Active versus Passive Coherent Equalization of Passive Linear Quantum Systems

V. Ugrinovskii

M. R. James

Abstract—The paper considers the problem of equalization of passive linear quantum systems. While our previous work was concerned with the analysis and synthesis of passive equalizers, in this paper we analyze coherent quantum equalizers whose annihilation (respectively, creation) operator dynamics in the Heisenberg picture are driven by both quadratures of the channel output field. We show that the characteristics of the input field must be taken into consideration when choosing the type of the equalizing filter. In particular, we show that for thermal fields allowing the filter to process both quadratures of the channel output may not improve mean square accuracy of the input field estimate, in comparison with passive filters. This situation changes when the input field is ‘squeezed’.

I. INTRODUCTION

Quantum communication systems are subject to fundamental quantum mechanical limits which restrict their capacity to transfer information. Due to these limitations, the problem of correcting distortions in quantum communication systems differs significantly from its classical counterparts. This point has been demonstrated in [8] where we highlighted some conceptual differences which arise when Wiener’s paradigm of mean-square error optimization [5] is applied in the derivation of coherent quantum filters. In particular, optimal coherent equalizing filters may require a noise field to be injected into the filter, and for optimal performance, the filter must be tuned to balance this noise against the noise in the channel output.

In this paper, we continue the analysis of the coherent equalization problem for quantum communication channels introduced in [8]. Although the problem resembles the problem of optimal Wiener filtering, the coherent equalizer must satisfy the laws of quantum physics in that it must preserve certain operator commutation relations. This leads to additional requirements on the synthesized equalizing filter, known as physical realizability, which do not arise in the classical filtering theory [3], [7]. It has been shown in [8] that even in the simplified case concerned with equalization of passive quantum channels using passive coherent filters, these additional requirements translate into nontrivial optimization constraints, and the problem of optimal coherent filtering reduces to a challenging nonconvex optimization problem. It has been observed in [8] that an optimal passive coherent equalizer is not always able to reduce the mean-square error between the channel input and output fields. This naturally leads to the question as to whether expanding the class of filters to include more general active filters can help to resolve this issue.

As it turns out, the potentially greater flexibility in shaping the filter output offered by active filters is not always easy to realize — our first result identifies a class of coherent equalization problems involving thermal input field in which active coherent filters have no advantage over passive (noncausal) filters. On the other hand, our second result demonstrates that when the input field is ‘squeezed’, the mean-square optimal coherent filter utilizes both quadratures of the channel output.

The paper is organized as follows. In the next section we present the necessary basics of linear quantum systems. The quantum equalization problem is reviewed in Section III. Next, Section IV presents the results of the paper. Concluding remarks are presented in Section V.

Notation: For an operator a in a Hilbert space $\mathcal{H}$, $a^*$ denotes the Hermitian adjoint of a, and $a^\dagger$ denotes the complex conjugate transpose of a. The notation $\mathbf{col}(a_1, \ldots, a_n)$ denotes the column vector of operators obtained by concatenating operators $a_1, \ldots, a_n$. Let $\mathbf{col}(a_1, \ldots, a_n)$ be a column vector comprised of n operators (i.e., $a_i$ is an operator $\mathcal{H} \rightarrow \mathcal{H}$); then $a^\# = \mathbf{col}(a_1^*, \ldots, a_n^*)$ denotes the column vector transposed and conjugated. Let $A = \mathbf{col}(A_1, \ldots, A_n)$ be a matrix.

This work was supported by the Australian Research Council and the ARC Centre for Quantum Computation and Communication Technology. V. Ugrinovskii is with the School of Engineering and Information Technology, University of New South Wales Canberra, Canberra, ACT 2600, Australia, v.ougrinovskii@adfa.edu.au. M. R. James are with the ARC Centre for Quantum Computation and Communication Technology, Research School of Engineering, the Australian National University, Canberra, ACT 2601, Australia, matthew.james@anu.edu.au.

For any two complex matrices $X_-, X_+$, we write $\Delta(X_-, X_+) \equiv \begin{bmatrix} X_- & X_+ \\ X_-^\# & X_+^\# \end{bmatrix}$. When $X_-, X_+$ are complex...
transfer functions $X_-(s)$, $X_+(s)$, the corresponding stacking operation defines the transfer function $\Delta(X_-(s), X_+(s)) \triangleq \begin{bmatrix} X_-(s) & X_+(s) \\ (X_+(s^*))^\# & (X_-(s^*))^\# \end{bmatrix}$.

