STRONGLY IRREDUCIBLE SURFACE AUTOMORPHISMS

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Abstract. A surface automorphism is strongly irreducible if every essential simple closed curve in the surface has nontrivial geometric intersection with its image. We show that a three-manifold admits only finitely many inequivalent surface bundle structures with strongly irreducible monodromy.

1. Introduction

A surface automorphism \( h : F \to F \) is strongly irreducible if every essential simple closed curve \( \gamma \subset F \) has nontrivial geometric intersection with its image, \( h(\gamma) \). This paper shows that a three-manifold admits only finitely many inequivalent surface bundle structures with strongly irreducible monodromy. This imposes a serious restriction; for example, any three-manifold which fibres over the circle and has \( b_2(M) \geq 2 \) admits infinitely many distinct surface bundle structures.

The main step is an elementary proof that all weakly acylindrical surfaces inside of an irreducible triangulated manifold are isotopic to fundamental normal surfaces. As weakly acylindrical surfaces are a larger class than the acylindrical surfaces this strengthens a result of Hass \[5\]; an irreducible three-manifold contains only finitely many acylindrical surfaces.

Section 2 gives necessary topological definitions, examples of strongly irreducible automorphisms, and precise statements of the theorems. The required tools of normal surface theory are presented in Section 3. Section 4 defines weakly acylindrical and proves that every such surface is isotopic to a fundamental surface. In the spirit of the Georgia Topology Conference the paper ends by listing several open questions.

Many of the ideas and terminology discussed come from the study of Heegaard splittings as in \[2\] and in my thesis \[11\]. This paper, in particular Theorem 4.2, owes an obvious debt to \[3\] by Jaco and Oertel. I thank Ian Agol for simplifying my original proof of Proposition 2.3 and Dave Bachman for critiquing an early version of this paper.

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2. Definitions and Examples

This section lays out the necessary definitions, states the main theorems precisely, and gives examples of families of strongly irreducible surface automorphisms.

Let $F$ be a closed, orientable, genus($F$) > 1 surface. Let $h : F \to F$ be an automorphism of $F$. If $\gamma_0$ and $\gamma_1$ are simple closed curves in $F$ then the geometric intersection number, $i(\gamma_0, \gamma_1)$, is the minimum of $|\gamma'_0 \cap \gamma'_1|$ taken over all $\gamma'_i$ isotopic to $\gamma_i$.

**Definition.** The map $h$ is strongly irreducible if $i(h(\gamma), \gamma) > 0$ for every essential simple closed curve $\gamma \subset F$.

If $h$ is not strongly irreducible then $h$ is weakly reducible.

**Remark 2.1.** A reducible surface automorphism is one which admits an invariant set of disjoint essential simple closed curves, up to isotopy. Thus reducible maps are also weakly reducible.

**Remark 2.2.** There exist weakly reducible pseudo-Anosov maps with arbitrarily high stretch factor — the main ideas required for the construction may be found in [9].

**Example.** Let $P$ be a regular $4g$-gon in the hyperbolic plane with angle at the vertices equal to $2\pi/4g$. Here $g$ is assumed to be two or larger. Glue opposite sides of $P$ by isometries to obtain $F$, an orientable surface of genus $g$. Let $h : P \to P$ be a counter-clockwise rotation of $P$ about its center, $O$, through an angle of $2\pi/4g$. Let $h'$ be the induced isometry of $F$.

**Proposition 2.3.** The periodic map $h'$ is strongly irreducible.

All of the hyperbolic trigonometry needed in the proof may be found in Chapter 7 of [1].

**Proof.** Suppose that $R$ is the distance between $O$ and $V$, where $V$ is a vertex of $P$. Then $\cosh(R) = \cot^2(2\pi/8g)$. Both $O$ and $V$ give fixed points, $x_O, x_V \in F$, of $h'$. Suppose that $\gamma \subset F$ is a simple closed geodesic. There is a point of $\gamma$ which lies within $R/2$ of either $x_O$ or $x_V$. This last holds because the points of $F$ which are not this close to one of $x_O$ or $x_V$ form a disjoint union of disks.

