Towards conformally flat isothermic metrics

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Abstract

According to [8] if the stationary Schrödinger equation on \( n \)-dim. Riemann space admits \( R \)-separation of variables (i.e. separation of variables with a factor \( R \)), then the underlying metric is necessarily isothermic. An important sub-class of isothermic metrics are the so called binary metrics. In this paper we study conditions for vanishing of components \( C_{ijkl} \) of Weyl tensor of arbitrary 4-binary metrics. In particular all 4-binary metrics for which \( C_{ijij} \) are the only non-vanishing components are classified into four classes. Finally, conformally flat metrics of the last class are isolated.

1 Introduction

The method of separation of variables is one of the most useful method of solving linear partial differential equations. The theory of \( R \)-separability of variables (separability with a factor \( R(x) \)) in the 3-dimensional Laplace equation were laid in the second part of nineteenth century with the work of G. Darboux [2]. The detailed study of orthogonal coordinates in Euclidean spaces has also been given by him in [1] and [3]. The contributions to separability of Schrödinger equation in \( n \)-dimensional Euclidean space were made by H.P. Robertson [7] and L.P. Eisenhart [4]. In twentieth century W. Miller and E.G. Kalnins studied separability theory also in connection with symmetries (see e.g. [6]).

It is also of great importance to find and analyze exact solutions on non-flat spaces. The method of separation allows to construct such solutions and moreover imposes the restrictions on the metrics.

This article is a part of the project of describing the geometry of isothermic metrics.

1.1 \( R \)-separability

We assume that 4-dimensional space \( \mathcal{R}^4 \) admits local orthogonal coordinates \( x = (x^1, x^2, x^3, x^4) \) in which the metric has the following form

\[
g = \sum_{i=1}^{4} H_i^2 (dx^i)^2, \quad H_i = H_i(x). \quad (1)
\]

In [8] we investigated the problem of \( R \)-separability of \( n \)-dimensional stationary Schrödinger equation

\[
\Delta \psi + (k^2 - V(x)) \psi = 0, \quad \Delta = h^{-1} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} h \frac{\partial}{\partial x^i}, \quad h = \prod_{i=1}^{n} H_i, \quad k = \text{const}, \quad (2)
\]
i.e. the existence of solution of the form \( \psi(x) = R(x) \prod_i \psi_i(x^i) \), where \( R(x) \) is a non-vanishing function and functions of one variable \( \psi_i(x^i) \) satisfy

\[
\psi'' + p_i \psi'_i + q_i \psi_i = 0
\]  

(3)

for some \( p_i(x^i), q_i(x^i) \). It has also been shown that the stationary Schrödinger equation (2) is \( R \)-separable in \( \mathbb{R}^n \) if the metric is isothermic and the \( R \)-equation is satisfied (for details see Theorem 3 in [8]).

The isothermic 4-dimensional metric is given by (1) with

\[
H_i^2 = R^{-2} G_{(i)}^{-2} f_i^{-2} \prod_{k=1}^4 G_{(k)},
\]

(4)

where \( G_{(i)} \) does not depend on \( x^i \) while \( f_i \) depends only on \( x^i \). Given six functions \( G_{ij}(x^i, x^j) \) \((i, j = 1, 2, 3, 4 \text{ and } i < j)\). In (4) we put

\[
G_{(i)} = \prod_{p,q \neq i} G_{pq}
\]

(5)

then \( H_i \) is given explicitly as

\[
H_1 = \frac{G_{12}G_{13}G_{14}}{M f_1}, \quad H_2 = \frac{G_{12}G_{23}G_{24}}{M f_2}, \quad H_3 = \frac{G_{13}G_{23}G_{34}}{M f_3}, \quad H_4 = \frac{G_{14}G_{24}G_{34}}{M f_4},
\]

(6)

where \( G_{ij} \) depend only on two variables \( x^i \) and \( x^j \) and \( M \) stands for \( R \). The metric (1) where \( H_i \) are given by (6) is called binary metric.

1.2 Summary convention

It is difficult to maintain the summation convention when working with diagonal metrics. Therefore Einstein summary convention is not used throughout this paper.