II. AN OPEN LINEAR SYSTEM MODEL OF A QUANTUM COMMUNICATION CHANNEL

In the Heissenberg picture of quantum mechanics, an open quantum system can be modeled as a linear system governed by an input field $\hat{b} = \text{col}(b, b^\#)$ where $b$ is a column vector of $n$ quantum noise processes, $b = \text{col}(b_1, \ldots, b_n)$ [2], [10]. The noise processes can be represented as annihilation operators on an appropriate Fock space [2], but from the system theory viewpoint they can be treated as quantum stochastic processes. In this paper, it will be assumed that these input processes represent Gaussian white noise processes with zero mean, $\langle b(t) \rangle = 0$, and the covariance

$$\begin{bmatrix} b(t) \\ b^\dagger(t') \end{bmatrix} \begin{bmatrix} \Sigma_b & \Pi_b \\ \Pi_b^\dagger & \Sigma_b \end{bmatrix} \delta(t - t'),$$

where $\Sigma_b$, $\Pi_b$ are complex matrices with the properties that $\Sigma_b = \Sigma_b^\dagger$, $\Pi_b = \Pi_b^\dagger$, and $\delta(t - t')$ is the $\delta$-function.

Using this notation, dynamics of an open quantum system without scattering are described by a quantum stochastic differential equation

$$\begin{align*}
\dot{a} &= \tilde{A}a + \tilde{B}b, \\
\dot{y} &= \tilde{C}a + \tilde{D}b.
\end{align*}$$

(3)

The column vector $\tilde{a} = \text{col}(a, a^\#)$ is composed of the column vector $a = \text{col}(a_1, \ldots, a_m)$ of annihilation operators on a certain Hilbert space $\mathcal{H}$ and the column vector $a^\# = \text{col}(a_1^*, \ldots, a_m^*)$ of the corresponding creation operators on the same Hilbert space. Also, $\tilde{y} = \text{col}(y, y^\#)$ denotes the output field of the system that carries away information about the system interacting with the input field $b$. The matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are partitioned accordingly, as

$$\begin{align*}
\tilde{A} &= \Delta(A_-, A_+), \\
\tilde{B} &= \Delta(B_-, B_+), \\
\tilde{C} &= \Delta(C_-, C_+), \\
\tilde{D} &= \Delta(D_-, D_+).
\end{align*}$$

A detailed discussion about open linear quantum systems can be found in references [4], [1], [3], [10].

In this paper, we are concerned with the situation where the system [3] models a quantum communication channel, and $b$ and $y$ describe the input and output signals of this channel. Furthermore, similarly to [8] we consider a class of passive communication channels [3] whose properties make them analogous to classical passive systems [4]. In a passive quantum system, $A_+ = 0$, $B_+ = 0$, $C_+ = 0$, and $D_+ = 0$. That is, the dynamics of $a$ are governed by the input $b$ consisting of annihilation operators only, and the dynamics of $a^\#$ are governed by the input $b^\#$ consisting of creation operators only. Passivity reflects the fact that the Hamiltonian of the system and its coupling with the environment only allow dissipation of energy.

With the above assumptions, the output field of the passive system [3] can be written as

$$\begin{align*}
y(t) &= C_- e^{A_- (t-t_0)} a(t_0) + \int_{t_0}^t g(t - \tau) b(\tau) d\tau, \\
y^\#(t) &= C_-^\# e^{A_-^\# (t-t_0)} a^\#(t_0) + \int_{t_0}^t g^\#(t - \tau) b^\#(\tau) d\tau.
\end{align*}$$

(4)

Here we introduced the notation for the impulse response, associated with the annihilation part of the system [10],

$$g(t) = \begin{cases} C_- e^{A_- t} B_- + \delta(t) I, & t \geq 0, \\
0, & t < 0. \end{cases}$$

(5)

The transfer function of the passive system [3] is then

$$\Gamma(s) = \begin{bmatrix} G(s) & 0 \\
0 & G(s^*)^\# \end{bmatrix},$$

where $G(s) = C_- (sI - A_-)^{-1} B_- + I$. Since $B_- = -C_-^\dagger$, the transfer functions $G(s)$ and $\Gamma(s)$ are square matrices.

Not every system of the form [3] corresponds to physical quantum dynamics. For this to be true, the system must preserve the canonical commutation relations during its evolution [7], [3]. This property translates to a formal requirement [7], [10] that for a physically realizable system [3] it must hold that

$$G(s)G(-s^*)^\dagger = I, \quad \Gamma(s)J\Gamma(-s^*)^\dagger = J.$$

(6)

We will be concerned with stationary behaviours of the systems under consideration. Suppose that the matrix $A_-$ is stable, then the stationary component of the system output is obtained from [3] by letting $t_0 \to -\infty$:

$$\begin{align*}
y(t) &= \int_{-\infty}^{+\infty} g(t - \tau) b(\tau) d\tau, \\
y^\#(t) &= \int_{-\infty}^{+\infty} g^\#(t - \tau) b^\#(\tau) d\tau.
\end{align*}$$

(7)

