Oriented expressions of graph properties.*

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Abstract

Several graph properties are characterized as the class of graphs that admit an orientation avoiding finitely many oriented structures. For instance, if \( F_k \) is the set of homomorphic images of the directed path on \( k + 1 \) vertices, then a graph is \( k \)-colourable if and only if it admits an orientation with no induced oriented graph in \( F_k \). There is a fundamental question underlying this kind of characterizations: given a graph property, \( \mathcal{P} \), is there a finite set of oriented graphs, \( \mathcal{F} \), such that a graph belongs to \( \mathcal{P} \) if and only if it admits an orientation with no induced oriented graph in \( \mathcal{F} \)? We address this question by exhibiting necessary conditions upon certain graph classes to admit such a characterization. Consequently, we exhibit an uncountable family of hereditary classes, for which no such finite set exists. In particular, the class of graphs with no holes of prime length belongs to this family.

Keywords: Forbidden subgraph characterization, generalized colouring, forbidden orientations

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1 Introduction

All graphs and digraphs considered in this work are loopless and without parallel edges or parallel arcs. For basic terminology and notation, we refer the reader to [1]. Moreover, for any concepts related to graph and digraph homomorphisms, we refer the reader to [10].

Given a pair of (oriented) graphs, \( G \) and \( H \), we will write \( H < G \) if \( H \) is an induced (oriented) subgraph of \( G \). A natural way to characterize hereditary properties is by finding a minimal set of forbidden induced subgraphs. Such a set is often called the set of minimal obstructions of the associated hereditary property. Most of the time these sets turn out to be infinite, hard to describe, or very difficult to find. Nonetheless, characterizing such properties through alternative forbidden structures has usually led to a finite set of forbidden

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structures. For instance, consider the class of chordal graphs, and denote by $B_1$ the oriented graph $\{(1, 2), (1, 3)\}$. Clearly, the family of chordal graphs is a hereditary property with an infinite set of minimal obstructions, but chordal graphs are characterized as those graphs that admit a $B_1$-free acyclic orientation [16], i.e. a graph $G$ is chordal if and only if it admits an acyclic orientation $G'$ such that $B_1$ is not an induced subdigraph of $G'$. Similarly, the Roy-Gallai-Hasse-Vitaver Theorem [6, 11, 15, 17] states that a graph is $k$-colourable if and only if it admits an orientation with no directed walk on $k + 1$ vertices. This work studies these kind of characterizations, that is, characterizations of hereditary properties by forbidding certain orientations.

For a set of oriented graphs, $F$, Skrien defined the class of $F$-graphs as those graphs that admit an $F$-free orientation [16]. We believe this definition might be misleading in the sense that the class of $F$-graphs is negatively defined with respect to $F$. We propose to invert this definition. Given a class of oriented graphs $\mathcal{O}$ we define the class of $\mathcal{O}$-graphs as the family of underlying graphs of $\mathcal{O}$. In other words, a graph $G$ is an $\mathcal{O}$-graph if and only if there is an orientation $G'$ of $G$ such that $G' \in \mathcal{O}$. For instance, in a basic graph theory course [1] the student learns that a graph $G$ is 2-edge-connected, if and only if it admits a strongly connected orientation. So if $\mathcal{O}$ is the class of strongly connected oriented graphs, then the class of $\mathcal{O}$-graphs is the family of 2-edge-connected graphs. We are interested in classes of $\mathcal{O}$-graphs where $\mathcal{O}$ is a hereditary property of (acyclic) oriented graphs with finitely many forbidden substructures.

Consider a pair of (oriented) graphs, $G$ and $H$, if $G$ is homomorphic to $H$, we will write $G \to H$, and we say that $G$ is $H$-colourable. A homomorphism class is a class of graphs defined as those graphs homomorphic to some fixed graph $H$. Given a set of oriented graphs, $F$, we denote by $\text{Forb}(F)$ the class of oriented graphs, $G$, such that $H \not\to G$ for every $H \in F$. For this work, an embedding is a homomorphism $\varphi : G \to H$ such that $G$ is isomorphic to its image, $\varphi[G]$. So $G$ embeds in $H$ if and only if $G < H$. We extend the previously introduced notation and denote by $\text{Forb}_c(F)$ the class of oriented graphs, $G$, such that $H \not< G$ for every $H \in F$. Note that the class of $\text{Forb}_c(F)$-graphs corresponds to the class of $F$-graphs in the sense of Skrien [16]. We denote by $\text{Forb}_c^e(F)$ the subclass of acyclic oriented graphs in $\text{Forb}_c(F)$. In any of these cases, if $F$ is the singleton $\{L\}$, we will simply write $\text{Forb}(L)$, $\text{Forb}_c(L)$ or $\text{Forb}_c^e(L)$. Going back to our previous examples, the class of $\text{Forb}(\overrightarrow{P}_{k+1})$-graphs is the class of $k$-colourable graphs, and a graph $G$ is a $\text{Forb}_c^e(B_1)$-graph if and only if $G$ is a chordal graph.

We say that a hereditary property, $\mathcal{P}$, is expressible by $\text{Forb}$-graphs, if there is a finite set of oriented graphs, $F$, such that $\mathcal{P}$ corresponds to the class of $\text{Forb}(F)$-graphs. In this case we say that $\text{Forb}(F)$ is an expression of $\mathcal{P}$. Now suppose that $\text{Forb}(F)$ is an expression of $\mathcal{P}$, and let $F'$ be the set of all homomorphic images of oriented graphs in $F$. Then, $\mathcal{P}$ equates the class of $\text{Forb}_c(F')$-graphs. So the expressive power of $\text{Forb}_c(F)$-graph classes is more robust than the expressive power of $\text{Forb}(F)$-graph classes. We say that $\mathcal{P}$ is expressible by forbidden orientations if there is a finite set, $F$, of oriented graphs such that $\mathcal{P}$ coincides with the class of $\text{Forb}_c(F)$-graphs. With a simple cardinality argument one can notice that not every hereditary property is expressible by forbidden orientations; there are countably many
finite sets of oriented graphs, while there is an uncountable class of hereditary properties. This simple observation raises the fundamental question from which this paper stems.

**Question 1.** Which hereditary properties are expressible by forbidden orientations?

It is very likely that Skrien did not have this question in mind when working on [16]. Nonetheless, his work provides the first partial answer to this question: he provides a list of graph classes that coincide to a class of $Forb_e(F)$-graphs when $F$ is a set of orientations of $P_3$. Recently, we extended his work by considering all sets of oriented graphs on three vertices. The aforementioned manuscripts illustrate one way to tackle Question 1: fix a finite set of oriented graphs, $F$, and characterize the class of $Forb_e(F)$-graphs. A second way to tackle this question is by fixing a hereditary property, $\mathcal{P}$, and then (be lucky enough to) find a finite set of oriented graphs, $F$, such that $\mathcal{P}$ corresponds to the class of $Forb_e(F)$-graphs. We believe that the Roy-Gallai-Hasse-Vitaver Theorem can be considered the first result that aligns with this approach. We certainly did so in [8] where we showed that for every odd cycle, $C$, there is an oriented path, $P_C$, such that a graph $G$ is homomorphic to $C$ if and only if $G$ is a $Forb(P_C)$-graph. Following the perspective behind the previously mentioned results, in Section 4 we propose a characterization of those graphs, $H$, for which the class of $H$-colourable graphs is expressible by $Forb$-orientations. A third proceeding towards solving Question 1 is exhibiting hereditary properties that are not expressible by forbidden orientations. As far as we are concerned, this is the first write up to follow this path. Moreover, we believe that the main contribution of this work is proposing strong necessary condition upon certain graph classes to be expressible by forbidden orientations. As a consequence of this result, we obtain an uncountable family of hereditary properties that are not expressible by forbidden orientations.

