On tau functions for orthogonal polynomials and matrix models

Gordon Blower
Department of Mathematics and Statistics
Lancaster University
Lancaster, LA1 4YF
England
g.blower@lancaster.ac.uk
13th August 2010

Abstract. Let \( v \) be a real polynomial of even degree, and let \( \rho \) be the equilibrium probability measure for \( v \) with support \( S \); so that, \( v(x) \geq 2 \int \log |x-y| \rho(dy) + C_v \) for some constant \( C_v \) with equality on \( S \). Then \( S \) is the union of finitely many bounded intervals with endpoints \( \delta_j \), and \( \rho \) is given by an algebraic weight \( w(x) \) on \( S \). The system of orthogonal polynomials for \( w \) gives rise to the Magnus–Schlesinger differential equations. This paper identifies the \( \tau \) function of this system with the Hankel determinant \( \det \left[ \int x^{j+k} \rho(dx) \right]_{j,k=0}^{n-1} \) of \( \rho \). The solutions of the Magnus–Schlesinger equations are realised by a linear system, which is used to compute the tau function in terms of a Gelfand–Levitan equation. The tau function is associated with a potential \( q \) and a scattering problem for the Schrödinger operator with potential \( q \). For some algebro-geometric \( q \), the paper solves the scattering problem in terms of linear systems. The theory extends naturally to elliptic curves and resolves the case where \( S \) has exactly two intervals.

MSC (2000) classification: 60B20 (37K15)

Keywords: Random matrices, Scattering theory

1. Introduction

This paper concerns systems of orthogonal polynomials that arise in random matrix theory, specifically in the theory of the generalized unitary ensemble [26], and may be described in terms of electrostatics. We consider a unit of charge to be distributed along an infinite conducting wire in the presence of an electrical field. The field is represented by a real polynomial \( v(x) = \sum_{j=0}^{2N} a_j x^j \) such that \( a_{2N} > 0 \), while the charge is represented by a Radon probability measure on the real line.

Boutet de Monvel et al [5, 29] prove the existence of the equilibrium distribution \( \rho \) that minimises the electrostatic energy. Under general conditions which include the above \( v \), they prove that there exists a constant \( C_v \) such that

\[
v(x) \geq 2 \int_S \log |x-y| \rho(dy) + C_v \quad (x \in \mathbb{R})
\] (1.1)
and that equality holds if and only if $x$ belongs to a compact set $S$. Furthermore, there exists $g \geq 0$ and

$$-\infty < \delta_1 < \delta_2 \leq \delta_3 < \ldots < \delta_{2g+2} < \infty$$

such that

$$S = \bigcup_{j=1}^{g+1} [\delta_{2j-1}, \delta_{2j}]$$

(1.3)

It is a tricky problem, to find $S$ for a given $v$, and [10, Theorem 1.46 and p. 408] contains some significant results including the bound $g + 1 \leq N + 1$ on the number of intervals. When $v$ is convex, a relatively simple argument shows that $g = 0$, so there is a single interval [21].

**Definition.** The $n^{th}$ order Hankel determinant for $\rho$ is

$$D_n = \det \left[ \int_S x^{j+k} \rho(dx) \right]_{j,k=0}^{n-1}.$$  

(1.4)

In section 3 we introduce the system of orthogonal polynomials for $\rho$, and in section 4, we regard $D_n$ as a function of $\delta = (\delta_1, \ldots, \delta_{2g+2})$, and derive a system for differential equations for $\log D_n$, known as Schlesinger’s equations. Let $A(z)$ be a proper rational $2 \times 2$ matrix function with simple poles at $\delta_j$; let $\alpha_j$ be the residue at $\delta_j$, and suppose that the eigenvalues of $\alpha_j$ are distinct modulo the integers for $j = 1, \ldots, M$. Consider the differential equation

$$\frac{d}{dz} \Phi = A(z) \Phi(z),$$

(1.5)

and introduce the 1-form

$$\Omega(\delta) = \frac{1}{2} \sum_{j=1}^{M} \text{trace} \text{ residue}(A(z)^2 : z = \delta_j) d\delta_j$$

(1.6)

to describe its deformations. Then $\Omega$ turns out to be closed by [19].

**Definition.** (i) The tau function of the deformation equations associated with (1.5) is $\tau : \mathbb{C}^M \setminus \{\text{diagonals}\} \to \mathbb{C}$ such that $d \log \tau = \Omega$.

(ii) Given a self-adjoint and trace-class operator $K : L^2(\rho) \to L^2(\rho)$ such that $0 \leq K \leq I$, and $P_{(t, \infty)}$ the orthogonal projection $f \mapsto I_{[t, \infty)} f$, the tau function of $K$ is $\tau(t) = \det(I - P_{[t, \infty)} K)$. (The definitions in (i) and (ii) are reconciled by Proposition 3.3.)

In section 5 use the results from preliminary sections to prove that $D_n$ gives the appropriate $\tau$ function for Schlesinger’s equations. As an illustration which is of importance in random matrix theory, we calculate the tau function explicitly when $\rho$ is the semicircular
law. When $S$ is the union of two intervals, the Schlesinger equations reduce to the Painlevé VI equation, as we discuss on section 7. Okamoto derived $\tau$ functions for other Painlevé equations in [27]. See also [15] and [2].

In sections 2, 3 and 4, we develop standard arguments, then our analysis follows that of Chen and Its [8], who considered the $\rho$ that is analogous to the Chebyshev distribution on multiple intervals. Chen and Its found their tau function explicitly in terms of theta functions on a hyperelliptic Riemann surface; in this paper, we show by means of scattering theory why such solutions emerge.

In terms of definition (ii), we have a real $\tau(x)$ which is associated with a compactly supported real potential $q(x) = -2 \frac{d^2}{dx^2} \log \tau(2x)$ and hence the Schrödinger differential operator $-\frac{d^2}{dx^2} + q(x)$. One associates with each smooth $q$ a scattering function $\phi$; then one analyses the spectral data of $-\frac{d^2}{dx^2} + q(x)$ in terms of $\phi$, with a view to recovering $q$. The Gelfand–Levitan integral equation links $\phi$ with $q$.

In random matrix theory, tau functions are introduced alongside integrable kernels that describe the distribution of eigenvalues of random matrices, especially the generalized unitary ensemble; see [14, 26, 31]. Let $X$ be a $n \times n$ complex Hermitian matrix, let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n)$ be the corresponding eigenvalues, listed according to multiplicity, and consider the potential $V(X) = n^{-1} \sum_{j=1}^{n} v(\lambda_j)$. Now let $dX$ be the product of Lebesgue measure on the entries that are on or above the leading diagonal of $X$; then there exists $0 < Z_n < \infty$ such that

$$\nu_n^{(2)}(dX) = Z_n^{-1} \exp(-n^2 V(X))dX$$

defines a probability measure on the $n \times n$ complex Hermitian matrices. There is a natural action of the unitary group $U(n)$ on $M_n$ given by $(U, X) \mapsto UXU^\dagger$, which leaves $\nu_n^{(2)}$ invariant. Hence $\nu_n^{(2)}$ is the generalized unitary ensemble with potential $v$.

There exists a constant $\zeta_n$ such that $\rho_n(dx) = \zeta_n^{-1} e^{-n v(x)} dx$ defines a probability measure on $\mathbb{R}$; then we let $E_k^{\rho_n}$ be the orthogonal projection onto span$\{x^j : j = 0, \ldots, k-1\}$ in $L^2(\rho_n)$. The eigenvalue distribution satisfies

$$\int \frac{1}{n} \mathbb{1}_{\{j : \lambda_j(X) \leq t\}} \nu_n^{(2)}(dX) = \det(I - I_{(t, \infty)} E_k^{\rho_n})$$

(1.8)

where the right-hand side can be expressed in terms of Hankel determinants. For large $n$, most of the eigenvalues actually lie in $S$ by results of [5, 26]. Moreover, there exists a trace-class operator $K$ on $L^2(\rho)$ such that $0 \leq K \leq I$ and

$$\det(I - I_{(t, \infty)} E_k^{\rho_n})) \to \det(I - I_{(t, \infty)} K) \quad (n \to \infty);$$

(1.9)
call this limit $\tau(t)$. Tracy and Widom [31] showed how to express such determinants in terms of systems of differential equations and Hankel operators.

We introduce the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and apply a simple gauge transformation to (1.5). Then for a sequence of real symmetric $2 \times 2$ matrices $J\beta_k(n)$, we consider solutions of the differential equation

$$J \frac{dZ}{dx} = \sum_{k=1}^{2g+2} \frac{J\beta_k(n)}{x - \delta_k} Z,$$

$$Z(x) \to 0 \quad (x \to \delta_j),$$

and form the kernel

$$K(x, y) = \frac{Z(y)^\dagger JZ(x)}{y - x}.$$  \hfill (1.11)

We show that the properties of $K$ depend crucially upon the sequence of signatures of the matrices $(\delta_j - \delta_k)J\beta_k(n)$. In Theorem 8.3, we introduce a symbol function $\phi$ from $Z$, a constant signature matrix $\sigma$ and a Hankel operator $\Gamma_{\phi}$ such that

$$K = \Gamma_{\phi}^\dagger \sigma \Gamma_{\phi}.$$  \hfill (1.12)

In section 9 we introduce $\phi$ from (1.11), express $\phi$ in terms of a linear system as in [4] and hence obtain a matrix Hamiltonian $H(x)$ such that

$$\tau(2x) = \exp\left( - \int_x^{\infty} \text{trace} \ H(u) \, du \right),$$

and prove that $q(x)$ is meromorphic on a region. We regard $-\frac{d^2}{dx^2} + q$ as integrable if $-f'' + qf = \lambda f$ can be solved by quadratures for typical $\lambda$. This imposes severe restrictions upon $q$; indeed, Gelfand, Dikij and Its [12, 6] showed that the integrable cases arise from finite-dimensional Hamiltonian systems. In sections 10, 11 and 12 we consider cases in which $-f'' + qf = \lambda f$ has a meromorphic general solution for all $\lambda$, and $q$ satisfies one of the following conditions:

(i) $q$ is rational and bounded at infinity;

(ii) $q$ is of rational character on an elliptic curve;

(iii) $q$ is the restriction of an abelian function to a straight line in the Jacobian of a hyperelliptic Riemann surface.

In cases (ii) and (iii), the corresponding Schrödinger equation has a spectrum with only finitely many gaps. In (i) and (ii), we introduce a linear system $(-A, B, C)$ so as to realise $\phi(x) = Ce^{-xA}B$, and use the operators $A, B, C$ to solve the Gelfand–Levitan
equation. Thus we obtain explicit expressions for $\phi$ and $\tau$. In (iii), we can do likewise under further hypotheses.

2. The equilibrium measure

Given the special form of the potential, the equilibrium measure and its support satisfy special properties. To describe these, we introduce the polynomial $u$ of degree $2N - 2$ by

$$u(z) = \int_S \frac{v'(z) - v'(x)}{z - x} \rho(dx)$$

and the Cauchy transform of $\rho$ by

$$R(z) = \int_S \frac{\rho(dx)}{x - z} \quad (z \in \mathbb{C} \setminus S)$$

and the weight

$$w(x) = 2Na_2N \left( -Q(x) \prod_{j=1}^{2g+2} (x - \delta_{2j-1})(x - \delta_{2j}) \right)^{1/2}$$

where $Q(x)$ is a product of monic irreducible quadratic factors such that $w(x)^2 = 4u(x) - v'(x)^2$.

**Proposition 2.1** (i) The Cauchy transform is the algebraic function that satisfies

$$R(z)^2 + v'(z)R(z) + u(z) = 0$$

and $R(z) \to 0$ as $z \to \infty$. There exist nonzero polynomials $u_0, u_1, u_2$ such that $u_0R' = u_1R + u_2$.

(ii) The support of $\rho$ is

$$S = \{ x \in \mathbb{R} : 4u(x) - v'(x)^2 \geq 0 \}.$$  

(iii) $\rho$ is absolutely continuous and the Radon–Nikodym derivative satisfies

$$\frac{d\rho}{dx} = \frac{1}{2\pi} \mathbf{1}_S(x)w(x)$$

where $2\pi = \int_S w(t)dt$ and $w(x) \to 0$ as $x$ tends to an endpoint of $S$.

**Proof.** (i) The quadratic equation is due to Bessis, Itzykson and Zuber, and is proved in the required form in [28]. One can easily deduce that $R$ satisfies a first-order linear differential equation with polynomial coefficients.
(ii) Pastur [28] shows that the support is those real $x$ such that

$$|v'(x) + \sqrt{v'(x)^2 - 4u(x)}|^2 = 4u(x),$$  \hspace{1cm} (2.7)

and this condition reduces to $4u(x) \geq v'(x)^2$ and $u(x) \geq 0$, where the former inequality implies the latter. The polynomial $4u(x) - v'(x)^2$ has real zeros $\delta_1, \ldots, \delta_{2g+2}$, and may additionally have pairs of complex conjugate roots, which we list as $\delta_{2g+3}, \ldots, \delta_{4N-2}$ with regard to multiplicity. Hence we can introduce $w$ as above such that $4u(x) - v'(x)^2 = w(x)^2$.

