A Polynomial-Time Algorithm for the Equivalence between Quantum Sequential Machines

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Abstract

Quantum sequential machines (QSMs) are a quantum version of stochastic sequential machines (SSMs). Recently, we showed that two QSMs $\mathcal{M}_1$ and $\mathcal{M}_2$ with $n_1$ and $n_2$ states, respectively, are equivalent iff they are $(n_1 + n_2)^2$–equivalent (Theoretical Computer Science 358 (2006) 65-74). However, using this result to check the equivalence likely needs exponential expected time. In this paper, we consider the time complexity of deciding the equivalence between QSMs and related problems. The main results are as follows: (1) We present a polynomial-time algorithm for deciding the equivalence between QSMs, and, if two QSMs are not equivalent, this algorithm will produce an input-output pair with length not more than $(n_1 + n_2)^2$. (2) We improve the bound for the equivalence between QSMs from $(n_1 + n_2)^2$ to $n_1^2 + n_2^2 - 1$, by employing Moore and Crutchfield’s method (Theoretical Computer Science 237 (2000) 275-306). (3) We give that two MO-1QFAs with $n_1$ and $n_2$ states, respectively, are equivalent iff they are $(n_1 + n_2)^2$–equivalent, and further obtain a polynomial-time algorithm for deciding the equivalence between two MO-1QFAs. (4) We provide a counterexample showing that Koshiba’s method to solve the problem of deciding the equivalence between MM-1QFAs may be not valid, and thus the problem is left open again.

Keywords: Quantum computing; Stochastic sequential machines; Quantum sequential

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1. Introduction

Over the past two decades, quantum computing has attracted wide attention in the academic community [21, 29]. To a certain extent, this was motivated by the exponential speed-up of Shor’s quantum algorithm for factoring integers in polynomial time [33] and afterwards Grover’s algorithm of searching in database of size $n$ with only $O(\sqrt{n})$ accesses [19].

Quantum computers—the physical devices complying with the rules of quantum mechanics were first considered by Benioff [5], and then suggested by Feynman [15]. By elaborating and formalizing Benioff and Feynman’s idea, in 1985, Deutsch [13] re-examined the Church-Turing Principle and defined quantum Turing machines (QTMs). Subsequently, Deutsch [14] considered quantum network models. In 1993, Yao [35] demonstrated the equivalence between QTMs and quantum circuits. Quantum computation from the viewpoint of complexity theory was first studied systematically by Bernstein and Vazirani [12].

Another kind of simpler models of quantum computation is quantum finite automata (QFAs), that can be thought of as theoretical models of quantum computers with finite memory. This kind of computing machines was firstly studied independently by Moore and Crutchfield [28], as well as Kondacs and Watrous [26]. Then it was deeply dealt with by Ambainis and Freivalds [1], Brodsky and Pippenger [11], and the other authors (e.g., name only a few, [2-4,7-10,18], and for the details we may refer to [21]). The study of QFAs is mainly divided into two ways: one is one-way quantum finite automata (1QFAs) whose tape heads only move one cell to right at each evolution, and the other is two-way quantum finite automata (2QFAs), in which the tape heads are allowed to move towards right or left, or to be stationary. (Notably, Amano and Iwama [3] dealt with an intermediate form called 1.5QFAs, whose tape heads are allowed to move right or to be stationary, and, particularly, they showed that the emptiness problem for this restricted model is undecidable.) Furthermore, by means of the measurement times in a computation, 1QFAs have two fashions: measure-once 1QFAs (MO-1QFAs) proposed by Moore and Crutchfield [28], and, measure-many 1QFAs (MM-1QFAs) studied first by Kondacs and Watrous [26].
The characteristics of quantum principles can essentially strengthen the power of some models of quantum computing, but the unitarity and linearity of quantum physics also lead to some weaknesses. We briefly state some essential differences between the QFAs stated above and their classical counterparts by two aspects. One is from their power. The class of languages recognized by MM-1QFAs with bounded error probabilities is strictly bigger than that by MO-1QFAs, but both MO-1QFAs and MM-1QFAs recognize proper subclass of regular languages with bounded error probabilities [10,11,26,28]. (Also, the class of languages recognized by MM-1QFAs with bounded error probabilities is not closed under the binary Boolean operations [11,10].) Kondacs and Watrous [26] proved that some 2QFA can recognize non-regular language \( L_{eq} = \{a^n b^n | n > 0 \} \) with one-sided error probability in linear time (Freivalds [16] proved that two-way probabilistic finite automata can recognize non-regular language \( L_{eq} \) with arbitrarily small error, but it requires exponential expected time [23]. As it is well-known, classical two-way finite automata can accept only regular languages [24]).

The other difference is from the viewpoint of decidability. By \( P_A(x) \) we denote the probability of the automaton \( A \) accepting input string \( x \). Then the four cut-point languages, recognized by \( A \) with cut-point \( \lambda \in [0,1] \), are defined by \( L_\lambda = \{ x : P_A(x) \gg \lambda \} \), for \( \gg \in \{<, \leq, >, \geq \} \). When \( A \) is an MM-1QFA, Blondel et al. [9] proved that the problems of determining whether \( L_\lambda \) (\( \gg \in \{<, >\} \)) are empty are decidable, but when \( \gg \in \{\leq, \geq \} \), such problems are undecidable. In contrast, when \( A \) is a probabilistic automaton, all these emptiness problems for \( \gg \in \{<, \leq, >, \geq \} \) are undecidable [30].

Recently, another finding is concerning an essential difference between quantum sequential machines (QSMs) and stochastic sequential machines (SSMs). SSMs [30] may be viewed as a generalization of probabilistic automata [32,30], since an SSM that has only one output element and some accepting states are assigned, reduces to a probabilistic automaton. Two SSMs, say \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) having \( n_1 \) and \( n_2 \) states, respectively, and the same input and output alphabets, are called equivalent if they have equal accepting probability for any input-output pair \((u,v)\). As was known, a crucial result concerning SSMs by Paz [30] is that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are equivalent iff they are \((n_1 + n_2 - 1)\)-equivalent (that is, their accepting probabilities are equal for any input-output pair whose length is not more than \( n_1 + n_2 - 1 \)). Recently, Gudder [20] first defined sequential quantum machines (SQMs), a quantum analogue of SSMs, and Gudder asked whether or not such an equivalence consequence also holds for SQMs. Then,
Qiu [31] re-defined an equivalent version called *quantum sequential machines* (QSMs), that were formally a quantum counterpart of SSMs, just as *quantum finite automata* (QFAs) to probabilistic automata. Qiu [31] demonstrated that there are two QSMs with \( n_1 \) and \( n_2 \) states, respectively, such that they are \((n_1 + n_2 - 1)\)-equivalent, but *not* equivalent, after all. Hence, Gudder’s problem was given a negative answer.

