Abstract.

In the present paper, we give the notion of $k$-type hyperbolic slant helices in $H^3$, where $k \in \{0, 1, 2, 3\}$. We give the necessary and sufficient conditions for hyperbolic curves to be $k$-type slant helices in terms of their hyperbolic curvature functions.

1. Introduction

The notion of a slant helix was due to Izumiya and Takeuchi ([6]). A curve $\gamma$ with non-zero curvature is called a slant helix in Euclidean 3-space $\mathbb{R}^3$ if the principal normal line of $\gamma$ makes a constant angle with a fixed vector in $\mathbb{R}^3$. Also some characterizations of such curves were presented in [1, 7, 8, 14]. Slant helices are the successor curves of the general helices. In particular, they are geodesics of the helix surfaces.

Further, $k$-type slant helices emerged and attracted attention of researchers. Ergüt et al ([5]) studied $k$-slant helices in Minkowski 3-space, $\mathbb{R}^3_1$. Also curves of such a type were studied in Minkowski space-time by some researchers such as [2, 10]. Lastly, in [12, 13], the authors studied $k$-slant helices for null curves in lightlike cone in Minkowski space-time and $k$-type spacelike slant helices lying on lightlike surfaces.

On the other hand, in [9], the author considered hyperbolic curves in 3-dimensional hyperbolic space, and construct the hyperbolic frame of the hyperbolic space curves. Also, the author studied the associated curve of a hyperbolic curve in $H^3$. Hyperbolic curves in $H^3$ according to their Frenet frame, are characterized in [4].

In this paper, we introduce the notion of $k$-type hyperbolic slant helices in $H^3$, where $k \in \{0, 1, 2, 3\}$. We give the necessary and sufficient conditions for hyperbolic curves to be $k$-type slant helices in terms of their hyperbolic curvature functions. Finally, we give the related examples.

2. Preliminaries

The Minkowski space-time $\mathbb{R}^4_1$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$
where \((x_1, x_2, x_3, x_4)\) is a rectangular coordinate system of \(\mathbb{E}^4_1\). Recall that a vector \(v \in \mathbb{E}^4_1\setminus\{0\}\) can be spacelike if \(\langle v, v \rangle > 0\), timelike if \(\langle v, v \rangle < 0\) and null (lightlike) if \(\langle v, v \rangle = 0\). In particular, the vector \(v = 0\) is said to be a spacelike. The norm of a vector \(v\) is given by \(\|v\| = \sqrt{\langle v, v \rangle}\). Two vectors \(v\) and \(w\) are said to be orthogonal, if \(\langle v, w \rangle = 0\). An arbitrary curve \(\alpha(s)\) in \(\mathbb{E}^4_1\), can locally be spacelike, timelike or null (lightlike), if all its velocity vectors \(\alpha'(s)\) are respectively spacelike, timelike or null [11].

A null curve \(\alpha\) is parameterized by pseudo-arc \(s\) if \(\langle \alpha''(s), \alpha''(s) \rangle = 1\) [3]. On the other hand, a non-null curve \(\alpha\) is parametrized by the arc-length parameter \(s\) if \(\langle \alpha'(s), \alpha'(s) \rangle = \pm 1\).

Let \(m\) be a fixed point and \(r > 0\) be a constant. The pseudo-Riemannian hyperbolic space is defined by

\[
\mathbb{H}^3(m, r) = \{ u \in \mathbb{E}^4_1 : \langle u - m, u - m \rangle = -r^2 \}.
\]

When \(m = 0\) and \(r = 1\), we denote \(\mathbb{H}^3(0, 1)\) by \(\mathbb{H}^3\).

For the regular curve \(x(s) \subset \mathbb{H}^3 \subset \mathbb{E}^4_1\) with hyperbolic Frenet frame \(\{x(s), \alpha(s), \beta(s), y(s)\}\) and hyperbolic curvature functions \(\kappa(s), \tau(s)\), the Frenet formulas of hyperbolic space curve \(x(s)\) in \(\mathbb{H}^3\) can be written as

\[
\begin{align*}
\frac{d}{ds}x' &= \alpha(s), \\
\frac{d}{ds}\alpha' &= x(s) + \kappa(s)y(s), \\
\frac{d}{ds}\beta' &= \tau(s)y(s), \\
y' &= -\kappa(s)\alpha(s) - \tau(s)\beta(s),
\end{align*}
\]

where for all \(s\),

\[
\begin{align*}
\langle x(s), x(s) \rangle &= -1, & \langle \alpha(s), \alpha(s) \rangle &= \langle \beta(s), \beta(s) \rangle &= \langle y(s), y(s) \rangle &= 1, \\
\langle x(s), \alpha(s) \rangle &= \langle x(s), \beta(s) \rangle = \langle x(s), y(s) \rangle &= 0, \\
\langle \alpha(s), \beta(s) \rangle &= \langle \alpha(s), y(s) \rangle = \langle \beta(s), y(s) \rangle &= 0.
\end{align*}
\]

