DE RHAM COHOMOLOGY
OF CONFIGURATION SPACES
WITH POISSON MEASURE

SERGIO ALBEVERIO
Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany;
SFB 256, Univ. Bonn, Germany;
SFB 237, Bochum–Düsseldorf–Essen, Germany;
CERFIM (Locarno); Acc. Arch. (USI), Switzerland;
BiBoS, Univ. Bielefeld, Germany

ALEXEI DALETSKII
Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany;
SFB 256, Univ. Bonn, Germany;
Institute of Mathematics, Kiev, Ukraine;
BiBoS, Univ. Bielefeld, Germany

EUGENE LYTVYNOV
Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany;
SFB 256, Univ. Bonn, Germany;
BiBoS, Univ. Bielefeld, Germany

Abstract
The space $\Gamma_X$ of all locally finite configurations in a Riemannian manifold $X$ of infinite volume is considered. The deRham complex of square-integrable differential forms over $\Gamma_X$, equipped with the Poisson measure, and the corresponding deRham cohomology are studied. The latter is shown to be unitarily isomorphic to a certain Hilbert tensor algebra generated by the $L^2$-cohomology of the underlying manifold $X$.

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1 Introduction

Let $\Gamma_X$ denote the space of all locally finite configurations in a complete, stochastically complete, connected, oriented Riemannian manifold $X$ of infinite volume. In this paper, we define and study the deRham complex of square-integrable differential forms over the configuration space $\Gamma_X$ equipped with the Poisson measure.

The growing interest in geometry and analysis on the configuration spaces can be explained by the fact that these naturally appear in different problems of statistical mechanics and quantum physics. In [7, 8, 9], an approach to the configuration spaces as infinite-dimensional manifolds was initiated. This approach was motivated by the theory of representations of diffeomorphism groups, see [29, 47, 31] (these references as well as [9, 11] also contain discussion of relations with quantum physics). We refer the reader to [10, 11, 45, 36] and references therein for further discussion of analysis on the configuration spaces and applications.

On the other hand, stochastic differential geometry of infinite-dimensional manifolds, in particular, their (stochastic) cohomologies and related questions (Hodge–deRham Laplacians and harmonic forms, Hodge decomposition), has been a very active topic of research in recent years. It turns out that many important examples of infinite-dimensional nonflat spaces (loop spaces, product manifolds, configuration spaces) are naturally equipped with probability measures (Brownian bridge, Gibbs measures, Poisson measures). The geometry of these measures interplays in a nontrivial way with the differential geometry of the underlying spaces themselves, and plays therefore a significant role in their study. Moreover, in many cases the absence of a proper smooth manifold structure makes it more natural to work with $L^2$-objects (such as functions, sections, etc.) on these infinite-dimensional spaces, rather than to define analogs of the smooth ones.
Thus, the concept of an $L^2$-cohomology has an important meaning in this framework. The study of $L^2$-cohomologies for finite-dimensional manifolds, initiated in [17], was a subject of many works (whose different aspects are treated in e.g. [24, 23, 27], see also the review papers [39, 37]). In the infinite-dimensional case, loop spaces have been most studied [32, 34, 26, 35], the last two papers containing also a review of the subject. The deRham complex on infinite product manifolds with Gibbs measures (which appear in connection with problems of classical statistical mechanics) was constructed in [1, 2] (see also [19] for the case of the infinite-dimensional torus). We should also mention the papers [46, 14, 15, 16, 6], where the case of a flat Hilbert state space is considered (the $L^2$-cohomological structure turns out to be nontrivial even in this case due to the existence of interesting measures on such a space).

In [3, 4], the authors started studying differential forms over the infinite-dimensional space $\Gamma_X$, with $X$ as above, and the corresponding Laplacians (of Bochner and deRham type).

The structure of the present paper is as follows. Section 2 has an introductory character. We recall the definition of the space $L^2_\pi \Omega^n$ of differential forms over $\Gamma_X$ that are square integrable with respect to the Poisson measure $\pi$, and the construction of the unitary isomorphism

$$I^n : L^2_\pi \Omega^n \to L^2_\pi(\Gamma_X) \otimes \left[ \bigoplus_{m=1}^n L^2_\Psi \Psi_{sym}^{n}(X^m) \right],$$

given in [4]. Here, $L^2_\pi(\Gamma_X)$ is the space of square-integrable functions over $\Gamma_X$ and $L^2_\Psi \Psi_{sym}^{n}(X^m)$ is a space of square-integrable $n$-forms over $X^m$ which satisfy some additional conditions.

We consider only the case of the Poisson measure with intensity given by the Riemannian volume of $X$, which, according to [9], can be thought of as the volume measure on the configuration space $\Gamma_X$, in the sense that the natural lifting of the gradient and divergence on the underlying manifold $X$ become dual operators.

In Section 3, we define the $L^2$-deRham complex over $\Gamma_X$ and the corresponding spaces $H^{(n)}_\pi$ of (reduced) $L^2$-cohomologies. We introduce the Hodge–deRham Laplacian $H^{(n)}$ acting in $L^2_\pi \Omega^n$ and study the space $K^{(n)} := \text{Ker} H^{(n)}$ of harmonic forms. We show, in particular, that $H^{(n)}$ can be expressed, under the action of the isomorphism $I^n$, in terms of the Laplacian operator on functions on $\Gamma_X$ and the Hodge–deRham Laplacians $H^{(n,m)}$ acting respectively in the spaces $L^2_\Psi \Psi_{sym}^{n}(X^m)$. The application of the fact [9] that the Dirichlet form of the Poisson measure is irreducible gives us the possibility to express the harmonic forms on $\Gamma_X$ in terms of harmonic forms on $X$. Our main result here is the construction of the isomorphism

$$\bigoplus_{n=0}^{\infty} K^{(n)} \simeq A_{sym}(K^{(1)}, \ldots, K^{(\dim X)}),$$

where $A_{sym}(K^{(1)}, \ldots, K^{(\dim X)})$ is a supercommutative Hilbert tensor algebra generated by the spaces $K^{(m)} := \text{Ker} H^{(m)}$, $H^{(m)}$ denoting the Hodge–deRham Laplacian in the $L^2$-space of $m$-forms on $X$, $m = 1, \ldots, \dim X$. The spaces $K^{(n)}$ appear to be finite-dimensional,
provided so are all the $K^{(m)}$ spaces. Using the weak Hodge–deRham decomposition, we identify the spaces of harmonic forms with the spaces $H^n_\pi$ of (reduced) $L^2$-cohomologies. In the case where $\beta_m := \dim K^{(m)} < \infty$, $m = 1, \ldots, \dim X$, we give an explicit formula for the dimension $b_n$ of $H_\pi^n$:

$$b_n = \sum_{m=1}^{\infty} \sum_{1 \leq k_1 < \cdots < k_m \leq \dim X} \sum_{s_1, \ldots, s_m \in \mathbb{N}}: s_1 k_1 + \cdots + s_m k_m = n \quad \beta_{k_1}^{(s_1)} \cdots \beta_{k_m}^{(s_m)},$$

where

$$\beta_k^{(s)} := \begin{cases} \binom{\beta_k}{s}, & k = 1, 3, \ldots, \\ \binom{\beta_k + s - 1}{s}, & k = 2, 4, \ldots \end{cases}$$

We remark that this formula has the following interesting consequence: although the spaces $H^{(n)}_\pi$ can be, in general, nontrivial for any $n \in \mathbb{N}$, they vanish for $n$ big enough, provided the cohomologies of $X$ of the even order do.

Finally, let us outline some links and open problems related to the subject of the present paper.

1. Homology and homotopy of the spaces of finite configurations (as topological spaces) were studied by many authors (see e.g. [28, 21, 30, 48]). An intriguing question is to understand the relation between the results of these authors and our results.

2. Any differential form $W \in L^2_\pi \Omega^n$ defines an antisymmetric $n$-linear $L^2_\pi(\Gamma_X)$-valued functional on the Lie algebra $\text{Vect}_0(X)$ of compactly supported vector fields over $X$. On the other hand, there exists a natural representation of $\text{Vect}_0(X)$ in $L^2_\pi(\Gamma_X)$ generated by the action of the diffeomorphism group $\text{Diff}_0(X)$ on $\Gamma_X$ (see [47, 9, 11]). It seems that the $L^2$-cohomology of $\Gamma_X$ is related to a cohomology of the Lie algebra $\text{Vect}_0(X)$ with coefficients in this representation.

3. In the present paper, we consider the case of the Poisson measure with intensity given by Riemannian volume of $X$. This approach can easily be extended to the case of a more general intensity measure (the corresponding Hodge–deRham Laplacian is defined in [4]). An important problem is to consider the case of a Gibbs measure (for analysis and geometry on configuration spaces equipped with Gibbs measures and their relations to the statistical mechanics of continuous systems, see [10] and the review paper [45]). The corresponding $L^2$-cohomologies could give, in this case, invariants of such measures and related models of statistical mechanics.

A different approach to the construction of differential forms and related objects over Poisson spaces, based on the “transfer principle” from Wiener spaces, is proposed in [42], see also [40] and [41].

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2 Differential forms over a configuration space

The aim of this section is to recall some definitions and known facts concerning the differential structure of a configuration space and differential forms over it. For more details and proofs, we refer the reader to [9, 3, 4].

Let $X$ be a complete, stochastically complete, connected, oriented, $C^\infty$ Riemannian manifold of infinite volume. Let $d$ denote the dimension of $X$, $\langle \cdot, \cdot \rangle_x$ the inner product in the tangent space $T_xX$ to $X$ at a point $x \in X$. The associated norm will be denoted by $| \cdot |_x$. Let $\nabla^X$ stand for the gradient on $X$.

The configuration space $\Gamma_X$ over $X$ is defined as the set of all locally finite subsets (configurations) in $X$:

$$\Gamma_X := \{ \gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X \}.$$  

Here, $|A|$ denotes the cardinality of a set $A$.

We can identify any $\gamma \in \Gamma_X$ with the positive, integer-valued Radon measure $\sum_{x \in \gamma} \varepsilon_x < M(X)$, where $\varepsilon_x$ is the Dirac measure with mass at $x$, $\sum_{x \in \emptyset} \varepsilon_x :=$ zero measure, and $M(X)$ denotes the set of all positive Radon measures on the Borel $\sigma$-algebra $\mathcal{B}(X)$. The space $\Gamma_X$ is endowed with the relative topology as a subset of the space $M(X)$ with the vague topology, i.e., the weakest topology on $\Gamma_X$ with respect to which all maps $\Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x) \gamma(dx) \equiv \sum_{x \in \gamma} f(x)$ are continuous. Here, $f \in C_0(X) := \text{the set of all continuous functions on } X \text{ with compact support}$. Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel $\sigma$-algebra.

The tangent space to $\Gamma_X$ at a point $\gamma$ is defined as the Hilbert space

$$T_{\gamma} \Gamma_X := L^2(X \to TX; d\gamma) = \bigoplus_{x \in \gamma} T_x X.$$  \hfill (2.1) 

The scalar product and the norm in $T_{\gamma} \Gamma_X$ will be denoted by $\langle \cdot, \cdot \rangle_\gamma$ and $\|\cdot\|_\gamma$, respectively. Thus, each $V(\gamma) \in T_{\gamma} \Gamma_X$ has the form $V(\gamma) = (V(\gamma)_x)_{x \in \gamma}$, where $V(\gamma)_x \in T_x X$, and

$$\|V(\gamma)\|^2_\gamma = \sum_{x \in \gamma} |V(\gamma)_x|^2_x.$$ 

Vector fields and first order differential forms on $\Gamma_X$ will be identified with sections of the bundle $TT\Gamma_X$. Higher order differential forms will be identified with sections of the tensor bundles $\wedge^n(T\Gamma_X)$ with fibers

$$\wedge^n(T\Gamma_X) = \wedge^n \left( \bigoplus_{x \in \gamma} T_x X \right).$$  \hfill (2.2)
where $\wedge^n(\mathcal{H})$ (or $\mathcal{H}^{\wedge^n}$) stands for the $n$th antisymmetric tensor power of a Hilbert space $\mathcal{H}$. Thus, under a differential form $W$ of order $n$, $n \in \mathbb{N}$, over $\Gamma_X$, we will understand a mapping
\[
\Gamma_X \ni \gamma \mapsto W(\gamma) \in \wedge^n(T_\gamma \Gamma_X).
\] (2.3)

We will now recall how to introduce a covariant derivative of a differential form $W: \Gamma_X \to \wedge^n(TT_X)$.

