Estimation of critical speeds of a rotor supported on ball bearings

Pavuluri Uday1*, J Prabu1, M Solairaju2, C Shravankumar1, K Jegadeesan1 and T V V L N Rao1

1Department of Mechanical Engineering, SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India.
2Combat Vehicles Research and Development Establishment, Avadi, Tamil Nadu, India.

E-mail: uday_sn@srmuniv.edu.in

Abstract. In this work study of a rotor supported on ball bearings and containing a fatigue crack is carried out. The primary objective of the work is to obtain the load-distribution of the ball bearings from which the support stiffness in two transverse directions are obtained. The other objective is the rotor-dynamic analysis of a shaft supported on these bearings. A mathematical model of a rigid rotor supported on flexible bearings is considered for the purpose. Effects of bearing stiffness, damping and gyroscopic effects are included in the study. The governing equations of motion are obtained. The whirl frequencies of the rotor system are obtained using Eigen value formulation of the problem. A Campbell diagram is plotted which is used to obtain the critical speeds of the rotor system. These results are supported with the results from frequency response function plots.

1. Introduction
In the rotor system, the whirling phenomenon take place due to the masses which are not evenly distributed in the system. It gets higher at the critical speeds. The precession takes place when the shaft is rotated which leads to gyroscopic effect. In the forward and the backward whirling takes place due to the effect. This leads to the damage to the system. Thus determining the critical speed is essential. The critical speed is also referred to as peak response speeds. At the peak response speed the critical speeds will occur, when the whirl frequency is excited by the uneven of the shaft or by any other means.

In a rolling element bearing the contact forces at the rolling element – raceway interface varies nonlinearly with the rolling element deflection. When the bearings have a non-linear characteristics, the accurate rotor response calculation requires the bearing properties since the rotor dynamic response strongly depends on bearing properties. Fleming [1] in his paper has used the nonlinear bearing characteristics for range of speeds. The bearing stiffness was determined as a load and speed. In the analysis of rotor dynamic indicated that the unbalanced responses in the system varied nonlinearly with the amount of rotor imbalance. The bearing stiffness increases, caused a huge increment in the rotor response amplitude as the critical speed approached.
Swanson [2] in his paper explained about the critical speeds and its relation to the resonance and the natural frequencies. He also explained how the natural frequencies change as the rotational speed of the shaft changes and also he has given a view about the difference of the rotational natural frequencies with the more familiar natural frequencies and modes in the structures.

Malgol [3] in his paper modelled the rotor system using ANSYS environment and the analysis of the system has been done. In this paper, the author modelled the rotor system which consists of shaft and disk by considering 2D beam elastic element. The author considered the three different disk positions in the modal analysis and determined the natural frequency and mode shapes. In the contact zone of a lubricated ball bearing, a lubricated film is present which supports the load and provide the damping. In case of dry contact ball bearing, the damping is almost negligible. Hence the damping influences the system stability significantly.

Sarangi [4] presented the paper describing stiffness and damping characteristics of the lubricated ball bearing. The gyroscopic and mass effects have been considered and the paper comes to the result that the gyroscopic effect which are not influenced from the cylindrical rotor modes which remains fairly a constant frequency and the rotor speed and conversely the conical rotor modes where increased the rotor speed the components are split into the forward and backward which the increase and decrease in the frequency.

This paper work presents the study of rotor supported on ball bearings. The load distribution of the ball bearings are obtained. A rigid shaft bearing system is developed mathematically. The governing equations of motions are obtained using analytical approach. Based on the simulation, the whirl frequencies of the system are obtained using Eigen value formulation. Even in the preliminary design stages it is important to determine the critical speeds of a rotor-bearing system in order to select the safe operating speed range and to avoid resonance condition for the system.

2. Load distribution and Stiffness coefficients of a Ball Bearing

The following assumptions are considered while studying ball bearing model: there is steady state rotation of the ball elements and the inner raceway; there are no centrifugal forces or gyroscopic moments in the balls and there is no influence of frictional forces on load distribution.

