Quantum Orders and Spin Liquids in Cs$_2$CuCl$_4$

Yi Zhou$^1$ and Xiao-Gang Wen$^{2,\dagger}$

$^1$Center for Advanced Study, Tsinghua University, Beijing 100084, P. R. China
$^2$Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Motivated by experiments on Cs$_2$CuCl$_4$ samples, we studied and classified the symmetric spin liquids on triangular lattice. We identified 63 $Z_2$ spin liquids, 30 $U(1)$ spin liquids and 2 $SU(2)$ spin liquids. All those spin liquids have the same symmetry but different quantum orders. We calculated the spin spectral functions in some simple spin liquids and compared them to the one measured by experiments on Cs$_2$CuCl$_4$. We find that the $U1C_{1+}C_{1}^{-}r_{1}^{+}$ spin liquid or one of its relatives is consistent with observed properties of the spin liquid state in Cs$_2$CuCl$_4$. We discussed the fine distinctions among those possible spin liquids and the spin liquids proposed using slave-fermion approach, so that future experiments can determine which spin liquids actually describe the spin state in Cs$_2$CuCl$_4$.

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I. INTRODUCTION

Our understanding of states of matter and their internal orders has been dominated by Landau’s symmetry breaking theory. We used to believe the all possible orders are described by various symmetry breaking states. However, after the discovery of fractional quantum Hall (FQH) states, we started to realize that FQH states contain a new kind of order - topological order (which was first proposed to describe spin liquids found in research of high $T_c$ superconductors). What is new about topological orders is that topological orders cannot be characterized by symmetry breaking and the related local order parameters and long range correlations. Topological orders are characterized by new set of universal quantum numbers, such as ground state degeneracy, fractional statistics, edge excitations, etc.

Recently, a concept of quantum order was proposed to describe non-symmetry breaking order that generally appear in a quantum state. The quantum order generalizes the topological order to gapless states. To have a concrete description of quantum order without using order parameters, a new mathematical object - projective symmetry group (PSG) - was introduced. The concept of quantum order and its PSG characterization allow us to understand quantum phases and quantum phase transitions in a systematic way.

Just like the group theory allows us to classify symmetry breaking orders, the PSG characterization of quantum states allows us to classify different quantum orders and can distinguish two different quantum phases even when they have exactly the same symmetry.

Also just like the symmetry description of classical order allows us to obtain low energy properties of system without knowing the details of the systems, the PSG description of quantum orders also allows us to obtain low energy properties of system without knowing the details of the systems. However, unlike symmetries which produce and protect gapless Nambu-Goldstone bosons, the quantum orders can produce and protect gapless collective modes which behave like light and other gapless gauge bosons. Those gapless collective modes can also be gapless/massless fermions, even when the original theory is purely bosonic. More recently, it was realized that the quantum ordered states described by the PSG are actually string-net condensed states. The emerging gauge bosons are the fluctuations of condensed string-nets and emerging fermions are the end of condensed strings.

In this paper, we would like to further develop the PSG description of quantum orders. In Ref. using PSG, a classification of quantum orders in symmetric spin liquids on 2D square lattice was given. Here, we would like to expand the results of Ref. to 2D spin liquids which break the parities $P_x$: $x \rightarrow -x$ and $P_y$: $y \rightarrow -y$. We will assume the spin liquids have the following symmetries: two translation symmetries $T_x, T_y$, two parities $P_{xy}$: $(x, y) \rightarrow (y, x)$ and $P_{y}$: $(x, y) \rightarrow (-y, -x)$, and a time reversal symmetry. Such type of 2D spin liquids was observed recently in Cs$_2$CuCl$_4$ sample. By classifying the quantum orders in those spin liquids, we hope to identify the quantum order in the Cs$_2$CuCl$_4$ sample.

The spin dynamics in Cs$_2$CuCl$_4$ can be described by a 2D spin-1/2 system on a square lattice with nearest neighbor coupling $J' = 0.125\text{meV}$ and one diagonal coupling $J = 0.375\text{meV}$ in the $\hat{x} + \hat{y}$ direction. At temperature $T > T_c = 0.62K$, the 2D spin system was found to be in a liquid state. Since $T_c$ is a small energy scale, we will regard the finite temperature spin liquid state as a zero temperature quantum state. (More precisely, we assume that we can add additional frustrations to lower $T_c$ to zero.) The theoretical spin-wave calculations and series expansion calculations also suggest the existence of spin liquids in the above $J'$ model. In this paper, we are going concentrate on the
following physical issue: what is this spin liquid in the Cs$_2$CuCl$_4$ sample. There are several possibilities.

Since $J'$ is small, one possibility is that the spin liquid, at low energies, just behaves like a decoupled 1D spin-1/2 chains. In other words, the spin liquids is in the same universality class of the decoupled 1D spin-1/2 chains. It is also possible the spin liquid is not purely 1D and has intrinsic 2D correlations. The 2D correlation can lead to 2D dispersion for low lying excitations. Furthermore, there can be several different types of 2D spin liquids that have the same symmetry as the purely 1D spin liquid. Since all those possible spin liquids have the same symmetry, it is difficult to study them without knowing how to characterize them.

In this paper, using quantum order and its PSG, we construct and characterize a large class of 2D spin liquids that have the same symmetry as the purely 1D spin liquid. In section IV we introduce mean-field ansatz that describe the symmetric spin liquids - the spin liquids that do not break any symmetries. In section IV we discuss how to use PSG’s to characterize different mean-field phases (or the universal classes of the mean-field ansatz). In section IV we find all the $Z_2$ and $SU(2)$ PSG’s and a large class of $U(1)$ PSG’s within the $SU(2)$ slave-boson approach. This tells us the possible spin liquids that can be constructed use the $SU(2)$ slave-boson theory.

Constructing a large class of symmetric spin liquids and obtaining their PSG characterization are useful in the following sense. If a spin liquid state is found in Cs$_2$CuCl$_4$ or some other samples which do not break any symmetry, then the spin liquid has a good chance to be in the class that we obtained. Identifying the PSG that characterizes the constructed spin liquid will allow us to identify many universal properties of the spin liquid. We can check those universal properties experimentally which allow us to identify the PSGs for the experimentally observed spin liquids. The classification will also help us the study the phase transitions between symmetric spin liquids that do not change any symmetries.

In section IV we discuss some simple ansatz that realize some of the classified mean-field spin liquids. We calculate the mean-field energies of the constructed spin liquids to determine which spin liquids are likely to appear in the Cs$_2$CuCl$_4$ system. We also calculate spin correlation in those likely spin liquids in section IV. This allows experimentists to determine which spin liquid actually describe Cs$_2$CuCl$_4$ system (above $T_c$) using neutron scattering and other techniques. Section VII discusses connection of our results with experiments and previous theoretical results.

The main point of our calculation is to identify universal properties of various spin liquid states. This point is highly non-trivial since the involved spin liquids all have the same symmetry. We show that the crystal momenta of gapless spin-1 excitations can be used to experimentally distinguish different spin liquids.

We would like to pointed out that there are two ways to study possible spin liquid states in Cs$_2$CuCl$_4$. The first approach is the slave-fermion approach, where one start with a spin ordered state (such as the spiral state) and then study the spin liquid induced by strong spin wave fluctuations. One can obtain the spin spectral function from the slave-fermion approach. The calculated spin spectral function agrees well with observed spin spectral function. In the slave-fermion approach, the spin disordered state (the spin liquid state) always has a gap, with low lying bosonic excitations.

In this paper, we are going to use slave-boson approach to study spin liquid states. In the slave-boson approach, the spin liquid state can either be gapped or gapless. The low lying excitations are fermions. In general slave-boson approach can generate more exotic states then the slave-fermion approach. We will see later that the spin spectral function obtained from the slave-boson approach also agrees well with observed spin spectral function.

However, the spin liquid obtained in Ref. and the spin liquids obtained in this paper are really different. For one thing, the spin liquid obtained in Ref. has a gap, while the spin liquids obtained in this paper is gapless. Because the gap is small, the two spin liquids behave similarly at finite temperatures.

What is the relation between the slave-boson approach and the slave-fermion approach? To understand the relation, we need use the result in Ref. where it was shown that the spin liquids obtained from the slave-boson approach and the slave-fermion approach have a condensation of nets of closed strings. The quantum orders in the spin liquids obtained by both slave-fermion and slave-boson approaches can be described by PSG’s. PSG’s simply characterize different string-net condensations.

In the spin liquids obtained by slave-fermion approach, we concentrate on one type of condensed strings whose ends are bosons (we will call those strings bosonic strings). The PSG is nothing but the symmetry group of the hopping Hamiltonian of the ends of the condensed strings. For different string condensations, the PSG for the ends of the condensed strings are different. Hence PSG can be used to characterize different string condensations (or different quantum orders). In slave-fermion approach, we concentrate on spin liquids with condensation of fermionic strings. The ends of fermionic strings are fermions. We can use the PSG for the ends of fermionic strings to characterize different string condensed (or quantum ordered) states.

Some spin liquids contain condensation of both bosonic strings and fermionic strings. Those states can be characterized by either the PSG for the ends of the bosonic strings or the PSG for the ends of the fermionic strings. Those states can be constructed either by slave-fermion approach or slave-boson approach.
In this paper, we will only consider spin liquids with spin rotation symmetry, and we will assume that the ansatz satisfy the above conditions.

If we regard the ansatz \( (u_{ij}, a_0^\tau) \) as a label of physical wave function, \( \Psi_{\text{spin}}(u_{ij}, a_0^\tau, \{r_i\}) \), then such a label is not a one-to-one label. It has a \( SU(2) \) gauge structure. i.e. two ansatz, \( (u_{ij}, a_0^\tau) \) and \( (W(u_{ij}), W(a_0^\tau)) \), related by a \( SU(2) \) gauge transformation label the same physical wave function:

\[
\Psi_{\text{spin}}(\{\alpha_i\}) = \langle 0 | \prod_i f_{\alpha_i} | \Psi_{\text{mean}}(W(u_{ij}), W(a_0^\tau)) \rangle = \langle 0 | \prod_i f_{\alpha_i} | \Psi_{\text{mean}}(u_{ij}, a_0^\tau) \rangle
\]

where \( W(u_{ij}) = W_i u_{ij} W_i^\dagger \), \( W(a_0^\tau) = W_i a_0^\tau W_i^\dagger \), and \( W_i \in SU(2) \).

