GROUND STATES SOLUTIONS FOR A NON-LINEAR EQUATION INVOLVING A PSEUDO-RELATIVISTIC
SCHRÖDINGER OPERATOR

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Abstract. In this paper we prove the existence, regularity and symmetry of a ground state for a nonlinear equation in the whole space, involving a pseudo-relativistic Schrödinger operator.

1. Introduction

Recently, the study of fractional and nonlocal operators of elliptic type has attracted the attentions of many mathematicians. These operators arises in many different areas of research such as optimization, finance, minimal surfaces, phase transitions, quasi-geostrophic flows, crystal dislocation, anomalous diffusion, conservation laws and ultra-relativistic limits of quantum mechanics. For more details and applications we refer to [3, 6, 8, 10, 12, 17, 19, 25, 26, 27, 23, 33, 34, 37] and references therein.

In this paper we consider the following nonlinear fractional equation

\[(−∆ + m^2)^s − m^2s]u + μu = |u|^{p−2}u \text{ in } \mathbb{R}^N \tag{1.1}\]

where \(s \in (0, 1), N > 2s, p \in (2, \frac{2N}{N−2s}), m \geq 0 \) and \(μ > 0\).

The fractional operator

\[\mathcal{F}(-∆ + m^2)^s u(k) = (|k|^2 + m^2)^s \mathcal{F}u(k) \tag{1.2}\]

which appears in (1.1) is defined in the Fourier space by setting

\[\mathcal{F}(-∆ + m^2)^s u(x) = c_{N,s}m^{\frac{N+2s}{2}} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) − u(y)}{|x − y|^{\frac{N+2s}{2}}} K^{\frac{N+2s}{2}}(m|x−y|)dy + m^{2s}u(x)\]

for every \(x \in \mathbb{R}^N\); see [22]. Here P.V. stands for the Cauchy principal value,

\[c_{N,s} = 2^{−\frac{N+2s}{2} + 1} \pi^{−\frac{N}{2}} \frac{2s(1−s)}{Γ(2−s)}\]

and \(K_ν\) denotes the modified Bessel function of the second kind with order \(ν\).

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When \( s = 1/2 \) the operator (1.2) has a clear meaning in quantum mechanics: it corresponds to the free Hamiltonian of a free relativistic particle of mass \( m \). We remind that the study of \( \sqrt{-\Delta + m^2} \) has been strongly influenced by papers \([28, 29]\) of Lieb and Yau on the stability of relativistic matter. For more recent works about this topic one can see \([1, 16, 18, 24, 25, 27]\).

Let us point out that the operator (1.2) has a deep connection with the Stochastic Process theory: in fact \( (-\Delta + m^2)^s - m^{2s} \) is an infinitesimal generator of a Levy process called the 2s-stable relativistic process: see \([14, 31]\).

In the present paper we are interested in the study of the ground states of (1.1). Such problems are motivated in particular by the search for certain kinds of solitary waves (stationary states) in nonlinear equations of the Klein-Gordon or Schrödinger type.

In the celebrated papers \([4, 5]\) Berestycki and Lions studied the existence of ground state of the following problem

\[
\begin{align*}
\left\{ \\
-\Delta u &= g(u) \quad \text{in } \mathbb{R}^N \\
 u &\in H^1(\mathbb{R}^N) \quad \text{and } u \not\equiv 0
\end{align*}
\] (1.3)

where \( N \geq 3 \), \( g \) is a real continuous function such that

1. \( \lim_{t \to 0} \frac{g(t)}{t} = -\mu < 0 \);
2. \( \lim_{t \to +\infty} \frac{g(t)}{t^{N+2}} \leq 0 \);
3. there exists \( t_0 > 0 \) such that \( G(t_0) = \int_0^{t_0} g(z)dz > 0 \).

In \([4]\), solutions are constructed by solving the minimization problem

\[
\inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} G(u)dx = 1 \right\}.
\]

Then a rescaling from \( u \) to a suitable \( u(t^{-1}x) \) produces a solution of (1.3) by absorbing the Lagrange multiplier.

More recently, in \([19]\) authors studied, in a similar way, a non-local version of the above problem with a particular nonlinearity:

\[
\begin{align*}
\left\{ \\
(-\Delta)^s u &= -\mu u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N \\
 u &\in H^s(\mathbb{R}^N) \quad \text{and } u \not\equiv 0
\end{align*}
\] (1.4)

Our aim is to prove that their result holds again when we replace \( (-\Delta)^s \) by \( (-\Delta + m^2)^s - m^{2s} \). Unfortunately the method used in \([19]\) does not work in the case of the Bessel operator (1.2). The most important difference between \( (-\Delta)^s \) and (1.2) is that the first has some scaling properties that the latter does not have. In fact, it is not hard to check that one can solve the minimum problem

\[
\inf \left\{ \int_{\mathbb{R}^N} |(-\Delta + m^2)^{s/2}u|^2 dx : \int_{\mathbb{R}^N} G(u)dx = 1 \right\}
\]
in the space of radially symmetric functions. But, in this case, we are not able
to absorb the Lagrange multiplier because of the lack of scaling invariance for
the Bessel fractional operator.

It is worth remembering that, by using different variational techniques and a
Pohozaev type identity, Chang and Wang [15] proved the existence of a ground
state to the problem (1.4) with a more general nonlinearity. In a future work we
will try to extend their result to the more general operator \((-\Delta + m^2)^s - m^{2s}\).

Our first result can be stated as follows

**Theorem 1.** Let \(m > 0\) and \(\mu > 0\). Then there exists a positive ground state
solution \(u \in H^s_m(\mathbb{R}^N)\) to (1.1) which is radially symmetric.

The main difficulty of studying of problem (1.1) is the non-local character
of the involved operator. To overcome this difficulty, we use a technique very
common in the recent literature: the Caffarelli-Silvestre extension method [11].
Such method consists to write a given nonlocal problem in a local way via the
Dirichlet-Neumann map: this allow us to apply known variational techniques
to these kind of problems; see for instance [2] [13] [9] [36]. More precisely, for
any \(u \in H^s_m(\mathbb{R}^N)\) there exists a unique \(v \in H^1_m(\mathbb{R}^{N+1}, y^{1-2s})\) weakly solving

\[
\begin{cases}
-\text{div}(y^a \nabla v) + m^2 y^a v = 0 & \text{in } \mathbb{R}^{N+1}_+
\vspace{0.2cm}
v(x, 0) = u(x) & \text{on } \partial \mathbb{R}^{N+1}_+
\end{cases}
\]

and such that

\[
\frac{\partial v}{\partial y^a} := -\lim_{y \to 0} y^a \frac{\partial v}{\partial y}(x, y) = (-\Delta + m^2)^s u(x) \text{ in } H^{-s}_m(\mathbb{R}^N)
\]

where \(a = 1 - 2s \in (-1, 1)\).

We will exploit this fact, and we will prove the existence, regularity results
and qualitative properties of solutions to

\[
\begin{cases}
-\text{div}(y^a \nabla v) + m^2 y^a v = 0 & \text{in } \mathbb{R}^{N+1}_+
\vspace{0.2cm}
\frac{\partial v}{\partial y^a} = \kappa_s [m^{2s} u - \mu u + |u|^{p-2} u] & \text{on } \partial \mathbb{R}^{N+1}_+ \ .
\end{cases}
\]

(1.5)

When we assume \(m\) sufficiently small we are able to rediscover the result
obtained by [19] (and [15]). In fact, we are able to pass to the limit in (1.1) as
\(m \to 0\) and we find a nontrivial solution to

\((-\Delta)^s u + \mu u = |u|^{p-2} u \text{ in } \mathbb{R}^N\).

