Euclidean Grid Structures in Banach Spaces

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Abstract

We study the way in which the Euclidean subspaces of a Banach space fit together, somewhat in the spirit of the Kašin decomposition. For a certain class of spaces, that we characterize exactly as those having nontrivial cotype, sparse vectors act as though they were in a Hilbert space. This allows us to transfer results that hold in Euclidean space, such as the restricted isometry property of random matrices and the Johnson-Lindenstrauss lemma, to a more general setting.

1 Introduction

A fundamental result in the geometry of Banach spaces is Dvoretzky’s theorem (see e.g. [22, 29]), which states that any Banach space \(X\) of dimension \(n \in \mathbb{N}\) is richly endowed with approximately Euclidean subspaces of dimension \(k \geq c \log n\). Besides knowing that there are many Euclidean subspaces, it is not known precisely how these subspaces are arranged within \(X\). In the case \(X = \ell_1^n\), it was shown by Kašin [14] that there exist mutually orthogonal subspaces \(E_1, E_2 \subset \mathbb{R}^n\) such that \(\mathbb{R}^n = E_1 \oplus E_2\) and for all \(i \in \{1, 2\}\) and all \(x \in E_i\),

\[
c_1|x| \leq \frac{1}{\sqrt{n}}||x||_1 \leq c_2|x|
\]

where \(c_1, c_2 > 0\) are universal constants, \(|x| = (\sum_{i=1}^n x_i^2)^{1/2}\) and \(||x||_1 = \sum_{i=1}^n |x_i|\). The same decomposition was shown to hold for spaces with universally bounded volume ratio [30, 32] (see Section 3 for more details). In this paper we study how the Euclidean subspaces of more general Banach spaces fit together, somewhat in the spirit of the Kašin decomposition.

For a real Banach space \((X, ||-||)\) of dimension \(n \in \mathbb{N}\), we seek a decomposition into high dimensional approximately Euclidean subspaces that respects the structure of \(X\). This means that the Euclidean subspaces should be orthogonal with respect to a non-degenerate ellipsoid \(\mathcal{E}\) such as the John ellipsoid. Furthermore, the unit ball of \(X\) in each subspace should be about the same size, and not too eccentric when compared to \(\mathcal{E}\). We refer to Theorem 1 and Theorem 2 for precise statements. In the general setting, there is no hope of a decomposition into two subspaces, and the best one can hope for is \(O(n/\log(n))\). The focus is not as much on the number of subspaces in the decomposition, but rather on the dimension of these subspaces.
In the second part of the paper we seek a richer collection of Euclidean subspaces that fit together like a grid in such a way so that, with respect to a particular basis, sparse vectors act as though they were in a Hilbert space. This allows us to transfer results that hold in Euclidean space, such as the restricted isometry property for random matrices and the Johnson-Lindenstrauss lemma, to a more general setting.

The most interesting example is $\ell^n_\infty$, which has a decomposition of the first kind, but not of the second.

## 2 Main results

**Theorem 1** There exist universal constants $c, c_1, c_2 > 0$ with the following property. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that the ellipsoid of maximum volume in $B_X = \{ x : \|x\| \leq 1 \}$ is the standard Euclidean ball $B^n_2$. Then there exists a decomposition $X = \oplus_{i=1}^N H_i$ into mutually orthogonal subspaces $(H_i)_i^N$ of $\dim(H_i) \geq c \log n$ such that for all $1 \leq i \leq N$ and all $x \in H_i$, 

$$c_1 M |x| \leq \|x\| \leq c_2 M |x|$$

where $M$ is the average value of $\|\cdot\|$ on $S^{n-1}$. Moreover, the result holds with high probability for a random decomposition, using a random orthogonal matrix uniformly distributed in $O(n)$.

Theorem 1 may be converted to a result similar to Theorem 2 below, but with the John ellipsoid of $B_X$ a universally bounded perturbation of $B^n_2$. However for many spaces such as those with a symmetric basis and those that come with a pre-packaged coordinate system, such as $\ell^n_\infty$, the Euclidean structure associated to the (exact) John ellipsoid is of particular importance. Answering a question that we posed in an earlier draft of this paper, Konstantin Tikhomirov gave a proof of the following result which we discuss further in Section 5.2. We thank him for allowing us to include it here.

**Theorem 2** There exists a universal constant $c > 0$ with the following property. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that the ellipsoid of maximum volume in $B_X = \{ x : \|x\| \leq 1 \}$ is the standard Euclidean ball $B^n_2$ and let $c(\log \log n)^{3/2}/(\log n)^{1/2} < \varepsilon < 1/2$. Then there exists a decomposition $X = \oplus_{i=1}^N H_i$ into mutually orthogonal subspaces $(H_i)_i^N$ of $\dim(H_i) \geq c\varepsilon^2(\log \varepsilon^{-1})^{-1} \log n$ such that for all $1 \leq i \leq N$ and all $x \in H_i$, 

$$(1 - \varepsilon) M^\sharp |x| \leq \|x\| \leq (1 + \varepsilon) M^\sharp |x|$$

where $M^\sharp$ is the median of $\|\cdot\|$ on $S^{n-1}$.

Theorem 2 guarantees the existence of at least one decomposition, and we do not know whether it holds for a typical decomposition. In Corollary 19 we consider the infinite dimensional case.
**Definition 3** Let $X$ be an infinite dimensional Banach space over $\mathbb{R}$. We shall say that $X$ satisfies Definition 3 if for all $\varepsilon > 0$ and all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ with the following property. For any finite dimensional subspace $E \subset X$ with $\dim(E) > N$, there exists a basis $(e_i)_{i=1}^n$ for $E$ such that for any $\Lambda \subset \{1, 2 \ldots n\}$ with $|\Lambda| \leq k$ and any sequence of coefficients $(a_i)_{i \in \Lambda}$,

$$
(1 - \varepsilon) \left( \sum_{i \in \Lambda} |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i \in \Lambda} a_i e_i \right\|_X \leq (1 + \varepsilon) \left( \sum_{i \in \Lambda} |a_i|^2 \right)^{1/2} \quad (1)
$$

The space $c_0$, which is the most interesting space that we study, does not have this property (see Lemma 22). Going back to the work of Kwapień [10] and Figiel, Lindenstrauss and Milman [9], it is well known that there are intimate connections between the Euclidean structures within a Banach space and the notions of type and cotype. The $\ell_\infty^n$ spaces have the smallest possible collection of Euclidean subspaces. By the Maurey-Pisier theorem, these spaces are excluded as subspaces of $X$ precisely when $X$ has nontrivial cotype. This leads naturally into the following result.

**Theorem 4** An infinite dimensional real Banach space $X$ satisfies Definition 3 if and only if it has nontrivial cotype.

