Neuromorphic Functions of Light in Parity-Time-Symmetric Systems

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As an elementary processor of neural networks, a neuron performs exotic dynamic functions, such as bifurcation, repetitive firing, and oscillation quenching. To achieve ultrafast neuromorphic signal processing, the realization of photonic equivalents to neuronal dynamic functions has attracted considerable attention. However, despite the nonconservative nature of neurons due to energy exchange between intra- and extra-cellular regions through ion channels, the critical role of non-Hermitian physics in the photonic analogy of a neuron has been neglected. Here, a neuromorphic non-Hermitian photonic system ruled by parity-time symmetry is presented. For a photonic platform that induces the competition between saturable gain and loss channels, dynamical phases are classified with respect to parity-time symmetry and stability. In each phase, unique oscillation quenching functions and nonreciprocal oscillations of light fields are revealed as photonic equivalents of neuronal dynamic functions. The proposed photonic system for neuronal functionalities will become a fundamental building block for light-based neural signal processing.

1. Introduction

Ionic mechanisms in a neuron\(^1\) are governed by the state-dependent gating of ion channels (Na\(^+\), K\(^+\), and leak) and their nonlinear competition satisfying the law of current conservation, which derive unique neuronal functions of bifurcation,\(^2,3\) repetitive firing,\(^4\) and oscillation quenching.\(^5,6\) To achieve neuromorphic signal processing, important clues regarding the analogy of a neuron are therefore in electronic,\(^7\) spintronic,\(^8\) and photonic\(^9-11\) implementations of the state-dependent channel gating and interchannel interactions. In particular, with the advantage of ultrafast signal processing in photonics, the realization of suitable “light” channels that mimic “ion” channels within a neuron has been a key issue of practical importance in the field of neuromorphic photonics.\(^9-11\)

For neuromorphic photonics systems, we can envisage the substitution of neuronal ion channels that connect intra- and extra-cellular regions by nonlinear wave amplification and dissipation, due to the similarity between biological and optical state-dependent dynamics. The study of state-dependent channels in wave systems is readily found in traditional photonic devices, such as laser with gain saturation\(^12\) and mode locking\(^13\) with saturable absorption. The emergence of metamaterial concepts, nanofabrication technologies, and innovative material platforms has also allowed the enhanced strength and tunability of nonlinear amplification or absorption for photonic devices, as shown in coherent amplification in lossy plasmonic metamaterials\(^14\) ultrafast pulsed lasing in graphene structures,\(^15\) or black phosphorus,\(^16\) and nonlinear activations in photonic deep-learning circuits.\(^17\)

Non-Hermitian photonics\(^18-20\) inspired by parity-time (PT) symmetry\(^21\) has rejuvenated the utilization of nonlinear wave channels for photonic\(^22-24\) or microwave\(^25\) functionalities. The concept of PT symmetry\(^21\) has allowed the access to real observables in non-Hermitian Hamiltonians. Due to the design flexibility of photonic platforms and the Schrödinger-like paraxial wave equation for light, photonic structures have been employed as a testbed for examining wave phenomena in PT-symmetric systems. Early studies have mostly focused on linear wave phenomena near the exceptional point (EP),\(^20,26\) such as asymmetric\(^27\) and sensitive\(^28\) excitations, unidirectional invisibility,\(^29\) enhanced spin,\(^30,31\) or orbital angular momentum,\(^32\) and asymmetric modal conversion,\(^33\) which have been demonstrated in waveguides,\(^27,33\) fibers,\(^29\) metamaterials,\(^30,31\) and resonators.\(^28,32\) Recently, a research focus in this field has been extended to the interpretation of nonlinear wave phenomena. The effects of nonlinear wave channels near the EP\(^26\) have been studied for nonreciprocal transparency from directional resonator excitations,\(^22,23\) robust power transfer using gain saturation,\(^25\) and dynamical encircling for polarization conversion which is robust to nonlinear effects.\(^34\)

However, in spite of the inherent non-Hermitian nature of neurons as shown in energy exchange between intra- and extra-cellular regions of a neuron, an analytical framework to address the critical role of non-Hermitian phenomena in neuromorphic photonics is still absent.

