BRAIDED GEOMETRY AND THE INDUCTIVE CONSTRUCTION OF LIE ALGEBRAS AND QUANTUM GROUPS

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Abstract Double-bosonisation associates to a braided group in the category of modules of a quantum group, a new quantum group. We announce the semiclassical version of this inductive construction.

1 Introduction

A question usually overlooked in deformation theory is that of uniformity: we can quantise this or that Poisson manifold, but do our individual quantisations fit together into a coherent 'quantum geometry'? In classical geometry the co-ordinate rings are assumed uniformly to be commutative. When we relax this, each object has many 'directions' in which to become non-commutative and we need to know how to pick these in a coherent way.

This problem is addressed by braided geometry, introduced by the author through about 60 papers since 1989. Rather than deforming one algebra at a time, we deform the tensor product itself; we do group theory and geometry in a braided category in place of Vec. Then all mathematical concepts founded in linear algebra are q-deformed uniformly as we switch on the braiding. Braided geometry has its own method of proofs in which algebraic information 'flows' along braid and tangle diagrams like information in a computer, except that under and over crossings of wires are nontrivial braiding operators $\Psi$. In physical terms, braided geometry is a generalisation of supergeometry with $-1$ in Bose-Fermi statistics replaced by $-1$.
by braid statistics (e.g. by \( q \)). This is conceptually quite different from the usual quantisation picture where \( q = e^{2\pi i} \). But braided-commutative with respect to some \( \otimes_q \) still means non-commutative with respect to the usual \( \otimes \), so we generate noncommutative algebras, which we can then ‘semiclassicalise’ via such an expansion; we do not start with Poisson brackets but rather we generate them, i.e. this is a deeper point of view.

The starting point is the concept of braided group\(^1\) or Hopf algebra in a braided category. This means an algebra and coalgebra \( B \) in the category for which the coproduct \( \Delta : B \to B \otimes B \) is an algebra homomorphism, where \( \otimes \) is the braided tensor product of algebras\(^1\) in a braided category. In concrete terms, \( B \otimes B \) has product \((a \otimes b)(c \otimes d) = a \Psi(b \otimes c)d\).

The simplest example\(^4\) is the tensor algebra \( TV \) on a finite-dimensional vector space \( V \) equipped with a braiding \( \Psi : V \otimes V \to V \otimes V \). Write \( TV = \mathbb{C}\langle x_i \rangle \) and \( \Psi(x_i \otimes x_j) = x_b \otimes x_a R_{i \ j}^{a \ b} \) (with summation of indices), where \( R \) obeys the Yang-Baxter equation. Then the coproduct has the form\(^4\):

\[
\Delta x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{r=0}^{m} x_{j_1} \cdots x_{j_r} \otimes x_{j_{r+1}} \cdots x_{j_m} \left[ m \ ; R \right]_{j_{1 \cdots i_m}}^{j_1 \cdots j_m}.
\]

In his talk, Rosso\(^8\) mentioned the ‘quantum shuffle algebra’ but this is just the graded dual of \( TV \). Its product has just the structure of the coproduct of the latter. Writing \( y^{i_m \cdots i_1} \) for the dual basis to \( x_{i_1} \cdots x_{i_m} \), clearly

\[
y^{i_m \cdots i_{r+1}} y^{i_r \cdots i_1} = \left[ m \ ; R \right]_{j_1 \cdots j_m}^{i_1 \cdots i_m} y^{j_m \cdots j_1}, \quad \Delta y^{i_m \cdots i_1} = \sum_{r=0}^{m} y^{i_m \cdots i_{r+1}} \otimes y^{i_r \cdots i_1}.
\]

From standard properties\(^4\) of these braided binomial matrices \([ m \ ; R ]\),

\[\pi : T(V^*) \to (TV)^*, \quad \pi(y^{i_m} y^{i_{m-1}} \cdots y^{i_1}) = [ m ; R ]_{j_1 \cdots j_m}^{i_1 \cdots i_m} y^{j_m \cdots j_1}\]

is a homomorphism of braided groups, where \([ m ; R ]!\) are the braided factorial matrices and \( T(V^*) = \mathbb{C}\langle y^i \rangle \). Hence ev : \( T(V^*) \otimes TV \to \mathbb{C},\)

\[\text{ev}(f(y), g(x)) = \pi(f(y))(g(x)) = f(\partial)g(x) \mid_{x=0} = f(y)g(\partial) \mid_{y=0} \]

is a duality pairing of braided groups. Here \( \partial \) denotes braided differentiation

