ON A NONLINEAR EIGENVALUE PROBLEM FOR GENERALIZED LAPLACIAN IN ORLICZ-SOBOLEV SPACES

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Abstract. We consider a nonlinear eigenvalue problem for some elliptic equations governed by general operators including the \( p \)-Laplacian. The natural framework in which we consider such equations is that of Orlicz-Sobolev spaces. We exhibit two positive constants \( \lambda_0 \) and \( \lambda_1 \) with \( \lambda_0 \leq \lambda_1 \) such that \( \lambda_1 \) is an eigenvalue of the problem while any value \( \lambda < \lambda_0 \) cannot be so. By means of Harnack-type inequalities and a strong maximum principle, we prove the isolation of \( \lambda_1 \) on the right side. We emphasize that throughout the paper no \( \Delta_2 \)-condition is needed.

Key words and phrases: Orlicz-Sobolev spaces; Nonlinear eigenvalue problems; Harnack inequality.

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1. Introduction

Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^N \), \( N \geq 2 \), having the segment property. In this paper we investigate the existence and the isolation of an eigenvalue for the following weighted Dirichlet problem

\[
\begin{aligned}
-\text{div}(\phi(|\nabla u|)\nabla u) &= \lambda \rho(x)\phi(|u|)u \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \phi : (0, \infty) \to (0, \infty) \) is a continuous function, so that defining the function \( m(t) = \phi(|t|)t \) we suppose that \( m \) is strictly increasing and satisfies \( m(t) \to 0 \) as \( t \to 0 \) and \( m(t) \to \infty \) as \( t \to \infty \). The weight function \( \rho \in L^\infty(\Omega) \) is such that \( \rho \geq 0 \) a.e. in \( \Omega \) and \( \rho \neq 0 \) in \( \Omega \).

If \( \phi(t) = |t|^{p-2} \) with \( 1 < p < +\infty \) the problem (1.1) is reduced to the eigenvalue problem for the \( p \)-Laplacian

\[
\begin{aligned}
-\text{div}(\nabla u|u|^{p-2}\nabla u) &= \lambda \rho(x)|u|^{p-2}u \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.2)

while for \( p = 2 \) and \( \rho = 1 \) it is reduced to the classical eigenvalue problem for the Laplacian

\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.3)

It is known that the problem (1.3) has a sequence of eigenvalues \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) such that \( \lambda_n \to \infty \) as \( n \to \infty \). Moreover, the eigenvalues of the problem (1.3) have multiplicities and the first one is simple. Anane [2] proved the existence, simplicity and isolation of the first eigenvalue \( \lambda_1 > 0 \) of the problem (1.2) assuming some regularity on the boundary \( \partial \Omega \). The simplicity of the first eigenvalue of the problem (1.2) with \( \rho = 1 \) was proved later by Lindqvist.
without any regularity on the domain $\Omega$. For more results on the first eigenvalue of the $p$-Laplacian we refer for example to \[14] [16].

In the general setting of Orlicz-Sobolev spaces, the following eigenvalue problem
$$\begin{aligned}
\begin{cases}
-\text{div}(A(|\nabla u|^2)\nabla u) = \lambda \psi(u), & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}$$
(1.4)

was studied in \[5\] in the Orlicz-Sobolev space $W^1_0 L_\Phi(\Omega)$ where $\Phi(s) = \int_0^s A(|t|^2)t\,dt$ and $\psi$ is an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$. In \[5\] the authors proved the existence of a minimum of the functional $u \to \int_{\Omega} \Phi(|\nabla u|)\,dx$ which is subject to a constraint and they proved the existence of principal eigenvalues of the problem \[1.4\] by using a non-smooth version of the Ljusternik theorem and by assuming the $\Delta_2$-condition on the N-function $\Phi$ and its complementary $\overline{\Phi}$. Mustonen and Tienari \[13\] studied the eigenvalue problem
$$\begin{aligned}
\begin{cases}
-\text{div}\left(\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda \rho(x)\frac{m(|u|)}{|u|}u, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}$$
(1.5)
in the Orlicz-Sobolev space $W^1_0 L_M(\Omega)$, where $M(s) = \int_0^s m(t)\,dt$ with $m(t) = \phi(|t|)t$ and $\rho = 1$, without assuming the $\Delta_2$-condition neither on $M$ nor on its conjugate N-function $\overline{M}$. Consequently, the functional $u \to \int_{\Omega} M(|\nabla u|)\,dx$ is not necessarily continuously differentiable and so classical variational methods can not be applied. They prove the existence of eigenvalues $\lambda_r$ of problem \[1.5\] with $\rho = 1$ and for every $r > 0$, by proving the existence of a minimum of the real valued functional $\int_{\Omega} M(|\nabla u|)\,dx$ under the constraint $\int_{\Omega} M(u)\,dx = r$.

By the implicit function theorem they proved that every solution of such minimization problem is a weak solution of the problem \[1.5\]. This result was then extended in \[9\] to \[1.3\] with $\rho \neq 1$ and without assuming the $\Delta_2$-condition by using a different approach based on a generalized version of Lagrange multiplier rule. The problem \[1.1\] was studied in \[12\] under the restriction that both the corresponding N-function and its complementary function satisfy the $\Delta_2$-condition.

In the present paper we define
$$\lambda_0 = \inf_{u \in W^1_0 L_M(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \phi(|\nabla u|)|\nabla u|^2\,dx}{\int_{\Omega} \rho(x)\phi(|u|)|u|^2\,dx}$$
(1.6)
and
$$\lambda_1 = \inf \left\{ \int_{\Omega} M(|\nabla u|)\,dx \, | \, u \in W^1_0 L_M(\Omega), \int_{\Omega} \rho(x)M(|u|)\,dx = 1 \right\}.\quad (1.7)$$

In the particular case where $\phi(t) = |t|^{p-2}$, $1 < p < +\infty$, we obtain $\lambda_0 = \lambda_1$ and so $\lambda_0 = \lambda_1$ is the first isolated and simple eigenvalue of the problem \[1.2\] (see \[2\]).

However, in the non reflexive Orlicz-Sobolev structure the situation is more complicated since we can not expect that $\lambda_0 = \lambda_1$. Nonetheless, we show that $\lambda_0 \leq \lambda_1$ and that any value $\lambda < \lambda_0$ can not be an eigenvalue of the problem \[1.1\]. Following the lines of \[12\] we also show that $\lambda_1$ is an eigenvalue of problem \[1.1\] associated to an eigenfunction $u$ which is a weak solution of \[1.1\] (see Definition \[2\] below). It is in our purpose in this paper to prove that $\lambda_1$ is isolated from the right-hand side. To do so, we first prove some Harnack-type inequalities that enable us to show that $u$ is Hölder continuous and then by a strong maximum principle
we show that $u$ has a constant sign. Besides, we prove that any eigenfunction associated to another eigenvalue than $\lambda_1$ necessarily changes its sign. This allows us to prove that $\lambda_1$ is isolated from the right hand side.

Let $\Omega$ be an open subset in $\mathbb{R}^N$ and let $M(t) = \int_0^{|t|} m(s)ds$, $m(t) = \phi(|t|)t$. The natural framework in which we consider the problem (1.1) is the Orlicz-Sobolev space defined by

$$W^1L_M(\Omega) = \left\{ u \in L_M(\Omega) : \partial_i u := \frac{\partial u}{\partial x_i} \in L_M(\Omega), i = 1, \cdots, N \right\},$$

where $L_M(\Omega)$ stands for the Orlicz space defined as follows

$$L_M(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right)dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The spaces $L_M(\Omega)$ and $W^1L_M(\Omega)$ are Banach spaces under their respective norms

$$\|u\|_M = \inf \{ \lambda > 0 : \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right)dx \leq 1 \} \text{ and } \|u\|_{1,M} = \|u\|_M + \|\nabla u\|_M.$$

The closure in $L_M$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The complementary function $\overline{M}$ of the $N$-function $M$ is defined by

$$\overline{M}(x,s) = \sup_{t \geq 0} \{ st - M(x,t) \}.$$

Observe that by the convexity of $M$ follows the inequality

$$\|u\|_M \leq \int_{\Omega} M(|u(x)|)dx + 1 \text{ for all } u \in L_M(\Omega). \quad (1.8)$$

Denote by $W^1_0L_M(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^1L_M(\Omega)$ with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_M)$. It is known that if $\Omega$ has the segment property, then the four spaces

$$(W^1_0L_M(\Omega), W^1_0E_M(\Omega); W^{-1}L_{\overline{M}}(\Omega), W^{-1}E_{\overline{M}}(\Omega))$$

form a complementary system (see [6]). If $\Omega$ is bounded in $\mathbb{R}^N$ then by the Poincaré inequality [6 lemma 5.7], $\|u\|_{1,M}$ and $\|\nabla u\|_M$ are equivalent norms in $W^1_0L_M(\Omega)$.

Let $J : D(J) \to \mathbb{R} \cup \{ +\infty \}$ and $B : W^1_0L_M(\Omega) \to \mathbb{R}$ are the two functionals defined by

$$J(u) = \int_{\Omega} M(|\nabla u|)dx$$

and

$$B(u) = \int_{\Omega} \rho(x)M(|u|)dx,$$ \quad (1.10)

respectively. The functional $J$ takes values in $\mathbb{R} \cup \{ +\infty \}$. Since $W^1_0L_M(\Omega) \subset E_M(\Omega)$ (see [9]), then the functional $B$ is real valued on $W^1_0L_M(\Omega)$. Set

$$K = \{ u \in W^1_0L_M(\Omega) : B(u) = 1 \}.$$

In general, the functional $J$ is not finite nor of class $C^1$. 
2. Main results

We will show that $\lambda_1$ given by relation (1.7) is an eigenvalue of the problem (1.1) and isolated from the right hand side, while any $\lambda < \lambda_0$ is not an eigenvalue of (1.1). In the sequel we assume that $\Omega$ is an open bounded subset in $\mathbb{R}^N$ having the segment property.

