Projective Superspace as a Double–Punctured Harmonic Superspace

Sergei M. Kuzenko
Department of Physics, Tomsk State University
Lenin Ave. 36, Tomsk 634050, Russia
kuzenko@phys.tsu.ru

Abstract

We analyse the relationship between the $N = 2$ harmonic and projective superspaces which are the only approaches developed to describe general $N = 2$ super Yang-Mills theories in terms of off-shell supermultiplets with conventional supersymmetry. The structure of low-energy hypermultiplet effective action is briefly discussed.

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1 Introduction

The $N = 2$ harmonic superspace \cite{1} is the only manifesty supersymmetric approach developed which allows us to describe general $N = 2$ super Yang-Mills theories and $N = 2$ supergravity \cite{2} in terms of unconstrained superfields. It is the harmonic superspace which makes it possible to realize most general matter self-couplings in $N = 2$ supersymmetry \cite{4} as well as to develop a general setting for $N = 2$ rigid supersymmetric field theories with gauged central charge \cite{3}. Since all known $N = 2$ supersymmetric theories naturally emerge in the harmonic superspace approach, this is a universal or ‘master’ formalism for $N = 2$ supersymmetry.

Harmonic superspace is also enough powerful for the study of quantum aspects of $N = 2$ super Yang-Mills theories. The Feynman rules in harmonic superspace \cite{3} have been successfully applied to compute the holomorphic corrections to the effective action of $N = 2$ Maxwell multiplet coupled to the matter $q^+$ hypermultiplet \cite{6} as well as the induced hypermultiplet self-coupling \cite{7}. The background field formalism in harmonic superspace \cite{8} has already been utilized to derive Seiberg’s holomorphic action for $N = 2$ $SU(2)$ SYM theory \cite{9} from $N = 2$ supergraphs, to rigorously prove the $N = 2$ non-renormalization theorem \cite{10} as well as to compute the leading non-holomorphic quantum correction for $N = 4$ $SU(2)$ SYM theory \cite{11}. On the other hand, the quantum harmonic formalism has yet to be further elaborated. The main virtue of this approach – its universality – often turns into technical difficulties when making quantum loop calculations. For example, to compute leading corrections to the effective action we have to very carefully integrate out infinitely many auxiliary degrees of freedom contained in the unconstrained analytic superfields used to describe the charge hypermultiplet and $N = 2$ vector multiplet, and this is a non-trivial technical problem. But it seems that there exist more economical techniques for computing the low-energy action, which could be deduced from the first principles of the quantum harmonic formalism.

Recently, a new manifestly $N = 2$ supersymmetric approach to quantum $N = 2$ super Yang-Mills theories has intensively been developed \cite{13, 14, 15, 16}. This approach is based on the concept of $N = 2$ projective superspace \cite{17} having its origin in a remarkable paper \cite{18}. A nice feature of the projective multiplets is that they can easily be decomposed into a sum of well-known $N = 1$ multiplets. Therefore, the quantum calculations in $N = 2$ projective superspace can be controlled by comparing their results with those known for $N = 1$ supersymmetric models. On the other hand, it is far from obvious how to reduce,

\footnote{The non-holomorphic action for $N = 4$ $SU(2)$ SYM was first computed in \cite{12, 16}.}
say, the harmonic matter $q^+$ hypermultiplet to $N = 1$ superfields in an elegant way. A drawback of the projective superspace approach is that it allows us to manifestly realize only a $U(1)$ subgroup of the automorphism group $SU(2)_A$ of $N = 2$ supersymmetry. In the harmonic superspace approach, however, $SU(2)_A$ is manifestly realized.

As far as we know, no detailed discussion has been given in literature on the relationship between the harmonic and projective superspaces. One can find a few comments in [18] in the context of $N = 2$ tensor multiplet models. It was also mentioned [17] that the polar multiplet might be closely related to the $q^+$ hypermultiplet as well as pointed out that ‘the two approaches are presumably essentially equivalent’. More recently, the two formalisms were considered as alternative ones [14, 15]. In our opinion, these approaches are certainly related and, in a sense, complementary to each other. Therefore, it is worth combining their powerful properties for the study of quantum $N = 2$ supersymmetric field theories. The purpose of the present paper is just to reveal such a relationship.

