THE 2-DIMENSIONAL JACOBIAN
CONJECTURE VIA KLEIN’S PROGRAM

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Abstract. We investigate the 2-dimensional jacobian conjecture via Klein’s program.

§0. F. Klein noticed that we can reformulate some algebraic geometrical problems into invariant theoretical problems and then try to solve these invariant theoretical problems. Note that Klein’s program was considered as pure speculative idea (no one algebraic geometrical problem was solved by Klein’s program). The purpose of our investigation is to develop Klein’s program for algebraic geometrical problems. The 2-dimensional Jacobian conjecture is taken as test problem for the development of Klein’s program. This preliminary paper contains a description of two approaches to the 2-dimensional Jacobian conjecture via Klein’s program.

Klein’s program consists of the following two steps.

(1) A reformulation of an algebraic geometrical problem into an invariant theoretical problem.

(2) A solution of the invariant theoretical problem.

Note that usually there are many invariant theoretical reformulations of an algebraic geometrical problem.

Let us recall the Jacobian conjecture (see [2], [3]).

The Jacobian conjecture. Suppose the jacobian of a polynomial mapping

\[ F : \mathbb{C}^n \rightarrow \mathbb{C}^n \]

is equal to 1; then \( F \) is an isomorphism.

In §1 we formulate two general invariant theoretical problems. Some invariant theoretical reformulations of some algebraic geometrical problems are partial cases of these invariant theoretical problems. In §2 we define covariant \( Q \) (quasiresultant). In §3 we recall some facts about representations of the groups \( SL_2 \) and \( SL_3 \). In §4 we define group \( G \), some \( G \)-moduli, and covariants. In §5 we formulate conjecture 1. This conjecture is true iff the 2-dimensional Jacobian conjecture is true. Conjecture 1 is a partial case of general invariant theoretical problem * from §1. In §6 we formulate conjecture 2. This conjecture is true iff the 2-dimensional Jacobian conjecture is true. Conjecture 1 is a partial case of general invariant theoretical problem ** from §1.
Remark 0.1. Let us describe Klein’s program for the 2-dimensional Jacobian conjecture. Let $V$ be the linear space of pairs $(F_1, F_2)$ of polynomials of degree $\leq n$ in the variables $z_1, z_2,$

$$X_{\neq 0} = \{(F_1, F_2) \in V \mid \text{the jacobian of } (F_1, F_2) \text{ is equal to a nonzero constant}\},$$

$$X_0 = \{(F_1, F_2) \in V \mid \text{the jacobian of } (F_1, F_2) \text{ is equal to } 0\},$$

$I(X_{\neq 0}) \subset \mathbb{C}[V]$ be the ideal of $X_{\neq 0},$ and $I(X_0) \subset \mathbb{C}[V]$ be the ideal of $X_0.$ There exist elements $s \in \mathbb{C}[V]$ such that $s \in I(X_{\neq 0}) \setminus I(X_0)$ iff the 2-dimensional Jacobian conjecture is true for polynomial mappings of polynomial degree $\leq n.$ At the present moment there exists only one tool (Puiseux expansions) to construct elements of $I(X_{\neq 0}) \setminus I(X_0)$ (these elements give, for example, Folk-theorem mentioned in §6). Klein’s program is the way to obtain elements of $I(X_{\neq 0}) \setminus I(X_0)$ by means of invariant theory. (Even simple invariant theoretical constructions give some elements of $I(X_{\neq 0}) \setminus I(X_0),$ but these elements do not give a solution of the 2-dimensional Jacobian conjecture.)

We use in the paper some facts of representation theory of semisimple linear algebraic groups (see [1]).

§1. In this section we formulate two general invariant theoretical problems.