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $\mathcal{O}_{\gamma,x}$ we will denote an arbitrary open neighborhood of $x$ in $X$ such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. We define the mapping
\[
\mathcal{O}_{\gamma,x} \ni y \mapsto W_x(\gamma, y) := W(\gamma_y) \in \wedge^n(T_{\gamma_y} \Gamma_X), \quad \gamma_y := (\gamma \setminus \{x\}) \cup \{y\}.
\]
This is a section of the Hilbert bundle
\[
\wedge^n(T_{\gamma_y} \Gamma_X) \ni y \in \mathcal{O}_{\gamma,x}.
\] (2.4)

The Levi-Civita connection on $TX$ generates in a natural way a connection on this bundle. We denote by $\nabla^X_{\gamma,x}$ the corresponding covariant derivative and use the notation
\[
\nabla^X_x W(\gamma) := \nabla^X_{\gamma,x} W_x(\gamma, x) \in T_x X \otimes (\wedge^n(T_\gamma \Gamma_X))
\]
if the section $W_x(\gamma, \cdot)$ is differentiable at $x$.

We say that the form $W$ is differentiable at a point $\gamma$ if for each $x \in \gamma$ the section $W_x(\gamma, \cdot)$ is differentiable at $x$, and
\[
\nabla^T W(\gamma) := (\nabla^X_x W(\gamma))_{x \in \gamma} \in T_\gamma \Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X)).
\]
The mapping
\[
\Gamma_X \ni \gamma \mapsto \nabla^T W(\gamma) := (\nabla^X_x W(\gamma))_{x \in \gamma} \in T_\gamma \Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X))
\]
will be called the covariant gradient of the form $W$.

Analogously, one can introduce higher order derivatives of a differential form $W$, the $m$th derivative $(\nabla^T)^{(m)} W(\gamma) \in (T_\gamma \Gamma_X)^{\otimes m} \otimes (\wedge^n(T_\gamma \Gamma_X))$.

Let us note that, for any $\eta \subset \gamma$, the space $\wedge^n(T_\eta \Gamma_X)$ can be identified in a natural way with a subspace of $\wedge^n(T_\gamma \Gamma_X)$. In this sense, we will use the expression $W(\gamma) = W(\eta)$ without additional explanations.

A form $W: \Gamma_X \to \wedge^n(TT_X)$ is called local if there exists a compact $\Lambda = \Lambda(W)$ in $X$ such that $W(\gamma) = W(\lambda \gamma)$ for each $\gamma \in \Gamma_X$.

Let $\mathcal{F}\Omega^n$ denote the set of all local, infinitely differentiable forms $W: \Gamma_X \to \wedge^n(TT_X)$ which are polynomially bounded, i.e., for each $W \in \mathcal{F}\Omega^n$ there exist a function $\varphi \in C_0(X)$ and $k \in \mathbb{N}$ such that
\[
\|W(\gamma)\|_{\wedge^n(T_\gamma \Gamma_X)}^2 \leq \langle \varphi^{\otimes k}, \gamma^{\otimes k} \rangle \quad \text{for all } \gamma \in \Gamma_X.
\] (2.5)
Below, we will give an explicit construction of a class of forms from $\mathcal{F}\Omega^n$.

Our next goal is to give a description of the space of $n$-forms that are square-integrable with respect to the Poisson measure.

Let $dx$ denote the volume measure on $X$, and let $\pi$ denote the Poisson measure on $\Gamma_X$ with intensity $dx$. This measure is characterized by its Laplace transform

$$
\int_{\Gamma_X} e^{(f,\gamma)} \pi(d\gamma) = \exp \left( \int_X (e^{f(x)} - 1) \, dx \right), \quad f \in C_0(X).
$$

If $F : \Gamma_X \to \mathbb{R}$ is integrable with respect to $\pi$ and local, i.e., $F(\gamma) = F(\gamma_\Lambda)$ for some compact $\Lambda \subset X$, then one has

$$
\int_{\Gamma_X} F(\gamma) \pi(d\gamma) = e^{-\text{vol}(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) \, dx_1 \cdots dx_n.
$$

(2.6)

We define on the set $\mathcal{F}\Omega^n$ the $L^2$-scalar product with respect to the Poisson measure:

$$
(W_1, W_2)_{L^2, \Omega^n} := \int_{\Gamma_X} \langle W_1(\gamma), W_2(\gamma) \rangle_{\wedge^n(T, \Gamma_X)} \pi(d\gamma).
$$

(2.7)

The integral on the right hand side of (2.7) is finite, since the Poisson measure has all moments finite. Moreover, $(W, W)_{L^2, \Omega^n} > 0$ if $W$ is not identically zero. Hence, we can define a Hilbert space $L^2_{\pi}(\Omega^n) = L^2(\Gamma_X \to \wedge^n(T\Gamma_X); \pi)$ as the completion of $\mathcal{F}\Omega^n$ with respect to the norm generated by the scalar product (2.7).

We will now give an isomorphic description of the space $L^2_{\pi}(\Omega^n)$ via the space $L^2(\Gamma_X : \pi)$ and some special spaces of square-integrable forms on $X^m$, $m = 1, \ldots, n$.

We first need some preparations. For $x_1, \ldots, x_n \in X$, the space $T_{x_1}X \wedge T_{x_2}X \wedge \cdots \wedge T_{x_n}X$ will be understood as a subspace of the Hilbert space $(T_{y_1}X \oplus T_{y_2}X \oplus \cdots \oplus T_{y_k}X)^{\otimes n}$, where $\{y_1, \ldots, y_k\}$ is the set of the different $x_j$'s, $j = 1, \ldots, n$. We remark that

$$
(T_{y_1}X \oplus T_{y_2}X \oplus \cdots \oplus T_{y_k}X)^{\otimes n} \simeq (T_{y_{\nu(1)}}X \oplus T_{y_{\nu(2)}}X \oplus \cdots \oplus T_{y_{\nu(n)}}X)^{\otimes n}, \quad \nu \in S_k
$$

(2.8)

(where $\simeq$ means isomorphism), and moreover $T_{x_1}X \wedge T_{x_2}X \wedge \cdots \wedge T_{x_n}X$ and $T_{x_{\sigma(1)}}X \wedge T_{x_{\sigma(2)}}X \wedge \cdots \wedge T_{x_{\sigma(n)}}X$, $\sigma \in S_n$, coincide as subspaces of the space (2.8).

Let

$$
\tilde{X}^m := \{ (x_1, \ldots, x_m) \in X^m \mid x_i \neq x_j \text{ if } i \neq j \}.
$$

Then, for $(x_1, \ldots, x_m) \in \tilde{X}^m$, we evidently have

$$
\wedge^n(T_{(x_1, \ldots, x_m)}X^m) = \bigoplus_{0 \leq k_1, \ldots, k_m \leq d, \ k_1 + \cdots + k_m = n} (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m}X)^{\wedge k_m}.
$$

(2.9)
For a form \( \omega : X^m \to \wedge^n (TX^m) \) and \((x_1, \ldots, x_m) \in \tilde{X}^m \), we denote by \( \omega(x_1, \ldots, x_m)_{k_1, \ldots, k_m} \) the corresponding component of \( \omega(x_1, \ldots, x_m) \) in the decomposition (2.9).

We introduce the set \( \Psi^m_{\text{sym}} (X^m) \) of smooth forms \( \omega : X^m \to \wedge^n (TX^m) \) which have compact support and satisfy the following assumptions on \( \tilde{X}^m \):

(i) \( \omega(x_1, \ldots, x_m)_{k_1, \ldots, k_m} = 0 \) if \( k_j = 0 \) for some \( j \in \{1, \ldots, m\} \);

(ii) \( \omega \) is invariant with respect to the action of the group \( S_m \):

\[
\omega(x_1, \ldots, x_m) = \omega(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \quad \text{for each } \sigma \in S_m. \tag{2.10}
\]

For example, let \( f \in C^\infty_0 (X^2) \) be antisymmetric and let \( v : X \to TX \) be a smooth, compactly supported vector field on \( X \). Then, the form \( \omega : X^2 \to \wedge^2 (TX^2) \) given by

\[
\omega(x_1, x_2) := f(x_1, x_2) v(x_1) \wedge v(x_2) + f(x_2, x_1) v(x_2) \wedge v(x_1)
\]

belongs to \( \Psi^m_{\text{sym}} (X^2) \).

Let us denote by \( L^2 \Psi^m_{\text{sym}} (X^m) \) the Hilbert space obtained as the completion of \( \Psi^m_{\text{sym}} (X^m) \) with respect to the \( L^2 \)-norm determined the measure \( dx_1 \cdots dx_m \).

We will use the notation

\[
\mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m := \bigoplus_{1 \leq k_1, \ldots, k_m \leq d} \bigoplus_{k_1 + \cdots + k_m = n} (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m}X)^{\wedge k_m}. \tag{2.11}
\]

By virtue of (2.2), we have

\[
\wedge^n (T_{\gamma} \Gamma_X) = \bigoplus_{m=1}^{n} \bigoplus_{\{x_1, \ldots, x_m\} \subset \gamma} \mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m. \tag{2.12}
\]

For \( W \in \mathcal{F} \Omega^n \), we denote by \( W_m(\gamma) \in \bigoplus_{\{x_1, \ldots, x_m\} \subset \gamma} \mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m \) the corresponding component of \( W(\gamma) \) in the decomposition (2.12). Thus, for \( \{x_1, \ldots, x_m\} \subset \gamma \), \( W_m(\gamma, x_1, \ldots, x_m) \) is equal to the projection of \( W(\gamma) \in \wedge^n (T_{\gamma} \Gamma_X) \) onto the subspace \( \mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m \).

**Proposition 2.1** [4] Setting, for \( W \in L^2_{\pi} \Omega^n \),

\[
(I^n W)(\gamma, x_1, \ldots, x_m) := (m!)^{-1/2} W_m(\gamma \cup \{x_1, \ldots, x_m\}, x_1, \ldots, x_m), \quad m = 1, \ldots, n, \tag{2.13}
\]

one gets the unitary operator

\[
I^n : L^2_{\pi} \Omega^n \to \bigoplus_{m=1}^{n} L^2_{\pi} (\Gamma_X) \otimes L^2 \Psi^m_{\text{sym}} (X^m).
\]
Remark 2.1 Actually, formula (2.13) makes sense only for \((x_1, \ldots, x_m) \in \tilde{X}^m\). However, since the set \(X^m \setminus \tilde{X}^m\) is of zero \(dx_1 \cdots dx_m\) measure, this does not lead to a contradiction.

Sketch of the proof. That \(I^n\) is an isometric operator from \(L^2\pi\Omega^n\) into \(\bigoplus_{m=1}^n L^2\pi(\Gamma_X) \otimes L^2\Psi_{\text{sym}}^n(X^m)\) follows from the definition of \(L^2\Psi_{\text{sym}}^n(X^m)\), (2.11)–(2.13), and the generalized Mecke identity:

\[
\int_{\Gamma_X} \sum_{\{x_1, \ldots, x_m\} \subset \gamma} f(\gamma, x_1, \ldots, x_m) \pi(d\gamma) = (m!)^{-1} \int_{\Gamma_X} \int_{X^m} f(\gamma \cup \{x_1, \ldots, x_m\}, x_1, \ldots, x_m) dx_1 \cdots dx_m \pi(d\gamma),
\]

where \(f : \Gamma_X \times X^m \to \mathbb{R}\) is a measurable function for which at least one of the integrals in (2.14) exists (this formula can be proved by a repeated application of the Mecke identity, see [43]).

