\[ \mathcal{N} = 1 \text{ Geometric Supergravity and Chiral Triples on Riemann Surfaces} \]

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**Abstract:** We construct a global geometric model for the bosonic sector and Killing spinor equations of four-dimensional \( \mathcal{N} = 1 \) supergravity coupled to a chiral non-linear sigma model and a Spin\(^c\) structure. The model involves a Lorentzian metric \( g \) on a four-manifold \( M \), a complex chiral spinor and a map \( \varphi : M \rightarrow \mathcal{M} \) from \( M \) to a complex manifold \( \mathcal{M} \) endowed with a novel geometric structure which we call a **chiral triple**. Using this geometric model, we show that if \( M \) is spin then the Kähler-Hodge condition on \( \mathcal{M} \) suffices to guarantee the existence of an associated \( \mathcal{N} = 1 \) chiral geometric supergravity. This positively answers a conjecture proposed by D. Z. Freedman and A. V. Proeyen. We dimensionally reduce the Killing spinor equations to a Riemann surface \( X \), obtaining a novel system of partial differential equations for a harmonic map with potential \( \varphi : X \rightarrow \mathcal{M} \). We characterize all Riemann surfaces \( X \) admitting supersymmetric solutions with vanishing superpotential, showing that such solutions \( \varphi \) are holomorphic maps satisfying a certain condition involving the canonical bundle of \( X \) and the chiral triple of the theory. Furthermore, we determine the biholomorphism type of all Riemann surfaces carrying supersymmetric solutions with complete Riemannian metric and finite-energy scalar map \( \varphi \).

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1. Introduction

Supergravity theories are supersymmetric theories of gravity which, at least from a differential-geometric point of view, can be viewed as a system of partial differential equations coupling a Lorentzian metric to a number of other objects, such as connections on principal bundles, spinors or forms of different degrees arising as curvatures on (possibly higher) Abelian gerbes [25,52,55]. The equations defining a supergravity theory are strongly constrained by the requirement of invariance under local supersymmetry transformations. The local structure of supergravity theories has been the object of intense study since the discovery of supersymmetry in the early seventies and nowadays there exists a reasonably complete local classification of (ungauged) supergravities in Lorentzian signature, see [4–6,55] and references therein for more details. However, geometric models which implement and extend at a global level the mathematical structures and symmetries appearing in the local formulation of such theories have not been investigated systematically. The motivations to construct global geometric formulations of supergravity theories are manifold. For instance, understanding the global character of general supergravity solutions (for example, their so-called U-fold structure [16,35]) requires precise knowledge of the global formulation of the supergravity theory under consideration. In particular, extending a local supergravity solution demands a priori knowledge of the global character of the physical fields of the theory. A second important motivation arises from the study of moduli spaces of solutions of supergravity theories, which requires a precise description of the equations of motion in terms of globally well-defined differential operators. In this context, the choice of a global geometric model of the theory can have a dramatic impact on the moduli space of solutions.

A global geometric model for the universal bosonic sector of four-dimensional supergravity was constructed in [43,44]. This involves a flat submersion $\pi : E \to M$ over space-time and a flat symplectic vector bundle $S$ over $E$. Whereas this model subsumes the bosonic sector of any ungauged four-dimensional supergravity theory, it does not yet implement supersymmetry. One of the goals of the present paper is to contribute a solution of this problem by giving a global construction of the Killing spinor equations for (ungauged) $\mathcal{N} = 1$ supergravity coupled to chiral matter. This corresponds to the ‘untwisted’ case when $\pi$ and $S$ are trivial. Whereas it is desirable to have a geometric model for the complete $\mathcal{N} = 1$ supergravity theory, including its fermionic sector, we shall focus on the simpler problem of constructing a geometric model for the bosonic sector together with the associated Killing spinor equations. This suffices for the differential-geometric applications which we have in mind. The Killing spinor equations are are crucial in this context, since they lead to a global definition of the notion of supersymmetric solution and give rise to well-defined moduli spaces which generalize well-known moduli problems, such as that of generalized instantons [19,53] and, most notably, the moduli problem of pseudo-holomorphic maps [48]. From a Physics perspective, $\mathcal{N} = 1$ four-dimensional supergravity is of great importance given its many applications to particle physics, cosmology and string phenomenology (see [17,18] and references therein). The present article forms part of wider program aimed at developing the mathematical foundations of geometric supergravity, which we define as the global geometric theory of bosonic supergravity together with its globally-formulated Killing spinor equations.

The geometric model of $\mathcal{N} = 1$ supergravity which we discuss in this paper involves the notion of a chiral triple, which we introduce in Definition 3.11. This encodes a set of sufficient conditions for a complex manifold $\mathcal{M}$ to be an admissible target space for the non-linear sigma model of the theory. Strictly speaking, existence of a chiral
triple is not equivalent with the Kähler-Hodge condition which is usually considered in the literature [4,25,58], in that the former contains more information and in particular requires the existence of an isomorphism between the pull-back of the holomorphic line bundle of the chiral triple and the determinant line bundle of the Spin$^c_0(3, 1)$ structure $Q$. However, we prove that if the space-time $M$ is spin then every Kähler-Hodge manifold $\mathcal{M}$ admits a chiral triple and thus $\mathcal{M}$ can occur as the scalar manifold of an $\mathcal{N} = 1$ chiral supergravity theory, thereby giving a positive answer to a question raised in [25, Section 17.5].

We couple the theory to a Spin$^c_0(3, 1)$ structure $Q$ by using a delicate interplay between the chiral triple and the determinant line bundle of $Q$ which exists thanks to very specific properties of the Clifford algebra in signature $(3, 1)$. Consequently, we formulate the geometric model using complex chiral spinors associated to $Q$. We illustrate the construction with several examples. We find that the Killing spinor equations of chiral $\mathcal{N} = 1$ supergravity yield a very rich system of spinorial equations in which the superpotential plays a crucial role, as does its interplay with the complex conjugation operation on the spinor bundle. To the best of our knowledge, this system was not explored previously in the Mathematics literature and it gives rise to a number of outstanding open problems which we summarize in Sect. 5. Reducing the Killing spinor equations with non-trivial superpotential on a Riemann surface $X$ produces a variant of the moduli theory of the perturbed pseudoholomorphicity equations for maps from $X$ into a Kähler manifold. When the scalar manifold is a point, our equations reduce to a particular instance of generalized Killing spinor equation of references [28,29], in which complex conjugation on the spinor bundle plays a pivotal role.

As an application of the geometric model discussed in this paper, we reduce the theory to a Riemann surface and characterize all supersymmetric solutions with vanishing superpotential, obtaining the following result.

**Theorem 1.1.** Supersymmetric solutions of chiral $\mathcal{N} = 1$ ungauged supergravity with vanishing superpotential on a Riemann surface $X$ consist of holomorphic maps $\varphi: X \to \mathcal{M}$ satisfying a stability condition with respect to the linearization given by the canonical bundle of $X$ and the chiral triple of the theory.

A detailed account of this result can be found in Theorem 4.13. Using a seminal result by A. Huber, see [38, Pages 1–2], we can exploit the previous theorem to prove the following classification result.

**Theorem 1.2.** If $X$ admits a supersymmetric solution with complete Riemannian metric and finite-energy scalar map, then it is biholomorphic with either $\mathbb{P}^1$, a complex elliptic curve $E$, the complex plane $\mathbb{C}$ or the punctured complex plane $\mathbb{C}^*$, with prescribed metric singularities on the one-point and two-point compactification of $\mathbb{C}$ or $\mathbb{C}^*$, respectively.

A detailed account of this result can be found in Theorem 4.21. Taking the Riemann surface $X$ to be compact, the previous theorem recovers the black-hole horizon topologies classified in reference [33], see also [49]. In contrast to the $\mathcal{N} > 1$ case, supersymmetric solutions of local chiral $\mathcal{N} = 1$ supergravity have not been completely classified even at the local level. Supersymmetric solutions with vanishing superpotential were considered in [51], where the local generic form of the metric was obtained and the problem of constructing supersymmetric solutions was reduced to solving a minimal set of partial differential equations. Local supersymmetric solutions with non-vanishing superpotential were studied in [32], where they were again characterized in terms of a minimal set of PDEs. Other aspects of supersymmetric solutions of $\mathcal{N} = 1$ supergravity were
explored in [33, 37, 49]. We hope that the geometric model presented here can constitute a stepping stone toward a better understanding of the global structure of supersymmetric solutions of $\mathcal{N} = 1$ supergravity as well as of the associated moduli spaces. From a different point of view, a previous contribution to the development of the geometric formulation of supergravity was made in reference [26], which introduced a formulation of supersymmetry that is covariant with respect to coordinate transformations on the scalar manifold. We note that the $\mathcal{N} = 1$ Killing spinor equations presented in the present paper are automatically covariant by construction in the sense of op. cit.

It is a key aspect of supersymmetry that it provides a partial integration of the bosonic equations of the theory in terms of (usually) first-order spinorial equations. One may also consider different spinorial equations whose solutions produce non-supersymmetric solutions of $\mathcal{N} = 1$ supergravity. Inspired by such ideas, Sect. 4.2 introduces the notion of anti-supersymmetric solution, which yields a family of non-supersymmetric solutions of the supergravity theory which are defined on space-times given by the direct product of $\mathbb{R}^2$ with a hyperbolic Riemann surface. These turn out to correspond to solutions of the supergravity theory where supersymmetry is broken spontaneously.\(^1\) In view of the results of [33], one may wonder if some of these non-supersymmetric solutions can appear as near-horizon geometries of non-supersymmetric black holes in $\mathcal{N} = 1$ supergravity.

The paper is organized as follows. Section 2 gives a detailed account of Clifford algebras in four dimensions with Lorentzian signature and of the associated real and complex irreducible modules. In Sect. 3, we introduce the notion of chiral triple as well as the geometric model for the bosonic sector and Killing spinor equations of chiral $\mathcal{N} = 1$ supergravity, giving several examples. In Sect. 4, we reduce this geometric model to a Riemann surface and characterize all supersymmetric solutions with vanishing superpotential. Furthermore, we introduce the notion of anti-supersymmetric solution using a natural variation of the Killing spinor equations which still yields solutions of chiral $\mathcal{N} = 1$ supergravity. We provide numerous examples of supersymmetric and anti-supersymmetric solutions. Section 5 concludes with a summary of results and a list of open problems.

### 2. Clifford Algebras in Four Dimensions

In this section, we present the background material on four-dimensional Clifford algebras that shall be needed in the rest of the manuscript. Let $(V, h)$ be a four-dimensional real vector space equipped with a non-degenerate symmetric bilinear form $h$ of signature $(3, 1) = (+, +, +, -)$. We fix a metric volume form $\nu$ on $(V, h)$. We denote by $\text{Cl}(V, h)$ the real Clifford algebra associated to $(V, h)$. We use the following convention for the Clifford relation:

$$v^2 = h(v, v), \quad v \in V.$$  

The real Clifford algebra $\text{Cl}(V, h)$ admits a unique irreducible real representation:

$$\gamma : \text{Cl}(V, h) \xrightarrow{\sim} \text{End}_{\mathbb{R}}(\Sigma_{\mathbb{R}}),$$

where $\Sigma_{\mathbb{R}}$ is the associated real spinor space. The representation $\gamma$ is in fact an isomorphism of unital, associative, real algebras, and hence, upon a choice of a basis in $\Sigma_{\mathbb{R}}$, $\gamma$

\(^1\) We thank the anonymous referee for clarifying this point.
yields an isomorphism $\text{Cl}(V, h) \simeq \text{Mat}(4, \mathbb{R})$, where $\text{Mat}(4, \mathbb{R})$ denotes the unital and associative algebra of real four by four square matrices. Using the canonical isomorphism of vector spaces $\text{Cl}(V, h) \simeq \Lambda(V)$ between the Clifford algebra $\text{Cl}(V, h)$ and the exterior algebra of $V$, we can consider the volume form $\nu$ as an element $\nu = e_0 e_1 e_2 e_3$ in $\text{Cl}(V, h)$, which for simplicity we denote with the same symbol. Here $\{e_0, \ldots, e_3\}$ denotes a positively oriented orthonormal basis. We have:

$$\nu^2 = -\text{Id}, \quad \nu \in Z(\text{Cl}^{eu}(V, h)),$$

where $\text{Cl}^{eu}(V, h) \subset \text{Cl}(V, h)$ denotes the even subalgebra of $\text{Cl}(V, h)$ and $Z(\text{Cl}^{eu}(V, h))$ denotes the center of $\text{Cl}^{eu}(V, h)$. Hence, $\gamma(\nu)$ defines an almost complex structure on $\Sigma_\mathbb{R}$ which however does not commute with Clifford multiplication. Restricting $\gamma$ to $\text{Cl}^{eu}(V, h)$ we obtain the unique irreducible real representation of $\text{Cl}^{eu}(V, h)$, which is again of real dimension four. Since $\nu \in Z(\text{Cl}^{eu}(V, h))$, the image of $\text{Cl}^{eu}(V, h)$ in $\text{End}(\Sigma_\mathbb{R})$ consists on endomorphisms complex-linear with respect to the complex structure $\gamma(\nu)$. In fact, it can be shown that the restriction of $\gamma$ to $\text{Cl}^{eu}(V, h)$ gives an isomorphism of unital and associative algebras:

$$\gamma^+ \overset{\text{def.}}{=} \gamma|_{\text{Cl}^{eu}(V, h)} : \text{Cl}^{eu}(V, h) \tilde{\rightarrow} \text{End}(\Sigma_\mathbb{R}, \gamma(\nu)) \simeq \text{End}_\mathbb{C}(\Sigma_0^+),$$

where $\text{End}(\Sigma_\mathbb{R}, \gamma(\nu))$ denotes the $\gamma(\nu)$-linear endomorphisms of $\Sigma_\mathbb{R}$ and $\text{End}_\mathbb{C}(\Sigma_0^+)$ denotes the complex endomorphisms of $\Sigma_0^+ \overset{\text{def.}}{=} (\Sigma_\mathbb{R}, \gamma(\nu))$, where the later is understood as a two-dimensional complex vector space with complex structure $\gamma(\nu)$. Defining $\Sigma_0^- \overset{\text{def.}}{=} (\Sigma_\mathbb{R}, -\gamma(\nu))$ with the opposite complex structure with respect to $\Sigma_0$, we obtain the complex-conjugate representation:

$$\gamma^- \overset{\text{def.}}{=} \gamma|_{\text{Cl}^{eu}(V, h)} : \text{Cl}^{eu}(V, h) \tilde{\rightarrow} \text{End}(\Sigma_\mathbb{R}, -\gamma(\nu)) \simeq \text{End}_\mathbb{C}(\Sigma_0^-).$$

This way we obtain the two complex-conjugate irreducible representations of $\text{Cl}^{eu}(V, h)$, inducing equivalent real irreducible representations, which correspond to the two chiral irreducible complex representations of $\text{Cl}^{eu}(V, h)$. The spin group $\text{Spin}(V, h)$ injects in $\text{Cl}^{eu}(V, h)$, and the restriction of $\gamma^+$ and $\gamma^-$ to the image of $\text{Spin}(V, h)$ in $\text{Cl}^{eu}(V, h)$ yields two, complex-conjugate, irreducible spinorial complex representations of $\text{Spin}(V, h)$, which are again equivalent as real representations. Since we are interested in irreducible representations of

$$\text{Spin}^c(V, h) = (\text{Spin}(V, h) \times \text{U}(1))/\{1, -1\},$$

we complexify the previous set up. Let $V^c \overset{\text{def.}}{=} V \otimes_\mathbb{R} \mathbb{C}$ denote the complexification of $V$. Extending $h$ by $\mathbb{C}$-linearity to $V^c$ we obtain a complex quadratic space $(V^c, h^c)$. We have the following isomorphism of complex unital and associative algebras:

$$\text{Cl}(V, h) \overset{\text{def.}}{=} \text{Cl}(V, h) \otimes \mathbb{C} \simeq \text{Cl}(V^c, h^c),$$

where $\text{Cl}(V^c, h^c)$ denotes the complex Clifford algebra associated to $(V^c, h^c)$. Hence $\text{Cl}(V, h) \simeq \text{Mat}(4, \mathbb{C})$. Let $\Sigma \overset{\text{def.}}{=} \Sigma_\mathbb{R} \otimes \mathbb{C}$. Extending $\gamma$ to $\text{Cl}(V, h)$ by $\mathbb{C}$-linearity we obtain the unique complex irreducible representation $\gamma^c_\mathbb{C}$ of $\text{Cl}(V, h)$:

$$\gamma^c_\mathbb{C} : \text{Cl}(V, h) \rightarrow \text{End}(\Sigma).$$
Note that we have a canonical inclusion $\text{Spin}^c(V, h) \subset \text{Cl}(V, h)$ and therefore $\text{Spin}^c(V, h)$ has an induced action on $\Sigma$. Restricting $\gamma_C$ to $\text{Cl}(V, h) \subset \text{Cl}(V, h)$ we obtain the unique complex irreducible representation of $\text{Cl}(V, h)$, which for simplicity we denote again by $\gamma_C$. Since the complex irreducible Clifford module $\Sigma$ is the complexification of the irreducible real Clifford module $\Sigma_{\mathbb{R}}$, $\Sigma$ admits a canonical $\text{Cl}(V, h)$ equivariant real structure defined in the usual way:

$$c : \Sigma \rightarrow \Sigma, \quad \xi \otimes z \mapsto \xi \otimes \bar{z}, \quad \xi \in \Sigma_{\mathbb{R}}, \quad z \in \mathbb{C}.$$ 

Furthermore, the results of Reference [1,3] (see also [42]), show that the real representation vector space $\Sigma_{\mathbb{R}}$ admits a unique non-degenerate symmetric inner product $\langle -, - \rangle$ such that:

$$\langle \gamma(v)\xi_1, \xi_2 \rangle = \langle \xi_1, \gamma(v)\xi_2 \rangle,$$

for all $\xi_1, \xi_2 \in \Sigma_{\mathbb{R}}$ and all $v \in V$. In particular, $\langle -, - \rangle$ is $\text{spin}(V, h)$-invariant and thus also $\text{Spin}_0(V, h)$-invariant, where $\text{Spin}_0(V, h) \subset \text{Spin}(V, h)$ denotes the connected component of $\text{Spin}(V, h)$ containing the identity. We $\mathbb{C}$-linearly extend $\langle -, - \rangle$ to $\Sigma$. As a $\mathbb{C}$-bilinear inner product in $\Sigma$, $\langle -, - \rangle$ is still $\text{Spin}_0(3, 1)$-invariant but fails to be $\text{Spin}_0^0(3, 1)$-invariant, since:

$$\langle [g, z]\xi, [g, z]\xi \rangle = z^2\langle \xi, \xi \rangle, \quad \forall [g, z] \in \text{Spin}_0^0(3, 1), \quad \forall \xi \in \Sigma.$$

The simultaneous existence of $\langle -, - \rangle$ and $c(-)$ gives rise to a canonical $\text{Spin}_0^0(3, 1)$-invariant Hermitian scalar product, defined as follows:

$$\langle \xi_1, \xi_2 \rangle = \langle \xi_1, c(\xi_2) \rangle, \quad \forall \xi_1, \xi_2 \in \Sigma.$$

From its definition and the properties of $\langle -, - \rangle$ we immediately conclude that:

$$\langle \gamma(v)\xi_1, \xi_2 \rangle = \langle \xi_1, \gamma(v)\xi_2 \rangle,$$

for all $\xi_1, \xi_2 \in \Sigma$ and all $v \in V$. We split $\Sigma = \Sigma^{1,0} \oplus \Sigma^{0,1}$ in terms of eigenspaces of the complex structure $\gamma(v)$, which we assume to be extended $\mathbb{C}$-linearly to $\Sigma$. We have the following isomorphisms of complex vector spaces:

$$\Sigma^{1,0} \simeq \Sigma_0^+, \quad \Sigma^{0,1} \simeq \Sigma_0^-.$$

Note that $\Sigma^{1,0}$ and $\Sigma^{0,1}$ are maximally isotropic complex subspaces of $\Sigma$ with respect to $\langle -, - \rangle$ and hence $\langle -, - \rangle$ is of split signature $(2, 2)$.