**Definition 2.1.** A function $u$ is said to be a weak solution of (1.1) associated with $\lambda \in \mathbb{R}$ if

$$
\begin{aligned}
&u \in W_0^1 L_M(\Omega), m(|\nabla u|) \in L_M^+(\Omega) \\
&\int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla \psi \, dx = \lambda \int_{\Omega} \rho(x) \phi(|u|) u \psi \, dx, \quad \text{for all } \psi \in W_0^1 L_M(\Omega)
\end{aligned}
$$

(2.1)

In this definition, both of the two integrals in (2.1) make sense. Indeed, for all $u \in W_0^1 L_M(\Omega)$ since $m(|\nabla u|) = \phi(|\nabla u|)|\nabla u| \in L_M^+(\Omega)$, the first term is well defined. From the Young inequality and the integral representation of $M$, we easily get $M(m(u)) \leq um(u) \leq M(2u)$. So that since $u \in E_M(\Omega)$ the integral on the right-hand side also makes sense.

**Definition 2.2.** We said that $\lambda$ is an eigenvalue of the problem (1.1), if there exists a function $v \neq 0$ belonging to $W_0^1 L_M(\Omega)$ such that $(\lambda, v)$ satisfy (2.1). The function $v$ will be called an eigenfunction associated to the eigenvalue $\lambda$.

2.1. Existence result. We start with the next result that can be found in [13, Lemma 3.2]. For the convenience of the reader we give here a slightly different proof.

**Lemma 2.1.** Let $J$ and $B$ be defined by (1.9) and (1.10). Then
(i) $B$ is $\sigma(\Pi L_M, \Pi E_M^\ast)$ continuous,
(ii) $J$ is $\sigma(\Pi L_M, \Pi E_M^\ast)$ lower semi-continuous.

Proof. (i) Let $u_n \to u$ for $\sigma(\Pi L_M, \Pi E_M^\ast)$ in $W_0^1 L_M(\Omega)$. By the compact embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$, $u_n \to u$ in $E_M(\Omega)$ in norm. Hence $M(2(u_n - u)) \to 0$ in $L^1(\Omega)$. By the dominated convergence theorem, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ with $u_n \to u$ a.e. in $\Omega$ and there exists $h \in L^1(\Omega)$ such that

$$
M(2(u_n - u)) \leq h(x) \text{ a.e. in } \Omega
$$

for a subsequence. Therefore,

$$
|u_n| \leq |u| + \frac{1}{2} M^{-1}(h),
$$

so

$$
M(u_n) \leq \frac{1}{2} M(2u) + \frac{1}{2} h(x)
$$

and since $\rho \geq 0$ for a.e. in $\Omega$, then

$$
\rho(x) M(u_n) \leq \frac{1}{2} \rho(x) M(2u) + \frac{1}{2} \rho(x) h(x) \in L^1(\Omega).
$$

Thus, the assertion (i) follows from the dominated convergence theorem.

To show (ii) we assume that $u_n \to u$ for $\sigma(\Pi L_M, \Pi E_M^\ast)$ in $W_0^1 L_M(\Omega)$, that is

$$
\int_{\Omega} u_n v \, dx \to \int_{\Omega} u v \, dx \text{ and } \int_{\Omega} \partial_i u_n v \, dx \to \int_{\Omega} \partial_i u v \, dx,
$$

for all $v \in E_M^\ast$. This holds, in particular, for all $v \in L^\infty(\Omega)$. Hence,

$$
\partial_i u_n \to \partial_i u \text{ and } u_n \to u \text{ in } L^1(\Omega) \text{ for } \sigma(L^1, L^\infty).
$$

(2.2)
Since the embedding $W_0^1L_M(\Omega) \hookrightarrow L^1(\Omega)$ is compact, then \( \{ u_n \} \) is relatively compact in $L^1(\Omega)$. By passing to a subsequence, $u_n \rightarrow u$ strongly in $L^1(\Omega)$. In view of (2.2), $v = u$ and $u_n \rightarrow u$ strongly in $L^1(\Omega)$. Passing once more to a subsequence, we obtain that $u_n \rightarrow u$ almost everywhere on $\Omega$. Since $\zeta \mapsto M(|\zeta|)$ is convex for $\zeta \in \mathbb{R}^N$, we can use [3, Theorem 2.1, Chapter 8], to obtain

$$J(u) = \int_{\Omega} M(|\nabla u|)dx \leq \liminf_{n \to \infty} \int_{\Omega} M(|\nabla u_n|)dx = \liminf_{n \to \infty} J(u_n).$$

The first result of this paper is given by the following theorem.

**Theorem 2.1.** The infimum in (1.7) is achieved at some function $u \in K$ which is a weak solution of (1.1) and thus $u$ is an eigenfunction associated to the eigenvalue $\lambda_1$. Furthermore, $\lambda_0 \leq \lambda_1$ and each $\lambda < \lambda_0$ is not an eigenvalue of problem (1.1).

**Proof.** We split the proof of Theorem 2.1 into three steps.

**Step 1:** We show that the infimum in (1.7) is achieved at some function $u \in K$. By (1.8) we have

$$J(u) = \int_{\Omega} M(|\nabla u|)dx \geq \| \nabla u \|_M - 1.$$

So, $J$ is coercive. Let \( \{ u_n \} \subset W_0^1L_M(\Omega) \) be a minimizing sequence, i.e. $u_n \in K$ and $u_n \rightarrow \inf_{v \in K} J(v)$. The coercivity of $J$ implies that \( \{ u_n \} \) is bounded in $W_0^1L_M(\Omega)$ which is in the dual of a separable Banach space. By the Banach-Alaoglu-Bourbaki theorem, there exists $u \in W_0^1L_M(\Omega)$ such that for a subsequence still indexed by $n$, $u_n \rightarrow u$ for $\sigma(\Pi L_M, \Pi E_M)$ in $W_0^1L_M(\Omega)$. As a consequence of Lemma 2.1 the set $K$ is closed with respect to the topology $\sigma(\Pi L_M, \Pi E_M)$ in $W_0^1L_M(\Omega)$. Thus, $u \in K$. Since $J$ is $\sigma(\Pi L_M, \Pi E_M)$ lower semi-continuous, it follows

$$J(u) \leq \liminf_{v \in K} J(u_n) = \inf_{v \in K} J(v),$$

which shows that $u$ is a solution of (1.7).

**Step 2:** The function $u \in K$ found in Step 1 is such that $m(|\nabla u|) \in L_M(\Omega)$ and satisfies (2.1). This was already proved in [3, Theorem 4.2].

**Step 3:** Let $\lambda_0$ be given by (1.6). Any value $\lambda < \lambda_0$ cannot be an eigenvalue of problem (1.1). Indeed, suppose by contradiction that there exists a value $\lambda \in (0, \lambda_0)$ which is an eigenvalue of problem (1.1). It follows that there exists $u_\lambda \in W_0^1L_M(\Omega) \setminus \{ 0 \}$ such that

$$\int_{\Omega} \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla v dx = \lambda \int_{\Omega} \rho(x)\phi(|u_\lambda|)u_\lambda v dx \text{ for all } v \in W_0^1L_M(\Omega).$$

Thus, in particular for $v = u_\lambda$ we can write

$$\int_{\Omega} \phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 dx = \lambda \int_{\Omega} \rho(x)\phi(|u_\lambda|)|u_\lambda|^2 dx.$$
The fact that \( u_\lambda \in W_0^1 L_M(\Omega) \setminus \{0\} \) ensures that \( \int_\Omega \rho(x) \phi(|u_\lambda|)|u_\lambda|^2 \, dx > 0 \). By the definition of \( \lambda_0 \), we obtain

\[
\int_\Omega \phi(|\nabla u_\lambda|) |\nabla u_\lambda|^2 \, dx \geq \lambda_0 \int_\Omega \rho(x) \phi(|u_\lambda|)|u_\lambda|^2 \, dx \\
> \lambda \int_\Omega \rho(x) \phi(|u_\lambda|)|u_\lambda|^2 \, dx \\
= \int_\Omega \phi(|\nabla u_\lambda|) |\nabla u_\lambda|^2 \, dx.
\]

Which yields a contradiction. Therefore, we conclude that \( \lambda_0 \leq \lambda_1 \). The proof of Theorem 2.1 is now complete. \( \square \)

2.2. Isolation result. In this subsection we first show a maximum principle which enables us to prove that any eigenfunction associated to \( \lambda_1 \) has a constant sign in \( \Omega \). This property is then used to prove that \( \lambda_1 \) is isolated from the right-hand side.

Let \( w \) be an eigenfunction of problem (1.1) associated to the eigenvalue \( \lambda_1 \). Since \( |w| \in K \) it follows that \( |w| \) achieves also the infimum in (1.7), which implies that \( |w| \) is also an eigenfunction associated to \( \lambda_1 \). So we can assume that \( w \) is non-negative, that is \( w(x) \geq 0 \) for \( x \in \Omega \).

