Applications of Lodrigues Matrix in 3D Coordinate Transformation

YAO Jili  XU Yufei  XIAO Wei

Abstract  Three transformation models (Bursa-Wolf, Molodensky, and WTUSM) are generally used between two data systems transformation. The linear models are used when the rotation angles are small; however, when the rotation angles get bigger, model errors will be produced. In this paper, we present a method with three main terms: ① the traditional rotation angles \( \theta, \phi, \psi \) are substituted with \( a, b, c \) which are three respective values in the anti-symmetrical or Lodrigues matrix; ② directly and accurately calculating the formula of seven parameters in any value of rotation angles; and ③ a corresponding adjustment model is established. This method does not use the triangle function. Instead it uses addition, subtraction, multiplication and division, and the complexity of the equation is reduced, making the calculation easy and quick.

Keywords  3D transformation; linear model; transformation equation; Lodrigues matrix

CLC number  P226.3

Introduction

The coordinate transformation in surveying and mapping has two main tasks: coordinate datum’s transformation and point’s coordinate transformation between two coordinate systems. The Bursa-Wolf, Molodensky, and WTUSM models are generally used in 3D coordinate transformation[1]. The seven-parameter model of 3D coordinate transformation is improved and a nonlinear model of 3D coordinate transformation is presented in Reference [2] in the case where the rotation angles are bigger than 50°. In the adjustment model, the initial values of three rotation angles are zero. The first approximations of the seven parameters and model linearization are not presented. With the development of the attitude determination of movement carriers (airborne, ships and automobiles), the models are being used more widely[3-7]. Spatial position, attitude and rotation angle of the carriers are facultative, therefore accurate determination of the transformation parameter is necessary. With the Lodrigues matrix being used widely in photogrammetry, the workload of calculation is deduced and rigorous mathematics models are established. In this paper, the Lodrigues matrix is used in 3D coordinate transformation and formulas of direct calculation of seven parameters and an adjustment model are derived. This model is rigorously, simply and rapidly proved in theory and practice, and fits 3D coordinate transformation with any angle.

1 Mathematics model of 3D coordinate transformation

On the basis of physics processing of coordinate transformation, there exists an equation:

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}_T = \begin{bmatrix}
X_{\hat{t}} \\
Y_{\hat{t}} \\
Z_{\hat{t}}
\end{bmatrix} + \lambda \mathbf{R} \begin{bmatrix}
X_S - X_{\hat{b}} \\
Y_S - Y_{\hat{b}} \\
Z_S - Z_{\hat{b}}
\end{bmatrix}
\]  \hspace{1cm} (1)

Received on June 11, 2007.

YAO Jili, School of Architecture Engineering, Shandong University of Technology, 12 Zhangzhou Road, Zibo 255049, China.
E-mail: ysy_941123@sdut.edu.cn
which can be reduced to:

\[
\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \lambda \begin{bmatrix} \Delta X' \\ \Delta Y' \\ \Delta Z' \end{bmatrix} + \lambda \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \]  

(2)

The meaning of the symbols above are taken from Reference [2]. \( \Delta X, \Delta Y, \Delta Z, \lambda, \theta, \phi, \psi \) are usually called the seven parameters.

### 2 Direct calculation of seven parameters

Eq.(2) can be solved uniquely when three control points exist (i.e., common points that coordinates are known in the two systems). According to the transformation mathematical model, the value of scale factor \( \lambda \) is first computed, followed by the value of the rotation matrix and finally the three translations factor. Thus, the key problem is determining the rotation matrix and finally the three translations.

The rotation matrix and Lodrigues matrix are:

\[
\lambda(I+M) = (I-S)(I+S)^{-1} \]

(3)

Eq.(3) is called the Lodrigues matrix, in which \( A=1+a^2+b^2+c^2 \).

\[
M = \begin{bmatrix} 1+a^2-b^2-c^2 & -2c-2ab & -2b+2ac \\ 2c-2ab & 1-a^2+b^2-c^2 & -2a-2bc \\ 2b+2ac & 2a-2bc & 1-a^2-b^2+c^2 \end{bmatrix} \]

In this paper, \( \theta, \phi, \psi \) are substituted for \( a,b,c \), and a set of equations of 3D coordinate transformation are established. The characteristics of the anti-symmetry matrix and Lodrigues matrix are:

1. \( S^T = -S \), \( R = (I+S)(I-S)^{-1} \)
2. \( (I-S)^T = I+S, (I+S)^T = I-S \)
3. \( R^T = R^{-1} = (I+S)^{T}(I-S) \)
4. \( (I+S)^{-1} = \frac{1}{2}(I+R^T)^{-1}, S = 2(I+R^T)^{-1} - I \)

where \( I \) is a 3×3 unit matrix.