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In this paper, we develop the photonic analogy of a neuron to realize neuromorphic dynamic functions of bifurcation, firing, and quenching by exploiting state-dependent wave channels that satisfy PT symmetry. Comparing different types of optical nonlinearities to the Hodgkin–Huxley (HH) model,[1] we show that PT-symmetric coupled resonators of saturable gain and loss satisfy the criteria of interacting state-dependent wave channels. We then perform the classification of PT-symmetric phases in terms of the stability.[2] The emergence of oscillation quenching functions[5] in the PT-symmetric system is demonstrated in relation to PT-symmetric phases, revealing the transition between amplitude death (AD)[34] and oscillation death (OD)[35] across the EP. We also show the directional repetitive firing of light, allowed in the coexistence regime of unbroken and broken PT symmetry. With its multifaceted neuromorphic functions, the proposed PT-symmetric photonic neuron will serve as a fundamental building block for light-based artificial neural networks, especially providing functional robustness and directionality.

2. Analogy of a Neuron in Nonlinear Photonic Systems

To draw the photonic analogy of a neuron, we examine neuronal ion channels using the HH model.[1] The dynamics of the membrane potential $V$ is described by $dV/dt = \rho_+(V) + \rho_-(V)$ with $\rho_+(V) = g_{\text{Na}}(V)(V - V_{\text{Na}})/C_m$ and $\rho_-(V) = -g_K(V)(V - V_K) + g_{\text{leak}}(V - V_{\text{leak}})/C_m$, where $C_m$ is the membrane capacitance; $g_{\text{Na}}$ and $g_K$ are the nonlinear conductance of sodium ($Na^+$) and potassium ($K^+$) ion channels, respectively, and $g_{\text{leak}}$ is the constant leak conductance; and $V_{\text{Na}}$, $V_K$, and $V_{\text{leak}}$ are the sodium, potassium, and leak reversal potentials, respectively. Figure 1a shows the calculated $g_{\text{Na}}$, $g_K$, and $g_{\text{leak}}$ at steady state (see the Experimental Section). Owing to different reversal potentials, $dV/dt$ is determined by the contrasting contributions (Figure 1b) of (i) $Na^+$ source channel ($\rho_+, > 0$) with channel strength $g_{\text{Na}}(V) = C_m[\rho_+(V)/(V - V_{\text{Na}})]$ and (ii) $K^+$ channel and leak sink channel ($\rho_-, < 0$) with channel strengths of $g_K(V) = -C_m[\rho_-(V)/(V - V_K)]$ and negligible $g_{\text{leak}}$ (Note S1 in the Supporting Information for the channel strength). Neuronal operations[1,3–6] are thus primarily governed by the competition between $Na^+$ and $K^+$ channels, which experience “saturating” when $V \rightarrow V_{\text{Na}}$, and $V \rightarrow V_K$, respectively (dashed arrows in Figure 1a).

The contrasting and saturable behaviors of ion channels provide guidance for designing nonlinear wave structures. Because the neuron is an open system interacting with the extracellular region, an open system in non-Hermitian photonic systems[18–20] is a suitable platform for realizing photonic neurons. The core of this analogy will then be the construction of state-dependent “source” and “sink” channels for light. To reproduce the membrane-potential-dependent nonlinearity, we utilize light-intensity-dependent nonlinearities in photonics, including the Kerr effect,[16] two-photon absorption (TPA),[17] and saturable amplification[13,38] or absorption.[15,39,40] Using such nonlinearities, we can assign potential-dependent “source” and “sink” ion channels to intensity-dependent “amplifying” and “dissipating” wave channels in the coupled resonator platform (Figure 1c).

