On the Dirichlet problem for fractional Laplace equation on a
genral domain

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Abstract

In this paper, we study the weak strong uniqueness of the Dirichlet type problems of fractional Laplace (Poisson) equations. We construct the Green’s function and the Poisson kernel. We then provide a somewhat sharp condition for the solution to be unique. We also show that the solution under such condition exists and must be given by our Green’s function and Poisson kernel. In doing these, we establish several basic and useful properties of the Green’s function and Poisson kernel. Based on these, we obtain some further a priori estimates of the solutions. Surprisingly those estimates are quite different from the ones for the local type elliptic equations such as Laplace equations. These are basic properties to the fractional Laplace equations and can be useful in the study of related problems.

Keywords: fractional Laplacian, Dirichlet problem, Green function, existence and uniqueness of solutions.

MSC: 35A01; 35B45; 35J08; 35S05.

1 Introduction and results

We are interested in the existence and uniqueness of solutions for the following Dirichlet problem with fractional Laplacian:

\[
\begin{aligned}
(-\Delta)^s u + cu &= f & \text{in } \Omega, \\
0 &= \text{on } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \(0 < s < 1\), \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) satisfying the uniform exterior ball condition.

In the classical work of Laplacian, the Dirichlet problem of \(W^{2,p}\)-type plays an essential part: Assume \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary, \(f \in L^p(\Omega)\), \(\bar{b}, c \in L^\infty(\Omega)\) and \(c \geq 0\) in \(\Omega\), consider the problem \[25\]:

\[
\begin{aligned}
-\Delta u + \bar{b} \cdot \nabla u + cu &= f & \text{in } \Omega, \\
0 &= \text{on } \partial \Omega.
\end{aligned}
\]

It is well known that \(1.2\) has a unique solution \(u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\), which admits the following apriori estimate:

\[
\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.
\]

Dirichlet problem has been utilized in nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, gravitational equilibrium of stars and elsewhere, more details on the application of Dirichlet problem can be found in \[21, 22, 23, 24\]. Many fruitful results on such problems for

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Laplacian were obtained. Brezis et al. [6] and Figueiredo et al. [20] (see also Berestycki and Lions [4] for a particular case) worked on a priori bound and existence for solutions of semilinear equations. Chen and Li [13] investigated a priori estimates for solutions to nonlinear elliptic equations.

The fractional Laplacian has caught the researchers’ attentions because of the applications in physics, astrophysics, mechanics and economics in recent years (see [2, 3, 15, 16, 40]). In contrast to the situation with local Laplacian, essential tools such as maximum principle, Harnack principle and Hopf lemma are not at our disposal when dealing with solutions in the setting of the nonlocal fractional Laplacian. Rather, these tools need to be reconstructed based on the nonlocal properties of fractional operators. For $c = 0$ in (1.1), Ros-Oton and Serra [37] studied the regularity up to the boundary of solutions to (1.1) by developing fractional analog of the Krylov boundary Harnack method. As $\Omega$ being a ball, $c = 0$ in (1.1), Bucur [9] provided the representation formulas of solutions for (1.1). For more results on the fractional Laplacian, please see [1, 5, 10–12, 14, 23, 26, 32–34] and the references therein.

It is a common idea to represent the solutions of (1.1) as the convolution of the Green function with the forcing term for $c = 0$. The construction of Green functions on $\Omega$ plays an important role in our study.

It is known that (see for [30])

$$\lim_{\epsilon \to 0} \epsilon^{-1} \int_{\Omega \setminus \Omega_\epsilon} |u(x)| \, dx = 0$$

indicates that the trace of $u$ is 0 on $\partial \Omega$. Hence, (1.4) can be a uniqueness condition for the classical second order elliptic PDE. Here we denote

$$\Omega_\epsilon = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \epsilon \} = \{ x \in \Omega \mid B_\epsilon(x) \subset \Omega \},$$

(1.5)

Similar to the Laplace problem, our main observation is that the following condition (1.6) plays an important role in the existence, uniqueness and the maximum principle for problem (1.1).

$$\lim_{\epsilon \to 0} \epsilon^{-s} \int_{\Omega \setminus \Omega_\epsilon} |u(x)| \, dx = 0.$$  

(1.6)

Indeed, we have

**Theorem 1.1** (Maximum Principle). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, $0 < s < 1$, $c \geq 0$ in $\Omega$. If $u \in L_2(\Omega)$ satisfies

$$\begin{cases}
(-\Delta)^s u + cu \geq 0 & \text{in } \Omega, \\
u \geq 0 & \text{on } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

(1.7)

Then we conclude

$$u \geq 0 \quad \text{in } \mathbb{R}^n.$$  

(1.9)

Here, we say $\Omega$ satisfies the exterior ball condition if for any $X \in \partial \Omega$, there exists a ball $B_r(c)$ such that

$$B_r(c) \cap \Omega = \emptyset \quad \text{and} \quad X \in \overline{B_r(c)} \cap \overline{\Omega}.$$  

(1.10)

We say $\Omega$ satisfies the uniform exterior ball condition with uniform radius $r$, if the radius of exterior ball corresponding to any $X \in \partial \Omega$ can be chosen uniformly as $r$. 

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Theorem 1.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, $0 < s < 1$, $c = 0$ in $\Omega$. If $f \in L^1_s(\Omega)$, then problem (1.1) has a unique solution $u = G_{\Omega} * f$ satisfying condition (1.6).

Here and hereafter, the weighted Lebesgue spaces $L^p_\sigma$ is defined as:

$$L^p_\sigma(\Omega) = \{ f \in L^1_{\text{loc}}(\Omega) \mid \text{dist}(\cdot, \mathbb{R}^n \setminus \Omega)^\sigma f \in L^p(\Omega) \},$$

with a nature norm: $\|f\|_{L^p_\sigma(\Omega)} = \|\text{dist}(\cdot, \mathbb{R}^n \setminus \Omega)^\sigma f\|_{L^p(\Omega)}$.

Remark 1. If condition (1.6) is not satisfied, then the solution is not unique. Indeed, even for $\Omega = B_1$, $f \equiv 0$ and $g \equiv 0$, there exists a nontrivial solution

$$u(x) = \begin{cases} (1 - |x|^2)^{s-1} & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases} \quad (1.12)$$

One observe that such $u$ nearly satisfies the condition (1.6).

This kind of weak-strong uniqueness and very weak non-uniqueness phenomena is also found for other equations such as the Navier-Stokes equations and some transport equations. For the Navier-Stokes equations, the weak-strong uniqueness was established by Fabes, Ladyzhenskaya, Prodi, Serrin and etc. (see for [22], [28], [36], [38], · · · ). Non-uniqueness of weaker solutions was first proved by Buckmaster in [8], see also [7, 35]. Recently, Cheskidov and Luo showed the critical regularity criteria for the non-uniqueness of very weak solution to Navier-Stokes equations [19] [17] and the transport equation in [18].

Theorem 1.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, $0 < s < 1$, $|\sigma| \leq s$, $c \geq 0$ in $\Omega$. If $f \in L^p_\sigma(\Omega)$, then problem (1.1) has a unique solution $u$ satisfying condition (1.6).

Moreover, the solution $u$ satisfies the following estimate:

1. If $p = 1$, then for any $q$ satisfying $1 \leq q < \frac{n}{n-2s}$, $u \in L^q_\sigma(\Omega)$ with

$$\|u\|_{L^q_\sigma(\Omega)} \leq C \|f\|_{L^1_\sigma(\Omega)}.$$

(1.13)

2. If $1 < p < \frac{n}{2s}$, then $u \in L^{\frac{np}{n-2sp}}_\sigma(\Omega)$ with

$$\|u\|_{L^{\frac{np}{n-2sp}}_\sigma(\Omega)} \leq C \|f\|_{L^p_\sigma(\Omega)}.$$

(1.14)

3. If $p > \frac{n}{2s}$, then $u \in L^{\infty}_\sigma(\Omega)$ with

$$\|u\|_{L^{\infty}_\sigma(\Omega)} \leq C \|f\|_{L^p_\sigma(\Omega)}.$$

(1.15)

The following well known result can also be seen as a corollary of our estimate in Theorem 1.3.

Corollary 1.4. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, $0 < s < 1$, $c \geq 0$ in $\Omega$. If $f \in L^\infty(\Omega)$ in (1.1), and $u$ satisfies (1.6) then $u \in C^s(\Omega)$.

We organize this paper as follows. In Section 2 we introduce some basic preliminaries about the fractional Laplacian. Section 3 is devoted to the construction of Green’s function on general domains. Some basic properties and estimates are established in Section 4. We construct the Poisson kernel in Section 5. And we finally solve the Dirichlet problem in Section 6.

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2 Preliminaries about the fractional Laplacian

For $0 < s < 1$ and $u \in C_0^\infty(\mathbb{R}^n)$, the fractional Laplacian $(-\Delta)^s u$ is given by

$$(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad 0 < s < 1,$$

(2.1)

where P.V. stands for the Cauchy principal value (see for [29]). In order that $(-\Delta)^s w$ make sense as a distribution, it is common to define:

$$L_\alpha = \{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\alpha}} \, dy < +\infty \}.$$

It is easy to see that for $w \in L_2$, $(-\Delta)^s w$ as a distribution is well-defined:

$$\forall \varphi \in C_\infty^0(\mathbb{R}^n), \quad (-\Delta)^s w(\varphi) = \int_{\mathbb{R}^n} w(x)(-\Delta)^s \varphi(x) \, dx.$$

(2.2)

The fundamental solution $\Phi$ for $s$-Laplacian is defined as:

$$\Phi(x) = a(n,s)|x|^{2s-n}.$$

(2.3)

One has the following equation in sense of distribution:

$$(-\Delta)^s \Phi = \delta_0.$$

(2.4)

The Green’s functions and Poisson kernels of fractional Laplacian on the unit ball is introduced in [9]. Define the Poisson kernel $P(\cdot, \cdot)$ and Green’s function $G(\cdot, \cdot)$ for $s$-Laplacian in $B_1$ [9] as:

$$P(x,y) = c(n,s) \left( \frac{1 - |x|^2}{|y|^2 - 1} \right)^s \frac{1}{|x - y|^n}, \quad \text{for } x \in B_1, y \in \mathbb{R}^n \setminus B_1,$$

(2.5)

$$G(x,y) = \frac{n(n,s)}{|x - y|^{n-2s}} \int_0^{\rho(x,y)} t^{s-1} \left( \frac{1}{1 + t} \right)^{\frac{n}{2}} \, dt, \quad \text{for } x, y \in B_1,$$

(2.6)

where $\rho(x,y)$ is defined as

$$\rho(x,y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}.$$

(2.7)

Let $P * g$ and $G * f$ be defined as

$$P * g(x) = \begin{cases} \int_{\mathbb{R}^n \setminus B_1} P(x,y) g(y) \, dy & \text{if } x \in B_1, \\ g(x) & \text{if } x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

(2.8)

$$G * f(x) = \begin{cases} \int_{B_1} f(y) G(x,y) \, dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases}$$

(2.9)

The following two results are introduced in [9]:

**Lemma 2.1.** Let $0 < s < 1$, $f \in C^{2s+\epsilon}(B_1) \cap C(\overline{B_1})$. Then there exists a unique continuous solution: $u = G * f$ that solves the following problem pointwisely.

