KÄHLER-RICCI SOLITONS ON SOME WONDERFUL GROUP COMPACTIFICATIONS

DELGOVE FRANÇOIS

1. Introduction

The founding article on the Kähler-Ricci solitons is Hamilton’s article [Ham88]. The Kähler-Ricci solitons are natural generalizations of the Kähler-Einstein metrics and appear as fixed points of the Kähler-Ricci flow. On a Fano compact Kähler manifold $M$, a Kähler metric $g$ is a Kähler-Ricci soliton if its Kähler form $\omega_g$ satisfies:

$$\text{Ric}(\omega_g) - \omega_g = L_X \omega_g,$$

where $\text{Ric}(\omega_g)$ is the Ricci form of $g$ and $L_X \omega_g$ is the Lie derivative of $\omega_g$ along a holomorphic vector field $X$ on $M$. Usually, we denote the Kähler-Ricci soliton by the pair $(g, X)$ and $X$ is called the solitonic vector field. We immediately note that if $X = 0$ then $g$ is a Kähler-Einstein metric.

Firstly, the study of the solitonic vector field $X$ was done in the article [TZ00, TZ02]. Thanks to the Futaki function, the authors discovered an obstruction to the existence of Kähler-Ricci soliton and proved that $X$ is in the center of a reductive Lie subalgebra $\eta_r(M)$ of $\eta(M)$, which is the set of all holomorphic vector fields. This study also gives us a result about the Kähler-Ricci soliton’s unicity (theorem 0.1 in [TZ00]).

Subsequently, the study was supplemented by Wang, Zhu in [WZ04] where they show the existence of Kähler-Ricci solitons on toric varieties using the continuity method. This work was supplemented by a study of the Ricci flow by Zhu in [Zhu12] on the toric varieties which showed that the Kähler-Ricci flow converges to the Kähler-Ricci soliton of the toric variety. The result about existence of Kähler-Ricci solitons has been extended to cases of toric fibrations by Podesta and Spiro in [PS10]. Recently, the result concerning the convergence of the Ricci flow has been also extended in [Hua17].

In 2015, Delcroix used the approach of Zhu and Wang in the case of Kähler-Einstein metrics on some wonderful compactifications of reductive groups. In his paper [Del15], the main result is a necessary and sufficient condition for the existence of a Kähler-Einstein metric in some group compactifications. The condition is that the barycenter of the polytope associated to the group compactification must lie in a particular zone of the polytope. The first tool used in the proof is a study of the $K \times K$-invariant functions (for the $KAK$ decomposition), in particular he computes the complex Hessian of a $K \times K$-invariant function. And the second tool is an estimate of the convex potential associated to a $K \times K$-invariant metric on ample line bundles. Then he proves the main result by reducing the problem to a real Monge-Ampère equation and by obtaining $C^0$ estimates along the continuity method. In our paper, we extend this result to some wonderful group compactifications in the following way:
Theorem 1.1. Let’s assume $M$ is a Fano smooth wonderful $G \times G$-compactification on a complex algebraic connected and reductive group $G$ such that $G \times G$ acts faithfully. There is a Kähler-Ricci soliton on $M$ excepted for exceptional cases cited in [Pez09] (the exceptional cases are the cases where $M$ does not satisfy $\text{Aut}^0(M) = G \times G$).

In our case, we do not use the barycenter of the polytope to get the a priori estimate. We will use the Futaki invariant (as in the articles [WZ04]) and compute this invariant in wonderful compactifications to get integral equations (similar to the toric case):

Theorem 1.2. Let’s assume $M$ satisfies the conditions of the theorem 1.1. Then $X$ is a solitonic vector field (modulo the action of $\text{Aut}^0(M)$) if and only if it is written

$$X = \sum_{j=1}^{r} a_j^1(\hat{a}_j^1 + \hat{s}_j^1) + a_j^2(\hat{a}_j^2 + \hat{s}_j^2),$$

where the vector fields $\hat{a}_j^1$ and $\hat{s}_j^1$ are induced by the action of $G \times G$ (see section 3.3 for more details). Moreover, the constants $a_j^i$ satisfy:

$$0 = \int_{\text{p}^+} p_i \exp \left( \sum_{j=1}^{r} a_j^i p_j \right) \prod_{\alpha \in \varphi^+} \alpha(p)^2 dp, \ \forall i = 1, \cdots, r$$

where we denote $a_j := a_j^1 + a_j^2$.

Using this result and general results of [Del15] on Monge-Ampère equations, we can prove the a priori estimate and extend the result of the existence to these wonderful compactifications thanks to the continuity method. In particular, we do not need any notion of K-stability to prove this result.

2. Structure of a reductive Lie algebra

Let $G$ be a complex algebraic group i.e. an algebraic subgroup of $GL_N(\mathbb{C})$ for $N > 0$. Furthermore, we suppose $G$ is reductive, that means if $K$ is a maximal compact subgroup of $G$ then $G$ is reductive if and only if $G$ is isomorphic to the complexification of $K$. Thus, if we denote the Lie Algebra of $G$ and $K$ by $\mathfrak{g}$ and $\mathfrak{k}$ respectively then we have $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$.

Moreover, $\mathfrak{g}$ admits the following decomposition (see for instance [Kna13]):

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

where $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of the derived subgroup $D(G)$ of $G$ and $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. We recall the Killing form $B$ is nondegenerate on $[\mathfrak{g}, \mathfrak{g}]$ and zero on $Z(\mathfrak{g})$. We can extend globally this form on $\mathfrak{g}$ by putting any nondegenerate bilinear form on $Z(\mathfrak{g})$ and assuming that $Z(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}]$ are orthogonal. We denote it by $\langle \cdot, \cdot \rangle$.

2.1. Root System.
2.1.1. Semisimple case. If \( \mathfrak{g} \) is a semisimple complex Lie algebra and \( \mathfrak{t} \) a Cartan subalgebra of \( \mathfrak{g} \) then \( \mathfrak{g} \) admits the following Cartan decomposition:

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,
\]

where \( \Phi \) is a finite subset of \( \mathfrak{t}^* \) called the root system of \( \mathfrak{g} \). Moreover, we have \( \mathfrak{g}_0 = \mathfrak{t} \) and \( \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid \text{ad}(h)(x) = \alpha(h) x \ \forall h \in \mathfrak{t} \} \) (see for instance [Kna13]). Also, using the nondegeneracy of the Killing form \( B \), we can associate to each root \( \alpha \in \Phi \) a unique element \( h_\alpha \in \mathfrak{t} \) such that \( \alpha(h) = B(h_\alpha, h) \) for all \( h \in \mathfrak{t} \). We set \( (\alpha, \beta) := B(h_\alpha, h_\beta) \). Finally, the root system satisfies the following properties:

1. \( E := \text{span}_\mathbb{R}(\Phi) \) is a real vector subspace of \( \mathfrak{t}^* \) and \( \dim_{\mathbb{R}} E = \dim_{\mathbb{C}} \mathfrak{t} \).
2. \( (\cdot, \cdot) \) is a positive definite form on \( E \).
3. \( \forall (\lambda, \alpha) \in \mathbb{R} \times \Phi, \lambda \alpha / \langle \lambda, \alpha \rangle \in \Phi \Rightarrow \lambda \in \{-1, 1\} \).
4. \( \forall (\beta, \alpha) \in \Phi^2, \beta - 2(\beta, \alpha) \alpha / (\alpha, \alpha) \in \Phi \).
5. \( \forall (\beta, \alpha) \in \Phi^2, 2(\beta, \alpha) / (\alpha, \alpha) \in \mathbb{Z} \).