(iii) From (i) we deduce that

$$R(\lambda) = \frac{1}{2\pi i} \left( \int_S \frac{\sqrt{4u(t) - v'(t)^2}}{t - \lambda} \, dt \right)$$

since both sides are holomorphic on $\mathbb{C} \setminus S$, vanish at infinity and have the same jump across $S$. By Plemelj’s formula, we deduce that

$$v'(\lambda) = 2\text{p.v.} \int_S \frac{\sqrt{4u(t) - v'(t)^2}}{(\lambda - t)2\pi} \, dt \quad (\lambda \in S).$$  \hspace{1cm} (2.9)

See [28, 5, 29]. This gives the required expression for $\rho$.

\[\square\]

3. Orthogonal polynomials

First we introduce orthogonal polynomials for $\rho$, then the corresponding differential equations. Let $(p_j)_{j=0}^\infty$ be the sequence of monic orthogonal polynomials in $L^2(\rho)$, where $p_j$ has degree $j$ and let $h_j$ be the constants such that

$$\int_S p_j(x)p_k(x)\rho(dx) = h_j \delta_{jk};$$  \hspace{1cm} (3.1)

and let $(q_j)_{j=1}^\infty$ be the monic polynomials of the second kind, where

$$q_j(z) = \int_S \frac{p_j(z) - p_j(x)}{z - x} \rho(dx)$$  \hspace{1cm} (3.2)

has degree $j - 1$. On account of Proposition 2.1, the orthogonal polynomials are semi classical in Magnus’s sense [22], although the weight typically lives on several intervals. The following result is standard in the theory of orthogonal polynomials; see [8].

**Lemma 3.1** Let $c_n = h_n/h_{n-1}$ and $b_n = h_n^{-1} \int_S x p_n(x)^2 \rho(dx)$. Then

(i) the polynomials $(p_n)_{n=0}^\infty$ satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_{n+1}p_n(x) + c_n p_{n-1}(x);$$  \hspace{1cm} (3.3)
(ii) the polynomials \((q_j)_{j=1}^\infty\) likewise satisfy (3.3);
(iii) the Hankel determinant of (1.4) satisfies

\[
D_n = h_0 h_1 \ldots h_{n-1}. \tag{3.4}
\]

We introduce also

\[
Y_n(z) = \begin{bmatrix}
\frac{p_n(z)}{h_{n-1}} & \frac{\int_S \frac{p_n(t)\rho(dt)}{z-t}}{h_{n-1}} \\
\frac{\int_S \frac{p_{n-1}(t)\rho(dt)}{z-t}}{h_{n-1}} & \frac{h_{n-1}}{h_{n-1}}
\end{bmatrix} \tag{3.5}
\]

and

\[
V_n(z) = \begin{bmatrix}
z - b_{n+1} & -h_n \\
1/h_n & 0
\end{bmatrix}. \tag{3.6}
\]

**Proposition 3.2** (i) The matrices satisfy the recurrence relation

\[
Y_{n+1}(z) = V_n(z)Y_n(z). \tag{3.7}
\]

(ii) The matrix \(Y_n(z)\) is invertible, and \(\det Y_n(z) = 1\).

**Proof.** (i) This follows from (i) and (ii) of the Lemma 3.1.

(ii) This follows by induction, where the induction step follows from the recurrence relation in (i).

\[
\square
\]

We restrict \(\rho\) restrict to \((-\infty, t) \cap S\) and let

\[
\mu_j(t) = \int_{S \cap (-\infty, t)} x^j \rho(dx) \tag{3.8}
\]

be the \(j^{th}\) moment; the corresponding Hankel determinant is

\[
D_{n+1}(t) = \det[\mu_{j+k}(t)]_{j,k=0}^n. \tag{3.9}
\]

Let \(E_n : \mathcal{L}_2(\rho) \rightarrow \text{span}\{x^k : k = 0, \ldots, n-1\}\) be the orthogonal projection; we also introduce the projection \(P_{(t,\infty)}\) on \(\mathcal{L}_2(\rho)\) given by multiplication \(f \mapsto I_{(t,\infty)} f\), where \(I_{(t,\infty)}\) denotes the indicator function of \((t, \infty)\).

**Proposition 3.3** The tau function of \(E_{n+1}\) satisfies

\[
\det(I - E_{n+1}P_{(t,\infty)}) = \frac{D_{n+1}(t)}{D_{n+1}}. \tag{3.10}
\]
**Proof.** We introduce an upper triangular matrix \([a_{\ell,j}]_{j,\ell=0}^n\) with ones on the leading diagonal such that \(p_j(x) = \sum_{\ell=0}^n a_{\ell,j} x^\ell\). Then we can compute

\[
\det[\mu_{j+k}(t)]_{j,k=0}^n = \det[a_{\ell,j}]_{j,\ell=0}^n \det[\mu_{j+k}(t)]_{j,k=0}^n \det[a_{k,m}]_{k,m=0}^n \\
= \det \left[ \int_{-\infty}^t p_j(x)p_k(x)\rho(dx) \right]_{j,k=0}^n \tag{3.11}
\]

We can also express the operators on \(L^2(\rho)\) as matrices with respect to the orthonormal basis \((p_j/\sqrt{h_j})_{j=0}^n\), and we find

\[
E_{n+1} - E_{n+1}P_{(t,\infty)}E_{n+1} \leftrightarrow \left[ \frac{1}{\sqrt{h_j h_k}} \int_{-\infty}^t p_j(z)p_k(z)\rho(dz) \right]_{j,k=0}^n \tag{3.12}
\]

so that

\[
\det \left[ \int_{-\infty}^t p_j(x)p_k(x)\rho(dx) \right]_{j,k=0}^n = \det(E_{n+1} - E_{n+1}P_{(t,\infty)}E_{n+1}) h_0 \ldots h_n. \tag{3.13}
\]

We deduce that

\[
\det[\mu_{j+k}(t)]_{j,k=0}^n = \det(E_{n+1} - E_{n+1}P_{(t,\infty)}E_{n+1}) D_{n+1}. \tag{3.14}
\]  

4. **Schlesinger’s equations and recurrence relations**

Invoking Proposition 3.2(ii), we introduce the matrix function

\[
A_n(z) = Y_n'(z)Y_n(z)^{-1} + Y_n(z) \begin{bmatrix} 0 & 0 \\ 0 & -w'(z)/w(z) \end{bmatrix} \begin{bmatrix} 0 \\ w(z) \end{bmatrix} \tag{4.1}
\]

The basic properties of \(A_n(z)\) are stated in (i) of the following Lemma, while (ii) gives detailed information that we need in the subsequent proof of Theorem 5.1.

**Lemma 4.1** (i) Let \(v'(z)^2 - 4u(z)\) have zeros at \(\delta_j\) for \(j = 1, \ldots, 4N - 2\). Then \(A_n(z)\) is a proper rational function so that

\[
A_n(z) = \sum_{j=1}^{4N-2} \frac{\alpha_j(n)}{z - \delta_j}, \tag{4.2}
\]

where the residue matrices \(\alpha_j(n)\) depend implicitly upon \(\delta\).
(ii) The $(1, 2)$ and diagonal entries of the residue matrices satisfy

\[
\begin{align*}
\sum_{k=1}^{4N-2} \alpha_k(n)_{12} &= 0; \quad (4.3) \\
\sum_{k=1}^{4N-2} (\alpha_k(n)_{11} - \alpha_k(n)_{22}) &= 2(n + N) - 1; \quad (4.4) \\
\sum_{k=1}^{4N-2} \delta_k \alpha_k(n)_{12} &= -2h_n(n + N). \quad (4.5)
\end{align*}
\]

**Proof.** (i) The defining equation (4.1) for $A_n(z)$ may be written more explicitly as

\[
A_n(z) = \begin{bmatrix}
p_n(z) & \int_S p_n(t)w(t)dt/z-t \\
p_n'(z) & -\int_S p_n(t)w(t)dt/(z-t)^2
\end{bmatrix}
= \begin{bmatrix}
p_n(z) & \int_S p_n(t)w(t)dt/z-t \\
p_n'(z) & -\int_S p_n(t)w(t)dt/(z-t)^2
\end{bmatrix} + \begin{bmatrix}
0 & \int_S p_n(t)w(t)dt/z-t \\
0 & \int_S p_n(t)w(t)dt/(z-t)^2
\end{bmatrix}
\]

(iii) First we compute the $(1, 2)$ entry of $A_n(z)$, namely

\[
A_n(z)_{12} = -p_n'(z) \int_S p_n(t)w(t)dt/z-t - p_n(z) \int_S p_n(t)w(t)dt/(z-t)^2 - \frac{w'(z)}{w(z)} \int_S p_n(t)w(t)dt/z-t
\]

\[
= -\frac{n}{z^2} \int_S t^n p_n(t)w(t)dt - \frac{(n+1)p_n(z)}{z^{n+2}} \int_S t^n p_n(t)w(t)dt - \frac{w'(z)}{w(z)} \frac{p_n(z)}{z^{n+1}} \int_S t^n p_n(t)w(t)dt + O\left(\frac{1}{z^3}\right)
\]

and we can reduce these terms to

\[
A_n(z)_{12} = -\frac{h_n n}{z^2} - \frac{h_n (n+1)}{z^2} - \frac{h_n (2N-1)}{z^2} + O\left(\frac{1}{z^3}\right),
\]

which gives (4.3) and (4.5).
Next, the \((2,2)\) entry of \(A_n(z)\) is

\[
A_n(z)_{22} = -\frac{p'_n(z)}{h_{n-1}} \int_S \frac{p_n(t)w(t)dt}{z-t} - \frac{p_n(z)}{h_{n-1}} \int_S \frac{p_{n-1}(t)w(t)dt}{(z-t)^2} - \frac{w'(z)p_n(z)}{w(z)h_{n-1}} \int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \\
= -\frac{(n-1)z^{n-2}}{h_{n-1}z^{n+1}} \int_S t^n p_n(t)w(t)dt - \frac{p_n(z)}{h_{n-1}z^{n+1}} \int_S nt^{n-1}p_{n-1}(t)w(t)dt \\
= -\frac{2N-1}{z} \frac{p_n(z)}{h_{n-1}z^n} \int_S t^{n-1}p_{n-1}(t)w(t)dt + O\left(\frac{1}{z^2}\right) + O\left(\frac{1}{z^2}\right) \\
= \frac{1}{z} - n - 2N + O\left(\frac{1}{z^2}\right) \\
\tag{4.10}
\]

Similarly, the \((1,1)\) entry is

\[
A_n(z)_{11} = -\frac{p'_n(z)}{h_{n-1}} \int_S \frac{p_n(t)w(t)dt}{z-t} - \frac{p_n(z)}{h_{n-1}} \int_S \frac{p_{n-1}(t)w(t)dt}{(z-t)^2} - \frac{w'(z)p_n(z)}{w(z)h_{n-1}} \int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \\
= -\frac{p'_n(z)}{h_{n-1}z^n} \int_S t^{n-1}p_n(t)w(t)dt - \frac{p_n(z)}{h_{n-1}z^{n+1}} \int_S nt^{n-1}p_{n-1}(t)w(t)dt \\
= -\frac{2N-1}{z} \frac{p_n(z)}{h_{n-1}z^n} \int_S t^{n-1}p_{n-1}(t)w(t)dt + O\left(\frac{1}{z^2}\right) \\
= \frac{n}{z} + O\left(\frac{1}{z^2}\right) \quad (z \to \infty). \\
\tag{4.11}
\]

By comparing the coefficients of \(1/z\) in \((4.7)\) with \((4.9), (4.10)\) and \((4.11)\), we obtain

\[
\sum_{k=1}^{4N-2} \alpha_k(n) = \begin{bmatrix}
    n & 0 \\
    0 & 1 - n - 2N
\end{bmatrix}, \\
\tag{4.12}
\]

which leads to \((4.4)\).

\[\square\]

Let

\[
\Phi_n(z) = \begin{bmatrix}
\sqrt{2\pi i}p_n(z) & -\frac{i\pi w(z)p_n(z)+q_n(z)}{w(z)\sqrt{2\pi i}} \\
\sqrt{2\pi i}p_{n-1}(z) & -\frac{i\pi w(z)p_{n-1}(z)+q_{n-1}(z)}{w(z)p_{n-1}(z)\sqrt{2\pi i}}
\end{bmatrix}, \\
\tag{4.13}
\]

which is a matrix function with entries in \(\mathbb{C}[z][w]\); note that \(\Phi_n\) also depends upon the \(\delta_j\).

**Lemma 4.2** The functions \(\Phi_n\) satisfy

(i) the basic differential equation

\[
\frac{d\Phi_n(z)}{dz} = A_n(z)\Phi_n(z), \\
\tag{4.14}
\]

10
(ii) the deformation equation
\[ \frac{\partial \Phi_n}{\partial \delta_j} = -\frac{\alpha_j(n)}{z - \delta_j} \Phi_n(z), \quad (4.15) \]

(iii) and the recurrence relation \( \Phi_{n+1}(z) = V_n(z)\Phi_n(z); \)
(iv) moreover, \( \Phi_n \) is invertible since \( \det \Phi_n(z) = 1/w(z). \)

**Proof.** (i) We can write
\[ \Phi_n(z) = Y_n(z) \begin{bmatrix} \sqrt{2\pi i} & 0 \\ 0 & 1/(w(z)\sqrt{2\pi i}) \end{bmatrix}, \quad (4.16) \]
and then the property (i) follows from (4.1).