Suppose there is a point of $\gamma$ within distance $R/2$ of the point $x_O$. (The other case is similar.) Pick $\tilde{\gamma} \subset \mathbb{H}^2$, a lift of $\gamma$, which lies within $R/2$ of $O$. Let $L$ be the distance between $\tilde{\gamma}$ and $O$. Let $\theta$ be the visual angle which $\tilde{\gamma}$ occupies, as viewed from $O$. (When $\tilde{\gamma}$ meets $x_O$...
take $\theta = \pi$.) Then $L \leq R/2$ and $\cosh(L) = 1/\sin(\theta/2)$. Applying a “double-angle” formula to $\cosh(R)$ yields
\[
cosh(R/2) = 1/\sqrt{2}\sin(2\pi/8g).
\]
Hyperbolic cosine is an increasing function on the positive reals so
\[
1/\sin(\theta/2) \leq 1/\sqrt{2}\sin(2\pi/8g).
\]
As sine is increasing in the interval $[0, \pi/2]$ we deduce that $\theta > 2\pi/4g$. Thus the visual angle of $\tilde{\gamma}$ is greater than $2\pi/4g$ and $h(\tilde{\gamma}) \cap \tilde{\gamma} \neq \emptyset$. This gives the desired conclusion as the intersection and geometric intersection numbers agree for geodesics.

\[\square\]

**Proposition 2.4.** If $h$ is a pseudo-Anosov map then there is an $n \in \mathbb{N}$ such that $h^n$ is strongly irreducible.

Here we only sketch a proof; the numbers in parentheses refer to theorems in Kapovich’s book \[8\] which we take as our reference for $PML(F)$, the space of projectively measured laminations of $F$. Recall that geometric intersection extends to a continuous function on $ML \times ML$ (11.26).

Let $\lambda^\pm$ be the stable and unstable laminations for $h$. Let $U, V$ be small neighborhoods of $\lambda^\pm$ (respectively) in $PML(F)$. Choose $U$ and $V$ so that all $x \in U$, $y \in V$ have $i(x,y) > 0$. This is possible because the geometric intersection between $\lambda^+$ and $\lambda^-$ is non-zero (11.49). In particular, $U \cap V = \emptyset$.

There is an $m \in \mathbb{N}$ such that if $x \in PML(F) \setminus V$ then $h^m(x) \in U$ (11.47). Now, $n = m + 1$ gives the desired conclusion. To see this, pick an essential simple closed curve $y \subset F$. Denote the corresponding element of $PML(F)$ again by $y$. There is a integer $k$ such that $h^k(y) \in V$ but $h^{k+1}(y) \notin V$. (Again, 11.47.) So $i(y, h^n(y)) = i(h^k(y), h^{k+1+n}(y)) > 0$. This completes the proof sketch.

**Remark 2.5.** The notions irreducible and strongly irreducible may be generalized by introducing the curve complex of $F$, $C(F)$, and defining the translation distance, $\tau(h)$, of $h$’s action on $C(F)$. The deep results of \[10\] give a positive integer $n(g)$ depending only on $g = \text{genus}(F)$ such that: If $h$ is pseudo-Anosov then $\tau(h^{n(g)}) \geq 2$ and so $h^{n(g)}$ is strongly irreducible. This greatly improves upon Proposition 2.4.

We now turn from examples to the main objects of interest: surface bundles over the circle.
Definition. If $h : F \to F$ is a surface automorphism then let $M_h$ be the mapping torus of $h$. So

$$M_h \cong F \times I/(x, 1) \sim (h(x), 0).$$

The mapping torus admits a natural map to the circle, $\pi_h : M_h \to S^1$. The embedded surfaces $F \times \{t\} \subset M$ are called the fibres of this map while $h$ is the monodromy and $g(F) = \text{genus}(F) \geq 2$ is the genus of the bundle. As an example, if $h$ is a strongly irreducible periodic automorphism, then $M_h$ is an atoroidal Seifert fibred space.

