Separation of variables in multi–Hamiltonian systems: an application to the Lagrange top.

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Abstract

Starting from the tri-Hamiltonian formulation of the Lagrange top in a six-dimensional phase space, we discuss the reduction of the vector field and of the Poisson tensors. We show explicitly that, after the reduction on each one of the symplectic leaves, the vector field of the Lagrange top is separable in the sense of Hamilton–Jacobi.

Keywords: Lagrange top, Hamiltonian formulation, Separability.
AMS 2000 Subject classifications: 37K10, 37J35, 53D17, 70E40, 70H06.
1 Introduction

This paper completes the analysis of the Lagrange top (LT) as a quasi-bi-Hamiltonian (qbH) system started in [1], to which we refer for more details. Summarizing, we showed in [1] that the tri–Hamiltonian structure of LT, defined on a six-dimensional phase space by three compatible Poisson tensors $P_0$, $P_1$, $P_2$, can be reduced onto a four-dimensional phase space. When one tries to eliminate the Casimirs of the Poisson tensors by fixing their values, one is faced with a typical situation, occurring also for other finite-dimensional integrable systems [2, 3, 4]: to each one of the symplectic leaves $S_0$, $S_1$, $S_2$ one can restrict only the vector field $X_L$ and the corresponding Poisson tensor, but not the entire tri–Hamiltonian structure, which is lost under restriction. Nevertheless, the LT vector field $X_L$, restricted to the symplectic leaf $S_0$ of the Poisson tensor $P_0$, can be given a qbH formulation, and this fact yields its separability in the sense of Hamilton–Jacobi (HJ).

In this paper, we show that this property of the vector field $X_L$ is more general. In fact, the qbH property can be recovered also if the restriction is performed to any symplectic leaf of the second Poisson tensor $P_1$; moreover, exploiting some properties of the qbH model, we can explicitly give the separation variables for the restriction of $X_L$ to any symplectic leaf of the third Poisson tensor $P_2$. It is remarkable that in both cases the separation variables are obtained by means of non-symplectic maps.

The paper is organized as follows. In Section 2 and 3, we discuss some properties of a generic tri–Hamiltonian structure with deformation and of the qbH model, in view of application to LT. In Sections 4 and 5 the tri–Hamiltonian structure of LT, the deformation field and their reduction are briefly reviewed. At last, the results of Section 3 are applied in Section 6 to show that $X_L$ can be written in a separable form when its restriction is performed to any symplectic leaf of the three Poisson tensors.

2 Deformation of multi–Hamiltonian structures and hereditary operators

Let us assume that:

i) $(M, P_2, \tau)$ is a Poisson manifold with a deformation, i.e., $M$ is a differential manifold endowed with a Poisson tensor $P_2$ and with a vector field $\tau$ in such a way that the Lie derivative $P_1 = L_\tau(P_2)$ of $P_2$ w.r.t. $\tau$ is itself a Poisson
tensor. This assumption implies that $P_2 - \lambda P_1$ is a Poisson pencil.

ii) The Poisson tensor $P_1$ is exact w.r.t. $\tau$, i.e., $L_\tau^2(P_1) = 0$. This implies that $P_0 = L_\tau(P_1) = L_\tau^2(P_2)$ is itself a Poisson tensor, compatible with both $P_1$ and $P_2$.

So, under the above assumptions it follows that:

2.1 Lemma The manifold $(M, P_0, P_1, P_2)$ is a tri–Hamiltonian manifold, i.e., the linear combination $P_0 - \lambda P_1 - \mu P_2$ is itself a Poisson tensor for every value of the constants coefficients $\lambda$ and $\mu$.

Remark As one can easily verify, if there is a tri–Hamiltonian structure $(\tilde{P}_0, \tilde{P}_1, \tilde{P}_2)$ such that $L_\tau(\tilde{P}_0) = 0$, $L_\tau(\tilde{P}_1) = \alpha \tilde{P}_0$, $L_\tau(\tilde{P}_2) = \beta \tilde{P}_1 + \gamma \tilde{P}_0$, with $\alpha$, $\beta$ and $\gamma$ constant parameters, then $P_0 = \tilde{P}_0$, $P_1 = (1/\alpha)\tilde{P}_1 + (\gamma/\alpha\beta)\tilde{P}_0$ and $P_2 = (a/\alpha\beta)\tilde{P}_2$ fulfil, for any given $a$, the deformation relations

$$L_\tau(P_2) = a P_1 \quad L_\tau(P_1) = P_0, \quad L_\tau(P_0) = 0.$$  \hspace{1cm} (2.1)

The following Lemma gives a sufficient condition for the projection of the tri–Hamiltonian structure along a submersion.

