Log-symmetric regression models for correlated errors with an application to mortality data

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Abstract

Log-symmetric regression models are particularly useful when the response variable is continuous, strictly positive and asymmetric. In this paper, we proposed a class of log-symmetric regression models in the context of correlated errors. The proposed models provide a novel alternative to the existing log-symmetric regression models due to its flexibility in accommodating correlation. We discuss some properties, parameter estimation by the conditional maximum likelihood method and goodness of fit of the proposed model. We also provide expressions for the observed Fisher information matrix. A Monte Carlo simulation study is presented to evaluate the performance of the conditional maximum likelihood estimators. Finally, a full analysis of a real-world mortality data set is presented to illustrate the proposed approach.

Keywords Log-symmetric distributions; Time series; Maximum likelihood methods; Model selection criteria; Monte Carlo simulation; R software.

1 Introduction

Log-symmetric distributions are obtained when a random variable follows the same distribution as its reciprocal, or when the distribution of a logged random variable is symmetric; see Vanegas and Paula (2016a). The log-symmetric family of distributions has as special cases the log-normal, log-Student-$t$ and log-power-exponential distributions, among others. Some of its recent applications are in survival analysis, finance and movie industry; see, for example, Vanegas and Paula (2016c), Saulo and Leão (2017) and Ventura et al. (2018).

Recently, some works have been published on log-symmetric regression models; see Vanegas and Paula (2016a), Vanegas and Paula (2016c, 2017) and Medeiros and Ferrari (2017). This class of regression models arises when the distribution of the random errors is a member of the log-symmetric family, being particularly useful when the response variable is strictly positive and follows an asymmetric distribution. Moreover, in these models, either the median or skewness of the response variable can be modeled; see Vanegas and Paula (2016a).

A major drawback of using traditional (Gaussian) or log-symmetric regression models arises when the errors are correlated with each other. In this context, the true standard deviation of the estimated regression coefficients may be underestimated by the standard error of the regression coefficients, and the inferential procedures are no longer strictly applicable. Therefore, methods
that take into account or remove autocorrelation are necessary. In this scenario, we introduce in this work a class of log-symmetric regression models capable of accommodating correlation, named log-symmetric-autoregressive and moving average (log-symmetric-ARMAX) models. We obtain the conditional maximum likelihood estimators of the proposed model parameters and evaluate their performance by a Monte Carlo simulation study. We also fit the proposed models to a real-world data set for illustrative purpose.

The rest of the paper proceeds as follows. In Section 2, we describe the log-symmetric distribution and its corresponding regression model. In Section 3, we introduce the log-symmetric regression model for correlated data. Moreover, we discuss stationary conditions, parameter estimation, Fisher information and residual analysis. In Section 4, we carry out a Monte Carlo simulation study to evaluate the behavior of the estimators of the proposed log-symmetric-ARMAX model parameters. In Section 5, we apply the proposed models to a real-world mortality data set which is used to study the possible effects of temperature and pollution on mortality in Los Angeles County. Finally, in Section 6, we discuss some concluding remarks and future research.

2 Log-symmetric distribution and its regression model

The class of log-symmetric distributions is obtained by taking the exponential of a symmetric random variable; see Vanegas and Paula (2016b). In other words, let $V$ be a continuous random variable following a symmetric distribution with location parameter $\mu \in \mathbb{R}$, scale parameter $\phi > 0$ and a density generating kernel $g$, denoted by $V \sim S(\mu, \phi, g)$, and with probability density function (PDF) given by $f_V(v; \mu, \phi) = \frac{\xi_{nc}}{\sqrt{\phi}} g\left(\frac{(v - \mu)^2}{\phi}\right)$, where $v \in \mathbb{R}$, $g(u) > 0$ for $u > 0$ such that $\int_{-\infty}^{+\infty} g(z^2) \, dz = 1/\xi_{nc}$ and $\xi_{nc}$ is a normalizing constant; see Fang et al. (1990).

Then, the random variable $Y = \exp(V)$ follows a log-symmetric distribution with PDF

$$f_Y(y; \lambda, \phi) = \frac{\xi_{nc}}{\sqrt{\phi} \lambda y} g\left(\frac{1}{\phi} \left(\log \left(\frac{y}{\lambda}\right)\right)^2\right), \quad y > 0,$$

where $\lambda = \exp(\mu) > 0$ and $\phi > 0$ are the scale and shape parameters and they represent, respectively, the median and skewness (or relative dispersion) of the $Y$ distribution. $g$ is a density generating kernel which may be associated with an additional parameter $\vartheta$ (or vector $\vartheta$). In this case, we use the notation $Y \sim LS(\lambda, \phi, g)$. Some special log-symmetric distributions are the log-normal, log-power-exponential, log-Student-$t$ and log-slash, among others; see Crow and Shimizu (1988) and Vanegas and Paula (2016b).

A regression model based on (1) was studied by Vanegas and Paula (2016a, 2017), where for a set of $n$ independent random variables, $Y_1, \ldots, Y_n$ say, such that $Y_i \sim LS(\lambda_i, \phi_i, g)$, $i = 1, \ldots, n$, $Y_i$ satisfies the following functional relation

$$Y_i = \lambda_i \epsilon_i^{\sqrt{\phi_i}}, \quad \epsilon_i \sim LS(1, 1, g),$$

or in logarithm terms,

$$V_i = \log(Y_i) = \mu_i + \sqrt{\phi_i} \epsilon_i, \quad i = 1, \ldots, n,$$
where \( \mu_i = \log(\lambda_i), \varepsilon_i = \log(\epsilon_i), \lambda_i = \Lambda^{-1}(x_i^\top \beta) \) and \( \phi_i = \Lambda^{-1}(w_i^\top \tau) \), with \( \beta = (\beta_0, \ldots, \beta_k)^\top \) and \( \tau = (\tau_0, \ldots, \tau_l)^\top \) being vectors of unknown parameters and \( x_i^\top = (1, x_{i1}, \ldots, x_{ik})^\top \) and \( w_i^\top = (1, w_{i1}, \ldots, w_{il})^\top \) are the values of \( k \) and \( l \) covariates associated with the median \( \lambda_i \) and skewness \( \phi_i \), respectively. \( \Lambda \) is an invertible link function and its inverse function is \( \Lambda^{-1} \). Note that \( \varepsilon_i \sim S(0, 1, g) \) and \( V_i \sim S(\mu_i, \phi_i, g) \).