(ii) This follows from (i) by standard results in the theory of Fuchsian differential equations as in [14, 16].

(iii) The recurrence relation from Proposition 3.2(i).

(iv) Given (iii), this identity follows from Proposition 3.2(ii).

Lemma 4.2 states several properties that the \( \Phi_n \) satisfy simultaneously, and hence generates several consistency conditions. By taking (i), (ii) and (ii i) pairwise, we obtain three Lax pairs, which we state in the following three propositions.

**Proposition 4.3** The residue matrices satisfy Schlesinger’s equations
\[ \frac{\partial \alpha_k(n)}{\partial \delta_j} = \frac{[\alpha_j(n), \alpha_k(n)]}{\delta_j - \delta_k} \quad (j \neq k) \quad (4.17) \]

and
\[ \frac{\partial \alpha_j(n)}{\partial \delta_j} = -\sum_{k=1; j \neq k}^{4N-2} \frac{[\alpha_j(n), \alpha_k(n)]}{\delta_j - \delta_k}. \quad (4.18) \]

**Proof.** We can express the consistency condition \( \frac{\partial^2 \Phi_n(z)}{\partial \delta_j \partial z} = \frac{\partial^2 \Phi_n(z)}{\partial z \partial \delta_j} \) as the Lax pair
\[ \frac{\partial A_n(z)}{\partial \delta_j} - A_n(z) \frac{\alpha_j(n)}{z - \delta_j} = \frac{\alpha_j(n)}{(z - \delta_j)^2} - \frac{\alpha_j(n)A_n(z)}{z - \delta_j} \quad (4.19) \]
and then one can simplify the resulting system of differential equations. See [14, 19].

**Proposition 4.4** The basic differential equation (4.14) and the recurrence relation in Lemma 4.2 are consistent, so
\[ A_{n+1}(z)V_n(z) - V_n(z)A_n(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.20) \]
Proof. The Lax pair associated with the these conditions gives

\[ A_{n+1}(z)\Phi_{n+1}(z) = \frac{d}{dz} \Phi_{n+1}(z) = \frac{d}{dz} \left( V_n(z)\Phi_n(z) \right). \tag{4.21} \]

\[ \text{Proposition 4.5 (i) The deformation equation (4.15) and the recurrence relation in Lemma 4.2 are consistent, so} \]

\[-\alpha_j(n+1) V_n(z) + V_n(z) \alpha_j(n) \frac{\alpha_j(n)}{z - \delta_j} = \frac{\partial V_n(z)}{\partial \delta_j}. \tag{4.22} \]

(ii) In particular, the (1, 2) entry satisfies

\[ \frac{\partial}{\partial \delta_j} \log h_n = -h_n^{-1} \alpha_j(n)_{12}. \tag{4.23} \]

Proof. (i) This is the Lax pair associated with Lemma 4.2.

(ii) By letting \( z \to \infty \) in (4.22), we deduce

\[-\alpha_j(n+1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \alpha_j(n) = \begin{bmatrix} -\frac{\partial b_{n+1}}{\partial \delta_j} & -\frac{\partial h_n}{\partial \delta_j} \\ -\frac{1}{h_n} \frac{\partial h_n}{\partial \delta_j} & 0 \end{bmatrix} \]

which implies that \( \alpha_j(n)_{12} = -\frac{\partial h_n}{\partial \delta_j} \).

\[ \text{5. The tau function} \]

We introduce the differential 1-form on \( \mathbb{C}^{4N-2} \setminus \{\text{diagonals}\} \) by

\[ \Omega_n = \sum_{j,k=1; j \neq k}^{4N-2} \text{trace} \left( \frac{\alpha_j(n)\alpha_k(n)}{\delta_j - \delta_k} \right) d\delta_j. \tag{5.1} \]

**Theorem 5.1** The Hankel determinant \( D_n \) gives the tau function, so

\[ \Omega_n = d \log D_n. \tag{5.2} \]

**Proof.** By Proposition 4.3 and results of Jimbo et al [20], \( \Omega_n \) is an exact differential form, so \( d\Omega_n = 0 \); hence there exists a function \( \tau_n \) such that \( d\log \tau_n = \Omega_n \), and so we proceed to identify \( \tau_n \). By Lemma 3.1(iii), we have \( \log h_n = \log D_{n+1}/D_n \), hence we consider

\[ \Omega_{n+1} - \Omega_n = \sum_{j \neq k; j, k = 1}^{4N-2} \text{trace} \left( \frac{\alpha_j(n+1)\alpha_k(n+1) - \alpha_j(n)\alpha_k(n)}{\delta_j - \delta_k} \right) d\delta_j \tag{5.3} \]
where by Proposition 4.5(i) \( \alpha_j(n+1) = V_n(\delta_j)\alpha_j(n)V_n(\delta_j)^{-1} \) so

\[
\text{trace}(\alpha_j(n+1)\alpha_k(n+1) - \alpha_j(n)\alpha_k(n)) = \text{trace}(\alpha_j(n)V_n(\delta_j)^{-1}V_n(\delta_k)\alpha_k(n)V_n(\delta_k)^{-1}V_n(\delta_j) - \alpha_j(n)\alpha_k(n)).
\] (5.4)

We have

\[
V_n(\delta_j)^{-1}V_n(\delta_k) = \begin{bmatrix}
\frac{1}{h_n} & 0 \\
\delta_j - \delta_k & 1
\end{bmatrix}
\] (5.5)

so by direct calculation

\[
\Omega_{n+1} - \Omega_n = \sum_{j \neq k; j, k = 1}^{4N-2} \left\{ h_n^{-1}\alpha_j(n)_{12}(\alpha_k(n)_{11} - \alpha_k(n)_{22}) + h_n^{-1}\alpha_k(n)_{12}(\alpha_j(n)_{22} - \alpha_j(n)_{11}) - h_n^{-2}(\delta_j - \delta_k)\alpha_j(n)_{12}\alpha_k(n)_{12} \right\} d\delta_j
\] (5.6)

In this sum we have taken \( j \neq k \); but the expression is unchanged if we include the corresponding terms for \( j = k \); hence the coefficient of \( d\delta_j \) is

\[
\alpha_j(n)_{12} \sum_{k=1}^{4N-2} 4N-2 h_n^{-1}(\alpha_k(n)_{11} - \alpha_k(n)_{22}) - (\alpha_j(n)_{11} - \alpha_j(n)_{22}) \sum_{k=1}^{4N-2} h_n\alpha_k(n)_{12} - \frac{\delta_j\alpha_j(n)_{12}}{h_n^2} \sum_{k=1}^{4N-2} \alpha_k(n)_{12} + \frac{\alpha_j(n)_{12}}{h_n^2} \sum_{k=1}^{4N-2} \delta_k\alpha_k(n)_{12}.
\] (5.7)

We use Lemma 4.1 to reduce this to \( -h_n^{-1}\alpha_j(n)_{12} \), so

\[
\Omega_{n+1} - \Omega_n = -\sum_{j=1}^{4N-2} h_n^{-1}\alpha_j(n)_{12}d\delta_j
\]

\[
= \sum_{j=1}^{4N-2} \frac{\partial}{\partial \delta_j} \log h_n d\delta_j.
\] (5.8)

Hence \( \Omega_n = d\sum_{k=0}^{n-1} \log h_k. \)

\[ \square \]

Following [19], we interpret (5.1) in terms of integrable systems and Hamiltonian mechanics. Let \( M = M_2(\mathbb{R})^{4N-2} \) be the product space of matrices, and let \( G = GL_2(\mathbb{R}) \) act on \( M \) by conjugating each matrix in the list

\[
(X_1, \ldots, X_n) \mapsto (UX_1U^{-1}, \ldots, U X_n U^{-1}).
\]
The Lie algebra $g$ of $G$ has dual $g^*$, and for each $\xi \in g^*$ the symplectic structure at $\xi$ on $g \times g$ is given by $\omega_\xi(X, Y) = \xi([X, Y])$. Given

$$A(z) = \sum_{k=1}^{2N-2} \frac{\alpha_k}{z - \delta_k}$$

as in (4.2), we introduce

$$\omega(X, Y) = \sum_{k=1}^{2N-2} \text{trace}(\alpha_k[X_k, Y_k])$$

for $X = (X_k)_{k=1}^{2N-2}$ and $Y = (Y_k)_{k=1}^{2N-2}$ in $g^{2N-2}$. Given $f, g : M \to \mathbb{C}$, their Poisson bracket is $\{f, g\} = X_f(g)$, and the corresponding vector field satisfies $X_{\{f, g\}} = [X_f, X_g]$. The spectral curve of $A(z)$ is the algebraic variety

$$\Sigma_A = \left\{(z, w) \in \mathbb{C}^2 : \det (w I - A(z)) = 0 \right\}.$$
With $H_k$, we associate the Hamiltonian vector field $X_{H_j} = (X_{H_j}^{(k)})_{k=1}^{2N-2}$ such that

$$Y(H_j) = \omega(X_{H_j}, Y) = \sum_{k=1}^{2N-2} \text{trace}(\alpha_k[(X_{H_j}^{(k)}, Y_k)]).$$  \tag{5.15}$$

We deduce that

$$X_{H_j}^{(k)} = \frac{\alpha_j}{\delta_j - \delta_k} \quad (k \neq j) \tag{5.16}$$

$$X_{H_j}^{(j)} = \sum_{k: k \neq j} \frac{\alpha_k}{\delta_j - \delta_k}. \tag{5.17}$$

It is then a simple calculation to check that $\dot{\alpha}_k = [X_{H_j}^{(k)}, \alpha_k]$ extends to give (5.13) for $(d/dt)A(z)$.

(ii) Given the vector fields $(X_{H_k}^{(j)})_j$ corresponding to $H_k$ and $(X_{H_\ell}^{(j)})_j$ corresponding to $H_\ell$ from (5.12), one can compute

$$\{H_k, H_\ell\} = \sum_j \text{trace}([X_{H_k}^{(j)}, A_j]X_{H_\ell}^{(j)}) \tag{5.18}$$

and reduce the expression to zero by an elementary calculation.

(iii) One can check that for each positive integer $m$, the $\frac{d}{dt}\text{trace}A(z)^m = 0$, and hence $\det(wI - A(z))$ is invariant under the flow.

\[\square\]

6. Orthogonal polynomials on a single interval

In this section we consider the Chebyshev polynomials.

- First suppose that $v = 0$. Then the corresponding equilibrium distribution is the Chebyshev distribution $(1/\pi)(1-x^2)^{-1/2}$. In this case, the orthogonal polynomials are the Chebyshev polynomials of the first kind and, unsurprisingly, our results reduce to those of Chen and Its [8].

- For $a < b$, let

$$v(z) = \frac{8}{(b-a)^2} \left(z - \frac{a+b}{2}\right)^2, \tag{6.1}$$

so, by standard results used in random matrix theory [26], the equilibrium measure is the semicircular law on $[a, b]$, as given by

$$\rho(dx) = \frac{8}{\pi(b-a)^2} \sqrt{(b-x)(x-a)} I_{[a,b]}(x) dx \tag{6.2}$$
Proposition 6.1 The tau function for the semicircular distribution is

\[ \tau(a, b) = (a - b)^{(2n^2+2n+1)/4}e^{n(n+2)(a-b)^2/32}. \] (6.3)

Proof. Let \( U_n \) be the Chebyshev polynomial of the second kind of degree \( n \), which satisfies

\[ U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \] (6.4)

and let

\[ p_n(x) = 2^{−2n}(b - a)^nU_n\left(\frac{2x -(a + b)}{b - a}\right) \] (6.5)

which is monic and of degree \( n \), and the \( p_n \) are orthogonal with respect to the measure \( \rho \). By elementary calculations involving trigonometric functions, one can show that \( h_n = 2^{−4n}(b - a)^{2n} \) and

\[ A_n(x) = \frac{1}{(x-b)(x-a)} \left[ \frac{n(x -(a + b)/2) - (n + 1)(b - a)^2h_{n-1}/8}{n(b-a)^2/(2h_{n-1}) - (n + 1)(x -(a + b)/2)} \right], \] (6.6)

which has poles at \( a \) and \( b \), as expected. One verifies that

\[ \Omega_n = \left(\frac{n^2 + (n + 1)^2}{4}\right)\frac{da - db}{a - b} + \frac{n(n + 2)}{16}(a - b)(da - db) \] (6.7)

so that (6.3) follows by integration.

7. Painlevé equations for pairs of intervals

Akhiezer considered a generalization of the Chebyshev polynomials to the pair of intervals \([-1, \alpha] \cup [\beta, 1]\), and investigated their properties by conformal mapping. Chen and Lawrence [9] used the theory of elliptic functions to investigate these polynomials and in (8.18) expressed the Hankel determinant in terms of Jacobi’s elliptic theta functions. In this section we obtain the differential equation where \( S \) is two intervals, and obtain a differential equation for the endpoints that is related to the one from [9].

