Complete classification and nondegeneracy of minimizers for the fractional Hardy-Sobolev inequality, and applications

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Abstract
We study linear and non-linear equations related to the fractional Hardy–Sobolev inequality. We prove nondegeneracy of ground state solutions to the basic equation and investigate existence and qualitative properties, including symmetry of solutions to some perturbed equations.

Keywords: Fractional Laplacian, fractional Hardy–Sobolev inequality, nondegeneracy, symmetry preserving.

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1 Introduction

Our starting point is the nonlocal problem

\[
\begin{cases}
(-\Delta)^s u = |x|^{-bq} u^{q-1}, & u \in D^s(\mathbb{R}^n) \\
\quad u > 0.
\end{cases}
\] (P_0)

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Here $n \geq 2$, $(-\Delta)^s$ is the fractional Laplacian of order $s \in (0, 1)$, the exponents $q, b$ satisfy
\[
2 < q < 2^*_s := \frac{2n}{n - 2s}, \quad \frac{n}{q} - b = \frac{n}{2} - s
\] (1.1)
and $\mathcal{D}^s(\mathbb{R}^n)$ is the natural Sobolev-type function space. Problem $\mathcal{P}_0$ is related to the fractional Hardy–Sobolev inequality
\[
S_q \cdot \| |x|^{-b}u \|_q^2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2^2, \quad u \in \mathcal{D}^s(\mathbb{R}^n),
\] (1.2)
that plainly follows via Hölder interpolation between the Hardy and Sobolev inequalities.

The best constant $S_q$ in (1.2) is attained by a nonnegative radially symmetric function
\[
z_1 \in \mathcal{D}^s(\mathbb{R}^n)
\] (see [16]) which is a weak solution to $(-\Delta)^s u = |x|^{-b} u^{q-1}$. Since $(-\Delta)^s z_1 \geq 0$ in the sense of distributions, then the strong maximum principle (see [19] Section 2] and [15 Corollary 4.2]), ensures that $z_1$ is lower semicontinuous and positive on $\mathbb{R}^n$. Hence, $z_1$ solves $\mathcal{P}_0$. Further, by adapting the moving plane argument in [5] or [7] one can prove that $z_1$ is radially symmetric about the origin and radially decreasing.

We agree that the minimizer $z_1$ is fixed, form now on.

By direct computations one can check that for any $t > 0$, the radial function
\[
z_t(x) = t^{\frac{2b}{q-1}} z_1 \left( \frac{x}{t} \right)
\] achieves $S_q$ and solves $\mathcal{P}_0$. However, we emphasise the fact that, differently from the critical case $q = 2^*_s$ and from the local case $s = 1$ (see [11, 13], respectively), the minimizers for $S_q$ are not explicitly known, nor classified.

We are in position to state our first main result.

**Theorem 1.1 (Regularity, decay estimates and uniqueness)**

1) $z_1 \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^\infty(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ for any $\alpha \in [0, 2s - bq)$; moreover, there exist positive constants $C_1, C_2$ such that
\[
\frac{C_1}{1 + |x|^{-2s}} \leq z_1(x) \leq \frac{C_2}{1 + |x|^{-2s}} \quad \text{for any } x \in \mathbb{R}^n;
\]
ii) if \( u \in \mathcal{D}^s(\mathbb{R}^n) \) is a solution to \((\mathcal{P}_0)\) then \( u = z_t \) for some \( t > 0 \);

iii) the function \( t \mapsto z_t \) is a regular curve in \( \mathcal{D}^s(\mathbb{R}^n) \) of class \( \mathcal{C}^2 \).

The proof of Theorem 1.1 is based on some preliminary results on eigenvalue problems of the form

\[
(-\Delta)^s \varphi = \mu V(x) \varphi, \quad \varphi \in \mathcal{D}^s(\mathbb{R}^n),
\]

where \( V > 0 \) is a given measurable weight satisfying suitable integrability assumptions. Our results on \((1.3)\), see Section 2, might have an independent interest. The proof of Theorem 1.1 is carried out in Section 3.

Our next focus is the problem

\[
(-\Delta)^s v = (q - 1)|x|^{-bq} z_t^{q-2} v, \quad v \in \mathcal{D}^s(\mathbb{R}^n),
\]

which is obtained by linearizing \((\mathcal{P}_0)\) at \( z_t \). Let us denote by a "dot" the differentiation with respect to \( t \). Thanks to part \( iii \) in Theorem 1.1, it is easily seen that \( \dot{z}_t \) is a weak solution to \((\mathcal{L}_t)\). In Section 4 we prove the next uniqueness result.

**Theorem 1.2 (Nondegeneracy)** If a function \( v \in \mathcal{D}^s(\mathbb{R}^n) \) solves \((\mathcal{L}_t)\), then \( v \) is proportional to \( \dot{z}_t \).

Nondegeneracy in the limiting case \( q = 2^*_s \) has been proved in [6], by taking advantage of the explicit knowledge of the minimizer \( z_t \).

As a first consequence of Theorem 1.2 we obtain a symmetry result for ground state (i.e. least energy) solutions to the nonlocal problem

\[
\begin{align*}
\begin{cases}
(-\Delta)^s u + \lambda |x|^{-2s} u &= |x|^{-bq} u^{q-1} & u \in \mathcal{D}^s(\mathbb{R}^n) \\
u > 0.
\end{cases}
\end{align*}
\]

If \( \lambda \leq 0 \), then the moving plane method can be applied to show that any weak solution to \((\mathcal{P}_\lambda)\) is radially symmetric about the origin. In particular, letting \( H_s \) to be the fractional Hardy constant (see [12] for its explicit value), we have that any minimizer for the best constant

\[
S^\lambda_q = \inf_{\substack{u \in \mathcal{D}^s(\mathbb{R}^n) \\ u \neq 0}} \frac{\|(-\Delta)^s u\|_2^2 + \lambda \| |x|^{-s} u\|_2^2}{\| |x|^{-b} u\|_q^2}
\]

(1.4)
is radial, provided that $-H_s < \lambda \leq 0$ (existence has been proved in \[16\]). On the other hand, symmetry breaking occurs: if $\lambda > 0$ is large, then no extremal for $S^\lambda_q$ is radially symmetric (see \[16\] Theorem 1.1).

In the next theorem, which is proved in Section 5, we show that symmetry persists also for small positive values of $\lambda$.

**Theorem 1.3 (Symmetry preserving)** There exists $\lambda_s^R > 0$ such that for every $\lambda \in (-H_s, \lambda_s^R)$, any minimizer for $S^\lambda_q$ is radially symmetric about the origin.

As a further consequence of Theorem 1.2, in Section 6 we use a Lyapunov-Schmidt argument inspired by \[9\] Sections 3 and 4 to obtain sufficient conditions on a prescribed weight $k(x)$ on $\mathbb{R}^n$ which guarantee the existence of solutions to the perturbative model problem

\[
\begin{cases}
(-\Delta)^s u = (1 + \varepsilon k(x))|x|^{-b_q}u^{q-1} & u \in D^s(\mathbb{R}^n) \\
u > 0.
\end{cases}
\]

(\(P^\varepsilon_k\))

For instance, we obtain the following extension of \[9\] Theorem 1.3).

**Theorem 1.4** Let $k \in L^\infty(\mathbb{R}^n)$. If $\lim_{x \to 0} k(x) = \lim_{|x| \to \infty} k(x)$, then problem $P^\varepsilon_k$ has at least a solution for any $\varepsilon$ close enough to 0.

**Notation.** The fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$, $n \geq 2$, is formally defined by

\[
\mathcal{F}[(-\Delta)^s u] = |\xi|^{2s} \mathcal{F}[u],
\]

where $\mathcal{F} = \mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx$ is the Fourier transform. Thanks to the Sobolev inequality, the space

\[
D^s(\mathbb{R}^n) = \{ u \in L^{2^*_s}(\mathbb{R}^n) \mid (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \}
\]

naturally inherits a Hilbertian structure from the relations

\[
(u, v)_{D^s} = ((-\Delta)^s u, v) = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx, \quad \|u\|_{D^s}^2 = (u, u)_{D^s}.
\]
From now on, we will always use the shorter notation $D^s$ instead of $D^s(\mathbb{R}^n)$, and we let $(D^s)'$ be its dual space. By elementary arguments, any $w \in (D^s)'$ can be identified with the distribution $(-\Delta)^s v$, where $v \in D^s$ is uniquely determined by $w$.

Denote by $\| \cdot \|_p$ the norm in $L^p(\mathbb{R}^n)$.

For $0 < \alpha < 1$, $C^\alpha$ stands for standard Hölder space. For $1 < \alpha < 2$, we denote by $C^\alpha$ the space of continuously differentiable functions with $\nabla u \in C^{\alpha-1}$.

## 2 Preliminaries on eigenvalue problems

In this section we study the linear problem (1.3) under the assumption $V > 0$. We use the following regularity results within the classical theory for Riesz potentials.

