1. Introduction

It is well known that prime numbers play a central role in number theory. It has been known, since Riemann’s famous memoir [26] in 1859, that the distribution of prime numbers can be described by the zero-free region of the Riemann zeta function $\zeta(s)$. This function is a meromorphic function of the complex variable $s$. It has infinitely many zeros and a unique pole at $s = 1$ with residue 1. Let $\mathbb{C}$ denote the set of complex numbers. It is customary to denote $s = \sigma + it$, with $\sigma$ and $t$ real, for any $s \in \mathbb{C}$. For $\sigma > 1$, the Riemann zeta function can be defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}},$$

where $\mathbb{P}$ is the set of all prime numbers, with the second equality above being Euler’s identity. One may verify Euler’s identity from the fundamental theorem of arithmetic, which asserts

$$n = \prod_{l=1}^{k} p_l^{a_l},$$

for every $n \in \mathbb{N}$, with $k \in \mathbb{N}$ and $p_l^{a_l} \in \mathbb{P}^\mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers and $\mathbb{P}^\mathbb{N}$ is that of all prime powers, that is $n \in \mathbb{P}^\mathbb{N}$ if and only if $n = p^k$ for some prime $p$ and integer $k \geq 1$. 

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For $\sigma > 1$, we may use the logarithmic differentiation of Euler’s identity to obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Here, the Mangoldt function $\Lambda$ is an arithmetic function defined by

$$\Lambda(n) = \begin{cases} \log p, & n \in \mathbb{P}^N; \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbb{P}^N$ is defined after (1.1).

We also use the notation $\mathbb{R}^+$ for the set of all positive real numbers. We shall use the symbol $\epsilon \in \mathbb{R}^+$ for an arbitrary small positive real number, not necessarily the same at each occurrence in a given statement. Suppose that $g(x)$ and $h(x)$ are complex functions of the variable $x$ and $f(x)$ is a non-negative real-valued function of $x$. The notation $g(x) \leq h(x) + B f(x)$ represents the fact that $|g(x) - h(x)| \leq Bf(x)$ where $B > 0$ is a constant, whenever $x$ is sufficiently large, or $x \geq x_0$ for some fixed positive number $x_0$. We use the notation $g(x) \geq h(x) + B f(x)$ in the similar way for the inequality in the opposite direction.

The Riemann zeta function has real zeros at $s = -2, -4, -6, \ldots$, called trivial zeros. Non-real zeros are known as non-trivial zeros. It is not very difficult to show that non-trivial zeros of $\zeta(s)$ are located in the commonly referred to as the critical strip $0 < \sigma < 1$. Some other results in this direction are zero-free regions in the form of

$$\sigma > 1 - h(t), \quad |t| > 3,$$

where $h(t)$, with $0 < h(t) \leq \frac{1}{2}$, is a decreasing function of $t$. This function $h(t)$ includes $\frac{C}{\log |t|}$, $\frac{C \log \log |t|}{\log |t|}$, $\frac{C}{\log^{3/4 + \epsilon}|t|}$, and $\frac{C}{\log^{2/3}|t| \log \log |t|}t^{1/3}$, where $C$ is a positive constant, which may be different in each situation. One may refer to any standard literature, e.g., [4], [17], [21], [28], and/or [19].

We adopt the notation $\sum_{n \leq x} f(n)$, which means that we use the half-maximum convention to the sum function of the arithmetic function $f(n)$. Therefore, $f(x_0) = \frac{1}{2}(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x))$. Let $x \geq 2$. We define the $\psi$-function similar to that in the literature, but with this half-maximum convention. We define the $\varpi$-function with this convention, as well. That is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \quad \varpi(x) = \sum_{n \leq x} (\Lambda(n) - 1),$$

where $\Lambda(n)$ is the Mangoldt function.
where $n$ runs through the set of positive integers not greater than $x$. We remark here that any estimate on $\psi(x)$ may be converted to an estimate on $\varpi(x)$, and vice versa; the latter of which is needed later on. We notice that

$$\left(1.4\right) \quad \sum_{n \leq x} 1 = \begin{cases} x - \{x\}, & x \notin \mathbb{N}, \\ x - \frac{1}{2}, & x \in \mathbb{N}, \end{cases}$$

where $\{x\}$ is the fractional part of $x$. From this and

$$\sum_{n \leq x} \Lambda(n) = \sum_{n \leq x} (\Lambda(n) - 1) + \sum_{n \leq x} 1,$$

we see that

$$\left(1.5\right) \quad \varpi(x) + x - 1 \leq \psi(x) \leq \varpi(x) + x.$$

It is well known that a zero-free region of $\zeta(s)$, in the form of $\sigma > 1 - h(t)$ and $|t| \geq 3$, implies the prime number theorem in the following equivalent $\psi$-form and $\varpi$-form:

$$\left(1.6\right) \quad \psi(x) = x + O\left(x^{1-H(x)} \log^2 x\right), \quad \varpi(x) \leq B x^{1-H(x)} \log^2 x,$$

with an absolute positive constant $B$, where the function $H(x)$ is connected to $h(t)$ in a certain way. Less known is that the converse is also true. Actually, Turán proved in 1950 that $\pi(x) = Li(x) + O\left(x \exp\left(-C x \log^{1+a} x\right)\right)$, for an $a$ with $0 < a < 1$, implies the above zero-free region in (1.2) of $\zeta(s)$ with $h(t) = \frac{C_1}{\log^2 |t|}$ for $|t| \geq C_0$ with positive constants $C_x$, $C_t$, and $C_0$. See [21] and [30].

Corresponding to the definition of $\varpi(x)$ in (1.3) and the estimate on $\varpi(x)$ in (1.6), we may instead study the function

$$\left(1.7\right) \quad - \frac{\zeta'(s)}{\zeta(s)} - \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s},$$

as in [16]; this is the reason for us to have put Theorem 1 in the $\varpi$-form, although the $\psi$-form is well-known and used in the literature. The series in (1.7) is convergent when $\sigma > 1$. For $\sigma > 0$, we have

$$\zeta(s) = \frac{s}{s - 1} - s \int_1^\infty \frac{v - \lfloor v \rfloor}{v^{s+1}} \, dv,$$

where $\lfloor v \rfloor$ is the integer part of $v$.

In this article, we use the notation $\mathbb{Z}$ for the set of non-trivial zeros of $\zeta(s)$, which are denoted customarily by $\rho = \beta + i \gamma$ with $0 < \beta < 1$. 

ANALYTIC IMPLICATIONS FROM THE P.N.T.
Let \( \gamma_0 (\approx 0.577215) \) denote the Euler-Mascheroni constant (also called Euler’s constant). It is well known that

\[
- \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho \in \mathbb{Z}} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) - \log \pi
\]

(1.8)

\[
+ \frac{\Gamma'(\frac{1}{2} s + 1)}{2\Gamma\left(\frac{1}{2} s + 1\right)} + 1 + \frac{\gamma_0}{2} - \log 2.
\]

where the Gamma-function \( \Gamma(s) \) has neither zeros nor poles for \( \sigma > 1 \). Also, we remark that there is no pole at \( s = 1 \) for the function on the left hand side of (1.7), since the pole of \( -\frac{\zeta'(s)}{\zeta(s)} \) at \( s = 1 \) and that of \( \zeta(s) \) at the same point, cancel on the right hand side of (1.7). The set \( \mathbb{Z} \) is the same as the set of poles of the function \( -\frac{\zeta'(s)}{\zeta(s)} \).

It is known from [18] that the Riemann zeta function for \(|t| < 14\) does not have any non-trivial zeros. From the computational perspective, we mention here that Xavier Gourdon uses an optimization of the Odlyzko and Schönhage algorithm in [22] and verifies in [20] that the \( 10^{13} \) first non-trivial zeros of the Riemann zeta function are all simple and located on \( \sigma = \frac{1}{2} \). Let \( N(T) \) denote the number of non-trivial zeros of the Riemann zeta function in the region \( 0 < \sigma < 1 \) and \( 0 \leq t \leq T \). We use this result defined as above, if \(|\gamma| < 2445999556029\), as in [5]. Henceforth, we let \( T_0 = 2445999556027 \) in this article, where the difference of 2 is used for convenience with other related issues in the articles [5], [6], and [16].

We may adopt the notation \( a \mathbb{N} + b \) for \( a \) and \( b \in \mathbb{Z} \), to denote the subset \( \{a n + b : n \in \mathbb{N}\} \) of \( \mathbb{Z} \). Also, we denote

\[
\mathbb{N}_7 := \{0, 1, 2, 3, 4, 5, 6\} \subseteq \mathbb{N},
\]

in this article. Let \( w \geq 5 \). For \( j \in \mathbb{N}_7 \), we let \( X_0 = 28.99 \) and

\[
H_j(x) = \begin{cases} 
\frac{1}{2}, & 1 \leq x < X_0, \\
\frac{2}{\log^{(7-j)/2} x}, & x \geq X_0,
\end{cases}
\]

and

\[
h_j(t) = \begin{cases} 
\frac{1}{2}, & |t| < T_0, \\
\frac{2 (7-j)^w /12}{2}, & |t| \geq T_0.
\end{cases}
\]

It is easy to see that both \( H_j(x) \) and \( h_j(t) \) are two-piece piece-wise differentiable functions for each \( j \in \mathbb{N}_7 \). They are monotonically decreasing and tend to 0, respectively, as \( x \) and \( t \) tends to \( \infty \), respectively for each fixed \( j \in \mathbb{N}_7 \). The function \( H_j(x) \) is continuous for all \( x \geq 1 \) and the function \( h_j(t) \) has a unique jump discontinuity at its change in
definition at \( t = T_0 \), for all \( j \in \mathbb{N}_7 \). We remark that from the definitions 
\[
\begin{align*}
\frac{2}{\log((7-j)/12)x} \leq H_j(x) & \leq \frac{1}{2}, \quad \text{for } x > X_0, \\
\frac{1}{2U t((7-j)/12)x} \leq h_j(t) & \leq \frac{1}{2}, \quad \text{for } t > T_0,
\end{align*}
\]
for all \( j \in \mathbb{N} \); otherwise, we have both \( H_j(x) = \frac{1}{2} \) and \( h_j(t) = \frac{1}{2} \) by their definitions in (1.10) and (1.11). The domain of \( H_j(x) \) is \( x \in [1, \infty) \) and that of \( h_j(t) \) is \( t \in (-\infty, \infty) \), even though we may only use the function for \( t \in [0, \infty) \) conveniently with the symmetry property of the Riemann zeta function by the Schwarz principle.

