TOLEDO INVARIANTS OF HIGGS BUNDLES ON ELLIPTIC
SURFACES ASSOCIATED TO BASE ORBIFOLDS OF SEIFERT
FIBERED HOMOLOGY 3-SPHERES

MIKE KREBS

ABSTRACT. To each connected component in the space of semisimple representations from
the orbifold fundamental group of the base orbifold of a Seifert fibered homology 3-sphere
into the Lie group $U(2,1)$, we associate a real number called the “orbifold Toledo invariant.”
For each such orbifold, there exists an elliptic surface over it, called a Dolgachev surface.
Using the theory of Higgs bundles on these Dolgachev surfaces, we explicitly compute all
values taken on by the orbifold Toledo invariant.

INTRODUCTION

In this paper, we investigate the space $\mathcal{R}^+_{U(2,1)}(O) = \frac{\text{Hom}^+(\pi_{\text{orb}}(O),U(2,1))}{U(2,1)}$ of semisimple
representations from the orbifold fundamental group of a certain 2-orbifold $O$ into the Lie
group $U(2,1)$, modulo conjugation. To each connected component in $\mathcal{R}^+_{U(2,1)}(O)$, we asso-
ciate a number that we call the “orbifold Toledo invariant.” Our main result (Thm. 7.1)\nexplicitly computes all values that the orbifold Toledo invariant takes on. One thereby
obtains (Cor. 7.2a) a lower bound for the number of connected components in $\mathcal{R}^+_{U(2,1)}(O)$.\nThe orbifolds we consider are quotients of certain 3-manifolds $Y$—namely, Seifert fibered
homology 3-spheres—by the action of $S^1$. Our results also yield (Cor. 7.2b) a lower bound
for the number of connected components in $\mathcal{R}^*_{\text{PU}(2,1)}(Y) = \frac{\text{Hom}^*(\pi_1(Y),\text{PU}(2,1))}{\text{PU}(2,1)}$, the space of
irreducible representations from the fundamental group of $Y$ into $\text{PU}(2,1)$, modulo conjugation.

In [33], Toledo introduces an invariant $\tau$ for representations of the fundamental group
of a oriented 2-manifold $M$ into $\text{PU}(p,1)$. This invariant can be viewed as a map $\tau : \text{Hom}(\pi_1(M),\text{PU}(p,1)) \to \mathbb{R}$. As discussed in [11] the construction of the Toledo invariant
is quite general: one may replace $M$ by an arbitrary topological space and $\text{PU}(p,1)$ by any
topological group $G$. Moreover, representations which take on distinct Toledo invariants
necessarily lie in distinct components of the corresponding representation space. In the case where $M$ is a compact Riemann surface of genus $g > 1$, previously established results about the space $\mathcal{R}_G^+(M) = \frac{\text{Hom}^+(\pi_1(M),G)}{G}$ of semisimple representations of $\pi_1(M)$ into $G$, modulo conjugation, include:

- The Toledo invariant gives a bijection between the set of all $\tau \in \frac{2}{3}\mathbb{Z}$ with $|\tau| \leq 2g - 2$ and the set of all connected components in $\mathcal{R}_{\text{PU}(2,1)}^+(M)$ \cite{18}, \cite{10}.
- If $\tau$ is sufficiently large and $c$ is any integer, then the subset of $\mathcal{R}_{\text{PU}(p,p)}^+(M)$ corresponding to representations with Toledo invariant $\tau$ and Chern class $c$ is connected \cite{29}.
- The Toledo invariant gives a bijection between the set of even integers $\tau$ with $|\tau| \leq 2(g - 1)$ and the set of connected components in $\mathcal{R}_{\text{U}(p,1)}^+(M)$ \cite{11}.
- The subset $\mathcal{R}(\tau,c)$ of $\mathcal{R}_{\text{PU}(p,q)}^+(M)$ corresponding to representations with Toledo invariant $\tau$ and Chern class $c$ is non-empty if and only if

$$\tau = \frac{|qa - p(c - a)|}{p + q} \leq (g - 1) \cdot \min\{p, q\}$$

for some integer $a$. Moreover, if this inequality is satisfied and $p + q$ and $c$ are coprime, then $\mathcal{R}(\tau,c)$ is connected \cite{7}.

Other results concerning Toledo invariants can be found in \cite{8}, \cite{9}, \cite{10}, \cite{17}, \cite{18}, \cite{20}, \cite{21}, \cite{38}, and \cite{39}.

To our orbifold $O$ we associate a complex surface $X$, called a Dolgachev surface, whose fundamental group is isomorphic to $\pi_1^{\text{orb}}(O)$. The reason for doing so is that for complex algebraic manifolds $M$, we have a correspondence between representations of $\pi_1(M)$ and certain algebro-geometric objects on $M$ called Higgs bundles. (A Higgs bundle on $M$ consists of a holomorphic vector bundle plus some extra data; see \S 5 for the definition and basic properties.) The relationship between representations of $\pi_1(M)$ and holomorphic vector bundles on $M$ has been developed over the last forty years by Narasimhan and Seshadri \cite{30}, Atiyah and Bott \cite{11}, Hitchin \cite{22}, Donaldson \cite{13}, Corlette \cite{11}, Simpson \cite{33}, and others.
In §6, we obtain detailed information about those Higgs bundles on the Dolgachev surface $X$ that correspond to semisimple representations $\rho : \pi_1(X) \to U(2,1)$. In [10], Xia computes the Toledo invariant of such a representation in terms of the associated Higgs bundle. This computation, together with the results of §6, enables us to determine all Toledo invariants that arise from semisimple representations $\rho$.

In §4, we define “orbifold Toledo invariants” for representations of the orbifold fundamental group of the 2-orbifold $O$ into $PU(2,1)$ and show that these are in one-to-one correspondence with Toledo invariants on the Dolgachev surface $X$. In §7, we put the pieces together, obtaining numerical conditions which completely determine whether or not a real number $\tau$ represents an orbifold Toledo invariant:

**Main Theorem.** Let $O$ be the base orbifold of Seifert fibered homology 3-sphere $Y$ such that $\pi_1^\text{orb}(O)$ is infinite. Let $n$ equal the number of cone points that $O$ has, and let $m_1, \ldots, m_n$ denote the orders of these cone points. Let $\tau \in \mathbb{R}$. Then there exists a semisimple representation $\rho : \pi_1^\text{orb}(O) \to U(2,1)$ such that $\tau$ is the orbifold Toledo invariant of $\rho$ if and only if $\tau = \pm (y + \sum \frac{y_k}{m_k})$ for some integers $(y, y_1, \ldots, y_n)$ with $0 \leq y_k < m_k$ such that at least one of the numerical conditions (i)–(iv) holds:

(i) There exist integers $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ with $0 \leq a_k, b_k < m_k$ such that $b \leq -2$, and $a + \# \{k | a_k \neq 0 \} \geq 2$, and $2A < B$, and $A < 2B$, and $(\star)$ below holds.

(ii) There exist integers $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ with $0 \leq a_k, b_k < m_k$ such that $-B < A < \frac{1}{2}B$, and $d_2 \leq -2$, and $b \leq -2$, and $(\star)$ below holds, and $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$ for every $(n + 1)$-tuple of integers $(c, c_1, \ldots, c_n)$ such that $0 \leq c_k < m_k$ for all $k$ and $d_1 \geq 0$ and $C \geq \frac{2}{3}(A + B)$.

(iii) $y + \sum \frac{y_k}{m_k} > 0$, and $2y + \# \{y_k \geq \frac{m_k}{2} \} \leq -2$.

(iv) $y = y_k = 0$ for all $k$.

$(\star)$ $3y + \sum \lfloor \frac{3y_k}{m_k} \rfloor = a + b$, and $3y_k - \lfloor \frac{3y_k}{m_k} \rfloor m_k = a_k + b_k$ for $k = 1, \ldots, n$. 
Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$; $B = b + \sum \frac{b_k}{m_k}$; $C = c + \sum \frac{c_k}{m_k}$; $d_1 = b - c - \#\{b_k < c_k\}$; $d_2 = a - b - \#\{a_k < b_k\}$; and $d_3 = a - c - \#\{a_k < c_k\}$.

As a corollary, we obtain a lower bound for the number of connected components in $\mathcal{R}^+_U(2,1)(O)$. In §2, we show that irreducible PU(2,1) representations of $\pi_1(Y)$ are in one-to-one correspondence with irreducible PU(2,1) representations of $\pi_1^{orb}(O)$. The main theorem therefore also furnishes a lower bound for the number of connected components in $\mathcal{R}^+_{PU(2,1)}(Y)$.

The space of irreducible SU(2) representations of $\pi_1(Y)$ has been studied in detail by Fintushel and Stern [14], Bauer and Okonek [3], Kirk and Klassen [26], Furuta and Steer [16], Bauer [2], and Boden [6]. (The motivation of these authors was the study of the SU(2) Casson’s invariant and Floer homology for such spaces $Y$.) In many of these papers, the method is to associate to $Y$ an auxiliary object whose fundamental group is closely related to that of $Y$. In [16] and [6], the auxiliary object is a 2-orbifold; they study parabolic Higgs bundles on the orbifold’s underlying Riemann surface, i.e. $\mathbb{C}P^1$. In this paper, as in [4], the auxiliary object is a Dolgachev surface. (Along the same lines, in [5], an elliptic surface is used to study vector bundles on a 2-orbifold; it is not clear, however, that the associated elliptic surface is algebraic, as claimed—see, for example, [2, Example 13.2].)

One motivation for studying PU(2,1) representations of the fundamental groups of 3-manifolds comes from spherical CR geometry. A spherical CR structure on a 3-manifold $M$ is a system of coordinate charts into $S^3$ so that the transition functions are elements of PU(2,1). (Here we regard PU(2,1) as the isometry group of the complex ball in $\mathbb{C}^2$ and the conformal group of its boundary $S^3$. See [17].) In [24], Kamishima and Tsuboi classify those closed orientable 3-manifolds that admit $S^1$-invariant spherical CR structures; these include the Seifert-fibered homology 3-spheres considered here. The space $\text{Hom}(\pi_1(M),\text{PU}(2,1))/\text{PU}(2,1)$ provides a local model for the deformation space of spherical CR structures on $M$ [28].

Our lower bound for the number of components in $\mathcal{R}^+_{PU(2,1)}(Y)$ takes into account only those PU(2,1) representations which lift to U(2,1) representations. Moreover, for $\mathcal{R}^+_U(2,1)(O)$, we conjecture that the number of components is in general strictly greater than the number of orbifold Toledo invariants that occur. We plan to continue investigating these
representation spaces, with the goal of precisely determining the number of components in
them.

The author is grateful to I. Dolgachev, E. Falbel, H. Ren, R. Seyyedali, S. Zrebiec, G. Tinaglia, and most especially his thesis advisor, Richard Wentworth, for many helpful
discussions.

1. Toledo invariants

Given a manifold (or topological space) $M$ and a topological group $G$, one may wish to
study the representation variety $\mathcal{R} = \frac{\text{Hom}(\pi_1(M), G)}{G}$. The goal of this section is to define a
family of invariants, called Toledo invariants, that can be used to distinguish components
of $\mathcal{R}$. We then describe one such Toledo invariant more specifically in the case where
$G = U(2, 1)$.

1.1. The “abstract nonsense” of Toledo invariants. Let $B$ be a solid topological
space \[^{36}\] (Euclidean space $\mathbb{R}^n$ is solid, for example.) Let $G$ be a topological group acting
continuously on $B$ on the left. We now take $\omega$ to be a fixed $G$-invariant representative of
a cohomology class in $H^*(B, \mathbb{C})$. (If $B$ is a manifold, we may regard $\omega$ as a closed singular
cochain or as a closed differential form, depending which is more convenient.)

Let $M$ be a $C^\infty$ manifold. We define a map $\tau^{B,G,\omega}$ from $\text{Hom}(\pi_1(M), G)$ to $H^*(M, \mathbb{C})$
as follows. Let $\rho \in \text{Hom}(\pi_1(M), G)$. Let $\tilde{M}$ be the universal cover of $M$. Note that $\pi_1(M)$
acts on $\tilde{M} \times B$ by $\gamma \cdot (m, x) = (\gamma \cdot m, \rho(\gamma) \cdot x)$. Let $E_\rho$ be the flat $B$-bundle on $M$
obtained by taking $\tilde{M} \times B$ modulo the action of $\pi_1(M)$. Let $\pi_B : \tilde{M} \times B \to B$ be the projection
map onto the second factor, and let $\varphi$ be the natural map from $\tilde{M} \times B$ to $E_\rho$. Since $\pi_1(M)$
acts freely on $\tilde{M}$ and $\omega$ is $G$-invariant and closed, the pullback $\pi_B^* \omega$ descends to $E_\rho$, where
it represents a cohomology class $[\varphi_* \pi_B^* \omega] \in H^*(E_\rho, \mathbb{C})$. Since the fibre $B$ is solid, $E_\rho$ has
a section; moreover, any two sections are homotopic \[^{36}\] Theorem 12.2]. Consequently,
$[s^* \varphi_* \pi_B^* \omega]$ is a well-defined cohomology class in $H^*(M, \mathbb{C})$.

**Definition 1.1.** The Toledo invariant $\tau^{B,G,\omega}(\rho)$ is defined by

$$\tau^{B,G,\omega}(\rho) = [s^* \varphi_* \pi_B^* \omega]$$
for any section \( s \) of \( E_\rho \).

**Lemma 1.2.** Let \( M \) be a \( C^\infty \) manifold, let \( \rho \in \text{Hom}(\pi_1(M), G) \), let \( g \in G \), and define \( \rho' : \pi_1(M) \to \pi_1(M) \) by \( \rho'(\gamma) = g\rho(\gamma)g^{-1} \). Then \( \tau^{B,G,\omega}(\rho) = \tau^{B,G,\omega}(\rho') \). In other words, the Toledo invariant is invariant under conjugation.

**Proof.** We define a map \( \psi : \tilde{M} \times B \to \tilde{M} \times B \) by \( \psi(x,b) = (x,g \cdot b) \). Let \( E_\rho = \tilde{M} \times B / \pi_1(M) \) (where the action is induced by \( \rho \)), and let \( E_{\rho'} = \tilde{M} \times B / \pi_1(M) \) (where the action is induced by \( \rho' \)). Then \( \psi \) descends to a map from \( E_\rho \) to \( E_{\rho'} \); we denote this new map by \( \psi \) as well. If \( s \) is a section of \( E_\rho \), then \( s' = \psi \circ s \) is a section of \( E_{\rho'} \). In summary, we have that the following diagram commutes.

\[
\begin{array}{ccc}
\tilde{M} \times B & \xrightarrow{\pi_B} & B \\
\uparrow{\varphi} & & \uparrow{id} \\
M & \xrightarrow{s} & E_\rho \\
\downarrow{id} & & \downarrow{s'} \\
M & \xrightarrow{\pi_B} & B \\
\end{array}
\]

The lemma follows from chasing this diagram. \( \square \)

Let \( G \) act on \( \text{Hom}(\pi_1(M), G) \) by conjugation. Lemma 1.2 shows that \( \tau^{B,G,\omega}(\rho) \) defines a map from \( \text{Hom}(\pi_1(M), G) \) to \( H^*(M, \mathbb{C}) \). Topologize \( \text{Hom}(\pi_1(M), G) \) by giving \( \text{Hom}(\pi_1(M), G) \) by the point-open topology and giving \( \text{Hom}(\pi_1(M), G) \) the quotient topology. Note that if \( t_1, \ldots, t_n \) are generators for \( \pi_1(M) \), then \( \text{Hom}(\pi_1(M), G) \) is homeomorphic to the closed subspace \( \{ (x_1, \ldots, x_n) \in G^n \mid r_\alpha(x_1, \ldots, x_n) = 1 \} \) of \( G^n \), where the \( r_\alpha \)'s range over all relations between the \( t \)'s.
Lemma 1.3. Suppose that $B$ and $M$ are $C^\infty$ manifolds, that $M$ is compact, that $G$ is a Lie group, and that $\omega$ is a closed $G$-invariant $k$-form on $B$. Topologize $H^k(M; \mathbb{C})$ as a finite-dimensional vector space. Then $\tau^{B,G,\omega}$ defines a continuous function from $\frac{\text{Hom}(\pi_1(M), G)}{G}$ to $H^k(M; \mathbb{C})$.