**Proposition 2.1** Let $\alpha \in (0, n)$ be given.

i) Let $f \in L^p(\mathbb{R}^n)$ with $p < \frac{n}{\alpha}$, then $f \ast |x|^{\alpha-n} \in L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)$ (the Hardy–Littlewood–Sobolev theorem, [20, Ch. V, Theorem 1]);

ii) Let $f \in L^p(\mathbb{R}^n)$ with $p > \frac{n}{\alpha}$; if $\alpha > 1$ assume in addition that $p < \frac{n}{\alpha-1}$. Then $f \ast |x|^{\alpha-n} \in C^{\alpha-\frac{n}{\alpha}}(\mathbb{R}^n)$ ([20, Ch. V, Theorem 5] and [20, Ch. V, 6.7a]).

We say that a nontrivial function $\varphi \in D^s$ is an eigenfunction for (1.3) if it is a weak solution to (1.3), namely

$$((-\Delta)^s \varphi, \psi) = \mu \int_{\mathbb{R}^n} V(x) \varphi \psi \, dx \quad \text{for any } \psi \in C_0^\infty(\mathbb{R}^n).$$

**Lemma 2.2** 1. Let $V \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$. Then the spectrum of (1.3) is discrete. We denote by $\mu_j$ a non-decreasing unbounded sequence of eigenvalues counting with multiplicities. The corresponding eigenfunctions $\varphi_j$ form a complete orthogonal system in $D^s$. Moreover,

$$\mu_j = \min_{\varphi \in D^s, \varphi \neq 0} \frac{((-\Delta)^s \varphi, \varphi)}{\int_{\mathbb{R}^n} V(x)|\varphi(x)|^2 \, dx}. \quad (2.1)$$

The first eigenvalue $\mu_1 > 0$ is simple, and it is the only eigenvalue admitting a positive eigenfunction.
2. If in addition \( V \in L^{\frac{n}{2s}}(\Omega) \) for some \( \varepsilon \in (0, 2s), \varepsilon \neq 1 \), then \( \varphi_j \in C^\varepsilon_{\text{loc}}(\Omega) \) for any \( j \geq 1 \).

3. If in addition \( V \in C^\infty(\Omega) \), then \( \varphi_j \in C^\infty(\Omega) \) for any \( j \geq 1 \).

**Proof.**

1. The quadratic form \( Q(\varphi) := \int_{\mathbb{R}^n} V(x)|\varphi(x)|^2 \, dx \) satisfies

\[
|Q(\varphi)| \leq \|V\|_{\frac{n}{2s}} \|\varphi\|_{2^*}^2
\]

by Hölder’s inequality. Hence, \( Q \) is bounded in \( D^s \). If \( V \in C^\infty_0(\mathbb{R}^n) \) then \( Q \) generates a compact operator in \( D^s \) by the Rellich theorem. Since any arbitrary \( V \in L^{\frac{n}{2s}}(\mathbb{R}^n) \) can be approximated in \( L^{\frac{n}{2s}}(\mathbb{R}^n) \) by smooth and compactly supported functions, the corresponding operator is compact as well, because of by \( (2.2) \). So, the discreteness of the spectrum and the completeness of \( (\varphi_j) \) follow by the Hilbert–Schmidt theorem. The equalities \( (2.1) \) hold by well known variational principle, see e.g. [2, Sec. 10.2].

Now we invoke the Green representation formula for \( (1.3) \),

\[
\varphi_j(x) = C(n,s)((-\Delta)^s \varphi_j) \ast |x|^{2s-n} = C(n,s)\mu_j \int_{\mathbb{R}^n} \frac{V(\xi)\varphi_j(\xi)}{|x-\xi|^{n-2s}} \, d\xi.
\]

Since the kernel is positive, the principal eigenfunction \( \varphi_1 \) is positive, and the corresponding eigenvalue \( \mu_1 \) is simple [11]. On the other hand, for any \( j > 1 \) we have

\[
\mu_j \int_{\mathbb{R}^n} V(x)\varphi_j(x)\varphi_1(x) \, dx = ((-\Delta)^s \varphi_j, \varphi_1) = 0,
\]

thus \( \varphi_j \) can not have constant sign.

2. We split the integral in \( (2.3) \) into two parts:

\[
\varphi_j(x) = C(n,s)\mu_j \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{V(\xi)\varphi_j(\xi)}{|x-\xi|^{n-2s}} \, d\xi + \int_{\Omega} \frac{V(\xi)\varphi_j(\xi)}{|x-\xi|^{n-2s}} \, d\xi \right).
\]

Since the first integral is a smooth function of \( x \in \Omega \), we only have to deal with the second one.

We know that \( \varphi \in L^{2^*}(\mathbb{R}^n) \). If \( \varepsilon < 1 \), we use \( (2.3) \) and statement \( i) \) in Proposition 2.1 with \( \alpha = 2s \), to improve the integrability exponent for \( \varphi_j \) which, in turns, improves the integrability exponent of \( V\varphi_j \). A bootstrap procedure provides, in a finite
number of steps, $V \phi_j \in L^p(\Omega)$ for some $p > \frac{n}{2s}$. Then statement ii) in Proposition 2.1 gives $\phi_j \in C^0(\Omega)$ and thus $V \phi_j \in L^{\frac{n}{2s}}(\Omega)$. Finally, part ii) in Proposition 2.1 gives $\phi_j \in C^{\epsilon}_{\text{loc}}(\Omega)$.

If $\epsilon > 1$ then we can repeat the same steps up to obtain $\phi_j \in C^0(\Omega)$. Then we differentiate (3.1), put $\alpha = 2s - 1$ and apply part ii) in Proposition 2.1 to obtain $\nabla \phi \in C^{\epsilon-1}_{\text{loc}}(\Omega)$, that again gives $\phi \in C^{\epsilon}_{\text{loc}}(\Omega)$.

3. The last claim follows from [20, Ch. V, Theorem 4] and the bootstrap argument. □

Lemma 2.3 Let a positive weight $V \in L^{\frac{n}{2s}}_{\text{loc}}(\mathbb{R}^n)$ be symmetric-decreasing. For $s \leq \frac{1}{2}$, assume in addition that $V \in C^\beta_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ with $\beta > 1 - 2s$. Then for any $\mu \in \mathbb{R}$, the problem (1.3) has at most one linearly independent radial eigenfunction.

Proof. We follow the outline of the proof in [10, Theorem 1]. Notice that the argument in [10] cannot be applied directly because the weight in the right-hand side of (1.3) might be singular at the origin.

We introduce the Caffarelli–Silvestre extension [3] of any function $\phi \in D^s$, that is the solution $\Phi$ of the boundary value problem

$$-\text{div}(y^{1-2s}\nabla \Phi) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+, \quad \Phi|_{y=0} = \phi,$$

satisfying

$$C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \Phi|^2 \, dx \, dy = \langle (-\Delta)^s \phi, \phi \rangle$$

for some explicitly known constant $C_s$. The eigenvalue problem (1.3) can be rewritten as follows,

$$-C_s \cdot \lim_{y \to 0^+} y^{1-2s} \partial_y \Phi(x, y) = \mu V(x) \phi(x), \quad x \in \mathbb{R}^n,$$

so that

$$\mu = C_s \frac{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \Phi|^2 \, dx \, dy}{\int_{\mathbb{R}^n} V(x)|\phi|^2 \, dx}.$$

1In fact, this assumption restricts only the behavior of $V$ at zero.
In general, problem (2.4)–(2.5) admits separation of variables; we can write its solutions in the form
\[
\Phi(x, y) = W(r, y)Y(\Theta), \quad \varphi(x) = h(r)Y(\Theta),
\]
where \((r, \Theta)\) are spherical coordinates in \(\mathbb{R}^n\) and \(Y\) is a spherical harmonic.

Now we turn to the proof of the Lemma. If \(\varphi(x)\) is a radially symmetric eigenfunction for (1.3), then its extension \(\Phi(x, y)\) is radially symmetric in the \(x\)-variable as well, and we have
\[
\Phi(x, y) = W(r, y); \quad \varphi(x) = h(r).
\]

Since \(V\) is positive and symmetric-decreasing, it is bounded outside of the origin. By [10, Proposition B.1], we have \(\varphi \in C^{1+\delta}(\mathbb{R}^n \setminus \{0\})\) for some \(\delta > 0\). Next, Lemma 2.2 gives \(\varphi \in C^\beta_{\text{loc}}(\mathbb{R}^n)\) and therefore \(h \in C^\beta_{\text{loc}}(\mathbb{R}^+)\). So, to prove the Lemma it is sufficient to show that if \(h(0) = 0\) then \(W \equiv 0\).

We rewrite the representation formula (2.3) as follows,
\[
h(|x|) = \mu C(n, s) \int_{\mathbb{R}^n} \frac{V(|\xi|)h(|\xi|)}{|x - \xi|^{n-2s}} d\xi.
\]
The inclusion \(h \in C^\beta_{\text{loc}}(\mathbb{R}^+)\) and the assumption \(h(0) = 0\) reduces the order of singularity of the integrand at \(\xi = 0\). In turns, this gives a better Hölder estimate for \(h\). Repeating this argument we obtain \(h \in C^\beta_{\text{loc}}(\mathbb{R}^+)\) for any \(\beta_1 < 2s\).