Concerning the application of this setup of \( H_j(x) \) and \( h_j(t) \) later on, we let \( f(x) = \log((j+5)/12)x - \frac{e(j+5)}{12} \log \log x \) for \( x \in (e, \infty) \) here. We have \( f'(x) = \frac{j+5}{12x \log((7-j)/12)x} \left( 1 - \frac{e}{\log((j+5)/12)x} \right) = 0 \) for the unique critical point \( x_0 = e^{e^{12/(j+5)}} \). Note for \( x = x_0 \), we have \( f(x) = 0 \). Checking the sign of \( f'(x) \), we see that \( f(x) \geq 0 \) for all \( x \geq e \). Hence,
\[
\log((j+5)/12)x \geq \frac{e(j+5)}{12} \log \log x, \quad x \in (e, \infty).
\]
From the last inequality, here we remark that
\[
\frac{\log((j+5)/12)x}{\log x} = \frac{1}{\log((7-j)/12)x} \geq \frac{e(j+5) \log \log x}{12 \log x},
\]
with the last expression corresponding to that used in the definition of \( H_j(x) \).

Our main result in this article is as follows. It is needed in [16] when we study the Riemann hypothesis, which states that the real parts of all non-trivial zeros of the Riemann zeta function are equal to \( \frac{1}{2} \).

**Theorem 1.** Let \( j \in \mathbb{N}_7 \). If
\[
(1.14) \quad \varpi(x) \leq D x^{1 - H_j(x)} \log^2 x, \quad x \geq X_0,
\]
with \( D = 9 \), then \( \zeta(s) \) does not vanish when \( \sigma > 1 - h_j(t) \), with \( w \geq (j - 3)^2 + 4 \).

We sketch the proof of Theorem 1 in the next section, and provide the details afterwards. We remark here that the result in Theorem 1 translates estimates on the remainder term of the prime number theorem as an algebraic object into analytic descriptions for the zero-free region of the Riemann zeta function, and it plays a pivotal role in [16] with a proof of the Riemann hypothesis based on the results in other article mentioned in the sequence [4], [10], [3], [1], [14], [11], [9], [5], [12], [6], [15], [7], this article, and [16]; with [8] as a summary of insight and [2] as a sketch of the technical tools.
2. Proof of Theorem 1

We shall show Theorem 1 by contradiction in this section. Assume to the contrary that there is a non-trivial zero \( \rho' = \beta' + i\gamma' \in \mathbb{Z} \) such that

\[
\gamma' > T_0, \quad 1 - h_j(\gamma') < \beta' < 1.
\]

We let

\[
\beta_0 = \frac{1}{2(T_0)^{13/12}} < \frac{1}{5 \times 10^{13}},
\]

and have

\[
\beta' > 1 - h_j(T_0) \geq 1 - h_6(T_0) = 1 - \beta_0,
\]

from (2.1), noting that \( h_j(t) \) is a monotonously decreasing function of \( t \) and increasing with respect to \( j \in \mathbb{N}_7 \), as \( \gamma' \geq T_0 \), where \( T_0 \) is defined in the last paragraph before (1.10).

We prove (3.12) after some preparations in Section 3. We define \( H, H_0, H_1, H_2, \) and \( H_3 \) as specific subsets of \( \mathbb{Z} \), and sums on these subsets \( S, S_0, S_1, S_2, \) and \( S_3 \), respectively. We give upper bounds for \( |S_0|, |S_1|, |S_2|, \) and \( |S_3| \), in Section 4, by means of the inequality in (3.12). An upper bound on \( |S| \) is given in (2.21), making use of the relationship between \( S \) and other four subset \( S_0, S_1, S_2, \) and \( S_3 \) given in (2.20). We then take the advantage of Lemma 2 to acquire a lower bound for \( S \) as in (2.23). By our assumption in (2.1), we obtain a lowerbound on \( |S| \) that is greater than our upper bound. Therefore, we have proved Theorem 1 by contradiction.

We define some independent constants whose values will be chosen with respect to each of the corresponding sub-intervals. First of all, we let

\[
\sigma_0 > 1, \quad s_0 = \sigma_0 + i\gamma',
\]

where \( \beta' \) and \( \gamma' \) are subject to (2.1).

Next, we let

\[
1 < \hat{x} \leq 1 + \frac{1 - 2\beta_0}{2(\sigma_0 - 1 + \beta_0)},
\]

for a horizontal restraint, which is needed in (2.8). Recalling (2.3), one sees that the second restriction in the above implies \( \hat{x} \leq \frac{\sigma_0 - 1/2}{\sigma_0 - \beta'} \). Also, we let

\[
\hat{u} = \frac{1}{\sigma_0 - \beta'},
\]

as a restraint for directions both horizontal and vertical directions. The values of these constants \( \sigma_0 \) and \( \hat{x} \) will be chosen later on. For a vertical restriction, we let

\[
1 = \hat{u}(\sigma_0 - \beta') < \hat{y} < T_0, \quad \hat{y} \in \mathbb{N} + 1.
\]
The values of the following constants \( \sigma_0, \hat{x}, \bar{u}, \) and \( \hat{y} \) will be chosen later on. We notice that
\[
(2.8) \quad \sigma_0 - \hat{x}(\sigma_0 - \beta') > \frac{1}{2},
\]
from (2.5), recalling the remark afterwards.

Then, we define
\[
(2.9) \quad H = \{ \rho \in \mathbb{Z} : |\gamma - \gamma'| \leq \bar{u}(\sigma_0 - \beta') \quad \text{and} \quad \beta > \sigma_0 - \hat{x}(\sigma_0 - \beta') \},
\]
and \( L \) to be the number of the zeros of the Riemann zeta function in the region \( H \). For convenience, we also let \( H_0 = \mathbb{Z} \) and define

\[
\mathbf{H}_0 = \mathbf{H}_1 \cup \mathbf{H}_2 \cup \mathbf{H}_3 \cup \mathbf{H}
\]

\[
\mathbf{H}_1 = \{ \rho \in \mathbb{Z} : |\gamma - \gamma'| > \hat{y} \},
\]

\[
(2.10) \quad \mathbf{H}_2 = \{ \rho \in \mathbb{Z} : \bar{u}(\sigma_0 - \beta') < |\gamma - \gamma'| \leq \hat{y} \},
\]

\[
\mathbf{H}_3 = \{ \rho \in \mathbb{Z} : |\gamma - \gamma'| \leq \bar{u}(\sigma_0 - \beta') , \beta \leq \sigma_0 - \hat{x}(\sigma_0 - \beta') \}.
\]

We remark here that \( \mathbf{H}_3 \) is on the left side of \( \mathbf{H} \), whereas \( \mathbf{H} \) is on the right side of the half line \( \sigma = \frac{1}{2} \), which can be seen from (2.8). Also, \( \mathbf{H}_2 \) covers regions above and below the union of \( \mathbf{H}_3 \) and \( \mathbf{H} \), and \( \mathbf{H}_1 \) covers the outside regions above and below \( \mathbf{H}_2 \). In particular, we emphasize that the region \( \mathbf{H} \) is completely located in the open half plane to the right of the line \( \sigma = \frac{1}{2} \), by (2.8). We shall need this property when we give the lower bound of \( S \) as in (2.23) in Section 5.

Let \( N(\lambda, T) \) be the number of zeros of \( \zeta(s) \) when \( \Re(s) \geq \lambda \) and \( 0 \leq \Im(s) \leq T \), with \( \frac{1}{2} \leq \lambda \leq 1 \) and \( T \geq 0 \). In [6], there is a stronger
estimate on the zero-growth rate equivalent to the Lindelöf hypothesis, which is proved with the newly introduced pseudo-Gamma function by Y. Cheng, C. B. Pomerance, G. J. Fox, and S. W. Graham, in [14]. We quote the result of Theorem 1 from [6] as the lemma below.

