Cubic interaction vertex of higher-spin fields with external electromagnetic field

I.L. Buchbinder\textsuperscript{a,b,}\footnote{ joseph@tspu.edu.ru }\quad T.V. Snegirev\textsuperscript{b,}\footnote{ snegirev@tspu.edu.ru }\quad Yu.M. Zinoviev\textsuperscript{c,}\footnote{ Yurii.Zinoviev@ihep.ru }\quad

\textsuperscript{a} The Erwin Schrodinger International Institute for Mathematical Physics
A-1090, Vienna, Austria

\textsuperscript{b} Department of Theoretical Physics,
Tomsk State Pedagogical University,
Tomsk 634061, Russia

\textsuperscript{c} Institute for High Energy Physics,
Protvino, Moscow Region, 142280, Russia

Abstract

We fulfill the detailed analysis of coupling the charged bosonic higher spin fields to external constant electromagnetic field in first order in external field strength. Cubic interaction vertex of arbitrary massive and massless bosonic higher spin fields with external field is found. Construction is based on deformation of free Lagrangian and free gauge transformations by terms linear in electromagnetic field strength. In massive case a formulation with Stueckelberg fields is used. We begin with most general form of deformations for Lagrangian and gauge transformations, admissible by Lorentz covariance and gauge invariance and containing some number of arbitrary coefficients, and require the gauge invariance of the deformed theory in first order in strength. It yields the equations for the coefficients which are exactly solved. As a result, the complete interacting Lagrangian of arbitrary bosonic higher spin fields with constant electromagnetic field in first order in electromagnetic strength is obtained. Causality of massive spin-2 and spin-3 fields propagation in the corresponding electromagnetic background is proved.
1 Introduction

Construction of interacting Lagrangians, which describe coupling of the higher-spin fields to each other or to low-spin fields or to external fields is a central line of modern development in higher-spin field theory. As known, the standard procedure of switching on the minimal interactions, which are usually used in low-spin field models, do not work for higher-spin fields. The attempts of naive adaptations of this procedure to build the interactions of higher-spin fields yield the inconsistency problems such as a possibility of propagating non-physical auxiliary fields, violation of causality, breaking the gauge invariance of free theory, appearance of ghosts and so on. Some aspects of modern state of the higher spin field theory are discussed in the reviews [1].

In general, two types of interaction problems are considered in field theory, interactions among the dynamical fields and couplings of dynamical fields to external background. In conventional field theory, these problems are closely related. However, in higher spin theory, where the generic interaction Lagrangians are not established so far, these two types of interactions can be studied as independent problems (the second one being much simpler).

Consistency problems of higher spin fields couplings to external electromagnetic field have first been studied in refs. [2]. Inconsistency of higher spin coupling to gravity was investigated in [3] for example of spin-2 field. The first attempts to build the higher spin field Lagrangian interaction have been undertaken in the refs. [4] and [5] where the massless higher spin fields were considered and the consistency aspects were pointed out. The substantial progress in understanding the massless higher spin field coupling to gravity has been attained in refs. [6] where the it was shown that such coupling demand to involve the fields of all spins, the higher spin symmetry has been introduced, the cubic vertex of all spin fields interacting with gravity has been constructed and it was shown that such a vertex exists in \((A)dS\) space only. Some later, the consistent interacting equations of motion for all massless higher spin fields have been found [7]. Recently, the arguments have been given that these equations can be Lagrangian [8].

By the present time, different approaches to constructing the higher spin field interactions were developed and some progress has been attained for building the cubic vertex for massless and massive higher spin fields (see e.g. the recent papers [9], [10], [11], [12], [13] and references therein).

The aim of this paper is a generic construction of lowest order interaction vertex for massless and massive bosonic arbitrary spin fields with constant electromagnetic field. As we pointed out above, the Lagrangian formulation of higher spin fields in external background is an independent problem. Just this problem is discussed in the given paper.

Various aspects of Lagrangian formulation for higher spin fields in electromagnetic background are discussed in the recent papers [14], [15], [16], [17]. It is worth noting the related attempts to derive the equations of motion for higher spin fields in external electromagnetic and gravitational backgrounds from string theory (see e.g. [18], [19], [20]). Generic problem of constructing the cubic coupling vertex of arbitrary higher spin fields to electromagnetic field is open.

The paper is organized as follows. Section 2 is devoted to description of general method to derive the vertices for massless and massive higher spin fields coupled to constant electromagnetic field. In Section 3 we solve the problem of cubic coupling of arbitrary massless integer higher spin field to constant electromagnetic background and in Section 4 the analogous problem is solved for massive integer higher spin fields. Conclusion is devoted to summary of the results obtained. Some technical aspects of calculations are putted into Appendix.
2 Procedure of interacting Lagrangian construction for higher spin fields in electromagnetic background

In this section we describe a generic scheme for constructing the cubic interaction vertex of higher spin fields with external electromagnetic field.

We want to construct the interaction of higher-spin fields with external constant electromagnetic field strength $F_{\mu\nu}$. As usual to provide the invariance under the $U(1)_{em}$ group, the e/m potential $A_\mu$ should enter into a Lagrangian either through covariant derivative

$$D_\mu = \partial_\mu - \varepsilon_{U(1)} A_\mu$$

where $\varepsilon_{U(1)}$ is the $U(1)_{em}$ generator, or through the electromagnetic field strength $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

In the case of external constant electromagnetic field the strength $F_{\mu\nu}$ is simply a constant antisymmetric matrix and since the electromagnetic field is external we will not include the kinetic term for vector field $A_\mu$ into the Lagrangian.

The approach to the vertex construction is based on two points. First is gauge invariance, the Lagrangian for given model $\mathcal{L}$ is constructed to be invariant with respect to gauge transformation $\delta$, i.e. the vanishing of variation $\delta \mathcal{L} = 0$. Second point is perturbative consideration where the interacting Lagrangian in constructed as a sum of terms, which are linear, quadratic and so on in external field strength. It means that the interacting Lagrangian is represented as a series in powers of the strength $F$ in the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + ...$$

where $\mathcal{L}_0$ is the free Lagrangian of dynamical fields, $\mathcal{L}_1$ is quadratic in dynamical fields and linear in strength $F$ and so on. Also, the gauge transformations are written as the series

$$\delta = \delta_0 + \delta_1 + ...$$

where $\delta_0$ are gauge transformations of free theory, $\delta_1$ are linear in strength $F$ one and so on.

The aim of this paper is to construct in explicit form the first interacting contribution to the complete Lagrangian $\mathcal{L}$, i.e. we consider a first correction to Lagrangian $\mathcal{L}_1$ and first correction to gauge transformation $\delta_1$. Both these corrections are linear in strength $F$. The Lagrangian $\mathcal{L}_1$, being quadratic in dynamical fields and linear in external field, defines the cubic coupling of higher spin fields to external electromagnetic field. Gauge variation of action in the case under consideration has the form

$$\delta S = (\delta_0 + \delta_1)(\mathcal{L}_0 + \mathcal{L}_1) = \delta_0 \mathcal{L}_0 + \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 + \delta_1 \mathcal{L}_1 = 0 \quad (1)$$

Since the variation $\delta_1 \mathcal{L}_1$ is quadratic on $F$, it can be omitted in first approximation. To find $\mathcal{L}_1$ in explicit form we will implement the following procedure. We write down the most general expressions for gauge transformations $\delta_1$ and Lagrangian $\mathcal{L}_1$ on the base of Lorentz symmetry and e/m gauge invariance up to the numerical coefficients. Then the relation (1) yields the equations for the coefficients which can be solved in principle, although, as we will see bellow, the solutions are quite non-trivial.

\emph{1} Up to the total divergence
Let us note that there is an important difference between massless and massive cases. As is known (see e.g. last reference in [1]), for massless fields with spin $s \geq 3/2$ in a flat Minkowski space it is impossible to switch on minimal e/m interactions (while it become possible in (A)dS space [15]). At the same time for massive fields such possibility does exist even in a Minkowski space with the addition of appropriate non-minimal corrections with the coefficients proportional to inverse powers of mass $m$. In this, it is possible to consider a limit when both mass $m$ and e/m charge $e_0$ simultaneously go to zero in such a way that only non-minimal terms survive. So our strategy here will be as follows. In Section 3 we will consider massless case and construct non-minimal interactions which exist even in Minkowski space. Then, in Section 4 we will turn to the massive case where as it will be seen the very same non-minimal interactions play crucial role.

In general, we begin with the free Lagrangian $L_0$, which is invariant under free gauge transformation $\delta_0$. Then we add all admissible corrections to Lagrangian and gauge transformations $L_1$ and $\delta_1$ respectively and require fulfillment of the relation (1). In the massive case we also include the minimal interactions. Essential element of the approach under consideration is the use of Stueckelberg fields to provide the gauge invariance in the massive theories (see e.g. [21] for metric-like formalism and [22] for frame-like one). It is interesting to point out that the Stueckelberg fields are automatically arose in the BRST approach to Lagrangian formulation for higher spin fields [23].

3 Massless theory

Before to start a generic analysis we consider the particular cases of spin-2 and spin-3 fields. It allows us to get some experience of constructing the interactions of higher spin fields with external field and apply this experience to coupling of arbitrary higher spin fields to external electromagnetic fields.

3.1 Spin-2 field

We consider the charged massless spin-2 field propagating in external constant electromagnetic background. Such a field is described in terms of doublet of rank-2 real symmetric tensor fields $h_{\mu\nu}^i$, $i = 1, 2$. Free Lagrangian for such theory is well known and in flat space has the form

$$L_0 = \frac{1}{2} \partial^\alpha h_{\mu\nu}^i \partial_\alpha h_{\mu\nu}^i - (\partial h)^\mu \partial_\mu h^i - \frac{1}{2} \partial^\mu h^i \partial_\mu h^i$$

where $(\partial h)^\mu = \partial^\nu h_{\mu\nu}^i$, $h^i = g_{\mu\nu} h_{\mu\nu}^i$. Lagrangian (2) is invariant under standard gauge transformations with gauge parameter $\xi_\mu^i$

$$\delta_0 h_{\mu\nu}^i = \partial_\mu \xi_\nu^i + \partial_\nu \xi_\mu^i$$

The equations of motion corresponding to Lagrangian (2) are written as follows

$$\left( \frac{\delta S_0}{\delta h_{\mu\nu}^i} \right) = -\partial^2 h_{\mu\nu}^i + \partial^\mu (\partial h)^\nu \partial_\mu h^i + \partial^\nu (\partial h)^\mu \partial_\nu h^i - \partial^\mu \partial^\nu h^i - g_{\mu\nu}^\nu (\partial \partial h)^i + g_{\mu\nu} \partial^2 h^i$$

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2 We work in $d$-dimensional Minkowski space with metric $g_{\mu\nu} = (+, -, -, -)$. The gauge group is realized as $SO(2)$. 

4
Condition of gauge invariance of free theory is

\[ \delta_0 L_0 = \delta_0 h_{\mu\nu}^i \left( \frac{\delta S_0}{\delta h_{\mu\nu}^i} \right) = 0 \]

Let us consider the interacting theory in linear approximation in external field, adding the first order corrections \( L_1 \) and \( \delta_1 \) to free Lagrangian and free gauge transformation respectively. Then the condition of gauge invariance (1) in given approximation takes the form

\[ \delta_0 h_{\mu\nu}^i \left( \frac{\delta S_1}{\delta h_{\mu\nu}^i} \right) + \delta_1 h_{\mu\nu}^i \left( \frac{\delta S_0}{\delta h_{\mu\nu}^i} \right) = 0 \tag{4} \]

where \( S_1 \) is the correction action corresponding to the Lagrangian \( L_1 \).

