Minimax state estimation for linear continuous differential-algebraic equations

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Abstract

This paper describes a minimax state estimation approach for linear Differential-Algebraic Equations (DAE) with uncertain parameters. The approach addresses continuous-time DAE with non-stationary rectangular matrices and uncertain bounded deterministic input. An observation’s noise is supposed to be random with zero mean and unknown bounded correlation function. Main results are a Generalized Kalman Duality (GKD) principle and sub-optimal minimax state estimation algorithm. GKD is derived by means of Young-Fenhel duality theorem. GKD proves that the minimax estimate coincides with a solution to a Dual Control Problem (DCP) with DAE constraints. The latter is ill-posed and, therefore, the DCP is solved by means of Tikhonov regularization approach resulting a sub-optimal state estimation algorithm in the form of filter. We illustrate the approach by an synthetic example and we discuss connections with impulse-observability.

Key words: Minimax; Robust estimation; Descriptor systems; Time-varying systems; Optimization under uncertainties; Tikhonov regularization.

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1 Introduction

This paper presents a generalization of the minimax state estimation approach to linear Differential-Algebraic Equations (DAE) in the form

\[ \frac{d(Fx)}{dt} = C(t)x(t) + f(t), \quad Fx(t_0) = x_0 \]  (1)

where \( F \in \mathbb{R}^{m \times n} \) is a rectangular \( m \times n \)-matrix and \( t \mapsto C(t) \in \mathbb{R}^{m \times n} \) is a continuous matrix-valued function. The research presented here may be thought of as a continuation of the paper [Zhuk, 2010], where the case of discrete time DAEs with time-depending coefficients was investigated. We stress that the DAE with \( F \in \mathbb{R}^{m \times n} \) is non-causal (the matrix pencil \( F - \lambda C(t) \) is singular [Gantmacher, 1960]) if \( m \neq n \). Also the coefficient \( C(t) \) depends on time. Therefore the state estimation problem for DAE in the form (1) can not be directly addressed by parameter estimation methods (see for instance [Gerdin et al., 2007], [Darouach et al., 1997] and citations there), based on the transformation of the regular matrix pencil \( F - \lambda C \) (\( \det (F - \lambda C) \neq 0 \)) to the Weierstrass canonical form [Gantmacher, 1960]. As it was mentioned in [Gerdin et al., 2007], the latter transformation allows to convert DAE (with regular pencil) into Ordinary Differential Equation (ODE), provided the unknown input \( f \) is smooth enough and \( C(t) \equiv \text{const} \). On the other hand, in applications \( f \) is often modelled as a realization of some random process or as a measurable squared-integrable function with bounded \( L_2 \)-norm. One way to go is to take Sobolev-Shvartz derivative of \( f \), allowing the state \( x(t) \) of DAE to be discontinuous function. If the latter is not acceptable, it is natural to ask if it is possible to derive a state estimation algorithm for DAE in the form (1) with measurable \( f \) without transforming the pencil \( F - \lambda C \) into a canonical form? The same question arises if \( C(t) \) is not constant as in this case it may be impossible (see [Campbell, 1987]) to transform DAE to ODE even if the pencil \( F - \lambda C(t) \) is regular for all \( t \). In this paper we give a positive answer to this question for the following state estimation problem: given observations \( y(t), t \in [t_0, T] \) of \( x(t) \), to reconstruct \( Fx(T) \), provided \( x \) is a weak solution to (1). We note, that many authors (see [Gerdin et al., 2007], [Darouach et al., 1997] and citations there) assume the state vector \( x(t) \) of (1) to be a differentiable (in the classical sense) function. In contrast, we only assume...
that $t \mapsto Fx(t)$ is an absolutely continuous function. In this setting $x(T)$ is not necessary well defined. Hence, it makes sense to estimate $Fx(T)$ only. In what follows, we assume that $f$ is an unknown squared-integrable function, which belongs to a given bounded set $\mathcal{S}$. We will also assume that observations $y(t)$ may be incomplete and noisy, that is $y(t) = H(t)x(t) + \eta(t)$, where $\eta$ is a realization of a random process with zero mean and unknown but bounded correlation function. Following [Nakonechny, 1978] we will be looking for the minimax estimate $\ell(x)$ of a linear function $\ell(x) := (\ell, Fx(T))$ among all linear functions of observations $u(y)$. Main notions of deterministic minimax state estimation approach [Milanese and Tempo, 1985; Cherrousko, 1994; Kurzhanski and Vályi, 1997] are reachability set, minimax estimate and worst-case error. By definition, reachability set contains all states of the model which are consistent with observed data and uncertainty description. Given a point $P$ within the reachability set one defines a worst-case error as the maximal distance between $P$ and other points of the reachability set.

Then the minimax estimate of the state is defined as a point minimizing the worst-case error (a Tchebysheff center of the reachability set). In this paper we deal with random noise in observations. This prevents us from describing the reachability set. Instead, we derive a dynamic mean-squared minimax estimate minimizing mean-squared worst-case error.

The contributions of this paper are a Generalized Kalman Duality (GKD) and sub-optimal minimax estimation algorithm, both for DAE in the form (1). As it was previously noted in [Gerdin et al., 2007] the need to differentiate an unknown input posed a problem of mathematical justification of the filtering framework based on DAE with classical derivatives. In [Gerdin et al., 2007] the authors propose a solution, provided det$(F - \lambda C) \neq 0$ for any $\lambda$ [Gerdin et al., 2007]. Here we apply GKD in order to justify the minimax filtering framework for the case of DAEs (1) with any rectangular $F$ and time-varying $C(t)$. We do not use the theory of matrix pencils so that the condition of differentiability of the unknown input $f$ in (1) is not necessary for our derivations. Applying GKD we arrive to the Dual Control Problem (DCP) with DAE constraint, which has a unique absolutely continuous solution, provided $\ell$ belongs to the minimax observable subspace $\mathcal{L}(T)$. Otherwise, the solution of DCP is represented in terms of the impulsive control. In this sense the minimax observable subspace generalizes impulse observability condition (see [Gerdin et al., 2007]) to the case of DAE with rectangular time-varying coefficients.

