Are multi-factor Gaussian term structure models still useful? An empirical analysis on Italian BTPs

Michele Leonardo Bianchi
Regulation and Macroprudential Analysis Directorate, Bank of Italy, Rome, Italy

ABSTRACT
In this paper, we empirically study models for pricing Italian sovereign bonds under a reduced form framework, by assuming different dynamics for the short-rate process. We analyze classical Cox-Ingersoll-Ross and Vasicek multi-factor models, with a focus on optimization algorithms applied in the calibration exercise. The Kalman filter algorithm together with a maximum likelihood estimation method are considered to fit the Italian term-structure over a 17-year horizon, including the global financial crisis, the euro area sovereign debt crisis and the Italian political turmoil in 2018. Analytic formulas for the gradient vector and the Hessian matrix of the likelihood function are provided.

ARTICLE HISTORY
Received 29 April 2019
Accepted 21 January 2020

KEYWORDS
Cox-Ingersoll-Ross processes; Gaussian Ornstein-Uhlenbeck processes; Bond pricing; Kalman filter; Maximum likelihood estimation; Non-linear optimization

1. Introduction
This paper provides a detailed comparative assessment of a number of affine interest rate models applied to the term structure of Italian government securities.

Having reliable estimates of these models allows to conduct scenario analyses that can be used by banks for risk management purposes, e.g., to perform a risk-return analysis or to compute the value-at-risk of a bond portfolio. Furthermore, as indicated by Abdymomunov and Gerlach (2014), instead of assuming deterministic scenarios, supervisory authorities may consider hypothetical stochastic scenarios to stress test bank balance sheets.

The present paper contributes to the relevant literature from three different perspectives: (i) systematically assessing different specifications of affine models, applied to data from one of the largest and most liquid European government bond markets over a 17-year horizon which includes structural breaks; (ii) providing the reader with a detailed tutorial of the technicalities related to the practical implementation of such models (especially in the Appendix A);1 (iii) providing new insight into the modeling of the Italian term structure of interest rates, relying on a well-established and simple mathematical framework.

Multi-factor models have been widely studied in the literature (see Aït-Sahalia and Kimmel (2010) and Duffee and Stanton (2012) for a complete overview). Duan and Simonato (1999) analyze monthly yield series for the U.S. Treasury debt securities with
maturities 3, 6, 12 and 60 months spanning the period from April 1964 to December 1997 and conclude for a rejection of exponential affine models. Geyer and Pichler (1999) have calibrated and tested multi-factor versions of the Cox-Ingersoll-Ross (CIR) model on U.S. interest rates observed monthly from January 1964 to December 1993. Although the specification of multi-factor CIR models is sufficiently flexible for the shape of the term structure, they find strong evidence against the adequacy of the CIR model. They have shown that in the two-factor model the first factor corresponds to the general level of interest rates and the second to the spread between long and short rates. Therefore, they conclude that the assumption that all factors follow a square-root process hinders the explanatory power of additional factors (as it restricts admissible values for these factors to be nonnegative). This assumption also affects the decision about the number of factors to be included. As a consequence, they suggest to relax the square-root assumption for all factors, in particular in the case of multi-factor models. De Jong (2000) shows that a three-factor affine model is able to provide an adequate fit of the cross-section and the dynamics of the term structure. The three factors can be given the usual interpretation of level, steepness, and curvature. Almeida (2005) explains the relation between principal components obtained assuming no dynamic restrictions and the dynamic factors estimated using multi-factor Gaussian term structure models. The author finds that the linear structure embedded in dynamic affine term structure models directly translates into an approximation of the non-negligible principal components by linear transformations of the state vector. Date and Wang (2009) have studied the out-of-sample forecasting ability of linear Gaussian interest rate models with unobservable underlying factor by considering both the Kalman filter and a non-linear filter. Wüthrich and Merz (2013) estimated the Swiss franc yield curve under a dynamic framework and with the Kalman filter technique. They showed that the multi-factor Vasicek model often provides a reasonable term structure model and it is rather flexible in modeling and predicting yield curves with different shapes. Recently, Rebonato, Saroka, and Putyatin (2014) have investigated the theoretical framework for the affine evolution of mean-reverting principal component models, and they have shown how the model can be calibrated using both risk-neutral and historical measure information.

As regards the choice of relying on short rate processes, note that neither the Libor nor the swap market models (see Brigo and Mercurio (2006) and Dempster, Evans, and Medova (2014)) can be applied in our context, for two main reasons. First, these market models have been developed to calibrate the quotes of the most liquid interest rate derivatives traded in the market (i.e., caps, floors and swaptions), with the purpose of pricing and hedging exotic derivatives, usually traded over the counter. Even if the information derived by the implied volatilities of these liquid derivatives may be considered to capture market expectations on future rates, the analysis of these derivatives is beyond the scope of this paper. Second, these models are usually calibrated under a static perspective, in the sense that at each given point in time model prices have to be as close as possible to the prices traded in the market. Even if the market is quoting unreasonable prices, the trader has to find the parameters that replicate those prices. As explained by Nawalkha and Rebonato (2011), traders need to achieve static consistency in order to provide at each given point in time arbitrage-free prices or to hedge a derivatives portfolio. Conversely, our model is calibrated following a dynamic approach, in
order to analyze the behavior of yield curves over time. In this respect, we assume that the structural parameters remain constant over time, i.e., we assume that the models considered are time homogeneous while fitting them to observed bond yield curves, taking into account both the cross sectional and the time-series dimensions of Italian sovereign bond rates (i.e., pooling yield curves in a single panel dataset).

Similar to Bams and Schotman (2003) and Ang and Longstaff (2013), we analyze multi-factor Gaussian term structure models under the risk-neutral measure, where the dynamics of the short-rate factors is defined as the sum of Vasicek and CIR factors. Note that the dynamics of the factors of the short rate process under the historical measure, as well as the associated market price of risk, is left unspecified and is not investigated in this paper. That is, only the risk-neutral dynamics is inferred by the observed bond yield curves. Indeed, as the factors driving the short rate process are non-tradable assets, it is not possible to perform a double-calibration in which the historical and risk-neutral model parameters can be jointly estimated. A joint calibration is possible if both the underlying and the derivative are observable, such as for example in the case of equity options (see e.g., Tassinari and Bianchi (2014) and Bianchi and Tassinari (2018) for an example of double-calibration).\(^2\) As a consequence, it is not possible to check whether a given historical dynamics of the factors driving the short rate process is feasible or not, and for this reason we prefer not to follow this path. Under the taxonomy introduced by Nawalkha, Beliaeva, and Soto (2010), the model we analyze can be defined as single-plus.

Fitting sovereign bond yields without modeling risk premia nor regime shifts is particularly challenging while analyzing Italian government bond yields over almost two decades which include three crisis periods. In this sense, the comparative analysis of affine models presented in this paper can be viewed as an assessment of widely used models, aiming to highlight their strengths as well as their limitations. At the same time, the analysis contributes to the literature on Italian government securities with a thorough examination of different interest models, indicating some viable approaches along with possible pitfalls of well-known models. In the relevant literature, the Italian government bond market has been studied under different perspectives. Barone, Cuoco, and Zautzik (1989) tested the CIR model using the prices of Italian government bonds in the secondary market in order to obtain useful indications regarding the efficiency of the secondary market and the consistency between the primary and the secondary markets. The authors have conducted a static calibration and have assessed the parameter stability. D’Ecclesia and Zenios (1994) developed a multi-factor model for the yields of Italian bonds and, to control the sensitivity of portfolio returns to movements in the risk factors, implemented appropriate factor immunization models. Based on the observation that short-term bonds interest-rates are characterized by a trend basically driven by the European Central Bank (ECB) key interest rates, Maggi and Infortuna (2008) have proposed to filter out the effect of the ECB key interest rates from the term structure of Italian government bonds first, and then, for each maturity, to estimate a CIR model on the transformed data. Musti and D’Ecclesia (2008) have investigated the informational content of the yield curve in the European market using data on the Italian

\(^2\)There are options on Euro-BTP futures traded in the market, but they have not been traded during the entire time period analyzed in this work.
term structures and tested the expectation hypothesis theory. Girardi and Impenna (2013) have conducted an analysis on two regulated markets supervised by the Bank of Italy in cooperation with Consob (Commissione Nazionale per la Società e la Borsa – Companies and Stock Exchange Commission). They analyzed the price discovery process and the informational role of trading in the interdealer business-to-business (B2B) trading venue (MTS cash), and the business-to-customer (B2C) BondVision market. More recently, Pelizzon et al. (2016, 2018) have examined the relation between the Italian sovereign bond market and both the credit default swap and the future markets during the eurozone crisis and the subsequent ECB interventions.

Finally, the present paper contributes to the literature with a view at the implementation of the models analyzed, providing a detailed description of the mathematical details that practitioners have to work out to efficiently implement the gradient and the Hessian matrix of the objective function of the optimization problem for the calibration of those models. The dynamic models we study can be casted in a state-space form, where the state is given by the unobservable factors and the observations are given by the term structure. A state and parameter estimation can then be obtained using the Kalman filter together with a maximum likelihood estimation method. However, finding an optimal solution to this type of problems is nontrivial and optimization packages should be handled with care. In particular, in order to have reliable optimization algorithms and to compute the standard errors of the estimates exactly, we compute the analytic expression for the gradient vector and the Hessian matrix of the likelihood function considered in the factors and parameter estimation. Full details are provided in the Appendix A.

The remainder of the paper is organized as follows. Section 2 reviews the Vasicek and CIR multi-factor models considered in the empirical study, including the derivation of bond prices and yields in these models. The estimation algorithms together with the main empirical results are discussed in Sec. 3. Section 4 concludes. The Appendix A provides analytic formulas for the gradient and the Hessian of the likelihood function.

2. The model

2.1. Evaluate zero-coupon bonds

We consider the short-rate approach to model the dynamics of the Italian interest rate term structure. There is a general consensus in assuming a stochastic short-rate instead of a deterministic short-rate to model uncertainty about the future dynamics of the interest rate and credit risk of a given bond. The building block of short-rate models is the integrated process $R_t$, i.e.,

$$R_t = \int_0^t r_s \, ds$$

where $r_t$ is the instantaneous short-rate process, that is generally assumed to be a stationary affine process. It follows that the price at time 0 of a zero coupon bond maturing at $t$

$$B(0, t) = \mathbb{E}[\exp \{-R_t\}] = \phi_{R_t}(i),$$

(1)
where the expectation is taken under the risk-neutral measure, $\phi_{R_t}(u)$, with $u \in \mathbb{C}$, is the characteristic function of the random variable $R_t$ and $i$ is the imaginary unit. The previous equality easily follows from the definition of the characteristic function of a random variable $X$

$$
\phi_X(u) = E[\exp (iuX)],
$$

where $u \in \mathbb{C}$. Knowing the characteristic function of the process $R_t$, it is straightforward to compute the expectation in Eq. (1), and hence the yield to maturity at time 0 of a zero-coupon bond maturing at $t$

$$
y(0, t) = -\frac{1}{t} \log B(0, t) = -\frac{1}{t} \log \left( E[\exp (-R_t)] \right) = -\frac{1}{t} \log \left( \phi_{R_t}(i) \right).
$$

As a consequence, we can compute the entire yield curve by assuming a model for the short rate process $r_t$. In the following, we explore several options, assuming that $r_t$ is a linear combination of stochastic processes that follow either the Vasicek or the CIR model.