Now, following to scheme, described in Section 2, we should write down the admissible form of the \( L_1 \) and \( \delta_1 \) up to numerical coefficients. The most general ansatz for the first correction to Lagrangian looks like

\[ L_1 = \varepsilon^{ij} F^{\alpha\beta} [a_1 \partial^\alpha h_\alpha^{i}\partial_\mu h_\nu^{j} + a_2 (\partial h)_\alpha^{i}(\partial h)_\beta^{j} + a_3 \partial_\alpha h_\alpha^{i}(\partial h)_\mu^{j} + a_4 (\partial h)_\alpha^{i}\partial_\beta h^j] \tag{5} \]

where \( \varepsilon^{ij} = -\varepsilon^{ji}, \varepsilon^{12} = 1 \) and \( a_1, a_2, a_3, a_4 \) are the arbitrary real coefficients. The Lagrangian (5) rises the following contribution to equations of motion

\[ \left( \frac{\delta S_1}{\delta h_{\mu\nu}^i} \right) = \varepsilon^{ij} [a_1 F^{\alpha\beta} \partial^\alpha h_\alpha^{i}\partial_\mu h_\nu^{j} + a_2 F^{\alpha\beta}(\partial h)_\alpha^{i}(\partial h)_\beta^{j} + a_3 \partial_\alpha h_\alpha^{i}(\partial h)_\mu^{j} + a_4 (\partial h)_\alpha^{i}\partial_\beta h^j] \]

where the parentheses denote symmetrization of the indices without normalization. It is easy to see that the Lagrangian (5) is not invariant under free gauge transformations (3). To recover the gauge invariance we deform the free gauge transformation by the additional term \( \delta_1 \). The most general ansatz for such deformation is written as follows

\[ \delta_1 h_{\mu\nu}^i = \varepsilon^{ij} \left[ \gamma_1 F_{(\mu}^{a\alpha}(\partial_\alpha \xi_\nu)\right]^{j} + \gamma_2 F_{(\mu}^{a\alpha}(\partial_\nu)\xi_\alpha^{j} + \gamma_3 g_{\mu\nu} F^{\alpha\beta} \partial_\alpha \xi_\beta\right] \tag{6} \]

where \( \gamma_1, \gamma_2, \gamma_3 \) are the arbitrary real coefficients. Recall that in all cases when interacting Lagrangian has the same or higher number of derivatives as the free one, there always exists a possibility to make some fields and gauge parameters redefinitions. In this, all Lagrangians related by such redefinitions are completely equivalent. In the case at hands we have a two-parameter arbitrariness associated with the following reparametrization of fields and gauge parameters

\[ h_{\mu\nu} = h_{\mu\nu}^i \Rightarrow \quad \h_{\mu\nu}^i + \kappa_1 \varepsilon^{ij} F_{(\mu}^{a\alpha} h_\nu^{j} \]

\[ \xi_\mu^i \rightarrow \quad \xi_\mu^i + \kappa_2 \varepsilon^{ij} F_{\mu}^{a\alpha} \xi_\alpha^{j} \]

Here \( \kappa_1, \kappa_2 \) are the arbitrary real coefficients. This arbitrariness allows us to vanish the terms with coefficients \( \gamma_1 \) and \( \gamma_2 \). Indeed, taking \( \kappa_1 = -\gamma_1, \kappa_2 = \gamma_1 - \gamma_2 \) we obtain that first two

\footnote{As we have already noted, for the massless fields in Minkowski space it is impossible to switch on minimal e/m interactions so that electric charge is zero. That is why throughout this section we use ordinary partial derivatives instead of covariant ones.}
terms in (6) are absent. Calculating the variations \( \delta_0 \mathcal{L}_1, \delta_1 \mathcal{L}_0 \) and substituting the results into (4) we obtain the following relations for the arbitrary coefficients

\[
2a_1 - a_3 = 0, \quad 2a_1 + 2a_2 = 0, \quad 2a_2 - a_3 + 2a_4 = 0
\]

\[
a_3 - \gamma \nu (d - 2) = 0, \quad a_4 - \gamma \nu (d - 2) = 0
\]

These equations can be easily solved and we find all the coefficients \( a_1 - a_4 \) in terms of a single parameter \( \gamma \nu \)

\[
a_1 = -a_2 = \frac{1}{2} \gamma \nu (d - 2), \quad a_3 = a_4 = \gamma \nu (d - 2)
\] (7)

As a result we have constructed the consistent cubic interaction Lagrangian (5) of spin-2 field with electromagnetic field and the corresponding correction to gauge transformations (6). To be more precise, we obtained the one-parametric family of Lagrangians and gauge transformations. It is important to note that the parameter \( \gamma \nu \) has dimension of inverse mass square.

### 3.2 Spin-3 field

Now let us consider one more example of coupling the higher spin field to constant external electromagnetic field. It will allow us to get additional experience in constructing the interaction before to go to generic case.

The charged spin-3 field is described by doublet of total symmetric real tensor rank-3 fields \( \phi_{\mu \nu \sigma}^i, i = 1, 2 \). The free dynamics is described by Lagrangian

\[
\mathcal{L}_0 = -\frac{1}{2} \partial^\rho \phi_{\mu \nu \sigma}^i \partial_{\rho} \phi_{\mu \nu \sigma}^i + \frac{3}{2} (\partial \phi)^{\mu \nu} (\partial \phi)^{\alpha \beta} \phi_{\nu \alpha \sigma}^i + 3(\partial \partial \phi)^{\mu \nu} \phi_{\nu \sigma}^i + \frac{3}{2} \partial^\rho \phi^{\mu \nu \sigma} \partial_{\rho} \partial \phi^i + \frac{3}{4} (\partial \phi)^i (\partial \phi)^j
\]

with gauge transformations of the form

\[
\delta_0 \phi_{\mu \nu \sigma}^i = \partial (\mu \xi_{\nu \sigma})^i, \quad \xi_{\mu \nu}^i = 0
\]

here a tilde means a trace of tensor. As in previous subsection, we begin with writing down the most general first order deformations of the Lagrangian and gauge transformations.\(^4\) For the Lagrangian we have

\[
\mathcal{L}_1 = -\varepsilon^{ij} F_{\alpha \beta}^{\nu \sigma} (a_1 \partial^\mu \phi_{\alpha \mu}^{\nu \sigma} + a_2 (\partial \phi)_\alpha^{\nu \sigma} (\partial \phi)_\beta^{\nu \sigma} + a_3 \partial_\alpha \phi_{\nu \sigma}^j + a_4 (\partial \phi)_\alpha^{\nu \sigma} + a_5 \partial_\alpha \phi_{\nu \sigma}^j + a_6 \partial^\mu \phi_{\mu \nu}^{\nu \sigma} + a_7 \partial_\alpha \phi_{\beta \sigma}^j (\partial \phi)^j]
\] (8)

The Lagrangian (8) depends on seven real arbitrary numerical coefficients \( a_1, a_2, ..., a_7 \). For gauge transformations one can use, as for the spin-2 case, the arbitrariness in redefinition of the fields \( \phi_{\mu \nu \sigma}^i \) and parameters \( \xi_{\mu \nu}^i \). It allows us to set

\[
\delta \phi_{\mu \nu \sigma}^i = \gamma \varepsilon^{ij} \xi_{\mu \nu} F_{\alpha \beta}^{\nu \sigma} \partial_\alpha \xi_{\nu \sigma}^j
\]

\(^4\)In general case with dynamical e/m field, the higher the spin of particles we want to consider the higher will be the number of derivatives we will have to introduce (see e.g third reference in [10]) so that the cubic vertex for massless spin \( s \) particle contains \((2s - 1)\) derivatives. But in the case of constant e/m field the problem turns out to be less restricted and as a result has more solutions. In particular, for the spin 3 case (as well as for arbitrary spin as will be seen later on) it is enough to consider non-minimal interactions with three derivatives only.
The gauge variation \( \delta_1 \) contains a single arbitrary real parameter \( \gamma \). Gauge invariance in linear approximation in \( F \) is written as
\[
\delta_0 \phi_{\mu\nu}^i \left( \frac{\delta S_1}{\delta \phi_{\mu\nu}^i} \right) + \delta_1 \phi_{\mu\nu}^i \left( \frac{\delta S_0}{\delta \phi_{\mu\nu}^i} \right) = 0 \tag{9}
\]
Equation (9) completely defines all unknown coefficients \( a_1, a_2, \ldots, a_7 \) in terms parameter \( \gamma \). The system of equations for these coefficients follows from (9) and has the form
\[
2a_1 - a_3 = 0, \quad 2a_1 + a_2 = 0, \quad a_2 - a_3 + a_4 = 0 \\
a_2 + a_5 = 0, \quad 2a_3 - 3\gamma d = 0, \quad a_4 - 3\gamma d = 0 \\
a_4 - 2a_7 - \frac{3}{2} \gamma d = 0, \quad a_5 - 2a_7 = 0, \quad a_5 + 2a_6 = 0
\]
The solution to this system is
\[
a_3 = 2a_1 = \frac{3}{2} \gamma d, \quad a_4 = -2a_2 = 3\gamma d, \quad a_4 = -2a_6 = 2a_7 = \frac{3}{2} \gamma d. \tag{10}
\]
Arbitrary parameter \( \gamma \) has dimension of inverse square mass.

### 3.3 Arbitrary integer spin \( s \)

Now we generalize the results obtained in previous subsections to general massless integer spin-\( s \) field.

For description of massless charged field of arbitrary integer spin \( s \) we use a doublet of totally symmetric real tensor rank-\( s \) fields \( \Phi_{\mu_1\mu_2\ldots\mu_s}^i, \ i = 1, 2, \) satisfying also the double traceless condition
\[
\Phi_{\alpha\beta} = 0
\]
Further we will use the following compact notations
\[
\Phi_{\mu_1\mu_2\ldots\mu_s}^i = \Phi_s^i, \quad \partial^{\mu_1} \Phi_{\mu_2\ldots\mu_s-1}^i = (\partial \Phi)_{s-1}^i, \quad g^{\mu_1\mu_2} \Phi_{\mu_1\mu_2\ldots\mu_s-2}^i = \tilde{\Phi}_{s-2}^i \tag{11}
\]
Free theory in these terms is described by Fronsdal Lagrangian [24]
\[
\mathcal{L}_0 = (-1)^s \frac{1}{2} [\partial^{\mu} \Phi^i \partial_\mu \Phi_s^i - s(\partial \Phi)^{s-1, i}(\partial \Phi)_{s-1}^i + s(s-1)(\partial \Phi)^{\mu_1 s-2, i} \partial_\mu \tilde{\Phi}_{s-2}^i - s(s-1)(s-2) \frac{s}{4} (\partial \Phi)^{s-3, i} (\partial \Phi)_{s-3}^i] \tag{12}
\]
which is invariant under the gauge transformations
\[
\delta_0 \Phi_s^i = \partial_{(\mu_1} \xi_{s-1)}^i, \quad \tilde{\xi}_{s-3}^i = 0 \tag{13}
\]
where \( \xi_{s-1}^i \) is symmetric traceless rank-(\( s - 1 \)) tensor field and the tilde means a trace.

Further we follow the procedure what was used in two previous subsections for the cases of spin-2 and spin-3 fields. We write down the first order corrections \( \mathcal{L}_1 \) and \( \delta_1 \) to Lagrangian and gauge transformation respectively admissible from Lorentz covariance and e/m gauge invariance. These corrections contain some number of arbitrary real coefficients. First order gauge invariance condition has the form
\[
\delta_0 \Phi_s^i \left( \frac{\delta S_1}{\delta \Phi_s^i} \right) + \delta_1 \Phi_s^i \left( \frac{\delta S_0}{\delta \Phi_s^i} \right) = 0 \tag{14}
\]
It is easy to verify that the solution to these equations is

\[ \delta \text{Calculating variations} \]

imposes the constraints on the coefficients.

We begin with cubic coupling to electromagnetic field. The most general ansatz for such interacting Lagrangian has the form

\[
\mathcal{L}_1 = ( -1 )^{s + i j} F^{\alpha \beta} [ a_1 \partial^\mu \Phi_\alpha \hspace{1pt} s^{-1} - i \partial_\mu \Phi \beta s - 1] + a_2 (\partial \Phi)_\alpha s^{2 - i} (\partial \Phi)_{\beta s - 2} + \\
+ a_3 \partial_\alpha \Phi s^{-1 - i} (\partial \Phi)_{s - 1} + a_4 (\partial \Phi)_\alpha s^{2 - i} \partial_\beta \Phi_{s - 2}^2 + \\
+ a_5 (\partial \Phi)_\alpha s^{1 - 2} i \partial_\mu \Phi \beta s - 3 + a_6 (\partial \Phi)_\alpha s^{3 - i} \partial_\mu \Phi \beta s - 3 + \\
+ a_7 \partial_\alpha \Phi s^{-3 - i} (\partial \Phi)_{s - 3}^2 + a_8 (\partial \Phi)_\alpha s^{4 - i} (\partial \Phi)_{\beta s - 4}^2 ] 
\]

(15)

It contains eight terms, where the coefficients \( a_1, a_2, ..., a_8 \) are the unknown parameters.