The paper is organized as follows. In Sections 2 and 3 we give a brief introduction to projective and harmonic superspaces, respectively. In Section 4 we suggest an approximation of the projective superfields by smooth analytic superfields on harmonic superspace. Embedded into the $q^+$ hypermultiplet, for instance, is a global analytic superfield which coincides with the arctic multiplet inside a disk of radius $R$. The pure arctic multiplet emerges in the limit $R \to \infty$. We derive projective actions from harmonic superspace and show, as an example, how to obtain the polar hypermultiplet propagator from that corresponding to the $q^+$ hypermultiplet. The results of Section 4 imply in fact that projective superspace provides us with a minimal truncation of unconstrained analytic superfields, which inevitably breaks $SU(2)_A$ but is nicely suited for $N = 1$ reduction. In Section 5 we discuss the projective truncation of a general low-energy $q^+$ hypermultiplet action and show that the leading contributions to special truncated action describe the so-called chiral–non-minimal nonlinear sigma-model which was recently proposed to describe a supersymmetric low-energy QCD action [19]. In Appendix A we review the well-known correspondence between tensor fields on $S^2$ and functions over $SU(2)$ possessing definite $U(1)$-charges. Appendix B is devoted to the harmonic superspace description [4] of $O(2k)$ multiplets.
2 Projective superspace

Superfields living in the \( N = 2 \) projective superspace \([17]\) are parametrized by a complex bosonic variable \( w \) along with the coordinates of \( N = 2 \) global superspace \( \mathbb{R}^{4|8} \)

\[
z^M = (x^m, \theta^i, \bar{\theta}^\dot{i}) \quad \bar{\theta}^i = \bar{\theta}^{\dot{i}} \
i = 1, 2 .\tag{2.1}
\]

A superfield of the general form

\[
\Xi(z, w) = \sum_{n=-\infty}^{+\infty} \Xi_n(z) w^n \tag{2.2}
\]

is said to be projective if it satisfies the constraints

\[
\nabla_\alpha(w) \Xi(z, w) = 0 \quad \bar{\nabla}_{\dot{\alpha}}(w) \Xi(z, w) = 0 \tag{2.3}
\]

which involve the operators

\[
\nabla_\alpha(w) \equiv w D^1_\alpha - D^2_\alpha \quad \bar{\nabla}_{\dot{\alpha}}(w) \equiv \bar{D}^{\dot{\alpha}1} + w \bar{D}^{\dot{\alpha}2} \tag{2.4}
\]

constructed from the \( N = 2 \) covariant derivatives \( D_M = (\partial_m, D^i_\alpha, \bar{D}^{\dot{i}}_{\dot{\alpha}}) \). The operators \( \nabla_\alpha(w) \) and \( \bar{\nabla}_{\dot{\alpha}}(w) \) strictly anticommute with each other, as a consequence of the covariant derivative algebra

\[
\{D^i_\alpha, D^j_\beta\} = \{\bar{D}^{\dot{i}}_{\dot{\alpha}}, \bar{D}^{\dot{j}}_{\dot{\beta}}\} = 0 \quad \{D^i_\alpha, \bar{D}^{\dot{j}}_{\dot{\beta}}\} = -2i \delta^i_j \partial_{\alpha\dot{\beta}} .\tag{2.5}
\]

With respect to the inner complex variable \( w \), the projective superfields are holomorphic functions on the punctured complex plane \( \mathbb{C}^* \)

\[
\partial_{\bar{w}} \Xi(z, w) = 0 \tag{2.6}
\]

Constraints \([2.3]\) rewritten in components

\[
D^2_\alpha \Xi_n = D^1_\alpha \Xi_{n-1} \quad D^{\dot{\alpha}2} \Xi_n = -D^{\dot{\alpha}1} \Xi_{n+1} \tag{2.7}
\]

determine the dependence of the component \( N = 2 \) superfields \( \Xi \)'s on \( \theta^i_2 \) and \( \bar{\theta}^{\dot{i}}_2 \) in terms of their dependence on \( \theta^i_1 \) and \( \bar{\theta}^{\dot{i}}_1 \). Therefore, the components \( \Xi_n \) are effectively superfields over the \( N = 1 \) superspace parametrized by

\[
\theta^\alpha = \theta^\alpha_1 \quad \bar{\theta}_{\dot{\alpha}} = \bar{\theta}^{\dot{i}}_1 .\tag{2.8}
\]
If the power series in (2.2) terminates somehow, several $N = 1$ superfields satisfy constraints involving the $N = 1$ covariant derivatives

$$D_\alpha = D^1_\alpha, \quad \bar{D}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}}_1.$$  \hspace{1cm} (2.9)

A natural operation of conjugation, which brings every projective superfield into a projective one, reads as follows

$$\tilde{\Xi}(y, w) = \sum_n (-1)^n \Xi_{-n}(z) w^n \hspace{1cm} (2.10)$$

with $\Xi_{-n}$ being the complex conjugate of $\Xi_n$. A real projective superfield is constrained by

$$\tilde{\Xi} = \Xi \iff \tilde{\Xi}_{-n} = (-1)^n \Xi_{-n}.$$  \hspace{1cm} (2.11)

The component $\Xi_0(z)$ is seen to be real.

