ON THE GENERALIZED CONVEXITY AND CONCAVITY

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ABSTRACT. In this paper, authors study the convexity and concavity properties of real-valued function with respect to the classical means, and prove a conjecture posed by Bruce Ebanks in [12].

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1. Introduction

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $[m_1, m_2]$-convex (concave) if $f(m_1(x, y)) \leq (\geq)m_2(f(x), f(y))$ for all $x, y \in \mathbb{R}_+ = (0, \infty)$ and $m_1, m_2 \in \mathcal{M}$, where $\mathcal{M}$ denotes the family of all mean values of two numbers in $\mathbb{R}_+$. Some examples of mean values of two distinct positive real numbers are given below:

Arithmetic mean: $A = A(x, y) = \frac{x + y}{2}$,

Geometric mean: $G = G(x, y) = \sqrt{xy}$,

Harmonic mean: $H = H(x, y) = \frac{1}{A(1/x, 1/y)}$,

Logarithmic mean: $L = L(x, y) = \frac{x - y}{\log(x) - \log(y)}$,

Identric mean: $I = I(x, y) = \frac{1}{e} \left( \frac{x^x}{y^y} \right)^{1/(x-y)}$,

Alzer mean: $J_p = J_p(x, y) = \frac{p}{p+1} \frac{x^{p+1} - y^{p+1}}{x^p - y^p}$, $p \neq 0, -1$,

Power mean: $M_t = M_t(x, y) = \begin{cases} \left( \frac{x^t + y^t}{2} \right)^{1/t}, & t \neq 0, \\ \sqrt[2]{xy}, & t = 0. \end{cases}$
It is easy to see that \( J_1(x, y) = A(x, y), J_0(x, y) = L(x, y), J_{-2}(x, y) = H(x, y) \). For the historical background of these means we refer the reader to see \[4, 5, 11, 15, 16\] and the bibliography of these papers.

Before we introduce the earlier results from the literature we recall the following definition, see \[2, 6\].

1.1. Definition. Let \( f : I \to (0, \infty) \) be continuous, where \( I \) is a sub-interval of \((0, \infty)\). Let \( M \) and \( N \) be two any mean functions. We say that the function \( f \) is \( MN \)-convex \((concave)\) if

\[
f(M(x, y)) \leq (\geq)N(f(x), f(y)) \quad \text{for all } x, y \in I.
\]

In \[2\], Anderson, Vamanamurthy and Vuorinen studied the convexity and concavity of a function \( f \) with respect two mean values, and gave the following detailed result:

1.2. Theorem. Let \( I \) be an open sub-interval of \((0, \infty)\) and let \( f : I \to (0, \infty) \) be differentiable. In parts (4)(9), let \( I = (0, b), 0 < b < \infty \).

(1) \( f \) is \( AA \)-convex \((concave)\) if and only if \( f'(x) \) is increasing \( (decreasing) \),

(2) \( f \) is \( AG \)-convex \((concave)\) if and only if \( f'(x)/f(x) \) is increasing \( (decreasing) \),

(3) \( f \) is \( AH \)-convex \((concave)\) if and only if \( f'(x)/f(x)^2 \) is increasing \( (decreasing) \),

(4) \( f \) is \( GA \)-convex \((concave)\) if and only if \( xf'(x) \) is increasing \( (decreasing) \),

(5) \( f \) is \( GG \)-convex \((concave)\) if and only if \( xf'(x)/f(x) \) is increasing \( (decreasing) \),

(6) \( f \) is \( GH \)-convex \((concave)\) if and only if \( xf'(x)/f(x)^2 \) is increasing \( (decreasing) \),

(7) \( f \) is \( HA \)-convex \((concave)\) if and only if \( x^2f'(x) \) is increasing \( (decreasing) \),

(8) \( f \) is \( HG \)-convex \((concave)\) if and only if \( x^2f'(x)/f(x) \) is increasing \( (decreasing) \),

(9) \( f \) is \( HH \)-convex \((concave)\) if and only if \( x^2f'(x)/f(x)^2 \) is increasing \( (decreasing) \).

After the publication of \[2\] many authors have studied generalized convexity. For a partial survey of the recent results, see \[3\].

In \[9\], the following inequalities were studied:

1.3. Theorem. Let \( f : I \to (0, \infty) \) be a continuous and \( I \subseteq (0, \infty) \), then

(1) \( f \) is \( LL \)-convex \((concave)\) if \( f \) is increasing and \( \log \)-convex \((concave)\),

(2) \( f \) is \( AL \)-convex \((concave)\) if \( f \) is increasing and \( \log \)-convex \((concave)\).