III. QUANTUM ORDER AND PROJECTIVE SYMMETRY GROUP

For the symmetry breaking states, the symmetry is a universal property shared by all the states in the same phase. Because of this, the symmetry provides a quantum number that characterize different symmetry breaking orders. Hence we can say symmetry group (SG) describes the internal orders of symmetry breaking states. To characterize and classify different quantum orders which contain no symmetry breaking, we should construct some universal quantum numbers which describe different classes of quantum entanglement in the many-body ground state wave function. Ref. 27 and 28 propose that the symmetry of the mean-field ansatz \( (u_{ij}, a_0^\tau) \) is a universal property and serves as a quantum number that characterize the quantum order in spin liquids. The symmetry of group of the ansatz will be called the projective symmetry group (PSG). An element of PSG is a combined operation of a symmetry transformation followed by a gauge transformation. By definition, a PSG is formed by all the combined operations that leaves the ansatz unchanged.

Because of the \( SU(2) \) gauge structure, the \( (u_{ij}, a_0^\tau) \) labeling of the physical spin wave function \( \Psi_{\text{spin}}(\{\alpha_i\}) \) is not a one-to-one labeling. Two mean-field ansatz differed by a \( SU(2) \) gauge transformation give rise to the same spin wave function. Because of this, it is a non-trivial task to find out the symmetry of a spin liquid label by an ansatz \( (u_{ij}, a_0^\tau) \). In order for a spin wave function to have a symmetry, its corresponding ansatz is only required to be invariant under the symmetric transformation followed by a proper \( SU(2) \) gauge transformation. Thus given two spin wave functions with the same symmetry, their ansatz can be invariant under the same symmetric transformations followed by different gauge transformations. In this case, the two spin liquids with the same symmetry can have different PSG’s. We see

II. SYMMETRIC SPIN LIQUIDS ON A SPIN-1/2 SYSTEM

We consider the spin-1/2 system on a 2D square lattice with nearest-neighbor coupling \( J' \) and next-nearest-neighbor coupling \( J \) on only one of the diagonal link \( (i_x, i_y) \rightarrow (i_x + 1, i_y + 1) \). Such a lattice can also be regarded as a triangular lattice which does not have the 60° rotation symmetry. Within the \( SU(2) \) slave-boson approach 26 27, a general spin wave function can be constructed by introducing a mean-field Hamiltonian

\[
H_{\text{mean}} = -\sum_{\langle i,j \rangle} \left( \psi_i^\dagger u_{ij} \psi_j + h.c. \right) + \sum_i a_0^l \psi_i^\dagger \tau^l \psi_i \tag{1}
\]

where \( \psi^T = (\psi_1, \psi_2) \), \( u_{ij}^l = u_{ij} \), \( \tau^{1,2,3} \) are the Pauli matrices, and \( u_{ij} \) are \( 2 \times 2 \) complex matrices. The collection \( (u_{ij}, a_0^\tau) \) is called a mean-field ansatz 7, 8. For each mean-field ansatz, we can obtain a mean-field ground state by filling the negative energy levels of \( H_{\text{mean}} \) with the spinons \( \psi : |\Psi_{\text{mean}}(u_{ij}, a_0^\tau)\rangle \). The physical spin wave function \( \Psi_{\text{spin}} \) can now be obtained by performing a projection (see 27):

\[
\Psi_{\text{spin}}(u_{ij}, a_0^\tau, \{r_i\}) = \langle 0 | \prod_i (\psi_{1,r_i} \psi_{2,r_i})^\dagger | \Psi_{\text{mean}}(u_{ij}, a_0^\tau) \rangle \tag{2}
\]

where \( r_i \) is the coordinate of the \( i^{th} \) up-spin and \( N_{up} \) is the total number of the up-spin. Here an empty site of \( \psi \) represent a down-spin, and a double occupied site represent a up-spin. Single occupied sties are unphysical states and are projected out.

Our representation does not have explicitly spin rotation symmetry and it is hard to see spin rotation symmetry from the mean-field ansatz. This problem is solved in Ref. 7. We find that a spin liquid is spin-rotation symmetric iff \( u_{ij} \) satisfies

\[
\begin{align*}
    u_{ij} &= i \rho_{ij} W_{ij} \\
    \rho_{ij} &= \text{non-negative real number} \\
    W_{ij} &\in SU(2)
\end{align*}
\]

FIG. 1: The nearest neighbor spin model on a triangular lattice.
that the PSG characterization is more refined than the symmetry group characterization. PSG contains some information about the phase of spin wave function.

To understand the precise relation between PSG and SG, we need to introduce a subgroup of PSG - the invariant gauge group (IGG). IGG is a special subgroup of PSG, which is formed by pure gauge transformations that leave the ansatz unchanged.

\[ IGG = \left\{ W_i | W_i u_i W_j^{-1} = u_{ij}, W_i \in SU(2) \right\} \]

Then SG and PSG are related by \( SG = PSG/IGG \).

Spin liquids may support a special kind of low energy collective excitations - gauge fluctuations. It was shown that the gauge group of those low energy gauge fluctuations is nothing but the IGG of the corresponding ansatz. We see that IGG of an ansatz is very important. In this paper, we will consider only three kinds of IGG, \( SU(2) \), \( U(1) \) and \( Z_2 \). Corresponsibly, we call the corresponding spin liquids \( SU(2) \), \( U(1) \) and \( Z_2 \) spin liquids.

Mathematically, PSG is defined as,

\[ PSG = \{ G_U | G_U U(u_{ij}) = u_{ij}, G_U(i) \in SU(2) \} \]

where \( U(u_{ij}) = \tilde{u}_{ij} = u_{ij}(u_{ij})_i, G_U U(\tilde{u}_{ij}) = G_U(i)\tilde{u}_{ij}(G_U(\tilde{u}_{ij}))_i, U \) generates the symmetry transformation and \( G_U \) is the associated gauge transformation.

A PSG may change under a gauge transformation \( W \). From \( WG_U u_{ij} = W(u_{ij}) \), where \( W(u_{ij}) = W_i u_{ij} W_j^{-1} \), we find that \( WG_U W^{-1} W(u_{ij}) = W(u_{ij}) \). Therefore if \( G_U U \) is in the PSG of ansatz \( u_{ij} \), then \( W G_U W^{-1} \) is in the PSG of the gauge transformed ansatz \( W(u_{ij}) \). We see that the gauge transformation \( G_U \) associated with the transformation \( U \) changes in the following way

\[ G_U(i) \rightarrow W(i) G_U(i) W(U(i))^{-1} \]

under a \( SU(2) \) gauge transformation.

Since PSG is a property of an ansatz, we can group all ansatz sharing the same PSG together to form a class. Such a class is the universal class of quantum states that corresponds a quantum phase.

Now let us consider some simple examples of spin liquids described by the ansatz \( (u_{ij}, a_l^b_{ij} \tau_l) \). The first example is

\[ u_{i,i+m} = u_m = u_m^l \tau^l \]

where \( u_m^l \) are real, \( l = 1,2,3 \). It is easy to obtain the spinon dispersion of such an ansatz,

\[ E_{\pm}(k) = \pm \sqrt{\sum (u_k^l - a_k^l)^2} \]

where \( u_k^l = \sum u_m^l e^{ik \cdot m} \). The Brillouin zone is \( k_x, k_y \in (-\pi, \pi) \).

The second example is

\[ u_{i,i+x} = i \chi \tau^0 + \eta \tau^3, \]
\[ u_{i,i+y} = (-)^i (i \chi \tau^0 + \eta \tau^3), \]
\[ u_{i,i+x+y} = (-)^i \lambda \tau^1, \]

where \( \chi, \eta \) and \( \lambda \) are real. Its spinon spectrum is determined by

\[ H(k) = \left( \chi \sin k_x \tau^0 + \eta \cos k_x \tau^3 \right) \tau^1 \]
\[ + \left( \chi \sin k_y \tau^0 + \eta \cos k_y \tau^3 \right) \tau^3 \]
\[ + \lambda \tau^1 \tau^2 \sin(k_x + k_y) + a_0^l \tau^l \tau^0 \]

where the Brillouin zone is \( k_x \in (-\pi/2, \pi/2), k_y \in (-\pi, \pi). \) The four bands of spinon dispersion have a form of \( \pm E_1(k), \pm E_2(k) \). It seems strange to find that the spinon spectrum is defined only on half of the lattice Brillouin zone. However, this is not inconsistent with the translation symmetry since the single spinon excitation is not physical. Only two-spinon excitations correspond to physical excitations and their spectrum should be defined on the full Brillouin zone. The two-spinon spectrum defined on the full Brillouin zone can be constructed form single-spinon spectrum.

\[ E_{2s}(k) = E_{\alpha_1}(k_1) + E_{\alpha_2}(k_2) \]
\[ k = k_1 + k_2 + n \pi \hat{x} \]

where \( n = 0,1 \) and \( \alpha_{1,2} = 1,2 \) is the sub-index of the single spinon dispersion \( \pm E_1(k), \pm E_2(k) \). We note that the physical spin-1 excitations are formed by two-spinon excitations in the mean-field theory. Thus the The two-spinon spectrum \( E_{2s} \) is also the spectrum of physical spin-1 excitation and can be measured in experiments.

