Granular impact model as an energy-depth relation

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Abstract - Velocity-squared drag forces are common in describing an object moving through a granular material. The resulting force law is a nonlinear differential equation, and closed-form solutions of the dynamics are typically obtained by making simplifying assumptions. Here, we consider a generalized version of such a force law which has been used in many studies of granular impact. We show that recasting the force law into an equation for the kinetic energy vs. depth, $K(z)$, yields a linear differential equation, and thus general closed-form solutions for the velocity vs. depth. This approach also has several advantages in fitting such models to experimental data, which we demonstrate by applying it to data from 2D impact experiments. We also present new experimental results for this model, including shape and depth dependence of the velocity-squared drag force.

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Introduction. - A dense granular material which is struck by a high-speed object exerts a decelerating force on the intruder, the nature of which is important for many applications, such as soil penetration tests [1,2], meteor impacts [3], and ballistic applications [4,5]. Additionally, understanding the dynamics of intruder motion in a granular material, as well as the granular flow around it, is a fundamental problem in granular physics. To probe this process, it is common to drop or push an intruder into a granular material and record the dynamics and/or the force, $F$, exerted on the intruder. In general, the dynamics depend on the microscale physical characteristics of the grains and intruder, and may show large fluctuations, as in [6].

A common approach [6–14] dating back to the times of Euler and Poncelet is to use space-time averaged macroscopic force laws, where the various terms in the law are empirical expressions based on assumptions of relevant physical principles. These terms are typically assumed to depend on the depth, $z$, as well as on the intruder velocity, $v$, taken to be strictly vertical, i.e., $v = \dot{z}$. Most of these models have the following generalized form:

$$F = m\ddot{z} = mg - f(z) - h(z)\dot{z}^2, \quad (1)$$

where dots denote time derivatives. This force law contains three terms: gravity; a depth-dependent static term, $f(z)$, often taken to be linear in $z$; and a collisional term proportional to the square of the intruder speed, $h(z)\dot{z}^2$, where $h(z)$ is often assumed to be constant. Here, $z$ is measured from the original unperturbed surface, with $z = 0$ as the point of initial contact. These force laws are intended as coarse-grained descriptions of local granular processes, similar in spirit to the static and collisional terms in a general coarse-graining description, such as that formulated by Goldenberg and Goldhirsch [15]. Regardless of the justification, they are often quite successful in describing the average dynamics of the intruder trajectories.

However, theoretical or experimental application of these models reveals several difficulties. First, eq. (1) cannot be integrated to obtain the closed-form solutions for the trajectory, $z(t)$ and $v(t)$. To obtain the final stopping time and depth, one must assume specific forms for $f(z)$ and $h(z)$, typically that they are constants [5,9,10,12]. But, this assumption is inconsistent with several experimental studies [11–13]. Additionally, experimental comparison to these models requires depth, velocity, and acceleration data for the intruder. Typically, this is done by examining data from trajectories with many different initial velocities [6,11–14]. For a specific depth, $z = \zeta$, eq. (1) becomes:

$$F = m\ddot{z} = mg - f(\zeta) - h(\zeta)\dot{z}^2. \quad (2)$$

If this model is valid, plotting $\ddot{z}$ vs. $\dot{z}^2$, calculated at $z = \zeta$, yields an approximately straight line of slope $h(\zeta)$ and intercept $g - f(\zeta)/m$. However, it is often difficult to obtain accurate acceleration data at short
time scales, since accelerometers have limited time resolution, and discrete differentiation from position data greatly amplifies measurement noise. Finally, acceleration measurements often contain large fluctuations at short time scales [5,6,12,13]. Recent work [6] has shown that these fluctuations are a physical aspect of the dynamics, connected to acoustic activity beneath the intruder. These large fluctuations make the precise determination of \( f(z) \) and \( h(z) \) difficult, as shown in Fig. 1. In contrast, the fluctuations in velocity data are considerably smaller, which is a crucial point in the analysis presented here.