Let \( v \) be a polynomial of degree \( 2N \geq 4 \) such that \( S = [\delta_1, \delta_2] \cup [\delta_3, \delta_4] \). There exists a Möbius transformation \( \varphi \) such that \( \varphi(\delta_1) = 0, \varphi(\delta_2) = 1 \) and \( \varphi(\delta_4) = \infty \); then we let \( t = \varphi(\delta_3) \). Having fixed three of the endpoints, we can introduce the differential equations from section 4 that describe the effect of varying the endpoint \( t \), namely

\[ \frac{d}{dx} \Phi(x) = \left(\frac{\alpha_0}{x} + \frac{\alpha_1}{x - 1} + \frac{\alpha_t}{x - t}\right)\Phi \] (7.1)

and

\[ \frac{\partial \Phi}{\partial t} = \frac{-\alpha_t}{x - t} \Phi. \] (7.2)
Let $A(x,t)$ be the matrix $(\alpha_0/x + \alpha_1/(x-1) + \alpha_t/(x-t))$ and let $A(x,t)_{12}$ be its top right entry. Then we introduce $x = \lambda(t)$ such that $A(x,t)_{12} = 0$; then by [18, p. 1333], the corresponding Schlesinger equations give a version of the nonlinear Painlevé equation $P_{VI}$ in terms of $\lambda$, namely

$$
\frac{d^2 \lambda}{dt^2} + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda - t}\right) \frac{d\lambda}{dt} - \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t}\right) \left(\frac{d\lambda}{dt}\right)^2 = \frac{1}{2} \frac{\lambda (\lambda - 1) (\lambda - t)}{t^2 (t-1)^2} \left[ k_\infty - \frac{k_0 t}{\lambda^2} + \frac{k_1 (t-1)}{(\lambda - 1)^2} - \frac{(k_t - 1) t (t-1)}{(\lambda - t)^2} \right].
$$

(7.3)

The Hamiltonian and tau function satisfy

$$
H_t = \frac{d}{dt} \log \tau = \text{trace} \left(\frac{\alpha_0 \alpha_t}{t} + \frac{\alpha_1 \alpha_t}{t-1}\right).
$$

(7.4)

Having transformed $S$ to $[0, 1] \cup [t, \infty]$ we can lift this to the portions of the real axis that are covered by the elliptic curve $E = \{(\lambda, w) : w^2 = 4\lambda(\lambda - 1)(\lambda - t)\}$ which has parameters \( \lambda = \mathcal{P}(u/2) \) and \( w = \mathcal{P}'(u/2) \) in terms of Weierstrass’s function $\mathcal{P}$ with $e_1 = t$, $e_2 = 1$ and $e_3 = 0$. Hence we transform to the dependent variables

$$
u = \int_0^\lambda \frac{ds}{\sqrt{s(s-1)(s-t)}},
$$

(7.5)

and make the substitution $u = u(\lambda(t), t)$. Fuchs [16] observed that

$$
\frac{d^2 u}{dt^2} + \frac{2t-1}{t(t-1)} \frac{du}{dt} + \frac{u}{4t(t-1)} = \frac{\sqrt{\lambda (\lambda - 1) (\lambda - t)}}{2t^2 (t-1)^2} \left[ k_\infty - \frac{k_0 t}{\lambda^2} + \frac{k_1 (t-1)}{(\lambda - 1)^2} - \frac{(k_t - 1) t (t-1)}{(\lambda - t)^2} \right].
$$

(7.6)

To solve this in a special case, we introduce the complete elliptic integral

$$
K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}.
$$

By comparing terms on power series, one can recover the following result.

**Proposition 7.1** (Poincaré) Suppose that $k_0 = k_1 = k_t = k_\infty = 0$ and that $u(t) = c_1 K(\sqrt{t}) + c_2 K(\sqrt{1-t})$ for constants $c_1$ and $c_2$. Then $u$ satisfies Legendre’s equation

$$
t(t-1) \frac{d^2 u}{dt^2} + (2t-1) \frac{du}{dt} + \frac{u}{4} = 0,
$$

(7.7)

so $\lambda(t) = \mathcal{P}(u(t)/2; 0, 1, t)$ gives a solution of $P_{VI}$. 

17
8. Kernels associated with Schlesinger’s equations

In this section, we introduce kernels that are associated with Schlesinger’s equations, and then factorize them in terms of Hankel operators. First we let \( \nu_j = -2^{-1} \text{trace} \alpha_j(n) \) and observe that \( \nu_j \) does not depend upon \( n \). Indeed, by multiplying (4.22) by \( V_n^{-1} \), one deduces that \( \text{trace} A_{n+1}(z) = \text{trace} A_n(z) \), and since \( \text{trace} \alpha_j(n) = \lim_{z \to \delta_j} (z - \delta_j) \text{trace} A_n(z) \), we deduce that \( \text{trace} \alpha_j(n) \) is constant with respect to \( n \). By (4.12), we have

\[
\sum_{j=1}^{4N-2} \nu_j \text{trace} \alpha_j(n) = 1 - 2N.
\]

Now, given \( \Phi_n \) as in (4.13), let

\[
\Psi_n(z) = \prod_{j=1}^{4N-2} (z - \delta_j)^{\nu_j} \Phi_n(z).
\]

We next introduce the matrix valued kernel

\[
M_n(z, \zeta) = \frac{\Psi_n(z)^\dagger J \Psi_n(\zeta)}{-2\pi i (z - \zeta)};
\]

we aim to show that \( M_n \) is positive definite as an integral operator on \( L^2(S) \), and we observe that this property does not change if we introduce weights on \( S \).

**Proposition 8.1** Let \( E_n(z, \zeta) \) be the kernel of the orthogonal projection onto \( \text{span}\{x^j : j = 0, \ldots, n-1\} \) in \( L^2(\rho) \). Then the top left entry of \( M_n(z, \zeta) \) equals

\[
M_n(z, \zeta)_{11} = \frac{h_n}{h_{n-1}} \prod_{j=1}^{4N-2} (z - \delta_j)^{\nu_j} \prod_{j=1}^{4N-2} (\zeta - \delta_j)^{\nu_j} E_n(z, \zeta).
\]

**Proof.** The Christoffel–Darboux formula gives

\[
E_n(z, \zeta) = \frac{p_n(z)p_{n-1}(\zeta) - p_{n-1}(z)p_n(\zeta)}{h_n(z - \zeta)}.
\]

One can find \( \Psi_n(z)^\dagger J \Psi_n(\zeta) \) by direct calculation, and compare with this. 

\[\square\]

Let \( \beta_j(n) = \alpha_j(n) + \nu_j I_2 \), which has zero trace. Furthermore, if \( \Phi_n \) is a solution of the basic differential equation (4.14), then

\[
\frac{d}{dz} \Psi_n(z) = B_n(z) \Psi_n(z)
\]

where

\[
B_n(z) = \sum_{j=1}^{4N-2} \frac{\beta_j(n)}{z - \delta_j}.
\]
We pause to note an existence result for solutions of the matrix system (8.5).

**Lemma 8.2** Suppose that \( \beta_j(n) \) has eigenvalues \( \pm \kappa_j(n) \) where \( 2\kappa_j(n) \) is not an integer. Then on a neighbourhood of \( \delta_j \), there exists an analytic matrix function \( \Xi_{n,j} \) such that

\[
\Psi_n(z) = \Xi_{n,j}(z)(z - \delta_j)^{\beta_j(n)}
\]

satisfies (8.5).

**Proof.** This follows from Turrittin’s theorem; see [3]. \( \square \)

For notational simplicity, we consider the interval \((\delta_1, \delta_2)\) and assume that \( \delta_1 = 0 \) and \( 1 < \delta_2 \); the general case follows by scaling. For a continuous function \( \phi : (0,1) \to \mathbb{R}^{8N-6} \), the Hankel operator \( \Gamma_\phi : L^2((0,1); dy/y; \mathbb{R}) \to L^2((0,1); dy/y; \mathbb{R}^{8N-6}) \) is given by

\[
\Gamma_\phi f(x) = \int_0^1 \phi(xy)f(y)\frac{dy}{y}.
\]

Since \( \beta_k(n) \) has zero trace, the matrix \((\delta_1 - \delta_k)J\beta_k(n)\) is real symmetric and hence is congruent to either

\[
\sigma_k = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};
\]

let \( \sigma = \text{diagonal}[\sigma_k]_{k=2}^{4N-2} \) be the block diagonal sum of these matrices.

**Theorem 8.3** (i) Let \( \beta_1(n) \) be as in Lemma 8.2. Then there exists \( Z_n \), a \( 2 \times 1 \) real vector solution of (8.5) such that \( Z_n(x) \to 0 \) as \( x \to \delta_1 \).

(ii) The integral operator on \( L^2((0,1); dx/x) \) with kernel

\[
K_n(z, \zeta) = \frac{\sqrt{z\zeta}Z_n(\zeta)^\dagger JZ_n(z)}{z - \zeta}
\]

is of trace class; moreover, there exists a real vector Hankel operator \( \Gamma_{\psi_n} \) on \( L^2((0,1), dy/y) \) such that

\[
K_n = \Gamma_{\psi_n}^\dagger \sigma \Gamma_{\psi_n}.
\]

(iii) If \( \sigma \geq 0 \), then \( K_n \geq 0 \).

**Proof.** (i) There exists an invertible constant \( 2 \times 2 \) matrix \( S_n \) such that

\[
S_n\zeta^{\beta_1(n)}S_n^{-1} = \begin{bmatrix} z^{\kappa_1(n)} & 0 \\ 0 & z^{-\kappa_1(n)} \end{bmatrix}.
\]
where \( \kappa_1(n) > 0 \). Hence by Lemma 8.2, there exists a constant \( 2 \times 1 \) matrix \( C \) such that 
\[ Z_n(z) = \Psi_n(z)C \]
is a solution of (8.5), and \( Z_n(z) = O(|z - \delta_1|^{\kappa_1(n)}) \) as \( z \to \delta_1 \).

(ii) Hence we can introduce \( K_n \) by (8.10), and next we prove that the kernel satisfies

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) K_n(x, y) = \sum_{k=2}^{4N-2} \frac{-\delta_k \sqrt{xy}}{(x - \delta_k)(y - \delta_k)} Z_n(y) \beta_k(n) Z_n(x). \tag{8.13}
\]

First note that by homogeneity \( (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) \sqrt{xy}/(x - y) = 0 \). Since the \( \beta_k(n) \) have zero trace, we have \( J\beta_k(n) + \beta_k(n) J = 0 \) and hence the differential equation gives

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) Z_n(y) \beta_k(n) Z_n(x)
= Z_n(y) B_n(y) \beta_k(n) Z_n(x) + Z_n(y) J B_n(x) Z_n(x)
= \sum_{k=2}^{4N-2} \sqrt{xy} Z_n(y) \beta_k(n) Z_n(x) \left( \frac{x}{x - \delta_k} - \frac{y}{y - \delta_k} \right); \tag{8.14}
\]
on dividing by \( x - y \), we obtain

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \sqrt{xy} Z_n(y) \beta_k(n) Z_n(x)
= \sum_{k=2}^{4N-2} \frac{\delta_k \sqrt{xy}}{(x - \delta_k)(y - \delta_k)} Z_n(y) \beta_k(n) Z_n(x) \tag{8.15}
\]
as in (8.13).

Noting the shape of the final factor in (8.12), we choose

\[
\phi_n(x) = \text{column} \left[ \frac{\sqrt{x} Z_n(x)}{x - \delta_k} \right]_{k=2,\ldots,4N-2} \tag{8.16}
\]
which has a \( 2 \times 1 \) entry for each endpoint \( \delta_k \) of \( S \) after \( \delta_1 \), and the block diagonal matrix

\[
\beta(n) = \text{diagonal} \left[ -\delta_k \beta_k(n) \right]_{k=2,\ldots,4N-2} \tag{8.17}
\]
with \( 2 \times 2 \) blocks, and we consider

\[
\tilde{K}_n(x, y) = \int_0^1 \phi_n(yz) \beta(n) \phi_n(zx) \frac{dz}{z}. \tag{8.18}
\]

First note that since \( \kappa_1(n) > 0 \), we have \( \tilde{K}(x, y) \to 0 \) as \( x, y \to 0 \). Then

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \tilde{K}_n(x, y) = \int_0^1 \left( y \phi_n'(yz) \beta(n) \phi_n(zx) + x \phi_n(yz) \beta(n) \phi_n'(xy) \right) dz
= \phi_n(y) \beta(n) \phi_n(x) - \phi_n(0) \beta(n) \phi_n(0). \tag{8.19}
\]
We have $\phi_n(0) = 0$, so

$$K_n(x, y) = \bar{K}_n(x, y) + \xi(x/y)$$  \hspace{1cm} (8.20)

for some function $\xi$. But $Z_n(z)/(z - \delta_1)^{\kappa_1(n)}$ is analytic on a neighbourhood of $\delta_1$, so it is clear that $K_n(x, y) \to 0$ and $\bar{K}_n(x, y) \to 0$ as $x \to 0$ or $y \to 0$; hence $\xi = 0$.