Since by Theorem 2.5 the eigenfunction \( w \) is bounded, we set

\[
0 \leq \delta := \sup_{\Omega} w < +\infty.
\]

Let \( f \) be the continuous strictly increasing function defined on \([0, \delta]\) by \( f(t) = \phi(t)t \) and let \( F(s) = \int_0^s f(t) \, dt \). Assume that

\[
\int_0^\delta \frac{ds}{H^{-1}(F(s))} = +\infty,
\]

where \( H \) is the function defined for all \( t \geq 0 \) by

\[
H(t) = tm(t) - M(t).
\]

The proof of the strong maximum principle will be given after proving the following two Lemmas.

**Lemma 2.2.** Denote by \( B(y, R) \) an open ball in \( \Omega \) of radius \( R \) and centered at \( y \in \Omega \) and consider the annulus

\[
E_R = \{ x \in B(y, R) : \frac{R}{2} \leq |y - x| < R \}.
\]

Assume that \((2.3)\) holds. Then there exists a function \( v \in C^1 \) with \( 0 < v < \delta, v' < 0 \) in \( E_R \) and \( w \geq v \) on \( \partial E_R \). Moreover, \( v \) satisfies

\[
-\int_\Omega \phi(|\nabla v|) \nabla v \nabla \psi \, dx \leq \int_\Omega f(v) \psi \, dx,
\]

for every \( \psi \in W_0^1 L_M(\Omega) \) and \( \psi \leq 0 \).

**Proof.** Let \( r = |y - x| \) for \( x \in \overline{E}_R \). The function \( v(x) = v(r) \) given by [15, Lemma 2] satisfies for every positive numbers \( k, l, \)

\[
[m(|v'|)]' + \frac{k}{r} m(|v'|) + lf(v) \leq 0,
\]
0 < v < δ, v' < 0 in ER and v(x) = 0 if |y − x| = R. In addition, for x ∈ ER with |y − x| = \frac{R}{2} we have v(x) < ε < \inf\{w(x) < δ\}. Hence, follows w ≥ v on ∂ER. Moreover, by the radial symmetric expression of div(\phi(|\nabla v|)w), we have
\[ \text{div}(\phi(|\nabla v|)w) - f(v) = -m(|v'|) - \frac{(N - 1)}{r}m(|v'|) - f(v) ≥ 0, \]
where we recall that v' < 0 and use \cite{15} Lemma 2. Multiplying the above inequality by \psi \in W^1_0L_M(Ω) with ψ ≤ 0 and then integrating over Ω we obtain (2.4). The proof is achieved.

\textbf{Lemma 2.3 (Weak comparison principle).} Assume that (2.3) holds. Let v be the C^1-function given by Lemma 2.2 with 0 < v < δ in Ω and w ≥ v on ∂Ω. Then w ≥ v in Ω.

\textit{Proof.} Let h = w − v in Ω. Assume by contradiction that there exists x_1 ∈ Ω such that h(x_1) < 0. Fix \epsilon > 0 so small that h(x_1) + \epsilon < 0. By Theorem 2.4 (see Appendix) the function \psi is continuous in Ω, then so is the function h. Since h ≥ 0 on ∂Ω, the support Ω_0 of the function h_\epsilon = \min\{h + \epsilon, 0\} is a compact subset in Ω. By Theorem 2.4 (see Appendix), the function h_\epsilon belongs to W^1_0L_M(Ω). Taking it as a test function in (2.1) and (2.4) it yields
\[ \int_{Ω_0} \phi(|\nabla \psi|)\nabla \psi \cdot (\nabla w - \nabla v)\,dx = \lambda_1 \int_{Ω_0} \rho(x)\phi(|w|)wh_\epsilon\,dx \]
and
\[ -\int_{Ω_0} \phi(|\nabla \psi|)\nabla \psi \cdot (\nabla w - \nabla v)\,dx ≤ \int_{Ω_0} \phi(|\psi|)vh_\epsilon\,dx. \]
Summing up the two formulations, we obtain
\[ \int_{Ω_0} [\phi(|\nabla \psi|)\nabla \psi - \phi(|\nabla \psi|)\nabla \psi] \cdot (\nabla w - \nabla v)\,dx ≤ \int_{Ω_0} (\lambda_1\rho(x)m(w) + m(v))h_\epsilon\,dx. \] (2.5)
The left hand side of (2.5) is positive due to Lemma 2.4 (given in Appendix), while the right hand side of (2.5) is non positive, since h_\epsilon < 0 in Ω_0. Therefore,
\[ \int_{Ω_0} [\phi(|\nabla \psi|)\nabla \psi - \phi(|\nabla \psi|)\nabla \psi] \cdot (\nabla w - \nabla v)\,dx = 0 \]
implying \nabla h_\epsilon = 0 and so h + \epsilon > 0 which contradicts the fact that h(x_1) + \epsilon < 0.

Now we can prove our maximum principle.

\textbf{Theorem 2.2 (Maximum principle).} Assume that (2.3) holds. Then, if w is a non-negative eigenfunction associated to \lambda_1, then w > 0 in Ω.

\textit{Proof.} Let B(y, R) be an open ball of Ω of radius R and centered at a fixed arbitrary y ∈ Ω. We shall prove that w(x) > 0 for all x ∈ B(y, R). Let v be the C^1-function given by Lemma 2.2 with w ≥ v on ∂ER where
\[ E_R = \{x ∈ B(y, R) : \frac{R}{2} ≤ |y − x| < R\}. \]
Applying Lemma 2.3 we get w ≥ v > 0 in E_R. For |y − x| < \frac{R}{2} we consider
\[ E_{\frac{R}{2}} = \{x ∈ B(y, R) : \frac{R}{4} ≤ |y − x| < \frac{R}{2}\}. \]
We can use similar arguments as in the proof of Lemma 2.2 to obtain that there is \( v \in C^1 \) in \( E_R \), with \( v > 0 \) in \( E_R \) and \( w \geq v \) on \( \partial E_R \). Applying again Lemma 2.3 we obtain \( w \geq v > 0 \) in \( E_R \). So, by the same way we can conclude that \( w(x) > 0 \) for any \( x \in B(y, R) \).

Now we are ready to prove that the associated eigenfunction of \( \lambda_1 \) has necessarily a constant sign in \( \Omega \).

**Proposition 2.1.** Assume that (2.3) holds. Then, every eigenfunction \( u \) associated to the eigenvalue \( \lambda_1 \) has constant sign in \( \Omega \), that is, either \( u > 0 \) in \( \Omega \) or \( u < 0 \) in \( \Omega \).

**Proof.** Let \( u \) be an eigenfunction associated to the eigenvalue \( \lambda_1 \). Then \( u \) achieves the infimum in (1.7). Since \( |u| \in K \) it follows that \( |u| \) achieves also the infimum in (1.7), which implies that \( |u| \) is also an eigenfunction associated to \( \lambda_1 \). Therefore, applying Theorem 2.2 with \( |u| \) instead of \( w \), we obtain \( |u| > 0 \) for all \( x \in \Omega \) and since \( u \) is continuous (see Theorem 2.7 in Appendix), then, either \( u > 0 \) or \( u < 0 \) in \( \Omega \).

Before proving the isolation of \( \lambda_1 \), we shall prove that every eigenfunction associated to another eigenvalue \( \lambda > \lambda_1 \) changes in force its sign in \( \Omega \). Denote by \( |E| \) the Lebesgue measure of a subset \( E \) of \( \Omega \).

**Proposition 2.2.** Assume that (2.3) holds. If \( v \in W^1_0 L_M(\Omega) \) is an eigenfunction associated to an eigenvalue \( \lambda > \lambda_1 \), then \( v^+ \not\equiv 0 \) and \( v^- \not\equiv 0 \) in \( \Omega \). Moreover, if we set \( \Omega^+ = \{ x \in \Omega : v(x) > 0 \} \) and \( \Omega^- = \{ x \in \Omega : v(x) < 0 \} \), then

\[
\min\{|\Omega^+|, |\Omega^-|\} \geq \min\left\{ \frac{1}{M(\min(a, 1))^d}, \frac{1}{M(\min(b, 1))^d} \right\}
\]

(2.6)

where \( a = \int \Omega v^+(x)dx \), \( b = \int \Omega v^-(x)dx \), \( c = c(\lambda, |\Omega|, \|v\|_\infty, \|\rho\|_\infty) \) and \( d \) is the constant in the Poincaré norm inequality (see [6, Lemma 5.7]).

