Non-disturbance criteria of quantum measurements*

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Abstract

In 2004, Kirkpatrick discussed three ways (I), (II) and (III) of describing non-disturbance between quantum measurements $X$ and $Y$, and showed that they are all equivalent to the compatibility of $X$ and $Y$ if they are both sharp measurements. In 2005, based on a special sequential product on the standard effect algebra, Gudder showed that if $X$ and $Y$ are unsharp measurements, then (I) holds if and only if $X$ and $Y$ are compatible and $Y$ is sharp measurement; compatibility of $X$ and $Y$ implies (II), but the converse does not hold, and only (III) is equivalent to the compatibility of $X$ and $Y$. In 2009, Liu and Shen and Wu in [J. Phys. A: Math. Theor. 42, 185206 (2009), J. Phys. A: Math. Theor. 42, 345203 (2009)] showed that there are many sequential products on the standard effect algebra. In this paper, we obtain the same conclusions as Gudder’s conclusions for all these sequential products of the standard effect algebra.

Key Words: Quantum measurements, non-disturbance, sequential product

1 Introduction

The fact that quantum measurements can disturb each other is manifested first by Heisenberg uncertainty principle. This disturbance is due to the non-commutativity of the position and momentum operators. In fact, the concepts of non-disturbance, compatibility, commutativity, coexistence and joint measurability are closely related to each other and is studied by many authors.

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In order to state our results, we first need to fix the notations. Let \( H \) be a complex Hilbert space which represents a quantum-mechanical system \( S \). A bounded self-adjoint operator \( A \) on \( H \) such that \( 0 \leq A \leq I \) is called a quantum effect on \( H \) ([1], [2]). We denote the set of all quantum effects on \( H \) by \( \mathcal{E}(H) \) and call it a standard effect algebra, the set of all orthogonal projection operators on \( H \) by \( \mathcal{P}(H) \). Orthogonal projection operators represent sharp yes-no measurements, while quantum effects represent yes-no measurements that may be unsharp. Let \( S(H) \) denote the set of density operators, i.e., the trace class positive operators on \( H \) of unit trace, which represent the states of quantum system \( S \). An operation is a positive linear mapping \( \Phi : S(H) \to S(H) \) such that for each \( T \in S(H) \), \( 0 \leq \text{tr}[\Phi(T)] \leq 1 \) ([3-5]).

Each orthogonal projection operator \( P \in \mathcal{P}(H) \) is associated with a so-called Lüders operation \( \Phi^P_L : T \to PTP \), moreover, when the quantum-mechanical system \( S \) is in state \( W \in S(H) \), the probability that \( P \) is observed is given by \( p_W(P) = \text{tr}(\Phi^P_L(W)) = \text{tr}(PWP) = \text{tr}(PW) \), and the resulting state after \( P \) is observed is \( W_P = \frac{PWP}{\text{tr}(PWP)} \) whenever \( \text{tr}(PWP) \neq 0 \). If \( P,Q \) are two orthogonal projection operators, then the conditional probability that \( P \) is observed given that \( Q \) has been observed is \( p_W(P|Q) = \frac{p_W(QPQ)}{p_W(Q)} = \frac{\text{tr}(QPQW)}{\text{tr}(QWQ)} \) whenever \( \text{tr}(QWQ) \neq 0 \). These operations arise in the context of sharp measurements ([4-5]). In general, each quantum effect \( A \in \mathcal{E}(H) \) gives rise to a general Lüders operation \( \Phi^A_L : T \to A\frac{1}{2}TA\frac{1}{2} \), moreover, when quantum-mechanical system \( S \) is in state \( W \in S(H) \), the probability that the effect \( A \) is observed is given by \( p_W(A) = \text{tr}(\Phi^A_L(W)) = \text{tr}(A\frac{1}{2}WA\frac{1}{2}) = \text{tr}(AW) \), and the resulting state after \( A \) is observed is \( W_A = \frac{A\frac{1}{2}WA\frac{1}{2}}{\text{tr}(AW)} \) whenever \( \text{tr}(AW) \neq 0 \). If \( A,B \) are two effects, then the conditional probability that \( B \) is observed given that \( A \) has been observed is \( p_W(B|A) = \frac{p_W(A\frac{1}{2}BA\frac{1}{2})}{p_W(A)} = \frac{\text{tr}(A\frac{1}{2}BA\frac{1}{2}W)}{\text{tr}(AW)} \) whenever \( \text{tr}(AW) \neq 0 \). These operations arise in the context of unsharp measurements ([4-6]).