\[\partial^j x_{i_1} \cdots x_{i_m} = x_{j_2} \cdots x_{j_m} [ m ; R ]_{j_{1 \cdots i_m}}^{j_2 \cdots j_m}\]

where \([ m ; R ]\) is the braided integer matrix. Similarly for \( \overleftarrow{\partial} \). These are some rudiments of braided geometry on free algebras\(^4\).
The kernels of \( ev \) may be non-zero; quotienting by them gives new braided groups such as the quantum planes \( \mathbb{C}_q^n \). Another choice \( R = R^i_j k_l = \delta^i_j \delta^k_l q^{\beta_{ij}} \), where \( \beta \) is a bilinear form. This is the case considered in [8] and Fronsdal’s talk [10]. When \( \beta \) comes from a Cartan matrix, Lusztig [12] computed \( \ker \pi \) as the \( q \)-Serre relations, i.e. \( T(V^*)/\ker \pi = \text{image}(\pi) = U_q(n_+) \).

## 2 Transmutation and Bosonisation; Induction Principle

General theorems about braided groups are the following. We use Sweedler notation \( \Delta h = h_{(1)} \otimes h_{(2)} \) for coproducts, \( S \) for the antipode and \( \mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \) for Drinfeld’s quasitriangular structure (summations understood).

1. **Transmutation** (SM 1990). Let \( H \) be a quantum group (quasitriangular Hopf algebra). Its *transmutation* is the braided group \( H \in H\mathcal{M} \), the braided category of modules. \( H \) is \( H \) as a module-algebra by \( \text{Ad} \), and

\[
\Delta h = h_{(1)} S \mathcal{R}^{(2)} \otimes \text{Ad}\mathcal{R}^{(1)}(h_{(2)}), \quad \Psi(h \otimes g) = \text{Ad}\mathcal{R}^{(2)}(g) \otimes \text{Ad}\mathcal{R}^{(1)}(h).
\]

2. **Bosonisation** (SM 1991). Let \( B \in H\mathcal{M} \) be a braided group with \( \Delta b = b_{(1)} \otimes b_{(2)} \) and action \( \triangleright \) of \( H \). Its *bosonisation* is the Hopf algebra \( B \triangleright H \) generated by \( H \) as a Hopf algebra, \( B \) as an algebra, and

\[
hb = (h_{(1)} \triangleright b) h_{(2)}, \quad \Delta b = b_{(1)} \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b_{(2)}. \tag{2}
\]

3. **Biproducts** (cf. Radford 1985, SM 1992). Let \( B \in F\mathcal{M} \), the crossed modules over a Hopf algebra \( F \) with bijective \( S \). There is a *biproduct* Hopf algebra \( B \triangleright F \) projecting to \( F \). Every projection to \( F \) is of this form.

4. **Double-bosonisation** (SM 1995). Let \( B' \) be dually paired to \( B \in H\mathcal{M} \) via \( \text{ev} : B \otimes B' \to \mathbb{C} \). There is a quantum group \( B \triangleright H_b \triangleleft B'^{\text{op}} \) containing \( B \triangleright H \) and \( H_b \triangleleft B'^{\text{op}} \) as subHopf algebras, defined by \( (2) \) and

\[
b_{(1)} \mathcal{R}^{(2)} c_{(1)} \text{ev}(\mathcal{R}^{(1)} \triangleright b_{(2)}), c_{(2)} = \text{ev}(b_{(1)}), \mathcal{R}^{(2)} \triangleright c_{(1)} c_{(2)} \mathcal{R}^{(1)} b_{(2)}
\]

\[
hc = (h_{(2)} \triangleright c) h_{(1)}, \quad \Delta c = \mathcal{R}^{(2)} \triangleright c_{(1)} c_{(2)} \mathcal{R}^{(1)}, \quad \mathcal{R}^{\text{new}} = \mathcal{R} \exp^{-1},
\]

where \( \mathcal{R}^{\text{new}} \) needs a canonical element (coevaluation) \( \exp \in B' \otimes B \) for \( \text{ev} \).

