Finite reflection groups and the Dunkl-Laplace
differential-difference operators in conformal
group

P. Somberg

Abstract

For a finite reflection subgroup \( G \leq O(n + 1, 1, \mathbb{R}) \) of the conformal
group of the sphere with standard conformal structure \((S^n, [g_0])\), we geo-
metrically derive differential-difference Dunkl version of the series of con-
formally invariant differential operators with symbols given by powers of
Laplace operator. The construction can be regarded as a deformation of
the Fefferman-Graham ambient metric construction of GJMS operators.

Key words: Conformal geometry, Finite reflection groups, Fefferman-
Graham ambient metric construction, conformal Dunkl-Laplace opera-
tors.

MSC classification: 51F15, 53A30, 20F55, 58E09.

1 Introduction

Invariant theory for finite reflection groups \( G \) is usually considered in the
framework \( G \leq O(n) \), where \( O(n) \) is the orthogonal group associated to
the Hilbert space \((\mathbb{R}^n, <, >)\). The action of \( G \) then carries over to function
spaces on \( \mathbb{R}^n \) and leads to the concept of Dunkl differential-difference
operators and special function theory for finite reflection groups, see e.g.
\[4\], \[9\], \[10\] and extensive references therein.

The aim of the present article is to change the geometrical perspective
and study similar questions (or at least, some of them) as in the classical
well-known case. In particular, our point of view is to consider the con-
formal compactification \( S^n \) of \( \mathbb{R}^n \) and finite reflection subgroups of the
conformal group \( O(n + 1, 1, \mathbb{R}) \) defined by finite number of reflecting sub-
spheres diffeomorphic to spheres \( S^{n-1} \) of codimension one. The collection
of reflecting sub spheres is an analog of the collection of reflecting hyper-
planes in \( \mathbb{R}^n \). Moreover, a reflection subgroup of the conformal group is
induced from a reflection group of a root system in \( \mathbb{R}^{n+1} \leq \mathbb{R}^{n+1,1} \), the
ambient vector space of signature \((n + 1, 1)\). The conformal sphere \( S^n \)
is realized by projectivization of the cone \( C \) of null length vectors in \( \mathbb{R}^{n+1,1} \)
and is the flat version of the curved Fefferman-Graham ambient metric
construction.
In the article we focus on the construction of differential-difference Dunkl modification of conformal Laplacian type invariants (also termed conformal Dunkl-Laplace operators) associated to the structure \((S^n, [g_0], G)\), the conformal sphere with conformal class of the round metric and its finite reflection subgroup. This collection of differential-difference operators can be regarded as a deformation by multiplicity function on \(G\) of conformally invariant powers of the Laplace operator (corresponding to the trivial multiplicity function) and yields \(G\)-equivariant intertwining operators acting on principal series representations of \(O(n + 1, 1, \mathbb{R})\) induced from characters of the Levi factor of the conformal (maximal) parabolic subgroup \(P \leq O(n + 1, 1, \mathbb{R})\).

In summary, the present article should be understood as a special instance of general research program to construct and study properties of differential-difference invariants for the couple given by a manifold with geometric structure and a finite subgroup of the group of its automorphisms, combining techniques from the differential geometry, algebra and representation theory of both Lie and finite reflection groups.

2 Homogenous model of flat conformal geometry

Ambient metric construction in conformal geometry associates to an \(n\)-dimensional conformal manifold \((M, [g])\) of signature \((p, q)\) a (up to certain order) pseudo-Riemannian space \((\tilde{M}, \tilde{g})\) of two dimensions higher, see e.g., [6].

In the article we shall need just the flat version of the ambient metric construction, see e.g., [5]. The sphere \(S^n\) is conformally flat manifold \((S^n, [g_0])\) of signature \((n, 0)\), realized as the projectivization of the cone of null-vectors \(C\) in \(\mathbb{R}^{n+1,1}\), with the flat metric

\[
\tilde{g}(X^0, X^1, \ldots, X^n, X^\infty) = dX^0 \otimes dX^\infty + dX^\infty \otimes dX^0 + \sum_{i=1}^n dX^i \otimes dX^i
\]

of signature \((n + 1, 1)\) in the coordinates \(X^0, X^1, \ldots, X^n, X^\infty\).

The ambient space \(\tilde{M}\) is a small neighborhood of \(C\) in \(\mathbb{R}^{n+1,1}\) and the ambient metric is the pull-back of the flat metric \(\tilde{g}(X^0, X^1, \ldots, X^n, X^\infty)\) from \(\mathbb{R}^{n+1,1}\) to this neighborhood.