More precisely we obtain

**Theorem 2.** There exists a nontrivial ground state solution \(u \in H^s(\mathbb{R}^N)\) to (1.4) such that \(u\) is radially symmetric.
2. PRELIMINARIES

In this section we collect some notations and basic facts we will use later. Let $s \in (0, 1)$ and $m > 0$.

Let

$$\mathbb{R}_+^{N+1} = \{(x, y) \in \mathbb{R}^{N+1} : y > 0\}.$$  

We use the notation $|(x, y)| = \sqrt{|x|^2 + y^2}$ to denote the euclidean norm in $\mathbb{R}_+^{N+1}$. Let $R > 0$. We denote by

$$B_R^+ = \{(x, y) \in \mathbb{R}_+^{N+1} : |(x, y)| < R\},$$

$$\Gamma_R^0 = \{(x, 0) \in \partial \mathbb{R}_+^{N+1} : |x| < R\}$$

and

$$\Gamma_R^+ = \{(x, y) \in \mathbb{R}_+^{N+1} : |(x, y)| = R\}.$$  

Let $q \in [1, \infty]$. We denote by $| \cdot |_{L^q(A)}$ the $L^q$-norm in $A \subset \mathbb{R}^N$ and $|| \cdot ||_{L^q(B)}$ the $L^q$-norm in $B \subset \mathbb{R}_+^{N+1}$, respectively.

We define $H^s_m(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$|u|^2_{H^s_m(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|k|^2 + m^2)^s |\mathcal{F}u(k)|^2 dk < \infty,$$

where $\mathcal{F}u(k)$ is the Fourier transform of $u$. When $m = 1$ we write $H^s(\mathbb{R}^N) = H^s_1(\mathbb{R}^N)$ and $| \cdot |_{H^s(\mathbb{R}^N)} = | \cdot |_{H^s_1(\mathbb{R}^N)}$.

We define $H^s_m(\mathbb{R}_+^{N+1}, y^{1-2s})$ the completion of $C_c^\infty(\mathbb{R}_+^{N+1})$ with respect to the norm

$$||v||^2_{H^s_m(\mathbb{R}_+^{N+1}, y^{1-2s})} := \iint_{\mathbb{R}_+^{N+1}} y^{1-2s} (|\nabla v|^2 + m^2 v^2) dx dy < \infty.$$  

As before, when $m = 1$, we use the notation $H^1(\mathbb{R}_+^{N+1}, y^{1-2s}) = H^1_1(\mathbb{R}_+^{N+1}, y^{1-2s})$ and $|| \cdot ||_{H^1(\mathbb{R}_+^{N+1}, y^{1-2s})} = || \cdot ||_{H^1_1(\mathbb{R}_+^{N+1}, y^{1-2s})}$.

It is possible to prove (see [22]) that it holds the following result:

**Theorem 3.** There exists a trace operator $Tr : H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s}) \to H^s_m(\mathbb{R}^N)$ such that

1. $Tr(v) = v(x, 0)$ for any $v \in C_c^\infty(\mathbb{R}_+^{N+1})$

2. $\kappa_s |Tr(v)|^2_{H^s_m(\mathbb{R}^N)} \leq ||v||^2_{H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s})}$ for any $v \in H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s})$,

where $\kappa_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s)$.

Equality holds in (2) for some function $v \in H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s})$ if and only if $v$ is a weak solution to

$$-\text{div}(y^{1-2s} \nabla v) + m^2 y^{1-2s} v = 0 \text{ in } \mathbb{R}_+^{N+1}.$$  

As a consequence we can deduce
Theorem 4. Let \( u \in H^s_m(\mathbb{R}^N) \). Then there exists a unique \( v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) \) which solves

\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
v = u & \text{on } \partial \mathbb{R}^{N+1}_+.
\end{cases}
\]  

(2.1)

In particular

\[- \lim_{y \to 0} y^{1-2s} \frac{\partial v}{\partial y}(x, y) = \kappa_s (-\Delta + m^2)^s u(x) \text{ in } H^{-s}_m(\mathbb{R}^N). \]

Since \( H^s_m(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \) for any \( q \leq \frac{2N}{N-2s} \), we can prove that

Theorem 5. For any \( v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) \) and for any \( q \in \left[ 2, \frac{2N}{N-2s} \right] \)

\[ C_{q,s,N} |u|^2_{L^q(\mathbb{R}^N)} \leq \kappa_s \int_{\mathbb{R}^N} (|k|^2 + m^2)^s |F_u(k)|^2 dk \]

\[ \leq \int \int_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dxdy \]  

(2.2)

where \( u(x) = v(x, 0) \) is the trace of \( v \) on \( \partial \mathbb{R}^{N+1}_+ \).

In the sequel we will exploit Theorem 4 and we will look for the solutions to the following problem

\[
\begin{cases}
-\text{div}(y^a\nabla v) + m^2 y^a v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
\frac{\partial v}{\partial \nu^a} = \kappa_s [m^2 u - \mu u + |u|^{p-2} u] & \text{on } \partial \mathbb{R}^{N+1}_+ 
\end{cases}
\]  

(2.3)

where

\[ \frac{\partial v}{\partial \nu^a} := - \lim_{y \to 0} y^a \frac{\partial v}{\partial y}(x, y) \]

and \( a = 1 - 2s \in (-1, 1) \).

For simplicity we will assume that \( \kappa_s = 1 \). Finally we recall the following compact embedding (see [30]):

Theorem 6. Let

\[ X^m_{rad} = \{ v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) : v \text{ is radially symmetric with respect to } x \}. \]

Then \( X^m_{rad} \) is compactly embedded in \( L^q(\mathbb{R}^N) \) for any \( q \in \left( 2, \frac{2N}{N-2s} \right) \).

3. Some results on elliptic problems involving \((-\Delta + m^2)^s\): Schauder estimates and maximum principles

In this section we give some results about local Schauder estimates and maximum principle for problems involving the operator

\[-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v.\]

Firstly we give the following definition:
Definition 1. Let $R > 0$ and $h \in L^1(\Gamma^+_R)$. We say that $v \in H^1_m(B^+_R)$ is a weak solution to
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s}v = y^{1-2s}d & \text{in } B^+_R \\
\frac{\partial v}{\partial \nu^{1-2s}} = h & \text{on } \Gamma^0_R
\end{cases}
\]
if
\[
\iint_{B^+_R} y^{1-2s}[\nabla v \cdot \varphi + m^2 y^{1-2s} \varphi] dxdy = \int_{\Gamma^+_R} h \varphi dx
\]
for any $\varphi \in C^1(B^+_R)$ such that $\varphi = 0$ on $\Gamma^+_R$.

Now, we state the following several regularity results whose proof can be found in [22].

Theorem 7. Let $a, b \in L^q(\Gamma^+_R)$ for some $q > \frac{N}{2s}$ and $c, d \in L^r(B^+_R, y^{1-2s})$ for some $r > \frac{N+2-2s}{2}$. Let $v \in H^1_m(B^+_R, y^{1-2s})$ be a weak solution to
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s}v = y^{1-2s}d & \text{in } B^+_R \\
\frac{\partial v}{\partial \nu^{1-2s}} = a(x)v + b(x) & \text{on } \Gamma^0_R
\end{cases}
\]
Then $v \in C^{0,\alpha}(B^+_{R/2})$.

Theorem 8. Let $a, b \in C^k(\Gamma^+_R)$ for some $q > \frac{N}{2s}$ and $\nabla c, \nabla d \in L^\infty(B^+_R)$ for some $k \geq 1$. Let $v \in H^1_m(B^+_R, y^{1-2s})$ be a weak solution to
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s}v = y^{1-2s}d & \text{in } B^+_R \\
\frac{\partial v}{\partial \nu^{1-2s}} = a(x)v + b(x) & \text{on } \Gamma^0_R
\end{cases}
\]
Then $v \in C^{i,\alpha}(B^+_{R/2})$ for $i = 1, \ldots, k$.

