Complex and real Hermite polynomials and related quantizations

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Abstract

It is known that the anti-Wick (or standard coherent state) quantization of the complex plane produces both canonical commutation rule and quantum spectrum of the harmonic oscillator (up to the addition of a constant). In this work, we show that these two issues are not necessarily coupled: there exists a family of separable Hilbert spaces, including the usual Fock–Bargmann space, and in each element in this family there exists an overcomplete set of unit-norm states resolving the unity. With the exception of the Fock–Bargmann case, they all produce non-canonical commutation relation whereas the quantum spectrum of the harmonic oscillator remains the same up to the addition of a constant. The statistical aspects of these non-equivalent coherent state quantizations are investigated. We also explore the localization aspects in the real line yielded by similar quantizations based on real Hermite polynomials.

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1. Introduction

It is well known that the anti-Wick (or Klauder–Berezin–Toeplitz) quantization (see for instance [1] and references therein) of the complex plane equipped with the Lebesgue measure yields both canonical commutation rule $[\hat{q}, \hat{p}] = i\hbar I$, and quantum spectrum of the harmonic oscillator (namely $\hbar \omega (\hat{N} + 1/2)$ up to the addition of $1/2$). The aim of this paper is to prove that these two issues are not necessarily coupled: there exists a discrete family of separable Hilbert subspaces $\mathcal{K}_s$, $s \in \mathbb{N}$, in $L^2(\mathbb{C}, d^2z/\pi)$, including the ‘canonical’ subspace Fock–Bargmann, and in each element in this family there exists an overcomplete set of states resolving the unity and producing, with the exception of the Fock–Bargmann case, non-canonical commutation relation and the same quantum spectrum of the harmonic oscillator up to the addition of the
constant \(2s + 1/2\). Each \(\mathcal{K}_s\) is the closure of the linear span of complex Hermite polynomials \([2, 3]\) weighted by a Gaussian, \(e^{-|z|^2/2} h^{s+1/2}(z, \bar{z})\).

The organization of the paper is as follows. In section 2 we recall some well-known facts about the anti-Wick or standard coherent state quantization and make comparisons with the canonical quantization. Then, in section 3, we present a general construction of coherent states (CS) and we describe the corresponding CS quantization. In the following sections, the procedure is worked out with CS based on complex and real Hermite polynomials. The complex Hermite polynomials are defined in section 4 and the corresponding quantization of the complex plane is implemented in section 5. Its remarkable feature is the appearance of a new commutation rule for the lowering and raising operators, and so for the position and momentum operator, where an extra term proportional to the projector on the ground state is involved. Notwithstanding, we obtain for the energy spectrum of the CS quantized harmonic oscillator the same as for the usual one up to the addition of a constant defined by the class of considered complex Hermite polynomials. We examine in section 6 a possible connection of our results with supersymmetric quantum mechanics (SUSYQM). Some statistical aspects of the complex Hermite polynomial CS and the corresponding quantization are examined in section 7. In the same vein, we explore in section 8 the quantization of the real line with CS in finite-dimensional Hilbert spaces constructed with real Hermite polynomials and we study the resulting localization properties. It turns out that for a given dimension the position operator is the same as the position operator derived from the corresponding finite-dimensional approximation of the usual quantum mechanics. We give in section 9 some indications for future developments issued from our work.

2. Anti-Wick or coherent state versus canonical quantization

The anti-Wick quantization, to which we prefer the name of coherent state quantization, consists in starting from the plane \(\mathbb{R}^2 \simeq \mathbb{C} = \{z = \frac{1}{\sqrt{2}}(q + ip)\}\), where we put \(\hbar = 1\) for convenience, equipped with its Lebesgue measure \(\mu(dz \, d\bar{z}) = \frac{1}{\pi} \, d^2 \bar{z}\) with \(d^2 \bar{z} = d\Re z \, d\Im z\), and viewed as the phase space for the motion of a particle on the line. In the Hilbert space \(L^2(\mathbb{C}, \mu(dz \, d\bar{z}))\) of all complex-valued functions on the complex plane which are square-integrable with respect to this measure, we choose the orthonormal set formed of the normalized powers of the conjugate of the complex variable \(z\) weighted by the Gaussian, i.e. \(\phi_n(z) \equiv e^{-|z|^2/2} \frac{z^n}{\sqrt{n!}}\) with \(n \in \mathbb{N}\). This set is an orthonormal basis for the so-called Fock–Bargmann Hilbert subspace, here denoted by \(\mathcal{K}_0\), in \(L^2(\mathbb{C}, \mu(dz \, d\bar{z}))\). Let \(\mathcal{H}\) be a separable Hilbert space (e.g. a Fock space) with the orthonormal basis \(\{|e_n\rangle, n \in \mathbb{N}\}\) (e.g. the ‘number states’ \(|n\rangle\)). We then consider the following infinite linear superposition in \(\mathcal{H}\):

\[
|z\rangle = \sum_n \phi_n(z) |e_n\rangle = e^{-|z|^2/2} \sum_n \frac{z^n}{\sqrt{n!}} |e_n\rangle.
\]

