From Euclidean to Lorentzian loop quantum gravity via a positive complexifier

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Received 27 August 2018, revised 7 November 2018
Accepted for publication 21 November 2018
Published 12 December 2018

Abstract

We construct a positive complexifier, differentiable almost everywhere on the classical phase space of real triads and SU(2) connections, which generates a Wick transform from Euclidean to Lorentzian gravity everywhere except on a phase space set of measure zero. This Wick transform assigns an equal role to the self dual and anti-self dual Ashtekar variables in quantum theory. We argue that the appropriate quantum arena for an analysis of the properties of the Wick rotation is the diffeomorphism invariant Hilbert space of loop quantum gravity (LQG) rather than its kinematic Hilbert space. We examine issues related to the construction, in quantum theory, of the positive complexifier as a positive operator on this diffeomorphism invariant Hilbert space. Assuming the existence of such an operator, we explore the possibility of identifying physical states in Lorentzian LQG as Wick rotated images of physical states in the Euclidean theory. Our considerations derive from Thiemann’s remarkable proposal to define Lorentzian LQG from Euclidean LQG via the implementation in quantum theory of a phase space ‘Wick rotation’ which maps real Ashtekar–Barbero variables to Ashtekar’s complex, self dual variables.

Keywords: loop quantum gravity, Wick rotation, Lorentzian

1. Introduction

The Hamiltonian dynamics of classical general relativity is generated by constraints, i.e. by functions on the gravitational phase space which vanish on-shell. Canonical quantization seeks to implement the constraints as operators and identify their kernel with the space of physical states. The complicated nonpolynomial form of the constraints in terms of the traditional ADM phase space variables [1] provides an impediment to the development of a rigorous canonical quantization based on these variables. In contrast, the Ashtekar variables, comprising of a complex connection and a conjugate triad render these constraints polynomial. Their
discovery [2] inspired a renewed attempt at canonical quantization of gravity [3, 4] leading to the development of the loop quantum gravity (LQG) approach [5–9].

However, it has proven difficult to construct quantum representations based directly on the Ashtekar variables due to their complex valued nature. As a result most developments in LQG are based on the related real Ashtekar–Barbero variables [10]. These variables are naturally adapted to Euclidean gravity and render its constraints polynomial, this polynomiality being the analog of the polynomiality of the constraints of the Lorentzian theory in the complex Ashtekar variables. In contrast, the expressions for the constraints of the Lorentzian theory in terms of the real variables are non-polynomial. However, due to the impressive development of the quantum kinematics based on the real variables there are concrete proposals which confront this non-polynomiality [11–13]. Despite these advances in the construction of the constraint operators for the Lorentzian theory based on their classical description in the real Ashtekar–Barbero variables, we are of the opinion that it would be preferrable to transit to a formalism which ascribes a central role to the original complex Ashtekar variables. The reason is that the complex Ashtekar variables have an immediate spacetime interpretation: the connection is the spatial pull back of the self dual part of the spacetime spin connection and the (densitized) triad is naturally related to the spacetime tetrad. In constrast the real Ashtekar–Barbero connection has a purely spatial character with no direct spacetime interpretation [14]. Hence, it is natural to anticipate that the construction of a spacetime covariant quantum theory for Lorentzian gravity based entirely on the real variables would be significantly more complicated than a construction which incorporates the power and elegance of the self dual description.

In a remarkable set of papers in the mid-nineties [11, 15–17], Thiemann proposed a formalism which combines the rigor of the quantum framework based on the real variables with the simplicity of the classical description in terms of the complex variables. He noted that the complex canonical transformation from real to complex variables is generated by a complexifier function. This complexifier is proportional to the spatial integral of the trace of the extrinsic curvature, the proportionality constant being imaginary and equal to $-i\pi$. $G$ being Newton’s constant. The finite complex canonical transformation generated by this complexifier maps the real connection to its self dual counterpart. Since the Lorentzian constraints are obtained by substituting the real connection and triad in the Euclidean expressions by their images through this complex canonical transformation, it follows that the complex canonical transformation maps the Euclidean constraints to their Lorentzian counterparts. Thiemann also noted that the complexifier could be expressed as the Poisson bracket between the total volume $V$ of the spatial slice and the Euclidean Hamiltonian constraint, $H_E(N = 1)$, smeared by a unit (and hence constant) lapse function $N$. Since the operator correspondent of $V$ is rigorously constructed as an operator in LQG [18–21] and since proposals are available for the construction of the Euclidean Hamiltonian constraint operator [11, 22], it is possible to attempt the construction of Thiemann’s complexifier in the quantum theory as the commutator between the operators corresponding to $V$ and $H_E(N = 1)$. The finite transformation is then mediated in quantum theory by the exponential of $\frac{1}{\hbar}$ times the operator correspondent of the complexifier. More precisely, if the integral of the trace of the extrinsic curvature multiplied by a real factor of $\frac{\pi}{\hbar}$ is denoted by $C_T$, the canonical transformation is generated by $-iC_T$ and the action of this transformation in quantum theory on any operator $\hat{O}$ is $e^{\frac{i}{\hbar}C_T} \hat{O} e^{-\frac{i}{\hbar}C_T}$ where $\hat{C}_T$ is the operator correspondent of $C_T$ [15]. In particular, the Lorentzian Hamiltonian constraint $\hat{h}_L(x)$ is obtained from its Euclidean counterpart $\hat{h}_E(x)$ as $\hat{h}_L(x) = e^{\frac{\pi}{\hbar} C_T} \hat{h}_E(x) e^{-\frac{\pi}{\hbar} C_T}$ and the physical states, $\Psi_L$, which are annihilated by the dual action of the Lorentzian constraints can be obtained from their Euclidean counterparts $\Psi_E$ as $\Psi_L = \Psi_E$. Thus as suggested by
Ashtekar [23] one may attempt to construct the physical states of the Lorentzian theory within the Hilbert space of the Euclidean theory.

The key issue to be resolved with regard to the above proposal is that of the well definedness of the map between the Euclidean and Lorentzian quantum theories. Clearly, the well definedness or lack thereof hinges on the properties of the operator \( \hat{C}_T \) corresponding to the complexifier and of its exponential \( e^{-\frac{\hat{C}_T}{\hbar}} \). In this regard, we note the following. First, as emphasised by Thiemann, the complexifier function \( C_T \) is not endowed with a particular sign, i.e. it can be positive or negative as it varies over phase space. As a result the action of the exponential operator \( e^{-\frac{\hat{C}_T}{\hbar}} \) is not sufficiently under control and its well definedness on (putative) physical states of the Euclidean theory is not clear. If instead, a complexifier function \( C \) could be found which was positive, \( e^{-\frac{\hat{C}_T}{\hbar}} \) would be expected to be a bounded operator and hence much better behaved.

Second, the Euclidean Hamiltonian constraint is constructed in LQG through a 2 step procedure. In the first step, a discrete finite triangulation approximant to the constraint is constructed as an operator on the kinematic Hilbert space of LQG spanned by spin network states. In the second step, the finite triangulation operator action is evaluated in the ‘continuum’ limit of infinitely fine triangulation. It is important to note that this limiting action of finite triangulation operators does not exist in an operator topology defined by the kinematic Hilbert space norm; a different topology must be used [7, 11, 24]. Since \( C_T \) is constructed as the commutator between the volume and Hamiltonian constraint operators, it is also defined first at finite triangulation and then in the continuum limit. Similar procedures must be employed to construct its exponential. The main problem which arises is that adjointness properties of the finite triangulation operators on the kinematic Hilbert space do not necessarily survive the continuum limit [22] and hence it is difficult to exercise control on the adjointness properties of continuum limit operators by controlling the adjointness properties of their finite triangulation approximants at the kinematic level.