Theorem 9. Let $g \in C^{0,\gamma}(\Gamma^+_R)$ for some $\gamma \in [0, 2-2s)$. Let $v \in H^1_m(B^+_R, y^{1-2s})$ be a weak solution to
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s}v = 0 & \text{in } B^+_R \\
\frac{\partial v}{\partial \nu^{1-2s}} = g & \text{on } \Gamma^0_R
\end{cases}
\]
Then, for any $t_0 > 0$ sufficiently small, $y^{1-2s} \partial_\nu v \in C^{0,\alpha}([0, t_0] \times \Gamma_{R/8})$.

In the spirit of the paper [7] we can prove the following maximum principles:

Theorem 10. (weak maximum principle) Let $v \in H^1_m(B^+_R, y^{1-2s})$ be a weak solution to
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s}v \geq 0 & \text{in } B^+_R \\
\frac{\partial v}{\partial \nu^{1-2s}} \geq 0 & \text{on } \Gamma^0_R \\
v \geq 0 & \text{on } \Gamma^+_R
\end{cases}
\]
Then $v \geq 0$ in $B^+_R$.

Proof. It is enough to multiply the weak formulation of above problem by $v^-$. \(\square\)
Remark 1. We can deduce also the strong maximum principle: either \( v \equiv 0 \) or \( v > 0 \) in \( B_R^+ \cup \Gamma_R^0 \). In fact, \( v \) can’t vanish at an interior point by the classical strong maximum principle for strictly elliptic operators. Finally the fact that \( v \) can’t vanish at a point in \( \Gamma_R^0 \) follows by the Hopf principle that we will proved below.

**Theorem 11.** Let \( C_R = B_R(0) \times (0, 1) \) and \( v \in H_m^1(C_R, y^{1-2s}) \cap C(\overline{C_R}) \) be a weak solution to

\[
\begin{cases}
  - \text{div}(y^{1-2s} \nabla v) + m^2 y^{1-2s} v \leq 0 & \text{in } C_R \\
  v > 0 & \text{in } C_R \\
  v(0, 0) = 0
\end{cases}
\]  

Then

\[
\limsup_{y \to 0^+} -y^{1-2s} \frac{v(0, y)}{y} < 0.
\]

If \( y^{1-2s} v_y \in C(\overline{C_R}) \) then

\[
\frac{\partial v}{\partial y^{1-2s}}(0, 0) < 0.
\]

**Proof.** We consider the function

\[
w_A(x, y) = y^{-1+2s}(y + Ay^2)\varphi(x)
\]

where \( A > 0 \) is a constant that will be chosen later and \( \varphi(x) \) is the first eigenfunction of \( -\Delta_x + m^2 \) in \( B_{R/2}(0) \) with zero boundary condition. Then we can conclude proceeding as in the proof of Proposition 4.11 in [7]. \( \square \)

**Theorem 12.** Let \( d \in C^{0, \alpha}(\Gamma_R^0) \) and \( v \in H_m^1(B_R^+, y^{1-2s}) \) be a weak solution to

\[
\begin{cases}
  - \text{div}(y^{1-2s} \nabla v) + m^2 y^{1-2s} v = 0 & \text{in } B_R^+ \\
  \frac{\partial v}{\partial y^{1-2s}} + d(x)v = 0 & \text{on } \Gamma_R^0 \\
  v \geq 0 & \text{on } B_R^+
\end{cases}
\]  

Then \( v > 0 \) in \( B_R^+ \cup \Gamma_R^0 \) unless \( v \equiv 0 \) in \( B_R^+ \).

**Proof.** By using Theorem 7 and Theorem 9 we know that \( v \) and \( y^{1-2s} v_y \) are \( C^{0, \alpha} \) up to the boundary. So the equation \( \frac{\partial v}{\partial y^{1-2s}} + d(x)v = 0 \) is satisfied pointwise on \( \Gamma_R^0 \). If \( u \) is not identically 0 in \( B_R^+ \) then \( u > 0 \) in \( B_R^+ \) by the strong maximum principle for the operator \( L(v) = -\text{div}(y^{1-2s} \nabla v) + m^2 y^{1-2s} v \). If \( u(x_0, 0) = 0 \) at some point \( (x_0, 0) \in \Gamma_R^0 \), then \( \frac{\partial u}{\partial y^{1-2s}}(x_0, 0) < 0 \). This gives a contradiction. \( \square \)
4. EXISTENCE OF GROUND STATE

In this section we prove the existence of a ground state to \((1.1)\). Let us consider the following functional

\[
I_m(v) = \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy - \frac{m^{2s}}{2} \int_{\mathbb{R}^N} v^2 dx + \frac{\mu}{2} \int_{\mathbb{R}^N} v^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |v|^p dx
\]

(4.1)

defined for any \(v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})\).

Firstly we note that

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx
\]

is equivalent to the standard norm in \(H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})\)

\[
||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})} = \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy.
\]

In fact, if \(\mu \geq m^{2s}\) then we have

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx \geq ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}
\]

and by using (2.2) we get

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx \leq (1 + \frac{\mu - m^{2s}}{m^{2s}}) ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}.
\]

Now, we suppose \(\mu < m^{2s}\).

Then

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx \leq ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}
\]

and by (2.2)

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx \geq (1 + \frac{\mu - m^{2s}}{m^{2s}}) ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}.
\]

Thus

\[
C_1(m, s, \mu) ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})} \leq ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})} + (\mu - m^{2s}) ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}
\]

\[
\leq C_2(m, s, \mu) ||v||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}
\]

(4.2)

and we set

\[
||v||^2_{e,m} := \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy + (\mu - m^{2s}) \int_{\mathbb{R}^N} v^2 dx.
\]
Now, in order to prove Theorem 1, we minimize \( I_m \) on the following Nehari manifold
\[
N_m = \{ v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) \setminus \{0\} : J_m(v) = 0 \},
\]
where \( J_m(v) = I'_m(v) \) that is
\[
J_m(v) = \int_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) dx dy - m^2 \int_{\mathbb{R}^N} v^2 dx - \int_{\mathbb{R}^N} |v|^p dx.
\]
\[
= ||v||^2_{\epsilon,m} - \int_{\mathbb{R}^N} |v|^p dx.
\]
Finally we define
\[
c_m = \min_{v \in N_m} \mathcal{I}_m(v).
\]

**Proof.** (proof of Theorem 1) We divide the proof into several steps.

**Step 1** The set \( N_m \) is not empty.

Fix \( v \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) \setminus \{0\} \). Then
\[
h(t) := \mathcal{I}_m(tv) = \frac{t^2}{2} ||v||^2_{\epsilon,m} - \frac{t^p}{p} |v|^p_{L^p(\mathbb{R}^N)}
\]
achieves its maximum in some \( \tau > 0 \). By differentiating \( h \) with respect to \( t \) we have \( \mathcal{I}'_m(\tau v)v = 0 \) and \( \tau v \in N_m \).

**Step 2** Selection of an adequate minimizing sequence.

We prove that there exists a radially symmetric function \( v \in N_m \) such that \( \mathcal{I}_m(v) = c_m \).