Let $\text{Cl}^{ev}(V, h) = \text{Cl}^{ev}(V, h) \otimes \mathbb{C}$ denote the even Clifford subalgebra of $\text{Cl}(V, h)$. The restriction of $\gamma_C$ to $\text{Cl}^{ev}(V, h)$:

$$\gamma_C : \text{Cl}^{ev}(V, h) \rightarrow \text{End}(\Sigma^{1,0} \oplus \Sigma^{0,1}),$$

preserves both $\Sigma^{1,0}$ and $\Sigma^{0,1}$. Therefore, the restriction of $\gamma_C$ to $\text{Cl}^{ev}(V, h)$ splits as a sum of the irreducible representations

$$\gamma_C^+ : \text{Cl}^{ev}(V, h) \rightarrow \text{End}(\Sigma^{1,0}), \quad \gamma_C^- : \text{Cl}^{ev}(V, h) \rightarrow \text{End}(\Sigma^{0,1}),$$

defined by projection of $\gamma_C$ on the corresponding factor. Note that we have complex isomorphisms:

$$\text{End}(\Sigma^{1,0}) \simeq \text{End}_\mathbb{C}(\Sigma_0^+), \quad \text{End}(\Sigma^{0,1}) \simeq \text{End}_\mathbb{C}(\Sigma_0^-).$$
We define the complex volume form \( \nu_C \stackrel{\text{def.}}{=} i \nu \in \Cl(V, h) \), which satisfies:
\[
\nu_C^2 = \text{Id}, \quad \nu_C \in Z(\Cl^{ev}(V, h)).
\]

In terms of \( \nu_C \), \( \Cl^{ev}(V, h) \) splits as the direct sum of two unital, associative complex algebras as follows:
\[
\Cl^{ev}(V, h) = \Cl^{ev}_+(V, h) \oplus \Cl^{ev}_-(V, h), \quad \Cl^{ev}_\pm(V, h) = \frac{1}{2} (1 \mp i\nu) \Cl^{ev}(V, h).
\]

For ease of notation we define the projectors \( P_\pm \stackrel{\text{def.}}{=} \frac{1}{2} (1 \mp i\gamma(v)) : \Cl^{ev}(v, h) \to \Cl^{ev}_\pm(V, h) \). We have the following isomorphisms of complex unital and associative algebras:
\[
\Cl^{ev}_+(V, h) \simeq \text{End}_C(\Sigma^+_0), \quad \Cl^{ev}_-(V, h) \simeq \text{End}_C(\Sigma^-_0).
\]

The restriction of \( \gamma_C \) to \( \Cl^{ev}(V, h) \), which splits as a direct sum in terms of the two inequivalent irreducible complex representations \( \gamma^{\pm}_C \) of \( \Cl^{ev}(V, h) \), factors through projection on the given factor of \( \Cl^{ev}(V, h) \) as follows:
\[
\gamma^{\pm}_C : \Cl^{ev}(V, h) \to \Cl^{ev}_\pm(V, h) \to \text{End}_C(\Sigma^{\pm}_0),
\]
\[
x \otimes z \mapsto z P_\pm(x) \mapsto (\text{Re}z \pm i\text{Im} \gamma(v)) \circ \gamma(x).
\]

Note that \( \gamma^{+}_C \) and \( \gamma^{-}_C \) are not complex conjugate of each other, although they induce isomorphic real representations of \( \Cl^{ev}(V, h) \) and are in fact complex-conjugate representations of the latter. Restricting \( \gamma^{\pm}_C \) to \( \text{Spin}^c(V, h) \subset \Cl^{ev}(V, h) \) we obtain two of the four irreducible representations of \( \text{Spin}^c(V, h) \) that will play a role in the mathematical formulation of four-dimensional chiral \( \mathcal{N} = 1 \) supergravity. We define:
\[
\tau^{\pm} \stackrel{\text{def.}}{=} \gamma_C|_{\text{Spin}^c(V, h)} : \text{Spin}^c(V, h) \to \text{Aut}_C(\Sigma),
\]
\[
\tau^{\pm}_C \stackrel{\text{def.}}{=} \gamma_C^{\pm}|_{\text{Spin}^c(V, h)} : \text{Spin}^c(V, h) \to \text{Aut}_C(\Sigma^{\pm}_0).
\]

The real structure \( c : \Sigma \to \Sigma \) intertwines the complex representations \( \tau^{\pm} \) through complex conjugation in the second factor, that is:
\[
c \circ \tau^{\pm}[g, z] = \tau^{\mp}[g, \bar{z}] \circ c, \quad \forall [g, z] \in \text{Spin}^c(V, h).
\]

The failure for \( \tau^+ \) and \( \tau^- \) to be complex-conjugate of each other is given by the outer-automorphism of \( \text{Spin}^c(V, h) \) defined by complex conjugation on the \( \text{U}(1) \) factor. For future reference, we introduce the following irreducible representations:
\[
\tau^{\pm}_c : \Cl^{ev}(V, h) \to \text{End}_C(\Sigma^{\pm}_0), \quad \tau^{\pm}_c([g, z]) \stackrel{\text{def.}}{=} \tau^{\pm}([g, \bar{z}]), \quad \forall [g, z] \in \text{Spin}^c(V, h),
\]

by composition of \( \tau^{\pm} \) with the complex-conjugation automorphism of \( \text{Spin}^c(V, h) \). With this definition we have \( \tau^{\pm}[g, z] = c \circ \tau^{\mp}_c[g, z] \circ c \) and hence \( \tau^{\mp}_c \) is the complex conjugate representation of \( \tau^{\pm} \). With respect to Clifford multiplication, the irreducible representations \( \tau^{\pm} \) and \( \tau^{\mp}_c \) relate as follows:
\[
\tau^{\mp}([g, z]) \circ \gamma(v) = \gamma(\text{Ad}_{g^{-1}}(v)) \circ \tau^{\pm}([g, z]),
\]
\[
\tau^{\mp}_c([g, z]) \circ \gamma(v) = \gamma(\text{Ad}_{g^{-1}}(v)) \circ \tau^{\pm}_c([g, z]),
\]
for all \( v \in V \) and \( [g, z] \in \text{Spin}^c(V, h) \). Therefore, Clifford multiplication by non-zero elements of \( V \) does not preserve the chiral splitting, or, in other words, Clifford multiplication by non-zero elements of \( V \) defines a complex linear map form \( \Sigma_0^\pm \) to \( \Sigma_0^\mp \). To summarize, we have introduced four inequivalent irreducible complex representations \( \tau^\pm \) and \( \tau_c^\pm \) of \( \text{Spin}^c(3, 1) \). Note that \( \tau^\pm \) is complex conjugate to \( \tau_c^\mp \) and therefore are equivalent as real representations.

**Remark 2.1.** In order to make contact with common physics jargon, we note that the real Clifford module \( \Sigma_{\mathbb{R}} \) is denoted as the *Majorana representation* in the physics literature, whereas the complex Clifford modules \( \Sigma \) and \( \Sigma^\pm \) are respectively called the (Lorentzian) four-dimensional *Dirac representation* and *Weyl representations*. Correspondingly, elements of \( \Sigma_{\mathbb{R}}, \Sigma \) and \( \Sigma^\pm \) are respectively called *Majorana spinors*, *Dirac spinors* and *Weyl spinors* in physics notation. The real structure \( c: \Sigma \rightarrow \Sigma \) corresponds with *charge conjugation* and it is extensively used in the physics literature to define Majorana spinors from Dirac spinors by imposing the appropriate reality condition. On the other hand, \( \gamma(v) \), namely the image of the volume form in \( \text{End}(\Sigma) \), is traditionally denoted as \( \gamma^{(5)} \), modulo a (possible imaginary) constant.

### 3. Bosonic Sector and Killing Spinor Equations

In this section we construct a global geometric model for the Killing spinor equations and bosonic sector of four-dimensional chiral \( \mathcal{N} = 1 \) supergravity \([14, 15]\), that is, \( \mathcal{N} = 1 \) supergravity coupled to an arbitrary number of chiral multiplets. We will refer to such theory as \( \mathcal{N} = 1 \) *chiral supergravity* or chiral supergravity for short.

#### 3.1. Chiral spinor bundles

Let \((M, g)\) be a connected and oriented Lorentzian four-manifold, with Lorentzian metric \( g \). We denote by \( \text{Cl}(M, g) \) the bundle of real Clifford algebras associated to \((M, g)\), whose typical fiber is the real Clifford algebra \( \text{Cl}(3, 1) \cong \text{Mat}(4, \mathbb{R}) \) of signature \((3, 1)\). The construction of supergravity crucially relies on the existence of a vector bundle over \( M \) equipped with Clifford multiplication, whose sections correspond with the *supersymmetry generators or parameters* of the theory. Hence, we will assume that \((M, g)\) is equipped with a bundle of irreducible complex Clifford modules, that is, a pair \((S, \gamma)\) where \( S \) is a complex vector bundle over \( M \) and:

\[
\gamma : \text{Cl}(M, g) \rightarrow \text{End}(S),
\]

is a morphism of bundles of unital associative algebras such that, for every point \( p \in M \), the restriction \( \gamma_p : \text{Cl}(M, g)_p \rightarrow \text{End}(S_p) \) of \( \gamma \) to the fiber of \( \text{Cl}(M, g) \) over \( p \) is an irreducible complex Clifford representation. The irreducibility condition is required in order to match the local degrees of freedom of the \( \mathcal{N} = 1 \) supersymmetry generator of the theory. Existence of \((S, \gamma)\) is obstructed. The obstruction was computed in References \([30, 45]\), where it was shown that \((S, \gamma)\) exists on \((M, g)\) if and only if \((M, g)\) admits a \( \text{Pin}^c(3, 1) \) structure, in which case there exists a unique (modulo isomorphisms) \( \text{Pin}^c(3, 1) \) structure \( Q^\text{Pin} \) on \((M, g)\) such that \( S \) is the vector bundle associated to \( Q^\text{Pin} \) through the tautological representation of \( \text{Pin}^c(3, 1) \subset \text{Cl}(3, 1) \otimes \mathbb{C} \). Recall that \((M, g)\) admits a \( \text{Pin}^c(3, 1) \) structure \( Q^\text{Pin} \) if and only if there exists a \( U(1) \) principal bundle \( P_{Q^\text{Pin}} \) such that:

\[
w_2(M) + w_1^-(M)^2 + w_1^-(M)w_1^+(M) = w_2(P_{Q^\text{Pin}}),
\]
where $w_2(M)$ denotes the second Stiefel-Whitney class of $M$, $w_1^-(M)$ denotes the first Stiefel-Whitney class of the bundle of time-like lines of $(M, g)$, $w_1^+(M)$ denotes the first Stiefel-Whitney class of the bundle of space-like planes of $(M, g)$ and $w_2(P_{\text{Spin}^c})$ denotes the second Stiefel-Whitney class of $P_{\text{Spin}^c}$, understood as an $\text{SO}(2) \simeq \text{U}(1)$ principal bundle. Let $\nu$ be the Lorentzian volume form on $(M, g)$. The complex volume form $\nu_C = i \nu$ acts as an involution on $S$ and hence induces a splitting

$$S = S^+ \oplus S^-,$$

of $S$ in terms of the so-called chiral bundles $S^+$ and $S^-$. Note that this splitting is not preserved by the full Clifford algebra but only by its even part. In fact, $S^+$ and $S^-$ are inequivalent bundles of irreducible complex Clifford modules over $\text{Cl}^{\text{ev}}(M, g)$, where $\text{Cl}^{\text{ev}}(M, g) \subset \text{Cl}(M, g)$ denotes the bundle of even Clifford algebras over $(M, g)$, with typical fiber isomorphic to $\text{Cl}^{\text{ev}}(3, 1) \simeq \text{Mat}(2, \mathbb{C})$. Furthermore, the $\text{Pin}^c(3, 1)$ structure $Q_{\text{Spin}^c}$ reduces to a $\text{Spin}^c(3, 1)$ structure $Q$ and we can understand $S^+$ and $S^-$ as being vector bundles associated to $Q$ by means of the two inequivalent tautological representations $\tau^\pm$ of $\text{Spin}^c(3, 1) \subset \text{Cl}^{\text{ev}}(3, 1) \otimes \mathbb{C}$ introduced in Sect. 2. Therefore, we can write:

$$S = Q \times_\tau \Sigma, \quad S^\pm = Q \times_{\tau^\pm} \Sigma_0^\pm.$$

**Remark 3.1.** The previous discussion shows that a necessary condition to formulate chiral $\mathcal{N} = 1$ supergravity on an oriented Lorentzian manifold $(M, g)$ is for $(M, g)$ to admit a $\text{Spin}^c(3, 1)$ structure.

For future reference, we define:

$$S^+_c \overset{\text{def}}{=} Q \times_{\tau^+_c} \Sigma_0^+, \quad S^-_c \overset{\text{def}}{=} Q \times_{\tau^-_c} \Sigma_0^-.$$

**Remark 3.2.** We have introduced four complex spinor bundles, namely $S^+$, $S^-$, $S^+_c$ and $S^-_c$, all of which are associated to the same $\text{Spin}^c(3, 1)$ structure $Q$ and all of which will play a relevant role in the formulation of chiral supergravity.

Given a $\text{Spin}^c(3, 1)$ structure $Q$ on $M$, we will denote by $P_Q$ its associated characteristic $\text{U}(1)$-bundle and by $L_Q$ its associated determinant complex line bundle. These are defined in terms of $Q$ as follows:

$$P_Q \overset{\text{def}}{=} Q \times_I \text{U}(1), \quad L_Q \overset{\text{def}}{=} Q \times_I \mathbb{C},$$

where $I : \text{Spin}^c(3, 1) \to \text{U}(1)$ is the homomorphism of groups defined through $I([g, z]) = z^2$ acting by left-multiplication on $\text{U}(1)$ and $\mathbb{C}$, respectively.

**Remark 3.3.** The space-time manifold $(M, g)$ may admit non-equivalent $\text{Spin}^c(3, 1)$ structures. The set of isomorphism classes of $\text{Spin}^c(3, 1)$ structures can be shown to have the structure of a torsor over the Abelian group $H^2(M, \mathbb{Z})$, see for example [27]. Different choices of $\text{Spin}^c(3, 1)$ structure yield, in principle, inequivalent chiral supergravity theories.

**Lemma 3.4.** The real structure $c : \Sigma_0^\pm \to \Sigma_0^{\mp}$ induces the following anti-isomorphism of complex bundles:

$$c : S^\pm \to S_c^{\mp}, \quad [q, \xi] \mapsto [g, c(\xi)],$$

which yields an isomorphism of bundles of real Clifford modules.
Proof. The fact that $c$ is well-defined as an anti-isomorphism of complex vector bundles follows from the anti-linearity of $c : \Sigma_0^\pm \to \Sigma_0^\mp$ and the following computation:

$$[q \cdot [g, z], c(\tau^\pm([g, z]^{-1})\xi)] = [q \cdot [g, z], \tau^\mp([g, z]^{-1})c(\xi)] = [q, c(\xi)],$$

which holds for all $[g, z] \in Spin^c(3, 1)$. Additionally, $c$ induces an isomorphism of bundles of real Clifford modules since $\gamma(v) \circ c = c \circ \gamma(v)$ for all $v \in TM$. □

**Proposition 3.5.** The following isomorphisms of complex line bundles hold:

$$L_Q \simeq \Lambda^2(S^+) \simeq \Lambda^2(S^-), \quad L_Q^{-1} \simeq \Lambda^2(S^+_c) \simeq \Lambda^2(S^-_c).$$

**Proof.** The result follows from the explicit form of the determinant representation of $\tau^\pm: Spin^c(3, 1) \to Aut(\Sigma_0^\pm)$ and $\tau^\pm_c: Spin^c(3, 1) \to Aut(\Sigma_0^\pm)$, which reads:

$$\det(\tau^\pm([g, z])) = z^2 \in U(1),$$
$$\det(\tau^\pm_c([g, z])) = z^2 \in U(1),$$

where we have used that $\det(\gamma(g)) = 1$ for all $g \in Spin(3, 1)$. □

To construct chiral supergravity it is convenient to endow $S$ with the sesquilinear pairing $\langle -, - \rangle$ and the bilinear pairing $\langle -, - \rangle$ introduced in Sect. 2 on the representation spaces $\Sigma$ and $\Sigma_0^\pm$. Since they are respectively invariant under $Spin_0^c(3, 1)$ and $Spin_0(3, 1)$ transformations, for this to be possible we need the $Spin^c(3, 1)$ structure $Q$ to further reduce to a $Spin_0^c(3, 1)$ structure. This is in general obstructed.

**Proposition 3.6.** A $Spin^c(3, 1)$ structure $Q$ on $(M, g)$ reduces to a $Spin_0^c(3, 1)$ structure if and only if:

$$w_1^-(M, g) = 0,$$

where $w_1^-(M, g)$ denotes the first Stiefel-Whitney class of the $O(1) \times O(3)$ reduction of the orthonormal frame bundle.

**Proof.** The group $Spin_0(3, 1)$ is the universal cover of $SO_0(3, 1)$, where $SO_0(3, 1)$ denotes the connected component of $SO(3, 1)$ containing the identity. The group $SO_0(3, 1)$ is in particular time- and space-orientation preserving. If $Q$ is a $Spin^c_0(3, 1)$ structure on $(M, g)$, its image inside the orthonormal frame bundle induces a reduction to $SO_0(3, 1)$, implying that $(M, g)$ is time-oriented and space-oriented. Hence $w_1^-(M, g) = 0$. To prove the converse, we use that $(M, g)$ is oriented and note that if in addition $w_1^-(M, g) = 0$, then the frame bundle admits a reduction to $SO_0(3, 1)$. This reduction lifts to a $Spin^c_0(3, 1)$ structure by taking its preimage through the projection of the $Spin^c(3, 1)$ structure $Q$ to the oriented orthonormal frame bundle of $(M, g)$. □

We will assume that the obstruction is absent and use for simplicity the same symbol $Q$ to denote the reduced $Spin^c_0(3, 1)$ structure. Note that $w_1^-(M, g) = 0$ is equivalent to $(M, g)$ being time-orientable, a condition which is usually assumed in Lorentzian geometry as part of the definition of a space-time, see for example [11]. In particular we will assume that, given a choice of Lorentzian metric $g$, $M$ is endowed with a fixed time-orientation and space-orientation.
We define:

\[ \mathcal{O} : S \times S \to L_Q, \quad [q, \xi_0 \otimes z] \times [q, \eta_0 \otimes w] \mapsto \langle \xi_0, \eta_0 \rangle z w \]  

\[ \mathcal{B} : S \times S \to \mathbb{C}, \quad [q, \xi_0 \otimes z] \times [q, \eta_0 \otimes w] \mapsto (\xi_0 \otimes z, \eta_0 \otimes w) = (\xi_0, \eta_0) z \bar{w}, \]  

for all \( q \in Q, \xi_0, \eta_0 \in \Sigma_{\mathbb{R}} \) and \( z, w \in \mathbb{C} \). Note that \( \mathcal{O} \) is complex bilinear and \( \mathcal{B} \) is a non-degenerate Hermitian inner product of split signature \((2, 2)\) for which the chiral bundles \( S^\pm \) are maximally isotropic vector sub-bundles of \( S \).

The tensor product connection of the Levi-Civita connection \( \nabla^g \) on \((M, g)\) with any connection \( \nabla_A \) on \( P_Q \) defines a connection on \( \text{Fr}_0^g(M) \times_M P_Q \) which we denote by \( \nabla^g_A \), where \( \text{Fr}_0^g(M) \) denotes principal \( SO_0(3, 1) \) bundle of restricted \( g \)-orthonormal frames. Hence \( \text{Fr}_0^g(M) \times_M P_Q \) is an \( SO_0(3, 1) \times U(1) \) principal bundle. The connection \( \nabla^g_A \) on \( \text{Fr}_0^g(M) \times_M P_Q \) canonically lifts to a connection on the \( \text{Spin}_0^c(3, 1) \) bundle \( Q \) which in turn induces a connection on \( S \) which preserves the chiral splitting \( S^\pm \). For simplicity, we denote the lifted connection by the same symbol.

We introduce now the description of \( \text{Spin}_0^c(3, 1) \) structures that will be used in the definition of chiral triple. Let \( M \) be an oriented four-manifold, and let \( P_{\text{Gl}_+} \) denote the associated principal \( \text{Gl}_+(4, \mathbb{R}) \) bundle of oriented frames, where \( \text{Gl}_+(4, \mathbb{R}) \) denotes the maximal connected subgroup of \( \text{Gl}(4, \mathbb{R}) \). We have \( \pi_1(\text{Gl}_+(4, \mathbb{R})) = \mathbb{Z}_2 \) and \( \text{Gl}_+(4, \mathbb{R}) \) fits in the following short exact sequence

\[ 1 \to \mathbb{Z}_2 \to \tilde{\text{Gl}}_+(4, \mathbb{R}) \xrightarrow{\tilde{\lambda}} \text{Gl}_+(4, \mathbb{R}) \to 1, \]

where \( \tilde{\text{Gl}}_+(4, \mathbb{R}) \) is the universal covering of \( \text{Gl}_+(4, \mathbb{R}) \) and \( \tilde{\lambda} \) denotes the associated cover map. We define the group \( \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \) as:

\[ \tilde{\text{Gl}}_+^c(4, \mathbb{R}) = \tilde{\text{Gl}}_+(4, \mathbb{R}) \cdot U(1) = (\tilde{\text{Gl}}_+(4, \mathbb{R}) \times U(1)) / \mathbb{Z}_2, \]

which fits into the following short exact sequence:

\[ 1 \to \mathbb{Z}_2 \to \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \xrightarrow{\tilde{\lambda}_c} \text{Gl}_+(4, \mathbb{R}) \times U(1) \to 1, \]

where \( \tilde{\lambda}_c \) denotes a \( \mathbb{Z}_2 \) cover map given by \([a, u] \mapsto (\tilde{\lambda}(a), u^2)\) for every \([a, u] \in \tilde{\text{Gl}}_+^c(4, \mathbb{R})\).

**Remark 3.7.** Note that \( \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \) is not a matrix group, since it does not admit any finite-dimensional faithful representation.

**Definition 3.8.** A \( \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \) structure on \( M \) is a pair \((P_{\tilde{\text{Gl}}_+^c}, \tilde{\lambda}_c)\) where \( P_{\tilde{\text{Gl}}_+^c} \) is principal bundle with structure group \( \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \) and \( \tilde{\lambda}_c \) is a principal bundle map:

\[ \tilde{\lambda}_c : P_{\tilde{\text{Gl}}_+^c} \to P_{\text{Gl}_+}, \]

satisfying:

\[ \tilde{\lambda}_c(qu) = \tilde{\lambda}_c(q) \tilde{\lambda}_c(u), \]

for every \( q \in P_{\tilde{\text{Gl}}_+^c} \) and every \( u \in \tilde{\text{Gl}}_+^c(4, \mathbb{R}) \).
To every $\tilde{\text{GL}}^c_+(4, \mathbb{R})$ structure we can associate a determinant complex line bundle $\tilde{L}$ and a characteristic principal $\text{U}(1)$ bundle $\tilde{P}$, just in the same way as we did for $\text{Spin}^c(3, 1)$ structures.

