The modified \( q \)-Bessel functions and the \( q \)-Bessel-Macdonald functions of the first and second kind are introduced. Their definition is based on representations as power series. Recurrence relations, the \( q \)-Wronskians, asymptotic decompositions and \( q \)-integral representations are received. In addition, the \( q \)-Bessel-Macdonald function of kind 3 is determined by its \( q \)-integral representation.

1 Introduction

The \( q \)-Bessel-Jackson functions of kinds 1, 2 and 3 were introduced at the very beginning of the century \([1]\). Their properties were considered in \([2, 3, 6, 7]\). The definitions of the modified \( q \)-Bessel-Jackson functions (\( q \)-MBF) and \( q \)-Macdonald functions (\( q \)-MF) and their properties were given in \([4]\).

The \( q \)-analogs of the modified Bessel functions and Macdonald functions are interesting because they arise in the harmonic analysis on the quantum homogeneous spaces as in the classical case. If we suppose that the commutation relations in the universal enveloping algebra and the commutation relations for the generators of the quantum Lobachevsky space \([5]\) are determined by different parameters than the eigenvalue problem for the second Casimir operator leads us to the difference equation depending on one parameter. This equation is the second order one for three values of this parameter. These values of parameter correspond the \( q \)-Bessel-Jackson functions of kind 1 \( J^{(1)}_\nu \), kind 2 \( J^{(2)}_\nu \), and kind 3 \( J^{(3)}_\nu \). This fact allows to consider these functions with uniform point of view. Same authors name the \( q \)-Bessel-Jackson functions of kind 3 by the Hahn-Exton ones.

In this work we start from the definition of the \( q \)-MBF as a solution of the second order difference equation. The \( q \)-MBFs are connected with \( q \)-Bessel functions as in the classical case. We determine the actions of the difference operators, the recurrence relations, and the \( q \)-Wronskians for them. The Laurent series for the \( q \)-MBF of kinds 1 and 2 are very important for a determination of the \( q \)-MFs. At last we represent the \( q \)-MFs of kinds 1 and 2 by the Jackson \( q \)-integral. We determine the \( q \)-MFs of kind 3 by their \( q \)-integral representation, and then we receive the expression of these functions by the \( q \)-MBFs of kind 3.

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2 Preliminary results

2.1. The modified $q$-Bessel functions

In [1] the $q$-Bessel functions were defined as follows:

\begin{align}
J_{
u}^{(1)}(z, q) &= \frac{(q^{
u+1}, q)_{\infty}}{(q, q)_{\infty}}(z/2)^{\nu} 2\Phi_1(0, 0; q^{\nu+1}; q, -\frac{z^2}{4}), \\
J_{
u}^{(2)}(z, q) &= \frac{(q^{
u+1}, q)_{\infty}}{(q, q)_{\infty}}(z/2)^{\nu} 0\Phi_1(-; q^{\nu+1}; q, -\frac{z^2q^{\nu+1}}{4}), \\
J_{
u}^{(3)}(z, q) &= \frac{(q^{
u+1}, q)_{\infty}}{(q, q)_{\infty}}(z/2)^{\nu} 1\Phi_1(-; q^{\nu+1}; q, -\frac{z^2q^{\nu+1}}{4}).
\end{align}

where $\Phi_s$ is basic hypergeometric function [2],

\[ \Phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_r, q)_n}{(q, q)_n(b_1, q)_n \cdots (b_s, q)_n} \left[ (-1)^n q^n(n-1)/2\right]^{1+s-r} z^n. \]

It allows to introduce the modified $q$-Bessel functions (q-MBFs) using (2.1), (2.2) and (2.3) similarly to the classical case [10].

**Definition 2.1** The modified $q$-Bessel functions are the functions

\[ I_{\nu}^{(j)}(z, q) = \frac{(q^{
u+1}, q)_{\infty}}{(q, q)_{\infty}}(z/2)^{\nu} \, \delta\Phi_1 \left( 0, \ldots, 0; q^{\nu+1}; \frac{z^2q^{\nu+1}(2-\delta)}{4} \right). \]

Here

\[ \delta = \begin{cases} 
2 & \text{for } j = 1 \\
0 & \text{for } j = 2 \\
1 & \text{for } j = 3.
\end{cases} \]

