δ-Invariants, Inequalities of Submanifolds and Their Applications

Dedicated to Prof. Leopold Verstraelen on the occasion of his 60th birthday

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Abstract

The famous Nash embedding theorem was aimed for in the hope that if Riemannian manifolds could be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. However, as late as 1985 (see [142]) this hope had not been materialized. The main reason for this is due to the lack of controls of the extrinsic properties of the submanifolds by the known intrinsic invariants. In order to overcome such difficulties as well as to provide answers to an open question on minimal immersions, we introduced in the early 1990’s new types of Riemannian invariants, known as the δ-invariants or the so-called Chen invariants, different in nature from the “classical” Ricci and scalar curvatures. At the same time we also able to establish general optimal relations between the new intrinsic invariants and the main extrinsic invariants for Riemannian submanifolds. Since then many results concerning these invariants, inequalities, related subjects, and their applications have been obtained by many geometers.

The main purpose of this article is thus to provide an extensive and comprehensive survey of results over this very active field of research done during the last fifteen years. Several related inequalities and their applications are presented in this survey article as well.

Topics in Differential Geometry, 29-155, Ed. Acad. Române, Bucharest, 2008 (Edited by A. Mihai, I. Mihai and R. Miron).
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1 Motivation to introduce $\delta$-invariants

Curvature invariants are the $N^0$ 1 Riemannian invariants and the most natural ones. Curvature invariants also play key roles in physics. For instance, the magnitude of a force required to move an object at constant speed, according to Newton’s laws, is a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein’s general theory of relativity, by the curvatures of space time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures (cf. [218]). Borrowing a term from biology, Riemannian invariants are the DNA of Riemannian manifolds. Classically, among the Riemannian curvature invariants, people have been studying sectional, scalar and Ricci curvatures in great detail.

One of the most fundamental problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to the 1956 celebrated embedding theorem of J. F. Nash [208], every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension.

The Nash embedding theorem was aimed for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. However, this hope had not been materialized as late as 1985 according to M. Gromov [142] (see also [70]). The main reason for this is due to the lack of controls of the extrinsic properties of the submanifolds by the known intrinsic invariants.

In view of Nash’s theorem, to study embedding problems it is natural to impose some suitable condition(s) on the immersions. For example, if one imposes the minimality condition, it leads to

**Problem 1.** Given a Riemannian manifold $M$, what are necessary conditions for $M$ to admit a minimal isometric immersion in a Euclidean $m$-space $\mathbb{E}^m$?
It is well-known that for a minimal submanifold in $\mathbb{E}^m$, the Ricci tensor satisfies $\text{Ric} \leq 0$. For many years this was the only known necessary Riemannian condition for a general Riemannian manifold to admit a minimal isometric immersion in a Euclidean space regardless of codimension. That is why S. S. Chern asked in his 1968 monograph to search for further Riemannian obstructions for $M$ to admit an isometric minimal immersion into a Euclidean space. Also, no solutions to Chern’s problem were known for many years before the invention of the $\delta$-invariants.

In order to overcome those difficulties, we need to introduce certain new types of Riemannian invariants, different in nature from the “classical” invariants. Moreover, we also need to establish general optimal relationships between the main extrinsic invariants with the new intrinsic invariants on the submanifolds. These are the author’s original motivation in 1990’s to introduce his so-called $\delta$-invariants on Riemannian manifolds.

The $\delta$-invariants are very different in nature from the “classical” scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are “total sum” of sectional curvatures on a Riemannian manifold. In contrast, all of the non-trivial $\delta$-invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures.

## 2 Definition of $\delta$-invariants

Let $M$ be a Riemannian $n$-manifold. Denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_pM$, the scalar curvature $\tau$ at $p$ is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let $L$ be a subspace of $T_pM$ of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. We define the scalar curvature $\tau(L)$ of the $r$-plane section $L$ by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

Given an orthonormal basis $\{e_1, \ldots, e_r\}$ of the tangent space $T_pM$, we simply denote by $\tau_{1\ldots r}$ the scalar curvature of the $r$-plane section spanned by $e_1, \ldots, e_r$. The scalar curvature $\tau(p)$ of $M$ at $p$ is nothing but the scalar
curvature of the tangent space of $M$ at $p$; and if $L$ is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of $L$.

Geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of $L$ at $p$ under the exponential map at $p$.

For an integer $k \geq 0$ denote by $S(n,k)$ the finite set consisting of unordered $k$-tuples $(n_1,\ldots,n_k)$ of integers $\geq 2$ satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $S(n)$ the set of unordered $k$-tuples with $k \geq 0$ for a fixed $n$.

For each $k$-tuple $(n_1,\ldots,n_k) \in S(n)$ the Riemannian invariant $\delta(n_1,\ldots,n_k)$ is defined to be

$$\delta(n_1,\ldots,n_k)(p) = \tau(p) - \inf \{ \tau(L_1) + \cdots + \tau(L_k) \},$$

where $L_1,\ldots,L_k$ run over all $k$ mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1,\ldots,k$.

Similarly, we have also defined $\hat{\delta}(n_1,\ldots,n_k)(p)$ by

$$\hat{\delta}(n_1,\ldots,n_k)(p) = \tau(p) - \sup \{ \tau(L_1) + \cdots + \tau(L_k) \},$$

where $L_1,\ldots,L_k$ run over all $k$ mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1,\ldots,k$.

Let $\#S(n)$ denote the cardinal number of $S(n)$. Then $\#S(n)$ increases quite rapidly with $n$. For instance, for

$$n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots, 20, \ldots,$$

$$50, \ldots, 100, \ldots, 200, \ldots,$$

$\#S(n)$ are given respectively by

$$1, 2, 4, 6, 10, 14, 21, 29, 41, 54, 76, \ldots, 626, \ldots,$$

$$204225, \ldots, 190569291, \ldots, 3972999029387, \ldots.$$

In general, the cardinal number $\#S(n)$ is equal to $p(n) - 1$, where $p(n)$ denotes the partition function. The asymptotic behavior of $\#S(n)$ is given by

$$\#S(n) \approx \frac{1}{4n\sqrt{3}} \exp \left[ \pi \sqrt{\frac{2n}{3}} \right] \text{ as } n \to \infty.$$
In terms of the $\delta$-invariants, the scalar curvature $\tau$ is nothing but $\delta(\emptyset)$ or $\hat{\delta}(\emptyset)$ (with $k = 0$). The simplest non-trivial $\delta$-invariants are $\delta(2)$ and $\hat{\delta}(2)$. The scalar curvature and the $\delta$-invariants $\delta(n_1, \ldots, n_k)$ with $k > 0$ differ greatly in nature.

Obviously, one has

$$\delta(n_1, \ldots, n_k) \geq \hat{\delta}(n_1, \ldots, n_k)$$

for any $k$-tuple $(n_1, n_2, \ldots, n_k) \in S(n)$. A Riemannian $n$-manifold $M$ is called an $S(n_1, \ldots, n_k)$-space if it satisfies

$$\delta(n_1, \ldots, n_k) = \hat{\delta}(n_1, \ldots, n_k)$$

identically for a fixed $k$-tuple $(n_1, \ldots, n_k) \in S(n)$.

In this article, some other invariants of a similar nature, i.e., those invariants obtained from the scalar curvature by deleting certain amount of sectional curvature, are also called $\delta$-invariants. Those invariants have a similar nature as $\delta(n_1, \ldots, n_k)$ or $\hat{\delta}(n_1, \ldots, n_k)$ with $k > 0$. For instance, we have the so-called affine $\delta$-invariants, Kählerian $\delta$-invariants, normal $\delta$-invariant, ..., etc.

3 Relations between $\delta$-invariants and Einstein and conformally flat manifolds

The $S(n_1, \ldots, n_k)$-spaces are completely determined by the following two propositions.

**Proposition 3.1.** Let $M$ be a Riemannian $n$-manifold with $n > 2$. Then

(1) For any integer $j$ with $2 \leq j \leq n - 2$, $M$ is an $S(j)$-space if and only if $M$ is a Riemannian space form.

(2) $M$ is an $S(n - 1)$-space if and only if $M$ is an Einstein space.

**Proposition 3.2.** Let $M$ be a Riemannian $n$-manifold such that $n$ is not a prime and $k$ an integer $\geq 2$. Then

(1) if $M$ is an $S(n_1, \ldots, n_k)$-space, then $M$ is a Riemannian space form unless $n_1 = \ldots = n_k$ and $n_1 + \cdots + n_k = n$, and
\( \delta \)-invariants

(2) \( M \) is an \( S(n_1, \ldots, n_k) \)-space with \( n_1 = \ldots = n_k \) and \( n_1 + \cdots + n_k = n \) if and only if \( M \) is a conformally flat space.

By using the notion of \( \delta \)-invariant, we have the following simple characterization of Einstein spaces which generalizes the well-known characterization of Einstein 4-manifolds given by I. M. Singer and J. A. Thorpe [244].

**Theorem 3.1.** [91] Let \( M \) be a Riemannian 2\( r \)-manifold. Then \( M \) is an Einstein space if and only if we have

\[
\tau(L) = \tau(L^\perp)
\]

for any \( r \)-plane section \( L \subset T_pM, p \in M \).

Moreover, also by using the notion of the scalar curvature of \( r \)-plane sections, we have the following simple characterization of conformally flat spaces which generalizes a well-known result of R. S. Kulkarni [170].

**Theorem 3.2.** [91] Let \( M^n \) be a Riemannian manifold with \( n \geq 4 \), and let \( s \) be an integer satisfying \( 2 < 2s \leq n \). Then \( M \) is a conformally flat manifold if and only if, for any orthonormal set \( \{e_1, \ldots, e_{2s}\} \) of vectors, one has

\[
\tau_1 \cdots s + \tau_{s+1} \cdots 2s = \tau_1 \cdots s-1 + \tau_{s+2} \cdots 2s.
\]

In general, the \( \delta \)-invariants \( \delta(n_1, \ldots, n_k) \) are independent invariants. However, Theorem 3.1 implies that, for a 2\( r \)-dimensional Einstein manifold, we have the following relations:

\[
2\delta(r) - \delta(r, r) = 2\hat{\delta}(r, r).
\]

For any \( k \)-tuple \( (n_1, \ldots, n_k) \in S(n) \), let us put

\[
\Delta(n_1, \ldots, n_k) = \frac{\delta(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)},
\]

where \( c(n_1, \ldots, n_k) \) is defined by

\[
c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}.
\]

Since a Riemannian \( n \)-manifold with \( n \geq 3 \) satisfies inequality \( \Delta(2) > \Delta(\emptyset) = \tau \) if and only if inf \( K < \tau/(n - 1)^2 \). Thus, a Riemannian \( n \)-manifold \( (n \geq 3) \) with vanishing scalar curvature satisfies

\[
\Delta_0(2) > \Delta_0(\emptyset)
\]
automatically, unless $M$ is flat.

For compact homogeneous Einstein Kähler manifolds, we also have the following relationship between the $\delta$-invariants and scalar curvature.

**Proposition 3.3.** [50] Let $M$ be a compact homogeneous Einstein Kaehler manifold with positive scalar curvature. Then, for each $(n_1, \ldots, n_k) \in S(n)$, we have

$$\Delta(n_1, \ldots, n_k) \leq \left(2 - \frac{2}{n}\right)\Delta(\emptyset),$$

where $n$ denotes the real dimension of $M$.

### 4 Fundamental inequalities involving $\delta$-invariants

Let $M$ be an $n$-dimensional submanifold of a Riemannian $m$-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame

$$e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$$

in $\tilde{M}^m$ such that, restricted to $M$, the vectors $e_1, \ldots, e_n$ are tangent to $M$ and hence $e_{n+1}, \ldots, e_m$ are normal to $M$. Let $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denote respectively the sectional curvatures of $M$ and $\tilde{M}^m$ of the plane section spanned by $e_i$ and $e_j$.

For the submanifold $M$ in $\tilde{M}^m$ we denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections of $M$ and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [32])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for any vector fields $X, Y$ tangent to $M$ and vector field $\xi$ normal to $M$, where $h$ denotes the second fundamental form, $D$ the normal connection, and $A$ the shape operator of the submanifold.

Let $\{h^r_{ij}\}$, $i, j = 1, \ldots, n$; $r = n + 1, \ldots, m$, denote the coefficients of the second fundamental form $h$ with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then we have

$$h^r_{ij} = \langle h(e_i, e_j), e_r \rangle = \langle A e_i, e_j \rangle,$$

where $\langle \ , \ \rangle$ denotes the inner product.
The mean curvature vector $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n} \text{trace } h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \quad (4.3)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle $TM$ of $M$. The squared mean curvature is then given by

$$H^2 = \langle \vec{H}, \vec{H} \rangle.$$ 

A submanifold $M$ is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Denote by $R$ and $\tilde{R}$ the Riemann curvature tensors of $M$ and $\tilde{M}^m$, respectively. Then the equations of Gauss and Codazzi are given respectively by

$$R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle h(X, W), h(Y, Z) \rangle$$

$$- \langle h(X, Z), h(Y, W) \rangle, \quad (4.4)$$

$$(\tilde{R}(X, Y)Z)^\perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z),$$

where $X, Y, Z, W$ are tangent to $M$ and $\tilde{\nabla} h$ is defined by

$$(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (4.5)$$

for vectors $X, Y, Z, W$ tangent to $M$.

A submanifold $M$ is called a parallel submanifold if we have $\tilde{\nabla} h = 0$ identically.

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, let $b(n_1, \ldots, n_k)$ denote the constant given by

$$b(n_1, \ldots, n_k) = \frac{1}{2} n(n-1) - \frac{1}{2} \sum_{j=1}^{k} n_j(n_j - 1). \quad (4.6)$$

For any isometric immersion from a Riemannian submanifold into another Riemannian manifold, we have the following general optimal inequality.

**Theorem 4.1.** [77] Let $\phi : M \to \tilde{M}$ be an isometric immersion of a Riemannian $n$-manifold into a Riemannian $m$-manifold. Then, for each point $p \in M$ and each $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have the following inequality:

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k) H^2(p) + b(n_1, \ldots, n_k) \max \tilde{K}(p), \quad (4.7)$$
where \( \max \tilde{K}(p) \) denotes the maximum of the sectional curvature function of \( \tilde{M}^m \) restricted to 2-plane sections of the tangent space \( T_p M \) of \( M \) at \( p \).

The equality case of inequality (4.7) holds at \( p \in M \) if and only if the following conditions hold:

(a) There exists an orthonormal basis \( e_1, \ldots, e_m \) at \( p \), such that the shape operators of \( M \) in \( \tilde{M}^m \) at \( p \) take the following form:

\[
A_{e_r} = \begin{pmatrix}
A^r_1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & A^r_k \\
0 & \ldots & \mu_r I
\end{pmatrix}, \quad r = n + 1, \ldots, m, \tag{4.8}
\]

where \( I \) is an identity matrix and \( A^r_j \) is a symmetric \( n_j \times n_j \) submatrix such that

\[
\text{trace } (A^r_1) = \cdots = \text{trace } (A^r_k) = \mu_r. \tag{4.9}
\]

(b) For any \( k \) mutual orthogonal subspaces \( L_1, \ldots, L_k \) of \( T_p M \) which satisfy

\[
\delta(n_1, \ldots, n_k) = \tau - \sum_{j=1}^k \tau(L_j)
\]

at \( p \), we have \( \tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p) \) for any \( \alpha_i \in \Gamma_i, \alpha_j \in \Gamma_j \) with \( 0 \leq i \neq j \leq k \), where

\[
\begin{align*}
\Gamma_0 &= \{1, \ldots, n_1\}, \\
\cdots \cdots \\
\Gamma_{k-1} &= \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}, \\
\Gamma_k &= \{n_1 + \cdots + n_k + 1, \ldots, n\}.
\end{align*}
\]

5 Special cases of fundamental inequalities

5.1 Submanifolds in real, complex and quaternionic space forms

The following results are special cases of Theorem 4.1.
Theorem 5.1. [40, 54] For each \( k \)-tuple \( (n_1, \ldots, n_k) \in \mathcal{S}(n) \) and for each \( n \)-dimensional submanifold \( M \) in a Riemannian space form \( R^m(\epsilon) \) of constant sectional curvature \( \epsilon \), we have

\[
\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)H^2 + b(n_1, \ldots, n_k)\epsilon. \tag{5.1}
\]

The equality case of (5.1) holds at a point \( p \in M \) if and only if there exists an orthonormal basis \( e_1, \ldots, e_m \) at \( p \) such that the shape operators of \( M \) at \( p \) take the forms (4.8) and (4.9).

In particular, for any submanifold \( M \) of a Euclidean \( m \)-space, we have the following general optimal inequality

Theorem 5.2. [40, 54] For any \( k \)-tuple \( (n_1, \ldots, n_k) \in \mathcal{S}(n) \) and any \( n \)-dimensional submanifold \( M \) of a Euclidean space \( \mathbb{E}^m \) with arbitrary codimension, we have

\[
\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)H^2. \tag{5.2}
\]

Since the sectional curvatures of a complex projective space \( CP^m(4\epsilon) \) (or quaternion projective space \( QP^m(4\epsilon) \)) satisfies \( \epsilon \leq K \leq 4\epsilon \), Theorem 4.1 implies

Theorem 5.3. Let \( M \) be an \( n \)-dimensional submanifold of the complex projective \( m \)-space \( CP^m(4\epsilon) \) of constant holomorphic sectional curvature \( 4\epsilon \) (or the quaternionic projective \( m \)-space \( QP^m(4\epsilon) \) of quaternionic sectional curvature \( 4\epsilon \)). Then we have

\[
\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + 4b(n_1, \ldots, n_k)\epsilon \tag{5.3}
\]

for any \( k \)-tuple \( (n_1, \ldots, n_k) \in \mathcal{S}(n) \).

Since the sectional curvatures of a complex hyperbolic space \( CH^m(4\epsilon) \) (or quaternion hyperbolic space \( QH^m(4\epsilon) \)) satisfies

\[
4\epsilon \leq K \leq \epsilon,
\]

Theorem 4.1 also gives the following.

Theorem 5.4. Let \( M \) be an \( n \)-dimensional submanifold of the complex hyperbolic \( m \)-space \( CH^m(4\epsilon) \) of constant holomorphic sectional curvature \( 4\epsilon \)
(or the quaternionic hyperbolic m-space $QH^m(4\epsilon)$ of quaternionic sectional curvature $4\epsilon$). Then we have
\[ \delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\epsilon \] (5.4)
for any k-tuple $(n_1, \ldots, n_k) \in S(n)$.

5.2 Submanifolds in Sasakian space forms

A $(2m+1)$-dimensional manifold is said to be almost contact if it admits a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying
\[ \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \] (5.5)
where $I$ is the identity endomorphism. It is well-known that $\phi \xi = 0$, $\eta \circ \phi = 0$.

An almost contact manifold $(\tilde{M}, \phi, \xi, \eta)$ is called an almost contact metric manifold if it admits a Riemannian metric $g$ such that
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \] (5.6)
for vector fields $X, Y$ tangent to $\tilde{M}$. Setting $Y = \xi$ we have $\eta(X) = g(X, \xi)$.

By a contact manifold we mean a $(2m+1)$-manifold $\tilde{M}$ together with a global 1-form $\eta$ satisfying
\[ \eta \wedge (d\eta)^m \neq 0 \]
on $\tilde{M}$. If $\eta$ of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is a contact form and if $\eta$ satisfies
\[ d\eta(X, Y) = g(X, \phi Y) \]
for all vectors $X, Y$ tangent to $\tilde{M}$, then $\tilde{M}$ is called a contact metric manifold.

A contact metric manifold is called $K$-contact if its characteristic vector field $\xi$ is a Killing vector field. A $K$-contact manifold is called Sasakian if we have
\[ N_\phi + 2d\eta \otimes \xi = 0, \]
where $N_\phi$ is the Nijenhuis tensor associated to $\phi$. A plane section $\sigma$ in $T_p\tilde{M}^{2m+1}$ of a Sasakian manifold $\tilde{M}^{2m+1}$ is called $\phi$-section if it is spanned by $X$ and $\phi(X)$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature with respect to a $\phi$-section $\sigma$ is called a $\phi$-sectional curvature. If a
Sasakian manifold has constant $\phi$-sectional curvature, it is called a Sasakian space form.

An $n$-dimensional submanifold $M^n$ of a Sasakian space form $\tilde{M}^{2m+1}(\epsilon)$ is called a $C$-totally real submanifold of $\tilde{M}^{2m+1}(\epsilon)$ if $\xi$ is a normal vector field on $M^n$. A direct consequence of this definition is that $\phi(TM^n) \subset T^\perp M^n$, which means that $M^n$ is an anti-invariant submanifold of $\tilde{M}^{2m+1}(\epsilon)$.

It is well-known that the Riemannian curvature tensor of a Sasakian space form $\tilde{M}^{2m+1}(\epsilon)$ of constant $\phi$-sectional curvature $\epsilon \geq 1$ is given by [12]:

\[
\tilde{R}(X,Y)Z = \epsilon + \frac{3}{4} \langle (Y,Z) X - \langle X,Z \rangle Y \rangle \\
+ \frac{\epsilon - 1}{4} \langle \eta(X)\eta(Z)Y - \eta(Y)\eta(X)X + \langle X,Z \rangle \eta(Y)\xi \rangle \\
- \langle X,Z \rangle \eta(X)\xi \langle (Y,Z) \phi X - \langle \phi X,Z \rangle \phi Y - 2 \langle \phi X,Y \rangle \phi Z \rangle
\]

(5.7)

for $X, Y, Z$ tangent to $\tilde{M}^{2m+1}(\epsilon)$. Hence if $\epsilon \geq 1$, the sectional curvature function $\tilde{K}$ of $\tilde{M}^{2m+1}(\epsilon)$ satisfies

\[
\frac{\epsilon + 3}{4} \leq \tilde{K}(X,Y) \leq \epsilon
\]

(5.8)

for $X, Y \in \ker \eta$; if $\epsilon < 1$, the inequalities are reversed.

From Theorem 4.1 and these sectional curvature properties (5.7) and (5.8) of Sasakian space forms, we obtain the following results for arbitrary Riemannian submanifolds in Sasakian space forms.

**Corollary 5.1.** If $M$ is an $n$-dimensional submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant $\phi$-sectional curvature $\epsilon \geq 1$, then, for any $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have

\[
\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\epsilon. \tag{5.9}
\]

**Corollary 5.2.** If $M$ is an $n$-dimensional submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant $\phi$-sectional curvature $\epsilon < 1$, then, for any $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have

\[
\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k). \tag{5.10}
\]

**Corollary 5.3.** If $M$ is an $n$-dimensional $C$-totally real submanifold of a Sasakian space form $\tilde{M}(\epsilon)$ of constant $\phi$-sectional curvature $\epsilon \leq 1$, then, for any $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have

\[
\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\frac{\epsilon + 3}{4}. \tag{5.11}
\]
Corollary 5.3 has been obtained in [116].

5.3 Lagrangian and totally real submanifolds in complex space forms

Since the proof of inequality (5.1) is based only on the equation of Gauss, the same inequality holds for Lagrangian submanifolds (or more generally, totally real submanifolds) in a complex space form. In fact, we have the following inequality for totally real submanifolds (see [57, 90]).

**Theorem 5.5.** Let $M$ be a totally real submanifold of a Kähler manifold $\tilde{M}^m(4\epsilon)$ of constant holomorphic sectional curvature $4\epsilon$. Then we have

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\epsilon \quad (5.12)$$

for any $k$-tuple $(n_1, \ldots, n_k) \in S(n)$.

When the totally real submanifolds are Lagrangian, we have the following result [57, 87].

**Theorem 5.6.** Let $M$ be a Lagrangian submanifold of a Kähler manifold $\tilde{M}^n(4\epsilon)$ of constant holomorphic sectional curvature $4\epsilon$. Then we have

$$\delta(n_1, \ldots, n_k)(p) \leq c(n_1, \ldots, n_k)H^2(p) + b(n_1, \ldots, n_k)\epsilon \quad (5.13)$$

for any $k$-tuple $(n_1, \ldots, n_k) \in S(n)$.

If the equality case of (5.13) holds identically on $M$, then $M$ is a minimal Lagrangian submanifold of $\tilde{M}^n(4\epsilon)$.

A Lagrangian immersion is said to have full first normal bundle if the first normal space of $M^n$ equals to the normal space at each point $p \in M^n$, i.e. $\text{Im} h = T^\perp M^n$.

In [57,], the author has determined ideal Lagrangian submanifolds in complex space forms as follows (see, also [79]) (see section 6 for the definition of ideal immersions).

**Theorem 5.7.** If $x : M^n \to \mathbb{C}^n$ is a Lagrangian immersion of a Riemannian $n$-manifold into the complex Euclidean $n$-space $\mathbb{C}^n$ with full first normal bundle, then $x$ is an ideal Lagrangian immersion if and only if $x$ is locally the product of some minimal Lagrangian immersions with full first normal bundle.
It is known that there exist ample examples of ideal Lagrangian sub-
manifolds in complex projective and complex hyperbolic spaces. On the
contrast, we had proved the following two non-existence results in [57].

**Theorem 5.8.** There do not exist ideal Lagrangian submanifolds in a com-
plex projective space with full first normal bundle.

**Theorem 5.9.** There do not exist ideal Lagrangian submanifolds in a com-
plex hyperbolic space with full first normal bundle.

A submanifold $M$ in a Riemannian manifold $N$ is called ruled if at each
point $p \in M$, $M$ contains a geodesic $\gamma_p$ of $N$ through $p$.

**Theorem 5.10.** Let $M^n$ be a Lagrangian submanifold of $\mathbb{C}^n$ such that
$\text{Im} h_p \neq T_p^\perp M^n$ at each point $p \in M^n$. If $M^n$ is ideal, then it is a ruled
minimal submanifold.

**Theorem 5.11.** Let $M^n$ be a Lagrangian submanifold of a complex space
form $\tilde{M}^n(4c)$ with $c \neq 0$. If $M^n$ is ideal, then it is a ruled minimal subman-
ifold.

6 Ideal immersions–best ways of living

The fundamental inequalities (4.7) and (5.1) give prima controls on the
most important extrinsic curvature; namely, the squared mean curvature
$H^2$, by the initial intrinsic curvatures, the $\delta$-invariants $\delta(n_1, \ldots, n_k)$, of the
Riemannian manifold.

6.1 A maximum principle

In general there do not exist direct relationship between $\delta$-invariants $\delta(n_1, \ldots, n_k)$.
On the other hand, we have the following.

**Maximum Principle.** Let $M$ be an $n$-dimensional submanifold of a Eu-
clidean $m$-space $\mathbb{E}^m$. If it satisfies the equality case of (5.2), i.e., it satisfies

$$H^2 = \Delta(n_1, \ldots, n_k)$$

(6.1)

for any $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, then

$$\Delta(n_1, \ldots, n_k) \geq \Delta(m_1, \ldots, m_j).$$

(6.2)
For any isometric immersion $x: M \to \mathbb{E}^m$ of a Riemannian $n$-manifold $M$ in $\mathbb{E}^m$. Theorem 5.1 yields

$$H^2(p) \geq \hat{\Delta}_0(p),$$

(6.3)

where $\hat{\Delta}_0$ is the Riemannian invariant on $M$ defined by

$$\hat{\Delta}_0 = \max \{ \Delta(n_1, \ldots, n_k) : (n_1, \ldots, n_k) \in S(n) \}.$$

### 6.2 Ideal immersions

Inequality (6.3) enables us to introduce the notion of ideal immersions.

**Definition 6.1.** An isometric immersion of a Riemannian $n$-manifold $M$ in $\mathbb{E}^m$ is called an ideal immersion if it satisfies the equality case of (6.3) identically.

The above maximum principle yields the following important fact:

**Theorem 6.1.** [50, 54] If an isometric immersion $x: M \to \mathbb{E}^m$ of a Riemannian $n$-manifold into $\mathbb{E}^m$ satisfies equality (6.3) for a given $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, then it is an ideal immersion automatically.

**Remark 6.1.** (Physical Interpretation of Ideal Immersions) An isometric immersion $x: M \to \mathbb{E}^m$ is an ideal immersion means that $M$ receives the least possible amount of tension (given by $\hat{\Delta}_0(p)$) from the surrounding space at each point $p$ on $M$. This is due to (6.3) and the well-known fact that the mean curvature vector field is exactly the tension field for an isometric immersion of a Riemannian manifold in another Riemannian manifold; thus the squared mean curvature at each point on the submanifold simply measures the amount of tension the submanifold is receiving from the surrounding space at that point.

For this reason, an ideal immersion is also called a best way of living.

Although a standard $n$-sphere $S^n$ does admit an ideal immersion in $\mathbb{E}^{n+1}$, the following results show that other compact rank one symmetric spaces do not admit ideal immersions in any Euclidean space.

**Proposition 6.1.** Let $FP^n (n > 1)$ denote a projective space over real, complex, or quaternion field equipped with a standard Riemannian metric, where the real dimension of $FP^n$ is $n, 2n$ or $4n$, according to $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. 
Then $FP^n$ doesn't admit ideal immersions in a Euclidean space, regardless of codimension.

**Proposition 6.2.** The Cayley plane $OP^2$ with a standard Riemannian metric does not admit ideal immersions in a Euclidean space, regardless of codimension.

### 6.3 Two world problems

A goal of research in the direction is to solve the following.

**World Problem 1.** Determine those individuals (those Riemannian manifolds) who admit a best way of living (ideal immersions) in a best world (in a space form).

**World Problem 2.** Determine the best ways of living for those individuals who admit an ideal immersion in a best world.

Here, by a “best world” we mean a space with the highest degree of homogeneity. According to work of Lie, Klein and Killing, the family of Riemannian manifolds with the highest degree of homogeneity consists of Euclidean spaces, Riemannian spheres, real projective spaces, and real hyperbolic spaces.

Such spaces have the highest degree of homogeneity simply because they have the largest groups of isometries. Hence, a best world in the terminology of differential geometry is nothing but a Riemannian space form.

### 6.4 Ideal immersions as stable critical points of total tension functional

Ideal immersions are closely related with critical point problem in the theory of total mean curvature. This can be seen as follows:

Let $M$ be a compact Riemannian manifold (with or without boundary). Denote by $I(M, R^m(\epsilon))$ the family of isometric immersions of $M$ into a real space form $R^m(\epsilon)$. For each $\phi \in I(M, R^m(\epsilon))$, we define its total tension (or the total squared mean curvature, or Willmore) functional by the formula

$$T(\phi) = \int_M H_{\phi}^2 dV,$$
where $H^2_\phi$ denotes the squared mean curvature of $\phi : M \to R^m(\epsilon)$.

It follows from Theorem 5.1 that an ideal immersion of $M$ into $R^m(\epsilon)$ is a critical point of the total tension functional within the class of $I(M, R^m(\epsilon))$ automatically. Clearly, every ideal immersion of $M$ in $R^m(\epsilon)$ is also stable, i.e., the second variation of $T(\phi)$ is nonnegative for each variation of $\phi$ in the class of $I(M, R^m(\epsilon))$.

### 6.5 Size of the smallest ball containing an ideal submanifold

According to Nash’s embedding theorem, every compact Riemannian $n$-manifold can be isometrically embedded in any small portion of Euclidean space if the codimension is large enough. In contrast the following theorem states that ideal compact submanifolds cannot be contained in a very small ball of the Euclidean space. In fact, by applying Theorem 5.1 we can estimate the radius of the smallest ball in the Euclidean space which contains a given compact ideal submanifold in terms of $\delta$-invariants.

**Theorem 6.2.** [50] Let $\phi : M \to E^m$ be an ideal immersion from a compact Riemannian $n$-manifold $M$ into a Euclidean $m$-space. Then, regardless of codimension, the radius $R$ of the smallest ball $B(R)$ containing $\phi(M)$ satisfies

$$R^2 \geq \frac{\text{vol}(M)}{\int_M \Delta_0 dV},$$

with the equality sign holding if and only if $x$ is a $1$-type ideal immersion.

### 7 Applications of $\delta$-invariants to estimates of eigenvalues of the Laplacian

#### 7.1 Type number of immersions

For an isometric immersion $x : M \to E^m$ of $M$ in $E^m$, let

$$x = x_0 + \sum_{t=p}^q x_t, \quad \Delta x_t = \lambda_t x_t$$

denote the spectral resolution of $x$, where $x_0$ is center of mass of $M$ in $E^m$ and $\Delta$ is the Laplacian of $M$. The set

$$T(x) = \{ t \in \mathbb{Z} : x_t \neq \text{constant map} \}$$
is called the order of the submanifold. The smallest element \( p \) in \( T(x) \) is called the lower order of \( x \) and the supremum \( q \) of \( T(x) \) is called the upper order of \( x \). The immersion is said to be of finite type if the upper order \( q \) is finite; and it is said to be of infinite type if the upper order \( q \) is infinite. Moreover, the immersion is said to be of \( k \)-type if \( T(x) \) contains exactly \( k \) elements.

Clearly, the immersion is of \( 1 \)-type if and only if \( p = q \). In this case, the immersion is called a 1-type immersion of order \( \{ p \} \) (see [45] for a comprehensive survey on submanifolds of finite type).

### 7.2 \( \lambda_1 \) of compact homogeneous spaces and \( \delta \)-invariants

By applying the inequalities (5.2) and the theory of finite type submanifolds [33, 41], we can establish the following new intrinsic results concerning intrinsic spectral properties of homogeneous spaces via extrinsic data.

**Theorem 7.1.** If \( M \) is a compact homogeneous Riemannian \( n \)-manifold with irreducible isotropy action, then the first nonzero eigenvalue \( \lambda_1 \) of the Laplacian on \( M \) satisfies

\[
\lambda_1 \geq n \Delta(n_1, \ldots, n_k)
\]

for any \( k \)-tuple \((n_1, \ldots, n_k) \in \mathcal{S}(n)\).

The equality sign of (7.1) holds if and only if \( M \) admits a 1-type ideal immersion in a Euclidean space.

**Remark 7.1.** If \( k = 0 \), inequality (7.1) reduces to the well-known result of T. Nagano on \( \lambda_1 \) obtained in [207]; namely

\[
\lambda_1 \geq n\rho,
\]

where \( \rho = \tau/n_2 \) is the normalized scalar curvature. In general, we have

\[
\Delta(n_1, \ldots, n_k) \geq \rho.
\]

Moreover, we have \( \Delta(n_1, \ldots, n_k) > \rho \) for \( k > 0 \) on most Riemannian manifolds.

For \( \delta \)-invariants on a compact homogeneous space, we also have the following.
Theorem 7.2. The following statements hold.

(1) A compact homogeneous Riemannian $n$-manifold $M$ with irreducible isotropy action admits an ideal immersion in a Euclidean space if and only if it satisfies $\lambda_1 = n \hat{\Delta}_0$.

(2) Every ideal immersion of a compact homogeneous Riemannian manifold with irreducible isotropy action in a Euclidean space is a 1-type immersion of order $\{1\}$.

(3) If a compact homogeneous Riemannian $n$-manifold with irreducible isotropy action admits an ideal immersion in a Euclidean space, then

$$\hat{\Delta}_0 = \Delta(n_1, \ldots, n_1)$$

for some $(n_1, \ldots, n_1) \in S(n, k)$ with $kn_1 = n$.

(4) If a compact homogeneous Riemannian $n$-manifold $M$ with irreducible isotropy action admits an ideal immersion in a Euclidean $m$-space such that the image of $M$ is contained in a hypersphere with radius $r$, then we have

$$\lambda_1 = \frac{n}{r^2}, \quad \hat{\Delta}_0 = \frac{1}{r^2}.$$ 

Remark 7.2. For a compact irreducible homogeneous $n$-manifold $M$, Theorem 7.2 can be applied to determine whether $M$ admits an ideal immersion. In principle, $\lambda_1$ (using Freudenthal’s formula for Casimir’s operator) and the invariant $\hat{\Delta}$ are both “computable” for every compact irreducible homogeneous Riemannian manifold.

For many compact irreducible symmetric spaces $M = G/H$ with $G$ being a classical group, $\lambda_1$ of $M$ has been computed by T. Nagano in [207].

Remark 7.3. Besides Riemannian spheres, there do exist other compact homogeneous Riemannian manifolds which admit ideal immersions in the Euclidean space. For instance, the following three compact homogeneous Riemannian manifolds:

$$SU(3)/T^2, \quad Sp(3)/Sp(1)^3, \quad \text{and} \quad F_4/\text{Spin}(8)$$

admit ideal immersions in $E^8$, $E^{14}$, and $E^{26}$ of codimension 2 associated with

$$(3,3) \in S(6), \quad (3,3,3,3) \in S(12), \quad \text{and} \quad (12,12) \in S(24),$$

respectively.
These ideal immersions of $SU(3)/T^2$, $Sp(3)/Sp(1)^3$, and $F_4/\text{Spin}(8)$ in $\mathbb{E}^8$, $\mathbb{E}^{14}$ and $\mathbb{E}^{26}$ arise from their isometric immersions in $S^7$, $S^{13}$ and $S^{25}$ respectively as minimal isoparametric hypersurfaces.

7.3 Estimate of eigenvalues of $\Delta$ and $\delta$-invariants

For $\delta$-invariants on a general compact Riemannian manifold, we have the following general results.

**Theorem 7.3.** Let $\phi : M \to \mathbb{E}^m$ be an isometric immersion from a compact Riemannian $n$-manifold into a Euclidean $m$-space. Then

$$\lambda_q \geq \frac{n}{\text{vol}(M)} \int_M \Delta(n_1, \ldots, n_k) dV$$

for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, where $q$ is the upper order of the immersion.

The equality case holds for a $k$-tuple $(n_1, \ldots, n_k) \in S(n)$ if and only if $\phi$ is a 1-type ideal immersion of order $\{q\}$ associated with $(n_1, \ldots, n_k)$, i.e., $\phi$ is a 1-type immersion of order $\{q\}$ satisfying

$$\Delta(n_1, \ldots, n_k) = H^2$$

identically.

**Theorem 7.4.** Let $M$ be a compact Riemannian $n$-manifold. We have

1. If $M$ admits an ideal immersion in a Euclidean space associated with a $k$-tuple $(n_1, \ldots, n_k)$, then

$$\lambda_1 \leq \frac{n}{\text{vol}(M)} \int_M \Delta(n_1, \ldots, n_k) dV.$$  \hspace{1cm} (7.3)

2. If $M$ satisfies

$$\lambda_p \leq \frac{n}{\text{vol}(M)} \int_M \hat{\Delta}_0 dV,$$

then every order $\{p\}$, 1-type isometric immersion of $M$ in a Euclidean space is an ideal immersion.

3. An ideal immersion of $M$ satisfies the equality case of (7.3) if and only if the immersion is a 1-type ideal immersion of order $\{1\}$. 
**Theorem 7.5.** Let $M$ be a compact Riemannian manifold which admits a 1-type isometric immersion of order $\{1\}$ in a Euclidean space. If $M$ admits an ideal immersion into some Euclidean space, then

$$\lambda_1 = \frac{n}{\text{vol}(M)} \int_M \hat{\Delta}_0 dV.$$  \hspace{1cm} (7.4)

In particular, if a compact strongly harmonic $n$-manifold admits an ideal immersion in a Euclidean space, then (7.4) holds.

By applying Theorem 7.5 we obtain the following simple obstruction to ideal immersions for compact Riemannian manifolds in terms of the $\delta$-invariant $\hat{\Delta}_0$.

**Theorem 7.6.** Let $M$ be a compact Riemannian $n$-manifold. If $M$ satisfies

$$\lambda_1 > \frac{n}{\text{vol}(M)} \int_M \hat{\Delta}_0 dV,$$  \hspace{1cm} (7.5)

then $M$ admits no ideal immersion in a Euclidean space for any codimension.

In particular, every compact Riemannian manifold with non-positive sectional curvatures admits no ideal immersion in a Euclidean space for any codimension.

For ideal immersions we have the following relationship between the $\delta$-invariant and the first two eigenvalues of the Laplacian $\Delta$ on compact Riemannian manifolds.

**Theorem 7.7.** Let $x : M \to \mathbb{E}^m$ be an ideal immersion of a compact Riemannian $n$-manifold in Euclidean $m$-space $\mathbb{E}^m$. Then

$$\int_M \hat{\Delta}_0 dV \geq \frac{1}{n^2} \left\{ n(\lambda_1 + \lambda_2) - R^2 \lambda_1 \lambda_2 \right\} \text{vol}(M),$$  \hspace{1cm} (7.6)

where $R$ denotes the radius of the smallest ball $B(R)$ in $\mathbb{E}^m$ which contains the image of $M$.

The equality sign of (7.6) holds if and only if the image of $M$ is contained in the boundary $S^{m-1}$ of the ball $B(R)$ and the immersion $x$ is a 1-type ideal immersion of order $\{1\}$, or a 1-type ideal immersion of order $\{2\}$, or a 2-type ideal immersion of order $\{1, 2\}$.