The log-likelihood function (without the constant) associated with the log-symmetric regression model defined by (2) and (3) is given by

\[
\ell(\theta) = -\frac{1}{2} \sum_{i=1}^{n} \log(\phi_i) + \sum_{i=1}^{n} \log(\phi_i^2),
\]

where \( \theta = (\beta, \zeta)^\top \) and \( z_i = (v_i - \mu_i)/\sqrt{\phi_i} \), for \( i = 1, \ldots, n \). The maximum likelihood estimate of \( \theta \) must be obtained numerically with an iterative method for non-linear optimization problems. For example, by the Broyden-Fletcher-Goldfarb-Shanno quasi-Newton method; see Mittelhammer et al. (2000).

3 Log-symmetric regression model for correlated data

Let \( \{Y_t\} \) be random variables and \( A_t = \sigma(Y_t, Y_{t-1}, \ldots) \) be the \( \sigma \)-field generated by the information up to time \( t \). We assume that the conditional distribution of \( Y_t \) given \( A_{t-1} \) follows a log-symmetric distribution, denoted by \( Y_t|A_{t-1} \sim LS(\lambda_t, \phi_t, g) \), with density

\[
f_{Y_t|A_{t-1}}(y_t; \lambda_t, \phi_t|A_{t-1}) = \frac{\xi_{nc}}{\sqrt{\phi_t} y_t} g \left( \frac{1}{\phi_t} \left( \log \left( \frac{y_t}{\lambda_t} \right) \right)^2 \right), \quad y_t > 0,
\]

where \( \lambda_t = \exp(\mu_t) > 0 \) and \( \phi_t > 0 \) are the corresponding scale and shape parameters, respectively. By using the relation in (2), we can write

\[
h(Y_t) = \lambda_t \epsilon_t^{\sqrt{\phi_t}}
\]

and set \( h(Y_t) = \log(Y_t) \), to obtain

\[
h(Y_t) = \mu_t + \sqrt{\phi_t} \varepsilon_t, \quad t = 1, \ldots, n,
\]

where \( h(Y_t)|B_{t-1} \sim S(\mu_t, \phi_t, g), \phi_t = \Lambda^{-1}(w_t^\top \tau) \) and

\[
\mu_t = E[h(Y_t)|B_{t-1}] = x_t^\top \beta + \varphi_t, \quad t = 1, \ldots, n,
\]

with \( B_t = \sigma(h(Y_t), h(Y_{t-1}), \ldots) \) being the \( \sigma \)-field generated by the information up to time \( t \), and \( \varphi_t \) denoting a dynamic element with ARMA structure, that is,

\[
\varphi_t = \sum_{i=1}^{p} \kappa_i \left( h(Y_{t-i}) - x_{t-i}^\top \beta \right) + \sum_{j=1}^{q} \zeta_j r_{t-j},
\]
where \( r_t := h(Y_t) - \mu_t \) is a martingale difference sequence (MDS), i.e., \( \mathbb{E}|r_t| < \infty \), and \( \mathbb{E}[r_t | \mathcal{B}_{t-1}] = 0 \), a.s., for all \( t \). This implies that \( \mathbb{E}[r_t] = 0 \) for all \( t \), and \( \text{Cov}[r_s, r_t] = 0 \) (uncorrelatedness of the sequence) for all \( t \neq s \).

By adding \( h(Y_t) - \mu_t \) to both sides of (7), we have

\[
h(Y_t) = x_t^\top \beta + \sum_{l=1}^{p} \kappa_l \left( h(Y_{t-l}) - x_{t-l}^\top \beta \right) + \sum_{j=1}^{q} \zeta_j r_{t-j} + r_t. \tag{9}
\]

In (7), (8) and (9), \( h, x_t, \beta, w_t \) and \( \tau \) are as in (3), \( \eta \in \mathbb{R} \), \( \kappa = (\kappa_1, \ldots, \kappa_p)^\top \in \mathbb{R}^p \) and \( \zeta = (\zeta_1, \ldots, \zeta_q)^\top \in \mathbb{R}^q \). Note that (7) and (8) lead to the notation log-symmetric-ARMAX(\( p, q \)), as usual in ARMA models.

### 3.1 Stationarity conditions

**Theorem 1.** The marginal mean of \( h(Y_t) \) in the log-symmetric-ARMAX(\( p, q \)) model is given by

\[
\mathbb{E}[h(Y_t)] = x_t^\top \beta,
\]

provided that \( \Phi(B) : \mathbb{R} \to \mathbb{R} \) is an invertible operator (the autoregressive polynomial) defined by \( \Phi(B) = -\sum_{i=0}^{p-1} \kappa_i B^i \) with \( \kappa_0 = -1 \), and \( B^i \) is the lag operator, i.e., \( B^i y_t = y_{t-i} \).