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_1, \\ u = 0 & \text{on } \mathbb{R}^n \setminus B_1. \end{cases}$$

(2.10)
Lemma 2.2. Let $0 < s < 1$, $g \in L^2_s \cap C(\mathbb{R}^n)$. Then there exists a unique continuous solution: $u = P \ast g$ that solves the following problem (2.11) pointwisely.
\[
\begin{aligned}
& (-\Delta)^s u = 0 \quad \text{in } B_1, \\
& u = g \quad \text{on } \mathbb{R}^n \setminus B_1.
\end{aligned}
\] (2.11)

The classical maximum principle for fractional Laplacian is given in [39] Lemma 2.17:

**Theorem 2.3** (Maximum principle). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $u$ be a lower semi-continuous function in $\Omega$ such that
\[
\begin{aligned}
& (-\Delta)^s u \geq 0 \quad \text{in } D'(\Omega), \\
& u \geq 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\] (2.12)

Then $u \geq 0$ in $\mathbb{R}^n$.

The following technique also comes from [39]:

**Definition 2.4.** Let $\Gamma$ be defined as follow
\[
\Gamma(x) = \begin{cases}
\Phi(x) = a(n,s)|x|^{2s-n} & |x| \geq 1, \\
\frac{n-2s+2}{2}a(n,s) - \frac{n-2s}{2}a(n,s)|x|^2 & |x| < 1.
\end{cases}
\] (2.13)

It is easy to check: $\Gamma \in C^{1,1}(\mathbb{R}^n) \cap L^2_s$. Therefore, one can define $\gamma = (-\Delta)^s \Gamma \in C(\mathbb{R}^n)$.

**Proposition 2.5** (Proposition 2.11 in [39]). $\gamma \geq 0$ and
\[
\int_{\mathbb{R}^n} \gamma(x) \, dx = 1.
\] (2.14)

**Proposition 2.6** (Proposition 2.13 in [39]). Define $\gamma_\lambda(x) = \frac{1}{\lambda^n} \gamma(\frac{x}{\lambda})$. Then $\gamma_\lambda$ is an approximation of Dirac $\delta$ as $\lambda \to 0^+$.

**Theorem 2.7** (Proposition 2.15 in [39]). Let $\Omega$ be a domain in $\mathbb{R}^n$. If $u \in L^2_s$ satisfies
\[
(-\Delta)^s u \leq 0 \quad \text{in } D'(\Omega).
\] (2.15)

Then, there exists a version $\tilde{u}$ of $u$ (i.e. $u = \tilde{u}$ a.e. $\mathbb{R}^n$), such that $\tilde{u}$ is upper semi-continuous in $\Omega$. Indeed,
\[
\tilde{u} = \lim_{\lambda \to 0^+} u \ast \gamma_\lambda.
\] (2.16)

The following result proved in [31] is also crucial in our progress.

**Theorem 2.8** (Theorem 5.4 in [31]). Let $\Omega$ be a domain in $\mathbb{R}^n$. Assume $u,v \in L^2_s$, $f,g \in L^1_{loc}(\Omega)$, and satisfy
\[
\begin{aligned}
& (-\Delta)^s u(x) + \tilde{b}(x) \cdot \nabla u(x) + c(x)u(x) \leq f(x) \\
& (-\Delta)^s v(x) + \tilde{b}(x) \cdot \nabla v(x) + c(x)v(x) \leq g(x)
\end{aligned}
\] in $D'(\Omega)$, (2.17)

where $\|\tilde{b}(x)\|_{C^1(\Omega)} + \|c(x)\|_{L^\infty(\Omega)} < \infty$. Then for $w(x) = \max\{u(x), v(x)\}$, it holds that
\[
(-\Delta)^s w(x) + \tilde{b}(x) \cdot \nabla w(x) + c(x)w(x) \leq f(x)\chi_{u>v} + g(x)\chi_{u<v} + \max\{f(x), g(x)\}\chi_{u=v} \quad \text{in } D'(\Omega).
\] (2.18)
3 The construction of Green function for general domains

In this section, we will use Perron’s method to show the existence of Green functions for domains with sufficiently regular boundary. Indeed, we aim to prove the following theorem:

**Theorem 3.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition. Assume that $0 < s < 1$. Then for any $y \in \Omega$, there exists a function $h_y \in C(\mathbb{R}^n) \cap C^\infty(\Omega)$ such that

$$\begin{cases}
(-\Delta)^s h_y = 0 & \text{in } \mathcal{D}'(\Omega), \\
h_y = g_y & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

for $g_y(x) = \frac{a(n,s)}{|x-y|^{n-2s}}$.

Mollifier is a basic tool in our analysis. In the following, we denote by $J_\epsilon[u]$ the mollification of $u$:

$$J_\epsilon[u](x) = (j_\epsilon * u)(x) = \int_{B_\epsilon(x)} j_\epsilon(x-y)u(y) \, dy,$$

where $j \in C^\infty_0(B_1)$ is a positive smooth radially symmetric function supported in $B_1$ satisfying $\int_{\mathbb{R}^n} j(x) \, dx = 1$. Moreover, $j_\epsilon(x) = \frac{1}{\epsilon^n} j(\frac{x}{\epsilon})$.

**Definition 3.2.** With $h_y$ defined, one can define the Green function corresponding to the domain $\Omega$ as:

$$G_\Omega(x,y) = \Phi(x-y) - h_y(x).$$

**Remark 2.** $G_\Omega(x,y) = 0$, if either one of $x, y$ is not in $\Omega$.

In order to prove Theorem 3.1, we need the following definitions and lemmas to apply the Perron’s method.

**Definition 3.3.** We define the Perron sub-solution class corresponding to (3.1) as:

$$\mathcal{P} = \left\{ p \in C(\mathbb{R}^n) \mid \begin{cases} (-\Delta)^s p \leq 0 & \text{in } \mathcal{D}'(\Omega), \\ p \leq g_y & \text{on } \mathbb{R}^n \setminus \Omega, \\ p \geq 0 & \text{in } \mathbb{R}^n. \end{cases} \right\}.$$ (3.4)

**Remark 3.** Set $\mathcal{P}$ is nonempty.

Indeed, let $p_0 : \mathbb{R}^n \to \mathbb{R}$ be defined as $p_0(x) \equiv 0$. Then, one easily verifies $p_0 \in \mathcal{P}$.

**Remark 4.** By maximum principle 2.3 each $p \in \mathcal{P}$ satisfies, for all $x \in \mathbb{R}^n$

$$0 \leq p(x) \leq g_y(x) \quad \text{and} \quad 0 \leq p(x) \leq \max_{\xi \in \mathbb{R}^n \setminus \Omega} g_y(\xi).$$ (3.5)

**Remark 5.** The Perron sub-solution class $\mathcal{P}$ is separable, with respect to the metric

$$\rho(p_1, p_2) = \max_{x \in \mathbb{R}^n} |p_1(x) - p_2(x)| = \|p_1 - p_2\|_{C(\mathbb{R}^n)}.$$ (3.6)

Indeed, $\mathcal{P} \subset C_0(\mathbb{R}^n)$ (the Banach space consists of continuous functions on $\mathbb{R}^n$ which vanishes at infinity with norm $\| \cdot \|_{C(\mathbb{R}^n)}$) and $C_0(\mathbb{R}^n)$ is separable.

**Definition 3.4.** The Perron solution $S$ of (3.1) is defined to be:

$$S(x) = \sup_{p \in \mathcal{P}} p(x).$$ (3.7)
Remark 6. Seeing Remark 4, we have the following estimate on $S$:

$$0 \leq S(x) \leq g_y(x) \quad \text{and} \quad 0 \leq S(x) \leq \max_{\xi \in \mathbb{R}^n \setminus \Omega} g_y(\xi) \quad \text{for all} \quad x \in \mathbb{R}^n. \quad (3.8)$$

**Lemma 3.5.** $S$ is lower semi-continuous.

**Proof of Lemma 3.5.** Indeed, for any $x \in \mathbb{R}^n$, $\epsilon > 0$, there exists $p \in \mathcal{P}$ such that

$$p(x) > S(x) - \epsilon. \quad (3.9)$$

However, $p$ is continuous, then there exists $\delta > 0$ such that for any $\xi \in B_\delta(x)$,

$$p(\xi) > p(x) - \epsilon. \quad (3.10)$$

Therefore, for any $\xi \in B_\delta(x)$

$$S(\xi) \geq p(\xi) > p(x) - \epsilon > S(x) - 2\epsilon. \quad (3.11)$$

I.e. $S$ is lower semi-continuous. \qed

**Lemma 3.6.** Let $p_1, p_2 \in \mathcal{P}$, $p_3$ defined as follow:

$$p_3(x) = \max\{p_1(x), p_2(x)\} \quad \text{for any} \quad x \in \mathbb{R}^n. \quad (3.12)$$

Then $p_3 \in \mathcal{P}$.

**Proof.** It is easy to see: $p_3 \in C(\mathbb{R}^n) \cap L_{2s}$ from the fact that $p_1, p_2 \in C(\mathbb{R}^n) \cap L_{2s}$.

It is also clear that $p_3 \leq g_y$ on $\mathbb{R}^n \setminus \Omega$.

