A NOTE ON CERTAIN COEFFICIENTS OF ERGODICITY

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Abstract. We investigate the ergodicity coefficients introduced by Rothblum et al in [6] and [3]. A complex matrix version of the farmer is proposed in [5] and we provide a limit property for it, accordingly with a known limit property of the original coefficient. Then we propose a generalization of the coefficient in [3] and we show that it can be used to relate the eigenvector problem $Ax = x$ to the solution of a particular $M$-matrix linear system.

1. Introduction. In this work we analyze two particular ergodicity coefficients essentially introduced by Rothblum et al. in [6] and [3] and subject of the recent comprehensive study made by Ipsen and Selee in [5]. The paper is divided into two main sections:

In Section 2 we consider a possible extension of the ergodicity coefficients $\mu_{\|\|}$ and $\tau_{\|\|}$, introduced respectively by Seneta in [8] and Rothblum and Tan in [6], defined over complex matrices $A$. We denote such generalized coefficient by $\phi_{\|\|}$. A similar coefficient has been recently considered by Ipsen and Selee who show, in particular, that it provides an upper bound for the absolute value of a generic eigenvalue of $A$ [5, Thm. 7.4]. We obtain the same bound but in a slightly different way, moreover in Theorem 2.4 we observe that the modulus of a generic eigenvalue of $A$ can be represented as a limit involving $\phi_{\|\|}$, accordingly with an analogous limit property holding for the original functionals $\mu_{\|\|}$ and $\tau_{\|\|}$ (see [8, pp. 579-580] and [6, Thm. 3.3] respectively).

In Section 3 we analyze another coefficient of ergodicity $\tau$ which was defined by Rothblum et al. in [3] for nonnegative matrices having a positive dominant eigenvector. We observe that such $\tau$ can be defined in a different way and propose a possible generalization, moreover we show that when $A$ is stochastic, $\tau(A)$ can be used to compute the dominant eigenvector $x$ such that $Ax = x$, by solving of a suitable $M$-matrix linear system of equations.

1.1. Notations and Preliminaries. To avoid ambiguity and repetitions we briefly fix here our notation. The symbol $M_n(F)$ denotes the set of square $n \times n$ matrices over the field $F$. For our purposes $F$ will be either $\mathbb{C}$ or $\mathbb{R}$. For $A \in M_n(F)$ we let $\sigma(A)$ be the set of its eigenvalues, $\lambda(A)$ be the set of its distinct eigenvalues and $\rho(A) = \max_{\lambda \in \lambda(A)} |\lambda|$ be its spectral radius. For $\lambda \in \lambda(A)$ we let $g_A(\lambda)$ and $a_A(\lambda)$ denote the geometric and algebraic multiplicities, respectively.

e denotes the vector of all ones.

If $\|\|$ is any norm on the vector space $V$, we write $B^1_{\|\|}(V)$ to denote the unit ball in $V$ with respect the norm $\|\|$, that is $B^1_{\|\|}(V) = \{x \in V \mid \|x\| \leq 1\}$.

A real matrix $A$ is called

- nonnegative if its entries are nonnegative numbers, in symbols $A \geq O$ (analogous notation is used for vectors)
- positive if its entries are positive numbers, in symbols $A > O$ (analogous notation is used for vectors)
- reducible if there exists a permutation $P$ such that $PAP^T$ is block triangular
- irreducible if it is not reducible

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We shall make freely use of the well known Perron-Frobenius theorem for nonnegative matrices, which is briefly and partially recalled for completeness: If \( A \geq O \) then \( \rho(A) \in \lambda(A) \) and there exists nonnegative left and right eigenvectors of \( \rho(A) \). If \( A \geq O \) is irreducible, then \( \rho(A) \) is simple, positive and its corresponding left and right eigenvectors can be chosen positive. Finally, \( A \) is primitive\(^1\) if and only if \( A \) is nonnegative, the left end right eigenvectors relative to \( \rho(A) \) can be chosen positive and no other eigenvalues of \( A \) have modulus \( \rho(A) \), except for \( \rho(A) \) itself.