Next, we rewrite problem (2.4)–(2.5) in polar coordinates to obtain that the pair \(W, h\) solve
\[
-\partial^2_{rr} W - \frac{n-1}{r} \partial_r W - \partial^2_{yy} W - \frac{1 - 2s}{y} \partial_y W = 0, \quad y > 0;
\]
\[
W(r, 0) = h(r); \quad C_s \lim_{y \to 0^+} y^{1-2s} \partial_y W(r, y) + \mu V(r)h(r) = 0;
\]
\[
\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}(|\partial_r W|^2 + |\partial_y W|^2) dxdy < \infty.
\]

Following [10], we introduce the function
\[
H(r) = C_s \int_0^\infty y^{1-2s} [\partial_r W|^2 - |\partial_y W|^2] dy + \mu V(r)h^2(r).
\]
Proposition B.2 in [10] gives
\[ H(\infty) := \lim_{r \to \infty} H(r) = 0. \]
Moreover, \( h(0) = 0 \) implies \( |h(r)| \leq C_s r^\beta \), thus \( \lim_{r \to 0} V(r) h^2(r) = 0 \) because of the summability of assumption on \( V \). In addition, \( \partial_r W(0, y) \equiv 0 \) by symmetry and thus
\[ H(0) = -C_s \int_0^\infty y^{1-2s} (\partial_y W(0, y))^2 \, dy \leq 0. \]
Finally, repeating the proof of [10, Lemma 4.1] we conclude that
\[ H'(r) \leq -2C_s \frac{n-1}{r} \int_0^\infty y^{1-2s}|\partial_r W|^2 \, dy \leq 0 \]
in the sense of distributions. Therefore, \( H \) is non-increasing. Since \( H(0) \leq H(\infty) \), we infer that \( H \equiv 0 \). This gives \( \partial_r W \equiv 0 \) a.e. and hence \( W = W(t) \). But this implies \( h(r) = const \), therefore \( h \equiv 0 \) and \( W \equiv 0 \). \( \square \)

3 Proof of Theorem 1.1

Let \( u \) be a weak solution to (\( P_0 \)). As mentioned in the introduction, \( u \) is radially symmetric about the origin and radially decreasing. Also, notice that \( u \) solves (1.3) for \( \mu = 1 \) with weight \( V(x) = |x|^{-bq} u^{-2} \in L^{\frac{n}{2}}(\mathbb{R}^n) \), so that we can write the Green representation formula for (\( P_0 \)),
\[ u(x) = C(n, s) \int_{\mathbb{R}^n} \frac{|\xi|^{-bq} u^{-1}(\xi)}{|x - \xi|^{n-2s}} \, d\xi. \]
By repeating literally the proof of Lemma 6 in [21] one first obtains that \( u \in L^\infty(\mathbb{R}^n) \). Then, using [8, Proposition 2.6] one infers that the s-Kelvin transform
\[ x \mapsto y = \frac{x}{|x|^2}, \quad u(x) \mapsto \tilde{u}(y) = \frac{1}{|y|^{n-2s}} u\left(\frac{y}{|y|^2}\right) \]
maps a solution of \( \mathcal{P}_0 \) to a solution of \( \mathcal{P}_0 \). This gives
\[
\frac{C_1}{1 + |x|^{n-2s}} \leq u(x) \leq \frac{C_2}{1 + |x|^{n-2s}}.
\] (3.3)

Notice that the constants in (3.3) and in the estimates that follow depend on the choice of \( u \).

Thanks to Lemma 2.2, from (3.3) we infer that \( u \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C^\varepsilon(\mathbb{R}^n) \) for any \( \varepsilon < 2s - bq \). Thus \( i \) in Theorem 1.1 follows by choosing \( u = z_t \).

Now we prove \( ii \). Let \( u \) be a solution to \( \mathcal{P}_0 \). Then \( u \) is radially symmetric, radially decreasing and continuous on \( \mathbb{R}^n \). Take \( t > 0 \) such that \( u(0) = z_t(0) \) and put
\[
\varphi(x) = u(x) - z_t(x), \quad V(x) = \begin{cases} |x|^{-bq} \frac{(u(x))^q - (z_t(x))^q}{u(x) - z_t(x)} & \text{if } u(x) \neq z_t(x) \\ |x|^{-bq} (q - 1)(z_t(x))^{q-2} & \text{if } u(x) = z_t(x) \end{cases}
\]
Then \( V \) is radial and satisfies the regularity assumptions in Lemma 2.3. It turns out that \( V \) is symmetric-decreasing, thanks to the next calculus lemma.

**Lemma 3.1** Let \( f \) be a convex function on \( \mathbb{R}_+ \). If \( u \) and \( v \) are decreasing (increasing) positive functions on \( \mathbb{R}_+ \), then \( g = \frac{f(u) - f(v)}{u - v} \) is decreasing (increasing) on \( \mathbb{R}_+ \).

**Proof.** It is sufficient to assume all functions smooth. We calculate
\[
g' = \frac{u'}{(u-v)^2} (f(v) - f(u) - f'(u)(v-u)) + \frac{v'}{(u-v)^2} (f(u) - f(v) - f'(v)(u-v)),
\]
and the statement follows. \( \square \)

We can now continue the proof of the Theorem. Since \( V \) is symmetric-decreasing and \( \varphi \) is a radial solution of
\[
(-\Delta)^s \varphi = V(x) \varphi, \quad \varphi(0) = 0,
\]
then Lemma 2.3 applies and gives \( \varphi \equiv 0 \). Thus \( ii \) is proved.

Before proving \( iii \) it is convenient to point out the next observation.
Remark 3.2 Let us notice that by \( \text{ii) \ } \), the transform \( \theta \) maps \( z_t \) to \( z_{\tau} \) for some \( \tau > 0 \). From now on we assume that \( z_1 \) is a fixed point of the \( s \)-Kelvin transform.

To go further we study in detail the action of the group of isometries \( D^s \to D^s \) parametrized by \( t > 0 \) and given by

\[
\mathcal{I}(t)u(x) := t^{\frac{2s-n}{2}} u\left(\frac{x}{t}\right).
\]

Notice that \( z_t = \mathcal{I}(t)z_1 \). Since \( z_t \) is a smooth function on \( \mathbb{R}^n \setminus \{0\} \), we can differentiate the identity \( z_t = \mathcal{I}(t)z_1 \) with respect to \( t \) to obtain

\[
\dot{z}_t = \frac{1}{t} \mathcal{I}(t) \dot{z}_1 , \tag{3.4}
\]

where the radial function \( \dot{z}_1 \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) is given by

\[
\dot{z}_1(x) = -x \cdot \nabla z_1(x) - \frac{n-2s}{2} z_1(x). \tag{3.5}
\]

For \( x \neq 0 \), consider the integral

\[
\int_{\mathbb{R}^n} \frac{\xi \cdot \nabla z_1(\xi)}{|x - \xi|^{n-2s}} \; d\xi = I_1 + I_2 + I_3 := \int_{|\xi| < \frac{|x|}{2}} \int_{|\xi| < 2|x|} \int_{|\xi| > 2|x|} \phi \; d\xi + \int_{|\xi| < \frac{|x|}{4}} \int_{|\xi| < 2|x|} \int_{|\xi| > 2|x|} \phi \; d\xi.
\]

Easily, the integral \( I_2 \) converges. Furthermore, the estimate \( 3.3 \) implies

\[
I_1 + I_3 \leq C(x) \int_0^\infty \frac{r^{n-1}r^{1-bq}|\nabla z_1(r)|}{(1 + r^{n-2s})^{q-1}} \; dr \leq C(x) \int_0^\infty |\nabla z_1(r)| \; dr.
\]

(the inequality \( * \) follows from \( 1.1 \)). Since \( z_1 \) is symmetric-decreasing, the last integral converges. Moreover, this convergence is uniform with respect to \( x \) in any compact set bounded away from the origin.

This allows us to differentiate the equality \( 3.1 \) for \( u = z_t \) with respect to \( t \). We arrive at

\[
\dot{z}_t(x) = (q-1)C(n, s) \int_{\mathbb{R}^n} \frac{\xi \cdot \nabla z_t^2(\xi) \dot{z}_1(\xi)}{|x - \xi|^{n-2s}} \; d\xi. \tag{3.6}
\]
We infer that \( \dot{z}_t \) is an eigenfunction to (2.3) with weight \( V(x) = |x|^{-bq}z_t^{q-2}(x) \).