In this letter, we demonstrate a new approach to applying this force-law model, where eq. (1), a nonlinear differential equation for force vs. time, is reformulated into a linear differential equation in kinetic energy. We show that this approach addresses all of the aforementioned issues. Mathematically, it allows formal closed-form trajectory solutions without specific assumptions on \( f(z) \) and \( h(z) \). These solutions then provide a natural way to experimentally measure \( f(z) \) and \( h(z) \) using only velocity and depth data, with no assumptions about the functional form of these terms. Using high-speed video data from two-dimensional impact experiments with bronze intruders and photoelastic disks, we study the dynamics using this new approach. This yields several important results, including a significantly higher collisional force at the point of impact for circular intruders, an effect which vanishes for intruders with elongated noses.

Kinetic energy formulation. – We recast eq. (1) from second order in time for \( z \) to first order in depth for the kinetic energy of the intruder, \( K = \frac{1}{2} m \ddot{z}^2 \), by using a relation that is familiar from the work-energy theorem of mechanics: \( m \ddot{z} = dK/dz \).

\[
\frac{dK}{dz} = mg - f(z) - \frac{2h(z)}{m} K. \tag{3}
\]

This yields an inhomogeneous linear ordinary differential equation with (potentially) nonconstant coefficients, by contrast to the nonlinear equation of motion, eq. (1), for \( \ddot{z}(t) \). The fact that eq. (3) is a linear ODE means that standard ODE methods immediately yield formal solutions for \( K(z) \):

\[
K(z) = K_0(z)(K_0 + \phi(z)), \tag{4}
\]

where \( K_0 \) is the kinetic energy at impact,

\[
K_0(z) = \exp \left( - \int_{z_0}^z \frac{2}{m} h(z') \, dz' \right), \tag{5}
\]

and

\[
\phi = \int_{z_0}^z \, dj \left[ (mg - f(z')) / K_0(z') \right]. \tag{6}
\]

This reduces the problem for the trajectory to a quadrature. The velocity can be written as

\[
\ddot{z} = \frac{dz}{dt} = \left[ \frac{2}{m} K_0(z)(K_0 + \phi(z)) \right]^{1/2}, \tag{7}
\]

and \( z(t) \) follows by integrating and inverting:

\[
t(z) = \int_0^z \, dz' \left[ \frac{2}{m} K_0(z')(K_0 + \phi(z')) \right]^{-1/2}. \tag{8}
\]

If the forms of \( f(z) \) and \( h(z) \) are simple, much of the calculation of these integrals can by done explicitly. For example, using the commonly assumed forms \( f(z) = f_0 + kz \) and \( h(z) = b \), we obtain \( K(z) \) as

\[
K(z) = (K_0 - c_1) \exp(-c_2 z) + c_1 - c_3 z. \tag{9}
\]

Here, the constants are \( c_1 = (mg - f_0) c_2 + k c_2 \), \( c_2 = 2b/m \), and \( c_3 = k/c_2 = km/(2b) \).

Even without integrating eq. (8), it is possible to find the stopping distance by setting \( K(z_{stop}) = 0 \), or \( \phi(z_{stop}) = -K_0 \), yielding the stopping depth as a function of impact energy, \( K_0 \). Specifically, for the common case described by eq. (9), the stopping depth, \( z_{stop} \) satisfies

\[
z_{stop} = \frac{m}{2b} \log \left[ \frac{2b K_0 + f_0 + km}{f_0 + k z_{stop} + km} - mg \right]. \tag{10}
\]

Note that if we take \( f(z) \) as roughly constant, \( f(z) = f_0 \) and \( k = 0 \), then \( z_{stop} \) increases logarithmically with \( K_0 \), as in [5,10,12].

\[
z_{stop} = \frac{m}{2b} \log \left[ 1 + \frac{2b}{m} \left( \frac{K_0}{f_0 - mg} \right) \right]. \tag{11}
\]
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This approximation is relevant in the limit of high impact energy, where \( b^2 \) dominates. In the low-energy limit, it predicts \( z_{stop}(K_0 = 0) = 0 \). However, an intruder must be at least partially submerged to be supported by frictional grains. This occurs at a depth which increases with intruder size. So, when \( K_0 \rightarrow 0 \), we expect \( z_{stop}(K_0 = 0) \sim D \). Note that eqs. (9) and (10) may yield a nonzero upward force, \( F = \frac{\Delta K}{\rho h} \), as the intruder comes to a stop, and we return later to this issue.