They are the well-known Schrödinger–Klauder–Glauber–Sudarshan, or simply standard, coherent states. From the numerous properties of these states \([4, 5]\) we retain here two features, namely normalization and resolution of the unity in \(\mathcal{H}\):

\[
\langle z | z \rangle = 1, \quad \frac{1}{\pi} \int_{\mathbb{C}} |z\rangle \langle z| d^2 \bar{z} = 1_{\mathcal{H}}.
\]

CS quantization means that a classical observable \(f\), that is a (usually supposed to be smooth) function of phase-space variables \((q, p)\) or equivalently of \((z, \bar{z})\), is transformed through the operator integral

\[
\frac{1}{\pi} \int_{\mathbb{C}} f(z, \bar{z}) |z\rangle \langle z| d^2 \bar{z} = A_f,
\]
into an operator $A_I$ acting on the Hilbert space $\mathcal{H}$. We get for the most basic one,

$$\frac{1}{\pi} \int_{\mathbb{C}} |z| |\phi_I(z)|^2 \, d^2z = \sum_n \sqrt{n+1} |\phi_I(e_n)|^2 = \alpha,$$

which is the lowering operator, $a(\alpha) = \sqrt{\alpha} |\alpha\rangle$. We easily check that the CS are eigenvectors of $a$: $a|\alpha\rangle = \sqrt{\alpha} |\alpha\rangle$. The adjoint $a^\dagger$ is obtained by replacing $z$ by $\bar{z}$ in (1), and we get the factorization $\hat{N} = a^\dagger a$ for the number operator, $\hat{N}|\alpha\rangle = n|\alpha\rangle$, together with the commutation rule $[a, a^\dagger] = \mathbb{I}_{\mathcal{H}}$. The lower symbol or expected value of the number operator $\langle z|\hat{N}|z\rangle$ is precisely $|z|^2$. From $q = \frac{1}{\sqrt{2}}(z + \bar{z})$ and $p = \frac{i}{\sqrt{2}}(z - \bar{z})$, one easily infers by linearity that the canonical position $q$ and the momentum $p$ map to the quantum observables $\frac{1}{\sqrt{2}}(a + a^\dagger) \equiv Q$ and $\frac{1}{2\sqrt{2}}(a - a^\dagger) \equiv P$, respectively. In consequence, the self-adjoint operators $Q$ and $P$ obey the canonical commutation rule $[Q, P] = i\mathbb{I}_{\mathcal{H}}$, and for this reason fully deserve the name of position and momentum operators of the usual (Galilean) quantum mechanics, together with all localization properties specific to the latter. Let us now CS quantize the classical harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + q^2) = |z|^2$:

$$A_H = A_{|z|^2} = \hat{N} + \mathbb{I}_{\mathcal{H}}.$$  

We see with this elementary example that the CS quantization does not fit exactly with the ‘canonical’ one, which consists in just replacing $q$ by $Q$ and $p$ by $P$ in the expressions of the observables $f(q, p)$ and next proceeding with a symmetrization in order to comply with self-adjointness. In fact, the quantum Hamiltonian obtained through this usual ansatz is equal to $\hat{H} = \frac{1}{2}(p^2 + q^2) = \hat{N} + (1/2)\mathbb{I}_{\mathcal{H}}$. In the present case, there is a shift by 1/2 between the spectrum of $\hat{H}$ and the CS quantized Hamiltonian $A_H$. Actually, no physical experiment can discriminate between those two spectra that differ from each other by a simple shift (for a thorough discussion on this point, see for instance [6]).

3. Coherent state quantization: the general setting

Let $\Sigma$ be a set of parameters (e.g. a phase space for a classical motion on a manifold) equipped with a measure $\mu$ and its associated Hilbert space $L^2(\Sigma, \mu)$ of complex-valued square integrable functions with respect to $\mu$. Let us choose in $L^2(\Sigma, \mu)$ a finite or countable orthonormal set $O = \{\phi_n, n = 0, 1, \ldots\}$:

$$\langle \phi_m | \phi_n \rangle = \int_{\Sigma} \overline{\phi_m(\alpha)} \phi_n(\alpha) \mu(d\alpha) = \delta_{mn}.$$ (2)

In the case of infinite countability, this set must obey the (crucial) finiteness condition

$$\sum_n |\phi_n(\alpha)|^2 \overset{\text{def}}{=} \mathcal{N}(\alpha) < \infty \quad \text{a.e.}$$ (3)

Let $\mathcal{H}$ be a separable complex Hilbert space with an orthonormal basis $\{|e_n\rangle, n = 0, 1, \ldots\}$, in one-to-one correspondence with the elements of $O$. From conditions (2) and (3) there results that the family of normalized ‘coherent’ states $\mathcal{F}_\alpha = \{|\alpha\rangle, \alpha \in \Sigma\}$ in $\mathcal{H}$, which are defined by

$$|\alpha\rangle = \frac{1}{\sqrt{\mathcal{N}(\alpha)}} \sum_n \phi_n(\alpha) |e_n\rangle,$$ (4)

resolves the identity in $\mathcal{H}$:

$$\int_{\Sigma} \mu(d\alpha) \mathcal{N}(\alpha) |\alpha\rangle \langle \alpha| = \mathbb{I}_{\mathcal{H}}.$$ (5)
Such a relation allows us to implement a coherent state or frame quantization of the set of parameters $\Sigma$ by associating with a function $\Sigma \ni \alpha \mapsto f(\alpha)$ that satisfies appropriate conditions the following operator in $H$:

$$f(\alpha) \mapsto A_f \overset{\text{def}}{=} \int_{\Sigma} \mu(\text{d}\alpha) N(\alpha) f(\alpha) |\alpha\rangle \langle \alpha|.$$  