2.1.2. Reductive case. We suppose \( G \) being reductive with maximal compact group \( K \). We can choose a maximal compact torus \( S \) of \( K \) such that we denote \( T \) its complexification. We can observe immediately that \( T \) is a maximal torus of \( G \). Then we define the rank \( r := \text{rank}(G) \) of the group \( G \) as the complex dimension of \( T \) (which is the real dimension of \( S \)) and we define the root system \( \Phi \) associated with the pair \( (G, T) \) as the root system of the semisimple part \( [\mathfrak{g}, \mathfrak{g}] \) of \( \mathfrak{g} \) with Cartan algebra \( \mathfrak{t}_{ss} \) corresponding to \( \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}] \). We obtain the same decomposition

\[
\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.
\]

We set \( \mathfrak{a} := \sqrt{-1} \mathfrak{s} \) and \( \mathfrak{A} := \exp \mathfrak{a} \). It can then be shown that

\[
\mathfrak{t} = \mathfrak{s} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{t}_\alpha,
\]

where \( \Phi^+ \) is the set of postives roots (set the section 2.2 for more details) and \( \mathfrak{t}_\alpha = \{ x \in \mathfrak{t} \mid \text{ad}(h)^2(x) = \alpha(h)^2 x, \ \forall h \in \mathfrak{g}_0 \} \) (see for instance [Kna13]).

2.2. Weyl Chamber. We keep the notations introduced in the previous section. We say that \( \Delta \subset \Phi \) is a set of simple roots if \( \Delta \) is a base of \( E \) such that for any root \( \alpha \in \Phi \) then the coordinates of \( \alpha \) in the base \( \Delta \) are all positive or negative. This implies a decomposition of \( \Phi \) into two disjoint sets \( \Phi = \Phi^+ \sqcup \Phi^- \). We then define the Weyl chamber \( \overline{\alpha}^+ \) by

\[
\overline{\alpha}^+ := \{ x \in \mathfrak{a} \mid \forall \alpha \in \Phi^+, \ \alpha(x) \geq 0 \}.
\]

We denote \( \mathfrak{A}^+ \) the image of \( \overline{\alpha}^+ \) by the exponential map. We observe that \( \mathfrak{A}^+ \) is a subspace of \( \mathfrak{A} \). We define the open Weyl chamber \( \alpha^+ \) as the interior of \( \overline{\alpha}^+ \). Moreover, for the rest of the article we define \( 2 \rho := \sum_{\alpha \in \Phi^+} \alpha \). Finally, we define Weyl group (with respect to the maximum torus \( T \)) by \( \mathcal{W} := N_G(T)/T \). This group acts on \( T \) and thus induces an action on \( \mathfrak{a} \) such that Weyl chamber \( \overline{\alpha}^+ \) is a fundamental domain for the action of \( \mathcal{W} \) on \( \mathfrak{a} \).
2.3. KAK decomposition. We always consider a reductive group $G$ and we keep the notations introduced previously. We recall here the decomposition $KAK$ (see for instance [Kna13]):

**Proposition 2.1.** [Kna13] Let $g \in G$. There is a triplet $(k_1, k_2, t) \in K^2 \times A$ such that $g = k_1 t k_2$. Furthermore in this decomposition, the element $t$ is unique modulo the action of $W$.

There are several consequences of this result. First of all, using the fact that $\mathfrak{a}^+$ is a fundamental domain for the action of $W$, we obtain that we can write $g = k_1 \exp(a) k_2$ where $a \in \mathfrak{a}^+$. It also means that the quotient of $G$ by the action both left and right of $K$ on $G$ is isomorphic to $A^+$.

Secondly, any $K \times K$-invariant function $f$ on $G$ depends only on its values on $A$. Thus, we can define a function $\tilde{f}$ on $\mathfrak{a}$ such that $\tilde{f}(a) = f(\exp a)$. We then have the following proposition:

**Proposition 2.2.** [FJ78] The correspondence $f \mapsto \tilde{f}$ gives a bijection between $K \times K$-invariant functions on $G$ (resp. smooth $K \times K$-invariant functions on $G$) and $W$-invariant functions on $\mathfrak{a}$ (resp. smooth $W$-invariant function on $\mathfrak{a}$).

Similarly, there is another result:

**Proposition 2.3.** [AL92] The correspondence $f \mapsto \tilde{f}$ gives a bijection between $K \times K$-invariant plurisubharmonic functions on $G$ (resp. smooth strictly plurisubharmonic $K \times K$-invariant functions on $G$) and $W$-invariant convex functions on $\mathfrak{a}$ (resp. smooth strictly convex $W$-invariant function on $\mathfrak{a}$).

Now, we can easily compute the integration of a $K \times K$-invariant function thanks to the following proposition:

**Proposition 2.4.** [Kna13] Let $dg$ denote a Haar measure on $G$, and $dx$ a Lebesgue measure on $\mathfrak{a}^+$, then there exists a constant $C > 0$ such that for all $K \times K$-invariant function $f$ on $G$:

$$\int_G f(g) dg = C \int_{\mathfrak{a}^+} J(x) f(\exp x) dx,$$

where

$$J(x) := \prod_{\alpha \in \Phi^+} \sinh^2(\alpha(x)).$$

Moreover, if $(l_i)_{i=1}^n$ is the base of $\mathfrak{g}$ of the section 2.1.2 then in a neighborhood $U_g$ of every point $g$, there exists a holomorphic coordinate system given by

$$(z_1, \cdots, z_n) \in \mathbb{C}^n \mapsto e^{z_1 l_1 + \cdots + z_n l_n} g \in U_g,$$

then we can compute the Hessian matrix $\text{Hess}_C(f)(g)$ and its determinant $\text{MA}_C(f)(g)$ for every $K \times K$-invariant function $f$ in this basis. Before summing up the result, we denote for each $f$ defined on $\mathfrak{a}$ the determinant of its real hessian matrix by $\text{MA}_R(f)$ and its gradient for the scalar product $\langle \cdot, \cdot \rangle$ by $\nabla f$.

**Theorem 2.5.** [Del15] Let a $K \times K$-invariant function $f$ on $G$. In the previous holomorphic coordinate system and for all $a \in \mathfrak{a}^+$, we have

$$\text{MA}_C(f)(\exp a) = \frac{1}{4^{r+p}} \text{MA}_R(\tilde{f})(a) \prod_{\alpha \in \Phi^+} \alpha(\nabla f(a))^2 \frac{1}{J(a)}.$$

Moreover, renormalizing the basis, we obtain

\[ MA_{\mathbb{C}}(f)(\exp a) = MA_{\mathbb{R}}(\tilde{f})(a) \prod_{\alpha \in \Phi^+} \alpha(\nabla f(a))^2 \frac{1}{J(a)}. \]

3. Group compactification

Let G an algebraic group. A variety X is a G-variety if X is equipped with an action of the algebraic group G which induces a morphism of variety.

3.1. Linearized line bundles on reductive groups.

Definition 3.1. A G-linearization of a line bundle L on a G-variety is a G-action on L such that the G-action on L lifts the G-action on X and the map between the fibers \( L_x \) and \( L_{gy} \) defined by the action of \( g \in G \) is linear.