Moreover, if the equality case of (7.6) holds, then $M$ is mass-symmetric in $S^{m-1}$ and $\hat{\Delta}_0$ is a constant on $M$.

For further applications of $\delta$-invariants to eigenvalues of Laplacian on Riemannian manifolds, see [50, 54, 56].
8 Applications of $\delta$-invariants to minimal immersions

Since the fundamental inequalities provide us the simplest relationship between the $\delta$-invariants and the squared mean curvature, one important immediate application of the $\delta$-invariants and the fundamental inequalities is to provide many solutions to Problem 1 proposed by S. S. Chern.

**Theorem 8.1.** Let $M$ be a Riemannian $n$-manifold. If there exists a point $p \in M$ and a $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that

$$\delta(n_1, \ldots, n_k)(p) > \frac{1}{2}n(n-1)\epsilon - \frac{1}{2} \sum n_j(n_j - 1)\epsilon,$$

then $M$ admits no minimal isometric immersion into a Riemannian space form $R^m(\epsilon)$, regardless of codimension.

In particular, we have the following solution to Problem 1.

**Theorem 8.2.** Let $M$ be a Riemannian $n$-manifold. If there exists a point $p \in M$ and a $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that

$$\delta(n_1, \ldots, n_k)(p) > 0,$$

then $M$ admits no minimal isometric immersion into Euclidean space for any codimension.

**Remark 8.1.** There exist many Riemannian manifolds with scalar $\tau < 0$ but with some positive $\delta$-invariants (see, for instance [247]).

**Remark 8.2.** It is important to mention that, for each integer $n \geq 2$ and each $k$-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the condition on $\delta(n_1, \ldots, n_k)$ given in Theorem 8.2 is sharp. This can be seen as follows:

Let $f_j : M_j^{n_j} \to \mathbb{E}^{m_j},$ $j = 1, \ldots, k,$ be $k$ minimal submanifolds and $\iota$ a totally geodesic immersion of a Euclidean $(n - \sum n_j)$-space into a Euclidean space. Then the invariant $\delta(n_1, \ldots, n_k)$ of the Riemannian product $M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^{n-\sum n_j}$ vanishes identically. Clearly, the product immersion $f_1 \times \cdots \times f_k \times \iota$ is a minimal immersion.

**Corollary 8.1.** Let $M_1^{n_1}, \ldots, M_k^{n_k}$ be Riemannian manifolds of dimensions $\geq 2$ with scalar curvatures $\leq 0$. Then every minimal isometric immersion

$$f : M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^{n-\sum n_j} \to \mathbb{E}^m$$
of $M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^{n-\sum n_j}$ in any Euclidean $m$-space is a product immersion $f_1 \times \cdots \times f_k \times \iota$ of $k$ minimal immersions $f_j : M_j^{n_j} \to \mathbb{E}^{m_j}$, $j = 1, \ldots, k$, and a totally geodesic immersion $\iota$.

For a submanifold $M$ in a real space form $R^m(\epsilon)$, if we consider $\delta(2)$ on $M$, then inequality (5.1) reduces to

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2} (n+1)(n-2) \epsilon. \quad (8.1)$$

This inequality implies that if $M$ admits an isometric minimal immersion into some Euclidean space, one has

$$K \geq \tau. \quad (8.2)$$

9 Applications of $\delta$-invariants to Lagrangian and slant immersions

An immersion of a Riemannian $n$-manifold $M$ in a Hermitian manifold $\tilde{M}$ is called totally real (or isotropic in symplectic geometry) if the almost complex structure $J$ of $\tilde{M}$ maps each tangent space of $M$ into its corresponding normal space. In particular, a totally real immersion is said to be Lagrangian if $\dim M = \dim_{\mathbb{C}} \tilde{M}$.

For Lagrangian immersions in complex Euclidean $n$-space $\mathbb{C}^n$, a result of Gromov states that a compact $n$-manifold $M$ admits a Lagrangian immersion (not necessarily isometric) into $\mathbb{C}^n$ if and only if the complexification $TM \otimes \mathbb{C}$ of the tangent bundle of $M$ is trivial. Gromov’s result [141] implies that there is no topological obstruction to Lagrangian immersions for compact 3-manifolds in $\mathbb{C}^3$, because the tangent bundle of a 3-manifold is always trivial.

The class of Lagrangian immersions is included in the class of slant immersions which are defined as follows:

Let $M$ be a Riemannian manifold isometrically immersed in almost Hermitian manifold $\tilde{M}$ with almost complex structure $J$ and almost Hermitian metric $g$. For any nonzero vector $X$ tangent to $M$ at a point $p \in M$, the angle $\theta(X)$ between $JX$ and the tangent space $T_p M$ is called the Wirtinger angle of $X$. 
A submanifold $M$ of $\tilde{M}$ is called slant if the Wirtinger angle $\theta(X)$ is a constant (which is independent of the choice of $x \in M$ and of $X \in T_xN$). The Wirtinger angle of a slant submanifold is called the slant angle of the slant submanifold.

Complex submanifolds and totally real submanifolds are nothing but slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is called proper if it is neither complex nor totally real. There exist abundant examples of slant submanifolds in $\mathbb{C}^n$ (cf. [39]). The author obtained in [39] a topological obstruction to slant immersions, namely:

A compact $2k$-manifold $M$ with $H^{2i}(M; \mathbb{R}) = 0$ for some $1 \leq i \leq k$ admits no slant immersion in any Kählerian manifold $\tilde{M}^m$ unless it is totally real (or Lagrangian when $m = 2k$). Moreover, Chen and Y. Tazawa proved in [100] that there exist no slant immersions of a compact $n$-manifold in $\mathbb{C}^m$ unless it is totally real. On the other hand, there do exist compact slant submanifolds in a complex $n$-torus.

From the Riemannian point of view, it is natural to ask the following basic question.

**Problem 2** What are necessary conditions for a compact Riemannian manifold to admit a Lagrangian (or more generally, slant) immersion in $\mathbb{C}^n$?

Another important application of the $\delta$-invariants is the following solution to Problem 2.

**Theorem 9.1.** Let $M$ be a compact Riemannian $n$-manifold with null first Betti number $b_1$ or with finite fundamental group $\pi_1$. If there is a $k$-tuple $(n_1, \ldots, n_k) \in S(n)$ such that

$$\delta(n_1, \ldots, n_k) > 0,$$

(9.1)

then $M$ admits no slant isometric immersion in a complex $n$-torus $CT^n$ or in the complex Euclidean $n$-space $\mathbb{C}^n$.

In particular, if (9.1) holds for some $(n_1, \ldots, n_k) \in S(n)$, then $M$ admits no Lagrangian isometric immersion in a complex $n$-torus or in complex Euclidean $n$-space.

**Remark 9.1.** The assumptions on the finiteness of $\pi_1(M)$ and vanishing of $b_1(M)$ given in Theorem 9.1 are both necessary for $n \geq 3$. This can be seen from the following example:
Let $F : S^1 \to \mathbb{C}$ be the unit circle in the complex plane given by $F(s) = e^{is}$ and let $\iota : S^{n-1} \to E^n \ (n \geq 3)$ be the unit hypersphere in $E^n$ centered at the origin. Denote by $f : S^1 \times S^{n-1} \to \mathbb{C}^n$ the complex extensor defined by

$$f(s, p) = F(s) \otimes \iota(p), \ p \in S^{n-1}.$$  

Then $f$ is an isometric Lagrangian immersion of $M = S^1 \times S^{n-1}$ into $C^n$ which carries each pair $\{(u, p), (-u, -p)\}$ of points in $S^1 \times S^{n-1}$ to a point in $C^n$. Clearly, we have $\pi_1(M) = \mathbb{Z}$ and $b_1(M) = 1$. Moreover, for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, the $\delta$-invariant $\delta(n_1, \ldots, n_k)$ on $M$ is a positive constant.

This example shows that both the conditions on $\pi_1(M)$ and $b_1(M)$ cannot be removed.

For Lagrangian immersions in complex space forms, we also have the following two results.

**Theorem 9.2.** Let $M$ be a compact Riemannian $n$-manifold with finite fundamental group or with null first Betti number. If

$$\delta(n_1, \ldots, n_k) > \frac{1}{2} \left( n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1) \right)$$

holds for some $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, then $M$ admits no Lagrangian isometric immersion into the complex projective $n$-space $CP^n(4)$.

**Theorem 9.3.** Let $M$ be a compact Riemannian $n$-manifold either with finite fundamental group or with null first Betti number. If

$$\delta(n_1, \ldots, n_k) > \frac{1}{2} \left( \sum_{j=1}^{k} n_j(n_j - 1) - n(n-1) \right)$$

holds for some $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, then $M$ admits no Lagrangian isometric immersion into the complex hyperbolic $n$-space $CH^n(-4)$.

10 Applications to rigidity problems

Although the ideal immersions of a given Riemannian manifold in a Euclidean space is not necessarily unique, very often the $\delta$-invariants and Theorem 5.1 can be applied to obtain the rigidity for isometric immersion of arbitrary codimension from a Riemannian manifold into a Riemannian space
form; in particular, to obtain a rigidity theorem for open portions of a homogeneous Riemannian manifold isometrically immersed in a Euclidean space, regardless of codimension.

The philosophy of the rigidity comes from the fact that, for a given Riemannian manifold $M$, inequality (5.1) provides us a lower bound of the squared mean curvature. When the inequality is actually an equality, the submanifold is an ideal submanifold according to our maximum principle. In this case Theorem [5.1] shows that the shape operators of the submanifold must take a special simple form. In many cases, this information on the Riemannian structure of $M$ and on the shape operators is sufficient to conclude the rigidity of the submanifold without any global assumption and regardless of codimension.

Here we provide three of many such applications.

**Proposition 10.1.** Let $M$ be an open portion of a unit $n$-sphere $S^n(1)$. Then, for any isometric immersion of $M$ into $\mathbb{E}^m$, we have

$$H^2 \geq 1$$

regardless of codimension.

The equality case of (10.1) holds identically if and only if $M$ is immersed as an open portion of an ordinary hypersphere in an affine $(n+1)$-subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.

**Proposition 10.2.** Let $M$ be an open portion of $S^k(1) \times \mathbb{E}^{n-k}$, $k > 1$. Then, for any isometric immersion of $M$ into $\mathbb{E}^m$, we have

$$H^2 \geq \left(\frac{k}{n}\right)^2$$

regardless of codimension.

The equality case of (10.2) holds identically if and only if $M$ is immersed as an open portion of an ordinary spherical hypercylinder in an affine $(n+1)$-subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.

**Proposition 10.3.** Let $M$ be an open portion of $S^k(1) \times S^{n-k}(1), 1 < k < n - 1$. Then for any isometric immersion of $M$ into $\mathbb{E}^m$, we have

$$H^2 \geq \left(\frac{k}{n}\right)^2 + \left(\frac{n-k}{n}\right)^2$$

(10.3)
regardless of codimension.

The equality case of (10.3) holds identically if and only if $M$ is embedded in a standard way in an affine $(n+2)$-subspace of $\mathbb{E}^m$.

Theorem 5.1 and Moore’s lemma can be applied to provide some decomposition results. For instance, we have

**Proposition 10.4.** Let $M_1^{n_1}, \ldots, M_k^{n_k}$ ($k \geq 2$) be $k$ Riemannian manifolds satisfying $n_1 + \cdots + n_k \leq n$. Then, regardless of codimension, we have

1. Every isometric immersion of $M_1^{n_1} \times \cdots \times M_k^{n_k} \times H^{n-\sum n_j}(-\epsilon)$ into the hyperbolic $m$-space $H^m(-\epsilon)$, $\epsilon > 0$, satisfies

   \[ H^2 \geq \frac{b(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)} \epsilon. \]

   In particular, if

   \[ H^2 = \frac{b(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)} \epsilon \]

   identically, then the immersion is a product immersion.

2. Every minimal isometric immersion from $M_1^{n_1} \times \cdots \times M_k^{n_k} \times \mathbb{E}^{n-\sum n_j}$ into a Euclidean space is a product immersion.

Statement (2) of Proposition 10.4 was due to N. Ejiri [132].

11 Applications to warped products

Let $N_1, \ldots, N_k$ be Riemannian manifolds and let $N = N_1 \times \cdots \times N_k$ be the Cartesian product of $N_1, \ldots, N_k$. For each $i$, denote by $\pi_i : N \to N_i$ the canonical projection of $N$ onto $N_i$. When there is no confusion, we identify $N_i$ with a horizontal lift of $N_i$ in $N$ via $\pi_i$.

If $f_2, \ldots, f_k : N_1 \to \mathbb{R}_+$ are positive-valued functions, then

\[ \langle X, Y \rangle := \langle \pi_1 X, \pi_1 Y \rangle + \sum_{i=2}^{k} (f_i \circ \pi_1)^2 \langle \pi_i X, \pi_i Y \rangle \]

defines a Riemannian metric $g$ on $N$, called a multiply warped product metric. The product manifold $N$ endowed with this metric is denoted by

\[ N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k. \]
For a multiply warped product manifold $N_1 \times_f N_2 \times \cdots \times_f N_k$, let $D_i$ denote the distributions obtained from the vectors tangent to $N_i$ (or more precisely, vectors tangent to the horizontal lifts of $N_i$). Assume that

$$
\phi : N_1 \times_f N_2 \times \cdots \times_f N_k \to \tilde{M}
$$

is an isometric immersion of a multiply warped product $N_1 \times_f N_2 \times \cdots \times_f N_k$ into a Riemannian manifold $\tilde{M}$. Denote by $h$ the second fundamental form of $\phi$. Then the immersion $\phi$ is called \textit{mixed totally geodesic} if

$$
h(D_i, D_j) = \{0\}
$$

holds for distinct $i, j \in \{1, \ldots, k\}$.

Let $\phi : N_1 \times_f N_2 \times \cdots \times_f N_k \to \tilde{M}$ be an isometric immersion of a multiply warped product $N_1 \times_f N_2 \times \cdots \times_f N_k$ into an arbitrary Riemannian manifold $\tilde{M}$. Denote by $h_i$ the trace of $h$ restricted to $N_i$, that is

$$
\text{trace } h_i = \sum_{\alpha=1}^{n_i} h(e_{\alpha}, e_{\alpha})
$$

for some orthonormal frame fields $e_1, \ldots, e_{n_i}$ of $D_i$.

By considering a special $\delta$-invariant, we have the following general optimal result for any isometric immersion of a warped into a real space form for any codimension:

\textbf{Theorem 11.1.} \cite{64} Let $\phi : N_1 \times_f N_2 \to R^m(\epsilon)$ be an isometric immersion of a warped product into a Riemannian $m$-manifold of constant sectional curvature $\epsilon$. Then we have

$$
\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 \epsilon,
$$

(11.1)

where $n_i = \dim N_i$, $i = 1, 2$, $H^2$ is the squared mean curvature of $\phi$, and $\Delta$ is the Laplacian operator of $N_1$.

The equality sign of (11.1) holds identically if and only if $\phi : N_1 \times_f N_2 \to R^m(\epsilon)$ is a mixed totally geodesic immersion satisfying $\text{trace } h_1 = \text{trace } h_2$.

This result was extended to multiply warped products as follows.
Theorem 11.2. \[84, 105\] Let \( \phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \rightarrow \tilde{M}^m \) be an isometric immersion of a multiply warped product \( N := N_1 \times f_2 N_2 \times \cdots \times f_k N_k \) into an arbitrary Riemannian \( m \)-manifold. Then we have

\[
\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \leq \frac{n_1^2}{4} H^2 + n_1 (n - n_1) \max \tilde{K}, \quad n = \sum_{j=1}^{k} n_j, \tag{11.2}
\]

where \( \max \tilde{K}(p) \) denotes the maximum of the sectional curvature function of \( \tilde{M}^m \) restricted to 2-plane sections of the tangent space \( T_pN \) of \( N \) at \( p = (p_1, \ldots, p_k) \).

The equality sign of (11.2) holds identically if and only if the following two statements hold:

1. \( \phi \) is a mixed totally geodesic immersion satisfying \( \text{trace } h_1 = \cdots = \text{trace } h_k \);

2. At each point \( p \in N \), the sectional curvature function \( \tilde{K} \) of \( \tilde{M}^m \) satisfies \( \tilde{K}(u,v) = \max \tilde{K}(p) \) for each unit vector \( u \) in \( T_{p_1}(N_1) \) and each unit vector \( v \) in \( T_{(p_2, \ldots, p_k)}(N_2 \times \cdots \times N_k) \).

The following example shows that inequality (11.2) is sharp.

Example 11.1. Let \( M_1 \times \rho_2 M_2 \times \cdots \times \rho_k M_k \) be a multiply warped product representation of a real space form \( R^m(\epsilon) \). Assume that \( \psi_1 : N_1 \rightarrow M_1 \) is a minimal immersion of \( N_1 \) into \( M_1 \) and let \( f_2, \ldots, f_k \) be the restriction of \( \rho_2, \ldots, \rho_k \) on \( N_1 \). Then the following warped product immersion:

\[
\psi = (\psi_1, \text{id}, \ldots, \text{id}) : N_1 \times f_2 M_2 \times \cdots \times f_k M_k \rightarrow M_1 \times \rho_2 M_2 \times \cdots \times \rho_k M_k \subset R^m(\epsilon)
\]

is a mixed totally geodesic warped product submanifold of \( R^m(\epsilon) \) which satisfies the condition:

\[
\text{trace } h_1 = \cdots = \text{trace } h_k = 0. \tag{11.3}
\]

Thus, the immersion \( \psi \) satisfies the equality case of (11.2) according to Theorem 11.2. Therefore, inequality (11.2) is optimal.

As an immediate consequence of Theorem 11.2, we have the following.

Corollary 11.1. \[84, 105\] Let \( \phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \rightarrow R^m(\epsilon) \) be an isometric immersion of a multiply warped product \( N_1 \times f_2 N_2 \times \cdots \times f_k N_k \) into
a Riemannian $m$-manifold $R^m(\epsilon)$ of constant curvature $\epsilon$. Then we have
\[
\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \leq \frac{n^2}{4} \epsilon^2 + n_1(n - n_1)\epsilon, \quad n = \sum_{j=1}^{k} n_j.
\] (11.4)

The equality sign of (11.4) holds identically if and only if $\phi$ is a mixed totally geodesic immersion satisfying $\text{trace } h_1 = \cdots = \text{trace } h_k$.

Combining Theorem 11.1 and Nölker’s theorem gives immediately the following.

**Corollary 11.2.** Let $\phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \to R^m(\epsilon)$ be an isometric immersion of the multiply warped product $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ into a Riemannian $m$-manifold of constant curvature $\epsilon$. If we have
\[
\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} = \frac{n^2}{4} \epsilon^2 + n_1(n - n_1)\epsilon,
\]
then $\phi$ is a warped product immersion.

By applying Theorem 11.1 we have the following.

**Corollary 11.3.** If $N_1$ is a compact Riemannian manifold, then every warped product $N_1 \times f_2 N_2$ does not admit an isometric minimal immersion into a Euclidean space or a hyperbolic space for any codimension.

**Corollary 11.4.** If $f_2, \ldots, f_k$ are harmonic functions on $N_1$ or eigenfunctions of the Laplacian $\Delta$ on $N_1$ with positive eigenvalues, then the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ cannot be isometrically immersed into any Riemannian manifold of negative sectional curvature as a minimal submanifold.

**Corollary 11.5.** If $f_2, \ldots, f_k$ are eigenfunctions of the Laplacian $\Delta$ on $N_1$ with nonnegative eigenvalues and at least one of $f_2, \ldots, f_k$ is non-harmonic, then the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ cannot be isometrically immersed into any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.

By applying Theorem 11.1 we also have the following.
Corollary 11.6. If $f_2, \ldots, f_k$ are harmonic functions on $N_1$, then every isometric minimal immersion of the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times f_k N_k$ into a Euclidean space is a warped product immersion.

Since the proof of Theorem 11.1 bases only on the equation of Gauss, the same proof as Theorem 11.1 yields the following.

Corollary 11.7. Let $\phi : N_1 \times f_2 N_2 \times \cdots \times f_k N_k \rightarrow \tilde{M}^m(4\epsilon)$ be a totally real isometric immersion of the multiply warped product manifold $N_1 \times f_2 N_2 \times \cdots \times \sigma_k N_k$ into a complex space form of constant holomorphic sectional curvature $4\epsilon$ (or a quaternionic space form of constant quaternionic sectional curvature $4\epsilon$). Then we have

$$\sum_{j=2}^{k} n_j \Delta f_j \leq \frac{n^2}{4} H^2 + n_1(n - n_1)\epsilon, \quad n = \sum_{j=0}^{k} n_j.$$

Remark 11.1. In view of Corollary 11.3, it is interesting to point out that there do exist many isometric minimal immersions from $N_1 \times f N_2$ into Euclidean space with compact $N_2$. For example, a hypercatenoid in $\mathbb{E}^{n+1}$ is a minimal hypersurface which is isometric to a warped product $R \times f S^{n-1}$. Also, for any compact minimal submanifold $N_2$ of $S^{m-1} \subset \mathbb{E}^m$, the minimal cone $C(N_2)$ is a warped product $R_+ \times s N_2$ which is also a such example.

Remark 11.2. In view of Corollary 11.4, it is interesting to point out that there exist many minimal submanifolds in Euclidean space which are warped products with harmonic warping function.

For example, if $N_2$ is a minimal submanifold of the unit $(m-1)$-sphere $S^{m-1} \subset \mathbb{E}^m$, the minimal cone $C(N_2)$ over $N_2$ with vertex at the origin of $\mathbb{E}^m$ is the warped product $R_+ \times_s N_2$ whose warping function $f = s$ is a harmonic function. (Here $s$ is the coordinate function of the positive real line $R_+$).

Remark 11.3. In view of Corollary 11.4, it is interesting to point out that there exist isometric minimal immersions from warped products $N_1 \times f N_2$ into a hyperbolic space such that the warping function $f$ is an eigenfunction with negative eigenvalue. For example, $\mathbb{R} \times e_x \mathbb{E}^{n-1}$ admits an isometric minimal immersion into the hyperbolic space $H^{n+1}(-1)$ of constant sectional curvature $-1$. 
Remark 11.4. In contrast to Euclidean and hyperbolic spaces, the standard $m$-sphere $S^m$ admits warped product minimal submanifolds $N_1 \times_f N_2$ such that $N_1, N_2$ are both compact. The simplest such examples are minimal Clifford tori $M_{k,n-k}$, $k = 2, \ldots, n-1$, in $S^{n+1}$ defined by

$$M_{k,n-k} = S^k \left( \sqrt{\frac{k}{n}} \right) \times S^{n-k} \left( \sqrt{\frac{n-k}{n}} \right).$$

12 Growth estimates of warping functions

In this section, we provide some growth estimates of the warping function $f$ of the warped product $N_1 \times_f N_2$ given in the last section. By applying these, we know in particular that when the warping function $f$ is an $L^q$ function on a complete noncompact Riemannian manifold $N_1$ for some $q > 1$, then for any Riemannian manifold $N_2$ the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any Riemannian manifold with non-positive sectional curvature.

To see this, let us assume that $N_1$ is a complete noncompact Riemannian manifold and $B(x_0; r)$ denotes the geodesic ball of radius $r$ centered at $x_0 \in N_1$.

We recall some definitions from [105, 238]

**Definition 12.1.** A function $f$ on $N_1$ is said to have $p$-finite growth (or, simply, is $p$-finite) if there exists $x_0 \in N_1$ such that

$$\lim_{r \to \infty} \inf \frac{1}{r^p} \int_{B(x_0; r)} |f|^q \, dv < \infty; \quad (12.1)$$

it has $p$-infinite growth (or, simply, is $p$-infinite) otherwise.

**Definition 12.2.** A function $f$ has $p$-mild growth (or, simply, is $p$-mild) if there exists $x_0 \in N_1$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=l_0}^{\infty} \left( \frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1}) \setminus B(x_0;r_j)} |f|^q \, dv} \right)^\frac{1}{p} = \infty; \quad (12.2)$$

and has $p$-severe growth (or, simply, is $p$-severe) otherwise.
Definition 12.3. A function $f$ has $p$-obtuse growth (or, simply, is $p$-obtuse) if there exists $x_0 \in N_1$ such that for every $a > 0$, we have
\[
\int_a^\infty \left( \frac{1}{\int_{\partial B(x_0,r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty; \tag{12.3}
\]
and has $p$-acute growth (or, simply, is $p$-acute) otherwise.

Definition 12.4. A function $f$ has $p$-moderate growth (or, simply, is $p$-moderate) if there exist $x_0 \in N_1$ and
\[
F(r) \in \mathcal{F} = \left\{ F : [a, \infty) \to (0, \infty) : \int_a^\infty \frac{dr}{rF(r)} = +\infty \text{ for some } a \geq 0 \right\},
\]
such that
\[
\limsup_{r \to \infty} \frac{1}{r^{pF_p^{-1}(r)}} \int_{B(x_0,r)} |f|^q dv < \infty. \tag{12.4}
\]
And it has $p$-immoderate growth (or, simply, is $p$-immoderate) otherwise.
(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

Definition 12.5. A function $f$ has $p$-small growth (or, simply, is $p$-small) if there exists $x_0 \in N_1$, such that for every $a > 0$, we have
\[
\int_a^\infty \left( \frac{r}{\int_{B(x_0,r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty; \tag{12.5}
\]
and has $p$-large growth (or, simply, is $p$-large) otherwise.

The above definitions of “$p$-finite, $p$-mild, $p$-obtuse, $p$-moderate, $p$-small” and their counter-parts “$p$-infinite, $p$-severe, $p$-acute, $p$-immoderate, $p$-large” growth depend on $q$, and $q$ will be specified in the context in which the definition is used.

From now on, we assume that $N_1$ is a complete noncompact Riemannian $n_1$-manifold and $f$ is a $C^2$-function on $N_1$. Denote by $M^m_\epsilon$ a Riemannian $m$-manifold with sectional curvatures $\hat{K} \leq \epsilon$ for some real number $\epsilon \leq 0$.

In [103], the author and S. W. Wei study the growth of warping functions for the warping functions of warped products. They obtain the following results.
**Theorem 12.1.** If $f$ is nonconstant and 2-finite for some $q > 1$, then for any Riemannian $n_2$-manifold $N_2$ and any isometric immersion $\phi$ of the warped product $N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}_m^\epsilon$ with $\epsilon \leq 0$, the mean curvature $H$ of $\phi$ satisfies

$$H^2 > \frac{4n_1n_2|\epsilon|}{(n_1 + n_2)^2}$$

(12.6)

at some point.

**Corollary 12.1.** Suppose the squared mean curvature of the isometric immersion $\phi : N_1 \times f N_2 \to \tilde{M}_m^\epsilon$ satisfies

$$H^2 \leq \frac{4n_1n_2|\epsilon|}{(n_1 + n_2)^2}$$

(12.7)

everywhere on $N_1 \times f N_2$. Then the warping function $f$ is either a constant or for every $q > 1$, $f$ has 2-infinite growth, i.e., for every $x_0 \in N_1$,

$$\lim_{r \to \infty} \frac{1}{r^2} \int_{B(x_0;r)} |f|^q dv = \infty.$$  

(12.8)

**Theorem 12.2.** If $f$ is nonconstant and 2-mild for some $q > 1$, then for any isometric immersion of $N_1 \times f N_2$ into a Riemannian manifold $\tilde{M}_m^\epsilon$ with $\epsilon \leq 0$ we have (12.6) at some point.

**Corollary 12.2.** Suppose that the squared mean curvature of the isometric immersion $\phi : N_1 \times f N_2 \to \tilde{M}_m^\epsilon$ satisfies (12.7) everywhere on $N_1 \times f N_2$. Then the warping function $f$ is either a constant or has 2-severe growth for every $q > 1$.

**Theorem 12.3.** If $f$ is nonconstant and 2-obtuse for some $q > 1$, then for any isometric immersion of $N_1 \times f N_2$ into a Riemannian manifold $\tilde{M}_m^\epsilon$ with $\epsilon \leq 0$ we have (12.6) at some point.

**Corollary 12.3.** Suppose the squared mean curvature of the isometric immersion $\phi : N_1 \times f N_2 \to \tilde{M}_m^\epsilon$ satisfies (12.7) everywhere on $N_1 \times f N_2$. Then the warping function $f$ is either a constant or has 2-acute growth for every $q > 1$. 
Theorem 12.4. If \( f \) is nonconstant and 2-moderate for some \( q > 1 \), then for any isometric immersion of \( N_1 \times fN_2 \) into a Riemannian manifold \( \tilde{M}^m_\epsilon \) with \( \epsilon \leq 0 \) we have \((12.6)\) at some point.

Corollary 12.4. Suppose the squared mean curvature of the isometric immersion \( \phi : N_1 \times fN_2 \to \tilde{M}^m_\epsilon \) satisfies \((12.7)\) everywhere on \( N_1 \times fN_2 \). Then the warping function \( f \) is either a constant or has 2-immoderate growth for every \( q > 1 \).

Theorem 12.5. If \( f \) is nonconstant and 2-small for some \( q > 1 \), then for any isometric immersion of \( N_1 \times fN_2 \) into a Riemannian manifold \( \tilde{M}^m_\epsilon \) with \( \epsilon \leq 0 \) we have \((12.6)\) at some point.

Remark 12.1. The assumption on the function \( f \) in Theorems \([12.4][12.5]\) above cannot be dropped. Otherwise, we have counter-examples that violate \((12.6)\).

Corollary 12.5. Suppose that the squared mean curvature of the isometric immersion \( \phi : N_1 \times fN_2 \to \tilde{M}^m_\epsilon \) satisfies \((12.7)\) everywhere on \( N_1 \times fN_2 \). Then the warping function \( f \) is either a constant or has 2-large growth for every \( q > 1 \).

Remark 12.2. Corollaries \([12.4][12.5]\) lead to a dichotomy between constancy and “infinity” of the warping functions on complete noncompact Riemannian manifolds for isometric immersions of the warped products.

In particular, we have the following result of the author and Wei from \([105]\).

Theorem 12.6. Let \( f \) be a nonconstant, \( L^q \) function on \( N_1 \) for some \( q > 1 \), then for any isometric immersion of \( N_1 \times fN_2 \) into a Riemannian manifold \( \tilde{M}^m_\epsilon \) with \( \epsilon \leq 0 \) we have \((12.6)\) at some points.

From Theorem \([12.6]\) we have

Corollary 12.6. If \( f \) is an \( L^q \) function on \( N_1 \) for some \( q > 1 \), then for any Riemannian manifold \( N_2 \) the warped product \( N_1 \times fN_2 \) does not admit any isometric minimal immersion into any Riemannian manifold with non-positive sectional curvature.
Theorem 12.7. Suppose $q > 1$ and the warping function $f$ is one of the following: 2-finite, 2-mild, 2-obtuse, 2-moderate, and 2-small. If $N_2$ is compact, then there does not exists an isometric minimal immersion from $N_1 \times f N_2$ into any Euclidean space.

Remark 12.3. In views of the above results, it is interesting to point out that there do exist isometric minimal immersions from a warped product $N_1 \times f N_2$ into $\tilde{M}_\epsilon^n$ with $\epsilon \leq 0$ such that the warping function $f$ is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any $q > 1$.

A simple such example is the warped product $\mathbb{R} \times e^x \mathbb{E}^{n-1}$ (or $\mathbb{R} \times e^{-x} \mathbb{E}^{n-1}$) of constant sectional curvature $-1$ which can be isometrically immersed in $H^{n+1}(-1)$ as a totally geodesic (hence minimal) submanifold.

Remark 12.4. The inequality (12.6) (resp. inequality (12.7)) on $H^{2}$ as the assumption for Theorems 12.1–12.5 (resp. assumption of Corollaries 12.1–12.5) is sharp. This can be seen from the following two examples (cf. [32]):

First, let us regard the Euclidean $2k$-space $\mathbb{E}^{2k}$ as the warped product $\mathbb{E}^k \times f \mathbb{E}^k$ with a constant warping function $f$. Then $\mathbb{E}^k \times f \mathbb{E}^k$ can be isometrically immersed in $H^{2k+1}(-1)$ as a totally umbilical hypersurface with $H^2 = 1$. Since $n_1 = n_2 = k$, the right hand side of (12.7) is also equal to 1. Thus, this example satisfies the equality case of (12.7). For the case $k \leq 2$, this example also shows that the nonconstant assumption in Theorems 12.1–12.5 cannot be dropped.

The second example is the warped product $\mathbb{R} \times \cosh bx \mathbb{R}$ of constant negative curvature $-b^2$. This warped product admits an isometric immersion in $H^3(-1)$ as totally umbilical surface, for $0 < b < 1$. The squared mean curvature of the immersion satisfies

$$H^2 = 1 - b^2 < \frac{4n_1 n_2 |\epsilon|}{(n_1 + n_2)^2} = 1$$

and $H^2 \to 1$ as $b \to 0$.

The warping function $\cosh bx$ with $0 < b < 1$ is a nonconstant and non-harmonic function which is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any real number $q > 1$.

Remark 12.5. Let $\varphi$ be the function on $\mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1; \\ x^2 & \text{if } |x| > 1. \end{cases}$$
Denote by \( f \) the smooth out function of \( \varphi \) at \( \pm 1 \). Then \( f \) is a subharmonic function on \( \mathbb{R} \) which is 2-finite, 2-mild, 2-obtuse, 2-moderate, and 2-small for any \( q \leq 1 \); but it is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for \( q > 1 \).

The sectional curvature \( K \) of the warped product \( N = \mathbb{R} \times_f \mathbb{E}^{n-1} \) with this subharmonic warping function \( f \) satisfies \( K \leq 0 \). Let \( \hat{M}_0^{n+1} = \mathbb{R} \times N \) denote the Riemannian product of the real line and \( N \). Clearly, the sectional curvatures of \( \hat{M}_0^{n+1} \) is bounded above by 0 and the warped product \( N \) can be trivially isometrically imbedded in \( \hat{M}_0^{n+1} \) as a totally geodesic hypersurface.

This isometric imbedding of \( N \) in \( \hat{M}_0^{n+1} \) satisfies \( H^2 = \epsilon = 0 \), which shows that the condition “\( q > 1 \)” given in Theorems 12.1–12.5 and Corollaries 12.1–12.5 is sharp as well.

**Remark 12.6.** The assumption on the warping function \( f \) given in Theorem 12.7 cannot be dropped, since there do exist minimal hypersurfaces in \( \mathbb{E}^{n+1} \) which are isometric to some warped products \( N_1 \times_f N_2 \) with compact \( N_2 \). A simple such example is the hypercaternoid in \( \mathbb{E}^{n+1} \). The hypercaternoid is isometric to a warped product product \( \mathbb{R} \times_f S^{n-1} \) with compact \( N_2 = S^{n-1} \) whose warping function is 2-infinite, 2-severe, 2-acute, 2-immoderate and 2-large for any \( q > 1 \) according to Theorem 12.1.

### 13 A \( \delta \)-invariant and its applications to Riemannian submersions

#### 13.1 Riemannian submersions

Let \( M \) and \( B \) denote Riemannian manifolds with \( n = \dim M > \dim B = b > 0 \). A **Riemannian submersion** \( \pi : M \to B \) is a mapping of \( M \) onto \( B \) satisfying the following two axioms:

1. **(S1)** \( \pi \) has maximal rank;
2. **(S2)** the differential \( \pi_* \) preserves lengths of horizontal vectors.

For each \( p \in B \), \( \pi^{-1}(p) \) is an \((m-b)\)-dimensional submanifold of \( M \). The submanifolds \( \pi^{-1}(p), p \in B \), are called fibers. A vector on \( M \) is called **vertical** if it is tangent to fibers; and it is called **horizontal** if it is orthogonal to fibers. We use corresponding terminology for individual tangent vectors as well. Let \( \mathcal{H} \) and \( \mathcal{V} \) denote the horizontal and vertical distributions.
The simplest type of Riemannian submersions is the projection of a Riemannian product manifold on one of its factors. For such Riemannian submersions, both horizontal and vertical distributions are *totally geodesic distributions*, i.e., both distributions are completely integrable and their leaves are totally geodesic submanifolds.

A Riemannian manifold $M$ is said to admit a *non-trivial Riemannian submersion* if there exists a Riemannian submersion $\pi : M \to B$ from $M$ onto another Riemannian manifold $B$ such that the horizontal and vertical distributions of the submersion are not both totally geodesic distribution.

### 13.2 A submersion $\delta$-invariant $\tilde{\delta}_\pi$ and its applications

For a Riemannian submersion $\pi : M \to B$, we consider a $\delta$-invariant on $M$ defined by

$$\tilde{\delta}_\pi(p) = \tau(p) - \tau(H_p) - \tau(V)_p.$$  \hfill (13.1)

By applying this *submersion $\delta$-invariant*, we have the following results.

**Theorem 13.1.** \cite{75} If a Riemannian manifold admits a non-trivial Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.

If $\phi_F : F \to \mathbb{E}^{m_1}$ and $\phi_B : B \to \mathbb{E}^{m_2}$ are minimal isometric immersions of Riemannian manifolds $F$ and $B$ into Euclidean spaces, then the product immersion of $\phi_F$ and $\phi_B$ is the immersion:

$$(\phi_F, \phi_B) : F \times B \to \mathbb{E}^{m_1} \oplus \mathbb{E}^{m_2}$$  \hfill (13.2)

which carries $(q, p) \in F \times B$ to $(\phi_F(q), \phi_B(b))$. The product immersion $(\phi_F, \phi_B)$ is also a minimal isometric immersion.

**Theorem 13.2.** \cite{75} Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. If $M$ admits a minimal isometric immersion $\phi$ into a Euclidean space, then locally $M$ is the Riemannian product of a fiber $F$ and the base manifold $B$ and $\phi$ is the product immersion $(\phi_F, \phi_B)$ of some minimal isometric immersions $\phi_F : F \to \mathbb{E}^{m_1}$ and $\phi_B : B \to \mathbb{E}^{m_2}$ into some Euclidean spaces.
The proof of these two theorems is based on the following.

**Theorem 13.3.** [75] Let \( \pi : M \to B \) be a Riemannian submersion with totally geodesic fibers. Then, for any isometric immersion of \( M \) into a Riemannian \( m \)-manifold \( R^m(\epsilon) \) of constant sectional curvature \( \epsilon \), the submersion invariant \( \tilde{\delta}_\pi \) on \( M \) satisfies the following inequality:

\[
\tilde{\delta}_\pi \leq \frac{n^2}{4}H^2 + b(n - b)\epsilon.
\]  

(13.3)

If the target manifold in Theorem 13.1 is negatively curved, we have

**Corollary 13.1.** [75] If a Riemannian manifold admits a Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of negative sectional curvature as a minimal submanifold.

**Corollary 13.2.** [75] Every Riemannian manifold which admits a non-trivial Riemannian submersion with totally geodesic fibers cannot be isometrically immersed in any Hermitian symmetric space of non-compact type as a minimal submanifold.

**Remark 13.1.** The results obtained above can be applied to various very large families of Riemannian manifolds, since Riemannian submersions with totally geodesic fibers occur widely in geometry. For examples, we have:

(i) The well-known Hopf fibrations:

\( \pi : S^{2n+1} \to CP^n(4) \) and \( \pi : S^{4n+3} \to HP^n(4) \)

are Riemannian submersions with totally geodesic fibers.

(ii) Let \( \pi : M \to B \) be a Riemannian submersion with totally geodesic fibers. If \( B' \) is a submanifold of \( B \), then the restriction of \( \pi \) to \( \pi^{-1}(B') \):

\( \pi : \pi^{-1}(B') \to B' \)

is a Riemannian submersion with totally geodesic fibers. For instance, for any submanifold \( N \) of the complex projective \( n \)-space \( CP^n(4) \) of constant holomorphic sectional curvature \( 4 \), \( \pi : \pi^{-1}(N) \to N \) is a Riemannian submersion with totally geodesic fibers. For this submersion, the invariant \( \tilde{\delta}_\pi \) is given by

\[
\tilde{\delta}_\pi = ||P||^2,
\]  

(13.4)
\(\delta\)-invariants

where \(P : \mathcal{H} \to \mathcal{H}\) is the endomorphism such that \(PX\) is the projection of \(\phi X\) onto \(\mathcal{H}\), \(\phi\) being the (1,1)-tensor of the natural Sasakian structure on \(S^{2n+1}\).

(iii) If \(G\) is a Lie group equipped with a bi-invariant Riemannian metric and \(H\) is a closed subgroup, then the usual Riemannian structure on the homogeneous space \(G/H\) is characterized by the fact that the natural mapping

\[\pi : G \to G/H\]

is a Riemannian submersion.