A conformal density on \(S^n\) is a section of homogeneous line bundle \(L \rightarrow S^n\), induced from a character of the reductive Levi factor of the conformal parabolic subalgebra \(P\) of \(O(n + 1, 1, \mathbb{R})\). The parabolic subalgebra \(P \leq O(n + 1, 1, \mathbb{R})\) is the stabilizer of a real line in \(C\).

A section of \(L\) can be identified with \(\mathbb{R}_+\)-homogeneous function on the null-cone \(C\) in the ambient space. Let us remark that the construction of conformally invariant differential operators of Laplace type acting between densities on \(S^n\) proceeds by application of \(\tilde{\Delta}^j, j \in \mathbb{N}\), to an extension of a density off \(C\) and the resulting section of different homogeneity restricts back to \(C\).

Notice that for the explicit computation of invariants in a coordinate chart and in the ambient metric, we need the following preparatory result
Lemma 2.1 The two coordinate systems on $\mathbb{R}^{n+1,1}$; $(t, x_1, \ldots, x_n, \rho)$, the coordinate system adapted to the null-cone $\mathcal{C}$ and $(X^0, X^1, \ldots, X^n, X^\infty)$, the euclidean coordinate system, are related by

$$
t = X^0, \quad x_i = \frac{X^i}{X^0}, \quad \rho = \frac{X^\infty}{X^0} + \frac{1}{2X^0}||X^i||^2, \\
X^0 = t, \quad X^i = tx_i, \quad X^\infty = t(\rho - \frac{1}{2}||x_i||^2).$$

(1)

The coordinate vector fields transform as

$$
\frac{\partial}{\partial X^0} = \frac{\partial}{\partial t} - \frac{1}{t} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} - \frac{1}{t}(\rho + \frac{1}{2}||x_i||^2) \frac{\partial}{\partial \rho}, \\
\frac{\partial}{\partial X^i} = \frac{1}{t} \frac{\partial}{\partial x_i} + \frac{x_i}{t} \frac{\partial}{\partial \rho}, \quad i = 1, \ldots, n \\
\frac{\partial}{\partial X^\infty} = \frac{1}{t} \frac{\partial}{\partial \rho}.
$$

(2)

3 Finite reflection groups in conformal geometry on $S^n$ and ambient metric construction

In what follows we restrict to the vector space $\mathbb{R}^{n+1,1}$ with non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature $(n+1, 1)$. If not stated otherwise, the coordinates of a vectors in $\mathbb{R}^{n+1,1}$ are considered with respect to the canonical orthonormal basis $\{e_0, e_1, \ldots, e_n, e_\infty\}$. Recall that the couple $(\mathbb{R}^{n+1,1}, \langle \cdot, \cdot \rangle)$ plays the role of the flat (or homogeneous) case of the Fefferman-Graham ambient space. [6]. We shall adopt for our purposes a few basic definitions related to finite reflection groups, see e.g. [2], [10], [4], [9], from euclidean signature to the signature $(n+1, 1)$ and introduce the notion of an induced reflecting subsphere $S_n^{a-1}$.

Let us denote by $\mathcal{C} \subset \mathbb{R}^{n+1,1}$ the cone of null-vectors, i.e. $\mathcal{C} := \{X \in \mathbb{R}^{n+1,1} | \langle X, X \rangle = 0\}$. For $\alpha \in \mathbb{R}^{n+1,1} \setminus \mathcal{C}$ let $R_\alpha$ be the reflection along the hyperplane orthogonal to $\alpha$, i.e.

$$R_\alpha X = X - 2\frac{\langle \alpha, X \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad X \in \mathbb{R}^{n+1,1}.$$ 

(3)

Definition 3.1 A finite subset $R \subset (\mathbb{R}^{n+1,1} \setminus \mathcal{C})$ is called root system provided

1. $R \cap \mathbb{R} \alpha = \{\alpha, -\alpha\}$,
2. $R_\alpha(R) = R$

for all $\alpha \in R$.

Let $G \leq O(\mathbb{R}^{n+1,1}, \langle \cdot, \cdot \rangle) \simeq O(n+1,1, \mathbb{R})$ be the finite reflection group (generated by $\{R_\alpha | \alpha \in R\}$) associated to the root system $R$. We use the notation $[R]$ for the space of $G$-orbits of the root system $R$. An element
of the vector space of all functions $k : [R] \to \mathbb{C}$ is called the multiplicity function, and its value at $\alpha \in [R]$ is denoted $k(\alpha)$.