(6)

Operator $A_f$ is symmetric if $f(\alpha)$ is real valued, and is bounded if $f(\alpha)$ is bounded. The original $f(\alpha)$ is an ‘upper symbol’ in the sense of Lieb [7] or contravariant in the sense of Berezin [8], usually non-unique, for the operator $A_f$. It will be called a classical observable with respect to the family $\mathcal{F}_\eta$ if the so-called lower symbol in the sense of Lieb [7] or covariant in the sense of Berezin [8], $\hat{A}_f(\alpha) \overset{\text{def}}{=} \langle \alpha | A_f | \alpha \rangle$ of $A_f$, has mild functional properties to be made precise (e.g. smooth function) according to further topological properties granted to the original set $\Sigma$ (e.g. symplectic manifold).

4. Complex Hermite polynomials

Let $r$ and $s$ be the nonnegative integers. Complex Hermite polynomials are defined as [2, 3]

$$h^{r,s}(z, \bar{z}) \equiv (-1)^r s! e^{|z|^2} \frac{\partial^r}{\partial z^r} \frac{\partial^s}{\partial \bar{z}^s} e^{-|z|^2} = \sum_{k=0}^{\min(r,s)} \frac{(-1)^k k!}{(r-k)! (s-k)!} r^s |z|^{2k}.$$  

(7)

They form a complete orthogonal system in the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} d^2z)$ with $\nu > 0$. Suppose now that $r \geq s$. Then the corresponding polynomials can be written in terms of confluent hypergeometric functions or in terms of associate Laguerre polynomials:

$$h^{s+n,s}(z, \bar{z}) = s!(s+n)! z^n \sum_{k=0}^{s} \frac{(-1)^{s-k} (s-k)!}{n! (s-k)! k! (s+k)!} |z|^{2k}.$$  

(8)

where $r = s = n \in \mathbb{N}$. In particular, for $s = 0$ and 1, expression (7) reduces, respectively, to $\bar{z}^n$ and $\bar{z}^n (|z|^2 - n - 1)$. For a fixed $s$ we have an infinite family of complex polynomials of degree $n + 2s$ in the variables $z$ and $\bar{z}$, and which are pairwise orthogonal. Precisely, by using relation (2.20.1.19) in [9], we obtain

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2z e^{-|z|^2} h^{s+n,s} h^{s+n,s} = \begin{cases} s! (s+n)! & \text{if } n = n' \\ 0 & \text{if } n \neq n'. \end{cases}$$  

(9)

The functions $h^{r,s+n}$ are related through the ladder operators

$$\begin{align*}
\left( \frac{\partial}{\partial z} + \bar{z} \right) h^{s+n,s} &= h^{s+n+1,s} \\
\left( -\frac{\partial}{\partial \bar{z}} + z \right) h^{s+n,s} &= h^{s+n+1,s}.
\end{align*}$$

Let us fix $s$ and introduce the Hilbert subspace $\mathcal{K}_s$ in $L^2(\mathbb{C}, d^2z/\pi)$ as the closure of the linear span of the set of orthonormal functions defined as

$$\phi_{n,s}(z) \overset{\text{def}}{=} \frac{1}{\sqrt{s!(s+n)!}} e^{-|z|^2/2} h^{s+n,s}.$$  

(10)
The functions $\phi_{n,s}$ are related through the ladder operators
\begin{align}
\left(\frac{\partial}{\partial \xi} + \frac{\xi}{2}\right) \phi_{n+1,s} &= \sqrt{s + n + 1} \phi_{n,s}, \\
\left(-\frac{\partial}{\partial z} + \frac{z}{2}\right) \phi_{n,s} &= \sqrt{s + n + 1} \phi_{n+1,s}.
\end{align}

We thus obtain a countably infinite family of orthogonal Hilbert subspaces $\mathcal{H}_s$. The 'canonical' Fock–Bargmann subspace corresponds to $s = 0$. Note that, at the exception of this case, the lowest state $\phi_{0,s}$ is not canceled by $(\frac{\partial}{\partial \xi} + \frac{\xi}{2})$.

