The Logotropic Dark Fluid: Observational and Thermodynamic Constraints

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We have considered a spatially flat, homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) Universe filled with a single fluid, known as logotropic dark fluid (LDF), whose pressure evolves through a logarithmic equation of state. We use the latest Hubble parameter $H(z)$ dataset to constrain the parameters of this model, the present fraction of dark matter $\Omega_{\text{dm}}$ and the Hubble constant $H_0$. We find that the best-fit values of these parameters are $\Omega_{\text{dm}} = 0.254_{-0.027}^{+0.031}$ (1σ) and $H_0 = 70.35_{-2.50}^{+2.49}$ (1σ) $\text{km s}^{-1} \text{Mpc}^{-1}$, which is approximately the mean value of the global and local measurements of $H_0$ at the 1σ confidence level. The best-fit values obtained from this dataset are then applied to examine the evolutionary history of the logotropic equation of state and the deceleration parameter. Our study shows that the Universe is indeed undergoing an accelerated expansion phase following the decelerated one. We also measure the redshift of this transition (i.e., the cosmological deceleration-acceleration transition) $z_t = 0.81\pm0.04$ (1σ error) and is well consistent with the present observations. Interestingly, we find that the Universe will settle down to a $\Lambda$CDM model in future and there will not be any future singularity in the LDF model. Furthermore, we compare the LDF and $\Lambda$CDM models. We notice that there is no significant difference between the LDF and $\Lambda$CDM models at the present epoch, but the difference (at the percent level) between these models is found as the redshift increases. These dynamical features of the LDF can be effective in determining the late-time evolution of the Universe and thus may provide an answer to the coincidence problem. We have also studied the generalized second law of thermodynamics at the dynamical apparent horizon for the LDF model with the Bekenstein and Viaggiu entropies. Our analysis has yielded a thermodynamically allowable range for the dimensionless logotropic temperature $B$, $0 \leq B \leq 0.339$, thereby supporting the value, $B = 3.53 \times 10^{-3}$ obtained by P.H. Chavanis from galactic observations [16].

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I. INTRODUCTION

Many cosmological observations [1–10] have supported that the expansion of the current Universe is accelerating and the alleged acceleration is rather a recent phenomenon. In this context, the most accepted idea is that an exotic component of the matter sector with long range anti-gravity properties, dubbed as dark energy, is responsible for this acceleration mechanism. However, the true nature of dark energy (DE) and dark matter (DM) is still unknown and a plethora of theoretical models has been introduced to account for the observation of cosmic acceleration (for review, one can look into Refs. [11–13]). In the context of DE, the concordance Lambda-Cold-Dark-Matter ($\Lambda$CDM) model is the simplest model and is consistent with most of the observations [10]. However, this model suffers from the fine tuning and the cosmological coincidence problems [14, 15]. Till now, we do not have a concrete cosmological model that can provide a satisfactory solution to all the problems.

Recently, Logotropic Dark Fluid (LDF), a robust and natural candidate for unifying DE and DM, has gained immense interest in the literature [16–22]. An important advantage lies on the fact that it is a consequence of the first principle of thermodynamics. The LDF, proposed by P.H. Chavanis [16, 17], is an attempt towards unification of DM and DE. It belongs to the class of modified matter models in an otherwise flat, homogeneous and isotropic FLRW Universe. It is worth noting that the Logotropic model is almost indistinguishable from the $\Lambda$CDM model for a substantial part of the late-time evolution of the Universe right up to the present time. However, the difference between the two models will be reflected at some point in the future (about 25 Gyrs [18] from now) when the LDF

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behaves as a phantom fluid, while the ΛCDM model will enter in a de Sitter phase. The advantages of considering a LDF is three fold [18] — (a) The speed of sound and the Jeans length are both non-zero in a Logotropic model which might alleviate the cusp problem and the missing satellite problem of the ΛCDM model, (b) Such a model is consistent with the empirical Burkert profile of galaxy rotation curves [23] which are characteristic of most observed DM halos. This is not the case with the ΛCDM model [24], and (c) The universality of the surface density of DM halos [25], the universality of the mass of dwarf spheroidal galaxies [26], and the Tully-Fisher relation [27] are explained neatly by the Logotropic model, as confirmed by analysis of observational data [16, 17]. These remarkable features have placed the LDF in a unique spot amongst other unified models [30–34] which attempt to unify DM and DE.