### 2.2. The Vasicek process

As proposed by Vasicek (1977), a Gaussian Ornstein-Uhlenbeck (OU) process can be used to model the dynamics of the instantaneous short-rate, that is

$$
dr_t = \kappa(\eta - r_t)dt + \theta dW_t,
$$

with $r_0 > 0$, $\kappa$ and $\theta$ positive parameters, $\eta \in \mathbb{R}$, and $W_t$ is a Brownian motion. Usually, the parameter $\eta$ is chosen to be positive. In order to reduce the number of parameters, in our empirical study we will assume that some Vasicek factors will have $\eta = 0$. We recall that, even if $\eta$ is positive, the process can be negative with positive probability (see Brigo and Mercurio (2006)). The conditional density $p(r_{t+1}|r_t)$ is normal with

$$
E[r_{t+1}|r_t] = \eta(1 - e^{-\kappa \Delta t}) + e^{-\kappa \Delta t} r_t
$$

$$
Var[r_{t+1}|r_t] = \frac{\theta^2}{2\kappa}(1 - e^{-2\kappa \Delta t}).
$$

Under the Vasicek model there exists a closed-form expression for the characteristic function of the integrated process (see Proposition 2.6.2.1 in Jeanblanc, Yor, and Chesney (2009)) and the integral in Eq. (1) can be computed as follows

$$
B(0, t) = e^{A(t)+B(t)r_0},
$$

where

$$
A(t) = -\eta t + \frac{1 - e^{-\kappa t}}{\kappa} - \frac{\theta^2}{4\kappa^3}(1 - e^{-\kappa t})^2 + \frac{\theta^2}{2\kappa^2} \left(t - \frac{1 - e^{-\kappa t}}{\kappa}\right),
$$

$$
B(t) = -\frac{1 - e^{-\kappa t}}{\kappa}.
$$

For a description of the possible shape of the yield curve under the Vasicek model see Zeytun and Gupta (2007) and Keller-Ressel and Steiner (2008).
2.3. The Cox-Ingersoll-Ross process

A well-known way to model the instantaneous short-rate process is to assume the Cox-Ingersoll-Ross (CIR) dynamics (see Cox, Ingersoll, and Ross (1985)), by considering the following mean-reverting process

$$dr_t = \kappa(\eta - r_t)dt + \vartheta\sqrt{r_t}dW_t,$$

under the initial condition $r_0 > 0$, where $W_t$ is a standard Brownian motion and $\kappa$, $\eta$ and $\vartheta$ are positive parameters satisfying the additional condition $2\kappa\eta > \vartheta^2$ in order to ensure that the origin is inaccessible, i.e., that $r_t$ remains positive for all $t$. The conditional density $p(r_{t+1}|r_t)$ of this process has the following form (see e.g., Brigo and Mercurio (2006))

$$p(r_{t+1}|r_t) = cf_{\chi^2(v,l)}(cz)$$

where $f_{\chi^2(v,l)}$ is the probability density function of a noncentral $\chi^2$ random variable with parameters $v$ and $l$, and

$$c = \frac{4\kappa}{\vartheta^2(1 - e^{-\kappa\Delta t})}, \quad v = \frac{4\kappa\eta}{\vartheta^2}, \quad l = cr_te^{-\kappa\Delta t}.$$  

Strictly speaking, the CIR process is not Gaussian, since its conditional distribution is not normal. However, with a slight abuse of terminology, we will refer to it as a Gaussian process, taking into account that its dynamic is driven by a Brownian motion. It can be proven that the conditional mean and variance are

$$E[r_{t+1}|r_t] = \eta(1 - e^{-\kappa\Delta t}) + e^{-\kappa\Delta t}r_t$$

$$Var[r_{t+1}|r_t] = \frac{\eta\vartheta^2}{2\kappa}(1 - e^{-\kappa\Delta t})^2 + r_t \frac{\vartheta^2}{\kappa}(e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}).$$

Under the CIR assumption there exists a closed-form expression for the characteristic function of the integrated CIR process and the bond price in Eq. (1) can be computed as follows (see Schoutens and Cariboni (2009), Teng, Ehrhardt, and Günther (2013) and references therein for the derivation of the formula)

$$B(0,t) = e^{A(t)+B(t)r_0},$$

where

$$A(t) = \frac{2\kappa\eta}{\vartheta^2} \left( \log(2) + \log \left( \frac{b/(\kappa - b)}{ae^{bt} - 1} \right) \right),$$

$$B(t) = -\frac{2}{\kappa - b} \left( \frac{e^{bt} - 1}{ae^{bt} - 1} \right),$$

$$a = \frac{\kappa + b}{\kappa - b} \quad \text{and} \quad b = \sqrt{\kappa^2 + 2\vartheta^2}.$$  

For a description of the possible shape of the yield curve under the CIR model see Zeytun and Gupta (2007) and Keller-Ressel and Steiner (2008).
2.4. A multi-factor short-rate model

The extension of Eq. (1) to the sum of two or more short-rate factors \( r_1, \ldots, r_d \) is straightforward when pairwise independence is assumed between factors. In this case the formula in Eq. (1) becomes

\[
B(0,t) = \phi_{R_1}(i) \cdots \phi_{R_d}(i).
\] (6)

However, if one considers dependent factors, the decomposition in Eq. (6) no longer holds. Normal-based models may still have a closed-form solution, but in general, if one assumes a richer dependence structure (for example, a multivariate model or a copula allowing for tail dependence), Monte Carlo simulations or numerical methods are needed to evaluate bond prices. Feldhütter and Lando (2008) proposed a model with six independent factors to calibrate Treasury bonds, corporate bonds, and swap rates using both cross-sectional and time-series properties of the observed yields. This independence assumption may be restrictive, although the advantage is that pricing formulas have explicit solutions, and the model is more parsimonious with fewer parameters to estimate. Following a similar approach, in our empirical analysis on the Italian yield curve we will consider different multi-factor models with independent CIR or Gaussian OU (Vasicek) factors. This assumption allows us to find a balance between computational tractability and model flexibility. We assume that the short-rate \( r_t \) at time \( t \) is defined as

\[
r_t = \sum_i r^i_t,
\] (7)

where \( r^i_t \) may be a CIR or a Vasicek factor. As already observed, when more than one factor is considered, we assume that the Brownian motions driving these factors are independent. We analyze two 1-factor models (1-CIR and 1-Vasicek), three 2-factor models (2-CIR, 2-Vasicek, and 1-CIR&1-Vasicek), and three 3-factor models (3-CIR, 3-Vasicek, and 2-CIR&1-Vasicek).

Also in this case, the price of a (defaultable) sovereign zero-coupon bond can be written as

\[
B(0,t) = \mathbb{E} \left[ e^{-\int_0^t r_s ds} \right].
\] (8)

and the model can be calibrated to the observed yield curve, that is,

\[
y(0,t) = -\frac{1}{t} \log (B(0,t)).
\] (9)

By Eqs. (5) and (3), it follows that the yield is a linear function of \( r_0, \ldots, r_0^d \), that is the actual level of the short-rate. As we will see in the following, this property allows one to estimate the model with a Kalman filter.

3. Fitting market data in practice

A Buono del Tesoro Poliennale (BTP) is an Italian government bond with semiannual coupon interest payments, principal repaid on maturity and a minimum denomination of EUR 1000. The main institutional investor secondary market for these sovereign
bonds is managed by MTS S.p.a. and the only Italian regulated secondary market for retail investors dedicated to the trading of Italian government securities is managed by Borsa Italiana (MOT).

We obtain zero coupon yield data of Italian government securities from the Bloomberg F905 curve series. The zero coupon yield is computed by stripping the par coupon curve. We considered daily time series from January 1, 2002 to December 31, 2018 for a total of 4435 trading days and 15 maturities (from 3 months to 30 years). The time period in this study includes the high volatility period after the Lehman Brothers filing for Chapter 11 bankruptcy protection (September 15, 2008), the euro area sovereign debt crisis, during which, in November 2011, the spread between the 10-year Italian BTP and the German Bund with the same maturity was higher than 500 basis point, and the political turmoil in May 2018 when the short-term Italian bond yields suffered the biggest one-day jump since 1992. We are aware of the fact that this empirical study can be affected by the expanded asset purchase program announced by the European Central Bank in January 2015.

3.1. Kalman filter

The stochastic risk-free component $r_t$ is calibrated by fitting the interest rate term structure to the term structure of Italian government bonds. That is, at each trading day we consider the yield defined in Eq. (9). There are two possible methodologies to estimate a short-rate model: (1) one can fit the model to the yields observed daily in the market and check both the model flexibility and the parameter stability; or (2) one can extract the unobservable short-rate process (or processes) by using a filter as described and empirically tested for instance by Chen and Scott (2003), Sanford and Martin (2005), O’Sullivan (2008), Bellini and Riani (2012), Wüthrich and Merz (2013) or Bianchi and Rocco (2016). While the former is a static estimation, the latter is a dynamic approach which captures the behavior of the yield curve over time, taking into account intertemporal consistency of parameter estimates. In this paper we will follow a dynamic approach. In all the cases we are interested in, the model can be written in the following form

$$
\begin{align*}
x_t &= f(x_{t-1}, \Theta, \nu_{t-1}) \\
z_t &= h(x_t, \Theta, \epsilon_t)
\end{align*}
$$

where $t$ is the day counter, $x_t$ is the $d$-dimensional state variable (also referred to as the latent or unobservable factor; $r_t$ in the notation of Sec. 2), $\nu_{t-1}$ is the randomness from the state variable with covariance matrix $Q$, and $\Theta$ are the parameters. The state variable follows the dynamics described by $f$. The variable $z_t$ represents the set of $n$ observations (in our case the Italian bond term structures observed in the market; $y_t$ in the notation of Sec. 2), while $h$ is the so-called measurement function, which in our case is given by the yield pricing formula (9), and it depends on the state variable, model parameters and measurement noise $\epsilon_t$. It is standard to assume that this measurement noise is normally distributed: since we consider more than one yield observations each day, we have a multivariate normally distributed error. Although in general the measurement error covariance matrix $R$ can be a non-diagonal matrix, for the sake of
simplicity it is assumed to be diagonal in this study. Therefore the covariance structure of short-rates is represented only by the model itself and not by the measurement error covariance matrix. Note that, if some component of \( x_t \) follows a CIR dynamics, the model proposed in Eq. (10) is non-Gaussian with respect to the state variable. Nonetheless, since it is linear with respect to the measurement function, the classical Kalman filter can be used (see Chen and Scott (2003)). More details on the maximum likelihood estimation (MLE) method implemented to calibrate market data can be found in O’Sullivan (2008). Using the notation of O’Sullivan (2008), the model proposed in Eq. (10) can be written as

\[
x_t = \Phi^0 e + \Phi^1 x_{t-1} + v_{t-1}
\]

\[
z_t = H^0 e + H^1 x_t + e_t
\]

