REGULARITY OF TRANSITION DENSITIES AND ERGODICITY FOR AFFINE JUMP-DIFFUSION PROCESSES

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ABSTRACT. In this paper we study the transition density and exponential ergodicity in total variation for an affine process on the canonical state space \( \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n \). Under a Hörmander-type condition for diffusion components as well as a boundary non-attainment assumption, we derive the existence and regularity of the transition density for the affine process and then prove the strong Feller property. Moreover, we also show that under these and the additional subcritical conditions the corresponding affine process on the canonical state space is exponentially ergodic in the total variation distance. To prove existence and regularity of the transition density we derive some precise estimates for the real part of the characteristic function of the process. Our ergodicity result is a consequence of a suitable application of a Harris-type theorem based on a local Dobrushin condition combined with the regularity of the transition densities.

1. Introduction and main results

1.1. Affine processes. The general notion of affine processes was first introduced by Duffie, Filipović, and Schachermayer [3] (2003) and it provides a unified treatment of Ornstein-Uhlenbeck type processes on \( \mathbb{R}^n \) and CBI (continuous-state branching processes with immigration) processes on \( \mathbb{R}_{\geq 0}^m \). Such processes have been widely used in mathematical finance. In the following we will recall the framework of affine processes on the canonical state space, mainly following [3]. Denote by \( D := \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n \) the canonical state space, where \( m, n \in \mathbb{N}_0 \) with \( m + n > 0 \). For \( D \), we write \( I = \{1, \ldots, m\} \) and \( J = \{m + 1, \ldots, m + n\} \) for the index sets of the \( \mathbb{R}_{\geq 0}^m \)-valued components and the \( \mathbb{R}^n \)-valued components, respectively. For \( x \in D \), let \( x_I = (x_i)_{i \in I} \) and \( x_J = (x_j)_{j \in J} \). Throughout this paper, we use the notation

\[
A = \begin{pmatrix}
A_{II} & A_{IJ} \\
A_{JI} & A_{JJ}
\end{pmatrix}
\tag{1.1}
\]

for a \( d \times d \)-matrix \( A \), where \( A_{II} = (a_{ij})_{i,j \in I} \), \( A_{IJ} = (a_{ij})_{i \in I, j \in J} \), \( A_{JI} = (a_{ij})_{i \in J, j \in I} \), and \( A_{JJ} = (a_{ij})_{i,j \in J} \). We endow \( D \) with the usual inner product \( \langle \cdot, \cdot \rangle \) and denote by \( \|x\| \) the induced Euclidean norm of a vector \( x \in D \). Finally, let \( \mathbb{S}^+_d \) stand for the cone of symmetric positive semidefinite \( d \times d \)-matrices.

Definition 1.1. We call \((a, \alpha, b, \beta, m, \mu)\) a set of admissible parameters for the state space \( D \) if

(i) \( a \in \mathbb{S}^+_d \) and \( a_{kl} = 0 \) for \( k \in I \) or \( l \in I \);
(ii) \( \alpha = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i = (\alpha_{i,kl})_{1 \leq k,l \leq d} \in \mathbb{S}^+_d \) and \( \alpha_{i,kl} = 0 \) if \( k \in I \setminus \{i\} \) or \( l \in I \setminus \{i\} \);
(iii) \( \nu \) is a Borel measure on \( D \setminus \{0\} \) satisfying

\[\int_{D \setminus \{0\}} \left| \frac{d\nu}{\sqrt{\det(A_{II})}} \right| < \infty, \]

\[\int_{D \setminus \{0\}} \frac{d\nu}{\sqrt{\det(A_{II})}} < \infty, \]

\[\int_{D \setminus \{0\}} \frac{d\nu}{\sqrt{\det(A_{IJ})}} < \infty, \]

\[\int_{D \setminus \{0\}} \frac{d\nu}{\sqrt{\det(A_{JI})}} < \infty, \]

\[\int_{D \setminus \{0\}} \frac{d\nu}{\sqrt{\det(A_{JJ})}} < \infty. \]
\[
\int_{D\setminus\{0\}} \left( 1 \wedge \|\xi\|^2 + \sum_{i \in I} (1 \wedge \xi_i) \right) \nu(d\xi) < \infty;
\]

(iv) \( \mu = (\mu_1, \ldots, \mu_m) \) where every \( \mu_i \) is a Borel measure on \( D\setminus\{0\} \) satisfying
\[
\int_{D\setminus\{0\}} \left( \|\xi\| \wedge \|\xi\|^2 + \sum_{k \in I\setminus\{i\}} \xi_k \right) \mu_i(d\xi) < \infty;
\]

(v) \( b \in D \);
(vi) \( \beta \in \mathbb{R}^{d \times d} \) with \( \beta_{ki} - \int_{D\setminus\{0\}} \xi_k \mu_i(d\xi) \geq 0 \) for all \( i \in I \) and \( k \in I \setminus \{i\} \), and \( \beta_{IJ} = 0 \).

In contrast to the definition of admissible parameters of Duffie et al. [3], here we neglect parameters corresponding to killing and our definition includes an additional first moment condition on the jump measures \( \mu_i \). Let
\[
\mathcal{U} = \mathbb{C}_{\leq 0}^m \times \mathbb{R}^n = \{ u = (u_I, u_J) \in \mathbb{C}^m \times \mathbb{R}^n : \text{Re}(u_I) \leq 0, \text{Re}(u_J) = 0 \}.
\]

Note that the function \( D \ni x \mapsto \exp(\langle u, x \rangle) \) is bounded for any \( u \in \mathcal{U} \). We denote the Banach space of continuous functions that vanish at infinity by \( C_0(D) \). Moreover, \( C^2_\xi(D) \) stands for the function space of twice continuously differentiable functions on \( D \) with compact support and \( C^\infty_\xi(D) \) for the space of smooth functions on \( D \) with compact support, respectively.

**Theorem 1.1** ([3]). Let \((a, \alpha, b, \beta, m, \mu)\) be a set of admissible parameters. Then there exists a unique conservative Feller transition semigroup \((P_t)_{t \geq 0}\) acting on \( C_0(D) \) such that its infinitesimal generator \((A, \text{dom} A)\) satisfies \( C^2_\xi(D) \subset \text{dom} A \) and, for \( f \in C^2_\xi(D) \) and \( x \in D \),
\[
Af(x) = \langle b + \beta x, \nabla f(x) \rangle + \sum_{k,l=1}^d \left( a_{kl} + \sum_{i=1}^m \alpha_{i,kl}x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l}
\]
\[
+ \int_{D\setminus\{0\}} \left( f(x + \xi) - f(x) - \langle \xi, \nabla f(x) \rangle 1_{\{\|\xi\| \leq 1\}} \right) \nu(d\xi)
\]
\[
+ \sum_{i=1}^m x_i \int_{D\setminus\{0\}} (f(x + \xi) - f(x) - \langle \xi, \nabla f(x) \rangle) \mu_i(d\xi),
\]

where \( \nabla J = (\partial / (\partial x_j))_{j \in J} \). Moreover, \( C^\infty_\xi(D) \) is a core for the generator and the Fourier transform of the transition semigroup \((P_t)_{t \geq 0}\) has the representation
\[
\int_D e^{\langle u, \xi \rangle} P_t(x, d\xi) = \exp(\langle \phi(t, u) + \langle x, \psi(t, u) \rangle \rangle), \quad t \geq 0, u \in \mathcal{U},
\]
where \( \phi(t, u) \) and \( \psi(t, u) = (\psi_I(t, u), \psi_J(t, u)) \) solve the generalized Riccati differential equations, for each \( u = (u_I, u_J) \in \mathcal{U} \),
\[
\begin{align*}
\partial_t \phi(t, u) &= F(\psi(t, u)), \quad \phi(0, u) = 0, \\
\partial_t \psi_I(t, u) &= R(\psi(t, u)), \quad \psi_I(0, u) = u_I \\
\psi_J(t, u) &= e^{\beta_{IJ} t} u_J
\end{align*}
\]
and the function \( R \) and \( F \) are given by
\[
F(u) = \langle u, au \rangle + \langle b, u \rangle + \int_{D\setminus\{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle u_J, \xi_J \rangle 1_{\{\|\xi\| \leq 1\}} (\xi) \right) \nu(d\xi),
\]
we denote by \( C \) subspace of \( p \) where we have used the notation (1.1). For a given nonnegative integer \( P \) based on the latter results, we show that regularity of the transition density including the strong Feller property. On the other hand, the purpose of this work is twofold: on the one hand, we investigate sufficient conditions for the admissible parameters (\( a, \alpha, b, \beta, \nu, \mu \)) is indeed conservative. Note also that in Theorem 1.1, the form of \( \mathcal{A} f \) looks slightly different compared with the one given in [3, Theorem 2.7, formula (2.12)]. In fact, we used different compensation in the integrand of the integral with respect to \( \nu \) and \( \mu_i \). However, we may modify the drift parameters \( b \) and \( \beta \) accordingly so that the presentation provided here is actually equivalent to that of [3].

**Definition 1.2.** A conservative Markov process with state space \( D \) and with transition semigroup \( (P_t)_{t \geq 0} \) given in Theorem 1.1 is called an affine process with admissible parameters \( (a, \alpha, b, \beta, \nu, \mu) \).