Obviously,

\[ I_{\nu}^{(j)}(z, q) = e^{-i\pi z} J_{\nu}^{(j)}(e^{i\pi/2} z, q), \quad j = 1, 2, 3. \]

In the sequel we consider the functions

\begin{align}
I_{\nu}^{(1)}((1 - q^2)z; q^2) &= \sum_{k=0}^{\infty} \frac{(1 - q^2)^k(z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2(\nu + k + 1)}, \quad |z| < \frac{2}{1-q^2}, \\
I_{\nu}^{(2)}((1 - q^2)z; q^2) &= \sum_{k=0}^{\infty} \frac{q^{2k(\nu+1)}(1 - q^2)^k(z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2(\nu + k + 1)}, \\
I_{\nu}^{(3)}((1 - q^2)z; q^2) &= \sum_{k=0}^{\infty} \frac{q^{k(\nu+1)}(1 - q^2)^k(z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2(\nu + k + 1)}.
\end{align}

If $|q| < 1$, the series (2.5) and (2.6) are absolutely convergent for all $z \neq 0$. Consequently, $I_{\nu}^{(1)}((1 - q^2)z; q^2)$ and $I_{\nu}^{(3)}((1 - q^2)z; q^2)$ are holomorphic functions outside a neighborhood of zero.
Remark 2.1
\[ \lim_{q \to 1^-} I_{\nu}^{(j)}((1 - q^2)z; q^2) = I_{\nu}(z), \quad j = 1, 2, 3. \]

Proposition 2.1 The function \( I_{\nu}^{(j)}((1 - q^2)z; q^2) \) is a solution of the difference equation
\[ f(q^{-1}z) - (q^{-\nu} + q^{\nu})f(z) + f(qz) = q^{-\delta(1 - q^2)^2}z^2 f(q^{1-\delta}z). \] (2.8)
\( j = 1, 2, 3 \) are connected with \( \delta = 2, 0, 1 \) by relations \([2, 4]\).

Corollary 2.1 The function \( I_{-\nu}^{(j)}((1 - q^2)z; q^2) \) satisfies equation \([2, 8]\).

Proposition 2.2 The functions \( I_{\nu}^{(j)}((1 - q^2)z; q^2) \) satisfies the relations
\[ \frac{2}{(1 + q)z} \partial_q z^{\nu} I_{\nu}^{(j)}((1 - q^2)z; q^2) = q^{-\frac{2+\delta}{2}(\nu-1)} z^{\nu-1} I_{\nu-1}^{(j)}((1 - q^2)q^{\frac{\delta}{1-q^2}} z; q^2), \]
\[ \frac{2}{(1 + q)z} \partial_q z^{\nu} I_{\nu}^{(j)}((1 - q^2)z; q^2) = q^{-\frac{2+\delta}{2}(\nu-1)} z^{\nu-1} I_{\nu+1}^{(j)}((1 - q^2)q^{\frac{\delta}{1-q^2}} z; q^2), \]
where the operator \( \partial_q \) is defined as \( \partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \).

Proposition 2.3 The functions \( I_{\nu}^{(j)}((1 - q^2)z; q^2) \) satisfy the recurrence relations
\[ q^{-\frac{2+\delta}{2}} I_{\nu-1}^{(j)}((1 - q^2)z; q^2) - q^{\frac{2-\delta}{2} \nu} I_{\nu+1}^{(j)}((1 - q^2)z; q^2) = \]
\[ = \frac{2}{(1 - q^2)z} (q^{-\nu} - q^{\nu}) I_{\nu}^{(j)}((1 - q^2)q^{\frac{\delta}{1-q^2}} z; q^2), \]
\[ q^{-\frac{2+\delta}{2}} I_{\nu-1}^{(j)}((1 - q^2)z; q^2) + q^{\frac{2-\delta}{2} \nu} I_{\nu+1}^{(j)}((1 - q^2)z; q^2) = \frac{4}{(1 - q^2)z} I_{\nu}^{(j)}((1 - q^2)q^{-\frac{2-\delta}{2}} z; q^2) - \]
\[ - \frac{2}{(1 - q^2)z} (q^{-\nu} + q^{\nu}) I_{\nu}^{(j)}((1 - q^2)q^{\frac{\delta}{1-q^2}} z; q^2). \]