The upper limit of integration has been changed to $+\infty$ since $g(t)$ is causal by definition. Since the matrix $A_-$ is Hurwitz, $P_{y,y^\#}(s)$ is well defined on the imaginary axis and $P_{y,y^\#}(s)|_{s=\omega} = P_{y,y^\#}(i\omega)$, where the expression on the left-hand side refers to the bilateral Laplace transform and the expression on right-hand side is the Fourier transform of $R_{y,y^\#}(t)$. Both expressions are usually referred to as the cross power spectrum density (cross PSD) [5]. It is easy to obtain that the power spectrum density matrix of the output $y(t)$, $P_{y,y}(s) = (P_{y,y^\#}(s))^\#_{j,k=1}$ is related to the power spectrum density matrix of the noise $b$ in the standard manner [10]:

$$P_{y,y}(s) = \Gamma(s) \begin{bmatrix} I + \Sigma_b^\dagger \Pi_b \Pi_b^\dagger \Sigma_b \end{bmatrix} [\Gamma(-s^*)]^\dagger.$$

(8)

III. ACTIVE EQUALIZATION OF PASSIVE QUANTUM COMMUNICATION CHANNELS

In this section, we review the general equalization scheme introduced in [8] and introduce the class of quantum systems which serve as candidate coherent equalizers.
Consider the system in Fig. 1 consisting of a quantum channel \( \Gamma(s) \) and a second quantum system acting as an equalizer. The input field \( b \) plays the role of a message signal transmitted through the channel, and \( w \) denotes the vector comprised of quantum noises. It includes the noise inputs that are necessarily present in the physically realizable system [3], [9], as well as noises introduced by routing devices such as beam splitters. In terms of the notation adopted in the previous section, we have \( b = \text{col}(b, w) \). This combined input and its adjoint signal are transmitted through the quantum channel with the transfer function \( \Gamma(s) \), as described in the previous section, to produce the output \( \hat{y} = \text{col}(y_b, y_w, y_w^\#) \). The input to the filter \( \Xi(s) \) is comprised of the channel output field components \( \hat{y}_b = \text{col}(y_b, y_b^#) \). We emphasize that in contrast to [8], we consider filters that process both quadratures of the channel output \( \hat{y}_b \).

Unlike the classical Wiener equalization problem, a coherent filter must be realizable as a quantum system, and therefore it must preserve canonical commutation relations. For this, the filter system must satisfy the physical realizability conditions analogous to condition (6). It is known [3], [9] that for this, additional noise inputs may need to be injected into the filter; the input \( z \) in Fig. 1 symbolizes those additional noise inputs. As in [8], the added noise \( z \) will be assumed to be in the Gaussian vacuum state, i.e., the corresponding mean and covariance of \( z \) are

\[
\langle z(t) \rangle = 0, \quad \left[ \begin{array}{c} z(t) \\ z^\dagger(t') \end{array} \right] = \left[ \begin{array}{rr} I & 0 \\ 0 & 0 \end{array} \right] \delta(t - t'). \tag{9}
\]

With the additional noise \( z \) injected into the filter, the filter system can be regarded as a mapping \( \hat{u} \rightarrow \hat{u}_w \), where \( u = \text{col}(y_b, z) \), \( \hat{u} = \text{col}(\hat{b}, \hat{z}) \). The filter transfer function \( \Xi(s) \) can be partitioned accordingly:

\[
\Xi(s) = \Delta(H(s), T(s)) = \left[ \begin{array}{cc} H(s) & T(s) \\ T(s)^\# & H(s)^\# \end{array} \right]. \tag{10}
\]

The physical realizability condition for the filter is analogous to (6).

\[
\Xi(s)J\Xi(-s^\#)^\dagger = J. \tag{11}
\]

The set of physically realizable equalizers \( \Xi(s) \), i.e., filters satisfying condition (11) will be denoted \( \mathcal{H} \). In the sequel, we will also consider a subset of the set \( \mathcal{H} \) consisting of constant complex \( J \)-symplectic matrices \( \Xi = \Delta(H, T) \). The set of such matrices will be denoted \( \mathcal{H}_p \).

Reference [8] proposed an approach to the design of coherent equalizers which was analogous to the classical mean-square equalization scheme. It aimed to compute a physically realizable transfer function of a filter by minimizing the power spectrum density \( P_{e,c}(\omega) \) of the error operator \( e(t) = b(t) - b(t) \). In this paper, the optimal equalization objective is defined in a similar manner, as

\[
\min_{\Xi \in \mathcal{H}} \sup_\omega \text{tr} P_{e,c}(\omega). \tag{12}
\]

The outer optimization operation is to be carried out over the set \( \mathcal{H} \) of filters which satisfy (11). Thus, (12) represents a constrained optimization problem.

