Strict Log-Subadditivity for Overpartition Rank

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Abstract. Bessenrodt and Ono initially found the strict log-subadditivity of partition function $p(n)$, that is, $p(a + b) < p(a)p(b)$ for $a, b > 1$ and $a + b > 9$. Many other important partition statistics are proved to enjoy similar properties. Lovejoy introduced the overpartition rank as an analog of Dyson’s rank for partitions from the $q$-series perspective. Let $\mathcal{N}(a, c, n)$ denote the number of overpartitions with rank congruent to $a$ modulo $c$. Ciolan computed the asymptotic formula of $\mathcal{N}(a, c, n)$ and showed that $\mathcal{N}(a, c, n) > \mathcal{N}(b, c, n)$ for $0 \leq a < b \leq \lfloor \frac{c}{2} \rfloor$ and $n$ large enough if $c \geq 7$.

In this paper, we derive an upper bound and a lower bound of $\mathcal{N}(a, c, n)$ for each $c \geq 3$ by using the asymptotics due to Ciolan. Consequently, we establish the strict log-subadditivity of $\mathcal{N}(a, c, n)$ analogous to the partition function $p(n)$.

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1. Introduction

Strict log-subadditivity phenomena of partition statistics originated from the work of Bessenrodt and Ono [3]. Recall that a partition of a positive integer $n$ is a sequence of non-increasing positive integers whose sum equals $n$. Let $p(n)$ denote the number of partitions of $n$. Bessenrodt and Ono showed that $p(n)$ satisfies the strict log-subadditivity result, that is, for $a, b > 1$ and $a + b > 9$

$$p(a + b) < p(a)p(b).$$

Since then different partition functions are proved to enjoy the strict log-subadditivity as $p(n)$, for instance, the overpartition function [16], the $k$-regular partition function [2] and the spt-function [6,10].

Strict log-subadditivity of other statistics concerned with partitions comes into sight. Dyson [11] introduced the rank of a partition to interpret Ramanujan’s congruences for $p(n)$ combinatorially. This is defined as the largest part...
of the partition minus the number of parts. Let \( N(a, c, n) \) denote the number of partitions of \( n \) with rank congruent to \( a \) modulo \( c \). Hou and Jagadeesan [15] showed that

\[
N(r, 3, a + b) < N(r, 3, a)N(r, 3, b)
\]

when \( r = 0 \) (resp. \( r = 1, 2 \)) and \( a + b \geq 12 \) (resp. 11, 12). They also conjectured the general strict log-subadditivity: for \( 0 \leq r < t \) and \( t \geq 2 \),

\[
N(r, t, a + b) < N(r, t, a)N(r, t, b)
\] (1.1)

holds for sufficiently large \( a \) and \( b \). This conjecture was confirmed by Males [18]. Gomez and Zhu [13] proved that (1.1) holds for \( t = 2 \) and \( a, b \geq 12 \). The strict log-subadditivity for other statistics are referred to [1, 5, 14].

In this paper, we show that the rank of overpartitions also satisfies strict log-subadditivity. Recall that an overpartition of a nonnegative integer \( n \) is a partition of \( n \) where the first occurrence of each distinct part may be overlined. The total number of overpartitions of \( n \) is denoted by \( p(n) \). Corteel and Lovejoy [9] gave a generating function of \( p(n) \)

\[
F(q) := \sum_{n \geq 0} p(n)q^n = \prod_{n \geq 1} \frac{1 + q^n}{1 - q^n},
\] (1.2)

where \( q = e^{2\pi iz} \) and \( z = x + iy \) with \( y > 0 \). Engel [12] provided an estimation for the error term of the convergent series for the overpartition function, due to Zuckerman [20] which reads

\[
p(n) = \frac{1}{2\pi} \sum_{1 \leq k \leq N} \sqrt{k} \sum_{0 \leq h \leq k} \frac{\omega(h, k)^2}{\omega(2h, k)} e^{-\frac{2\pi i nh}{k}} \frac{d}{dn} \left( \frac{\sinh \left( \frac{\pi \sqrt{n}}{k} \right)}{\sqrt{n}} \right) + R_2(n, N),
\]

where

\[
|R_2(n, N)| \leq \frac{N \frac{5}{2}}{\pi n \frac{3}{2}} \sinh \left( \frac{\pi \sqrt{n}}{N} \right).
\]

In particular, when \( N = 3 \), we have

\[
p(n) = \frac{1}{8n} \left[ \left( 1 + \frac{1}{\pi \sqrt{n}} \right) e^{-\pi \sqrt{n}} + \left( 1 - \frac{1}{\pi \sqrt{n}} \right) e^{\pi \sqrt{n}} \right] + R_2(n, 3),
\] (1.3)

where

\[
|R_2(n, 3)| \leq \frac{3 \frac{5}{2}}{\pi n \frac{3}{2}} \sinh \left( \frac{\pi \sqrt{n}}{3} \right) \leq \frac{3 \frac{5}{2} e^{\frac{\pi \sqrt{n}}{3}}}{2\pi n \frac{3}{2}} \leq \frac{\pi e^{\pi \sqrt{n}}}{16\pi n \frac{3}{2}}.
\]

Let \( \ell(\lambda) \) denote the largest part of \( \lambda \), \( \#(\lambda) \) denote the number of parts of \( \lambda \). Lovejoy [17] defined the rank of an overpartition \( \lambda \) as

\[
\text{rank}(\lambda) = \ell(\lambda) - \#(\lambda).
\]

Let \( \overline{N}(a, c, n) \) be the number of overpartitions of \( n \) with rank congruent to \( a \) modulo \( c \). Our main result is the following.
Theorem 1. Given any residue $a \pmod{c}$, we have
\[ \overline{N}(a, c, n_1 + n_2) < \overline{N}(a, c, n_1)\overline{N}(a, c, n_2) \]  
(1.4)
if $3 \leq c \leq 5$ and $n_1, n_2 \geq 9$ or if $c \geq 6$ and $n_1, n_2 \geq M_c$, where
\[ M_c := 1.691 \times 10^{13} c^{20} \exp \left( \frac{16c^2(2c^2 + \pi)}{\pi^2} \right). \]

This paper is organized as follows. In Sect. 2, we state the asymptotic behavior of $\overline{N}(a, c, n)$ due to Ciolan. In Sect. 3, we estimate the upper bounds for both main terms and error terms of the asymptotic formula. Based on these estimates, we finally prove Theorem 1 in Sect. 4.

2. Asymptotic Formula Due to Ciolan

Our approach to the strict log-subadditivity of overpartition rank relies on the asymptotic behavior of the generating function of $\overline{N}(a, c, n)$ in [7]. In this section, we recall the asymptotic formula due to Ciolan [7,8].