**Proof.** Let \( \Theta(B) = \sum_{i=0}^{q} \xi_i B^i \) with \( \xi_0 = 1 \), be the moving averages polynomial. Since \( \Theta(B) \Phi(B)^{-1} = \sum_{i=0}^{\infty} \psi_i B^i \) with \( \psi_0 = 1 \), using (9), the log-symmetric-ARMAX(\( p, q \)) model can be rewritten as

\[
w_t = \sum_{l=1}^{p} \kappa_l w_{t-l} + \sum_{j=1}^{q} \zeta_j r_{t-j} + r_t = \Theta(B) \Phi(B)^{-1} r_t, \tag{10}
\]

where the error \( r_t = h(Y_t) - \mu_t \) is a MDS and \( w_t = h(Y_t) - x_t^\top \beta \). Then

\[
\mathbb{E}[h(Y_t)] = x_t^\top \beta + \mathbb{E}[w_t] = x_t^\top \beta + \Theta(B) \Phi(B)^{-1} \mathbb{E}[r_t] = x_t^\top \beta,
\]

whenever the series \( \Theta(B) \Phi(B)^{-1} r_t \) converges absolutely. \( \square \)

**Theorem 2.** Assuming that \( \Theta(B) \Phi(B)^{-1} = \sum_{i=0}^{\infty} \psi_i B^i \) and \( \Phi(B) \) is invertible, we have that the marginal variance of \( h(Y_t) \) in the log-symmetric-ARMAX(\( p, q \)) model is given by

\[
\text{Var}[h(Y_t)] = \xi \sum_{i=0}^{\infty} \psi_i^2 \phi_{t-i}^{1/2},
\]

where \( \xi > 0 \) is a constant not depending on the parameters. The quantity \( \xi \) for some distributions is presented in Table 1 of Medeiros and Ferrari (2017).

**Proof.** Since \( \mathbb{E}[r_t | \mathcal{B}_{t-1}] = 0 \), a.s., for all \( t \), and \( \text{Cov}[r_s, r_t] = 0 \) for all \( t \neq s \), following the
notation of Theorem 1, we have

\[
\text{Var}[h(Y_t)] = \text{Var}[w_t] = \text{Var}[\Theta(B) \Phi(B)^{-1}r_t] = \text{Var} \left[ \sum_{i=0}^{\infty} \psi_i B^i r_t \right] = \sum_{i=0}^{\infty} \psi_i^2 \text{Var}[r_{t-i}] . \tag{11}
\]

On the other hand, the law of total variance states that

\[
\text{Var}[r_t] = E[\text{Var}[r_t|B_{t-1}]] + \text{Var}[E[r_t|B_{t-1}]]
= E[\text{Var}[h(Y_t)|B_{t-1}]] . \tag{12}
\]

Since \(\text{Var}[h(Y_t)|B_{t-1}] = \xi \phi_i^{1/2} \text{ a.s.}\), combining (11) and (12), the proof follows. \(\square\)

**Theorem 3.** The covariance and correlation of \(h(Y_t)\) and \(h(Y_{t-k})\) in the log-symmetric-ARMAX \((p, q)\) model are given by

\[
\text{Cov}[h(Y_t), h(Y_{t-k})] = \xi \sum_{i=0}^{\infty} \psi_i \psi_{i-k} \phi_i^{1/2}, \quad k > 0,
\]

\[
\text{Corr}[h(Y_t), h(Y_{t-k})] = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i-k} \phi_i^{1/2}}{\prod_{j \in \{0, k\}} \sqrt{\sum_{i=0}^{\infty} \psi_i^2 \phi_i^{1/2}}},
\]

respectively.

**Proof.** Since \(w_t = h(Y_t) - x_t^T \beta\) and \(\text{Cov}[r_s, r_t] = 0\) for all \(t \neq s\),

\[
\text{Cov}[h(Y_t), h(Y_{t-k})] = \text{Cov}[w_t, w_{t-j}] \overset{(10)}{=} \text{Cov}[\Theta(B) \Phi(B)^{-1}r_t, \Theta(B) \Phi(B)^{-1}r_{t-k}]
= \sum_{i=0}^{\infty} \psi_i \psi_{i-k} \text{Var}[r_{t-i}] .
\]

Using (12) the expression on the right side is equal to \(\sum_{i=0}^{\infty} \psi_i \psi_{i-k} E[\text{Var}[h(Y_{t-i})|B_{t-i-1}]].\)

Since \(\text{Var}[h(Y_t)|B_{t-1}] = \xi \phi_i^{1/2} \text{ a.s.}\), the proof follows. \(\square\)

**Remark 1.** If the parameter \(\phi_t = \phi\) is constant, \(\text{Var}[r_t|B_{t-1}] = \text{Var}[h(Y_t)|B_{t-1}] = \xi \phi^{1/2} \text{ a.s.}\), for all \(t\) (then the MDS would be a white noise). Then of Theorems 2 and 3, the following stationarity conditions follows (see Maior and Cysneiros (2018))

\[
\text{Var}[h(Y_t)] = \xi \phi^{1/2} \sum_{i=0}^{\infty} \psi_i^2, \quad \text{Cov}[h(Y_t), h(Y_{t-k})] = \xi \phi^{1/2} \sum_{i=0}^{\infty} \psi_i \psi_{i-k} \quad \text{and}
\]

\[
\text{Corr}[h(Y_t), h(Y_{t-k})] = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i-k}}{\sum_{i=0}^{\infty} \psi_i^2}, \quad k > 0.
\]
3.2 Estimation and inference

The conditional maximum likelihood method can be used to obtain the model parameter estimates based on the first \( m \) observations. Consider the parameter vector \( \theta = (\beta^T, \tau^T, \kappa^T, \zeta^T)^T \) and \( m = \max\{p, q\} \), for \( n > m \). Then, the conditional likelihood function is given by

\[
L_{m,n}(\theta) = \prod_{t=m+1}^{n} f_{\text{log}(Y_t)|B_{t-1}}(v_t; \mu_t, \phi_t|B_{t-1}), \quad v_t \in \mathbb{R},
\]

which implies the following conditional log-likelihood function (without the constant)

\[
\ell_{m,n}(\theta) = -\frac{1}{2} \sum_{t=m+1}^{n} \log(\phi_t) + \sum_{t=m+1}^{n} \log(g(z_t^2)), \quad \text{(13)}
\]

where \( z_t = (v_t - \mu_t)/\sqrt{\phi_t} \), for \( t = m + 1, \ldots, n \), \( \phi_t = \Lambda^{-1}(w_t^T \tau) \) and

\[
\mu_t = \sum_{r=0}^{k} \beta_r x_{tr} + \sum_{s=1}^{p} \kappa_s \left( v_{t-s} - \sum_{i=0}^{k} \beta_i x_{(t-s)i} \right) + \sum_{j=1}^{q} \zeta_j r_{t-j}. \quad \text{(14)}
\]