Our proof shows that when $X$ has cotype $q < \infty$ and corresponding cotype constant $\beta \in (0, 1]$, one can take $k \geq c \beta^2 \varepsilon^2 (\log \varepsilon^{-1})^{-1} (\log n)^{-1} n^{2/q}$, provided $0 < \varepsilon < 0.99$.

This result is closely connected to the restricted isometry property of Candès and Tao [7], which plays a fundamental role in compressed sensing. A vector $x \in X$ is said to be $k$-sparse with respect to a basis $(e_i)_{i=1}^n$ if it can be expressed as a linear combination of no more than $k$ basis vectors. A vector $a \in \mathbb{R}^n$ is said to be $k$-sparse (without reference to a basis) if it is $k$-sparse with respect to the standard basis.

**Corollary 5** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces of dimensions $n, m \in \mathbb{N}$ respectively, with cotype $q_1, q_2 < \infty$ and corresponding cotype constants $\beta_1, \beta_2 > 0$. Let $0 < \varepsilon < 0.99$ and

$$
k \leq c \varepsilon^2 (\log \varepsilon^{-1})^{-1} (\log n)^{-1} \min \{ \beta_1^2 n^{2/q_1}, \beta_2^2 m^{2/q_2} \}
$$

Then there exist bases $(e_i)_{i=1}^n$ and $(f_i)_{i=1}^m$ in $X$ and $Y$ respectively with the following property. Let $G$ be an $m \times n$ random matrix with i.i.d. $N(0, 1)$ entries. With probability at least

$$
1 - c_1 \exp \left( -C\beta_2^2 m^{2/q_2} \varepsilon^2 \right)
$$

the following event occurs. For all $k$-sparse vectors $a \in \mathbb{R}^n$,

$$
(1 - \varepsilon) \left\| \sum_{j=1}^n a_j e_j \right\|_X \leq \left\| \frac{1}{\sqrt{m}} \sum_{i=1}^m \sum_{j=1}^n G_{ij} a_j f_i \right\|_Y \leq (1 + \varepsilon) \left\| \sum_{j=1}^n a_j e_j \right\|_X
$$

Here, $c, c_1, c_2, c_3, C > 0$ are universal constants.
The Johnson-Lindenstrauss lemma \[12\] does not hold in a general Banach space, even spaces with nontrivial cotype \[13\]. For sparse vectors, however, the situation is very different.

**Corollary 6** Let \((X, \|\cdot\|)\) be a real Banach space of dimension \(n \in \mathbb{N}\), with cotype \(q \in [2, \infty)\) and corresponding cotype constant \(\beta > 0\). Let \(0 < \varepsilon < 0.99\) and let \(k \leq c\beta^2 \varepsilon^2 (\log \varepsilon^{-1})^{-1}(\log n)^{-1} n^{2/q}\). Then there exists a basis \((e_i)_{i=1}^{n}\) for \(X\) with the following property. Let \(\Omega \subset X\) be a finite collection of vectors that are each \(k\)-sparse with respect to the given basis and let \(m = [c\varepsilon^{-2} \log |\Omega|]\). Then there exists a linear operator \(T : X \to \ell_m^2\) such that for all \(x, y \in \Omega\),

\[
(1 - \varepsilon) \|x - y\| \leq |Tx - Ty| \leq (1 + \varepsilon) \|x - y\|
\]

If in the above Corollary one insists on a map \(Q : X \to E\), where \(E\) is a subspace of \(X\), then we may use Dvoretzky’s theorem and modify the bound on \(m\).

Our proof that \(c_0\) does not satisfy Definition \[3\] only works for \(0 < \varepsilon < (\sqrt{2} - 1)^4\), and thus the following problem remains unsolved.

**Problem 7** Does there exist a universal constant \(C > 0\) and a sequence \((\omega_n)_{n=1}^{\infty}\) with \(\lim_{n \to \infty} \omega_n = \infty\) such that the following is true? For every \(n \in \mathbb{N}\) and any real Banach space \((X, \|\cdot\|)\) of dimension \(n\), there is a basis \((e_i)_{i=1}^{n}\) for \(X\) such that for any \(\Lambda \subset \{1, 2, \ldots, n\}\) with \(|\Lambda| \leq \omega_n\) and any sequence of coefficients \((a_i)_{i \in \Lambda}\),

\[
\left(\sum_{i \in \Lambda} a_i^2 \right)^{1/2} \leq \left\| \sum_{i \in \Lambda} a_i e_i \right\| \leq C \left(\sum_{i \in \Lambda} a_i^2 \right)^{1/2}
\]

In this direction we present the following local version of John’s theorem, which also resembles the isomorphic Dvoretzky theorem \[23, 24\].

**Theorem 8** Let \((E, \|\cdot\|)\) be a real Banach space of dimension \(n \in \mathbb{N}\) and let \(k \leq c(\log \log n / \log n)^2 n\). Then \(E\) has a basis \((e_i)_{i=1}^{n}\) such that for any \(\Lambda \subset \{1, 2, \ldots, n\}\) with \(|\Lambda| \leq k\) and any sequence of coefficients \((a_i)_{i \in \Lambda}\),

\[
\left(\sum_{i \in \Lambda} |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i \in \Lambda} a_i e_i \right\| \leq c \left(1 - \frac{\log k}{\log n}\right)^{-1/2} \sqrt{k} \left(\sum_{i \in \Lambda} |a_i|^2 \right)^{1/2}
\]

Note that when \(k \leq n^{0.99}\), say, the distortion is \(O(\sqrt{k})\), which is exactly what we get from the Dvoretzky-Rogers factorization \[5, 31, 10\]. However the Dvoretzky-Rogers factorization only applies to a subspace of proportional dimension.