In this work we address both these points. We address the first through the construction of a positive complexifier function \( C \) which is differentiable everywhere on phase space except on a set of measure zero\(^1\) and the second by shifting the quantum arena from the kinematic Hilbert space to the diffeomorphism invariant Hilbert space of LQG\(^2\).

The layout of this paper is as follows. In section 2 we construct the new positive complexifier on phase space and discuss its properties. In section 3 we argue that it is necessary to shift the primary quantum arena for the analysis of the properties of quantum complexifiers from the kinematic to the diffeomorphism invariant Hilbert space. Next, we comment on the construction of the operator version of the positive complexifier on the diffeomorphism invariant Hilbert space as well as on the implementation of the Wick rotation proposal with this choice of complexifier. In our analysis we assume that the continuum limit of the finite triangulation Euclidean Hamiltonian constraint operator smeared with unit lapse is a densely defined operator on the diffeomorphism invariant Hilbert space (see assumption A, section 3.2 for details).

\(^1\)It is difficult to anticipate the repercussions of this lack of classical differentiability in quantum theory. However, since the quantum theory hints at a discrete microstructure, classical configurations are expected to be obtained from quantum states only after some sort of coarse graining. Moreover, the relevant set of (Liouville) measure zero is characterised by \( C \) taking a sharp classical value. In the quantum theory one expects fluctuations about sharp classical values. For these reasons a view from quantum theory of classical configurations may assign a diminished significance to classical properties of a zero measure set of classical configurations. Hence it seems reasonable to press on regardless of classical pathologies on sets of measure zero.

\(^2\)Two alternatives to the route advocated in this paper exist to address the adjointness issue: the first based on the new Hilbert space introduced by Lewandowski and Sahlmann [25] and the second on Thiemann’s symmetric operator [26]. We comment on these in section 5. We also note that the route advocated in this paper may already have been implicitly suggested in earlier works such as [26].
In section 4 we discuss issues related to the existence of the zero measure set of complexifier non-differentiability. Section 5 contains a summary of our results, a discussion of possible alternatives to tackle the self adjointness issue and speculations on the path ahead.

Before we proceed to the next section we briefly comment on related work. Livine and Ben Achour have explored the construction of the Thiemann complexifier in the context of loop quantum cosmology [27]. The key difference with the considerations of our work here is the absence of a diffeomorphism constraint in the finite dimensional cosmological mini-superspace setting and the consequent presence of a regularization length scale in that work. Nevertheless the construction of the ‘complexifier-volume-Hamiltonian constraint’ algebra and the emphasis on its representations in that work constitutes an important development, and a rare instance, of an application of Thiemann’s complexifier ideas. A preliminary exploration of Wick rotation ideas in the context of $2+1$ gravity has been initiated by Hartmann and Wisniewski in [28]. Finally, there is an independent body of work which also stems from the desire to use Hamiltonian structures which interact better with the property of spacetime covariance but which does not, to our knowledge, employ complexifier ideas. More in detail, the explicit 4D spacetime covariance of the Lagrangian description of classical gravity is broken in the transition to the $3+1$ Hamiltonian formalism. This transition entails a number of steps in line with the standard Dirac–Bergmann algorithm. These steps typically mutilate the spacetime covariant properties of the Lagrangian variables. As emphasized by Barros e Sa [29], starting from the tetrad based Holst action [30], one such step partially gauge fixes the internal Lorentz rotations of the tetrad by choosing a certain ‘time gauge’. This results in the residual internal $SU(2)$ rotations of the Ashtekar–Barbero description. Clearly, retaining the full internal Lorentz group results in a more covariant formalism. Pioneering work on exactly such a formalism was initiated by Alexandrov and developed further in many works (see for example [31–33]).

2. A positive complexifier

2.1. The real and complex phase space variables: notation and review

We provide a brief review of the Ashtekar–Barbero variables and the complex Ashtekar variables mainly to set up notation and define conventions. The reader may consult [2, 10] for details. We use the conventions of [15]. The real Ashtekar–Barbero phase space variables are an $SU(2)$ connection $A^i_a$ and its conjugate unit density weighted electric field $\tilde{E}^i_b$ with non-vanishing Poisson bracket:

$$\{A^i_a(x), \tilde{E}^j_b(y)\} = G\delta^i_j\delta^a_b\delta(x,y) \tag{2.1}$$

where $a$ is a tangent space index on the Cauchy slice $\Sigma$, $i$ is an $SU(2)$ internal index and $G$ is Newton’s constant. The Cauchy slice $\Sigma$ is assumed to be compact without boundary. We shall raise and lower internal indices by the Kronecker delta $\delta^i_i$ (in other words $-\delta_a^a$ is the Cartan–Killing form for $SU(2)$). Here $\tilde{E}^i_b$ has the interpretation of a densitized triad field so that $\tilde{E}^i_b\tilde{E}^b_i = qq_{ab}$ where $q$ is the determinant of the metric. It is easy to see that $q_{ab}$ can be re-constructed from $\tilde{E}^i_b$.

The $SU(2)$ connection takes the form:

$$A^i_a = \Gamma^i_a + K^i_a \tag{2.2}$$

where $\Gamma^i_a$ is the spin connection compatible with the triad and $K^i_a$ is related to the the extrinsic curvature $K_{ab}$ of the slice $\Sigma$ through:
where $e'_{ab} = \frac{q_{ab} \tilde{E}}{\sqrt{q}}$ is the cotriad (and where in this relation $q_{ab}$ and its determinant are constructed from $\tilde{E}^{a}_i$). The Gauss law, Vector and Hamiltonian constraints of Euclidean gravity expressed in these real phase space variables take the form

\[ G_{\tilde{E}} = \mathcal{D}_a \tilde{E}^a_i \]  
\[ V_{\tilde{E}a} = \tilde{E}^a_i F^i_{ab} \]  
\[ h_{\tilde{E}} = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} \]

where $\epsilon^{ijk}$ is the alternating tensor (related to the structure constants of the Lie algebra of $SU(2)$), $\mathcal{D}$ is the gauge covariant derivative associated with $A'_a$ and $F_{ab}$ is the curvature of $A'_{ab}$.

Next, we turn to Lorentzian gravity and the Ashtekar variables. The Ashtekar momentum variable is, as in the real case, the real densitised triad $\tilde{E}^a_i$. The connection variable is complex and given by

\[ (+) A^a_j = \Gamma^a_j - iK^a_j \]  
\[ (+) \tilde{E}^a_i = \tilde{E}_i^a \]  
\[ (+) h = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} \]

where $i$ in the above equation refers to the square root of $-1$ and should not be confused with the symbol for an $SU(2)$ index (in what follows the context will make it amply clear and there will be no room for confusion). Here $\Gamma^a_j, K^a_j$ are exactly the same fields as employed in the discussion of the real variables above.