2.2 Lemma Let $\pi : M \rightarrow M'$ be a surjective submersion onto a manifold $M'$. If both $P_2$ and $\tau$ are projectable onto $M'$, the whole tri–Hamiltonian structure is preserved under the submersion $\pi$. Denoting the reduced tensors as $P'_0$, $P'_1$, $P'_2$ and the reduced vector field as $\tau'$, it still holds that

$$L_{\tau'}(P'_2) = a P'_1 \quad L_{\tau'}(P'_1) = P'_0, \quad L_{\tau'}(P'_0) = 0.$$  \hspace{1cm} (2.2)

As is known, if the reduced Poisson tensor $P'_0$ is kernel-free, so that its inverse is symplectic, then on $M'$ the operator $N = P'_1 P'_0^{-1}$ is a hereditary operator, i.e., it has a vanishing Nijenhuis torsion \[5\]. We now search for some conditions on the deformation $\tau$, assuring that $N$ acts as a recursion operator (in a direction opposite to the one of the Lie derivatives w.r.t. $\tau$) for the reduced tri–Hamiltonian structure on $M'$, mapping also $P'_1$ to $P'_2$ (possibly, up to a constant factor): $NP'_1 = \lambda P'_2$ for some constant $\lambda$; to this purpose, the following result can be used.

2.3 Lemma Given a vector field $\tau$ on $M$ and its reduction $\tau'$ on $M'$, let us consider the equation $L_{\tau'}(Q') = 0$ and look for a solution $Q'$ which is a
(2,0) skew-symmetric tensor. If $P'_0$ is the general solution (up to a constant factor): $L_{\tau'}(Q') = 0 \Rightarrow Q' = \alpha P'_0$, then the tensor $N := \tau'_1 P'_0^{-1}$ defined on the manifold $M'$ is such that

$$NP'_1 = \frac{2}{a} P'_2 + \beta P'_0 \quad (\beta = \text{constant}) . \quad (2.3)$$

**Proof** On account of (2.2) and the previous assumptions, the equations $L_{\tau'}(Q') = P'_0$ and $L_{\tau'}(Q') = P'_1$ have the solutions $Q' = P'_1 + \alpha P'_0$ and $Q' = (1/a) P'_2 + \alpha P'_0$, respectively; since it is also $L_{\tau'}(N) = I$, we have

$$\frac{1}{2-a} L_{\tau'}(NP'_1 - P'_2) = P'_1 \quad \Rightarrow \quad \frac{1}{2-a} (NP'_1 - P'_2) = \frac{1}{a} P'_2 + \alpha P'_0$$

yielding (2.3) with $\beta = (2-a) \alpha$.

**2.4 Lemma** Let a tri–Hamiltonian structure $(P'_0, P'_1, P'_2)$ be given, with $\tau'$ fulfilling (2.2) and a recursion operator $N$ such that $N P'_0 = P'_1$ and $N P'_1 = \lambda P'_2 + \mu P'_0$. Then the tensors $Q'_0 = P'_0, Q'_1 = P'_1, Q'_2 = P'_2 + (\mu/\lambda) P'_0$ are such that the deformation relations (2.2) still hold and $N Q'_0 = Q'_1, N Q'_1 = \lambda Q'_2$.

**Proof** A trivial computation.

On account of the above results, given a tri–Hamiltonian structure $(P_0, P_1, P_2)$ with a deformation $\tau$ fulfilling (2.1), if the deformation and the tri–Hamiltonian structure are preserved under the submersion $\pi$, then on the reduced manifold there is also (possibly after a rescaling) a recursion structure defined by $N$.

## 3 Some properties of the quasi-bi-Hamiltonian model

The $qbH$ model was introduced in [6, 7] and developed in [8, 9]. Let $Q_0, Q_1$ be two compatible Poisson tensors; a vector field $X$ admits a $qbH$ formulation if there are three functions $\rho, H, K$ such that

$$X = Q_0 dH = \frac{1}{\rho} Q_1 dK . \quad (3.1)$$

If $M$ is even-dimensional, dim $M = 2n$, the $qbH$ formulation is said to be of maximal rank if $Q_0, Q_1$ are non degenerate at each point $m \in M$ and
the associated tensor $N = Q_1 Q_0^{-1}$ (with vanishing Nijenhuis torsion) has $n$ distinct eigenvalues $\lambda_1(m),\ldots,\lambda_n(m)$; the $qbH$ formulation is said to be of Pfaffian type if $\rho = \prod_{i=1}^{n} \lambda_i$. For a $qbH$ structure of maximal rank, one can introduce a Darboux-Nijenhuis chart $(\lambda_i, \mu_i)$ ($i = 1, 2, \ldots, n$) such that $Q_0$, $Q_1$ and $N$ take the canonical form

$$Q_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

(3.2)

where $I$ is the $n \times n$ unit matrix and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

3.1 Proposition [8] In a Darboux-Nijenhuis chart, the general solution of Eq.(3.1) for the Pfaffian case is given by the functions

$$H = \sum_{i=1}^{n} \frac{f_i}{\Delta_i}, \quad K = \sum_{i=1}^{n} \frac{\rho f_i}{\lambda_i \Delta_i}$$

(3.3)

where $\Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j)$ and $f_i$ are arbitrary functions, depending at most on the pair $(\lambda_i, \mu_i)$. Moreover, the HJ equations for both $H$ and $K$ are separable in the chart $(\lambda, \mu)$.

