EVALUATION AND NORMALIZATION OF JACK SUPERPOLYNOMIALS

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Abstract. Two evaluation formulas are derived for the Jack superpolynomials. The evaluation formulas are expressed in terms of products of fillings of skew diagrams. One of these formulas is nothing but the evaluation formula of the Jack polynomials with prescribed symmetry, which thereby receives here a remarkably simple formulation. Among the auxiliary results required to establish the evaluation formulas, the determination of the conditions ensuring the non-vanishing coefficients in a Pieri-type rule for Jack superpolynomials is worth pointing out. An important application of the evaluation formulas is a new derivation of the combinatorial norm of the Jack superpolynomials. We finally mention that the introduction of a simpler version of the dominance ordering on superpartitions is fundamental to establish our results.
1. Introduction

The Jack superpolynomials were introduced in 2003 [8] as the orthogonal eigenfunctions of a quantum mechanical many-body problem that had been formulated a decade before, namely the supersymmetric Calogero-Moser-Sutherland model [18, 3] (see also [6] for more results and references on this model). As their name suggests, the Jack superpolynomials generalize Jack’s symmetric polynomials [15, 19] by incorporating both commuting and anticommuting variables. The presence of anticommuting variables obviously makes computations and demonstrations more involved than in the classical theory of symmetric polynomials. But despite this apparent complexity, the Jack superpolynomials share many elegant properties with their classical counterparts [8, 10], such as orthogonality with respect to two different scalar products, and duality. The aim of the article is to further develop the strong analogy between the properties of the Jack superpolynomials and those of the Jack polynomials. Before presenting the most relevant results, let us review some elements of the theory of symmetric superpolynomials.
1.1. Superpartitions. Superpartitions were first introduced in 2001 [6], but it was later noticed [9] that they could be interpreted as overpartitions [5] or as MacMahon standard diagrams [17]. Here we adopt the following definition:

**Definition 1.** A superpartition $\Lambda$ of degree $(n|m)$ and length $\ell$ is a pair $(\Lambda^\circ, \Lambda^*)$ of partitions $\Lambda^\circ$ and $\Lambda^*$ such that

1. $\Lambda^* \subseteq \Lambda^\circ$
2. the degree of $\Lambda^*$ is $n$
3. the length of $\Lambda^\circ$ is $\ell$
4. the skew diagram $\Lambda^\circ/\Lambda^*$ is both a horizontal and a vertical $m$-strip.

Note that we follow Macdonald’s notation for partitions, diagrams and skew-diagrams (see Section 2 and [15]). Obviously, if $\Lambda^\circ = \Lambda^* = \lambda$, then $\Lambda = (\lambda, \lambda)$ can be interpreted as the partition $\lambda$.

A very convenient way to represent superpartitions was introduced in [9]. Concretely, the Ferrers diagram of a superpartition $\Lambda = (\Lambda^\circ, \Lambda^*)$ is obtained by

1. drawing the diagram of $\Lambda^\circ$, and
2. replacing the cells that belong to $\Lambda^\circ/\Lambda^*$ by circles.

Figure 1 illustrates this procedure for the case $\Lambda^\circ = (4, 3, 3, 1, 1)$ and $\Lambda^* = (3, 3, 2, 1)$. To distinguish them from the circles, the cells corresponding to those of $\Lambda^*$ in the diagram of $\Lambda$ will be called squares. Because the circles form a horizontal and a vertical strip, two circles cannot appear in the same column nor in the same row. In other words, two rows or two columns ending with a circle cannot have the same length. This situation is clearly reminiscent of the Pauli exclusion principle for fermionic states in quantum physics. For this reason, rows and columns that terminates with a circle are called fermionic, the other ones being said to be bosonic. The bosonic content of a superpartition, denoted by $B\Lambda$, is defined as the set of squares in the diagram of $\Lambda$ that do not belong at the same time to a fermionic row and a fermionic column. The fermionic content of $\Lambda$ is given by the complement of the bosonic content in the diagram $\Lambda^\circ$, that is, $F\Lambda = \Lambda^\circ/B\Lambda$. See Figure 2.

**Figure 1.** Diagram of a superpartition $\Lambda = (\Lambda^\circ, \Lambda^*)$

In the following paragraphs, we shall extensively make use of a partial order on superpartitions that generalizes naturally the usual dominance order. Let us recall that for any pair of partitions $\lambda$ and $\nu$ of $n$, $\lambda \geq \nu$ in the dominance order if and only if $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \nu_i$ for all $k$. We now equip the set of all superpartitions of a given degree $(n|m)$ with the following dominance order:

$$\Lambda \geq \Omega \iff \Lambda^\circ \geq \Omega^\circ \quad \text{and} \quad \Lambda^* \geq \Omega^*.$$  \hspace{1cm} (1)
1.2. Jack symmetric superpolynomials. Let \( x = (x_1, \ldots, x_N) \) and \( \theta = (\theta_1, \ldots, \theta_M) \) be two sets of indeterminates that satisfy the following commutation relations:

\[
x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i, \quad \theta_i^2 = 0,
\]

for all indices \( i, j \). A superpolynomial, or polynomial in superspace, is an element of the ring of polynomials in \( x \) and \( \theta \) over a ring \( R \). Equivalently, a superpolynomial is an element of the Grassmann algebra generated by \( (\theta_1, \ldots, \theta_M) \) over the polynomial ring \( R[x_1, \ldots, x_N] \). Following the terminology used in physics, the variables \( x \) and \( \theta \) will be respectively called bosonic and fermionic.

From now on, we set \( N = M \) and assume that \( R \) is the field \( \mathbb{Q}(\alpha) \) of rational functions in the indeterminate \( \alpha \). A symmetric superpolynomial \([6]\) is a superpolynomial \( f(x, \theta) \) such that

\[
f(x_1, \ldots, x_N, \theta_1, \ldots, \theta_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}, \theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)})
\]

for any permutation \( \sigma \) of \( \{1, \ldots, N\} \). Notice that if \( f(x, \theta) \) is symmetric and homogeneous both in \( x \) and in \( \theta \), then it can be decomposed as follows:

\[
f(x_1, \ldots, x_N, \theta_1, \ldots, \theta_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} \theta_{i_1} \cdots \theta_{i_m} f_{i_1, \ldots, i_m}(x),
\]

where \( f_{i_1, \ldots, i_m}(x) \) is a homogeneous polynomials antisymmetric in the variables \( x_{i_1}, \ldots, x_{i_m} \) and symmetric in the remaining variables. In fact, \( f_{i_1, \ldots, i_m}(x) \) is an example of a polynomial with prescribed symmetry \([1, 2, 12]\).

The set of homogeneous symmetric superpolynomials of degree \( n \) in \( x \) and \( m \) in \( \theta \) obviously forms a finite vector space over \( \mathbb{Q}(\alpha) \), which will be denoted \( \mathcal{R}^{n,m}_N \). As explained in \([6, 9]\), there exists a bijective map between any basis of \( \mathcal{R}^{n,m}_N \) and the set of superpartitions \( \Lambda \) of degree \( (n|m) \) and of length not larger than \( N \). We will see in Section 2.2 that the symmetric monomials \( m_{\Lambda}(x, \theta) \) form a simple basis of \( \mathcal{R}^{n,m}_N \). For the moment, the only additional information we need concerning the symmetric monomials is the following stability property:

\[
\rho_{M,N} : m_{\Lambda}(x_1, \ldots, x_M, \theta_1, \ldots, \theta_M) \mapsto m_{\Lambda}(x_1, \ldots, x_N, \theta_1, \ldots, \theta_N) \quad \forall M \geq N,
\]

where \( \rho_{M,N} \) is the homomorphism \( \mathcal{R}^{n,m}_M \to \mathcal{R}^{n,m}_N \) that sends the indeterminates \( x_{N+1}, \theta_{N+1}, \ldots, x_M, \theta_M \) to zero and acts as the identity on the remaining ones. Note that by definition, \( m_{\Lambda}(x, \theta) \) is zero whenever the length of the superpartition is greater than the number of bosonic variables.

The stability property together with the fact that \( \rho_{N,N} = \text{id} \) and \( \rho_{M,N} \circ \rho_{L,M} = \rho_{L,N} \) for all \( N \leq M \leq L \), enable us to take the inverse limit:

\[
m_{\Lambda} = \lim_{\leftarrow} m_{\Lambda}(x, \theta) = (m_{\Lambda}(x_1, \theta_1), m_{\Lambda}(x_1, x_2, \theta_1, \theta_2), m_{\Lambda}(x_1, x_2, x_3, \theta_1, \theta_2, \theta_3), \ldots).
\]
We can then identify $\text{span}_{\mathbb{Q}(\alpha)} \{ m_{\Lambda} : \Lambda \text{ is a superpartition} \}$ with the following bi-graded vector space:

$$\mathcal{R} = \bigoplus_{n \geq 1, m \geq 0} \mathcal{R}^{n,m}, \quad \mathcal{R}^{n,m} = \lim_{\leftarrow} \mathcal{R}^{n,m}_N. \quad (7)$$

Given that the symmetric superpolynomials in $N$ bosonic and $N$ fermionic variables form a ring, the componentwise product between elements of $\mathcal{R}$ is well defined, and so $\mathcal{R}$ also carries the structure of a bi-graded algebra. The elements of the latter will be called symmetric superfunctions. $\mathcal{R}$ is moreover equipped with a surjective homomorphism $\rho_N : \mathcal{R} \to \bigoplus_{n,m} \mathcal{R}^{n,m}_{N}$ that maps all the $x_i$ and $\theta_i$ with $i > N$ to zero. To sum up, any element $f$ of $\mathcal{R}$ is a symmetric superfunction; it is equal to a finite linear combination of the monomials $m_{\Lambda}$; and to any such $f$ corresponds a symmetric superpolynomial in $N$ bosonic and $N$ fermionic indeterminates, $f(x,\theta) = \rho_N(f)$, which is nonzero if $N$ is large enough.

It was shown in [9, 10] that the algebra $\mathcal{R}$ of symmetric superfunctions can be endowed with a natural scalar product $\langle \langle | \rangle \rangle : \mathcal{R} \times \mathcal{R} \to \mathbb{Q}(\alpha)$, which generalizes the usual Hall scalar product for symmetric polynomials [15] (see (26) for an explicit definition of the scalar product).

We are now in a position to define the Jack superpolynomials.

**Definition 2.** Let $\Lambda$ be superpartitions of degree $(n|m)$. The monic Jack superfunction $P_{\Lambda}$ is the unique element of $\mathcal{R}$ that satisfies

$$P_{\Lambda} = m_{\Lambda} + \sum_{\Omega <_{\Lambda} \Lambda} c_{\Lambda\Omega} m_{\Omega} \quad \text{(triangularity)} \quad (9)$$

$$\langle \langle P_{\Lambda} | P_{\Omega} \rangle \rangle = 0 \quad \text{if } \Lambda \neq \Omega \quad \text{(orthogonality)} \quad (10)$$

where the coefficients $c_{\Lambda\Omega}$ in the triangularity relation belong to $\mathbb{Q}(\alpha)$. The monic Jack superpolynomial $P_{\Lambda}(x,\theta)$ with $N$ bosonic and $N$ fermionic indeterminates is equal to $\rho_N(P_{\Lambda})$.

The existence of the superpolynomials $P_{\Lambda}(x,\theta)$ was proved in [10]. It was also shown in [10] that the $P_{\Lambda}(x;\theta)$’s are equivalent to the Jack superpolynomials previously defined in [8] as the orthogonal solutions of a quantum mechanical eigenvalue problem (the orthogonality being with respect to a distinct scalar product). Note that the usual Jack symmetric polynomials $P_{\lambda}(x)$ are recovered by setting $\Lambda = (\lambda,\lambda)$, which corresponds to letting the degree $m$ in the Grassmann variables $\theta$ be equal to zero.

To conclude this review section, a precision is in order. Definition 2 is in fact a slightly more precise version than the one presented in [10] in that the dominance order controlling the triangular decomposition is now more restrictive. Indeed, the partial order $\succeq$ used in [10] was defined as follows: For $\Lambda$ and $\Omega$ two superpartitions of degree $(n|m)$,

$$\Lambda \succeq \Omega \quad \iff \quad \Lambda^* > \Omega^* \quad \text{or} \quad \Lambda^* = \Omega^* \quad \text{and} \quad \Lambda^\circ \geq \Omega^\circ, \quad (11)$$

where again the order on partitions is the dominance order. Observe that the order $\succeq$ is clearly less restrictive than the order $\geq$. We shall nevertheless prove in Appendix B that

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1We stress that in our previous works [8,10], we have denoted the monic Jack superpolynomials by $J_{\Lambda}$. Here we model our notation on the standard one [15,19] for the monic case and reserve the symbol $J_{\Lambda}$ for a different normalization – see Section 5.
the two orders lead to the same symmetric polynomials in superspace, which will allow us to exploit all the properties of the Jack superpolynomials obtained in \[8, 10\].

1.3. Main results. A combinatorial formula for the norm squared \(\langle P_\Lambda | P_\Lambda \rangle = \| P_\Lambda \|^2\) was conjectured in \[10\]. With \(\Lambda = (\Lambda^\flat, \Lambda^\ast)\) a superpartition of degree \((n|m)\), the conjecture given in \[10\] is equivalent to

\[
\| P_\Lambda \|^2 = \alpha^m \prod_{s \in B\Lambda} \frac{l_{\Lambda^\flat}(s) + \alpha(1 + a_{\Lambda^\ast}(s))}{1 + l_{\Lambda^\ast}(s) + \alpha a_{\Lambda^\flat}(s)}, \tag{12}
\]

where we stress that the arm- and leg-lengths are evaluated with respect to two different diagrams (for the definitions of \(a_{\Lambda}(s)\) and \(l_{\Lambda}(s)\) we refer to Section 2 or \[15\]).

This formula was proved in \[14\] using a characterization of the Jack superpolynomials in terms of the non-symmetric Jack polynomials – cf. \[8\] Sect. 9. The norm expression is reduced in \[14\] to an identity on partitions whose proof relies on the Gessel-Viennot lemma. Here we provide an alternative proof of (12). Our proof essentially follows Stanley’s method \[19\] in which the norm formula for a Jack polynomials \(P_\Lambda(x)\) is obtained as a consequence of the evaluation formula and the duality property of the \(P_\Lambda(x)\)’s. In the superpolynomial case, the proof relies on the duality, established in \[10\] Sect. 6.1, and two new evaluation formulas.

The precise statement of these evaluation formulas requires some more notation. Let \(f(x, \theta)\) be an element of \(\mathcal{R}^{n,m}_N\), that is, \(f(x, \theta)\) is a bi-homogeneous symmetric superpolynomials that can thus be expanded as in \[14\]. The evaluation of such symmetric superpolynomials is defined as the map

\[
E_{N,m} : \mathcal{R}^{n,m}_N \longrightarrow \mathbb{Q}(\alpha)
\]

such that

\[
E_{N,m}(f) = \left[ \prod_{1 \leq i < j \leq m} (x_i - x_j)^{-1} f_{1, \ldots, m}(x) \right]_{x_1 = \ldots = x_N = 1}. \tag{14}
\]

Our central result are the following two evaluation formulas.

**Theorem 3.** Let \(\Lambda = (\Lambda^\flat, \Lambda^\ast)\) be a superpartition of degree \((n|m)\) such that \(\ell(\Lambda) \leq N\). Let \(S\Lambda\) be the skew-diagram \(\Lambda^\flat/\delta_{m+1}\) where \(\delta_{m+1}\) stands for the diagram associated to the partition \((m, m-1, \ldots, 0)\). Finally, as in Figure 2, let \(B\Lambda\) denote the bosonic content of \(\Lambda\). Then the evaluation of the monic Jack polynomial \(P_\Lambda(x, \theta)\) is given by

\[
E_{N,m}(P_\Lambda) = \prod_{s \in S\Lambda} \left( N - l'_{\Lambda^\flat}(s) + \alpha a'_{\Lambda^\ast}(s) \right) / \prod_{s \in B\Lambda} \left( 1 + l_{\Lambda^\ast}(s) + \alpha a_{\Lambda^\flat}(s) \right). \tag{15}
\]

**Theorem 4.** Let \(\Lambda = (\Lambda^\flat, \Lambda^\ast)\) be a superpartition of degree \((n|m)\) such that \(m > 0\) and \(\ell(\Lambda) \leq N\). Let \(S\Lambda\) be the skew-diagram \(\Lambda^\ast/\delta_m\). The evaluation of

\[
F_\Lambda = \left[ (-1)^{m-1} \partial_{\theta_N} P_\Lambda(x, \theta) \right]_{x_N = 0}
\]

is given by

\[
E_{N-1,m-1}(F_\Lambda) = \prod_{s \in S\Lambda} \left( N - 1 - l'_{\Lambda^\ast}(s) + \alpha a'_{\Lambda^\ast}(s) \right) / \prod_{s \in B\Lambda} \left( 1 + l_{\Lambda^\ast}(s) + \alpha a_{\Lambda^\ast}(s) \right). \tag{17}
\]
Let us emphasize some unusual aspects of the evaluation. We first stress that in the evaluation of a superpolynomial, only the commuting variables $x_i$ are specialized at 1. Clearly, the anticommuting variables cannot be set equal to a common anticommuting value since every fermionic monomial of degree larger than 1 would then vanish. The necessity of factorizing a Vandermonde determinant is also easily understood. A homogeneous symmetric superpolynomial is of the form (4) where $f_{i_1 \ldots i_m}$ is antisymmetric with respect to $x_{i_1}, \ldots, x_{i_m}$ so that these variables cannot be set equal to 1 without causing the direct vanishing of the whole expression. Hence, before specializing each term, one has to factorize its antisymmetric core, that is, divide it by a Vandermonde determinant of order $m$.

1.4. Organization of the article. Before plunging into the different steps leading to the proof of Theorems 3 and 4 we need to review further results concerning superpartitions and symmetric superpolynomials. This is the subject of Section 2. The derivation of the evaluation formula relies on establishing in Section 3 the necessary conditions for the non-vanishing of the Pieri-type coefficients. The relevant results in that regard are Propositions 10 and 11 (proved in Appendix A).

Another required new tool is what might be called the analogue of the “column-by-column” decomposition of a Jack polynomial (cf. [19, Prop. 5.1]). In the present context, where a column might be either fermionic or bosonic, this requires the introduction of two distinct operations described in Section 4: the stripping of a bosonic column and the transmutation of a fermionic column into a bosonic one (see Figure 7). At the core of these column decompositions is the following remarkable property: removing/transmuting a leftmost column of a Jack superpolynomial in the right number of variables generates another Jack superpolynomial, up to a proportionality factor in the non-monic case. These factors are the building blocks of the expression for the combinatorial norm as shown in Section 6. Such proportionality factors, being the ratio of two polynomials, are most readily computed when the polynomials are specialized to particular values of their variables.

The proof of the evaluation formula given in Theorem 3 is presented in Section 5.1. As explained above, before implementing the evaluation, one must first divide by a Vandermonde determinant of order $m$. Remarkably, when $m > 0$ this order can be reduced from $m$ to $m-1$, which leads to the second non-trivial evaluation formula given in Theorem 4 and whose proof is presented in Section 5.2.