The problem in [8] can be regarded as a special case of the problem (12) in which optimizers \( \Xi(s) \) are constrained to those of the form \( \Xi(s) = \Delta(H(s), 0) \). Let \( \mathcal{H}_p \) denote the class of such filters,

\[
\mathcal{H}_p = \{ \Xi \in \mathcal{H} : \Xi(s) = \Delta(H(s), 0) \}. \tag{13}
\]

Note that for \( \Xi \in \mathcal{H}_p \), the condition (11) reduces to the condition \( H_p(s)H_p(-s^\#)^\dagger = I \). Clearly, we have

\[
\min_{\Xi \in \mathcal{H}} \sup_\omega \text{tr} P_{e,c}(\omega) \leq \min_{\Xi \in \mathcal{H}_p} \sup_\omega \text{tr} P_{e,c}(\omega). \tag{13}
\]

In the next section we will consider a situation where the optimal values of the two problems are equal. Also, we will discuss a version of the coherent equalization problem where the inequality in (13) is a strict inequality.

Consider the partitions of the transfer functions \( H(s) \) and \( T(s) \) compatible with the partitions of the filter input and output operators \( \text{col}(y_b, z) \) and \( \text{col}(b, \hat{z}) \) and the corresponding partitions of the adjoint operators:

\[
H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}, \quad T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}. \tag{14}
\]

Also, consider the partition of the transfer functions \( G(s) \) compatible with the partition \( b = \text{col}(b, w), y = \text{col}(y_b, y_w) \):

\[
G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}. \tag{15}
\]

The covariance matrix of the input \( \hat{b} \) is assumed to be partitioned accordingly, as

\[
\begin{bmatrix} I + \Sigma_b^T \Pi_b \\ \Pi_b^T \Sigma_b \end{bmatrix} = \begin{bmatrix} I + \Sigma_b^T \Pi_b \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Pi_b^T \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Pi_b^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Pi_b & 0 \\ 0 & 0 \end{bmatrix}. \tag{16}
\]

The matrix on the right-hand side reflects a standard assumption that the message signal \( b \) and the noise signal \( w \) are not correlated. Also (16) reflects the standing assumption in this paper that \( \langle w(t) w^\dagger(t') \rangle = \langle w(t)^\# w^\dagger(t') \rangle = 0 \). This assumption about the noise field \( w \) is the same as the corresponding assumption made in [8]. However, in contrast to [8] here we do not generally assume that \( \Pi_b = 0 \).
With these assumptions, we obtain using (13) that
\[
P_{e,e}(s, \Xi(s)) = (H_{11}(s)G_{11}(s) - I)(I + \Sigma_b^T)G_{12}(s)H_{11}(s)^\dagger - I + H_{11}(s)G_{12}(s)(I + \Sigma_b^T)G_{12}(s)H_{11}(s)^\dagger + H_{12}(s)H_{12}(s)^\dagger + T_{11}(s)G_{11}(s)^\dagger \Sigma_b G_{11}(s)^T T_{11}(s)^\dagger
\]
\[
+ T_{11}(s)G_{12}(s)^\dagger \Sigma_b G_{12}(s)^T T_{11}(s)^\dagger + T_{11}(s)G_{11}(s)^\dagger \Pi_b^T(11)(-s)^T T_{11}(s)^\dagger + (H_{11}(s)G_{11}(s) - I) \Pi_b G_{11}(s)^T T_{11}(s)^\dagger. \tag{17}
\]

Here, we used the notation \(P_{e,e}(s, \Xi(s))\) to specify the transfer function \(\Xi(s)\) of the system used as the filter in the system in Fig. 1. Also, the constraint (11) can be expanded as follows,
\[
H_{11}(s)H_{11}(s)^\dagger + H_{12}(s)H_{12}(s)^\dagger - T_{11}(s)T_{11}(s)^\dagger - T_{12}(s)T_{12}(s)^\dagger = I, \tag{18}
\]
\[
H_{11}(s)H_{21}(s)^\dagger + H_{12}(s)H_{22}(s)^\dagger - T_{11}(s)T_{21}(s)^\dagger - T_{12}(s)T_{22}(s)^\dagger = 0, \tag{19}
\]
\[
H_{11}(s)T_{11}(s)^{T}s - H_{12}(s)T_{12}(s)^{T}s - T_{11}(s)H_{11}(s)^{-1} - T_{12}(s)H_{12}(s)^{-1} = 0, \tag{20}
\]
\[
H_{11}(s)T_{21}(s)^{T}s + H_{12}(s)T_{22}(s)^{T}s - T_{11}(s)H_{21}(s)^{-1} - T_{12}(s)H_{22}(s)^{-1} = I, \tag{21}
\]
\[
H_{11}(s)T_{21}(s)^{T}s - H_{12}(s)T_{22}(s)^{T}s - T_{11}(s)H_{21}(s)^{-1} - T_{12}(s)H_{22}(s)^{-1} = 0, \tag{22}
\]
\[
H_{11}(s)T_{21}(s)^{T}s + H_{12}(s)T_{22}(s)^{T}s - T_{11}(s)H_{21}(s)^{-1} - T_{12}(s)H_{22}(s)^{-1} = 0. \tag{23}
\]