If \(\langle x''(s), x''(s) \rangle = -1\), together with \(\langle x(s), x(s) \rangle = \langle x(s), x''(s) \rangle = -1\) we know that \(x''(s) = x(s)\). So we assume that \(\langle x''(s), x''(s) \rangle > -1\) and call the curve regular ([9]).

3. **k-type hyperbolic slant helices in 3-dimensional hyperbolic space \(\mathbb{H}^3\)**

In this section, we study \(k\)-type hyperbolic slant helices in hyperbolic space \(\mathbb{H}^3\). Let us set that

\[
V_0 = x, \quad V_1 = \alpha, \quad V_2 = \beta, \quad V_3 = y.
\]

In the following definition, we introduce the \(k\)-type slant helices lying in pseudohyperbolic space \(\mathbb{H}^3\).

**Definition 3.1.** A hyperbolic space curve \(x(s)\) parametrized by arc-length \(s\) with hyperbolic Frenet frame \(\{V_0, V_1, V_2, V_3\}\) in pseudohyperbolic space \(\mathbb{H}^3\) is called a \(k\)-type hyperbolic slant helix for \(k \in \{0, 1, 2, 3\}\) if there exists a non-zero fixed vector \(U \in \mathbb{E}^4_1\) such that the following holds

\[
\langle V_k, U \rangle = \text{constant}.
\]

**Firstly, we consider 0-type hyperbolic slant helices in \(\mathbb{H}^3\).**

**Theorem 3.2.** Let \(x(s)\) be a hyperbolic space curve in \(\mathbb{H}^3\) parametrized by arc-length \(s\) with non-zero curvatures \(\kappa, \tau\). Then \(x(s)\) is a 0-type hyperbolic slant helix if and only if

\[
\left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)' + \frac{1}{\tau} \left( \frac{1}{\kappa} \right)'' + \frac{\tau}{\kappa} = 0.
\]
Proof. Assume that $x(s)$ is a $0$-type hyperbolic slant helix in $\mathbb{H}^3$ parametrized by arc-length $s$ with non-zero curvatures $\kappa, \tau$. Then there exists a non-zero fixed vector $U \in \mathbb{R}^3$ such that

$$\langle x, U \rangle = \kappa, \quad c \in \mathbb{R}. \quad (3)$$

Taking derivative of the equation (3) with respect to $s$ and using Frenet equations (1), we get

$$\langle x, U \rangle = 0, \quad \langle y, U \rangle = -\frac{c}{\kappa}. \quad (4)$$

By using (4), we can write $U$ with respect to the frame $\{x, \alpha, \beta, \gamma\}$ as follows

$$U = -cx + x\beta - \frac{c}{\kappa}y, \quad (5)$$

where $\lambda$ is some differentiable function of $s$ and $c \in \mathbb{R} \setminus \{0\}$. Taking derivative of the equation (5) with respect to $s$ and using Frenet equations (1), we have

$$\left(\frac{1}{\kappa} + c\frac{\tau}{\kappa}\right) \beta + \left(\lambda \tau - c\left(\frac{1}{\kappa}\right)\right) y = 0$$

which implies that

$$\left(\frac{1}{\kappa}\right)' \left(\frac{1}{\kappa}\right) + \frac{1}{\kappa} \left(\frac{1}{\kappa}\right)'' + \frac{\tau}{\kappa} = 0. \quad (6)$$

Conversely, assume that (2) holds. Choosing the vector $U$ as

$$U = -c \left[ x - \frac{1}{\tau \kappa} \left(\frac{1}{\kappa}\right)' \beta + \frac{1}{\kappa} y \right],$$

we get $U' = 0$ and $\langle x, U \rangle = c$ (constant). Thus $x(s)$ is a $0$-type hyperbolic slant helix. \hfill \Box