As an aside, we mention that before obtaining the evaluation formula (15) expressed in terms of skew diagrams, a quite different-looking version had been obtained by experimentation. Since this might be of independent interest, it is presented in Appendix C where the connection between the two formulas is also sketched.

Finally, it should be clear from the remark following Theorem 3 that the evaluation formula (15) for Jack polynomials in superspace is actually an evaluation formula for ordinary Jack polynomials with mixed symmetry (or with prescribed symmetry in the terminology of [1, 2, 12]). This implies that our evaluation formula must agree with the one presented in [12, Prop. 3.6] (yet another expression is given in [11]). It is remarkable that the very complicated-looking form of the latter evaluation formula can be reexpressed in the simple form presented here.

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2. Definitions

Let us first recall some definitions related to partitions [15]. A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of degree \( d \) is a vector of non-negative integers such that \( \lambda_i \geq \lambda_{i+1} \) for \( i = 1, 2, \ldots \) and such that \( \sum_i \lambda_i = d \). The length \( \ell(\lambda) \) of \( \lambda \) is the number of non-zero entries of \( \lambda \). Each partition \( \lambda \) has an associated Ferrers diagram with \( \lambda_i \) lattice squares in the \( i \)th row, from the top to bottom. Any lattice square in the Ferrers diagram is called a cell, where the cell \((i, j)\) is in the \( i \)th row and \( j \)th column of the diagram. Given a partition \( \lambda \), its conjugate \( \lambda' \) is the diagram obtained by reflecting \( \lambda \) about the main diagonal. Given a cell \( s = (i, j) \) in \( \lambda \), we let

\[
\begin{align*}
    a_\lambda(s) &= \lambda_i - j, \\
    a'_\lambda(s) &= j - 1, \\
    l_\lambda(s) &= \lambda'_j - i, \\
    l'_\lambda(s) &= i - 1.
\end{align*}
\]

(18)

The quantities \( a_\lambda(s), a'_\lambda(s), l_\lambda(s) \) and \( l'_\lambda(s) \) are respectively called the arm-length, arm-colength, leg-length and leg-colength. For instance, if \( \lambda = (8, 5, 5, 3, 1) \)

we have that \( a_\lambda(3, 2) = 3, a'_\lambda(3, 2) = 1, l_\lambda(3, 2) = 1 \) and \( l'_\lambda(3, 2) = 2 \). We say that the diagram \( \mu \) is contained in \( \lambda \), denoted \( \mu \subseteq \lambda \), if \( \mu_i \leq \lambda_i \) for all \( i \). Finally, \( \lambda/\mu \) is a horizontal (resp. vertical) \( n \)-strip if \( \mu \subseteq \lambda, |\lambda| - |\mu| = n \), and the skew diagram \( \lambda/\mu \) does not have two cells in the same column (resp. row).

We now review some basic results concerning superpartitions and the objects for which they provide the proper labeling, namely the symmetric polynomials in superspace. The functions of interest here are the superspace version of the Jack polynomials, which are introduced in Section 2.3. This material is essentially lifted from [6, 8, 9, 10].

2.1. Operations on superpartitions. Let us first go back to Definition [11]. When considering the superpartition \( \Lambda \) as a diagram such as in Figure 1, \( \Lambda \odot \) corresponds to the diagram obtained by replacing the circles in \( \Lambda \) by cells. Similarly, \( \Lambda^* \) corresponds to the diagram obtained by removing all the circles in \( \Lambda \). This allows us to consider the circled star \( \odot \) and the star * as operations on superpartitions (see Figure 3).
The bosonic degree $|\Lambda|$ of the superpartition $\Lambda$ is equal to $|\Lambda^\circ|$ (the number of cells in the diagram of $\Lambda^\circ$). The fermionic degree $\underline{\Lambda}$ of $\Lambda$ is the number of circles in the diagram of $\Lambda$, that is, $\underline{\Lambda} = |\Lambda^\circ| - |\Lambda^\ast|$. We say that $\Lambda$ is a superpartition of degree $\binom{n}{m}$ if $|\Lambda| = n$ and $\Lambda$ has fermionic degree $m$. The length $\ell(\Lambda)$ of the superpartition $\Lambda$ is equal to the length of $\Lambda^\circ$ (the number of rows in the diagram of $\Lambda^\circ$).

Though very practical for many purposes, such as to define the dominance order (1) on superpartitions, Definition 1 turns out to be less effective when working directly on symmetric superpolynomials. This is why we shall occasionally return to the original definition of a superpartition given in [6].

**Definition 5.** A superpartition $\Lambda$ of length $\ell$ is a pair of partitions $(\Lambda^a; \Lambda^s)$, the first one of which contains at most one 0 and does not have repeated entries. Explicitly,

$$\Lambda = (\Lambda^a; \Lambda^s) = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_\ell),$$

where

$$\Lambda_1 > \ldots > \Lambda_m \geq 0 \quad \text{and} \quad \Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_\ell > 0.$$ 

Note that $m$ corresponds in this definition to the fermionic degree of $\Lambda$. When $m = 0$, we simply omit the semi-column in $\Lambda = (\emptyset; \Lambda^s)$ and identify $\Lambda$ with $\Lambda^s$.

The equivalence between the two definitions is quite obvious: the parts of $\Lambda$ that belong to $\Lambda^a$ are the parts of $\Lambda^\ast$ such that $\Lambda^\circ_k - \Lambda_k^\ast = 1$. Going back to the example given in Figure 1, we see that if $\Lambda$ is such that $\Lambda^\circ = (4, 3, 3, 1, 1)$ and $\Lambda^\ast = (3, 3, 2, 1)$, then we have $\Lambda = (3, 2, 0; 3, 1)$. It is clear that $a : \Lambda \mapsto \Lambda^a$ can be viewed as a map that sends superpartitions of degree $\binom{n}{m}$ to strictly decreasing partitions of length $m$ and with at most one part equal to zero. In the same vein, $s : \Lambda \mapsto \Lambda^s$ maps superpartitions of degree $\binom{n}{m}$ end length $\ell$ to partitions of length $n - m$.

We finally define an important involution on the set of superpartitions: the conjugation. It is actually simpler to define the conjugation diagrammatically: the conjugate of a superpartition $\Lambda$, denoted by $\Lambda'$, is the superpartition whose diagram is obtained by reflecting the diagram of $\Lambda$ with respect to the main diagonal. As shown in Figure 4, reflecting for instance the diagram of $\Lambda = (3, 1, 0; 5, 4, 3)$ gives $\Lambda' = (5, 4, 2; 4, 1)$.

**2.2. Classical bases for symmetric polynomials in superspace.** As already mentioned in the introduction, a polynomial in superspace (a superpolynomial for short) is a polynomial in $x = (x_1, \ldots, x_N)$ and $\theta = (\theta_1, \ldots, \theta_N)$, where $\theta$ denotes a set of $N$ Grassmann variables. Rephrasing the information contained in [3], a polynomial in superspace is said to be symmetric if it is invariant under the simultaneous interchange of $x_i \leftrightarrow x_j$ and $\theta_i \leftrightarrow \theta_j$ for all
The set of all homogeneous symmetric polynomials of degree \(n\) in \(x\) and degree \(m\) in \(\theta\), forms a vector space \(\mathcal{R}^{n,m}_N\) over \(\mathbb{Q}(\alpha)\).

In what follows, we adopt the notation of Definition 5 and suppose that the superpartition \(\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_{\ell})\) is of degree \((n|m)\) and length \(\ell\). We always assume \(\ell \leq N\). In the case where \(\ell < N\), we set:

\[
\Lambda_{\ell+1} = \ldots = \Lambda_N = 0. \tag{22}
\]

A simple basis for the space \(\mathcal{R}^{n,m}_N\) is furnished by the following extension to superspace of the usual monomial symmetric functions \(m_\Lambda(x)\):

\[
m_\Lambda(x; \theta) = \frac{1}{n_\Lambda!} \sum_{\sigma \in S_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} x^{\Lambda_1}_{\sigma(1)} \cdots x^{\Lambda_m}_{\sigma(m)} \tag{23},
\]

where \(\Lambda\) is a superpartition of degree \((n|m)\) and of length \(\ell\leq N\), and

\[
n_\Lambda! := \prod_{i \geq 1} n_{\Lambda^i}(i)!, \tag{24}
\]

with \(n_{\Lambda^i}(i)\) being the number of parts in \(\Lambda^i = (\Lambda_{m+1}, \ldots, \Lambda_N)\) that are equal to \(i\). The factor \(n_\Lambda!\) is introduced to guarantee that the distinct non-symmetric monomials of the form \(\theta_{i_1} \cdots \theta_{i_m} x^{\Lambda_1}_{i_1} \cdots x^{\Lambda_m}_{i_m}\) appear in \(m_\Lambda\) with coefficients equal to \(\pm 1\). As explained in the introduction, the set of all monomials \(m_\Lambda = \lim_{\leftarrow} m_\Lambda(x; \theta)\) forms a linear basis for the algebra \(\mathcal{R}\) of symmetric superfunctions.

Another basis of symmetric superpolynomials is given by the power-sums

\[
p_\Lambda(x, \theta) := \tilde{p}_{\Lambda_1}(x, \theta) \cdots \tilde{p}_{\Lambda_m}(x, \theta) p_{\Lambda_{m+1}}(x) \cdots p_{\Lambda_{\ell}}(x), \tag{25}
\]

where

\[
\tilde{p}_n(x, \theta) := \sum_i \theta_i x^n_i \quad \text{and} \quad p_n(x) := \sum_i x^n_i. \tag{26}
\]

Now let \(p_\Lambda\) denote the inverse limit of the superpolynomial \(p_\Lambda(x, \theta)\). Then \(\mathcal{R}\) is equal to \(\text{span}_{\mathbb{Q}(\alpha)}\{p_\Lambda : \Lambda\text{ is a superpartition}\}\). The relevance of the power sums \(p_\Lambda\) in this article is rooted in the natural scalar product on \(\mathcal{R}\) defined as

\[
\langle p_\Lambda | p_\Omega \rangle = (-1)^{(n)\over(2)} \alpha^{(\lambda)} z_{\Lambda^\ast} \delta_{\Lambda, \Omega}, \tag{27}
\]

where

\[
z_{\Lambda^\ast} = \prod_i i^{n_{\Lambda^i}(i)} n_{\Lambda^i}(i)!. \tag{28}
\]

The sign \((-1)^{(n)\over(2)}\) arises in all scalar products of symmetric superfunctions of fermionic degree \(m\). It is thus convenient to define:

\[
\tilde{F} = (-1)^{(n)\over(2)} F \tag{29}
\]

on any homogeneous superfunction \(F\) of fermionic degree \(m\). In fact, the left-arrow is the involution in the Grassmann algebra generated by \(\theta\) that reverses the order of the variables \(\theta\), that is, \(\tilde{\theta}_{i_1} \cdots \tilde{\theta}_{i_m} = \theta_{i_m} \cdots \theta_{i_1}\). For cosmetic reasons, we also introduce

\[
\hat{F} = F \tag{30}
\]

which allows to write the scalar product (27) in a symmetrical fashion:

\[
\langle \tilde{p}_\Lambda | \tilde{p}_\Omega \rangle = \alpha^{(\lambda)} z_{\Lambda^\ast} \delta_{\Lambda, \Omega}. \tag{31}
\]
We shall also make use of the elementary superpolynomials $e_{\Lambda}(x, \theta)$, which provide another multiplicative basis for $\mathcal{R}_N^{n,m}$. They are defined as follows:

$$e_{\Lambda}(x, \theta) := \tilde{e}_{\Lambda_1}(x, \theta) \cdots \tilde{e}_{\Lambda_m}(x, \theta) e_{\Lambda_{m+1}}(x) \cdots e_{\Lambda_{\ell}}(x),$$

where $\Lambda$ is again a superpartition of fermionic degree $m$ and length $\ell(\Lambda) = \ell$, and where

$$\tilde{e}_n(x, \theta) := m_{(0,1^n)}(x, \theta) \quad \text{and} \quad e_n(x) := m_{(1^n)}(x).$$

### 2.3. Jack polynomials in superspace.

The basis of symmetric polynomials in superspace of concern here is the generalization of the Jack polynomials. They are most naturally defined as solutions of a double eigenvalue problem [8, 10]. Theorems 22 and 31 in [8] together with the discussion in Appendix B readily establish the following.

**Theorem 6.** Let $D$ and $\Delta$ be the two following algebraically independent and commuting differential operators:

$$D = \frac{1}{2} \sum_{i=1}^{N} \alpha x_i^2 \partial_{x_i}^2 + \sum_{1 \leq i \neq j \leq N} \frac{x_i x_j}{x_i - x_j} \left( \partial_{x_i} - \frac{\theta_i - \theta_j}{x_i - x_j} \partial_{\theta_i} \right),$$

and

$$\Delta = \sum_{i=1}^{N} \alpha x_i \partial_{x_i} \partial_{\theta_i} + \sum_{1 \leq i \neq j \leq N} \frac{x_i \theta_j + x_j \theta_i}{x_i - x_j} \partial_{\theta_i}.$$

Let also

$$\epsilon_{\Lambda}(\alpha) = \alpha b(\Lambda^*) - b(\Lambda),$$

where $b(\Lambda) = \sum_{i=1}^{\ell(\Lambda^*)} (i - 1) \Lambda_i^*$. Finally, let

$$\epsilon_{\Lambda}(\alpha) = \alpha |\Lambda^a| - |\Lambda'^a|.$$

Then, there exists a unique monic symmetric polynomial in superspace,

$$P_{\Lambda}(x, \theta) = m_{\Lambda}(x, \theta) + \sum_{\Omega < \Lambda} c_{\Lambda \Omega}(\alpha) m_{\Omega}(x, \theta),$$

satisfying

$$DP_{\Lambda}(x, \theta) = \epsilon_{\Lambda}(\alpha) P_{\Lambda}(x, \theta) \quad \text{and} \quad \Delta P_{\Lambda}(x, \theta) = \epsilon_{\Lambda}(\alpha) P_{\Lambda}(x, \theta).$$

As is the case for the Jack polynomials, the coefficients $c_{\Lambda \Omega}$ in the expansion (38) of $P_{\Lambda}$ do not depend on the number of variables. It then easily follows that the $P_{\Lambda}$’s behave well under the obvious extension $\rho_{N,N-1} : \mathcal{R}_N^{n,m} \to \mathcal{R}_{N-1}^{n,m}$ of the standard homomorphism that restricts the number of variables (see [15] p. 18 and the introduction). This allow us to take the inverse limit of any Jack superpolynomial and obtain a Jack symmetric superfunction $P_{\Lambda}$. In other words, it makes sense to work with $P_{\Lambda}$ even if the latter contains an infinite number of variables since it is equal to a finite and stable sum of monomials $m_{\Lambda}$.

As mentioned in the introduction, Jack superfunctions have been shown to be orthogonal with respect to the scalar product [31] [10]:

$$\langle P_{\Lambda} | P_{\Omega} \rangle = \|P_{\Lambda}\|^2 \delta_{\Lambda,\Omega},$$

with $\|P_{\Lambda}\|^2 \neq 0$ a certain rational function in $\alpha$ that does not depend on the number of variables $N$ (the non-vanishing of $\|P_{\Lambda}\|^2$ follows from the fact that the scalar product [31] is positive definite when $\alpha > 0$).
Directly related to this orthogonality relation, we have the Cauchy formula
\[
\prod_{i,j} \frac{1}{(1 - x_i y_j - \theta_i \phi_j)^{1/\alpha}} = \sum \frac{1}{\|P_{\Lambda}\|^2} \overrightarrow{P}_{\Lambda}(x, \theta) \overrightarrow{P}_{\Lambda}(y, \phi),
\] (41)
where \(\|P_{\Lambda}\|^2\) was defined in (40).

We conclude this section by mentioning a useful duality property of \(P_{\Lambda}\). Let \(\hat{\omega}_\alpha\) stand for the endomorphism of the space of symmetric polynomials in superspace defined on the power sums as
\[
\hat{\omega}_\alpha(p_n) = (-1)^{n-1} \alpha p_n \quad \text{and} \quad \hat{\omega}_\alpha(\tilde{p}_n) = (-1)^n \alpha \tilde{p}_n.
\] (42)
It was shown in [10, Theo. 27] that
\[
\hat{\omega}_\alpha(\overrightarrow{P}_{\Lambda}) = \|P_{\Lambda}\|^2 \overrightarrow{P}_{\Lambda}^{(1/\alpha)},
\] (43)
where \(P_{\Lambda}^{(1/\alpha)}\) stands for \(P_{\Lambda'}\) with \(\alpha\) replaced by \(1/\alpha\).

2.4. Complementary remarks on the eigenfunction characterization of the Jack superpolynomials. For completeness, we provide some clarification comments on the description of the Jack superpolynomials as eigenfunctions of a quantum \(N\)-body problem. In that regard, we clear up some discrepancies between the notations used in the current paper and those of previously-quoted articles. None of these comments is used in the rest of the article, so that this subsection can be safely skipped.

The version of Theorem 6 presented in [10, Theo. 14] differs slightly from the one presented here. The eigenvalue problem in [10] is given in terms of operators \(H\) and \(I\) that are related to \(D\) and \(\Delta\) through the relations
\[
D = \frac{1}{2} \alpha \mathcal{H} - \frac{(\alpha + N - 1)}{2} \mathcal{H}_1 \quad \text{and} \quad \Delta = \alpha \mathcal{I} + \frac{1}{2} (\mathcal{I}_0^2 - \mathcal{I}_0),
\] (44)
where
\[
\mathcal{H}_1 = \sum_{i=1}^{N} x_i \partial_{x_i} \quad \text{and} \quad \mathcal{I}_0 = \sum_{i=1}^{N} \theta_i \partial_{\theta_i}.
\] (45)
Given that \(\mathcal{H}_1\) and \(\mathcal{I}_0\) are constant on polynomials in superspace of a given fermionic and bosonic degree, the theorem still holds (after an obvious modification of the eigenvalues).

In the \(\theta \to 0\) limit, \(D\) becomes the operator \(D^S\) used in [19, Eq. 11], up to a rescaling and minor modifications that remove the dependence upon \(N\) in the eigenvalues:
\[
\lim_{\theta \to 0} D = 2D^S - 2(N - 1)\mathcal{H}_1,
\] (46)
where \(\mathcal{H}_1\) is defined in (45). In physics, \(D\) is interpreted as the Hamiltonian (energy operator). The operator \(\Delta\) is a conserved quantity that disappears in the non-supersymmetric case (i.e., when \(\theta_i \to 0\)).