A resonator with intensity-dependent nonlinearities is modeled by the temporal equation,[41] $d\psi/dt = i\omega_0\psi + M(|\psi|^2)\psi$, where $\psi$ and $\omega_0$ are the field amplitude and resonant frequency, respectively, and $M(|\psi|^2)$ is the function that determines the type of nonlinearity. The forms of $M(|\psi|^2)$ are classified as (i) a saturable function $M_{\text{Sat}}(|\psi|^2) = \gamma_{\text{Sat}}/1 + |\psi|^2/|\psi_0|^2$ with the characteristic intensity $|\psi_0|^2$ or (ii) a Stuart-Landau (SL) oscillator[13,35,36,41,42,43] $M_{\text{SL}}(|\psi|^2) = \gamma_{\text{SL}}|\psi|^2$, where $\gamma_{\text{Sat}}$ and $\gamma_{\text{SL}}$ are the strength coefficients. While $M_{\text{SL}}$ includes the Kerr effect[16] with imaginary-valued $\gamma_{\text{SL}}$ and TPA[17] with real-valued $\gamma_{\text{Sat}}$, $M_{\text{Sat}}$ covers the gain saturation[13,38] with real-valued $\gamma_{\text{Sat}} > 0$ and saturable absorption[15,39,40] with real-valued $\gamma_{\text{Sat}} < 0$.

As the photonic equivalent of the channel strengths $g_{\text{Na}}(V)$ and $g_K(V)$, we define the “photonic channel strength” as $g(I) = d(I)/dt$, where $I = |\psi|^2$ is the light intensity (see Note S1 for the experimental section). Owing to different reversal potentials, we have $g_{\text{Na}}(V)$ and $g_K(V)$ are the nonlinear conductance of sodium ($Na^+$) and potassium, and leak reversal potentials, respectively.
in the Supporting Information for comparison). Each type of optical nonlinearity then supports the channel strength \( g_{\text{sat}}(I) = 2 \text{Re}[\gamma_{\text{sat}}][I(1 + I/I_{\text{th}})] \) with \( I_{\text{th}} = |\psi_G|^2 \) and \( g_{\text{sat}}(I) = 4 \text{Re}[\gamma_{\text{sat}}]I \). Considering saturable ion channels, \( g_{\text{sat}}(I) \) provides a more suitable fit to \( g_{\text{Na}}(V) \) and \( g_{K}(V) \) than \( g_{\text{sat}}(I) \) by assigning gain and loss resonator excitations to \( V \to V_{\text{Na}} \) and \( V \to V_K \), respectively (Figure 1d, in comparison with upper and lower illustrations in Figure 1a). To achieve the maximum power transfer between resonators, we set an identical resonant frequency \( \omega_0 \) to both resonators,[41] which leads to the photonic neuron satisfying PT symmetry[18–21] with saturable nonlinearities (Figure 1c). The competition between ion channels mediated by Kirchhoff’s law (Experimental Section) is then reproduced by the electromagnetic coupling between resonators.

3. Stability Analysis

Saturable gain and loss are quantified by the gain[48] and loss[44] coefficients \( \gamma_{G,L} = \gamma_{G,L,0}(1 + |\psi_{G,L}|^2/|\psi_{G,L,0}|^2) \), where \( \gamma_{G,L,0} \) are constant coefficients, and \( |\psi_{G,L,0}|^2 \) are saturation intensities. The photonic neuron in Figure 1c is then modeled by the platform-transparent temporal coupled mode theory (TCMT),[41] as

\[
\begin{align*}
\frac{d}{dt}[\psi_G(t)] &= \begin{bmatrix}
\gamma_G
\frac{\psi_G}{1 + |\psi_G|^2/|\psi_G|^2} & i\kappa
\end{bmatrix}
\begin{bmatrix}
\psi_G(t)
\psi_L(t)
\end{bmatrix},
\end{align*}
\]

where \( \psi_G(t) \) and \( \psi_L(t) \) are the field amplitudes in gain and loss resonators, respectively, and \( \kappa \) is the evanescent coupling coefficient between them. Equation (1) can be applied to any types of weakly coupled resonant elements in photonics and microwaves, including microcavities,[45] nanoparticles,[46] and metamaterials.[47] We also note that Equation (1) is the generalization of linear two-level PT-symmetric systems[27] to nonlinear domains (\( \psi_{G,L} \to \infty \) for linear systems), covering saturable responses of nonlinear gain and loss coefficients. This extension also allows for the photonic analogy of the competition between saturable ion channels using the coupling coefficient \( \kappa \) between saturable gain and loss elements.

