SOME NIP-LIKE PHENOMENA IN NTP\textsubscript{2}

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Abstract. We introduce the notion of an NTP\textsubscript{2}-smooth measure and prove that they exist assuming NTP\textsubscript{2}. Using this, we propose a notion of distality in NTP\textsubscript{2} that unfortunately does not intersect simple theories trivially. We then prove a finite alternation theorem for a subclass of NTP\textsubscript{2} that contains resilient theories. In the last section we prove that under NIP, any type over a model of singular size is finitely satisfiable in a smaller model, and ask if a parallel result (with non-forking replacing finite satisfiability) holds in NTP\textsubscript{2}.

1. Introduction

In recent years, a lot of attention has been given to unstable classes of first order theories. In particular, NIP and to a lesser extent NTP\textsubscript{2}. The former, NIP, is a very important class of theories which was studied extensively, see [Sim15]. The latter, NTP\textsubscript{2}, is a class of theories which contains both simple and NIP theories. In recent years many examples of NTP\textsubscript{2} theories were discovered. For example, the ultraproduct of the \(p\)-adics [Che14], bounded PRC fields [Mon17] and valued fields with a generic automorphism [CH13]. Though it is a very large class of theories, some general nontrivial results were nonetheless attained. For example, in [CK12] it is proved that forking and dividing agree over models, and [BYC14] contains an independence theorem for NTP\textsubscript{2}. Under the assumption of groups or fields, more has been done. See for example [CKSI15] (about groups and fields in general NTP\textsubscript{2}), [HO17] (about definable envelopes of subgroups), and more recently [MOSI16] (about groups definable in bounded PRC fields).

Roughly speaking the ideology guiding our results on NTP\textsubscript{2} is that it is NIP up to non-forking. We exhibit this in two instances.

1) In Section 3 we introduce the notion of an NTP\textsubscript{2}-smooth Keisler measure. For any theory \(T\) and \(M \models T\), a Keisler measure \(\mu\) over \(M\) is smooth if for every \(N \succ M\), there is a unique extension of \(\mu\) to \(\mathcal{M}_x(N)\). This notion turned out to be very important in the study of measures in NIP theories (see [Sim15] Section 7.3) so it is natural to find a parallel notion for NTP\textsubscript{2}. As per our guiding ideology, we say that \(\mu \in \mathcal{M}_x(M)\) is NTP\textsubscript{2}-smooth if for every extension \(\mu' \in \mathcal{M}_x(N)\) of \(\mu\), if \(\varphi(x,a)\) forks over \(M\) then \(\mu'(\varphi(x,a)) = 0\). We then prove that such measures exist: every

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Keisler measure over a model in an NTP$_2$ theory can be extended to an NTP$_2$-smooth one. In the last part of this section, Subsection 3.2, we try to define a suitable notion of NTP$_2$-distality, and provide two equivalent definitions (one of them using NTP$_2$-smooth measures). However, it intersects simple theories, and thus this definition will probably have to be refined.

2) In Section 4 we prove a finite alternation result. Recall that in NIP, just by definition, there is no indiscernible sequence $\langle a_i | i < \omega \rangle$, a formula $\varphi(x, y)$ and $b$ such that $\varphi(b, a_i)$ holds iff $i$ is even. We introduce a subclass of NTP$_2$, which we call $\omega$-resilient theories (and contain resilient theories, for which it is unknown whether it is a proper subclass of NTP$_2$), and prove that if $\langle a_i | i < \omega \rangle$ is an indiscernible sequence and $\varphi(x, b)$ divides over $I = \langle a_{2i} | i < \omega \rangle$, then for all but finitely many $i$'s, $\neg \varphi(a_i, b)$ holds. Note that this holds if $T$ is simple (see just below Theorem 4.4).

Finally, in Section 5 we move to NIP theories, and prove that a kind of local character result holds there, which we call “singular local character”. Namely, if $p \in S(M)$ and $|M|$ is singular with cofinality greater than $|T|$, then $p$ is finitely satisfiable over $N \prec M$ of smaller cardinality. In particular, $p$ does not fork over $N$. Since the last statement is trivially true for simple theories, it is natural to ask whether this is true for NTP$_2$.

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2. Preliminaries

We recall the basic definitions of NIP and NTP$_2$. For a thorough discussion of NIP and its importance, we refer the reader to [Sim15]. The class NTP$_2$ is also discussed there, but we also add [Che14].

Definition 2.1. A complete theory $T$ is NIP if there is no formula $\varphi(x, y)$ with the independence property (IP), where $\varphi$ has IP if, in some $M \models T$ there are $\langle a_i | i < \omega \rangle$ and $\langle b_s | s \subseteq \omega \rangle$ such that $M \models \varphi(a_i, b_s)$ iff $i \in s$.

Definition 2.2. A formula $\varphi(x, y)$ has the tree property of the second kind (TP$_2$) if there is an array $\langle a_{i,j} | i, j < \omega \rangle$ and some $k < \omega$ such that every vertical path is consistent (for every $\eta : \omega \rightarrow \omega$, $\{ \varphi(x, a_{i,\eta(i)}) | i < \omega \}$ is consistent) and every row is $k$-inconsistent ($\{ \varphi(x, a_{i,j}) | j < \omega \}$ is $k$-inconsistent). A complete theory $T$ is NTP$_2$ if no formula has TP$_2$.

Our notations are standard, e.g., $T$ will denote some complete first-order theory and $C$ will be its monster model.

3. On NTP$_2$-smooth measures and a possible definition for NTP$_2$-distal theories

Recall that an additive probability measure on a Boolean algebra $B$ is a function $\mu : B \rightarrow [0, 1]$ such that $\mu(1) = 1$, $\mu(x^c) = 1 - \mu(x)$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \land y = 0$. 

Definition 3.1. Suppose that $A$ is a set of parameters in some model $M$. A Keisler measure (or just a measure) over $A$ in the variable $x$ is a finitely additive probability measure on $L_x(A)$: the Boolean algebra of definable sets in $x$ over $A$. We denote the space of measures in $x$ over $A$ by $\mathfrak{M}_x(A)$.

Fact 3.2. \cite{Sim15} Lemma 7.3] Let $\Omega \subseteq L_x(A)$ be a sub-algebra of $L_x(A)$ (i.e., closed under intersection, union and complement, and contains $x = x$). Let $\mu$ be a finitely additive probability measure on $\Omega$. Then $\mu$ extends to a Keisler measure over $A$.

3.1. NTP$_2$-smooth measures. In this section we will define an analog notion to smooth measures from NIP in the NTP$_2$-context. The main result is that every measure can be extended to an NTP$_2$-smooth measure, assuming NTP$_2$.

Remark 3.3. Recall that if $T$ is NIP and $M \models T$ then $\mu \in \mathfrak{M}_x(M)$ is smooth if for every $N \succ M$, there is a unique extension of $\mu$ to $\mathfrak{M}_x(N)$. If $\mu \in \mathfrak{M}_x(N)$ and $M \prec N$ then $\mu$ is smooth over $M$ if the restriction $\mu|_M$ is smooth. We can also extend this definition to any set of parameters, working in $\mathfrak{C}$: $\mu \in \mathfrak{M}_x(A)$ is smooth if it has a unique extension to $\mathfrak{M}_x(\mathfrak{C})$.

For a global measure $\mu \in \mathfrak{M}_x(\mathfrak{C})$, $\mu$ is called $A$-invariant for some set $A$ if whenever $b \equiv_A c$ we have $\mu(\varphi(x,b)) = \mu(\varphi(x,c))$. In NIP, by \cite{Sim15} Proposition 7.15], $\mu \in \mathfrak{M}_x(\mathfrak{C})$ is invariant over a model $M$ iff for every formula $\varphi(x,c)$ which forks (or divides) over $M$, $\mu(\varphi(x,c)) = 0$ (in this case we say that $\mu$ does not fork over $M$).

Definition 3.4. A Keisler measure $\mu \in \mathfrak{M}_x(A)$ is called NTP$_2$-smooth if for every $A \subseteq N$ and any extension of $\mu$ to $\mu' \in \mathfrak{M}_x(N)$, if $\varphi(x,b)$ divides over $A$ then $\mu'(\varphi(x,b)) = 0$. When $\mu \in \mathfrak{M}_x(N)$ and $A \subseteq N$ We say that $\mu$ is NTP$_2$-smooth over $A$ if $\mu|_A$ is smooth.