The Hamiltonian and the conserved quantity just mentioned refer to the supersymmetric extension of the trigonometric Calogero-Moser-Sutherland (stCMS) model [18, 4, 6]. Let us digress for a moment to comment on this a priori curious situation that eigenfunctions of the stCMS Hamiltonian only (that is, the eigenfunctions of \(D\)) can fail to be orthogonal. It is clear from (46) that the Hamiltonian eigenvalues are insensitive to the fermionic or bosonic nature of the parts in the superpartition parametrizing the eigenfunction. There is thus a residual degeneracy. The way to lift this degeneracy is, however, clear from the point of view of integrable systems. Recall that the usual trigonometric CMS model, being
integrable, has \(N\) (the number of degrees of freedom) independent and mutually commuting conservation laws – the Hamiltonian being one of them. But since the stCMS model has \(2N\) degrees of freedom, it must have an extra set of \(N\) commuting conservation laws – disappearing when the anticommuting variables are set equal to zero. Select the first non-trivial representative of this second tower, called the partner Hamiltonian (this is essentially \(\Delta\)). The common eigenfunctions of the Hamiltonian and its partner turn out to be have non-degenerate eigenvalues; in addition, they are orthogonal [8]. These are the Jack polynomials in superspace.

3. Linear expansion of products of Jack superpolynomials

As a preliminary step toward the derivation of the evaluation formula for the Jack superpolynomials, the following two technical problems must be addressed:

1. Identify the Jack superpolynomials that can appear in the expansion of \(P_R \cdot P_\Lambda\), where \(R\) is a single row (bosonic or fermionic) superpartition.

2. Obtain \(P_\Lambda(x_1, x_2, \ldots, y_1, y_2; \theta_1, \theta_2, \ldots, \phi_1, \phi_2, \ldots)\) as a linear combination of Jack superpolynomials in \(x\) and \(\theta\) with coefficients in \(y\), \(\phi\), and \(\alpha\).

In Sections 3.1 and 3.2 below, we address the first point by carefully studying the coefficients \(g^\Lambda_{\Omega, \Gamma}\), which are defined as the rational function in \(\alpha\) satisfying

\[
P_\Omega P_\Gamma = \sum_{\Lambda} \frac{1}{\|P_\Lambda\|^2} g^\Lambda_{\Omega, \Gamma} P_\Lambda.
\]

By orthogonality, the latter equation is equivalent to

\[
g^\Lambda_{\Omega, \Gamma} = \langle P_\Lambda | \overline{P_\Omega P_\Gamma} \rangle.
\]

The second issue is considered in Section 3.3, where we introduce skew Jack polynomials in superspace by using the coefficients \(g^\Lambda_{\Omega, \Gamma}\).

3.1. Necessary conditions for the non-vanishing of \(g^\Lambda_{\Omega, \Gamma}\). The following lemma is an immediate consequence of the duality (43) induced by \(\omega_\alpha\).

**Lemma 7.** We have that

\[
g^\Lambda_{\Omega, \Gamma} \neq 0 \quad \text{if and only if} \quad g^{\Lambda'}_{\Omega', \Gamma'} \neq 0.
\]

Our first non-trivial result on the coefficients \(g^\Lambda_{\Omega, \Gamma}\) shows that they behave quite like their non-superspace counterparts. It also neatly illustrates the efficiency of the ordering (11) on superpartitions. Recall from [15, p.5-6] that given two partitions \(\lambda\) and \(\mu\), \(\lambda \cup \mu\) stands for the partition whose parts are those of \(\lambda\) and \(\mu\) arranged in weakly decreasing order, while \(\lambda + \mu\) stands for the partition whose \(i^{th}\) part is \(\lambda_i + \mu_i\). The two notions are related by the formula \((\lambda \cup \mu)' = \lambda' + \mu'\). The following proposition is a direct generalization of [19 Prop. 4.1].

**Proposition 8.** If \(g^\Lambda_{\Omega, \Gamma} \neq 0\) then

\[
\Omega^* \cup \Gamma^* \leq \Lambda^* \leq \Omega^* + \Gamma^* \quad \text{and} \quad \Omega^\oplus \cup \Gamma^\oplus \leq \Lambda^\oplus \leq \Omega^\oplus + \Gamma^\oplus.
\]

Moreover, if there exists a superpartition \(\Lambda\) such that \(\Lambda^* = \Omega^* \cup \Gamma^*\) and \(\Lambda^\oplus = \Omega^\oplus \cup \Gamma^\oplus\), then \(g^\Lambda_{\Omega, \Gamma} \neq 0\). Similarly, if there exists a superpartition \(\Lambda\) such that \(\Lambda^* = \Omega^* + \Gamma^*\) and \(\Lambda^\oplus = \Omega^\oplus + \Gamma^\oplus\), then \(g^\Lambda_{\Omega, \Gamma} \neq 0\).
Proof. We first prove that
\[ m_\Omega m_\Gamma = \sum_{\Delta} t^{\Delta}_{\Omega\Gamma} m_\Lambda \] (51)
is such that \( t^{\Delta}_{\Omega\Gamma} \) is non-zero only if \( \Lambda^* \leq \Omega^* + \Gamma^* \) and \( \Lambda^\circ \leq \Omega^\circ + \Gamma^\circ \). Furthermore, we prove that if there exists a superpartition \( \Lambda \) such that \( \Lambda^* = \Omega^* + \Gamma^* \) and \( \Lambda^\circ = \Omega^\circ + \Gamma^\circ \), then \( t^{\Delta}_{\Omega\Gamma} \) is non-zero.

Since \( \tilde{\xi}_\Delta = \lim_{\alpha \to 0} \tilde{P}_\Delta \) [10, Eq. 6.22] we have from (38) that
\[ e_{\Delta'} = \pm m_\Lambda + \sum_{\Omega < \Lambda} w_{\Omega\Lambda} m_\Lambda, \] (52)
which implies that
\[ m_\Lambda = \pm e_{\Delta'} + \sum_{\Omega < \Lambda} w'_{\Omega\Lambda} e_{\Omega\Gamma}. \] (53)

This immediately gives that
\[ m_\Omega m_\Gamma = \sum_{\Delta \leq \Omega, \Gamma} w'_{\Omega\Delta} w'_{\Gamma\Omega} e_{\Delta'} e_{\Gamma'} \] (54)
where the coefficient of \( e_{\Omega\Gamma} e_{\Gamma'} \) is equal to \( \pm 1 \). Now, \( e_{\Delta'} e_{\Gamma'} = 0 \) if \( \Delta' \) and \( \Gamma' \) have fermionic rows of the same lengths. Otherwise \( e_{\Delta'} e_{\Gamma'} = \pm e_{\Theta'} \), where \( \Theta \) is the unique superpartition such that \( \Theta^* = \Delta^* + \Gamma^* \) and \( \Theta^\circ = \Delta^\circ + \Gamma^\circ \). This implies, from (52), that if \( m_\Lambda \) appears in \( e_{\Delta'} e_{\Gamma'} \) then \( \Lambda \leq \Theta \). Hence, if \( m_\Lambda \) appears in \( m_\Omega m_\Gamma \) then
\[ \Lambda^* \leq \Theta^* \leq \Delta^* + \Gamma^* \leq \Omega^* + \Gamma^* \] and
\[ \Lambda^\circ \leq \Theta^\circ \leq \Delta^\circ + \Gamma^\circ \leq \Omega^\circ + \Gamma^\circ, \] (55)
which proves (51). The fact that if there exists a superpartition \( \Lambda \) such that \( \Lambda^* = \Omega^* + \Gamma^* \) and \( \Lambda^\circ = \Omega^\circ + \Gamma^\circ \), then \( t^{\Delta}_{\Omega\Gamma} \) is non-zero is immediate since, as already observed, the coefficient of \( e_{\Omega\Gamma} e_{\Gamma'} \) in (54) is equal to \( \pm 1 \).

From the triangularity relation (38), the previous result implies that \( g^{\Lambda}_{\Omega\Gamma} = 0 \) unless \( \Lambda^* \leq \Omega^* + \Gamma^* \) and \( \Lambda^\circ \leq \Omega^\circ + \Gamma^\circ \). From Lemma 7, we have
\[ g^{\Lambda}_{\Omega\Gamma} \neq 0 \quad \text{if and only if} \quad g^{\Lambda'}_{\Gamma\Omega} \neq 0. \] (56)
But if \( g^{\Lambda'}_{\Omega\Gamma} \neq 0 \) then \( \Lambda'' \leq \Omega'' + \Gamma'' \) and \( \Lambda''^\circ \leq \Omega''^\circ + \Gamma''^\circ \) which is equivalent to \( \Omega'' + \Gamma'' \leq \Lambda^* \) and \( \Omega'' + \Gamma'' \leq \Lambda^\circ \).

As we have seen, if there exists a superpartition \( \Lambda \) such that \( \Lambda^* = \Omega^* + \Gamma^* \) and \( \Lambda^\circ = \Omega^\circ + \Gamma^\circ \), then \( t^{\Delta}_{\Omega\Gamma} \) is non-zero. By the triangularity (38), we have \( g^{\Delta}_{\Omega\Gamma} \neq 0 \) in those cases. The final claim follows from (56). \( \square \)

3.2. Necessary conditions for the non-vanishing of coefficients in the Pieri rule: horizontal and vertical strips. Let \( n \) and \( \tilde{n} \) refer respectively to the superpartitions \((n)\) and \((\tilde{n})\), i.e., associated respectively to the following diagrams both containing \( n \) squares:
\[ n = \boxed{ } \cdots \boxed{ } \quad \text{and} \quad \tilde{n} = \boxed{ } \cdots \boxed{ } . \] (57)

We now obtain necessary conditions for the non-vanishing of the coefficients \( g^{\Lambda}_{\Omega,n} \) and \( g^{\Lambda}_{\Omega,\tilde{n}} \).

These results specify – without evaluating them explicitly – the coefficients that can appear in a Pieri-type rule for Jack polynomials in superspace.

When no fermions are involved (in which case superpartitions \( \Lambda \) and \( \Omega \) are usual partitions \( \lambda \) and \( \mu \)), it is known that the coefficient \( g^{\Lambda}_{\mu,n} \neq 0 \) if and only if \( \lambda/\mu \) is a horizontal \( n \)-strip. The concept of horizontal or vertical strip can be easily generalized to superpartitions.
Definition 9. We say that $\Lambda/\Omega$ is a horizontal $n$-strip if $\Lambda^* \nleq \Omega^*$ and $\Lambda^\circ \nleq \Omega^\circ$ are both horizontal $n$-strips. Similarly, we say that $\Lambda/\Omega$ is a horizontal $\tilde{n}$-strip if $\Lambda^* \nleq \Omega^*$ is a horizontal $n$-strip and $\Lambda^\circ \nleq \Omega^\circ$ is a horizontal $n+1$-strip. The definitions are similar for vertical strips.

Consider for example, $\Lambda = (4,1;2,1)$ and $\Omega = (2,0;3,1)$. Then, as illustrated in Figure 5, $\Lambda/\Omega$ is a horizontal 3-strip, but it is not a vertical 3-strip. Similarly, it is readily seen from Figure 6 that $(3,0;2,1)/(2;2)$ is a vertical $\tilde{2}$-strip.

The proofs of the next two propositions rely on properties of non-symmetric Jack polynomials. As the latter are not used elsewhere and the demonstrations are rather involved, they are relegated to Appendix A. Note that the equivalences in the statements follow from Lemma 7.

Proposition 10. The coefficient $g_{\lambda_{1,n}}^\Lambda \neq 0$ only if $\Lambda/\Omega$ is a horizontal $n$-strip. Equivalently, the coefficient $g_{\lambda_{1,1}}^\Lambda \neq 0$ only if $\Lambda/\Omega$ is a vertical $n$-strip.

Proposition 11. The coefficient $g_{\lambda_{1,\tilde{n}}}^\Lambda \neq 0$ only if $\Lambda/\Omega$ is a horizontal $\tilde{n}$-strip. Equivalently, the coefficient $g_{\lambda_{1,1}}^\Lambda \neq 0$ only if $\Lambda/\Omega$ is a vertical $\tilde{n}$-strip.

Remark 12. Contrary to what occurs in the Pieri rule of Jack polynomials [19, Prop 5.3], the only if in Propositions 10 and 11 cannot be replaced by a if and only if. For example, if $\Omega = (2;1)$, $\Lambda = (3;1)$ and $\tilde{n} = (1;)$ then it can be checked that $g_{\lambda_{1,\tilde{n}}}^\Lambda = 0$ even though $\Lambda/\Omega$ is a horizontal $\tilde{1}$-strip.

Recall that the diagram $\mu$ is contained in $\lambda$, denoted $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all $i$. For superpartitions we define $\Omega \subseteq \Lambda$ as follows:

$$\Omega \subseteq \Lambda \quad \text{if and only if} \quad \Omega^* \subseteq \Lambda^* \quad \text{and} \quad \Omega^\circ \subseteq \Lambda^\circ. \quad (58)$$

For instance, $(0;3,2) \subseteq (3,0;3,1)$ but $(2,1;3) \not\subseteq (3,0;3,1)$. Since the $e_{\lambda}$’s form a multiplicative basis of $R$, the previous propositions have the following corollary.

Corollary 13. We have that $g_{\Omega,\Gamma}^\Lambda$ is zero unless $\Omega \subseteq \Lambda$ and $\Gamma \subseteq \Lambda$. 

3.3. Skew Jack polynomials in superspace. The skew Jack polynomial \( P_{\Lambda/\Omega} \) is defined as the unique symmetric superfunction in \( x \) and \( \theta \) such that

\[
g^{\Lambda}_{\Omega} = \langle \overrightarrow{P}_{\Lambda/\Omega} | \overrightarrow{P}_{\Gamma} \rangle = \langle \overrightarrow{P}_{\Lambda} | \overrightarrow{P}_{\Omega} \overrightarrow{P}_{\Gamma} \rangle.
\]  

(59)

Observe that this definition is equivalent to

\[
P_{\Lambda/\Omega} = \sum_{\Gamma} \frac{g^{\Lambda}_{\Gamma}}{\|P_{\Gamma}\|^2} P_{\Gamma}.
\]  

(60)

Lemma 14.

\[
\sum_{\Omega} \frac{1}{\|P_{\Omega}\|^2} \overrightarrow{P}_{\Omega}(x; \theta) \overrightarrow{P}_{\Omega/\Lambda}(y; \phi) = \sum_{\Omega} \frac{1}{\|P_{\Omega}\|^2} \overrightarrow{P}_{\Lambda}(x; \theta) \overrightarrow{P}_{\Omega}(x; \theta) \overrightarrow{P}_{\Omega}(y; \phi).
\]  

(61)

Proof. We denote the left-hand side and right-hand side of the equation by \( F_{\Lambda}(x; \theta|y; \phi) \) and \( G_{\Lambda}(x; \theta|y, \phi) \), respectively. By the linear independence and the orthogonality of the Jack polynomials, it is sufficient to prove that the following equation holds true for all superpartitions \( \Gamma \) and \( \Delta \):

\[
\langle F_{\Lambda}(x; \theta|y; \phi) | \overrightarrow{P}_{\Gamma}(x; \theta) \overrightarrow{P}_{\Delta}(y; \phi) \rangle = \langle G_{\Lambda}(x; \theta|y; \phi) | \overrightarrow{P}_{\Gamma}(x; \theta) \overrightarrow{P}_{\Delta}(y; \phi) \rangle,
\]  

(62)

where it is understood that two independent scalar products are taken, first with respect to the indeterminates \( x \) and \( \theta \), and then with respect to \( y \) and \( \theta \). Thus, the left-hand side (LHS) is

\[
\text{LHS} = \sum_{\Omega} \frac{1}{\|P_{\Omega}\|^2} \langle \overrightarrow{P}_{\Omega}(x; \theta) | \overrightarrow{P}_{\Gamma}(x; \theta) \rangle (\langle \overrightarrow{P}_{\Omega/\Lambda}(y; \phi) | \overrightarrow{P}_{\Delta}(y; \phi) \rangle - \langle \overrightarrow{P}_{\Omega}(y; \phi) | \overrightarrow{P}_{\Lambda}(y; \phi) \rangle)
\]  

(63)

The sign in the first equality comes from the reordering of the two terms in the product \( \overrightarrow{P}_{\Omega/\Lambda} \overrightarrow{P}_{\Gamma} \). To get the second line, we used the obvious equality \( \langle \overrightarrow{P}_{\Omega} | \overrightarrow{P}_{\Gamma} \rangle = \langle \overrightarrow{P}_{\Omega} | \overrightarrow{P}_{\Gamma} \rangle \).

Finally, since the scalar product in the third line is non-zero only if \( \Gamma = \Sigma + \Delta \), this is used to simplify the phase factor. Similarly for the right-hand side, we have

\[
\text{RHS} = \sum_{\Omega} \frac{1}{\|P_{\Omega}\|^2} \langle \overrightarrow{P}_{\Lambda}(x; \theta) | \overrightarrow{P}_{\Omega}(x; \theta) \rangle (\langle \overrightarrow{P}_{\Omega/\Lambda}(y; \phi) | \overrightarrow{P}_{\Delta}(y; \phi) \rangle - \langle \overrightarrow{P}_{\Omega}(y; \phi) | \overrightarrow{P}_{\Lambda}(y; \phi) \rangle)
\]  

(64)

which proves the lemma. \( \Box \)

Proposition 15. Let \((x, y; \theta, \phi)\) denote the ordered set \((x_1, x_2, \ldots, y_1, y_2, \ldots; \theta_1, \theta_2, \ldots, \phi_1, \phi_2, \ldots)\). Then, we have

\[
P_{\Gamma}(x, y; \theta, \phi) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} P_{\Lambda}(x; \theta) P_{\Gamma/\Lambda}(y; \phi).
\]  

(65)

Moreover, the following generalization holds

\[
P_{\Gamma/\Omega}(x, y; \theta, \phi) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} P_{\Lambda/\Omega}(x; \theta) P_{\Gamma/\Omega}(y; \phi),
\]  

(66)

reducing to the previous identity when \( \Omega \) is the empty superpartition.
Proof. Let us first point out that throughout the proof, we use of the obvious identity
\((-1)^{\Xi} = (-1)^{\Xi}\), which is true for any superpartition since \(n^2 = n \mod 2\) for any integer \(n\).