Equation (1) can be divided by separating the amplitude and phase components of light fields[48] \( \psi_{G,L} = (\psi_{G,L})_{1/2} \exp(i\theta_{G,L}) \). According to PT-symmetric phases,[22] we then derive two real-valued equations from Equation (1), each for the eigenmodes of “unbroken” and “broken” PT symmetry (see the Experimental Section)

\[
\begin{align*}
&\frac{d}{dt}I_G = \frac{2\gamma_G}{1 + (I_G/I_{\text{th}})} I_G - \sqrt{I_G} I_L + \frac{\gamma_{\text{th}}}{1 + (I_G/I_{\text{th}})} I_L,
&\frac{d}{dt}L = - \frac{2\gamma_L}{1 + (I_L/I_{\text{th}})} I_L + \sqrt{I_L} I_G + \frac{\gamma_{\text{th}}}{1 + (I_L/I_{\text{th}})} I_G,
\end{align*}
\]

where \( I_{G,L} = |\psi_{G,L}|^2 \). It is noted that the resonant frequency \( \omega_0 \) does not affect the dynamics of light intensities inside the photonic neuron.

With Equations (2) and (3), we conduct the bifurcation analysis[23] to examine the stability of PT-symmetric phases (see the Experimental Section). From Equation (2) with the equilibrium condition \( dI_{G,L}/dt = 0 \), we look for the nontrivial equilibrium of the unbroken PT symmetry, which leads to the homogeneous steady state[14] (HSS) of \( I_G = I_L = I_{\text{H}} \): the same light intensity level in gain and loss resonators (Experimental Section). The existence of \( I_{\text{H}} \) is determined by \( I_{\text{H}} \geq 0 \) (Figure 2a) which automatically satisfies unbroken PT symmetry. On the other hand, Equation (3) with \( dI_{G,L}/dt = 0 \) results in a nontrivial equilibrium of the broken PT-symmetric phase, which leads to \( I_G = I_L = I_{\text{L}} \) at the EP but \( I_G \neq I_L \) in the broken PT-symmetric phase (Experimental Section). In contrast to the HSS in the unbroken phase or at the EP, the broken phase thus supports the inhomogeneous steady state[5] (IHSS) in the regime of \( \gamma_{G,L,0} \geq \kappa^2 \) for nonnegative \( I_G \) and \( I_L \) (Figure 2b): the different light intensity level in gain and loss resonators.

![Figure 2. Stability of the equilibria in PT-symmetric phases. a,b) Equilibrium intensities of a) unbroken \( (I_{\text{H}}) \) and b) broken \( (I_G, I_L) \) phases \( (I_L \) is not shown). Black dashed contours in (b) denote \( I_G/I_{\text{th}}, I_L/I_{\text{th}} \) Jacobian eigenvalues for the stability classification: c) \( \text{Max}(-\text{Re}[\lambda_{I}], \text{Re}[\lambda_{I}]) \) for the broken phase and d) the maximum value of \( -\text{Re}[\lambda_{I}] \) for the broken phase. Red dashed lines in (c,d) represent the zero level, and white dashed line in (d) denotes the transition between nodes and foci (Note S2, Supporting Information). Points 4a and 4b in (a–d) denote the cases of Figure 4a,b, respectively. \( \kappa = 2 \times 10^{-3} \), and \( \gamma_{G,L} = 2 \) for all cases.](image)
From the first Lyapunov criterion, we classify the stability of each equilibrium in Figure 2c,d using the Jacobian matrices of Equations (2,3). First, the Jacobian matrix of the unbroken phase has only one eigenvalue \( \lambda \) at equilibrium (Experimental Section), forming the hyperbolic equilibrium with \( \gamma_{G0} \neq \gamma_{L0} \). Because the stability of hyperbolic equilibria is defined by the sign of the real parts of Jacobian eigenvalues, \[2\] the phase transition at the unbroken PT symmetry occurs at \( \gamma_{G0} = \gamma_{L0} \) (Figure 2c): asymptotically stable with \( \gamma_{G0} > \gamma_{L0} \) owing to \( \text{Re}\{\lambda\} < 0 \) and unstable with \( \gamma_{G0} < \gamma_{L0} \). In contrast, equilibrium in the broken phase supports two Jacobian eigenvalues \( \lambda_1, \lambda_2 \) (Experimental Section). Except for the boundary (red dashed lines in Figure 2d), \( \text{Re}\{\lambda_{1,2}\} \) is nonzero, again corresponding to hyperbolic equilibria. The stability of the equilibria is then classified\[2\] as asymptotically stable nodes or foci for \( \text{Re}\{\lambda_{1,2}\} < 0 \) and unstable foci for \( \text{Re}\{\lambda_{1,2}\} > 0 \) (Figure 2d, see also Note S2 in the Supporting Information).