Measuring \( f(z) \) and \( h(z) \). This formulation also provides a way to measure \( f(z) \) and \( h(z) \) in terms of experimental data for \( z(t) \) and \( \dot{z}(t) \), without specific assumptions for \( f(z) \) and \( h(z) \). Subtracting two different trajectories, with different \( K_0 \)'s (not necessarily close), \( K_i(z) = K_p(K_0 + \phi) \) and \( K_j(z) = K_p(K_0 + \phi) \), yields

\[
\frac{K_i(z) - K_j(z)}{K_{i0} - K_{j0}} \equiv K_p(z) = e^{-\int_0^{z} \frac{2K_i(z')}{m} \, dz'},
\]

which gives

\[
h(z) = -\frac{d}{dz} \left[ \frac{m}{2} \log K_p(z) \right].
\]

To avoid numerical differentiation, we also use \( \int h(z)dz \) in our discussion below.

Since the kinetic energy goes to zero when the intruder stops, we can set eq. (4) equal to zero at \( z = z_{stop}, \) i.e., \( K_0 = -\phi(z_{stop}) \). The expression for \( f(z) \) follows by then differentiating with respect to \( z_{stop} \), which yields

\[
f(z_{stop}) = K_p(z_{stop}) \left( \frac{dK_0}{dz_{stop}} \right) + mg.
\]

This analysis requires a determination of \( K_p \) (i.e., \( h(z) \) is determined), and \( z_{stop}(K_0) \), where the latter is generally straightforward.

Application to experimental data. – To test the approach outlined above (based on kinetic energy), and to compare to the approach based on determining the acceleration (shown in fig. 1), we use data from two-dimensional granular impact experiments [6], where disk and ogive intruders (i.e., intruders with an elliptical-like nose), cut from bronze sheet, are normally incident on a collection of approximately 25000 bidisperse, hard, photoelastic disks (diameters of 6 mm and 4.3 mm) confined between two 0.91 m × 1.22 m × 1.25 cm acrylic sheets. Particles are constructed from PSM-1, a stiff photoelastic polymer, made by Vishay Precision Group (bulk density of 1.28 g/cm³, elastic modulus of 2.5 GPa, and Poisson ratio of 0.38). Before each impact, a long rod is used to stir the particles and smooth out the top surface, producing a fairly consistent packing fraction, \( \phi = 0.82 \). During impact, some local compaction occurs beneath the intruder, but the global packing fraction does not change significantly (\( \Delta \phi < 0.005 \)). Intruders have initial velocities between 0 and 6 m/s, which are in the subsonic regime, since videos show the granular sound speed to be about 300 m/s [6]. We record results with a high-speed camera, typically at 40000 frames per second, which allows us to determine the position of the intruder in each frame, as well as the forces on the photoelastic particles. Here, we focus on the intruder dynamics only; sample trajectories are shown in fig. 2.

Once the intruder trajectory is known, we differentiate to find the velocity and acceleration of the intruder at each frame. To avoid noise amplification, some filtering is required with each derivative. This is accomplished by fitting a linear function to \( W \) frames of the position data, centered at \( z(t) \) for the frame of interest, which yields the velocity, with time resolution reduced by a factor of \( W \).
The same procedure is repeated to obtain the acceleration from velocity data. We choose $W = 300$, which is the smallest value for $W$ such that the signal-to-noise ratio is 10:1 for acceleration data. This ensures that the observed fluctuations in velocity and acceleration are physical. As discussed in Clark et al. [6], the acceleration data for the intruder, obtained in this manner, exhibits fluctuations that are intrinsic to the emission of acoustic pulses at the interface between the intruder and grains.