A key ingredient in both Theorem \[1\] and Theorem \[8\] is the following lemma which uses a result of Vershynin \[34\] on contact points of \(\partial B_X\) with the John ellipsoid, see Lemma \[18\].
Lemma 9 There exist universal constants \( c, c', c_1, c_2, c_3, c_4 > 0 \) with \( c' > c_2^2 \) such that the following holds. Consider any Banach space \( X \) of dimension \( n \in \mathbb{N} \) and let \( t \in \mathbb{R} \) with \( c_1 \leq t \leq c_2 \sqrt{n} \). Then there exists an identification of \( X \) with \( \mathbb{R}^n \) such that the John ellipsoid of \( B_X \) obeys \( c_3 B_2^n \subseteq \mathcal{E} \subseteq c_4 B_2^n \) and such that the following is true. Let \( K_t = \text{conv}(tB_2^n, B_X) \). Let \( M_t \) and \( b_t \) denote the mean and maximum of the Minkowski functional of \( K_t \) on \( S^{n-1} \). Then

\[
\frac{M_t}{b_t} \geq ct \sqrt{\frac{1}{n} \log \left( \frac{c'n}{t^2} \right)}
\]

3 Background

Most of the background material relevant to the paper can be found in [1], [9], [17], [18], [19], [22], [25] and [26]. The letters \( c, c_1, c_2, c', C \) etc. denote universal constants that take on different values from one line to the next. They are not arbitrary, but have very specific numerical values that we do not always have control over. The symbols \( \mathbb{P} \) and \( \mathbb{E} \) denote probability and expected value. For \( p \in [1, \infty) \), let \( \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \) denote the \( p \)-norm of a vector \( x \in \mathbb{R}^n \) and let \( |\cdot| = \|\cdot\|_2 \) be the standard Euclidean norm. For \( p = \infty \), \( \|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\} \). Let \( B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\} \) and \( S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \). The Grassmannian manifold \( G_{n,k} \) consists of all \( k \)-dimensional linear subspaces of \( \mathbb{R}^n \), while the orthogonal group \( O(n) \) is the space of all orthogonal \( n \times n \) matrices. The spaces \( S^{n-1}, G_{n,k} \) and \( O(n) \) each have a unique rotational invariant probability measure called Haar measure, that will be denoted in each case as \( \sigma_n \). Of fundamental importance is Levy’s concentration inequality, which follows from the isoperimetric inequality on \( S^{n-1} \).

Theorem 10 Let \( f : S^{n-1} \to \mathbb{R} \) be a Lipschitz function and let \( M_f = \int_{S^{n-1}} f \, d\sigma_n \). Then for all \( t > 0 \),

\[
\sigma_n \left\{ \theta \in S^{n-1} : |f(\theta) - M_f| < t \cdot \text{Lip}(f) \right\} \geq 1 - c_1 e^{-c_2 nt^2} \tag{2}
\]

where \( c_1, c_2 > 0 \) are universal constants. The same result holds with the mean \( M_f \) replaced with anything between (say) the 10\(^{th} \) and 90\(^{th} \) percentile of \( f \), such as the median, and \( \text{Lip}(f) \) is measured with respect to either the Euclidean metric on \( S^{n-1} \) or the geodesic distance.

A symmetric convex body \( K \subset \mathbb{R}^n \) is a compact, convex set with non empty interior such that \( x \in K \) if and only if \(-x \in K \). The associated Minkowski and dual Minkowski functionals are the norms defined by

\[
\|x\|_K = \inf \{ t \geq 0 : x \in tK \}
\]
\[
\|y\|_K^\circ = \max \{ \langle x, y \rangle : x \in K \}
\]

where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product. The body \( K^\circ = \{ y : \|y\|_{K^\circ} \leq 1 \} \).
is known as the polar of $K$.

The John ellipsoid of a convex body $K$, denoted $E_K$, is the ellipsoid of maximal volume contained within $K$. It can be shown via a compactness argument that such an ellipsoid exists. It is also known that $E_K$ is unique. When $E_K = B^n_2$, we say that $K$ is in John’s position, and in this case

$$B^n_2 \subseteq K \subseteq \sqrt{n}B^n_2$$

Two parameters of particular importance are the mean and the maximum,

$$M(K) = \int_{S^{n-1}} ||\theta||_K d\sigma_n(\theta)$$

$$b(K) = \max \{||\theta||_K : \theta \in S^{n-1}\}$$

When $K$ is in John’s position $b \leq 1$, and it can be shown using the Dvoretzky-Rogers lemma that

$$M \geq c \sqrt{\frac{\log n}{n}}$$

Let $(\Omega, \rho)$ be a compact metric space and $0 < \varepsilon < 1$. An $\varepsilon$-net $N \subset \Omega$ is a set such that for all $\theta \in \Omega$ there exists $\omega \in N$ such that $\rho(\theta, \omega) < \varepsilon$ and for all $\omega_1, \omega_2 \in N$, $\rho(\omega_1, \omega_2) \geq \varepsilon$. Sometimes the latter condition is dropped. Such a set can easily be shown to exist using a greedy algorithm. In the case $\Omega = S^{n-1}$, a volumetric argument yields

$$|N| \leq \left( \frac{3}{\varepsilon} \right)^n$$

By homogeneity, any $x \in \mathbb{R}^n$ can be expressed as $x = |x|\omega_0 + x'$, where $\omega_0 \in N$ and $|x'| < \varepsilon|x|$. Iterating this expression yields

$$x/|x| = \omega_0 + \sum_{i=1}^{\infty} \varepsilon_i \omega_i$$

where $(\omega_i)_0^\infty$ is a sequence in $N$ and $0 \leq \varepsilon_i < \varepsilon^i$. Applying the triangle inequality then leads to the following lemma.

**Lemma 11** Let $|| \cdot ||$ be a norm on $\mathbb{R}^n$ and $\delta \in (0, 1/4)$. Let $M > 0$ and let $N$ be a $\delta$-net in $S^{n-1}$. Suppose that for all $\omega \in N$, $(1 - \delta)M \leq ||\omega|| \leq (1 + \delta)M$. Then for all $x \in \mathbb{R}^n$,

$$(1 - 4\delta)M \leq ||x|| \leq (1 + 4\delta)M.$$  

**Theorem 12 (The general Dvoretzky theorem)** Let $|| \cdot ||$ be a norm on $\mathbb{R}^n$ with parameters $M, b$ as defined by (3) and (4) respectively. Let $0 < \varepsilon < 0.99$ and $k \leq c_1\varepsilon^2(\log \varepsilon^{-1})^{-1}M^2b^{-2}n$, and let $E \in G_{n,k}$ be any fixed subspace. Let $T$ be a random orthogonal matrix uniformly distributed in $O(n)$ and let $E = TF$. Then with probability at least $1 - c_1 \exp(-c_2\varepsilon^2M^2b^{-2}n)$ we have that for all $x \in E$, $(1 - \varepsilon)M|x| \leq ||x|| \leq (1 + \varepsilon)M|x|$.
Lemma 11. Then with probability at least 1 in every infinite dimensional Banach space, the unit ball \( B \) with respect to this basis, the unit ball \( (\text{uniformly distributed with respect to the Haar measure on } k) \) and \( c \leq \) ratio theorem states that if \( 1 \leq k \leq n \) and \( E \in G_{n,k} \) is a random subspace of dimension \( k \) (uniformly distributed with respect to the Haar measure on \( G_{n,k} \) corresponding to \( E_K \)), then with probability at least \( 1 - 2^{-n} \),

\[
(\mathcal{E}_K \cap E) \subseteq (K \cap E) \subseteq (4\pi vr(K))^{4/(n-k)} (\mathcal{E}_K \cap E)
\]

For spaces with universally bounded volume ratio, such as \( \ell_1^n \) where \( vr(B_1^n) \leq \sqrt{2\pi/e} \), this gives a version of Dvoretzky’s theorem with proportional dimension.