From now on, functions of the form (3.3) in a given chart will be said to have a normal form. The above Proposition yields straightforwardly the following.

3.2 Corollary Let $X = Q_0 \, dH$ be a Hamiltonian vector field, $Q_0$ and $H$ taking the form (3.2), (3.3); then there exist $Q_1$ and $K$ of the form (3.2) and (3.3), respectively, such that $X = (1/\rho) \, Q_1 \, dK$.

Viceversa, let $X = (1/\rho) \, Q_1 \, dK$, with $\rho = \prod_{i=1}^{n} \lambda_i$, $Q_1$ and $K$ of the form (3.2) and (3.3); then there exist $Q_0$ and $H$ of the form (3.2) and (3.3), respectively, such that it is also $X = Q_0 \, dH$.

In view of applications to LT, let us consider in more detail a four-dimensional manifold ($n = 2$); in this case, we have some more general conditions assuring that a Hamiltonian vector field is separable.

3.3 Proposition On a four-dimensional manifold, let $X = Q_0 \, d\tilde{H}$ be a Hamiltonian vector field, with $Q_0$ of the canonical form (3.2) in a chart...
Let the Hamiltonian $\tilde{H}$ be a linear combination of two functions $\hat{H}$, $\hat{K}$ possessing the normal form (3.3), i.e.,
\begin{align*}
\tilde{H}(x; y) &= \alpha \hat{H}(x; y) + \beta \hat{K}(x; y) \quad (\alpha, \beta = \text{const.}) , \\
\hat{H}(x; y) &= \frac{1}{x_1 - x_2} \left( \hat{f}_1(x_1, y_1) - \hat{f}_2(x_2, y_2) \right) , \\
\hat{K}(x; y) &= \frac{1}{x_1 - x_2} \left( x_2 \hat{f}_1(x_1, y_1) - x_1 \hat{f}_2(x_2, y_2) \right) .
\end{align*}

Then the map $\Phi_0 : (x; y) \mapsto (\lambda; \mu)$
\begin{align*}
\lambda_i &= \frac{\beta}{\alpha + \beta x_i} , \quad \mu_i = -\frac{1}{\beta^2} (\alpha + \beta x_i)^2 y_i \quad (i = 1, 2) \quad (3.5)
\end{align*}
is symplectic for $Q_0$ (i.e., $Q_0$ is preserved under $\Phi_0$); $\tilde{H}$ is transformed under $\Phi_0$ into a function $H$ of the normal form (3.3), with
\begin{align*}
f_i(\lambda_i, \mu_i) = -\beta \lambda_i \hat{f}_i \left( \frac{1}{\lambda_i} - \frac{\alpha}{\beta} , -\lambda_i \mu_i \right) \quad (i = 1, 2) . \quad (3.6)
\end{align*}

So, the vector field $X$ admits a qbH formulation and the Hamiltonian $\tilde{H}$ is separable in the chart $(\lambda; \mu)$.

**Proof** By straightforward computations one checks that the map $\Phi_0$ is symplectic for $Q_0$ and that the Hamiltonian $\tilde{H}$ takes the form
\begin{align*}
\tilde{H} \left( x(\lambda; \mu); y(\lambda; \mu) \right) = \frac{1}{\lambda_1 - \lambda_2} \left( f_1(\lambda_1, \mu_1) - f_2(\lambda_2, \mu_2) \right) , \quad (3.7)
\end{align*}
with $f_i(\lambda_i; \mu_i)$ given by (3.6). On account of Corollary 3.2, the vector field $X = Q_0 dH$ admits also the qH formulation $X = 1/\rho Q_1 dK$; the separability of $\tilde{H}$ in the chart $(\lambda; \mu)$ follows from Proposition 3.1.

### 3.4 Corollary
$\tilde{H}$ is separable also in the chart $(x; y)$; the corresponding HJ equation $\tilde{H}(x; \partial W/\partial x) = \tilde{h}$ has the complete solution $W = W_1 + W_2$, $W_1$ and $W_2$ fulfilling the Jacobi separation equations
\begin{align*}
\hat{f}_i(x_i, W'_i(x_i)) = x_i \hat{h} - \hat{k} , \quad \alpha \hat{h} + \beta \hat{k} = \tilde{h} \quad (i = 1, 2) . \quad (3.8)
\end{align*}
Indeed, the map $\Phi_0$ is a separated map \cite{12}, i.e., it maps separated coordinates into separated ones. So, taking into account the form (3.7) of the function $\tilde{H}$, it is easily checked that the HJ equation has a complete solution $W(x_1, x_2; \hat{h}, \hat{k}) = W_1(x_1; \hat{h}, \hat{k}) + W_2(x_2; \hat{h}, \hat{k})$, with $\alpha \hat{h} + \beta \hat{k} = \tilde{h}$ and $W_1, W_2$ fulfilling the Jacobi separation equations (3.8) for the HJ equations $\hat{H}(x; \partial W/\partial x) = \hat{h}$, $\hat{K}(x; \partial W/\partial x) = \hat{k}$.