Remark 3.5. A measure $\mu \in \mathfrak{M}_x(A)$ is NTP$_2$-smooth iff for every $A \subseteq N$ and any extension of $\mu$ to $\mu' \in \mathfrak{M}_x(N)$, if $\varphi(x,b) \in L_x(N)$ forks over $A$ then $\mu'(\varphi(x,b)) = 0$. To see this, note that if $\varphi(x,b)$ forks over $A$, then we can extend the measure $\mu'$ to include in its domain the dividing formulas that $\varphi(x,b)$ implies, so all of these must have measure zero.

Fact 3.6. \cite{Sim13} Lemma 7.5] If $M$ is a model, $\langle b_i \mid i < \omega \rangle$ is an indiscernible sequence of tuples in $M$ and $\mu \in \mathfrak{M}_x(M)$ is such that $\mu(\varphi(x,b_i)) \geq \varepsilon > 0$ for all $i$, then $\{\varphi(x,b_i) \mid i < \omega\}$ is consistent.

Remark 3.7. If $T$ is NIP and $M \models T$, then $\mu \in \mathfrak{M}_x(M)$ is smooth iff it is NTP$_2$-smooth.

Indeed, suppose that $\mu$ is smooth, and $\mu'$ extends $\mu$ to $\mathfrak{M}_x(N)$, and $\mu(\varphi(x,b)) > 0$ with $\varphi(x,b)$ dividing over $M$. Assuming that $N$ is $|M|^+$-saturated, it contains an indiscernible sequence $\langle b_i \mid i < \omega \rangle$ over $M$ which witnesses dividing. As $\mu$ is smooth, it follows that $\mu(\varphi(x,b_i)) = \mu(\varphi(x,b))$ for all $i$. Together we get a contradiction to Fact 3.6.
On the other hand, if \( \mu \) is NTP\(_2\)-smooth then it is smooth: first, by [Sim15] Proposition 7.9, we know that \( \mu \) can be extended to a smooth measure in \( \mathcal{M}_x (N) \) for some \( N \succ M \). Extended it even further to \( \mu' \in \mathcal{M}_x (\mathcal{C}) \). By definition \( \mu' \) is NTP\(_2\)-smooth over \( M \) and by NIP (and the second statement of Remark 3.3) \( \mu' \) is \( M \)-invariant. Now, [Sim15] Lemma 7.17 tells us that that \( \mu' \) is smooth over \( M \) so we are done.

As was said in the proof of Remark 3.7 in NIP, every measure can be extended to a smooth one. The analogous statement in NTP\(_2\) is then the following.

**Theorem 3.8.** Suppose that \( T \) is NTP\(_2\) and \( M \models T \). Any Keisler measure \( \mu \in \mathcal{M}_x (M) \) can be extended to an NTP\(_2\)-smooth measure over some \( N \succ M \).

**Proof.** Suppose not. Construct an increasing continuous sequence of measures and models

\[
\left\langle (M_\alpha, \mu_\alpha) \right| \alpha < (|T| + 2^{|\aleph_0|})^+\right.
\]

such that \( M_0 = M, \mu_0 = \mu \) and for every \( \alpha < (|T| + 2^{|\aleph_0|})^+ \), there are \( \varphi_\alpha (x, y_\alpha) \in L \) and \( b_\alpha \in M_{\alpha+1} \) such that \( \varphi_\alpha (x, b_\alpha) \) divides over \( M_\alpha \) and \( \mu_{\alpha+1} (\varphi_\alpha (x, b_\alpha)) = \varepsilon_\alpha > 0 \). Also, for each \( \alpha \) there is some \( \mathcal{M}_\alpha \)-indiscernible sequence \( b_\alpha = \langle b_{\alpha, i} \mid i \in \mathbb{Z} \rangle \subseteq M_{\alpha+1} \) such that \( b_{\alpha, 0} = b_\alpha \) and \( \{ \varphi (x, b_{\alpha, i}) \mid i \in \mathbb{Z} \} \) is \( k_\alpha \)-inconsistent. Also, we ask that:

\[
(\ast) \text{ For each formula } \psi (x, y_\alpha) \text{ over } M_\alpha, \text{ and every } i, j \in \mathbb{Z}, \text{ we have that } \mu_{\alpha+1} (\psi (x, b_{\alpha, i})) > 0 \text{ iff } \mu_{\alpha+1} (\psi (x, b_{\alpha, j})) > 0. \text{ Moreover, if } \mu_{\alpha+1} (\psi (x, b_{\alpha, i})) > \varepsilon \text{ then } \mu_{\alpha+1} (\psi (x, b_{\alpha, j})) > \varepsilon \cdot 2^{-|i-j|}.
\]

How? By our assumption toward contradiction, in stage \( \alpha < (|T| + 2^{|\aleph_0|})^+ \) in the construction we can find \( M'_{\alpha+1}, \varphi_\alpha, b_\alpha, \langle b_{\alpha, i} \mid i \in \mathbb{Z} \rangle, \varepsilon_\alpha \) and some \( \mu'_{\alpha+1} \) which satisfy everything except \((\ast)\). To get \((\ast)\), we let \( \sigma \in \text{Aut} (\mathcal{C}/M_\alpha) \) take \( \langle b_{\alpha, i} \mid i \in \mathbb{Z} \rangle \) to \( \langle b_{\alpha, i+1} \mid i \in \mathbb{Z} \rangle \), and extend \( M'_{\alpha+1} \) to \( M_{\alpha+1} \supset M'_{\alpha+1} \) such that \( \sigma \upharpoonright M_{\alpha+1} \in \text{Aut} (M_{\alpha+1}/M_\alpha) \). Now let \( \mu_{\alpha+1} = \sum \{ 2^{-|i|} \cdot 2^{|\sigma (\mu_{\alpha+1})|} (\mu'_{\alpha+1} \mid i \in \mathbb{Z} \setminus \{0\}) + \frac{1}{2}\mu'_{\alpha+1} \rangle \text{ Note that } \mu_{\alpha+1} \in \mathcal{M}_x (M_{\alpha+1}) \text{ and it extends } \mu_\alpha \text{. Let us check that } (\ast) \text{ holds. Suppose that } \mu_{\alpha+1} (\psi (x, b_{\alpha, i})) > \varepsilon. \text{ Then without loss of generality } \sum_{k=0}^{\infty} 2^{-k-2} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+i})) > \varepsilon \text{ (this is the “positive side” of this sum). If } j < i \text{ then the positive side of the sum which calculates } \mu_{\alpha+1} (\psi (x, b_{\alpha, j})) \text{ is } \geq \sum_{k=i-j}^{\infty} 2^{-k-2} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+j})) > 2^{i-j-1} \varepsilon. \text{ If } j > i \text{, then as } \mu_{\alpha+1} (\psi (x, b_{\alpha, j})) \geq \sum_{k=i-j}^{\infty} 2^{-k-2} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+i})) \text{, we get that it is } \geq 2^{i-j} \sum_{k=0}^{\infty} 2^{-k-1} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+i})) + 2^{j-i} \sum_{k=0}^{\infty} 2^{-k-2} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+1})) \text{ which is } \geq 2^{i-j} \sum_{k=0}^{\infty} 2^{-k-2} \mu'_{\alpha+1} (\psi (x, b_{\alpha, k+i})) > 2^{i-j-1} \varepsilon. \text{ Now extract a sequence } \langle (M_i, \mu_i, b_i) \mid i < \omega \rangle \text{ such that for some fixed formula } \varphi (x, y), \varphi (x, b_{i,0}) \text{ divides over } M_i \text{ and even } k-\text{divides for a fixed } k \text{ as witnessed by } b_i, \text{ and } \mu_{i+1} (\varphi (x, b_{i,0})) > \varepsilon_\ast \text{ for some fixed } \varepsilon_\ast. \text{ Let } \mu^* = \bigcup_{i < \omega} \mu_i. \]
Next we extract a sequence $\langle \bar{b}'_i \mid i < \omega \rangle$ where $\bar{b}'_i = \langle b'_{i,j} \mid j \in \mathbb{Z} \rangle$ such that $\langle \bar{b}'_i \mid i < \omega \rangle$ is indiscernible with respect to $\mu^*$ with the same $\mu^*$-EM-type as $\langle \bar{b}_i \mid i < \omega \rangle$. Indiscernible with respect to $\mu^*$ means this sequence is indiscernible and for all $i_0 < \ldots < i_{n-1}$, $\mu^*(\psi(x, \bar{b}'_{i_1}, \ldots, \bar{b}'_{i_n})) = \mu^*(\psi(x, \bar{b}_{i_1}, \ldots, \bar{b}_{i_n-1}))$. Having the same $\mu^*$-EM-type means having the same EM-type and moreover, if $\mu^*(\psi(x, \bar{b}_{i_0}, \ldots, \bar{b}_{i_{n-1}})) = \varepsilon$, then for every $\delta > 0$ there is an increasing tuple $i_0 < \ldots < i_{n-1} < \omega$ with $|\mu^*(\psi(x, \bar{b}_{i_0}, \ldots, \bar{b}_{i_{n-1}})) - \varepsilon| < \delta$.