By the estimate (3.3) we can apply part 2 of Lemma 2.2. So, \( \dot{z}_t \) is bounded and Hölder continuous in \( \mathbb{R}^n \). Also it is smooth outside the origin. Finally, the \( s \)-Kelvin transform gives

\[
|\dot{z}_1(x)| \leq C_3 \frac{1}{1 + |x|^{n-2s}}. \tag{3.7}
\]

The estimates (3.3) and (3.7) show that

\[
|x|^{-bq}z_t^{q-2}(x)\dot{z}_1(x) \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n) \subset (\mathcal{D}^s(\mathbb{R}^n))^{'},
\]

and (3.6) gives \( \dot{z}_1 \in \mathcal{D}^s(\mathbb{R}^n) \).

Repeating this procedure we can differentiate (3.6) with respect to \( t \) once more. This gives the integral equation for \( \ddot{z}_t \), from which we derive, similarly to previous steps,

\[
\ddot{z}_1 \in \mathcal{D}^s(\mathbb{R}^n) \cap \mathcal{C}^\alpha(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}); \quad |\ddot{z}_1(x)| \leq C_4 \frac{1}{1 + |x|^{n-2s}}.
\]

Since \( \dot{z}_t \in \mathcal{D}^s(\mathbb{R}^n) \setminus \{0\} \), we obtain \( iii) \) in Theorem 1.1. \( \Box \)

### 4 The linearized problem and proof of Theorem 1.2

Consider the functional on \( \mathcal{D}^s \),

\[
E_0[u] = \frac{1}{2}((-\Delta)^su, u) - \frac{1}{q} \int_{\mathbb{R}^n} |x|^{-bq}u_+^q \, dx,
\]

where \( u_+ = \max\{u, 0\} \). Recalling that the truncation operator \( u \mapsto u_+ \) is continuous in \( \mathcal{D}^s \) for \( s \in (0, 1] \), see [18, Theorem 5.5.2/3], and using (1.2), one can prove in a standard way that \( E_0 \) is of class \( \mathcal{C}^2 \), with first and second order differentials given by distributional equalities

\[
E_0'[u] = (-\Delta)^s u - |x|^{-bq}u_+^{q-1},
\]

\[
E_0''[u] \varphi = (-\Delta)^s \varphi - (q - 1)|x|^{-bq}u_+^{q-2} \varphi.
\]

**Remark 4.1** Let \( u, \varphi, \psi \in \mathcal{D}^s \). The next identities for \( t > 0 \) can be checked by elementary change of variables:

\[
E_0[\mathcal{I}(t)u] = E_0(u); \quad (E_0'[\mathcal{I}(t)u], \varphi) = (E_0'[u], \mathcal{I}(t^{-1})\varphi);
\]

\[
E_0''[\mathcal{I}(t)u](\varphi, \psi) = E_0''[u](\mathcal{I}(t^{-1})\varphi, \mathcal{I}(t^{-1})\psi). \tag{4.1}
\]
For any \( t > 0 \) we have that \( E'_0[z_t] = 0 \) and the kernel of \( E''(z_t) \) is the set of solutions to the linearized problem \( (L_t) \). By the results in Section 2 (with weight \( V(x) = |x|^{-bq}z_q^{-2} \)), the related eigenvalue problem
\[
(-\Delta)^s \varphi = \mu |x|^{-bq}z_t^{-2} \varphi, \quad \varphi \in \mathcal{D}^s,
\]
has a discrete, non decreasing sequence \( (\mu_j) \) of eigenvalues that admit a variational characterization \( (2.1) \). Since the energy \( E_0 \) is invariant with respect to the action of the transforms \( \mathcal{I}(t) \), the eigenvalues \( \mu_j \) do not depend on \( t > 0 \).

Clearly \( \mu_1 = 1 \), and the first eigenfunction is \( z_t \). Next, we deal with the second eigenvalue.

**Lemma 4.2** The eigenvalue \( \mu_2 \) equals \( q - 1 \).

**Proof.** By part iii) of Theorem 1, \( \dot{z}_t \in \mathcal{D}^s \) for any \( t > 0 \), hence \( E''_0[z_t]\dot{z}_t = 0 \). Thus \( \mu = q - 1 \) is an eigenvalue for \( (E_t) \), and \( \mu_2 \leq q - 1 \).

Next, recall that \( z_t \) solves \( (P_0) \) and achieves the best constant \( S_q \). Thus
\[
\int_{\mathbb{R}^n} |x|^{-bq}z_t^{-1} \varphi \, dx = \langle (-\Delta)^s z_t, \varphi \rangle, \quad J''[z_t](\varphi, \varphi) \geq 0 \quad \text{for any } \varphi \in \mathcal{D}^s,
\]
where \( J[u] = \frac{\langle (-\Delta)^s u, u \rangle}{\Vert |x|^{-b} u \Vert_q^2} \) for \( u \in \mathcal{D}^s \setminus \{0\} \). By direct computation (see for instance [17, Lemma 3.1]) we obtain
\[
\frac{\Vert |x|^{-b} z_t \Vert_q^2}{2} J''[z_t](\varphi, \varphi) = \langle (-\Delta)^s \varphi, \varphi \rangle - (q - 1) \int_{\mathbb{R}^n} |x|^{-bq}z_t^{-2} \varphi^2 \, dx + \frac{q - 2}{\Vert |x|^{-b} z_t \Vert_q^2} \langle (-\Delta)^s z_t, \varphi \rangle^2.
\]
We infer that if \( \varphi \) is orthogonal to \( z_t \) then \( \langle (-\Delta)^s \varphi, \varphi \rangle - (q - 1) \int_{\mathbb{R}^n} |x|^{-bq}z_t^{-2} \varphi^2 \, dx \geq 0 \), hence \( \mu_2 \geq q - 1 \). This completes the proof. \( \square \)
To prove Theorem 1.2 we need the following auxiliary statement.

**Lemma 4.3** Let $W_0(x,y) = W_0(r,y)$ be the Caffarelli–Silvestre extension of $z_t$. Then $\partial_r W_0 < 0$ for $r > 0$, $y > 0$.

**Proof.** We use the Green representation for the Caffarelli–Silvestre extension, see [3]:

$$W_0(x,y) = C(n,s) \int_{\mathbb{R}^n} \frac{y^{2s} z_t(\xi) \, d\xi}{(|x - \xi|^2 + y^2)^{n+2s/2}}.$$  

The fact that the convolution of two symmetric-decreasing functions is symmetric-decreasing is well known. We give the proof for the reader’s convenience.

Since $\partial_r W_0 = \sum_{i=1}^{n} r^{-1} x_i \partial_{x_i} W_0$, it suffices to prove that $\partial_{x_n} W_0(x,y) < 0$ for $x_n > 0$. Using the notation $x = (x', x_n)$, we derive

$$\partial_{x_n} W_0(x,y) = C(n,s)(n+2s) \int_{\mathbb{R}^n} \frac{y^{2s} (\xi_n - x_n) z_t(\xi) \, d\xi}{(|x' - \xi'|^2 + |x_n - \xi_n|^2 + y^2)^{n+2s/2}} = C(n,s)(n+2s) \int_{0}^{\infty} \int_{\mathbb{R}^{n-1}} \frac{y^{2s} \eta (z_t(\xi' + \eta) - z_t(\xi' - \eta)) \, d\xi' \, d\eta}{(|x' - \xi'|^2 + \eta^2 + y^2)^{n+2s+2/2}}.$$  

From $|x_n + \eta| > |x_n - \eta|$ we infer $z_t(\xi', x_n + \eta) - z_t(\xi', x_n - \eta) < 0$, since $z_t$ is symmetric-decreasing, and the lemma follows. □

**Proof of Theorem 1.2** By Lemma 2.3, $z_t$ is the only radial eigenfunction corresponding to the eigenvalue $\mu_2 = q - 1$. Now we exclude the existence of the eigenfunctions with non-trivial spherical harmonic $Y$. In fact, we put $V(x) = |x|^{-bq} z_t^{-q-2}$ in (2.4)–(2.5) and show that if $Y \not\equiv 1$ in (2.7) the quotient in (2.6) is strictly greater than $q - 1$.

Given $h(r)$, we can minimize the quotient in (2.6) with respect to $Y$. This gives, modulo rotations, $Y(\Theta) = \frac{r_n}{r}$.

Since $\partial_r z_t < 0$ and $\partial_r W_0 < 0$ for $r > 0$, we can write

$$W(r,y) = g(r,y) \partial_r W_0; \quad h(r) = g(r) \partial_r z_t,$$

where $g = g|_{y=0}$. 