We refer to [3, 7, 15] for extensive surveys on the developments of affine processes. The purpose of this work is twofold: on the one hand, we investigate sufficient conditions for the regularity of the transition density including the strong Feller property. On the other hand, the latter results, we show that \( P_t(x, \cdot) \) converges in total variation exponentially fast to its unique invariant measure.

### 1.2. Existence and regularity of transition densities.

For given \( n \times n \)-matrices \( A_1, \ldots, A_n \) we write \( [A_1, \ldots, A_n] \) for the \( n \times n^2 \)-block matrix which is obtained by putting the matrices next to each other. Let us then introduce the \( n \times n^2 \)-matrix \( K \) given by

\[
K = [a_{JJ}, \beta_{JJ}a_{JJ}, \ldots, \beta_{JJ}^{n-1}a_{JJ}],
\]

where we have used the notation (1.1). For a given nonnegative integer \( p \), let \( C_0^p(D) \) be the subspace of \( C_0(D) \) of all \( p \)-times continuously differentiable functions whose derivatives up to order \( p \) all belong to \( C_0(D) \). For given multi-indices \( q = (q_1, \ldots, q_d) \), \( \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_d) \in \mathbb{N}_0^d \) we denote by

\[
\partial^{(q, \tilde{q})}_{(x,y)} = \frac{\partial^{\left| q \right| + \left| \tilde{q} \right|}}{\partial x_1^{q_1} \ldots \partial x_d^{q_d} \partial y_1^{\tilde{q}_1} \ldots \partial y_d^{\tilde{q}_d}}
\]

the corresponding mixed partial derivatives of orders \( \left| q \right| = q_1 + \ldots + q_d \) and \( \left| \tilde{q} \right| = \tilde{q}_1 + \ldots + \tilde{q}_d \) acting on functions in the variables \( (x, y) \in D \times D \). The following is our main result on the existence and regularity of the transition densities.

**Theorem 1.2.** Assume that \( K \) given in (1.5) has full rank, that \( \min_{i \in \{1, \ldots, m\}} \alpha_{i,ii} > 0 \), and there exists a nonnegative integer \( p \) with

\[
p < \min_{i \in I} \frac{b_i}{\alpha_{i,ii}} - m.
\]

Then the following assertions hold:

(a) For each \( x \in D \) and \( t > 0 \), \( P_t(x, \cdot) \) admits a density \( f_t(x, \cdot) \) of class \( C_0^p(D) \) given by

\[
f_t(x, y) = \int_{\mathbb{R}^d} e^{-(y,nu)e^\phi(t,nu)+(x,\phi(t,nu))} \frac{du}{(2\pi)^d}, \quad y \in D.
\]

The function \( (0, \infty) \times D \times D \ni (t, x, y) \mapsto f_t(x, y) \) defined by (1.7) is jointly continuous and, for each \( t > 0 \), the mapping \( x \mapsto f_t(x, \cdot) \in L^1(D) \) is continuous.
For each pair of multi-indices \( q, \tilde{q} \in \mathbb{N}_0^d \) satisfying
\[
q_1 = \cdots = q_m = 0 \quad \text{and} \quad \sum_{i=1}^{m} \tilde{q}_i \leq p,
\]
the derivative \( \partial^{(q,\tilde{q})}_{(x,y)} f_t \) exists and is jointly continuous in \((t, x, y) \in (0, \infty) \times D \times D\). Moreover, it holds that for all \( t_1 > t_0 > 0 \)
\[
\sup_{(t,x,y) \in [t_0, t_1] \times D \times D} \left| \partial^{(q,\tilde{q})}_{(x,y)} f_t(x, y) \right| < \infty.
\]

(c) Assume that
\[
\int_{\{0 < \|\xi\| \leq 1\}} \|\xi\| \mu_i(d\xi) < \infty, \quad i = 1, \ldots, m.
\]
Then for each pair of multi-indices \( q, \tilde{q} \in \mathbb{N}_0^d \) satisfying
\[
\sum_{i=1}^{m} (q_i + \tilde{q}_i) \leq p,
\]
the derivative \( \partial^{(q,\tilde{q})}_{(x,y)} f_t \) exists for \( x \in D^o \), \( y \in D \), \( t > 0 \) and is jointly continuous in \((t, x, y) \in (0, \infty) \times D^o \times D\), where \( D^o \) denotes the interior of \( D \). Moreover, it holds that for any \( t_1 > t_0 > 0 \) and any compact set \( K \subset D^o \),
\[
\sup_{(t,x,y) \in [t_0, t_1] \times K \times D} \left| \partial^{(q,\tilde{q})}_{(x,y)} \mu_t(x, y) \right| < \infty.
\]

Note that the continuity of \( x \mapsto f_t(x, \cdot) \in L^1(D) \) stated in part (a) implies that the corresponding affine process has the strong Feller property. The restrictions on the multi-indices \( (q, \tilde{q}) \) imposed in part (b) assert that \( (x, y) \mapsto f_t(x, y) \) is, for \( t > 0 \), smooth in \((x_J, y_J)\) and \( p \)-times continuously differentiable in \( y_I \). Part (c) asserts that if we assume finite first moment for the small jumps of the measures \( \mu_i \) and restrict ourselves to the interior of \( D \), then the transition density is also differentiable with respect to the \( x_I \) variables. Note that in all cases differentiability with respect to the variables \( x_I, y_I \) is determined by the index \( p \) given by \((1.6)\), while the density is smooth in the variables \( x_J, y_J \).

The proof of Theorem 1.2 is given in Section 3 and is essentially based on an estimate on the characteristic function of the affine process shown in Section 2. As a consequence of the proof, we find that the derivatives \( \partial^{(q,\tilde{q})}_{(x,y)} f_t \) can be obtained from \((1.7)\) by differentiating under the integral. To derive the desired estimates on the characteristic function of the affine process, we extend the methods developed in [1] and [4] to general affine processes on the canonical state space \( D \). Below we shall comment on the conditions imposed in the above statements.

First, note that \( \min_{i \in \{1, \ldots, m\}} \alpha_{i,i} > 0 \) implies that the underlying diffusion component restricted to \( \mathbb{R}^m_{\geq 0} \) is non-degenerate except on the boundary \( \partial \mathbb{R}^m_{\geq 0} \), which can be seen from the particular structure of the generator \( A \). Condition \((1.6)\) is a multi-dimensional version of the Feller condition for diffusions. Such condition guarantees that the affine process does not charge the boundary of its state space and that its law has a density being continuous on the whole \( D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) including the boundary \( \partial \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \). Finally, note that without condition \((1.6)\) we already find that in dimension \( d = m = 1 \) an affine process whose law
has a density on \((0, \infty)\) with \(\{0\}\) being a singular point, see, e.g., the remark given after [4, Theorem 4.1].

Second, the assumption that \(K\) given by (1.5) has full rank guarantees that the underlying component on \(\mathbb{R}^n\) drives the process to each point of the state space. Such condition is an analogue of the well-known Hörmander condition for diffusion operators and is, for instance, satisfied if \(a_{JJ}\) is invertible. However, depending on the particular form of the drift \(\beta_{JJ}\) one may also find examples where \(a_{JJ} \neq 0\) is not invertible.

While the \(C^p_0(D)\) regularity of the density \(f_t(x,\cdot)\) from statement (a) partially extends a known result as formulated in [4, Theorem 4.1] for affine processes with state-dependent jumps of finite variation, continuity and differentiability with respect to the variable \(x\) as discussed in statements (b) and (c) have not been investigated yet in this generality. For affine processes with state space \(D = \mathbb{R}_{\geq 0}\) (i.e., one-dimensional CBI processes) existence and regularity of transition densities, including the strong Feller property, was studied by analytic techniques in [2]. In [17], the strong Feller property of one-dimensional CBI processes was established, provided Grey’s condition is satisfied, based on a precise analysis of the Laplace transform from which the construction of a successful coupling was derived. Note that Grey’s condition is relatively mild and applies to diffusions as well as processes with non-trivial jump behavior. Based on a precise convergence rate for the one-step Euler approximations combined with a discrete integration by parts formula, existence and Besov regularity of transition densities were studied for multi-type CBI processes on state space \(D = \mathbb{R}^m_{\geq 0}, m \geq 1\), in [8]. Such a result can be used to provide a simple proof of the strong Feller property for multi-dimensional settings, see [2] where the anisotropic stable JCIR process was studied. Certainly, this method is also applicable to general affine processes through a combination of the methods used in [8] with the stochastic equation for affine processes derived in [7]. Since one would only get a density in a weighted Besov space instead of \(C^p_0(D)\), the expected results would be less optimal than our Theorem 1.2. In contrast, this work is restricted to affine processes for which the underlying diffusion component is non-degenerate, while the aforementioned works investigate models for which the diffusion part is allowed to be degenerate or even absent.