The fibers of such a submersion are the left cosets of \(G \mod H\) which are totally geodesic. The invariant \(\delta_\pi\) is given by

\[
\delta_\pi = \frac{1}{4} \sum_{i,j=1}^{b} \sum_{b+1}^{n} \langle [e_i, e_j], e_s \rangle^2, \quad b = \dim H, \tag{13.5}
\]

where \(e_1, \ldots, e_b\) are orthonormal left-invariant horizontal vector fields and \(e_{b+1}, \ldots, e_n\) an orthonormal basis of the vertical distribution \(\mathcal{V}\).

(iv) On an oriented Riemannian 4-manifold \(N\), there exists an \(S^2\)-bundle \(Z\), called the twistor space of \(N\), whose fiber over any point \(x \in N\) consists of all almost complex structures on \(T_x N\) that are compatible with the metric and the orientation. It is known that there is one-parameter family of metrics \(g^t\) on \(Z\), making the projection \(Z \to N\) into a Riemannian submersion with totally geodesic fibers.

13.3 Riemannian submersions satisfying the equality case

In this subsection, we provide Riemannian submersions \(\pi : M \to B\) with totally geodesic fibers and the isometric immersions of \(M\) in \(S^N\) which satisfy the equality case of inequality \((13.3)\) identically. In order to do so, we recall briefly the definition of Hopf’s fibration.

Consider \(S^{2n+1}\) as the unit hypersphere in \(\mathbb{C}^{n+1}\) centered at the origin and let \(z\) be its unit outward normal. Let \(\tilde{J}\) be the natural almost complex structure on \(\mathbb{C}^{n+1}\). Then \(\tilde{J}z\) defines an integrable distribution on \(S^{2n+1}\) with totally geodesic leaves. Identifying the leaves as points we obtain the complex projective \(n\)-space \(\mathbb{C}P^n\). By taking as the horizontal distribution, the orthogonal complements to \(\tilde{J}z\) in \(TS^{2n+1}\), one can make this into a
Riemannian submersion, known as the Hopf fibration:

$$\pi_C : S^{2n+1} \to CP^n$$

with great circles as fibers.

Similarly, consider $S^{4k+3}$ as the unit hypersphere in $Q^{k+1}$ and let $z$ its unit outward normal. Let $J_1, J_2, J_3$ be the natural almost complex structures on $Q^{k+1}$ with

$$J_1J_2 = J_3, \quad J_2J_3 = J_1, \quad J_3J_1 = J_2.$$  \hfill (13.6)

Then $J_1z, J_2z, J_3z$ define an integrable distribution on $S^{4k+3}$ with totally geodesic leaves. Identifying the leaves as points we obtain the quaternionic projective $k$-space $QP^k$ which can be made into a Riemannian submersion:

$$\pi_Q : S^{4k+3} \to QP^k$$  \hfill (13.7)

by taking as the horizontal distribution, the orthogonal complements to $J_iz, i = 1, 2, 3$, in $T S^{4k+3}$. Fibers of $\pi_Q$ are totally geodesic 3-spheres in $S^{4k+3}$. The projection (13.7) is also known as the Hopf fibration.

The following result shows that there exist many Riemannian manifolds which admit Riemannian submersions with totally geodesic fibers and which also admit isometric immersions satisfying the equality case of (13.3) identically into some unit spheres.

**Theorem 13.4.** \cite{78} We have:

(a) Let $B$ be a Kähler submanifold of $CP^n$ and let $\pi_C^B : \pi_C^{-1}(B) \to B$ be the restriction of Hopf’s fibration $\pi_C$ to $\pi_C^{-1}(B) \subset S^{2n+1}$. Then the inclusion map $\iota_C : \pi_C^{-1}(B) \to S^{2n+1}$ is an isometric immersion such that

(a.1) the fibers of $\pi_C^B$ are fibers of the Hopf fibration $\pi_C$ and

(a.2) $\iota_C$ satisfies the equality case of (13.3) identically with $\epsilon = 1$ and $m = 1 + b; b = \dim\mathbb{R} B$.

(b) Let $B$ be an open portion of a totally geodesic $QP^k \subset QP^k$ and let $\pi_Q^B : \pi_Q^{-1}(B) \to B$ be the restriction of Hopf’s fibration $\pi_Q$ to $\pi_Q^{-1}(B) \subset S^{4k+3}$. Then the inclusion map $\iota_Q : \pi_Q^{-1}(B) \to S^{4k+3}$ is an isometric immersion such that

(b.1) the fibers of $\pi_Q^B$ are fibers of Hopf’s fibration $\pi_Q$ and

(b.2) $\iota_Q$ satisfies the equality case of (13.3) identically with $\epsilon = 1$ and $m = 3 + b; b = \dim\mathbb{R} B$.\hfill \Box
Let $\pi : M \to B$ and $\pi' : M' \to B'$ be two Riemannian submersions with totally geodesic fibers. Then $\pi$ and $\pi'$ are said to be equivalent provided that there exists an isometry $f : M \to M'$ which induces an isometry $f_B : B \to B'$ so that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
\downarrow & & \downarrow \\
B & \xrightarrow{f_B} & B'.
\end{array}
\]

As a converse to Theorem 13.4, we have the following.

**Theorem 13.5.** [78] Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. Then we have:

(i) If $M$ admits an isometric imbedding $\phi : M \to S^{2n+1}$ which carries fibers of $\pi$ to fibers of $\pi_C : S^{2n+1} \to CP^n$ and $\phi$ satisfies the equality sign of (13.3), then there exists a Kähler submanifold $B_1 \subset CP^n$ such that $\pi : M \to B$ is equivalent to $\pi_C : \pi_C^{-1}(B_1) \to B_1$ and $\phi$ is congruent to the inclusion map:

\[
\iota_C : \pi_C^{-1}(B_1) \to S^{2n+1}.
\]

(ii) If $M$ admits an isometric imbedding $\phi : M \to S^{4k+3}$ which carries fibers of $\pi$ to fibers of $\pi_Q : S^{4k+3} \to QP^k$ and $\phi$ satisfies the equality sign of (13.3), then there exists an open portion $B_2$ of some totally geodesic $QP^k \subset QP^k$ such that $\pi : M \to B$ is equivalent to $\pi_Q : \pi_Q^{-1}(B_2) \to B_2$ and $\phi$ is congruent to the inclusion map:

\[
\iota_Q : \pi_Q^{-1}(B_2) \to S^{4k+3}.
\]

Let $\pi : M \to B$ be a Riemannian submersion. An immersion $\phi : M \to S^N$ is called mixed-totally geodesic if its second fundamental form $h$ satisfies

\[
h(X, V) = 0
\]

for any horizontal vector $X$ and vertical vector $V$ on $M$.

In view of Theorem 13.5 we mention the following.

**Theorem 13.6.** [78] Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. If an isometric immersion $\phi : M \to S^N$ carries the
fibers of \( \pi \) to totally geodesic submanifolds of \( S^N \) and satisfies the equality case of (13.3) with \( \epsilon = 1 \), then we have

1. \( \dim M \geq 3 \) and
2. \( M \) is immersed as a minimal mixed-totally geodesic submanifold of \( S^N \).

13.4 Characterization of Cartan hypersurfaces in terms of \( \delta_\pi \)

The Cartan hypersurface in \( S^4 \subset \mathbb{R}^5 \) is defined by the equation:

\[
2x_5^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 + 6\sqrt{3}x_1x_2x_3 = 2.
\]

\( \acute{E}. \) Cartan proved that this hypersurface is the homogeneous Riemannian manifold \( SU(3)/SO(3) \) (equipped with a suitable metric) and its principal curvatures in \( S^4 \) are given by

\[ 0, \sqrt{3}, -\sqrt{3}. \]

It is also known that the Cartan hypersurface is a tubular hypersurface about the Veronese surface with radius \( r = \frac{\pi}{2} \).

The next theorem classifies hypersurfaces in \( S^4 \) satisfying the hypothesis of Theorem 13.6. This result provides us a new characterization of the Cartan hypersurface in terms of the submersion invariant \( \delta_\pi \).

**Theorem 13.7.** \([78]\) Let \( \pi : M^3 \to B \) be a Riemannian submersion with totally geodesic fibers. If \( \phi : M^3 \to S^4 \) is a non-totally geodesic isometric immersion which carries fibers of the submersion \( \pi \) to totally geodesic submanifolds of \( S^4 \), then \( \phi \) satisfies the equality case of (13.3) (with \( \epsilon = 1 \)) if and only if \( \phi \) is congruent to the Cartan hypersurface.

13.5 A canonical cohomology class for Riemannian submersions

There is a canonical cohomology class, denoted by \( c_\pi(M) \), associated with each Riemannian submersion \( \pi : M \to B \) with orientable base manifold \( B \) as follows.

Let \( b = \dim B, n = \dim M \), and let \( e_1, \ldots, e_n \) be a local orthonormal frame on \( M \) which satisfies the following two conditions:
(i) $e_{b+1}, \ldots, e_n$ are vertical vector fields and 
(ii) $e_1, \ldots, e_b$ are basic horizontal vector fields such that $(e_1)^* \ldots, (e_b)^*$ gives rise to the positive orientation of $B$.

Let $\omega^1, \ldots, \omega^n$ be the dual frame of $e_1, \ldots, e_n$. Consider the $b$-form $\omega$ on $M$ defined by 

$$\omega = \omega^1 \wedge \cdots \wedge \omega^b.$$  

Then we have $d\omega = 0$, since $\omega$ is the pull back of the volume form of $B$. Thus $\omega$ defines a cohomology class: 

$$c_\pi(M) = [\omega] \in H^b(M; \mathbb{R}).$$

By applying this cohomology class, the author proves the following result.

**Theorem 13.8.** [75] Let $b = \dim B$ and $\pi : M \to B$ be a Riemannian submersion with minimal fibers and orientable base manifold $B$. If $M$ is a closed manifold with $H^b(M; \mathbb{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is never integrable. Thus the submersion is always non-trivial.

Since each nonzero harmonic form represents a non-trivial cohomology class, Theorem 13.8 follows from the following.

**Theorem 13.9.** [75] Let $\pi : M \to B$ be a Riemannian submersion from a closed manifold $M$ onto an orientable base manifold $B$. Then the pull back of the volume element of $B$ is harmonic if and only if the horizontal distribution $\mathcal{H}$ is integrable and fibers are minimal.

### 14 A $\delta$-invariants and its applications to Einstein manifolds

Let $p$ be a point in a Riemannian $n$-manifold $M$ and $q$ a natural number $\leq n/2$. For a given point $p \in M$, let $\pi_1, \ldots, \pi_q$ be $q$ mutually orthogonal plane sections in $T_pM$. Define the invariant $K^\inf_q(p)$ to be the infinimum of the average of the sectional curvatures $K(\pi_1), \ldots, K(\pi_q)$, i.e., 

$$K^\inf_q(p) = \inf_{\pi_1 \perp \cdots \perp \pi_q} \frac{K(\pi_1) + \cdots + K(\pi_q)}{q},$$

(14.1)
where \( \pi_1, \ldots, \pi_q \) run over all mutually orthogonal \( q \) plane sections in \( T_pM \).

For a natural number \( q \leq n/2 \), we define a special \( \delta \)-invariant \( \hat{\delta}_{Ric}^q \) by

\[
\hat{\delta}_{Ric}^q(p) = \sup_{X \in T^1_p M} \text{Ric}(X, X) - \frac{2q}{n} K^\inf_q(p),
\]

(14.2)

where \( n = \text{dim} M \) and \( X \) runs over vectors in \( T^1_p M := \{ X \in T_p M : |X| = 1 \} \).

Recall that a submanifold \( M \) of a Riemannian manifold is called pseudo-umbilical if its mean curvature vector \( H \) is nonzero and its shape operator \( A_H \) at the mean curvature vector is proportional to the identity map (cf. [32]).

For the family of Einstein manifolds, we have the following results from [72].

**Theorem 14.1.** For any integer \( k \geq 2 \) and any isometric immersion of an Einstein \( 2k \)-manifold \( M \) into \( R^m(\epsilon) \) with arbitrary codimension, we have

\[
\hat{\delta}_{Ric}^k \leq 2(k - 1) \left( H^2 + \epsilon \right).
\]

(14.3)

The equality sign of (14.3) holds identically if and only if one of the following two cases occurs:

1. \( M \) is a minimal Einstein submanifold such that, with respect to some suitable orthonormal frame \( \{ e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_m \} \), we have

\[
A_r = \begin{pmatrix}
A_1^r & 0 \\
\vdots & \ddots \\
0 & A_k^r
\end{pmatrix}, \quad r = 2k + 1, \ldots, m,
\]

where \( A_j^r, j = 1, \ldots, k \), are symmetric \( 2 \times 2 \) submatrices satisfying

\[
\text{trace} (A_1^r) = \cdots = \text{trace} (A_k^r) = 0.
\]

2. \( M \) is a pseudo-umbilical Einstein submanifold such that, with respect to some suitable orthonormal frame \( \{ e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_m \} \), we have

\[
A_r = \begin{pmatrix}
A_1^r & 0 \\
\vdots & \ddots \\
0 & A_k^r
\end{pmatrix}, \quad r = 2k + 2, \ldots, m,
\]
where $A_j^k$, $j = 1, \ldots, k$, are symmetric $2 \times 2$ submatrices satisfying
\[
\text{trace} (A_1^k) = \cdots = \text{trace} (A_k^k) = 0.
\]

Remark 14.1. The following example illustrates that inequality (14.3) does not hold for arbitrary submanifolds in general.

Example 14.1. Consider the following spherical hypercylinder:
\[
M := S^2(1) \times E^{2k-2} \subset E^{2k+1}.
\]
We have $\delta_k^Ric = 1$ and $H^2 = 1/k^2$ on $M$ which imply that
\[
\delta_k^Ric = 1 > \frac{2(k-1)}{k^2} = 2(k-1)H^2
\]
for $k \geq 2$.

Theorem 14.2. Let $\phi : M \to R^m(\epsilon)$ be an isometric immersion of an Einstein $n$-manifold $M$ into $R^m(\epsilon)$. Then, for every natural number $q < \frac{n}{2}$, we have
\[
\delta_q^Ric \leq \frac{n(n-q-1)}{n-q}H^2 + \left( n - 1 - \frac{2q}{n} \right) \epsilon. \tag{14.4}
\]

The equality sign of (14.4) holds identically if and only if $M$ is totally geodesic.

Remark 14.2. The next example shows that inequality (14.4) does not hold for arbitrary submanifolds in general as well.

Example 14.2. For the spherical hypercylinder: $S^{n-q}(1) \times E^q \subset E^{n+1}$, we have
\[
\delta_q^Ric = n-q-1, \quad H^2 = \frac{(n-q)^2}{n^2}
\]
for $q < n/2$ which imply that
\[
\delta_q^Ric > \frac{n(n-q-1)}{n-q}H^2.
\]

Some consequences of Theorem 14.1 and Theorem 14.2 are the following.
Corollary 14.1. If a Riemannian manifold $M$ admits an isometric immersion into a Euclidean space which satisfies
\[ \delta_q^{\text{Ric}} > \frac{n(n - q - 1)}{n - q} H^2, \quad n = \dim M, \quad (14.5) \]
for some natural number $q \leq n/2$ at some points, then $M$ is not Einstein.

This corollary applies to a large family of Riemannian manifolds. For instance, Example 14.1 and Corollary 14.1 imply immediate that $S^2 \times \mathbb{E}^{2k-2}$ is not Einstein.

Theorem 14.1 and Theorem 14.2 also imply the following.

Corollary 14.2. If an Einstein $n$-manifold satisfies
\[ \delta_q^{\text{Ric}} > \left( n - 1 - \frac{2q}{n} \right) \epsilon \quad (14.6) \]
for some natural number $q \leq n/2$ at some points, then it admits no minimal isometric immersion into $\mathbb{R}^m(\epsilon)$ regardless of codimension.

Corollary 14.3. Let $M$ be a compact Einstein $n$-manifold with finite fundamental group $\pi_1$ or with null first Betti number, i.e., $b_1 = 0$. If there is a natural number $q \leq n/2$ such that $\delta_q^{\text{Ric}} > 0$, then $M$ admits no Lagrangian isometric immersion into any complex $n$-torus or complex Euclidean $n$-space.

15 A $\delta$-invariant and its applications to conformally flat manifolds

Let $\text{Ric}$ denote the Ricci tensor of a Riemannian $n$-manifold $M$. For each $\ell$-subspace $L$ of $T_p(M)$, $p \in M$, we define the Ricci curvature $S(L)$ of $L$ as the trace of the restriction of the Ricci tensor $\text{Ric}$ on $L$, i.e.,
\[ S(L) = \text{Ric}(e_1, e_1) + \ldots + \text{Ric}(e_\ell, e_\ell) \quad (15.1) \]
for an orthonormal basis $\{e_1, \ldots, e_\ell\}$ of $L$.

For each $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$, we consider in [74] a special $\delta$-invariant $\sigma(n_1, \ldots, n_k)$ for conformally flat $n$-manifolds which is defined by
\[ \sigma(n_1, \ldots, n_k) = \tau - \inf \left\{ \frac{(n - 1) \sum_{j=1}^k (n_j - 1)}{(n - 1)(n - 2) + \sum_{j=1}^k n_j(n_j - 1)} S(L_j) \right\}, \quad (15.2) \]
where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

The following general optimal inequality for the family of conformally flat submanifolds was obtained by the author in [74].

**Theorem 15.1.** Let $M$ be a conformally flat $n$-manifold isometrically immersed in a Riemannian $m$-manifold $R^m(\epsilon)$. Then, for each $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$, we have

$$\sigma(n_1, \ldots, n_k) \leq \alpha(n_1, \ldots, n_k)H^2 + \beta(n_1, \ldots, n_k)\epsilon,$$

(15.3)

where

$$\alpha(n_1, \ldots, n_k) = \frac{n^2(n - 1)(n - 2)\left(n + k - 1 - \sum_{j=1}^{k} n_j\right)}{2\left((n - 1)(n - 2) + \sum_{j=1}^{k} n_j(n_j - 1)\right)\left(n + k - \sum_{j=1}^{k} n_j\right)},$$

$$\beta(n_1, \ldots, n_k) = \frac{(n - 1)(n - 2)(n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1))}{2(n - 1)(n - 2) + 2\sum_{j=1}^{k} n_j(n_j - 1)}.$$

The equality case of inequality (15.3) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $e_1, \ldots, e_m$ at $p$, such that the shape operators of $M$ in $R^m(\epsilon)$ at $p$ take the following forms:

$$A_r = \begin{pmatrix} A_{1}^r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{k}^r \end{pmatrix}, \quad r = n + 1, \ldots, m,$$

(15.4)

where $I$ is an identity matrix and $A_j^r$’s are symmetric $n_j \times n_j$ submatrices satisfying

$$\text{trace} (A_j^1) = \cdots = \text{trace} (A_j^k) = \mu.$$  

(15.5)

**Remark 15.1.** Inequality (15.3) does not hold for arbitrary submanifolds in general. This can be seen from the following example.

**Example 15.1.** Let

$$M := S^{n-2}(1) \times E^2 \subset E^{n+1} = E^{n-1} \times E^2.$$
denote the standard isometrically embedding of \( S^{n-2}(1) \times \mathbb{E}^2 \) in \( \mathbb{E}^{n+1} \) as a spherical hypercylinder over the unit \((n - 2)\)-sphere. If we choose \( k = 1, n_1 = 2 \), then we have \( \inf S(\pi) = 0 \), where \( \pi \) runs over all 2-planes of \( T_q M \) at a given point \( q \in M \). Hence, by (15.2), we have

\[
\sigma(2) = \tau - \frac{n - 1}{n^2 - 3n + 4} \inf S(\pi) = \frac{(n - 2)(n - 3)}{2}.
\]  

(15.6)

On the other hand, we have

\[
\alpha(2) = \frac{n^2(n - 2)^2}{2(n^2 - 3n + 4)}, \quad H^2 = \frac{(n - 2)^2}{n^2}.
\]

(15.7)

Hence, we obtain

\[
\frac{(n - 2)(n - 3)}{2} = \sigma(2) > \alpha(2)H^2 = \frac{(n - 2)^4}{2(n^2 - 3n + 4)}
\]

for \( n > 4 \), which shows that the inequality (15.3) does not hold for an arbitrary submanifold in a Euclidean space in general.

An immediate application of Theorem 15.1 is the following.

**Corollary 15.1.** If a Riemannian \( n \)-manifold \( M \) admits an isometric immersion into a Euclidean space whose \( \sigma \)-invariant satisfies

\[
\sigma(n_1, \ldots, n_k) > \alpha(n_1, \ldots, n_k)H^2
\]

at some points in \( M \) for some \((n_1, \ldots, n_k) \) in \( S(n) \), then \( M \) is not conformally flat.

For instance, by applying this corollary, we conclude from Example 15.1 that, for any \( n \geq 4 \), the Riemannian product \( S^{n-2} \times \mathbb{E}^2 \) is not conformally flat. On the other hand, it is well-known that \( S^{n-1} \times \mathbb{R} \) is a conformally flat space.

Two other consequences of Theorem 15.1 are the following obstructions to minimal and Lagrangian immersions.

**Corollary 15.2.** Let \( M \) be a conformally flat \( n \)-manifold \( M \). If there exist a \( k \)-tuple \((n_1, \ldots, n_k) \) in \( S(n) \) such that

\[
\sigma(n_1, \ldots, n_k) > 0
\]

at some points in \( M \), then \( M \) does not admit any minimal isometric immersion into a Euclidean space.
Corollary 15.3. Suppose that $M$ is a compact conformally flat $n$-manifold either with finite fundamental group $\pi_1$ or with null first betti number $b_1$. If there exists a $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$ such that the $\sigma$-invariant
$$\sigma(n_1, \ldots, n_k) > 0$$

at some points in $M$, then $M$ admits no Lagrangian isometric immersion into any complex $n$-torus or into the complex Euclidean $n$-space.

Remark 15.2. The condition on $\sigma(n_1, \ldots, n_k)$ given in Corollary 15.3 is sharp. This is illustrated by the following example.

Example 15.2. Consider the Whitney immersion $w_a : S^n \to \mathbb{C}^n$ defined by
$$w_a(y_0, y_1, \ldots, y_n) = \frac{a(1 + i y_0)}{1 + y_0^2}(y_1, \ldots, y_n), \quad a > 0,$$

with $y_0^2 + y_1^2 + \cdots + y_n^2 = 1$.

The Whitney $n$-sphere $W^n_a$ is the topological $n$-sphere $S^n$ endowed with the induced metric via (15.8). The Whitney $n$-sphere is a conformally flat space and the Whitney immersion is a Lagrangian immersion which has a unique self-intersection point at $w_a(-1, 0, \ldots, 0) = w_a(1, 0, \ldots, 0)$.

For any $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$, we have
$$\sigma(n_1, \ldots, n_k) \geq 0$$
on $W^n_a$ with respect to the induced metric. Moreover, $\sigma(n_1, \ldots, n_k) = 0$ holds only at the unique point of self-intersection. From these one may conclude that the condition on the $\sigma$-invariant in Corollary 15.3 is sharp.

Remark 15.3. Let $F : S^1 \to \mathbb{C}$ be the unit circle in the complex plane defined by $F(s) = e^{is}$ and let $\iota : S^{n-1} \to E^n$ $(n \geq 3)$ be the unit hypersphere in $E^n$ centered at the origin. Denote by
$$f : S^1 \times S^{n-1} \to \mathbb{C}^n$$
the complex extensor given by
$$f(s, p) = F(s) \otimes \iota(p), \quad p \in S^{n-1}.$$

Then $f$ defines an isometric Lagrangian immersion of the conformally flat space $M =: S^1 \times S^{n-1}$ into $\mathbb{C}^n$ which carries each pair $\{(u, p), (-u, -p)\}$
of points in $S^1 \times S^{n-1}$ to a point in $\mathbb{C}^n$. Clearly, we have $\pi_1(M) = \mathbb{Z}$ and $b_1(M) = 1$. Moreover, for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, $\sigma(n_1, \ldots, n_k)$ is a positive constant on $M$. This example illustrates that both conditions on $\pi_1(M)$ and $b_1(M)$ in Corollary 15.3 are necessary.

A Riemannian submanifold $M$ is minimal if its second fundamental form $h$ satisfies $\sum_{i=1}^{n} h(e_i, e_i) = 0$ with respect to some orthonormal frame $e_1, \ldots, e_n$. On the other hand, the notion of coordinate-minimality is introduced in [93] as follows: A Riemannian submanifold is said to be coordinate-minimal with respect to a coordinate system $\{x_1, \ldots, x_n\}$ (or with respect to $g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j$) if its second fundamental form satisfies

$$\sum_{i=1}^{n} h(\partial x_i, \partial x_i) = 0,$$

where $\partial x_i$ denotes the coordinate vector $\partial/\partial x_i$.

Although coordinate-minimal surfaces are not necessary minimal, they do share some important properties with minimal surfaces in real space forms.

**Proposition 15.1.** [93] Let $\varphi : N \to \mathbb{R}^m(\epsilon)$ be a coordinate-minimal surface with respect to a coordinate system $\{s, t\}$. Then we have:

(a) At each point $p \in N$, the second fundamental form at $p$ takes the form:

$$h(\partial s, \partial s) = -h(\partial t, \partial t) = \lambda \hat{e}_3 + \phi \hat{e}_4, \quad h(\partial s, \partial t) = \mu \hat{e}_3 \quad (15.9)$$

for some real numbers $\lambda, \mu, \phi$ and orthonormal normal vectors $\hat{e}_3, \hat{e}_4$.

(b) The Gauss curvature $G$ of $N$ satisfies $G \leq \epsilon$.

(c) $G = \epsilon$ holds identically if and only if $\varphi : N \to \mathbb{R}^m(\epsilon)$ is totally geodesic.

The following result of the author and O. J. Garay [93] completely classifies conformally flat $n$-manifolds ($n \geq 4$) in a Euclidean space which satisfy the equality case of (15.3) with $k = 1$ and $n_1 = 2$.

**Theorem 15.2.** Let $\phi : M \to \mathbb{E}^m$ be an isometric immersion of a conformally flat $n$-manifold with $n \geq 4$ into Euclidean $m$-space $\mathbb{E}^m$ with arbitrary
\[ -\text{invariants} \]

Then we have
\[
\sigma(2) \leq \frac{n^2(n-2)^2}{2(n^2-3n+4)}H^2. \quad (15.10)
\]

The equality case of (15.10) holds identically if and only if one of the following holds:

1. \( M \) is an open part of a totally geodesic \( n \)-plane.
2. \( M \) is an open part of a spherical hypercylinder \( S^{n-1} \times \mathbb{R} \) in an affine \((n+1)\)-subspace of \( \mathbb{E}^m \).
3. \( M \) is an open part of a round hypercone in an affine \((n+1)\)-subspace of \( \mathbb{E}^m \).
4. \( m \geq n + 3 \) and \( M \) is the loci of \((n-2)\)-spheres defined by
   \[
   \phi = \left( \Psi(s,t), \left( \frac{1}{4c^2} - c^2(s^2 + t^2) \right)F \right), \quad (15.11)
   \]
   where \( c \) is a positive number, \( F : S^{n-2} \rightarrow \mathbb{E}^{n-1} \) is a unit hypersphere in \( \mathbb{E}^{n-1} \), and \( \Psi : N_1 \rightarrow \mathbb{E}^{m-n+1} \) is a coordinate-minimal isometric immersion with respect to
   \[
   g_{N_1} = (1 - 4c^4s^2)ds^2 - 8c^4stdsdt + (1 - 4c^4t^2)dt^2. \quad (15.12)
   \]
5. \( m \geq n + 3 \) and \( M \) is the loci of \((n-2)\)-spheres defined by
   \[
   \phi = (P(s,t), f(s,t)F),
   \]
   where \( F \) is a unit hypersphere in \( \mathbb{E}^{n-1} \) and \( P : N_2 \rightarrow \mathbb{E}^{m-n+1} \) is a coordinate-minimal surface with respect to
   \[
   g_P = (f\Delta f + f_t^2)ds^2 - 2f_sf_tdsdt + (f\Delta f + f_s^2)dt^2,
   \]
   where \( \Delta = - (\partial_s^2 + \partial_t^2) \) and \( f \) is a positive solution of the system:
   \[
   (\Delta f)K_s = (\Delta f)_s - f_s, \quad (\Delta f)K_t = (\Delta f)_t + f_t, \quad \Delta f > 0 \quad (15.14)
   \]
with \( K = \ln(f^2\Delta \ln f) \).

Remark 15.4. It is proved in [93] that there exist many coordinate-minimal isometric immersion \( \Psi : N_1 \rightarrow \mathbb{E}^{m-n+1} \) with respect to the metric (15.12). Moreover, it is also proved in [93] that there exist coordinate-minimal isometric immersion \( PN_2 \rightarrow \mathbb{E}^{m-n+1} \) with respect to the metric (15.13) which satisfies system (15.14).
16 A $\delta$-invariant for contact manifolds and its applications

Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric $(2n + 1)$-manifold. For any natural number $k \in [2, 2n]$, we define a contact $\delta$-invariant $\delta^c(k)$ by (cf. [97])

$$\delta^c(k)(p) = \tau(p) - \inf_{L^k_\xi} \tau(L^k_\xi),$$  \hspace{1cm} (16.1)

where $L^k_\xi$ runs over all linear $k$-subspace of $T_xM$ containing $\xi$.

As application of the invariant $\delta^c(k)$ we have the following results of Chen and I. Mihai from [97].

**Theorem 16.1.** Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric $(2n + 1)$-manifold for which $\eta$ is a contact structure. Then, for any integer $k \in [n + 1, 2n]$ and any isometric immersion of $M$ into a real space form $R^m(\epsilon)$, we have:

$$\delta^c(k) \leq \frac{(2n + 1)^2(2n - k + 1)}{2(2n - k + 2)} \|H\|^2 + \frac{1}{2}(2n(2n + 1) - k(k - 1))\epsilon.$$  \hspace{1cm} (16.2)

Moreover, the equality case holds identically if and only if we have:

(1) $n + 1 \leq k \leq 2n$;

(2) With respect to some suitable orthonormal basis $\{e_1, ..., e_{2n+1}, e_{2n+2}, ..., e_m\}$ with $\xi = e_1$, the shape operator of $M$ takes the following form:

$$A_r = \begin{pmatrix} A^r_{k+1} & 0 \\ 0 & \mu_r I \end{pmatrix}, \hspace{1cm} r \in \{2n + 2, ..., m\},$$  \hspace{1cm} (16.3)

where $A^r_{k+1}$ are symmetric $(k + 1) \times (k + 1)$ submatrices satisfying

$\text{trace} \ A^r_{k+1} = \mu_r$ for $r = 2n + 2, ..., m$.

**Theorem 16.2.** If $f : M \rightarrow R^m(\epsilon)$ is an isometric immersion of a $K$-contact $(2n + 1)$-manifold $M$ into a real space form $R^m(\epsilon)$ which satisfies the equality case of (16.2) for some integer $k \in [n + 1, 2n]$, then we have $\epsilon \geq 1$.

In particular, when $\epsilon = 1$, the $K$-contact structure on $M$ is Sasakian.

**Theorem 16.3.** If a $K$-contact $(2n + 1)$-manifold $M$ admits an isometric immersion into an $m$-sphere $S^m(\epsilon)$ of constant curvature $\epsilon$ which satisfies the equality case of (16.2) with $k = 2$ identically, then we have
δ-invariants

(1) $\epsilon = n = 1$.
(2) $M$ is a Sasakian manifold of constant curvature one.
(3) The immersion is totally geodesic.

**Theorem 16.4.** If a $K$-contact $(2n + 1)$-manifold $M$ admits an isometric immersion into $S^m(1)$ which satisfies the equality case of (16.2) with $k = n + 1$, then $M$ is a minimal submanifold of $S^m(1)$.

**Theorem 16.5.** If a $K$-contact $(2n + 1)$-manifold $M$ admits an isometric immersion into $S^m(1)$ which satisfies the equality case of (16.2) with $k = 3$, then $n = 2$ and we have either

(a) $M$ is a Sasakian manifold of constant curvature one isometrically immersed in $S^m(1)$ as a totally geodesic submanifold, or

(b) $M$ is a Sasakian 5-manifold foliated by Sasakian 3-manifolds of constant curvature one and $M$ is isometrically immersed in $S^m(1)$ as a minimal submanifold of codimension at least two. Moreover, leaves of the foliation are immersed as totally geodesic submanifolds of $S^m(1)$.

**Remark 16.1.** Let $\hat{\mathcal{f}} : N \to CP^m(4)$ be a Kähler immersion of a Kähler surface $N$ into $CP^m(4)$ with relative nullity two at each point. Then the pre-image $M^5 := \pi^{-1}(N)$ of $N$ via the Hopf fibration $\pi$ is a non-totally geodesic Sasakian 5-manifold in $S^{2m+1}(1)$ which satisfies the equality case of (16.2) for $k = 3$.

**Remark 16.2.** It follows from Theorem 16.2 that there do not exist $K$-contact manifolds in Euclidean spaces which satisfies the equality case of (16.2) for any integer $k \in [n + 1, 2n]$.

In contrast, the following example shows that there do exist almost contact metric manifolds in Euclidean spaces which satisfy the equality case of (16.2) identically.

**Example 16.1.** Consider the cylindrical hypersurface

$$f : M := \mathbb{R} \times S^2(1) \to \mathbb{E}^4$$

defined by

$$f(t, \theta, \varphi) = (t, \cos \theta \cos \varphi, \sin \theta \cos \varphi, \sin \varphi), \quad (16.4)$$
where \( \mathbb{E}^4 \) is the Euclidean 4-space endowed with the flat Riemannian metric:

\[
g = dt^2 + d\varphi^2 + \cos^2 \varphi d\theta^2. \tag{16.5}
\]

Define an almost contact metric structure \((\phi, \xi, \eta, g)\) on \( M \) by

\[
\eta = \cos \theta dt + \sin \theta d\varphi,
\xi = \cos \theta \frac{\partial}{\partial t} + \sin \theta \frac{\partial}{\partial \varphi},
\phi \left( \frac{\partial}{\partial t} \right) = -\tan \theta \frac{\partial}{\partial \theta},
\phi \left( \frac{\partial}{\partial \varphi} \right) = \frac{\partial}{\partial \theta},
\phi \left( \frac{\partial}{\partial \theta} \right) = \cos \varphi \left( \sin \theta \frac{\partial}{\partial t} - \cos \theta \frac{\partial}{\partial \varphi} \right).
\]

Consider the orthonormal frame \( \{e_1, e_2, e_3\} \) on \( M \) given by

\[
e_1 = \xi, \quad e_2 = -\sin \theta \frac{\partial}{\partial t} + \cos \theta \frac{\partial}{\partial \varphi}, \quad e_3 = \sec \varphi \frac{\partial}{\partial \theta}.
\]

With respect to this frame, we have

\[
\phi e_1 = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2.
\]

It is easy to verify that \( g \) satisfies \( g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \). Moreover, we also have \( \eta \wedge d\eta = dt \wedge d\theta \wedge d\varphi \neq 0 \) on \( M \). So, \((M, \phi, \xi, \eta, g)\) is an almost contact metric manifold. It is easy to verify that

\[
\nabla_{\partial_t} \partial_t = \nabla_{\partial_t} \partial_\varphi = \nabla_{\partial_t} \partial_\theta = \nabla_{\partial_\varphi} \partial_\varphi = 0,
\nabla_{\partial_\varphi} \partial_\theta = -\tan \varphi \partial_\theta,
\nabla_{\partial_\theta} \partial_\theta = \sin \varphi \cos \varphi \partial_\varphi.
\tag{16.6}
\]

which implies \( \nabla_\xi \xi = 0 \).

This almost contact metric hypersurface \((M, \phi, \xi, \eta, g)\) in \( \mathbb{E}^4 \) is non-\( K \)-contact and it satisfies the equality case of \([16.2]\) with \( k = 2 \) identically.
17 \( \delta(2) \) and Lagrangian submanifolds

If \( k = 1 \) and \( n_1 = 2 \), inequality (5.12) reduces to

\[
\delta(2) \leq \frac{n^2(n - 2)}{2(n - 1)} H^2 + \frac{1}{2} (n + 1)(n - 2) \epsilon.
\] (17.1)

Recall that if a Lagrangian submanifold in complex space form \( \tilde{M}^n(4\epsilon) \) satisfies the equality case of (17.1), then it is minimal (cf. Theorem 5.6). Hence, we have the following equality:

\[
\delta(2) = \frac{1}{2} (n + 1)(n - 2) \epsilon.
\] (17.2)

For an \( n \)-dimensional Lagrangian submanifold in \( CP^n(4) \) satisfying (17.2), we define for every \( p \in M \) the kernel of the second fundamental form by

\[
\mathcal{D}(p) = \{ X \in T_pM \mid \forall Y \in T_pM : h(X, Y) = 0 \}.
\]

If the dimension of \( \mathcal{D}(p) \) is constant, then it follows from [40] that either \( M \) is totally geodesic or that the distribution \( \mathcal{D} \) is an \( (n - 2) \)-dimensional completely integrable distribution.

Denote by \( \mathcal{D}^\perp \) the complementary orthogonal distribution of \( \mathcal{D} \) in \( TM \). Contrast to \( \mathcal{D} \), the distribution \( \mathcal{D}^\perp \) is not necessary integrable in general.

17.1 Lagrangian submanifolds with constant scalar curvature satisfying (17.2)

To state the next result, we provide an example of minimal Lagrangian submanifold \( M \) with constant scalar curvature in \( CP^3(4) \) which satisfies equality (17.2) with \( n = 3 \) as follows.

Example 17.1. Define two complex structures on \( \mathbb{C}^4 \) by

\[
I(v_1, v_2, v_3, v_4) = (iv_1, iv_2, iv_3, iv_4)
\]
\[
J(v_1, v_2, v_3, v_4) = (-\bar{v}_4, \bar{v}_3, -\bar{v}_2, \bar{v}_1).
\]

Clearly \( I \) is the standard complex structure. The corresponding Sasakian structures on \( S^7(1) \) have characteristic vector fields \( \xi_1 = -I(x) \) and \( \xi_2 = -J(x) \). Since we consider two complex structures on \( \mathbb{C}^4 \), we can consider
two different Hopf fibrations $\pi_j : S^7(1) \to CP^3(4)$. The vector field $\xi_j$ is vertical for $\pi_j$.

Now we consider the Calabi curve $C_3$ of $CP^1$ into $CP^3(4)$ of constant Gauss curvature $4/3$, given by

$$C_3(z) = [1, \sqrt{3}z, \sqrt{3}z^2, z^3]$$

Since $C_3$ is holomorphic with respect to $I$, there is a circle bundle $\pi : M^3 \to CP^1$ and an isometric minimal immersion $I : M^3 \to S^7(1)$ such that $\pi_1(I) = C_3(\pi)$. It is known in [90] that $M$ has constant scalar curvature and $I$ is horizontal with respect to $\pi_2$ such that the immersion

$$J : M^3 \to CP^3(4),$$

defined by $J = \pi_2(I)$ is a minimal Lagrangian immersion which satisfies the equality (17.2) with $n = 3$.

For Lagrangian submanifolds with constant scalar curvature, we have

**Theorem 17.1.** [90] Let $\phi : M^n \to \tilde{M}^n(4\epsilon)$, $\epsilon \in \{-1, 0, 1\}$ and $n \geq 3$ be a Lagrangian minimal immersion with constant scalar curvature. Then $M^n$ satisfies

\[(1.2) \quad \delta_M = \frac{1}{2}(n+1)(n-2)\epsilon,\]

if and only if either

1. $M^n$ is a totally geodesic immersion, or
2. $n = 3$, $\epsilon = 1$ and $\phi$ is locally congruent to the immersion $J : M^3 \to CP^3(4)$ defined above.

The next example shows that there exists a 3-dimensional totally real submanifold in $CP^3$ with non-constant scalar curvature which satisfies the equality (17.2). From that example, it follows that the condition of constant scalar curvature in Theorem 17.1 cannot be omitted.

**Example 17.2.** Consider a totally geodesic $S^5(1)$ in $S^7(1)$. Let $N$ be a unit vector orthogonal to the linear subspace containing $S^5(1)$. Let $N^2$ be any minimal surface in $S^5(1)$, immersed by $f$. We define an isometric immersion from the warped product manifold $M^3 = (-\pi/2, \pi/2) \times_{\cos t} N^2$ into $S^7(1)$ by

$$x : M^3 \to S^7(1) : (t, p) \mapsto (\sin t)N + (\cos t)f(p).$$
This immersion is minimal and satisfies equality (17.2). In order for \( x \) to be Legendrian, we have to assume that \( S^5(1) \) is contained in a complex hyperplane \( C^3 \) of \( C^4 \supset S^7(1) \) and that \( f \) is Legendrian. It is easy to check that \( M^3 \) does not have constant scalar curvature if \( N^2 \) is not totally geodesic.