3.5 Proposition On a four-dimensional manifold, let $X = Q_1 \, d\tilde{H}$ be a Hamiltonian vector field, with $Q_1$ of the form (3.2) in a chart $(x, y)$; let the Hamiltonian $\tilde{H}$ be a linear combination of two functions $\hat{H}, \hat{K}$ with the normal form (3.3), i.e.,

$$\tilde{H}(x; y) = \alpha \hat{H}(x; y) + \beta \hat{K}(x; y) \quad (\alpha, \beta = \text{const.}) ,$$

$$\hat{H}(x; y) = \frac{1}{x_1 - x_2} \left( \hat{f}_1(x_1, y_1) - \hat{f}_2(x_2, y_2) \right) ,$$

$$\hat{K}(x; y) = \frac{1}{x_1 - x_2} \left( x_2 \hat{f}_1(x_1, y_1) - x_1 \hat{f}_2(x_2, y_2) \right) .$$

Then the map $\Phi_1 : (x; y) \mapsto (\lambda, \mu)$ given by

$$\lambda_i = \frac{1}{\alpha + \beta x_i} , \quad \mu_i = \frac{-1}{\beta} \left( \alpha + \beta x_i \right)^2 \frac{y_i}{x_i} \quad (i = 1, 2) \quad (3.9)$$

is a Darboux map for $Q_1$ (i.e., $Q_1$ is mapped to $Q_0$ under $\Phi_1$); $\tilde{H}$ is transformed under $\Phi_1$ into a function $H$ of the normal form (3.3), with

$$f_i(\lambda_i, \mu_i) = -\beta \hat{f}_i \left( \frac{1}{\lambda_i} - \alpha \right) , -\left( \frac{1}{\lambda_i} - \alpha \right) \lambda_i^2 \mu_i \quad (i = 1, 2) . \quad (3.10)$$

So, the vector field $X$ admits a qbH formulation and the Hamiltonian $\tilde{H}$ is separable in the chart $(\lambda; \mu)$.

Proof A straightforward computation allows one to check that $\Phi_1$ is a Darboux map for $Q_1$ and that $\tilde{H}$ is mapped into a function $H$ of normal form, with $f_i$ given by (3.10). Corollary 3.2 and Proposition 3.1 assure that $\tilde{H}$ is separable in the chart $(\lambda; \mu)$.

3.6 Proposition On a four-dimensional manifold, let $X = Q_2 \, d\tilde{G}$ be a Hamiltonian vector field, with $Q_2$ of the form (3.2) in a chart $(x; y)$ and

$$\tilde{G} = \alpha \hat{K} + \beta \hat{H}^2 ,$$
\( \hat{H} \) and \( \hat{K} \) being in the normal form \((3.3)\). Then the map \( \Phi_2 : (x, y) \mapsto (\lambda, \mu) \)

\[
\lambda_i = -\frac{1}{x_i}, \quad \mu_i = y_i \quad (i = 1, 2)
\]

(3.11)
is a Darboux map for \( Q_2 \) (i.e., it maps \( Q_2 \) into \( Q_0 \)); \( \tilde{G} \) is mapped under \( \Phi_2 \) into the function

\[
G = \alpha H + \beta K^2
\]

(3.12)
with \( H \) and \( K \) in the normal form \((3.5)\) and

\[
f_i(\lambda_i, \mu_i) = -\lambda_i \dot{f}_i(-\frac{1}{\lambda_i}, \mu_i) \quad (i = 1, 2).
\]

(3.13)
The function \( \tilde{G} \) is separable in the chart \((\lambda; \mu)\).

**Proof** It is straightforward to verify that \( \Phi_2 \) is a Darboux map for \( Q_2 \) and that \( \tilde{G} \) is mapped into \( G \). Let us consider the \( HJ \) equation \( G(\lambda; \partial W/\partial \lambda) = g \) for \( G \); one can easily verify that it is \( W = W_1 + W_2 \) with \( W_1 \) and \( W_2 \) solutions of the separation equations

\[
f_i(\lambda_i, W'_i(\lambda_i)) = \lambda_i h - k \quad (\alpha h + \beta k^2 = g) \quad (i = 1, 2).
\]

(3.14)

4 The tri–Hamiltonian structure of the Lagrange top

In the comoving frame, whose axes are the principal inertia axes of the top, with fixed point \( O \), the \( LT \) is parametrized by the pair \( m = (\omega; \gamma) \), where \( \omega = (\omega_1, \omega_2, \omega_3)^T \) and \( \gamma = (\gamma_1, \gamma_2, \gamma_3)^T \) are the angular velocity and the vertical unit vector, respectively. If \( \mu \) is the mass of the top, \( g \) the acceleration of gravity, \( J = \text{diag}(A, A, cA) \) the principal inertia matrix \((c \neq 1)\) and \( G = (0, 0, a)^T \) the center of mass, normalisations are chosen so that \( \mu ag = A \).