Motivated by the above facts, in this paper, we consider that the Universe is made of a single dark fluid described by a logotropic equation of state. The cosmological aspects of this model has already been studied in Refs. [16, 17]. In the present work, using the latest H(z) dataset, we try to constrain the model parameters to study the different properties of this model extensively. By considering the Universe as a thermodynamical system, we also study the thermodynamics of the model at the dynamical apparent horizon, particularly, the generalized second law (GSL) of thermodynamics. For this purpose, we consider two different entropies, viz, Bekenstein and Viaggiu entropies [35–37]. With this thermodynamic analysis, we also try to constrain the dimensionless logotropic temperature $B$ whose value has been obtained from the surface density of DM halos by Chavanis [16]. The paper has been organized as follows: The LDF model has been reviewed briefly in Section II. Section III concerns with the observational data analysis and the results of the analysis are presented in detail in Section IV. The GSL has been studied in Section V. Finally, a short discussion with conclusions can be found in Section VI.

II. THE LOGOTROPIC DARK FLUID MODEL

In this section, we study the basic structure of the LDF Model. We assume a homogeneous and isotropic FLRW universe filled with a perfect fluid having energy density $\epsilon(t)$, rest mass density $\rho(t)$, and isotropic pressure $p(t)$. Further, we consider the Universe to be spatially flat as indicated by the anisotropy of the CMBR measurement [28]. Then, the Einstein’s field equations yield the Friedmann and the acceleration equations respectively given by [29]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\epsilon$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\epsilon + 3\frac{p}{c^2}\right)$$

respectively. In the above equation, $a(t)$ denotes the scale factor of the Universe, $H(t) = \left(\frac{\dot{a}}{a}\right)$ denotes the Hubble parameter and an overhead dot represents derivative with respect to the cosmic time $t$. Also, the constant parameters $G$ and $c$ represent the universal gravitational constant and the velocity of light respectively. Now, the energy conservation equation can be obtained as [29]

$$\frac{d\epsilon}{dt} + 3\left(\frac{\dot{a}}{a}\right)\left(\epsilon + \frac{p}{c^2}\right) = 0.$$  

Among the above three equations (equations (1), (2) and (3)), only two are independent equations with three unknown parameters $H$, $\epsilon$ and $p$. So we still have freedom to choose one parameter to close the above system of equations. For the present work, we assume that the Universe is filled with a single dark fluid satisfying an equation of state (EoS) [16–19]

$$p = A \ln\left(\frac{\rho}{\rho_*}\right), \quad A \geq 0$$

which is known as the logotropic equation of state (EoS) and the fluid which obeys this EoS will be called the logotropic dark fluid (LDF). Here, $\rho$ is again the rest mass density, $A$ is the logotropic temperature (see Sec. 3 of Ref. [16]), and

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1 We humbly point out here that there are typos in the equations (2.1), (2.2), and (2.3) of Refs. [16, 18], although the subsequent analyses are not affected by these typos.
\( \rho \) has been identified with the Planck density, \( \rho_\text{p} = 5.16 \times 10^{99} \text{ g m}^{-3} \) (see Sec. 6 of Ref. [16]). The relation between the energy density \( \epsilon \) and the rest mass density \( \rho \) can be evaluated as [16]

\[
\epsilon = \rho c^2 + u(\rho) = \rho c^2 - A \ln \left( \frac{\rho}{\rho_\text{p}} \right) - A,
\]

where \( \rho c^2 \) is the rest mass-energy and \( u(\rho) = -A \ln \left( \frac{\rho}{\rho_\text{p}} \right) - A \), is the internal energy of the LDF respectively. Again, the pressure is related to the internal energy by the relation \( p = -u - A \). Noting that a pressureless matter \( (p = 0) \) gives \( \rho = \rho_0 \left( \frac{a_0}{a} \right)^3 \) from equation (3), we have [16]

\[
\epsilon = \rho_0 c^2 \left( \frac{a_0}{a} \right)^3 - A \ln \left( \frac{\rho_0}{\rho_\text{p}} \left( \frac{a_0}{a} \right)^3 \right) - A,
\]