2 Geometric quantities for binary metrics

We will investigate the binary metric (1), (6) with specific Weyl tensor. We introduce new functions \( \varphi_{ij}(x^i, x^j) \) defined by

\[
G_{ij} = \exp(\varphi_{ij})
\]

and use them to rewrite binary metric in the form

\[
g = \frac{1}{M^2} \left[ e^{\varphi_{12}+\varphi_{13}+\varphi_{14}} \frac{(dx^1)^2}{F_1} + e^{\varphi_{12}+\varphi_{23}+\varphi_{24}} \frac{(dx^2)^2}{F_2} + e^{\varphi_{13}+\varphi_{23}+\varphi_{34}} \frac{(dx^3)^2}{F_3} + e^{\varphi_{14}+\varphi_{24}+\varphi_{34}} \frac{(dx^4)^2}{F_4} \right],
\]

(7)

where \( F_i = f_i^2(x^i) \). From now on we assume that \( F_i \) are arbitrary functions which could be also negative. The Weyl tensor of binary metric (7) is given by (all indices \( i, j, k, l \) are different and
range from 1 to 4) \(C_{ijkl} = 0, \quad i, j, k, l - \text{different}, \) (8)\n\n\[C_{ikj} = \varphi_{ij,i} \varphi_{jk,j} + \varphi_{ki,i} \varphi_{ij,j} - \varphi_{ki,i} \varphi_{ij,j} - \frac{1}{2} \sum_{l \neq i,j} (\varphi_{ij,i} \varphi_{lj,j} + \varphi_{li,i} \varphi_{ij,j} - \varphi_{li,i} \varphi_{lj,j}), \quad i, j, k - \text{different}, \]

(9)\n\n\[C_{ij} = -\frac{1}{3} M^2 \left[ \frac{1}{2} G_i^{-2} E_{ij,i} F_i + \frac{1}{2} G_j^{-2} E_{ji,j} F_j - \frac{1}{2} \sum_{k \neq i,j} G_k^{-2} \partial_k (E_{ki} + E_{kj}) F_k \right. \]
\[+ G_i^{-2} \left( E_{ij,i} - \varphi_{ij,i} \sum_{k \neq i,j} \varphi_{ik,i} + \sum_{l \neq i,j,k \neq i,j,l} \varphi_{il,i} \varphi_{ik,i} \right) F_i \]
\[+ G_j^{-2} \left( E_{ji,j} - \varphi_{ij,j} \sum_{k \neq i,j} \varphi_{jk,j} + \sum_{l \neq i,j,k \neq i,j,l} \varphi_{jl,j} \varphi_{jk,j} \right) F_j \]
\[\left. - \sum_{k \neq i,j} G_k^{-2} \left( \partial_k^2 (E_{ki} + E_{kj}) - 3 \varphi_{ik,k} \varphi_{jk,k} + \frac{1}{2} \sum_{l \neq k, r \neq k} \varphi_{kl,k} \varphi_{kr,k} \right) F_k \right] \quad i \neq j, \]

where
\[E_{ij} = \varphi_{ij} - \frac{1}{2} \sum_{k \neq i,j} \varphi_{ik}. \]

3 Special binary metrics

As can be seen from (8), for any binary metric the components of Weyl tensor with all indices different are zero (in fact, this is true for any diagonal metric). We classify all metrics (7) for which in addition
\[C_{ikj} = 0. \] (11)

If conditions (11) hold then there exists at most two non-zero independent components of Weyl tensor (eg. \(C_{1212}, C_{1313}\)). To find all metrics for which (11) holds we consider the derivaties of components of the Weyl tensor (9), namely
\[C_{il} = -C_{ikj} \left( \varphi_{ij,k} + \varphi_{ik,j} \right) - \frac{1}{2} \left( (\varphi_{ij} - \varphi_{ki})_i \varphi_{jk,jk} + (\varphi_{ij} - \varphi_{kj})_j \varphi_{ik,ik} \right), \quad k \neq i, j, l. \] (12)

Introducing the following quantities
\[\lambda_{ijk} = -\lambda_{jik} = (\varphi_{jk} - \varphi_{ik},k) \varphi_{ij,ij} \]
(13)
we can rewrite \(C_{ikj} = \lambda_{k(ij)}\). Therefore, the necessary condition for (11) to be satisfied is
\[\lambda_{(ijk)} = 0, \quad i, j, k - \text{different}. \]
(14)

Equations (14) can be rewritten as
\[\lambda_{ijk} = \lambda_{jki} = \lambda_{kij} \]
(15)
for any three different indices $i, j, k = 1, 2, 3, 4$. Surprisingly all solutions of (15) are known [4]. They are listed in Table 1, where functions $U_i, V_i$ and $Q_i$ depend only on $x^i$ and $m$ is constant.