### IV. CLASSIFICATION OF SYMMETRIC SPIN LIQUIDS

In this section, we classify symmetric spin liquids on triangular lattices. We consider the spin liquids which are invariant under translation transformation \( T_x(i \rightarrow i + \hat{x}) \) and \( T_y(i \rightarrow i + \hat{y}) \), parity transformation \( P_{xy}((i_x,i_y) \rightarrow (i_y,i_x)) \) and \( P_{xy}((i_x,i_y) \rightarrow (-i_y,-i_x)) \), spin rotation transformation, spin-parity transformation \( T^* (S_x \rightarrow S_x, S_y \rightarrow -S_y, S_z \rightarrow S_z) \). The spin rotation symmetry requires the ansatz take the form of Eq. 5. Under the spin-parity transformation \( T^* \), the three components of spin change as \( S_x \rightarrow S_x, S_y \rightarrow -S_y, S_z \rightarrow S_z \). This transformation can be realized through wavefunction transformation \( \Phi \rightarrow \Phi^* \),

\[ \langle \tilde{S} \rangle = \frac{1}{2} \int \Phi^* f^l \bar{\sigma} f \Phi \rightarrow \frac{1}{2} \int \Phi^* f^l \bar{\sigma} f \Phi^*, \]

under the \( S_z \) representation \( f \) and \( f^l \) are real, then

\[ \langle \tilde{S} \rangle^* = \langle \tilde{S} \rangle = \frac{1}{2} \int \Phi^* f^l \bar{\sigma} f \Phi \]
\[
\rightarrow \frac{1}{2} \int \Phi^f \Phi^* = \left( \frac{1}{2} \int \Phi^f \Phi^* \right)^*,
\]

which leads to \( S_x \rightarrow S_x, S_y \rightarrow -S_y, S_z \rightarrow S_z \). The ansatz \( u_{ij} \) transfer as \( u_{ij} \rightarrow u_{\hat{ij}} \), combined with a gauge transformation \( u_{ij} \rightarrow (i \tau^2) u_{ij} (-i \tau^2) \), we denote this combined transformation as

\[ T^* : u_{ij} \rightarrow -u_{ij}. \tag{11} \]

Let us firstly discuss the method to classify the PSG briefly and leave the details to the appendix. For two given symmetry transformations, whose corresponding elements in a PSG, must satisfy some algebraic relations. Solving these equations allows us to construct a PSG which will be called the algebraic PSG. The new name algebraic PSG is introduced to distinguish them from the invariant PSG defined in the previous section. Any invariant PSG will be algebraic PSG. But an algebraic PSG may not be an invariant PSG unless there does exist an ansatz such that the algebraic PSG is the total symmetry group of the ansatz.

As an example, let us consider the two translations \( T_x \) and \( T_y \), which satisfy the following relation

\[
T_x T_y T_x^{-1} T_y^{-1} = 1. \tag{12}
\]

From the definition of PSG, we find that the two elements of PSG, \( G_x T_x \) and \( G_y T_y \), must satisfy

\[
G_x T_x G_y T_y (G_x T_x)^{-1} (G_y T_y)^{-1} = G_x T_x G_y T_y T_x^{-1} G_x^{-1} T_y^{-1} G_y^{-1} = G_x (i) G_x (i - \hat{x}) G_x^{-1} (i - \hat{y}) G_y^{-1} (i) \in G. \tag{13}
\]

Hereafter we will denote the IGG as \( G \). Each solution of the equation (13) corresponds to an algebraic PSG for \( T_x \) and \( T_y \).

By adding other symmetry transformations, we can find and classify all the algebraic PSG associated with a given symmetry group. Since an invariant PSG is always an algebraic PSG, we can check whether the algebraic PSG is an invariant PSG through constructing an explicit ansatz \( u_{ij} \). If an algebraic PSG supports an ansatz \( u_{ij} \) with no addition symmetry, then it is an invariant PSG. Through this method, we can classify the symmetric spin liquids through PSG.

In the appendix, we classify spin liquids with the above mentioned symmetries through PSG’s. We limit ourselves to \( SU(2), U(1) \) and \( Z_2 \) spin liquids (we only consider those PSG’s whose IGG is one of \( SU(2), U(1) \) and \( Z_2 \)). We found that there are 63 kinds of \( Z_2 \) spin liquids, 30 kinds of \( U(1) \) spin liquids and 2 kinds of \( SU(2) \) spin liquids with \( T_{x,y}, P_{xy,x\bar{y}}, T^* \) and spin rotation symmetries.

V. APPLICATION TO NEAREST-NEIGHBOR SPIN COUPLING MODEL

In this section, we present our classification results via some simple examples. We will assume only \( u_{i,i+x}, u_{i,i+x+y}, \) and \( u_{i,i+x+y} \) are no zero. In this case, we have the following 7 kinds of \( Z_2 \) spin liquids and 3 kinds of \( U(1) \) (Other \( Z_2 \) and \( U(1) \) spin liquids require non-vanishing \( u_{ij} \) on longer bonds.)

The 7 \( Z_2 \) spin liquids are:

1. \( 2A\tau^0 \tau^3 \) spin liquid

\[
u_{i,i+x} = \chi^1 + \eta \tau^2, \quad u_{i,i+y} = \chi^1 + \eta^2, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]

2. \( 2A\tau^1 \tau^3 \) spin liquid

\[
u_{i,i+x} = \chi^1 + \eta \tau^2, \quad u_{i,i+y} = \chi^1 + \eta^2, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]

3. \( 2A\tau^0 \tau^1 \tau^3 \) spin liquid

\[
u_{i,i+x} = i \chi^0 + \eta \tau^3, \quad u_{i,i+y} = i \chi^0 + \eta^3, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]

4. \( 2A\tau^1 \tau^3 \tau^3 \) spin liquid

\[
u_{i,i+x} = i \chi^0 + \eta \tau^3, \quad u_{i,i+y} = i \chi^0 + \eta^3, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]

5. \( 2B\tau^3 \tau^0 \tau^3 \) spin liquid

\[
u_{i,i+x} = \chi^1 + \eta \tau^2, \quad u_{i,i+y} = \chi^1 + \eta^2, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]

6. \( 2B\tau^1 \tau^2 \tau^3 \) spin liquid

\[
u_{i,i+x} = \chi^1 + \eta \tau^2, \quad u_{i,i+y} = \chi^1 + \eta^2, \quad u_{i,i+x+y} = \lambda^1, \quad a_0^1 = a_1, a_0^2 = a_2, a_0^3 = 0
\]
We will call such a state SUIGG is SU states, we need to find their PSGs. 

The labeling scheme of these spin liquids is defined in the appendix. We have performed the self-consistent mean-field calculation for our J-J’ model. We found many self-consistent mean-field solutions. The energies of four of them are plotted in Fig. 2, where we have set J + J’ = 1. To understand the physical properties of those mean-field states, we need to find their PSGs.

The A phase in Fig. 2 is a one-dimensional state. Its IGG is SU∞(2), one SU(2) for each decoupled 1D chain. We will call such a state SU∞(2) spin liquid. Its ansatz is given by

\[ u_{i,i+\hat{x}} = \chi \tau^3, \]
\[ u_{i,i+\hat{y}} = - (-)^i \chi \tau^3, \]
\[ u_{i,i+\hat{x}+\hat{y}} = (-)^i \lambda \tau^3, \]
\[ a^{1,2,3}_0 = 0, a^3 = a_3. \]

The 3 U(1) spin liquids are:

1. U1A\(\tau^0_+ \tau^0_+ \tau^1_+\) spin liquids:

\[ u_{i,i+\hat{x}} = \chi \tau^3, \]
\[ u_{i,i+\hat{y}} = - (-)^i \chi \tau^3, \]
\[ u_{i,i+\hat{x}+\hat{y}} = (-)^i \lambda \tau^3, \]
\[ a^{1,2,3}_0 = 0, a^3 = a_3. \]

2. U1B\(\tau^1_+ \tau^0_+ \tau^1_+\) spin liquids:

\[ u_{i,i+\hat{x}} = \chi \tau^3, \]
\[ u_{i,i+\hat{y}} = - (-)^i \chi \tau^3, \]
\[ u_{i,i+\hat{x}+\hat{y}} = (-)^i \lambda \tau^3, \]
\[ a^{1,2,3}_0 = 0. \]

3. U1C\(\tau^0_+ \tau^0_+ \tau^1\) spin liquids:

\[ u_{i,i+\hat{x}} = \chi \tau^2, \]
\[ u_{i,i+\hat{y}} = - \chi \tau^2, \]
\[ u_{i,i+\hat{x}+\hat{y}} = \lambda \tau^3, \]
\[ a^{1,2,3}_0 = 0, a^3 = a_3. \]

The labeling scheme of these spin liquids is defined in the appendix.

The B phase in Fig. 2 is the one-dimensional state. Its IGG is SU∞(2), one SU(2) for each decoupled 1D chain. We will call such a state SU∞(2) spin liquid. Its ansatz is given by

\[ u_{i,i+\hat{x}+\hat{y}} = \lambda \tau^3, a^{1,2,3}_0 = 0, \]

with spinon dispersion

\[ E(k) = \pm 2\lambda \cos(k_x + k_y) \] (24)

This phase, favored by small J’, corresponds to the phase of decoupled 1D spin chain mentioned earlier.

The B phase is the U1B\(\tau^1_+ \tau^0_+ \tau^1_+\) spin liquid. Its four spinon bands are given by

\[ \pm 2 \sqrt{\chi^2 \cos^2 k_x + \cos^2 k_y} + \lambda^2 \sin^2(k_x + k_y) \] (25)

The spinon dispersion of the \(E_+(k)\) and the lower edge of the spectrum \(E_{2s}(k)\) of the physical spin-1 excitations are plotted as functions of \((k_x/\pi, k_y/\pi)\) in Fig. 3. This phase is one of 2D spin liquid phases.