Now let \((z; \tau) = (z_1, z_2, \ldots; \tau_1, \tau_2, \ldots)\). The use of the Cauchy formula allows us to write

\[
\sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P} (x, y; \theta, \phi) \hat{P}_{\Gamma} (z; \tau) = \prod_{i \neq j} \prod_{k} (1 - x_j z_k - \theta_j \tau_k)^{-1/\alpha} (1 - y_i z_k - \phi_i \tau_k)^{-1/\alpha}
\]

\[
= \sum_{\Lambda, \Gamma} \frac{1}{\|P_{\Lambda}\|^2}\|P_{\Gamma}\|^2 \hat{P}_{\Lambda} (x; \theta) \hat{P}_{\Lambda}^\times (z; \tau) \hat{P}_{\Gamma} (y; \phi) \hat{P}_{\Gamma}^\times (z; \tau)
\]

\[
= \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Lambda} (x; \theta) \sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P}_{\Gamma} (z; \tau) \hat{P}_{\Gamma/\Lambda} (y; -\phi)
\]

\[
= \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Lambda} (x; \theta) \sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P}_{\Gamma/\Lambda} (y; \phi) \hat{P}_{\Gamma} (z; \tau) (-1)^{\Xi - \Xi}
\]

(67)

where in the last step, the last two terms have been interchanged. We then use the identity

\((-1)^{\Xi} \hat{P}_{\Gamma} (y; \phi) = \hat{P}_{\Gamma} (y, -\phi)\)

(68)

and Lemma 14 to get

\[
\sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P} (x, y; \theta, \phi) \hat{P}_{\Gamma} (z; \tau) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Lambda} (x; \theta) \sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P}_{\Gamma/\Lambda} (y; -\phi)
\]

\[
= \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Lambda} (x; \theta) \sum_{\Gamma} \frac{1}{\|P_{\Gamma}\|^2} \hat{P}_{\Gamma/\Lambda} (y; \phi) \hat{P}_{\Gamma} (z; \tau) (-1)^{\Xi - \Xi}
\]

(69)

where the last line has been simplified thanks to \((-1)^{\Xi - \Xi} + (\Xi - \Xi) = (-1)^{\Xi - \Xi}\). Again, the linear independence of the Jack superpolynomials allow us to equate the coefficients of \(\hat{P}_{\Gamma} (z; \tau)\) on both sides. We finally permute \(\hat{P}_{\Lambda} (x; \theta)\) with \(\hat{P}_{\Gamma/\Lambda} (y; \phi)\) in order to cancel out the factor \((-1)^{\Xi - \Xi}\) and get

\[
\hat{P}_{\Gamma} (x, y; \theta, \phi) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Gamma/\Lambda} (y; \phi) \hat{P}_{\Lambda} (x; \theta).
\]

(70)

which is equivalent to (65).

To prove the second part, we first observe that the symmetry property of \(\hat{P}_{\Lambda}\) allows us to interchange the variables \((x; \theta)\) and \((y; \phi)\) in (70), so that

\[
\hat{P}_{\Gamma} (x, y; \theta, \phi) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Gamma/\Lambda} (x; \theta) \hat{P}_{\Lambda} (y; \phi).
\]

(71)

Writing (71) in terms of three sets of variables yields

\[
\hat{P}_{\Gamma} (x, y, z; \theta, \phi, \tau) = \sum_{\Lambda} \frac{1}{\|P_{\Lambda}\|^2} \hat{P}_{\Gamma/\Lambda} (x; \theta) \hat{P}_{\Lambda} (y, z; \phi, \tau).
\]

(72)

Using (71) to expand \(\hat{P}_{\Lambda} (y, z; \phi, \tau)\), we obtain

\[
\hat{P}_{\Gamma} (x, y, z; \theta, \phi, \tau) = \sum_{\Lambda, \Omega} \frac{1}{\|P_{\Lambda}\|^2 \|P_{\Omega}\|^2} \hat{P}_{\Gamma/\Lambda} (x; \theta) \hat{P}_{\Lambda/\Omega} (y, \phi) \hat{P}_{\Omega} (z, \tau).
\]

(73)
However, there is another way to write \((71)\) in terms of three sets of variables:

\[
\tilde{P}_\Gamma(x, y, z; \theta, \phi, \tau) = \sum_\Omega \frac{1}{\|P_\Omega\|^2} \tilde{P}_{\Gamma/\Omega}(x, y; \theta, \phi) \tilde{P}_\Omega(z; \tau).
\]

(74)

Equating the coefficients of \(\tilde{P}_\Omega(z; \tau)\) in the last two expressions of \(\tilde{P}_\Gamma(x, y, z; \theta, \phi, \tau)\) gives

\[
\tilde{P}_{\Gamma/\Omega}(x, y; \theta, \phi) = \sum_\Lambda \frac{1}{\|P_\Lambda\|^2} \tilde{P}_{\Gamma/\Lambda}(x; \theta) \tilde{P}_{\Lambda/\Omega}(y; \phi).
\]

(75)

Finally, \((66)\) is established by interchanging \((x; \theta)\) and \((y; \phi)\), and by reversing the ordering of all the Grassmann variables. \(\square\)

4. Decomposition of Jack Superpolynomials

Our main results, presented in Sections 5.1 and 5.2, rely in an essential way on certain column-wise decomposition properties of the Jack polynomials, presented in Section 4.1, that generalize known properties of Jack polynomials. The analogous row-wise decompositions, worked out in Section 4.2, are given for completeness.

4.1. Column operations. If the first column of the diagram of \(\Lambda\) does not contain a circle, we introduce the “column-removal” operation \(C\) defined such that \(C \Lambda\) is the superpartition whose diagram is obtained by removing the first column of the diagram of \(\Lambda\) (the operation is illustrated in Fig. 7).

If the first column of the diagram of \(\Lambda\) contains a circle, we define the “circle-removal” operation \(\tilde{C}\) such that the diagram of \(\tilde{C} \Lambda\) is obtained from that of \(\Lambda\) by removing the circle in the first column of the diagram of \(\Lambda\) (also illustrated in Fig. 7).

**Figure 7.** Operators \(C\) and \(\tilde{C}\)

Proposition 16. Let \(\Lambda\) be a superpartition having no parts equal to zero, i.e., \(\ell(\Lambda) = \ell(\Lambda^*) = \ell\). Then

\[
P_\Lambda(x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell) = x_1 \cdots x_\ell P_{C\Lambda}(x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell).
\]

(76)

**Proof.** To simplify the notation, we will assume throughout the proof that the polynomials are polynomials in the variables \((x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell)\). Since in that case \(m_{1\ell} = x_1 \cdots x_\ell\), we have to show that \(m_{1\ell} P_{C\Lambda} = P_\Lambda\). According to Theorem 6 this amounts to prove that \(m_{1\ell} P_{C\Lambda}\) is: (i) triangular with leading term \(m_\Lambda\) and, (ii) an eigenfunction of \(D\) and \(\Delta\) with eigenvalues \(\epsilon_\Lambda\) and \(\epsilon_\Lambda\) respectively.

From the definition of the monomial symmetric functions, we immediately get

\[
m_{1\ell} m_{C\Gamma} = m_\Gamma,
\]

(77)
Moreover, one readily shows that whenever $\Lambda = \ell(\Gamma) = \ell$. From the diagrammatic representation of superpartitions, it is also obvious that if $\ell(\Gamma) = \ell(\Gamma') = \ell$ then

$$
\Lambda \geq \Gamma \quad \iff \quad \mathcal{C}\Lambda \geq \mathcal{C}\Gamma.
$$

Hence, we obtain

$$
m_{1\ell}P_{\mathcal{C}\Lambda} = m_{1\ell}(m_{\mathcal{C}\Lambda} + \sum_{i < \mathcal{C}\Lambda} c_{\mathcal{C}\Lambda,\Omega}(\alpha) m_{\Omega}) = m_{1\ell}(m_{\mathcal{C}\Lambda} + \sum_{i < \mathcal{C}\Lambda} c_{\mathcal{C}\Lambda,\mathcal{C}\Gamma}(\alpha) m_{\Gamma})
$$

$$
= m_{\Lambda} + \sum_{i < \Lambda} c_{\mathcal{C}\Lambda,\mathcal{C}\Gamma}(\alpha) m_{\Gamma},
$$

which proves the triangularity of $m_{1\ell}P_{\mathcal{C}\Lambda}$.

We now compute the action of $D$ on $m_{1\ell}P_{\mathcal{C}\Lambda}$:

$$
D(m_{1\ell}P_{\mathcal{C}\Lambda}) = D(m_{1\ell})P_{\mathcal{C}\Lambda} + m_{1\ell}D(P_{\mathcal{C}\Lambda}) + \alpha \sum_{i} x_{i}\partial_{x_{i}}(m_{1\ell})x_{i}\partial_{x_{i}}(P_{\mathcal{C}\Lambda})
$$

$$
= (\varepsilon_{1\ell}(\alpha) + \varepsilon_{\mathcal{C}\Lambda}(\alpha) + \alpha|\mathcal{C}\Lambda|)m_{1\ell}P_{\mathcal{C}\Lambda}.
$$

Using

$$
b(\mathcal{C}\Lambda) = b(\Lambda) - \frac{\ell(\ell - 1)}{2}, \quad b((\mathcal{C}\Lambda)^{\alpha}) = b(\Lambda^{\alpha}) - |\mathcal{C}\Lambda|\quad \text{and} \quad \varepsilon_{1\ell}(\alpha) = -\frac{\ell(\ell - 1)}{2},
$$

which implies that $\varepsilon_{1\ell}(\alpha) + \varepsilon_{\mathcal{C}\Lambda}(\alpha) + \alpha|\mathcal{C}\Lambda| = \varepsilon_{\Lambda}(\alpha)$, we get

$$
D(m_{1\ell}P_{\mathcal{C}\Lambda}) = \varepsilon_{\Lambda}(\alpha)m_{1\ell}P_{\mathcal{C}\Lambda}.
$$

Similarly, the action of $\Delta$ on $m_{1\ell}P_{\mathcal{C}\Lambda}$ gives

$$
\Delta(m_{1\ell}P_{\mathcal{C}\Lambda}) = \Delta(m_{1\ell})P_{\mathcal{C}\Lambda} + m_{1\ell}\Delta(P_{\mathcal{C}\Lambda}) + \alpha \sum_{i} x_{i}\partial_{x_{i}}(m_{1\ell})\theta_{i}\partial_{\theta_{i}}(P_{\mathcal{C}\Lambda})
$$

$$
= (0 + \varepsilon_{\mathcal{C}\Lambda}(\alpha) + \alpha m)m_{1\ell}P_{\mathcal{C}\Lambda}.
$$

Using $|\mathcal{C}\Lambda^{a}| = |\Lambda^{a}| - m$ and $|(\mathcal{C}\Lambda)^{a}| = |\Lambda^{a}|$, we are led to

$$
\Delta(m_{1\ell}P_{\mathcal{C}\Lambda}) = \varepsilon_{\Lambda}(\alpha)m_{1\ell}P_{\mathcal{C}\Lambda}.
$$

We have established that $m_{1\ell}P_{\mathcal{C}\Lambda}$ has the right triangularity property and satisfies the required eigenvalue problems. We can thus conclude that $m_{1\ell}P_{\mathcal{C}\Lambda} = P_{\Lambda}$. \hfill \Box

**Proposition 17.** Let $\Lambda$ be a superpartition such that $\Lambda_{m} = 0$, i.e., $\ell(\Lambda) = \ell(\Lambda^{*}) + 1$. Then

$$
(-1)^{m-1}\left[\partial_{\theta_{1}}P_{\Lambda}(x_{1}, \ldots, x_{\ell}; \theta_{1}, \ldots, \theta_{\ell})\right]_{x_{\ell}=0} = P_{\mathcal{C}\Lambda}(x_{1}, \ldots, x_{\ell-1}; \theta_{1}, \ldots, \theta_{\ell-1}).
$$

**Proof.** Let $(x; \theta) = (x_{1}, \ldots, x_{\ell}; \theta_{1}, \ldots, \theta_{\ell})$ and $(x_{-}; \theta_{-}) = (x_{1}, \ldots, x_{\ell-1}; \theta_{1}, \ldots, \theta_{\ell-1})$. As in Proposition 16, we will prove that $\partial_{\theta_{1}}P_{\Lambda}(x; \theta)$ evaluated at $x_{\ell} = 0$ has the right triangularity and satisfies the required eigenvalue problems (see Theorem 6).

We have, for any superpartition $\Omega$ with $\ell(\Omega^{\theta}) = \ell$,

$$
(-1)^{m-1}\left[m_{\Omega}(x; \theta)\right]_{x_{\ell}=0} = \begin{cases} 
\theta_{\ell} m_{\mathcal{C}\Omega}(x_{-}; \theta_{-}) & \text{if } \Omega_{m} = 0 \\
0 & \text{if } \Omega_{m} > 0.
\end{cases}
$$

Moreover, one readily shows that whenever $\Lambda_{m} = \Omega_{m} = 0$, we have

$$
\Lambda \geq \Omega \quad \iff \quad \mathcal{C}\Lambda \geq \mathcal{C}\Omega.
$$


Therefore, from the expansion (86) of $P_\Lambda(x, \theta)$ in terms of monomials, we immediately get

$$( -1)^{m-1} \left[ \partial_{\theta_1} P_\Lambda(x; \theta) \right]_{x_\ell=0} = m_{\bar{\mathcal{C}}\Lambda}(x_-; \theta_-) + \sum_{\Omega<\Lambda \atop \Omega_m=0} c_{\Lambda, \Omega}(\alpha) m_{\bar{\mathcal{C}}\Omega}(x_-; \theta_-)$$

$$( -1)^{m-1} \left[ \partial_{\theta_1} P_\Lambda(x; \theta) \right]_{x_\ell=0} = m_{\bar{\mathcal{C}}\Lambda}(x_-; \theta_-) + \sum_{\tilde{\mathcal{C}}\Omega<\mathcal{C}\Lambda \atop \Omega_m=0} c_{\Lambda, \Omega}(\alpha) m_{\tilde{\mathcal{C}}\Omega}(x_-; \theta_-), \quad (88)$$

which gives the desired triangularity. Observe that we have used the fact that if $\Omega \leq \Lambda$ then $\ell(\Omega^\circ) \geq \ell(\Lambda^\circ) = \ell$.

Let $D_\ell$ and $D_{\ell-1}$ stand for the operator $D$ in the variables $(x, \theta)$ and $(x_-, \theta_-)$ respectively, and similarly for $\Delta_\ell$ and $\Delta_{\ell-1}$. For any superpolynomial $f(x, \theta)$, straightforward calculations yield

$$\left[ \partial_{\theta_1} D_\ell f(x, \theta) \right]_{x_\ell=0} = \left[ \partial_{\theta_1} D_{\ell-1} f(x, \theta) \right]_{x_\ell=0} = D_{\ell-1} \left[ \partial_{\theta_1} f(x, \theta) \right]_{x_\ell=0}.$$  

Therefore

$$D_{\ell-1} \left[ \partial_{\theta_1} P_\Lambda(x, \theta) \right]_{x_\ell=0} = \left[ \partial_{\theta_1} D_\ell P_\Lambda(x, \theta) \right]_{x_\ell=0} = \varepsilon_{\bar{\mathcal{C}}\Lambda}(\alpha) \left[ \partial_{\theta_1} P_\Lambda(x, \theta) \right]_{x_\ell=0}, \quad (90)$$

since $\varepsilon_\Lambda(\alpha) = \varepsilon_{\bar{\mathcal{C}}\Lambda}(\alpha)$.

The second eigenvalue problem is somewhat more involved. We have, for any superpartition $\Omega$ with $\ell(\Omega^\circ) = \ell$,

$$\left[ \partial_{\theta_1} (\Delta_{\ell-1} - \Delta_\ell) m_\Omega(x; \theta) \right]_{x_\ell=0} = \partial_{\theta_1} \left[ (\Delta_{\ell-1} - \Delta_\ell) m_\Omega(x; \theta) \right]_{x_\ell=0} = \partial_{\theta_1} \left[ \sum_{j=1}^{\ell-1} \theta_1 (\partial_{\theta_1} - \partial_{\theta_j}) m_\Omega(x; \theta) \right]_{x_\ell=0}$$

$$= \left\{ \begin{array}{ll} (\ell - 1)(-1)^{m-1} m_{\bar{\mathcal{C}}\Omega}(x_-; \theta_-) & \text{if } \Omega_m = 0 \\ 0 & \text{if } \Omega_m > 0 \end{array} \right.$$  

$$= (\ell - 1) \left[ \partial_{\theta_1} m_\Omega(x; \theta) \right]_{x_\ell=0}, \quad (91)$$

where we have used (86) in the next to last step. This leads to

$$\Delta_{\ell-1} \left[ \partial_{\theta_1} P_\Lambda(x, \theta) \right]_{x_\ell=0} = \left[ \partial_{\theta_1} (\Delta_\ell + (\Delta_{\ell-1} - \Delta_\ell)) P_\Lambda(x, \theta) \right]_{x_\ell=0}$$

$$= (\varepsilon_\Lambda(\alpha) + \ell - 1) \left[ \partial_{\theta_1} P_\Lambda(x, \theta) \right]_{x_\ell=0} = \varepsilon_{\bar{\mathcal{C}}\Lambda}(\alpha) \left[ \partial_{\theta_1} P_\Lambda(x, \theta) \right]_{x_\ell=0}, \quad (92)$$

since $|\bar{\mathcal{C}}\Lambda|^\circ = |\Lambda^\circ|$, $|\bar{\mathcal{C}}\Lambda'|^\circ = |\Lambda^\circ| - \Lambda'_1$, and $\Lambda'_1 = \ell - 1$.

Using Eqs (88), (90) and (92), we conclude from Theorem 5 that

$$(-1)^{m-1} \left[ \partial_{\theta_1} P_\Lambda(x; \theta) \right]_{x_\ell=0} = P_{\bar{\mathcal{C}}\Lambda}(x_-; \theta_-). \quad (93)$$

\square
4.2. **Row operations.** Similarly to the column case, we can introduce two row operations whose actions on diagrams is illustrated in Fig. 8. The following two propositions show how the polynomial \( P_\Lambda \) can be row-wise deconstructed (at the expense of losing the first variable).