For polynomial mappings $\alpha, \beta : V \to W$ we write $\alpha(v)|_{\beta(v)=0} = 0$ iff $v \in V, \beta(v) = 0$ imply $\alpha(v) = 0.$

**Problem **. Let $G$ be a reductive linear algebraic group, $G : V,U,\tilde{U}$ be linear representations,

$$\varphi : V \to U, \quad \tilde{\varphi} : V \to \tilde{U}$$

be homogeneous covariants. Is it true that $\varphi(v)|_{\tilde{\varphi}(v)=0} = 0$?

**Definition 1.1.** Let $G$ be a linear algebraic group, $G : V,U,\tilde{U}$ be linear representations,

$$\varphi : V \to U, \quad \tilde{\varphi} : V \to \tilde{U}$$

be covariants. A covariant

$$\gamma : V \times U \to W$$

is called $(\varphi, \tilde{\varphi})$-identity if

$$\gamma(v, \varphi(v))|_{\tilde{\varphi}(v)=0} = 0.$$

**Problem **. Let $G$ be a reductive linear algebraic group, $G : V,U,\tilde{U}$ be linear representations,

$$\varphi : V \to U, \quad \tilde{\varphi} : V \to \tilde{U}$$

be homogeneous covariants, and $v_0 \in V, u_0 \in U.$ Does exist a bihomogeneous $(\varphi, \tilde{\varphi})$-identity $\gamma$ such that $\gamma(v_0, u_0) \neq 0?$
§2. In this section we define covariant $Q$.

Consider the group $SL_m$. The group $SL_m$ acts canonically in the spaces $\mathbb{C}^m$, $\mathbb{C}^{m*}$, $S^n\mathbb{C}^{m*}$, ...

Consider the resultant

$$R : S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_m}\mathbb{C}^{m*} \to \mathbb{C}.$$ 

Set $N = n_1 \cdots n_m$. The resultant $R$ is a polyhomogeneous (of polydegree $(\frac{N}{n_1}, \ldots, \frac{N}{n_m})$) $SL_m$-covariant. Consider the case $n_m = 1$. In this case the resultant $R$ defines canonically the polyhomogeneous (of polydegree $(\frac{N}{n_1}, \ldots, \frac{N}{n_{m-1}})$) $SL_m$-covariant

$$Q : S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_{m-1}}\mathbb{C}^{m*} \to (S^N\mathbb{C}^{m*})^* = S^N\mathbb{C}^m.$$ 

We have

$$R(f_1, \ldots, f_{m-1}, h) = \langle Q(f_1, \ldots, f_{m-1}), h^N \rangle$$

for $(f_1, \ldots, f_{m-1}, h) \in S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_{m-1}}\mathbb{C}^{m*} \times \mathbb{C}^m$. It follows from this formula that

1. if $(f_1, \ldots, f_{m-1}) \in S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_{m-1}}\mathbb{C}^{m*}$, then $Q(f_1, \ldots, f_{m-1}) = 0$ iff $\dim(V(f_1) \cap \cdots \cap V(f_{m-1})) \geq 1$,

2. if $(f_1, \ldots, f_{m-1}) \in S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_{m-1}}\mathbb{C}^{m*}$, $|V(f_1) \cap \cdots \cap V(f_{m-1})| = N$, then $Q(f_1, \ldots, f_{m-1}) = l_1 \cdots l_N$, there $l_i \in \mathbb{C}^m$ and

$$V(f_1) \cap \cdots \cap V(f_{m-1}) = \{l_1, \ldots, l_N\}.$$

It follows that if $(f_1, \ldots, f_{m-1}) \in S^{n_1}\mathbb{C}^{m*} \times \cdots \times S^{n_{m-1}}\mathbb{C}^{m*}$ and $Q(f_1, \ldots, f_{m-1}) \neq 0$, then $Q(f_1, \ldots, f_{m-1}) = l_1 \cdots l_N$ and $l_i \in V(f_i) \cap \cdots \cap V(f_{m-1})$, $1 \leq i \leq N$.