Proof. It suffices to show that $\tau^{B,G,\omega}$ is continuous on $\text{Hom}(\pi_1(M), G)$. Let $C = \text{Hom}(\pi_1(M), G) \times \tilde{M} \times B$. An action of $\pi_1(M)$ on $C$ is given by

$$\gamma \cdot (\rho, m, x) = (\rho, \gamma \cdot m, \rho(\gamma) \cdot x).$$

Then $\frac{C}{\pi_1 M}$ is a fibre bundle over $\text{Hom}(\pi_1(M), G) \times M$ with fibre $B$. Since $B$ is solid and since $\text{Hom}(\pi_1(M), G)$ is a subspace of $G^n$, there exists a section $s : \text{Hom}(\pi_1(M), G) \times M \to \frac{C}{\pi_1 M}$. Lift $s$ to a map $\tilde{s} : \text{Hom}(\pi_1(M), G) \times \tilde{M} \to C$. By Tietze extension and by inclusion of $C$ into $G^n \times \tilde{M} \times B$, we have that $\tilde{s}$ extends to a map $\tilde{s}$ from $G^n \times \tilde{M}$ to $G^n \times \tilde{M} \times B$. Let $\pi_B : G^n \times \tilde{M} \times B \to B$ denote projection onto the third factor. Given $\rho \in G^n$, define $\iota_\rho : \tilde{M} \to G^n \times M$ by $\iota_\rho(m) = (\rho, m)$. Then $\iota_\rho^* \tilde{s}^* \pi_B^* \omega$ defines a closed $\pi_1(M)$-invariant singular $k$-cochain on $\tilde{M}$, which therefore defines a closed cochain $\tau_0(\rho)$ on $M$. Let $\tau(\rho)$ be the associated cohomology class in $H^k(M; \mathbb{C})$. Note that $\tilde{s} \circ \iota_\rho$ defines a section of $E_\rho$, the fibre bundle from the definition of the Toledo invariant. Therefore, we have that $\tau^{B,G,\omega}(\rho) = \tau(\rho)$. We now show that $\tau(\rho)$ varies continuously with $\rho$.

Let $r$ be the dimension over $\mathbb{C}$ of $H^k(M; \mathbb{C})$. Given open sets $U_1, \ldots, U_r$ of $\mathbb{C}$ and closed cochains $\sigma_1, \ldots, \sigma_r \in C^k(M, \mathbb{C})$ such that the associated cohomology classes $[\sigma_1], \ldots, [\sigma_r]$ are linearly independent, let $IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) = \{ \sum a_\ell \sigma_\ell \mid a_\ell \in U_\ell \}$. Then $\tau$ is continuous if and only if $\tau_0^{-1}(IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r))$ is open for all $\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r$.

Now fix $r$ closed cochains $\sigma_1, \ldots, \sigma_r \in C^k(M, \mathbb{C})$ such that the associated cohomology classes $[\sigma_1], \ldots, [\sigma_r]$ are linearly independent. Let $C_1, \ldots, C_r$ be singular $k$-simplices in $M$ such that the map from $\phi : IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) \to \mathbb{C}^r$ defined by $\phi(\sigma) = (\sigma, C_1 >$
, \ldots, <\sigma, C_r>) is bijective. Define $\phi_\ell : IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) \to C$ by $\phi_\ell(\sigma) = <\sigma, C_\ell>$.

It now suffices to show that $\tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$ is open for every open set $U_\ell$ of $C$.

By subdividing $C_\ell$ into small enough pieces, we can assume that the image of $C_\ell$ is a subset of an open set $V$ of $M$ such that $V$ is homeomorphic, via the natural covering map, to an open set $\tilde{V}$ of $\tilde{M}$. We may then regard $C_\ell$ as a map from the standard $k$-simplex $\Delta_k$ to $\tilde{M}$.

Let $\rho_0 \in \tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$. Endow $G^n$ with a Riemannian metric. We now show that for sufficiently small $\delta$, the ball of radius $\delta$ centered at $\rho_0$ lies entirely within $\tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$; this will conclude our proof. Let $\rho_1 \in G^n$. Let $c(t)$ be a geodesic in $G^n$ with $c(0) = \rho_0$ and $c(1) = \rho_1$. Let $h : \Delta_k \times [0, 1] \to B$ be a piecewise smooth function homotopic to $\pi_B \circ \hat{s} \circ \iota_{c(t)} \circ (C_\ell \times \text{id})$. Note that $\phi_\ell(\tau(\rho_j)) = \int_{\Delta_k \times \{j\}} h^* \omega$ for $j = 0, 1$. Stokes’ Theorem then guarantees, for any $\epsilon > 0$, the existence of a $\delta > 0$ such that $|\phi_\ell(\tau(\rho_1)) - \phi_\ell(\tau(\rho_0))| < \epsilon$ if the distance from $\rho_0$ to $\rho_1$ is less than $\delta$. \hfill $\square$

**Remark.** If the image of $\tau^{B,G,\omega}$ is discrete, then Lemma 1.3 shows that $\tau^{B,G,\omega}$ is constant on connected components of $\frac{\text{Hom}(\pi_1(M), G)}{G}$. This will be the case in our main theorem (Thm. 7.1); the number of distinct values of $\tau^{B,G,\omega}$ therefore provides, in this case, a lower bound for the number of connected components in $\frac{\text{Hom}(\pi_1(M), G)}{G}$. Lemma 1.3 is used (implicitly) in this manner in [40, 41, 29, and 7].

**Example.** A simple example shows that $\tau^{B,G,\omega}$ is not always constant on connected components of $\text{Hom}(\pi_1(M), G)$. Let $M$ be the unit circle $S^1$, let $G = B = \mathbb{R}$ (where $G$ acts on $B$ by translation), and let $\omega = dx$. Let $t$ be the standard generator of $\pi_1(M)$, and identify $\text{Hom}(\pi_1(M), G)$ with $\mathbb{R}$ by $\rho \mapsto \rho(t)$. Since $\text{Hom}(\pi_1(M), G)$ has a single connected component, it suffices to show that the Toledo invariant is not a constant function. Identifying $\tilde{M}$ with $\mathbb{R}$ in the usual way, a $\rho$-equivariant section of $\tilde{M} \times B$ is given by $x \mapsto (x, \rho(t)x)$. One can then compute that the Toledo invariant $\tau^{B,G,\omega}(\rho)$ is the cohomology class defined by $\rho(t)d\theta$. \hfill $\square$
1.2. The $U(2,1)$ Toledo invariant. We now turn our attention to the special case of this construction that will be the focus of the remainder of this paper. Define $g : \mathbb{C}^3 \to \mathbb{C}$ by $g(z_0, z_1, z_2) = |z_0|^2 - |z_1|^2 - |z_2|^2$. Let $U(2,1) = \{ A \in \text{GL}(3, \mathbb{C}) \mid g(Az) = g(z) \text{ for all } z \in \mathbb{C}^3 \}$.

Let $G = U(2,1)$. Define $B$ by:

$$ B = H^2_{\mathbb{C}} = \{ (1, z_1, z_2) \in \mathbb{C}^3 : 1 - |z_1|^2 - |z_2|^2 < 1 \}.$$  

($B$ is the ball model of 2-dimensional complex hyperbolic space [17].) Note that $B$ is homeomorphic to $\mathbb{R}^4$, hence solid. $G$ acts on $B$ as follows. Let $z \in B$ and $A \in G$. Define the action of $A$ on $z$ by $A \cdot z = \lambda \cdot (Az)$, where the $Az$ on the right hand side is given by ordinary matrix multiplication (regarding $z$ as a column vector), and $\lambda$ is the unique complex number such that $\lambda \cdot (Az) \in B$. (We know that $\lambda$ exists since $U(2,1)$ preserves the indefinite form $|z_0|^2 - |z_1|^2 - |z_2|^2$.) Let $\omega = \frac{i}{2\pi} \partial \bar{\partial} \log g$. Note that $\omega$ is invariant under multiplication by elements of $U(2,1)$ (since $g$ is) and is invariant under multiplication by scalars. By the definition of the action of $G$ on $B$, then, the restriction of $\omega$ to $B$ is $G$-invariant. The center $Z(U(2,1))$ of $U(2,1)$ equals $\{ \lambda I \mid \lambda \in U(1) \}$. Let $PU(2,1) = \frac{U(2,1)}{Z(U(2,1))}$. Then there is an action of $PU(2,1)$ on $B$, inherited from the $U(2,1)$ action. It follows that $\omega$ is $PU(2,1)$-invariant. From now on, all Toledo invariants will have $B$ and $\omega$ as in this paragraph and $G = U(2,1)$ or $G = PU(2,1)$.

2. $PU(2,1)$ representations of fundamental groups of Seifert fibered homology 3-spheres

The goal of this section is to note the relationship between $PU(2,1)$ representations of the fundamental group of a Seifert fibered homology 3-sphere and $PU(2,1)$ representations of the fundamental group of a certain elliptic surface called a Dolgachev surface.

Let $Y$ be a Seifert fibered homology 3-sphere. (Following Lemma 2.1 we shall impose some additional constraints on $Y$.) For the definition of Seifert fibered spaces and basic facts about them, we refer to [31]. A $(2n + 1)$-tuple $(-c_0; (m_1, c_1), \ldots, (m_n, c_n))$ of integers, with $m_k$ positive for all $k$, is associated to $Y$. (These integers are called the Seifert invariants of $Y$; we may think of $m_k$ as the degree of twisting of the $k$th singular fibre of $Y$.) To be a
homology 3-sphere, we must have that \( \gcd(m_j, m_k) = 1 \) whenever \( j \neq k \). The notations \( Y, n \), and \( (-c_0; (m_1, c_1), \ldots, (m_n, c_n)) \) will be fixed throughout the rest of this paper.

The fundamental group of \( Y \) has the following presentation \([31, \text{section 5.3}]\):

\[
\pi_1(Y) = \langle t_1, \ldots, t_n, h | t_k^{m_k} h^c_k = t_1 \cdots t_n h^{c_0} = [h, t_k] = 1 \rangle
\]

If \( G \) is any group, then let \( Z(G) \) denote its center. We have that \( Z(\pi_1(Y)) \) is generated by \( h \) \([31, \text{section 5.3}]\), so

\[
\frac{\pi_1(Y)}{Z(\pi_1(Y))} = \langle t_1, \ldots, t_n | t_k^{m_k} = t_1 \cdots t_n = 1 \rangle.
\]

We now construct a complex surface \( X \), called a Dolgachev surface. The following description of this construction is taken from \([31\text{]}\). A generic cubic pencil in \( \mathbb{CP}^2 \) has nine base points. Blowing up at these nine points, we obtain an algebraic surface \( X_0 \) along with an elliptic fibration \( \pi_0 : X_0 \to \mathbb{CP}^1 \). Apply logarithmic transformations \([19]\) along \( n \) disjoint nonsingular fibres of \( X_0 \) with multiplicities \( m_1, \ldots, m_n \). The result is an elliptic fibration \( \pi : X \to \mathbb{CP}^1 \), where \( X \) is the desired complex surface. Throughout this paper, \( X \) will denote a Dolgachev surface whose invariants are \( (m_1, \ldots, m_n) \).

**Lemma 2.1.** \( \pi_1(X) = \frac{\pi_1(Y)}{Z(\pi_1(Y))} \). If \( n \leq 2 \), then \( \pi_1(X) \) is trivial. If \( n = 3 \) and \( \{m_1, m_2, m_3\} = \{2, 3, 5\} \), then \( \pi_1(X) \) is the alternating group \( A_5 \).

**Proof.** \([12, \text{Chapter II, \S 3}]\) or \([31\text{]}\ Prop. 1.2 and subsequent discussion]. \( \square \)

Because of Lemma 2.1 we will impose the restrictions that \( n \geq 3 \) and that if \( n = 3 \), then \( \{m_1, m_2, m_3\} \neq \{2, 3, 5\} \).

**Definition 2.2.** The Lie group \( \text{PU}(2,1) \) acts on its Lie algebra \( \mathfrak{g} \) via the adjoint representation. Consequently, if \( H \) is a group and \( \rho : H \to \text{PU}(2,1) \) is a representation, then \( \rho \) induces an action of \( H \) on \( \mathfrak{g} \). We say that \( \rho \) is irreducible (resp. reducible) if this induced action is irreducible (resp. reducible). We denote the set of irreducible representations \( \rho : H \to \text{PU}(2,1) \) by \( \text{Hom}^*(H, \text{PU}(2,1)) \).

**Lemma 2.3.** Let \( H \) be a group, and let \( \rho \in \text{Hom}^*(H, \text{PU}(2,1)) \). Then no points and no complex geodesics in \( H^2_\mathbb{C} \) are invariant under the action of \( H \) on \( H^2_\mathbb{C} \) induced by \( \rho \).
Proof. First, suppose that there exists \( x \in \mathbb{H}^2_C \) such that \( \rho(h) \cdot x = x \) for all \( h \in H \). Let \( K = \{ \phi \in \text{PU}(2,1) | \phi(x) = x \} \). Then \( K \) is a Lie subgroup of \( \text{PU}(2,1) \); in fact, \( K \) is conjugate to \( \text{P}(\text{U}(2) \times \text{U}(1)) \). Let \( \mathfrak{g} \) be the Lie algebra of \( \text{PU}(2,1) \), and let \( \mathfrak{f} \) be the Lie subalgebra of \( \mathfrak{g} \) corresponding to \( K \). Since \( \rho(H) \subseteq K \), we have that \( \mathfrak{f} \) is invariant under the action of \( H \) on \( \mathfrak{g} \)—but this is a contradiction, since \( \rho \) is irreducible.

Similarly, suppose that \( P \) is a complex geodesic in \( \mathbb{H}^2_C \) such that \( \rho(h) \cdot x \in P \) for all \( h \in H \) and \( x \in P \). In this case, we take \( K \) to be the set of all elements in \( \text{PU}(2,1) \) that preserve \( P \). Again, \( K \) is a Lie subgroup of \( \text{PU}(2,1) \); this time, \( K \) is conjugate to \( \text{P}(\text{U}(1) \times \text{U}(1,1)) \). Again, we find that \( \mathfrak{f} \) is invariant under \( H \), contradicting \( \rho \)'s irreducibility. \( \Box \)

Lemma 2.4. There exists a bijection \( \varphi : \text{Hom}^*(\pi_1(Y), \text{PU}(2,1)) \to \text{Hom}^*(\pi_1(X), \text{PU}(2,1)) \).