The cost function of DCP describes the mean-squared worst-case error and its minimizer represents a minimax estimate. However, Pontryagin Maximum Principle (PMP) cannot be applied directly to solve DCP: a straightforward application of the PMP to the dual problem could reduce the minimax observable subspace to the trivial case $\mathcal{L}(T) = \{0\}$ (see example in Subsection 2.2). In order to preserve the structure of $\mathcal{L}(T)$ we apply Tikhonov regularization approach. As a result (Proposition 4) we represent a sub-optimal minimax state estimation algorithm as a unique solution of a well-posed Euler-Lagrange system with a small parameter. This solution converges to the minimax estimate.

This paper is organized as follows. At the beginning of section 2 we describe the formal problem statement and introduce definitions of the minimax mean-squared estimates and errors. The rest of this section consists of two subsections. Subsection 2.1 presents the GKD (Theorem 2). In subsection 2.2 we discuss optimality conditions and derive regularization scheme (Proposition 4) along with the representation of the sub-optimal minimax estimate in the sequential form (Corollary 5). Also we present an example. Section 3 contains conclusion. Appendix contains proofs of technical statements.

Notation: $E\eta$ denotes the mean of the random element $\eta$; int $G$ denotes the interior of $G$; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; $\mathbb{L}_2(t_0, T, \mathbb{R}^m)$ denotes a space of square-integrable functions with values in $\mathbb{R}^m$ (in what follows we will often write $\mathbb{L}_2(t_0, T, \mathbb{R}^k)$ where the dimension $k$ will be defined by the context); $\mathbb{H}_1(t_0, T, \mathbb{R}^m)$ denotes a space of absolutely continuous functions with $\mathbb{L}_2$-derivative and values in $\mathbb{R}^m$; the prime $'$ denotes the operation of taking the adjoint; $L'$ denotes adjoint operator, $F'$ denotes the transposed matrix; $c(G, \cdot)$ denotes the support function of a set $G$; $\langle \cdot, \cdot \rangle$ denotes the inner product in a Hilbert space $H$, $\|x\|^2_H := \langle x, x \rangle$, $\|\cdot\|$ denotes norm in $\mathbb{R}^n$; $S > 0$ means $\langle Sx, x \rangle > 0$ for all $x$; $F^+$ denotes the pseudoinverse matrix; $Q^{1/2}$ denotes the square-root of the symmetric non-negative matrix $Q$. $I_n$ denotes $n \times n$-identity matrix, $0_{n \times m}$ denotes $n \times m$-zero matrix, $I_0 := 0$; $tr$ denotes the trace of the matrix.

2 Linear minimax estimation for DAE

Consider a pair of systems

$$\frac{d(Fx)}{dt} = C(t)x(t) + f(t), \quad Fx(t_0) = x_0,$$

$$y(t) = H(t)x(t) + \eta(t),$$

1 Note that in order to reconstruct $Fx(T)$ it is enough to reconstruct a linear function $\ell(x) := (\ell, Fx(T))$ for any $\ell \in \mathbb{R}^m$. Having the estimate of $\ell(x)$ for any $\ell \in \mathbb{R}^m$, one can set $\ell := e_i = (0, \ldots, 1, \ldots, 0)^T$ in order to reconstruct $i$-th component of $Fx(T)$. The cost function of DCP describes the mean-squared worst-case error and its minimizer represents a minimax estimate. However, Pontryagin Maximum Principle (PMP) cannot be applied directly to solve DCP: a straightforward application of the PMP to the dual problem could reduce the minimax observable subspace to the trivial case $\mathcal{L}(T) = \{0\}$ (see example in Subsection 2.2). In order to preserve the structure of $\mathcal{L}(T)$ we apply Tikhonov regularization approach. As a result (Proposition 4) we represent a sub-optimal minimax state estimation algorithm as a unique solution of a well-posed Euler-Lagrange system with a small parameter. This solution converges to the minimax estimate. We represent the sub-optimal estimate in the classical sequential form: as a solution to a Cauchy problem for a linear stochastic ODE, driven by a realization of observations $y(t)$, $t \in [t_0, T]$. We recall that $y(t)$ is perturbed by a “random noise”, which can be a realization of any random process (not necessary Gaussian as in [Gerdin et al., 2007]) with zero mean and unknown but bounded correlation function.
where \( x(t) \in \mathbb{R}^n, f(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, \eta(t) \in \mathbb{R}^p \) represent the state, input, observation and observation's noise respectively. As above, we assume \( F \in \mathbb{R}^{m \times n}, f \in L^2(t_0, T, \mathbb{R}^m), \) and \( C(t) \) and \( H(t) \) are continuous\(^2\) matrix-valued functions of \( t \) on \([t_0, T], t_0, T \in \mathbb{R}\). Now let us describe our assumptions on uncertain functions then we can consider new measurements of state, input, observation and observation's noise respectively. As above, we assume \( \hat{\eta} \) to be just measurable.

**Definition 1** Given \( T < +\infty, u \in L^2(t_0, T, \mathbb{R}^p) \) and \( \ell \in \mathbb{R}^m \) define a mean-squared worst-case estimation error

\[
\sigma(T, \ell, u) := \sup_{x, (x_0, f) \in \mathcal{G}, R_\eta \in \mathcal{W}} E[(\ell, Fx(T)) - u(y)]^2
\]

A function \( \hat{u}(y) = \int_{t_0}^T \hat{u}(t), y(t) dt \) is called an \( \ell \)-estimate if \( \inf_{\sigma} \sigma(T, \ell, u) = \sigma(T, \ell, \hat{u}) \). The number \( \hat{\sigma}(T, \ell) = \sigma(T, \ell, \hat{u}) \) is called a minimax mean-squared a priori error in the direction \( \ell \) at time-instant \( T \) \( \ell \)-error). The set \( \mathcal{L}(T) = \{ \ell \in \mathbb{R}^m : \hat{\sigma}(T, \ell) < +\infty \} \) is called a minimax observable subspace.

2.1 Generalized Kalman Duality Principle

Definition 1 reflects the procedure of deriving the minimax estimation. The first step is, given \( \ell \) and \( u \) to calculate the worst-case error \( \sigma(T, \ell, u) \) by means of the suitable duality concept.

