Holographic Entanglement Entropy for noncommutative Anti-de Sitter space

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A metric is proposed to explore the noncommutative form of the Anti-de Sitter space due to quantum effects. It is proven that the noncommutativity in AdS spacetime induces a single component gravitoelectric field. The holographic Ryu-Takayanagi algorithm is then used to compute the entanglement entropy in dual CFT. This can be used for example to compute central charge of the certain noncommutative CFT2.

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Introduction The challenge now is to move the string theory from "feasible" to "practical" in order to describe strongly coupled quantum systems in high energy and condensed matter physics. The fundamental key of the path is a gauge-gravity pattern called Anti-de Sitter space/Conformal Field Theory (AdS/CFT) [1]. Acknowledging the presence of AdS boundary of gravitational bulk, asymptotic values of certain fields could act as dual quantum operators in CFT. This law will become a fundamental part of that ever string/fields duality. We can actually do computations in CFT with needing a suitable dictionary in AdS bulk. From a common sense point of view, evidences suggest that we are being modified significantly with quantum effects at Planck length, \( \lambda_p = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33} \) cm, that modify our geometry of spacetime. The Planck scale physics offered to modify the Riemannian manifold to eliminate the singularities after a quantum epoch. To provide efficient geometry of the spacetime, noncommutative geometry (NCG) have been proposed to deliver the quantum effects to the geometry. The first is the fundamental question whether it is right in principle to modify Lorentz algebra(symmetry) generally? [2]. A. Connes formulated his now famous framework of \( C^* \) algebras on spacetimes [3]. This could include deformation quantization of Poisson manifolds which were formulated to replace lost parts [4]. These noncommutativity of the coordinates of spacetime have been successfully formulated to support and maintain the anti-symmetric tensor field arising from massless states of strings [3]:

\[ [x^\mu, x^\nu] = i\Theta^{\mu\nu} \tag{1} \]

where \( \Theta^{\mu\nu} \) denotes a constant skew-symmetric tensor. The latter point has to be stressed since we did not address the question of Moyal product at all:

\[ (f \ast g) = f(x) \exp\left(\frac{i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \right) g(x). \tag{2} \]

While noncommutative spacetime with the commutation relation have proven Lorentz violation over manifolds [6, 7], they are severely improved via the twisted Poincaré algebra [8]. The most suitable abelian twist element for this is the following:

\[ \mathcal{F} = \exp\left(\frac{-i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \right). \tag{3} \]

The abelian twist elements are used in universal enveloping algebra of the Poincaré algebra describing the noncommutative multiplication of our functions. A paper describing a fundamental and systematic study of non-commutative Riemannian geometry (NCRG) has been published [9]. The isometric embeddings of a curved commutative spacetime in a flat higher dimensional spacetime [8, 10], is a term describing professional-algorithm that is extremely applicable. They introduce some principles of good noncommutativity that must be applied if embedding of commutative metric is to be used adequate. NRG transformations may be applied to alter the shape of the known metrics [11, 12]. We’ve extended the AdS/CFT even further and come up with the noncommutative NCAdS/CFT [13], gravity dual to NC gauge theories [13] and even for Holography of NC geometries (NCG) [16].

One of most important quantum measurement is entanglement entropy (EE) of the dual quantum system in CFT via AdS/CFT principle. As an example, consider the density matrix \( \rho \equiv |\Psi \rangle \langle \Psi | \) for the dual CFT system in a pure quantum state \( |\Psi \rangle \). The EE of systems never exist independent of the von Neumann entropy \( S_X = -Tr_X(\rho X \log \rho X) \). When \( S_A \) is computed, a complement observer \( B \) might be inclined to say that

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remnant will be calculated for the end value of the von Neumann entropy. This is the sense in which the von Neumann entropy define "EE" \[ \text{EE} = \sum_{\text{all states}} \langle \rho | H | \rho \rangle - \langle \rho | H | \rho \rangle^2 \]

We were astonished to discover that EE has become a geometrical, boundary entropy via AdS/CFT \[ \text{[18] [19]}, \] called holographic entanglement entropy (HEE) (see for a review \[ \text{[20]} \]. In this scheme, you'll discover the surface area entropy, one of magic’s most enduring classics \[ \text{[21]} \] in gravitational physics. This has been used recently to produce a quantitative model for critical phenomena with some success \[ \text{[22] [23]}. \]

A major goal of this letter is to understand how HEE deformed on \[ \text{[22] [29]}, \] A major goal of this letter is to understand how HEE deformed on \[ \text{[22] [29]}. \]

It helped define the coordinate chart NCRG behavior in metric? Objectives need to be a definite and real representation of this document as if you were reading it naturally.

