2)_{i_1 k_1} S(p_1, -p_2)_{j_1 j_1} \mathcal{R}^R (p_1)_{i_1 k_1} S(-p_2, -p_1)_{j_2 j_2}.
\] (12)

We suppose that the right boundary \( S \)-matrix has the same form as the bulk \( S \)-matrix, so we have

\[
\mathcal{R}^R (p) = \begin{pmatrix}
a(p) & 0 & 0 & 0 \\
0 & b(p) & c(p) & 0 \\
0 & d(p) & e(p) & 0 \\
0 & 0 & 0 & f(p)
\end{pmatrix}.
\] (13)

Substituting the boundary matrix equation (13) and the bulk \( S \)-matrix equation (9) into (12), we get 64 equations. We find out that, in the 64 equations, there are 48 identical equations. The remaining 16 equations are the following:

\[
(x_1^+ + x_2^+) (x_2^+ - x_2^-) a_1 e_2 + (x_1^+ + x_2^-) (x_1^- - x_2^+) d_1 c_2 + (x_1^- - x_2^+) (x_2^- - x_2^-) e_1 e_2
- (x_1^- + x_2^-) (x_2^+ - x_2^-) e_1 a_2 - (x_1^- + x_2^-) (x_2^- - x_2^-) a_1 a_2 = 0,
\] (14)

\[
(x_1^- + x_2^-) (x_2^+ - x_2^-) a_1 d_2 + (x_1^+ + x_2^-) (x_1^- - x_2^+) d_1 b_2
+ (x_1^- - x_2^+) (x_2^- - x_2^-) e_1 d_2 - (x_1^- + x_2^-) (x_1^- - x_2^-) d_1 a_2 = 0,
\] (15)

\[
(x_1^- - x_2^+) (x_1^+ - x_1^-) a_1 a_2 + (x_1^+ + x_2^-) (x_1^- - x_1^-) e_1 a_2 - (x_1^+ + x_2^-) (x_1^- + x_2^-) c_1 d_2
- (x_1^- + x_2^-) (x_1^- - x_1^-) a_1 e_2 - (x_1^- + x_2^-) (x_1^- + x_2^-) e_1 e_2 = 0,
\] (16)

\[
(x_1^+ + x_2^-) (x_1^+ - x_1^-) d_1 c_2 - (x_1^+ + x_2^-) (x_2^- - x_2^-) c_1 d_2 = 0,
\] (17)

\[
(x_1^+ + x_2^+) (x_1^- - x_2^+) a_1 d_2 + (x_1^+ + x_2^+) (x_1^- - x_1^-) d_1 b_2 + (x_1^- - x_1^-) (x_2^- - x_2^-) e_1 d_2
- (x_1^+ + x_2^-) (x_1^- - x_2^-) b_1 d_2 - (x_1^- + x_2^-) (x_1^- + x_2^-) d_1 e_2 = 0,
\] (18)

\[
(x_1^+ + x_2^-) (x_1^- - x_2^-) c_1 a_2 - (x_1^+ + x_2^+) (x_1^- + x_2^-) c_1 b_2
- (x_1^- + x_2^-) (x_1^- - x_1^-) a_1 c_2 - (x_1^- + x_2^-) (x_1^- + x_2^-) e_1 c_2 = 0,
\] (19)
Then we will get 16 equations about $x$:

\[(x^+_1 + x^+_2) (x^+_1 - x^-_2) b_1 c_2 + (x^+_1 - x^-_2) (x^+_2 - x^-_2) c_1 e_2 - (x^+_1 + x^+_2) (x^+_2 - x^-_2) c_1 b_2
- (x^-_1 + x^+_2) (x^-_1 - x^-_2) a_1 c_2 - (x^-_1 + x^-_2) (x^+_2 - x^-_2) e_1 c_2 = 0,\]  
\[(x^+_1 - x^-_2) (x^+_2 - x^-_2) c_1 d_2 - (x^+_1 - x^-_1) (x^+_1 - x^-_2) d_1 c_2 = 0,\]
\[(x^+_1 + x^-_2) (x^+_1 - x^-_2) f_1 b_2 + (x^-_1 - x^-_2) (x^+_1 - x^-_1) b_1 b_2 - (x^+_1 + x^-_2) (x^+_1 - x^-_2) c_1 d_2
- (x^-_1 + x^+_2) (x^-_1 - x^-_1) b_1 f_2 - (x^+_1 - x^-_1) (x^+_1 - x^-_2) f_1 f_2 = 0,\]
\[a \text{ boundary matrix for AdS/CFT } SU(1|1) \text{ spin chain} \]