**Proof.** By contradiction, we assume that there exists an eigenfunction \( v \) associated to \( \lambda > \lambda_1 \) such that \( v > 0 \). The case \( v < 0 \) being completely analogous so we omit it. Let \( u > 0 \) be an eigenfunction associated to \( \lambda_1 \). Let \( \Omega_0 \) be a compact subset of \( \Omega \) and define the two functions

\[
\eta_1(x) = \begin{cases} 
  u(x) - v(x) + \sup_{\Omega} v & \text{if } x \in \Omega_0 \\
  0 & \text{if } x \notin \Omega_0
\end{cases}
\]

and

\[
\eta_2(x) = \begin{cases} 
  v(x) - u(x) - \sup_{\Omega} v & \text{if } x \in \Omega_0 \\
  0 & \text{if } x \notin \Omega_0
\end{cases}
\]

Pointing out that \( v \) is bounded (Theorem 2.5 in Appendix), the two functions \( \eta_1 \) and \( \eta_2 \) are admissible test functions in (2.1) (see Theorem 2.4 in Appendix). Thus, we have

\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla \eta_1 dx = \lambda_1 \int_{\Omega} \rho(x)\phi(|u|) u \eta_1 dx
\]

and

\[
\int_{\Omega} \phi(|\nabla v|) \nabla v \cdot \nabla \eta_2 dx = \lambda \int_{\Omega} \rho(x)\phi(|v|) v \eta_2 dx.
\]
By summing up and using Lemma 2.4 (in Appendix), we get
\[
0 \leq \int_{\Omega} \left[ \phi(|\nabla u|) \nabla u - \phi(|\nabla v|) \nabla v \right] \cdot (\nabla u - \nabla v) dx \\
= \int_{\Omega} \rho(x) \left( \lambda_1 m(u) - \lambda m(v) \right) (u - v + \sup_{\Omega} v) dx.
\]

We claim that
\[
\lambda_1 m(u) \leq \lambda m(v).
\]

Indeed, suppose that \(\lambda_1 m(u) > \lambda m(v)\) and let us define the two admissible test functions
\[
\eta_1(x) = \begin{cases} 
 u(x) - v(x) - \sup_{\Omega} u & \text{if } x \in \Omega_0, \\
 0 & \text{if } x \notin \Omega_0
\end{cases}
\]
and
\[
\eta_2(x) = \begin{cases} 
 v(x) - u(x) + \sup_{\Omega} u & \text{if } x \in \Omega_0, \\
 0 & \text{if } x \notin \Omega_0
\end{cases}
\]
As above, inserting \(\eta_1\) and \(\eta_2\) in (2.1) and then summing up we obtain
\[
0 \leq \int_{\Omega} \left[ \phi(|\nabla u|) \nabla u - \phi(|\nabla v|) \nabla v \right] \cdot (\nabla u - \nabla v) dx \\
= \int_{\Omega} \rho(x) \left( \lambda_1 m(u) - \lambda m(v) \right) (u - v - \sup_{\Omega} v) dx \leq 0,
\]

implying by Lemma 2.4 that \(v = u\), but such an equality can not occur since \(\lambda > \lambda_1\) which proves our claim. Finally, we conclude that the function \(v\) can not have a constant sign in \(\Omega\).

Next we prove the estimate (2.6). According to the above \(v^+ > 0\) and \(v^- > 0\). Choosing \(v^+ \in W^1_0 L_M(\Omega)\) as a test function in (2.1), we get
\[
\int_{\Omega} m(|\nabla v^+|) |\nabla v^+| dx = \lambda \int_{\Omega} \rho(x) m(v^+) v^+ dx.
\]
Since \(M(t) \leq m(t) t \leq M(2t)\) for \(t \geq 0\), we obtain
\[
\int_{\Omega} M(|\nabla v^+|) dx \leq \lambda \|\rho(\cdot)\|_{\infty} \int_{\Omega} M(2v^+) dx.
\]
We already know that by Theorem 2.5 (in Appendix) the function \(v\) is bounded, then we get
\[
\int_{\Omega} M(|\nabla v^+|) dx \leq \lambda \|\rho(\cdot)\|_{\infty} M(2\|v\|_{\infty}) |\Omega|.
\]
(2.7)

So, (1.8) and (2.7) imply that there exists a positive constant \(c\), such that
\[
\|\nabla v^+\|_M \leq c.
\]
(2.8)

On the other hand, by the Hölder inequality [10] and the Poincaré type inequality [6] Lemma 5.7, we have
\[
\int_{\Omega} v^+(x) dx \leq \|\chi_{\Omega^+}\|_M \|v^+\|_M \leq d \|\chi_{\Omega^+}\|_M \|\nabla v^+\|_M,
\]
d being the constant in Poincaré type inequality. Hence, using (2.8) to get
\[
\int_{\Omega} v^+(x) dx \leq cd \|\chi_{\Omega^+}\|_M.
\]
(2.9)
We have to distinguish two cases, the case \( \int_\Omega v^+(x)dx > 1 \) and \( \int_\Omega v^+(x)dx \leq 1 \).

**Case 1**: Assume that \( \int_\Omega v^+(x)dx > 1 \).

Thus, by (2.9) we have
\[
\frac{1}{d} \leq \|\chi_+\|_M. \tag{2.10}
\]

**Case 2**: Assume that \( \int_\Omega v^+(x)dx \leq 1 \).

Recall that by Theorem 2.7 (in Appendix) the function \( v^+ \) is continuous and as \( v^+ > 0 \) in \( \Omega \) then \( \int_\Omega v^+(x)dx > 0 \). Therefore, by using (2.9) we obtain
\[
\frac{a}{d} \leq \|\chi_+\|_M, \tag{2.11}
\]

where \( a = \int_\Omega v^+(x)dx \). So, by (2.10) and (2.11), we get
\[
\min\{a, 1\} \leq \|\chi_+\|_M,
\]

where \( \|\chi_+\|_M = \frac{1}{M^{-1}\left(\frac{1}{|\Omega^+|}\right)} \) (see [10, page 79]). Hence,
\[
|\Omega^+| \geq \frac{1}{M\left(\frac{1}{\min\{a, 1\}}\right)}.
\]

Such an estimation with \( v^- \) can be obtained following exactly the same lines above. Then follows the inequality (2.6).

Finally, we prove that the eigenvalue \( \lambda_1 \) given by the relation (1.7) is isolated from the right-hand side.

**Theorem 2.3.** Assume that (2.3) holds. Then, the eigenvalue \( \lambda_1 \) is isolated from the right-hand side, that is, there exists \( \delta > 0 \) such that in the interval \( (\lambda_1, \lambda_1 + \delta) \) there are no eigenvalues.

**Proof.** Assume by contradiction that there exists a non-increasing sequence \( \{\mu_n\} \) of eigenvalues of (1.1) with \( \mu_n > \lambda_1 \) and \( \mu_n \to \lambda_1 \). Let \( u_n \) be an associated eigenfunction to \( \mu_n \) and let
\[
\Omega^+_n = \{x \in \Omega : u_n > 0\} \quad \text{and} \quad \Omega^-_n = \{x \in \Omega : u_n < 0\}.
\]

By (2.6), there exists \( c_n > 0 \) such that
\[
\min\{|\Omega^+_n|, |\Omega^-_n|\} \geq c_n. \tag{2.12}
\]

Since \( b_n := \int_\Omega \rho(x)M(|u_n(x)|)dx > 0 \) we define
\[
v_n(x) = \begin{cases} 
M^{-1}\left(\frac{M(u_n(x))}{b_n}\right) & \text{if } x \in \Omega^+_n, \\
-M^{-1}\left(\frac{M(-u_n(x))}{b_n}\right) & \text{if } x \in \Omega^-_n.
\end{cases} \tag{2.13}
\]
On the other hand, we have

\[ |\nabla v_n| \leq \left| (M^{-1})' \left( \frac{M(|u_n|)}{b_n} \right) \right| \frac{m(|u_n|)|\nabla u_n|}{b_n}, \]

since \( u_n \) is continuous, then there exists \( d_n > 0 \) such that \( \inf_{x \in \Omega_n^+ \cup \Omega_n^-} |u_n(x)| \geq d_n \). Let \( b = \min\{b_n\} \) and \( d = \min\{d_n\} \). Being \( \{u_n\} \) uniformly bounded (Theorem 2.5 in Appendix), there exists a constant \( c_\infty > 0 \), not depending on \( n \), such that

\[ \|u_n\|_\infty \leq c_\infty, \text{ for all } n \in \mathbb{N}. \quad (2.14) \]

Using the fact that \( (M^{-1})'(\cdot) \) is decreasing, we get

\[ |\nabla v_n| \leq \left| (M^{-1})' \left( \frac{M(d)}{\|\rho\|_\infty M(c_\infty)\Omega} \right) \right| \frac{m(c_\infty)}{b} |\nabla u_n| = C_0 |\nabla u_n|, \quad (2.15) \]

where \( C_0 = \left| (M^{-1})' \left( \frac{M(d)}{\|\rho\|_\infty M(c_\infty)\Omega} \right) \right| \frac{m(c_\infty)}{b} \). On the other hand, taking \( u_n \) as test function in (2.1) and using (2.14) and the inequality \( M(t) \leq m(t)t \) for \( t > 0 \), one has

\[ \int_\Omega M(|\nabla u_n|)dx \leq \mu_n \|\rho\|_\infty m(c_\infty)c_\infty |\Omega|. \]

Since \( \mu_n \) converges to \( \lambda_1 \), there exists a constant \( C_1 > 0 \), such that

\[ \int_\Omega M(|\nabla u_n|)dx \leq C_1. \quad (2.16) \]

Therefore, by (2.15) and (2.16) we obtain that \( \{v_n\} \) is uniformly bounded in \( W^1_0 L_M(\Omega) \).

Alaoglu’s theorem ensures the existence of a function \( v \in W^1_0 L_M(\Omega) \) and a subsequence of \( v_n \), still indexed by \( n \), such that \( v_n \to v \) for \( \sigma(\Pi_M, \Pi E^\perp_M) \). By (2.13), \( v_n \in K \) and since \( B \) is \( \sigma(\Pi_M, \Pi E^\perp_M) \) continuous (see Lemma 2.1), then

\[ \int_\Omega \rho(x)M(|v(x)|)dx = B(v) = \lim_{n \to \infty} B(v_n) = 1. \]

Therefore, \( v \in K \). Since by Lemma 2.1 the functional \( J \) is \( \sigma(\Pi_M, \Pi E^\perp_M) \) lower semi-continuous, we get

\[ J(v) = \int_\Omega M(|\nabla v|)dx \leq \lim \inf J(v_n) = \inf_{w \in K} J(w). \]

So that \( v \) is an eigenfunction associated to \( \lambda_1 \). Applying Proposition 2.4, we have either \( v > 0 \) or \( v < 0 \) in \( \Omega \). Assume that \( v < 0 \) in \( \Omega \) with \( v^- \not\equiv 0 \). By Egorov’s Theorem, \( v_n \) converges uniformly to \( v \) except on a subset of \( \Omega \) of null Lebesgue measure. Thus, \( v_n \leq 0 \) a.e. in \( \Omega \) with \( v_n^- \not\equiv 0 \) outside a subset of \( \Omega \) of null Lebesgue measure, which implies that

\[ |\Omega_n^-| = 0, \]

which is a contradiction with the estimation (2.12). \( \square \)
Appendix

We prove here some important lemmas that are necessary for the accomplishment of the proofs of the above results.