Definition 2.2 The q-Wronskian of two solutions \( f_{\nu}^{1}(z) \) and \( f_{\nu}^{2}(z) \) of a second-order difference equation is defined as follows:
\[ W(f_{\nu}^{1}, f_{\nu}^{2})(z) = f_{\nu}^{1}(z)f_{\nu}^{2}(qz) - f_{\nu}^{1}(qz)f_{\nu}^{2}(z). \]

If the q-Wronskian does not vanish, then any solution of the second-order difference equation can be written in form
\[ f_{\nu}(z) = C_{1} f_{\nu}^{1}(z) + C_{2} f_{\nu}^{2}(z). \]

In this case the functions \( f_{\nu}^{1}(z) \) and \( f_{\nu}^{2}(z) \) form a fundamental system of the solutions of the given equation.
Proposition 2.4 If \( \nu \neq k + \frac{im}{lnq} \), \( k \) and \( m \) are integers, then the functions \( I_{\nu}^{(j)}((1-q^2)z; q^2) \) and \( I_{-\nu}^{(j)}((1-q^2)z; q^2) \) form a fundamental system of the solutions of equation (2.3) \( (z \neq \pm \frac{2q-r}{1-q^2}, r = 0, 1, \ldots) \).

This Proposition is following from

\[
W(z) = \begin{cases} 
\frac{q^{-\nu}(1-q^2)}{\nu^{(n)}(1-q)\nu} e^{q^2} (\frac{(1-q^2)^2}{4} z^2) & \text{for } \delta = 2 \\
\frac{1}{\nu^{(n)}(1-q)\nu} (1-q^2) & \text{for } \delta = 1 \\
\frac{1}{\nu^{(n)}(1-q)\nu} E_{\nu}(\frac{(1-q^2)^2}{4} q^2 z^2) & \text{for } \delta = 0.
\end{cases}
\]

Obviously, this function is defined for \( z \neq \pm \frac{2q-r}{1-q^2}, r = 0, 1, \ldots \) and does not vanish.

If \( \nu = n \) is an integer, then in view of (2.3) - (2.7)

\[
I_{\nu}^{(j)}((1-q^2)z; q^2) = I_{\nu}^{(j)}((1-q^2)z; q^2), \quad j = 1, 2, 3.
\]

The following relations take place

\[
I_{\nu}^{(1)}((1-q^2)z; q^2) = e^{q^2} (\frac{1-q^2}{4} z^2) I_{\nu}^{(2)}((1-q^2)z; q^2).
\]

\[
I_{\nu}^{(2)}((1-q^2)z; q^2) = E_{\nu}(\frac{(1-q^2)^2}{4} q^2 z^2) I_{\nu}^{(1)}((1-q^2)z; q^2).
\]

Proposition 2.5 The function \( I_{\nu}^{(1)}((1-q^2)z; q^2) \) is a meromorphic function outside a neighborhood of zero, with simple poles at the points \( z = \pm \frac{2q-r}{1-q^2}, r = 0, 1, \ldots \).

Remark 2.2 If \( q \to 1 - 0 \), then the poles of the function \( I_{\nu}^{(1)}((1-q^2)z; q^2) \)

\[
z_r = \pm \frac{2q-r}{1-q^2}, \quad r = 0, 1, \ldots
\]

tend to infinity along the real axis.

2.2. Laurent type series for \( q \)-MBFs of kinds 1 and 2

Proposition 2.6 For \( z \neq 0 \), and \( j = 1, 2 \) \( q \)-MBF can be represented by form

\[
I_{\nu}^{(j)}((1-q^2)z; q^2) = \frac{a_{\nu}}{\sqrt{z}} \left[ \phi^{(j)}(z) \Phi_{\nu}(z) + i e^{i\nu\pi} \phi^{(j)}(-z) \Phi_{\nu}(-z) \right],
\]

where

\[
\phi^{(j)}(z) = \begin{cases} 
e_q \left( \frac{1-q^2}{2} z \right) & \text{for } j = 1 \\
E_q \left( \frac{1-q^2}{2} z \right) & \text{for } j = 2,
\end{cases}
\]

\[
\Phi_{\nu}(z) = 2 \Phi_1 \left( q^{\nu+\frac{1}{2}}, q^{-\nu+\frac{1}{2}}; q^{-1}; q, \frac{2q}{1-q^2}z \right),
\]

\[
a_{\nu} = \sqrt{2} \left[ \frac{I_{\nu}^{(2)}(2; q^2)}{1-q^2 (-1, q) \infty \Phi_{\nu}(\frac{2}{1-q^2})} \right].
\]