The C phase has a PSG U1C\(\tau^0_+ \tau^0_+ \tau^1\). The spinon dispersion is given by

\[ E_+(k) = 2 \sqrt{\chi^2 \cos^2 k_x + \cos^2 k_y} + \lambda^2 \sin^2(k_x + k_y) \] (26)

The spinon dispersion \(E_+(k)\) and the lower edge of the spin-1 spectrum \(E_{2s}(k)\) are plotted as functions of \((k_x/\pi, k_y/\pi)\) in Fig. 4. This phase is another 2D spin liquid phase.

The D phase in Fig. 2 is the \(\pi\)-flux phase (SU2B spin
The spinon dispersion is of \((k, E)\) the spin-1 spectrum energy dispersion of the one is the \(Z\) the spin-1 spectrum a function of \((k, k_y/\pi)\) for the \(U1C\)\(_1^0\) \(C\)\(_2\) \(C\)\(_3\) state.

FIG. 4: (a) Contour plot of the spinon dispersion \(E_+(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(U1C\)\(_1^0\) \(C\)\(_2\) \(C\)\(_3\) state. (b) Lower edge of the two-spinon spectrum \(E_{2s}(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(U1C\)\(_1^0\) \(C\)\(_2\) \(C\)\(_3\) state.

\[ u_{i,i+\hat{x}} = \chi x, \]
\[ u_{i,i+\hat{y}} = \chi y, \]
\[ a_{1,2,3} = 0. \]

The spinon dispersion is

\[ E_+(k) = 2\chi \sqrt{\cos^2 k_x + \cos^2 k_y} \]

The spinon dispersion \(E_+(k)\) and the lower edge of the spin-1 spectrum \(E_{2s}(k)\) are plotted as functions of \((k_x/\pi, k_y/\pi)\) in Fig. 5.

FIG. 5: a) Contour plot of the spinon dispersion \(E_+(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(\pi\)-flux state. (b) Lower edge of the two-spinon spectrum \(E_{2s}(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(\pi\)-flux state.

Now we consider two other \(Z_2\) spin liquids. The first one is the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^1\) spin liquid state in Eq. (14). The energy dispersion of the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^1\) state is given by

\[ E_+(k) = 2\sqrt{\epsilon_1^2 + \epsilon_2^2} \]
\[ \epsilon_1 = \chi (\cos k_x + \cos k_y) + \lambda \cos (k_x + k_y) - a_1 \]
\[ \epsilon_2 = \eta (\cos k_x - \cos k_y) \]

The spinon dispersion \(E_+(k)\) and the lower edge of the spin-1 spectrum \(E_{2s}(k)\) are plotted as functions of \((k_x/\pi, k_y/\pi)\) in Fig. 6.

FIG. 6: (a) Contour plot of the spinon dispersion \(E_+(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^1\) state. (b) Lower edge of the two-spinon spectrum \(E_{2s}(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^1\) state.

The second one is the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^3\) spin liquid state in Eq. (15). Energy dispersion of the \(Z2A\)\(_1^0\) \(\tau_+^0\) \(\tau_+^3\) state is given by

\[ E_+(k) = 2\sqrt{\epsilon_1^2 + \epsilon_2^2} \]
\[ \epsilon_1 = \chi (\cos k_x + \cos k_y) + \lambda \cos (k_x + k_y) - a_1 \]
\[ \epsilon_2 = \eta (\cos k_x - \cos k_y) \]

The spinon dispersion \(E_+(k)\) and the lower edge of the spin-1 spectrum \(E_{2s}(k)\) are plotted as functions of \((k_x/\pi, k_y/\pi)\) in Fig. 7.

FIG. 7: (a) Contour plot of the spinon dispersion \(E_+(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(Z2A\)\(_1^1\) \(\tau_+^1\) \(\tau_+^1\) state. (b) Lower edge of the two-spinon spectrum \(E_{2s}(k)\) as a function of \((k_x/\pi, k_y/\pi)\) for the \(Z2A\)\(_1^1\) \(\tau_+^1\) \(\tau_+^1\) state.

If the mean-field state is stable against the gauge fluctuations, we expect that the mean-field spin-1 spectrum \(E_{2s}\) should qualitatively agree with the real spin-1 spectrum.

VI. SPIN SPECTRAL FUNCTION

The quantity of interest for comparison with experiments is the spin spectral function in \((q, \omega)\) space \(S(q, \omega)\). \(S(q, \omega)\) can be calculated from the Fourier transformation of two spin correlation function in real space.
Due to the spin-rotation invariance, we obtain that
\[ \langle \vec{S}_i(t) \cdot \vec{S}_j(0) \rangle = 3 \langle S_i^r S_j^r \rangle = 3 \langle S_i^\theta S_j^\theta \rangle \]
The spin spectral function at zero temperature is given by
\[ S(q, \omega) = 3 S_{zz}^z(q, \omega) \] (32)
where
\[ S_{zz}^z(q, \omega) = \sum_{\lambda} 2\pi | \langle \lambda | S^z(q) | G \rangle |^2 \delta(E_G + \omega - E_\lambda) \] (33)
\[ S^z(q) = \sum_j e^{iq \cdot j} S^z_j = \frac{1}{2} \left( \sum_{k,\alpha} \psi^\dagger_{k+q,\alpha} \psi_{k,\alpha} - N \delta_{q,0} \right) \] (34)
the summation is over all the eigenstates |\lambda\rangle and N is the number of lattice sites.
Numerical calculation indicates that the neutron scattering spectrum predicted by the dynamics spin correlation contains continuous modes besides the main sharp peak. The Fig. 8-11 show the spin spectral function \( S_{zz}^z(q, \omega) \) in the q-space. The point \( \Gamma \) is (0,0), \( M \) is (0,\( \pi \)), and \( X \) is (\( \pi \),\( \pi \)). Darker shade represents a larger spectral function. These results can be compared with the inelastic neutron scattering experiments. The scan from \( \Gamma \) to \( X \) corresponds to the scan from \( k = 0 \) to \( k = 2\pi/b \) in Fig. 2a of Ref. [18].

VII. COMPARISON WITH EXPERIMENTS AND PREVIOUS THEORETICAL RESULTS

Our \( J-J' \) model is designed to describe the spin fluctuations in the Cs\(_2\)CuCl\(_4\) sample. The spin dynamics in Cs\(_2\)CuCl\(_4\) at low temperatures were measured by neutron scattering. [18] To the first order approximation, the measured spin spectral function is similar to that of decoupled 1D spin chain (or the \( SU^\infty(2) \) spin liquid) (see Fig. 8). The decoupled 1D spin chain has gapless spin-1 excitations along the lines \( k_x + k_y = \pi \) and \( k_x + k_y = 2\pi \). If we examine the measured spectral function more closely to determine the low energy boundary of spin-1 excitation \( \varepsilon_1(k_x, k_y) \), we find the gapless spin-1 excitations actually only appear at certain isolated points, implying that the spin state is 2-dimensional. The
The interaction due to fluctuations enhances the spin quantitatively with experimental measured results\cite{18,19} we eventually describes the sample. Further experiments can determine which spin liquid ac-

tions open a small gap near (0,0), (π, π), (π/2 + e, π/2 + e) and ±(π/2 - e, π/2 - e). The Z2Aτ^1 r^3 τ^3 spin liquid only has gapless spin-1 excitations at (0,0), ±(π - e', π - e'), ±(π/2 + e, π/2 + e) and ±(π/2 - e, π/2 - e). The Z2Aτ^2 r^3 τ^3 and Z2Aτ^0 r^1 τ^3 spin liquids only have gapless spin-1 excitations in small patches near (0,0), (π, π), and ±(π/2, π/2). By measuring the locations of gapless spin-1 excitations and their spectral functions, we hope further experiments can determine which spin liquid actually describes the sample.

However, to compare the spin spectral function quantitatively with experimental measured results\cite{18,19} we need to include fluctuations around the mean-field state. The interaction due to fluctuations enhances the spin fluctuations at (π/2, π/2). The strong enough interac-

tion can drive the transition from the spin liquid state to the spiral state.

To include the interaction enhancement of the (π/2, π/2) spin fluctuations, we consider the effects of the J-term (the diagonal coupling term): \[\sum_{i} J S_{i} \cdot S_{i+x+y}.\] Within the random phase approximation (RPA), the spin correlation function \(\pi(k, \omega)\) can be obtained from the mean-field spin correlation function \(\pi_{0}(k, \omega)\):

\[
\pi(k, \omega) = \frac{\pi_{0}(k, \omega)}{1 - J \cos(k + k_{y}) \pi_{0}(k, \omega)}
\]  

The imaginary part of \(\pi(k, \omega)\) gives us the RPA spin spectral function. To be definite, we start with the mean-field \(U1C_{x}^{0} r^{0} r^{1}\) spin liquid state and choose \(\pi_{0}(k, \omega)\) to be the mean-field spin correlation function of the \(U1C_{x}^{0} r^{0} r^{1}\) state. The experiments\cite{18,19} indicate that the system is close to the spiral phase. So here we choose the value of \(J\) to make the system close to the spiral phase instability. The resulting RPA spin spectral function is plotted in Fig. 12 and Fig. 13. The spin spectral functions along the lines A, C, and D in Fig. 12 were measured by experiments.\cite{18} The simple RPA results are quite similar to the measured results (see Fig. 3 in Ref. 18).