Example 3.3. The hyperbolic curvature functions

$$\kappa = \frac{\sqrt{s^4 + 6s^2 + 10}}{s^2 + 2} \quad \text{and} \quad \tau = \frac{2s^2}{s^4 + 6s^2 + 10}$$

satisfy (2). The hyperbolic curve $x(s)$ with the hyperbolic curvature functions $\kappa$ and $\tau$ can be written as

$$x(s) = \left( \frac{s}{\sqrt{s^2 + 2}}, \cos A - \frac{s \sin A}{\sqrt{s^2 + 2}}, 0, \sin A + \frac{s \cos A}{\sqrt{s^2 + 2}} \right),$$

with

$$\alpha(s) = \left( \begin{array}{c} s \cos A - \frac{s \sin A}{\sqrt{s^2 + 2}} \\ 0, \sin A + \frac{s \cos A}{\sqrt{s^2 + 2}} \end{array} \right),$$

$$\beta(s) = \left( \begin{array}{c} -s^4 - 4s^2 - 2 \\ -s \sqrt{s^2 + 2} \left(3 + s^2\right) \cos A - \left(4 + s^2\right) \sin A \end{array} \right),$$

$$\gamma(s) = \left( \begin{array}{c} 2 \sqrt{s^2 + 2} \\ s \sqrt{s^2 + 2} \cos A - \left(s^2 + 2\right) \sin A \end{array} \right),$$

$$\delta(s) = \left( \begin{array}{c} \frac{4 + s^2}{\sqrt{s^2 + 2} \sqrt{s^2 + 6s^2 + 10}} \\ \frac{4 + s^2}{\sqrt{s^2 + 2} \sqrt{s^2 + 6s^2 + 10}} \end{array} \right).$$
where \( A = \text{arcsinh} \frac{s}{\sqrt{2}} \). So we get
\[
U = -c \left[ x - \frac{1}{\tau} \left( \frac{1}{\kappa} \right)' \beta + \frac{1}{\kappa} y \right] = (0, 0, c, 0)
\]
and \( (x, U) = c \) (constant). Thus \( x(s) \) is a \( 0 \)-type hyperbolic slant helix.

**Example 3.4.** The following hyperbolic curvature functions satisfy (2).
(i) \( \kappa = 1/\cos s \), \( \tau = 1 \)  
(ii) \( \kappa = 1/\cos (\ln s) \), \( \tau = 1/s \)

**Corollary 3.5.** The axis of a \( 0 \)-type hyperbolic slant helix is given by
\[
U = -c \left[ x - \frac{1}{\tau} \left( \frac{1}{\kappa} \right)' \beta + \frac{1}{\kappa} y \right]
\]  
where \( c \in \mathbb{R} \backslash \{0\} \).

**Corollary 3.6.** Let \( x(s) \) be a hyperbolic space curve in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa \), \( \tau \). Then \( x(s) \) is a \( 0 \)-type hyperbolic slant helix if and only if
\[
\frac{1}{\tau^2} \left( \left( \frac{1}{\kappa} \right)'^2 \right) + \frac{1}{\kappa^2} = \text{constant}.
\]  
(8)

**Proof.** Assume that \( x(s) \) is a \( 0 \)-type hyperbolic slant helix in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa \), \( \tau \). Then there exists a non-zero fixed vector \( U \in \mathbb{E}_4^1 \) such that
\[
\langle \alpha, U \rangle = c_1, \quad c_1 \in \mathbb{R}.
\]  
(10)

Then we can write \( U \) with respect to the frame \( \{x, \alpha, \beta, y\} \) as follows
\[
U = \lambda_1 x + c_1 \alpha + \lambda_3 \beta + \lambda_4 y
\]  
(11)
where \( \lambda_1, \lambda_3 \) and \( \lambda_4 \) are some differentiable functions of \( s \). Differentiating the equation (11) with respect to \( s \) and using Frenet equations (1), we get

\[
0 = \left( \lambda'_1 + c_1 \right) x + \left( \lambda_1 - \kappa \lambda_4 \right) \alpha + \left( \lambda'_3 - \tau \lambda_4 \right) \beta + \left( c_1 \kappa + \lambda_3 \tau + \lambda'_4 \right) y
\]

which implies that

\[
\begin{cases}
\lambda'_1 + c_1 = 0, \\
\lambda_1 - \kappa \lambda_4 = 0, \\
\lambda'_3 - \tau \lambda_4 = 0, \\
c_1 \kappa + \lambda_3 \tau + \lambda'_4 = 0.
\end{cases}
\]

Solving (12), we get

\[
c_1 \left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)' \left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)' (c_1 s + c_2) + c_1 \left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)' - \frac{1}{\tau} \left( \frac{1}{\kappa} \right)'' - \left( c_1 s + c_2 \right) - \frac{1}{\kappa} (-c_1 s + c_2) = 0,
\]

where \( c_1, c_2 \in \mathbb{R} \) and \( (c_1, c_2) \neq (0, 0) \).