(2.10)
Proposition 2.7 The coefficients $a_\nu$ satisfy the relations

\begin{align}
    a_{\nu+1} &= a_\nu q^{-\nu-\frac{1}{2}}, \\
    a_\nu a_{-\nu} &= \frac{q^{-\nu+\frac{1}{2}}}{2\Gamma q^2(\nu)\Gamma q^2(1-\nu) \sin \nu \pi}.
\end{align}

If $\nu = n$, then from (2.11) we have

$$a_n = a_{n-k} q^{-k/2(2n-k)(\frac{1}{2} - |\delta - 1|)}.$$

Assume $k = 2n$. Then $a_n = a_{-n}$. From (2.12) and [9] (1.10.6) we get

$$(a_n)^2 = \frac{q^{-n^2+1/2} \ln q^{-2}}{2\pi(1 - q^2)}.$$

2.3. The $q$-Macdonald functions of kinds 1 and 2

Definition 2.3 We define the $q$-Macdonald function ($q$-MF) for $j = 1, 2, 3$ as follows:

\begin{align}
    K^{(j)}_\nu ((1 - q^2)z; q^2) &= \frac{1}{2} q^{-\nu^2+\nu} \Gamma q^2(\nu) \Gamma q^2(1-\nu) \left[ \frac{a_\nu}{a_{-\nu}} I^{(j)}_\nu ((1 - q^2)z; q^2) - \sqrt{\frac{a_{-\nu}}{a_\nu}} I^{(j)}_- ((1 - q^2)z; q^2) \right], \tag{2.13}
\end{align}

where $a^{(j)}_\nu$ is determined by (2.10).

As in the classical case, this definition must be extended to integral values of $\nu = n$ by passing to the limit in (2.13).

Proposition 2.8 The $q$-MF $K^{(1)}_\nu ((1 - q^2)z; q^2)$ is represented by form

$$K^{(1)}_\nu ((1 - q^2)z; q^2) = \frac{q^{-\nu^2+1/2}}{2 \sqrt{a_\nu a_{-\nu}} \sqrt{z}} F_q \left( -\frac{1 - q^2}{2} z \right) \Phi \nu(-z),$$

and hence it is a holomorphic function in the region $\Re z > \frac{2q}{1 - q^2}$.

Proposition 2.9 The $q$-MF $K^{(2)}_\nu ((1 - q^2)z; q^2)$ is represented by form

$$K^{(2)}_\nu ((1 - q^2)z; q^2) = \frac{q^{-\nu^2+1/2}}{2 \sqrt{a_\nu a_{-\nu}} \sqrt{z}} E_q \left( -\frac{1 - q^2}{2} z \right) \Phi \nu(-z),$$

and hence it is a holomorphic function in the region $z \neq 0$. 5
Proposition 2.10 The function \( K^{(j)}_{\nu}((1 - q^2)z; q^2) \) satisfies the relations
\[
\frac{2}{(1 + q)z} \partial_q z^\nu K^{(j)}_{\nu}((1 - q^2)z; q^2) = -q^{\frac{2-\delta}{2}(\nu - 1)} z^{\nu - 1} K^{(j)}_{\nu - 1}((1 - q^2)q^{\frac{2-\delta}{2}} z; q^2), \tag{2.14}
\]
\[
\frac{2}{(1 + q)z} \partial_q z^{-\nu} K^{(j)}_{\nu}((1 - q^2)z; q^2) = -q^{\frac{2-\delta}{2}(\nu + 1)} z^{\nu - 1} K^{(j)}_{\nu + 1}((1 - q^2)q^{\frac{2-\delta}{2}} z; q^2). \tag{2.15}
\]