**Lemma 2.** Let $T \geq T_0$ with $T_0$ designed in the paragraph before (1.10). Then, for $\frac{1}{2} < \lambda < 1$ and $1 \leq d \leq \frac{5}{4}$, we have

(2.11) \[ N(\lambda, T + d) - N(\lambda, T - d) \leq 3. \]

Still, we use the constants $b$ and $c$, which are related to $\gamma'$, satisfying

(2.12) \[ b \geq \frac{U}{45}, \]

in which, the restriction on $c$ is to guarantee that the value of $\tau_0$ in (2.22) below is positive; we actually let

(2.13) \[ c \log \gamma' - b \log \gamma' \geq U, \]

for the application in (5.7). This choice of $c$ also place a restriction on the choice of $b$. The values of $b$ is determined later to restrict the choice of a constant $k$, noting that $\gamma' \geq T_0$, such that

(2.14) \[ b \log \gamma' \leq k \leq c \log \gamma', \quad k \in \mathbb{N} + 6. \]

Recall the definition of $N_7$ and $j \in N_7$ in (1.9) so that $0 \leq j \leq 6$. We let $J(t)$ as below and note

(2.15) \[ J(t) = \left[6t(t+4)\right]^{\frac{6}{(j+5)e-12}} \geq \log \frac{11t^2+1}{10}, \]

as $\frac{6}{(j+5)e-12} \geq \frac{6}{11e-12} > \frac{1}{3}$, $\left[6t(t+4)\right]^{\frac{6}{(j+5)e-12}} \geq t^{2/3}$, $t^{2/3} > 3 \log t$, and $3 \log t > \log \frac{11t^2+1}{10}$ for $t \geq T_0$, noting that the denominator $(j+5)e - 12 \geq 5e - 12 > 1$ in the exponent of the first component. The above two components in the maximum in (2.15) will be used for (3.8) and (3.6), respectively. Then, we use a rational constant $\hat{h} \in \mathbb{Q}$ for adjustment, such that

(2.16) \[ \frac{1}{6k} \leq \hat{h} \leq \frac{2}{6k}. \]

Like other constants with convenient values that will be determined later conveniently, this constant $\hat{h}$ would look superfluous once we decide on the values of $b$ and $k$. But, it is convenient at this point for us to use this extra constant in order to derive some statement more easily, especially in the sake of (2.18).

For a technical convenience, we also let

\[ \delta = \delta(t) := \begin{cases} 0, & \text{if } e^{\hat{h}bk J(t)} \notin \mathbb{Q}, \\ 1, & \text{if } e^{\hat{h}bk J(t)} \in \mathbb{Q}. \end{cases} \]
Then, we let

\[
W := W(k; t) = e^{h_k J(t)} \left(1 + \frac{\pi}{Q}\right)^{\delta}.
\]

where \(Q \in \mathbb{Q}\) is a sufficiently large positive rational number so that

\[
Q \geq \frac{\pi e^{h_k J(t)} \eta}{\eta},
\]

with respect to \(\eta > 0\). Furthermore, we see that \(W\) is always an irrational number by the design of \(\delta\), and

\[
11t^2 + 10 < e^{2/3} < W \leq (1 + \eta) e^{2h_k \log \gamma' [6(t(t+4))^{1/2} - 2]},
\]

for any \(\eta > 0\).

Now, we consider the sums \(S\) and \(S_j\) for \(j = 0, 1, 2, 3\), where

\[
S = \sum_{\rho \in H} W_{\rho - \rho'} (s_0 - \rho')^k, \quad S_j = \sum_{\rho \in H_j} W_{\rho - \rho'} (s_0 - \rho')^k,
\]

where \(W = W(k; \gamma')\) with \(W(k; t)\) defined in (2.17) above. Moreover, \(H_0 = H_1 \cup H_2 \cup H_3 \cup H\) with the union being disjoint, for which one may refer to Figure 1 for a rough visualization. Hence,

\[
S = S_0 - S_1 - S_2 - S_3.
\]

We also define a constant \(\tau'\), which depends on \(\tau_0, \tau_1, \tau_2, \) and \(\tau_3\) in (2.22) below, where \(\tau_0, \tau_1, \tau_2, \) and \(\tau_3\) concerns the above sum \(S_0, S_1, S_2, \) and \(S_3, \) respectively, whose values are determined by the choices of the above independent constants later on.

We make a preparation by giving an upper bound for a sum related to \(S\) in Section 3. In Section 4, we give the estimates for \(S_0, S_1, S_2, \) and \(S_3.\) Recalling the definition of \(\tau'\) in (2.22), we obtain an upper bound for the sum \(S,\) under the assumption of (2.1), in the form of

\[
S = \sum_{\rho \in H} \left(\frac{e^{\omega(\rho - \rho')} s_0 - \rho'}{s_0 - \rho}\right)^k \leq \frac{CW^{1-\beta'} \log \gamma'}{(\gamma')^{\tilde{\tau}}},
\]

where

\[
\tilde{\tau} = \min\{\tau_0, \tau_1, \tau_2, \tau_3\}, \quad C = \frac{c_0}{28.525} + C_1 + C_2 + C_3,
\]

with

\[
\begin{align*}
\tau_0 &= 1 - \frac{c_0}{s_0 - 1}, \quad C_0 = 6.7, \quad \tau_1 = b \log \hat{y}, \quad C_1 = 1.3, \\
\tau_2 &= b \log \hat{u} - \frac{\log(2.081 b_2 (\hat{y} - 1))}{\log T}, \quad C_2 = 2.081 (1 - \delta_2)(\hat{y} - 1), \\
\tau_3 &= b \log \hat{x}, \quad C_3 = 2.081, \quad C = 10.081 + 2.081 (1 - \delta_2)(\hat{y} - 1),
\end{align*}
\]
as \( \gamma' \geq T_0 \) and \( \log T_0 \geq 28.525 \), where \( W = W'(k; t) \) with \( t = \gamma' \) in (2.1) and the restrictions of \( \hat{y} \) in (2.7), that of \( \hat{u} \) in (2.6), and \( \hat{\tau} \) is defined in (2.25), by \( |S| \leq |S_0| + |S_1| + |S_2| + |S_3| \) from (2.20) with the upper bounds of \( S_j \) for \( j = 0, 1, 2, \) and \( 3 \) from (1.4), (4.11), (4.12), and (4.14).

In Section 5, we find a lower bound for \( S \) under the assumption of (2.1), under the condition in (2.12). Therefore, there exists at least one such sum, corresponding to a \( k \) satisfying (2.14), such that

\[
|S| \geq \frac{1}{A' \log^U \gamma'},
\]

where \( A' = \left( \frac{42e^2(b+U/\log T_0)}{U} \right)^U \), as \( \gamma' \geq T_0 \).

Combining (2.21) and (2.23), we get

\[
C A' W^{1-\beta'} \log^U \gamma' \geq (\gamma')^\tau,
\]
or equivalently, we would have

\[
-\hat{\tau} \log \gamma' + \left[ \frac{6\gamma' (\gamma'+4)}{2(\gamma')^{(j-3)w/2}} \right]
+ \log \left[ 10.081 + 2.081(1-\delta_2)\hat{y} \right]
+ 3(\log \log \gamma' + \log \left[ 42e^2(b/3 + 1/\log T_0) \right]) \geq 0,
\]
in which, we have used the assumption in (2.1) with the definition of \( h_j(t) \) in (1.11) so that \( 1 - \beta' < h_j(\gamma') = \frac{1}{2(\gamma')^{(j-3)w/2}} \) with \( w \geq (j-3)^2+4 \), which makes the second term on the left hand side of the last inequality sufficiently small and becomes a constant without involving \( \gamma' \). Hence, the above inequality is roughly equivalent to the expression

\[
-\hat{\tau} + 0.686 + \frac{\log \hat{y} + \log 2}{\log T_0} \geq 0,
\]
noting

\[
\log \gamma' \approx 0.686
\]
and \( 10.081 + 2.081(1-\delta_2)\hat{y} = 2.081(\hat{y} - 1)(1 - \delta_2 + \frac{10.081}{2.081(\hat{y} - 1)}) \approx 2\hat{y} \), if we let \( 1 - \delta_2 + \frac{10.081}{2.081(\hat{y} - 1)} \approx 1 \) by letting \( \delta_2 \approx \frac{5}{\hat{y}} \). To finish the proof of Theorem 1, we need to reach a contradiction by managing to have the inequality in (2.26) invalid, for which we need to acquire the largest possible value for \( \hat{\tau} \) with suitable choices of all constant “variables”.

It is not difficult to see that we would conveniently hope to get \( \hat{\tau} = \frac{999}{1000} \) is only possible. It follows that \( \sigma_0 = 1 + \frac{c_0}{1-\gamma_0} \) by the definition of \( \hat{\tau} \) with all other involved variables known. We may tentatively choose \( b = \frac{3}{40} \), which is the least value allowed for \( b \). Therefore, we let \( c = 0.231 \geq b + \frac{U}{\log T_0} \geq b + \frac{U}{\log \gamma'} \) as required by our remark after (2.12). After some experiment, we actually get \( \hat{\tau} = \frac{999}{1000} \) by setting \( \sigma_0 = 1 + \frac{3}{2.5 \times 10^{12}} \).
Since the choice of \( \hat{y} \) is critical for the value of \( \tau_1 \) and affects the value of \( \tau_2 \), we would like to choose \( \hat{y} \) as small as possible in order to keep the wanted value for \( \tilde{\tau} \). Thus, we choose \( \hat{y} = 2864.073 \) so that 
\[
\log \hat{y} = \frac{995}{1000} \geq \frac{199}{25} = \frac{2.5 \times 10^{12}}{3}
\]
and 
\[
\tau_1 > \frac{995}{1000}
\]
but very closely. We recall (2.6) to have \( \check{u} = 1 - \frac{1 - 2 \beta}{1 + \beta} \geq 1 - \frac{1}{13} \sigma_0 = 2.5 \times 10^{12} \).