Conversely, assume that the relation (9) holds. Then choosing the vector \( U \) as follows

\[
U = (-c_1 s + c_2) x + c_1 \alpha + \frac{1}{\tau} \left[ c_1 \left( \frac{1}{\kappa} \right)' \left( \frac{1}{\kappa} \right)' - \left( \frac{1}{\kappa} \right)' \left( \frac{1}{\kappa} \right)' \right] \beta + \frac{1}{\kappa} (-c_1 s + c_2) y,
\]

we get \( U' = 0 \) and \( \langle \alpha, U \rangle = c_1 \) (constant). Thus \( x(s) \) is a 1-type hyperbolic slant helix.

**Example 3.8.** The following hyperbolic curvature functions satisfy (9).

(i) \( c_1 = 0, c_2 = 1, \kappa = 1/\sin s, \tau = 1 \).

**Corollary 3.9.** The axis of a 1-type hyperbolic slant helix is given by

\[
U = (-c_1 s + c_2) x + c_1 \alpha + \frac{1}{\tau} \left[ c_1 \left( \frac{1}{\kappa} \right)' \left( \frac{1}{\kappa} \right)' - \left( \frac{1}{\kappa} \right)' \left( \frac{1}{\kappa} \right)' \right] \beta + \frac{1}{\kappa} (-c_1 s + c_2) y,
\]

where \( c_1, c_2 \in \mathbb{R} \) and \( (c_1, c_2) \neq (0, 0) \).

Assume that \( c_1 = 0 \) in (9). Then we have \( c_2 \neq 0 \) and

\[
\left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)' + \frac{\tau}{\kappa} = 0.
\]

Then \( x(s) \) is a 0-type hyperbolic slant helix. Thus we give the following corollary.

**Corollary 3.10.** Let \( x(s) \) be a hyperbolic space curve in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then \( x(s) \) is a 0-type hyperbolic slant helix if and only if \( x(s) \) is a 1-type hyperbolic slant helix whose axis \( U \) satisfies \( \langle \alpha, U \rangle = 0 \).

Thirdly, we consider 2-type hyperbolic slant helices in \( \mathbb{H}^3 \).

**Theorem 3.11.** Let \( x(s) \) be a hyperbolic space curve in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then \( x(s) \) is a 2-type hyperbolic slant helix if and only if

\[
\left( \frac{\tau}{\kappa} \right)'' = \frac{\tau}{\kappa} = 0,
\]

or equivalently

\[
\frac{\tau}{\kappa} = c_1 e^s + c_2 e^{-s}.
\]
Proof. Assume that \( x(s) \) is a 2-type hyperbolic slant helix in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then there exists a non-zero fixed vector \( U \in \mathbb{H}^3 \) such that
\[
\langle \beta, U \rangle = c, \quad c \in \mathbb{R}.
\] (14)
Assume that \( c = 0 \). Then \( U = 0 \) which is a contradiction. So \( c \neq 0 \).

Taking derivative of the equation (14) with respect to \( s \) and using Frenet equations (1), we get
\[
\langle \alpha, U \rangle = -\frac{\tau}{\kappa} c, \quad \langle y, U \rangle = 0.
\] (15)
By using (15), we can write \( U \) with respect to the frame \( \{ x, \alpha, \beta, y \} \) as follows
\[
U = \lambda x - \frac{\tau}{\kappa} \alpha + c \beta,
\] (16)
where \( \lambda \) is some differentiable function of \( s \). Differentiating the equation (16) with respect to \( s \) and using Frenet equations (1), we get
\[
0 = \left( \lambda' - \frac{\tau}{\kappa} c \right) x + \left( \lambda - c \left( \frac{\tau}{\kappa} \right)' \right) \alpha
\]
which implies that
\[
\left( \frac{\tau}{\kappa} \right)' - \frac{\tau}{\kappa} = 0,
\]
or equivalently
\[
\frac{\tau}{\kappa} = c_1 e^s + c_2 e^{-s}.
\]
Conversely, assume that the relation (13) holds. Then choosing the vector \( U \) as follows
\[
U = c \left( \frac{\tau}{\kappa} \right)' x - \frac{\tau}{\kappa} \alpha + c \beta,
\]
where \( c \in \mathbb{R} \setminus \{ 0 \} \), we get \( U' = 0 \) and \( \langle \beta, U \rangle = c \) (constant). Thus \( x(s) \) is a 2-type hyperbolic slant helix.