By the choice of $\sigma$, there exists a block diagonal matrix $\gamma(n)$ such that $\gamma(n)\sigma \gamma(n) = \beta_n$, so we can introduce $\psi_n(x) = \gamma(n)\phi_n(x)$ such that $\phi_n(x)\sigma \phi_n(y) = \psi_n(x)\sigma \psi_n(y)$. For this symbol function $\psi_n$ we have

$$K_n(x, y) = \int_0^1 \psi_n(yz)\sigma \psi_n(zx) \frac{dz}{z},$$  \hspace{1cm} (8.21)

or in terms of Hankel operators $K_n = \Gamma_p^\dagger \sigma \Gamma_p$. We have

$$\int_0^1 \log \frac{1}{u} \|\psi_n(u)\|^2 \frac{du}{u} < \infty,$$

so $\Gamma_p$ is Hilbert–Schmidt and hence $K_n$ is trace class.

(iii) If $\sigma_k \geq 0$ for all $k$ or equivalently $\sigma \geq 0$, then $K_n \geq 0$.

Corollary 8.4 Suppose that $Z$ is a $2 \times 1$ solution of

$$\frac{d}{dx}Z = \left(\frac{\beta_0}{x} + \frac{\beta_1}{x - 1} + \frac{\beta_t}{x - t}\right)Z$$  \hspace{1cm} (8.22)

such that the entries satisfy $Z(\bar{x}) = \bar{Z}(x)$ and where

(i) $\beta_0$ is as in Lemma 8.2, and $Z(x) \to 0$ as $x \to 0$;

(ii) $J\beta_1$ is positive definite;

(iii) $J\beta_t \geq 0$.

Then there exist an invertible real matrix $S$ and a real diagonal matrix $D$ such that

$$\psi(x) = \begin{bmatrix} \sqrt{xSZ(x)} \\ \frac{x}{x - 1} \sqrt{xDSZ(x)} \\ x^{-t} \end{bmatrix}$$  \hspace{1cm} (8.23)

satisfies $\psi(\bar{x}) = \overline{\psi(x)}$ and

$$\frac{\sqrt{xy}Z(y)\dagger JZ(x)}{y - x} = \int_0^1 \psi(xz)\dagger \psi(zy) \frac{dz}{z}.$$  \hspace{1cm} (8.24)

Proof. We simultaneously reduce the quadratic forms associated with $J\beta_1$ and $J\beta_t$, and introduce an invertible real matrix $S$ such that $J\beta_1 = SS^\dagger$ and $J\beta_t = SD^2S^\dagger$, where $D$
is a real diagonal matrix such that the diagonal entries $\kappa$ of $D^2$ satisfy $\det(\beta_1 - \kappa \beta_t) = 0$. Then we can write

$$
\psi(y)\dagger \psi(x) = \sqrt{xy}Z(y)\dagger \left( \frac{J\beta_1}{(x-1)(y-1)} + \frac{J\beta_t}{(x-t)(y-t)} \right)Z(x).
$$

(8.25)

Now we can follow the proof of Theorem 8.3, and deduce that

$$
-\psi(y)\dagger \psi(x) = \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \frac{\sqrt{xy}Z(y)\dagger JZ(x)}{x-y};
$$

(8.26)

hence we can obtain the result by integrating and using (i).

Example 8.5 For the semicircle law on $[a, b]$, as in Proposition 6.1, we have

$$
J\beta_a(n) = \begin{bmatrix}
\frac{n(b-a)}{2n-1} & \frac{2n+1}{4} \\
\frac{2n+1}{4} & \frac{(n+1)(b-a)h_{n-1}}{8}
\end{bmatrix}
$$

(8.27)

and

$$
J\beta_b(n) = \begin{bmatrix}
\frac{-n(b-a)}{2n-1} & \frac{2n+1}{4} \\
\frac{2n+1}{4} & \frac{-(n+1)(b-a)h_{n-1}}{8}
\end{bmatrix},
$$

(8.28)

so that

$$
\det J\beta_a(n) = \det J\beta_b(n) = \frac{n(n+1)(b-a)^2}{16} - \frac{(2n+1)^2}{16}.
$$

(8.29)

In particular, when $a = -1$ and $b = 1$, the matrices $J\beta_{-1}(n)$ and $J\beta_1(n)$ are indefinite.

9. The tau function realised by a linear system

In this section, we express the tau function of $K_n$ from Theorem 8.3 as a Fredholm determinant, and then obtain this from the solution of an integral equation of Gelfand–Levitan type. The first step is introduce a scattering function $\psi$ and then to realise this by a linear system, so that we can solve the Gelfand–Levitan equation.

The differential equation

$$
\frac{dZ_n}{dx} = B_n(x)Z_n(x)
$$

(9.1)

has a solution from which we constructed a symbol function

$$
\psi_n(x) = \text{column} \left[ \sqrt{x\gamma(n)}Z(x) \right]_{k=2}^{4N-2}.
$$

(9.2)

Suppressing $n$ for simplicity, we change $x \in (0, 1)$ to $t \in (0, \infty)$ by letting $x = \delta_1 + e^{-t}$ and in the new variables write

$$
\psi(t) = \sum_{\ell=0}^{\infty} \chi_\ell e^{-(\kappa_1+\ell+1/2)t}.
$$

(9.3)
where $\sum_{\ell=0}^{\infty} \|\chi_\ell\| < \infty$. Likewise, we write $\tau(t)$ for $\tau(\delta_1 + e^{-t})$.

Let $\Omega = \{ z : \Re z \geq 0 \}$ be the open right half-plane, let

$$
\Psi(x) = \begin{bmatrix} 0 & \psi(x) \\ \psi(x)^\dagger & 0 \end{bmatrix}
$$

(9.4)

and extend $\Psi$ to an analytic function $\Psi : \Omega \to M_{8N-5}(\mathbb{C})$ such that $\Psi(x) = \Psi(x)^\dagger$ for $x > 0$. Let $\Psi(s) = \Psi(x + 2s)$ and $\Psi^*(s)(x) = \Psi(x + 2s)^\dagger$ and let $\sigma$ be a constant matrix; then let $K_s = \Gamma_{\Psi(s)} \sigma \Gamma_{\Psi(s)}$ be a family of operators on $L^2(0, \infty)$.

**Proposition 9.1** (i) The $\tau$ function associated with $K = \Gamma_{\Psi^*} \sigma \Gamma_{\Psi}$ is $\tau(2s) = \det(I - K_s)$, which gives an analytic function on $\Omega$.

(ii) Let $q(s) = -2 \frac{d^2}{ds^2} \log \tau(2s)$. Then $q(s)$ is meromorphic on $\Omega$, and analytic where

$$
\int_0^{\infty} x \|\Psi(x + s)\|^2 dx < 1.
$$

(iii) If $0 \leq K \leq I$, then $\tau(s)$ is non-negative for $0 < s < \infty$, increasing and converges to one as $s \to \infty$.

**Proof.** (i) The kernel of the Hankel operator $\Gamma_{\Psi(s)}$ has a nuclear expansion

$$
\Gamma_{\Psi(s)} \leftrightarrow \sum_{\ell=0}^{\infty} e^{-(\kappa_1 + \ell + 1/2)(x + y + 2s)} \begin{bmatrix} 0 & \chi_\ell \\ \chi_\ell^\dagger & 0 \end{bmatrix}
$$

(9.5)

where $\sum_{\ell=0}^{\infty} \|\chi_\ell\| \int_0^{\infty} e^{-2(\kappa_1 + \ell + 1/2)(x + \Re s)} dx < \infty$, so the Fredholm determinants are well defined. As in Schwarz’s reflection principle, $s \mapsto \Psi^*(s)$ is analytic, and $\Gamma_{\Psi(s)}$ is Hilbert–Schmidt, so $K_s$ is an analytic trace-class valued function on $\Omega$. Using unitary equivalence, one checks that

$$
\det(I - K_s) = \det(I - P_{(2s, \infty)}) K \quad (s > 0).
$$

(9.6)

(ii) Except on the discrete set of zeros of $\tau(2s)$, the operator $I - K_s$ is invertible and

$$
q(s) = 2 \frac{d}{ds} \text{trace}\left((I - K_s)^{-1} \frac{dK_s}{ds}\right).
$$

(9.7)

(iii) This follows from (9.6).

 district, we obtain an alternative formula for $q$ by realising $\Psi$ via a linear system. The technique is suggested by the inverse scattering transform. Let $H_0 = \mathbb{C}^{8N-6}$ be the column vectors, $H = \ell^2$ be Hilbert sequence space, written as infinite columns, and introduce an infinite row of column vectors $C \in \ell^2(H_0)$ by $C = (\chi_\ell/\|\chi_\ell\|^{1/2})_{\ell=0}^{\infty}$ and a column $B \in \ell^2$ by $B = (\|\chi_\ell\|^{1/2})_{\ell=0}^{\infty}$ and the infinite square matrix $A = \text{diagonal}[\ell + \kappa_1 + 1/2]_{\ell=0}^{\infty}$. While
A is real and symmetric, we will write $A^\dagger$ in some subsequent formulas, so as to emphasize their symmetry.

In the following result we use the $(8N-5) \times (8N-5)$ block matrices

$$W(x,y) = \begin{bmatrix} U(x,y) & v(x,y) \\ w(x,y) & z(x,y) \end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} 0 & \psi(x) \\ \psi(x)^\dagger & 0 \end{bmatrix},$$

so that $\Psi(\bar{x}) = \Psi(x)^\dagger$ and the matrix Hamiltonian

$$H(x) = \begin{bmatrix} U(x,x)\sigma & v(x,x) \\ w(x,x)^{\dagger}\sigma & z(x,x) \end{bmatrix}$$

where $v, w \in H_0$, $U$ operates upon $H_0$ and $z$ is a scalar. To simplify the statements of results, we use a special non-associative product $\ast$, involving $\sigma$, that is defined by

$$\begin{bmatrix} U & v \\ w^\dagger & z \end{bmatrix} \ast \begin{bmatrix} 0 & \psi \\ \psi^\dagger & 0 \end{bmatrix} = \begin{bmatrix} v\psi^\dagger & U\sigma\psi \\ z\psi^\dagger & w^\dagger\sigma\psi \end{bmatrix}.$$ (9.10)

**Theorem 9.2** (i) The symbol $\psi$ is realised by the linear system $(-A,B,C)$, so

$$\psi(t) = Ce^{-tA}B.$$ (9.11)

(ii) There exists a solution of the Gelfand–Levitan equation

$$W(x,y) + \Psi(x+y) + \int_x^\infty W(x,s) \ast \Psi(s+y) \, ds = 0 \quad (0 < x < y)$$ (9.12)

such that the tau function of Proposition 9.1(i) satisfies

$$\frac{d}{dx} \log \tau(2x) = \text{trace} H(x) \quad (x > 0).$$ (9.13)

(iii) Suppose moreover that $\int_0^\infty x\|\Psi(x)\|^2 dx < 1$. Then

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) W(x,y) = -2 \frac{dH}{dx} W(x,y).$$ (9.14)

**Proof.** (i) This identity follows from (9.3). Since $\kappa_1 + \ell + 1/2 > 0$, the semigroup $e^{-tA} = \text{diagonal} \left[ e^{-t(\kappa_1+\ell+1/2)} \right]_{\ell=0}^\infty$ consists of trace class operators, and the integrals in the remainder of the proof are convergent.