Here we use eight different intruders, with width $D$ and varying nose shape, as shown in Fig. 2. Four circular intruders, with diameters, $D$, of 6.35, 10.16, 12.70, and 20.32 cm, were used to test size effects, and four ogive intruders were constructed to test shape effects. The shapes of the ogives consisted of a continuous piece of material, where the leading part is a half-ellipse truncated along the minor axis, with semi-major axis $a$ and semi-minor axis $b = D/2$, terminated by a rectangular tail of length $L$. Three different ellipses were used, with $a/b = 1$ (half-circle), $a/b = 2$, and $a/b = 3$. The width of the ogives was held constant, $D = 9.3$ cm, and $L$ was varied to keep the intruder mass constant ($L = b = 4.15$ cm for $a/b = 3$ case, longer for other ogives). By keeping the width and mass constant, we isolate shape effects. Additionally, we used one smaller ogive with $a/b = 1$, $b = 3$ cm, and $L = 7.7$ cm.

As discussed above, fitting to the force law of eq. (1) requires plotting the acceleration vs. velocity squared. The fluctuations in acceleration, shown in Fig. 2, are very large, so determining a clear value for $f(z)$ and $h(z)$ at each depth is difficult, as shown in Fig. 1. However, with the kinetic energy approach, only velocity data is needed to determine $f(z)$ and $h(z)$. Using eq. (12), and averaging over all pairs of trajectories (omitting trajectory pairs with very similar initial velocities), we obtain a clear average value for $K_p(z)$, and thus for $h(z)$, as shown in Fig. 3.

Data for $-m/2 \log K_p = \int f(z) h(z) dz$ are approximately linear in $z$. The slope gives the collisional term $h(z)$ (shown in Fig. 3), which scales approximately with the intruder size, as discussed below. Here, calculating $h(z)$ requires taking a spatial derivative, which amplifies the fluctuations in $K_p(z)$. However, especially when comparing data from multiple intruders, we are now able to separate these fluctuations from systematic variation in the functional form of $h(z)$: we observe an initial transient regime, after which $h(z)$ approaches a nearly constant value. For circles, the collisional term is stronger at impact, which may be surprising, since the area of impact is smallest then. This effect is reminiscent of surface tension or so-called “added mass” effects for fluid impacts [16–18], but it is not obvious what physical mechanisms are at play in the granular case. For intruders with more elongated noses, this effect is greatly weakened, and even reversed in the $a/b = 3$ case shown in Fig. 3. We also note that the nature of $h(z)$ — constant, after an initial transient — supports the omission of as force law term which is linear in the velocity (i.e., proportional to $K^{1/2}$), at least for the experiments discussed here.

Once $h(z)$ is specified, eq. (14) provides a solution for $f(z)$ with adequate data for $z_{stop}(K_0)$. First, we plot the final depth, $z_{stop}$, vs. $K_0$, as shown in Fig. 4. Note that the data for the higher energies are consistent with the logarithmic behavior in eq. (11), and that the data for low energies deviate from this curve with a nonzero intercept which scales with intruder diameter ($z_{stop}(K_0 = 0) \approx D$). Only circular intruders are used, since ogive intruders penetrate much deeper and may interact with the lower boundary of the experiment.

Finally, by fitting a smooth function to the curve for each circular intruder and differentiating, we solve for $f(z)$, wherever $z_{stop}(K_0)$ is defined. This yields a linear function for all circular intruders with a nonzero intercept, $f_0$, which scales linearly with the intruder mass (Fig. 4), i.e., $f_0 \approx 1.35mg$. We note that impact experiments performed by Goldman and Umbanhowar [12] show a similar result for $f_b$, but with a $f_0$ which increases from 0 to $2mg$ during $0 < z < D$, then saturates at $f_0 \approx 2mg$.
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Fig. 4: (Color online) Top: plot of $z_{\text{stop}}$ vs. $K_0$, with fits of the form $a \log(bK_0 + 1) + c$. Bottom: plot of $f(z)$ for circular intruders. Linear fits are $f_0 + kz$, where the slope, $k$, corresponds to hydrostatic pressure. Note that $f(z)$ is dominated by the offset, $f_0$, for our data. Inset: plot of $f_0$ vs. $mg$, with a linear fit through the origin, with slope 1.35.

