A noncommutative anomaly through Seiberg-Witten map and non-locally regularized BV quantization

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Abstract

Anomalies are one essential concept for the renormalization of noncommutative (NC) gauge theories. A NC space can be visualized as a deformation of the usual spacetime with the $\ast$-product and can be constructed after the quantization of a given space with its symplectic structure. The Seiberg-Witten (SW) map connects NC fields, transformations parameters and gauge potential to their commutative analogs. In this work we used the SW map to calculate the NC version of the anomaly of the BV quantized chiral Schwinger model with nonlocal regularization.

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1 Introduction

It is a well known fact that quantum field theory has its main basis in the principle of gauge symmetry \[1\]. The gauge theory, constructed with the principle of gauge symmetry, encompassing the symmetries and their corresponding conservation laws, has underlying role in the description of the fundamental forces in nature. Nevertheless, we have also to consider that a specific conservation law that is true in a certain classical theory, can be broken when the theory is quantized. In this case we have what is known as an anomaly (for a review see \[2\]). Anomalies have their importance in physics where they are needed to describe certain experimental facts, for example. The anomaly cannot be considered just a perturbation effect, which results from the regularization of some divergent diagrams, it shows the deep laws of quantum physics. So, as we can see, it is important to find methods to compute the anomaly by the quantized the primary theory.

After quantization, we have the classical dynamical variables of the theory becoming noncommuting operators. This fact makes us to believe that the classical manifold framework of spacetime at the quantum (Planck) scale should have some kind of non-commutative (NC) structure. So, to deal with this quantum gravity theory we have to consider a quantum field theory beyond a structure that depends on locality. To understand these and other issues, it is a common sense, nowadays, in theoretical physics that NC geometry have the proper, precise and rigorous formalism to accomplish this target (see \[3\] for reviews).

Concerning NC manifolds, the anomalies in gauge theories has been explored with some intensity. It was realized the importance of the structure of anomalies in this scenario. The map developed by Seiberg and Witten \[4\] establish the validity of classical gauge transformations for theoretical systems constructed in NC and ordinary commutative spacetimes. Through this analysis, we have an alternative procedure to investigate NC gauge theories via their commutative analogs.

In this paper we compute the anomaly of the chiral Schwinger model (CSM) in a NC bi-dimensional spacetime manifold. After reviewing the non-local regularization BV formalism, we have applied the SW map to obtain the NC version of the anomaly. Although the CSM anomaly is a well known result in the current literature, its NC analog is a new one.

The method developed by Batalin-Vilkovisky (BV) \[5\] showed itself to be a very powerful way to quantize the most difficult field theories. A two dimensional gauge theory, the string theory, is one of these examples. For a review see \[6, 7, 8\]. The BV, or field-antifield, formalism provides, at Lagrangian level, a general framework for covariant path integral quantization of gauge theories. This formalism uses interesting mathematical objects like a Poisson-like bracket (the antibracket), canonical transformations, ghosts for the BRST transformations, etc. The most important object of this method at the classical level is an equation called classical master equation (CME).

It is important to say that the CSM has been constructed and completely solved by Jackiw and Rajaraman \[9\]. The organization of the paper is as follows. In section 2 we described the main relation between the SW maps and the anomaly. In section 3 a brief review of the field-antifield formalism and its regularization has been made. The computation of the NC CSM anomaly at one-loop has been calculated in section 4. The conclusions and final remarks were accomplished in section 5.
2 Noncommutativity and the Seiberg-Witten map

In few words we can say that the noncommutativity used follows the idea of the deformation of the Minkowsky space with a real antisymmetric and constant parameter $\theta^{\mu\nu}$ such that $[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, where the $\star$-product will be defined in a jiffy. It is known also as the Moyal-Weyl product.

To obtain the SW maps we have to take two different limits of string theory. Hence, we can define a map between $N_c$ fields and theirs ordinary analogs. A gauge equivalence relation can be written as

$$\hat{A}_\mu(A, \theta) + \delta_\lambda \hat{A}_\mu(A, \theta) = \hat{A}_\mu(A + \delta_\alpha A, \theta)$$  (1)

where $\alpha$ and $A$ are the ordinary gauge parameter and gauge field respectively. $\delta_\alpha$ is the ordinary gauge transformation,

$$\delta_\alpha A_\mu = \partial_\mu \alpha - i[A_\mu, \alpha] = D_\mu \alpha$$  (2)

So, we can write (1) as

$$\hat{\delta}_\lambda A_\mu(A, \theta) = \hat{A}_\mu(A + \delta_\alpha A, \theta) - \hat{A}_\mu(A, \theta)$$  (3)

It is important to say that the NC gauge field $\hat{A}$ and the NC gauge parameter $\hat{\lambda}$ can be defined obeying the following dependence,

$$\hat{A}_\mu = \hat{A}_\mu(A, \theta)$$

$$\hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}(A, \theta)$$

$$\hat{\lambda} = \hat{\lambda}(\alpha, A, \theta)$$  (4)

Hence, we have to solve (1) simultaneously for $\hat{A}_\mu$ and $\hat{\lambda}_\lambda$ which is a difficult task. However, this difficulty can be solved by generalizing the ordinary gauge condition

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha, \beta]}$$  (5)

to the NC case

$$i\delta_\alpha \hat{\lambda}_\beta - i\delta_\beta \hat{\lambda}_\alpha - [\hat{\lambda}_\alpha, \hat{\lambda}_\beta]_\star = i\hat{\lambda}_{-i[\alpha, \beta]}$$  (6)

where this equation focuses only on the parameter $\hat{\lambda}_\alpha$ [10] and the solutions can be computed order by order [11]. To find the SW maps, one have to solve (6) and (3), respectively, order by order in $\theta$.

After that explanation we will explain the main steps of SW work [4] which objective was to construct a bridge between commutative and NC field. Nevertheless, our aim is to use the SW map to calculate the NC version of CSM from its commutative analog, as we said above.