Given a real projective superfield $\mathcal{L}(z, w)$, $\tilde{\mathcal{L}} = \mathcal{L}$, we can construct a $N = 2$ supersymmetric invariant by the following rule

$$S = \int \text{d}^4x \, D^4 \mathcal{L}_0(z) \big|_{D^4 = \frac{1}{16} D^1_\alpha \bar{D}^{\dot{\alpha}}_1 D^{\dot{\alpha}}_1 \bar{D}^{\dot{\alpha}}_1}$$  \hspace{1cm} (2.12)

where $D^4$ is the $N = 1$ superspace measure and $U|_{\theta}$ means the $\theta$-independent component of a superfield $U$. Really, from the standard supersymmetric transformation law

$$\delta \mathcal{L} = i \left( \varepsilon^\alpha_i Q^i_\alpha + \varepsilon^{\dot{\alpha}}_i \bar{Q}^{\dot{\alpha}}_i \right) \mathcal{L}$$  \hspace{1cm} (2.13)

we get

$$\delta S = i \int \text{d}^4x \, \left( \varepsilon^\alpha_i Q^i_\alpha + \varepsilon^{\dot{\alpha}}_i \bar{Q}^{\dot{\alpha}}_i \right) D^4 \mathcal{L}_0 \big|_{\theta} = - \int \text{d}^4x \, \left( \varepsilon^\alpha_i \bar{D}^{\dot{\alpha}}_i + \varepsilon^{\dot{\alpha}}_i \bar{D}^{\dot{\alpha}}_i \right) D^4 \mathcal{L}_0 \big|_{\theta}$$

$$= - \int \text{d}^4x \, \left( \varepsilon^\alpha_i \bar{D}^{\dot{\alpha}}_i + \varepsilon^{\dot{\alpha}}_i \bar{D}^{\dot{\alpha}}_i \right) D^4 \mathcal{L}_0 \big|_{\theta} = - \int \text{d}^4x \, D^4 \left( \varepsilon^\alpha_i \bar{D}^{\dot{\alpha}}_i \right) D^4 \mathcal{L}_0 \big|_{\theta}$$

$$= - \int \text{d}^4x \, D^4 \left( \varepsilon^\alpha_i \bar{D}^{\dot{\alpha}}_i \mathcal{L}_0 \big|_{\theta} - \bar{\varepsilon}^{\dot{\alpha}}_i \bar{D}^{\dot{\alpha}}_i \mathcal{L}_1 \big|_{\theta} \right) = 0.$$  \hspace{1cm}

The action can be rewritten in the form \[^{[17]}\]

$$S = \frac{1}{2 \pi i} \oint_C \frac{d w}{w} \int \text{d}^4x \, D^4 \mathcal{L} \big|_{\theta}$$  \hspace{1cm} (2.14)

where $C$ is a contour around the origin.
Let us review several multiplets which can be realized in projective superspace \cite{17}. It is worth starting with the so-called polar multiplet (or $\Upsilon$ hypermultiplet) describing a charged $N = 2$ scalar multiplet \footnote{An off-shell $N = 2$ hypermultiplet is said to be charged or complex if it possesses an internal $U(1)$ symmetry that couples to complex Yang-Mills, and neutral or real otherwise; neutral hypermultiplets can transform only in real representations of the gauge group.}:

$$
\Upsilon(z, w) = \sum_{n=0}^{\infty} \Upsilon_n(z) w^n \quad \bar{\Upsilon}(z, w) = \sum_{n=0}^{\infty} (-1)^n \bar{\Upsilon}_n(z) \frac{1}{w^n} . \tag{2.15}
$$