In our case, we are interested in the case where G is a connected reductive group equipped with the action of \( G \times G \) acting by right and left translation i.e. \( (g_1, g_2) \cdot g = g_1 g g_2^{-1} \). Thus we have the following lemma :

Lemma 3.2. Any \( G \times G \) linearized line bundle L on G admits a \( G \times \{ e \} \)-equivariant trivialisation.

Proof. It is sufficient to take a non-zero element \( x \) in the fiber \( L_e \) over the neutral element \( e \) of G. The section \( s(g) := (g, e) \cdot x \) checks the requested properties. \( \square \)

3.2. Group compactifications. Let G be a connected complex reductive group.

Definition 3.3. A normal irreductible projective G×G-variety is called a G×G-equivariant compactification of G if X admits an open dense orbit under G×G equivariantly isomorphic to G on which G×G acts by left and right multiplication.

Now, we consider a G-linearized ample line bundle L on X. We can then associate to the couple \( (X, L) \) a unique polytope using the theory of toric varieties. We have, in fact, the following result from [AK05].

Proposition 3.4. [AK05] Let \( \tilde{T} \) be a reductive group which we denote a maximal torus of group of characters M and X a G×G-compactification. If we put Z the closure of \( \tilde{T} \) into X then \( (Z, L|_Z) \) is a polarized toric variety and we can associate it a polytope \( P \subset M \otimes \mathbb{R} \simeq \mathbb{R}^a \) W-invariant. Conversely, if we have a polytope \( P \) of complete dimension and W-invariant then there exists a G×G-compactification of G whose associated polytope is \( P \).

Now, we will be interested in wonderful compactifications.

Definition 3.5. A G×G-compactification X is called wonderful if it satisfies the conditions :

1. X is smooth,
2. \( X \setminus G \) is the union of smooth normal crossing prime divisors, with non-empty intersections,
3. the G×G-orbits in X are precisely the intersections of families of these divisors.
3.3. Automorphims of a wonderfull compactifications. Let’s first recall that the group of automorphisms of a compact complex manifold $M$ is a Lie group of finite dimension whose Lie algebra is $\eta(M)$.

We fix a connected and complex reductive group $G$. We consider a wonderful $G$-compactification $M$ such that $G \times G$ acts faithfully. This additional hypothesis make it possible, using the 3.1.2 theorem from [Pez09], to determine the group of automorphisms of $M$:

**Theorem 3.6.** [Pez09] If $M$ is a wonderful $G$-compactification such that $G \times G$ acts faithfully. Then we have $\text{Aut}^0(M) = G \times G$ except for exceptional cases cited in [Pez09].

Now, we assume that $\text{Aut}^0(M) = G \times G$ and therefore exclude the exceptions of our reasoning. The first consequence of this result is that the Lie algebra of the holomorphic vector fields is reductive. Indeed, $\text{Aut}^0(X)$ is a Lie group whose Lie algebra $\eta(M)$ is formed by the holomorphic vectors fields. Thus, we obtain

$$ (2) \quad \eta(M) \simeq \mathfrak{g} \oplus \mathfrak{g}, $$

where $\mathfrak{g}$ is the Lie algebra of $G$. We then conclude remembering that $G$ is reductive.

Recall that if $\text{Aut}^0(M)$ is reductive then it means that there exists a maximal compact subgroup $K(M)$ of $\text{Aut}^0(M)$ such that $\text{Aut}^0(M)$ is the complexification of $K(M)$. Let’s also take a maximal torus $S(M)$ in $K(M)$ and denote by $T(M)$ its complexification, we notice immediately that it is a maximal torus for $\text{Aut}^0(M)$.

These properties can be translated on the Lie algebras $\mathfrak{k}(M), \mathfrak{s}(M), \mathfrak{t}(M)$ of $K(M), S(M)$ and $T(M)$ respectively. In particular, we write $\mathfrak{a}(M) := \sqrt{-1}\mathfrak{s}(M)$. In addition, as $\eta(M) \simeq \mathfrak{g} \oplus \mathfrak{g}$, we have that every element $X \in \mathfrak{g}$ induces two vector fields $\tilde{X}^1, \tilde{X}^2 \in \eta(M)$ corresponding to one of the two factors of the previous decomposition.

3.4. Hermitian metric on line bundles. Let $X$ be a compact Kähler manifold and $L$ a line bundle on $X$. Let’s recall that a hermitian metric is the data for all $x \in X$ of a hermitian metric $h_x$ on the $L_x$ fiber of $L$. Moreover, we say that the metric is smooth if the application $x \mapsto h_x$ is smooth.

3.4.1. Local potential. Let’s now take a trivialization $s$ above an open $U \subset X$. This means that for all $x \in X$ the vector $s(x)$ is a base of $L_x$ and hermitian $h_x$ is summarized to give itself a positive real $a_x$ that will be equal to the squared norm of $s(x)$ with respect to the hermitian form $h_x$ i.e. $a_x = |s(x)|^2_{h_x}$. We then define the local potential of $h$ (with respect to the trivialization $s$) by $\varphi : x \in U \mapsto -\ln(|s(x)|^2_{h_x}) \in \mathbb{R}$. Let’s note that the metric $h$ is entirely determined by all its local potentials and that $h$ is smooth if and only if all its local potentials are smooth.

Let’s finish by saying that we can associate to a hermitian smooth metric a $(1,1)$-form $\omega_h$ called curvature of $h$. To do this, we define locally $\omega_h|_U = -\sqrt{-1} \partial \bar{\partial} \varphi$ where $\varphi$ is the local potential associated with the trivialization $s$ above $U$. We verify that $\omega_h|_U$ does not depend on the trivialization and thus define a global $(1,1)$-form. Moreover, we can show that $\omega_h \in c_1(L)$. We will also say that $L$ is positive curvature if there exists a metric $h$ such that $\omega_h$ is a Kähler form ([Dem]).

3.4.2. Global Potential. There is also a notion of global potential. To define it, we set a reference hermitian metric $h^0$ and for any hermitian metric $h$ we define the
function \( \psi \) on \( X \), called \textit{global potential of} \( h \) \textit{with respect to} \( h^0 \) by the following formula:

\[
|\xi|_h^2 = e^{-\psi(x)}|\xi|_{h^0}^2.
\]

Note that the function \( \psi \) satisfies the following relation (thanks to the \( \partial \bar{\partial} \)-lemma):

\[
\omega_{h^0} = \omega_h + \sqrt{-1} \partial \bar{\partial} \psi.
\]

3.4.3. \textit{Potentials in the case of compactifications.} Let \( G \) be a complex reductive group and a smooth \( G \times G \)-compactification \( X \) of \( G \). Let \( L \) be an ample \( G \times G \)-linearized line bundle. We will then identify the dense orbit of \( X \) with \( G \).

If we consider a Kähler form in \( c_1(L) \) then we can define the moment map \( \mu \) of the action of \( K \times K \) on \( (X,\omega) \) and we can show that \( \text{im}(\mu) \cap (a \oplus a)^* \) is a convex polytope known as polytope moment identified by Brion’s work (see \[Bri87\]) with the intersection of the polytope \( P \subset a^* \) with \( (a + a)^* \) where the latter are embedded in \( (a \oplus a)^* \) by identifying \( a^* \) with the antidiagonal of \( (a \oplus a)^* \).