We would like to remark that the low energy effective theory for the \(U1C_{x}^{0} r^{0} r^{1}\) state contain gapless fermions interacting with \(U(1)\) gauge field. Depending on that strength of the \(U(1)\) gauge fluctuations, the \(U(1)\) gauge interaction may or may not destabilize the \(U1C_{x}^{0} r^{0} r^{1}\) state. It is shown that if the spinons has a very anisotropic dispersion, the instanton effect of the 2+1 \(U(1)\) gauge theory is irrelevant.\cite{17} In this case, the \(U1C_{x}^{0} r^{0} r^{1}\) state is an algebraic spin liquid,\cite{34} which contains gapless spin-1 excitations, but non of the gapless excitations are described by well defined quasiparticles. If the spinons has a more isotropic dispersion, the instanton effect is relevant, which lead to a confinement of the
In this paper, we use the projective symmetry group (PSG) to characterize quantum orders in symmetric spin liquids on 2D triangular lattice. We classify these symmetric spin liquids through PSG, and find that there are 63 kinds of $Z_2$ spin liquids, 30 kinds of $U(1)$ spin liquids and 2 kinds of $SU(2)$ spin liquids. The mean-field phase diagram Fig. 2. for the nearest neighbor spin coupling system is calculated. One-spinon and two-spinon excitation spectrum for some $Z_2$ and $U(1)$ spin liquids are inspected. We show that the gapless spinon-1 excitation spectrum can be used to physically measure the quantum order. By examining the measured spin spectral function in Cs$_2$CuCl$_4$ sample, we find that it is possible that the spin liquid state in Cs$_2$CuCl$_4$ is described by the $U1C\tau_0^0\tau_1^0\tau_1^1$ spin liquid or one of its relatives.

We would like to point out that the RPA spin spectral function for the $U1C\tau_0^0\tau_1^0\tau_1^1$ spin liquid is very close to the spin spectral function obtained from the slave-fermion approach. In the experiments, the spin liquid was observed only at a low but finite temperature. At that temperature, it is hard to distinguish to the two proposed states. However, if we find a spin liquid state at zero temperature, then we can tell which of the two proposed spin liquids can describe the experiments. This is because at zero temperature, the $U1C\tau_0^0\tau_1^0\tau_1^1$ spin liquid (and its relatives) and the spin liquid from the slave-fermion approach are qualitatively different. The $U1C\tau_0^0\tau_1^0\tau_1^1$ spin liquid (and its relatives) contains gapless spin-1 excitations while the spin liquid from the slave-fermion approach is fully gapped.

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APPENDIX A: CLASSIFICATION OF $Z_2$ SPIN LIQUIDS

In this appendix we consider the connected ansatz only, which is interesting and cover a wide range of spin liquids in fact. For any two sites $i$ and $j$, a connected ansatz will offer a path $(i_{1}i_{2}\cdots i_{n})$, which satisfy that $u_{i_{k}i_{k+1}} \neq 0$, $k = 1, 2, \ldots, n$, $i_{1} = i$ and $i_{n} = j$. Due to the translation symmetry of the ansatz, we can choose a gauge in which all the loop operators of the ansatz are translation invariant. We will call such a gauge uniform gauge. Hereafter we will work in uniform gauge. According to the definition of PSG and uniform gauge,

$$G_{x}(i)P_{TC}G_{x}^\dagger(i) = G_{x}(i)P_{C}G_{x}^{\dagger}(i) = P_{C},$$

$$G_{x}(i)P_{C} = P_{C}G_{x}(i).$$

For $Z_2$ spin liquids, different loop operators basing at the same base point do not commute. We see that translation invariance of $P_{C}$ in the uniform gauge requires that

$$G_{x}(i), G_{y}(i) \in G = \{ \pm \tau_{0} \}.$$

Gauge transformations $W(i) = (-)^{f(i)} \tau_{0}$ ($f(i) = \pm 1$) do not change the translation invariant property of the loop operators. We can always choose a gauge which satisfy that :

$$G_{y}(i) = \tau_{0}.$$

(A1)

Gauge transformation $W(i) = W(iz)$ does not change the condition $G_{y}(i) = \tau_{0}$, hence we can gauge fix that

$$G_{x}(iz, iy = 0) = \tau_{0}.$$  

(A2)

From the relations (A3), it is easy to see that

$$G_{x}(i)G_{y}(i - \hat{x})G_{x}^{-1}(i - \hat{y})G_{y}^{-1}(i) = \pm \tau_{0}.$$  

(A3)

Substituting Eqs. (A1) and (A2) into the above, one sees that

$$G_{x}(i)G_{x}^{-1}(i - \hat{y}) = \pm \tau_{0}.$$  

(A4)

Therefore there are two classes of PSG belong to translation symmetry,

$$G_{x}(i) = G_{y}(i) = \tau_{0}$$  

(A5)

and

$$G_{x}(i) = (-)^{\nu} \tau_{0}, G_{y}(i) = \tau_{0}.$$  

(A6)
We will label the former \(Z2A\) and the later \(Z2B\).
Then we consider the spin parity symmetry \(T^*\) which satisfies that
\[
T^*T_xT^{-1}x^{-1} = T^*T_yT^{-1}y^{-1} = T^*T_xT_y = \pm \mathbb{I}.
\]
(A7)

In a similar way, we have the following relations
\[
G_{T^*}T^*G_xT_x(G_{T^*}T_x)^{-1}(G_xT_x)^{-1} \in \mathcal{G},
G_{T^*}T^*G_yT_y(G_{T^*}T_y)^{-1}(G_yT_y)^{-1} \in \mathcal{G},
G_{T^*}T^*G_T^* \in \mathcal{G},
\]
(A8)

then
\[
G_{T^*} (i) G_x (i) G_{T^*} (i - \check{z}) G_x^{-1} = \eta_{zt} T^0,
G_{T^*} (i) G_y (i) G_{T^*} (i - \check{y}) G_y^{-1} = \eta_{yt} T^0,
G_{T^*} (i)^2 = \pm T^0,
\]
(A9)

\[
G_{T^*} (i) = \eta_{ix} \eta_{iy} T^0
\]
(A10)

where \(\eta_{zt} = \pm 1\), \(\eta_{yt} = \pm 1\) and \(g_{T^*}^2 = \pm T^0\). Two gauge inequivalent choices of \(g_{T^*}\) are \(g_{T^*} = T^0\) and \(g_{T^*} = i T^3\).

Next we add two types parity transformations \(P_{xy} (i_x, i_y) = (i_y, i_x)\) and \(P_{xy} (i_x, i_y) = (-i_y, -i_x)\), which satisfy that
\[
T_x P_{xy} T_y^{-1} P_{xy} = 1,
T_y P_{xy} T_x^{-1} P_{xy} = 1,
T_x P_{zy} T_y P_{zy} = 1,
T_y P_{zy} T_x^{-1} P_{zy} = 1,
T^* T_x P_{xy} T_y^{-1} P_{xy} = 1,
T^* T_y P_{xy} T_x^{-1} P_{xy} = 1,
\]
(A11)

From the above equations, we find that
\[
(G_x T_x)(G_{P_{xy}} T_x)(G_{P_{xy}} T_y)^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
(G_y T_y)(G_{P_{xy}} P_{xy})^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
(G_x T_x)(G_{P_{xy}} P_{xy})^{-1}(G_x T_x)^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
(G_y T_y)(G_{P_{xy}} P_{xy})^{-1}(G_y T_y)^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
G_{T^*} P_{xy}(G_{P_{xy}} P_{xy})(G_{T^*} P_{xy})^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
G_{T^*} P_{xy}(G_{T^*} P_{xy})^{-1}(G_{P_{xy}} P_{xy})^{-1} \in \mathcal{G},
\]
(A12)

and
\[
G_x (i) G_{P_{xy}} (i - \check{z}) G_x^{-1} (P_{xy} (i)) G_{P_{xy}}^{-1} (i) = \eta_{xpxy} T^0,
G_y (i) G_{P_{xy}} (i - \check{y}) G_y^{-1} (P_{xy} (i)) G_{P_{xy}}^{-1} (i) = \eta_{ypyxy} T^0,
G_x (i) G_{P_{xy}} (i - \check{z}) G_y^{-1} (P_{xy} (i)) G_{P_{xy}}^{-1} (i) = \eta_{xpxy} T^0,
G_y (i) G_{P_{xy}} (i - \check{y}) G_x^{-1} (P_{xy} (i)) G_{P_{xy}}^{-1} (i) = \eta_{ypyxy} T^0,
G_{T^*} (i) G_{P_{xy}} (i) G_{T^*} P_{xy} (i) G_{P_{xy}}^{-1} (i) = \eta_{xpxy} T^0,
G_{T^*} (i) G_{P_{xy}} (i) G_{T^*} P_{xy} (i) G_{P_{xy}}^{-1} (i) = \eta_{ypyxy} T^0,
\]
(A13)

where \(\eta_{xpxy} = \pm 1\...\)

For \(Z2A\) type PSG’s,
\[
G_{P_{xy}} (i) = \eta_{xpxy} \eta_{ypxy} g_{P_{xy}},
G_{P_{xy}} (i) = \eta_{xpxy} \eta_{ypxy} g_{P_{xy}}.
\]
(A14)

For \(Z2B\) type PSG’s,
\[
G_{P_{xy}} (i) = (-)^{i x + i y} \eta_{xpxy} \eta_{ypxy} g_{P_{xy}},
G_{P_{xy}} (i) = (-)^{i x + i y} \eta_{xpxy} \eta_{ypxy} g_{P_{xy}}.
\]
(A15)

We also have that
\[
P_{xy}^2 = P_{xy}^2 = P_{xy} P_{xy} P_{xy} P_{xy} = 1
\]
(A16)

which leads to
\[
G_{P_{xy}} (i) G_{P_{xy}} (P_{xy} (i)) = \pm T^0,
G_{P_{xy}} (i) G_{P_{xy}} (P_{xy} (i)) = \pm T^0,
\]
and
\[
(\eta_{xpxy} \eta_{ypxy})^{i x + i y} g_{P_{xy}}^2 = \pm T^0,
(\eta_{xpxy} \eta_{ypxy})^{i x + i y} g_{P_{xy}}^2 = \pm T^0.
\]

The above implies that
\[
\eta_{xpxy} = \eta_{ypxy} = \eta_{pxy},
\eta_{xpxy} = \eta_{ypxy} = \eta_{pxy},
g_{P_{xy}}^2 = \pm T^0, g_{P_{xy}}^2 = \pm T^0.
\]

By the similar way, one sees that
\[
\eta_{zt} = \eta_{yt} = \eta_t.
\]
(A17)

Gauge transformation \(W(i) = (-)^i z\) will change \(\eta_{pxy}\) to \(-\eta_{pxy}\), therefore we can chose a gauge in which \(\eta_{pxy} = 1\).