5. Complex Hermite polynomial quantization

Following the guideline indicated in section 3, for a fixed $s$, we construct the CS based on complex Hermite polynomials as the infinite linear combination of the orthonormal elements $|n; s\rangle$ of some separable Hilbert space $\mathcal{H}_s$,
\begin{equation}
|z; s\rangle = \frac{1}{\sqrt{e^{-|z|^2} \mathcal{N}_s(|z|^2)}} \sum_{n=0}^\infty \phi_{n,s}(z) |n; s\rangle = \frac{1}{\sqrt{\mathcal{N}_s(|z|^2)}} \sum_{n=0}^\infty \frac{h^{n,s}(z, \bar{z})}{\sqrt{s! (s+n)!}} |n; s\rangle,
\end{equation}
where the normalization factor is defined as
\begin{equation}
\mathcal{N}_s(|z|^2) = \sum_{n=0}^\infty \frac{|h^{n,s}(z, \bar{z})|^2}{s! (s+n)!}.
\end{equation}

Note the change of notation in regard to equation (3) in order to delete the Gaussian factor. Also, we could choose all spaces $\mathcal{H}_s$ as identical, e.g. the Fock space spanned by the number states $|n\rangle$, or the Hilbert space $L^2(\mathbb{R}, dx)$, in which case there is no need to specify the parameter $s$. On the other hand, we could choose $\mathcal{H}_s = \mathcal{K}_s$ and identify the states $|n; s\rangle$ with the functions $\phi_{n,s}$.

Series (13) can be easily summed for lower values of $s$, e.g. for $s = 0$ and 1: they are respectively equal to $e^{|z|^2}$ and $e^{|z|^2} - |z|^2$. If we use the definition of the complex Hermite polynomials (7) in equation (12), we obtain the alternative form
\begin{equation}
|z; s\rangle = \frac{(-1)^s}{\sqrt{\mathcal{N}_s(|z|^2)}} \sum_{n=0}^\infty \left(\frac{s + n}{s}\right)^{1/2} \frac{e^n}{\sqrt{n!}} L_s^{(n)}(|z|^2) |n; s\rangle,
\end{equation}
and for the normalization function,
\begin{equation}
\mathcal{N}_s(|z|^2) = \sum_{n=0}^\infty \frac{s^n}{(s+n)!} |z|^{2n} \left(L_s^{(n)}(|z|^2)\right)^2.
\end{equation}

With the help of this form it can be easily checked that for $s = 0$ they are the standard CS, but for the remaining values of $s$ we are in the presence of some deformation of the standard $|z; 0\rangle \equiv |z\rangle$. Therefore, we have with equation (14) an infinite family of CS, which is labeled by $s \in \mathbb{N}$.

We next proceed with the corresponding coherent state quantization, starting as usual with the simplest functions $f(z, \bar{z}) = z$ and $\bar{z}$. With the help of equations (6) and (2.20), (2.19.23.6) in [9], we get
\begin{align}
A_{z,s} &= \sum_{n=0}^\infty \sqrt{s + n + 1} |n; s\rangle \langle n + 1; s|, \\
A_{\bar{z},s} &= \sum_{n=0}^\infty \sqrt{s + n + 1} |n + 1; s\rangle \langle n; s|.
\end{align}
In the realization $\mathcal{H}_s = K_s$, we see from (10)–(11) that $A_{z,s}$ is identified with $(-\frac{\partial}{\partial z} + \frac{i}{2})$ whereas $A_{\bar{z},s}$ is not equal to $(\frac{\partial}{\partial \bar{z}} + \frac{i}{2})$, since $A_{z,s}|0; s\rangle = 0$. The latter are identical on the subspace generated by the states $|n; s\rangle$ with $n > 0$. Note also that the state $|\tilde{z}; s\rangle$ is not an eigenstate of the operator $A_{z,s}$. This means that it differs from the coherent state $|\tilde{z}\rangle$ defined as

$$\tilde{z} = \frac{1}{\sqrt{\mathcal{N}(\tilde{z})^2}} \sum_{n=0}^{\infty} \frac{\tilde{z}^n}{\sqrt{(s + 1)n}} |n\rangle, \quad \mathcal{N}(\tilde{z})^2 = \sum_{n=0}^{\infty} \frac{|\tilde{z}|^{2n}}{(n+s)!}. \tag{17}$$

which fulfills $A_{z,s}|\tilde{z}\rangle = z|\tilde{z}\rangle$. (For more details about the states (17), e.g. measure for resolution of the unity, see [10].)

The lowering $A_{z,s}$ and raising $A_{\bar{z},s}$ fulfill a new commutation relation

$$[A_{z,s}, A_{\bar{z},s}] = \sum_{n=0}^{\infty} (n + s + 1) \langle n; s| \langle n + 1; s| - \langle n + 1; s| \langle n; s| \rangle \langle n; s| + \langle n + 1; s| \langle n; s| \rangle = \mathbb{I}_{\mathcal{H}_s} + s|0; s\rangle \langle 0; s| \rangle. \tag{18}$$

Equation (18) for $s = 0$ leads to the usual commutation rule $[A_{z,0}, A_{\bar{z},0}] = \mathbb{I}_{\mathcal{H}_s}$. For other values of $s$, there is an extra term proportional to the orthogonal projector on the ground state $|0; s\rangle$. The appearance of this projector makes the Lie algebra generated by the triplet $\{A_{z,s}, A_{\bar{z},s}, \mathbb{I}_{\mathcal{H}_s} + s|0; s\rangle \langle 0; s|\}$ infinite dimensional. Such a structure deserves further attention on a more mathematically oriented setting, e.g. in the spirit of [11] and references therein.