Next step is construction of ansatz for correction to gauge transformations. Recall that, as discussed in above examples, we have an arbitrariness in redefinition of fields \( \Phi_{s}^i \) and gauge parameters \( \xi_{s-1}^i \). Using that, one can show that the correction to gauge transformations can be reduced to

\[
\delta_1 \Phi_{s}^i = \gamma s^{ij} g_{(\mu \nu \mu \nu)} F^{\alpha \beta} \partial_\alpha \xi_{s-2}^j
\]

Calculating variations \( \delta_0 \mathcal{L}_1 \) and \( \delta_1 \mathcal{L}_0 \) and using the gauge invariance relation (14), we obtain the system of algebraic equations for the unknown coefficients

\[
2 a_1 - a_3 = 0, \hspace{1cm} 2 (s - 1) a_1 + 2 a_2 = 0, \hspace{1cm} 2 a_2 - a_3 (s - 1) + 2 a_4 = 0 \\
(s - 2) a_2 + a_5 = 0, \hspace{1cm} a_3 - \frac{s}{2} \gamma [d + 2s - 6] = 0 \\
a_4 - \frac{s (s - 1)}{2} \gamma [d + 2s - 6] = 0, \hspace{1cm} a_4 - 2 a_7 - \frac{s (s - 1)}{4} \gamma [d + 2s - 6] = 0 \\
a_5 - 2 a_7 = 0, \hspace{1cm} a_5 + 2 a_6 = 0, \hspace{1cm} (s - 3) a_5 + 4 a_8 = 0
\]

It is easy to verify that the solution to these equations is

\[
a_3 = 2 a_1 = \frac{1}{2} \gamma s (d + 2s - 6) \\
a_4 = -2 a_2 = \frac{1}{2} \gamma s (s - 1) (d + 2s - 6) \\
a_4 = -2 a_6 = 2 a_7 = \frac{1}{4} \gamma s (s - 1) (s - 2) (d + 2s - 6) \\
a_8 = -\frac{1}{16} \gamma s (s - 1) (s - 2) (s - 3) (d + 2s - 6)
\]

(16)

The relations (16) contains the single arbitrary parameter \( \gamma \) of dimension of inverse mass square. In particular, we see that for \( s = 2, 3 \) the coefficients (16) coincides with the ones for spin-2 and spin-3 fields given in two previous subsections. Note that the constructed vertex will be useful for finding the cubic vertex of the massive theory as well.

4 Massive theory

Notations. To simplify the details of calculations we introduce convenient notations. The Lagrangian and gauge transformations of the theories under consideration have a common structure

\[
\mathcal{L} = \mathcal{L}_{00} + \mathcal{L}_{01} + \mathcal{L}_{02} + \mathcal{L}_{10} + \mathcal{L}_{11} + \ldots
\]
\[ \delta = \delta_{00} + \delta_{01} + \delta_{10} + \delta_{11} + \ldots \]

Here the first index in gauge transformations denotes the power of fields, and one in Lagrangian means the power of fields higher than quadratic. The second index both in Lagrangian and in transformations denotes the number of derivatives. That is, in these general notations we have \( \delta_{kn} \sim \partial^n \Phi^k \xi, L_{kn} \sim \partial^n \Phi^{k+2} \), where \( \xi \) is gauge parameter. In such a case the variation of Lagrangian is written as follows

\[
\delta L = \delta_{00} L_{00} + (\delta_{00} L_{01} + \delta_{01} L_{00}) + (\delta_{00} L_{02} + \delta_{01} L_{11}) + \delta_{01} L_{02} +
\]

\[
+ (\delta_{00} L_{10} + \delta_{10} L_{00}) + (\delta_{00} L_{11} + \delta_{01} L_{10} + \delta_{10} L_{01} + \delta_{11} L_{00}) + \ldots
\]

(17)

Here in the right-hand side the variations are grouped in such a way that the sum of the first and second indices for each group in braces has the same value. Therefore the gauge invariance \( \delta L = 0 \) requires vanishing for each group of variations independently that allows us to construct the interaction vertices, in principle, in any order. We are interested only in cubic interaction, which corresponds in our notations to the sum of the first indices of variations to be equal or less than one. For example, in this notations, the Lagrangian and gauge transformations constructed in Section 3 for massless theory have the structure

\[ L = L_{02} + L_{13} \quad \delta = \delta_{01} + \delta_{12} \]

Further all considerations are carried out with help of above notations. As in massless case, we begin with particular case of spin-2 and spin-3 fields and then will go to general massive case.

### 4.1 Spin 2

**Free theory.** For gauge-invariant description of free massive spin-2 field theory we need a set of fields \( \Phi^a = \{ h_{\mu\nu}, b_\mu, \varphi \} \), first of them is symmetric where \( b_\mu \) and \( \varphi \) are the auxiliary Stueckelberg fields. With notations given above, the Lagrangian for such a theory has the form:

\[ L_0 = L_{00} + L_{01} + L_{02} \quad (18) \]

\[
L_{02} = \frac{1}{2} \partial^\rho h^{\mu\nu} \partial_\rho h_{\mu\nu} - (\partial h)^\mu (\partial h)_\mu + (\partial h)^\mu \partial_\mu h - \frac{1}{2} \partial^\rho h \partial_\rho h -
\]

\[
- \frac{1}{2} \partial^\mu b^\nu \partial_\mu b_\nu + \frac{1}{2} (\partial b)(\partial b) + \frac{1}{2} \partial^\alpha \varphi \partial_\alpha \varphi
\]

\[
L_{01} = m [\alpha_1 h^{\mu\nu} \partial_\mu b_\nu - \alpha_1 h (\partial b) + \alpha_0 b_\mu \partial_\mu \varphi] \]

\[
L_{00} = m^2 \left[ - \frac{1}{2} h^{\mu\nu} h_{\mu\nu} + \frac{1}{2} hh + \frac{1}{2} \alpha_1 \alpha_0 h \varphi + \frac{d}{2(d-2)} \varphi^2 \right]
\]

The Lagrangian (18) is invariant under the gauge transformations

\[
\delta_0 = \delta_{00} + \delta_{01}
\]

\[
(\delta_{01} + \delta_{00}) h_{\mu\nu} = \partial_(\mu \xi_\nu) + \frac{m \alpha_1}{d-2} g_{\mu\nu} \xi
\]

\[
(\delta_{01} + \delta_{00}) b_\mu = \partial_\mu \xi + m \alpha_1 \xi_\mu
\]

\[
\delta_0 \varphi = -m \alpha_0 \xi
\]
where

$$(\alpha_1)^2 = 2, \quad (\alpha_0)^2 = 2 \frac{d-1}{d-2}$$

Note that in the massless limit $m \to 0$ this Lagrangian decomposes into the sum of the Lagrangians describing the massless fields with spins 2, 1 and 0. Equations of motion corresponding to Lagrangian (18) are written as follows

$$\left( \frac{\delta S_0}{\delta \Phi^a} \right) = \left( \frac{\delta S_{02}}{\delta \Phi^a} \right) + \left( \frac{\delta S_{01}}{\delta \Phi^a} \right) + \left( \frac{\delta S_{00}}{\delta \Phi^a} \right)$$

The gauge invariance means vanishing of the variation

$$\delta_0 L_0 = \delta_0 \Phi^a \left( \frac{\delta S_0}{\delta \Phi^a} \right) = 0$$

**Minimal interaction.** We work in terms of real doublets $\Phi^a = \{ h_{\mu \nu}^i, b_{\mu}^i, \varphi^i \}, \ i = 1, 2$. Let us introduce a minimal electromagnetic interaction replacing the ordinary derivatives by covariant ones

$$\partial_\mu \to D_\mu^{ij} = \delta^{ij} \partial_\mu + e_0 e^{ij} A_\mu, \quad e^{ij} = -e^{ji}, \quad e^{12} = 1,$$  \hspace{1cm} (19)

Their commutator proportional to $e/m$ field strength $[D_\mu^{ij}, D_\nu^{kj}] = e_0 e^{ij} F_{\mu \nu}$ ($F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). The parameter $e_0$ is a charge. As a consequence, gauge invariance is violated, and non-invariant part in linear approximation on $F_{\mu \nu}$ is equal to

$$\delta_0 L_0 = (\delta_0 L_{02} + \bar{\delta}_1 L_{01}) + \bar{\delta}_01 L_{02}$$  \hspace{1cm} (20)

$$\begin{align*}
(\delta_0 L_{02} + \bar{\delta}_1 L_{01}) &= m e_0 e^{ij} \xi_\mu^i [-\alpha_1 F^{\alpha \mu} b^j_\alpha]; \\
\bar{\delta}_0 L_{02} &= e_0 e^{ij} \xi_\mu^i [-4 F^{\alpha \beta} \partial_\alpha h^\mu_{\beta} - 2 F^{\alpha \mu}(\partial h)^i_{\beta} + 3 F^{\alpha \mu} \partial_\alpha h^j_{\beta}] + \\
&\quad + e_0 e^{ij} \xi_\mu^i [2 F^{\alpha \beta} \partial_\alpha h^j_{\beta}];
\end{align*}$$

where the bar means that we have replaced the ordinary derivatives by covariant ones.

**Non-minimal interaction.** To recover gauge invariance we add the non-minimal terms to Lagrangian and to gauge transformations. As we have already mentioned, we assume that the free theory is deformed in such a way that the massless limit $e_0 \to 0, \ m^2 \to 0, \ \frac{e_0}{m^2} = const$ exists. Therefore it is natural to search the corrections to Lagrangian and gauge transformations in the form

$$\begin{align*}
\mathcal{L}_1 = \mathcal{L}_{13} & \quad \delta_1 = \delta_{12} \\
\delta_{12} h^i_{\mu \nu} &= \gamma_2 \frac{m}{m^2} e^{ij} g_{\mu \nu} F^{\alpha \beta} \partial_\alpha \xi^j_{\beta} \\
\mathcal{L}_{13} &= \frac{e_0}{m^2} e^{ij} F^{\alpha \beta} [a_1 \partial^\mu \xi_{\alpha}^i \partial^\nu_{\beta} h^j_{\mu \nu} + a_2 (\partial h)^i_{\alpha} (\partial h)^j_{\beta} + a_3 \partial_\alpha h^i_{\beta} (\partial h)^j_{\mu} + \\
&\quad + a_4 (\partial h)^i_{\alpha} \partial_\beta h^j_{\mu} + b_1 \partial^\mu b^i_{\alpha} \partial_{\mu} b^j_{\beta} + b_2 \partial_\alpha b^i_{\beta} (\partial h)^j_{\mu}]
\end{align*}$$

\(^3\)Note that in this section in all terms linear in $e/m$ field strength $F_{\mu \nu}$, i.e. cubic terms in the Lagrangians and linear terms in gauge transformations, we will use ordinary partial derivatives because their replacement by covariant ones produce corrections of order $F^2$ that go beyond the approximation considered.
Here $\gamma_2, a_1, a_2, a_3, a_4, b_1, b_2$ are the real coefficients. Condition of gauge invariance (1) looks in this case like
\[
\tilde{\delta}_0 L_0 + \delta_0 L_1 + \delta_1 L_0 = 0
\]
or if rewrite it in details
\[
(\delta_{00} \tilde{L}_{02} + \tilde{\delta}_{01} \tilde{L}_{01}) + \tilde{\delta}_{01} \tilde{L}_{02} + (\tilde{\delta}_{00} + \delta_{01}) L_{13} + \delta_{12} (L_{00} + L_{01} + L_{02}) = \\
= (\delta_{00} \tilde{L}_{02} + \tilde{\delta}_{01} \tilde{L}_{01}) + (\tilde{\delta}_{01} \tilde{L}_{02} + \delta_{12} L_{00}) + \\
+ (\delta_{00} L_{13} + \delta_{12} L_{01}) + (\delta_{01} L_{13} + \delta_{12} L_{02}) = 0
\]
(21)