Moreover, we know that there exists a \( G \)-equivariant trivialization (see lemma \[Del15\]) of \( L|_G \) and therefore we have a potential \( h \) on \( G \) with respect to \( s^0 \) defined by

\[
\Psi(z) := -\ln(|s^0(z)|_{h}^2).
\]

We note that if \( h \) is \( K \times K \)-invariant then \( \psi \) also is. We can then define a function \( \varphi : a \rightarrow \mathbb{R} \), called \textit{convex potential of} \( h \), by the formula \( \varphi(x) = \psi(\exp(x)) \). We remark that \( \varphi \) is indeed a convex function (which justifies terminology). We then have the following properties for the convex potential on the anticanonical bundle \(-K_X\) (see \[Del15\] for more details):

**Proposition 3.7.** \[Del15\] \textbf{Let} \( f : a \rightarrow \mathbb{R} \) \textbf{a convex potential of a} \( K \times K \)-\textit{invariant smooth hermitian metric} \( h \) \textbf{of positive curvature on} \(-K_X\). \textbf{Then we have}

1. \( f \) \textit{is} \( W \)-\textit{invariant}
2. \( \nabla f(a) = \text{int}(2P) \)
3. \( \nabla f(a^+) = \text{int}(2P^+) \)
4. \( |\nabla f| \leq d \) \textit{for some constant} \( d \) \textit{independent of} \( f \)
5. \( f(x) \leq v(x-x_0) + f(x_0) \) \textit{for any} \( x_0 \in a \) \textit{and where} \( v \) \textit{is the support function of the polytope} \( 2P \).
6. \( f(x) \geq v(x) + C_1 \) \textit{for some constant} \( C_1 \) \textit{depending on} \( f \).

4. \textbf{The vector field of a Kähler-Ricci soliton}

In this section, a holomorphic invariant is defined and calculated in the case of wonderful compactification. This invariant is an obstruction to the existence of a Kähler-Ricci Soliton.

**Definition 4.1.** \textbf{Let} a \( n \)-\textit{dimensional compact Kähler manifold} \((M,g)\) \textbf{with positive first Chern class} \( c_1(M) \) \textbf{such that} its Kähler form \( \omega_g \in c_1(M) \). \textbf{Then, for any holomorphic vector field} \( X \in \eta(M) \), \textbf{we define the linear functional} \( F_X \), \textbf{called Futaki invariant, by}

\[
F_X : v \in \eta(M) \mapsto \int_M v(h_g - \theta_X) e^{\theta_X} \omega^n_g \in \mathbb{C},
\]

\textit{where we denote :}
• \( h_g \) is the unique function in \( C^\infty(M, \mathbb{R}) \) such that
\[
\text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} h_g, \quad \int_M e^{h_g} \omega_g^n = \int_M \omega_g^n,
\]

• \( \theta_X \) is the unique function in \( C^\infty(M, \mathbb{R}) \) such that
\[
i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \theta_X, \quad \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n.
\]

A first remark is that, according to proposition 1.1 of [TZ02], the invariant does not depend on the chosen metric \( g \).

Now, if \( K(M) \) is a maximal compact subgroup of \( \text{Aut}^e(M) \) which is the identity component of \( \text{Aut}(M) \), then the decomposition of Chevalley gives us that
\[
\text{Aut}^e(M) = \text{Aut}_r(M) \times \mathbb{R}_u,
\]
where \( \text{Aut}_r(M) \) is a reductive subgroup of \( \text{Aut}^e(M) \) and the complexification of \( K(M) \) and \( \mathbb{R}_u(M) \) the unipotent radical of \( \text{Aut}^e(M) \). Moreover, if \( \eta(M) \), \( \eta_r(M) \), \( \eta_u(M) \) and \( \kappa(M) \) are the Lie algebras of \( \text{Aut}(M) \), \( \text{Aut}_r(M) \), \( \mathbb{R}_u(M) \) and \( K(M) \) respectively, then we have
\[
\eta(M) = \eta_r(M) \oplus \eta_u(M).
\]

After these notations, we have this fundamental proposition:

**Proposition 4.2.** There exists a unique holomorphic vector field \( X \in \eta_r(M) \) with \( \text{Im}(X) \in \kappa(M) \) such that the holomorphic invariant \( F_X \) vanishes on \( \eta_r(M) \). Moreover, \( X \) is either zero or an element of the center of \( \eta_r(M) \) and
\[
F_X([u,v]) = 0, \quad \forall (u,v) \in \eta_r(M) \times \eta(M).
\]

Now, if \( M \) is a smooth wonderful \( G \times G \)-compactification such that \( G \times G \) acts faithfully then we have
\[
\eta(M) = \eta_r(M).
\]
Furthermore, we have \( \mathfrak{z}(\eta(M)) \subset \mathfrak{t}(M) = \mathfrak{s}(M) \oplus \mathfrak{a}(M) \). Now, using the logarithmic coordinates \((w_1, w_2, \ldots, w_d) = (x_1 + \sqrt{-1} \theta_1, \ldots, x_d + \sqrt{-1} \theta_d)\), we obtain that
\[
\mathfrak{s}(M) = \bigoplus_{i=1}^d \mathbb{R} \cdot \frac{\partial}{\partial \theta_i},
\]
so
\[
X = \sum_{i=1}^d c_i \frac{\partial}{\partial w_i}, \quad c_i = r_i + \sqrt{-1} t_i \in \mathbb{C}.
\]

Thus, we get
\[
X = \sum_l (r_l + \sqrt{-1} t_l) \left( \frac{\partial}{\partial x^l} + \sqrt{-1} \frac{\partial}{\partial y^l} \right)
\]
\[
= \sum_l \left( r_l \frac{\partial}{\partial x^l} - t_l \frac{\partial}{\partial y^l} \right) + \sqrt{-1} \sum_l \left( t_l \frac{\partial}{\partial x^l} + r_l \frac{\partial}{\partial y^l} \right).
\]

Finally, we have
\[
\text{Im}(X) = \sum_l \left( t_l \frac{\partial}{\partial x^l} + r_l \frac{\partial}{\partial y^l} \right).
Yet $\text{Im}(X) \in s(M)$ which is generated by the family $(\partial_{\alpha})$, so we have $t_i = 0$ for all $i \in \{1, \cdots, n\}$. This is summarized in the following proposition:

**Proposition 4.3.** Keeping the previous notations, the vector field $X$ canceling the Futaki invariant belongs to $\mathfrak{t}(M)$ and is written in the form

$$X = \sum_{i=1}^{d} a_i (\hat{A}_i + \sqrt{-1} J \hat{A}_i), \quad a_i \in \mathbb{R},$$

where $(\hat{A}_i)_{i=1,\cdots,d}$ is a real basis of $\mathfrak{a}(M)$ and $J$ the complex structure of $M$.

Now, using the structure of the $G$ group studied in the section 2.1.2, we get

$$\mathfrak{a}(M) = \mathfrak{a} \oplus \mathfrak{a},$$

and thus by setting $(a_i)_{i=1}^{r}$ a real basis of $\mathfrak{a}$, we obtain that

$$X = A^1 + A^2,$$

where for $i = 1, 2$

$$A^i = \sum_{j=1}^{r} a^i_j (a_j + \sqrt{-1} a_j), \quad a^i_j \in \mathbb{R}.$$

### 4.1. Futaki invariant in the wonderful case.

In this section, we want to compute the Futaki invariant in the wonderful case. To do this, we first must compute $\theta_X$.