Figure 3a shows the phase classification of the photonic neuron, achieved from Figure 2. The entire diagram is classified into five phases according to the stability of equilibria and PT symmetry: (i) the OD phase with stable IHSS of broken PT symmetry (Figure 3b), (ii) the AD phase with stable HSS of unbroken PT symmetry (Figure 3c), (iii) the C1 (Figure 3d) and (iv) C2 (Figure 3e) phases with the coexisting nontrivial equilibrium of unbroken and broken PT-symmetric phases, and (v) the U unstable phase without any nontrivial equilibrium. Each phase boundary originates from a different physical origin with respect to stability and PT symmetry. Phase transitions for the given PT-symmetric phase (yellow arrows) occur at the interfaces between the AD and C1–C2 phases (unbroken) and between the OD–C1 and C2 phases (broken). The interface between the AD and OD phases (black arrow) leads to both stability and PT-symmetric phase transitions across the EP. While the OD phase can be divided into two subclasses for stable nodes and foci (white arrow), these subclasses exhibit the same stability condition (Note S2, Supporting Information).

4. Photonic Oscillation Quenching

In the phase classification of Figure 3, we examine wave behaviors in AD and OD phases, each representing distinct oscillation quenching phenomena. In the regime of \( \gamma_{G0} > \gamma_{L0} \) with the stable HSS \( I_H \) (e.g., point 4a in Figure 2a,c), the calculated phase portrait shows the robust convergence of \( I_G \) and \( I_L \) to the same value of \( I_H \) (Figure 4a), which derives AD oscillation quenching\[34\] in unbroken PT symmetry: the identical constant excitation of resonators with \( I_G = I_L \). In contrast, in the regime of \( \gamma_{G0} < \gamma_{L0} \) with the stable IHSS \( I_G \neq I_L \) (e.g., point 4b in Figure 2b,d), numerical analysis proves the robust convergences to the equilibrium of \( I_G \neq I_L \) independent from initial \( I_G \) and \( I_L \) (Figure 4b), which allows OD oscillation quenching\[35\] in broken PT symmetry: the constant excitation of resonators with different intensity states \( I_G \neq I_L \).

We demonstrate these oscillation quenching functions in Figure 4c,d with the time domain simulation, showing photonic
AD and OD, respectively. The results in Figure 2a–d provide the design criteria of AD and OD phenomena in the photonic neuron in terms of the intensity level (Figure 2a,b) and the types of the phase portrait defined by Jacobian eigenvalues (Figure 2c,d). It is also noted that the AD-OD transition boundary (black arrow in Figure 3a) at the EP represents the complete suppression of light fields inside resonators (Note S3, Supporting Information). The transition between the EP and the stable equilibrium of the unbroken PT-symmetric phase and the unstable equilibrium of the unbroken phase. The PT-symmetric phase between being stable broken and unstable unbroken is dependent on the initial condition (Note S4 in the Supporting Information for point B in Figure 3a), allowing the turning “on” and “off” behavior of the OD oscillation quenching functions.

We now focus on the coexisting phase C2, offering nonreciprocal oscillations which are absent from other dynamic phases (AD, OD, C1, and U phases). This reveals the critical role of the saturable absorption (finite value of \( \psi_{G_0} \)) for nonreciprocal oscillations, because the gain saturation only (finite \( \psi_{G_0} \) and \( \psi_{G_0} \to \infty \)) cannot lead to the C2 phase. In contrast to the C1 phase, the C2 phase possesses unstable equilibria of both the unbroken and broken PT-symmetric phases, and this instability allows the dynamic transition between PT-symmetric phases.