A Banach space \( E \) embeds (linearly) into a space \( Y \) with distortion \( \gamma \geq 1 \) if there exists \( \alpha, \beta > 0 \) and a linear map \( T : E \to Y \) such that \( \alpha \beta^{-1} = \gamma \) and for all \( x \in E \),

\[
\beta \|x\|_E \leq \|Tx\|_Y \leq \alpha \|x\|_E
\]

In this case we shall write \( E \hookrightarrow \gamma Y \). When \( Y \) is a Hilbert space we write \( d(E) \leq \gamma \). A space \( X \) is finitely representable in \( Y \) if for all finite dimensional subspaces \( E \subset X \) and all \( \varepsilon > 0 \), \( E \hookrightarrow_{1+\varepsilon} Y \). By Dvoretzky’s theorem, every Hilbert space is finitely representable in every infinite dimensional Banach space.

The notions of type and cotype capture the spirit of the \( L_p \) spaces in an abstract setting. Let \((\varepsilon_i)_1^n\) denote an i.i.d. sequence of Bernoulli random variables with \( \mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2 \). A Banach space \( X \) is said to have type \( p \in [1, 2] \) if there exists \( \alpha \in [1, \infty) \) such that for all finite sequences \((x_i)_1^m\) in \( X \),

\[
\mathbb{E}\left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\| \leq \alpha \left( \sum_{i=1}^{m} \|x_i\|^p \right)^{1/p}
\]

Similarly, \( X \) is said to have cotype \( q \in [2, \infty] \) if there exists \( \beta \in (0, 1) \) such that for all \((x_i)_1^n\) in \( X \),

\[
\beta \left( \sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq \mathbb{E}\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|
\]
with the appropriate interpretation when $q = \infty$. Any Banach space $X$ has type 1 and cotype $\infty$, and these are referred to as trivial type/cotype. If $X$ has type $p$ and cotype $q$, then it has type $p'$ and cotype $q'$ for all $p' \in [1, p]$ and $q' \in [2, q]$. Type and cotype are inherited by subspaces, and the space $L_p$ ($1 \leq p < \infty$) has type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$. If $E$ is a finite dimensional space with cotype $q < \infty$ and corresponding constant $\beta$, then with respect to the John ellipsoid of $B_E$ inequality (5) can be improved to

$$M \geq c\beta n^{\frac{1}{2} - \frac{1}{2}}$$

(8)

where $n = \dim(E)$, and the general Dvoretzky theorem guarantees the existence of Euclidean subspaces of dimension $c\varepsilon^2(\log\varepsilon^{-1})^{-1}\beta^2 n^{2/q}$.

Let $p_X$ and $q_X$ be the supremum (resp. infimum) over all values of $p$ and $q$ such that $X$ has type $p$ and cotype $q$. One of the most significant results in the theory of type and cotype is the Maurey-Pisier theorem, which builds on work by Brunel and Sucheston [6] and Krivine [15].

**Theorem 13** If $X$ is infinite dimensional, then $\ell_{p_X}$ and $\ell_{q_X}$ are finitely representable in $X$.

The result of Kwapień alluded to in Section 2 is that a Banach space $X$ is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2, and in this case $d(X) \leq c\alpha\beta^{-1}$, where $\alpha$ and $\beta$ are (respectively) the type 2 and cotype 2 constants of $X$. Bourgain and Milman [4] (following [32]) proved that spaces with cotype 2 have uniformly bounded volume ratio. Milman and Pisier [21] then studied the related notion of weak cotype 2. An infinite dimensional space $X$ has weak cotype 2 if and only if there exists $C > 0$ such that $\vr(E) < C$ for all finite dimensional subspaces $E \subset X$. This turns out to be equivalent to the following property (among others): every finite dimensional subspace $E \subset X$ has a further subspace $F \subset E$ of proportional dimension that is approximately Euclidean.

## 4 Preparatory lemmas

In this section we shall use the following notation. Let $|| \cdot ||$ be a norm on $\mathbb{R}^d$ with parameters $M, b$ defined by (3) and (4), and let $(F_i)^N$ be any collection of subspaces with $\dim(F_i) \in \{k - 1, k\}$ for some $k \leq d$. Let $T$ be a random orthogonal matrix uniformly distributed in $O(d)$ and set $E_i = TF_i$.

**Lemma 14** Let $0 < \varepsilon < 0.99$ and use the notation defined above. With probability at least

$$1 - c_1 \exp \left( -c_2 dM^2b^{-2}\varepsilon^2 + \log N + k \log(12\varepsilon^{-1}) \right)$$

we have for all $i \in \{1, 2 \ldots N\}$ and all $x \in E_i$,

$$M - \varepsilon)x_1 | \leq ||x|| \leq (1 + \varepsilon)x_1$$

(9)
Proof. We follow the proof of Theorem 12. For each $1 \leq i \leq N$ let $N_i$ be an $\varepsilon/4$-net in $S(F_i) = \{x \in F_i : |x| = 1\}$ and let $\mathcal{N} = \cup_{i=1}^N N_i$. The epsilon net bound (9) yields $|\mathcal{N}| \leq N(12/\varepsilon)^k$. Furthermore, $|| \cdot ||$ is Lipschitz with $\text{Lip}(|| \cdot ||) = b$. For each $\omega \in \mathcal{N}$, $T\omega$ is uniformly distributed in $S^{d-1}$. The result then follows from Levy’s inequality (2) with $t = 4^{-1}\varepsilon Mb^{-1}$, the union bound, and Lemma 14. 

Corollary 15 Let $0 < \varepsilon < 0.99$, $d, k \in \mathbb{N}$ and $q \in [2, \infty)$. Let $|| \cdot ||$ be a norm on $\mathbb{R}^d$ such that the associated unit ball is in John’s position and let $\beta(d)$ denote the cotype $q$ constant of $(\mathbb{R}^d, || \cdot ||)$. If $k \leq c\beta^2 d^2 (\log \varepsilon^{-1})^{-1} (\log d)^{-1} d^{2/q}$, then there exists an orthogonal matrix $T \in O(d)$ with the following property. For all $k$-sparse vectors $x \in \mathbb{R}^d$, $(1 - \varepsilon)M|x| \leq ||Tx|| \leq (1 + \varepsilon)M|x|$. 

