An annotation on the prime graph of an integral domain

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Abstract
We introduce the prime graph of the product ring $R_1 \times R_2$ where $R_1$, $R_2$ are integral domains, which is an extension of study on prime graph of an integral domain. We prove that, if $R_1, R_2$ are two integral domains, the graph obtained by removing the isolated vertices from $\text{PG}(R_1 \times R_2)$ is a bipartite graph. We obtain some consequences.

Keywords
Associative ring, Integral Domain, Graph, Prime Graph.

AMS Subject Classification
05C20, 05C25, 13E15, 68R10, 05C99.

1. Introduction
The prime graph of an associative ring, a concept from algebraic graph theory was introduced by Satyanarayana et.al [11] has shown a new path for the researchers to explore and extend the study in their fields of interest. Satyanarayana et. al [4, 5], studied prime graphs related to a ring of integers modulo $n$. The complement of a prime graph of a ring was studied by Power and Joshi [2]. These studies motivated us to derive few results in the prime graph of an integral domain which is an extension to the work of Satyanarayana et. al [5].

Our study is presented in three small sections. Section 1, is a collection of necessary definitions, and results from the literature. Section 2 and 3 contains new findings.

Definition 1.1. [7] An algebraic system with a non-empty set $R$ together with two binary operations addition and multiplication is said to be a ring (or an associative ring) if $(R, +)$ is an abelian group; $(R, \cdot)$ is a semigroup and multiplication is distributive over the addition among the elements of $R$. If in addition $R$ satisfies commutative property with multiplication, then it is called a commutative ring. Further ring containing multiplicative identity is called a ring with unity.

Definition 1.2. [7] A non-empty subset $I$ of a ring $R$ is called a left ideal if $(I, +)$ is subgroup of $(R, +)$ and for any element $r$ of $R$ and $i$ of $I$, $ri \in I$. It is called right ideal if $ir \in I$ for all elements $r$ of $R$ and $i$ of $I$ with $(I, +)$ being subgroup of $(R, +)$.

Definition 1.3. [7] (i) An ideal $P$ of a ring $R$ is said to be prime for any two ideals $A, B$ of $R$, and $AB \subseteq P$ imply $A \subseteq P$ or $B \subseteq P$ (equivalently, $a, b \in R$ and $aRb \subseteq P \Rightarrow a \in P$ or $b \in P$).

(ii) Let $I, J$ be two ideals of $R$ such that $I \subseteq J$. We say that $I$ is essential (or ideal essential) in $J$ if it satisfies the following condition: $K \subseteq R, K \subseteq J, I \cap K = (0)$ imply $K = (0)$.

(iii) Given two distinct ideals $I$ and $J$ of $R$, if $I$ is essential in $J$, then we say that $J$ is proper essential extension of $I$. We use $I \leq e J$ to represent $I$ is essential in $J$.

Definition 1.4. [7] (i) A non-zero ideal $I$ of $R$ is said to be uniform if for any other non-zero ideal $J$ or $R$ contained in $I$ imply $I \leq e J$.

(ii) A non-zero ideal $K$ of $R$ is said to have finite dimension on ideals of $R$ (FDIR, in short) if $K$ does not contain an infinite number of non-zero ideals of $R$ whose sum is direct. It is clear that if $R$ has FDI, then every non-zero ideal of $R$ has FDIR.

Definition 1.5. A commutative ring with unity is said to be an integral domain if for any two elements $a$ and $b$, $ab = 0$ implies...
either \( a = 0 \) or \( b = 0 \).

**Theorem 1.6.** [7] Suppose \( H \) is a non-zero ideal of a ring \( R \) and \( H \) has finite dimension on ideals of \( R \). Then there exist ideals \( U_1, U_2, \ldots, U_n \) of \( R \) which are uniform whose sum is direct and essential in \( H \) and further these are unique in number.