#### 4.1.1. Computation of $\theta_X$.

We fix a smooth $G \times G$-wonderful Fano compactification $M$ such that $G \times G$ acts faithfully. We consider a $K \times K$-invariant Kähler form $\omega$. In particular, we know that on the dense open $M^0$ isomorphic to $G$ there exists a potential $K \times K$-invariant $\varphi \in C^\infty(G, \mathbb{R})$ such that

$$\omega|_G = \sqrt{-1} \partial \bar{\partial} \varphi.$$

Moreover, if we denote $(l_i)_{i=1}^{n}$ a complex basis of $\mathfrak{g}$ then in a neighborhood $U_g$ of any point $g$, there is a holomorphic coordinate system given by

$$(z_1, \cdots, z_n) \in \mathbb{C}^n \mapsto e^{z_1 l_1 + \cdots + z_n l_n} \quad g \in U_g,$$

then

$$\omega_g = \sum_{i=1}^{n} \sqrt{-1} u_{ij} \, dl^i \wedge d\bar{l}^j,$$

where $u : \mathfrak{a} \to \mathbb{R}$ is the convex potential $\varphi$ i.e. $u(a) = \varphi(\exp a)$. Now, if $v \in \eta(M)$ then it can be written locally $v = v_i \frac{\partial}{\partial l_i}$ and

$$i_v \omega_g = \sum_{i,j=1}^{r} \sqrt{-1} u_{ij} v_j \, d\bar{l}^i.$$

Now, we take $v = \hat{l}_1$ then $v = \frac{\partial}{\partial \bar{l}^1}$ on these coordinates and we have for $j = 1, \cdots, r$

$$\frac{\partial \theta_{\hat{l}_1}}{\partial \bar{l}^j} = \frac{\partial}{\partial \bar{l}^j} \left( \frac{\partial \varphi}{\partial \bar{l}^1} \right).$$
In particular, as \( \theta_{\hat{l}i} \) is bounded on \( M \) thus on \( G \) and \( \text{im}(\nabla \varphi) = 2P \) (so \( \nabla \varphi \) is bounded too), we obtain for \( i = 1, \cdots, n \)
\[
\theta_{\hat{l}i}|_G = \hat{l}^1_i(\varphi) + \phi^1_i, \ \phi^1_i \in \mathbb{R}.
\]
Similarly, by taking the coordinate system:
\[
(z_1, \cdots, z_n) \in \mathbb{C}^n \mapsto g e^{zi_1 + \cdots + z_n} \in U_g,
\]
we have
\[
\theta_{\hat{l}2i}|_G = \hat{l}^2_i(\varphi) + \phi^2_i, \ \phi^2_i \in \mathbb{R}.
\]
Now recall \( g \) is the complexification of \( k \) and so the complexification of \( \sqrt{-1}k \) i.e. \( g = \sqrt{-1}k \otimes \mathbb{C} \). Moreover, \( G \) admits the following decomposition :
\[
\mathfrak{g} = \mathfrak{s} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{f}_\alpha,
\]
thus
\[
\mathfrak{g} = \left( \mathfrak{a} \oplus \mathfrak{t} \right) \otimes \mathbb{C}.
\]
Now, taking a basis of \( \mathfrak{g} \) adapted of this decomposition, we can suppose \( l_i := a_i \otimes 1 \) for \( i = 1, \cdots, r \) where the elements \( a_i \) form a basis of \( \mathfrak{a} \). Now, thanks to the \( K \times K \)-invariance of \( \varphi \), we obtain
\[
\hat{l}^j_i(\varphi) = 0, \ \forall i = r + 1, \cdots, n, \ j = 1, 2
\]
and by \( K \times K \)-invariance
\[
\hat{l}^j_i(\varphi) = \frac{\partial u}{\partial a_i}, \ \forall i = 1, \cdots, r, \ j = 1, 2.
\]

4.1.2. Futaki invariant reformulation. For this section, the reference article is [TZ02].

To compute the Futaki invariant, it is preferable to renormalize the function \( \theta_X \) to a function \( \tilde{\theta}_X \) by requesting that it checks
\[
i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\theta}_X, \ \Delta \tilde{\theta}_X + X(h_g) = -\tilde{\theta}_X.
\]
This condition is equivalent to
\[
\int_M \tilde{\theta}_X e^{\tilde{\theta}_X} \omega^n_g = 0,
\]
and we obtain
\[
F_X(v) = -\int_M \tilde{\theta}_v e^{\tilde{\theta}_X} \omega^n_g.
\]
In particular, it can be seen that in the case where \( X = \hat{l}^j_i \) for \( i = 1, 2 \) and \( j = 1, \cdots, r \) then
\[
\tilde{\theta}_{\hat{l}^j_i}|_G = \tilde{\theta}^j_i(\varphi).
\]
4.1.3. Computation of the Futaki Invariant. We can then compute the Futaki invariant directly for the vectors $\tilde{l}^i_j$ with $i = 1, \cdots, r$:

$$F_X(\tilde{l}^i_j) = \int_M \tilde{\theta} e^{\delta x} \omega^n$$

$$= \int_G \tilde{a}^i_j(\varphi) \exp \left[ \sum_{i=1}^r \sum_{j=1}^r \tilde{a}^i_j(\varphi) \right] MAC(\varphi) dg$$

$$= C \int_{a^+} \left( \frac{\partial u}{\partial a_i} \exp \left[ \sum_{j=1}^r (a^1_j + a^2_j) \frac{\partial u}{\partial a_i} \right] \right) MAC(\varphi) J(a) da$$

$$= C \int_{2a^+} \left[ \sum_{j=1}^r a_j p_j \prod_{\alpha \in \phi^+} \alpha(p)^2 dp, \forall \alpha = 1, \cdots, r \right]$$

where $a_j := a^1_j + a^2_j$. Now, using the proposition [4.2], we obtain:

**Theorem 4.4.** Let’s assume $M$ is a Fano smooth wonderful $G \times G$-compactification on a complex algebraic connected and reductive groupe $G$ such that $G \times G$ acts faithfully and $M$ is not an exceptional case cited in [Pez09]. Then $X \in \eta(M)$ is a solitonic vector field (modulo the action of $Aut^0(M)$) if and only if it is written

$$X = \sum_{j=1}^r a_j^1 (\tilde{a}^1_j + \tilde{s}^1_j) + a_j^2 (\tilde{a}^2_j + \tilde{s}^2_j),$$

where $(a_j)_{j=1,\ldots,r}$ is a real basis of $a$, $s_j = \sqrt{-1} \alpha_j \in s$ for all $j \in \{1, \cdots, r\}$ and the constants $a_j^i$ satisfy:

$$0 = \int_{2a^+} p_i \exp \left[ \sum_{j=1}^r a_j p_j \prod_{\alpha \in \phi^+} \alpha(p)^2 dp, \forall \alpha = 1, \cdots, r \right]$$

where we denote $a_j := a^1_j + a^2_j$.

5. Existence of soliton in the wonderful case

In this section, we want to prove the following theorem:

**Theorem 5.1.** Let’s assume $M$ is a Fano smooth wonderful $G \times G$-compactification on a complex algebraic connected and reductive groupe $G$ such that $G \times G$ acts faithfully. There is a Kähler-Ricci soliton on $M$ excepted for exceptional cases cited in [Pez09] (the exceptional cases are the cases where $M$ does not satisfy $Aut^0(M) = G \times G$).