In conclusion, we have the following 60 kinds of \(Z2A\) algebraic PSG’s and 60 kinds of \(Z2B\) algebraic PSG’s on our triangular lattices:

\[Z2Ag_{P_{xy}}(g_{P_{xy}})(g_{T^*})(\eta_{pxy})(\eta_{t})\]
\[G_x (i) = G_y (i) = T^0,
G_{T^*} (i) = \eta_{ix} \eta_{iy} T^0, \eta_t = \pm 1,
G_{P_{xy}} (i) = g_{P_{xy}},
G_{P_{xy}} (i) = \eta_{ix} \eta_{iy} g_{P_{xy}}, \eta_{pxy} = \pm 1,
\]
(A18)

and

\[Z2Bg_{P_{xy}}(g_{P_{xy}})(g_{T^*})(\eta_{t})\]
\[G_y (i) = T^0, G_x (i) = (-)^{i x} T^0,
G_{T^*} (i) = \eta_{ix} \eta_{iy} T^0, \eta_t = \pm 1,
G_{P_{xy}} (i) = (-)^{i x + i y} g_{P_{xy}},
G_{P_{xy}} (i) = (-)^{i x + i y} \eta_{pxy} g_{P_{xy}}, \eta_{pxy} = \pm 1,
\]
(A19)
where $g_T$, $g_{P_{xy}}$, and $g_{P_{xy}}$ satisfy that
\[
\begin{align*}
g_T^2 &= \pm \tau^0, \\
g_{P_{xy}}^2 &= \pm \tau^0, \\
g_{P_{xy}}^2 &= \pm \tau^0, \\
g_{T \cdot g_{P_{xy}} g_{T}^{-1} g_{P_{xy}}^{-1}} &= \pm \tau^0, \\
g_{T \cdot g_{P_{xy}} g_{T}^{-1} g_{P_{xy}}^{-1}} &= \pm \tau^0, \\
g_{P_{xy}} g_{P_{xy}} g_{P_{xy}}^{-1} g_{P_{xy}}^{-1} &= \pm \tau^0.
\end{align*}
\]
\[
(A20)
\]
All the gauge inequivalent $g_T$, $g_{P_{xy}}$, and $g_{P_{xy}}$ are given by the following table:

| $g_T$ | $g_{P_{xy}}$ | $g_{P_{xy}} g_{P_{xy}}$ |
|-------|-------------|---------------------|
| $\tau^0$ | $\tau^0$ | $\tau^0$ |
| $\tau^0$ | $i \tau^3$ | $i \tau^3$ |
| $\tau^0$ | $i \tau^3$ | $i \tau^3$ |
| $i \tau^3$ | $\tau^0$ | $\tau^0$ |
| $i \tau^3$ | $i \tau^1$ | $i \tau^1$ |
| $i \tau^3$ | $i \tau^3$ | $i \tau^3$ |
| $i \tau^3$ | $i \tau^1$ | $i \tau^1$ |
| $i \tau^3$ | $i \tau^3$ | $i \tau^3$ |
| $i \tau^3$ | $i \tau^1$ | $i \tau^1$ |

After finding all the PSGs, next we need to find the ansatz that is invariant under those PSGs. This way we classify (mean-field) $Z_2$ spin liquids through PSG’s. Let us first consider the translation symmetry. We have two classes of spin liquids, $Z2A$ spin liquids
\[
G_y(i) = \tau^0, \quad G_x(i) = \tau^0, \\
u_{i,i+m} = u_m,
\]
and $Z2B$ spin liquids
\[
G_y(i) = \tau^0, \quad G_x(i) = (-)^{\bar{s}} \tau^0, \\
u_{i,i+m} = (-)^{n_y i \tau^0} u_m.
\]
\[
(A21)
\]
Considering the spin parity symmetry $T^\ast$, we have
\[
G_T \cdot T^\ast \left( u_{i,i+m} \right) = u_{i,i+m},
\]
\[
(A23)
\]
which reads
\[
- u^m_i g_T \cdot u_m g_T^{-1} = u_m
\]
\[
(A24)
\]
Therefore, for different $(g_T)_{\eta}$, we have the following table:

| $\eta$ | $u_m$ |
|-------|-------|
| $\tau^0$ | $u_m = 0$ |
| $\tau^0$ | $u_m = 0$, if $m = \text{even}$ |
| $\tau^3$ | $u_m = u^1_m \tau^1 + u^2_m \tau^2$ |
| $\tau^3$ | $u_m = u^0_m \tau^0 + u^1_m \tau^1$, if $m = \text{even}$ |
| $\tau^3$ | $u_m = u^0_m \tau^0 + u^1_m \tau^1$, if $m = \text{odd}$ |

Now we consider the $180^\circ$ rotation symmetry, which can be constructed through combining $P_{xy}$ and $P_{x\bar{y}}$,
\[
G_{P_{xy}} P_{x\bar{y}} G_{P_{xy}} P_{x\bar{y}} (u_{i,i+m}) = u_{i,i+m}
\]
\[
(A25)
\]
the above leads to
\[
y^m_{p_{xy}} g_{P_{xy}} g_{P_{xy}} g_{P_{xy}}^{-1} g_{P_{xy}}^{-1} = u^\dagger_m,
\]
\[
(A26)
\]
for the $Z2A$ spin liquids, and
\[
(-)^{m_x m_y} y^m_{p_{xy}} g_{P_{xy}} g_{P_{xy}} g_{P_{xy}} g_{P_{xy}}^{-1} g_{P_{xy}}^{-1} = u^\dagger_m,
\]
\[
(A27)
\]
for the $Z2B$ spin liquids.

By writing all the ansatz, we find that although there are 120 kinds $Z_2$ algebraic PSG’s, only 23 of them lead to $Z2A$ spin liquids and 40 of them lead to $Z2B$ spin liquids. Other algebraic PSG’s lead to vanishing ansatz, $U(1)$ or $SU(2)$ spin liquids.

**APPENDIX B: CLASSIFICATION OF U(1) SPIN LIQUIDS**

For $U(1)$ spin liquids, we can choose a gauge so that $u_{ij}$ take the form
\[
u_{ij} = i \rho_{ij} e^{i \theta_{ij}} \tau^3 = u^0_{ij} \tau^0 + u^3_{ij} \tau^3
\]
\[
(B1)
\]
We will call this gauge canonical gauge. In the canonical gauge, IGG has a form of $G = \left\{ e^{i \theta \tau^3}, \theta \in [0, 2\pi) \right\}$, where $\theta$ is real for each $i$. Due to the translation symmetry of the ansatz, the loop operator $P_C$ have a form
\[
P_{T,C} = (i \tau^1)^{n_x} P_C (i \tau^1)^{n_x}
\]
\[
(B2)
\]
where $n_x = 0, 1$. The gauge transformation $G_{x,y}$ associated with the translation take the following form
\[
G_x(i) = (-i \tau^1)^{n_x} e^{i \theta_{x} (i \tau^3)} (i \tau^1)^{n_x-i}
\]
\[
G_y(i) = (-i \tau^1)^{n_y} e^{i \theta_{y} (i \tau^3)} (i \tau^1)^{n_y-i}
\]
\[
(B3)
\]
in the canonical gauge. We note that a gauge transformations keep $u_{ij}$ to have the form in the canonical gauge must have one of the following two forms
\[
W_i = e^{i \theta_i \tau^3}
\]
\[
W_i = e^{i \theta_i \tau^3} (i \tau^1)
\]
\[
(B4)
\]
\[
(B5)
\]
For spin liquids with connected $u_{i,j}$, $G_{x,y}$ must take one of the above two forms in the canonical gauge. Thus $n_i$ can only be one of the following four choices: $n_i = 0$, $n_i = (1 - (-)^i)/2$, $n_i = (1 - (-)^i y)/2$, and $n_i = (1 - (-)^i y)/2$. In these four cases, $G_{x,y}$ take one of the above two forms and $u_{i,j}$ can be connected.

Let us consider those cases in turn. We will work in the canonical gauge.

When $n_i = 0$, $G_{x,y}$ have a form

$$G_x(i) = e^{i\theta_x(i)\tau^3}$$
$$G_y(i) = e^{i\theta_y(i)\tau^3}$$

(B6)

by (B4), we can gauge fix them up to $\theta_y(i_x,0) = 0$ and $\theta_y(i) = 0$. The relation (B7) requires that

$$\theta_x(i) + \theta_y(i - x) - \theta_x(i - y) - \theta_y(i) = \varphi$$

(B7)

Therefore $G_{x,y}$ have a form

$$G_x(i) = e^{i(\varphi + \theta_x)\tau^3}$$
$$G_y(i) = e^{i\theta_y\tau^3}$$

(B8)

The ansatz with translation symmetry has a form

$$u_{i,i+m} = i\rho_m e^{i(-m_y i_x \varphi + \phi_m)\tau^3}$$

(B9)

When $n_i = (1 - (-)^i)/2$,

$$G_x(i) = e^{i\theta_x(i)\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y(i)\tau^3}(i\tau^1)$$

(B10)

then

$$G_x(i) = e^{i((-)^y \varphi + \theta_x)\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y\tau^3}(i\tau^1)$$

(B11)

Using gauge transformation $W_i = e^{i((-)^y \varphi + \varphi/2)\tau^3}$, the above change to

$$G_x(i) = e^{i\theta_y\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y\tau^3}(i\tau^1)$$

(B12)

The ansatz with translation symmetry has a form

$$u_{i,i+m} = i\rho_m e^{i(-)^y \phi_m \tau^3}$$

(B13)