The contact between a ball and the raceways can be considered as a point contact, though ideally the contact is a small elliptical area. The load-deflection relation is governed by the nonlinear relation given in Eqn. (1). Here, the normal/reaction load acting at the ball-raceway is represented by \( Q \), the load-deflection constant is \( K_p \) and \( \delta \) is the deflection/contact-deformation. \( n = 1.5 \) for ball bearing.

\[
Q = K_p \delta^n
\]

The total normal load-deflection constant is defined as the sum of the load-deformation constants at the inner and outer race ways. It is expressed as in Eqn. (2), in a manner similar to stiffness of springs in series. The load-deflection constants at the inner and outer raceways are represented as \( K_i \) and \( K_o \). The deflection of the bearing raceway is shown in figure 1.

\[
K_o = \left[ \frac{1}{K_i^n} + \frac{1}{K_o^n} \right]^{-\frac{1}{n}}
\]

Eqn. (3) gives the radial deflection \( \delta_p \) at a given ball angular position \( \varphi \), in a rigidly supported bearing with applied radial load \( F_r \), and \( P_d \) being the diametrical clearance.

\[
\delta_p = \delta_c \cos \varphi - \frac{P_d}{2} = \delta_{\max} \left[ 1 - \frac{1}{2 \epsilon} (1 - \cos \varphi) \right]
\]
Figure 1. Bearing rings after radial deflection.

In terms of maximum radial deformation $\delta_{max}$ (which occurs at $\varphi = 0$) and load-distribution factor $\varepsilon$, the radial deflection can also be expressed as shown in Eqn. (3). Here, $\delta_{max} = \delta_r - \frac{P_d}{2}$ and $\varepsilon = \frac{1}{2} \left(1 - \frac{P_d}{2\delta_r}\right)$.

Load zone is defined as the angular extent inside which the ball elements share the load. It is represented as $\pm \varphi_l$, which is given in Eqn. (4). At the boundaries of load zone, $\delta_{\varphi} = 0$. In case $P_d = 0$, then $\pm \varphi_l = \frac{\pi}{2}$.

$$\varphi_l = \cos^{-1} \left( \frac{P_d}{2\delta_r} \right) \quad (4)$$

With $Q$ being the reaction load at the ball elements, the relation between $Q_{\varphi}$ and $Q_{max}$ is presented in Eqn. (5).

$$\frac{Q_{\varphi}}{Q_{max}} = \left( \frac{\delta_{\varphi}}{\delta_{max}} \right)^n = \left[1 - \frac{1}{2\varepsilon} (1 - \cos \varphi)\right]^n \quad (5)$$

Let $F_r$ be the applied radial load which must be equal to the summation of the various vertical reaction loads $Q$ at the ball elements for static equilibrium condition, i.e., the residual of the two becomes zero. This is presented in Eqn. (6).

$$F_r = \sum_{\varphi} Q_{\varphi} \cos \varphi = Q_{max} \sum_{\varphi} \left[1 - \frac{1}{2\varepsilon} (1 - \cos \varphi)\right]^n \cos \varphi \quad (6)$$

The same applied radial load $F_r$ can be related to the reaction loads $Q_{\varphi}$ or $Q_{max}$, $\varepsilon$, $\varphi$ and written in an integral form as given in Eqn. (7).

$$F_r = \frac{ZQ_{max}}{2\pi} \int_{-\pi}^{\pi} \left[1 - \frac{1}{2\varepsilon} (1 - \cos \varphi)\right]^n \cos \varphi d\varphi = ZQ_{max} J, (\varepsilon) \quad (7)$$
For a particular value of load-distribution factor, the residual $F_r - ZQ_{max} J_r(\epsilon)$ becomes equal to zero, from which $J_r(\epsilon)$ and $F_r$ are determined for the given problem. The value of $\epsilon$ is obtained by trial and error in the range of $0 < \epsilon < 0.5$.

3. Calculation of bearing stiffness coefficients

The bearing specifications from manufacturer’s catalogue and some of the other bearing parameters are presented in tables 1 and 2.

| Table 1. Ball bearing specification. |
|-------------------------------|--------|-----------------|
| Specification                  | Value  |
| Bearing Number                 | 209    |
| Inner diameter raceway($d_i$)  | 52.29 mm |
| Outer diameter raceway($d_o$)  | 77.71 mm |
| Bearing pitch diameter($d_m$)  | 65 mm  |