Note that $G$ acts linearly on $\mathbb{R}^{n+1,1}$ and preserves $\mathcal{C}$, i.e.

$$< X, X > = 0 \implies < R_\alpha X, R_\alpha X > = 0, \ \forall \alpha \in R. \quad (4)$$

In particular, $G$ preserves the lines in $\mathcal{C}$ and so induces a map on $S^n$, the projectivization of $\mathcal{C}$. The canonical (ambient) metric on $\mathbb{R}^{n+1,1}$ written in the coordinates $X^0, X^1, \ldots, X^n, X^\infty$ with respect to canonical orthonormal basis,

$$\tilde{g}(X^0, X^1, \ldots, X^n, X^\infty) = dX^0 \odot dX^\infty + dX^\infty \odot dX^0 + \sum_{i=1}^{n} dX^i \odot dX^i,$$

can be conveniently rewritten in the coordinates

$$\tilde{X}^0 := \frac{1}{\sqrt{2}}(X^0 + X^\infty), \ \tilde{X}^\infty := \frac{1}{\sqrt{2}}(X^0 - X^\infty), \ \tilde{X}^i = X^i (i = 1, \ldots, n) \quad (5)$$
as

$$\tilde{g}(\tilde{X}^0, \tilde{X}^1, \ldots, \tilde{X}^n, \tilde{X}^\infty) = d\tilde{X}^0 \odot d\tilde{X}^0 + \sum_{i=1}^{n} d\tilde{X}^i \odot d\tilde{X}^i - d\tilde{X}^\infty \odot d\tilde{X}^\infty.$$

A finite (reflection) group $G$ is contained in the maximal compact subgroup $K = O(n, \mathbb{R}) \times O(1, \mathbb{R})$ of $O(n+1,1, \mathbb{R})$, where $O(1, \mathbb{R}) \simeq \mathbb{Z}_2$. The elements in $G \cap O(n, \mathbb{R})$ are generated by a root system in the hyperplane $X^\infty = 0$ (i.e. $X^0 - X^\infty = 0$), while the non-trivial element in $G \cap O(1, \mathbb{R})$ is generated by reflection along the root in the coordinate axis $X^\infty$. If there was a root for $O(1, \mathbb{R})$ lying in the axis $X^\infty$, it would have a negative norm and its reflecting hyperplane $H_\alpha$ would not intersect the positive null cone, i.e. it would not induce any fixed point on the projectivisation of the null-cone. In conclusion, we restrict the root system for $G$ to lie in the euclidean (with respect to the induced metric) subspace $X^\infty = 0$.

**Example 3.2** Let us consider the irreducible root system on $\mathbb{R}^{n+1} \leq \mathbb{R}^{n+1,1}$ of type $B_{n+1}$,

$$B_{n+1} := \{ \pm e_i \pm e_j \} | 0 \leq j < i \leq n \} \cup \{ \pm e_i | 0 \leq i \leq n \}. \quad (6)$$

The inclusion $\mathbb{R}^n \leq \mathbb{R}^{n+1} \leq \mathbb{R}^{n+1,1}$ ($\mathbb{R}^n \subset S^n$) implies that the set of roots $B_{n+1}$ splits on two subsets, $B_{n+1} = B_n \cup S$:

1. The subset $B_n$ is the set of vectors in $\mathbb{R}^{n+1,1}$ with $\alpha_0 = \alpha_{n+1} = 0$,

$$B_n = \{ (0, \pm e_i \pm e_j, 0) | 1 < j < i \leq n \} \cup \{ (0, \pm e_i, 0) | 1 < i \leq n \}, \quad (7)$$

    corresponding to standard reflections in the Levi subgroup $O(n, 1, \mathbb{R}) \leq O(n, \mathbb{R})$.

2. The subset $S$,

$$S = \{ (\pm 1, \pm e_i \pm e_j, \pm 1) | 1 < i \leq n \} \cup \{ (1, 0, \ldots, 0, 1) \}, \quad (8)$$

are the roots characterized by $\alpha_0 = \alpha_{n+1} \neq 0$. The elements in $O(n+1, \mathbb{R}) \setminus O(n, \mathbb{R})$ are responsible for non-trivial rational factor in the conformal Dunkl-Laplace operator.
So let \( \alpha \in R \) be a root such that \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_0) \) in canonical orthonormal basis of \( \mathbb{R}^{n+1,1} \). Then

\[
< \alpha, \alpha > = 2\alpha_0^2 + \sum_{i=1}^{n} \alpha_i^2,
\]
i.e. the fixed length \( < \alpha, \alpha > \) of \( \alpha \) allows to express

\[
\alpha_0 = \sqrt{\frac{1}{2} ( < \alpha, \alpha > - \sum_{i=1}^{n} \alpha_i^2 )}.
\]

The following result is a straightforward computation.