5. **Double-biproducts** (SM 1995). Let \( B' \in F\mathcal{M} \) be dually paired to \( B \in F\mathcal{M} \) in 3. as in [6]. There is a Hopf algebra \( B \triangleright F_b \triangleleft B'^{\text{op}} \). A functor \( H\mathcal{M} \to H\mathcal{M} \) allows 2. & 4. to be viewed as special cases of 3. & 5.
Bosonisation has been used to construct inhomogeneous Hopf algebras $C_q^n \triangleright \triangleleft U_q(\tilde{su}_n)$. The $\triangleright \triangleleft$ denotes a central extension. On the other hand, double-bosonisation can be iterated to provide a graph of quantum groups, including the standard families of $U_q(g)$ as well as new quantum groups without classical limit. At each node $H$, the branches are the inequivalent $B \in H\mathcal{M}$. The new node is $B \triangleright \triangleright H \triangleright \triangleright B^{\text{op}}$. The initial node is the quantum group $C$. Its central extension is the quantum line $U_q(1)$. Adjoining the braided line $C_q$ to this yields $U_q(su_2)$. There are several braided groups in the category of $U_q(su_2)$-modules, each yielding a new quantum group. The quantum-braided plane $C_q^2$ gives us $U_q(su_3)$. There are some technicalities, see [3].

The required quantum-braided planes for induction up the A,B,C,D series $U_q(g)$ are known, while the exceptional series are currently under investigation. As there is surely some braided group $B$ in the category of $U_q(e_8)$-modules, we obtain at least one quantum group $B \triangleright \triangleright U_q(e_8) \triangleright \triangleright B^{\text{op}}$ which could be called $U_q(e_9)$! Presumably it does not survive as $q \to 1$. Also, building up $U_q(g)$ inductively by a series of triple products yields automatically a natural inductive block basis for it, which becomes a basis when we fix bases for the braided planes $B$ which are adjoined at each stage. For example,

$$U_q(su_n) = C_q^{n-1} \triangleright \triangleright C_q^{n-2} \triangleright \cdots \triangleright U_q(\beta) \triangleright \triangleright C_q \triangleright \cdots \triangleright C_q^{n-2} \triangleright \triangleright C_q^{n-1}$$

(3)

where the central extensions are collected together as $U_q(\beta) = U(1)^{\otimes n}$ generated by $H_i$ with a quasitriangular structure $R_\beta = q^{\sum \beta_i^j H_i \otimes H_j}$. Here $\beta$ is the symmetrised Cartan matrix. This proves the PBW theorem for $U_q(g)$ and explicitly constructs $U_q(n_+) = C_q^{n-1} \triangleright \triangleright C_q^{n-2} \cdots \triangleright \triangleright C_q$. Choosing bases for the $C_q^i$ gives us a basis for $U_q(n_+)$, as well as all the relations between them (including the $q$-Serre relations when expressed in terms of the simple roots). The inductive basis is coherent across the graph of quantum groups. Moreover, its restriction to any substring of factors gives a sub-braided or quantum group. In (3), $C_q^{n-1} \triangleright \triangleright \cdots \triangleright \triangleright U_q(\beta) \triangleright \triangleright C_q = C_q^{n} \triangleright \triangleright U_q(su_2)$, etc. If one is interested in only half the story, i.e. only in constructing $U_q(b_+)$, one can also do it by iterated biproducts. Thus, $C_q^n \triangleright \triangleright U_q(b_+)$ gives the $q$-Borel of $U_q(su_{n+1})$.

Double-bosonisation also generalises Lusztig’s construction. Any $\beta$ defines a quantum group $U_q(\beta)$ with generators $h_i$ and $R_\beta = q^{\sum \beta_i h_i \otimes h_j}$. $B = \mathbb{C}\langle y^i \rangle$, $B^* = \mathbb{C}\langle x_i \rangle$, paired by (4), live in the category of $U_q(\beta)$-modules by $h_i \triangleright y^i = \delta_{ij} y^j$, $h_i \triangleright x_j = -\delta_{ij} x_j$. So $\mathbb{C}\langle y^i \rangle \triangleright \triangleright U_q(\beta) \triangleright \triangleright \mathbb{C}\langle x_i \rangle$ is a Hopf
algebra. Quotienting by the kernels of ev we obtain a quantum group \( U_q(n_+) > \triangleright U_q(\beta) \triangleleft U_q(n_-) \) with \( R = R_\beta \exp^{-1} \). For generic \( \beta \) (or generic \( R \)-matrix in Section 1) \([m; R]\) are invertible and the coevaluation for (1) is

\[
\exp = \sum_{m=0}^{\infty} x_{i_m} \cdots x_{i_1} (\exp^{-1})_{j_1 \cdots j_m} y_{j_m} \cdots y_{j_1} \in B^\text{op} \otimes B.
\]

Otherwise, quotienting by the kernels is nontrivial but we still have \( \partial, \bar{\partial} \) and the braided exponential \( \exp \) is characterised as their eigenfunction\([7]\).

