Tilting the Primordial Power Spectrum with Bulk Viscosity

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Within the context of the cold dark matter model, current observations suggest that inflationary models which generate a tilted primordial power spectrum with negligible gravitational waves provide the most promising mechanism for explaining large scale clustering. The general form of the inflationary potential which produces such a spectrum is a hyperbolic function and is interpreted physically as a bulk viscous stress contribution to the energy-momentum of a perfect baryotropic fluid. This is equivalent to expanding the equation of state as a truncated Taylor series. Particle physics models which lead to such a potential are discussed.

Key words: galaxies: clustering – cosmology: theory – early Universe – large-scale structure of Universe

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1 Introduction

The origin and evolution of large scale structure is one of the most important problems in cosmology today. It is widely accepted that the growth of small fluctuations by gravitational instability leads to structure formation. The inflationary paradigm, whilst providing possible solutions to a number of other problems associated with the hot big bang model, also produces a Gaussian, adiabatic fluctuation spectrum which is nearly, though not exactly, scale-invariant (Guth 1981; Albrecht & Steinhardt 1982; Linde 1982; Olive 1990; Liddle & Lyth 1993). Based on this prediction, the standard Cold Dark Matter (CDM) model of galaxy formation employs the flat, Harrison-Zel’dovich spectrum as an input parameter (Efstathiou 1990). The CDM model successfully accounts for small ($\leq 10h^{-1}\text{Mpc}$) and intermediate ($10h^{-1}\text{Mpc} - 100h^{-1}\text{Mpc}$) scale observations, if one introduces a bias in the distribution of luminous to dark matter (Davies et al. 1985).

However, standard CDM has come under severe pressure from a number of recent observations (for a detailed review see Liddle & Lyth 1993). In particular, the APM angular galaxy-galaxy correlation function (Maddox et al. 1991) and the IRAS QDOT redshift survey (Efstathiou et al. 1991) indicate that there exists more large scale structure than that predicted by standard CDM. One possible resolution to this problem is to consider tilted CDM models. Here the primordial power spectrum is assumed to be of the form $P(k) \propto A_S^2(k)k \propto k^n$, where $k$ is the comoving wavenumber of the Fourier expansion of the perturbation, $A_S$ is the amplitude of the quantum fluctuation when it crosses the Hubble radius during the matter- or radiation-dominated eras and $n$ is the power spectrum. Other possibilities involve the addition of a cosmological constant or a hot dark matter component (Liddle & Lyth 1993).

Inflation also produces a spectrum of gravitational wave (tensor) perturbations, whose amplitude may or may not be comparable to that of the scalar fluctuations. In this paper we shall concentrate on models which lead to tilted power spectra with a negligible gravitational wave component. There exists a wide range of observational constraints on the tilt arising from large angle ($\theta \geq 3^\circ$) microwave background anisotropies (Smoot et al. 1992), galaxy clustering (Maddox et al. 1990; Efstathiou et al. 1990), peculiar velocity flows (Bertschinger & Dekel 1989; Dekel, Bertschinger & Faber 1990; Bertschinger et al. 1990), high redshift quasars (Efstathiou & Rees 1988) and the red shift of structure formation (Adams et al. 1993). When combined together these observations strongly limit the allowed value of $n$. It has been shown that tilted CDM can not fit all of the current data simultaneously (Adams et al. 1993; Liddle & Lyth 1993). For inflationary models in which gravitational wave production is negligible, a lower limit of $n > 0.7$ is partially consistent with the COBE 2-sigma upper limit and the bulk flow data, if the clustering and pairwise velocity data are ignored. In models where the gravitational wave contribution to the microwave back-

\textsuperscript{1}The current value of the expansion rate is $H_0 = 100h \text{ km s}^{-1} \text{Mpc}^{-1}$, where $0.4 \leq h \leq 1$. 

1
ground anisotropy is important, however, this limit is strengthened to $n > 0.84$. This is clearly inconsistent with the APM galaxy correlation function, which indicates that $0.3 < n < 0.7$ provides a good fit to the excess clustering data.

On the other hand, there is growing observational evidence for a departure from a pure power law at a scale $\lambda \approx 150 \pm 50 h^{-1} \text{Mpc}$ (Einasto et al. 1993). The power spectrum of clusters of galaxies has spectral index $-2 \leq n \leq -1$ on intermediate scales, whilst it is consistent with $n = 1$ on large scales (Peacock 1991).

In short, the current status of the observations is far from conclusive and it is therefore important to consider all the theoretical options available. In this paper we investigate the general circumstances in which a tilted power law scalar spectrum and a negligible gravitational wave amplitude arise in inflationary models. In Section 2, we summarize the details of a powerful framework which allows the general form of the inflaton potential to be derived in a straightforward manner. The form of such a potential is shown to be a hyperbolic secant function in Section 3. In Section 4, it is further shown that such a potential arises when a bulk viscous stress is added to the energy-momentum tensor of a perfect baryotropic fluid. It is illustrated how a number of plausible particle physics models, such as the quantum creation of fundamental strings (Turok 1988) and $N = 2$ supergravity (Salam & Sezgin 1984), lead to a potential of this form.