17.2 Lagrangian submanifolds of \( CP^n \) with integrable \( D^\perp \) satisfying (17.2)

Lagrangian submanifolds of \( CP^n \) with integrable \( D^\perp \) satisfying (17.2) are completely determined by the author, Dillen, Verstraelen and Vrancken in [87].

**Theorem 17.2.** Let \( \phi : M^n \longrightarrow CP^n(4) \) be a Lagrangian immersion satisfying equality (17.2) and

1. the dimension of \( D \) is constant,
2. \( D^\perp \) is an integrable distribution.

Then either \( \phi \) is totally geodesic or \( \phi \) has no totally geodesic points and, up to holomorphic transformations, \( \phi(M) \) is contained in the image under the Hopf fibration \( \pi : S^{2n+1}(1) \rightarrow CP^n(4) \) of the image of one of the immersions described in the next proposition.

**Proposition 17.1.** Let \( S^{2n+1}(1) \) be the unit hypersphere of \( C^{n+1} \) and consider the orthogonal decomposition \( C^{n+1} = C^3 \oplus J(E^{n-2}) \oplus E^{n-2} \). Let \( f : M^2 \rightarrow S^5(1) \subset C^3 \) be a minimal Legendrian immersion and consider the hypersphere \( S^{n-3}(1) \) in \( E^{n-2} \). Then

\[ F : (0, \pi/2) \times \cos t M^2 \times \sin t S^{n-3}(1) \rightarrow S^{2n+1}(1) : (t, p, q) \mapsto \cos tf(p) + \sin tq \]

is a minimal Legendrian immersion satisfying equality (17.2). Moreover, if \( f \) has no totally geodesic points, then the dimension of \( D \) is exactly \( n - 2 \).

Finally, if we extend \( F \) to a map

\[ \tilde{F} : (-\pi/2, \pi/2) \times M^2 \times S^{n-3}(1) \rightarrow S^{2n+1}(1) : (t, p, q) \mapsto \cos tf(p) + \sin tq. \]

Then \( \tilde{F} \) fails to be immersive at \( t = 0 \), but the image of \( \tilde{F} \) is an immersed minimal Legendrian submanifold. If \( f \) is not totally geodesic, then this image can not be extended further.
17.3 Improved inequality for Lagrangian submanifolds

For Lagrangian submanifolds in $M^n(4\epsilon)$, T. Oprea [216] improves inequality (17.1) to the following.

$$\delta(2) \leq \frac{n^2(2n-3)}{2n+3}H^2 + \frac{1}{2}(n+1)(n-2)\epsilon. \quad (17.3)$$

The improved inequality (17.3) was proved in [216] by using a method of optimizations. Recently, a purely algebraic proof of (17.3) is obtained in [16]. Moreover, it is also proved in [16,17] that an $n$-dimensional Lagrangian submanifold in a complex space form $M^n(4\epsilon)$ with $n \geq 4$ is minimal if the equality case of (17.3) is attained at all points. Notice that in the minimal case, inequality (17.3) reduces to (17.1).

Three-dimensional non-minimal Lagrangian submanifolds in $CP^3(4)$ which attain the equality case of (17.3) have been constructed in [21]. More precisely, J. Bolton and L. Vrancken prove the following.

**Theorem 17.3.** Let $M$ be a 3-dimensional non-minimal Lagrangian submanifold of $CP^3(4)$ which attains equality at every point in (17.3). Then there is a minimal Lagrangian surface $\tilde{W}$ in $CP^2(4)$ such that $M$ can be locally written as $[E_0]$, where

$$E_0 = \frac{e^{it/3}}{\sqrt{1 + b_1^2 + \lambda_2^2}}(0,W) + \frac{(-b_1 + i\lambda_2)}{\sqrt{1 + b_1^2 + \lambda_2^2}}(e^{it},0,0,0),$$

where $b_1$ and $\lambda_2$ are solutions of the following system of ordinary differential equations:

$$\frac{db_1}{dt} = -\frac{1 + 3\lambda_2^2 + b_1^2}{3\lambda_2}, \quad \frac{d\lambda_2}{dt} = \frac{2}{3}b_1.$$

Conversely, any 3-dimensional Lagrangian submanifold $M$ obtained in this way attains equality at each point in (17.3).

18 $\delta(2)$ and CR-submanifolds

For CR-submanifolds in complex space forms, there exists a sharp relationship between the invariant $\delta(2)$ and the squared mean curvature $H^2$:
Theorem 18.1. Let $M$ be an $n$-dimensional CR-submanifold in a complex space form $\tilde{M}^m(4\epsilon)$. Then we have

$$\delta(2) \leq \begin{cases} 
\frac{n^2(n-2)}{2(n-1)}H^2 + \left\{ \frac{1}{2}(n+1)(n-2) + 3h \right\} \epsilon, & \text{if } \epsilon > 0; \\
\frac{n^2(n-2)}{2(n-1)}H^2, & \text{if } \epsilon = 0; \\
\frac{n^2(n-2)}{2(n-1)}H^2 + \frac{1}{2}(n+1)(n-2)\epsilon, & \text{if } \epsilon < 0,
\end{cases} \tag{18.1}$$

where $h$ is the complex dimension of the holomorphic distribution.

There exist many CR-submanifolds in complex space forms which satisfy the equality cases of the above inequalities.

For instance, for Hopf hypersurfaces in $CP^m$, we have the following classification theorem.

Theorem 18.2. \[44\] Let $M$ be a Hopf hypersurface of $CP^m(4)$ ($m \geq 2$). Then $M$ satisfies

$$\delta(2) = \frac{(2m-1)^2(2m-3)}{4(m-1)}H^2 + 2m^2 - 3$$

if and only if

1. $M$ is an open part of a geodesic sphere of radius $\frac{4}{m}$ in $CP^m(4)$ or
2. $m = 2$ and $M$ is an open part of a tube over a complex quadric curve $Q_1$ with radius $r_1 = \arctan \left( \frac{1}{2} \left( 1 + \frac{\sqrt{5} - \sqrt{2} + 2\sqrt{5}}{\sqrt{2}} \right) \right)$.

By a Hopf hypersurface in $CP^m$ we mean a real hypersurface such that $J\xi$ is an eigenvector of the shape operator $A_\xi$, where $\xi$ is a unit normal vector field.

Up to rigid motions of $CH^m(-4)$, a horosphere in $CH^m(-4)$ is a real hypersurface defined by the equation

$$|z_1 - z_0| = 1. \quad (18.2)$$

For hypersurfaces in $CH^m(-4)$, we obtain the following characterization of horospheres in $CH^2(-4)$ in terms of $\delta(2)$.

Theorem 18.3. \[44\] Let $M$ be a real hypersurface of $CH^m(-4)$ ($m \geq 2$). Then

$$\delta(2) = \frac{(2m-1)^2(2m-3)}{4(m-1)}H^2 - (2m^2 - 6) \quad (18.3)$$
holds identically if and only if \( m = 2 \) and \( M \) is an open part of a horosphere in \( \mathbb{CH}^2(-4) \).

Proper \( CR \)-submanifolds of complex hyperbolic spaces satisfying the equality case of \( (18.3) \) were completely determined by the author and Vrancken [103]. More precisely, they obtained the following:

**Theorem 18.4.** Let \( U \) be a domain of \( \mathbb{C} \) and \( \Psi : U \to \mathbb{C}^{m-1} \) be a nonconstant holomorphic curve in \( \mathbb{C}^{m-1} \). Define \( z : E^2 \times U \to \mathbb{C}^{m+1}_1 \) by

\[
 z(u, t, w) = e^{it}(iu - 1 - \frac{1}{2}\Psi(w)\overline{\Psi}(w), iu - \frac{1}{2}\Psi(w)\overline{\Psi}(w), \Psi(w)) .
\] (18.4)

Then \( \langle z, z \rangle = -1 \) and the image \( z(E^2 \times U) \) in \( H^{2m+1}_1 \) is invariant under the group action of \( H^1_1 \). Moreover, away from points where \( \Psi'(w) = 0 \), the image \( \pi(E^2 \times U) \), under the projection

\[
 \pi : H^{2m+1}_1(-1) \to CH^m(-4),
\]

is a proper \( CR \)-submanifold of \( CH^m(-4) \) which satisfies

\[
 \delta_M = \frac{n^2(n-2)}{2(n-1)}H^2 + \frac{1}{2}(n+1)(n-2)c .
\] (18.5)

Conversely, up to rigid motions of \( CH^m(-4) \), every proper \( CR \)-submanifold of \( CH^m(-4) \) satisfying the equality is obtained in such way.

Theorem [18.4] can be regarded as a natural extension of Theorem [18.3].

19 \( \delta(2) \) and \( CMC \) hypersurfaces

A hypersurface in the unit round sphere \( S^{n+1}(1) \) is called isoparametric if it has constant principal curvatures. It is known that an isoparametric hypersurface in \( S^{n+1}(1) \) is either an open portion of a 3-sphere or an open portion of the product of a circle and a 2-sphere, or an open portion of a tube of constant radius over the Veronese embedding.

Because every isoparametric hypersurface in \( S^{n+1}(1) \) has constant mean curvature (CMC) and constant scalar curvature, it is an interesting problem to determine all CMC hypersurfaces with constant scalar curvature. Many results in this directions have been obtained (see, for instance, [31, 106]).
On the other hand, every isoparametric hypersurface in $S^{n+1}(1)$ or in $\mathbb{E}^{n+1}$ has constant mean curvature and constant $\delta(2)$-invariant. Thus, it is also a very natural problem to study CMC hypersurfaces with constant $\delta(2)$-invariant in a real space form $R^{n+1}(\epsilon)$.

In this respect, we mention the following results of the author and O. J. Garay proved in [92].

**Theorem 19.1.** A CMC hypersurface in the Euclidean 4-space $\mathbb{E}^4$ has constant $\delta(2)$-invariant if and only if it is one of the following:

1. An isoparametric hypersurface;
2. A minimal hypersurface with relative nullity greater than or equal to 1;
3. An open portion of a hypercylinder $N \times \mathbb{R}$ over a surface $N$ in $\mathbb{E}^3$ with constant mean curvature and nonpositive Gauss curvature.

**Theorem 19.2.** A CMC hypersurface $M$ in the unit 4-sphere $S^4(1)$ has constant $\delta(2)$-invariant if and only if one of the following two statements holds:

1. $M$ is an isoparametric hypersurface;
2. There is an open dense subset $U$ of $M$ and a non-totally geodesic isometric minimal immersion $\phi: B^2 \to S^4(1)$ from a surface $B^2$ into $S^4(1)$ such that $U$ is an open subset of $NB^2 \subset S^4(1)$, where $NB^2$ is defined by

$$N_pB^2 = \{ \xi \in T_{\phi(p)}S^4(1) : \langle \xi, \xi \rangle = 1, \langle \xi, \phi_*(T_pB^2) \rangle = 0 \}. $$

As an immediate application of Theorem 19.1, we have the following corollary for complete CMC hypersurfaces.

**Corollary 19.1.** Let $M$ be a non-minimal, complete, CMC hypersurface of the Euclidean 4-space $\mathbb{E}^4$. Then $M$ has constant $\delta(2)$-invariant if and only if $M$ is one of the following hypersurfaces:

1. An ordinary hypersphere;
2. A spherical hypercylinder: $\mathbb{R} \times S^2$;
3. A hypercylinder over a circle: $\mathbb{E}^2 \times S^1$. 
20 \ δ(2) and submanifolds of nearly Kähler $S^6$

20.1 General results of nearly Kähler $S^6$

It was proved by E. Calabi [25] in 1958 that any oriented submanifold $M^6$ of the hyperplane $\text{Im} \ O$ of the imaginary octonions carries a $U(3)$-structure (that is, an almost Hermitian structure). For instance, let $S^6 \subset \text{Im} \ O$ be the sphere of unit imaginary vectors; then the right multiplication by $u \in S^6$ induces a linear transformation

$$J_u : O \rightarrow O,$$

which is orthogonal and satisfies $(J_u)^2 = -I$. The operator $J_u$ preserves the 2-plane spanned by 1 and $u$ and therefore preserves its orthogonal 6-plane which may be identified with $T_u S^6$. Thus $J_u$ induces an almost complex structure on $T_u S^6$ which is compatible with the inner product induced by the inner product of $O$ and $S^6$ has an almost complex structure.

The almost complex structure $J$ on $S^6$ is a nearly Kähler structure in the sense that the (2,1)-tensor field $G$ on $S^6$, defined by

$$G(X, Y) = (\bar{\nabla}_X J)(Y),$$

is skew-symmetric, where $\bar{\nabla}$ denotes the Riemannian connection on $S^6$.

The group of automorphisms of this nearly Kähler structure is the exceptional simple Lie group $G_2$ which acts transitively on $S^6$ as a group of isometries.

A. Gray [140] proved the following:

1. Every almost complex submanifold of the nearly Kähler $S^6$ is a minimal submanifold, and
2. The nearly Kähler $S^6$ has no 4-dimensional almost complex submanifolds.

A 3-dimensional submanifold $M$ of the nearly Kähler $S^6$ is called Lagrangian if the almost complex structure $J$ on the nearly Kähler 6-sphere carries each tangent space $T_x M$, $x \in M$ onto the corresponding normal space $T^\perp_x M$.

N. Ejiri proved in [133] that a Lagrangian submanifold $M$ in the nearly Kähler $S^6$ is always minimal and orientable. He also proved that if $M$ has constant sectional curvature, then $M$ is either totally geodesic or has
constant curvature 1/16. The first nonhomogeneous examples of Lagrangian submanifolds in the nearly Kähler 6-sphere were described in [134].

20.2 $G_2$-equivariant Lagrangian submanifolds and their characterizations

K. Mashimo [185] classified the $G_2$-equivariant Lagrangian submanifolds $M$ of the nearly Kähler 6-sphere. It turns out that there are five models, and every equivariant Lagrangian submanifold in the nearly Kähler 6-sphere is $G_2$-congruent to one of the five models.

These five models can be distinguished by the following curvature properties:

1. $M^3$ is totally geodesic ($\delta(2) = 2$),
2. $M^3$ has constant curvature 1/16 ($\delta(2) = 1/8$),
3. the curvature of $M^3$ satisfies $1/16 \leq K \leq 21/16$ ($\delta(2) = 11/8$),
4. the curvature of $M^3$ satisfies $-7/3 \leq K \leq 1$ ($\delta(2) = 2$),
5. the curvature of $M^3$ satisfies $-1 \leq K \leq 1$ ($\delta(2) = 2$).

F. Dillen, L. Verstraelen and L. Vrancken [125] characterized models (1), (2) and (3) as the only compact Lagrangian submanifolds in $S^6$ whose sectional curvatures satisfy $K \geq 1/16$. They also obtained an explicit expression for the Lagrangian submanifold of constant curvature 1/16 in terms of harmonic homogeneous polynomials of degree 6. Using these formulas, it follows that the immersion has degree 24. Further, they also obtained an explicit expression for model (3).

It follows from the fundamental inequality in Theorem 5.1 and Ejiri’s result that the invariant

$$\delta(2) = \tau - \inf K$$

always satisfies

$$\delta(2) \leq 2,$$  \hspace{1cm} (20.2)

for every Lagrangian submanifold of the nearly Kähler $S^6$.

From above, we know that the models (1), (4) and (5) satisfy the basic equality: $\delta(2) = 2$ identically. On the other hand, Chen, F. Dillen, L. Verstraelen and L. Vrancken proved in [89] that models (1), (2) and (3)
are the only Lagrangian submanifolds of the nearly Kähler $S^6$ with constant scalar curvature that satisfy the equality $\delta(2) = 2$.

Many further examples of Lagrangian submanifolds in the nearly Kähler $S^6$ satisfying the equality $\delta(2) = 2$ have been constructed in [88, 89].

Recall that a Riemannian $n$-manifold $M$ whose Ricci tensor has an eigenvalue of multiplicity at least $n - 1$ is called quasi-Einstein.

R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken [119] proved that Lagrangian submanifolds of the nearly Kähler 6-sphere satisfying $\delta(2) = 2$ are always quasi-Einstein.

### 20.3 Classification of Lagrangian submanifolds in the nearly Kähler $S^6$

The complete classification of Lagrangian submanifolds in the nearly Kähler 6-sphere satisfying $\delta(2) = 2$ was established by Dillen and Vrancken in [127]. More precisely, they proved the following:

1. Let $\phi : N_1 \to CP^2(4)$ be a holomorphic curve in $CP^2(4)$, $PN_1$ the circle bundle over $N_1$ induced by the Hopf fibration $\pi : S^5(1) \to CP^2(4)$, and $\psi$ the isometric immersion such that the following diagram commutes:

   $$
   \begin{array}{ccc}
   PN_1 & \xrightarrow{\psi} & S^5 \\
   \downarrow & & \downarrow \pi \\
   N_1 & \xrightarrow{\phi} & CP^2(4).
   \end{array}
   $$

   Then, there exists a totally geodesic embedding $i$ of $S^5$ into the nearly Kähler 6-sphere such that the immersion $i \circ \psi : PN_1 \to S^6$ is a 3-dimensional Lagrangian immersion in $S^6$ satisfying equality $\delta(2) = 2$.

2. Let $\bar{\phi} : N_2 \to S^6$ be an almost complex curve (with second fundamental form $h$) without totally geodesic points. Denote by $UN_2$ the unit tangent bundle over $N_2$ and define a map

   $$
   \bar{\psi} : UN_2 \to S^6 : v \mapsto \bar{\phi}_*(v) \times \frac{h(v,v)}{\|h(v,v)\|} \quad (20.3)
   $$

   Then $\bar{\psi}$ is a (possibly branched) Lagrangian immersion into $S^6$ satisfying equality $\delta(2) = 2$. Moreover, the immersion is linearly full in $S^6$. 

(3) Let $\tilde{\phi} : N_2 \to S^6$ be a (branched) almost complex immersion. Then, $SN_2$ is a 3-dimensional (possibly branched) Lagrangian submanifold of $S^6$ satisfying equality $\delta(2) = 2$.

(4) Let $f : M \to S^6$ be a Lagrangian immersion which is not linearly full in $S^6$. Then $M$ automatically satisfies equality $\delta(2) = 2$ and there exists a totally geodesic $S^5$, and a holomorphic immersion $\phi : N_1 \to CP^2(4)$ such that $f$ is congruent to $\psi$, which is obtained from $\phi$ as in (1).

(5) Let $f : M \to S^6$ be a linearly full Lagrangian immersion of a 3-dimensional manifold satisfying equality $\delta(2) = 2$. Let $p$ be a non totally geodesic point of $M$. Then there exists a (possibly branched) almost complex curve $\tilde{\phi} : N_2 \to S^6$ such that $f$ is locally around $p$ congruent to $\tilde{\psi}$, which is obtained from $\tilde{\phi}$ as in (3).

Let $f : S \to S^6$ be an almost complex curve without totally geodesic points. Define

$$F : T_1S \to S^6(1) : v \mapsto \frac{h(v,v)}{||h(v,v)||},$$

where $T_1S$ denotes the unit tangent bundle of $S$.

Also, Vrancken [255] showed that the following:

(i) $F$ given by (20.4) defines a Lagrangian immersion if and only if $f$ is superminimal, and

(ii) If $\psi : M \to S^6(1)$ be a Lagrangian immersion which admits a unit length Killing vector field whose integral curves are great circles. Then there exist an open dense subset $U$ of $M$ such that each point $p$ of $U$ has a neighborhood $V$ such that $\psi : V \to S^6$ satisfies $\delta(2) = 2$, or $\psi : V \to S^6$ is obtained as in (1).

20.4  $CR$-submanifolds in the nearly Kähler $S^6$

M. Djorić and Vrancken [129] study 3-dimensional $CR$-submanifolds in $S^6$ and proved the following.

Theorem 20.1. Let $M$ be a 3-dimensional minimal $CR$-submanifold in the nearly Kähler $S^6$ satisfying $\delta(2) = 2$. Then $M$ is a totally real submanifold.
or locally $M$ is congruent with the immersion:
\[
    f(t, u, v) = \left( \cos t \cos u \cos v, \sin t, \cos t \sin u \cos v, \cos t \cos u \sin v, 0, -\cos t \sin u \sin v, 0 \right).
\]

Notice that (20.5) can also be described algebraically by the equations:
\[
    x_5 = x_7 = 0, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 = 1, \quad x_3x_4 + x_1x_6 = 0,
\]
from which we see that the CR-submanifold is a hypersurface lying in a totally geodesic $S^4(1)$.

Very recently, Djorić and Vrancken proved in [130] that 3-dimensional CR-submanifolds in the nearly Kähler $S^6$ satisfying $\delta(2) = 2$ must be minimal.

For a 4-dimensional submanifold $M$ of $S^6(1)$, the fundamental inequality (5.1) in Theorem 5.1 yields the following inequality for $\delta(2)$:
\[
    \delta(2) \leq \frac{16}{3} H^2 + \frac{15}{2}.
\]

Recently, Antić, Djorić and Vrancken [5] study 4-dimensional CR-submanifolds in $S^6$ and proved the following.

**Theorem 20.2.** Let $M$ be a 4-dimensional minimal CR-submanifold in the nearly Kähler $S^6$ satisfying $\delta(2) = 15/2$. Then $M$ is locally congruent with the following immersion:
\[
    f(x_1, x_2, x_3, x_4) = \left( \cos x_4 \cos x_1 \cos x_2 \cos x_3, \sin x_4 \sin x_1 \cos x_2 \cos x_3, \sin 2x_4 \sin x_3 \cos x_2 + \cos 2x_4 \sin x_2, 0, \sin x_4 \cos x_1 \cos x_2 \cos x_3, \cos x_4 \sin x_1 \cos x_2 \cos x_3, \cos 2x_4 \sin x_3 \cos x_2 - \sin 2x_4 \sin x_2 \right).
\]

## 21 Ricci curvature and its applications

For a Riemannian $n$-manifold $M$ and for each unit vector $X \in T_pM$, $p \in M$, the Ricci curvature $\text{Ric}(X)$ of $X$ can be regarded as a $\delta$-invariant:
\[
    \text{Ric}(X) = \tau(p) - \tau(L_X^\perp), \tag{21.1}
\]
where $L_X^\perp$ is the hyperplane of $T_pM$ with $X$ as its hyperplane normal.

For the Ricci curvature we have the following result from [51].
**Theorem 21.1.** Let \( x : M^n \to R^m(\epsilon) \) be an isometric immersion of a Riemannian \( n \)-manifold \( M^n \) into a Riemannian space form \( R^m(\epsilon) \). Then

1. For each unit tangent vector \( X \in T_pM^n \), we have
   \[
   \text{Ric}(X) \leq \frac{n^2}{4} H^2 + (n - 1) \epsilon. \tag{21.2}
   \]

2. If \( H(p) = 0 \), then a unit tangent vector \( X \) at \( p \) satisfies the equality case of (21.2) if and only if \( X \) lies in the relative null space \( N_p \) at \( p \).

3. The equality case of (21.2) holds identically for all unit tangent vectors at \( p \) if and only if either \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.

In particular, we have the following.

**Corollary 21.1.** Let \( x : M^n \to E^m \) be an isometric immersion of a Riemannian \( n \)-manifold \( M^n \) in a Euclidean \( m \)-space with arbitrary codimension. Then we have

\[
H^2(p) \geq \left( \frac{4}{n^2} \right) \max_X \text{Ric}(X), \tag{21.3}
\]

where \( X \) runs over all unit tangent vectors at \( p \).

**Remark 21.1.** There exist many examples of submanifolds in a Euclidean \( m \)-space which satisfy the equality case of (21.3) identically. Two simple examples are spherical hypercylinder \( S^2(r) \times \mathbb{R} \) and round hypercone in \( \mathbb{E}^4 \).

For a Riemannian \( n \)-manifold \( M \), denote by \( K(\pi) \) the sectional curvature of a 2-plane section \( \pi \subset T_pM \), \( p \in M \). Suppose \( L^k \) is a \( k \)-plane section of \( T_pM \) and \( X \) a unit vector in \( L^k \). We choose an orthonormal basis \( \{e_1, \ldots, e_k\} \) of \( L^k \) such that \( e_1 = X \). Define the Ricci curvature \( \text{Ric}_{L^k} \) of \( L^k \) at \( X \) by

\[
\text{Ric}_{L^k}(X) = K_{12} + \cdots + K_{1k},
\]

where \( K_{ij} \) is the sectional curvature of the 2-plane section spanned by \( e_i, e_j \). We simply called such a curvature a \( k \)-Ricci curvature.

For each integer \( k, 2 \leq k \leq n \), we have defined in [51] a Riemannian invariant on a Riemannian \( n \)-manifold \( M^n \) by

\[
\theta_k(p) = \left( \frac{1}{k - 1} \right) \inf_{L^k, X} \text{Ric}_{L^k}(X), \quad p \in M^n,
\]
where \( L^k \) runs over all \( k \)-plane sections at \( p \) and \( X \) runs over all unit vectors in \( L^k \).

The following result was also proved in [51].

**Theorem 21.2.** Let \( x : M^n \to R^m(\epsilon) \) be an isometric immersion of a Riemannian \( n \)-manifold \( M^n \) in a real space form \( R^m(\epsilon) \) of constant sectional curvature \( c \). Then, for any integer \( k, 2 \leq k \leq n \), we have

\[
H^2(p) \geq \frac{4(n-1)}{n^2} \left( \frac{\theta_k(p)}{k-1} - \epsilon \right).
\]

(21.4)

The equality case of (21.4) holds identically for all unit tangent vectors at \( p \) if and only if either \( p \) is a totally geodesic point or \( k = n = 2 \) and \( p \) is a totally umbilical point.

**Remark 21.2.** In general, given an integer \( k, 2 \leq k \leq n-1 \), there does not exist a positive constant, say \( C(n,k) \), such that

\[
H^2(p) \geq C(n,k) \max_{L^k \times X} \text{Ric}_{L^k}(X),
\]

(21.5)

where \( L^k \) runs over all \( k \)-plane sections in \( T_pM^n \) and \( X \) runs over all unit tangent vectors in \( L^k \). This fact can be seen from the following example:

Let \( x : M^3 \to \mathbb{E}^4 \) be a minimal hypersurface whose shape operator is non-singular at some point \( p \in M^3 \). Then by the minimality there exist two principal directions at \( p \), say \( e_1, e_2 \), such that their corresponding principal curvatures \( \kappa_1, \kappa_2 \) are of the same sign. This implies that the sectional curvature \( K_{12} \) at \( p \) is positive. Now, consider the minimal hypersurface in \( \mathbb{E}^{n+1} \) which is given by the product of \( x : M^3 \to \mathbb{E}^4 \) and the identity map \( \iota : \mathbb{E}^{n-3} \to \mathbb{E}^{n-3} \).

It is clear that, for any integer \( k, 2 \leq k \leq n-1 \), the maximum value of the \( k \)-th Ricci curvatures of \( M^n := M^3 \times \mathbb{E}^{n-3} \) at a point \( (p, q) \), \( q \in \mathbb{E}^{n-3} \) is given by \( K_{12} = \kappa_1 \kappa_2 \) which is positive. Since \( H = 0 \), this shows that there does not exist any positive constant \( C(n,k) \) which satisfies (21.5).

## 22 Kählerian δ-invariants and applications to complex geometry

Let \( M^n \) be a Kähler manifold of complex dimension \( n \). Denote by \( J \) the complex structure on the Kähler manifold. For each plane section \( \pi \subset \)
$T_xM$, $x \in M$, we denote by $K(\pi)$ the sectional curvature of the plane section $\pi$ as before. A plane section $\pi \subset T_xM$ is called \textit{totally real} if $J\pi$ is perpendicular to $\pi$.

For a (real) $2n$-dimensional Kähler submanifold $M$ of a Kählerian space form $\tilde{M}^m(4\epsilon)$ with constant holomorphic sectional curvature $4\epsilon$, the scalar curvature $\tau$ of $M$ satisfies

\[ \tau \leq 2n(n+1)\epsilon, \tag{22.1} \]

with equality holding if and only if $M$ is a totally geodesic Kähler submanifold.

### 22.1 Kählerian $\delta$-invariants $\delta^c(2n_1, \ldots, 2n_k)$

Let $M$ be a real $2n$-dimensional Kähler manifold. The \textit{Kählerian $\delta$-invariants} $\delta^c(2n_1, \ldots, 2n_k)$ is defined in [50] as:

\[ \delta^c(2n_1, \ldots, 2n_k) = \tau - \inf \{ \tau(L_1^c) + \cdots + \tau(L_k^c) \} \]

for each $k$-tuple $(2n_1, \ldots, 2n_k) \in S(2n)$, where $L_1^c, \ldots, L_k^c$ run over all $k$ mutually orthogonal complex subspaces of $T_pM$, $p \in M$, with real dimensions $2n_1, \ldots, 2n_k$, respectively.

For a Kähler submanifold in a complex space form, we have the following general result from [50].

**Proposition 22.1.** Let $M$ be a (real) $2n$-dimensional Kähler submanifold of a complex space form $\tilde{M}^m(4\epsilon)$. Then, for each $k$-tuple $(2n_1, \ldots, 2n_k) \in S(2n)$, the complex $\delta$-invariant $\delta^c(2n_1, \ldots, 2n_k)$ satisfies

\[ \delta^c(2n_1, \ldots, 2n_k) \leq 2 \left( n(n+1) - \sum_{j=1}^k n_j(n_j + 1) \right) \epsilon. \tag{22.2} \]

The equality case of inequality (22.2) holds at a point $p \in M$ if and only if, there exists an orthonormal basis

\begin{align*}
e_1, \ldots, e_{n_1}, Je_1, \ldots, Je_{n_1}, \ldots, e_{2(n_1+\cdots+n_{k-1})+1}, \ldots, e_{2(n_1+\cdots+n_{k-1})+n_k}, \\Je_{2(n_1+\cdots+n_{k-1})+1}, \ldots, Je_{2(n_1+\cdots+n_{k-1})+n_k}, e_{2n+1}, \ldots, e_{2m}
\end{align*}
at $p$, such that the shape operators of $M$ in $\tilde{M}^m(4\epsilon)$ at $p$ take the following form:

$$A_r = \begin{pmatrix}
A_r^1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \\
0 & \cdots & A_r^k & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \\
0 & \cdots & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad r = 2n + 1, \ldots, 2m,$$

where each $A_r^j$ is a symmetric $(2n_j) \times (2n_j)$ submatrix with zero trace.

Remark 22.1. Contrast to inequality (22.1), besides totally geodesic Kähler submanifolds there do exist Kähler submanifolds of $\tilde{M}^m(4\epsilon)$ which satisfy the equality case of (22.2) identically.

For instance, let $Q_n$ denote the complex hyperquadric in $CP^{n+1}(4)$ defined by

$$\{(z_0, z_1, \ldots, z_{n+1}) \in CP^{n+1}(4) : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}. \quad (22.4)$$

For $Q_n$ we have

$$\tau = 2n^2, \quad \delta_k^{(c)} := \delta_k^{(c)}(2, \ldots, 2) = 2n(n - 1), \quad (22.5)$$

where $2$ in $\delta_k^{(c)}(2, \ldots, 2)$ repeats $n$ times.

The complex quadric $Q_n$ in $CP^{n+1}(4\epsilon)$ satisfies the equality case of (22.2) identically for the $n$-tuple $(2, \ldots, 2) \in S(2n)$.

Also, direct products of Kähler submanifolds of complex Euclidean spaces are Kähler submanifolds of complex Euclidean spaces which satisfy the equality case of (22.2) for some suitable $k$-tuples.

In views of the above facts, it is an interesting problem to classify all Kähler submanifolds of Kählerian space forms which satisfy either the equality case of inequality (22.2)) or the equality case of inequality (22.6).

22.2 Totally real $\delta$-invariants $\delta^{\tau}(n_1, \ldots, n_k)$

For $(n_1, \ldots, n_k)$ in $S(2n)$ we also introduced in [50] the totally real $\delta$-invariants $\delta^{\tau}(n_1, \ldots, n_k)$ by

$$\delta^{\tau}(n_1, \ldots, n_k) = \tau - \inf\{\tau(L_1^1) + \cdots + \tau(L_k^1)\},$$
where $L_1^r, \ldots, L_k^r$ run over all $k$ mutually orthogonal totally real subspaces of $T_pM$, $p \in M$, with dimensions $n_1, \ldots, n_k$, respectively.

For totally real $\delta$-invariants $\delta^r(n_1, \ldots, n_k)$ of a Kähler submanifold in a complex space form, we have the following.

**Proposition 22.2.** Let $M$ be a (real) $2n$-dimensional Kähler submanifold of a complex space form $\tilde{M}^m(4\epsilon)$. Then, for each $k$-tuple $(n_1, \ldots, n_k) \in S(2n)$, the totally real $\delta$-invariant $\delta^r(n_1, \ldots, n_k)$ satisfies

$$\delta^r(n_1, \ldots, n_k) \leq \left(2n(n+1) - \frac{1}{2} \sum_{j=1}^{k} n_j(n_j+1)\right)\epsilon. \quad (22.6)$$

The equality case of inequality (22.6) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $e_1, \ldots, e_{n_1}, e_{n_1+n_2}, \ldots, e_{n_1+n_2+n_3} \ldots e_{n_1+\ldots+n_k}$ at $p$, such that

$$\text{Span}\{e_1, \ldots, e_{n_1}\}, \ldots, \text{Span}\{e_{n_1+\ldots+n_k-1}, \ldots, e_{n_1+\ldots+n_k}\}$$

are totally real subspaces of $T_pM$ and the shape operators of $M$ in $\tilde{M}^m(4\epsilon)$ at $p$ take the following form:

$$A_r = \begin{pmatrix}
A_1^r & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & A_k^r & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 0 \\
\end{pmatrix},$$

where each $A_j^r$ is a symmetric $n_j \times n_j$ submatrix with zero trace.

### 22.3 Kählerian $\delta$-invariant $\delta_k^r$ and strongly minimal Kähler submanifolds

For each real number $k$ we may also define a Kählerian $\delta$-invariant $\delta_k^r$ by

$$\delta_k^r(p) = \tau(p) - k \inf K^r(p), \quad p \in M, \quad (22.7)$$
where
\[
\inf K^r(p) = \inf_{\pi^r} \{K(\pi^r)\}
\]
and \(\pi^r\) runs over all totally real plane sections in \(T_p M\).

Let \(M\) be a Kähler submanifold of a Kähler manifold \(\tilde{M}^{n+p}\). Just like in the real case, we denote by \(h\) and \(A\) the second fundamental form and the shape operator of \(M^n\) in \(\tilde{M}^{n+p}\), respectively.

For the Kähler submanifold we consider an orthonormal frame
\[
e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n
\]
of the tangent bundle and an orthonormal frame
\[
\xi_1, \ldots, \xi_p, \xi_1^* = J\xi_1, \ldots, \xi_p^* = J\xi_p
\]
of the normal bundle. With respect to such an orthonormal frame, the complex structure \(J\) on \(M\) is given by
\[
J = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix}, \quad (22.8)
\]
where \(I_n\) denotes an identity matrix of degree \(n\).

For a Kähler submanifold \(M^n\) in \(\tilde{M}^{n+p}\) the shape operator of \(M^n\) satisfies (see, for instance, [211])
\[
A_{\xi r} = JA_r, \quad JA_r = -A_r J, \quad \text{for} \quad r = 1, \ldots, n, 1^*, \ldots, p^*, \quad (22.9)
\]
where \(A_r = A_{\xi r}\). From these it follows that the shape operator of \(M^n\) takes the form:
\[
A_{\alpha} = \begin{pmatrix}
A'_{\alpha} & A''_{\alpha} \\
A''_{\alpha} & -A'_{\alpha}
\end{pmatrix}, \quad A_{\alpha}^* = \begin{pmatrix}
-A''_{\alpha} & A'_{\alpha} \\
A'_{\alpha} & A''_{\alpha}
\end{pmatrix}, \quad \alpha = 1, \ldots, p, \quad (22.10)
\]
where \(A'_\alpha\) and \(A''_\alpha\) are \(n \times n\) matrices. This condition implies that every Kähler submanifold \(M^n\) is minimal, i.e.,
\[
\text{trace} A_\alpha = \text{trace} A_{\alpha}^* = 0, \quad \alpha = 1, \ldots, p.
\]

The notion of strongly minimal Kähler submanifolds was first introduced in [62].
Definition 22.1. A Kähler submanifold $M^n$ of a Kähler manifold $\tilde{M}^{n+p}$ is called strongly minimal if it satisfies
\[
\text{trace } A'_\alpha = \text{trace } A''_\alpha = 0, \text{ for } \alpha = 1, \ldots, p, \tag{22.11}
\]
with respect to some orthonormal frame:
\[
e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n, \xi_1, \ldots, \xi_p, \xi_1^* = J\xi_1, \ldots, \xi_p^* = J\xi_p.
\]

For Kähler submanifolds, we have the following sharp general results.

Theorem 22.1. \cite{62} For any Kähler submanifold $M^n$ of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+p}(4\epsilon)$, the following statements hold.

1. For each $k \in (-\infty, 4]$, $\delta_k^r$ satisfies
\[
\delta_k^r \leq (2n^2 + 2n - k)\epsilon. \tag{22.12}
\]

2. Inequality \eqref{22.12} fails for every $k > 4$.

3. $\delta_k^r = (2n^2 + 2n - k)\epsilon$ holds identically for some $k \in (-\infty, 4)$ if and only if $M^n$ is a totally geodesic Kähler submanifold of $\tilde{M}^{n+p}(4\epsilon)$.

4. The Kähler submanifold $M^n$ satisfies $\delta_4^r = (2n^2 + 2n - 4)\epsilon$ at a point $x \in M^n$ if and only if there exists an orthonormal basis
\[
e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n, \xi_1, \ldots, \xi_p, \xi_1^* = J\xi_1, \ldots, \xi_p^* = J\xi_p
\]
of $T_x\tilde{M}^{n+p}(4\epsilon)$ such that, with respect to this basis, the shape operator of $M^n$ takes the following form:
\[
A'_\alpha = \begin{pmatrix} A'_{\alpha} & A''_{\alpha} \\ A''_{\alpha} & -A'_\alpha \end{pmatrix}, \quad A''_{\alpha} = \begin{pmatrix} -A''_{\alpha} & A'_\alpha \\ A'_\alpha & A''_{\alpha} \end{pmatrix}, \tag{22.13}
\]
\[
A'_\alpha = \begin{pmatrix} a_\alpha & b_\alpha & 0 \\ b_\alpha & -a_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A''_{\alpha} = \begin{pmatrix} a^*_\alpha & b^*_\alpha & 0 \\ b^*_\alpha & -a^*_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22.14}
\]

for some $n \times n$ matrices $A'_\alpha, A''_{\alpha}$, $\alpha = 1, \ldots, p$. 
Theorem 22.2. [62] A complete Kähler submanifold $M^n (n \geq 2)$ in $CP^{n+p}(4\epsilon)$ satisfies

$$\delta_4^e = 2(n^2 + n - 2)\epsilon$$

identically if and only if

1. $M^n$ is a totally geodesic Kähler submanifold, or

2. $n = 2$ and $M^2$ is a strongly minimal Kähler surface in $CP^{2+p}(4\epsilon)$.

Theorem 22.3. A complete Kähler submanifold $M^n (n \geq 2)$ of $C^{n+p}$ satisfies $\delta_4^e = 0$ identically if and only if

(1) $M^n$ is a complex $n$-plane of $C^{n+p}$, or

(2) $M^n$ is a complex cylinder over a strongly minimal Kähler surface $M^2$ in $C^{n+p}$ (i.e., $M$ is the product submanifold of a strongly minimal Kähler surface $M^2$ in $C^{p+2}$ and the identity map of the complex Euclidean $(n - 2)$-space $C^{n-2}$).

22.4 Examples of strongly minimal Kähler submanifolds

Every totally geodesic Kähler submanifold of a complex space form is trivially strongly minimal. There also exist nontrivial examples of strongly minimal Kähler submanifolds.

Example 22.1. Consider the complex quadric $Q_2$ in $CP^3(4\epsilon)$ defined by

$$Q_2 = \left\{ (z_0, z_1, z_2, z_3) \in CP^3(4\epsilon) : z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \right\},$$

(22.16)

where $\{z_0, z_1, z_2\}$ is a homogeneous coordinate system of $CP^3(4\epsilon)$.

It is known that the scalar curvature $\tau$ of $Q_2$ is equal to $8\epsilon$ and $\inf K^r = 0$. Thus, we obtain $\delta_4^e = 8\epsilon$. Hence, $Q_2$ is a non-totally geodesic Kähler submanifold which satisfies (22.15) with $n = 2$. Therefore, according to Theorem 22.2, $Q_2$ is a strongly minimal Kähler surface in $CP^3(4\epsilon)$.