The Gauss law, vector and Hamiltonian constraints of Lorentzian gravity expressed in these Ashtekar variables are:

\[ (+) G = (+) \mathcal{D}_a \tilde{E}^a_i \]  
\[ (+) V_{ab} = \tilde{E}^b_{i} (+) F^i_{ab} \]  
\[ (+) h = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j (+) F_{abk} \]

where $(+)$ $\mathcal{D}$ is the gauge covariant derivative associated with $(+)$ $A^a_j$ and $(+)$ $F_{ab}$ is the curvature of $A'_{ab}$. The variables $(+)$ $A^a_j, (+) \tilde{E}^a_i$ are canonically related to the real variables so that the only non-vanishing Poisson bracket is:

\[ \{ (+) A^a_j(x), (+) \tilde{E}^b_i(y) \} = -iG \delta^a_b \delta_j^i \delta(x,y). \]  
\[ (2.11) \]

The connection (2.7) is the pull back of the self dual part of the 4D spin connection [2] and on-shell is one of the Sen connections [2, 34]. As noted by Ashtekar [2] one could equally well use a second connection which is the pull back of the anti-self dual part of the 4D spin connection (and is on-shell the second Sen connection [2, 34]). We shall refer to this ‘anti self-dual’ connection as $(−)$ $A'_{ab}$ and it is given by

\[ (−) A'_{a} = \Gamma'_{a} + iK'_{a}. \]  
\[ (2.12) \]

In obvious notation, the polynomial constraints (2.8)–(2.10) are equivalent to the set of polynomial constraints:

\[ (−) G = (−) \mathcal{D}_a \tilde{E}^a_i \]  
\[ (2.13) \]
$$(-)V_{ab} = \tilde{E}^b_i(-)F_{ab}$$

$$(-)h_\ell = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j(-)F_{abk}$$

with (2.11) replaced by

$$\{(-)A'_i(x), \tilde{E}^b_j(y)\} = iG\delta^a_b \delta_i^j \delta(x,y).$$

As remarked in [5], while there is no reason to prefer the self dual variables over the anti-self dual ones in classical theory, one seems to be forced to make a choice when one proceeds to a ‘connection representation’ in the quantum theory with the deeper reason for the necessity of such a choice not understood. As we shall see below, the employment of a positive complexifier function to define the Lorentzian theory from the Euclidean one restores the democratic use of the self dual and anti-self dual variables in quantum theory.

### 2.2. Review of Thiemann’s complexifier

We briefly review the essential features of Thiemann’s remarkable work. The reader is urged to consult Thiemann [15, 16] for details. Thiemann’s complexifier is:

$$T_+ = \frac{\pi}{2G} \int_\Sigma K^a_i \tilde{E}^a_i.$$  \hspace{1cm} (2.17)

This function multiplied by a factor of $-i$ generates a complex canonical transformation from the real variables $(A'_i, E^b_k)$ to the complex canonical pair $(+ A'_i, i\tilde{E}^b_k)$:

$$(+)A'_i = \sum_{n=0}^{\infty} \frac{\{A'_i, (-iT_+)\}_0(n)}{n!}$$

$$i\tilde{E}^b_k = \sum_{n=0}^{\infty} \frac{\{E^b_k, (-iT_+)\}_0(n)}{n!}$$

where $\{A, B\}_0$ refers to the $n$th order Poisson bracket $\{\ldots \{A, B\}, B\ldots B\}$ with $B$ appearing $n$ times and with $\{A, B\}_0$ defined to be equal to $A$. Equations (2.18) and (2.19) follow from the remarkable facts that $(K^a_i, E^b_k)$ are related to $(A'_i, E^b_k)$ through a canonical transformation [5] and that $T_+$ Poisson commutes with the spin connection $\Gamma^a_i$ [15].

Upto operator ordering this implies that the operator correspondent $\hat{O}_E$ of any function $O_E(A'_i, E^b_k)$ of the real variables is mapped to the operator correspondent $(+)\hat{O}_L$ of the same function of the self dual variables $O_E(+)A'_i, i\tilde{E}^b_k)$ through:

$$(+)\hat{O}_L = e^{T_+} \hat{O}_E e^{-T_+}.$$  \hspace{1cm} (2.20)

It is then straightforward to see that the classical Euclidean constraints (2.4)–(2.6) are mapped, upto overall factors to their Lorentzian counterparts (2.8)–(2.10) by the canonical transformation (2.18) and (2.19). The idea is to then define the Lorentzian constraint operators in quantum theory as images of their Euclidean counterparts using equation (2.20). Following Thiemann [15] and Ashtekar [23], we refer to the canonical transformation (2.18) and (2.19) and its quantum counterpart (2.20) as a Wick rotation.

It then follows that, formally, physical states of Lorentzian quantum gravity (i.e. states which lie in the kernel of the Lorentzian constraints) can be obtained by Wick rotating physical states of Euclidean quantum gravity via
\[ \Psi_{\text{phys},L} = e^{-\frac{\pi}{\mathcal{T}^+}} \Psi_{\text{phys},E}, \]  

(2.21)

where we have used obvious notation. The equation follows from the definition of the Lorentzian constraints as Wick rotated images of the Euclidean constraints, together with the assumptions that \( \mathcal{T}^+ \) is Hermitian (since \( \mathcal{T}^+ \) is real), that the solutions lie in the algebraic dual space to an appropriately chosen dense subspace of the kinematic Hilbert space of LQG and that the constraints are represented on this dual by their adjoints which act by dual action.

Equation (2.21) is formal because of the lack of adequate control on the well-definedness of the operators \( \mathcal{T}^+ \) and \( e^{-\frac{\pi}{\mathcal{T}^+}} \). As remarked earlier, this situation would improve if we could construct complexifiers of definite signs. We do this in the next section.

2.3. The new complexifiers

Our starting point is the observation that replacing the complexifier \( \mathcal{T}^+ \) with its negative results in a canonical transformation to anti-self dual variables. Accordingly we define

\[ \mathcal{T}^- = -\mathcal{T}^+ = -\frac{\pi}{2G} \int_\Sigma K^a_i \bar{E}^a_i. \]  

(2.22)

It is then immediate to see that \( \mathcal{T}^- \) generates the canonical transformation:

\[ (-)^j A^j = \sum_{n=0}^{\infty} \left\{ A^j, (-i\mathcal{T}^-) \right\}_{(n)} \frac{n!}{n!}, \]  

(2.23)

\[ -i\bar{E}^a_j = \sum_{n=0}^{\infty} \left\{ \bar{E}^a_j, (-i\mathcal{T}^-) \right\}_{(n)} \frac{n!}{n!}. \]  

(2.24)

As noted in section 2.1, the Lorentzian constraints can equally well be obtained by replacing the real variables in the expressions for the Euclidean constraints by the anti-self dual variables. Hence we may equally well define the Lorentzian quantum constraints and their kernel as the images of their Euclidean counterparts by replacing the ‘+’ super- and sub-scripts in (2.20) and (2.21) by ‘−’ ones respectively. Here, \( \mathcal{T}^- \) also suffers from not having a definite sign.

In order to obtain a complexifier with definite sign we define a new complexifier \( \mathcal{T} \) as follows:

\[ \mathcal{T} = \mathcal{T}^+ \quad \text{when } \mathcal{T}^+ > 0 \]  

(2.25)

\[ = \mathcal{T}^- = -\mathcal{T}^+ \quad \text{when } \mathcal{T}^+ < 0 \]  

(2.26)

\[ = 0 \quad \text{when } \mathcal{T}^+ = 0. \]  

(2.27)

In other words we set

\[ \mathcal{T} := \frac{\pi}{2G} \int_\Sigma K^a_i \bar{E}^a_i. \]  

(2.28)

From (2.25) and (2.26) \( \mathcal{T} \) generates a canonical transformation to the self dual Ashtekar variables on those parts of phase space where \( \mathcal{T}^+ \) is positive definite and a canonical transformation to the anti-self dual Ashtekar variables on those parts of phase space where \( \mathcal{T}^+ \) is negative definite. On the part of phase space where \( \mathcal{T}^+ \) vanishes, the complexifier function \( \mathcal{T} \) is not
differentiable. This part of phase space is of co-dimension one and hence of measure zero in accordance with our claim in section 1.