Let \( (v_j) \subset N_m \) be a minimizing sequence for \( \mathcal{I}_m \) and \( u_j \) its trace. Let \( \tilde{u}_j \) be the symmetric-decreasing rearrangement of \( u_j \). It is known (see [32]) that
\[
\int_{\mathbb{R}^N} (|k|^2 + m^2)^s |\mathcal{F}\tilde{u}_j(k)|^2 dk \leq \int_{\mathbb{R}^N} (|k|^2 + m^2)^s |\mathcal{F}u_j(k)|^2 dk \quad (4.3)
\]
and \( ||\tilde{u}_j||_{L^q(\mathbb{R}^N)} = ||u_j||_{L^q(\mathbb{R}^N)} \) for any \( q \geq 1 \). Now, let \( \tilde{v}_j \) be the unique solution to
\[
\begin{cases}
- \text{div}(y^{1-2s} \nabla \tilde{v}_j) + m^2 \tilde{v}_j = 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\
\tilde{v}_j(x, 0) = \tilde{u}_j(x) \quad \text{on } \partial \mathbb{R}^{N+1}_+.
\end{cases}
\]

We recall that
\[
||\tilde{v}_j||_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})} = ||\tilde{u}_j||_{H^1_m(\mathbb{R}^N)},
\]
by using (2) in Theorem 3.
Taking into account (2.2), (4.3) and (4.5) we get
\[ \int \int_{R_{N+1}} y^{1-2s} (|\nabla \tilde{v}_j|^2 + m^2 \tilde{v}_j^2) \, dx\, dy = \int_{R^N} (|k|^2 + m^2) |\mathcal{F} \tilde{u}_j(k)|^2 \, dk \]
\[ \leq \int_{R^N} (|k|^2 + m^2) |\mathcal{F} u_j(k)|^2 \, dk \]
\[ \leq \int \int_{R_{N+1}} y^{1-2s} (|\nabla v_j|^2 + m^2 v_j^2) \, dx\, dy, \]
so we deduce that
\[ \mathcal{J}_m(\tilde{v}_j) \leq \mathcal{J}_m(v_j) = 0 \] and \[ \mathcal{I}_m(\tilde{v}_j) \leq \mathcal{I}_m(v_j). \]

Proceeding as in the proof of Step 1, we can find \( t_j > 0 \) such that \( \mathcal{J}_m(t_j \tilde{v}_j) = 0 \). By using the fact that \( \mathcal{J}_m(\tilde{v}_j) \leq 0 \) we can see that \( t_j \leq 1 \).

Since \( \mathcal{J}_m(v) = \mathcal{I}_m'(v) v \) we have
\[ 0 = \mathcal{J}_m(t_j \tilde{v}_j) \]
\[ = \left[ \mathcal{I}_m(t_j \tilde{v}_j) + \frac{1}{p} \int_{R^N} |t_j \tilde{v}_j|^p \, dx - \frac{1}{2} \int_{R^N} |t_j \tilde{v}_j|^p \, dx \right] \]
and by using the fact that \( 0 < t_j \leq 1 \) we obtain
\[ \mathcal{I}_m(t_j \tilde{v}_j) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{R^N} |t_j \tilde{v}_j|^p \, dx \]
\[ \leq \left( \frac{1}{2} - \frac{1}{p} \right) \int_{R^N} |\tilde{v}_j|^p \, dx \]
\[ = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{R^N} |v_j|^p \, dx \]
\[ = \mathcal{I}(v_j). \]

Then \( w_j := t_j \tilde{v}_j \) is a minimizing sequence, radially symmetric with respect to \( x \), of \( \mathcal{I}_m \) on \( \mathcal{N}_m \). By using (4.2) we get
\[ \left( \frac{1}{2} - \frac{1}{p} \right) C_1(m, s, \mu) \left| w_j \right|_{H^1_{m}(R_{N+1}, y^{1-2s})}^2 \leq \left( \frac{1}{2} - \frac{1}{p} \right) \left| w_j \right|_{e,m}^2 = \mathcal{I}_m(w_j) < C \]
then, by using Theorem 6 we can assume that
\[ w_j \rightharpoonup w \text{ in } H^1_{m}(R_{N+1}, y^{1-2s}) \] \[ w_j(\cdot, 0) \rightharpoonup w(\cdot, 0) \text{ in } L^q(R^N) \quad \forall q \in \left( 2, \frac{2N}{N-2s} \right). \] Then we deduce that \( \mathcal{I}_m(w) \leq c \) and \( \mathcal{J}_m(w) \leq 0 \). Now we claim that \( w \) is not identically zero. We check this, we assume by contradiction that \( w = 0 \). By
using the fact that $J_m(w_j) = 0$ and (2.2), we can see that
\[
\int_{\mathbb{R}^N} |w_j|^p \, dx = \iint_{\mathbb{R}^{N+1}^+} y^{1-2s}(|\nabla w_j|^2 + m^2 w_j^2) \, dxdy + (\mu - m^2s) \int_{\mathbb{R}^N} w_j^2 \, dx \\
\geq C_1(m, s, \mu) ||w_j||^2_{H^s_b(\mathbb{R}^{N+1}, y^{1-2s})} \\
\geq C(m, s, \mu, N, p) \left( \int_{\mathbb{R}^N} |w_j|^p \, dx \right)^{\frac{2}{p}}.
\]

By using (4.7) we deduce that $|w|_{L^p(\mathbb{R}^N)} \geq C(m, s, \mu, N, p)^{1/p-2} > 0$, which gets a contradiction. Then, we can find $\tau \in (0, 1]$ such that $I_m(\tau w) \leq c$ and $J_m(\tau w) = 0$.

**Step 3 Conclusion.** Let $v$ be the minimizer obtained above. By using the fact that $v \in N_m$ we have
\[
J_m'(v) v = 2 \left( \iint_{\mathbb{R}^{N+1}^+} y^{1-2s}(|\nabla v|^2 + m^2 v^2) \, dxdy + (\mu - m^2s) \int_{\mathbb{R}^N} v^2 \, dx \right) - p \int_{\mathbb{R}^N} |v|^p \, dx \\
= (2 - p) \int_{\mathbb{R}^N} |v|^p \, dx \neq 0.
\]

As a consequence we can find a Lagrange multiplier $\lambda \in \mathbb{R}$ such that
\[
I_m'(v) \varphi = \lambda J_m'(v) \varphi \tag{4.8}
\]
for any $\varphi \in X_{rad}^m$. Taking $\varphi = v$ in (4.8) we deduce that $\lambda = 0$ and $v$ is a nontrivial solution to (1.5).

\[\square\]

5. **REGULARITY AND SYMMETRY OF SOLUTION TO (1.1)**

**Lemma 1.** Let $v \in H^1_m(\mathbb{R}_{+}^{N+1}, y^{1-2s})$ be a weak solution to
\[
\left\{ \begin{array}{ll}
- \text{div}(y^{1-2s} \nabla v) + m^2y^{1-2s}v = 0 & \text{in } \mathbb{R}_{+}^{N+1} \\
\frac{\partial v}{\partial n} = m^2s v + f(x, v) & \text{on } \partial \mathbb{R}_{+}^{N+1},
\end{array} \right. \tag{5.1}
\]
where $f(x, v) = -\mu v + |v|^{p-2} v$. Then $v(\cdot, 0) \in L^q(\mathbb{R}^N)$ for all $q \in [2, \infty]$.

**Proof.** We proceed as in the proof of Theorem 3.2. in [18]. Since $v$ is a weak solution to (1.1), we know that
\[
\iint_{\mathbb{R}_{+}^{N+1}} y^{1-2s}(\nabla v \nabla \eta + m^2 v \eta) \, dxdy = \int_{\mathbb{R}^N} m^2 s v \eta + f(x, v) \eta \, dx \tag{5.2}
\]
for all $\eta \in H^1_m(\mathbb{R}_{+}^{N+1}, y^{1-2s})$.  

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Let \( w = v_K^{2\beta} \in H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s}) \) where \( v_K = \min\{|v|, K\} \), \( K > 1 \) and \( \beta \geq 0 \). Taking \( \eta = w \) in (5.2) we deduce that
\[
\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} v_K^{2\beta} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + \int\int_{D_{K,T}} 2\beta y^{1-2s} v_K^{2\beta} |\nabla v|^2 \, dx \, dy
= m^2 \int_{\mathbb{R}^N} v^2 v_K^{2\beta} \, dx + \int_{\mathbb{R}^N} f(x,v) vv_K^{2\beta} \, dx
\] (5.3)
where \( D_{K,T} = \{(x,y) \in \mathbb{R}^{N+1}_+ : |v(x,y)| \leq K\} \).