Fix $M$, a closed orientable three-manifold.

Definition. A surface bundle structure on $M$ is a pair $(h, \phi)$, where $h$ is a surface automorphism and $\phi$ is a homeomorphism from $M_h$ to $M$.

Let $(h, \phi)$ and $(h', \phi')$ be two surface bundle structures on $M$. Suppose that $\psi : M_h \to M_{h'}$ is a homeomorphism such that $\pi_h = \pi_{h'} \circ \psi$ and $\phi$ is isotopic to $\phi' \circ \psi$. Then the two bundle structures on $M$ are equivalent. Thus $(h, \phi)$ is equivalent to $(h', \phi')$ if and only if $h'$ is conjugate to $h$ and the fibres of the two bundle structures are isotopic in $M$.

Here is a precise statement of the theorem alluded to in the introduction.

**Theorem 4.3.** Suppose that $M$ is a closed, orientable three-manifold. Then $M$ admits only finitely many inequivalent surface bundle structures with strongly irreducible monodromy.

Closely related is our:

**Theorem 4.4.** Suppose that $M$ is a closed, orientable three-manifold. There is a positive real number $c(M)$ such that if $(h, \phi)$ is a surface bundle structure on $M$ with genus $g > 1$ then $h^i$ is weakly reducible for all integers $i$ where $1 \leq i \leq c(M) \cdot (2g - 2)$.

If $c(M) \cdot (2g - 2) < 1$ then the theorem is vacuous as no such $i$ exists.

**Remark 2.6.** If $M$ is atoroidal then $M$ has only finitely many inequivalent surface bundle structures in each genus. (This simply because there are only finitely many incompressible surfaces, up to isotopy, of each genus. See [1].) Thus, in the atoroidal case, Theorem 4.4 implies Theorem 4.3.

**Remark 2.7.** Suppose $M$ is a closed, atoroidal three-manifold admitting infinitely many inequivalent surface bundle structures. Then this manifold and Theorem 4.4 provide another proof that the constant $n(g)$ of Remark 2.5 must tend to infinity as $g$ does.
Remark 2.8. Again, consider the translation distance $\tau(h)$ where $h$ is an automorphism of the closed, orientable, genus at least two, surface $F$. One obtains an analogue of a theorem of Hartshorn \cite{4} regarding Heegaard splittings: If $(h, \phi)$ is a surface bundle structure on $M$ and $G$ is a two-sided incompressible surface in $M$ then either $G$ is isotopic to a fibre, $G$ is a torus (and $h$ is reducible), or $\tau(h) \leq -\chi(G)$.

As a corollary, deduce that if $h$ has translation distance greater than $-\chi(F)$, where $F$ is the fibre, then $F$ is the unique minimal genus incompressible surface in $M$, up to isotopy. Weak conclusions about the shape of the Thurston norm ball and on the structure of $M$’s symmetry group follow.

3. Normal surfaces

This section presents the required bare minimum of normal surface theory. For a more complete treatment consult \cite{3} or \cite{4}.

Fix a closed orientable three-manifold $M$ and choose $T$, a triangulation of $M$. Suppose $F \subset M$ is a closed embedded surface. The weight of $F$, $w(F)$, is the number of intersections between $F$ and the one-skeleton of $T$. The surface $F$ is normal if $F$ intersects every tetrahedron of $T$ in a disjoint collection of normal triangles and quadrilaterals. See Figure 1. In each tetrahedron there are four types of normal triangle and three types of normal quad.