Latterly, we [27] further proved that two QSMs \( M_1 \) and \( M_2 \) having \( n_1 \) and \( n_2 \) states, respectively, and the same input and output alphabets \( I \) and \( O \), are equivalent iff they are \((n_1 + n_2)^2\)-equivalent, a new feature in contrast to the \((n_1 + n_2 - 1)\)-equivalence for SSMs [30]. However, using this result to check the equivalence between QSMs needs exponential expected time \((O(m^{(n_1+n_2)^2}))\) where \( m = |I| \times |O| \).

The remainder of the paper is organized as follows. In Section 2, we recall the definitions of SSMs and QSMs, and related results. Section 3 is the main part. In Subsection 3.1, we detail a polynomial-time algorithm \((O(m.(n_1 + n_2)^{12}))\) for deciding the equivalence between QSMs. In Subsection 3.2, we improve the bound for the equivalence between QSMs from \((n_1 + n_2)^2\) to \( n_1^2 + n_2^2 - 1 \), by employing Moore and Crutchfield’s method [28]. Section 4 is concerning the equivalence between *one-way* QFAs. In Subsection 4.1, we provide a polynomial-time algorithm for deciding the equivalence between MO-1QFAs. In Subsection 4.2, we provide a counterexample showing that the method stated in [25] to decide the equivalence between MM-1QFAs is not valid. Finally, some remarks are made in Section 5 to conclude this paper.

### 2. Preliminaries

In this section, we briefly review some definitions and related properties that will be used in the sequel.

Firstly, we explain some notations. An \( n \)-dimensional row vector \((a_1\ a_2\ldots\ a_n)\) is called *stochastic* if \( a_i \geq 0 \ (i = 1, 2, \ldots, n) \), and \( \sum_{i=1}^{n} a_i = 1 \); in particular, it is called a *degenerate stochastic vector* if it has 1 only in one entry and else 0s. A matrix is called *stochastic* if its each row is a stochastic vector. As usual, for non-empty set \( I \), by \(|I|\) we mean the cardinality of \( I \), and by \( I^* \) we mean the set of all finite length strings over \( I \). For \( u \in I^* \), \(|u|\) denotes the length of \( u \); when \(|u| = 0\), \( u \) is an empty string, denoted by \( \epsilon \). We denote \( I^+ = I^* - \{\epsilon\} \). For
input alphabet $I$ and output alphabet $O$, the set of all input-output pairs is defined as

$$\{(u, v) \in I^* \times O^* : |u| = |v|\}.$$  

For any input-output pair $(u, v)$, we denote by $l(u, v)$ the length of $u$ or $v$.

For the details on stochastic sequential machines (SSMs), we can refer to [30]. Next, we recall the definition of quantum sequential machines (QSMs), a quantum counterpart of SSMs [30].

**Definition 1 ([31]).** A QSM is a 5-tuple $M = (S, \eta_{i_0}, I, O, \{A(y|x) : x \in I, y \in O\})$ where $S = \{s_1, s_2, \ldots, s_n\}$ is a finite set of internal states; $\eta_{i_0}$ is an $n-$dimensional degenerate stochastic row vector; $I$ and $O$ are input and output alphabets, respectively; $A(y|x)$ is an $n \times n$ complex matrix satisfying $\sum_y A(y|x)A(y|x)^\dagger = I$ for any $x \in I$, where the symbol $\dagger$ denotes Hermitian conjugate operation and $I$ is unit matrix. In particular, for input-output pair $(\epsilon, \epsilon)$, $A(\epsilon|\epsilon) = I$.

**Remark 1.** It is worth pointing out that, before the definition of QSMs [31], Gudder [20] first defined sequential quantum machines (SQMs), an equivalent version of QSMs. The equivalence between QSMs and SQMs was proved by Qiu [31]. The reader can refer to [20,31] for more information.

In brief, we may denote QSM $M$ as $(S, \eta_{i_0}, I, O, \{A(y|x)\})$. In the QSM $M$ defined above, if matrix $A(y|x) = [a_{ij}(y|x)]$, then $a_{ij}(y|x)$ (resp. $|a_{ij}(y|x)|^2$) represents the amplitude (resp. the probability) of the machine entering state $s_j$ and yielding $y$ after $x$ being inputted with the present state $s_i$. Thus, given the QSM $M$ above, we let $P_M(v|u)$ denote the probability of $M$ printing the word $v$ after having been fed with the word $u$, and it is defined as follows:

$$P_M(v|u) = \left\| \eta_{i_0} A(v|u) \right\|^2,$$

(1)

where $\eta_{i_0}$ is the initiation-state distribution of $M$, $(u, v) = (x_1 x_2 \ldots x_m, y_1 y_2 \ldots y_m)$ denotes an input-output pair, and $A(v|u) = A(y_1|x_1)A(y_2|x_2) \cdots A(y_m|x_m)$. Clearly, we have

$$P_M(v|u) = \eta_{i_0} A(v|u) A(v|u)^\dagger \eta_{i_0}^\dagger.$$

(2)

If the initiation-state distribution $\eta_{i_0}$ in QSM $M$ is omitted, then we call $M$ uninitiated QSM (UQSM). For a UQSM $M$, by $P_M^{\eta_{i_0}}(v|u)$ we mean the probability that, with the initiation-state distribution $\eta_{i_0}$ being specified, $M$ prints $v$ after $u$ being inputted.

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Definition 2. Let $\mathcal{M}$ be a UQSM. Any two initiation-state distributions $\eta_{i0}$ and $\eta_{j0}$ of $\mathcal{M}$ are said to be equivalent (resp. $k$–equivalent) with respect to $\mathcal{M}$, if $P_{\mathcal{M}}^{\eta_{i0}}(v|u) = P_{\mathcal{M}}^{\eta_{j0}}(v|u)$ for any input-output pair $(u,v)$ (resp. for any input-output pair $(u,v)$ with $l(u,v) \leq k$).

In the following, we define the equivalence between machines.

Definition 3. Two machines (SSMs, SQMs, or QSMs) $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same input and output alphabets are called equivalent (resp. $k$–equivalent) if $P_{\mathcal{M}_1}(v|u) = P_{\mathcal{M}_2}(v|u)$ for any input-output pair $(u,v)$ (resp. for any input-output pair $(u,v)$ with $l(u,v) \leq k$).

Remark 2. Given a QSM $\mathcal{M}$, from Eq. (1), we know $P_{\mathcal{M}}(\epsilon|\epsilon) \equiv 1$. Therefore, in Definition 3, we can require $l(u,v) \geq 1$. In what follows, for verifying the equivalence between QSMs, we only consider the input-output pair $(u,v)$ with $l(u,v) \geq 1$.

A crucial result concerning SSMs is that two SSMs with $n_1$ and $n_2$ states, respectively, are equivalent iff they are $(n_1 + n_2 - 1)$–equivalent [30]. Therefore, Gudder asked whether it holds for SQMs. Then Qiu [31] gave its negative answer. Recently, we further showed that two QSMs $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent iff they are $(n_1 + n_2)^2$–equivalent [27]. The two results are stated in the following theorem.