It is easy to see that
\[
\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla (vv_K^{\beta})|^2 \, dx \, dy
= \int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} v_K^{2\beta} |\nabla v|^2 \, dx \, dy + \int\int_{D_{K,T}} (2\beta + \beta^2) y^{1-2s} v_K^{2\beta} |\nabla v|^2 \, dx \, dy.
\] (5.4)

Then, putting together (5.3) and (5.4) we get
\[
||vv_K^{\beta}||^2_{H^1_m(\mathbb{R}^{N+1}_+, y^{1-2s})}
= \int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} (|\nabla (vv_K^{\beta})|^2 + m^2 v^2 v_K^{2\beta}) \, dx \, dy
= \int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} v_K^{2\beta} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + \int\int_{D_{K,T}} 2\beta \left( 1 + \frac{\beta}{2} \right) y^{1-2s} v_K^{2\beta} |\nabla v|^2 \, dx \, dy
\leq c_\beta \left[ \int\int_{\mathbb{R}^{N+1}_+} y^{1-2s} v_K^{2\beta} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + \int\int_{D_{K,T}} 2\beta y^{1-2s} v_K^{2\beta} |\nabla v|^2 \, dx \, dy \right]
= c_\beta \int_{\mathbb{R}^N} m^2 v^2 v_K^{2\beta} + f(x,v) vv_K^{2\beta} \, dx
\] (5.6)
where \( c_\beta = 1 + \frac{\beta}{2} \). Then we deduce that
\[
m^2 v^2 v_K^{2\beta} + f(x,v) vv_K^{2\beta} \leq (m^2 + C) v^2 v_K^{2\beta} + C |v|^{p-2} v^2 v_K^{2\beta} \quad \text{on } \mathbb{R}^N.
\]

Now, we prove that
\[
|v|^{p-2} \leq 1 + h \quad \text{on } \mathbb{R}^N
\]
for some \( h \in L^{N/2s}(\mathbb{R}^N) \). Firstly, we observe that
\[
|v|^{p-2} = \chi_{\{|v| \leq 1\}} |v|^{p-2} + \chi_{\{|v| > 1\}} |v|^{p-2} \leq 1 + \chi_{\{|v| > 1\}} |v|^{p-2} \quad \text{on } \mathbb{R}^N.
\]

If \((p-2)\frac{N}{2s} < 2\) then
\[
\int_{\mathbb{R}^N} \chi_{\{|v| > 1\}} |v|^\frac{N}{2s}(p-2) \, dx \leq \int_{\mathbb{R}^N} \chi_{\{|v| > 1\}} |v|^2 \, dx < \infty
\]
while if \(2 \leq (p-2)\frac{N}{2s}\) we have that \((p-2)\frac{N}{2s} \in [2, \frac{2N}{N-2s}]\).
Therefore, there exist a constant $c > 0$ and a function $h \in L^{N/2s}(\mathbb{R}^N)$, $h \geq 0$ and independent of $K$ and $\beta$, such that

$$m^{2s}v^2 \nu^2 v_{K} + f(x,v) \nu^2 v_{K} \leq (c + h)v^2 v_{K}$$

on $\mathbb{R}^N$. \hfill (5.7)

Taking into account (5.5) and (5.7) we have

$$||v\nu^2 \nu^2 v_{K}||_{L^1(\mathbb{R}^N)} \leq c \beta \int_{\mathbb{R}^N} (c + h)v^2 v_{K} \, dx,$$

and by Monotone Convergence Theorem ($v_{K}$ is increasing with respect to $K$) we have as $K \to \infty$

$$||v||_{L^1(\mathbb{R}^N)} \leq c \beta \int_{\mathbb{R}^N} |v|^2(\beta + 1) \, dx + c \beta \int_{\mathbb{R}^N} h|v|^2(\beta + 1) \, dx.$$ \hfill (5.8)

Fix $M > 0$ and let $A_1 = \{ h \leq M \}$ and $A_2 = \{ h > M \}$.

Then

$$\int_{\mathbb{R}^N} h|v(\cdot,0)|^2(\beta + 1) \, dx \leq M||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1} + \varepsilon(M)||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1}.$$ \hfill (5.9)

where $\varepsilon(M) = \left( \int_{A_2} h^{N/2s} \, dx \right)^{\frac{2s}{N}} \to 0$ as $M \to \infty$. Taking into account (5.8), (5.9), we get

$$||v||_{L^1(\mathbb{R}^N)} \leq c \beta(c + M)||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1}.$$ \hfill (5.10)

By using (2.2) we know that

$$||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1} \leq C_{2\beta}^2 ||v||_{H^1(\mathbb{R}^N)}^{\beta + 1}.$$ \hfill (5.11)

Then, choosing $M$ large so that $\varepsilon(M)c \beta C_{2\beta}^2 < \frac{1}{2}$, and by using (5.10) and (5.11) we obtain

$$||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1} \leq 2C_{2\beta}^2 c \beta(c + M)||v(\cdot,0)||_{L^2(\mathbb{R}^N)}^{\beta + 1}.$$ \hfill (5.12)

Then we can start a bootstrap argument: since $v(\cdot,0) \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ we can apply (5.12) with $\beta_1 + 1 = \frac{N}{N-2s}$ to deduce that $v(\cdot,0) \in L^{\frac{2(N+1)}{N-2s}}(\mathbb{R}^N) = L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$. Applying (5.12) again, after $k$ iterations, we find $v(\cdot,0) \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, and so $v(\cdot,0) \in L^q(\mathbb{R}^N)$ for all $q \in [2, \infty)$.

Now we prove that $v(\cdot,0) \in L^\infty(\mathbb{R}^N)$. Since $v(\cdot,0) \in L^q(\mathbb{R}^N)$ for all $q \in [2, \infty)$ we have that $h \in L^{\frac{N}{N-2s}}(\mathbb{R}^N)$.
By using generalized Hölder inequality, we can see that
\[
\int_{\mathbb{R}^N} h|v|^{2(\beta+1)} \, dx \leq |h|_{L^{\frac{N}{s}}(\mathbb{R}^N)} \left( |v|^{\beta+1} \right)_{L^2(\mathbb{R}^N)} ||v|^{\beta+1} \right)_{L^2(\mathbb{R}^N)} 
\leq \frac{1}{\lambda} ||v|^{\beta+1} \right)_{L^2(\mathbb{R}^N)} \left( \right)_{L^2(\mathbb{R}^N)} (5.13)
\]
and by using (5.8) we deduce
\[
|||v|^{\beta+1}|\right)_{L^2(\mathbb{R}^N)} \leq \frac{c_{\beta}}{\lambda} ||h|_{L^{\frac{N}{s}}(\mathbb{R}^N)} ||v|^{\beta+1} \right)_{L^2(\mathbb{R}^N)} (5.14)
\]
Taking \(\lambda\) such that
\[
\frac{c_{\beta}}{\lambda} = \frac{1}{2}
\]
and by using (2.2), we obtain
\[
||v|^{\beta+1}|\right)_{L^2(\mathbb{R}^N)} \leq 2 \frac{c_{\beta}}{\lambda} ||h|_{L^{\frac{N}{s}}(\mathbb{R}^N)} ||v|^{\beta+1} \right)_{L^2(\mathbb{R}^N)} := M_\beta ||v|^{\beta+1}|\right)_{L^2(\mathbb{R}^N)} (5.15)
\]
Now we can control the dependence on \(\beta\) of \(M_\beta\) as follows
\[
M_\beta \leq C \frac{c_{\beta}^2}{\lambda} \leq C (1 + \beta)^2 \leq M_0^2 e^{2\sqrt{1+1}},
\]
which implies that
\[
||v|_{L^{2(\beta+1)}(\mathbb{R}^N)} \leq M_0 e^{\sqrt{1+1}} ||v|_{L^{2(\beta+1)}(\mathbb{R}^N)}.
\]
Iterating this last relation and choosing \(\beta_0 = 0\) and \(2(\beta_{n+1} + 1) = 2^\frac{1}{s}(\beta_n + 1)\),
we deduce that
\[
||v|_{L^{2(\beta_n+1)}(\mathbb{R}^N)} \leq M_0 \sum_{i=0}^{n} \frac{1}{\beta_i + 1} \sum_{i=0}^{n} \frac{1}{\sqrt{\beta_i + 1}} ||v|_{L^{2(\beta_0+1)}(\mathbb{R}^N)}.
\]
We note that \(1 + \beta_n = (\frac{N}{N-2s})^n\), so the series
\[
\sum_{i=0}^{\infty} \frac{1}{\beta_i + 1} \text{ and } \sum_{i=0}^{\infty} \frac{1}{\sqrt{\beta_i + 1}}
\]
are finite and we get
\[
||v|_{L^{\infty}(\mathbb{R}^N)} = \lim_{n \to \infty} ||v|_{L^{2(\beta_n+1)}(\mathbb{R}^N)} < \infty.
\]
\[\square\]