The value of \( \delta_2 \) should be chosen as large as possible while we keep \( \tau_2 \geq \frac{995}{1000} \), for which, we set
\[
(2.27) \quad \delta_2 = 0.143 \leq \frac{e^{6 \log \hat{y} - 995/1000} \log T_0}{2.081(\hat{y} - 1)}.
\]

Using of \( \delta_2 \) does make the final result a little better. It is easy to see that we should take \( \hat{x} = 2864.073 \), as the same choice for \( \hat{y} \). This choice of \( \hat{x} \) also satisfies (2.25), as 
\[
1 + \frac{1 - 2 \beta}{1 + \beta} > 5.297 \times 10^{11}.
\]

What remains is to compute the real final result in (2.25) with this choice of all constants \( b, c, \sigma_0, \hat{x}, \hat{y}, \) and \( \check{u} \) to see that we really have reached a contradiction. In fact, we only need to check the value of the second term in (2.25) with
\[
\left[ \frac{6 \gamma' (\gamma' + 4) \gamma^6}{2 (\gamma')^{(7-j)w/12}} \right] \leq \frac{[(\gamma')^5]^{12/4.983}}{2(\gamma')^{13}} \leq \frac{1}{\gamma'} \leq \frac{1}{T_0} < \frac{1}{10^{13}},
\]
and
\[
- \frac{995}{1000} + 0.686 + 0.3 = -0.009 < 0,
\]
with
\[
\frac{10^{-13} + 10.081 + 2.081(1 - \delta_2)(\hat{y} - 1)}{\log T_0} < 0.3.
\]
This proves that the expression on the left hand side of (2.25) is actually negative, and this is a contradiction to (2.25) and finish the proof of Theorem 1.

3. A preparatory estimate by Turán

In this section, we prove an inequality shown in (3.12) below by following Turán’s argument in [29]. We first prove the inequality (3.1) involving \( W = W(k; t) \) below. For \( s \in \mathbb{C} \) with \( \sigma > 1 \) and \( t \geq 25 \), we have
\[
(3.1) \quad \left| \frac{W^{1-s}}{(s-1)^k} \sum_{\rho \in \mathbb{Z}} W^{\rho-s} \left( \frac{1}{s-\rho} \right)^k - \sum_{n=1}^{\infty} W^{-2n-s} (s+2n)^k \right| \leq \beta t \left( \frac{2^{\sigma-1}}{(2^{\sigma-1}-1)(\sigma-1)^{k-1}} \right),
\]
under the assumption (1.14) in Theorem 1 stated in Section 1.

We recall the remark before (2.18) that \( W \not\in \mathbb{Q} \). Here, the positive integer \( k \) is not less than 4, or \( k \in \mathbb{N} + 3 \), as stipulated in (2.14). A similar result to (3.1) without the explicit constant, under a similar
assumption to (1.14), is found in [29]. From the inequality in (3.1), one proves (3.12), which is used in Section 4 for estimating $S$ in (2.21).

In order to prove (3.1), we first cite the following lemma from [29]. The result in this lemma exhibits why we are interested in that kind of expression on the left hand side of (3.1).

**Lemma 3.** Let $W \in \mathbb{R}^+ \setminus \mathbb{Q}$ and $k \in \mathbb{N} + 3$. Then we have

\begin{equation}
\sum_{n \geq W} \frac{\Lambda(n)}{n^s} \log^{k-1} \frac{n}{W} = (k - 1)! \left[ \frac{W^{1-s}}{(s-1)^k} - \sum_{\rho \in \mathbb{Z}} \frac{W^{s-\rho}}{(s-\rho)^k} - \sum_{n=1}^{\infty} \frac{W^{-2n-s}}{(s+2n)^k} \right],
\end{equation}

for $\sigma > 1$. \hfill \Box

From Lemma 3, one sees that we need to study the sum on the left hand side expression of (3.2). To do so, we first consider the corresponding sum without “the logarithmic factors” on the left side of (3.2). Dividing the sum into infinitely many finite subsums, we have

\begin{equation}
F_W(s) = \sum_{n=N}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{\tau=1}^{N_{\tau+1}} G_\tau(s),
\end{equation}

with

\begin{equation}
G_\tau(s) = \sum_{n=N_{\tau}}^{N_{\tau+1}+1} \frac{\Lambda(n)}{n^s},
\end{equation}

where $N = \lceil W \rceil$ and $N_{\tau} = 2^{\tau-1}N$ for all $\tau \in \mathbb{N}$. For each $\tau$, one has

\begin{align*}
G_\tau(it) &= \sum_{n=N_{\tau}}^{N_{\tau+1}+1} \frac{\psi(n) - \psi(n-1)}{n^it} \\
&= \sum_{n=N_{\tau}}^{N_{\tau+1}+1} \psi(n) \left( \frac{1}{n^it} - \frac{1}{(n+1)^it} \right) - \left( \frac{\psi(N_{\tau} - 1)}{N_{\tau}^it} - \frac{\psi(N_{\tau+1} - 1)}{N_{\tau+1}^it} \right),
\end{align*}

by the partial summation method. Here, we use the inequalities in (1.5) with the assumption in (1.14). It results

\begin{align*}
G_\tau(it) &\leq \left| -\frac{N_{\tau} - 1}{N_{\tau}^it} + \frac{N_{\tau+1} - 1}{N_{\tau+1}^it} + \sum_{n=N_{\tau}}^{N_{\tau+1}-1} n \left( \frac{1}{n^it} - \frac{1}{(n+1)^it} \right) \right| \\
&\quad + \log^2 \frac{N_{\tau+1}}{N_{\tau+1}^{\gamma_{\tau+1}/(\gamma_{\tau+1})}} \left( 2N_{\tau+1} + \sum_{n=N_{\tau}}^{N_{\tau+1}-1} n \left| \frac{1}{n^it} - \frac{1}{(n+1)^it} \right| \right),
\end{align*}
recalling the definition of $H_j(x)$ from (1.10) and noting that it is a decreasing function of $x$ for every $j \in \mathbb{N} + 2$. The sum of the first three terms in the last expression is equal to $\sum_{n=Nr}^{N_{r+1}-1} \frac{1}{m^n}$ by the partial summation method. Therefore, recalling the definition of $G_r(s)$ in (3.4), we have

$$G_r(it) \leq \sum_{n=N_r}^{N_{r+1}-1} \frac{1}{n^it} + \frac{\log^2 N_{r+1}}{N_{r+1}}$$

(3.5)

$$\times \left( 2N_{r+1} + \sum_{n=N_r}^{N_{r+1}-1} n \left| \frac{1}{n^it} - \frac{1}{(n+1)^it} \right| \right).$$

We need to estimate the two sums as above.

We estimate the first sum in (3.5) by tricky use of the inequality $|e^z - 1 - z| \leq |z|^2$ when $\Re(z) \leq 1$ with $z = (1 - it) \log\left(1 + \frac{1}{n}\right)$. We also use $|\log(1 + u) - u| \leq \frac{3}{2} u^2$ and $\log(1 + u) \leq u$ for $0 < u < 1$. We have

$$\left| (n + 1)^{1-it} - n^{1-it} - (1-it) n^{-it} \right| = n \left| (1 + \frac{1}{n})^{1-it} - 1 - \frac{1-it}{n} \right|$$

$$\leq n \left| (1 + \frac{1}{n})^{1-it} - 1 - (1-it) \log\left(1 + \frac{1}{n}\right) \right| + n|1-it| \left| \log\left(1 + \frac{1}{n}\right) - \frac{1}{n} \right|$$

$$\leq n (t^2 + 1) \log^2 \left(1 + \frac{1}{n}\right) + \frac{\sqrt{t^2+1}}{2n} \leq \frac{t^2+1}{n} (1 + \frac{1}{2\sqrt{t^2+1}}) < \frac{11(t^2+1)}{10n},$$

with $z = (1 - it) \log(1 + \frac{1}{n})$ and $u = \frac{1}{n}$, noting that $W > \frac{11(t^2+1)}{10}$ from (2.18), if only $t \geq 25$.

Summarizing the last inequality with its end terms from $n = N_r$ to $N_{r+1} - 1$, we acquire an inequality involving the first sum in (3.5) with all other expressions estimable. That is,

$$\left| N_{r+1}^{1-it} - N_r^{1-it} - (1-it) \sum_{n=N_r}^{N_{r+1}-1} \frac{1}{n^it} \right| < \frac{11(t^2+1)}{10} \sum_{n=N_r}^{N_{r+1}-1} \frac{1}{n}.$$
using $|a| - |b| \leq |a - b|$, noting $N_{r+1} = 2N_r$ and $|N_{r+1}^{1-it} - N_{r}^{1-it}| \leq N_{r+1} + N_r$, and recalling $N_r \geq N > W(k;t) > \frac{11t^2+1}{10}$ with the designation of $N$ in (3.4).

As for the second sum in (3.5), we note

$$\frac{1}{n^{it}} - \frac{1}{(n+1)^{it}} = \frac{1}{(n+1)^{it}} \left((1 + \frac{1}{n})^{it} - 1\right) = \frac{1}{(n+1)^{it}} \left(e^{it\log(1+1/n)} - 1\right).$$

We apply the mean value theorem to the difference in the last expression, getting $e^{it\log(1+\frac{1}{n})} - 1 = it \log(1+\frac{1}{n})e^{im}$ with $0 < m \leq t \log(1+\frac{1}{n})$. Note again that $\log(1+u) \leq u$ for $0 < u < 1$. Hence,

$$\left|\frac{1}{n^{it}} - \frac{1}{(n+1)^{it}}\right| \leq \frac{t}{n}.$$ 

Therefore,

$$\sum_{n=N_r}^{N_{r+1}-1} n \left|\frac{1}{n^{it}} - \frac{1}{(n+1)^{it}}\right| \leq t \sum_{n=N_r}^{N_{r+1}-1} 1 \leq N_r t,$$

as we have $N_r$ terms in the sum.