Consider the group $SL_m \times SL_{m-1}$. The group $SL_m \times SL_{m-1}$ acts canonically in the spaces $\mathbb{C}^m, \mathbb{C}^{m-1}, \mathbb{C}^{m*}, S^m\mathbb{C}^m, S^m\mathbb{C}^{m*} \otimes \mathbb{C}^{m-1}, \ldots$. Let $a_1, \ldots, a_{m-1}$ be the standart basis of $\mathbb{C}^{m-1}$.

Consider the homogeneous (of degree $(m-1)n^{m-2}$) polynomial mapping

$$S^n\mathbb{C}^{m*} \otimes \mathbb{C}^{m-1} \to S^{n_{m-1}}\mathbb{C}^m,$$

$$f_1 \otimes a_1 + \cdots + f_{m-1} \otimes a_{m-1} \mapsto Q(f_1, \ldots, f_{m-1}).$$

It follows from the previous notes that this mapping is $SL_m \times SL_{m-1}$-covariant.

§3. In this section we recall some facts about representations of the groups $SL_2$ and $SL_3$.

The group $SL_2$ acts canonically in the space $\mathbb{C}^2$. Let $a_1, a_2$ be the standard basis of $\mathbb{C}^2$ and $y_1, y_2$ be the dual basis of $\mathbb{C}^{2*}$. The group $SL_3$ acts canonically in the space $\mathbb{C}^3$. Let $e_1, e_2, e_3$ be the standard basis of $\mathbb{C}^3$ and $x_1, x_2, x_3$ be the dual basis of $\mathbb{C}^{3*}$.

Recall that $SL_2$-module $V(b) = S^b\mathbb{C}^{2*}$ is irreducible and $\dim V(b) = b+1$. Every irreducible $SL_2$-module is isomorphic to $V(b)$ for some $b$. For example,

$$V(b) = S^b\mathbb{C}^{2*} \to S^b\mathbb{C}^2,$$

$$f(y_1, y_2) \mapsto f(a_1, a_2).$$
is the isomorphism of $SL_2$-moduli. The decomposition
\[ V(b) \otimes V(b') \simeq V(b+b') \oplus V(b+b'-2) \oplus \cdots \oplus V(b+b'-2 \min \{b,b'\}). \]
holds. Therefore, there exists a unique (up to a scalar factor) nontrivial bilinear covariant (transvectant)
\[ \psi_i : V(b) \times V(b') \rightarrow V(b+b'-2i), \quad 0 \leq i \leq \min \{b,b'\}. \]
The explicit form of $\psi_i$ is
\[ \psi_i(f_1,f_2) = \frac{(b-i)!(b'-i)!}{b!b'!} \sum_{0 \leq j \leq i} (-1)^j \binom{i}{j} \frac{\partial^i f_1}{\partial y_1^{i-j} \partial y_2^j} \frac{\partial^i f_2}{\partial y_1^{i-j} \partial y_2^j}. \]
where $f_1 \in V(b)$, $f_2 \in V(b')$.
Consider the linear $SL_3$-mapping
\[ \Delta = \sum \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} : \mathbb{C}[\mathbb{C}^{3*}] \otimes \mathbb{C}[\mathbb{C}^3] \rightarrow \mathbb{C}[\mathbb{C}^{3*}] \otimes \mathbb{C}[\mathbb{C}^3]. \]
For $b,c \geq 0$ set
\[ V(b,c) = \text{Ker}(\Delta|_{S^b \mathbb{C}^3 \otimes S^c \mathbb{C}^{3*}}). \]
Recall that $SL_3$-module $V(b,c)$ is irreducible and $\dim V(b,c) = \frac{1}{2}(b+1)(c+1) (b+c+2)$. Every irreducible $SL_3$-module is isomorphic to $V(b,c)$ for some $b,c$. Let
\[ \pi_{b,c} : S^b \mathbb{C}^3 \otimes S^c \mathbb{C}^{3*} \rightarrow V(b,c) \]
be $SL_3$-projection. The decomposition
\[ V(b,c) \otimes V(b',c') \simeq V(b+b',c+c') \oplus V(b+b'+1,c+c'-2) \oplus \cdots \]
\[ \oplus V(b+b'+\min \{c,c'\},c+c'-2\min \{c,c'\}) \oplus \cdots. \]
holds. Therefore, there exists a unique (up to a scalar factor) nontrivial bilinear covariant
\[ \rho_i : V(b,c) \times V(b',c') \rightarrow V(b+b'+i,c+c'-2i), \quad 0 \leq i \leq \min \{c,c'\}. \]
Let us give the explicit form of $\rho_1$:
\[ \rho_1(f_1,f_2) = \pi_{b+b'+1,c+c'-2} \left( \sum_{\sigma \in S_3} \text{sgn}(\sigma)e_{\sigma(1)} \frac{\partial f_1}{\partial x_{\sigma(2)}} \frac{\partial f_2}{\partial x_{\sigma(3)}} \right), \]
where $f_1 \in V(b,c)$, $f_2 \in V(b',c')$. Similarly, there exists a unique (up to a scalar factor) nontrivial bilinear covariant
\[ \tau_i : V(b,c) \times V(b',c') \rightarrow V(b+b'-2i,c+c'+i), \quad 0 \leq i \leq \min \{b,b'\}. \]
Lemma 3.1. Consider the polyhomogeneous (of polydegree \((2n, 1, 1)\)) covariant
\[
\eta : \mathbb{C}^3^* \times \mathbb{S}^n \mathbb{C}^3^* \times \mathbb{S}^n \mathbb{C}^3^* \to V(2n - 4, 2),
\]
\[
(h, f_1, f_2) \mapsto \tau_2(\rho_n(h^n, f_1), \rho_n(h^n, f_2)).
\]
Then
\[
\eta(x_3, f_1, f_2) = c_0 \left( \psi_2(f_1(y_1, y_2, 0), f_2(y_1, y_2, 0)) \right)_{y_1 = y_2 = -e_1} x_3^2,
\]
where \(c_0 \in \mathbb{C}, c_0 \neq 0\).