Let us begin with the well known maps in a $U(1)$ gauge theory given by

$$\hat{A}_{\mu
u} = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + O(\theta^2)$$  (7)
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} - \theta^{\alpha\beta} (A_{\alpha \beta} F_{\mu\nu} + F_{\mu\alpha} F_{\beta\nu}) + O(\theta^2) \] (8)
\[ \hat{\lambda} = \lambda - \frac{1}{2} \theta^{\alpha\beta} A_{\alpha \beta} \lambda + O(\theta^2) \] (9)

where the hat indicates that the variable is NC. It is easy to see that this map is a gauge equivalence between the NC gauge theory and its ordinary analog. Also, the map (8) is a direct result from map (7) since we have also that

\[ \hat{F}_{\mu\nu} = \partial_{\mu} \hat{A}_{\nu} - \partial_{\nu} \hat{A}_{\mu} - i[\hat{A}_{\mu}, \hat{A}_{\nu}] \] (10)

where \([\hat{A}_{\mu}, \hat{A}_{\nu}] = \hat{A}_{\mu} \ast \hat{A}_{\nu} - \hat{A}_{\nu} \ast \hat{A}_{\mu}\) and the \(\ast\)-product is the so-called Moyal-Weyl product or star product which, for two fields \(A(x)\) and \(B(y)\) is given by

\[ (A \ast B)(x) = \exp \left( \frac{i}{2} \theta^{\alpha\beta} \partial_{\alpha} A(x) \partial_{\beta} B(y) \right) \] (11)

So, \(\hat{F}_{\mu\nu}\) in (10) can be written as,

\[ \hat{F}_{\mu\nu} = \partial_{\mu} \hat{A}_{\nu} - \partial_{\nu} \hat{A}_{\mu} + \theta^{\alpha\beta} \partial_{\alpha} \hat{A}_{\mu} \partial_{\beta} \hat{A}_{\nu} + O(\theta^2) \] (12)

However, notice that, since we know that \(F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\) is gauge invariant, we have that \(\hat{F}_{\mu\nu}\) transforms covariantly under the star gauge transformation \([12, 13]\) given by

\[ \delta_{\hat{\lambda}} \hat{F}_{\mu\nu} = i[\hat{\lambda}, \hat{F}_{\mu\nu}] \] (13)

which stability is granted by the gauge transformations given in (7)-(9) such that

\[ \delta_{\hat{\lambda}} \hat{A}_{\mu} = \hat{D}_{\mu} \ast \hat{\lambda} \equiv \partial_{\mu} \hat{\lambda} + i[\hat{\lambda}, \hat{A}_{\mu}] \] (14)

and \(\delta_{\lambda} A_{\mu} = \partial_{\mu} \lambda\).

To analyze NC gauge theories we have to make the transition between commutative and NC gauge theories. For example, if we are treating a NC action defined by \([12, 13]\)

\[ \hat{S}(\hat{A}, \hat{\Psi}) = -\frac{1}{4} \int d^4 x \hat{F}_{\mu\nu} \ast \hat{F}_{\mu\nu} + \hat{S}_M(\hat{\Psi}, \hat{A}) \] (15)

where the first term \((\hat{S}_A)\) is for the gauge field alone and the second one is for the charged matter fields \(\hat{\Psi}\). The equation of motion for \(\hat{A}_{\mu}\) is

\[ \frac{\delta \hat{S}_A}{\delta A_{\mu}} = \hat{D}_{\nu} \ast \hat{F}^{\nu\mu} = \hat{j}^\mu \] (16)

where

\[ \hat{j}^\mu = -\frac{\delta \hat{S}_M}{\delta A_{\mu}} |_{\hat{\Psi}} \] (17)
If we substitute the SW map (7)-(9) into the action in (15) we will have the NC parameters within the commutative action as we can see in

\[ \hat{S}(\hat{A}, \hat{\Psi}) \rightarrow S(A, \Psi, \theta) = S_A(A, \theta) + S_M(A, \Psi, \theta) \] (18)

where

\[ S_A(A, \theta) = -\frac{1}{4} \int d^4x \left[ F_{\mu\nu}^2 + \theta^{\alpha\beta} F_{\mu\nu}^\alpha (2F_{\mu\alpha}F_{\nu\beta} \frac{1}{2} F_{\beta\alpha}F_{\mu\nu}) + O(\theta^2) \right] \] (19)

From Eq. (18) we have that

\[ \frac{\delta S_A(A, \theta)}{\delta A_\mu} = J_\mu = -\frac{\delta S_M(A, \theta)}{\delta A_\mu} \] (20)

and, directly, we obtain the conservation law

\[ \partial_\mu J_\mu = 0 \] (21)

From (8) we can write,

\[ \hat{J}_\mu = J_\mu - \theta^{\alpha\beta} A_\alpha \partial_\beta J_\mu + O(\theta^2) \] (22)

Notice that in (22) we have the freedom of adding more \( O(\theta) \) terms such that these extra terms are invariant under ordinary gauge transformation. The most general expression is given by,

\[ \hat{J}_\mu = J_\mu - \theta^{\alpha\beta} A_\alpha \partial_\beta J_\mu + c_1 \theta^{\mu\alpha} F_{\alpha\beta} J_\beta + c_2 \theta^{\alpha\beta} F_{\alpha\beta} J_\mu + c_3 \theta^{\alpha\beta} F_{\mu\alpha} J_\beta + O(\theta^2) \] (23)

where \( c_1, c_2 \) and \( c_3 \) are parameters to be determined [12, 13]. For instance, for a simultaneous conservation we can write that

\[ \hat{D}_\mu \ast \dot{J}_\mu = \partial_\mu J_\mu \] (24)

which fixes \( c_1 = 2c_2 = 1 \) and \( c_3 = 0 \), and

\[ \dot{J}_\mu = J_\mu - \theta^{\alpha\beta} \partial_\beta (A_\alpha J^\mu) + \theta^{\mu\alpha} F_{\alpha\beta} J_\beta + O(\theta^2) \] (25)

And using (7) and (25), the covariant divergence of \( \dot{J}_\mu \),

\[ \hat{D}_\mu \ast \dot{J}_\mu = \partial_\mu \dot{J}_\mu + i[\dot{J}_\mu, \dot{A}_\mu] = \partial_\mu \dot{J}_\mu - \theta^{\alpha\beta} \partial_\alpha \dot{J}_\mu \partial_\beta \dot{A}_\mu + O(\theta^2) = \partial_\mu J_\mu + \theta^{\alpha\beta} \partial_\alpha (A_\beta \partial_\mu J_\mu) + O(\theta^2) \] (26)

and we see clearly in this expression that the covariant conservation of \( \dot{J}_\mu \) relies on the conservation of \( J_\mu \).