Putting (3.6) and (3.7) in (3.5), we obtain

$$|G_r(it)| \leq \frac{5.1N_r}{t} + \frac{\log^2 N_{r+1}}{N_r^{it}N_{r+1}^{it}} \left(2N_{r+1} + N_r t\right)$$

$$= \frac{N_{r+1}}{2t} \left(5.1 + \frac{t(t+4) \log N_{r+1}}{N_r^{it}N_{r+1}^{it}}\right) < \frac{2^{r+1}N}{t},$$

by

$$\frac{t(t+4) \log^2 N_{r+1}}{N_r^{it}N_{r+1}^{it}} \leq \frac{1}{6}, \text{ or } e^{\frac{2(t+4)-12}{6} \log \log N_{r+1} \geq 6 t (t+4)},$$

and $\frac{5.1+1/6}{2} < 4$?; recalling $N_{r+1} = 2^\tau N$ for $\tau \in \mathbb{N}$ and the first inequality in (1.11) and $N_\tau > N$ for all $\tau \in \mathbb{N}$ after (3.3) with $N > W$ and

$W = W(k;t) \geq k e^{6(t+2) \frac{\tau(t+4)-12}{6}}$ from (2.18).

With this estimate in (3.8), it then follows from $N_r = 2^{r-1} N$ that

$$G_r(s) = \sum_{n=2^{r-1}N}^{2^rN-1} \Lambda(n) \frac{n^{\sigma+it}}{n^{\sigma+it}} \leq \frac{G_r(it)}{(2^{r-1}N)^\sigma} \leq \frac{4}{2^{(\sigma-1)\tau-1}N^{\sigma-1}t}.$$

Recalling the definition of $F_W(s)$ in (3.3) with the definition of $G_W(s)$ in (3.4) with (3.9), we acquire

$$|F_W(s)| \leq \frac{4}{t N^{\sigma-1}} \sum_{j=1}^{\infty} (2^{\sigma-1})^{j-1} = \frac{2^{\sigma+1}}{t (2^{\sigma-1} - 1) N^{\sigma-1}}.$$

Next, we use the following lemma, which is from the context on page 161 in [30], in estimating the expression on the left side of (3.2).
Lemma 4. Let $W \in \mathbb{R}^+ \setminus \mathbb{Q}$ and $k \in \mathbb{N} + 3$. Then,

\begin{equation}
\sum_{n \geq W} \frac{\Lambda(n)}{n^s} \log^{k-1} \frac{n}{W} = (k-1) \int_W^\infty F_u(s) \frac{\log^{k-2} u}{u} \, du,
\end{equation}

for $\sigma > 1$.

From (3.10) and (3.11), one has

\begin{equation}
\left| \sum_{n \geq W} \frac{\Lambda(n)}{n^s} \log^{k-1} \frac{n}{W} \right| \leq \frac{4(k-1)2^{\sigma-1}}{t(2^{\sigma-1} - 1)} \int_W^\infty \log^{k-2} \frac{u}{W} \, du
= \frac{4(k-1)!2^{\sigma-1}W^{1-\sigma}}{t(2^{\sigma-1} - 1)(\sigma-1)^{k-1}}.
\end{equation}

By Lemma 3 with $(k-1)!$ in both the last expression and (3.2) canceled, one obtains (3.1).

To finish the proof of (3.12), we use (3.1) to get the estimate as shown in (3.12) as below. Multiplying (3.1) with $s = s_0 = \sigma_0 + i\gamma'$, or, $\sigma = \sigma_0$, and $t = \gamma'$ and $W = W(k; \gamma')$ defined in (2.4), by a factor $W^{\sigma_0-\beta'}(s_0 - \rho')^k$ whose absolute value being $W^{-\sigma_0-\beta'}(s_0 - \beta')^k$, recalling that $\rho' = \beta' + i\gamma'$, we acquire

\begin{equation}
W^{1-\rho'} \left( \frac{s_0 - \rho'}{s_0 - 1} \right)^k - \sum_{\rho \in \mathbb{Z}} W^{\sigma_0-\rho'} \left( \frac{s_0 - \rho'}{s_0 - \rho} \right)^k - \sum_{n=1}^\infty W^{-2\rho-\beta'} \left( \frac{s_0 - \rho'}{s_0 + 2n} \right)^k
\end{equation}

with $\tau_0 = 1 - \frac{\beta_0}{\sigma_0 - 1}$ as in (2.22), since $\frac{\sigma_0 - \beta'}{\sigma_0 - 1} \leq \log \left( 1 + \frac{\beta_0}{\sigma_0 - 1} \right) \leq \frac{\beta_0}{\sigma_0 - 1}$, recalling (2.3).

One may notice that the first sum on the left side of the last inequality runs over the set $\mathbb{Z}$ of all zeros while that sum in (2.21) involves only a subset $\mathbb{H}$ defined in (2.9) of zeros for the Riemann zeta function. That is the reason we have to deal with the set outside of $\mathbb{H}$ in the next section.

4. Estimates on Sums

In this section, we estimate $S_j$’s defined in (2.19) for $j = 0, 1, 2, 3$.

To estimate $S_0$, we use (3.12), in which $S_0$ is the second sum on its left hand side. Recall the designations of $\sigma_0$ and $s_0$ in (2.4), and, the restriction on $k$ in (2.14). For the first sum on the left hand side of (3.12), which is the major term in this regard, we get

\begin{equation}
\left| W^{1-\rho'} \left( \frac{s_0 - \rho'}{s_0 - 1} \right)^k \right| \leq W^{1-\beta'} \left( \frac{s_0 - 1/2}{\gamma'} \right)^k \leq \frac{W^{1-\beta'}}{(\gamma')^b \log \frac{\sigma_0 - 1/2}{\gamma'}}.
\end{equation}
as $s_0 - \rho' = \sigma_0 - \beta' \leq \sigma_0 - \frac{1}{2}$ by $\beta' > \frac{1}{2}$ in (2.1), $|s_0 - 1| > \gamma'$, and, $k \geq b \log \gamma'$. For the third sum on the left side of (3.12), we have

$$
(4.2) \quad \left| \sum_{n=1}^{\infty} W^{-2n-\rho'} \left( \frac{s_0 - \rho'}{s_0 + 2n} \right)^k \right| \leq \frac{1}{W^2} \left( \frac{\sigma_0 - 1/2}{\gamma'} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^{k-2}} \leq \frac{5}{4W^2} \leq \frac{5}{4e^{(\gamma')^{2/3}}},
$$

as $W^{-2n-\rho'} \leq \frac{1}{W^2}$, $s_0 + 2n \geq \gamma'$, $s_0 + 2n \geq n$, noting that $\sigma_0 - \frac{1}{2} < \gamma'$ from (2.4), $k \geq 7$ from the stipulation after (2.14), and $\sum_{n=1}^{\infty} \frac{1}{n^{k-2}} \leq 1 + \int_{1}^{\infty} \frac{dx}{x^3} = 1 + \frac{1}{e-3} \leq \frac{5}{4}$, and recalling (2.18) for the larger lower bound of $W$.

With (3.12), (4.1), and (4.2), one sees that

$$
(4.3) \quad |S_0| \leq \frac{4\sqrt{2} W^{1-\beta'}}{(\gamma')^2 \sigma_0^{-1/2}} + \frac{W^{1-\beta'}(\gamma')^{2/3} \log \frac{1}{\sigma_0-1/2}}{4e^{(\gamma')^{2/3}}} + \frac{5W^{1-\beta'}}{(\gamma')^6} \leq \frac{C_0 W^{1-\beta'}}{(\gamma')^6},
$$

where $C_0 = 6.7$ and $\tau_0$ is defined in (2.22), noting that $4\sqrt{2} + 1 + \frac{5T_0}{4e^{\gamma_0}} \leq 6.7$, $b \log \frac{T_0}{\sigma-1/2} > 1$, and $\tau_0 < 1$ from its definition, as $\gamma \geq T_0$.

For the estimate of $S_1$, we consider it by estimating its two sub-sums separately. One is over the set $\{ \rho \in \mathbb{Z} : \gamma > \gamma' + \hat{y} \}$; another over the set $\{ \rho \in \mathbb{Z} : \gamma < \gamma' - \hat{y} \}$. One may estimate both sub-sums similarly with the same upper bound, therefore, we give the details only for the first one.

We need an estimate on the number of zeros in certain regions, for which we recall from [27] that for $T \geq 2$

$$
(4.4) \quad \left| N(T) - M(T) - \frac{T}{8} \right| \leq Q(T),
$$

where $M(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ and $Q(T) = 0.137 \log T + 0.443 \log \log T + 1.588$. From this formula, for $T \geq 8 > \frac{T}{2\pi} + 1$ we derive for $0 < t_1 < t_2$

$$
(4.5) \quad N(t_2) - N(t_1) \leq M(t_2) - M(t_1) + Q(t_2) + Q(t_1).
$$

We now only consider the special case, in which we use $t_2 = t$ and $t_1 = t - 1$. Then, we apply the mean-value theorem for the function $f(x) = M(x)$ with the above function $M$ in the interval $[t-1, t]$, getting that the derivative $f'(x) = \frac{1}{2\pi} \log \frac{x}{2\pi}$ and

$$
(4.6) \quad M(t_2) - M(t_1) \leq \frac{1}{2\pi} \log \frac{t}{2\pi}.
$$

From (4.5) and (4.6), we see that $N(t) - N(t - 1) \leq \frac{1}{2\pi} \log \frac{t}{2\pi} + 0.137 \log t + 0.443 \log \log t + 1.588 < 0.16 \log t + 0.137 \log t + 0.443 \log \log t + 1.296$ as $1.588 - \frac{1}{2\pi} \log(2\pi) < 1.296$. We have the following lemma.
Lemma 5. Let $t \geq 25$. Then,

$$\sum_{\rho \in \mathbb{Z}: t-1 < \gamma - \rho \leq t} 1 \leq 1.04 \log t. \quad (4.7)$$

We now estimate $S_j$ defined in (2.19) with $H_j$ defined in (2.10) for $j = 1, 2, 3.$

For $S_1$, we notice that the number of zeros $\zeta(s)$ for $\gamma' + n < \gamma \leq \gamma' + n + 1$ is not greater than $1.04 \log (\gamma' + n + 1)$, by the last lemma. For each such an $n \geq \hat{y}$, we note $\sigma_0 - \beta' \leq 1$. Also, we have $s_0 - \rho = \sigma_0 + i \gamma' - \beta - i \gamma$ so that $|s_0 - \rho| \geq |\gamma' - \gamma| \geq n$, recalling the definition of $s_0$ in (2.4). From the definition of $S_1$ in (2.19) with the definition of $H_1$ in (2.10), we get

$$|S_1| \leq 2W^{1-\beta'} \sum_{\rho \in \mathbb{Z}: \gamma' + n < \gamma \leq \gamma' + n + 1} \left( \frac{\sigma_0 - \beta'}{|s_0 - \rho|} \right)^k \leq 1.04W^{1-\beta'} \sum_{n=\hat{y}+1}^{\infty} \frac{1}{n^k} \log (\gamma' + n + 1), \quad (4.8)$$

recalling that $\hat{y} \in \mathbb{N}$ from (2.7).