Proof. Since \( \pi_1(X) = \frac{\pi_1(Y)}{Z(\pi_1(Y))} \), we have a surjection \( \sigma : \pi_1(Y) \to \pi_1(X) \), which in turn induces an injection \( \overline{\varphi} : \text{Hom}(\pi_1(Y), \text{PU}(2,1)) \to \text{Hom}(\pi_1(Y), \text{PU}(2,1)) \). Now, \( \rho \) and \( \overline{\varphi}(\rho) = \sigma \circ \rho \) have the same image, so \( \rho \) is irreducible if and only if \( \overline{\varphi}(\rho) \) is irreducible. Restricting \( \overline{\varphi} \) to the irreducible representations then gives us an injection \( \varphi \) from \( \text{Hom}^*(\pi_1(X), \text{PU}(2,1)) \) to \( \text{Hom}^*(\pi_1(Y), \text{PU}(2,1)) \). We must now show that \( \varphi \) surjects onto \( \text{Hom}^*(\pi_1(Y), \text{PU}(2,1)) \).

Let \( \tilde{\rho} : \pi_1(Y) \to \text{PU}(2,1) \) be an irreducible representation. We must find

\[
\rho : \frac{\pi_1(Y)}{Z(\pi_1(Y))} \to \text{PU}(2,1)
\]

such that \( \tilde{\rho} = \sigma \circ \rho \). Recalling that the center of \( \pi_1(Y) \) is generated by the single element \( h \), we see that it suffices to prove that \( \tilde{\rho} \) maps \( h \) to the identity element in \( \text{PU}(2,1) \).

Regard \( \text{PU}(2,1) \) as the group of isometries of \( \mathbb{H}^2_C \). Our first goal is to show that \( \tilde{\rho}(h) \) has three linearly independent fixed points \( x_1, x_2, x_3 \). Goldman [17, p. 203] shows that \( \tilde{\rho}(h) \) has a fixed point \( x_1 \in \mathbb{H}^2_C \cup \partial \mathbb{H}^2_C \). There must exist \( f \in \tilde{\rho}(\pi_1(Y)) \) such that \( x_2 = f(x_1) \neq x_1 \), else \( \tilde{\rho} \) would not be irreducible, by Lemma 2.3. Since \( h \) is central, \( \tilde{\rho}(h) \) commutes with \( f \). Thus, \( \tilde{\rho}(h)(x_2) = f(\tilde{\rho}(h)(x_1)) = x_2 \). That is, \( x_2 \) is another fixed point of \( \tilde{\rho}(h) \). Let \( P \) be the complex geodesic spanned by \( x_1 \) and \( x_2 \). By linearity, \( P \) is invariant under \( \tilde{\rho}(h) \). So, there must exist \( g \in \tilde{\rho}(\pi_1(Y)) \) and \( x \in \{ x_1, x_2 \} \) such that \( x_3 = g(x) \notin P \), else \( \tilde{\rho} \) would not
be irreducible, again by Lemma 2.3. As before, we find that $x_3$ is a fixed point of $\tilde{\rho}(h)$. By construction, $x_1, x_2,$ and $x_3$ are linearly independent.

Choose a lift of $\tilde{\rho}(h)$ to $U(2,1)$; denote the lift by $\tilde{h}$. The three linearly independent fixed points $x_1, x_2, x_3$ yield three linearly independent eigenvectors of $\tilde{h}$. We now prove by contradiction that $\tilde{h}$ has exactly one eigenvalue.

First, suppose that $\tilde{h}$ has 3 distinct eigenvalues. In this case, we have that $x_1, x_2,$ and $x_3$ are exactly the three one-dimensional eigenspaces of $\tilde{h}$. For each $k \in \{1, \ldots, n\}$, lift $\tilde{\rho}(t_k)$ to $U(2,1)$, and denote the lift by $\tilde{t}_k$. Now, as before, we find that $\tilde{\rho}(t_k)$ maps fixed points of $\tilde{\rho}(h)$ to fixed points of $\tilde{\rho}(h)$. In other words, $\tilde{t}_k$ permutes $x_1, x_2,$ and $x_3$. Let $\eta_k$ be this permutation, regarded as an element of the symmetric group $S_3$. The relation $t_k^m h c_k = 1$ in $\pi_1(Y)$ implies that $\eta_k^m = 1$. Consequently, the order $\text{ord}(\eta_k)$ of $\eta_k$ divides $m_k$. Now, $\text{ord}(\eta_k) \in \{1, 2, 3\}$, and the $m_k$’s are pairwise coprime. Therefore, there are at most 2 $k$’s such that $\text{ord}(\eta_k) \neq 1$. Moreover, $\text{ord}(\eta_k)$ is relatively prime to $\text{ord}(\eta_{k'})$ whenever $k \neq k'$. The relation $t_1 \ldots t_n h c_0 = 1$ in $\pi_1(Y)$ implies that $\eta_1 \ldots \eta_n = 1$. Therefore no $\eta_k$ has order 3; for if so, then $\eta_1 \ldots \eta_n$ is an odd permutation. We must then have that $\eta_k = 1$ for each $k$, for otherwise $\text{ord}(\eta_1 \ldots \eta_n) = 2$. However, $\eta_k = 1$ if and only if $\tilde{\rho}(t_k)$ fixes $x_1, x_2,$ and $x_3$. So every element in the image of $\tilde{\rho}$ fixes, say, $x_1$. By Lemma 2.3, this contradicts irreducibility of $\tilde{\rho}$.

Suppose now that $\tilde{h}$ has exactly 2 distinct eigenvalues. Without loss of generality, suppose that $x_1$ and $x_2$ belong to the same 2-dimensional eigenspace $P$ and that $x_3$ is the 1-dimensional eigenspace of $\tilde{h}$. Let $f$ be in the image of $\tilde{\rho}$, and let $\tilde{f}$ be a lift of $f$ to $U(2,1)$. We claim that $P$ is invariant under $f$. As before, $f$ commutes with $\tilde{h}$, so $\tilde{f}$ maps eigenvectors of $\tilde{h}$ to eigenvectors of $\tilde{h}$. In particular, if $P$ is not invariant under $\tilde{f}$, then $\tilde{f}$ maps either $x_1$ or $x_2$ to $x_3$. Let $e_1, e_2,$ and $e_3$ be nonzero vectors in $x_1, x_2,$ and $x_3$, respectively. Without loss of generality, assume that $\tilde{f}(e_1) \in x_3$. Since $\tilde{f}$ is nondegenerate, we must then have that $\tilde{f}(e_2) \in P$ and $\tilde{f}(e_3) \in P$. But then $\tilde{f}(e_2 + e_3)$ is an eigenvector of $\tilde{h}$ which is neither in $P$ nor in $x_3$—a contradiction. So $P$ is invariant under an arbitrary element in the image of $\tilde{\rho}$, once again violating irreducibility.
So, $\tilde{h}$ has three linearly independent eigenvectors and exactly one eigenvalue. Consequently, $\tilde{h}$ is of the form $\lambda I$, which implies that $\tilde{\rho}(h)$ is the identity in $\text{PU}(2,1)$. □

3. DOLGACHEV SURFACES

In this section, we collect facts about our Dolgachev surface $X$ that will be useful later. Recall the construction of $X$ from [2]. We may choose our pencil of curves such that each singular fibre is a rational curve with an ordinary double point. There are, then, 12 such singular fibres in this fibration [15, p. 192]. Denote these 12 fibres by $E_1, \ldots, E_{12}$. Denote the generic fibre of $X$ by $F$ and the multiple fibres of $X$ by $F_1, \ldots, F_n$, where $F_k$ has multiplicity $m_k$. For all $j,k$, we have that $E_j$ is linearly equivalent to $F$ is linearly equivalent to $m_k F_k$.

We say a divisor $D$ on $X$ is vertical if $mD$ is linearly equivalent to $\pi^*(D')$ for some divisor $D'$ on $\mathbb{CP}^1$. Note that a multiple fibre $F_k$ is vertical, but it is not the pullback of a divisor on $\mathbb{CP}^1$. (Note: this definition of a vertical divisor $D$ is not equivalent to the condition $D \cdot F = 0$, contrary to what one sees occasionally in the literature.) A divisor $D$ is vertical if and only if it is linearly equivalent to $aF + \sum a_k F_k$ for some integers $a, a_1, \ldots, a_n$. If we write a vertical divisor in this form, we will always assume that $0 \leq a_j < m_j$ for all $j = 1, \ldots, n$, unless otherwise noted.

Lemma 3.1 (I. Dolgachev). The surface $X$ has the following numerical invariants: the topological Euler characteristic $e_X = 12$; the irregularity $q = 0$; the geometric genus $p_g = 0$. Also, the canonical bundle $K_X = \mathcal{O}_X(-F + \sum_k (m_k - 1)F_k)$.

Lemma 3.2 (I. Dolgachev). $X$ is projective.

Lemma 3.3.

(i) $h^0(\mathcal{O}_X(\ell F + \sum \ell_k F_k)) = \max(\ell + 1, 0)$, and

(ii) $h^1(\mathcal{O}_X(\ell F + \sum \ell_k F_k)) = \max(\ell, -\ell - 1)$.

Proof. [1, Lemma 1.1]

Lemma 3.4. If $s$ is a global section of the locally free sheaf $\mathcal{O}_X(aF + \sum a_k F_k)$, then $s$ is constant on fibres.
Proof Let $w_0 = \pi(F) \in \mathbb{CP}^1$. Choose a local coordinate $w$ on $\mathbb{CP}^1$ centered at $w_0$. In order that $H^0(O_X(aF + \sum a_kF_k)) \neq 0$, we must have $a \geq 0$, by Lemma 3.3. Let $f_j = w^{-j}$, for $j = 0, \ldots, a$. The $f_j$'s are linearly independent, so $\{f_j \circ \pi\}$ is a set of $a + 1$ linearly independent elements in $H^0(O_X(aF + \sum a_kF_k)) \neq 0$. By Lemma 3.3, $s$ must be a linearly combination of $f_j \circ \pi$'s. Since each $f_j \circ \pi$ is constant on fibres, so is $s$. □

Lemma 3.5. Let $F_k$ be a multiple fibre. Then there exists a collection $\{U_\alpha\}$ of open sets of $X$ such that the $U_\alpha$'s cover $F_k$; each $U_\alpha$ is a coordinate neighborhood on $X$; each $U_\alpha$ is disjoint from the singular fibres and from the other multiple fibres; and, denoting the coordinates on $U_\alpha$ by $(w_\alpha, z_\alpha)$ and those on $U_\beta$ by $(w_\beta, z_\beta)$, we have that $w_\alpha = \zeta_{\alpha\beta}w_\beta$ and $z_\alpha = z_\beta + t_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ for some complex numbers $\zeta_{\alpha\beta}$ with $\zeta^{m_k}_{\alpha\beta} = 1$ and some functions $t_{\alpha\beta}$; the fibration map $\pi$ locally takes the form $(w_\alpha, z_\alpha) \mapsto w = w^{m_k}_\alpha$, where $w$ is the local coordinate on $\mathbb{CP}^1$; and $\{w_\alpha = 0\}$ is a set of local defining equations for the divisor $F_k$.

Proof. The result follows directly from the definition of the logarithmic transformation [19]. See [28] for more details. □

In the sequel, we will not distinguish between a vector bundle and its associated locally free sheaf of holomorphic sections, if no confusion is likely to result. Two exceptions will come in Lemmas 3.6 and in §6.2, where we will make use of the following system of trivializations for vertical line bundles.

Let $V$ be a small coordinate disc in $\mathbb{CP}^1$, with coordinate $w$ centered at 0, such that $\pi_0(E_j) \notin V$ for $j = 1, \ldots, 12$. Without loss of generality, assume that $V$ contains the points 0, $\infty$, and $\pi(F_k)$ for each multiple fibre $F_k$; that $\pi(F_k) \notin \{0, \infty\}$ for all $k$; and that $F = \pi^{-1}(0)$. Cover $\pi^{-1}(V - \infty) - \bigcup F_k$ by coordinate neighborhoods $V_\gamma$ so that there are coordinates $(w_\gamma, z_\gamma)$ on $V_\gamma$, and the map $\pi$ is given by $\pi(w_\gamma, z_\gamma) = w$ on $V_\gamma$, where $w$ is the coordinate on $\mathbb{CP}^1$ centered at 0. For each multiple fibre $F_k$, let $\{U_{\alpha,k}\}$ be a system of coordinate neighborhoods covering $F_k$, where $U_{\alpha,k}$ has coordinates $(w_{\alpha,k}, z_{\alpha,k})$. Cover $\pi^{-1}(V - 0) - \bigcup_{\alpha,k} \overline{U_{\alpha,k}}$ by coordinate neighborhoods $W_\xi$ so that there are coordinates $(w_\xi, z_\xi)$ on $W_\xi$, and the map $\pi$ is given by $\pi(w_\xi, z_\xi) = \frac{1}{w_\xi}$ on $W_\xi$. The relationships between
the \( w \)'s are as follows:

On \( U_{\alpha_1} \cap U_{\alpha_2} \), we have \( w_{\alpha_1,k} = \zeta_{\alpha_1\alpha_2,k} w_{\alpha_2,k} \) for some \( m_k \)th root of unity \( \zeta_{\alpha_1\alpha_2,k} \).

On \( U_\alpha \cap V_\gamma \), we have \( w_\gamma = w_{\alpha,k}^{m_k} + t_{\alpha,k} \) for some complex number \( t_{\alpha,k} \).

On \( V_\gamma \cap W_\xi \), we have \( w_\xi = \frac{1}{w_\gamma} \).

Let \( L = \mathcal{O}_X(aF + \sum a_k F_k) \) be a vertical line bundle. Local trivializations for \( L \) are given by the maps \( f \cdot w_{\alpha,k}^{-a_k} \mapsto f \) on \( U_{\alpha,k} \); \( f \cdot w_\gamma^{-a} \mapsto f \) on \( V_\gamma \); and \( f \mapsto f \) on \( W_\xi \). From now on, the notations \( U_{\alpha,k}, V_\gamma, W_\xi, w_{\alpha,k}, w_\gamma, w_\xi \) will be fixed. Moreover, sections of a vertical line bundle \( L \) will be written locally on \( U_{\alpha,k}, V_\gamma, \) and \( W_\xi \) with respect to these trivializations.

**Lemma 3.6.** Let \( L = \mathcal{O}_X(aF + \sum a_k F_k) \) be a vertical line bundle.

(i) Suppose \( a \geq 0 \). If \( 0 \leq j \leq a \), then there exists a section \( s_j \in H^0(L) \) such that \( s_j \) is given by \( s_\xi = w_\xi^{a-j} \) on \( W_\xi \); \( s_\gamma = w_\gamma^j \) on \( V_\gamma \); and \( s_{\alpha,k} = (w_{\alpha,k}^{m_k} + t_{\alpha,k})^j w_{\alpha,k}^{a_k} \) on \( U_{\alpha,k} \). Moreover, \( \{s_j | 0 \leq j \leq a \} \) is a basis for \( H^0(L) \).

(ii) Suppose \( a \leq -2 \). If \( a < j < 0 \), then there exists a \( \check{C}ech \) 1-cocycle \( \sigma_j \in C^1(L) \) such that \( \sigma_j \) is given by \( \sigma_\gamma = w_\gamma^j \) on \( V_\gamma \cap W_\xi \) with respect to the trivialization on \( V_\gamma \), and \( \sigma_{\xi_1\xi_2}, \sigma_{\gamma_1\gamma_2}, \sigma_{\alpha,k;\gamma} \), and \( \sigma_{\alpha_1,k;\alpha_2,k} \) vanish on \( W_{\xi_1} \cap W_{\xi_2}, V_{\gamma_1} \cap V_{\gamma_2}, U_{\alpha,k} \cap V_\gamma \), and \( U_{\alpha_1,k} \cap U_{\alpha_2,k} \), respectively. Moreover, identifying \( \sigma_j \) with its image in \( H^1(L) \), we have that \( \{\sigma_j | a < j < 0 \} \) is a basis for \( H^1(L) \).