**Lemma 3.3** The reflecting hyperplane \( H_\alpha \subseteq \mathbb{R}^{n+1,1} \) associated to the root \( \alpha \in R \) is given by linear span of \((n+1)\)-tuple of (linearly independent) vectors

\[
(-\alpha_1, \alpha_0, 0, \ldots, 0),
(-\alpha_2, 0, \alpha_0, 0, \ldots, 0),
\ldots,
(-\alpha_n, 0, \ldots, 0, \alpha_0, 0),
\]

\[
(-\frac{\alpha_0}{\alpha_0} \sum_{i=1}^{n} \alpha_i^2, \alpha_1, \ldots, \alpha_n, \alpha_0).
\]

orthogonal to \( \alpha \).

**Definition 3.4** The reflecting subsphere \( S^{n-1}_\alpha \) corresponding to the (positive length) root \( \alpha \) is the projectivization of the intersection of \( H_\alpha \) and \( \mathcal{C} \).

In fact, in the local chart \( U := \{(1, x_1, \ldots, x_n, -\frac{1}{2} \sum_{i=1}^{n} x_i^2) | (x_1, \ldots, x_n) \in \mathbb{R}^n \subset S^n \} \) of \( S^n \), we have

\[
S^{n-1}_\alpha \cap U = \{(1, x_1, \ldots, x_n, -\frac{1}{2} \sum_{i=1}^{n} x_i^2) \in \mathcal{C} | < (1, x_1, \ldots, x_n, -\frac{1}{2} \sum_{i=1}^{n} x_i^2), (\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_0) > = 0 \}.
\]

The explicit form of the quadric \( S^{n-1}_\alpha \) in the local chart \( U \) is

\[
\sqrt{\frac{1}{2} ( < \alpha, \alpha > - \sum_{i=1}^{n} \alpha_i^2 )} (1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i = 0.
\]

On the sphere \( S^n \), the reflection of \((1, x_1, \ldots, x_n, -\frac{1}{2} \sum_{i=1}^{n} x_i^2) \in U \) along
$S^n_{α}$ is a point on $S^n$ given by

$$X \rightarrow R_α X = X - \frac{2 \langle α, X \rangle}{\langle α, α \rangle} α,$$

\begin{equation}
(1, x_1, \ldots, x_n, -\frac{1}{2} \sum_{i=1}^n x_i^2) \mapsto \left(1 - \frac{2α_0}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{i=1}^n x_i^2) + \sum_{i=1}^n α_i x_i), \right.

x_1 = \frac{2α_1}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{i=1}^n x_i^2) + \sum_{i=1}^n α_i x_i), \ldots,

\left. x_n = \frac{2α_n}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{i=1}^n x_i^2) + \sum_{i=1}^n α_i x_i), \right.

-\frac{1}{2} \sum_{i=1}^n x_i^2 = \frac{2α_0}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{i=1}^n x_i^2) + \sum_{i=1}^n α_i x_i)). \right.
\end{equation}

Let us summarize the previous considerations in the following Lemma.

**Lemma 3.5** In the local chart $U \subset S^n$, $X \in U$, the rational map of the underlying reflection $R_α$ associated to the root $α = (α_0, α_1, \ldots, α_n, α_0)$ is

$$X \rightarrow R_α X,$$

\begin{equation}
X \mapsto x_i - \frac{2α_i}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{j=1}^n x_j^2) + \sum_{j=1}^n α_j x_j) / (1 - \frac{2α_0}{\langle α, α \rangle} (α_0(1 - \frac{1}{2} \sum_{j=1}^n x_j^2) + \sum_{j=1}^n α_j x_j)) \quad i = 1, \ldots, n.
\end{equation}

Notice that Equation (13) applies to all vectors $X \in \mathbb{R}^{n+1,1}$, but the rational transformation realized by an element of the reflection group does not preserve the local chart $U$. So either the expression in the denominator of the rational transformation is non-zero and the image of a given point in $U$ rests in $U$, or it is zero and the point maps to the point at infinity $S^n \setminus U$. Observe that it is just the component $α_0$ of the root $α = (α_0, α_1, \ldots, α_n, α_0)$ responsible for the rational factor of the map $R_α$.

We shall now slightly change our focus and introduce the main algebraic tool, encapsulating the interplay between finite reflection group $G$ and conformal geometry of $(S^n, [g_0])$. Our definition is basically a signature modification of certain structure in euclidean (i.e., positive definitive) signature, see e.g. [4], to vector spaces of any signature, e.g. $(\mathbb{R}^{n+1,1}, <, >)$.