The Euler-Poisson equations \( dL_0/dt = M_0 \) and \( d\gamma/dt = 0 \) take the form \( dm/dt = X_L(m) \), where \( X_L \) is given by

\[
X_L(m) = \left(-(c-1)\omega_2\omega_3 - \gamma_2,(c-1)\omega_3\omega_1 + \gamma_1,0; \gamma_2\omega_3 - \gamma_3\omega_2, \gamma_3\omega_1 - \gamma_1\omega_3, \gamma_1\omega_2 - \gamma_2\omega_1 \right)^T.
\]

The \( LT \) vector field \( X_L \) can be given a tri–Hamiltonian formulation

\[
X_L = P_0dh_0 = P_1dh_1 = P_2dh_2;
\]
written in matrix block form, the compatible Poisson tensors are:

\[
P_0 = \begin{pmatrix} 0 & B \\ B & C \end{pmatrix}, \quad P_1 = \begin{pmatrix} -B & 0 \\ 0 & \Gamma \end{pmatrix}, \quad P_2 = \begin{pmatrix} T & R \\ -RT & 0 \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \omega_3 & -\omega_2 \\ -c \omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix},
\]

\[
T = \begin{pmatrix} 0 & -c \omega_3 & \omega_2/c \\ c \omega_3 & 0 & -\omega_1/c \\ -\omega_2/c & \omega_1/c & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2/c & \gamma_1/c & 0 \end{pmatrix}.
\]

The Hamiltonian functions are

\[
h_0 = \frac{1}{2} F_4 + (c-1) F_1 F_3, \quad h_1 = \frac{1}{2} c(c-1) F_1^3 - F_3 - (c-1) F_1 F_2, \quad h_2 = F_2
\]

\[
F_1 = \omega_3, \quad F_2 = \frac{1}{2} (\omega_1^2 + \omega_2^2 + c \omega_3^2) - \gamma_3, \quad F_3 = \omega_1 \gamma_1 + \omega_2 \gamma_2 + c \omega_3 \gamma_3, \quad F_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2.
\]

The functions \((F_1, F_2)\) are Casimirs of \(P_0\), \((F_1, F_3)\) of \(P_1\) and \((F_3, F_4)\) of \(P_2\). The Hamiltonian formulation of \(LT\) w.r.t. \(P_2\) is classical (see, e.g., [13]); the bH formulation w.r.t. \((P_0, P_2)\) was introduced in [14] in the semidirect product \(\mathfrak{so}(3) \times \mathfrak{so}(3)\), and was later recovered in [15] in an algebraic-geometric setting; the tri–Hamiltonian formulation w.r.t. \((P_0, P_1, P_2)\) was constructed in [16], by a suitable reduction of the Lie-Poisson pencil defined in the direct sum of three copies of \(\mathfrak{so}(3)\).

As shown in [16], the tri–Hamiltonian structure of \(LT\) admits the deformation \(L_\tau(P_2) = 2P_1 + L_\tau(P_1) = P_0\), \(L_\tau(P_0) = 0\), where \(\tau\) is given, in the chart \((\omega; \gamma)\), by \(\tau = (0, 0, -2/c; \omega_1, \omega_2, c \omega_3)^T\); on the contrary, a recursion operator \(N\) relating the Poisson tensors does not exist in \(M\).

The Poisson pencils \(P_1 - \lambda P_0\), \(P_2 - \lambda P_1\), \(P_2 - \lambda P_0\) are three Poisson pencils of Gelfand–Zakharevich (GZ) type: more precisely, they belong to the class of complete torsionless GZ systems of rank 2 [17]. Each one of them has two polynomial Casimir functions, whose coefficients form two Lenard chains for each pencil which can be constructed by means of the deformation field \(\tau\). Graphically, the Lenard chains of \((P_0, P_1)\) can be represented in the following
where \( G = F_2 + c(c - 1)F_1^2/2 \). The Lenard chains of \((P_1, P_2)\) are:

So, we can state that \( P_1 - \lambda P_0 \) and \( P_2 - \lambda P_1 \) are Poisson pencils of rank 2 and type \((1, 5)\). Finally, the Lenard chains of \((P_0, P_2)\) are:

implying that \( P_2 - \lambda P_0 \) is Poisson pencil of rank 2 and type \((3, 3)\).
5 The reduction of the tri–Hamiltonian and deformation structures