2. Limit properties of the ergodicity coefficients for complex matrices.

In [5, Sec. 7] Ipsen and Selee give a generalization of some known results [8],[6], concerning the second largest eigenvalue of a nonnegative matrix, essentially extending to complex matrices the definition and the applicability of the coefficient of ergodicity \( \tau \), defined by a specific vector norm \( \| \| \), firstly proposed by Seneta in [8] and then deeply investigated by Rothblum and Tan in [6]. In particular, Thm. 3.3 in the latter paper it is shown that the modulus of the second larger eigenvalue of a nonnegative irreducible matrix can be expressed in terms of a limit involving the powers of \( \tau \).

In this section we complete [5] by essentially obtaining the same limit property of [6, Thm. 3.3] for complex matrices, by means of a generalized version of \( \tau \) which we denote hereafter by \( \phi \). To this end we shall introduce some preliminary results.

Given a \( n \times n \) complex matrix \( A \) let \( \nu \subseteq \lambda(A) \) be any subset of distinct eigenvalues of \( A \), and let \( \overline{\nu} = \lambda(A) \setminus \nu \) be its complement in \( \lambda(A) \). For \( \lambda \in \lambda(A) \) let \( J(\lambda) \) be the direct sum of the elementary Jordan blocks corresponding to \( \lambda \). Define

\[
\rho(\nu) = \max_{\lambda \in \nu} |\lambda|
\]

and let \( J(\nu) \) be the direct sum of the blocks \( J(\lambda) \) relative to the eigenvalues in \( \nu \), namely \( J(\nu) = \bigoplus_{\lambda \in \nu} J(\lambda) \). Thus, for any \( \nu \subseteq \lambda(A) \), the Jordan canonical form of \( A \) can be assumed to be as follows

\[
A = XJX^{-1} = \left[ \begin{array}{cc} W_{\nu} & W_{\overline{\nu}} \\ \ \end{array} \right] \left[ \begin{array}{c} J(\nu) \\ J(\overline{\nu}) \end{array} \right] \left[ \begin{array}{cc} W_{\nu} & W_{\overline{\nu}} \end{array} \right]^{-1}
\]

where \( W_{\nu} = W_{\nu}(A) \) is \( n \times \sum_{\lambda \in \nu} g_A(\lambda) \) and its columns are linearly independent.

Note that the Jordan space range(\( W_{\nu} \)) contains the invariant space \( \bigoplus_{\lambda \in \nu} \ker(A - \lambda I) \) spanned by \( \sum_{\lambda \in \nu} g_A(\lambda) \) columns of \( W_{\nu} \).

We shall commonly write \( W_{\nu} = W_{\nu}(A) \) to underline the matrix \( A \) to which \( W_{\nu} \) corresponds, even if it might be easily derived by the context.

**Definition 2.1.** Given \( A \in M_n(\mathbb{C}) \) let \( \nu \) be any subset of \( \lambda(A) \) and let \( W_{\nu} = W_{\nu}(A) \) be the matrix whose columns are the Jordan vectors relative to the elements in \( \nu \) (see Sec. 1.1). For any norm \( \| \| \) over \( \mathbb{C}^n \) we define

\[
\phi_{\|W_{\nu}\|}(W_{\nu}, A) = \max_{x \in B_1(\mathbb{C}^n) \cap \ker W_{\nu}^*} \|x^*A\|
\]

Such \( \phi_{\|W_{\nu}\|} \) is a continuous scalar function and is the probably the most general form of an ergodicity coefficient defined by a vector norm. It is sometimes denoted by \( \tau_{\|W_{\nu}\|} \) ([5, Sec. 7], [6, Sec. 7] or [2, Sec. 3] for instance) but here we prefer to use a different symbol for the sake of clearness, whereas we use the symbol \( \tau_{\|W_{\nu}\|} \) to denote the coefficient introduced in [6, p. 59].

\(^1\) A matrix \( M \) is primitive if \( M \geq O \) and there exists \( k \geq 1 \) such that \( M^k \) is positive.
In what follows \( \mathcal{K} \) is any closed bounded subset of \( \mathbb{C}^n \) with nonempty interior and containing properly the null vector (i.e. the null vector is in \( \mathcal{K} \) but does not lie on its boundary). Also \( \| \| \) is any norm over \( \mathbb{C}^n \).