Getting such a sequence is standard using Ramsey and compactness, see e.g., [Sim15, Proof of Lemma 7.5].

We forgot $M_i$, but we still retain that $\{\varphi(x, b'_{i,j}) \mid j \in \mathbb{Z}\}$ is inconsistent (because we fixed $k$) and (*) still holds (for $\varphi(x, y)$ defined over $b'_{<1}$), by the “moreover” part. Rename $b'_{i,j}$ to $b_{i,j}$.

Suppose that there was some $K < \omega$ with $\mu^*(\bigwedge_{i < K} \varphi(x, b_{i,i})) > 0$ but $\mu^*(\bigwedge_{i < K+1} \varphi(x, b_{i,i})) = 0$. Then by (2), $\mu^*(\bigwedge_{i < K} \varphi(x, b_{i,i}) \land \varphi(x, b_{K,0})) = 0$, so letting $\bar{c}_i = \langle b_{i,i} \mid i < K \rangle$ (for $l < \omega$) and $\psi(x, \bar{y}) = \bigwedge_{i < K} \varphi(x, b_{i,i})$ we get that $|\mu^*(\psi(x, \bar{c}_{K}))| < \omega < l$ is constant and positive, while $\mu^*(\psi(x, \bar{c}_{K}) \land \psi(x, \bar{c}_{(l+1)K})) = 0$ for all $l < \omega$, but the total measure is 1 so this is impossible.

By NTP$_2$ and compactness there is some $N < \omega$ such that there is no array $\langle a_{i,j} \mid i < N, j < N \rangle$ such that for every $i < N$, $\{\varphi(x, a_{i,j}) \mid j < N\}$ is $k$-inconsistent and for every $\eta : N \to N$, $\{\varphi(x, a_{i,\eta(i)}) \mid i < N\}$ is consistent.

Suppose that the measure of the diagonal $\mu^*(\bigwedge_{i < N} \varphi(x, b_{i,i}))$ is positive.

In this case, by $\mu^*$-indiscernibility of $\langle \bar{b}_i \mid i < \omega \rangle$ and Fact 3.6 it follows that the set of all $N$-diagonals $\{\bigwedge_{i < N} \varphi(x, b_{kN+i,i}) \mid k < \omega\}$ is consistent. In particular, for any $\eta : N \to N$, $\{\varphi(x, b_{kN+i,\eta(i)}) \mid k < N\}$ is consistent. Hence, by indiscernibility of $\langle \bar{b}_i \mid i < \omega \rangle$ it follows that $\{\varphi(x, b_{i,\eta(i)}) \mid i < N\}$ is consistent.

This is a contradiction, hence $\mu^*(\bigwedge_{i < N} \varphi(x, b_{i,i})) = 0$. But $\mu^*(\varphi(x, b_{0,0})) > \varepsilon$, so we can find some $K$ as above — contradiction.

\hfill $\square$

### 3.2. On a possible definition of NTP$_2$-distal theories

Distal theories form an important class of NIP theories. Defined and studied in [Sim13], they were studied further in [CS15] where some surprising combinatorial results were discovered. Distal theories were given a “set-theoretic” characterization in terms of the existence of saturated models in [KSS17].

We would like to suggest a possible definition of NTP$_2$-distal. In the context of NIP, several equivalent definitions of distality can be given. We will use the one which relates it to smooth measures. We will see in the end that our proposed definition lacks an important property of distal theories: in the NIP context, distal theories can never be stable. Here we would like to have that NTP$_2$-distal theories are never simple. This is not the case in our definition, which raises the question of possible refinements. We will not deal with this here.

First let us give the more familiar definition.
**Definition 3.9.** A theory $T$ is *distal* if whenever $I_1 + a + I_2$ is indiscernible, $I_1$, $I_2$ have no endpoints and $I_1 + I_2$ is indiscernible over $A$, $I_1 + a + I_2$ is indiscernible over $A$.

Note that if $T$ is distal then $T$ is NIP (if not, then we can find a formula $\varphi(x, y)$, an indiscernible sequence $\langle a_i \mid i \in \mathbb{Z} \rangle$ and $b$ such that $\varphi(a_i, b)$ holds iff $i$ is even. Extracting, we may assume that the sequence of pairs $\langle (a_{2i}, a_{2i+1}) \mid i \in \mathbb{Z} \rangle$ is indiscernible over $b$ and in particular $\langle a_{2i} \mid i \in \mathbb{Z} \rangle$ is indiscernible over $b$, $\langle a_i \mid i \in 2\mathbb{Z} \cup \{1\} \rangle$ is indiscernible, but not over $b$).

Given an indiscernible sequence $I = \langle a_i \mid i \in [0, 1] \rangle$ where $a_i$ has the same length as the variable $x$, let $\Omega \subseteq L_x(\mathcal{C})$ be the family of all definable sets $D \subseteq \mathcal{C}$ such that $D_I = \{ i \in [0, 1] \mid a_i \in D \}$ is a Borel measurable set. Note that $\Omega$ is a Boolean algebra (i.e., closed under intersections, unions and complements, and contains $x = x$). Let $\lambda$ be the Lebesgue measure on $[0, 1]$. Then $\lambda$ induces a probability measure $Av_I$ on $\Omega$ by setting $Av_I(D) = \lambda(D_I)$. Note that if $D$ is definable over $I$, then $D_I$ is a finite union of intervals, so Borel, hence $Av_I$ defines a Keisler measure on $L_x(I)$, which we will naturally denote by $Av_I|_I$ (the restriction of $Av_I$ to $I$).

In NIP, every $D_I$ is a finite union of intervals, so that $Av_I$ is a global Keisler measure.

**Fact 3.10.** [Sim13] Proposition 2.21|A complete theory $T$ is distal iff it is NIP and for all indiscernible sequences $I = \langle a_i \mid i \in [0, 1] \rangle$ and any model $M$, $Av_I|_M$ is smooth.

In fact, reading the proof in [Sim13] Proposition 2.21], we get that $T$ is distal iff it is NIP and for all such $I$'s, $Av_I|_I$ is smooth. Thus we propose the following definition.

**Definition 3.11.** Say that a theory $T$ is *NTP$_2$-distal* if it is NTP$_2$ for every indiscernible sequence $I = \langle a_i \mid i \in [0, 1] \rangle$, $Av_I|_I$ is NTP$_2$-smooth.

**Question 3.12.** Do we need to assume NTP$_2$ in Definition 3.11?

As with distal theories, we would like to have an equivalent definition that does not use measures.

For this we will use the following fact on extensions of measures.

**Fact 3.13.** [Sim15] Lemma 7.4| If $A$ is any set, $\mu \in \mathfrak{M}_x(A)$ and $\varphi(x, b)$ is some formula over $\mathcal{C}$, then for every $r \in [0, 1]$ such that

$$\inf \{ \mu(\theta) \mid \theta \in L_x(A), \varphi(x, b) \vdash \varphi(x, b) \} \leq r \leq \sup \{ \mu(\theta) \mid \theta \in L_x(A), \varphi(x, b) \vdash \theta \}$$

there is an extension $\nu$ of $\mu$ to $\mathfrak{M}_x(\mathcal{C})$ such that $\nu(\varphi(x, b)) = r$.

Actually, the lemma in [Sim15] Lemma 7.4] is stated only when $A$ is a model, but the same proof goes through.

**Theorem 3.14.** An NTP$_2$ theory $T$ is NTP$_2$-distal iff for every dense indiscernible sequence $I$ which we write as $I_1 + I_2$, with $I_1$ endless and $I_2$ with no first element, if $I$ is $d$-indiscernible, $I_1 + b + I_2$ is indiscernible and $\varphi(x, d)$ divides over $I$ then $\neg \varphi(b, d)$ holds.
Proof. Suppose first that \( T \) is NTP\(_2\)-distal and let \( I = I_1 + I_2 \), \( b \) and \( \varphi(x,d) \) as in there and assume that \( \varphi(b,d) \) holds. Let \( k < \omega \) witness that \( \varphi(x,d) \) divides over \( I \). We may assume that \( I \) is countable (by taking a subsequence) and ordered by \((0,1) \cap \mathbb{Q} \) where \( b \) is in some irrational spot \( \alpha \in (0,1) \). By compactness, we can extend \( I \) to have order type \([0,1] \setminus \{\alpha\}\) where \( I_1 = I_{<\alpha} \) and \( I_2 = I_{>\alpha} \) (note that the conditions involved, that \( I \) is \( d \)-indiscernible, that \( \varphi(x,d) \) \( k \)-divides over \( I \) and that \( I_1 + b + I_2 \) is indiscernible are type definable). Write \( I = \{a_i \mid i \in [0,1] \setminus \{\alpha\}\} \).