Let \(\mathcal{FC}_b^\infty(D, \Gamma_X)\) denote the set of smooth cylinder functions that is defined in Appendix A. For \(F \in \mathcal{FC}_b^\infty(D, \Gamma_X)\) and \(\omega \in \Psi_{\text{sym}}^n(X^m), m \in \{1, \ldots, n\}\), we define a form \(W\) by setting

\[
W_k(\gamma, x_1, \ldots, x_k) := \begin{cases} 
0, & k \neq m, \\
(m!)^{1/2} F(\gamma \setminus \{x_1, \ldots, x_m\}) \omega(x_1, \ldots, x_m), & k = m. 
\end{cases}
\]

As easily seen, \(W\) is a local, infinitely differentiable \(n\)-form over \(\Gamma_X\) such that, for some \(\varphi \in C_0(X), \varphi \geq 0,\)

\[
\|W(\gamma)\|_{\Lambda^n(T, \Gamma_X)}^2 \leq \langle \varphi^{\otimes n}, \gamma^{\otimes n} \rangle \quad \text{for all } \gamma \in \Gamma_X,
\]

and hence we have the inclusion \(W \in \mathcal{F}\Omega^n\). Moreover,

\[
(I^nW)(\gamma, x_1, \ldots, x_k) = \begin{cases} 
0, & k \neq m, \\
F(\gamma)\omega(x_1, \ldots, x_m), & k = m,
\end{cases}
\]

for each \(\gamma \in \Gamma_X\) and each \((x_1, \ldots, x_m) \in \tilde{X}^m\) such that \(\{x_1, \ldots, x_m\} \cap \gamma = \emptyset\). Since \(\gamma\) is a set of zero \(dx\) measure and since the linear span of \(F \otimes \omega\) with \(F\) and \(\omega\) as above, is dense in \(L^2(\Gamma_X) \otimes L^2\Psi_{\text{sym}}^n(X^m)\), we obtain the desired result. ■

In what follows, we will denote by \(D\Omega^n\) the linear span of the forms defined by (2.15) with \(m = 1, \ldots, n\). As we already noticed in the proof of Proposition 2.1, \(D\Omega^n\) is a subset of \(\mathcal{F}\Omega^n\) and is dense in \(L^2\pi\Omega^n\).
3 De Rham complex over a configuration space

3.1 Exterior differentiation and $L^2$-cohomologies

For $n \in \mathbb{N}$, let $E\Omega^n$ denote the subset of $F\Omega^n$ consisting of all forms $W \in F\Omega^n$ such that all derivatives of $W$ are polynomially bounded, that is, for each $k \in \mathbb{N}$ there exist $\varphi \in C_0(X)$, $\varphi \geq 0$, and $l \in \mathbb{N}$ (depending on $W$) such that

$$\|(\nabla^k W(\gamma))\|^2_{(T_\gamma \Gamma X) \otimes \bigotimes \Lambda^n (T_\gamma \Gamma X)} \leq \langle \varphi_l, \varphi_l \rangle$$

for all $\gamma \in \Gamma X$, (3.1)

and additionally, for each fixed $\gamma \in \Gamma X$ and $r \in \mathbb{N}$, the mapping

$$(X \setminus \gamma)^r \cap \tilde{X}^r \ni (x_1, \ldots, x_r) \mapsto W(\gamma + \varepsilon x_1 + \cdots + \varepsilon x_r) \in \Lambda^n (T_\gamma \Gamma X \oplus T_{x_1} X \oplus \cdots \oplus T_{x_r} X)$$

extends to a smooth, compactly supported form

$$X^r \ni (x_1, \ldots, x_r) \mapsto \omega(x_1, \ldots, x_r) \in \Lambda^n (T_\gamma \Gamma X \oplus T_{x_1} X \oplus \cdots \oplus T_{x_r} X).$$

(Notice that the locality of a form, together with the above condition of extension, will automatically imply the infinite differentiability of the form.)

As easily seen, $D\Omega^n$ is a subset of $E\Omega^n$, and so we get the following chain of inclusions

$$D\Omega^n \subset E\Omega^n \subset F\Omega^n.$$

Absolutely analogously, we define the set $E\Omega^0$ of all local, smooth functions $F: \Gamma X \to \mathbb{R}$ which, together with all their derivatives, are polynomially bounded. We have $FC^\infty_0(D, \Gamma X) \subset F\Omega^0$ (see Appendix A).

We define linear operators

$$d_n: E\Omega^n \to E\Omega^{n+1}, \quad n \in \mathbb{Z}_+,$$

by

$$(d_n W)(\gamma) := (n + 1)^{1/2} \text{AS}_{n+1} (\nabla^\Gamma W(\gamma)), \quad (3.3)$$

where

$$\text{AS}_{n+1}: (T_\gamma \Gamma X)^{(n+1)} \to \Lambda^{n+1} (T_\gamma \Gamma X) \quad (3.4)$$

is the antisymmetrization operator. (We notice that the polynomial boundedness of the form $d_n W$ and its derivatives follows from the corresponding boundedness of $\nabla^\Gamma W$ and the fact that the norm of the operator (3.4) for each $\gamma \in \Gamma X$ is equal to one).

Let us now consider $d_n$ as an operator acting from the space $L^2_\pi \Omega^n$ into $L^2_\pi \Omega^{n+1}$. We denote by $d_n^*$ the adjoint operator of $d_n$.

**Proposition 3.1** $d_n^*$ is a densely defined operator from $L^2_\pi \Omega^{n+1}$ into $L^2_\pi \Omega^n$ with domain containing $E\Omega^{n+1}$. 


Proof. Let $\gamma \in \Gamma_X$ and $x \in \gamma$ be fixed. Let $C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^n(\mathcal{T}_\gamma \Gamma_X))$ denote the space of all smooth sections of the Hilbert bundle (2.4). We define the operator

$$d_{x,n} : C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^n(\mathcal{T}_\gamma \Gamma_X)) \to C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^{n+1}(\mathcal{T}_\gamma \Gamma_X))$$

whose action, in local coordinates on the manifold $X$, is given as follows:

$$d_{x,n} \phi(y) h_1 \wedge \cdots \wedge h_n = (n + 1)^{1/2} \nabla^X \phi(y) \wedge h_1 \wedge \cdots \wedge h_n,$$

$\phi \in C^\infty(\mathcal{O}_{\gamma,x} \to \mathbb{R})$, $h_k \in T_{x_k}X$, $x_k \in \gamma$, $k = 1, \ldots, n$. It easily follows from the definition of $d_n$ and $\nabla^\Gamma$ that, for $W \in \mathcal{F}\Omega^n$,

$$(d_n W)(\gamma) = \sum_{x \in \gamma} d_{x,n} W_x(\gamma, x). \quad (3.5)$$

Analogously, we define the operator

$$\delta_{x,n} : C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^{n+1}(\mathcal{T}_\gamma \Gamma_X)) \to C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^n(\mathcal{T}_\gamma \Gamma_X))$$

setting

$$\delta_{x,n} \phi(y) h_1 \wedge \cdots \wedge h_{n+1} :=
\begin{cases} 
- (n + 1)^{-1/2} \sum_{i=1}^{n+1} (-1)^{i-1} \varepsilon_{x,x_i} \langle \nabla^X \phi(y), h_i \rangle_x h_1 \wedge \cdots \wedge \check{h}_i \wedge \cdots \wedge h_{n+1}, \quad (3.6) 
\end{cases}$$

where $\phi \in C^\infty(\mathcal{O}_{\gamma,x} \to \mathbb{R})$, $h_k \in T_{x_k}X$, $x_k \in \gamma$, $k = 1, \ldots, n + 1$,

$$\varepsilon_{x,x_i} := \begin{cases} 1, & x = x_i, \\
0, & x \neq x_i, \end{cases}$$

and $\check{h}_i$ denotes the absence of $h_i$. We now set for $W \in E\Omega^{n+1}$

$$\delta_n W(\gamma) = \sum_{x \in \gamma} \delta_{x,n} W_x(\gamma, x). \quad (3.7)$$

By using (2.5), (3.6), and (3.7), we conclude that

$$\delta_n : E\Omega^{n+1} \to E\Omega^n.$$

Moreover, from (2.6) and the definition of $d_n$ and $\delta_n$, we derive, for arbitrary $V \in \mathcal{F}\Omega^n$ and $W \in \mathcal{F}\Omega^{n+1}$,

$$\int_{\Gamma_X} ((d_n V)(\gamma), W(\gamma))_{\wedge^{n+1}(\mathcal{T}_\gamma \Gamma_X)} \pi(d\gamma) = \int_{\Gamma_X} (V(\gamma), (\delta_n W)(\gamma))_{\wedge^n(\mathcal{T}_\gamma \Gamma_X)} \pi(d\gamma),$$

which proves the proposition. □
Corollary 3.1 The operator $d_n : L^2_\pi \Omega^n \to L^2_\pi \Omega^{n+1}$, $\text{Dom } d_n = \mathcal{E} \Omega^n$, is closable.

We denote by $\bar{d}_n$ the closure of $d_n$. The space $Z^n := \text{Ker } \bar{d}_n$ is then a closed subspace of $L^2_\pi \Omega^n$. Let $B^n$ denote the closure in $L^2_\pi \Omega^n$ of the subspace $\text{Im } d_{n-1}$ (of course, $B^n = \text{the closure of Im } d_{n-1}$).

We obviously have $d_n \bar{d}_n - 1 = 0$, which implies

$$\text{Im } d_{n-1} \subset \text{Ker } d_n \subset Z^n.$$

Hence $B^n \subset Z^n$ and

$$\bar{d}_n \bar{d}_n - 1 = 0. \quad (3.8)$$

Thus, we have the infinite complex

$$\cdots \xrightarrow{\bar{d}_{n-1}} L^2_\pi \Omega^n \xrightarrow{\bar{d}_n} L^2_\pi \Omega^{n+1} \xrightarrow{\bar{d}_{n+1}} \cdots,$$

and the associated Hilbert complex

$$\cdots \xrightarrow{\bar{d}_{n-1}} L^2_\pi \Omega^n \xrightarrow{\bar{d}_n} L^2_\pi \Omega^{n+1} \xrightarrow{\bar{d}_{n+1}} \cdots. \quad (3.9)$$

Our next goal is to study the (reduced) $L^2$-cohomologies of $\Gamma_X$, that is, the homologies of the complex (3.9). We set in a standard way

$$\mathcal{H}^n = Z^n / B^n, \quad n \in \mathbb{N}.$$

Below, we will introduce the Hodge–deRham Laplacian acting in the space $L^2_\pi \Omega^n$, and identify $\mathcal{H}^n$ with the space of harmonic forms. This will give us a possibility to express $\mathcal{H}^n$ in terms of the cohomology spaces of the initial manifold $X$.

3.2 Hodge–deRham Laplacian of the Poisson measure

For $n \in \mathbb{N}$, we define a bilinear form $\mathcal{E}^{(n)}_\pi$ on $L^2_\pi \Omega^n$ by

$$\mathcal{E}^{(n)}_\pi(W_1, W_2) := \int_{\Gamma_X} [\langle d_n W_1(\gamma), d_n W_2(\gamma) \rangle \wedge_{n+1} (T \Gamma_X)]$$

$$+ \langle d_{n-1}^* W_1(\gamma), d_{n-1}^* W_2(\gamma) \rangle \wedge_{n-1} (T \Gamma_X) \pi(d\gamma), \quad (3.10)$$

where $W_1, W_2 \in \text{Dom } \mathcal{E}^{(n)}_\pi := \mathcal{E} \Omega^n$. The function under the sign of integral in (3.10) is polynomially bounded, so that the integral exists.

Theorem 3.1 For any $W_1, W_2 \in \mathcal{E} \Omega^n$, we have

$$\mathcal{E}^{(n)}_\pi(W_1, W_2) = \int_{\Gamma_X} \langle H^{(n)} W_1(\gamma), W_2(\gamma) \rangle \wedge^n (TT \Gamma_X) \pi(d\gamma).$$
Here, \( H^{(n)} = d_{n-1} d^*_n + d^*_n d_n \) is an operator in the space \( L^2_\pi \Omega^n \) with domain \( \text{Dom} H^{(n)} := \mathcal{E} \Omega^n \). It can be represented as follows:

\[
H^{(n)} W(\gamma) = \sum_{x \in \gamma} H^{(n)}_x W(\gamma) = \langle H^{(n)}_x W(\gamma), \gamma \rangle, \quad W \in \mathcal{E} \Omega^n,
\]

where

\[
H^{(n)}_x = d_{x,n-1} \delta_{x,n-1} + \delta_{x,n} d_{x,n}.
\]

Proof. The statement follows from (3.2), (the proof of) Proposition 3.1, and the equality

\[d_{x,n-1} \delta_{y,n-1} + \delta_{y,n} d_{x,n} = 0\]

holding for all \( x, y \in \Gamma, x \neq y \).