We note here that $\gamma' + n + 1 = \gamma'(1 + \frac{n+1}{\gamma'}) \leq 2\gamma'$ if $n \leq \gamma' - 1$ and $\log \gamma' + \log 2 = \log \gamma'(1 + \frac{\log 2}{\log \gamma'}) \leq 1.025 \log \gamma'$ as $\gamma' \geq T_0$, recalling the value of $T_0$ from the end of the paragraph before (1.9). This is used for the first inequality in (4.9). For the sake of the second inequality in (4.9), we have $\log (\gamma' + n + 1) \leq n \log \gamma'$ if $(\gamma')^{-1} + \frac{n+1}{\gamma'} + 1$ as $\gamma' \geq T_0$; and if only $n \geq 2$. Also, by $u^n - u - n - 1 > 0$ for any $u > 3$ with $u = \gamma$, we see that $\gamma' + n + 1 \leq (\gamma')^n$. We get

$$\log (\gamma' + n + 1) \leq \begin{cases} 1.025 \log \gamma', & \text{if } \hat{y} + 1 \leq n < \gamma' - 1, \\ n \log \gamma', & \text{if } \hat{y} + 1 \leq n < \infty, \end{cases} \quad (4.9)$$

the latter in which only used in the case when $n \geq |\gamma'|$ in the following paragraph.

Recall the designation that $\hat{y} \geq 2$ from (2.7). We let $\gamma' - 1 < g := \lfloor \gamma' \rfloor \leq \gamma'$.

$$\sum_{n=\hat{y}+1}^{\infty} = \sum_{n=\hat{y}+1}^{g-1} + \sum_{n=g}^{\infty} = ???. \quad (4.10)$$
\[\sum_{n=\hat{y}+1}^{\gamma-1} \frac{1}{n^k} \leq \frac{1}{\hat{y}^k} + \int_{\hat{y}}^{\gamma-1} \frac{dv}{v^k} \leq \frac{1}{\hat{y}^k} \left( 1 + \frac{\hat{y}}{k-1} \right) \text{ and } \sum_{n=\gamma'}^{\infty} \frac{1}{n^k} \leq \left( \frac{\gamma'}{\gamma' - 1} \right)^{k-2} \left( \frac{1}{\gamma' - 1} \right)^{k-1}, \]
as 
\frac{\gamma'}{\gamma' - 1} = 1 + \frac{1}{\gamma'} \text{ and } \gamma' - 1 + \frac{1}{k-2} < \gamma'.

Recalling (4.8) with (4.9) and the above remarks, we acquire
\[
|S_1| \leq 1.066 \left( 1 + \frac{\hat{y}}{k-1} \right) W^{1-\beta'} \frac{\log \gamma'}{\hat{y}^k} + 1.04 W^{1-\beta'} \frac{\log \gamma'}{(\gamma')^{k-1}} \leq C_1 W^{1-\beta'} \frac{\log \gamma'}{(\gamma')^{k-1}},
\]
noting that
\[
1.066 \left( 1 + \frac{\hat{y}}{k-1} \right) + (1 + \frac{1}{\gamma' - 1}) \left( \frac{1}{\gamma'} \right)^{k-2} \frac{1}{(\gamma')^{k-1}} \text{ is less than}
\]
\[
1.066 \left( 1 + \frac{\hat{y}}{b \log \gamma' - 1} \right) + (1 + \frac{1}{\gamma' - 1}) b \log \gamma' - 2 \frac{1}{(\gamma')^{b \log \gamma' - 1}} = C_3 = 1.3,
\]
as in (2.14), and \(\tau_1\) is defined after (2.22).

As for the estimate of \(S_2\), we recall the definition of \(S_2\) from (2.19) with the definition of \(\hat{y}\) and the restriction of \(\hat{y}\) in (2.7). We use (4.7) from Lemma 5 to each of the 2 subsets subject to \(t - 1 < \gamma' \leq t\) with \(t = \gamma' - \hat{y} + j\) for \(j = 1, 2, \ldots, \hat{y} - \hat{\mu} - 1, \hat{y} + \hat{\mu}(\sigma_0 - \beta'), \hat{y} + \hat{\mu}(\sigma_0 - \beta'), \ldots, 2\hat{y} - 1\), we see that
\[
|H_2| \leq 1.04 \times 1.000001 \times 2 \left( \hat{y} - \hat{\mu}(\sigma_0 - \beta') \right) \log \gamma' \leq 2.081 \left( \hat{y} - \hat{\mu}(\sigma_0 - \beta') \right) \log \gamma',
\]
recalling the set up and restrictions on \(\hat{\mu}\) and \(\hat{\mu}\) in (2.6) and (2.7), respectively.

Noting \(\sigma_0 - \beta' = \sigma_0 - \beta'\) again, recalling the definition of \(\hat{\mu}\) in (2.6)
and \(\sigma_0 - \rho = \sigma_0 + i \gamma' - \beta - i \gamma \geq |\gamma' - \gamma| \geq \hat{\mu}(\sigma_0 - \beta')\), we acquire
\[
|S_2| \leq 2.081 \left( y - \hat{\mu}(\sigma_0 - \beta') \right) W^{1-\beta'} \frac{\log \gamma'}{\hat{\mu}^k} \leq C_2' W^{1-\beta'} \frac{\log \gamma'}{(\gamma')^{k-2}},
\]
where \(C_2' = 2.081 \left( \hat{y} - \hat{\mu}(\sigma_0 - \beta') \right)\) and \(\tau_2' = \frac{b}{2} \log \hat{\mu}\) as in (2.22).

For the sake of sub-optimization, we let \(0 < \delta_2 < 1\) and rewrite the last estimate in the form of
\[
|S_2| \leq \frac{C_2 W^{1-\beta'}}{(\gamma')^{\tau_2'}},
\]
where \(C_2 = 2.081(1 - \delta_2)(\hat{y} - 1)\) and \(\tau_2 = \tau_2' - \frac{\log[2.081 \delta_2(\hat{y} - 1)]}{\log \hat{y}}\) for \(\gamma' \geq T_0\).

This is the result in (2.22).

We estimate \(S_3\) similarly. Recall the definition of \(S_3\) in (2.19) and noting that
\[
\frac{s_0 - \beta'}{s_0 - \rho} = \frac{\sigma_0 - \beta'}{\sigma_0 - \beta' + i (\gamma' - \gamma)} \leq \frac{1}{\tau_3},
\]
from (2.10) with \(\sigma_0 - \beta \geq \hat{x}(\sigma_0 - \beta')\) by the definition of \(H_3\), and Lemma 5 with \(t = \gamma'\) and \(t = \gamma' + \hat{\mu}(\sigma_0 - \beta')\) with the design of \(\hat{\mu}\) in (2.6), so
that the number of zeros in $H_3$ is not greater than $2.08 \log(\gamma' + 1) \leq 2.081 \log \gamma'$, as $\gamma' \geq T_0$. We have

$$|S_3| \leq \frac{C_3 W^{1 - \beta' \log \gamma'}}{x \log \gamma'},$$

where $C_3 = 2.081$ and $\tau_3 = b \log \hat{x}$, as in (2.22).

We finish the proof of (2.21) by collecting (4.3), (4.11), (4.12), and (4.14), recalling the definition of $\hat{\tau}$ in (2.22).

5. Applying the power sum method lemmas

Turán created the power sum method while investigating the Riemann zeta function and used this method to prove results about its zeros. Using his power sum method, Turán proved the following lemma 6 in [30]. A similar result was used in [30] by Turán in proving similar results to Theorem 1 with respect to a different designation for the functions $H_j(x)$ and $h_j(t)$, but only in the sub-interval close to 1.

**Lemma 6.** Let $L \in \mathbb{N} + 1$ and $z_1, z_2, \ldots, z_L$ be complex numbers with

$$\min_{1 \leq l \leq L} |z_l| \geq M.$$  

Then for all $D \in \mathbb{R}^+$ such that $D \geq 1$,

$$\max_{D \leq \nu \leq D + L} \left| z_1^{\nu} + z_2^{\nu} + \ldots + z_L^{\nu} \right| > M^D \left( \frac{M L}{e(M + 1)(D + L)} \right)^L.$$ 

□

In Section 2, we followed Turán’s proof by using Lemma 6 in case (i). However, we need a slightly different version of Lemma 6 in case (ii). This slightly different version is stated as Lemma 7.

The next lemma is a modified version of Lemma T, from Turan’s paper [31], with minor improvements and explicit constants. Using Lemma 7 instead of Lemma 6 in our application in this section below, we have a lesser restriction on $D$ and an improved constant in the lower bound.