**Theorem 2 (Generalized Kalman duality)** Take \( \ell \in \mathbb{R}^n \). The \( \ell \)-error \( \hat{\sigma}(T, \ell) \) is finite iff

\[
\frac{d(F'z)}{dt} = -C'(t)z(t) + H'(t)u(t), \quad F'z(T) = F'\ell \quad (6)
\]

for some \( z \in L^2(0, T, \mathbb{R}^m) \) and \( u \in L^2(0, T, \mathbb{R}^p) \). If \( \hat{\sigma}(T, \ell) < +\infty \) then

\[
\sigma(T, \ell, u) = \min_{v,d} \{ ||\hat{Q}_0^{-\frac{1}{2}}(z(t_0) - v(t_0)) - Q_0^{-\frac{1}{2}}d||^2 + \int_{t_0}^T ||Q^{-\frac{1}{2}}(z - v)||^2 dt \}
\]

\[
= ||\hat{Q}_0^{-\frac{1}{2}}(z(t_0) - \hat{v}(t_0)) - Q_0^{-\frac{1}{2}}d||^2 + \int_{t_0}^T ||Q^{-\frac{1}{2}}(z - \hat{v})||^2 + ||R^{-\frac{1}{2}}u||^2 dt \quad (7)
\]

where \( \hat{Q}_0^{-\frac{1}{2}} = Q_0^{-\frac{1}{2}}F^{t}F' \), min in (7) is taken over all \( d \) such that \( F'd = 0 \) and all \( v \) verifying (6) with \( u = 0 \) and \( \ell = 0 \), and min is attained at \( \hat{v}, \hat{d} \).

**Remark 3** An obvious corollary of Theorem 2 is an expression for the minimax observable subspace

\[
\mathcal{L}(T) = \{ \ell \in \mathbb{R}^m : (6) \text{ holds for some } z, u \} \quad (8)
\]

In the case of stationary \( C(t) \) and \( H(t) \) the minimax observable subspace may be calculated explicitly, using the canonical Kronecker form [Gantmacher, 1960].

**PROOF.** Take \( \ell \in \mathbb{R}^m \) and \( u \in L^2(t_0, T, \mathbb{R}^p) \). Using \( E\eta(t) = 0 \) we compute

\[
E[\ell(x) - u(y)]^2 = E[\int_{t_0}^T \langle \ell, Fx(T) \rangle]dt^2 := \gamma^2
\]

\[
+ \left( \langle \ell, Fx(T) \rangle - \int_{t_0}^T \langle H'u, x \rangle dt \right)^2 := \mu^2
\]

Let us transform \( \mu^2 \). There exists \( w \in L^2(t_0, T, \mathbb{R}^m) \) such that: (W) \( F'w \in \mathbb{H}^1(t_0, T, \mathbb{R}^m) \) and \( F'w(T) = F'\ell \).

Noting that [Albert, 1972] \( F = FF' \) we have

\[
\langle \ell, Fx(T) \rangle = \langle F'f, F'Fx(T) \rangle = \langle F'w(T), F'Fx(T) \rangle.
\]

The latter equality, an integration-by-parts formula

\[
\langle F'w(T), F'Fx(T) \rangle - \langle F'w(t_0), F'Fx(t_0) \rangle = \int_{t_0}^T \langle \frac{d(Fx)}{dt}, w \rangle + \langle \frac{d(F'w)}{dt}, x \rangle dt \quad (10)
\]

(proved in [Zhuk, 2007] for \( Fx \in \mathbb{H}^1(t_0, T, \mathbb{R}^m) \) and \( F'w \in \mathbb{H}^1(t_0, T, \mathbb{R}^m) \) and (2) gives that

\[
\mu = \langle (F')'Fw(t_0), Fx_0 \rangle + \int_{t_0}^T \langle f, w \rangle dt
\]

\[
+ \int_{t_0}^T \langle \frac{d(F'w)}{dt}, x \rangle dt + C'w - H'u, x dt \quad (11)
\]
By (9) and (11): $\sigma(T, \ell, u) = \sup_{\mathbb{R}^n} \gamma^2 + \sup_{f,x_0,x} \mu^2$. By Cauchy inequality $\gamma^2 \leq \int_0^T E(\mathcal{R}_\eta, \eta)dt + \int_0^T (R^{-1} u, u) dt$. As $E(\mathcal{R}_\eta, \eta) = \text{tr} (R^2 R_{\eta})$, it follows from (4)

$$\sigma(T, \ell, u) = \int_0^T \langle R^{-1} u, u \rangle dt + \sup_{f,x_0,x} \mu^2 \quad (12)$$

Assume $\bar{\sigma}(T, \ell) > -\infty$. Then $\sigma(T, \ell, u) < -\infty$ for at least one $u$ so that $\sup_{f,x_0,x} \mu^2 < -\infty$. The term $\int_0^T (f, w) dt$ in the first line of (11) is bounded due to (5). Thus

$$\sup_x \left\{ \langle F'w(t_0), F^+ F x(t_0) \rangle + \int_0^T \frac{d(F'(w)}{dt} + C'w - H'u, x) dt \right\} < \infty$$

(13) allows us to prove that there exists $z$ such that (6) holds for the given $t$ and $u$. To do so we apply a general duality result $^3$ from [Zhu, 2009]:

$$\sup_{x \in \mathcal{D}(L)} \left\{ \{ F, x \}, Lx \in G = \inf_{b \in \mathcal{D}(L)} \{ c(G, b), L'b = F \} \right\}$$

provided (A1) $L : \mathcal{D}(L) \subset H_1 \to H_2$ is a closed dense-defined linear mapping, (A2) $G \subset H_2$ is a closed bounded convex set and $H_{1,2}$ are abstract Hilbert spaces. Define

$$\mathcal{D}(L)(t) = \left\{ \frac{d(Fz)}{dt} - C(t)x(t), Fx(t_0) \right\}, x \in \mathcal{D}(L) \quad (15)$$

Then $L$ is a closed dense defined linear mapping [Zhu, 2007] and

$$\mathcal{D}(L) := \left\{ \{ b = (z, t_0), F'z(t_0) \right\} \in \mathcal{D}(L) \}$$

and

$$\mathcal{D}(L) := \left\{ \{ b = (z, t_0) \in H_1(t_0, T, \mathbb{R}^m), F'z(T) = 0, z_0 = F'z(t_0) + d, F'd = 0 \right\} \}$$

(16)

Setting $F := (F' + F'w(t_0), \frac{d(F'w)}{dt} + C'w - H'u)$ we see from (13) that the right-hand part of (14) is finite. Hence, there exists at least one $b \in \mathcal{D}(L)$ such that $L'b = F$ or (using (16)) $-\frac{d(F'z)}{dt} - C'z = \frac{d(F'w)}{dt} + C'w - H'u$. Setting $\hat{z} := (w + z)$ we obtain $\frac{d(F'\hat{z})}{dt} + C'\hat{z} - H'u = 0$ and $F'\hat{z} = F'\ell$. This proves (6) has a solution.