As we will see, it turns out that some natural properties are not expressible by forbidden orientations, but they do correspond to some class of $Forb^*_e(F)$-graphs for some finite set $F$. If such a finite set, $F$, exists for some property, $\mathcal{P}$, we will say that $\mathcal{P}$ is expressible by forbidden acyclic orientations. Some readers might consider this to be a devious tactic to obtain a finite expression of the corresponding graphs class, as we are forbidding an infinite set of oriented graphs beforehand (all directed cycles). Rather than dwelling on the validity of such technique, we will notice that most of the results regarding forbidden orientations can be adapted to forbidden acyclic orientations.

The structure of this work is as follows. In Section 2, we introduce some concepts and prove a couple of results of language theory that we will use in Section 3. Consequently, in Section 4 we present our main results; we propose some necessary conditions for a hereditary property to be expressible by forbidden (acyclic) orientations, and we exhibit an uncountable family of hereditary properties that are not expressible by forbidden (acyclic) orientations. In Section 4 we propose a characterization of those homomorphism classes expressible by $Forb$-graphs. Finally, in Section 5 we present conclusions and some problems that we think would be an interesting follow up in this subject.
2 Languages

Consider a finite set \( \mathcal{A} \), which we will call an alphabet. A word over \( \mathcal{A} \) is a finite sequence of elements in \( \mathcal{A} \). Whenever there is no ambiguity on the alphabet, we will call a word over \( \mathcal{A} \) only a word. The length of a word \( a \) is the number of elements in the sequence, and we denote it by \( |a| \). If \( |a| = k \), we say that \( a \) is a \( k \)-word. We denote by \( \mathcal{A}_k \) the set of \( k \)-words over \( \mathcal{A} \), and by \( \mathcal{A}^* \) the union of \( \{\mathcal{A}_k\}_{k \geq 0} \), where the only word in \( \mathcal{A}_0 \) is denoted by \( \epsilon \) and it is called the empty word. A language over \( \mathcal{A} \) is a subset \( \mathcal{L} \) of \( \mathcal{A}^* \). Consider a pair words, \( a \) and \( b \), \( a = a_1 a_2 \ldots a_k \) and \( b = b_1 b_2 \ldots b_l \), we denote by \( ab \) the word \( a_1 a_2 \ldots a_k b_1 b_2 \ldots b_l \). For \( n \geq 1 \), we denote by \( a^n \) the \( n \)th-power of a word \( a \), and it is defined recursively; \( a^0 = \epsilon \) and \( a^n = a^{n-1} a \). We say that a word \( a \in \mathcal{L} \) is \( k \)-periodic in \( \mathcal{L} \), if \( a^n \in \mathcal{L} \) for every \( n \geq 1 \), and \( |a| = k \). We say that \( a \) is a constant word if all the symbols in \( a \) are the same.

We say that a word \( a \) is a factor of a word \( b \) if there are two (possibly empty) words \( c \) and \( d \) such that \( cad = b \); in this case we write \( a < b \). It is not hard to notice that the relation induced by factors is a partial order in \( \mathcal{A}^* \). Let \( \mathcal{A} \) be a set of words, we say that \( b \) is \( \mathcal{A} \)-free if \( b \) contains no factor in \( \mathcal{A} \). We denote the language of \( \mathcal{A} \)-free words by \( \mathcal{L}_\mathcal{A} \). A language is hereditary if it is closed under factors. Note that for any set of words \( \mathcal{A} \) the language of \( \mathcal{A} \)-free words is hereditary. Let \( m \) be a positive integer, we say that a language \( \mathcal{L} \) is \( m \)-synchronizing, if for any choice of words\( a,b,d \) such that \( ba, ad \in \mathcal{L} \), if \( |a| = m \), then \( bad \in \mathcal{L} \). Clearly, \( \mathcal{L} \) is \( m \)-synchronizing if and only if for any pair of words \( ba, ad \in \mathcal{L} \), such that \( |a| \geq m \) then \( bad \in \mathcal{L} \).

**Observation 2.** Let \( \mathcal{A} \) be a finite set of words over any alphabet, and let \( m \) a positive integer, such that for every \( a \in \mathcal{A} \), \( |a| \leq m \), then \( \mathcal{L}_\mathcal{A} \) is \( m \)-synchronizing.

We can naturally translate words over the alphabet \( \{\leftarrow, \rightarrow\} \) to oriented paths. Denote by \( W_k \) the set of oriented paths on \( k \) edges, and by \( W^* \) the union of \( \{W_k\}_{k \geq 0} \). Consider the surjective function \( t : \{\leftarrow, \rightarrow\}^* \rightarrow W^* \), \( t(a_1 a_2 \ldots a_k) = P \), where \( P = v_1 v_2 \ldots v_{k+1} \) such that \( v_i \rightarrow v_{i+1} \) if \( a_i = \leftarrow \), and \( v_{i+1} \rightarrow v_i \) otherwise. Clearly \( t \) is a monotone function, i.e., for any \( \{\leftarrow, \rightarrow\} \)-words, \( a \) and \( b \), if \( a < b \) then \( t(a) < t(b) \). Moreover, for any oriented paths, \( P \) and \( Q \), if \( P < Q \), then there are \( \{\leftarrow, \rightarrow\} \)-words, \( p \) and \( q \), such that \( p < q \), \( t(p) = P \), and \( t(q) = Q \). The first item of the following lemma follows from the three previous observations.

**Lemma 3.** Let \( F \) be a finite set of connected oriented graphs. Then, there is a set \( \mathcal{A} \) of words over \( \{\leftarrow, \rightarrow\} \) and a positive integer \( m \), such that for any positive integer \( k \) the following statements hold:

1. if \( k \geq m \), the path on \( k \) edges admits an \( F \)-free orientation, if and only if there is a \( k \)-word in \( \mathcal{L}_\mathcal{A} \),
2. if \( k \geq \max\{m,4\} \), the cycle on \( k \) edges admits an \( F \)-free orientation, if and only if there is a \( k \)-periodic word in \( \mathcal{L}_\mathcal{A} \), and
3. if \( k \geq \max\{m,4\} \), the cycle on \( k \) edges admits an \( F \)-free acyclic orientation, if and only if there is a non-constant \( k \)-periodic word in \( \mathcal{L}_\mathcal{A} \).
Proof. Let \( m \) be the integer \( \max\{|V(G)| : G \in F\} + 1 \), and let \( A \) be the set \( \{a \in \{\leftarrow, \rightarrow\}^*: t(a) \in F\} \). The first statement follows directly using the observations preceding this lemma. Now we prove the third statement. Let \( k \) be an integer, \( k \geq \max\{m, 4\} \), and let \( C = (c_1, c_2, \ldots, c_k, c_1) \) be non-directed oriented cycle. First note that, since \( k \geq \max\{m, 4\} \), and by the choice of \( m \), if there is a graph \( P \in F \) such that \( P < C \), then \( P \) is an oriented path. Consider the \( k \)-word \( l = l_1 \ldots l_k \), where for every \( i \in \{1, \ldots, k-1\} \), \( l_i = \rightarrow \) if \( c_i \rightarrow c_{i+1} \), and \( l_k = \leftarrow \) otherwise, and \( l_k = \rightarrow \) if \( c_k \rightarrow c_0 \) and \( l_k = \leftarrow \) otherwise. Since \( C \) is not a directed cycle, then \( l \) is not a constant word. Moreover, it is not hard to notice that there is a path \( P \in F \) such that \( P < C \) if and only if there is word \( a \in A \) such that \( a < l^2 \). Similarly, if \( h \) is a non-constant \( k \)-word such that \( h^2 \in \mathcal{L}_A \) we can find an \( F \)-free acyclic orientation of the cycle on \( k \)-edges. Thus, the \( k \)-cycle \( C_k \) admits an \( F \)-free acyclic orientation, if and only if there is a non-constant \( k \)-word, \( l \), such that \( l^2 \in \mathcal{L}_A \). Now, note that if there is a non-constant \( k \)-periodic word in \( \mathcal{L}_A \), in particular there is a non-constant \( k \)-word \( l \) such that \( l^2 \in \mathcal{L}_A \). The converse implication also holds since every word in \( A \) is bounded by \( m \), and \( k \geq m \), so by Observation \( \ref{lem:periodic} \) if \( l^2 \in \mathcal{L}_A \) then \( l^3 \in \mathcal{L}_A \), so inductively we show that a word \( l \) is \( k \)-periodic in \( \mathcal{L}_A \) if and only if \( l^2 \in \mathcal{L}_A \). Thus, the third statement holds, and the second one follows an analogous proof. \( \square \)