**Proposition 18.** Let \((x_1; \theta_1) = (x_2, x_3, \ldots; \theta_2, \theta_3, \ldots)\). Let also \( \Lambda \) be a superpartition whose fermionic degree is \( m \). If the first row of the diagram of \( \Lambda \) is bosonic (that is, \( \Lambda_1 = \Lambda^\circ_1 = k \)), then

\[
\text{coeff}_{x_1^k} P_\Lambda(x; \theta) \propto P_{\mathcal{R} \Lambda}(x_1; \theta_1). \tag{94}
\]

**Proof.** We have from Proposition 15 that

\[
P_\Lambda(x, \theta) = \sum_{\Omega} \frac{1}{\|P_\Omega\|^2} P_{\Omega}(x_1, \theta_1) P_{\Lambda/\Omega}(x_1, \theta_1). \tag{95}
\]

Using

\[
P_{\Omega}(x_1, \theta_1) = \begin{cases} x_1^r & \text{if } \Omega = r \\ \theta_1 x_1^r & \text{if } \Omega = \tilde{r} \\ 0 & \text{otherwise,} \end{cases} \tag{96}
\]

and the fact that \( P_{\Lambda/\Omega} = 0 \) unless \( \Omega \subseteq \Lambda \) from Corollary 13 (that is, unless \( \Omega^\circ_1 \leq k \) and \( \Omega_1^\circ \leq k \)), we obtain

\[
P_\Lambda(x, \theta) = \sum_{r=1}^{k} \frac{1}{\|P_r\|^2} x_1^r P_{\Lambda/r}(x_1, \theta) + \frac{1}{\|P_{\tilde{r}}\|^2} \theta_1 x_1^r P_{\Lambda/\tilde{r}}(x_1, \theta), \tag{97}
\]

which immediately gives

\[
\text{coeff}_{x_1^k} P_\Lambda(x; \theta) \propto P_{\mathcal{R} \Lambda}(x_1; \theta_1). \tag{98}
\]

Moreover, we have from \([59]\) that

\[
P_{\Lambda/k}(x_1, \theta) = \sum_{\Omega} \frac{g_{\Omega, k}^\Lambda}{\|P_\Omega\|^2} P_{\Omega}(x_1, \theta), \tag{99}
\]

where we recall from Proposition 10 that \( g_{\Omega, k}^\Lambda \neq 0 \) only if \( \Lambda/\Omega \) is a horizontal \( k \)-strip. Now, the only superpartition \( \Omega \) such that \( \Lambda/\Omega \) is a horizontal \( k \)-strip is \( \mathcal{R} \Lambda \), and therefore

\[
\text{coeff}_{x_1^k} P_\Lambda(x; \theta) \propto P_{\mathcal{R} \Lambda}(x_1; \theta_1). \tag{100}
\]

Finally, since

\[
\text{coeff}_{x_1^k} m_\Lambda(x; \theta) = m_{\mathcal{R} \Lambda}(x_1, \theta_1), \tag{101}
\]
we have that \( \text{coeff} \ P_\Lambda(x; \theta) \) is monic and the proposition follows. \( \square \)

**Proposition 19.** Let \((x_1; \theta_1) = (x_2, x_3, \ldots; \theta_2, \theta_3, \ldots)\). Let also \( \Lambda \) be a superpartition whose fermionic degree is \( m \). If the first row of the diagram of \( \Lambda \) is fermionic (that is, \( \Lambda_1^\dagger = \Lambda_1^\circ - 1 = k \)), then

\[
\text{coeff} \ \partial_{\theta_1} \ P_\Lambda(x; \theta) = P_{\mathcal{R}_R \Lambda}(x_-; \theta_-).
\]

**Proof.** The proof is essentially the same as that of Proposition 18. Using Eqs. (95) and (96), and the fact that \( P_\Lambda/\Omega = 0 \) unless \( \Omega \subseteq \Lambda \) (that is, unless \( \Omega_1^\dagger \leq k \) and \( \Omega_1^\circ \leq k + 1 \)), we obtain

\[
P_\Lambda(x, \theta) = \sum_{r=0}^{k} \frac{1}{\|P_r\|^2} x_r^1 P_{\Lambda/r}(x_-; \theta_-) + \sum_{r=0}^{k} \frac{1}{\|P_r\|^2} \theta_1 x_r^1 P_{\Lambda/r}(x_-; \theta_-),
\]

which immediately gives

\[
\text{coeff} \ \partial_{\theta_1} \ P_\Lambda(x; \theta) \propto P_{\mathcal{R}_R \Lambda}(x_-; \theta_-).
\]

Now

\[
P_{\Lambda/k}(x_-; \theta_-) = \sum_{\Omega} \frac{g^\Lambda_{\Omega, k}}{\|P_\Omega\|^2} P_\Omega(x_-; \theta_-)
\]

is such that \( g^\Lambda_{\Omega, k} = 0 \) unless \( \Lambda/\Omega \) is a horizontal \( k \)-strip, which readily yields

\[
\text{coeff} \ \partial_{\theta_1} \ P_\Lambda(x; \theta) \propto P_{\mathcal{R}_R \Lambda}(x_-; \theta_-).
\]

Finally, since

\[
\text{coeff} \ \partial_{\theta_1} \ m_\Lambda(x; \theta) = m_{\mathcal{R}_R \Lambda}(x_-; \theta_-),
\]

we have that \( \text{coeff} \ \partial_{\theta_1} \ P_\Lambda(x; \theta) \) is monic and the proposition follows. \( \square \)

### 5. Evaluation formulas

We now come to our first main results: the derivation of evaluation formulas for the Jack superpolynomials. In what follows, it will prove convenient to work with a different normalization of the Jack polynomials in superspace. Let \( \Lambda_{\min} \) be the lowest superpartition of degree \((n|m)\) in the dominance ordering, namely:

\[
\Lambda_{\min} := (\delta_m; 1^{\ell_{n,m}}),
\]

where

\[
\ell_{n,m} := n - |\delta_m| \quad \text{and} \quad \delta_m := (m - 1, m - 2, \ldots, 0).
\]

Let also \( c^\Lambda_{\min}(\alpha) \) stand for the coefficient of \( \ell_{n,m}! m_{\Lambda_{\min}} \) in the monomial expansion of \( P_\Lambda \).

**Definition 20.** We define the non-monic Jack symmetric function in superspace as

\[
J_\Lambda := v_\Lambda(\alpha) P_\Lambda = \frac{1}{c^\Lambda_{\min}(\alpha)} P_\Lambda.
\]
This normalization, which is such that the coefficient of $m_{\Lambda_{\min}}$ in $J_{\Lambda}$ is $\ell_{n,m}!$, reduces to the integral form of the Jack polynomials [19] when $m = 0$. We define the expansion coefficients of $J_{\Lambda}$ as

$$J_{\Lambda} = \sum_{\Omega \subseteq \Lambda} v_{\Lambda \Omega}(\alpha) m_{\Omega},$$

(111)

with the identification:

$$v_{\Lambda}(\alpha) \equiv v_{\Lambda \Lambda}(\alpha).$$

(112)

**Remark 21.** It has been conjectured in [10, Conj. 33] that the coefficients $v_{\Lambda \Omega}(\alpha)$ belong to $\mathbb{Z}[\alpha]$. This conjecture is still open.

5.1. **First evaluation formula.** Let $F(x; \theta)$ be a polynomial in superspace of fermionic degree $m$. The evaluation of $F(x; \theta)$ is defined as

$$E_{N,m}[F(x; \theta)] := \left[ \frac{\partial_{\theta_m} \cdots \partial_{\theta_1} F(x; \theta)}{V_m(x)} \right]_{x_1=\ldots=x_N=1},$$

(113)

where $m \leq N$ and where

$$V_m(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j)$$

(114)

is the Vandermonde determinant in the variables $x_1, \ldots, x_m$. Note that it is understood that $V_m(x) = 1$ when $m = 0$ or 1, and thus this evaluation reduces to the standard evaluation at $x_1 = \ldots = x_N = 1$ when $m = 0$.

It will prove useful to reexpress the division by the Vandermonde determinant $V_m(x)$ in the evaluation (113) as a differentiation with respect to the operator

$$\partial^\delta_x := \left[ \prod_{i=1}^{m-1} \frac{1}{(m-i)!} \right] \partial_{x_1}^{m-1} \partial_{x_2}^{m-2} \cdots \partial_{x_{m-1}}.$$  

(115)

**Lemma 22.** Let $F(x; \theta)$ be a polynomial in superspace of fermionic degree $m$. Then

$$E_{N,m}[F(x; \theta)] = \left[ \partial^\delta_x \partial_{\theta_m} \cdots \partial_{\theta_1} F(x; \theta) \right]_{x_1=\ldots=x_N=1}$$

(116)

**Proof.** If $f(x_1, \ldots, x_N)$ is a polynomial antisymmetric in the variables $x_1, \ldots, x_m$, then

$$f(x_1, \ldots, x_N) = g(x_1, \ldots, x_N)V_m(x)$$

(117)

for some polynomial $g(x_1, \ldots, x_N)$. This implies that

$$\left[ \frac{f(x_1, \ldots, x_N)}{V(x)} \right]_{x_1=\ldots=x_N=1} = \left[ g(x_1, \ldots, x_N) \right]_{x_1=\ldots=x_N=1},$$

(118)

from which we have

$$\left[ \frac{f(x_1, \ldots, x_N)}{V_m(x)} \right]_{x_1=\ldots=x_N=1} = \left[ g(x_1, \ldots, x_N) \partial^\delta_x \prod_{1 \leq i < j \leq m} (x_i - x_j) \right]_{x_1=\ldots=x_N=1}$$

(119)

$$= \left[ \partial^\delta_x g(x_1, \ldots, x_N) \prod_{1 \leq i < j \leq m} (x_i - x_j) \right]_{x_1=\ldots=x_N=1}$$

$$= \left[ \partial^\delta_x f(x_1, \ldots, x_N) \right]_{x_1=\ldots=x_N=1}. $$
If \( F(x, \theta) \) is a polynomial in superspace of fermionic degree \( m \), then \( \partial_{\theta_m} \cdots \partial_{\theta_1} F(x, \theta) \) is a polynomial in \( x_1, \ldots, x_N \) antisymmetric in \( x_1, \ldots, x_m \). This proves the lemma. □

**Lemma 23.** Let \( \Lambda = (\Lambda^a; \Lambda^s) \) be a superpartition of length \( \ell \leq N \) and of fermionic degree \( m \). Let also \( \gamma = \Lambda^a - \delta_m \) where \( \delta_m = (m-1, m-2, \ldots, 0) \). Then

\[
E_{N,m} [m_{\Lambda}] = \frac{(N-\ell+1)_{\ell-m}}{n_{\Lambda^s}!} \prod_{(i,j) \in \gamma} \frac{m+j-i}{l_{\gamma}(i,j) + a_{\gamma}(i,j) + 1},
\]

where \( (a)_k := a(a+1) \cdots (a+k-1) \), with \( (a)_0 = 1 \).

**Proof.** It is easy to deduce that

\[
\frac{\partial_{\theta_m} \cdots \partial_{\theta_1} m_{\Lambda}(x; \theta)}{V_m(x)} = s_{\gamma}(x_1, \ldots, x_m) m_{\Lambda^s}(x_{m+1}, \ldots, x_N)
\]

where \( s_{\gamma}(x_1, \ldots, x_m) \) is a Schur function. The result is then immediate from the evaluations of Schur [15, Ex. 4, p. 45] (or [19, Theo. 5.4 and Prop. 1.2]) and monomial functions [19, Eq. 33]. □

**Corollary 24.** Let \( \Lambda_{\text{max}} = \Lambda_{\text{min}}' \) be the largest superpartitions of degree \( (n|m) \) in the dominance ordering (i.e., \( \Lambda_{\text{max}}^a = \delta_m + (\ell_{n,m}) \) and \( \Lambda_{\text{max}}^s = 0 \)). Then

\[
E_{N,m} [m_{\Lambda_{\text{min}}}^a] = \frac{(N-m-\ell_{n,m}+1)_{\ell_{n,m}}}{\ell_{n,m}!}
\]

and

\[
E_{N,m} [m_{\Lambda_{\text{max}}}^s] = \frac{(m)_{\ell_{n,m}}}{\ell_{n,m}!}
\]

The following technical result will be needed in the proof of the main theorem of this section.

**Lemma 25.** For \( r, s \) nonnegative integers such that \( r \leq s \), we have that

\[
\left[ \partial_{x_1}^r \partial_{\theta_1} J_s(x; \theta) \right]_{x_1=\cdots=x_N=1} = \frac{s!}{(s-r)!} \alpha^s(1/\alpha + 1)_r (N/\alpha + r + 1)_{s-r}
\]

**Proof.** From [10, Eq. 3.11] we have

\[
\sum_{s \geq 0} t^s [g_s(x) + \tau \tilde{g}_s(x; \theta)] = \prod_{i \geq 1} \frac{1}{(1-tx_1 - \tau \theta_1)_1^{1/\alpha}}
\]

where the polynomials \( g_s(x) \) and \( \tilde{g}_s(x; \theta) \) are up to a constant equal to \( J_s(x) \) and \( J_s(x; \theta) \) respectively:

\[
J_s(x) = s! \alpha^s g_s(x) \quad \text{and} \quad J_s(x; \theta) = s! \alpha^{s+1} \tilde{g}_s(x; \theta)
\]

Taking the coefficient of \( \tau \) on each side of \( (125) \) and using

\[
\frac{\partial}{\partial \theta_1} (1-tx_1 - \tau \theta_1)^{-1/\alpha} = \frac{-\tau}{\alpha} \left( 1-tx_1 - \tau \theta_1 \right)^{-1/\alpha - 1},
\]

we obtain

\[
\sum_{s \geq 0} \frac{t^s}{s! \alpha^{s+1}} \partial_{\theta_1} J_s(x; \theta) = \frac{1}{\alpha} (1-tx_1)^{-1/\alpha - 1}(1-tx_2)^{-1/\alpha} \cdots (1-tx_N)^{-1/\alpha}.
\]
This readily implies
\[
\left[ \sum_{s \geq 0} \frac{t^s}{s! \alpha^s} \partial_{x_1}^r \partial_{\theta_1} \partial_{x_2} \partial_{\theta_2} \cdots \partial_{x_N} \partial_{\theta_N} J_{s}(x; \theta) \right]_{x_1=\cdots=x_N=1} = \frac{t^r}{\alpha}(1/\alpha + 1)r(1-t)^{-N/\alpha - r - 1}
\]
(129)
\[
= \frac{1}{\alpha}(1/\alpha + 1) \sum_{m \geq 0} \frac{(N/\alpha + r + 1)m}{m!} t^{m+r}
\]
(130)
The lemma then follows by taking the coefficient of \(t^s\) on both sides of the equation. \(\square\)

In contradistinction to the evaluation of Jack polynomials without fermions, the presence of fermions requires the introduction of skew diagrams. If \( \Lambda \) is a superpartition of fermionic degree \( m \), let \( S \) be such that
\[
S \Lambda = \Lambda \otimes / (m, m-1, \ldots, 1).
\]
(131)

Fig. 9 illustrates the action of \( S \) on the diagram of \( \Lambda \). A rationale for the addition of the \( m \) circles and the removal of the staircase \( (m, m-1, \ldots, 1) \) is that by dividing the polynomial by the Vandermonde determinant, we decrease the degree in \( x \) by \( m(m-1)/2 \).

**Figure 9.** Operator \( S \)

![Figure 9](image)

**Theorem 26.** Let \( \Lambda \) be a superpartition of fermionic degree \( m \), and define
\[
b^{(\alpha,N)}_{\Lambda}(i,j) := \prod_{(i,j) \in \Lambda} (N - (i-1) + \alpha(j-1)).
\]
(132)

Then, for all \( N \geq \ell = \ell(\Lambda) \),
\[
E_{N,m}[J_{\Lambda}] = b^{(\alpha,N)}_{\Lambda}.
\]
(133)

**Proof.** We proceed by induction on \( m \) and on the dominance ordering for fixed \( m \). The base case \( m = 0 \) being known [19], we can assume that \( m > 0 \). We have from Lemma 23 (122) and the normalization (110) of the Jack polynomials in superspace \( J_{\Lambda} \) that \( E_{N,m}[J_{\Lambda}] \) is a monic polynomial in \( N \) of degree \( |\Lambda| - m(m-1)/2 \).

From Proposition 15, we have that
\[
J_{\Lambda}(x; \theta) = \sum_{\Omega} k_{\Omega}(\alpha) J_{\Lambda/\Omega}(x_1; \theta_1) J_{\Omega}(x_-; \theta_-),
\]
(134)
where \( k_{\Omega}(\alpha) \) are coefficients that do not depend on \( N \), and where \( (x_-; \theta_-) \) stands for the variables \( (x_2, \cdots, x_N; \theta_2, \cdots, \theta_N) \). Since \( m > 0 \), we have from (113) that
\[
E_{N,m}[J_{\Lambda}] = \sum_{\Omega} (-1)^{m-1} k_{\Omega}(\alpha) \left[ \frac{\partial_{x_1}^{m-1} \partial_{\theta_1} J_{\Lambda/\Omega}(x_1; \theta_1)}{(m-1)!} \right]_{x_1=1} E_{N-1,m-1}[J_{\Omega}],
\]
(135)
It is easy to see that the monomial $m_{\Gamma}(x_1, \theta_1) = 0$ if $\ell(\Gamma) > 1$. From the triangularity \cite{[38]}, we thus also have that $J_{\Gamma}(x_1, \theta_1) = 0$ if $\ell(\Gamma) > 1$. Therefore $J_{\Lambda/\Omega}$ can only be equal (up to a constant) to $J_r$ or $J_f$, where $r = |\Lambda| - |\Omega|$. Hence, for $\partial_{\theta_i} J_{\Lambda/\Omega}(x_1; \theta_1)$ to be non-zero, $J_{\Lambda/\Omega}$ must be equal (up to a constant) to $J_r$. Proposition \ref{prop:1} and \cite{[59]} then imply that $\partial_{\theta_i} J_{\Lambda/\Omega}(x_1; \theta_1) \neq 0$ only if $\Lambda/\Omega$ is a horizontal $\tilde{r}$-strip. Therefore $S\Omega$ contains all the cells of $S\Lambda$ that do not lie at the bottom of a column along with cells $(1, m), (2, m-1),\ldots,(m,1)$ (since $\Omega$ has one less fermionic row than $\Lambda$). By induction on the number of fermions $m$ we thus have that $E_{N-1,m-1}[J_{\Omega}]$ is a polynomial in $N$ divisible by

$$
\prod_{i=1}^{m} (N - 1 - (i - 1) + \alpha(m - i)) \prod_{(i,j) \in S\Lambda \atop i \notin \Lambda^{\ast}} (N - 1 - (i - 1) + \alpha(j - 1))
= \prod_{(i,j) \in S\Lambda \atop (i,j) \notin \{(1,m+1),\ldots,(1,\Lambda^{\ast}_1)\}} (N - (i - 1) + \alpha(j - 1)) \tag{136}
$$

and therefore $E_{N,m}[J_{\Lambda}]$ is also divisible by \cite{[136]}. We thus have left to show that $E_{N,m}[J_{\Lambda}]$ is divisible by

$$
\prod_{(i,j) \in \{(1,m+1),\ldots,(1,\Lambda^{\ast}_1)\}} (N - (i - 1) + \alpha(j - 1)) = (N + m\alpha) \cdots (N + (\Lambda^{\ast}_1 - 1)\alpha) \tag{137}
$$

Suppose that the first row of $\Lambda^{\ast}$ is fermionic ($\Lambda^{\ast}_1 = \Lambda^{\ast}_1 - 1 = s$) and let $\Omega$ be such that $\Omega^{\ast} = (\Lambda^{\ast}_2, \Lambda^{\ast}_3,\ldots)$ and $\Omega^{\ast} = (\Lambda^{\ast}_2, \Lambda^{\ast}_3,\ldots)$. Then

$$
J_{\kappa} J_{\Omega} = g_{\Lambda}^{\Gamma \Delta} \frac{v_{\Omega}v_{\kappa}}{|P_{\Delta}|^2 v_{\Lambda}} J_{\Lambda} + \sum_{\Gamma > \Lambda} g_{\Gamma \Delta}^{\Gamma} \frac{v_{\Omega}v_{\kappa}}{|P_{\Delta}|^2 v_{\Gamma}} J_{\Gamma} \tag{138}
$$

where $g_{\Omega \kappa}^{\Lambda} \neq 0$ by Proposition \cite{[8]} since $\Omega^{\ast} \cup s = \Lambda^{\ast}$ and $\Omega^{\ast} \cup (s + 1) = \Lambda^{\ast}$.