On the contrary, let $z$ verify (6) for the given $t$ and $u$.

Then $z$ verifies conditions (W), therefore we can plug $z$ into (11) instead of $w$. Define $^4 G_1 := \mathcal{G} \cap \mathcal{R}(L)$, where $\mathcal{R}(L)$ is the range of the linear mapping $L$ defined by (15) and set $S := \sup_{(x_0, f) \in G_1} (F'z(t_0), F^+ x_0) + \int_0^T (z, f) dt$.

Now, using (12) one derives easily

$$\sigma(T, \ell, u) = \int_0^T \langle R^{-1} u, u \rangle dt + S^2$$

(17)

Since $G_1$ is bounded, it follows that $\sigma(T, \ell, u)$ is finite. Let us prove (7). Note that $S$ is a value of the support function of the set $G_1 = \mathcal{G} \cap \mathcal{R}(L)$ on $(F^+ F'z(t_0), z)$. To compute $S$ we note that $L, \mathcal{G}$ verify (A1), (A2) and int $G_1 \neq \emptyset$. Thus (see [Zhu, 2009]):

$$S = \min_{(z_0, z) \in N(L')} \left\{ \min_{c \in \mathcal{D}(L)} \{ c(G, F^+ F'z(t_0) - z_0, z - v) \} \right\}$$

(18)

and the min in (18) is attained on some $(\tilde{z}_0, \tilde{z}) \in N(L')$. Recalling the definition of $L'$ (formula (16)) and noting that $\|z - b\|^2 = |Q_0^{-\frac{1}{2}}(F^+ F'(z(t_0) - v(t_0)) - d)|^2 + \int_0^T \|Q^{-\frac{1}{2}}(z - v)\|^2 dt$ we derive (7) from (17)-(18). This completes the proof.

### 2.2 Optimality conditions

Assume $t \in L(T)$. By definition 1 and due to Generalized Kalman Duality (GKD) principle (see Theorem 2) the $\ell$-estimate $\hat{u}$ is a solution of the Dual Control Problem (DCP), that is the optimal control problem with cost (7) and DAE constraint (6) for any constant $F \in \mathbb{R}^{m \times n}$ and continuous $t \mapsto C(t) \in \mathbb{R}^{m \times n}$, $t \in [t_0, T]$. If $F = I_{m \times n}$ then $\hat{u} = RHp$ where $p$ may be found from the following optimality conditions (Euler-Lagrange System in the Hamilton form [Ioffe and Tikhomirov, 1974]):

$$\frac{dp}{dt} = C p + Q^{-1} z, F(p(t_0)) = \tilde{Q}_0^z(t_0),$$

$$\frac{dF'}{dt} = -C' z + H'RH p, F'(z) = F'.$$

(19)

with $\tilde{Q}_0 = Q_0^{-\frac{1}{2}}F^+ F'$. In the general case $F \in \mathbb{R}^{m \times n}$, let us assume that (AS) the system (19) is solvable. One can prove using direct variational method (see [Ioffe and Tikhomirov, 1974]) that $\hat{u} = RHp$ solves the DCP with cost (7) and DAE constraint (6). Although the assumption (AS) allows one to solve the optimal control problem with DAE constraints, it may be very restrictive for state estimation problems. To illustrate this, let us consider an example. Define

$$F' = \left[ \begin{array}{c} 1 \\
0 \\
0 \\
0 \\
0 \\
0 \
\end{array} \right], C'(t) = \left[ \begin{array}{c} 0 \\
-1 \\
0 \\
0 \\
0 \\
-1 
\end{array} \right], H'(t) = \left[ \begin{array}{c} 1 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{array} \right]$$

(20)

$^3$ The proof of (14) for bounded $L$ and Banach spaces $H_{1,2}$ can be found in [Ioffe and Tikhomirov, 1974]

$^4$ $G_1$ is a set of all $x_0, f$ such that $(x_0, f) \in \mathcal{G}$ and (2) has a solution $x$. 

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and take \( Q_0 = Q(t) = I_{2 \times 2}, R(t) = I_{4 \times 4} \). In this case (19) reads as:

\[
\begin{align*}
\frac{dz_1}{dt} &= z_2 + p_1, \\
\frac{dz_2}{dt} &= p_2, \\
\frac{dp_1}{dt} &= z_1 + p_1(t_0), \\
\frac{dp_2}{dt} &= -p_1 - p_2 = z_2(t_0).
\end{align*}
\]

We claim that (21) has a solution iff \( \ell_1 = \ell_2 = 0 \). Really, \( z_1(t) \equiv 0 \) implies \( z_1(T) = \ell_1 = 0 \), \( z_2(t) = -z_2 = p_1 = p_4 \) and \( \frac{d}{dt} p_1 = p_3 \). According to this we rewrite (21) as follows:

\[
\begin{align*}
\frac{dp_1}{dt} &= -p_2, \\
\frac{dp_2}{dt} &= -p_1 - p_2 = 0.
\end{align*}
\]

It is clear that (22) has a solution iff \( \ell_2 = 0 \). Thus, the assumption (AS) leads to the trivial minimax observability subspace: \( \mathcal{L}(T) = \{0 \} \times \{0 \} \). However, \( \mathcal{L}(T) = \{0 \} \times \mathbb{R} \).

To see this, take \( u_3 \in L_2, \ell_2 \in \mathbb{R} \) and \( \ell_1 = 0 \), and define \( z_1 = 0, u_{1,2} = -z_2 = z_2 = -f \int_T u_3(s) ds \). By direct substitution one checks that \( z_{1,2} \) and \( u_{1,2,3} \) solve (6). Therefore \( \mathcal{L}(T) = \{0 \} \times \mathbb{R} \) due to (8). We see that classical optimality condition (Euler-Lagrange system in the form (19)) may be inefficient for solving the minimax state estimation problems for DAEs. In the next proposition we prove that optimal control problem with cost (7) and DAE constraint (6) has a unique solution \( \tilde{u}, \tilde{z} \), provided \( \ell \in \mathcal{L}(T) \), and we present one possible approximation of \( \tilde{u}, \tilde{z} \) based on the Tikhonov regularization method [Tikhonov and Arsenin, 1977].