From Theorem 3.1 we conclude that the bilinear form \( \mathcal{E} \pi^{(n)} \) is closable in the space \( L^2_\pi \Omega^n \). The generator of its closure (being actually the Friedrichs extension of the operator \( H^{(n)} \), for which we preserve the same notation) will be called the Hodge–deRham Laplacian on \( \Gamma \) (corresponding to the Poisson measure \( \pi \)). By (3.11) and (3.12), \( H^{(n)} \) is the lifting of the Hodge–deRham Laplacian on \( X \).

For linear operators \( A \) and \( B \) acting in Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively, we introduce an operator \( A \oplus B \) in \( \mathcal{H} \otimes \mathcal{K} \) by

\[
A \oplus B := A \otimes 1 + 1 \otimes B, \quad \text{Dom}(A \oplus B) := \text{Dom}(A) \otimes_a \text{Dom}(B),
\]

where \( \otimes_a \) stands for the algebraic tensor product. If the operators \( A \) and \( B \) are closable, then so is \( A \oplus B \), and we will preserve the same notation for its closure.

Next, for operators \( A_1, \ldots, A_n \) acting in Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_n \), respectively, let \( \bigoplus_{i=1}^n A_i \) denote the operator in \( \bigoplus_{i=1}^n \mathcal{H}_i \) given by

\[
\left( \bigoplus_{i=1}^n A_i \right) (f_1, \ldots, f_n) = (A_1 f_1, \ldots, A_n f_n), \quad f_i \in \text{Dom}(A_i).
\]

Theorem 3.2 1) On \( D \Omega^n \) we have

\[
H^{(n)} = (I^n)^{-1} \left[ H^{(0)} \bigoplus \left( \bigoplus_{m=1}^n H^{(n,m)}_{\text{sym}} \right) \right] I^n,
\]

where \( H^{(0)} \) is the Laplacian in the space \( L^2_\pi (\Gamma_X) \) (see Appendix A), and \( H^{(n,m)}_{\text{sym}} \) is the restriction of the Hodge–deRham Laplacian \( H^{(n,m)} \) acting in the space \( L^2 \Omega^n(X^m) := L^2(X^m \to \bigwedge^n (TX^m)); \ dx_1 \cdots dx_m \) to the subspace \( L^2 \Psi^n_{\text{sym}}(X^m) \).

2) \( D \Omega^n \) is a domain of essential selfadjointness of \( H^{(n)} \), and the equality (3.13) holds for the closed operators \( H^{(n)} \) and \( H^{(0)} \bigoplus \left( \bigoplus_{m=1}^n H^{(n,m)}_{\text{sym}} \right) \) (where the latter operator is closed from its domain of essential selfadjointness \( I^n(D \Omega^n) \)).
Proof. This theorem was proved in [4] in a more general setting. Here, we present a simplified version of this proof adapted to our special case of the volume measure on \(X\).

1) Let \(W \in \mathcal{D}\Omega^n\) be given by formula (2.15). Then, using Theorem 3.1 and Appendix A, we get

\[
(H^{(n)}W)_k(\gamma) = 0 \quad \text{for } k \neq m,
\]

\[
(H^{(n)}W)_m(\gamma, \bar{x}^m) = \left(\sum_{x \in \gamma} H^xW\right)_m(\gamma, \bar{x}^m)
\]

\[
= \left(\sum_{x \in \gamma \setminus \{\bar{x}^m\}} H^xW\right)_m(\gamma, \bar{x}^m) + \left(\sum_{x \in \{\bar{x}^m\}} H^xW\right)_m(\gamma, \bar{x}^m)
\]

\[
= (m!)^{1/2} \left[ \left(\sum_{x \in \gamma \setminus \{\bar{x}^m\}} H_x F(\gamma \setminus \{\bar{x}^m\})\omega(\bar{x}^m) + F(\gamma \setminus \{\bar{x}^m\}) \left(\sum_{x \in \{\bar{x}^m\}} H^x\omega\right)(\bar{x}^m) \right] \right]
\]

where \(\bar{x}^m := (x_1, \ldots, x_m)\), \(\{\bar{x}^m\} := \{x_1, \ldots, x_m\}\), and \(\{\bar{x}^m\} \subset \gamma\). (Notice that the Hodge-deRham Laplacian in the space \(L^2\Omega^n(X^n)\) leaves the set \(\Psi^n_{\text{sym}}(X^n)\) invariant.) Therefore,

\[
(I^n H^{(n)}W)(\gamma, \bar{x}^k) = \begin{cases} 0, & k \neq m, \\ (H^{(0)}F)(\gamma)\omega(\bar{x}^m) + F(\gamma)(H_{\text{sym}}^{(n,m)}\omega)(\bar{x}^m), & k = m. \end{cases}
\]  

Hence, by virtue of (2.16), we get

\[
\left( \begin{array}{c} H^{(0)} \\ \bigoplus_{m=1}^n H^{(n,m)}_{\text{sym}} \end{array} \right) I^n \left( \begin{array}{c} H^{(n)}W \\ I^n \end{array} \right)(\gamma, \bar{x}^k) = (I^n H^{(n)}W)(\gamma, \bar{x}^k), \quad k = 1, \ldots, n,
\]

which proves (3.13).

2) Let \(\Omega^n(X^n)\) denote the space of all smooth forms \(\omega: X^n \to \wedge^n(TX^n)\) with compact support, and let \(L^2\Omega^n_{\text{sym}}(X^n)\) denote the subspace of \(L^2\Omega^n(X^n)\) consisting of all forms invariant with respect to the action of the symmetric group \(S_m\), i.e., the forms \(\omega \in L^2\Omega^n(X^n)\) for which the equality (2.10) holds for a.a. \((x_1, \ldots, x_m) \in X^n\). Evidently, the orthogonal projection \(P^m_n\) onto this subspace is given by the formula

\[
(P^m_n\omega)(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \omega(x_{\sigma(1)}, \ldots, x_{\sigma(m)})
\]

and

\[
P^m_n\Omega^n(X^n) = \Omega^n_{\text{sym}}(X^n),
\]

where \(\Omega^n_{\text{sym}}(X^n)\) denotes the subspace of \(\Omega^n(X^n)\) consisting of all \(S_m\)-invariant forms.
It is known that the Hodge–deRham Laplacian \( H^{(n,m)} \) in \( L^2\Omega^n(X^m) \) is essentially self-adjoint on \( \Omega^n(X^m) \) (e.g. [25]). Then, the nonnegative definiteness of \( H^{(n,m)} \) yields that the set \( (H^{(n,m)} + 1)\Omega^n(X^m) \) is dense in \( L^2\Omega^n(X^m) \), see e.g. [44, Section 10.1]. Therefore, the set \( \{ P_m(H^{(n,m)} + 1)\Omega^n(X^m) \} \) is dense in \( L^2\Omega_{\text{sym}}^n(X^m) \). But upon (3.16) and (3.17),

\[
P_m(H^{(n,m)} + 1)\Omega^n(X^m) = (H^{(n,m)} P_m + P_m)\Omega^n(X^m) = (H^{(n,m)} + 1)\Omega_{\text{sym}}^n(X^m),
\]

which implies that the restriction \( H_{\text{sym}}^{(n,m)} \) of the operator \( H^{(n,m)} \) to the subspace \( L^2\Omega_{\text{sym}}^n(X^m) \) is essentially self-adjoint on \( \Omega_{\text{sym}}^n(X^m) \).

Because \( H_{\text{sym}}^{(n,m)} \) acts invariantly on the subspace \( L^2\Psi_{\text{sym}}^n(X^m) \) and its orthogonal complement in \( L^2\Omega_{\text{sym}}^n(X^m) \), we conclude that \( H_{\text{sym}}^{(n,m)} \) considered as an operator in \( L^2\Psi_{\text{sym}}^n(X^m) \) is essentially self-adjoint on \( \Psi_{\text{sym}}^n(X^m) \). Consequently, the operator \( \bigoplus_{m=1}^n H_{\text{sym}}^{(n,m)} \) is essentially self-adjoint on the direct sum of the sets \( \Psi_{\text{sym}}^n(X^m) \), \( m = 1, \ldots, n \).

Finally, remarking that the operator \( H^{(0)}_0 \) is essentially self-adjoint on \( \mathcal{F}C_0(X, \Gamma_X) \) ([9, Theorem 5.3], see also Appendix A), we conclude from the theory of operators admitting separation of variables (e.g. [20, Ch. 6]) that \( I_n(D\Omega^n) \) is a domain of essential self-adjointness for the operator \( H^{(0)} \equiv \left( \bigoplus_{m=1}^n H_{\text{sym}}^{(n,m)} \right) \) in the space \( L^2_0(\Gamma_X) \otimes \left[ \bigoplus_{m=1}^n L^2\Psi_{\text{sym}}^n(X^m) \right] \). Thus, from (3.13) we deduce the remaining statements of the theorem. \( \blacksquare \)

### 3.3 Harmonic forms

In this section, we study the spaces \( K^{(n)} := \text{Ker} H^{(n)} \) of harmonic forms over \( \Gamma_X \). We give their description in terms of the spaces of harmonic forms of the underlying manifold \( X \). For this, we need some auxiliary facts concerning Hilbert tensor algebras with certain commutation relations.

#### Some Hilbert tensor algebras

Let \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) be the free Hilbert tensor algebra generated by real separable Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_l, l \in \mathbb{N} \). That is,

\[
\mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \bigoplus_{m=0}^{\infty} \mathcal{A}_m(\mathcal{H}_1, \ldots, \mathcal{H}_l),
\]

\[
\mathcal{A}_0(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \mathbb{R},
\]

\[
\mathcal{A}_m(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \bigoplus_{i_1, \ldots, i_m \in \{1, \ldots, l\}} \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_m}, \quad m \in \mathbb{N},
\]

with the usual addition and tensor product of elements.

To each space \( \mathcal{H}_i, i = 1, \ldots, l \), we associate a parameter \( p(i) \equiv p(\mathcal{H}_i) \in \mathbb{N} \) (degree). Let \( \Theta \) be the closure of the ideal in \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) generated by the elements

\[
h \otimes f - (-1)^{p(i)p(j)} f \otimes h, \quad h \in \mathcal{H}_i, f \in \mathcal{H}_j, i, j \in \{1, \ldots, l\}.
\]
That is,

$$\Theta := \text{c.l.s.} \left\{ a \otimes [h \otimes f - (-1)^{p(i)p(j)} f \otimes h] \otimes b \mid a, b \in \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l), \ h \in \mathcal{H}_i, \ f \in \mathcal{H}_j, \ i, j \in \{1, \ldots, l\} \right\},$$

where c.l.s. means the closed linear span.

Let us define the quotient Hilbert space

$$\mathcal{A}_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l)/\Theta.$$  

As usual, we can identify \( \mathcal{A}_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) with the orthogonal complement of \( \Theta \) in \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \).

**Lemma 3.1** Let the linear continuous operator \( P \) in \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) be defined through the relation

$$P(h_1 \otimes \cdots \otimes h_m) := \frac{1}{m!} \sum_{\sigma \in S_m} \text{sign} (\sigma, i_1, \ldots, i_m) h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(m)},$$

(3.18)

$$h_k \in \mathcal{H}_{i_k}, \ k = 1, \ldots, m, \ i_1, \ldots, i_m \in \{1, \ldots, l\}.$$  

Here,

$$\text{sign} (\sigma, i_1, \ldots, i_m) := \prod_{k < r: \sigma(k) > \sigma(r)} (-1)^{p(i_{\sigma(k)})p(i_{\sigma(r)})},$$

with \( \prod_{x \in \emptyset} a_x := 1 \). Then, \( P \) is the orthogonal projection of \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) onto \( \mathcal{A}_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \).

**Proof.** See Appendix B. ■

Let

$$\Theta_m := \Theta \cap \mathcal{A}_m(\mathcal{H}_1, \ldots, \mathcal{H}_l)$$

and

$$\mathcal{A}_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \mathcal{A}_m(\mathcal{H}_1, \ldots, \mathcal{H}_l)/\Theta_m.$$  

Evidently,

$$\mathcal{A}_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) = \bigoplus_{m=0}^{\infty} \mathcal{A}_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l).$$

The following lemma gives an isomorphic description of the spaces \( \mathcal{A}_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \).