**Proof.** Let \( f_j \circ \pi \) be as in the proof of Lemma 3.4. Let \( s_j = f_j \circ \pi \). From Lemma 3.4 we know that \( \{s_j\} \) is a basis for \( H^0(L) \). In local coordinates, \( s_j \) has the form required in (i). The \( \sigma_j \)'s in (ii) are obtained by pulling back a basis for \( H^1(\mathcal{O}_{\mathbb{P}^1}(a)) \) via \( \pi \). \( \square \)

**Definition 3.7.** Let \( H_0 \) be a fixed ample divisor on \( X \).

Let \( k_0 = 1 + 3 \left( \max \left\{ 1, -2 + \sum \frac{m_k-1}{m_k} \right\} \right) (H_0 \cdot F) \).

Let \( H = H_0 + k_0 F \).

Note that \( H \) is ample. Throughout this document, the degree of a coherent sheaf—and all related concepts (e.g., stability)—will be with respect to \( H \).
Lemma 3.8. There exists a short exact sequence

\[ 0 \to O_X(-2F + \sum_k (m_k - 1)F_k) \to \Omega^1_X \to I_Z \otimes O_X(F) \to 0, \]

where \( \Omega^1_X \) denotes the sheaf of holomorphic 1-forms on \( X \), where \( Z \) is the reduced subscheme associated to the set of singular points of singular fibres of \( X \), and where \( I_Z \) is the ideal sheaf of \( Z \).

Proof. Pullback of holomorphic 1-forms via \( \pi \) gives rise \( \text{[2, p. 98]} \) to an injection of sheaves

\[ 0 \to \pi^*\Omega^1_{\mathbb{CP}^1} \to \Omega^1_X. \]

Let \( \Omega^1_{X/\mathbb{CP}^1} \) denote the sheaf of relative differentials (i.e., the cokernel of this map). Since \( \pi^*\Omega^1_{\mathbb{CP}^1} = O_X(-2F) \), we compute that \( \det \left( \Omega^1_{X/\mathbb{CP}^1} \right) = O_X(F + \sum (m_k - 1)F_k) \).

Let \( T = \text{Tor} \left( \Omega^1_{X/\mathbb{CP}^1} \right) \), where \( \text{Tor}(S) \) denotes the torsion part of a sheaf \( S \). We claim that \( T \) is isomorphic to \( \bigoplus_{k=1}^n O_{(m_k-1)F_k}((m_k - 1)F_k) \). To prove this claim, we first observe that the support of \( T \) is contained in the union of the multiple fibres of \( X \) \( \text{[2, p. 98]} \). Let \( F_k \) be a multiple fibre, and let \( \{U_\alpha\} \) be a collection of coordinate neighborhoods as in Lemma \( \text{[2, p. 98]} \). It suffices to show that \( T|_{\bigcup U_\alpha} \) is isomorphic to \( O_{(m_k-1)F_k}((m_k - 1)F_k) \).

Let \( V \) be an open subset of \( \bigcup U_\alpha \). A section \( s \) of \( \Omega^1_{X/\mathbb{CP}^1}(V) \) is given by a collection \( \{(V_\alpha, s_\alpha)\} \) where \( \bigcup V_\alpha = V \), \( s_\alpha \in \Omega^1_X(V_\alpha) \), and \( s_\beta - s_\alpha \in \pi^*\Omega^1_{\mathbb{CP}^1}(V_\alpha \cap V_\beta) \). Without loss of generality, we assume that \( V_\alpha \subset U_\alpha \) for each \( \alpha \). For coordinates on \( V_\alpha \), we take the coordinates \( (w_\alpha, z_\alpha) \) from \( U_\alpha \), as in Lemma \( \text{[2, p. 98]} \). Now, \( \Omega^1_X(V_\alpha) \) is free; its generators are \( dw_\alpha \) and \( dz_\alpha \). Also, \( \pi^*\Omega^1_{\mathbb{CP}^1}(V_\alpha) \) is free, with generator \( \pi^*(du) = d(w^{m_k}_\alpha) = (m_k - 1)w^{m_k-1}_\alpha dw_\alpha \), where \( u \) is the local coordinate on \( \mathbb{CP}^1 \). We see then that locally, \( \Omega^1_{X/\mathbb{CP}^1} \) has two generators, \( dw_\alpha \) and \( dz_\alpha \), subject to the relation \( w^{m_k-1}_\alpha dw_\alpha = 0 \). Therefore, \( T \) is given locally by the one generator \( dw_\alpha \) subject to the relation \( w^{m_k-1}_\alpha dw_\alpha = 0 \).

Similarly, we find that \( O_{(m_k-1)F_k}((m_k - 1)F_k) \) is given locally by one generator, \( w^{1-m_k}_\alpha \), subject to the rather odd-looking relation \( w^{m_k-1}_\alpha \cdot w^{1-m_k}_\alpha = 0 \). Consequently, the map from \( T|_{\bigcup U_\alpha} \) to \( O_{(m_k-1)F_k}((m_k - 1)F_k) \) that sends \( dw_\alpha \) to \( w^{1-m_k}_\alpha \) is a well-defined isomorphism of sheaves.
We can then compute that \( \det(Q) = \det(T)^{\ast} \otimes \det(\Omega_{X/\mathbb{CP}^1}^{1}) = \mathcal{O}_X(F). \) We have a natural map \( \Omega_X^{1} \rightarrow Q, \) which is surjective. Let \( N \) be the kernel of this map. We then have a short exact sequence

\[
0 \rightarrow N \rightarrow \Omega_X^{1} \rightarrow Q \rightarrow 0.
\]

We then find that \( N = \mathcal{O}_X(-2F + \sum(m_k - 1)F_k). \) Since \( Q \) is torsion-free, we have that \( Q = I_Z \otimes \det(Q) = I_Z \otimes \mathcal{O}_X(F) \) for some codimension 2 subscheme \( Z \) \cite[p. 33]{15}. Now, \( \Omega_{X/\mathbb{CP}^1}^{1} \) fails to be locally free precisely where \( \pi \) is singular. Since \( T \) is supported on the union of the multiple fibres, \( Q = \Omega_{X/\mathbb{CP}^1}^{1} \) will fail to be locally free at every singular point of \( \pi \) outside of the multiple fibres. In particular, \( Z \) contains the set of singular points of the 12 singular fibres. From \( \Omega_{X/\mathbb{CP}^1}^{1} \) and the equation \cite[p. 29]{15}

\[
c_2(\Omega_X^{1}) = c_1(N) \cdot c_1(\mathcal{O}_X(F)) + \ell(Z)
\]

(where \( \ell(Z) \) is the length of \( Z \)), we find that \( \ell(Z) = c_2(\Omega_X^{1}) = 12. \) We conclude that \( Z \) is the subscheme of \( X \) associated to the set of singular points of the singular fibres, each point taken with multiplicity one. The exact sequence \( \llbracket 11 \rrbracket \) then has the desired form. \( \square \)

From now on, let \( N, Q, \) and \( Z \) be as in Lemma \( \llbracket 3.8 \rrbracket \)

**Lemma 3.9.** Let \( A = aF + \sum a_kF_k \) be a vertical divisor. If \( H^0(\mathcal{O}_X(-A) \otimes Q) \neq 0, \) then \( H^0(\mathcal{O}_X(-A) \otimes N) \neq 0 \) and \( \deg(\mathcal{O}_X(A)) < 0. \)

*Proof.* A nonzero global section \( s \) of \( \mathcal{O}_X(-A) \otimes Q \) is a nonzero global section of \( \mathcal{O}_X(-A + F) \) that vanishes on the total space of \( Z. \) Since \( -A + F \) is vertical, \( s \) is constant on fibres, by Lemma \( \llbracket 3.3 \rrbracket \) Thus \( s \) vanishes identically on each singular fibre of \( X, \) and hence can be regarded as a nonzero global section of \( \mathcal{O}_X(-A + F - \sum_{j=1}^{12}(E_j)). \) Now, \( -A + F - \sum_{j=1}^{12}(E_j) \) is linearly equivalent to \( (-11 - a - \#\{k|a_k \neq 0\})F + \sum_{a_k \neq 0}(m_k - a_k)F_k, \) so by Lemma \( \llbracket 3.3 \rrbracket \)

\[
a \leq -11 - \#\{k|a_k \neq 0\} \leq -2.
\]

Again by Lemma \( \llbracket 3.3 \rrbracket \)

\[
h^0(\mathcal{O}_X(-A) \otimes N) = (-2 - a) + 1 > 0,
\]

as desired. Moreover, \( \deg(\mathcal{O}_X(A)) = \left(a + \sum \frac{a_k}{m_k}\right) \deg(F) \leq \left(a + \#\{k|a_k \neq 0\}\right) \deg(F) \leq -11 \deg(F) < 0. \) \( \square \)

**Lemma 3.10.** Let \( B = bF + \sum b_kF_k. \) Then \( H^0(\mathcal{O}_X(-B) \otimes \Omega_X^{1}) \neq 0 \) if and only if \( b \leq -2. \)
Proof. First assume that \( b \leq -2 \). Tensoring the exact sequence (11) from Lemma \ref{lemma} with \( \mathcal{O}_X(-B) \), we see that \( H^0(\mathcal{O}_X(-B) \otimes N) \neq 0 \). The nonvanishing of \( H^0(\mathcal{O}_X(-B) \otimes N) \) follows from the effectiveness of \(-B + (-2F + \sum_k (m_k - 1)F_k)\). (Recall the convention that \( b_k < m_k \) for all \( k \).)

We now assume that \( H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0 \) and show that \( b \leq -2 \). We must have \( H^0(\mathcal{O}_X(-B) \otimes Q) \neq 0 \) or \( H^0(\mathcal{O}_X(-B) \otimes N) \neq 0 \). Either way, \( H^0(\mathcal{O}_X(-B) \otimes N) \neq 0 \), by Lemma \ref{lemma}. But then \((-2 - b)F + \sum(m_k - 1 - b_k)F_k\) is linearly equivalent to an effective divisor. Therefore \( b \leq -2 \). \( \square \)

Remark: In fact, we can compute that \( h^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) = \max\{0, -2-b\} \). To do so, let \( L = \mathcal{O}_X(B) \), and consider the exact sequence \( 0 \to \pi_*(L^* \otimes N) \to \pi_*(L^* \otimes \Omega^1_X) \to \pi_*(L^* \otimes Q) \).

Then show that \( \pi_*(L^* \otimes N) \) is a line bundle on \( \mathbb{CP}^1 \), that \( \pi_*(L^* \otimes \Omega^1_X) \) is a coherent sheaf of rank 1 on \( \mathbb{CP}^1 \), and that \( \pi_*(L^* \otimes Q) \) is torsion-free. It follows that

\[
\max\{0, -2-b\} = h^0(L^* \otimes N) = h^0(\pi_*(L^* \otimes N)) = h^0(\pi_*(L^* \otimes \Omega^1_X)) = h^0(L^* \otimes \Omega^1_X).
\]

4. Toledo invariants on 2-orbifolds and Dolgachev surfaces

In this section, we associate to our Seifert fibered space \( Y \) a 2-orbifold \( O \). The goal of this section is to show how Toledo invariants on the Dolgachev surface \( X \) correspond to “orbifold” Toledo invariants which arise from representations of the orbifold fundamental group of \( O \).

Let \( O \) be the hyperbolic 2-orbifold such that the underlying space \(|O|\) of \( O \) is the sphere \( S^2 \) and \( O \) has \( n \) elliptic points \( p_1, \ldots, p_n \) (also known as cone points) of orders \( m_1, \ldots, m_n \), respectively. (We refer to \[6, 16, 25, 32, \text{ and } 37\] for details of this construction and for basic facts about orbifolds.) The orbifold fundamental group of \( O \) has the following presentation:

\[
\pi^\text{orb}_1(O) = \langle u_1, \ldots, u_n \mid u^m_k = u_1 \ldots u_n = 1 \rangle
\]

We may think of \( u_j \) as a small loop that travels once around the cone point \( p_j \).

In our elliptic fibration \( \pi : X \to \mathbb{CP}^1 \), we identify \( \mathbb{CP}^1 \) with \(|O|\), and we assume that \( p_j = \pi(F_j) \) for each multiple fibre \( F_j \). Let \( \tilde{X} \) be the universal cover of our Dolgachev surface \( X \). The restrictions we imposed on the \( m \)'s following Lemma \ref{lemma} imply that the
orbifold universal cover \( \tilde{O} \) of \( O \) is the upper half-plane \( \mathbb{H}^2 \). Fix a base point \( x_0 \) in \( X \) and a base point \( y_0 \) in \( O \) such that \( y_0 = \pi(x_0) \) and \( x_0 \notin \{E_1, \ldots, E_{12}, F_1, \ldots, F_n\} \). We may regard the elements of \( \tilde{X} \) (resp. \( \tilde{O} \)) as equivalence classes of paths in \( X \) (resp. \( O \)) beginning at \( x_0 \) (resp. \( y_0 \)). (Caution: One must be careful as to what is meant by a path in \( O \). See \[16] \( \S 2 \).) Pushing forward paths in \( X \) to paths in \( O \), we obtain a map \( \tilde{\pi} : \tilde{X} \to \tilde{O} \) that covers \( \pi \). If \( \gamma \) is an element of \( \pi_1(X) \), then denote the action of \( \gamma \) on \( \tilde{X} \) by \( L_\gamma \). (Similarly for \( \tilde{O} \).) Recall that \( t_1, \ldots, t_n \) are the generators of \( \pi_1(X) \). Then \( \tilde{\pi} \circ L_{t_j} = L_{u_j} \circ \tilde{\pi} \). It follows that \( \pi_s(t_j) = u_j \), and so \( \pi_s \) is an isomorphism from \( \pi_1(X) \) to \( \pi_1^{\text{orb}}(O) \).

**Lemma 4.1.** Let \( \rho \in \text{Hom}(\pi_1(X), U(2,1)) \), and let \( E_\rho = \frac{\tilde{X} \times \mathbb{H}^2}{\pi_1(X)} \). Let \( q : \tilde{X} \times \mathbb{H}^2 \to \mathbb{H}^2 \) be projection onto the second factor. Then there exists a section \( s_0 \) of the fibre bundle \( E_\rho \) (as in Definition \[11] \( \S 2 \)) and a lift \( \tilde{s}_0 : \tilde{X} \to \tilde{X} \times \mathbb{H}^2 \) of \( s_0 \) such that for each point \( x \in \tilde{O} \), we have that \( q \circ \tilde{s}_0 \) is constant on \( \tilde{\pi}^{-1}(x) \).

**Definition 4.2.** Let \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(O), PU(2,1)) \). We say that a map \( s : \tilde{O} \to \tilde{O} \times \mathbb{H}^2 \) is \( \rho \)-equivariant if \( s(\gamma \cdot x) = \rho(\gamma) \cdot s(x) \) for all \( \gamma \in \pi_1^{\text{orb}}(O) \) and \( x \in \tilde{O} \). If \( s_1 \) and \( s_2 \) are two \( \rho \)-equivariant maps, then we say that \( s_1 \) and \( s_2 \) are \( \rho \)-equivariantly homotopic if there exists a homotopy \( F : [0,1] \times \tilde{O} \to [0,1] \times \tilde{O} \times \mathbb{H}^2 \) from \( s_1 \) to \( s_2 \) such that \( F(t, \cdot, \cdot) \) is \( \rho \)-equivariant for all \( t \in [0,1] \).