Proof. Since \(\rho_n \neq o\), we have
\[
(3.1) \quad \rho_n(x_3^n, f_i) = c' f_i(e_2, -e_1, 0), \quad i = 1, 2,
\]
where \(c' \in \mathbb{C}, c' \neq 0\). Since \(\tau_2 \neq 0\), we have
\[
(3.2) \quad c'' \left( \psi_2(r_1(-y_2, y_1, 0), r_2(-y_2, y_1, 0)) \right)_{y_1 = y_2 = -e_1} x_3^2,
\]
where \(r_i \in \mathbb{S}^n \mathbb{C}^3, c'' \in \mathbb{C}, c'' \neq 0\). By (3.1) and (3.2), it follows the Lemma.

\(\S 4\). In this section we define group \(G\), some \(G\)-moduli, and covariants. We use the notations of \(\S 3\).

Let \(h \in \mathbb{C}^3^*, h \neq 0\). Define the 2-dimensional affine space
\[
\mathbb{A}(h) = P\mathbb{C}^3 \setminus \{\pi \in P\mathbb{C}^3 | \langle h, x \rangle = 0\}.
\]
Define 2-form
\[
\omega_h = \frac{\Omega_h}{h^2} \Big|_{\mathbb{A}(h)}
\]
on \(\mathbb{A}(h)\), where \(\Omega_h \in \wedge^2 \mathbb{C}^3\), \(\Omega_h \wedge dh = dx_1 \wedge dx_2 \wedge dx_3\).