**Lemma 3.2** For each \( m \in \mathbb{N} \), there exists a unitary isomorphism

$$U_m : \mathcal{A}_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \to \bigoplus_{s_1, \ldots, s_l \in \mathbb{Z}_+}^{s_1 + \cdots + s_l = m} \bigotimes_{i=1}^{l} \mathcal{H}_{i}^{p(i) s_i}.$$
Here, for each \( i \in \{1, \ldots, l\} \), \( p(i) \) denotes the antisymmetric tensor product \( \bigwedge \) if \( p(i) \) is odd and the symmetric tensor product \( \bigotimes \) if \( p(i) \) is even. The unitary operator \( U_m \) is constructed through the relation

\[
U_m \left( \mathbf{P} ( f_1^{(1)} \otimes \cdots \otimes f_{r_1}^{(1)} \otimes \cdots \otimes f_1^{(l)} \otimes \cdots \otimes f_{r_l}^{(l)}) \right) :=
\left( \frac{m!}{r_1! \cdots r_l!} \right)^{1/2} \left( f_1^{(1)} \bigotimes \cdots \bigotimes f_{r_1}^{(1)} \right) \otimes \cdots \otimes \left( f_1^{(l)} \bigotimes \cdots \bigotimes f_{r_l}^{(l)} \right), \tag{3.19} \]

if \( f_k^{(i)} \in \mathcal{H}_i, \ k = 1, \ldots, r_i, \ r_1, \ldots, r_l \in \mathbb{Z}_+, \ r_1 + \cdots + r_l = m \),

the resulting operator \( U_m \) being independent of the representation of a vector from \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \).

Proof. See Appendix B. \[ \blacksquare \]

Now, for each \( n \in \mathbb{N} \), we define the subspace \( \mathcal{A}^n(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) of \( \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) by setting

\[
\mathcal{A}^n(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \bigoplus_{m=1}^n \mathcal{A}^n_m(\mathcal{H}_1, \ldots, \mathcal{H}_l),
\]

\[
\mathcal{A}^n_m(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \bigoplus_{i_1, \ldots, i_m \in \{1, \ldots, l\}} \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_m}.
\]

Let also

\[
\Theta^n := \Theta \cap \mathcal{A}^n(\mathcal{H}_1, \ldots, \mathcal{H}_l),
\]

\[
\mathcal{A}^n_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \mathcal{A}^n(\mathcal{H}_1, \ldots, \mathcal{H}_l) / \Theta^n.
\]

Evidently,

\[
\mathcal{A}^n_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) = \bigoplus_{m=1}^n \mathcal{A}^n_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l), \tag{3.20} \]

\[
\mathcal{A}^n_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) := \mathcal{A}^n_m(\mathcal{H}_1, \ldots, \mathcal{H}_l) / \Theta^n_m, \quad \Theta^n_m := \Theta \cap \mathcal{A}^n_m(\mathcal{H}_1, \ldots, \mathcal{H}_l).
\]

By Lemma 3.1, the orthogonal projection \( P^n_m \) of \( \mathcal{A}^n_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) onto \( \mathcal{A}^n_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \) is the restriction of \( P \) to \( \mathcal{A}^n_m(\mathcal{H}_1, \ldots, \mathcal{H}_l) \), and by (3.20) and Lemma 3.2 the restrictions of the \( U_m \)'s, \( m = 1, \ldots, n \), define the unitary operator

\[
U^n : \mathcal{A}^n_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l) \to \bigoplus_{m=1}^n \bigoplus_{s_1, \ldots, s_l \in \mathbb{Z}_+} \prod_{i=1}^l \mathcal{H}_{i}^{\bigotimes p(i)s_i} \tag{3.21}
\]

\[
\bigotimes_{s_1+\cdots+s_l=m}^n \mathcal{H}_{i}^{p(1)s_1+\cdots+p(l)s_l=n}
\]
Remark 3.1 Actually, \( U^n \) is a natural isomorphism generated by the passage to summation in ordered families of indices in the definition of \( A^n_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_t) \), which uses the commutation relation
\[
h \otimes f = (-1)^{p(i)p(j)} f \otimes h, \quad h \in \mathcal{H}_i, \quad f \in \mathcal{H}_j, \quad i, j \in \{1, \ldots, l\}.
\]

Remark 3.2 Setting \( A^0_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_t) := \mathbb{R} \), one gets the orthogonal decomposition
\[
A_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_t) = \bigoplus_{n=0}^{\infty} A^n_{\text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_t).
\]

The Kernel of the Hodge–deRham Laplacian

Our next goal is to investigate the kernel of \( H^{(n)} \). We first need the following general result.

Lemma 3.3 Let \( A \) and \( B \) be self-adjoint, non-negative operators in separable Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Then, we have
\[
\text{Ker}(A \boxplus B) = \text{Ker} A \otimes \text{Ker} B,
\]
where \( A \boxplus B \) is the closure of the operator \( A \otimes I + I \otimes B \) from the set \( \text{Dom} A \otimes_a \text{Dom} B \).

Proof. \( \text{Ker} A \) and \( \text{Ker} B \) are closed subspaces of \( \mathcal{H} \), resp. \( \mathcal{K} \), and so their tensor product \( \text{Ker} A \otimes \text{Ker} B \) is a closed subspace of the space \( \mathcal{H} \otimes \mathcal{K} \). The inclusion \( \text{Ker} A \otimes \text{Ker} B \subset \text{Ker}(A \boxplus B) \) is trivial. Let \( f \in \text{Ker}(A \boxplus B) \). Using the theory of operators admitting separation of variables (e.g. [20, Ch. 6]), we have
\[
0 = (A \boxplus Bf, f) = \int_{\mathbb{R}^2_+} (x_1 + x_2) d(E(x_1, x_2)f, f)
= \int_{\mathbb{R}^2_+} x_1 d(E(x_1, x_2)f, f) + \int_{\mathbb{R}^2_+} x_2 d(E(x_1, x_2)f, f)
= (A \otimes If, f) + (I \otimes Bf, f), \tag{3.22}
\]
where \( E \) is the joint resolution of the identity of the commuting operators \( A \otimes I \) and \( I \otimes B \). Since both operators \( A \otimes I \) and \( I \otimes B \) are non-negative, we conclude from (3.22) that
\[
f \in \text{Ker}(A \otimes I) \cap \text{Ker}(I \otimes B) = \text{Ker} A \otimes \text{Ker} B. \quad \blacksquare
\]

Let us fix any \( i_1, \ldots, i_m \in \{1, \ldots, d\} \), \( i_1 + \cdots + i_m = n \). For any \( \omega_r \in \Omega^r(X) \), \( r = 1, \ldots, m \), we define the form
\[
X^m \ni (x_1, \ldots, x_m) \mapsto \tilde{\omega}_r(x_1, \ldots, x_m) := \omega_r(x_r) \in \wedge^r(T_x X) \subset \wedge^r(T_{(x_1, \ldots, x_m)} X^m).
\]
Now, we set
\[
U_{i_1, \ldots, i_m}(\omega_1 \otimes \cdots \otimes \omega_m) := \left( \frac{n!}{i_1! \cdots i_m!} \right)^{1/2} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_m \in \Omega^n(X^m).
\]
(We use here the convention that the exterior product of two forms, \( \omega \) and \( \nu \), is given by \( \omega \wedge \nu := \text{AS}(\omega \otimes \nu) \), where \( \text{AS} \) denotes the antisymmetrization operator). It is easy to see that \( U_{i_1, \ldots, i_m} \) can be extended by linearity and continuity to a linear isometric operator
\[
U_{i_1, \ldots, i_m} : L^2\Omega^i(X) \otimes \cdots \otimes L^2\Omega^m(X) \to L^2\Omega^n(X^m)
\]
with the image
\[
\text{Im} U_{i_1, \ldots, i_m} = L^2\Psi_{i_1, \ldots, i_m}(X^m),
\]
where \( L^2\Psi_{i_1, \ldots, i_m}(X^m) \) denotes the space of the forms
\[
X^m \ni (x_1, \ldots, x_m) \mapsto \omega(x_1, \ldots, x_m) \in (T_{x_1}X)^{\wedge i_1} \wedge \cdots \wedge (T_{x_m}X)^{\wedge i_m}
\]
that are square integrable with respect to \( dx_1 \cdots dx_m \).

Setting
\[
L^2\Psi^n(X^m) := \bigoplus_{i_1, \ldots, i_m \in \{1, \ldots, d\}} L^2\Psi_{i_1, \ldots, i_m}(X^m),
\]
(3.23)
we construct, by using the \( U_{i_1, \ldots, i_m} \)'s, the unitary isomorphism
\[
U^n_m : \bigoplus_{i_1, \ldots, i_m \in \{1, \ldots, d\}} L^2\Omega^{i_1}(X) \otimes \cdots \otimes L^2\Omega^{i_m}(X) \to L^2\Psi^n(X^m),
\]
or equivalently
\[
U^n_m : A^n_m(L^2\Omega^1(X), \ldots, L^2\Omega^d(X)) \to L^2\Psi^n(X^m),
\]
(3.24)
where \( p(i) = p(L^2\Omega^i(X)) := i \).

We notice that the restriction of the orthogonal projection
\[
P^n_m : L^2\Omega^n(X^m) \to L^2\Omega^n_{\text{sym}}(X^m)
\]
to the subspace \( L^2\Psi^n(X^m) \) determines the orthogonal projection
\[
P^n_m : L^2\Psi^n(X^m) \to L^2\Psi^n_{\text{sym}}(X^m).
\]

**Lemma 3.4** We have
\[
P^n_m U^n_m = U^n_m P^n_m,
\]
where \( P^n_m \) is the orthogonal projection of \( A^n_m(L^2\Omega^1(X), \ldots, L^2\Omega^d(X)) \) onto \( A^n_{m,\text{sym}}(L^2\Omega^1(X), \ldots, L^2\Omega^d(X)) \).
Proof. For any $\omega_r \in \Omega^i(X)$, $r = 1, \ldots, m$, $i_1, \ldots, i_r \in \{1, \ldots, d\}$, $i_1 + \cdots + i_m = n$, we get by using Lemma 3.1
\[
(P^n_m U^n_m \omega_1 \otimes \cdots \otimes \omega_m)(x_1, \ldots, x_m) = \left(\frac{n!}{i_1! \cdots i_m!}\right)^{1/2} \sum_{\sigma \in S_m} \omega_1(x_{\sigma(1)}) \wedge \cdots \wedge \omega_m(x_{\sigma(m)})
\]
\[
= \left(\frac{n!}{i_1! \cdots i_m!}\right)^{1/2} \sum_{\sigma \in S_m} \text{sign}(\sigma, i_1, \ldots, i_m) \omega_{\sigma(1)}(x_1) \wedge \cdots \wedge \omega_{\sigma(m)}(x_m)
\]
\[
= \sum_{\sigma \in S_m} \text{sign}(\sigma, i_1, \ldots, i_m) \left(U^n_m \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(m)}\right)(x_1, \ldots, x_m)
\]
\[
= \left(U^n_m \mathbf{P}^n_m \omega_1 \otimes \cdots \otimes \omega_m\right)(x_1, \ldots, x_m). \tag*{\blacksquare}
\]

Since $P^n_m$ is the orthogonal projection of $L^2 \Psi^n(X^m)$ onto $L^2 \Psi^n_{\text{sym}}(X^m)$ and $\mathbf{P}^n_m$ is the orthogonal projection of $\mathcal{A}^n_m(L^2 \Omega^1(X), \ldots, L^2 \Omega^d(X))$ onto $\mathcal{A}^n_{m, \text{sym}}(L^2 \Omega^1(X), \ldots, L^2 \Omega^d(X))$, we conclude from (3.24) and Lemma 3.4 that the restriction of $U^n_m$ to $\mathcal{A}^n_{m, \text{sym}}(L^2 \Omega^1(X), \ldots, L^2 \Omega^d(X))$ defines the unitary isomorphism
\[
U^n_m : \mathcal{A}^n_{m, \text{sym}}(L^2 \Omega^1(X), \ldots, L^2 \Omega^d(X)) \to L^2 \Psi^n_{\text{sym}}(X^m).
\]

Finally, setting
\[
U^n := \bigoplus_{m=1}^n U^n_m, \tag{3.25}
\]
we get the unitary mapping
\[
U^n : \mathcal{A}^n_{\text{sym}}(L^2 \Omega^1(X), \ldots, L^2 \Omega^d(X)) \to \bigoplus_{m=1}^n L^2 \Psi^n_{\text{sym}}(X^m).
\]

We denote by $\mathcal{K}^{(i)}$ the kernel of the Hodge–deRham Laplacian $H^{(i)}$ in the space $L^2 \Omega^i(X)$, $i = 1, \ldots, d$. Each $\mathcal{K}^{(i)}$ as a closed subspace of the Hilbert space $L^2 \Omega^i(X)$ is itself a Hilbert space. Let also $\mathbf{K}^{(n)}$ denote the kernel of the operator $H^{(n)}$.