We define the set of (non-constant) periods of \( \mathcal{L} \) as the positive integers, \( k \), such that there is a (non-constant) \( k \)-periodic word in \( \mathcal{L} \). We denote these sets by \( \text{per}(\mathcal{L}) \) and \( \text{per}^*(\mathcal{L}) \) respectively. Lemma \( \ref{lem:periodic} \) (\( \ref{lem:periodic compliment} \)) shows that if a property \( \mathcal{P} \) is expressible by forbidden (acyclic) orientations then, there is a set of words \( A \) such that for any large enough integer, \( k \), the \( k \)-cycle belongs to \( \mathcal{P} \), if and only if \( k \in \text{per}(\mathcal{L}_A) \) (\( k \in \text{per}^*(\mathcal{L}_A) \)). For a set of oriented graphs \( F \) we denote by \( A_F \) the set of words \( \{a \in \{\leftarrow, \rightarrow\}^*: t(a) \in F\} \). For instance, let \( F = \{TT_3, \rightarrow_3, \leftarrow_3\} \) (note that the class of \( \text{Forb}_v(F) \)-graphs is the class of bipartite graphs due to the Roy-Gallai-Hasse-Vitaver Theorem). In this case, \( A_F = \{\leftarrow\leftarrow\rightarrow\rightarrow\} \), and thus the binary language of \( A_F \)-free words corresponds to those words such that no two consecutive letters are the same.\(^1\)

\begin{lemma}
\label{lem:forbidding}
Let \( F \) be a finite set of connected oriented graphs, and \( m \) the maximum order of a graph in \( F \). Then, the following statements are equivalent,
\begin{itemize}
  \item there is a positive integer \( k \), \( k \geq \max\{4, m\} \), such that the \( k \)-cycle admits an acyclic \( F \)-free orientation,
  \item there is a positive integer \( k \), \( k \geq \max\{4, m\} \), such that for every multiple of \( k \), \( r \), the \( r \)-cycle admits an acyclic \( F \)-free orientation, and
  \item there is a infinite set of cycles that admit an acyclic \( F \)-free orientation.
\end{itemize}
\end{lemma}

\textbf{Proof.} Clearly the first item is a particular case of the third one, while the latter is an implication of the second one. We now prove the first item implies the second one. Let

\(^1\)Notice that if \( F \) is a finite set, then \( A_F \) is a finite set as well, and so the binary language corresponding to the \( F \)-free orientations of paths is a regular language.
A = A_F$, and suppose that a cycle on $k$ edges, $k \geq \max\{4, m\}$, admits an $F$-free acyclic orientation. By Lemma 3.3, there is a non-constant $k$-periodic word $p \in \mathcal{L}_A$. Thus, for every $n \geq 1$, $p^n \in \mathcal{L}_A$, and since, for every positive integer, $l$, the equality $(p^l)^n = p^{ln}$ holds, then $p^l$ is periodic in $\mathcal{L}_A$. Clearly $p^l$ is not a constant word, and $|p^l| = kl$. So by Lemma 3.3, for every multiple of $k$, $r = kl$, the cycle on $kl$-edges admits an $F$-free acyclic orientation.

The equivalent statement of Lemma 4 can be translated (with the same proof) to $F$-free (not necessarily acyclic) orientations of cycles. Moreover, there are two more equivalent statements when we do not restrict ourselves to acyclic orientations.

**Lemma 5.** Let $F$ be a finite set of oriented graphs, and $m$ the maximum order of a graph in $F$. Then, the following statements are equivalent,

- there is a positive integer $k$, $k \geq \max\{4, m\}$, such that the $k$-cycle admits an $F$-free orientation,
- there is a positive integer $k$, $k \geq \max\{4, m\}$, such that for every multiple of $k$, $r$, the $r$-cycle admits an $F$-free orientation,
- there is an infinite set of cycles that admit an $F$-free orientation,
- every path admits an $F$-free orientation, and
- $\mathcal{L}_A$ is infinite, where $A = A_F$.

**Proof.** To prove the equivalence between the first three items one can follow an analogous proof to Lemma 4. The final two statements are equivalent due to Lemma 3.1. To show that the third item implies the fourth one, it suffices to notice that if a graph $G$ admits an $F$-free orientation and $H < G$, $H$, admits an $F$-free orientation. Since every path can be embedded in any sufficiently large cycle, we conclude that the third item implies the fourth one. Finally, we show that the last statement implies the first one. So we assume there are arbitrarily large words in $\mathcal{L}_A$. Since there is only a finite amount of $m$-words, by taking a large enough word $w \in \mathcal{L}_A$, we can find a factor of $w$ of the form $aba$, where $|a| \geq m$ and $b$ is possibly an empty word. Thus, by Observation 2 $ababa \in \mathcal{L}_A$, so $abab \in \mathcal{L}_A$. By recursively using Observation 2 we prove that for any $n \geq 1$, $(ab)^n \in \mathcal{L}_A$. Thus $(ab)^4$ is a periodic word such that $|(ab)^4| \geq 4m \geq \max\{4, m\}$, and so by Lemma 3.2, there is an $F$-free orientation of a cycle on at least $\max\{4, m\}$ edges.

Lemmas 4 and 5 are the first results that yield necessary conditions for a hereditary property to be expressible by forbidden orientations, and forbidden acyclic orientations, respectively.

The following statement is a basic arithmetic result. The reader could prove it as an exercise to not forget our basic courses of algebra and number theory, or can refer to Appendix D [2] for a proof. For a set of positive integers $A$, we denote the greatest common divisor of $A$ by $gcd(A)$. An integer, $l$, is a positive combination of a set of numbers $\{a_1, \ldots, a_k\}$, if $l = m_1a_1 + \cdots + m_ka_k$ where $m_i$ is a positive integer for every $i \in \{1, \ldots, k\}$.
Lemma 6. For any infinite set of positive integers, $A$, with greatest common divisor $r$, there is a finite subset $B \subseteq A$ such that $\gcd(B) = r$. Moreover, if $A$ is closed under addition, then $A$ is cofinite in $r\mathbb{Z}^+$, i.e., the complement of $A$ in $r\mathbb{Z}^+$ is finite.

These are all the results we need to proceed to Section 3. The three remaining results of this section build up to a language theoretic result that is within reach now.

We say that a set of positive integers $A$ satisfies the weak addition property, if there are: a finite subset $B \subseteq A$, $B = \{b_1, \ldots, b_k\}$, such that $\gcd(B) = \gcd(A)$, and a multiple of $\gcd(A)$, $l$, such that for every positive combination of elements in $B$, $c = m_1b_1 + \cdots + m_kb_k$, the integer $l + c$ belongs to $A$.

Lemma 7. Let $A$ be a set of positive integers with greatest common divisor $r$. If $A$ satisfies the weak addition property, then $A$ is cofinite in $r\mathbb{Z}^+$.