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Let \( h \in C^\infty_0(0, +\infty) \). Then \( g \) is smooth. Using (2.7) we rewrite (4.2) as follows:

\[
\Phi(x, y) = g(r, y)\partial_{x_n} W_0; \quad \varphi(x) = g(r)\partial_{x_n} z_t,
\]

and therefore

\[
C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \Phi|^2 \, dxdy
\]

\[
= C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}\left(|\nabla \partial_{x_n} W_0|^2 g^2 + \partial_{x_n} W_0 \nabla \partial_{x_n} W_0 \cdot \nabla (g^2) + (\partial_{x_n} W_0)^2 |\nabla g|^2\right) \, dxdy.
\]

Integrating by parts in the second term and using the equation \(-\text{div}(y^{1-2s}\nabla W_0) = 0\) we obtain

\[
C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \Phi|^2 \, dxdy = \int_{\mathbb{R}^n} g^2 \partial_{x_n} z_t (-\Delta)^s \partial_{x_n} z_t \, dx
\]

\[
+ C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}(\partial_{x_n} W_0)^2 |\nabla g|^2 \, dxdy.
\]

Since \((-\Delta)^s \partial_{x_n} z_t = \partial_{x_n}(|x|^{-b_q} z_t^{q-1})\), we arrive at

\[
C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla \Phi|^2 \, dxdy - (q - 1) \int_{\mathbb{R}^n} |x|^{-b_q} z_t^{q-2} \varphi^2 \, dx
\]

\[
= C_s \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}(\partial_{x_n} W_0)^2 |\nabla g|^2 \, dxdy - bq \int_{\mathbb{R}^n} g^2 z_t^{q-1} \partial_r z_t \frac{x_n^2}{|x|^{b_q+3}} \, dx.
\]

The right-hand side here is positive. Moreover, it is bounded away from zero if we approximate an arbitrary function \( \varphi(x) = h(r)\frac{z_t}{r} \in \mathcal{D}^s(\mathbb{R}^n) \) by functions the supports of which are bounded and separated from the origin. This completes the proof. \(\square\)
Corollary 4.4 There exists $\kappa > 0$ independent of $t > 0$ such that
\[
E''_0[z_t](\varphi, \varphi) \geq \kappa \|\varphi\|_{D^s}^2
\] (4.3)
for any $\varphi \in \langle z_t, \dot{z}_t \rangle^\perp$ and for any $t > 0$. Moreover, the following facts hold:

i) If $\varphi \in D^s$ solves $E''_0[z_t] \varphi = \gamma (-\Delta)^s \dot{z}_t$ for some $\gamma \in \mathbb{R}$, then $\varphi \in \langle \dot{z}_t \rangle$, hence $\gamma = 0$;

ii) For any $v \in \langle \dot{z}_t \rangle^\perp$ there exists a unique $\varphi \in \langle \dot{z}_t \rangle^\perp$ such that $E''_0[z_t] \varphi = (-\Delta)^s v$. Moreover,
\[
\kappa_* \|\varphi\|_{D^s} \leq \|v\|_{D^s},
\]
where $\kappa_* = \min\{\kappa, q - 2\}$.

In particular, the operator $E''_0[z_t] : \langle \dot{z}_t \rangle^\perp \mapsto (-\Delta)^s (\langle \dot{z}_t \rangle^\perp)$ is isomorphism.

Proof. We already noticed that the eigenvalues $\mu_j$ of $E''_0$ do not depend on $t > 0$ because of the invariance of $E_0$ with respect to the transforms $I(t)$. Let $\mu_3$ be the third eigenvalue of $E_0$. Then $\mu_3 > \mu_2$ by Theorem 1.2. Thus (4.3) holds, with $\kappa = 1 - \frac{q - 1}{\mu_3} > 0$. The last conclusions are immediate. 

5 Proof of Theorem 1.3

We introduce the $C^1$ function
\[
F(\lambda, u) = E'_0[u] + \lambda |x|^{-2s} u - ((-\Delta)^s \dot{z}_1, u) (-\Delta)^s \dot{z}_1, \quad F : \mathbb{R} \times D^s \to (D^s)'\]
and notice that $F(0, z_1) = 0$. We claim that $\partial_u F(0, z_1)$ is invertible. Explicitly, we have
\[
\partial_u F(0, z_1) \varphi = E''_0[z_1] \varphi - ((-\Delta)^s \dot{z}_1, \varphi) (-\Delta)^s \dot{z}_1, \quad \partial_u F(0, z_1) : D^s \to (D^s)'.
\]

By Corollary 4.4, $\partial_u F(0, z_1)$ maps isomorphically $\langle \dot{z}_t \rangle^\perp$ onto $(-\Delta)^s (\langle \dot{z}_t \rangle^\perp)$. Since evidently it maps $\langle \dot{z}_t \rangle$ onto $(-\Delta)^s (\langle \dot{z}_t \rangle)$, it isomorphically maps the space $D^s$ onto $(-\Delta)^s (D^s) = (D^s)'$, and the claim follows.
Thanks to the implicit function theorem, there exist \( \lambda_0 > 0 \) and a neighbourhood \( \mathcal{U} \) of \( z_1 \) such that for any \( \lambda \in (-\lambda_0, \lambda_0) \), the equation \( F(\lambda, u) = 0 \) has a unique solution \( u \in \mathcal{U} \). Of course, \( u \) must be radially symmetric, precisely because of the uniqueness given by the implicit function theorem. To conclude the proof it suffices to show that any minimizer for \( S_\lambda^q \) can be properly rescaled to obtain a function \( u_\lambda \in \mathcal{U} \) such that \( F(\lambda, u_\lambda) = 0 \), provided that \( \lambda > 0 \) is small enough (we already noticed that any minimizer for \( S_\lambda^q \) is radially symmetric if \( \lambda \leq 0 \)).

We start by taking any \( \lambda > 0 \) and any minimizer \( u_\lambda \) for \( S_\lambda^q \). Since replacing \( u \rightarrow |u| \) decreases the quotient in the right-hand side of (1.4), see [14, Theorem 3], we can assume \( u_\lambda \) nonnegative. We normalize \( u_\lambda \) so that it solves \((P_\lambda)\), that is

\[
\|(-\Delta)^{\frac{1}{2}} u_\lambda\|_2^2 + \lambda \|x\|^{-s} u_\lambda^2 = (S_\lambda^q)^{\frac{q}{2}}. 
\]

Inspired by the Emden-Fowler transform, we introduce the functions \( v, w : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
v(\zeta) = e^{\frac{2n-2n-2}{2} \zeta} z_1(e^{-\zeta}), \quad w(\zeta) = e^{\frac{2n-2n-2}{2} \zeta} \int_{\mathbb{S}^{n-1}} u_\lambda(e^{-\zeta} \sigma) \, d\sigma
\]

(here we identified the radial function \( z_1 \) with a function of \( r = |x| \)). Using Hölder inequality and (1.1) we obtain

\[
\int_{-\infty}^{\infty} v^q \, d\zeta \leq c \int_{-\infty}^{\infty} e^{(bq-n-2)\zeta} \int_{\mathbb{S}^{n-1}} u_\lambda^q(e^{-\zeta} \sigma) \, d\sigma \, d\zeta = c \int_{\mathbb{R}^n} |x|^{-bq} u_\lambda^q \, dx,
\]

(here \( c \) depends only on \( n, q \)), that gives \( w \in L^q(\mathbb{R}) \).

Further, by our choice in Remark 3.2 we have that \( v(\zeta) \equiv v(-\zeta) \); Theorem 1.1 and formula (3.5) give us

\[
v \in C^\infty(\mathbb{R}), \quad 0 \leq v(\zeta) \leq \frac{C_2}{\cosh(\frac{n-2n-2}{2} \zeta)}.
\]

We infer that \( \lim_{|\zeta| \to \infty} v(\zeta) = 0 \) and \( v \in L^p(\mathbb{R}) \) for any \( p \in [1, \infty) \). In particular, there
exists \( t_\lambda > 0 \) such that \( t_\lambda \) achieves the maximum of the smooth function

\[
f(t) := (v^{q-1} * w)(\log t) = \int_{-\infty}^{\infty} (v(\zeta - \log t))^{q-1}w(\zeta) \, d\zeta
\]

\[
= \int_{\mathbb{R}^n} t^{n-2q(q-1)}|x|^{-bq}z_1^{-q-1}(x)u_\lambda(x) \, dx = \int_{\mathbb{R}^n} |x|^{-bq}z_1^{-q-1}u_\lambda \, dx.
\]

Recall that \( z_{1/t} = \mathcal{I}(1/t)z_1 = (\mathcal{I}(t))^{-1}z_1 \) solves \([P_0]\) and that \( \mathcal{I}(1/t) \) is isometry in \( \mathcal{D}^s \). Thus we have

\[
f(t) = ((-\Delta)^s z_{1/t}, u_\lambda) = ((-\Delta)^s z_1, \mathcal{I}(t)u_\lambda);
\]

\[
f'(t) = -\frac{1}{t^2} ((-\Delta)^s z_{1/t}, u_\lambda) = -\frac{1}{t} ((-\Delta)^s z_1, \mathcal{I}(t)u_\lambda).
\]

Since \( t_\lambda \) achieves the maximum of \( f \), we have

\[
((-\Delta)^s z_1, \mathcal{I}(t_\lambda)u_\lambda) = 0
\]

and

\[
((-\Delta)^s z_1, \mathcal{I}(t_\lambda)u_\lambda) = f(t_\lambda) \geq f(\tau^{-1}t_\lambda) = ((-\Delta)^s z_1, \mathcal{I}(\tau^{-1}t_\lambda)u_\lambda) = \langle (-\Delta)^s z_\tau, \mathcal{I}(t_\lambda)u_\lambda \rangle
\]

for any \( \tau > 0 \). We see that, eventually replacing \( u_\lambda \) with \( \mathcal{I}(t_\lambda)u_\lambda \), we can assume

\[
\langle (-\Delta)^s z_1, u_\lambda \rangle = 0; \quad \tag{5.2}
\]

\[
\|u_\lambda - z_1\|_{\mathcal{D}^s} = \min_{\tau > 0} \|u_\lambda - z_\tau\|_{\mathcal{D}^s}. \quad \tag{5.3}
\]

From (5.2) and since \( u_\lambda \) solves \([P_0]\) we infer that \( F(\lambda, u_\lambda) = 0 \) for any \( \lambda > 0 \). To conclude the proof we only need to show that \( u_\lambda \in \mathcal{U} \) for \( \lambda \) small enough.