Recently, Baricz \[7\] took one step further and studied the \( MN \)-convexity(concavity) of a function \( f \) in a generalized way, and gave the following result:

1.4. Lemma. \[7\] Lemma 3] Let \( p, q \in \mathbb{R} \) and let \( f : [a, b] \to (0, \infty) \) be a differentiable function for \( a, b \in (0, \infty) \). The function \( f \) is \((p, q)\)-convex \(( (p, q)\)-concave) if and only if \( x \mapsto x^\alpha f'(x)/(f(x))^{q-1} \) is increasing \( (decreasing) \).

It can be observed easily that \((1, 1)\)-convexity means the \( AA \)-convexity, \((1, 0)\)-convexity means the \( AG \)-convexity, and \((0, 0)\)-convexity means \( GG \)-convexity.
1.5. **Lemma.** [7] Theorem 7] Let $a, b \in (0, \infty)$ and $f : [a, b] \to (0, \infty)$ be a differentiable function. Denote $g(x) = \int_1^x f(t) dt$ and $h(x) = \int_0^b f(t) dt$. Then

(a) If for all $p \in [0, 1]$ the function $f$ is $(p, 0)$-concave, then the function $g$ is $(p, q)$-concave for all $p \in [0, 1]$ and $q \leq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is increasing for all $p \in [0, 1]$, then $g$ is $(p, q)$-concave for all $p \in [0, 1]$ and $q \in (0, 1)$. Moreover, if for all $p \in \mathbb{R}$ the function $x \mapsto x^{1-p}f(x)$ is increasing, then $g$ is $(p, q)$-convex for all $p \in \mathbb{R}$ and $q \geq 1$.

(b) If for all $p \in [0, 1]$ the function $f$ is $(p, 0)$-concave, then the function $g$ is $(p, q)$-concave for all $p \in [0, 1]$ and $q \leq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is decreasing for all $p \in [0, 1]$, then $g$ is $(p, q)$-concave for all $p \in [0, 1]$ and $q \in (0, 1)$. Moreover, if for all $p \in \mathbb{R}$ the function $x \mapsto x^{1-p}f(x)$ is decreasing, then $g$ is $(p, q)$-convex for all $p \in \mathbb{R}$ and $q \geq 1$.

(c) If for all $p \notin (0, 1)$ we have $a^{1-p}f(a) = 0$ and the function $f$ is $(p, 0)$-convex, then $g$ is $(p, q)$-convex for all $p \notin (0, 1)$ and $q \geq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is increasing for all $p \notin (0, 1)$, then $g$ is $(p, q)$-convex for all $p \notin (0, 1)$ and $q < 0$.

(d) If for all $p \notin (0, 1)$ we have $b^{1-p}f(b) = 0$ and the function $f$ is $(p, 0)$-convex, then $g$ is $(p, q)$-convex for all $p \notin (0, 1)$ and $q \geq 0$. If, in addition the function $x \mapsto x^{1-p}f(x)$ is decreasing for all $p \notin (0, 1)$, then $g$ is $(p, q)$-convex for all $p \notin (0, 1)$ and $q < 0$.

In this paper we make a contribution to the subject by giving the following theorems.

1.6. **Theorem.** Let $f : I \to (0, \infty)$ and $I \subseteq (0, \infty)$. Then the following inequality holds true:

\[
I(f(x), f(y)) \geq f(I(x, y))
\]

\[
I(f(x), f(y)) \leq f(A(x, y)),
\]

if the function $f(x)$ is a continuously differentiable, increasing and log-convex(concave).

1.7. **Theorem.** Let $f$ be a continuous real-valued function on $(0, \infty)$. If $f$ is strictly increasing and convex, then

\[
(1.8) \quad P_f(x, y) \leq R_f(x, y)
\]

where

\[
P_f(x, y) = f \left( (xy)^{1/4} \left( \frac{x + y}{2} \right)^{1/2} \right)
\]

and

\[
R_f(x, y) = \frac{1}{y - x} \int_x^y f(t) dt.
\]

1.9. **Remark.** In [12], Ebanks defined $P_f(x, y)$ and $R_f(x, y)$, and proposed an open problem for a continuous and strictly monotonic real-valued function $f$ on $(0, \infty)$ as follows:

Problem. Does $f$ strictly increasing and convex (or $f'' > 0$) imply $P_f \leq R_f$?
It is obvious that the Theorem 1.7 gives an affirmative answer to the Ebanks’ problem.

1.10. **Theorem.** Let \( f : I \to (0, \infty) \) and \( I \subseteq (0, \infty) \).