It is noted that the Weyl tensor of Lorentzian binary metrics for which (11) holds might be algebraically special of Petrov type $D$ or $O$.

In the next subsection we concentrate on the case iv) where $m$ is integer or half integer.

### 3.1 Case iv)

In the case iv) we assume that $U_i$ ($i = 1, 2, 3, 4$) are non-constant function and choose $U_i = x^i$. Then the metric (7) becomes

$$g = \frac{1}{M^2} \left[ (x^1 - x^2)^{2m}(x^1 - x^3)^{2m}(x^1 - x^4)^{2m} F_1(x^1) (dx^1)^2 + (x^1 - x^2)^{2m}(x^2 - x^3)^{2m}(x^2 - x^4)^{2m} F_2(x^2) (dx^2)^2 + (x^1 - x^3)^{2m}(x^2 - x^3)^{2m}(x^3 - x^4)^{2m} F_3(x^3) (dx^3)^2 + (x^1 - x^4)^{2m}(x^2 - x^4)^{2m}(x^3 - x^4)^{2m} F_4(x^4) (dx^4)^2 \right].$$

(16)

We do not assume that functions $F_i$ are positive, therefore in general the latter can have Riemannian, Lorentzian or neutral (+ + --) signature in some regions. All conformally flat metrics (16) with $m = \frac{N}{2}$ ($N \in \mathbb{Z}$) can be found.

**Lemma.** The metric (16) with integer or half-integer $m$ is conformally flat if and only if

a) $m = -1$ and $F_i(x^i)$ are constant functions,

b) $m = -\frac{1}{2}$ and $F_i(x^i)$ are polynomials of second order,

c) $m = 0$ and $F_i(x^i)$ are arbitrary functions of one variable,

d) $m = \frac{1}{2}$ and $F_i(x^i)$ are polynomials of sixth order.

**Proof:** The condition (11) is automatically satisfied for all metrics (16). Hence there exist only 2 independent components of Weyl tensor, e.g. $C^{12}_{12}, C^{13}_{13}$, which depend linearly on $F_i$ and $F_i'$. It turns out that

$$[x^2(x^1 + x^3 - 2x^4) + x^3(x^1 - 2x^4) + x^1x^4)] C^{12}_{12} - [x^3(x^1 + x^2 - 2x^4) + x^2(x^4 - 2x^1) + x^1x^4] C^{13}_{13}$$

$$= \frac{M^2}{2}m(2m - 1) \prod_{k<l}(x^k - x^l)^{-2m-1} \sum_{i=1}^{4} \left( \prod_{k<l\neq i}(x^k - x^l)^{2(m+1)} \right) F_i.$$  

(17)

| \(\varphi_{12}\) | \(\varphi_{13}\) | \(\varphi_{14}\) | \(\varphi_{23}\) | \(\varphi_{24}\) | \(\varphi_{34}\) |
|---|---|---|---|---|---|
| i) arbitrary | \(U_1 + U_3\) | \(U_1 + U_4\) | \(U_2 + U_3\) | \(U_2 + U_4\) | arbitrary |
| ii) arbitrary | \(U_1 + U_3\) | \(V_1 + V_4\) | \(U_2 + U_3\) | \(V_2 + V_4\) | \(Q_3 + Q_4\) |
| iii) \(U_1 + U_2\) | \(V_1 + U_3\) | \(Q_1 + U_4\) | \(V_2 + V_3\) | \(Q_2 + V_4\) | \(Q_3 + Q_4\) |
| iv) \(m \ln |U_1 - U_2|\) | \(m \ln |U_1 - U_3|\) | \(m \ln |U_1 - U_4|\) | \(m \ln |U_2 - U_3|\) | \(m \ln |U_2 - U_4|\) | \(m \ln |U_3 - U_4|\) |

Table 1: Components of the binary metrics (7) for which $C^{k \ i k j} = 0$. 