We note that we could equally have chosen $-T$ as a complexifier, or indeed any complexifier of the form $\pm |T_+ - B|$ for any real constant $B$. The latter set of complexifiers are then differentiable and satisfactory everywhere except on the set defined by $T_+ = B$. In what follows we shall restrict attention to the choice $B = 0$.

3. Wick rotation in quantum theory

In section 3.2 we argue that the appropriate arena to define and analyse the properties of the complexifier in quantum theory is the diffeomorphism invariant Hilbert space rather than the kinematic Hilbert space. For concreteness we phrase our discussion in terms of the Thiemann complexifier $T_+$. In section 3.3 we comment on the construction of the positive complexifier $T$ as a positive operator on the diffeomorphism invariant Hilbert space. In section 3.4 we assume the existence of such an operator and show that its positivity enables the implementation of the ‘domain changing’ strategy suggested by Thiemann [15], the aim of which is a precise specification of the distributional properties of the Lorentzian physical states obtained by Wick rotation of Euclidean physical states. We start in section 3.1 with a quick review of our notation for the various spaces which are used to house quantum states in LQG.

Before we start, we mention a caveat to our considerations in sections 3.2–3.4. In these sections we have used properties of operators which are known to hold for separable Hilbert spaces. Due to our lack of knowledge, we are not sure if these properties hold for the case of non-separable Hilbert spaces considered here. However, even if some of our analysis is questionable in the non-separable context, we believe that it may be possible to restrict attention to separable subspaces of physical interest and that our analysis would then be applicable to these separable sectors.

3.1. Review and notation

We denote the finite span of spin network states by $\mathcal{D}$. The space $\mathcal{D}$ is dense in the kinematic Hilbert space $\mathcal{H}_{\text{kin}}$. The algebraic dual to any dense set is denoted by a ‘$*$’ superscript so that $\mathcal{D}^*$ is the algebraic dual space to $\mathcal{D}$ i.e. $\mathcal{D}^*$ is the space of complex linear maps on $\mathcal{D}$. The diffeomorphism invariant space is constructed by the group averaging of states in $\mathcal{D}$ [6, 9, 35]. The finite span of states, each such state being the group average of a spin net state, is denoted by $\mathcal{D}_{\text{diff}}$. The completion of $\mathcal{D}_{\text{diff}}$ in the inner product defined by the group averaging map yields the diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. The algebraic dual space to $\mathcal{D}_{\text{diff}}$ is denoted by $\mathcal{D}_{\text{diff}}^*$. Given a representation of a * algebra of operators on the dense domain $\mathcal{D}$ of a Hilbert space, the dual action of any operator $\hat{A}$ in this algebra on an element $\Psi$ of the algebraic dual $\mathcal{D}^*$ is defined to be $\hat{A}\Psi(\phi) = \Psi(\hat{A}^\dagger\phi)$, $\forall \phi \in \mathcal{D}$ where we have assumed that $\hat{A},\hat{A}^\dagger$ map $\mathcal{D}$ to itself.

3.2. An appropriate arena for the quantum complexifier

We focus for concreteness on the Thiemann complexifier $T_+$. As noted by Thiemann [7, 11], this complexifier can be expressed as the Poisson bracket between the functions $H_E$ and $V$ where $V$ is the total volume of space,

$$V = \int_\Sigma \sqrt{q}$$  \hfill (3.1)
and \( H_E \) is the Euclidean Hamiltonian constraint of density one integrated over \( \Sigma \) against a
unit lapse:

\[
H_E = \int_{\Sigma} \frac{h_E}{\sqrt{q}} \tag{3.2}
\]

where \( h_E \) is defined in equation (2.6). Following Thiemann [7, 11] and using the conventions
of section 2 and [15], we have:

\[
T_+ := \frac{\pi}{2G} \{ H_E, V \}. \tag{3.3}
\]

It follows [11] that the operator \( \hat{T}_+ \) can be defined up to overall factors as the commutator
between the operator correspondents of \( \hat{V}, \hat{H}_E \) of \( V, H_E \):

\[
\hat{T}_+ := \frac{\pi}{2G^2} \frac{[\hat{H}_E, \hat{V}]}{i\hbar}. \tag{3.4}
\]

While the volume operator \( \hat{V} \) is well defined [19] on the kinematic Hilbert space \( \mathcal{H}_{\text{kin}} \) of LQG, the
Euclidean Hamiltonian constraint operator \( \hat{H}_E \) is not. Instead [7, 11], the operator is first
defined at finite triangulation on the kinematic Hilbert space, and then an appropriate conti-
nuum limit is taken. Thus if we were to continue to work in the arena provided by the kinematic
Hilbert space and define physical states as states in an appropriate algebraic dual, we would
have to work first at a finite triangulation characterised by some coarseness parameter \( \delta \) with
operators \( \hat{H}_{E,\delta}, \hat{T}_{+,\delta} \) and then take appropriate \( \delta \to 0 \) limits in some operator topology such as
the ‘Uniform Rovelli Smolin Operator Topology’ (URST) [7].

Since \( H_E, T_+ \) are real functions we would like their operator correspondents to be self
adjoint. Since \( \hat{V} \) is self adjoint [19], from (3.4) a construction of \( \hat{H}_E \) as a self adjoint (or even
symmetric) operator would provide a starting point for the construction of \( \hat{T}_+ \) as a self adjoint
operator. If we could construct \( \hat{T}_+ \) as a self adjoint operator, its exponential could be defined
through spectral theory and the properties of the Wick rotation map (2.20) and (2.21) could
then be analysed. If we adopt the URST view of the continuum limit, then we would like to
control the adjointness properties of \( \hat{H}_E \) at finite triangulation by choosing its finite trian-
gulation approximants to be self adjoint on \( \mathcal{H}_{\text{kin}} \). This may be done by choosing any finite
triangulation approximant to \( \hat{H}_E \) and setting the desired finite triangulation approximant to be
half the sum of the the chosen approximant and its adjoint on \( \mathcal{H}_{\text{kin}} \). We may then hope that the
continuum limit operator is also self adjoint. Unfortunately, as shown in the beautiful work of
[22], the action of the adjoint of the chosen approximant typically vanishes in the continuum
limit so that the kinematic self adjointness property at finite triangulation does not survive the
continuum limit.

The underlying reason for this trivialization of the action of the kinematic adjoint opera-
tors in the continuum limit is as follows. The action of finite triangulation constraint operators
on a spin net state typically creates states with new ‘offspring’ vertices in a \( \delta \) vicinity of the
‘parent’ vertex. Their adjoints when acting on a state check if the state has a vertex configura-
tion corresponding to offspring vertices separated by \( \delta \) and contract such vertices to yield a
parent vertex. Thus for a nontrivial action of the adjoint at parameter value \( \delta \) on a given state,
the state vertex configuration must exactly match an offspring vertex configuration at that pre-
cise value of \( \delta \). Clearly on a fixed state for sufficiently small values of \( \delta \) there will be no such
configuration so that the continuum limit action of the kinematic adjoint typically vanishes
(see [22] for details). Preliminary calculations show that key contributions to the kinematic
adjoint of the finite triangulation Hamiltonian constraint also vanish in the continuum limit in
the context of the toy model of PFT [36]. Since the underlying reason is robust, we expect that this situation will persist even if the action of the Euclidean Hamiltonian constraint is modified (relative to that considered in [11, 22]) so as to incorporate the lessons of [37].