Proposition 4 (optimality conditions) Let \( \varepsilon > 0 \).

The DAE boundary-value problem

\[
\begin{align*}
\frac{d(F' z)}{dt} &= -C' z + H' \tilde{u} + \tilde{p}, \\
\frac{d(F p)}{dt} &= C p + \varepsilon \eta^{-1} z, \varepsilon \tilde{u} = RH p, \\
F'(T) + F' p(T) &= F'(t), F' d = 0, \\
1 - \varepsilon F p(t_0) &= \eta^{-1} F'(T) \eta^{-1} z(t_0) - d.
\end{align*}
\]

has a unique solution \( \tilde{u}, \tilde{z}, \tilde{\varepsilon}, \tilde{d} \). If \( \ell \in \mathcal{L}(T) \) then there exists \( \tilde{u}, \tilde{z} \) such that 1) \( \tilde{d} \rightarrow d \) in \( \mathbb{R} \) and \( \tilde{u} \rightarrow \tilde{u}, \tilde{z} \rightarrow \tilde{z} \) in \( L_{2,1} \), 2) \( \tilde{u} \) and \( \tilde{z} \) verify (6) and 3) \( \tilde{u} \) is the \( \ell \)-estimate and \( \tilde{\varepsilon} \)-error is given by

\[
\begin{align*}
\sigma(T, \ell, \tilde{u}) = \Omega(\tilde{u}, \tilde{z}, \tilde{d}) := \|R^{-\frac{1}{2}} \tilde{u}\|_{L_2}^2 + \|Q_0^{-\frac{1}{2}} (F' p(T) - d)\|_{L_2}^2 + 1 - \varepsilon F p(t_0) = \eta^{-1} F'(T) \eta^{-1} z(t_0) - d.
\end{align*}
\]

PROOF. Define \( r := \text{rang} F \) and \( D = \text{diag}(\lambda_1 \ldots \lambda_r) \) where \( \lambda_i, i = 1, r \) are positive eigenvalues of \( FF' \). If \( r = 0 \) then (23) is obviously uniquely solvable. Assume \( r > 0 \). It is easy to see, applying the SVD decomposition [Albert, 1972] to \( F \), that \( F = U' S V \), where \( U U' = I_m \), \( V' V = I_n \) and \( S = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \).

Thus multiplying the first equation of (23) by \( U \), the second – by \( V \), and changing variables one can reduce the general case to the case of DAE (23) with \( F = \begin{bmatrix} \eta & \tilde{I} \\ 0_{m-r \times m} & 0_{r \times r} \end{bmatrix} \). In what follows, therefore, we can focus on this case only. Having in mind the above 4-block representation for \( F \) we split the coefficients of (23) as follows:

\[
\begin{align*}
C(t) &= \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, \\
Q &= \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}, \\
Q_0 &= \begin{bmatrix} Q_0^1 & Q_0^2 \\ Q_0^3 & Q_0^4 \end{bmatrix}, \\
H' R H &= \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}, \\
\ell &= \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}, \\
d &= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.
\end{align*}
\]

If \( n - r = 0 \) and \( m - r = 0 \) we set

\[
\begin{align*}
C_2 &= \begin{bmatrix} 0 & r_1 \end{bmatrix}, \\
C_3 &= \begin{bmatrix} 0 & r_2 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} 0 & Q_2^3 \\ Q_2^4 & 0 \end{bmatrix}, \\
Q_3 &= \begin{bmatrix} Q_3^1 & 0 \\ Q_3^2 & 0 \end{bmatrix}, \\
S_1 &= \begin{bmatrix} S_1^1 & S_2^1 \\ S_3^1 & S_4^1 \end{bmatrix}.
\end{align*}
\]

And algebraic part:

\[
\bar{Q}_4 := Q_0^1 - Q_0^2 (Q_0^2)^{-1} (Q_0^2)' + (Q_0^2)'
\]

and set

\[
\begin{align*}
W(t, \varepsilon) &= \varepsilon I_{m-r} + S_4 + C_4 Q_4 - C_4, \\
A(t) &= (C_1^2 - C_4^2) C_4, \\
B(t) &= (C_2 - C_4^2) Q_2, \\
C(t) &= -C_1^2 + C_3 Q_4 - C_2 B(t), \\
Q_E(t) &= -\frac{1}{\varepsilon} A(t) M(t, \varepsilon) A(t) + I_r + \frac{1}{\varepsilon} [S_1 + C_4^2 Q_4 C_3], \\
S(t) &= \varepsilon (Q_1 - Q_2 Q_4 - Q_2 B(t) M(t, \varepsilon) B(t)).
\end{align*}
\]

Solving the algebraic equations for \( z_2, p_2, d_2 \) we find:

\[
\begin{align*}
Q_4 z_2 &= (Q_2 - C_4 M B) z_1 + \frac{1}{\varepsilon} (C_2 M A' - C_3) p_1, \\
p_2 &= \varepsilon M B z_1 - M A' p_1, d_2 = (Q_0^1 - (Q_0^2)' z_1(t_0).
\end{align*}
\]

Substituting (25) into differential equations for \( p_1, z_1 \) we obtain

\[
\begin{align*}
\frac{dz_1}{dt} &= C_{\varepsilon} z_1 + Q_2 z_1, \\
\frac{dp_1}{dt} &= -C_{\varepsilon} p_1 + S_{\varepsilon} z_1, \\
&= \varepsilon \bar{Q}_4 z_1(t_0).
\end{align*}
\]
We claim that (26) has a unique solution for any $\ell_1 \in \mathbb{R}^r$ and $\varepsilon > 0$. Let us prove uniqueness. Note that $Q_1 - Q_2 - \varepsilon Q_2^*$, $Q_2^*$ and $Q_2(t) > 0$ as $Q(t) > 0$ (see [Albert, 1972] for details) and thus $S_r(t) > 0$ for $\varepsilon > 0$ (as $B'MB \geq 0$). Applying simple matrix manipulations one can prove (see, for instance, [Kurina, 1986]) that $Q_2(t) \geq 0$ for $\varepsilon > 0$. Assume $z_1, p_1$ solve (26) for $\ell_1 = 0$. Then, integrating by parts and using (26) we obtain

$$-\langle z_1(T), z_1(T) \rangle - \langle z_1(t_0), \varepsilon \bar{Q} \bar{z}_1(t_0) \rangle = \int_{t_0}^T \langle Q \bar{z}_1(t), p_1 \rangle + \langle S \bar{z}_1(t), z_1 \rangle \, dt$$