**Lemma 2.2.** Let \( A \in \mathbb{M}_n(\mathbb{C}) \), \( \mathcal{K} \subseteq \mathbb{C}^n \) be any compact set that contains properly the origin, \( \nu \subseteq \lambda(A) \) and \( W_{\nu} \) defined as above. Then

\[
\rho(\mathcal{V}) \geq \limsup_{k \to \infty} \left( \max_{x \in \mathcal{K} \cap \ker W_{\nu}^*} \| x^* A^k \| \right)^{1/k}
\]

**Proof.** Set \( \Omega = \mathcal{K} \cap \ker W_{\nu}^* \). If \( x \in \Omega \) then

\[
x^* A^k = x^* X J^k X^{-1} = \begin{bmatrix} 0 & x^* W_{\nu} \end{bmatrix} J^k X^{-1}
\]

since of course \( x \in \ker W_{\nu}^* \) if and only if \( x^* W_{\nu} = 0 \). It is clear therefore that replacing the Jordan block \( J(\nu)^k \) inside \( J^k \) with the zero matrix, does not affect the product \( x^* A^k \). Now let \( \mu > \rho(\mathcal{V}) \) and define the functions

\[
f_k : \mathbb{C}^n \to \mathbb{R}^+, \quad f_k(x) = \left\| \frac{x^* A^k}{\mu^k} \right\|
\]

Note that \( f_k(x) \) converges punctually to zero for any \( x \in \Omega \). Also note that \( x_0^{(k)} = \arg\left( \sup_{x \in \Omega} f_k(x) \right) \) belongs to \( \Omega \), therefore \( f_k(x) \) converges uniformly as well. As a consequence \((f_k(x))_k\) is bounded in \( \Omega \), i.e. there exists \( M > 0 \) such that

\[
\forall \mu > \rho(\mathcal{V}), \quad \sup_{x \in \Omega} \left\| \frac{x^* A^k}{\mu^k} \right\| = \left( \sup_{\Omega} f_k \right) \leq M
\]

Now, taking the infimum over \( \mu \) and then the \( \limsup \) over \( k \), we eventually get

\[
\rho(\mathcal{V}) = \limsup_{k \to \infty} \left( \max_{x \in \mathcal{K} \cap \ker W_{\nu}^*} \| x^* A^k \| \right)^{1/k}
\]

and this finally proves the statement. \( \square \)

The following result is an easy generalization of [5, Thm. 7.4]

**Lemma 2.3.** Let \( A \in \mathbb{M}_n(\mathbb{C}) \), \( \nu \subseteq \lambda(A) \) and let \( \mathcal{K} \) be any compact set in \( \mathbb{C}^n \) containing the origin. If \( w_1, \ldots, w_p \) are some columns of \( W_{\nu} \), \( p \geq 1 \), set \( W_{\nu,p} = \begin{bmatrix} w_1 & \cdots & w_p \end{bmatrix} \). Then

\[
\left( \min_{y \in \partial \mathcal{K}} \| y \| \right) \cdot \rho(\mathcal{V})^k \leq \max_{x \in \mathcal{K} \cap \ker W_{\nu,p}^*} \| x^* A^k \|
\]

for any integer \( k \geq 1 \).

**Proof.** Let \( \lambda_* \) be such that \( |\lambda_*| = \rho(\mathcal{V}) \). The set \( \partial \mathcal{K} \cap \ker W_{\nu,p}^* \) obviously contains an eigenvector \( u \) of \( \lambda_* \), that is there exists \( u \in \partial \mathcal{K} \cap \ker W_{\nu,p}^* \) such that \( u^* A = \lambda_* u \). Then \( u^* w_i = 0 \) for any \( i \), due to the fact that the left (right) eigenvectors of one eigenvalue are orthogonal to the right (left) eigenvectors of any other eigenvalue. The proof is now straightforward since

\[
\rho(\mathcal{V})^k \min_{y \in \partial \mathcal{K}} \| y \| = \min_{y \in \partial \mathcal{K}} \| \lambda_*^k y \| \leq \| \lambda_*^k u \|
\]

and of course \( u^* A^k = \lambda_*^k u \). \( \square \)
We are ready to state the limit property for $\phi_{\| \|}$ that we announced.