(ii) We introduce the observability Gramian

$$Q_x^\sigma = \int_x^\infty e^{-sA^\dagger} C^\dagger \sigma C e^{-sA} \, ds \quad (x > 0),$$ (9.15)
modified to take account of $\sigma$, and the usual controllability Gramian
\[ L_x = \int_0^\infty e^{-sA}BB^\dagger e^{-sA^\dagger} \, ds, \]
both of which define trace class operators on $\ell^2$, and where $L_x \geq 0$. The controllability operator $\Xi_x : L^2((0, \infty); H_0) \to H$ is
\[ \Xi_x f = \int_0^\infty e^{-tA}Bf(s) \, ds \]  
while the observability operator is $\Theta_x : L^2((0, \infty); H_0) \to H$ is
\[ \Theta_x f = \int_0^\infty e^{-sA^\dagger}C^\dagger f(s) \, ds. \]
Finally, we let $\psi(x)(s) = \psi(s + 2x)$, so that $\psi(x)$ is realised by $(-A, e^{-xA}B, Ce^{-xA})$. In terms of these operators, we have the basic identities
\[ \Gamma_{\psi(x)} = \Theta_x^\dagger \Xi_x, \quad \Gamma_{\psi(x)}^\dagger = \Xi_x^\dagger \Theta_x \]
while
\[ L_x = \Xi_x \Xi_x^\dagger \quad \text{and} \quad Q_x^\sigma = \Theta_x \sigma \Theta_x^\dagger. \]
Hence we can rearrange the factors in the Fredholm determinants
\[ \det(I - \lambda \Gamma_{\psi(x)}^\dagger \sigma \Gamma_{\psi(x)}) = \det(I - \lambda \Xi_x \Theta_x \sigma \Theta_x^\dagger \Xi_x) \]
\[ = \det(I - \lambda \Xi_x \Xi_x^\dagger \Theta_x \sigma \Theta_x^\dagger) \]
\[ = \det(I - \lambda L_x Q_x^\sigma). \]
We deduce that
\[ \log \tau(2x) = \log \det(I - \Gamma_{\psi(x)}^\dagger \sigma \Gamma_{\psi(x)} P_{(2x, \infty)}) \]
\[ = \log \det(I - \sigma \Gamma_{\psi(x)} P_{(2x, \infty)} \Gamma_{\psi(x)}^\dagger) \]
\[ = \log \det(I - \sigma \Gamma_{\psi(x)} \Gamma_{\psi(x)}^\dagger) \]
\[ = \log \det(I - \Gamma_{\psi(x)} \sigma \Gamma_{\psi(x)}^\dagger) \]
\[ = \text{trace} \log(I - L_x Q_x^\sigma), \]
and hence
\[ \frac{d}{dx} \log \tau(2x) = \text{trace} \left( (I - L_x Q_x^\sigma)^{-1} (e^{-xA}BB^\dagger e^{-xA} C^\dagger \sigma Ce^{-xA}) \right) \]
\[ = B^\dagger e^{-xA} Q_x^\sigma (I - L_x Q_x^\sigma)^{-1} e^{-xA} B \]
\[ + \text{trace} \sigma Ce^{-xA} (I - L_x Q_x^\sigma)^{-1} L_x e^{-xA} C^\dagger. \]
The integral equation
\[
\begin{bmatrix}
U(x, y) & v(x, y) \\
w(x, y)^\dagger & z(x, y)
\end{bmatrix}
+ \int_x^\infty \begin{bmatrix}
U(x, s) & v(x, s) \\
w(x, s)^\dagger & z(x, s)
\end{bmatrix} \ast \begin{bmatrix}
0 & \psi(s + y) \\
\psi(s + y)^\dagger & 0
\end{bmatrix} ds = 0 \tag{9.24}
\]
reduces to the identities
\[
U(x, y) = -\int_x^\infty v(x, s)\psi(s + y)^\dagger ds,
\]
\[
z(x, y) = -\int_x^\infty w(x, s)^\dagger \sigma \psi(s + y) ds, \tag{9.25}
\]
and the pair of integral equations
\[
v(x, y) + \psi(x + y) - \int_x^\infty \int_x^\infty v(x, t)\psi(t + s)^\dagger \sigma \psi(s + y) dsdt = 0 \tag{9.26}
\]
and
\[
w(x, y) + \psi(x + y) - \int_x^\infty \int_x^\infty \psi(s + y)\psi(t + s)^\dagger \sigma w(x, t) dtds = 0. \tag{9.27}
\]
To solve these integral equations, we let
\[
v(x, y) = -Ce^{-xA}(I - L_xQ_x^\sigma)^{-1}e^{-yA}B \tag{9.28}
\]
and
\[
w(x, y) = -Ce^{-yA}(I - L_xQ_x^\sigma)^{-1}e^{-xA}B; \tag{9.29}
\]
then by substituting these into (9.25) we obtain the diagonal blocks of the solution $W$, namely
\[
U(x, y) = Ce^{-xA}(I - L_xQ_x^\sigma)^{-1}L_xe^{-yA}C^\dagger \tag{9.30}
\]
and
\[
z(x, y) = B^\dagger e^{-yA}Q_x^\sigma(I - L_xQ_x^\sigma)^{-1}e^{-xA}B. \tag{9.31}
\]
Hence we can identify the trace of the solution (9.9) as
\[
\text{trace } H(x) = \text{trace } \sigma U(x, x) + z(x, x)
\]
\[
= \text{trace } \sigma Ce^{-xA}(I - L_xQ_x^\sigma)^{-1}e^{-yA}C^\dagger
\]
\[
+ B^\dagger e^{-yA}Q_x^\sigma(I - L_xQ_x^\sigma)^{-1}e^{-xA}B
\]
\[
= \frac{d}{dx} \log \tau(2x). \tag{9.32}
\]
(iii) By integrating by parts, we obtain the identity

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) W(x, y) - 2 \frac{dH}{dx} \Psi(x + y) + \int_{x}^{\infty} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial s^2} \right) W(x, s) \ast \Psi(s + y) ds = 0 \tag{9.33}
\]

for \(0 < x < y\). One can easily verify that the product \(\ast\) and the standard matrix multiplication satisfy \((QW) \ast \Psi = Q(W \ast \Psi)\), hence the formula

\[
-2 \frac{dH}{dx} W(x, y) - 2 \frac{dH}{dx} \Psi(x + y) - \int_{x}^{\infty} \left( 2 \frac{dH}{dx} W(x, s) \right) \ast \Psi(s + y) ds = 0 \tag{9.34}
\]

follows from multiplying (9.33) by \(-2 \frac{dH}{dx}\), and this shows that both \(-2 \frac{dH}{dx} W(x, y)\) and \((\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) W(x, y)\) are solutions of the same integral equation. By uniqueness of solutions, they are equal.

\(\square\)

10. Integrability of the tau function of a linear system

Let \(q(x) = -2 \frac{d}{dx} \text{trace} H(x)\) and \(\tau\) be as in (9.32). In this section we describe the properties of \(\tau\) in terms of the algebraic theory of differential equations [30].

Let \(F\) be a field (of complex functions) with differential \(\partial\) that contains the subfield \(\mathbb{C}\) of constants and adjoin an element \(h\) to form \(F(h)\), where either:

(i) \(h = \int g\) for some \(g \in F\), so \(\partial h = g\);

(ii) \(h = \exp \int g\) for some \(g \in F\); or

(iii) \(h\) is algebraic over \(F\).

**Definition.** Let \(F_j\) \((j = 1, \ldots, n)\) be fields with differential \(\partial\) that contain the subfield \(\mathbb{C}\) of constants and suppose that

\[
F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n, \tag{10.1}
\]

where \(F_j\) arises from \(F_{j-1}\) by applying some operation (i), (ii) or (iii). Then \(F_n\) is a Liouvillian extension of \(F_1\).

**Example.** The tau function (6.3) belongs to some Liouvillian extension of \(\mathbb{C}(x)\).

**Lemma 10.1** Let \(A_n : \mathbb{C}^n \rightarrow \mathbb{C}^n\), \(B_n : \mathbb{C}^m \rightarrow \mathbb{C}^n\) and \(C_n : \mathbb{C}^n \rightarrow \mathbb{C}^k\) be finite matrices, such that \(MA_n + A_n^t M\) is positive definite for some positive definite \(M\). Let \(\psi_n(t) = Ce^{-tA_n} B\), and define the corresponding terms as in the proof of Theorem 9.2. Then \(\tau_n(x)\) belongs to some Liouvillian extension field \(F\) of \(F_0 = \mathbb{C}(t, e^{-t\kappa_j}, e^{-t\bar{\kappa}_j}; j = 1, \ldots, n)\), where \((\kappa_j)_{j=1}^n\) is the spectrum of \(A_n\).

**Proof.** By Lyapunov’s criterion [7], all of the eigenvalues \(\kappa_j\) of \(A_n\) satisfy \(\Re \kappa_j > 0\), hence \(\|e^{-tA_n}\|\) is of exponential decay as \(t \rightarrow \infty\). By considering the Jordan canonical form
of $A_n$, we obtain matrix polynomials $p_j(t)$ such that $e^{-tA_n} = \sum_{j=1}^n p_j(t)e^{-\kappa_j t}$. Observe that $F_0$ contains all the entries of $e^{-tA_n}B_nB_n^\dagger e^{-tA_n}$ and $e^{-tA_n^1}C_n^1 \sigma C_n e^{-tA_n}$. The operator $L_x$ is an indefinite integral of $e^{-tA_n}B_nB_n^\dagger e^{-tA_n}$, while the operator $Q_x^\sigma$ is an indefinite integral of $e^{-tA_n^1}C_n^1 \sigma C_n e^{-tA_n}$, hence $L_x$ and $Q_x^\sigma$ have entries in $F_0$; moreover, the entries of $(I - L_x Q_x^\sigma)^{-1}$ are quotients of determinants with elements in $F_0$. Hence by (9.32), $\frac{d}{dx} \log \tau_n(2x)$ gives an element of $F_0$, so $\tau_n(x)$ itself is in a Liouvillian extension $F$ of $F_0$.

\[ \square \]

**Theorem 10.2** Let $\psi$ be as in Theorem 9.2.

(i) There exists a sequence of finite rank matrices $(A_n)_{n=1}^\infty$, with corresponding tau functions $\tau_n$, such that $\frac{d}{dt} \log \tau_n(2t)$ belongs to $C(e^{-(\kappa_1+1/2)t}, e^{-t})$ and $\tau_n(2t) \to \tau(2t)$ as $n \to \infty$, uniformly on compact subsets of \{ $t : \Re t > 0$ \}.

(ii) Suppose further that $\kappa_1$ is rational. Then there exists a positive integer $N$ such that $\frac{d}{dt} \log \tau(2t)$ is periodic with period $2\pi i N$, and $\tau_n(2t)$ is given by elementary functions as in (10.4) below.

**Proof.** (i) We introduce the finite rank matrices

\[ A_n = \text{diagonal } [\kappa_1 + 1/2, \kappa_1 + 3/2, \ldots, \kappa_1 + n + 1/2, 0, 0, \ldots] \quad (10.2) \]

so that $\|e^{-tA} - e^{-tA_n}\|_1 \leq e^{-(\kappa_1+n+1)\Re t}/(1 - e^{-\Re t})$. Similarly, we cut down $B$ and $C$ and follow through the proof of Theorem 9.2 to produce the appropriate choice of $W_n(t, t)$ by the prescription of (9.8). Evidently, the eigenvalues of $e^{-tA_n}$ have the form $e^{-t(\kappa_1+\ell+1/2)}$ where $\ell = 0, \ldots, n$. We observe that $e^{-t(\kappa_1+\ell+1/2)}$ belongs to $C(e^{-(\kappa_1+1/2)t}, e^{-t})$ for all $\ell$, so $W_n(t, t)$ likewise belongs to $C(e^{-(\kappa_1+1/2)t}, e^{-t})$.

(ii) In this case, the set $\{m\kappa_1 + m/2 + n\ell; m, n \in \mathbb{Z}; \ell = 0, 1, 2, \ldots\}$ is a finitely generated subgroup of the rationals, and hence has a smallest positive element $M/N$, where $M, N \in \mathbb{N}$ with $M < N$. Then all the terms $N(\kappa_1 + \ell + 1/2)$ are positive integers, so $\exp(-t(2\pi N i)A) = \exp(-tA)$ for all $\Re t > 0$, hence $\tau'(2t)/\tau(2t)$ is periodic.

By Lemma 10.1, there exists a rational function $r_n$ such that

\[ \frac{d}{dt} \log \tau_n(2t) = r_n(e^{-t/N}). \quad (10.3) \]

Suppose for simplicity that $r_n(z)/z$ has only simple poles; then from the partial fractions decomposition, there exist coefficients $\alpha_j, \beta_j$ and $b_j, c_j$ such that $b_j^2 < c_j$, real poles $a_k$ and a polynomial $q_n(z)$ such that (10.3) integrates to

\[ \log \tau_n(2t) = q_n(e^{-t/N}) + \sum_k \alpha_k \log |e^{-t/N} - a_k| + \sum_j \beta_j \log \left( e^{-2t/N} + 2b_j e^{-t/N} + c_j \right) \]

\[ + \sum_j \frac{\gamma_j}{\sqrt{c_j - b_j^2}} \tan^{-1} \frac{e^{-t/N} + b_j}{\sqrt{c_j - b_j^2}}. \quad (10.4) \]

28
When $r_n(z)/z$ has higher order poles, one likewise obtains expressions that are similar but more complicated.

\[\frac{z}{r} = \frac{z}{x} \]

\[\text{In view of Theorem 10.2(ii) and the results of [8], it is natural to consider how the properties of } \tau \text{ relate to integrability of Schrödinger's equation } -f'' + qf = \lambda f.\]

**Definition.** Let $q$ be meromorphic on $\mathbb{C}$. As in [6,17], we say that $q$ is algebro-geometric if there exists a non-zero $R : \mathbb{C}^2 \to \mathbb{C} \cup \{\infty\}$ such that $x \mapsto R(x; \lambda)$ is meromorphic, $\lambda \mapsto R(x; \lambda)$ is a polynomial, and

\[-R'''' + 4(q - \lambda)R' + 2q'R = 0. \tag{10.5}\]

Drach [6] observed that Schrödinger’s equation is integrable by quadratures for all $\lambda$ only if $q$ is algebro-geometric. Conversely, if $R(x; \lambda)$ is as above, then

\[f(x) = \sqrt{R(x; \lambda)} \exp\left(-\int \frac{dt}{R(t; \lambda)}\right) \tag{10.6}\]

gives a solution to Schrödinger’s equation. In [6], Brezhnev catalogues several known special functions which give integrable forms of Schrödinger’s equation. The theory extends to meromorphic potentials on compact Riemann surfaces by [13, p. 235] and [23, p. 1122].

The following result of Gesztesy and Weikard summarizes various sufficient conditions for a potential to be algebro-geometric. The initial hypothesis rules out variants of Bessel’s equation.

**Theorem 10.3** [17] Suppose that $-f'' + qf = \lambda f$ has a meromorphic fundamental solution for each $\lambda$ and that either

(i) $q$ is rational and bounded at infinity;

(ii) $q$ is elliptic, that is, doubly periodic; or

(iii) $q$ is periodic, with purely imaginary period, and $q$ is bounded on $\{z \in \mathbb{C} : |\Re z| > r\}$ for some $r > 0$.