1.3. Exponential ergodicity in total variation. We now turn to the investigation of the long-time behavior of affine processes on \(D\). Following [12, Theorem 2.7], for an affine transition kernel \(P_t(x,d\xi)\) on \(D\) with an admissible parameter set \((a,\alpha,b,\beta,m,\mu)\) such that \(\beta\) has only eigenvalues with strictly negative real-parts, and the state-independent jump-measure \(\nu\) satisfies \(\int_{\{\|\xi\|>1\}} \log \|\xi\| \nu(d\xi) < \infty\), there exists a unique invariant distribution\(^1\) \(\pi\) and \(P_t(x,\cdot) \to \pi\) weakly as \(t \to \infty\). After having established the existence of a unique invariant distribution for affine processes, it is natural to study the convergence rate towards this limiting distribution in different distances on the space of probability measures. The particular choice of Wasserstein distances was recently treated in [7] where an exponential rate of convergence was established. Below we provide a similar statement for the total variation distance.

For two measures \(\varrho, \tilde{\varrho}\) on \(P(D)\), the class of probability measures on \(D\), the total variation distance is defined by

\[
\|\varrho - \tilde{\varrho}\|_{TV} := \sup \{ |\varrho(A) - \tilde{\varrho}(A)| : A \subset D \text{ Borel set} \}.
\]

\(^1\)A probability measure \(\pi\) on \(D\) is said to be an invariant distribution if

\[
\int_D P_t(x,d\xi)\pi(d\xi) = \pi(d\xi), \quad t \geq 0, \quad x \in D.
\]
Our second main result is built on Theorem 1.2 and yields the existence and regularity of the density for the unique invariant measure \( \pi \) as well as convergence of the transition kernel towards \( \pi \) in total variation.

**Theorem 1.3.** Let \( P_t(x, \cdot) \) be the transition semigroup of an affine process on \( D \) with an admissible parameter set \((a, \alpha, b, \beta, m, \mu)\). Suppose that \( \beta \) has only eigenvalues with strictly negative real-parts, \( \int_{\{\|\xi\| > 1\}} \log \|\xi\| m(d\xi) < \infty \), that \( K \) defined by (1.5) has full rank, \( \min_{i \in \{1, \ldots, m\}} \alpha_{i,ii} > 0 \), and (1.6) holds for some nonnegative integer \( p \). Then the unique invariant measure \( \pi \) has a density \( f^\pi \) of class \( C^p_0(D) \). Moreover, for each multi-index \( q \in \mathbb{N}_0^d \) satisfying \( \sum_{i \in I} q_i \leq p \), the derivative \( \partial^{q_1}(f^\pi)/\partial_1 \ldots \partial_d \) exists and is uniformly bounded. Finally, there exist constants \( c, C > 0 \) such that

\[
\|P_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq C e^{-ct} (1 + \log (1 + \|x\|)) , \quad t \geq 0, \; x \in D.
\]

The proof of Theorem 1.3 is deferred to Section 4. Existence and regularity of the density \( f^\pi \) is a consequence of the estimates for the characteristic function of the affine processes established in Section 2. The ergodicity statement is based on a suitable application of a Harris-type theorem combined with a coupling argument and follows some ideas taken from [2].

Existing literature on the exponential ergodicity in total variation distance for affine processes on \( \mathbb{R}^m_0 \times \mathbb{R}^n \) are often limited to specific affine models, see, e.g., [1, 5, 6, 10, 11, 13, 14, 17, 18, 22]. Compared to Corollary 1.3 the most relevant result in the literature are those of Zhang and Glynn [23] and of Mayerhofer, Stelzer and Vestweber [19], where analogous results are obtained under similar conditions, but with essentially more restrictive assumptions on the state-dependent and state-independent jump measures. At this point it is worthwhile to mention that, in addition to the ergodicity results the authors also studied the central limit theorem in [23]. Also, in [19] the authors study affine processes on symmetric cones which includes affine processes on \( D = \mathbb{R}^m_0 \) as a particular case. Let us also mention that the ergodicity results in the works [5, 6, 10, 17, 18, 22], where the diffusion component is allowed to be degenerate, are limited to particular classes of affine processes.

1.4. **Structure of the work.** In Section 2 we prove a pointwise estimate for the characteristic function of the affine process on the canonical state space. Based on this estimate we prove Theorem 1.2 as well as the regularity of the density for the invariant measure in Section 3. Finally, ergodicity in the total variation distance is shown in Section 4.

2. **Estimate on the characteristic function**

2.1. **Main statement.** The following is our main result for this section.

**Theorem 2.1.** Let \( P_t(x, \cdot) \) be the transition probability of an affine process on \( D \) with an admissible parameter set \((a, \alpha, b, \beta, m, \mu)\). Let \( \phi, \psi \) be the corresponding solutions to (1.4). Assume that \( K \) given in (1.5) has full rank and \( \min_{i \in \{1, \ldots, m\}} \alpha_{i,ii} > 0 \). Then, for each \( t_0 > 0 \) and each \( \vartheta > 0 \) there exist constants \( C_{t_0, \vartheta}, M_{t_0, \vartheta}, \delta_{t_0, \vartheta} > 0 \) such that

\[
e^{\text{Re}(\phi(t, u))} \leq C_{t_0, \vartheta} (1 + \|u_I\|)^{-\lambda(\vartheta)} e^{-\delta_{t_0, \vartheta} \|u_I\|^2}
\]

holds for all \( u \in \mathbb{R}^d \) with \( \|u\| \geq M_{t_0, \vartheta} \) and all \( t \geq t_0 \), where

\[
\lambda(\vartheta) = \min_{i \in I} b_i / \hat{\alpha}_{i,ii}(\vartheta) \geq 0
\]
and

\[ \hat{\alpha}_{i,i}(\vartheta) = \alpha_{i,i} + \int_{\{||\xi|| \leq \vartheta\}} ||\xi||^2 \mu_i(d\xi), \quad i \in I. \]  

(2.1)

Note that Re(\psi(t,iu)) \leq 0 holds for t \geq 0 and u \in \mathbb{R}^d. In particular, it holds that

\[ \left| \int_D e^{i(u,\xi)} P_t(x,d\xi) \right| = \left| e^{(x,\psi(t,iu)) + \phi(t,iu)} \right| \leq e^{\text{Re}(\psi(t,iu))}. \]

The rest of this subsection is devoted to the proof of Theorem 2.1. It is obtained as a consequence of several technical lemmas proven below.

2.2. Technical lemmas. Set

\[ U := \{ \text{Re} u : u \in \mathcal{U} \} = \left\{ (x_I,0) \in \mathbb{R}^d : x_I \in \mathbb{R}^m_{\leq 0} \right\}. \]

For (x, y) \in U \times \mathbb{R}^d and u \in \mathbb{R}^d\{0\}, we define

\[ \begin{align*}
H_1^i(x,y;u) & = \langle x, \alpha_i x \rangle - \langle y, \alpha_i y \rangle + \frac{1}{||u||} \sum_{k=1}^d \beta_{ki} x_k \\
& \quad + \frac{1}{||u||^2} \int_{D \setminus \{0\}} \left( e^{||u||\langle \xi, x \rangle} \cos \left( ||u||\langle \xi, y \rangle \right) - 1 - ||u||\langle \xi, x \rangle \right) \mu_i(d\xi), \\
H_2^i(x,y;u) & = 2\langle x, \alpha_i y \rangle + \frac{1}{||u||} \sum_{k=1}^d \beta_{ki} y_k \\
& \quad + \frac{1}{||u||^2} \int_{D \setminus \{0\}} \left( e^{||u||\langle \xi, x \rangle} \sin \left( ||u||\langle \xi, y \rangle \right) - ||u||\langle \xi, y \rangle \right) \mu_i(d\xi).
\end{align*} \]

These functions will serve as the vector fields for the Riccati equations when decomposed into their real- and imaginary parts and rescaled appropriately.

Lemma 2.2. The functions H_1^k(x,y;u), k = 1, 2, are continuous in (x, y, u) \in U \times \mathbb{R}^d \times \mathbb{R}^d\{0\} for each i \in I.

Proof. Fix i \in I and let (x, y, u) \in U \times \mathbb{R}^d \times \mathbb{R}^d\{0\} be arbitrary. It suffices to prove continuity in (x, y, u) of the integrals with respect to \mu_i. We have

\[ \left| e^{||u||\langle \xi, x \rangle} \cos \left( ||u||\langle \xi, y \rangle \right) - 1 - ||u||\langle \xi, x \rangle \right| \]

\[ \leq \left| e^{||u||\langle \xi, x \rangle} \cos \left( ||u||\langle \xi, y \rangle \right) - e^{||u||\langle \xi, x \rangle} \right| + \left| e^{||u||\langle \xi, x \rangle} - 1 - ||u||\langle \xi, x \rangle \right| \]

\[ \leq 1_{\{||\xi|| \leq 1\}}(\xi)||u||^2||\xi||^2 \left( ||x||^2 + ||y||^2 \right) + 1_{\{||\xi|| > 1\}}(\xi) \left( 2||u||||\xi|| ||x|| + 2 \right), \]

where we used that x \in U and \left| \cos(||u||\langle \xi, y \rangle) - 1 \right| = 2 \sin^2\left( ||u||\langle \xi, y \rangle / 2 \right) \leq 2 \left( ||u||^2 ||\xi||^2 ||y||^2 \right).