Lemma 3.1. (Haken \cite{3}) Suppose that $(M, T)$ is closed and irreducible. If $F \subset M$ is embedded and incompressible then $F$ is isotopic to a normal surface $F'$ with $w(F') \leq w(F)$.

A normal isotopy of $M$ fixes every simplex of $T$ setwise. Two normal surfaces, $G$ and $H$, are compatible if in each tetrahedron $G$ and $H$ have the same types of quad (or one or both have no quads.) In this situation we form the Haken sum $F = G + H$ as follows:
Normally isotope $H$ to make $G$ transverse to $H$, to make $\Gamma = G \cap H$ transverse to the skeleta of $T$, and to minimize the number of curves in $\Gamma$. The components of $\Gamma$ are the exchange curves. For every such $\gamma \subset \Gamma$ let $R(\gamma)$ be the closure (taken in $M$) of $\eta_M(\gamma)$, an open regular neighborhood of $\gamma$. The set $R(\gamma)$ is a solid torus containing $\gamma$ as a core curve.

Then $(\partial R(\gamma)) \setminus (G \cup H)$ is a union of annuli. Taking closures divide these into two sets, the regular and irregular annuli, $A_r(\gamma)$ and $A_i(\gamma)$, as indicated by Figure 2. Finally, as in Figure 3, form the surface

$$F = \left( (G \cup H) \setminus \bigcup_{\gamma} \{\eta_M(\gamma)\} \right) \cup \bigcup_{\gamma} \{A_r(\gamma)\}.$$ 

Each connected component of $(G \cup H) \setminus \bigcup_{\Gamma} \{\eta_M(\gamma)\}$ is a patch of the sum $G + H$ while each closed annulus $A_r(\gamma)$ is a seam. Note that $F$ is again a normal surface which, up to normal isotopy, does not depend on the choices made in the above construction.

A normal surface is fundamental if it admits no such decomposition. A fundamental result due to Haken is:

**Lemma 3.2.** (See [7]) A closed orientable triangulated three-manifold $(M^3, T)$ contains only finitely many fundamental normal surfaces, up to normal isotopy.

**Remark 3.3.** The proof of Lemma 3.2 requires only that the manifold $M$ admit a finite triangulation. We restrict ourselves to the closed and orientable case to avoid an unnecessarily technical discussion about normal surfaces.

Each “cut-and-paste” operation involved in the Haken sum $F = G + H$ may be recorded by an embedded exchange band $(C(\gamma), \partial C(\gamma)) \subset (N(\gamma), A_r(\gamma))$. That is, the band is embedded in $N(\gamma)$ with boundary inside of $A_r(\gamma)$. See Figure 3. Each $C(\gamma)$ is either an annulus or
a Mobius band. The exchange bands record enough information to reverse the sum. Note also that each seam is a regular neighborhood (in $F$) of a boundary component of some $C(\gamma)$.

A sum $F = G + H$ is reduced if $F$ cannot be realized as a sum $G' + H'$ where $G'$ and $H'$ are again normal, isotopic to $G$ and $H$ respectively, with $|G' \cap H'| < |G \cap H|$. Note that the isotopy between $G$ and $G'$ ($H$ and $H'$) need not be normal. Lemma 3.4 is a key technical result for [6] and our Theorem 4.2.

**Lemma 3.4.** (Jaco and Oertel [6]) Suppose that $(M^3, T)$ is a closed, orientable, irreducible, triangulated three-manifold. Suppose that $F \subset M$ is an incompressible normal surface which is least weight in its isotopy class. If the sum $F = G + H$ is reduced then no patch of $G + H$ is a disk.

4. **Weakly acylindrical surfaces**

Here the weakly acylindrical surfaces are defined. Equipped with this definition and Theorem 4.2 we will prove Theorem 4.3, the main goal of the paper.

Suppose now that $N$ is a compact, orientable three-manifold with non-empty boundary.

**Definition.** An embedded annulus $(A, \partial A) \subset (N, \partial N)$ is essential in $N$ if $A$ is incompressible and boundary-incompressible.