Then $q$ is algebro-geometric.

We proceed to consider the cases (i),(ii) and (iii) of Theorem 10.3, the linear systems $(-A,B,C)$ that give rise to them, and the corresponding $\tau$ functions.

**Proposition 10.4** Suppose that $q$ satisfies Theorem 10.3(i). Then $f$ has a rational Laplace transform and hence is the transfer function of a linear system $(-A_n,B,C)$ with a finite matrix $A_n$. 
Proof. By a theorem of Halphen [17], the general solution of $-f'' + qf = \lambda f$ has the form $f(x) = \sum_{j=1}^{n} q_j(x)e^{-\kappa_j x}$, where $q_j(x)$ are polynomials. Hence there exist constants $a_k$, not all zero, such that $\sum_{k=0}^{N} a_k f^{(k)}(x) = 0$; so by taking the Laplace transform, and introducing the initial conditions, we can recover the rational function $\hat{f}(s)$; see [7, p.15]. We recall that any proper rational function arises as the transfer function of a linear system that has a finite matrix $A_n$, so $\hat{f}(\lambda) = C(\lambda I + A_n)^{-1}B$.

Example (iii). In Theorem 10.2 (ii), $q$ is a rational function of $e^{-t/N}$ and under certain conditions gives rise to case (iii) of Theorem 10.3. In particular, $q(t) = -2\text{sech}^2 t$ is algebro-geometric, has period $2\pi i$, is bounded on $\{z : |\Re z| > r\}$ for all $r > 0$ and arises from $\tau(2t) = 1 + e^{-2t}$. This potential appears in the theory of solitons [4].

11. Realising linear systems for elliptic potentials

Suppose that $q$ is real, smooth and periodic with period one; introduce Hill’s operator $-\frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbb{R})$. Then we introduce the Bloch spectrum, which is

$$S_B = \{\lambda \in \mathbb{C} : \text{the general solution of } -f'' + qf = \lambda f \text{ has } f \in L^\infty(\mathbb{R}; \mathbb{C})\}. \quad (11.1)$$

One can show that when $q$ is an algebro-geometric potential, $S_B$ has only finitely many gaps; see [1,6, 24]. So we suppose that

$$S_B = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \ldots \cup [\lambda_{2g}, \infty) \quad (11.2)$$

with $g$ gaps. The $\lambda_j$ are the points of the simple periodic spectrum, such that $-f'' + qf = \lambda_j f$ has a unique solution, up to scalar multiples, that is one or two periodic. Let $\Phi$ be the $2 \times 2$ fundamental solution matrix that satisfies

$$\frac{d}{dx} \Phi(x) = \begin{bmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{bmatrix} \Phi(x), \quad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (11.3)$$

and let $\Delta(\lambda) = \text{trace } \Phi(1)$ be the discriminant of Hill’s equation. We can characterize $S_B$ as $\{\lambda \in \mathbb{R} : \Delta(\lambda)^2 \leq 4\}$, and its components are known as the intervals of stability.

The (hyper) elliptic curve $\mathcal{C} : y^2 = -\prod_{j=0}^{2g} (x - \lambda_j)$ has genus $g$, and we can form the (hyper) elliptic function field $\mathcal{E}_g = \mathcal{C}(x)[y]$. We therefore have a situation quite analogous to (2.5) and the Riemann surface $\mathcal{E}$ of section 7. In this section, we consider the case of $g = 1$, where $\mathcal{C}$ is an elliptic curve which is parametrized by $\mathcal{P}$. Hochstadt proved that $g = 1$ if and only if $q(x) = c_1 + 2\mathcal{P}(x + c_2)$ where $c_1$ and $c_2$ are constants; see [17].

Starting finite matrices, we formulate a version of the Gelfand–Levitan equation that is appropriate when $\phi(x) = C e^{-xA}B$ is periodic. (The equation (9.12) does not converge
when the functions are periodic.) A variant of this was used in [11] to solve the matrix nonlinear Schrödinger equation. Thus we will realise elliptic tau functions from linear systems.

**Definition.** (Periodic linear system \((-A, B, C; E)\)) Let \(A, B, C\) and \(E\) be finite square matrices of equal size; let \(\varepsilon = \pm 1\), and suppose that \(BC = \varepsilon(AE + EA), BE = EB, EA = AE\) and \(\exp 2\pi A = I\). Define \(\phi(x) = Ce^{-xA}B\) to be the scattering function for \((-A, B, C)\) and then introduce

\[
W(x, y) = Ce^{-xA} \left( I - e^{-xA}Ee^{-xA} \right)^{-1} e^{-yA}B. \tag{11.4}
\]

We define the tau function to be

\[
\tau(x) = \exp \left( \int_0^x \text{trace} W(y, y) \, dy \right) \tag{11.5}
\]

and let \(q(x) = -2\frac{d^2}{dx^2} \log \tau(x)\) be the potential function.

We define \(\tau\) indirectly so as to accommodate the most significant applications. The definition retains the spirit of Theorem 9.2, on account of the following result.

**Lemma 11.1** (i) The matrices satisfy the Gelfand–Levitan equation

\[
-\phi(x + y) + W(x, y) - \varepsilon \int_x^{2\pi} W(x, z)\phi(z + y) \, dz = W(x, y)E \quad (0 < x < y < 2\pi), \tag{11.6}
\]

and

\[
\frac{d}{dx} \log \det(I - e^{-xA}Ee^{-xA}) = \varepsilon \text{trace} W(x, x). \tag{11.7}
\]

(ii) Let \(F\) be a differential field that contains all the entries of \(e^{-xA}\). Then \(\tau(x)\) belongs to a Liouvillian extension of \(F\), and \(\tau(x + 2\pi) = \kappa \tau(x)\) where \(\kappa = \exp \int_0^{2\pi} \text{trace} W(y, y) \, dy\).

(iii) Suppose moreover that \(\varepsilon = 1\) and \(2\pi \|\phi\|_\infty < 1\). Then

\[
\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} = -2 \left( \frac{d}{dx} W(x, x) \right) W(x, y). \tag{11.8}
\]

**Proof.** (i) One can check that

\[
\int_x^{2\pi} e^{-zA}BCe^{-zA} \, dz = \varepsilon e^{-xA}Ee^{-xA} - \varepsilon E \tag{11.9}
\]

and it is then a simple matter to verify the integral equation (11.6).
By rearranging terms, one checks that

\[
\text{trace } W(x, x) = \text{trace } \left( (I - e^{-x A} E e^{-x A})^{-1} e^{-x A} BC e^{-x A} \right)
\]

\[
= \varepsilon \frac{d}{dx} \text{trace} \log (I - e^{-x A} E e^{-x A})
\]

\[
= \varepsilon \frac{d}{dx} \log \text{det} (I - e^{-x A} E e^{-x A}). \quad (11.10)
\]

(ii) By (i), \( \tau \) is given by exponential integrals of the entries of \( e^{-xA} \). Note that \( W(x, y) \) is periodic in both \( x \) and \( y \), so \( W(x, x) \) is periodic and hence \( \int_0^x \text{trace} W(y, y) \, dy \) increases by the same amount as \( x \) increases through any interval of length \( 2\pi \).

(iii) By repeatedly differentiating (11.6), and using periodicity, one derives the identity

\[
\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} + 2 \left( \frac{d}{dx} W(x, x) \right) \phi(x + y) + W(x, 0) \phi'(y) - \frac{\partial W}{\partial y}(x, 0) \phi(y)
\]

\[- \int_x^{2\pi} \left( \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} \right) \phi(z + y) \, dz = \frac{\partial^2 W}{\partial x^2} E - \frac{\partial^2 W}{\partial y^2} E \quad (11.11)\]

Since \( ABC - CBA = 0 \), we obtain

\[
W(x, 0) \phi'(y) - \frac{\partial W}{\partial y}(x, 0) \phi(y) = 0, \quad (11.12)
\]

so (11.11) is a multiple of the original integral equation by \( -2 \frac{d}{dx} W(x, x) \). By the assumptions on \( \|\phi\|_{\infty} \), the solutions are unique, hence the differential equation is satisfied.

\[\square\]

By introducing infinite block matrices, we can extend the scope of Lemma 11.1. Clearly we can replace \( \varepsilon \) in (11.6) by a diagonal matrix with blocks of \( \pm 1 \) entries on the diagonal.

One can interpret the following result as saying that Lamé’s operator \(-\frac{d^2}{dx^2} + 2P\) has the scattering function proportional to \( \sin x \). Let \( \omega_1 \) and \( \omega_2 \) be the periods, so that \( \omega = \omega_2/\omega_1 \) has \( 3\omega > 0 \); then let \( e_1 = P(\omega_1/2), e_2 = P((\omega_1 + \omega_2)/2) \) and \( e_3 = P(\omega_2/2) \); then let Jacobi’s modulus be \( m^2 = (e_2 - e_3)/(e_1 - e_3) \) and \( q = e^{i\omega\pi} \). To be specific, we choose \( w_1 = 2\pi \) and \( w_2 = 2\pi i \). Let \( A, B \) and \( C \) be the infinite block diagonal matrices with \( 2 \times 2 \) diagonal blocks

\[
A = \text{diagonal}[J]_{n=-\infty}^{\infty}, \quad C = A, \quad (11.13)
\]

\[
E = \text{diagonal}[q^{2n}I_2]_{n=-\infty}^{\infty}, \quad B = 2E. \quad (11.14)
\]
Proposition 11.2 (i) The functions $\phi(x) = Ce^{-xA}B$ and $W(x,y)$ of (11.4) satisfy the Gelfand–Levitan equation (11.6) and

$$\text{trace } \phi(x) = 4 \frac{1 + q^2}{1 - q^2} \sin x; \quad (11.15)$$

(ii) The corresponding tau function is entire, belongs to a Liouvillian extension of the standard elliptic function field and satisfies

$$2\mathcal{P}(x) = -\frac{d^2}{dx^2} \log \tau(x). \quad (11.16)$$

Proof. (i) The matrices satisfy $EB = BE, AE = EA$ and $BC = AE + EA$, so Lemma 11.1(i) applies. Note that the entries of $E$ are summable, so $E$ defines a trace class operator, hence the trace exists and a simple calculation gives (11.15).

(ii) Observe also that

$$\det(I - e^{-xA}Ee^{-xA}) = \det \begin{bmatrix} 1 - q^{2|n|} \cos 2x & -q^{2|n|} \sin 2x \\ q^{2|n|} \sin 2x & 1 - q^{2|n|} \cos 2x \end{bmatrix} = 1 - 2q^{2|n|} \cos 2x + q^{4|n|}, \quad (11.17)$$

so one has

$$\det(I - e^{-xA}Ee^{-xA}) = 4 \sin^2 x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2x + q^{4n})^2; \quad (11.18)$$

for comparison, by [25, p 135] the Jacobi elliptic function satisfies

$$\theta_1(x) = 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2x + q^{4n})(1 - q^{2n}). \quad (11.19)$$

So we have an entire function

$$\tau(x) = \det(I - e^{-xA}Ee^{-xA}) = \frac{\theta_1(x)^2}{q^{1/2} \prod_{n=1}^{\infty} (1 - q^{2n})^2}. \quad (11.20)$$

Moreover, we have [25, p. 132]

$$\mathcal{P}(x) = -\frac{d^2}{dx^2} \log \theta_1(x) + e_1 + \frac{d^2}{dx^2} \log \theta_1(x)|_{x=1/2}, \quad (11.21)$$

hence we obtain (11.16). Let $\mathcal{E}$ be the elliptic function field of functions of rational character on the complex torus $C/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$. Then $\mathcal{E} = C(\mathcal{P}|\mathcal{P}')$, and by (11.21) $\mathcal{E}$ has a Liouvillian extension $\mathcal{E}_\theta$ that contains $\theta_1$. 

33
Theorem 11.3 Let \( \tau \) be an elliptic function. Then there exists a periodic linear system \((-A, B, C; E)\), where \(A, B, C\) and \(E\) are infinite block diagonal matrices with \(2 \times 2\) blocks, such that
\[
\frac{d}{dx} \log \tau(x) = \text{trace} W(x, x). \tag{11.22}
\]

**Proof.** Any elliptic function is of rational character on \(\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})\), and is the ratio of theta functions by [25, p 105], so
\[
\tau(x) = \prod_{j=1}^{m} \frac{\theta(x-a_j)}{\theta(x-b_j)} \tag{11.23}
\]
where \(a_1 + \ldots + a_m = b_1 + \ldots + b_m\).

First we construct a periodic linear system with \(\theta\) as its tau function. For \(n = 0\), let \(A_0 = J/2, E_0 = -iJ, B_0 = iI\) and \(C_0 = I\), then \((-A_0, B_0, C_0; E_0)\) is a periodic linear system such that \(\det(I - e^{-x A_0} E_0 e^{-x A_0}) = 2i \sin x\).

For \(n = 1, 2, \ldots\), let \(A_n = C_n = J, E_n = q^{2n} I\) and \(B_n = 2E_n\); then \((-A_n, B_n, C_n; E_n)\) is a periodic linear system such that \(\det(I - e^{-x A_n} E_n e^{-x A_n}) = 1 - 2q^{2n} \cos 2x + q^{4n}\).