We begin with some notations that we need. Let
\[ \omega_{h,k} := \exp(\pi is(h,k)), \]
where the Dedekind sum $s(h,k)$ is defined by
\[ s(h,k) := \sum_{u \pmod{k}} \left( \left( \frac{u}{k} \right) \left( \frac{hu}{k} \right) \right); \]
and $(\cdot)$ is the sawtooth function defined by
\[ ((x)) := \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases} \]

In what follows, $0 \leq h < k$ are coprime integers, and $h' \in \mathbb{Z}$ is defined by the congruence $hh' \equiv -1 \pmod{k}$. Further, let $0 < a < c$ are coprime positive integers with $c > 2$, $c_1 := \frac{c}{(c,k)}$ and $k_1 := \frac{k}{(c,k)}$. Let the integer $0 \leq l < c_1$ be the solution to $l \equiv ak_1 \pmod{c_1}$.

We then introduce several Kloosterman sums. Here and throughout we write $\sum'_{h}$ to denote summation over the integers $0 \leq h < k$ that are coprime to $k$.

If $c | k$, let
\[ A_{a,c,k}(n, m) := (-1)^{k_1+1} \tan \left( \frac{\pi a}{c} \right) \sum'_{h} \frac{\omega_{h,k}^2}{\omega_{h,k}/2} \cdot \cot \left( \frac{\pi ah'}{c} \right) \cdot e^{-\frac{2\pi i a}{c}} \cdot e^{\frac{2\pi i (nh + mh')}{c}}, \]
and
\[ B_{a,c,k}(n, m) := -\frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum'_{h} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot \frac{1}{\sin \left( \frac{\pi ah'}{c} \right)} \cdot e^{-\frac{2\pi i a}{c}} \cdot e^{\frac{2\pi i (nh + mh')}{c}}. \]
If \( c \nmid k \) and \( 0 < \frac{l}{c_1} \leq \frac{1}{4} \), let
\[
D_{a,c,k}(n,m) := \frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum_h \omega_{h,k}^{2} \cdot e^{2\pi i (nh+mh')},
\]
and if \( c \nmid k \) and \( \frac{3}{4} < \frac{l}{c_1} < 1 \), let
\[
D_{a,c,k}(n,m) := -\frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum_h \omega_{h,k}^{2} \cdot e^{2\pi i (nh+mh')},
\]
Finally, if \( c \nmid k \), set
\[
\delta_{c,k,r} := \begin{cases} 
\frac{1}{16} - \frac{l^2}{2c_1} - \frac{l}{c_1}, & \text{if } 0 < \frac{l}{c_1} \leq \frac{1}{4}, \\
0, & \text{if } \frac{1}{4} < \frac{l}{c_1} \leq \frac{3}{4}, \\
\frac{1}{16} - \frac{3l}{2c_1} + \frac{l^2}{c_1} + \frac{1}{2} - r \left( 1 - \frac{1}{c_1} \right), & \text{if } \frac{3}{4} < \frac{l}{c_1} < 1,
\end{cases}
\]
and
\[
m_{a,c,k,r} := \begin{cases} 
-\frac{1}{2c_1} (2(a k_1 - l)^2 + c_1 (a k_1 - l) + 2 r c_1 (a k_1 - l)), & \text{if } 0 < \frac{l}{c_1} \leq \frac{1}{4}, \\
0, & \text{if } \frac{1}{4} < \frac{l}{c_1} \leq \frac{3}{4}, \\
-\frac{1}{2c_1} (2(a k_1 - l)^2 + 3 c_1 (a k_1 - l) - 2 r c_1 (a k_1 - l) - c_1^2 (2r - 1)), & \text{if } \frac{3}{4} < \frac{l}{c_1} < 1.
\end{cases}
\]
Let \( \overline{N}(m,n) \) denote the number of overpartitions of \( n \) with rank \( m \). We adopt the notation \( O(u; q) \) in [17] to denote the generating function of \( \overline{N}(m,n) \), that is,
\[
O(u; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} \overline{N}(m,n) u^m q^n. \tag{2.1}
\]
If \( 0 < a < c \) are coprime positive integers, and \( \zeta_n := e^{2\pi i/n} \) denotes the primitive \( n \)-th root of unity, we define
\[
O \left( \frac{a}{c}; q \right) := O \left( \zeta_c^a; q \right) = 1 + \sum_{n \geq 1} A \left( \frac{a}{c}; n \right) q^n. \tag{2.2}
\]
Ciolan obtained the following asymptotic formula for \( A \left( \frac{a}{c}; n \right) \).

**Theorem 2.** [7, Theorem 1] If \( 0 < a < c \) are coprime positive integers with \( c > 2 \), and \( \varepsilon > 0 \) is arbitrary, then
\[
A \left( \frac{a}{c}; n \right) = i \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{\frac{\pi}{c_1 k_1}}} B_{a,c,k}(-n,0) \cdot \sinh \left( \frac{\pi \sqrt{m}}{k} \right) \\
+ 2 \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{\frac{\pi}{c_1 k_1}}} D_{a,c,k}(-n,m_{a,c,k,r},r) \cdot \sinh \left( \frac{4\pi \sqrt{\delta_{c,k,r,m}}}{k} \right) + O(n^\varepsilon). \tag{2.3}
\]
To obtain the inequality $N(a, c, n) > N(b, c, n)$ for $0 \leq a < b$, Ciolan [8] bounded the main term and the error term in above theorem. Here we list these bounds respectively.

- **Bounds in the main terms:** Ciolan [8, § 4.2] gave upper bounds for Kloosterman sums and hyperbolic sine function in (2.3).

\[
|B_{j,c,k}(-n,0)| \leq \cot \left( \frac{\pi}{2c} \right) \frac{2k(1 + \log \left( \frac{n-1}{c} \right))}{\pi \left( 1 - \frac{n^2}{24} \right)},
\]

(2.4)

\[
|D_{j,c,k}(-n,m_{j,c,k,r})| \leq \frac{k}{\sqrt{2}} \cot \left( \frac{\pi}{2c} \right),
\]

(2.5)

\[
\sinh \left( \frac{4\pi \sqrt{\delta_{c,k,r}n}}{k} \right) \leq \sinh \left( \pi \sqrt{n} \left( 1 - \frac{4}{c} \right) \right).
\]

(2.6)

- **Bounds in the error terms:** Two different kinds of error terms are considered here. The first kind arises from the Circle Method. We follow the notations of Ciolan in [8, § 4.3.1], which are denoted by $S_1, S_2, \ldots, S_6$. However, $\sum_7$ and $\sum_8$ in his paper are rewritten as $S_7$ and $S_8$ instead to keep consistent. Note that estimations $S_1, S_2, \ldots, S_6$ are obtained by Ciolan and $S_7$ and $S_8$ are given in Appendix.