A basic device in the harmonic analysis for finite reflection groups $G$ on $\mathbb{R}^{n+1,1}$ are the Dunkl differential-difference operators,

$$T_ξ(k)(f)(X) := \partial_ξ f(X) + \sum_{α \in R_+} k(α) \frac{\langle α, ξ \rangle}{\langle α, X \rangle} (1 - R_α) f(X),$$

$ξ \in \mathbb{R}^{n+1,1}, f \in C^∞(\mathbb{R}^{n+1,1}).$ \hspace{1cm} (15)

Recall that $k$ is the multiplicity function on the root system $R$. They form mutually commuting family of $G$-equivariant operators acting on
\[ C^\infty(\mathbb{R}^{n+1,1}). \] Consequently, expanded in the canonical orthonormal basis \( \{ e_0, e_1, \ldots, e_n, e_\infty \} \) the Dunkl-Laplace operator

\[ \tilde{\Delta}_k := \sum_{i=1}^{n+2} T_{e_i}(k)T^{e_i}(k) = \sum_{i=1}^{n+2} < T_{e_i}(k), T_{e_i}(k) > \quad (16) \]

fulfills \( \tilde{\Delta}_k \circ g = g \circ \tilde{\Delta}_k \) for all \( g \in G \). Let us consider three homogeneous differential-difference \( G \)-invariant operators in \( \text{End}(C^\infty(\mathbb{R}^{n+1,1})) \), written in signature independent notation as

\[ E := -\frac{1}{4} < X, X >, \]
\[ F := \tilde{\Delta}_k, \]
\[ H := \frac{n+2}{2} + \gamma_k + < X, \partial >, \quad (17) \]

where \( < X, \partial > \) is the Euler homogeneity operator on \( \mathbb{R}^{n+1,1} \) and \( \tilde{\Delta}_k \) the Dunkl-Laplace operator on \( \mathbb{R}^{n+1,1} \):

\[ (\tilde{\Delta}_k f)(X) = (\tilde{\Delta} f)(X) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \frac{< \tilde{\nabla} f(X), \alpha >}{< \alpha, X >} - \frac{f(X) - f(\rho_\alpha X)}{< \alpha, X >^2} \right). \quad (18) \]

Here we used the notation \( \tilde{\Delta} \) for the Laplace operator on \( \mathbb{R}^{n+1,1} \) and

\[ \gamma_k := \sum_{\alpha \in R_+} k(\alpha). \quad (19) \]

The proof of the next Lemma is just a signature modification of the proof known in the euclidean case.

**Lemma 3.6** The three operators \( E, F, G \) fulfill the \( \text{sl}(2) \) commutation relations:

\[ [E, F] = H, \]
\[ [E, H] = 2E, \]
\[ [F, H] = 2F. \quad (20) \]

The space of smooth \( w \)-homogeneous functions \( C^w_\infty(\mathcal{C}) \) on the null-cone \( \mathcal{C} \) is isomorphic to the space of \( w \)-densities \( C^\infty(S^n, \mathcal{L}_w) \) on \( S^n \) via

\[ \iota : C^\infty(S^n, \mathcal{L}_w)|_{\mathbb{R}^n} \to C^w_\infty(\mathcal{C}) \]
\[ \iota(f)(X^0, X^1, \ldots, X^n, X^\infty) = (X^0)^w f\left(\frac{X^1}{X^0}, \ldots, \frac{X^n}{X^0}\right). \quad (21) \]

The ambient Dunkl-Laplacian \( \tilde{\Delta}_k \) and its powers act on smooth \( w \)-densities \( C^\infty(S^n, \mathcal{L}_w) \) on \( S^n \) as follows. A \( w \)-density on \( S^n \) corresponds to a \( w \)-homogeneous function on \( \mathcal{C} \), which is then arbitrarily extended out of \( \mathcal{C} \) to its neighborhood in \( \mathbb{R}^{n+1,1} \). After the action of \( \tilde{\Delta}_k^j \) we restrict back to \( \mathcal{C} \). For each power of \( \tilde{\Delta}_k \) there exists a specific (termed critical) weight \( w \) for which the previous procedure is independent of the extension chosen.
Theorem 3.7 Let \( j \in \mathbb{N} \) and set \( w := -\frac{n}{2} + j - \gamma_k \). Moreover, let \( \tilde{f} \) be an extension to \( \mathbb{R}^{n+1,1} \) of a smooth \( w \)-density \( f \in C^\infty(S^n, \mathcal{L}_w) \). Then the composition

\[
    f \rightarrow \tilde{f} \rightarrow \tilde{\Delta}^l_k(\tilde{f}) \rightarrow \tilde{\Delta}^l_k(\tilde{f})|_{C}
\]

depends on \( f \) only and not on its extension \( \tilde{f} \), and consequently induces a nontrivial \( G \)-invariant differential-difference operator

\[
    \tilde{\Delta}^l_k|_{C} : C^\infty(S^n, \mathcal{L}_w) \rightarrow C^\infty(S^n, \mathcal{L}_{w-2j})
\]

of order \( 2j \) and symbol \( \Delta^j \).