Figure 5 shows the phase portraits of the C2 phase with gain (Figure 5a–c) and loss (Figure 5d–f) resonator excitations. The limit cycle solutions are achieved at low intensity (red lines in Figure 5a,d), while the system diverges at high intensity (Figure 5c,f). The limit cycle that allows the repetitive firing of light is a unique feature of the C2 phase originating from the “oscillatory” PT-symmetric phase transition across the EP (Note S5, Supporting Information). The transition between the limit cycle and instability occurs at different intensity levels of the excitation port (Figure 5b,e), which imposes directionality on the photonic neuron. We also note that the convergence “speed” to the limit cycle in the repetitive firing is manipulated dependent on the initial intensity (Figure 5d,e). Therefore, with identical limit cycles that have different convergence times, we can develop the tuning of the temporal phase of photonic repetitive firing with the full coverage (Notes S6 and S7 in the Supporting Information).

6. Conclusion

In summary, we investigated the dynamic functions in photonic neurons with saturable gain and loss channels. We developed the phase diagram of photonic neuronal dynamics in terms of its phase portrait. Each phase exhibits different stability and equilibrium conditions in relation to the PT-symmetric phase. The connection between the PT-symmetric phase transition and AD-OD transition was revealed, providing the design criteria of photonic oscillation quenching functions. In the coexisting phase, we also demonstrated the repetitive firing of light with directionality and tunable time delay.

As shown in the platform-transparent TCMT equation and general stability theory, our analysis can be applied to various optical elements in the weak coupling regime. This
enables the utilization of well-established saturable media and structures, such as pumped media,\textsuperscript{[23,38]} organic dyes,\textsuperscript{[39]} graphene layers,\textsuperscript{[15]} and artificial realizations\textsuperscript{[40]} for the construction of nonlinear PT-symmetric photonics.

Our proposal of PT-symmetric dynamics will inspire new approaches for the design of a photonic neuron or its network, as demonstrated in the robust stability nature of photonic oscillation quenching and strong nonreciprocal repetitive firing. It is expected that neuronal functionalities of electromagnetic waves could be used as building blocks for neuromorphic wave circuits, underpinning the critical role for oscillation quenching, network synchronizations,\textsuperscript{[50]} and weighted and directional graph networks.\textsuperscript{[51]}

7. Experimental Section

Nonlinear Conductance of the Neuron: According to the HH model\textsuperscript{[1]} the Na\textsuperscript+ and K\textsuperscript+ ion channel strengths are given by $g_{Na}(V) = g_{Na}0 \cdot m^3 h$ and $g_{K}(V) = g_{K} \cdot n^4$, respectively, where

$$\frac{dX}{dt} = \alpha_X(V)(1-X) - \beta_X(V)X$$  \hspace{1cm} (4)

with $X = m$, $n$, and $h$, and

$$\alpha_m = \frac{0.1(25-V)}{\exp[(25-V)/10] - 1} \hspace{1cm} \beta_m = 4\exp(-V/18)$$  \hspace{1cm} (5)

$$\alpha_n = \frac{0.01(10-V)}{\exp[(10-V)/10] - 1} \hspace{1cm} \beta_n = 0.125\exp(-V/80)$$

$$\alpha_h = 0.07\exp(-V/20), \hspace{1cm} \beta_h = \frac{1}{\exp[(30-V)/10] + 1}$$

The steady-state condition $(d/dt \rightarrow 0)$ leads to $X(V) = \alpha_X(V)/[\alpha_X(V) + \beta_X(V)]$, clarifying the state dependency of the ion channel strengths. To calculate the result in Figure 1a, the parameters used were: $g_{Na}0 = 120 \text{ mS}\cdot\text{cm}^{-2}$, $g_{K0} = 36 \text{ mS}\cdot\text{cm}^{-2}$, and $g_{NaK0} = 0.3 \text{ mS}\cdot\text{cm}^{-2}$, $V_{Na} = 115 \text{ mV}$, $V_{K} = -12 \text{ mV}$, $V_{NaK0} = 10.599 \text{ mV}$, and $C_{m0} = 1 \mu\text{F}\cdot\text{cm}^{-2}$.