Here we have grouped the variations in their index dimensions. Then to vanish all variations we should vanish each group independently. First, note that the condition
\[
(\delta_{01} L_{13} + \delta_{12} L_{02}) = 0
\]
corresponds exactly to the cubic vertex for the massless theory. Using the result (7) we have
\[
a_1 = -a_2 = \frac{1}{2} \gamma_2 (d - 2) \quad a_3 = a_4 = \gamma_2 (d - 2)
\]
Also it is easy to see that
\[
b_2 = 2b_1
\]
Let us now look at the rest of the relations (21). First group contains only non-invariant part, remaining after the minimal interaction, and is not compensated, thus we can immediately conclude that the constructed corrections are not enough. Besides, it is easy to check that the third group does not vanish, thus it also requires some new corrections for compensation. This can be done only by means of contributions from the variations $\tilde{\delta}_{01} L_{12}$ and $\delta_{11} \tilde{L}_{02}$. Therefore we should build the corrections $L_{12}$ and $L_{11}$. And, at last, the further calculations show that we need also the corrections of the form $L_{11}$ and $\delta_{11}$. We construct all the corrections in the most general form. As to the gauge transformations, the only possibility is
\[
\delta_{11} b^i_\mu = \frac{\delta_1}{m} \varepsilon^{ij} F^\alpha \delta_{11} c^j_\alpha
\]
Additional terms to the Lagrangian are chosen as follows
\[
L_{12} = \frac{1}{m} \varepsilon^{ij} F^\alpha [c_1 \partial_\alpha h^\mu_\beta i^j_\mu | + c_2 (\partial h^i_\alpha b^j_\beta | + c_3 \partial_\alpha h^i_\beta b^j_\beta | + c_4 \partial_\alpha i^j_\beta |]
\]
\[
L_{11} = \varepsilon^{ij} F^\alpha [d_1 h^\mu_\alpha i^j_\mu + d_2 i^j_\alpha b^j_\beta |
\]
where $c_{1,2,3,4}, d_{1,2}$ are the arbitrary coefficients. Further problem is reduced to finding all variations and fulfillment of three relations
\[
\delta_{00} L_{13} + \delta_{01} L_{12} + \delta_{11} L_{02} + \delta_{12} L_{01} = 0
\]
(22)
\[
\delta_{00} \tilde{L}_{02} + \tilde{\delta}_{01} \tilde{L}_{01} + \delta_{00} L_{11} + \delta_{11} L_{00} = 0
\]
(23)
\[
\tilde{\delta}_{01} \tilde{L}_{02} + \delta_{00} L_{12} + \delta_{01} L_{11} + \delta_{11} L_{01} + \delta_{12} L_{00} = 0
\]
(24)
where we have taken into account the contribution from minimal interaction (19). Relations (22) form the system of algebraic equations, which allows us to express the coefficients $c_{1,2,3}$ through parameters of gauge transformations $\gamma_2, \delta_1, b_1$
\[
c_1 = 2a_1 b_1 - \delta_1 + \alpha_1 \gamma_2 (d - 1), \quad c_2 = -2a_1 b_1 - \delta_1, \quad c_3 = 2a_1 b_1 + \frac{\delta_1}{2}
\]
The other two relations \((23), (24)\) yield the following result

\[
\begin{align*}
\delta_1 &= \frac{2\gamma_2(d-1) - 8b_1 + 6e_0}{3\alpha_1}, \\
\delta_2 &= \frac{2\gamma_2(d-1) + 2b_1 - 3e_0}{3}, \\
c_4 &= \frac{\alpha_0(\gamma_2(d+2) + 8b_1 - 6e_0)}{6}, \\
d_1 &= \frac{-2\gamma_2(d-1) + 2b_1 - 3e_0}{3}, \\
d_2 &= \frac{-e_0}{2}.
\end{align*}
\]

Parameters \(\gamma_2, b_1, e_0\) are arbitrary. As a result the cubic vertex and the corresponding gauge transformations are found.

**Gauge fixing and constraints.** Now we investigate the causal aspects of constructed Lagrangian, namely we show that the equations of motion for the field \(h_{\mu \nu}^i\) contains higher derivatives only in form of d’Alambertian. First, we fix the gauge transformations and eliminate the auxiliary fields \(b_\mu\) and \(\varphi\). After that the Lagrangian takes the form

\[
L_0 = \frac{1}{2} D^\alpha h^{\mu \nu} D_\alpha h_{\mu \nu} - (Dh)^\mu(Dh)_\mu - (DDh)h - \frac{1}{2} D^\mu h D_\mu h - \frac{m^2}{2} h_{\mu \nu} h_{\mu \nu} + \frac{m^2}{2} hh
\]

where we have omitted the terms quadratic in \(h\).

\[
L_1 = \frac{1}{m^2} \varepsilon^{ij} F^{\alpha \beta}[a_1 \partial^\mu h_{\alpha \beta}^i \partial_\mu h_{\nu \beta}^j + a_2 (\partial h)^{ij}_\alpha (\partial h)^{ij}_\beta + a_3 \partial_\alpha h_{\beta 1}^i h_{\mu}^j + a_4 (\partial h)^i_\alpha \partial_\beta h_{\mu}^j + m^2 d_1 h_{\alpha 1}^i h_{\beta 1}^j]
\]

It is easy to verify by direct calculations that the equations of motion yield the usual algebraic constraint \(h^i = 0\). Indeed, acting by the following second order differential operator on equations of motion ones obtain

\[
\left(D^\mu D^\nu - \frac{m^2}{d-2} \delta^{ij} g_{\mu \nu} + \frac{2\delta_1}{m^2 \alpha_1} \varepsilon^{ij} F^{\alpha} \partial_\alpha \partial_\nu \right) \left(\frac{\delta S}{\delta h_{\mu \nu}^j}\right) = -m^4 \frac{d-1}{d-2} h^i = 0
\]

where we have omitted the terms quadratic in \(F\). To see what happens with the differential constraint in the case under consideration, we act on the equations of motion by the following first order differential operator

\[
\left(g_{\mu \alpha} D^\nu - \frac{\gamma_2}{2m^2 \varepsilon^{ij} g_{\mu \nu} F^{\sigma} \partial_{\sigma}} \right) \left(\frac{\delta S}{\delta h_{\mu \nu}^j}\right) = 0.
\]

Taking into account that \(h^i = 0\) one get

\[
-m^2 (Dh)^i_\alpha + \varepsilon^{ij} [(2e_0 + d_1) F^{\sigma} \partial_{\sigma} h_{\alpha \rho}^j + (1 - d_1) F^{\sigma} (\partial h)^j_\sigma] = 0
\]

We see that the differential constraint is modified and as a result \((Dh) \sim F, (DDh) \sim F^2\). Therefore, up to terms of the second order in \(F\) the equation for \(h_{\mu \nu}^i\) can be rewritten in the form

\[
-(D^2 h_{\mu \nu})^i - m^2 h_{\mu \nu}^i - \frac{1}{m^2} \varepsilon^{ij} F^\alpha_{(\mu} (a_1 \partial^2 h_{\nu)\alpha}^j - m^2 d_1 h_{\nu\alpha}^j) + \varepsilon^{ij} (2e_0 + d_1 + \frac{a_3}{2}) F^{\alpha \beta} \partial_{\alpha} (\partial h)^j_{\beta} = 0
\]

(25)

If we choose the free parameters so that the coefficient at the last term is zero, the higher derivatives in this equation form the d’Alambertian, what guarantees the causality. This can be done by selecting the appropriate parameter \(b_1\)

\[
b_1 = \frac{\gamma_2(d+2) - 6e_0}{4}
\]
4.2 Spin 3

**Free theory.** For gauge invariant description of free massive spin-3 field theory we need a set of fields $\Phi^a = \{\phi_{\mu\nu}, h_{\mu\nu}, b_\mu, \varphi\}$, first two of them are totally symmetric. Here $h_{\mu\nu}, b_\mu, \varphi$ are the auxiliary Stueckelberg fields. With notations given at beginning of section 3, the Lagrangian for such a theory has the form

$$L_0 = L_{00} + L_{01} + L_{02}$$

$L_{02} = -\frac{1}{2} \partial^\alpha \phi^{\mu\nu} \partial_\alpha \phi_{\mu\nu} + \frac{3}{2} (\partial \phi)^j_{\mu\nu} (\partial \phi)^{\mu\nu} + 3(\partial \partial \phi)^\mu_\mu + \frac{3}{2} \partial^\alpha \tilde{\phi}^\mu_\mu \partial_\alpha \tilde{\phi}_\mu + \frac{3}{4} (\partial \tilde{\phi})(\partial \tilde{\phi}) + \frac{1}{2} \partial^\alpha h^{\mu\nu} \partial_\alpha h_{\mu\nu} - (\partial h)^\mu_\mu (\partial h)_\mu - (\partial \partial h) h - \frac{1}{2} \partial^\mu h \partial_\mu h - \frac{1}{2} \partial^\mu b_\mu b_\mu + \frac{1}{2} (\partial b)(\partial b) + \frac{1}{2} \partial^\alpha \varphi \partial_\alpha \varphi$

$$L_{01} = m[\alpha_2 \phi^{\mu\nu} \partial_\mu h_{\nu\sigma} - 2\alpha_2 \delta^{\mu\nu}(\partial h)_\mu + \frac{\alpha_2}{2} \delta^{\mu\nu} h \partial_\mu h + \alpha_1 h^{\mu\nu} \partial_\nu b_\mu - \alpha_1 h (\partial b) + \alpha_0 b^\mu \partial_\mu \varphi]$$

$$L_{00} = m^2\left[\frac{1}{2} \phi^{\mu\nu} \phi_{\mu\nu} - \frac{3}{2} \delta^{\mu\nu} \phi_\mu + \frac{3}{2} h \phi + \frac{3}{4} h^2 + \frac{\alpha_1}{2} h \varphi - \frac{d+2}{2d} b^\mu b_\mu + \frac{d+1}{d-2} \varphi^2\right]$$

Lagrangian $[26]$ is invariant under the gauge transformations

$$\delta_0 = \delta_{00} + \delta_{01}$$

$$(\delta_0 + \delta_{00}) \phi_{\mu\sigma} = \partial_{(\mu} \xi_{\sigma)} - \frac{2m\alpha_2}{3d} g_{\mu\nu} \xi_{\sigma}$$

$$(\delta_0 + \delta_{00}) h_{\mu\nu} = \partial_{(\mu} \xi_{\nu)} - m(\alpha_2 \xi_{\mu\nu} - \frac{\alpha_1}{d-2} g_{\mu\nu} \xi)$$

$$(\delta_0 + \delta_{00}) b_\mu = \partial_\mu \xi + m\alpha_1 \xi_{\mu}$$

$$\delta_{00} \varphi = -m\alpha_0 \xi$$

where

$$(\alpha_2)^2 = 3, \quad (\alpha_1)^2 = 4 \frac{d+1}{d}, \quad (\alpha_0)^2 = \frac{3d}{d-2}$$

gauge parameter $\xi_{\mu\nu}$ is symmetric and traceless, $g_{\mu\nu} \xi_{\mu\nu} = 0$. As for the spin-2 case, note that in the massless limit $m \to 0$ the Lagrangian is decomposed into the sum of the Lagrangians describing massless fields with spins 3, 2, 1 and 0 that corresponds to a conservation of the number of physical degrees of freedom.

**Minimal interaction.** We formulate the charged spin-3 field in terms of real doublets $\Phi^a = \{\phi_{\mu\nu}^i h_{\mu\nu}^i, b_\mu^i, \varphi^i\}$, $i = 1,2$. Minimal electromagnetic interaction violates the gauge invariance. Non vanishing gauge variations of the Lagrangian in the linear approximation in $F_{\mu\nu}$ are

$$\delta_0 L_0 = (\delta_{00} L_{02} + \delta_{01} L_{01}) + \bar{\delta}_{01} \bar{L}_{02}$$

$$\delta_{00} \bar{L}_{02} + \delta_{01} \bar{L}_{01} = me_0 \varepsilon^{ij} \xi^{i}_{\mu\nu} [2\alpha_2 F^{\alpha\beta} h_{\alpha}^i + \frac{d+1}{d} F^{\alpha\beta} \tilde{b}_i^j + \alpha_1 F^{\alpha\beta} b_\alpha^i]$$

$$\bar{L}_{01} \bar{L}_{02} = e_0 \varepsilon^{i}_{\mu\nu} \xi^{j}_{\mu\nu} [6F^{\alpha\beta} \partial_\alpha h_{\beta}^i + 6F^{\alpha\beta} (\partial \phi)_\alpha^i - 6F^{\alpha\beta} \partial^\nu \phi_{\alpha}^i - 9F^{\alpha\beta} \partial_\alpha \phi_{\beta}^i + 3F^{\alpha\beta} \partial_\alpha \phi_{\beta}^i]$$
Non-minimal interactions. To recover gauge invariance we add non-minimal terms to the Lagrangian and gauge transformations. Analogously to the spin-2 case, we write down all possible additional terms. Corrections to gauge transformations are

\[
\delta_1 = \delta_{11} + \delta_{12}
\]

\[
\delta_{12} \phi^i_{\mu\nu} = \frac{\gamma^3}{m^2} \varepsilon^{ij} g(\mu \nu) F^{\alpha\beta} \partial_{\alpha} \xi^j_{\beta} \\
\delta_{12} h^i_{\mu\nu} = \frac{\gamma^2}{m^2} \varepsilon^{ij} g_{\mu\nu} F^{\alpha\beta} \partial_{\alpha} \xi^j_{\beta}
\]

Additional terms to the Lagrangian look like

\[
\mathcal{L}_1 = \mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{13}
\]

Both the corrections to gauge transformations and corrections to Lagrangian contain some number of arbitrary parameters. Before to go further let us make a few comments on above expressions for Lagrangian.