## 2 Inflaton dynamics and primordial fluctuations

Inflation proceeds if the potential energy of a scalar field dominates the energy-momentum tensor of the universe, since this leads to a violation of the strong energy condition (Hawking & Ellis 1973). One can model self-interacting scalar fields in terms of perfect fluids, and vice-versa. Within the context of the isotropic and homogeneous Friedmann-Robertson-Walker (FRW) cosmologies, the complete history of the universe is determined by three independent equations. The Friedmann equation

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2}$$

is the first integral of the Raychaudhuri equation (Raychaudhuri 1955) and describes the conservation of energy. Here, $\rho(t)$ represents the total energy density of the universe, $k = \{-1, 0, +1\}$ for open, flat, and closed spatial sections respectively, $a(t)$ is the scale factor, and $H \equiv \dot{a}/a$ is the expansion rate, where a dot denotes differentiation with respect to cosmic time $t$. Units are chosen such that $\hbar = c = 8\pi G = 1$. For a matter content with pressure $p(t)$, the Bianchi identity (equivalent to the local conservation of energy-momentum) is

$$\dot{\rho} + 3H(\rho + p) = 0,$$
and, in principle, the general solution to these equations follows once the equation of state

\[ p = p(\rho) \]  

is specified. This equation describes the particle physics sector of the model.

However, analytical solutions have only been found for a limited number of specific examples, such as the special case \(p/\rho = \text{constant}\) (Barrow 1990). For an arbitrary equation of state which satisfies the dominant energy condition (\(\rho - p \geq 0\)), it proves convenient to redefine the sum and difference of \(\rho\) and \(p\) in terms of the new functions

\[ \dot{\phi}^2 \equiv \rho + p \quad \iff \quad \phi(t) = \int^t dt' \sqrt{\rho(t') + p(t')} \]  

\[ 2V \equiv \rho - p. \]  

We may then rewrite equations (1) and (2) as

\[
H^2 = \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + V \right) - \frac{k}{a^2} \\
\dot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0,
\]

which are the Einstein field equations for a minimally coupled scalar field, \(\phi\). Specifying the equation of state now becomes a question of choosing an appropriate functional form for the potential \(V(\phi)\), and vice-versa. For example, an exponential potential is equivalent to \(p/\rho = \text{constant}\) when \(k = 0\).

Recent advances in the treatment of equations (8) and (9) have been made by viewing the scalar field as the effective dynamical variable of the system (Muslimov 1990; Salopek & Bond 1990, 1991; Lidsey 1991, 1993). From the definition \(\rho \equiv \dot{\phi}^2/2 + V\), the time dependence can be eliminated by rewriting the scalar field equation (8) as

\[
\rho' = -3H\dot{\phi}, \quad \dot{\phi} \neq 0,
\]

where a prime denotes differentiation with respect to \(\phi\). This is consistent if \(\phi\) does not oscillate (i.e. \(\dot{\phi}\) does not pass through zero). It follows that \(6H^2 = -\rho'X'/X\), where \(X(\phi) \equiv a^2(\phi)\), and the Friedmann equation becomes

\[
\rho'X' + 2\rho X = 6k.
\]

The potential can be found immediately from the expression

\[
V(\phi) = \rho(\phi) - \frac{1}{18} \frac{(\rho')^2}{H^2(\phi)}
\]

once the forms of \(\rho(\phi)\) and \(X(\phi)\) are known.
When $k = 0$, $3H^2 = \rho$, and these field equations take the particularly simple form
\begin{equation}
2H'a' = -Ha, \quad 2H' = -\dot{\phi},
\end{equation}
thereby allowing the general solution, $a(\phi)$, to be expressed in terms of quadratures with respect to $\phi$ (Salopek & Bond 1991; Lidsey 1993). The expression for the potential reduces to an Hamilton-Jacobi differential equation in $H(\phi)$ of the form
\begin{equation}
V(\phi) = 3H^2(\phi) - 2(H')^2.
\end{equation}

This framework is useful for determining the primordial power spectrum. During inflation short-wavelength quantum fluctuations in the inflaton and graviton fields are redshifted beyond the Hubble radius when the comoving wavenumber of the perturbation satisfies $k = aH$. Once outside, their amplitude remains frozen until they re-enter the Hubble radius during the radiation-dominated or matter-dominated eras. The density perturbation (scalar) spectrum is given by (Copeland et al. 1993)
\begin{equation}
A_S(k) = \left( \frac{mH^2}{4\pi^{3/2}|\phi|} \right)_{k=aH} = \left( \frac{mH^2}{8\pi^{3/2}|H'|} \right)_{k=aH}
\end{equation}
where the quantities on the right are calculated when the fluctuation first crosses the Hubble radius. In the uniform Hubble constant gauge the constant $m = 4$ if the perturbation re-enters during the radiation-dominated era, whereas $m = 2/5$ if the perturbation re-enters during matter domination.