Dynamic Intensity Equations for PT-Symmetric Phases: From Equation (1) and the relation of $d|\psi|^2/dt = 2Re(|\psi|^2d\psi/dt)$, the following real-valued intensity equation was derived

$$\frac{d}{dt} \begin{bmatrix} I_c(t) \\ I_s(t) \end{bmatrix} = \begin{bmatrix} \frac{2\gamma cl_{ca}}{l_{cs} + l_{c}(t)} & \frac{2\gamma cl_{cl}}{l_{cs} + l_{c}(t)} \\ -2\kappa \sqrt{l_{cs}(t)} l_{c}(t) & -\frac{2\gamma sl_{ca}}{l_{ss} + l_{s}(t)} \end{bmatrix} \begin{bmatrix} I_c(t) \\ I_s(t) \end{bmatrix}$$  \hspace{1cm} (6)

where $\theta_c$ and $\theta_s$ are the real-valued phase of the field amplitude inside each optical resonator, as $\psi_{c,s} = (r_{c,s})^{1/2} \exp(\imath \theta_{c,s})$, resulting in the relation of the local phase difference $\theta_c - \theta_s = \text{Im}[\log(\psi_{c,s})]$. Equation (6) can be simplified for the eigenmodes of the system, which are obtained from the harmonic approximation of Equation (1). To achieve this, the instantaneous eigenfrequencies were derived: $\omega_{c,s}(t) = \omega_0 - \imath \kappa \theta_{c,s}(t) \pm \kappa |D(t)|^{1/2}$ of the system at time $t$, from $d\psi/dt \rightarrow i\omega_0 \psi$, where

$$\gamma_{c,s}(t) = \frac{\gamma cl_{ca}}{l_{cs} + l_{c}(t)} \pm \frac{\gamma cl_{cl}}{l_{cs} + l_{c}(t)}, \hspace{1cm} D(t) = 1 - \frac{\gamma cl_{ca}}{l_{cs} + l_{c}(t)} \pm \frac{\gamma cl_{cl}}{l_{cs} + l_{c}(t)}$$  \hspace{1cm} (7)

The time-varying function $\gamma_{c,s}(t)$ determines the instantaneous gauge of the PT-symmetric system, as $\gamma_{c,s} > 0$ for the active regime, $\gamma_{c,s} < 0$ for the passive regime, and $\gamma_{c,s} = 0$ for the normal PT symmetry. On the other hand, $D(t)$ defines the phase of PT symmetry\textsuperscript{[27]} $D(t) > 0$ for the unbroken phase, $D(t) < 0$ for the broken phase, and $D(t) = 0$ for the PT-symmetric phase transition at the EP. Notably, the local phase difference $\theta_c - \theta_s$ of the eigenmode is uniquely defined by the phase of PT symmetry\textsuperscript{[27]} Using the instantaneous eigenmodes of the system obtained from Equation (1) for both eigenfrequencies $\omega_{c,s}(t)$

$$\sin(\theta_c - \theta_s) = \frac{\gamma cl_{ca}}{l_{cs} + l_{c}(t)} \pm \frac{\gamma cl_{cl}}{l_{cs} + l_{c}(t)}$$  \hspace{1cm} (8)

$$\sin(\theta_c - \theta_s) = -1$$  \hspace{1cm} (9)

each for the unbroken and broken PT-symmetric phases, deriving Equations (2) and (3) in the main text from Equation (6), respectively.
Nontrivial Equilibrium in PT-Symmetric Phases: From $dl/dt = 0$ in Equations (2.3) for the condition of the equilibrium, the following relations are obtained for unbroken and broken phases, respectively

$$2\gamma \frac{\partial \alpha}{\partial t} = 2\gamma \frac{\partial \beta}{\partial t} = \gamma \frac{\partial \alpha}{\partial t} + \gamma \frac{\partial \beta}{\partial t}$$

(10)

$$2\gamma \frac{\partial \alpha}{\partial t} = 2\gamma \frac{\partial \beta}{\partial t} = 2\gamma \frac{\partial \alpha}{\partial t} + 2\gamma \frac{\partial \beta}{\partial t}$$

(11)