- First of all, note that corrections \( \mathcal{L}_{13} \) and \( \delta_{12} \) should have the same form as the cubic vertex and gauge transformations in massive theory. It thus provides the correct massless limit \( \epsilon_0 \to 0, \frac{m^2}{\epsilon_0} = \text{const} \).

- Corrections to Lagrangian \( \mathcal{L}_{12} \) is constructed in such a way that cross-structures contain the fields describing the neighboring spins only.

- To construct the \( \mathcal{L}_{11} \) we have considered all possible combinations of fields. The same concerns the gauge transformations.
Finding the unknown coefficients is based on gauge invariance of the full Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with respect to gauge transformations $\delta = \delta_0 + \delta_1$. Also we do not forget about non-invariant part remaining after the minimal interaction (27). Hence we have

$$\bar{\delta}_0 \bar{\mathcal{L}}_0 + \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0$$

or if to rewrite in more details

$$\begin{align*}
(\delta_{00} \bar{\mathcal{L}}_{02} + \delta_{01} \bar{\mathcal{L}}_{01}) + \delta_{01} \bar{\mathcal{L}}_{02} + \\
+ (\delta_{00} + \delta_{01})(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{13}) + (\delta_{11} + \delta_{12})(\mathcal{L}_{00} + \mathcal{L}_{01} + \mathcal{L}_{02}) = \\
= (\delta_{00} \bar{\mathcal{L}}_{02} + \delta_{01} \bar{\mathcal{L}}_{01} + \delta_{00} \mathcal{L}_{11} + \delta_{11} \mathcal{L}_{00}) + \\
+ (\delta_{01} \bar{\mathcal{L}}_{02} + \delta_{00} \mathcal{L}_{12} + \delta_{01} \mathcal{L}_{11} + \delta_{11} \mathcal{L}_{01} + \delta_{12} \mathcal{L}_{00}) + \\
+ (\delta_{00} \mathcal{L}_{13} + \delta_{01} \mathcal{L}_{12} + \delta_{11} \mathcal{L}_{02} + \delta_{12} \mathcal{L}_{01}) + (\delta_{01} \mathcal{L}_{13} + \delta_{12} \mathcal{L}_{02}) = 0
\end{align*}$$

(29)

Here on the last step, we have grouped the variations by dimensions, what means in our notations equality the sums of first and second indices in individual variations. Vanishing the all variations requires vanishing of each group independently. The rest consideration is quire direct but rather tedious, therefore we formulate here only the final results. The intermediate calculations are given in the Appendix.

Thus, the condition

$$\delta_{01} \mathcal{L}_{13} + \delta_{12} \mathcal{L}_{02} = 0$$

corresponds to cubic vertices of massless theory for fields $\phi_{\mu\nu\sigma}^i$ and $h_{\mu\nu}^i$. Coefficients in Lagrangian $\mathcal{L}_{13}$ are given by (7) and (10). Following relation

$$\delta_{00} \mathcal{L}_{13} + \delta_{01} \mathcal{L}_{12} + \delta_{11} \mathcal{L}_{02} + \delta_{12} \mathcal{L}_{01} = 0$$

allow us to express the coefficients in Lagrangian $\mathcal{L}_{12}$ (except $c_{31}$, which in this approximation does not rise the contribution) through the parameters $\gamma_3, \gamma_2, \eta_3, \delta_1$ (see Appendix (37))

$$\begin{align*}
c_{11} &= \frac{e_1 \gamma_3 (d + 2) + 3 \eta_3 d}{2}, \\
c_{12} &= -\alpha_2 \gamma_3 d + 3 \eta_3 d, \\
c_{13} &= \frac{\alpha_2 \gamma_3 (d - 4) - 6 \eta_3 d}{4}, \\
c_{14} &= \alpha_2 \gamma_3 (d - 2) - 3 \eta_3 d, \\
c_{15} &= \alpha_2 \gamma_3 d - 3 \eta_3 d
\end{align*}$$

Besides, it yields one equation on the parameters of gauge transformations

$$\alpha_2 \gamma_3 d - \alpha_2 \gamma_2 (d - 2) - 3 \eta_3 d + 2 \delta_2 = 0$$

(30)

From the remaining two relations

$$\begin{align*}
\bar{\delta}_0 \mathcal{L}_{02} + \delta_{01} \mathcal{L}_{01} + \delta_{00} \mathcal{L}_{11} + \delta_{11} \mathcal{L}_{00} &= 0, \\
\bar{\delta}_{01} \mathcal{L}_{02} + \delta_{00} \mathcal{L}_{12} + \delta_{01} \mathcal{L}_{11} + \delta_{11} \mathcal{L}_{01} + \delta_{12} \mathcal{L}_{00} &= 0
\end{align*}$$

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which lead to the system of equations (38), (39) (see Appendix), one can express the other coefficients in terms of parameters of gauge transformations

\[ c_{31} = \frac{\alpha_0 \gamma_2 d}{2} - \frac{\alpha_0 \delta_1}{\alpha_1}, \quad d_3 = \frac{2\alpha_2 \alpha_1 b_1 + \alpha_1 \delta_2 - \alpha_2 \delta_1}{2}, \]

\[ d_1 = \frac{3\gamma_3 (d + 2) + 3\alpha_2 \eta_3 d + 12 \epsilon_0}{4}, \quad d_4 = \epsilon_0, \]

\[ d_2 = \frac{-3\gamma_3 d + 3\alpha_2 \eta_3 d + 2\alpha_2 \delta_2 + 6 \epsilon_0}{4}, \quad d_5 = \frac{12b_1 (d + 2) + 3\alpha_2 \eta_3 d (d + 2) + 2\alpha_2 \delta_2 (d + 2) - 12 \epsilon_0}{12d}. \]

The above two relations impose the conditions on the parameters of gauge transformations, which together with (30) give us the following solution

\[
\begin{align*}
\gamma_3 &= \frac{\alpha_2 (\gamma_3 (d - 2) - \gamma_2 (d - 2) - 6 \epsilon_0)}{9d}, \\
\delta_2 &= \frac{-\gamma_3 (d + 1) + \gamma_2 (d - 2) - 3 \epsilon_0}{\alpha_2}, \\
\delta_1 &= \sqrt{\frac{d + 1}{d}} \left( \frac{2\gamma_3 (d + 4) - 2\gamma_2 (d + 1) + 6 \epsilon_0}{3} \right)
\end{align*}
\]

The parameters \( \gamma_3, \gamma_2 \) and \( \epsilon_0 \) are arbitrary. As a result the first order corrections to Lagrangian and gauge transformations are found.

**Gauge fixing and constraints.** Now we turn to study the causal aspects of the constructed Lagrangian. The problem is more complicated in comparison with spin-2 case due to the fact that the auxiliary scalar field is not eliminated. Indeed, because of tracelessness of parameter \( \xi_{\mu \nu} \) we cannot kill all auxiliary fields. For example, one can fix the gauge transformations so that the fields \( \phi, b_j \) are eliminated as well as the traceless part of tensor \( h_{\mu \nu} \), which is determined by the decomposition

\[ h_{\mu \nu} = h'_{\mu \nu} + \frac{1}{d} g_{\mu \nu} h. \]

Here \( h'_{\mu \nu} \) is traceless part, \( g^{\mu \nu} h'_{\mu \nu} = 0 \) and \( h \) is a trace of \( h_{\mu \nu} \), \( g^{\mu \nu} h_{\mu \nu} = h \). Then, after such a gauge fixing the Lagrangian takes the form

\[
\mathcal{L}_0 = -\frac{1}{2} D^\alpha \phi^{\mu \sigma} D_{\alpha} \phi_{\mu \sigma} + \frac{3}{2} (D \phi)^{\mu \nu} (D \phi)_{\mu \nu} + 3 (D D \phi)^{\mu} \tilde{\phi}_\mu + \frac{3}{2} D^\alpha \tilde{\phi}^\mu D_{\alpha} \tilde{\phi}_\mu + \\
+ \frac{3}{4} (D \tilde{\phi}) (D \tilde{\phi}) - \frac{(d - 2)(d - 1)}{2d^2} D^\mu h D_\mu h + \\
+ m \alpha_2 \frac{d}{2d} \tilde{\phi}^\mu D_\mu h + m^2 \left[ \frac{1}{2} \phi^{\mu \sigma} \phi_{\mu \sigma} - \frac{3}{2} \tilde{\phi}^\mu \tilde{\phi}_\mu + \frac{3}{4} h h \right]
\]

\[
\mathcal{L}_1 = -\frac{1}{m^2} \epsilon^{ij} \Gamma^\alpha_{\beta \gamma} \left( \frac{3 \gamma_3 d}{4} [\partial^\mu \phi^{\alpha \sigma}] i \partial^i j \phi_{i j} - 2 (\partial \phi)^{\mu} \phi_{\mu \beta} + 2 \partial_\alpha \phi^{\mu \nu} \partial^i j [\phi_{i j}] \right) + \\
+ 4 (\partial \phi)^{\mu} \phi_{\mu \beta} + 2 \partial_\alpha \phi^{\mu \nu} \partial^i j \phi_{i j} - \partial^\mu \tilde{\phi}^i j - \partial_\alpha \tilde{\phi}^i j \right) - \\
- m \frac{c_{11} + c_{13} + c_{14} d - c_{15}}{d} \partial_\alpha \tilde{\phi}^i j h^i j - m^2 \left[ d_1 \phi^{\mu \nu} \phi_{\mu \beta} + d_2 \tilde{\phi}^i j \tilde{\phi}^i j \right];
\]
First, we show that the scalar field $h^i$ is auxiliary, i.e. the equations of motion lead to $h^i = 0$ as their consequence. To see that, we act by the following operators on equations of motion for the fields $\phi_{\mu\nu\sigma}^i$, $h^i$ and combine them

$$A_{\mu\nu\sigma}^{ij} \left( \frac{\delta S}{\delta \phi_{\mu\nu\sigma}^j} \right) - \frac{m}{\alpha_2} (D^{2ij} - 2m^2 \frac{d+1}{d-2} \delta^{ij}) \left( \frac{\delta S}{\delta h^j} \right) = m^2 \frac{\alpha_2(d+1)}{d-2} h^i = 0 \quad \text{(31)}$$

Here the operator $A_{\mu\nu\sigma}^{ij}$ has the form

$$A_{\mu\nu\sigma}^{ij} = (D^{i k} D_{\rho}^{k l} - \frac{1}{d} g_{\mu \nu} D^{i l} - \frac{m^2}{d} g_{\mu \sigma} \delta^{i l}) D_{\sigma}^{j} + \frac{1}{m^2} \varepsilon^{i j} \left( \beta_1 g_{\mu \nu} F_{\alpha}^{\sigma} \delta_{\sigma} \partial^2 + \beta_2 \partial_{\mu} \partial_{\nu} F_{\alpha}^{\sigma} \partial_{\sigma} + m^2 \beta_3 g_{\mu \nu} F_{\alpha}^{\sigma} \partial_{\sigma} \right)$$

with the following parameters

$$\beta_1 = \frac{\gamma_3}{d}, \quad \beta_2 = \frac{\gamma_3}{2d} - \frac{\gamma_2}{2}$$

$$\beta_2 = -\gamma_3 (d + 2) + \gamma_2 (d - 1) - 3 \epsilon_0 = \alpha_2 \delta_2 + \gamma_3 + \gamma_2$$

Thus, it follows from (31) that $h^i = 0$ when $m \neq 0$. Further we set the scalar field to be zero.