We first prove that $E_{N,m}[J_{\kappa}J_{\Omega}]$ is divisible by \cite{[137]}. The proof is not entirely straightforward and will rely on Lemma \cite{[25]}

Using \cite{[116]}, we can write

$$
E_{N,m}[J_{\kappa}J_{\Omega}] = \left[ \partial_{\theta_1}^{\alpha} \partial_{\theta_2} \cdots \partial_{\theta_m} \left( J_{\kappa}J_{\Omega} \right) \right]_{x_1=\cdots=x_N=1}
= \left[ \partial_{\theta_1}^{\alpha} \sum_{i=1}^{N} (-1)^{m-i} (\partial_{\theta_i} J_{\kappa}) \partial_{\theta_m} \cdots \widehat{\partial_{\theta_i}} \cdots \partial_{\theta_1} J_{\Omega} \right]_{x_1=\cdots=x_N=1} \tag{139}
$$

where $\widehat{\partial_{\theta_i}}$ means that $\partial_{\theta_i}$ is omitted. Consider the term $i = 1$. The expression $\partial_{\theta_m} \cdots \partial_{\theta_2} J_{\Omega}$ is antisymmetric in $x_2,\ldots,x_m$ and thus divisible by $\prod_{2 \leq i < j \leq m} (x_i - x_j)$. This implies that

$$
\partial_{\theta_m} \cdots \partial_{\theta_2} J_{\Omega} = H(x_1,\ldots,x_N) \prod_{2 \leq i < j \leq m} (x_i - x_j) \tag{140}
$$
for a certain polynomial \( H(x_1, \ldots, x_N) \), from which we get that

\[
\left[ \partial^R_x (\partial_1 J_s \cdots \partial_m J_\Omega) \right]_{x_1=\cdots=x_N=1} = \left[ \partial^R_x \left( \partial_1 (J_s H(x_1, \ldots, x_N) \prod_{2 \leq i < j \leq m} (x_i - x_j) \right) \right]_{x_1=\cdots=x_N=1} = \left[ \frac{1}{(m-1)!} \partial_{x_1}^{m-1} \left( \partial_1 (J_s H(x_1, \ldots, x_N) \prod_{2 \leq i < j \leq m} (x_i - x_j) \right) \right]_{x_1=\cdots=x_N=1}
\]

where

\[
\partial^R_x := \left[ \prod_{i=1}^{m-2} \frac{1}{(m-i-1)!} \right] \partial_{x_2}^{m-2} \partial_{x_3}^{m-3} \cdots \partial_{x_m-1}.
\]

By Lemma 25, we have that \( \partial^R_x \partial_1 J_s \) is divisible by (137) for any \( r \leq m - 1 \). Hence from (141)

\[
\left[ \partial^R_x (\partial_1 J_s \cdots \partial_m J_\Omega) \right]_{x_1=\cdots=x_N=1}
\]

is also divisible by (137). By symmetry of \( J_s J_\Omega \) under the exchange \( x_1 \leftrightarrow x_i, \theta_1 \leftrightarrow \theta_i \), it follows that every term in the right-hand side of (139) is divisible by (137), and consequently so is \( E_{N,m}[J_s J_\Omega] \).

Let \( \Lambda = \Lambda^{\max} = \Lambda^{\min} \) be the largest superpartition of \((n|m)\) in dominance order. We now show that \( E_{N,m}[J_s J_\Omega] \) is divisible by (137), the starting point in our induction process for fixed \( m > 0 \). Observe that the first row of the diagram of \( \Lambda^{\max} \) is fermionic since the first column of the diagram of \( \Lambda^{\min} \) is fermionic. If \( m = 1 \), then \( \Lambda^{\max} = \bar{n} \) and

\[
E_{N,1}[J_{\bar{n}}] = [\partial_1 J_{\bar{n}}]_{x_1=\cdots=x_N=1} = (N + \alpha) \cdots (N + n\alpha)
\]

by Lemma 25. Therefore \( E_{N,1}[J_s J_\Omega] \) is divisible by (137) in that case. If \( m > 1 \), using (138) with \( \Lambda = \Lambda^{\max} \) gives that \( J_s J_\Omega \) is equal to \( J_s J_\Omega \) up to a non-zero constant. Since we have established that \( E_{N,m}[J_s J_\Omega] \) is divisible by (137), our claim follows.

Now we consider again the general situation in (138). The polynomials \( J_\Gamma \) that appear in the sum in the right-hand-side are such that \( \Gamma \succ \Lambda \). The first row of \( \Lambda^{\circ} \) is then not smaller than the first row of \( \Lambda^{\circ} \). By induction on the dominance ordering we thus have that \( E_{N,m}[J_\Gamma] \) is divisible by (137). Since we have established that \( E_{N,m}[J_s J_\Omega] \) is divisible by (137), so is \( E_{N,m}[J_\Lambda] \) given that its coefficient in (138) is non-zero.

Finally, suppose that the first row of \( \Lambda^{\circ} \) is bosonic \( (\Lambda^*_1 = \Lambda^*_1 = s) \) and let \( \Omega \) be such that \( \Omega^* = (\Lambda^*_2, \Lambda^*_3, \ldots) \) and \( \Omega^\circ = (\Lambda^\circ_2, \Lambda^\circ_3, \ldots) \). Then

\[
J_s J_\Omega = g^\Lambda_{\Omega^\circ} \frac{v^T v_s}{\|P_\Lambda\|^2 v^A_j} J_\Lambda + \sum_{\Gamma \succ \Omega} g^\Gamma_{\Omega^\circ} \frac{v^T v_s}{\|P_\Gamma\|^2 v^\Gamma} J_\Gamma
\]

where \( g^\Lambda_{\Omega^\circ} \neq 0 \) by Proposition 8 since \( \Omega^* \cup (s) = \Lambda^* \) and \( \Omega^\circ \cup (s) = \Lambda^\circ \). Again we have by induction on the dominance ordering that \( E_{N,m}[J_\Gamma] \) is divisible by (137) for every \( J_\Gamma \) appearing in the sum. Since \( E_{N,m}[J_s J_\Omega] = E_{N,m}[J_s] E_{N,0}[J_\Omega] \) and \( E_{N,0}[J_s] \) is divisible by (137) (see (132) in the case \( \Lambda = s \), we have that \( E_{N,m}[J_\Lambda] \) is also divisible by (137).

The polynomials (136) and (137) are relatively prime (since their zeroes are distinct for generic values of \( \alpha \)) and we have shown that \( E_{N,m}[J_\Lambda] \) is divisible by both. Therefore \( E_{N,m}[J_\Lambda] \)
is divisible by their product which is equal to $\Lambda - m(m - 1)/2$. Since we have seen that $E_{N,m}[J_\Lambda]$ is monic and of degree $|\Lambda| - m(m - 1)/2$, the theorem follows.

Let us end this subsection with an example illustrating formula (133). Consider for instance $\Lambda = (3, 1; 2)$, for which

$$S(3, 1; 2) = \begin{array}{cccc}
1 & & & \\
& 0+2\alpha & & \\
& & 0+3\alpha & \\
& & & 1+\alpha \\
& & & \tilde{2}+\alpha \\
& & & \tilde{2}+2+\alpha \\
\end{array}$$

(146)

By filling the cells $(i, j)$ of $S\Lambda$ with the factors $N - (i - 1) + \alpha(j - 1)$, one gets

$$\begin{array}{cccc}
\tilde{6}+2\alpha & \tilde{6}+3\alpha \\
\tilde{1}+\alpha & \\
\tilde{2} & \tilde{2}+\alpha \\
\end{array}$$

(147)

where $\tilde{k} = N - k$ with the understanding that $N \geq 3$. Thus

$$E_{N,2}[J_{(3,1,2)}] = b^{(\alpha,N)}_{(3,1,2)} = (N + 2\alpha)(N + 3\alpha)(N - 1 + \alpha)(N - 2)(N - 2 + \alpha).$$

(148)

5.2. Second evaluation formula. Remarkably, if the Jack superpolynomial $J_\Lambda(x, \theta)$ has non-zero fermionic degree, the evaluation $E_{N-1,m-1}$ of $[\partial_{\theta_N} J_\Lambda]_{x_N=0}$ is very similar to that of $J_\Lambda$ even though its expansion in terms of Jack superpolynomials can involve many terms. We will refer to the evaluation of $[\partial_{\theta_N} J_\Lambda]_{x_N=0}$ as our second evaluation formula. To simplify the notation, we will define

$$\tilde{E}_{N,m}[F(x, \theta)] := E_{N-1,m-1} [(-1)^{m-1} \partial_{\theta_N} F(x; \theta)]_{x_N=0}.$$ (149)

Before getting to the derivation of the explicit form of the second evaluation formula on Jack polynomials in superspace, we must introduce another operation on partition. Let $\Lambda$ be a superpartition of fermionic degree $m$. Then $\tilde{S}$ on $\Lambda$ is defined as

$$\tilde{S}\Lambda = \Lambda^* / (m - 1, m - 2, \ldots, 0).$$

(150)

See Fig. 10 for a diagrammatic illustration of this definition.

Figure 10. Operator $\tilde{S}$
Lemma 27. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_\ell)$ be a superpartition. Then

$$\bar{E}_{N,m}[m_\Lambda] = \begin{cases} E_{N-1,m-1}[m_{\Lambda_-}] & \text{if } \Lambda_m = 0 \\ 0 & \text{otherwise} \end{cases}$$

(151)

where $\Lambda_- = (\Lambda_1, \ldots, \Lambda_{m-1}; \Lambda_{m+1}, \ldots)$.

Proof. The lemma follows immediately from

$$[\partial_{\theta_N} m_\Lambda(x; \theta)]_{x_N=0} = \begin{cases} (-1)^{m-1}m_{\Lambda_-}(x_-; \theta_-) & \text{if } \Lambda_m = 0 \\ 0 & \text{otherwise} \end{cases}$$

(152)

where $(x_-; \theta_-) = (x_1, \ldots, x_{N-1}; \theta_1, \ldots, \theta_{N-1})$.

Theorem 28. Let $\Lambda$ be a superpartition of fermionic degree $m > 0$. Let also

$$\hat{b}_\Lambda^{\alpha,N} := \prod_{(i,j) \in S_\Lambda} \hat{b}_\Lambda^{\alpha,N}(i,j) := \prod_{(i,j) \in S_\Lambda} (N - 1 - (i - 1) + \alpha(j - 1)).$$

Then, for all $N \geq \ell = \ell(\Lambda),

$$\bar{E}_{N,m}[J_\Lambda] = \hat{b}_N^{\alpha,N}.$$ 

(154)

Proof. As in the proof of Theorem 26, we first prove that $\bar{E}_{N,m}[J_\Lambda]$ is a monic polynomial in $N$ of degree $\ell_{n,m} = |\Lambda| - m(m - 1)/2$. From Lemmas 23 and 27, we have that the degree in $N$ of $\bar{E}_{N,m}[J_\Lambda]$ is given by the degree in $N$ of $\bar{E}_{N,m}[m_{\Lambda_{\min}}]$. Using

$$\bar{E}_{N,m}[m_{\Lambda_{\min}}] = E_{N-1,m-1}[m_{(\Lambda_{\min})-}] = \frac{(N - \ell_{n,m} + m)\ell_{n,m}}{\ell_{n,m}!}$$

(155)

the claim then follows from the normalization (110) of $J_\Lambda$. From Proposition 15, we have that

$$\bar{E}_{N,m}[J_\Lambda(x, \theta)] = \sum_\Omega k_\Omega(\alpha) E_{N-1,m-1}[J_\Omega(x_-; \theta_-)][\partial_{\theta_N} J_\Lambda(\Omega)(x_N; \theta_N)]_{x_N=0}$$

(156)

where $k_\Omega(\alpha)$ are coefficients that do not depend on $N$. Since

$$[\partial_{\theta_N} J_\Gamma(x_N; \theta_N)]_{x_N=0} \neq 0 \iff \Gamma = \tilde{0}$$

(157)

we have that $[\partial_{\theta_N} J_\Lambda(\Omega)(x_N; \theta_N)]_{x_N=0} \neq 0$ only if $g^{\Lambda_{\Omega}\Lambda}_{\tilde{0}} \neq 0$, that is only if $\Lambda/\Omega$ is a horizontal $\tilde{0}$-strip by Proposition 11. If $\Lambda/\Omega$ is a horizontal $\tilde{0}$-strip then the diagram of $\Omega$ corresponds to the diagram of $\Lambda$ with one of its circles removed. It is then immediate that $E_{N-1,m-1}[J_\Omega(x_-, \theta_-)]$ is divisible by $\hat{b}_\Lambda^{\alpha,N}$ given that $\tilde{S}(\Lambda)$ is contained in $S_\Omega = \Omega^\circ/(m - 1, \ldots, 1)$. Since every term in the right-hand side of (156) is divisible by $\hat{b}_\Lambda^{\alpha,N}$, so is $\bar{E}_{N,m}[J_\Lambda]$. The theorem then follows given that $\hat{b}_\Lambda^{\alpha,N}$ is a monic polynomial in $N$ of degree $|\Lambda| - m(m - 1)/2$.

Let us illustrate the second evaluation formula with the superpartition $\Lambda = (3, 0; 2, 1)$. The evaluation amounts to filling the cells of $\tilde{S}(3, 0; 2, 1)$ with the numbers $\hat{b}_\Lambda^{\alpha,N}(3, 0; 2, 1)(i,j)$. Using
$k = N - k$, this corresponds to the filling

\[
\begin{array}{ccc}
1 + \alpha & 1 + 2\alpha \\
2 & 2 + \alpha \\
3 & \\
\end{array}
\]

which gives

\[
\tilde{E}_{N,2}[J_{(3,0,2,1)}] = (N - 1 + \alpha)(N - 1 + 2\alpha)(N - 2 + \alpha)(N - 3).
\]  

5.3. Homomorphisms. Evaluations over rings are usually defined in algebra as ring homomorphisms. As an example, consider the ring Sym formed by the symmetric functions in the indeterminates $x = (x_1, x_2, \ldots)$ with coefficients in the field $Q(\alpha)$. Then, Macdonald [15, Section I.2] defines the standard evaluation of symmetric functions as the homomorphism $\varepsilon_X : \text{Sym} \to Q(\alpha)[X]$ such that

\[
\varepsilon_X(p_n) = X, \quad \forall n \geq 1.
\]

(160)

In the case of the usual Jack symmetric functions $J_\lambda$, it is possible to show that [15, Eq. 10.25]

\[
\varepsilon_X(J_\lambda) = \prod_{(i,j) \in \lambda} (X - (i - 1) + \alpha(j - 1)).
\]

(161)

Setting $X = N$ in the last equation, we recover Stanley’s result for the evaluation of a Jack symmetric function at $x_1 = \ldots = x_N = 1$ and $x_{N+1} = x_{N+2} = \ldots = 0$ [19, Theo. 5.4].

Obviously, the evaluation operator $E_{N,m}$ introduced in (14) is not a homomorphism. Rather, it is a linear map from the vector space $R_{N,m}$ to the ring $Q(\alpha)[N]$. It is nevertheless possible to define an evaluation homomorphism $E_{X,Y}$ on the whole ring $R$ of symmetric superfunctions with coefficient in $Q(\alpha)$. As we show below, $E_{X,Y}$ generalizes the homomorphism $\varepsilon_X$ defined above and connects with $E_{N,m}$ in the special case where $X = N$ and $Y = M = m$.

Now let $R$ and $R[\phi_1, \ldots, \phi_M]$ respectively denote the polynomial ring $Q(\alpha)[X,Y]$ and the ring of polynomials in the Grassmann variables $\phi_1, \ldots, \phi_M$ with coefficients in $R$. We define

\[
E_{X,Y} : R \to R[\phi_1, \ldots, \phi_M]
\]

(162)
as the homomorphism such that

\[
E_{X,Y}(p_{n+1}) = X, \quad E_{X,Y}(\tilde{p}_n) = \sum_{i=1}^{M} \phi_i \varepsilon_Y(h_{n+i-M}), \quad \forall n \geq 0,
\]

(163)

where $h_r$ denotes the complete symmetric function of degree $r$ [15 p. 21]. By convention, $h_r$ is equal to 1 for $r = 0$ and to 0 whenever $r < 0$. On readily shows [15] that if $r > 0$,

\[
\varepsilon_Y(h_r) = \binom{Y + r - 1}{r} = (-1)^r \binom{-Y}{r}.
\]

(164)

**Proposition 29.** For any symmetric function in superspace $F(x, \theta)$ of fermionic degree $m$, we have

\[
E_{N,m}(F(x, \theta)) = \phi_1 \cdots \phi_m E_{N,m}(F(x, \theta)).
\]

(165)
Proof. It is enough to show that the proposition holds when $F(x, \theta) = p_\Lambda$, where $\Lambda = (\Lambda^a; \Lambda^s)$. Let us first recall a basic lemma: if $F_i = \sum_j \phi_j A_{i,j}$, where $\phi_j$ and $A_{i,j}$ respectively denote anticommutative and commutative variables, then

$$F_1 \cdots F_m = \sum_{1 \leq j_1 < \ldots < j_m \leq M} \phi_{j_1} \cdots \phi_{j_m} \det[A_{i,j}]_{i=1,\ldots,m \atop j=j_1,\ldots,j_m}. \quad (166)$$

Applying formula (166) to the case $F_1 \cdots F_m = \tilde{p}_\Lambda a_1^\Lambda \cdots \tilde{p}_\Lambda a_m^\Lambda \quad (167)$

which corresponds to $\phi_i = \theta_i$ and $A_{i,j} = x_j^\Lambda$, we get

$$p_\Lambda(x; \theta) = \sum_{1 \leq j_1 < \ldots < j_m \leq m} \theta_{j_1} \cdots \theta_{j_m} a_\Lambda(x_{j_1}, \ldots, x_{j_m}) p_{\Lambda^s}(x_1, \ldots, x_N) \quad (168)$$

$$= \sum_{1 \leq j_1 < \ldots < j_m \leq m} \theta_{j_1} \cdots \theta_{j_m} V_m(x_{j_1}, \ldots, x_{j_m}) s_{\Lambda^a - \delta_m}(x_{j_1}, \ldots, x_{j_m}) p_{\Lambda^s}(x_1, \ldots, x_N), \quad (169)$$

where $a_\Lambda(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} \text{sgn}(\sigma)x_{\lambda_1}^{\sigma(1)} \cdots x_{\lambda_m}^{\sigma(m)}$ and where the second equality follows from the standard definition of Schur functions. Consequently

$$E_{N,m}(p_\Lambda) = N^\ell(\Lambda^s) s_{\Lambda^a - \delta_m}(1^m). \quad (170)$$

It thus remains to prove that

$$E_{N,m}(p_\Lambda) = \phi_1 \cdots \phi_m N^\ell(\Lambda^s) s_{\Lambda^a - \delta_m}(1^m). \quad (171)$$

Returning to (166) and setting $A_{i,j} = \varepsilon_Y(h_\Lambda^a + j - M)$, we get

$$E_{X,Y}^M(p_\Lambda) = X^\ell(\Lambda^s) \sum_{1 \leq j_1 < \ldots < j_m \leq M} \phi_{j_1} \cdots \phi_{j_m} \det[\varepsilon_Y(h_\Lambda^a + j - M)]_{i=1,\ldots,m \atop j=j_1,\ldots,j_m}. \quad (172)$$

When $M = m$, the sum on the RHS produces only one term, which is equal to

$$\phi_1 \cdots \phi_m \det[\varepsilon_Y(h_\Lambda^a + j - M)]_{i=1,\ldots,m \atop j=j_1,\ldots,j_m}. \quad (173)$$

Now, according to the Jacobi-Trudi formula [15, Eq. 3.4],

$$s_{\lambda - \delta_m} = \det[h_{\lambda_i + j - m}]_{i=1,\ldots,m \atop j=1,\ldots,m}. \quad (174)$$

where it is understood that we are working with symmetric polynomials in $m$ variables. Hence

$$E_{X,Y}^m(p_\Lambda) = X^\ell(\Lambda^s) \phi_1 \cdots \phi_m \varepsilon_Y(s_{\Lambda^a - \delta_m}), \quad (175)$$

which reduces to the desired equation for $X = N$ and $Y = m$. □
6. Normalization of the Jack polynomials in superspace

The simplest way of writing the normalization of the standard Jack polynomials is in terms of the upper and lower hook-lengths [19]. The same is true for the norm of the Jack superpolynomials: it is expressed in terms of the superpartition hook-lengths. Recall that leg-lengths and arm-lengths were defined in Section 2.