This equality is possible only if $p_1 = 0$, $z_1$ is a unique solution for $\ell_1 = 0$, as (26) is a Noether Boundary Value Problem (BVP), which has a unique solution for $\ell_1 = 0$ and $S_r(t), Q_2(t) \geq 0$. Assume $z_1, p_1$ solve (26) for $\ell_1 = 0$. Then, integrating by parts and using (26) we obtain

$$-\langle z_1(T), z_1(T) \rangle - \langle z_1(t_0), \varepsilon \bar{Q} \bar{z}_1(t_0) \rangle = \int_{t_0}^T \langle Q \bar{z}_1(t), p_1 \rangle + \langle S \bar{z}_1(t), z_1 \rangle \, dt$$

The solution for any $\ell_1$ is unique and it is bounded by (24). We claim (see appendix for the details) that

$$\Omega(u, \hat{z}, \hat{d}) = \inf_{u, z, d} T_\varepsilon(u, z, d) := T_\varepsilon^*, \quad \forall \varepsilon > 0 \quad (27)$$

Take any $\ell \in L(T)$. By (8) there exists $u$ and $z$ such that

$$0 \leq T_\varepsilon^* \leq T_\varepsilon(u, z, d) \leq \delta(u, \hat{z}), \quad \varepsilon > 0$$

for any $d : F^*d = 0$. Due to (28), $T_\varepsilon^* = \delta(u, \hat{z}), \quad \varepsilon > 0$ and $\Omega(u, z, d)$ so that

$$\Omega(u, \hat{z}, \hat{d}) = \Omega(u, z, d), \quad \forall \varepsilon > 0 \quad (29)$$

We claim that (see appendix for technical details)

$$\Omega(u, \hat{z}, \hat{d}) = \lim_{\varepsilon \to 0} \Omega(u, \hat{z}, \hat{d})$$

By (31) and (29) we get:

$$\Omega(u, \hat{z}, \hat{d}) = \Omega^* := \inf_{(u, z) : \delta(u, z) = 0} \Omega(u, z, d) \quad (32)$$

Note that $\Omega$ is strictly convex, therefore $\Omega$ has a unique minimizer $w^*$, which coincides with $(\hat{u}, \hat{z}, \hat{d})$ by (32). This proves that $w^*$ is a unique weak limit point for the bounded sequence $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$. Then, $(\hat{d}_n)$ converges to $d$ in $\mathbb{R}^n$ as $\varepsilon \to 0$. Thus, we claim that $(\hat{u}_n, \hat{z}_n)$ verifies (6) and $(\hat{u}_n, \hat{z}_n)$ is a unique weak limit point for the bounded sequence $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$. Therefore, we refer $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$ as sub-optimal $(\hat{u}, \hat{z}, \hat{d})$. Hence, $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$ converges to $(\hat{u}, \hat{z}, \hat{d})$ in $\mathbb{R}^n$. The latter proves 1) as $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$ converges to $(\hat{u}, \hat{z}, \hat{d})$ in $\mathbb{R}^n$. Thus, $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$ converges to $(\hat{u}, \hat{z}, \hat{d})$ in $\mathbb{R}^n$. Hence, $(\hat{u}_n, \hat{z}_n, \hat{d}_n)$ converges to $(\hat{u}, \hat{z}, \hat{d})$ in $\mathbb{R}^n$. On the other hand, we get by 1):

$$\delta(T, \ell) \leq \sigma(T, \ell, u) \leq \min_{v} \Omega(u, \hat{z}, \hat{d})$$

where we obtained the 4th line notating $\|Q^{-\frac{1}{2}}(\hat{z}_v - v)\|_{\mathbb{L}^2}^2 = \|Q^{-\frac{1}{2}}(\hat{z}_v)\|_{\mathbb{L}^2}^2 + \|Q^{-\frac{1}{2}}v\|_{\mathbb{L}^2}^2 - 2 \int_{t_0}^T \langle Q^{-\frac{1}{2}}\hat{z}_v, v \rangle dt$ and

$$\int_{t_0}^T \langle Q^{-\frac{1}{2}}\hat{z}_v, v \rangle dt = \frac{1}{\varepsilon} \int_{t_0}^T \langle \varepsilon \hat{p}_v, \hat{p}_v \rangle dt - C \hat{p}_v, \hat{p}_v dt$$

where the latter equality follows from (10), definition of $w$ (see notes after (7)) and (23). Thus

$$\delta(T, \ell) \leq \sigma(T, \ell, u) \leq \mu(u) = \Omega(u, \hat{z}, \hat{d})$$

(24) implies, in turn, $u$ is the $\ell$-estimate by definition. This completes the proof.

We will refer $u_n$ as sub-optimal $\ell$-estimate. Let us represent $u_n(y)$ in the form of the minimax filter. Recalling definitions of $M, A, B$ introduced at the beginning of the proof of Proposition 4, and splittings for $\ell, Q, R, H, C$ we
The sub-optimal \( \ell \)-error is given by
\[
\sigma^+(T, \ell) := \frac{1}{\varepsilon} [(z_1(T), z_1(T))] - \int_{t_0}^{T} \| \Phi(t, \varepsilon)z_1 \|^2 dt
\]