In order to overcome this problem we propose that the arena for an analysis of the continuum limit operators be changed from the kinematic Hilbert space $\mathcal{H}_{\text{kin}}$ to the diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. Viewed in this way, the continuum limit operators $\hat{H}_E, \hat{T}_+$ are to be seen as operators on $\mathcal{H}_{\text{diff}}$. The problem is then that kinematic self adjointness of finite triangulation approximants to the Euclidean Hamiltonian constraint on $\mathcal{H}_{\text{kin}}$ does not necessarily translate to self adjointness of their continuum limit operator on $\mathcal{H}_{\text{diff}}$. Indeed, in view of the above discussion, we feel that it may be impossible to construct a self adjoint/symmetric $\hat{H}_E$ operator on $\mathcal{H}_{\text{diff}}$ as the continuum limit of finite triangulation approximants. If, as we expect, this turns out to be the case in Euclidean LQG, then a simple way to construct a self adjoint/symmetric $\hat{H}_E$ operator is as follows: (i) construct the continuum limit operator, $\hat{H}_E, \delta \to 0$, of suitable finite triangulation approximants as an operator on $\mathcal{H}_{\text{diff}}$ (ii) compute the adjoint of this continuum limit operator on $\mathcal{H}_{\text{diff}}$ and (iii) define $\hat{H}_E$ to be half the sum of (i) and (ii).

In the above discussion we have implicitly assumed that $\hat{H}_E, \delta \to 0$ can be constructed as an operator on $\mathcal{H}_{\text{diff}}$. Note that this assumption does not follow from the existence of a URST continuum limit of $\hat{H}_E, \delta \to 0$. The existence of such a URST limit implies that $\hat{H}_E, \delta \to 0$ maps any diffeomorphism invariant distribution which lies in $\mathcal{D}^*$ to an element of $\mathcal{D}^*$. However, since $\hat{H}_E$ is a diffeomorphism invariant function, it is reasonable to require that its operator correspondent $\hat{H}_E, \delta \to 0$ maps any diffeomorphism invariant element of $\mathcal{D}^*$ to another such diffeomorphism invariant element. Note, however, that such elements of $\mathcal{D}^*$ could in principle lie outside $\mathcal{H}_{\text{diff}}$; for example they could be infinite linear combinations of states in $\mathcal{D}_{\text{diff}}$, which are not normalizable in the inner product on $\mathcal{H}_{\text{diff}}$. Here we make the following simplifying assumption:

**Assumption A.**

A.1 The continuum limit operator $\hat{H}_{E,\delta \to 0}$ can be constructed as a densely defined operator on $\mathcal{H}_{\text{diff}}$.

A.2 $\mathcal{D}_{\text{diff}}$ is a dense domain for $\hat{H}_{E,\delta \to 0}$.

A.3 $\hat{H}_{E,\delta \to 0}$ maps $\mathcal{D}_{\text{diff}}$ to itself.

One possibility to construct $\hat{T}_+$ as a symmetric operator on $\mathcal{H}_{\text{diff}}$ is to construct $\hat{H}_E$ as a symmetric operator through (i)–(iii) and to define $\hat{T}_+$ through

$$\hat{T}_+ := \frac{\pi}{2G^2} \frac{[\hat{H}_E, \hat{V}]}{i\hbar}. \tag{3.5}$$

This is well defined provided there are no domain issues. More in detail $\hat{V}$ maps $\mathcal{D}_{\text{diff}}$ to itself and from Assumption A above, so does $\hat{H}_{E,\delta \to 0}$. However, a domain problem could occur if $\hat{H}_{E,\delta \to 0}$ does not preserve $\mathcal{D}_{\text{diff}}$. A way to avoid this possible problem is to first define the right hand side of (3.5) using only the contribution (i), and define $\hat{T}_+$ as half of the sum of this right hand side and its adjoint:

$$\hat{T}_{+1} := \frac{\pi}{2G^2} \frac{[\hat{H}_{E,\delta \to 0}, \hat{V}]}{i\hbar}. \tag{3.6}$$

$^3$We shall discuss this assumption further in section 5.
\[ \hat{T}_+ := \frac{\hat{T}_{+1} + \hat{T}_{+1}^\dagger}{2}. \] (3.7)

The existence of \( \hat{T}_+ \) as a symmetric operator depends on the domain of \( \hat{T}_{+1}^\dagger \). If \( \hat{T}_+ \) is symmetric, one may attempt to find a self-adjoint extension and define its exponential through its spectral decomposition.

### 3.3. On the existence of \( \hat{T} \) as a positive operator

In this section we comment on the construction of the operator correspondent \( \hat{T} \) of the positive complexifier \( T \). If either of equations (3.5) or (3.7) define \( \hat{T}_+ \) as a symmetric operator one may seek to construct a self-adjoint extension and define \( \hat{T} \) as the square root of \( \hat{T}_+^2 \) so constructed.

A simpler way which may turn out to be of more practical value is as follows. Define \( \hat{T}_+ \) simply as \( \hat{T}_{+1} \) (see equation (3.6)). From Assumption A, \( \hat{T}_{+1} \) maps \( \mathcal{D}_{\text{diff}} \) to itself. Let us assume that the domain \( \mathcal{D}(\hat{T}_{+1}^\dagger) \) of \( \hat{T}_{+1}^\dagger \) contains \( \mathcal{D}_{\text{diff}} \). Preliminary calculations indicate that this assumption is satisfied for the appropriate analogs of \( \hat{T}_{+1} \) in PFT [36, 37].

Next note that for any \( \psi, \phi \in \mathcal{D}_{\text{diff}} \),
\[
(T_+ \psi, T_+ \phi) = (\psi, T_+^\dagger T_+ \phi) = (T_+^\dagger T_+ \psi, \phi) \tag{3.8}
\]
where we have used that \( T_+ \psi, T_+^\dagger \psi \in \mathcal{D}_{\text{diff}} \subset \mathcal{D}(T_+^\dagger) \) equation (3.8) implies that \( \hat{T}_+^\dagger \hat{T}_+ \) is a positive symmetric operator on \( \mathcal{D}_{\text{diff}} \). Hence we may construct its (positive) Friedrich’s extension [38]. Finally, we may define \( \hat{T} \) to be the positive square root of this Friedrich’s extension.

### 3.4. Wick rotation with positive \( \hat{T} \)

In this section we assume that \( \hat{T} \) can be constructed as a densely defined positive self-adjoint operator on \( \mathcal{H}_{\text{diff}} \). Since under this assumption \( e^{-\hat{T}} \) is a bounded operator, it may turn out that the Wick rotation is better defined than with \( \hat{T}_+ \). More in detail, any Euclidean physical state \( \Psi_{E,\text{phys}} \in \mathcal{D}_{\text{diff}}^* \) can be Wick rotated to the Lorentzian physical state \( \Psi_{L,\text{phys}} \) through:
\[
e^{-\hat{T}} \Psi_{E,\text{phys}} = \Psi_{L,\text{phys}} \tag{3.9}
\]

\[
\Rightarrow \Psi_{L,\text{phys}}(\phi) := \Psi_{E,\text{phys}}(e^{-\hat{T}} \phi) \forall \phi \in \mathcal{D}_{\text{diff}}. \tag{3.10}
\]

Here we have assumed, consistent with the lessons from work on toy models [36, 39], that the physical states of Euclidean LQG lie in \( \mathcal{D}_{\text{diff}}^* \). If the right hand side of (3.10) is finite, \( \Psi_{L,\text{phys}} \) also resides in \( \mathcal{D}_{\text{diff}} \). One may hope that the bounded operator \( e^{-\hat{T}} \) is well behaved enough that this indeed is the case. However, this can only be ascertained after the Euclidean theory physical states are constructed so we are unable to say anything more in this direction.