One can expect that the algebraic constraint is same as for free theory, namely $\tilde{\phi}_{\mu}^i = \phi_{\alpha \mu}^i = 0$. We show that this is actually true. To do that, we act on the equations of motion by the following operator

$$B_{\rho\mu\nu\sigma}^{ij} \left( \frac{\delta S}{\delta \phi_{\mu\nu\sigma}^i} \right) + (- \frac{m}{\alpha_2} D^{ij} + \varepsilon^{ij} \frac{1}{m} \rho_0 F_{\rho}^{\alpha} \partial_{\alpha}) \left( \frac{\delta S}{\delta h^j} \right) = m^2 \frac{\alpha_2(d+1)}{d-2} \phi_{\rho}^j = 0 \quad \text{(32)}$$

where

$$B_{\rho\mu\nu\sigma}^{ij} = (g_{\mu \nu} D_{\rho}^{k l} D_{\sigma}^{k l} - \frac{1}{d} g_{\mu \nu} D_{\rho}^{k l} D_{\sigma}^{k l} - \frac{m^2}{d} g_{\mu \sigma} \delta^{i l}) + \frac{1}{m^2} \varepsilon^{i j} \left( \rho_1 g_{\mu \nu} F_{\alpha}^{\sigma} \partial_{\alpha} - \frac{1}{d} \rho_1 g_{\mu \nu} F_{\alpha}^{\sigma} \partial_{\alpha} + \frac{\gamma_3}{d} g_{\mu \nu} F_{\alpha}^{\sigma} \partial_{\sigma} + \rho_2 F_{\rho}^{\alpha} \partial_{\sigma} - \frac{m^2}{2} \gamma_3 d g_{\mu \nu} F_{\sigma \rho} \right)$$

with the parameters

$$\rho_1 = -\frac{1}{3} (\gamma_3 (d + 1) - \gamma_2 (d - 2) + 3 \epsilon_0) = \frac{\delta_2}{\alpha_2}$$

$$\rho_2 = \frac{1}{3d} (2d_2 - 3 \epsilon_0) = \frac{3 \gamma_3 - 3 \alpha_2 \eta_3 - 2 \alpha_2 \delta_2 d}{6}$$

$$\rho_0 = \frac{1}{6 \alpha_2} (5 \gamma_3 (d + 2) + \gamma_2 (d + 4) + 6 \epsilon_0) = \frac{2 \gamma_3 d + 2 \gamma_2 - \alpha_2 \eta_3 d}{2 \alpha_1}$$

From (32) we have the algebraic constraint $\tilde{\phi}_{\rho}^i = 0$, what will be taken into account later.

Let us study now how the differential constraint is modified. For that purpose we act on the equations of motion by the first order differential operator $C_{\rho\lambda\mu\nu\sigma}^{ij}$

$$C_{\rho\lambda\mu\nu\sigma}^{ij} \left( \frac{\delta S}{\delta \phi_{\mu\nu\sigma}^j} \right) = 0$$
where
\[
C^i_{j\rho\lambda\mu\sigma} = (g_{\mu\rho}g_{\nu\lambda} - \frac{1}{d}g_{\rho\lambda}g_{\mu\nu})D^j_{\sigma} - \frac{1}{m^2}\varepsilon^{ij}\left(\frac{1}{2}\gamma_3 g_{\mu\nu}g_{\sigma(\rho}F^\alpha_{\lambda)}\partial_\alpha - \frac{1}{d}\gamma_3 g_{\mu\nu}g_{\rho\lambda}F^\alpha_{\sigma}\partial_\alpha\right)
\]

As a result we have the differential constraint
\[
m^2(D\phi)_{\rho\lambda}^i - \varepsilon^{ij}(\frac{2}{3}c_1 - 2\epsilon_0)F^{\alpha\beta}\partial_{\beta}\phi_{\alpha\rho\lambda}^j + \varepsilon^{ij}(\frac{2}{3}c_1 + \epsilon_0)F_{\alpha(\rho(\partial\phi)_{\lambda)}^j = 0
\]

Taking into account all constraints, we can rewrite the equations of motion (up to the terms quadratic in \(F\)) in the form
\[
D^2\phi_{\mu\nu\sigma, i} + m^2\phi_{\mu\nu\sigma, i} - \frac{1}{m^2}\varepsilon^{ij}\left[\gamma_3 d\partial^2 F^{\alpha(\mu\phi_{\nu\sigma}), j} + m^2 \frac{2c_1}{3} F^{\alpha(\mu\phi_{\nu\sigma}), j}\right] - \frac{1}{m^2}\varepsilon^{ij}\left(\frac{\gamma_3 d}{2} - \frac{2}{3}d_1 + 2\epsilon_0\right)F^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\phi_{\mu\nu\sigma, i} = 0
\]

The last term can be canceled by suitable choice of the free parameter \(\gamma_2\)
\[
\gamma_2 = \frac{\gamma_3(d+4) - 6\epsilon_0}{d-2}
\]

Then we see that the higher derivatives in the equations of motion (33) form the d’Alambertian, what ensures the causality. Note that if we express \(\gamma_3\) from (34) and substitute in the formula for \(b_1\), we get
\[
b_1 = -\frac{\gamma_2(d+2) - 6\epsilon_0}{4}
\]
that is the same for spin-2 one up to sign redefinition.

### 4.3 Arbitrary integer spin

**Free theory.** As before, we use the gauge invariant approach to massive higher spin theories proposed in [21] (also see first ref. in [14]). In this approach the theory of integer spin \(s\) is described by a set of totally symmetric double traceless tensor fields \(\Phi^k, \tilde{\Phi}^{k-4} = 0\) where \(0 \leq k \leq s\). We again use the compact notation (11). Free Lagrangian has form
\[
\mathcal{L} = \mathcal{L}_{02} + \mathcal{L}_{01} + \mathcal{L}_{00}
\]

\[
\mathcal{L}_{02} = \sum_{k=0}^{s}(-1)^k\left[\frac{1}{2}\partial^\mu\Phi^k\partial_\mu\Phi - \frac{k}{2}(\partial\Phi)^{k-1}(\partial\Phi)_{k-1} + \frac{k(k-1)}{2}\partial_{\mu_1}\Phi^k_{\mu_2}\partial_{\mu_2}\Phi_{k-2} - \frac{k(k-1)(k-2)}{4}\partial_\nu\Phi^k_{\mu_2}\partial_\mu_2\Phi_{k-2} - \frac{k(k-1)(k-2)}{8}(\partial\Phi)^{k-3}(\partial\Phi)_{k-3}\right]
\]

\[
\mathcal{L}_{01} = \sum_{k=0}^{s}(-1)^k m \left[a_{1k}(\partial\Phi)^{k-1}(\partial\Phi)_{k-1} + a_{2k} \Phi^{k-2}(\partial\Phi)_{k-2} + a_{3k}(\partial\Phi)^{k-3}\Phi_{k-3}\right]
\]

\[
\mathcal{L}_{00} = \sum_{k=0}^{s}(-1)^k m^2 \left[b_{1k}(\Phi^k)^2 + b_{2k}(\tilde{\Phi}^{k-2})^2 + b_{3k}\tilde{\Phi}^{k-2}\Phi_{k-2}\right]
\]
The gauge transformations are written in the form

\[(\delta_{01} + \delta_{00})\Phi_k = \partial_{(\mu_1} \xi_{k-1)} + m \left( \alpha_k \xi_k + \beta_k g(\mu_1\mu_2 \xi_{k-2}) \right), \quad \tilde{\xi}_{k-3} = 0\]

All unknown coefficients are uniquely determined by the parameters \(\alpha_k\) from gauge invariance

\[a_{1k} = -\alpha_{k-1}, \quad a_{2k} = -\alpha_{k-1}(k-1), \quad a_{3k} = -\alpha_{k-1} \frac{(k-1)(k-2)}{4}\]

\[b_{1k} = -\frac{1}{2(k+2)}(\alpha_{k+1})^2 \frac{d+2k}{d+2k-2}\]

\[b_{2k} = \frac{k-1}{4}(\alpha_{k-1})^2 - \frac{k(k-1)}{8(k+1)}(\alpha_k)^2 \frac{d+2k}{d+2k-4}\]

\[b_{3k} = -\frac{1}{2} \alpha_{k-1} \alpha_{k-2}, \quad \beta_k = \frac{2\alpha_{k-1}}{k(d+2k-6)}\]

Parameter \(\alpha_k\) is fixed from the requirement that the coefficient \(b_{1s}\) for basic field \(\Phi^s\) has the standard canonical value \(b_{1s} = -\frac{1}{2}\). Then

\[(\alpha_k)^2 = (s-k)(k+1) \frac{(d+s+k-3)}{(d+2k-2)}\]

Again, note that in the massless limit \(m \to 0\) this Lagrangian is decomposed in sum of correct Lagrangians describing the massless fields of all spins below \(s\) inclusively.

**Minimal interaction.** After switching on the minimal interactions non-invariant part of variations in linear approximation on \(F^{\mu\nu}\) is equal to

\[\delta \mathcal{L} = (\delta_{00} + \tilde{\delta}_{01})(\mathcal{L}_{00} + \tilde{\mathcal{L}}_{01} + \tilde{\mathcal{L}}_{02})\]

(35)

The bar means that we have replaced the usual derivatives by covariant ones. Here

\[\delta_{00} \tilde{\mathcal{L}}_{02} + \tilde{\delta}_{01} \mathcal{L}_{01} = \sum_{k=1}^{s} (-1)^k m e_0 \varepsilon^{ij} \xi_{k-1} \left[ -\alpha_{k-1}(k-1) F^{\alpha\mu_1} \Phi_{\alpha_{k-2},j} + \right.\]

\[\left. + (-1)^k m e_0 \varepsilon^{i j} \xi_{k-2} \left[ -\alpha_{k-1}(k-2) \frac{d+2k-5}{d+2k-6} F^{\alpha\mu_1} \Phi_{\alpha_{k-3},j} \right] \right] \]

\[\tilde{\delta}_{01} \mathcal{L}_{02} = \sum_{k=1}^{s} (-1)^k e_0 \varepsilon^{i j} \xi_{k-1} \left[ -2k F^{\alpha\beta} \partial_{\alpha} \Phi_{\beta}^{k-1,j} - k(k-1) F^{\alpha\mu_1} (\partial \Phi)_{\alpha_{k-2},j} + \right.\]

\[\left. + \frac{3}{2} k(k-1) F^{\alpha\mu_1} \partial_{\alpha} \tilde{\Phi}_{k-2,j} + k(k-1)(k-2) F^{\alpha\mu_1} \partial_{\mu_2} \tilde{\Phi}_{\alpha_{k-3},j} \right] \]

**Non-minimal interaction.** As in previous subsections to recover a broken invariance of the Lagrangian we should add the appropriate corrections to Lagrangian and gauge transformations. The form and structure of these corrections we have discussed in two previous subsections for examples of spin-2 and spin-3 fields. Therefore here we just write down the corresponding deformation terms for arbitrary spin. The additional Lagrangian has the form

\[\mathcal{L}_1 = \mathcal{L}_{13} + \mathcal{L}_{12} + \mathcal{L}_{11}\]
\[ \mathcal{L}_{13} = \sum_{k=0}^{s} (-1)^k \frac{1}{m^2} \varepsilon^{ij} F_{\alpha \beta} F_{\alpha \beta k-1}^{k-1, i \partial \mu \Phi_{\alpha k}^{k-1, j} + c_{2k}(\partial \Phi)_\alpha^{k-2, i}(\partial \Phi)_{\beta k-2}^j + + c_{3k} \partial \mu \Phi_{\beta k-1}^{k-1, j} + c_{4k}(\partial \Phi)_\alpha^{k-2, i} \partial \beta \Phi_{k-2}^{k-2, j} + c_{5k}(\partial \Phi)_\alpha^{\mu k-3, i} \partial \mu_1 \Phi_{\beta k-3}^{k-3, j} + + c_{6k} \partial \mu_2 \Phi_{\beta k-3}^{k-3, j} + c_{7k} \partial \mu_3 \Phi_{\beta k-3}^{k-3, j} + c_{8k}(\partial \Phi)_\alpha^{k-4, i}(\partial \Phi)_{\beta k-4}^j] \]

\[ \mathcal{L}_{12} = \sum_{k=0}^{s} (-1)^k \frac{1}{m} \varepsilon^{ij} F_{\alpha \beta} [d_{1k} \Phi_{\alpha k-1}^{k-1, i} \partial \beta \Phi_{k-1}^{k-1, j} + d_{2k} \Phi_{\alpha k-2}^{k-2, i} \partial \beta \Phi_{k-2}^{k-2, j} + d_{3k} \Phi_{\alpha k-2}^{k-2, i} \partial \beta \Phi_{k-2}^{k-2, j} + + d_{4k} \Phi_{\alpha k-3}^{k-3, i} \partial \beta \Phi_{k-3}^{k-3, j} + d_{5k} \Phi_{\alpha k-3}^{k-3, i} \partial \beta \Phi_{k-3}^{k-3, j} + + d_{6k} \Phi_{\alpha k-4}^{k-4, i} \partial \beta \Phi_{k-4}^{k-4, j}] \]

\[ \mathcal{L}_{11} = \sum_{k=0}^{s} (-1)^k \varepsilon^{ij} F_{\alpha \beta} [e_{1k} \Phi_{\alpha k-1}^{k-1, i} \partial \beta \Phi_{k-1}^{k-1, j} + e_{2k} \Phi_{\alpha k-3}^{k-3, i} \partial \beta \Phi_{k-3}^{k-3, j} + e_{3k} \Phi_{\alpha k-3}^{k-3, i} \partial \beta \Phi_{k-3}^{k-3, j}] \]