where \( e \) is a \( d \)-dimensional column vector with all components equal to 1, \( H^0 \) and \( H^1 \) are \( n \times d \) rectangular matrices, \( \Phi^0 \) and \( \Phi^1 \) are \( d \)-dimensional diagonal matrices, and for each maturity \( T_j \), with \( j = 1, \ldots, n \), and for each factor \( i \), with \( i = 1, \ldots, d \),

\[
\Phi^0_{i,i} = (1 - e^{-\kappa_i \Delta t}), \quad \Phi^1_{i,i} = e^{-\kappa_i \Delta t},
\]

\[
H^0_{j,i} = -\frac{1}{T_j} A_i(T_j), \quad H^1_{j,i} = -\frac{1}{T_j} B_i(T_j) \quad \text{and} \quad \Sigma = \text{diag}(\sigma^2_i)
\]

where \( A_i(t) \) and \( B_i(t) \) are defined in Eqs. (3) and (5), respectively. Then, we have the diagonal element \( i \) of the matrix \( Q_t \) equal to

\[
Q^i_t = \frac{\vartheta^2_i}{2 \kappa_i} (1 - e^{-2\kappa_i \Delta t}),
\]

in the Vasicek case, and

\[
Q^i_t = \frac{\vartheta^2_i \eta_i}{2 \kappa_i} (1 - e^{-\kappa_i \Delta t})^2 + \tau^i_{t-1} \frac{\vartheta^2_i}{\kappa_i} (e^{-\kappa_i \Delta t} - e^{-2\kappa_i \Delta t}),
\]

in the CIR case. With a notation similar to O’Sullivan (2008), the Kalman filter algorithm to calibrate the model defined in Eq. (11) reads as follows. The so-called prediction step is given by

\[
x_t^- = \Phi^0 e + \Phi^1 x_{t-1}
\]

\[
P_t^- = \Phi^1 P_{t-1} (\Phi^1)' + Q_t.
\]

The predicted measurement, the prediction error, and its covariance are

\[
z_t^- = H^0 e + H^1 x_t^-
\]

\[
u_t = z_t - z_t^-
\]

\[
P_{zz,t}^- = H^1 P_t^- (H^1)' + \Sigma.
\]

The filtered updates are given by

\[
K_t = P_t^- (H^1)' (P_{zz,t})^{-1}
\]

\[
x_t = x_t^- + K_t u_t
\]

\[
P_t = (I - K_t H^1) P_t^-.
\]
Finally, the log-likelihood function for the maximum likelihood estimation of the model proposed in Eq. (11) is

$$\log L(\Theta) = -\frac{nM}{2} \log 2\pi - \frac{1}{2} \sum_{t=0}^{M} \log |P_{zz,t}| - \frac{1}{2} \sum_{t=0}^{M} u_t P_{zz,t}^{-1} u_t'$$ (14)

where $\Theta$ is the set of model parameters, $M$ is the number of trading days (equal to 4435), and $n$ is the number of maturities (equal to 15). Since we are considering daily observations $\Delta t = 1/250$, that is $\Delta t M$ is equal to 17 years.

As observed by Chen and Scott (2003), if one considers a CIR factor, Eq. (11) is an approximation of the true dynamics, because the innovations in the CIR model have a noncentral $\chi^2$ distributions, in contrast to the normal distribution that is assumed for maximum likelihood estimation of the Kalman filter. For this reason, the model (10) should in principle be calibrated taking into account the non-Gaussian nature of the CIR process. For example, the expectation maximization (EM) algorithm proposed by Schön, Wills, and Ninness (2011) could be applied. This approach is not pursued in this paper.

### 3.2. The optimization algorithm

It is well-known that finding an optimal solution that maximizes the likelihood function in Eq. (14) is not a simple task. Most of the theoretical literature on this subject does not report explicitly the algorithm applied to solve the optimization problem above nor the computational time needed for the calibration. Note that in optimization packages first and second derivatives are usually approximated with finite differences, if the analytic gradient vector and the Hessian matrix are not provided. In order to speed up the optimization algorithm and to compute the standard error of the estimates in an exact way, we use the analytic expression for the gradient vector and the Hessian matrix (see Appendix A). As shown in the appendix, the analytical calculation of derivatives involving the CIR factor are more cumbersome compared to those involving the Vasicek factor.

In the optimization algorithm we constrain the parameters $(\kappa, \eta, \vartheta)$ of each factor in the region between $(1e^{-4}, 1e^{-4}, 1e^{-4})$ and $(5, 0.1, 0.5)$ for the CIR factors, or $(5, 0.1, 0.1)$ for the Vasicek factors. The parameter $\sigma_z$ ranges between $1e^{-4}$ and 0.5. The optimization algorithm applied in this study is the sequential quadratic programing method implemented in the `fmincon` MATLAB function in which the option `sqp` is selected. The analytic gradient vector of the likelihood is provided in order to speed up the algorithm. As an alternative the interior point algorithm that considers the analytic Hessian matrix can be used. As the computation of the analytic Hessian is time-consuming, we only use it to compute the standard errors of the parameters and not in the optimization algorithm. From a practical standpoint, multi-factor models based only on Vasicek factors are simpler to implement, particularly when one wants to use the analytic Hessian matrix.

---

*In a maximum likelihood estimation standard errors can be computed by taking the square root of the diagonal elements of the inverse Hessian matrix at the optimal point.*
As a starting point of the algorithm we select a random point in the parameter region. This approach seems to work for all models except for those involving more than two CIR factors. Given the presence of low (or even negative) rates, the model does not seem to perform well (see Orlando, Mininni, and Bufalo 2019a, 2019b). One should add a negative constant to the short-rate process in Eq. (7), as done for the pricing of interest rate derivatives (see Kienitz and Caspers (2017)) to raise the mean of the factors and give greater flexibility to the model. Alternatively, one can add a Vasicek factor. In this paper, we follow the latter approach. A similar approach has been considered by market practitioners to price plain-vanilla interest rate derivatives: in periods with negative interest rates the standard log-normal market model can no longer be used and the Bachelier model is implemented instead to prices caps (see Eberlein, Gerhart, and Grbac (2018)).

### 3.3. Empirical results

In Tables 1 and 2 we report, for each maturity, parameter estimates and calibration errors for different models of the Italian government bond term structure over the time horizon considered (January 1, 2002 to December 31, 2018). We present results for eight different multi-factor short-rate models with CIR and (or) Vasicek factors. In addition, Table 2 reports the value of the log-likelihood function and its gradient, as well as the first order optimality condition at the optimal point.\(^4\)

For comparative purposes, the models performance across maturities and different observation dates is evaluated with the average percentage error (APE)

\[
APE = \frac{1}{\bar{y}_{\text{market}}} \sum_{t} \sum_{T_i} \frac{|y_{t,T_i}^{\text{market}} - y_{t,T_i}^{\text{model}(\theta)}|}{\text{number of observations}},
\]

and the root mean square error (RMSE)

\[
RMSE = \sqrt{\sum_{t} \sum_{T_i} \frac{(y_{t,T_i}^{\text{market}} - y_{t,T_i}^{\text{model}(\theta)})^2}{\text{number of observations}}},
\]

where \(y_{t,T_i}\) denotes the yield to maturity \(T_i\) observed at time \(t\), and \(\bar{y}_{\text{market}}\) is the average yield across time and maturities.

Based on the APE and the RMSE evaluated over the entire sample on successive cross-sections of bond yields multi-factor models with at least a Vasicek factor perform better than one-factor models. The APE and the RMSE are larger for both CIR and Vasicek 1-factor models and smaller for three-factor models with at least a Vasicek factor. While the overall APE ranges between 16.86 (3-CIR) and 2.76% (3-Vasicek), the overall RMSE ranges between 82.92 (2-CIR) and 14.98 basis points (3-Vasicek). Both APE and RMSE values reported in Table 1 show that the calibration error depends on the maturity. For all models, the error is usually larger for the shortest maturities. Beside the CIR model and the Vasicek 1-factor show a similar dynamic for the

\(^4\)For the definition of the first order optimality condition we refer to the MATLAB documentation.
### Table 1. Calibration error of the multi-factor Gaussian models.

| Years to maturity | 0.25 | 0.50 | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 15  | 20  | 30  | Total |
|-------------------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|
| 1-CIR APE         | 24.50| 21.40| 20.65| 19.99| 18.64| 16.36| 14.95| 13.88| 13.63| 14.95| 17.51| 18.41| 23.33| 27.28| 34.88| 13.85 |
| RMSE              | 92.55| 79.10| 64.24| 42.07| 31.79| 29.89| 33.18| 35.28| 40.05| 45.92| 52.20| 54.31| 68.49| 69.73| 77.93| 57.72 |
| 1-Vasicek APE     | 25.21| 24.72| 17.84| 9.64 | 5.14 | 4.80 | 6.75 | 7.86 | 9.46 | 12.27| 13.79| 17.41| 15.56| 20.17| 24.84| 57.72 |
| RMSE              | 80.68| 70.44| 53.90| 31.18| 19.94| 17.79| 22.37| 25.37| 30.45| 35.91| 41.71| 44.61| 62.29| 66.31| 65.93| 48.84 |
| 2-CIR APE         | 19.04| 16.28| 16.58| 15.37| 15.04| 15.37| 16.04| 15.66| 16.15| 16.81| 17.43| 17.43| 14.28| 13.98| 10.72| 57.72 |
| RMSE              | 71.89| 60.56| 51.88| 39.68| 37.21| 37.29| 38.61| 37.73| 38.44| 40.25| 42.16| 42.58| 49.04| 51.67| 70.28| 48.63 |
| 2-Vasicek APE     | 6.08 | 6.02 | 3.15 | 4.91 | 5.97 | 5.84 | 5.16 | 4.00 | 3.51 | 4.00 | 4.61 | 4.37 | 4.99 | 5.61 | 8.52 | 3.93 |
| RMSE              | 26.24| 20.42| 17.71| 19.69| 19.85| 17.95| 16.83| 15.13| 14.79| 14.70| 14.13| 16.63| 21.82| 24.66| 18.62|       |
| 1-CIR + 1-Vasicek APE | 6.40 | 6.27 | 3.32 | 4.99 | 6.13 | 6.06 | 5.34 | 4.13 | 3.68 | 3.68 | 4.69 | 4.54 | 5.16 | 5.89 | 8.99 | 4.04 |
| RMSE              | 27.16| 21.06| 18.00| 19.87| 20.15| 18.39| 17.14| 15.29| 15.02| 15.52| 15.03| 14.61| 16.46| 16.00| 20.69| 18.98 |
| 3-CIR APE         | 15.84| 13.27| 17.14| 21.59| 25.45| 28.89| 31.36| 32.09| 33.58| 34.96| 35.50| 37.84| 49.41| 60.07| 86.06| 16.86 |
| RMSE              | 50.17| 38.96| 40.79| 47.06| 55.01| 61.55| 66.31| 68.24| 71.33| 73.65| 74.13| 79.58| 102.71| 123.82| 176.44| 82.92 |
| 3-Vasicek APE     | 3.33 | 3.21 | 3.45 | 3.46 | 3.98 | 2.74 | 2.65 | 2.67 | 2.97 | 2.74 | 3.41 | 2.94 | 3.15 | 3.19 | 4.64 | 2.76 |
| RMSE              | 18.17| 14.53| 17.48| 16.77| 15.35| 13.35| 12.19| 12.30| 13.58| 12.77| 13.59| 12.69| 14.29| 17.62| 17.76| 14.98 |
| 2-CIR + 1-Vasicek APE | 3.83 | 3.01 | 7.09 | 3.47 | 2.97 | 4.27 | 4.94 | 4.84 | 4.54 | 3.22 | 3.13 | 3.06 | 4.52 | 6.00 | 5.48 | 3.36 |
| RMSE              | 17.80| 14.20| 21.46| 15.44| 13.44| 13.64| 14.85| 15.84| 16.30| 13.49| 12.45| 11.93| 16.75| 23.24| 18.85| 16.28 |