In view of (12), we can apply dominated convergence theorem to obtain the continuity of H_1^i. Similarly, we estimate

\[ \left| e^{||u||\langle \xi, x \rangle} \sin \left( ||u||\langle \xi, y \rangle \right) - ||u||\langle \xi, y \rangle \right| \]

\[ \leq \left| \sin \left( ||u||\langle \xi, y \rangle \right) - ||u||\langle \xi, y \rangle \right| + \left| e^{||u||\langle \xi, x \rangle} - 1 \right| ||u||\langle \xi, y \rangle \]

\[ = \int_0^1 ||\xi||\langle \xi, y \rangle \cos \left( r||u||\langle \xi, y \rangle \right) dr - ||u||\langle \xi, y \rangle \left| e^{||u||\langle \xi, x \rangle} - 1 \right| ||u||\langle \xi, y \rangle. \]
Then we can find an open neighbourhood satisfying $\|F\| = \|\xi\| \leq 1(\|\xi\| \leq 1)(\xi) + 1(\|\xi\| > 1)(\xi) (1 + 2\|u\|\|\xi\|\|y\|).$

Once again we can apply dominated convergence theorem to obtain the continuity of $H^2_\gamma.$

**Lemma 2.3.** For any $u \neq 0$, let

$$F(t, u) = \frac{1}{\|u\|} \text{Re} \psi \left( \frac{t}{\|u\|}, iu \right) \quad \text{and} \quad G(t, u) = \frac{1}{\|u\|} \text{Im} \psi \left( \frac{t}{\|u\|}, iu \right). \quad (2.2)$$

Fix $i \in I$. Then for each $\varepsilon > 0$ there exist some constants $t_0 > 0$, $g > 0$, and $M > 0$ such that

$$F_i(t, u) \leq -gt, \quad t \in (0, t_0],$$

for all $u \in \mathbb{R}^d$ with $\|u\| \geq M$ and $\langle u, \alpha, u \rangle \geq \varepsilon \|u\|^2$.

**Proof.** It is easy to see that $F$ and $G$ satisfy

$$\begin{cases}
\partial_t F_i(t, u) = H^1_i(F(t, u), G(t, u); u), \\
\partial_t G_i(t, u) = H^2_i(F(t, u), G(t, u); u),
\end{cases}$$

$F(0, u) = 0,$ and $G(0, u) = \frac{u}{\|u\|}$.

Moreover, it holds $F_J \equiv 0$ and $G_J(t, iu) = \exp(\beta_J^* t, \|u\|)u_J^1 = 1$.

**Step 1:** Choose a small open neighbourhood $O_1 \subset \mathbb{R}^d$ of 0, and also a large enough $M > 0$ such that for $x \in O_1 := O_1 \cap U$ and $\|u\| \geq M$,

$$\left| \langle x, \alpha, x \rangle + \frac{1}{\|u\|} \sum_{k=1}^d \beta_{ki}x_k + \frac{1}{\|u\|^2} \int_{D \setminus \{0\}} \left( e^{\|u\|\|\xi, x\|} - 1 - \|u\|\langle \xi, x \rangle \right) \mu_i(d\xi) \right| \leq \frac{\varepsilon}{4}.$$

**Step 2:** Define

$$K := \left\{ u \in \mathbb{R}^d : \langle u, \alpha, u \rangle \geq \varepsilon, \|u\| = 1 \right\}.$$

Then we can find an open neighbourhood $O_2$ of $K$ such that

$$\langle y, \alpha, y \rangle \geq \frac{\varepsilon}{2}, \quad y \in O_2.$$

**Step 3:** Based on the last two steps, we now have, for $x \in O_1$, $y \in O_2$, and $u \in \mathbb{R}^d$ satisfying $\|u\| \geq M$,

$$H^1_i(x, y; u) = \langle x, \alpha, x \rangle - \langle y, \alpha, y \rangle + \frac{1}{\|u\|} \sum_{k=1}^d \beta_{ki}x_k$$

$$+ \frac{1}{\|u\|^2} \int_{D \setminus \{0\}} \left( e^{\|u\|\|\xi, x\|} \cos \langle \|u\|\langle \xi, x \rangle \rangle - 1 - \|u\|\langle \xi, x \rangle \right) \mu_i(d\xi)$$

$$\leq -\langle y, \alpha, y \rangle + \langle x, \alpha, x \rangle + \frac{1}{\|u\|} \sum_{k=1}^d \beta_{ki}x_k$$

$$+ \frac{1}{\|u\|^2} \int_{D \setminus \{0\}} \left( e^{\|u\|\|\xi, x\|} - 1 - \|u\|\langle \xi, x \rangle \right) \mu_i(d\xi)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{\varepsilon}{4}.$$

**Step 4:** It is easy to verify that $H^1_i(x, y; u)$ and $H^2_i(x, y; u)$ are both bounded for $(x, y, \|u\|) \in O_1 \times O_2 \times [M, \infty)$, i.e., there exists $\gamma > 0$ such that

$$|H^1_i(x, y; u)| + |H^2_i(x, y; u)| \leq \gamma,$$

for all $(x, y, \|u\|) \in O_1 \times O_2 \times [M, \infty)$. 

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Step 5: Note that \( O_1 \times O_2 \) is an open neighbourhood of \( \{0\} \times K \subset U \times \mathbb{R}^d \) in the relative topology w.r.t. \( U \times \mathbb{R}^d \). Since \( \{0\} \times K \) is compact, the boundary of \( O_1 \times O_2 \) (which is closed) has a positive distance \( l > 0 \) to the closed set \( \{0\} \times K \). So starting from a point in \( \{0\} \times K \), running according to the vector field \((H_1^1, H_2^1)\), we will have to wait at least for a positive time \( t_0 := l/\gamma \) to attain the boundary of \( O_1 \times O_2 \). At this point we implicitly use the fact that \( F_I(t, u) = \|u\|^{-1} \text{Re} \left( \psi(t/\|u\|,iu) \right) \leq 0 \). Recall that for \((x,y,\|u\|) \in O_1 \times O_2 \times [M,\infty)\),

\[
H_i^1(x,y;u) \leq -\frac{\varepsilon}{4}.
\]

Hence, we obtain for \( t \in (0,t_0) \)

\[
F_i(t,u) = \int_0^t \partial_t F_i(s,u)\,ds = \int_0^t H_i^1(F(s,u),G(s,u);u)\,ds \leq -\frac{\varepsilon}{4}t,
\]

for all \( u \in \mathbb{R}^d \) with \( \|u\| \geq M \) and \( \langle u,\alpha_iu \rangle \geq \varepsilon\|u\|^2 \).

The remainder of our proof is motivated by [3, Theorem 4.1], and we will extend it to the more general case where the Lévy measures \( \mu = (\mu_1,\ldots,\mu_m) \) do not have finite first moment for the small jumps.

**Proposition 2.4.** Fix \( i \in I \). Then for each \( \varepsilon > 0 \), \( \vartheta > 0 \), and \( t_0 > 0 \), there exist some constants \( M_{\varepsilon,t_0}, C_{\varepsilon,t_0,\vartheta} > 1 \) such that

\[
\exp \left( \vartheta \int_0^t \text{Re}(\psi_i(s,iu))\,ds \right) \leq C_{\varepsilon,t_0,\vartheta} \left( 1 + \|u\| \right)^{-b_i/\widetilde{\alpha}_{ii}(\vartheta)}, \quad t \geq t_0,
\]

for all \( u \in \mathbb{R}^d \) with \( \|u\| \geq M_{\varepsilon,t_0} \) and \( \langle u,\alpha_iu \rangle \geq \varepsilon\|u\|^2 \).

**Proof.** Define \( f(t,iu) = \text{Re} \psi(t,iu) \) and \( g(t,iu) = \text{Im} \psi(t,iu) \). Let \( i \in I \) be fixed. Noting that \( f_j(t,iu) \equiv 0 \) and \( g_j(t,iu) = e^{\theta_j t}f_j \), an easy computation shows that

\[
\partial_t f_i = \alpha_{ii} f_i^2 - \langle g,\alpha_i g \rangle + \sum_{k=1}^d \beta_{ki} f_k + \int_{D\setminus\{0\}} \left( e^{\xi,f_i} \cos(\xi,g) - 1 - \xi_i f_i \right) \mu_i(\,d\xi),
\]

\[
\partial_t g_i = 2f_i \alpha_{ii} g_i + \sum_{k=1}^d \beta_{ki} g_k + \int_{D\setminus\{0\}} \left( e^{\xi,f_i} \sin(\xi,g) - \langle g,\xi \rangle \right) \mu_i(\,d\xi)
\]

with initial conditions \( f_i(0) = 0 \) and \( g_i(0) = u_i \), where

\[
\widetilde{\beta}_{ki} = \begin{cases} 
\beta_{ki} - \int_{D\setminus\{0\}} \xi_k \mu_i(\,d\xi), & k \neq i \\
\beta_{ii}, & k = i.
\end{cases}
\]