The gravitational wave (tensor) spectrum is calculated in the transverse-traceless gauge, where $+$ and $\times$ denote the two independent polarization states of the metric perturbation. The classical amplitude of the fluctuation satisfies the massless Klein-Gordon equation (Grishchuk 1974, 1977; White 1992), so the graviton can be viewed as a massless, minimally coupled scalar field, which has two degrees of freedom $\psi_{+,\times}$. The spectrum of tensor perturbations is then given by (Abbott & Wise 1984)
\begin{equation}
A_G(k) = \left( \frac{H}{4\pi^{3/2}} \right)_{k=aH}
\end{equation}

One may define two parameters (Copeland et al. 1993)
\begin{equation}
\epsilon \equiv 3\frac{\dot{\phi}^2/2}{V + \dot{\phi}^2/2} = 2\left( \frac{H'}{H} \right)^2
\end{equation}
and
\begin{equation}
\eta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = 2\frac{H''}{H}.
\end{equation}

Up to a numerical factor, $\epsilon$ measures the relative contribution of the field’s kinetic energy to its total energy density and may be referred to as the energy parameter.
The quantity \( \eta \) measures the ratio of the field’s acceleration relative to the friction contribution. We refer to it as the friction parameter.

These parameters may be recast in an alternative form by defining new variables

\[
x \equiv H^2, \quad y \equiv (H')^2.
\]

It follows from the identity \( 2H'' = d(H')^2/dH \) that

\[
\epsilon = \frac{2y}{x}, \quad \eta = 2 \frac{dy}{dx}.
\]

Inflation proceeds in the region of parameter space for which \( \epsilon < 1 \) and the coasting solution, or Milne universe, corresponds to \( \epsilon = 1 \) (i.e. \( 2y = x \)). It is interesting to note that the friction parameter does not directly determine whether inflation occurs. The ‘slow-roll’ approximation is valid when \( \{ \epsilon, |\eta| \} \ll 1 \).

One may also write the (scale-dependent) spectral indices of the scalar and tensor fluctuations in terms of these two quantities. It is easy to show that

\[
n - 1 \equiv \frac{d\ln[A_S^2(k)]}{d\ln k} = 2 \left( \frac{2\epsilon_* - \eta_*}{\epsilon_* - 1} \right),
\]

\[
n_G \equiv \frac{d\ln[A_G^2(k)]}{d\ln k} = \frac{2\epsilon_*}{\epsilon_* - 1},
\]

where * indicates that \( \epsilon \) and \( \eta \) should be evaluated when a particular scale first crosses the Hubble radius. The flat Harrison-Zel’dovich spectrum is equivalent to \( n = 1 \).

It follows from the definitions of \( A_S \) and \( A_G \) that

\[
\frac{A_G}{A_S} = \sqrt{2\epsilon}.
\]

It is often stated that inflation leads to a Harrison-Zel’dovich scalar spectrum with a negligible gravitational wave contribution. However such a conclusion follows because the slow-roll approximation is assumed a priori. Equation (21) implies that the gravity wave amplitude can be comparable to \( A_S \) if \( \epsilon \) is sufficiently large. There exists a table of correspondences which summarizes the four possibilities (Barrow & Liddle 1993).

In the following section we shall employ this framework to derive the form of the inflaton potential.

### 3 The general potential leading to tilted spectra

Exact results may be derived by solving equation (19). After substitution of equation (18) this becomes the first-order differential equation

\[
\frac{dy}{dx} + \left( \frac{n - 5}{2} \right) \frac{y}{x} + \left( \frac{1 - n}{4} \right) = 0.
\]
Scalar spectrum | Gravitational waves important | Gravitational waves negligible
--- | --- | ---
Small tilt | $e$ large | $e$ small
$2e \approx \eta$ | $|\eta|$ small
Significant tilt | $\epsilon$ large | $\epsilon$ small
$|\eta|$ large | $|\eta|$ large

Table 1 - The table of correspondences illustrating the connection between tilt and the magnitude of the energy and friction parameters. The description ‘large’ implies significantly larger than zero (but still less than unity) and ‘small’ implies the parameter is very close to zero.

The solution for constant $n$ is

$$y = \frac{1}{2} \left( \frac{1-n}{3-n} \right) x + Cx^{(5-n)/2}$$

where $C$ is an arbitrary integration constant. The case $C < 0$ is of interest here because it leads to the general solution

$$H(\phi) = \lambda \left[ \sech \left( \sqrt{\frac{(n-1)(n-3)}{8}} \phi \right) \right]^{2/(3-n)}, \quad n < 1,$$

where

$$\lambda = \left[ \frac{1}{2|C|} \frac{n-1}{n-3} \right]^{1/(3-n)}$$

Without loss of generality the integration constant has been removed by performing a linear translation on the value of $\phi$. The constant $|C|$ determines the energy scale at which these processes are occurring (Carr & Lidsey 1993; Lidsey & Tavakol 1993). From equation (12) the potential is

$$V(\phi) = \lambda^2 \left[ \sech(\omega\phi) \right]^{4/(3-n)} [3 - \left( \frac{n-1}{n-3} \right) \tanh^2(\omega\phi)],$$

where

$$\omega \equiv \sqrt{\frac{(n-1)(n-3)}{8}}.$$  

The solution to equation (24) when $C = 0$ is the exponential potential

$$H(\phi) \propto \exp \left[ \pm \frac{1}{2} \left( \frac{n-1}{n-3} \right) \phi \right].$$