Theorem 1 ([31,27]). (1) There exist QSMs (or SQMs) $\mathcal{M}_1$ and $\mathcal{M}_2$ with $n_1$ and $n_2$ states, respectively, and the same input and output alphabets, such that though $\mathcal{M}_1$ and $\mathcal{M}_2$ are $(n_1 + n_2 - 1)$–equivalent, they are not equivalent.

(2) Two machines (SQMs or QSMs) $\mathcal{M}_1$ and $\mathcal{M}_2$ with $n_1$ and $n_2$ states, respectively, and the same input and output alphabets, are equivalent iff they are $(n_1 + n_2)^2$–equivalent.

3. Equivalence between QSMs

In this section, we consider further the equivalence between QSMs. In Subsection 3.1, based on the way stated in [27], we give a polynomial-time algorithm that takes as input two QSMs and determines whether they are equivalent. In Subsection 3.2, by employing Moore and Crutchfield’s method [28], we give a better bound for the equivalence between QSMs.
3.1. A polynomial-time algorithm for the equivalence between QSMs

As stated before, directly testing Theorem 1 (2) for the equivalence between QSMs needs exponential expected time. Therefore, in this subsection, we present a polynomial-time algorithm for the equivalence between QSMs.

Before presenting the algorithm, we recall the definition of direct sum of two matrices. Suppose that \( A_{mn} \) and \( B_{kl} \) are \( m \times n \) and \( k \times l \) matrices, respectively. Then the direct sum \( A_{mn} \oplus B_{kl} \) is an \((m + k) \times (n + l)\) matrix, defined as

\[
A_{mn} \oplus B_{kl} = \begin{bmatrix} A_{mn} & 0 \\ 0 & B_{kl} \end{bmatrix}.
\]

Now, we can present the main theorem as follows.

**Theorem 2.** There is a polynomial-time algorithm that takes as input two QSMs \( M_1 \) and \( M_2 \) and determines whether \( M_1 \) and \( M_2 \) are equivalent. Furthermore, if the two QSMs are not equivalent, then the algorithm outputs an input-output pair \((u, v)\) satisfying that \( P_{M_1}(v|u) \neq P_{M_2}(v|u) \), and \( l(u, v) \leq (n_1 + n_2)^2 \), where \( n_1 \) and \( n_2 \) are the numbers of states in \( M_1 \) and \( M_2 \), respectively.

**Proof.** Given two QSMs having the same input and output alphabets:

\[
M_1 = (S_1, \eta_{i_0}, I, O, \{A_1(y|x)\}) \quad \text{and} \quad M_2 = (S_2, \eta_{j_0}, I, O, \{A_2(y|x)\}),
\]

where \(|S_1| = n_1\) and \(|S_2| = n_2\). We construct UQSM \( M = (S, I, O, \{A(y|x) : x \in I, y \in O\}) \), where \( S = S_1 \cup S_2 \), \( A(y|x) = A_1(y|x) \oplus A_2(y|x) \). For any input-output pair \((u, v)\), denote

\[
D(v|u) = A(v|u)A(v|u)^\dagger.
\]

Then for any input-output pairs \((u, v)\) and \((x, y)\) where \( l(x, y) = 1 \), we have

\[
D(yv|xu) = A(yv|xu)A(yv|xu)^\dagger
= A(y|x)A(v|u)(A(y|x)A(v|u))^\dagger
= A(y|x)D(v|u)A(y|x)^\dagger.
\]

Let

\[
D = \{D(v|u) : (u, v) \text{ is input–output pair, and } l(v, u) \geq 1\},
\]

and \( \rho = (\eta_{i_0}, 0) \) and \( \rho^* = (0, \eta_{j_0}) \), where \( \rho \) and \( \rho^* \) are \((n_1 + n_2)\)-dimensional row vectors, and can serve as two different initiation-state distributions of \( M \). Then for any input-output pair
(u, v), we have

\[ P_{M_1}^\rho(v|u) = \rho A(v|u) A(v|u)^\dagger \rho^\dagger = \eta_i A_1(v|u) A_1(v|u)^\dagger \eta_i^\dagger = P_{M_1}(v|u), \] (6)

and similarly,

\[ P_{M_2}^{\rho'}(v|u) = P_{M_2}(v|u). \] (7)

Therefore, \( M_1 \) and \( M_2 \) are equivalent (i.e., \( P_{M_1}(v|u) = P_{M_2}(v|u) \) for any input-output pair \( (u, v) \)) if and only if \( \rho \) and \( \rho' \) are equivalent with regard to \( M \), that is, for any \( D(v|u) \in \mathcal{D} \),

\[ \rho D(v|u) \rho^\dagger = \rho' D(v|u) \rho'^\dagger. \] (8)

Let \( \Phi(\mathcal{D}) \) be the linear subspace spanned by \( \mathcal{D} \), and let \( \mathcal{B} \) be a basis for the subspace \( \Phi(\mathcal{D}) \). Clearly, the total space as to \( \Phi(\mathcal{D}) \) consists of all \((n_1 + n_2)\)—order complex square matrices, together with the usual operations of matrices, whose dimension is \((n_1 + n_2)^2\). Hence \( \mathcal{B} \) has at most \((n_1 + n_2)^2\) elements, and the two QSMs \( M_1 \) and \( M_2 \) are equivalent if and only if Eq. (8) holds for every vector \( D(v|u) \in \mathcal{B} \).

**Design of the algorithm.** Without loss of generality, we assume that \( I = \{ a \} \) and \( O = \{ 0, 1 \} \). Then we define binary tree \( T \) as follows. Tree \( T \) has a corresponding node \( D(v|u) \) (defined in Eq. (3)) for every input-output pair \((u, v) \in I^* \times O^*\). The root of \( T \) is \( D(\epsilon|\epsilon) \) that is identity matrix \( I \). In the definition of \( \mathcal{D} \) (Eq. (5)), we notice that \( l(v, u) \geq 1 \), so, we exclude \( D(\epsilon|\epsilon) \) when searching for the basis \( \mathcal{B} \) of subspace \( \Phi(\mathcal{D}) \). However, we still make it act as the root of tree \( T \) for the sake of convenience. Every node \( D(v|u) \) in \( T \) has two children \( D(0v|au) \) and \( D(1v|au) \). For any \( x \in I \), and \( y \in O \), \( D(yv|xu) \) can be calculated from its parent \( D(v|u) \) by Eq. (4).

Our algorithm is described in Figure 1 which is to efficiently search for the basis \( \mathcal{B} \) of \( \Phi(\mathcal{D}) \) by pruning tree \( T \). In the algorithm, \textit{queue} denotes a queue, and \( \mathcal{B} \), the basis stated above, is initially set to be the empty set. We visit tree \( T \) by breadth-first order. At each node \( D(v|u) \), we verify whether it is linearly independent of \( \mathcal{B} \). If it is, we add it to \( \mathcal{B} \). Otherwise, we prune the subtree rooted at \( D(v|u) \). We stop traversing tree \( T \) after every node in \( T \) has been either visited or pruned. The vectors in the resulting set \( \mathcal{B} \) will form a basis for \( \Phi(\mathcal{D}) \),
which will be proven later on. At the end of the algorithm, we verify whether Eq. (8) holds for every vector in $B$. If yes, then the two QSMs are equivalent. Otherwise, the algorithm returns an input-output pair $(u,v)$ satisfying that $P_{M_1}(v|u) \neq P_{M_2}(v|u)$.