\[
(x^+_1 + x^-_2) (x^+_1 - x^-_1) f_1 e_2 + (x^-_1 - x^-_2) (x^+_1 - x^-_1) b_1 c_2
- (x^-_1 + x^-_2) (x^-_1 - x^-_2) c_1 e_2 + (x^-_1 + x^+_2) (x^-_1 - x^-_2) c_1 f_2 = 0,
\]
\[a = a(p_i), \text{ so do } b_1, c_1, d_1, e_1, f_1. \text{ The property } x^+(−p) = −x^+(p) \text{ is used } [17].\]

In order to solve $R^R$, we differentiate the above 16 equations with $p_2$, and let $p_2 = p_0$. Then we will get 16 equations about $p$, from which we can solve $R^R$. During this process, we introduce the following 16 parameters: $x^+(p_0)$, $x^−(p_0)$, $x^+(p_0)'$, $x^−(p_0)'$, $a(p_0)$, $a(p_0)'$, $b(p_0)$, $b(p_0)'$, $c(p_0)$, $c(p_0)'$, $d(p_0)$, $d(p_0)'$, $e(p_0)$, $e(p_0)'$, $f(p_0)$ and $f(p_0)'$. We believe that these 16 parameters will be self-consistent, which offers the chance to simplify the equations.

If we let the initial condition be $R^R(p_0) = I$, where $I$ is a $4 \times 4$ identity matrix, we will find at once that

\[x^+(p_0) = −x^−(p_0), \quad [x^+(p)]_{p=p_0} = [x^−(p)]_{p=p_0}.\]

(30)

Considering that [17]

\[x^±(−p) = −x^±(p), \quad x^±(p) = \frac{e^{±(ip/2)}}{4g \sin (p/2)} \left(1 + \sqrt{1 + 16g^2 \sin^2 \left(\frac{p}{2}\right)}\right),\]

(31)

we find $p_0 = (2k + 1)\pi$. Without losing generality, we choose $p_0 = \pi$. Then we have

\[a(\pi) = b(\pi) = e(\pi) = f(\pi) = 1, \quad c(\pi) = d(\pi) = 0\]

\[x^+(\pi) = −x^−(\pi) = x_B, \quad x^±(p)'_{|p=\pi} = x^−(p)'_{|p=\pi} = \frac{i}{2} x_B,\]

(32)

\[\text{doi:10.1088/1742-5468/2009/12/P12001}\]
where $x_B = (i/4g)(1 + \sqrt{1 + 16g^2})$. With the help of equation (32), we find that equations (17), (21), (23) and (29) are identical, and so are equations (14) and (16), equations (15) and (19), equations (18) and (20), equations (22) and (24), equations (25) and (27), and equations (26) and (28). Simplifying these equations, we finally obtain one solution to the right reflecting equation (2):

$$
\begin{align*}
a &= -\frac{\xi \eta (x^+ - x_B) [(x^+)^2 - (x_B)^2] + 2(x^+)^2 (2x^+ - x_B)}{\xi \eta (x^+ + x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^- + x_B)} f, \\
b &= -\frac{\xi \eta (x^+ - x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^+ - x_B)}{\xi \eta (x^- + x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^- + x_B)} f, \\
c &= \frac{i\xi x_B (x^+ + x^-) (x^+ - x^-)}{\xi \eta (x^- + x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^- + x_B)} f, \\
d &= \frac{2i\eta (x_B)^2 (x^+ + x^-)}{\xi \eta (x^- + x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^- + x_B)} f, \\
e &= \frac{\xi \eta (x^- + x_B) [(x^+)^2 - (x_B)^2] + 2(x^+)^2 (2x^- + x_B)}{\xi \eta (x^- + x_B) [(x^-)^2 - (x_B)^2] + 2(x^-)^2 (2x^- + x_B)} f,
\end{align*}
$$

where $\xi$ and $\eta$ are arbitrary boundary parameters.

4. Discussion

By directly solving the reflection equation (2), we obtain a boundary matrix with two free parameters for the $SU(1|1)$ spin chain model. Here, we only suppose the boundary takes the form like equation (13). Whether there are other boundary matrices, we need to explore further.

The open $SU(1|1)$ quantum spin chain with its right boundary matrix satisfying the reflection equation (1) is solved by an analytical Bethe ansatz method [28]. It would be interesting to investigate how to solve this spin chain with its boundary matrix satisfying equation (2).

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