Proof. Let $B \subseteq A$ be a finite subset such that $\gcd(B) = r$, let $S$ be the set of positive combinations of elements in $B$, and let $l$ be a multiple of $r$ such that for every $s \in S$, $l + s \in A$. Clearly, $S$ is closed under addition and it is also not hard to notice that $\gcd(S) = \gcd(B) = r$. Thus, by Lemma 6, $S$ is cofinite in $r\mathbb{Z}^+$, and since $l$ is a multiple of $r$, then $l + S = \{l + s : s \in S\}$ is cofinite in $r\mathbb{Z}^+$. Recall that, by the choice of $A$ and $l$, $l + S \subseteq A$, so $A$ is also cofinite in $r\mathbb{Z}^+$. \hfill $\square$

We say that a language $\mathcal{L}$ is transitive if for every two words $a, b \in \mathcal{L}$, there is a third (possibly empty) word $d$, such that $adb \in \mathcal{L}$. As a temporary and convenient definition, we say that the greatest common divisor of an empty set is 0.

Lemma 8. Let $m$ be a positive integer, let $\mathcal{L}$ be a hereditary, transitive, $m$-synchronizing language, and let $r = \gcd(\per(\mathcal{L}))$. Then, $\per(\mathcal{L})$ is a cofinite subset of $r\mathbb{Z}^+$.

Proof. The case when $\per(\mathcal{L})$ is empty is clear. So, we assume that $\per(\mathcal{L})$ is not empty. We will show that $\per(\mathcal{L})$ satisfies the weak addition property, and thus conclude by Lemma 6. By Lemma 6, we can choose a finite set $A = \{a_1, \ldots, a_k\} \subseteq \per(\mathcal{L})$, such that $\gcd(A) = \gcd(\per(\mathcal{L}))$. We can assume that $\min\{a_1, \ldots, a_k\} \geq m$. If this was not the case, let $p_1, \ldots, p_k$ be distinct primes greater than $\max\{a_1, \ldots, a_k\}$, such that $p_ia_i \geq m$ for every $i \in \{1, \ldots, k\}$. Clearly, $\gcd(p_1a_1, \ldots, p_ka_k) = \gcd(A)$, and since for every $1 \leq i \leq k$, there is a periodic word $\alpha_i$ such that $|\alpha_i| = a_i$, then $\alpha_i^n$ is a periodic word of length $a_ip_i$, i.e., $a_ip_i \in \per(\mathcal{L})$. Thus, without loss of generality we will assume that $\min\{a_1, \ldots, a_k\} \geq m$. Now, let us observe that there is a positive integer $l$ such that for any positive combination of elements of $A$, $c = m_1a_1 + \cdots + m_ka_k$, there is a $(c + l)$-periodic word in $\mathcal{L}$. For every $i \in \{1, \ldots, k\}$ let $\alpha_i$ by an $a_i$-periodic word in $\mathcal{L}$. Since $\mathcal{L}$ is a transitive language, for any $\alpha_i$ with $1 \leq i < k$ there is a word $\beta_i$ such that $\alpha_i\beta_i\alpha_{i+1} \in \mathcal{L}$, and there is a word $\beta_k$ such that $\alpha_k\beta_k\alpha_1 \in \mathcal{L}$. Let $m_1a_1 + \cdots + m_ka_k$ be a positive combination of $A$, that is $m_i \geq 1$. Recall that $\mathcal{L}$ is $m$-synchronizing, and since $\min\{a_1, \ldots, a_k\} \geq m$, then $\alpha_1^{m_1}\beta_1\alpha_2^{m_2}\beta_2\cdots\alpha_k^{m_k}\beta_k\alpha_1 \in \mathcal{L}$. Let $\gamma = \alpha_1^{m_1-1}\beta_1\alpha_2^{m_2}\beta_2\cdots\alpha_k^{m_k}\beta_k$, we proceed to show that $\alpha_1\gamma$ is periodic in $\mathcal{L}$. Since $|\alpha_1| \geq m$ and $\alpha_1\gamma\alpha_1 = \alpha_1^{m_1}\beta_1\alpha_2^{m_2}\beta_2\cdots\alpha_k^{m_k}\beta_k\alpha_1 \in \mathcal{L}$, then $\alpha_1\gamma\alpha_1\gamma \in \mathcal{L}$. Recall that $\mathcal{L}$ is hereditary, so $\alpha_1\gamma\alpha_1\gamma \in \mathcal{L}$, hence, we inductively conclude that $(\alpha_1\gamma)^n \in \mathcal{L}$. So
\(|\alpha_1\gamma| \in \text{per}(L)\). By construction of \(\gamma\), \(|\alpha_1\gamma| = m_1a_1 + \cdots + m_ka_k + l\), where \(m_1a_1 + \cdots + m_ka_k\) is any positive combination of \(A\), and \(l = |\beta_1| + \cdots + |\beta_k|\). Let \(S\) be the set of positive combinations of \(A\). We have shown that \(l + S = \{l + s : s \in S\} \subseteq \text{per}(L)\). Thus, \(\text{per}(L)\) satisfies the weak addition property, so by Lemma 7, \(\text{per}(L)\) is a cofinite subset of \(r\mathbb{Z}^+\).

**Theorem 9.** Let \(A\) be a finite set of words over any alphabet. If \(L_A\) is a transitive language, then there is a positive integer, \(r\), such that \(\text{per}(L_A)\) is a cofinite subset of \(r\mathbb{Z}^+\).

**Proof.** As noted before, \(L_A\) is a hereditary language. Since \(A\) is finite, by Observation 2, \(L_A\) is \(m\)-synchronizing for some \(m \geq 1\). Thus we conclude using Lemma 8.

3 Expressions by forbidden (acyclic) orientations

A graph (digraph) homomorphism, \(\varphi : H \to G\), is an overlap if and only if \(\varphi\) restricted to each connected component of \(H\) is an embedding. In this case we say that \(G\) contains an overlap of \(H\). Let \(\mathcal{F}\) be a set of graphs, we say that a graph \(G\) is \(\mathcal{F}\)-overlap free if for every \(H \in \mathcal{F}\), \(G\) does not contain an overlap of \(H\). Note that if \(\mathcal{F}\) consists of connected graphs, then \(G\) is \(\mathcal{F}\)-free if and only if \(G\) is \(\mathcal{F}\)-overlap free. Let \(F\) be a set of oriented graphs, we say that a graph \(G\) admits an \(F\)-overlap free orientation if there is an orientation \(G'\) such that \(G'\) is \(F\)-overlap free as a digraph. A property is called additive if it is closed under disjoint unions. We denote the disjoint union of a pair of graphs, \(G\) and \(H\), by \(G + H\).

**Lemma 10.** Let \(\mathcal{P}\) be a hereditary graph property with set of minimal obstructions \(\mathcal{F}_\mathcal{P}\) and let \(F\) be a set of oriented graphs. If \(\mathcal{P}\) is additive, then the following hold:

1. \(\mathcal{F}_\mathcal{P}\) consists of connected graphs,
2. a graph \(G\) is \(\mathcal{F}_\mathcal{P}\)-free if and only if it is \(\mathcal{F}_\mathcal{P}\)-overlap-free,
3. if \(\text{Forb}_e(F)\) is an expression of \(\mathcal{P}\), then a graph \(G\) admits an \(F\)-free orientation if and only if \(G\) admits an \(F\)-overlap free orientation, and
4. if \(\text{Forb}_a^+(F)\) is an expression of \(\mathcal{P}\), then a graph \(G\) admits an acyclic \(F\)-free orientation if and only if \(G\) admits an acyclic \(F\)-overlap free orientation.

**Proof.** We will prove the first statement by contrapositive. Assume that there is a disconnected graph \(H = H_1 + H_2 \in \mathcal{F}_\mathcal{P}\), then \(H_1, H_2 \in \mathcal{P}\) but \(H_1 + H_2 \notin \mathcal{P}\), so \(\mathcal{P}\) is not additive. Hence, \(\mathcal{F}_\mathcal{P}\) consists of connected graphs. The second statement is a straightforward implication of the first one.