Using that \( f_i \leq 0 \), we have

\[
\int_{D\setminus\{0\}} \left( e^{\xi,f_i} \cos(\xi,g) - 1 - \xi_i f_i \right) \mu_i(\,d\xi)
\]

\[
= \int_{D\setminus\{0\}} \left( e^{\xi,f_i} \cos(\xi,g) - \xi_i f_i \right) \mu_i(\,d\xi) + \int_{D\setminus\{0\}} \left( e^{\xi_i f_i} - 1 - \xi_i f_i \right) \mu_i(\,d\xi)
\]

\[
\leq \int_{D\setminus\{0\}} \left( e^{\xi_i f_i} - 1 - \xi_i f_i \right) \mu_i(\,d\xi)
\]

\[
\leq f_i^2 \int_{\{\|\xi\| \leq \vartheta\}} \xi_i^2 \mu_i(\,d\xi) - 2f_i \int_{\{\|\xi\| > \vartheta\}} \xi_i \mu_i(\,d\xi),
\]
yielding that \\
\[ \partial_t f_i \leq \hat{\alpha}_{i,i}(\vartheta) f_i^2 - \langle g, \alpha_i g \rangle + \hat{\beta}_{i,i}(\vartheta) f_i, \]
where we define \( \hat{\beta}_{i,i}(\vartheta) = \beta_{i,i} - 2 \int_{\{||\xi|| > \vartheta\}} e\xi_i \mu_i(d\xi) \) and we have used that \\
\[ \hat{\beta}_{ki} = \beta_{ki} - \int_{D \setminus \{0\}} e\xi_k \mu_i(d\xi) \geq 0, \quad k \neq i. \]

Moreover, for any \( u \neq 0 \), we define the scaled functions \( F(t, u) \) and \( G(t, u) \) as in Lemma 2.3. Then \( F_i \) satisfies the following differential inequality \\
\[ \partial_t F_i \leq \hat{\alpha}_{i,i}(\vartheta) F_i^2 - \langle G, \alpha_i G \rangle + \frac{1}{\|u\|} \hat{\beta}_{i,i}(\vartheta) F_i \leq \hat{\alpha}_{i,i}(\vartheta) F_i^2 + \frac{1}{\|u\|} \hat{\beta}_{i,i}(\vartheta) F_i \]
(2.4) with initial condition \( F_i(0) = 0 \), where we used that \( \langle G, \alpha_i G \rangle \geq 0 \), since \( \alpha_i \in \mathcal{S}_i^+ \). Applying Lemma 2.3 yields that for each \( \varepsilon > 0 \) and \( t_0 > 0 \), there exist constants \( \delta_{\varepsilon,t_0} = \delta \in (0, t_0) \), \( \varphi_{\varepsilon,t_0} = \varphi > 0 \), and \( M_{\varepsilon,t_0} = M > 0 \) such that \\
\[ F_i(\delta, u) \leq -\varphi \quad \text{for all} \quad u \in \mathbb{R}^d \quad \text{with} \quad \|u\| \geq M \]
and \( \langle u, \alpha_i u \rangle \geq \varepsilon \|u\|^2 \). Using [4, Lemma C.3], we arrive at the following differential inequality \\
\[ F_i(t, u) \leq F_i(t - \delta, u) \]
for all \( t \geq \delta \), where \( F_i \) solves \
\[ F_i(t, u) = \hat{\alpha}_{i,i}(\vartheta) F_i(t, u)^2 + \frac{1}{\|u\|} \hat{\beta}_{i,i}(\vartheta) F_i(t, u), \quad t > 0, \]
\[ F_i(0, u) = -\varphi. \]

Without loss of generality we may and do assume that \( \hat{\beta}_{i,i}(\vartheta) < 0 \); indeed, if \( \hat{\beta}_{i,i}(\vartheta) \geq 0 \), we can replace \( \hat{\beta}_{i,i}(\vartheta) \) in (2.4) by \(-1\) and adjust the equation for \( F_i \) accordingly. So \\
\[ F_i(t, u) \leq F_i(t - \delta, u) = \frac{-e^{\frac{\hat{\beta}_{i,i}(\vartheta)}{\|u\|}(t-\delta)}}{\|u\| \hat{\alpha}_{i,i}(\vartheta) \hat{\beta}_{i,i}(\vartheta)^{-1} \left(e^{\frac{\hat{\beta}_{i,i}(\vartheta)}{\|u\|}(t-\delta)} - 1 \right) + \frac{1}{\varphi}}, \quad t \geq \delta, \]
for all \( u \in \mathbb{R}^d \) with \( \|u\| \geq M \) and \( \langle u, \alpha_i u \rangle \geq \varepsilon \|u\|^2 \), where we used [4, Lemma C.5] to derive an explicit solution for \( F_i(t - \delta, u) \). Noting that \( f_i(t, u) = \|u\| F_i(t\|u\|, iu) \), by rescaling and then integrating we get \\
\[ \int_0^t f_i(s, iu) ds \leq \int_0^t f_i(s, iu) ds = -\frac{1}{\hat{\alpha}_{i,i}(\vartheta)} \log \left(\varphi \|u\| \hat{\alpha}_{i,i}(\vartheta) \hat{\beta}_{i,i}(\vartheta)^{-1} \left(e^{\frac{\hat{\beta}_{i,i}(\vartheta)}{\|u\|}(t-\delta)} - 1 \right) + 1 \right), \]
(2.5) for all \( t \geq \delta \) (thus \( t \geq \delta \geq \delta/M \geq \delta/\|u\|^{-1} \)) and for all \( u \in \mathbb{R}^d \) with \( \|u\| \geq M \) and \( \langle u, \alpha_i u \rangle \geq \varepsilon \|u\|^2 \) (see the derivation of (C.8) in the proof of [4] p.109, Theorem 4.1 for details on (2.4)). Note that \( \delta \in (0, t_0) \) and \( M > 1 \). This yields, for \( t \geq t_0 \) and \( u \in \mathbb{R}^d \) with \( \|u\| \geq M \) and \( \langle u, \alpha_i u \rangle \geq \varepsilon \|u\|^2 \), \\
\[ e^{b_i} \int_0^t f_i(s, iu) ds \leq \left(1 + \varphi \|u\| \hat{\alpha}_{i,i}(\vartheta) \hat{\beta}_{i,i}(\vartheta)^{-1} \left(e^{\frac{\hat{\beta}_{i,i}(\vartheta)}{\|u\|}(t_0-\delta)} - 1 \right) \right)^{-b_i/\hat{\alpha}_{i,i}(\vartheta)} \leq \left(1 + \varphi \|u\| \hat{\alpha}_{i,i}(\vartheta) \hat{\beta}_{i,i}(\vartheta)^{-1} \left(e^{\frac{\hat{\beta}_{i,i}(\vartheta)}{\|u\|}(t_0-\delta)} - 1 \right) \right)^{-b_i/\hat{\alpha}_{i,i}(\vartheta)} \leq C_{\varepsilon, t_0, \vartheta} (1 + \|u\|)^{-b_i/\hat{\alpha}_{i,i}(\vartheta)}, \]
where \( C_{\varepsilon, t_0, \vartheta} > 0 \) is some constant independent of \( t \). This proves the assertion. \( \square \)
The next lemma provides a decomposition of the state space into cones for which (2.3) can be applied.

**Lemma 2.5.** Let $t_0 > 0$ be arbitrary. Then there exists a constant $\varepsilon_{t_0} > 0$ such that $\bigcup_{i=0}^m A_i = \mathbb{R}^d$, where

$$A_0 := \left\{ u \in \mathbb{R}^d \mid \left\langle u_J, \left( \int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds \right) u_J \right\rangle \geq \varepsilon_{t_0} \| u \|^2 \right\} \quad (2.6)$$

and

$$A_i = \{ u \in \mathbb{R}^d \mid \langle u, \alpha_i u \rangle \geq \varepsilon_{t_0} \| u \|^2 \}, \quad i \in \{ 1, \ldots, m \}.$$

**Proof.** Let $t_0 > 0$ be fixed. Since the matrix $K$ defined by (1.5) has full rank, the same arguments as given in the proof of [4, Lemma C.2] for the equivalence of (i) and (ii) also show that the matrix $\int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds$ is non-singular and hence positive definite. So there exists large enough $\kappa_{t_0} > 1$ such that

$$\kappa_{t_0}^{-1} \| u_J \|^2 \leq \left\langle u_J, \left( \int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds \right) u_J \right\rangle \leq \kappa_{t_0} \| u_J \|^2, \quad u_J \in \mathbb{R}^n.$$

Suppose, by contradiction, that $\bigcup_{i=0}^m A_i \neq \mathbb{R}^d$. Then for each $\varepsilon > 0$, there exists $u^\varepsilon = (u_1^\varepsilon, \ldots, u_n^\varepsilon) \in \mathbb{R}^d$ such that $\langle u^\varepsilon, \alpha_i u^\varepsilon \rangle < \varepsilon \| u^\varepsilon \|^2$ holds for all $i = 1, \ldots, m$ and

$$\langle u^\varepsilon, \left( \int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds \right) u^\varepsilon \rangle < \varepsilon \| u^\varepsilon \|^2.$$