Theorem 3.3 We have
\[
I^n \mathbf{K}^{(n)} = \{\text{const}\} \otimes \left[ U^n \mathcal{A}^n_{\text{sym}}(\mathcal{K}^{(1)}, \ldots, \mathcal{K}^{(d)}) \right], \tag{3.26}
\]
where $p(i) := i$, $i = 1, \ldots, d$.

Proof. By [9, Theorem 4.3],
\[
\mathbf{K}^{(0)} := \text{Ker} \, H^{(0)} = \{\text{const}\}, \tag{3.27}
\]

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and hence by Theorem 3.2 and Lemma 3.3

\[ I^n K^{(n)} = \{\text{const}\} \otimes \left[ \bigoplus_{m=1}^{n} \text{Ker} H_{\text{sym}}^{(n,m)} \right]. \] (3.28)

Let us find the kernel of the Hodge–deRham Laplacian \( H^{(n,m)} \) in the space \( L^2 \Psi^m(X^m) \). The operator \( H^{(n,m)} \) acts invariantly in each space in the direct sum (3.23), so that it suffices to find the kernel of each restriction \( H_{i_1,\ldots,i_m}^{(n,m)} \) of \( H^{(n,m)} \) to the subspace \( L^2 \Psi_{i_1,\ldots,i_m}(X^m) \).

By using the operator \( U_n^{(n,m)} \), we easily conclude that

\[ (U_n^{(n,m)})^{-1} H_{i_1,\ldots,i_m}^{(n,m)} U_n^{(n,m)} = (c_1 H^{(i_1)}) \oplus \cdots \oplus (c_m H^{(i_m)}), \]

where \( c_1, \ldots, c_m \) are non-zero constants. Therefore, by Lemma 3.3

\[ \text{Ker} H_{i_1,\ldots,i_m}^{(n,m)} = U_n^{(n,m)} (K^{(i_1)} \otimes \cdots \otimes K^{(i_m)}), \]

which yields that

\[ \text{Ker} H^{(n,m)} = U_n^{(n,m)} \bigoplus_{i_1,\ldots,i_m \in \{1,\ldots,d\}} (K^{(i_1)} \otimes \cdots \otimes K^{(i_m)}) \]

\[ = U_n^{(n,m)} A_{n,sym} (K^{(1)}, \ldots, K^{(d)}). \]

Since

\[ H_{\text{sym}}^{(n,m)} P_m^n = P_m^n H^{(n,m)}, \]

we get

\[ \text{Ker} H_{\text{sym}}^{(n,m)} = P_m^n \text{Ker} H^{(n,m)}, \]

which implies by Lemma 3.4 that

\[ \text{Ker} H_{\text{sym}}^{(n,m)} = U_n^{(n,m)} A_{n,sym} (K^{(1)}, \ldots, K^{(d)}). \] (3.29)

Combining (3.25), (3.28) and (3.29), we get the conclusion of the theorem. \( \blacksquare \)

**Corollary 3.2** The isomorphisms \( I^n, U^n \) and the equality (3.27) generate the unitary isomorphism of the Hilbert spaces

\[ \bigoplus_{n=0}^{\infty} K^{(n)} \simeq A_{\text{sym}}^{(1)}, \ldots, K^{(d)}). \]

**Proof.** For each \( n \in \mathbb{N} \), we get from (3.26) the unitary isomorphism of the spaces

\[ K^{(n)} \simeq A_{\text{sym}}^{(1)}, \ldots, K^{(d)}). \]

Moreover, it follows from (3.27) that \( K^{(0)} \simeq \mathbb{R} \). Hence, the conclusion of the corollary follows from Remark 3.2. \( \blacksquare \)
Remark 3.3 Formula (3.26) is wrong in the case where the manifold $X$ has finite volume (in that case the Poisson measure $\pi$ is concentrated on the space of finite configurations over $X$). Instead of (3.26), one then gets

$$I^n K^{(n)} = \text{Ker} \mathbf{H}^{(0)} \otimes \left[ U^n A_{\text{sym}}^n (\mathcal{K}^{(1)}, \ldots, \mathcal{K}^{(d)}) \right],$$

the space $\text{Ker} \mathbf{H}^{(0)}$ being infinite-dimensional.

### 3.4 Structure of $L^2$-cohomologies

The aim of this section is to study the structure of the spaces $\mathcal{H}^n_\pi$ of $L^2$-cohomologies of $\Gamma_X$ using the representation of the kernel of $H^{(n)}$ given by Theorem 3.3. The following proposition reflects a quite standard fact in the $L^2$-theory.

**Proposition 3.2** The natural isomorphism between $\mathcal{H}^n_\pi$ and the orthogonal complement of $B^n$ to $Z^n$ is the isomorphism of the Hilbert spaces

$$\mathcal{H}^n_\pi \simeq \text{Ker} \mathbf{H}^{(n)}.$$ \hspace{1cm} (3.30)

**Proof.** Using [13, Proposition A.1], we conclude from Proposition 3.1 and formula (3.8) that

$$L^2_\pi \Omega^n = \text{Ker} \mathbf{H}^{(n)} \oplus \text{Im} d_{n-1} \oplus \text{Im} d_n$$ \hspace{1cm} (3.31)

(weak Hodge–deRham decomposition). For the closed operator $\bar{d}_n$ we have the standard decomposition

$$L^2_\pi \Omega^n = \text{Ker} \bar{d}_n \oplus \text{Im} \bar{d}_n,$$

which together with (3.31) implies the result. \hfill ■

Due to the Hodge–deRham theory of the underlying manifold $X$, we have the isomorphisms

$$\mathcal{K}^k \simeq \mathcal{H}^{(k)}_\pi (X), \quad k = 1, \ldots, d,$$

(3.32)

where $\mathcal{H}^{(k)}_\pi (X) := \text{Ker} d_k / \text{Im} d_{k-1}$ ($d_j$, $j = 1, \ldots, d$, denoting the Hodge differential of $X$) is the corresponding space of (reduced) $L^2$-cohomologies of $X$.

**Remark 3.4** Because of the elliptic regularity of the Hodge–deRham Laplacian on $X$, there exists a canonical map $\mathcal{H}^*_\pi (X) \to \mathcal{H}^*(X)$, where $\mathcal{H}^*(X)$ is the deRham cohomology of $X$. In general, this map is neither surjective, nor injective.

**Theorem 3.4** 1) The isomorphisms (3.30), $I^n$, $U^n$, $U^n$, and (3.32) generate the unitary isomorphism of the Hilbert spaces

$$\mathcal{H}^n_\pi \simeq \bigoplus_{m=1}^n \bigoplus_{1 \leq k_1 < \cdots < k_m \leq d} \bigoplus_{s_1, \ldots, s_m \in \mathbb{N}} \left( \mathcal{H}^{k_1}_\pi (X) \right)^{k_1 s_1} \otimes \cdots \otimes \left( \mathcal{H}^{k_m}_\pi (X) \right)^{k_m s_m}.$$ \hspace{1cm} (3.33)
2) Let \( \beta_k := \dim \mathcal{H}^k_2(X) < \infty, k = 1, \ldots, d \). Then, all the spaces \( \mathcal{H}^n_\pi, n \in \mathbb{N} \), are finite-dimensional, and we have the following formula for their dimensions \( b_n \):

\[
b_n = \sum_{m=1}^{n} \sum_{1 \leq k_1 < \cdots < k_m \leq d} \sum_{s_1, \ldots, s_m \in \mathbb{N} \atop k_1 s_1 + \cdots + k_m s_m = n} \beta^{(s_1)}_{k_1} \cdots \beta^{(s_m)}_{k_m}, \tag{3.34}
\]

where

\[
\beta^{(s)}_k := \begin{cases} 
\binom{\beta_k}{s}, & k = 1, 3, \ldots, \\
\binom{\beta_k + s - 1}{s}, & k = 2, 4, \ldots \tag{3.35}
\end{cases}
\]

Proof. 1) Follows from Theorem 3.3. Actually, (3.33) is a more explicit form of (3.25). 2) It is easy to see that, for a finite-dimensional space \( \mathcal{H} \), we have

\[
\dim \mathcal{H}^s = \frac{(s + 1)(s + 2) \cdots (s + \dim \mathcal{H} - 1)}{(\dim \mathcal{H} - 1)!} = \binom{\dim \mathcal{H} + s - 1}{s},
\]

\[
\dim \mathcal{H}^\wedge s = \binom{\dim \mathcal{H}}{s}.
\]

The statement follows now from (3.33). ■

**Corollary 3.3** Let \( \beta_1, \ldots, \beta_d \) be finite, and moreover let \( \beta_k = 0 \) for all \( k \) even. Then:

\[
b_k = 0, \quad \text{for all } k > K_0 := \sum_{i=1}^{d} i \beta_i,
\]

\[
b_{K_0} = 1.
\]

Proof. The condition \( \beta_k = 0 \) for all \( k \) even implies that

\[
\mathcal{H}^n_\pi \cong \bigoplus_{m=1}^{n} \bigoplus_{1 \leq k_1 < \cdots < k_m \leq d} \bigoplus_{s_1, \ldots, s_m \in \mathbb{N} \atop k_1 s_1 + \cdots + k_m s_m = n} (\mathcal{H}^{k_1}_{(2)}(X))^\wedge s_1 \otimes \cdots \otimes (\mathcal{H}^{k_m}_{(2)}(X))^\wedge s_m.
\]

Obviously \((\mathcal{H}^{k}_{(2)}(X))^\wedge s = 0\) for \( s > \beta_k \) and \((\mathcal{H}^{k}(X))^\wedge s = \mathbb{R}^1\) for \( s = \beta_k \), which implies the result. ■

**Example 3.1** Let \( X \) be a manifold with a cylindrical end (that is, \( X = M \cup (N \times \mathbb{R}^1) \) for some compact manifold \( M \) with boundary \( N \)). It is proven in [18] that \( \mathcal{H}^{k}_{(2)}(X) \) is isomorphic to the image of the canonical map \( \mathcal{H}^{k}_{0}(X) \rightarrow \mathcal{H}^{k}(X) \), where \( \mathcal{H}^{k}_{0}(X) \) is the space of the compactly supported deRham cohomologies of \( X \). By e.g. [22], the spaces \( \mathcal{H}^{k}(X) \) are finite-dimensional. Thus, all \( \mathcal{H}^{k}_{(2)}(X) \) are finite-dimensional and, in general, non-trivial, and hence so are all spaces \( \mathcal{H}^n_\pi \). For a bigger class of examples of manifolds \( X \) with finite-dimensional spaces \( \mathcal{H}^{k}_{(2)}(X) \) see [38].
Example 3.2 Let $d = 2$. Then, $\beta_0 = \beta_2 = 0$ (see e.g. [12]), and if $X$ is as in Example 3.1, we also have $\beta_1 < \infty$. Thus, $X$ satisfies the conditions of Corollary 3.3, and we have $b_k = 0$ for all $k > \beta_1$ and $b_k = \binom{\beta_1}{k}$ for $k \leq \beta_1$.