**Lemma 4.3.** Let \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(O), PU(2,1)) \). Then there exists a \( \rho \)-equivariant map \( s : \tilde{O} \to \tilde{O} \times \mathbb{H}^2 \). Moreover, if \( s_1 \) and \( s_2 \) are any two such \( \rho \)-equivariant maps, then \( s_1 \) and \( s_2 \) are \( \rho \)-equivariantly homotopic.

**Proof.** Existence: push forward \( \tilde{s}_0 \) from Lemma \[11] \( \S 2 \). Invariance: push forward a homotopy.

**Definition 4.4.** Let \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(O), PU(2,1)) \). Let \( \Sigma \) be a fundamental domain for the action of \( \pi_1^{\text{orb}}(O) \) on \( \tilde{O} \). Take \( q \) and \( \omega \) as in Lemma \[11] \( \S 2 \). Then we define the orbifold Toledo invariant \( \tau_{\text{orb}}(\rho) \) by

\[
\tau_{\text{orb}}(\rho) = \int_{\Sigma} s^* q^* \omega
\]

where \( s : \tilde{O} \to \tilde{O} \times \mathbb{H}^2 \) is any \( \rho \)-equivariant map.
Lemma 4.3 implies that $\tau_{orb}(\rho)$ is defined and that it is independent of the choice of $s$. The $\rho$-equivariance of $s$ implies that $\tau_{orb}(\rho)$ is independent of the choice of $\Sigma$. We now fix $\tilde{s}_0$ as in Lemma 4.1 and let $s$ be its $\rho$-equivariant push-forward, as in Lemma 4.3.

Let $H^2_{orb}(O, \mathbb{Z})$ be the orbifold second cohomology group of $O$ with integer coefficients [16]. (Note that [16] uses the notation “V” in place of “orb,” since they use the older terminology “V-manifold” in place of “orbifold.”) Let $H^1_{vert}(X, \mathcal{O}'_X)$ be the subgroup of $H^1(X, \mathcal{O}'_X)$ consisting of vertical line bundles on $X$. Let $H^2_{vert}(X, \mathbb{Z}) = c_1(H^1_{vert}(X, \mathcal{O}'_X))$ be the group of first Chern classes of vertical line bundles on $X$.

**Lemma 4.5.** The map $\pi$ induces an isomorphism $\pi^*: H^2_{orb}(O, \mathbb{Z}) \rightarrow H^2_{vert}(X, \mathbb{Z})$.

Let $\text{Pic}^t_{orb}(O)$ be the set of topological isomorphism classes of orbifold line bundles on $O$. We have that $\text{Pic}^t_{orb}(O)$ is a group, where the group law is given by the tensor product.

**Lemma 4.6.** $\text{Pic}^t_{orb}(O) \cong H^2_{orb}(O, \mathbb{Z})$

**Proof.** [16] Theorem 2.2(ii) □

**Lemma 4.7.** Let $\rho \in \text{Hom}(\pi^u_{orb}(O), \text{PU}(2, 1))$. If $\tau(\rho \circ \pi_x) = c_1(\mathcal{O}_X(aF + \sum a_kF_k))$, then $\tau_{orb}(\rho) = a + \sum \frac{a_k}{m_k}$.

**Proof.** Let $p \in |O| - \{p_1, \ldots, p_n\}$. Let $L_p$ be the holomorphic point bundle determined by $p$. Then $L_p$ is an orbifold line bundle on $O$ with $c_1(L_p) = 1$ [16]. Let $\sigma_k : \frac{\pi}{m_k \mathbb{Z}} \rightarrow U(1)$ be the standard representation. Let $L_{pk}$ be the orbifold line bundle on $O$ with first Chern class $c_1(L_{pk}) = \frac{1}{m_k}$ and trivial isotropy except at $p_k$, where it is $\sigma_k$. Then $\pi^*L_p = \mathcal{O}_X(F)$ and $\pi^*L_{pk} = \mathcal{O}_X(F_k)$. Let $L = L_p^\otimes a \otimes (\bigotimes_k L_{pk}^\otimes a_k)$. The following diagrams commute:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{s}_0} & \tilde{X} \times H^2_C \xrightarrow{q_X} H^2_C \\
\tilde{\pi} \downarrow & & \downarrow \\
\tilde{\mathcal{O}} & \xrightarrow{s} & \tilde{\mathcal{O}} \times H^2_C \xrightarrow{q_{\mathcal{O}}} H^2_C \\
\end{array}
$$

and

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{\mathcal{O}} \\
\varphi_X \downarrow & & \downarrow \varphi_{\mathcal{O}} \\
X & \xrightarrow{\pi} & O
\end{array}
$$
and
\[ \tilde{X} \to \tilde{X} \times H^2_C \]
\[ \varphi X \downarrow \downarrow \varphi \]
\[ X \to E_{\rho \circ \pi} \]

From these diagrams, we find that \( \tilde{\pi}^* c_1(\varphi^* L) = c_1(\tilde{\pi}^* \varphi^* O) = c_1(\varphi^* \tilde{\pi}^* L) = \varphi_X c_1(\mathcal{O}_X(aF + \sum a_k F_k) = \varphi_X^* s_0^*(\varphi_X^* q_0^* \omega = s_0^* q_0^* \omega = \tilde{s}_0^* q_0^* \omega). \]

Let us identify \( H^2_{\text{orb}}(O, \mathbb{Z}) \) with the set of all \( \pi_1^\text{orb}(O) \)-invariant elements of \( H^2(\tilde{O}, \mathbb{Z}) \).

Lemma 4.5 then implies that \( c_1(\varphi^* O L) = s^* q_0^* \omega \). Therefore

\[ \tau_{\text{orb}}(\rho) = \int \sum s^* q_0^* \omega = \int \sum c_1(\varphi^* O L) = c_1(L) = a + \sum \frac{a_k}{m_k}. \]

5. U(2,1) Higgs bundles

Hitchin, Simpson, et al. \([22, 33, 35]\) have shown that representations of the fundamental group of a compact Kähler manifold are closely related to holomorphic objects called Higgs bundles. The goal of this section is to describe the Higgs bundles that arise from U(2,1) representations of the fundamental group of a Dolgachev surface, then describe the Toledo invariant of such a representation in terms of the Chern classes of the associated Higgs bundle.

**Definition 5.1.** Let \( M \) be a complex algebraic manifold, and let \( H \) be a fixed ample line bundle on \( M \). A Higgs bundle on \( M \) is a pair \((V, \theta)\), where \( V \) is a holomorphic vector bundle on \( M \); \( \theta \in H^0(\text{End}(V) \otimes \Omega^1_M) \); and \( \theta \wedge \theta = 0 \). \( \theta \) is called the Higgs field. A subsheaf \( S \) of \( V \) is said to be \( \theta \)-invariant if \( \theta(S) \subset S \otimes \Omega^1_M \). The slope \( \mu(S) \) of a coherent sheaf \( S \) on \( M \) with \( \text{rank}(S) > 0 \) is defined by \( \mu(S) = \frac{\text{deg}(S)}{\text{rank}(S)} \), where \( \text{deg}(S) \) is the degree of \( S \) with respect to \( H \). A Higgs bundle \((V, \theta)\) is stable if \( \mu(S) < \mu(V) \) for all coherent \( \theta \)-invariant subsheaves \( S \) of \( V \) with \( \text{rank}(S) > 0 \). A Higgs bundle \((V, \theta)\) is polystable if it is a direct sum of stable Higgs bundles, each with the same slope. (One forms the direct sum in the obvious way.) A Higgs bundle \((V, \theta)\) is reducible if it is a direct sum of Higgs bundles and is irreducible otherwise. We say that a Higgs bundle \((V, \theta)\) is a U(2,1)-Higgs bundle if \( V = V_P \oplus V_Q \) (where \( V_P \) and \( V_Q \) are vector bundles of rank 2 and 1, respectively), and \( \theta \) maps \( V_P \) to \( V_Q \otimes \Omega^1_M \) and \( V_Q \) to \( V_P \otimes \Omega^1_M \).
If $H$ is any group, then let $\text{Hom}^+(H,\text{U}(2,1))$ denote the space of semisimple representations from $H$ into $\text{U}(2,1)$.

**Lemma 5.2.** There exists a surjective function $\phi : \mathcal{H} \to \text{Hom}^+(\pi_1(X),\text{U}(2,1))$, where $\mathcal{H}$ is the set of all polystable $\text{U}(2,1)$ Higgs bundles $(V,\theta)$ on $X$ whose summands have vanishing Chern classes.

*Proof.* Let $\mathcal{H}'$ be the set of all polystable rank 3 Higgs bundles $(V,\theta)$ on $X$ whose summands have vanishing Chern classes. By Lemma 3.2, $X$ is algebraic, hence compact Kähler. In [33], Simpson shows that there is a surjective function $\phi : \mathcal{H}' \to \text{Hom}^+(\pi_1(X),\text{GL}(3,\mathbb{C}))$. In [40, Proposition 3.1], Xia shows that $\mathcal{H} = \phi^{-1}(\text{Hom}^+(\pi_1(X),\text{U}(2,1)))$. (Xia’s proof is for Riemann surfaces, but it goes through for any compact Kähler manifold.) □

**Lemma 5.3.** Let $\mathcal{H}$ and $\phi$ be as in Lemma 5.2 and let $(V,\theta) \in \mathcal{H}$. Write $V = V_P \oplus V_Q$ as in Def. 5.1. Then $\tau(\phi(V,\theta)) = c_1(V_P)$.

*Proof.* [40]

**Remark:** Lemmas 5.2 and 5.3 serve as a bridge from the world of semisimple representations and Toledo invariants to the world of polystable Higgs bundles and Chern classes. Consequently, while the definition of the Toledo invariant is topological in nature, these two lemmas enable us to use algebraic geometry in order to compute which Toledo invariants actually occur.

**Definition 5.4.** If $(V,\theta) = (V_P \oplus V_Q,\theta)$ is a $\text{U}(2,1)$ Higgs bundle as in Def. 5.1, then we define the Higgs bundle Toledo invariant $\tau(V,\theta)$ by $\tau(V,\theta) = \frac{1}{3}(c_1(V_P) - 2c_1(V_Q))$.

Note that if $(V,\theta) \in \mathcal{H}$, then $V$ is flat, in which case Def. 5.4 is consistent with Lemma 5.3.

**Remark:** Consider a semisimple representation $\rho : \pi_1(X) \to \text{PU}(2,1)$. The corresponding principal $\text{PU}(2,1)$ bundle on $X$ lifts to a principal $\text{U}(2,1)$ bundle with an associated vector bundle $V = V_P \oplus V_Q$. One can show that $\tau(\rho) = \frac{1}{3}(c_1(V_P) - 2c_1(V_Q))$; the proof is similar to that of Lemma 5.3. This is the motivation for Def. 5.4.
Lemma 5.5. Let \((V, \theta) \in \mathcal{H}\), and let \(L\) be a line bundle. Then:

(i) \((V \otimes L, \theta \otimes 1)\) is a polystable \(U(2,1)\) Higgs bundle with \(\tau_{(V \otimes L, \theta \otimes 1)} = \tau_{(V, \theta)}\).

(ii) \((V^*, \theta) \in \mathcal{H}\), and \(\tau_{(V^*, \theta)} = -\tau_{(V, \theta)}\).

Proof. These statements follow directly from the definitions. See [40] for more details.

6. Systems of Hodge bundles on Dolgachev surfaces

The results of §5 imply that to compute Toledo invariants of semisimple \(U(2,1)\) representations of the fundamental group of a Dolgachev surface, it suffices to compute Chern classes of the summands of certain polystable \(U(2,1)\) Higgs bundles. The goal of this section is to show that we may restrict our attention to a special class of these Higgs bundles, namely systems of Hodge bundles. The method is due to Simpson. (See [33] and [34]).

Following Xia [40], we then divide these systems of Hodge bundles into two types, binary and ternary.

Definition 6.1 ([34]). Let \(M\) be a complex algebraic manifold. A system of Hodge bundles on \(M\) is a Higgs bundle \((V, \theta)\) such that \(V = \bigoplus V^{r,s}\) and \(\theta : V^{r,s} \to V^{r-1,s+1} \otimes \Omega^1_M\).

Lemma 6.2. (a) There exists a quasiprojective variety \(\mathcal{M}_{\text{Dol}}\) whose points parametrize direct sums of stable Higgs bundles with vanishing Chern classes.

(b) Let \(f\) be the map from \(\mathcal{M}_{\text{Dol}}\) to the space of polynomials with coefficients in symmetric powers of the cotangent bundle which takes \((V, \theta)\) to the characteristic polynomial of \(\theta\). Then \(f\) is proper.

(c) Let \(\mathcal{M}_{\text{Dol}}(U(2,1))\) denote the subspace of \(\mathcal{M}_{\text{Dol}}\) whose points parametrize polystable \(U(2,1)\) bundles. Then every connected component of \(\mathcal{M}_{\text{Dol}}(U(2,1))\) contains a system of Hodge bundles.

(d) \(\mathcal{M}_{\text{Dol}}(U(2,1))\) is homeomorphic to \(\mathcal{R}^+_{U(2,1)}(X)\).

Proof. (a) [34] Prop. 1.4]

(b) [34] Prop. 1.4]

(c) In [34] Theorem 3], Simpson proves that every component of \(\mathcal{M}_{\text{Dol}}\) contains a system of Hodge bundles, as follows. Let \(\mathbb{C}^*\) act on \(\mathcal{M}_{\text{Dol}}\) by \(t \cdot (V, \theta) = (V, t\theta)\). As \(t \to 0\), we have
Since $f$ is proper, $t \cdot (V, \theta)$ converges to a limit Higgs bundle $(V_0, \theta_0)$. Since $\mathcal{M}_{\text{Dol}}$ is Hausdorff, the limit is unique. Consequently, $(V_0, \theta_0)$ is fixed under the action of $\mathbb{C}^*$ and is therefore a system of Hodge bundles \[\text{[34, Lemma 4.1].}\]

Since $U(2,1)$ is closed in $\text{GL}(3, \mathbb{C})$, we have that $\mathcal{M}_{\text{Dol}}(U(2,1))$ is closed in $\mathcal{M}_{\text{Dol}}$. Therefore, $f$ restricted to $\mathcal{M}_{\text{Dol}}(U(2,1))$ is still proper, and the above proof goes through unchanged. □

**Definition 6.3** \[\text{[40].}\] We say a Higgs bundle $(V, \theta)$ is binary if $V = V_P \oplus V_Q$ where $V_P$ and $V_Q$ are vector bundles of rank 2 and 1, respectively; and $\theta$ maps $V_P$ to $V_Q \otimes \Omega^1_X$ and $V_Q$ to 0. In this situation, denote $(V, \theta)$ by $V_P \xrightarrow{\theta} V_Q$ (omitting $\theta$ if it’s clear from the context).