This yields

$$\min\{ \kappa_{t_0}^{-1}, \alpha_{1,1}, \ldots, \alpha_{m,m} \} \| u^\varepsilon \|^2 \leq \min_{i \in \{1, \ldots, m\}} \alpha_{i,i} \| u^\varepsilon_i \|^2 + \kappa_{t_0}^{-1} \| u^\varepsilon \|^2$$

$$\leq \sum_{i=1}^m \langle u^\varepsilon_i, \alpha_i u^\varepsilon \rangle + \langle u^\varepsilon, \left( \int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds \right) u^\varepsilon \rangle$$

$$\leq \varepsilon (m + 1) \| u^\varepsilon \|^2. \quad (2.7)$$

Since $\min_{i \in \{1, \ldots, m\}} \alpha_{i,i} > 0$ we may choose $\varepsilon > 0$ small enough so that

$$\varepsilon (m + 1) < \min\{ \kappa_{t_0}^{-1}, \alpha_{1,1}, \ldots, \alpha_{m,m} \},$$

which contradicts (2.7). \qed

**Lemma 2.6.** For each $t_0 > 0$ it holds that

$$\int_0^t \Re(\langle \psi(s, iu), a\psi(s, iu) \rangle) ds \leq -\delta_{t_0} \| u_J \|^2, \quad u \in \mathbb{R}^d, \quad t \geq t_0 \quad (2.8)$$

where

$$\delta_{t_0} := \inf_{\| u_J \| = 1} \int_0^{t_0} \langle u_J, e^{\beta_J J} a_J e^{\beta_J J} u_J \rangle ds > 0.$$

**Proof.** Using first that $a_{JJ} = 0$, $a_{JJ} = 0$, $a_{JJ} = 0$ and then $\psi_J(s, iu) = e^{\beta_J J} i u_J$ we find

$$\int_0^t \langle \psi(s, iu), a\psi(s, iu) \rangle ds = - \int_0^t \langle u_J, e^{\beta_J J} a_J e^{\beta_J J} u_J \rangle ds \leq -\delta_{t_0} \| u_J \|^2. \quad (2.9)$$

As shown in Lemma 2.5, the matrix $\int_0^{t_0} e^{\beta_J J} a_J e^{\beta_J J} ds$ is positive definite which yields $\delta_{t_0} > 0$. This proves the assertion. \qed
2.3. Proof of Theorem 2.1. We are now prepared to prove Theorem 2.1.

Proof of Theorem 2.1. Fix \( t_0, \vartheta > 0 \) and let \( x \in D, u \in \mathbb{R}^d \) be arbitrary. According to Lemma 2.6, there exists \( \varepsilon_{t_0} > 0 \) such that \( \bigcup_{i=0}^m A_i = \mathbb{R}^d \), where \( A_i = A_i(t_0) \) is as in Lemma 2.6. Using for \( i = 0 \) equation (2.9) and the definition of \( A_0 \) in (2.8), and for \( i = 1, \ldots, m \) equations (2.3) and (2.8), we find constants \( C_{t_0, \vartheta}, M_{t_0, \vartheta} \geq 1 \) such that for all \( \|u\| \geq M_{t_0, \vartheta} \) and all \( (t,x) \in (t_0, \infty) \times D \),

\[
\left| \mathbb{E} \left( \Re(\phi(t,iu)) \right) \right| = \sum_{i=0}^m \mathbb{1}_{A_i}(u) e^{\Re(\phi(t,iu))} \leq \mathbb{1}_{A_0}(u) e^{-f_0(t,0,\vartheta)^{\frac{1}{2}} + C_{t_0, \vartheta} \sum_{i=1}^m \mathbb{1}_{A_i}(u) (1 + \|u\|)^{-\lambda(\vartheta)} e^{-\delta_0 \|u\|^2}} \leq \tilde{C}_{t_0, \vartheta} (1 + \|u\|)^{-\lambda(\vartheta)} e^{-\delta \|u\|^2},
\]

where \( \delta := \varepsilon_{t_0} \wedge \delta_{t_0} \), \( \tilde{C}_{t_0, \vartheta} = C_{t_0, \vartheta} \) is a large enough constant and we have used that \( \Re(\psi(t,iu)) \leq 0 \) so that

\[
\Re(\phi(t,iu)) \leq \int_0^t \Re(\langle \psi(s,iu), a\psi(s,iu) \rangle) ds + \int_0^t \Re(\langle b, \psi(s,iu) \rangle) ds.
\]

This proves the assertion. \( \square \)

3. Existence and regularity of densities

3.1. Proof of Theorem 1.2. (a) Using (1.3), then Theorem 2.1 and finally (1.6), we conclude that

\[
\sup_{(t,x) \in [t_0, \infty) \times D} \int_{\mathbb{R}^d} \left| \mathbb{E} \left( u(t,x) \right) \right|^p \left| \mathbb{E} \left( u(t,x) \right) \right|^q \int_D e^{\langle u, \xi \rangle} P(x, d\xi) \right| du < \infty.
\]

holds for each \( t_0 > 0 \) and \( q \geq 0 \). By classical properties of the Fourier transform, see [20 Proposition 28.1], we find that \( P_t(x,\cdot) \) has a density \( f_t(x,\cdot) \) given by (1.7) and satisfies \( f_t(x,\cdot) \in C_0^\infty(D) \) for each \( t > 0 \) and \( x \in D \). Since the integrand in (1.7) is jointly continuous in \( (t,x,y) \in (0, \infty) \times D \times D \), by Theorem 2.1 and the dominated convergence theorem we also find that \( D \times D \times (0, \infty) \ni (x,y,t) \mapsto f_t(x,y) \) is jointly continuous. Next, let us prove that \( x \mapsto f_t(x,\cdot) \in L^1(D) \) is continuous for each \( t > 0 \). So, let \( t > 0 \) and \( x \in D \) be fixed. Let \( (x_n)_{n \in \mathbb{N}} \subset D \) so that \( x_n \to x \) as \( n \to \infty \). We have

\[
\limsup_{n \to \infty} \|f_t(x_n,\cdot) - f_t(x,\cdot)\|_{L^1(D)} = \limsup_{n \to \infty} \int_{B_\delta(0)} |f_t(x_n,y) - f_t(x,y)| \, dy + \limsup_{n \to \infty} \int_{D \setminus B_\delta(0)} |f_t(x_n,y) - f_t(x,y)| \, dy,
\]

where \( B_\delta(0) \) denotes the ball with center 0 and radius \( \delta > 0 \). For the first integral on the right hand-side, by the joint continuity of \( f_t(x,y) \), we get

\[
\limsup_{n \to \infty} \int_{B_\delta(0)} |f_t(x_n,y) - f_t(x,y)| \, dy = 0.
\]
Turning to the second integral, we note that $P_t(x_n, \cdot) \rightarrow P_t(x, \cdot)$ weakly as $n \rightarrow \infty$ by (1.3). So we can use Portmanteau’s theorem to estimate
\[
\limsup_{n \rightarrow \infty} \int_{D \setminus B_\delta(0)} |f_t(x_n, y) - f_t(x, y)| \, dy \leq \limsup_{n \rightarrow \infty} \int_{D \setminus B_\delta(0)} f_t(x_n, y) \, dy + \int_{D \setminus B_\delta(0)} f_t(x, y) \, dy \\
\leq 2 \int_{D \setminus B_\delta(0)} f_t(x, y) \, dy.
\]

Note that the latter integral tends to zero as $\delta \rightarrow \infty$. This proves the desired continuity of $x \mapsto f_t(x, \cdot) \in L^1(D)$.

(b) Let $t_1 > t_0 > 0$ be arbitrary and consider two multi-indices $\mathbf{q}, \mathbf{q} \in \mathbb{N}_d^0$ satisfying (1.8). Differentiating formally under the integral in (1.7) with respect to $x_j$ and $y$, we conclude the assertions of Theorem 1.2.(b), provided that we can show that
\[
\sup_{t \in [t_0, t_1]} \int_D \|\psi_J(t, u)\| \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| \sum_{j \in J} \|\hat{q}_j\| e^{Re(\phi(t, u))} \, du < \infty. \tag{3.1}
\]

Now, we verify that (3.1) is true. Using $\psi_J(t, u) = i e^{tJ^*} u_J$ we obtain $\|\psi_J(t, u)\| \leq c_0(t_1) \|u_J\|$ for $t \in [0, t_1]$ with $c_0(t_1) := e^{C_d \|J\|_{HS}}$ where $\cdot \|_{HS}$ denotes the Hilbert-Schmidt norm. Hence we obtain
\[
\int_D \|\psi_J(t, u)\| \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| \sum_{j \in J} \|\hat{q}_j\| e^{Re(\phi(t, u))} \, du \\
\leq c_0(t_1) |\mathbf{q}| \int_D (1 + \|u_J\|)^{|\mathbf{q}|+|\mathbf{q}|} (1 + \|u_J\|)^p e^{Re(\phi(t, u))} \, du. \tag{3.2}
\]

It follows easily from (1.6) that there exists some $\vartheta \in (0, 1)$ small enough such that
\[
m + p < \lambda(\vartheta) = \min_{\mathbf{i} \in I} b_i / \hat{\alpha}_{i, ii}(\vartheta), \tag{3.3}
\]
where $\hat{\alpha}_{i, ii}$ is as in (2.1). By Theorem 2.1, we obtain
\[
e^{Re(\phi(t, u))} \leq C (1 + \|u_J\|)^{-\lambda(\vartheta)} e^{-\vartheta \|u_J\|^2}, \quad \|u\| \geq M, \quad t \geq t_0, \tag{3.4}
\]
where $C, \delta, M > 0$ are constants depending on $t_0$ and $\vartheta$. Thus (3.1) readily follows from (3.2), (3.3) and (3.4).