**Proof.** Assume \( \hat{p}_\varepsilon \) solves (23). We split \( p_\varepsilon = (p_1^1, p_2^1) \) where \( p_1^1 \) solves (26) and \( p_2^1 \) is defined by (25). It can be checked by direct calculation that \( p_1^1(t) = K(t, \varepsilon)z_1(t) \) where \( z_1 \) is defined by (33). Using this and (25) we deduce \( \hat{p}_\varepsilon = \Phi(t, \varepsilon)z_1 \). (23) implies \( \hat{u}_\varepsilon = \frac{1}{\varepsilon} RH \hat{p}_\varepsilon \). Finally, using the obtained representations for \( \hat{p}_\varepsilon, \hat{u}_\varepsilon \) and (34) we obtain integrating by parts that
\[
\hat{u}_\varepsilon(y) = \int_{t_0}^{T} (y, -\varepsilon RH \hat{p}_\varepsilon) dt = \int_{t_0}^{T} (\frac{1}{\varepsilon} \Phi'H'Ry, z_1) dt
\]
\[
= (1-1) (I_\varepsilon + K(T, \varepsilon))^{-1} \hat{\varepsilon}(T) \int_{t_0}^{T} (\hat{u}, y) dt = \hat{u}(y)
\]
By (29)-(31) \( \Omega(\hat{u}_\varepsilon, \hat{z}_\varepsilon, \hat{d}_\varepsilon) \rightarrow \sigma(T, \ell) \). It is easy to compute using (23) that \( \Omega(\hat{u}_\varepsilon, \hat{z}_\varepsilon, \hat{d}_\varepsilon) = \varepsilon^{-1} ((\ell, F \hat{p}_\varepsilon) + \| \hat{p}_\varepsilon \|_{L_2})^2 ) \). To conclude it is sufficient to substitute \( \hat{p}_\varepsilon = \Phi(t, \varepsilon)z_1 \) into the latter formula.

**Example.** In order to demonstrate main benefits of Proposition 4 we will apply it to the example presented above: assume that the bounding set, state equation and observation operator are defined by (20). Note that \( F'F \) and \( F'F' \) are defined. Thus \( F' = 0 \) implies \( d = 0 \). According to

\[
\text{Theorem 2} \ 	ext{the exact } \ell \text{-estimate } \hat{u} = \hat{u}_1/\hat{u}_2 \text{ may be obtained minimizing } \sigma(T, \ell, u) = \sum_{j=1}^{2} \| u_j(t_0) \|^2 + \| z_1 \|^2_{L_2} + \sum_{j=1}^{3} \| u_j \|^2_{L_2} \text{ over solutions of the DAE}
\]

\[
\frac{dz_1}{dt} - z_2 - u_1 = 0, z_1(T) = \ell_1, -z_1 = 0
\]

\[
\frac{dz_2}{dt} - u_3 - z_2 = 0, z_2(T) = \ell_2, -z_2 - u_2 = 0
\]

Assume \( \ell_1 = 0 \) so that \( \ell = (\ell_1, \ell_2) \in \mathcal{L}(T) \). If \( \hat{u}_1, \hat{u}_2 \) solves (35) then \( u_1, u_2 \) may found minimizing \( \sigma(T, \ell, u) = \| z_2(t_0) \|^2 + \int_{t_0}^{T} z_2^2 + u_3^2 dt \) over

\[
\frac{dz_3}{dt} = u_3, z_3(T) = \ell_3
\]

the optimality condition takes the following form: \( \hat{u}_3 = p, \frac{dz_3}{dt} = p, z_2(T) = \ell_2, \frac{dp}{dt} = 3\varepsilon^2, p(t_0) = z_2(t_0) \). Let us represent the estimate in the form of the minimax filter. Introducing \( k \) as a solution of the Riccati equation \( \frac{dk}{dt} = k^2_3 - 3, k(0) = 1 \) we find that \( \hat{u}_2 = k z_2 \). Where \( z_2 \) solves the following Cauchy problem:

\[
\frac{dz_2}{dt} = - k \varepsilon - y_1 - y_2 + y_3, z_2(t_0) = 0
\]

is then to easy to see that \( \hat{u}(y) = \int_{t_0}^{T} \langle \hat{u}, y \rangle dt = \ell_2 \hat{z}(T), y = (y_1, y_2, y_3) \) denotes a realization of the random process representing observed data. Let us compute the sub-optimal \( \ell \)-estimate. (34) reads as

\[
\frac{d}{dt} (\frac{x_1}{x_2}) \Rightarrow \frac{d}{dt} [\begin{array}{c} k_1 k_2 1 0 \\
0 1 1 0 \end{array}] 
= \left[ \begin{array}{c} k_1 k_2 1 0 \\
0 1 1 0 \end{array} \right] \left[ \begin{array}{c} k_1 k_2 1 0 \\
0 1 1 0 \end{array} \right] \left( \frac{1}{\varepsilon} \right) \left( \frac{1}{\varepsilon} \right)
\]

Define \( \hat{u}_1 \) as \( \hat{u}_1(t) = \langle \hat{u}, x \rangle dt = \frac{1}{\varepsilon} \left( \hat{u}_1, x(T) \right) \). If \( \ell_1 = 0 \) then the sub-optimal \( \ell \)-error is given by \( \sigma^+(T, \ell) \rightarrow \sigma^+(T, \ell) = \frac{1}{\varepsilon} \left( \hat{u}_1, x_3(t_0) \right) \). Take \( t_0 = 0, T = 1 \) and assume that \( F \) and \( C \) are defined by (20). In the corresponding DAE \( x_{3,4} \) are free components. For simulations we choose \( x_3 = \cos(t) \) and \( x_4 = \sin(t) \), \( x_1(0) = 0.1, x_2(0) = -0.1, f_1 = f_2 = 0 \). In order to generate artificial observations we take \( \eta(t) = (-0.3, 0.3)^T \). In Figure 1 the optimal \( \ell \)-estimate, sub-optimal \( \ell \)-estimate and sub-optimal \( \ell \)-error are presented, provided \( \ell_1 = 0, \ell_2 = 1 \). As \( \mathcal{L}(t) = \{0\} \times \mathbb{R} \) we see that \( x_1 \) is not observable in the minimax sense. This can be explained as follows. The derivative \( x_3 \) of \( x_1 \) may be any element of \( L_2 \). As we apply integration by parts formula in order to compute \( \sigma(T, \ell, u) \) (see (10)), the expression for \( \sigma(T, \ell, u) \) contains
needs to solve the DCP, that is a linear-quadratic control problem with DAE constraints. Application of the classical optimality conditions (Euler-Lagrange equations) imposes additional constraints onto the minimax observability subspace $L(T)$. To avoid this we apply a Tikhonov regularization approach allowing to construct sub-optimal solutions of DCP or, sub-optimal estimates.