Then by assumption, \( \text{Av}_{\pi'}|_{\nu'} \) is NTP\(_2\)-smooth, which means that in every extension \( \mu \) of \( \text{Av}_{\pi'}|_{\nu'} \), \( \mu(\varphi(x,d)) = 0 \). By Fact \([5.13]\) for every \( 0 < \varepsilon \) there is some formula \( \theta_{\varepsilon}(x) \in L_x(I') \) such that \( \varphi(x,d) \vdash \theta_{\varepsilon} \) and \( \text{Av}_{\pi'}(\theta_{\varepsilon}) < \varepsilon \). Take \( \varepsilon \) small enough so that \( 0 < \alpha - \varepsilon < \alpha + \varepsilon < 1 \). The formula \( \theta_{\varepsilon} \) is over \( I' \) so it defines a union of intervals in \( I' \), defined by the parameters defining \( \theta_{\varepsilon} \). By moving those parameters, we can ensure that \( \theta_{\varepsilon} \) is over \( I_{<\alpha-\varepsilon} + I_{>\alpha+\varepsilon} \), and since \( I' \) is indiscernible over \( d \), we did not lose anything. Since \( \varphi(b,d) \) holds, \( \theta_{\varepsilon}(b) \) holds, and as \( I_1 + b + I_2 \) is indiscernible, \( \theta_{\varepsilon}(a_i) \) holds for all \( i \in ([\alpha-\varepsilon, \alpha+\varepsilon]) \). Hence \( \text{Av}_{\pi'}(\theta_{\varepsilon}) \geq 2\varepsilon \), contradiction.

On the other hand, suppose that that the condition on the right hand side hold and we want to show that \( T \) is NTP\(_2\)-distal. Let \( I = \{a_i \mid i \in [0,1]\} \) be indiscernible. We want to show that \( \mu = \text{Av}_I|_{\nu} \) is NTP\(_2\)-smooth. If \( I \) is constant then \( \mu \) has a unique extension (the unique realized type of the element in \( I \)), so we may assume it is not. The proof is similar to what is done in the NIP case \([Sim13]\) Proposition 2.21], and like there it relies on several steps.

Let \( \text{supp}(\mu) = \{p \in S(I) \mid \varphi \in p \Rightarrow \mu(\varphi) > 0\} \), in other words, the set of all weakly random types over \( I \).

**Claim 3.15.** If \( p \in \text{supp}(\mu) \) then \( p = \lim^+(\alpha) \) or \( \lim^-(\alpha) \) for some \( \alpha \in [0,1] \). That is, either [there is some \( 0 \leq \alpha < 1 \) such that for every formula \( \varphi(x) \) over \( I \), \( \varphi \in p \) iff for some \( \alpha < \beta \), for all \( \alpha < \gamma < \beta \), \( \varphi(a_{\gamma}) \) holds] or [there is \( 0 < \alpha \leq 1 \) such that for every formula \( \varphi(x) \) over \( I \), \( \varphi \in p \) iff for some \( \beta < \alpha \), for all \( \beta < \gamma < \alpha \), \( \varphi(a_{\gamma}) \) holds].

**Proof of Claim.** Note that for every formula \( \varphi(x) \) over \( I \), \( \varphi(I) \) is a union of intervals. Let \( r \in \bigcap_{\varphi \in p} \text{cl} \{\alpha \in [0,1] \mid \varphi(a_{\alpha})\} \), which exists by compactness of \([0,1]\). For every formula \( \varphi \in p \), either \( r \) is an isolated point of \( \varphi(I) \) or \( \varphi(I) \) contains an open interval to the left or right of \( r \). Since \( p \) is closed under finite intersection and contains \( x \neq a_r \), we can assume that for \( \varphi \in p \), \( \varphi \) contains an interval to, say, the right of \( r \), and as \( p \) is complete, \( p = \lim^+(r) \). \( \square \)

Suppose for contradiction that \( \mu \) is not NTP\(_2\)-smooth. Then there is a formula \( \varphi(x,d) \) which divides over \( I \) and some extension of \( \mu \) which gives it positive measure.

Let \( \Sigma = \{\theta \in L_x(I) \mid \varphi(x,d) \vdash \theta\} \). By Fact \([5.13]\) \( \inf \{\mu(\theta) \mid \theta \in \Sigma\} \) is positive. This means that we can find \( p \in \text{supp}(\mu) \) such that \( p \) contains \( \Sigma \). According to \([Sim15]\) Beginning of Section 7.1], \( \mu \) can be thought of as a \( \sigma \)-additive Borel measure on the space of types \( S_x(I) \) such that for every
closed set $F$, $\mu(F) = \inf \{D \mid F \subseteq D, D \text{ clopen}\}$. In particular, $\mu(\Sigma)$ is positive (where we identify $\Sigma$ with the set of types containing it).

**Claim 3.16.** For every type $p \in S_x(I)$, $\mu(\{p\}) = 0$.

**Proof of Claim.** Note that $I$ is not totally indiscernible, as otherwise for every $\alpha \in (0,1)$, $I_{<\alpha} + I_{>\alpha}$ is indiscernible over $a_\alpha$ and $x = a_\alpha$ divides over $I_{<\alpha} + I_{>\alpha}$, contradiction. Hence for any $\varepsilon > 0$ there is a formula $\psi_\varepsilon(x,y)$ over $\{a_\alpha \mid \alpha \in [0,\varepsilon] \cup [1-\varepsilon,1]\}$ which defines the order relation on $(\varepsilon,1-\varepsilon)$. Partition $(\varepsilon,1-\varepsilon)$ into intervals $J_k$, each of length $\varepsilon$, which are then definable by $\psi_k$. Then $\mu(\neg \bigvee \psi_k) \leq 2\varepsilon$, and $\mu(\psi_k) \leq 3\varepsilon$ for each $k$ (if $\psi_k(a_\alpha)$ holds, then if $\alpha \in (\varepsilon,1-\varepsilon)$ it must be in the interval defined by $\psi_k$). Since $p$ is complete, it follows by the definition of $\mu$ on closed sets that $\mu(\{p\}) = 0$. \hfill $\square$

It follows that for any closed set $F$ with $\mu(F) > 0$, we can find infinitely many types $p_i \in \text{supp}(\mu) \cap F$ (let $\varepsilon = \mu(F)$). Find $p \in \text{supp}(\mu) \cap F$, and a clopen set containing $p$ of measure $\varepsilon/2$. Removing this set we still have a closed set with measure at least $\varepsilon/2$, so we can go on).

Thus, since $\mu(\Sigma) > 0$, we can find infinitely many $\alpha$’s in $(0,1)$ such that $\lim^+(\alpha)$ or $\lim^-(\alpha)$ satisfy $\Sigma$. Without loss they are all of the form $\lim^-(\alpha)$. Enumerate them as $\langle \alpha_i \mid i < \omega \rangle$. By definition of $\Sigma$, for each $i < \omega$ we can find $b_{\alpha_i}$ such that $\langle a_\alpha \mid \alpha < \alpha_i \rangle + b_{\alpha_i} + \langle a_\alpha \mid \alpha \geq \alpha_i \rangle$ is indiscernible and $\varphi(b_{\alpha_i},d)$ holds.

Now find a dense indiscernible sequence $\langle (a'_i b'_i) \mid i \in \mathbb{Q} \rangle$ which is indiscernible over $d$ and has the same EM-type as $\langle (a_\alpha b_{\alpha_i}) \mid i < \omega \rangle$ over $d$. Let $I' = \langle a'_i \mid i \in \mathbb{Q} \rangle$, then, for every $i \in \mathbb{Q}$, $I'_{<i} + I'_{>i}$ is indiscernible over $d$, $I'_{<i} + b_{i} + I'_{>i}$ is indiscernible, $\varphi(b_i,d)$ holds, and $\varphi(x,d)$ divides over $I'_{<i} + I'_{>i}$ — contradiction. \hfill $\square$

**Proposition 3.17.** The theory $T$ is distal iff it is $\text{NTP}_2$-distal and $\text{NIP}$.

**Proof.** Suppose that $T$ is $\text{NTP}_2$-distal and $\text{NIP}$. By Fact 3.10 we have to show that for any model $M$ and $I \subseteq M$ indiscernible indexed by $[0,1]$, $\text{Av}_I |_M$ is smooth. By Remark 3.10 it is enough to show that $\text{Av}_I |_M$ is $\text{NTP}_2$-smooth. Suppose that $\varphi(x,c)$ divides over $M$ and that $\mu(\varphi(x,c)) > 0$ for some $\mu$ extending $\text{Av}_I |_M$. In particular $\mu$ extends $\text{Av}_I |_I$ and $\varphi(x,c)$ divides over $I$, so the latter is not $\text{NTP}_2$-smooth — contradiction.