where the parameters with suffix '0' are their corresponding values at the present epoch. Chavanis [16] has shown that the first term in equation (6) mimics DM and the remaining terms mimics DE. We also observe that the early Universe \( (a \rightarrow 0, \rho \rightarrow \infty) \) was dominated by the rest mass-energy (DM), while the late Universe \( (a \rightarrow \infty, \rho \rightarrow 0) \) is dominated by the internal energy (DE). If we now introduce the dimensionless logotropic temperature \( B = \frac{A}{e_\Lambda} \) and the normalized scale factor \( R = \frac{a}{a_0} \), then equation (6) takes the equivalent form [16]

\[
\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m0}}{R^3} + (1 - \Omega_{m0})(1 + 3B \ln R)
\]

where, \( \epsilon_0 = \frac{3H_0^2 c^2}{8\pi G} \) is the present energy density of the Universe in which \( H_0 \) indicates the present value of the Hubble parameter. \( \epsilon_\Lambda = (1 - \Omega_{m0})\epsilon_0 = \Omega_{\Lambda0}\epsilon_0 \) is the present DE density, with \( \Omega_{m0} \) and \( \Omega_{\Lambda0} \) as the fractions of DM and DE at the present epoch, respectively.

Again, the pressure is related to the scale factor as [16]

\[
p = -\epsilon_0(1 - \Omega_{m0})(B + 1 + 3B \ln R).
\]

Finally using equations (7) and (8), one can obtain the expression for evolution equation of the EoS parameter \( \omega \) for the LDF as

\[
\omega = \frac{p}{\epsilon} = -\frac{(1 - \Omega_{m0})(1 + B + 3B \ln R)}{\frac{\Omega_{m0}}{R^3} + (1 - \Omega_{m0})(1 + 3B \ln R)}.
\]

The deceleration parameter also plays an important role in studying the evolutionary history of the Universe. It is defined as

\[
q = -\frac{\dot{a}}{aH^2} = -\frac{\ddot{H}}{H^2} - 1
\]

with the convention that the Universe will decelerate \( (\ddot{a} < 0) \) for \( q > 0 \), while it will accelerate \( (\ddot{a} > 0) \) for \( q < 0 \). Now, using equations (1) and (7), the expression for the Hubble parameter can be obtained as

\[
H(a) = H_0 \sqrt{\frac{\Omega_{m0}}{R^3} + (1 - \Omega_{m0})(1 + 3B \ln R)}.
\]

Then, using equations (10) and (11), the expression for \( q \) is obtained as

\[
q = \frac{\Omega_{m0} - (1 - \Omega_{m0})(2 + 3B + 6B \ln R)}{2 \left[ \frac{\Omega_{m0}}{R^3} + (1 - \Omega_{m0})(1 + 3B \ln R) \right]}. \tag{12}
\]

In terms of redshift \( z \), equation (11) can be written as

\[
H(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0})(1 - 3B \ln(1 + z))} \tag{13}
\]
where \( R = (1 + z)^{-1} \). It is notable that for \( B = 0 \), the logotropic model reduces to the standard ΛCDM model. It is important to mention here that the parameter \( B \) depends on all the fundamental constants of physics and from now on, we shall regard \( B \) as a fundamental constant (for details, see [16]). As a result, the present model only depends on two cosmological parameters \( H_0 \) and \( \Omega_{m0} \), like the ΛCDM model. This interesting feature allows us to make a very accurate comparison between the two models in order to determine how close they are.

Clearly, the cosmological characteristics of the LDF model given in equation (13) strongly depend on values of the parameters \( H_0 \) and \( \Omega_{m0} \). In the next section, we have constrained these parameters (\( H_0 \) and \( \Omega_{m0} \)) using the latest \( H(z) \) data.

III. OBSERVATIONAL CONSTRAINTS ON THE MODEL PARAMETERS

Being independent observational data, \( H(z) \) determinations have been frequently used in cosmological research. In this section, we shall fit the LDF model with the \( H(z) \) dataset. For completeness, we have also described the latest \( H(z) \) dataset used in our analysis and the \( \chi^2 \) method used to analyze them.