Equation (10) leads to $I_c = I_s = (\gamma \alpha - \gamma \beta) I_c / (\gamma \alpha - \gamma \beta)$ for unbroken PT symmetry, while Equation (11) derives the equilibrium for broken PT symmetry, as

$$I_c = \frac{1}{2} \frac{\gamma \alpha}{\gamma \beta} I_c + \frac{\gamma \beta}{\gamma \alpha} \left( \gamma \frac{\partial \alpha}{\partial t} + \gamma \frac{\partial \beta}{\partial t} \right)$$

Equation (11) leads to $I_c = I_s = (\gamma \alpha - \gamma \beta) I_c / (\gamma \alpha - \gamma \beta)$ for the broken PT-symmetric phase at equilibrium, as

$$I_c = \frac{1}{2} \frac{\gamma \alpha}{\gamma \beta} I_c + \frac{\gamma \beta}{\gamma \alpha} \left( \gamma \frac{\partial \alpha}{\partial t} + \gamma \frac{\partial \beta}{\partial t} \right)$$

(12)

It is noted that the condition of broken PT symmetry except the EP, $\kappa < F(I_c, I_s)$, where $F(I_c, I_s) = [\gamma \alpha I_c / (I_c + I_s) + \gamma \beta I_s / (I_c + I_s)] / 2$, enforces the IHSS $I_c \neq I_s$ from Equations (11) and (12).

Stability Analysis from the First Lyapunov Criterion: To examine the stability of the obtained equilibria, the Jacobian matrix is derived as the linearization of the PT-symmetric system. From Equation (2) of the unbroken PT-symmetric phase, the Jacobian matrix $A$ at equilibrium $I_c = I_s = \gamma \alpha - \gamma \beta I_c / (\gamma \alpha - \gamma \beta)$ becomes

$$A = \begin{bmatrix} \frac{M^2}{\gamma \alpha} & \frac{M^2}{\gamma \beta} \\ \frac{M^2}{\gamma \alpha} & \frac{M^2}{\gamma \beta} \end{bmatrix}$$

(13)

where $M = \gamma \alpha I_c / (I_c + I_s) = \gamma \alpha I_c / (I_c + I_s)$. Therefore, the Jacobian matrix of the unbroken PT-symmetric phase becomes 1-dimensional due to the linear dependence, and its unique eigenvalue $\lambda_1$ becomes

$$\lambda_1 = \left( \gamma \alpha - \gamma \beta \right) \frac{I_c / I_s}{(I_c + I_s / (I_c + I_s))}$$

(14)

Due to the first Lyapunov criterion, the PT-symmetric neuron at the unbroken phase becomes asymptotically stable with $\text{Re}[\lambda_1] < 0$, which is satisfied by $\gamma \alpha > \gamma \beta$, $I_c \neq 0$ and $I_s \neq 0$. In contrast, Equation (3) derives the Jacobian matrix $A$ for the broken PT-symmetric phase at equilibrium, as

$$A = \begin{bmatrix} \frac{2\gamma \alpha I_c^2}{(I_c + I_s)^2} - \frac{\kappa}{\gamma \alpha} I_c + \frac{\kappa}{\gamma \beta} I_s \\ - \frac{2\gamma \alpha I_c^2}{(I_c + I_s)^2} + \frac{\kappa}{\gamma \alpha} I_c + \frac{\kappa}{\gamma \beta} I_s \end{bmatrix}$$

(15)

where $I_c$ and $I_s$ at equilibrium are derived by Equation (12). The matrix $A$ of Equation (15) then supports two eigenvalues, as

$$\lambda_{1,2} = \frac{1}{2} \left( \frac{\kappa}{\gamma \alpha} I_c + \frac{\kappa}{\gamma \beta} I_s \right) \pm \sqrt{ \left( \frac{\kappa}{\gamma \alpha} I_c + \frac{\kappa}{\gamma \beta} I_s \right)^2 - \frac{2\gamma \alpha I_c^2}{(I_c + I_s)^2} (I_c + I_s)^2}$$

(16)

The PT-symmetric neuron at the broken phase then becomes asymptotically stable with $\text{Re}[\lambda_1] < 0$ and $\text{Re}[\lambda_2] < 0$.

Supporting Information
Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest
The authors declare no conflict of interest.

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amplitude death, neuromorphic function, optical nonlinearity, oscillation death, parity-time symmetry

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