We say a Higgs bundle $(V, \theta)$ is ternary if $V = V_2 \oplus V_3 \oplus V_1$ where $V_1$, $V_2$, and $V_3$ are line bundles; and $\theta$ maps $V_2$ to $V_3 \otimes \Omega^1_X$, maps $V_3$ to $V_1 \otimes \Omega^1_X$, and maps $V_1$ to 0. In this situation, denote $(V, \theta)$ by $V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$. (In this case, we have $V_P = V_1 \oplus V_2$ and $V_Q = V_3$.)

It follows from Def. 6.3 and Lemma 5.5 that if a polystable $U(2,1)$ Higgs bundle is a system of Hodge bundles, then it is either ternary, binary, or dual to a binary bundle. Also, every polystable Higgs bundle is either stable or reducible. We therefore investigate the following four types of polystable $U(2,1)$ Higgs bundles: stable ternary, stable binary, reducible ternary, and reducible binary.

6.1. The case of the stable ternary Higgs bundle.

**Proposition 6.4.** Let $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$ and $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$. Then there exists a Higgs field $\theta$ such that $(V, \theta) = V_2 \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} V_1$ is a stable ternary Higgs bundle if and only if:

(i) $b \leq -2$, and

(ii) $a + \# \{k \mid a_k \neq 0\} \geq 2$, and

(iii) $2A < B$, and

(iv) $A < 2B$.

Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$ and $B = b + \sum \frac{b_k}{m_k}$. 

Proof. First assume that such a Higgs field $\theta$ exists. Stability then implies that $\theta|_{V_2}$ and $\theta|_{O_X}$ are nonzero. Hence $H^0(V'_2 \otimes \Omega^1_X) \neq 0$ and $H^0(V_1 \otimes \Omega^1_X) \neq 0$. Conditions (i) and (ii) then follow from Lemma 3.10. Conditions (iii) and (iv) follow from the fact that the $\theta$-invariant subsheaves $O_X \oplus V_1$ and $V_1$ are not destabilizing.

Conversely, if (i) and (ii) hold, then let $\theta_2$ be a nonzero global map from $V_2$ to $N$ and $\theta_1$ a nonzero global map from $O_X$ to $V_1 \otimes N$. (Lemma 3.10 shows that $\theta_2$ and $\theta_1$ exist.) Let $(w_\gamma, z_\gamma)$ be coordinates on $V_\gamma$, as in the discussion following Lemma 3.5. On $V_\gamma$, then, $\theta_1$ has the form $g_1 dw_\gamma$ for some meromorphic function $g_1$, and $\theta_2 = g_2 dw_\gamma$ on $V_\gamma$ for some meromorphic $g_2$. Define $\theta$ by $\theta|_{V_2} = \theta_2$, $\theta|_{O_X} = \theta_1$, and $\theta|_{V_1} = 0$. Then $\theta \wedge \theta = \theta_1 \wedge \theta_2 = 0$ on $V_\gamma$. Similarly, we find that $\theta \wedge \theta$ vanishes outside the union of the singular fibres and the multiple fibres. Hence $\theta \wedge \theta = 0$ everywhere. Moreover, conditions (iii) and (iv), together with the nonvanishing of $\theta_1$ and $\theta_2$, guarantee that $(V, \theta)$ is stable. □

Lemma 6.5. Suppose that $(V, \theta) = V_2 \oplus \rightarrow O_X \oplus \rightarrow V_1$ is a stable ternary Higgs bundle. Then $V_2$ and $V_1$ are vertical.

Proof. Choose divisors $D_1$ and $D_2$ such that $V_1 = O_X(D_1)$ and $V_2 = O_X(D_2)$. As in the proof of Lemma 6.4 we see that $H^0(O_X(-D_2) \otimes \Omega^1_X) \neq 0$. From the short exact sequence in Lemma 3.8 we find that either $-D_2 - 2F + \sum (m_k - 1)F_k$ or $-D_2 + F$ is linearly equivalent to an effective divisor. Consequently, we find that $D_2 \cdot F \leq 0$, with equality iff $D_2$ is vertical. We also find that $H_0 \cdot D_2 \leq \frac{k_0}{3}$, where $H_0$ and $k_0$ are as in Def. 3.7. Similarly, we find that $D_1 \cdot F \geq 0$, with equality iff $D_1$ is vertical, and that $H_0 \cdot D_1 \geq -\frac{k_0}{3}$. Therefore $H_0 \cdot (D_1 - 2D_2) \geq -k_0$. Suppose that either $D_1$ or $D_2$ is nonvertical. Then, from Def. 3.7 we find that $H \cdot (D_1 - 2D_2) \geq 0$. But this violates (iv) of Prop. 6.4. □

6.2. The case of the stable binary Higgs bundle with $\text{rank}(\text{im}(\theta)) = 1$.

Let $(V, \theta) = V_P \oplus \rightarrow O_X$ be a stable projectively flat binary Higgs bundle. When restricted to $V_P$, the Higgs field $\theta|_{V_P}$ is a map from $V_P$ to $\Omega^1_X$. The image $\text{im}(\theta|_{V_P})$ of this map is a subsheaf of $\Omega^1_X$. Stability implies that $\theta|_{V_P}$ cannot be the zero map. It follows that $\text{im}(\theta|_{V_P})$ has rank 1 or rank 2. We shall take these cases separately, beginning with the rank 1 case.
Proposition 6.6. If \((V,\theta) = V_P \overset{\Theta}{\to} \mathcal{O}_X\) is a stable projectively flat binary Higgs bundle with rank(\(\text{im}(\theta|V_P)\)) = 1, then \(V_P\) can be written as an extension of the form

\[
0 \to V_1 \to V_P \overset{\beta}{\to} V_2 \to 0,
\]

where \(V_1 = \mathcal{O}_X(aF + \sum a_kF_k)\) and \(V_2 = \mathcal{O}_X(bF + \sum b_kF_k)\) with the \(a\)'s and \(b\)'s subject to the following numerical conditions:

(i) \(-B < A < \frac{1}{2}B\), and

(ii) \(d_2 \leq -2\), and

(iii) \(b \leq -2\), and

(iv) \((c, c_1, \ldots, c_n)\) is an \((n+1)\)-tuple of integers such that \(0 \leq c_k < m_k\) for all \(k\) and \(d_1 \geq 0\) and \(C \geq \frac{2}{3}(A + B)\), then \(d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)\).

Here we have used the notations \(A = a + \sum \frac{a_k}{m_k}\); \(B = b + \sum \frac{b_k}{m_k}\); \(C = c + \sum \frac{c_k}{m_k}\); \(d_1 = b - c - \#\{b_k < c_k\}\); \(d_2 = a - b - \#\{a_k < b_k\}\); and \(d_3 = a - c - \#\{a_k < c_k\}\).

Conversely, given \(a\)'s and \(b\)'s satisfying (i)–(iv), there exists a stable projectively flat binary Higgs bundle \(V_P \overset{\Theta}{\to} \mathcal{O}_X\) with \(V_P\) given as an extension of the form \(2\).

Before proving this proposition, we first prove several preliminary lemmas.

Lemma 6.7. Let \((V,\theta) = V_P \overset{\Theta}{\to} \mathcal{O}_X\) be a binary Higgs bundle such that \(\text{im}(\theta|V_P)\) has rank 1. Let \(V_1 = \ker(\theta|V_P)\). Then \((V,\theta)\) is stable if and only if:

(SB1) \(\deg(V_1) < \frac{1}{3} \deg(V_P)\), and

(SB2) \(\deg(S) < \frac{2}{3} \deg(V_P)\) for every rank 1 subsheaf \(S\) of \(V_P\), and

(SB3) \(\deg(V_P) > 0\).

Proof. If \((V,\theta)\) is stable, then (SB1)–(SB3) follow directly from the fact that the \(\theta\)-invariant subsheaves \(V_1, S \oplus \mathcal{O}_X,\) and \(\mathcal{O}_X,\) respectively, do not destabilize \(V\). Conversely, if (SB1)–(SB3) hold, then any proper \(\theta\)-invariant subsheaf \(S'\) of \(V\) must be a rank 1 subsheaf of \(V_1\), a rank 1 subsheaf of \(\mathcal{O}_X\), or of the form \(S \oplus \mathcal{O}_X\), where \(S\) is a rank 1 subsheaf of \(V_P\), in which case (SB1)–(SB3) imply that \(S'\) is not destabilizing. □
Lemma 6.8. Let \((V, \theta) = V_P \rightarrow O_X\) be a stable projectively flat binary Higgs bundle such that \(\text{im}(\theta|V_P)\) has rank 1. Let \(V_1 = \ker(\theta|V_P)\) and \(V_2 = \text{im}(\theta|V_P)\). Then \(V_1\) and \(V_2\) are vertical line bundles.

Proof. We have an exact sequence \(0 \rightarrow V_1 \rightarrow V_P \rightarrow V_2 \rightarrow 0\). It follows that there exist divisors \(D_1\) and \(D_2\) and a dimension 0 subscheme \(\tilde{Z}\) such that \(V_1 = O_X(D_1)\) and \(V_2 = I_{\tilde{Z}} \otimes O_X(D_2)\), where \(I_{\tilde{Z}}\) is the ideal sheaf associated to \(\tilde{Z}\).

We first show that \(D_2\) is a vertical divisor. Since \(V_2\) is the image of \(\theta|V_P\), which maps to \(\Omega^1_X\), we find from the short exact sequence in Lemma 3.8 that either \(\text{Hom}(I_{\tilde{Z}} \otimes O_X(D_2), N) \neq 0\) or \(\text{Hom}(I_{\tilde{Z}} \otimes O_X(D_2), Q) \neq 0\). Since \(\tilde{Z}\) has codimension 2, we then deduce that either \(H^0(O_X(-D_2 - 2F + \sum (m_k - 1)F_k)) \neq 0\) or \(H^0(O_X(-D_2 + F)) \neq 0\). Let \(H_0, H,\) and \(k_0\) be as in Def. 3.7. Since either \(-D_2 - 2F + \sum (m_k - 1)F_k\) or \(-D_2 + F\) is linearly equivalent to an effective divisor, we have that \(H_0 \cdot D_2 < \frac{1}{2}k_0 < k_0\). We also find that \(D_2 \cdot F \leq 0\), with equality iff \(D_2\) is vertical. Suppose that \(D_2 \cdot F < 0\). This would imply that \(H \cdot D_2 = H_0 \cdot D_2 + k_0 F \cdot D_2 < k_0 - k_0 = 0\). But conditions (SB1)–(SB3) in Lemma 3.7 imply that \(H \cdot D_2 > 0\).

We now show that \(D_1\) is a vertical divisor. We begin to do so by showing that \(F \cdot D_1 = 0\). Suppose that \(F \cdot D_1 > 0\). Since \(V\) is projectively flat, we have \((D_1 + D_2)^2 = 3(\ell(\tilde{Z}) + D_1 \cdot D_2)\), where \(\ell(\tilde{Z})\) denotes the length of \(\tilde{Z}\). Since \(D_2\) is vertical, we have that \(D_1 \cdot D_2 = \frac{(H_0-D_2)(D_1-F)}{H_0 \cdot F} > 0\). It follows that \(D_1^2 > 0\). The Hodge index theorem, applied to \((H \cdot F)D_1 - (H \cdot D_1)F\), then shows that \(H \cdot D_1 \geq 0\). But conditions (SB1)–(SB3) in Lemma 3.7 imply that \(H \cdot D_1 < 0\).

Now suppose that \(F \cdot D_1 < 0\). This time, we apply the Hodge index theorem to \((H_0 \cdot F)D_1 - (H_0 \cdot D_1)F\) to find that \(H_0 \cdot D_1 \leq \frac{(H_0 \cdot F)(3(\ell(\tilde{Z}) + D_1 \cdot D_2))}{2(D_1 \cdot F)}\). It follows that \(H \cdot D_1 \leq \frac{H_0 - D_2}{2} - k_0 < -H \cdot D_2\). But conditions (SB1)–(SB3) in Lemma 3.7 imply that \(H \cdot D_1 > -H \cdot D_2\).

Hence \(F \cdot D_1 = 0\). The Hodge index theorem now implies that \(D_1^2 \leq 0\) with equality iff \(D_1\) is vertical. Projective flatness implies that \(D_1^2 = 3\ell(\tilde{Z}) \geq 0\). Therefore \(D_1\) is vertical.

Finally, since \(D_1\) is vertical, we have that \(0 = D_1^2 = 3\ell(\tilde{Z})\), which implies that \(V_2\) is a line bundle. □
Lemma 6.9. Let $V_1 = \mathcal{O}_X(aF + \sum a_kF_k)$ and $V_2 = \mathcal{O}_X(bF + \sum b_kF_k)$ be vertical line bundles such that $d_2 \leq -2$, where $d_2 = a - b - \#\{a_k < b_k\}$.

If there is a nonsplit extension of the form
\[
0 \to V_1 \to V_P \overset{\beta}{\to} V_2 \to 0,
\]
and $L = \mathcal{O}_X(cF + \sum c_kF_k)$ is a vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$, where $d_1 = b - c - \#\{b_k < c_k\}$ and $d_3 = a - c - \#\{a_k < c_k\}$, such that $H^0(L^* \otimes V_3) = 0$, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

Conversely, there exists a nonsplit extension such that if $L = \mathcal{O}_X(cF + \sum c_kF_k)$ is any vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ such that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$, then $H^0(L^* \otimes V_2) = 0$.

Proof. First, we show that if $V_P$ and $L$ are subject to the given conditions, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

Observe that $L^* \otimes V_1 = \mathcal{O}_X(d_3F + \sum r_kF_k)$ and $L^* \otimes V_2 = \mathcal{O}_X(d_1F + \sum r'_kF_k)$ for some $r_k, r'_k \geq 0$. Consider the short exact sequence
\[
0 \to L^* \otimes V_1 \to L^* \otimes V_P \to L^* \otimes V_2 \to 0.
\]

The associated long exact sequence in cohomology then implies that the coboundary map $\delta : H^0(L^* \otimes V_2) \to H^1(L^* \otimes V_1)$ is injective. Consequently, $h^0(L^* \otimes V_2) \leq h^1(L^* \otimes V_1)$, and so by Lemma 3.3, we have that $d_1 + 1 \leq -d_3 - 1$.

Suppose now that $d_1 + 1 > -d_2 - 1$. Let $\sigma$ be an element of $H^1(V_P^* \otimes V_1)$ which defines the extension. Note that $V_P^* \otimes V_1 = \mathcal{O}_X(d_2F + \sum r_kF_k)$ for some $r_k$ with $r_k \geq 0$. Taking notation from Lemma 3.6(ii), we have that $\sigma$ equals $\sigma_{-1} w_{\gamma}^{-1} + \cdots + \sigma_{d_2+1} w_{\gamma}^{d_2+1}$ on $V_\gamma \cap W_{\xi}$ and 0 elsewhere for some $\sigma_{-1}, \ldots, \sigma_{d_2+1}$. Let $\{\phi'_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_1$, and let $\{\phi''_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_2$. We may regard $\sigma$ as the extension class of $\mathcal{L}$. Transition matrices for $L^* \otimes V_P$ are then given by
\[
\begin{pmatrix}
\phi'_{\alpha\beta} & \phi''_{\alpha\beta} \\
0 & \phi''_{\alpha\beta}
\end{pmatrix}.
\]

Let $s \in H^0(L^* \otimes V_2)$ be the nonzero section such that with respect to the trivialization on $V_\gamma$, we have $s_{\gamma} = w_{\gamma}^{d_1}$, as in Lemma 3.6(i). Then $\delta(s) = \sigma_{-1} w_{\gamma}^{-1} + \cdots + \sigma_{d_2+1} w_{\gamma}^{d_2+1}$ on
\(V_γ ∩ W_ξ\) and 0 elsewhere. So, by Lemma 3.6(ii) and the inequality \(d_1 + 1 > -d_2 - 1\), we have that \(δ(s) = 0 ∈ H^1(L^* ⊗ V_1)\). But since \(δ\) is injective, this yields the desired contradiction.