On the other hand, it is also well-known that $Q_2$ is an Einstein-Kähler surface with Ricci tensor $S = 4\epsilon g$, where $g$ is the metric tensor of $Q_2$. Thus, the equation of Gauss yields

$$g(A^2_2X, Y) = cg(X, Y), \quad X, Y \in TQ_2.$$  

(22.17)
\( \delta \text{-invariants} \)

Hence, with respect to a suitable choice of \( e_1, e_2, Je_1, Je_2, \xi_1, J\xi_1 \), we have

\[
A_1 = \begin{pmatrix} A'_1 & A''_1 \\ A''_1 & -A'_1 \end{pmatrix}, \quad A_1^* = \begin{pmatrix} -A''_1 & A'_1 \\ A'_1 & A''_1 \end{pmatrix},
\]

(22.18)

where

\[
A'_1 = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & -\sqrt{\epsilon} \end{pmatrix}, \quad A''_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(22.19)

This also shows that \( Q_2 \) is strongly minimal in \( CP^3(4\epsilon) \).

Two non-trivial examples of strongly minimal Kähler surfaces in \( C^3 \) are the following.

**Example 22.2.** The complex surface:

\[
\left\{ z \in C^3 : z_1^2 + z_2^2 + z_3^2 = 1 \right\}
\]

(22.20)

is a strongly minimal complex surface in \( C^3 \).

**Example 22.3.** The Kähler surfaces:

\[
N^2_k = \left\{ z \in C^3 : z_1 + z_2 + z_3^2 = k \right\}, \quad k \in C,
\]

(22.21)

are strongly minimal Kähler surfaces in \( C^3 \).

**22.5 Framed-Einsteinian and strongly minimality**

A Riemannian \( n \)-manifold \( M \) is called **framed-Einstein** if there exist a function \( \gamma \) and an orthonormal frame \( \{ e_1, \ldots, e_n \} \) on \( M \) such that the Ricci tensor \( Ric \) of \( M \) satisfies

\[
Ric(e_i, e_i) = \gamma g(e_i, e_i)
\]

for \( i = 1, \ldots, n \). Clearly, every Einstein manifold is framed-Einstein, but not the converse in general.

The following result provides a simple relationship between strongly minimal surfaces and framed-Einsteinian.

**Proposition 22.3.** Let \( M^2 \) be a strongly minimal Kähler surface in a complex space form. Then \( M^2 \) is a framed-Einstein Kähler surface.
22.6 Invariants $\delta^r_{\ell,k}$

For a Kähler manifold $M$ of complex dimension $n$, one may extend the invariant $\delta^r_k$ to $\delta^r_{\ell,k}$ as

$$\delta^r_{\ell,k}(x) = \tau(x) - \frac{k}{\ell - 1} \inf_{L^r_{\ell}} \tau(L^r_{\ell}), \quad x \in M,$$

(22.22)

where $L^r_{\ell}$ runs over all totally real $\ell$-subspaces of $T_xM$.

For each integer $\ell \in [2,n]$, the inequality (22.12) was extended by B. Suceavă [248] to the following inequality:

$$\delta^r_{\ell,k}(x) \leq \left\{ 2n^2 + 2n - \frac{k}{4} \binom{\ell}{2} \right\} \epsilon$$

(22.23)

for Kähler submanifolds in a complex space form $\tilde{M}^m(4\epsilon)$. However, for $\ell \geq 3$, the equality sign of (22.23) occurs only for totally geodesic Kähler submanifolds.

22.7 $\delta(2)$ and Kähler hypersurfaces

For Kähler hypersurfaces in $\mathbb{C}^{n+1}$, Z. Sentürk and L. Verstraelen [242] proved the following.

Proposition 22.4. Let $M^n$ be a Kähler hypersurface $M^n$ in $\mathbb{C}^{n+1}$. The we have

(1) $\delta(2)$ satisfies $\delta(2) \leq 2$;

(2) At any point of $M^n$, $\delta(2) = 0$ holds if and only if (real) rank $(A) \leq 2$ at that point.

It follows from Proposition 22.4 and Abe’s complex version [4] of the Hartman-Nirenberg cylinder theorem that

Theorem 22.4. [242] The complex hypercylinders $\mathcal{C}^n$ in $\mathbb{C}^{n+1}$, i.e. the products of any complex curve $C$ in a complex 2-plane $\mathbb{C}^2$ in $\mathbb{C}^{n+1}$ with complex $(n-1)$-dimensional complex linear subspaces $\mathbb{C}^{n-1}$ of $\mathbb{C}^{n+1}$ which are perpendicular to the plane $\mathbb{C}^2$ of the curve $C$, are the complete Kähler hypersurfaces $M^n$ in $\mathbb{C}^{n+1}$ for which the $\delta(2)$ curvature vanishes identically.
23 Applications to affine differential geometry (I) : δ\# -invariants

23.1 Basics of affine differential geometry

If $M$ is an $n$-dimensional manifold, let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface of the affine $(n+1)$-space whose position vector field is nowhere tangent to $M$. Then $f$ can be regarded as a transversal field along itself. We call $\xi = -f$ the centroaffine normal. Following Nomizu, we call $f$ together with this normalization a centroaffine hypersurface.

The centroaffine structure equations are given by

\begin{align*}
D_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi, \quad (23.1) \\
D_X \xi &= -f_*(X), \quad (23.2)
\end{align*}

where $D$ denotes the canonical flat connection of $\mathbb{R}^{n+1}$, $\nabla$ is a torsion-free connection on $M$, called the induced centroaffine connection, and $h$ is a non-degenerate symmetric $(0,2)$-tensor field, called the centroaffine metric.

From now on we assume that the centroaffine hypersurface is definite, i.e., $h$ is definite. In case that $h$ is negative definite, we shall replace $\xi = -f$ by $\xi = f$ for the affine normal. In this way, the second fundamental form $h$ is always positive definite. In both cases, (23.1) holds. Equation (23.2) change sign. In case $\xi = -f$, we call $M$ positive definite; in case $\xi = f$, we call $M$ negative definite.

Denote by $\hat{\nabla}$ the Levi-Civita connection of $h$ and by $\hat{R}$ and $\hat{\kappa}$ the curvature tensor and the normalized scalar curvature of $h$, respectively. The difference tensor $K$ is then defined by

\begin{equation}
K_X Y = K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \quad (23.3)
\end{equation}

which is a symmetric $(1,2)$-tensor field. The difference tensor $K$ and the cubic form $C$ are related by

\begin{equation}
C(X, Y, Z) = -2h(K_X Y, Z).
\end{equation}

Thus, for each $X$, $K_X$ is self-adjoint with respect to $h$.

The Tchebychev form $T$ and the Tchebychev vector field $T^\#$ of $M$ are
defined respectively by
\[ T(X) = \frac{1}{n} \text{trace } K_X, \quad (23.4) \]
\[ h(T^#, X) = T(X). \quad (23.5) \]

If \( T = 0 \) and if we consider \( M \) as a hypersurface of the equiaffine space, then \( M \) is a so-called proper affine hypersphere centered at the origin.

If the difference tensor \( K \) vanishes, then \( M \) is a quadric, centered at the origin, in particular an ellipsoid if \( M \) is positive definite and a two-sheeted hyperboloid if \( M \) is negative definite.

An affine hypersurface \( \phi : M \to \mathbb{R}^{n+1} \) is called a graph hypersurface if the transversal vector field \( \xi \) is a constant vector field. A result of [209] states that a graph hypersurface \( M \) is locally affine equivalent to the graph immersion of a certain function \( F \). Again in case that \( h \) is nondegenerate, it defines a semi-Riemannian metric, called the Calabi metric of the graph hypersurface. If \( T = 0 \), a graph hypersurface is a so-called improper affine hypersphere.

Let \( M_1 \) and \( M_2 \) be two improper affine hyperspheres in \( \mathbb{R}^{p+1} \) and \( \mathbb{R}^{q+1} \) defined respectively by the equations:
\[ x_{p+1} = F_1(x_1, \ldots, x_p), \quad y_{q+1} = F_2(y_1, \ldots, y_q). \]
Then one can define a new improper affine hypersphere \( M \) in \( \mathbb{R}^{p+q+1} \) by
\[ z = F_1(x_1, \ldots, x_p) + F_2(y_1, \ldots, y_q), \]
where \((x_1, \ldots, x_p, y_1, \ldots, y_q, z)\) are the coordinates on \( \mathbb{R}^{p+q+1} \). The Calabi normal of \( M \) is \((0, \ldots, 0, 1)\). Obviously, the Calabi metric on \( M \) is the product metric. Following [126] we call this composition the Calabi composition of \( M_1 \) and \( M_2 \).

### 23.2 Affine \( \delta \)-invariants and general fundamental inequalities

Analogous to the \( \delta \)-invariants \( \delta(n_1, \ldots, n_k) \) we defined in [86] affine \( \delta \)-invariants \( \delta^#(n_1, \ldots, n_k) \) as follows:
\[ \delta^#(n_1, \ldots, n_k)(p) = \hat{\tau}(p) - \sup \{ \hat{\tau}(L_1) + \cdots + \hat{\tau}(L_k) \}, \quad (23.6) \]
where $L_1, \ldots, L_k$ run over all $k$ mutually $h$-orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$ and $\hat{\tau}(L)$ is the scalar curvature of $L$ with respect to induced affine metric $h$.

We have the following theorem from [86].

**Theorem 23.1.** Let $M$ be a definite centroaffine hypersurface in $\mathbb{R}^{n+1}$. Then, for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, we have

$$\delta^\#(n_1, \ldots, n_k) \geq \frac{1}{2} n(n-1) \varepsilon - \frac{1}{2} \sum_{j=1}^{k} n_j(n_j - 1) \varepsilon$$

$$- \frac{n^2 \left( n + k - 1 - \sum_{j=1}^{k} n_j \right)}{2 \left( n + k - \sum_{j=1}^{k} n_j \right)} h(T^\#, T^\#),$$

(23.7)

where $\varepsilon = 1$ or $-1$, according to $M$ is positive definite or negative definite.

The equality case of inequality (23.7) holds at a point $p \in M$ if and only if $T^\# = 0$ at $p$ and there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ at $p$ such that with respect this basis every linear map $K_X$, $X \in T_p M$, takes the following form:

$$K_X = \begin{pmatrix} A_1^X & \cdots & 0 \\ \cdots & \cdots & \cdots \\ A_k^X & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix},$$

(23.8)

where $\{A_j^X\}_{j=1}^{k}$ are symmetric $n_j \times n_j$ submatrices satisfying

$$\text{trace} (A_1^X) = \cdots = \text{trace} (A_k^X) = 0.$$  

(23.9)

### 23.3 Some immediate applications

Two immediate consequences of Theorem 23.1 are the following.

**Corollary 23.1.** [86] Let $M$ be a Riemannian $n$-manifold. If there is a $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$ such that

$$\delta^\#(n_1, \ldots, n_k) < \frac{1}{2} n(n-1) - \frac{1}{2} \sum_{j=1}^{k} n_j(n_j - 1)$$

(23.10)
at some point, then $M$ cannot be realized as an elliptic proper affine hypersphere.

In particular, if there is a $k$-tuple $(n_1, \ldots, n_k)$ such that $\delta^#(n_1, \ldots, n_k) \leq 0$ at some point in $M$, then $M$ cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.

**Corollary 23.2.** [86] Let $M$ be a Riemannian $n$-manifold. If there is a $k$-tuple $(n_1, \ldots, n_k)$ in $\mathcal{S}(n)$ such that

$$\delta^#(n_1, \ldots, n_k) < \frac{1}{2} \sum_{j=1}^{k} n_j(n_j - 1) - \frac{1}{2} n(n - 1)$$

(23.11)

at some point, then $M$ cannot be realized as a hyperbolic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Recall that a hyperovaloid in $\mathbb{R}^{n+1}$ is a compact strictly convex hypersurface embedded in $\mathbb{R}^{n+1}$. The inequalities (23.7) give rise to some global centroaffine curvature invariants for hyperovaloids. Moreover, they provide simple characterizations of hyperellipsoids in terms of these global invariants.

**Corollary 23.3.** [86] Consider a centroaffine hyperovaloid $f : M \to \mathbb{R}^{n+1}, n \geq 3$, with normalization as in (2.1) with $\xi = -f$. Then, for any $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have the following global inequality:

$$\int_M \left( \delta^#(n_1, \ldots, n_k) + \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)} h(T^#, T^#) \right) \omega(h) \geq \frac{1}{2} \left( n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1) \right) \text{vol}(M, h),$$

(23.12)

where $\omega(h)$ is the volume form and $\text{vol}(M, h)$ is the volume of $(M, h)$.

The equality sign of (23.12) holds if and only if $M$ is a hyperellipsoid centered at the origin.

Moreover, we have the following theorems from [86].

**Theorem 23.2.** [86] If $M$ is a definite centroaffine hypersurface in $\mathbb{R}^{n+1}, \ n \geq 3$, which satisfies the equality case of one of the inequalities in (23.7), then we have $\dim(\text{Im} \ K) < n$. 


**Theorem 23.3.** [86] Let $M$ be a positive definite centroaffine hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$, satisfying the equality case of $(23.7)$ for some $k$-tuple $(n_1, \ldots, n_k) \in S(n)$. If $\dim (\text{Im} \ K)$ is constant on $M$, then $M$ is foliated by $q$-dimensional ellipsoids centered at the origin of $\mathbb{R}^{n+1}$, where $q = n - \dim (\text{Im} \ K)$.

**Theorem 23.4.** [86] Let $M$ be a negative definite centroaffine hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$, satisfying the equality case of $(23.7)$ for a $k$-tuple $(n_1, \ldots, n_k) \in S(n)$. If $\dim (\text{Im} \ K)$ is constant, then $M$ is foliated by $q$-dimensional two-sheeted hyperboloids centered at the origin, where $q = n - \dim (\text{Im} \ K)$.

Similarly, for graph hypersurfaces we have

**Theorem 23.5.** [128] Let $M$ be a definite graph hypersurface in $\mathbb{R}^{n+1}$. Then, for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, we have

$$\delta^#(n_1, \ldots, n_k) \geq -\frac{n^2 (n + k - 1 - \sum_{j=1}^{k} n_j)}{2 (n + k - \sum_{j=1}^{k} n_j)} h(T^#, T^#),$$

where $\varepsilon = 1$ or $-1$, according to $M$ is positive definite or negative definite.

The equality case of inequality $(23.13)$ holds at a point $p \in M$ if and only if $T^# = 0$ at $p$ and there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ at $p$ such that with respect this basis every linear map $K_X$, $X \in T_p M$, takes the following form:

$$K_X = \begin{pmatrix} A_1^X & 0 \\ \vdots & \ddots \\ 0 & A_k^X \\ 0 & 0 \end{pmatrix},$$

where $\{A_j^X\}_{j=1}^{k}$ are symmetric $n_j \times n_j$ submatrices satisfying

$$\text{trace} (A_1^X) = \cdots = \text{trace} (A_k^X) = 0.$$

An immediate consequence of Theorem 23.5 is the following.

**Corollary 23.4.** [128] If $(M, h)$ is a Riemannian manifold and for some $k$-tuple $(n_1, \ldots, n_k) \in S(n)$ the $\delta$-invariant satisfies $\delta^#(n_1, \ldots, n_k) < 0$ at some point, then $(M, h)$ cannot be realized as improper affine sphere in some affine space.
23.4 Remarks and examples

Remark 23.1. For each $k$-tuple $(n_1, \ldots, n_k)$ in $S(n)$, the inequality (23.7) is sharp. For instance, for each $k$-tuple $(n_1, \ldots, n_k)$, any ellipsoid in $\mathbb{R}^{n+1}$ centered at the origin satisfies $K = 0$ identically. Hence, it satisfies the equality case of the inequality (23.7) trivially.

Remark 23.2. For any $k$-tuple $(n_1, n_2, \ldots, n_k) \in S(n)$, there also exist many definite centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ centered at the origin, which satisfy the equality case of the inequality (23.7) identically for the $k$-tuple $(n_1, n_2, \ldots, n_k)$.

Example 23.1. Let $\phi_1 : M^{n_1} \to \mathbb{R}^{n_1+1} \times \{0\}$ and $\phi_2 : M^{n_2} \to \{0\} \times \mathbb{R}^{n_2+1}$ be two elliptic affine hyperspheres centered at the origin. We put $n = n_1 + n_2 + 1$ and consider the immersion: $\phi^+ : M = M_1 \times M_2 \times \mathbb{R} \to \mathbb{R}^{n+1}$ defined by

$$\phi^+(u_1, u_2, t) = (\cos t)\phi_1(u_1) + (\sin t)\phi_2(u_2), \quad (23.16)$$

for $u_1 \in M_1$, $u_2 \in M_2$, $t \in \mathbb{R}$. Then a straightforward long computation shows that $\phi^+$ is again an elliptic affine hypersphere, centered at the origin, and that the linear map $K_X$, $X \in TM$, of $\phi^+$ satisfies

$$K_{X_1} = \begin{pmatrix} K_{X_1}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_{X_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{X_2}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_V = 0 \quad (23.17)$$

for $X_1, X_2, V$ tangent to $M_1, M_2, \mathbb{R}$, respectively, where $K_{X_1}^1$, $K_{X_2}^2$ denote the corresponding linear maps of $\phi_1$ and $\phi_2$, respectively, which satisfy $\text{trace}(K_{X_1}^1) = \text{trace}(K_{X_2}^2) = 0$. Therefore, the immersion $\phi^+$ gives rise to a positive definite centroaffine hypersurface in $\mathbb{R}^{n+1}$ which satisfies the equality case of the inequality (23.7) for the 2-tuple $(n_1, n_2) \in S(n)$ identically.

In particular, if we choose $\phi_2$ to be an ellipsoid centered at the origin, then $K^2 = 0$. Hence, the immersion $\phi^+$ also satisfies the inequality (23.7) for the 1-tuple $(n_1) \in S(n)$ identically.

More general, we can use the same procedure again many times and construct in this way a positive definite centroaffine hypersurface in $\mathbb{R}^{n+1}$.
which satisfies the equality case of the inequality (23.7) for any $k$-tuple $(n_1, n_2, \ldots, n_k)$ in $S(n)$ identically, at least if $n - (n_1 + n_2 + \cdots + n_k) \geq k - 1$.

**Remark 23.3.** Similarly, if we choose $\phi_1$ to be a hyperbolic affine hypersphere centered at the origin, and $\phi_2$ an elliptic affine hypersphere centered at the origin, then the corresponding immersion defined by

$$\phi^-(u_1, u_2, t) = (\cosh t)\phi_1(u_1) + (\sinh t)\phi_2(u_2), \quad u_1 \in M_1, u_2 \in M_2, \ t \in \mathbb{R}$$

gives rise to a negative definite centroaffine hypersurface in $\mathbb{R}^{n+1}$ which satisfies the equality case of the inequality (23.7) for the 2-tuple $(n_1, n_2) \in S(n)$.

Also here we can use the same procedure again many times and construct in this way a negative definite centroaffine hypersurface in $\mathbb{R}^{n+1}$ which satisfies the equality case of the inequality (23.7) for any $k$-tuple $(n_1, n_2, \ldots, n_k) \in S(n)$ identically, at least if $n - (n_1 + n_2 + \cdots + n_k) \geq k - 1$.

## 24 Applications to affine differential geometry (II):
warped products

### 24.1 A realization problem

For a Riemannian $n$-manifold $(M, g)$ with Levi-Civita connection $\nabla$, É. Cartan and A. P. Norden studied nondegenerate affine immersions $f : (M, \nabla) \to \mathbb{R}^{n+1}$ with a transversal vector field $\xi$ and with $\nabla$ as the induced connection.

The well-known Cartan-Norden theorem states that if $f$ is such an affine immersion, then either $\nabla$ is flat and $f$ is a graph immersion or $\nabla$ is not flat and $\mathbb{R}^{n+1}$ admits a parallel Riemannian metric relative to which $f$ is an isometric immersion and $\xi$ is perpendicular to $f(M)$ (cf. for instance, [210, p. 159]) (see, also [123]).

In [76, 82], we study Riemannian manifolds in affine geometry from a viewpoint different from Cartan-Norden. More precisely, we investigate the following:

**Realization Problem:** Which Riemannian manifolds $(M, g)$ can be immersed as affine hypersurfaces in an affine space in such a way that the fundamental form $h$ (e.g. induced by the centroaffine normalization or a constant transversal vector field) is the given Riemannian metric $g$?
We say that a Riemannian manifold \((M, g)\) can be realized as an affine hypersurface if there exists a codimension one affine immersion from \(M\) into some affine space in such a way that the induced affine metric \(h\) is exactly the Riemannian metric \(g\) of \(M\) (notice that we do not put any assumption on the affine connection).

In [82] we prove that Robertson-Walker space times can be realized as centro-affine and graph hypersurfaces in some affine space.

24.2 Existence results

Also, we show in [76] that there exist many warped product Riemannian manifolds which can be realized either as graph or centroaffine hypersurfaces. More precisely, we prove the following.

**Theorem 24.1.** Let \(f = f(s)\) be a positive function defined on an open interval \(I\). Assume that \(\mathbb{R}, S^n(a^2), H^n(-a^2), \) and \(E^n\) are equipped with their canonical metrics. Then we have:

(a) Every warped product surface \(I \times_f \mathbb{R}\) can be realized as a graph surface in the affine 3-space \(\mathbb{R}^3\).

(b) For each integer \(n > 2\), the warped product manifold \(I \times_f H^{n-1}(-a^2)\) can be realized as a graph hypersurface in \(\mathbb{R}^{n+1}\).

(c) If \(f'(s) \neq 0\) on \(I\), then the warped product manifold \(I \times_f E^{n-1}, n > 2,\) can be realized as a graph hypersurface in \(\mathbb{R}^{n+1}\).

(d) If \(f'(s)^2 > a^2\) on \(I\) for some positive number \(a\), then the warped product manifold \(I \times_f S^{n-1}(a^2), n > 2,\) can be realized as a graph hypersurface in \(\mathbb{R}^{n+1}\).

**Theorem 24.2.** The following results hold.

(a) If \(n > 2\) and \(f = f(s)\) is a positive function defined on an open interval \(I\), then we have:

(a.1) If \(f'(s)^2 > f^2(s) - a^2\) on \(I\) for some positive number \(a\), then \(I \times_f H^{n-1}(-a^2)\) can be realized as a centroaffine hypersurface in \(\mathbb{R}^{n+1}\).

(a.2) If \(f'(s)^2 > f(s)^2\) on \(I\), then \(I \times_f E^{n-1}\) can be realized as a centroaffine hypersurface in \(\mathbb{R}^{n+1}\).

(a.3) If \(f'(s)^2 > f(s)^2 + a^2\) on \(I\) for some positive number \(a\), then \(I \times_f S^{n-1}(a^2)\) can be realized as a graph hypersurface in \(\mathbb{R}^{n+1}\).
(b) If \( n = 2 \) and \( f = f(s) \) is a positive function defined on a closed interval \([\alpha, \beta]\), then the warped product surface \( J \times_f \mathbb{R}, J = (\alpha, \beta) \), can always be realized as a centroaffine surface in \( \mathbb{R}^3 \).

### 24.3 An inequality for graph hypersurfaces and its application

We apply in [76] an affine \( \delta \)-invariant and prove the following results.

**Theorem 24.3.** [76] If a warped product manifold \( N_1 \times_f N_2 \) can be realized as a graph hypersurface in \( \mathbb{R}^{n+1} \), then the warping function satisfies

\[
\frac{\Delta f}{f} \geq -\frac{(n_1 + n_2)^2}{4n_2} h(T^\#, T^\#),
\]

(24.1)

where \( n = n_1 + n_2 \), \( n_1 = \dim N_1 \) and \( n_2 = \dim N_2 \).

The following result characterizes affine hypersurfaces which verify the equality case of inequality (24.1).

**Theorem 24.4.** [76] Let \( \phi : N_1 \times_f N_2 \to \mathbb{R}^{n+1} \) be a realization of a warped product manifold as a graph hypersurface. If the warping function satisfies the equality case of (24.1) identically, then we have:

(a) The Tchebychev vector field \( T^\# \) vanishes identically.

(b) The warping function \( f \) is a harmonic function.

(c) \( N_1 \times_f N_2 \) is realized as an improper affine hypersphere.

An application of Theorem 24.3 is the following.

**Corollary 24.1.** If \( N_1 \) is a compact Riemannian manifold, then every warped product manifold \( N_1 \times_f N_2 \) cannot be realized as an improper affine hypersphere in \( \mathbb{R}^{n+1} \).

As an application of Theorems 24.3 and 24.4 we have the following.

**Theorem 24.5.** [76] If the Calabi metric of an improper affine hypersphere in an affine space is the Riemannian product metric of \( k \) Riemannian manifolds, then the improper affine hypersphere is locally the Calabi composition of \( k \) improper affine spheres.

Theorem 24.3 also implies the following.
Corollary 24.2. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( \Delta f < 0 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as an improper affine hypersphere in \( \mathbb{R}^{n+1} \).

24.4 An inequality for centro-affine hypersurfaces and its application

Similarly, for centro-affine hypersurfaces we have the following \[76\].

Theorem 24.6. If a warped product manifold \( N_1 \times_f N_2 \) can be realized as a centroaffine hypersurface in \( \mathbb{R}^{n+1} \), then the warping function satisfies

\[
\frac{\Delta f}{f} \geq n_1 \varepsilon - \frac{(n_1 + n_2)^2}{4n_2} h(T^#, T^#),
\]

where \( n = n_1 + n_2 \), \( n_i = \dim N_i \), \( i = 1, 2 \), \( \Delta \) is the Laplace operator of \( N_1 \), and \( \varepsilon = 1 \) or \( -1 \) according to whether the centroaffine hypersurface is elliptic or hyperbolic.

Theorem 24.7. Let \( \phi : N_1 \times_f N_2 \to \mathbb{R}^{n+1} \) be a realization of a warped product manifold \( N_1 \times_f N_2 \) as a centroaffine hypersurface. If the warping function satisfies the equality case of (24.2) identically, then we have:

1. The Tchebychev vector field \( T^# \) vanishes identically.
2. The warping function \( f \) is an eigenfunction of the Laplacian \( \Delta \) with eigenvalue \( n_1 \varepsilon \).
3. \( N_1 \times_f N_2 \) is realized as a proper affine hypersphere centered at the origin.

Two immediate consequences of Theorem 24.6 are the following.

Corollary 24.3. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( \Delta f \leq 0 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as an elliptic proper affine hypersphere in \( \mathbb{R}^{n+1} \).

Corollary 24.4. If the warping function \( f \) of a warped product manifold \( N_1 \times_f N_2 \) satisfies \( (\Delta f)/f < -\dim N_1 \) at some point on \( N_1 \), then \( N_1 \times_f N_2 \) cannot be realized as a hyperbolic proper affine hypersphere in \( \mathbb{R}^{n+1} \).

Another interesting application of Theorem 24.6 is the following.
Corollary 24.5. If $N_1$ is a compact Riemannian manifold, then every warped product manifold $N_1 \times_f N_2$ with arbitrary warping function cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Another application of Theorem 24.6 is the following.

Corollary 24.6. If $N_1$ is a compact Riemannian manifold, then every warped product manifold $N_1 \times_f N_2$ cannot be realized as an improper affine hypersphere in an affine space $\mathbb{R}^{n+1}$.

24.5 Remarks and examples

The following examples show that the results of this section are optimal.

Example 24.1. Let $M = N_1 \times_{\cos s} N_2$ be the warped product of the open interval $N_1 = (-\pi, \pi)$ and an open portion $N_2$ of the unit $(n-1)$-sphere $S^{n-1}(1)$ equipped with the warped product metric:

$$h = ds^2 + \cos^2 s \left( du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 u_j du_n^2 \right). \quad (24.3)$$

Consider the immersion of $M$ into the affine $(n+1)$-space defined by

$$\left( \sin s, \sin u_2 \cos s, \ldots, \sin u_n \cos s \prod_{j=2}^{n-1} \cos^2 u_j, \cos s \prod_{j=2}^{n-1} \cos u_j \right). \quad (24.4)$$

Then $M$ is a centroaffine elliptic hypersurface whose centroaffine metric is the warped product metric (24.3) and it satisfies $T^\# = 0$. Moreover, the warping function $f = \cos s$ satisfies

$$\frac{\Delta f}{f} = 1 = \varepsilon n_1.$$

Hence, this centroaffine hypersurface satisfies the equality case of (24.2) identically. Consequently, the estimate given in Theorem 24.6 is optimal for centroaffine elliptic hypersurfaces.

Example 24.2. Let $M = \mathbb{R} \times_{\cosh s} H^{n-1}(-1)$ be the warped product of the real line and the unit hyperbolic space $H^{n-1}(-1)$ equipped with warped product metric:

$$h = ds^2 + \cosh^2 s \left( du_2^2 + \cosh^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cosh^2 u_j du_n^2 \right). \quad (24.5)$$
Consider the immersion of \( M \) into the affine \((n + 1)\)-space defined by

\[
\left( \sinh s, \sinh u_2 \cosh s, \ldots, \sinh u_n \cosh s \prod_{j=2}^{n-1} \cosh^2 u_j, \cosh s \prod_{j=2}^{n} \cosh u_j \right).
\]

Then \( M \) is a centroaffine hyperbolic hypersurface whose centroaffine metric is the warped product metric \((24.5)\) and it satisfies \( T^\# = 0 \). Moreover, the warping function \( f = \cosh s \) satisfies

\[
\frac{\Delta f}{f} = -1 = \varepsilon n_1.
\]

Therefore, this centroaffine hypersurface satisfies the equality case of \((24.2)\) identically. Consequently, the estimate given in Theorem \(24.6\) is optimal for centroaffine hyperbolic hypersurfaces as well.

**Example 24.3.** Let \( M = \mathbb{R} \times s N_2 \) be the warped product of the real line and an open portion \( N_2 \) of \( S^{n-1}(1) \) equipped with the warped product metric:

\[
h = ds^2 + s^2 \left( du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 u_j du_n^2 \right). \tag{24.6}
\]

Consider the immersion of \( M \) into the affine \((n + 1)\)-space defined by

\[
s \left( \sin u_2, \sin u_3 \cos u_2, \ldots, \sin u_n \prod_{j=2}^{n-1} \cos^2 u_j, \prod_{j=2}^{n} \cos u_j, s \right). \tag{24.7}
\]

Then \( M \) is a graph hypersurface with Calabi normal given by \( \xi = (0, \ldots, 0, 1) \) and it satisfies \( T^\# = 0 \). Moreover, the Calabi metric of this graph hypersurface is given by the warped product metric \((24.6)\). Clearly, the warping function is a harmonic function. Therefore, this warped product graph hypersurface satisfies the equality case of \((24.1)\) identically. Consequently, the estimate given in Theorem \(24.3\) is also optimal.

**Remark 24.1.** Example \(24.1\) shows that the conditions

\[\Delta f \leq 0\]

in Corollary \(24.3\) and the “harmonicity” in Corollary \(24.4\) are both necessary.
Remark 24.2. Example [24.1] implies that the condition “$N_1$ is a compact Riemannian manifold” given in Corollary [24.6] is necessary.

Remark 24.3. Example [24.2] illustrates that the condition

$$(\Delta f)/f < - \dim N_1$$

given in Corollary [24.5] is sharp.

Remark 24.4. Example [24.3] shows that the condition

$$\Delta f < 0$$

in Corollary [24.1] is optimal as well.

25 Applications to affine differential geometry (III): eigenvalues

For each integer $k \in [2, n]$, we introduce in [81] the affine invariant $\hat{\theta}_k$ on $M$ by

$$\hat{\theta}_k(p) = \left( \frac{1}{k-1} \right) \sup_{L^k, X} \hat{S}_{L^k}(X), \quad p \in T_p M,$$

where $L^k$ runs over all linear $k$-subspaces in the tangent space $T_p M$ at $p$ and $X$ runs over all $h$-unit vectors in $L^k$.

The relative $K$-null space $N^K_p$ of $M$ in $\mathbb{R}^{n+1}$ is defined by

$$N^K_p = \left\{ X \in T_p M : K(X,Y) = 0 \text{ for all } Y \in T_p M \right\}.$$

When $\dim N^K_p$ is constant, $N^K = \cup_{p \in M} N^K_p$ defines a subbundle of the tangent bundle, called the relative $K$-null subbundle.

For affine hypersurfaces in $\mathbb{R}^{n+1}$ we have the following results.

Theorem 25.1. [81] Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$. Then, for any integer $k \in [2, n]$, we have:

1. If $\hat{\theta}_k \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of $K_{T^\#}$ at $p$ is greater than $\left( \frac{n-1}{n} \right) (\varepsilon - \hat{\theta}_k(p))$.

2. If $\hat{\theta}_k(p) = \varepsilon$, every eigenvalue of $K_{T^\#}$ at $p$ is $\geq 0$. 

(3) A nonzero vector $X \in T_p M$ is an eigenvector of the operator $K_{T^\#}$ with eigenvalue $(\frac{n-1}{n}) (\varepsilon - \hat{\theta}_k(p))$ if and only if $\hat{\theta}_k(p) = \varepsilon$ and $X$ lies in the relative $K$-null space $N_p^K$ at $p$, where $\varepsilon = 1$ or $-1$ according to $M$ being of elliptic or hyperbolic type.

**Theorem 25.2.** [81] Let $f : M \to \mathbb{R}^{n+1}$ be a graph hypersurface in $\mathbb{R}^{n+1}$ with positive definite Calabi metric. Then, for any integer $k \in [2,n]$, we have:

1. If $\hat{\theta}_k \neq 0$ at a point $p \in M$, then every eigenvalue of $K_{T^\#}$ at $p$ is greater than $(\frac{1-n}{n}) \hat{\theta}_k(p)$.
2. If $\hat{\theta}_k = 0$ at $p$, then every eigenvalue of $K_{T^\#}$ at $p$ is $\geq 0$.
3. A nonzero vector $X \in T_p M$ is an eigenvector of the operator $K_{T^\#}$ with eigenvalue $(\frac{1-n}{n}) \hat{\theta}_k(p)$ if and only if we have $\hat{\theta}_k(p) = 0$ and $X \in N_p^K$.

Examples were given in [81] to show that the estimates of the eigenvalues given in Theorems 25.1 and 25.2 are both sharp.

As applications of Theorems 25.1 and 25.2 we have:

**Corollary 25.1.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$. If $\sup \hat{K} \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $(\frac{n-1}{n}) (\varepsilon - \sup \hat{K}(p))$.

**Corollary 25.2.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$. If $\sup \hat{S} \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $(\frac{n-1}{n}) (\varepsilon - \sup \hat{S}(p))$.

**Corollary 25.3.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface in $\mathbb{R}^{n+1}$. If we have $\hat{\theta}_k < \varepsilon$ on $M$ for some integer $k \in [2,n]$, then every eigenvalue of $K_{T^\#}$ is positive.

**Corollary 25.4.** An elliptic centroaffine hypersurface $M$ in $\mathbb{R}^{n+1}$ is centroaffinely equivalent to an open portion of a hyperellipsoid if and only if we have

$$nK_{T^\#} = (n-1)(1 - \hat{\theta}_k)I$$

on $M$ for some integer $k \in [2,n]$.

**Corollary 25.5.** A hyperbolic centroaffine hypersurface $M$ in $\mathbb{R}^{n+1}$ is centroaffinely equivalent to an open portion of a two-sheeted hyperboloid if and only if, for some integer $k \in [2,n]$, we have $nK_{T^\#} = (1 - n)(1 + \hat{\theta}_k)I$ identically on $M$. 

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Corollary 25.6. Let $f : M \rightarrow \mathbb{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If we have either $\sup \hat{K} \neq 0$ or $\sup \hat{S} \neq 0$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ is greater than $(1 - \frac{n}{n}) \sup \hat{K}$ at $p$.

Corollary 25.7. Let $f : M \rightarrow \mathbb{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k < 0$ holds on $M$, then every eigenvalue of $K_{T^\#}$ is positive.

Corollary 25.8. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 1$ at some point $p \in M$, then $M$ cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Corollary 25.9. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < -1$ at some point $p \in M$, then $M$ cannot be realized as a hyperbolic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Corollary 25.10. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 0$ at some point $p \in M$, then $M$ cannot be realized as an improper affine hypersphere in $\mathbb{R}^{n+1}$.

26 $\delta^\#(2)$ and affine hypersurfaces

From (23.7) we see that the simplest affine $\delta$-invariant different from the affine scalar curvature $\tilde{\tau}$ is $\delta^\#(2)$, which is defined by

$$\delta^\#(2) = \tilde{\tau} - \sup \hat{K}.$$  \hspace{1cm} (26.1)

In this section we survey some results on affine hypersurfaces involving $\delta^\#(2)$.

26.1 An inequality involving $\delta^\#(2)$

For $\delta^\#(2)$ we have the following general optimal inequality (see Theorem [23.1] for the general affine $\delta$-invariants).

Theorem 26.1. Let $M$ be an $n$-dimensional definite centroaffine hypersurface in an affine $(n + 1)$-space $\mathbb{R}^{n+1}$. Then we have

$$\delta^\#(2) \geq \frac{\epsilon}{2}(n + 1)(n - 2) - \frac{n^2(n - 2)}{2(n - 1)}h(T^\#, T^\#),$$ \hspace{1cm} (26.2)
where $T^\#$ is the Tchebychev vector field, $h$ is the induced affine metric on $M$ and $\epsilon = 1$ or $-1$ according to $M$ is positive-definite or negative-definite, respectively.

Theorem 26.2. \cite{235} Let $M$ be an $n$-dimensional definite centroaffine hypersurface in an affine $(n+1)$-space $\mathbb{R}^{n+1}$. If $M$ realizes the equality in (26.2) at a point, then the Tchebychev vector field $T^\#$ vanishes at $p$ and the equiaffine support function is stationary at $p$. Moreover, if $M$ realizes the equality for every $p \in M$, then $M$ is a proper affine hypersphere centered at the origin.

When $n = 3$, inequality (26.2) reduces to

$$\delta^\#(2) \geq 2\epsilon - \frac{9}{4} h(T^\#, T^\#).$$

(26.3)

Of course also for definite affine hyperspheres in equiaffine (Blaschke) differential geometry the above inequality (but with $T = 0$) remains valid.

Note that $M$ is called an affine hypersphere if and only if the affine shape operator $S$ satisfies $S = H I$, i.e., all affine normals are parallel or pass through a fixed point.

We recall that as $M$ is definite, we can choose the Blaschke normal such that the affine metric is positive definite. This allows us to make still the following distinction between affine hyperspheres:

1. elliptic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H > 0$,
2. hyperbolic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H < 0$,
3. parabolic affine hyperspheres, i.e. all the affine normals are parallel ($H = 0$).

In \cite{168}, Kriele, Scharlach and Vrancken classified 3-dimensional elliptic affine spheres for which the equality in (26.3) is assumed. This classification is achieved through reducing the problem to the problem of classifying those 3-dimensional minimal surfaces in the unit pseudo-sphere $S_3^3(1)$ of index 3 whose ellipses of curvature are circles.

In \cite{168} they also investigated 2-dimensional minimal surfaces in $S_3^3(1)$ whose ellipses of curvature are circles. Also, 3-dimensional hyperbolic affine spheres satisfying the equality in (26.3) are classified by Kriele and Vrancken.
Here, the integrability conditions were solved directly. A crucial point, in both the elliptic and the hyperbolic case, is the existence of an adapted frame such that the difference tensor $K$ has a special simple form.

## 26.2 $\delta^\#(2)$ and affine hypersurfaces admitting a pointwise symmetry

It is known that if an affine 3-sphere satisfies the equality case of inequality \[26.3\], then it admits a pointwise $S_3$-symmetry, but this family is bigger.

The idea to study pointwise symmetries comes from Bryant’s article [24] on special Lagrangian 3-manifolds. In [256], Vrancken studied affine hypersurfaces $M$ admitting a pointwise symmetry, i.e., there exists a subgroup $G$ of $\text{Aut}(T_p M)$ for all $p \in M$ which preserves the affine metric $h$, the difference tensor $K$ and the affine shape operator $S$, i.e., for any $X,Y \in T_p M$ and $g \in G$, we have

\[
\begin{align*}
    h(gX, gY) &= h(X,Y), \\
    K(gX, gY) &= g(K(X,Y)), \\
    S(gX) &= g(SX).
\end{align*}
\]

L. Vrancken introduces the following symmetric polynomial

\[
    f_p(x, y, z) = h(K(xe_1 + ye_2 + ze_3, xe_1 + ye_2 + ze_3), xe_1 + ye_2 + ze_3),
\]

where $\{e_1, e_2, e_3\}$ is an orthonormal basis at the point $p$. The apolarity condition implies that the trace of this polynomial with respect to the metric vanishes.