**Theorem 2.4.** Let $A \in M_n(C)$, for any $\nu \subseteq \lambda(A)$ we have

$$\rho(\mathcal{F}) = \lim_{k \to \infty} (\phi_{\| \|}(W_{\nu}, A^k))^\frac{1}{k}$$

**Proof.** Using Lemma 2.3 choosing $p$ as the whole number of columns in $W_{\nu}$, $\mathcal{X} = B_1^{\|}(C^n)$, and taking the infimum over $k$, one gets $\rho(\mathcal{F}) \leq \liminf_k (\phi_{\| \|}(W_{\nu}, A^k))^\frac{1}{k}$. Analogously we get $\rho(\mathcal{F}) \geq \limsup_k (\phi_{\| \|}(W_{\nu}, A^k))^\frac{1}{k}$ by Lemma 2.2 with the same choice for $\mathcal{X}$. Combining the two bounds we conclude. \[\Box\]

The functional $\phi_{\| \|}$ is a generalized version of the ergodicity coefficients introduced by Seneta and Rothblum [8, 6]. Given an irreducible non-negative matrix $A$ they considered the following functionals

$$\mu_{\| \|}(x, A) = \max_{\|y\| \leq 1} \|y^T A\|, \quad \tau_{\| \|}(x, A) = \max_{\|y\| \leq 1} \|y^T A\|$$

where $x$ is a real non-negative right eigenvector of $A$ corresponding to the eigenvalue $\rho(A)$, i.e. $Ax = \rho(A)x$, $x \geq 0$. In [8, pp. 579-580] Seneta observed that $\mu_{\| \|}(x, A)$ is a bound for the second largest modulus of the eigenvalues of $A$. One can easily derive this fact by noting that $\phi_{\| \|}(W_{\rho(A)}, A) \leq \mu_{\| \|}(x, A)$, for any non-negative matrix $A$, whereas the equality holds when $A$ is irreducible.

More precisely, Lemma 2.3 and Theorem 2.4 imply that

$$\max_{\lambda \in \sigma(A) \setminus \{\rho(A)\}} |\lambda| = \phi_{\| \|}(W_{\rho(A)}, A^k)^{1/k} \leq \mu_{\| \|}(x, A)^{1/k}, \quad \forall k \geq 1$$

and

$$\max_{\lambda \in \sigma(A) \setminus \{\rho(A)\}} |\lambda| = \lim_{k \to \infty} \phi_{\| \|}(W_{\rho(A)}, A^k)^{1/k} \leq \lim_{k \to \infty} \mu_{\| \|}(x, A)^{1/k}$$

(2.1)

Again, equality holds in (2.1) when $A$ is irreducible.

Rothblum and Tan [6, Thm. 3.3] have shown that the same limit property holds for $\tau_{\| \|}$ (see Corollary 2.6 below). Using their approach here we observe that their result follows from Theorem 2.3.

Given any vector $x \in C^n$, decompose it as the sum $x = \text{Re} \ x + i \text{Im} \ x$, where $\text{Re} \ x$ and $\text{Im} \ x$ are the unique real vectors defined in the obvious way by the real and imaginary parts of the entries of $x$. Consider the following functional over $C^n$

$$\|x\|_\square = \sup_{\alpha \in \mathbb{R}} \|\text{Re} \ x \cos \alpha + \text{Im} \ x \sin \alpha\|,$$

and observe that $\| \|_\square$ is a norm, if and only if $\| \|$ is a norm.