**Foundations of NCRG** Due to standard RG convention each element of the \( (N^{1,n-1}, g) = n \)-dimensional Lorentzian manifold is labeled in the most comprehensible manner for the metric \( g \) with signature \((-1,1,\cdots,1)\)\[ \text{[8]}. \] NCRG is achieved by enriching and extending the Nash's isometric embedding of pseudo-Riemannian manifolds. Extending this just a little further allows denitions with a set of smooth function \( X^1,\cdots, X^p, X^{p+1}, \cdots, X^{p+q} \) on \( N^{1,n-1} \), such that

\[
g = -\sum_{i=1}^p (dX^i)^2 + \sum_{p+1}^q (dX^i)^2. \tag{4}\]

It helped define the coordinate chart \( U \) of \( N^{1,n-1} \) and natural coordinates \( = \{x^0, x^1, \cdots, x^n\} \) for it. How would we define NC behavior in metric? Objectives need to be a definite and real representation of this document as if you were reading it naturally.

\[
\hat{g}_{ij} = E_1 \cdot E_j.
\]

We denote these metrics as \( N \) \( g = (\hat{g}_{ij}) \) the \( n \times n \) matrix with entries \( \hat{g}_{ij} \in A \). Let \( (\hat{g}^{ij}) \) denote the standard inverse :

\[
\hat{g}^{ij} \cdot \hat{g}_{jk} = \hat{g}^{ik} \cdot \hat{g}_{jk} = \delta^k_l.
\]

**Embedding technique**: To find out more about the form on NCAdS follow the embedding scheme for \( AdS_3 \) spacetime. This showed that \( AdS_3 \) were already embedding into the \( (3 + 1)D \) flat spacetime \( R^{2,2} \):

\[
g = -(dX^2_1 + dX^2_2) + (dX^2_3 + dX^2_4). \tag{11}\]

We will be comparing \( \text{[11] [13]} \) with \( \text{[8]} \). We observe that \( p = 2, q = d - 1 \). This embedding is invariant under \( SO(2,1) \), which is the precise isometry group of \( AdS_3 \), with conformal boundary. An appropriate rescaling casts the boundary in hyperboloid form in \( (2)D \) form, which is universal for systems supporting \( SO(1,2) \) isometry group, with one dilatation and two special conformal transformations. Now, we find the embedding coordinates \( (t, \rho, \theta) \) for all of the hyperboloids:

\[
\begin{align*}
X_1 &= l \cos \rho \sin \theta, \quad (12) \\
X_2 &= l \cos \rho \cos \theta, \quad (13) \\
X_3 &= l \sin \rho \sin \theta, \quad (14) \\
X_4 &= l \sin \rho \cos \theta \quad (15)
\end{align*}
\]

on \( U \) is defined by:

\[
A \cdot B = -\sum_{i=1}^p a_i \cdot b_i + \sum_{j=p+1}^{p+q} a_i \cdot b_i
\]

We can always define the NC metric in terms of functions \( X \in A^m, E_i = \partial_i X \):

\[
\hat{g}_{ij} = E_i \cdot E_j.
\]

Results will define the connection of the metric in response to spacetime deformation:

\[
\nabla_i E_j = \Gamma^k_{ij} \cdot E_k,
\]

here \( \Gamma^k_{ij} = \partial_i E_j \cdot E_k \) and

\[
\Gamma^k_{ij} = \partial_j E_i \cdot E_k.
\]

and the associated noncommutative Riemann and Ricci scalars are defined as:

\[
R^l_{ijk} = \partial_i \partial_j \Gamma^l_{ik} - \partial_j \partial_i \Gamma^l_{ik} - \Gamma^l_{ik} \cdot \Gamma^p_{jp} - \Gamma^p_{kp} \cdot \Gamma^l_{jp}, \tag{8}\]

\[
R^l_{ij} = \hat{g}^{ik} \cdot \hat{R}^p_{kpj}, \quad \Theta^l_{ij} = \hat{g}^{ik} \cdot \hat{R}^l_{kpj}, \quad R = R^l_{ij}. \tag{9}\]

In our case the appropriate addressee of Einstein equations with cosmological constant from the auditor to those modifications with NCRG is the following:

\[
R^l_{ij} + \Theta^l_{ij} - \delta^l_{ij} R + 2 \delta^l_{ij} \lambda = 2 T^l_{ij}; \tag{10}\]

where \( T^l_{ij} \) denotes the generalized energy-momentum tensor, \( \lambda \) is the cosmological constant. Exact results have been reported in a few papers \( \text{[11] [13]} \). The goal of next section is to realize NCAdS via the mentioned formalism.