The projective superfields $\Upsilon$ and $\bar{\Upsilon}$ are called arctic and antarctic \cite{13}, respectively. If we treat the components of $\Upsilon$ as $N = 1$ superfields, then $\Upsilon_0$ is a chiral superfield, $\Upsilon_1$ a complex linear superfield, and $\Upsilon_2, \Upsilon_3, \ldots$, complex unconstrained superfields

$$
\bar{D}_\alpha \Upsilon_0 = 0 \quad \bar{D}^2 \Upsilon_1 = 0 . \tag{2.16}
$$

The corresponding super Lagrangian reads

$$
\mathcal{L} = \bar{\Upsilon} \Upsilon \quad \mathcal{L}_0 = \sum_{n=0}^{\infty} (-1)^n \bar{\Upsilon}_n \Upsilon_n . \tag{2.17}
$$

Cutting off the power series in (2.15) at some finite stage $p > 2$, one results in the so-called complex $O(p)$ multiplet

$$
\Lambda^{[p]}(z, w) = \sum_{n=0}^{p} \Lambda_n(z) w^n \quad \bar{\Lambda}^{[p]}(z, w) = \sum_{n=0}^{p} (-1)^n \bar{\Lambda}_n(z) \frac{1}{w^n} . \tag{2.18}
$$

Its component superfields are constrained as follows:

$$
\bar{D}_\alpha \Lambda_0 = 0 \quad \bar{D}^2 \Lambda_1 = 0 \\
D_\alpha \Lambda_p = 0 \quad D^2 \Lambda_{p-1} = 0 \tag{2.19}
$$

and the rest components are unconstrained. The case $p = 1$ corresponds to the on-shell hypermultiplet, while for $p = 2$ we obtain two tensor multiplets.

The next multiplet of principal interest is called \cite{13} tropical and looks as follows

$$
V(z, w) = \bar{V}(z, w) = \sum_{n=-\infty}^{+\infty} V_n(z) w^n \quad \bar{V}_n = (-1)^n V_{-n} \tag{2.20}
$$
with all the components being unconstrained $N = 1$ superfields but $V_0$ real. This projective superfield describes a free massless $N = 2$ vector multiplet provided the corresponding gauge invariance is \( \delta V = i (\hat{\Sigma} - \Sigma) \)

\[ (2.21) \]

where $\Sigma$ is an arbitrary arctic superfield. The gauge invariant action reads \( \[15\] \( S[V] = -\frac{1}{2} \int d^{12} z \int \frac{dw_1}{2\pi i} \frac{dw_2}{2\pi i} \frac{V(z, w_1)V(z, w_2)}{(w_1 - w_2)^2} \). \)

\[ (2.22) \]

Cutting off the power series in \( (2.20) \) at some finite stage $k > 1$ but preserving the reality condition, one results in the so-called real $O(2k)$ multiplet

\[ \Omega^{[2k]}(z, w) = \bar{\Omega}^{[2k]}(z, w) = \sum_{n=-k}^{+k} \Omega_n(z) w^n \quad \bar{\Omega}_n = (-1)^n \Omega_{-n} . \]

\[ (2.23) \]

The components are constrained by

\[ \bar{D}_\alpha \Omega_{-k} = 0 \quad \bar{D}^2 \Omega_{-k+1} = 0 \quad \bar{\Omega}_0 = \Omega_0 \]

\[ (2.24) \]

and $\Omega_{-n+2}, \ldots, \Omega_{-1}$ are unconstrained complex superfields. The super Lagrangian

\[ \mathcal{L} = \frac{1}{2} (-1)^k (\Omega^{[2k]})^2 \quad \mathcal{L}_0 = \sum_{n=0}^{k} (-1)^{k-n} \bar{\Omega}_{-n} \Omega_{-n} \]

\[ (2.25) \]

describes a real off-shell hypermultiplet. The case $k = 1$, which was excluded from our consideration, corresponds to the free $N = 2$ tensor multiplet.

Complex $O(p)$ multiplets \((2.18)\) are of little importance by themselves. In the even case, $p = 2k$, we can write $\Lambda^{[2k]}(z, w) = w^k \lambda^{[2k]}(z, w)$, where $\lambda^{[2k]}$ is seen to be a complex combination of two real $O(2k)$ hypermultiplets \((2.23)\). When $p$ is odd, on the other hand, we cannot define a supersymmetric action with the correct kinetic terms for all the chiral and complex linear superfields contained in $\Lambda^{[k]}$ \([13]\). However, the polar or $O(\infty)$ multiplet is of principal importance, since it provides us with a realization of charged hypermultiplet and can be coupled to the $N = 2$ gauge field in arbitrary representations of the gauge group. A single charged hypermultiplet must inevitably possess infinitely many auxiliary fields \([20]\) in $N = 2$ supersymmetry without central charge.