**Lemma 2.5.** Let $A \in M_n(\mathbb{R})$, $\nu \subseteq \lambda(A)$, and assume that $p$ columns of $W_{\nu}$ are real, for some $p \geq 1$. Let $w_1, \ldots, w_p$ be such columns. If $W_{\nu,p} = \left[ \begin{array}{c} w_1 \cdots w_p \end{array} \right]$, then

$$\max_{x \in B_1^{\|}(C^n) \cap \ker W_{\nu,p}^T} \|x^* A\|_\square = \max_{x \in B_1^{\|}(\mathbb{R}^n) \cap \ker W_{\nu,p}^T} \|x^T A\|_\square.$$
Proof. Since \( W_{\nu,p} \) is real, we have \( x \in \ker W_{\nu,p}^T, y \Rightarrow \Re x, \Im x \in \ker W_{\nu,p}^T, \) therefore \((\Re x \cos \alpha + \Im x \sin \alpha) \in \ker W_{\nu,p}^T,\) As a consequence, for any \( x \in \ker W_{\nu,p}, \) \( \|x\|_{\Box} \leq 1,\)

\[
\|x^* A\|_{\Box} = \sup_{\alpha \in \mathbb{R}} \|\Re x \cos \alpha + \Im x \sin \alpha\|^T A \leq \max_{x \in B_1^\|\|} \|z^T A\|
\]

and the thesis follows by taking the maximum over \( x \in B_1^\|\| (\mathbb{C}^n) \cap \ker W_{\nu,p}^T. \)

As announced, we conclude this section by noting that a straightforward application of Lemma 2.2, for \( K = B_1^\|\| (\mathbb{R}^n), \) Theorem 2.3 and Lemma 2.5 with \( \| \cdot \| = \| \cdot \|_{\Box}, \) implies

**Corollary 2.6** ([6], Thm. 3.1, Thm. 3.3 and Thm. 7.1). Let \( A \) be real and \( \nu \subseteq \lambda(A). \) Assume that \( p \) columns of \( W_{\nu} \) are real, for some \( p \geq 1. \) Let \( w_1, \ldots, w_p \) be such columns. If \( W_{\nu,p} = \begin{bmatrix} w_1 & \cdots & w_p \end{bmatrix}, \) then

\[
\rho(\mathcal{X}) \leq \max_{x \in B_1^\|\| (\mathbb{R}^n) \cap \ker W_{\nu,p}^T} \|x^T A\|.
\]

In particular, if \( A \geq 0, \) then

\[
\max_{\lambda \in \sigma(A) \setminus \{\rho(A)\}} |\lambda| \leq \max_{z \in B_1^\|\| (\mathbb{R}^n) \cap \ker x^T} \|z^T A\| = \tau_{\|\|}(x, A)
\]

where \( x \) is a real nonnegative eigenvector of \( A \) corresponding to the eigenvalue \( \rho(A). \) Moreover, if \( A \) is irreducible, then

\[
\max_{\lambda \in \sigma(A) \setminus \{\rho(A)\}} |\lambda| = \lim_{k \to \infty} \tau_{\|\|}(x, A^k)^{1/k}
\]

### 3. Nonnegative matrices with a positive dominant eigenvector

Consider now a nonnegative matrix \( A \) and let \( y \geq 0 \) be a left dominant eigenvector, i.e. \( y^T A = \rho(A) y^T. \) When \( y > 0 \) \(^2\) we may consider another functional \( \tau(A, y) \) introduced by Haviv, Ritov and Rothblum in [3], given by

\[
\tau(A, y) = \rho(A) - \sum_{i=1}^n \left( \min_{j=1,\ldots,n} y_j^{-1} a_{ij} \right) y_i \tag{3.1}
\]

which has properties somehow analogous to those of \( \tau_{\|\|}, \) since

\[
|\lambda_2(A)| \leq \tau(A, x) \quad \text{and} \quad |\lambda_2(A)| = \lim_{k \to \infty} \tau(A^k, x)^{1/k}. \tag{3.2}
\]

Now let \( S^+ \) be the set of nonnegative column stochastic matrices, that is \( S \in S^+ \) if and only if \( S \geq O \) and \( e^T S = e^T, \) and let \( A^+ \) be the set of nonnegative and nonnull matrices having a positive dominant left eigenvector. It is known that any \( A \in A^+ \) is similar, via a positive definite diagonal similarity, to a nonnegative matrix whose columns sum up to \( \rho(A) \) (see f.i. [10, Cor. 4.15]). Thus

\[
A^+ = \{ \mu D S^+ D^{-1} \mid \mu > 0, D \text{ diagonal}, d_{ii} > 0 \ \forall i \}
\]