Consider an affine space \(\mathbb{A}(h)\), where \(h \in \mathbb{C}^3^*, h \neq 0\). Let \(\mathbb{C}[\mathbb{A}(h)]_{\leq n}\) be the linear space of polynomials (of degree \(\leq n\)) on \(\mathbb{A}(h)\). The space \(\mathbb{C}[\mathbb{A}(h)]_{\leq n} \otimes \mathbb{C}^2\) is the linear space of polynomial mappings (of polynomial degree \(\leq n\)) of affine space \(\mathbb{A}(h)\) to \(\mathbb{C}^2\). Fix the following isomorphisms of the linear spaces:
\[
i_h : \mathbb{S}^n \mathbb{C}^3^* \to \mathbb{C}[\mathbb{A}(h)]_{\leq n},
\]
\[
f(x) \mapsto \frac{f(x)}{(h(x))},
\]
\[
I_h : \mathbb{S}^n \mathbb{C}^3^* \otimes \mathbb{C}^2 \to \mathbb{C}[\mathbb{A}(h)]_{\leq n} \otimes \mathbb{C}^2,
\]
\[
f_1(x) \otimes a_1 + f_2(x) \otimes a_2 \mapsto \frac{f_1(x)}{(h(x))} \otimes a_1 + \frac{f_2(x)}{(h(x))} \otimes a_2.
\]
Set
\[ G = SL_3 \times SL_2. \]
The group \( G \) acts canonically in the spaces \( \mathbb{C}^3, \mathbb{C}^2, \mathbb{C}^{3*}, S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2, \ldots \). Consider the representation
\[ G : \mathbb{C}^{3*} \times S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2. \]

The decomposition
\[ \mathbb{C}^{3*} \otimes S^2(S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2) \simeq S^{2n-2} \mathbb{C}^{3*} \oplus \ldots \]
holds. Therefore, there exists a unique (up to a scalar factor) nontrivial polyhomo-
geneous (of polydegree \((1,2)\)) covariant
\[ J : \mathbb{C}^{3*} \times S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2 \rightarrow S^{2n-2} \mathbb{C}^{3*}. \]
Let us give the explicit form of \( J \):
\[ J(h, f_1(x) \otimes a_1 + f_2(x) \otimes a_2) = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial h} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial h} \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{pmatrix}. \]

**Lemma 4.1.** Suppose \((h, f) \in \mathbb{C}^{3*} \times S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2, h \neq 0\); then \(i_h(J(h, f))\) is the jacobian of the polynomial mapping \(I_h(f)\).

**Proof.** It is sufficient to consider the case \(h(x) = x_3\). The form \(\omega_h\) on \(\Lambda(h)\) is \(\frac{dx_1 \wedge dx_2}{x_3}\). We have
\[
I_h(f_1(x) \otimes a_1 + f_2(x) \otimes a_2)(x) = \frac{f_1(x)}{x_3^n} a_1 + \frac{f_2(x)}{x_3^n} a_2,
\]
\[
df{x^n}{f_1(x)} \wedge df{x^n}{f_2(x)} = \frac{1}{x_3^{2n-2}} \det \left( \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \frac{\partial f_1}{\partial h} \right) dx_1 \wedge dx_2,
\]
\[
i_h^{-1} \left( \frac{1}{x_3^{2n-2}} \det \left( \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \frac{\partial f_1}{\partial h} \right) \right) = \det \left( \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \frac{\partial f_1}{\partial h} \right) = J(x_3, f_1(x) \otimes a_1 + f_2(x) \otimes a_2).
\]

Set
\[ J_c : \mathbb{C}^{3*} \times S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2 \rightarrow V(1, 2n - 3), \]
\[ (h, f) \mapsto \rho_1(h, J(h, f)). \]

\(J_c\) is the covariant of polydegree \((2,2)\).

**Lemma 4.2.** Suppose \((h, f) \in \mathbb{C}^{3*} \times S^n \mathbb{C}^{3*} \otimes \mathbb{C}^2, h \neq 0\); then \(J_c(h, f) = 0\) iff the jacobian of the polynomial mapping \(I_h(f)\) is equal to a constant.