The $O(p)$ multiplets, $p > 2$, have been intensively studied. They were originally formulated in the standard $N = 2$ superspace \([21]\) (see also \([21]\)) in terms of symmetric isotensors $\Omega^{[i_1 i_2 \cdots i_p]}(z)$ constrained by

\[ D^{(i_1}_{\alpha} \Omega^{i_2 \cdots i_p+1)} = \bar{D}^{(i_1}_{\bar{\alpha}} \Omega^{i_2 \cdots i_p+1)} = 0 , \]

\[ (2.26) \]
then described in harmonic superspace [4] and finally realized in projective superspace [17]. The $O(2k)$ multiplets provide us with different off-shell realizations for real hypermultiplet. Their harmonic superspace formulation [4] is briefly discussed in Appendix B.

Since the harmonic and projective descriptions of the $O(2k)$ multiplets are completely equivalent, in what follows we will concentrate on answering to the question whether there is a room for the polar and tropical multiplets in the harmonic superspace approach.

3 Harmonic superspace

Harmonic superspace $\mathbb{R}^{4|8} \times S^2$ is a homogeneous space of the $N = 2$ Poincaré supergroup. The most useful in practice global parametrization of $S^2 = SU(2)/U(1)$ is that in terms of the harmonic variables $u_i^-, u_i^+$ which parametrize $SU(2)$, the automorphism group of $N = 2$ supersymmetry,

$$(u_i^-, u_i^+) \in SU(2)$$

$$u_i^+ = \varepsilon_{ij} u^+_j, \quad u_i^- = u_i^+, \quad u^+ u^-= 1.$$  \hspace{1cm} (3.1)

As is demonstrated in Appendix A, tensor fields over $S^2$ are in a one-to-one correspondence with functions on $SU(2)$ possessing definite harmonic $U(1)$-charges. A function $\Psi^{(p)}(u)$ is said to have the harmonic $U(1)$-charge $p$ if

$$\Psi^{(p)}(e^{i\varphi} u^+, e^{-i\varphi} u^-) = e^{ip\varphi} \Psi^{(p)}(u^+, u^-), \quad |e^{i\varphi}| = 1.$$  

A function $\Psi^{(p)}(z, u)$ on $\mathbb{R}^{4|8} \times S^2$ with $U(1)$-charge $p$ is called a harmonic $N = 2$ superfield.

When working with harmonic superfields, it is advantageous to make use of the operators

$$D^{\pm\pm} = u^{\pm i} \partial/\partial u^{\pm i}, \quad D^0 = u^+ \partial/\partial u^+ - u^- \partial/\partial u^-$$

$$[D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}, \quad [D^{++}, D^{--}] = 0$$  \hspace{1cm} (3.2)

being left-invariant vector fields on $SU(2)$. Here $D^{\pm\pm}$ are two independent harmonic covariant derivatives on $S^2$, while $D^0$ is the $U(1)$-charge operator, $D^0 \Phi^{(p)} = p \Phi^{(p)}$.

Using the harmonics, one can convert the spinor covariant derivatives into $SU(2)$-invariant operators on $\mathbb{R}^{4|8} \times S^2$

$$D^\pm_{\alpha} = D^i_{\alpha} u^\pm_i, \quad \bar{D}^\pm_{\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}} u^\pm_i.$$  \hspace{1cm} (3.3)
Then the covariant derivative algebra (2.5) implies the existence of the following anticommuting subset \((D^+_{\alpha}, \bar{D}^+_{\dot{\alpha}})\),

\[
\{D^+_{\alpha}, D^+_{\beta}\} = \{\bar{D}^+_{\dot{\alpha}}, \bar{D}^+_{\dot{\beta}}\} = \{D^+_{\alpha}, \bar{D}^+_{\dot{\alpha}}\} = 0 .
\] (3.4)

As a consequence, one can define an important subclass of harmonic superfields constrained by

\[
D^+_{\alpha}\Phi = \bar{D}^+_{\dot{\alpha}}\Phi = 0 .
\] (3.5)

Such superfields are functions over the so-called analytic subspace of the harmonic superspace parameterized by

\[
\{\zeta, u^\pm\} \equiv \{x^m, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^\pm_i\} \quad \Phi^{(p)}(z, u) \equiv \Phi^{(p)}(\zeta, u)\]

where \(x^m = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u^+_iu^-_j\) \(\theta^{\pm}_{\alpha} = \theta^{i\alpha}u^\pm_i\) \(\bar{\theta}^{\pm}_{\dot{\alpha}} = \bar{\theta}^{i\dot{\alpha}}u^\pm_i\). (3.6)

That is why such superfields are called analytic.