(28)
Eqs. (24) and (28) lead to the same scalar perturbation spectrum and a determination of \( \mathcal{A}_S(k) \) alone will not lift the degeneracy. What distinguishes the two solutions is the relative contribution of the gravitational wave spectrum, as determined by equation (21). We find

\[
\epsilon = \left( \frac{n-1}{n-3} \right) \quad (29)
\]

and

\[
\epsilon = \left( \frac{n-1}{n-3} \right) \left[ \tanh \left( \sqrt{\frac{(n-1)(n-3)}{8}} \phi \right) \right]^2 \quad (30)
\]

for the exponential and hyperbolic cases respectively. If cosmological scales crossed the Hubble radius when \( |\phi| \ll 1 \), the amplitude of tensor perturbations is exponentially suppressed in the latter example.

Since equation (24) is an exact solution, it is valid for all values of \( \phi \). In particular, for sufficiently small \( \phi \) the Taylor expansion

\[
H(\phi) = \lambda \left[ 1 - \left( \frac{1-n}{8} \right) \phi^2 + \mathcal{O}(\phi^4) \right] \approx \sqrt{\frac{V(\phi)}{3}} \quad (31)
\]

will also lead to a constant spectral index. This implies that a power spectrum with \( n = \text{constant} < 1 \) will arise from any function of \( H(\phi) \) which is identical to equation (31) in the small \( \phi \) approximation. It is well known that deviations from scale invariance, without significant gravitational wave production, are possible whenever the potential resembles an inverted harmonic oscillator (Steinhardt & Turner 1984), but the above calculation provides further insight. Such a result follows because the inverted oscillator resembles the hyperbolic secant function to lowest order in \( \phi \). Equation (31) is very useful because it directly relates the effective imaginary mass of the field to the scalar spectral index.

These results are summarized pictorially in Figs. 1a and 1b, which are representations of the class of solutions (23) in the \( x - y \) plane. (\( x \) and \( y \) are defined in equation (18)). In Fig. 1a the coasting solution \( y = x/2 \) is shown as the dashed line and the strong energy condition is violated to the right of this line. The \( x \)-axis represents the de Sitter solution, \( H = \text{constant} \), and the origin is Minkowski space, which itself may be viewed as de Sitter space with an infinite radius of curvature (Hawking & Ellis 1973). The solid lines represent solutions of constant \( n \) when \( C = 0 \). In these models \( x \) is a measure of the energy density of the universe and decreases as time increases. The trajectories of these constant \( n \) universes are indicated by the arrows and they all asymptotically approach Minkowski space at \( t \to +\infty \).

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**Figure 1**

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\(^2\)In reality the shape of the potential must change at some point \((x, y)\) to allow for an exit from inflation.
Fig. 1b illustrates the trajectories for finite values of $C < 0$ and $n = 0.7$. This class of universe begins in a de Sitter phase at $t = -\infty$ and evolves towards the $C = 0$ asymptote at $t = +\infty$ in such a way that the scalar spectral index remains constant at all times. The magnitude of $C$ determines the amplitude of the scalar quantum fluctuations but not the scale dependence.

Having found the form of the potential required, it is now necessary to consider the physics which may lead to such a model.

4 Modelling the potential as a bulk viscous stress

In this section we employ the techniques discussed in Section 2 for an equation of state which has a number of physical applications in the early universe. Equation (3) may be rewritten in the form $p \equiv \Gamma(\rho)\rho$ for some arbitrary function $\Gamma$. The simplest form for $\Gamma(\rho)$ is the baryotropic equation of state

$$\Gamma \equiv \gamma - 1 = \text{constant}, \quad 0 \leq \gamma \leq 2,$$

which is equivalent at the classical level to the subset of solutions

$$\dot{\phi}^2 = \mu V, \quad \mu \equiv \frac{2\gamma}{2 - \gamma}.$$

Hence $0 \leq \mu \leq \infty$, where $\mu = 0$ ($\gamma = 0$) corresponds to vacuum energy and $\mu = \infty$ ($\gamma = 2$) represents a stiff fluid. In particular, radiation and matter dominated universes are characterized by $\mu = 4$ ($\gamma = 4/3$) and $\mu = 2$ ($\gamma = 1$) respectively. Furthermore, the equation of state for a universe dominated by topological defects such as domain walls or a gas of cosmic strings is also given by equation (32). It can be shown that $\gamma_W = v_W^2 + 1/3$ for a gas of domain walls moving with constant velocity $v_W$, whereas $\gamma_S = 2(1 + v_S^2)/3$ for a string gas (Kolb & Turner 1990).