\[
g = -(dX^2_1 + dX^2_2) + (dX^2_3 + dX^2_4). \tag{11}\]

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\]

It is wise to obtain a copy of dot-product for an element

\[
A = (a_1, \ldots, a_m) : A^m \otimes_R [\mathbb{K}] A^m \rightarrow A^m \quad (6)
\]
We know we need an AdS radius, so we should define it by \( l^2 = -\frac{3}{\Lambda} \). These coordinates represent a special universal covering of the \( \text{AdS}_3 \) spacetime:

\[
g = l^2(-\cosh^2 \rho d\rho^2 + d\rho^2 + \sinh^2 \rho d\theta^2) .
\]

The domain \( \rho = 0 \) represent the the AdS boundary of the AdS metric. The aim of the next section is to compute the noncommutative version of \( (16) \) from \( (4) \).

**Explicit form of NC \( \text{AdS}_3 \):** Evaluation techniques of \( (12) \) include star product and skew symmetric matrix \( \theta_{ij} \). The starting point for the computation of \( (3) \) is the functions multiplying \( (17) \), which we can write it in the following simple form:

\[
A(y) \ast B(y') = \lim_{y \rightarrow y'} e^{k(\partial_y \partial_{y'} - \partial_{y'} \partial_y)} A(y)B(y'), \quad (17)
\]

\[
y \equiv (t, \rho, \theta),
\]

where

\[
(\theta_{ij})_{3 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]

The computation of the \( (7) \) is performed continuously for all orders of \( \hbar \), using the Zassenhaus formula:

\[
e^{k(\partial_y \partial_{y'} - \partial_{y'} \partial_y)} e^{-k(\partial_y \partial_{y'})} = e^{k(\partial_y \partial_{y'})} e^{-k(\partial_y \partial_{y'})}.
\]

This allows routine computation of the components of NCAdS. Here is the complete form of the nonvanishing components of the noncommutative version of \( \text{AdS}_3 \) spacetime:

\[
\begin{align*}
\dot{g}_{tt} &= -l^2 \cosh^2 \rho \\
\dot{g}_{\rho \rho} &= l^2 + 2l^2 \sin^2 \left(\frac{\hbar}{2}\right) \left(1 - \cos \hbar \cosh(2\rho)\right) \\
\dot{g}_{\theta \theta} &= l^2 \sin^2 \left(\frac{\hbar}{2}\right) \left(e^{2\rho} \cos \hbar - \sin(2\theta - \hbar)\right) + e^{-2\rho} \sin(2\theta) \\
\dot{g}_{\rho \theta} &= l^2 \sin^2 \rho - \frac{l^2}{\sqrt{2}} \sin^2 \left(\frac{\hbar}{2}\right) \left[e^{2\rho} \cos \hbar - \frac{1}{\sqrt{2}} \left(\sqrt{2} \sinh(\rho) \cos(2\theta - \frac{\pi}{4}) + 2 \cos(2\theta - \frac{\pi}{4}) \cos \hbar \right) \right] \\
&= -2 \cos(2\theta - \frac{\pi}{4}) \cos \hbar \\
\end{align*}
\]

The metric components are reduced to a standard \( \text{AdS}_3 \) in limit with \( \hbar = 0 \). Deformation of \( (15) \) using \( (10, 20, 22, 23) \) is given by the following metric:

\[
\dot{g} = g + (\delta \dot{g}_{tt} d\rho^2 + 2 \delta \dot{g}_{\rho \rho} d\rho d\theta + \delta \dot{g}_{\theta \theta} d\theta^2).
\]

where \( \delta \dot{g}_{ij} \equiv |\dot{g}_{ij} - g_{ij}| \ll g_{ij} \). There is a dimensional reduction in noncommutative part of metric. It was the first observation in the NCRG to use NC to reduce dimensions of the commutative bulk and is fully surprising.