Although the Type Ia Supernova or BAO or CMB dataset is powerful in constraining various cosmological models, but the integration in its formula makes it hard to reflect the precise measurement of the expansion rate of the universe as a function of redshift, i.e., \( H(z) \) \[38\]. This is very challenging as any noise on the measured quantities will be magnified in the integrations. However, the problem can be lessened if direct \( H(z) \) datasets are used because this is the most direct and model independent observable of the dynamics of the universe. Therefore, the structure of the expansion history of the universe can be well indicated by the \( H(z) \) dataset. From the observational point of view, the ages of the most massive and passively evolving galaxies will provide direct measurements of \( H(z) \) at different redshifts \[39\]. It should be noted that \( H(z) \) measurements are always obtained from two different techniques: radial BAO size and galaxy differential age (also known as cosmic chronometer) methods. The Hubble parameter depending on the differential ages as a function of redshift \( z \) can be written in the form of

\[
H(z) = -\frac{1}{(1+z)} \frac{dz}{dt}
\]

It is evident from the above equation that \( H(z) \) can be obtained directly if \( \frac{dz}{dt} \) is known \[40\]. In this work, we have used the latest observational \( H(z) \) dataset consisting of 41 data points in the redshift range, \( 0.07 \leq z \leq 2.36 \), larger than the redshift range that covered by the Type Ia supernova dataset \[41–52\]. To sum up, we have used the following \( H(z) \) data sets:

**Cosmic Chronometer (CC):** 31 CC data points (including 5 new data points obtained by Moresco et al. \[41\]). For more details, one can look into \[41–45\] and the references therein.

**BAO:** 10 radial BAO data points \[46–52\]. Among them, 3 measurements of Hubble parameter are determined based upon radial BAO features in the Lyman–α forest and were compiled by Busca et al.\[50\], Delubac et al.\[51\] and Font-Ribera et al.\[52\].

For this dataset, the \( \chi^2 \) function is defined as

\[
\chi^2_H = \sum_{i=1}^{41} \frac{[H^{obs}(z_i) - H^{th}(z_i, \theta_m)]^2}{\sigma_H^2(z_i)}
\]

where \( \sigma_H(z_i) \) represents the error associated with the \( i^{th} \) data point and \( \theta_m \) denotes the model parameters. In equation (15), the subscript “obs” refers to observational quantities and subscript “th” refers to the corresponding theoretical ones. Then, we use the maximum likelihood method and take the likelihood function as

\[
\mathcal{L} = e^{-\frac{\chi^2}{2}}
\]

\[2\] We assume \( a_0 = 1 \), without any loss of generality.
It should be noted that the best-fit parameter values (say, $\theta^*_m$) are those that maximize the likelihood function (or minimize the $\chi^2$ function)

$$\mathcal{L}(\theta^*_m) = e^{-\frac{1}{2} \left[ \sum_{i=1}^{41} \frac{|H^{obs}(z_i)-H^{th}(z_i,\theta^*_m)|^2}{\sigma_\theta(z_i)} \right]}$$  \hspace{1cm} (17)$$

We can now plot the contours for different confidence levels. The confidence levels 1$\sigma$(68.3%) and 2$\sigma$(95.4%) are taken proportional to $\Delta \chi^2 = 2.3$ and 6.17 respectively, where $\Delta \chi^2 = \chi^2_H(\theta_m) - \chi^2_H(\theta^*_m)$ and $\chi^2_{min}$ is the minimum value of $\chi^2_H$. In what follows, we describe the main observational consequences for the Logotropic model.

**IV. RESULTS OF THE DATA ANALYSIS**

In this section, we have discussed the results obtained from the $\chi^2$ analysis method (as described in the previous section). We have obtained the constraints on the model parameters $\Omega_{m0}$ and $H_0$ by using the latest $H(z)$ dataset. It is important to mention here that for the present analysis and in all the figures (Fig. 1-5), we have considered the value $B = 3.53 \times 10^{-3}$, as predicted by the theory in Refs.[16, 17]. The 1$\sigma$ and 2$\sigma$ contours in $\Omega_{m0} - H_0$ plane for the LDF model is shown in figure 1. The best-fit values for the model parameters are obtained as $\Omega_{m0} = 0.253^{+0.031}_{-0.027}$ and $H_0 = 70.35^{+2.49}_{-2.50}$ km s$^{-1}$ Mpc$^{-1}$ (with $\chi^2_{min} = 30.7$). It has been observed that the value of $\Omega_{m0}$ obtained in this work is lower than the value obtained by the Planck analysis [55], which puts the limit on $\Omega_{m0}$ as $\Omega_{m0} = 0.315 \pm 0.017$ with 1$\sigma$ errors [55]. Interestingly, it has been found that the best-fit value of the parameter $H_0$ obtained in the present analysis is almost same with the value $H_0 = 70.5^{+0.5}_{-0.5}$ km s$^{-1}$ Mpc$^{-1}$, obtained by the Lin et al. [53], using the Pantheon compilation of type Ia supernovae and the non-parametric method. On the other hand, it is well known that there is more than 3$\sigma$ tension between the values of $H_0$ measured from the global CMB radiation ($H_0 = 67.4^{+0.5}_{-0.5}$ km s$^{-1}$ Mpc$^{-1}$ [54], $H_0 = 67.3^{+1.2}_{-1.2}$ km s$^{-1}$ Mpc$^{-1}$ [55]) and that from the local distance ladders ($H_0 = 73.24^{+1.74}_{-1.74}$ km s$^{-1}$ Mpc$^{-1}$ [56]).