Choose $\bar{b}, c, f, g$ to be zero in Theorem 2.8, one derives

$$(-\Delta)^s p_3 \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \quad (3.13)$$

Therefore, $p_3 \in \mathcal{P}$. \qed

**Lemma 3.7.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the exterior ball condition, $S$ be the Perron solution corresponding to (3.1). Then, there exists a monotonically increasing sequence $\{p_i\}_{i=1}^\infty \subset \mathcal{P}$ such that

$$S(x) = \lim_{i \to \infty} p_i(x). \quad (3.14)$$

**Proof.** From Remark 5, there exists a countable subset $\{p_j^*\}_{j=1}^\infty$ of $\mathcal{P}$ that is dense in $\mathcal{P}$. As a consequence, we can write $S(x)$ as:

$$S(x) = \sup_{j \in \mathbb{N}_+} p_j^*(x). \quad (3.15)$$

Define

$$p_i(x) = \max_{1 \leq j \leq i} p_j^*(x). \quad (3.16)$$

Clearly, sequence $\{p_i\}_{i=1}^\infty$ is monotonically increasing, and

$$S(x) = \lim_{i \to \infty} p_i(x). \quad (3.17)$$

Moreover, by Lemma 3.6 one derives $p_i \in \mathcal{P}$. \quad \qed

**Corollary 3.8.** $S(x)$ satisfies

$$(-\Delta)^s S \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \quad (3.18)$$
Proof. For any test function $\varphi \in \mathcal{D}(\Omega)$, there exist constants $C_1, C_2, M > 0$ such that

$$
|(-\Delta)^s \varphi (x)| \leq C_1 \quad \text{for} \quad x \in \mathbb{R}^n
$$

$$
|(-\Delta)^s \varphi (x)| \leq \frac{C_2}{|x|^{n+2s}} \quad \text{for} \quad |x| \geq M.
$$

Then

$$
p_i(x)|(-\Delta)^s \varphi (x)| \leq |(-\Delta)^s \varphi (x)| \max_{\xi \in \mathbb{R}^n \setminus \Omega} g_y(\xi) \leq \begin{cases} C_3 & \text{for} \quad |x| < M, \\ C_4 & \text{for} \quad |x| \geq M. \end{cases}
$$

Therefore, $\psi \in L^1(\mathbb{R}^n)$ and by Lebesgue dominated convergence theorem, we have

$$
\int_{\mathbb{R}^n} S(x)(-\Delta)^s \varphi (x) \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} p_i(x)(-\Delta)^s \varphi (x) \, dx \leq 0.
$$

I.e.,

$$
(-\Delta)^s S \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
$$

\[ \square \]

Remark 7. As a consequence of Corollary 3.8 and Theorem 2.7, $S$ has a version $\tilde{S}$ that is upper semi-continuous in $\Omega$. Indeed, for $x \in \Omega$,

$$
\tilde{S}(x) = \lim_{\lambda \to 0^+} S * \gamma_\lambda(x).
$$

Lemma 3.9 (Barrier function). Let $X \in \partial \Omega, B_r(c)$ be an exterior ball of $\Omega$ at $X$. Then there exists function $p \in \mathcal{P}$ such that

$$
p(x) = g_y(x) \quad \text{for} \quad x \in \overline{B_r(c)}.
$$

Moreover, one estimates:

$$
|p(x) - p(X)| \leq C|x - X|^s.
$$

Here $C$ depends only on $s, n, r$ and dist$(y, \partial \Omega)$.

Proof. Doing the Kelvin transform to $g_y$ along $\partial B_r(c)$, one gets

$$
\hat{g}_y(x) = \frac{r^{n-2s}}{|x - c|^{n-2s}} g_y \left( \frac{r^2(x - c)}{|x - c|^2} + c \right) = a(n, s) \frac{r^{n-2s}}{|x - c|^{n-2s}} \frac{r^2(x - c) + c - y}{|x - c|^2} + c - \eta \right)^{2s-n}
$$

$$
= a(n, s) \frac{|\eta - c|^{n-2s}}{r^{n-2s}} \frac{1}{|x - \eta|^{n-2s}},
$$

where

$$
\eta = r^2 \frac{(y - c)}{|y - c|^2} + c \in B_r(c).
$$

One observes $\hat{g} \in L_{2s} \cap C(\mathbb{R}^n \setminus B_r(c))$. Therefore one can apply Lemma 2.2 to derives the existence of function $\hat{p} \in C(\mathbb{R}^n)$ such that

$$
\begin{cases} 
(-\Delta)^s \hat{p} = 0 & \text{in} \ B_r(c), \\
\hat{p} = \hat{g}_y & \text{on} \ \mathbb{R}^n \setminus B_r(c).
\end{cases}
$$

In particular, by maximum principle 2.3, we know $\hat{p} \leq \hat{g}_y$ in $\mathbb{R}^n$. 

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Do the Kelvin transform to \( \hat{p} \) along \( \partial B_r(c) \), one obtains function \( p \in C(\mathbb{R}^n) \cap \mathcal{L}_{2s} \) with:

\[
\begin{cases}
(-\Delta)^s p = 0 & \text{in } \mathbb{R}^n \backslash B_r(c), \\
p = g_y & \text{on } B_r(c), \\
p \leq g_y & \text{in } \mathbb{R}^n.
\end{cases}
\] (3.29)

Note that \( \Omega \subset \mathbb{R}^n \backslash B_r(c) \), we conclude \( p \in \mathcal{P} \). The estimate (3.25) comes from the fact that

\[
\int_{\mathbb{R}^n \backslash B_r(c)} \frac{\hat{g}_y(x)}{|x - c|^{n+2s}} \, dx \leq C_y \quad \text{and} \quad [\hat{g}_y]_{C^1(\mathbb{R}^n \backslash B_r(c))} \leq C_y.
\] (3.30)

\[ \blacksquare \]

**Lemma 3.10.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the exterior ball condition, \( S \) be the Perron solution corresponding to (3.1). Then, \( S \) is continuous on \( \mathbb{R}^n \setminus \Omega \). Moreover,

\[
S(x) = g_y(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega.
\] (3.31)

**Proof.** For \( x_1 \in \mathbb{R}^n \setminus \overline{\Omega} \), say \( \text{dist}(x_1, \Omega) = 4\epsilon > 0 \). Then we do the following construction:

Let \( \Omega^{2\epsilon} = \{ x \in \mathbb{R}^n | \text{dist}(x, \Omega) < 2\epsilon \} \) and

\[
\eta(x) = J_\epsilon[\chi_{\Omega^{2\epsilon}}](x)
\] (3.32)

Then \( \eta \in C^\infty(\mathbb{R}^n) \) and

\[
\eta(x) = \begin{cases}
0 & \text{if } \text{dist}(x, \Omega) \geq 3\epsilon \\
1 & \text{if } \text{dist}(x, \Omega) \leq \epsilon.
\end{cases}
\] (3.33)

Let \( p_1(x) = g_y(x)(1 - \eta(x)) \). One easily verifies \( p_1 \in \mathcal{P} \). While, \( p_1(x) = g_y(x) \) for \( x \in B_r(x_1) \). Therefore, \( S(x) \geq g_y(x) \) for \( x \in B_r(x_1) \). On the other hand, \( S(x) \leq g_y(x) \) for \( x \in B_r(x_1) \). Hence, \( S(x) = g_y(x) \) for \( x \in B_r(x_1) \) and \( S \) is continuous at \( x_1 \).

For \( X \in \partial \Omega \), there exists an exterior ball \( B_r(c) \) of \( \Omega \) at \( X \). According to Lemma 3.9, there exists \( p \in \mathcal{P} \) such that \( p(x) = g_y(x) \) for \( x \in B_r(c) \). In particular, \( p(X) = g_y(X) \). On the other hand, \( S(x) \leq g_y(x) \). Hence, \( S(X) = g_y(X) \). Moreover, in a neighborhood of \( X \),

\[
p(x) \leq S(x) \leq g_y(x),
\] (3.34)

with both \( p \) and \( g_y \) continuous near \( X \). As a consequence, \( S \) is continuous at \( X \).

Summing up the above results, \( S \) is continuous on \( \mathbb{R}^n \setminus \Omega \). Moreover,

\[
S(x) = g_y(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega.
\] (3.35)

\[ \blacksquare \]

**Lemma 3.11.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the uniform exterior ball condition, \( S \) be the Perron solution corresponding to (3.1). Then, \( S \) is continuous in \( \Omega \).

**Proof.** \( S \) is lower semi-continuous and as a \( s \)-subharmonic function, \( S \) has an upper semi-continuous version in \( \Omega \) (see Remark 7):

\[
\tilde{S}(x) = \lim_{\lambda \to 0^+} S \ast \gamma_\lambda(x)
\] (3.36)

Hence, it suffices to show that

\[
S(x) = \tilde{S}(x) \quad \text{for all } x \in \Omega.
\] (3.37)
We readily know (see Proposition 2.5)
\[ S(x) = \tilde{S}(x) \quad \text{for a.e. } x \in \Omega. \] (3.38)

Then, by the semi-continuity of \( S \) and \( \tilde{S} \), we know:
\[ S(x) \leq \tilde{S}(x). \] (3.39)

We now prove that (3.37) is true by contradiction. Assume that (3.37) is not true, then there exists \( x_0 \in \Omega \) such that
\[ S(x_0) < \tilde{S}(x_0) = \lim_{\lambda \to 0^+} S \ast \gamma(x_0). \] (3.40)

We complete the contradiction by the following two claims:

Claim 1: Assuming (3.40) is true, then
\[ \limsup_{\lambda \to 0^+} J_\lambda[S](x_0) > S(x_0). \] (3.41)

Indeed, if (3.41) is not true, then
\[ \limsup_{\lambda \to 0^+} J_\lambda[S](x_0) \leq S(x_0). \] (3.42)

I.e., for any \( \epsilon > 0 \), if \( \lambda \) is small enough, then
\[ J_\lambda[S](x_0) - S(x_0) \leq \epsilon. \] (3.43)

On the other hand, \( S \) is lower semi-continuous, therefore, for sufficiently small \( \lambda > 0 \),
\[ S(x) \geq S(x_0) - \epsilon \quad \text{for } x \in B_\lambda(x_0). \] (3.44)

Now, if we denote \( E_\mu = \{ x \in B_\mu(x_0) | S(x) > \frac{S(x_0) + \tilde{S}(x_0)}{2} \} \), then one also calculates that:
\[ J_\lambda[S](x_0) - S(x_0) = \int_{B_\lambda(x_0)} j_\lambda(x_0 - \eta)(S(\eta) - S(x_0)) \, d\eta \]
\[ \geq \int_{E_{\lambda/2}} j_\lambda(x_0 - \eta) \frac{\tilde{S}(x_0) - S(x_0)}{2} \, d\eta - \int_{B_\lambda(x_0) \setminus E_{\lambda/2}} \epsilon j_\lambda(x_0 - \eta) \, d\eta \]
\[ \geq \frac{C(\tilde{S}(x_0) - S(x_0))}{2\lambda^n} m(E_{\lambda/2}) - \epsilon. \] (3.45)

Combining (3.43) and (3.45) and choosing \( \epsilon \) sufficiently small, one derives
\[ \frac{m(E_{\lambda/2})}{m(B_\lambda)} < \frac{1}{2} \quad \text{for } \lambda < \lambda_0. \] (3.46)

Denote \( F_\lambda = B_\lambda(x_0) \setminus E_\lambda \). Then
\[ S(x) \leq \frac{S(x_0) + \tilde{S}(x_0)}{2} \quad \text{for } x \in F_\lambda \quad \text{and} \quad \frac{m(F_{\lambda/2})}{m(B_\lambda)} \geq \frac{1}{2} \quad \text{when } \lambda < \lambda_0. \] (3.47)

By the upper semi-continuity of \( \tilde{S} \), one derives that for \( \mu < \mu_0 \) and \( x \in B_\mu(x_0) \),
\[ \tilde{S}(x) \leq \tilde{S}(x_0) + \epsilon. \] (3.48)
While, notice that
\[ \int_{\mathbb{R}^n \setminus B_\mu} \gamma_\lambda(\eta) \, d\eta = \int_{\mathbb{R}^n \setminus B_{\mu/\lambda}} \gamma_1(\eta) \, d\eta. \] (3.49)

Hence, \( \int_{\mathbb{R}^n \setminus B_{\mu/\lambda}} \gamma_1(\eta) \, d\eta \) vanishes as \( \mu/\lambda \to \infty \).