(c) Consider $t_1 > t_0 > 0$ and let $\mathbf{q}, \mathbf{q} \in \mathbb{N}_d^0$ be two multi-indices satisfying (1.10). Further, let $K \subset D_0$ be a compact set. The proof of this assertion follows a similar approach to assertion (b). Indeed, by formally differentiating under the integral in (1.7), the assertion follows from the dominated convergence theorem, provided that we can show the existence of constants $c, \delta, M > 0$ such that for all $(t, x, y) \in [t_0, t_1] \times K \times \mathbb{R}^d$ and all $u \in \mathbb{R}^d$ with $\|u\| \geq M$,
\[
\|\psi_J(t, u)\| \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| \sum_{j \in J} \|\hat{q}_j\| e^{Re(\phi(t, u))} + Re(x, \psi(t, u)) \\
\leq c (1 + \|u_J\|)^{p-\lambda(\vartheta)} (1 + \|u_J\|)^{|\mathbf{q}|+|\mathbf{q}|} e^{-\vartheta \|u_J\|^2}. \tag{3.5}
\]
Since $K$ is a compact set with $K \cap \partial D = \emptyset$, we find $\varepsilon > 0$ such that $x_i \geq \varepsilon$ holds for each $x \in K$ and $i \in \{1, \ldots, m\}$. Then
\[
\|\psi_J(t, u)\| \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| \psi_J(t, u) \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| e^{Re(\phi(t, u))} + Re(x, \psi(t, u)) \\
\leq \|\psi_J(t, u)\| \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| \psi_J(t, u) \sum_{j \in J} \|u_j\| \sum_{j \in J} \|\hat{q}_j\| e^{Re(\phi(t, u))} + \varepsilon \sum_{i=1}^m Re(\psi_i(t, u)). \tag{3.6}
\]
The troubling term can be estimated as follows:

\[
\left\| \psi_I(t, iu) \right\| \sum_{i \in I} q_i e^{\sum_{j=1}^m \text{Re}(\psi_j(t, iu))} \leq (1 + \left\| \psi_I(t, iu) \right\| \sum_{i \in I} q_i e^{\sum_{j=1}^m \text{Re}(\psi_j(t, iu))} \\
\leq (1 + \left\| \psi_I(t, iu) \right\| \sum_{i \in I} q_i e^{\sum_{j=1}^m \text{Re}(\psi_j(t, iu))} + 2^p (1 + \left\| \text{Re}(\psi(t, iu)) \right\|) \sum_{i \in I} q_i e^{\sum_{j=1}^m \text{Re}(\psi_j(t, iu))} + c_1 (1 + \left\| \text{Im}(\psi(t, iu)) \right\|) \sum_{i \in I} q_i ,
\]

where \( c_1 > 0 \) is a constant. Note that the function

\[
(-\infty, 0] \ni r \mapsto (1 + |r|)^p e^{e r}
\]

is bounded. (3.8)

By \( 1.9 \) and \( 4 \) p.108, (C.5], we can find a constant \( c_2 > 0 \) such that for all \( t \in [t_0, t_1] \) and \( u \neq 0 \),

\[
\left\| G(t, iu) \right\|^2 \leq \exp \left( \frac{c_2 t}{\| u \|} \right),
\]

where \( G(t, iu) \) is as in (2.2). This gives

\[
|\text{Im}(\psi(t, iu))| = |u| |G_i(t\| u\|, iu)| \leq |u| \exp \left( \frac{c_2 t_1}{2} \right).
\]

Combining (3.7), (3.8), and (3.9) gives

\[
\left\| \psi_I(t, iu) \right\| \sum_{i \in I} q_i e^{\sum_{j=1}^m \text{Re}(\psi_j(t, iu))} \leq c_3 (1 + \| u \|) \sum_{i \in I} q_i.
\]

In view of (3.10), we finally get

\[
\left\| \psi_I(t, iu) \right\| \sum_{i \in I} q_i \| u \| \sum_{j \in I} \hat{q}_j \left\| \psi_I(t, iu) \right\| \sum_{j \in I} q_j \| u_j \| \sum_{j \in I} \hat{q}_j \text{e}^{\text{Re}(\phi(t, iu)) + \text{Re}(e \psi(t, iu))} \leq c_4 (1 + \| u \|)^{p} \left( 1 + \| u_j \| \right) \sum_{j \in I} \hat{q}_j \text{e}^{\text{Re}(\phi(t, iu))}
\]

for another constant \( c_4 > 0 \). In view of (3.11), we thus arrive at (3.12) and the rest of the proof goes then exactly as in part (b). This completes the proof.

\[ \square \]

3.2. Existence and regularity of the invariant measure. Below we prove the existence and regularity for the invariant measure.

Lemma 3.1. Suppose that the same conditions as in Theorem 1.3 are satisfied. Then the unique invariant measure \( \pi \) has a density \( f^\pi \) of class \( C_0^p(D) \). Moreover, for each multi-index \( q \in \mathbb{N}_0^d \) satisfying \( \sum_{i \in I} q_i \leq p \), the derivative \( \partial_q f^\pi / \partial y_1^{q_1} \ldots \partial y_d^{q_d} \) is given by

\[
\frac{\partial_q f^\pi}{\partial y_1^{q_1} \ldots \partial y_d^{q_d}}(y) = (-i)^{|q|} \int_{\mathbb{R}^d} \prod_{k=1}^d (u_k)^{q_k} e^{-\langle y, iu \rangle} e^{\phi(t, iu)} \frac{du}{(2\pi)^d}, \quad y \in D.
\]

Proof. Similarly as in the proof of Theorem 1.2, we first find small enough \( \vartheta \in (0, 1) \) such that

\[
m + p < \lambda(\vartheta) = \min_{i \in I} b_i/\delta_{i, \vartheta}(\vartheta),
\]

where \( \delta_{i, \vartheta}(\vartheta) \) is as in (2.1), and then apply Theorem 2.1 to get

\[
e^{\text{Re}(\phi(t, iu))} \leq C(1 + \| u_j \|)^{-\lambda(\vartheta)} e^{-\delta_{i, \vartheta}^2 |u_j|^2}, \quad \| u \| \geq M, \quad t \geq 1,
\]
where $C$, $\delta$, $M > 0$ are constants depending on $\vartheta$. So
\[
\left| \int_D e^{(iu, \xi)} P_t(x, d\xi) \right| \leq C(1 + \|u \|) e^{-\lambda(\vartheta) e^{-\delta\|u\|}}
\]
holds for all $x \in D$, $u \in \mathbb{R}^d$ with $\|u\| \geq M$, and $t \geq 1$. Following [12, Theorem 2.7] it holds
\[
\int_D e^{(iu, \xi)} \pi(d\xi) = \lim_{t \to \infty} \int_D e^{(iu, \xi)} P_t(x, d\xi), \quad u \in \mathbb{R}^d, \quad x \in D,
\]
and hence we obtain, for $u \in \mathbb{R}^d$ with $\|u\| \geq M$,
\[
\left| \int_D e^{(iu, \xi)} \pi(d\xi) \right| \leq C(1 + \|u \|) e^{-\lambda(\vartheta) e^{-\delta\|u\|}}.
\]
In view of [3,10], the existence and differentiability of the density $f^n$ follows by classical properties of the Fourier transform.

\section{Exponential ergodicity in the total variation norm}

In this section we prove the exponential ergodicity statement in Theorem [12,3]. For this purpose, we first prove a similar result for the affine process with transition semigroup $(Q_t)_{t \geq 0}$ and admissible parameters $(a, \alpha, b, \beta, \nu = 0, \mu)$, i.e.,
\[
\int_D e^{(u, \xi)} Q_t(x, d\xi) = \exp \left( \int_0^t \langle \psi(s, u), \omega \psi(s, u) \rangle ds + \langle b, \int_0^t \psi(s, u) ds \rangle + \langle x, \psi(t, u) \rangle \right)
\]
for all $u \in U$, where $\psi(t, u)$ is obtained from [14]. The assertion is then deduced by a convolution argument similar to [7,12]. According to Theorem [1,1] the generator $A_Q$ of $(Q_t)_{t \geq 0}$ is given by
\[
A_Q f(x) = A_Q^0 f(x) + A_Q^1 f(x),
\]
for all $x \in D$ and defined for every $f \in C^2_c(D)$, where
\[
A_Q^0 f(x) = \langle b, \nabla f(x) \rangle + \sum_{k,l=1}^d a_{kl} \frac{\partial^2 f(x)}{\partial x_k \partial x_l},
\]
\[
A_Q^1 f(x) = \langle \beta x, \nabla f(x) \rangle + \sum_{k,l=1}^d \left( \sum_{i=1}^m \alpha_{i,k} x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \sum_{i=1}^m x_i \int_{D \setminus \{0\}} \left( f(x + \xi) - f(x) - \langle \xi, \nabla f(x) \rangle \right) \mu(d\xi).
\]
Proceeding as in [12, p. 10], for $\beta$ whose eigenvalues have strictly negative real-parts, we define the following norms
\[
\|x_I\|_I := \sqrt{\langle x_I, x_I \rangle_I} = \sqrt{\langle x_I, M_I x_I \rangle} \quad \text{and} \quad \|x_J\|_J := \sqrt{\langle x_J, x_J \rangle_J} = \sqrt{\langle x_J, M_J x_J \rangle},
\]
where
\[
M_I := \int_0^\infty e^{\beta_I t} e^{\beta_I t} dt \quad \text{and} \quad M_J := \int_0^\infty e^{\beta_J t} e^{\beta_J t} dt.
\]
Since $\beta_{II}$ and $\beta_{JJ}$ have only eigenvalues with strictly negative real-parts, the matrices $M_I$ and $M_J$ are well-defined. Note that both $M_I$ and $M_J$ are symmetric positive definite matrices.
Proposition 4.1. Assume \( d = m + n \geq 1 \) and let \( (Q_t)_{t \geq 0} \) be the affine semigroup given by (4.1) with admissible parameters \((a, \alpha, b, \beta, \nu = 0, \mu)\) such that \( \beta \) has only eigenvalues with strictly negative real-parts. Let \( V_\varepsilon \in C^2(D) \) be given by
\[
V_\varepsilon(x) = (1 + \langle x_I, x_I \rangle_I + \varepsilon \langle x_J, x_J \rangle_J)^{1/2}, \quad x \in \mathbb{R}_0^m \times \mathbb{R}^n,
\]
where \( \varepsilon > 0 \). Then \( V_\varepsilon \) belongs to the domain of the extended generator \( \mathcal{A}_Q \). If \( \varepsilon > 0 \) is small enough, then there exist positive constants \( c \) and \( C \) such that
\[
\mathcal{A}_Q V_\varepsilon(x) \leq -cV_\varepsilon(x) + C, \quad \text{for all } x \in D. \tag{4.3}
\]