Conversely, we now show that there exists a nonsplit extension \(\mathfrak{M}\) such that if \(L = \mathcal{O}_X(cF + \sum c_k F_k)\) is any vertical line bundle with \(d_1 ≥ 0\) and \(d_3 ≤ -2\) such that \(d_1 + 1 ≤ \min(-d_2 - 1, -d_3 - 1)\), then \(H^0(L^* ⊗ V_P) = 0\). Let \((σ_{-1}, σ_{-2}, \ldots, σ_{d_2 + 1})\) be a \((-d_2 - 2)\)-tuple of complex numbers such that for any \(ℓ_1, ℓ_3\) with \(ℓ_1 ≥ 0\) and \(ℓ_3 ≤ -2\) such that \(ℓ_1 + 1 ≤ \min(-d_2 - 1, -d_3 - 1)\), the matrix

\[
Θ_{ℓ_1, ℓ_3} = \begin{pmatrix}
σ_{ℓ_1 + 1} & σ_{ℓ_3} & \ldots & σ_{d_2 + 1} & 0 & 0 & \ldots & 0 \\
σ_{ℓ_1 + 2} & σ_{ℓ_3 + 1} & \ldots & σ_{d_2 + 2} & σ_{d_2 + 1} & 0 & \ldots & 0 \\
\vdots & & & & & \ddots & & \vdots \\
σ_{ℓ_3 + d_2 + 1} & σ_{ℓ_3 + d_2} & \ldots & \ldots & \ldots & σ_{d_2 + 1} \\
\vdots & & & & & & \vdots \\
σ_{-1} & σ_{-2} & \ldots & \ldots & \ldots & σ_{-ℓ_1 - 1}
\end{pmatrix}
\]

has maximal rank. (One may construct such a sequence of \(σ\)'s by induction on \(-d_2 - 1\); given \(σ_{-1}, σ_{-2}, \ldots, σ_{d_2}\), choose \(σ_{d_2 + 1}\) so that every square matrix of the above form has nonzero determinant. This is possible because there are only finitely many such matrices, and for each such matrix, the determinant is zero for only finitely many values of \(σ_{d_2 + 1}\).)

Let \(σ\) be the element in \(H^1(V_2^* ⊗ V_1)\) represented by a 1-cocycle which equals \(σ_γξ = σ_{-1} w_γ^{-1} + \cdots + σ_{d_2 + 1} w_γ^{d_2 + 1}\) on \(V_γ ∩ W_ξ\) and 0 elsewhere. Let \(V_P\) be the rank 2 bundle given as an extension as in \(\mathfrak{M}\) whose extension class is determined by \(σ\). Since \(σ\) is nonzero, \(\mathfrak{M}\) does not split. Let \(L = \mathcal{O}_X(cF + \sum c_k F_k)\) be a vertical line bundle with \(d_1 ≥ 0\) and \(d_3 ≤ -2\) such that \(d_1 + 1 ≤ \min(-d_2 - 1, -d_3 - 1)\). We must show that \(H^0(L^* ⊗ V_P) = 0\).

The condition \(d_3 ≤ -2\) guarantees that \(H^0(L^* ⊗ V_1) = 0\). It therefore suffices to show that the coboundary map \(δ : H^0(L^* ⊗ V_2) → H^1(L^* ⊗ V_1)\) is injective. Let \(s ∈ H^0(L^* ⊗ V_2)\). We now show that if \(δ(s) = 0\), then \(s = 0\).

From Lemma 3.6(i), we know that on \(V_γ\), the section \(s\) is of the form \(s_γ = s_0 + s_1 w_γ + \cdots + s_{d_1} w_γ^{d_1}\) with respect to the trivialization on \(V_γ\). From Lemma 3.6(ii), we know that if \(c\) is the 1-cocycle given by \(w^j\) on \(V_γ ∩ W_ξ\) and 0 elsewhere, then \([c] = 0 ∈ H^1(L^* ⊗ V_1)\) if and only if \(j ≥ 0\) or \(j ≤ -d_3\). Since \(δ(s)\) equals \(s_γ σ_γξ\) on \(V_γ ∩ W_ξ\) and 0 elsewhere, we have
that $\delta(s) = 0$ if and only if the following equalities hold:

$$\sigma_{d_1+1}s_0 + \sigma_{d_2}s_1 + \cdots + \sigma_{d_3}d_3 - d_2 = 0$$

$$\sigma_{d_3+1}s_0 + \sigma_{d_3+2}s_1 + \cdots + \sigma_{d_3+1}s_{d_3-1} = 0$$

$$\cdots$$

$$\sigma_{d_3+d_2+1}s_0 + \sigma_{d_3+d_2}s_1 + \cdots + \sigma_{d_3+1}s_{d_2} = 0$$

$$\cdots$$

$$\sigma_{-1}s_0 + \sigma_{-2}s_1 + \cdots + \sigma_{-d_1-1}s_{d_1} = 0$$

Since $\Theta_{d_1,d_3}$ has maximal rank and $d_1 + 1 \leq -d_3 - 1$ (which is to say, regarding the $s$’s as variables, that there are at least as many equations as variables), we conclude that $s = 0$. □

**Proof of Prop. 6.6.** We first show that if $(V, \theta) = V_P \xrightarrow{\oplus} O_X$ is a stable projectively flat binary Higgs bundle with rank$(im(\theta|V_P)) = 1$, then $V_P$ has the stated form.

Lemma 6.8 implies that $V_1 = ker(\theta|V_P)$ and $V_2 = im(\theta|V_P)$ are vertical line bundles; we therefore obtain the extension (2). Condition (i) follows from (SB1) and (SB3) of Lemma 6.7.

Stability implies that (2) does not split; therefore $h^1(V_2^* \otimes V_1) > 0$. It follows from (i) that $d_2 < 0$. Condition (ii) then follows from Lemma 3.3 (ii).

Since $V_2$ is a subsheaf of $\Omega^1_X$, we must have that $H^0(V_2^* \otimes \Omega^1_X) \neq 0$. Condition (iii) then follows from Lemma 3.10.

Let $(c, c_1, \ldots, c_n)$ be an $(n + 1)$-tuple of integers such that $0 \leq c_k < m_k$ for all $k$ and $d_1 \geq 0$ and $C \geq \frac{2}{3}(A + B)$. Let $L = O_X(cF + \sum c_kF_k)$. From (SB2) of Lemma 6.7 we know that $H^0(L^* \otimes V_P) = 0$. Note that $L^* \otimes V_1 = O_X(d_3F + \sum r_kF_k)$ for some $r_k$ with $0 \leq r_k < m_k$ for all $k$. Arguing as in the proof that condition (ii) holds, we see that $d_3 < 0$. From the long exact sequence in cohomology associated to (4), we find that $H^1(L^* \otimes V_1) \neq 0$. Lemma 3.8 then implies that $d_3 \leq -2$. Condition (iv) then follows from Lemma 6.9.

Conversely, suppose that we are given $a$’s and $b$’s satisfying conditions (i)–(iv), and let $V_1 = O_X(aF + \sum a_kF_k)$ and $V_2 = O_X(bF + \sum b_kF_k)$. We will show that there exists a stable projectively flat binary Higgs bundle $V_P \xrightarrow{\oplus} O_X$ with rank$(im(\theta|V_P)) = 1$ and $V_P$ as in (2).
Lemma 6.10 and condition (ii) guarantee the existence of a rank 2 bundle \( V_P \) and a nonsplit extension \([2]\) such that if \( L = \mathcal{O}_X(cF + \sum c_k F_k) \) is any vertical line bundle with \( d_1 \geq 0 \) and \( d_3 < 0 \) such that \( d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1) \), then \( H^0(L^* \otimes V_P) = 0 \). By Lemma 3.10 and condition (iii), there exists a nonzero map \( \alpha : V_2 \to \Omega^1_X \). Let \( V = V_P \oplus \mathcal{O}_X \).

Define a Higgs field \( \theta \) by \( \theta|_{V_P} = \alpha \circ \beta \) and \( \theta|_{\mathcal{O}_X} = 0 \). Note that \( \theta \wedge \theta = 0 \). Then \((V, \theta)\) is a binary Higgs bundle with \( \text{rank}(\text{im}(\theta|_{V_P})) = 1 \). Moreover, \( V \) is projectively flat, since \( 0 = c_1^2(V) = 3c_2(V) \).

It remains to be shown that \((V, \theta)\) is stable. \((\text{SB1})\) and \((\text{SB3})\) from Lemma 6.7 follow from condition (i). Let us now verify that \((\text{SB2})\) holds. Suppose to the contrary that there exists a rank 1 subsheaf \( S \) of \( V_P \) such that \( \text{deg}(S) \geq \frac{2}{3} \text{deg}(V_P) \). Let \( L \) be the kernel of the natural map \( V_P \to \frac{V_P}{\text{Tor}(\frac{V_P}{S})} \). Then \( L \) is a line bundle, \( \text{deg}(L) \geq \text{deg}(S) \), and \( H^0(L^* \otimes V_P) \neq 0 \). (See [27].) Stability, together with Def. 3.7, implies that \( L \) is vertical.

Write \( L = \mathcal{O}_X(cF + \sum c_k F_k) \). Dividing both sides of \( \text{deg}(L) \geq \frac{2}{3} \text{deg}(V_P) \) by \( H \cdot F \), we find that \( C \geq \frac{2}{3}(A + B) \). Note that \( L^* \otimes V_2 = \mathcal{O}_X(d_1 F + \sum r_k F_k) \), and so \( H^0(L^* \otimes V_2) \neq 0 \) implies that \( d_1 \geq 0 \). It now follows from condition (iv) that \( d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1) \). Moreover, \( d_3 < 0 \) since \( H^0(L^* \otimes V_1) = 0 \). Our choice of \( V_P \) then implies that \( H^0(L^* \otimes V_P) = 0 \), contradicting our earlier assertion that \( H^0(L^* \otimes V_P) \neq 0 \). Therefore \((V, \theta) = V_P \to \mathcal{O}_X \) is stable, as desired. \( \square \)

6.3. The case of the stable binary Higgs bundle with \( \text{rank}(\text{im}(\theta)) = 2 \).

In this subsection, we show that there does not exist a stable binary Higgs bundle \((V, \theta)\) on \( X \) with \( \text{rank}(\text{im}(\theta)) = 2 \). Throughout this section, let \( N = \mathcal{O}_X(-2F + \sum (m_k - 1)F_k) \) and \( Q = I_Z \otimes \mathcal{O}_X(F) \), as in Lemma 3.8.

**Lemma 6.10.** Suppose that \((V, \theta) = V_P \to \mathcal{O}_X \) is a stable projectively flat binary Higgs bundle with \( \text{rank}(\text{im}(\theta)) = 2 \). Then there exists an exact sequence

\[
0 \to V_1 \to V_P \to V_2 \to 0,
\]

where \( V_1 \) and \( V_2 \) are vertical line bundles and \( H^0(V_2^* \otimes Q) \neq 0 \).

**Proof.** Let \( \beta \) be the map in the exact sequence of Lemma 3.8 from \( \Omega^1_X \) to \( Q \). Let \( V_2 = \text{im}(\beta \circ (\theta|_{V_P})) \), and let \( V_1 = \ker(\beta \circ (\theta|_{V_P})) \). This gives us an exact sequence \( 0 \to \)
$V_1 \to V_P \to V_2 \to 0$. Since $\text{rank}(\text{im}(\theta)) = 2$, we see that $1 = \text{rank}(V_2) = \text{rank}(V_1)$. The proof of Lemma 6.8 shows that $V_1$ and $V_2$ are vertical line bundles. Moreover, the inclusion map $i : V_2 \hookrightarrow Q$ yields a nonzero element of $H^0(V_2^* \otimes Q)$. □

**Proposition 6.11.** If $(V, \theta)$ is a stable binary Higgs bundle, then $\text{im}(\theta)$ has rank 1.

**Proof.** By tensoring with a line bundle, as in Lemma 5.5, we may assume that $(V, \theta)$ is of the form $V_P \oplus O_X$. Then $\text{im}(\theta)$ is a subsheaf of $\Omega^1_X$ and so has rank 0, 1, or 2. As noted in the introduction to §6.2, $\text{im}(\theta)$ cannot have rank 0.

Suppose $\text{im}(\theta)$ has rank 2. By Lemma 6.10, we have an exact sequence

$$0 \to V_1 \to V_P \to V_2 \to 0,$$

where $V_1$ and $V_2$ are vertical line bundles and $H^0(V_2^* \otimes Q) \neq 0$. By Lemma 3.9 we have that $\text{deg}(V_2) < 0$. As in Lemma 6.7 stability implies that $\text{deg}(V_P) > 0$, whence we see that $0 < \text{deg}(V_P) = \text{deg}(V_1) + \text{deg}(V_2) < \text{deg}(V_1)$. The proof of Lemma 6.7 also shows that $\text{deg}(V_1) < \frac{2}{3} \text{deg}(V_P)$, whereby one obtains the contradictory inequality

$$0 < \text{deg}(V_1) < 2 \text{deg}(V_2) < 0.$$

□

### 6.4. The case of the reducible ternary Higgs bundle.

We now consider reducible, polystable, ternary Higgs bundles of the form $(V, \theta) = V_2 \oplus V_3 \oplus V_1$. In this case, either $\theta|V_2$ or $\theta|V_3$ must be the zero map. (For if not, then $V$ is not reducible.) We divide into three cases accordingly, depending whether the first map only is zero, the second map only is zero, or both are.

#### Case 1: $\theta|V_2 = 0$ and $\theta|V_3 \neq 0$

**Proposition 6.12.** There exists a polystable ternary Higgs bundle $(V, \theta) = V_2 \oplus V_3 \oplus V_1$ with $\theta|V_2 = 0$ and $\theta|V_3 \neq 0$ and $c_1(V_2) = c_1(V_3 \oplus V_1) = c_2(V_3 \oplus V_1) = 0$ if and only if $V_2 = O_X$ and $V_3 = O_X(bF + \sum b_k F_k)$ and $V_1 = V_3^*$, where the $b$’s are subject to the following numerical conditions:

1. $B = b + \sum \frac{b_k}{m_k} > 0$, and
2. $2b + \#\{b_k \geq \frac{m_k}{2}\} \leq -2$. 


Case 2: $\theta|V_2 \neq 0$ and $\theta|V_3 = 0$.

This case is the same as Case 1, with the $V$’s relabeled.

Case 3: $\theta|V_2 = \theta|V_3 = 0$

This case is trivial; there exists a polystable Higgs bundle $V_2 \xrightarrow{\theta_2} V_3 \xrightarrow{\theta_3} V_1$ with $c_1(V_2) = c_1(V_3) = c_1(V_1) = 0$ and $\theta|V_2 = \theta|V_3 = 0$ if and only if $V_2 = V_3 = V_1 = \mathcal{O}_X$.