Proposition 2.11 The functions \( K^{(j)}_{\nu}((1 - q^2)z; q^2) \) satisfy the functional relations:
\[
q^{\frac{2-\delta}{2}\nu} K^{(j)}_{\nu - 1}((1 - q^2)z; q^2) - q^{\frac{2-\delta}{2}\nu} K^{(j)}_{\nu + 1}((1 - q^2)z; q^2) = -\frac{2}{(1 - q^2)z} (q^{-\nu} - q^\nu) K^{(j)}_{\nu}((1 - q^2)q^{\frac{2-\delta}{2}} z; q^2), \tag{2.16}
\]
\[
q^{\frac{2-\delta}{2}\nu} K^{(j)}_{\nu - 1}((1 - q^2)z; q^2) + q^{\frac{2-\delta}{2}\nu} K^{(j)}_{\nu + 1}((1 - q^2)z; q^2) = -\frac{4}{(1 - q^2)z} K^{(j)}_{\nu}((1 - q^2)q^{\frac{2-\delta}{2}} z; q^2), + \frac{2}{(1 - q^2)z} (q^{-\nu} + q^\nu) K^{(j)}_{\nu}((1 - q^2)q^{\frac{2-\delta}{2}} z; q^2). \tag{2.17}
\]

Proposition 2.12 For any \( \nu \) the functions \( I^{(j)}_{\nu}((1 - q^2)z; q^2) \) and \( K^{(j)}_{\nu}((1 - q^2)z; q^2) \) form a fundamental system of the solutions of equation (2.9).

The \( q \)-Wronskian \( W(I^{(j)}_{\nu}, K^{(j)}_{\nu})(z) \) differs from the \( q \)-Wronskian \( \text{(2.3)} \) by constant multiplier.
\[
W(I^{(j)}_{\nu}, K^{(j)}_{\nu})(z) = \begin{cases} 
\frac{q^{-\nu^2(1-q^2)}}{2} \sqrt{\frac{a_{\nu}}{\Delta_{\nu}}} e^{\nu \Gamma_q(1)} \frac{(1-q^2)z^2}{4} & \text{for } j = 1 \\
\frac{q^{-\nu^2(1-q^2)}}{2} \sqrt{\frac{a_{\nu}}{\Delta_{\nu}}} E_q \frac{(-1-q^2)q^2z^2}{4} & \text{for } j = 2.
\end{cases}
\]

Hence, this function does not vanish.

2.4. The \( q \)-integral representations

In \( \text{(8)} \) the representations of \( q \)-MFs (for \( j = 1, 2 \)) by the Jackson \( q \)-integral were received. Let \( z \) and \( s \) be non commuting variables
\[
zs = qsz.
\]

Proposition 2.13 If \( \nu > \frac{3}{2} \) the \( q \)-MFs can be represented by \( q \)-integrals
\[
K^{(1)}_{\nu}((1 - q^2)z; q^2) = \frac{q^{-\nu^2 + \frac{1}{2}} \Gamma_q(\nu + \frac{1}{2}) \Gamma_q(\nu)}{4Q_{\nu}} \sqrt{\frac{a_{\nu}}{\Delta_{\nu}}} \left( \frac{z}{2} \right)^{-\nu} \times
\]

\[ \times \int_{-\infty}^{\infty} E_q \left( i \frac{1-q^2}{2} z s \right) \frac{(-q^2 s^2, q^2)_q}{(-q^{-2\nu+1}s^2, q^2)_\infty} d_q s, \quad (2.18) \]

\[ K^{(2)}_\nu((1-q^2)z; q^2) = \frac{q^{-\nu^2+1+\frac{1}{2}} \Gamma_q(\nu+\frac{1}{2}) \Gamma_q(\frac{1}{2})}{4Q^{\nu+\frac{1}{2}}} \left( \frac{z}{2} \right)^{-\nu} \times \]

\[ \times \int_{-\infty}^{\infty} \Phi_1 \left( \frac{1}{2}; 0, q, i \frac{1-q^2}{2} z s \right) \frac{(-q^{\nu+1}s^2, q^2)_\infty}{(-s^2, q^2)_\infty} d_q s, \quad (2.19) \]

where

\[ Q_\nu = (1-q) \sum_{m=-\infty}^{\infty} q^{m+\nu-\frac{1}{2}} + q^{-m+\nu+\frac{1}{2}}. \quad (2.20) \]

### 3 The q-Macdonald functions of kind 3

**Definition 3.1** Starting from \((2.18), (2.19)\) we define the q-Macdonald function of kind 3 by its q-integral representation for \( \nu > \frac{3}{2} \).