We report, for each model and for each maturity, the bond term structure calibration error in the period from January, 1 2002 to December, 31 2018. The average percentage error (APE) in percentage points and the root mean square error (RMSE) in basis points are reported.
Table 2. Estimated parameters of the multi-factor Gaussian models in the period from January 1, 2002 to December 31, 2018.

| Model                  | $\kappa_1$  | $\eta_1$  | $\psi_1$  | $\kappa_2$  | $\eta_2$  | $\psi_2$  | $\kappa_3$  | $\eta_3$  | $\psi_3$  | $\sigma_\epsilon$ | $\text{LL}$ | opt.con. | AIC | $\text{AIC}/\text{LL}$ |
|------------------------|-------------|------------|------------|-------------|------------|------------|-------------|------------|------------|---------------------|------------|-----------|-----|------------------|
| 1-CIR                  | 0.0526      | 0.1000     | 0.0636     | 4.89        | -5.98e+4   | -2.23      | 4.42e-4     | 6.43e-3    | 8.21e-6    | 0.0866              | 2.486e+5   | 6.79e-1   | 4.89 |                   |
| 1-Vasicek              | 0.0854      | 0.1000     | 0.0247     | -6.54e-4    | -1.35e+5   | 1.32e-3    | 1.04e-4     | 1.27e-4    | 1.72e-5    | 0.0768              | 2.587e+5   | -6.58e-4 | 1.32e-3 |                   |
| 2-CIR                  | 0.0001      | 0.0465     | 0.0361     | -1.73e-3    | -4.67e+4   | -7.66e-2   | 1.04e-8     | 1.16e-2    | 3.73e-6    | 0.0750              | 2.470e+5   | 0.0750    | 2.407e+5 |                   |
| 2-Vasicek              | 0.2521      | 0.1000     | 0.0525     | -7.46e+1    | -1.23e+4   | 2.66e+1    | 3.95e-6     | 1.50e-4    | 6.86e-6    | 0.0750              | 2.536e+5   | 0.0750    | 2.385e+5 |                   |
| 1-CIR + 1-Vasicek      | 0.0268      | 0.1000     | 0.0543     | -8.23e+4    | -1.46e+5   | -8.60e+4   | 8.79e-9     | 3.59e-2    | 1.45e-5    | 0.0999              | 3.699e+5   | 0.0999    | 3.699e+5 |                   |
| 3-CIR                  | 0.0001      | 0.0803     | 0.1656     | -2.54e+6    | -2.48e+3   | -1.46e+5   | 8.78e-9     | 3.59e-2    | 1.45e-5    | 0.0803              | 3.699e+5   | 0.0344    | 0.4977 |                   |
| 3-Vasicek              | 0.2120      | 0.1000     | 0.0113     | -3.72e+5    | -8.60e+4   | -1.80e+6   | 2.99        | -1.63e+4   | -5.09e-1   | 0.0113              | 0.4977      | 0.0113    | 0.358e+5 |                   |
| 2-CIR + 1-Vasicek      | 0.4668      | 0.0077     | 0.2223     | -6.35e+4    | -5.74e+3   | -1.20e+5   | 1.12e+3     | 7.17e-4    | 4.37e-6    | 0.0492              | 3.356e+5   | 0.0126    | 0.358e+5 |                   |

We report, for each model the estimated parameters, the corresponding value of the gradient and the standard error. The value of the likelihood (LL), of the first order optimality condition at the optimal point (opt.con), and of the AIC are shown in the last column.
unobservable factor and a similar calibration error, the estimated parameter differs (see Table 2).

As expected, the analysis shows that the models with more than one factor have a better performance in terms of calibration error. On the one hand, those including a Vasicek factor have a smaller calibration error, due to a higher degree of flexibility. However, the use of the Vasicek factor does not ensure that the short-rate remains positive, even if by construction its mean value is strictly positive. On the other hand, while multi-factor CIR models generate only positive factors, they are less flexible, particularly

Figure 1. Estimated factors and short-rate process $r_t$ on the Italy bond yields from January 1, 2002 to December 31, 2018. Bond maturities range from 3 months to 30 years. Recall that the black line is the sum of the unobservable factors, that is the estimated short rate $r_t$. We report the estimated factors for all models investigated. The Kalman filter is considered to extract the unobservable short-rate process. For the model 1-CIR (2-CIR) + 1-Vasicek, the CIR factor is the positive one (the CIR factors are the positive ones).
when interest rates approach the zero lower bound or become negative. In this case it may happen that for some multi-factor models the optimal parameters hit the boundaries and this results in a value of the gradient that is far from zero. In addition, for CIR factors some numerical errors (i.e., floating-point approximation errors) may appear in the evaluation of the inverse of the Hessian matrix and this results in negative values (although small in absolute value) for the square of the standard errors. As expected, the model with three CIR factors shows a very poor performance.

To assess the performance of different models, we apply the Akaike information criterion (AIC), defined as

$$AIC = 2np - 2 \log L$$

where $np$ is the number of parameters and $\log L$ is the model log-likelihood. According to the AIC reported in Table 2, the 3-factor Vasicek model performs better than the other models considered in our analysis, i.e., it has the smallest AIC value. The Bayesian information criterion (BIC) gives the same ranking of models is therefore omitted.

Figure 1 shows the dynamics of the factors for the eight models analyzed in this paper. The value of sum of the factors (in black in Figure 1) is the short-rate $r_t$ in Eq. (7) estimated on the entire term structure (with maturities from 3 months to 30 years) by following the approach described in Sec. 3.1. The short-rate captures the level of the

Figure 2. Calibration of the best performing short-rate model (3-Vasicek) on the Italy bond yields from January 1, 2002 to December 31, 2018. We report the observed bond rates and the estimated ones for all maturities investigated. The Kalman filter is considered to extract the unobservable short-rate process.
shortest maturity rates while the other parameters capture the average slope and curvature of the term structure throughout the sample period. It appears clear that if one considers more than one CIR factor, there are factors that hit the zero lower bound. This makes the calibration of these models computationally more challenging, as confirmed by the fact that the optimal parameters depend on the starting point in the optimization. Conversely, if one considers models with at least one Vasicek factor, this becomes negative, even if the sum of the factors (the black line) is positive or slightly negative. In these cases the optimization algorithm is more robust and the optimal solution does not depend on the initial point.

In order to have a visual assessment of the fitting exercise, in Figure 2 the estimated rates of the best performing model (i.e., the model with three Vasicek factors) are compared with the market ones. Furthermore, in Figure 3 we show the corresponding calibration errors. On the last trimester of 2011 and on the second trimester of 2018 the model reaches the largest calibration error. We are aware of the fact that Gaussian models are not able to explain sharp movements in the market, even if the performance over the entire observation period is satisfactory.

In addition to estimating the eight models, we empirically study their forecasting performance over time, by considering 10,000 Monte Carlo simulations. Based on the estimates in the period from January 1, 2002 to December 31, 2017, we compare the

\[\text{Figure 3. Calibration errors of the best performing short-rate model (3-Vasicek) on the Italy bond yields from January 1, 2002 to December 31, 2018. We report the calibration errors for all maturities investigated. The Kalman filter is considered to extract the unobservable short-rate process.}\]

\[\text{3700 M. L. BIANCHI}\]

\[\text{The estimated parameters and the performance of the calibration conducted on this subsample are not reported.}\]
simulated yield with the data observed in the market until December 31, 2018. In Figure 4 we show the 5 (10, 20, 60, 120, and 260) trading days ahead forecasts. For each model, we report the mean, the 5th and the 95th percentile based on 10,000 simulated scenarios. Figure 4 shows that the forecasting performance of the 1-factor model is poor. The same is true for the models with only CIR factors. The addition of at least one Vasicek factor in multi-factor models largely improve the forecasting performance, even if negative values for short-term maturities are allowed. As expected, the forecasting performance decreases and the volatility of the forecast increases after 60 trading days. At least for the data considered in this study, some models show a satisfactory 3-month ahead forecasts. The 6- and 12-month ahead forecast are far from the yield curve observed in the market, even if they are still satisfactory for the multi-factor models with at least a Vasicek factor.

4. Conclusion

The objective of this paper is twofold. First, we provide a maximum likelihood optimization algorithm based on the Kalman filter in which both the gradient vector and the Hessian matrix can be computed in closed-form. Second, we explore the calibration performance of multi-factor models driven by independent univariate CIR and Vasicek processes under the risk-neutral measure in fitting the Italian term structure. As already observed in other empirical studies, at least three factors are necessary for a satisfactory representation of the behavior of yield curves. This seems a good compromise between

Figure 4. Simulated and observed Italy bond yields. We report the 5 (10, 20, 60, 120, and 260) trading days ahead forecast in the period from December 31, 2017 to December 31, 2018. For each model, we consider the mean, the 5th and the 95th percentile based on 10,000 simulated scenarios.
a satisfactory performance in terms of calibration error and parameter parsimony. Based on the data analyzed in this paper, we show that at least a Vasicek factor is needed to properly calibrate the dynamics of the yield curve. If even the Vasicek factor may become negative, the short-rate process, defined as the sum of the factors, remains positive (or slightly negative) and its expected mean is strictly positive. The calibration of multi-factor CIR models is affected by the presence of low level (near zero) rates, and these observed patterns complicate the convergence properties of the optimization algorithm. Finally, the multi-factor models with at least a Vasicek factor seem to show a better forecasting performance.