To attack the real issue here, the anomaly, we can use the study developed until now for the vector current to derive a map for the axial current. As we know, at the quantum level, the axial currents are not conserved. The well known ABJ current [14] is not modified by noncommutativity and is written as
\[ \partial_\mu J_5^\mu = A = \frac{1}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \]  

(27)

and the anomaly in the NC manifold is given by what we saw above so that

\[ \hat{A} = \hat{D}_\mu \star \hat{J}_5^\mu = \frac{1}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \hat{F}^{\mu\nu} \star \hat{F}^{\lambda\rho} \]  

(28)

Finally, the cherished map for anomalies, obtained from (26), is given by,

\[ \hat{A} = A + \theta^{\alpha\beta} \partial_\alpha (A_\beta A) + O(\theta^2) \]  

(29)

which was demonstrated to be valid [13], although derivative corrections are necessary at higher orders. So, now we can establish the axial current at the quantum level, so that

\[ \hat{J}_5^\mu = J_5^\mu - \theta^{\alpha\beta} \partial_\alpha (A_\beta J_5^\mu) + \theta^{\mu\alpha} F_{\alpha\beta} J_5^\beta + O(\theta^2) \]  

(30)

which can be used to investigate anomalous commutators in NC electrodynamics, for example [12].

3 The Field-Antifield Formalism

The basic idea of BV formalism is BRST invariance. The ingredients are the fields \( \Phi^A \), i.e., the classical fields of the theory, the ghosts, the auxiliary fields and their canonically conjugated antifields \( \Phi^*_A \). With all this elements we construct the so called BV action. At the classical level, the BV action becomes the classical action when all the antifields are put to be zero. A gauge-fixed action can be obtained by a canonical transformation. At this time we can say that the action is in a gauge-fixed basis. The other way to fix the gauge is through the choice of a gauge fermion and to make the antifields to be equal to the functional derivative of this fermion.

The method can be applied to gauge theories which have an open algebra (the algebra of gauge transformations closes only on shell), to closed algebras, to gauge theories that have structure functions rather than constants (soft algebras), and to the case where the gauge transformations may or may not be independent, reducible or irreducible algebras respectively. Zinn-Justin introduced the concept of sources of BRST-transformations [15]. These sources are the antifields in the BV formalism. It was shown also that the geometry of the antifields have a natural origin [16].

At the quantum level, the field-antifield formalism also works at one-loop anomalies [17, 18]. There, with the addition of extra degrees of freedom, which origins an extension of the original configuration space, we have a solution for the regularized quantum master equation (QME) at one-loop, that has been obtained as an independent part of the antifields inside the anomaly.

Let us construct the complete set of fields, including in this set the classical fields, the ghosts for all gauge symmetries and the auxiliary fields. This complete set will be denoted by \( \Phi^A \). Now, let us extend this space with the same number of fields, but at
this time, one will define the antifields \( \Phi^*_A \), which is the canonical conjugated variables with respect to the antibracket structure. This can be written as

\[
(X, Y) = \frac{\delta X}{\delta \Phi} \frac{\delta Y}{\delta \Phi^*} - (X \longleftrightarrow Y),
\]

(31)

where the indices \( r \) and \( l \) denote right and left derivation respectively.

Concerning the antibrackets, one can write the canonical conjugation relations

\[
(\Phi^*_A, \Phi^*_B) = \delta^A_B, \quad (\Phi^*_A, \Phi^*_B) = (\Phi^*_A, \Phi^*_B) = 0.
\]

(32)

The antifields \( \Phi^*_A \) have opposite statistics than their conjugated fields \( \Phi^*_A \). The antibracket is a fermionic operation so that the statistics of the antibracket \( \langle X,Y \rangle \) is opposite to that of \( XY \). The antibracket also satisfies some graded Jacobi relations,

\[
\langle X, (Y,Z) \rangle + (-)^{\epsilon_X \epsilon_Y + \epsilon_X + \epsilon_Y} \langle Y, (X,Z) \rangle = \langle \langle X, Y \rangle, Z \rangle.
\]

(33)

where \( \epsilon_X \) is the statistics of \( X \), i.e. \( \epsilon(X) = \epsilon_X \).

We will define a quantity named ghost number for fields and antifields. These are integers numbers such that

\[
gh(\Phi^*) = -1 - gh(\Phi).
\]

(34)

One can then construct an action of ghost number zero so that it is an extended action, the so called BV action, also called classical proper solution, so that

\[
S(\Phi, \Phi^*) = S_0(\Phi) + \Phi^*_A R^A(\Phi) + \frac{1}{2} \Phi^*_A \Phi^*_B R^{AB}(\Phi) + \cdots + \frac{1}{n!} \Phi^*_A \cdots \Phi^*_A R^{A_1 \cdots A_n} + \cdots
\]

(35)

This equation contains the whole algebra of the theory, the gauge invariance of the classical action \( (S_{cl} = S_{BV}(\Phi^*_A, \Phi^*_A) = 0) \), Jacobi identities,... Gauge fixing is obtained either by a canonical transformation or by choosing a fermion \( \Psi^A \) and writing

\[
\Phi^* = \frac{\delta \Psi^A}{\delta \Phi^A}
\]

(36)

At the quantum level the quantum action can be defined by

\[
W = S + \sum_{p=1}^{\infty} \hbar^p M_p,
\]

(37)

where the \( M_i \) are the quantum corrections, the Wess-Zumino terms, to the quantum action. The expansion (37) is not the only one, but is the usual one. An expansion in \( \sqrt{\hbar} \) [19] can be made. This will originate the so called background charges, which is useful in conformal field theory [20].