The conditional maximum likelihood estimates can be obtained by maximizing the expression defined in (13) by equating the score vector \( \ell'(\theta) \), which contains the first derivatives of \( \ell(\theta) \), to zero, providing the likelihood equations. Inference for \( \theta \) of the log-symmetric-\( \text{ARMA}(p, q) \) model can be based on the asymptotic distribution of the conditional maximum likelihood estimator \( \hat{\theta} \). For \( n \) sufficiently large and considering usual regularity conditions (Efron and Hinkley, 1978), the conditional maximum likelihood estimator converges in distribution to a normal distribution

\[
\sqrt{n} \left[ \hat{\theta} - \theta \right] \xrightarrow{D} N_{2+k+l+p+q}(0, \mathcal{J}(\theta)^{-1}),
\]

as \( n \to \infty \), where \( \xrightarrow{D} \) means “convergence in distribution” and \( \mathcal{J}(\theta) \) is the corresponding expected Fisher information matrix. In this case, we approximate the expected Fisher information matrix by its observed version obtained from the Hessian matrix

\[
\ddot{\ell}(\theta) = \begin{bmatrix}
\frac{\partial^2 \ell_{0,1}(\theta)}{\partial \beta_{r}^2} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \beta_{r} \partial \tau_{s}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \beta_{r} \partial \kappa_{l}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \beta_{r} \partial \zeta_{j}} \\
\frac{\partial^2 \ell_{0,1}(\theta)}{\partial \tau_{s} \partial \beta_{r}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \tau_{s}^2} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \tau_{s} \partial \kappa_{l}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \tau_{s} \partial \zeta_{j}} \\
\frac{\partial^2 \ell_{0,1}(\theta)}{\partial \kappa_{l} \partial \beta_{r}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \kappa_{l} \partial \tau_{s}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \kappa_{l} \partial \kappa_{l}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \kappa_{l} \partial \zeta_{j}} \\
\frac{\partial^2 \ell_{0,1}(\theta)}{\partial \zeta_{j} \partial \beta_{r}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \zeta_{j} \partial \tau_{s}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \zeta_{j} \partial \kappa_{l}} & \frac{\partial^2 \ell_{0,1}(\theta)}{\partial \zeta_{j} \partial \zeta_{j}}
\end{bmatrix},
\]

where \( r = 0, \ldots, k; s = 0, \ldots, l; l = 1, \ldots, p \) and \( j = 1, \ldots, q \). Since the function \( \ell_{0,1}(\theta) \) has continuous second partial derivatives at a given point \( \theta \) in \( \mathbb{R}^4 \), by Schwarz’s Theorem follows
that the partial differentiations of this function are commutative at that point, that is,
\[
\frac{\partial^2 \ell_{0,1} \left( \theta \right)}{\partial a \partial b} = \frac{\partial^2 \ell_{0,1} \left( \theta \right)}{\partial b \partial a}, \quad \text{for } a \neq b \text{ in } \{ \beta_r, \tau_s, \kappa_l, \zeta_j \}.
\]

It can easily be seen that the first derivatives of \( \ell_{0,1} \) are
\[
\frac{\partial \ell_{0,1} \left( \theta \right)}{\partial a} = \frac{1}{g(z_t^2)} \frac{\partial g(z_t^2)}{\partial a}, \quad a \in \{ \beta_r, \kappa_l, \zeta_j \},
\]
the second derivatives are
\[
\frac{\partial^2 \ell_{0,1} \left( \theta \right)}{\partial a^2} = \frac{1}{g(z_t^2)^2} \frac{\partial g(z_t^2)}{\partial a} \frac{\partial^2 g(z_t^2)}{\partial a^2} + \frac{1}{g(z_t^2)} \frac{\partial^2 g(z_t^2)}{\partial a^2}, \quad a \in \{ \beta_r, \kappa_l, \zeta_j \},
\]
and the mixed derivatives are given by
\[
\frac{\partial^2 \ell_{0,1} \left( \theta \right)}{\partial a \partial b} = \frac{1}{g(z_t^2)^2} \frac{\partial g(z_t^2)}{\partial a} \frac{\partial g(z_t^2)}{\partial b} + \frac{1}{g(z_t^2)} \frac{\partial^2 g(z_t^2)}{\partial a \partial b}, \quad a, b \in \{ \beta_r, \kappa_l, \zeta_j \},
\]

Let
\[
\eta_t := \frac{z_t}{\phi_t} = \frac{v_t - \mu_t}{\phi_t}.
\]

The first derivatives of \( g \) are
\[
\frac{\partial g(z_t^2)}{\partial a} = -2 \eta_t \frac{\partial \mu_t}{\partial a} \frac{g(z_t^2)}{\partial a}, \quad a \in \{ \beta_r, \kappa_l, \zeta_j \}, \quad \frac{\partial g(z_t^2)}{\partial \tau_s} = -\eta_t \frac{d \phi_t}{d \tau_s} \frac{g(z_t^2)}{\partial \tau_s},
\]
the second derivatives are
\[
\frac{\partial^2 g(z_t^2)}{\partial a^2} = 2 \left( \frac{1}{\phi_t} \left( \frac{\partial \mu_t}{\partial a} \right)^2 - \eta_t \frac{\partial^2 \mu_t}{\partial a^2} \right) \frac{\partial g(z_t^2)}{\partial a} - 2 \eta_t \frac{\partial \mu_t}{\partial a} \frac{\partial^2 g(z_t^2)}{\partial a^2}, \quad a \in \{ \beta_r, \kappa_l, \zeta_j \},
\]
\[
\frac{\partial^2 g(z_t^2)}{\partial \tau_s^2} = \eta_t \left( \frac{2}{\phi_t^2} \left( \frac{d \phi_t}{d \tau_s} \right)^2 - \eta_t \frac{d^2 \phi_t}{d \tau_s^2} \right) \frac{\partial g(z_t^2)}{\partial \tau_s} - \eta_t \frac{d \phi_t}{d \tau_s} \frac{\partial^2 g(z_t^2)}{\partial \tau_s^2},
\]
and the mixed derivatives are given by