Therefore, by choosing \( \mu < \mu_0 \) and \( \lambda \ll \mu \), one estimates:
\[
S^*_{\lambda}(x_0) = \int_{\mathbb{R}^n} S(\eta) \gamma_\lambda(x_0 - \eta) \, d\eta
\leq \int_{\mathbb{R}^n \setminus B_{\mu}(x_0)} S(\eta) \gamma_\lambda(x_0 - \eta) \, d\eta
+ \int_{B_{\mu}(x_0) \setminus F_\lambda} S(\eta) \gamma_\lambda(x_0 - \eta) \, d\eta
+ \int_{F_\lambda} \gamma_\lambda(x_0 - \eta) \, d\eta
\leq \epsilon + \int_{F_\lambda} \gamma_\lambda(x_0 - \eta) \, d\eta \frac{S(x_0) + \tilde{S}(x_0)}{2}
+ \int_{B_{\mu}(x_0) \setminus F_\lambda} \gamma_\lambda(x_0 - \eta) \, d\eta (\tilde{S}(x_0) + \epsilon)
\leq \tilde{S}(x_0) + 2\epsilon - \frac{Cm(F_\lambda)}{\lambda^n} \tilde{S}(x_0) - S(x_0) + \epsilon.
\] (3.50)

Now, by choosing \( \epsilon \) small enough, one can derive
\[ S^*_{\lambda}(x_0) < \tilde{S}(x_0) - \epsilon, \] (3.51)
and thus contradicts the assumption (3.40). This contradiction shows that under the assumption (3.40), one must have (3.41).

Claim 2: There exists a function \( p \in P \) such that
\[ p(x_0) > S(x_0). \] (3.52)

Indeed, from Claim 1, we can assume:
\[
\limsup_{\lambda \to 0^+} J_\lambda[S](x_0) = \lim_{k \to \infty} J_{\lambda_k}[S](x_0) = S(x_0) + 4h \quad (h > 0). \] (3.53)

As a consequence, there exists \( \lambda_1 > 0 \) such that for any \( \lambda < \lambda_1 \),
\[ J_\lambda[S](x_0) > S(x_0) + 3h. \] (3.54)

We now begin the construction of \( p \): 
Seeing Lemma 3.9, there exists \( \sigma > 0 \) such that for any \( X \in \partial \Omega \), the corresponding barrier function constructed in Lemma 3.9, denote as \( p_X \), satisfies:
\[ g_y(x) - p_X(x) \leq |g_y(x) - g_y(X)| + |p_X(X) - p_X(x)| < h \quad \text{for any} \quad x \in B_{2\sigma}(X). \] (3.55)

For such \( \sigma \), there exists \( \lambda_2 > 0 \) such that for any \( \lambda < \lambda_2 \),
\[ J_\lambda[S](x) \leq J_\lambda[g_y](x) \leq g_y(x) + h \quad \text{for any} \quad x \in \mathbb{R}^n \setminus \Omega_\sigma. \] (3.56)

Since \( \overline{\Omega} \setminus \Omega_\sigma \) is compact, and clearly
\[ \bigcup_{X \in \partial \Omega} B_{2\sigma}(X) \supseteq \overline{\Omega} \setminus \Omega_\sigma, \] (3.57)
then there exists a finite subcover, say \( \{ B_{2\sigma}(X_m) \}_{m=1}^M \) that covers \( \Omega \setminus \Omega_\sigma \). We denote by \( p_m = p_{X_m} \) the barrier function corresponding to the boundary points \( X_m \).

We can define a Perron sub-solution:

\[
p(x) = \max \{ J_\lambda[S](x) - 2h, 0, p_1(x), \ldots, p_M(x) \}.
\] (3.58)

One easily observe that for \( \lambda < \lambda_1 \),

\[
p(x_0) \geq J_\lambda[S](x_0) - 2h > S(x_0) + 3h - 2h = S(x_0) + h > S(x_0).
\] (3.59)

We now verify that for \( \lambda < \lambda_0 =: \min \{ \lambda_1, \lambda_2, \sigma \} \), \( p \) indeed is a Perron subsolution:

- \( p \) is continuous, since it is the maximum of finitely many continuous functions.
- Clearly, \( p \geq 0 \).
- In fact, from (3.56), it clearly holds that:

\[
J_\lambda[S](x) - 2h \leq g_y(x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega.
\] (3.60)

As a consequence,

\[
p(x) \leq g_y(x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega.
\] (3.61)

- Observe the fact that for any \( x \in \Omega \setminus \Omega_\sigma \), there is \( 1 \leq m \leq M \), such that \( x \in B_{2\sigma}(X_m) \), and hence there is \( p_m \), such that

\[
J_\lambda[S](x) - 2h \leq g_y(x) - h < p_m(x).
\] (3.62)

In sight of Theorem 2.8, one derives the following:

\[
(-\Delta)^s p \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\] (3.63)

Claim 2 is proved now.

Claim 2 clearly contradicts the definition of \( S \). This contradiction indicates that (3.37) must be true.

Now we are ready to prove Theorem 3.1 indeed, we are going to show \( h_y = S \) is exactly the solution of problem (3.1).

**Proof of Theorem 3.1.** Seeing Lemma 3.11, we only to prove the following two claims:

- \( S \) is \( s \)-harmonic in \( \Omega \)
- \( S \) is smooth in \( \Omega \)

Seeing the fact that,

\[
(-\Delta)^s S \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega),
\] (3.64)

there exists a non-negative Radon measure \( \mu \) on \( \Omega \) such that:

\[
(-\Delta)^s S = -\mu \quad \text{in} \quad \mathcal{D}'(\Omega).
\] (3.65)

We prove \( \mu = 0 \) by contradiction. Suppose \( \mu \neq 0 \), then there exist a ball \( B_{3r}(x_0) \subset \Omega \) such that \( \mu(B_r(x_0)) > 0 \).
For $\lambda < r$, we can define

$$v_\lambda(x) = J_\lambda[S](x) + \int_{B_r(x_0)} J_\lambda[\Phi](x - y) \, dy.$$  \hfill (3.66)

One calculates:

$$v_\lambda(x_0) \geq J_\lambda[S](x_0) + C_s r^{2s-n} \mu(B_r(x_0)).$$  \hfill (3.67)

Clearly,

$$(-\Delta)^s v_\lambda \leq 0 \quad \text{in} \quad D'(\Omega_\lambda).$$  \hfill (3.68)

While

$$v_\lambda(x) \leq J_\lambda[S](x) + C_s (2r)^{2s-n} \mu(B_r(x_0)) \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega.$$  \hfill (3.69)

Hence,

$$v_\lambda(x) - C_s (2r)^{2s-n} \mu(B_r(x_0)) \leq J_\lambda[S](x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega.$$  \hfill (3.70)

and

$$v_\lambda(x) - C_s (2r)^{2s-n} \mu(B_r(x_0)) \geq J_\lambda[S](x_0) + C_s C_s^r r^{2s-n} \mu(B_r(x_0)).$$  \hfill (3.71)

Then, one can use the same technique as in Claim 2 of Lemma 3.11 to construct a Perron sub-solution $p$ such that $p(x_0) > S(x_0)$. A contradiction!

This contradiction shows that $S$ must be $s$ harmonic in $\Omega$.

The smoothness of $S$ in $\Omega$ follows immediately according to Lemma 4.1. \hfill \QED

4 Properties of Green functions

In this section, we always assume $\Omega$ to be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, with $r$ being the uniform radius of exterior balls.

With Green function $G_\Omega(\cdot, \cdot)$ readily defined, it immediately follows from the Definition 3.2 that

**Corollary 4.1.** For any fixed $y \in \Omega$, $G_\Omega(\cdot, y) \in C(\mathbb{R}^n \setminus \{y\}) \cap C^\infty(\Omega \setminus \{y\})$ and satisfies

$$(-\Delta)^s G_\Omega(\cdot, y) = \delta_y \quad \text{in} \quad D'(\Omega).$$  \hfill (4.1)

**Proposition 4.2.** For any fixed $y \in \Omega$ and any $1 \leq p < \frac{n}{n-2s}$, $G_\Omega(\cdot, y) \in L^p(\Omega)$ with

$$\|G_\Omega(\cdot, y)\|_{L^p(\Omega)} \leq C(n, s, p, \text{diam}(\Omega)).$$  \hfill (4.2)

Proposition 4.2 follows immediately from the following inequality:

$$0 \leq G_\Omega(x, y) \leq \Phi(x - y).$$  \hfill (4.3)

**Lemma 4.3** (Barrier function). Let $X \in \partial \Omega$, $B_r(c)$ be an exterior ball of $\Omega$ at $X$. Then for any $x, y \in \Omega$, it holds that

$$G_\Omega(x, y) \leq G_{\mathbb{R}^n \setminus B_r(c)}(x, y).$$  \hfill (4.4)

Here

$$G_{\mathbb{R}^n \setminus B_r(c)}(x, y) = \frac{\kappa(n, s)}{|x - y|^{n-2s}} \int_0^{\rho_E(x, y)} \frac{t^{s-1}}{(1 + t)^\frac{n}{2}} \, dt,$$  \hfill (4.5)

for

$$\rho_E(x, y) = \frac{|x - c|^2 - r^2}{r^2 |x - y|^2}.$$  \hfill (4.6)
Proof. A Kelvin transformation criteria shows that to $G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(x, y)$ is the Green function of $\mathbb{R}^n \setminus \overline{B}_r(c)$ in the sense that

$$(-\Delta)^s G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(\cdot, y) = \delta_y \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \setminus \overline{B}_r(c)). \quad (4.7)$$

Note the fact that both $G_{\mathbb{R}^n \setminus \Omega}(\cdot, y)$ and $G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(\cdot, y)$ are continuous away from $y$, and satisfies:

$$(-\Delta)^s G_{\Omega}(\cdot, y) = \delta_y = (-\Delta)^s G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(\cdot, y) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \setminus \Omega),$$

$G_{\Omega}(x, y) = 0 \leq G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(x, y) \quad \text{for} \quad x \in \mathbb{R}^n \setminus \Omega. \quad (4.8)$

Then (4.4) is guaranteed by the maximum principle 2.3. \qed

**Proposition 4.4.** For any $x, y \in \Omega$, the following inequality holds:

$$G_{\Omega}(x, y) \leq C(n, s, r, \text{diam}(\Omega)) \frac{\text{dist}(x, \partial \Omega)^s}{|x - y|^{n-s}}. \quad (4.9)$$