**Proposition 3.9.** A fixed choice of $\tilde{\text{GL}}^c_+(4, \mathbb{R})$ structure $(P_{\tilde{\text{GL}}^c_+}, \tilde{\Lambda}_c)$ on $M$ induces, for every Lorentzian metric $g$ such that $w^{-1}_1(M, g) = 0$, a canonical $\text{Spin}^c_0(3, 1)$ structure $Q$ on $(M, g)$. Furthermore the associated characteristic and determinant bundles are isomorphic:

$$\tilde{P} \simeq P_Q, \quad \tilde{L} \simeq L_Q.$$

**Proof.** A Lorentzian metric $g$ on $M$ induces a $\text{SO}^c_0(3, 1)$ reduction of the bundle of oriented frames of $M$, which, using that by assumption $w^{-1}_1(M, g) = 0$, further reduces to a $\text{SO}_0(3, 1)$ structure $\text{Fr}^g_0(M)$. Then:

$$\iota: \text{Fr}^g_0(M) \hookrightarrow P_{\text{GL}^c_+},$$

is a connected embedded submanifold and:

$$Q \overset{\text{def}}{=} \tilde{\Lambda}_c^{-1}(\text{Fr}^g_0(M)) \hookrightarrow P_{\tilde{\text{GL}}^c_+},$$

defines a reduction of $P_{\tilde{\text{GL}}^c_+}$ to a $\text{Spin}^c_0(3, 1)$ principal bundle $Q$. The restriction of $\tilde{\Lambda}_c$ to $Q$ defines a bundle covering map:

$$\Lambda_c: Q \rightarrow \text{Fr}^g_0(M),$$

satisfying the equivariance property with respect to $\lambda_c: \text{Spin}^c_0(3, 1) \rightarrow \text{SO}_0(3, 1)$ which makes the pair $(Q, \Lambda_c)$ into a $\text{Spin}^c_0(3, 1)$ structure on $(M, g)$. The isomorphisms $\tilde{P} \simeq P_Q$ and $\tilde{L} \simeq L_Q$ follow from the fact that the homomorphism:

$$\text{Spin}^c_0(3, 1) \rightarrow \text{U}(1), \quad [g, z] \mapsto z^2,$$

factors through the inclusion $\text{Spin}^c_0(3, 1) \hookrightarrow \tilde{\text{GL}}^c_+(4, \mathbb{R}).$ $\square$

The choice of a $\tilde{\text{GL}}^c_+(4, \mathbb{R})$ structure allows to define a canonical $\text{Spin}^c_0(3, 1)$ structure associated to every Lorentzian metric $g$. This fact will play a relevant role in the definition of chiral supergravity and the notion of chiral triple, see Definition 3.11.

### 3.2. Chiral triples and the scalar potential

Once we have described the spinor bundles that shall be used to formulate the geometric model for the Killing spinor equations of chiral supergravity, we need to introduce the concept of chiral triple on a complex manifold $\mathcal{M}$, which plays the role of target space for the non-linear sigma model of the theory. This is the so-called scalar manifold in the physical literature. Let us fix a complex manifold $\mathcal{M}$ of Kähler type, that is, admitting a Kähler metric, of real dimension $2n$. When necessary, we will denote the complex structure on $\mathcal{M}$ by $\mathcal{I}$. 

Definition 3.10. Let \((\mathcal{L}, \mathcal{H})\) be a Hermitian holomorphic line bundle over \(M\), with Hermitian structure \(\mathcal{H}\). We say that \((\mathcal{L}, \mathcal{H})\) is positive (or has positive curvature) if the Chern curvature \(\Theta\) associated to \(\mathcal{H}\) defines a Riemannian metric \(G\) on \(M\) through the following formula:

\[
2\pi G \overset{\text{def.}}{=} i \Theta \circ (\text{Id} \otimes I).
\]

We say, that \((\mathcal{L}, \mathcal{H})\) is negative (or has negative curvature) if its dual, equipped with the induced holomorphic and Hermitian structures, is positive. If \((\mathcal{L}, \mathcal{H})\) is negative, we define the associated positive-definite metric as follows:

\[
2\pi G \overset{\text{def.}}{=} -i \Theta \circ (\text{Id} \otimes I).
\]

To ease the notation we will sometimes denote the holomorphic Hermitian line bundle \((\mathcal{L}, \mathcal{H})\) simply by \(\mathcal{L}\).

Recall that:

\[
\mathcal{V} \overset{\text{def.}}{=} \frac{i}{2\pi} \Theta \in \Omega^{1,1}_\mathbb{Z}(M),
\]

defines a real \((1, 1)\) integral closed form whose associated de Rham cohomology class \([\mathcal{V}]\) is equal to:

\[
[\mathcal{V}] = (j \circ c_1)(\mathcal{L}) \in j(H^2(M, \mathbb{Z})).
\]

Here \(c_1 : \text{Pic}(M) \to H^2(M, \mathbb{Z})\) denotes the first Chern class map and \(j : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})\) denotes the canonical homomorphism mapping the singular integral cohomology of \(M\) into its de Rahm cohomology with coefficients in \(\mathbb{C}\). Given a holomorphic Hermitian line bundle \(\mathcal{L}_\mathcal{H}\) we denote the associated principal \(U(1)\) bundle by \(\mathcal{P}_\mathcal{H}\). Recall that the Chern connection \(D\) of \((\mathcal{L}, \mathcal{H})\) is induced by a unique connection on \(\mathcal{P}_\mathcal{H}\), which we denote by \(A\).

Definition 3.11. Let \(M\) be a complex manifold of Kähler type and \(M\) an oriented four-manifold. A chiral triple on \((M, \mathcal{M})\) is a tuple \(Q \overset{\text{def.}}{=} (\mathcal{L}, \mathcal{H}, \mathcal{W})\), where:

- \(\mathcal{L}_\mathcal{H} \overset{\text{def.}}{=} (\mathcal{L}, \mathcal{H})\) is a negative holomorphic Hermitian line bundle on \(M\) and \(\mathcal{P}_{\tilde{G}_\mathfrak{l}^c}\) is a \(\tilde{G}_\mathfrak{l}^c(4, \mathbb{R})\) structure on \(M\) for which there exists a smooth map \(\varphi : M \to M\) such that:

\[
\tilde{\mathcal{L}} \simeq \mathcal{L}^\varphi.
\]

That is, the determinant line bundle \(\tilde{\mathcal{L}}\) of \(\mathcal{P}_{\tilde{G}_\mathfrak{l}^c}\) is isomorphic with the pull back \(\mathcal{L}^\varphi\) by \(\varphi\) of the holomorphic line bundle \(\mathcal{L}\).

- \(\mathcal{W} \in H^0(\mathcal{L})\) is a holomorphic section of \(\mathcal{L}\), which is usually referred to as the superpotential in the physics literature.

For future reference, we note the following. There is a canonical embedding:

\[
\iota : \mathcal{P} = \mathcal{P}_{\tilde{G}_\mathfrak{l}^c} \times \mathbb{C} \to \tilde{\mathcal{L}} = \mathcal{P}_{\tilde{G}_\mathfrak{l}^c} \times \mathbb{C}, \quad [p, u] \mapsto [p, u],
\]
which is constructed by using the presentation of $\tilde{P}$ and $\tilde{L}$ as bundles associated to $P_{\tilde{G}_c}$. Furthermore, there is a canonical projection:

$$pr_{\mathcal{H}}^\varphi: \mathcal{L}^\varphi_{\mathcal{H}} \to \mathcal{P}^\varphi_{\mathcal{H}}, \quad l \mapsto \frac{l}{H^\varphi(l, l)^{\frac{1}{2}}},$$

and, for every choice of isomorphism $\Psi: \tilde{L} \xrightarrow{\sim} \mathcal{L}^\varphi$, there exists a canonical isomorphism of principal bundles:

$$\Psi_{\mathcal{H}}: \tilde{P} \xrightarrow{\sim} \mathcal{P}^\varphi_{\mathcal{H}},$$

defined as follows:

$$\Psi_{\mathcal{H}} \overset{\text{def.}}{=} pr_{\mathcal{H}}^\varphi \circ \Psi \circ \iota.$$

Conversely, every isomorphism $\Psi_{\mathcal{H}}: \tilde{P} \xrightarrow{\sim} \mathcal{P}^\varphi_{\mathcal{H}}$ induces a canonical isomorphism of complex line bundles $\Psi: \tilde{L} \xrightarrow{\sim} \mathcal{L}^\varphi$ in the usual way by using that $\tilde{L}$ and $\mathcal{L}^\varphi$ are complex line bundles associated to $P$ and $\mathcal{P}^\varphi_{\mathcal{H}}$. For simplicity we will denote $\Psi_{\mathcal{H}}$ simply by $\Psi$.

**Remark 3.12.** A chiral triple $\mathcal{Q}$ induces, for any choice of Lorentzian metric $g$ on the oriented four-manifold $M$ such that $\omega^{-1}(M, g) = 0$, a canonical Spin$^c(3, 1)$ structure $Q_g$, whose determinant line bundle and characteristic bundles are respectively isomorphic to those of $P_{\tilde{G}_c}$, see Proposition 3.9. For simplicity, we denote the Spin$^c(3, 1)$ structure $Q$ on $(M, g)$ simply by $Q$, with the implicit understanding that, given a Lorentzian metric $g$ on $M$ with $\omega^{-1}(M, g) = 0$, the chosen Spin$^c(3, 1)$ structure $Q$ on $(M, g)$ corresponds with the canonical Spin$^c(3, 1)$ structure determined by $\mathcal{Q}$ on $(M, g)$. In particular, we have:

$$P_Q \simeq \mathcal{P}^\varphi_{\mathcal{H}}, \quad L_Q \simeq \mathcal{L}^\varphi.$$

We will only consider Lorentzian metrics whose associated bundle of time-like lines is trivializable, that is, satisfying $\omega^{-1}(M, g) = 0$.

**Remark 3.13.** Note that a given Kähler manifold $\mathcal{M}$ need not admit, in general, any chiral triple for any four-manifold $M$. If $\mathcal{M}$ is compact, Kodaira’s embedding theorem implies that a necessary condition for $\mathcal{M}$ to admit a chiral triple is that it be projective, since it must admit a positive line bundle. Therefore, non-algebraic complex manifolds of Kähler type, such as non-algebraic tori or K3 surfaces, give a large number of compact Kähler manifold not admitting chiral triples. The existence of a chiral triple is also obstructed for non-compact Kähler manifolds. To see this, we use an argument of Verbitsky on the non-existence of positive holomorphic line bundles on certain non-compact Kähler manifolds. Consider for instance a K3 surface with no non-zero integral $(1, 1)$ classes and remove a point $p$. This is a non-compact Kähler manifold which does not admit positive line bundles. Indeed, by the previous assumptions, a positive line bundle would imply the existence of an exact positive form $\omega$. By Sibony’s Lemma, see [56, Theorem 5.1], such $\omega$ would be locally integrable around $p$. Therefore, applying Skoda-El Mir Theorem, the trivial extension of $\omega$ to the original K3 surface is a closed, positive and hence exact current. This is impossible, because the K3 surface is closed and Kähler.
In the following we will assume that \((M, \mathcal{M})\) does admit chiral triples. Given a chiral triple \(\Omega\) we denote by \(\text{Map}(\Omega) \subseteq C^\infty(M, \mathcal{M})\) the set of all smooth maps from \(M\) to \(\mathcal{M}\) satisfying condition (4) for \(\Omega\). Note that if \(\varphi \in \text{Map}(\Omega)\) then all maps in the same homotopy class as \(\varphi\) also belong to \(\text{Map}(\Omega)\). We will refer to the elements of \(\text{Map}(\Omega)\) as scalar maps. The differential of a scalar map is a real vector bundle map:

\[ d\varphi : TM \to T\mathcal{M}, \]

from \(TM\) to the real tangent bundle \(T\mathcal{M}\) of \(\mathcal{M}\). We can consider \(d\varphi\) as a map from \(TM\) to the complexification \(T\mathbb{C}\mathcal{M} = TM \otimes \mathbb{C}\) of \(T\mathcal{M}\) by composition with the canonical injection \(TM \hookrightarrow T\mathbb{C}\mathcal{M}\). Splitting:

\[ T\mathbb{C}\mathcal{M} = T^{1,0}\mathcal{M} \oplus T^{0,1}\mathcal{M}, \]

in the usual way by using \(\mathcal{I}\), we define:

\[ d\varphi^{1,0} : TM \to T^{1,0}\mathcal{M}, \quad d\varphi^{0,1} : TM \to T^{0,1}\mathcal{M}, \]

respectively by projection onto the factors \(T^{1,0}\mathcal{M}\) and \(T^{0,1}\mathcal{M}\). As we will see later, these projections will be necessary for the global formulation of the Killing spinor equations of chiral supergravity. A choice of chiral triple will unambiguously define a bosonic sector of \(\mathcal{N} = 1\) chiral supergravity and its associated Killing spinor equations. Let \(|\cdot|_{\mathcal{H}}\) denote the norm induced by \(\mathcal{H}\) on \(\mathcal{L}\). The Hermitian metrics \(\mathcal{G}\) and \(\mathcal{H}\) induce a Hermitian metric on the vector bundle \(T^*\mathbb{C}\mathcal{M} \otimes \mathcal{L}\), whose norm we denote by \(|\cdot|_{\mathcal{H}, \mathcal{G}}\).

**Definition 3.14.** The scalar potential associated to the chiral triple \(\Omega\) is the smooth real-valued function \(\Phi_k \in C^\infty(\mathcal{M})\) defined through:

\[ \Phi_k \overset{\text{def.}}{=} |D\mathcal{W}|^2_{\mathcal{H}, \mathcal{G}} - k |\mathcal{W}|^2_{\mathcal{H}}, \]

for a real constant \(k > 0\), where \(D\) denotes the Chern connection of \(\mathcal{L}\mathcal{H}\).

Given a scalar map \(\varphi : M \to \mathcal{M}\), we denote by \(\Phi_k^{\varphi} \overset{\text{def.}}{=} \Phi_k \circ \varphi\) the pull-back of \(\Phi_k\).

### 3.3. Bosonic sector and Killing spinor equations.

We are ready to introduce the equations of motion and Killing spinor equations defining the bosonic sector of \(\mathcal{N} = 1\) chiral supergravity associated to a particular chiral triple \(\Omega\). The equations of motion follow from a variational principle involving a Lorentzian metric \(g\) and a scalar map \(\varphi : M \to \mathcal{M}\).

**Definition 3.15.** The configuration pre-sheaf \(\text{Conf}_\Omega\) of \(\mathcal{N} = 1\) chiral supergravity associated to \(\Omega\) is defined as the pre-sheaf of sets that assigns to every open set \(U \subset M\) the following set:

\[ \text{Conf}_\Omega(U) \overset{\text{def.}}{=} \{(g, \varphi) \mid g \in L_0(U), \varphi \in \text{Map}_U(\Omega)\}, \]

where \(L_0(U)\) denotes the space of Lorentzian metrics on \(U \subset M\) such that \(w^{-1}(U, g) = 0\) and \(\text{Map}_U(\Omega)\) denotes the set of smooth maps from \(U\) to \(\mathcal{M}\) that satisfy condition (4).
Given a scalar map $\varphi \in \text{Map}(\Omega)$, we denote by $T\mathcal{M}^\varphi$ the pull-back of $T\mathcal{M}$ by $\varphi$. We equip $T\mathcal{M}^\varphi$ with the bundle pull-back metric $G^\varphi$ of $G$. Likewise, we respectively denote by $\mathcal{W}^\varphi$, $\mathcal{H}^\varphi$ and $\mathcal{D}^\varphi$ the bundle pull-backs of $\mathcal{W}$, $\mathcal{H}$ and $\mathcal{D}$ by $\varphi$. In general, for any section $s$ of a bundle over $\mathcal{M}$ we denote by $s^\varphi \triangleq s \circ \varphi$ the corresponding section of the pull-back bundle. We denote by $|\cdot|_g, G$ the norm induced by $g$ and $G^\varphi$ on $TM \otimes T\mathcal{M}^\varphi$ and associated tensor powers. Inspired by the local formulation of standard chiral $\mathcal{N} = 1$ supergravity \cite{25,52} we introduce the following definition.

**Definition 3.16.** The Lagrange density $\mathcal{Lag} : \text{Conf}_\Omega(U) \to \Omega^4(U)$ of chiral $\mathcal{N} = 1$ ungauged supergravity associated to the scalar manifold $(\mathcal{M}, \Omega)$ is given by:

$$\mathcal{Lag}[g, \varphi] = \left[ R_g - \frac{1}{2} |d\varphi|^2_{g, G} + \Phi_k^\varphi \right] \text{vol}_g, \quad (g, \varphi) \in \text{Conf}_\Omega(U),$$  \hfill (5)

for every open subset $U$ of $\mathcal{M}$, where $\text{vol}_g$ is the Lorentzian volume form of $(M, g)$ and $R_g$ is the scalar curvature of $g$.

The following Proposition follows from direct computation by using standard theory of variations, so we leave its proof to the reader.

**Proposition 3.17.** The Euler-Lagrange equations associated to the Lagrangian (5) is given by:

- The Einstein equations:

$$\mathcal{E}_E(g, \varphi) \triangleq G(g) - T(g, \varphi) = 0$$  \hfill (6)

where the energy-momentum tensor $T(g, \varphi) \in \Gamma(M, \otimes^2 T^*M)$ given by:

$$T(g, \varphi) = \varphi^*G - \frac{1}{2} \left( |d\varphi|^2_{g, G} + \Phi_k^\varphi \right) g.$$

and $G(g) = \text{Ric}^g - \frac{R^g}{2} g$ is the Einstein tensor.

- The scalar equations:

$$\mathcal{E}_S(g, \varphi) \triangleq \text{Tr}_g \nabla d\varphi - \frac{1}{2} (\text{grad}_G \Phi_k)^\varphi = 0,$$  \hfill (7)

where $\nabla$ denotes the connection induced by the Levi Civita connections on $(M, g)$ and $(\mathcal{M}, G)$,

for pairs $(g, \varphi) \in \text{Conf}_\Omega(M)$.

**Remark 3.18.** We denote by $\text{Sol}_\Omega \subset \text{Conf}_\Omega$ the pre-sheaf of solutions of (6) and (7).

We proceed now to introduce the Killing spinor equations of chiral $\mathcal{N} = 1$ supergravity.

**Proposition 3.19.** For every isomorphism $\Psi : \mathcal{L}_Q \cong \mathcal{L}^\varphi$ of complex line bundles, there exists canonical isomorphisms of complex vector bundles:

$$\mathcal{S}_\Psi : \mathcal{L}^\varphi \otimes S^\pm_c \cong S^\pm.$$
Proof. Using the identification $\Psi: L_Q \xrightarrow{\sim} \mathcal{L}^\phi$, such canonical isomorphism is given by:

$$\mathcal{T}_\Psi: \mathcal{L}^\phi \otimes S^\pm_c \xrightarrow{\sim} S^\pm, \{q, z\} \otimes \{g, \xi\} \mapsto \{q, z \xi\},$$

for all $\{q, \xi\} \in S^\pm_c = Q \times_{c^\pm} \Sigma^\pm_0$ and $\{q, z\} \in \mathcal{L}^\phi = Q \times I \mathbb{C}$. This is well-defined since:

$$\{q \{g, w\}, (w^{-2}z) (w g^{-1}) \xi\} = \{q \{g, w\}, w^{-1}g^{-1}z \xi\} = \{q, z \xi\},$$

where we have used that any representative of the class $\{q, z\} \in \mathcal{L}^\phi$ is of the form $(q \{g, w\}, w^{-2}z)$ and any representative of the class $\{q, \xi\} \in S^\pm_c$ is of the form $(q \{g, w\}, \tau^\pm_c ([g, w]^{-1}) \xi) = (q \{g, w\}, w g^{-1} \xi)$ for an element $\{g, w\} \in \text{Spin}_0(3, 1)$. \hfill \Box

Let $\varphi \in \text{Map}(\mathcal{Q})$ be a scalar map. For every superpotential $\mathcal{W}$, scalar map $\varphi$, Chern connection $\mathcal{D}$ and isomorphism $\Psi: \mathcal{L}^\phi \xrightarrow{\sim} L_Q$ we define the morphisms of real vector bundles:

$$\mathcal{C}^\Psi_{\mathcal{W}, \varphi}: TM \otimes S^\pm \rightarrow S^\pm, \quad \mathcal{C}^\psi_{\mathcal{W}, \varphi}: S^\pm \rightarrow \Lambda^{1,0}T^*\mathcal{M}^\phi \otimes S^\pm,$$

as follows:

$$\mathcal{C}^\psi_{\mathcal{W}, \varphi}(v \otimes \epsilon) \overset{\text{def.}}{=} v \cdot \mathcal{T}_\Psi (\mathcal{W}^\phi \otimes c(\epsilon)),$$

$$\mathcal{C}^\psi_{\mathcal{W}, \varphi}(\epsilon) = \mathcal{T}_\Psi ((\mathcal{D}\mathcal{W})^\phi \otimes c(\epsilon)), \quad \forall v \in TM, \forall \epsilon \in S^\pm,$$

where in the definition of $\mathcal{C}^\psi_{\mathcal{W}, \varphi}$ we have trivially extended $\mathcal{T}_\Psi$ to sections of $\mathcal{L}^\phi \otimes S^\pm_c$ taking values on $\Lambda^{1,0}T^*\mathcal{M}^\phi$. Here the dot denotes Clifford multiplication $TM \otimes S^\pm \rightarrow S^\pm$, which exchanges chirality. For every $\epsilon \in S^\pm$, evaluation in $\mathcal{C}^\psi_{\mathcal{W}, \varphi}$ defines the following vector bundle map:

$$\mathcal{C}^\psi_{\mathcal{W}, \varphi}(\epsilon): TM \rightarrow S^\pm, \quad v \mapsto \mathcal{C}^\psi_{\mathcal{W}, \varphi}(v \otimes \epsilon),$$

which we will consider as a section of $T^*M \otimes S^\pm$ in Eq. (8).

Remark 3.20. Recall that the superscript $\phi$ in $(\mathcal{D}\mathcal{W})^\phi$ denotes pull-back of $\mathcal{D}\mathcal{W}$ as a section of an abstract vector bundle instead of a one-form taking values on a complex line bundle. Hence, $(\mathcal{D}^{1,0}\mathcal{W})^\phi \in \Gamma(\Lambda^{1,0}T^*\mathcal{M}^\phi \otimes \mathcal{L}^\phi)$, where $\Lambda^{1,0}T^*\mathcal{M}^\phi$ denotes the vector bundle over $M$ obtained from $\Lambda^{1,0}T^*\mathcal{M}$ via bundle pull-back by $\varphi$.