On the other hand, if $T$ is distal, then it is $\text{NIP}$ (see after Definition 3.9). If the reader believes that Fact 3.10 is also true over $I$ (which is not stated but follows from the proof of this fact), then there is a unique global extension of $\text{Av}_I |_I$, namely $\text{Av}_I |_c$, which is also finitely satisfiable in $I$ (if $\text{Av}_I(\varphi(x,c)) > 0$ then $\varphi(I,c) \neq \emptyset$) so does not divide over $I$. In particular every extension of $\text{Av}_I |_I$ does not divide over $I$ so it is $\text{NTP}_2$-smooth.

For the skeptic reader, we also give an alternative proof using Theorem 3.11 if $I = I_1 + I_2$ is an indiscernible sequence with $I_1$ endless and $I_2$ with no first element and $I$ is $d$-indiscernible,
\[ I_1 + b + I_2 \] is indiscernible then by distality \( I_1 + b + I_2 \) is indiscernible. Hence, if \( \varphi(x, d) \) divides over \( I \) then \( \neg\varphi(b, d) \) holds. \( \square \)

**Example 3.18.** The ordered random graph is \( \text{NTP}_2 \)-distal. The ordered random graph is the model companion of the theory of ordered graphs in the language \( \{R, <\} \) where the order and the graph are independent. The restriction to the order part is just DLO and hence distal, and the restriction to the graph \( R \) is the random graph. Note the following easy facts:

- If \( p_<(x) \) and \( p_R(x) \) are non-algebraic (i.e., equations free) types over any set \( A \) in \( \{<\} \), \( \{R\} \) respectively then their union \( p \) is a complete consistent type over \( A \). Here \( x \) is any (finite) tuple of variables.

- It follows that if \( p(x) \) is a complete non-algebraic type over \( A \) which divides over some \( B \) then either its restriction to \( \{<\} \) or its restriction to \( \{R\} \) divides over \( B \).

- As non-algebraic types in the random graph do not divide, it follows that in such a case, \( p_\prec = p \mid \{<\} \) must divide.

- If we are in the situation of Theorem 3.14, i.e., \( I = I_1 + I_2 \), \( I_1 + b + I_2 \) is indiscernible over \( A \) and the intersection of any two tuples is empty (we choose \( A \) to ensure this), \( I \) is \( Ad \)-indiscernible and \( \varphi(x, d) \) divides over \( AI \), but \( \varphi(b, d) \) holds then: if \( p(x) = tp(b/dAI) \) is non-algebraic then \( p_\prec \) must divide over \( I \), but as \( DLO \) is distal \( I_1 + b + I_2 \) is indiscernible over \( Ad \) and hence \( p_\prec(a) \) holds for all \( a \in I \) and in particular \( p_\prec \) is finitely satisfiable in \( I \) so cannot divide over it.

- It follows that in such a situation \( p(x) \) is algebraic, so one of the points \( b' \) in the tuple \( b \) is equal to some \( d' \in d \) (it is impossible that \( b' \in AI \)). As \( I_1 + b + I_2 \) is \( Ad \)-indiscernible in \( \{<\} \), it follows that in the coordinate of \( b' \), \( I \) must be constant, contradiction.

**Example 3.19.** The random tournament is \( \text{NTP}_2 \)-distal. This theory \( T \) is the model companion of the theory of tournaments: it is a universal theory in the language \( \{R\} \) where \( R \) is a binary relation whose only axiom is \( \forall xyR(x, y) \leftrightarrow \neg R(y, x) \). The theory \( T \) is supersimple of \( U \)-rank 1. In other words, if \( \varphi(x, a) \) forks over \( A \) where \( x \) is a singleton then \( \varphi \) is algebraic (i.e., \( \varphi \vdash \bigvee_{i<k} x = c_i \)).

In fact, any union of non-algebraic complete types \( p_i(x) \) with \( x \) any finite tuple over \( A_i \) for \( i < \omega \) such that \( A_i \cap A_j = A \) for all \( i < j < \omega \) is consistent.

Suppose that \( I = I_1 + I_2 \) is \( Ad \)-indiscernible, \( I_1 + b + I_2 \) is \( A \)-indiscernible, the intersection of any two distinct tuples from \( I \) is empty and \( \varphi(x, d) \) divides over \( AI \). Suppose that \( \varphi(b, d) \) holds. It follows that \( p(x) = tp(b/dAI) \) is algebraic. Since \( b \cap AI = \emptyset \), there must be some \( b' \in b \) and \( d' \in d \) such that \( b' = d' \). Suppose that \( a_1 \in I_1 \) and \( a'_1 \in a_1 \) is in the same coordinate as \( b' \), and \( R(a'_1, b') \), then it must be that \( R(b', a'_2) \) for any \( a_2 \in I_2 \). But as \( I \) is indiscernible over \( d \), it follows that \( R(a'_2, b') \) as well — contradiction.
Example 3.19 shows that as opposed to the distal NIP case, where no distal stable theory exists, it is not true that there are no NTP₂-distal simple theories. This raises questions about this definition, so we leave this definition as a proposition.

Problem 3.20. It would be interesting to see more examples of NTP₂-distal. Some natural candidates include the theory of bounded PRC fields, see [Mon17].

4. On ω-resilience and a finite alternation theorem

Recall that T is called resilient if whenever (a_i | i ∈ ℤ) is indiscernible, and ϕ(x, a_0) divides over a_≠0, then ⟨ϕ(x, a_i) | i ∈ ℤ⟩ is inconsistent. This notion was introduced in [BYC14]. All NIP and simple theories are resilient, and all resilient theories are NTP₂. It is conjectured that NTP₂ theories are all resilient.

Definition 4.1. Say that T is ω-resilient if whenever (a_i | i < ω) and (b_i | i < ω) are such that:

- Both ⟨a_i | i < ω⟩ and ⟨b_i | i < ω⟩ are indiscernible.
- For every k < ω, ⟨a_i | i ≤ k⟩ + ⟨b_k⟩ + ⟨a_i | k < i < ω⟩ is indiscernible (in particular, b_i and a_j are all tuples of the same length).

Then for every formula ϕ(x, y), if {ϕ(x, a_i) | i < ω} is consistent, then so is {ϕ(x, b_i) | i < ω}.

Remark 4.2. (1) In Definition 4.1 using Ramsey we could replace the first bullet by asking that ⟨a_ib_i | i < ω⟩ is indiscernible.

(2) If T is resilient then T is ω-resilient. Why? Suppose that ⟨a_ib_i | i < ω⟩ is indiscernible and as in the definition and {ϕ(x, a_i) | i < ω} is consistent, but {ϕ(x, b_i) | i < ω} is inconsistent.

Then for any N < ω, (b_{N+1} | i < N) is a sequence which witnesses the dividing of ϕ(x, b_N) over a_{<N} ∪ a_{>N+N}. Hence by compactness we find a sequence contradicting resilience.

(3) By [BYC14 Proposition 4.5], if T is ω-resilient and ⟨a_ib_i | i < ω⟩ are as in the definition then we can find an array (with mutually indiscernible rows) ⟨c_{ij} | i, j < ω⟩ such that every row c_i ≡ ⟨b_i | i < ω⟩ and for every η: ω → ω, ⟨c_{i,η(i)} | i < ω⟩ ≡ ⟨a_i | i < ω⟩.

Question 4.3. Does is ω-resilience implies NTP₂?

The following is the main theorem for this section.

Theorem 4.4. Suppose that T is ω-resilient and NTP₂. Suppose that ⟨a_i | i < ω⟩ is an indiscernible sequence and that ϕ(x, b) divides over ⟨a_{2i} | i < ω⟩. Then for all but finitely many i’s, ¬ϕ(a_i, b) holds.