Note that Fronsdal in his talk and \([10]\) considered recursion relations for an ansatz of the form \( R_\beta f(x, y) \) to obey the Yang-Baxter equation, with resulting Hopf algebra being coboundary. By contrast, double-bosonisation already provides a closed expression for \( R \) via a braided-exponential, proves that quotienting by kernels of ev yields a Hopf algebra and proves that it is quasitriangular. \([6]\) has been circulated in October 1995.

### 3 Braided-Lie Bialgebras and Lie Induction

We now announce a semiclassical concept of braided groups. Let \( g, \delta : g \to g \otimes g, r \in g \otimes g \) be a quasitriangular Lie bialgebra as per Drinfeld\([13]\). Let \( 2r_+ = r + \tau(r) \) where \( \tau \) is transposition. Let \( \triangleright \) denote an action of \( g \).

0. A **braided-Lie bialgebra** \( b \in gM \) is a \( g \)-covariant Lie algebra and \( g \)-covariant Lie coalgebra with cobracket \( \hat{\delta} : b \to b \otimes b \) obeying \( \forall x, y \in b, \)

\[
\hat{\delta}([x, y]) = \text{ad}_x \delta y - \text{ad}_y \delta x - \psi(x \otimes y); \quad \psi = 2r_+ (\triangleright \otimes \triangleright) \circ (\text{id} - \tau),
\]

i.e., \( \text{d} \hat{\delta} = \psi \) where \( \text{d} \) is the Lie coboundary on \( \hat{\delta} \in C^1_{\text{ad}}(b, b \otimes b) \) and \( \text{d} \psi \equiv 0 \).

1. Let \( i : g \to f \) be a map of Lie bialgebras. The **transmutation** of \( f \) is a braided-Lie bialgebra \( \hat{f} \in gM \) with Lie algebra \( f \) and for all \( x \in f, \)

\[
\hat{\delta} x = \delta x + r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x, \quad \triangleright = \text{ad} \circ i.
\]

In particular, \( g \) has a braided version \( \hat{g} \in gM \) by \( \text{ad} \), the same bracket, and

\[
\hat{\delta} x = 2r_+^{(1)} \otimes [x, r_+^{(2)}]. \tag{4}
\]

2. Let \( b \in gM \) be a braided-Lie bialgebra. Its **bosonisation** is the Lie bialgebra \( b \triangleright g \) with \( g \) as sub-Lie bialgebra, \( b \) as sub-Lie algebra and

\[
[x, \xi] = \xi \triangleright x, \quad \delta x = \hat{\delta} x + r^{(2)} \otimes r^{(1)} \triangleright x - r^{(1)} \triangleright x \otimes r^{(2)}, \quad \forall \xi \in g, \ x \in b. \tag{5}
\]
3. Let \( f \) be a Lie bialgebra and \( \overline{f}M \) its category of Lie crossed modules (=modules of the Drinfeld double \( D(\overline{f}) \)). Objects \( b \) are simultaneously \( f \)-modules \( \triangleright \) and \( f \)-comodules \( \beta : b \to f \otimes b \) obeying \( \forall \xi \in f, x \in b, \)

\[
\beta(\xi \triangleright x) = ([\xi, ] \otimes \text{id} + \text{id} \otimes [\xi \triangleright ]) \beta(x) + (\delta \xi) \triangleright x.
\]

Writing \( \beta(x) = x^{(1)} \otimes x^{(2)} \), the infinitesimal braiding in this category is

\[
\psi(x \otimes y) = y^{(1)} \triangleright x \otimes y^{(2)} - x^{(1)} \triangleright y \otimes x^{(2)} - y^{(2)} \otimes y^{(1)} \triangleright x + x^{(2)} \otimes x^{(1)} \triangleright y.
\]

Let \( b \in \overline{f}M \) be a braided-Lie bialgebra. The \textit{bism} Lie bialgebra \( b \triangleright g \) has semidirect Lie bracket/cobracket and projects onto \( f \). Any Lie bialgebra projecting onto \( f \) is of this form. A functor \( gM \to \overline{g}M \) relates 2. & 3.