Lemma 2.4. Let \( \xi \) and \( \eta \) be vectors in \( \mathbb{R}^N \). Then
\[
[\phi(|\xi|) \xi - \phi(|\eta|) \eta] \cdot (\xi - \eta) > 0, \quad \text{whenever } \xi \neq \eta.
\]
Proof. Since \( \phi(t) > 0 \) when \( t > 0 \) and \( \xi \cdot \eta \leq |\xi| \cdot |\eta| \), there follows by a direct calculation
\[
[\phi(|\xi|) \xi - \phi(|\eta|) \eta] \cdot (\xi - \eta) \geq [m(|\xi|) - m(|\eta|)] \cdot (|\xi| - |\eta|)
\]
and the conclusion comes from the strict monotonicity of \( m \). \( \square \)

The following result can be found in [3, Lemma 9.5] in the case of Sobolev spaces.

Theorem 2.4. Let \( A \) be an \( N \)-function (cf. [1]). If \( u \in W^1 L_A(\Omega) \) has a compact support in an open \( \Omega \) having the segment property, then \( u \in W^1_0 L_A(\Omega) \).

Proof. Let \( u \in W^1 L_A(\Omega) \). We fix a compact set \( \Omega' \subset \Omega \) such that \( \text{supp } u \subset \Omega' \) and we denote by \( \tilde{u} \) the extension by zero of \( u \) to the whole of \( \mathbb{R}^N \). Let \( J \) be the Friedrichs mollifier kernel defined on \( \mathbb{R}^N \) by
\[
\rho(x) = ke^{-\frac{1}{1-\|x\|^2}} \quad \text{if } \|x\| < 1 \quad \text{and } 0 \quad \text{if } \|x\| \geq 1,
\]
where \( k > 0 \) is such that \( \int_{\mathbb{R}^N} \rho(x)dx = 1 \). For \( \epsilon > 0 \), we define \( \rho_\epsilon(x) = n^N J(\epsilon x) \). By [4], there exists \( \lambda > 0 \) large enough such that \( A \left( \frac{|u(x)|}{\lambda} \right) \in L^1(\Omega) \), \( A \left( \frac{\|\partial u/\partial x_i(x)\|}{\lambda} \right) \in L^1(\Omega) \), \( i \in \{1, \cdots , N\} \), and
\[
\int_{\mathbb{R}^N} A \left( \frac{|\rho_\epsilon \ast \tilde{u}(x) - \tilde{u}(x)|}{\lambda} \right) dx \to 0 \quad \text{as } n \to +\infty
\]
and hence
\[
\int_{\Omega} A \left( \frac{|\rho_\epsilon \ast \tilde{u}(x) - u(x)|}{\lambda} \right) dx \to 0 \quad \text{as } n \to +\infty. \quad (2.17)
\]
Choosing \( n \) large enough so that \( 0 < \frac{1}{n} < \text{dist}(\Omega', \partial \Omega) \) one has \( \rho_\epsilon \ast \tilde{u}(x) = \rho_\epsilon \ast u(x) \) for every \( x \in \Omega' \). Hence, \( \partial(\rho_\epsilon \ast \tilde{u})/\partial x_i = \rho_\epsilon \ast (\partial u/\partial x_i) \) on \( \Omega' \) for every \( i \in \{1, \cdots , N\} \). As \( \partial u/\partial x_i \in L_A(\Omega') \) we have
\[
\partial(\rho_\epsilon \ast \tilde{u})/\partial x_i \in L_A(\Omega').
\]
Therefore,
\[
\int_{\Omega} A \left( \frac{|\partial(\rho_\epsilon \ast \tilde{u})/\partial x_i(x) - \partial u/\partial x_i(x)|}{\lambda} \right) dx \to 0 \quad \text{as } n \to +\infty. \quad (2.18)
\]
Observe that the functions \( w_\epsilon = \rho_\epsilon \ast \tilde{u} \) do not necessary lie in \( C_0^\infty(\Omega) \). Let \( \eta \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 < \eta < 1 \), \( \eta(x) = 1 \) for all \( x \) with \( \|x\| \leq 1 \), \( \eta(x) = 0 \) for all \( x \) with \( \|x\| \geq 2 \) and \( |\nabla \eta| \leq 2 \). Let further \( \eta_n(x) = \eta \left( \frac{x}{n} \right) \) for \( x \in \mathbb{R}^N \). We claim that the functions \( v_\epsilon = \eta_n w_\epsilon \) belong to \( C_0^\infty(\mathbb{R}^N) \) and satisfy
\[
\int_{\Omega} A \left( \frac{|v_\epsilon(x) - u(x)|}{4\lambda} \right) dx \to 0 \quad \text{as } n \to +\infty \quad (2.19)
\]
and
\[
\int_{\Omega'} A \left( \frac{|\partial v_\epsilon/\partial x_i(x) - \partial u/\partial x_i(x)|}{12\lambda} \right) dx \to 0 \quad \text{as } n \to +\infty \quad (2.20)
\]
Indeed, by (2.17) there exist a subsequence of \( \{w_n\} \) still indexed by \( n \) and a function \( h_1 \in L^1(\Omega) \) such that

\[
    w_n \to u \text{ a.e. in } \Omega
\]

and

\[
    |w_n(x)| \leq |u(x)| + \lambda A^{-1}(h_1)(x); \text{ for all } x \in \Omega \tag{2.21}
\]

which together with the convexity of \( A \) in force for the weak topology

Consequently, the sequence \( \{w_n\} \) converges weakly with respect to the weak \( \ast \) topology for any weak solution \( u \) of \( (1.1) \).

Being the functions \( A \) equi-integrable on \( \Omega \) and since \( \{w_n\} \) converges to \( u \) a.e. in \( \Omega \), we obtain (2.19) by applying Vitali's theorem.

By (2.18) there exists a subsequence, relabeled again by \( n \), and a function \( h_2 \in L^1(\Omega') \) such that

\[
    \partial w_n/\partial x_i \to \partial u/\partial x_i \text{ a.e. in } \Omega'
\]

and

\[
    |\partial w_n/\partial x_i(x)| \leq |\partial u/\partial x_i(x)| + h_2(x), \text{ for all } x \in \Omega'. \tag{2.22}
\]

Therefore, using (2.21) and (2.22) for all \( x \in \Omega' \) we arrive at

\[
    A\left(\frac{|\partial w_n/\partial x_i(x)|}{\lambda}\right) \leq \frac{1}{6} A\left(\frac{|u(x)|}{\lambda}\right) + A\left(\frac{|\partial u/\partial x_i(x)|}{\lambda}\right) + h_1(x) + \frac{1}{2} h_2(x)
\]

and by Vitali's theorem we obtain (2.20).

Finally, let \( K \subset \Omega' \) be a compact set such that \( \text{supp}(u) \subset K \). There exists a cut-off function \( \zeta \in C_0^\infty(\Omega') \) satisfying \( \zeta = 1 \) on \( K \). Denoting \( u_n = \zeta u_n \), we can deduce from (2.19) and (2.20)

\[
    \int_\Omega A\left(\frac{|u_n(x) - \zeta u(x)|}{12 \lambda}\right) dx \to 0 \text{ as } n \to +\infty
\]

and

\[
    \int_\Omega A\left(\frac{|\partial u_n/\partial x_i(x) - \partial(\zeta u)/\partial x_i(x)|}{12 \lambda}\right) dx \to 0 \text{ as } n \to +\infty.
\]

Consequently, the sequence \( \{u_n\} \subset C_0^\infty(\Omega) \) converges modularly to \( \zeta u = u \) in \( W^1 L_\lambda(\Omega) \) and in force for the weak topology \( \sigma(\Pi L_\lambda, \Pi L_\lambda^\ast) \) (see [7, Lemma 6]) which in turn imply the convergence with respect to the weak* topology \( \sigma(\Pi L_\lambda, \Pi E^\ast_\lambda) \). Thus, \( u \in W^1_0 L_\lambda(\Omega) \). \( \square \)

**Theorem 2.5.** For any weak solution \( u \in W^1_0 L_\lambda(\Omega) \) of (1.1) there exists a constant \( c > 0 \), not depending on \( u \), such that

\[
    \|u\|_{L^\infty(\Omega)} \leq c.
\]

**Proof.** For \( k > 0 \) we define the set \( A_k = \{x \in \Omega : |u(x)| > k\} \) and the two truncation functions \( T_k(s) = \max(-k, \min(s,k)) \) and \( G_k(s) = s - T_k(s) \). By Hölder's inequality we get

\[
    \int_{A_k} |G_k(u(x))| dx \leq |A_k|^{\frac{1}{N}} \left( \int_{A_k} |G_k(u(x))|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}}
\]

\[
    \leq C(N) |A_k|^{\frac{1}{N}} \int_{A_k} |\nabla u| dx,
\]
where $C(N)$ is the constant in the embedding $W^{1,1}_0(A_k) \hookrightarrow L^{\frac{N}{N-1}}(A_k)$. We shall estimate the integral $\int_{A_k} |\nabla u| dx$; to this aim we distinguish two cases: the case $m(|\nabla u|)|\nabla u| < \lambda \| \rho \|_\infty$ and $m(|\nabla u|)|\nabla u| \geq \lambda \| \rho \|_\infty$.