The tri–Hamiltonian structure \((P_0, P_1, P_2)\) of \(LT\) and the deformation field \(\tau\) admit a reduction on a four-dimensional manifold \(M'\) (see [1] for an interpretation of this process in terms of the Marsden-Ratiu reduction theorem). Let \(M'\) be a four-dimensional manifold parametrized by a chart \((x; y) = (x_1, x_2; y_1, y_2)\), and let \(\pi : M \mapsto M'/\pi, : (\omega; \gamma) \mapsto (x; y)\) be the surjective submersion given by:

\[
\begin{align*}
x_{1,2} &= -\frac{1}{2} (c\omega_3 - i\omega_2) \mp \frac{1}{2} \sqrt{(c\omega_3 - i\omega_2)^2 + 4(\gamma_3 - i\gamma_2)} \\
y_{1,2} &= -\gamma_1 - \frac{1}{2} \omega_1 (c\omega_3 - i\omega_2) \mp \frac{1}{2} \omega_1 \sqrt{(c\omega_3 - i\omega_2)^2 + 4(\gamma_3 - i\gamma_2)}.
\end{align*}
\]

A straightforward calculation allows one to conclude the following.

5.1 Proposition The Poisson tensor \(P_0\) and the deformation field \(\tau\) can be reduced under \(\pi\): the projected tensor fields take the form

\[
P'_0 = -i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \tau' = (1, 1, 0, 0)^T
\]

(5.1)

where \(I\) is the \(2 \times 2\) unit matrix. On account of Proposition 2.2, also \(P_1\) and \(P_2\) are projectable: the reduced tensors \(P'_1, P'_2\) take the form

\[
P'_1 = -i \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad P'_2 = -i \begin{pmatrix} 0 & X^2 \\ -X^2 & 0 \end{pmatrix},
\]

(5.2)

where \(X = \text{diag}(x_1, x_2)\).

Moreover, the deformation relations are maintained under \(\pi\). Since \(P'_0\) is clearly kernel-free, one has a torsionless tensor \(N = P'_1 P'_0^{-1}\); it can be easily checked that \(N P'_0 = P'_1\), \(N P'_1 = P'_2\). So, we are just in the situation discussed in Lemma 2.3 with \(a = 2, \beta = 0\).

6 The reduction of the vector field \(X_L\) on the symplectic leaves

In this section we consider the reduction of \(LT\) on the symplectic leaves \(S_i\) of the Poisson tensors \(P_i (i = 0, 1, 2)\). Each \(S_i\) is a four-dimensional submanifold
of $M$, being characterized as a level set of two Casimirs functions of $P_i$. On account of Eq. (4.1), the symplectic leaves are defined as

$$S_0 = \{ m \in M \mid \omega_3 = C_1, \quad \omega_1^2 + \omega_2^2 + c\omega_3^2 - 2\gamma_3 = 2C_2 \}, \quad (6.1)$$

$$S_1 = \{ m \in M \mid \omega_3 = C_1, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = C_4 \},$$

$$S_2 = \{ m \in M \mid \omega_1 \gamma_1 + \omega_2 \gamma_2 + \omega_3 \gamma_3 = C_3, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = C_4 \}$$

where $C_1, C_2, C_3$ and $C_4$ are fixed values of the Casimirs $F_1, F_2, F_3$ and $F_4$, respectively. As these integrals of motion are in involution w.r.t. each $P_i$, a level set, on $S_i$, of the other two integrals of motion is a leaf $\Lambda_i$ of a Lagrangian foliation of $S_i$. Let us note that for any $m \in M$, the Lagrangian leaves $\Lambda_0, \Lambda_1, \Lambda_2$, passing through $m$, coincide. Moreover, using the Marsden-Ratiu reduction theorem, one can prove the following result.

**6.1 Proposition** [8] The symplectic leaves $S_0, S_1, S_2$ are (locally) diffeomorphic to the four-dimensional manifold $M' = M/\pi$.

Explicitly, let $(x_1, x_2; y_1, y_2)$ be a chart of $M'$; one can verify that the symplectic leaves admit the following parametrizations.