Instead, in the remainder of this section, we show that it is possible to implement (our interpretation of) Thiemann’s suggestion in [15] to define a new dense domain \( \mathcal{D}_{\text{diff}} \subset \mathcal{H}_{\text{diff}} \) such that physical states of Lorentzian LQG are precisely characterised as elements of the algebraic...
dual space $\tilde{D}_{\text{diff}}^*$. In order to do this we shall, counter intuitively, use $-T$ as our complexifier. Accordingly we have that

$$e^{\hat{T}} \Psi_{\text{E,phys}} =: \Psi_{\text{L,phys}}.$$  \hfill (3.11)

We define the set

$$\tilde{D}_{\text{diff}} := e^{-\hat{T}} D_{\text{diff}}.$$  \hfill (3.12)

We now show that $\tilde{D}_{\text{diff}}$ is dense in $\mathcal{H}_{\text{diff}}$. First note that since $e^{-\hat{T}}$ is bounded, it is continuous on $\mathcal{H}_{\text{diff}}$ so that the image of any Cauchy sequence by $e^{-\hat{T}}$ is Cauchy. Hence we have that

$$e^{-\hat{T}} D_{\text{diff}} \supseteq e^{-\hat{T}} \mathcal{H}_{\text{diff}}.$$  \hfill (3.13)

Next, let $D'_{\text{diff}} \subset \mathcal{H}_{\text{diff}}$ be a dense domain for $e^{\hat{T}}$ so that:

$$e^{\hat{T}} : D'_{\text{diff}} \to R \subseteq \mathcal{H}_{\text{diff}}$$ \hfill (3.14)

where $R$ is the image of $D'_{\text{diff}}$ by $e^{\hat{T}}$ so that $e^{-\hat{T}} R = D_{\text{diff}}'$ (recall that since $e^{-\hat{T}}$ is bounded it is defined on all of $\mathcal{H}_{\text{diff}}$). This in turn implies that:

$$e^{-\hat{T}} \mathcal{H}_{\text{diff}} \supseteq D'_{\text{diff}}.$$  \hfill (3.15)

Using (3.13) in the above equation yields:

$$e^{-\hat{T}} D_{\text{diff}} \supseteq D'_{\text{diff}}.$$  \hfill (3.16)

Since $e^{-\hat{T}} D_{\text{diff}}$ closed and $D'_{\text{diff}}$ is dense it follows that $e^{-\hat{T}} D_{\text{diff}} = \mathcal{H}_{\text{diff}}$. Equation (3.12) then implies that $\tilde{D}_{\text{diff}}$ is dense.

Next, we show that $\Psi_{\text{L,phys}}$ in (3.9) is an element of $\tilde{D}_{\text{diff}}^*$. To do so, let $\tilde{\phi} \in \tilde{D}_{\text{diff}}$. From equation (3.12) we have that $\tilde{\phi} = e^{-\hat{T}} \phi$ for some $\phi \in D_{\text{diff}}$. It follows that:

$$\Psi_{\text{L,phys}}(\tilde{\phi}) = e^{\hat{T}} \Psi_{\text{E,phys}}(\tilde{\phi}) = \Psi_{\text{E,phys}}(e^{\hat{T}} \tilde{\phi}) = \Psi_{\text{E,phys}}(\phi).$$  \hfill (3.17)

Thus $\Psi_{\text{L,phys}}$ resides in $\tilde{D}_{\text{diff}}$.

To summarise: choosing the complexifier to be $-T$ enables us to define the Wick rotated image of any Euclidean solution as a distribution in the algebraic dual to the new dense set $\tilde{D}_{\text{diff}}$.

4. Comments on the zero measure set of complexifier non-differentiability

4.1. Behaviour of the positive complexifier $T$ on the $T = 0$ surface

In the main body of the paper we ignored the lack of differentiability of $T$ on the zero measure, co-dimension one surface $T = 0$. Here we attempt to quantify this lack of differentiability by the formal manipulations in the following, and to interpret the transformation generated on this surface.

Let $\Theta(x)$ be the step function on the real line:

$$\Theta(x) = 1, x > 0$$ \hfill (4.1)

$$= 0, x = 0$$ \hfill (4.2)

$$= -1, x < 0.$$ \hfill (4.3)
It follows from (2.25)–(2.27) that
\[ T = T_+ \Theta(T_+). \] (4.4)
Let \( f \) be any smooth function on phase space. It follows that:
\[ \{ f, T \} = \{ f, T_+ \} \left( \frac{d}{dT_+}(T_+ \Theta(T_+)) \right) = \{ f, T_+ \} \alpha(T_+). \] (4.5)
where we have set
\[ \alpha(T_+) := \frac{d}{dT_+}(T_+ \Theta(T_+)) = 2T_+ \delta(T_+) + \Theta(T_+). \] (4.6)
This implies
\[ \{ f, T \} (2) = \{ f, T_+ \} (2)(\alpha(T_+))^2 + \{ f, T_+ \} \{ \alpha(T_+), T_+ \Theta(T_+) \}. \] (4.7)
If \( \alpha, \Theta \) were smooth functions of \( T_+ \) rather than distributions, the last Poisson bracket in (4.7) would vanish. If we assume that there is a way to regulate their Poisson bracket so that it vanishes, we have that
\[ \{ f, T \} (2) = \{ f, T_+ \} (2)(\alpha(T_+))^2. \] (4.8)
We may compute higher order Poisson brackets in a similar fashion. In doing so we encounter Poisson brackets between pairs of distributions each of which depend solely on \( T_+ \). If we assume that there is a way to regulate such Poisson brackets so that they vanish, we have that
\[ \{ f, T \} (n) = \{ f, T_+ \} (n)(\alpha(T_+))^n. \] (4.9)
Finally, from its definition (4.6) it seems plausible that a regularization exists such that we may set \( \alpha = 0 \) at \( T_+ = 0 \). This would mean that \( \{ f, T \} (n) \) vanishes for all \( n \geq 1 \). Assuming this is true and using (2.18) and (2.19) with \( T_+ \) replaced by \( T \), we arrive at the conclusion that \( T \) generates a canonical transformation to self dual variables when \( T_+ > 0 \), to anti-self dual variables when \( T_+ < 0 \) and the identity transformation when \( T_+ = 0 \). Hence, if we are to take the above formal arguments seriously the Wick transformation generated by \( T \) maps the Euclidean theory to a pair of ‘Lorentzian phases’ separated by a ‘Euclidean phase’ boundary. If we use the complexifier \(|T_+ - B|\) (see the last paragraph of section 2.3), it is straightforward to repeat our formal manipulations with the substitution \( T_+ \to T_+ - B \) and conclude that the Euclidean phase boundary shifts to the codimension one surface \( T_+ = B \). We note here that Euclidean phases do appear in prior discussions of quantum gravity [42, 43].

4.2. Quantum theory viewpoints on the zero measure set of complexifier non-differentiability

For the reasons spelt out in footnote 1, we believe that it is a useful strategy to pursue the construction of the positive complexifier as an operator in quantum theory despite its classical non-differentiability on an infinite dimensional surface of codimension one, and hence of measure zero, in phase space. Nevertheless, one must be open to the possibility that this classical lack of differentiability may give rise to unphysical features in the putative quantum

\[ ^5 \text{We have not explored the physical consequences of the Euclidean phase encountered in our work. Our intention in referring to the Euclidean phases encountered in [42, 43] is not to create the impression that the resulting physics in these cases is related to that of the Euclidean phase of our work. Rather we merely wish to note that Euclidean phases do manifest in other analyses of quantum gravity.} \]
theory especially because (a large subset of) classical initial data on this codimension one surface qualify as benign from a classical point of view.