Remark 3.21. For future reference, it is convenient to explicitly write the local form of the complex isomorphism $\mathcal{T}_\Psi$ defined above. Let $E: U \rightarrow Q$ denote a local section of $Q$, with $U \subset M$ open. The local section $E$ defines local frames $u$, $\{e^\pm_a\}_{a=1,2}$ and $\{e^\pm_a\}_{a=1,2}$ of $\mathcal{P}_{\mathcal{H}^\phi}$, $S^\pm$ and $S^\pm_c$, respectively. The local frames $\{e^\pm_a\}_{a=1,2}$ and $\{e^\pm_a\}_{a=1,2}$ can be chosen so that the following equation is satisfied:

$$c(e^\pm_a) = e^\mp_a, \quad a = 1, 2.$$

Let $l^\phi$ denote the pull-back by $\varphi$ of a local holomorphic frame $l$ of $\mathcal{L}$. Define $H^\phi_l \overset{\text{def.}}{=} \mathcal{H}^\phi(l^\phi, l^\phi)$. The local frame $u: U \rightarrow \mathcal{L}$ can be chosen such that:

$$u^\phi = (H^\phi_l)^{-\frac{1}{2}} l^\phi.$$
where we are considering $u$ as a unitary section of $(\mathcal{L}, \mathcal{H})$. The isomorphism of complex vector bundles $\mathcal{I}_{\psi}$ constructed in Proposition 3.19 can now be evaluated at the homogeneous element $l^\psi \otimes e_a^{\pm}$, yielding:

$$\mathcal{I}_{\psi}(l^\psi \otimes e_a^{\pm}) = (H_l^\psi)^{\frac{1}{2}} \mathcal{I}_{\psi}(u^\psi \otimes e_a^{\pm}) = (H_l^\psi)^{\frac{1}{2}} e_a^{\pm}.$$ 

Any section $\eta \in \Gamma(S^\pm)$ can be locally written as:

$$\eta = \eta^a e_a^{\pm},$$

for some local complex valued smooth functions $\eta^a$, $a = 1, 2$. By complex linearity of $\mathcal{I}_{\psi}$ we obtain:

$$\mathcal{I}_{\psi}(l^\psi \otimes \eta) = (H_l^\psi)^{\frac{1}{2}} \mathcal{I}_{\psi}(\eta^a u^\psi \otimes e_a^{\pm}) = (H_l^\psi)^{\frac{1}{2}} \eta^a e_a^{\pm},$$

which gives the local expression of $\mathcal{I}_{\psi}$. Similarly, let now $\epsilon \in \Gamma(S^\pm)$ be a section of $S^\pm$, which we write as $\epsilon = e_a^\pm e_a^\mp$ for some complex functions $e_a^\pm$ for $a = 1, 2$. We have:

$$\mathcal{I}_{\psi}(l^\psi \otimes c(\epsilon)) = (H_l^\psi)^{\frac{1}{2}} \mathcal{I}_{\psi}(u^\psi \otimes c(e^\mp e_a^\pm))$$

$$= (H_l^\psi)^{\frac{1}{2}} e_a^\pm \mathcal{I}_{\psi}(u^\psi \otimes e_a^{\pm})$$

$$= (H_l^\psi)^{\frac{1}{2}} \epsilon a e_a^\pm \in \Gamma(S^\mp),$$

This expression will be used in the proof of Proposition 3.27.

Given a chiral triple $\Omega$, a choice of isomorphism $\Psi : L_Q \to \mathcal{L}^\psi$ and a Lorentzian metric $g$ we have a canonical choice $\nabla^g_{\Psi^*, A^\psi}$ of connection on the spinor bundle $S$, which is constructed as follows. The choice of isomorphism $\Psi : L_Q \to \mathcal{L}^\psi$ of complex line bundles induces a canonical isomorphism of principal $U(1)$ bundles (which we denote by the same symbol) $\Psi : P_Q \to \mathcal{P}^\psi$. We use this isomorphism to take the pull-back of $A^\psi$ and define a connection $\Psi^*, A^\psi$ on $P_Q$. The connection $\Psi^*, A^\psi$ on $P_Q$ together with the Levi-Civita connection of $(M, g)$ yield, in the usual way through tensor product and lifting, a unique associated connection $\nabla^g_{\Psi^*, A^\psi}$ on $S$. For ease of notation we denote $\nabla^g_{\Psi^*, A^\psi}$ simply by $\nabla^\psi$. Recall that we defined $A$ as the unique unitary connection on $\mathcal{H}$ to which the Chern connection $\mathcal{D}$ on $\mathcal{L}_{\mathcal{H}}$ is associated in the standard way, and that the superscript $\psi$ denotes pull-back by $\psi$.

**Definition 3.22.** We define the extended configuration pre-sheaf $\text{Conf}_{\Omega, E}$ as the pre-sheaf of sets which assigns, to every open set $U \subset M$, the following set:

$$\text{Conf}_{\Omega, E} \overset{\text{def.}}{=} \{(g, \varphi, \epsilon, \Psi) \in \text{Conf}_\Omega(U) \times \Gamma(S^\mp|U) \times \text{Iso}(L_Q|U, L^\psi|U)\},$$

where $\Psi \in \text{Iso}(L_Q|U, L^\psi|U)$ is a $C^\infty$-isomorphism of complex line bundles over $U$.

**Definition 3.23.** Let $\Omega$ be a chiral triple on $(M, \mathcal{M})$. The Killing spinor equations (KSE) associated to $\Omega$ are defined as follows:

$$\nabla^\psi \epsilon = \mathcal{C}_{\Psi^*, \psi}(\epsilon), \quad (d_{\varphi^0, 1})^b \cdot \epsilon = \mathcal{C}_{\Psi^*, \mathcal{D}}(\epsilon),$$

for tuples $(g, \varphi, \epsilon, \Psi) \in \text{Conf}_{\Omega, E}$. Here the dot denotes Clifford multiplication $T^* M \otimes S \to S$ and $b$ the musical isomorphism $T^{0,1}_{\mathcal{M}} \Psi^\psi \simeq \Lambda^{1,0}_{T^* \mathcal{M}} \mathcal{G}^\psi$. A spinor $\epsilon \in \Gamma(S^\mp)$ satisfying Eq. (8) is called a supersymmetry spinor or supersymmetry generator.
Remark 3.24. In the definition of Killing spinor equations we have allowed only for supersymmetry spinors $\epsilon \in \Gamma(S^-)$ of negative chirality. In principle, either choice of chirality is allowed. The right choice exclusively depends on the conventions used: once the conventions have been fixed only one of the two chiralities, in our conventions negative chirality, will be compatible with the supersymmetric structure of the theory. This is the main reason why we have chosen to call this supergravity theory chiral, the other one being that the matter content of the theory we consider consists exclusively of chiral supermultiplets. The term chiral theory is however sometimes used in the physics literature with a different meaning which is worth clarifying. In the physics literature the term chiral supersymmetry/supergravity is used for cases where only supercharges of one chirality are used in formulating a theory, or, more generally, an unequal number of supercharges of both chiralities. Examples include type-IIB supergravity in ten dimensions and $(p, q)$-supersymmetry with $p \neq q$ in six and in two dimensions. Four-dimensional supersymmetry cannot be chiral in this sense, because when writing the superbracket in terms of chiral spinors one needs supercharges of both chiralities to have a non-trivial algebra. Formulating four-dimensional $\mathcal{N} = 1$ supersymmetry using Weyl spinors rather than Majorana spinors involves a conventional choice of which chirality represents the independent degrees of freedom. However the opposite chirality will also occur in the theory, albeit through dependent quantities. Thus $\mathcal{N} = 1$ supergravity would not be chiral in the sense applicable to some theories in two, six and ten dimensions. On the other hand, the term chiral in chiral supermultiplet refers to the fact that when formulating such a multiplet in superspace, the chiral superspace derivative is employed to impose a differential constraint on a general (super)function in order to obtain an irreducible representation. The resulting supermultiplet can then be written as a function on chiral superspace, involving only half of the odd coordinates. While this involves a choice of chirality, the opposite chirality will always be present in the full theory through complex-conjugated fields.

Definition 3.25. We define the pre-sheaf of supersymmetric configurations as the pre-sheaf of sets $\text{Conf}_{\Omega, S}$ which to every open subset $U \subset M$ assigns the following set:

$$\text{Conf}_{\Omega, S}(U) \overset{\text{def}}{=} \{(g, \varphi, \epsilon, \Psi) \in \text{Conf}_{\Omega, E}(U) \mid \text{KSE are satisfied}\}$$

We say that $(g, \varphi) \in \text{Sol}_{\Omega}(M)$ is a supersymmetric solution if there exists at least one spinor $\epsilon \in \Gamma(S^-)$ and isomorphism $\Psi \in \text{Iso}(L_Q, L^{\varphi})$ such that $(g, \varphi, \epsilon, \Psi) \in \text{Conf}_{\Omega, S}(M)$. We denote the pre-sheaf of supersymmetric solutions by $\text{Sol}_{\Omega, S}$.

Remark 3.26. Notice that we have not used the condition $w_{1^-}(M, g) = 0$. Indeed, this is not required to formulate neither the equations of motion nor the Killing spinor equations. However, it is crucial in order to construct the full supersymmetric Lagrangian, since it requires the use the sesquilinear pairing $\mathcal{B}$ introduced in Eq. (1), which is only invariant under $\text{Spin}_{10}(3, 1)$, compare Proposition 3.6.

The following Proposition shows that chiral $\mathcal{N} = 1$ supergravity and its associated Killing spinor equations, as introduced in Definitions 3.16 and 3.23, reproduce the well-known local formulation of four-dimensional ungauged $\mathcal{N} = 1$ supergravity coupled to chiral multiplets, which the reader can find explained in detail in References [25,52]. This result summarizes the underlying motivation for the structures and definitions introduced.

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2 We thank an anonymous referee of Communications in Mathematical Physics for writing the following detailed explanation.
so far, showing in addition that we have constructed an allowed global extension of the local formulas defining ungauged $\mathcal{N} = 1$ chiral supergravity in four Lorentzian dimensions.

**Proposition 3.27.** Let $\Omega$ be a chiral triple on $(M, \mathcal{M})$, $(U, x^\mu)$ a local coordinate chart on $M$ and $(V, w^i)$ a holomorphic coordinate chart on $\mathcal{M}$. The restriction of the bosonic Lagrangian

$$\mathcal{L}_{\text{ag}} : \text{Conf}_\Omega(U) \to C^\infty(U)$$

and Killing spinor equations to the chart $U$ are respectively given by:

$$\mathcal{L}_{\text{ag}}[g, \varphi]|_U = \left[ R_g - 2 \mathcal{G}_{ij}(z, \bar{z}) \partial_\mu z^i \partial_\mu \bar{z}^j - \Phi_k(z, \bar{z}) \right] \text{vol}_g,$$

and:

$$\nabla^g_{\mu} \epsilon = e^{K(z, \bar{z})/2} W(z) \gamma_\mu \bar{\epsilon}, \quad \mathcal{G}_{ik}(\partial_\mu \bar{z}^k) \gamma^\mu \epsilon = e^{K(z, \bar{z})/2} D_i W(z) \bar{\epsilon}, \quad (9)$$

where $\{z^i\}$ denotes the local coordinate expression of $\varphi$, $\mathcal{G}_{ij} = \mathcal{G}(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^j}) = \partial_{w^i} \partial_{\bar{w}^j} \mathcal{K}$ denotes the restriction of $\mathcal{G}$ to $U$, and $\mathcal{K}$ is an associated Kähler potential. Here we assume $\varphi(U) \subset V$. Hence, Definition 3.16 and Definition 3.23 locally reproduce the standard local bosonic sector and Killing spinor equations of $\mathcal{N} = 1$ chiral supergravity.

**Proof.** The complexified tangent bundle $T_C M$ is locally the complex span of $\{\partial w^i, \partial \bar{w}^i\}$, whereas the tangent bundle $TM$ is locally the real span of $\{\partial x^\mu\}$. We have:

$$d\varphi(\partial_\mu) = \partial_\mu z^i \partial_{w^i} + \partial_\mu \bar{z}^i \partial_{\bar{w}^i},$$

where for simplicity we denote by:

$$z^i \overset{\text{def}}{=} w^i \circ \varphi: U \subset \mathbb{R}^4 \to \mathbb{C},$$

the local components of $\varphi$. Hence:

$$(d\varphi)^{1,0}(\partial_\mu) = \partial_\mu z^i \partial_{w^i}, \quad (d\varphi)^{0,1}(\partial_\mu) = \partial_\mu \bar{z}^i \partial_{\bar{w}^i}.$$ 

Explicit computation gives:

$$|d\varphi|^2_{g, \mathcal{G}}|_U = 2 g^{\mu\nu} \partial_\mu z^i \partial_\nu z^k \mathcal{G}_{ik}(z, \bar{z}),$$

which corresponds with the standard sigma model appearing in the local formulation of chiral supergravity. Let $l: V \to \mathcal{L}$ be a holomorphic local trivializing section of $\mathcal{L}$. We set $K_l(w, \bar{w}) \overset{\text{def}}{=} \mathcal{H}(l, l) \in C^\infty(V)$ and write $\mathcal{W}|_V = W(w) l$ for some local holomorphic complex function $W(w)$ on $V$. We have:

$$|\mathcal{W}|^2_{\mathcal{H}(l)}|_V = K_l(w, \bar{w}) |W(w)|^2.$$

Furthermore, we write $Dl = Q \otimes l$ for the Chern connection $D$, where $Q \in \Omega^{1,0}(M)$ is a local $(1, 0)$ form, which in local coordinates $\{w^i\}$ is given by:

$$Q = \partial \log K_l(w, \bar{w}) = \partial_i \log K_l(w, \bar{w}) dw^i.$$
The associated Chern curvature is locally given by $\Theta(D) = -\partial\bar{\partial} \log K_I(w, \bar{w})$. By the definition of chiral triple, the two-form:

$$\frac{i}{2\pi} \partial\bar{\partial} \log K_I(w, \bar{w}),$$

is the Kähler form associated to $G$ and $I$. Hence, we can write:

$$\frac{i}{2\pi} \partial\bar{\partial} \log K_I(w, \bar{w}) = \frac{i}{2\pi} \partial\bar{\partial} \mathcal{K}(w, \bar{w}),$$

for some local Kähler potential $\mathcal{K}(w, \bar{w}) \in C^\infty(V)$. Integrating, we can express $K_I(w, \bar{w})$ uniquely up to the usual local Kähler transformations of $\mathcal{K}(w, \bar{w})$.

Let $\Phi_k = e^{\mathcal{K}(w, \bar{w})} [G_{ij}(w, \bar{w}) \mathcal{D}_i W(w) \bar{\mathcal{D}}_j \bar{W}(\bar{w}) - k W(w) \bar{W}(\bar{w})]$, which, for $k = 3$, corresponds with the standard potential of local chiral supergravity. Since by definition $\nabla^\phi$ is constructed by lifting the product connection of the Levi-Civita connection associated to $g$ and:

$$\nabla^\phi_{\mu}\epsilon = \nabla^g_{\mu}\epsilon + \frac{1}{2} A^\phi_{\mu}(z, \bar{z})\epsilon,$$

where $\nabla^g_{\mu}$ is the local lift to the spinor bundle of the Levi-Civita connection associated to $g$ and:

$$A^\phi_{\mu} = \frac{1}{2} (\partial_\mu z^i \partial_i \mathcal{K}(z, \bar{z}) - \partial_\mu \bar{z}^i \bar{\partial}_i \mathcal{K}(z, \bar{z})).$$

Note that the formula above gives the local form of the Chern connection with respect to a unitary frame instead of a holomorphic frame. This gives the local form of the standard $U(1)$-coupled covariant derivative appearing in the supersymmetry transformation of the gravitino of chiral supergravity. Let now $E : U \to Q$ denote a local section of $Q$, with $U \subset M$ open subset. As explained in Remark 3.21, the local section $E$ canonically defines local frames $u$, $\{e^\pm_a\}$ and $\{e^+_c\}$ of $S^\pm$ and $S^c$, respectively. We choose $E$ such that the following is satisfied:

$$u = \frac{l}{\mathcal{H}(l, l)^{\frac{1}{2}}}.$$

Let $l^\phi : U \to L^\phi$ be the pull-back by $\phi$ of the local holomorphic frame $l$ over $V \subset M$. Upon the use of Proposition 3.19 and Remark 3.21 we obtain:

$$\nabla^\phi_{\psi}(W^\phi \otimes c(\epsilon))|_U = W(z) \nabla^\phi_{\psi}(l^\phi \otimes c(e^a e^{-}_a))$$

$$= W(z) \tilde{e}^a \nabla^\phi_{\psi}(l^\phi \otimes e^+_a)$$

$$= W(z) e^{\mathcal{K}(z, \bar{z})/2} \tilde{e}^a e^+_a.$$
compare Remark 3.21. Since Clifford multiplication is complex linear and changes chirality we have:

\[ \gamma(\partial_\mu)(\bar{\epsilon}^a e_a^+ + a) = \bar{\epsilon}^a (\gamma_\mu)^b_a e_b^- = \gamma_\mu \bar{\epsilon}, \]

whence the first equation in (9) follows. Similarly, we have:

\[ T/\Psi_1((DW)\phi^c(\bar{\epsilon}))|U = D_i W(z) \bar{\epsilon}^a D_i (\gamma_\mu) b e_b^- = \bar{\epsilon}^a (\gamma_\mu) b e_b^- = \gamma_\mu \bar{\epsilon}. \]

Using now that:

\[ (d\phi^{0,1})^b \cdot \epsilon|_U = G_{ik} (\partial_\mu \bar{z}^k) \gamma^\mu \epsilon, \]

we conclude. □

**Remark 3.28.** Proposition 3.27 shows that the map \( \phi: M \to M \) encodes what in the physics literature is usually called the scalar fields of the theory, which for \( \mathcal{N} = 1 \) supergravity consist on an arbitrary number of complex scalar fields. Furthermore, it shows that the complex spinor \( \epsilon \) can be locally understood as a U(1)-coupled complex spinor field, a type of spinor field happening for instance in Quantum Electrodynamics.

### 3.4. Chiral supergravity on a spin four-manifold.

Under the assumption that the Spin\(_c(3,1)\) structure \( Q \) considered so far can be reduced to Spin\(_0(3,1)\) structure \( Q_0 \), we will show how now to formulate the Killing spinor equations in terms of the Dirac spinor bundle \( S_0 \) and a holomorphic line bundle \( R \) over \( M \). In the formalism of the previous subsection, the latter corresponds to a square root of the line bundle \( L \). As a consequence, the pull-back \( R^\phi \) is the square root of \( L^\phi \). The existence of such square root follows from the relation \( 0 = w_2(M) = c_1(L_Q) \mod 2 \).

Hence, let \( Q_0 \) be a Spin\(_0(3,1)\) structure on \( (M, g) \) and let \( S_0 \) denote the associated Dirac complex spinor bundle, which splits as \( S_0 = S_0^+ \oplus S_0^- \) in terms of the chiral spinor bundles \( S_0^\pm \). We denote by \((\cdot, \cdot)_0\) the Spin\(_0(3,1)\)-invariant Hermitian product on \( S_0 \). Contrary to what happened in the Spin\(_c(3,1)\) case, the spinor bundle \( S_0 \) admits now a real structure, that is, an antilinear and involutive isomorphism \( c_0: S_0^\pm \to S_0^\mp \). In this set up, the notion of chiral triple can be simplified in a way which we proceed to discuss. For the benefit of the reader we recall that, using the notion introduced in Sect. 2, \( S^0, S^+ \) and \( S^- \) are associated with the Spin\(_0(3,1)\) modules \( \Sigma, \Sigma^{0,1} \) and \( \Sigma^{1,0} \), respectively.

**Definition 3.29.** Let \( M \) be a complex manifold of Kähler type. A chiral triple defined on \( M \) is a triple \((\mathcal{R}, \mathcal{H}_\frac{1}{2}, \mathcal{W})\), where \((\mathcal{R}, \mathcal{H}_\frac{1}{2})\) is a negative (in the sense of Definition 3.10) Hermitian holomorphic line bundle, with Hermitian structure \( \mathcal{H}_\frac{1}{2} \), and \( \mathcal{W} \) is a holomorphic section of \( \mathcal{L} \stackrel{\text{def}}{=} \mathcal{R} \otimes \mathcal{R} \).

**Remark 3.30.** Note that the previous definition does not impose any conditions on the admissible scalar maps \( \phi: M \to M \) and, in fact, it is independent of \( M \) as long as the latter \( M \) is spin.
We equip $\mathcal{L}$ with the Hermitian form $\mathcal{H} \overset{\text{def}}{=} \mathcal{H}_1 \otimes \mathcal{H}_1$ induced by $\mathcal{H}_1$. Given any scalar map $\varphi : M \to \mathcal{M}$, we define the (chiral) supergravity spinor bundle associated to $\varphi$ as follows:

$$S^{\pm} \overset{\text{def}}{=} S_0 \otimes R^\varphi,$$

where $R^\varphi$ denotes the pull-back of $R$ by $\varphi$, which we endow with the pull-back Hermitian form $\mathcal{H}_1 = \mathcal{H}_1$. Likewise, we define:

$$S_c^{\pm} \overset{\text{def}}{=} S_0 \otimes (R^\varphi)^{-1}, \quad S_c^{\pm} \overset{\text{def}}{=} S_0^{\pm} \otimes (R^\varphi)^{-1},$$

Here we have used the symbols $S^{\pm}$ and $S_c^{\pm}$ to denote the spinor bundles that correspond to those denoted by same symbols in the general $\text{Spin}_0(3, 1)$ case. Using the Hermitian form $(-, -)_{0}$ present on $S_{0}$ together with the Hermitian form $\mathcal{H}_{1/2}$ on $R$, we define the Hermitian form $(-, -) = (-, -)_{0} \otimes \mathcal{H}_{1/2}^\varphi$ on $S$. In addition, we define an extension of $c_0 : S_0^{\pm} \to S_c^{\pm}$ to the supergravity spinor bundles $S^{\pm}$ and $S_c^{\pm}$ as follows:

$$c : S^{\pm} \to S_c^{\pm}, \quad \xi_0 \otimes l^\varphi \mapsto c_0(\xi_0) \otimes \mathcal{H}_{1/2}^\varphi (-, l^\varphi),$$

acting on homogeneous sections of $S^{\pm} = S_0^{\pm} \otimes R^\varphi$. With these definitions, we reproduce the action of $c$ as defined in the $\text{Spin}_0(3, 1)$ case, with the advantage that we can now isolate how it acts on each of the factors appearing in the definition of $S^{+}$ and $S^{-}$. The Killing spinor equations are now given formally by the same expression as in the general $\text{Spin}_0(3, 1)$ case, that is:

$$\nabla^\varphi \epsilon = \mathcal{C}_W^\Psi (\epsilon), \quad (d\varphi^{0, 1})^b \epsilon = \mathcal{C}_W^\varphi (\epsilon).$$

Note that in this situation the connection $\nabla^\varphi$ is an honest tensor product connection on the complex spinor bundle $S_0$ tensorized with a complex a line bundle.