Proof. Take $(F_i)_1^N$ to be the collection of all $k$ dimensional coordinate subspaces of $\mathbb{R}^d$. The result follows from the estimate $M \geq c\beta d^{1/q - 1/2}$, see (8), Lemma 14, and the bound $(\begin{array}{c} d \\ k \end{array}) \leq d^k$, which is rather crude but sufficient.

Corollary 16 We refer to the notation above. For all $R > 0$ there exist $c_R, \tilde{c}_R > 0$ such that the following is true. Suppose that $R^{-1}|x| \leq ||x|| \leq R|x|$ for all $x \in \mathbb{R}^d$ and that $\mathbb{R}^d = \bigoplus_{i=1}^N F_i$. Let $d, k \in \mathbb{N}$ and $0 < \varepsilon < 0.99$ such that $c_R \log d \leq k \leq \tilde{c}_R \varepsilon^2 (\log \varepsilon^{-1})^{-1} d$. With nonzero probability, (9) holds for all $x \in E_i$, $i = 1, 2, \ldots, N$. 

Lemma 17 Let $(X_i)_1^m$ be an i.i.d. $N(0, 1)$ sequence and $(\log m)^{-1/2} \leq s \leq 1 - c(\log m)^{-1}$. With probability at least 0.52, 

$$ \left| \left\{ i : s\sqrt{\log m} \leq X_i \leq 3\sqrt{\log m} \right\} \right| \geq cm^{1-s^2} $$

Proof. Let $\phi$ denote the cumulative standard normal distribution function. For all $t \geq 1$ (see e.g. [8]), 

$$ \frac{\phi(t)}{2t} \leq 1 - \Phi(t) \leq \frac{\phi(t)}{t} $$

where $\phi$ is the standard normal density. This implies that 

$$ \mathbb{P} \left\{ s\sqrt{\log m} \leq X_i \leq 3\sqrt{\log m} \right\} \geq cm^{-s^2} $$

Whether we are in the normal or the Poisson regime (depending on the value of $m^{1-s^2}$), the mean and the median (as well as the $52^{nd}$ percentile) of the random variable in question are of the same order of magnitude and the result follows.

Lemma 18 Let $(X, || \cdot ||_X)$ be a real Banach space of dimension $n \in \mathbb{N}$ and let $m = \lceil n/2 \rceil$. Then there exists an inner product $\langle \cdot, \cdot \rangle_2$ on $X$ and a sequence $(u_i)_1^m$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle_2$ such that for all $1 \leq i \leq m$, 

$$ C^{-1} \leq ||u_i||_X \leq C $$

$$ C^{-1} \leq ||u_i||_{X^*} \leq C $$

where $|| \cdot ||_{X^*}$ is the dual norm on $X$ under the duality corresponding to $\langle \cdot, \cdot \rangle_2$, i.e. $||x||_{X^*} = \sup \{ \langle x, y \rangle_2 : ||y||_X \leq 1 \}$. Furthermore, the ellipsoid of maximum volume in $B_X$, denoted, $\mathcal{E}$, satisfies $c_1 \mathcal{E}^2 \subseteq \mathcal{E} \subseteq c_2 \mathcal{E}^2$, where $\mathcal{E}^2 = \{ x \in X : \langle x, x \rangle_2 \leq 1 \}$. 

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Proof. Identify $X$ with $\mathbb{R}^n$ so that $K = B_X$ is in John’s position. By a result of Vershynin [34], there exists a sequence $(v_i)_n$ of contact points between $B^n_2$ and $\partial K$ such that for any sequence of coefficients $(a_i)_n^m \in \mathbb{R}^n$,

$$c_1 \left( \sum_{i=1}^m a_i^2 \right)^{1/2} \leq \left| \sum_{i=1}^m a_i v_i \right| \leq c_2 \left( \sum_{i=1}^m a_i^2 \right)^{1/2}$$

By construction, $\|v_i\|_K = |v_i| = 1$ for all $1 \leq i \leq n$. These vectors are linearly independent, and can be extended to a basis for $\mathbb{R}^n$, $(v_i)_n^m$, such that $(v_i)_m^{m+1}$ are orthonormal and $\text{span}\{v_i\}_n^m$ is orthogonal to $\text{span}\{v_i\}_m^{m+1}$. Using these properties, for any sequence of coefficients $(a_i)_n^m \in \mathbb{R}^n$,

$$c_3 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \leq \left| \sum_{i=1}^n a_i v_i \right| \leq c_4 \left( \sum_{i=1}^n a_i^2 \right)^{1/2}$$

which implies $c \leq ||A||_{2 \rightarrow 2} \leq c'$ and $c \leq ||A^{-1}||_{2 \rightarrow 2} \leq c'$, where $A$ is the $n \times n$ matrix with the vectors $(v_i)_n^m$ as columns and $\|\cdot\|_{2 \rightarrow 2}$ denotes the operator norm of a matrix from $\ell_n^n$ to $\ell_2^n$. Define $\langle x, y \rangle_x = \langle A^{-1} x, A^{-1} y \rangle$. By the Hahn-Banach theorem, there exists $(w_i)_n^m$ such that $\|w_i\|_{K^o} = 1$ and $\langle v_i, w_i \rangle = 1$. Since $v_i \in B_2^n$ and $B_2^n \subseteq K$,

$$\sup \{ \langle w_i, x \rangle : x \in B_2^n \} = 1$$

and we conclude that $|w_i| = 1$. Since $\langle v_i, w_i \rangle = 1$ and $v_i, w_i \in S^{n-1}$, $v_i = w_i$. Therefore $\|v_i\|_{K^o} = 1$. Lastly,

$$\|v_i\|_{X^*} = \sup \{ \langle v_i, x \rangle_x : x \in K \}
= \sup \{ \langle v_i, y \rangle : y \in (A^{-1})^T A^{-1} K \}$$

so $c \leq \|v_i\|_{X^*} \leq c'$. The result now follows with $u_i = \langle v_i, v_i \rangle_x^{-1/2} v_i$. ■

Proof of Lemma 9. Let $K = B_X = \{ x \in X : \|x\| \leq 1 \}$ and $m = \lceil n/2 \rceil$. Identify $X$ with $\mathbb{R}^n$ so that the inner product $\langle \cdot, \cdot \rangle_x$ from Lemma 18 is the standard Euclidean inner product, and consider the vectors $(u_i)_n^m$ as in Lemma 18. After applying an orthogonal transformation, we may assume that $u_i = e_i$ for all $1 \leq i \leq m$, where $(e_i)_n^m$ are the standard basis vectors of $\mathbb{R}^n$. Let $K_t = \text{conv}\{K, tB_2^n\}$. Then $\|y\|_{K_t} = \max\{\|y\|_{K^o}, t|y|\}$. Let $\theta \in S^{n-1}$ be a random point uniformly distributed on the sphere. We can simulate $\theta = X/|X|$, where $(X_i)_n^m$ are i.i.d. $N(0,1)$ variables. Set $s = (1 - 2 \log(ct)/\log m)^{1/2}$ in which case $t = cm^{(1-s^2)/2}$. By Lemma 17 with probability at least 0.52, $|\Omega| \geq cm^{1-s^2} = ct^2$, where

$$\Omega = \left\{ 1 \leq i \leq m : s\sqrt{\log m} \leq X_i \leq 3\sqrt{\log m} \right\}$$