**Proof.** Fix $x \in \Omega$, there exists $X \in \partial \Omega$ such that

$$|x - X| = \text{dist}(x, \partial \Omega). \quad (4.10)$$

Let $B_r(c)$ be an exterior ball of $\Omega$ at $X$. Then, we estimate as follow:

1. If $|x - y| \leq |x - X|$, then

$$|y - c|^2 - r^2 \leq C(r, \text{diam}(\Omega))(|y - c| - r) \leq C(r, \text{diam}(\Omega))(|x - X| + |x - y|) \leq C(r, \text{diam}(\Omega))|x - y|. \quad (4.11)$$

Then

$$\rho_E(x, y) \leq C(r, \text{diam}(\Omega)) \frac{|x - X|}{|x - y|}. \quad (4.12)$$

As a consequence of Lemma 4.3

$$G_{\Omega}(x, y) \leq G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(x, y) \leq C(n, s) \frac{1}{|x - y|^{n-2s}} (\rho_E(x, y))^s \leq C(n, s, r, \text{diam}(\Omega)) \frac{|x - X|^s}{|x - y|^{n-s}}. \quad (4.13)$$

2. If $|x - y| > |x - X|$, then

$$G_{\Omega}(x, y) \leq G_{\mathbb{R}^n \setminus \overline{B}_r(c)}(x, y) \leq \kappa(n, s) \frac{1}{|x - y|^{n-2s}} \int_0^\infty \frac{t^{s-1}}{(1 + t)^{\frac{n}{2}}} \, dt \leq C(n, s) \frac{|x - X|^s}{|x - y|^{n-s}}. \quad (4.14)$$

Noting $|x - X| = \text{dist}(x, \partial \Omega)$, we finish the proof. \qed

**Remark 8.** Similarly, one can also prove for any $x, y \in \Omega$:

$$G_{\Omega}(x, y) \leq C(n, s, r, \text{diam}(\Omega)) \frac{\text{dist}(y, \partial \Omega)^s}{|x - y|^{n-s}}. \quad (4.15)$$
**Definition 4.5.** Analogously to the definition (2.9), we denote

\[ G_\Omega * f(x) := \int_{\mathbb{R}^n} f(y)G_\Omega(x,y) \, dy = \int_{\Omega} f(y)G_\Omega(x,y) \, dy. \] (4.16)

**Theorem 4.6.** For any \( f \in C(\overline{\Omega}) \), problem

\[
\begin{aligned}
(-\Delta)^s u &= f \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}
\] (4.17)

has a unique continuous solution: \( u = G_\Omega * f \).

**Proof.** For arbitrary \( \varphi \in \mathcal{D}(\Omega) \), one calculates:

\[
(-\Delta)^s u[\varphi] = \int_{\mathbb{R}^n} G_\Omega * f(x) \cdot (-\Delta)^s \varphi(x) \, dx
= \int_{\Omega} G_\Omega * f(x) \cdot (-\Delta)^s \varphi(x) \, dx
= \int_{\Omega} \int_{\Omega} G_\Omega(x,y)f(y)(-\Delta)^s \varphi(x) \, dy \, dx
= \int_{\Omega} \left[ \int_{\Omega} G_\Omega(x,y)(-\Delta)^s \varphi(x) \, dx \right] f(y) \, dy
= \int_{\Omega} \varphi(y)f(y) \, dy.
\] (4.18)

I.e.,

\[ (-\Delta)^s u = f \quad \text{in} \quad \mathcal{D}'(\Omega). \] (4.19)

It is easy to see that \( u \) is continuous on \( \mathbb{R}^n \setminus \overline{\Omega} \) and \( \Omega \). Hence, we only need to show the continuity of \( u = G_\Omega * f \) on \( \partial \Omega \).

Indeed, for \( x \in \Omega \),

\[
|u(x)| = \left| \int_{\Omega} G_\Omega(x,y)f(y) \, dy \right|
\leq \left| \int_{\Omega} G_\Omega(x,y) \, |f(y)| \, dy \right|
\leq C \text{dist}(x,\partial\Omega)^s \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-s}} \, dy
\leq C \text{dist}(x,\partial\Omega)^s \rightarrow 0 \quad \text{as} \quad x \rightarrow \partial \Omega.
\] (4.20)

The continuity is proved. The maximum principle 2.3 ensures the uniqueness of solution. \( \square \)

**Lemma 4.7.** For fixed \( x \in \Omega \),

\[ (-\Delta)^s G_\Omega(x,\cdot) = \delta_x \quad \text{in} \quad \mathcal{D}'(\Omega). \] (4.21)

**Proof.** For any \( \varphi \in \mathcal{D}(\Omega) \), we have \( (-\Delta)^s \varphi \in C(\overline{\Omega}) \). Let \( u = G_\Omega * (-(\Delta)^s \varphi) \). Then Theorem 4.6 shows that \( u \) is the only continuous function satisfying

\[
\begin{aligned}
(-\Delta)^s u &= -(\Delta)^s \varphi \quad \text{in} \quad \Omega, \\
u &= 0 = \varphi \quad \text{on} \quad \partial \Omega.
\end{aligned}
\] (4.22)

Then \( u(x) = \varphi(x) \), which indicates (4.21). \( \square \)
Theorem 4.8. $G_{\Omega} (\cdot, \cdot)$ is symmetric, i.e.
\begin{equation}
G_{\Omega} (x, y) = G_{\Omega} (y, x).
\end{equation}

Proof. For fixed $y \in \Omega$, let $v(x) = G_{\Omega} (x, y) - G_{\Omega} (y, x)$. Then Corollary 4.1 and Lemma 4.7 shows that
\begin{equation}
(-\Delta)^s v = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\end{equation}
As a consequence, $v$ is smooth in $\Omega$. Note that $v(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. And Remark 8 also ensures the continuity of $v$ on $\partial \Omega$. Then we know $v$ is continuous in $\mathbb{R}^n$.

The maximum principle 2.3 then shows that, $v(x) = 0$, i.e. (4.23) holds. □

5 Poisson kernel

In this section, we still assume $\Omega$ to be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, with $r$ being the uniform radius of exterior balls. We also need the following terminologies of domain shifting: Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, we denote
\begin{equation}
\Omega_\epsilon = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \epsilon \} = \{ x \in \Omega \mid \overline{B}_\epsilon(x) \subset \Omega \},
\end{equation}
\begin{equation}
\Omega^\epsilon = \{ x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \epsilon \} = \bigcup_{x \in \Omega} B_\epsilon(x) = \bigcup_{x \in \Omega} \overline{B}_\epsilon(x).
\end{equation}

Definition 5.1. Define the Poisson kernel of $\Omega$ to be
\begin{equation}
P_{\Omega} (x, y) = C_{n, s} \int_{\Omega} \frac{G_{\Omega}(x, z)}{|z - y|^{n+2s}} \, dz \quad x \in \Omega \quad y \in \mathbb{R}^n \setminus \Omega.
\end{equation}

Definition 5.2. Analogously to the definition (2.8), we denote
\begin{equation}
P_{\Omega} \ast g(x) := \begin{cases} 
\int_{\mathbb{R}^n \setminus \Omega} g(y) P_{\Omega}(x, y) \, dy & \text{for} \quad x \in \Omega, \\
g(x) & \text{for} \quad x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\end{equation}

Then one instantly observes that:

Theorem 5.3. For any $g \in C^\infty (\mathbb{R}^n) \cap L_{2s}$ with $g = 0$ in $\Omega$, problem
\begin{equation}
\begin{cases}
(-\Delta)^s u = 0 & \text{in} \quad \Omega, \\
u = g & \text{on} \quad \partial \Omega.
\end{cases}
\end{equation}
has a unique continuous solution: $u = P_{\Omega} \ast g$.

Proof. Indeed, one can calculate for $x \in \Omega$:
\begin{equation}
(-\Delta)^s g(x) = C_{n, s} \int_{\mathbb{R}^n} \frac{g(x) - g(y)}{|x - y|^{n+2s}} \, dy \\
= -C_{n, s} \int_{\mathbb{R}^n \setminus \Omega} \frac{g(y)}{|x - y|^{n+2s}} \, dy.
\end{equation}
Clearly $(-\Delta)^s g \in C(\overline{\Omega})$. Let $v = G_{\Omega} \ast [(-\Delta)^s g]$ then Theorem 4.3 shows that $v \in C(\mathbb{R}^n)$ with
\begin{equation}
\begin{cases}
(-\Delta)^s v = (-\Delta)^s g & \text{in} \quad \Omega, \\
v = 0 & \text{on} \quad \mathbb{R}^n \setminus \Omega.
\end{cases}
\end{equation}
Hence letting \( u = g - v \), we know \( u \) is a continuous solution of (5.5). For \( x \in \Omega \), one calculates
\[
  u(x) = g(x) - v(x) = 0 - \int_{\Omega} G(x, z)(-\Delta)^s g(z) \, dz \\
  = \int_{\Omega} G(x, z) \left[ C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{g(y)}{|z - y|^{n+2s}} \, dy \right] \, dz \\
  = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} g(y) \left[ \int_{\Omega} \frac{G_{\Omega}(x, z)}{|z - y|^{n+2s}} \, dz \right] \, dy \\
  = \int_{\mathbb{R}^n \setminus \Omega} g(y) P_{\Omega}(x, y) \, dy. 
\] (5.8)

The uniqueness comes immediately from the maximum principle 2.3.

With Theorem 5.3 proved, we can estimate:

**Lemma 5.4** (Barrier function). Let \( B_r(c) \) be an exterior ball of \( \Omega \), for \( x \in \Omega \), \( y \in B_r(c) \), one estimates:
\[
  P_{\Omega}(x, y) \leq c(n,s) \left( \frac{|x - c|^2 - r^2}{r^2 - |y - c|^2} \right)^s \frac{1}{|x - y|^n}. 
\] (5.9)

**Proof.** A Kelvin transform argument instantly shows that
\[
  P_{\mathbb{R}^n \setminus B_r(c)}(x, y) = c(n,s) \left( \frac{|x - c|^2 - r^2}{r^2 - |y - c|^2} \right)^s \frac{1}{|x - y|^n} 
\] (5.10)

performs as the Poisson kernel of \( \mathbb{R}^n \setminus B_r(c) \).