Now fix $M$, a closed orientable three-manifold. Let $F \subset M$ be a closed, embedded, incompressible, two-sided surface with genus at least two.

**Definition.** The surface $F$ is cylindrical if $N = M \setminus \eta_M(F)$ admits an essential annulus. If $N$ does not admit an essential annulus then $F$ is acylindrical.
**Definition.** The surface $F$ is *strongly cylindrical* if there exists an embedded annulus $(A, \partial A) \subset (M, F)$ such that $A \setminus \eta_M(F)$ is essential in $N = M \setminus \eta_M(F)$. On the other hand, if no such annulus exists then $F$ is *weakly acylindrical*.

**Remark 4.1.** If $F$ is acylindrical then $F$ is weakly acylindrical. Note that if $F$ is a fibre of a surface bundle then $F$ is never acylindrical, as the complement of $F$ is homeomorphic to $F \times I$. However, if the monodromy is strongly irreducible then $F$ is weakly acylindrical. For separating surfaces the notions acylindrical and weakly acylindrical coincide.

**Theorem 4.2.** Suppose that $(M, T)$ is a closed, orientable, irreducible, triangulated three-manifold. Suppose $F \subset M$ is weakly acylindrical. Then $F$ is isotopic to a fundamental normal surface.

Before proceeding with the proof of Theorem 4.2 we remark that the normal surface techniques used may easily be replaced by the methods of branched surface theory.

**Proof of Theorem 4.2.** Fix $F \subset M$ as in the hypothesis. Recall that weakly acylindrical includes the property of being incompressible. Applying Lemma 3.1 isotope $F$ to be normal and least weight in its isotopy class. Set $N = M \setminus \eta_M(F)$. Recall also that $F$ is two-sided, and has genus two or greater.

Suppose that $F$ is not fundamental. Pick a reduced decomposition $F = G + H$. This sum admits some exchange band $A$. Let $A_N = A \cap N$. If $A$ is a Mobius band then $A$’s *double*, defined below, must be compressible or boundary-compressible in $N = M \setminus \eta_M(F)$. If $A$ is an annulus then $A_N$ is itself compressible or boundary-compressible in $N$. The proof deals with each of these possibilities in turn, showing that all lead to contradiction.

Suppose that $A$ is an exchange Mobius band. As $M$ is orientable, $A$ is one-sided. Let $X$ be a closed regular neighborhood of $A_N$, taken in $N$. Let $B = X \cap \partial N$. Then $B$ is an annulus on the boundary of the solid torus $X$. The *double* of $A$, $\tilde{A}$, is the closure of $(\partial X) \setminus B$. Note that $\tilde{A}$ is an embedded annulus with $(\tilde{A}, \partial \tilde{A}) \subset (N, \partial N)$. As $F$ is weakly acylindrical $\tilde{A}$ must compress or boundary-compress in $N = M \setminus \eta_M(F)$.

Suppose that $\tilde{A}$ compresses along a disk $E$. Note that $E \cap X = \partial E$. Compress $\tilde{A}$ along $E$ to obtain a pair of disks $C$ and $D$. Then $S = B \cup C \cup D$ is a two-sphere bounding $\mathbb{R}P^3 \setminus B^3$ on the side which meets...
A. Thus $M \cong \mathbb{RP}^3$, contradicting the fact that $F$ was two-sided and incompressible.

Suppose that $\tilde{A}$ instead boundary-compresses along a disk $E$. Let $Y$ be the closure of $\eta_{N \setminus X}(E)$. Boundary-compressing $\tilde{A}$ along $E$ gives a disk $D$. That is, $D$ is the closure of $(\partial (X \cup Y)) \setminus \partial N$. Since $F$ is incompressible and two-sided, $\partial D$ bounds a disk $D' \subset \partial N$. Note that $D'$ meets $X \cup Y$ along $\partial D'$ only, as $\partial N \cap (X \cup Y)$ is nonplanar. Now, as $F$ is incompressible, the two-sphere $D \cup D'$ bounds a three-ball, $Z$, on the side not meeting $A$. It follows that $X \cup Y \cup Z$ is a solid torus with boundary equal to a component of $\partial N$. However this is impossible, as every component of $\partial N$ has genus equal to that of $F$.