If Planck scale quantum effects are sufficient to deform the commutative bulk, then NC to reduce bulk’s dimensions will usually be arisen. Ascending to the NC metric \( (28) \), the portion of noncommutativity escaping from the purely \( \rho \) dependency to the \( (\rho, \theta) \) dependent metric in one lower dimension. The NC portion of the metric gets sent to the spacelike metric due to the absence of \( \delta g_{tt} \). The deformed metric can write ever more wellness to produce a gravitoelectric field.

\[
E_g \equiv \frac{1}{2} \nabla_{\rho \theta} \log g_{11}, \quad (24)
\]

where we define the length of a strip in the Euclidean coordinates \( \tilde{\rho}, \phi \) as \( -i\vartheta \):

\[
\gamma \equiv (-g_{11} \dot{A}^2 + g_{22}) d\phi^2, \quad (25)
\]

where

\[
\begin{align*}
g_{11} &= \delta g_{\rho \rho} \{|\rho \rightarrow i\tilde{\rho}, \theta \rightarrow -i\phi\}, \\
g_{22} &= \delta g_{\rho \theta} \{|\rho \rightarrow i\tilde{\rho}, \theta \rightarrow -i\phi\}, \\
\dot{A} &= \frac{\delta g_{\rho \phi}}{\delta g_{\rho \rho}} \{|\rho \rightarrow i\tilde{\rho}, \theta \rightarrow -i\phi\}
\end{align*}
\]

There is no doubt about a fact that *noncommutativity in AdS spacetime induces a single component gravitoelectric field.*

**HEE for NC \( \text{AdS}_3 \):** The HEE is defined as the entropy of a region of space \( \tilde{A} \) and its complement on the minimal surfaces in \( \text{AdS}_{d+1} \). The HEE expression (29) can then be corrected which indeed, is infinitely long and divergent.

\[
S_{\tilde{A}} \equiv S_{\text{HEE}} = \frac{\text{Area}(\gamma_{\tilde{A}})}{4G_{d+1}}. \quad (30)
\]

What we need is to compute the \((d-1)D\) Planck minor surface \( \gamma_{\tilde{A}} \) on equal time patches. We suppose that we can extend \( \gamma_{\tilde{A}} \) inside the bulk, and we are also limited, to a certain boundary condition, by keeping the boundaries same \( \partial \gamma_{\tilde{A}} = \partial \tilde{A} \). Several options for the parametrization of the boundary surfaces are available as well as different choices for the form of minimal surface functional in the (29). Here the assumption for HEE purpose is that none of the minimal surfaces on the boundaries is noncommutative under (11). We consider that the assumption of commutativity is correct and may validate some of the arguments. Often the implicit assumption was that minimal surface appeared into a strip from which the UV boundary of AdS ever cutoff by \( \rho = \rho_0 \):
here $S_{HEE}^{AdS} = \frac{l}{4G_3} \log \left( e^{2\rho_0} \sin \left( \frac{\rho}{2} \right) \right)$ was calculated in [18] in terms of central charge of (1 + 1) CFT and and geometrical lengths of spacelike sector of AdS spacetime. Consequently, what major has been done is to write an expression for NCHEE as follows:

$$\Delta S_{HEE}^{nmk} = \frac{1}{4G_3} \int_{0}^{\rho_0} d\rho \left[ (\rho')^{2m+n-k} \right]$$

$$(l^2 \rho^2 + l^2 \sinh^2 \rho) \left[ (\delta g_{\rho\rho})^m (\delta g_{\theta\theta})^{n-k} (\delta g_{\phi\phi})^{k-m} \right].$$

We define a Lagrangian density as the following:

$$L^{nmk}(\rho, \theta) \equiv \left[ (\rho')^{2m+n-k} (l^2 \rho^2 + l^2 \sinh^2 \rho) \right]$$

$$\times (\delta g_{\rho\rho})^m (\delta g_{\theta\theta})^{n-k} (\delta g_{\phi\phi})^{k-m}.$$ 

Unfortunately, this Lagrangian doesn't satisfy Beltrami's identity. The static geodesic satisfies the following equation:

$$\partial_\rho (\rho' L^{nmk}) - \partial_\rho L^{nmk} = 0.$$ 

Our problem is to minimize the following functional:

$$\text{Minimize} \{ I^{nmk}[\rho(\theta)] \} = \int_{0}^{\rho_0} d\rho L^{nmk}(\rho, \theta),$$

that connects the boundary points $\rho(0) = \rho(\theta_0) = \rho_0 \gg 1$. Leading the $m = n = k = 1$ of integral to look at $\Delta S_{HEE}^{nmk}$ is really appropriate here. An ideal solution here is to choose a phase-space like solution for $\Delta S_{HEE}^{nmk}$ in the form $\rho = \rho'(\rho)$ [33]:

$$\rho'^2 = \left\{ \begin{array}{ll}
-\frac{2lq_0}{\sqrt{2}} - \sqrt{\frac{2}{2}} \cos \left( \frac{\pi}{4} \sinh^{-1} \left( \frac{2lq_0}{\sqrt{2}} \right) \right), & p < 0 \\
-2\sqrt{2} \sinh \left( \frac{\pi}{4} \sinh^{-1} \left( \frac{2lq_0}{\sqrt{2}} \right) \right), & p > 0
\end{array} \right.$$ 