There are two special cases: \( m = 0 \) and \( m = \frac{1}{2} \) which must be treated separately. If \( m = 0 \) then metric (16) is obviously conformally flat for any functions \( F_i(x^i) \),

\[
g = \frac{1}{M^2} \left[ \frac{(dx^1)^2}{F_1(x^1)} + \frac{(dx^2)^2}{F_2(x^2)} + \frac{(dx^3)^2}{F_3(x^3)} + \frac{(dx^4)^2}{F_4(x^4)} \right]. \tag{18}
\]

In the case \( m = \frac{1}{2} \) it can be shown that

\[
\partial_i^7 F_i = 0, \quad i = 1, 2, 3, 4,
\]

so \( F_i(x^i) \) are polynomials at most of the sixth degree. Further analysis of the components \( C^{12}_{12} \), \( C^{13}_{13} \) leads to conclusion that the coefficients of the polynomials are equal up to sign. Finally conformally flat metric in this case can be written in the following form

\[
g = \frac{1}{M^2} \sum_{i=1}^{4} \prod_{j \neq i} (x^i - x^j)^2 \sum_{k=0}^{m} a_k (x^i)^k (dx^i)^2, \tag{20}
\]

where \( a_k \) are arbitrary constants.

If \( m \notin \{0, \frac{1}{2}\} \) then

\[
L := \sum_{i=1}^{4} \left( \prod_{k \neq i} (x^k - x^i)^2 (m + 1) \right) F_i = 0. \tag{21}
\]

By differentiate the latter with respect \( x^1, x^2, x^3 \) and \( x^4 \) we can express each \( F_i \) in terms of \( F_1, F_2, F_3, F_4 \) and then eliminate \( F_i \) from \( C^{12}_{12} \) and \( C^{13}_{13} \). The calculation ends up with the result

\[
C^{12}_{12} = f_1 L, \quad C^{13}_{13} = f_2 L, \tag{22}
\]

where \( f_1, f_2 \) are two non-vanishing functions. This shows that (21) is a necessary and sufficient condition for metric (16) to be conformally flat. We show below that there exist non-trivial solutions for (21) only if \( m = -1 \) or \( m = -\frac{1}{2} \) (we omit cases \( m = 0, \frac{1}{2} \)).

First let us assume that \( m < -1 \), then \( \alpha = -2(m + 1) > 0 \) and it is more convenient to use equation \( K = 0 \), where

\[
K = L \prod_{k<l} (x^k - x^l)^{-2(m+1)}
\]

\[
= [(x^1 - x^2)(x^1 - x^3)(x^1 - x^4)]^{-2(m+1)} F_1 + [(x^1 - x^2)(x^2 - x^3)(x^2 - x^4)]^{-2(m+1)} F_2
\]

\[
+ [(x^1 - x^3)(x^2 - x^3)(x^3 - x^4)]^{-2(m+1)} F_3 + [(x^1 - x^4)(x^2 - x^4)(x^3 - x^4)]^{-2(m+1)} F_4. \tag{23}
\]

We can differentiate \( \alpha \) times equation \( K = 0 \) with respect \( x^1, x^2, x^3 \) and \( x^4 \) and obtain

\[
\alpha !^3 \left[ (-1)^\alpha F_1^{(\alpha)} + F_2^{(\alpha)} + (-1)^\alpha F_3^{(\alpha)} + F_4^{(\alpha)} \right] = 0, \tag{24}
\]

where \( \alpha \) in subscript denotes derivative of order \( \alpha \). This shows that \( F_i \) must be polynomials at most of the degree \( \alpha \). We proceed further with \( F_1 \). Calculating the derivative \( \partial_1^{\alpha+1} \partial_2^{\alpha-1} \partial_3^\alpha \partial_4^\alpha K \) one gets

\[
(x^1 - x^2) F_1^{(\alpha+1)} + (\alpha + 1) F_1^{(\alpha)} = 0 \tag{25}
\]
and so, $F_1$ is a polynomial at most of the degree $\alpha - 1$. Considering now $\partial_1^{\alpha+1} \partial_2^{\alpha-1} \partial_3 \partial_4 K$, where $l = 1, 2, \ldots, \alpha$ we conclude by induction that $F_1 = \text{const}$. An analogous treatment with $F_2, F_3, F_4$ leads to the result

$$F_i = \text{const}, \quad i = 1, 2, 3, 4. \tag{26}$$

But then

$$K = (x^1)^{-6(m+1)} F_1 + (-x^2)^{-6(m+1)} F_2 + (x^3)^{-6(m+1)} F_3 + (-x^4)^{-6(m+1)} F_4 + \mathcal{O}, \quad F_i = \text{const}, \tag{27}$$

where $\mathcal{O}$ contains terms with $x^i$ to power less then $-6(m + 1)$. We obtain from this $F_i = 0, \quad i = 1, 2, 3, 4$. This means that no solutions exist for $m < -1$.