\[
S_1 < 4C_3e^{2\pi} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}}, \quad S_2 < 4C_1e^{2\pi} \sqrt{2} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}},
\]

(2.7)

\[
S_3 < 2C_4e^{2\pi} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}}, \quad S_4 < C_5e^{2\pi} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}},
\]

\[
S_5 < C_2e^{2\pi} \sqrt{2} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}}, \quad S_6 < C_2e^{2\pi} \frac{1}{\sqrt{2}} \cot \left( \frac{\pi}{2c} \right) \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}},
\]

\[
S_7 < \frac{4e^{2\pi} n^{\frac{1}{2}} \log \left( \frac{n}{c} \right)}{\pi \left( 1 - \frac{n^2}{24} \right) \sin \left( \frac{\pi}{c} \right)}, \quad S_8 < \frac{4\sqrt{2}e^{2\pi} n^{\frac{1}{2}} \log \left( \frac{n}{c} \right)}{\pi \left( 1 - \frac{n^2}{24} \right) \sin \left( \frac{\pi}{c} \right)},
\]

where

\[
C_1 := \sum_{r \geq 1} p(r) \left( e^{-\frac{(16r-1)\pi}{16}} + e^{-\frac{(16r+7)\pi}{16}} \right), \quad C_2 := 2 \sum_{r \geq 1} p(r) e^{-\frac{(2^2-8)\pi r}{16c^2}},
\]

(2.8)

\[
C_3 := \sum_{r \geq 1} p(r) e^{-\pi r}, \quad C_4 := \sum_{r \geq 1} p(r) e^{-\frac{\pi r}{2c^2}}, \quad C_5 := \sum_{r \geq 1} p(r) e^{-\frac{\pi (2c+1)}{8}}.
\]

(2.9)

The other kind of error terms arises from integrating over the remaining parts of the interval and integrating along a smaller arc. We adopt the following notations. The errors arising from integrating over the remaining parts of the interval are denoted by $S_{2err}, S_{5err}, S_{6err}$, and errors introduced by integrating along a smaller arc are denoted by $I_{2err}, I_{5err}, I_{6err}$. Ciolan [8, § 4.3.3—4.3.4] provided the following bounds on each of those
pieces:

\[ S_{2err} < \sqrt{2} e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) n^{-\frac{1}{2}} \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}}, \]

\[ S_{5err} < 2\sqrt{2} e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) n^{-\frac{1}{2}} \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}}, \]

\[ S_{6err} < \sqrt{2} e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) n^{-\frac{1}{2}} \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}} \]  

(2.10)

and

\[ I_{2err} < 2\sqrt{2} \left( \frac{4}{3} + 2^5 \right) e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} n^{\frac{1}{4}}, \]

\[ I_{5err} < 4\sqrt{2} \left( \frac{4}{3} + 2^5 \right) e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} n^{\frac{1}{4}}, \]

\[ I_{6err} < 2\sqrt{2} \left( \frac{4}{3} + 2^5 \right) e^{2\pi + \frac{\pi}{2}} \cot \left( \frac{\pi}{2c} \right) \frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} n^{\frac{1}{4}}. \]

(2.11)

3. Bounds for \( \overline{N}(a, c, n) \)

In this section, we show that \( \overline{N}(a, c, n) \) can be bounded by a constant multiple of \( \overline{p}(n) \) for each \( c \geq 3 \).

**Theorem 3.** We have

\[ 0.0019 \overline{p}(n) < \overline{N}(a, 3, n) < 0.6648 \overline{p}(n), \quad n \geq 2089, \]  

(3.1)

\[ 0.0091 \overline{p}(n) < \overline{N}(a, 4, n) < 0.4909 \overline{p}(n), \quad n \geq 272, \]  

(3.2)

\[ 0.0103 \overline{p}(n) < \overline{N}(a, 5, n) < 0.3897 \overline{p}(n), \quad n \geq 449. \]  

(3.3)

For \( c \geq 6 \), we have

\[ \left( \frac{1}{2c} \right) \overline{p}(n) < \overline{N}(a, c, n) < \left( \frac{3}{2c} \right) \overline{p}(n) \]  

for all \( n \geq M_c \),

(3.4)

where

\[ M_c := 1.691 \times 10^{13} c^{20} \exp \left( \frac{16c^2(2c^2 + \pi)}{\pi^2} \right). \]  

(3.5)

Our main mission of this section is to prove the above theorem. We sketch our strategy here. Setting \( u = \zeta_c^a \) in (2.1), we have

\[ \mathcal{O} \left( \frac{a}{c} \right) = \sum_{n \geq 0} \sum_m \overline{N}(m, n) \zeta_c^{an} q^n. \]  

(3.6)
Comparing the coefficient of $q^n$ in (2.2) and (3.6), along with the orthogonality of the roots of unity, we obtain

$$\overline{N}(a, c, n) = \frac{1}{c} \overline{p}(n) + \frac{1}{c} \sum_{1 \leq j \leq c-1} \zeta_c^{-aj} A \left( \frac{j}{c}; n \right). \quad (3.7)$$

The second summation in right-hand side of (3.7) is denoted by $\overline{R}(a, c, n)$, and we obtain

$$\overline{N}(a, c, n) = \frac{1}{c} \overline{p}(n) + \overline{R}(a, c, n). \quad (3.8)$$

With the help of (3.8), we estimate $\overline{R}(a, c, n)$ and $\overline{p}(n)$ separately to get bounds for $\overline{N}(a, c, n)$. For $\overline{R}(a, c, n)$, we give a bound for the main terms as well as the error terms in view of Theorem 2. These will be shown in Theorem 4 and Theorem 5 respectively. Next, we give an upper bound and lower bound for $\overline{p}(n)$ in Theorem 6 based on the asymptotic formula for $\overline{p}(n)$ due to Engel [12].

From the view of Theorem 2, we have the following estimates to bound the main terms of $\overline{R}(a, c, n)$.

**Lemma 1.** For integers $c > 2$ and $n \geq 1$, then we have

$$|B_{j,c,k}(-n,0)| \leq 0.3444kc^2, \quad (3.9)$$

$$|D_{j,c,k}(-n,m_{j,c,k,r})| \leq 0.4503kc, \quad (3.10)$$

$$\sinh \left( \frac{4\pi \sqrt{\delta_{c,k,r,n}}}{k} \right) \leq \frac{1}{2} e^{\pi \sqrt{n} (1 - \frac{1}{c})}. \quad (3.11)$$

**Proof.** From [4, p. 159], for all $a, x > 0$, we have

$$\log x \leq a(x^{1/a} - 1). \quad (3.12)$$

Applying (3.12) with $x = \left( \frac{c-1}{2} \right)$ and $a = 1$, it follows that

$$\frac{1 + \log \left( \frac{c-1}{2} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right)} < 0.2704c. \quad (3.13)$$

Substituting (3.13) and

$$\cot \left( \frac{\pi}{2c} \right) \leq \frac{2c}{\pi} < 0.6367c \quad (3.14)$$

into (2.4), we get the first inequality (3.9). By means of the estimates in (2.5) and (3.14), we arrive at the second inequality (3.10). The third inequality (3.11) immediately follows from the definition of hyperbolic sine function and (2.6). This completes the proof. \(\square\)

Now we can give a bound for the main term of $\overline{R}(a, c, n)$.