As a consequence the spectrum of any \( A \in A^+ \) is the spectrum of an element of \( S^+ \) scaled by a positive factor. It follows that, if \( y > 0 \) is a left dominant eigenvector of

\(^2\text{Note that this implies } \rho(A) > 0, \text{ unless } A = O.\)
A ∈ A⁺, and S ∈ S⁺ is such that DSD⁻¹ = ρ(A)⁻¹A (note that such an S may be not uniquely defined since g_A(ρ(A)) ≥ 1 in general), then
\[ \tau(A, y) = ρ(A)τ(DSD⁻¹, y) = ρ(A)τ_{n−1}(S) \]

where \( τ_{n−1}(S) = τ(S, e) = 1 − \sum_i \min_j s_{ij} \).

So we reduce our attention to \( S⁺ \) and to \( τ_{n−1} : S⁺ → ℝ \) rather than on (3.1). Such \( τ_{n−1} \) is sometimes denoted by \( β \) (see f.i. [5, p. 157] or [9, p. 137]) and is one of the earlier proper ergodicity coefficients. A proper ergodicity coefficient was originally defined as a continuous scalar function \( f \) from \( S⁺ \) to the real interval \([0, 1]\) and such that \( f(M) = 0 \) if and only if \( M \) is a rank-one matrix in \( S⁺ \) [7, p. 509]. In order to justify our choice of the symbol \( τ_{n−1} \) and to recall that \( τ_{n−1} \) is a proper coefficient, we state the following

**Proposition 3.1.** Let \( A \) be a matrix of \( S⁺ \). Then

- \( τ_{n−1}(A) = \max_{j=1,...,n} \sum_{i\in V} a_{ij} − \sum_{i\in V} \min_{k=1,...,n} a_{ik} \), for any subset \( V \subset \{1, ..., n\} \) such that \(|V| = n−1\).
- \( τ_{n−1} : S⁺ → [0, 1] \). In particular \( τ_{n−1}(A) = 0 \) if and only if \( A = \frac{1}{n} eeᵀ \), whereas \( τ_{n−1}(A) = 1 \) if and only if each row of \( A \) has at least one zero entry (or, in other words, \( τ_{n−1}(A) < 1 \) if and only if \( \max_i \min_j a_{ij} > 0 \)).

**Proof.** Let \( A ∈ S⁺ \) and \( \{1, ..., n\} = V ∪ \{h\} \), then \( \max_j \sum_{i\in V} a_{ij} = 1 − \min_j a_{hj} \). This proves the first statement. Obviously \( 0 ≤ τ_{n−1}(A) ≤ 1 \) and \( τ_{n−1}(A) = 1 \iff \max_i \min_j a_{ij} = 0 \). Finally, if \( τ_{n−1}(A) = 0 \) then we see from its definition that \( 1 = \sum_i a_{ij} = \sum_i \min_j a_{ij} \) for all \( j = 1, ..., n \). Therefore \( A \) must be entrywise constant, i.e. \( A = \frac{1}{n} eeᵀ \). The vice versa is straightforward.

Therefore we can rewrite \( τ_{n−1}(A) \) as the maximum of a certain function of the entries of \( A \) over the subsets \( V \subset \{1, ..., n\} \) of cardinality \( n−1 \). This fact immediately suggests a generalization to \( τ_m \) as

\[ τ_m(A) = \max_{V \subset \{1, ..., n\}} \max_{|V| = m} \sum_{i\in V} \left( a_{ij} − \min_{k\in\{1, ..., n\}} a_{ik} \right) \]  

(3.3)

and we observe that

**Proposition 3.2.** For any \( A ∈ S⁺, k ≤ m \implies τ_k(A) ≤ τ_m(A) \)

**Proof.** Let \( V_k \subset \{1, ..., n\}, |V_k| = k \), be the subset realizing \( τ_k(A) \). For any \( V_m ≥ V_k \) we have

\[ τ_k(A) = \max_{j} \sum_{i\in V_k} (a_{ij} − \min_j a_{ij}) \]
\[ ≤ \max_{j} \left\{ \sum_{i\in V_k} (a_{ij} − \min_j a_{ij}) + \sum_{i\in V_m \setminus V_k} (a_{ij} − \min_k a_{ik}) \right\} \]
\[ = \max_{j} \sum_{i\in V_m} (a_{ij} − \min_j a_{ij}) \]
\[ ≤ τ_m(A) \]

since \( a_{ij} ≥ \min_k a_{ik} \) for any \( i \) and \( j \).