The most interesting result of our analysis is that the inferred Hubble constant is approximately the mean value of the global and local measurements of $H_0$, thus alleviating the tension between the global and local measurements. Also, the marginalized likelihoods of individual parameters are shown in figure 2. It is clear from the likelihood plots that the likelihood functions are well fitted to a Gaussian distribution function for the $H(z)$ dataset.

The plot of the deceleration parameter $q(z)$, as given in figure 3, clearly shows that the LDF model successfully generates late time cosmic acceleration ($q < 0$) along with a decelerated ($q > 0$) expansion phase in the past. This is essential for the structure formation of the Universe. It is observed that $q(z)$ shows a signature flip at the transition redshift $z_f = 0.81 \pm 0.04$ (within 1$\sigma$ error). This result is in good agreement with the recent estimate found in Refs. [57-66]. Furthermore, the functional behavior of the equation of state parameter $\omega$ is displayed in figure 4. From this figure, one can clearly observe that for the best-fit model, the value of $\omega$ is close to zero at the high redshifts, at the current epoch (i.e., $z = 0$) it is close to $-0.75$ and settles to a value $-1$ in future. In other words, the figure 4 indicates that model is always non-phantom ($\omega > -1$) at 1$\sigma$ confidence level and thus there is no future singularity in this model. These scenarios also agree very well with the results obtained in Refs. [6, 67, 68].

For a comprehensive analysis, in figure 5, we have also plotted the percentage deviation in the normalized Hubble parameter ($\Delta h(\%) = \Delta \left( \frac{H(z)}{H_0} \right) = \frac{h(z) - H_{\Lambda CDM}(z)}{H_{\Lambda CDM}(z)} \times 100$) for the above model as compared to a $\Lambda$CDM model, and the corresponding deviation is found to be 1.8% at $z \sim 0.2$, 4.5% at $z \sim 0.5$ and 8.6% at $z \sim 1.5$. Again, we have also found that the two models are indistinguishable at present. Therefore, these deviations in the present model also need further attention because the dark components (DE and DM) are two manifestations of the same dark fluid. As a result, it may solve or at least alleviate the cosmological coincidence problem.

**V. GENERALIZED SECOND LAW IN THE LOGOTROPIC MODEL**

This section deals with a study of the *generalized second law* (GSL) of thermodynamics at the dynamical apparent horizon in the Logotropic model. We consider the actual expression (i.e., the non-truncated version) of the Hawking
FIG. 1: This figure shows the 1σ (magenta region) and 2σ (gray region) confidence contours in the $\Omega_{m0}$-$H_0$ plane using the latest $H(z)$ dataset. In this plot, the black dot represents the best-fit values of the pair ($\Omega_{m0}$, $H_0$).

FIG. 2: Left panel shows the marginalized likelihood function vs. $H_0$ and the right panel shows the marginalized likelihood function vs. $\Omega_{m0}$ for the present model.

FIG. 3: The dynamical evolution of $q$ as a function of $z$ is shown in 1σ confidence region. Here, the central dark line denotes the best-fit curve resulting from our joint analysis, while the horizontal line denotes $q(z) = 0$. The intersection of the best-fit curve with the horizontal line corresponds to the point at which the Universe starts accelerating.

Recall that in a flat FLRW universe, the radius $R_{AL}$ of the apparent horizon is simply $R_{AL} = \frac{1}{\pi}$.\footnote{\textsuperscript{3}}
FIG. 4: The evolution of the logotropic equation of state parameter $\omega$ against $z$ is shown in 1σ confidence region for the present model. Here, the central dark line denotes the best-fit curve, while the horizontal line (red dashed) represents the $\Lambda$CDM ($\omega_\Lambda = -1$) model.