**Theorem 4.** For integers $c > 2$ and $n \geq 1$, a bound for the main term of $\overline{R}(a, c, n)$ is

$$0.1624 e^{\frac{\pi \sqrt{\pi}}{c} n^{\frac{1}{2}} c + (0.0266c + 0.2123)e^{\pi \sqrt{\pi}(1 - \frac{1}{c})} n^{\frac{1}{2}} c}.$$
Proof. Immediately, the main term of (2.3) gives the main term of \( R(a, c, n) \)
\[
\frac{1}{c} \sum_{1 \leq j \leq c-1} \zeta_c^{-aj} \int \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k} \frac{B_{j,c,k}(-n,0)}{\sqrt{k}} \cdot \sinh \left( \frac{\pi \sqrt{n}}{k} \right) \\
+ \frac{1}{c} \sum_{1 \leq j \leq c-1} \zeta_c^{-aj2} \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k, c_1 \neq 4 \atop r \geq 0, \delta_{c,k,r} > 0} \frac{D_{j,c,k}(-n,m_{j,c,k,r})}{\sqrt{k}} \cdot \sinh \left( \frac{4\pi \sqrt{c_{k,r} n}}{k} \right).
\]
\[(3.15)\]

Let \( G_1 \) and \( G_2 \) denote the two sums of (3.15). Using (3.9), we have
\[
|G_1| \leq \sqrt{\frac{2}{n}} \sinh \left( \frac{\pi \sqrt{n}}{c} \right) \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k} 0.3444k^{\frac{1}{2}} c^2 \leq 0.1624e^{\frac{\pi \sqrt{n}}{c}} n^{\frac{1}{4}} c, \quad (3.16)
\]

where the last inequality is due to
\[
\sum_{1 \leq k \leq \sqrt{n} \atop c \mid k} k^{\frac{1}{2}} \leq c^{\frac{1}{2}} \sum_{1 \leq m \leq \left\lfloor \frac{\sqrt{n}}{c} \right\rfloor} m^{\frac{1}{2}} \leq c^{\frac{1}{2}} \int_1^{\left\lfloor \frac{\sqrt{n}}{c} \right\rfloor} x^{\frac{1}{2}} dx \leq \frac{2}{3c} n^{\frac{3}{4}}.
\]

Now we give the bound of \( |G_2| \). From [8, p. 480], for fixed \( k \), we have
\[
\sum_{r \geq 0 \atop c \mid k, 2 \mid k, c_1 \neq 4 \atop \delta_{c,k,r} > 0} 1 \leq \frac{c+8}{16}. \quad (3.17)
\]

Then by (3.10), (3.11) and (3.17), we get
\[
|G_2| \leq 0.4503c \sqrt{\frac{2}{n}} e^{\pi \sqrt{n}(1-\frac{2}{3})} \frac{c+8}{16} \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k} k^{\frac{1}{2}}.
\]

Since
\[
\sum_{1 \leq k \leq \sqrt{n} \atop c \mid k} k^{\frac{1}{2}} \leq \int_1^{\sqrt{n}} x^{\frac{1}{2}} dx \leq \frac{2}{3} n^{\frac{3}{4}},
\]

we have
\[
|G_2| \leq 0.4503c \sqrt{\frac{2}{n}} e^{\pi \sqrt{n}(1-\frac{2}{3})} \frac{c+8}{16} \cdot \frac{2}{3} n^{\frac{3}{4}}
\leq (0.0266c + 0.2123)e^{\pi \sqrt{n}(1-\frac{2}{3})} n^{\frac{1}{4}} c. \quad (3.18)
\]

Plugging (3.16) and (3.18) into (3.15), we complete the proof. \( \square \)

Next we shall give a bound for the error terms of \( R(a, c, n) \). To this end, we need to analyse the terms in (2.7), (2.10) and (2.11) in details.
Lemma 2. For integers $c > 2$ and $n \geq 1$, then we have

$S_1 \leq 1496.9n^{\frac{3}{4}}c$, $S_2 \leq 3111.36n^{\frac{3}{4}}c$, $S_3 \leq 1363.79C_4n^{\frac{3}{4}}c,$

$S_4 \leq 8246.98n^{\frac{3}{4}}c$, $S_5 \leq 964.35C_2n^{\frac{3}{4}}c$, $S_6 \leq 482.18C_2n^{\frac{3}{4}}c,$

$S_7 \leq 3895.2n^{\frac{3}{4}}c$, $S_8 \leq 5508.6n^{\frac{3}{4}}c,$

and

$S_{2err} \leq 386.18n^{-\frac{1}{4}}c^2$, $S_{5err} \leq 772.36n^{-\frac{1}{4}}c^2$, $S_{6err} \leq 386.18n^{-\frac{1}{4}}c^2,$

$I_{2err} \leq 1433.39n^{\frac{1}{4}}c^2$, $I_{5err} \leq 2866.78n^{\frac{1}{4}}c^2$, $I_{6err} \leq 1433.39n^{\frac{1}{4}}c^2,$

where

$C_2 := 2 \exp\left(\frac{32c^2(16c^2 + (c^2 - 8)\pi)}{\pi^2(c^2 - 8)^2}\right)$, $C_4 := \exp\left(\frac{4c^2(2c^2 + \pi)}{\pi^2}\right).$

Proof. First, we give the estimations of $C_1, C_2, C_3, C_4, C_5$ in (2.8) and (2.9).

From the following fact

$p(n) < e^{\pi\sqrt{n}}$ for all $n \geq 1$, \hspace{1cm} (3.19)

we have

$C_1 \leq \int_1^{\infty} e^{\pi\sqrt{r}} \left(e^{-\frac{(16r-1)x}{16}} + e^{-\frac{(16r+7)x}{16}}\right) dr \leq 0.8066. \hspace{1cm} (3.20)$

Similarly, it can be proved that

$C_3 \leq 0.5488$, $C_5 \leq 120.942. \hspace{1cm} (3.21)$

To estimate $C_2$, we claim that

$\sum_{n \geq 0} p(n)e^{-2\pi ny} \leq \exp\left(\frac{2e^{-2\pi y}}{(1 - e^{-2\pi y})^2}\right), \hspace{1cm} (3.22)$

where $y$ is a positive real number.