Let \( \Phi_1, \Phi_2 \) be two operations. The composition \( \Phi_2 \circ \Phi_1 \) is a new operation, called a sequential operation as it is obtained by performing first \( \Phi_1 \) and then \( \Phi_2 \). In general, \( \Phi_2 \circ \Phi_1 \neq \Phi_1 \circ \Phi_2 \). Note that for any two quantum effects \( A,B \in \mathcal{E}(H) \) we have \( (\Phi^B_L \circ \Phi^A_L) = \Phi^{A\frac{1}{2}BA\frac{1}{2}}_L ([5, P_{36-37}]). This showed that the new quantum effect \( A\frac{1}{2}BA\frac{1}{2} \) yielded by \( A \) and \( B \) has important physical meaning, that is, \( A\frac{1}{2}BA\frac{1}{2} \) can be used to describe the effect by first measuring \( A \) and then measuring \( B \). Professor Gudder called it the sequential product of \( A \) and \( B \), and denoted it by \( A \circ B \), moreover, \( \circ \) has the following properties ([6-8]):

(S1). The map \( B \to A \circ B \) is additive for each \( A \in \mathcal{E}(H) \), that is, if \( B + C \leq I \), then \((A \circ B) + (A \circ C) \leq I \) and \((A \circ B) + (A \circ C) = A \circ (B + C) \).

(S2). \( I \circ A = A \) for all \( A \in \mathcal{E}(H) \).

(S3). If \( A \circ B = 0 \), then \( A \circ B = B \circ A \).

(S4). If \( A \circ B = B \circ A \), then \( A \circ (I - B) = (I - B) \circ A \) and \( A \circ (B \circ C) = (A \circ B) \circ C \) for all \( C \in \mathcal{E}(H) \).
Theorem 1.1 showed that there are many sequential products on complicated process where we allow a duration between the measurement.

A is completed at once after the measurement if and only algebraic operation on answered the problem negatively (10).

Thus, we need to consider the following general sequential product to guarantee (1).

Note that for each A, B ∈ E of f, defined on the spectra we define f ∈ B(sp(A)) such that for each A ∈ E, there exists a complex constant ξ such that |ξ| = 1 and f_A(A)f_B(B) = ξf_{AB}(AB).

Professor Gudder presented the following open problem in 9: Is A ○ B = A_B A_B the only algebraic operation on E(H) which satisfies properties (S1)-(S5)? In 2009, Liu and Wu answered the problem negatively (10).

We would like to point out that the sequential product A ○ B = A_B A_B = A_B(B(A_B)^*) of A and B can only describe the instantaneous measurement, that is, the measurement B is completed at once after the measurement A is performed. In order to describe a more complicated process where we allow a duration between the measurement A with the measurement B, then we need to replace A_B with f(A), (A_B)^* with (f(A))^*, where f(A) is a function of A which describe the change of A was made by the duration between B with A. Thus, we need to consider the following general sequential product f(A)B(f(A))^*. In order to guarantee (f(A))^* = f(A), we ask f to be a bounded complex Borel function which is defined on the spectra sp(A) of A.

By the above motivation, in 11, Shen and Wu proved the following result:

**Theorem 1.1.** Let H be a finite dimensional complex Hilbert space, C the set of complex numbers, R the set of real numbers, for each A ∈ E(H), sp(A) the spectra of A and B(sp(A)) the set of all bounded complex Borel functions on sp(A). Take f_A ∈ B(sp(A)) and B ∈ E(H), we define

A ○ B = f_A(A)B(f_A(A))^*.

Then ○ has properties (S1)-(S5) if and only if the set \( \{f_A\}_{A \in E(H)} \) satisfies the following two conditions:

(i) For each A ∈ E(H) and t ∈ sp(A), \( |f_A(t)| = \sqrt{t} \);

(ii) For any A, B ∈ E(H), if AB = BA, then there exists a complex constant ξ such that |ξ| = 1 and \( f_A(A)f_B(B) = ξf_{AB}(AB) \).

Note that for each A ∈ E(H), we can take many \( f_A \in B(sp(A)) \) satisfies (i) and (ii), so, Theorem 1.1 showed that there are many sequential products on E(H).

Henceforth, H is always a finite dimensional complex Hilbert space and the set \( \{f_A\}_{A \in E(H)} \) satisfies conditions (i) and (ii).