Suppose $f \equiv 2l^2 \sinh^{2} \left( \frac{p}{2} \right) \left( 1 - \cos h \cosh(2\rho) \right)$ and $4p^3 + 27q^2 > 0$, we define:

$$p = -\frac{E^2 f^2 \sinh^{3} \rho}{3} \left( E^2 f^2 \sinh^{3} \rho - 6 \right),$$

$$q = -\frac{E^2 f^2 \sinh^{3} \rho}{27} \left( 2E^4 f^4 \sinh^{6} \rho \right)$$

$$-18E^2 f^2 \sinh^{3} \rho + 27 \right).$$

We observe a generic same scenario with the $p < 0, q < 0$ in our model.

However rewriting $\Delta S_{HEE}^{nmk}$ in terms of $\rho$ really produce exceptionally pretty links between $\rho, \rho'$:

$$\Delta S_{HEE}^{nmk} = \frac{l}{2G_3} \sin^{2} \left( \frac{h}{2} \right) \lim_{\rho \to 0} \int_{e}^{\rho_0} dp \left[ \rho' \right]$$

$$\times (\rho^2 + \sinh^2 \rho) \left[ 1 - \cos h \cosh(2\rho) \right].$$

The phase-space solution [37] is used to evaluate the difference [10] between pure commutative AdS and NC one. The numeric integration will be applied to evaluate [10] the effect of various forms of NC terms:

$$\Delta S_{HEE}^{nmk} \simeq \frac{l}{2G_3} \sin^{2} \left( \frac{h}{2} \right) \lim_{\rho \to 0} \int_{e}^{\rho_0} dp \left[ \rho' \right]$$

$$\times (\rho^2 + \sinh^2 \rho) \left[ 1 - \cos h \cosh(2\rho) \right].$$

here $\gamma_1 = \frac{16 \sqrt{6} l^{10} E^5}{G_3} = L^{111} - \rho' \partial_\rho L^{111} \equiv \text{constant}$. Finally we compute the "physical" (the $\Re$ part) of the leading integral, we obtain:

$$\Delta S_{HEE}^{nmk} \simeq \left( \frac{4 \pi E^5}{G_3} \right) e^{\rho_0}. \right.$$ (42)

We found a mathematical formula that worked, much more anything capable of becoming the basis for the HEE of a NCAdS $\gamma$. The first term in this formula becomes divergent completely in agreement to the commutative case of $AdS\gamma$, it is safe to use on all higher dimensional extensions. The HEE for a NCAdS$\gamma \geq 4$ can be calculated in an analogous manner to the $AdS\gamma$ in NCRG. The first term, known as noncommutative area law, presents the ruling in holographic formula beginning if the spacetime becomes noncommutative just slightly. However, it manages to be surprising or deviate from the Ryu-Takayanagi formula of the [18]. In our NC case, the growth in the value of HEE is proportional to the instantaneous value of the HEE. The rate may be positive or negative. By comparing experiences in Ryu-Takayanagi formula and NC case, we discover some possible perspectives in interpretation of NCAdS spaces. By comparing results from the two studies, we hope to understand more about the impact of HEE being reared in Planck scale.

Summary This letter offers a glimpse of how one can use holography to realize entanglement entropy for the noncommutative AdS space. We first compute the noncommutative Anti-de Sitter space using a fully consistent version of noncommutative Riemannian geometry. Later we use the Ryu-Takayanagi formula to compute holographic entanglement entropy of a commutative region, needs to derive an expression for entropy of a possible noncommutative CFT. A correction for the entanglement entropy was applied to the minimal case $n = m = k = 1$ value because this was generally large compared to the other higher terms. Planck scale correction for entropy of noncommutative system of order $h^{111}$ contained the following parameters: a constant of motion $G_3$, the powers of AdS radii $l$ as $l^{111}$ and the Newtonian constant $G_3^{-1}$. Recommendation is to allow noncommutative corrections of $\Delta S_{HEE}^{nmk}$ that requires further research.

\footnote{It stated: these Lagrangians do not satisfy a simple identity in the form $L - \rho' \partial_\rho L \not= C$. The only Beltrami’s case came from $m = n = k$ term, where the geometric $\rho(\theta)$ obtained from [35] was not analytical.}
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