The scale factor \( \lambda \) can be obtained by comparing the length of correspondence in the two coordinate systems, that is:

\[
\lambda = 1 + m = \frac{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \]  

(5)

Eq.(2) is nonlinear and can be solved uniquely when three control points exist. This is because in each point, an equation set in the form of Eq.(2) can be written and three points yield 9 equations involving 7 unknowns. After eliminating translations and substituting Eq.(4), for point 1 and 2, the combined form is:

\[
\lambda(I+S)[X_{S_1} - Y_{S_1} \quad Y_{S_1} - Y_{S_1} \quad Z_{S_1} - Z_{S_1}] = (I - S)[X_{R_1} - X_{R_1} \quad Y_{R_1} - Y_{R_1} \quad Z_{R_1} - Z_{R_1}] \]

Simplification form is given as:

\[
\lambda(I+S)[X_{S_1} \quad Y_{S_1} \quad Z_{S_1}] = (I - S)[X_{R_1} \quad Y_{R_1} \quad Z_{R_1}] \]

(6)

extending and arranging, Eq.(6) becomes:

\[
\begin{bmatrix}
X_{S_1} - \lambda X_{S_1} & -\lambda Y_{S_1} - Y_{S_1} & -\lambda Z_{S_1} - Z_{S_1} \\
-\lambda Z_{S_1} - Z_{S_1} & \lambda X_{S_1} + X_{S_1} & 0 \\
0 & \lambda Y_{S_1} + Y_{S_1} & \lambda Z_{S_1} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} X_{R_1} - \lambda X_{R_1} \\ Y_{R_1} - \lambda Y_{S_1} \\ Z_{R_1} - \lambda Z_{S_1} \end{bmatrix} \]

(7)

Eq.(7) involves two independent equations only and cannot compute 3 unknowns. For points 1 and 3, a set of equations that are similar to Eq.(7) can be written.

\[
\begin{bmatrix}
0 & -w_2 & -v_2 \\
-w_2 & 0 & u_3 \\
v_3 & u_3 & 0 \\
\end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} X_{S_1} - \lambda X_{S_1} \\ Y_{S_1} - \lambda Y_{S_1} \\ Z_{S_1} - \lambda Z_{S_1} \end{bmatrix} \]

where \( u_3 = \lambda X_{R_3} + X_{R_3} \); \( v_2 = \lambda Y_{S_3} + Y_{S_3} \); \( w_2 = \lambda Z_{S_3} + Z_{S_3} \); \( u_3 = \lambda X_{R_3} + X_{R_3} \); \( v_2 = \lambda Y_{S_3} + Y_{S_3} \). Then three parameters \( a,b,c \) can be computed by the following equations.

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} u_3 & -u_3 v_2 & u_3 w_2 \\ u_3 v_2 & v_2 & u_3 w_2 \\ -u_3 w_2 & -v_2 & -w_2 \end{bmatrix} \begin{bmatrix} X_{S_1} - \lambda X_{S_1} \\ Y_{S_1} - \lambda Y_{S_1} \\ Z_{S_1} - \lambda Z_{S_1} \end{bmatrix} \]

(8)

where \( \Delta = u_3 v_2 w_2 - u_3 v_3 w_3 \). Rotation matrix \( R \) can be computed by Eq.(3) and translations by Eq.(2).
\[
\begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix}
X_t \\
Y_t \\
Z_t
\end{bmatrix} - R \begin{bmatrix}
X_s \\
Y_s \\
Z_s
\end{bmatrix}
\]  
(9)