**Corollary 1.7.** [7] If \( R \) is a ring with FDI, then there exist uniform ideals \( U_1, U_2, \ldots, U_n \) in \( R \) whose sum is direct and essential in \( R \); and if \( V_1, 1 \leq i \leq k \), possessing the same property as of \( U_j, 1 \leq j \leq n \) mentioned above, then \( k = n \).

**Definition 1.8.** The number \( n \), obtained above, is called the dimension of \( H \), and is denoted by \( \text{dim } H \).

For further developments in this dimension concept in ring theory, we refer [3, 7, 9].

Now we present some Graph theoretic concepts: A graph is a system \( G(V, E, \varphi) \) consist of non-empty set \( V \) of elements called vertices; another set \( E \) of elements called edges and incidence relation \( \varphi \) from \( E \) to \( v_i, v_j \) of \( V \). If in \( G \), both \( |V| \) and \( |E| \) are finite, then \( G \) is called a finite graph. If edge set in graph becomes empty then \( G \) is called an empty graph or a null graph. A simple graph is a graph in which no edge incident to same end vertices and no two edges share the same end vertices. A complete graph is a simple graph in which every vertex is adjacent to every other vertex in the graph. We use \( K_n \) to denote a complete graph with \( n \) vertices. The degree of a vertex \( d(v) \) is the count of number edges incident to it. A component of a graph is a subgraph which is maximally connected. The distance between any two vertices \( u \) and \( v \) of a graph \( G \) is denoted by \( d(u, v) \). In this paper we study only simple graphs. For a graph \( G(V, E, \varphi) \) if there is graph \( G_1 \) with vertex set \( X \) which is a non-empty subset of \( V \) and edge set which are exclusively connecting the vertices of \( X \) is called the subgraph generated by \( X \) or the maximal subgraph with vertex set \( X \).

A star graph is a graph having a fixed vertex \( v \) and edge set containing only edges which are incident with \( v \) and are not forming loop with the fixed vertex. An \( n \)-star graph is a star graph having \( n \) vertices in it.

We refer Herstein [1], and Satyanarayana and Syam Prasad [10] for further readings in ring theory and graph theory.

**Definition 1.9.** [11] A prime graph of a ring \( R \) is a graph \( G(V, E) \) having the vertex set as \( R \) and edge set contains only edges which satisfied either \( xRy = 0 \) or \( yRx = 0 \) for all distinct \( x, y \) from \( V \). It is denoted by \( PG(R) \).

**Example 1.10.** The prime graph of a ring of integers modulo 6 is given in following diagram 1.1.

![Figure 1. PG(\( \mathbb{Z}_6 \))](image)

**Observation 1.11.** [11] (i) Every prime graph of a ring is a simple graph. (ii) The degree of an additive identity element of a ring is always one less than number of elements of the ring. We can find a \( n \)-star graph as a sub graph of it as there always an edge between fixed vertex 0 to any other non-zero vertex of \( V \) together with edge connecting any two non-zero vertices satisfying the property mentioned in the definition. It is always a connected graph with distance from a vertex 0 to any other vertex is 1 and maximum distance from any two vertices 2. (iii) The distance between any two vertices of \( PG(R) \) becomes 2 if and only if when \( xRy \neq 0 \). (iv) The domination number of a prime graphs is 1 as \( \{0\} \) is a dominating set.

For further developments in prime graphs of a ring, we refer [2, 4–6, 9].

## 2. PG(\( R \)) where \( R \) is an integral domain

**Lemma 2.1.** [6] If the ring \( R \) becomes an integral domain, then prime graph of it is a star graph with number of vertices \( |R| \).

**Theorem 2.2.** [6] Given a prime number \( p \), the set of integers modulo \( p, \mathbb{Z}_p \) is a field and hence it is an integral domain. \( PG(\mathbb{Z}_p) \) is a star graph with number of vertices \( p \) and centre ‘0’. Conversely any star graph with \( p \) vertices is isomorphic to the graph \( PG(\mathbb{Z}_p) \).