Moreover, in 11, Shen and Wu still proved that:

**Theorem 1.2.** (1) \( f_A(A)f_A(A) = f_A(A)f_A(A) = A, (f(A))^* = f(A) \).

(2) If 0 ∈ sp(A), then \( f_A(0) = 0 \).

(3) If A = \( \sum_{k=1}^{n} λ_kE_k \), where \( \{E_k\}_{k=1}^{n} \) are pairwise orthogonal projections, then \( f_A(A) = \).
\[
\sum_{k=1}^{n} f_A(\lambda_k)E_k.
\]

(4) For each \( E \in \mathcal{P}(H) \), \( f_E(E) = f_E(0)(I - E) + f_E(1)E = f_E(1)E \).

(5) For any \( A, B \in \mathcal{E}(H) \), \( A \circ B \in \mathcal{E}(H) \).

Take \( A \in \mathcal{E}(H) \) and \( f_A \in \{ f_A \}_{A \in \mathcal{E}(H)} \), we define \( \psi^A : T \rightarrow \overline{f_A(A)Tf_A(A)} \) for \( T \in \mathcal{S}(H) \). Now, we can define the probability and conditional probability which bases on the general sequential product \( A \circ B = f_A(A)B\overline{f_A(A)} \). For example, when the system is in state \( W \in \mathcal{S}(H) \), the probability that the effect \( A \in \mathcal{E}(H) \) is observed is given by \( p_W(A) = tr(\psi^A(W)) = tr(\overline{f_A(A)Wf_A(A)}) \), and the resulting state after the effect \( A \) is observed is \( W_A = \frac{\psi^A(W)}{tr(\psi^A(W))} \) whenever \( tr(\psi^A(W)) \neq 0 \), the conditional probability that \( B \) is observed given that \( A \) has been observed is \( p_W(B|A) = p_{W_A}(B) = \frac{tr(\overline{f_A(A)B\overline{f_A(A)}W})}{tr(\psi^A(W))} \) whenever \( tr(\psi^A(W)) \neq 0 \). Thus, it follows from the definition and Theorem 1.2 that

\[
p_W(C|A \circ B) = \frac{p_{(\overline{\psi^B \cdot \psi^A})(W)C}}{tr(\overline{\psi^B \cdot \psi^A}W)} = \frac{tr(C\overline{f_B(B)\overline{f_A(A)}Wf_A(A)})}{tr(\overline{f_B(B)Wf_A(A)})} \tag{1}
\]

whenever \( tr(\overline{f_B(B)Wf_A(A)}) \neq 0 \).

In [12], Kirkpatrick discussed three ways of describing non-disturbance between quantum measurements as follows:

Let \( X \) and \( Y \) be two discrete POVMs, i.e., \( X = \{ A_i \}_{i=1}^{m}, Y = \{ B_j \}_{j=1}^{n} \), where \( A_i, B_j \in \mathcal{E}(H), i = 1, 2, \cdots \), \( m, j = 1, 2, \cdots, n \), and \( \sum_{i=1}^{m} A_i = I, \sum_{j=1}^{n} B_j = I \). Kirkpatrick discussed the following three ways of describing non-disturbance between quantum measurements \( X \) and \( Y \):

(I) The probability of an established value of \( Y \) is unchanged by the later occurrence of a value of \( X \).

(II) The probability of occurrence of a \( Y \) value is unchanged by a preceding execution of \( X \).

(III) If \( p \) and \( q \) are \( X \) and \( Y \) values, respectively, then the probability of \( p \) followed by \( q \) coincides with the probability of \( q \) followed by \( p \).

Kirkpatrick showed that (I), (II) and (III) are equivalent to the compatibility of \( X \) and \( Y \) if they are sharp measurements, i.e., when \( A_i, B_j \in \mathcal{P}(H), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \), then (I), (II) and (III) are equivalent to the compatibility of \( X \) and \( Y \), that is, \( A_iB_j = B_jA_i \) for all \( i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \).

In [13], based on the special sequential product \( A \circ B \) of the standard effect algebra \( \mathcal{E}(H) \), Gudder showed that if \( X \) and \( Y \) are unsharp measurements, i.e., when \( A_i, B_j \in \mathcal{E}(H), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \), then (I) holds if and only if \( X \) and \( Y \) are compatible and \( Y = \{ B_j \}_{j=1}^{n} \subseteq \mathcal{P}(H) \); compatibility of \( X \) and \( Y \) implies (II), but the converse does not
hold, and only (III) is equivalent to the compatibility of \(X\) and \(Y\).