**Theorem 13.** Let \(u \in H^s(\mathbb{R}^N)\) be a solution to (1.1). Then \(u \in C^{1,\beta}(\mathbb{R}^N)\) for some \(\beta \in (0, 1)\) and \(u(x) \to 0\) as \(|x| \to \infty\).
Theorem 14. Every ground state $u$ of (1.1) has one sign.

Proof. Let $v$ be a unique solution to (2.1) with data $u$. Then $v \in \mathcal{N}_m$ and $\mathcal{I}_m(v) = c_m$. In particular $|v| \in \mathcal{N}_m$ and $\mathcal{I}_m(|v|) = \mathcal{I}_m(v)$, that is $|v|$ is a weak solution to (1.5). Then $|v|$ and $y^{1-2s} \partial_y|v|$ are continuous up to the boundary. Let assume by contradiction that $|v|$ achieves its global minimum at $(x_0, 0) \in \partial \mathbb{R}^{N+1}_+$. By using Hopf principle we can deduce that $-y^{1-2s} \frac{\partial |v|}{\partial y}(x_0, 0) < 0$. This gives a contradiction since

$$-y^{1-2s} \frac{\partial |v|}{\partial y}(x_0, 0) = [m^{2s} - \mu]|v|(x_0, 0) + |v|^{p-1}(x_0, 0) = 0.$$ 

Then $u > 0$ or $u < 0$ in $\mathbb{R}^N$. \hfill \Box

Theorem 15. Every positive solution $u \in H^s(\mathbb{R}^N)$ of (1.1) is radially symmetric with respect to some point $x_0 \in \mathbb{R}^N$.

Proof. We proceed as in [16]. Let $v$ be a unique solution of (2.1) with boundary data $u$.

Let $\lambda > 0$ and we consider the sets

$$R_\lambda = \{(x_1, \ldots, x_N, y) : x_1 > \lambda, y \geq 0\}$$

and

$$T_\lambda = \{(x_1, \ldots, x_N, y) : x_1 = \lambda, y \geq 0\}.$$ 

Let $v_\lambda(x, y) = v(2\lambda - x_1, \ldots, x_N, y)$ and $w_\lambda = v - v_\lambda$. Then $w_\lambda$ satisfies

$$\begin{cases} -\text{div}(y^{1-2s} \nabla w_\lambda) + m^2 y^{1-2s} w_\lambda = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
\frac{\partial w_\lambda}{\partial y} = (C_\lambda(x) + m^{2s} - \mu)w_\lambda(x, 0) & \text{on } \partial \mathbb{R}^{N+1}_+ , \end{cases}$$

(5.16)

where

$$C_\lambda(x) := \begin{cases} \frac{v^{p-1}(x, 0) - v^{p-1}(x, 0)}{v(x, 0) - v(x, 0)} & \text{if } v_\lambda(x, 0) \neq v(x, 0) \\
0 & \text{if } v_\lambda(x, 0) = v(x, 0). \end{cases}$$

(5.17)

Let $w_\lambda^- := \min\{0, w_\lambda\}$. Note that as $\lambda \to \infty$, $C_\lambda(x) \to 0$ uniformly for $x$ such that $w_\lambda \neq 0$ because of $\lim_{|x| \to \infty} |v(x, 0)| = 0$ and $0 \leq v_\lambda < v$. Multiplying the
weak formulation of (3.16) by \( w^-_\lambda \) and applying inequality (2.2), we get for for \( \lambda > 0 \) sufficiently large

\[
\iint_{R_\lambda} y^{1-2s} ||\nabla w^-_\lambda||^2 + m^2(w^-_\lambda)^2 \, dx \, dy = \int_{\{x_1 > \lambda\}} (C_\lambda(x) + m^2 - \mu)(w^-_\lambda)^2 \, dx
\]

\[
\leq \int_{\{x_1 > \lambda\}} C_\lambda(x)(w^-_\lambda)^2 \, dx + A(m, \mu, s) \iint_{R_\lambda} y^{1-2s} ||\nabla w^-_\lambda||^2 + m^2(w^-_\lambda)^2 \, dx \, dy
\]

where \( A(m, \mu, s) = (1 - \frac{\mu}{m^{2s}}) \) if \( m^{2s} - \mu > 0 \) and \( A(m, \mu, s) = 0 \) otherwise. Then \( w^-_\lambda \equiv 0 \) on \( R_\lambda \) and as a consequence

\[
w\lambda(x, y) \geq 0 \text{ on } R_\lambda
\]

for \( \lambda > 0 \) sufficiently large.

Now, we define

\[
\nu := \inf\{\tau > 0 : w_\lambda \geq 0 \text{ on } R_\lambda \text{ for every } \lambda \geq \tau\}.
\]

We distinguish two cases. We begin assuming \( \nu > 0 \). We want to prove that \( w_\nu \equiv 0 \). We argue by contradiction.

By continuity, \( w_\nu \geq 0 \) on \( R_\nu \), and by the strong maximum principle \( w_\nu > 0 \) on the set

\[
R'_\nu := \{(x_1, \ldots, x_N, y) : x_1 > \nu, x_i \in \mathbb{R} (i = 1, \ldots, N) y > 0 \}.
\]

We also have \( w_\nu(x, 0) \geq 0 \) on the set \( \{x \in \mathbb{R}^N : x_1 \geq \nu\} \) by continuity. Furthermore, by Hopf principle \( w_\nu(x, 0) > 0 \) on the set \( \{x \in \mathbb{R}^N : x_1 > \nu\} \).

Take \( \lambda_j < \nu \) such that \( \lambda_j \to \nu \) as \( j \to \infty \). Let \( r_0 > 0 \) such that \( |C_{\nu}(x)| \leq \frac{\nu}{4} \) for every \( |x| > r_0 \). Since \( ||w_{\lambda_j}||_{C^1(\mathbb{R}^{N+1})} \) is uniformly bounded, \( D := |C_{\nu}|_{L^\infty(\mathbb{R}^N)} < \infty \) and \( |C_{\lambda_j}(x)| \leq \frac{\nu}{2} \) for every \( |x| > r_0 \) and \( j \in \mathbb{N} \).