Mathematically, a natural and plausible extension towards a more realistic effective equation of state is to view the function $\Gamma$ as a truncated Taylor series

$$\Gamma(\rho) = (\gamma - 1) - \beta \rho^\alpha,$$

for positive definite constants $\{\alpha, \beta\}$. The second term in this expression can be treated as a first-order perturbative correction to $\Gamma = \text{constant}$. Physically, when $k = 0$, equation (34) is equivalent to a fluid with bulk viscosity $\eta(\rho) \equiv \Pi \rho^m$ and a perfect fluid $p = (\gamma - 1)\rho$, where $\beta = \sqrt{3}\Pi$ and $m = \alpha + \frac{1}{2}$ (Barrow 1988). It is known that the effects of particle production in the early universe can be modelled in terms of a classical bulk viscosity of this form. The polarization (trace anomaly) of quantized fields in curved spacetimes also leads to vacuum viscosity effects (Zel’dovich 1980; Hu 1982; Waga, Falcião & Chanda 1986). Exact cosmological solutions based on equation (34) have been found by Murphy (1973) and Barrow (1988), but in this
section we derive them in a much simpler parametric form. This allows the primordial fluctuation spectra to be calculated.

With this choice of $\Gamma(\rho)$ the Bianchi identity (4) integrates exactly for all $k$ to yield

$$\rho(a) = \left[ \frac{\beta}{\gamma} + \left( \frac{a}{a_0} \right)^{3\alpha\gamma} \right]^{-1/\alpha},$$  \hspace{1cm} (35)

where the constant of integration is expressed in terms of $a_0$ and $\{\alpha, \gamma\} \neq 0$. When $\beta = 0$, $\rho \propto a^{-3\gamma}$ as expected. Unfortunately analytical solutions to the Friedmann equation (9) have not been found when $k = \pm 1$, but this equation simplifies to

$$\left( \frac{\rho'}{\rho} \right)^2 = 3 \left( \gamma - \beta \rho^\alpha \right)$$  \hspace{1cm} (36)

when $k = 0$ and this has the exact solution

$$1 - \frac{\beta}{\gamma} \rho^\alpha(\phi) = \tanh^2(\omega \phi)$$  \hspace{1cm} (37)

$$\omega \equiv \sqrt{\frac{3\gamma \alpha^2}{4}}.$$  \hspace{1cm} (38)

The dominant energy condition is violated when $\beta < 0$, so this case is not considered further. The expressions for $H(\phi)$, $a(\phi)$ and $V(\phi)$ follow from equations (11) and (12) as

$$H(\phi) = \frac{1}{\sqrt{3}} \left( \frac{\gamma}{\beta} \right)^{1/2\alpha} [\text{sech}(\omega \phi)]^{1/\alpha}$$  \hspace{1cm} (39)

$$a(\phi) = a_0 \left( \frac{\beta}{\gamma} \right)^{1/3\alpha\gamma} |\sinh(\omega \phi)|^{2/3\alpha\gamma}$$  \hspace{1cm} (40)

$$V(\phi) = \left( \frac{\gamma}{\beta} \right)^{1/\alpha} [\text{sech}(\omega \phi)]^{2/\alpha} \left( 1 - \frac{\gamma}{2} \tanh^2(\omega \phi) \right).$$  \hspace{1cm} (41)

It should be emphasized that these solutions are exact and no ‘slow-roll’ approximations, such as $|\dot{\phi}| \ll H|\phi|$ and $\dot{\phi}^2 \ll V$, have been made. These parametric solutions are plotted schematically in Fig. 2.

**Figure 2**

For completeness, we include the well known solutions for $\beta = 0$, which lead to the exponential potential

$$H(\phi) \propto \exp \left( \sqrt{\frac{3\gamma}{4}} \phi \right), \quad V(\phi) \propto \exp \left( \sqrt{3\gamma} \phi \right)$$  \hspace{1cm} (42)
and power law expansion $a(t) \propto t^{2/3\gamma}$.

Hence, when $k = 0$, a perfect baryotropic fluid with bulk viscosity can be modelled as a self-interacting scalar field with potential ($t^4$). The inclusion of bulk viscosity alters the structure of the potential away from an exponential form in the neighbourhood of $|\phi| \approx 0$. This is illustrated in Fig. 2c. Near the origin, we find that the equation of state ($\beta$) can be adequately described as an inverted harmonic oscillator. The exponential potentials are recovered in the asymptotic limit as $|\phi| \to \infty$, because the viscous effects decay faster than the perfect fluid contribution at large $|\phi|$. The general advantage of rewriting equation ($t^4$) in terms of a scalar field is that the qualitative history of the universe is easily determined by considering the evolution of the scalar field along its interaction potential. Here the field is initially placed at $\phi = 0$, which corresponds to a de Sitter expansion with $H = (\gamma/\beta)^{1/2\alpha}/\sqrt{3}$. As the field rolls either to the left or right of the potential, the universe expands and approaches the power law attractor solution $a \propto t^{2/3\gamma}$ with $H \to 0$ monotonically. This behaviour was first noted by Barrow (1988).