From (17), we observe that the spectral density function \(P_{e,e}(s, \Xi(s))\) depends only on the transfer functions \(H_{11}(s), H_{12}(s)\) and \(T_{11}(s)\) within \(\Xi(s)\). Therefore, similarly to [8] a two-step procedure can be employed to solve the constrained optimization problem (12). The first step of this procedure is to minimize the power spectrum density objective \(\sup_{s} P_{e,e}(\omega, \Xi(\omega))\) subject to the following relaxed version of (18) as the optimization constraint,
\[
H_{11}(\omega)H_{11}(\omega)^{\dagger} + H_{12}(\omega)H_{12}(\omega)^{\dagger} - T_{11}(\omega)T_{11}(\omega)^{\dagger} - T_{12}(\omega)T_{12}(\omega)^{\dagger} = I. \tag{24}
\]
Indeed, if an optimal equalizer exists in the problem (12), it must necessarily satisfy (24). Let \(\mathcal{H}_p\) denote the set of transfer functions of the form \(\Xi(s) = \Delta(H(s), T(s))\) which satisfy the condition (24) for a given \(\omega\). Also, let \(\mathcal{H} = \bigcap_{\omega} \mathcal{H}_p\), i.e., \(\mathcal{H}\) is a set of transfer functions of the form \(\Xi(s) = \Delta(H(s), T(s))\) which satisfy the condition (24) for every \(\omega\). Thus, \(\mathcal{H} \subseteq \mathcal{H}_p\), and we obtain the following lower bound on (12),
\[
\min_{\Xi \in \mathcal{H}} \sup_{\omega} \text{tr} P_{e,e}(\omega) \leq \min_{\Xi \in \mathcal{H}} \sup_{\omega} \text{tr} P_{e,e}(\omega). \tag{25}
\]

In the second step, the set of solutions of the problem on the left-hand side of (25) must be reduced to select only those \(\Xi(s)\) which satisfy all of the constraints (18)-(23). If such \(\Xi(s)\) can be selected, then the lower bound (25) is tight. In the next section, we will use this procedure to investigate whether expanding the class of filters from passive filters of the form \(\Delta(H(s),0)\) (as considered in [8]) to filters of the form \(\Delta(H(s),T(s))\) leads to an improved mean-square error.

IV. THE MAIN RESULTS

This section presents the main results of the paper. In Section IV-A, an equalization problem is presented in which expanding the set of filters from \(\mathcal{H}_p\) to \(\mathcal{H}\) does not reduce the optimal power spectrum density guaranteed by passive filters. In this problem \(\langle b(t)b^T(t') \rangle = 0\), i.e., \(\Pi_b = 0\). Next, in Section IV-B, the coherent equalization problem for static channels will be analyzed in which the input filed \(b\) is squeezed, i.e., \(\langle b(t)b^T(t') \rangle = \Pi_b \delta(t-t')\), \(\Pi_b \neq 0\). It will be shown that any optimal equalizer arising in this problem utilizes both inputs \(y_b\) and \(y_b^\ast\).

To present these results, we restrict the class of systems under consideration to systems with scalar operator inputs. Accordingly, the transfer functions \(G_{ij}, H_{ij}, T_{ij}\) are assumed to be scalar. The constraint (24) reduces to
\[
|H_{11}^2(\omega)|^2 + |H_{12}(\omega)|^2 - |T_{11}(\omega)|^2 - |T_{12}(\omega)|^2 = 1. \tag{26}
\]

Also, \(\Sigma_b, \Sigma_w, \Pi_b, \ldots\), are scalars. To emphasize the latter fact, we will use the lower case notation, i.e., \(\Sigma_b = \sigma_b^2, \Sigma_w = \sigma_w^2, \Pi_b = \pi_b, \ldots\). Since, \(\sigma_b, \sigma_w, \pi_b, \ldots\) are constant, and \(\sigma_b, \sigma_w, \pi_b, \ldots\) are real. In this case, \(P_{e,e}(\omega, \Xi(\omega))\) is scalar,
\[
P_{e,e}(\omega, \Xi(\omega)) = (1 + \sigma_b^2)H_{11}^2(\omega)G_{11}(\omega) - 1^2 + (1 + \sigma_w^2)|H_{12}(\omega)|^2G_{12}(\omega)^2 + |H_{12}(\omega)|^2
\]
\[
+ (\sigma_b^2G_{11}(\omega))^2 + (\sigma_w^2G_{12}(\omega))^2 |T_{11}(\omega)|^2. + 2\text{Re}[\pi_b(H_{11}(\omega)G_{11}(\omega) - 1)G_{11}(\omega)T_{11}(\omega)^\ast]. \tag{27}
\]

A. Equalization of scalar passive quantum channels when \(\pi_b = 0\)

When an equalizing filter is restricted to have \(T_{11}(s) = 0, T_{12}(s) = 0\), the function (27) reduces to the function used as an optimization objective in [8]. Indeed, letting \(T_{11}(s) = 0, T_{12}(s) = 0\) means that the output channel \(b\) of the filter is \(b = H_{11}(s)b + H_{12}(s)z\). The following results confirm that in this case optimization of the error power spectrum density can be reduced to optimization over the set \(\mathcal{H}_p\).