**Example 3.31.** Let $(M, g)$ be a four-dimensional Lorentzian spin four manifold and let us take $\mathcal{M}$ to be the complex projective line $\mathbb{P}^1$. We have that $T\mathbb{P}^1 \simeq \mathcal{O}(2)$ is positive with respect to the Hermitian structure induced by the standard Fubini-Study metric $\mathcal{H}$ and thus $(T\mathbb{P}^1, \mathcal{H})$ satisfies the positivity notion given in Definition 3.10. Therefore, its dual bundle, that is, the canonical bundle $K_{\mathbb{P}^1}$ is negative and furthermore has Chern number $(-2)$, whence it admits a (unique) holomorphic square root $K^{1/2}_{\mathbb{P}^1}$. A series of chiral triples parameterized by a natural number $n \in \mathbb{N}_*$ is then given by $((K^{1/2}_{\mathbb{P}^1})^n, \mathcal{H}^{1/2}, \mathcal{W} = 0)$. We are forced to take $\mathcal{W}$ to be the zero-section since a negative line bundle over a closed complex manifold is well-known to not admit any non-zero holomorphic sections. In this situation, it is sometimes physically admissible to generalize the notion of superpotential and allow for meromorphic sections.

**Example 3.32.** Let $(M, g)$ be a four-dimensional Lorentzian spin four manifold and let $\mathcal{M}_\ell$ be a hyperbolic Riemann surface of genus $\ell$. We have deg$(T^{1,0}\mathcal{M}_\ell) = 2 - 2\ell < 0$, and thus $T^{1,0}\mathcal{M}_\ell$ is a negative holomorphic line bundle with respect to any Hermitian metric $\mathcal{H}_{1/2}$ of constant curvature $(-1)$. A family of chiral triples parameterized by a natural number $n \in \mathbb{N}_*$ is then given by $(\mathcal{R} = (T\mathcal{M}_\ell)^n, \mathcal{H}^{1/2}, \mathcal{W})$, where $\mathcal{W}$ is any holomorphic section of $(T\mathcal{M}_\ell)^{2n}$. 

\[ \mathcal{N} = 1 \text{ Geometric Supergravity and Chiral Triples on Riemann Surfaces} \]
We are now in disposition to show that every (possibly non-compact) complex manifold $\mathcal{M}$ admitting integral Kähler forms can be endowed with a chiral triple and thus can be considered as the target space of chiral $\mathcal{N} = 1$ supergravity on a spin manifold $M$. Let $(\mathcal{M}, \omega)$ be a Kähler-Hodge manifold, which we define as a complex manifold equipped with an integral Kähler form. Then, using a classical theorem of Weil [57], there exists a complex Hermitian line bundle $(L, \mathcal{H}, D)$ equipped with a unitary connection $D$ such that:

$$\omega = -\frac{i}{2\pi} \Theta(D),$$

where $\Theta(D)$ denotes the curvature of $D$. Since $\omega$ is of type $(1, 1)$ the previous equation implies $\Theta(D)^{0,2} = 0$ whence $D^{0,1}$ defines a holomorphic structure on $L$. Denote by $\mathcal{L}$ the corresponding holomorphic line bundle. Then $D$ is the Chern connection of $(\mathcal{L}, \mathcal{H})$, which becomes a negative Hermitian holomorphic line bundle, as required in order to define a chiral triple on $(M, \mathcal{M})$. Hence, if $(M, g)$ is spin, $\mathcal{M}$ can be considered as the target space of the non-linear sigma model of chiral $\mathcal{N} = 1$ supergravity.

**Remark 3.33.** The argument given above may not work if $(M, g)$ is not spin. If $(M, g)$ is not spin, we have to prove that there exists at least one integral Kähler two-form on $\mathcal{M}$ whose associated negative Hermitian holomorphic line bundle $(\mathcal{L}, \mathcal{H})$ is isomorphic via pull-back by some map $\varphi$ to the determinant line bundle of some $\text{Spin}^c_0(3,1)$ structure $Q$ on $(M, g)$. This may not be possible, as the following example shows.

**Example 3.34.** Take $(M, g)$ to be a non-spin Lorentzian manifold admitting $\text{Spin}^c_0(3,1)$ structures (for instance, $\mathbb{P}^2$ minus a point $p$), and let $\mathcal{M}$ be any non-compact Riemann surface. Recall that every holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$ is holomorphically trivial by [31, Theorem 30.3] and, on the other hand, by [31, Corollary 26.8] every such $\mathcal{M}$ is Stein and thus admits global subharmonic functions. Hence, every holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$ admits a Hermitian metric $\mathcal{H}$ of negative Chern-curvature. We conclude that the pull-back of any holomorphic line bundle $\mathcal{L}$, in particular any negative line bundle $(\mathcal{L}, \mathcal{H})$, is topologically trivial. Since $(M, g)$ is not spin, the trivial line bundle can never be isomorphic to the determinant line bundle of a $\text{Spin}^c_0(3,1)$ structure on $(M, g)$. Hence, even though $\mathcal{M}$ admits negative line bundles, the pair $(M, \mathcal{M})$ with $M$ as above does not admit any chiral triple. In particular, this means that the scalar manifold of a supergravity theory on a non-spin Lorentzian manifold cannot be an open Riemann surface.

### 3.5. Trivial scalar manifold

As an example of the general formulation introduced in the previous sections, we consider now the simplest case of chiral supergravity, which goes under the name of pure (AdS)$_{\mathcal{N}} = 1$ supergravity in the physics literature. Despite being the simplest case of chiral $\mathcal{N} = 1$ supergravity, the associated Killing spinor equations pose an interesting problem involving generalized Killing spinors [28,29] of a particular type, which we describe in this section. Pure (AdS)$_{\mathcal{N}} = 1$ supergravity is defined as the unique $\mathcal{N} = 1$ supergravity not coupled to any matter content, whence exclusively containing the gravitational supermultiplet. Accordingly, we take the scalar manifold $\mathcal{M} = \{p\}$ to be a point. Let $\Omega = (\mathcal{L}, \mathcal{H}, Q, \mathcal{W})$ be a chiral triple over $(M, \{p\})$. The fact that $\mathcal{M}$ is a point implies that $\mathcal{L}_\mathcal{H}$ is holomorphically trivial and can be identified with the one-dimensional Hermitian complex vector space $(\mathbb{C}, H)$, where $H$ denotes the standard Hermitian form on $\mathbb{C}$. Furthermore, the superpotential $\mathcal{W}$ becomes a complex number which we denote by $w$. In addition, the scalar map $\varphi$ is necessarily constant and the
pull-back of $\mathcal{L}_H$ is the trivial Hermitian line bundle over $M$, which we denote again by $(\mathbb{C}, H)$. Since, by definition of chiral triple, we must have $P_Q \simeq D^0_H$, we conclude that $L_Q$ is trivial and $D^0$ is the trivial connection, which in turn implies that the determinant bundles of $S^+$ and $S^-$ are trivial, compare Proposition 3.5. In particular, $w_2(L_Q) = 0$ whence $w_2(M) = 0$, and the Spin$^c_0(3, 1)$ structure $Q$ reduces to a Spin$^c_0(3, 1)$ structure $Q_0$. Hence, we consider that $S = S^+ \oplus S^-$ is the complex spinor bundle associated to $Q_0$ in the usual manner. With these provisos in mind, the scalar potential $\Phi = -k |w|^2$ becomes a non-positive constant and hence the Lagrangian of the theory reduces to the Hilbert-Einstein Lagrangian coupled to a non-positive cosmological constant, that is:

$$\mathcal{L}_{ag}[g] = R_g + k |w|^2,$$

where $w \in \mathbb{C}$ denotes the superpotential and $k$ is a positive real constant. The equations of motion associated to the previous functional read:

$$\text{Ric}(g) = -\frac{k}{2} |w|^2 g,$$

which are the standard Einstein equations coupled to a non-positive cosmological constant. The Killing spinor equations in turn reduce to:

$$\nabla^g_v \epsilon = w v \cdot \epsilon, \quad \forall v \in \mathfrak{X}(M),$$

where $\nabla^g$ denotes the lift of the Levi-Civita connection to the spinor bundle. Therefore, the set $\text{Sol}_S(M)$ of supersymmetric solutions on $M$ consists of pairs $(g, \epsilon)$, with $g$ a Lorentzian metric and $\epsilon$ a chiral spinor, such that:

$$\text{Sol}_S(M) = \left\{ (g, \epsilon) \mid \text{Ric}(g) = -\frac{k}{2} |w|^2 g, \quad \nabla^g_v \epsilon = w v \cdot \epsilon, \quad \forall v \in \mathfrak{X}(M) \right\}.$$

It is important to point out that Eq. (10) does not correspond to a standard Killing spinor equation [7,9,10,13] (for neither a real nor an imaginary Killing spinor) even if $w$ is real. This is due to the complex-conjugate bundle map $c$ appearing in (10). In order to see this explicitly, we define:

$$\epsilon_1 \overset{\text{def.}}{=} \epsilon \in \Gamma(S^-), \quad \epsilon_2 \overset{\text{def.}}{=} c(\epsilon) \in \Gamma(S^+).$$

From (10) we deduce that the spinors $\epsilon_1$ and $\epsilon_2$ satisfy:

$$\nabla^g_v \epsilon_1 = w v \cdot \epsilon_2, \quad \nabla^g_v \epsilon_2 = \bar{w} v \cdot \epsilon_1.$$  \hspace{1cm} (11)

Crucially, the second equation above involves the complex conjugate $\bar{w}$ of $w$ instead of $w$. This in turn implies that $|w|^2$, instead of $w^2$, appears in the Einstein constant of the corresponding integrability condition, which allows for $w$ to be any complex number instead of only real or purely imaginary, as it happens in the standard theory of Killing spinors [7,9,10,13].

Defining now the following complex endomorphism of $S$:

$$T_w : \Omega^0(S) \rightarrow \Omega^1(S), \quad T_w(\epsilon_1 \oplus \epsilon_2)(v) = v \cdot (w \epsilon_2 \oplus \bar{w} \epsilon_1),$$

we can rewrite the Killing spinor equations (11) as a particular case of a generalized Killing spinor equation:

$$\nabla^g \eta = T_w(\eta).$$  \hspace{1cm} (12)
where $\eta = \epsilon_1 \oplus \epsilon_2 \in \Gamma(S)$. Solutions to the Killing spinor equation (12) which satisfy $\epsilon_1 = \epsilon(\epsilon_2)$ are admissible supersymmetric solutions of $\mathcal{N} = 1$ supergravity with trivial scalar manifold. The study and classification of such solutions will be considered in a separate publication.

**Remark 3.35.** Note that Eq. (10) is required by supersymmetry, which motivates (11) as natural Killing spinor equations to study on a four-dimensional Lorentzian manifold.

### 3.6. Vanishing superpotential

Another particularly important special case of chiral supergravity is given by taking the superpotential $\mathcal{W} \in H^0(M, L)$ to be the zero section of the holomorphic line bundle $L$. When $\mathcal{W}$ is taken to be the zero section, the action functional of the theory reduces to:

$$\mathcal{L}_{ag}[g, \varphi] = R_g - |d\varphi|_{\mathcal{G}, g}^2 \ , \quad (g, \varphi) \in \text{Conf}_\Omega(M) ,$$

and therefore the theory reduces to Einstein gravity coupled to a non-linear sigma model with target space given by the complex manifold $\mathcal{M}$. The Killing spinor equations reduce in turn to:

$$\nabla^\varphi \epsilon = 0 \ , \quad d\varphi^{0,1} \cdot \epsilon = 0 ,$$

**Remark 3.36.** Note that taking $\mathcal{W} = 0$ the complex-conjugate map $c$ disappears from the Killing spinor equations and no longer plays a role in the formulation of the theory. This is the main source of simplification in this particular case, as the role played by $\epsilon$ in the formulation of the general theory is one of the genuine aspects brought by chiral local supersymmetry in four Lorentzian dimensions.

When $\mathcal{W} = 0$ Lorentzian manifolds $(M, g)$ equipped with a solution $(g, \varphi, \epsilon)$ to the equations above are particular instances of Lorentzian Spin$^c(3, 1)$ manifolds admitting parallel spinors. Simply connected and complete Lorentzian manifolds of this type have been studied and classified in the literature, see Reference [50] for the Riemannian case and Reference [39] for the pseudo-Riemannian case. We do not expect every Spin$^c(3, 1)$ manifold admitting a parallel spinor to admit a solution to the Killing spinor equations and, on the other hand, assuming completeness of $(M, g)$ rules out many physically interesting Lorentzian four-manifolds. Adapting the main Theorem of [39] to our situation we obtain the following result.

**Proposition 3.37.** Let $M$ be a geodesically complete and simply-connected Lorentzian four-manifold admitting a supersymmetric solution $(g, \varphi, \epsilon)$ to $\mathcal{N} = 1$ chiral supergravity with vanishing superpotential. Then we can have at most the following possibilities:

1. $(M, g)$ is isometric to four-dimensional flat Minkowski space.
2. $(M, g)$ is isometric to $(M, g) \simeq (\mathbb{R}^2 \times X, \delta_{1,1} \times h)$, where $\delta_{1,1}$ is the flat two-dimensional Minkowski metric and $X$ is a Riemann surface equipped with a Kähler metric $h$.
3. The holonomy group $H$ of $(M, g)$ is a subgroup of the parabolic subgroup $SO(2) \ltimes \mathbb{R}^2 \subset SO_0(3, 1)$. 


Remark 3.38. Every geodesically complete and simply connected supersymmetric solution must be of the form described by the previous proposition. However, the converse may not be true, since a supersymmetric solution requires \((M, g)\) to admit a parallel spinor with respect to the specific connection \(\nabla^\psi\), which is coupled to the scalar map \(\varphi\), which is in turn required to satisfy its corresponding Killing spinor equation. It is indeed an interesting open problem to classify which of the Lorentzian four-manifolds specified above can carry supersymmetric solutions of chiral supergravity. We will consider this problem in detail in Sect. 4 for the specific case (2) above, namely for the case in which \(M = \mathbb{R}^2 \times X\). Case (1) admits the obvious solution given by taking \(\varphi\) as a constant map.

4. Reduction to a Riemann Surface

Motivated by the structure of the candidates to supersymmetric solutions presented in Proposition 3.37, especially case (2), in this section we consider the reduction of \(\mathcal{N} = 1\) chiral supergravity to an oriented two-manifold \(X\) by assuming that the space-time manifold \(M\) is of the form:

\[
M = \mathbb{R}^2 \times X,
\]

and it is equipped with the product metric \(g_4 = \delta_{1,1} \times g\), where \(\delta_{1,1}\) denotes the flat Minkowski metric on \(\mathbb{R}^2\). The reduction is natural in the sense that the type of spinorial structures and Clifford modules coincide in dimension \((3, 1)\) and \((2, 0)\) by Clifford periodicity, since:

\[
(3 - 1) = (2 - 0) = 2 \mod 8,
\]

which corresponds to the real case of simple type in the standard classification of real Clifford algebras and their modules. Hence, all the structures introduced in the formulation of chiral \(\mathcal{N} = 1\) supergravity on a Lorentzian four-manifold \(M\) exist also on \(X\), and we can consider directly the formulation of the theory on \(X\). This would be equivalent to perform a reduction trivially along the \(\mathbb{R}^2\) factor in \(M\). We leave the details to the reader and proceed instead by directly formulating chiral supergravity on \(X\). Using the fixed orientation on \(X\), every Riemannian metric \(g\) on \(X\) defines a canonical complex structure \(J_g\) given by point-wise counter-clockwise rotation. We define \(Q_g\) to be the anti-canonical \(\text{Spin}^c(2)\) structure associated to \(g\), which means that the determinant line bundle of \(Q_g\) is the canonical bundle of \((X, g)\), compare [27] for the terminology. See Remarks 4.12 and 4.16 for the reasons behind this choice of \(\text{Spin}^c(2)\) structure. For each Riemannian metric \(g\), we define \(S\) to be the complex spinor bundle canonically associated to \(Q_g\). The spinor bundle \(S\) admits an explicit model given by:

\[
S = \Lambda^{*,0}(X),
\]

where the splitting is performed with respect to the complex structure \(J_g\). Clifford multiplication is given by:

\[
\beta \cdot \alpha = 2\beta^{1,0} \wedge \alpha + i(\beta^\sharp)^{1,0} \alpha,
\]

for all \(\alpha \in \Omega^{*,0}(X)\) and all \(\beta \in \Omega^1(X)\). Here we have set:

\[
\beta^{1,0} = \frac{1}{2}(\beta - i J_g \beta), \quad (\beta^\sharp)^{1,0} = \frac{1}{2}(\beta^\sharp - i J \beta^\sharp)
\]
with the musical isomorphism \( \sharp \) taken with respect to the metric \( g \). The determinant line bundle associated to the anti-canonical Spin\(^c\)(2) structure \( Q_g \) is given by the canonical bundle \( K_g \) of \((X, g)\):

\[
L_{Q_g} = K_g = \Lambda^{1,0}(X).
\]

The complex spinor bundle \( S \) is thus a complex vector bundle of rank two, which splits in the usual way:

\[
S = S^+ \oplus S^-,
\]

in terms of the chiral bundles \( S^+ \) and \( S^- \), of complex rank one. In the presentation \( S = \Lambda^{*,0}(X) \) of the spinor bundle, the chiral spinor bundles \( S^\pm \) respectively correspond with:

\[
S^+ \simeq \Lambda^{even,0}(X) \simeq \Lambda^{0,0}(X), \quad S^- \simeq \Lambda^{odd,0}(X) \simeq \Lambda^{1,0}(X),
\]

whereas the chiral spinor bundles \( S^\pm_c \) correspond with:

\[
S^+_c \simeq \Lambda^{0,1}(X), \quad S^-_c \simeq \Lambda^{0,0}(X).
\]

As required, we have:

\[
S^+ \simeq S^+_c \otimes K_g, \quad S^- \simeq S^-_c \otimes K_g.
\]

**Remark 4.1.** In the standard terminology used in the literature, \( S \) corresponds to the complex spinor bundle associated to the anti-canonical Spin\(^c\)(2) structure of \((X, g)\), whereas \( S_c \) corresponds to the complex spinor bundle associated to the canonical Spin\(^c\)(2) structure of \((X, g)\), see for example [27] for more details.

**Remark 4.2.** Recall that we have a canonical isomorphism of real vector bundles:

\[
K_g = \Lambda^{1,0}(X) \simeq T^*X,
\]

where \( T^*X \) denotes the real cotangent bundle of \( X \). Hence, the isomorphism type of \( K_g \) as a real vector bundle does not depend on \( g \). Not only this, the isomorphism type of \( K_g \) as a \( C^\infty \) complex line bundle does not depend on \( g \) either. To see this, note that if \( X \) is compact the Chern number of \( K_g \) is minus the Euler characteristic of \( X \) whereas if \( X \) is open every holomorphic line bundle over \((X, g)\), in particular \( K_g \), is holomorphically trivial.

In the set-up introduced above, the notion of chiral triple, see Definition 3.11, simplifies. This is due to the fact that, by assumption, we have established a canonical choice of complex structure and Spin\(^c\)(2) structure for every Riemannian metric \( g \) on \( X \), whose associated characteristic line bundle is \( K_g \). More precisely, since the isomorphism class of \( K_g \) as a complex line bundle does not depend on \( g \) and furthermore the definition of chiral triple only requires \( L^\psi \) to be \( C^\infty \) isomorphic to \( K_g \), we can define the isomorphism of complex line bundles \( \Psi \) between \( L^\psi \) and the determinant line bundle \( K_g \) of the given Spin\(^c\)(2) structure independently of the metric \( g \). Consequently, we arrive to the following simplification of a chiral triple, which we proceed to define.
**Definition 4.3.** A chiral triple \((\mathcal{L}, \mathcal{H}, \mathcal{W})\) at \((X, \mathcal{M})\) consists of a negative Hermitian holomorphic line bundle \((\mathcal{L}, \mathcal{H})\) and a holomorphic section \(\mathcal{W} \in H^0(\mathcal{M}, \mathcal{L})\) such that there exists a map \(\phi : X \to \mathcal{M}\) and a metric \(g\) on \(X\) for which:

\[ K_g \simeq \mathcal{L}^g, \]

as complex line bundles.

We fix a chiral triple \(\Omega = (\mathcal{L}, \mathcal{H}, \mathcal{W})\) on \((X, \mathcal{M})\) and consider the associated chiral supergravity on \(X\). The fact that \(L_Q = K_g\) implies, directly from the definition of chiral triple, that the pull-back of the holomorphic line bundle \(\mathcal{L}\) by a scalar map must be \(C^\infty\)-isomorphic to \(L_Q\):

\[ L_Q = K_g \simeq \mathcal{L}^g. \]

For every choice of isomorphism \(\Psi : K_g \to \mathcal{L}^g\), we endow \(K_g\) with the pull-back connection \(\Psi^\ast D^g\) with respect to \(\Psi\), where \(D\) denotes the Chern connection on \(\mathcal{L}_\mathcal{H}\) and \(D^g\) its pull-back by \(\phi\). For ease of notation, we will sometimes denote \(\Psi^\ast D^g\) simply by \(D^g\). Recall that the definition of chiral triple also establishes the existence of a \(C^\infty\) isomorphism:

\[ P_Q \simeq P_{\mathcal{H}^g}. \]

Indeed, every choice of isomorphism \(\Psi : K_g \to \mathcal{L}^g\) induces a canonical isomorphism between \(P_Q\) and \(P_{\mathcal{H}^g}\), which we denote for simplicity by the same symbol, namely \(\Psi : P_Q \to P_{\mathcal{H}^g}\). For the type of complex spinor \(S\) we are considering, which is associated to the anti-canonical Spin\(^c\)(2) structure of \((X, g)\), \(P_Q\) corresponds with the principal bundle of unitary coframes defined by \(g^\ast\) (the dual of \(g\)) on \(T^\ast X\). On the other hand, the principal U(1) bundle \(P_{\mathcal{H}^g}\) corresponds to the U(1) reduction induced on \(L_Q\) by the metric \(\Psi^\ast H^g\). The isomorphism of principal U(1) bundles \(\Psi : P_Q \to P_{\mathcal{H}^g}\) gives an isomorphism between these two reductions.