Take any sequence \( \lambda_h \downarrow 0 \). By (5.1), \( u_{\lambda_h} \) is a bounded minimizing sequence for \( S_q \), so we can suppose that \( u_{\lambda_h} \) converges weakly in \( \mathcal{D}^s \). Arguing as in the proof of [10] Lemma 4.2], we can rescale \( u_{\lambda_h} \) so that its weak limit \( u \) is non-zero, hence \( u \) is a (nonnegative) solution of \([P_0]\) and \( u_{\lambda_h} \to u \) strongly in \( \mathcal{D}^s \). Thanks to the uniqueness result in Theorem [11] we see that there exists \( \tilde{\tau} > 0 \) and a sequence \( \tau_h > 0 \) such that \( \mathcal{I}(\tau_h)u_{\lambda_h} - z_{\tilde{\tau}} \to 0 \) in \( \mathcal{D}^s \). But then (5.3) gives

\[
\|u_{\lambda_h} - z_1\|_{\mathcal{D}^s} \leq \|u_{\lambda_h} - z_{\tilde{\tau}^{-1}}\|_{\mathcal{D}^s} = \|u_{\lambda_h} - (\mathcal{I}(\tau_h^{-1})z_{\tilde{\tau}})\|_{\mathcal{D}^s} = \|\mathcal{I}(\tau_h)u_{\lambda_h} - z_{\tilde{\tau}}\|_{\mathcal{D}^s} = o(1).
\]

Hence \( u_{\lambda_h} \to z_1 \), that is enough to conclude. \( \square \)
Remark 5.1 In fact, the assumption that $u_\lambda$ is a minimizer for (1.4) is used only to show that $u_\lambda$ are bounded for $\lambda \searrow 0$. We conjecture that there is a $\lambda_0 > 0$ such that for $\lambda \in (-H_s, \lambda_0]$ any nonnegative solution to $(P_\lambda)$ is radially symmetric. However, this problem is open.

6 Dimension reduction and proof of Theorem 1.4

Given $k \in L^\infty(\mathbb{R}^n)$, we put

$$G[u] = \frac{1}{q} \int_{\mathbb{R}^n} k(x)|x|^{-bq}u_+^q \, dx.$$  

For any $\varepsilon \in \mathbb{R}$ we introduce the energy functional on $\mathcal{D}^s$ given by

$$E_\varepsilon[u] = E_0[u] - \varepsilon G[u] = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^n} |x|^{-bq}(1 + \varepsilon k(x))u_+^q \, dx.$$  

Evidently, $E_\varepsilon \in C^2$, and any critical point $u$ for $E_\varepsilon$ is a weak solution to

$$(-\Delta)^s u = (1 + \varepsilon k(x))|x|^{-bq}u_+^{q-1}.$$  

If $u \neq 0$ and $|\varepsilon||k|_\infty \leq 1$, then $u$ is positive by the strong maximum principle [19]. Hence, $u$ solves $(P_\varepsilon)$.

In order to face the problem $E_\varepsilon'[u] = 0$ for $\varepsilon$ close to zero we combine variational methods with a Lyapunov-Schmidt technique, in the spirit of [1]. The next lemma is the crucial step.

Lemma 6.1 (Dimension reduction) There exist $\varepsilon_0 > 0$ such that the problem

$$E_\varepsilon'[u] = \frac{(E_\varepsilon'[u], \hat{z}_t)}{\|(-\Delta)^{\frac{s}{2}}\hat{z}_t\|_2^2} (-\Delta)^{\frac{s}{2}}\hat{z}_t, \quad u \in \langle \hat{z}_t \rangle^\perp,$$  

has a nontrivial solution $u = U_\varepsilon$ for any $(\varepsilon, t) \in (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}_+$. Moreover, this solution is unique in a neighbourhood of $z_t$, the function $(\varepsilon, t) \mapsto U_\varepsilon$ is $C^1$-smooth, and the following facts hold:
i) \( \|(-\Delta)^{s/2}(U^\varepsilon_t - z_t)\|_2 = O(\varepsilon) \) as \( \varepsilon \to 0 \), uniformly with respect to \( t \in \mathbb{R}_+ \);

ii) For any \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), the curve \( t \mapsto U^\varepsilon_t \) is a natural constraint for \( E_\varepsilon \), that is,

\[
\frac{d}{dt}E_\varepsilon[U^\varepsilon_t]_{t=t_*} = 0 \quad \text{for some} \quad t_* > 0 \quad \iff \quad E_\varepsilon[U^\varepsilon_{t_*}] = 0;
\]

iii) Assume in addition \( \lim_{x \to 0} k(x) = \lim_{|x| \to \infty} k(x) = 0 \). Then \( U^\varepsilon_t - z_t \to 0 \) in \( D^s \) as \( t \to 0 \) and as \( t \to \infty \), uniformly with respect to \( \varepsilon \).

**Proof.** We basically follow the outline of the arguments in [9, Sections 3 and 4] but we considerably simplify the proofs there. Moreover, since the solution \( z_1 \) to the unperturbed problem is not explicitly known, in some of the steps the proof needs more care with respect to [9].

In order to shorten formulas we denote by \( \|\cdot\| \) the norm of \( \varphi \in D^s \), instead of \( \|\cdot\|_{D^s} \). The norm in dual space \( (D^s)' \) is denoted by \( \|\cdot\|' \). Thus \( \|(-\Delta)^s \varphi\|' = \|\varphi\| \) for any \( \varphi \in D^s \).

If \( X, Y \) are Banach spaces, we denote by \( \|\cdot\|_{X \to Y} \) the standard norm in \( B(X, Y) \), which is the space of linear and continuous operators \( X \to Y \). If \( X \) and \( Y \) are clear by definition, we write simply \( \|\cdot\| \). For instance, if \( J : D^s \to \mathbb{R} \) is a smooth functional, then for any \( u \in D^s \) we have \( J'[u] \in (D^s)' \), \( J''[u] \in B(D^s, (D^s)') \) and we have

\[
\|J'[u]\|' = \sup_{\varphi \in D^s, \|\varphi\| = 1} |(J'[u], \varphi)|, \quad \|J''[u]\|' = \sup_{\psi \in D^s, \|\psi\| = 1} \|J''[u] \psi\|' = \sup_{\psi \in D^s, \|\psi\| = 1} |J''[u](\psi, \varphi)|.
\]

We introduce also the extended space

\[
D^s_\times = D^s \times \mathbb{R} \quad \text{with norm} \quad \|(\eta, \gamma)\|_\times^2 := \|\eta\|^2 + \gamma^2,
\]

and its dual space \( (D^s_\times)' = (D^s)' \times \mathbb{R} \), with norm \( \|((-\Delta)^s \eta, \gamma)\|'_\times = \|(\eta, \gamma)\|_\times \).

Consider the map \( \mathfrak{F} = [\mathfrak{F}_1, \mathfrak{F}_2] : (\mathbb{R} \times \mathbb{R}_+) \times D^s_\times \to (D^s_\times)' \):

\[
\mathfrak{F}_1(\varepsilon, t; \eta, \gamma) := E_\varepsilon'[z_t + \eta] + t \gamma (-\Delta)^s z_t \in (D^s)', \\
\mathfrak{F}_2(\varepsilon, t; \eta, \gamma) := t((-\Delta)^s z_t, \eta) \in \mathbb{R}
\]
(the multiplier $t$ in both entries is a normalization factor; notice that $t\|\dot{z}_t\| = \|\dot{z}_1\|$ does not depend on $t$ by (3.4)).