One way to avoid this particular surface is to use the complexifier $|T + B|$ (see the end of section 2.3). This moves the pathological surface from $T_+ = 0$ to $T_+ = B$. One may then treat $|B|$ as a regularization parameter to be taken to infinity. It is instructive to examine the location of the pathological surface for large $|B|$. To this end, consider a point $p = (\tilde{E}_a, K^b_i)$ in phase space on the surface $T_+ = B$. Let $V$ be the volume of the spatial slice computed from $\tilde{E}_a$. Define the spatial average of the trace of the extrinsic curvature at $p$ as $\int \Sigma \tilde{E}_a K^b_i V$. Then for $|B|$ large enough that $G|B|V \gg l_P$ (here $l_P$ is the Planck length), it follows that the spatial average of the trace of the extrinsic curvature at $p$ is trans-Planckian. Thus as $|B| \rightarrow \infty$, we may ascribe any features arising from non-differentiability of $|T + B|$ to deep quantum gravity physics.

Indeed, from the considerations of section 4.1, one may perhaps conceive of the existence of a trans Planckian Euclidean phase as one of the physical imprints of the $|B| \rightarrow \infty$ limit.

Reverting to the choice $B = 0$, one may envisage an attempt at combining the Dirac quantization strategy adopted in LQG with some sort of gauge fixing. One popular gauge choice fixes the trace of the extrinsic curvature to be constant on the spatial slice [47]. If this constant is non-zero then such gauge fixed configurations lie away from the pathological surface of complexifier non-differentiability. On the other hand if the trace of the extrinsic curvature vanishes, these configurations lie exactly on the pathological surface. In this regard we note the curious fact that most Lorentzian spacetimes with topology $\Sigma \times R$, $\Sigma$ either closed or asymptotically flat, which satisfy the Einstein equations with matter sources obeying the weak energy condition, do not admit spatial slices with vanishing trace of extrinsic curvature due to topological obstructions [48].

5. Summary, open issues and possible strategies

5.1. Summary

The two main points made in this work are:

1. It is possible to construct a positive complexifier which ascribes an equal role to the anti-self dual and self dual Ashtekar variables. This comes at the cost of introducing an infinite dimensional surface of non-differentiability. Since the surface is of measure zero, it is still a useful strategy to pursue the use of this complexifier in constructing Lorentzian LQG via a Wick rotation of Euclidean LQG as proposed by Thiemann.

2. The Thiemann complexifier $\hat{T}_+$ is constructed as the commutator between the Hamiltonian constraint operator smeared with a unit lapse, $\hat{H}_E$, and the total volume operator $V$. Hence, it is necessary to first construct $\hat{H}_E$. However, $\hat{H}_E$ cannot be constructed directly on the LQG kinematic Hilbert space $\mathcal{H}_{\text{kin}}$. Instead it is necessary to construct $\hat{H}_E$ as a continuum limit of kinematically well defined finite triangulation approximant operators. Since $\hat{H}_E$ is real, it is desireable that $\hat{H}_E$ be represented as a self adjoint operator. While it may seem reasonable to expect that a choice of approximants which are self adjoint on $\mathcal{H}_{\text{kin}}$ leads to a continuum limit operator $\hat{H}_E$ which is self adjoint, the analysis of [22] together with a study of the PFT toy model suggests that this need not be true. The reason is that certain contributions to the action of kinematically self adjoint approximants vanish in the

Note that the $|B| \rightarrow \infty$ limit is distinct from the $\frac{|B|}{V} \rightarrow \infty$ limit in the sense that the latter is phase space dependent. Indeed, in quantum theory, $V^{-1}$ would be an operator. Our discussion above is too qualitative to take account of any ensuing subtleties.
continuum limit. Since $T_+$ is real and since $\hat{T}_+$ is constructed from $\hat{H}_E$, it is expected that a similar adjointness problem afflicts its construction.

We propose to get around this problem of the destruction of kinematic adjointness properties in the continuum limit by changing the arena from $\mathcal{H}_{\text{kin}}$ to the diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. Since $\hat{H}_E$ is diffeomorphism invariant, we assume that its continuum limit operator correspondent is well defined on the diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. The problem of adjointness now takes the form: the continuum limit of kinematically self adjoint finite triangulation approximants is not necessarily self adjoint on $\mathcal{H}_{\text{diff}}$. The problem admits a ready solution: construct the continuum limit operator, compute its adjoint with respect to the Hilbert space structure of $\mathcal{H}_{\text{diff}}$ and construct the desired self adjoint operator $\hat{H}_E$ as half the sum of the continuum limit operator and its adjoint. Since $T_+$ is diffeomorphism invariant and since $\hat{T}_+$ is constructed from $\hat{H}_E$, it follows from our proposal that the appropriate arena to analyse quantum complexifier adjointness properties should be $\mathcal{H}_{\text{diff}}$ rather than $\mathcal{H}_{\text{kin}}$. A similar conclusion holds for the analysis of self adjointness and positivity properties of the operator correspondent of the positive complexifier.

Underlying our analysis of complexifier operator properties is Assumption A of section 3. This regard note that $\mathcal{D}_{\text{diff}}$ and $\mathcal{H}_{\text{diff}}$ are enormous spaces and presumably contain many states which are not physically relevant. Even if assumption A is not valid, the hope is that we may be able to restrict attention to some suitable subspace of states within $\mathcal{D}_{\text{diff}}$ for which an appropriate version of assumption A would be valid. Related to this is our view of LQG itself. We view it as a conservative effort rooted in the continuum which provides glimpses of a discrete microstructure. At some stage it may be necessary to use the intuition for this discrete microstructure provided by this conservative effort to make radical jumps which take this microstructure as fundamental with continuum structures being emergent. This may entail an enlargement of the gauge group of diffeomorphisms [40, 41, 44]. It may be that on the resulting replacement for $\mathcal{H}_{\text{diff}}, \mathcal{D}_{\text{diff}}$, some version of Assumption A is valid.

5.2. Comparison with alternatives: open issues

Our viewpoint on the continuum limit definition of operator correspondents of diffeomorphism invariant functions is that the continuum limit operator is defined on $\mathcal{H}_{\text{diff}}$ rather than on $\mathcal{H}_{\text{kin}}$, the latter being the ‘URST’ viewpoint articulated in [7]. As mentioned in section 3, from [22] and from preliminary calculations in the PFT toy model, the URST viewpoint does not allow us to circumvent the adjointness problem and we expect that even with a modified Euclidean constraint action of the type in [39] this situation may not change. It would be useful to see if one could suitably refine/modify the URST viewpoint in a consistent and practically useful way so as to analyse the properties of the quantum complexifier staying within $\mathcal{H}_{\text{kin}}$.

Absent this, a change in arena from $\mathcal{H}_{\text{kin}}$ seems to be necessary. One possibility is to work in $\mathcal{H}_{\text{diff}}$. One may do this in the manner advocated here. Another possibility is to use Thiemann’s symmetric constraint [26]. As far as we understand, the regulated symmetric constraint is defined on a certain modification $\mathcal{H}_{\text{kin,mod}}$ of $\mathcal{H}_{\text{kin}}$. The modification seems to require an extension of the piecewise analytic category of spin network graphs appropriate to $\mathcal{H}_{\text{kin}}$ [26] to

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7 We note here that our proposal to get around the adjointness problem may already have been implicit in earlier works (for e.g. see second paragraph before definition 3.1 in [26]). Indeed, one of the purposes of this work is to argue that the complexifier ideas of Thiemann [15] and their suggested application by Ashtekar [23], both formulated in the early days of LQG, when re-examined in the light of developments since then, provide an eminently viable path to the construction of Lorentzian LQG.
graphs with not only piecewise analytic edges but also certain ‘marked’ edges which are $C^\infty$.