Proof: Let us consider an extension \( \tilde{f} \) of a \( w \)-homogeneous function \( f \) on \( C \). Any other extension \( \tilde{f}_1 \) differs from \( \tilde{f} \) by \( E \) for some \( g \) of homogeneity \( w-2 \). We have

\[
    F\tilde{f}_1 = F\tilde{f} + Efg = F\tilde{f} + EFg - \left( \frac{n+2}{2} + \gamma_k + w - 2 \right)g,
\]

and so for \( w = -\frac{n}{2} - \gamma_k \) we have \( F\tilde{f}_1|_{C} = F\tilde{f}|_{C} \), exactly as claimed. The case of an integral power of \( F \) is based on the commutator \([E,F]\) and is analogous to \( j = 1 \). The proof is complete. \( \blacksquare \)

The operators constructed in Theorem 3.7 are expected to be related to the obstructions to harmonic extension of a \( t \)-homogeneous function on \( C \) to a solution of the Dunkl-Laplace equation \( \tilde{\Delta}_k \tilde{f} = 0 \) on \( \mathbb{R}^{n+1,1} \), but we shall not pursue this line of considerations.

It would be also extremely useful to construct the Dunkl type first order differential-difference operators on the sphere \( S^n \), whose sum of squares yields the conformal Dunkl-Laplace differential-difference operators.

4 The explicit form of second order conformal Dunkl-Laplace operator

In the previous section we proved the existence of Dunkl version of conformally invariant Laplace operator and its powers. An explicit form of these operators is another matter and amounts to the computation for a chosen finite reflection group \( G \) and in a chosen representative metric. In the present section, we compute explicit form of the conformal Dunkl-Laplace operator in a local coordinate chart \( U \) of \( S^n \).

A smooth \( w \)-density \( f = f(x_1, \ldots, x_n) \) on \( S^n \) can be regarded as a \( w \)-homogeneous function on \( C \). The construction described in Theorem 3.7 is independent on the extension of \( t^w f(x_1, \ldots, x_n) \) off the null-cone \( C \) and so we choose the constant, i.e. \( \rho \)-independent, extension, denoted by \( \tilde{f} \),

\[
    \tilde{f}(t, x_1, \ldots, x_n, \rho) = \tilde{f}(t, x_1, \ldots, x_n) = t^w f(x_1, \ldots, x_n)
\]

for the critical value \( w = -\frac{n}{2} - \gamma_k + 1 \). We determine the explicit form of the operator on \( S^n \) induced by descending \( \tilde{\Delta}_k \) (recall equation (22)):
1. It follows from Lemma 2.1

\[ \tilde{f}(X)|_{t=1} = (2 \frac{\partial}{\partial X_0} \frac{\partial}{\partial X_0} + \sum_{i=1}^{n} \frac{\partial^2}{\partial X_i^2}) \tilde{f}(X)|_{t=1} = \sum_{i=1}^{n} \frac{\partial^2}{\partial X_i^2} f(x_1, \ldots, x_n), \]

i.e. \( \tilde{\Delta} \) descends in the local chart \( U \) to the Laplace operator \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) acting on \( f \).

2. It follows from the formula (14) and comments beyond Lemma 3.5

\[ \nabla = \left((\frac{n}{2} - \gamma_k + 1 - \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}), \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, 0\right), \]

so we get for \( \alpha_0 = \sqrt{\frac{1}{2}(<\alpha, \alpha> - \sum_{i=1}^{n} \alpha_i^2)} \)

\[ <\alpha, \nabla> = \alpha_0 \left(\frac{n}{2} - \gamma_k + 1 - \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} \alpha_i x_i. \tag{25} \]

Moreover,

\[ <\alpha, X> = \sqrt{\frac{1}{2}(<\alpha, \alpha> - \sum_{i=1}^{n} \alpha_i^2)(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i}, \]

and so \( \frac{\alpha_i}{<\alpha, X>} \) descends to the operator

\[ <\alpha, \nabla> = \frac{\alpha_0 \left(\frac{n}{2} - \gamma_k + 1 - \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} \alpha_i x_i}{\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i}. \tag{26} \]

acting on \( f \).