The function $\mathfrak{F}$ is continuously differentiable (for the derivative with respect to $t$ use part $iii$) in Theorem 1.1) and $\mathfrak{F}(0,t;0,0) \equiv 0$. We fix $t > 0$ and solve the equation $\mathfrak{F}(\varepsilon,t;\eta,\gamma) = 0$ in a neighbourhood of $(0,t;0,0)$. To this goal we define

$$L(\varepsilon,t;\eta,\gamma) := \partial_{(\eta,\gamma)} \mathfrak{F}(\varepsilon,t;\eta,\gamma) \in B(D_s^\ast,(D_s^\ast)')$$

and

$$L(t) := L(0,t;0,0).$$

In matrix form, we have

$$L(\varepsilon,t;\eta,\gamma) = \begin{bmatrix} E''_{\varepsilon}[z_t + \eta] & t(-\Delta)^s \dot{z}_t \\ t(-\Delta)^s \dot{z}_t & 0 \end{bmatrix}$$

and

$$L(t) = \begin{bmatrix} E''_{0}[z_t] & t(-\Delta)^s \dot{z}_t \\ t(-\Delta)^s \dot{z}_t & 0 \end{bmatrix}.$$  

First, we claim that $L(t)$ is invertible, and the norm of $L(t)^{-1}$ admits the estimate independent of $t$. Indeed, for $\varphi \in D^s$ and $\zeta \in \mathbb{R}$ we have

$$L(t) \begin{bmatrix} \varphi \\ \zeta \end{bmatrix} = \begin{bmatrix} E''_{0}[z_t] \varphi + t\zeta(-\Delta)^s \dot{z}_t \\ t((-\Delta)^s \dot{z}_t,\varphi) \end{bmatrix}.$$  

Assume $L(t)[\varphi,\zeta]^T = 0$. The vanishing of the first entry implies that $\varphi \in \langle \dot{z}_t \rangle$ and $\zeta = 0$ by part $i$) in Corollary 4.4; on the other hand, the vanishing of the second entry gives $\varphi \in \langle \dot{z}_t \rangle^\perp$. So, $L(t)$ is injective.

To prove that $L(t)$ is surjective we take $v \in D^s$, $\gamma \in \mathbb{R}$, and seek for $\varphi \in D^s$, $\zeta \in \mathbb{R}$ such that

$$E''_{0}[z_t] \varphi = -t\zeta(-\Delta)^s + (-\Delta)^s v,$$

$$t((-\Delta)^s \dot{z}_t,\varphi) = \gamma.  \quad (6.2)$$

We choose

$$\zeta = \frac{((-\Delta)^s \dot{z}_t,v)}{t\|\dot{z}_t\|^2},$$

so that $v - t\zeta \dot{z}_t \in \langle \dot{z}_t \rangle^\perp$. By part $ii$) in Corollary 4.4, we find a unique $\varphi^\perp \in \langle \dot{z}_t \rangle^\perp$ such that

$$E''_{0}[z_t] \varphi^\perp = (-\Delta)^s (v - t\zeta \dot{z}_t),$$

It is easy to check that the (unique) solution to (6.2) is given by

$$\varphi = \frac{\gamma}{t\|\dot{z}_t\|^2} \dot{z}_t + \varphi^\perp.$$
We recall that $t\|\dot{z}_t\| = \|\dot{z}_1\|$ and use part (ii) in Corollary 4.4 to infer
\[
|\zeta| \leq \frac{\|v\|}{\|\dot{z}_1\|}, \quad \|\varphi\|^2 \leq \frac{\|v\|^2}{\kappa_s^2} + \frac{\gamma^2}{\|\dot{z}_1\|^2}.
\]
Thus,
\[
\|\mathcal{L}(t)^{-1}\| \leq c_* := \frac{1}{\min\{\kappa_s, \|\dot{z}_1\|\}}, \quad (6.3)
\]
and the claim follows.

Thanks to the implicit function theorem, for any $t > 0$ and any $\varepsilon$ close to zero the equation $\mathfrak{F}(\varepsilon, t; \eta, \gamma) = 0$ is uniquely solvable in a neighbourhood of $(0, t; 0, 0)$. We denote this solution by $[\eta^\varepsilon_t, \gamma^\varepsilon_t]^\top$ and put
\[
U^\varepsilon_t := z_t + \eta^\varepsilon_t.
\]
The equality $\mathfrak{F}_2(\varepsilon, t; \eta^\varepsilon_t, \gamma^\varepsilon_t) = 0$ gives $\eta^\varepsilon_t \in \langle \dot{z}_t \rangle^\perp$ and thus $U^\varepsilon_t \in \langle \dot{z}_t \rangle^\perp$. Further, the equality $\mathfrak{F}_1(\varepsilon, t; \eta^\varepsilon_t, \gamma^\varepsilon_t) = 0$ reads
\[
E'_\varepsilon[U^\varepsilon_t] = -t\gamma^\varepsilon_t (-\Delta)^s \dot{z}_t.
\]
Testing this equation with $\dot{z}_t$ we see that it solves (6.1). The $C^1$ regularity of the function $(\varepsilon, t) \mapsto (U^\varepsilon_t, \gamma^\varepsilon_t)$ is given by the implicit function theorem.

To prove (i) we need some estimates. We begin with
\[
\|\mathcal{L}(\varepsilon, t; \eta, \gamma) - \mathcal{L}(t)\| \leq \|E''_0[z_t + \eta] - E''_0[z_t]\| + |\varepsilon| \|G''[z_t + \eta]\|. \quad (6.4)
\]
We define
\[
C_0(\rho) := \sup_{\|\eta\| \leq \rho} \|E''_0[z_t + \eta] - E''_0[z_t]\|
\]
and notice that $C_0(\rho) \to 0$ as $\rho \to 0$, because $E_0$ is of class $C^2$. Moreover, $C_0(\rho)$ does not depend on $t$. Indeed, since $\mathcal{I}(t)$ is an isometry in $\mathcal{D}^s$, the relation (4.1) gives
\[
C_0(\rho) = \sup_{\|\mathcal{I}(t^{-1})\eta\| \leq \rho} \|E''_0[\mathcal{I}(t)(z_1 + \mathcal{I}(t^{-1})\eta)] - E''_0[\mathcal{I}(t)z_1]\| = \sup_{\|\eta\| \leq \rho} \|E''_0[z_1 + \eta] - E''_0[z_1]\|.
\]
Thus we can fix a small $\rho_0 > 0$ such that if $\|\eta\| \leq \rho_0$ then the first term in the right-hand side of (6.4) does not exceed $\frac{1}{\kappa_{s^*}}$, where $c_*$ is defined in (6.3).
Further, by the Hölder inequality and (1.2) we obtain for \( \| \eta \| \leq \rho_0 \)

\[
\| G''[z_t + \eta]\| \leq (q-1) \| k \|_{\infty} \sup_{\| \varphi \|, \| \psi \| = 1} \int_{\mathbb{R}^n} |x|^{-bq} |z_t + \eta|^{q-2} |\varphi| | \psi| \, dx \leq c_1 \| z_t + \eta \|^{q-2} \leq c_2
\]

where \( c_2 \) does not depend on \( t \). Therefore, there is \( \varepsilon_0 \) independent of \( t \) such that for \( |\varepsilon| < \varepsilon_0 \) and \( \| \eta \| \leq \rho_0 \) the second term in the right-hand side of (6.4) also does not exceed \( \frac{1}{3c_*} \).

By the Banach inverse mapping theorem, for any \( t > 0, \gamma \in \mathbb{R}, |\varepsilon| < \varepsilon_0 \) and \( \| \eta \| \leq \rho_0 \) the operator \( \mathcal{L}(\varepsilon, t; \eta, \gamma) \) is invertible, and

\[
\| \mathcal{L}(\varepsilon, t; \eta, \gamma)^{-1} \| = \| \mathcal{L}(t)^{-1} (I + (\mathcal{L}(\varepsilon, t; \eta, \gamma) - \mathcal{L}(t)) \mathcal{L}(t)^{-1})^{-1} \| \leq 3c_\ast.
\]

We are allowed to differentiate the implicit function and obtain

\[
\partial_\varepsilon \left[ \eta_t^\varepsilon \right] = -\mathcal{L}(\varepsilon, t; \eta_t^\varepsilon, \gamma_t^\varepsilon)^{-1} \partial_\varepsilon \mathcal{L}(\varepsilon, t; \eta_t^\varepsilon, \gamma_t^\varepsilon) = \mathcal{L}(\varepsilon, t; \eta_t^\varepsilon, \gamma_t^\varepsilon)^{-1} \left[ G'[z_t + \eta_t^\varepsilon] \right]. \tag{6.5}
\]

Using again the Hölder inequality and (1.2) we get for \( \| \eta \| \leq \rho_0 \)

\[
\| G'[z_t + \eta] \|' \leq \| k \|_{\infty} \sup_{\| \varphi \| = 1} \int_{\mathbb{R}^n} |x|^{-bq} |z_t + \eta|^{q-1} |\varphi| \, dx \leq c_3 \| z_t + \eta \|^{q-1} \leq c_4 \tag{6.6}
\]

with \( c_4 \) independent of \( t \). Therefore, the relation (6.5) gives \( \| \partial_\varepsilon \eta_t^\varepsilon \| \leq c_5 := 3c_*c_4 \) which implies \( \| U_t^\varepsilon - z_t \| = \| \eta_t^\varepsilon \| \leq c_5 \varepsilon \). Thus, \( i) \) is proved.

Reducing \( \varepsilon_0 \) if needed we arrive at \( c_5\varepsilon_0 \leq \rho_0 \). Now \( \eta_t^\varepsilon \) (and thus \( U_t^\varepsilon \)) is well-defined in the whole strip \( t > 0, |\varepsilon| < \varepsilon_0 \).