Let $z = \sum_{i \in \Omega} e_i$. Then $\langle z, X \rangle \geq s(\log m)^{1/2}|\Omega|$. By the triangle inequality, $\|z\|_{K^o} \leq c|\Omega|$ and $|z| = |\Omega|^{1/2}$. By the bound on $|\Omega|$, $c|\Omega| \geq t|\Omega|^{1/2}$, which implies that $\|z\|_{K^o_t} \leq c|\Omega|$. We thus have $\|X\|_{K_t} \geq \langle z, X \rangle / \|z\|_{K^o_t} \geq cs(\log m)^{1/2}$. The result now follows because the median and the mean of $\|\cdot\|_{K_t}$ on $S^{n-1}$ have the same order of magnitude, and with probability at least 0.99, $|X| \leq c\sqrt{n}$. ■
5 Euclidean decompositions

5.1 Typical decompositions

Proof of Theorem 1. We use Lemma 9 under the same identification $X = \mathbb{R}^n$, with $t = c'$ (a universal constant) large enough so that $M_i/b_i \geq ct\sqrt{n^{-1}\log(c'^{nt^{-2}})} \geq C\sqrt{\log n/n}$, where $C > 0$ is a sufficiently large constant that will appear shortly. Let $\mathbb{R}^n = E_1 \oplus E_2 \ldots \oplus E_N$ be an orthogonal decomposition into subspaces of dimension $\Theta(\log n)$. Let $U \in O(n)$ be a random orthogonal matrix. Consider the event $\Delta$, that for all $x \in E_i$, $1/2M_i|x| \leq \|Ux\|_{K_i} \leq 3/2M_i|x|$, and $\Delta = \cap_i \Delta_i$. By the general Dvoretzky theorem (Theorem 12), $\Pr(\Delta_i) \geq 1 - c_1 \exp(-c_2c^2\log n) > 1 - c_1n^{-101}$ and $\Pr(\Delta) \geq 1 - c_1n^{-100}$. If $\Delta$ occurs, then for all $x \in E_i (i = 1, 2 \ldots N)$, $1/2M_i|x| \leq \|Ux\| \leq 3/2tM_i|x|$ and the result follows by changing the coordinate structure so that $B_X$ is in John’s position. □

Corollary 19 Let $X$ be an infinite dimensional Banach space over $\mathbb{R}$ with a Schauder basis $(e_i)_{i=1}^\infty$. For any $N \in \mathbb{N}$ and $\varepsilon > 0$, $X$ admits an FDD (finite dimensional decomposition) $(E_n)_{i=1}^\infty$ where $\dim(E_i) \geq N$ and $d(E_i) \leq (1 + \varepsilon)$.

Proof. For all $n \in \mathbb{N}$ define $U_n = \text{span}\{e_j : (n - 1) \exp(cN) < j \leq n \exp(cN)\}$. Then apply Theorem 1 and Corollary 16 to $U_n$ to obtain $U_n = V_1^{(n)} \oplus V_2^{(n)} \ldots \oplus V_{N(n)}^{(n)}$. We now claim that

$$X = V_1^{(1)} \oplus V_2^{(1)} \ldots \oplus V_{N(1)}^{(1)} \oplus V_1^{(2)} \oplus V_2^{(2)} \ldots \oplus V_{N(2)}^{(2)} \oplus \ldots$$

The main subtlety here is convergence, however the norms of the partial sum projections of $U_n$ onto $\oplus_{i=1}^k V_i^{(n)}$ for $1 \leq k \leq N(n)$ are all bounded above by $e^{cN}$. □

5.2 Almost-isometric decompositions in John’s position

For the remainder of this section, let $(X, \|\|)$ denote a real normed space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that $B_X$ is in John’s position. For any subspace $E \subset X$ let $b(E)$ denote the Lipschitz constant of $\|\|$ restricted to $E$, and $M^2(E)$ the median of $\|\|$ restricted to $S^{n-1} \cap E$. Let $M^2 = M^2(X)$. All random objects that we consider (points, subspaces and orthogonal matrices) are distributed according to Haar measure on the appropriate space. Lemmas 20 and 21 as well as the proof of Theorem 2 are taken (with permission) from [33].

Lemma 20 There exists $c > 0$ such that the following is true. Let $k \geq (M^2)^2n$ and $F \in G_{n,k}$. Let $U \in O(n)$ be a random orthogonal matrix and let $E = UF$. Then with probability at least $1 - 2^{-k}$, $b(E) \leq c\sqrt{k/n}$.

Proof. Let $N$ be a 1/2-net in $S^{n-1} \cap F$ with $|N| \leq 6^k$. The result follows by applying Levy’s inequality [2] with $t = c_3\sqrt{k/n}$ and applying the series representation (7), the union bound, and the triangle inequality. □
Lemma 21 There exists a universal constant $c_1 > 0$ such that the following is true. Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ with the unit ball in John’s position, let $\theta \in S^{n-1}$ be a random point and $E \in G_{n,k}$ a random subspace (for some $k < n$). Let $\varepsilon > 0$ be such that

$$\mathbb{P}\{|\|\theta\|-M^\sharp| \geq \varepsilon M^\sharp\| \leq 1/4$$

Then

$$\mathbb{P}\{|M^\sharp(E) - M^\sharp| \leq \varepsilon M^\sharp\| \geq 1 - 2\exp(-c_1 k)$$

Proof. Let $(v_i)_i^k$ be i.i.d. random points on $S^{n-1}$. Let $\alpha = \mathbb{P}\{M^\sharp(E) \leq (1 - \varepsilon)M^\sharp\}$ and $M = \{H \in G_{n,k} : M^\sharp(H) \leq (1 - \varepsilon)M^\sharp\}$

By definition of $M$, for all $1 \leq i \leq k$ and any $H \in M$ we have the 'conditional' probability

$$\mathbb{P}\{|\|v_i\| \leq (1 - \varepsilon)M^\sharp\| : v_i \in H\} \geq 1/2,$$

and therefore

$$\mathbb{P}\{|\{i : \|v_i\| \leq (1 - \varepsilon)M^\sharp\| \} \geq k/2\} \geq \alpha/2$$