**Lemma 7.** Let $L \in \mathbb{N} + 1$, $l = 1, 2, \ldots, L$, and $z_l \in \mathbb{C}$ satisfy the condition

$$\max_{1 \leq l \leq L} |z_l| \geq 1.$$  

Then for every $D \in \mathbb{R}^+$ such that $D \geq \frac{L}{40}$,

$$\max_{\nu: D \leq \nu \leq D + L} \left| z_1^{\nu} + z_2^{\nu} + \ldots + z_L^{\nu} \right| \geq \left( \frac{L}{42e^2(D + L)} \right)^L.$$
We shall give the proof of Lemma 7 in Section 6. Here, we apply Lemma 7 to validate the lower bound of $|S|$ in (2.23) for at least one $k$ subject to (2.14).

We let $l$ be a one-to-one map from $H$ to $\{1, 2, \ldots, L\}$ such that $l = l(\rho)$ and denote

$$z_l = z_l(\rho) = e^{\omega(\rho - \rho') \frac{s_0 - \rho'}{s_0 - \rho}}.$$  

(5.5)

Recalling the definition of $H$ in (2.9) with (2.8), we know that $H$ is completely located on the right half plane $\sigma > \frac{1}{2}$. We apply Lemma 2 and have

$$L = |H| \leq U,$$

(5.6)

where $U$ is defined in Lemma 2. We recall that $L > 0$ from the definition of $H$ with the assumption on contrary in (2.1). We notice that $z_l(\rho') = 1$ so that the condition in (5.3) is satisfied.

We remark here that the estimate in (5.4) is better if $D$ is as small as possible, because $(\frac{L}{42e^2(D+U)})^L$ is a decreasing function of $D$ when $L$ is fixed. We would take $b$ to be the least possible, which also depends heavily on the choice of $b$, with the value required to be not less than $\frac{L}{40}$, to determine the value of $\tilde{\tau}$ in (2.22). For this reason, we will not decide on the value of $b$ until we make a conclusion by putting everything together as in (2.25).

We then apply Lemma 7. We choose the quantities $D = b \log \gamma'$, which meets the requirement in Lemma 7, and $D + U \leq c \log \gamma'$, with $b$ and $c$ subject to the conditions in (2.12). We notice that the expression on the right hand side of (5.4) is a decreasing function of $L$ when $D$ is fixed. Therefore, we have $(\frac{L}{42e^2(D+U)})^L \geq \left(\frac{U}{42e^2(b \log \gamma' + U)}\right)^U$. By Lemma 7, we see that there exist a $\nu = k$ satisfying $b \log \gamma' \leq \nu \leq b \log \gamma' + U = c \log \gamma'$ such that

$$(5.7) \quad \sum_{l=1}^{L} z_l^\nu = \sum_{l=1}^{L_a} e^{\omega(\rho - \rho') \frac{s_0 - \rho'}{s_0 - \rho}} \nu \geq \left(\frac{U}{42e^2(b \log \gamma' + U)}\right)^U,$$

by (5.6) and (2.14). However, the sum on the left hand side of (5.7) is the same as $S$ defined in (2.19) with our choice of $z_l$, where $\nu = k$, recalling the definition of $H$ in (2.9). Therefore, we have proved (2.23).

This ends this section.

6. The power sum method lemmas

In this section, we prove Lemma 7.
Note that
\[ \max_{\nu \geq 0} \left| z_1^\nu + z_2^\nu + \ldots + z_L^\nu \right| \geq \max_{\nu \in \mathbb{N} : D \leq \nu \leq D + L} \left| z_1^\nu + z_2^\nu + \ldots + z_L^\nu \right|, \]
and \( \left( \frac{N}{D+N} \right)^N \) is a decreasing function with respect to \( N \) for any fixed value of \( D \). We see that the result in Lemma 7 with \( N \) being replaced by \( L \) is actually stronger in the case that \( N > L \). In the remain of this section, we use Lemma 6 to prove Lemma 7 with \( N \) being replaced by \( L \). For convenience, we use the notations
\[
M_0 = \max_{1 \leq j \leq L} |z_j|,
\]
\[
M_1 = \max_{D \leq \nu \leq D+L} \left| z_1^\nu + z_2^\nu + \ldots + z_L^\nu \right|,
\]
\[
M_2 = \max_{D+1 \leq \nu \leq D+L} \left| z_1^\nu + z_2^\nu + \ldots + z_L^\nu \right|
\]
from now on.

First of all, note that we may assume that \( M_0 = 1 \) without loss of generality. To justify this claim, we only need to apply Lemma 7 with respect to the assumption that \( M_0 = 1 \) to the case in which \( M_0 > 1 \) and using \( z_j/M_0 \) in place of \( z_j \) for \( j = 1, 2, \ldots, L \).

Secondly, we may assume that \( D \) is an integer in Lemma 7 with \( M_1 \) defined in (6.1) being replaced by \( M_2 \) defined in (6.1). One may justify that the lemma is valid for any \( D \in \mathbb{R}^+ \) by using the integer part \( \lfloor D \rfloor \) in place of \( D \) and noting that \( M_1 \geq M_2 \).

We also may assume that \( z_j \)'s for \( 1 \leq j \leq L \) are all distinct in proving Lemma 7. Otherwise, we justify the lemma by constructing an infinite sequence of the list \( [z_{1k}, z_{2k}, \ldots, z_{Lk}] \) with respect to all \( k \in \mathbb{N} \) such that the sequence converges to the list \( [z_1, z_2, \ldots, z_L] \) and all \( z_{jk} \)'s are distinct for any fixed \( k \) and use the limit \( \lim_{k \to \infty} \max_{D+1 \leq \nu \leq D+L} |z_1^\nu + z_2^\nu + \ldots + z_L^\nu| \), as in [31].

Therefore, we only need to prove Lemma 7 under the assumption that \( z_j \)'s are all distinct for \( j = 1, 2, \ldots, L \), \( M_0 = 1 \), and \( D \) is an integer with \( M_1 \) being replaced by \( M_2 \).

We choose
\[
\hat{U} = \frac{1}{4e} \left( \frac{1}{1 + \frac{D}{L}} \right) = \frac{L}{4e(D+L)},
\]
and we have
\[
1 = 4e \hat{U} = 1 - \frac{L}{D+L} = \frac{L}{D+L} > 0.
\]

We need a lemma in [25] from the analytic theory of polynomials.
Lemma 8. Let \( w \in \mathbb{C} \) and for any prescribed \( U \in \mathbb{R}^+ \) the inequality \( |f(w)| \geq U^L \) holds outside at most \( L \) discs \( |w - z_j| \leq r_j \) such that \( r_1 + r_2 + \ldots + r_L \leq 2eU \).

By Lemma 8, we have \( |f(w)| \geq U^L \) on the circle \( |w| = r \) for some \( r \) satisfying the above condition. From \( |w - z_j| \leq 2 \) for every \( j = 1, 2, \ldots, L \), we see that

\[
|w - z_{i_1}| |w - z_{i_2}| \cdots |w - z_{i_\lambda}| \geq \left( \frac{U}{L} \right)^L,
\]

on \( |w| = r \) for every choice of \( \{i_1, i_2, \ldots, i_\lambda\} \) from \( \{1, 2, \ldots, L\} \). We rearrange the set \( \{1, 2, \ldots, L\} \) so that we have two cases.

Case (i). \( 1 = |z_1| \geq |z_2| \geq \ldots \geq |z_L| > r \).

Here, we use Lemma 6 but with \( M_1 \) being replaced by \( M_2 \), which is valid when \( D \in \mathbb{N} \) from [31]. With \( M = 1 - 4e\hat{U} \) in Lemma 6, one gets

\[
M_2 \geq (1 - 4e\hat{U})^D \left( \frac{(1 - 4e\hat{U})L}{2e(1 - 2e\hat{U})(D + L)} \right)^L \geq (1 - 4e\hat{U})^D \left( \frac{L}{42e(D + L)} \right)^L.
\]

We recall (6.3) and note that

\[
(1 - 4e\hat{u})^D = \left( \frac{D}{D + L} \right)^D = \frac{1}{\left( 1 + \frac{L}{D} \right)^{D/L}} \geq \frac{1}{e^L},
\]

with \( z = \frac{D}{L} \) by \( (1 + z)^{1/z} \leq e \) from the fact that the function \( (1 + z)^{1/z} \) is monotonically decreasing for all \( z \in (0, \infty) \) with \( \lim_{z \to 0} (1 + z)^{1/z} = e \). Also we recall \( \frac{D}{L} \geq \frac{1}{40} \) from the statement of Lemma 7 have

\[
\frac{1 - 4e\hat{U}}{1 - 2e\hat{u}} = \frac{1 - \frac{1}{1 + D/L}}{1 - \frac{1}{2(1 + D/L)}} = \frac{1}{1 + \frac{1}{2(1 + D/L)}} \geq \frac{1}{21}.
\]

Using these two inequalities (6.6) and (6.7) in (6.5), we deduce the estimate stated in the lemma.

Case (ii). \( 1 = |z_1| \geq |z_2| \geq \ldots \geq |z_l| > r > |z_{l+1}| \geq \ldots \geq |z_L| \), where \( l \in \{1, 2, \ldots, L - 1\} \). Let

\[
P(w) = \prod_{j=l+1}^{L} (w - z_j) = \sum_{j=0}^{L-l} a_j w^{L-l-j}.
\]

For the coefficients of the polynomial \( P(w) \), we have

\[
a_j = \sum_{l+1 \leq k_1 < k_2 < \ldots < k_j \leq L} z_{k_1} z_{k_2} \cdots z_{k_j} \leq \binom{L - l}{j}.
\]
Now, we need the following lemma, which is a classical result from the theory of Newton-interpolation, see page 48 in [31].