3. It follows from the formula (13) and comments beyond Lemma 3.5

\[ \tilde{f}(X)|_{t=1} = f(x_1, \ldots, x_n), \]

\[ \tilde{f}(R_\alpha X)|_{t=1} = \tilde{f} \left( X^0 - \frac{2\alpha_0}{<\alpha, \alpha>} \left(\alpha_0(X^0 - \frac{1}{2} \sum_{i=1}^{n} X_i^2) + \sum_{i=1}^{n} \alpha_i X_i\right), \right) \]

\[ X^1 - \frac{2\alpha_1}{<\alpha, \alpha>} \left(\alpha_0(X^0 - \frac{1}{2} \sum_{i=1}^{n} X_i^2) + \sum_{i=1}^{n} \alpha_i X_i\right), \]

\[ \ldots \]

\[ X^n - \frac{2\alpha_n}{<\alpha, \alpha>} \left(\alpha_0(X^0 - \frac{1}{2} \sum_{i=1}^{n} X_i^2) + \sum_{i=1}^{n} \alpha_i X_i\right), \]

\[ = \left(1 - \frac{2\alpha_0}{<\alpha, \alpha>} \left(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i\right)\right)^{-\frac{1}{2} - \gamma_k + 1} \]

\[ f \left( x_1 - \frac{2\alpha_1}{<\alpha, \alpha>} \left(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i\right) \right) \]

\[ \ldots \]

\[ x_n - \frac{2\alpha_n}{<\alpha, \alpha>} \left(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i\right) \]

\[ = \left(1 - \frac{2\alpha_0}{<\alpha, \alpha>} \left(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i\right)\right)^{-\frac{1}{2} - \gamma_k + 1} \]
and so

\[
\frac{(1 - R_0)f(X)}{<\alpha, X>^2}|_{t=1} = \frac{\dot{f}(X) - \dot{f}(R_0X)}{<\alpha, X>^2}|_{t=1} =
\]

\[
\frac{1}{(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i)^2} \left( f(x_1, \ldots, x_n) - \left( 1 - \frac{2\alpha_0}{<\alpha, \alpha>} (\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i) \right) \cdot \frac{1}{2} \gamma - \frac{1}{2} \right). \]

Theorem 4.1 Let \( G \leq O(n+1, \mathbb{R}) \leq O(n+1, \mathbb{R}) \) be a finite reflection group associated to the root system \( R = \{ \alpha | \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_0) \} \subset \mathbb{R}^{n+1}. \) Then the conformal Dunkl-Laplace operator is in the local chart \( U \cong \mathbb{R}^n \subset S^n \) given by

\[
\Delta_k f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n) + 2 \sum_{\alpha \in R_+} k(\alpha)
\]

\[
\left( \frac{1}{(\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i)^2} \cdot \left( f(x_1, \ldots, x_n) - \left( 1 - \frac{2\alpha_0}{<\alpha, \alpha>} (\alpha_0(1 - \frac{1}{2} \sum_{i=1}^{n} x_i^2) + \sum_{i=1}^{n} \alpha_i x_i) \right) \cdot \frac{1}{2} \gamma - \frac{1}{2} \right) \right),
\]

when acting on \( f \in C^\infty(S^n, L_w) \) for the critical weight \( w. \) ■

The explicit formulas for conformal powers of the second order Dunkl-Laplace operator are notationally less trivial due to the iterative composition of rational terms. Because of complexity of the number of descending terms we do not attempt to write down extensive formulas explicitly.
5 Open questions

There are several interesting aspects of the construction, leading to its generalization in various directions. First of all, the present construction works in any signature. Also, one does not need to restrict to the principal series representations of $O(n + 1, 1, \mathbb{R})$ induced from character, but inducing from a finite dimensional representation (e.g. the spinor representation) leads to the canonical $G$-equivariant version of a given conformally invariant operator (e.g. the conformally invariant powers of the Dirac operator.) Another direction for possible generalization stems from the existence of a curved version of the Fefferman-Graham ambient metric construction of the manifold with conformal structure $(\tilde{M}, [\tilde{g}])$, see the review [8] and the references therein. In particular, the ambient metric is an Einstein metric of constant scalar curvature, and so finite reflection subgroups of the (compact subgroup of) the isometry group $Isom(\tilde{M}, [\tilde{g}])$ of the Fefferman-Graham metric of signature $(n + 1, 1)$ together with its action on the Fefferman-Graham ambient space can be geometrically analyzed along the lines of e.g., [1]. Similar considerations might be applied to finite reflection groups realized in the group of automorphisms of another geometric structure, e.g. the complex reflection groups leading to the Dunkl version of so called CR-Laplace operator and its powers, see [7]. Another source of examples involves quaternion reflection groups and related parabolic geometries of quaternion type.