FIG. 5: This figure shows the percentage deviation in the normalized Hubble parameter $h$ as function of $z$ compared to the $\Lambda$CDM model. In this plot, we have taken $\Omega_{m0} = 0.315$ [55] for the $\Lambda$CDM model.

where $R_{A_L}$ is the proper radius of the apparent horizon in the Logotropic model. Although, the truncated expression

$$ T_{A_L} = \frac{1}{2\pi R_{A_L}} $$

(19)

is generally used in gravitational thermodynamics but Biétruy and Helou [70, 71] has put forward several strong arguments against using this type of formalism. Also, the use of the former expression has led to some promising results recently [72]. We shall consider two types of entropies on the dynamical apparent horizon, the most widely used Bekenstein entropy [35] and the recently proposed Viaggiu entropy [36, 37]. It is interesting to note that although the Viaggiu entropy is simply a correction to the Bekenstein entropy due to the dynamic nature of the Universe, yet it has yielded a very nice result for a constant EoS parameter [72] which is in striking contrast with that obtained with the Bekenstein entropy [73–75]. With this thermodynamic analysis, we aim to garner support for the choice of the value of the free parameter $B$ in the Logotropic model. Chavanis has already obtained the exact value $3.53 \times 10^{-3}$ for $B$ from the measurement of surface density of DM halos [16] but here we employ ourselves in finding a range of values for $B$ purely by thermodynamic means. If we succeed, the Logotropic model will be put in a much stronger footing. This is due to the well-known fact that there exists an intimate connection between thermodynamics and General Relativity [76–80].

Let us first note that the Bekenstein entropy on the horizon $R_{A_L}$ has the expression [35]

$$ S_{A_L}^B = \left( \frac{c^3}{G \hbar} \right) \frac{A_{A_L}}{4} = \pi R_{A_L}^2 $$

(20)

where $A_{A_L} = 4\pi R_{A_L}^2$ is the proper area bounded by the apparent horizon, while the Viaggiu entropy on the horizon

4 Henceforth, in this section, we assume gravitational units $G = c = \hbar = \kappa_B = 1$. 
\( R_{AL} \) is expressed as \([36, 37, 72]\)

\[
S_{AL}^V = \left(\frac{1}{4L_p^2}\right) A_{AL} + \left(\frac{3\kappa B}{2cL_p^2}\right) V_{AL} H
= \pi R_{AL}^2 + 2\pi R_{AL}^2,
\]

(21)

where \( L_p = \sqrt{\frac{G}{c^2}} \) is the Planck length and \( V_{AL} = \frac{4}{3}\pi R_{AL}^3 \) is the volume bounded by the apparent horizon. The time-derivative of entropy of the fluid inside the horizon, \( \dot{S}_{fAL} \), is evaluated by using the Clausius relation

\[
T_{fAL} dS_{fAL} = dU + pdV_{AL},
\]

(22)

where \( T_{fAL} \) and \( S_{fAL} \) are, respectively, the temperature and the entropy of the fluid, while

\[
U = \frac{4}{3}\pi R_{AL}^3 \epsilon
\]

is the internal energy of the fluid, evaluated at the dynamical apparent horizon.

Using equations (20) and (22), we arrive at the total entropy\(^5\) (for the case with Bekenstein entropy and non-truncated Hawking temperature; equation (19) of Ref. [72])

\[
\dot{S}_{TAL}^B = 18\pi R_{AL} \frac{(1 + w)^2}{(1 - 3w)},
\]

(23)

which shows that the GSL is true for \( w \leq \frac{1}{3} \). On the other hand, using equations (21) and (22), the expression for total entropy becomes (for the case with Viaggiu entropy and non-truncated Hawking temperature; equation (21) of Ref. [72])

\[
\dot{S}_{TAL}^V = 6\pi R_{AL} \frac{(1 + w)(8 + 3w)}{(1 - 3w)},
\]

(24)

from which it has been established [72] that the GSL holds only for \(-1 \leq w \leq \frac{1}{3}\).