Concerning the regularization framework, the definition of a path integral is missing, which can be seen as a way to define the measure. Anomalies represent the non conservation of classical symmetries at quantum level.
For a theory to be free of anomalies, the quantum action $W$ has to be a solution of the QME,

$$ (W, W) = 2i\hbar \Delta W $$

where

$$ \Delta \equiv (-1)^{A+1} \frac{\partial r}{\partial \Phi^A} \frac{\partial r}{\partial \Phi^*}. $$

In Eq. (39) one can see that:

$$ \mathcal{A} \equiv \left[ \Delta W + \frac{i}{2\hbar} (W, W) \right] (\Phi, \Phi^*). $$

And computing a $\hbar$ expansion,

$$ \mathcal{A} = \sum_{p=0}^{\infty} \hbar^{p-1} M_p $$

one have the form of the $p$-loop BRST anomalies,

$$ \begin{align*}
\mathcal{A}_0 &= \frac{1}{2} (S, S) \equiv 0 \\
\mathcal{A}_1 &= \Delta S + i (M_1, S) \\
\mathcal{A}_2 &= \Delta M_{p-1} + i \sum_{q=1}^{p-1} (M_q, M_{p-q}) + i (M_p, S) , p \geq 2
\end{align*} $$

The first equation is the known CME. The second one is an equation for $M_1$. If the second equation does not have a solution for $M_1$ then $\mathcal{A}$ is called anomaly. The anomaly is not uniquely determined since $M_1$ is arbitrary. and it satisfies the Wess-Zumino consistency condition [21], $(\mathcal{A}, S) = 0$.

### 3.1 The Non-local Regularization

When the Wess-Zumino terms, which cancel the anomaly, can not be found, the theory can be said to have a genuine anomaly. A few years back, a method was developed to handle with global anomalies [22], i.e., when a quantity that is conserved classically is not conserved at quantum level.

However, the solution of the QME is not easily obtained because there is a divergence when the $\Delta$ operator, a two order differential operator defined below, is applied on local functionals, a $\delta(0)$-like divergence. Therefore, a regularization method has to be used to cut the divergence in the QME. One of these methods is the well known Pauli-Villars (PV) regularization [23, 24, 25], where new fields, the PV fields, and an arbitrary mass matrix are introduced. However, this method is very useful only at one-loop level. At higher orders, the PV method is still mysterious. The BPHZ renormalization [26] of the BV formalism was formulated [27, 28]. A dimensional regularization method in the quantum aspect of the field-antifield quantization has been studied in ref. [29].

The non-local regularization (NLR) [30, 31, 32] gives a consistent way to compute anomalies at higher order levels of $\hbar$. The main ideas were based on Schwinger’s proper time method [33]. The preliminary results [34, 35] were very well received. The NLR
separates the original divergent loop integrals in a sum over loop contribution in such a way that the loops, now composed of a set of auxiliary fields, contain the original singularities. To regularize the original theory one has to eliminate these auxiliary fields by putting them on shell. In this way the theory is free of the quantum fluctuations. An extension of the NLR method to the BV framework has been formulated in [36].

As we explained at the introduction, the non-local regularization can be applied only to theories which have a perturbative expansion, i.e. for actions that can be decomposed into a free and an interacting part. For much more details, including the diagrammatic part, the interested reader can see the references [30, 31, 32, 36].

Let us define an action $S(\Phi)$ where $\Phi$ is the set $\Phi^A$ of the fields, $A = 1, \ldots, N$, and with statistics $\epsilon(\Phi^A) \equiv \epsilon_A$.

$$S(\Phi) = F(\Phi) + I(\Phi), \quad (46)$$

$F(\Phi)$ is the kinetic part and $I(\Phi)$ is the interacting part. $F(\Phi)$ can be written as $F(\Phi) = \frac{1}{2} \Phi^A \mathcal{F}_{AB} \Phi^B$ and $I(\Phi)$ is an analytic function in $\Phi^A$ around $\Phi^A = 0$. $\mathcal{F}_{AB}$ is called the kinetic operator.

We have now to introduce a cut-off or regulating parameter $\Lambda^2$. An arbitrary and invertible matrix $T_{AB}$ has to be introduced too. With the combination between $\mathcal{F}_{AB}$ and $(T^{-1})^{AC}$ we can define a second order derivative regulator

$$R^A_B = (T^{-1})^{AC} \mathcal{F}_{AB}, \quad (47)$$

which will help in the construction of two important operators. The first one is the smearing operator

$$\epsilon^A_B = \exp \left( \frac{R^A_B}{2\Lambda^2} \right), \quad (48)$$

and the second one is the shadow kinetic operator

$$O^{-1}_{AB} = T_{AC} (\hat{\Omega}^{-1})^C_B = \left( \frac{F}{\epsilon^2 - 1} \right)^A_{AB}, \quad (49)$$

where $(\hat{\Omega})^A_B$ is defined as

$$\hat{\Omega}^A_B = \left( \frac{\epsilon^2 - 1}{R} \right)^A_B = \int_0^1 dt \frac{1}{\Lambda^2} \exp \left( t \frac{R^A_B}{\Lambda^2} \right). \quad (50)$$