Appendix A

A.1. Gradient of the likelihood function

Here we provide the formulas to compute the gradient of the likelihood function to use in the optimization algorithm. For a generic parameter $\xi \in \Theta$, we have that
\[
\frac{\partial \log L(\Theta)}{\partial \xi} = -\frac{1}{2} \sum_{t=0}^{T} \left\{ 2 \frac{\partial u_t}{\partial \xi} P_{zt}^{-1} u'_t + \text{tr}\left( P_{zt}^{-1} \frac{\partial P_{zt}}{\partial \xi} P_{zt}^{-1} P_{zt}^{-1} u'_t \right) \right\}.
\] (A.1)

The first gradient in Eq. (A.1) is given by
\[
\frac{\partial u_t}{\partial \xi} = -\frac{\partial H_0}{\partial \xi} e - \frac{\partial H_1}{\partial \xi} x_t - \frac{\partial x_t}{\partial \xi} e + \frac{\partial \Phi^1}{\partial \xi} x_{t-1} + \Phi^1 \frac{\partial x_{t-1}}{\partial \xi}.
\]

In both the Vasicek and the CIR case we have
\[
\frac{\partial \Phi^0}{\partial \xi} = (\eta \Delta t e^{-\kappa \Delta t} \ 1 - e^{-\kappa \Delta t} \ 0)^' \] and
\[
\frac{\partial \Phi^1}{\partial \xi} = (-\Delta t e^{-\kappa \Delta t} \ 0 \ 0)^'.
\]

The partial derivatives of $H_0$ and $H_1$ are given in Sec. A.3. More challenging is the computation of the partial derivatives of $P_{zt}$, These involves both the factors parameters and the measurement error parameters.
\[
\frac{\partial P_{zt}}{\partial \xi} = \frac{\partial H_1}{\partial \xi} P_t (H_1)' + H_1 \frac{\partial P_t}{\partial \xi} (H_1)' + H_1 P_t \left( \frac{\partial H_1}{\partial \xi} \right)' + \frac{\partial \Sigma}{\partial \xi},
\]
where the partial derivatives of $H_1$ are given in Sec. A.3,
\[
\frac{\partial P_t}{\partial \xi} = \Phi^1 \frac{\partial P_{t-1}}{\partial \xi} (\Phi^1)' + \Phi^1 \frac{\partial P_{t-1}}{\partial \xi} (\Phi^1)' + \Phi^1 P_{t-1} \left( \frac{\partial \Phi^1}{\partial \xi} \right)' + \frac{\partial Q_t}{\partial \xi},
\]
and the derivatives of $\Sigma$ can be computed by considering that the matrix is diagonal with all element equal to $\sigma^2_t$. Furthermore, the filtered updates are given by
\[
\frac{\partial x_t}{\partial \xi} = \frac{\partial x_t}{\partial \xi} + \frac{\partial K_t}{\partial \xi} u_t + K_t \frac{\partial u_t}{\partial \xi},
\]
\[
\frac{\partial P_t}{\partial \xi} = \frac{\partial P_t}{\partial \xi} - \frac{\partial K_t}{\partial \xi} H_1 P_t' - K_t \frac{\partial H_1}{\partial \xi} P_t' - K_t H_1 \frac{\partial P_t}{\partial \xi},
\]
with
\[
\frac{\partial K_t}{\partial \xi} = \frac{\partial P_t}{\partial \xi} (H^t)^{(P_{zz,t})^{-1}} + P_t \frac{\partial (H^t)^{(P_{zz,t})^{-1}}}{\partial \xi} (P_{zz,t})^{-1} - P_t (H^t)^{(P_{zz,t})^{-1}} \frac{\partial P_{zz,t}}{\partial \xi} (P_{zz,t})^{-1}.
\]

The partial derivatives of \( Q_t \) for a Vasicek factor \( r \) is
\[
\frac{\partial Q_t}{\partial \xi} = \frac{\partial}{\partial \xi} \left\{ \frac{\varrho^2}{2\kappa} \left( 1 - e^{-2\kappa \Delta t} \right) \right\} = \frac{\partial q_v}{\partial \xi},
\]
where
\[
\frac{\partial q_v}{\partial \kappa} = -\frac{\varrho^2}{2\kappa^2} (1 - e^{-2\kappa \Delta t}) + \frac{\varrho^2 \Delta t}{\kappa} e^{-2\kappa \Delta t}
\]
\[
\frac{\partial q_v}{\partial \eta} = 0
\]
\[
\frac{\partial q_v}{\partial \theta} = \frac{\varrho}{\kappa} (1 - e^{-2\kappa \Delta t})
\]

The partial derivatives of \( Q_t \) for a CIR factor \( r \) is
\[
\frac{\partial Q_t}{\partial \xi} = \frac{\partial}{\partial \xi} \left\{ \frac{\varrho^2 \eta}{2\kappa} \left( 1 - e^{-\kappa \Delta t} \right)^2 + r_t - 1 \frac{\varrho^2}{\kappa} \left( e^{-\kappa \Delta t} - e^{-2\kappa \Delta t} \right) \right\}
\]
\[
= \frac{\partial}{\partial \xi} \left\{ q_{c1} + r_t - 1 q_{c2} \right\} = \frac{\partial q_{c1}}{\partial \xi} + \frac{\partial r_t - 1}{\partial \xi} q_{c2} + r_t - 1 \frac{\partial q_{c2}}{\partial \xi}
\]
where
\[
\frac{\partial q_{c1}}{\partial \kappa} = -\frac{\varrho^2 \eta}{2\kappa} (1 - e^{-\kappa \Delta t})^2 + \frac{\varrho^2 \eta \Delta t}{\kappa} (e^{-\kappa \Delta t} - e^{-2\kappa \Delta t})
\]
\[
\frac{\partial q_{c1}}{\partial \eta} = \frac{\varrho^2}{2\kappa} (1 - e^{-\kappa \Delta t})^2
\]
\[
\frac{\partial q_{c1}}{\partial \theta} = \frac{\varrho \eta}{\kappa} (1 - e^{-\kappa \Delta t})^2
\]
and
\[
\frac{\partial q_{c2}}{\partial \kappa} = -\frac{\varrho^2}{\kappa^2} (e^{-\kappa \Delta t} - e^{-2\kappa \Delta t}) - \frac{\varrho^2}{\kappa} (\Delta t e^{-\kappa \Delta t} - 2 \Delta t e^{-2\kappa \Delta t})
\]
\[
\frac{\partial q_{c2}}{\partial \eta} = 0
\]
\[
\frac{\partial q_{c2}}{\partial \theta} = \frac{2\varrho}{\kappa} (e^{-\kappa \Delta t} - e^{-2\kappa \Delta t}).
\]

By considering that the initial point of the KF algorithm are
\[
x_0 = \eta \quad \text{and} \quad P_0 = \frac{\varrho^2}{2\kappa}
\]
in the Vasicek case, and
\[
x_0 = \eta \quad \text{and} \quad P_0 = \frac{\eta \varrho^2}{2\kappa}
\]
in the CIR case, the derivatives with respect to a generic parameter \( \xi \) are simple to compute, that is in both the Vasicek and the CIR case we have
\[
\frac{\partial x_0}{\partial \xi} = \left( \begin{array}{ccc} 0 & 1 & 0 \end{array} \right)
\]
and
\[
\frac{\partial P_0}{\partial \xi} = \left( -\frac{\psi^2}{2\kappa^2} \ 0 \ \frac{\psi}{\kappa} \right)'
\]
in the Vasicek case, and
\[
\frac{\partial P_0}{\partial \xi} = \left( -\frac{\partial \eta}{2\kappa} \ \frac{\partial^2 \eta}{2\kappa} \ 0 \ \frac{\partial \eta}{\kappa} \right)'
\]
in the CIR case. These derivatives are needed as initial point of the algorithm that compute the gradient on the likelihood function to maximize.

### A.2. Hessian of the likelihood function

Here we provide the formulas to compute the Hessian of the likelihood function to use in the optimization algorithm. For generic parameters \(\xi_1, \xi_2 \in \Theta\), we have that
\[
\frac{\partial \log L(\Theta)}{\partial \xi_1 \partial \xi_2} = -\frac{1}{2} \sum_{t=0}^{n} \left\{ 2 \frac{\partial u_t}{\partial \xi_1} P_{z,t}^{-1} u_t' - 2 \frac{\partial u_t}{\partial \xi_2} P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} u_t' + 2 \frac{\partial u_t}{\partial \xi_1} P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_1} u_t' \right. \\
- \text{tr} \left( P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} \right) + \text{tr} \left( P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_1} \frac{\partial P_{z,t}}{\partial \xi_2} \right) \\
- 2 \frac{\partial u_t}{\partial \xi_2} P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} u_t' - u_t \frac{\partial}{\partial \xi_2} \left( P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} \right) u_t' \left\} \right.
\]
(A.2)

where
\[
\frac{\partial}{\partial \xi_2} \left( P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} P_{z,t}^{-1} \right) = -P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} - P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} P_{z,t}^{-1}
\]
\[
+ P_{z,t}^{-1} \frac{\partial P_{z,t}}{\partial \xi_2} P_{z,t}^{-1}
\]

Then, we need to compute only the following second derivatives \(\frac{\partial u_t}{\partial \xi_1 \partial \xi_2}\) and \(\frac{\partial u_t}{\partial \xi_1 \partial \xi_2}\), since all other derivatives have been already reported in Sec. A.1. By Sec. A.1 we have that
\[
\frac{\partial u_t}{\partial \xi_1 \partial \xi_2} = -\frac{\partial H^0}{\partial \xi_1 \partial \xi_2} e - \frac{\partial H^1}{\partial \xi_1 \partial \xi_2} x_t - \frac{\partial H^2}{\partial \xi_1 \partial \xi_2} x_t - \frac{\partial H^3}{\partial \xi_1 \partial \xi_2} x_t - \frac{\partial H^4}{\partial \xi_1 \partial \xi_2} x_t - \frac{\partial H^5}{\partial \xi_1 \partial \xi_2} x_t + \Phi^0 \frac{\partial x_t}{\partial \xi_1 \partial \xi_2} + \Phi^1 \frac{\partial x_t}{\partial \xi_1 \partial \xi_2} + \Phi^2 \frac{\partial x_t}{\partial \xi_1 \partial \xi_2} + \Phi^3 \frac{\partial x_t}{\partial \xi_1 \partial \xi_2}
\]

In both the Vasicek and the CIR case we have
\[
\frac{\partial \Phi^0}{\partial \xi_1 \partial \xi_2} = \begin{pmatrix}
-\eta \Delta t e^{-\kappa \Delta t} & te^{-\kappa \Delta t} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and
\[
\frac{\partial \Phi^1}{\partial \xi_1 \partial \xi_2} = \begin{pmatrix}
\Delta t^2 e^{-\kappa \Delta t} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The second derivatives of \(H^0\) and \(H^1\) are given in Sec. A.3. More challenging is the computation of the Hessian of \(P_{z,t}\). These involves both the factors parameters and the measurement error parameters.
\[
\frac{\partial P_{zz,t}}{\partial \xi_1 \partial \xi_2} = \frac{\partial H^1}{\partial \xi_1 \partial \xi_2} P_t^\prime (H^1) + \frac{\partial H^1}{\partial \xi_1} \frac{\partial P_t}{\partial \xi_2} (H^1) + \frac{\partial H^1}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\frac{\partial H^1}{\partial \xi_2}) + H^1 \frac{\partial P_t}{\partial \xi_1} (\frac{\partial H^1}{\partial \xi_2}) + \frac{\partial}{\partial \xi_2} (\frac{\partial H^1}{\partial \xi_2}) + \frac{\partial}{\partial \xi_1 \partial \xi_2}
\]

where, as already observed, the second derivatives of \( H^1 \) are given in Sec. A.3, and the derivatives of \( \Sigma \) can be computed by considering that the matrix is diagonal with all elements equal to \( \sigma_e^2 \).