Fig. 5: (Color online) Main plot: average velocity of all four circular intruders as they come to a stop. All intruders decelerate at the same rate, $a \approx 0.75g$. The upward force required would be approximately $1.75mg$, which is consistent with $f(z_{\text{stop}}) = 1.35mg + kz_{\text{stop}}$, from fig. 4. Inset: end of all trajectories, with initial velocities between 1 and 6 m/s, for the $D = 12.7$ cm circular intruder. Note that the stopping time is very weakly dependent on the initial velocity, since these trajectories all end at approximately the same time.

Fig. 6: (Color online) Top: plot of the average $h(z)$ for circular-nosed intruders vs. the diameter of the nose, $D$, which shows that the two are directly proportional, with $h(z) \sim 5.5D$. Bottom: plot of the average $h(z)/D$ vs. the intruder aspect ratio, $a/b$, which shows a substantial decrease in the collisional force as the intruder nose is elongated.

value of $f(z)$ from fig. 4), and the acceleration jumps to zero at $v = 0$, as in [12,13].

To examine size and shape effects, we also plot the depth-averaged $h(z)$ as a function of intruder size and shape, as shown in fig. 6. The top plot shows that $h(z)$ is directly proportional to the intruder size. The bottom plot shows the average $h(z)$ vs. the aspect ratio of the elliptical nose, which falls off substantially as aspect ratio is increased. Thus, for equal intruder widths, a more elongated nose has a substantial effect in decreasing the collisional force. We also note that the two circular-nosed intruders which have a slightly larger collisional force are the larger, circular ogive and the largest disk. We believe that the deviation of these two intruders relates to their larger mass-to-width ratio, $m/D \approx 3$ kg/m (for all other circular-nosed intruders, $m/D < 2$ kg/m). These two intruders generate photoelastic activity that extends considerably farther into the material (the larger, elliptical-nosed intruders with $m/D \approx 3$ kg/m generate far less photoelastic activity, perhaps keeping them in the same regime as the smaller circles). Thus, as a potential explanation, we suggest that these intruders are...

As discussed previously, the net force must go to zero as the intruder stops. With this in mind, we examine trajectories for $v < 0.3$ m/s (shown in fig. 5), where $h(z)v^2 \ll mg$. This shows approximately constant deceleration as the intruder comes to a stop (consistent with the expected...
effectively interacting with a larger mass of grains. Similar collective effects were used in [14], as well as suggested by Waitukaitis and Jaeger to explain the more extreme case of shear thickening of suspensions which are subjected to impact [19].

Conclusion. – We have shown a new approach to a commonly used model for describing the dynamics of an intruder impacting on a granular material. By reformulating the model into a linear ODE, we obtain formal solutions of the trajectory in terms of the initial kinetic energy, as well as a systematic way of calculating the different terms in the model — namely \( f(z) \) and \( h(z) \) — using only position and velocity data, which are more easily obtained experimentally than data for acceleration.

Additionally, we have used this approach to measure \( f(z) \) and \( h(z) \) for experimental data. The high level of agreement between the experimental data and the model in eq. (1), as well as the sensible behavior and scaling of the \( f(z) \) and \( h(z) \) terms, validate the use of the model. However, the grain-scale origins of the force-law terms are not well understood, and this will be the subject of future study. We also observe that the usual assumption that \( h(z) \) is constant applies only after an initial transient at impact, which varies with intruder shape. This result could be important in engineering and control applications. We also note that terms linear in velocity have been proposed in the context of this model [12], but our data shows no need to implement them here. We also note a substantial reduction in the collisional term, \( h(z) \), as the intruder nose is elongated. Thus, future work should include investigation of other intruder shapes (e.g. triangular/conical noses). Finally, it is not clear under what conditions the force-law model is valid. Further study might explore the limits of this model, such as intruder velocities approaching the granular sound speed or connection to the slow drag regime [20–22].

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