Stress state analysis of a semi-infinite region with a hole by the combined method

N V Chernysheva and B E Melnikov
Peter the Great St. Petersburg Polytechnic University, 29 Polytechnicheskaya, St.Petersburg, 195251, Russia
E-mail: chernysheva_nv@spbstu.ru

Abstract. A computational algorithm based on the FEM and Somigliana’s integral formula for the boundary problem in 3D medium is considered. The algorithm is modified for the case of inhomogeneous medium. Some practical boundary problems of geological and mechanical earth models are solved by algorithms discussed.

1. Introduction
Computational analysis of three-dimensional exterior boundary problem is a matter of great practical importance. Using of the finite element method (FEM) has some peculiar properties for the second boundary problem (i.e. boundary problem with force boundary condition) as well as for the infinite domain. These peculiarities cause some computational difficulties, for example see Refs. [1–3]. Various algorithms offer to avoid such problems due to combining different methods and equations, for example see Refs. [4–6]. An algorithm dealt with matching of FEM and Somigliana’s integral formula – combined method (CM) – is considered.

2. Sample and experiment
Efficiency gains for the difficult to recover reserves development is involved with accuracy of geological material mechanics properties measurement. The relevant practical objective is oil-and-gas parent deep-seated rock effective mechanics properties measuring according to the data of mechanical laboratory testing of core sample (see figure 1) pulled out of underground hole. Computed simulation model for the multilevel sequential (iteratively refined) analysis has to be rather elaborated and adequate to satisfy the design requirements of problems in question.

The influence of interaction stresses between a core sample and surrounding massive (see figure 2) on the sample texture should be allowed and is of independent practical interest. These stresses are also considered as acting on the hole from core sample and can be worked out iteratively by solution sequence of stress-strain analysis.
3. Materials and Methods

The method for setting up an exterior boundary problem in the semi-infinite space domain $\Omega$ bounded by a surface $S$ in the absence of the volume forces is as follows [7]: the displacement vector $u_\xi = (u_1(\xi), u_2(\xi), u_3(\xi))^T$ at arbitrary point $\xi \in \Omega$ should satisfy the Navier’s equilibrium equations and force boundary conditions:

$$t_i(\xi) = \sigma_{ij}(\xi)n_i(\xi) = p_i(\xi), \quad \xi \in S.$$  \hspace{1cm} (1)

Here $\sigma_{ij}(\xi)$ are the components of stress tensor, $p_i(\xi)$ is given components of the force vector, $n_i(\xi)$ are the components of the unit surface normal.

The solution algorithm of the combined method (CM for the problem formulated is as follows. Let a sub-space $\Omega_0$ bounded by a surface $S_0$ is selected in the semi-infinite space $\Omega$ (see figure 4). Boundary condition $u_i^{(0)} = 0$ at the surface $S_0$ is initially established. Then the boundary problem in the sub-space $\Omega_0$ can be solved by FEM to establish displacement boundary conditions (Eq. 1) on the surface $S$. This allows to determine the first approximation $u_i^{(1)}$ using the Somigliana’s integral formula:

$$u_i^{(1)}(\xi) = \int_S (t_i(\eta) G_{ij}(\eta, \xi) - F_j(\eta, \xi) u_i(\eta)) dS(\eta),$$  \hspace{1cm} (2)

where $G_{ij}(\eta, \xi)$ and $F_j(\eta, \xi)$ are the fundamental solution of Navier’s equilibrium equations.

Thus the $(k-1)$-th approximation $u_i^{(k-1)} = u_i^{(k-1)}(\xi), \quad \xi \in S_0$ is a boundary condition for the boundary problem in the sub-space $\Omega_0$. This problem can be solved by the FEM analysis, and then the $k$-th approximation $u_i^{(k)}(\xi), \xi \in S_0$ can be found by applying the formula (Eq. 2). The process should be...
ongoing until the difference between the conterminous approximations $u_i^{(k)}(\xi)$ and $u_i^{(k+1)}(\xi)$ ($\xi \in S$) achieves the required accuracy.

The CM algorithm also can be applied for the inhomogeneous medium problem by essential modification [8]. If all the inhomogeneous peculiarities having a substantial influence on the strain-stress state of the structure can be encase in an outer limited surface $L$. The remaining past of space $\Omega_L$ is considered as a homogeneous one. A surface $S_0$ constrains the space $\Omega_L$ (see figure 4) and a closed sub-space $\Omega_0$ between the surfaces $S$ and constrain $S_0$ is considered.