**Lemma 9.** Let \( w \in \mathbb{C} \) and \( \mathcal{C} \) be a simple closed curve consisting of analytic arcs on the \( w \)-plane and \( G(w) \) a regular function outside and on \( \mathcal{C} \) so that \( G(w) \to 0 \) uniformly if \( |w| \to \infty \). Let \( l \in \mathbb{N} \), \( w_1, w_2, \ldots, w_l \) be different points outside \( \mathcal{C} \), and \( g(w) \) be a polynomial of degree \( l-1 \). If \( g(w) = G(w) \) when \( w = w_j \) for all \( j = 1, 2, \ldots, l \), then

\[
g(w) = \sum_{j=0}^{l-1} b_j \prod_{k=1}^{j} (w - w_k),
\]

with the coefficients

\[
b_j = \frac{1}{2\pi i} \int_{|z|=r} \frac{G(z)}{\prod_{k=1}^{l}(z - w_k)} \, dz,
\]

where the product \( \prod_{k=1}^{l}(z - w_k) \) is regarded to be 1. \( \square \)

Let \( Q(w) \) be the polynomial of degree \( l-1 \) such that \( Q(z_j) = \frac{1}{z_j^{p+1} P(z_j)} \) for every \( j = 1, 2, \ldots, l \). Then, by Lemma 9, we have

\[
(6.9) \quad Q(w) = \sum_{j=0}^{l-1} b_j \prod_{k=1}^{j} (w - z_k) = \sum_{j=0}^{l-1} c_j w^j,
\]

with

\[
b_j = \frac{1}{2\pi i} \int_{|z|=r} \frac{d z}{z^{D+1} P(z) \prod_{k=1}^{l-1}(z - z_k)},
\]

for \( j = 0, 1, \ldots, l-1 \). From this, one gets

\[
(6.10) \quad |b_j| \leq \frac{1}{r^D} \left( \frac{2}{U} \right)^L \leq \frac{1}{(1 - 4eD)^D} \left( \frac{2}{U} \right)^L,
\]

recalling (6.4). Expressing \( c_j \) in terms of \( b_j \) in (6.9) by the above lemma, we see

\[
c_j = b_j - b_{j+1} \sum_{1 \leq i_1 \leq j+1} z_{i_1} + b_{j+2} \sum_{1 \leq i_1, i_2 \leq j+1} z_{i_1} z_{i_2} - \ldots
\]

\[
+ (-1)^{l-j-1} b_{l-1} \sum_{1 \leq i_1, i_2, \ldots, i_{l-j-1} \leq j+1} z_{i_1} z_{i_2} \cdots z_{i_{l-j-1}},
\]

for \( j = 0, 1, \ldots, l-2 \) and \( c_{l-1} = b_{l-1} \). By this inequality and (6.10), we acquire

\[
(6.11) \quad |c_j| \leq \left( \frac{l}{j+1} \right) \frac{1}{(1 - 4eD)^D} \left( \frac{2}{U} \right)^L
\]
recalling $|z_j| \leq 1$ for $j = 0, 1, \ldots, l - 2$ and noting that $1 + (\frac{j+1}{2}) + \ldots + (\frac{l-1}{l-j-1}) = (\frac{l}{j+1})$.

Finally we let

$$R(w) = w^{D+1}P(w)Q(w) = \sum_{j=D+1}^{D+L} d_j w^j.$$  

It follows from the definition of $P(w)$ and $Q(w)$ and $z_j \neq z_k$ for $j \neq k$ that $R(z_j) = 1$ for $j = 1, 2, \ldots, l$ and $R(z_j) = 0$ for $j = l + 1, l + 2, \ldots, L$. Replacing 1 by $R(z_j)$ for all $j = 1, 2, \ldots, l$ in (6.12) and adding the results together, one gets

$$M_2 \sum_{j=D+1}^{D+L} |d_j| \geq 1.$$  

By (6.12) with (6.11) and (6.8), we obtain

$$\sum_{j=D+1}^{D+L} |d_j| \leq \left( \sum_{j=0}^{L-1} |c_j| \right) \left( \sum_{j=0}^{L-l} |a_j| \right) \leq \frac{1}{(1 - 4e\hat{U})^D} \left( \frac{4}{\hat{U}} \right)^L,$$

from which and (6.13), we deduce that

$$M_2 \geq (1 - 4e\hat{U})^D \left( \frac{\hat{U}}{4} \right)^L.$$  

Finally, we recall the choice of $\hat{U}$ in (6.2) and we see that the last expression is the same as the last expression in (6.5) as in Case (i). This finishes estimating in Case (ii).

This ends the proof of Lemma 7. \hfill $\square$

References

1. C. Caldwell and Yuanyou Cheng, Determining Mills’ Constant and a Note on Honaker’s Problem, Journal of Integer Sequences, Article 05.4.1, Vol. 8 2005, pp. 1-9.
2. Y. Cheng, How to prove the Riemann hypothesis, 2020. To appear, Journal of Mathematics and System Science.
3. Y. Cheng, Estimates on primes between consecutive cubes, Rocky Mtn. J. Math., 40(1), 2010, pp. 117–153.
4. Y. Cheng, An explicit zero-free region for the Riemann zeta-function, Rocky Mtn. J. Math. , 30(1), 2000, pp. 135–148.
5. Yuanyou Cheng, S. Albeverio, R. L. Graham, S. W. Graham, and C. B. Pomerance, Proof of the strong density hypothesis, 2020. Submitted to Annals of Mathematics.
6. Yuanyou Cheng, S. Albeverio, R. L. Graham, C. B. Pomerance, and J. Wang, *Proof of the strong Lindelöf hypothesis*, 2020. Submitted to Annals of Mathematics.
7. Yuanyou Cheng, G. J. Fox, and M. Hassani, *Estimates on prime numbers*, 2020. To appear, Mathematica Aeterna.
8. Yuanyou Cheng and S. W. Graham, *The universe for prime numbers is random but symmetric – A panorama in proving the Riemann Hypothesis*, The International Journal of Artificial Intelligence and Applications, Vol ?, No. ?, May 2021; the 7th International Conference on Computer Science, Information Technology and Applications and the 7th International Conference on Artificial Intelligence, May 22-23, Zurich, Switzerland.
9. Yuanyou Cheng and S. W. Graham, *Estimates on the Riemann ξ-function via pseudogamma functions*, 2020. Submitted to the Journal of Analysis.
10. Yuanyou Cheng and S. W. Graham, *Estimates on the Riemann zeta function*, Rocky Mountain Journal of Mathematics, 34(4), 2004, pp. 1261-1280.
11. Yuanyou Cheng, S. W. Graham, and Bill Z. Yang, *Devising Pseudogamma Functions with Mathematica*, 2021. International Journal of Information Technology and Management, ???, ???. March 2021; the 10th International conference on Parallel, Distributed Computing and Applications and the 8th International Conference on Computer Science and Information Technology, April 24-25, 2021, Copenhagen, Denmark.
12. Yuanyou Cheng and Gongbao Li, *Estimates on ratios of the Riemann Xi-function via pseudo-Gamma functions*, 2020. Submitted to the Journal of Analysis.
13. Y. Cheng and C. Pomerance, *On a conjecture of R. L. Graham*, Rocky Mountain J. Mathematics, 24(3), pp. 961-965, 1994.
14. Yuanyou Cheng, C. B. Pomerance, G. J. Fox, and S. W. Graham, *A family of pseudogamma functions*, 2020. Submitted to the Journal of Analysis.
15. Yuanyou Cheng, C. B. Pomerance, R. L. Graham, and S. W. Graham, *Application of the Mellin Transform in the Distribution of Prime Numbers*, 2020. Submitted to American J. of Math.
16. Yuanyou Cheng, C. B. Pomerance, R. L. Graham, and S. W. Graham, *Proof of the Riemann Hypothesis from the density and Lindelöf hypotheses via a power sum method*, 2020. Submitted to Annals of Mathematics.
17. H. Davenport, *Multiplicative Number Theory*, V.74, Graduate Texts in Mathematics. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by Hugh L. Montgomery.
18. H. M. Edwards, *Riemann’s zeta function*, Academic Press, New York, London, 1974.
19. K. Ford, *Zero-free regions for the Riemann zeta function*, Number theory for the millennium, II (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 25–56.
20. X. Gourdon, *The 1013 first zeros of the Riemann zeta function, and zeros computation at very large height*, http://mathworld.wolfram.com/RiemannZetaFunctionZeros.html
21. A. Ivić, *The Riemann zeta function – Theory and Applications*, Dover Publications, Inc., Mineola, New York, 1985.
22. A. Odlyzko and A. Schönhage, *Fast algorithms for multiple evaluations of the Riemann zeta function*, Trans. Amer. Math. Soc. **309** (2): pp. 797–809, 1988.

23. K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

24. http://planetmath.org/encyclopedia/ValueOfTheRiemannZetaFunctionAtS0.html

25. Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, Oxford University Press, 2002.

26. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Akad., Berlin, pp. 671–680, 1859.

27. J. B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math., **63** (1941), pp. 211-232.

28. E. C. Titchmarsh, *The Theory of the Riemann zeta function*, Oxford University Press, Oxford, 1951; 2nd ed. revised by D. R. Heath-Brown, 1986.

29. P. Turán, *On Riemann’s hypothesis*, Bull. de l’Acad. des Sciences de l’URSS, Série Math., **11** (1947), pp. 197-262.

30. P. Turán, *On the remainder term of the prime number formula, II*, Acta Math. Acad. Sci. Hung. **1** (1950), pp. 155-166.

31. P. Turán, *On Carlson’s theorem in the theory of the zeta-function of Riemann*, Acta Math. Hung., No. 1-2, **2** (1951), pp. 39–73.

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