In this paper, we obtain the same conclusions as Gudder’s conclusions for all the sequential products of the standard effect algebra \(E(H)\).

2 Some Lemmas

In this section, we present some useful lemmas such that to prove our main results in section 3.

**Lemma 2.1 ([11]).** Let \(A, B \in E(H)\). If \(AB = BA\), then \(A \odot B = B \odot A = AB\). If \(A \odot B = B \odot A\) or \(A \odot B = f_B(B)Af_B(B)\), then \(AB = BA\).

**Lemma 2.2 ([14 Corollary 4.1.2]).** If \(A\) is a normal element of a \(C^*\)-algebra \(\mathcal{U}\), and \(A^k = 0\) for some positive integer \(k\), then \(A = 0\).

**Lemma 2.3 ([15]).** Let \(A \in \mathcal{B}(H)\) have the following operator matrix form

\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\]

with respect to the space decomposition \(H = H_1 \oplus H_2\). Then \(A \succeq 0\) if and only if

1. \(A_{ii} \in \mathcal{B}(H_i)\) and \(A_{ii} \geq 0\), \(i = 1, 2\);
2. \(A_{21} = A_{12}^*\);
3. there exists a linear operator \(D\) from \(H_2\) into \(H_1\) such that \(|D|\leq 1\) and \(A_{12} = A_{11}^{\frac{1}{2}}DA_{22}^{\frac{1}{2}}\).

The following lemma is important in establishing the first non-disturbance criteria.

**Lemma 2.4.** Let \(A \in \mathcal{B}(H)\) be a normal operator and \(B \in E(H)\). If \(AB = BAB\), then \(AB = BA\).

**Proof.**

*Step 1.* Suppose that \(A\) is an invertible operator. It follows from \(AB = BAB\) that \(BA^* = BA^*B\), so

\[
ABA^* = ABA^*B.
\]

Since \(H\) is a finite dimensional space and \(0 \leq B \leq I\), by spectral decomposition, \(B\) can be represented as \(B = \sum_{i=1}^{n} \lambda_i E_i\), where \(0 \leq \lambda_i \leq 1\), \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\), \(\{E_k\}_{k=1}^{n}\) is pairwise orthogonal projection operators and \(\sum_{k=1}^{n} = I\). Thus, we have

\[
ABA^*E_i = \lambda_iABA^*E_i.
\]
That is,
\[(1 - \lambda_i)ABA^*E_i = 0.\]

So, it is easily to obtain that
\[ABA^* = ABA^*P,\]
where \(P\) denotes the orthogonal projection operator corresponding to the eigenvalue \(\lambda_1 = 1.\)

It follows from \(A\) is invertible that \(\text{rank}(ABA^*) = \text{rank}(B).\) So from the above equality we can easily obtain \(\text{rank}(P) = \text{rank}(B),\) which means that \(B = P.\) By the condition that \(AB = BAB\) and \(B = P\) we have
\[AP = PAP.\]

Since \(A\) is normal, by functional calculus we can easily get \(A^*P = PA^*P.\) Thus we obtain \(AP = PA.\) That is, \(AB = BA.\)

**Step 2.** Suppose that \(A \in \mathcal{B}(H).\) Since \(\dim H < \infty,\) denote \(\dim H = n,\) then \(A\) can be represented as
\[A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\]
with respect to the space decomposition \(H = R(A^n) \oplus N(A^n),\) where \(A_1\) is an invertible operator and \(A_2\) is a nilpotent operator.

By Lemma 2.2, it is easy to see that \(A_2 = 0.\) And by Lemma 2.3, \(B\) can be represented by
\[B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix},\]
where \(0 \leq B_{11} \leq I, 0 \leq B_{22} \leq I.\) Then from \(AB = BAB,\) we get
\[\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix},\]
it implied that
\[
\begin{align*}
A_1B_{11} &= B_{11}A_1B_{11} \\
A_1B_{12} &= B_{11}A_1B_{12} \\
B_{12}^*A_1B_{11} &= 0 \\
B_{12}^*A_1B_{11} &= 0
\end{align*}
\]
(3)

It follows from Step 1 and the first equality that \(A_1B_{11} = B_{11}A_1.\) Then from the third equality we have \(B_{12}^*B_{11}A_1 = 0.\) Since \(A_1\) is invertible, \(B_{12}^*B_{11} = 0,\) so \(B_{11}B_{12} = 0.\) Also from the second equality we have \(A_1B_{12} = A_1B_{11}B_{12},\) thus \(A_1B_{12} = 0,\) and so \(B_{12} = 0.\) Therefore,
\[B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.\]