Figure 1. Algorithm for the equivalence between QSMs.

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Input: $M_1 = (S_1, \eta_{i_0}, \{a\}, \{0,1\}, \{A_1(y|x)\})$, and $M_2 = (S_2, \eta_{j_0}, \{a\}, \{0,1\}, \{A_2(y|x)\})$

Set $B$ to be the empty set;
queue ← $D(\epsilon|\epsilon)$;
while queue is not empty do
    begin take an element $D(v|u)$ from queue;
    if $D(v|u) \notin \text{span}(B)$ then
        begin add vector $D(v|u)$ to $B$; // $D(\epsilon|\epsilon)$ is not added to $B$.
        add $D(0v|au)$ and $D(1v|au)$ to queue;
    end;
end;
if $\forall B \in B$, $(\eta_{i_0}, 0)B(\eta_{i_0}, 0)^\dagger = (0, \eta_{j_0})B(0, \eta_{j_0})^\dagger$ then return (yes)
else return (the pair $(u,v)$: $(\eta_{i_0}, 0)B(\eta_{i_0}, 0)^\dagger \neq (0, \eta_{j_0})B(0, \eta_{j_0})^\dagger$);
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Remark 3. The basic idea regarding our algorithm is to efficiently search for the basis $B$ of $\Phi(D)$. The foundation of our algorithm is the breadth-first search for traversing a tree. The method used in our algorithm is to prune tree $T$. By pruning some unwanted subtrees, we do not need to visit all nodes with height not more than $(n_1 + n_2)^2$ (here the height of root is defined as 0), such that we can greatly reduce the number of nodes to be visited.

Validity of the algorithm. Now we explain why the resulting set $B$ form a basis for $\Phi(D)$. From the analysis above, we know that after running the algorithm, we will get a pruned tree. We denote the resulting tree by $T_P$ which is formed by the nodes in the following set

$$B \cup \{D(\sigma_o v|\sigma_i u) : D(v|u) \in B, D(\sigma_o v|\sigma_i u) \in \text{span}(B), \sigma_i \in I, \sigma_o \in O\},$$

where the former part $B$ consists of the internal nodes of tree $T_P$, and the latter part comprises the leaf nodes of tree $T_P$. 

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For $i \geq 0$, we let
\[ B_i = \{ D(yv|xu) : D(v|u) \text{ is a leaf in } T_P, \ l(x, y) = i \}, \]
(9)
where when $i \geq 1$, set $B_i$ consists of unvisited nodes in tree $T$ which have distance $i$ from a leaf in tree $T_P$; when $i = 0$, set $B_0$ is the set of leaves of $T_P$. Then it can be readily seen that
\[ D = B \cup \bigcup_{i=0}^{\infty} B_i. \]
(10)
Proving that $B$ forms a basis for $\Phi(D)$ amounts to showing that $\operatorname{span}(B) \equiv \operatorname{span}(D)$. Equivalently, we only need to prove the following proposition.

**Proposition 3.** For all $i \geq 0$, $B_i \subseteq \operatorname{span}(B)$.

**Proof.** Let $B = \{ B_1, B_2, \ldots, B_m \}$ for some $m \leq (n_1 + n_2)^2$. We show the proposition by induction on $i$. The basic case $B_0 \subseteq \operatorname{span}(B)$ follows straightforward from our analysis above. Now assume that $B_i \subseteq \operatorname{span}(B)$. Then for any input-output pairs $(u, v), (x, y)$, and, $(\sigma_o, \sigma_I)$, where $l(\sigma_o, \sigma_I) = 1$, such that $D(v|u)$ is a leaf and $l(x, y) = i$, by Eq. (9) we know $D(yv|xu) \in B_i$, and, with the assumption, $D(yv|xu) = \sum_{j=1}^{m} \alpha_j B_j$ for some $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \ldots, m$); furthermore, for any $B_j \in B$, say $B_j = D(v_j|u_j)$ for some input-output pair $(u_j, v_j)$, we have $A(\sigma_o|\sigma_I)B_j A(\sigma_o|\sigma_I)\dagger = D(\sigma_o v_j|\sigma_I u_j) \in \operatorname{span}(B \cup B_0)$. Therefore, we get that
\[
D(\sigma_o yv|\sigma_I xu) = A(\sigma_o|\sigma_I)D(yv|xu)A(\sigma_o|\sigma_I)\dagger \\
= A(\sigma_o|\sigma_I) \left( \sum_{j=1}^{m} \alpha_j B_j \right) A(\sigma_o|\sigma_I)\dagger \\
= \sum_{j=1}^{m} \alpha_j \left( A(\sigma_o|\sigma_I)B_j A(\sigma_o|\sigma_I)\dagger \right) \\
\in \operatorname{span}(B \cup B_0) \equiv \operatorname{span}(B).
\]
This shows that $B_{i+1} \subseteq \operatorname{span}(B)$, and the proposition is proved. \qed

**Complexity of the algorithm.** Firstly we assume that all the inputs consist of complex numbers whose real and imaginary parts are rational numbers and that each arithmetic operation on rational numbers can be done in constant time. Because the basis $B$ has at most
\((n_1 + n_2)^2\) elements, the nodes to be visited will be at most \(O((n_1 + n_2)^2)\). (Here we need recall a result that to verify whether a set of \(n\)-dimensional vectors is linearly independent needs time \(O(n^3)\) \([17]\).) At every visited node \(D(v|u)\) the algorithm may do two things: (i) verifying whether or not the \((n_1 + n_2)^2\)-dimensional vector \(D(v|u)\) is linearly independent of the set \(B\), which needs time \(O((n_1 + n_2)^6)\) according to the result in \([17]\) just stated above; (ii) calculating its children nodes by Eq. (4) (if \(D(v|u) \notin B\)), which can be done in time \(O((n_1 + n_2)^4)\). Thus the total runtime is \(O((n_1 + n_2)^{12})\).

So far, we have completed the proof of Theorem 2.

Remarks 4. In the algorithm above, for convince we consider only the case where \(|I| = 1\) and \(|O| = 2\). In general, let \(m = |I| \times |O|\). Then the algorithm almost keeps on except that the total nodes to visit will be at most \(O(m.(n_1 + n_2)^2)\), and as a result, the time complexity will be \(O(m.(n_1 + n_2)^{12})\).

3.2. An improved bound for the equivalence between QSMs

In this subsection, we give an improved bound for the equivalence between QSMs, using the bilinearization technique given by Moore and Crutchfield \([28]\). Firstly, we define a new model as follows.