Taking logarithm of both sides of (1.2), we have

$log F(|q|) < \frac{2|q|}{(1 - |q|)^2}.$

And taking the exponential function on both sides of this inequality along with the substitution $|q| = e^{-2\pi y}$, we get (3.22). Note that for all $c > 2, c^2 - 8 > 0.$

Taking $y = \frac{c^2 - 8}{32c^2}$ in (3.22), we have

$C_2 \leq 2 \exp\left(\frac{2e^{-\frac{(c^2-8)\pi}{16c^2}}}{\left(1 - e^{-\frac{(c^2-8)\pi}{16c^2}}\right)^2}\right).$

For $x > -1$, it can be easily derived from

$\frac{x}{1+x} < 1 - e^{-x} < x$


that

\[
\frac{e^{-x}}{(1-e^{-x})^2} < \frac{1+x}{x^2}.
\]

Therefore

\[
C_2 \leq 2 \exp \left( \frac{32c^2(16c^2+(c^2-8)\pi)}{\pi^2(c^2-8)^2} \right) := \overline{C}_2.
\]  

From (3.22) and (3.23), following the same argument as above, one can derive

\[
C_4 \leq \exp \left( \frac{4c^2(2c^2+\pi)}{\pi^2} \right) := \overline{C}_4.
\]

Since

\[
\sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}} \leq \int_{1}^{\sqrt{n}} x^{-\frac{1}{2}} \, dx \leq 2n^{\frac{1}{4}},
\]  

substituting (3.14), (3.26) and the bounds for \(C_1, \ldots, C_5\) as given in (3.20), (3.21), (3.24) and (3.25) into (2.7), finally we derive estimates for \(S_1, \ldots, S_6\) in Lemma 2.

For \(S_7\) and \(S_8\), first we note that when \(x \in \left(0, \frac{\pi}{2}\right)\),

\[
\sin x \geq \frac{2x}{\pi}.
\]

Then plugging \(x = \frac{\pi}{c}\) in the above inequality, we have

\[
\sin \left(\frac{\pi}{c}\right) \geq \frac{2}{c}.
\]  

Meanwhile, setting \(x = \frac{n}{4}\) and \(a = 8\) in (3.12), for all \(n \geq 1\), we have

\[
\log \left(\frac{n}{4}\right) \leq 8 \left(\frac{n}{4}\right)^{\frac{1}{8}}.
\]

Thus \(S_7\) and \(S_8\) are both bounded by 0.9093\(\sqrt[7]{c}\) by applying (3.27) and (3.28) to (2.7). The asymptotic estimates \(S_{2\text{err}}, S_{5\text{err}}, S_{6\text{err}}, I_{2\text{err}}, I_{5\text{err}}\) and \(I_{6\text{err}}\) easily follow from (3.13), (3.14) and (3.26). This completes the proof. \(\square\)

**Theorem 5.** For integers \(c > 2\) and \(n \geq 1\), a bound for the error term of \(\overline{R}(a, c, n)\) is

\[
1544.72n^{-\frac{1}{4}}c^2 + 87078.1n^{\frac{1}{2}}c + 5733.56n^{\frac{3}{4}}c^2 + 9403.8n^{\frac{7}{8}}c + 1363.79\overline{C}_4n^{\frac{3}{2}}c + 1446.53\overline{C}_2n^\frac{5}{4}c.
\]

**Proof.** Adding the estimates in Lemma 2 together and applying to (2.3), we get the bound for the error term of \(\overline{R}(a, c, n)\) as desired. \(\square\)

The following bounds for \(\overline{p}(n)\) serve as a connection between \(\overline{p}(n)\) and \(\overline{R}(a, c, n)\).
Theorem 6. For all $n \geq 1$, we have
\[
\frac{1}{8n} \left( 1 - \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}} \leq p(n) \leq \frac{1}{8n} \left( 1 + \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}}. \tag{3.29}
\]

Proof. By (1.3), for all $n \geq 1$, we get the following upper bound for $p(n)$
\[
p(n) \leq \frac{1}{8n} \left( 1 + \frac{1}{\pi \sqrt{n}} \right) e^{\pi \sqrt{n}} + \frac{\pi e^{\pi \sqrt{n}}}{16\pi n^{\frac{1}{2}}}. \tag{3.30}
\]

The right hand side of above inequality can be rewritten as
\[
\frac{1}{8n} \left( 1 + \frac{1}{\pi \sqrt{n}} \right) e^{\pi \sqrt{n}} + \frac{\pi e^{\pi \sqrt{n}}}{16\pi n^{\frac{1}{2}}} = \frac{1}{8n} \left( 1 + \frac{2 + \pi}{2\pi \sqrt{n}} \right) e^{\pi \sqrt{n}},
\]
which arrives at the right-hand side of (3.29). The left hand-side of (3.29) can be verified easily. This completes the proof. \qed

We are now ready to complete the proof of Theorem 3.

Proof of Theorem 3. Combining Theorems 4 and 5, for integers $c > 2$ and $n \geq 1$, we have
\[
\left| \bar{R}(a, c, n) \right| \leq 0.1624 e^{\frac{\pi}{c}} n^{\frac{1}{c}} c + (0.0266c + 0.2123)e^{\pi \sqrt{n}} (1 - \frac{4}{c}) n^{\frac{1}{c}} c
\]
\[+ 1544.72n^{-\frac{1}{2}} c^2 + 87078.1 n^{\frac{1}{4}} c + 5733.56 n^{\frac{1}{4}} c^2 + 9403.8 n^{\frac{1}{4}} c
\]
\[+ 1363.79 C_4 n^{\frac{1}{4}} c + 1446.53 C_2 n^{\frac{1}{4}} c. \tag{3.30}
\]

We note that for $n \geq 2$,
\[
\frac{\sqrt{n}}{\sqrt{n} - 1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \leq 3.415.
\]

By Theorem 6, the upper bound of $p(n)^{-1}$ can be estimated as
\[
p(n)^{-1} \leq 8n \frac{\sqrt{n}}{\sqrt{n} - 1} e^{\pi \sqrt{n}} \leq 27.32 ne^{-\pi \sqrt{n}}. \tag{3.31}
\]

From (3.30) and (3.31), for all $c > 2$ and $n \geq 2$, we have
\[
\left| \bar{R}(a, c, n) \right| \leq 4.44 c e^{\frac{1}{c} - 1} \pi \sqrt{n} n^{\frac{3}{4}} + (0.73c + 5.81) c e^{-\pi \sqrt{n}} n^{\frac{3}{4}} + e^{-\pi \sqrt{n}}
\]
\[\times \left[ 42201.8c^2 n^{\frac{3}{4}} + 156641 c^2 n^{\frac{3}{4}} + 2.379 \times 10^6 cn^{\frac{3}{4}} + 256912cn^{\frac{3}{4}}
\]
\[+ 37258.7c C_4 n^{\frac{3}{4}} + 39519.2c C_2 n^{\frac{3}{4}} \right]. \tag{3.32}
\]