For each field $\Phi^A$ an auxiliary field $\Psi^A$ can be constructed, i.e., the shadow field, with the same statistics. A new auxiliary action couple both sets of fields

$$\tilde{S}(\Phi, \Psi) = F(\Phi) - A(\Psi) + I(\Phi + \Psi). \quad (51)$$

The second term of this auxiliary action is called kinetic term,

$$A(\Psi) = \frac{1}{2} \Psi^A (O^{-1})_{AB} \Psi^B. \quad (52)$$

The fields $\hat{\Phi}^A$, the smeared fields, which make part of the auxiliary action are defined by

$$\hat{\Phi}^A \equiv (\epsilon^{-1})^A_B \Phi^B. \quad (53)$$
It can be proved that, to eliminate the quantum fluctuations associated with the shadow fields at the path integral level one has to accomplish this by putting the auxiliary fields $\Psi$ on shell. So, the classical shadow fields equations of motion are

$$\frac{\partial_r \tilde{S}(\Phi, \Psi)}{\partial \Psi} = 0 \implies \Psi^A = \left( \frac{\partial_r I}{\partial \Phi^B}(\Phi + \Psi) \right) \mathcal{O}^{BA}. \quad (54)$$

These equations can be solved in a perturbative fashion. The classical solutions $\bar{\Psi}_0(\Phi)$ can now be substituted in the auxiliary action (51). This substitution modify the auxiliary action so that a new action, the non-localized action appear,

$$S_\Lambda(\Phi) \equiv \tilde{S}(\Phi, \bar{\Psi}_0(\Phi)), \quad (55)$$

which can be expanded in $\bar{\Psi}_0$. As a result, we see the appearance of the smeared kinetic term $F(\hat{\Phi})$, the original interaction term $I(\Phi)$ and an infinite series of new non-local interaction terms. But all these interaction terms are $O\left(\frac{1}{\Lambda^2}\right)$. When the limit $\Lambda^2 \to \infty$ is taken, we will have that $S_\Lambda(\Phi) \to S(\Phi)$, and the original theory is recovered. Equivalently to this limit, the same result can be obtained with the limits

$$\epsilon \to 1, \quad \mathcal{O} \to 0, \quad \bar{\Psi}_0(\Phi) \to 0. \quad (56)$$

With all this framework, when we introduce the smearing operator, any local quantum field theory can be made ultraviolet finite. But a question about symmetry can appear. Obviously this form of non-localization destroy any kind of gauge symmetry or its associated BRST symmetry. The final consequence is the damage of the corresponding Ward identities at the tree level. If the original action (46) is invariant under the infinitesimal transformation

$$\delta \Phi^A = R^A(\Phi), \quad (57)$$

the auxiliary action is invariant under the auxiliary infinitesimal transformations

$$\tilde{\delta} \Phi^A = (\epsilon^2)^A_B R^B(\Phi + \Psi),$$
$$\tilde{\delta} \Psi^A = (1 - \epsilon^2)^A_B R^B(\Phi + \Psi). \quad (58)$$

However, the non-locally regulated action (55) is invariant under the transformation

$$\delta_\Lambda(\Phi^A) = \left( \epsilon^2 \right)^A_B R^B(\Phi + \bar{\Psi}_0(\Phi)), \quad (59)$$

where $\bar{\Psi}_0(\Phi)$ are the solutions of the classical equations of motions (54).

Hence, any of the original continuous symmetries of the theory are preserved at the tree level, even the BRST transformations, and consequently, the original gauge symmetry. The reader can see [30, 31, 32] for details.

### 3.2 The Extended (BV) Non-local Regularization

Using the construction, described in the last section, of the NLR and the BV results, one can build a regulated BRST classical structure of a general gauge theory from the original one. Consequently, a non-locally regularized BV formalism comes out.
The BV configuration space has to be enlarged introducing the antifields \( \{ \Psi^A, \Psi^*_A \} \). Note that the shadow fields have antifields too. Then, an auxiliary proper solution, which incorporates the auxiliary action (51), corresponding to the gauge-fixed action \( S(\Phi) \), its BRST symmetry (58) and the unknown associated higher order structure functions. The auxiliary BRST transformations (58), are modified by the presence of the term \( \Phi^*_A R^A(\Phi) \) in the original proper solution. Then it can be written that the BRST transformations are

\[
\left[ \Phi^*_A(\epsilon^2)^A_B + \Psi^*_A(1 - \epsilon^2)^A_B \right] R^B (\Phi + \Psi)
\]

which are originated from the substitution

\[
\Phi^*_A \rightarrow \left[ \Phi^*_A(\epsilon^2)^A_B + \Psi^*_A(1 - \epsilon^2)^A_B \right] \equiv \Theta^*_A
\]

For higher orders, the natural way would be

\[
R^{A_n...A_1}(\Phi) \rightarrow R^{A_n...A_1}(\Phi + \Psi) = R^{A_n...A_1}(\Theta)
\]

and an obvious ansatz for the auxiliary proper solution is

\[
\tilde{S}(\Phi, \Phi^*; \Psi, \Psi^*) = \tilde{S}(\Phi, \Psi) + \Theta^*_A R^A(\Theta) + \Theta^*_A \Theta^*_B R^{AB}(\Theta) + \ldots
\]

It is intuitive to see that the same canonical conjugation relations, equations (32), should be obtained, i.e.

\[
(\Theta^A, \Theta^*_B) = \delta^A_B.
\]

Consequently, we have to construct a new set of fields and antifields \( \{ \Sigma^A, \Sigma^*_A \} \) defined by

\[
\Sigma^A = \left[ (1 - \epsilon^2)^A_B \Phi^B - (\epsilon^2)^A_B \Psi^B \right],
\]

and

\[
\Sigma^*_A = \Phi^*_A - \Psi^*_A.
\]

Now we have that the linear transformation

\[
\{ \Phi^A, \Phi^*_A; \Psi^A, \Psi^*_A \} \rightarrow \{ \Theta^A, \Theta^*_A; \Sigma, \Sigma^*_A \}
\]

is canonical in the antibracket sense. And the auxiliary action (51) is the original proper solution (35) with arguments \( \{ \Theta^A, \Theta^*_A \} \).