To our understanding, the continuum limit is not taken but the dual operator corresponding to
the symmetric constraint smeared with unit lapse has a well defined action on the space of dif-
fefomorphism invariant states $\mathcal{H}_{\text{diff,mod}}$ obtained by group averaging spin net states in $\mathcal{H}_{\text{lin,mod}}$
with respect to diffeomorphisms$^8$, and, moreover, this dual operator is symmetric on $\mathcal{H}_{\text{diff,mod}}$.
Hence, it may be considered as a candidate for $H_E$. However, it is not clear to us if the action
of the symmetric constraint is consistent with the requirement of a non-trivial anomaly free
representation of the Poisson bracket between a pair of (higher density) Hamiltonian const-
straints in the sense of [39]. Another possibility is to work in the new Hilbert space defined by
Lewandowski and Sahlmann [25]. The Lewandowski–Sahlmann Hilbert space, remarkably,
supports the continuum limit action of the Hamiltonian constraint discussed in their work for
any choice of lapse, and hence, in particular, for unit lapse. However, it is not clear to us if
their construction of this Hilbert space can be generalised to support a constraint action of the
type [39] which changes the vertex set of the state acted upon. Clearly, a deeper investigation
of the constructions of [25, 26] with regard to the non-trivial anomaly free requirement would
be very useful.

On the other hand, while our proposal to shift the arena for the implementation of the Wick
rotation (2.21) from the kinematic to a diffeomorphism invariant one is applicable to physical
states annihilated by the nontrivial anomaly free constraint actions of the type [39], the follow-
ing issue arises. The complexifier and its exponential are defined now only on $D_{\text{diff}}, \mathcal{H}_{\text{diff}}$. On
the other hand the constraint $h_E$ smeared with an arbitrary lapse does not preserve $D_{\text{diff}}, \mathcal{H}_{\text{diff}}$.
Hence the relation (2.20) with $\mathcal{O}_E$ set equal to this smeared Hamiltonian constraint is not well
defined on the diffeomorphism invariant space $D_{\text{diff}}$ (nor on its algebraic dual $\mathcal{H}_{\text{diff}}$). As
a result, the relation (2.21) which does not suffer from this issue acquires the status of a
proposal with formal motivation deriving from (2.20).

One possible route to the construction of physical states through (2.20) and (2.21), using
only diffeomorphism invariant arenas, which seems worth exploring, is to use the Master
Constraint formalism [12]. Define the diffeomorphism invariant Euclidean and Lorentzian
Master Constraints as $H_{\text{EM}} := \int \frac{d^3x}{\sqrt{\gamma}} \left( \frac{\hbar}{\sqrt{\gamma}} \right)^2$, $H_{\text{LM}} := \int \frac{d^3x}{\sqrt{\gamma}} \left( \frac{\hbar}{\sqrt{\gamma}} \right)^2$. Using $T_+$ (or $T$), the vanish-
ing of the Wick rotated image of $H_{\text{EM}}$ implies $H_{\text{LM}} = 0$ (or $H_{\text{LM}} = 0$ away from the $T = 0$
surface). More precisely the Wick rotated image is proportional to $H_{\text{LM}}$ with the exact propor-
tionality depending on how we choose the roots of $i$ coming from the Wick rotation of the $\sqrt{\gamma}$
actors in $H_{\text{EM}}$ by $T_+$ (or the roots of $\pm i$ with the phase space dependent $\pm$ signs depending
on whether these factors are rotated by $T$ at $T_+ > 0$ or $T_+ < 0$). Our suggestion is to set $\mathcal{O}_E$
in (2.20) to be the Euclidean Master constraint operator and to identify Lorentzian solutions
with the kernel of the Wick rotated image $\mathcal{O}_L$ which then implies (2.21). The consideration of
the many subtleties involved in the Master constraint program [12, 49] and their implications
for our suggestion here are beyond the scope of this work and should be analysed. To articu-
late the key issue which arises, it is useful to denote the Lorentzian Hamiltonian constraint
$h_L$ smeared with lapse $N$ by $h_L(N)$. The key issue is then as follows: does the structure of $\mathcal{O}_L$

$^8$To our understanding, the type of diffeomorphisms employed in [26] are ones which preserve analyticity. The
current state of art often employs semianalytic diffeomorphisms [7, 45] which preserve only piecewise analyticity
of graphs and which play a key role in the ‘LOST’ uniqueness theorem [45]. Nevertheless, we believe that it should
be possible to find a way to apply the basic idea of ‘marking’ edges in [26] to the semianalytic context. We also note
here that there is an independent uniqueness theorem by Fleischhack [46] which uses a notion of ‘stratified analytic
diffeomorphisms’. Regrettably, due to the limitations of our technical expertise we are not able to comment further
on this category of diffeomorphisms and their relation to semianalytic diffeomorphisms. We refer the interested
reader to Fleischhack’s clear exposition [46] and the references therein.
and its solutions obtained by Wick rotation suggest a construction of the constraint operators \( \hat{h}_L(N) \), \( N \) arbitrary, such that that these solutions lie in the kernel of the \( \hat{h}_L(N) \) so constructed?

We end this section with a general remark pertaining to Wick rotated operators. Dynamically important diffeomorphism invariant operators of the Euclidean theory which arise as counterparts of classically real functions such as the Euclidean Master constraint or real Dirac observables are expected to be represented as self adjoint operators on \( \mathcal{H}_{\text{diff}} \). Their Wick rotated counterparts are then generically not self adjoint and it would seem that their spectral analysis would be extremely involved. In this regard, we note that such Wick rotated images of self adjoint operators seem to resemble a class of well studied operators known as pseudo Hermitian operators whose spectral properties, despite their non-Hermiticity, are under mathematical control: for example a large class of such operators have eigen values which are either real or occur in complex conjugate pairs [50]. While many of these studies are in the context of quantum mechanical operators with discrete spectra with a slightly restrictive notion of pseudo Hermiticity, there is a vast literature on the subject (see for e.g. [51, 52]) which may prove useful for a spectral analysis of Wick rotated operators relevant to Lorentzian LQG.

5.3. Future work

First, the complexifier strategies, outlined in this work and in Thiemann’s earlier works, should be tested on simpler systems which have a diffeomorphism constraint. In this regard 2+1 gravity provides an excellent testing ground for which a beautiful first analysis already exists [28] and should be built upon. Second, as remarked in section 5.2, a possible connection to the rich literature on pseudo Hermitian operators [51] should be explored. Third, it is imperative to make progress in the construction of Euclidean LQG. We believe the state of the art now permits a serious re-engagement with this problem.

An ever present question is what to do even if we have constructed the Euclidean LQG solution space. Since there are no Dirac observables, how must we interpret these states and how do we construct the correct inner product on this space? It seems to us that any interpretation rests on a satisfactory interpretation of states in \( \mathcal{D}_{\text{diff}} \). If we could construct semiclassical states in \( \mathcal{D}_{\text{diff}} \) and interpret them as quantum Cauchy slices one could hope that semiclassical physical states could be built as linear combinations of such states and that such physical states could be interpreted as representing quantum spacetimes (in the immensely simpler context of Parameterised Field Theory this is exactly what happens). A beautiful suggestion for how to construct an inner product on physical states has been made by Thiemann in the mid nineties in [17]. It would be good to revisit this suggestion in the light of developments since then. One would then like to Wick rotate these states and construct Lorentzian states. It is difficult to anticipate at this stage what the structure of such states could be and how to confront the problems of the inner product and interpretation in the Lorentzian theory. Hence, we end further speculation as well as this paper here.

Acknowledgments

We thank Eduardo Villaseñor for help with functional analytic issues. We are very grateful to Fernando Barbero and Thomas Thiemann for useful conversations and comments on a draft version of this manuscript.
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