5.1. Monge Ampère equation in the wonderful case. We fix a compact Fano manifold $(M, g^0)$ with $\omega^0 \in c_1(M)$ such that $(X, g)$ is a Kähler-Ricci soliton i.e.

$$Ric(\omega) - \omega = L_X \omega.$$

Thanks to the $\partial \bar{\partial}$-lemma, there is a function $\psi$ such that

$$\omega = \omega^0 + \sqrt{-1} \partial \bar{\partial} \psi.$$
Noting \( \theta_X(g) = \theta_X(g^0) + X(\phi) \), it is shown (WZ04 for instance) that solving the Kähler-Ricci soliton equation is equivalent to finding a potential \( \psi \) solution of the following Monge-Ampère equation:

\[
\begin{cases}
\det(g_{i\overline{j}}^0 + \psi_{i\overline{j}}) = \det(g_{i\overline{j}}^0) \exp(h - \theta_X(g^0) - X(\psi) - \psi) \\
(g_{i\overline{j}}^0 + \psi_{i\overline{j}}) > 0.
\end{cases}
\]

Moreover, if we fix a hermitian metric \( m^0 \) on \( -K_M \) such that \( \omega_{m^0} = \omega_{g^0} \) then we can define a volume form \( dV \) given in a local trivialisation \( s \) of \( -K_M \) by \( dV = |s|_{m^0}s^{-1} \wedge \overline{s}^{-1} \) then modulo a constant we obtain that \( h \) is equal to the logarithm of the potential of \( dV \) with respect to \( \omega_{g^0}^m \), so we renormalize to match it. Another way to write the first equation of (5) is then:

\[
(\omega_{g^0}^n + \partial \overline{\partial} \psi)^n = e^{h - \psi - \theta_X(g^0) - X(\psi)} \omega_{g^0}^n.
\]

We keep the previous notations but we consider that in addition the variety \( M \) is a variety of Fano which is a smooth \( G \times G \)-compactification of \( G \) such that \( G \times G \) acts faithfully and we denote by \( P \) the polytope associated with the bundle \( -K_M \). By continuity of the metrics and the potentials, it suffices to study the equation of Monge-Ampère in restriction to the dense open isomorphic to \( G \). In this case, we have the existence (see lemma 3.2) of a \( G \)-invariant section \( s \) of the line bundle \( -K_X \) on \( G \) such that

\[
(\omega_{g^0}^n)|_G = (\sqrt{-1} \partial \overline{\partial} \psi_0)^n = MA_C(\psi_0)s^{-1} \wedge \overline{s}^{-1}
\]

so

\[
(\omega_{g}^n)|_G = MA_C(\psi^0 + \psi)s^{-1} \wedge \overline{s}^{-1}.
\]

Moreover, we consider only the \( K \times K \)-invariant metrics. It is therefore assumed that \( m^0 \) is \( K \times K \)-invariant and we note \( u^0 \) the convex potential associated with \( m^0 \) i.e. \( u^0(a) = \psi(\exp(a)) \). Similarly, we suppose that \( \varphi \) is \( K \times K \)-invariant and we denote by \( \varphi \) its associated convex potential i.e.

\[
\varphi(a) := \psi(\exp(a)).
\]

Note then that \( \omega_g \) is \( K \times K \)-invariant and that its convex potential \( u \) is then \( u = u^0 + \varphi \). Thus we obtain

\[
\omega_g^n|_G = MA_R(u(a)) \frac{1}{J(a)} \prod_{\alpha \in \Phi^+} \alpha (\nabla u(a))^2 s^{-1} \wedge \overline{s}^{-1}.
\]

Moreover, by definition of \( h \) and by the renormalisation chosen, we have

\[
e^{h} \omega_g^n|_G = e^{-\psi_0} s^{-1} \wedge \overline{s}^{-1},
\]

so we finally get

\[
e^{h - \psi - \theta_X(g^0) - X(\psi)} \omega_{g^0}^n = \exp \left[-u - \sum_{l=1}^{r} a_l \frac{\partial u}{\partial a_l}\right].
\]

So we must solve the following equation:

\[
MA_R(u(a)) \cdot \prod_{\alpha \in \Phi^+} (\alpha (\nabla u(a))^2 = J(a) \cdot \exp \left[-u - \sum_{l=1}^{r} a_l \frac{\partial u}{\partial a_l}\right].
\]
5.2. The continuity method. We now want the existence of Kähler-Ricci solitons in the wonderful case. To do this, we will use the method of continuity which we now recall the approach.

To begin with, we introduce into the Monge-Ampère equation a parameter \( t \in [0,1] \) :

\[
\begin{cases}
\det(g^{\alpha\beta}_{ij} + \psi g_{ij}) = \det(g^{\alpha\beta}_{ij}) \exp(h - \theta X - X(\psi) - t\psi) \\
(g^{\alpha\beta}_{ij} + \psi g_{ij}) > 0.
\end{cases}
\]

We note that the equation \( [5] \) is the previous equation with \( t = 1 \). Moreover, if a solution exists at time \( t \), we denote it by \( \psi_t \). Now, if \( \psi_t \) is \( K \times K \)-invariant, it has a convex potential \( \varphi_t \). Thus, setting \( u_t = u^0 + \varphi_t \) and \( w_t = t \cdot u_t + (1-t) \cdot u^0 \), we can write this equation on the dense orbit as :

\[
\begin{align*}
(9) & \quad \text{MA}_R(u_t) \cdot \prod_{\alpha \in \Phi^+} (\alpha(\nabla u_t(a)) \cdot J(a)^2 = J(a) \cdot \exp \left[ -w_t(a) - \sum_{i=1}^r \alpha_i \frac{\partial u_t}{\partial a_i}(a) \right].
\end{align*}
\]

Moreover, setting \( j(a) = -\ln(J(a)) \) and \( \nu_t(a) = w_t(a) - j(a) \), we get finally :

\[
(10) & \quad \text{MA}_R(u_t) \cdot \prod_{\alpha \in \Phi^+} (\alpha(\nabla u_t(a)) \cdot J(a)^2 = \exp \left[ -\nu_t(a) - \sum_{i=1}^r \alpha_i \frac{\partial u_t}{\partial a_i}(a) \right].
\]

The method of continuity consists in considering the set \( S \) of times when there exists a solution:

\[
S := \{ t \in [0,1] / \text{There is a solution } u_t \text{ of the equation } (5) \text{ at the time } t \},
\]

and showing that \( S \) is a close open and nonempty set of \([0,1]\).

The openness and existence of a solution at time \( t = 0 \) comes from the study of the Monge-Ampère equations made in [Anb78, Yau78]. We can also consult [TZ00] for a study made in the case of the Kähler-Ricci solitons. Moreover, thanks to the Arzelà-Ascoli theorem, it suffices to have an a priori estimate \( C^3 \) of the potentials \( \psi_t \) and thus the potentials \( \varphi_t \) to obtain that \( S \) is close. Now, thanks to the works of Yau and Calabi made in appendix A of [Yau78], we can reduce this estimate \( C^3 \) to an estimate \( C^0 \). Moreover, by the following Harnack inequality (see [TZ00, WZ04] for instance)

\[
-\inf_M \varphi_t \leq C(1 + \sup_M \varphi_t),
\]

we can reduce to a uniform upper bound for the \( \varphi_t \).

6. Proof of the a priori estimation

We must a priori find an estimate for \( t \in [0,1] \). Now, using the fact that \( 0 \in S \) and \( S \) is open, one can reduce to show an estimate on \([t_0,1]\) for \( t_0 > 0 \). We set such a \( t_0 \) for the rest. Moreover, we denote by \( t \) the notations introduced in the 5.1 section for the solutions of the equations \([5.9, 10]\).