All possible linear in external field strength corrections to gauge transformations (up to possible redefinitions) are written as follows:

\[ \delta_{12} \Phi_k^i = \frac{\gamma_k}{m^2} \varepsilon^{ij} g_{(\mu_1 \mu_2} F_{\alpha \beta \mu \nu)} \partial_\alpha \xi_{\beta k-2}^j \]

\[ \delta_{11} \Phi_k^i = \frac{1}{m} \varepsilon^{ij} \left( \delta_{k} F_{\mu \nu}^{\alpha \beta} \xi_{\alpha k-3}^j + \eta_k g_{(\mu_1 \mu_2} F_{\mu \nu)}^{\alpha \beta} \xi_{\alpha k-3}^j \right) \]

It remains to vary the above Lagrangian according to rule (29) and find unknown coefficients. First of all, note that the variations with three derivatives (\( \delta_{01} \mathcal{L}_{13} + \delta_{12} \mathcal{L}_{02} \)), correspond to cubic vertex of massless theory. Therefore we immediately can write down the expressions for the coefficients \( c_{1-8,k} \), they are given by relations (16).

Let us consider the variations with two derivatives \( \delta_{00} \mathcal{L}_{13} + \delta_{01} \mathcal{L}_{12} + \delta_{11} \mathcal{L}_{02} + \delta_{12} \mathcal{L}_{01} \)

They yield the following system of linear equations (40) (see Appendix), allowing to express coefficients \( d_{1-6,k} \) through the parameters of gauge transformations \( \gamma \) and \( \delta \)

\[ d_{1k} = \alpha_k^{k-1} \gamma_k (d + 2k - 5) - \frac{1}{2} \alpha_k^{k-1} \gamma_k (d + 2k - 8) - \delta_k \]

\[ d_{2k} = -\frac{1}{2} \alpha_k^{k-1} \gamma_k (k - 1) (d + 2k - 8) - \delta_k (k - 1) \]

\[ d_{3k} = -\frac{1}{2} \alpha_k^{k-1} \gamma_k (k - 1) (d + 2k - 8) - \frac{1}{2} (k - 1) \delta_k \]

\[ d_{4k} = -\frac{1}{2} \alpha_k^{k-1} \gamma_k (k - 1) (d + 2k - 8) - \delta_k (k - 1) (k - 2) \]

\[ d_{5k} = -\frac{1}{8} \alpha_k^{k-1} \gamma_k (k - 1) (k - 2) (d + 2k - 8) + \frac{1}{4} \alpha_k^{k-1} \gamma_k (k - 1) (k - 2) (d + 2k - 8) + + \frac{1}{2} \delta_k (k - 1) (k - 2) \]

\[ d_{6k} = -\frac{1}{8} \alpha_k^{k-1} \gamma_k (k - 1) (k - 2) (d + 2k - 8) - \frac{1}{4} \delta_k (k - 1) (k - 2) (k - 3) \]

Besides, we have one more equation on parameters what allows to express \( \eta_k \)

\[ \eta_k = \frac{\alpha_k^{k-1} \gamma_k (d + 2k - 8) - \alpha_k^{k-1} \gamma_k (d + 2k - 6) + 2 \delta_k}{k (d + 2k - 6)} \]
Let us group the variations with one derivative, and also take into account the contributions obtained from minimal interaction (35)

\[ \delta_{01} \mathcal{L}_{02} + \delta_{00} \mathcal{L}_{12} + \delta_{01} \mathcal{L}_{11} + \delta_{11} \mathcal{L}_{01} + \delta_{12} \mathcal{L}_{00} \]

They yield the system of equations (41) (see Appendix), allowing us to find the coefficients \(e_{1-3,k}\) through parameters \(\gamma\) and \(\delta\)

\[
e_{1k} = -\frac{1}{4}(\alpha_{k-1})^2 \gamma_{k-1}(d + 2k - 8) + \frac{k}{k + 1} (\alpha_k)^2 \gamma_k \frac{(d + 2k - 6)(d + 2k - 3)}{d + 2k - 4}
\]

\[-\frac{1}{2k + 1}(\alpha_k)^2 \gamma_{k+1}(d + 2k - 3) + 2 \frac{k}{k + 1} \alpha_k \delta_k \frac{d + 2k - 3}{d + 2k - 4} - \alpha_{k-1} \delta_{k-1} + \frac{k}{2} e_0\]

\[
e_{2k} = \frac{1}{4}(k - 1)(k - 2) \left[\frac{1}{2} (\alpha_{k-1})^2 \gamma_{k-1}(d + 2k - 8) - \frac{k}{2k + 1} (\alpha_k)^2 \gamma_k \frac{(d + 2k - 6)(d + 2k - 3)}{d + 2k - 4} + \frac{k}{4k + 1} (\alpha_k)^2 \gamma_{k+1}(d + 2k - 3)
\]

\[+ \frac{k}{k + 1} \alpha_k \delta_k \frac{d + 2k - 3}{d + 2k - 4} + 2 \alpha_{k-1} \delta_{k-1} - k e_0]\]

\[
e_{3k} = -(k - 2) \left[\frac{1}{4} \alpha_{k-1} \alpha_{k-2} \gamma_{k-2}(d + 2k - 10) + \frac{1}{2} \alpha_{k-1} \delta_{k-2} + \frac{1}{2} \alpha_{k-2} \delta_{k-1}\right]\]

However we did not use yet three equations for the parameters of gauge transformations (41). Taking into account these three equations for parameters \(\gamma_k\) and \(\delta_k\) together with equations which follow from variations without derivatives (42), we obtain finally the recurrent relations for parameters \(\gamma_k\) and \(\delta_k\). As a result we have the extremely complicated system of recurrent relations. Solution to these relations is written as follows. First of all we introduce the notations

\[\hat{\gamma}_s = (d + 2s - 2) \gamma_s, \quad \hat{\gamma}_{s-1} = (d + 2s - 8) \gamma_{s-1}, \quad \delta_k = \alpha_k \delta_k\]

Then

\[
\gamma_k = \frac{-(s - k - 1) [3(d + 2s - 8)(d + 2k - 2) + 4(s - k - 3)(s - k - 2)] \hat{\gamma}_s}{3(d + 2k - 2)(d + 2k - 4)(d + 2k - 6)}
\]

\[+ \frac{(s - k) [3(d + 2s - 6)(d + 2k - 2) + 4(s - k - 2)(s - k - 1)] \hat{\gamma}_{s-1}}{3(d + 2k - 2)(d + 2k - 4)(d + 2k - 6)}\]

\[\hat{\delta}_k = \frac{(d + s + k - 4) [(d + s + k - 4)^2 + 3(s - k - 4)(s - k) + 8] \hat{\gamma}_s}{3(d + 2k)(d + 2k - 2)(d + 2k - 4)}
\]

\[- \frac{(d + s + k - 3) [(d + s + k - 3)^2 + 3(s - k - 3)(s - k + 1) + 8] \hat{\gamma}_{s-1}}{3(d + 2k)(d + 2k - 2)(d + 2k - 4)} + e_0\]

The parameters \(\gamma_s, \gamma_{s-1}\) are arbitrary. For the cases \(s = 2\) and \(s = 3\) the above relations coincide with expressions for the parameters given in subsection 2 and subsection 3 respectively. As a result, the first order corrections to Lagrangian and gauge transformations for arbitrary spin \(s\) are finally found.

5 Conclusion

We have constructed the cubic vertex for arbitrary bosonic massless and massive higher spin fields in \(d\) dimensional flat space coupled to constant electromagnetic background. Construction
is based on gauge-invariant description as a fundamental principle that determine as basic properties of the free theory as well as possible forms of interaction. In case of massive theories, to provide the gauge invariance we have used a formulation with corresponding Stueckelberg fields.

The cubic interaction vertex is a deformation of free Lagrangian by the terms linear in electromagnetic field strength. Such a deformation violates the free gauge invariance and to restore the gauge invariance of the theory we also must deform the free gauge transformations by the terms linear in strength. The deformations of free Lagrangian and free gauge transformation are obtained in explicit form and gauge invariance of the total Lagrangian has been tested up to the first order in strength.

The above procedure works in massless theory directly. In the massive theory, we started with the formulation of the free theory as the gauge-invariant model. The problem of constructing the vertex is immediately complicated by the fact that it was necessary to introduce the auxiliary Stueckelberg fields. Further, the introduction of the minimal interaction with electromagnetic field violates gauge invariance. We have constructed the necessary deformations of the Lagrangian and gauge transformations, so that to achieve the invariance under the new gauge transformation in first order in strength.

Both in massless theory and in massive theory the calculations have been carried out first for spin 2 and spin 3 cases and then the results were generalized for the fields with arbitrary spins. Also we have studied the aspects of causality of the interacting theory. We have proved that the interacting equations of motion for massive spin 2 and spin 3 fields can be identically transformed in such a way that the higher (second) derivatives form the d’Alambertian. This circumstance guarantees the causal propagation. Generalization for arbitrary spin field can be in principle realized in the same way as for spin 2 and spin 3 fields although is more complicated technically.

As a result, the general vertex describing cubic coupling of the arbitrary spin massless and massive fields to constant electromagnetic field is completely constructed\textsuperscript{6}.

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\textsuperscript{6} Some time ago Metsaev developed a general classification of cubic vertices in higher spin theories (see first paper in ref. \cite{9}) using light-cone formulation. From formal point of view, the vertices constructed in our paper could be correspond to the case s-s-1 interaction, where two fields with spin s are both massive or massless and spin-1 field massless. But similarly to the massless case massive vertex even restricted to the case of constant e/m field will still contain terms with up to (2s − 1) derivatives so it will not coincide with the vertex constructed in our paper. As we have already noted the reason is that for the constant e/m field the problem turns out to be less restricted and has more solution than the one for dynamical field. We grateful to R. Metsaev for discussion of this issue.
Appendix

A Calculating of the variations

During the most part of the paper there was necessary to calculate the gauge variations of the various first order corrections to free higher spin Lagrangians. Below we explain and summarize the details of such calculations.

Let a theory is given in terms of fields $\Phi^a$ and Lagrangian $L$ and let the gauge transformations of the fields are $\delta \Phi^a$. Condition of gauge invariance can be written as

$$\delta L = \delta \Phi^a \left( \frac{\delta S}{\delta \Phi^a} \right) = 0 \quad (36)$$

Further we present the results of calculating the variations on the base of (36) in the massive theory. These results are formulated as the systems of equations for the arbitrary parameters from Lagrangians and gauge transformations.