The region of parameter space in which the strong energy condition ($\rho + 3p > 0$) is violated can be determined from equations ($\beta^4$) and ($\beta^7$). Defining the quantity $\Delta \equiv \rho + 3p$, we find

$$\Delta = 3\gamma \rho \left( \tanh^2(\omega\phi) - \frac{2}{3\gamma} \right).$$

(43)

It follows that inflation occurs for all $\phi$ when $\gamma < 2/3$ and there exists a graceful exit problem. An identical problem is encountered when $\beta = 0$. The potential must be modified in such a way that allows the expansion to become subluminal. For $\gamma > 2/3$, on the other hand, the universe inflates initially, but deflates once

$$\tanh^2 \omega \phi > \frac{2}{3\gamma}$$

(44)

is satisfied. The end of inflation can be defined as the point where this becomes an equality.

The epoch during inflation which is relevant for large-scale structure observations can then be determined and we now proceed to discuss models which lead to a potential of the form given by equation ($\beta^4$).

5 Particle physics models

The purpose of this section is to discuss a number of particle physics models in which the features described above may arise.

A. Bulk Viscosity
A direct comparison of Eqs. (24) and (39) indicates that tilted spectra with constant spectral index and negligible gravitational waves arise in a class of bulk viscosity models if we identify
\[ \alpha = \frac{3 - n}{2} \] (45)
\[ \gamma = \frac{2}{3} \left( \frac{n - 1}{n - 3} \right) \iff n = \frac{9\gamma - 2}{3\gamma - 2} \] (46)
or equivalently
\[ \alpha = \frac{2}{2 - 3\gamma} \] (47)
The constraint \( n \geq 0.7 \) now becomes the upper limit \( \gamma \leq 0.09 \). (We require \( \gamma < 2/3 \) for consistency in this class, since equation (39) is only valid for positive \( \alpha \)). The parameter \( \beta \) of equation (34) plays the role of \( |C| \) in determining the amplitude of the fluctuations, whereas \( \alpha \) determines the tilt of the spectrum.

When \( \{\alpha, \gamma\} \) are not related by equation (47) the tilt is not exactly scale invariant, but \( n \approx \text{constant} \) is an excellent approximation if \( \omega|\phi| \ll 1 \). For standard reheating, scales of astrophysical interest first crossed the Hubble radius approximately 50 e-foldings before the end of inflation. Defining the value of the field at this point as \( \phi_{50} \), we find from equation (44) that
\[ \sinh^2(\omega\phi_{50}) = \left( \frac{2}{3\gamma - 2} \right) e^{-3\alpha\gamma N_{50}} \] (48)
for \( \gamma > 2/3 \), where \( N_{50} \approx 50 \). Hence, if \( \gamma \) is not too close to \( \gamma = 2/3 \), \( \omega\phi_{50} \ll 1 \) is valid and it is consistent to expand equation (39) to lowest order. It follows from equation (31) that
\[ n = 1 - 3\alpha\gamma \] (49)
and \( n \geq 0.7 \) leads to the constraint
\[ \alpha\gamma \leq 0.1. \] (50)

B. Quantum creation of fundamental strings

Turok (1988) has considered the quantum production of infinitely thin Witten strings on super-horizon size scales (Green, Schwarz & Witten 1988). He suggested that a deflationary expansion follows naturally from a quasi de Sitter phase in the early universe. When \( \alpha = 1 \) and \( 2/3 \leq \gamma \leq 1 \), it was shown that the equation of state (34) provides a good phenomenological description of the quantum creation of these strings after compactification to four dimensions (Barrow 1988; Turok 1988). The parameter \( \beta \) depends on the fundamental string tension and the fractal dimension of
the string. The lower and upper limits on $\gamma$ correspond to long strings with negligible velocity and a highly convoluted, relativistic string distribution respectively.

Therefore, the parametric solutions (39)-(41) describe the evolution of the flat Friedmann universe when dominated by fundamental strings created on super-horizon scales. It follows from equation (49) that $-2 \leq n \leq -1$ and the model in its present form gives far too much power on large scales. However, it is interesting that this predicted range for the spectral index corresponds precisely to that observed on intermediate scales $1 \leq \lambda \leq 200h^{-1}$ Mpc in the distribution of clusters of galaxies (Einasto et al. 1993).