Definition 4. A bilinear machine (BLM) is a four-tuple \(M = (S, \pi, \{M(\sigma)\}_{\sigma \in \Sigma}, \eta)\) over alphabet \(\Sigma\), where \(S\) with \(|S| = n\) is a finite state set, \(\pi \in \mathbb{C}^{1 \times n}\), \(\eta \in \mathbb{C}^{n \times 1}\) and \(M(\sigma) \in \mathbb{C}^{n \times n}\) for \(\sigma \in \Sigma\).

Associated to a BLM \(M\), the word function \(f_M : \Sigma^* \rightarrow \mathbb{C}\) is defined in the form: \(f_M(w) = \pi M(w_1) \cdots M(w_n) \eta\), where \(w = w_1 \cdots w_n \in \Sigma^*\).

A probabilistic automaton (PA) is a BLM with the restriction that \(\pi\) is a stochastic vector, \(\eta\) consists of 0’s and 1’s only, and the matrices \(M(\sigma)\) (\(\sigma \in \Sigma\)) are stochastic. Then, the word function \(f_M\) associated to PA \(M\) has domain in \([0, 1]\).

Definition 5. Two BLMs (include PAs) \(M_1\) and \(M_2\) over the same alphabet \(\Sigma\) are said to be equivalent (resp. \(k\)-equivalent) if \(f_{M_1}(w) = f_{M_2}(w)\) for any \(w \in \Sigma^*\) (resp. for any input string \(w\) with \(|w| \leq k\)).
As stated in Paz [30], the result with regard to the equivalence between SSMs can also be applied to PAs. Therefore, based on Paz [30], Tzeng [34] considered further the equivalence between PAs, giving a polynomial-time algorithm to the problem. Now, the results are stated in the following.

**Theorem 4 ([30,34])**. Two PAs $M_1$ and $M_2$ with $n_1$ and $n_2$ states, respectively, are equivalent if and only if they are $(n_1 + n_2 - 1)$-equivalent. Furthermore, there is a polynomial-time algorithm that takes as input two PAs $M_1$ and $M_2$ and determines whether $M_1$ and $M_2$ are equivalent.

**Remark 5.** In fact, one can readily find that Paz’s way [30] can also be applied to BLMs, and the algorithm given by Tzeng [34] still works for BLMs. Therefore, Theorem 4 holds for the more general model—BLMs.

Next, we transform a QSM to a BLM by the way given by Moore and Crutchfield [28], and then obtain an improved result for the equivalence between QSMs. That is the following theorem.

**Theorem 5.** Two QSMs $M_1$ and $M_2$ with $n_1$ and $n_2$ states, respectively, are equivalent if and only if they are $(n_1^2 + n_2^2 - 1)$-equivalent.

**Proof.** Given an $n$-state QSM $M = (S, \eta_0, I, O, \{A(y|x)\})$, let $h_j (j = 1, \ldots n)$ be a column vector that has only 1 in the $j$th element and else 0s. Then we have

$$P_M(v|u) = \left\| \eta_0 A(v|u) \right\|^2 = \sum_{j=1}^{n} |\eta_0 A(v|u) h_j|^2$$

$$= \sum_{j=1}^{n} (\eta_0 \otimes \eta_0^*) [A(v|u) \otimes A(v|u)^*] (h_j \otimes h_j^*)$$

$$= (\eta_0 \otimes \eta_0^*) [A(v|u) \otimes A(v|u)^*] \sum_{j=1}^{n} (h_j \otimes h_j^*).$$

Here we can construct a BLM $M' = (S', \pi, M(\sigma)_{\sigma \in \Sigma}, \eta)$ as follows:

- $|S'| = n^2$, $\pi = \eta_0 \otimes \eta_0^*$;
- $\Sigma = \{(y|x) : y \in O, x \in I\}$, and $M((y|x)) = A(y|x) \otimes A(y|x)$;
• \( \eta = \sum_{j=1}^{n} (h_j \otimes h_j^*) \).

Then we get \( P_{M}(y_1 \ldots y_m|x_1 \ldots x_m) = f'_{M'}((y_1|x_1) \ldots (y_m|x_m)) \). Hence every \( n \)-state QSM can be transformed to an equivalent \( n^2 \)-state BLM, and by Remark 5, we have proved the theorem.

\( \square \)

**Remark 6.** Considering the equivalence between two QSMs, we have improved the bound from \((n_1+n_2)^2\) to \((n_1^2+n_2^2-1)\) by the way of Moore and Crutchfield [28], which seems to imply that the way used in Subsection 3.1 is unwanted. Nevertheless, the way used in Subsection 3.1 offers us a different insight to QSMs and even other quantum computing models, and maybe can be used to solve some new problems concerning quantum computing models.

### 4. Equivalence between one-way QFAs

In this section, we consider the equivalence between one-way QFAs. More specifically, in Subsection 4.1, we present a polynomial-time algorithm for the equivalence between MO-1QFAs, by means of the idea in Subsection 3.1; in Subsection 4.2, we provide a counterexample showing that the method used in [25] to decide the equivalence between MM-1QFAs may be not valid.

#### 4.1 Equivalence between MO-1QFAs

First, we review the definition of MO-1QFAs [28,11].

An MO-1QFA \( \mathcal{A} \) is a 5-tuple \( \mathcal{A} = (Q, \Sigma, q_0, \{A(x) : x \in \Sigma\}, F) \), where \( Q \) is a finite set of states (let \( |Q| = n \)); \( q_0 \) is the initial state; \( \Sigma \) is a finite set of input symbols; \( A(x) \) denotes an \( n \times n \) unitary evolution matrix for each \( x \in \Sigma \); \( F \subseteq Q \) is the set of accepting states, with corresponding projection matrix \( P_{\text{acc}} = \text{diag}(p_0 \ldots p_{n-1}) \) where for \( i = 0, \ldots, n-1 \), \( p_i \) equals to 1 if \( q_i \in F \) else 0.

As usual, let \( \langle q_i \rangle \) denote the \( n \)-dimensional row vector \((0 \cdots 1 \cdots 0)\) whose \((i+1)\)th component is 1 and the others 0s \((i = 0, 1, \ldots, n-1)\). Any configuration of \( \mathcal{A} \) is described by a unit row vector in the superposition form \( \langle \psi \rangle = \sum_{i=0}^{n-1} \alpha_i \langle q_i \rangle \), with that \( \sum_{i=0}^{n-1} |\alpha_i|^2 = 1 \), and
\( \alpha_i \) denoting the amplitude of \( A \) being in state \( q_i \). If \( A \) is in configuration \( |\psi\rangle \) and reads an input symbol \( \sigma \in \Sigma \), then the new configuration of \( A \) becomes \( |\psi'\rangle = |\psi\rangle A(\sigma) \).