6.5. The case of the reducible binary Higgs bundle. Let $(V, \theta) = V_P \xrightarrow{\theta} V_Q$ be a reducible polystable binary Higgs bundle whose summands have vanishing Chern classes, where $\text{rank}(V_P) = 2$ and $\text{rank}(V_Q) = 1$. The rank $R$ of the image of $\theta$ in $V_Q \otimes \Omega^1_X$ is either 2, 1,
or 0. If $R = 2$, then $(V, \theta)$ can not be reducible. If $R = 1$, then we must have $V_P = V_1 \oplus V_2$, where $V_1 = \ker(\theta|_{V_P})$; this case was discussed in §6.4. If $R = 0$, then $\theta$ is the zero map. In this case, we must have $V_Q = \mathcal{O}_X$ and $V_P$ stable.

Remark: An explicit description of all stable rank 2 bundles on $X$ with vanishing Chern classes can be found in [4, Proposition 4.1]. (The method of proof of Prop. 6.6 also yields such a description.)

7. Main Theorem and an Example

Putting together the pieces from the previous sections, we have the following explicit description of all orbifold Toledo invariants that arise from semisimple $U(2, 1)$ representations of the orbifold fundamental group of the 2-orbifold associated to a Seifert fibered homology 3-sphere.

**Theorem 7.1.** Let $O$ be the base orbifold of a large Seifert fibered homology 3-sphere. Let $n$ equal the number of cone points that $O$ has, and let $m_1, \ldots, m_n$ denote the orders of these cone points. Then there exists a semisimple representation $\rho : \pi_1^{\text{orb}}(O) \to U(2, 1)$ such that $\tau = \tau_{\text{orb}}(\rho)$ if and only if $\tau = \pm(a + b + \sum a_k/b_k)$ for some $(2n + 2)$-tuple $(a, a_1, \ldots, a_n, b, b_1, \ldots, b_n)$ of integers with $0 \leq a_k, b_k < m_k$ for all $k = 1, \ldots, n$ such that at least one of (i)–(iv) holds:

(i) The a’s and b’s satisfy (i)–(iv) from Prop. 6.3 as well as (⋆) below; or

(ii) The a’s and b’s satisfy (i)–(iv) from Prop. 6.6 as well as (⋆) below; or

(iii) The b’s satisfy (i) and (ii) from Prop. 6.12 and $a = a_k = 0$ for all $k$; or

(iv) $a = b = a_k = b_k = 0$ for all $k$.

(⋆) There exist integers $y, y_1, \ldots, y_n, s_1, \ldots, s_n$ such that $3y + (s_1 + \cdots + s_n) = a + b$ and $3y_k - m_k s_k = a_k + b_k$ for $k = 1, \ldots, n$.

Proof. By Lemmas 5.3, 5.5(ii), 6.2 and 4.7 as well as the discussion following Def. 6.3, it suffices to show that $c_1(\mathcal{O}_X((a + b)F + \sum (a_k + b_k)F_k))$ equals the Higgs bundle Toledo invariant of a stable ternary, stable binary, reducible ternary, or reducible binary Higgs bundle whose summands have vanishing Chern classes if and only if the a’s and b’s satisfy one of (i)–(iv).
Suppose \((V, \theta) = V_2 \oplus V_3 \oplus V_1\) is a stable ternary Higgs bundle with vanishing Chern classes. By Lemma 5.3 tensoring with \(V_3^*\) yields a stable ternary Higgs bundle \((V', \theta') = (V \otimes V_3^*, \theta \otimes 1) = (V_2 \otimes V_3^*) \oplus \mathcal{O}_X \oplus (V_2 \otimes V_3^*)\) with \(\tau_{(V, \theta)} = \tau_{(V', \theta')}\). By Prop. 6.1 and Definition 5.4 we then have that \(\tau_{(V', \theta')} = c_1(\mathcal{O}_X((a + b)F + \sum(a_k + b_k)F_k))\), where the \(a\)'s and \(b\)'s satisfy (i)–(iv) from Prop. 6.4. Moreover, \(\mathcal{O}_X((a + b)F + \sum(a_k + b_k)F_k) = \text{det}(V') = V_3^* \otimes V_3^* \otimes V_3^*\) is vertical. Thus \(V_3^*\) is of the form \(\mathcal{O}_X(yF + \sum y_kF_k)\) with \(3(yF + \sum y_kF_k)\) linearly equivalent to \((a + b)F + \sum(a_k + b_k)F_k\). Condition (\(\ast\)), which is equivalent to the condition that \((a + b)F + \sum(a_k + b_k)F_k\) is “divisible by 3,” therefore holds.

Conversely, given \(a\)'s and \(b\)'s satisfying (i), Prop. 6.4 and Def. 5.4 guarantee the existence of a stable projectively flat ternary Higgs bundle \((V', \theta')\) with \(\tau_{(V', \theta')} = \mathcal{O}_X((a + b)F + \sum(a_k + b_k)F_k)\). Condition (\(\ast\)) is then equivalent to the existence of a vertical line bundle \(V_3 = \mathcal{O}_X(yF + \sum y_kF_k)\) such that \(c_1(V' \otimes V_3) = c_2(V' \otimes V_3) = 0\). By Lemma 5.5 \(V' \otimes V_3\) is a stable ternary Higgs bundle with \(\tau_{(V, \theta)} = \tau_{(V', \theta')}\).

To summarize: \(c_1(\mathcal{O}_X((a + b)F + \sum(a_k + b_k)F_k))\) equals the Higgs bundle Toledo invariant of a stable flat ternary Higgs bundle on \(X\) if and only if the \(a\)'s and \(b\)'s satisfy (i).

A similar argument, using Prop. 6.6 instead of Prop. 6.4, shows that \(c_1(\mathcal{O}_X((a + b)F + \sum(a_k + b_k)F_k))\) equals the Higgs bundle Toledo invariant of a stable binary Higgs bundle \((V, \theta)\) with \(c_1(V) = c_2(V) = 0\) and \(\text{rank}(\text{im}(\theta)) = 1\) if and only if the \(a\)'s and \(b\)'s satisfy (ii). Prop. 6.11 shows that there are no stable binary Higgs bundles with \(\text{rank}(\text{im}(\theta)) = 2\). Condition (iii) covers Cases 1 and 2 from 5.5 and (iv) covers Case 3 from 6.4 as well as the reducible binary case with \(\theta = 0\) (as discussed in 5.5), since the Toledo invariant vanishes in both of these cases. □

**Corollary 7.2.** (a) A lower bound for the number of distinct connected components in the representation space \(\mathcal{R}^+_{U(2,1)}(O) = \frac{\text{Hom}^+(\pi^\text{vb}(O), U(2,1))}{U(2,1)}\) is given by the number of distinct values \(\pm(a + b + \sum \frac{a_k + b_k}{m_k})\), where the \(a\)'s and \(b\)'s satisfy one of (i)–(iv) from Thm. 7.1.

(b) A lower bound for the number of distinct connected components in the representation space \(\mathcal{R}^+_{PU(2,1)}(Y) = \frac{\text{Hom}^+(\pi_1(Y), PU(2,1))}{PU(2,1)}\) is given by the number of distinct values \(\pm(a + b + \sum \frac{a_k + b_k}{m_k})\), where the \(a\)'s and \(b\)'s satisfy (i) or (ii) from Thm. 7.1.
Proof. We prove (b) only; the proof of (a) is similar. Lemma 2.4 shows that we may replace $Y$ by $X$ in the statement of this theorem. Lemma 1.3 shows that (equivalence classes of) $PU(2,1)$ representations with distinct Toledo invariants lie in distinct components of $R^*_{PU(2,1)}(X)$. If $\rho \in \text{Hom}^*(\pi_1(X), U(2,1))$, then $\varphi \circ \rho \in \text{Hom}^*(\pi_1(X), PU(2,1))$, where $\varphi : U(2,1) \to PU(2,1)$ is the canonical homomorphism. Lemmas 5.2 and 5.3 show that the number of distinct Toledo invariants arising from irreducible $U(2,1)$ representations of $\pi_1(X)$ exactly equals the number of distinct Higgs bundle Toledo invariants of stable $U(2,1)$ system of Hodge bundles on $X$ with vanishing Chern classes. There exist $a$’s and $b$’s satisfying (i) or (ii) from Thm. 7.1 if and only if $\pm c_1(O_X((a+b)F + \sum(a_k+b_k)F_k))$ equals the Higgs bundle Toledo invariant of a stable $U(2,1)$ Higgs bundle on $X$ with vanishing Chern classes—in which case, by Lemma 4.7, the corresponding orbifold Toledo invariant is $\pm(a + b + \sum \frac{a_k+b_k}{m_k})$. □

Example: Let $n = 3$, and let $(m_1, m_2, m_3) = (2, 3, 11)$. Departing from our previous notations, let $F_{m_k}$ (instead of $F_k$) denote the multiple fibre on $X$ of multiplicity $m_k$.

Let $(V_1, \theta_1) = O_X(-2F + F_2 + 2F_3 + 10F_{11}) \oplus O_X \to O_X(-F + 2F_2 + 3F_3 + F_{11})$ be a stable ternary Higgs bundle.

Let $(V_2, \theta_2) = O_X \to O_X \to O_X$, where $\theta_2$ is the zero map.

Let $(V_3, \theta_3)$ be a stable binary Higgs bundle of the form $V_P \oplus O_X$, where $V_P$ is given by a nontrivial extension

$$0 \to O_X(-F + F_3 + 7F_{11}) \to V_P \to O_X(-2F + F_2 + 2F_3 + 10F_{11}) \to 0.$$ 

Let $(V_4, \theta_4) = O_X(-2F + F_2 + 2F_3 + 10F_{11}) \oplus O_X \to O_X(-F + 2F_2 + 3F_3 + 2F_{11})$ be a stable ternary Higgs bundle.

Theorem 7.1 guarantees that all orbifold Toledo invariants arise from these four Higgs bundles and their duals. Let $\tau_k$ be the orbifold Toledo invariant corresponding to $(V_k, \theta_k)$. Then $0 = \tau_1 = \tau_2$, $0.0455 \approx \tau_3$, and $0.0909 \approx \tau_4$. We conclude that in this case, $R^+_{U(2,1)}(O)$ contains at least 5 distinct connected components.

Though $(V_1, \theta_1)$ and $(V_2, \theta_2)$ have the same Higgs bundle Toledo invariant, it is unclear whether they lie in the same component of $M_{Dol}(U(2,1))$. We hope to address this question in a future paper.
REFERENCES

[1] M. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A, 308(1505):523–615, 1983.

[2] W. Barth, C. Peters, and A. Van de Ven. Compact Complex Surfaces. Springer-Verlag, 1984.

[3] S. Bauer. Parabolic bundles, elliptic surfaces and SU(2)-representation spaces of genus zero Fuchsian groups. Mathematische Annalen, 290(3):509–526, 1991.

[4] S. Bauer and C. Okonek. The algebraic geometry of representation spaces associated to Seifert fibered homology 3-spheres. Mathematische Annalen, 286:45–76, 1990.

[5] I. Biswas. Orbifold principal bundles on an elliptic fibration and parabolic principal bundles on a Riemann surface. Collect. Math., 54(3):293–308, 2003.

[6] H. Boden. Representations of orbifold groups and parabolic bundles. Comment. Math. Helv., 66(3):389–447, 1991.

[7] S. Bradlow, O. García-Prada, and P. Gothen. Surface group representations and U(p,q)-Higgs bundles. Journal of Differential Geometry, 64(1):111–170, 2003.

[8] S. Bradlow, O. García-Prada, and P. Gothen. Maximal surface group representations in isometry groups of classical hermitian symmetric spaces, 2005. Preprint. arxiv.org/abs/math/0511415.

[9] M. Burger, A. Iozzi, F. Labourie, and A. Wienhard. Maximal representations of surface groups: Symplectic anosov structures, 2005. Preprint. arxiv.org/abs/math/0506079.

[10] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. C. R. Math. Acad. Sci. Paris, 336(5):387–390, 2003.

[11] K. Corlette. Flat G-bundles with canonical metrics. J. Differential Geom., 28(3):361–382, 1988.

[12] I. Dolgachev. Algebraic surfaces with $q = p_g = 0$. In Algebraic surfaces : III ciclo, 1977, Villa Monastero Varenna-Como. 1981. Centro internazionale matematico estivo. ISBN: 8820711087.

[13] S. Donaldson. Twisted harmonic maps and the self-duality equations. Proc. London Math. Soc. (3), 55(1):127–131, 1987.

[14] R. Fintushel and R. Stern. Instanton homology of Seifert fibred homology three spheres. Proc. London Math. Soc. (3), 61(1):109–137, 1990.

[15] R. Friedman. Algebraic Surfaces and Holomorphic Vector Bundles. Springer, 1998.

[16] M. Furuta and B. Steer. Seifert fibered homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points. Advances in Mathematics, 96:38–102, 1992.

[17] W. Goldman. Complex Hyperbolic Geometry. Clarendon Press, 1999.

[18] W. Goldman, M. Kapovich, and B. Leeb. Complex hyperbolic manifolds homotopy equivalent to a Riemann surface. Communications in Analysis and Geometry, 9(1):61–95, 2001.

[19] P. Griffiths and J. Harris. Principles of Algebraic Geometry. John Wiley & Sons, Inc., 1978.
[20] N. Gusevskii and J. Parker. Representations of free Fuchsian groups in complex hyperbolic space. *Topology*, 39(1):33–60, 2000.

[21] N. Gusevskii and J. Parker. Complex hyperbolic quasi-Fuchsian groups and Toledo’s invariant. *Geom. Dedicata*, 97:151–185, 2003. Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999).

[22] N. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[23] Y. Kamishima and S. Tan. Deformation spaces on geometric structures. In *Aspects of low-dimensional manifolds*, volume 20 of *Adv. Stud. Pure Math.*, pages 263–299. Kinokuniya, Tokyo, 1992.

[24] Y. Kamishima and T. Tsuboi. CR-structures on Seifert manifolds. *Invent. Math.*, 104(1):149–163, 1991.

[25] M. Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.

[26] P. Kirk and E. Klassen. Representation spaces of Seifert fibered homology spheres. *Topology*, 30(1):77–95, 1991.

[27] S. Kobayashi. *Differential Geometry of Complex Vector Bundles*. Iwanami Shoten, Publishers and Princeton University Press, 1987.

[28] M. Krebs. *Toledo invariants on 2-orbifolds*. PhD thesis, Johns Hopkins University, 2005.

[29] E. Markman and E. Xia. The moduli of flat PU(p, p)-structures with large Toledo invariants. *Math. Z.*, 240(1):95–109, 2002.

[30] M. Narasimhan and C. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.

[31] P. Orlik. *Seifert manifolds*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 291.

[32] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[33] C. Simpson. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *Journal of the American Mathematical Society*, 1(4):867–918, 1988.

[34] C. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.

[35] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. *Inst. Hautes Études Sci. Publ. Math.*, (80):5–79 (1995), 1994.

[36] N. Steenrod. *The topology of fibre bundles*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.

[37] W. Thurston. *The geometry and topology of 3-manifolds*. Princeton University Notes, 1988.

[38] D. Toledo. Harmonic maps from surfaces to certain Kaehler manifolds. *Math. Scand.*, 45(1):13–26, 1979.

[39] D. Toledo. Representations of surface groups in complex hyperbolic space. *J. Differential Geom.*, 29(1):125–133, 1989.
[40] E. Xia. The moduli of flat $PU(2,1)$ structures on Riemann surfaces. \textit{Pacific Journal of Mathematics}, 195(1):231–256, 2000.

[41] E. Xia. The moduli of flat $U(p,1)$ structures on Riemann surfaces. \textit{Geom. Dedicata}, 97:33–43, 2003.

Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999).