**6.2 Lemma** A generic symplectic leaf $S_0$ is parametrized by the mapping $\Psi_0 : M' \mapsto M : (x; y) \mapsto (\omega; \gamma)$ given by

$$\omega_1 = \frac{y_2 - y_1}{x_2 - x_1}, \quad \omega_2 = -i(x_1 + x_2 + cC_1), \quad \omega_3 = C_1$$

$$\gamma_1 = \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}, \quad \gamma_2 = \frac{i}{2}[x_1^2 + x_2^2 + g_1(x; y)], \quad \gamma_3 = -\frac{1}{2}[(x_1 + x_2)^2 + g_1(x; y)]$$

$$g_1(x; y) = -\frac{(y_2 - y_1)^2}{(x_2 - x_1)^2} + 2cC_1(x_1 + x_2) + 2C_2 + c(c - 1)C_1^2.$$

A symplectic leaf $S_1$ is parametrized by the mapping $\Psi_1$ given by

$$\omega_1 = \frac{y_2 - y_1}{x_2 - x_1}, \quad \omega_2 = -i(x_1 + x_2 + cC_1), \quad \omega_3 = C_1$$

$$\gamma_1 = \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}, \quad \gamma_2 = \frac{-i}{2}[x_1 x_2 + g_2(x; y)], \quad \gamma_3 = \frac{-1}{2}[x_1 x_2 + g_2(x; y)]$$

$$g_2(x; y) = -\frac{(x_2 y_1 - x_1 y_2)^2}{x_1 x_2 (x_2 - x_1)^2} + \frac{C_4}{x_1 x_2}.$$
A symplectic leaf $S_2$ is parametrized by the mapping $\Psi_2$ given by

$$
\omega_1 = \frac{y_2 - y_1}{x_2 - x_1}, \quad \omega_2 = -\frac{i}{2} [x_1 + x_2 + g_3(x; y)], \quad \omega_3 = -\frac{1}{2c} [x_1 + x_2 - g_3(x; y)]
$$

$$
\gamma_1 = \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}, \quad \gamma_2 = -\frac{i}{2} [x_1 x_2 - g_2(x; y)], \quad \gamma_3 = -\frac{1}{2} [x_1 x_2 + g_2(x; y)]
$$

$$
g_3(x; y) = \frac{x_1^2 y_2^2 - x_2^2 y_1^2}{x_1 x_2 (x_2 - x_1)} - \frac{2 C_3}{x_1 x_2} + \frac{x_1 + x_2}{x_1^2 x_2^2} C_4.
$$

By means of these parametrizations, we can show that the LT admits a qbH formulation on each one of the symplectic leaves. The following three Propositions are easily proved by straightforward computations.

**6.3 Proposition** The vector field $X_L$, restricted to $S_0$, takes the form

$$
X_L = P'_0 \, d\tilde{H},
$$

with $P'_0$ given by (5.1). The Hamiltonian $\tilde{H} = h_0|_{S_0}$ is given by

$$
\tilde{H}(x; y) = (c - 1) C_1 \, \tilde{H}(x; y) + \tilde{K}(x; y)
$$

$$
\tilde{H}(x; y) = F_3|_{S_0} = \frac{1}{x_1 - x_2}\left(\hat{f}(x_1, y_1) - \hat{f}(x_2, y_2)\right),
$$

$$
\tilde{K}(x; y) = F_4|_{S_0} = \frac{1}{x_1 - x_2}\left(x_2 \hat{f}(x_1, y_1) - x_1 \hat{f}(x_2, y_2)\right),
$$

$$
\hat{f}(\xi, \eta) = -\frac{1}{2} \eta^2 + \frac{1}{2} \xi^4 + c C_1 \xi^3 + \left(C_2 + \frac{1}{2} (c - 1) C_1^2\right) \xi^2.
$$

So, we are in the situation discussed in Proposition 3.3 with $\alpha = (c - 1) C_1$ and $\beta = 1$; this allows us to conclude that in the chart $(\lambda, \mu)$ given by (3.4) $X_L|_{S_0}$ is a separable qbH vector field.

**Remark** The vector field $X_L|_{S_0}$ is separable also in the chart $(x; y)$. Indeed, from Eq. (3.8) of Corollary 3.4 and from the expression (6.3) of $\hat{f}$ it follows that the solution $W$ of the HJ equation for $\tilde{H}$ is $W = W_1 + W_2$, where $W_1$ and $W_2$ can be computed solving by quadratures the ODEs obtained replacing $y_i$ by $\partial W/\partial x_i$ in the equation

$$
y_i^2 = x_i^4 + 2c C_1 x_i^3 + [2 C_2 + c(c - 1) C_1^2] x_i^2 - 2 \hat{h} x_i + 2 \hat{k} \quad (i = 1, 2);
$$
here, $\tilde{h} = (c - 1)C_1 \hat{h} + \hat{k}$ and $\tilde{h}$, $\hat{h}$, $\hat{k}$ are the values of $\tilde{H}$, $\hat{H}$ and $\hat{K}$, respectively, on a Lagrangian leaf $\Lambda_0(C_3, C_4)$. Now, we are able to make contact with the Sklyanin method of SoV \[11\]. Indeed, comparing \[6.4\] with the spectral curve coming from the Lax pair \[14\], it immediately follows that the separation variables $(x; y)$ satisfy the equation of the spectral curve restricted to $\Lambda_0(C_3, C_4)$.

Now, let us consider the reduction on a generic symplectic leaf $S_1$.