The probability of \( A \) accepting input string \( u = x_1 x_2 \ldots x_m \) is defined as

\[
P_A^{q_0}(u) = \| \langle q_0 | A(u) P_{acc} \| ^2 \tag{11}
\]

where \( A(u) = A(x_1) A(x_2) \ldots A(x_m) \). For simplicity, we often write \( P_A^{q_0}(u) \) by \( P_A(u) \) if no confusion results.

We introduce two definitions regarding the equivalence between MO-1QFAs as follows.

**Definition 6.** Two MO-1QFAs \( A_1 \) and \( A_2 \) having the same set of input symbols are called equivalent (resp. \( k \)-equivalent) if for any input string \( u \) (resp. for any input string \( u \) with \(|u| \leq k \)), they have equal accepting probability, i.e., \( P_{A_1}(u) = P_{A_2}(u) \).

**Definition 7.** Given an MO-1QFA \( A \) whose initial state is not specified, then two states \( q_1 \) and \( q_2 \) in \( A \) are called equivalent (resp. \( k \)-equivalent) if for any input string \( u \) (resp. for any input string \( u \) with \(|u| \leq k \)), \( P_{A}^{q_1}(u) = P_{A}^{q_2}(u) \).

Concerning the equivalence between MO-1QFAs, we have the following theorem.

**Theorem 6.** Two MO-1QFAs \( A_1 \) and \( A_2 \) are equivalent if and only if they are \((n_1 + n_2)^2\)-equivalent, where \( n_1 \) and \( n_2 \) are the numbers of states in \( A_1 \) and \( A_2 \), respectively. Furthermore, there is a polynomial-time algorithm that takes as input two MO-1QFAs \( A_1 \) and \( A_2 \) and determines whether \( A_1 \) and \( A_2 \) are equivalent.

**Proof.** We give a brief proof by four steps below.

1. Let MO-1QFA \( A = (Q, \Sigma, q_0, \{A(x) : x \in \Sigma\}, F) \). Then for any \( u \in \Sigma^* \), we have

\[
P_A^{q_0}(u) = \| \langle q_0 | A(u) P_{acc} \| ^2
\]

\[
= \langle q_0 | A(u) P_{acc} P_{acc}^\dagger A(u)^\dagger | q_0 \rangle
\]

\[
= \langle q_0 | A(u) P_{acc} A(u)^\dagger | q_0 \rangle,
\]

where notation \( |\cdot\rangle \) denotes the conjugate transpose of \( \langle \cdot | \) .

Denote \( F(u) = A(u) P_{acc} A(u)^\dagger \). Then

\[
P_A^{q_0}(u) = \langle q_0 | F(u) | q_0 \rangle, \tag{12}
\]

\[
F(xu) = A(x) F(u) A(x)^\dagger. \tag{13}
\]
2. Denote
\[ F = \{ F(u) : u \in \Sigma^* \}, \tag{14} \]
\[ F(k) = \{ F(u) : u \in \Sigma^*, |u| \leq k \}. \tag{15} \]

As we did in the proof of Theorem 1, a set of linearly independent vectors can be found in \( F(n^2) \) \((n = |Q|)\) such that any vector in \( F \) is a linearly combination of these vectors. Therefore, by Eq. (12), we obtain that two initial states \( q_0 \) and \( q'_0 \) for \( A \) are equivalent iff they are \( n^2 \)-equivalent.

3. As in the proof of Theorem 2, given two MO-1QFAs \( A_1 = (Q_1, \Sigma, q_0, \{A_1(x)\}, F_1) \) and \( A_2 = (Q_2, \Sigma, p_0, \{A_2(x)\}, F_2) \), we let \( A = (Q_1 \cup Q_2, \Sigma, \{A_1(x) \oplus A_2(x) : x \in \Sigma\}, F_1 \cup F_2) \) (we assume \( Q_1 \cap Q_2 = \emptyset \)). Then the equivalence between \( A_1 \) and \( A_2 \) amounts to the equivalence between the initial states \( \rho \) and \( \rho' \) with regard to \( A \), where \( \rho \) and \( \rho' \) have corresponding vectors \( (q_0, 0) \) and \( (0, p_0) \), respectively.

4. In virtue of the above considerations and the idea in Subsection 3.1, we describe an algorithm in Figure 2. Analogous to Theorem 2 and Remark 4, the time-complexity of this algorithm is \( O(m.(n_1 + n_2)^2) \), where \( n_1 = |Q_1|, n_2 = |Q_2| \) and \( m = |\Sigma| \).

---

**Figure 2.** Algorithm for the equivalence between MO-1QFAs.

**Input:** \( A_1 = (Q_1, q_0, \{0, 1\}, \{A_1(x) : x = 0, 1\}, F_1) \) and \( A_2 = (Q_2, p_0, \{0, 1\}, \{A_2(x) : x = 0, 1\}, F_2) \)

Set \( B \) to be the empty set;

\( \text{queue} \leftarrow \text{node}(\epsilon) \);

\textbf{while} \( \text{queue} \) is not empty \textbf{do}

\textbf{begin} take an element \( F(u) \) from \( \text{queue} \);

\hspace{1cm} \textbf{if} \( F(u) \notin \text{span}(B) \) \textbf{then}

\hspace{2cm} \textbf{begin} add vector \( F(u) \) to \( B \);

\hspace{3cm} \text{add} \( F(0u) \) and \( F(1u) \) to \( \text{queue} \);

\hspace{2cm} \textbf{end};

\textbf{end};

\hspace{1cm} \textbf{if} \( \forall B \in B, ((q_0, 0)B(q_0), 0) = (0, p_0)B(0, p_0) \) \textbf{then} return \( \text{yes} \)

\hspace{1cm} \textbf{else} return \( \text{the string} u: ((q_0, 0)B(q_0), 0) \neq (0, p_0)B(0, p_0)) \);
Remark 7. It is worth indicating that Brodsky and Pippenger [11] and Koshiba [25] also considered the equivalence problem concerning MO-1QFAs. Their methods can be described by two steps: (i) firstly using the bilinearization technique [28] to convert MO-1QFAs to generalized stochastic finite automata [28]; (ii) secondly determining the equivalence of generalized stochastic finite automata. Their difference is regarding step (ii): Koshiba [25] applied the tree pruning technique [34] to determine the generalized stochastic systems’ equivalence, while Brodsky and Pippenger [11] employed Paz [30]’s method to do that. Therefore, Koshiba [25] gave a polynomial-time algorithm for the problem, but Brodsky and Pippenger [11] did not consider its efficiency. One can find that our method is different from [25, 11].

4.2 A counterexample for the equivalence between MM-1QFAs

Gruska [22] proposed as an open problem that is it decidable whether two MM-1QFAs are equivalent. Then Koshiba [25] tried to solve the problem. For any MM-1QFA, Koshiba [25] wanted to construct an equivalent MO-g1QFA (like MO-1QFA but with evolution matrices not necessarily unitary) and then decided the equivalence between MO-g1QFAs using the known way on MO-1QFAs. Nevertheless, we find that the construction technique stated in [25, Theorem 3] may be not valid, and as a result, the problem is in fact not solved there. We will give a counterexample to show its invalidity. In the following, we adopt the definitions of QFAs stated in [11] where only the right end-marker symbol $ is considered. So the reader can refer to [11] for the definitions and we do not detail them here.