Of course the event $\{v_i \in H\}$ has measure zero and this does not fit into the classical definition of conditional probability. However, it can be justified using a construction involving Fubini’s theorem on $O(n) \times O(k)$. On the other hand, from the condition imposed on $\varepsilon$, $\mathbb{P}\{|\|v_i\| \leq (1 - \varepsilon)M^\sharp\| \leq 1/4$ and

$$\mathbb{E}\{|\{i : \|v_i\| \leq (1 - \varepsilon)M^\sharp\| \} \| \leq k/4$$

By Hoeffding’s inequality,

$$\mathbb{P}\{|\{i : \|v_i\| \leq (1 - \varepsilon)M^\sharp\| \} \| \geq k/2\} \leq \exp(-c_1 k)$$

The bound on $\mathbb{P}\{M^\sharp(E) \geq (1 + \varepsilon)M^\sharp\}$ follows similar lines. ■

Proof of Theorem 2. If $M^\sharp \geq c_1 \varepsilon^{-1}(\log(n)/n)^{1/2}$, the statement follows from Lemma 14 and we may assume without loss of generality that $M^\sharp < c_1 \varepsilon^{-1}(\log(n)/n)^{1/2}$. Let

$$N = \left\lfloor n^{1 - \varepsilon^2(\log \varepsilon^{-1})^{-2}} \right\rfloor$$

and let $H_1 \oplus H_2 \ldots \oplus H_N$ be a decomposition of $\mathbb{R}^n$ into mutually orthogonal subspaces of dimension either $k$ or $k+1$, with $k \approx n^{\varepsilon^2(\log \varepsilon^{-1})^{-2}}$ and let $U \in O(n)$ be a random orthogonal matrix. From Lemmas 20 and 21 it follows that with high probability, for all $i$, $M^\sharp(UH_i)/b(UH_i) \geq c_4 \varepsilon^{-1}(\log \varepsilon^{-1})\sqrt{(\log k)/k}$. It now follows by Lemma 14 that each $UH_i$ can be decomposed yet again into approximately Euclidean subspaces of dimension $c\varepsilon^2(\log \varepsilon^{-1})^{-1} \log n$. ■

6 Grid structures

Lemma 22 The space $c_0$ does not satisfy Definition 3.
Proof. Assume for the sake of a contradiction that \( c_0 \) satisfies Definition \( 3 \). Let \( n \geq 3 \), \( k = 2 \) and consider any \( 0 < \varepsilon < (\sqrt{2} - 1)^4 \). This ensures that both of the following inequalities hold
\[
(1 + \varepsilon + 2\sqrt{\varepsilon}) < \sqrt{2}(1 - \varepsilon) \tag{10}
\]
\[
(1 + \varepsilon)^2 < 2(1 - \varepsilon)^2 \tag{11}
\]
Let \( (f_i)_{1}^{n} \) be a basis for \( \ell_{\infty}^{n} \) such that for any \( \Lambda \subset \{1, 2 \ldots n\} \) with \( |\Lambda| \leq 2 \), (11) holds. Let \( T \) be the \( n \times n \) matrix with the vectors \( (f_i)_{1}^{n} \) as columns. For all 2-sparse vectors \( x \in \mathbb{R}^n \), (11) can be written as
\[
(1 - \varepsilon)|x| \leq ||Tx||_{\infty} \leq (1 + \varepsilon)|x| \tag{12}
\]
It follows by setting \( x = e_j \) (the \( j^{th} \) standard basis vector of \( \mathbb{R}^n \)) that for all \( 1 \leq j \leq n \)
\[
1 - \varepsilon \leq \max_{1 \leq i \leq n} |T_{i,j}| \leq 1 + \varepsilon
\]
In particular, there exists \( \nu = \nu(j) \) such that \( |T_{\nu,j}| \geq 1 - \varepsilon \). Since this holds for each column, there are at least \( n \) entries of the matrix such that \( |T_{i,j}| \geq 1 - \varepsilon \). By duality, for all \( 1 \leq i \leq n \) and any \( j, l \in \{1, 2 \ldots n\} \),
\[
T_{i,j}^{2} + T_{i,l}^{2} \leq (1 + \varepsilon)^2
\]
By (11), there can be at most one such entry per row, and we conclude that there is exactly one in every row. Hence \( \nu \in S_n \) is a permutation of the set \{1, 2 \ldots n\}. For all \( 1 \leq i \leq n \), if \( j \neq \nu^{-1}(i) \) then \( T_{i,j}^{2} = T_{i,j}^{2} + T_{i,\nu^{-1}(j)}^{2} - T_{i,\nu^{-1}(i)}^{2} \leq (1 + \varepsilon)^2 - (1 - \varepsilon)^2 = 4\varepsilon \) and \( |T_{i,j}| \leq 2\sqrt{\varepsilon} \). The matrix \( T \) is therefore a small perturbation of a permutation matrix. If we now take \( x = e_1 + e_2 \), then \( |x| = \sqrt{2} \) and for all \( 1 \leq i \leq n \),
\[
|T_{i,1} + T_{i,2}| \leq 1 + \varepsilon + 2\sqrt{\varepsilon} < \sqrt{2}(1 - \varepsilon)
\]
by (10), which implies that \( ||Tx||_{\infty} < \sqrt{2}(1 - \varepsilon) \). This contradicts (12) and the result follows.  

Proof of Theorem 4. If \( X \) has cotype \( q \) for some \( q < \infty \), then it follows by Corollary 15 that \( X \) satisfies Definition 3. If \( X \) fails to have nontrivial cotype, then it follows from the Maurey-Pisier theorem that \( \ell_{\infty} \) (equivalently \( c_0 \)) is finitely representable in \( X \), and the result follows from Lemma 22.  