**Case 1**: Assume that

\[ m(|\nabla u|)|\nabla u| < \lambda \| \rho \|_\infty. \tag{2.23} \]

Let $k_0 > 0$ be fixed and let $k > k_0$. Using (2.23) we can write

\[
\int_{A_k} |\nabla u| dx \leq \int_{A_k \cap \{|\nabla u| \leq 1\}} |\nabla u| dx + \int_{A_k \cap \{|\nabla u| > 1\}} |\nabla u| dx
\]

\[
\leq |A_k| + \frac{1}{m(1)} \int_{A_k} m(|\nabla u|)|\nabla u| dx
\]

\[
\leq (1 + \frac{\lambda \| \rho \|_\infty}{m(1)}) |A_k|.
\]

Thus,

\[
\int_{A_k} |G_k(u(x))| dx \leq C(N) \left(1 + \frac{\lambda \| \rho \|_\infty}{m(1)}\right) |A_k|^\frac{1}{N} + 1. \tag{2.24}
\]

**Case 2**: Assume now that

\[ m(|\nabla u|)|\nabla u| \geq \lambda \| \rho \|_\infty. \tag{2.25} \]

Since $u \in W^{1,1}_0 L_M(\Omega)$ is a weak solution of problem (1.1), we have

\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla \varphi dx = \lambda \int_{\Omega} \rho(x) \phi(|u|) u \varphi dx, \tag{2.26}
\]

for all $\varphi \in W^{1,1}_0 L_M(\Omega)$. Let $s, t, k > 0$ and let $v = \exp(M(u^+)) T_s(G_k(T_t(u^+)))$. So, from [8, Lemma 2] we can use $v$ as a test function in (2.26) to obtain

\[
\int_{\Omega} m(|\nabla u|)|\nabla u|m(u^+) \exp(M(u^+)) T_s(G_k(T_t(u^+))) dx
\]

\[+ \int_{\{k < T_t(u^+) \leq k+s\}} \phi(|\nabla u|) \nabla u \cdot \nabla T_t(u^+) \exp(M(u^+)) dx
\]

\[\lambda \int_{\Omega} \rho(x) \phi(|u|) u \exp(M(u^+)) T_s(G_k(T_t(u^+))) dx = 0.
\]

Since we integrate on the set $\{u > 0\}$, by (2.25) we have

\[
\lambda \rho(x) \phi(|u|) u \leq \phi(|u|) u m(|\nabla u|)|\nabla u|
\]

and so obtain

\[
\int_{\Omega} m(|\nabla u|)|\nabla u|m(u^+) \exp(M(u^+)) T_s(G_k(T_t(u^+))) dx
\]

\[\lambda \int_{\Omega} \rho(x) \phi(|u|) u \exp(M(u^+)) T_s(G_k(T_t(u^+))) dx \geq 0.
\]

Therefore, we only have

\[
\int_{\{k < T_t(u^+) \leq k+s\}} \phi(|\nabla u|) \nabla u \cdot \nabla T_t(u^+) \exp(M(u^+)) dx = 0
\]

and since $\exp(M(u^+)) \geq 1$ we have

\[
\int_{\{k < T_t(u^+) \leq k+s\}} \phi(|\nabla u|) \nabla u \cdot \nabla T_t(u^+) dx = 0.
\]
Pointing out that
\[ \int_{\{k < T_t(u^+) \leq k + s\}} \phi(|\nabla u|) \nabla u \cdot \nabla T_t(u^+) dx = \int_{\{k < u \leq k + s\} \cap \{0 < u < t\}} \phi(|\nabla u|) \nabla u \cdot \nabla u dx. \]
We can apply the monotone convergence theorem as \( t \to +\infty \) obtaining
\[ \int_{\{k < u \leq k + s\}} \phi(|\nabla u|) \nabla u \cdot \nabla u dx = 0. \]
Applying again the monotone convergence theorem as \( k \to +\infty \) we get
\[ \int_{\{u > -k\}} \phi(|\nabla u|) \nabla u \cdot \nabla u dx = 0. \]
Thus, since \( m(t) = \phi(|t|) t \) we conclude that
\[ \int_{A_k} m(|\nabla u|) |\nabla u| dx = 0. \tag{2.27} \]
On the other hand, by the monotonicity of the function \( m^{-1} \) and by (2.27), we can write
\[ \int_{A_k} |\nabla u| dx = \int_{A_k \cap \{m(|\nabla u|) < 1\}} |\nabla u| dx + \int_{A_k \cap \{m(|\nabla u|) \geq 1\}} |\nabla u| dx \leq m^{-1}(1)|A_k| + \int_{A_k} m(|\nabla u|) |\nabla u| dx = m^{-1}(1)|A_k|. \]
Hence,
\[ \int_{A_k} |G_k(u(x))| dx \leq C(N)|A_k|^\frac{1}{N+1}. \tag{2.28} \]
Finally, we note that the two obtained inequalities (2.24) and (2.28) are exactly the starting point of Stampacchia’s \( L^\infty \)-regularity proof (see [17]), so that in both cases there exists a constant \( c \), not depending on \( u \), such that
\[ \|u\|_{L^\infty(\Omega)} \leq c. \]
\[ \square \]
Lemma 2.5. Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^N \). Let \( B_R \subset \Omega \) be an open ball of radius \( 0 < R \leq 1 \). Suppose that \( g \) is a non-negative function such that \( g^\alpha \in L^\infty(B_R) \), where \( \alpha \geq 1 \). Assume that
\[ \left( \int_{B_R} g^{\alpha k} dx \right)^{\frac{1}{k}} \leq C \int_{B_R} g^\alpha dx, \tag{2.29} \]
where \( q, k > 1 \) and \( C \) is a positive constant. Then for any \( p > 0 \) there exists a positive constant \( c \) such that
\[ \sup_{B_R} g^\alpha \leq \frac{c}{R^{(k-1)p}} \left( \int_{B_R} g^p dx \right)^{\frac{1}{p}}. \]
Proof. Let $q = pk^\nu$ where $\nu$ is a non-negative integer. Then using (2.29) and the fact that $R \leq 1$ we can have

$$
\left( \int_{B^R} g^{\alpha pk^{\nu+1}} dx \right)^{\frac{1}{pk^{\nu+1}}} \leq \left( \frac{C}{R} \right)^{pk^\nu} \left( \int_{B^R} g^{\alpha pk^\nu} dx \right)^{\frac{1}{pk^\nu}}.
$$

An iteration of this inequality with respect to $\nu$ yields

$$
\|g^\alpha\|_{L^{pk^{\nu+1}}(B^R)} \leq \left( \frac{C}{R} \right)^{p \sum_{i=0}^{\nu} \frac{1}{k^i}} \left( \int_{B^R} g^{\alpha pk^\nu} dx \right)^{\frac{1}{pk^\nu}}.
$$

(2.30)

For $\beta \geq 1$, we consider $\nu$ large enough such that $pk^{\nu+1} > \beta$. Then, there exists a constant $c_0$ such that

$$
\|g^\alpha\|_{L^\beta(B^R)} \leq c_0 \|g^\alpha\|_{L^{pk^{\nu+1}}(B^R)}.
$$

Since the series in (2.30) are convergent and $g^\alpha \in L^\infty(B^R)$, Theorem 2.14 in [1] implies that

$$
\sup_{B^R} g^\alpha \leq \frac{c}{R^N} \left( \int_{B^R} g^{\alpha pk^\nu} dx \right)^{\frac{1}{pk^\nu}}.
$$

□

As we need to get a Hölder estimate for weak solutions of (1.1), we use the previous lemma to prove Harnack-type inequalities. To do this, we define for a bounded weak solution $u \in W^1_0 L^M(\Omega)$ of (1.1) the two functions $v = u - \inf_{B^R} u$ and $w = \sup_{B^R} u - u$. We start by proving the following two lemmas.

**Lemma 2.6.** Let $B^r \subset \Omega$ be an open ball of radius $0 < r \leq 1$. Then for every $p > 0$, there exists a positive constant $C$, depending on $p$, such that

$$
\sup_{B^r_{\frac{r}{2}}} v \leq C \left( \frac{1}{r^N} \int_{B^r_{\frac{r}{2}}} v^p dx \right)^{\frac{1}{p}} + r,
$$

(2.31)

where $B^r_{\frac{r}{2}}$ is the ball of radius $r/2$ concentric with $B^r$.