### 3 Adjustment model of 3D coordinate transformation

In real measurements, for the purpose of error detection and higher accuracy, more than three common points are available and a least-squares solution can be obtained. Let \( H = [\Delta X \ \Delta Y \ \Delta Z \ \lambda \ a \ b \ c]^T \) represent vectors, by Eq.(5), Eq.(8), Eq.(9), the initial approximations of unknowns can be computed and Eq.(2) can be linearized. The linear expression is as follows:

\[
\begin{align*}
C_1 \frac{dX_r}{dY_r} + C_2 \frac{dY_s}{dZ_s} + C_3 \frac{dH}{W} - W &= 0 \\
\end{align*}
\]
(10)

where \( W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \lambda \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \); \( C_i = -I \); \( C_3 = \lambda R \); \( C_3 \) represents the coefficient matrix of transformation parameters.

\[
C_3 = \begin{bmatrix}
\lambda & 0 & 0 & X_r / \lambda \\
0 & \lambda & 0 & Y_r / \lambda & A & B & C \\
0 & 0 & \lambda & Z_r / \lambda 
\end{bmatrix}
\]

After developing, three coefficient matrices are given:

\[
\begin{align*}
A &= \lambda \frac{\partial R}{\partial a} \begin{bmatrix}
X_s \\
Y_s \\
Z_s
\end{bmatrix} = \begin{bmatrix}
a & -b & c \\
2a & \lambda a + \lambda & -b & -a & 0 \\
2b & \lambda b & \lambda c & \lambda & 0
\end{bmatrix} \begin{bmatrix}
X_s \\
Y_s \\
Z_s
\end{bmatrix} \\
B &= \lambda \frac{\partial R}{\partial b} \begin{bmatrix}
X_s \\
Y_s \\
Z_s
\end{bmatrix} = \begin{bmatrix}
a & b & -c & 0 \\
2b & -a & 0 & \lambda & 0 \\
2a & 0 & -c & 0 & \lambda
\end{bmatrix} \begin{bmatrix}
X_s \\
Y_s \\
Z_s
\end{bmatrix} \\
\end{align*}
\]

The linear form of the 3D transformation model in the matrix is represented as:

\[
C_1 \lambda \frac{dX_r}{dY_r} + C_2 \lambda \frac{dY_s}{dZ_s} + C_3 \lambda \frac{dH}{W} = 0
\]
(11)

### 4 Example and analysis

In this example, the coordinates in both coordinate systems are simulative, the true values of the seven parameters are known, the average distances are about 20 km, rotating angles can range from 0.005° to 80°. There are a total of 140 sets of coordinates of the common points simulated in both coordinate systems. Assuming that in the two coordinates the point-error of every point is ±3 cm, and the error of coordinates are symmetrical in the three coordinates, errors are all ±1.73 cm. We can use two schemes for changing coordinates. Scheme 1 uses the linear model. Scheme 2 is the 3D coordinate transformation model based on Lodrigues matrix using MSE to describe the errors of seven parameters in the two methods.

\[
u = \text{MSE} = \left[ \frac{\partial h}{\partial \delta} \right] = \delta^T \delta 
\]
(12)

\[
\text{MSE} = \delta X^2 + \delta Y^2 + \delta Z^2 + \delta \lambda^2 \cdot S^2
\]

while \( \| \delta \| \) represents 2-norm and \( S \) represents the mean distance. The analysis of the seven parameters in the two projects are as follows.

From Table 1, it can be seen that the difference between the parameter solutions and real values is