Using \(\Psi\) we equip \(P_Q\) with the pull-back connection \(\Psi^\ast A^g\), where \(A^g\) denotes the pull-back by \(\phi\) of the U(1) connection \(A\) associated to the Chern connection \(D\). Again, for ease of notation, we will sometimes denote \(\Psi^\ast A^g\) simply by \(A^g\). Lifting the Levi-Civita connection \(\nabla^g\) associated to \(g\) and \(A^g\) to the spinor bundle \(S\) we obtain a connection \(\nabla^g\) on \(S\) which we denote by \(\nabla^g\). Furthermore, using the isomorphism \(\Psi : K_g \to \mathcal{L}^g\), the pull-backed superpotential \(\mathcal{W}^g\) can be identified with a complex one-form on \(X\) of \((1, 0)\) type. This complex one-form is in principle not holomorphic since \(\phi\) is only assumed to be a \(C^\infty\)-map. The formulas defining the action functional and Killing spinor equations of the theory are formally the same as in \((3, 1)\) dimensions, namely:

\[ \mathcal{L}_{\text{lag}}[g, \phi] = R_g - |d\phi|^2_{\mathcal{L}_g} - \Phi^g_k \quad \text{for tuples} \quad (g, \phi) \in \text{Conf}_\Omega(X), \]

\[ \nabla^g \epsilon = C_{W, \phi}^\Psi(\epsilon), \quad (d\phi^0)^b \cdot \epsilon = C_{W, \phi}^{\Psi, D}(\epsilon), \quad (14) \]

for tuples \((g, \phi, \epsilon, \Psi) \in \text{Conf}_{\Omega, E}\). We remind the reader that the vector bundle maps \(C_{W, \phi}^{\Psi} : S^+ \to T^\ast X \otimes S^\ast\) and \(C_{W, \phi}^{\Psi, D} : S^+ \to \Lambda^1 \Lambda^1 T^\ast \mathcal{M}^\phi \otimes S^\ast\) are defined as follows:

\[ C_{W, \phi}^{\Psi} \epsilon(v) \overset{\text{def}}{=} v \cdot \mathcal{L}_\phi ((\mathcal{W}^g \otimes \epsilon)) \quad \forall v \in \mathcal{X}(X), \quad C_{W, \phi}^{\Psi, D} \epsilon = \mathcal{L}_\phi ((D\mathcal{W})^g \otimes \epsilon), \]
for $\epsilon \in \Gamma(S^+)$. Here $\Sigma_\Psi : \mathcal{L}^\theta \otimes S^\pm_c \rightarrow S^\pm_c$ denotes the canonical isomorphism constructed in Proposition 3.19. For the reasons behind the choice of positive chirality for the supersymmetry generator $\epsilon$ we refer the reader to Remark 4.16. The first Killing spinor equation in (14) can be written simply as:

$$D \epsilon = 0.$$ 

in terms of the following real-linear connection $D$ on the real rank-two vector bundle $S$:

$$D \equiv \nabla^\theta - \mathcal{C}^\Psi_{\mathcal{W},s} : S \rightarrow S.$$ 

Hence, if $\epsilon$ is non-zero at some point (which we will assume in the following) it will be non-zero everywhere. Note however that given such $\epsilon$, the section $i\epsilon \in \Gamma(S)$ may be non-parallel even if $\epsilon$ is. Using the fact that $X$ is a Riemann surface, the Einstein equations (6) of the theory drastically simplify and can be written as follows:

$$\varphi^* G = \frac{|d\varphi|^2_{g, g}}{2} g, \quad \Phi_\varphi = 0,$$

where we have used that $G(g)$ is identically zero and $\text{Tr}_g (\varphi^* G) = |d\varphi|^2_{G, g}$. Vanishing of the potential $\Phi_\varphi$ is equivalent to:

$$|DW|^2_{\mathcal{H}, g} \circ \varphi = c |\mathcal{W}|^2_{\mathcal{H}} \circ \varphi.$$

Condition $\Phi_\varphi = 0$ does not imply in general neither $\Phi_k = 0$ nor $d\Phi_k = 0$ identically on $\mathcal{M}$. Hence, the equation of motion for $\varphi$ does not reduce in general to the standard harmonicity condition and instead we have:

$$(\text{Tr}_g \nabla d\varphi)^b = \frac{1}{2} d\Phi_k \circ \varphi$$

for $\varphi : X \rightarrow \mathcal{M}$. Therefore, maps $\varphi : X \rightarrow \mathcal{M}$ satisfying the equations of chiral $\mathcal{N} = 1$ supergravity on $X$ are particular instances of harmonic maps with potential [22,47]. Because of this, we will call solutions $\varphi$ to the previous equation harmonic maps with potential $\Phi_k$ (rather than $\frac{1}{2} \Phi_k$ as in Reference [22]). We obtain:

**Corollary 4.4.** Let $\Omega$ be a chiral triple and let $(X, \mathcal{M})$ be an oriented two-manifold. A pair $(g, \varphi)$ satisfies the equations of chiral supergravity associated to $\Omega$ on $X$ if and only if:

$$\varphi^* G = \frac{|d\varphi|^2_{G, g}}{2} g, \quad |DW|^2_{\mathcal{H}, g} \circ \varphi = c |\mathcal{W}|^2_{\mathcal{H}} \circ \varphi, \quad (\text{Tr}_g \nabla d\varphi)^b = \frac{1}{2} d\Phi_k \circ \varphi.$$ 

(15)

In particular, if $d\Phi_k \circ \varphi = 0$ then $\varphi : X \rightarrow \mathcal{M}$ is a harmonic map.

**Remark 4.5.** Recall that condition $\Phi_k^\varphi = 0$ implies:

$$\varphi(X) \subset \Phi_k^{-1}(0) \subset \mathcal{M},$$

In principle, the zero level set of $\Phi_k$ may not be a smooth $(2n - 1)$-dimensional submanifold of $\mathcal{M}$, since it is not guaranteed that 0 be a regular value of $\Phi_k$. However, regularity of the critical points of $\Phi_k$ is indeed of physical relevance in relation for example with the stabilization of moduli in string theory compactifications.
The following proposition settles the classification of solutions to chiral supergravity on $X$ in the simple case in which $\phi$ is the constant scalar map.

**Proposition 4.6.** Let $k > 0$ and $X$ connected. A pair $(g, \phi)$ with $d\phi = 0$ is a solution to Equations (15) if and only if $\Phi_k(q) = 0$ and $d\Phi_k|_q = 0$, where $q$ is the constant value of $\phi$. In particular, a $\mathcal{N} = 1$ chiral supergravity with chiral triple $\Omega$ admits solutions with constant scalar map if and only if $0 \in \mathbb{R}$ is a critical value of the scalar potential $\Phi_k : M \to \mathbb{R}$.

**Proof.** Condition $d\phi = 0$ implies that the first equation (15) is automatically solved for any metric $g$ on $X$. The second equation in (15) is equivalent with $\Phi_k|_q = 0$, whereas the third equation in (15) is equivalent with $d\Phi_k|_q = 0$. $\square$

The previous corollary recovers the well-known fact that gravity in two dimensions is topological, which translates into the fact that in dimension two every metric satisfies the vacuum Einstein equations. In the following we will use the symbol $|d\phi|^2_{g,\mathcal{G}} > 0$ to denote that $|d\phi|^2_{g,\mathcal{G}}$ is nowhere vanishing.

**Lemma 4.7.** Let $|d\phi|^2_{g,\mathcal{G}} > 0$. If $(g, \phi)$ is a solution to the first equation in (15) then $\phi: (X, g) \to (M, \mathcal{G})$ is a conformal immersion.

**Proof.** Since by assumption $|d\phi|^2_{g,\mathcal{G}} \neq 0$ everywhere, the first equation in (15) implies that $\phi^*\mathcal{G}$ is conformal to $g$ and therefore $\phi$ is a conformal immersion. $\square$

**Proposition 4.8.** Let $\Omega$ be a chiral triple with vanishing superpotential $\mathcal{W}$. A pair $(g, \phi)$ with $|d\phi|^2_{g,\mathcal{G}} > 0$ is a solution to the associated chiral supergravity if and only if $\phi$ is a minimal immersion of $X$ into $M$.

**Remark 4.9.** We remind the reader that a minimal immersion is defined as a critical point of the area functional among compactly supported variations, see for example [20].

**Proof.** If $W$ is the zero section then $\Phi_k$ is identically zero and thus $(\text{grad}_g \Phi_k)^\phi = 0$, implying that the scalar equation for $\phi$ reduces to the harmonicity condition:

$$\text{Tr}_g \nabla d\phi = 0.$$ 

Hence, using Lemma 4.7, it follows that $\phi: X \to M$ is a harmonic conformal immersion, which for a two-dimensional source is equivalent with $\phi$ being a minimal immersion [20,21]. $\square$

For every solution $(g, \phi)$ that does not satisfy condition $|d\phi|^2_{G,\mathcal{G}} > 0$ there exists a non-empty closed subset $C \subset X$ of $X$ at which $|d\phi|^2_{G,\mathcal{G}}|_C = 0$. Since $g$ is by assumption a non-degenerate Riemannian metric, the Einstein equation implies that the real rank of $d\phi$ is zero over $C$ and two over $X \setminus C$. The critical set of $\phi$ coincides with $C$ and does not contain points at which the rank of $\phi$ is one. When restricted to $C$, the Einstein equation of chiral supergravity is automatically satisfied independently of the restriction of $g$ to $C$:

$$\phi^*\mathcal{G}|_C = 0, \quad |d\phi|^2_{G,\mathcal{G}}|_C = 0.$$

Fixing $\phi$, the Einstein equation becomes a quadratic algebraic equation $\phi^*\mathcal{G} = \frac{|d\phi|^2_{G,\mathcal{G}}}{2} g$ for $g$ and can in principle be used to determine $g$ on $X \setminus C$. However, over $C$ the Einstein is
For chiral supergravities of possibly non-vanishing superpotential $W$ and references therein. On existence and non-existence of such maps are by now available, see for example and holomorphic maps have been extensively studied in the literature, and many results on a Riemann surface into Kähler manifold is a minimal immersion. Minimal immersions if it is a minimal immersion. It is well-known that every holomorphic immersion of a Riemannian manifold $\phi$ that equations (15) are satisfied on the interior of $X$ such that $\phi$. The formulation introduced in this Section of chiral supergravity on $X$ such that $\phi$ and the set of non-constant conformal immersions are to be the associated complex spinor bundle through the tautological representation of Spin$^c(2)$. It is in principle possible

\[ L_{ab}(p) \overset{\text{def.}}{=} \partial_a \phi^A(p) \partial_b \phi^B(p) G_{AB}(p), \]

where $\phi^A$, $G^A$ are the local expressions of the components of $\phi$ and $G$ in the given coordinates $\{x^a\}$ and $\{\phi^A\}$. Note that $\{L_{ab}(p)\}$ is a matrix of real numbers. After a quick manipulation, the Einstein equation evaluated at $p$ can be written as follows:

\[ L_{ab}(p) \det(g(p)) = \frac{g_{ab}(p)}{2} \left( (g_{22}(p)L_{11}(p) + g_{11}(p)L_{22}(p) - 2g_{12}(p)L_{12}(p)) \right), \]

which is a non-trivial system of quadratic equations for the components of $g_{ab}$ if $p \in X\setminus C$.

Let $\phi : X \to M$ be a smooth map with critical set $C$. Let $g^\alpha$ be a smooth metric on $X\setminus C$ such that equations (15) are satisfied on the interior of $X\setminus C$. A natural question is: can $g^\alpha$ be extended to a smooth metric $g$ on $X$ such that $(g, \phi)$ is a solution of equations (15) on the whole $X$? The answer to this question highly depends on the properties enjoyed by $C$. If $\mathcal{W}$ is not the zero section, we have little control over $C$, which in principle could even have interior points. Nonetheless, in Sect. 4.1 we will see that for supersymmetric solutions there always exists a smooth extension to $X$ canonically induced by the chiral triple of the theory.

Remark 4.10. As previously mentioned, a smooth map $\phi : X \to M$ from a Riemann surface into a Riemannian manifold is a harmonic conformal immersion if and only if it is a minimal immersion. It is well-known that every holomorphic immersion of a Riemann surface into Kähler manifold is a minimal immersion. Minimal immersions and holomorphic maps have been extensively studied in the literature, and many results on existence and non-existence of such maps are by now available, see for example [20,21] and references therein.

For chiral supergravities of possibly non-vanishing superpotential $W^\phi \neq 0$ we obtain the following corollary, which characterizes the space of solutions of chiral $\mathcal{N} = 1$ supergravity on $X$ satisfying $|d\phi|^2_{G,g} > 0$ as maps $\phi : X \to M$.

Corollary 4.11. There is a canonical bijection between the set of solutions of chiral $\mathcal{N} = 1$ supergravity on $X$ such that $|d\phi|^2_{G,g} > 0$ and the set of non-constant conformal harmonic immersions $\phi : (X, g) \to (M, G)$ with potential $\Phi_k$ vanishing along $\phi$.

If $\mathcal{W}$ is the zero section, the $\Phi_k = 0$ on $M$, and hence given a chiral triple $\Omega$ on $(X, M)$ with vanishing superpotential, solutions to chiral supergravity with $|d\phi|^2_{G,g} > 0$ correspond to minimal immersions $\phi : (X, g) \to (M, G)$. A systematic study of Riemann surfaces admitting supersymmetric solutions to chiral $\mathcal{N} = 1$ supergravity with possibly non-vanishing superpotential is beyond the scope of this manuscript and will be considered elsewhere. In Sect. 4.1 we will consider the classification problem of supersymmetric solutions with vanishing superpotential.

Remark 4.12. The formulation introduced in this Section of chiral $\mathcal{N} = 1$ supergravity on $X$ fixes the Spin$^c(2)$ structure of the theory to be, given a Riemannian metric $g$ on $X$, the anti-canonical one on $(X, g)$, and takes $S$ to be the associated complex spinor bundle through the tautological representation of Spin$^c(2)$. It is in principle possible
to use a different (inequivalent) complex spinor bundle to construct the theory. If we change the complex spinor bundle, we should expect to obtain a non-equivalent chiral $\mathcal{N} = 1$ supergravity on $X$, see for example [23]. However, if we assume the existence of a non-vanishing chiral complex spinor $\epsilon \in \Gamma(S)$, then the allowed choices of complex spinor bundle $S$ are restricted and we can construct a canonical isomorphism of complex spinor bundles between $S$ and either $\Lambda^{0,*}(X)$ or $\Lambda^{*,0}(X)$ (note that every chiral spinor in two-dimensions is automatically pure). Using the map

$$\Lambda^{*,*}(X) \to S, \quad \alpha \mapsto \alpha \cdot \epsilon,$$

it can be seen that if there exists an everywhere non-zero spinor $\epsilon$ of positive chirality then we have an isomorphism of complex spinor bundles:

$$S \simeq \Lambda^{*,0}(X),$$

with Clifford multiplication given by (13). On the other hand, if the complex spinor $\epsilon$ is of negative chirality, we obtain the isomorphism:

$$S \simeq \Lambda^{0,*}(X),$$

with Clifford multiplication given by:

$$\beta \cdot \alpha = 2\beta^{0,1} \wedge \alpha + \iota(\beta^* \gamma^0 \gamma^1)\alpha,$$

for all $\alpha \in \Omega^{*,0}(X)$ and all $\beta \in \Omega^1(X)$. The determinant line bundle is in this case the anti-canonical line bundle $K_g^*$ of $(X, g)$ and the resulting complex spinor bundle corresponds to the canonical Spin$^c(2)$ structure on $(X, g)$. Hence, if $X$ admits supersymmetric solutions (a condition that implies the existence of a non-vanishing $\epsilon \in \Gamma(S^\pm)$), there is no loss of generality in assuming that the complex spinor bundle $S$ is associated to either the canonical Spin$^c(2)$ structure on $(X, g)$ if $\epsilon$ has negative-chirality or the anti-canonical Spin$^c(2)$ structure on $(X, g)$ if $\epsilon$ has positive-chirality.

### 4.1. Supersymmetric solutions with vanishing superpotential

We consider the Killing spinor equations on $X$ with vanishing superpotential. The local structure of the supersymmetric solutions of chiral $\mathcal{N} = 1$ supergravity has been considered in References [32,51], where the generic local form of the supersymmetric solutions of the theory in four dimensions has been partially characterized in terms of a minimal set of partial differential equations. The goal of this sub-section is to obtain a global classification result in the special case in which the superpotential $\mathcal{W}$ is zero. In the course of the proof of Theorem 4.13 we will see that every supersymmetric configuration is actually a solution and therefore it is not necessary to consider the equations of motion explicitly in order to classify supersymmetric solutions. Let $Q$ be a chiral triple on $(X, M)$ such that $\mathcal{W} = 0$. The Killing spinor equations of chiral supergravity associated to $(M, Q)$ on $X$ reduce to:

$$\nabla^g \epsilon = 0, \quad d\varphi^{0,1} \cdot \epsilon = 0,$$

for $\epsilon \in \Gamma(S^+)$.  

**Theorem 4.13.** Let $Q$ be a chiral triple on $(X, M)$ such that $\mathcal{W} = 0$. A triple $(g, \varphi, \Psi)$ with non-constant $\varphi$ is a supersymmetric solution of the chiral supergravity associated to $(M, Q)$ if and only if the following conditions hold:
The smooth map $\varphi : (X, g) \rightarrow (\mathcal{M}, \mathcal{G})$ is a holomorphic map with respect to $J_g$ and the fixed complex structure $\mathcal{I}$ on $\mathcal{M}$.

(2) $\Psi : K_g \rightarrow L^\varphi$ is an isomorphism of holomorphic line bundles such that:

$$g^*_c = \kappa \Psi^* \mathcal{H}^\varphi,$$

for a constant $\kappa \in \mathbb{R}_{>0}$, where $g^*_c$ denotes the Hermitian metric induced by $g$ on $\Lambda^{1,0}(X)$.

These conditions imply that:

$$R_g = |d\varphi|^2_{\mathcal{G},g} g = \frac{R_g}{2} g,$$

and that the Kähler metric $\mathcal{G}$, the Riemannian metric $g$ and the map $\varphi$ satisfy:

$$\varphi^* \mathcal{G} = \frac{|d\varphi|^2_{\mathcal{G},g}}{2} g = \frac{R_g}{2} g.$$

Hence, $\varphi$ is a conformal immersion of $X \setminus C$ into $\mathcal{M}$, where $C \subset X$ denotes the critical set of $\varphi$. Furthermore, if $(X, g)$ is compact, then it is biholomorphic with the Riemann sphere $\mathbb{P}^1$.

**Proof.** Let $(g, \varphi, \Psi)$ be a supersymmetric solution on $X$ with non-constant $\varphi$. The Riemannian metric $g$ induces a complex structure $J_g$ upon use of the fixed orientation of $X$. Assuming that $\epsilon$ is non-zero at a point, the first Killing spinor equation implies that $\epsilon$ is everywhere non-vanishing. Therefore, we obtain an isomorphism of complex spinor bundles $S \simeq \Lambda^{0,0}(X) \oplus \Lambda^{1,0}(X)$ which maps $\epsilon$ to $1 \in \Lambda^{0,0}(X)$. Hence, the second Killing spinor equation is equivalent with:

$$\partial \varphi^{0,1} = 0$$

implying that a map $\varphi : X \rightarrow \mathcal{M}$ satisfying the Killing spinor equations is necessarily holomorphic, a condition that immediately implies harmonicity of $\varphi$. Using that $\varphi$ is holomorphic it follows that $(L^\varphi, \mathcal{H}^\varphi)$ is a holomorphic line bundle over $(X, J_g)$ and in addition $\mathcal{D}^\varphi$ coincides with the Chern connection on $(L^\varphi, \mathcal{H}^\varphi)$. Let $\{U, w\}, U \subset X$ open, be a local holomorphic coordinate on $X$ and let $\{V, z^i\}, V \subset \mathcal{M}$ open, $i = 1, \ldots, n$, be local complex coordinates on $\mathcal{M}$ such that $\varphi(U) \subset V$. We write $g$ as follows:

$$g = e^F dw \otimes d\bar{w},$$

for a smooth function $F \in C^\infty(V)$. Using that $\epsilon = 1$, the first Killing spinor equation in (16) can be locally written as:

$$i A^{g^*} = \Psi^* A^\varphi,$$

where $i A^{g^*}$ denotes the Chern connection of the Hermitian metric $g^*(\cdot, \cdot)$ on $K_g$ and $A^\varphi$ denotes the connection one-form associated to the Chern connection $\mathcal{D}^\varphi$. By type decomposition, the previous equation locally reads:

$$\partial_w z^i \partial_{\bar{z}}^i K(z(w), \bar{z}(\bar{w})) dw = -\partial_w F(w, \bar{w}) dw,$$

(17)
where we are using a local holomorphic trivialization $l$ of $\mathcal{L}$ in which $\mathcal{H}(l, l) = e^K$. Since this equation must hold on any pair of complex coordinate charts $\{U, w\}$ and $\{V, z^i\}$ as above, we conclude that:

$$\nabla^C_{g^*} = \Psi^* D^\varphi,$$

(18)

globally on $X$, where $\nabla^C_{g^*}$ denotes the Chern connection on $K_g$. Therefore, the Chern connection on $(\mathcal{L}^\varphi, \mathcal{H}^\varphi)$ is mapped, through the $C^\infty$ diffeomorphism $\Psi$, to the Chern connection $\nabla^C_{g^*}$ on the holomorphic cotangent bundle of $X$ equipped with the Hermitian inner product induced by $g^*$. Since $K_g$ and $\mathcal{L}^\varphi$ come equipped with the holomorphic structures induced respectively by $\nabla^C_{g^*}$ and $D^\varphi$, we conclude that $\Psi: K_g \xrightarrow{\sim} \mathcal{L}^\varphi$ is in fact an isomorphism of holomorphic line bundles. By integrating (18), it can be seen that the fact that the Chern connections associated to $g^*$ and $\Psi^* \mathcal{H}^\varphi$ are equal is equivalent with:

$$g_c^* = \kappa \Psi^* \mathcal{H}^\varphi, \quad \kappa \in \mathbb{R}_{>0},$$

which in local coordinates implies $F(w, \bar{w}) = -K(z(w), \bar{z}(\bar{w})) - \log \kappa$ and gives the following local expression for $g$:

$$g = \frac{2}{\kappa} e^{-K(z(w), \bar{z}(\bar{w}))} dw \circ d\bar{w}.$$ 

Taking the derivative of Eq. (17) with respect to $\bar{w}$ we obtain:

$$\partial_w \partial_{\bar{w}} F = -\partial_w z^t \partial_{\bar{w}} \bar{z}^j \partial_z^i \partial_{\bar{z}}^j K(z(w), \bar{z}(\bar{w})), $$

where we have used that $\varphi$ is holomorphic. This equation directly implies, after suitable identifications, the following relation:

$$\frac{R_g}{2} g = \varphi^* \mathcal{G},$$

where we have used that the Riemannian scalar curvature $R_g$ of $g$ is locally explicitly given by $R_g = -\Delta_g F$ in terms of the Laplacian $\Delta_g$ on $(X, g)$. Taking the trace of the previous equation, we obtain:

$$R_g = |d\varphi|^2_{G, g},$$

and thus the curvature of $g$ is non-negative and prescribed by the norm of $d\varphi$. If $X$ is compact, this implies that the Euler characteristic $\chi(X)$ of is strictly positive (recall that we are assuming that $\varphi$ is non-constant) and thus $X$ must be biholomorphic with the Riemann sphere $\mathbb{P}^1$. In particular, for any $p \in X$ we have $R_g|_p = 0$ if and only if $|d\varphi|^2_{G, g}|_p = 0$. Since, $\varphi$ is holomorphic, $C \subset X$ is a discrete subset of $X$, whence finite if $X$ is compact. Note that $R_g|_C = 0$ and $R_g|_{X \setminus C} > 0$. Combining the previous two equations we conclude:

$$\frac{|d\varphi|^2_{G, g}}{2} g = \varphi^* \mathcal{G},$$
whence every supersymmetric configuration is in fact a supersymmetric solution. On $X\setminus \mathbb{C}$ the scalar curvature $R_g$ of $g$ as well as $|d\varphi|^2_{S, g}$ are both nowhere vanishing. In particular:

$$g = \frac{2}{|d\varphi|^2_{S, g}} \varphi^* G,$$

whence $\varphi$ is a holomorphic conformal immersion of $X\setminus \mathbb{C}$ into $\mathcal{M}$. Direct computation in local coordinates on $X\setminus \mathbb{C}$ shows (we take $\kappa = 2$ for simplicity):

$$2 |d\varphi|^2_{S, g} \varphi^* G = \frac{2}{4} e^K e^{\frac{1}{2} \partial_u \partial_{\bar{u}} \partial_z \partial_{\bar{z}}} K dw \otimes d\bar{w} = e^{-K} dw \otimes d\bar{w},$$

as expected. For the converse, we just note that any pair $(g, \varphi, \Psi)$ obeying conditions (1) and (2) of the theorem satisfies the Killing spinor equations (16) for $\epsilon = 1$. □

**Remark 4.14.** If $\varphi$ is constant, then (16) implies that $g$ is flat and $K_g$ must be holomorphically trivial. In particular, if $X$ is compact then it is biholomorphic with an elliptic curve $\mathbb{E}$.