\[
\frac{\partial^2 g(z_t^2)}{\partial \beta_r \partial \tau_s} = \eta_t \left( \frac{2 \partial \mu_t}{\phi_t} \frac{\partial g}{\partial \beta_r}(z_t^2) - \eta_t \frac{\partial^2 g}{\partial \beta_r \partial \tau_s}(z_t^2) \right) \frac{d\phi_t}{d\tau},
\]

\[
\frac{\partial^2 g(z_t^2)}{\partial \beta_r \partial a} = \left( \frac{2 \partial \mu_t}{\phi_t} \frac{\partial^2 g}{\partial \beta_r \partial a}(z_t^2) - 2\eta_t \frac{\partial^2 g}{\partial a}(z_t^2) \right) \frac{d\mu_t}{d\alpha}, \quad a \in \{\kappa_t, \zeta_j\},
\]

\[
\frac{\partial^2 g(z_t^2)}{\partial \tau_s \partial b} = 2\eta_t \left( \frac{1}{\phi_t} \frac{\partial g}{\partial \tau_s}(z_t^2) - \frac{\partial^2 g}{\partial \tau_s \partial b}(z_t^2) \right) \frac{d\mu_t}{d\beta}, \quad b \in \{\kappa_t, \zeta_j\},
\]

\[
\frac{\partial^2 g(z_t^2)}{\partial \kappa_l \partial \zeta_j} = \left( \frac{2 \partial \mu_t}{\phi_t} \frac{\partial \mu_t}{\partial \kappa_l}(z_t^2) - 2\eta_t \frac{\partial \mu_t}{\partial \zeta_j}(z_t^2) \right) \frac{d\mu_t}{d\beta}, \quad j = 1, \ldots, q,
\]

With

\[
\frac{d\phi_t}{d\tau} = w_{ts} \left( \frac{\partial \Lambda}{\partial \tau_s}(\phi_t) \right)^{-1}, \quad \frac{d^2 \phi_t}{d\tau^2} = -w_{ts} \left( \frac{\partial \Lambda}{\partial \tau_s}(\phi_t) \right)^{-2} \frac{\partial^2 \Lambda}{\partial \tau_s^2}(\phi_t).
\]

By (14), the first derivatives of \(\mu_t\) are

\[
\frac{\partial \mu_t}{\partial \beta_r} = x_{tr} - \sum_{l=1}^p \kappa_l x_{(t-l)r} - \sum_{j=1}^q \zeta_j \frac{\partial \mu_{t-j}}{\partial \beta_r},
\]

\[
\frac{\partial \mu_t}{\partial \kappa_l} = \nu_{t-l} - \sum_{i=0}^k \beta_i \nu_{(t-l)i} - \sum_{j=1}^q \frac{\partial \mu_{t-j}}{\partial \kappa_l},
\]

\[
\frac{\partial \mu_t}{\partial \zeta_j} = \nu_{t-j} - \nu_{t-j} - \sum_{j=1}^q \frac{\partial \mu_{t-j}}{\partial \zeta_j},
\]

the second derivatives are given by

\[
\frac{\partial^2 \mu_t}{\partial \beta_r^2} = -\sum_{j=1}^q \zeta_j \frac{\partial^2 \mu_{t-j}}{\partial \beta_r^2}, \quad \frac{\partial^2 \mu_t}{\partial \kappa_l^2} = -\sum_{j=1}^q \zeta_j \frac{\partial^2 \mu_{t-j}}{\partial \kappa_l^2}, \quad \frac{\partial^2 \mu_t}{\partial \zeta_j^2} = -\frac{\partial \mu_{t-j}}{\partial \zeta_j} - \sum_{j=1}^q \zeta_j \frac{\partial^2 \mu_{t-j}}{\partial \zeta_j^2},
\]

with mixed derivatives

\[
\frac{\partial^2 \mu_t}{\partial \beta_r \partial \kappa_l} = -x_{(t-l)r} - \sum_{j=1}^q \zeta_j \frac{\partial^2 \mu_{t-j}}{\partial \beta_r \partial \kappa_l}, \quad \frac{\partial^2 \mu_t}{\partial \beta_r \partial \zeta_j} = -\frac{\partial \mu_{t-j}}{\partial \beta_r} - \sum_{j=1}^q \zeta_j \frac{\partial^2 \mu_{t-j}}{\partial \beta_r \partial \zeta_j}, \quad a \in \{\beta_r, \kappa_l\}.
\]

### 3.3 Residual analysis

We assess goodness of fit and departures from the assumptions of the model by using the quantile residual, which is given by

\[
r_t^Q = \Phi^{-1}(\hat{S}(t|B_{t-1})), \quad t = m + 1, \ldots, n,
\]
where $\Phi^{-1}$ is the inverse function of the standard normal cumulative distribution function (CDF) and $\hat{S}$ the fitted survival function. The quantile residual has a standard normal distribution when the model is correctly specified. Note that this residual is usually applied to generalized additive models for location, scale and shape; see Dunn and Smyth (1996).

4 Monte Carlo simulation

A Monte Carlo simulation study is carried out to evaluate the performance of the conditional maximum likelihood estimators for the log-symmetric-ARMAX(1, 1) model under the log-normal (LogN), log-Student-t (Logt) and log-power-exponential (LogPE) cases. The simulation scenario considered the following model

$$\log(Y_t) = \beta_0 + \beta_1 x_{t-1} + \kappa_1 \left( \log(Y_{t-1}) - \beta_0 - \beta_1 x_{t-1} \right) + \zeta_1 r_{t-1} + r_t \quad t = 2, \ldots, n,$$

where $n \in \{100, 300, 500\}$, $\phi_t = \phi \in \{1.00, 2.00, 3.00\}$ for all $t$, $\beta_0 = 1$, $\beta_1 = 0.7$, $\kappa_1 = 0.6$, $\zeta_1 = 0.3$, $\vartheta = 0.5$ (LogPE) and $\vartheta = 4$ (Logt). The conditional maximum likelihood estimation results are presented in Tables 1–3. In particular, bias and mean squared error (MSE) are reported in these tables. Note that the results allow us to conclude that, as the sample size increases, the bias and MSE of all the estimators decrease, as expected. In general, bias and MSE associated with the conditional maximum likelihood estimates of the Logt-ARMAX(1, 1) model, present the lowest values.