Then we have

\[
\frac{\partial P_t}{\partial \xi_1 \partial \xi_2} = \frac{\partial \Phi^1}{\partial \xi_1 \partial \xi_2} P_{t-1}(\Phi^1) + \frac{\partial \Phi^1}{\partial \xi_1} \frac{\partial P_{t-1}}{\partial \xi_2} (\Phi^1) + \frac{\partial \Phi^1}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (\frac{\partial \Phi^1}{\partial \xi_2}) + \Phi^1 \frac{\partial P_{t-1}}{\partial \xi_1} (\frac{\partial \Phi^1}{\partial \xi_2}) + \frac{\partial}{\partial \xi_2} (\frac{\partial \Phi^1}{\partial \xi_2}) + \frac{\partial Q_t}{\partial \xi_1 \partial \xi_2}
\]

and, furthermore, the filtered updates are given by

\[
\frac{\partial \alpha_t}{\partial \xi_1 \partial \xi_2} = \frac{\partial \alpha_t}{\partial \xi_1 \partial \xi_2} + \frac{\partial K_t}{\partial \xi_1} \frac{\partial u_t}{\partial \xi_2} + \frac{\partial K_t}{\partial \xi_2} \frac{\partial u_t}{\partial \xi_1} + K_t \frac{\partial u_t}{\partial \xi_1 \partial \xi_2}
\]

\[
\frac{\partial P_t}{\partial \xi_1 \partial \xi_2} = \frac{\partial P_t}{\partial \xi_1 \partial \xi_2} - \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} (H^1) P_t - \frac{\partial}{\partial \xi_1} \frac{\partial H^1}{\partial \xi_2} P_t - \frac{\partial}{\partial \xi_2} \frac{\partial H^1}{\partial \xi_1} P_t - \frac{\partial}{\partial \xi_1 \partial \xi_2} (H^1) P_t - \frac{\partial}{\partial \xi_1 \partial \xi_2} (H^1) P_t
\]

\[
\frac{\partial}{\partial \xi_1 \partial \xi_2} (H^1) P_t - \frac{\partial}{\partial \xi_2} \frac{\partial H^1}{\partial \xi_1} P_t - \frac{\partial}{\partial \xi_2} \frac{\partial H^1}{\partial \xi_2} P_t - \frac{\partial}{\partial \xi_1 \partial \xi_2} \frac{\partial H^1}{\partial \xi_1} P_t - \frac{\partial}{\partial \xi_1 \partial \xi_2} \frac{\partial H^1}{\partial \xi_2} P_t
\]

The second derivatives of \( q_v \), for a Vasicek factor \( r \) are

\[
\frac{\partial q_v}{\partial \kappa} = \frac{\kappa}{\kappa^2} (1 - e^{-2\kappa \Delta t}) - \frac{2 \kappa^2 \Delta t e^{-2\kappa \Delta t}}{\kappa^2} - \frac{2 \kappa^2 \Delta t^2 e^{-2\kappa \Delta t}}{\kappa}
\]

and all other second derivatives are zero. The second derivatives of \( Q_t \), for a CIR factor \( r \) are

\[
\frac{\partial Q_t}{\partial \xi_1 \partial \xi_2} = \frac{\partial q_{x_1}}{\partial \xi_1 \partial \xi_2} + \frac{\partial r_{x_1-1}}{\partial \xi_1 \partial \xi_2} \frac{\partial q_{x_2}}{\partial \xi_2} + \frac{\partial r_{x_1-2}}{\partial \xi_1 \partial \xi_2} \frac{\partial q_{x_2}}{\partial \xi_2} + \frac{\partial r_{x_1-1} \partial q_{x_2}}{\partial \xi_1 \partial \xi_2} + \frac{\partial q_{x_2}}{\partial \xi_1 \partial \xi_2}
\]
As observed in Eq. (13) we have that

\[ H^0_{i,j} = -\frac{1}{T_j} A_i(T_j), \quad H^1_{i,j} = -\frac{1}{T_j} B_i(T_j) \]

Therefore, to compute the gradient and the Hessian with respect to the model parameters it is enough to compute the gradient and the Hessian of \( A(t) \) and \( B(t) \).
A.3.1. Vasicek model
In the Vasicek case, we have
\[
\begin{align*}
z_{\text{model}} &= -\frac{1}{t} (A(t) + B(t)r_0) \\
A(t) &= -\eta t - \eta g(t) - \frac{\theta^2}{4\kappa} (g(t))^2 + \frac{\theta^2}{2\kappa^2} \left( t + g(t) \right), \\
B(t) &= g(t) = -\frac{1 - e^{-\kappa t}}{\kappa}.
\end{align*}
\]
It follows that the first derivatives to compute the gradient are
\[
\begin{align*}
\frac{\partial g(t)}{\partial \kappa} &= \frac{1 - e^{-\kappa t} - t \kappa e^{-\kappa t}}{\kappa^2}, \\
\frac{\partial g(t)}{\partial \eta} &= \frac{\partial g(t)}{\partial \vartheta} = 0, \\
\frac{\partial A(t)}{\partial \kappa} &= -\eta \frac{\partial g(t)}{\partial \kappa} - \frac{\theta^2}{2\kappa} g(t) \frac{\partial g(t)}{\partial \kappa} + \frac{\theta^2}{4\kappa^2} \left( g(t) \right)^2 \\
&\quad + \frac{\theta^2}{2\kappa^2} \frac{\partial g(t)}{\partial \kappa} - \frac{\theta^2}{\kappa^3} \left( t + g(t) \right), \\
\frac{\partial A(t)}{\partial \eta} &= -t - g(t), \\
\frac{\partial A(t)}{\partial \vartheta} &= -\frac{\vartheta}{2\kappa} \left( g(t) \right)^2 + \frac{\vartheta}{\kappa^2} \left( t + g(t) \right)
\end{align*}
\]
The second derivatives to compute the Hessian are
\[
\frac{\partial^2 g(t)}{\partial \kappa^2} = \frac{t^2 \kappa^2 e^{-\kappa t} - 2 \left( 1 - e^{-\kappa t} \right) + 3 \kappa \kappa e^{-\kappa t}}{\kappa^3},
\]
all other second derivatives of \( g(t) \) are zero, and
\[
\begin{align*}
\frac{\partial A(t)}{\partial \kappa^2} &= -\eta \frac{\partial g(t)}{\partial \kappa^2} + \frac{\theta^2}{2\kappa^2} g(t) \frac{\partial g(t)}{\partial \kappa} - \frac{\theta^2}{2\kappa} \left( g(t) \right) \frac{\partial g(t)}{\partial \kappa} - \frac{\theta^2}{2\kappa^2} \left( g(t) \right)^2 \\
&\quad + \frac{\theta^2}{2\kappa^2} g(t) \frac{\partial g(t)}{\partial \kappa} - \frac{\theta^2}{2\kappa^2} \frac{\partial g(t)}{\partial \kappa} + \frac{\theta^2}{2\kappa^2} \frac{\partial g(t)}{\partial \kappa} + \frac{3\theta^2}{\kappa^4} \left( t + g(t) \right) - \frac{\vartheta^2}{\kappa^3} \frac{\partial g(t)}{\partial \kappa}, \\
\frac{\partial A(t)}{\partial \kappa \partial \eta} &= -\frac{\partial g(t)}{\partial \kappa}, \\
\frac{\partial A(t)}{\partial \kappa \partial \vartheta} &= -\frac{\vartheta}{\kappa} g(t) \frac{\partial g(t)}{\partial \kappa} + \frac{\vartheta}{2\kappa^2} \left( g(t) \right)^2 + \frac{\vartheta}{\kappa^2} \frac{\partial g(t)}{\partial \kappa} - \frac{2\vartheta}{\kappa^3} \left( t + g(t) \right), \\
\frac{\partial A(t)}{\partial \eta \partial \vartheta} &= \frac{\partial A(t)}{\partial \eta \partial \vartheta} = 0, \\
\frac{\partial A(t)}{\partial \eta^2} &= -\frac{1}{2\kappa} \left( g(t) \right)^2 + \frac{1}{\kappa^2} \left( t + g(t) \right).
\end{align*}
\]

A.3.2. CIR model
In the CIR case, we have
\[
\begin{align*}
z_{\text{model}} &= -\frac{1}{t} (A(t) + B(t)r_0) \\
A(t) &= -\eta t - \eta g(t) - \frac{\vartheta^2}{4\kappa} (g(t))^2 + \frac{\vartheta^2}{2\kappa^2} \left( t + g(t) \right), \\
B(t) &= g(t) = -\frac{1 - e^{-\kappa t}}{\kappa}.
\end{align*}
\]
where
\[
A(t) = \frac{2k\eta}{\partial^2} \left( \log(2) + \log(b) - \log(\kappa - b) + (\kappa + b)t/2 - \log(ae^{bt} - 1) \right),
\]
\[
B(t) = -\frac{2}{\kappa - b} \left( \frac{e^{bt} - 1}{ae^{bt} - 1} \right),
\]
\[
a = \frac{\kappa + b}{\kappa - b} \quad \text{and} \quad b = \sqrt{\kappa^2 + 2\eta^2}.
\]
It follows that
\[
\frac{\partial b}{\partial \kappa} = \frac{\kappa}{\sqrt{\kappa^2 + 2\eta^2}} \quad \frac{\partial b}{\partial \eta} = 0 \quad \frac{\partial b}{\partial \theta} = \frac{2\theta}{\sqrt{\kappa^2 + 2\theta^2}}
\]
and
\[
\frac{\partial a}{\partial \kappa} = \frac{(\kappa - b) \left( 1 + \frac{\partial b}{\partial \kappa} \right) - (\kappa + b) \left( 1 - \frac{\partial b}{\partial \kappa} \right)}{(\kappa - b)^2} = \frac{2 \left( \frac{\partial b}{\partial \kappa} - b \right)}{(\kappa - b)^2}
\]
\[
\frac{\partial a}{\partial \eta} = 0
\]
\[
\frac{\partial a}{\partial \theta} = \frac{(\kappa - b) \frac{\partial b}{\partial \kappa} + (\kappa + b) \frac{\partial b}{\partial \eta}}{(\kappa - b)^2} = \frac{2k \frac{\partial b}{\partial \kappa}}{(\kappa - b)^2}
\]
Thus, by setting
\[
f = \log(2) + \log(b) - \log(\kappa - b) + (\kappa + b)t/2 - w
\]
\[
w = \log(ae^{bt} - 1),
\]
the equalities
\[
\frac{\partial f}{\partial \kappa} = \frac{1}{b} \frac{\partial b}{\partial \kappa} - \frac{1}{k - b} \left( 1 - \frac{\partial b}{\partial \kappa} \right) + \left( 1 + \frac{\partial b}{\partial \kappa} \right) \frac{t}{2} - \frac{\partial w}{\partial \kappa}
\]
\[
\frac{\partial f}{\partial \eta} = 0
\]
\[
\frac{\partial f}{\partial \theta} = \frac{1}{b} \frac{\partial b}{\partial \theta} + \frac{1}{k - b} \frac{\partial b}{\partial \theta} + \frac{t}{2} \frac{\partial b}{\partial \theta} - \frac{\partial w}{\partial \theta}
\]
\[
\frac{\partial w}{\partial \kappa} = \frac{\partial b}{\partial \kappa} \frac{e^{bt}}{ae^{bt} - 1}
\]
follow and we obtain for \(A(t)\) the following partial derivatives
\[
\frac{\partial A(t)}{\partial \kappa} = \frac{2\eta}{\partial^2} f + \frac{2k\eta}{\partial^2} \frac{\partial f}{\partial \kappa}
\]
\[
\frac{\partial A(t)}{\partial \eta} = \frac{2k}{\partial^2} f
\]
\[
\frac{\partial A(t)}{\partial \theta} = -\frac{4k\eta}{\partial^2} f + \frac{2k\eta}{\partial^2} \frac{\partial f}{\partial \theta},
\]
and for \(B(t)\)
\[
\frac{\partial B(t)}{\partial \kappa} = \frac{2h}{(\kappa - b)^2} \left( 1 - \frac{\partial b}{\partial \kappa} \right) - \frac{2}{\kappa - b} \frac{\partial h}{\partial \kappa}
\]
\[
\frac{\partial B(t)}{\partial \eta} = 0
\]
\[
\frac{\partial B(t)}{\partial \vartheta} = -\frac{2h}{(\kappa - b)^2} \frac{\partial b}{\partial \vartheta} - \frac{2}{\kappa - b} \frac{\partial h}{\partial \vartheta}
\]

where

\[
h = \frac{e^{bt} - 1}{ae^{bt} - 1}
\]

and for the generic parameter \( \zeta \in \{ \kappa, \vartheta \} \)

\[
\frac{\partial h}{\partial \zeta} = \frac{e^{bt} \left( -t \frac{\partial h}{\partial \zeta} - e^{bt} \frac{\partial a}{\partial \zeta} + \frac{\partial a}{\partial \zeta} + at \frac{\partial h}{\partial \zeta} \right)}{(ae^{bt} - 1)^2}.
\]