Then, for any non-negative function \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) with \( \text{supp}(\varphi) \subset \subset B_r(c) \), one has:
\[
  \begin{cases}
    (-\Delta)^s P_{\Omega} * \varphi = 0 = (-\Delta)^s P_{\mathbb{R}^n \setminus B_r(c)} * \varphi & \text{in } \mathcal{D}'(\Omega), \\
    P_{\Omega} * \varphi = \varphi = P_{\mathbb{R}^n \setminus B_r(c)} * \varphi & \text{in } B_r(c), \\
    P_{\Omega} * \varphi = 0 \leq P_{\mathbb{R}^n \setminus B_r(c)} * \varphi & \text{in } \mathbb{R}^n \setminus (B_r(c) \cup \Omega). 
  \end{cases} 
\] (5.11)

Moreover, seeing Theorem 5.3 we know both \( P_{\Omega} * \varphi \) and \( P_{\mathbb{R}^n \setminus B_r(c)} * \varphi \) are continuous. Then the maximum principle 2.3 gives that, for any given \( x \in \Omega \),
\[
  \int_{B_r(c)} P_{\Omega}(x, y) \varphi(y) \, dy \leq \int_{B_r(c)} P_{\mathbb{R}^n \setminus B_r(c)}(x, y) \varphi(y) \, dy. 
\] (5.12)

This indicates that
\[
  P_{\Omega}(x, y) \leq P_{\mathbb{R}^n \setminus B_r(c)}(x, y) \quad \text{for any } x \in \Omega \text{ and } y \in B_r(c). 
\] (5.13)

**Lemma 5.5.** For \( x \in \Omega \), \( y \in \Omega' \setminus \Omega \), it holds that
\[
  P_{\Omega}(x, y) \leq \frac{C(n, s, r, \text{diam } \Omega)}{\text{dist}(y, \partial \Omega)^s |x - y|^{n-s}}. 
\] (5.14)
Proof. Let $X \in \partial \Omega$ such that $|y - X| = \text{dist}(y, \partial \Omega)$, let $B_r(c)$ be the exterior ball of $\Omega$ at $X$. Then clearly we have $y \in B_r(c)$, and hence Lemma 5.4 shows that

$$P_\Omega(x, y) \leq c(n, s) \left( \frac{|x - c|^2 - r^2}{r^2 - |y - c|^2} \right)^s \frac{1}{|x - y|^{n-s}}. \quad (5.15)$$

Here

$$r^2 - |y - c|^2 \geq r(r - |y - c|) = r|y - X| = r \text{dist}(y, \partial \Omega). \quad (5.16)$$

$$|x - c|^2 - r^2 \leq (2r + \text{diam}(\Omega))(|x - c| - r) \leq (2r + \text{diam}(\Omega))|x - y|. \quad (5.17)$$

Then we have

$$P_\Omega(x, y) \leq C \frac{1}{\text{dist}(y, \partial \Omega)^s} \frac{1}{|x - y|^{n-s}}. \quad (5.18)$$

\[ \blacksquare \]

**Lemma 5.6.** For $x \in \Omega$, $y \in \mathbb{R}^n \setminus \Omega$, it holds that

$$P_\Omega(x, y) \leq C(n, s, \Omega) \frac{\text{dist}(x, \partial \Omega)^s}{\text{dist}(y, \partial \Omega)^{2s} |x - y|^{n-s}}. \quad (5.19)$$

**Proof.** Let $r = \frac{|x-y|}{2}$ From Proposition 4.4 we has

$$G_\Omega(x, z) \leq C(n, s, \Omega) \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-s}}. \quad (5.20)$$

Therefore

$$P_\Omega(x, y) = \int_\Omega G_\Omega(x, z) \frac{1}{|z - y|^{n+2s}} \, dz$$

$$\leq C \int_\Omega \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-s}|z - y|^{n+2s}} \, dz$$

$$= C \left[ \int_{\Omega \setminus B_r(x)} \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-s}|z - y|^{n+2s}} \, dz + \int_{B_r(x)} \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-s}|z - y|^{n+2s}} \, dz \right] \quad (5.21)$$

$$\leq C \left[ \int_{\Omega \setminus B_r(x)} \frac{\text{dist}(x, \partial \Omega)^s}{r^{n-s} \text{dist}(y, \partial \Omega)^{2s}} \, dz + \frac{\text{dist}(x, \partial \Omega)^s}{r^{n-s}} \int_{B_r(x)} \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-s}(2r - |x - z|)^{n+2s}} \, dz \right] \quad (5.22)$$

$$\leq C \frac{\text{dist}(x, \partial \Omega)^s}{r^{n-s} \text{dist}(y, \partial \Omega)^{2s}} \leq C \frac{\text{dist}(x, \partial \Omega)^s}{r^{n-s} \text{dist}(y, \partial \Omega)^{2s}} \quad (5.23)$$

\[ \blacksquare \]

**Lemma 5.7.** For $x \in \Omega$, $y \in \mathbb{R}^n \setminus \Omega^r$, one estimates:

$$P_\Omega(x, y) \leq C(n, s, \Omega) \frac{\text{dist}(x, \partial \Omega)^s}{|x - y|^{n+2s}}. \quad (5.22)$$

**Proof.** For $x \in \Omega$ and $y \in \mathbb{R}^n \setminus \Omega$ with $\text{dist}(y, \partial \Omega) > r$, one can choose $X \in \partial \Omega$ such that $\text{dist}(y, \partial \Omega) = |y - X|$, then

$$|y - x| \leq |y - X| + |X - x| \leq \left( 1 + \frac{\text{diam}(\Omega)}{r} \right) |y - X| = \left( 1 + \frac{\text{diam}(\Omega)}{r} \right) \text{dist}(y, \partial \Omega). \quad (5.24)$$
and also $|y - z| \geq \text{dist}(y, \partial \Omega)$ for any $z \in \Omega$. Therefore

$$P_\Omega(x, y) = C_{n, s} \int_\Omega \frac{G_\Omega(x, z)}{|z - y|^{n+2s}} \, dz$$

$$\leq C(n, s, \Omega) \int_\Omega \frac{1}{\text{dist}(y, \partial \Omega)^{n+2s}} \frac{\text{dist}(x, \partial \Omega)^s}{|x - z|^{n-2s}} \, dz$$

$$\leq \frac{\text{dist}(x, \partial \Omega)^s}{|x - y|^{n+2s}} \int_{B_{\text{diam}(\Omega)}(x)} \frac{1}{|x - z|^{n-2s}} \, dz$$

$$= C(n, s, \Omega) \frac{\text{dist}(x, \partial \Omega)^s}{|x - y|^{n+2s}}.$$

(5.24)

Lemma 5.8. Let $R \geq 1$, $c$ be a point in $\mathbb{R}^n$ with $|c| = 1 + R$, $v \in L^\infty(\mathbb{R}^n)$ satisfies:

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } B_1 = B_1(0), \\ v = 0 & \text{on } B_R(c). \end{cases}$$

(5.25)

Then for a universal positive constant $\eta_0 < 1$,

$$|v(0)| \leq \eta_0 \|v\|_{L^\infty(\mathbb{R}^n)}.$$  

(5.26)

Proof. We only need to prove for $R = 1$ and $|c| = 2$. Let $w = P * v$, as defined in (2.8), then

$$\begin{cases} (-\Delta)^s w = 0 = (-\Delta)^s v & \text{in } B_1, \\ w = v & \text{on } \mathbb{R}^n \setminus B_1. \end{cases}$$

(5.27)

Moreover, both $w$ and $v$ are bounded. Hence, Theorem indicates that

$$w \equiv v.$$

(5.28)

$$|v(0)| = |P * v(0)|$$

$$\leq \int_{\mathbb{R}^n \setminus B_1} P(0, y)|v(y)| \, dy$$

$$\leq \int_{\mathbb{R}^n \setminus (B_1 \cup B_1(c))} P(0, y) \, dy \|v\|_{L^\infty(\mathbb{R}^n)}$$

$$= \left(1 - \int_{B_1(c)} P(0, y) \, dy\right) \|v\|_{L^\infty(\mathbb{R}^n)}.$$  

(5.29)

$$\eta_0$$

Proposition 5.9. For any $x \in \Omega$, the following identity holds:

$$\int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y) \, dy = 1.$$  

(5.30)

Proof. Let

$$u(x) = P_\Omega * \chi_{\mathbb{R}^n \setminus \Omega} = \begin{cases} \int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y) \, dy & \text{for } x \in \Omega, \\ 1 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

(5.31)
It suffices to show that \( u \equiv 1 \).

Indeed, one constructs

\[
g_k(x) = \int_{\text{dist}(y, \Omega) > 1/k} j_{1/k}(x - y) \, dy,
\]

where \( j \) is the mollifier. Then each \( g_k \in C^\infty(\mathbb{R}^n) \cap L_2 \), \( g_k = 0 \) in \( \Omega \) and

\[
g_k \rightharpoonup \chi_{\mathbb{R}^n \setminus \overline{\Omega}} \quad \text{as} \quad k \to \infty.
\]

The monotone convergence theorem immediately shows that

\[
u = \lim_{k \to \infty} P_\Omega \ast g_k.
\]

Theorem 5.3 implies that

\[
(-\Delta)^s [P_\Omega \ast g_k] = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

then the maximum principle 2.3 also gives:

\[
0 \leq P_\Omega \ast g_k \leq 1.
\]

As the point-wise limit of \( P_\Omega \ast g_k \), \( u \) satisfies:

\[
(-\Delta)^s u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \quad \text{and} \quad 0 \leq u \leq 1.
\]

Consider \( v = 1 - u \), we know:

\[
0 \leq v \leq 1, \quad (-\Delta)^s v = 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{and} \quad v = 0 \quad \text{on} \quad \mathbb{R}^n \setminus \Omega.
\]

As a consequence, \( v \in C^\infty(\Omega) \). We now prove \( v = 0 \).

For any \( x \in \Omega \) with \( \text{dist}(x, \partial \Omega) \leq r \), we may say \( X \in B_{\text{dist}(x, \partial \Omega)}(x) \cap \partial \Omega \). Let \( B_r(c) \) be the exterior ball at \( X \), then we have:

\[
(-\Delta)^s v = 0 \quad \text{in} \quad \mathcal{D}'(B_{\text{dist}(x, \partial \Omega)}(x)) \quad \text{and} \quad v = 0 \quad \text{on} \quad B_r(c).
\]

Doing a translation and scaling, one concludes from Lemma 5.8 that

\[
v(x) \leq \eta_0 \|v\|_{L^\infty(\Omega)} = \eta_0 \|v\|_{L^\infty(\Omega)}.
\]

Hence, \( v \leq \eta_0 \|v\|_{L^\infty(\Omega)} \) on \( \mathbb{R}^n \setminus \Omega_r \). Since \( v \) is continuous in \( \Omega \supset \overline{\Omega}_r \), one applies maximum principle 2.3 to \( v \) on \( \Omega_r \) to derive

\[
v(x) \leq \eta_0 \|v\|_{L^\infty(\Omega)} \quad \text{for} \quad x \in \Omega_r.
\]

Therefore

\[
\|v\|_{L^\infty(\Omega)} \leq \eta_0 \|v\|_{L^\infty(\Omega)}.
\]

Seeing that \( \eta_0 < 1 \), we know \( v = 0 \). Hence, \( u \equiv 1 \).