First let us recall the method stated in [25, Theorem 3] for constructing MO-g1QFAs from MM-1QFAs. Given an MM-1QFA $\mathcal{M} = (Q, \Sigma, \{U_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}, q_0, Q_{acc}, Q_{rej})$, an MO-g1QFA $\tilde{\mathcal{M}} = (Q', \Sigma, \{U'_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}, q_0, F)$ is constructed as follows:

- $Q' = Q \cup \{q_\sigma : \sigma \in \Sigma \cup \{\$\}\}\setminus Q_{acc}$, and $F = \{q_\sigma : \sigma \in \Sigma \cup \{\$\}\}$;
- $U'_\sigma|q\rangle = \cdots + \alpha_i|q_i\rangle \cdots + \alpha_A|q_A\rangle$ when $U_\sigma|q\rangle = \cdots + \alpha_i|q_i\rangle \cdots + \alpha_A|q_A\rangle$ and $q_A \in Q_{acc}$; and
- add the rules: $U'_\sigma|q_\sigma\rangle = |q_\sigma\rangle$ for all $|q_\sigma\rangle \in F$. 

From the above four steps we have completed the proof.
Koshiba [25] deemed that the construction technique stated above can ensure that for any input word, the accepting probability in $M$ is preserved in $M'$, which will be shown to be not so.

Now we turn to the following counterexample provided by us, showing the invalidity of the above method.

A counterexample  Let MM-1QFA $M = (Q, \Sigma, \{U_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}, q_0, Q_{acc}, Q_{rej})$, where $Q = \{q_0, q_1, q_{acc}, q_{rej}\}$ with the set of accepting states $Q_{acc} = \{q_{acc}\}$ and the set of rejecting states $Q_{rej} = \{q_{rej}\}$; $\Sigma = \{a\}$; $q_0$ is the initial state; $\{U_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}$ are described below.

$$U_a(|q_0)\rangle = \frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_{acc}\rangle,$$

$$U_a(|q_1)\rangle = \frac{1}{2}|q_0\rangle - \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_{acc}\rangle,$$

$$U_\$ (|q_0\rangle) = |q_{acc}\rangle, U_\$ (|q_1\rangle) = |q_{rej}\rangle.$$ 

Next, we show how this automaton works on the input word $aa\$.

1. The automaton starts in $|q_0\rangle$. Then $U_a$ is applied, giving $\frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_{acc}\rangle$.

   This state is observed with two possible results. With probability $(\frac{1}{2})^2$, the accepting state is observed and then the computation terminates. Otherwise, a non-halting state $\frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle$ (unnormalized) is observed and then the computation continues.

2. After the second $a$ is fed, the state $\frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle$ is mapped to $\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_0\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})|q_1\rangle + \frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_{acc}\rangle$. This is observed with two possible results. With probability $[\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})]^2$, the computation terminates in the accepting state $q_{acc}$. Otherwise, the computation continues with a new no-halting state $\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_0\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})|q_1\rangle$ (unnormalized).

3. After the last symbol $\$ is fed, the automaton’s state turns to $\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_{acc}\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})|q_{rej}\rangle$. This is observed. The computation terminates in the accepting state $|q_{acc}\rangle$ with probability $[\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})]^2$ or in the rejecting state $|q_{rej}\rangle$ with probability $[\frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})]^2$.

The total accepting probability is $(\frac{1}{2})^2 + [\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})]^2 + [\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})]^2 = \frac{5}{8} + \frac{1}{2\sqrt{2}}$. 

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Now according to the construction technique \cite[Theorem 3]{25} stated before, we get an MO-g1QFA $M' = (Q', \Sigma, \{U'_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}, q_0, F)$ where $Q' = \{q_0, q_1, q_{\text{rej}}, q_a, q_\$$\}$, $F = \{q_a, q_\$$\}$ and $\{U'_\sigma\}_{\sigma \in \Sigma \cup \{\$\}}$ are described below.

\[
U'_a(|q_0\rangle) = \frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_a\rangle, \\
U'_a(|q_1\rangle) = \frac{1}{2}|q_0\rangle - \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_a\rangle, \\
U'_\$(|q_0\rangle) = |q_\$$\rangle, \quad U'_\$(|q_1\rangle) = |q_{\text{rej}}\rangle, \\
U'_a(|q_a\rangle) = |q_a\rangle, \quad U'_\$(|q_a\rangle) = |q_a\rangle.
\]

When the input word is $aa\$$, the automaton works as follows. Starting from state $|q_0\rangle$, when the first $a$ is fed, the automaton turns to state $\frac{1}{2}|q_0\rangle + \frac{1}{\sqrt{2}}|q_1\rangle + \frac{1}{2}|q_a\rangle$. After the second $a$ is fed, the state is mapped to $\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_0\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})|q_1\rangle + \frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_a\rangle$. After the last symbol $\$ is fed, the state is mapped to $\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_0\rangle + \frac{1}{\sqrt{2}}(\frac{1}{2} - \frac{1}{\sqrt{2}})|q_{\text{rej}}\rangle + \frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})|q_a\rangle$.

The total accepting probability is $\left[\frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})\right]^2 + \left[\frac{1}{2} + \frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})\right]^2 = \frac{7}{8} + \frac{1}{\sqrt{2}}$.

Now it turns out that the accepting probability in the original MM-1QFA is not preserved in the constructed machine as expected in \cite{25}. Therefore, the invalidity of the method \cite[Theorem 3]{25} has been shown.

**Remark 8.** (1) The essential reason for the invalidity of the way in \cite{25} is that the accepting state set $F$ in $M'$ does not cumulate the accepting probabilities in the original MM-1QFA. Instead, it accumulates just the accepting amplitudes. In addition, we know that in general, $|a|^2 + |b|^2 \neq |a + b|^2$. Therefore, the way in \cite{25} leads to invalidity. (2) Due to the complex accepting behavior of MM-1QFAs, it is likely no longer valid to decide the equivalence between MM-1QFAs as we did for MO-1QFAs. To our knowledge, so far there seems to be no existing valid solution to this problem. Therefore, the equivalence between MM-1QFAs is worth considering further.

5. Concluding remarks

In this paper, based on the results in \cite{27, 31}, we presented a polynomial-time algorithm ($O(m.(n_1+n_2)^{12})$) for determining the equivalence between two QSMs with $n_1$ and $n_2$ states, respectively, and, if they are not equivalent, this algorithm will produce an input-output pair.
with length not more than \((n_1+n_2)^2\). Furthermore, by using the way of Moore and Crutchfield [28], we obtained that two QSMs \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are equivalent iff they are \((n_1^2 + n_2^2 - 1)\)-equivalent, which improves the result in [27].