**Lemma 1**: If \(T_{11}(s) = 0, T_{12}(s) = 0\) in the partition (14) of \(\Xi(s)\) which satisfies condition (18), then, there exists \(\Xi_p(s) \in \mathcal{H}_p\), such that
\[
P_{e,e}(\omega, \Xi_0(\omega)) = P_{e,e}(\omega, \Xi_p(\omega)) \quad \forall \omega \in \mathbb{R}^1. \tag{28}
\]

**Remark 1**: The transfer function \(\Xi_p\) may not be causal.

We now discuss a method for obtaining a passive physically realizable filter which attains an optimal value in the problem (12).

Since \(\pi_b = 0\), the scalar function \(P_{e,e}(\omega, \Xi(\omega))\) becomes
\[
P_{e,e}(\omega, \Xi(\omega)) = (1 + \sigma_b^2)H_{11}^2(\omega)G_{11}(\omega)^2 - 1^2
\]
\[
+ (1 + \sigma_w^2)|H_{11}(\omega)|^2|G_{12}(\omega)|^2 + |H_{12}(\omega)|^2
\]
\[
+ (\sigma_b^2G_{11}(\omega)|^2 + (\sigma_w^2G_{12}(\omega))^2 |T_{11}(\omega)|^2. \tag{29}
\]
First we establish a result about an auxiliary point-wise optimization problem

\[ V_\omega \triangleq \min_{\Xi \in \mathcal{H}_p \omega} P_{e,e}(i\omega, \Xi). \tag{30} \]

Here \( \Xi = \Delta(H,T) \) is a complex constant matrix composed of complex matrices \( H, T \), partitioned in the same way as in \( \Omega \), and the notation \( \mathcal{H}_p \omega \) refers to the set of constant matrices \( \Xi = \Delta(H,T) \) which satisfy the condition

\[ |H_{11}|^2 + |H_{12}|^2 - |T_{11}|^2 - |T_{12}|^2 = 1. \tag{31} \]

Furthermore, \( P_{e,e}(i\omega, \Xi) \) refers to the value on the right-hand side of (29) in which the components of \( \Xi(i\omega) \) are replaced with the corresponding components of \( \Xi \).

Without loss of generality, it will be assumed that

\[ \psi(i\omega) \triangleq \sigma_b^2(G_{11}(i\omega))^2 + \sigma_w^2(G_{12}(i\omega))^2 > 0, \]

\[ |G_{11}(i\omega)| > 0 \quad \forall \omega \in \mathbb{R}^1. \tag{32} \]

**Lemma 2:** Suppose \( \pi_0 = 0 \). Let \( \omega \) be fixed and let a matrix \( \Xi_\omega = \Delta(H_\omega, T_\omega) \) attain the minimum in the problem (30). Then \( T_{\omega,11} = 0 \). Furthermore, the following statements hold.

1) If

\[ \frac{(1 + \sigma_b^2)(G_{11}(i\omega))}{1 + \psi(i\omega)} > 1, \tag{33} \]

then \( H_{\omega,12} = 0 \), \( H_{\omega,11} = \frac{(1 + \sigma_b^2)(G_{11}(i\omega))^*}{1 + \psi(i\omega)} \), \( |T_{\omega,12}|^2 = \frac{(1 + \sigma_b^2)(G_{12}(i\omega))^2}{(1 + \psi(i\omega))^2} - 1 > 0 \), and

\[ V_\omega = (1 + \sigma_b^2) \left( 1 - \frac{(1 + \sigma_b^2)(G_{11}(i\omega))^2}{1 + \psi(i\omega)} \right). \tag{34} \]

2) If

\[ \frac{\psi(i\omega)}{1 + \psi(i\omega)} < \frac{(1 + \sigma_b^2)(G_{11}(i\omega))}{1 + \psi(i\omega)} \leq 1, \tag{35} \]

then \( H_{\omega,11} = \frac{G_{11}(i\omega)^*}{(G_{11}(i\omega))^2} \), \( H_{\omega,12} = 0 \), \( T_{\omega,12} = 0 \). Furthermore,

\[ V_\omega = (1 + \sigma_b^2) + \psi(i\omega) - 2(1 + \sigma_b^2)G_{11}(i\omega). \tag{36} \]