Remark 3.5 The vanishing of the spaces $H^n_\pi$ does not, in general, imply the absence of non-exact closed forms. Suppose, for example, that $X = \mathbb{R}^1$. Clearly, there are no $L^2$-harmonic forms on $\mathbb{R}^1$, which implies that all the spaces $H^n_\pi(\Gamma_{\mathbb{R}^1})$ are trivial. Let us consider a 1-form $\varphi(x) = g(x) \, dx$ on $\mathbb{R}^1$ such that $g(x)$ has a compact support and $\int_{\mathbb{R}^1} \varphi \neq 0$. The latter implies that $\varphi \neq d_0 f$ for any $f \in L^2(\mathbb{R}^1)$. We now define $\Phi \in L^2_\pi(\Omega^1(\Gamma_{\mathbb{R}^1}))$ setting $\Phi(\gamma)|_x := \varphi(x)$. It is easy to see that $\Phi \neq d_0 F$ for any $F \in L^2_\pi(\Gamma_{\mathbb{R}^1})$ and $d_1 \Phi = 0$.

Example 3.3 Marked configuration spaces. Let $Y = X \times M$, where $M$ is a compact Riemannian manifold. We note that $\Gamma_Y$ coincides up to a set of zero $\pi$ measure with the marked configuration space $\Gamma_X(M)$, see e.g. [33]. Let us recall that the latter space is defined as follows:

$$\Gamma_X(M) := \{ \gamma \in \Gamma_{X \times M} : \forall (x_1, m_1), (x_2, m_2) \in \gamma : (x_1, m_1) \neq (x_2, m_2) \Rightarrow x_1 \neq x_2 \}.$$  

The K"unneth formula implies

$$H^n_{(2)}(X \times M) = \bigoplus_{m=0}^n H^m_{(2)}(X) \otimes H^{n-m}_{(2)}(M).$$

We remark that, for each $k$, $H^k_{(2)}(M) = H^k(M)$ and is finite-dimensional. Thus, all the spaces $H^n_\pi(\Gamma_X(M))$ are finite-dimensional, provided so are all $H^k_{(2)}(X)$.

4 Appendix

4.1 Appendix A: Laplacian on the configuration space

We recall here the definition of the Laplacian on the configuration space and some facts about it from [9], which we present in a form adapted to the aims of the present paper (see also [3, 4]).

Let $F : \Gamma_X \to \mathbb{R}$. For fixed $\gamma \in \Gamma_X$ and $x \in \gamma$, we define the function

$$\mathcal{O}_{\gamma,x} \ni y \mapsto F_x(\gamma, y) := F((\gamma \setminus \{x\}) \cup \{y\}) \in \mathbb{R}.$$  

We say that $F$ is differentiable at $\gamma \in \Gamma_X$ if, for each $x \in \gamma$, the function $F_x(\gamma, \cdot)$ is differentiable at $x$ and

$$\nabla^\Gamma F(\gamma) := (\nabla^X F_x(\gamma, x))_{x \in \gamma} \in T_\gamma \Gamma_X.$$  

Analogously, the higher order derivatives of $F$ are defined, $(\nabla^\Gamma)^{(m)} F(\gamma) \in (T_\gamma \Gamma_X)^{\otimes m}$, $m \in \mathbb{N}$.  

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A function $F : \Gamma_X \rightarrow \mathbb{R}$ is called local if there exists a compact $\Lambda \subset X$ such that $F(\gamma) = F(\gamma \Lambda)$ for each $\gamma \in \Gamma_X$.

We define $\mathcal{FC}_b^\infty(D, \Gamma_X)$ as the set of all functions $F : \Gamma_X \rightarrow \mathbb{R}$ of the form

$$F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle),$$

(4.1)

where $g_F \in C_b^\infty(\mathbb{R}^N)$ and $\varphi_1, \ldots, \varphi_N \in \mathcal{D} := C_0^\infty(X)$ (:=the set of all infinitely differentiable functions on $X$ with compact support). Each function $F \in \mathcal{FC}_b^\infty(D, \Gamma_X)$ is evidently bounded, local, and infinitely differentiable with derivatives satisfying the estimate

$$\|(\nabla^\Gamma)^{(m)}F(\gamma)\|_{(T, \tau, \Gamma_X)^\otimes m} \leq \langle \varphi^{\otimes m}, \gamma^{\otimes m} \rangle$$

for all $\gamma \in \Gamma_X$, with some $\varphi \in C_0(X)$ depending on $F$ and $m \in \mathbb{N}$.

On the space $L^2_\pi(\Gamma_X)$ we consider the pre-Dirichlet form

$$\mathcal{E}_\pi^{(0)}(F_1, F_2) := \int_{\Gamma_X} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma F(\gamma) \rangle_{T, \tau, \Gamma_X} \pi(d\gamma)$$

with domain $\text{Dom} \mathcal{E}_\pi^{(0)} := \mathcal{FC}_b^\infty(D, \Gamma_X)$, which is dense in $L^2_\pi(\Gamma_X)$.

The following theorem can be proved by using formula (2.6).

**Theorem 4.1** For any $F_1, F_2 \in \mathcal{FC}_b^\infty(D, \Gamma_X)$, we have

$$\mathcal{E}_\pi^{(0)}(F_1, F_2) = \int_{\Gamma_X} \langle \mathbf{H}^{(0)}(\gamma) F_1(\gamma) F_2(\gamma) \rangle \pi(d\gamma).$$

Here, $\mathbf{H}^{(0)} = -\Delta^\Gamma$ is the operator in $L^2_\pi(\Gamma_X)$ with domain $\text{Dom} \mathbf{H}^{(0)} := \mathcal{FC}_b^\infty(D, \Gamma_X)$ that is given by the formula

$$(\mathbf{H}^{(0)} F)(\gamma) := -\sum_{x \in \gamma} \Delta^X F_x(\gamma, x), \quad F \in \mathcal{FC},$$

(4.2)

$\Delta^X$ denoting the Laplacian on $X$.

From Theorem 4.1 we conclude that the bilinear form $\mathcal{E}_\pi^{(0)}$ is closable in the space $L^2_\pi(\Gamma_X)$. The generator of its closure (being actually the Friedrichs extension of the operator $\mathbf{H}^{(0)}$, for which we preserve the same notation) will be called the Laplacian on $\Gamma_X$. By (4.2), $\mathbf{H}^{(0)}$ is the lifting of the Laplacian on $X$.

**Theorem 4.2** The operator $\mathbf{H}^{(0)}$ is essentially self-adjoint on $\mathcal{FC}_b^\infty(D, \Gamma_X)$.

**Proof.** See [9, Theorem 5.3]. ■

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4.2 Appendix B: Proof of Lemmas 3.1 and 3.2

We first prove Lemma 3.1. Extending the relation (3.18) by linearity and continuity, we get a linear continuous operator $P$ in $\mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l)$.

Let us show that the operator $P$ is self-adjoint. For arbitrary $f_k \in \mathcal{H}_{i_k}$ and $g_k \in \mathcal{H}_{j_k}$, $i_k, j_k \in \{1, \ldots, l\}$, $k = 1, \ldots, m$, we get from (3.18):

$$
(P(f_1 \otimes \cdots \otimes f_m), g_1 \otimes \cdots \otimes g_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \text{sign}(\sigma, i_1, \ldots, i_m) \prod_{k=1}^{m} (f_{\sigma(k)}, g_k)
$$

$$
= \frac{1}{m!} \sum_{\sigma \in S_m} \text{sign}(\sigma^{-1}, i_1, \ldots, i_m) \prod_{k=1}^{m} (f_k, g_{\sigma(k)})
$$

$$
= \frac{1}{m!} \sum_{\sigma \in S_m: i_1 = j_{\sigma(1)}, \ldots, i_m = j_{\sigma(m)}} \prod_{r<s; \sigma^{-1}(r) > \sigma^{-1}(s)} (-1)^{p(i_{\sigma^{-1}(r)})p(i_{\sigma^{-1}(s)})} \prod_{k=1}^{m} (f_k, g_{\sigma(k)})
$$

$$
= \frac{1}{m!} \sum_{\sigma \in S_m: i_1 = j_{\sigma(1)}, \ldots, i_m = j_{\sigma(m)}} \prod_{r<s; \sigma^{-1}(r) > \sigma^{-1}(s)} (-1)^{p(j_r)p(j_s)} \prod_{k=1}^{m} (f_k, g_{\sigma(k)})
$$

$$
= \frac{1}{m!} \sum_{\sigma \in S_m: i_1 = j_{\sigma(1)}, \ldots, i_m = j_{\sigma(m)}} \prod_{r<s; \sigma(r) > \sigma(s)} (-1)^{p(j_r)p(j_s)} \prod_{k=1}^{m} (f_k, g_{\sigma(k)})
$$

and so $P$ is indeed self-adjoint.

Next, it follows from the definition of $P$ that, for $f_k \in \mathcal{H}_{i_k}$, $i_k \in \{1, \ldots, l\}$, $k = 1, \ldots, m$,

$$
P(f_1 \otimes \cdots \otimes f_r \otimes f_{r+1} \otimes \cdots \otimes f_m)
$$

$$
= (-1)^{p(i_r)p(i_{r+1})}P(f_1 \otimes \cdots \otimes f_{r-1} \otimes f_{r+1} \otimes f_r \otimes f_{r+2} \otimes \cdots \otimes f_m). \quad (4.3)
$$

The latter formula implies that, for each $\sigma \in S_m$,

$$
P(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)}) = \text{sign}(\sigma, i_1, \ldots, i_m)P(f_1 \otimes \cdots \otimes f_m), \quad (4.4)
$$

and hence

$$
P^2(f_1 \otimes \cdots \otimes f_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \text{sign}(\sigma, i_1, \ldots, i_m)P(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)})
$$

$$
= \frac{1}{m!} \sum_{\sigma \in S_m} \text{sign}(\sigma, i_1, \ldots, i_m)^2 P(f_1 \otimes \cdots \otimes f_m) = P(f_1 \otimes \cdots \otimes f_m).
$$

Thus, $P$ is a bounded self-adjoint operator in $\mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l)$ satisfying $P^2 = P$, and so $P$ is an orthogonal projection. Hence, it remains only to show that

$$
\Theta = \text{Ker} P. \quad (4.5)
$$
The inclusion $\Theta \subset \ker P$ follows from (4.3). Moreover, we have
\[ \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) = \ker P \oplus \text{Im} P. \]

Hence, to prove (4.5) it suffices to show that
\[ \mathcal{A}(\mathcal{H}_1, \ldots, \mathcal{H}_l) = \Theta \oplus \text{Im} P. \] (4.6)

Let us fix arbitrary vectors $f_k \in \mathcal{H}_{i_k}$, $i_k \in \{1, \ldots, l\}$, $k = 1, \ldots, m$, $m \geq 2$. We will now show that the vector $f_1 \otimes \cdots \otimes f_m$ can be represented as a sum of vectors from $\Theta$ and $\text{Im} P$, which will imply (4.6) (notice that $P \mid \mathcal{A}_i(\mathcal{H}_1, \ldots, \mathcal{H}_l) = 1$, $i = 0, 1$).

It is enough to show that, for each $\sigma \in S_m$, the vector
\[ F_\sigma := (f_1 \otimes \cdots \otimes f_m) - \text{sign}(\sigma, i_1, \ldots, i_m)(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)}) \] (4.7)
belongs to $\Theta$, because (4.7) yields
\[ P(f_1 \otimes \cdots \otimes f_m) + \frac{1}{m!} \sum_{\sigma \in S_m} F_\sigma = f_1 \otimes \cdots \otimes f_m. \]

But the inclusion $F_\sigma \in \Theta$ can be proved by recurrent application of the following identity
\[
\text{sign}(\sigma, i_1, \ldots, i_m) \left[ (f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)}) - Q_{\sigma(s)}(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)}) \right] \\
= \text{sign}(\tau, i_1, \ldots, i_m)(f_{\tau(1)} \otimes \cdots \otimes f_{\tau(m)}),
\]
where
\[ s := \max \left\{ r : r \in \{1, \ldots, l\}, \sigma(r) \neq r \right\}, \]
\[ \tau(1, \ldots, m) := (\sigma(1), \ldots, \sigma(s-1), \sigma(s+1), \sigma(s), \sigma(s+2), \ldots, \sigma(m)), \]
and by definition
\[ Q_\tau(g_1 \otimes \cdots \otimes g_m) := (g_1 \otimes \cdots \otimes g_m) - \\
- (-1)^{p(j_r)p(j_{r+1})}(g_1 \otimes \cdots \otimes g_{r-1} \otimes g_{r+1} \otimes g_r \otimes g_{r+2} \otimes \cdots \otimes g_m), \]
\[ g_k \in \mathcal{H}_{j_k}, j_k \in \{1, \ldots, l\}, k \in \{1, \ldots, m\}, r \in \{1, \ldots, m-1\}. \]

Thus, Lemma 3.1 is proven.