When $n_i = (1 - (-)^i y)/2$,

$$G_x(i) = e^{i\theta_x(i)\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y(i)\tau^3}$$

(B14)

then

$$G_x(i) = e^{i((-)^y \varphi + \theta_x)\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y\tau^3}$$

(B15)

Using gauge transformation $W_i = e^{i((-)^y \varphi/2)\tau^3}$, the above change to

$$G_x(i) = e^{i\theta_x\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y\tau^3}$$

(B16)

The ansatz with translation symmetry has a form

$$u_{i,i+m} = i\rho_m e^{i(-)^y \phi_m \tau^3}$$

(B17)

When $n_i = (1 - (-)^i y)/2$,

$$G_x(i) = e^{i\theta_x(i)\tau^3}(i\tau^1)$$
$$G_y(i) = e^{i\theta_y(i)\tau^3}(i\tau^1)$$

(B18)

then

$$G_x(i) = e^{i((-)^y \varphi + \theta_x)\tau^3}$$
$$G_y(i) = e^{i\theta_y\tau^3}(i\tau^1)$$

(B19)

Using gauge transformation $W_i = e^{i((-)^y \varphi + \varphi/2)\tau^3}$, the above change to

$$G_x(i) = e^{i\theta_x\tau^3}$$
$$G_y(i) = e^{i\theta_y\tau^3}(i\tau^1)$$

(B20)

The ansatz with translation symmetry has a form

$$u_{i,i+m} = i\rho_m e^{i(-)^y \phi_m \tau^3}$$

(B21)

Now we add parity symmetry $P_{xy}$. Considering the parity in a plaquette, the parity symmetry requires $\varphi = 0$ or $\pi$ in (B4). From the relations

$$G_x(i) G_{P_{xy}}(i - x) G_y^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) = e^{i\theta_{p_{xy}}}$$
$$G_y(i) G_{P_{xy}}(i - y) G_x^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) = e^{i\theta_{p_{xy}}}$$

$$G_x(i) G_{P_{xy}}(i - x) G_y^{-1}(P_{xy}(i) - x) G_{P_{xy}}^{-1}(i) = e^{i\theta_{p_{xy}}}$$
$$G_y(i) G_{P_{xy}}(i - y) G_x^{-1}(P_{xy}(i) - y) G_{P_{xy}}^{-1}(i) = e^{i\theta_{p_{xy}}}$$

$$G_{P_{xy}}(i) G_{P_{xy}}(P_{xy}(i)) G_{P_{xy}}^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) = e^{i\theta_{p_{xy}}}$$

(B22)

we will find only $n_i = 0$ and $n_i = (1 - (-)^i)/2$ lead to the existence of $G_{P_{xy}}(i)$. Then we can classify the $U(1)$ spin liquids with parity symmetry into three classes according to their translation property.

$U1A$ spin liquid:

$$G_x(i) = e^{i\theta_x\tau^3}$$
$$G_y(i) = e^{i\theta_y\tau^3}$$

(B23)

$$u_{i,i+m} = i\rho_m e^{i\phi_m \tau^3}.$$
and does not change the forms of $G_{x,y}$.

b) $G_{pxy}(i) = e^{i\phi_{pxy} x^3}$

\[ G_{pg}(i) = \eta_{pxy} e^{i\phi_{pxy} x^3} (i\tau^1), \eta_{pxy} = \pm 1 \quad \text{(B31)} \]

c) $G_{pxy}(i) = e^{i\phi_{pxy} x^3} (i\tau^1)$

\[ G_{pg}(i) = \eta_{pxy} e^{i\phi_{pxy} x^3}, \eta_{pxy} = \pm 1 \quad \text{(B32)} \]

d) $G_{pxy}(i) = e^{i\phi_{pxy} x^3} (i\tau^1)$

\[ G_{pg}(i) = e^{i\phi_{pxy} x^3} (i\tau^1) \quad \text{(B33)} \]

Gauge transformation $W(i) = e^{-i(i\tau-y)\varphi_{pxy}/2\tau^3}$ transforms the above into

\[ G_{pxy}(i) = e^{i\phi_{pxy} x^3} (i\tau^1), \quad G_{pg}(i) = e^{i\phi_{pxy} x^3} (i\tau^1) \quad \text{(B34)} \]

Hence we have that all the $U1A$ type PSG’s:

\[ G_{pxy}(i) = e^{i\phi_{pxy} x^3} \quad \text{(B35)} \]

\[ G_{pg}(i) = e^{i\phi_{pxy} x^3} \quad \text{(B36)} \]

\[ G_{pxy}(i) = e^{i\phi_{pxy} x^3} \quad \text{(B37)} \]
The ansatz gives rise to corresponding to \( g \). We consider the form of ansatz that in invariant under the above PSG’s. The translation symmetry requires that

\[
 u_{i,i+m} = u_m = u_m^0 \tau^0 + u_m^3 \tau^3. \tag{B39}
\]

The 180° rotation symmetry requires that for \( g \),

\[
 u_m = u_{-m} = u_m^1 \tag{B40}
\]

and for \( g \),

\[
 u_m = \eta_{pxy}^m \tau^1 u_{-m} \tau^1 = -\eta_{pxy}^m u_m \tag{B41}
\]

The spin parity symmetry \( T^* \) requires that for \( g \),

\[
 u_m = -(-)^m u_m \tag{B42}
\]

and for \( g \),

\[
 u_m = -\eta_{pxy}^m u_m^0 \tau^0 + \eta_{pxy}^m u_m^3 \tau^3 \tag{B43}
\]

We find that the following 6 sets of ansatz that give rise to \( U(1) \) symmetric spin liquids:

\[
 U1A[\tau^0_1, \tau^1_1, \tau^0_0] \tau^0
\]

\[
 u_{i,i+m} = u_m^0 \tau^0 + u_m^3 \tau^3
\]

\[
 u_m^0 = 0, \text{ if } m = \text{even} \tag{B44}
\]

Here we have used the notation \( U1A[ab, cd]e \) to represent the collection \( U1Adeb \) and \( U1Ace \).

\[
 U1A[\tau^0_1, \tau^1_1, \tau^1_0] \tau^1
\]

\[
 u_{i,i+m} = u_m^3 \tau^3 \tag{B45}
\]

\[
 U1A[\tau^0_1, \tau^1_1, \tau^0_0] \tau^1
\]

\[
 u_{i,i+m} = u_m^3 \tau^3
\]

\[
 u_m^3 = 0, \text{ if } m = \text{odd} \tag{B46}
\]

The ansatz gives rise to \( U(1) \times U(1) \) spin liquids since \( u_{ij} \) only connect points within two different sublattices. Other PSG’s lead to vanishing ansatz or \( SU(2) \) spin liquids and can be dropped.

2) \( U1B \) spin liquid, \( G_x(i) = (-)^i e^{i\theta x \tau^3}, G_y(i) = e^{i\theta y \tau^3} \). From relations \([B20]\), one can obtain all the \( U1B \) PSG’s easily,

\[
 G_{px} (i) = (-)^{i+1} e^{i\theta px \tau^3},
\]

\[
 G_{px} (i) = (-)^{i} e^{i\theta px \tau^3}, \tag{B47}
\]

\[
 G_{Ty} (i) = \eta_{pxy}^i e^{i\theta \tau^3} \tag{B48}
\]

\[
 G_{pxy} (i) = (-)^{i} e^{i\theta px \tau^3} \tag{B49}
\]

\[
 G_{px} (i) = (-)^{i} e^{i\theta px \tau^3} \tag{B50}
\]

\[
 G_{Ty} (i) = \eta_{pxy}^i e^{i\theta \tau^3} \tag{B51}
\]

We can label these spin liquids as \( U1B \). Now we consider the form of ansatz that in invariant under the above PSG’s. The translation symmetry requires that

\[
 u_{i,i+m} = (-)^m u_m \tag{B51}
\]

The 180° rotation symmetry requires that for \( g \),

\[
 u_m = u_{-m} = u_m^1 \tag{B52}
\]

and for \( g \),

\[
 u_m = -\eta_{pxy}^m u_m^0 \tau^0 + \eta_{pxy}^m u_m^3 \tau^3 \tag{B53}
\]

The spin parity symmetry \( T^* \) requires that for \( g \),

\[
 u_m = -(-)^m u_m \tag{B54}
\]

and for \( g \),

\[
 u_m = -\eta_{pxy}^m u_m^0 \tau^0 + \eta_{pxy}^m u_m^3 \tau^3 \tag{B55}
\]

We find that the following 10 sets of ansatz that give rise to \( U(1) \) symmetric spin liquids:
$U_1B[\tau^0\tau^1_-,\tau^1\tau^0_+]\tau_-^0$

$$u_{1,i+m} = (-)^{m_{xz}}(u_{m,0}^0 + u_m^3\tau^3)$$

$$u_{0,3}^m = 0, \text{ if } m = \text{ even} \quad (B56)$$

$U_1B[\tau^0\tau^1_+,\tau^1\tau^0_+]\tau_+^1$

$$u_{1,i+m} = (-)^{m_{xy}}u_m^3\tau^3$$

$$u_0^3 = 0, \text{ if } m_x = \text{ even and } m_y = \text{ odd} \quad (B57)$$

$U_1B[\tau^0\tau^1_+,\tau^1\tau^0_+]\tau_+^1$

$$u_{1,i+m} = (-)^{m_{xy}}u_m^3\tau^3$$

$$u_0^3 = 0, \text{ if } m_x = \text{ even and } m_y = \text{ odd} \quad (B58)$$

$U_1B[\tau^0\tau^1_-,\tau^1\tau^0_+]\tau_-^1$

$$u_{1,i+m} = (-)^{m_{xy}}(u_{m,0}^0 + u_m^3\tau^3)$$

$$u_0^3 = 0, \text{ if } m = \text{ even}$$

$$u_0^3 = 0, \text{ if } m_x = \text{ even or } m_y = \text{ even} \quad (B60)$$

The ansatz gives rise to $(U(1))^4$ spin liquids since $u_{ij}$ only connect points within four different sublattices.