The position $\hat{q}_s$ and momentum $\hat{p}_s$ operators are easily obtained by using the quantized version of the relations $q = (z + \bar{z})/\sqrt{2}, p = -i(z - \bar{z})/\sqrt{2}$, where the coordinates $q, p$ and $z, \bar{z}$ are replaced by the operators $\hat{q}_s, \hat{p}_s$ and $A_{z,s}, A_{\bar{z},s}$, respectively. Now, with the help of equations (15) we have

$$\hat{q}_s = \sum_{n=0}^{\infty} \sqrt{\frac{s + n + 1}{2}} \langle n; s| \langle n + 1; s| + \langle n + 1; s| \langle n; s|), \tag{19}$$

$$\hat{p}_s = -i \sum_{n=0}^{\infty} \sqrt{\frac{s + n + 1}{2}} \langle n; s| \langle n + 1; s| - \langle n + 1; s| \langle n; s|). \tag{20}$$

In the explicit matrix form we have for $\hat{q}_s$ and $\hat{p}_s$

$$\hat{q}_s = \begin{pmatrix} 0 & \sqrt{\frac{s+1}{2}} & 0 & \cdots \\ \sqrt{\frac{s+1}{2}} & 0 & \sqrt{\frac{s+2}{2}} & \cdots \\ 0 & \sqrt{\frac{s+2}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \hat{p}_s = i \begin{pmatrix} 0 & \sqrt{\frac{s+1}{2}} & 0 & \cdots \\ -\sqrt{\frac{s+1}{2}} & 0 & \sqrt{\frac{s+2}{2}} & \cdots \\ 0 & -\sqrt{\frac{s+2}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
\[ H = (q^2 + p^2)/2 = |z|^2. \] Its quantum version \( A_{[z]}; s \) is easily calculated and reads as the diagonal operator
\[
A_{[z]}; s = A_{z,s} + s \sum_{n=0}^{\infty} (n + 2s + 1)|n; s\rangle\langle n; s|.
\] (21)

This entails that the lowest state \( |0; s\rangle \) has energy \((2s + 1)\) and that the energy levels are equidistant by 1, like for the energy levels of the canonical case.

The alternative to this direct CS quantization is to use the standard ansatz which consists in replacing \( \hat{q} \) by \( \hat{q}_s \) and \( \hat{p} \) by \( \hat{p}_s \) in the expression of the classical observable \( H = (q^2 + p^2)/2 \). This leads to the operator \( \hat{H}_s = (\hat{q}_s^2 + \hat{p}_s^2)/2 \), where \( \hat{q}_s \) and \( \hat{p}_s \) are given by (19) and (20), respectively. Now, we obtain
\[
\hat{H}_s = \frac{s + 1}{2} |0; s\rangle\langle 0; s| + \sum_{n \geq 1} (n + s + 1/2)|n; s\rangle\langle n; s|.
\] (22)

The distance between the first and second level is \( s/2 + 1 \), whereas the distance between the upper levels (e.g., third and second level and so on) is constant and equal to 1. It is obvious that for \( s = 0 \) equations (21) and (22) are the same. The distinctions between them hold for \( s \geq 1 \), for which there is a shift of the ground-state energy.

Note that the difference between the operators \( \hat{H}_s \) and \( A_{[z]}; s \) is given by
\[
\hat{H}_s = A_{[z]}; s - (s + 1/2)I_{2\pi} - \frac{s}{2} |0; s\rangle\langle 0; s|.
\] (23)

This is again a byproduct of the appearance of the ground-state projector in the commutation rule (18).

It is interesting to examine the respective lower symbols of \( \hat{q}_s \) and \( \hat{p}_s \): \( \hat{q}_s = |z; s\rangle\langle q|z; s| \), \( \hat{p}_s = |z; s\rangle\langle p|z; s| \). We get
\[
\hat{q}_s = \frac{q}{N_s(|z|^2)} \sum_{n=0}^{\infty} \left( s + n + 1 \right) |z|^{2n} n! \left( \frac{1}{1 + e^{-|z|^2}} \right) \; F_1(-s-n+1; |z|^2) \; F_1(-s-n+2; |z|^2),
\]
\[
\hat{p}_s = \frac{p}{N_s(|z|^2)} \sum_{n=0}^{\infty} \left( s + n + 1 \right) |z|^{2n} n! \left( \frac{1}{1 + e^{-|z|^2}} \right) \; F_1(-s-n+1; |z|^2) \; F_1(-s-n+2; |z|^2).
\]

In the simplest cases \( s = 0 \) and \( s = 1 \) we obtain respectively
\[
\hat{q}_0 = q, \quad \hat{p}_0 = p, \quad \hat{q}_1 = q \left( 1 + \frac{1}{e^{-|z|^2} - |z|^2} \right), \quad \hat{p}_1 = p \left( 1 + \frac{1}{e^{-|z|^2} - |z|^2} \right).
\]
The first case \( s = 0 \) yields a well-known result, while the second case displays an interesting deformation of the complex plane essentially concentrated around the origin.