**Example 2.3.** [6] (Prime graph of \( R \times \mathbb{Z}_2 \)) Suppose \( R \) is an integral domain and \( \mathbb{Z}_2 \) is a ring of integers modulo 2. For \( (a, b), (c, d) \in R \times \mathbb{Z}_2 \), we define addition and multiplication component wise. Then \( R \times \mathbb{Z}_2 \) becomes the product ring, and the zero element of \( R \times \mathbb{Z}_2 \) in \( (0, 0), (0, 0) \times (1, 0) \) and \( (0, 0) \times (0, 1) \) are two elements in \( R \times \mathbb{Z}_2 \) with \( (1, 0) \neq (0, 1) \neq (0, 0) \). So \( R \times \mathbb{Z}_2 \) is not an integral domain.

**Theorem 2.4.** [6] Let \( R \) contains \( n \) elements. Then \( PG(R \times \mathbb{Z}_2) \) contains two particular elements \( (0, 0) = a, (say), (0, 1) = b \) (say) such that \( |V(PG(R \times \mathbb{Z}_2))| = 2n \) and \( PG(R \times \mathbb{Z}_2) = \{ \text{the 2n-star graph with } R \times \mathbb{Z}_2 \text{ as vertex set and centre } a \} \) \( \cup \{ \text{the n-star graph with vertex set } \{(x, 0) : 0 \neq x \in R \text{ with centre } b \} \} \).

**Note 2.5.** In the proof of this theorem we arrived at two subgraphs \( H \) and \( K \) of \( PG(R \times \mathbb{Z}_2) \). We can state that \( E(H) \cap E(K) = \phi \) and \( a \notin V(K) \).
Remark 2.6 (6). The graph $PG(R \times \mathbb{Z}_2)$ where $R$ an integral domain, satisfy the following properties:

(i) $|V(G)| = 2n$ where $n = |R|$.
(ii) It contains two particular vertices $a, b \in V(G)$ with $a \neq b$.
(iii) There exists a subgraph $H$ of $G$ such that $H$ is a 2n-star graph (with centre $a$).
(iv) There exists a subgraph $K$ of $G$ such that $K$ is a n-star graph (with centre $b$).
(v) $G = H \cup K$.

Theorem 2.7. [6] Suppose $G$ is a graph satisfying the following conditions:

(i) $|V(G)| = 2p$, where $p$ is a prime number.
(ii) $G$ contains two particular vertices $a^*, b^*$ with $a^* \neq b^*$.
(iii) $H^*$ is a 2p-star graph (with centre $a^*$) which is a subgraph of $G$.
(iv) $K^*$ is a p-star graph of $G$ (with center $b^*$) and $a^* \notin V(K)$.
(v) $G = H^* \cup K^*$. Then $G$ is isomorphic to $PG(Z_p \times \mathbb{Z}_2)$.

Now we obtain the following new results:

Theorem 2.8. If $R$ is an integral domain, then

(i) $R$ is a uniform ideal and (ii) dim($R$) = 1.

Proof. Let $I$ be a non-zero ideal of $R$. We wish to prove that $I$ is essential in $R$. In a contrary way, suppose that $I$ is not essential in $R$. Then there exists a non-zero ideal $J$ of $R$ such that $I \cap J = (0)$. Let $0 \neq x \in I$ and $0 \neq y \in J$. Now $xy \in I \cap J = (0)$. We proved that $x, y$ are two non-zero elements such $xy = 0$, a contradiction (to the fact that $R$ is an integral domain). This shows that $I$ is essential in $R$. Therefore every non-zero ideal of $R$ is essential in $R$. By Theorem 7[8], we have that $R$ is Uniform and hence dim $R = 1$.

The proof of the following corollary from the fact that every field is an integral domain.