6.4 Proposition The vector field $X_L$, restricted to $S_1$, takes the form

$$X_L = P'_1 d\tilde{H}$$

with $P'_1$ given by \[5.1\]. The Hamiltonian $\tilde{H} = h_1|_{S_1}$ can be written as

$$\tilde{H}(x; y) = (c - 1)C_1 \hat{H}(x; y) + \hat{K}(x; y),$$

$$\hat{H}(x; y) = -G|_{S_1} = \frac{1}{x_1 - x_2} \left( \tilde{f}(x_1, y_1) - \tilde{f}(x_2, y_2) \right),$$

$$\hat{K}(x; y) = \frac{1}{x_1 - x_2} \left( x_2\tilde{f}(x_1, y_1) - x_1\tilde{f}(x_2, y_2) \right),$$

$$\tilde{f}(\xi, \eta) = -\frac{1}{2\xi} \eta^2 + \frac{1}{2} \xi^3 + cC_1 \xi^2 + \frac{C_4}{2\xi} - \frac{1}{2}c(c-1)(c-2)C_1^3.$$

We are in the situation discussed in Proposition \[3.5\] with $\alpha = (c - 1)C_1$ and $\beta = 1$; so, we can conclude that the chart $(\lambda; \mu)$ given by \[8.9\] provides the separation variables for $\tilde{H}$. A solution of the HJ equation for $\tilde{H}$ is given by:

$$W(\lambda_1, \lambda_2; \tilde{h}, \hat{k}) = \int^{\lambda_1}_{} \sqrt{\varphi_1(\xi)} \, d\xi + \int^{\lambda_2}_{} \sqrt{\varphi_1(\xi)} \, d\xi$$

$$\varphi_1(\xi) = \frac{1}{\xi^4} \left( \frac{1}{\xi - \alpha} \right)^2 + 2cC_1 \left( \frac{1}{\xi} - \alpha \right) - \frac{\alpha(c-2)C_1^2}{\xi - \alpha} - 2\hat{h} + \frac{2\hat{k}}{\xi} + \frac{C_4}{(\xi - \alpha)^2}$$

where $\tilde{h}$ and $\hat{k}$ are the values of $\tilde{H}$ and $\hat{H}$ on a Lagrangian leaf $\Lambda_1(C_2, C_3)$. At last, passing to the reduction on the symplectic leaf $S_2$, one has the following result.
6.5 Proposition The vector field $X_L$, restricted to $S_2$, takes the form

$$X_L = P' \, d\hat{G}$$

with $P'$ given by (6.1). The Hamiltonian $\hat{G} = h_2|_{S_2}$ can be written as

$$\hat{G} = -\frac{1}{2c} (c-1) \hat{H}^2 + \hat{K} \quad (6.7)$$

$$\hat{H}(x; y) = -c F_1|_{S_2} = \frac{1}{x_1 - x_2} \left( x_2 \hat{f}(x_1, y_1) - x_1 \hat{f}(x_2, y_2) \right)$$

$$\hat{K}(x; y) = G|_{S_2} = \frac{1}{x_1 - x_2} \left( x_2 \hat{f}(x_1, y_1) - x_1 \hat{f}(x_2, y_2) \right),$$

$$\hat{f}(\xi, \eta) = -\frac{1}{2} \left( \eta^2 \xi^2 - \xi^2 + 2 \frac{C_3}{\xi} - \frac{C_4}{\xi^3} \right).$$

So, we are just in the situation of Proposition 3.6 with $\alpha = -(c-1)/2c$ and $\beta = 1$; also in this case, the vector field $X_L|_{S_2}$ is separable in the chart $(\lambda; \mu)$ given by (3.11). A solution of the HJ equation for $\hat{K}$ is given by:

$$W(\lambda_1, \lambda_2; h, k) = \int_{\lambda_1}^{\lambda_2} \sqrt{\hat{\varphi}_2}(\xi) \, d\xi + \int_{\lambda_2}^{\hat{\varphi}_2}(\xi) \, d\xi \quad (6.8)$$

$$\hat{\varphi}_2(\xi) = \left( C_4 + \frac{2C_3}{\xi} + \frac{2k}{\xi^2} - \frac{2h}{\xi^3} + \frac{1}{\xi^4} \right)$$

where $\hat{h}$ and $\hat{k}$ are the values of $\hat{H}$ and $\hat{K}$ on a Lagrangian leaf $\Lambda_2(C_1, C_2)$.

Remark In contrast with what happens on $S_0$, let us observe that on $S_1$ and $S_2$ the LT vector field $X_L$ is separable in the chart $(\lambda; \mu)$ but not in the chart $(x; y)$, in which it does not admit a Hamiltonian formulation w.r.t. the canonical Poisson tensor $Q_0$.

Acknowledgments. This work was partially supported by the research project Geometry of Integrable Systems of M.I.U.R. and by G.N.F.M. of I.N.D.A.M.

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