Hence, we obtain \(AB = BA\) in the general case. This completes the proof.
3 Non-disturbance criteria

Our main results are the following:

**Theorem 3.1.** Let \( X = \{ A_k \}_{k=1}^m \) and \( Y = \{ B_j \}_{j=1}^n \) be two quantum measurements, where \( A_k, B_j \in \mathcal{E}(H), k = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \). Then

\[
p_W(B_j|B_j \circ A_k) = 1
\]

holds for any \( j, k \) and \( W \in \mathcal{S}(H) \) if and only if \( A_k B_j = B_j A_k \) and \( B_j \in \mathcal{P}(H) \) for all \( j \).

**Proof.** The sufficiency. By assumption, we have \( \bar{f}_{A_k}(A_k)B_j = B_j \bar{f}_{A_k}(A_k) \), and \( \bar{f}_{B_j}(B_j) = \bar{f}_{B_j}(1)B_j \) (Theorem 1.2(4)). Thus,

\[
p_W(B_j|B_j \circ A_k) = \frac{\text{tr}(B_j \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j)W f_{B_j}(B_j)f_{A_k}(A_k))}{\text{tr}(A_k\bar{f}_{B_j}(B_j)W f_{B_j}(B_j))}
\]

\[
= \frac{\text{tr}(\bar{f}_{A_k}(A_k)B_j \bar{f}_{B_j}(1)B_j f_{B_j}(B_j)f_{A_k}(A_k))}{\text{tr}(A_k\bar{f}_{B_j}(B_j)W f_{B_j}(B_j))}
\]

\[
= \frac{\text{tr}(A_k\bar{f}_{B_j}(1)B_jW f_{B_j}(B_j))}{\text{tr}(A_k\bar{f}_{B_j}(1)B_jW f_{B_j}(B_j))}
\]

\[
= 1.
\]

Necessity. Since conditional probability is countably additive in its first argument, so (4) implies

\[
p_W(B_i|B_j \circ A_k) = 0
\]

for \( i \neq j \). Thus we have

\[
\frac{\text{tr}(B_i \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j)W f_{B_j}(B_j)f_{A_k}(A_k))}{\text{tr}(A_k\bar{f}_{B_j}(B_j)W f_{B_j}(B_j))} = 0
\]

for all \( i \neq j \), whenever \( \text{tr}(A_k\bar{f}_{B_j}(B_j)W f_{B_j}(B_j)) \neq 0 \). We can write (6) as

\[
\text{tr}(f_{B_j}(B_j)f_{A_k}(A_k)B_i \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j)W) = 0.
\]

Now (7) holds even if \( \text{tr}(A_k\bar{f}_{B_j}(B_j)W f_{B_j}(B_j)) = 0 \) because in this case

\[
\bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j)W f_{B_j}(B_j)f_{A_k}(A_k) = 0.
\]

Since (7) holds for every \( W \) we conclude that

\[
f_{B_j}(B_j)f_{A_k}(A_k)B_i \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j) = 0
\]

for all \( i \neq j \). We then obtain

\[
(B_i^{1/2} \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j))^*(B_i^{1/2} \bar{f}_{A_k}(A_k)\bar{f}_{B_j}(B_j)) = 0
\]

for all \( i \neq j \).
for all \( i \neq j \). Hence, \( B_i^{1/2} \overline{f_{A_k}(A_k)f_{B_j}(B_j)} = 0 \) for all \( i \neq j \). So \( B_i \overline{f_{A_k}(A_k)}B_j = 0 \) for all \( i \neq j \).

Summing over \( i \neq j \) and using \( \sum_i B_i = I \), we have

\[
0 = (I - B_j) \overline{f_{A_k}(A_k)}B_j = \overline{f_{A_k}(A_k)}B_j - B_j \overline{f_{A_k}(A_k)}B_j.
\]

Thus

\[
\overline{f_{A_k}(A_k)}B_j = B_j \overline{f_{A_k}(A_k)}B_j.
\]

Note that \( \overline{f_{A_k}(A_k)} \) is a normal operator, by Lemma 2.4 we obtain that

\[
\overline{f_{A_k}(A_k)}B_j = B_j \overline{f_{A_k}(A_k)}.
\]

Taking adjoint, we have

\[
B_j \overline{f_{A_k}(A_k)} = f_{A_k}(A_k)B_j.
\]