We will prove the last two statements at once. Clearly, if a graph \(G\) admits an (acyclic) \(F\)-overlap free orientation, then it admits an (acyclic) \(F\)-free orientation. We will prove the remaining implication by contrapositive, assuming the negation of 3 (4) to reach that \(\mathcal{P}\) is not additive. So, suppose that \(\text{Forb}_e(F)\) (\(\text{Forb}_a^+(F)\)) is an expression of \(\mathcal{P}\) and there is a graph \(G\) that admits an (acyclic) \(F\)-free orientation, but not an (acyclic) \(F\)-overlap free orientation. Clearly \(G \in \mathcal{P}\). Let \(F_G = \{L \in F : |V(L)| \leq |V(G)|\}\), let \(k = |F_G|\), and let \(l = \max\{n \in \mathbb{N} : \text{there is an oriented graph } L \in F_G \text{ with } n \text{ connected components}\}\). Consider the disjoint union of \(G\) with itself \(lk\) times, \(H = \sum_{i=1}^{lk} G\), and any (acyclic) orientation \(H'\) of \(H\). Naturally, every connected component of \(H'\) is an (acyclic) orientation of \(G\). Since \(G\)
does not admit an (acyclic) $F$-overlap free orientation, every connected component of $H'$ is not $F_G$-overlap free. By thinking of the elements in $F_G$ as pigeonholes, and of each connected component of $H'$ as a pigeon, there must be an element $L \in F_G$ that occurs as an overlap in $l$ (acyclic) oriented copies of $G$ in $H'$. Since $L$ has at most $l$ connected components, then $L < H'$. Thus, no (acyclic) orientation of $H$ is $F$-free, hence $H \not\in \mathcal{P}$, and so $\mathcal{P}$ is not closed under disjoint unions.

The following lemma strengthens the last two items of Lemma 10.

**Lemma 11.** Let $\mathcal{P}$ be an additive and hereditary property, and let $F$ be a set of oriented graphs. The following statements hold:

- if $\text{Forb}_e(F)$ is an expression of $\mathcal{P}$, then there is a set of connected oriented graphs $F_1$, such that $\text{Forb}_e(F_1)$ is an expression of $\mathcal{P}$, and $|F_1| \leq |F|$, and

- if $\text{Forb}^*_e(F)$ is an expression of $\mathcal{P}$, then there is a set of connected oriented graphs $F_1$, such that $\text{Forb}^*_e(F_1)$ is an expression of $\mathcal{P}$, and $|F_1| \leq |F|$.

**Proof.** If $F$ is an infinite set, we choose $F_1$ to be the set of all (acyclic) orientations of the minimal obstructions of $\mathcal{P}$. The fact that every oriented graph in $F_1$ is connected follows from Lemma 10.1. Clearly, $|F_1| \leq |F|$, and $\text{Forb}_e(F_1)$ ($\text{Forb}^*_e(F_1)$) is an expression of $\mathcal{P}$. Now suppose that $F$ is finite, and let $f$ be the number of disconnected oriented graphs in $F$. If $f = 0$ there is nothing to prove. We will show that if $f > 0$, then there is a set, $F'$, of (acyclic) oriented graphs, such that $|F'| \leq |F|$, the number of disconnected oriented graphs in $F'$ is $f - 1$, and $\text{Forb}(F')$ ($\text{Forb}^*(F')$) is an expression of $\mathcal{P}$. Thus, the proof will follow inductively.

By Lemma 10.3 (10.4) we can think of $\mathcal{P}$ as the class of graphs that admit an $F$-overlap free (acyclic) orientation. For any positive integer $n$, denote by $G_n$ the disjoint union of graphs in $\mathcal{P}$ on at most $n$ vertices. The following three facts are not hard to verify: first $G_n \in \mathcal{P}$, also for any graph $G \in \mathcal{P}$ there is a positive integer, $n$, such that $G < G_n$, and finally $G_n < G_{n+1}$. From the first fact it follows that, for every positive integer, $n$, we can choose an $F$-overlap free (acyclic) orientation $G'_n$ of $G_n$. Let $H$ be a disconnected (acyclic) oriented graph in $F$. It is not hard to notice that an oriented graph $D$ is $F$-overlap free if and only if $D$ is $(F \setminus \{H\})$-overlap free and $D$ is $h$-free for some connected component, $h$, of $H$. Thus, there must be a connected component, $h < H$, such that an infinite subset of the (acyclic) orientations $\{G'_n\}_{n \geq 1}$ are $h$-free and $(F \setminus \{H\})$-overlap free. Let $\{n_k\}_{k \geq 1}$ be the infinite sequence of positive integers, such that the (acyclic) orientations $\{G'_{n_k}\}_{k \geq 1}$ are $h$-free and $(F \setminus \{H\})$-overlap free. Since any $(F \setminus \{H\})$-overlap free oriented graph is $(F \setminus \{H\})$-free, then all orientations $\{G'_{n_k}\}_{k \geq 1}$ are $(F \setminus \{H\} \cup \{h\})$-free. Denote by $F'$ the set $F \setminus \{H\} \cup \{h\}$. Since for any graph, $G \in \mathcal{P}$, there is an integer $m$ such that $G < G_{n_m}$, then we can obtain an $F'$-free (acyclic) orientation of $G$; anyone induced by $G'_{n_m}$. Therefore the class of $\text{Forb}_e(F')$-graphs ($\text{Forb}^*_e(F')$-graphs) contains $\mathcal{P}$. On the other hand, if a graph $G$ admits an $F'$-free (acyclic) orientation, then this (acyclic) orientation is $F$-free, so $G \in \mathcal{P}$. Therefore, $\text{Forb}_e(F')$ ($\text{Forb}^*_e(F')$) is an expression of $\mathcal{P}$. Clearly $|F'| \leq |F|$ and $F'$ has $f - 1$ disconnected graphs. As we previously mentioned, both claims follow inductively. \qed
For a hereditary property $\mathcal{P}$ we denote by $\text{cyc}(\mathcal{P})$ the set of lengths of cycles that belong to $\mathcal{P}$. We are now ready to state some necessary conditions for an additive hereditary property to be expressible by forbidden (acyclic) orientations.

**Lemma 12.** Let $\mathcal{P}$ be an additive hereditary property expressible by forbidden (acyclic) orientations. If $\text{cyc}(\mathcal{P})$ is an infinite set, then there is a positive integer $M$ such that for every integer $k$, $k \geq M$, the $k$-cycle belongs to $\mathcal{P}$, if and only if for every positive multiple, $l$, of $k$ the $l$-cycle belongs to $\mathcal{P}$.

**Proof.** Let $F$ be a finite set of oriented graphs such that $\mathcal{F} = \mathcal{P}$ is an expression of $\mathcal{P}$. By Lemma 11 we can assume that $F$ consists of connected oriented graphs. Let $m$ be the maximum order of an oriented graph in $F$, and let $M = \max\{4, m\}$. Since $\text{cyc}(\mathcal{P})$ is an infinite set, we conclude by the equivalence of the first two items of Lemma 4. The case when $\mathcal{F} = \mathcal{P}$ is an expression of $\mathcal{P}$ follows the same proof but using the equivalence of the items of Lemma 5.

Furthermore, we can add one more hypothesis and show that for any additive hereditary property $\mathcal{P}$ expressible by forbidden orientations, there must be arbitrarily large cycles in $\mathcal{P}$.

**Proposition 13.** Let $\mathcal{P}$ be an additive hereditary property expressible by forbidden orientations. If every path belongs to $\mathcal{P}$, then $\text{cyc}(\mathcal{P})$ is an infinite set. In particular, the classes of chordal graphs and forests are not expressible by forbidden orientations.

**Proof.** Assume that there is a finite set of oriented graphs $F$, such that $\mathcal{F} = \mathcal{P}$ is an expression of $\mathcal{P}$. Since $\mathcal{P}$ is additive, by Lemma 11 we can assume that every oriented graph in $F$ is connected. Since $F$ is finite, by Lemma 5 there is an infinite set of cycles that admit an $F$-free orientation.