Table 1: Empirical bias and MSE (in parentheses) from simulated data for the indicated conditional maximum likelihood estimators of the LogN-ARMAX(1, 1) model.

| $n$ | $\phi = 0.5$ | | | $\phi = 1$ | | | $\phi = 2$ | |
|-----|-------------|---|---|-------------|---|---|-------------|---|
|     | Bias | MSE | | Bias | MSE | | Bias | MSE |
| 100 | $\hat{\phi}$ | -0.0257 | 0.0055 | | -0.0514 | 0.0218 | | -0.1028 | 0.0873 |
|     | $\hat{\beta}_0$ | -0.0072 | 0.0631 | | -0.0102 | 0.1261 | | -0.0144 | 0.2522 |
|     | $\hat{\beta}_1$ | 0.0021 | 0.0705 | | 0.0030 | 0.1410 | | 0.0042 | 0.2820 |
|     | $\hat{\kappa}_1$ | -0.0394 | 0.0150 | | -0.0394 | 0.0150 | | -0.0394 | 0.0150 |
|     | $\hat{\zeta}_1$ | 0.0326 | 70.0197 | | 0.0326 | 0.0197 | | 0.0326 | 0.0197 |
| 300 | $\hat{\phi}$ | -0.0093 | 0.0018 | | -0.0185 | 0.0070 | | -0.0371 | 0.0281 |
|     | $\hat{\beta}_0$ | 0.0063 | 0.0224 | | 0.0089 | 0.0449 | | 0.0126 | 0.0897 |
|     | $\hat{\beta}_1$ | -0.0136 | 0.0234 | | -0.0192 | 0.0468 | | -0.0272 | 0.0937 |
|     | $\hat{\kappa}_1$ | -0.0156 | 0.0046 | | -0.0156 | 0.0046 | | -0.0156 | 0.0046 |
|     | $\hat{\zeta}_1$ | 0.0118 | 0.0066 | | 0.0118 | 0.0066 | | 0.0118 | 0.0066 |
| 500 | $\hat{\phi}$ | -0.0055 | 0.0010 | | -0.0110 | 0.0041 | | -0.0221 | 0.0162 |
|     | $\hat{\beta}_0$ | 0.0029 | 0.0138 | | 0.0041 | 0.0275 | | 0.0059 | 0.0550 |
|     | $\hat{\beta}_1$ | -0.0106 | 0.0129 | | -0.0149 | 0.0258 | | -0.0211 | 0.0516 |
|     | $\hat{\kappa}_1$ | -0.0077 | 0.0025 | | -0.0077 | 0.0025 | | -0.0077 | 0.0025 |
|     | $\hat{\zeta}_1$ | 0.0072 | 0.0039 | | 0.0072 | 0.0039 | | 0.0072 | 0.0039 |
Table 2: Empirical bias and MSE (in parentheses) from simulated data for the indicated conditional maximum likelihood estimators of the Logit-ARMAX(1, 1) model.

| n   | φ = 0.5 |          |          |          |          |          |          |
|-----|---------|----------|----------|----------|----------|----------|----------|
|     | Bias    | MSE      | Bias     | MSE      | Bias     | MSE      | Bias     | MSE      |
| 100 |         |          |          |          |          |          |          |          |
| φ   | -0.0162 | 0.0091   | -0.0323  | 0.0364   | -0.0646  | 0.1456   |          |          |
|     | -0.0107 | 0.1113   | -0.0152  | 0.2226   | -0.0215  | 0.4453   |          |          |
|     | -0.0032 | 0.0981   | -0.0046  | 0.1961   | -0.0065  | 0.3923   |          |          |
|     | -0.0389 | 0.0125   | -0.0389  | 0.0125   | -0.0389  | 0.0125   |          |          |
|     | 0.0302  | 0.0157   | 0.0302   | 0.0157   | 0.0302   | 0.0157   |          |          |
| 300 |         |          |          |          |          |          |          |          |
| φ   | -0.0067 | 0.0031   | -0.0134  | 0.0124   | -0.0269  | 0.0496   |          |          |
|     | -0.0022 | 0.0378   | -0.0032  | 0.0757   | -0.0045  | 0.1513   |          |          |
|     | -0.0038 | 0.0328   | -0.0054  | 0.0656   | -0.0076  | 0.1312   |          |          |
|     | -0.0129 | 0.0033   | -0.0129  | 0.0033   | -0.0129  | 0.0033   |          |          |
|     | 0.0101  | 0.0047   | 0.0101   | 0.0047   | 0.0101   | 0.0047   |          |          |
| 500 |         |          |          |          |          |          |          |          |
| φ   | -0.0049 | 0.0018   | -0.0099  | 0.0072   | -0.0198  | 0.0289   |          |          |
|     | -0.0059 | 0.0219   | -0.0083  | 0.0437   | -0.0117  | 0.0875   |          |          |
|     | -0.0040 | 0.0175   | -0.0057  | 0.0350   | -0.0081  | 0.0701   |          |          |
|     | -0.0083 | 0.0020   | -0.0083  | 0.0020   | -0.0083  | 0.0020   |          |          |
|     | 0.0066  | 0.0029   | 0.0066   | 0.0029   | 0.0066   | 0.0029   |          |          |

Table 3: Empirical bias and MSE (in parentheses) from simulated data for the indicated conditional maximum likelihood estimators of the LogPE-ARMAX(1, 1) model.