Proof of Theorem 8. We may assume without loss of generality that \( k \geq k_0 \), where \( k_0 \) is a sufficiently large universal constant. Consider the identification of \( X \) with \( \mathbb{R}^n \) and the norm \( \|\cdot\|_{K_t} \) defined in Lemma 9. Let \( (F_i)_{1}^{N} \) be the collection of all \( k \)-dimensional coordinate subspaces of \( \mathbb{R}^n \). We use Lemma 14 with \( \varepsilon = 1/2 \). Our task is to choose the smallest value of \( t \) allowable in Lemma 9; in particular \( c_1 \leq t \leq c_2\sqrt{n} \), so that the probability bound in Lemma 14 is positive. Define
\[
s = \left(1 - \frac{2 \log(c^{-1}t)}{\log n}\right)^{1/2}
\]
where \( c > 0 \) is a suitably small numerical constant. Then
\[
t = cn^{(1-s^2)/2}
\]
Consider the function $\xi : [0, 1] \to [0, \infty)$ defined by $\xi(x) = cx^2n^{1-x^2} = cn \exp(\log x^2 - x^2 \log n)$, where $c > 0$ is a suitably small numerical constant. Note that $\xi$ is continuous, strictly increasing on $0 < x < (\log n)^{-1/2}$, reaches a maximum of $cn / \log n$ at $x = (\log n)^{-1/2}$, and is strictly decreasing on $(\log n)^{-1/2} < s < 1$. Also, $\xi(0) = 0$ and $\xi(1 - c(\log n)^{-1}) \leq k_0$. Therefore, we may define

$$s = \sup \{ x \in (0, 1) : c_1(\log n)^{-1/2} \leq x \leq 1 - c_2(\log n)^{-1}, \xi(x) \geq k \}$$

in which case

$$k = \xi(s) = cs^2t^2$$

(13)

and the probability bound from Lemma 14 is positive. The result is seen to hold with total distortion at most $3t$. Our remaining task is to find a more explicit bound for $s$, and then for $t$. The last equation can be written as

$$s^2 \log n + \log s^{-2} = \log(cnk^{-1})$$

(14)

Since $k \leq c(\log n / \log n)^2n$, it follows that $s \geq ((\log n - \log \log n) / \log n)^{1/2}$. This is because

$$\xi\left(\sqrt{\frac{\log n - \log \log n}{\log n}}\right) = \frac{cn(\log n)(\log n - \log \log n)}{(\log n)^2} = c(1 + o(1))(\log n / \log n)^2n$$

The function $x \mapsto x^2 \log n$ is increasing on $[0, 1]$ while $x \mapsto \log x^{-2}$ is decreasing. This implies that $\log s^{-2} \leq 2s^2 \log n$ and from (14) we get $s^2 \log n \leq \log(cnk^{-1}) \leq 3s^2 \log n$. Equation (14) then reduces to $qs^2 \log n = \log(cnk^{-1})$ for some $q \in [1, 3]$. This is easily solved with $s \geq c(1 - \log k / \log n)^{1/2}$. By (13), the resulting distortion is then $3t = cs^{-1}\sqrt{k}$. ■

**Remark 23** If we consider a parameter $\alpha$ that ranges over $(\log n)^{-1/2} \leq \alpha \leq 1$ and we set $s = \alpha(\log \log n / \log n)^{1/2}$ (referring to the proof above) then we get

$$k = c\frac{\alpha^2 \log \log n}{(\log n)^{\alpha^2 + 1}}n$$

and the corresponding distortion becomes

$$\frac{c\sqrt{n}}{(\log n)^{\alpha^2/2}}$$

although this is not the regime that we are most interested in.

We give the following result for completeness and because we could not find a precise enough reference in the literature. See [2] for a similar statement.

**Lemma 24** Consider any $m, n \in \mathbb{N}$ and let $U$ be a random $m \times n$ matrix with i.i.d. $N(0, 1)$ entries, $0 < \varepsilon < 0.99$ and $k \leq c\varepsilon^2(\log \varepsilon^{-1})^{-1}(\log n)^{-1}m$. Then with probability at least $1 - c_1 \exp(-c_2 m \varepsilon^2)$, the inequality $(1 - \varepsilon)|x| \leq m^{-1/2}|Ux| \leq (1 + \varepsilon)|x|$ holds simultaneously for all $k$-sparse vectors $x \in \mathbb{R}^n$.  

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Proof. For each $k$ dimensional coordinate subspace $E \subset \mathbb{R}^n$ let $\mathcal{N}_E$ be an $\varepsilon/4$ net in $S_E = \{x \in E : |x| = 1\}$. Let $\mathcal{N}$ be the union of all these nets. For any $\omega \in S^{n-1}$, $U\omega$ is a normally distributed random vector with mean zero and identity covariance in $\mathbb{R}^m$. Since $| \cdot |$ is a 1-Lipschitz function, standard Gaussian concentration implies that with probability at least $1 - c_1 \exp(-c_2 m \varepsilon^2)$, $(1 - \varepsilon/4) \leq m^{-1/2} |U\omega| \leq (1 + \varepsilon/4)$. Applying (6) and the union bound, this holds simultaneously for all $\omega \in \mathcal{N}$ with probability at least $1 - c_1 n^k (12/\varepsilon)^k \exp(-c_2 m \varepsilon^2)$. The result follows by using the series expansion (7) and the triangle inequality.

Proof of Corollary 5. We know from Theorem 4 that there exists a basis in $X$, say $(e_i)_1^n$ such that for all $k$-sparse vectors $a \in \mathbb{R}^n$,

$$(1 - \varepsilon/6)|a| \leq \left\| \sum_{i=1}^n a_i e_i \right\|_X \leq (1 + \varepsilon/6)|a|$$

Identify $X$ with $\mathbb{R}^n$ using this basis. Identify $Y$ with $\mathbb{R}^m$ in such a way so that $B_Y$ is in John’s position, and then readjust the coordinate structure by scalar multiplication so that the mean of $\| \cdot \|_Y$ in $S^{m-1}$ obeys $M(\| \cdot \|_Y) = 1$. Let $G$ be a random $m \times n$ matrix with i.i.d. $N(0,1)$ entries. For each $k$-dimensional coordinate subspace $E \subset X$, $GE$ has dimension $k$ with probability 1 and is uniformly distributed in $G_{m,k}$. Therefore, using Lemma 14 and (8), with probability at least $1 - c_1 \exp(-c_2^2 m^2/44^2 \varepsilon^2)$, for all $y \in GE$, $(1 - \varepsilon/6)|y| \leq \|y\|_Y \leq (1 + \varepsilon/6)|y|$. By applying the union bound, with probability at least $1 - c_1 n^k \exp(-c_2^2 m^2/44^2 \varepsilon^2)$, for all $k$-sparse vectors $x \in X$ we have $(1 - \varepsilon/6)||Gx|| \leq ||Gx||_Y \leq (1 + \varepsilon/6)||Gx||$. The result now follows from Lemma 24 and the inequality $1 - \varepsilon \leq (1 - \varepsilon/6)(1 + \varepsilon/6)^{-2} \leq (1 - \varepsilon/6)^{-2}(1 + \varepsilon/6) \leq 1 + \varepsilon$. ■

Proof of Corollary 6. The difference between any two $k$-sparse vectors is $2k$-sparse. The result now follows from Theorem 4 and the comment following the statement of the theorem (by readjusting the constants involved) as well as the usual form of the Johnson-Lindenstrauss lemma. ■

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