Proposition 13 cannot be extended to properties expressible by forbidden acyclic orientations. The intuition behind this fact is that, depending on the set of oriented graphs $F$, a directed path could be an $F$-free acyclic orientation of a path, while a directed cycle is clearly not an $F$-free acyclic orientation of a cycle (regardless of the set of oriented graphs). For instance, chordal graphs and forests satisfy that every path belongs to these properties, but only a finite amount of cycles do as well, and both classes are expressible by forbidden acyclic orientations [16]. This previous observation together with Proposition 13 show that there are some properties expressible by forbidden acyclic orientations that are not expressible by forbidden orientations. Now we show that there are some natural graph classes that are not expressible by forbidden orientations nor by forbidden acyclic orientations. To do so, recall that given a graph, $G$, a hole in $G$ is an induced cycle in $G$ of length greater than three.

**Proposition 14.** For an integer $k$, $k \geq 2$, let $\mathcal{P}_k$ be the class of graphs defined as those graphs with no holes of length a multiple of $k$. Then, $\mathcal{P}_k$ is not expressible by forbidden (acyclic) orientations. In particular, the class of even-hole free graphs is not expressible by forbidden (acyclic) orientations.
Proof. It suffices to notice that $\mathcal{P}_k$ is an additive hereditary property such that $\text{cyc} (\mathcal{P})$ is an infinite set. By the choice of $\mathcal{P}_k$, we can choose an arbitrarily large integer $m$ such that the $m$-cycle belongs to $\mathcal{P}_k$, but the $km$-cycle does not belong to $\mathcal{P}_k$. So by Lemma 12, $\mathcal{P}_k$ is not expressible by forbidden (acyclic) orientations.

We proceed to strengthen the necessary conditions proposed in Lemma 12. To do this, we introduce a technical but not so rare property of graph classes. Consider a pair of cycles $C$ and $C'$, we denote by $CC'$ the graph obtained by taking the disjoint union of $C$ and $C'$, and then identifying a vertex in $C$ with a vertex in $C'$. We call the graph $CC'$ the coupling of $C$ and $C'$. We say that a property is closed under couplings, if for every pair of cycles, $C, C' \in \mathcal{P}$, the coupling $CC'$ belongs to $\mathcal{P}$. Finally, given a set of integers $B$ and an integer $m$, we denote by $B_m$ the integers in $B$ greater than or equal to $m$, and recall that given a set of oriented graphs $F$ we denote by $A_F$ the set of words $\{ a \in \{\rightarrow, \leftarrow\}^* : t(a) \in F\}$.

**Lemma 15.** Let $\mathcal{P}$ be an additive hereditary property expressible by forbidden (acyclic) orientations. If $\text{cyc} (\mathcal{P})$ if an infinite set and $\mathcal{P}$ is closed under couplings, then there is pair of integers, $M$ and $r$, such that $\text{cyc} (\mathcal{P})_M \subseteq r\mathbb{Z}^+$, and $\text{cyc} (\mathcal{P})_M$ is cofinite in $r\mathbb{Z}^+$.

**Proof.** Let $F$ be a finite set of connected oriented graphs such that $\text{Forb}_k (F)$ is an expression of $\mathcal{P}$, let $m$ be the maximum order of an oriented graph in $F$, and let $A = A_F$. By Lemma 4.2, the positive integer $M$, $M = \max\{4, m + 1\}$, satisfies that if $k \geq M$, then $k \in \text{cyc} (\mathcal{P})$ if and only if $k \in \text{per} (\mathcal{L}_A)$. Thus, $\text{cyc} (\mathcal{P})_M = \text{per} (\mathcal{L}_A)_M$, so for any $r, s \in \text{per} (\mathcal{L}_A)_M$, the cycles $C_r$ and $C_s$ belong to $\mathcal{P}$. Since $\mathcal{P}$ is closed under couplings then there is an $F$-free orientation, $C_rC_s'$, of the coupling $C_rC_s$. From this orientation, and from the fact that $r, s \geq M > m$, it is not hard to obtain a periodic word in $\mathcal{L}_A$ of length $r + s$. Indeed, by traversing $C_rC_s'$ starting by the unique vertex that belongs to both cycles, then traversing $C_r$ and then $C_s$, we obtain an $(s + r)$-periodic word in $\mathcal{L}_A$. Hence, $\text{per} (\mathcal{L}_A)_M$ is closed under addition. Let $r = \gcd (\text{per} (\mathcal{L}_A)_M)$, so by Lemma 6 the set $\text{per} (\mathcal{L}_A)_M$ is cofinite in $r\mathbb{Z}$, and since $\text{cyc} (\mathcal{P})_M = \text{per} (\mathcal{L}_A)_M$, then $\text{cyc} (\mathcal{P})_M \subseteq r\mathbb{Z}^+$, and $\text{cyc} (\mathcal{P})_M$ is cofinite in $r\mathbb{Z}^+$.

The remaining case, when $\mathcal{P}$ is expressible by forbidden acyclic orientations, follows an analogous proof.

A natural way of defining a graph class is by forbidding a set of holes.

**Theorem 16.** Let $\mathcal{C}$ be a set of positive integers and let $\mathcal{P}$ be the set of graphs with no holes of lengths in $\mathcal{C}$. If $\mathcal{P}$ is expressible by forbidden acyclic orientations, then one of the following statements hold:

- $\mathcal{C}$ is a finite set,
- $\mathcal{C}$ is a cofinite subset of $\mathbb{Z}^+$; equivalently $\text{cyc} (\mathcal{P})$ is a finite set, or
- there is a positive integer $M$ such that $\mathcal{C}_M$ is the set of odd integers greater than or equal to $M$; equivalently, $\text{cyc} (\mathcal{P})_M$ the set of even integers greater than or equal to $M$.

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Proof. Assume that $C$ is not a finite set, nor a cofinite subset of $\mathbb{Z}^+$. First note that $\text{cyc}(P)$ is the complement of $C$ in the set of integers greater than or equal to 3. As $C$ is not cofinite in $\mathbb{Z}^+$, then $\text{cyc}(P)$ is an infinite set. Let $F$ be a set of connected oriented graphs such that $\text{Forb}^*_e(F)$ is an expression of $P$. By definition of $P$, $\mathcal{P}$ is closed under couplings, thus by Lemma 15 there is a pair of positive integers, $r$ and $m$, such that $\text{cyc}(\mathcal{P})_m$ is the set of multiples of $r$ greater or equal to $m$. Furthermore, we can assume that every oriented graph in $F$ has less that $m$ vertices (otherwise let $m'$ be a large enough multiple of $r$ that satisfies our assumption). We proceed to prove that $r = 2$ by contradiction. Since $C$ is infinite, then $r > 1$, so we will assume that $r > 2$. Let $\alpha$ be an integer such that $r\alpha > m + 1$, and consider the cycle on $n$ vertices, $C_n$, where $n = 2r\alpha - 2$. By the choice of $\alpha$, we know that $n > m$, and since $r$ is greater than 2, then $n$ is not a multiple of $r$, thus $C_n$ does not belong to $\mathcal{P}$. Let $x, y \in V(C_n)$ be two antipodal vertices and let $G = C_n + xy$. Clearly $G$ has two holes each of length $r\alpha$. Again, by the choice of $\alpha$, $C_{r\alpha} \in \mathcal{P}$, so $G$ contains no holes of length in $C$, thus $G \in \mathcal{P}$ and it admits an $F$-free acyclic orientation $G'$. The induced acyclic orientation of $C_n$ by $G'$ is $F$-free since $F$ consists of connected oriented graphs of size at most $m < r\alpha - 1$. Which contradicts the fact that $C_n \notin \mathcal{P}$ and $\text{Forb}^*_e(F)$ is an expression of $\mathcal{P}$.