If $\ell \in L(T)$ then the sequence of sub-optimal estimates converges to the $\ell$-estimate which belongs to $L_2$. Otherwise sub-optimal estimates weakly converge to the linear combination of delta-functions. The $L_2$-norms of the sub-optimal $\ell$-estimates grow infinitely in this case.

**Appendix.** Let us prove (27). Integrating by parts (formulae (10)) one finds

$$\int_{t_0}^T \langle \frac{dF \hat{p}_z}{dt} - C \hat{p}_z, z \rangle dt = - \int_{t_0}^T \langle \hat{p}_z, \frac{dF^t_z}{dt} + C' z \rangle dt + \langle F \hat{p}_z(T), F'^t z(T) \rangle - \langle F \hat{p}_z(t_0), F'^t F' z(t_0) \rangle$$

In particular

$$\int_{t_0}^T \langle \frac{dF \hat{p}_z}{dt} - C \hat{p}_z, \hat{z}_f \rangle dt = \langle F' \hat{z}_f(T) - F' t, F' \hat{z}_f(T) \rangle - \langle \hat{p}_z, \hat{z}_f(T) \rangle$$

$$- \langle Q_0^{-1} (F'^t F' \hat{z}_f(t_0) - d), F'^t F' \hat{z}_f(t_0) - \hat{d} \rangle$$

Having this in mind it is straightforward to check that

$$\frac{\varepsilon}{2} \int_{t_0}^T \langle \hat{z} R^{-\frac{1}{2}} u, \hat{z} R^{-\frac{1}{2}} u \rangle dt + \langle z Q^{-\frac{1}{2}} z, \hat{z}_f(T) \rangle = \int_{t_0}^T \langle H \hat{p}_z(T), u - \hat{\hat{u}}_e \rangle dt$$

$$\geq \int_{t_0}^T \langle \hat{z} R \hat{u}_e, u - \hat{\hat{u}}_e \rangle + \langle Q^{-\frac{1}{2}} z, \hat{z}_f(T) \rangle$$

$$+ \int_{t_0}^T \langle \frac{dF \hat{p}_z}{dt} - C \hat{p}_z, \hat{z}_f(T) \rangle dt - \int_{t_0}^T \langle \frac{dF \hat{p}_z}{dt} - C \hat{p}_z, \hat{z}_f(T) \rangle dt$$

$$\int_{t_0}^T \langle \hat{z} \hat{p}_z(T), F'^t z(T) \rangle - \langle F \hat{p}_z(t_0), F'^t F' z(t_0) \rangle - \langle F \hat{p}_z(T), F'^t F' \hat{z}_f(T) \rangle + \langle F \hat{p}_z(t_0), F'^t F' \hat{z}_f(t_0) \rangle$$

where we have applied the sub-gradient inequality [Rockafellar, 1970] to pass from the first line to the second line. Using this inequality, the definition of $\mathcal{T}_\varepsilon$ and (23) it is straightforward to check that

$$\mathcal{T}_\varepsilon(u, z, d) - \mathcal{T}_\varepsilon(\hat{u}_e, \hat{z}_f, \hat{d}_e) \geq \| F' z(T) - F' \hat{u}_e + F' \hat{p}_z(T) \|^2$$

$$+ \| Q_0^{-\frac{1}{2}} (F'^t F' z(t_0) - F'^t F' \hat{z}_f(t_0) + \hat{d}_e - d) \|^2$$

$$+ \int_{t_0}^T \| \frac{dF' z}{dt} + C' z - H' u - \hat{p}_z \|^2 dt \geq 0$$
Let us prove (31). We proved that \( \hat{u}_c, \hat{z}_c \) converges weakly to \( \hat{u}, \hat{z} \) and \( \{d_c\} \to d \) in \( \mathbb{R}^n \). As the norm in \( L_2 \) is weakly low semi-continuous, it follows that

\[
\Omega(\hat{u}_c, \hat{z}_c, d_c) - \|Q_0^{-\frac{1}{2}}(F^+ f'(t) - d_c)\|^2 \\
\leq \Omega(\hat{u}, \hat{z}, d) - \|Q_0^{-\frac{1}{2}}(F^+ f'(t) - d)\|^2
\]

Therefore it is sufficient to show that \( F'z_c(t_0) \to F'\hat{z}(t_0) \) in \( \mathbb{R}^n \). Noting that \( F'q(t_0) = F^2q(T) - \int_{t_0}^T \frac{dF'q(t)}{dt} dt \) for any \( q \in L_1 \) we write

\[
\langle F'z_c(t_0) - F'\hat{z}(t_0), v \rangle \leq \|F'z_c(T) - F'\hat{z}(T)\| \times \\
\times \|v\| + \int_{t_0}^T \left| \frac{dF'\hat{z}_c}{dt} - \frac{dF'\hat{z}}{dt} \right| dt \quad (1)
\]

(30) implies \( \|F'z_c(T) - F'\hat{z}(T)\| \to 0 \) and

\[
\int_{t_0}^T \left| \left( \frac{dF'\hat{z}_c}{dt} + C'\hat{z}_c - H'\hat{u}_c \right) \right|^2 dt < +\infty, \forall \epsilon > 0 \quad (2)
\]

As \( \hat{z}_c \) and \( \hat{u}_c \) converge weakly, it follows that \( \lim \{C'\hat{z}_c - H'\hat{u}_c\} = C'\hat{z} - H'\hat{u} \). This and \( (2) \) implies \( \frac{dF'\hat{z}_c}{dt} \) is bounded. Therefore, the weak convergence of \( \hat{z}_c \) gives:

\[
\lim_{\epsilon \to 0} \int_{t_0}^T \langle \frac{dF'\hat{z}_c}{dt}, v \rangle dt = \int_{t_0}^T \langle \frac{dF'\hat{z}}{dt}, v \rangle dt, v \in L_2 \quad (3)
\]

(3) implies \( F'z_c(t_0) - F'\hat{z}(t_0), v \) in \( (1) \) converges to zero for any \( v \in \mathbb{R}^n \) implying \( F'z_c(t_0) \to F'\hat{z}(t_0) \).

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