4. Let \( b^* \in gM \) be a braided-Lie bialgebra dually paired with \( b \) by invariant \( \text{ev} : b \otimes b^* \to \mathbb{C} \). Its \textit{double-bosonisation} is the Lie bialgebra \( b \triangleright g \triangleright b^* \text{op} \) with \( g \) as sub-Lie bialgebra, \( b, b^* \text{op} \) sub-Lie algebras, \( \{f^a\} \) and

\[
[\xi, \phi] = \xi \triangleright \phi, \quad [x, \phi] = \text{ev}(x_{(1)}, \phi)x_{(2)} + \text{ev}(x, \phi_{(1)})\phi_{(2)} + 2r_+^{(1)}\text{ev}(x, r_+^{(2)} \triangleright \phi)
\]

\[
\delta \phi = \delta \phi + r^{(2)} \triangleright \phi \otimes r^{(1)} - r^{(1)} \otimes r^{(2)} \triangleright \phi, \quad r^{\text{new}} = r - \sum_a f^a \otimes e_a,
\]

\( \forall x \in b, \xi \in g \) and \( \phi \in b^* \). Here \( \delta x = x_{(1)} \otimes x_{(2)}, \) etc., and \( r^{\text{new}} \) assumes that \( \text{ev} \) has a coevaluation, i.e. if \( \{e_a\} \) is a basis of \( b \) then \( \{f^a\} \) is dual w.r.t. \( \text{ev} \).

Double bosonisation provides an inductive construction for quasitriangular Lie bialgebras, preserving factorisability (nondegeneracy of \( r_+ \)). It is a co-ordinate free version of the idea of adjoining a node to a Dynkin diagram (adjoining a simple root vector in the Cartan-Weyl basis). Moreover, building up \( g \) iteratively like this also builds up the quasitriangular structure \( r \). The braided-Lie bialgebra used in the induction could be trivial:

**Proposition.** Let \( g \) be a semisimple factorisable (s.s.f) Lie bialgebra and \( b \) a faithful isotypical representation such that \( \Lambda^2 b \) is isotypical. Then \( b \) with zero bracket and zero cobracket is a braided-Lie bialgebra in \( \overline{g}M \), and \( b \triangleright \overline{g} \triangleright b^* \text{op} \) is another s.s.f. Lie bialgebra. Here \( \overline{g} \) is a central extension.

The induction also works at the simple strictly quasitriangular level (with \( b \) irreducible). For example, the 2-dimensional and 3-dimensional representations of \( su_2 \) have the required property
(ensuring $\psi \propto (\text{id} - \tau)$ in $gM$). They give $su_3$ and $so_5$, taking us up the $A$ and $B$ series respectively.

Finally, just as Lie bialgebras extend to Poisson-Lie groups, so braided-Lie bialgebra structures generally extend to the associated Lie group of $g$. The resulting Poisson bracket does not, however, respect the group product in the usual way but rather up to a ‘braiding’ obtained from $\psi$.

**Example.** The transmutation (8) of the Drinfeld-Sklyanin (or other factorisable) Lie cobracket on semisimple $g$ is the Kirillov-Kostant Lie cobracket. Moreover, this *Kirillov-Kostant braided-Lie bialgebra* extends, in principle, to a braided-Poisson Lie group. Details are to appear elsewhere.

**References**

[1] S. Majid. Braided groups and algebraic quantum field theories. *Lett. Math. Phys.*, 22:167–176, 1991.

[2] S. Majid. Transmutation theory and rank for quantum braided groups. *Math. Proc. Camb. Phil. Soc.*, 113:45–70, 1993.

[3] S. Majid. Cross products by braided groups and bosonization. *J. Algebra*, 163:165–190, 1994.

[4] S. Majid. Free braided differential calculus, braided binomial theorem and the braided exponential map. *J. Math. Phys.*, 34:4843–4856, 1993.

[5] S. Majid. Algebras and Hopf algebras in braided categories. volume 158 of *Lec. Notes in Pure and Appl. Math*, pages 55–105. Marcel Dekker, 1994.

[6] S. Majid. Double bosonisation and the construction of $U_q(g)$. Preprint [q-alg/9511001], 1995.

To appear *Math. Proc. Camb. Phil. Soc.*

[7] S. Majid. *Foundations of Quantum Group Theory*. Cambridge Univ. Press, 1995.

[8] M. Rosso. Groupes quantiques et algébres de battage quantiques. *C. R. Acad. Sci.*, 320:145–148, 1995.
[9] S. Majid. C-statistical quantum groups and Weyl algebras. *J. Math. Phys.*, 33:3431–3444, 1992.

[10] C. Fronsdal. Generalisation and exact deformations of quantum groups. Preprint [alg/9606020](https://arxiv.org/abs/alg/9606020), 1996.

[11] D. Radford. The structure of Hopf algebras with a projection. *J. Algebra*, 92:322–347, 1985.

[12] G. Lusztig. *Introduction to Quantum groups*. Birkhauser, 1993.

[13] V.G. Drinfeld. Quantum groups. In *Proc. ICM*, pages 798–820. AMS, 1987.