\[ K^{(3)}_\nu((1-q^2)z; q^2) = C(\nu) \frac{q^{-\nu^2+1+\frac{1}{2}} \Gamma_q(\nu+\frac{1}{2}) \Gamma_q(\frac{1}{2})}{4Q^{\nu+\frac{1}{2}}} \left( \frac{z}{2} \right)^{-\nu} \times \]

\[ \times \int_{-\infty}^{\infty} \Phi_2 \left( \frac{1}{2}; 0, -q^2 s^2, q, -1, -i \frac{1-q^2}{2} z s \right) \frac{(-q^{\nu+\frac{3}{2}} s^2, q^2)_\infty}{(-q^{-\nu+\frac{3}{2}} s^2, q^2)_\infty} d_q s. \quad (3.1) \]

Moreover

\[ \lim_{m \to \infty} K^{(3)}_\nu((1-q^2)z; q^2) \big|_{\frac{1-q^2}{2} z = q^m} = 0. \quad (3.2) \]

Multiplying both sides of \((3.1)\) by \((\frac{z}{2})^\nu\) and setting \( z = 0 \), we obtain

\[ \left( \frac{z}{2} \right)^\nu K^{(3)}_\nu((1-q^2)z; q^2) \big|_{z=0} = C(\nu) \frac{q^{-\nu^2+1+\frac{1}{2}} \Gamma_q(\nu+\frac{1}{2}) \Gamma_q(\frac{1}{2})}{4Q^{\nu+\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{(-q^{\nu+\frac{3}{2}} s^2, q^2)_\infty}{(-q^{-\nu+\frac{3}{2}} s^2, q^2)_\infty} d_q s. \]

Calculate the last q-integral using the properties of the q-binomial formula \([8]\).

\[ \text{Int} = \int_{-\infty}^{\infty} \frac{(-q^{\nu+\frac{3}{2}} s^2, q^2)_\infty}{(-q^{-\nu+\frac{3}{2}} s^2, q^2)_\infty} d_q s = 2(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(-q^{\nu+2m+\frac{3}{2}}, q^2)_\infty}{(-q^{-\nu+2m+\frac{3}{2}}, q^2)_\infty} = \]

\[ = 2(1-q) \sum_{m=-\infty}^{\infty} q^m \frac{(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-2\nu+1}, q^2)_k q^{(2\nu+1)k}}{(q^2, q^2)_k (1+q^{-\nu+2m+2k})}. \]

The inner series converges uniformly with respect to \( m \), and we can change the order of summing. Then we have using \((2.20)\)

\[ \text{Int} = \frac{2(1-q)(q^{2\nu+1}, q^2)_\infty}{(q^2, q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-2\nu+1}, q^2)_k q^{2nk}}{(q^2, q^2)_k} \sum_{m=-\infty}^{\infty} q^{m+k} = \]

\[ = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} q^{m+k} = \]
It follows from (3.3) and (3.5)

C

if

\[ \lim_{\nu \rightarrow -\infty} \quad (1-q^2)z; q^2) = 0. \]  

(3.3)

Obviously the last sum tends to unit if

\[ \lim_{\nu \rightarrow -\infty} \quad (1-q^2)z; q^2) = 0. \]  

(3.3)

Let \( \nu \neq n \). As \( K_{\nu}^{(3)}((1-q^2)z; q^2) \) is the solution of (2.8) for \( \delta = 1 \), \( I_{\nu}^{(3)}((1-q^2)z; q^2) \) and 

\[ I_{\nu}^{(3)}((1-q^2)z; q^2) \]

form the fundamental system of solutions of this equation, and 

\[ K_{\nu}^{(3)}((1-q^2)z; q^2) \]

we can write

\[ K_{\nu}^{(3)}((1-q^2)z; q^2) = \frac{1}{2} q^{-\nu^2 + \nu} \Gamma(q^2) \Gamma(q^2) (1-n) \]  

(3.4)

It follows from this equality

\[ \left( \frac{z}{2} \right)^{\nu} K_{\nu}^{(3)}((1-q^2)z; q^2) \mid_{z=0} = \frac{1}{2} q^{-\nu^2 + \nu} \Gamma(q^2) \Gamma(q^2) A_{\nu}. \]  

(3.5)

It follows from (1.3) and (3.3) \( C(\nu) = A_{\nu}. \)