6. A possible interpretation in terms of supersymmetric quantum mechanics

It is well known that the harmonic oscillator Hamiltonian \( H = -1/2 \left( \partial^2 / \partial x^2 \right) + 1/2 \right x^2 \) has the eigenvalue spectrum
\[
\frac{1}{2}, \quad 1 + \frac{1}{2}, \quad 2 + \frac{1}{2}, \quad \ldots,
\] (24)
and the normalized eigenfunctions
\[
\psi_n(x) = \frac{1}{\sqrt{n!} \sqrt{2^n \pi}} e^{-x^2/2} H_n(x).
\] (25)
The general solution of the equation $H u = \varepsilon u$ considered up to a constant factor is

$$u_\varepsilon(x) = e^{-\frac{x^2}{2}} \left[ 1 F_1 \left( \frac{1}{4} - \frac{\varepsilon^2}{2} \right) + 2 \mu x \frac{\Gamma \left( \frac{3}{2} - \frac{\varepsilon}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{\varepsilon}{2} \right)} 1 F_1 \left( \frac{3}{4} - \frac{\varepsilon}{2} ; \frac{3}{2} ; x^2 \right) \right],$$  \hspace{1cm} (26)

where $\mu$ is an arbitrary constant [15]. For $\varepsilon < \frac{1}{2}$ and $|\mu| < 1$ the solution $u_\varepsilon$ is nodeless and $1/u_\varepsilon$ is normalizable. From the relation $H u_\varepsilon = \varepsilon u_\varepsilon$, that is,

$$-\frac{1}{2} u''_\varepsilon + \frac{1}{2} x^2 u_\varepsilon = \varepsilon u_\varepsilon,$$  \hspace{1cm} (27)

it follows the factorization

$$H = A^*_s A_s,$$  \hspace{1cm} (28)

where

$$A_s = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + \frac{u'_s}{u_s} \right), \quad A^*_s = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \frac{u'_s}{u_s} \right).$$

The supersymmetric partner

$$H_s = H - \frac{d^2 \ln u_s}{dx^2} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{d^2 \ln u_s}{dx^2}$$  \hspace{1cm} (29)

defined by the relation $H_s \varepsilon - \varepsilon = A_s A^*_s$ has the eigenvalue spectrum

$$\varepsilon, \quad \frac{1}{2}, \quad 1 + \frac{1}{2}, \quad 2 + \frac{1}{2}, \ldots,$$  \hspace{1cm} (30)

and the corresponding normalized eigenfunctions

$$|0, \varepsilon\rangle = \frac{\sqrt{1}}{\int_{-\infty}^{\infty} u_\varepsilon(x) dx}, \quad |1, \varepsilon\rangle = \frac{A_s \psi_0}{\sqrt{1 - \varepsilon}}, \quad |2, \varepsilon\rangle = \frac{A_s \psi_1}{\sqrt{1 + 1/2 - \varepsilon}}, \ldots$$  \hspace{1cm} (31)

If we write relation (22) as

$$\hat{H}_s - s - 1 = -\frac{s - 1}{2} \left| 0; s \right\rangle \left\langle 0; s \right| + \sum_{n \geq 1} (n - 1/2) |n; s\rangle \left\langle n; s \right|,$$

and choose

$$\varepsilon = -\frac{s - 1}{2}, \quad |0; s\rangle = \left| 0, -\frac{s - 1}{2} \right\rangle, \quad |1; s\rangle = \left| 1, -\frac{s - 1}{2} \right\rangle, \ldots,$$

then we get

$$\hat{H}_s - s - 1 = H_{s-1}.$$

On the other hand for equation (21) up to an isometry we have

$$A_{s, \varepsilon} = 2s - \frac{1}{2} = H,$$

where $H$ is the harmonic oscillator Hamiltonian. So up to an isometry and a translation, the operator $\hat{H}_s$ is a supersymmetric partner of $A_{s, \varepsilon}^S$.

7. Some statistical properties

Let us now examine some basic statistical aspects of the CS (14). Like the standard CS are connected with the Poisson distribution, the complex Hermite CS are connected with the following generalization of the latter:

$$n \mapsto P_s(n; \lambda) = \frac{1}{N_s(\lambda)} \left( \frac{s + n}{s} \right)^n \frac{\lambda^n}{n!} \Gamma \left( \frac{1}{2} + \frac{n}{s} \right) \Gamma \left( \frac{1}{2} - \frac{n}{s} \right) = \frac{s!}{N_s(\lambda)} \frac{\lambda^n}{(s + n)!} \left( F_s^{(n)}(\lambda) \right)^2.$$  \hspace{1cm} (32)
The parameter $\lambda \in \mathbb{R}$ is equal to $|z|^2$. For $s = 0$ the distribution (32) reduces to the Poisson distribution with the parameter $\lambda$. For $s \neq 0$, a quantitative estimate of the deviation from Poisson statistics is provided by the so-called Mandel parameter $Q_M \equiv (\Delta n)^2 / \langle n \rangle - 1$, where $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$ is a variance calculated for a given distribution. It is well known that in the Poissonian case we have $Q_M = 0$ whereas for $Q_M < 0$ (resp. $Q_M > 0$) we say that the distribution is sub-Poissonian (resp. super-Poissonian). Without loss of generality let us consider the probability distribution and the Mandel parameter for $s = 1$. In this case from equation (32), we get the following expression:

$$P_1(n; \lambda) = e^{-\lambda} \left( \frac{\lambda^n}{n!} \right) \frac{n + 1}{1 - e^{-\lambda}} \left( 1 - \frac{\lambda}{n + 1} \right)^2,$$

where we can identify the corrective factor to the Poisson distribution, and the following Mandel parameter:

$$Q_{M,1}(\lambda) = \frac{2 e^\lambda + 2 e^4 + 4 e^4 \lambda - 2 e^{2\lambda} - \lambda}{(e^\lambda - \lambda)(1 + e^\lambda)}.$$  

The behavior of the distribution $P_1(n; \lambda)$ for three values of the parameter $\lambda$, namely 1, 3 and 10, is shown in figure 1. The behavior of the parameter $Q_{M,1}$ is shown in figure 2. There we note the sub-Poissonian character of the distribution for $\lambda < 1.81$. The latter becomes super-Poissonian for $\lambda > 1.81$ while going smoothly to zero as $\lambda$ becomes large.

8. Hermite quantization of the real line

Since we examine in this paper some aspects of complex Hermite polynomials related to quantization, it is interesting to explore as well the same aspects for real Hermite polynomials.
It is well known that the Hermite polynomials $H_0, H_1, H_2, \ldots$ form an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, dx)$:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x) \, dx = \begin{cases} n!2^n & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$ (35)

Now, $L^2(\mathbb{R}, dx)$ is not a reproducing kernel Hilbert space, a required property for building CS resolving the unity [16]. This reflects in the fact that $\sum_{n=0}^{\infty} (H_n(x))^2 = \infty$. The most we can do here is to deal with finite subsets of such polynomials. Since they satisfy the Christoffel–Darboux formula

$$\sum_{n=0}^{N} \frac{1}{n!2^n} H_n(x)H_n(y) = \frac{H_{N+1}(x)H_N(y) - H_N(x)H_{N+1}(y)}{N!2^{N+1}(x-y)},$$ (36)

and its direct consequence (see [9] and [17])

$$\sum_{n=0}^{N} \frac{1}{n!2^n} H_n^2(x) = \left[ H_{N+1}^2(x) - H_N(x)H_{N+2}(x) \right] / (N!2^{N+1}),$$ (37)

let us take the most of this formula in exploring ‘real Hermite’ quantization of the real line. Let $\mathcal{E}_N$ be a real $(N + 1)$-dimensional Hilbert space and $\{e_0, e_1, \ldots, e_N\}$ an orthonormal basis in $\mathcal{E}_N$. The system of unit vectors

$$|x\rangle = \frac{1}{\sqrt{N_N(x)}} \sum_{n=0}^{N} \frac{1}{\sqrt{n!2^n}} H_n(x)|e_n\rangle, \quad x \in \mathbb{R},$$ (38)

with

$$N_N(x) = \sum_{n=0}^{N} \frac{1}{n!2^n} H_n^2(x) = \frac{H_{N+1}^2(x) - H_N(x)H_{N+2}(x)}{N!2^{N+1}},$$ (39)
satisfies the resolution of the identity
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} N_N(x) |x\rangle \langle x| dx = \mathbb{I}_{\mathcal{E}_N},
\]
and the overlapping relation
\[
\langle x | y \rangle = \frac{1}{\sqrt{N_N(x)N_N(y)}} \sum_{n=0}^{N} \frac{1}{n! 2^n} H_n(x) H_n(y) = \frac{H_{N+1}(x) H_N(y) - H_N(x) H_{N+1}(y)}{N! 2^{N+1}(x-y) [N_N(x) N_N(y)]^{1/2}}.
\]
The system \(|x\rangle_{x \in \mathbb{R}}\) is a continuous frame in \(\mathcal{E}_N\). It allows us to associate with each function \(f : \mathbb{R} \rightarrow \mathbb{R}\) satisfying certain conditions a linear operator, namely
\[
A_f : \mathcal{E}_N \rightarrow \mathcal{E}_N, \quad A_f = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |x\rangle e^{-x^2} \mathcal{N}_N(x) f(x) \langle x| dx. \tag{40}
\]
The lower symbol \(\tilde{f} : \mathbb{R} \rightarrow \mathbb{R},\)
\[
\tilde{f}(t) = \langle t | A_f | t \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \mathcal{N}_N(x) f(x) |\langle x| t \rangle|^2 dx,
\]
is given by
\[
\tilde{f}(t) = \frac{1}{(N!)^2 4^N \sqrt{\pi} N_N(t)} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{(t-x)^2} f(x) [H_{N+1}(t) H_N(x) - H_N(t) H_{N+1}(x)]^2 dx. \tag{41}
\]
The integrals (41) can be easily calculated for \(N = 0\) and \(f(x) = x^r (r \in \mathbb{N})\). The lower symbol vanishes for all odd \(r = 2k_1 + 1 (k_1 \in \mathbb{N})\), whereas is equal to 1 or \((2k_2 - 1)! / 2^{k_2}\), respectively, for \(r = 0\) or \(r = 2k_2 (k_2 = 1, 2, \ldots)\). For odd \(r\) and for \(N \neq 0\) the first non-zero value of \(\tilde{f}\) in equation (41) is for \(r = 1\), and for \(N = 1\) it is \(2t/(1 + 2t^2)\). The behavior of lower symbols for \(r = 1\) and for \(N = 1, 2, 3\) and 10 is shown in figure 3. The dashed lines are the classical quantities, \(f(x) = x\), while the full line denotes \(\tilde{f}(t)\). We can see that the graph of \(\tilde{f}(t)\) wraps its classical counterparts only in a median sector, which enlarges as \(N\) increases.