The elimination of the auxiliaries fields of BV method is the next step. The shadow fields have to be substituted by the solutions of their classical equations of motion. At the same time, their antifields goes to zero. In this way we can write that

\[
S_A(\Phi, \Phi^*) = \tilde{S}(\Phi, \Phi^*; \Psi, \Psi^* = 0),
\]

and the classical equations of motion are

\[
\frac{\delta_r \tilde{S}(\Phi, \Phi^*; \Psi, \Psi^*)}{\delta \Psi^A} = 0
\]

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with solutions $\tilde{\Psi} \equiv \tilde{\Psi}(\Phi, \Phi^*)$, which explicitly read

$$\tilde{\Psi}^A = \left[ \frac{\delta_r I}{\delta \Phi^B} (\Phi + \Psi) + \Phi^*_C \left( \epsilon^2 \right)^C_D R^D_B (\Phi + \Psi) + O \left( (\Phi^*)^2 \right) \right] \quad (70)$$

with

$$R^A_B = \frac{\delta_r R^A (\Phi)}{\delta \Phi^B}. \quad (71)$$

The lowest order of equation (70) is,

$$\tilde{\Psi}^A = \left( \frac{\delta_r I}{\delta \Phi^B} (\Phi + \Psi) \right) O^{BA} \quad (72)$$

and one can obtain an expression for $\tilde{\Psi}(\Phi, \Phi^*)$ at any desired order in antifields [36].

To quantize the theory, it is necessary to add extra counterterms $M_p$ to preserve the quantum counterpart of the classical BRST scheme. It is the same as to substitute the classical action $S$ by a quantum action $W$. It can be proved that in the field-antifield framework, in general, two and higher order loop corrections should also be considered [36].

The complete interaction term $I(\Phi, \Phi^*)$ of the original proper solution can be written as

$$I(\Phi, \Phi^*) \equiv I(\Phi) + \Phi^*_A R^A (\Phi) + R^{AB} (\Phi) \quad (73)$$

The non-localization of this interaction part furnishes a way to regularize interactions from counterterms $M_p$. To construct the auxiliary free and interaction parts we have that

$$\tilde{F} (\Phi + \Psi) = F(\Phi) - A(\Psi), \quad I(\Phi, \Phi^*; \Psi, \Psi^*) = I(\Theta, \Theta^*) \quad (74)$$

with $\{\Theta, \Theta^*\}$ already known.

Now one have to put the auxiliary fields on shell and its antifields to zero, so that

$$F_A (\Phi, \Phi^*) = \tilde{F} (\Phi, \tilde{\Psi}_0), \quad I_A (\Phi, \Phi^*) = \tilde{I} (\Phi + \tilde{\Psi}_0, \Phi^* \epsilon^2), \quad (75)$$

then $S_A = F_A + I_A$.

The quantum action $W$ can be expressed by

$$W = F + I + \sum_{p=1}^{\infty} \hbar M_p \equiv F + Y \quad (76)$$

where $Y$ now is the generalized quantum interaction. A decomposition in its divergent part and its finite part when $\Lambda^2 \rightarrow \infty$ can be accomplished in the regulated QME.

It can be shown that the expression of the anomaly is the value of the finite part in the limit $\Lambda^2 \rightarrow \infty$ of

$$A = \left[ \left( \Delta W \right)_R + \frac{i}{2 \hbar} \left( W, W \right) \right] (\Phi, \Phi^*) \quad (77)$$

and the regularized value of $\Delta W$ defined as

$$\left( \Delta W \right)_R \equiv \lim_{\Lambda^2 \rightarrow \infty} [\Omega_0] \quad (78)$$
where
\[ \Omega_0 = \left[ S^A_B \left( \delta \Lambda \right)_C \left( \epsilon^2 \right)_A \right] . \] (79)

and \((\delta \Lambda)^A_B\) is defined by
\[
\begin{align*}
(\delta \Lambda)^A_B &= \left( \delta^A_B - \mathcal{O}^{AC} \mathcal{I}_{CB} \right)^{-1} \\
&= \delta^A_B + \sum_{n=1} \left( \mathcal{O}^{AC} \mathcal{I}_{CB} \right)^n ,
\end{align*}
\] (80)

with
\[
\begin{align*}
S^A_B &= \frac{\delta_r \delta_l S}{\delta \Phi^B \delta \Phi^A} , \\
\mathcal{I}_{AB} &= \frac{\delta_r \delta_l \mathcal{I}}{\delta \Phi^A \delta \Phi^B} .
\end{align*}
\] (81)

Applying the limit \(\Lambda^2 \to \infty\) in (78), it can be shown that
\[
(\Delta S)_R \equiv \lim_{\Lambda^2 \to \infty} [\Omega_0]_0
\] (82)

And finally that
\[
\mathcal{A}_0 \equiv \frac{\left( \Delta S \right)_R}{\mathcal{I}} = \lim_{\Lambda^2 \to \infty} [\Omega_0]_0
\] (83)

All the higher orders loop terms of the anomaly can be obtained from equation (77), but this will not be analyzed in this paper.

4 The NC anomaly of the CSM

We know that the CSM is a \(U(1)\) gauge field coupled to chiral fermions and in spite of being anomalous, constitute a consistent unitary theory. Moreover, the fermionic determinant and the anomaly have some arbitrariness relative to the regularization of fermionic radiative contributions (see [9] for a review).

The classical action for the chiral Schwinger model is
\[
S = \int d^2 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^\mu \gamma^5 A_\mu \psi + \frac{e}{2} \bar{\psi} \gamma_\mu (1 - \gamma^5) A_\mu \psi \right] ,
\] (84)

which obviously has a perturbative expansion.