**Definition 30.** The upper and lower hook-lengths of \( s \in \Lambda \) are respectively given by

\[
h^\Lambda_{\alpha}(s) = l^\Lambda(s) + \alpha(a^\Lambda_\alpha(s) + 1) \quad \text{and} \quad h^\Lambda_{(\alpha)}(s) = l^\Lambda(s) + 1 + \alpha a^\Lambda(s).
\]

**Lemma 31.** For any diagram \( \Lambda \), the two hook-lengths are related by

\[
h^\Lambda_{\alpha}(i, j) = \alpha h^{(1/\alpha)}_{\Lambda^\alpha}(j, i).
\]

**Proof.** This is an immediate consequence of the identity \( l_{\lambda}(i, j) = a_{\lambda}(j, i) \). \( \square \)

Now recall Definition 20 relating the non-monic \( J^\Lambda \) to its monic counterpart \( P^\Lambda \). If we want to re-express Proposition 16 in terms of \( J^\Lambda \), this will necessarily involve a non-trivial proportionality factor since the Jack superpolynomials on the two sides of the equation are different. Stated precisely, there must exist a rational function \( r^\Lambda(\alpha) \) such that

\[
J^\Lambda(x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell) = r^\Lambda(\alpha)x_1 \cdots x_\ell J^C_{\alpha}(x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell),
\]

if \( \Lambda \) is a superpartition of length \( \ell \) whose first column is bosonic. Similarly, it follows from Proposition 17 that there exists a \( \tilde{r}^\Lambda(\alpha) \) such that

\[
(-1)^{m-1}\left[ \partial_{\theta_i} J^\Lambda(x_1, \ldots, x_\ell; \theta_1, \ldots, \theta_\ell) \right]_{x_\ell = 0} = \tilde{r}^\Lambda(\alpha) J^C_{\alpha}(x_1, \ldots, x_{\ell-1}; \theta_1, \ldots, \theta_{\ell-1}).
\]

when \( \ell(\Lambda^\alpha) = \ell(\Lambda^\ast) + 1 \). The rationale for introducing \( r^\Lambda(\alpha) \) and \( \tilde{r}^\Lambda(\alpha) \) is that they are the building blocks of the norm expression. Actually, the hook-lengths do appear in the norm via these proportionality factors.

The next two propositions give the relation between \( r^\Lambda(\alpha) \) and \( \tilde{r}^\Lambda(\alpha) \) with the lower and upper hook-lengths respectively. They imply in particular that \( r^\Lambda(\alpha) \) and \( \tilde{r}^\Lambda(\alpha)^{-1} \) are polynomials in \( \alpha \).

**Proposition 32.** Let \( \Lambda \) be a superpartition such that \( \Lambda_i > 0 \) for all \( 1 \leq i \leq \ell \) and let \( r^\Lambda(\alpha) \) be defined by (178). Then

\[
r^\Lambda(\alpha) = \prod_{i=1}^\ell h^\Lambda_{(\alpha)}(i, 1).
\]

**Proof.** We apply the evaluation \( E_{\ell,m} \) on both sides of (178) with \( m \) standing for the fermionic degree of \( \Lambda \). This yields

\[
r^\Lambda(\alpha) = \frac{E_{\ell,m}[J^\Lambda]}{E_{\ell,m}[J^C^\Lambda]}.
\]

Here it is crucial that the first column of \( \Lambda \) be bosonic to insure that the action of \( C \) on \( \Lambda \) is well defined. Now, Theorem 26 (i.e., (133)) implies that

\[
r^\Lambda(\alpha) = \frac{b^\Lambda_{(\alpha, \ell)}}{b^\Lambda_{(\alpha, \ell)}},
\]
From the definition of \( b^{(\alpha,N)}_A \) given in (132), it then follows that
\[
r_A(\alpha) = \prod_{(i,j) \in \mathcal{S} \setminus S(\mathcal{C} \Lambda)} \left( \ell - (i - 1) + \alpha(j - 1) \right) = \prod_{(i,j) \in \mathcal{S} \setminus S_j(\Lambda)} \left( \ell - (i - 1) + \alpha(j - 1) \right),
\]
where \( \Lambda_i^{\oplus} \) denotes the number of cells in the \( i \)th row of the diagram \( \Lambda^{\oplus} \). However, \( \ell - (i - 1) = l_A^\star(i,1) + 1 \) while \( j - 1 = a_{\Lambda^\oplus}(i,1) \) if \( j = \Lambda_i^{\oplus} \). Thus,
\[
r_A(\alpha) = \prod_{i=1}^\ell (l_A^\star(i,1) + 1 + \alpha a_{\Lambda^\oplus}(i,1)),
\]
and the proof is complete. \( \square \)

**Proposition 33.** Let \( \Lambda \) be a superpartition such that \( \ell(\Lambda^{\oplus}) = \ell(\Lambda^\star) + 1 \). Moreover, let \( \text{fr}(\Lambda) \) denote the set of fermionic rows in the diagrams of \( \Lambda \) and let \( \tilde{r}_A(\alpha) \) be the function introduced in (179). Then
\[
\tilde{r}_A(\alpha) = \prod_{(i,1) \in \text{fr}(\Lambda)} \frac{1}{b^A_{\Lambda}(i,1)},
\]
where \( \tilde{b}^{(\alpha,N)}_A \) is defined in (132). Applying the evaluation \( E_{\ell-1,m-1} \) on both sides of (179) and using relation (149), we get
\[
\tilde{r}_A(\alpha) = \frac{\tilde{E}_{\ell,m}[J_A]}{E_{\ell-1,m-1}[J_{\mathcal{C} \Lambda}]} = \frac{\tilde{b}^{(\alpha,\ell)}_A}{b^{(\alpha,\ell-1)}_{\Lambda^\star}}.
\]

**Theorem 34.** Let \( \mathcal{B} \Lambda \) denote the bosonic content of \( \Lambda \), i.e., the set of squares in \( \Lambda \) that do not appear at the same time in a row containing a circle and in a column containing a circle. Then the coefficient of \( m_\Lambda \) in \( J_\Lambda \) is
\[
v_\Lambda(\alpha) = \prod_{(i,j) \in \mathcal{B} \Lambda} h^A_{\Lambda}(i,j).
\]
Proof. We proceed by induction on the size of $\Lambda^\circ$. If $\Lambda$ is the empty partition, then $J_\emptyset = 1$ and the result holds since $v_\emptyset = 1$. If the first column of $\Lambda$ is bosonic, then from (178), Proposition 16 and Definition 20, we have

$$v_\Lambda(\alpha) = v_\Lambda(\alpha) v_{C\Lambda}(\alpha). \quad (190)$$

Using Proposition 32 and setting $\ell = \ell(\lambda)$, we thus have by induction that

$$v_\Lambda(\alpha) = \prod_{i=1}^{\ell} h^\Lambda_{(\alpha)}(i, 1) \prod_{(i, j) \in B(C\Lambda)} h^\Lambda_{(\alpha)}(i, j) = \prod_{(i, j) \in B\Lambda} h^\Lambda_{(\alpha)}(i, j), \quad (191)$$

and the result holds in that case. If the first column of $\Lambda$ is fermionic, we have from (179), Proposition 17 and Definition 20 that

$$v_\Lambda(\alpha) = \tilde{r}_\Lambda(\alpha) v_{\tilde{C}\Lambda}(\alpha). \quad (192)$$

Using Proposition 33, we then have by induction that

$$v_\Lambda(\alpha) = \prod_{(i, 1) \in fr(\Lambda)} 1 \prod_{(i, j) \in B(\tilde{C}\Lambda)} h^\Lambda_{(\alpha)}(i, j) = \prod_{(i, j) \in B\Lambda} h^\Lambda_{(\alpha)}(i, j), \quad (193)$$

which proves the theorem. \hfill \square

To illustrate the last formula, we consider the superpartition

$$\Lambda = (4, 2, 0; 2) = \begin{array}{|c|c|c|}
\hline
3+3\alpha & 1+\alpha & \hline
\hline
2+\alpha & & \hline
\hline
1+\alpha & & 1 \\
\hline
\end{array} \quad (194)$$

The bosonic content of $\Lambda$ and its associated upper hook-lengths are given by

$$\begin{array}{|c|c|}
\hline
3+3\alpha & 1+\alpha \\
\hline
2+\alpha & \\
\hline
1+\alpha & 1 \\
\hline
\end{array} \quad (195)$$

From this we conclude that $v_{(4,2,0;2)} = (3 + 3\alpha)(2 + \alpha)(1 + \alpha)^2$.

**Theorem 35.** For any superpartition $\Lambda$,

$$\|J_\Lambda\|^2 := \langle \tilde{J}_\Lambda \mid \tilde{J}_\Lambda \rangle = \alpha^\Lambda \prod_{s \in B\Lambda} h^\Lambda_{(\alpha)}(s) h^\Lambda_{(\alpha)}(s). \quad (196)$$

Furthermore,

$$\|P_\Lambda\|^2 := \langle \tilde{P}_\Lambda \mid \tilde{P}_\Lambda \rangle = \alpha^\Lambda \prod_{s \in B\Lambda} h^\Lambda_{(\alpha)}(s) h^\Lambda_{(\alpha)}(s) = \alpha^\Lambda \prod_{s \in \Lambda} h^\Lambda_{(\alpha)}(s). \quad (197)$$

Proof. Set $n = |\Lambda|$ and $m = \frac{\Lambda}{\alpha}$. According to Proposition 31 in \cite{10}, which is a consequence of the duality property of Jack polynomials given in \cite{10}, we have

$$\langle \tilde{P}_\Lambda \mid \tilde{P}_\Lambda \rangle = \alpha^{m+\ell_n, m} \frac{v_\Lambda(1/\alpha)}{v_\Lambda(\alpha)}. \quad (198)$$
But since \( \overrightarrow{J}_\Lambda = v_\Lambda \overrightarrow{P}_\Lambda \), it readily follows that
\[
\langle J_\Lambda | \overrightarrow{J}_\Lambda \rangle = \alpha^{m + \ell_{n,m}} v_\Lambda (1/\alpha) v_\Lambda (\alpha).
\] (199)

Now, exploiting Lemma 31 and the obvious property \((\mathcal{B}\Lambda)' = \mathcal{B}(\Lambda')\), we get
\[
v_\Lambda (1/\alpha) = \prod_{(i,j) \in \mathcal{B}(\Lambda')} h^{(\alpha)}(i,j) = \prod_{(i,j) \in (\mathcal{B}\Lambda)'} \frac{1}{\alpha} h^{(\alpha)}(j,i) = \frac{1}{\alpha^{n - \binom{m}{2}}} \prod_{(i,j) \in \mathcal{B}\Lambda} h^{(\alpha)}(i,j)
\] (200)

The above expressions for \( \|J_\Lambda\|^2 \) and \( \|P_\Lambda\|^2 \) follow by substituting the latter equation into (199) and (198) respectively and using \( \ell_{n,m} = n - m(m - 1)/2 \). That \( \mathcal{B}\Lambda \) can be replaced by \( \Lambda \) in the expression of \( \|P_\Lambda\|^2 \) follows from the identity
\[
h^{(\alpha)}(s) = l_{\Lambda}(s) + \alpha(a_{\Lambda}(s) + 1) = l_{\Lambda}(s) + 1 + \alpha a_{\Lambda}(s) = h^{(\alpha)}(s)
\] (201)
whenever \( s \) belongs to both a fermionic row and a fermionic column (i.e., \( s \in \Lambda/\mathcal{B}\Lambda = \mathcal{F}\Lambda \)).

\[\square\]

**Appendix A. Proofs of Propositions 10 and 11**

The proofs of Propositions 10 and 11 rely on the relation between Jack polynomials in superspace and non-symmetric Jack polynomials presented in [8, Sect. 9].

The non-symmetric Jack polynomials, \( E_\eta(x; \alpha) \), are indexed by compositions \( \eta \in \mathbb{Z}^N_0 \) with \( N \) parts (some of them possibly equal to zero). They were first studied systematically in [16] although they had appeared earlier in the physics literature as eigenfunctions of the commuting Dunkl-type operators [3]
\[
D_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{k<i} \frac{x_i}{x_i - x_k} (1 - K_{i,k}) + \sum_{i<k \leq N} \frac{x_k}{x_i - x_k} (1 - K_{i,k}) + 1 - i,
\] (202)
where \( K_{i,k} \) is the operator that exchanges the variables \( x_i \) and \( x_k \). The non-symmetric Jack polynomial \( E_\eta(x; \alpha) \) can be characterized as the unique polynomial, whose coefficient of \( x^\eta \) is equal to 1, such that
\[
D_i E_\eta(x; \alpha) = \bar{\eta}_i E_\eta(x; \alpha) \quad \forall i = 1, \ldots, N,
\] (203)
where the eigenvalue \( \bar{\eta}_i \) is given by
\[
\bar{\eta}_i = \alpha \eta_i - \# \{ k < i \mid \eta_k \geq \eta_i \} - \# \{ k > i \mid \eta_k > \eta_i \}.
\] (204)

The following properties of non-symmetric Jack polynomials [13] will prove to be important:
\[
K_{i,i+1} E_\eta(x; \alpha) = E_\eta(x; \alpha) \quad \text{if} \quad \eta_i = \eta_{i+1},
\] (205)
and
\[
\left[ K_{i,i+1} + \frac{1}{(\bar{\eta}_i - \bar{\eta}_{i+1})} \right] E_{\eta'}(x; \alpha) = E_{\eta'}(x; \alpha) \quad \text{if} \quad \eta_i > \eta_{i+1},
\] (206)
where \( \eta' = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \ldots, \eta_N) \).
For our purposes it will be convenient to associate a diagram to \( \eta \) given by the set of cells in \( \mathbb{Z}^2_{\geq 1} \) such that \( 1 \leq i \leq N \) and \( 1 \leq j \leq \eta_i \). For instance, if \( \eta = (0, 1, 3, 0, 0, 6, 2, 5) \), the diagram of \( \eta \) is

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

where a \( \bullet \) represents an entry of length zero.

Suppose \( i_1, \ldots, i_p \) are distinct integers between 1 and \( N \). It is known [12] that the non-symmetric Jack polynomials satisfy the following Pieri-type expansion

\[
x_{i_1} \cdots x_{i_p} E_\eta(x; \alpha) = \sum_{\nu \in \mathcal{J}_{N,p}} c_{\eta \nu} E_\nu(x; \alpha),
\]

for certain coefficients \( c_{\eta \nu} \). The set \( \mathcal{J}_{N,p} \) is most easily described in terms of the diagram of \( \eta \). A cell is first added to each of the \( p \) rows \( i_1, \ldots, i_p \) of \( \eta \) to form a new diagram. Then \( \mathcal{J}_{N,p} \) consists of all the rearrangements of the rows of the new diagram such that the rows with a cell added can only move downwards or stay stationary, while the remaining rows can only move upwards or stay stationary. For instance, if \( p = 2, i_1 = 2, i_2 = 3 \) and \( \eta = (3, 1, 3, 0) \) we have that \( \mathcal{J}_{N,p} \) consists of the following diagrams

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

where the cells with thick frames correspond to the rows with a cell added.

Given a superpartition \( \Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N) \), define \( \tilde{\Lambda} \) to be the composition

\[
\tilde{\Lambda} := (\Lambda_m, \ldots, \Lambda_1, \Lambda_N, \ldots, \Lambda_{m+1}).
\]

It was established in [8] Eq. 107 and Theo. 41] that the Jack polynomials in superspace can be obtained from the non-symmetric Jack polynomials through the following relation:

\[
P_\Lambda = \frac{(-1)^{m(m-1)/2}}{n_\Lambda!} \sum_{w \in S_N} \mathcal{K}_w \theta_1 \cdots \theta_m E_{\tilde{\Lambda}}(x; \alpha),
\]

where we recall that \( n_\Lambda! \) is defined in (23), and where \( \mathcal{K}_w \) is such that

\[
\mathcal{K}_w f(x_1, \ldots, x_N; \theta_1, \ldots, \theta_N) = f(x_{w(1)}, \ldots, x_{w(N)}; \theta_{w(1)}, \ldots, \theta_{w(N)})
\]

on any polynomial \( f(x_1, \ldots, x_N; \theta_1, \ldots, \theta_N) \) in \( x \) and \( \theta \).

Note that the composition \( \tilde{\Lambda} \) is of a very special form. Its first \( m \) rows (resp. last \( N - m \) rows) are strictly increasing (resp. weakly increasing). Diagrammatically, it is made of two partitions (the first of which having no repeated parts) drawn in the French notation (largest
row at the bottom). For instance if \( \Lambda = (3, 1, 0; 5, 3, 3, 0, 0) \), we have \( \tilde{\Lambda} = (0, 1, 3, 0, 0, 3, 3, 5) \) whose diagram is given by

\[
\begin{array}{c}
\cdot \\
\begin{array}{c}
\cdot \\
\cdot
\end{array}
\end{array}
\]

(213)

The first \( m \) rows (resp. last \( N - m \) rows) of \( \tilde{\Lambda} \) will be said to be fermionic (resp. bosonic).

We can now proceed to the proof of Propositions 10 and 11.

**Proof of Proposition 10.** Let

\[
O_{\text{sym}} = \sum_{w \in S_N} K_w \theta_1 \cdots \theta_m.
\]  

(214)

It is easy to see that if \( f \in \mathbb{Q}[x_1, \ldots, x_N](\alpha) \) then \( O_{\text{sym}} K_{i,j+1} f = -O_{\text{sym}} f \) if \( i = 1, \ldots, m - 1 \) and \( O_{\text{sym}} K_{i,j+1} f = O_{\text{sym}} f \) if \( i = m + 1, \ldots, N - 1 \). Using (206), we can thus deduce that

\[
P_{\Lambda} \propto O_{\text{sym}} E_{\tilde{\Lambda}},
\]  

(215)

whenever the fermionic rows of \( \eta \) (its first \( m \) entries) are a rearrangement of \( \Lambda_1, \ldots, \Lambda_m \) and its bosonic rows (its last \( N - m \) entries) are a rearrangement of \( \Lambda_{m+1}, \ldots, \Lambda_N \).