Let us fix any orthonormal basis $(e_{k_1}^{(i)})_{k \geq 1}$ in $\mathcal{H}_i$, $i = 1, \ldots, l$. Then, the vectors
\[ e_{k_1}^{(i_1)} \otimes \cdots \otimes e_{k_m}^{(i_m)}, \quad i_1, \ldots, i_m \in \{1, \ldots, l\}, k_1, \ldots, k_m \geq 1, \]
constitute an orthonormal basis in $\mathcal{A}_m(\mathcal{H}_1, \ldots, \mathcal{H}_l)$, $m \in \mathbb{N}$. Therefore, by using (4.4), we conclude that the following vectors constitute an orthogonal basis in $\mathcal{A}_{m, \text{sym}}(\mathcal{H}_1, \ldots, \mathcal{H}_l)$:
\[ e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_1}^{(r_1)} \otimes \cdots \otimes e_{k_{r_1}}^{(l_1)} \otimes \cdots \otimes e_{k_{r_l}}^{(l_l)}; \quad r_1, \ldots, r_l \in \mathbb{Z}_+, r_1 + \cdots + r_l = m, \]

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where \( k_1^{(i)} < k_2^{(i)} < \cdots < k_r^{(i)} \) if \( p(i) \) is odd, and \( k_1^{(i)} \leq k_2^{(i)} \leq \cdots \leq k_r^{(i)} \) if \( p(i) \) is even. For any such vector, we get

\[
\left\| \mathbf{P} (e_{k_1^{(i)}}^{(1)} \otimes \cdots \otimes e_{k_r^{(i)}}^{(1)} \otimes \cdots \otimes e_{k_1^{(l)}}^{(l)} \otimes \cdots \otimes e_{k_r^{(l)}}^{(l)}) \right\|^2 =
\]

\[
= \frac{1}{m!} \prod_{j=1}^l \left[ \sum_{\sigma_j \in S_{r_j}} \text{sign} (\sigma_j, j, \ldots, j) \left( e_{k_{\sigma_j(1)}^{(j)}}^{(j)} \otimes \cdots \otimes e_{k_{\sigma_j(r_j)}^{(j)}}^{(j)} , e_{k_1^{(j)}}^{(j)} \otimes \cdots \otimes e_{k_r^{(j)}}^{(j)} \right) \right]
\]

\[
= \frac{1}{m!} \prod_{j=1}^l \left[ \sum_{\sigma_j \in S_{r_j}} \mathcal{G}(\sigma_j, j) \left( e_{k_{\sigma_j(1)}^{(j)}}^{(j)} \otimes \cdots \otimes e_{k_{\sigma_j(r_j)}^{(j)}}^{(j)} , e_{k_1^{(j)}}^{(j)} \otimes \cdots \otimes e_{k_r^{(j)}}^{(j)} \right) \right]
\]

\[
= \frac{r_1! \cdots r_l!}{m!} \prod_{j=1}^l \left\| e_{k_{\sigma_j(1)}^{(j)}}^{(j)} \otimes \cdots \otimes e_{k_{\sigma_j(r_j)}^{(j)}}^{(j)} \right\|^2 \mathcal{H}_{p(j)} \sigma_{r_j} ,
\]

(4.8)

where

\[
\mathcal{G}(\sigma_j, j) = \begin{cases} 
\text{sign} \sigma_j, & \text{if } p(j) \text{ is odd,} \\
1, & \text{if } p(j) \text{ is even.}
\end{cases}
\]

(4.9)

From (4.8) and (4.9) the conclusion of Lemma 3.2 trivially follows.

References

[1] S. Albeverio, A. Daletskii, and Yu. Kondratiev, Stochastic analysis on product manifolds: Dirichlet operators on differential forms, Preprint SFB 256 No. 598, Universität Bonn, 1999, to appear in *J. Funct. Anal.*

[2] S. Albeverio, A. Daletskii, and Yu. Kondratiev, De Rham complex over product manifolds: Dirichlet forms and stochastic dynamics, to appear in “Festschrift of L. Streit” (eds. S. Albeverio et al.), World Scientific, 2000.

[3] S. Albeverio, A. Daletskii, and E. Lytvynov, Laplace operators and diffusions in tangent bundles over Poisson spaces, Preprint SFB 256 No. 629, Universität Bonn, 1999, to appear in *Proc. KNAW*.

[4] S. Albeverio, A. Daletskii, and E. Lytvynov, Laplace operators on differential forms over configuration spaces, *J. Geom. Phys.* 37 (2001), 15–46.

[5] S. Albeverio and R. Høegh-Krohn, Dirichlet forms and Markov semigroups on \( C^* \)-algebras, *Comm. Math. Phys.* 56 (1977), 173–187.
[6] S. Albeverio and Yu. Kondratiev, Supersymmetric Dirichlet operators, *Ukrainian Math. J.* **47** (1995), 583–592.

[7] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Differential geometry of Poisson spaces, *C. R. Acad. Sci. Paris* **323** (1996), 1129–1134.

[8] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces, *C. R. Acad. Sci. Paris* **323** (1996), 1179–1184.

[9] S. Albeverio, Yu. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, *J. Funct. Anal.* **154** (1998), 444–500.

[10] S. Albeverio, Yu. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces: The Gibbsian case, *J. Funct. Anal.* **157** (1998), 242–291.

[11] S. Albeverio, Yu. Kondratiev, and M. Röckner, Diffeomorphism groups and current algebras: Configuration spaces analysis in quantum theory, *Rev. Math. Phys.* **11** (1999), 1–23.

[12] M. T. Anderson, $L^2$ harmonic forms on complete Riemannian manifolds, in “Geometry and Analysis on Manifolds (Katata/Kyoto, 1987),” pp. 1–19, Lecture Notes in Math., 1339, Springer, Berlin, 1998.

[13] A. Arai, A general class of infinite dimensional Dirac operators and path integral representation of their index, *J. Funct. Anal.* **105** (1992), 342–408.

[14] A. Arai, Supersymmetric extension of quantum scalar field theories, in “Quantum and Non-Commutative Analysis” (eds. H. Araki et al.), Kluwer Academic Publishers, Dordrecht, 1993, 73–90.

[15] A. Arai, Dirac operators in Boson–Fermion Fock spaces and supersymmetric quantum field theory, *J. Geom. Phys.* **11** (1993), 465–490.

[16] A. Arai and I. Mitoma, De Rham–Hodge–Kodaira decomposition in $\infty$-dimensions, *Math. Ann.* **291** (1991), 51–73.

[17] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras. *Colloque "Analyse et Topologie" en l’Honneur de Henri Cartan* (Orsay, 1974), pp. 43–72. *Astérisque*, No. 32-33, Soc. Math. France, Paris, 1976.

[18] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry. I, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69.

[19] A. Bendikov and R. Léandre, Regularized Euler–Poincaré number of the infinite dimensional torus, *IDAQP* **2** (1999) 617–626.
[20] Yu. M. Beresansky, “Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables,” Amer. Math. Soc., Providence, R.I., 1986.

[21] C. F. Bödigheimer, F. Cohen, and L. Taylor, On the homology of configuration spaces. Topology 28 (1989), 111–123.

[22] R. Bott and L. W. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[23] J. Brüning and M. Lesch, Hilbert complexes, J. Funct. Anal. 108 (1992), 88-132.

[24] J. Dodziuk, de Rham-Hodge theory for $L^p$-cohomology of infinite coverings, Topology 16 (1977), 157–165.

[25] K. D. Elworthy, Geometric aspects of diffusions on manifolds, in Lecture Notes in Math., Vol. 1362, pp. 276–425, Springer, Berlin, New York, 1988.

[26] K. D. Elworthy and X.-M. Li, Special Ito maps and an $L^2$ Hodge theory for one forms on path spaces, in ”Festschrift of S. Albeverio,”, eds. F. Gesztesy et al., Canadian Math. Soc. Conf. Proc., to appear.

[27] K. D. Elworthy, X.-M. Li, and S. Rosenberg, Bounded and $L^2$ harmonic forms on universal covers, Geom. Func. Anal. 8 (1998), 283–303.

[28] E. Fadell, Homotopy groups of configuration spaces and the string problem of Dirac, Duke Math. J. 29 (1962), 231–242.

[29] G. A. Goldin, J. Grodnik, R. T. Powers, and D. H. Sharp, Nonrelativistic current algebra in the $N/V$ limit, J. Math. Phys. 15 (1974), 88–100.

[30] M. A. Guest, A. Kozlowsky, and K. Yamaguchi, Homological stability of oriented configuration spaces, J. Math. Kyoto Univ. 36 (1996), 809–814.

[31] R. S. Ismagilov, “Representations of Infinite-Dimensional Groups,” Amer. Math. Soc., Providence, R. I., 1996.

[32] J. D. S. Jones and R. Léandre, A stochastic approach to the Dirac operator over the free loop space, in “Loop Spaces and the Group of Diffeomorphisms,” Proceedings of Steklov Institute, Vol. 217, 1997, 253–282.

[33] J. F. C. Kingman, “Poisson Processes,” Clarendon Press, Oxford, 1993.

[34] R. Léandre and S. S. Roan, A stochastic approach to the Euler–Poincaré number of the loop space of developable orbifold, J. Geom. Phys. 16 (1995), 71–98.

[35] R. Léandre, Analysis over loop spaces and topology. Preprint, Nancy University, 1999.
[36] V. Lipscher, Integration by parts formulae for point processes, in "Festschrift of S. Albeverio," eds. F. Gesztesy et al., Canadian Math. Soc. Conf. Proc., to appear.

[37] V. Mathai, $L^2$ invariants of covering spaces, in "Geometric analysis and Lie theory in mathematics and physics," Austral. Math. Soc. Lect. Ser., Vol. 11, pp. 209–242, Cambridge Univ. Press, Cambridge, 1998.

[38] W. Müller, On the $L^2$-index of Dirac operators on manifolds with corners of codimension two. I, J. Differential Geometry 44 (1996), 97–177.

[39] P. Pansu, Introduction to $L^2$ Betti numbers, in "Riemannian geometry" (Waterloo, ON, 1993), Fields Inst. Monogr., Vol. 4, pp. 53–86, Amer. Math. Soc., Providence, RI, 1996.

[40] J. J. Prat, and N. Privault, Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds, J. Funct. Anal. 167 (1999), 201–242.

[41] N. Privault, Equivalence of gradients on configuration spaces, Random Oper. Stoch. Eq. 7 (1999), 241–262.

[42] N. Privault, Connections and curvature in the Riemannian geometry of configuration spaces, La Rochelle preprint, 1999.

[43] A. L. Rebenko and G. V. Shchepan’uk, The convergence of the cluster expansion for continuous systems with many-body interaction, J. Stat. Phys. 88 (1997), 665–689.

[44] M. Reed and B. Simon, “Methods of Modern Mathematical Physics, Vol. 2. Fourier Analysis, Self-Adjointness,” Academic Press, New York, London, 1972.

[45] M. Röckner, Stochastic analysis on configuration spaces: Basic ideas and recent results, in “New Directions in Dirichlet Forms” (eds. J. Jost et al.), Studies in Advanced Mathematics, Vol. 8, pp. 157–232, American Math. Soc., 1998.

[46] I. Shigekawa, De Rham–Hodge–Kodaira’s decomposition on an abstract Wiener space, J. Math. Kyoto Univ. 26 (1986), 191-202.

[47] A. M. Vershik, I. M. Gel’fand, and M. I. Graev, Representations of the group of diffeomorphisms, Russian Math. Surv. 30 (1975), 1–50.

[48] J. Wu, On the homology of configuration spaces $C((M, M_0) \times \mathbb{R}^n; X)$, Math. Zeitschrift 229 (1998), 235–248.