Initially on the surface $S_0$ one prescribes boundary condition $u_i^{(0)} = 0$. Then boundary condition (Eq. 1) on the surface $L$ is obtained by the FEM analysis of the stress state in the sub-space $\Omega_0$. Integration in Eq. 2 is performed on the surface $L$ constraining homogeneous region $\Omega_L$, instead of the surface $S$. Then the boundary problem in the sub-space $\Omega_0$ is solved by FEM to obtain the next boundary condition on the surface $L$ and so far. It is a double-boundary algorithm of combined method (DCM).

![Figure 3](image1.png)  ![Figure 4](image2.png)

**Figure 3.** Selection of analyzed domain in a space $\Omega$

**Figure 4.** Scheme of the inhomogeneous space selection for analysis

### 4. Results and discussions

The software developed based on implementation of the CM algorithm has been tested in a series of the model problems having an exact solution. These were external boundary problems with self-balanced and non-self-balanced external loads applied to a sphere-shaped concavity. The DCM algorithm has also been tested in stress-strain analysis of sphere-shaped cavity surrounded by a spherical layer under the internal uniform pressure. These problems were also analyzed by the FEM and results are compared with the results by CM and DCM obtained as well as with the exact solution.

The stress-strain state of homogeneous massive with a cylindrical hole is analyzed by the algorithm at issue in view of internal pressure. The value of pressure $q_0$ is worked out in keeping with damage accumulation in the core sample. The damage parameter of core rock corresponds with the subsurface pressure value. Natural state of stress in a rock mass simulated by the principle of so-called removed stresses. To determine such a stresses the stress state of the massive without hole under gravity loading is analyzed and stresses on the proposed surface of hole are obtained. Initially this pressure of $q_0$ is exerted to the undamaged core sample model and then its value is corrected iteratively. In the second and following steps displacement boundary conditions received on the previous step by CM on the
hole surface in massive are established on the outer surface of the core sample, and the damage parameter of core rock is changed in keeping with the instantaneous value of pressure \( q_0 \). The steady-state value of \( q_0 \) obtained by this process is within the 15% of the initial value.

The DCM algorithm was applied to investigate a rock massive inhomogeneity which consists in existence of two layers with different Young's modulus \( E_1 \) and \( E_2 \) which perhaps caused by different fracture density. Young’s modulus of the homogeneous sub-space \( \Omega_L \) was accepted by an average \( E = (E_1 + E_2) / 2 \) (see figure 5), the core sample also supposed as the homogeneous one. The dependences of results on the domain \( \Omega_0 \setminus \Omega_L \) diameter \( d_L \) and ratio of Young's modulus \( E_1 / E_2 \) were investigated. As it is shown in figure 6, the steady-state value of \( q \) for the inhomogeneous massive differs from the steady-state value \( q_0 \) for the homogeneous one more for small values of \( E_1 / E_2 \). Diameter \( d_L \) does not influenced on the results from \( d_L = 3d \).

![Figure 5. Underground hole in inhomogeneous space](image)

![Figure 6. Ratio of the steady-state value of \( q \) in the inhomogeneous massive to the steady-state value \( q_0 \) in the homogeneous one](image)

5. Conclusion
Some geological and mechanical earth models were analyzed by the CM and DCM. Both algorithms can be used for design of such computed simulation model for multilevel (sequential) analysis.

Acknowledgments
This work was supported by RFBR according to the research project 19-08-01241.

References
[1] Shabana A 2008 Computational continuum mechanics (Cambridge, Cambridge University Press)
[2] Wohlmuth B 2001 Discretization Methods and Iterative Solvers Based on Domain Decomposition (Springer)
[3] Zienkiewicz O and Taylor R 2000 The Finite Element Method, 5th ed. (Butterworth-Heinemann)
[4] Semenov A and Melnikov B 2016 Procedia Engineering. 165 1748-1756
[5] Le-Zakharov S, Melnikov B and Semenov A 2017 Mater. Phys. and Mech. 31 1-2 32-35
[6] Rudaev Y and Kitaeva D 2018 Journal of Physics: Conference Series 1141(1) 012074
[7] Chernysheva N, Kolosova G and Rozin L 2016 Magazine of Civil Engineering 2 83-91
[8] Chernysheva N and Rozin L 2016 MATEC Web of Conferences 53 01042