We denote by

\[
B_{r_0}(p_j) = \{x \in \mathbb{R}^N : |x - p_j| < r_0\} \subset \mathbb{R}^N
\]

where \( p_j = (\lambda_j, 0, \ldots, 0) \). As above, we obtain

\[
\iint_{R_{\lambda_j}} y^{1-2s} ||\nabla w_{\lambda_j}^-||^2 + m^2(w_{\lambda_j}^-)^2 \, dx \, dy \leq \int_{\{x_1 > \lambda_j\}} (C_{\lambda_j} + m^2 - \mu)(w_{\lambda_j}^-)^2 \, dx
\]

\[
\leq (D + m^2 + \mu) \int_{\{x_1 > \lambda_j\} \cap B_{r_0}(p_j)} (w_{\lambda_j}^-)^2 \, dx + (m^2 - \frac{\mu}{2}) \int_{\{x_1 > \lambda_j\} \setminus B_{r_0}(p_j)} (w_{\lambda_j}^-)^2 \, dx
\]

\[
\leq (D + m^2 + \mu) \int_{\{x_1 > \lambda_j\} \cap B_{r_0}(p_j)} (w_{\lambda_j}^-)^2 \, dx + B(m, \mu, s) \int_{\{x_1 > \lambda_j\} \setminus B_{r_0}(p_j)} (w_{\lambda_j}^-)^2 \, dx.
\]

where \( B(m, \mu, s) = (m^2 - \frac{\mu}{2}) \) if \( m^2 - \frac{\mu}{2} > 0 \) and it is zero otherwise. Since \( w_\nu(x) > 0 \) on the set \( \{x \in \mathbb{R}^N : x_1 > \nu\} \), the measure of set \( E_j = \{x \in B_{r_0}(p_j) : w_{\lambda_j}^- (x, 0) \neq 0\} \) goes to 0 as \( j \to \infty \).
Then using Hölder and Sobolev inequality, we see
\[
\int_{\{x_1 > \lambda_j\} \cap B_{r_0}(p_j)} (w_{\lambda_j}^-)^2 dx = \int_{\{x_1 > \lambda_j\}} \chi_{E_j} (w_{\lambda_j}^-)^2 dx \\
\leq \|\chi_{E_j}\|_{L^{N/2s}}^2 |w_{\lambda_j}^-|^2_{L^{2N/N-2s}} \\
\leq o(1) \int_{R_{\lambda_j}} |\nabla w_{\lambda_j}^-|^2 dxdy.
\]

Therefore \(w_{\lambda_j} \geq 0\) on \(R_{\lambda_j}\), if \(j\) is large. This gives a contradiction because of the minimality of \(\nu\). Thus we can conclude that \(w_{\nu} \equiv 0\) on \(R_{\nu}\) and we get the symmetry with respect to the \(x_1\) direction.

Now assume \(\nu = 0\). By repeating the above argument for \(\lambda < 0\) and \(w_{\lambda} := v_{\lambda} - v\) defined on \(L_{\lambda} := \{(x_1, \ldots, x_N, y) \in \mathbb{R}^{N+1}_+ : x_1 < \lambda, x_i \in \mathbb{R} (i = 1, \ldots, n), y \geq 0\}\).

Then \(w_{\lambda} \geq 0\) for \(|\lambda|\) sufficiently large. Let
\[
\nu' := \sup\{\tau < 0 : W_{\lambda} \geq 0 \text{ on } L_{\lambda} \text{ for every } \lambda \leq \tau\}.
\]
If \(\nu' < 0\), we get the symmetry as above. If \(\nu' = 0\), by using \(\nu = 0\) we have
\[
v(-x_1, x_2, \ldots, x_N, y) \geq v(x_1, x_2, \ldots, x_N, y) \text{ on } \mathbb{R}^{N+1}_+.
\]
Consequently, by replacing \(x_1\) with \(-x_1\) we deduce that
\[
v(-x_1, x_2, \ldots, x_N, y) = v(x_1, x_2, \ldots, x_N, y) \text{ on } \mathbb{R}^{N+1}_+.
\]

Using the same approach in any arbitrary direction \(x_i\), we conclude the proof. \(\square\)

6. Passage to the limit as \(m \to 0\)

In this section we show that it is possible to pass to the limit in problem (1.5) and to find a nontrivial ground state to (1.4). In order to prove this, we estimate \(c_m\) defined in Section 1 from above and below uniformly in \(m\).

Fix \(0 < m < (\frac{4}{7})^{1/2s}\). By using the characterization of the infimum \(c_m\) on \(N_m\) we can see that
\[
c_m = \inf_{v \in N_m} \mathcal{I}_m(v) = \inf_{v \in X_{m, \text{rad}} \setminus \{0\}} \max_{t > 0} \mathcal{I}_m(tv).
\]

Now, we prove that there exists \(\lambda > 0\) and \(\delta > 0\) independent from \(m\) such that
\[
\lambda \leq c_m \leq \delta. \tag{6.1}
\]

Let
\[
w(x, y) = v_0(x) \frac{1}{y + 1}
\]
where \( v_0 \) is defined by setting

\[
v_0(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
2 - |x| & \text{if } 1 \leq |x| \leq 2 \\
0 & \text{if } |x| \geq 2
\end{cases}
\]  

(6.2)

Then \( w \in H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s}) \) and

\[
||w||^2_{H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s})} = \left( \int_0^\infty y^{1-2s} \frac{1}{(y + 1)^2} dy \right) \left[ \int_{\mathbb{R}^N} |\nabla_x v_0|^2 + m^2 v_0^2 dx \right] \\
+ \left( \int_0^\infty y^{1-2s} \frac{1}{(y + 1)^4} dy \right) \int_{\mathbb{R}^N} v_0^2 dx \\
\leq A \left[ \int_{\mathbb{R}^N} |\nabla_x v_0|^2 + \left( \frac{\mu}{2} \right)^{1/s} v_0^2 dx \right] + B \int_{\mathbb{R}^N} v_0^2 dx =: C. \quad (6.3)
\]

Hence we have

\[
\sup_{t > 0} \mathcal{I}_m(tw) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{||w||^{\frac{2}{2-p}}_{L^p(\mathbb{R}^N)}}{||w||^{\frac{2-p}{2}}_{L^p(\mathbb{R}^N)}} \\
\leq \left( \frac{1}{2} - \frac{1}{p} \right) \frac{||w||^{2}_{H^1_m(\mathbb{R}_+^{N+1}, y^{1-2s})} + \mu |v_0|^2_{L^2(\mathbb{R}^N)}^{\frac{p}{p-2}}}{|v_0|^\frac{2-p}{2}} \\
\leq \left( \frac{1}{2} - \frac{1}{p} \right) \frac{C + \mu |v_0|^2_{L^2(\mathbb{R}^N)}^{\frac{p}{p-2}}}{|v_0|^\frac{2-p}{2}} =: \delta
\]

Since \( \mathcal{I}_m(v_m) = c_m \) and \( \mathcal{J}_m(v_m) = 0 \) we know that

\[
c_m = \mathcal{I}_m(v_m) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |v_m|^p dx.
\]

Then to deduce a lower bound to \( c_m \) it is enough to estimate the \( L^p(\mathbb{R}^N) \) norm of \( v_m \). Since \( \mathcal{J}_m(v_m) = 0 \) we can see that

\[
|v_m|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}_+^{N+1}} y^{1-2s} (|\nabla v_m|^2 + m^2 v_m^2) dxdy + (\mu - m^2) \int_{\mathbb{R}^N} v_m^2 dx \\
\geq \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla v_m|^2 dxdy + \frac{\mu}{2} |v_m|^2_{L^2(\mathbb{R}^N)} \\
\geq \kappa_s |v_m|^2_{H^s(\mathbb{R}^N)} + \frac{\mu}{2} |v_m|^2_{L^2(\mathbb{R}^N)} \\
\geq C_{s,\mu} |v_m|^2_{H^s(\mathbb{R}^N)} \geq C'_{s,\mu} |v_m|^2_{L^p(\mathbb{R}^N)}
\]

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that is
\[ |v_m|_{L^p(\mathbb{R}^N)} \geq (C'_{s,\mu})^{1/p} \quad \text{and} \quad c_m \geq \left( \frac{1}{2} - \frac{1}{p} \right)(C'_{s,\mu})^{\frac{p}{p+2}} =: \lambda. \tag{6.4} \]

Now, by using (6.4), we are able to prove the following result

**Theorem 16.** There exists \( v \in H^1_{\text{loc}}(\mathbb{R}^{N+1}, y^{1-2s}) \) such that \( v_m \rightharpoonup v \) in \( L^2(\mathbb{R}^N \times [0, \varepsilon], y^{1-2s}) \) for any \( \varepsilon > 0 \), \( \nabla v_m \rightharpoonup \nabla v \) in \( L^2(\mathbb{R}^{N+1}, y^{1-2s}) \) and \( v_m(\cdot, 0) \to v(\cdot, 0) \) in \( L^q(\mathbb{R}^N) \) for any \( q \in (2, \frac{2N}{N-2s}) \), as \( m \to 0 \). In particular \( v(\cdot, 0) \) is a nontrivial weak solution to (1.4).