| Angle(°) | Model | Difference between parameter solution and real values |
|----------|-------|----------------------------------------------------|
| \( \delta X / m \) | \( \delta Y / m \) | \( \delta Z / m \) | \( \delta \lambda / \times 10^6 \) | \( \delta \theta / (°) \) | \( \delta \phi / (°) \) | \( \delta \psi / (°) \) |
| 0.000 / 5/0.013 | Scheme 1 | 0.058 7 | 0.054 9 | 0.071 4 | -0.062 | -0.000 02 | -0.000 02 | 0.000 01 |
| 0.000 / 5/0.005 | Scheme 2 | 0.009 0 | 0.055 8 | 0.007 4 | -0.054 | -0.000 01 | -0.000 01 | 0.000 01 |
| 0.005 / 0.491 | Scheme 1 | 0.069 9 | 0.062 5 | 0.072 7 | -1.092 | 0.000 00 | -0.000 04 | 0.000 04 |
| 0.005 / 0.409 | Scheme 2 | -0.013 7 | -0.004 9 | -0.111 4 | 0.145 | 0.000 01 | -0.000 01 | 0.000 01 |
| 0.05 / 0.491 | Scheme 1 | 0.069 9 | 0.062 5 | 0.072 7 | -1.092 | 0.000 00 | 0.000 03 | 0.000 03 |
| 0.05 / 0.004 | Scheme 2 | 0.038 3 | 0.028 3 | 0.038 7 | -0.025 | -0.000 01 | 0.000 01 | 0.000 00 |
| 0.5 / 0.171 | Scheme 1 | -0.360 0 | -0.389 8 | 0.099 8 | -4.21 | -0.004 36 | -0.004 50 | 0.099 92 |
| 0.5 / 0.001 | Scheme 2 | 0.001 1 | -0.022 0 | 0.002 4 | -0.009 | 0.000 02 | 0.000 00 | -0.000 02 |
| 5/2.18×10^7 | Scheme 1 | 45.88 9 | 66.89 7 | -1.461 | -737.6 | 0.186 23 | -0.470 14 | 0.420 28 |
| 5/0.02 | Scheme 2 | 0.014 9 | 0.026 5 | 0.070 8 | -0.187 | -0.000 13 | 0.000 17 | 0.000 00 |
| 50/7.1×10^9 | Scheme 1 | 1.199 2 | -4.580 9 | 89.1 | -419 231 | 30.577 1 | 2.092 62 | 44.645 7 |
| 50/0.170 | Scheme 2 | 0.059 6 | 0.206 9 | 0.064 4 | 0.546 | -0.000 01 | -0.000 03 | 0.000 01 |
small when the rotation angle is small. But when the rotation angles are above 0.5°, the MSE of the first scheme apparently increases, and exceeds 2 m². The difference between the parameter solutions of the second scheme and real values is always small. The differences are expressed in Fig.1.

Accuracy of the translation attains 0.15 m, the scale factor expressed in seconds and it’s accuracy higher than $0.4 \times 10^{-6}$. The precision of the rotation angle is bigger than 0.2°. When the rotation angle is big, the fitting analysis method is used for the solution of the linear model, which is solved by secondary polynomial and triple polynomial. The fitting code is expressed in Fig.2. From the figure we can learn that the MSE of the linear model is increasing when the rotation angle is above 0.5°; when the rotation angle is 2.5°, the MSE = 2 000 m²; when the rotation angle is 5°, the MSE = $2.18 \times 10^7$ m². The accuracy of the triple polynomial is higher than the secondary polynomial. When the average distance is about 20 km and the rotation angle is below 5°, the linear model can be used. From Table 1 we can obtain that the MSE can range from 0.001 m² to 0.17 m² in any angle, so the model presented in this paper is used widely.

References

[1] Liu Dajie, Shi Yimin, Guo Jingjun(1996) The principle and data processing of GPS[M]. Shanghai: Publishing House of Tongji University (in Chinese)

[2] Zeng Wenxian,Tao Benzao(2003) Non-linear model of 3D coordinate transformation[J]. Geomatics and Information Science of Wuhan University, 28(5):566-568 (in Chinese)

[3] Chinese Society for Geodesy Photogrammetry and Cartography(2005) Development of surveying and mapping subject blue book in China[M]. Beijing: Surveying and Mapping Publishing House (in Chinese)

[4] Liu Genyou(2003) A new method of determining attitude with GPS: damped LAMBDA algorithm with coordinates functional constraint[J]. Science of Surveying and Mapping, 28(3): 36-38 (in Chinese)

[5] Zhao Jianhu, Liu Jingnan, Zhou Fengnian(2000) Method in determining vessel attitude with GPS[J]. Journal of Wuhan Technical University of Surveying and Mapping, 25(4): 353-357 (in Chinese)

[6] Li Deren, Zheng Zhaobao(1992) Analysis photogrammetry[M]. Beijing: Surveying and Mapping Publishing House (in Chinese)

[7] Zhang Senlin(1987) Applications of rigor solving collinearity equations based on Lodrigues matrix[J]. Journal of Wuhan Technical University of Surveying and Mapping, 12(1): 81-91 (in Chinese)

[8] Wang Zhenjie(2003) Research on the regularization solutions of ill-posed problems in geodesy[D]. Wuhan: Institute of Geodesy and Geophysics, Chinese Academy of Sciences (in Chinese)