As far as such symmetric polynomials with vanishing trace on a 3-dimensional real vector space are concerned, we have the following result by Bryant [24]:

**Theorem 26.3.** Let $p \in M$ and assume that there exist an orientation preserving isometry which preserves $f_p$. Then there exists an orthonormal basis of $T_p M$ such that either

(i) $f_p = 0$, in this case $f_p$ is preserved by every isometry,

(ii) $f_p = \lambda(2x^3 - 3xy^2 - 3xz^2)$, for some positive number $\lambda$ in which case $f_p$ is preserved by a 1-parameter group of rotations,
(iii) \( f_p = 6\lambda xyz \) for some positive number \( \lambda \), in which case \( f_p \) is preserved by the discrete group \( A_4 \) of order 12,

(iv) \( f_p = \lambda(x^3 - 3xy^2) \) for some positive number \( \lambda \), in which case \( f_p \) is preserved by the discrete group \( S_3 \) of order 6,

(v) \( f_p = \lambda(2x^3 - 3xy^2 - 3xz^2) + 6\mu xyz \), for some \( \lambda, \mu > 0 \), with \( \lambda \neq \mu \), in which case \( f_p \) is preserved by the group \( \mathbb{Z}_2 \) of order 2,

(vi) \( f_p = \lambda(2x^3 - 3xy^2 - 3xz^2) + \mu(y^3 - 3xy^2) \) for some \( \lambda, \mu > 0 \), with \( \mu \neq \sqrt{2}\lambda \), in which case \( f_p \) is preserved by the group \( \mathbb{Z}_3 \).

Vrancken considers the special class of 3-dimensional affine hyperspheres, such that there exists an orientation preserving isometry which preserves \( f_p \), at every point \( p \). Thus he obtains six different types of expressions for the cubic form and he describes how to construct locally these affine hyperspheres of Type \( k \), from (i) to (vi). In particular, Type (i) corresponds to quadrics, Type (iii) to flat affine hyperspheres and Type (iv) realizes the equality of (26.3) in terms of the affine \( \delta \)-invariant \( \delta^\#(2) \).

In [182], Lu and Scharlach extend the above work and solve the corresponding algebraic problem for a general \( S \). They determine the nontrivial stabilizers \( G \) of the pair \( (K,S) \) under the action of \( \text{SO}(3) \) on a Euclidean vector space \( (V,h) \) and classify three-dimensional locally strongly convex affine hypersurfaces admitting a pointwise \( G \)-symmetry. Besides well-known examples, they obtain warped products of two-dimensional affine spheres and curves.

### 26.3 An improved inequality involving \( \delta^\#(2) \)

Recently, J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken [16] improve inequality (26.2) to the following.

**Theorem 26.4.** Let \( M \) be an \( n \)-dimensional definite centroaffine hypersurface in an affine \( (n+1) \)-space \( \mathbb{R}^{n+1} \). Then we have

\[
\delta^\#(2) \geq \frac{\epsilon}{2}(n+1)(n-2) - \frac{n^2(2n-3)}{2(2n+3)} h(T^\#, T^\#),
\]

where \( \epsilon = 1 \) or \(-1\) according to \( M \) being positive or negative definite.

It is proved in [16] that if \( n \geq 4 \), then the Tchebychev vector field \( T \) vanishes. Moreover, they obtain a classification theorem for definite cen-
troaffine hypersurfaces in $\mathbb{R}^4$ which satisfy the equality case of (26.4) with $T \neq 0$. More precisely, they obtain the following.

**Theorem 26.5.** Let $M$ be a 3-dimensional centroaffine hypersurface of $\mathbb{R}^4$ which attains equality at every point in (26.4). Then $M$ is locally given by the one of the following immersions:

1. \[ f = \frac{(3\lambda - b_1)e^{-3t}}{\sqrt{b_1^2 - 9\lambda^2 + \epsilon}} V + \frac{e^{-t}}{\sqrt{b_1^2 - 9\lambda^2 + \epsilon}} W, \]

where $V$ is a constant vector along the hypersurface, $W$ is a surface for which the Tchebychev form vanishes, and $b_1$ and $\lambda$ are solutions of the following system of ordinary differential equations:

\[
\frac{db_1}{dt} = \frac{b_1^2 - 9\lambda^2 + \epsilon}{3\lambda}, \quad \frac{d\lambda}{dt} = -\frac{2}{3} b_1
\]

and $b_1^2 - 9\lambda^2 + \epsilon \neq 0$.

2. \[ f(t, u, v) = (tu, tv, tg(u, v) + \gamma_2(t), t), \]

where $\gamma_2$ is a function satisfying

\[
t\gamma_2''\gamma_2 - t^2\gamma_2'\gamma_2' - t^2(\gamma_2')^2 + 4\gamma_2'\gamma_2 - 4t\gamma_2'\gamma_2'' = 0
\]

and $(u, v) \mapsto (u, v, g(u, v))$ defines an improper affine sphere with affine normal $(0, 0, 1)$.

27 Applications of $\delta$-invariants to general relativity

In 1916, Albert Einstein and David Hilbert independently constructed the equations for the pure gravitational field (cf. [131])

\[ G_{\lambda\mu} := \text{Ric}_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu}\rho = \kappa T_{\lambda\mu} \]

where Ric is the trace of the Ricci tensor, $T_{\lambda\mu}$ the energy-momentum tensor and $\kappa$ some dimension-transposing parameter.

The remaining part of his scientific career Einstein searched for an unification between his description of gravity and the other known forces, in particular electromagnetism (cf. [139]). Several attempts have been made by using e.g., a non-symmetric metric, a connection with torsion, ..., etc.
Shortly after the publication of the theory of General Relativity, Theodor Kaluza noticed in April 1919 that when he solved Einstein’s equations for General Relativity using five dimensions, Maxwell’s equations for electromagnetism emerged spontaneously. Kaluza wrote to Albert Einstein who encouraged him to publish. His theory was published in 1921 in [155] with Einstein’s support.

In 1926, Oskar Klein [166] suggested that this fifth dimension would be compactified and unobservable on experimentally accessible energy scales. However their work was neglected for many years as attention was directed towards quantum mechanics.

The idea that fundamental forces can be explained by additional dimensions did not re-emerge until string theory was developed. This idea of compactifying the extra dimension has now dominated the search for a unified theory and lead to the 11D supergravity theory and more recently 10D superstring theory. Recently, this strategy of using higher dimensions to unify different forces is also an active area of research in particle physics (see [219] for an overview).

Instead of compactifying the extra dimensions other approaches have also been developed during the last decade. For example, one particular variant of Kaluza–Klein theory is space-time-matter theory or induced matter theory, chiefly promulgated by Paul Wesson and other members of the so-called Space-Time-Matter Consortium (see, [http://astro.uwaterloo.ca/~wesson/]).

In this version of the theory, it is noted that solutions to the equation $R_{AB} = 0$ with $R_{AB}$ the 5D Ricci curvature, may be re-expressed so that in four dimensions, these solutions satisfy Einstein’s equation:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

with the precise form of the $T_{\mu\nu}$ following from the Ricci-flat condition on the 5D space.

Since the energy-momentum tensor $T_{\mu\nu}$ is normally understood to be due to concentrations of matter in 4D space, the above result can be interpreted as saying that 4D matter is induced from geometry in a Ricci-flat 5D space. In particular, the soliton solutions of

$$R_{AB} = 0$$

can be shown to contain the Robertson-Walker metric in both matter-dominated (early universe) and radiation-dominated (present universe) forms.
δ-invariants

The general equations can be shown to be sufficiently consistent with classical tests of general relativity to be acceptable on physical principles, while still leaving considerable freedom to also provide interesting cosmological models (see [259, 260]).

There is another approach proposed by Lisa Randall and Raman Sundrum in 1999. Randall-Sundrum models imagine our Universe as a 5D anti de Sitter space and the elementary particles except for the graviton are localized on a (3 + 1)-D brane or branes.

Their models attempt to address the hierarchy problem between the observed Planck and weak scales by embedding the 3-brane in a warped 5D metric; the warping of the extra dimension is analogous to the warping of space-time in the vicinity of a massive object, such as a black hole (see [225, 226] for details). The Randall–Sundrum scenario has gained a lot of support recently (see [82]).

More recently, S. Haesen and L. Verstraelen [149] extend δ-invariants to pseudo-Riemannian manifolds as follows:

For a given set of mutually orthogonal plane sections \( \{L_j\} \) with dimensions \( n_1, \ldots, n_k \) such that \( n_1 + \cdots + n_k \leq n \), the δ-curvatures in the semi-Riemannian case are given by

\[
\Lambda(n_1, \ldots, n_k) = \tau - \inf \{ \tau(L_1) + \cdots + \tau(L_k) : L_j \text{ a non-null plane section} \}
\]

and

\[
\hat{\Lambda}(n_1, \ldots, n_k) = \tau - \sup \{ \tau(L_1) + \cdots + \tau(L_k) : L_j \text{ a non-null plane section} \}.
\]

Clearly, when the pseudo-Riemannian manifolds are Riemannian, these reduce to \( \delta(n_1, \ldots, n_k) \) and \( \hat{\delta}(n_1, \ldots, n_k) \), respectively.

In [137, 149], Haesen and Verstraelen prove the following.

**Theorem 27.1.** Let \((M, g)\) be an n-dimensional Riemannian or Lorentzian manifold. Assume that \(M\) is locally and isometrically embedded in a \((n + 1)\)-dimensional semi-Riemannian manifold \((N, \tilde{g})\) with diagonalizable Ricci tensor \(\tilde{\text{Ric}}\), i.e., there exists an orthonormal basis \(\{e_\alpha\}\) of \(N\) such that \(\tilde{\text{Ric}} = \sum_{a=1}^{n+1} \lambda_a e_\alpha \otimes e_\alpha\). Then for any k-tuple \((n_1, \ldots, n_k)\) such that \(n_1 < m\) and \(n_1 + \cdots + n_k \leq n\), we have

\[
\|\tilde{H}\|_2^2 \geq c(n_1, \ldots, n_k) \Lambda(n_1, \ldots, n_k) - \frac{1}{2} c(n_1, \ldots, n_k) \left\{ \sum_{\alpha=1}^{n} \varepsilon_\alpha \lambda_\alpha - \lambda_{n+1} \right\},
\]
if \(\text{sign}(N) = (s_M, t_M + 1)\), where

\[
c(n_1, \ldots, n_k) = \frac{2(n + k - \sum_{j=1}^{k} n_j)}{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}.
\]

Equality holds if and only if the second fundamental form has the following form with respect to the eigenframe of the Ricci tensor \(\tilde{\text{Ric}}\):

\[
(\Omega_{\alpha\beta}) = A_{e_{\alpha}} = \begin{pmatrix}
A^r_1 & \ldots & 0 \\
\vdots & \ddots & \vdots & 0 \\
0 & \ldots & A^r_k \\
0 & \ldots & \mu I_s
\end{pmatrix},
\]

where \(s = n - \sum_{j=1}^{k} n_j\) and \(A_{n_j}\) is a symmetric \(n_j \times n_j\)-matrix with trace \((A_{n_j}) = \mu\).

Haesen and Verstraelen applied these \(\delta\)-invariants and inequalities in \cite{149} to investigate the embedding problem of space-times from the viewpoint of ideal embedding. Among others, they found that ideally embedded hypersurfaces in a pseudo-Euclidean space contain both the de Sitter space-time and a Robertson-Walker model. Embeddings of the de Sitter model and Robertson-Walker’s model were already considered by J. Ponce de León \cite{224} in 1988. It was later realized that his 5D embedding space was flat and this was used by S. Seahra and P. Wesson \cite{240} to study the structure of the Big Bang.

Furthermore, S. Haesen and L. Verstraelen \cite{149} also applied \(\delta\)-invariant to find a class of anisotropic perfect fluid models containing a timelike two-surface of constant curvature; also been shown to be ideally embedded. (For further results in this direction, see also \cite{147}.)

As we mentioned earlier in \textbf{24.1}, the author has recently constructed explicit embeddings of Robertson-Walker’s cosmological models as centroaffine and graph hypersurfaces in some flat affine space so that the induced affine
metrics are exactly the space-time metrics on the Robertson-Walker space-times (see 82 for details).

Since many physics quantities are represented by the metric and its curvatures, most physical quantities of the Robertson-Walker space-times are preserved via the realizations constructed in 82. These embeddings allow us to study Robertson-Walker space-times and their submanifolds using the method of affine differential geometry. Furthermore, in contrast to space-time-matter theory in which matter could be either space-like or time-like depending on embeddings, if additional physical quantities were induced from the extra dimension via our embeddings it would then be neutral (in the sense that it is neither space-like nor time-like).

28 \( k \)-Ricci curvature and shape operator

In this section we explain a sharp relationship between the \( k \)-Ricci curvature and the shape operator for arbitrary submanifolds in a real space form with arbitrary codimension (see 51 for details).

Recall that for a submanifold \( M \) in a Riemannian manifold, the relative null space of \( M \) at a point \( p \in M \) is defined by

\[
N_p = \{ X \in T_p M : h(X, Y) = 0 \text{ for all } Y \in T_p M \}.
\]

Moreover, for each integer \( k, 2 \leq k \leq n \), the invariant \( \theta_k \) on a Riemannian manifold is defined by

\[
\theta_k(p) = \left( \frac{1}{k-1} \right) \inf_{L^k, X} \operatorname{Ric}_{L^k}(X), \quad p \in T_p M^n,
\]

where \( L^k \) runs over all \( k \)-plane sections at \( p \) and \( X \) runs over all unit vectors in \( L^k \).

The following result from 51 provides a sharp relationship between the invariant \( \theta_k \) and the shape operator \( A_H \) for arbitrary submanifolds, regardless of codimension.

**Theorem 28.1.** Let \( x : M \to R^n(\epsilon) \) be an isometric immersion of a Riemannian \( n \)-manifold \( M \) into a real space form \( R^n(\epsilon) \) of constant sectional curvature \( \epsilon \). Then, regardless of codimension, for any integer \( k, 2 \leq k \leq n \), and any point \( p \in M \) we have:
(1) If $\theta_k(p) \neq \epsilon$, then the shape operator at the mean curvature vector $H$ satisfies
\[ A_H > \frac{n-1}{n}(\theta_k(p) - \epsilon)I \quad \text{at } p, \]
where $I$ denotes the identity map of $T_pM$.

(2) If $\theta_k(p) = \epsilon$, then $A_H \geq 0$ at $p$.

(3) A unit vector $X \in T_pM$ satisfies
\[ A_H X = \frac{n-1}{n}(\theta_k(p) - \epsilon)X \]
if and only if $\theta_k(p) = \epsilon$ and $X$ lies in the relative null space at $p$.

(4) We have
\[ A_H \equiv \frac{n-1}{n}(\theta_k - \epsilon)I \]
at a point $p \in M$ if and only if $p$ is a totally geodesic point, i.e., the second fundamental form vanishes identically at $p$.

Remark 28.1. Clearly, the estimate of $A_H$ given in statement (2) of Theorem 28.1 is sharp. Here we provide an example to illustrate that statement (1) of Theorem 28.1 is also sharp.

Consider a hyperellipsoid in $E^{n+1}$ defined by
\[ ax_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1, \]
where $0 < a < 1$. The principal curvatures of the hyperellipsoid are given by
\[ a_1 = \frac{a}{(1 + a(a-1)x_1^2)^{3/2}}, \]
\[ a_2 = \ldots = a_n = \frac{1}{(1 + a(a-1)x_1^2)^{1/2}}. \]
Therefore, for any $k$, $2 \leq k \leq n$, the $k$-Ricci curvatures at a point $p$ satisfies
\[ \text{Ric}_{L^k}(X) \geq (k-1)\theta_k(p) := \frac{(k-1)a}{(1 + a(a-1)x_1^2)^2} > 0 \]
for any $k$-plane section $L^k$ and any unit vector $X$ in $L^k$. Moreover, the eigenvalues $\kappa_1, \ldots, \kappa_n$ of the shape operator $A_H$ are given by
\[ \kappa_1 = \ldots = \kappa_{n-1} = \frac{a + (n-1)(1 + a(a-1)x_1^2)}{n(1 + a(a-1)x_1^2)^2}, \]
\[ \kappa_n = \frac{a(a + (n - 1)(1 + a(a - 1)x_1^2))}{n(1 + a(a - 1)x_1^2)^3}. \]

From these it follows that
\[ A_H > \frac{n - 1}{n} \theta_k(p) I_n, \]
and
\[ \kappa_1 = \frac{n - 1}{n} \theta_k(p) = \frac{a^2}{n(1 + a(a - 1)x_1^2)^3} \rightarrow 0, \text{ as } a \rightarrow 0. \]

Hence, this example shows that our estimate of \( A_H \) in statement (1) is also sharp.

One may also apply Theorem 28.1 to provide a lower bound for every eigenvalue of the shape operator \( A_H \) for all isometric immersions of a given Riemannian \( n \)-manifold, regardless of codimension. As the simplest example, Theorem 28.1 implies immediately the following.

**Corollary 28.1.** Let \( M \) be a Riemannian \( n \)-manifold. If there is a point \( p \in M \) such that every sectional curvature of \( M \) at \( p \) is equal to 1, then, for any isometric immersion of \( M \) in a Euclidean \( m \)-space with arbitrary codimension, each eigenvalue of the shape operator \( A_H \) at \( p \) is greater than \( (n - 1)/n \).

**Remark 28.2.** The estimate of the eigenvalues of \( A_H \) given in Corollary 28.1 is sharp. For instance, assume that \( M \) is a surface in \( \mathbb{E}^3 \) whose two principal curvatures at \( p \) are given respectively by \( a \) and \( 1/a \) with \( a \geq 1 \). Then the smaller eigenvalue of \( A_H \) at \( p \) is equal to \( (a^2 + 1)/2a^2 \) which approaches to 1/2 when \( a \) approaches \( \infty \).

For an \( n \)-dimensional submanifold \( M \) in \( \mathbb{E}^m \), let \( \mathbb{E}^{n+1} \) be the linear subspace of dimension \( n + 1 \) spanned by the tangent space at a point \( p \in M \) and the mean curvature vector \( H(p) \) at \( p \).

Geometrically, the shape operator \( A_{n+1} \) of \( M \) in \( \mathbb{E}^m \) is the shape operator of the orthogonal projection of \( M^n \) into \( \mathbb{E}^{n+1} \). Moreover, it is known that if the shape operator of a hypersurface in \( \mathbb{E}^{n+1} \) is definite at a point \( p \), then it is strictly convex at \( p \). For this reason a submanifold \( M \) in \( \mathbb{E}^m \) is said to be \( H \)-strictly convex if the shape operator \( A_H \) is positive-definite at each point in \( M \).

Theorem 28.1 implies immediately the following.
Corollary 28.2. Let $M$ be a submanifold of a Euclidean space. If there is an integer $k$, $2 \leq k \leq n$, such that $k$-Ricci curvatures of $M$ are positive, then $M$ is $H$-strictly convex, regardless of codimension.

By applying Theorem 28.1 and a classical result of W. Süss we have the following.

Corollary 28.3. If $M$ is a compact hypersurface of $\mathbb{E}^{n+1}$ with $\theta_k \geq 0$ (respectively, with $\theta_k > 0$) for a fixed $k$, $2 \leq k \leq n$, then $M$ is embedded as a convex (respectively, strictly convex) hypersurface in $\mathbb{E}^{n+1}$. In particular, if $M$ has constant scalar curvature, then $M$ is a hypersphere of $\mathbb{E}^{n+1}$.

Corollary 28.4. If $M$ is a compact hypersurface of $\mathbb{E}^{n+1}$ with nonnegative Ricci curvature (respectively, with positive Ricci curvature), then $M$ is embedded as a convex (respectively, strictly convex) hypersurface in $\mathbb{E}^{n+1}$. In particular, if $M$ has constant scalar curvature, then $M$ is a hypersphere of $\mathbb{E}^{n+1}$.

29 General inequalities (I): CR-products

Several general geometric optimal inequalities for submanifolds in complex space forms (or more generally in Kähler manifolds) have been discovered during the last decades. Although these inequalities do not relate directly with the $\delta$-invariants, however due to their simplicity we present them as well.

29.1 Segre embedding introduced in 1891

Let $(z_{i_0}^1, \ldots, z_{N_i}^1)$ ($1 \leq i \leq s$) denote the homogeneous coordinates of $CP^{N_i}$. Define a map:

$$S_{N_1 \cdots N_s} : CP^{N_1}(4) \times \cdots \times CP^{N_s}(4) \to CP^N(4), \quad N = \prod_{i=1}^{s} (N_i + 1) - 1,$$

which maps a point

$$(z_0^1, \ldots, z_{N_1}^1), \ldots, (z_0^s, \ldots, z_{N_s}^s)$$

of the product Kähler manifold $CP^{N_1}(4) \times \cdots \times CP^{N_s}(4)$ to the point

$$(z_1^i \cdots z_i^s)_{1 \leq i_1 \leq N_1, \ldots, 1 \leq i_s \leq N_s} \in CP^N(4).$$
The map $S_{N_1, \ldots, N_s}$, introduced by C. Segre [241] in 1981, is a Kähler embedding which is known as the Segre embedding. The Segre embedding plays some important role in algebraic geometry.

29.2 “Converse” of Segre embedding – 1981

In 1981, Chen [33] and Chen and W. E. Kuan [95] established the “converse” of the Segre embedding as follows:

**Theorem 29.1.** Let $M_1, \ldots, M_s$ be Kähler manifolds of dimensions $N_1, \ldots, N_s$, respectively. Then every Kähler immersion

$$\phi : M_1 \times \cdots \times M_s \rightarrow \mathbb{C}P^N(4), \quad N = \prod_{i=1}^{s} (N_i + 1) - 1,$$

of $M_1 \times \cdots \times M_s$ into $\mathbb{C}P^N(4)$ is locally given by the Segre embedding, i.e., $M_1, \ldots, M_s$ are open portions of $\mathbb{C}P^{N_1}(4), \ldots, \mathbb{C}P^{N_s}(4)$, respectively, and moreover, the Kähler immersion $\phi$ is congruent to the Segre embedding.

29.3 CR-submanifolds

Let $(\tilde{M}, g, J)$ be a Kähler manifold with complex structure $J$ and $M$ a Riemannian manifold isometrically immersed in $\tilde{M}$. For each tangent vector $X$ of $M$, we put

$$JX = PX + FX,$$

where $PX$ and $FX$ are the tangential and the normal components of $JX$. Then $P$ is an endomorphism of $TM$ and $F$ is a normal-bundle-valued 1-form.

For each point $x \in N$, we denote by $D_x$ the maximal holomorphic subspace of the tangent space $T_xM$ defined by

$$D_x = T_xM \cap J(T_xM).$$

If the dimension of $D_x$ is the same for all $x \in M$, $D_x$’s define a distribution $D$ on $M$, which is called the holomorphic distribution of $M$.

A submanifold $M$ in a Kähler manifold $\tilde{M}$ is called a CR-submanifold if there exists a holomorphic distribution $D$ on $M$ whose orthogonal complement $D^\perp$ is a totally real distribution, i.e., $JD^\perp \subset T^\perp N$ (cf. [11]). If
dim $\mathcal{D}_x = 0$ at each point $x \in M$, then the $CR$-submanifold $M$ is nothing but a totally real submanifold.

In [13], D. E. Blair and the author proved that every $CR$-submanifold of a Hermitian manifold is a $CR$-manifold.

In 1978, the author discovered that the totally real distribution $\mathcal{D}^\perp$ of a Kähler manifold is always completely integrable. This integrability theorem implies that every proper $CR$-submanifold of a Kähler manifold is foliated by totally real submanifolds. By applying this integrability theorem, A. Bejancu [10] proved in 1979 that a $CR$-submanifold of a Kähler manifold is mixed totally geodesic if and only if each leaf of the totally real distribution is totally geodesic in the $CR$-submanifold.

Chen’s integrability theorem was later extended to $CR$-submanifolds of various families of Hermitian manifolds by various geometers. For instance, this theorem was extended to $CR$-submanifolds of locally conformal symplectic manifolds by Blair and Chen in [13]. Furthermore, they constructed in [13] $CR$-submanifolds in some Hermitian manifolds with non-integrable totally real distributions.

For a $CR$-submanifold $M$ with Riemannian connection $\nabla$, let $e_1, \ldots, e_{2h}$ be an orthonormal frame field of the holomorphic distribution $\mathcal{D}$. Put $\hat{H} = \text{trace } \hat{\sigma}$, where $\hat{\sigma}(X, Y) = (\nabla_X Y)\perp$ is the component of $\nabla_X Y$ in the totally real distribution. The holomorphic distribution $\mathcal{D}$ is called a minimal distribution if $\hat{H} = 0$, identically.

Although the holomorphic distribution is not necessarily integrable in general, the author proved in [33] that the holomorphic distribution of a $CR$-submanifold is always a minimal distribution. He also discovered in [36] a canonical cohomology class $c(M) \in H^{2k}(M; \mathbb{R})$ for every compact $CR$-submanifold $M$ of a Kähler manifold. By applying this cohomology class, he proved the following:

Let $M$ be a compact $CR$-submanifold of a Kähler manifold. If the cohomology group $H^{2k}(M; \mathbb{R}) = \{0\}$ for some integer $k \leq h$, then either the holomorphic distribution $\mathcal{D}$ is not integrable or the totally real distribution $\mathcal{D}^\perp$ is not minimal.

The cohomology class $c(M)$ was applied by S. Dragomir in his study concerning the minimality of Levi distribution (cf. [131]).

A. Ros [225] proved that if $M$ is an $n$-dimensional compact minimal $CR$-submanifold of $CP^m(4)$, then the first nonzero eigenvalue of the Laplacian
\( \delta \)-invariants

of \( M \) satisfies

\[
\lambda_1 \leq \frac{2}{n} (n^2 + 4h + p).
\]

For further results on CR-submanifolds, see for instance [11, 33, 34, 36, 56, 262].

29.4 CR-products

Recall that a submanifold of a Kähler manifold is called a CR-product if it is the Riemannian product \( N_T \times N_\perp \) of a complex submanifold \( N_T \) and a totally real submanifold \( N_\perp \).

Let \( f : N_\perp \to CP^p(4) \) be a Lagrangian submanifold. Then the composition:

\[
CP^h(4) \times N_\perp \overset{i \times f}{\longrightarrow} CP^h(4) \times CP^p(4) \overset{S_{hp}}{\longrightarrow} CP^{h+p+hp}(4) \quad (29.2)
\]

is a CR-product in \( CP^{h+p+hp} \), where \( i : CP^h \to CP^h \) denotes the identity map and \( S_{hp} \) is the Segre embedding.

A CR-product \( M = N_T \times N_\perp \) in \( CP^m(4) \) is called a standard CR-product if \( m = h+p+hp \) and \( N_T \) is a totally geodesic Kähler submanifold of \( CP^m(4) \).

It is known that (see [33] for details):

**Theorem 29.2.** The following statements hold:

1. CR-products in complex hyperbolic spaces are either complex or totally real.

2. A submanifold in a complex Euclidean space is a CR-product if and only if it is the direct sum of a complex submanifold and a totally real submanifold of linear complex subspaces.

3. CR-products in the complex projective space \( CP^{h+p+hp}(4) \) are obtained from the Segre embedding in a natural way.

29.5 A general inequality for submanifolds in complex space forms

For an arbitrary submanifold \( M \) in a complex space form \( \tilde{M}^m(4\epsilon) \), we have the following general inequality.
Theorem 29.3. \[96\] Let \( M \) be a submanifold of a complex space form \( \tilde{M}^m(4\epsilon) \). Then we have
\[
||\nabla h||^2 \geq 2\epsilon^2 ||P||^2 ||F||^2,
\]
(29.3)
with equality sign holding if and only if \( M \) is a cyclic-parallel CR-submanifold.

Here, by a cyclic-parallel submanifold we mean a submanifold whose second fundamental form \( h \) satisfies
\[
(\bar{\nabla}_X h)(Y,Z) + (\bar{\nabla}_Y h)(Z,X) + (\bar{\nabla}_Z h)(X,Y) = 0
\]
(29.4)
for any vectors \( X, Y, Z \) tangent to \( M \).

It follows from polarization that a submanifold of a real space form is cyclic-parallel if and only if it is a parallel submanifold. For submanifolds in \( CH^m(-4) \), the author, G. D. Ludden and S. Montiel \[96\] showed that a CR-submanifold \( M \) of \( CH^m(-4) \) is cyclic-parallel if and only if the preimage \( \pi^{-1}(M) \) of \( M \) (via the Hopf fibration \( \pi : H^{2m+1}_{1}(-1) \to CH^m(-4) \)) has parallel second fundamental form in \( H^{2m+1}_{1}(-1) \). Similar result also holds for cyclic-parallel CR-submanifolds in \( CP^m(4) \) (see [262]). For the classification of cyclic-parallel CR-submanifolds of complex space forms, see [96, 262].

29.6 Two inequalities for CR-products

For CR-products in complex projective space \( CP^m(4) \), we have the following general inequality involving the squared norm of the second fundamental form.

Theorem 29.4. \[33\] Let \( M \) be a CR-product in \( CP^m(4) \). Then we have
\[
||h||^2 \geq 4hp,
\]
(29.5)
where \( h = \dim_{\mathbb{C}} H_x \) and \( p = \dim_{\mathbb{R}} H^\perp \).

If the equality sign of (29.5) holds, then \( N_T \) and \( N_\perp \) are both totally geodesic in \( CP^m(4) \). Moreover, the immersion is rigid and the CR-product is a standard one.

For minimal CR-products in \( CP^m(4) \), we also have the following.
Theorem 29.5. If $M$ is a minimal CR-product in $CP^m(4)$, then the scalar curvature of $M$ satisfies
\[ \tau \geq 2h^2 + 2h + \frac{1}{2}(p^2 - p), \]
with the equality holding when and only when $||h||^2 = 4hp$ holds.

30 General inequalities (II): CR-warped products

In [60], the author proved that there do not exist any warped product submanifold of the form: $N_T \times_f N_\perp$ in any Kähler manifold $\tilde{M}$ such that $N_T$ is a complex submanifold and $N_\perp$ is a totally real submanifold of $\tilde{M}$. Moreover, he combined in [60] the notion of warped products with the notion of CR-submanifolds to introduce the notion of CR-warped products as follows:

Definition 30.1. A submanifold of a Kähler manifold $M$ is called a CR-warped product if it is a warped product $N_T \times_f N_\perp$ of a complex submanifold $N_T$ and a totally real submanifold $N_\perp$.

30.1 Optimal inequalities involving $||h||^2$

For arbitrary CR-warped products in an arbitrary Kähler manifold, the author proved the following [60].

Theorem 30.1. Let $M = N_T \times_f N_\perp$ be a CR-warped product in an arbitrary Kähler manifold $\tilde{M}$. Then we have:

1. The squared norm of the second fundamental form of $M$ satisfies
   \[ ||h||^2 \geq 2p ||\nabla(\ln f)||^2, \quad (30.1) \]
   where $\nabla \ln f$ is the gradient of $\ln f$ and $p$ is the dimension of $N_\perp$.

2. If the equality sign of (30.1) holds identically, then $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$. Moreover, $M$ is a minimal submanifold in $\tilde{M}$.

3. If $M$ is anti-holomorphic and $p > 1$, then the equality sign of (30.1) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

4. If $M$ is anti-holomorphic and $p = 1$, then the equality sign of (30.1) holds identically if and only if the characteristic vector field $J\xi$ of $M$ is a
principal vector field with zero as its principal curvature. (Notice that in this case, \( M \) is a real hypersurface in \( \tilde{M} \).

Also, in this case, the equality sign of (30.1) holds identically if and only if \( M \) is a minimal hypersurface in \( \tilde{M} \).

CR-warped products in complex space forms attaining the equality case of (30.1) were classified in [61]. In fact, we have the following results.

Theorem 30.2. A non-trivial CR-warped product \( N_T \times_f N_\perp \) in \( CP^m(4) \) satisfies the basic equality case of inequality (30.1), i.e.,

\[
||h||^2 = 2p||\nabla(\ln f)||^2 \tag{30.2}
\]

if and only if we have

1. \( N_T \) is an open portion of complex Euclidean \( h \)-space \( C^h \),
2. \( N_\perp \) is an open portion of a unit \( p \)-sphere \( S^p \), and
3. up to rigid motions, the immersion \( \mathbf{x} \) of \( N_T \times_f N_\perp \) into \( CP^m(4) \) is the composition \( \pi \circ \tilde{\mathbf{x}} \), where

\[
\tilde{\mathbf{x}}(z, w) = \left( z_0 + (w_0 - 1)a_0 \sum_{j=0}^{h} a_j z_j, \ldots, z_h + (w_0 - 1)a_h \sum_{j=0}^{h} a_j z_j, w_1 \sum_{j=0}^{h} a_j z_j, \ldots, w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \right),
\]

where \( \pi \) is the projection \( \pi : C^{m+1}_x \to CP^m(4) \), \( z = (z_0, z_1, \ldots, z_h) \in C^{h+1} \) and \( w = (w_0, \ldots, w_p) \in S^p \subset E^{p+1} \), and \( a_0, \ldots, a_h \) are real numbers satisfying

\[
a_0^2 + a_1^2 + \ldots + a_h^2 = 1.
\]

Theorem 30.3. A CR-warped product \( N_T \times_f N_\perp \) in \( CH^m(-4) \) satisfies the equality case of (30.1) if and only if one of the following two cases occurs:

1. \( N_T \) is an open portion of complex Euclidean \( h \)-space \( C^h \), \( N_\perp \) is an open portion of a unit \( p \)-sphere \( S^p \) and, up to rigid motions, the immersion is the composition \( \pi \circ \tilde{\mathbf{x}} \), where \( \pi \) is the projection \( \pi : C^{m+1}_x \to CH^m(-4) \)
and

\[
\begin{align*}
\tilde{x}(z,w) &= \left(z_0 + a_0(1 - w_0) \sum_{j=0}^{h} a_j z_j, z_1 + a_1(w_0 - 1) \sum_{j=0}^{h} a_j z_j, \ldots , \\
z_h + a_h(w_0 - 1) \sum_{j=0}^{h} a_j z_j, w_1 \sum_{j=0}^{h} a_j z_j, \ldots , w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots , 0 \right),
\end{align*}
\]

where \(z = (z_0, \ldots , z_h) \in C_{h+1}^1\), \(w = (w_0, \ldots , w_p) \in S^p \subset E^{p+1}\) and \(a_0, \ldots , a_h\) are real numbers satisfying

\[
a_0^2 - a_1^2 - \cdots - a_h^2 = -1.
\]

(2) \(p := \dim N_{\perp} = 1, N_T\) is an open portion of \(C^h\) and, up to rigid motions, the immersion is the composition \(\pi \circ \tilde{x}\), where

\[
\begin{align*}
\tilde{x}(z,t) &= \left(z_0 + a_0(\cosh t - 1) \sum_{j=0}^{h} a_j z_j, z_1 + a_1(1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \\
&\quad \ldots , z_h + a_h(1 - \cosh t) \sum_{j=0}^{h} a_j z_j, \sinh t \sum_{j=0}^{h} a_j z_j, 0, \ldots , 0 \right)
\end{align*}
\]

for some real numbers \(a_0, a_1, \ldots , a_h\) satisfying \(a_0^2 - a_1^2 - \cdots - a_h^2 = 1\).

There exists another general optimal inequality involving \(\|h\|^2\) for CR-warped products in complex space forms. In fact, we have the following results form \([68]\).

**Theorem 30.4.** Let \(\phi : N_T \times_f N_{\perp} \to C^m\) be a CR-warped product in complex Euclidean m-space \(C^m\). Then we have

1. The squared norm of the second fundamental form of \(\phi\) satisfies

\[
\|h\|^2 \geq 2p\{\|\nabla (\ln f)\|^2 + \Delta (\ln f)\}.
\]

2. If the CR-warped product satisfies the equality case of (30.3), then we have

2.a) \(N_T\) is an open portion of \(C^h_*\);
2.b) \(N_{\perp}\) is an open portion of \(S^p\);
2.c) There exists a natural number \(\alpha \leq h\) and a complex coordinate system \(\{z_1, \ldots , z_h\}\) on \(C^h_*\) such that the warping function \(f\) is given by

\[
f = \sqrt{\sum_{j=1}^{\alpha} z_j \bar{z}_j}.
\]
(2.d) Up to rigid motions of $\mathbb{C}^m$, the immersion $\phi$ is given by $\phi^h_{\alpha}$ in a natural way; namely, we have

$$\phi(z, w) = (w_0z_1, \ldots, w_pz_1, \ldots, w_0z_\alpha, \ldots, w_pz_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0)$$

for $z = (z_1, \ldots, z_h) \in \mathbb{C}_h^h$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$.

**Theorem 30.5.** Let $\phi : N_T \times_f N_\perp \to CP^m(4)$ be a CR-warped product. Then

(1) The squared norm of the second fundamental form of $\phi$ satisfies

$$||h||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\} + 4hp.$$  \hspace{1cm} (30.4)

(2) The CR-warped product satisfies the equality case of (30.4) if and only if

(2.i) $N_T$ is an open portion of complex projective $h$-space $CP^h(4)$;

(2.ii) $N_\perp$ is an open portion of unit $p$-sphere $S^p$; and

(2.iii) There exists a natural number $\alpha \leq h$ such that, up to rigid motions, $\phi$ is the composition $\pi \circ \tilde{\phi}$, where

$$\tilde{\phi}(z, w) = (w_0z_0, \ldots, w_pz_0, \ldots, w_0z_\alpha, \ldots, w_pz_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0)$$

for $z = (z_0, \ldots, z_h) \in \mathbb{C}_s^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$, and $\pi$ being the projection $\pi : \mathbb{C}_s^{m+1} \to CP^m(4)$.

**Theorem 30.6.** Let $\phi : N_T \times_f N_\perp \to CH^m(-4)$ be a CR-warped product. Then

(1) The squared norm of the second fundamental form of $\phi$ satisfies

$$||h||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\} - 4hp.$$  \hspace{1cm} (30.5)

(2) The CR-warped product satisfies the equality case of (30.5) if and only if

(2.a) $N_T$ is an open portion of complex hyperbolic $h$-space $CH^h(-4)$;

(2.b) $N_\perp$ is an open portion of unit $p$-sphere $S^p$ (or $\mathbb{R}$, when $p = 1$); and

(2.c) up to rigid motions, $\phi$ is the composition $\pi \circ \tilde{\phi}$, where either $\tilde{\phi}$ is

$$\tilde{\phi}(z, w) = (z_0, \ldots, z_\beta, w_0z_\beta+1, \ldots, w_pz_\beta+1, \ldots, w_0z_h, \ldots, w_pz_h, 0, \ldots, 0)$$

for $z = (z_0, \ldots, z_h) \in \mathbb{C}_s^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$, and $\pi$ being the projection $\pi : \mathbb{C}_s^{m+1} \to CH^m(-4)$. 


\(\delta\)-invariants

for \(0 < \beta \leq h\), \(z = (z_0, \ldots, z_h) \in C_{s+1}^{h+1}\) and \(w = (w_0, \ldots, w_p) \in S^p\), or \(\phi\) is

\[\phi(z, u) = \left( z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \ldots, \right.\]
\[\left. z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0 \right)\]

for \(z = (z_0, \ldots, z_h) \in C_{s+1}^{h+1}\) and \(u \in \mathbb{R}\), and \(\pi\) being the projection

\[\pi : C_{s+1}^{m+1} \to CH^m(-4)\]

### 30.2 CR-warped products with compact holomorphic factor \(N_T\)

When the holomorphic factor \(N_T\) of a CR-warped product is a compact manifold, we have following additional results from [73] also involving the squared norm of the second fundamental form.

**Theorem 30.7.** For any CR-warped product \(N_T \times_f N_\perp\) in \(CP^m(4)\) with compact \(N_T\) and any \(q \in N_\perp\), we have

\[
\int_{N_T \times \{q\}} ||h||^2 dV_T \geq 4hp \text{vol}(N_T). \tag{30.6}
\]

The equality sign of (30.6) holds identically if and only if we have:

(i) The warping function \(f\) is constant.

(ii) \((N_T, g_{N_T})\) is holomorphically isometric to \(CP^h(4)\) and it is isometrically immersed in \(CP^m(4)\) as a totally geodesic complex submanifold.

(iii) \((N_\perp, f^2 g_{N_\perp})\) is isometric to an open portion of the real projective \(p\)-space \(RP^p(1)\) of constant sectional curvature one and it is isometrically immersed in \(CP^m(4)\) as a totally geodesic totally real submanifold.

(iv) \(N_T \times_f N_\perp\) is immersed linearly fully in a linear complex subspace \(CP^{h+p+h_\perp}(4)\) of \(CP^m(4)\).

Moreover, the immersion is rigid.

**Theorem 30.8.** If \(N_T \times_f N_\perp\) is a CR-warped product in \(CP^{h+p+h_\perp}(4)\) with compact \(N_T\), then \(N_T\) is holomorphically isometric to \(CP^h(4)\).