| n   | φ = 0.5 |          |          |          |          |          |          |
|-----|---------|----------|----------|----------|----------|----------|----------|
|     | Bias    | MSE      | Bias     | MSE      | Bias     | MSE      | Bias     | MSE      |
| 100 |         |          |          |          |          |          |          |          |
| φ   | -0.0177 | 0.0080   | -0.0434  | 0.0321   | -0.0868  | 0.1286   |          |          |
|     | 0.0067  | 0.1690   | 0.0095   | 0.3380   | 0.0135   | 0.6759   |          |          |
|     | -0.0193 | 0.1566   | -0.0273  | 0.3132   | -0.0387  | 0.6263   |          |          |
|     | -0.0432 | 0.0148   | -0.0432  | 0.0148   | -0.0432  | 0.0148   |          |          |
|     | 0.0348  | 0.0187   | 0.0348   | 0.0187   | 0.0348   | 0.0187   |          |          |
| 300 |         |          |          |          |          |          |          |          |
| φ   | -0.0076 | 0.0026   | -0.0151  | 0.0103   | -0.0302  | 0.0411   |          |          |
|     | 0.0019  | 0.0524   | 0.0027   | 0.1047   | 0.0039   | 0.2095   |          |          |
|     | 0.0049  | 0.0477   | 0.0069   | 0.0955   | 0.0097   | 0.1910   |          |          |
|     | -0.0135 | 0.0036   | -0.0135  | 0.0036   | -0.0135  | 0.0036   |          |          |
|     | 0.0118  | 0.0056   | 0.0118   | 0.0056   | 0.0118   | 0.0056   |          |          |
| 500 |         |          |          |          |          |          |          |          |
| φ   | -0.0057 | 0.0016   | -0.0114  | 0.0064   | -0.0229  | 0.0256   |          |          |
|     | 0.0028  | 0.0335   | 0.0040   | 0.0670   | 0.0056   | 0.1340   |          |          |
|     | -0.0021 | 0.0258   | -0.0029  | 0.0515   | -0.0041  | 0.1031   |          |          |
|     | -0.0080 | 0.0021   | -0.0080  | 0.0021   | -0.0080  | 0.0021   |          |          |
|     | 0.0070  | 0.0031   | 0.0071   | 0.0031   | 0.0071   | 0.0031   |          |          |

5 Illustrative example

The log-symmetric regression and log-symmetric-ARMAX models are now used to analyze a real-world data set, regarding the possible effects of temperature and pollution on weekly mortality in Los Angeles County over the 10 year period 1970-1979; see Shumway and Stoffer (2017). We have the following variables from this data set: cardiovascular mortality (response), temperature (covariate) and particulate levels (covariate). Figure 1 displays scatter-plots with...
their corresponding correlations for all these variables presented. From this figure, we detect adequate levels of correlation between the response and the covariates, justifying the use of a linear regression model.

Figure 1: Scatterplots and their correlations for the indicated variables with the mortality data.

Table 4 provides descriptive statistics for the mortality data set, including central tendency statistics, standard deviation (SD), coefficients of variation (CV), skewness (CS) and kurtosis (CK). From this table, note the presence of skewness and kurtosis in the data distribution; see Figure 2(centre). Note also the presence of autocorrelation; see Figure 2(right).

Table 4: Summary statistics for the mortality data.

| n   | Minimum | Median | Mean  | Maximum | SD    | CV    | CS   | CK   |
|-----|---------|--------|-------|---------|-------|-------|------|------|
| 508 | 68.11   | 87.33  | 88.699| 132.04 | 9.999 | 11.273%| 0.804| 0.981|

Figure 2: Timeplot (left), histogram (centre) and autocorrelation (right) function for the mortality data.
5.1 Log-symmetric regression results

We estimate three log-symmetric regression models based on the following special cases: LogN, Log$t$ and LogPE. Based on the scatterplots and timeplot shown in Figures 1 and 2(right) and Shumway and Stoffer (2017), we can set the following final variables: [response] $Y_1$ (mortality) and [covariates] $x_1$ (linear trend), $x_2$ (temperature), $x_3$ (squared temperature) and $X_4$ (particulates). We consider $\phi_i = \phi$ for $i = 1, \ldots, n$.

Table 5 reports the estimates, SEs and p-values of the t-test for the log-symmetric regression model parameters. Furthermore, we report the Akaike (AIC) and Bayesian information (BIC) criteria and the root mean square error (RMSE) to compare the fitted models. From Table 5, the three log-symmetric models provide virtually the same adjustments based on the values of RMSE, AIC and BIC. However, the QQ plots with simulated envelope of the quantile residuals for these models show good agreement with the N(0, 1) distribution only in the LogN and Log$t$ regression models; see Figure 3. Nevertheless, the three log-symmetric regression models produce autocorrelated quantile residuals, as shown in Figure 3. Note that the sample autocorrelation and partial autocorrelation functions of the quantile residuals shown in this figure suggest an AR(2) model for the residuals. Thus, a pure log-regression regression model is not adequate and an structure to accommodate correlation is necessary.

Table 5: Estimates (with SE in parentheses) and model selection measures for fit to the mortality data.