A little bit more tedious are the computation of the second derivatives. More in details, we have

\[
\frac{\partial b}{\partial \kappa^2} = \frac{2\vartheta^2}{(\kappa^2 + 2\vartheta^2)^2}
\]
\[
\frac{\partial b}{\kappa \vartheta} = \frac{\partial b}{\partial \vartheta} = \frac{\partial b}{\partial \kappa \vartheta} = \frac{2}{\kappa^2 + 2\vartheta^2} \frac{2\kappa \vartheta}{(\kappa - b)^3}
\]
\[
\frac{\partial b}{\partial \vartheta^2} = \frac{2\kappa^2}{(\kappa^2 + 2\vartheta^2)^2}
\]

with all other second derivatives equal to zero, and

\[
\frac{\partial a}{\partial \kappa^2} = \frac{2k \frac{\partial h}{\partial \kappa} (\kappa - b) - 4 \left( 1 - \frac{\partial b}{\partial \kappa} \right) \left( \frac{\partial b}{\partial \kappa} - b \right)}{(\kappa - b)^3}
\]
\[
\frac{\partial a}{\kappa \vartheta} = \frac{\partial a}{\partial \vartheta} = \frac{2 \left( \frac{\partial h}{\partial \kappa \vartheta} \right) (\kappa - b) + 2 \frac{\partial b}{\partial \vartheta} \left( 2k \frac{\partial h}{\partial \kappa} - b - k \right)}{(\kappa - b)^3}
\]
\[
\frac{\partial a}{\partial \vartheta^2} = \frac{2k \frac{\partial h}{\partial \kappa} (\kappa - b) + 4k \left( \frac{\partial h}{\partial \kappa} \right)^2}{(\kappa - b)^3}
\]

with all other second derivatives equal to zero. Then, by setting

\[
\frac{\partial f}{\partial \kappa^2} = -\frac{1}{b^2} \left( \frac{\partial b}{\partial \kappa} \right)^2 + \frac{1}{b} \frac{\partial b}{\partial \kappa^2} + \frac{1}{(k - b)^2} \left( 1 - \frac{\partial b}{\partial \kappa} \right)^2 + \frac{1}{k - b} \frac{\partial b}{\partial \kappa^2} + \frac{t}{2} \frac{\partial b}{\partial \kappa^2} - \frac{\partial w}{\partial \kappa^2}
\]
\[
\frac{\partial f}{\kappa \vartheta} = \frac{\partial f}{\partial \kappa \vartheta} = -\frac{1}{b} \frac{\partial b}{\partial \kappa \vartheta} + \frac{1}{b} \frac{\partial b}{\partial \kappa \vartheta} + \frac{1}{(k - b)^2} \left( 1 - \frac{\partial b}{\partial \kappa} \right) \frac{\partial b}{\partial \vartheta} + \frac{t}{2} \frac{\partial b}{\partial \kappa \vartheta} - \frac{\partial w}{\partial \kappa \vartheta}
\]
\[
+ \frac{1}{k - b} \frac{\partial b}{\partial \kappa \vartheta} + \frac{t}{2} \frac{\partial b}{\partial \kappa \vartheta} - \frac{\partial w}{\partial \kappa \vartheta}
\]
\[
\frac{\partial f}{\partial \vartheta^2} = -\frac{1}{b^2} \left( \frac{\partial b}{\partial \vartheta} \right)^2 + \frac{1}{b} \frac{\partial b}{\partial \vartheta^2} + \frac{1}{(k - b)^2} \left( \frac{\partial b}{\partial \vartheta} \right)^2 + \frac{1}{k - b} \frac{\partial b}{\partial \vartheta^2} + \frac{t}{2} \frac{\partial b}{\partial \vartheta^2} - \frac{\partial w}{\partial \vartheta^2}
\]

with
\[
\frac{\partial w}{\partial k^2} = \frac{\partial^2 e^{kt} + 2 \frac{\partial}{\partial t} e^{kt} \frac{\partial}{\partial k} + a t^2 e^{kt} \left( \frac{\partial}{\partial k} \right)^2 + a t e^{kt} \frac{\partial}{\partial k}}{(ae^{kt} - 1)}
- \left( \frac{\partial e^{kt} + a t e^{kt} \frac{\partial}{\partial k}}{ae^{kt} - 1} \right)^2
\]
\[
\frac{\partial w}{\partial k \partial \theta} = \frac{\partial w}{\partial \theta \partial k} = \frac{\partial^2 e^{kt} + \frac{\partial}{\partial t} e^{kt} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta} e^{kt} \frac{\partial}{\partial k} + a t^2 e^{kt} \left( \frac{\partial}{\partial \theta} \right)^2 + a t e^{kt} \frac{\partial}{\partial \theta} \frac{\partial}{\partial k}}{(ae^{kt} - 1)}
- \left( \frac{\partial e^{kt} + a t e^{kt} \frac{\partial}{\partial \theta}}{ae^{kt} - 1} \right)^2
\]
\[
\frac{\partial w}{\partial \theta^2} = \frac{\partial^2 e^{kt} + 2 \frac{\partial}{\partial \theta} e^{kt} \frac{\partial}{\partial \theta} + a t^2 e^{kt} \left( \frac{\partial}{\partial \theta} \right)^2 + a t e^{kt} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}}{(ae^{kt} - 1)}
- \left( \frac{\partial e^{kt} + a t e^{kt} \frac{\partial}{\partial \theta}}{ae^{kt} - 1} \right)^2
\]
we obtain for \(A(t)\) the following partial derivatives
\[
\frac{\partial A(t)}{\partial k^2} = \frac{4 \eta \frac{\partial f}{\partial k^2} - 2 \kappa \eta \frac{\partial f}{\partial k^2}}{\theta^2 \frac{\partial f}{\partial k^2}}
\]
\[
\frac{\partial A(t)}{\partial \theta \partial k} = \frac{\partial A(t)}{\partial k \partial \theta} = \frac{-4 \eta \frac{\partial f}{\partial k^2} + 2 \kappa \eta \frac{\partial f}{\partial k^2} - 4 \kappa \eta \frac{\partial f}{\partial k} + 2 \kappa \eta \frac{\partial f}{\partial k \partial \theta}}{\theta^2 \frac{\partial f}{\partial k \partial \theta}}
\]
\[
\frac{\partial A(t)}{\partial \theta^2} = \frac{12 \kappa \eta \frac{\partial f}{\partial \theta^2} - 8 \kappa \eta \frac{\partial f}{\partial \theta^2} + 2 \kappa \eta \frac{\partial f}{\partial \theta \partial \theta}}{\theta^2 \frac{\partial f}{\partial \theta \partial \theta}}
\]
and for \(B(t)\)
\[
\frac{\partial B(t)}{\partial k^2} = \frac{\partial}{\partial k} \left( \frac{2}{\theta^2 (\kappa - b)^2} \left( 1 - \frac{\partial b}{\partial k} \right) - \frac{4 h}{(\kappa - b)^3} \left( 1 - \frac{\partial b}{\partial k} \right)^2 \right) - \frac{2 h}{(\kappa - b)^2} \frac{\partial b}{\partial k^2}
\]
\[
+ \frac{2}{\theta^2 (\kappa - b)^2} \frac{\partial b}{\partial k} \left( 1 - \frac{\partial b}{\partial k} \right) \left( \frac{\partial b}{\partial k} \right) - \frac{2}{\theta^2 (\kappa - b)^3} \frac{\partial b}{\partial k^2}
\]
\[
\frac{\partial B(t)}{\partial k \partial \theta} = \frac{\partial B(t)}{\partial \theta \partial k} = \frac{\partial}{\partial \theta} \left( \frac{2}{\theta^2 (\kappa - b)^2} \left( 1 - \frac{\partial b}{\partial k} \right) + \frac{4 h}{(\kappa - b)^3} \left( 1 - \frac{\partial b}{\partial k} \right) \frac{\partial b}{\partial \theta} - \frac{2 h}{(\kappa - b)^2} \frac{\partial b}{\partial k \partial \theta} \left( \frac{\partial b}{\partial k} \right) \right)
\]
\[
- \frac{2}{\theta^2 (\kappa - b)^2} \frac{\partial \theta}{\partial \theta} \frac{\partial b}{\partial k \partial \theta} - \frac{2}{\theta^2 (\kappa - b)^3} \frac{\partial \theta}{\partial \theta} \frac{\partial b}{\partial k \partial \theta}
\]
\[
\frac{\partial B(t)}{\partial \theta^2} = -\frac{\partial}{\partial \theta} \left( \frac{2}{\theta^2 (\kappa - b)^2} \left( 1 - \frac{\partial b}{\partial \theta} \right) \frac{\partial b}{\partial \theta} - \frac{4 h}{(\kappa - b)^3} \left( \frac{\partial b}{\partial \theta} \right)^2 \right) - \frac{2 h}{(\kappa - b)^2} \frac{\partial b}{\partial \theta^2}
\]
\[
- \frac{2}{\theta^2 (\kappa - b)^2} \frac{\partial \theta}{\partial \theta} \frac{\partial b}{\partial \theta} - \frac{2}{\theta^2 (\kappa - b)^3} \frac{\partial \theta}{\partial \theta} \frac{\partial b}{\partial \theta}
\]
with
\[
\begin{align*}
\frac{\partial h}{\partial \kappa} &= t \frac{\partial h}{\partial \kappa} e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + at \frac{\partial h}{\partial \kappa} \right) \\
&\quad + \frac{e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - t \frac{\partial h}{\partial \tau} e^{bt} \frac{\partial a}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \tau} + at \frac{\partial h}{\partial \kappa} + at \frac{\partial h}{\partial \tau} \right)}{(ae^{bt} - 1)^2} \\
&\quad - 2e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + at \frac{\partial h}{\partial \kappa} \right) \left( \frac{\partial a}{\partial \kappa} e^{bt} + ate^{bt} \frac{\partial h}{\partial \kappa} \right) \\
\frac{\partial h}{\partial \kappa \partial \tau} &= t \frac{\partial h}{\partial \kappa} e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + at \frac{\partial h}{\partial \kappa} \right) \\
&\quad + \frac{e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - t \frac{\partial h}{\partial \tau} e^{bt} \frac{\partial a}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \tau} + at \frac{\partial h}{\partial \kappa} + at \frac{\partial h}{\partial \tau} \right)}{(ae^{bt} - 1)^2} \\
&\quad - 2e^{bt} \left( -t \frac{\partial h}{\partial \kappa} - e^{bt} \frac{\partial a}{\partial \kappa} + \frac{\partial a}{\partial \kappa} + at \frac{\partial h}{\partial \kappa} \right) \left( \frac{\partial a}{\partial \kappa} e^{bt} + ate^{bt} \frac{\partial h}{\partial \kappa} \right) \\
\frac{\partial h}{\partial \tau} &= t \frac{\partial h}{\partial \tau} e^{bt} \left( -t \frac{\partial h}{\partial \tau} - e^{bt} \frac{\partial a}{\partial \tau} + \frac{\partial a}{\partial \tau} + at \frac{\partial h}{\partial \tau} \right) \\
&\quad + \frac{e^{bt} \left( -t \frac{\partial h}{\partial \tau} - t \frac{\partial h}{\partial \tau} e^{bt} \frac{\partial a}{\partial \tau} - e^{bt} \frac{\partial a}{\partial \tau} + \frac{\partial a}{\partial \tau} + \frac{\partial a}{\partial \tau} + at \frac{\partial h}{\partial \tau} + at \frac{\partial h}{\partial \tau} \right)}{(ae^{bt} - 1)^2} \\
&\quad - 2e^{bt} \left( -t \frac{\partial h}{\partial \tau} - e^{bt} \frac{\partial a}{\partial \tau} + \frac{\partial a}{\partial \tau} + at \frac{\partial h}{\partial \tau} \right) \left( \frac{\partial a}{\partial \tau} e^{bt} + ate^{bt} \frac{\partial h}{\partial \tau} \right)
\end{align*}
\]