We will now use the Pieri-type rule given in (208) to show that the expansion

\[
e_n P_{\Lambda} = \sum_{\Omega} g_{\Omega, (1^n)} P_{\Omega}
\]  

(216)

is such that the coefficient \( g_{\Omega, (1^n)} \) is non-zero only if \( \Omega / \Lambda \) is a vertical \( n \)-strip. We have

\[
e_n P_{\Lambda} \propto \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} O_{\text{sym}} E_{\tilde{\Lambda}} = O_{\text{sym}} \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} E_{\tilde{\Lambda}},
\]  

(217)

where \( \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} \) commutes with \( O_{\text{sym}} \) since it is a symmetric function in \( x_1, \ldots, x_N \). Given a composition \( \eta \) whose first \( m \) entries are all distinct, let \( \Omega_{\eta} = (\Omega_{\eta}^a; \Omega_{\eta}^s) \) be the super-partition such that \( \Omega_{\eta}^a \) and \( \Omega_{\eta}^s \) are obtained respectively by rearranging the first \( m \) entries of \( \eta \) and the last \( N - m \) entries of \( \eta \). We thus simply need to show that the compositions \( \eta \) such that \( E_{\eta} \) appear in \( x_{i_1} \cdots x_{i_n} E_{\tilde{\Lambda}} \) are such that \( \Omega_{\eta} / \Lambda \) is a vertical \( n \)-strip.

We know from the rule given after (208) that \( \eta \) is obtained from \( \tilde{\Lambda} \) by adding \( n \) cells in distinct rows and then rearranging the rows. It is thus clear that \( \Omega_{\eta}^s / \Lambda^s \) is a vertical \( n \)-strip. Suppose that \( \omega \) is obtained from \( \tilde{\Lambda} \) by adding \( n \) cells in distinct rows. It is easily seen that \( \Omega_{\eta}^a / \Lambda^a \) is a vertical \( n \)-strip in that case. We will now see that if \( \eta \) is obtained by rearranging the rows of \( \omega \) then \( \Omega_{\eta}^s / \Lambda^s \) is still a vertical \( n \)-strip. First observe from (215) that the only rearrangements that matter are those that send a fermionic (resp. bosonic) row into a bosonic (resp. fermionic) row. Since fermionic rows lie above bosonic ones, for a bosonic row of \( \omega \) to become fermionic, the rule given after (208) imposes that the size of that row needs to be the same in \( \omega \) and in \( \tilde{\Lambda} \). Therefore this new fermionic row differs from the old bosonic row of \( \tilde{\Lambda} \) only by a circle, and thus when comparing the the two rows in \( \Omega_{\eta}^s / \Lambda^s \) we get a difference of one. The rule given after (208) imposes similarly that for a fermionic row of \( \omega \)
to become bosonic, the size of that row needs to be larger (by one) in \( \omega \) than in \( \tilde{\Lambda} \). Hence, when comparing the two rows in \( \Omega^\oplus_{\eta}/\Lambda^\oplus \) we get that they are of the same size (a circle was lost while a square was gained). We thus have that \( \Omega^\oplus_{\eta} \subseteq \Lambda^\oplus \) with the length of the rows of \( \Omega^\oplus_{\eta} \) and \( \Lambda^\oplus \) never differing by more than one. Consequently, \( \Omega^\oplus_{\eta}/\Lambda^\oplus \) is a vertical \( n \)-strip. \( \square \)

**Proof of Proposition 11.** We will prove the equivalent statement that

\[
\tilde{e}_n P_{\Lambda} = \sum_{\Omega} g^\Omega_{\Lambda,(0;1^n)} P_{\Omega} \tag{218}
\]

is such that the coefficient \( g^\Omega_{\Lambda,(0;1^n)} \) is non-zero only if \( \Omega/\Lambda \) is a vertical \( \tilde{n} \)-strip.

We use again the Pieri-type formula for non-symmetric Jack polynomials to get

\[
\tilde{e}_n P_{\Lambda} \propto \sum_{i,i_1<...<i_n} \theta_i x_{i_1} \cdots x_{i_n} O_{\text{sym}} E_{\tilde{\Lambda}} = O_{\text{sym}} \sum_{i,i_1<...<i_n} \theta_i x_{i_1} \cdots x_{i_n} E_{\tilde{\Lambda}}. \tag{219}
\]

Following the argument given after (217), it is immediate that we have again that if \( \Omega \) appears in \( \tilde{e}_n P_{\Lambda} \) then \( \Omega^\oplus/\Lambda^\oplus \) is a vertical \( n \)-strip. It remains to show that the terms \( \Omega \) occurring in \( \tilde{e}_n P_{\Lambda} \) are such that \( \Omega^\oplus/\Lambda^\oplus \) is a vertical \( (n+1) \)-strip. This is somewhat more subtle and we will use a different route to obtain the result.

Let \( d = \sum_{i=1}^\infty \theta_i \partial_{x_i} \) and \( d^\perp = \sum_{i=1}^\infty x_i \partial_{\theta_i} \). The operators \( d \) and \( d^\perp \) are adjoint of each other with respect to the scalar product (31), namely:

\[
\langle\langle d^\perp f \mid g \rangle\rangle = \langle\langle f \mid d g \rangle\rangle, \tag{220}
\]

for every symmetric functions in superspace \( f \) and \( g \). It is easy to see that the expansion

\[
d m_{\Lambda} = \sum_{\Omega} u_{\Lambda \Omega} m_{\Omega} \tag{221}
\]

is such that \( u_{\Lambda \Omega} \neq 0 \) only if \( \Omega^\oplus = \Lambda^\oplus \) (\( \theta_i \partial_{x_i} \) removes a square from a bosonic row and changes the row into a fermionic one). Similarly,

\[
d^\perp m_{\Lambda} = \sum_{\Omega} \tilde{u}_{\Lambda \Omega} m_{\Omega} \tag{222}
\]

is such that \( \tilde{u}_{\Lambda \Omega} \neq 0 \) only if \( \Omega^\oplus = \Lambda^\oplus \) (\( x_i \partial_{\theta_i} \) adds a square to a fermionic row and changes the row into a bosonic one).

By (38), we have that \( P_{\Lambda} = \sum_{\Omega \leq \Lambda} c_{\Lambda \Omega} m_{\Omega} \). By definition of the dominance order on superpartions, we get in particular that \( m_{\Omega} \) occurs in \( P_{\Lambda} \) only if \( \Omega^\oplus \leq \Lambda^\oplus \). Thus, from (221) and (222), we have

\[
d P_{\Lambda} = \sum_{\Omega^\oplus \leq \Lambda^\oplus} b_{\Lambda \Omega} P_{\Omega} \quad \text{and} \quad d^\perp P_{\Lambda} = \sum_{\Omega^\oplus \leq \Lambda^\oplus} \tilde{b}_{\Lambda \Omega} P_{\Omega}, \tag{223}
\]

Using the adjointness of \( d \) and \( d^\perp \) we have

\[
\langle\langle P_{\Gamma} \mid d P_{\Lambda} \rangle\rangle = \langle\langle d^\perp P_{\Gamma} \mid P_{\Lambda} \rangle\rangle. \tag{224}
\]

Then, from the orthogonality of the Jack polynomials in superspace, we have from (223) that the left-hand side is zero unless \( \Gamma^\oplus \leq \Lambda^\oplus \) while the right-hand side is zero unless \( \Gamma^\oplus \geq \Lambda^\oplus \). Therefore the expressions are zero unless \( \Gamma^\oplus = \Lambda^\oplus \), that is,

\[
d P_{\Lambda} = \sum_{\Omega^\oplus = \Lambda^\oplus} b_{\Lambda \Omega} P_{\Omega}. \tag{225}
\]
Now, an easy computation gives that \( d \epsilon_{n+1} = \tilde{\epsilon}_n \). Since \( d \) is a derivative, we have
\[
d(\epsilon_{n+1} P_\Lambda) = \tilde{\epsilon}_n P_\Lambda + \epsilon_{n+1} d(P_\Lambda),
\]
and thus
\[
\tilde{\epsilon}_n P_\Lambda = \epsilon_{n+1} d(P_\Lambda) - d(\epsilon_{n+1} P_\Lambda).
\]
Since, by (225), all the terms \( \Gamma \) that occur in \( dP_\Lambda \) are such that \( \Gamma^\circ = \Lambda^\circ \) we have by Proposition 10 (in the form demonstrated above) that all the terms \( \Omega \) that occur in \( \epsilon_{n+1} d(P_\Lambda) \) are such that \( \Omega^\circ/\Lambda^\circ \) is a vertical \((n+1)\)-strip. Similarly, all the terms \( \Gamma \) that occur in \( \epsilon_{n+1} P_\Lambda \) are such that \( \Gamma^\circ/\Lambda^\circ \) is a vertical \((n+1)\)-strip and thus by (225) all the terms \( \Omega \) that occur in \( d(\epsilon_{n+1} P_\Lambda) \) are such that \( \Omega^\circ/\Lambda^\circ \) is a vertical \((n+1)\)-strip. Therefore, all the terms \( \Omega \) that occur in \( \tilde{\epsilon}_n P_\Lambda \) are such that \( \Omega^\circ/\Lambda^\circ \) is a vertical \((n+1)\)-strip.

\[\square\]

**Appendix B. Orderings on superpartitions and Jack polynomials in superspace**

Let \( \leq \) and \( \preceq \) refer respectively to the orders on superpartitions defined in (1) and (11). The Jack polynomials in superspace were defined in [8] as in Theorem 6 but with the order \( \preceq \) instead of \( \leq \). We will show in this appendix that the two orders lead to the same family of Jack polynomials in superspace.

The Jack polynomials are known [8] to be such that
\[
\mathcal{I} P_\Lambda = \epsilon'_A P_\Lambda, \quad \text{and} \quad P_\Lambda = m_\Lambda + \sum_{\Omega \preceq \Lambda} c_{\Lambda \Omega} m_\Omega,
\]
where \( \alpha \epsilon'_A = \epsilon_A - m(m-1)/2 \), with \( \epsilon_A \) being defined in (37). Note that the relation between \( \mathcal{I} \) and \( \Delta \) is given in (44). A crucial step in the derivation of those results was to show that
\[
\mathcal{I} m_\Lambda = \epsilon'_A m_\Lambda + \sum_{\Omega \preceq \Lambda} d_{\Lambda \Omega} m_\Omega.
\]
Our main task here is to prove the stronger statement (given that \( \Gamma \leq \Lambda \) implies \( \Gamma \preceq \Lambda \))
\[
\mathcal{I} m_\Lambda = \epsilon'_A m_\Lambda + \sum_{\Omega \preceq \Lambda} b_{\Lambda \Omega} m_\Omega,
\]
where we emphasize that the order in the sum is now the order \( \preceq \). Since \( \epsilon'_\Gamma \neq \epsilon'_A \) if \( \Gamma \prec \Lambda \) (see [8]), (230) ensures that \( P_\Lambda = m_\Lambda + \sum_{\Omega \preceq \Lambda} c_{\Lambda \Omega} m_\Omega \), which is the result we are trying to establish. Suppose otherwise that there exists a \( c_{\Lambda \Gamma} \neq 0 \) with \( \Gamma \prec \Lambda \) and \( \Gamma \not\preceq \Lambda \). Pick \( \Gamma \) to be such that there is no \( c_{\Lambda \Omega} \neq 0 \) with \( \Omega \prec \Lambda \) and \( \Omega \not\preceq \Lambda \). Then we get the contradiction that the coefficient of \( c_{\Lambda \Gamma} \) in \( \mathcal{I} P_\Lambda \) is \( \epsilon'_\Gamma \neq \epsilon'_A \).

We now prove (230). We only need to show that there does not exist a \( \Gamma \) such that \( b_{\Lambda \Gamma} \neq 0 \) with \( \Gamma^\circ \not\preceq \Lambda^\circ \). In [8, Eq. 87], it is shown that the coefficient of \( \theta_1 \cdots \theta_m \) in \( \mathcal{I} m_\Lambda \) is given, up to a factor, by (using \( \beta = 1/\alpha \))
\[
\left[ \sum_{i=1}^m \Lambda_i - \beta m(m-1) \right] x^\Lambda + \frac{\beta}{n^\Lambda!} \sum_{w \in S_m} (-1)^{\text{sgn}(w)} K_w \sum_{i=1}^m \sum_{j=m+1}^N \frac{x_j}{(x_j-x_i)}(1-K_{ij})x^\Lambda,
\]
where \( n^\Lambda! \) is defined in (23). Observe that a term \( x^\beta \) in the resulting expression contributes to the coefficient of \( m_\Gamma \), where \( \Gamma = (\Gamma^\circ; \Gamma^\circ) \) is such that \( \Gamma^\circ \) is the reordering of the first \( m \) entries of \( \eta \) and \( \Gamma^\circ \) is the reordering of the remaining entries of \( \eta \). Since the \( \eta \)'s that can appear

\[2\text{Minor misprints in [8, Eq. 87] are corrected here.}\]
in the resulting expression differ from \( \Lambda \) in at most two entries, it suffices to consider the two-variable case. Let \( \Lambda = (a; b) \) (for the cases \( (a, b) \) and \( (a, b) \), the conclusion is immediate). We have

\[
\frac{x_2}{(x_1 - x_2)}(1 - K_{12})x_1^ax_2^b = \begin{cases} 
  x_1^{a-1}x_2^{b+1} + x_1^{a-2}x_2^{b+2} + \cdots + x_1^{b-1}x_2^{a+1} + x_1^bx_2^a & \text{if } a > b \\
  x_1^{b-1}x_2^{a+1} + x_1^{b-2}x_2^{a+2} + \cdots + x_1^{a-1}x_2^{b+1} + x_1^ax_2^b & \text{if } b > a.
\end{cases}
\]  

(232)

In the case where \( a > b \) we have \( \Lambda^\odot = (a + 1, b) \) and it is easy to see that it is larger in the dominance order than every term of the form \( (z + 1, y) \) where \( x_1^zx_2^y \) appears in (232).

Similarly, in the case where \( a < b \) we have \( \Lambda^\odot = (b + 1, a) \) and we see that it is larger or equal in the dominance order to every term of the form \( (z + 1, y) \) where \( x_1^zx_2^y \) appears in (232).

This implies that in the two-variable case every term \( m_\Gamma \) present in the action of \( \mathcal{I} \) on \( m_\Lambda \) is such that \( \Gamma^\odot \leq \Lambda^\odot \). As previously mentioned, the general case follows immediately.

APPENDIX C. ANOTHER COMBINATORIAL EXPRESSION FOR THE EVALUATION FORMULA

The evaluation formula of Theorem 26 is expressed in terms of the skew diagram \( S\Lambda \). It is possible to reexpress this evaluation formula directly in terms of the diagram of \( \Lambda \). This alternative expression involves what we will call the shadow of a cell in analogy with Viennot’s shadow in [20].

The shadow of \( s \) is made of all the cells weakly south-east of it, that is, the cells in the shadow of \( s = (i, j) \) are the cells \( s' = (i', j') \) such that \( i' \geq i \) and \( j' \geq j \). In the diagram of \( \Lambda \), we place a \( \bullet \) in the \( j \)-th cell or circle of the \( (m - j + 1) \)-th circled row (from top to bottom).

We then define

\[
\#_c s = \text{the number of circles in the shadow of } s, \\
\#_s s = \text{the number of } \bullet \text{ is the shadow of } s.
\]

For instance, for the superpartition \( (5, 3, 1; 2, 2, 2, 1) \), the position of the \( \bullet \) and the shadow of cell \( s = (3, 1) \) (indicated by \( x \)'s) are as follows:

\[
\begin{array}{cccccccc}
\textbf{ } & \textbf{ } & \textbf{ } & \bullet & \textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } & \textbf{ } \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} & \textbf{x} \\
\end{array}
\]

so that \( \#_c (3, 1) = 1 \) and \( \#_s (3, 1) = 1 \).

Recall that the definitions of arm-colengths and leg-colengths can be found in Section 2.

Proposition 36. Let

\[
E_{N,m} [J\Lambda] = \prod_{s \in \mathcal{B} \Lambda} f_\Lambda(s)
\]

(235)
with \( f_\Lambda(s) \) given by
\[
f_\Lambda(s) = N - l'_\Lambda(s) + \alpha (d'_\Lambda(s) + \#_s) \quad \text{if} \quad \Lambda_\circ - \Lambda_\bullet = 1
\]
\[
= N - l'_\Lambda(s) - \#_s + \alpha d'_\Lambda(s) \quad \text{if} \quad \Lambda_\circ - \Lambda_\bullet = 0
\]
where \( s = (i, j) \).

For instance, if \( \Lambda = (3, 1, 0; 4, 2, 1) \), filling the cells \( s \in B\Lambda \) with the values \( f_\Lambda(s) \) gives (using \( k = N - k \)):

\[
\begin{array}{ccc}
3 & 2 + \alpha & 1 + 2\alpha \\
& 1 + 3\alpha & \\
4 & 3 + \alpha & \\
& & \\
5 & \\
\end{array}
\]

(238)

In other words, we have:
\[
E_{N,3} [J_{(3,1,0;4,2,1)}] = (N-3)(N-2+\alpha)(N-1+2\alpha)(N-1+3\alpha)(N-4)(N-3+\alpha)(N-5). \quad (239)
\]

This can be compared with the result obtained from filling the skew tableau \( S\Lambda \) with the values \( b_{\alpha,N}^{(3,1,0;4,2,1)}(s) \) defined in Theorem 26:

\[
\begin{array}{ccc}
\emptyset + 3\alpha & \emptyset + 3\alpha & \\
& 1 + 2\alpha & 1 + 3\alpha \\
& 2 + \alpha & \\
3 & 3 + \alpha & \\
4 & \\
5 & \\
\end{array}
\]

(240)

The resulting expression for \( E_{N,3}[J_{(3,1,0;4,2,1)}] \), obtained by taking the products of the entries of the filled squares, is clearly equal to (239).

The relation between the two expressions for \( E_{N,m}[J_\Lambda] \) is simply described as follows. Notice at once that the number of filled squares is the same in the two representations: the number of squares in \( B\Lambda \) is \( |\Lambda| - m(m-1)/2 \), while there are \( |\Lambda| + m - m(m+1)/2 \) squares in \( S\Lambda \). Take the filling of the squares of \( \Lambda \) described by the factor \( f_\Lambda(s) \). Then replace the circles by squares, thus transforming the Ferrers diagram of \( \Lambda \) into that of \( \Lambda^\circ \). Finally, move the filled squares as follows: if the square \( s \) belongs to a fermionic (resp. bosonic) row of \( \Lambda \) then displace it to the right (resp. downward) by \( \#_\circ s \) (resp. \( \#_\bullet s \)) units. It is then straightforward to see that the two evaluations coincide.
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