Due to the above proposition and to the fact that \( |λ_2(A)| ≤ τ_{n−1}(A) \), one may guess that the inequality \( |λ_k(A)| ≤ τ_{n−k+1}(A) \) holds for any \( k ≥ 2 \). Moreover observe that \( τ_1 \) is a proper coefficient of ergodicity. In fact \( τ_1(A) = 0 \) if and only if \( A \) has all constant rows, that is \( τ_1(A) = 0 \) if and only if \( A \) is a rank one matrix \( A = xeᵀ \) where \( x ≥ 0 \) and \( \sum_i x_i = 1 \), thus \( λ_n(A) = 0 \). This seems to confirm the guess leaving us with a nontrivial open question. Nonetheless we firmly think that further investigations on this direction would be of significant interest. Observe that if in the definition (3.3)
of $\tau_m$ we replace $\max\nu$ with $\min\nu$, yet we have that $|\lambda_2(A)| \leq \tau_{n-1}(A)$, that $0$ is eigenvalue of $A$ if $\tau_1(A) = 0$, and also Proposition 3.2 still holds. However, $|\lambda_3(A)|$ may happen to be greater than $\tau_{n-2}(A)$ as it happens for

$$A = \begin{bmatrix} 0 & 0.29 & 0.55 \\ 0.63 & 0.4 & 0.12 \\ 0.37 & 0.31 & 0.33 \end{bmatrix},$$

where $|\lambda_3(A)| = 0.064 > \tau_1(A) = 0.06$ (instead, following our definition (3.3), $|\lambda_3(A)| = 0.064 < \tau_1(A) = 0.55$ for such $A$).

### 3.1. The computation of the dominant eigenvector.

In this subsection, given $A \in S^+$, we show that $x$ is a right nonnegative dominant eigenvector of $A$, such that $e^T x = 1$, if and only if there exist $B \in S^+$ and $\ell \geq 0$ such that $x$ is the solution of the linear system

$$M x = \ell$$

where $M$ is the M-matrix $I - \tau_{n-1}(A) B$. Moreover we observe that both $B$ and $\ell$ can be explicitly described in terms of the entries of $A$. We first need few lemmas

**Lemma 3.3.** Let $A \in S^+$ and $\alpha \in \mathbb{R}$. Given $h \in \{1, \ldots, n\}$ consider the vector $\ell = \ell(\alpha)$ such that $\ell_i = \min_{j \neq h} a_{ij}$, $i \neq h$, $\ell_h = 1 - (\sum_{i \neq h} \ell_i) - \alpha$, and note that $e^T \ell = 1 - \alpha$. Then

1. The columns of $\frac{1}{\alpha}(A - \ell \ell^T)$ sum up to one, for any $\alpha \neq 0$,
2. $C = A - \ell \ell^T$ is nonnegative, for any $\alpha \geq \tau_{n-1}(A)$.

**Proof.** (1) By definition we have $\frac{1}{\alpha}(A - \ell \ell^T)e = \frac{1}{\alpha}(e - (1 - \alpha)e) = e$. (2) Simply observe that if $i \neq h$, then $c_{ij} = a_{ij} - \min_k a_{ik} \geq 0$ and

$$c_{hj} = a_{hj} - \ell_h = 1 - \sum_{i \neq h} a_{ij} - (1 - \alpha - \sum_{i \neq h} \ell_i) \geq \alpha - \tau_{n-1}(A) \geq 0$$