For $c = 3$, from the definition of $C_2$ (resp. $C_4$) in (2.8) (resp. (2.9)) and (3.19), we have $C_2 < 4.5303 \times 10^{52}$ and $C_4 < 1.0535 \times 10^{58}$. So replacing $C_2$ and $C_4$ in (3.32) with $4.5303 \times 10^{52}$ and $1.0535 \times 10^{58}$, we obtain that
\[
\left| \bar{R}(a, 3, n) \right| \leq 13.32 e^{-\frac{2\pi \sqrt{n}}{4}} n^{\frac{3}{4}} + 24 e^{-\frac{\pi \sqrt{n}}{4}} n^{\frac{3}{4}} + e^{-\pi \sqrt{n}} \left( 379816.2n^{\frac{3}{4}}
\]
\[+ 5.3711 \times 10^{57} n^{\frac{3}{4}} + 770736n^{\frac{11}{4}} \right) := R_3(n). \tag{3.33}
\]
Plugging (3.33) into (3.8), we have
\[ \bar{p}(n) \left( \frac{1}{3} - R_3(n) \right) < \bar{N}(a, 3, n) < \bar{p}(n) \left( \frac{1}{3} + R_3(n) \right). \]  
(3.34)

Note that \( R_3(n) \) is decreasing for \( n > 1 \), then we have
\[ R_3(n) \leq R_3(2089) < 0.33142, \ n \geq 2089. \]  
(3.35)

Combining (3.34) and (3.35), we are led to (3.1).

Similar to \( c = 3 \), we first confirm the estimations of \( C_2 \) and \( C_4 \). When \( c = 4, C_2 < 2.977 \times 10^{13} \) and \( C_4 < 1.4885 \times 10^{13} \), for \( c = 5 \), we have \( C_2 < 2.4221 \times 10^{10} \) and \( C_4 < 4.0102 \times 10^{19} \). Then we obtain
\[ \left| \frac{\bar{R}(a, 4, n)}{\bar{p}(n)} \right| \leq 17.76 e^{\frac{-3\pi \sqrt{n}}{4} n^{\frac{3}{4}}} + e^{-\pi \sqrt{n}} \left( 675228.9n^{\frac{3}{4}} + 6.9244 \times 10^{18}n^{\frac{3}{4}} + 1.0277 \times 10^6n^{\frac{11}{8}} \right) := R_4(n), \]  
(3.36)

and
\[ \left| \frac{\bar{R}(a, 5, n)}{\bar{p}(n)} \right| \leq 69.5 e^{\frac{-4\pi \sqrt{n}}{3} n^{\frac{3}{4}}} + e^{-\pi \sqrt{n}} \left( 1.0551 \times 10^6n^{\frac{3}{4}} + 7.4708 \times 10^{24}n^{\frac{3}{4}} + 1.2846 \times 10^6n^{\frac{11}{8}} \right) := R_5(n). \]  
(3.37)

Substituting (3.36) and (3.37) into (3.8), respectively, we have
\[ \bar{p}(n) \left( \frac{1}{4} - R_4(n) \right) < \bar{N}(a, 4, n) < \bar{p}(n) \left( \frac{1}{4} + R_4(n) \right), \]  
(3.38)
\[ \bar{p}(n) \left( \frac{1}{5} - R_5(n) \right) < \bar{N}(a, 5, n) < \bar{p}(n) \left( \frac{1}{5} + R_5(n) \right). \]  
(3.39)

Since
\[ R_4(n) < 0.24084, \ n \geq 272, \]  
(3.40)
\[ R_5(n) < 0.1897, \ n \geq 449. \]  
(3.41)

Applying the estimates (3.40) into (3.38) and (3.41) into (3.39), we reach (3.2) and (3.3).

Next, we deal with the case \( c \geq 6 \). Since \( -\frac{4}{c} > \frac{1}{c} - 1 \) for \( c \geq 6 \), combining (3.32), for \( n \geq 2 \) we have
\[ \left| \frac{\bar{R}(a, c, n)}{\bar{p}(n)} \right| \leq e^{-\frac{4\pi \sqrt{n}}{c} n^{\frac{3}{4}}} [4.44c + (0.73c + 5.81)c + 42201.8c^2 + 156641c^2 \]  
\[ + 2.379 \times 10^6c + 37258.7cC_4 + 39519.2cC_2] + 256912ce^{-\pi \sqrt{n} n^{\frac{11}{8}}} \]  
\[ \leq 37259e^{\frac{4c^2(2c^2 + \pi)}{\pi^2}} e^{-\frac{4\pi \sqrt{n}}{c} n^{\frac{3}{4}}} + 256912ce^{-\pi \sqrt{n} n^{\frac{11}{8}} := R_c(n).} \]  
(3.42)

In view of (3.8) and (3.42), we get the following bounds of \( \bar{N}(a, c, n) \)
\[ \bar{p}(n) \left( \frac{1}{c} - R_c(n) \right) < \bar{N}(a, c, n) < \bar{p}(n) \left( \frac{1}{c} + R_c(n) \right). \]  
(3.43)
We claim that for all \( n \geq M_c \),
\[
R_c(n) < \frac{1}{2c}, \tag{3.44}
\]
where \( M_c \) is defined in (3.5).

To verify (3.44), we only need to show that
\[
37259c \exp\left(\frac{4c^2(2c^2 + \pi)}{\pi^2}\right) e^{-\frac{3}{4} \pi \sqrt{n} \frac{\pi}{4}} < \frac{1}{4c} \tag{3.45}
\]
and
\[
256912ce^{-\pi \sqrt{n} \frac{11}{8}} < \frac{1}{4c}, \tag{3.46}
\]
which are equivalent to prove that
\[
\frac{e^{\frac{4}{4} \pi \sqrt{n}}}{n^{\frac{5}{4}}} > 4 \times 37259c^2 \exp\left(\frac{4c^2(2c^2 + \pi)}{\pi^2}\right) \tag{3.47}
\]
and
\[
\frac{e^{\pi \sqrt{n}}}{n^{\frac{11}{8}}} > 4 \times 256912c^2. \tag{3.48}
\]
In addition, we recall the following inequality [19, p. 76]
\[
e^x > \left(1 + \frac{x}{y}\right)^y, \quad x, y > 0.
\]
Hence, letting \( x = \frac{4}{c} \pi \sqrt{n}, y = 3 \) and \( x = \pi \sqrt{n}, y = 4 \) respectively, we obtain
\[
\frac{e^{\frac{4}{4} \pi \sqrt{n}}}{n^{\frac{5}{4}}} > \frac{1}{n^{\frac{5}{4}}} \left(1 + \frac{4\pi \sqrt{n}}{3c}\right)^3 > \frac{4^3 \pi^3}{3^3 c^3 n^{\frac{5}{4}}} \tag{3.49}
\]
and
\[
\frac{e^{\pi \sqrt{n}}}{n^{\frac{11}{8}}} > \frac{1}{n^{\frac{11}{8}}} \left(1 + \frac{\pi \sqrt{n}}{4}\right)^4 > \frac{\pi^4}{4^4 n^{\frac{11}{8}}}. \tag{3.50}
\]
By (3.49) and (3.50), we know that the inequalities (3.47) and (3.48) hold when \( n \geq M_c \) and \( n \geq M'_c \), respectively, where
\[
M'_c := 1.952 \times 10^{10} c^\frac{46}{45}.
\]
Furthermore, we see that (3.45) and (3.46) hold when \( n \geq M_c \) and \( n \geq M'_c \), respectively. Therefore, (3.44) is verified for \( n \geq \max\{M_c, M'_c\} = M_c \). Plugging (3.44) into (3.43), we complete the proof. \( \square \)