Theorem 17. Let $C$ be a set of positive integers and let $\mathcal{P}$ be the set of graphs with no holes of lengths in $C$. If $\mathcal{P}$ is expressible by forbidden orientations, then one of the following statements hold:

• $C$ is a finite set, or

• there is a positive integer $M$ such that $C_M$ is the set of odd integers greater than or equal to $M$. Equivalently, $\text{cyc}(\mathcal{P})_M$ the set of even integers greater than or equal to $M$.

Proof. Once we prove that $C$ is not a cofinite subset of $\mathbb{Z}^+$, then we conclude following a proof analogous to the one used for Theorem 16. By definition of $\mathcal{P}$, every path belongs to $\mathcal{P}$. Moreover, $\mathcal{P}$ is closed under disjoint unions. Since $\mathcal{P}$ is expressible by forbidden orientations, then, by Proposition 13, $\text{cyc}(\mathcal{P})$, is an infinite set, and so $C$ cannot be a cofinite subset of $\mathbb{Z}^+$.

In particular, the class of graphs with no induced cycles of prime length is not expressible by forbidden (acyclic) orientations. Moreover, if $C$ is any infinite set of prime numbers, then the class of graphs with no cycles of lengths in $C$ is not expressible by forbidden (acyclic) orientations, and there are uncountable many such sets $C$. Actually, we can fix any other infinite set of positive integers (except for the set of odd integers), and apply the previous idea together with Theorem 17 to obtain an uncountable class of hereditary properties not expressible by forbidden orientations.

The downside of Theorems 16 and 17 is that they show that forbidden (acyclic) orientations have a rather weak expressive power regarding graph classes defined by forbidding induced cycles. But such strong necessary conditions raise our hopes of developing any of these theorems into a characterization.
4 \textit{Forb}-graphs

In the previous section we looked at hereditary properties defined by forbidden induced cycles and exhibited necessary conditions upon these classes to be expressible by forbidden orientations. In this section we study homomorphism classes and propose a characterization of those that are expressible by \textit{Forb}-graphs. Recall that every property expressible by \textit{Forb}-graphs is expressible by forbidden orientations, but not necessarily the other way around. So regarding Question 1 the characterization we propose in this section yields a sufficient condition for homomorphism classes to be expressible by forbidden orientations.

There are two main motivations to study homomorphism classes expressible by \textit{Forb}-graphs. On one hand, these expressions generalize the well-known and previously mentioned Roy-Gallai-Hasse-Vitaver Theorem. On the other hand, note that if a property, \( \mathcal{P} \), is expressible by \textit{Forb}-orientations, then \( \mathcal{P} \) is closed under homomorphic pre-images, and the most common properties closed under homomorphic pre-images are homomorphism classes, i.e., classes of \( H \)-colourable graphs for some fixed graph \( H \).

Dually to the definition of \( \text{Forb}(F) \), for a set of digraphs (graphs) \( M \) we denote by \( \text{CSP}(M) \) the class of digraphs (graphs) \( D \) such that \( D \rightarrow M \) for some \( M \in M \). We call \( \text{CSP}(M) \) the class of \( M \)-colourable digraphs (graphs). If \( M = \{M\} \), we will simply write \( \text{CSP}(M) \). A \textit{duality pair} in the digraph homomorphism order is an ordered pair of digraphs \( (A, B) \) such that \( \text{Forb}(A) = \text{CSP}(B) \). In [13] Nešetřil and Tardif characterize duality pairs as follows:

\textbf{Theorem 18.} [13] If \( (A, B) \) is a duality pair in the digraph homomorphism order then \( A \) is homomorphically equivalent to an oriented tree. Moreover, if \( T \) is an oriented tree, then there is a digraph \( D_T \) such that \( (T, D_T) \) is a duality pair.

We call \( D_T \) a \textit{dual} of \( T \) (any homomorphically equivalent digraph of \( D_T \) is a dual of \( T \)). It is not hard to observe that for every tree \( T \) any of its duals \( D_T \) is an oriented graph; simply note that \( T \) can be mapped to a symmetric arc, thus \( D_T \) has no symmetric arcs. With this observation we can immediately obtain a sufficient condition for a class of \( H \)-colourable graphs to be expressible by \textit{Forb}-orientations. If \( H \) is the underlying graph of a dual, \( D_T \), of an oriented tree, \( T \), then the class of \( \text{Forb}(T) \)-graphs equates the class of \( H \)-colourable graphs. Naturally, if \( H \) is such a graph, and \( H' \) is homomorphically equivalent to \( H \), then the class of \( H' \)-colourable graphs is expressible by \textit{Forb}-orientations. Turns out that the previous sufficient condition is close to be a characterization. We will derive this observation from a more general result.

A \textit{generalized duality} in the digraph homomorphism order, is an ordered pair of finite sets of incomparable digraphs \( (F, M) \) such that \( \text{Forb}(F) = \text{CSP}(M) \). For a set of digraphs, \( F \), we say that a digraph \( D \in F \) is \textit{minimal} (in \( F \)), if for every digraph \( D' \in F \) such that \( D' \rightarrow D \), it holds that \( D \rightarrow D' \). Generalized dualities have a similar characterization to that of duality pairs, due to Foniok, Nešetřil and Tardif.

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\footnote{Actually their result encompasses more general relational structures, but we state it only for the context of digraphs.}
Theorem 19. If \((F, M)\) is a generalized duality, then every digraph in \(F\) is homomorphic equivalent to an oriented forest. Conversely, for every finite incomparable set of oriented forests, \(F\), there is a finite set of incomparable oriented graphs \(M_F\) such that \((F, M_F)\) is a generalized duality.

As it happens with duality pairs, if \((F, M)\) is a generalized duality and \(M\) is the set of underlying graphs of \(M_F\), then \(\text{Forb}(F)\) is an expression of the class of \(M\)-colourable graphs. So by Theorem 19 for every set of oriented forests \(F\), there is a set of graphs \(M\) such that a graph \(G\) is a \(\text{Forb}(F)\)-graph if and only if \(G\) is \(M\)-colourable.

In general it does not hold that for any set of oriented graphs, \(F\), there is a finite set of graphs \(M\) such that \(\text{CSP}(M)\) corresponds to the class of \(\text{Forb}(F)\)-graphs. One would like to jump to the conclusion that there is a finite set of graphs \(M\) such that \(\text{Forb}(F)\) is an expression of \(M\)-colourable graphs, if and only if \(F\) is a set of oriented forests.

Well, this statement turns out to be true, but not at all obvious. This characterization is inspired on a result found in [12]. It differs enough so that we cannot simply cite their statement, but it is similar enough so that we can translate their proof to this context and nomenclature. Will use the Sparse Incomparability Lemma (for graphs) to do so. The version stated below is due to Nešetřil and Zhu [14].

Theorem 20 (Sparse Incomparability Lemma). Let \(k\) and \(l\) be positive integers and let \(G\) be a graph. Then, there is a graph \(G_0\) with the following properties:

- there is a homomorphism \(\varphi: G_0 \rightarrow G\),
- for any graph \(H\) on at most \(k\) vertices, \(G \rightarrow H\) if and only if \(G_0 \rightarrow H\), and
- \(G_0\) has girth at least \(l\).

The original version of the Sparse Incomparability Lemma asserts that for every pair of non-bipartite graphs, \(G\) and \(H\), such that \(G \rightarrow H\), and a positive integer \(l\), there is a graph \(H_0\) such that \(H_0 \rightarrow H\), \(G\) and \(H_0\) are incomparable, and \(H_0\) has girth at least \(l\). The version stated in Theorem 20 suits better our purpose.

Theorem 21. Let \(F\) be a finite set of oriented graphs. There is a finite set of graphs \(M\) such that \(\text{CSP}(M)\) is the class of \(\text{Forb}(F)\)-graphs if and only if \(F\) is a set of oriented forests.