3) If

\[ (1 + \sigma_b^2)(G_{11}(i\omega)) \leq \psi(i\omega), \tag{37} \]

then \( H_{\omega,11} = \frac{(1 + \sigma_b^2)(G_{11}(i\omega))^*}{\psi(i\omega)} \), \( T_{\omega,12} = 0 \) and \( H_{\omega,12} \) is such that \( |H_{\omega,12}|^2 = 1 - \frac{(1 + \sigma_b^2)(G_{11}(i\omega))^2}{\psi(i\omega)^2} \). Furthermore,

\[ V_\omega = (2 + \sigma_b^2) - \frac{(1 + \sigma_b^2)^2(G_{11}(i\omega))^2}{\psi(i\omega)}. \tag{38} \]

The proof of Lemma 2 is based on the method of Lagrange multiplier and is omitted for brevity.

We now establish a connection between the point-wise optimization problem (30) and the underlying problem (12) in the case where the channel \( G(s) \) satisfies the condition

\[ \frac{(1 + \sigma_b^2)(G_{11}(i\omega))}{1 + \psi(i\omega)} \leq 1, \quad \forall \omega \in \mathbb{R}^1. \tag{39} \]

According to Lemma 2 under this condition, \( T_{\omega,11} = T_{\omega,12} = 0 \) in any optimal point \( \Xi_\omega \) of the problem (30).

Note that since \( G_{11}(s) \) and \( G_{12}(s) \) are rational transfer functions, the expressions for \( H_{\omega,11}, H_{\omega,12} \) obtained in Lemma 2 are also rational functions of \( \omega \). Therefore using standard techniques, one can obtain transfer functions \( H_{11}(s), H_{12}(s) \) which match the frequency responses \( H_{\omega,11}, H_{\omega,12} \) obtained in Lemma 2. Secondly, under condition (39), \( T_{\omega,11} = T_{\omega,12} = 0 \), and we will show next that the only rational transfer functions \( T_{11}(s), T_{12}(s) \) that satisfy this requirement are \( T_{11}(s) = 0, T_{12}(s) = 0 \). The remaining entries of the matrix \( \Xi_\omega = \Delta(H_{11}, T_{12}) \) do not affect the optimal value of the objective function \( P_e(i\omega, \Xi_\omega) \). They can be selected so that a point-wise optimal solution \( \Xi_\omega \) of the problem (30) represents a frequency response of a rational transfer function.

**Theorem 1:** Suppose \( \pi_0 = 0 \) and condition (39) is satisfied. If \( \Xi_\omega = \Delta(H_{11}, T_{12}) \) attains the minimum in (30) for every \( \omega \in \mathbb{R}^1 \) and there exists a rational transfer function \( \Xi(s) = \Delta(H(s), T(s)) \) such that \( \Xi(i\omega) = \Xi_\omega \) \( \forall \omega \), then:

(i) \( T_{11}(s) = 0, T_{12}(s) = 0 \).

(ii) \( \Xi(s) \in \mathcal{H}_p \) and attains the minimum in the problem

\[ \min_{\Xi \in \mathcal{H}_p \omega} P_e(e, e(\Xi(i\omega)), \Xi(i\omega)), \tag{40} \]

(iii) In addition, if \( H_{11}(s), \bar{H}_{12}(s) \) satisfy

\[ H_{11}(s)\bar{H}_{12}(-s^*)^1 + \bar{H}_{12}(s)\bar{H}_{12}(-s^*)^1 = 1, \tag{41} \]

then an optimal filter in the problem (12) can be found within the class of passive filters \( \mathcal{H}_p \).

When either condition (35) or condition (37) hold for all \( \omega \), Theorem 1 provides a constructive method for deriving a passive (possibly noncausal) optimal transfer function which solves the underlying optimization problem (12). Under either of these conditions Lemma 2 yields single closed form expressions for \( H_{\omega,11} \) and \( H_{\omega,12} \). This allows to obtain transfer functions \( H_{11}(s), H_{12}(s) \) such that \( H_{11}(i\omega) = H_{\omega,11}, H_{12}(i\omega) = H_{\omega,12} \) using standard factorization techniques. Next, using the obtained \( H_{11}(s), H_{12}(s) \), a transfer function \( H_{\pi}(s) \) can be constructed as described in Lemma 1. The resulting transfer function \( \Xi_\pi(s) = \Delta(H_{\pi}(s), 0) \) is physically realizable, and \( \Xi_\omega = \Xi_{\pi}(i\omega) = \Delta(H_{\pi}(i\omega), 0) \) satisfies all conditions of Theorem 1. Therefore \( \Xi_\pi(s) = \Delta(H_{\pi}(s), 0) \) is an optimal passive equalizer for the problem (12).