6.1. The steps of the proof. The approach is inspired by the works [WZ04, Del15] and splits in several points:

- We show that the function \( u_t \) admits a unique minimum \( m_t \) reached in \( x_t \).
- We show that to find an estimate of \( u_t \) is equivalent to find a uniform estimate of \( |x_t| \) and \( |m_t| \).
- We find the estimates \( |x_t| \) and \( |m_t| \).
The first step comes from section 6.2.2 of [Del15] which does not depend on the solitonic Monge-Ampère equation. We just recall the results that will be needed later.

**Lemma 6.1.** [Del15] The function $j$ is strictly convex on $\mathfrak{a}^+$ and so the function $\nu_t$ is a strictly convex function which admits a minimum $m_t$ in $x_t$. Moreover, there is a constant $b_1 > 0$ independent of $t$ such that $x_t \in b_1 \rho + \mathfrak{a}^+$ and for any $b > 0$, there exists a constant $C$ such that for any $x \in b \rho + \mathfrak{a}^+$,

$$|\nabla(j)(x)| \leq C,$$

and for any $M > 0$, there exists a constant $b > 0$ independent of $t$ such that for any $x \in \mathfrak{a}^+$ satisfying $\alpha(x) < b \alpha(p)$ for some root $\alpha \in \Phi^+$ defining a wall of $\mathfrak{a}^+$, we have

$$\nu_t(x) \geq m_t + M.$$
To conclude, it suffices to note that $x_t$ is contained in the closed ball of radius $C_x$ independent of $t$ centered at the origin and therefore as $j$ and $u^0$ are continuous, we can bound these functions on this ball which will therefore not depend on $t$. □

6.3. Uniform estimate of $|m_t|$. In this section, we prove the following lemma:

**Lemma 6.3.** We have

$$m_t = \inf_{x \in \mathbb{R}^n} u_t(x) \leq C,$$

where $C > 0$ is independent of $t \in [\varepsilon_0, 1]$.

Before starting the proof, we recall a result concerning the convex domains which will be used in the proof:

**Lemma 6.4.** [WZ04, Guz75, Gut01] Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. Then there is a unique ellipsoid $E$, called the minimum ellipsoid of $\Omega$, which attains minimum volume among all ellipsoids containing $\Omega$, such that

$$\frac{1}{n} E \subset \Omega \subset E.$$

Let $T$ be a linear transformation with $|T| = 1$, which leaves the center $x_0$ of $E$ invariant, namely $T(x) = A(x - x_0) + x_0$ for some matrix $A$, such that $T(E)$ is a ball $B_R$. Then we have $B_{R/\varepsilon} \subset T(\Omega) \subset B_{R}$ for two balls with concentrated center.

Now we can prove the lemma 6.3.

**Proof.** We set

$$A_k := \{ x \in \mathbb{R}^n / m_t + k \leq \nu_t(x) \leq m_t + k + 1 \},$$

And then we have the following elementary properties:

- $A_k$ is bounded for $k \geq 0$ et $\cup_k A_k = \mathbb{R}^n$.
- $m_t \in A_0$.
- $\cup_{i=0}^k A_i$ is a convex set for $k \geq 0$.

Moreover, since $u$, $u^0$ and $j$ are convex, we have that $(u_{ij})$ and $(u^0_{ij})$ are positive matrices. In particular, there is a linear algebra result which gives us

$$\det(\nu_{ij}) = \det(tu_{ij} + (1 - t)u^0_{ij} + (j_{ij}) \geq \det(tu_{ij}) + \det((1 - t)u^0_{ij}) + \det(j_{ij}),$$

and

$$\det(\nu_{ij}) \geq \det(t \cdot u_{ij})$$

$$\geq t^n \cdot \det(u_{ij})$$

$$\geq t^n \cdot e^{-c-d} \cdot e^{-w}$$

(thanks to 10)

where $d = \sup\{c_t y_t / y \in 2P\}$ and $c > \log \sup\{\prod_{\alpha \in \Phi^+} (\alpha(p))^2 / p \in 2P\}$. But $t \in [\varepsilon_0, 1]$, so we get

$$\det(\nu_{ij}) \geq C_0 e^{-m_t} \text{ in } A_0,$$

where $C_0 = t^n_0 e^{-c-d-1}$. Using the lemma 6.4, there exists a linear transformation $y = T(x)$ with $|T| = 1$ and leaving the center of the minimal ellipsoid of $A_0$ invariant,

$$B_{R/\varepsilon} \subset T(A_0) \subset B_R,$$
and thus preserving the previous inequality. Moreover, we have
\[ R \leq \sqrt{2} r C_0^{-1/2} e^{m_t/2r}. \]

Indeed, we set the map
\[ v : y \in \mathbb{a}^+ \mapsto \frac{1}{2} C_0^{1/r} e^{m_t/r} [||y - y_t||^2 - \left( \frac{R}{r} \right)^2] + m_t + 1 \in \mathbb{R} \]
where \( y_t \) is the center of the minimal ellipsoid of \( A_0 \). A computation gives us that
\[ \det(v_{ij}) = C_0 e^{-m_t} \text{ on } T(A_0), \]
and \( v \geq \nu \) on \( \partial T(A_0) \) thus on \( T(A_0) \) thanks to the comparison principle. In particular, we get
\[ m_t \leq \nu_t \leq \nu(y_t) = -\frac{1}{2} C_0^{1/r} e^{m_t/r} \left( \frac{R}{r} \right)^2 + m_t + 1. \]

Now, thanks to the convexity of \( w \), we get
\[ T(A_k) \subset B_{2(k+1)R}, \]
and
\[ \bigcup_k A_k = \mathbb{a}^+. \]
Furthermore, \( T \) is affine isometry of \( \mathbb{R}^r \) so is isomorphism thus the family \( \{ T(A_k) \}_{k \in \mathbb{N}} \) is a cover of \( \mathbb{a}^+ \). Now, if we denote \( \omega_r \) the area of the sphere \( S_{r-1} \) then we have
\[ \int_{\mathbb{a}^+} e^{\nu_t} \leq \sum_k \int_{T(A_k)} e^{-\nu_t} \leq \sum_k e^{-m_t-k} |T(A_k)| \leq \omega_r \sum_k e^{-m_t-k} |2(k+1)R|^r = \omega_r \frac{(2R)^r}{e^{m_t}} \sum_k \frac{(k+1)^r}{e^k} \leq C e^{m_t/2}. \]

We note that the above integration is invariant under any linear transformation \( T \) with \( |T| = 1 \) so
\[ e^{m_t/2} \geq \frac{1}{C} \int_{\mathbb{a}^+} \prod_{\alpha \in \Phi^+} \alpha(\nabla u_t(a))^2 e^{-\nu_t(a)} da. \]
Moreover, using the equation \( 10 \) we get :
\[ e^{m_t/2} \geq \int_{\mathbb{a}^+} \prod_{\alpha \in \Phi^+} \alpha(\nabla u_t(a))^2 \det((u_t)_{ij}(a)) \exp(\sum_i a_i \frac{\partial u_t}{\partial a_l}(a)) da. \]
Finally, we obtain
\[ e^{m_t/2} \geq \frac{1}{C} \int_{2P^+} \prod_{\alpha \in \Phi^+} \alpha(p)^2 \cdot \exp(\sum_i a_i p_i) dp =: \beta, \]
is independent of $t$ (thanks to the last equality). In the end, we obtain, passing to the logarithm, that
\[ m_t \leq C, \]
where $C$ is a positive constant independent of $t$.