Note that we can observe the wrapping of the lower symbol along its classical counterparts also for the higher powers of \(x\). For instance, in the case of \(x^2\), the only difference appears in the lowest quantum level which starts from 1/2, see figures 4.

Now, let us calculate the operator \(A_f\) for classical quantities defined as an infinite sum of Hermite polynomials, \(f(x) = \sum_{n=0}^{\infty} a_n H_n(x)\), where \(a_n = \frac{1}{\sqrt{\pi}} \langle H_n | f \rangle\). For such a choice of \(f\), with the help of equation (40) and formula (2.20.17.2) in [9] the operator \(A_f\) reads
\[
A_f = \sum_{k,l=0}^{N} A^{k,l}_f |k\rangle \langle l|, \tag{42}
\]
where
\[
A^{k,l}_f = \sum_{r=0}^{\min(k,l)} a_{k+l-2r} 2^{k+l-2r} \frac{(k + l - 2r)! \sqrt{k! l!}}{(k-r)! (l-r)! r!} \tag{43}
\]
For the fixed value of the parameter \(N\) the operator \(A_f\) depends only on a few first coefficients \(a_r\). It means that an infinite set of classical quantities leads to the same operator and we lose a lot of information about classical systems.
Figure 3. The lower symbol (full line) as given in equation (41) for \( f(x) = x^r \) and \( r = 1 \), \( N = 1, 2, 3 \) and 10. The dashed line is the classical quantity \( f(x) = x \).

Figure 4. The lower symbol (full line) as given in equation (41) for \( f(x) = x^2 \) and \( N = 1 \) and 10. The dashed line is the classical quantity \( f(x) = x^2 \).

The real Hermite quantized version \( A_r \) of the classical position can be calculated by using equations (42), (43) and the well-known relations

\[
x' = (r!/2^r) \sum_{k=0}^{[r/2]} H_{r-2k}(x)/[k! (r - 2k)!].
\]  

(44)
The symbol \([\cdot]\) denotes the integer part of the involved number. Equation (44) for \(r = 1\) leads to \(x = (1/2)H_1(x)\). Thereby, in equation (43), only the coefficient \(a_1\) and the terms with \(k, k + 1\) or \(k + 1, k\) are different from zero. The explicit form of the operator \(A_x\) is given as the \((N + 1) \times (N + 1)\) matrix

\[
A_x = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\
\frac{1}{\sqrt{2}} & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \sqrt{\frac{N}{2}} \\
0 & \cdots & \cdots & \sqrt{\frac{N}{2}} & 0
\end{pmatrix}.
\] (45)

We note that (45) is the same as the finite approximation \(Q_N\) of the position operator \(Q\) in usual quantum mechanics (just truncate (19) at order \(N + 1\) and put \(s = 0\)). This \(Q_N\) is obtained from \(q = (z + \bar{z})/\sqrt{2}\) by quantization with finite approximation of standard CS [18].

The spectral properties of the position operator \(A_x\) are the same as for \(Q_N\) in [18]. The characteristic equation \(\Lambda_{N+1}(\lambda) = \det (A_N^\lambda - \lambda I_N)\) satisfies the following relation:

\[
\Lambda_{N+1}(\lambda) = \lambda \Lambda_N + (N/2) \Lambda_{N-1}(\lambda),
\]

which, for \(\Lambda_N = (-2)^{-N} H_N(\lambda)\), leads to the recurrence relation for the Hermite polynomials \(H_N(\lambda)\).

9. Conclusion

We have explored some unexpected features of the coherent state quantization of the complex plane and of the real line and of some functions living on them. The complex plane can be viewed as the phase space for the motion of a particle on the real line, and we have shown that there exist infinitely many ways to analyze it from a quantum perspective. The fundamental question that can now be addressed from our results is the existence or not of an actual ‘canonical’ or ‘privileged’ point of view among that infinite set of possibilities, uniquely discriminated on experimental bases. The answer goes far beyond the scope of this paper. Concerning our ‘quantum version’ of the real line, we have shown that coherent state quantization yields localization properties quite similar to those revealed by ordinary quantum mechanics. One is naturally led to conclude from these rather elementary facts that the (long!) quest for a univocal quantum version of a ‘classical’ object may reveal unexpected surprises, and open the way to a large field of future investigations.

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