**Theorem 30.9.** Let \(N_T \times_f N_\perp\) be a CR-warped product with compact \(N_T\) in \(CP^m(4)\). If the warping function \(f\) is non-constant, then, for each \(q \in N_\perp\),
we have
\[ \int_{N_T \times \{q\}} ||h||^2 dV_T \geq 2p\lambda_1 \int_{N_T} (\ln f)^2 dV_T + 4hp \text{vol}(N_T), \tag{30.7} \]
where $dV_T$, $\lambda_1$ and $\text{vol}(N_T)$ are the volume element, the first positive eigenvalue of the Laplacian $\Delta$ and the volume of $N_T$, respectively.

Moreover, the equality sign of (30.7) holds identically if and only if we have

(a) $\Delta \ln f = \lambda_1 \ln f$.

(b) The CR-warped product is both $N_T$-totally geodesic and $N_\perp$-totally geodesic.

The following example shows that Theorems 30.7, 30.8 and 30.9 are sharp.

**Example 30.1.** Let $\iota_1$ be the identity map of $CP^h(4)$ and let $\iota_2 : RP^p(1) \to CP^p(4)$ be a totally geodesic Lagrangian embedding of $RP^p(1)$ into $CP^p(4)$. Denote by $\iota = (\iota_1, \iota_2) : CP^h(4) \times RP^p(1) \to CP^h(4) \times CP^p(4)$ the product embedding of $\iota_1$ and $\iota_2$. Moreover, let $S_{h,p}$ be the Segre embedding of $CP^h(4) \times CP^p(4)$ into $CP^{h+p+h+p}(4)$. Then the composition $\phi = S_{h,p} \circ \iota$:

\[ CP^h(4) \times RP^p(1) \xrightarrow{(\iota_1, \iota_2)} CP^h(4) \times CP^p(4) \xrightarrow{S_{h,p}} CP^{h+p+h+p}(4) \]

is a CR-warped product in $CP^{h+p+h+p}$ whose holomorphic factor $N_T = CP^h(4)$ is a compact manifold.

Since the second fundamental form of $\phi$ satisfies the equation: $||\sigma||^2 = 4hp$, we have the equality case of (30.6) identically.

The next example shows that the assumption of compactness in Theorems 30.7, 30.8 and 30.9 cannot be removed.
**Example 30.2.** Let $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mathbb{C}^{m+1}_* = \mathbb{C}^{m+1} - \{0\}$. Denote by \{z_0, ..., z_h\} a natural complex coordinate system on $\mathbb{C}^{m+1}_*$. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{m+1}_*$ defined by
\[
\lambda \cdot (z_0, ..., z_m) = (\lambda z_0, ..., \lambda z_m)
\]
for $\lambda \in \mathbb{C}^*$. Let $\pi(z)$ denote the equivalent class containing $z$ under this action. Then the set of equivalent classes is the complex projective $m$-space $\mathbb{CP}^m(4)$ with the complex structure induced from the complex structure on $\mathbb{C}^{m+1}_*$.

For any two natural numbers $h$ and $p$, we define a map:
\[
\tilde{\phi} : \mathbb{C}^{h+1}_* \times S^p(1) \to \mathbb{C}^{h+p+1}_*
\]
by
\[
\tilde{\phi}(z_0, ..., z_h; w_0, ..., w_p) = (w_0z_0, w_1z_0, ..., w_pz_0, z_1, ..., z_h)
\]
for $(z_0, ..., z_h)$ in $\mathbb{C}^{h+1}_*$ and $(w_0, ..., w_p)$ in $S^p(1)$ with $\sum_{j=0}^{p} w_j^2 = 1$.

Since the image of $\tilde{\phi}$ is invariant under the action of $\mathbb{C}^*$, the composition:
\[
\pi \circ \tilde{\phi} : \mathbb{C}^{h+1}_* \times S^p(1) \xrightarrow{\tilde{\phi}} \mathbb{C}^{h+p+1}_* \xrightarrow{\pi} \mathbb{CP}^{h+p}(4)
\]
induces a CR-immersion of the product manifold $N_T \times S^p(1)$ into $\mathbb{CP}^{h+p}(4)$, where
\[
N_T = \{(z_0, ..., z_h) \in \mathbb{CP}^h(4) : z_0 \neq 0\}
\]
is a proper open subset of $\mathbb{CP}^h(4)$. Clearly, the induced metric on $N_T \times S^p(1)$ is a warped product metric and the holomorphic factor $N_T$ is non-compact.

Notice that the complex dimension of the ambient space is $h+p$; far less than $h+p+hp$.

### 30.3 Multiply CR-warped products

A multiply warped product
\[
N_T \times f_2 N_2 \times \cdots \times f_k N_k
\]
in a Kähler manifold is called a *multiply CR-warped product* if $N_T$ is a holomorphic submanifold and $N_\perp := f_2 N_2 \times \cdots \times f_k N_k$ is a totally real submanifold.

In [84], Theorem 30.1 was extended to the following.
Theorem 30.10. Let \( N = N_T \times f_2 N_2 \times \cdots \times f_k N_k \) be a multiply CR-warped product in an arbitrary Kaehler manifold \( \tilde{M} \). Then the second fundamental form \( h \) and the warping functions \( f_2, \ldots, f_k \) satisfy

\[
||h||^2 \geq 2 \sum_{i=2}^{k} n_i ||\nabla (\ln f_i)||^2. \tag{30.8}
\]

The equality sign of (30.8) holds identically if and only if the following statements hold:

(i) \( N_T \) is a totally geodesic holomorphic submanifold of \( \tilde{M} \);

(ii) For each \( i \in \{2, \ldots, k\} \), \( N_i \) is a totally umbilical submanifold of \( \tilde{M} \) with \(-\nabla (\ln f_i)\) as its mean curvature vector;

(iii) \( f_2 N_2 \times \cdots \times f_k N_k \) is immersed as mixed totally geodesic submanifold in \( \tilde{M} \); and

(iv) For each point \( p \in N \), the first normal space \( \text{Im} h_p \) is a subspace of \( J(T_p N) \), where \( J \) is the almost complex structure of \( \tilde{M} \).

The following example shows that inequality (30.8) is sharp.

Example 30.3. Assume that \( h \) and \( k \) are natural numbers with \( h \geq k \). Let \( N_T = C^h := \{(z_1, \ldots, z_h) : z_1, \ldots, z_h \in \mathbb{C}\} \) and let \( N_i = S^{n_i} \) denote the unit \( n_i \)-spheres for \( i = 2, \ldots, k \).

Consider the immersion \( \psi \) of \( N_T \times S^{n_2} \times \cdots \times S^{n_k} \) into \( C^{h+n_2+\cdots+n_k} \) defined by

\[
\psi = (z_1 w_2,0, \ldots, z_1 w_2,n_2, \ldots, z_k w_k,0, \ldots, z_k w_k,n_k, z_{k+1}, \ldots, z_h), \tag{30.9}
\]

where \((w_{i,0}, \ldots, w_{i,n_i}) \in \mathbb{R}^{n_i+1}\) satisfy \( \sum_{\alpha=0}^{n_i} w_{i,\alpha}^2 = 1 \) for \( i = 2, \ldots, k \).

It is easy to see that the product manifold \( C^h \times S^{n_2} \times \cdots \times S^{n_k} \) endowed with the induced metric via \( \psi \) is the multiply warped product manifold

\[
C^h \times f_2 S^{n_2} \times \cdots \times f_k S^{n_k}
\]

with \( f_i = |z_i| \). Moreover, with respect to the canonical complex structure of \( C^{h+n_2+\cdots+n_k} \), the immersion \( \psi \) is a multiply CR-warped product submanifold.
A straightforward computation shows that this example of multiply \( CR \)-warped product submanifold satisfies the equality case of (30.8). This examples shows that inequality (30.8) is optimal.

By applying Theorem 30.9 we have the following.

**Corollary 30.1.** If \( f_2, \ldots, f_k \) are harmonic functions on \( N_1 \) or eigenfunctions of the Laplacian \( \Delta \) on \( N_1 \) with positive eigenvalues, then the multiply warped product manifold \( N_1 \times f_2 N_2 \times \cdots \times f_k N_k \) cannot be isometrically immersed into any Riemannian manifold of negative sectional curvature as a minimal submanifold.

**Example 30.4.** Let \( M_1 \times f_2 M_2 \times \cdots \times f_k M_k \) be a multiply warped product representation of a Riemannian \( m \)-manifold \( R^m(\epsilon) \) of constant curvature \( \epsilon \). Assume that \( \psi^i : N_i \to M_i, \ i = 2, \ldots, k, \) are minimal immersions. Then the immersion:

\[
\psi : M_1 \times f_2 N_2 \times \cdots \times f_k N_k \to M_1 \times f_2 M_2 \times \cdots \times f_k M_k \quad (30.10)
\]

defined by \( \psi = (id, \psi_2, \ldots, \psi_k) \) is a minimal isometric immersion of the multiply warped product manifold \( M_1 \times f_2 M_2 \times \cdots \times f_k M_k \) into \( R^m(\epsilon) \).

On the other hand, since \( M_1 \times f_2 M_2 \times \cdots \times f_k M_k \) is of constant curvature \( \epsilon \), the warping functions \( f_2, \ldots, f_k \) are eigenfunctions of the Laplacian \( \Delta \) of \( M_1 \) with eigenvalues given by \( n_2\epsilon, \ldots, n_k\epsilon \), respectively. In particular, if \( \epsilon = 0 \) the warping functions \( f_2, \ldots, f_k \) are harmonic functions.

Example 30.4 illustrates that the warping functions \( f_2, \ldots, f_k \) in Corollary 30.1 cannot be replaced by eigenfunctions with negative eigenvalue. Moreover, the target space in Corollary 30.1 cannot be replaced either by Euclidean space or by spheres. Therefore, Corollary 30.1 is sharp.

### 31 Inequality involving normal scalar curvature and DDVV conjecture

#### 31.1 Inequalities of Chen and Guadelupe and Rodriguez

The \( \delta \)-invariant \( \delta(n_1, \ldots, n_k) \) reduces to the scalar curvature \( \tau \) when the \( k \)-tuple \( (n_1, \ldots, n_k) \) were chosen to be the empty set. Accordingly, Theorem 5.1 reduces to the following [43].
Corollary 31.1. Let \( x : M \to R^m(\epsilon) \) be an isometric immersion of a Riemannian \( n \)-manifold \( M \) with normalized scalar curvature \( \rho \) into an \( m \)-dimensional real space form \( R^m(\epsilon) \) of constant sectional curvature \( \bar{c} \). Then we have

\[
H^2 \geq \rho - \epsilon, \tag{31.1}
\]
equality holding at a point \( p \in M \) if and only if \( p \) is a totally umbilical point.

For a surface \( M \) in a Riemannian manifold, the **ellipse of curvature** \( E(x) \) at a point \( x \in M \) is defined by

\[
E(x) = \{ h(X, X) : X \in T_xM \text{ and } ||x|| = 1 \},
\]
where \( h \) is the second fundamental form of \( M \).

A surface \( M \) in \( \mathbb{R}^4 \) is called **superconformal** if all of its ellipses of curvature of \( M \) are circles.

On the other hand, Guadelupe and Rodriguez proved the following:

**Theorem 31.1.** [143] Let \( M^2 \) be a surface in a real space form \( R^{2+m}(\epsilon) \). Denote by \( K \) the Gaussian curvature of \( M^2 \) and by \( K^\perp \) the normal scalar curvature. Then we have

\[
K \leq H^2 - K^\perp + \epsilon \tag{31.2}
\]
at every point \( p \in M^2 \), with equality if and only if the ellipse of curvature at \( p \) is a circle.

### 31.2 DDVV conjecture

Consider a submanifold \( M^n \) of a real space form \( R^{n+m}(\epsilon) \), the normalized normal scalar curvature \( \rho^\perp \) is defined as

\[
\rho^\perp = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n, 1 \leq r < s \leq m} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2}.
\]

Since the normalized scalar curvature \( \rho \) is the higher-dimensional analogue of the Gaussian curvature \( K \) and the normalized normal scalar curvature \( \rho^\perp \) that of \( K^\perp \), De Smet, Dillen, Verstraelen and Vrancken thus made the following conjecture in [117]:
DDVV Conjecture. Let $M^n$ be a submanifold of a real space form $R^{n+m}(\epsilon)$ of constant sectional curvature $\epsilon$. Denote by $\rho$ the normalized scalar curvature and by $\rho^\perp$ the normalized normal scalar curvature. Then

$$\rho \leq H^2 - \rho^\perp + \epsilon. \quad (31.3)$$

DDVV Conjecture is true for submanifolds of dimension $n$ and codimension $m$ if, for every set $\{B_1, \ldots, B_m\}$ of symmetric $(n \times n)$-matrices with trace zero, the following inequality holds:

$$\sum_{\alpha, \beta=1}^m ||[B_\alpha, B_\beta]||^2 \leq \left( \sum_{\alpha=1}^m ||B_\alpha||^2 \right)^2. \quad (31.4)$$

For normally flat submanifolds, in particular for hypersurfaces, inequality (31.3) is nothing but inequality (31.1). Moreover, the conjecture was proven for immersions with codimension 2 in [118]; for immersions which are invariant with respect to the standard Kählerian and Sasakian structures on $E^{2k}$ and $S^{2k+1}(1)$ respectively in [120]; and for immersions which are totally real with respect to the nearly Kähler structure on $S^6(1)$ in [117].

Let $N$ be a submanifold of a Riemannian manifold $M$. Then, according to [145], $N$ is called austere if for each normal vector $\xi$ the set of eigenvalues of $A_\xi$ is invariant under multiplication by $-1$; this is equivalent to the condition that all the invariants of odd order of the Weingarten map at each normal vector of $N$ vanish identically. Of course every austere submanifold is a minimal submanifold.

Recently, Choi and Lu confirm in [108] that DDVV conjecture is also true for 3-dimensional submanifolds with arbitrary codimension. Moreover, they prove the following result for 3-dimensional submanifolds.

**Theorem 31.2.** If the equality in DDVV Conjecture is valid at any point and $M$ is minimal, then $M$ is an austere 3-fold.

Explicit construction of Euclidean submanifolds of codimension two, free of minimal and umbilical points, that attain equality case of (31.3) are constructed in [114]. In particular, in the two-dimensional case, the construction yields all superconformal surfaces in $E^4$.

The notion of $H$-umbilical Lagrangian submanifolds introduced in [47, 48] is defined as follows:
A Lagrangian submanifold $M^n$ of a Kähler manifold $\tilde{M}^n$ is called $H$-umbilical if it is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following simple form:

$$
\begin{align*}
    h(e_1, e_1) &= \lambda e_1, \\
    h(e_2, e_2) &= \cdots = h(e_n, e_n) = \mu e_1, \\
    h(e_1, e_j) &= \mu e_j, \\
    h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \ldots, n
\end{align*}
$$

(31.5)

for some suitable functions $\lambda$ and $\mu$ with respect to some suitable orthonormal local frame field $e_1, \ldots, e_n$. It is obvious that condition (31.5) is equivalent to

$$
\begin{align*}
    h(X, Y) &= \alpha \langle JX, H \rangle \langle JY, H \rangle H \\
    &\quad + \beta \langle H, H \rangle \{ \langle X, Y \rangle H + \langle JX, H \rangle JY + \langle JY, H \rangle JX \}
\end{align*}
$$

(31.6)

for any vectors $X, Y$ tangent to $M$, where

$$
\alpha = \frac{\lambda - 3\mu}{\gamma^3}, \quad \beta = \frac{\mu}{\gamma^3}, \quad \gamma = \frac{\lambda + (n - 1)\mu}{n}
$$

when $H \neq 0$.

According to [50], an $n$-dimensional submanifold $M$ of a Riemannian $m$-manifold $N^m$ is called ultra-minimal if, with respect to some suitable locally orthonormal frame fields, $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$, the shape operators $A_r$ take the form:

$$
A_r = \begin{pmatrix}
A_1^r & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & A_r^k & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \text{trace } A_j^r = 0,
$$

where $A_j^r, n + 1 \leq r \leq m, j = 1, \ldots, k$, are $n_j \times n_j$ symmetric submatrices.

It is known that every ultra-minimal submanifold is always an ideal submanifold (see [50]).

In [121], F. Dillen, J. Fastenakels and J. Van der Veken also show that the DDVV conjecture also holds for 4-dimensional ultra-minimal submanifolds in $\mathbb{C}^4$.

Also, Dillen, Fastenakels and Van der Veken prove in [120] the following.
Proposition 31.1. Let $M^{2n+1}$ be a submanifold of $S^{2m+1}(1)$ which is invariant with respect to the standard Sasakian structure on the unit sphere. Then we have $\rho + \rho^\perp \leq 1$.

Recently, the DDVV conjecture was proved to be true in general by Zhiqin Lu in [183].

31.3 Lorentzian version

Let $M$ be an $m$-dimensional, time-orientable, Lorentzian manifold and $\mathbb{E}^{m+2}_s$ an $(m+2)$-dimensional pseudo-Euclidean space of signature $(+, \cdots, +, -, \varepsilon_{m+1}, \varepsilon_{m+2})$ with $\varepsilon_A = \pm 1$, $A = m+1, m+2$.

We consider $M$ to be locally and isometrically embedded in $\mathbb{E}^{m+2}_s$. Let $\Omega_{\alpha\beta}$ and $\Lambda_{\alpha\beta}$ be the components of the second fundamental form with respect to $\xi_{m+1}, \xi_{m+2}$, respectively. Then the mean curvature vector $\vec{H}$ is defined as

$$\vec{H} = \frac{1}{m}(\varepsilon_{m+1}\Omega^\alpha_{\alpha}\xi_{m+1} + \varepsilon_{m+2}\Lambda^\alpha_{\alpha}\xi_{m+2}).$$

Let $\{e_1, \ldots, e_{m-1}, e_m\}$ be an orthonormal basis of $M$. Due to space-time applications in mind we take $M$ time-orientable such that there exists a global, nowhere zero, timelike vector field, denoted by $e_m$.

Let $h$ be the second fundamental form of $M$ in $\mathbb{E}^{m+2}_s$. Put

$$h(X, Y) = \sum_{r=m+1}^{m+2} \varepsilon_r \eta(\tilde{\nabla}X, \xi_r)\xi_r.$$

Then we have

$$\Omega_{\alpha m} = -\eta(e_m, \tilde{\nabla}_{e_\alpha} \xi_{m+1}) = -\eta(e_\alpha, \tilde{\nabla}_{e_m} \xi_{m+1})$$

with $\alpha = 1, \ldots, m-1$, and analogous relations hold for $\Lambda_{\alpha m}$.

Definition 31.1. An embedding $\phi : (M^m, g) \to \mathbb{E}^{m+2}_s$ with $\varepsilon_{m+1} = \varepsilon_{m+2} = 1$ is called causal-type preserving if and only if, with respect to some orthonormal basis $\{e_1, \ldots, e_m\}$, $\tilde{\nabla}_{e_\alpha} \xi_A$ is space-like, for $A = m+1, m+2$ and $\alpha = 1, \ldots, m-1$.

Definition 31.2. An embedding $\phi : (M^m, g) \to \mathbb{E}^{m+2}_s$ with $\varepsilon_{m+1} = \varepsilon_{m+2} = -1$ is called causal-type preserving if and only if, with respect to some orthonormal basis $\{e_1, \ldots, e_m\}$, $\tilde{\nabla}_{e_m} \xi_A$ is time-like, for $A = m+1, m+2$. 
Notice that causal-type preserving embeddings have $\Omega_{\alpha m} = \Lambda_{\alpha m} = 0$ for each $\alpha = 1, \ldots, m - 1$.

If $v = v_1 \xi_{m+1} + v_2 \xi_{m+2}$ is a vector in normal space, we define the norm:

$$||v||^2_\perp = \epsilon_{m+1} (v_1)^2 + \epsilon_{m+2} (v_2)^2.$$  

Using these definitions we define the scalar normal curvature as

$$\rho^\perp = \frac{\sqrt{2}}{m(m - 1)} ||[\Omega, \Lambda]||.$$  

In [122], F. Dillen, S. Haesen, M. Petrović and L. Verstraelen extended inequality (31.2) to Lorentzian manifolds in a pseudo-Euclidean space with codimension 2. They obtain the following.

**Theorem 31.3.** Let $\phi : (M^m, g) \to E^{m+2}_s$ be a causal-type preserving, local and isometric embedding of a Lorentzian manifold $M^m$ in a pseudo-Euclidean space $E^{m+2}_s$. Then we have

$$||\vec{H}||^2_\perp \geq \rho + \rho^\perp,$$  

if $\epsilon_{m+1} = \epsilon_{m+2} = 1$  

(31.7)

and

$$||\vec{H}||^2_\perp \leq \rho + \rho^\perp,$$  

if $\epsilon_{m+1} = \epsilon_{m+2} = -1$.  

(31.8)

In [122], they also determine the special form of the second fundamental form when the equality case of Theorem 31.3 occurs. Furthermore, they showed that space-times which realize the equality case of Theorem 31.3 are Petrov type D anisotropic fluid models with a time-like surface of constant curvature.

**31.4 An optimal inequality involving normal $\delta$-invariant $\hat{\delta}^\perp (2)$ for Kähler hypersurfaces**

Recall that the $\delta$-invariant $\hat{\delta}(2)$ is defined as

$$\hat{\delta}(2) = \tau - \max K.$$  

Similarly, Z. Sentürk and L. Verstraelen [212] consider the scalar curvature function defined by

$$\hat{\kappa}(2) = \tau - \max H(e),$$  

(31.9)
$\delta$-invariants

where $H(e)$ denotes the holomorphic sectional curvature with respect to a unit tangent vector $e$.

Let $R^\perp$ denote the normal curvature tensor of a Kähler hypersurface $M^n$ in $CP^{n+1}(4\epsilon)$. we put

$$K^\perp_{\alpha\beta} = R^\perp(e_\alpha, e_\beta, \xi, J\xi),$$

(31.10)

$$\delta^\perp = \sum_{\alpha<\beta} |K^\perp_{\alpha\beta}|,$$

(31.11)

where $\xi$ is a unit normal vector and $e_\alpha, e_\beta$ are orthogonal unit tangent vectors of $M^n$ in $CP^{n+1}(4\epsilon)$.

The scalar normal $\delta$-invariant $\hat{\delta}^\perp(2)$ is defined as

$$\hat{\delta}^\perp(2) = \delta^\perp - \max |K^\perp|.$$ 

(31.12)

It is proved by Sentürk and Verstraelen in [242] that

**Theorem 31.4.** For every Kähler hypersurface $M^n$ in the complex projective space $CP^{n+1}(4)$, the scalar valued curvatures $\hat{\kappa}(s)$ and $\hat{\delta}^\perp(2)$ always satisfy the following inequality:

$$\hat{\kappa}(2) \leq -\hat{\delta}^\perp(2) + 2(n - 1)(4n + 3).$$

(31.13)

And the only Kähler hypersurfaces $M^n$ in $CP^{n+1}(4)$ which are ideal in this respect, i.e. for which the equality holds at all of their points, are either open parts of the totally geodesic complex space forms $CP^n(4)$ or open parts of the complex quadrics $Q_n$ in $CP^{n+1}(4)$.

## 32 Inequalities involving scalar curvature

### 32.1 Lagrangian submanifolds

There exists a general inequality for Lagrangian submanifolds in a complex space form involving the scalar curvature. In fact, we have the following.

**Theorem 32.1.** The scalar curvature $\rho$ and the squared mean curvature $H^2$ of a Lagrangian submanifold $M$ in complex space form $M^n(4\epsilon)$ satisfy the following general sharp inequality:

$$H^2 \geq \frac{2(n + 2)}{n^2(n - 1)} r - \left(\frac{n + 2}{n}\right) \epsilon.$$ 

(32.1)
The equality sign holds if and only if, with respect to an adapted Lagrangian frame field \( e_1, \ldots, e_n, e_1^*, \ldots, e_n^* \) with \( e_1^* \) parallel to \( JH \), the second fundamental form \( h \) of \( M \) in \( M^n(4\epsilon) \) takes the following form:

\[
\begin{align*}
  h(e_1, e_1) &= 3\lambda e_1^*, \\
  h(e_2, e_2) &= \cdots = h(e_n, e_n) = \lambda e_1^*, \\
  h(e_1, e_j) &= \lambda e_j^*, \\
  h(e_j, e_k) &= 0, \quad 2 \leq j \neq k \leq n.
\end{align*}
\]

(32.2)

Inequality (32.1) with \( \epsilon = 0 \) and \( n = 2 \) was proved in [29]. Their proof relies on complex analysis which is not applicable to \( n \geq 3 \). The general inequality was established in [23] for \( \epsilon = 0 \) and arbitrary \( n \); and in [36] for \( \epsilon \neq 0 \) and arbitrary \( n \); (and independently in [30], for \( \epsilon \neq 0 \) with \( n = 2 \), also using the method of complex analysis).

If \( M^n(4\epsilon) = C^n \), the equality of (32.1) holds identically if and only if either the Lagrangian submanifold \( M \) is an open portion of a Lagrangian \( n \)-plane or, up to dilations, \( M \) is an open portion of the Whitney immersion [23, 229].

Let \( \text{cn}(x, k) \) and \( \text{dn}(x, k) \) be the usual Jacobi's elliptic functions with modulus \( k \). It is well-known that \( \text{cn}(x, k), \text{dn}(x, k) \) are doubly periodic functions. We put

\[
\begin{align*}
  \mu_a &= \frac{\sqrt{a^2-1}}{\sqrt{2}} \text{cn} \left( ax, \frac{\sqrt{a^2-1}}{\sqrt{2a}} \right), \quad a > 1, \\
  \eta_a &= \frac{\sqrt{a^2+1}}{\sqrt{2}} \text{dn} \left( \frac{\sqrt{a^2+1}}{\sqrt{2}}, \frac{\sqrt{2a}}{\sqrt{a^2+1}} \right), \quad 0 < a < 1, \\
  \rho_a &= \frac{\sqrt{a^2+1}}{\sqrt{2}} \text{cn} \left( ax, \frac{\sqrt{a^2+1}}{\sqrt{2a}} \right), \quad a > 1.
\end{align*}
\]

Let \( S^n(\epsilon) \) and \( H^n(-\epsilon) \) denote the \( n \)-sphere with constant sectional curvature \( \epsilon \) and the real hyperbolic \( n \)-space with constant sectional curvature \(-\epsilon \), respectively. For \( n \geq 3 \), we denote by \( P^n_\alpha, D^n_\alpha \) and \( C^n_\alpha \) the warped products \( I \times_{\mu_\alpha} S^{n-1}(\frac{a^2-1}{4}), \mathbb{R} \times_{\eta_\alpha} H^{n-1}(\frac{a^2-1}{4}) \) and \( I \times_{\rho_\alpha} S^{n-1}(\frac{a^2-1}{4}) \) with warped functions \( \mu_\alpha, \eta_\alpha \) and \( \rho_\alpha \), respectively, and \( I \) are the maximal open intervals containing 0 on which the corresponding warped functions are positive.

Moreover, we denote by \( F^n \) and \( L^n \) the warped products \( \mathbb{R} \times_{1/\sqrt{2}} H^{n-1}(\frac{a^2-1}{4}) \) and \( \mathbb{R} \times_{\text{sech}(x)} \mathbb{R}^{n-1} \), respectively. For \( n = 2 \) we shall replace \( S^{n-1}(\frac{a^2-1}{4}) \) or \( H^{n-1}(\frac{a^2-1}{4}) \) by the real line \( \mathbb{R} \) to define \( P^n_\alpha, D^n_\alpha, C^n_\alpha, F^2, \) and \( L^2 \).

It is easy to see that \( F^2 \) is a flat surface and \( D^n_\alpha, F^n \) and \( L^n \) are complete Riemannian \( n \)-manifolds, but \( P^n_\alpha \) and \( C^n_\alpha \) are not complete. Furthermore,
δ-invariants

these Riemannian n-manifolds are conformally flat. Topologically, $S^n$ is the two point compactification of both $P^n_a$ and $C^n_a$.

In [36], the author proved that the one-parameter family of Riemannian n-manifolds, $P^n_a$ ($a > 1$), admit Lagrangian isometric immersions into $CP^n(4)$ satisfying the equality case of the inequality (32.1) for $\epsilon = 1$; the two one-parameter families of Riemannian manifolds, $C^n_a$ ($a > 1$), $D^n_a$ ($0 < a < 1$), and the two exceptional n-spaces, $F^n$ and $L^n$, admit Lagrangian isometric immersion into $CH^n(4)$ satisfying the equality case of the inequality for $\epsilon = -1$.

It also proved in [46] that besides the totally geodesic ones, these are the only Lagrangian submanifolds in $CP^n(4)$ and in $CH^n(4)$ which satisfy the equality case of (32.1) (for the case $n = 2$, see also [30]).

The explicit expressions of those Lagrangian immersions of $P^n_a, C^n_a, D^n_a, F^n$ and $L^n$ satisfying the equality case of (32.1) were completely determined by B. Y. Chen and L. Vrancken in [102].

I. Castro and F. Urbano [30] showed that a Lagrangian surface in $CP^2$ satisfies the equality case of (32.1) for $n = 2$ and $c = 1$ if and only if the Lagrangian surface has holomorphic twistor lift.

32.2 Slant submanifolds

Proper slant submanifolds are even-dimensional. Such submanifolds do exist extensively for any even dimension greater than zero (cf. [39]).

A proper slant submanifold is called Kählerian slant if the endomorphism $P$ is parallel with respect to the Riemannian connection, that is, $\nabla P = 0$. A Kählerian slant submanifold is a Kähler manifold with respect to the induced metric and the almost complex structure defined by $\tilde{J} = (\sec \theta)P$. Kähler submanifolds, totally real submanifolds and slant surfaces in a Kähler manifold are examples of Kählerian slant submanifolds.

In general, let $M$ be a submanifold of a Kähler manifold $\tilde{M}$. Then $M$ satisfies $\nabla P = 0$ if and only if $M$ is locally the Riemannian product $M_1 \times \cdots \times M_k$, where each $M_i$ is a Kähler submanifold, a totally real submanifold or a Kählerian slant submanifold of $\tilde{M}$ (see [39]).

Slant submanifolds have the following topological properties:

Theorem 32.2. We have

1. Let $M$ be a compact 2k-dimensional proper slant submanifold of a
Kähler manifold, then:
\[ H^{2i}(M; \mathbb{R}) \neq \{0\} \]
for \( i = 1, \ldots, k \).

(2) Let \( M \) be a slant submanifold in a complex Euclidean space. If \( M \) is not totally real, then \( M \) is non-compact.

Statements (1) and (2) of Theorem 32.2 are due to [39] and [100], respectively.

An immediate consequence of statement (1) is the following.

**Corollary 32.1.** If \( M \) is a compact \( 2k \)-manifold with \( H^{2i}(M; \mathbb{R}) = \{0\} \) for some \( i \in \{1, \ldots, k\} \), then \( M \) cannot be immersed in any Kähler manifold as a proper slant submanifold.

Although there exist no compact proper slant submanifolds in complex Euclidean spaces, there do exist compact proper slant submanifolds in complex flat tori.

A submanifold \( N \) of a pseudo-Riemannian Sasakian manifold \((\tilde{M}, g, \phi, \xi)\) is called contact \( \theta \)-slant if the structure vector field \( \xi \) of \( \tilde{M} \) is tangent to \( N \) at each point of \( N \) and, moreover, for each unit vector \( X \) tangent to \( N \) and orthogonal to \( \xi \) at \( p \in N \), the angle \( \theta(X) \) between \( \phi(X) \) and \( T_pN \) is independent of the choice of \( X \) and \( p \).

Let \( H^{2m+1}_{1}(-1) \subset C^{m+1}_{1} \) denote the anti-de Sitter space-time and
\[ \pi: H^{2m+1}_{1}(-1) \to CH^{m}(-4) \]
the corresponding Hopf’s fibration. Then every \( n \)-dimensional proper \( \theta \)-slant submanifold \( M \) in \( CH^{m}(-4) \) lifts to an \( (n+1) \)-dimensional proper contact \( \theta \)-slant submanifold \( \pi^{-1}(M) \) in \( H^{2m+1}_{1}(-1) \) via \( \pi \).

Conversely, a proper contact \( \theta \)-slant submanifold of \( H^{2m+1}_{1}(-1) \) projects to a proper \( \theta \)-slant submanifold of \( CH^{m}(-4) \) via \( \pi \).

Similar correspondence also holds between proper \( \theta \)-slant submanifolds of \( CP^{m}(4) \) and proper contact \( \theta \)-slant submanifolds of the Sasakian \( S^{2m+1}(1) \) (see [101]).

For Kählerian \( \theta \)-slant submanifolds in complex space forms, we have the following inequality from [49, 59].

**Theorem 32.3.** Let \( \phi : M \to \tilde{M}^{n}(4\epsilon), \epsilon \in \{-1,0,1\} \), be a Kählerian \( \theta \)-slant submanifold of dimension \( n \) in a complete simply-connected complex
δ-invariants

space form \( \tilde{M}^n(4\epsilon) \). Then we have

\[
H^2 \geq \frac{2(n+2)}{n^2(n-1)} \tau - \frac{n+2}{n} \left( 1 + \frac{3\cos^2 \theta}{n-1} \right) \epsilon. \tag{32.3}
\]

The equality sign of (32.3) holds identically if and only if one of the following three cases occurs:

(a) \( \theta = 0 \) and \( M \) is a totally geodesic complex submanifold of \( \tilde{M}^n(4\epsilon) \), or

(b) \( \epsilon = 0 \) and \( M \) is a totally geodesic \( \theta \)-slant submanifold in \( \mathbb{C}^n \), or

(c) \( \epsilon = -1, n = 2, \theta = \cos^{-1} \left( \frac{1}{3} \right) \), and \( M \) is a surface of constant curvature \(-\frac{2}{3}\). Moreover, up to rigid motions, the immersion \( \phi \) is the composition \( \phi = \pi \circ z \), where \( \pi \) is the hyperbolic Hopf fibration \( \pi : H^5_1 \to CH^2(-4) \) and

\[
z : \mathbb{R}^3 \to H^5_1(-4) \subset \mathbb{C}^3
\]

is the immersion defined by

\[
z(u,v,t) = e^{it} \left( \frac{3}{2} \cosh av \frac{1}{2} + \frac{1}{6} u^2 e^{-av} - \frac{i}{6} \sqrt{6} u(1 + e^{-av}), \right.
\]

\[
\frac{1}{3} (1 + 2e^{-av})u + i\sqrt{6} \left( \frac{e^{av}}{4} + \frac{e^{-av}}{12} + \frac{e^{-av}}{18} u^2 - \frac{1}{3} \right), \tag{32.4}
\]

\[
\frac{\sqrt{2}}{6} (1 - e^{-av})u + i\sqrt{3} \left( \frac{1}{6} e^{av} + \frac{1}{12} e^{-av} + \frac{5e^{-av}}{12} \right)
\]

with \( a = \sqrt{2/3} \).

Inequality (32.3) has been extended by A. Oiag˘a [212] to the following.

**Theorem 32.4.** Let \( \phi : M \to \tilde{M}^n(4\epsilon), \epsilon \in \{-1,0,1\} \), be a purely real submanifold with \( \nabla P = 0 \) in a complex space form \( \tilde{M}^n(4\epsilon) \). Then we have

\[
H^2 \geq \frac{2(n+2)}{n^2(n-1)} \tau - \frac{n+2}{n} \left[ 1 + \frac{3||P||^2}{n(n-1)} \right] \epsilon. \tag{32.5}
\]

Here, a purely real submanifold is in the sense of [37], i.e., a submanifold \( M \) in \( \tilde{M}^n(4\epsilon) \) whose holomorphic distribution is trivial, or equivalently, \( D_x = \{0\} \) for each \( x \in M \).
After the invention of $\delta$-invariants, there are many articles which study one of the topics treated in this survey. Also, there is a book by Adela Mihai [196] which also provides a survey on this research area. However, to enable further study in this very active field of research we divide those articles into several categories according to their main results and applications.

1. $\delta$-invariants: [42, 50, 52, 54, 70, 72, 74, 75, 77, 86, 92, 97, 148, 196, 217, 234, 236, 247].

2. Inequalities involving $\delta$-invariants: [2, 3, 7, 16, 26, 27, 40, 42, 50, 52, 54, 62, 70, 74, 77, 78, 81, 84, 90, 97, 109, 110, 111, 112, 115, 116, 144, 148, 151, 156, 158, 162, 164, 175, 191, 196, 214, 215, 216, 242, 246, 247, 248, 250, 252, 254, 264].

3. Related inequalities: [23, 26, 27, 28, 46, 49, 50, 53, 59, 60, 61, 62, 63, 65, 66, 68, 69, 70, 71, 72, 73, 75, 76, 80, 81, 84, 94, 101, 103, 105, 111, 117, 118, 120, 121, 122, 143, 146, 150, 160, 163, 189, 190, 191, 192, 193, 194, 196, 214, 242, 245, 261, 265, 266].

4. Ideal immersions-equality involving $\delta$-invariants: [5, 15, 20, 21, 40, 41, 50, 54, 64, 65, 71, 76, 77, 78, 79, 80, 81, 84, 94, 101, 103, 104, 105, 114, 115, 124, 127, 128, 129, 130, 135, 137, 150, 154, 168, 169, 172, 173, 174, 182, 196, 199, 202, 220, 221, 223, 232, 233, 234, 236, 238, 239, 242, 248, 253].

5. Equality case of related inequalities: [49, 53, 57, 60, 61, 64, 65, 66, 68, 71, 76, 78, 79, 80, 81, 93, 101, 102, 108, 117, 118, 121, 196, 200, 204, 222, 248].

6. Applications to immersions: [26, 27, 50, 54, 64, 67, 72, 75, 77, 161].

7. General warped products: [14, 64, 66, 67, 69, 71, 76, 80, 84, 105, 137, 163, 186, 192, 194, 195, 196, 198, 201, 247, 267, 269].

8. $CR$-products: [11, 33, 34, 35, 56, 64, 96, 103, 196, 212, 196, 198, 201, 205, 206, 230, 231].

9. $CR$-warped products: [4, 6, 14, 22, 58, 60, 61, 66, 68, 73, 83, 146, 163, 196, 198, 201, 205, 206, 230, 231].

10. Ricci curvature: [8, 9, 51, 55, 65, 69, 136, 152, 153, 159, 160, 165].
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11. Shape operator: [43, 51, 54, 111, 157, 179, 180, 184, 187, 190, 196, 251, 263].

12. Contact and Sasakian manifolds: [3, 97, 109, 110, 111, 115, 116, 136, 158, 162, 164, 175, 176, 179, 180, 186, 196, 197, 199, 204, 250, 251, 252, 253, 266, 267, 268, 269].

13. Affine differential geometry: [16, 76, 81, 82, 86, 128, 168, 169, 182, 238, 239].

14. Lagrangian submanifolds of Kähler manifolds: [16, 17, 18, 19, 20, 23, 28, 33, 34, 44, 54, 55, 57, 63, 79, 87, 90, 102, 104, 196, 216].

15. Slant submanifolds: [3, 26, 27, 39, 53, 59, 112, 136, 144, 162, 174, 186, 187, 193, 200, 204, 230, 245, 254, 268].

16. Other applications: [62, 74, 91, 147, 148, 149, 242].

Acknowledgements: The author would like to express his many thanks to Professors D. E. Blair, F. Dillen, I. Dimitric, J. Fastenakels, S. Haesen, I. Mihai, B. Suceava, J. Van der Veken, L. Verstraelen, L. Vrancken and S. W. Wei for their valuable suggestions for the improvement of the presentation of this article.
Bibliography

[1] K. Abe, The complex version of Hartman-Nirenberg cylinder theorem, J. Differential Geometry 7 (1972), 453–460.

[2] N. Aktan, B.-Y. Chen’s inequality for semi-slant submanifolds in S-space forms, (to appear).

[3] P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon, B.-Y. Chen’s inequality for submanifolds of generalized space forms, Indian J. Pure Appl. Math. 38 (2007), 185–201.

[4] R. Al-Ghefari, F. Al-Solamy and M. H. Shahid, Contact CR-warped product submanifolds in generalized Sasakian space forms, Balkan J. Geom. Appl. 11 (2006), 1–10.

[5] M. Antić, M. Djorić and L. Vrancken, 4-dimensional minimal CR submanifolds of the sphere $S^6$ satisfying Chen’s equality, Differential Geom. Appl. 25 (2007), 290–298.

[6] K. Arslan, R. Ezentas, I. Mihai and C. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, J. Korean Math. Soc. 42 (2005), 1101–1110.

[7] K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Özgür, B.Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds, Bull. Inst. Math. Acad. Sinica 29 (2001), 231–242.

[8] K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Özgür, Ricci curvature of submanifolds in Kenmotsu space forms, Int. J. Math. Math. Sci. 29 (2002), 719–726.

[9] K. Arslan, R. Ezentas, I. Mihai and C. Murathan and C. Özgür, Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds, Math. J. Toyama Univ. 26 (2003), 13–24.