We carefully observe here that the upper bounds in both the cases are the same but the Viaggiu entropy, in addition, forces a lower bound on the value of the EoS parameter. After doing some algebra with inequalities, we can restrict the parameter \( B \):

\[
\frac{-4 + 3\Omega_{m0}}{3(1 - \Omega_{m0})} \leq B \leq \frac{\Omega_{m0}}{1 - \Omega_{m0}}
\]

(25)

at the present epoch, \( R = 1 \). For the best-fit model, i.e., \( \Omega_{m0} = 0.253 \), we finally obtain (using equation (25))

\[-1.446 \leq B \leq 0.339.\]

(26)

Since we have considered \( B = \frac{A}{\Lambda} \) and \( A \geq 0 \), we must have \( B \geq 0 \). This implies that \( 0 \leq B \leq 0.339 \). Therefore, the value \( B = 3.53 \times 10^{-3} \) obtained in Ref. [16] from galactic observations, is consistent with thermodynamics. We also observe that the upper bound on \( B \), as given in equation (26), is slightly higher than the corresponding bounds on \( B \) \((0 \leq B \leq 0.09425, 0 \leq B \leq 0.0262 \text{ and } 0 \leq B \leq 0.0379)\), as obtained in Ref. [16] from the galactic observations and from the measurements of the CMB shift parameter respectively. It is also interesting to see that we have obtained a range of allowable values of \( B \) purely by thermodynamic means.

**VI. CONCLUSIONS**

We have considered a spatially flat, homogeneous and isotropic FLRW Universe filled with a single dark fluid, whose pressure evolves through a logarithmic equation of state, as given in equation (4). The theoretical motivations and interesting features of this unified model have already been discussed in details in sections I & II. We have then

\(^5\) In calculating the total entropy, we assume that the temperature of the horizon and that of the fluid inside are equal, in accordance with the pioneering work by Mimoso and Pavón [81].
constrained the free parameters of the model by $\chi^2$ minimization technique using the latest observational dataset of $H(z)$ consisting of 41 data points in the redshift range of $0.07 \leq z \leq 2.36$ [41–52]. In particular, we have obtained $\Omega_{m0} = 0.253_{-0.027}^{+0.031}$ and $H_0 = 70.35_{-2.00}^{+2.49}$ km s$^{-1}$ Mpc$^{-1}$, which is in agreement with the recent estimate obtained $H_0 = 70.5_{-0.5}^{+0.5}$ km s$^{-1}$ Mpc$^{-1}$ in Ref. [53]. As mentioned in section IV, our detailed study shows that the deduced Hubble constant $H_0$ is approximately the mean value of the global and local measurements of $H_0$, and thus alleviating the tension between these measurements. Using the $H(z)$ dataset, we have also investigated the epoch of the DE dominance that drives the accelerated expansion of the Universe. It has been found that the values of the transition redshift (from decelerated to accelerated expansion) obtained within $1\sigma$ confidence level, are in good agreement with the previous results as reported in Refs. [57–66]. It has also been found that the LDF model is always non-phantom ($\omega > -1$) at 1$\sigma$ confidence level and will behave like a $\Lambda$CDM model in future. Additionally, we have also compared the Logotropic and $\Lambda$CDM models in order to determine quantitatively how much they differ. We have found that there is no significant difference between the Logotropic and $\Lambda$CDM models at the present epoch, but the difference between these models is evident at high redshifts (see figure 5). This may provide a possible solution to a number of cosmological problems.

Furthermore, we have undertaken a thermodynamic study of the Logotropic model at the dynamical apparent horizon by considering Bekenstein entropy [35] and Viaggiu entropy [36, 37]. We have restricted our study to the generalized second law of thermodynamics only. As mentioned earlier, the model studied in this work depends on the parameter $B$ (dimensionless logotropic temperature) in such a way that for $B = 0$, the $\Lambda$CDM model is recovered. Using the best-fit value of $\Omega_{m0}$, we have also obtained a thermodynamically allowable range for the parameter $B$, $0 \leq B \leq 0.339$. This result is interesting from both observational as well as cosmological points of view. It is important to note that these bounds support our earlier choice of its value, $B = 3.53 \times 10^{-3}$ for which we have plotted the graphs in section IV. We reiterate here that this particular value was obtained by P.H. Chavanis [16] from the surface density of DM halos.

According to the aforementioned results, we note that the present model is reliable for further study and is compatible with the latest observational dataset. Finally, we conclude that our model seems to represent a viable alternative to the $\Lambda$CDM model.

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