**Proof.** Suppose the jacobian of the polynomial mapping \(I_h(f)\) is equal to a constant, i.e. \(J(h, f) = ch^{2n-2}, c \in \mathbb{C}\) (Lemma 4.1). We have
\[ J_c(h, f) = \rho_1(h, J(h, f)) = \rho_1(h, ch^{2n-2}) = 0. \]
Suppose \(J_c(h, f) = 0\). From
\[ 0 = J_c(h, f) = \rho_1(h, J(h, f)) \]
it follows that \(J(h, f) = ch^{2n-2}, c \in \mathbb{C}\) and thus the jacobian of the polynomial mapping \(I_h(f)\) is equal to a constant (Lemma 4.1).
5. In this section we formulate conjecture 1. This conjecture is true iff the 2-dimensional jacobian conjecture is true. Conjecture 1 is a partial case of general invariant theoretical problem * (see §1). We use the notations of §3 and §4.

Set
\[ D_i : \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2 \to S^{n^2-i} \mathbb{C}^3, \quad i \geq 0, \]

\[(h, f_1(x) \otimes a_1 + f_2(x) \otimes a_2) \mapsto \Delta^i(h^i Q(f_1, f_2)), \]

(the definition of the covariant \(Q\) see in §2). \(D_i\) is the covariant of polydegree \((i, n^2)\).

Let \((h, f) = (h, f_1(x) \otimes a_1 + f_2(x) \otimes a_2) \in \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2, h \neq 0\). A polynomial mapping \(I_h(f)\) is called in general position if \(Q(f_1, f_2) \neq 0\). It is obvious that a polynomial mapping \(I_h(f)\) is not in general position iff \(\text{dim}(I_h(f)^{-1}(0)) \geq 1\) or the polynomial degree of \(I_h(f)\) is less than \(n\).

**Lemma 5.1.** Suppose \((h, f) = (h, f_1(x) \otimes a_1 + f_2(x) \otimes a_2) \in \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2, h \neq 0, \text{and } i \geq 0; \text{then } D_i(h, f) = 0 \text{ iff the polynomial mapping } I_h(f) \text{ is not in general position or } |I_h(f)^{-1}(0)| < i.\)

**Proof.** It is obvious that the lemma is true if the polynomial mapping \(I_h(f)\) is not in general position. Suppose the polynomial mapping \(I_h(f)\) is in general position. If \(|I_h(f)^{-1}(0)| < i\), then

\[ Q(f_1, f_2) = l_1 \cdots l_{i-1} \cdot \tilde{l}_1 \cdots \tilde{l}_{n^2-i+1}, \]

where \(\langle h, \tilde{l}_1 \rangle = 0, 1 \leq j \leq n^2 - i + 1. \) We have

\[ D_i(h, f) = \Delta^i(h^i Q(f_1, f_2)) = \]

\[ \Delta^i(h^i l_1 \cdots l_{i-1} \cdot \tilde{l}_1 \cdots \tilde{l}_{n^2-i+1}) = 0. \]

If

\[ 0 = D_i(h, f) = \Delta^i(h^i Q(f_1, f_2)), \]

then

\[ Q(f_1, f_2) = l_1 \cdots l_{i-1} \cdot \tilde{l}_1 \cdots \tilde{l}_{n^2-i+1}, \]

where \(\langle h, \tilde{l}_j \rangle = 0, 1 \leq j \leq n^2 - i + 1\) and thus \(|I_h(f)^{-1}(0)| < i.\)

**Conjecture 1.** Consider the \(G\)-module \(\mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2\) and the covariants \(J_c, D_2;\) then

\[ D_2(h, f)|_{J_c(h, f) = 0} = 0. \]

It follows from Lemma 4.2 and Lemma 5.1 that conjecture 1 is true iff the following conjecture is true.

**Conjecture 1’.** Suppose the jacobian of a polynomial mapping

\[ F : \mathbb{C}^2 \to \mathbb{C}^2 \]

is equal to a constant; then \(\text{dim}F^{-1}(0) \geq 1\) or \(|F^{-1}(0)| \leq 1.\)

Evidently, conjecture 1’ is true iff the 2-dimensional jacobian conjecture is true.
§6. In this section we formulate conjecture 2. This conjecture is true iff the 2-dimensional Jacobian conjecture is true. Conjecture 2 is a partial case of general invariant theoretical problem ** (see §1). We use the notations of §3 and §4.