**Proof.** Taking into account \( J(v_m) = 0 \), \( c_m \leq \delta \) and \( 0 < m < (\frac{\delta}{2})^{1/2s} \) we can see that

\[
\delta^{1/p} \left( \frac{1}{2} - \frac{1}{p} \right)^{-1/p} \geq c_m^{1/p} \left( \frac{1}{2} - \frac{1}{p} \right)^{-1/p} = |v_m|^p_{L^p(\mathbb{R}^N)}
\]

\[
= \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_m|^2 + m^2 v_m^2)\,dxdy + (\mu - m^2s) \int_{\mathbb{R}^N} v^2\,dx
\]

\[
\geq \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v_m|^2\,dxdy + \frac{\mu}{2} |v_m|^2_{L^2(\mathbb{R}^N)}
\]

\[
\geq \kappa_s [v_m]^2_{H^s(\mathbb{R}^N)} + \frac{\mu}{2} |v_m|^2_{L^2(\mathbb{R}^N)}
\]

\[
\geq C_{s,\mu} |v_m|^2_{H^s(\mathbb{R}^N)}. \tag{6.5}
\]

that is

\[
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v_m|^2\,dxdy \leq C_1
\]

and

\[
|v_m|^2_{H^s(\mathbb{R}^N)} \leq C_2.
\]

Now, fix \( \varepsilon > 0 \) and let \( v \in C_\infty_c(\mathbb{R}^{N+1}_+), \) such that \( ||v||_{H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, y^{1-2s})} < \infty \). For any \( x \in \mathbb{R}^N \) and \( y \in [0, \varepsilon] \), we have

\[
v(x, y) = v(x, 0) + \int_0^y \partial_y v(x, t)\,dt.
\]

By using \((a + b)^2 \leq 2a^2 + 2b^2\) for all \( a, b \geq 0 \) we obtain

\[
|v(x, y)|^2 \leq 2|v(x, 0)|^2 + 2\left( \int_0^y |\partial_y v(x, t)|\,dt \right)^2,
\]

and by applying the Hölder inequality we deduce

\[
|v(x, y)|^2 \leq 2 \left[ |v(x, 0)|^2 + \left( \int_0^y t^{1-2s}|\partial_y v(x, t)|^2\,dt \right) \frac{y^{2s}}{2s} \right].
\]
Multiplying both members by $y^{1-2s}$ we have

$$y^{1-2s}|v(x, y)|^2 \leq 2\left[y^{1-2s}|v(x, 0)|^2 + \left(\int_0^y t^{1-2s}|\partial_y v(x, t)|^2 dt\right)\frac{y}{2s}\right].$$  \hspace{1cm} (6.6)

Integrating (6.6) over $\mathbb{R}^N \times [0, \varepsilon]$ we have

$$||v||^2_{L^2(\mathbb{R}^N \times [0, \varepsilon], y^{1-2s})} \leq \frac{\varepsilon^{2-2s}}{1-s}||v(\cdot, 0)||^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon^2}{2s}||\partial_y v||^2_{L^2(\mathbb{R}^N_+, y^{1-2s})}.$$  \hspace{1cm} (6.7)

By density (6.7) holds for any $v \in H^1_{loc}(\mathbb{R}^N_+, y^{1-2s})$.

Then, replacing $\eta \psi$ by $v_m$ we infer that

$$||v_m||^2_{L^2(\mathbb{R}^N \times [0, \varepsilon], y^{1-2s})} \leq C(\varepsilon, s)K(\delta, p)^2$$

for any $0 < m < (\frac{\varepsilon}{2})^{1/2s}$. Then there exists $v \in H^1_{loc}(\mathbb{R}^N_+, y^{1-2s})$ such that

$$v_m \rightharpoonup v \text{ in } L^2(\mathbb{R}^N \times [0, \varepsilon], y^{1-2s}) \text{ for all } \varepsilon > 0 \hspace{1cm} (6.8)$$

$$\nabla v_m \rightharpoonup \nabla v \text{ in } L^2(\mathbb{R}^N_+, y^{1-2s}) \hspace{1cm} (6.9)$$

$$v_m(\cdot, 0) \rightarrow v(\cdot, 0) \text{ in } L^q(\mathbb{R}^N) \hspace{1cm} \forall q \in \left(2, \frac{2N}{N-2s}\right) \hspace{1cm} (6.10)$$

as $m \rightarrow 0$. Finally, we prove that $v(\cdot, 0)$ is a nontrivial weak solution to (1.4).

We proceed as in [2]. Fix $\eta \in C^\infty_{c}(\mathbb{R}^N_+)$ such that $\nabla \eta \in L^2(\mathbb{R}^N_+, y^{1-2s})$ and let $\psi \in C^\infty([0, \infty))$ defined by

$$\begin{cases} 
\psi = 1 & \text{ if } 0 \leq y \leq 1 \\
0 \leq \psi \leq 1 & \text{ if } 1 \leq y \leq 2 \\
\psi = 0 & \text{ if } y \geq 2.
\end{cases} \hspace{1cm} (6.11)$$

Let $\psi_R(y) = \psi(\frac{y}{R})$ for $R > 1$; then $\eta \psi_R \in H^1_m(\mathbb{R}^N_+, y^{1-2s})$.

Then putting $\eta \psi_R$ in the weak formulation of (1.5) we have

$$\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s}[\nabla v_m \nabla (\eta \psi_R) + m^2 v_m \eta \psi_R]dxdy$$

$$+ (\mu - m^2)\int_{\mathbb{R}^N} v_m \eta dx = \int_{\mathbb{R}^N} |v_m|^{p-1}\eta dx.$$  \hspace{1cm} (6.12)

Taking the limit as $m \rightarrow 0$ and by using (6.8)-(6.10) we find

$$\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla v \nabla (\eta \psi_R) dxdy + \mu \int_{\mathbb{R}^N} v \eta dx = \int_{\mathbb{R}^N} |v|^{p-1}\eta dx.$$  \hspace{1cm} (6.13)

By passing to the limit as $R \rightarrow \infty$ we deduce that

$$\int\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla v \nabla \eta dxdy + \mu \int_{\mathbb{R}^N} v \eta dx = \int_{\mathbb{R}^N} |v|^{p-1}\eta dx.$$  \hspace{1cm} (6.14)
for any $\eta \in C^\infty_c(\mathbb{R}_+^{N+1})$ such that $\nabla \eta \in L^2(\mathbb{R}_+^{N+1}, y^{1-2s})$. Finally $v(\cdot, 0)$ is not identically zero because of (6.4), (6.10) and $2 < p < \frac{2N}{N-2s}$. Proceeding as in [23] we can show that $v$ is a positive radially symmetric function. □

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