C. $N = 2$ supergravity in six dimensions

The Salam-Sezgin model is $N = 2$ supergravity in six dimensions compactified onto a two-sphere (Salam & Sezgin 1984). Although such a model does not reproduce a correct particle spectrum in four dimensions, it is thought to contain features generic to more complete theories and its cosmological implications have been studied in some detail (Liddle 1989). It has been shown that the theory is equivalent to two interacting scalar fields, $\xi$ and $\sigma$, whose full potential exhibits a global minimum in both $\xi$ and $\sigma$ directions if the scale-invariance of the theory is broken by quantum effects (Gibbons & Townsend 1987). If the $\xi$-field comes to rest, the potential for $\sigma$ becomes

$$V(\sigma) = V_0 \left[ \exp(-\sqrt{8}\sigma) - 2 \exp(-\sqrt{2}\sigma) + 1 \right].$$

The potential has a global minimum located at $\sigma = 0$ and an asymptotic form $V \propto \exp(-\sqrt{8}\sigma)$ for $\sigma \ll 0$. In this regime the system behaves as a baryotropic perfect fluid with $\gamma = 8/3$. In this unmodified form the model does not lead to inflation, but it is clear from Sections 3 and 4 that introducing a bulk viscosity into the system will violate the strong energy condition and lead to a tilted spectrum. By condition (50) observation requires $\alpha < 0.038$.

Moreover, the viscosity effects redshift faster than the perfect fluid contribution, implying that they will become negligible by the time the field has rolled towards $\sigma = 0$. Hence, the shape of the potential is not significantly altered within the vicinity of the global minimum and standard reheating can occur through rapid oscillations of the field about this minimum (Kolb & Turner 1990). This solves the graceful exit problem discussed in Section 2.

D. Natural Inflation

For completeness we remark that natural inflation, driven by a pseudo-Nambu-Goldstone boson with potential $V(\phi) \propto 1 + \cos(\phi/f)$, also produces a tilted spectrum. This model has been discussed in detail by Adams et al. (1993). In the small angle approximation the spectral index is given by $n = 1 - f^{-2}$ via equation (31).
6 Conclusions

The general inflationary potential which produces a measureable tilt in the primordial spectrum of scalar fluctuations, without producing a significant gravitational wave spectrum, is of the form (26). In this paper it has been shown that such a potential is physically equivalent to a bulk viscous stress modification to a baryotropic perfect fluid. Mathematically it is equivalent to expanding the equation of state as a truncated Taylor series. It was further shown that this potential arises in a number of particle physics theories. If future observations indicate that a) gravitational waves are not contributing to the anisotropy in the cosmic microwave background and b) a dark matter model based on a spectrum with significant tilt does provide a good fit to the data, then this would justify a more detailed study of the models discussed in Section 5.

A number of simplifying assumptions were made. In particular, the expressions (13) and (14) for the scalar and tensor amplitudes are strictly only valid in the slow roll regime, \( \{ \epsilon, \eta \} \ll 1 \), whereas equation (19) implies that a tilt away from the Harrison-Zel’dovich spectrum requires \( 0 < |\eta| \leq 1 \), approximately. However, it has been shown that the corrections away from slow-roll are not important near a local maximum, and although they slightly alter the amplitude of the fluctuations in the exponential regime, they do not change the spectral index (Stewart & Lyth 1993).

The physical interpretation of the potential in terms of a bulk viscosity is only valid in the spatially flat FRW cosmology. However, this paper has investigated the power spectra of such models and it is the last 60 e-foldings of inflationary expansion which are important for large-scale structure (Kolb & Turner 1990). In most chaotic scenarios the density parameter is very close to unity by this stage.

Furthermore, since all scales probed by large-scale structure correspond to a small \( \approx 9 \) number of e-foldings, it is reasonable to assume that the parameter \( \gamma \) is constant during this interval. These results may therefore have wider applications in models where the polytropic index is a function of cosmic time.

Finally we note that equation (34) with \( \gamma = 1 \) and \( \beta < 0 \) is the equation of state for a polytropic star, special cases of which include white dwarfs \( (\alpha = 5/3) \) and neutron stars \( (\alpha = 4/3) \) (Weinberg 1973). If the solution to equation (9) could be found for \( k = +1 \), the techniques described here could be relevant for stellar structure.

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References

Abbott L. F., Wise M. B., 1984, Nucl. Phys. B, 244, 541

Adams F. C., Bond J. R., Freese K., Frieman J. A., Olinto A. V., 1993, Phys. Rev. D, 47, 426

Albrecht A., Steinhardt P. J., 1982, Phys. Rev. Lett., 48, 1220

Barrow J. D., 1988, Nucl. Phys. B, 310, 743

Barrow J. D., 1990, Phys. Lett. B, 235, 40

Barrow J. D., Liddle A. R., 1993, ‘Perturbation Spectra from Intermediate Inflation, Sussex preprint SUSSEX-AST 93/2-1

Bertschinger E., Dekel A., 1989, ApJ Lett., 336, L5

Bertschinger E., Dekel A., Faber S. M., Dressler A., Burstein D., 1990, ApJ, 364, 370

Birrell N. D., Davies P. C. W., 1982, Quantum Fields in Curved Space. Cambridge Univ. Press, Cambridge