It remains to show a uniform minoration. For this, we remark
\[
\int_{a^+} e^{\nu_t(a)} da = \int_{a^+} \int_{\nu_t(x)}^{+\infty} e^{-s} ds da
\]
\[
= \int_{-\infty}^{+\infty} e^{-s} \int_{a^+} \mathbf{1}_{\nu_t(a) \leq s} ds da
\]
\[
= \int_{-\infty}^{+\infty} e^{-s} \text{Vol}\{\nu_t \leq s\} ds da
\]
\[
= \int_{-m_t}^{+\infty} e^{-s} \text{Vol}\{\nu_t \leq s\} ds da
\]
\[
= e^{-m_t} \int_0^{+\infty} e^{-s} \text{Vol}\{\nu_t \leq m_t + s\} ds da.
\]

Now, thanks to the convexity of $\nu_t$, we get $\{\nu_t \leq m_t + s\} \subset s \cdot A_0$, and thus
\[
\int_{a^+} e^{\nu_t(a)} da \geq e^{m_t} \int_0^{+\infty} e^{-s} S^r \text{Vol}(A_0) ds
\]
\[
\geq e^{m_t} \int_1^{+\infty} e^{-s} S^r \text{Vol}(A_0) ds
\]
\[
\geq e^{m_t} \text{Vol}(A_0) \int_1^{+\infty} e^{-s} ds.
\]

Moreover, thanks to the equation 9 we have
\[
\int_{a^+} e^{\nu_t(a)} da = \beta
\]
where $\beta$ is independent of $t$. Finally, we get
\[
m_t \geq \ln(\beta) - \ln(\text{Vol}(A_0)) \int_1^{+\infty} e^{-s} ds.
\]

To conclude, we must bound above uniformly $\text{vol}(A_0)$ and it is the result of the following lemma.

**Lemma 6.5.** [Del15] There exists a constant $c > 0$ independent of $t$ such that
\[
\text{Vol}(A_0) \geq c.
\]

**Proof.** The proof is taken from [Del15]. There exists a constant $b_2$ independent of $t$ such that $0 < b_2 < b_1$ and $A_0 \subset b_2 \rho + a^+$. This is a corollary of [6.1] taking $b_2$ corresponding to $M = 1$. Indeed, by lemma [6.1] and proposition [5.7] on $b_2 \rho + a^+$, $|\nabla(\nu_t)|$ is bounded independently of $t$, say by $M$. Then it is clear that the ball $B(x_t, M)$ is contained in $A_0$. So $\text{Vol}(A_0) \geq \text{Vol}(B(x_t, M) = \text{Vol}(B(0, M)) = c$, for some $c > 0$ independent of $t$.

Using the convexity of $\nu_t$, we obtain the following corollary:
Corollary 6.6. There are constants \( \kappa > 0 \) and \( C > 0 \) independent of \( t \) such that for every \( a \in a^+ \)

\[
\nu_t(a) \geq \kappa |x - x_t| - C.
\]

Thus, for all \( \varepsilon > 0 \), there is \( \delta > 0 \) independent of \( t \) such that

\[
\int_{a^+ \setminus B(x_t, \delta)} e^{-\nu_t(a)} da < \varepsilon.
\]

Proof. The proof is taken from [Del15]. The two previous lemmas tell us that there are two constants \( C_1 > 0 \) and \( C_2 > 0 \) independent of \( t \) such that

\[
C_2 \leq \text{Vol}(A_0) \leq C_1.
\]

By the proof of lemma 6.5, we know that \( \delta_0 > 0 \) is independent of \( t \) such that \( B(x_t, \delta_0) \subset A_0 \) so we can find \( \delta \) independent of \( t \) and depending only on \( \delta_0 \) and \( C_1 \) such that \( A_0 \subset B(x_t, \delta) \). By convexity of \( \nu_t \), one obtains that

\[
\nu_t(a) \geq \frac{1}{\delta} |x - x_t| + m_t, \quad \forall x \in a^+ \setminus B(x_t, \delta).
\]

We conclude by remarking that this last inequality can be extended to the whole space \( a^+ \) by subtracting 1 i.e.

\[
\nu_t(a) \geq \frac{1}{\delta} |x - x_t| + m_t - 1, \quad \forall x \in a^+.
\]

\[\square\]

6.4. Uniform estimate of \( |x_t| \). We have the following lemma:

Lemma 6.7. Let \( x^t = (x^t_1, \ldots, x^t_n) \) be the minimal point of \( \nu_t \). Then

\[
|x^t| \leq C,
\]

where \( C \) is a uniform constant.

Proof. Let’s first note that for \( i \in \{1, \ldots, r\} \), we have

\[
\int_{a^+} \frac{\partial u_t}{\partial a_i} e^{-\nu_t} dx = 0,
\]

and by linearity:

\[
0 = \int_{\mathbb{R}^n} \frac{\partial u_t}{\partial \xi} e^{-\nu_t} d\xi,
\]

for every unit vector \( \xi \in a^+ \). Indeed, we have, thanks to the equation 4 that:

\[
0 = \int_{\Omega} \prod_{\alpha \in \phi^+} \alpha(p)^2 \cdot p_i \exp(\sum_{l=1}^{n} a_i y_l) dp,
\]

\[
0 = \int_{a^+} \prod_{\alpha \in \phi^+} \alpha(\nabla(u_t)(x))^2 \cdot \frac{\partial u_t}{\partial a_i} \exp(\sum_{l=1}^{n} a_i \frac{\partial u_t}{\partial a_l}) \det((u_t)_{pq}) dx
\]

\[= \int_{a^+} \frac{\partial u_t}{\partial a_i} e^{-\nu_t} dx \quad \text{ (thanks to [10]).} \]
Moreover, by the corollary [6.6] we know that for every $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
\int_{a^+ \setminus B(x_1, \delta)} e^{-\nu_t(a)} da < \varepsilon.
\]
Thus, setting $d_0 := \sup_{x \in \mathbb{R}^n} \{ |x| / x \in 2P \}$ and remembering that $\text{im}(\nabla u) = 2P$, we get that for all $\xi \in a^+$:
\[
\forall \varepsilon > 0, \exists \delta > 0, \int_{a^+ \setminus B(x_1, \delta)} \partial u / \partial \xi e^{-\nu_t(a)} da < d_0 \varepsilon.
\]
We fix $\varepsilon$ and $\delta$ which verify the property above. We now argue by the absurd: we suppose therefore that
\[
\forall C > 0, \exists t \in [t_0, 1], |x^t| > C.
\]
As $\nabla u$ is a diffeomorphism of $a^+$ into $2P^+$ and $0 \in 2P^+$, there exists $t \in [t_0, 1]$ such that
\[
\partial u / \partial \xi(x) > \frac{1}{2} a_0, \forall x \in B(x_1, \delta)
\]
where $\xi = x_1/|x_1|$ and $a_0 = \inf \{|x| / x \in 2\partial P\}$. We obtain
\[
\int_{B(x_1, \delta)} \partial u / \partial \xi(x) e^{\nu_t} dx > 1/4 a_0 \beta.
\]
(We recall that
\[
\int_{a^+} e^{\nu_t(a)} da =: \beta
\]
is independant of $t$.) Thus for $\varepsilon$ small enough,
\[
\int_{a^+} \partial u / \partial \xi(x) e^{\nu_t} dx > 0.
\]
We reach a contradiction. This complete the proof.  \(\square\)

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