This action is invariant for the following gauge transformations:
\[
\begin{align*}
A^\mu(x) &\to A^\mu(x) + \partial_\mu \theta(x) \\
\psi(x) &\to \exp \left[ i e (1 - \gamma^5) \theta(x) \right] \psi(x)
\end{align*}
\] (85) (86)

The kinetic part of the action (84) is given by
\[
F = \int d^2 x \bar{\psi} i \gamma^\mu \gamma^5 A_\mu \psi = \int d^2 x \left[ \frac{1}{2} \bar{\psi} i \gamma^\mu \gamma^5 A_\mu \psi + \frac{1}{2} \bar{\psi} i \gamma^\mu \gamma^5 A_\mu \psi \right] .
\] (87)
Integrating by parts the second term we have that

\[ F = \int d^2x \left[ \frac{1}{2} \bar{\psi} i \partial \psi - \frac{1}{2} (i \partial^t \bar{\psi}) \psi \right] \tag{88} \]

The kinetic term has the form

\[ F = \frac{1}{2} \Psi^A F_{AB} \Psi^B, \tag{89} \]

where, \( \Psi = \left( \begin{array}{c} \bar{\psi} \\ \psi \end{array} \right) \) and

\[ F = \frac{1}{2} (\bar{\psi} \psi) \left( \begin{array}{cc} 0 & i \partial^t \\ i \partial^t & 0 \end{array} \right) \left( \begin{array}{c} \bar{\psi} \\ \psi \end{array} \right) \tag{90} \]

and we have that the kinetic operator \( (F_{AB}) \) is

\[ F_{AB} = \left( \begin{array}{cc} 0 & i \partial^t \\ i \partial^t & 0 \end{array} \right) \tag{91} \]

The regulator, a second order differential operator, is \( \mathcal{R}^\alpha_\beta = (T^{-1})^{\alpha\gamma} F_{\gamma\beta} \), where \( T \) is an arbitrary matrix, and one can make the following choice,

\[ \mathcal{R}^\alpha_\beta = - \partial^2 \tag{92} \]

Using the definition of the smearing operator,

\[ \epsilon^A_B = \exp \left( -\frac{\partial^2}{2\Lambda^2} \right), \tag{93} \]

and the smeared fields are defined by \( \hat{\Phi}^A = (\epsilon^{-1})^A_B \Phi^B \).

In the NLR scheme the shadow kinetic operator is

\[ \mathcal{O}^{-1}_{\alpha\beta} = \left( \frac{F}{\epsilon^2 - 1} \right)_{\alpha\beta} \tag{94} \]

then

\[ \mathcal{O} = \left( \begin{array}{cc} 0 & -i\mathcal{O}' \partial^t \\ -i\mathcal{O}' \partial^t & 0 \end{array} \right) \tag{95} \]

where

\[ \mathcal{O}' = \frac{\epsilon^2 - 1}{\partial^t \partial^t} = \int_0^1 dt \left[ t \frac{\partial^t \partial^t}{\Lambda^2} \right] \tag{96} \]

The interacting part of the action (84) is

\[ I \left[ A_\mu, \psi, \bar{\psi} \right] = e \bar{\psi} \gamma_\mu (1 - \gamma_5) A^\mu \psi \tag{97} \]

\[ I \left[ A_\mu, \psi + \Phi, \bar{\psi} + \bar{\Phi} \right] = e (\bar{\psi} + \bar{\Phi}) \gamma_\mu (1 - \gamma_5) A^\mu (\psi + \Phi) \tag{98} \]

where \( \Phi \) are the shadow fields.
The BRST transformations are given by

\[
\begin{align*}
\delta A_\mu &= \partial_\mu c, \\
\delta \psi &= i(1 - \gamma_5)\psi c, \\
\delta \bar{\psi} &= -i\bar{\psi}(1 + \gamma_5)c, \\
\delta c &= 0.
\end{align*}
\] (99)

Substituting (61) where the antifields are functions of the auxiliary fields,

\[
\begin{align*}
\psi^* &\rightarrow \left[\psi^*\epsilon^2 + \Phi^*(1 - \epsilon^2)\right] \\
\bar{\psi}^* &\rightarrow \left[\bar{\psi}^*\epsilon^2 + \bar{\Phi}^*(1 - \epsilon^2)\right].
\end{align*}
\] (100)

The generator of BRST transformations are

\[
\begin{align*}
R(\psi) &\rightarrow R(\psi + \Phi) = i(1 - \gamma_5)(\psi + \Phi)c \\
R(\bar{\psi}) &\rightarrow i(\bar{\psi} + \bar{\Phi})(1 + \gamma_5)c \\
R(c) &= 0.
\end{align*}
\] (101)

The non-local auxiliary proper action will be given in general by \( S_\Lambda(\Phi, \Phi^*) = \tilde{S}_\Lambda(\Phi, \Phi^*; \psi_s, \psi^* = 0) \) where \( \psi_s \) are the solutions of the classical equations of motion.

After an algebraic manipulation, one can write the non-localized action as

\[
\tilde{S}_\Lambda(\psi, \psi^*) = \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \bar{\psi}i \partial\psi + A^*_\mu\partial^\mu c + e^2 \bar{\psi}\gamma^\mu(1 - \gamma_5)A_\mu\psi + A^*_\mu\partial^\mu c
\]

\[
\psi^* \rightarrow \left[\psi^*\epsilon^2 + \Phi^*(1 - \epsilon^2)\right] \\
\bar{\psi}^* \rightarrow \left[\bar{\psi}^*\epsilon^2 + \bar{\Phi}^*(1 - \epsilon^2)\right].
\]

It can be seen easily that when one takes the limit \( \epsilon^2 \rightarrow 1 \), the original proper solution of the CSM, shown below, is obtained.