**Lemma 5.10.** For \( g \in C(\mathbb{R}^n \setminus \Omega) \cap L_2 \), \( P_\Omega \ast g \) is continuous in \( \mathbb{R}^n \).

**Proof.** \( P_\Omega \ast g = g \) is continuous in \( \mathbb{R}^n \setminus \Omega \). The continuity of \( P_\Omega \ast g \) in \( \Omega \) is also clear. It suffices to show that \( u \) is continuous on \( \partial \Omega \).
Indeed, for any $X \in \partial \Omega$ and $x \in \Omega$,

$$|P_\Omega \ast g(x) - g(X)| = \left| \int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y)g(y) \, dy - g(X) \right|$$

$$= \left| \int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y)(g(y) - g(X)) \, dy \right|$$

$$\leq \int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y)|g(y) - g(X)| \, dy$$

For any $\varepsilon > 0$, the continuity of $g$ assures that there exists $\sigma > 0$ such that

$$|g(y) - g(X)| < \frac{\varepsilon}{3}$$

for any $y \in B_{2\sigma}(X) \setminus \Omega$. Write

$$\int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y)|g(y) - g(X)| \, dy$$

$$= \left[ \int_{B_{2\sigma}(X) \setminus \Omega} + \int_{\Omega^{c}(\Omega \cup B_{2\sigma}(X))} + \int_{\mathbb{R}^n \setminus \Omega^r} \right] P_\Omega(x, y)|g(y) - g(X)| \, dy$$

$$= I_1 + I_2 + I_3.$$

For $I_1$, one instantly estimates:

$$I_1 \leq \frac{\varepsilon}{3} \int_{B_{2\sigma}(X) \setminus \Omega} P_\Omega(x, y) \, dy < \frac{\varepsilon}{3} \int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y) \, dy = \frac{\varepsilon}{3}.$$

Utilizing Lemma 5.7, one estimates:

$$I_3 \leq C \int_{\mathbb{R}^n \setminus \Omega^r} \frac{\text{dist}(x, \partial \Omega)^s (1 + |g(y)|)}{|x - y|^{n+2s}} \, dy < C \text{dist}(x, \partial \Omega)^s.$$ (5.47)

While for $I_2$, we restrict $x \in B_\sigma(X) \cap \Omega$ and estimate in the following way:

$$I_2 \leq C \int_{\Omega^r(\Omega \cup B_{2\sigma}(X))} P_\Omega(x, y) \, dy.$$ (5.48)

According to Lemma 5.5, for $x \in \Omega \cap B_\sigma(X)$ and $y \in \Omega^r(\Omega \cup B_{2\sigma}(X))$,

$$P_\Omega(x, y) \leq \frac{C}{\text{dist}(y, \partial \Omega)^s |x - y|^{n-s}} \leq \frac{C}{\sigma^{n-s}} \text{dist}(y, \partial \Omega)^{-s}.$$ (5.49)

One observes that $\text{dist}(y, \partial \Omega)^{-s}$ is integrable in $\Omega^r(\Omega \cup B_{2\sigma}(X))$.

On the other hand, According to Lemma 5.6, $P_\Omega(x, y) \to 0$ as $\text{dist}(x, \partial \Omega) \to 0^+$. Hence, the dominated convergence theorem ensures that

$$I_2 \to 0 \quad \text{as} \quad \text{dist}(x, \partial \Omega) \to 0^+.$$ (5.50)

Combining (5.47) and (5.50), one derives that there is $\delta > 0$ such that for any $x \in B_\sigma(X) \cap \Omega$ with $\text{dist}(x, \partial \Omega) < \delta$, we have $I_2 < \frac{\varepsilon}{3}$ and $I_3 < \frac{\varepsilon}{3}$, and hence

$$\int_{\mathbb{R}^n \setminus \Omega} P_\Omega(x, y)|g(y) - g(X)| \, dy = I_1 + I_2 + I_3 < \varepsilon.$$ (5.51)

This shows that $P_\Omega \ast g(x) \to g(X)$ if $x \to X$. 

\[\square\]
Theorem 5.11. For \( g \in C(\mathbb{R}^n \setminus \Omega) \cap \mathcal{L}_{2s} \), problem (5.5) has a unique continuous solution: \( u = P_\Omega \ast g \).

Proof. An approximation method instantly shows that \( u = P_\Omega \ast g \) satisfies:

\[
(-\Delta)^s u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

Lemma 5.10 shows that \( u = P_\Omega \ast g \) is continuous. The uniqueness comes immediately from the maximum principle 2.3.

6 Dirichlet problem

In this section, we first establish the existence and uniqueness theorem for the problem

\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]  

(6.1)

After that, we extend these results to problem (1.1).

One observes the following lemma, which describes the persistence of exterior ball condition when shifting the domain.

Lemma 6.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the uniform exterior ball condition, with \( r \) being the uniform radius of exterior balls. Then one conclude the following:

1. \( \Omega_\epsilon \) also satisfies the uniform exterior ball condition with uniform radius \( r + \epsilon \).

2. If \( \epsilon < r \) then \( \Omega^\epsilon \) satisfies the uniform exterior ball condition with uniform radius \( r - \epsilon \).

Hence, we are ready to show the following uniqueness theorem:

Theorem 6.2 (Uniqueness). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the uniform exterior ball condition, \( 0 < s < 1 \), \( f = 0 \) and suppose \( u \in \mathcal{L}_{2s} \) is a solution of (6.1) satisfying (1.6), then \( u = 0 \) in \( \Omega \).

Proof. First we do the mollify, let \( u_\epsilon(x) = J_\epsilon u(x) \). Then we know \( u_\epsilon \in C^\infty_0(\mathbb{R}^n) \) satisfies the following equations

\[
\begin{cases}
(-\Delta)^s u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\
u_\epsilon = 0 & \text{in } \mathbb{R}^n \setminus \Omega^\epsilon.
\end{cases}
\]  

(6.2)

Then we can represent \( u_\epsilon \) by using Theorem 5.11 on \( \Omega_\epsilon \), as

\[
u_\epsilon(x) = \int_{\mathbb{R}^n \setminus \Omega_\epsilon} P_{\Omega_\epsilon}(x,y) u_\epsilon(y) \, dy, \text{ for } x \in B_{1-\epsilon}.
\]  

(6.3)

Here \( P_{\Omega_\epsilon}(x,y) \) denotes the Poisson kernel of \( \Omega_\epsilon \).

Now for any given \( U \subset \subset \Omega \), let \( 3\sigma = \text{dist}(U, \partial \Omega) \). We choose \( 0 < \epsilon < \sigma \) and estimate \( \| u_\epsilon \|_{L^\infty(U)} \).

Indeed for \( x \in U \),

\[
|u_\epsilon(x)| = \left| \int_{\mathbb{R}^n \setminus \Omega_\epsilon} P_{\Omega_\epsilon}(x,y) u_\epsilon(y) \, dy \right|
\]  

\[
= \left| \int_{\Omega^\epsilon \setminus \Omega_\epsilon} P_{\Omega_\epsilon}(x,y) \left[ \int_{\Omega \setminus \Omega_2\epsilon} u(z) j_\epsilon(y - z) \, dz \right] \, dy \right|
\]  

\[
\leq \int_{\Omega \setminus \Omega_2\epsilon} \left[ \int_{\Omega^\epsilon \setminus \Omega_\epsilon} P_{\Omega_\epsilon}(x,y) j_\epsilon(y - z) \, dy \right] |u(z)| \, dz.
\]  

\]

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We claim that for a positive constant $M$ depending only on $n, s, \Omega$ and $U$, the following inequality holds:

$$A_\epsilon(x,z) \leq M \epsilon^{-s} \quad \text{for any } x \in U \quad \text{and} \quad z \in \Omega \setminus \Omega_2\epsilon.$$  \hspace{1cm} (6.5)

In fact, from Lemma 5.5 we know

$$P_{\Omega_\epsilon}(x,y) \leq \frac{C}{\operatorname{dist}(y,\partial \Omega_\epsilon)^{n-s}} \leq \frac{C}{\sigma^{n-s} \operatorname{dist}(y,\partial \Omega_\epsilon)^{s}} \quad \text{for } x \in U \quad y \in \Omega^c \setminus \Omega_\epsilon.$$  \hspace{1cm} (6.6)

Then

$$A_\epsilon(x,z) \leq M_0 \epsilon^{-n} \int_{\Omega \setminus \Omega_\epsilon} \operatorname{dist}(y,\partial \Omega_\epsilon)^{-s} \chi_{B_\epsilon(z)} \, dy
= M_0 \epsilon^{-n} \left( \int_{\Omega \setminus \Omega_\epsilon} + \int_{\Omega_\epsilon \setminus \Omega} \right) \operatorname{dist}(y,\partial \Omega_\epsilon)^{-s} \chi_{B_\epsilon(z)} \, dy
\leq M_0 \epsilon^{-n} \int_{0}^{\epsilon} \int_{\partial \Omega_\epsilon \cap B_\epsilon(z)} d\mathcal{H}_n^{n-1-t^{-s}} \, dt + M_0 \epsilon^{-n} \epsilon^{-s} |B_\epsilon(z)|
\leq M \epsilon^{-s}.$$  \hspace{1cm} (6.7)

We have proved the claim (6.5). Now, substituting (6.5) into (6.4), one has

$$\|u_\epsilon\|_{L^\infty(U)} \leq M \epsilon^{-s} \int_{\Omega \setminus \Omega_2\epsilon} |u(z)| \, dz.$$  \hspace{1cm} (6.8)

Noting the condition (1.6), by letting $\epsilon \to 0^+$ in (6.8), one obtain:

$$\|u\|_{L^\infty(U)} = 0.$$  \hspace{1cm} (6.9)

Since $U$ in (6.9) is arbitrary, one can derive $\|u\|_{L^\infty(\Omega)} = 0$. \hfill $\Box$

**Remark 9.** A similar approach also shows that under same conditions, if $f \leq 0$ then $u \leq 0$.

Based on this, one can also prove Theorem 1.1 as follow:

**Proof of Theorem 1.1.** Consider $u^- = \max\{-u, 0\}$, Theorem 2.8 shows that

$$\begin{cases}
(-\Delta)^s u^- \leq -c u^- \leq 0 & \text{in } D'(\Omega), \\
u^- = 0 & \text{on } \mathbb{R}^n \setminus \Omega.
\end{cases}$$  \hspace{1cm} (6.10)

Then Remark 9 instantly gives $u^- \leq 0$, hence $u^- = 0$ and $u \geq 0$ in $\Omega$. \hfill $\Box$

**Theorem 6.3** (Existence). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the uniform exterior ball condition, $0 < s < 1$ and suppose $f \in L^1_s(B_1)$. Then $u = G_\Omega * f$ solves the problem (2.10).