Note that this is true when T is simple: if not, by Ramsey, we may assume that ⟨a_{2i}, a_{2i+1} | i < ω⟩ is indiscernible over b and ϕ(a_{2i+1}, b) holds for all i < ω. We now extend I to have order type ω + ω, and let I_1 = ⟨a_{2i} | i < ω⟩ and I_2 = ⟨a_{2i+1} | ω ≤ i < ω + ω⟩. Then ϕ(x, b) divides over I_1.
so by symmetry \( b \not\subseteq I_1 \downarrow a_{2i+1} \) for every \( \omega \leq i \) as witnessed by some formula \( \psi(x, a_{2i+1}) \) over \( I_1 \) (by indiscernibility it is the same formula). However \( I_2 \) is a reversed Morley sequence over \( I_1 \) (in the sense that \( a_{2i+1} \downarrow I_1 \) \( a_{>2i+1} \)) so by Kim’s Lemma (see e.g., \[ Kim98\] Proposition 2.1]), \( \{ \psi(x, a_{2i+1}) \mid \omega \leq i < \omega \} \) is inconsistent — contradiction.

Note also that the conclusion of Theorem 4.4 is true if \( T \) is NTP-\( _2 \)-distal. This follows from Theorem 3.14 (we leave the details to the reader).

Before the proof let us recall a simple criterion for having TP-\( _2 \).

**Fact 4.5.** \[ LKS16\] Lemma 2.24 Suppose that \( A \) is some infinite set in \( \mathcal{C} \) and \( \varphi(x, y) \) is a formula such that for some \( k < \omega \), for every sequence \( \langle A_i \mid i < \omega \rangle \) of pairwise disjoint subsets of \( A \), there are \( \langle b_i \mid i < \omega \rangle \) such that \( A_i \subseteq \varphi(\mathcal{C}, b_i) \) and \( \{ \varphi(x, b_i) \mid i < \omega \} \) is \( k \)-inconsistent. Then \( T \) has TP-\( _2 \).

**Corollary 4.6.** Suppose that \( \langle a_{m,n} \mid m, n < \omega \rangle \) is an indiscernible sequence, ordered lexicographically and that \( \varphi(x, y) \) a formula such that for some \( k < \omega \), there is a sequence \( \langle b_m \mid m < \omega \rangle \) such that for every \( n, m < \omega \), \( \varphi(a_{m,n}, b_m) \) holds and \( \{ \varphi(x, b_m) \mid m < \omega \} \) is \( k \)-inconsistent. Then \( T \) has TP-\( _2 \).

**Proof of Corollary.** We show that \( A = \{ a_{n,0} \mid n < \omega \} \) has the property of Fact 4.5. Suppose that \( \langle A_m \mid m < \omega \rangle \) is a sequence of disjoint subsets of \( A \). For each \( m < \omega \), let \( a_m \) enumerate \( \langle a_{n,0} \mid n < \omega, a_{n,0} \in A_m \rangle \) and let \( a_m' \) enumerate \( \langle a_{n,m} \mid n < \omega, a_{n,0} \in A_m \rangle \). By indiscernibility, \( \langle a_m' \mid m < \omega \rangle \equiv \langle a_m \mid m < \omega \rangle \). Hence there are \( b_m' \) for \( n < \omega \) such that \( \langle b_m a_m' \mid m < \omega \rangle \equiv \langle b_m a_m \mid m < \omega \rangle \). These will satisfy the conditions of the Fact. \( \square \)

Finally we are ready to prove the theorem.

**Proof of Theorem 4.4.** For notational simplicity, write \( a_i \) for \( a_{2i} \) and \( c_i \) for \( a_{2i+1} \).

Assume that for every \( i < \omega \), \( \varphi(c_i, b) \) holds (obviously, \( \neg \varphi(a_i, b) \) holds for all \( i < \omega \)). Applying Ramsey and compactness, we may assume that \( \langle a_ic_i \mid i < \omega \rangle \) is indiscernible over \( b \). Next, we may assume its order type is \( \omega \times \omega \) (ordered lexicographically), so we have \( \langle a_{m,n}c_{m,n} \mid m, n < \omega \rangle \). Let \( \bar{a} = \langle a_{m,n} \mid m, n < \omega \rangle \) and similarly define \( \bar{c} \).

For \( k < \omega \), let \( \bar{a}_k = \langle a_{n,k} \mid n < \omega \rangle \) (the \( k \)-th column in \( \bar{a} \)), and similarly, let \( \bar{c}_k = \langle c_{n,k} \mid n < \omega \rangle \). Then for all \( N < \omega \), \( \langle \bar{a}_k \mid k \leq N \rangle \) \( \langle \bar{c}_N \rangle \) \( \langle \bar{a}_k \mid N < k \rangle \) is indiscernible.

Let \( \bar{b} = \langle b_i \mid i < \omega \rangle \) witness that \( \varphi(x, b) \) divides and even \( r \)-divides over \( \bar{a} \). For each \( k < \omega \), find \( \bar{c}_k \) such that \( b_k \bar{c}_k \equiv \bar{a} \bar{c}_k \). In particular, \( \langle \bar{a}_k \mid k \leq N \rangle \) \( \langle \bar{c}_N \rangle \) \( \langle \bar{a}_k \mid N < k \rangle \) is indiscernible for all \( N < \omega \). Note that \( \bar{c}_k b_k \equiv \bar{c}_0 b \) for all \( k < \omega \) because \( \bar{c} \) is \( b \)-indiscernible. Using Ramsey and compactness again, find an indiscernible sequence \( \langle \bar{c}_k f_k \mid k < \omega \rangle \) with the same EM-type as \( \langle \bar{a}_k \bar{c}_k \mid k < \omega \rangle \).
We still have that \( \langle \bar{e}_k \mid k \leq N \rangle + \langle \bar{f}_N \rangle + \langle \bar{e}_k \mid k > N \rangle \) is indiscernible for all \( N \). By \( \omega \)-resilience and Remark 4.2, there is an array \( \langle h_{n,m} \mid n, m < \omega \rangle \) such that for every \( n < \omega \), \( \langle h_{n,m} \mid m < \omega \rangle \equiv \langle f_k \mid k < \omega \rangle \) and for every \( \eta : \omega \to \omega \), \( \langle h_{n,\eta(n)} \mid n < \omega \rangle \equiv \langle \bar{e}_k \mid k < \omega \rangle \equiv \langle \bar{a}_k \mid k < \omega \rangle \).

For each \( n, m \) we can find \( b_{n,m} \) such that \( h_{n,m} b_{n,m} \equiv \bar{e}_0 b \) and \( \{ \varphi (x, b_{n,m}) \mid m < \omega \} \) is \( r \)-inconsistent (because this is a closed condition which holds for the sequence \( \langle \bar{e}_k \mid k < \omega \rangle \)). By extracting we may assume that the whole array \( \langle h_{n,m} b_{n,m} \mid n, m < \omega \rangle \) is indiscernible in the sense that the rows are mutually indiscernible, and even that the sequence of rows

\[
\langle \langle h_{n,m} b_{n,m} \mid m < \omega \rangle \mid n < \omega \rangle
\]

is itself indiscernible.

By NTP\(_2\), there is some \( \eta : \omega \to \omega \) such that \( \{ \varphi (x, b_{n,\eta(n)}) \mid n < \omega \} \) is inconsistent, so by indiscernibility, this is true for \( \eta \) being constantly 0 and hence it is \( l \)-inconsistent for some \( l \). As the sequence \( \langle h_{n,0} \mid n < \omega \rangle \equiv \langle \bar{a}_k \mid k < \omega \rangle \), we can find \( b'_k \) such that \( \{ \varphi (x, b'_k) \mid k < \omega \} \) is \( l \)-inconsistent and \( \varphi (a_{n,k}, b'_k) \) holds for all \( n < \omega \).

However this contradicts NTP\(_2\) by Corollary 4.6.

**Corollary 4.7.** Suppose that \( T \) is \( \omega \)-resilient and NTP\(_2\). Then the following is impossible:

There exists an infinite set \( A \), a formula \( \varphi (x, y) \) and some \( k < \omega \) such that for every subset \( s \subseteq A \), there is some \( b_s \) such that \( \varphi (x, b_s) \) divides and even \( k \)-divides over \( A \setminus s \) and for all \( a \in s \), \( a \models \varphi (x, b_s) \).

**Proof.** Suppose that there is such a set \( A \), a formula \( \varphi (x, y) \) and \( k \). Without loss of generality, \( A \) is countable. Enumerate \( A \) as \( \bar{a} = \langle a_i \mid i < \omega \rangle \). Let \( \bar{a}' = \langle a'_i \mid i < \omega \rangle \) be an indiscernible sequence with the same EM-type as \( \bar{a} \). Then there is some \( b \) such that \( \varphi (x, b) \) divides and even \( k \)-divides over \( \{ a'_i \mid i < \omega \} \) and \( \varphi (a_{2i+1}, b) \) holds for all \( i < \omega \). This contradicts Theorem 4.4.