| Table 2. Input parameters of ball bearing. |
|-------------------------------------------|--------|
| Bearing Parameters                        | Values |
| Diametric clearance, $P_o = d_o - d_i - 2D$ | 0.015 mm |
| $f_i$ = Inner raceway groove radius/D     | 0.52   |
| $f_o$ = Outer raceway groove radius/D     | 0.52   |
| $\gamma = D cos \alpha/d_m$               | 0.1954 |
| Mounted contact angle, $\alpha$           | $0^\circ$ |
| Curvature sum, $\sum \rho = (4/D)-(1/f_iD)+(2/D)(\gamma/1-\gamma)$ | 0.202 mm$^{-1}$ |
| Curvature difference, $F(\rho) = ((1/f_i)+2(\gamma/1-\gamma))/(4-(1/f_i)+2(\gamma/1-\gamma))$ | 0.138 mm$^{-1}$ |
| Curvature difference, $F(\rho_o) = ((1/f_o)-2(\gamma/1-\gamma))/(4-(1/f_o)-2(\gamma/1-\gamma))$ | 0.9399 |
| Dimensionless contact parameter($\delta\rho$) | 0.602 |
| Dimensionless contact parameter($\delta\rho_o$) | 0.658 |
| Load-deflection factor for inner raceway, $(K_{wo}) = 2.15e^{3} \sum \rho^{1/2} (\delta\rho)^{3/2}$ | 1.026 x 10$^{5}$ N/mm$^{1.5}$ |
| Load-deflection factor for inner raceway, $(K_{wo}) = 2.15e^{3} \sum \rho_o^{1/2} (\delta\rho_o)^{3/2}$ | 1.089 x 10$^{5}$ N/mm$^{1.5}$ |
| Total load-deflection factor, $K_w = [1/((1/K_o)^{0.667}+ (1/K_o)^{0.667})]^{1.5}$ | 3.735 x 10$^{6}$ N/mm$^{1.5}$ |

The diameter of shaft supported in the bearing is chosen as 45 mm. The load acting on each ball bearing $F_r$ for the rotor system is calculated as 167.03 N. It has been chosen as an input for further calculation of load-distribution and stiffness coefficients.

At the point of residual force becoming zero at first the load-distribution factor $\epsilon$ is obtained as 0.44 as shown in figure 2(a) Next, the load-distribution integral $J_r(\epsilon)$ is computed as 0.2191 from Equation (7) or from figure 2(b), the integration limits being $\pm \phi_b = 83.1^\circ$

Using Equation (6) maximum reaction load ($Q_{max}$ at $\phi = 0^\circ$) is calculated as 84.7 N. At angular intervals of 360$^\circ$/Z the load-distribution table is obtained for the chosen bearing. The deflections at ball locations $\phi$ are also calculated using Equation (4).
Using linear spring elements we make an assumption that the contact between the ball and raceways can be modelled. The stiffness values of these spring elements are determined using the radial load and deflection. The spring stiffness $k_\phi$ are also presented in table 3.

### Table 3. Load distribution, deformation and spring stiffness for bearing model.

| $\phi$ | $Q_\phi$ (N) | $\delta_\phi$ (mm) | $k_\phi$ (N/m) |
|-------|--------------|--------------------|----------------|
| 0     | 84.7         | 0.055              | $1540 \times 10^3$ |
| ±40   | 53.3         | 0.040              | $1332.5 \times 10^3$ |
| ±80   | 1.28         | 0.003              | $426.7 \times 10^3$ |
| ±120  | 0            | -                  | -              |
| ±160  | 0            | -                  | -              |

4. Mathematical model of a rigid rotor supported on flexible bearings

In this section, a shaft supported on ball bearing supports is considered for the study. The system can be described by a model of an inflexible rotor on flexible supports when the stiffness of the rotor supports is relatively small compared to the stiffness of the rotor. Equations of motion for such a model are derived in this section. At the left spring stiffness is represented by $k'_l, k'_l$, and at the right end is represented by $k'_r, k'_r$. Schematic diagram of a rotor bearing system is shown in figure 3.

![Figure 3. Schematic diagram of rigid rotor supported on flexible bearings.](image-url)
The z-axis of the coordinate system O-xyz coincides with the bearing centre line which connects the centre of the two bearings. The origin is at the position of the geometrical centre \( M \) of the cross-section with the centre of gravity \( G \). Let \( e \) be the eccentricity of \( G \). There is no dynamic unbalance i.e., the principal axes of the polar moment of inertia of the shaft coincides with the shaft centreline. It is inclined at an angle \( \theta \) to the bearing centre line during whirling motion. Let us assume that \( G \) is the distances of \( l_1 \) and \( l_2 \) from the left bearing and right bearing.

The system is modelled with four degree of freedom. They are two translational displacements and two rotational displacements: \( x \) and \( \theta \), in the \( x-z \) plane and \( y \) and \( \theta \), in the \( y-z \) plane. \( \theta_1 \) and \( \theta_2 \) are the projections of the inclination angle of the rigid shaft (i.e., \( \theta \)) onto the \( x-z \) plane and \( y-z \) plane respectively.

The governing equations of motion are written as given in Eqn. (8).