for $j = 1, \ldots, n$. \[\] Observe that due to the previous lemma, for any nonnegative column stochastic $A$ and any $\alpha \geq \tau_{n-1}(A)$, we have the decomposition $A = \alpha B + \ell \ell^T$, where $B$ is still nonnegative column stochastic. As a direct consequence of this fact we obtain an alternative proof of the well known upper bound in (3.2), that is $|\lambda| \leq \tau_{n-1}(A)$, for any $\lambda \in \sigma(A)$ such that $\lambda \neq 1$. In fact, given $e$, let $\{e, y_2, \ldots, y_n\}$ be a set of independent vectors and let $Y$ be the nonsingular matrix whose rows are such vectors. Then

$$Y A Y^{-1} = \alpha Y B Y^{-1} + Y \ell \ell^T Y^{-1} = \alpha \begin{bmatrix} 1 & 0 \\ w^T & Q \end{bmatrix} + \begin{bmatrix} 1 - \alpha & 0 \\ z & O \end{bmatrix}$$

which imply $\sigma(A) = \{1\} \cup \sigma(\alpha Q)$. Note that for all $\lambda \in \sigma(\alpha Q)$ we have $|\lambda| \leq \alpha$, since the eigenvalues of $Q$ are eigenvalues of $B$. The claimed bound now follows by taking the infimum over $\alpha$. Observe that, using the Perron-Frobenius theorem and recalling Proposition 3.1 one gets the following interesting consequence

**Corollary 3.4.** Any nonnegative irreducible column stochastic matrix $A \in S^+$ such that $\max_i \min_j a_{ij} > 0$ is primitive.

**Theorem 3.5.** Given $A \in S^+$ let $\ell$ be the vector $\ell_i = \min_j a_{ij}$ and $B \in S^+$ the matrix $\tau_{n-1}(A) B = A - \ell \ell^T$. If $\max_i \min_j a_{ij} > 0$ and $x$ is a right nonnegative dominant eigenvector of $A$ s.t. $e^T x = 1$, then $x = (I - \tau_{n-1}(A) B)^{-1} \ell$. Viceversa if
I − τ_{n-1}(A)B is invertible then max_i, min_j a_{ij} > 0 and x = (I − τ_{n-1}(A)B)^{-1}e is a right nonnegative dominant eigenvector of A such that e^T x = 1.

Proof. If max_i, min_j a_{ij} > 0 then τ_{n-1}(A) < 1 and I − τ_{n-1}(A)B is invertible. Moreover Ax = x and e^T x = 1 imply (I − τ_{n-1}(A)B)x = e. Viceversa I − τ_{n-1}(A)B invertible implies τ_{n-1}(A) ≠ 1, thus by Proposition 3.1 max_i, min_j a_{ij} > 0 and x = \sum_{k \geq 0} τ_{n-1}(A)^k B^k e \geq 0. Moreover, since e^T e = 1 − τ_{n-1}(A),

$$e^T x = e^T \sum_{k \geq 0} τ_{n-1}(A)^k B^k e = \sum_{k \geq 0} τ_{n-1}(A)^k − τ_{n-1}(A)^{k+1} = 1.$$  

Now use the identity (I − τ_{n-1}(A)B)x = e to observe that x is a nonnegative fixed point of A. □

Theorem 3.5 appears interesting from a computational point of view, indeed it shows the eigenvalue problem Ax = x can be interchanged with the linear system problem (I − τ_{n-1}(A)B)x = e, when dealing with nonnegative stochastic matrices A. This observation has several similarities and possibly several applications to some eigenvector centrality and random walk questions over finite graphs (see for instance [1]). A typical example is the pagerank index problem where an analogous of Theorem 3.5 arises from the particular structure of the model, see f.i. [1, 4, 11].

4. Concluding remarks. This brief note has two relevant tasks. It somehow completes the egregious review of Ipsen and Selee [5] and in particular Section 7 in it, by proposing a complex version of [6, Thm. 3.3] which represent the modulus of the second dominant eigenvalue of an irreducible nonnegative matrix as a limit involving the ergodicity coefficient τ_{n-1}. It revive τ_{n-1}, one of the earlier proper ergodicity coefficients [9, p. 137], proposing a new possible direction of investigation through its generalized version τ_n, and shows a theorem that relates the problem of the computation of the stationary distribution of a Markov chain with a M-matrix linear systems that involves τ_{n-1}.

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