**Remark 4.15.** When $X$ is open, the requirement that $K_g$ be negative does not pose any restriction on $X$, see Example 3.34 for more details.

**Remark 4.16.** The supersymmetric structure of chiral $\mathcal{N} = 1$ supergravity in four dimensions requires the supersymmetry generator to be of negative chirality. When reduced to two dimensions, negative chirality in four dimensions allows for either negative or positive chirality in two dimensions, depending on the decomposition chosen for the supersymmetry spinor in four dimensions. Here we have fixed the supersymmetry parameter $\epsilon$ in two dimensions to have positive chirality, see Remark 4.12. However, we could have chosen $\epsilon$ to have negative chirality and $(X, g)$ to be endowed with the canonical $\text{Spin}^c(2)$ structure. In this case, it can be shown that the Killing spinor equations require $\varphi$ to be an *anti-holomorphic* map, and that a supersymmetric solution with $\epsilon$ of negative chirality corresponds with a supersymmetric solution with $\epsilon$ of positive chirality after complex-conjugating the complex structure $J_g \mapsto -J_g$ on $X$. We have chosen to work with anti-canonical $\text{Spin}^c(2)$ structures and positive chirality supersymmetry generators in order to avoid working with supersymmetric solutions having anti-holomorphic, instead of holomorphic, scalar maps $\varphi$.

When $X$ admits supersymmetric solutions, the existence of a nowhere vanishing positive-chirality spinor $\epsilon \in \Gamma(S^+) \text{ determines } S \text{ as the complex spinor bundle associated to the canonical } \text{Spin}^c(2) \text{ structure on } (X, g). \text{ Hence, we obtain the following Corollary.}

**Corollary 4.17.** The spinor bundle $S$ of a supersymmetric solution is necessarily the complex spinor bundle associated to either the canonical or anti-canonical $\text{Spin}^c(2)$ structure on $(X, g)$ through the tautological representation of $\text{Spin}^c(2)$.

This corollary gives a very explicit example of how the existence of supersymmetric solutions depends on the choice of isomorphism class of the spinor bundle. Inspired by Theorem 4.13 we introduce the notion of chiral map.

**Definition 4.18.** Let $X$ be an oriented real two-manifold and let $(\mathcal{M}, \mathcal{L}, \mathcal{H})$ be a complex manifold equipped with a negative Hermitian holomorphic line bundle $(\mathcal{L}, \mathcal{H})$. We say
that a pair \((\varphi, \Psi)\) is a *chiral map* with respect to \((\mathcal{M}, \mathcal{L}, \mathcal{H})\) if there exists a complex structure \(J = J_\varphi\) on \(X\) such that:

\[
\varphi : (X, J) \rightarrow (\mathcal{M}, \mathcal{I}) ,
\]

is holomorphic and:

\[
\Psi : K \xrightarrow{\sim} \mathcal{L}^\varphi ,
\]

is an isomorphism of holomorphic line bundles, where \(K\) is the canonical bundle of \((X, J)\).

Associated to every chiral map \((\varphi, \Psi)\) we have a supersymmetric solution \((g, \varphi, \Psi)\) where \(g^* = \kappa \Psi^* \mathcal{H}^\varphi\) and, vice-versa, every supersymmetric solution of chiral supergravity on \(X\) gives rise to a chiral map. Aside from the role they play in chiral supergravity, chiral maps are interesting because they are particular instances of holomorphic maps of Riemann surfaces into Kähler manifolds and provide solutions to the *coupled* problem of prescribing the scalar curvature of a Riemann surface to:

\[
R_g = |d\varphi|^2_{g,G} .
\]

**Remark 4.19.** The problem of prescribing the Gaussian curvature of a compact Riemann surface has been extensively studied in the literature, see for example [40] and references therein. Reference [41] solves the problem in the case of \(X\) being a closed Riemann surface. In particular, it asserts that the sphere \(S^2\) admits a metric \(g\) with curvature \(\bar{f} \in C^\infty(X)\) if and only if the obvious Gauss-Bonnet sign condition is satisfied. Since, as shown in Theorem 4.13, the curvature of a supersymmetric solution \((g, \varphi, \Psi)\) is non-negative we conclude that supersymmetry imposes a non-trivial restriction on the allowed metrics appearing in a supersymmetric solution on \(S^2\).

When \(\mathcal{M}\) is also an oriented two-dimensional real manifold, Theorem 4.13 implies the following corollary.

**Corollary 4.20.** Let \(\mathcal{M}\) be connected and complex one-dimensional and let \((\varphi, \Psi)\) be a chiral map with respect to \((\mathcal{M}, \mathcal{L}, \mathcal{H})\). If \(\varphi\) is proper then it is a holomorphic branched covering of Riemann surfaces. If in addition \(X\) is compact, whence biholomorphic to the Riemann sphere \(\mathbb{P}^1\), then \(\mathcal{M}\) is a compact Riemann surface whose Euler characteristic \(\chi(\mathcal{M})\) satisfies:

\[
\chi(\mathcal{M}) = \frac{2 + k}{d} ,
\]

where \(d\) denotes the degree of \(\varphi\) and \(k\) denotes its total ramification index. In particular, \((2 + k)\) is divisible by \(d\).

**Proof.** By definition of chiral map, \(\varphi\) is holomorphic. Since by assumption \(\mathcal{M}\) is connected and \(\varphi\) is proper, \(\varphi\) must be surjective and thus it is a holomorphic branched covering, implying also that if \(X\) is compact then \(\mathcal{M}\) is also compact. Theorem 4.13 implies now that \(X = \mathbb{P}^1\) and the Riemann-Hurwitz formula implies the remaining statements. \(\square\)
Let \((g, \varphi, \Psi)\) be a supersymmetric solution. When \(X\) is non-compact the norm of \(\varphi\) may diverge and \(g\) may not be complete. In order to guarantee that \((X, g)\) is physically admissible we assume \(g\) is complete. Furthermore we assume finite \(L^2\) energy:

\[
\|d\varphi\|_{g, G}^2 < \infty
\]

Under these conditions, we can obtain the following, perhaps surprising, classification result on the possibly non-compact surfaces \(X\) admitting supersymmetric solutions with complete Riemannian metric \(g\) and finite energy.

**Theorem 4.21.** Let \(X\) be an oriented real two-manifold carrying a supersymmetric solution \((g, \varphi, \Psi)\) (with respect to some chiral triple \(Q\)) such that \(g\) is complete and \(\varphi\) has positive finite energy. Then, one of the following holds:

1. \((X, J_g)\) is biholomorphic to the complex projective line \(\mathbb{P}^1\) and:

\[
\|d\varphi\|_{g, G}^2 = 8\pi .
\]

2. \((X, J_g)\) is biholomorphic to the complex plane \(\mathbb{C}\), and there exists a neighborhood \(U(\infty)\) of infinity with complex coordinate \(w\) which is isomorphic to the punctured unit disk equipped with the metric:

\[
g|_{U(\infty)} = \frac{e^F}{|w|^2} dw \otimes d\bar{w}, \quad F \in L^1, \quad \Delta F \in L^1,
\]

and furthermore:

\[
\|d\varphi\|_{g, G}^2 = 4\pi .
\]

3. \((X, J_g)\) is biholomorphic to the complex plane \(\mathbb{C}\), and there exists a neighborhood \(U(\infty)\) of infinity with complex coordinate \(w\) which is isomorphic to the punctured unit disk equipped with the metric:

\[
g|_{U(\infty)} = \frac{e^F}{|w|^4} dw \otimes d\bar{w}, \quad F \in L^1, \quad \Delta F \in L^1,
\]

and \(\varphi\) is constant.

4. \((X, J_g)\) is biholomorphic to the punctured complex plane \(\mathbb{C}^*\), \(\varphi\) is constant and there exists neighborhoods \(U(\infty)\) of infinity with complex coordinate \(w_\infty\) and \(U(0)\) of zero with complex coordinate \(w_0\) which are isomorphic to the punctured unit disk equipped respectively equipped with the metric:

\[
\begin{align*}
g|_{U(\infty)} &= \frac{e^{F_\infty}}{|w_\infty|^2} dw_\infty \otimes d\bar{w}_\infty, \quad F_\infty \in L^1, \quad \Delta F_\infty \in L^1, \\
g|_{U(0)} &= \frac{e^{F_0}}{|w_0|^2} dw_0 \otimes d\bar{w}_0, \quad F_0 \in L^1, \quad \Delta F_0 \in L^1,
\end{align*}
\]

5. \((X, J_g)\) is biholomorphic to a complex elliptic curve and \(\varphi\) is constant.

**Proof.** Let \((g, \varphi, \Psi)\) be such a supersymmetric solution. Finiteness of the energy of \(\varphi\) imply, upon use of Theorem 4.13, that \((X, g)\) has non-negative and finite total curvature. Applying now A. Huber’s Theorem as stated in [38, Pages 1–2] we conclude. \(\square\)
Remark 4.22. The idea behind the previous Theorem is simple, once we have the results of A. Huber at our disposal: finiteness of the total curvature of \((X, g)\) implies that \(X\) has a natural compactification, that is, it is biholomorphic with a compact Riemann surface with a finite number of points removed. The fact that the total curvature not only finite but non-negative further restricts they type of Riemann surfaces \((X, g)\), since removing more point decreases the total finite curvature. Since the total finite curvature needs to remain non-negative, only the cases appearing in the previous Theorem can happen.

The four-dimensional Lorentzian manifold \((M_4, g_4)\) associated to the supersymmetric solutions considered in Theorem 4.13 is of the form:

\[
(M_4, g_4) = (\mathbb{R}^2 \times X, \delta_{1,1} \times g),
\]

where \((X, g)\) is a Riemann surface with the Riemannian metric \(g\) being part of a supersymmetric solution \((g, \varphi, \Psi)\) on \(X\). As we have said, if \(\varphi\) is non-constant and \(X\) is compact then it is biholomorphic with \(\mathbb{P}^1\) and in this case:

\[
(M_4, g_4) = (\mathbb{R}^2 \times \mathbb{P}^1, \delta_{1,1} \times g).
\]

Therefore, we recover the spherical near horizon geometry characterized in [33], see also [49]. Note that \(g\) may not be the round metric on \(S^2\). If \(\varphi\) is constant, \(X\) is biholomorphic with an elliptic curve and we recover the toroidal near horizon geometry characterized in [33].

4.2. Anti-supersymmetric solutions. As explained in Remark 4.16, had we chosen the canonical Spin\(_c(2)\) structure on \((X, g)\) to formulate chiral \(\mathcal{N} = 1\) supergravity, we would have found that scalar maps \(\varphi\) of supersymmetric solutions are anti-holomorphic and supersymmetric solutions in this case are equivalent to those characterized in Theorem 4.13 with the complex structure \(J_g\) replaced by \(-J_g\). Nonetheless, we can propose a natural modification of the Killing spinor equations which still yields solutions to chiral \(\mathcal{N} = 1\) supergravity with holomorphic scalar maps. These solutions are however not supersymmetric. In this section we introduce and classify these solutions, which we call anti-supersymmetric.

We assume \(X\) to be equipped with a fixed orientation. We define now \(Q_g\) to be the canonical Spin\(_c(2)\) structure associated to \(g\) and the complex structure \(J_g\). For each Riemannian metric \(g\), we define \(S\) to be the tautological complex spinor bundle associated to \(Q_g\). The spinor bundle \(S\) admits an explicit model given by:

\[
S = \Lambda^{0,*}(X),
\]

where the splitting is performed with respect to the complex structure \(J_g\). Clifford multiplication is given by:

\[
\beta \cdot \alpha = 2\beta^{0,1} \wedge \alpha + \iota(\beta^\sharp)^0,1 \alpha,
\]

for all \(\alpha \in \Omega^{0,*}(X)\) and all \(\beta \in \Omega^1(X)\). The determinant line bundle associated to the canonical Spin\(_c(2)\) structure \(Q_g\) is given by the complex-conjugate canonical bundle \(K^*_g\) of \((X, g)\):

\[
L_{Q_g} = K^*_g = \Lambda^{0,1}(X),
\]
which is complex-isomorphic to the anticanonical bundle $T^{1,0}X$ of $(X, g)$. The complex spinor bundle $S$ is thus a complex vector bundle of rank two, which splits in the usual way:

$$S = S^+ \oplus S^-,$$

in terms of the chiral bundles $S^+$ and $S^-$. In the polyform presentation $S = \Lambda^{0,*}(X)$ of the spinor bundle, the chiral spinor bundles $S^\pm$ respectively correspond with:

$$S^+ \simeq \Lambda^{0, \text{odd}}(X) \simeq \Lambda^{0,1}(X), \quad S^- \simeq \Lambda^{0, \text{even}}(X) \simeq \Lambda^{0,0}(X),$$

whereas the chiral spinor bundles $S^\pm_c$ correspond with:

$$S^+_c \simeq \Lambda^{0,0}(X), \quad S^-_c \simeq \Lambda^{1,0}(X).$$

As required, we have:

$$S^+ \simeq S^+_c \otimes K^*_g, \quad S^- \simeq S^-_c \otimes K^*_g.$$

The notion of chiral triple is modified accordingly.

**Definition 4.23.** A chiral triple $(\mathcal{L}, \mathcal{H}, \mathcal{W})$ on $(X, \mathcal{M})$ consists on a negative Hermitian holomorphic line bundle $(\mathcal{L}, \mathcal{H})$ and a holomorphic section $\mathcal{W} \in H^0(\mathcal{M}, \mathcal{L})$ such that there exists a map $\varphi : X \to \mathcal{M}$ and a metric $g$ on $X$ for which:

$$K^*_g \simeq \mathcal{L}^\varphi,$$

as complex line bundles.

We fix a chiral triple $\mathcal{Q}$ on $(X, \mathcal{M})$ with vanishing superpotential. Instead of considering the Killing spinor equations required by the supersymmetric structure of chiral $\mathcal{N} = 1$ supergravity, we consider the following equations:

$$\nabla^\varphi \epsilon = 0, \quad d\varphi^{0,1} \cdot \epsilon = 0, \quad (20)$$

for $\epsilon \in \Gamma(S^-)$. Note that the honest Killing spinor equations would require:

$$d\varphi^{0,1} \cdot \epsilon = 0,$$

instead of the second equation appearing in (20).

**Definition 4.24.** We call triples $(g, \varphi, \Psi)$ satisfying Eq. (20) anti-supersymmetric solutions.

The key point is that anti-supersymmetric solutions are not supersymmetric solutions yet they are honest solutions of chiral $\mathcal{N} = 1$ supergravity. The proof of the following theorem is completely analogous to the proof of Theorem 4.13.

**Theorem 4.25.** Let $\mathcal{Q}$ be a chiral triple on $(X, \mathcal{M})$ such that $\mathcal{W} = 0$. A triple $(g, \varphi, \Psi)$ with non-constant $\varphi$ is an anti-supersymmetric solution of the chiral supergravity associated to $(\mathcal{M}, \mathcal{Q})$ if and only if the following conditions hold:

1. The smooth map $\varphi : (X, g) \to (\mathcal{M}, G)$ is a holomorphic map with respect to $J_g$ and the fixed complex structure $\mathcal{I}$ on $\mathcal{M}$.
(2) $\Psi : K^*_g \xrightarrow{\sim} \mathcal{L}^\psi$ is an isomorphism of holomorphic line bundles such that:

$$g_c = \kappa \Psi^* \mathcal{H}^\psi,$$

for a constant $\kappa \in \mathbb{R}_{>0}$, where $g_c$ denotes the Hermitian metric induced by $g$ on $T^{1,0}X$.

These conditions imply that:

$$(\mathcal{G}, g, \phi),$$

and that the Kähler metric $\mathcal{G}$, the Riemannian metric $g$ and the map $\phi$ satisfy:

$$\phi^* \mathcal{G} = \frac{|d\phi|^2_{\mathcal{G}, g}}{2} g = -\frac{R_g}{2} g.$$

Hence, $\phi$ is a conformal immersion of $X \setminus C$ into $M$, where $C \subset X$ denotes the critical set of $\phi$. Furthermore, if $(X, g)$ is compact, then it is hyperbolic.

We can adapt the notion of chiral map to accommodate anti-supersymmetric solutions.

**Definition 4.26.** Let $X$ be an oriented real two-manifold and let $(M, \mathcal{L}, \mathcal{H})$ be a complex manifold equipped with a negative Hermitian holomorphic line bundle $(\mathcal{L}, \mathcal{H})$. We say that a pair $(\phi, \Psi)$ is a *anti-chiral map* with respect to $(M, \mathcal{L}, \mathcal{H})$ if there exists a complex structure $J = J_\phi$ on $X$ such that:

$$\phi : (X, J) \to (M, I),$$

is holomorphic and:

$$\Psi : K^*_g \xrightarrow{\sim} \mathcal{L}^\psi,$$

is an isomorphism of holomorphic line bundles, where $K^*$ is the anti-canonical bundle of $(X, J)$.

As it happened with chiral maps, aside from the role they play in chiral supergravity, anti-chiral maps are interesting because they are particular instances of holomorphic maps of Riemann surfaces into Kähler manifolds and provide solutions to the *coupled* problem of prescribing the scalar curvature of a Riemann surface to:

$$R_g = -\frac{|d\phi|^2_{\mathcal{G}, g}}{2} g = -\frac{R_g}{2} g.$$

When $\mathcal{M}$ is also an oriented two-dimensional real manifold, Theorem 4.13 implies the following corollary.

**Corollary 4.27.** Let $\mathcal{M}$ be connected and complex one-dimensional and let $\phi : X \to \mathcal{M}$ be a anti-chiral map with respect to $(\mathcal{L}, \mathcal{H})$. If $\phi$ is proper then it is a $d$-fold holomorphic branched covering of Riemann surfaces. If in addition $X$ is compact then $\mathcal{M}$ is necessarily compact and we have:

$$\deg(\mathcal{L}) = \chi(\mathcal{M}) - \frac{k}{d},$$

where $k$ is the total branching number of $\phi$. In particular:

$$|\deg(\mathcal{L})| \geq |\chi(\mathcal{M})|,$$

and if $\mathcal{L} \simeq T^{1,0} \mathcal{M}$ then $k = 0$ and $\phi$ is a holomorphic un-branched covering of compact Riemann surfaces.
Proof. By definition of anti-chiral map, $\varphi$ is holomorphic. Since by assumption $\mathcal{M}$ is connected and $\varphi$ is proper, $\varphi$ must be surjective and thus it is a holomorphic branched covering, implying also that if $X$ is compact then $\mathcal{M}$ is also compact. Assume now that $X$ (and thus also $\mathcal{M}$) is compact. The Riemann-Hurwitz formula implies:

$$\chi(X) = d \chi(\mathcal{M}) - k.$$ 

Using now that $\chi(X) = \deg(T^{1,0}X) = \deg(\mathcal{L}^\varphi) = d \deg(\mathcal{L})$ we obtain the first formula of the corollary. Now, the definition of chiral triple requires $\mathcal{L}$ to be negative and thus both $\deg(\mathcal{L})$ and $\chi(X)$ are negative. Hence:

$$-\deg(\mathcal{L}) \geq -\chi(\mathcal{M}),$$

and we conclude. \qed

The four-dimensional Lorentzian manifold $(M_4, g_4)$ associated to the anti-supersymmetric solutions considered in Theorem 4.25 is of the form:

$$(M_4, g_4) = (\mathbb{R}^2 \times X, \delta_{1,1} \times g),$$

where $(X, g)$ is a Riemann surface with the Riemannian metric $g$ being part of a supersymmetric solution $(g, \varphi, \Psi)$. As mentioned in Theorem 4.25, if $X$ is compact then it is biholomorphic with a hyperbolic Riemann surface. Reasoning by analogy with the situation for supersymmetric solutions, we wonder if solutions of this type, with $X$ compact hyperbolic, can appear as non-supersymmetric near horizon geometries of black hole solutions in four dimensions.