Example 3.12. The following hyperbolic curvature functions satisfy (13).
(i) \( \kappa = 1, \tau = e^s \)
(ii) \( \kappa = e^s, \tau = 1 \)

Corollary 3.13. The axis of a 2-type hyperbolic slant helix is given by
\[
U = c (c_1 e^s - c_2 e^{-s}) x - (c_1 e^s + c_2 e^{-s}) \alpha + c \beta,
\]
where \( c \in \mathbb{R} \setminus \{ 0 \} \).

Lastly, we consider 3-type hyperbolic slant helices in \( \mathbb{H}^3 \).

Theorem 3.14. Let \( x(s) \) be a hyperbolic space curve in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then \( x(s) \) is a 3-type hyperbolic slant helix if and only if
\[
\int \left( \frac{\tau}{\kappa} \int \tau ds \right) ds - \left( \frac{\tau}{\kappa} \right)' \int \tau ds = \frac{\kappa^2 + \tau^2}{\kappa}.
\] (17)
Proof. Assume that \( x(s) \) is a 3-type hyperbolic slant helix in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then there exists a non-zero fixed vector \( U \in \mathbb{H}^4 \) such that
\[
\langle y, U \rangle = c, \quad c \in \mathbb{R} \setminus \{0\}.
\]
Then we can write \( U \) with respect to the frame \( \{x, \alpha, \beta, y\} \) as follows
\[
U = \lambda_1 x + \lambda_2 \alpha + \lambda_3 \beta + cy
\]
where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are some differentiable functions of \( s \). Differentiating the equation (19) with respect to \( s \) and using Frenet equations (1), we get
\[
0 = (\lambda_1' + \lambda_2) x + (\lambda_1 + \lambda_2' - c\kappa) \alpha + (\lambda_3' - c\kappa) \beta + (\lambda_2\kappa + \lambda_3\tau) y,
\]
which implies that
\[
\begin{cases}
\lambda_1' + \lambda_2 = 0, \\
\lambda_1 + \lambda_2' - c\kappa = 0, \\
\lambda_3' - c\kappa = 0, \\
\lambda_2\kappa + \lambda_3\tau = 0.
\end{cases}
\]
Solving (20), we get
\[
\int \left( \frac{\tau}{\kappa} \int \tau ds \right) ds - \left( \frac{\tau}{\kappa} \right)' \int \tau ds = \frac{k^2 + \tau^2}{\kappa}.
\]
Conversely, assume that the relation (13) holds. Then choosing the vector \( U \) as follows
\[
U = \left( \int \left( \frac{\tau}{\kappa} \int \tau ds \right) ds - \left( \frac{\tau}{\kappa} \right)' \int \tau ds \right) x - \left( \frac{\tau}{\kappa} \right) \int \tau ds y + \int \tau ds \beta + y,
\]
we get \( U' = 0 \) and \( \langle y, U \rangle = 1 \) (constant). Thus \( x(s) \) is a 3-type hyperbolic slant helix. \( \square \)

**Example 3.15.** The following hyperbolic curvature functions satisfy (17).

(i) \( \kappa = s, \tau = 1 \)  
(ii) \( \kappa = -s, \tau = 1 \)

**Corollary 3.16.** The axis of a 3-type hyperbolic slant helix is given by
\[
U = c \left( \int \left( \frac{\tau}{\kappa} \int \tau ds \right) ds \right) x - c \left( \frac{\tau}{\kappa} \right) \int \tau ds \alpha + c \int \tau ds \beta + cy,
\]
where \( c \in \mathbb{R} \setminus \{0\} \).

Assume that \( c = 0 \) in (20), then we have
\[
\begin{cases}
\lambda_1' + \lambda_2 = 0, \quad \lambda_1 + \lambda_2' = 0, \\
\lambda_3' = 0, \quad \lambda_2\kappa + \lambda_3\tau = 0.
\end{cases}
\]
which implies that
\[
\frac{\tau}{\kappa} = c_1 e^s + c_2 e^{-s}.
\]
Then \( x(s) \) is a 2-type hyperbolic slant helix. Thus we give the following corollary.

**Corollary 3.17.** Let \( x(s) \) be a hyperbolic space curve in \( \mathbb{H}^3 \) parametrized by arc-length \( s \) with non-zero curvatures \( \kappa, \tau \). Then \( x(s) \) is a 2-type hyperbolic slant helix if and only if \( x(s) \) is a 3-type hyperbolic slant helix whose axis \( U \) satisfies \( \langle y, U \rangle = 0 \).
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