Thus the sum $G + H$ has no exchange Möbius bands. Instead, suppose that $A$ is an exchange annulus. As $M$ is orientable $A$ is two-sided. The weakly acylindrical hypothesis forces $A_N$ to be compressible or boundary-compressible in $N = M \setminus \eta_M(F)$.

Suppose that $A_N$ is compressible along a disk $E$. Compress $A$ along $E$ to obtain disks $C$ and $D$ with $\partial C \cup \partial D = \partial A$. As $F$ is incompressible $\partial C$ bounds a disk, $C' \subset F$. Now $\partial C = \partial C'$ is contained inside of a seam, say $A_p(\gamma)$. Thus $C''$, the closure of $C' \setminus A_p(\gamma)$, is a disk which is a union of patches and seams of $G + H$. An innermost disk in $C''$ must be a disk patch, contradicting Lemma 3.4.

Finally, suppose that $A_N$ is boundary-compressible along a disk $E$. Boundary-compress $A_N$ to obtain a disk $D$. By incompressibility $\partial D \subset \partial N$ bounds a disk $D' \subset \partial N$. Let $Z$ be the three-ball with boundary $D \cup D'$. If $E \subset Z$ then $A_N$ is compressible (see Figure 4) yielding contradiction as in the proceeding paragraph.

If $E \cap Z = \emptyset$ then the situation is more delicate. Let $W$ be the closure of the component of $N \setminus A_N$ containing $E$ and $Z$. Then $W$ is a solid torus having $E$ as a meridional disk. Note that $A_N \subset \partial W$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{A boundary-compressible tube}
\end{figure}
Figure 5. An irregular exchange along $A$

Figure 6. The weight-reducing isotopy

Let $X$ be the closure of $\eta_N(A_N)$. Let $\tilde{A}$ be the closure of $\partial X \setminus \partial N$. Take $F' = (\partial N \setminus X) \cup \tilde{A}$. Then $F' = T \cup F''$ where $T = \partial (W \setminus X)$ is a torus and $F$ is isotopic to $F''$. (The surface $F'$ is obtained by performing an irregular exchange along the annulus $A$.) To check that $F''$ is indeed isotopic to $F$ recall that $F$ is two-sided and note that the annulus $\partial N \cap (W \cup X)$ may be isotoped, relative to its boundary, across the solid torus $W \cup X$. See Figure 5 for a schematic picture of the cross-section of $W$.

Either $w(F'') < w(F)$ or $F''$ is not normal. (Again, see Figure 2.) In the latter case there is an isotopy of $F''$ to $F'''$, supported in a small neighborhood of some face in the two-skeleton, which reduces weight by two. See Figure 3. Both possibilities contradict Lemma 3.1 because $F$ is least weight in its isotopy class. \hfill $\square$

We now deduce our main theorem as a corollary of Theorem 4.2:

**Theorem 4.3.** Suppose that $M$ is a closed, orientable three-manifold. Then $M$ admits only finitely many inequivalent surface bundle structures with strongly irreducible monodromy.

**Proof.** It follows from Theorem 4.2 and Lemma 3.2 that there are finitely many isotopy classes of weakly acylindrical surfaces in $M$. As
in Remark 4.1, any bundle with strongly irreducible monodromy has a weakly acylindrical fibre. Finally, a given closed embedded surface in $M$ is isotopic to the fibre of at most two bundle structures, up to equivalence. This gives the desired conclusion.