We also proved that two MO-1QFAs \(\mathcal{A}_1\) and \(\mathcal{A}_2\) that have \(n_1\) and \(n_2\) states, respectively, and the same input alphabet \(\Sigma\) with \(|\Sigma| = m\), are equivalent if, and only if they are \((n_1+n_2)^2\)-equivalent. In terms of the idea of the algorithm for QSMs, we further provided a polynomial-time algorithm \((O(m.(n_1 + n_2)^{12}))\) for the equivalence between \(\mathcal{A}_1\) and \(\mathcal{A}_2\).

In addition, considering the problem of deciding the equivalence between MM-1QFAs, we provided a counterexample showing that the method stated in [25] to solve the problem may be not valid, and therefore the problem is left open again.

The further problems are regarding the minimization of states for QSMs [20,31]. As well, the equivalence concerning 2QFAs [26] is worthy of consideration.

References

[1] A. Ambainis, R. Freivalds, One-way quantum finite automata: strengths, weaknesses and generalizations, in: Proceedings of the 39th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Palo Alto, California, USA, 1998, pp. 332-341. Also [quant-ph/9802062] 1998.

[2] F. Ablayev, A. Gainutdinova, Complexity of Quantum Uniform and Nonuniform Automata, in: Proceedings of the 9th International Conference on Developments in Language Theory (DLT'2005), Lecture Notes in Computer Science, Vol. 3572 Springer-Verlag, Berlin, 2005, pp. 78-87.

[3] M. Amano, K. Iwama, Undecidability on Quantum Finite Automata, in: Proceedings of the 31st Annual ACM Symposium on Theory of Computing, Atlanta, Georgia, USA, 1999, pp. 368-375.

[4] A. Ambainis, A. Nayak, A. Ta-Shma, U. Vazirani, Dense quantum coding and quantum automata, Journal of the ACM 49 (4) (2002) 496-511.
[5] P. Benioff, The computer as a physical system: a microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines, Journal of Statistic Physics 22 (1980) 563-591.

[6] C. Bennett, E. Bernstein, G. Brassard, U. Vazirani, Strengths and weaknesses of quantum computation, SIAM Journal on Computing 26 (5) (1997) 1510-1523.

[7] A. Bertoni, M. Carpentieri, Analogies and differences between quantum and stochastic automata, Theoretical Computer Science 262 (2001) 69-81.

[8] A. Bertoni, M. Carpentieri, Regular Languages Accepted by Quantum Automata, Information and Computation 165 (2001) 174-182.

[9] V. D. Blondel, E. Jeandel, P. Koiran, N. Portier, Decidable and undecidable problems about quantum automata, SIAM Journal on Computing 34 (6) (2005) 1464-1473.

[10] A. Bertoni, C. Mereghetti, B. Palano, Quantum Computing: 1-Way Quantum Automata, in: Proceedings of the 9th International Conference on Developments in Language Theory (DLT’2003), Lecture Notes in Computer Science, Vol. 2710 Springer-Verlag, Berlin, 2003, pp. 1-20.

[11] A. Brodsky, N. Pippenger, Characterizations of 1-way quantum finite automata, SIAM Journal on Computing 31 (2002) 1456-1478. Also quant-ph/9903014, 1999.

[12] E. Bernstein, U. Vazirani, Quantum complexity theory, SIAM Journal on Computing 26 (5) (1997) 1411-1473.

[13] D. Deutsh, Quantum theory, the Church-Turing principle and the universal quantum computer, Proceedings of the Royal Society of London Series A 400 (1985) 97-117.

[14] D. Deutsh, Quantum computational networks, Proceedings of the Royal Society of London Series A 400 (1985) 73-90.

[15] R.P. Feynman, Simulating physics with computers, International Journal of Theoretical Physics 21 (1982) 467-488.
[16] R. Freivalds, Probabilistic two-way machines, in: Proceedings International Symposium on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science, Vol. 188 Springer-Verlag, Berlin, 1981, pp. 33-45.

[17] D.K. Faddeev, V.N. Faddeeva, Computational Methods of Linear Algebra, Freeman, San Francisco, 1963.

[18] M. Golovkins, Quantum Pushdown Automata, in: Proceedings of the 27th Conference on Current Trends in Theory and Practice of Informatics, Milovy, Lecture Notes in Computer Science, Vol. 1963, Spring-Verlag, Berlin, 2000, pp. 336-346.

[19] L.K. Grover, A fast quantum mechanical algorithm for database search, in: Proceedings of the 28th Annual ACM Symposium on Theory of Computing, Philadelphia, Pennsylvania, USA, 1996, pp. 212-219.

[20] S. Gudder, Quantum Computers, International Journal of Theoretical Physics 39 (2000) 2151-2177.

[21] J. Gruska, Quantum Computing, McGraw-Hill, London, 1999.

[22] J. Gruska, Descriptional complexity issues in quantum computing, Journal of Automata, Languages and combinatorics, 5(3): 191-218, 2000.

[23] A. Greenberg, A. Weiss, A lower bound for probabilistic algorithms for finite state machines, Journal of Computer and System Sciences 33 (1) (1986) 88-105.

[24] J.E. Hopcroft, J.D. Ullman, Introduction to Automata Theory, Languages, and Computation, Addision-Wesley, New York, 1979.

[25] T. Koshiba, Polynomial-time Algorithms for the Equivalence for One-way Quantum Finite Automata, in: Proceedings of the 12th International Symposium on Algorithms and Computation (ISAAC’2001), Christchurch, New Zealand, Lecture Notes in Computer Science, Vol. 2223, Spring-Verlag, Berlin, 2001, pp. 268-278.

[26] A. Kondacs, J. Watrous, On the power of finite state automata, in: Proceedings of the 38th IEEE Annual Symposium on Foundations of Computer Science, 1997, pp. 66-75.
[27] L.Z. Li and D.W. Qiu, Determination of equivalence between quantum sequential machines, Theoretical Computer Science 358 (2006) 65-74.

[28] C. Moore and J.P. Crutchfield, Quantum automata and quantum grammars, Theoretical Computer Science 237 (2000) 275-306. Also quant-ph/9707031 1997.

[29] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.

[30] A. Paz, Introduction to Probabilistic Automata, Academic Press, New York, 1971.

[31] D.W. Qiu, Characterization of Sequential Quantum Machines, International Journal of Theoretical Physics 41 (2002) 811-822.

[32] M.O. Rabin, Probabilistic automata, Information and Control 6 (1963) 230-244.

[33] P.W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, SIAM Journal on Computing 26 (5) (1997) 1484-1509.

[34] W.G. Tzeng, A Polynomial-time Algorithm for the Equivalence of Probabilistic Automata, SIAM Journal on Computing 21 (2) (1992) 216-227.

[35] A.C. Yao, Quantum circuit complexity, in: Proceedings of the 34th IEEE Symposium on Foundations of Computer science, 1993, pp. 352-361.