[10] A. Bejancu, On the geometry of leaves on a CR-submanifold, Ann. St. Univ. Al. I. Cuza Iasi 25 (1979), 393–398.

[11] A. Bejancu, Geometry of CR-Submanifolds, Dordrecht: Reidel Publ. 1986.
[12] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002.

[13] D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifold*, Israel J. Math. 34 (1979), 353–363.

[14] D.E. Blair and S. Dragomir, *CR products in locally conformal Kähler manifolds*, Kyushu J. Math. 56 (2002), 337–362.

[15] D.E. Blair, F. Dillen, L. Verstraelen and L. Vrancken, *Calabi curves as holomorphic Legendre curves and Chen’s inequality*, Kyungpook Math. J. 35 (1996), 407–416.

[16] J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken, *A best possible inequality for curvature-like tensor fields*, preprint.

[17] J. Bolton, C.R. Montealegre and L. Vrancken, *Characterizing warped product Lagrangian immersions in complex projective space*, preprint.

[18] J. Bolton, C. Scharlach, L. Vrancken and L.M. Woodward, *From certain Lagrangian minimal submanifolds of the 3-dimensional complex projective space to minimal surfaces in the 5-sphere*, in: Proceedings of the Fifth Pacific Rim Geometry Conference (Sendai, 2000), 23–31, Tohoku Math. Publ. 20 (2001).

[19] J. Bolton, C. Scharlach and L. Vrancken, *From surfaces in the 5-sphere to 3-manifolds in complex projective 3-space*, Bull. Austral. Math. Soc. 66 (2002), 465–475.

[20] J. Bolton and L. Vrancken, *Ruled minimal Lagrangian submanifolds of complex projective 3-space*, Asian J. Math. 9 (2005), 45–56.

[21] J. Bolton and L. Vrancken, *Lagrangian submanifolds attaining equality in the improved Chen’s inequality*, Bull. Belg. Math. Soc. 14 (2007), 311–315.

[22] V. Bonanzinga and K. Matsumoto, *Warped product CR-submanifolds in locally conformal Kähler manifolds*, Period. Math. Hungar. 48 (2004), 207–221.
[23] V. Borrelli, B.Y. Chen and J.-M. Morvan, Une caractérisation géométrique de la sphère de Whitney, C.R. Acad. Sci. Paris Sér. I Math. 321 (1995), 1485–1490.

[24] R.L. Bryant, Second order families of special Lagrangian 3-folds. Perspectives in Riemannian geometry, CRM Proc. Lecture Notes, 40 (2006), 63–98, Amer. Math. Soc., Providence, RI.

[25] E. Calabi, Construction and properties of some 6-dimensional almost complex manifolds, Trans. Amer. Math. Soc. 87 (1958), 407–438.

[26] A. Carriazo, A contact version of B.-Y. Chen’s inequality and its applications to slant immersions, Kyungpook Math. J. 39 (1999), 465–476.

[27] A. Carriazo, L.M. Fernandez and M.B. Hans-Uber, B.Y. Chen’s inequality for S-space-forms: applications to slant immersions, Indian J. Pure Appl. Math. 34 (2003), 1287–1298.

[28] A. Carriazo, Y.H. Kim and D.W. Yoon, Some inequalities on totally real submanifolds in locally conformal Kähler space forms, J. Korean Math. Soc. 4 (2004), 795–808.

[29] I. Castro and F. Urbano, Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form, Tôhoku Math. J. 45 (1993), 656–582.

[30] I. Castro and F. Urbano, Twistor holomorphic Lagrangian surface in the complex projective and hyperbolic planes, Ann. Global Anal. Geom. 13 (1995), 59–67.

[31] S. Chang, A closed hypersurface with constant scalar and mean curvatures in $S^4$ is isoparametric, Comm. Anal. Geom. 1 (1993), 71–100.

[32] B.Y. Chen, Geometry of Submanifolds, M. Dekker, New York, 1973.

[33] B.Y. Chen, CR-submanifolds of a Kähler manifold, J. Differential Geometry 16 (1981), 305–322.

[34] B.Y. Chen, CR-submanifolds of a Kähler manifold II, J. Differential Geometry 16 (1981), 493–509.
δ-invariants

[35] B.Y. Chen, Geometry of Submanifolds and Its Applications, Science University of Tokyo, Tokyo, 1981

[36] B.Y. Chen, Cohomology of CR-submanifolds Ann. Fac. Sc. Toulouse Math. Ser. V 3 (1981), 167–172.

[37] B.Y. Chen, Differential geometry of real submanifolds in a Kähler manifold, Monatsh. Math. 91 (1981), 257–274.

[38] B.Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, World Scientific Publ., New Jersey, 1984.

[39] B.Y. Chen, Geometry of Slant Submanifolds, Katholieke Universiteit Leuven, Leuven, Belgium 1990.

[40] B.Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math. 60 (1993), 568–578.

[41] B.Y. Chen, Tubular hypersurfaces satisfying a basic equality, Soochow J. Math. 20 (1994), 569–586.

[42] B.Y. Chen, A Riemannian invariant and its applications to submanifold theory, Results Math. 27 (1995), 17–28.

[43] B.Y. Chen, Mean curvature and shape operator of isometric immersions in real-space-forms, Glasgow Math. J. 38 (1996), 87–97.

[44] B.Y. Chen, A general inequality for submanifolds in complex space forms and its applications, Archiv Math. 67 (1996), 519–528.

[45] B.Y. Chen, A report of submanifolds of finite type, Soochow J. Math. 22 (1996), 117–337.

[46] B.Y. Chen, Jacobi’s elliptic functions and Lagrangian immersions, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 687–704.

[47] B.Y. Chen, Complex extensors and Lagrangian submanifolds in complex Euclidean spaces, Tohoku Math. J. 49 (1997), 277–297.

[48] B.Y. Chen, Interaction of Legendre curves and Lagrangian submanifolds, Israel J. Math. 99 (1997), 69–108.
[49] B.Y. Chen, *Special slant surfaces and a basic inequality*, Results Math. **33** (1998), 65–78.

[50] B.Y. Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, in: Third Pacific Rim Geom. Conf. (International. Press) 1998, pp. 7–60.

[51] B.Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension*, Glasgow Math. J. **41** (1999), 33–41.

[52] B.Y. Chen, *New types of Riemannian curvature invariants and their applications*, Geometry and Topology of Submanifolds, IX, 80–92, World Sci. Publ., River Edge, NJ, 1999.

[53] B.Y. Chen, *On slant surfaces*, Taiwanese J. Math. **3** (1999), 163–179.

[54] B.Y. Chen, *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan. J. Math. **26** (2000), 105–127.

[55] B.Y. Chen, *On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms*, Arch. Math. (Basel) **74** (2000), 154–160.

[56] B.Y. Chen, *Riemannian Submanifolds*, in: Handbook of Differential Geometry (eds. F. Dillen and L. Verstraelen), Vol. I, 2000, pp. 187–418.

[57] B.Y. Chen, *Ideal Lagrangian immersions in complex space forms*, Math. Proc. Cambridge Philos. Soc. **128** (2000), 511–533.

[58] B.Y. Chen, *Twisted product CR-submanifolds in Kähler manifolds*, Tamsui Oxford J. Math. Sci. **16** (2000), 105–121. (A special issue dedicated to N.C. Yeh).

[59] B.Y. Chen, *A general inequality for Kählerian slant submanifolds and related results*, Geom. Dedicata **85** (2001), 253–271.

[60] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kähler manifolds*, Monatsh. Math. **133** (2001), 177–195.

[61] B.Y. Chen, *Geometry of warped product CR-submanifolds in Kähler manifolds. II*, Monatsh. Math. **134** (2001), 103–119.
δ-invariants

[62] B.Y. Chen, A series of Kählerian invariants and their applications to Kählerian geometry, Beiträge Algebra Geom. 42 (2001), 165–178.

[63] B.Y. Chen, Riemannian geometry of Lagrangian submanifolds, Taiwanese J. Math. 5 (2001), 681–723.

[64] B.Y. Chen, On isometric minimal immersions from warped products into real space forms, Proc. Edinburgh Math. Soc. 45 (2002), 579–587.

[65] B.Y. Chen, Ricci curvature of real hypersurfaces in complex hyperbolic space, Arch. Math. (Brno) 38 (2002), 73–80.

[66] B.Y. Chen, Geometry of warped products as Riemannian submanifolds and related problems, Soochow J. Math. 28 (2002), 125–156.

[67] B.Y. Chen, Non-immersion theorems for warped products in complex hyperbolic spaces, Proc. Japan Acad. Ser. A Math. Sci. 79 (2002), 96–100.

[68] B.Y. Chen, Another general inequality for CR-warped products in complex space forms, Hokkaido Math. J. 32 (2003), 415–444.

[69] B.Y. Chen, A general optimal inequality for warped products in complex projective spaces and its applications, Proc. Japan Acad. Ser. A Math. 79 (2003), 89–94.

[70] B.Y. Chen, What can we do with Nash’s embedding theorem?, Soochow J. Math. 30 (2004), 303–338.

[71] B.Y. Chen, Warped products in real space forms, Rocky Mountain J. Math. 34 (2004), 551–563.

[72] B.Y. Chen, A Riemannian invariant and its application to Einstein manifolds, Bull. Austral. Math. Soc. 70 (2004), 55–65.

[73] B.Y. Chen, CR-warped products in complex projective spaces with compact holomorphic factor, Monatsh. Math. 141 (2004), 177–186.

[74] B.Y. Chen, A general inequality for conformally flat submanifolds and its applications, Acta Math. Hungar. 106 (2005), 239–252.
B.-Y. Chen, *Riemannian submersions, minimal immersions and cohomology class*, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), 162–167.

B.Y. Chen, *Geometry of affine warped product hypersurfaces*, Results Math. **48** (2005), 9–28.

B.Y. Chen, *A general optimal inequality for arbitrary Riemannian submanifolds*, J. Inequal. Pure Appl. Math. **6** (2005), no. 3, Article 77, 10 pp.

B.Y. Chen, *Examples and classification of Riemannian submersions satisfying a basic equality*, Bull. Austral. Math. Soc. **72** (2005), 391–402.

B.Y. Chen, *First normal bundle of ideal Lagrangian immersions in complex space forms*, Math. Proc. Camb. Phil. Soc. **138** (2005), 461–464.

B.Y. Chen, *On warped product immersions*, J. Geom. **82** (2005), 36–49.

B.Y. Chen, *Eigenvalues of a natural operator of centro-affine and graph hypersurfaces*, Beiträge Algebra Geom. **47** (2006), 15–27.

B.Y. Chen, *Realization of Robertson-Walker space-times as affine hypersurfaces*, J. Phys, A **40** (2007), 4241–4250.

B.Y. Chen, *Classification of marginally trapped Lorentzian flat surfaces in $\mathbb{E}^4_2$ and its application to biharmonic surfaces*, J. Math. Anal. Appl. **340** (2008), 861–875.

B.Y. Chen and F. Dillen, *Optimal inequalities for multiply warped product submanifold*, Intern. Electron. J. Geom. **1** (2008), 1–11.

B.Y. Chen, F. Dillen, J. Fastenakels and L. Verstraelen, *Conformally flat hypersurfaces in real space forms with least tension*, Taiwanese J. Math. **8** (2004), 285–325.

B.Y. Chen, F. Dillen and L. Verstraelen, *$\delta$-invariants and their applications to centroaffine geometry*, Differential Geom. Appl. **22** (2005), 341–354.
§-invariants

[87] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of \( \mathbb{CP}^n \) satisfying a basic equality*, Arch. Math. 63 (1994), 553–564.

[88] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Characterizing a class of totally real submanifolds of \( S^6 \) by their sectional curvatures*, Tohoku Math. J. 47 (1995), 185–198.

[89] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Two equivariant totally real immersions into the nearly Kähler 6-sphere and their characterization*, Japan. J. Math. (N.S.) 21 (1995), 207–222.

[90] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *An exotic totally real minimal immersion of \( S^3 \) in \( \mathbb{CP}^3 \) and its characterization*, Proc. Roy. Soc. Edinburgh Sec. A, 126 (1996), 153–165.

[91] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Characterizations of Riemannian space forms, Einstein spaces and conformally flat spaces*, Proc. Amer. Math. Soc. 128 (2000), 589–598.

[92] B.Y. Chen and O.J. Garay, *Constant mean curvature hypersurfaces with constant \( \delta \)-invariant*, Int. J. Math. Math. Sci. 2003, no. 67, 4205–4216.

[93] B.Y. Chen and O.J. Garay, *An extremal class of conformally flat submanifolds in Euclidean spaces*, Acta Math. Hungar. 111 (2006), 263–303.

[94] B.Y. Chen and S. Jiang, *Inequalities between volume, center of mass, circumscribed radius, order, and mean curvature*, Bull. Belg. Math. Soc. 2 (1995), 75–85.

[95] B.Y. Chen and W.E. Kuan, *The Segre imbedding and its converse*, Ann. Fac. Sc. Toulouse Math. Ser. V 7 (1981), 1–28.

[96] B.Y. Chen, G.D. Ludden and S. Montiel, *Real submanifolds of a Kähler manifold*, Algebra, Groups Geom. 1 (1984), 176–212.

[97] B.Y. Chen and I. Mihai, *Isometric immersions of contact Riemannian manifolds in real space forms*, Houston J. Math. 31 (2005), 743–764.
[98] B.Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. 193 (1974), 257–266.

[99] B.Y. Chen and K. Ogiue, *Two theorems on Kähler manifolds*, Michigan Math. J. 21 (1974), 257–266.

[100] B.Y. Chen and Y. Tazawa, *Slant submanifolds in complex Euclidean spaces*, Tokyo J. Math. 14 (1991), 101–120.

[101] B.Y. Chen and Y. Tazawa, *Slant submanifolds of complex projective and complex hyperbolic spaces*, Glasgow Math. J. 42 (2000), 439–454.

[102] B.Y. Chen and L. Vrancken, *Lagrangian submanifolds satisfying a basic equality*, Math. Proc. Cambridge Philos. Soc. 120 (1996), 291–307.

[103] B.Y. Chen and L. Vrancken, *CR-submanifolds of complex hyperbolic spaces satisfying a basic equality*, Israel J. Math. 110 (1999), 341–358.

[104] B.Y. Chen and L. Vrancken, *Lagrangian submanifolds of the complex hyperbolic space*, Tsukuba J. Math. 26 (2002), 95–118.

[105] B.Y. Chen and W.S. Wei, *Sharp growth estimates for warping functions and their geometric applications*, Glasg. Math. J. 51 (2009), no. 3, 579–592.

[106] Q.M. Cheng and Q.R. Wan, *Hypersurfaces of space forms $M^4(c)$ with constant mean curvature*, Geometry and Global Analysis (Sendai, 1993), Tohoku University, Sendai, 1993, pp. 437–442.

[107] S.S. Chern, *Minimal Submanifolds in a Riemannian Manifold*, University of Kansas, 1968.

[108] T. Choi and Z. Lu, *On the DDVV Conjecture and the Comass in Calibrated Geometry (I)*, preprint, [arXiv:math.DG/0610709](http://arxiv.org/abs/math.DG/0610709).

[109] D. Cioroboiu, *B. Y. Chen inequalities for bi-slant submanifolds in Sasakian space forms*, Demonstratio Math. 36 (2003), 179–187.

[110] D. Cioroboiu, *B.-Y. Chen inequalities for semislant submanifolds in Sasakian space forms*, Int. J. Math. Math. Sci. 2003 no. 27, 1731–1738.
δ-invariants

[111] D. Cioroboiu, *Shape operator $A_H$ for $C$-totally real submanifolds in Sasakian space forms*, Math. J. Toyama Univ. **26** (2003), 1–12.

[112] D. Cioroboiu and A. Oiagă, *B. Y. Chen inequalities for slant submanifolds in Sasakian space forms*, Rend. Circ. Mat. Palermo (2) **52** (2003), 367–381.

[113] M. Dajczer and L.A. Florit, *On Chen's basic equality*, Illinois J. Math. **42** (1998), 97–106.

[114] M. Dajczer and R. Tojeiro, *Submanifolds of codimension two attaining equality in the DDV inequality*, preprint.

[115] F. Defever, I. Mihai and L. Verstraelen, *B.Y. Chen’s inequality for $C$-totally real submanifolds in Sasakian space forms*, Boll. Un. Mat. Ital. Ser. B **11** (1997), 365–374.

[116] F. Defever, I. Mihai and L. Verstraelen, *B. Y. Chen's inequalities for submanifolds of Sasakian space forms*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), 521–529.

[117] P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *The normal curvature of totally real submanifolds of $S^6(1)$*, Glasgow Math. J. **40** (1998), 199–204.

[118] P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), 115–128.

[119] R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken, *Quasi-Einstein totally real submanifolds of $S^6(1)$*, Tohoku Math. J. **51** (1999), 461–478.

[120] F. Dillen, J. Fastenakels and J. Van der Veken, *A pinching theorem for the normal scalar curvature of invariant submanifolds*, J. Geom. Phys. **57** (2007), 833–840.

[121] F. Dillen, J. Fastenakels and J. Van der Veken, *Remarks on an inequality involving the normal scalar curvature*, arXiv:math/0610721v2.
[122] F. Dillen, S. Haesen, M. Petrović and L. Verstraelen, An inequality between intrinsic and extrinsic scalar curvature invariants for codimension 2 embeddings, J. Geom. Phys. 52 (2004), 101–112.

[123] F. Dillen, K. Nomizu and L. Verstraelen, Conjugate connections and Radons theorem in affine differential geometry, Monatsh. Math. 109 (1990), 221–235.

[124] F. Dillen, M. Petrović and L. Verstraelen, Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen’s equality, Israel J. Math. 100 (1997), 163–169.

[125] F. Dillen, L. Verstraelen and L. Vrancken, Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq 1/16$, J. Math. Soc. Japan 42 (1990), 565–584.

[126] F. Dillen and L. Vrancken, Calabi-type composition of affine spheres, Differential Geom. Appl. 4 (1994), 303–328.

[127] F. Dillen and L. Vrancken, Totally real submanifolds in $S^6(1)$ satisfying Chen’s equality, Trans. Amer. Math. Soc. 348 (1996), 1633–1646.

[128] F. Dillen and L. Vrancken, Improper affine spheres and $\delta$-invariants, PDEs, Submanifolds and Affine Differential Geometry, 157–162, Banach Center Publ., 69, Polish Acad. Sci., Warsaw, 2005.

[129] M. Djorić and L. Vrancken, Three-dimensional minimal CR submanifolds in $S^6$ satisfying Chen’s equality, J. Geom. Phys. 56 (2006), 2279–2288.

[130] M. Djorić and L. Vrancken, Geometric conditions in three dimensional CR submanifolds in $S^6$, Adv. Geom. (to appear).

[131] S. Dragomir and L. Ornea, Locally conformal Kähler geometry, Birkhäuser, Boston-Basel-Berlin, 1998.

[132] N. Ejiri, Minimal immersions of Riemannian products into real space forms, Tokyo J. Math. 2 (1979), 63–70.

[133] N. Ejiri, Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc. 83 (1981), 759–763.
δ-invariants

[134] N. Ejiri, *Calabi lifting and surface geometry in $S^4*}, Tokyo J. Math. **9** (1986), 297–324.

[135] J. Fastenakels, *Ideal tubular hypersurfaces in real space forms*, Arch. Math. (Brno) **42** (2006), 295–305.

[136] L.M. Fernández and M.B. Hans-Uber, *New relationships involving the mean curvature of slant submanifolds in $S$-space-forms*, J. Korean Math. Soc. **44** (2007), 647–659.

[137] S. Funabashi, Y.M. Kim and J.S. Pak, *On submanifolds of $L \times F$ satisfying Chen's basic equality*, Acta Math. Hungr. **99** (2003), 189–201.

[138] C.F. Gauss, *Disquisitiones generales circa superficies curvas*, Comment. Soc. Sci. Gotting. Recent. Classis Math. **6**, 1827.

[139] H. Goenner, *On the History of Unified Field Theories*, Living Rev. Relativity **7** no. 2 (2004). [http://www.livingreviews.org/llr-2004-2](http://www.livingreviews.org/llr-2004-2).

[140] A. Gray, *Almost complex submanifolds of the six sphere*, Proc. Amer. Math. Soc. **20** (1969), 277–279.

[141] M. Gromov, *A topological technique for the construction of solutions of differential equations and inequalities*, Intern. Congr. Math. (Nice 1970) **2** (1971), 221–225.

[142] M. Gromov, *Isometric immersions of Riemannian manifolds*, in: Elie Cartan et les Mathématiques d’Aujourd’hui, Astérisque 1985, pp. 129–133.

[143] I.V. Guadelupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. **106** (1983), 95–103.

[144] R.S. Gupta, I. Ahmad and S.M. Khursheed Haider, *B. Y. Chen’s inequality and its application to slant immersions into Kenmotsu manifolds*, Kyungpook Math. J. **44** (2004), 101–110.

[145] R. Harvey and H.B. Lawson, *Calibrated foliation*, Amer. J. Math. **104** (1982), 607–633.
[146] I. Hasegawa and I. Mihai, *Contact CR-warped product submanifolds in Sasakian manifolds*, Geom. Dedicata **102** (2003), 143–150.

[147] S. Haesen, *Optimal inequalities for embedded space-times*, Kragujevac J. Math. **28** (2005), 69–85.

[148] S. Haesen, A. Šebeković and L. Verstraelen, *Relations between intrinsic and extrinsic curvatures*, Kragujevac J. Math. **25** (2003), 139–145.

[149] S. Haesen and L. Verstraelen, *Ideally embedded space-times*, J. Math. Phys. **45** (2004), 1497–1510.

[150] A. Hinić, *Some curvature conditions of the type $2 \times 4$ on the submanifolds satisfying Chen’s equality*, Mat. Vesnik **59** (2007), 189–196.

[151] S. Hong, K. Matsumoto and M. M. Tripathi, *Certain basic inequalities for submanifolds of locally conformal Kähler space forms*, SUT J. Math. **41** (2005), 75–94.

[152] S. Hong and M. M. Tripathi, *On Ricci curvature of submanifolds*, Int. J. Pure Appl. Math. Sci. **2** (2005), 227–245.

[153] S. Hong and M. M. Tripathi, *On Ricci curvature of submanifolds of generalized Sasakian space forms*, Int. J. Pure Appl. Math. Sci. **2** (2005), 173–201.

[154] Y. Hong and C. S. Houh, *Lagrangian submanifolds of quaternion Kählerian manifolds satisfying Chen’s equality*, Beiträge Algebra Geom. **39** (1998), 413–421.

[155] T. Kaluza, *Zum Unitätsproblem der Physik*, Sitz. Preuss. Akad. der Wiss. Phys. Math. Berlin, 1921, pp. 966–972.

[156] J.-K. Kim, Y.-M. Song and M.M. Tripathi, *B.-Y. Chen inequalities for submanifolds in generalized complex space forms*, Bull. Korean Math. Soc. **40** (2003), 411–423.

[157] J.-K. Kim, Y.-M. Song and M.M. Tripathi, *Shape operator for slant submanifolds in generalized complex space forms*, Indian J. Pure Appl. Math. **34** (2003), 1153–1163.
δ-invariants

[158] J.S. Kim and J. Choi, A basic inequality for submanifolds in a cosymplectic space form, Int. J. Math. Math. Sci. 2003, no. 9, 539–547.

[159] J.-S. Kim, M.K. Dwivedi and M.M. Tripathi, Ricci curvature of integral submanifolds of an $S$-space form, Bull. Korean Math. Soc. 44 (2007), 395–406.

[160] J.S. Kim, M.M. Tripathi and J. Choi, Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds, Indian J. Pure Appl. Math. 35 (2004), 259–271.

[161] N.G. Kim and D.W. Yoon, An obstruction to totally real submanifolds in locally conformal Kähler space forms, Far East J. Math. Sci. (FJMS) 3 (2001), 645–655.

[162] Y.H. Kim and D.S. Kim, A basic inequality for submanifolds in Sasakian space forms, Houston J. Math. 25 (1999), 247–257.

[163] Y.H. Kim and D.S. Kim, Inequality for totally real warped products in locally conformal Kähler space forms, Kyungpook Math. J. 44 (2004), 585–592.

[164] Y.M. Kim, Chen’s basic equalities for submanifolds of Sasakian space form, Kyungpook Math. J. 43 (2003), 63–71.

[165] Y.M. Kim and J.S. Pak, On the Ricci curvature of submanifolds in the warped product $L \times_f F$, J. Korean Math. Soc. 39 (2002), 693–708.

[166] O. Klein, Quantentheorie und 5-dimensionale Relativittstheorie, Zeits. Phys. 37 (1926), 895–906.

[167] S. Kobayashi and K. Nomizu, Fondations of Differential Geometry, I, II, J. Wiley, New York, 1969.

[168] M. Kriele, C. Scharlach and L. Vrancken, An extremal class of 3-dimensional elliptic affine spheres, Hokkaido Math. J. 30 (2001), 1–23.

[169] M. Kriele and L. Vrancken, An extremal class of three-dimensional hyperbolic affine spheres, Geom. Dedicata 77 (1999), 239–252.
[170] R.S. Kulkarni, *Curvature structures and conformal transformations*, Bull. Amer. Math. Soc. **75** (1969), 91–94.

[171] P.F. Leung, *On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold*, Proc. Edinburgh Math. Soc. **28** (1985), 305–311.

[172] G. Li, *Semi-parallel, semi-symmetric immersions and Chen’s equality*, Results Math. **40** (2001), 257–264.

[173] G. Li, *Ideal Einstein, conformally flat and semi-symmetric immersions*, Israel J. Math. **132** (2002), 207–220.

[174] G. Li and C. Wu, *Slant immersions of complex space forms and Chen’s inequality*, Acta Math. Sci. Ser. B Engl. Ed. **25** (2005), 223–232.

[175] X. Li, G. Huang and J. Xu, *Some inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Soochow J. Math. **31** (2005), 309–319.

[176] X. Liu, *On Ricci curvature of C-totally real submanifolds in Sasakian space forms*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), 399–405.

[177] X. Liu, *On Ricci curvature of totally real submanifolds in a quaternion projective space*, Arch. Math. (Brno) **38** (2002), 297–305.

[178] X. Liu and W. Dai, *Ricci curvature of submanifolds in a quaternion projective space*, Commun. Korean Math. Soc. **17** (2002), 625–633.

[179] X. Liu and W. Su, *Shape operator of slant submanifolds in cosymplectic space forms*, Studia Sci. Math. Hungar. **42** (2005), 387–400.

[180] X. Liu, A. Wang and A. Song, *Shape operator of slant submanifolds in Kenmotsu space forms*, Bull. Iranian Math. Soc. **30** (2004), 81–96.

[181] A.A. Logunov, M.A. Mestvirishvili, V.A. Petrov, *How were the Hilbert-Einstein equations discovered?*, arXiv:physics/0405075.

[182] Y. Lu and C. Scharlach, *Affine hypersurfaces admitting a pointwise symmetry*, Results Math. **48** (2005), 275–300.

[183] Z. Lu, *Proof of the normal scalar curvature conjecture*, preprint.
δ-invariants

[184] D. Lupu, *The shape operator of C-totally real submanifolds of Sasakian space forms*, An. Univ. București Mat. Inform. 50 (2001), 87–91.

[185] K. Mashimo, *Homogeneous totally real submanifolds of S^6*, Tsukuba J. Math. 9 (1985), 185–202.

[186] K. Matsumoto and I. Mihai, *Warped product submanifolds in Sasakian space forms*, SUT J. Math. 38 (2002), 135–144.

[187] K. Matsumoto, I. Mihai and A. Oiagă, *Shape operator for slant submanifolds in complex space forms*, Bull. Yamagata Univ. Natur. Sci. 14 (2000), 169–177.

[188] K. Matsumoto, I. Mihai and A. Oiagă, *Ricci curvature of submanifolds in complex space forms*, Rev. Roumaine Math. Pures Appl. 46 (2001), 775–782.

[189] A. Mihai, *Certain Chen-like inequalities for slant submanifolds in generalized complex space forms*, Tensor (N.S.) 63 (2002), 101–112.

[190] A. Mihai, *Shape operator A_H for slant submanifolds in generalized complex space forms*, Turkish J. Math. 27 (2003), 509–523.

[191] A. Mihai, *An inequality for totally real surfaces in complex space forms*, Kragujevac J. Math. 26 (2004), 83–88.

[192] A. Mihai, *Warped product submanifolds in complex space forms*, Acta Sci. Math. (Szeged) 70 (2004), 419–427.

[193] A. Mihai, *B. Y. Chen inequalities for slant submanifolds in generalized complex space forms*, Rad. Mat. 12 (2004), 215–231.

[194] A. Mihai, *Warped product submanifolds in quaternion space forms*, Rev. Roumaine Math. Pures Appl. 50 (2005), 283–291.

[195] A. Mihai, *Warped product submanifolds in generalized complex space forms*, Acta Math. Acad. Paedagog. Nyhzi. (N.S.) 21 (2005), 79–87.

[196] A. Mihai, *Modern topics in submanifold theory*, Editura Universității din București, Bucharest, 2006.
[197] I. Mihai, *Ricci curvature of submanifolds in Sasakian space forms*, J. Aust. Math. Soc. **72** (2002), 247–256.

[198] I. Mihai, *Geometry of warped product submanifolds in Sasakian space forms*, Proceedings of the International Symposium on ”Analysis, Manifolds and Mechanics”, 55–61, M. C. Chaki Cent. Math. Math. Sci., Calcutta, 2003.

[199] I. Mihai, *Ideal C-totally real submanifolds in Sasakian space forms*, Ann. Mat. Pura Appl. (4) **182** (2003), 345–355.

[200] I. Mihai, *On Kählerian slant submanifolds in complex space forms satisfying a geometrical equality*, An. Univ. București Mat. **53** (2004), 77–84.

[201] I. Mihai, *Contact CR-warped product submanifolds in Sasakian space forms*, Geom. Dedicata **109** (2004), 165–173.

[202] I. Mihai, *Ideal Kählerian slant submanifolds in complex space forms*, Rocky Mountain J. Math. **35** (2005), 941–951.

[203] I. Mihai, F. Al-Solamy and M. H. Shahid, *On Ricci curvature of a quaternion CR-submanifold in a quaternion space form*, Rad. Mat. **12** (2003), 91–98.

[204] I. Mihai and Y. Tazawa, *On 3-dimensional contact slant submanifolds in Sasakian space forms*, Bull. Austral. Math. Soc. **68** (2003), 275–283.

[205] M.I. Munteanu, *Warped product contact CR-submanifolds of Sasakian space forms*, Publ. Math. Debrecen **66** (2005), 75–120.

[206] M.I. Munteanu, *Doubly warped product CR-submanifolds in locally conformal Kähler manifolds*, Monatsh. Math. **150** (2007), 333–342.

[207] T. Nagano, *On the minimum eigenvalues of the Laplacian in Riemannian manifolds*, Sci. Papers College Gen. Edu. Univ. Tokyo **11** (1961), 177–182.

[208] J.F. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.
δ-invariants

[209] K. Nomizu and U. Pinkall, *On the geometry of affine immersions*, Math. Z. **195** (1987), 165–178.

[210] K. Nomizu and T. Sasaki, *Affine Differential Geometry. Geometry of Affine Immersions*, Cambridge Tracts in Math. no. **111**, Cambridge University Press, 1994.

[211] K. Ogiue, *Differential geometry of Kähler submanifolds*, Adv. Math. **13** (1974), 73–114.

[212] A. Oiagă, *An estimate of scalar curvature for certain submanifolds in complex space forms*, Proc. Conf. in honor of Prof. Radu Rosca, 1999, PADGE **3**.

[213] A. Oiagă, *Ricci curvature of totally real submanifolds in locally conformal Kähler space forms*, Proceedings of the Centennial ”G. Vrânceanu” and the Annual Meeting of the Faculty of Mathematics (Bucharest, 2000), An. Univ. București Mat. Inform., **49** (2000), 69–76.

[214] A. Oiagă and I. Mihai, *B.Y. Chen inequalities for slant submanifolds in complex space forms*, Demonstratio Math. **32** (1999), 835–846.

[215] T. Oprea, *Generalization of Chen’s inequality*, Algebras Groups Geom. **23** (2006), 93–103.

[216] T. Oprea, *Chen’s inequality in the Lagrangian case*, Colloq. Math. **108** (2007), 163–169.

[217] T. Oprea, *On a Riemannian invariant of Chen type*, Rocky Mountain J. Math. **38** (2008), 568–587.

[218] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756.

[219] J.M. Overduina and P. Wesson, *Kaluza-Klein gravity*, Phys. Rep. **283** (1997), 303–378.

[220] C. Özgüür and K. Arslan, *On some class of hypersurfaces in \( \mathbb{E}^{n+1} \) satisfying Chen’s equality*, Turkish J. Math. **26** (2002), 283–293.
[221] C. Özgür and M. M. Tripathi, *On submanifolds satisfying Chen's equality in a real space form*, Arab. J. Sci. Eng., (to appear).

[222] M. Petrović and L. Verstraelen, *On Deszcz symmetries of Wintgen ideal submanifolds*, Arch. Math. (Brno) (to appear).

[223] M. Petrović-Torgašev and A. Hinić, *Some curvature conditions of the type $4 \times 2$ on the submanifolds satisfying Chen's equality*, Mat. Vesnik 59 (2007), 143–150.

[224] J. Ponce de León, *Cosmological models in a Kaluza-Klein theory with variable rest mass*, Gen. Relativity Gravitation 20 (1988), 539–550.

[225] L. Randall and R. Sundrum, *Large mass hierarchy from a small extra dimension*, Phys. Rev. Lett. 83 (1999), 3370–3373.

[226] L. Randall and R. Sundrum, *An alternative to compactification*, Phys. Rev. Lett. 83 (1999), 4690–4693.

[227] H. Reckziegel, *Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion*, Lecture Notes in Mathematics, Vol. 1156 (Springer, New York 1985) pp. 264–279.

[228] A. Ros, *Spectral geometry of CR-minimal submanifolds in the complex projective space*, Kodai Math. J. 6 (1983), 88–99.

[229] A. Ros, and F. Urbano, *Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Japan 50 (1998), 203–228.

[230] B. Sahin, *Nonexistence of warped product semi-slant submanifolds of Kähler manifolds*, Geom. Dedicata 117 (2006), 195–202.

[231] B. Sahin and R. Günes, *CR-warped product submanifolds of nearly Kaehler manifold*, preprint.

[232] T. Sasahara, *CR-submanifolds in a complex hyperbolic space satisfying an equality of Chen*, Tsukuba J. Math. 23 (1999), 565–583.

[233] T. Sasahara, *Three-dimensional CR-submanifolds in the nearly Kähler six-sphere satisfying B. Y. Chen’s basic equality*, Tamkang J. Math. 31 (2000), 289–296.
δ-invariants

[234] T. Sasahara, Chen invariant of CR-submanifolds, in: Geometry of Submanifolds, pp. 114–120, Kyoto, 2001.

[235] T. Sasahara, On Ricci curvature of CR-submanifolds with rank one totally real distribution, Nihonkai Math. J. 12 (2001), 47–58.

[236] T. Sasahara, On Chen invariant of CR-submanifolds in a complex hyperbolic space, Tsukuba J. Math. 26 (2002), 119–132.

[237] C. Scharlach, Affine geometry of surfaces and hypersurfaces in $R^4$, Symposium on Differential Geometry of Submanifolds, Valenciennes, France, pp. 251–256, July 2007.

[238] C. Scharlach, U. Simon, L. Verstraelen and L. Vrancken, A new intrinsic curvature invariant for centroaffine hypersurfaces, Beiträge Algebra Geom. 38 (1997), 437–458.

[239] C. Scharlach and L. Vrancken, A curvature invariant for centroaffine hypersurfaces. II, Geometry and Topology of Submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), 341–350, World Sci. Publ., River Edge, NJ, 1996.

[240] S. Seahra, and P. Wesson, The structure of the big bang from higher-dimensional embeddings, Classical Quantum Gravity 19 (2002), 1139–1155.

[241] C. Segre, Sulle varietà che rappresentano le coppie di punti di due piani o spazi, Rend. Cir. Mat. Palermo 5 (1891), 192–204.

[242] Z. Sentürk and L. Verstraelen, Chen Kähler hypersurface, Taiwanese J. Math. 12 (2008) (to appear).

[243] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, Introduction to the Affine Differential Geometry of Hypersurfaces, Science University of Tokyo, 1991.

[244] I.M. Singer and J.A. Thorpe, The curvature of 4-dimensional Einstein spaces, Global Analysis, Princeton University Press, 355–365 (1969).

[245] A. Song and X. Liu, Some inequalities of slant submanifolds in generalized complex space forms, Tamkang J. Math. 36 (2005), 223–229.
150

B.-Y. Chen

[246] B. Suceavă, Some remarks on B. Y. Chen’s inequality involving classical invariants, An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.) 45 (1999), 405–412.

[247] B. Suceavă, The Chen invariants of warped products of hyperbolic planes and their applications to immersibility problems, Tsukuba J. Math. 25 (2001), 311–320.

[248] B. Suceavă, Fundamental inequalities and strongly minimal submanifolds, Recent advances in Riemannian and Lorentzian geometries (Baltimore, MD, 2003), Amer. Math. Soc., Providence, Contemp. Math., 337 (2003), 155–170.

[249] H. Sungpyo and M.M. Tripathi, On Ricci curvature of submanifolds, Int. J. Pure Appl. Math. Sci. 2 (2005), 227–245.

[250] M.M. Tripathi, Certain basic inequalities for submanifolds in (κ,μ)-spaces, Recent advances in Riemannian and Lorentzian geometries (Baltimore, MD, 2003), Amer. Math. Soc., Providence, RI, Contemp. Math., 337 (2003), 187–202.

[251] M.M. Tripathi, J.S. Kim and S.B. Kim, Mean curvature and shape operator of slant immersions in a Sasakian space form, Balkan J. Geom. Appl. 7 (2002), 101–111.

[252] M.M. Tripathi, J.S. Kim and S.B. Kim, A basic inequality for submanifolds in locally conformal almost cosymplectic manifolds, Proc. Indian Acad. Sci. Math. Sci. 112 (2002), 415–423.

[253] M.M. Tripathi, J.S. Kim and S.B. Kim, A note on Chen’s basic equality for submanifolds in a Sasakian space form, Int. J. Math. Math. Sci. 2003, no. 11, 711–716.

[254] G.E. Vilcu, B.Y. Chen inequalities for slant submanifolds in quaternionic space forms, preprint.

[255] L. Vrancken, Killing vector fields and Lagrangian submanifolds of the nearly Kähler $S^6$, J. Math. Pure Appl. 77 (1998), 631–645.
[256] L. Vrancken, *Special classes of three dimensional affine hyperspheres characterized by properties of their cubic form*, Contemporary Geometry and Related Topics, 431–459, World Sci. Publ., River Edge, NJ, 2004.

[257] A.Q. Wang and X.M. Liu, *Ricci curvature of semi-invariant submanifolds in cosymplectic space forms*, J. Math. Res. Exposition 27 (2007), 195–200.

[258] S.W. Wei, J. Li, and L. Wu, *p-harmonic generalizations of the uniformization theorem and Bochner’s method, and geometric applications*, preprint.

[259] P. Wesson, *Space-Time-Matter, Modern Kaluza-Klein Theory*, World Scientific, River Edge, NJ. 1999.

[260] P. Wesson, *Five-Dimensional Physics: Classical and Quantum Consequences of Kaluza-Klein Cosmology*, World Scientific, River Edge, NJ., 2006.

[261] P. Wintgen, *Sur l’inégalité de Chen-Willmore*, C. R. Acad. Sci. Paris Sr. A-B 288 (1979), A993–A995.

[262] K. Yano and M. Kon, *CR-submanifolds of Kählerian and Sasakian manifolds*, Birkhäuser, Boston-Basel, 1983.

[263] M. Yildirim Yilmaz and M. Bekas, *Shape operator for real space forms in Hessian manifolds of constant Hessian sectional curvature*, Int. J. Contemp. Math. Sci. 2 (2007), 343–348.

[264] D.W. Yoon, *B.-Y. Chen’s inequality for CR-submanifolds of locally conformal Kähler space forms*, Demonstratio Math. 36 (2003), 189–198.

[265] D.W. Yoon, *Ricci curvature of submanifolds in quaternionic space forms*, Int. Math. J. 4 (2003), 377–384.

[266] D.W. Yoon, *Certain inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Bull. Inst. Math. Acad. Sinica 32 (2004), 263–283.
[267] D.W. Yoon, Some inequalities for warped products in cosymplectic space form, Differ. Geom. Dyn. Syst. 6 (2004), 51–58.

[268] D.W. Yoon, Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms, Turkish J. Math. 30 (2006), 43–56.

[269] D.W. Yoon and K.S. Cho, Inequality for warped products in generalized Sasakian space forms, Int. Math. J. 5 (2004), 225–235.