Now suppose that $M$ is a closed, orientable three-manifold with triangulation $T$. Let $\{F_i\}$ be the fundamental normal surfaces in $T$ with negative Euler characteristic while $\{T_i\}$ and $\{P_i\}$ are those fundamental surfaces of Euler characteristic zero and positive, respectively. By Lemma 3.2 each of these collections is finite. Let $K = |\{F_i\}|$ be the number of fundamental surfaces with negative Euler characteristic. Let $P = \max\{-\chi(F_i)\}$.

We end this section by sketching a proof of:

**Theorem 4.4.** If $(h, \phi)$ is a surface bundle structure on $M$ with genus $g > 1$ then $h^i$ is weakly reducible for all integers $i$ where $1 \leq i \leq c(M, T) \cdot (2g - 2)$ and $c(M, T) = 1/(3KP)$.

**Proof.** Suppose that $F \subset M$ is a fibre of a surface bundle structure with monodromy $h$. Isotope $F$ to be normal with respect to the triangulation $T$ and the least weight such. Suppose that $F$ decomposes as a Haken sum. Then, by Theorem 2.2 of [6], $F = \sum n_i F_i + \sum m_i T_i$ where the summands with nonzero coefficient are fundamental, incompressible normal surfaces. (Note that no fundamental surface with positive Euler characteristic appears as a summand; this follows directly from Lemma 3.4.)

Recall that $g(F) = \text{genus}(F) \geq 2$. Thus some of the $n_i$ are nonzero. Reindex to obtain $n_1 \geq n_i$ for all $i$. If $F_1$ is two-sided then rewrite the Haken sum as $F = nG + H$, with $n = n_1$, $G = F_1$, and $H$ equal to the sum of the remaining terms. If $F_1$ is one-sided then take $F = nG + H$ where $n$ is the integer part of $n_1/2$, $G = 2F_1$ is the double of $F_1$, and $H$ is the sum of the remaining terms.

Recall that Euler characteristic is additive under Haken sum. Now, if $n = 0$ then $n_i = 0$ or 1 for all $i$. It follows that $-\chi(F) \leq KP$ and thus $-\chi(F) \cdot c(M, T) < 1$. In this case the theorem is trivially satisfied.

Assume from now on that $n$ is positive. An easy estimate shows that $n \geq -\chi(F)/(3KP)$. Make $F = nG + H$ a reduced sum by isotoping $G$ and $H$ if necessary. The surface $nG$ is $n$ normally parallel copies of $G$. Label these, in order, $G_1, G_2, \ldots, G_n$. The surface $H$ is nonempty and intersects $G$ because $F$ is connected. So $H \cap nG$ decomposes into parallel (in $H$) families of curves, each family of size $n$. Let $\{\gamma_i\}_{i=1}^n$ be one such family. Each $\gamma_i \subset G_i$ yields an exchange annulus for the sum $F = nG + H$. An argument identical to the proof of Theorem 4.3 shows these annuli to be essential.
Pick $m \leq n$. The exchange annuli will lift to the $m^{th}$ cyclic cover of the surface bundle. Thus $h^m$ is not strongly irreducible.

5. Questions

Here are several questions, not all of which are necessarily difficult:

- Can a periodic surface automorphism be irreducible but not strongly irreducible?
- Are weakly acylindrical surfaces vertex surfaces? (See [7] for the definition of a vertex surface.)
- Give an algorithm to recognize strongly irreducible surface automorphisms or, more generally, compute translation distance.
- How are the strongly irreducible bundle structures on $M$ distributed among the fibred faces of the Thurston norm ball?
- Is there a meaningful stabilization theory for surface bundle structures on $M$?
- Suppose that $h$ is pseudo-Anosov. What does the translation distance of $h$ imply about the hyperbolic geometry of the mapping torus $M_h$?
- Can the function $n(g)$, as in Remark 2.5, be given more explicitly? Remark 2.7 only suggests a lower bound.

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