$U_1B[\tau^0\tau^1_-,\tau^1\tau^0_+]\tau_-^1$

$$u_{1,i+m} = (-)^{m_{xy}}(u_{m,0}^0 + u_m^3\tau^3)$$

$$u_0^3 = 0, \text{ if } m = \text{ even}$$

$$u_0^3 = 0, \text{ if } m_x = \text{ even or } m_y = \text{ even} \quad (B60)$$

Other PSG’s lead to vanishing ansatz or $SU(2)$ spin liquids and can be dropped.

3) $U1C$ spin liquid, $G_x(i) = e^{iφ_x\tau^3}(i\tau^1)$ and $G_y(i) = e^{iφ_y\tau^3}(i\tau^1)$.

$G_{xy}^x(i) = e^{iφ_{xy}(i)\tau^3}, e^{iφ_{xy}(i)\tau^3}(i\tau^1)$. For the former four equations in $[222]$, we have that

$$θ_{xy}^x(i) = (-)^iφ_{pxy} + φ_{xy}^x, \quad i_xπ + (-)^iφ_{pxy} + φ_{xy}^x$$

and the last three equations in $[222]$ require that

$$G_{xy}^x(i) = η_{xpy}^iη_{xy}^i e^{i(η_{xy}^xπ/4 + φ_{xy})\tau^3}, \quad \eta_{xpy}^i e^{i(-iφ_{pxy}+φ_{xy})\tau^3}(i\tau^1) \quad (B61)$$

and

$$G_{xy}^y(i) = η_{pxy}^iη_{xy}^i e^{i(η_{xy}^yπ/4 + φ_{xy})\tau^3}, \quad η_{pxy}^i e^{i(-iφ_{pxy}+φ_{xy})\tau^3}(i\tau^1) \quad (B62)$$

where $η_{pxy,pxy} = ±1$, $η_{pxy,pxy} = ±1$. Now we add the last symmetry spin parity $T^*$. In a similar way, through the relations $[12]$, we have that

$$G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}, \quad η_{xpy}^i e^{i(η_{xy}^xπ/4 + φ_{xy})\tau^3}(i\tau^1) \quad (B63)$$

where $η_{i} = ±1, η_{i} = ±1$.

It is noted that gauge transformation $W_x = e^{i(-iφ)^3}$ and $W_y = η_{xy}^i$ do not change $G_{xy}$, and can be used to simplify the forms of $G_{xy}$. We conclude all the $U1C$ type PSG’s:

$$G_{xy}^x(i) = η_{xpy}^iη_{xy}^i e^{i(η_{xy}^xπ/4 + φ_{xy})\tau^3} \quad (B64)$$

$$G_{xy}^y(i) = η_{pxy}^iη_{xy}^i e^{i(η_{xy}^yπ/4 + φ_{xy})\tau^3} \quad (B65)$$

$$G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1) \quad (B66)$$

$$G_{xy}^x(i) = η_{xpy}^iη_{xy}^i e^{i(η_{xy}^xπ/4 + φ_{xy})\tau^3} \quad (i\tau^1) \quad (B66)$$

$$G_{xy}^y(i) = η_{pxy}^iη_{xy}^i e^{i(η_{xy}^yπ/4 + φ_{xy})\tau^3} \quad (i\tau^1) \quad (B67)$$

$$G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1) \quad (B68)$$

The $180^\circ$ rotation symmetry requires that for $G_{xy}^x(i) G_{xy}^x(P_{xy}(i)) = η_{xy}^x u_{xy}^0$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = -η_{pxy}^m u_{xy}^3 \quad (B69)$$

for $G_{xy}^x(i) G_{xy}^x(P_{xy}(i)) = η_{pxy}^iη_{xy}^i u_{xy}^0 \quad (i\tau^1)$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = -η_{pxy}^m u_{xy}^3 \quad (B70)$$

The spin parity symmetry $T^*$ requires that for $G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1)$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = η_{pxy}^m u_{xy}^3 \quad (B71)$$

and for $G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1)$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = η_{pxy}^m u_{xy}^3 \quad (B72)$$

The translation symmetry requires that

$$u_{i,i+m} = u_{m}^0 r_0 + (-)^i u_{m}^3 r_3 \quad (B68)$$

The spin parity symmetry $T^*$ requires that for $G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1)$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = -η_{pxy}^m u_{xy}^3 \quad (B70)$$

The spin parity symmetry $T^*$ requires that for $G_{T^*}(i) = η_{pxy}^i e^{iφ_{xy}\tau^3}(i\tau^1)$

$$u_{m}^0 = -η_{pxy}^m u_{xy}^0, \quad u_{m}^3 = η_{pxy}^m u_{xy}^3 \quad (B72)$$
We find that the following 14 sets of ansatz that give rise to $U(1)$ symmetric spin liquids:

$$U1C[\tau_+^0 r^0, \tau_+^1 r^1, \tau_+^1 r^1] r^0$$

$$u_{i,i+m} = u_{m}^{0} r^0 + (-)^i u_{m}^{3} r^3$$
$$u_{m}^{3} = 0, \text{ if } m = \text{even};$$
(B73)

$$U1C[\tau_+^0 r^0, \tau_+^1 r^1] r^1$$

$$u_{i,i+m} = (-)^i u_{m}^{3} r^3$$
$$u_{m}^{3} = 0, \text{ if } m = \text{odd}$$
(B74)

The ansatz gives rise to $U(1) \times U(1)$ spin liquids.

$$U1C[\tau_+^0 r^0, \tau_+^1 r^1] r^1$$

$$u_{i,i+m} = (-)^i u_{m}^{3} r^3$$
(B75)

The ansatz gives rise to $(U(1))^4$ spin liquids.

$$U1C[\tau_+^0 r^0, \tau_+^1 r^1] r^1$$

$$u_{i,i+m} = u_{m}^{0} r^0 + (-)^i u_{m}^{3} r^3$$
$$u_{m}^{0} = 0, \text{ if } m_x = \text{even or } m_y = \text{odd}$$
$$u_{m}^{3} = 0, \text{ if } m_x = \text{odd}$$
(B77)

$$U1C[\tau_+^0 r^1, \tau_+^1 r^0] r^1$$

$$u_{i,i+m} = u_{m}^{0} r^0 + (-)^i u_{m}^{3} r^3$$
$$u_{m}^{0} = 0, \text{ if } m_x = \text{even}$$
$$u_{m}^{3} = 0, \text{ if } m_x = \text{odd or } m_y = \text{even};$$
(B78)

Other PSG’s lead to vanishing ansatz or $SU(2)$ spin liquids and can be dropped.

**APPENDIX C: CLASSIFICATION OF $SU(2)$ SPIN LIQUIDS**

We assume that for a $SU(2)$ spin liquid, one can always choose a gauge so that $u_{ij}$ has a form

$$u_{ij} = u_{ij}^{0} r^0$$
(C1)

Hereafter we call this gauge canonical gauge. In the canonical gauge, IGG has a form $G = SU(2)$. The gauge transformations that keep $u_{ij}$ to have the form in the canonical gauge are given by

$$W_i = \eta(i) g$$
(C2)

where $\eta(i) = \pm 1$ for each $i$ and $g \in SU(2)$. The gauge transformation $G_{x,y}$ associated with the translation also take the above form

$$G_x(i) = \eta_x(i) g_x$$
$$G_y(i) = \eta_y(i) g_y$$
(C3)

And we can gauge fix the above form as

$$G_x(i) = g_x$$
$$G_y(i) = \eta_y(i) g_y$$
(C4)

where $\eta_y(i_x = 0, i_y) = 1$. Now the relation reads

$$\eta_y(i - \hat{x}) \eta_y(i) \in SU(2)$$
(C5)

We find that there are only two different PSG’s for translation symmetric ansatz

$$G_x(i) = g_x, G_y(i) = g_y$$
(C6)

and

$$G_x(i) = g_x, G_y(i) = (-)^i g_y$$
(C7)

The two PSG’s lead to the following two translation symmetric ansatz

$$u_{i,i+m} = u_{m}^{0} r^0$$
(C8)

and

$$u_{i,i+m} = (-)^{m_x i_y} u_{m}^{0} r^0$$
(C9)

Similar to $Z_2$ case, adding more symmetries, we have all the $SU(2)$ PSG’s belonged to triangular lattices.

$SU2A \eta_{pxy} \eta_t$

$$G_x(i) = g_x$$
$$G_y(i) = g_y$$
$$G_{T^*}(i) = \eta_1^t g_{T^*}$$
$$G_{P_{xy}}(i) = g_{P_{xy}}$$
$$G_{P_{xy}}(i) = \hbar_{pxy} g_{P_{xy}}$$
(C10)

and

$SU2B \eta_{pxy} \eta_t$

$$G_x(i) = g_x$$
$$G_y(i) = (-)^i g_y$$
$$G_{T^*}(i) = \eta_1^t g_{T^*}$$
$$G_{P_{xy}}(i) = (-)^i \hbar_{pxy} g_{P_{xy}}$$
$$G_{P_{xy}}(i) = (-)^i \hbar_{pxy} g_{P_{xy}}$$
(C11)

where $g_x, g_y, g_{T^*}, g_{P_{xy}}, g_{P_{xy}} \in SU(2)$, $\eta_t = \pm 1$, and $\hbar_{pxy} = \pm 1$. Moreover, only $\eta_t = -1$ and $\eta_{pxy} = -1$ lead to non vanishing ansatz. So there are only two kinds of $SU(2)$ spin liquids,

$SU2A$ spin liquids:

$$u_{i,i+m} = u_{m}^{0} r^0, m = \text{odd},$$
(C12)

$SU2B$ spin liquids:

$$u_{i,i+m} = (-)^{m_x i_y} u_{m}^{0} r^0, m = \text{odd}.$$
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