Corollary 2.9. If $R$ is a field, then $R$ is uniform and dim $R = 1$.

We denote the set of all isolated points of graph $G$ by $Iso(G)$.

Theorem 2.10. If $R_1, R_2$ are two integral domains, then $PG(R_1 \times R_2)$ $Iso(PG(R_1 \times R_2))$ is a bipartite graph.

Proof. Write $R_1^* = \{(a,0)/0 \neq a \in R_1\}$ and $R_2^* = \{(0,b)/0 \neq b \in R_2\}$. Write $S = (R_1 \times R_1) \cup (R_1^* \times R_2)$). We wish to show that (i) $S = Iso(PG(R_1 \times R_2))$ and (ii) subgraph of $PG(R_1 \times R_2)$ generated by $R_1^* \cup R_2^*$ is a complete bipartite graph. It is clear that $S \subseteq (R_1 \times R_2) = V(PG(R_1 \times R_2))$.

Proof for (i): Let $(a,b) \in S$. If $(a,b) = (0,0)$ then it is isolated. Suppose $(a,b) \neq (0,0)$. We show that $d(a,b) = 0$ where $d(a,b)$ is the degree of the vertex $(a,b)$. Since $(0,0) \neq (a,b) \notin R_1^* \cup R_2^*$ we have that $a \neq 0 \neq b$. In a contrary way, suppose that $d(a,b) \neq 0$. Then there exists $(0,0) \neq (x,y) \in V(PG(R_1 \times R_2))$ such that $(a,b)$ and $(x,y)$ are adjacent. By the definition of prime graph $(a,b)(x,y) = (0,0)$ that implies $ax = 0$ and $by = 0$ implies that $x = 0$ and $y = 0$. (Since 0 $a \in R_1, b \in R_2, R_1$ and $R_2$ are integral domains). Implies that $(x,y) = (0,0)$, a contradiction. Hence $d(a,b) = 0$ and so $(a,b)$ is an isolated point. Hence $S \subseteq Iso(PG(R_1 \times R_2))$. Let $(a,b) \subseteq Iso(PG(R_1 \times R_2))$. If $(a,b) = (0,0)$ then $(a,b) \in S$. If $0 \neq a$ and $0 \neq b$ then $(a,b) \notin R_1 \cup R_2$ and so $(a,b) \notin S$. If $0 \neq a$ and $b = 0$ then $a = (0,0) \in R_1^*$ and $(a,0)(0,1) = 0$, so there is an edge between $(a,b)$ and $(0,1)$ hence $(a,b)$ is not an isolated point, a contradiction. (So the case $a \neq 0$ and $b = 0$ do not arise).

Now we proved that $Iso(PG(R_1 \times R_2)) \subseteq S$. Therefore, $Iso(PG(R_1 \times R_2)) = S = (R_1 \times R_2) \cup (R_1^* \cup R_2^*)$.

Proof of (ii): To show that the subgraph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph we show the following four conditions. (i) $R_1^* \cap R_2^* = \emptyset$. (ii) there is no edge between two vertices belonging to $R_1^*$. (iii) There is no edge between two vertices belonging to $R_2^*$. (iv) $(a,b) \in R_1^*, (c,d) \in R_2^*$ implies there is an edge between $(a,b)$ and $(c,d)$. $R_1^* \cap R_2^* = \{(a,0)/0 \neq a \in R_1\} \subseteq \{(0,b)/0 \neq b \in R_2\}$. Let $(a,0)(v,0) = (0,0)$ and so uv = 0. That implies $uv = 0$ or $v = 0$ (since $R_1$ is an integral domain) and hence $(0,0) \in R_1^*$, a contradiction. So we verified that there is no edge between any two vertices in $R_1^*$. A similar valid argument shows that there is no edge between any two vertices of $R_2^*$. Let $(a,0) \in R_1^*$ and $(b,0) \in R_2^*$. Then $(a,0) \neq (0,0) \neq (0,b)$ and $(a,0)(0,b) = (0,0)$ and so there is an edge between $(a,0)$ and $(0,b)$. Hence one can conclude that the graph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph.