A.1 Massive spin-3 field

Massive charged spin-3 fields is described by a set of fields $\Phi^a = \{\phi_{\mu\nu\sigma}^i, h_{\mu\nu}^i, b_\mu^i, \varphi^i\}$, $i = 1, 2$

Variations with two derivatives give the system of equations

$$\delta_{00}L_{13} + \delta_{01}L_{12} + \delta_{11}L_{02} + \delta_{12}L_{01} = 0$$

$$2\alpha_2 a_{21} + c_{12} - 2\delta_2 = 0$$

$$2\alpha_2 a_{22} + (c_{12} + 2c_{15}) + 2\delta_2 = 0$$

$$\alpha_2 a_{23} + 2c_{11} + \gamma_3(2\alpha_2 - 2\alpha_2[d + 2]) - 2\delta_2 = 0$$

$$\alpha_2 a_{23} - (c_{12} + 2c_{13}) = 0$$

$$\alpha_2 a_{24} - 2c_{14} - \frac{\gamma_3}{2}(2\alpha_2 + \alpha_2[d + 2]) - 2\delta_2 = 0$$

$$2\alpha_2 \left( \frac{2\alpha_1 + a_{15} + 2a_{16}[d + 2]}{3d} \right) + c_{15} + 3\eta_3 d = 0$$

$$2\alpha_2 \left( \frac{2\alpha_1 + a_{14}[d + 2]}{3d} \right) - (2c_{11} - c_{12}) = 0$$

$$2\alpha_2 \left( \frac{2\alpha_1 + a_{15}[d + 2]}{3d} \right) + c_{12} - 3\eta_3 d = 0$$

$$2\alpha_2 \left( \frac{a_{13} + a_{14} + a_{17}[d + 2]}{3d} \right) - c_{13} + \frac{\gamma_2}{2}(2\alpha_2 - 4\alpha_2 + \alpha_2 d) - \frac{3}{2}\eta_3 d = 0$$

$$2\alpha_2 \left( \frac{a_{13} + a_{14} - 2a_{15} + a_{17}[d + 2]}{3d} \right) + (2c_{14} - c_{15}) = 0$$

$$2\alpha_1 b_1 - c_{22} - \delta_1 = 0$$

$$\alpha_1 b_2 - c_{21} - \gamma_2(\alpha_1 - \alpha_1 d) - \delta_1 = 0$$

$$\alpha_1 b_2 + (c_{22} + 2c_{23}) = 0$$

$$\frac{\alpha_1}{d - 2}(2a_{22} + a_{24}d) - (c_{21} - c_{22}) = 0$$ \quad (37)

Variations with one derivative and contribution from minimal interaction give the system of equations

$$\delta_{01}L_{02} + (\delta_{00}L_{12} + \delta_{01}L_{11} + \delta_{12}L_{00}) = 0$$
\[\alpha_2 c_{11} - 2d_1 = -6e_0\]
\[\alpha_2 c_{12} + 4d_1 + 2\delta_2 \alpha_2 = -6e_0\]
\[\alpha_2 c_{13} + \gamma_3(3 - 3[d + 2]) - 2\delta_2 \alpha_2 = 9e_0\]
\[\alpha_2 c_{15} + 4d_2 - 2\delta_2 \alpha_2 = 6e_0\]
\[\alpha_2 c_{21} - \frac{\gamma_3}{2} \alpha_2 \alpha_1[d + 2] - \delta_2 \alpha_1 = 0\]
\[\alpha_2 c_{22} + 2d_3 - \delta_2 \alpha_1 = 0\]
\[\frac{2\alpha_2}{3d}(2c_{11} + c_{12} + c_{13}[d + 2]) - \alpha_1 c_{21} - 2d_4 = 4e_0\]
\[\frac{2\alpha_2}{3d}(c_{11} + c_{14}[d + 2]) - \alpha_1 c_{23} - \eta_3(2\alpha_2 + \alpha_2[d + 2]) + \frac{3}{2} \gamma_2 d - \delta_1 \alpha_1 = -3e_0\]
\[\frac{2\alpha_2}{3d}(c_{12} + c_{15}[d + 2]) + \alpha_1 c_{22} + 2\eta_3(\alpha_2 - \alpha_2[d + 2]) - 2d_4 - \delta_1 \alpha_1 = -2e_0\]
\[\alpha_1 c_{31} - \frac{\gamma_2}{2} \alpha_1 \alpha_0 d + \delta_1 \alpha_0 = 0\]
\[\frac{\alpha_1}{d - 2}(c_{11} + c_{13} + c_{14}d - c_{15}) + d_3 = 0\]
\[\frac{\alpha_1}{d - 2}(c_{21} + c_{22} + c_{23}d) - \alpha_0 c_{31} + 2d_5 = e_0\]  

Variations without derivatives with contribution from minimal interaction give three equations

\[\delta_{00}\hat{\mathcal{L}}_{02} + \delta_{01}\hat{\mathcal{L}}_{01} + (\delta_{00}\mathcal{L}_{11} + \delta_{11}\mathcal{L}_{00}) = 0\]

\[2\alpha_2 d_4 = -2\sqrt{3}e_0\]
\[\frac{2\alpha_2}{3d}(2d_1 + 2d_2[d + 2]) - \alpha_1 d_3 - 3\eta_3(d + 1) + \frac{\delta_1}{2} \alpha_2 \alpha_1 = 2\sqrt{3}\frac{d + 1}{d} e_0\]
\[\frac{2\alpha_2}{3d} d_3[d + 2] - 2\alpha_1 d_5 + \frac{\eta_3}{2} \alpha_2 \alpha_1[d + 2] + 2\delta_1 \frac{d + 2}{2d} = 2\sqrt{\frac{d + 1}{d}} e_0\]  

(38)

A.2 Massive field of arbitrary integer spin

Massive charged field of arbitrary integer spin is described by a set of fields \(\Phi^a = \{\Phi^a_i\}, k = 0, \ldots, s, i = 1, 2\).

Variations with two derivatives give the system of equations

\[\delta_{00}\mathcal{L}_{13} + \delta_{01}\mathcal{L}_{12} + \delta_{11}\mathcal{L}_{02}\delta_{12}\mathcal{L}_{01} = 0\]  

(39)
Variations with one derivatives plus contributions from minimal interaction give

\[2\alpha_{k-1}c_{1k-1} + d_{2k} - \delta_{k-1}(k - 1) = 0\]
\[\alpha_{k-1}c_{3k-1} + d_{1k}(k - 1) + \delta_{k-1}(k - 1) + \gamma_k(-a_{1k}(k - 1) + a_{2k}(d + 2k - 4)) = 0\]
\[\alpha_{k-1}c_{4k-1} - 2d_{5k} + \delta_{k-1}(k - 1)(k - 2) + \gamma_k(a_{1k}(k - 1)(k - 2)) - a_{3k}(d + 2k - 4) = 0\]
\[2\alpha_{k-1}c_{2k-1} - (-d_{2k}(k - 2) + 2d_{4k}) - \delta_{k-1}(k - 1)(k - 2) = 0\]
\[\alpha_{k-1}c_{5k-1} - 2d_{6k} + \delta_{k-1}(k - 1)(k - 2)(k - 3) = 0\]
\[\alpha_{k-1}c_{3k-1} - (d_{2k} - 2d_{3k}) = 0\]
\[\beta_k(2c_{2k}(k - 2) + c_{5k}(d + 2k - 4)) + d_{2k}(k - 2) + \eta_k\frac{k(k - 1)(k - 2)}{2}(d + 2k - 6) = 0\]
\[\beta_k(c_{1k}(k - 1)(k - 2) + c_{5k} + 2c_{6k}(d + 2k - 4)) - d_{4k} - \eta_k\frac{k(k - 1)(k - 2)}{2}(d + 2k - 6) = 0\]
\[\beta_k(c_{3k}(k - 1)(k - 2) + c_{4k}(k - 2) - c_{7k}(d + 2k - 4)) + d_{3k}(k - 2) + \eta_k\frac{k(k - 1)(k - 2)}{4}(d + 2k - 6) - \gamma_k^{-1}(a_{1k}(k - 1)(k - 2)) + a_{2k}(k - 2) - a_{3k}(d + 2k - 6) = 0\]
\[\beta_k(c_{2k}(k - 2)(k - 3) + 2c_{5k}(k - 3) + 2c_{8k}(d + 2k - 4)) - (d_{4k}(k - 3) - 2d_{6k}) - \eta_k\frac{k(k - 1)(k - 2)(k - 3)}{4}(d + 2k - 6) = 0\]
\[\beta_k(2c_{2k} - c_{4k}(d + 2k - 4)) - (d_{1k}(k - 1) - d_{2k}) = 0\]
\[\beta_k(c_{3k}(k - 1)(k - 2) + c_{4k}(k - 2) - 2c_{5k} + c_{7k}(d + 2k - 4)) + (d_{4k} + 2d_{5k}) = 0\] \hspace{1cm} (40)

Variations with one derivatives plus contributions from minimal interaction give

\[\delta_{01}L_{02} + (\delta_{00}L_{12} + \delta_{01}L_{11} + \delta_{12}L_{00}) = 0\]

Using notations

\[e_1 = (k - 1)e_0, \quad e_2 = (k - 1)(k - 2)e_0, \quad e_3 = (k - 1)(k - 2)(k - 3)e_0\]
we obtain:

\[
\begin{align*}
\alpha_{k-1}d_{1k-1} - \delta_{k-1}a_{1k-1} + \gamma_k b_{3k}(d + 2k - 4) &= 0 \\
\alpha_{k-1}d_{2k-1} - 2e_{3k} + \delta_{k-1}a_{1k-1}(k - 2) &= 0 \\
\alpha_{k-2}d_{1k-1} - \beta_k(d_{1k}(k - 1)) + a_{2k} - d_{3k}(d + 2k - 4)) + 2e_{1k-1} &= -e_1 \\
\alpha_{k-2}d_{2k-1} + \beta_k(d_{2k}(k - 2) - d_{4k}(d + 2k - 4)) - 2e_{1k-1}(k - 2) + \\
+ \delta_{k-2}a_{1k-1}(k - 2) + \eta_k(\alpha_{1k}(k - 1)(k - 2) - a_{2k}(k - 2)(d + 2k - 4)) &= -e_2 \\
\alpha_{k-2}d_{3k-1} + \beta_k(d_{1k}(k - 1)(k - 2)) - d_{5k}(d + 2k - 4)) + \delta_{k-2}a_{2k-1} - \\
- \eta_k(\alpha_{1k}(k - 1)(k - 2)) - a_{3k}(d + 2k - 4)) + \\
+ \gamma_k(1 - (k - 1)(k - 2) - 2b_{2k-1}(d + 2k - 6)) &= -\frac{3}{2}e_2 \\
\alpha_{k-2}d_{4k-1} - \beta_k(d_{2k}(k - 2)(k - 3)) - d_{6k}(d + 2k - 4)) + 4e_{2k-1} + \\
+ \delta_{k-2}a_{2k-1}(k - 3) - \eta_k(\alpha_{1k}(k - 1)(k - 2)(k - 3)) - \\
- a_{3k}(k - 3)(d + 2k - 4)) &= -e_3 \\
\beta_{k-1}(d_{1k}(k - 1)(k - 2)) + d_{3k}(k - 2) - d_{4k} + d_{5k}(d + 2k - 6)) - e_{3k} &= 0 \\
\beta_{k-1}(d_{2k}(k - 2)(k - 3)) + d_{4k}(k - 3) + d_{6k}(d + 2k - 6)) - e_{3k}(k - 3) + \\
+ \eta_{k-1}(\alpha_{1k}(k - 1)(k - 2)(k - 3)) + a_{2k}(k - 2)(k - 3) - \\
- a_{3k}(k - 3)(d + 2k - 6)) &= 0
\end{align*}
\]

Variations without derivatives with contribution from minimal interaction give three equations

\[
\delta_{00}\hat{L}_{02} + \tilde{\delta}_{01}\tilde{L}_{01} + (\delta_{00}L_{11} + \delta_{11}L_{00}) = 0
\]

\[
\begin{align*}
-2\alpha_{k-2}e_{1k-2} - \beta_k e_{3k}(d + 2k - 4) - 2\delta_{k-2}b_{1k-2}(k - 2) - \eta_k b_{3k}(k - 2)(d + 2k - 4) &= \\
= -\alpha_{k-2}(k - 2)e_0 \\
- \alpha_{k-2}e_{3k} - \beta_k(e_{1k}(k - 1)(k - 2) + 2e_{2k}(d + 2k - 4)) - \delta_{k-2}b_{3k}(k - 2) - \\
- \eta_k(b_{1k}k(k - 1)(k - 2) + 2b_{2k}(k - 2)(d + 2k - 4)) &= \alpha_{k-1}\frac{d + 2k - 5}{d + 2k - 6}e_2
\end{align*}
\]

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