**Conjecture 2.** Consider the linear representation \( G : \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2 \) and the elements

\[
(x_3, f_0) \in \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2, \quad x_3^{2n-2} \in S^{2n-2} \mathbb{C}^3,
\]

where

\[
f_0 = x_1^{n_1} x_2^{n_2} \otimes a_1 + x_1^{n_1} x_2^{n_2} \otimes a_2,
\]

\( n_1 \geq 1, n_2 \geq 1, n_1 + n_2 = n; \) then there exists a polyhomogeneous \((J, J_c)\)-identity

\[
\gamma : \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2 \times S^{2n-2} \mathbb{C}^3 \rightarrow W
\]

such that

\[
\gamma(x_3, f_0, x_3^{2n-2}) \neq 0.
\]

**Proposition 6.1.** Conjecture 2 is true iff the 2-dimensional Jacobian conjecture is true.

**Proof.** 1. Let us deduce the 2-dimensional Jacobian conjecture from conjecture 2.

If the 2-dimensional Jacobian conjecture is not true, then there exists a counterexample

\[
F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2,
\]

\[
z = (z_1, z_2) \mapsto (F_1(z), F_2(z))
\]

such that

\[
F_1(z) = z_1^{n_1} z_2^{n_2} + \tilde{F}_1(z), \quad F_2(z) = z_1^{n_1} z_2^{n_2} + \tilde{F}_2(z),
\]

where \( n_1 \geq 1, n_2 \geq 1, n_1 + n_2 = n, deg \tilde{F}_1 < n, deg \tilde{F}_2 < n \) (Folk-theorem). Consider the linear representation \( G : \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2 \) and the elements

\[
(x_3, f_0) \in \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2, \quad x_3^{2n-2} \in S^{2n-2} \mathbb{C}^3,
\]

where

\[
f_0 = x_1^{n_1} x_2^{n_2} \otimes a_1 + x_1^{n_1} x_2^{n_2} \otimes a_2.
\]

It follows from conjecture 2 that there exists a polyhomogeneous \((J, J_c)\)-identity

\[
\gamma : \mathbb{C}^3 \times S^n \mathbb{C}^3 \otimes \mathbb{C}^2 \times S^{2n-2} \mathbb{C}^3 \rightarrow W
\]

such that

\[
\gamma(x_3, f_0, x_3^{2n-2}) \neq 0.
\]

Set

\[
f = x_3^n F_1(\frac{x_1}{x_3}, \frac{x_2}{x_3}) \otimes a_1 + x_3^n F_2(\frac{x_1}{x_3}, \frac{x_2}{x_3}) \otimes a_2 = f_0 + \tilde{f},
\]

where

\[
\tilde{f} = x_3^n \tilde{F}_1(\frac{x_1}{x_3}, \frac{x_2}{x_3}) \otimes a_1 + x_3^n \tilde{F}_2(\frac{x_1}{x_3}, \frac{x_2}{x_3}) \otimes a_2.
\]
By Lemma 4.1, it follows that \( J(x_3, f) = x_3^{2n-2} \). Therefore, \( J_c(x_3, f) = 0 \) and

\[
0 = \gamma(x_3, f, J(x_3, f)) = \gamma(x_3, f_0 + \tilde{f}, x_3^{2n-2}). \tag{6.1}
\]

Let \((d_1, d_2, d_3)\) be the polydegree of the covariant \( \gamma \). Set

\[
N = 2d_1 - nd_2 + 2(2n - 2)d_3,
\]

\[
g(t) = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in G.
\]

From (6.1) we obtain

\[
0 = t^N g(t) \cdot \gamma(x_3, f_0 + \tilde{f}, x_3^{2n-2}) = \gamma(t^2 g(t) \cdot x_3, t^{-n} (g(t) \cdot f_0 + g(t) \cdot \tilde{f}), t^{2(2n-2)} g(t) \cdot x_3^{2n-2}) \to \gamma(x_3, f_0, x_3^{2n-2}) \neq 0
\]

as \( t \to \infty \). This contradiction concludes the proof.