Proof. Following the arguments given in the proof of [7, Proposition 5.1 (a)], we conclude that \( V_\varepsilon \) belongs to the domain of the extended generator and that \( \mathcal{A}_Q V_\varepsilon \) is given as in (4.2). It remains to prove (4.3). Now, it follows from the particular form of \( V_\varepsilon \) that there exists a constant \( c_1 > 0 \) such that \( |\mathcal{A}_Q^0 V_\varepsilon(x)| \leq c_1 \) for all \( x \in D \). Next, note that there exist constants \( c_2, c_3 > 0 \) and sufficiently small \( \varepsilon \in (0, 1/2) \) such that
\[
\mathcal{A}_Q^1 V_\varepsilon(x) \leq c_2 - c_3 (\|x_I\|^2 + \varepsilon \|x_J\|^2)^{1/2}, \quad \|x\| > 2. \tag{4.4}
\]

Indeed, if \( m \geq 1 \) and \( n \geq 1 \), then this follows from the inequalities shown in the proof of [12, Lemma 3.4]; the case \( m \geq 1, n = 0 \) follows by similar arguments to [12, Proposition 3.7]; while the case \( m = 0, n \geq 1 \) describes a Lévy driven OU-process and can be shown by similar (but essentially simpler) arguments to the previous two cases. Using (4.4) combined with \( |\mathcal{A}_Q^1 V_\varepsilon(x)| \leq c_4, \ x \in D \), where \( c_4 > 0 \) is some constant, we can easily show that
\[
\mathcal{A}_Q^1 V_\varepsilon(x) \leq c_5 - c_6 \varepsilon V_\varepsilon(x)
\]
where \( c_5, c_6 > 0 \) are some constants. Combining both estimates for \( \mathcal{A}_Q^0 V_\varepsilon(x) \) and \( \mathcal{A}_Q^1 V_\varepsilon(x) \) readily yields (4.3). \( \square \)

From the Lyapunov estimate we shall deduce a contraction estimate in total variation distance for the transition semigroup \( (Q_t)_{t \geq 0} \).

Proposition 4.2. Let \( (Q_t)_{t \geq 0} \) be the affine semigroup given by (4.1) with admissible parameters \((a, \alpha, b, \beta, \nu = 0, \mu)\) such that \( K \) has full rank, \( \min_{i \in \{1, \ldots, m\}} \alpha_{i,i} > 0 \), and \( \beta \) has only eigenvalues with strictly negative real-parts. Further, suppose that \( m < \min_{i \in I} b_i \alpha_{i,i}^{-1} \). Then for every \( M > 0 \) there exists \( h > 0 \) and \( \delta \in (0, 2) \) such that
\[
\|Q_h(x, \cdot) - Q_h(y, \cdot)\|_{TV} \leq 2 - \delta, \quad \text{for all } x, y \in D \text{ with } \|x\|, \|y\| \leq M.
\]

The proof of Proposition 4.2 goes along the very same lines as in the proof of [5, Proposition 5.3], see part (ii) therein. We omit the details here. We are ready to prove our second main result.

Proof of Theorem 5.3. Denote by \( (Q_t)_{t \geq 0} \) the transition semigroup given by (4.1) and let \( (R_t)_{t \geq 0} \) be the transition semigroup given by
\[
\int_D e^{(u, \xi)} R_t(x, d\xi)
= \exp \left( \int_0^t \int_{D \setminus \{0\}} \left( e^{(\psi(s, u), \xi)} - 1 - \langle \psi_J(s, u), \xi \rangle \mathbb{I}_{\{|\xi| \leq 1\}}(\xi) \right) \nu(d\xi) ds + \langle x, \psi(t, u) \rangle \right).
\]
for all $u \in U$, where $\psi(t,u)$ is given in (1.4). Then $R_t(x,d\xi)$ has admissible parameters $(a = 0, \alpha, b = 0, \beta, \nu, \mu)$. Using (1.3), we obtain

$$\int_D e^{(u,\xi)} P_t(x,d\xi) = \int_D e^{(u,\xi)} Q_t(x,d\xi) \cdot \int_D e^{(u,\xi)} R_t(0,d\xi), \quad t \geq 0, u \in U,$$

implying that $P_t(x,\cdot) = Q_t(x,\cdot) \ast R_t(0,\cdot)$ for all $t \geq 0$ and $x \in D$, where `$\ast$' denotes the usual convolution between measures. As a consequence of [6, Lemma 2.3] (see also [10, Corollary 2.8.3 and Theorem 3.2.3]), combining Propositions 4.1 and 4.2, yields

$$\|Q_t(x,\cdot) - Q_t(y,\cdot)\|_{TV} \leq c \min \left\{ 1, e^{-ct} (1 + \|x\| + \|y\|) \right\}, \quad t \geq 0, x \in D, \quad (4.5)$$

with some constant $c > 0$. In the following we let $H$ be any coupling\(^2\) of the Dirac measure $\delta_x$ concentrated in $x$ and the invariant distribution $\pi$. Let $C > 0$ be a generic constant that may vary from line to line. From [21, Theorem 4.8] combined with the invariance of $\pi$ and an obvious extension of [6, Lemma 2.3], we get

$$\|P_t(x,\cdot) - \pi(\cdot)\|_{TV} \leq \int_{D \times D} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{TV} H(dx,dy)$$

$$\leq \int_{D \times D} \|Q_t(x,\cdot) - Q_t(y,\cdot)\|_{TV} H(dx,dy)$$

$$\leq C \int_{D \times D} (1 \wedge e^{-ct} (1 + \|x\| + \|y\|)) H(dx,dy)$$

$$\leq C \int_{D \times D} \log (1 + e^{-ct} (1 + \|x\| + \|y\|)) H(dx,dy),$$

where the last two inequalities follow from (4.5) and the trivial inequality $1 \wedge c_1 \leq \log(2)^{-1} \log(1 + c_1)$ for all $c_1 > 0$. In [7, Lemma 8.5] the following inequality was shown

$$\log(1 + c_2 \cdot c_3) \leq c \min \left\{ c_2 \log(1 + c_3) \right\} + c \log(1 + c_3),$$

for any $c_2, c_3 \geq 0$, where $c > 0$ is a constant. So

$$\|P_t(x,\cdot) - \pi(\cdot)\|_{TV} \leq C \int_{D \times D} \min \left\{ e^{-ct}, \log (1 + \|x\| + \|y\|) \right\} H(dx,dy)$$

$$+ Ce^{-ct} \int_{D \times D} \log (1 + \|x\| + \|y\|) H(dx,dy)$$

$$\leq C \min \left\{ e^{-ct}, \int_{D \times D} \log (1 + \|x\| + \|y\|) H(dx,dy) \right\}$$

$$+ Ce^{-ct} \int_{D \times D} \log (1 + \|x\| + \|y\|) H(dx,dy)$$

$$\leq Ce^{-ct} \left( 1 + \int_{D \times D} \log (1 + \|x\| + \|y\|) H(dx,dy) \right).$$

\(^2\)A coupling $H$ of two Borel probability measures $(\xi, \tilde{\xi})$ on $D$ is a again Borel probability measure on $D \times D$ which has marginals $\xi$ and $\tilde{\xi}$, that is, for bounded Borel measurable functions $f$ and $g$ on $D$ it holds

$$\int_{D \times D} (f(x) + g(\tilde{x})) H(dx,d\tilde{x}) = \int_D f(x) \xi(dx) + \int_D g(x) \tilde{\xi}(dx).$$
Taking $H$ as the optimal coupling of $(\delta_x, \pi)$ and using the subadditivity of $\log$, we see that

$$\|P_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq C e^{-ct} \left( 1 + \log (1 + \|x\|) + \int_D \log (1 + \|y\|) \pi(dy) \right).$$

Notice that the integral on the right-hand side is indeed finite by [7, Theorem 1.5].

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