To prove \( ii) \) we test (6.1) with \( \dot{U}_t^\varepsilon \). This gives

\[
\frac{d}{dt} E_\varepsilon[U_t^\varepsilon] = (E_\varepsilon'[U_t^\varepsilon], \dot{U}_t^\varepsilon) = \left( \frac{E_\varepsilon'[U_t^\varepsilon]}{\| \dot{z}_t \|^2}, ((-\Delta)^s \dot{z}_t, \dot{U}_t^\varepsilon) \right)
\]

\[
= \left( 1 + \frac{((-\Delta)^s \dot{z}_t, \dot{U}_t^\varepsilon - \dot{z}_t)}{\| \dot{z}_t \|^2} \right) (E_\varepsilon'[U_t^\varepsilon], \dot{z}_t).
\]

Part \( iii) \) in Theorem [13] allows us to write

\[
((-\Delta)^s \dot{z}_t, \dot{U}_t^\varepsilon) = \frac{d}{dt} \left((-\Delta)^s \dot{z}_t, U_t^\varepsilon\right) - ((-\Delta)^s \ddot{z}_t, U_t^\varepsilon) - ((-\Delta)^s \dot{z}_t, U_t^\varepsilon) = -((-\Delta)^s \ddot{z}_t, U_t^\varepsilon)
\]

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(for the last equality use $U_t^e \in \langle \dot{z}_t \rangle^\perp$) and, in a similar way,
\[ ((-\Delta)^s \dot{z}_t, \dot{z}_t) = \frac{d}{dt} ((-\Delta)^s \dot{z}_t, z_t) - ((-\Delta)^s \dot{z}_t, \dot{z}_t) = -((\Delta)^s \ddot{z}_t, \dot{z}_t). \]

We differentiate with respect to $t$ the identity $t \dot{z}_t = \mathcal{I}(t) \dot{z}_1$ to get $t^2 \ddot{z}_t = \mathcal{I}(t)(\ddot{z}_1 - \dot{z}_1)$.

Since $\mathcal{I}$ is an isometry, we infer
\[ \left| \left( (-\Delta)^s \dot{z}_t, U_t^e - \dot{z}_t \right) \right| = \frac{\left| \left( (-\Delta)^s \ddot{z}_t, U_t^e - \dot{z}_t \right) \right|}{\| \dot{z}_t \|^2} \leq \frac{t^{-2} \| \ddot{z}_1 - \dot{z}_1 \|}{t^{-2} \| \dot{z}_1 \|^2} \| U_t^e - \dot{z}_t \| \leq c_6 \varepsilon \]
by i), with $c_6$ independent of $t > 0$. Therefore, if $c_6 \varepsilon_0 < 1$ then
\[ \frac{d}{dt} \mathcal{E}[U_t^e] = 0 \quad \iff \quad (\mathcal{E}[U_t^e], \dot{z}_t) = 0, \]
and the latter relation is equivalent to $\mathcal{E}[U_t^e] = 0$ by (6.1). Thus ii) in the statement holds.

To prove iii) we sharpen the estimate (6.6). The assumptions on $k$ imply that the function
\[ g(t) := \left( \int_{\mathbb{R}^n} |k(tx)||x|^{-bq} \,dz_1^q \right)^{\frac{q-1}{q}} = \left( \int_{\mathbb{R}^n} |k(x)||x|^{-bq} \,dz_1^q \right)^{\frac{q-1}{q}} \]
is bounded, continuous (use part iii) in Theorem 1.1), and satisfies
\[ \lim_{t \to 0} g(t) = \lim_{t \to \infty} g(t) = 0 \]
by the Lebesgue dominated convergence theorem.

For $\|\eta\| \leq \rho \leq \rho_0$ we write
\[ \|G'[z_t + \eta]\|' \leq \sup_{\|\varphi\| = 1} \int_{\mathbb{R}^n} |k(x)||x|^{-bq} |z_t + \eta|^{q-1} |\varphi| \,dx \leq I_1 + I_2 \]
\[ := \sup_{\|\varphi\| = 1} \int_{\mathbb{R}^n} |k(x)||x|^{-bq} |z_t|^{q-1} |\varphi| \,dx + c \sup_{\|\varphi\| = 1} \int_{\mathbb{R}^n} |k(x)||x|^{-bq} |\dot{z}_t + \eta|^{q-2} |\eta| |\varphi| \,dx. \]

We change the variable, use Hölder inequality and (1.2) once again and arrive at
\[ I_1 = \sup_{\|\varphi\| = 1} \int_{\mathbb{R}^n} |k(tx)||x|^{-bq} \dot{z}_1^{q-1} |\mathcal{I}(t^{-1})\varphi| \,dx \leq c_7 g(t). \]
In a similar way we get

\[ I_2 \leq c \| z_t + \eta \|^{q-2} \| \eta \| \leq c_8 \| \eta \| \]

(here \( c_7 \) and \( c_8 \) do not depend on \( t \)). Therefore, (6.5) gives

\[ \| \partial_\varepsilon \eta_\varepsilon \| \leq c_9 (g(t) + \rho), \quad c_9 := 3c_\ast \max\{c_7, c_8\}, \]

which implies \( \| \eta_\varepsilon \| \leq c_9 (g(t) + \rho) \varepsilon \). Thus, we obtain the implication

\[ \| \eta_\varepsilon \| \leq \rho \leq \rho_0 \quad \implies \quad \| \eta_\varepsilon \| \leq c_9 (g(t) + \rho) \varepsilon_0. \]

(6.8)

Reducing \( \varepsilon_0 \) if needed we arrive at \( c_9 \varepsilon_0 < 1 \). Then (6.8) yields

\[ \| U_\varepsilon^t - z_t \| = \| \eta_\varepsilon \| \leq \frac{c_9 \varepsilon_0}{1 - c_9 \varepsilon_0} g(t), \]

and iii) follows. The proof is complete. \( \square \)

**Proof of Theorem 1.4.** As in the previous proof, we let \( \| \cdot \| \) be the norm in \( \mathcal{D}^s \).

Up to multiplication of \( u \) by a proper constant we can assume without restriction that \( \lim_{x \to 0} k(x) = \lim_{|x| \to \infty} k(x) = 0 \).

Let \( U_\varepsilon^t \) be the function given by Lemma 6.1 and write

\[ E_\varepsilon[U_\varepsilon^t] = E_0(z_t) + \frac{1}{2} (\| U_\varepsilon^t \|^2 - \| z_t \|)^2 - \frac{1}{q} \int_{\mathbb{R}^n} |x|^{-bq}(1 + \varepsilon k(x))((U_\varepsilon^t)_+ - z_t^q) dx - \varepsilon G(z_t). \]

Recall that \( \| z_t \| = \| z_1 \| \) does not depend on \( t \). From the statement iii) in Lemma 6.1 we infer that \( U_\varepsilon^t \) is uniformly bounded in \( \mathcal{D}^s \), \( \| U_\varepsilon^t - z_t \| = o(1) \) as \( t \to 0 \) and as \( t \to \infty \) and therefore

\[ \left( \| U_\varepsilon^t \|^2 - \| z_t \|^2 \right) \leq (\| U_\varepsilon^t \| + \| z_t \|) \| U_\varepsilon^t - z_t \| = o(1). \]

Moreover, \( \| |x|^{-b((U_\varepsilon^t)_+ - z_t)} \|_q \leq \| |x|^{-b(U_\varepsilon^t - z_t)} \|_q = o(1) \) by (1.2). Using also Hölder inequality we plainly infer

\[ \int_{\mathbb{R}^n} |x|^{-bq}(1 + \varepsilon k(x))((U_\varepsilon^t)_+ - z_t^q) dx \leq c \int_{\mathbb{R}^n} |x|^{-bq}((U_\varepsilon^t)_+ - z_t^q) dx = o(1). \]
Finally, we already noticed that $|G(z_t)| \leq g(t)^q = o(1)$, where $g$ is the function in (6.7), and we can conclude that

$$
\phi^\varepsilon(t) := E_\varepsilon[U_\varepsilon^t] = E_0(z_t) + o(1) = E_0(z_1) + o(1) \text{ as } t \to 0 \text{ and as } t \to \infty.
$$

Thus, $\phi^\varepsilon$ has at least one critical point $t_\varepsilon$ (in fact, $\phi^\varepsilon$ might be constant). Hence $U_\varepsilon^{t_\varepsilon}$ is a critical point for $E_\varepsilon$ by the statement $ii)$ in Lemma 6.1. The conclusion follows.

□

**Remark 6.2** Theorem 1.4 in [9] can be extended to the fractional case as well, with minor modifications in the proof. Moreover, the above arguments apply to more general problems of the form

$$
(-\Delta)^s u + \lambda |x|^{-2s}u = |x|^{-b_q}u^{q-1} + \varepsilon f(x, u),
$$

where the perturbation term $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying suitable regularity and growth assumptions at 0 and at $\infty$.

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