We end this section with an open problem which we find extremely nice.

**Question 4.8.** (NTP\(_2\)) Suppose that \( \{ b_i \mid i < \omega \} \) is a Morley sequence over a model \( M \), and \( \varphi (x, b_0) \) divides over \( M \). Is it true that \( \{ \varphi (x, b_{2i}) \land \neg \varphi (x, b_{2i+1}) \mid i < \omega \} \) is inconsistent?

5. **On singular local character in NIP**

Here we prove two theorems on what we call singular local character in the setting of NIP. The idea is that local character for non-forking fails for general NIP theories, but we can still recover some version of it over sets of singular cardinality.

**Theorem 5.1.** Suppose that \( T \) is NIP. Suppose that \( A \subseteq \mathcal{C} \) is a small set of cardinality \( \mu \) where, \( |T| < \text{cof}(\mu) = \kappa < \mu \). Then for every (finitary) type \( p(x) \in S(A) \) there is some \( B \subseteq A \) of cardinality \( \kappa \) such that \( p \) does not divide over \( B \).
If $A$ is a model $M$, then there is some model $N < M$ with $|N| < \mu$ such that $p$ is finitely satisfiable in $N$.

Proof. We start with the first statement. We use the same ideas as in [Shel13, Theorem 2.12]. Write $A = \bigcup \{A_i \mid i < \kappa\}$ where $|A_i| < \mu$ for all $i < \kappa$ and $A_i \subseteq A_j$ for $i \leq j$.

Fix some $d \models p$. Let $X_{(A, i < \kappa)}$ be the set of sequences $\bar{c} = \langle (c^0_\alpha, c^1_\alpha) \mid \alpha < \gamma \rangle$ such that, letting $c_\alpha = (c^0_\alpha, c^1_\alpha)$:

- For every $\alpha < \gamma$, $c^0_\alpha = A\bar{c}_< \alpha$, $c^1_\alpha$; for some $i < \kappa$, $\tp(c_\alpha/Ac_<\alpha)$ is finitely satisfiable in $A_i$; $c^0_\alpha d \neq Ac_<\alpha c^1_\alpha d$.

We try to construct a maximal element in $X_{(A, i < \kappa)}$ of length $< |T|^+$, i.e., one that cannot be extended to a longer sequence. Suppose we cannot, i.e., there is $\bar{c} \in X_{(A, i < \kappa)}$ with $\gamma \bar{c} = |T|^+$. For each $\alpha < |T|^+$, there is a formula $\varphi_\alpha(x, y, z)$ over $\emptyset$ and $b_\alpha \in Ac_<\alpha$ such that $\varphi_\alpha(d, c^0_\alpha, b_\alpha) \land \neg \varphi_\alpha(d, c^1_\alpha, b_\alpha)$. Extracting we may assume that $\varphi_\alpha = \varphi$. By Fodor’s lemma, for some $\beta < |T|^+$ there is a stationary subset $S$ of $|T|^+ \setminus \beta$ such that for all $\alpha \in S$, $b_\alpha \in Ac_<\beta$. Now note that for any $\eta : S \to 2$, $\tp\left(\left\langle c^{(\alpha)}_\eta \mid \alpha \in S \right\rangle / Ac_<\beta\right)$ is constant regardless of $\eta$ (prove by induction that this is true for all finite subsets of $S$, using the fact that $\tp(\bar{c}/A)$ is finitely satisfiable in $A$). Hence we get that for any subset $s \subseteq S$, $\left\{ \varphi(x, c^0_\alpha, b_\alpha) \mid \alpha \in s \right\} \cup \left\{ \neg \varphi(x, c^1_\alpha, b_\alpha) \mid \alpha \notin s \right\}$ is consistent. But that clearly gives us IP for the formula $\varphi$.

Let $\bar{c} \in X_{(A, i < \kappa)}$ be maximal of length $< |T|^+$. Note that $\tp(\bar{c}/A)$ is finitely satisfiable in some $A_{i_0}$ for $i_0 < \kappa$, because we assumed that $\kappa \geq |T|^+$.

We get that $r(x) = \tp(d/A\bar{c})$ is weakly orthogonal to any type $q \in S(A\bar{c})$ which is finitely satisfiable in some $A_i$ for $i < \kappa$, in the sense that $q \cup r$ implies a complete type over $A\bar{c}$: if $e_1, e_2 \models q$ and $de_1 \neq A\bar{c} de_2$, then, taking a global finitely satisfiable in $A_i$ extension $q'$ of $q$, and letting $e' \models q'|A\bar{c}de_1e_2$, we may assume that $\tp(e_2/e_1dA\bar{c})$ is finitely satisfiable in $A_i$ and in particular $\tp(e_1e_2/A\bar{c})$ is finitely satisfiable in $A_i$, contradicting the maximality of $\bar{c}$.

Fix some formula $\varphi(x, y)$ over $\bar{c}$. For $i < \kappa$, let $Y_i$ be the set of all $q(y) \in S(A\bar{c})$ finitely satisfiable in $A_i$ such that $r(x) \cup q(y) \models \varphi(x, y)$. Then $Y_i$ is clopen (it is open in the space of all types over $A\bar{c}$ finitely satisfiable in $A_i$, but its complement is precisely those types $q \in S(A\bar{c})$ such that $r(x) \cup q(y) \models \neg \varphi(x, y)$, so also open). For each $q \in Y_i$ there is a formula $\zeta_q \in q$ and a formula $\psi_q \in r$ such that $\psi_q \land \zeta_q \models \varphi$. By compactness there are formulas $\zeta(y)$ and $\psi \in r$ such that $\psi \land \zeta \models \varphi$ and $\zeta$ covers $Y_i$. Let $E_i \subseteq A$ be the set of parameters appearing in all these formulas $\psi$ when we run over all formulas $\varphi(x, y)$. Then $|E_i| \leq |T|$ and $\tp(d/E_i\bar{c}) \models \tp(d/A, \bar{c})$. Let $E = \bigcup_{i < \kappa} E_i$. Then $|E| < \mu$ and $p = \tp(d/A)$ does not divide over $EA_{i_0} \subseteq A$: suppose that $\varphi(x, f) \in p$ divides over $EA_{i_0}$ where $f \in A_i$ for some $i < \kappa$. Note that as $\bar{c} \uparrow_{EA_{i_0}} Ef$ (i.e., $\tp(\bar{c}/EA_{i_0}Ef)$ is finitely satisfiable in $A_{i_0}$), $\varphi(x, f)$ also divides over $\bar{c}A_{i_0}E$ (any indiscernible sequence starting with $f$ over $EA_{i_0}$ can be moved so that it is also indiscernible over $\bar{c}$). Now,
\[ \text{tp}(d/E_i \bar{c}) \vdash \varphi(x, f) \] so some formula \( \psi(x, e, \bar{c}) \in \text{tp}(d/E_i \bar{c}) \) divides over \( EA_i \bar{c} \) and in particular over \( E_i \bar{c} \) which is impossible.

Now suppose that \( A = M \) is a model. Find a model \( N \prec M \) containing \( EA_i \bar{c} \) of size \( \leq |T| + |A_i E_i| \). We claim that \( p \) is finitely satisfiable in \( N \). Suppose that \( \varphi(x, f) \in p \). Let \( \psi(x, e, \bar{c}) \in \text{tp}(d/E \bar{c}) \) be such that \( \psi(x, e, \bar{c}) \vdash \varphi(x, f) \).

\[ \mathcal{C} \models \exists x \psi(x, e, \bar{c}) \wedge \forall x (\psi(x, e, \bar{c}) \rightarrow \varphi(x, f)). \]

As \( \bar{c} \not\subseteq_{fs} E \bar{f} \), there is some \( \bar{c}' \in N \) such that

\[ \mathcal{C} \models \exists x \psi(x, e, \bar{c}') \wedge \forall x (\psi(x, e, \bar{c}') \rightarrow \varphi(x, f)), \]

and as \( N \prec \mathcal{C}, N \models \exists x \psi(x, e, \bar{c}') \), so there is some \( d' \in N \) such that \( \psi(d', e, \bar{c}') \) holds. Hence, \( \varphi(d', f) \) holds as well.

The following questions seem natural.

**Question 5.2.** Is Theorem 5.1 true for \( \text{NTP}_2 \)? Namely, suppose that \( T \) is \( \text{NTP}_2 \) and that \( M \models T, |T| < \cof(|M|) < |M| \) and that \( p \in S(M) \). Is there some \( N \prec M \) over which \( p \) does not fork? The same question can be asked with \( M \) being a set (and forking replaced by dividing).