References

Abdymomunov, A., and J. Gerlach. 2014. Stress testing interest rate risk exposure. *Journal of Banking & Finance* 49:287–301. doi:10.1016/j.jbankfin.2014.08.013.

Ait-Sahalia, Y., and R. L. Kimmel. 2010. Estimating affine multifactor term structure models using closed-form likelihood expansions. *Journal of Financial Economics* 98 (1):113–44. doi:10.1016/j.jfineco.2010.05.004.

Almeida, C. I. R. 2005. A note on the relation between principal components and dynamic factors in affine term structure models. *Brazilian Review of Econometrics* 25 (1):89–114. doi:10.12660/bre.v25n12005.2673.

Ang, A., and F. A. Longstaff. 2013. Systemic sovereign credit risk: Lessons from the US and Europe. *Journal of Monetary Economics* 60 (5):493–510. doi:10.1016/j.jmoneco.2013.04.009.

Bams, D., and P. C. Schotman. 2003. Direct estimation of the risk neutral factor dynamics of Gaussian term structure models. *Journal of Econometrics* 117 (1):179–206. doi:10.1016/S0304-4076(03)00122-2.

Barone, E., D. Cuoco, and E. Zautzik. 1989. The term structure of interest rates: A test of the Cox, Ingersoll and Ross model on Italian treasury bonds. Working Paper, No. 128, Banca d’Italia, Roma, Italia.

Bellini, T., and M. Riani. 2012. Robust analysis of default intensity. *Computational Statistics & Data Analysis* 56 (11):3276–85. doi:10.1016/j.csda.2011.03.007.
Bianchi, M. L., and M. Rocco. 2016. Disentangling the information content of government bonds and credit default swaps: An empirical analysis on sovereigns and banks. *Frontiers in Applied Mathematics and Statistics* 2:22. doi:10.3389/fams.2016.00022.

Bianchi, M. L., and G. L. Tassinari. 2018. Forward-looking portfolio selection with multivariate non-Gaussian models and the Esscher transform. *Preprint*. https://arxiv.org/abs/1805.05584.

Brigo, D., and F. Mercurio. 2006. *Interest rate models: Theory and practice: With smile, inflation, and credit*. New York: Springer.

Chen, R. R., and L. Scott. 2003. Multi-factor Cox-Ingersoll-Ross models of the term structure: Estimates and tests from a Kalman filter model. *The Journal of Real Estate Finance and Economics* 27 (2):143–72.

Cox, J. C., J. E. Ingersoll, and S. A. Ross. 1985. A theory of the term structure of interest rates. *Econometrica* 53 (2):385–407. doi:10.2307/1911242.

Date, P., and C. Wang. 2009. Linear Gaussian affine term structure models with unobservable factors: Calibration and yield forecasting. *European Journal of Operational Research* 195 (1):156–66. doi:10.1016/j.ejor.2008.01.035.

de Jong, F. 2000. Time series and cross-section information in affine term-structure models. *Journal of Business & Economic Statistics* 18 (3):300–14. doi:10.2307/1392263.

D’Ecclesia, R. L., and S. A. Zenios. 1994. Risk factor analysis and portfolio immunization in the Italian bond market. *The Journal of Fixed Income* 4 (2):51–8. doi:10.3905/jfi.1994.408113.

Dempster, M. A. H., J. Evans, and E. Medova. 2014. Developing a practical yield curve model: An odyssey. In *Developments in macro-finance yield curve modelling*, ed. J. S. Chadha, A. C. J. Durré, M. A. S. Joyce, and L. Sarno, 252–92. Cambridge, UK: Cambridge University Press.

Duan, J. C., and J. G. Simonato. 1999. Estimating and testing exponential-affine term structure models by Kalman filter. *Review of Quantitative Finance and Accounting* 13 (2):111–35. doi:10.1023/A:1008304625054.

Duffee, G. R., and R. H. Stanton. 2012. Estimation of dynamic term structure models. *The Quarterly Journal of Finance* 2 (2):1250008. doi:10.1142/S2010139212500085.

Eberlein, E., C. Gerhart, and Z. Grbac. 2018. Multiple curve Lévy forward price model allowing for negative interest rates. https://arxiv.org/pdf/1805.02605.pdf.

Feldhütter, P., and D. Lando. 2008. Decomposing swap spreads. *Journal of Financial Economics* 88 (2):375–405. doi:10.1016/j.jfineco.2007.07.004.

Geyer, A. L. J., and S. Pichler. 1999. A state-space approach to estimate and test multifactor Cox-Ingersoll-Ross models of the term structure. *Journal of Financial Research* 22 (1):107–30. doi:10.1111/j.1475-6803.1999.tb00717.x.

Girardi, A., and C. Impenna. 2013. Price discovery in the Italian sovereign bonds market: The role of order flow. Working Paper, No. 906, Banca d’Italia, Roma, Italia.

Jeanblanc, M., M. Yor, and M. Chesney. 2009. *Mathematical methods for financial markets*. New York: Springer.

Kienitz, J., and P. Caspers. 2017. The SABR Model. In *Interest rate derivatives explained. Volume 2: Term structure and volatility modelling*, 87–121. London: Palgrave Macmillan.

Maggi, B., and F. Infortuna. 2008. Assessing Italian government bonds’ term structure with CIR model in the aftermath of EMU. *Applied Financial Economics Letters* 4 (3):163–70. doi:10.1080/17446540701689391.

Musti, S., and R. L. D’Ecclesia. 2008. Term structure of interest rates and the expectation hypothesis: The euro area. *European Journal of Operational Research* 185 (3):1596–606. doi:10.1016/j.ejor.2006.08.034.

Nawalkha, S. K., N. A. Beliaeva, and G. Soto. 2010. A new taxonomy of the dynamic term structure models. *Finance and Stochastics* 12 (2):149–72. doi:10.1007/s00780-007-0059-z.

Orlando, G., R. M. Mininni, and M. Bufalo. 2019a. A new approach to forecast market interest rates through the CIR model. *Studies in Economics and Finance*. doi:10.1108/SEF-03-2019-0116.
Orlando, G., R. M. Mininni, and M. Bufalo. 2019b. Interest rates calibration with a CIR model. *The Journal of Risk Finance* 20 (4):370–87. doi:10.1108/JRF-05-2019-0080.

O’Sullivan, C. 2008. Parameter uncertainty in Kalman-Filter estimation of the CIR term structure model. In *Numerical methods for finance*, ed. J. A. D. Appleby, D. C. Edelman, and J. H. Miller, 255–280. London: Chapman & Hall.

Pelizzon, L., M. G. Subrahmanyam, D. Tomio, and J. Uno. 2016. Sovereign credit risk, liquidity, and European Central Bank intervention: Deus ex machina? *Journal of Financial Economics* 122 (1):86–115. doi:10.1016/j.jfineco.2016.06.001.

Pelizzon, L., M. G. Subrahmanyam, D. Tomio, and J. Uno. 2018. Central bank-driven mispricing. SAFE Working Paper 226, Leibniz Institute for Financial Research SAFE, Frankfurt am Main, Germany.

Rebonato, R., I. Saroka, and V. Putyatin. 2014. A principal-component-based affine term structure model. Working paper. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2451130.

Sanford, A. D., and G. M. Martin. 2005. Simulation-based Bayesian estimation of an affine term structure model. *Computational Statistics & Data Analysis* 49 (2):527–54. doi:10.1016/j.csda.2004.05.026.

Schönbuch, T. B., A. Wills, and B. Ninness. 2011. System identification of nonlinear state-space models. *Automatica* 47 (1):39–49. doi:10.1016/j.automatica.2010.10.013.

Schoutens, W., and J. Cariboni. 2009. *Lévy processes in credit risk*. Boca Raton: Wiley.

Tassinari, G. L., and M. L. Bianchi. 2014. Calibrating the smile with multivariate time-changed Brownian motion and the Esscher transform. *International Journal of Theoretical and Applied Finance* 17 (4):1450023. doi:10.1142/S021902491450023X.

Teng, L., M. Ehrhardt, and M. Günther. 2013. Numerical evaluation of complex logarithms in the Cox-Ingersoll-Ross model. *International Journal of Computer Mathematics* 90 (5):1083–95. doi:10.1080/00207160.2012.749348.

Vasicek, O. 1977. An equilibrium characterization of the term structure. *Journal of Financial Economics* 5 (2):177–88. doi:10.1016/0304-405X(77)90016-2.

Wüthrich, M. V., and M. Merz. 2013. *Financial modeling, actuarial valuation and solvency in insurance*. New York: Springer.

Zeytun, S., and A. Gupta. 2007. A comparative study of the Vasicek and the CIR model of the short rate. Published report of Fraunhofer Institute for Industrial Mathematics ITWM, Kaiserslautern, Germany, No. 124.