Consider the restriction of function \( I_{\nu}^{(3)}((1-q^2)z; q^2) \) (2.7) on the lattice \( \{q^n\} \). e.i. assume

\[ \frac{1-q^2}{z} = q^n. \]

Then

\[ I_{\nu}^{(3)}((1-q^2)z; q^2) \mid_{1-q^2 = q^n} = I_{\nu}^{(3)}(q^n; q^2) = \frac{1}{(q^2, q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k(\nu+k)+n(\nu+2k)}(q^{2\nu+2k+2}, q^2)_{\infty}}{(q^2, q^2)_{k}} = \]

\[ = \frac{q^{\nu n}}{(q^2, q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)} q^{2m v} (q^{2\nu+2m+2}, q^2)_{\infty}}{(q^2, q^2)_{m}} = \]

\[ = \frac{q^{\nu n}}{(q^2, q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)} q^{2m v} (q^{2\nu+2m+2}, q^2)_{\infty}}{(q^2, q^2)_{m}} \]

\[ = \frac{q^{\nu n}}{(q^2, q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)} q^{2m v} (q^{2\nu+2m+2}, q^2)_{\infty}}{(q^2, q^2)_{m}}. \]

Obviously the last sum tends to unit if \( n \rightarrow -\infty. \)

Now consider the limit of quotient

\[ \frac{I_{\nu}^{(3)}((1-q^2)z; q^2)}{I_{\nu}^{(3)}((1-q^2)z; q^2)} \mid_{1-q^2 = q^n} \]

if \( n \rightarrow -\infty. \)

\[ \lim_{n \rightarrow -\infty} \frac{I_{\nu}^{(3)}(q^n; q^2)}{I_{\nu}^{(3)}(q^n; q^2)} = \frac{q^{2n v}(-q^{\nu^2+2n}, q^2)_{\infty}}{(-q^{\nu^2+2n}, q^2)_{\infty}} = \lim_{n \rightarrow -\infty} \frac{(-q^{\nu^2+2n}, q^2)_{n}}{(-q^{\nu^2+2n}, q^2)_{\infty}} = 1. \]
In order to (3.2) is fulfilled it is necessary $A_{-\nu} = A_{\nu}$. So $C(\nu) = A_{-\nu} = \text{constant}$. Assume this constant equal to unit. Then finally, if $\nu \neq n$

$$K_\nu^{(3)} \left( (1 - q^2) z; q^2 \right) = \frac{1}{2} q^{-\nu^2 + \nu} \Gamma_q(\nu) \Gamma_q(1 - \nu) \left[ I_\nu^{(3)} \left( (1 - q^2) z; q^2 \right) - I_{-\nu}^{(3)} \left( (1 - q^2) z; q^2 \right) \right]. \quad (3.6)$$

This definition must be extended to integral values of $\nu = n$ by passing to the limit in (3.4).

It is easily to verify a correctness of the following propositions.

**Proposition 3.1** The function $K_\nu^{(3)} \left( (1 - q^2) z; q^2 \right)$ satisfies (2.14), (2.15).

**Proposition 3.2** The function $K_\nu^{(3)} \left( (1 - q^2) z; q^2 \right)$ satisfies functional relations (2.16), (2.17).

**Proposition 3.3** For any $\nu$ $I_\nu^{(3)} \left( (1 - q^2) z; q^2 \right)$ and $K_\nu^{(3)} \left( (1 - q^2) z; q^2 \right)$ form the fundamental system of the solutions of equation (2.8) for $\delta = 1$, and

$$W(I_\nu^{(3)}, K_\nu^{(3)}) = \frac{1}{2} q^{-\nu^2} (1 - q^2).$$

**Remark 3.1**

$$\lim_{q \to 1-0} K_{\nu}^{(j)} \left( (1 - q^2) z; q^2 \right) = K_{\nu}(z).$$

It easily to show that $\lim_{q \to 1} Q_{\nu} = \frac{\pi}{2}$ for (2.20).

**Remark 3.2** If $q \to 1-0$ we have well-known integral representation for the classical Macdonald functions

$$K_\nu(z) = \frac{\Gamma(\nu + \frac{1}{2})}{2 \Gamma(\frac{1}{2})} \left( \frac{z}{2} \right)^{-\nu} \int_{-\infty}^{\infty} (z^2 + 1)^{-\nu - \frac{1}{2}} e^{i z s} ds.$$

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