| Model           | Parameter | ML estimate | p-value | RMSE | AIC          | BIC          |
|-----------------|-----------|-------------|---------|------|--------------|--------------|
| LogN regression model | $\beta_0$ | 35.4616(2.1630) | <0.0001 | 0.0692 | −1259.696 | −1234.313 |
|                 | $\beta_1$ | −0.0157(0.0011) | <0.0001 |       |              |              |
|                 | $\beta_2$ | −0.0051(0.0003) | <0.0001 |       |              |              |
|                 | $\beta_3$ | 0.0002(<0.0001) | <0.0001 |       |              |              |
|                 | $\beta_4$ | 0.0027(0.0002) | <0.0001 |       |              |              |
|                 | $\log(\phi)$ | −5.3412(0.0627) |       |       |              |              |
| Log$t$ regression model | $\beta_0$ | 35.4064(2.1630) | <0.0001 | 0.0692 | −1260.442 | −1235.059 |
|                 | $\beta_1$ | −0.0157(0.0011) | <0.0001 |       |              |              |
|                 | $\beta_2$ | −0.0051(0.0003) | <0.0001 |       |              |              |
|                 | $\beta_3$ | 0.0002(<0.0001) | <0.0001 |       |              |              |
|                 | $\beta_4$ | 0.0027(0.0002) | <0.0001 |       |              |              |
|                 | $\log(\phi)$ | −5.5567(0.0725) |       |       |              |              |
|                 | $\vartheta$ | 9 |       | | | |
| LogPE regression model | $\beta_0$ | 35.6571(2.1205) | <0.0001 | 0.0692 | −1260.376 | −1234.994 |
|                 | $\beta_1$ | −0.0158(0.0011) | <0.0001 |       |              |              |
|                 | $\beta_2$ | −0.0051(0.0003) | <0.0001 |       |              |              |
|                 | $\beta_3$ | 0.0002(<0.0001) | <0.0001 |       |              |              |
|                 | $\beta_4$ | 0.0027(0.0002) | <0.0001 |       |              |              |
|                 | $\log(\phi)$ | −5.7707(0.0725) |       |       |              |              |
|                 | $\vartheta$ | 0.24 |       | | | |

5.2 Log-symmetric-ARMAX results

Now, we present the results based on the proposed log-symmetric-ARMAX model. We also consider $\phi_i = \phi$ for $t = 1, \ldots, n$. Table 5 reports the estimates, SEs and p-values of the t-test for the log-symmetric ARMAX model parameters, as well as the values of AIC, BIC and RMSE. From this table, note that the LogN-ARMAX(2,0) model provides better adjustment compared...
where the errors are correlated with each other. The proposed approach is an autoregressive model. Note also that all three log-symmetric ARMAX models produce non-autocorrelated residuals according to the sample autocorrelation and partial autocorrelation functions. This result supports the importance of a model which takes into account serial correlation.

6 Concluding remarks

We have proposed a new class of log-symmetric regression models for dealing with cases where the errors are correlated with each other. The proposed approach is an autoregressive and moving average model with covariates and a log-symmetric conditional distribution. We
Table 6: Estimates (with SE in parentheses) and model selection measures for fit to the mortality data.

| Model                  | Parameter | ML estimate | p-value | RMSE  | AIC     | BIC     |
|------------------------|-----------|-------------|---------|-------|---------|---------|
| LogN-ARMAX(2,0) model  | $\kappa_1$ | 0.4050(0.0441) |         | 0.0547 | -1487.895 | -1454.051 |
|                        | $\kappa_2$ | 0.2789(0.0452) |         |        |         |         |
|                        | $\beta_0$  | 38.1026(2.1288) | <0.0001 |       |         |         |
|                        | $\beta_1$  | -0.0170(0.0011) | <0.0001 |       |         |         |
|                        | $\beta_2$  | -0.0017(0.0004) | <0.0001 |       |         |         |
|                        | $\beta_3$  | 0.0002(<0.0001) | <0.0001 |       |         |         |
|                        | $\beta_4$  | 0.0023(0.0002)  | <0.0001 |       |         |         |
|                        | $\phi$     | 0.0023(0.0003)  |         |       |         |         |
| Log$t$-ARMAX(2,0) model| $\kappa_1$ | 0.4527(0.0429) |         | 0.0550 | -1479.249 | -1445.406 |
|                        | $\kappa_2$ | 0.2637(0.0415) |         |        |         |         |
|                        | $\beta_0$  | 38.0715(2.0071) | <0.0001 |       |         |         |
|                        | $\beta_1$  | -0.0170(0.0010) | <0.0001 |       |         |         |
|                        | $\beta_2$  | -0.0015(0.0004) | <0.0001 |       |         |         |
|                        | $\beta_3$  | 0.0002(<0.0001) | <0.0001 |       |         |         |
|                        | $\beta_4$  | 0.0021(0.0002)  | <0.0001 |       |         |         |
|                        | $\phi$     | 0.0025(0.0003)  |         |       |         |         |
| LogPE-ARMAX(2,0) model | $\kappa_1$ | 0.3937(0.0452) |         | 0.0548 | -1486.880 | -1453.037 |
|                        | $\kappa_2$ | 0.2917(0.0465) |         |        |         |         |
|                        | $\beta_0$  | 38.1097(2.0483) | <0.0001 |       |         |         |
|                        | $\beta_1$  | -0.0170(0.0010) | <0.0001 |       |         |         |
|                        | $\beta_2$  | -0.0016(0.0004) | <0.0001 |       |         |         |
|                        | $\beta_3$  | 0.0002(<0.0001) | <0.0001 |       |         |         |
|                        | $\beta_4$  | 0.0023(0.0002)  | <0.0001 |       |         |         |
|                        | $\phi$     | 0.0020(0.0002)  |         |       |         |         |
|                        | $\theta$   | 0.24           |         |       |         |         |

have considered inference about the model parameters and a type of residual for these models. A Monte Carlo simulation study was carried out to evaluate the behavior of the conditional maximum likelihood estimators of the corresponding parameters. We have applied the proposed models to a real-world mortality data set. In general, the results have shown that the proposed models deal with serial correlation quite satisfactory and have great potential in many areas where the modelling of positive and autocorrelated data is necessary. As part of future research, it is of interest to discuss influence diagnostic tools and multivariate models. Related ARMA models based on the exponential family and the beta and symmetric distributions can be found in Benjamin et al. (2003), Rocha and Cribari-Neto (2009), Zheng et al. (2015) and Maior and Cysneiros (2018), and multivariate versions of these models can be proposed as well. Work on some of these issues is currently in progress and we hope to report some findings in future papers.
Figure 4: QQ plot and its envelope for the quantile residual, and sample autocorrelation and partial autocorrelation functions of the quantile residuals for the indicated model with the mortality data.
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