**Question 5.3.** (NIP) Assume that \( p \in S(\mathcal{C}) \) is a global type which is finitely satisfiable in a model \( M \) such that \( \mu = |M| \) is singular with \( |T| < \kappa = \cof(\mu) \). Does it follow that \( p \) is finitely satisfiable in some model \( M_0 \prec M \) such that \( |M_0| < \mu \), or even that \( p \) is finitely satisfiable in some \( M_0 \prec \mathcal{C} \) of size \( \leq \mu \)?

The next proposition seems to give us some hope in the direction of answering Question 5.3 positively.

**Proposition 5.4.** Suppose that \( T \) is NIP and that \( p \in S(\mathcal{C}) \) is a global type which is finitely satisfiable in a model \( M \) such that \( \mu = |M| \) and \( |T| < \kappa = \cof(\mu) \). Then there is some \( \lambda < \mu \) such that for every \( A \subseteq \mathcal{C} \) there is some \( M_0 \prec M \) of size \( \leq |A| + \lambda + |T| \) such that \( p \models (A \cup M) \) is finitely satisfiable in \( M_0 \).

Note that this proposition implies Theorem 5.1 (by taking \( A = \emptyset \)).

**Proof.** We use similar ideas to the ones in the proof of Theorem 5.1 but work in \( M^{Sh} \) (see below).

Let \( N \) be an \( |M|^+ \)-saturated model. It is enough to find such a \( \lambda \) that suffices for \( A \subseteq N \).

Let \( M^{Sh} \) be the Shelah expansion of \( M \) in the language \( L^{Sh} \), i.e., add predicates of the form \( R_{\varphi(x, c)}(x) \) for every \( \varphi(x, y) \) and \( c \in N \), and interpret them as \( \varphi(M, c) \).

Work in a monster model \( \mathcal{C}^{Sh} \) of \( M^{Sh} \), and let \( p^{Sh} = \{ R_{\varphi(x, c)}(x) \mid c \in N, \varphi(x, c) \in p \} \). By the assumption that \( p \) is finitely satisfiable in \( M \), \( p^{Sh} \) is a type over \( M^{Sh} \).

Write \( M = \bigcup \{ M_i \mid i < \kappa \} \) where \( M_i \prec M \) and \( |M_i| < \mu \).
Claim 5.5. Fix some $d \models p^{Sh}$. We cannot find a sequence $\bar{c} = \langle (c^0_\alpha, c^1_\alpha) \mid \alpha < |T|^+ \rangle$ in $\mathfrak{C}^{Sh}$ such that, letting $c_\alpha = (c^0_\alpha, c^1_\alpha)$:

- For every $\alpha < |T|^+$, $c^0_\alpha \models_{M\bar{c}_\alpha} c^1_\alpha$: For some $i < \kappa$, $\text{tp}(c_\alpha/M\bar{c}_\alpha)$ is finitely satisfiable in $M_i$; $c^0_i d \not\models_{M\bar{c}_\alpha} c^1_i d$. (All in the sense of $\mathfrak{C}^{Sh}$.)

Proof of Claim 5.5. The proof is similar to the one in Theorem 5.1. Suppose that $\bar{c}$ is such a sequence. As $M^{Sh}$ has quantifier elimination [She04, Sim15, Proposition 3.23], for each $\alpha < |T|^+$, there is a formula $\varphi_\alpha(x, y, z, e_\alpha)$ from $L(N)$ (i.e., $e_\alpha \in N$) and $\bar{b}_\alpha \in M\bar{c}_\alpha$ such that $R_{\varphi_\alpha}(d, c^0_\alpha, b_\alpha) \land \neg R_{\varphi_\alpha}(d, c^1_\alpha, b_\alpha)$ holds. Extracting, we may assume that $\varphi_\alpha(x, y, z, w)$ is constantly $\varphi$ (but of course, $e_\alpha$ may vary). By Fodor’s lemma, we find a stationary set $S \subseteq |T|^+$ and some $\beta < |T|^+$ such that $\bar{b}_\alpha \in M\bar{c}_\beta$ for all $\alpha \in S$.

Now note that, precisely as in the proof of Theorem 5.1, $\text{tp}(\langle c^\eta_\alpha \mid \alpha \in S \rangle/M\bar{c}_\beta)$ for $\eta : S \to 2$ is constant regardless of $\eta$ (all in $\mathfrak{C}^{Sh}$).

Let $\bar{e} = \langle e_\alpha \mid \alpha < |T|^+ \rangle$, and let $M' \models \mathfrak{C}^{Sh}$ be a model such that $M'$ contains $\bar{c}, d, M$. Find $\bar{e}' = \langle e'_\alpha \mid \alpha < |T|^+ \rangle$ in $\mathfrak{C}^{Sh}$ such that for every $\psi(x, y)$ in $L$, and every $f \in M'$, $\psi(f, \bar{e}')$ holds iff $R_{\varphi(x, \bar{e})}(f)$ holds. In other words, $\bar{e}'$ realizes the elementary extension of the type $\text{tp}(\bar{e}/M)$ to $M'$. In particular, we get that

- $\varphi_\alpha(d, c^0_\alpha, b_\alpha, e_\alpha) \land \neg \varphi_\alpha(d, c^1_\alpha, b_\alpha, e'_\alpha)$.

- If $f_1 \equiv f_2$ are from $M'$ (in the sense of $L^{Sh}$) then $f_1 \equiv^e f_2$ in $L$.

It follows that $\text{tp}(\langle c^\eta_\alpha \mid \alpha \in S \rangle/M\bar{c}_\beta)$ in $L$ is also constant, regardless of $\eta$.

Hence we get that for any subset $s \subseteq S$, $\{ \varphi(x, c^0_\alpha, b_\alpha, e'_\alpha) \mid \alpha \in s \} \cup \{ \neg \varphi(x, c^1_\alpha, b_\alpha, e'_\alpha) \mid \alpha \notin s \}$ is consistent. But that clearly gives us IP for the formula $\varphi$.

Now let $\bar{c}$ be a maximal sequence as in Claim 5.5. Since $|T| < \kappa$ and the (cardinality of the) length of $\bar{c}$ is at most $|T|$, $\text{tp}(\bar{c}/M^{Sh})$ is finitely satisfiable in some $M_i$ for $i_0 < \kappa$.

As in the proof of Theorem 5.1, $r(x) = \text{tp}(d/M^{Sh}\bar{c})$ is weakly orthogonal to every $q \in S$ ($M^{Sh}\bar{c}$) which is finitely satisfiable in some $M_i$, $i < \kappa$. From this we can deduce that for every $i < \kappa$ and for every formula $\varphi(x, y)$ in $L^{Sh}$, there is some formula $\psi(x) \in r$ such that $\psi(x) \vdash \text{tp}_\varphi(d/M_i\bar{c})$.

Let $\lambda = |M_i| + \kappa$. Given $A \subseteq N$, let $L^A$ be the restriction of $L^{Sh}$ to $\{ R_{\varphi(x, c)} \mid c \in A \}$ (and similarly, $M^A = M^{Sh} \upharpoonright L^A$). Then we get that for every $i < \kappa$ there are $A_i, E_i \subseteq M$ of size $\leq |A| + |T|$ such that $\text{tp}(d/E_i) \cap L^{A_i} \vdash \text{tp}(d/M_i\bar{c}) \cap L^A$. Let $B = A \cup \bigcup \{ A_i \mid i < \kappa \}$ and let $M_0 \prec M^B$ contain $\bigcup \{ E_i \mid i < \kappa \} \cup M_i$ be of size $\leq \lambda + |A| + |T|$. As in the proof of Theorem 5.1, $p \models (A \cup M)$ is finitely satisfiable in $M_0$: if $\varphi(x, f, a) \in p$ for $f \in M, a \in A$, then $R_{\varphi(x, f, a)}(x, f) \in \text{tp}(d/M^B)$ and for some $\psi(x, e, \bar{e}) \in \text{tp}(d/E_i)$ (in $L^B$), $\psi \vdash R_{\varphi}(x, f)$. Since $\bar{c} \downarrow_{M^{Sh}} M$ (in $L^{Sh}$), there is some $\bar{e}' \in N$ such that $\mathfrak{C}^{Sh} \models \exists x \psi(x, e, \bar{e}') \land \forall x (\psi(x, e, \bar{e}') \rightarrow R_{\varphi}(x, f))$. As $N \prec \mathfrak{C}^B$, there is some $d' \in N$ such that $\psi(d', e, \bar{e}')$ holds, so $R_{\varphi}(d', f)$ holds which means that $\varphi(d', f, a)$ holds. \qed
SOME NIP-LIKE PHENOMENA IN NTP$_2$

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