**Proof.** Since $u$ is a weak solution of problem (1.1) then $v$ satisfies the weak formulation

$$
\int_{\Omega} \phi(|\nabla v|) \nabla v \cdot \nabla \psi dx = \lambda \int_{\Omega} \rho(x) \phi(|v + \inf_{B^r} u|)(v + \inf_{B^r} u) \psi dx,
$$

(2.32)

for every $\psi \in W^1_0 L^M(\Omega)$. Let $\Omega_0$ be a compact subset of $\Omega$ such that $B^r_{\frac{r}{2}} \subset \Omega_0 \subset B^r$. Let $q > 1$ and let $\psi$ be the function defined by

$$
\psi(x) = \begin{cases} 
M(\bar{v}(x))^{q-1} \bar{v}(x) & \text{if } x \in \Omega_0, \\
0 & \text{if } x \notin \Omega_0
\end{cases}
$$

where $\bar{v} = v + r$. Observe that on $\Omega_0$

$$
\nabla \psi = M(\bar{v})^{q-1} \nabla \bar{v} + (q-1)M(\bar{v})^{q-2}m(\bar{v})\bar{v} \nabla \bar{v}
$$
and thus by Theorem 2.24 we have $\psi \in W_0^1 L_M(\Omega)$. So that $\psi$ is an admissible test function in (2.32). Taking it so it yields

$$
\int_{B_r} M(\tilde{v})^{q-1} m(|\nabla \tilde{v}|)|\nabla \tilde{v}|dx + (q - 1) \int_{B_r} M(\tilde{v})^{q-2} \tilde{m}(\tilde{v}) |\nabla \tilde{v}|dx = \lambda \int_{B_r} \rho(x) M(\tilde{v})^{q-1} \tilde{m}(\tilde{v} + \inf_{B_r} u) dx.
$$

Since $\tilde{m}(\tilde{v}) \geq M(\tilde{v})$ and $v + \inf_{B_r} u \leq \tilde{v} + \|u\|_{\infty}$, we get

$$
q \int_{B_r} M(\tilde{v})^{q-1} m(|\nabla \tilde{v}|)|\nabla \tilde{v}|dx \leq \lambda \|\rho\|_{\infty} \int_{B_r} M(\tilde{v})^{q-1} (\tilde{v} + \|u\|_{\infty}) \tilde{m}(\tilde{v} + \|u\|_{\infty}) dx.
$$

Let

$$
h(x) = \begin{cases} 
M(\tilde{v}(x))^q & \text{if } x \in \Omega_0, \\
0 & \text{if } x \notin \Omega_0.
\end{cases}
$$

Using the following inequality

$$
am(b) \leq b m(b) + am(a),
$$

with $a = |\nabla \tilde{v}|$ and $b = \tilde{v}$, we obtain

$$
\int_{B_r} |\nabla h| dx \leq q \int_{B_r} M(\tilde{v})^{q-1} m(|\nabla \tilde{v}|)|\nabla \tilde{v}|dx + q \int_{B_r} M(\tilde{v})^{q-1} \tilde{m}(\tilde{v}) dx
$$

$$
\leq q \int_{B_r} M(\tilde{v})^{q-1} m(|\nabla \tilde{v}|)|\nabla \tilde{v}|dx
$$

$$
+ q \int_{B_r} M(\tilde{v})^{q-1} (\tilde{v} + \|u\|_{\infty}) \tilde{m}(\tilde{v} + \|u\|_{\infty}) dx.
$$

In view of (2.33), we obtain

$$
\int_{B_r} |\nabla h| dx \leq C_2 \int_{B_r} M(\tilde{v})^q dx \leq C_2 M(2\|u\|_{\infty} + 1)^q |\Omega|,
$$

where $C_2 = \frac{(q + \lambda \|\rho\|_{\infty})(1 + 3\|u\|_{\infty})m(1 + 3\|u\|_{\infty})}{M(\tilde{r})}$. Therefore, $h \in W_0^{1,1}(B_r)$ and so we can write

$$
\left( \int_{B_r} M(\tilde{v})^{qN} dx \right)^{\frac{N - 1}{N}} \leq C_2 C(N) \int_{B_r} M(\tilde{v})^q dx,
$$

where $C(N)$ stands for the constant in the continuous embedding $W_0^{1,1}(B_r) \hookrightarrow L_{{\mathcal{N}} - \tau}^N(B_r)$. Then, applying Lemma 2.25 with $g = M(\tilde{v})$ and $\alpha = 1$ we obtain for any $p > 0$

$$
\sup_{B_r} M(\tilde{v}) \leq C_3 \left[ r^{-N} \int_{B_r} M(\tilde{v})^p dx \right]^\frac{1}{p},
$$

where $C_3 = (C_2 C(N))^{\frac{N}{p}}$. Hence, follows

$$
\sup_{B_{2r}} M(\tilde{v}) \leq C_3 \left[ r^{-N} \int_{B_r} M(\tilde{v})^p dx \right]^\frac{1}{p}.
$$
Since \( \frac{t}{2} m(\frac{t}{2}) \leq M(t) \leq tm(t) \) and \( \bar{v} = v + r = u - \inf_{B_r} u + r \) we have \( \sup_{B_r} M(\bar{v}) \geq m(\frac{r}{2}) \sup_{B_r} \frac{\bar{v}}{2} \) and \( M(\bar{v}) \leq \bar{v} m(1 + 2 \|u\|_\infty) \), which yields

\[
\sup_{B_r} \frac{\bar{v}}{2} \leq C \left[ r^{-N} \int_{B_r} \bar{v}^p dx \right]^{\frac{1}{p}},
\]

where \( C = (C_2 C(N)) \frac{2m(1 + 2 \|u\|_\infty)}{m(\frac{r}{2})} \). Hence, the inequality \( 2.31 \) is proved.

**Lemma 2.7.** Let \( B_r \subset \Omega \) be an open ball of radius \( 0 < r \leq 1 \). Then, there exist two constants \( C > 0 \) and \( p_0 > 0 \) such that

\[
\left( r^{-N} \int_{B_r} v^{p_0} dx \right)^{\frac{1}{p_0}} \leq C \left( \inf_{B_r} v + r \right),
\]

where \( B_r \) is the ball of radius \( r/2 \) concentric with \( B_r \).

**Proof.** Let \( \Omega_0 \) be a compact subset of \( \Omega \) such that \( B_r \subset \Omega_0 \subset B_r \). Let \( q > 1 \) and let \( \psi \) be the function defined by

\[
\psi(x) = \begin{cases} 
M(\bar{v}(x))^{-q-1} \bar{v}(x) & \text{if } x \in \Omega_0, \\
0 & \text{if } x \notin \Omega_0,
\end{cases}
\]

where \( \bar{v} = v + r \). On \( \Omega_0 \) we compute

\[
\nabla \psi = M(\bar{v})^{-q-1} \nabla \bar{v} + (-q - 1) M(\bar{v})^{-q-2} m(\bar{v}) \bar{v} \nabla \bar{v}.
\]

By Theorem 2.4 we have \( \psi \in W^1_0 L_M(\Omega) \). Thus, using the function \( \psi \) in (2.32) we obtain

\[
\lambda \int_{B_r} \rho(x) M(\bar{v})^{-q-1} \bar{m}(v + \inf_{B_r} u) dx = \int_{B_r} M(\bar{v})^{-q-1} |\nabla \bar{v}| m(|\nabla \bar{v}|) dx
\]

\[
+ (-q - 1) \int_{B_r} M(\bar{v})^{-q-2} m(\bar{v}) \bar{v} |\nabla \bar{v}| m(|\nabla \bar{v}|) dx.
\]

By the fact that \( \bar{m}(\bar{v}) \geq M(\bar{v}) \), we get

\[
\lambda \int_{B_r} \rho(x) M(\bar{v})^{-q-1} \bar{m}(v + \inf_{B_r} u) dx \leq -q \int_{B_r} M(\bar{v})^{-q-1} |\nabla \bar{v}| m(|\nabla \bar{v}|) dx.
\]

Thus, since on \( B_r \) one has \( |v + \inf_{B_r} u| \leq \bar{v} + \|u\|_\infty \) we obtain

\[
q \int_{B_r} M(\bar{v})^{-q-1} |\nabla \bar{v}| m(|\nabla \bar{v}|) dx \leq \lambda \|\rho\|_\infty \int_{B_r} M(\bar{v})^{-q-1} m(\bar{v} + \|u\|_\infty) (\bar{v} + \|u\|_\infty) dx. \quad (2.36)
\]

On the other hand, let \( h \) be the function defined by

\[
h(x) = \begin{cases} 
M(\bar{v}(x))^{-q} & \text{if } x \in \Omega_0, \\
0 & \text{if } x \notin \Omega_0.
\end{cases}
\]
Using once again (2.34) with $a = |\nabla \tilde{v}|$ and $b = \tilde{v}$, we obtain
\[
\int_{B_r} |\nabla h| \, dx \leq q \int_{B_r} M(\tilde{v})^{q-1}m(|\nabla \tilde{v}|)|\nabla \tilde{v}| \, dx + q \int_{B_r} M(\tilde{v})^{q-1}\tilde{v}m(\tilde{v}) \, dx
\leq q \int_{B_r} M(\tilde{v})^{q-1}m(|\nabla \tilde{v}|)|\nabla \tilde{v}| \, dx
+ q \int_{B_r} M(\tilde{v})^{q-1}(\tilde{v} + \|u\|_{\infty})m(\tilde{v} + \|u\|_{\infty}) \, dx,
\]
which together with (2.36) yield
\[
\int_{B_r} |\nabla h| \, dx \leq C_2 \int_{B_r} M(\tilde{v})^{-q} \, dx \leq C_2 M(r)^{-q} |\Omega|,
\]
with $C_2 = \frac{(q + \lambda \|\rho\|_{\infty})m(1 + 3\|u\|_{\infty})(1 + 3\|u\|_{\infty})}{M(r)}$.