4.3. Examples of chiral and anti-chiral maps. In this section we construct several examples of (anti) chiral maps and (anti) supersymmetric solutions.

Example 4.28. Take $X = \mathbb{P}^1$, $\mathcal{M} = \mathbb{P}^1$ and $\mathcal{L} = K_{\mathbb{P}^1}$. Denote by $g$ the round metric on $\mathbb{P}^1$ and take $\mathcal{H}$ to be the Hermitian metric induced by $g$ on $K_{\mathbb{P}^1}$. Then, any triple $(g, \varphi, \Psi)$, with $\varphi : (\mathbb{P}^1, g) \rightarrow (\mathbb{P}^1, g)$ holomorphic isometry and $\Psi$ induced by $\varphi$ gives a supersymmetric solution to chiral $N = 1$ supergravity.

Example 4.29. Let $(X, g)$ be a hyperbolic Riemann surface with $g$ of constant negative curvature $-1$ and admitting non-trivial holomorphic isometries (consider for example $X$ hyperelliptic curve and consider its hyperelliptic involution). Take $\mathcal{M} = X$ and define $\mathcal{L} = T^{1,0}X$ to be the holomorphic tangent bundle of $X$. Furthermore, we take $\mathcal{H} = g_c$, where $g_c$ denotes the Hermitian structure induced by $g$ on $T^{1,0}X$. With this choice of Hermitian structure $\mathcal{H}$, the Kähler metric associated to the Chern curvature of $\mathcal{H}$ in the sense of Definition 3.11 is again $g$. Hence $\mathcal{G} = g$. We claim that every isometry:

$$\varphi : (X, g) \rightarrow (X, g),$$

gives rise to an anti-supersymmetric solution $(g, \varphi, \Psi)$ with respect to the chiral triple $(T^{1,0}X, g_c, \Psi = 0)$ on the pair $(X, \mathcal{M} = X)$. To see this, note that we can define $\Psi = d\varphi$ since:

$$d\varphi : T^{1,0}X \sim (T^{1,0}X)^\varphi,$$
is an isomorphism of holomorphic line bundles. Hence, the conditions of Theorem 4.13 are satisfied and such \((g, \varphi, \Psi)\) is a solution of chiral \(N = 1\) supergravity on \(X\) associated to the chiral triple \((T^{1,0}X, g_c, 0)\). Since \(\varphi\) is assumed to be an isometry, direct computation shows that:

\[
|d\varphi|_{g,\mathcal{G}}^2 = 2,
\]

and thus:

\[
\varphi^*\mathcal{G} = g = \frac{g}{2} |d\varphi|_{g,\mathcal{G}}^2,
\]

as claimed in Theorem 4.25.

**Example 4.30.** Let \(X\) be a compact hyperbolic Riemann surface not of hyperelliptic type. Then, the canonical bundle \(K_X\) of \(X\) is very ample [34] and Kodaira’s embedding theorem implies that for an appropriate \(n > 1\) there exists a holomorphic embedding of \(X\) into \(n\)-dimensional projective space:

\[
\varphi : X \leftrightarrow \mathbb{P}^n,
\]

satisfying:

\[
K_X \simeq \varphi^*\mathcal{O}(1),
\]

where \(\mathcal{O}(1)\) denotes the tautological bundle of \(\mathbb{P}^n\). The previous equation implies that there exists an isomorphism \(\Psi\) of holomorphic line bundles:

\[
\Psi : T^{1,0}X \tilde{\rightarrow} \varphi^*\mathcal{O}(-1).
\]

Furthermore, the Hermitian structure \(\mathcal{H}\) on \(\mathcal{O}(-1)\) induced by the Fubini-Study metric on \(\mathbb{P}^n\) makes \((\mathcal{O}(-1), \mathcal{H})\) into a negative line bundle. Therefore, \((\mathcal{O}(-1), \mathcal{H}, 0)\) is a chiral triple on the pair \((X, \mathbb{P}^n)\) and \((g, \varphi, \Psi)\) is a supersymmetric solution, where \(g\) is constructed in terms of \(\mathcal{H}, \varphi\) and \(\Psi\) as prescribed by Theorem 4.25.

We present now a large family of supersymmetric solutions \((g, \varphi, \Psi)\) to chiral \(N = 1\) supergravity associated to a special class of scalar manifolds \((\mathcal{M}, \mathcal{Q})\) admitting plurisubharmonic functions and having vanishing superpotential.

Let \(X \subset \mathbb{C}\) be a complex domain in \(\mathbb{C}\) with complex coordinate \(w\) and let \(\mathcal{M}\) be an \(n\)-dimensional complex manifold admitting smooth strictly plurisubharmonic functions. For example, we can take \(\mathcal{M} = \mathbb{C}^n\), we can take \(\mathcal{M}\) to be any open Riemann surface, or more generally we can take \(\mathcal{M}\) to be a Stein complex \(n\)-manifold. Fix a smooth plurisubharmonic function \(\varphi : \mathcal{M} \rightarrow \mathbb{R}\) on \(\mathcal{M}\). Let \(\mathcal{L}\) be the holomorphically trivial complex line bundle over \(\mathcal{M}\), and let us fix a holomorphic trivialization \(\mathcal{L} = \mathcal{M} \times \mathbb{C}\). In this trivialization, we define a Hermitian structure \(\mathcal{H}\) as follows:

\[
\mathcal{H}(f_1, f_2) \overset{\text{def.}}{=} e^{\varphi}f_1\bar{f}_2,
\]

where \(f_1, f_2 : \mathcal{M} \rightarrow \mathbb{C}\) are smooth functions. Since the cotangent bundle of \(X\) is holomorphically trivial, the triple \(\mathcal{Q} \overset{\text{def.}}{=} (\mathcal{L}, \mathcal{H}, 0)\) is a chiral triple on \((X, \mathcal{M})\) for any holomorphic map:

\[
\varphi : X \rightarrow \mathcal{M}.
\]
In particular, \( \mathcal{L}^\varphi \) is holomorphically isomorphic to \( \Lambda^{1,0}(X) \). Let us trivialize \( \mathcal{L}^\varphi \) as \( \mathcal{L}^\varphi = X \times \mathbb{C} \) by using the pull-back of the fixed trivialization of \( \mathcal{L} \). The holomorphic cotangent bundle of \( X \) is the complex span of \( \{dw\} \). Using the previous trivializations and the complex coordinate \( w \), we define an isomorphism of holomorphic line bundles \( \Psi : \Lambda^{1,0}(X) \xrightarrow{\sim} \mathcal{L}^\varphi \) as follows:

\[
\Psi(dw) = 1.
\]

With these provisos in mind, Theorem 4.13 implies the following result.

**Corollary 4.31.** In the set-up introduced above, any triple \((g, \varphi, \Psi)\) with \( \varphi : X \rightarrow \mathcal{M} \) holomorphic and \( g \) given through its associated Hermitian metric on \( \Lambda^{1,0}(X) \) as follows:

\[
g^*_c = \Psi^* \mathcal{H}^\varphi.
\]

is a supersymmetric solution of the chiral \( N = 1 \) supergravity associated to \( \mathcal{Q} \).

**Remark 4.32.** This corollary can be easily adapted to yield anti-supersymmetric solutions instead of supersymmetric solutions. In order to do this, simply consider the holomorphic tangent bundle of \( X \) instead of its cotangent bundle. We leave the details to the reader.

Let us explore in more detail the solution provided by the previous corollary. Let \((g, \varphi, \Psi)\) be such a solution. The Hermitian structure \( \mathcal{H}^\varphi \) on \( \mathcal{L}^\varphi \) reads:

\[
\mathcal{H}^\varphi(f_1, f_2) = e^{\phi(\varphi)} f_1 \overline{f}_2,
\]

Using the explicit form of \( \Psi \), the two-dimensional metric \( g \) associated to \( \varphi \) as described in Theorem 4.13 reads (we take \( \kappa = 2 \) for simplicity):

\[
g = (\Psi^* \mathcal{H}^\varphi)^* = e^{-\phi(\varphi)(w)} dw \otimes d\bar{w}.
\]

Explicit computation gives the following formula for the scalar curvature:

\[
R_g = \Delta_g(\phi(\varphi)),
\]

showing that it is non-negative, as required by Theorem 4.13, since \( \phi \) is plurisubharmonic. Direct computation shows that:

\[
|d\varphi|^2_{g, \mathcal{G}} = \Delta_g(\phi(\varphi)),
\]

and thus, as required in order to have a supersymmetric solution, we have:

\[
R_g = |d\varphi|^2_{g, \mathcal{G}}.
\]

Furthermore:

\[
\varphi^* \mathcal{G} = \frac{e^{-\phi(\varphi)}}{2} \Delta_g(\phi(\varphi)) dw \otimes d\bar{w} = \frac{|d\varphi|^2_{g, \mathcal{G}}}{2} g,
\]

whence, as required by Theorem 4.13, the Einstein equation for \((g, \varphi)\) is satisfied.
Remark 4.33. Corollary 4.31 provides us with an infinite family of supersymmetry solutions to chiral $\mathcal{N}=1$ supergravity associated to the type of chiral triple introduced above. Remembering that the supergravity theory considered on $X$ is a reduction of the Lorentzian theory, we can easily reconstruct the associated family of Lorentzian solutions. We have $M = \mathbb{R}^2 \times X$ and:

$$g = -dt \otimes dt + dx \otimes dx + e^{-\phi(\varphi)} dw \otimes d\bar{w},$$

which gives, for fixed $X$ and $M$, a family of Lorentzian metrics on $M$ depending on the choice of plurisubharmonic function $\phi$ on $M$ and holomorphic map $\varphi: X \to \mathcal{M}$.

We present now two explicit examples of the previous construction.

Example 4.34. Let us set $X = \mathbb{C}$ and $\mathcal{M} = \mathbb{C}^*$ with its standard Kähler form. Let $L$ be the holomorphically trivial complex line bundle over $\mathcal{M}$, and let us fix a trivialization $L = \mathbb{C}^* \times \mathbb{C}$. In this trivialization, we define $H$ as follows:

$$H(f_1, f_2) \overset{\text{def}}{=} e^{|z|^2} f_1 \bar{f}_2,$$

where $z$ is a fixed complex coordinate of $\mathbb{C}^*$ and $f_1, f_2: \mathbb{C}^* \to \mathbb{C}$ are smooth functions. The curvature of the Chern connection associated to $H$ reads:

$$\Theta = -dz \wedge d\bar{z},$$

whence it is negative and its associated Kähler form $V$ and Kähler metric $G$ are given by:

$$V = \frac{i}{2\pi} dz \wedge d\bar{z}, \quad G = \frac{1}{2\pi} dz \otimes d\bar{z}$$

We define:

$$\varphi \overset{\text{def}}{=} \mathbb{C} \to \mathbb{C}^*, \quad w \mapsto e^w,$$

as a potential candidate for chiral map and in particular a solution of chiral $\mathcal{N}=1$ supergravity on $X$. Clearly, $d\varphi$ is an isomorphism of vector bundles. The complex line bundle $L^\varphi$ is holomorphically trivial over $X$ and we use the pull-back trivialization of $L$ to set $L^\varphi = X \times \mathbb{C}$. Let now $w$ denote a global complex coordinate on $X = \mathbb{C}$. The holomorphic tangent bundle of $\mathbb{C}$ is the complex span of $\{dw\}$. We define an isomorphism of holomorphic line bundles $\Psi: \Lambda^{1,0}(X) \xrightarrow{\sim} L^\varphi$ as follows:

$$\Psi(dw) = 1$$

in the chosen trivializations. With these provisos in mind, $(L, \mathcal{H}, 0)$ is a chiral triple on $(\mathbb{C}, \mathbb{C}^*)$. The Hermitian structure $\mathcal{H}^\varphi$ on $L^\varphi$ reads:

$$\mathcal{H}^\varphi(f_1, f_2) = e^{w+\bar{w}} f_1 \bar{f}_2.$$

Using the explicit form of $\Psi$, the two-dimensional metric $g$ associated to $\varphi$ as described in Theorem 4.13 reads:

$$g = (\Psi^* \mathcal{H}^\varphi)^* = e^{-e^{w+\bar{w}}} dw \otimes d\bar{w}.$$
Since $\varphi$ is a holomorphic immersion, in fact it is a holomorphic unramified covering, and $g$ is constructed as required by Theorem 4.13, we conclude that $(g, \varphi, \Psi)$ is a supersymmetric solution with respect to the pair $(\mathbb{C}, \mathbb{C}^*)$ and the chiral triple specified above. Explicit computation gives the following formula for the scalar curvature:

$$R_g = 4 \frac{e^{w+\bar{w}}}{e^{-e^{w+\bar{w}}}},$$

showing that it is positive definite and unbounded. Direct computation shows that:

$$|d\varphi|^2_{g,\mathcal{G}} = 4 \frac{e^{w+\bar{w}}}{e^{-e^{w+\bar{w}}}},$$

and thus, as required for a supersymmetric solution, we have:

$$R_g = |d\varphi|^2_{g,\mathcal{G}}.$$

Furthermore:

$$\varphi^* \mathcal{G} = 2 e^{w+\bar{w}} d\varphi \otimes d\bar{\varphi} = \frac{g}{2} |d\varphi|^2_{g,\mathcal{G}},$$

and thus, as expected, the Einstein equation for $(g, \varphi)$ is satisfied.

**Example 4.35.** Let us set $X = \mathbb{C}$ and $\mathcal{M} = \mathbb{C}$ with its standard Kähler form. Let $\mathcal{L}$ be the holomorphically trivial complex line bundle over $\mathcal{M}$, and let us fix a trivialization $\mathcal{L} = \mathbb{C} \times \mathbb{C}$. In this trivialization, we define $\mathcal{H}$ again as follows:

$$\mathcal{H}(f_1, f_2)^{\text{def.}} = e^{|z|^2} f_1 \tilde{f}_2,$$

where $z$ is the complex coordinate of $\mathcal{M} = \mathbb{C}$ and $f_1, f_2 : \mathbb{C} \to \mathbb{C}^*$ are smooth functions. Then, the curvature of the Chern connection associated to $\mathcal{H}$ reads:

$$\Theta = -dz \wedge d\bar{z},$$

whence the associated Kähler form $\mathcal{V}$ and Kähler metric $\mathcal{G}$ read:

$$\mathcal{V} = \frac{i}{2\pi} dz \wedge d\bar{z}, \quad \mathcal{G} = \frac{1}{2\pi} dz \otimes d\bar{z}.$$

We define:

$$\varphi_k^{\text{def.}} : \mathbb{C} \to \mathbb{C}, \quad w \mapsto w^k, \quad k \geq 1,$$

as a potential candidate for chiral map and in particular a solution of chiral $\mathcal{N} = 1$ supergravity on $X$, where $w$ denote a global complex coordinate on $X = \mathbb{C}$. The complex line bundle $\mathcal{L}^\varphi$ is holomorphically trivial over $X$ and we use the pull-back trivialization of $\mathcal{L}$ to set $\mathcal{L}^\varphi = X \times \mathbb{C}$. The holomorphic cotangent bundle of $\mathbb{C}$ is the complex span of $\{dw\}$. We define an isomorphism of holomorphic line bundles $\Psi : \Lambda^{1,0}(X) \xrightarrow{\sim} \mathcal{L}^\varphi$ as follows:

$$\Psi(dw) = 1.$$
in the chosen trivializations. With these provisos in mind, \((L, H, 0)\) is a chiral pair on the pair \((\mathbb{C}, \mathbb{C})\) for any holomorphic map \(\varphi: \mathbb{C} \rightarrow \mathbb{C}\). The Hermitian structure \(H^\varphi\) on \(L^\varphi\) reads (we take \(\kappa = 1\) for simplicity):

\[H^\varphi(f_1, f_2) = e^{(w\bar{w})^k} f_1 f_2.\]

Using the explicit form of \(\Psi\), the two-dimensional metric \(g\) associated to \(\varphi\) as described in Theorem 4.13 reads:

\[g = (\Psi^* H^\varphi)^* = e^{-(w\bar{w})^k} dw \odot d\bar{w}.\]

Since \(\varphi\) is a holomorphic map and \(g\) is constructed as required by Theorem 4.13, we conclude that \((g, \varphi, \Psi)\) is a supersymmetric solution with respect to the pair \((\mathbb{C}, \mathbb{C})\) and the chiral triple specified above. Explicit computation gives the following formula for the scalar curvature:

\[R_g = 4k^2(w\bar{w})^{k-1} e^{(w\bar{w})^k},\]

showing that it is positive semi-definite. Direct computation shows that:

\[|d\varphi|_{g, G}^2 = 4k^2(w\bar{w})^{k-1} e^{(w\bar{w})^k},\]

and thus as required for a supersymmetric solution we have:

\[R_g = |d\varphi|_{g, G}^2.\]

Furthermore:

\[\varphi^* G = 2k^2(w\bar{w})^{k-1} dw \odot d\bar{w} = \frac{g}{2} |d\varphi|_{g, G}^2,\]

and thus, as expected, the Einstein equation for \((g, \varphi)\) is satisfied. The only critical point of \(\varphi\) is \(0 \in \mathbb{C}\), which corresponds to the only point in \(\mathbb{C}\) at which the Gaussian curvature of \(g\) vanishes, as required by Theorem 4.13. Note that, crucially, although the symmetric bilinear form \(\varphi^* G\) is degenerate at \(0 \in \mathbb{C}\), the physical metric \(g\) constructed as prescribed by Theorem 4.13 is regular at \(0 \in \mathbb{C}\). Recall that \((g, \varphi, \Psi)\) cannot be smoothly extended to the one-point compactification of \(\mathbb{C}\), given by the Riemann sphere \(\mathbb{P}^1\). Furthermore, this example is not covered by Theorem 4.21 since the total curvature of \(g\) is not finite.

5. Conclusions and Open Problems

In this paper, we constructed a global geometric model for the bosonic sector and Killing spinor equations of four-dimensional \(\mathcal{N} = 1\) supergravity coupled to a Spin\(_c\)(3, 1) structure and to a non-linear sigma model, whose target space is given by a complex manifold admitting a novel geometric structure called a “chiral triple”. We dimensionally reduced the theory to a Riemann surface \(X\) and characterized all supersymmetric solutions on \(X\), classifying the allowed biholomorphism types of \(X\) for supersymmetric solutions with complete Riemannian metric and finite scalar energy. Furthermore, we introduced the notion of “anti-supersymmetric solution”, defined as a solution of a natural variant of the Killing spinor equations and we characterized all anti-supersymmetric solutions on a Riemann surface. More generally, we obtained a system of partial differential equations for a harmonic map with potential from a Riemann surface into a Kähler manifold which...
does not appear to have been studied before and which arises here as a consequence of $\mathcal{N} = 1$ chiral supersymmetry.

The geometric model which we constructed involves a number of new mathematical structures that have not been explored in the Mathematics literature. As such, the present work suggests several lines of research, such as:

- The geometric model presented in this article is based on the notion of chiral triple. Associated to this concept there are two natural problems. The first consists on characterizing which pairs $(X, M)$ consisting of a Riemann surface $X$ and a complex manifold $M$ admits chiral triples and thus give rise to admissible non-linear sigma models for $\mathcal{N} = 1$ chiral supergravity. The second problem concerns the classification of chiral triples for a given pair $(X, M)$, studying if these structures come in families and, if so, whether they form finite-dimensional moduli spaces.

- We characterized all supersymmetric solutions on Riemann surfaces $X$ for which the superpotential vanishes. However, the problem of classifying which Riemann surfaces (or, more ambitiously, which Lorentzian four-manifolds) admit supersymmetric solutions with non-vanishing superpotential (or even vanishing superpotential in the latter case) remains completely open. We expect the solution to this problem to depend markedly on the choice of scalar manifold and superpotential, whence a general solution will be probably out of reach. It is then reasonable to consider first the classification problem in the four-dimensional case for which the superpotential $W$ vanishes, as explained in Sect. 3.6 – a problem which is still open and where one may expect a complete classification result.

- An interesting class of supergravity solutions involves space-times which are globally hyperbolic Lorentzian four-manifolds. The main result of reference [12] characterizes such manifolds. Using this explicit presentation for the Lorentzian manifold, it would be very interesting to consider the classification problem in the four-dimensional case for which the superpotential $W$ vanishes, as explained in Sect. 3.6 – a problem which is still open and where one may expect a complete classification result.

- For certain space-times, such as the Riemannian product of two-dimensional Minkowski space and a Riemann surface, the Killing spinor equations of the theory give rise to well-defined moduli spaces of solutions. In such cases, one obtains novel moduli problems involving maps to a Kähler manifold, which may help understand the space of solutions of supergravity theories and could have further applications to differential topology. For the situation considered in Sect. 4.1, we found a variant of the moduli problem of holomorphic maps from a Riemann surface into a complex manifold.

- We considered a geometric model for which (in the language of reference [44]) the Lorentzian submersion $\pi$ describing the scalar sector of the theory is metrically trivial. It is an open problem to construct a geometric model for the Killing spinor equations in the case when $\pi$ is a non-trivial flat Lorentzian submersion.

- We considered a geometric model for chiral $\mathcal{N} = 1$ supergravity not coupled to background gauge fields, i.e. for which the ‘duality bundle’ of reference [43] is trivial. It would be interesting interesting to extend the construction to the case of non-trivial duality bundles.

- A particularly interesting open problem is to extend our geometric model to $\mathcal{N} = 2$ ungauged supergravity coupled to vector multiplets, understanding what variants of the notion of chiral triple and symplectic duality bundle are appropriate for this case, which should encode a version of projective Special Kähler geometry [2,24,54].
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