If $m = -1$ then from (21)

$$\sum_{i=1}^{4} F_i = 0. \tag{28}$$

Hence, the following metric

$$g = \frac{1}{M^2} \sum_{i=1}^{4} F_i \prod_{j \neq i} (x^i - x^j)^2, \tag{29}$$

is conformally flat provided $F_i$ are constants and (28) is satisfied. It is easily seen that the latter can have Riemannian, Lorentzian or neutral ($++-$) signature.

Let us assume that $m > -1$ and consider the identity (21). Calculating derivatives of $L = 0$ with respect to all $x^i$ we obtain four new equations which must be satisfied by $F_i$ and $F_i'$, namely

$$\partial_i L = 0, \quad i = 1, 2, 3, 4. \tag{30}$$

Now we solve the latter for $F_i'$ and insert them into

$$\partial_{12} L = 0, \quad \partial_{13} L = 0, \quad \partial_{14} L = 0. \tag{31}$$

By this procedure we get four homogeneous equations involving only $F_i$ (21 and 31). Non-zero solution is possible if and only if $\det M = 0$, where $M$ is $(4 \times 4)$ matrix containing coefficients of $F_i$ $(i = 1, 2, 3, 4)$ in (21) and (31). It can be shown that

$$\det M = -16(m + 1)^3(2m + 1)^3(x^2 - x^3)^2(x^2 - x^4)^2(x^3 - x^4)^2 \prod_{k<l} (x^k - x^l)^2(2m+1) \tag{32}$$

and hence $m = -1$ (which was considered before) or $m = -\frac{1}{2}$. Further analysis of the components $C_{12}^{12}, C_{13}^{13}$ for $m = -\frac{1}{2}$ leads to conclusion that $F_i$ are polynomials of second order. Finally we obtain

$$g = \frac{1}{M^2} \sum_{i=1}^{4} \frac{(dx^i)^2}{\prod_{j \neq i} (x^i - x^j) \sum_{k=0}^{2} a_k (x^i)^k}, \tag{33}$$

where $a_k$ are constants.\hfill\□
4 Conclusions

We found general forms of binary 4-dimensional metrics (see (7) and Table 1) for which \( C_{ijij} \) are the only non-vanishing components of the Weyl tensor. All conformally flat metrics of case iv) has been found in explicit form, see (18), (20), (29) and (33). All of them could be Riemannian, Lorentzian or neutral. Moreover in generic case neither of three solutions (20), (29), (33) are flat even for \( M = \text{const} \).

It is not difficult to construct an example of Lorentzian metric. Assume that polynomial \( \sum_{i=0}^6 a_k(x^i)^6 \) has exactly four real different roots \( b_i \) and \( a_6 > 0 \). If the coordinates \( (x^i) \) satisfy inequalities
\[
x^1 > b_1 > x^2 > b_2 > x^3 > b_3 > b_4 > x_4
\]
then metric (20) has signature \((++-+)\). The Lorentzian metrics could be potentially useful in general relativity where they represent spacetime. Metric (20) is flat if \( M \) is a constant function and \( a_5 = a_6 = 0 \). If in addition polynomial \( a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \) has four real different roots then \( (x^i) \) constitute elliptic coordinates in Euclidean flat space.

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References

[1] Darboux G., Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux, Annales Scientifiques de l’É.N.S. 7 (1878), 101–150 (I), 227–260 (II), 275–348 (III).

[2] Darboux G., Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux, Annales Scientifiques de l’É.N.S. 7 (1878), 275–348.

[3] Darboux G., Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes, Gauthier-Villars, Paris, 1910.

[4] L.P. Eisenhart, Triply Conjugate Systems with Equal Point Invariants, Annals of Mathematics 20, No. 4 (1919), 262

[5] Eisenhart L.P., Separable Systems of Stäckel, Annals of Mathematics 35 (1934), 284–305.

[6] Miller Jr W., The Technique of Variable Separation for Partial Differential Equations, ”Non-linear Phenomena”, Lecture Notes in Physics 189 (1983), 184–208.

[7] Robertson H.P., Bemerkung über separierbare Systeme in der Wellenmechanik, Mathematische Annalen 98 (1928), 749–752.

[8] A. Sym, A. Szereszewski, On Darboux’s Approach to R-Separability of Variables, SIGMA 7 (2011), 095