4. Strict Log-Subadditivity of \( \overline{N}(a, c, n) \)

In this section, we present a proof of Theorem 1 based on the intermediate inequalities given in the previous sections.
Proof of Theorem 1. We begin with the case $c = 3$. Substituting (3.29) into (3.1), we obtain that for $n \geq 2089$

$$(0.0019) \frac{1}{8n} \left( 1 - \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}} < N(a, 3, n) < (0.6648) \frac{1}{8n} \left( 1 + \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}}.$$  

(4.1)

Letting $n_2 = Cn_1$ for some $C \geq 1$, then by (4.1), we have

$N(a, 3; n_1)N(a, 3; n_2) > (0.0019)^2 \frac{1}{64Cn_1^2} \left( 1 - \frac{1}{\sqrt{n_1}} \right) \left( 1 - \frac{1}{\sqrt{Cn_1}} \right) e^{\pi(\sqrt{n_1} + \sqrt{Cn_1})},$

and

$N(a, 3; n_1 + n_2) < (0.6648) \frac{1}{8(n_1 + Cn_1)} \left( 1 + \frac{1}{\sqrt{n_1 + Cn_1}} \right) e^{\pi \sqrt{n_1 + Cn_1}}.$

It is sufficient to show that

$T_{n_1}(C) > \log (V_{n_1}(C)) + \log (S_{n_1}(C)),$

where

$T_{n_1}(C) := \pi(\sqrt{n_1} + \sqrt{Cn_1}) - \pi \sqrt{n_1 + Cn_1},$

$V_{n_1}(C) := \frac{0.6648 \times 8n_1}{(0.0019)^2(C + 1)},$

$S_{n_1}(C) := \frac{1 + \frac{1}{\sqrt{n_1 + Cn_1}}}{\left( 1 - \frac{1}{\sqrt{n_1}} \right) \left( 1 - \frac{1}{\sqrt{Cn_1}} \right)}.$

As function of $C$, it can be shown that $T_{n_1}(C)$ is increasing and $S_{n_1}(C)$ is decreasing for $C \geq 1$, and combining with

$V_{n_1}(C) < \frac{0.6648 \times 8n_1}{(0.0019)^2},$

it suffices to show that

$T_{n_1}(1) = 2\pi \sqrt{n_1} - \pi \sqrt{2n_1}$

$> \log \left( \frac{0.6648 \times 8n_1}{(0.0019)^2} \right) + \log \left( \frac{1 + \frac{1}{\sqrt{2n_1}}}{\left( 1 - \frac{1}{\sqrt{n_1}} \right)^2} \right).$  

(4.2)

By a short computation, we find that (4.2) holds for all $n_1 \geq 109$. Hence, if $n_1, n_2 \geq 2089$, we have

$N(a, 3, n_1 + n_2) < N(a, 3, n_1)N(a, 3, n_2).$

It is routine to check that (1.4) is true for $c = 3$ and $9 \leq n \leq 2088$.

The proofs of $c = 4, 5$ are similar to that of $c = 3$, and hence, they are omitted.

We proceed to prove the case for $c \geq 6$. Plugging (3.29) into (3.4), we obtain that
where $M_c$ is defined in (3.5).

Setting $n_2 = Cn_1$ for some $C \geq 1$, one can easily find that for $n_1 \geq M_c$

\[
\frac{1}{16nc} \left( 1 - \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}} < N(a, c, n) < \frac{3}{16nc} \left( 1 + \frac{1}{\sqrt{n}} \right) e^{\pi \sqrt{n}}, \quad n \geq M_c,
\]

and

\[
\frac{1}{256cn_1^2} \left( 1 - \frac{1}{\sqrt{n_1}} \right) \left( 1 - \frac{1}{\sqrt{n_2}} \right) e^{\pi (\sqrt{n_1} + \sqrt{Cn_1})},
\]

It remains to show that for $n_1 \geq M_c$

\[
T_{n_1}(C) > \log(W_{n_1}(C)) + \log(S_{n_1}(C)),
\]

where

\[
W_{n_1}(C) := (48c)\frac{Cn_1}{C + 1}.
\]

Obviously,

\[
W_{n_1}(C) < 48cn_1.
\]

Therefore, we only need show

\[
T_{n_1}(1) = 2\pi \sqrt{n_1} - \pi \sqrt{2n_1} > \log(48cn_1) + \log \left( \frac{1 + \frac{1}{\sqrt{2n_1}}}{\left( 1 - \frac{1}{\sqrt{n_1}} \right)^2} \right). \quad (4.3)
\]

If $n_1 \geq 2$, we obtain

\[
\log(48cn_1) + \log \left( \frac{1 + \frac{1}{\sqrt{2n_1}}}{\left( 1 - \frac{1}{\sqrt{n_1}} \right)^2} \right) < \log(48cn_1) + \log \left( \frac{1 + \frac{1}{\sqrt{4}}}{\left( 1 - \frac{1}{\sqrt{2}} \right)^2} \right) < \log(840cn_1).
\]

Moreover, if $n_1 \geq (840c)^2 \geq (840 \times 6)^2$, then

\[
T_{n_1}(1) = 2\pi \sqrt{n_1} - \pi \sqrt{2n_1} > 2 \log(n_1) \geq \log(n_1) + 2 \log(840c) > \log(840cn_1).
\]

Combining with (4.3), (4.4) and (4.5), choosing $n_1, n_2 \geq (840c)^2$, we complete the proof. \qed
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Appendix