Proof. As observed before, one implication is trivial. We prove the interesting implication; we assume that there is a finite set \(M\) such that \(\text{CSP}(M)\) is the class of \(\text{Forb}(F)\)-graphs.

Clearly, if \(H\) is a homomorphic image of an oriented graph in \(F\), then \(\text{Forb}(F \cup \{H\}) = \text{Forb}(F)\) and hence \(\text{CSP}(F \cup \{H\}) = \text{CSP}(M)\). So we close \(F\) under homomorphic images and then keep only the cores (we denote it by \(F\) again). Among all such sets we choose \(F\) to be of minimal cardinality.
If every minimal element of $F$ is an oriented forest there is nothing to prove. So suppose there is a minimal element $F_0 \in F$ that is not a forest. Note that by the minimality of $F$, there is a $\text{Forb}(F \setminus \{F_0\})$-graph $G$ that is not a $\text{Forb}(F)$-graph. So there is a $\text{Forb}(F \setminus \{F_0\})$-graph, $G$, such that $G \not \cong M$ for any $M \in \mathcal{M}$, and $G$ admits an orientation, $G'$, such that $F_0 \rightarrow G'$. Moreover, we claim that we can choose an orientation $G'$ of $G$ such that any homomorphism $\varphi: F_0 \rightarrow G'$ is injective. To prove it, denote by $h(F_0)$ the set of all non-injective homomorphic images of $F_0$. By the choice of $F$, for every oriented graph $D \in h(F_0)$ its core, $D_c$, belongs to $F$. The fact that $D_c$ is the core of a non-injective homomorphic image of $F_0$, implies that $|V(D_c)| < |V(F_0)|$. In particular, for every $D \in h(F_0)$ there is an oriented graph, $D' \in F \setminus \{F_0\}$, such that $D' \rightarrow D$. So let $H$ be a graph such that for any orientation, $H'$, of $H$, there is a non-injective homomorphism $\varphi: F_0 \rightarrow H'$. By considering the homomorphic image $\varphi[F_0]$ we conclude that there is an oriented graph $D \in F \setminus \{F_0\}$ such that $D \rightarrow \varphi[F_0] \rightarrow H'$. Since $F$ is minimal, our claim follows. So let $G$ be a $\text{Forb}(F \setminus \{F_0\})$-graph and $G'$ an orientation of $G$ such that any homomorphism $\varphi: F_0 \rightarrow G'$ is injective.

By the Sparse Incomparability Lemma (Theorem 20) for $G$, for $l > |V(F_0)|$, and for $k = \max\{|V(M)|: M \in \mathcal{M}\}$, there is a graph $G_0$ with girth at least $l$ such that $G_0 \rightarrow G$, and for any graph $M$ on at most $k$ vertices, $G_0 \rightarrow M$ if and only if $G \rightarrow M$. By the choice of $k$, and since $G \not \cong M$ for any $M \in \mathcal{M}$, then $G_0$ is not $\mathcal{M}$-colourable. By hypothesis, for any orientation, $G_0'$, of $G_0$, there is an oriented graph $L \in F$ such that $L \rightarrow G_0'$. Since $G_0 \rightarrow G$ choose $G_0'$ to be the orientation induced in $G_0$ by any homomorphism $G_0 \rightarrow G$ and the previously chosen orientation of $G$, $G'$. Clearly $G_0' \rightarrow G'$, and recall that $G$ is a $\text{Forb}(F \setminus \{F_0\})$-graph, thus there is no homomorphism from an oriented graph in $F \setminus \{F_0\}$ to $G_0'$. Which in turn implies that there is a homomorphism $\varphi: F_0 \rightarrow G_0'$, and the fact that the girth of $G_0'$ is greater than $|V(F_0)|$ implies that every cycle in $F_0$ must be mapped to an oriented path in $G_0'$. Hence there is non-injective homomorphism $F_0 \rightarrow G_0' \rightarrow G'$, contradicting the choice of $G'$. Concluding that if $F_0$ is a minimal element in $F$ it must be an oriented forest (or homomorphically equivalent to one).

Theorem 21 together with Theorem 19 have two immediate consequences.

**Corollary 22.** Let $H$ be any graph. The class of $H$-colourable graphs is expressible by $\text{Forb}$-graphs, if and only if there is a set of oriented graphs $\mathcal{M}_H$ whose underlying graphs are homomorphically equivalent to $H$, such that $(F, \mathcal{M}_H)$ is a generalized duality for some set of oriented forest $F$.

**Corollary 23.** Consider a graph $H$. There is an oriented graph, $T$, such that $\text{Forb}(T)$ is an expression of $H$-colourable graphs, if and only if $T$ is an oriented tree and $H$ is homomorphically equivalent to the underlying graph of the dual $D_T$.

In [8] we showed that for any odd cycle, $C$, there is an oriented path, $P_C$, such that a graph is $C$-colorable if and only if it is a $\text{Forb}(P_C)$-graph. We did so by finding an orientation, $C'$, of $C$ such that $C'$ is the dual of an oriented path $(P_C)$. Corollary 23 shows that the problem of characterizing the class of $H$-colorable graphs as $\text{Forb}(T)$-graphs, is almost equivalent to the previously mentioned technique, i.e., it is equivalent to finding a
graph, $R$, homomorphically equivalent to $H$, and then find an orientation, $R'$, of $R$ such that $R'$ is the dual of some oriented tree.

5 Conclusions

As we have already mentioned in Section 3, we believe that the necessary conditions exposed in Theorems 16 and 17 are quite close to be sufficient as well. So we believe that aiming to improve any of these theorems into a characterization is a feasible problem to pursue. Also, since a hereditary property might be expressible by forbidden orientations but not by $Forb$-graphs, we would like to know if there is a homomorphism class expressible by forbidden orientations, but not by $Forb$-graphs. In other words, does Corollary 22 holds if we replace “expressible by $Forb$-graphs” by “expressible by forbidden orientations”?

Similar to expressions by forbidden orientations, some authors have studied expressions by forbidden ordered graphs [3, 4, 9]. Amongst these, we would like to mention that the work of Feuilloley and Habib [4] stands out for being a recent, thorough and complete survey regarding such expressions. Question 1 can also be posed for forbidden ordered graphs. As far as we are concerned, there is no example of a hereditary property that cannot be expressed by finitely many forbidden ordered graphs, so we would like to propose this problem.

Problem 24. Find an example of a hereditary property that is not expressible by (finitely many) forbidden ordered graphs.

By means of a simple combinatorial argument, one can prove that if $P$ is expressible by forbidden acyclic orientations, then it is expressible by forbidden ordered graphs. Due to this observation, we believe that any property that is not expressible by forbidden acyclic orientations is a reasonable candidate to be a witness of the example required by Problem 24.

In particular we ask the following question.

Question 25. Is there a finite set of ordered graphs, $F$, such that a graph $G$ admits an $F$-free ordering, if and only if $G$ is an even-hole-free graph?

Finally as a side note to the reader familiar with symbolic dynamics, form Theorem 9 we obtain the following result.

Corollary 26. Let $X$ be a topological transitive shift of finite type, let $H$ be its set of periods and let $r = \gcd(H)$. Then, $H$ is a cofinite subset of $r\mathbb{Z}^+$. In particular, if $H$ is a set of relative primes, then $H$ is cofinite in $\mathbb{Z}^+$.

Proof. The reader familiar with shift spaces, can notice that $X$ is topological transitive if and only if the language, $L(X)$, of $X$ is transitive. Moreover, $X$ is of finite type if and only if there is a finite set of words $A$, such that $L(X) = \mathcal{L}_A$. Finally, the equalities $H = \text{per}(L(X)) = \text{per}(\mathcal{L}_A)$ hold, and $H$ is not an empty set for a shift of finite type. Thus, Theorem 9 implies the statement of this corollary.
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