Proof of (iii) By Part(i), we have that $R_1^* \cup R_2^* = R_1 \times R_2$ Iso $(PG(R_1 \times R_2))$. So vertex set of the subgraph generated by $R_1^* \cup R_2^* = V(PG(R_1 \times R_2))$ Iso $(PG(R_1 \times R_2))$. By; part (ii), the subgraph generated by $(R_1 \times R_2)$ is a complete bipartite graph. This shows that $PG(R_1 \times R_2)$Iso$(PG(R_1 \times R_2))$.

3. An application to $Z_p$, ring of integers modulo a prime number $p$

Let $p, q$ be two prime numbers. Then $Z_p, Z_q$ are two integral domains.

Lemma 3.1. $PG(Z_p \times Z_q)$ $Iso$ $(PG(Z_p \times Z_q))$ forms a complete bipartite graph $(K_{(p-1)(q-1)})$.

Proof. Write $R_1 = Z_p$ and $R_2 = Z_q$. Then the proof follows from Theorem 2.9.

Theorem 3.2. Suppose that $p, q$ are prime numbers. Then the subgraph $PG(Z_p \times Z_q)$ $Iso$ $(PG(Z_p \times Z_q))$ is complete bipartite graph $(K_{(p-1)(q-1)})$. Conversely any complete bipartite graph $(K_{(p-1)(q-1)})$ (where $p, q$ are primes) is isomorphic to a subgraph of $PG(R_1 \times R_2)$ that is generated by $R_1^* \cup R_2^*$ where $R_1 = Z_p$ and $R_2 = Z_q$. 884
Proof. Write $R_1 = Z_p$ and $R_2 = Z_q$. Then the first part is Lemma 3.1.

Converse: Consider the complete bipartite graph $(K_{(p-1)(q-1)})$ with $p,q$ are prime. Suppose the set of vertices of $(K_{(p-1)(q-1)})$ are divided into the partition $\{x_1,x_2,\ldots,x_n\}$ and $\{y_1,y_2,\ldots,y_n\}$. Now $V(K_{(p-1)(q-1)}) = \{x_1,x_2,\ldots,x_n\} \cup \{y_1,y_2,\ldots,y_n\}$. Write $R_1 = Z_p$, the integral domain of integer modulo $p$, and $R_2 = Z_q$ the integral domain of integer modulo $q$. Now $R_1^1 = \{(i,0)/1 \leq i \leq p-1\}$ and $R_2^j = \{(0,j)/1 \leq j \leq q-1\}$. Define $f : R_1^1 \cup R_2^j \rightarrow V(K_{(p-1)(q-1)})$ by $f((i,0)) = x_i$ for all $1 \leq i \leq p-1$ and $f((0,j)) = y_j$ for all $1 \leq j \leq q-1$. Also $f(((i,0)(j,0)) = xy_j = f(i,0)f(j,0)$. We proved that $K_{(p-1)(q-1)}$ is isomorphic to the subgraph $PG(Z_p \times Z_q)$. Iso $PG(Z_p \times Z_q)$ of $PG(Z_p \times Z_q)$. □

Example 3.3. $Z_p \times Z_q$

![Figure 2. PG(Zₖ × Z₃)](image)

Observation 3.4. $PG(Z_6 \times Z_3)$ is not a complete graph because there is no edge between $(1,0)$ and $(2,0)$. $PG(Z_6 \times Z_3)$ is not bipartite graph because it contains a triangle $\{(2,0),(0,2),(3,0)\}$. 

Note 3.5. Example 3.3. shows that Theorem 3.2 fails if $p$ is not a prime number. So our main result 3.2 of this section is not true if both $p,q$ are not prime numbers.

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