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
k_{xx}^l + k_{xx}^r & 0 & -k_{xx}^l & 0 \\
0 & k_{yy}^l + k_{yy}^r & 0 & -k_{yy}^l & 0 \\
-k_{xx}^l & 0 & k_{xx}^l & 0 \\
0 & -k_{yy}^l & 0 & k_{yy}^l & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\theta_1 \\
\theta_2
\end{bmatrix}
\]

In the final system equations of motion, the damping matrix and gyroscopic matrix can be combined together for an equivalent viscous damping model, which is represented as in Eqn. (9).

\[
[M][\dot{X}] + [(C) + [G]]\{\dot{X}\} + [K]\{X\} = \{0\}
\]

5. Numerical Problem

The equation of motion are written in the state space form. The Campbell diagram obtained. In between, the Eigen values and the Eigen vectors are obtained which corresponds to the whirl frequencies and the mode shapes separately. The input parameters of system are presented in the table 4.

| Table 4: Rotor-bearing system parameters. |
|-------------------------------------------|
| Shaft Mass | 34.05 kg |
| Shaft length | 2m |
| Shaft Diameter | 45mm |
| Shaft density | 7810 kg/m³ |
| Shaft Diometrical mass MOI | 4.30x10⁻³ kg/m² |
| Shaft Polar mass MOI | 8.6x10⁻³ kg/m³ |
| Ball bearing Stiffness \( K_{xx} \) | 2553.44x10⁻³ N/m |
| Ball bearing Stiffness \( K_{yy} \) | 3730.12x10⁻³ N/m |
| Ball bearing Damping \( C_{xx} \) | 2553.44Ns/m |
| Ball bearing Damping \( C_{yy} \) | 3730.12Ns/m |

5.1 Critical speeds

The Campbell diagram with the intersection of forward and backward precession mode order of line is shown in figure 4. The calculation of the critical speeds obtained from the Campbell diagram.
5.2 Modal analysis

The energy lost per cycle is dependent on the frequency of excitation in a viscous damping model, whereas experiments show that the hysteretic damping is independent of frequency. The governing equation of motion for a system with hysteretic damping is given in Eqn. (10).

\[
\begin{bmatrix} K + i[D] - \omega^2 [M] \end{bmatrix} \{X\} e^{i\omega t} = \{F\} e^{i\omega t}
\]

(10)

Here, \([K]\), \([D]\) and \([M]\) are the stiffness, damping and mass matrices. \(\omega\) is the response frequency/ excitation; \(\{X\}\) and \(\{F\}\) are displacement and force vectors; \(i = \sqrt{-1}\). \([D] = \alpha [K] + \beta [M]\), is the proportional hysteretic damping matrix, with \(\alpha\) and \(\beta\) as the proportionality constants.

Here, to obtain numerically \([\alpha (\omega)]\) is the flexibility matrix or FRF matrix, which in this form is difficult. The more convenient form of FRF matrix is given below in Eqn. (11).

\[
[\alpha(\omega)] = \varphi \left[ (\lambda^2 - \omega^2) \right]^{-1} \varphi^T
\]

(11)

6. Results and Discussions

From the figure 4, shows the Campbell diagram for a rigid rotor system. When the whirl frequency line intersects with the 1x harmonic of excitation, it gives the critical speed. In that 1x harmonic of excitation the shaft will be misalignment or the ball bearings will be misalignment in the rotor system.

![Figure 4. Whirl frequency vs spin frequency.](image)

The critical speeds obtained are as follows: 358.8 rad/s and 447.1 rad/s.

The Frequency Response Functions (FRFs) are obtained. The Magnitude-Frequency and Phase-Frequency plots are obtained for the direct flexibility coefficients of the system at the bearing locations. These are plotted as follows in the figure 5(a) and figure 5(b).
7. Conclusions

Thus in this paper, the rotor supported on ball bearings are studied. The load distributions of the ball bearings from which the support stiffness in two transverse directions are obtained. Mathematical modelling of the rotor system are carried out. The governing equations are obtained and based on the simulations, the whirl frequencies are obtained using the Eigen value formulation. The critical speeds are compared for the cases of with and without crack.

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Nomenclature

- \( Q \): Normal load (N)
- \( K_p \): Load-deflection factor
- \( \delta \): deflection
- \( n \): Load deflection exponent
- \( K_i \): Load deformation factor for inner raceway
- \( K_o \): Load deformation factor for outer raceway
- \( \delta_r \): Radial deflection (mm)
- \( F_r \): Radial load (N)
- \( Z \): number of balls
- \( D \): Ball diameter (mm)
- \( d_i \): Inner diameter raceway (mm)
- \( d_o \): Outer diameter raceway (mm)
- \( d_m \): Bearing pitch diameter (mm)
- \( r_i \): Inner groove radius (mm)
- \( r_o \): Outer groove radius (mm)
- \( P_d \): Bearing Diametral clearance (mm)
- \( \delta_o \): Radial deflection
- \( \varphi \): Angular position
- \( \delta_{max} \): Maximum radial deformation
- \( \epsilon \): Load distribution factor
- \( k_\varphi \): Spring stiffness
8. References

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