Thus, $h \in W^{1,1}_0(B_r)$ and so we can write
\[
\left( \int_{B_r} M(\tilde{v})^{-\frac{qN}{N-1}} \, dx \right)^{\frac{N-1}{N}} \leq C_2 C(N) \int_{B_r} M(\tilde{v})^{-q} \, dx,
\]
where $C(N)$ is the constant in the continuous embedding $W^{1,1}_0(B_r) \hookrightarrow L^{\frac{N}{N-1}}(B_r)$. Therefore, applying Lemma A.3 with $g = M(\tilde{v})$ and $\alpha = -1$ we get for any $p > 0$
\[
\sup_{B_r} M(\tilde{v})^{-1} \leq (C_2 C(N))^\frac{N}{p} \left( r^{-N} \int_{B_r} M(\tilde{v})^{-p} \, dx \right)^\frac{1}{p}.
\]
So that one has
\[
\left( r^{-N} \int_{B_r} M(\tilde{v})^{-p} \, dx \right)^\frac{1}{p} \leq (C_2 C(N))^\frac{N}{p} \inf_{B_r} M(\tilde{v})^p
\leq (C_2 C(N))^\frac{N}{p} \inf_{B_r^\#} M(\tilde{v})^p.
\]

The fact that $M(\tilde{v}) \geq m(\frac{r}{2}) \frac{\tilde{v}}{2}$ and $M(\tilde{v}) \leq m(2\|u\|_{\infty} + 1)\tilde{v}$, yields
\[
\left( r^{-N} \int_{B_r} \tilde{v}^{-p} \, dx \right)^\frac{1}{p} \leq C \inf_{B_r^\#} \tilde{v},
\]
where $C = (C_2 C(N))^\frac{N}{p} \frac{2m(2\|u\|_{\infty} + 1)}{m(\frac{r}{2})}$. Now, it only remains to show that there exist two constants $c > 0$ and $p_0 > 0$, such that
\[
\left( r^{-N} \int_{B_r} \tilde{v}^{p_0} \, dx \right)^\frac{1}{p_0} \leq c \left( r^{-N} \int_{B_r} \tilde{v}^{-p_0} \, dx \right)^\frac{1}{p_0}.
\]

Let $B_{r_1} \subset B_r$ and let $\Omega_0$ be a compact subset of $\Omega$ such that $B_{\frac{r_1}{2}} \subset \Omega_0 \subset B_{r_1}$. Let $\psi$ be the function defined by
\[
\psi(x) = \begin{cases} \tilde{v}(x) & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \notin \Omega_0. \end{cases}
\]

Then, inserting $\psi$ as a test function in (2.32) we obtain
\[
\int_{B_{r_1}} m(|\nabla \tilde{v}|)|\nabla \tilde{v}| \, dx \leq \lambda \|\rho\|_{\infty} \int_{B_{r_1}} m(|\nabla \tilde{v}|)(\tilde{v} + \|u\|_{\infty}) \, dx
\leq \lambda \|\rho\|_{\infty} \int_{B_{r_1}} m(\tilde{v} + \|u\|_{\infty})(\tilde{v} + \|u\|_{\infty}) \, dx.
\]
Since $\bar{v} \leq (2\|u\|_\infty + 1)$ and $|B_{r_1}| = r_1^N |B_1|$ we obtain

$$\int_{B_{r_1}} m(\|\nabla \bar{v}\|) |\nabla \bar{v}| dx \leq c_0 r_1^N,$$

(2.38)

where $c_0 = \lambda \|\rho\|_\infty m(3\|u\|_\infty + 1)(3\|u\|_\infty + 1)|B_1|$. On the other hand, we can use (2.34) with $a = |\nabla \bar{v}|$ and $b = \frac{\bar{v}}{r_1}$ obtaining

$$|\nabla \bar{v}| m(\frac{\bar{v}}{r_1}) \leq |\nabla \bar{v}| m(\frac{\bar{v}}{r_1}) + \frac{\bar{v}}{r_1}.$$

Pointing out that $\frac{\bar{v}}{r_1} m(\frac{\bar{v}}{r_1}) \geq M(\bar{v}) \geq m(1)$, we get

$$|\nabla \bar{v}| \leq \frac{1}{r_1 M(1)} m(|\nabla \bar{v}|) + \frac{1}{r_1}.$$

Integrating over the ball $B_{\frac{r}{2}}$ and using (2.38) we obtain

$$\int_{B_{\frac{r}{2}}} \frac{|\nabla \bar{v}|}{\bar{v}} dx \leq \frac{1}{r_1 M(1)} \int_{B_{\frac{r}{2}}} m(|\nabla \bar{v}|) |\nabla \bar{v}| dx + \frac{1}{r_1 |B_{\frac{r}{2}}|} \leq \left(\frac{c_0}{M(1)} + \frac{|B_1|}{2N}\right)^{r_1^{-1}}.$$

The above inequality together with [13, Lemma 1.2] ensure the existence of two constants $p_0 > 0$ and $c > 0$ such that

$$\left(\int_{B_r} \bar{v}^{p_0} dx\right) \left(\int_{B_r} \bar{v}^{-p_0} dx\right) \leq cr^{2N}.$$  

(2.39)

Finally, the estimate (2.35) follows from (2.37) with $p = p_0$ and (2.39).

□

Theorem 2.6 (Harnack-type inequalities). Let $u \in W^{1}_0 L_M(\Omega)$ be a bounded weak solution of (1.1) and let $B_{\frac{r}{2}}$, $0 < r \leq 1$, be a ball with radius $\frac{r}{2}$. There exists a large constant $C > 0$ such that

$$\sup_{B_{\frac{r}{2}}} v \leq C(\inf_{B_{\frac{r}{2}}} v + r)$$

(4.0)

and

$$\sup_{B_{\frac{r}{2}}} w \leq C(\inf_{B_{\frac{r}{2}}} w + r).$$

(4.1)

Proof. Putting together (2.31), with the choice $p = p_0$, and (2.35) we immediately get (4.0). In the same way as above, one can obtain analogous inequalities to (2.35) and (2.31) for $w$ obtaining the inequality (4.1).

□

We are now ready to prove the following Hölder estimate for weak solutions of (1.1).

Theorem 2.7 (Hölder regularity). Let $u \in W^{1}_0 L_M(\Omega)$ be a bounded weak solution of (1.1). Then there exist two constants $0 < \alpha < 1$ and $C > 0$ such that if $B_r$ and $B_R$ are two concentric balls of radii $0 < r \leq R \leq 1$, then

$$\text{osc}_{B_r} u \leq C \left(\frac{R}{r}\right)^{\alpha} \left(\sup_{B_R} |u| + C(R)\right),$$

where $\text{osc}_{B_r} u = \sup_{B_r} u - \inf_{B_r} u$ and $C(R)$ is a positive constant which depends on $R$. 
Proof. From (2.40) and (2.41) we obtain
\[
\sup_{B_r} u - \inf_{B_r} u = \sup_{B_r} v \leq C(\inf_{B_r} v + r) = C(\inf_{B_r} u - \inf_{B_r} u + r)
\]
and
\[
\sup_{B_r} u - \sup_{B_r} u = \sup_{B_r} w \leq C(\inf_{B_r} w + r) \leq C(\sup_{B_r} w + r) = C(\sup_{B_r} u - \sup_{B_r} u + r).
\]
Hence, summing up both the two first terms in the left-hand side and the two last terms in the right-hand side of the above inequalities, we obtain
\[
\sup_{B_r} u - \inf_{B_r} u \leq C \left( \sup_{B_r} u - \inf_{B_r} u + \inf_{B_r} u - \sup_{B_r} u + 2r \right),
\]
that is to say, what one still writes
\[
\text{osc}_{B_r} u \leq \left( \frac{C - 1}{C} \right) \text{osc}_{B_r} u + 2r. \tag{2.42}
\]
Let us fix some real number \( R_1 \leq R \) and define \( \sigma(r) = \text{osc}_{B_r} u \). Let \( n \in \mathbb{N} \) be an integer. Iterating the inequality (2.42) by substituting \( r = R_1, r = \frac{R_1}{2}, \ldots, r = \frac{R_1}{2^n} \), we obtain
\[
\sigma \left( \frac{R_1}{2^n} \right) \leq \gamma^n \sigma(R_1) + R_1 \sum_{i=0}^{n-1} \frac{\gamma^{n-1-i}}{2^{i-1}} \leq \gamma^n \sigma(R) + \frac{R_1}{1 - \gamma},
\]
where \( \gamma = \frac{C - 1}{C} \). For any \( r \leq R_1 \), there exists an integer \( n \) satisfying
\[
2^{-n-1}R_1 \leq r < 2^{-n}R_1.
\]
Since \( \sigma \) is an increasing function, we get
\[
\sigma(r) \leq \gamma^n \sigma(R) + \frac{R_1}{1 - \gamma}.
\]
Being \( \gamma < 1 \), we can write
\[
\gamma^n \leq \gamma^{-1} \gamma^{- \frac{\log(r)}{\log 2}} = \gamma^{-1} \left( \frac{r}{R_1} \right)^{- \frac{\log \gamma}{\log 2}}.
\]
Therefore,
\[
\sigma(r) \leq \gamma^{-1} \left( \frac{r}{R_1} \right)^{- \frac{\log \gamma}{\log 2}} \sigma(R) + \frac{R_1}{1 - \gamma}.
\]
This inequality holds for arbitrary \( R_1 \) such that \( r \leq R_1 \leq R \). Let now \( \alpha \in (0,1) \) and \( R_1 = R_{1-\alpha}r^\alpha \), so that we have from the preceding
\[
\sigma(r) \leq \gamma^{-1} \left( \frac{r}{R} \right)^{- (1-\alpha) \frac{\log \gamma}{\log 2}} \sigma(R) + \frac{R}{1 - \gamma} \left( \frac{r}{R} \right)^\alpha.
\]
Thus, the desired result follows by choosing \( \alpha \) such that \( \alpha = -(1-\alpha) \frac{\log \gamma}{\log 2} \), that is \( \alpha = \frac{- \log \gamma}{\log 2 - \log \gamma} \).
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