Carr B. J., Lidsey J. E., 1993, Phys. Rev. D, to appear

Copeland E. J., Kolb E. W., Liddle A. R., Lidsey J. E., 1993, ‘Reconstructing the Inflaton Potential - In Principle and in Practice’, Sussex/Fermilab preprint SUSSEX AST 93/3-1, FNAL-PUB-93/029-A, to appear Phys. Rev. D; and refs therein

Davies M., Efstathiou G., Frenk C. S., White S. D. M., 1985, ApJ, 292, 371

Dekel A., Bertschinger E., Faber S. M., 1990, ApJ, 364, 349

Efstathiou G., 1990, in Heavens A., Peacock J., Davies A., eds, The Physics of the Early Universe. SUSSP publications

Efstathiou G., Rees M. J., 1988, MNRAS, 230 , 5P

Efstathiou G., et al., 1990, MNRAS, 247 , 10P

Einasto J., Gramann M., Saar E., Tago E., 1993, MNRAS, 260, 705

Gibbons G. W., Townsend P. K., 1987, Nucl. Phys. B, 282, 610

Green M. B., Schwarz J. H., Witten E., 1988, Superstring Theory. Cambridge Univ. Press, Cambridge
Grishchuk L. P., 1975, Soviet Phys. JETP, 40, 409
Grishchuk L. P., 1977, Ann. N. Y. Acad. Sci., 302, 439
Guth A. H., 1981, Phys. Rev. D, 23, 347
Hawking S. W., Ellis G. F. R., The Large Scale Structure of Spacetime. Cambridge Univ. Press, Cambridge
Hu B. L., 1982, Phys. Lett. A, 90, 375
Kolb E. W., Turner M. S., 1990, The Early Universe. Addison-Wesley, New York
Liddle A. R., 1989, Phys. Lett. B, 220, 502; and refs therein
Liddle A. R., Lyth D. H., 1993, ‘The Cold Dark Matter Density Perturbation’. To appear Phys. Repts
Lidsey J. E., 1991 Phys. Lett. B, 273, 42
Lidsey J. E., 1993, Gen. Rel. Grav., 25, 399
Lidsey J. E., Tavakol R. K., 1993, ‘On the Correspondence between Theory and Observation in Inflationary Cosmology’, Fermilab preprint FNAL-PUB-93/034-A (1993). To appear, Phys. Lett. B
Linde A. D., 1982, Phys. Lett. B, 108, 389
Maddox S. J., Efstathiou G., Sutherland W. J., Loveday J., 1990, MNRAS, 242, 43P
Maddox S. J., Sutherland W. J., Efstathiou G., Loveday J., Peterson B. A., 1991, MNRAS, 247, 1P
Murphy G. L., 1973, Phys. Rev. D, 8, 4231
Muslimov A. G., 1990, Class. Quantum Grav., 7, 231
Olive K. A., 1990, Phys. Repts., 190, 307
Peacock, J. A., 1991, MNRAS, 253, 1P
Raychaudhuri A. K., 1955, Phys. Rev., 98, 1123
Salam A., Sezgin E., 1984, Phys. Lett. B, 147, 47
Salopek D. S., Bond J. R., 1990, Phys. Rev. D, 28, 2960
Salopek D. S., Bond J. R., 1991, Phys. Rev. D, 43, 1005
Saunders W., et al., 1991, Nat., 349, 32
Stewart E. D., Lyth D. H., 1993, Phys. Lett. B, 302, 171
Smoot G. F., et al., 1992, ApJ Lett., 396, L1
Steinhardt P. J., Turner M. S., 1984, Phys. Rev. D, 29, 2162
Turok N., 1988, Phys. Rev. Lett., 60, 549
Waga I., Falcão R. C., Chanda R., 1986, Phys. Rev. D, 33, 1839
Weinberg S. W., 1973, Gravitation and Cosmology. Wiley and Sons, New York
White M., 1992, Phys. Rev. D, 46, 4198
Wright E. L., et al., 1992, ApJ Lett., 396, L13
Zel’dovich Ya. B., 1980, Sov. Phys. JETP Lett., 12, 307
Figure Captions

*Figure 1a:* The trajectories of the constant $n$ universes in the $x - y$ place when $C = 0$. The dashed line represents the Milne universe and the strong energy condition is satisfied to the left of this line. All trajectories approach the origin at $t = +\infty$, which is de Sitter space with infinite radius of curvature.

*Figure 1b:* The trajectories of $n = 0.7$ universes when $C = \{0, -0.5, -1, -2\}$. The dashed line corresponds to $C = 0$. All solutions begin in a de Sitter phase at $t = -\infty$ and approach the $C = 0$ (power law) asymptote at $t = +\infty$.

*Figure 2:* Schematic plots of the parametric solution (38)-(40) as a function of the scalar field. In Fig. 2c the dashed lines represent the exponential potential corresponding to vanishing bulk viscosity. The inclusion of a bulk viscous stress results in a local maximum in the potential and a quasi de Sitter expansion. It is this feature which results in negligible gravitational wave production, but the curvature of the potential at the origin tilts the scalar spectrum.
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