**B. Equalization of scalar static passive quantum channels when \( \pi_0 \neq 0 \): An optimal filter is an active quantum system**

In this section, the optimization problem (12) is revisited for a squeezed noise input \( b \), i.e., when \( \pi_0 \neq 0 \). To demonstrate that in this case active filters may provide an advantage over passive filters, it will suffice to consider a static quantum channel. That is, in this section we assume that the transfer function \( G(s) \) is a constant unitary matrix:

\[ \begin{bmatrix} y_b \\ y_w \end{bmatrix} = G \begin{bmatrix} b \\ w \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} \\ -e^{i\theta}G_{12} \\ e^{i\theta}G_{11} \end{bmatrix}; \tag{42} \]

where \( \theta \in [0, 2\pi] \) and \( |G_{11}|^2 + |G_{12}|^2 = 1 \).

The quantity \( \psi = \sigma_b^2|G_{11}|^2 + \sigma_w^2|G_{12}|^2 \) is constant in this case. Also, as in the previous section, assume that \( |G_{11}| > 0 \).

Since the covariance matrix \( \begin{bmatrix} 1 + \sigma_b^2 \pi_b \\ \pi_b \sigma_b^2 \end{bmatrix} \) is positive definite,
and \( \pi_b \neq 0 \), then it must hold that \( \sigma_b^2 > 0 \). Together with the assumption that \( |G_{11}| > 0 \) this implies \( \psi > 0 \). That is, conditions (33) are satisfied in this section as well. In addition, we will assume in this section that \( |G_{11}|^2 < 1 \); this implies that \( |G_{12}|^2 > 0 \) since \( |G_{11}|^2 + |G_{12}|^2 = 1 \). These assumptions mean that we do not consider unrealistic situations where the channel is noiseless or blocks transmission of the field \( b \).

Since all coefficients in (42) are constants, in this section we will suppress the variable \( i\omega \) and write \( P_{e,e}(\Xi) \) or \( P_{e,e}(\Xi(\omega)) \) for a \( \Xi(s) = \Delta(H,T) \).

The main result of this section is as follows.

**Theorem 2:** Suppose \( 0 < |G_{11}| < 1 \) and \( \pi_b \neq 0 \). If a proper rational transfer function \( \Xi_0 = \Delta(H_0,T_0) \in \mathcal{H} \) is an optimal filter in the problem (12) for a static channel (42), then it must hold that \( T_{0,11}(s) \neq 0 \). Furthermore, the same optimal performance can be achieved using a static coherent filter in which \( T_{11} \neq 0 \) and at least one of the coefficients \( H_{12}, T_{12} \) is equal to 0.

From Theorem 2, it follows that a mean-square optimal estimate of a scalar squeezed input \( b \) transmitted via a static quantum channel can be obtained using one of the following expressions

\[
\hat{b} = H_{11}b + T_{11}b^* + T_{12}z^*
\]
or

\[
\hat{b} = H_{11}b + H_{12}z + T_{11}b^*.
\]
The coefficients of these filters are constant. They can be obtained from the auxiliary optimization problem

\[
V \triangleq \min_{\mathcal{H}_{e,0}} P_{e,e}(\Xi) \tag{43}
\]
which can be solved directly using the Lagrange multiplier technique. Since for a static channel \( G \) the cost of the equalization problem does not depend on the frequency variable \( \omega \) explicitly, the auxiliary problem (43) is not parameterized by \( \omega \). The optimization in (43) is carried out over the set of constant matrices \( \Xi = \Delta(H,T) \) subject to the constraint (31).

**V. Conclusions**

The paper has presented new results on the quantum counterpart of the classical Wiener filtering approach to equalization of quantum communication systems introduced in [8]. It has focused on the question as to whether the mean-square performance achievable by passive annihilation-only filters can be improved by driving the filter’s annihilation (respectively, creation) dynamics by both annihilation and creation components of the channel output. We have shown that in general, the answer to this question depends on the characteristics of the channel input field.

When the channel input field is in thermal state, i.e., \( \langle b(t)b^T(t') \rangle = 0 \), the paper has provided general conditions under which a physically realizable coherent mean-square optimal equalizing filter can be found within the class of passive (possibly noncausal) filters. In this case, optimal equalization of the channel distortion and noise may be accomplished passively, by dissipating energy in the filter output field. On the other hand, when the scalar input field to a static channel is in a squeezed state so that \( \langle b(t)b^T(t') \rangle = \pi_b \delta(t-t') \), with \( \pi_b \neq 0 \), the mean-square optimal estimate of the input field can only be obtained using an active filter.

The requirement for physical realizability of a filter introduces a critical constraint into the proposed optimization approach to quantum equalization. Due to this requirement, our result ascertaining the possibility of using passive equalizers for thermal input fields is limited in that the optimal passive filter constructed in Section [V-A] may not be causal. One of the possible directions for future work will be to address the requirement for causality of synthesized equalizers in a systematic manner.

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