Now we have to construct some very important matrices,

\[
S^A_B = \frac{\delta r \delta_i S_{BV}}{\delta \Phi^B \delta \Phi^*_A}
\] (103)

with the proper solution, the BV action, given by

\[
S_{BV} = \int d^2x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}i \partial\psi + \frac{e}{2} \bar{\psi}\gamma_\mu(1 - \gamma_5)A^\mu\psi + A^*_\mu\partial^\mu c \\
+ i\psi^*(1 - \gamma_5)\psi c - i\bar{\psi}^*(1 + \gamma_5)c \right\}
\] (104)

then

\[
S^A_B = \begin{pmatrix}
-ic(1 - \gamma_5) & 0 \\
0 & ic(1 + \gamma_5)
\end{pmatrix}
\] (105)

The operator \( \mathcal{I}_{AB} \) in this case is,

\[
\mathcal{I}_{AB} = \frac{\delta_\lambda \delta r \left[ I(\Phi) + \Phi^*_c R^c(\Phi) \right]}{\delta \Phi^A \delta \Phi^B}
\] (106)
and the result is,

$$\mathcal{I}_{AB} = \begin{pmatrix} 0 & -\frac{e}{2}\gamma_\mu (1 - \gamma_5) A^\mu \\ \frac{e}{2}\gamma_\mu (1 - \gamma_5) A^\mu & 0 \end{pmatrix}$$  \hspace{1cm} (107)

The one-loop anomaly is given by:

$$\mathcal{A} \equiv (\Delta S)_R \hspace{1cm} (108)$$

$$(\Delta S)_R = \lim_{\Lambda^2 \to \infty} [\Omega_0]_0 \hspace{1cm} (109)$$

$$\Omega_0 = \left[ \epsilon^2 S_A^A \right] + \left[ \epsilon^2 S_B^A O^{BC} T_{CA} \right] + O \left( \frac{(\Phi^*)^2}{A^2} \right)$$  \hspace{1cm} (110)

For the first term

$$\epsilon^2 S_A^A = \epsilon^2 tr S_B^A$$

$$= 0 \hspace{1cm} (111)$$

and we have that

$$(\Delta S)_R = \lim_{\Lambda^2 \to \infty} tr \left[ \epsilon^2 S_B^A O^{BC} T_{CA} \right]$$  \hspace{1cm} (112)

Using the $\gamma$ matrix representation

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (113)

$$\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$  \hspace{1cm} (114)

and

$$\gamma^5 = -i \gamma_1 \gamma_0$$  \hspace{1cm} (115)

in this representation we have that $\gamma^5 = \gamma_5$.

Finally, we have that,

$$(\Delta S)_R = \lim_{\Lambda^2 \to \infty} tr \left[ \epsilon^2 (-\epsilon c) \frac{\epsilon^2}{\partial^2} \partial \mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu \right] .$$  \hspace{1cm} (116)

But we know that

$$\lim_{\Lambda^2 \to \infty} tr \left[ \epsilon^2 F \partial^n \frac{\epsilon^2}{\partial^2} \partial G \partial^m \right] =$$  \hspace{1cm} (117)

$$= -\frac{i}{2\pi} \left[ \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{(-1)^k}{n+m+1-k} \left(1 - \frac{1}{2^{n+m+1-k}}\right) \right] \int d^2 x \frac{F}{\partial^n+1} G .$$

In our case

$$n = m = 0$$

$$F = 2\epsilon c$$

$$\partial G = \partial_\mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu$$  \hspace{1cm} (118)
and with Eq. (29) we can construct the NC version of the CSM, which can be written as,

\[ \hat{A} = \frac{ie}{2\pi} \left\{ \int d^2 x c \left( \partial_\mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu \right) + \theta^{\alpha\beta} \partial_\alpha \left[ A_\beta \int d^2 x c \left( \partial_\mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu \right) \right] \right\} + O(\theta^2) \]  

(119)

where in two dimensions we have that \( \theta^{01} = \theta^{10} = \theta \). The first term is the one loop ordinary anomaly of the CSM, i.e., \( A = (\Delta S)_R \), and the second term, of course, is the NC correction term. To investigate under what conditions the NC anomaly cancels is beyond the scope of this work. We can see easily form (119) that, like in NC QED, the anomaly shares terms with the commutative primary theory. It would be interesting to calculate the \( \theta \)-second order terms.

5 Conclusions

One of the greatest motivations to study noncommutativity is the introduction of a minimal length scale, which is one of the ingredients of quantum gravity. The non-commutation between the coordinates and momenta in quantum theory insinuates naturally that the same behavior can be performed by the coordinates. This would cause the introduction of a new scale in the theory through the NC parameter, which could be used, for example, to tame the divergences in QFT. Nevertheless, the renormalization procedures showed great success and the NC scenarios were put to sleep for more than fifty years until string theory bring them back recently.

Since the NC space can be understood as a deformation of the ordinary spacetime, one consequence of NC effects in QFT is to introduce a combination of IR and UV divergences through the appearing of phase factors in the vertex.

Concerning anomalies in NC scenarios, it was noticed that, resulting only from noncommutativity, two different currents can be defined even for a \( U(1) \) theory [37]. Another example is the description of the \( \theta \)-structure of the commutator anomalies in NC electrodynamics [12]. So, the treatment of NC anomalies can be considered a quite nontrivial issue and deserves more investigations.

There are two ways to accomplish NC anomalies. One of them is to introduce noncommutativity in the primary theory and after that to compute the anomaly [38, 39]. The second one is to use the SW map directly in the ordinary result of the anomaly, which was accomplished here for the first time.

In this work we decided in favour of the second one and we used the BV quantization to calculate the CSM anomaly. The regularization scheme used here was the non-local regularization formalism. The field-antifield framework exhibits a divergence on the application of the \( \Delta \) operator and hence it needs a regularization. This is a recent and a quite powerful method to regularize theories with a perturbative expansion which have higher loop order divergences. This arguably makes this method the ideal one to analyze NC theories with higher orders in \( \theta \).

The anomaly in (119) shows that it would be interesting to calculate superior orders in the \( \theta \)-parameter in order to see if higher terms shares something with the ordinary anomaly. Hence, as a direct perspective we can develop the SW map, concerning the anomaly, to calculate a general form for superior orders terms for the anomaly. This is a work in progress.
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