Hence we can introduce block diagonal matrices \(A = \text{diagonal}[A_0, A_1, \ldots]\) and \(E = \text{diagonal}[E_0, E_1, \ldots]\), and so on to give a periodic linear system \((-A, B, C; E)\) such that
\[
\det(I - e^{-x A} E e^{-x A}) = 2i \sin x \prod_{n=1}^{\infty} \frac{i\theta(x)}{q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})}. \tag{11.24}
\]

Next we replace \((-A, B, C; E)\) by the terms \((-A, e^{a_j} A B, C e^{a_j} A; e^{a_j} A e^{a_j} A)\) which give \(W_j\) by (11.4); likewise we introduce \((-A, e^{b_j} A B, -C e^{b_j} A; e^{b_j} A e^{b_j} A)\) which give \(\hat{W}_j\) by (11.4). We then form the block diagonal matrix
\[
\oplus_{j=1}^{m} \left( -A \oplus -A, e^{a_j} A B \oplus e^{b_j} A B, C e^{a_j} A \oplus (-C e^{b_j} A); e^{a_j} A e^{a_j} A \oplus e^{b_j} A e^{b_j} A \right) \tag{11.25}
\]
which gives the required \(W(x, y) = \oplus_{j=1}^{m} W_j(x, y) \oplus \hat{W}_j(x, y)\) by (11.4), and we verify
\[
\text{trace } W(x, x) = \sum_{j=1}^{m} \left( \text{trace } W_j(x, x) + \text{trace } \hat{W}_j(x, x) \right) = \frac{d}{dx} \sum_{j=1}^{m} \left( \log \theta(x-a_j) - \log \theta(x-b_j) \right) = \frac{d}{dx} \log \tau(x). \tag{11.26}
\]
One can check that \( W \) satisfies (11.6) with \( \varepsilon \) replaced by a diagonal matrix with diagonal entries \( \pm 1 \).

12. Linear systems for potentials on hyperelliptic curves

In this final section, we extend the analysis of section 11 to (11.2) in the case of \( g > 1 \).

To obtain a model for the Riemann surface of \( \mathcal{C} \), we choose a two-sheeted cover of \( \mathcal{C} \) with cuts along \( S_B \), and introduce the canonical homology basis consisting of:

- loops \( \alpha_j \) that start from \( \lambda_{2g} \), pass along the top sheet to \( \lambda_{2j-2}, \lambda_{2j-1} \), then return along the bottom sheet to the start on \( \lambda_{2g} \);
- loops \( \beta_j \) that go around the intervals of stability \( \lambda_{2j-2}, \lambda_{2j-1} \) that do not intersect with one another, for \( j = 1, \ldots, g \).

Then as in [14, p 61], we form the \( g \times 2g \) Riemann matrix \([I; \Omega]\) from the \( g \times g \) matrix blocks

\[
I = \left[ \int_{\alpha_k} \frac{x^{j-1} \, dx}{y} \right]_{j,k=1}^{g}, \quad \text{and} \quad \Omega = \left[ \int_{\beta_k} \frac{x^{j-1} \, dx}{y} \right]_{j,k=1}^{g}.
\]

Then the corresponding Riemann theta function is

\[
\Theta(s \mid \Omega) = \sum_{n \in \mathbb{Z}^g} \exp \left( i\pi \langle \Omega n, n \rangle + 2\pi i \langle s, n \rangle \right).
\]

Example. Suppose that \( g = 2 \) and let

\[
\Omega = \begin{bmatrix} a & b \\ b & d \end{bmatrix}
\]

where \( \Im a > 0, \Im d > 0 \) and \( b \in \mathbb{Q} \). Then choose \( p \in \mathbb{N} \) such that \( pb \in \mathbb{Z} \). One can easily check that

\[
\Theta(s, t \mid \Omega) = \sum_{r,\mu=0}^{p-1} e^{\pi (ar^2 + 2br\mu + d\mu^2)} e^{2\pi rs} e^{2\pi \mu t} \theta(ps + r \mid p^2 a) \theta(pt + \mu \mid p^2 d).
\]

Proposition 12.1 Suppose that \( q \) is a periodic potential with \( g \) spectral gaps, as above, and that \( \Theta(\cdot \mid \Omega) \) is a finite sum of products of Jacobi elliptic functions. Then there exist \( N < \infty, x_j \in \mathbb{R}, \sigma_j \in \mathbb{C} \) with \( \Im \sigma_j > 0 \); and block diagonal matrices \((A_j, B_j, C_j; E_j)\) with \( 2 \times 2 \) diagonal blocks for \( j = 1, \ldots, N \), such that \( \theta(x - x_j \mid \sigma_j) \) is the tau function of \((-A_j, B_j, C_j; E_j)\) and \( q \) belongs to the field \( \mathbb{C}(\theta(x - x_j \mid \sigma_j); j = 1, \ldots, N) \).

Proof. We introduce the product of real ovals

\[
T^g = \left\{ \frac{1}{2} \left( \Delta(x_j) + \sqrt{4 - \Delta(x_j)^2} \right)^g : \lambda_{2j-1} \leq x_j \leq \lambda_{2j} : j = 1, \ldots, g \right\}
\]
which has dimension \( g \). McKean and van Moerbeke [24, p260] considered the manifold \( \mathcal{M} \) of all the smooth real one-periodic potentials such that the corresponding Hill’s operator has simple spectrum \( \{ \lambda_1, \ldots, \lambda_{2g} \} \). Using the KdV hierarchy, they introduced a 1 to 1 differentiable map from \( \mathcal{M} \) onto \( T^g \), where the tangent vectors on \( \mathcal{M} \) are differential operators. Let \( \Lambda \) be the lattice generated by the columns of \([I; \Omega]\), and note that \( \mathbb{C}^g/\Lambda \) is the Jacobi variety of \( \mathbb{C} \). They showed that \( q \) extends to an abelian function on \( \mathbb{C}^g \) which is periodic with respect to \( \Lambda \), hence gives a function of rational character on \( \mathbb{C}^g/\Lambda \). The extended function \( q \) belongs to \( \mathcal{E}_g \), hence is a theta quotient. Moreover, translation on the potential is equivalent to linear motion on \( \mathbb{C}^g \) at constant velocity.

Thus they solved the inverse spectral problem explicitly, by showing on [24, p.262] that
\[
q(x) = \sum_{j=0}^{g} \varepsilon_j \frac{\Theta(X - \omega_j^*/2 \mid \Omega) \Theta(X - \omega_j^{**}/2 \mid \Omega)}{\Theta(X - \omega_\infty^*/2 \mid \Omega) \Theta(X - \omega_\infty^{**}/2 \mid \Omega)}
\]

(12.6)

where \( X = (x_1, \ldots, x_{g-1}, ax + b) \) has \( a, b, x_1, \ldots, x_{g-1} \) fixed, while \( x \) varies, and the constants \( \varepsilon_j, \omega_j^*, \omega_j^{**}, \omega_\infty^* \) and \( \omega_\infty^{**} \) are notionally computable.

By hypothesis, each factor \( \Theta(X - \omega^*/2 \mid \Omega) \) may be written a a finite sum of products of functions such as \( \theta(ax + c_j \mid d_j) \), and we can apply Theorem 11.3 to each such factor.

Weierstrass and Poincaré developed a systematic reduction procedure for such elliptic functions of higher genus, so we can describe the scope of Proposition 12.1. The Siegel upper half-space is
\[
S_g = \{ \Omega \in M_{g \times g}(\mathbb{C}) : \Omega = \Omega^t; \Im \Omega > 0 \}.
\]

(12.7)

Let \( X \) and \( J \) be the \( 2g \times 2g \) rational block matrices
\[
X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}
\]

(12.8)

such that \( XJX^t = J \); the set of all such \( X \) is the symplectic group \( Sp(2g; \mathbb{Q}) \). Now \( X \) is associated with the transformation \( \varphi_X \) of \( S_g \) given by
\[
\varphi_X(\Omega) = (\alpha I + \beta \Omega)^{-1}(\gamma I + \delta \Omega),
\]

(12.9)

thus \( Sp(2g; \mathbb{Q}) \) acts on \( S_g \).

**Proposition 12.2** [1] (i) Suppose that \( \Omega \) can be reduced to a diagonal matrix by the action of the symplectic group. Then \( \Theta(\mid \Omega) \) can be expressed as a sum of products of Jacobian elliptic theta functions.
(ii) Condition (i) is equivalent to \( C \) being a \( N \)-sheeted covering of the one-dimensional complex torus for some \( N \).

(iii) The orbit of \( \text{Sp}(2g, \mathbb{Q}) \) that contains \( iI \) is dense in \( S_g \).

References

1. E.D. Belokolos and V.Z. Enolskii, Reduction of abelian functions and algebraically integrable systems I. Complex analysis and representation theory. J. Math. Sci. (New York) 106 (2001), 3395–3486.
2. M. Bertola, B. Eynard, and J. Harnad, Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions, Comm. Math. Phys. 263 (2006), 401–437.
3. G. Blower, Integrable operators and the squares of Hankel operators, J. Math. Anal. Appl. 340 (2008), 943–953.
4. G. Blower, Linear systems and determinantal random point fields, J. Math. Anal. Appl. 355 (2009), 311–334.
5. A. Boutet de Monvel, L. Pastur and M. Shcherbina, On the statistical mechanics approach in the random matrix theory: integrated density of states, J. Statist. Physics 79 (1995), 585–611.
6. Y.V. Brezhnev, What does integrability of finite-gap or soliton potentials mean?, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 366 (2008), 923–945.
7. C.-T. Chen, Linear system theory and design, third edition, Oxford University Press, 1999.
8. Y. Chen and A.R. Its, A Riemann–Hilbert approach to the Akhiezer polynomials, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 366 (2008), 973–1003.
9. Y. Chen and N. Lawrence, A generalization of the Chebyshev polynomials, J. Phys. A 35 (2002), 4651–4699.
10. P. Deift, T. Kriecherbauer and K.T. -R. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field, J. Approx. Theory 95 (1998), 388–475.
11. F. Demontis and C. van der Mee, Explicit solutions of the cubic matrix nonlinear Schrödinger equation, Inverse Problems 24 (2008), 025020.
12. L.A. Dikij and I. M. Gelfand, Integrable nonlinear equations and the Liouville theorem, Funct. Anal. Appl. 13 (1979), 6–15.
13. H.M. Farkas and I. Kra, Riemann surfaces, Springer-Verlag, New York, 1980.
14. A.S. Fokas, A.R. Its, A.A. Kapaev and V. Yu. Novikshenov, Painlevé transcendents: the Riemann–Hilbert approach, American Mathematical Society, Providence Rhode Island, 2006.
15. P.L. Forrester and N.S. Witte, Random matrix theory and the sixth Painlevé equation, J. Phys. A 39 (2006), 12211–12233.
16. R. Fuchs, Über lineare homogone Differentialgleichungen zweiter Ordnung mit drei in Endlichen gelegenen wesentlich singulären Stellen, Math. Ann. 63 (1907), 301–321.
17. F. Gesztesy and R. Weikard, Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach, Bull. Amer. Math. Soc. (N.S.) 35 (1998), 271–317.
18. D. Guzzetti, The elliptic representation of the general Painlevé VI equation, Comm. Pure Appl. Math. 55 (2002), 1280–1363.
19. N.J. Hitchin, Riemann surfaces and integrable systems, pp 11–52 in N.J. Hitchin, G.B. Segal and R.S. Ward, Integrable systems: twistors, loop groups and Riemann surfaces, Oxford Science Publications, 1999.
20. M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I. General theory and \( \tau \) function, Phys. D 2 (1981), 306–352.
21. K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, Duke Math. J. 91 (1998), 151–204.
22. A.P. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, J.Comp. Appl. Math. 57 (1995), 215–237.
23. R. Maier, Lamé polynomials, hyperelliptic reduction and Lamé band structure, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 366 (2008), 1115–1153.
24. H.P. McKean and P. van Moerbeke, The spectrum of Hill’s equation, Invent. Math. 30 (1975), 217–274.
25. H. McKean and V. Moll, Elliptic curves: Function theory, geometry, arithmetic, Cambridge University Press, 1997.
26. M. L. Mehta, Random matrices, second edition (Academic Press, San Diego 1991).
27. K. Okamoto, On the \( \tau \)-function of the Painlevé equations, Physica D 2 (1981), 525–535.
28. L.A. Pastur, Spectral and probabilistic aspects of matrix models, pp 205–242, in Algebraic and Geometric Methods in Mathematical Physics, edrs. A. Boutet de Monvel and V.A. Marchenko, Kluwer Acad. Publishers, 1996.
29. E. B. Saff and V. Totik, Logarithmic potentials with external fields, Springer, Berlin 1997.
30. M.F. Singer, Introduction to the Galois theory of linear differential equations, pp. 1–83, in Algebraic theory of Differential Equations, Edrs M.A.H. McCallum and A.V. Mikhailov, London mathematical Society Lecture Notes, Cambridge, 2009.

38
31. C.A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models, Comm. Math. Phys. 163 (1994), 33–72.