2. Let us deduce conjecture 2 from the 2-dimensional Jacobian conjecture. Set

\[
\tilde{\eta} : \mathbb{C}^{3*} \times \mathbb{C}^{n} \mathbb{C}^{3*} \otimes \mathbb{C}^2 \to V(2n - 4, 2) \otimes S^2 \mathbb{C}^2,
\]

\[
(h, f_1 \otimes a_1 + f_2 \otimes a_2) \mapsto \eta(h, f_1, f_2) \otimes a_1^2 + 2\eta(h, f_1, f_2) \otimes a_1 a_2 + \eta(h, f_2, f_2) \otimes a_2^2
\]

(the definition of the covariant \( \eta \) see in §3). \( \tilde{\eta} \) is the polyhomogeneous (of polydegree \( (2n, 2) \)) covariant.

Suppose \( J(x_3, f) = cx_3^{2n-2} \), where \( c \in \mathbb{C}, c \neq 0 \), \( f = f_1 \otimes a_1 + f_2 \otimes a_2 \); then by Lemma 4.1 and 2-dimensional Jacobian conjecture, it follows that \( f_1(x_1, x_2, 0) = c_1 \lambda(x_1, x_2)^n, f_2(x_1, x_2, 0) = c_2 \lambda(x_1, x_2)^n \), where \( \lambda(x_1, x_2) \) is a linear form in the variables \( x_1, x_2 \), and by Lemma 3.1, it follows that \( \tilde{\eta}(x_3, f) = 0 \). Therefore,

\[
\tilde{\eta}(x_3, f)|_{J(x_3, f) = cx_3^{2n-2} = 0} = 0
\]

for \( c \in \mathbb{C}, c \neq 0 \).

By (6.2), it follows that

\[
\tilde{\eta}(h, f)|_{J(h, f) = ch^{2n-2} = 0} = 0
\]

for \( c \in \mathbb{C}, c \neq 0 \).

Consider the homogeneous (of polydegree \( (2n, 2, 1) \)) covariant

\[
\gamma : \mathbb{C}^{3*} \times \mathbb{C}^{n} \mathbb{C}^{3*} \otimes \mathbb{C}^2 \times S^{2n-2} \mathbb{C}^{3*} \to (V(2n - 4, 2) \otimes S^2 \mathbb{C}^2) \otimes S^{2n-2} \mathbb{C}^{3*}
\]

\[
(h, f, j) \mapsto \tilde{\eta}(h, f) \otimes j.
\]

By Lemma 4.2 and (6.3) it follows that

\[
\gamma(h, f, J(h, f))|_{J_c(h, f) = 0} = 0.
\]

This means that \( \gamma \) is a polyhomogeneous \( (J, J_c) \)-identity. Using Lemma 3.1, we get

\[
\tilde{\eta}(x_3, f_0) = \eta(x_3, x_1^{n_1} x_2^{n_2}, x_1^{n_1} x_2^{n_2}) \otimes (a_1^2 + 2a_1 a_2 + a_2^2) =
\]

\[
\eta_0 \frac{n_1 n_2 (1 - n_1 - n_2)}{n^2 (n - 1)^2} e_2^{n_1 - 2} e_1^{n_2 - 2} x_3^2 \otimes (a_1^2 + 2a_1 a_2 + a_2^2) \neq 0.
\]

Therefore,

\[
\gamma(x_3, f_0, x_3^{2n-2}) = \tilde{\eta}(x_3, f_0) \otimes x_3^{2n-2} \neq 0.
\]
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