Here we give a brief proof for the upper bounds of $\Sigma_7$ and $\Sigma_8$. Letting $\tilde{k} = 0$ if $k$ is even, and $\tilde{k} = 1$ if $k$ is odd, from Ciolan [8, p. 475], we know that

$$\begin{align*}
\Sigma_7 &= 4 \sin^2 \left( \frac{\pi a}{c} \right) \sum_{h,k \mid 2 \mid k} \omega_{h,k}^2 e^{-2 \pi i n_h / k} \sum_{\nu=0}^{k-1} (-1)^\nu e^{-2 \pi i h' \nu^2 / k} \int_{-\theta_{h,k}}^{\theta_{h,k}} \frac{1}{2 \pi i} I_{a,c,k,\nu}(z) d\Phi,
\end{align*}$$

and

$$\begin{align*}
\Sigma_8 &= 4 \sqrt{2} \sin^2 \left( \frac{\pi a}{c} \right) \sum_{h,k \nmid 2 \mid k} \omega_{h,k}^2 e^{-2 \pi i n_h / k} \sum_{\nu=0}^{k-1} e^{-\pi i h' (2\nu^2 - \nu) / k} \int_{-\theta_{h,k}}^{\theta_{h,k}} \frac{1}{2 \pi i} \frac{1}{2} I_{a,c,k,\nu}(z) d\Phi,
\end{align*}$$

where

$$I_{a,c,k,\nu}(z) := \int_{\mathbb{R}} e^{-\frac{2\pi z^2}{k}} H_{a,c} \left( \frac{2\pi i \nu}{k} - \frac{2\pi z x}{k} - \frac{\tilde{k} \pi i}{2k} \right),$$

with

$$H_{a,c}(x) := \frac{e^x}{1 - 2 \cos \left( \frac{2\pi a}{c} \right) e^x + e^{2x}}.$$
For $\Sigma_8$, following Ciolan’s proof in [8, p. 480], we note that $H_{a,c}(x) = H_{a,c}^+(x) + H_{a,c}^-(x)$, where

$$H_{a,c}^\pm = \pm \frac{i}{8} \cosh\left(\frac{x}{2}\right) \sin\left(\frac{\pi a}{c}\right) \sinh\left(\frac{x}{2} \pm \frac{\pi a}{c}\right).$$

And we denote the contributions of these functions to $I_{a,c,k,v}$ by $I_{a,c,k,v}^\pm$. From the proof of [7, Lemma 1] we obtain

$$z^\frac{1}{2} I_{a,c,k,v}^\pm \leq \sqrt{k} \left|\sin\left(\frac{\pi v}{k} \pm \frac{\pi a}{c}\right)\right| \cdot \sin\left(\frac{\pi a}{c}\right) \left|\Re\left(\frac{1}{z}\right)\right| \frac{1}{2} \leq \frac{2}{k(\sqrt{n} + 1)} < \frac{2}{k\sqrt{n}},$$

Combining the well-known facts from the theory of Farey arcs, such as

$$\Re(z) = \frac{k}{n}, \quad \vartheta'_{h,k} + \vartheta''_{h,k} \leq 2 \left(\frac{1}{k(N+1)} + 1\right) \leq \frac{2}{k\sqrt{n}},$$

we have

$$\Sigma_8 \leq \sqrt{2e^{2\pi}} \left|\sin\left(\frac{\pi a}{c}\right)\right| \sum_{\pm} \sum_{\nu=1}^{k} \sum_{1 \leq k \leq \sqrt{n}} \frac{1}{k\sqrt{k} \sin\left(\frac{\pi v}{k} \pm \frac{\pi a}{c}\right)}.$$  \hfill (4.6)

We next estimate the sum on $v$. We have

$$\sum_{\pm} \sum_{\nu=1}^{k} \left|\sin\left(\frac{\pi v}{k} \pm \frac{\pi a}{c}\right)\right| \leq 2 \sum_{\nu=1}^{k} \left|\sin\left(\frac{\pi v}{k} \mp \frac{\pi a}{4k}\right)\right|$$

$$= 2 \left(\sum_{v=1}^{\left\lfloor \frac{k}{4}\right\rfloor} \frac{1}{\sin\left(\frac{\pi v}{k} \mp \frac{\pi a}{4k}\right)} + \sum_{v=0}^{\left\lfloor \frac{k+1}{2}\right\rfloor} \frac{1}{\sin\left(\frac{\pi v}{k} \pm \frac{\pi a}{4k}\right)}\right).$$  \hfill (4.7)

By Taylor’s theorem, we have

$$\frac{1}{\sin\left(\frac{\pi v}{k} \mp \frac{\pi a}{4k}\right)} < \frac{k}{\pi (v - 1/4) \left(1 - \frac{1}{6} \left(\frac{\pi}{k}\left(\left\lfloor\frac{k}{2}\right\rfloor - \frac{1}{4}\right)\right)^2\right)},$$

$$\frac{1}{\sin\left(\frac{\pi v}{k} + \frac{\pi a}{4k}\right)} < \frac{k}{\pi (v + 1/4) \left(1 - \frac{1}{6} \left(\frac{\pi}{k}\left(\left\lfloor\frac{k+1}{2}\right\rfloor - \frac{3}{4}\right)\right)^2\right)}.$$  

These together with (4.7) give

$$\sum_{\pm} \sum_{\nu=1}^{k} \left|\sin\left(\frac{\pi v}{k} \pm \frac{\pi a}{c}\right)\right| \leq \frac{4k}{\pi} \log\left(\frac{k}{2}\right) \left(\frac{\pi a}{c}\right).$$  \hfill (4.8)

Substituting (4.8) into (4.6), we have

$$\Sigma_8 \leq \frac{4\sqrt{2e^{2\pi}} n^{1/4} \log\left(\frac{n}{4}\right)}{\pi (1 - \frac{\pi^2}{24}) \sin\left(\frac{\pi}{c}\right)}.$$.  

Here we use that
\[ \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}} < 2n^{\frac{1}{4}} \quad \text{and} \quad \log \left( \frac{k}{2} \right) < \frac{1}{2} \log \left( \frac{n}{4} \right). \]

We next give a bound for \( \Sigma_7 \). Repeating the above steps and combining Ciolan’s results in [8, p. 480], we have
\[
\Sigma_7 \leq e^{2\pi} \left| \sin \left( \frac{\pi a}{c} \right) \right| \left( \sum_{1 \leq k \leq m} \sum_{\nu=1}^{k} k^{\frac{1}{2}} \log \left( \frac{k}{2} \right) \right)
< \frac{4e^{2\pi}}{\pi \left( 1 - \frac{\pi^2}{24} \right) \sin \left( \frac{\pi}{c} \right)} \sum_{1 \leq k \leq \sqrt{n}} k^{-\frac{1}{2}} \log \left( \frac{k}{2} \right)
< \frac{4e^{2\pi} n^{\frac{1}{4}} \log \left( \frac{n}{4} \right)}{\pi \left( 1 - \frac{\pi^2}{24} \right) \sin \left( \frac{\pi}{c} \right)}.
\]

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