Thus, for all \( j \) and \( k \), we have

\[
A_kB_j = \overline{f_{A_k}(A_k)}f_{A_k}(A_k)B_j = \overline{f_{A_k}(A_k)}B_j f_{A_k}(A_k) = B_j \overline{f_{A_k}(A_k)}f_{A_k}(A_k) = B_j A_k.
\]

Now (8) becomes

\[
A_k f_{B_j}(B_j)B_i \overline{f_{B_j}(B_j)} = 0, \forall i \neq j. \tag{9}
\]

Summing (9) over \( k \) gives

\[
f_{B_j}(B_j)B_i \overline{f_{B_j}(B_j)} = 0, \forall i \neq j.
\]

Now summing over \( i \neq j \) we have

\[
f_{B_j}(B_j)(I - B_j) \overline{f_{B_j}(B_j)} = 0.
\]

Hence \( B_j = f_{B_j}(B_j)B_j \overline{f_{B_j}(B_j)} = f_{B_j}(B_j)\overline{f_{B_j}(B_j)}f_{B_j}(B_j)\overline{f_{B_j}(B_j)} = B_j^2 \). That is, \( B_j \in \mathcal{P}(H) \) for all \( j \).

**Theorem 3.2.** Let \( X = \{ A_k \}_{k=1}^n \) and \( Y = \{ B_j \}_{j=1}^n \) be two quantum measurements, where \( A_k, B_j \in \mathcal{E}(H), k = 1, 2, \cdots, m, j = 1, 2, \cdots, n \). If \( A_kB_j = B_j A_k \) for any \( k \) and \( j \), then

\[
p_W(B_j) = \sum_k p_W(A_k \circ B_j) \tag{10}
\]

holds for any \( j \) and \( W \in \mathcal{S}(H) \).

**Proof.** In terms of traces, (10) becomes

\[
tr(\overline{f_{B_j}(B_j)}Wf_{B_j}(B_j)) = \sum_k tr(B_j \overline{f_{A_k}(A_k)}Wf_{A_k}(A_k)).
\]

That is,

\[
tr(B_j W) = \sum_k tr(f_{A_k}(A_k)B_j \overline{f_{A_k}(A_k)}W). \tag{11}
\]
Since $A_kB_j = B_jA_k$, we get $f_{A_k}(A_k)B_j = B_jf_{A_k}(A_k)$. Then the right side of (11) becomes
\[
\sum_k tr(A_kB_jW) = tr((\sum_k A_k)B_jW) = tr(B_jW).
\]
So the theorem is proved.

Note that the converse of Theorem 3.2 does not hold even for the special sequential product $A \circ B = A^{\frac{1}{2}}BA^{\frac{1}{2}}$, so, it does also not hold for the general sequential product $A \circ B$.

**Theorem 3.3.** Let $X = \{A_k\}_{k=1}^m$ and $Y = \{B_j\}_{j=1}^n$ be two quantum measurements. Then
\[
p_W(A_k \circ B_j) = p_W(B_j \circ A_k)
\]
holds for any $k$ and $j$ and $W \in S(H)$ if and only if $A_kB_j = B_jA_k$ for any $k$ and $j$.

**Proof.** The sufficiency follows from Lemma 2.1 immediately. For necessity, in terms of traces, (12) becomes
\[
tr(B_jf_{A_k}(A_k)Wf_{A_k}(A_k)) = tr(A_kf_{B_j}(B_j)Wf_{B_j}(B_j)).
\]
that is,
\[
tr(f_{A_k}(A_k)B_jf_{A_k}(A_k)W) = tr(f_{B_j}(B_j)A_kf_{B_j}(B_j)W).
\]
Since (13) holds for all $W$, we have
\[
f_{A_k}(A_k)B_jf_{A_k}(A_k) = f_{B_j}(B_j)A_kf_{B_j}(B_j).
\]
That is,
\[
A_k \circ B_j = B_j \circ A_k.
\]
Then by Lemma 2.1, we obtain
\[
A_kB_j = B_jA_k.
\]
The theorem is proved.

**Remark 1.** For each $E \in \mathcal{P}(H)$, it follows from Theorem 1.2 that $f_E(E) = f_E(0)(I - E) + f_E(1)E = f_E(1)E$. Using this fact, we can easily see that if $P, Q \in \mathcal{P}(H)$, then
\[
P \circ Q = f_P(P)Qf_P(P) = f_P(1)PQf_P(1)P = PQP = P \circ Q.
\]
This showed that if $P$ and $Q$ are two sharp elements, then the instantaneous measurement and the duration measurement are same.
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