1. (If \( f(x) \) is continuously differentiable, strictly increasing (decreasing) and convex (concave) and \( f^{p-1}(x)f(x) \) is increasing on \( (0, 1) \), then

\[
J_p(f(x), f(y)) \geq f(J_p(x, y))
\]

\[
J_p(f(x), f(y)) \leq f(A(x, y))
\]

for \( p \leq 1 \).

2. (If \( f(x) \) is continuously differentiable, strictly decreasing (increasing) and convex (concave) and \( f^{p-1}(x)f(x) \) is decreasing on \( (0, 1) \), then

\[
J_p(f(x), f(y)) \geq f(J_p(x, y))
\]

\[
J_p(f(x), f(y)) \leq f(A(x, y))
\]

for \( p > 1 \).

2. **Lemmas and Proofs**

We recall the following lemmas which will be used in the proof of the theorems.

2.1. **Lemma.** [17] Let \( f, g : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Furthermore, let \( p : [a, b] \to \mathbb{R} \) be a positive, integrable function. Then

\[
\int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx.
\]

If one of the functions \( f \) or \( g \) is non-increasing and the other non-decreasing, then the inequality in (2.1) is reversed.

2.3. **Lemma.** [13] If \( f(x) \) is continuous and convex function on \( [a, b] \), and \( \varphi(x) \) is continuous on \( [a, b] \), then

\[
f \left( \frac{1}{b-a} \int_a^b \varphi(x)dx \right) \leq \frac{1}{b-a} \int_a^b f(\varphi(x))dx.
\]

If function \( f(x) \) is continuous and concave on \( [a, b] \), the inequality in (2.3) is reversed.

2.5. **Lemma.** [5] Fix two positive number \( a, b \). Then \( L(a, b) \leq I(a, b) \leq A(a, b) \).

2.6. **Lemma.** [13] The function \( p \mapsto J_p(x, y) \) is strictly increasing on \( \mathbb{R} \setminus \{0, -1\} \).

**Proof of Theorem 1.6.** Since the proof of part (2) is similar to part (1), we only prove the part (1) here. An easy computation and substitution \( t = f(u) \) yield

\[
\ln I(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} \ln t dt}{\int_{f(y)}^{f(x)} 1} = \frac{\int_y^x \ln f(u)f'(u)du}{\int_y^x f'(u)du}.
\]
Since the functions \( f(x) \) and \( f'(x) \) are increasing on \( I \subseteq (0, \infty) \), now by using Lemma 2.1 we have

\[
\int_y^x 1du \cdot \int_y^x \ln f(u)f'(u)du \geq \int_y^x f'(u)du \cdot \int_y^x \ln f(u)du.
\]

Combining (2.7) and (2.8), we obtain

\[
I(f(x), f(y)) \geq \frac{\int_y^x \ln f(u)du}{y - x}.
\]

Considering the log-convexity of the function \( f(x) \) and using Lemmas 2.3 and 2.5, we get

\[
I(f(x), f(y)) \geq \ln f \left( \frac{\int_y^x udu}{y - x} \right) = \ln f \left( \frac{x + y}{2} \right) \geq \ln f \left( I(x, y) \right).
\]

This completes the proof. \( \square \)

**Proof of Theorem 1.7.** Since \( f \) is strictly increasing and convex, by utilizing the Lemma 2.1 and the inequality \( G(x, y) \leq A(x, y) \) we obtain

\[
R_f(x, y) \geq \frac{\int_y^x f(u)du}{y - x} \geq f \left( \frac{\int_y^x udu}{y - x} \right) = f \left( \frac{x + y}{2} \right) \geq f \left( I(x, y) \right).
\]

This completes the proof. \( \square \)

**Proof of Theorem 1.10.** For the proof of part (1), letting \( t = f(u) \), we get

\[
J_p(f(x), f(y)) = \frac{\int_{f(y)}^{f(x)} t^p dt}{\int_{f(y)}^{f(x)} t^{p-1}} = \frac{\int_y^x f^p(u)f'(u)du}{\int_y^x f^{p-1}(u)f'(u)du}.
\]

By using Lemma 2.1, we obtain

\[
J_p(f(x), f(y)) \geq \frac{\int_y^x f(u)du}{y - x}.
\]

Considering convexity of the function \( f(x) \) and using Lemmas 2.3 and 2.6, we get

\[
J_p(f(x), f(y)) \geq f \left( \frac{\int_y^x udu}{y - x} \right) = f \left( \frac{x + y}{2} \right) \geq f \left( J_p(x, y) \right),
\]

this implies (1). The proof of part (2) follows similarly. \( \square \)
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