**Proof.** We first show that such $u$ satisfies the condition (1.6).

Indeed, for $\epsilon > 0$, one calculates:

$$\frac{1}{\epsilon^s} \int_{\Omega \setminus \Omega_\epsilon} |u(x)| \, dx \leq \int_{\Omega} \left[ \frac{1}{\epsilon^s} \int_{\Omega \setminus \Omega_\epsilon} G_\Omega(x,y) \, dx \right] |f(y)| \, dy.$$  \hspace{1cm} (6.11)

Utilizing Proposition 4.4 one estimates $A_\epsilon$ as follow:
\begin{itemize}
  \item If \( y \in \Omega \setminus \Omega_{2\epsilon} \)

  \[ A_\epsilon(y) \leq \frac{1}{\epsilon^s} \int_{\Omega \setminus \Omega_{2\epsilon}} G_\Omega(x, y) \, dx \]

  \[ \leq \frac{1}{\epsilon^s} \left[ \int_{\Omega \setminus (\Omega_{2\epsilon} \cup B_{\text{dist}}(y, \partial \Omega)(y))} G_\Omega(x, y) \, dx + \int_{B_{\text{dist}}(y, \partial \Omega)(y)} G_\Omega(x, y) \, dx \right] \]

  \[ \leq \frac{C}{\epsilon^s} \left[ \int_{\Omega \setminus (\Omega_{2\epsilon} \cup B_{\text{dist}}(y, \partial \Omega)(y))} \frac{\text{dist}(y, \partial \Omega)^s}{|x-y|^{n-s}} \, dx + \int_{B_{\text{dist}}(y, \partial \Omega)(y)} \frac{1}{4^n} \, dx \right] \]

  \[ \leq \frac{C}{\epsilon^s} \int_0^{4\epsilon} \int_{\partial \Omega \setminus B_{\text{dist}}(y, \partial \Omega)(y)} \frac{t^s}{|x-y|^{n-s}} \, d\mathcal{H}^{n-1}_x \, dt + C \text{dist}(y, \partial \Omega)^s \]

  \[ \left[ \int_0^{2\text{dist}(y, \partial \Omega)} \frac{t^s}{\text{dist}(y, \partial \Omega)^1 - s} \, dt + \int_0^{4\epsilon} \frac{t^s}{\text{dist}(y, \partial \Omega)^1 - s} \, dt \right] \]

  \[ + C \text{dist}(y, \partial \Omega)^s. \]

  \[ \leq C \text{dist}(y, \partial \Omega)^s. \]

  \[ (6.12) \]

  \[ \text{If } y \in \Omega_{2\epsilon} \]

  \[ A_\epsilon(y) \leq \frac{C}{\epsilon^s} \int_{\Omega \setminus \Omega_{2\epsilon}} \frac{\text{dist}(x, \partial \Omega)^s}{|x-y|^{n-s}} \, dx \]

  \[ = \frac{C}{\epsilon^s} \int_0^{\epsilon} \int_{\partial \Omega_{2\epsilon}} \frac{t^s}{|x-y|^{n-s}} \, d\mathcal{H}^{n-1}_x \, dt \]

  \[ \leq \frac{C}{\epsilon^s} \int_0^{\epsilon} \frac{t^s}{\epsilon^{1-s}} \, dt \quad (\text{since dist}(y, \partial \Omega_{\epsilon}) \geq \epsilon) \]

  \[ \leq C \epsilon^s \leq C \text{dist}(y, \partial \Omega)^s. \]

  \[ (6.13) \]

  Therefore, \( A_\epsilon(y) \leq C \text{dist}(y, \partial \Omega)^s \) and \( A_\epsilon(y) \to 0 \) as \( \epsilon \to 0 \).

  Now, we know \( A_\epsilon(y)|f(y)| \leq C \text{dist}(y, \partial \Omega)^s|f(y)| \in L^1(\Omega) \) and \( A_\epsilon(y)|f(y)| \to 0 \) as \( \epsilon \to 0 \), the Lebesgue convergence theorem assures that

  \[ \limsup_{\epsilon \to 0} \frac{1}{\epsilon^s} \int_{\Omega \setminus \Omega_{2\epsilon}} |u(x)| \, dx \leq \lim_{\epsilon \to 0} \int_{\Omega} A_\epsilon(y)|f(y)| \, dy = 0. \]

  \[ (6.14) \]

  I.e. \( u \) satisfies the uniqueness condition \( (1.6) \).

  The above calculation also gives us a basic estimate:

  \[ \|G * f\|_{L^1(\Omega)} \leq C\|f\|_{L^1(\Omega)}. \]

  \[ (6.15) \]

  Since \( f \in L^1(\Omega) \), we can choose \( f_k \in C^\infty(\overline{\Omega}) \cap L^1_s(B_1) \) that converges to \( f \) in \( L^1_s(\Omega) \) as \( k \to \infty \). According to Theorem \[ 4.6 \] there exists a corresponding solution sequence \( \{u_k = G_\Omega * f_k\} \). \[ (6.15) \] implies that \( u_k \) converges to \( u \) in \( L^1(\Omega) \) as \( k \to \infty \). Hence, \( u \) as the distributional limit of \( u_k \) satisfies \( (6.1) \).

  \[ \square \]

  \textbf{Remark 10.} Theorem \[ 1.2 \] is a direct corollary of Theorem \[ 6.2 \] and Theorem \[ 6.3 \]

\end{itemize}

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References

[1] N. Abatangelo, S. Jarohs, and A. Saldaña, Green function and Martin kernel for higher-order fractional Laplacians in balls, Nonlinear Anal., 175 (2018), pp. 173–190.

[2] D. Applebaum, Lévy processes—from probability to finance and quantum groups, Notices Amer. Math. Soc., 51 (2004), pp. 1336–1347.

[3] M. T. Barlow, R. F. Bass, Z.-Q. Chen, and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, Trans. Amer. Math. Soc., 361 (2009), pp. 1963–1999.

[4] H. Berestycki and P.-L. Lions, Une méthode locale pour l’existence de solutions positives de problèmes semi-linéaires elliptiques dans $\mathbb{R}^N$, J. Analyse Math., 38 (1980), pp. 144–187.

[5] M. Bhakta, A. Biswas, D. Ganguly, and L. Montoro, Integral representation of solutions using Green function for fractional Hardy equations, J. Differential Equations, 269 (2020), pp. 5573–5594.

[6] H. Brézis and R. E. L. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations, 2 (1977), pp. 601–614.

[7] T. Buckmaster, M. Colombo, and V. Vicol, Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1, J. Eur. Math. Soc. (JEMS), 24 (2022), pp. 3333–3378.

[8] T. Buckmaster and V. Vicol, Nonuniqueness of weak solutions to the Navier-Stokes equation, Ann. of Math. (2), 189 (2019), pp. 101–144.

[9] C. Bucur, Some observations on the Green function for the ball in the fractional Laplace framework, Commun. Pure Appl. Anal., 15 (2016), pp. 657–699.

[10] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations, 32 (2007), pp. 1245–1260.

[11] ———, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math., 62 (2009), pp. 597–638.

[12] H. Chen, M. Bhakta, and H. Hajaiej, On the bounds of the sum of eigenvalues for a Dirichlet problem involving mixed fractional Laplacians, J. Differential Equations, 317 (2022), pp. 1–31.

[13] W. Chen and C. Li, A priori estimates for solutions to nonlinear elliptic equations, Arch. Rational Mech. Anal., 122 (1993), pp. 145–157.

[14] W. Chen, C. Li, and Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math., 308 (2017), pp. 404–437.

[15] Z.-Q. Chen, P. Kim, and T. Kumagai, Global heat kernel estimates for symmetric jump processes, Trans. Amer. Math. Soc., 363 (2011), pp. 5021–5055.

[16] Z.-Q. Chen and R. Song, Estimates on Green functions and Poisson kernels for symmetric stable processes, Math. Ann., 312 (1998), pp. 465–501.

[17] A. Cheskidov and X. Luo, $L^2$-critical nonuniqueness for the 2d Navier-Stokes equations, 2021.
[18] ———, Nonuniqueness of weak solutions for the transport equation at critical space regularity, Ann. PDE, 7 (2021), pp. Paper No. 2, 45.

[19] ———, Sharp nonuniqueness for the Navier-Stokes equations, Invent. Math., 229 (2022), pp. 987–1054.

[20] D. G. de Figueiredo, P.-L. Lions, and R. D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. (9), 61 (1982), pp. 41–63.

[21] L. C. Evans, Partial differential equations, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[22] E. B. Fabes, B. F. Jones, and N. M. Rivière, The initial value problem for the Navier-Stokes equations with data in $L^p$, Arch. Rational Mech. Anal., 45 (1972), pp. 222–240.

[23] M. M. Fall and T. Weth, Nonexistence results for a class of fractional elliptic boundary value problems, J. Funct. Anal., 263 (2012), pp. 2205–2227.

[24] I. M. Gelfand, Some questions of analysis and differential equations, Amer. Math. Soc. Transl. (2), 26 (1963), pp. 201–219.

[25] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[26] Y. Guo, B. Li, A. Pistoia, and S. Yan, The fractional Brezis-Nirenberg problems on lower dimensions, J. Differential Equations, 286 (2021), pp. 284–331.

[27] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal., 49 (1972/73), pp. 241–269.

[28] O. A. Ladyzhenskaya, Uniqueness and smoothness of generalized solutions of Navier-Stokes equations, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 5 (1967), pp. 169–185.

[29] N. S. Landkof, Foundations of modern potential theory, Die Grundlehren der mathematischen Wissenschaften, Band 180, Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy.

[30] P. D. Lax, Functional analysis, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2002.

[31] C. Li, C. Liu, Z. Wu, and H. Xu, Non-negative solutions to fractional Laplace equations with isolated singularity, Adv. Math., 373 (2020), pp. 107329, 38.

[32] C. Li and L. Wu, Pointwise regularity for fractional equations, J. Differential Equations, 302 (2021), pp. 1–36.

[33] C. Li, Z. Wu, and H. Xu, Maximum principles and Bôcher type theorems, Proc. Natl. Acad. Sci. USA, 115 (2018), pp. 6976–6979.

[34] W. Long, S. Yan, and J. Yang, A critical elliptic problem involving fractional Laplacian operator in domains with shrinking holes, J. Differential Equations, 267 (2019), pp. 4117–4147.

[35] X. Luo, Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions, Arch. Ration. Mech. Anal., 233 (2019), pp. 701–747.
[36] G. Prodi, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4), 48 (1959), pp. 173–182.

[37] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9), 101 (2014), pp. 275–302.

[38] J. Serrin, *The initial value problem for the Navier-Stokes equations*, in Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), Univ. Wisconsin Press, Madison, Wis., 1963, pp. 69–98.

[39] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., 60 (2007), pp. 67–112.

[40] G. M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, Oxford University Press, Oxford, 2008. Reprint of the 2005 original.