On general abstract grounds, the problems discussed in the article are rather special instances of the classification scheme for $G$-invariant differential-difference operators on the sphere with standard round metric and its conformal structure $(S^n, [g_0])$, related to the branching problems for a (infinite dimensional) representations of a Lie group and finite subgroups $G \leq O(n + 1, 1, \mathbb{R})$. To our best knowledge, there is rather poor understanding of such problems. Moreover, in addition to the deformation of operators of elliptic type one should attempt to construct the Dunkl version of overdetermined (twistor type) operators and more generally Dunkl deformation of the geometric construction of Bernstein-Gelfand-Gelfand sequences, [3], or the invariant calculus of tractors for a wide class of parabolic geometries, [5]. Another generalization of the present results includes the variation on the theme of finite reflection groups. There is no need for such restrictive category, i.e. one can consider for a fixed background geometry a wider category of subgroups like the class of finite groups or even discrete groups.

The author expects that compatible structure consisting of a finite reflection group and the underlying conformal structure might be fruitful for the development of a conformal (or CR, quaternion, etc.) version of invariant theory for (complex, quaternion) reflection groups accompanied by algebraic structures of integrable systems, Cherednik and Hecke algebras, etc., in a systematic way.

The Fefferman-Graham ambient space $(\tilde{M}, \tilde{g})$ associated to a general conformal manifold $(M, [g])$ is a pseudo-Riemannian manifold equipped with an Einstein metric, i.e. a particularly nice class of constant scalar curvature manifolds on which the action of a finite reflection group can be studied.
The following definition is a natural generalization of a finite reflection group acting on unitary vector space.

**Definition 5.1** A reflection $s$ on a Riemannian manifold $(N, g)$ is an isometry, $s \in \text{Isom}(N)$, such that for some fixed point $x$ of $s$ the tangent map $T_x s$ is a reflection in the Hilbert space $(T_x N, g_x)$.

For a discrete subgroups of Isom$(N)$ generated by reflections, the classical geometrical concepts like its Dirichlet domain, Weyl chamber and for Coxeter manifolds a Riemannian chamber defined as a manifold with corners such that its walls are totally geodesic submanifolds and neighboring walls satisfy the Coxeter property, are studied, see e.g. [1] and the references therein. The techniques of Fefferman-Graham ambient metric construction are sufficiently flexible allowing in particular instances of finite reflection groups to construct Dunkl deformations of GJMS-operators or more generally, differential-difference invariants of $(M, [g], G)$ on manifolds with conformal structure $(M, [g])$. However, at the moment it is not clear what are the discrete subgroups of $\text{Isom}(\tilde{M}, \tilde{g})$ of most interest worth to be studied in details.

**Acknowledgment:** The author gratefully acknowledges the support of the grant GA CR P201/12/G028.

**References**

[1] D. V. Alekseevsky, A. Kriegl, M. Losik, P. W. Michor, Reflection groups on Riemannian manifolds, Annali Mat. Pura ed Applicata, v.187, No. 1, 25–58, 2007.

[2] C. T. Benson, L. C. Grove, Finite reflection groups, Springer, 1985, ISBN 0387960821, 9780387960821, 133 pages.

[3] A. Cap, J. Slovak, V. Soucek, Bernstein-Gelfand-Gelfand sequences, Annals of Mathematics, 154 (2001), 97–113.

[4] P. Etingof, Calogero-Moser Systems and Representation Theory, Zurich Lectures in Advanced Mathematics, 2007, vol. 4, ISBN-10: 3-03719-034-5, ISBN-13: 978-3-03719-034-0.

[5] A. Cap, A. R. Gover, Tractor calculi for parabolic geometries, Trans. Amer. Math. Soc. 354 (2002), 1511–1548.

[6] C. R. Graham, R. Jenne, L. Mason, and G. Sparling, Conformally invariant powers of the Laplacian, I: Existence, J. London Math. Soc. (2), 46 (1992) 557–565.

[7] C.R. Graham, A.R. Gover, CR-invariant powers of the sub-Laplacian, J. Reine Angew. Math. 583 (2005) 1–27.

[8] L.P. Hughston, T.R. Hurd, A $CP^5$ calculus for space-time fields, Physics Reports, Volume 100, Issue 5, November 1983, 273–326.

[9] Eric M. Opdam, Lecture notes on Dunkl operators for real and complex reflection groups, MSJ memoirs, vol. 8, Mathematical Society of Japan, 2000.
[10] M. Roesler, Dunkl operators: Theory and applications, Lecture Notes in Math., vol. 1817, Springer, 2003, 93–165.

Petr Somberg
Mathematical Institute of Charles University,
Sokolovská 83, 186 75 Praha
Czech Republic,
somberg@karlin.mff.cuni.cz