The analytic subspace (3.6) is closed under \(N = 2\) supersymmetry transformations and real with respect to the generalized conjugation (called in [3] the smile-conjugation) \(\gamma \equiv ^*\), where the operation \(^*\) is defined by

\[
(u^+_i)^* = u^-_i \quad (u^-_i)^* = -u^+_i \quad \Rightarrow \quad (u^\pm_i)^{*} = -u^\pm_i
\]

whence

\[
(u^+_i)^\gamma = -u^-_i \quad (u^-_i)^\gamma = u^-_i .
\] (3.8)

The analytic superfields with even \(U(1)\)-charge can therefore be chosen real.

Harmonic superspace provides us with the following universal, manifestly \(N = 2\) supersymmetric action

\[
S = \int du d\zeta (-4) \mathcal{L}^{(+4)}(\zeta, u) \quad \mathcal{L}^{(+4)} = \mathcal{L}^{(+4)}
\] (3.9)

with \(\mathcal{L}^{(+4)}\) being a real analytic superfield of \(U(1)\)-charge +4. Here the integration is carried out over the analytic subspace, \(d\zeta^{(-4)} = d^4x_A d^2\theta d^2\bar{\theta}\) and the integration over \(SU(2)\) is defined by \([4]\)

\[
\int du 1 = 1 \quad \int du \ u^+_i \ldots u^+_n u^-_j \ldots u^-_m = 0 \quad n + m > 0 .
\] (3.10)

Let us review the three basic harmonic multiplets which are used to realize general \(N = 2\) super Yang-Mills theories in terms of unconstrained superfields \([1, 3]\). The \(q^+\)
The hypermultiplet is formulated in terms of an unconstrained analytic superfield $q^+(\zeta, u)$ and its conjugate $\bar{q}^+(\zeta, u)$ with the action
\[
S[q^+] = - \int du d\zeta (-4) \bar{q}^+ D^{++} q^+ . \tag{3.11}
\]
The $q^+$ (charged) hypermultiplet can transform in arbitrary representations of the gauge group. Using this multiplet, one can construct most general matter self-couplings [4].

Further, the $\omega$ (real) hypermultiplet is formulated in terms of a real unconstrained analytic superfield $\omega(\zeta, u), \bar{\omega} = \omega$, with the free action
\[
S[\omega] = -\frac{1}{2} \int du d\zeta (-4) (D^{++} \omega)^2 . \tag{3.12}
\]
In eqs. (3.11) and (3.12) the operator $D^{++}$ is to be chosen in the analytic basis (3.7).

Finally, the free $N = 2$ vector multiplet is realized in terms of a real unconstrained analytic superfield $V^{++}(\zeta, u), V^{++} = V^{++}$, endowed with the gauge invariance
\[
\delta V^{++} = -D^{++} \lambda \tag{3.13}
\]
where $\lambda(\zeta, u)$ is an arbitrary real analytic scalar superfield. The gauge invariant action reads
\[
S[V^{++}] = \frac{1}{2} \int d^{12} z du_1 du_2 \frac{V^{++}(z, u_1)V^{++}(z, u_2)}{(u_1^+ u_2^+)^2} . \tag{3.14}
\]
The harmonic distributions such as $(u_1^+ u_2^+)^{-2}$ are defined in [3].

4 Embedding the projective superfields into analytic superfields

We turn to describing the precise relationship between the projective and analytic superfields.

4.1 Analytic superfields in local coordinates

To start with, it is worth rewriting the properties of analytic superfields in the local complex coordinates on $S^2$ introduced in Appendix A. Let $\Phi^{(p)}(z, u)$ be a smooth analytic superfield with non-negative $U(1)$-charge $p$. In the north chart it can be represented as follows
\[
\Phi^{(p)}(z, u) = (u^+)^p \Phi^{(p)}_N(z, w, \bar{w}) \tag{4.1}
\]
where $\Phi^{(p)}_{N}(z, w, \bar{w})$ is given as in eq. (A.10), but now the corresponding Fourier coefficients $\Phi^{(i_{1} \cdots i_{n} + j_{1} \cdots j_{n})}(z)$ are special $N = 2$ superfields. The fact that $\Phi^{(p)}(z, u)$ is a smooth function on $\mathbb{R}^{4|8} \times SU(2)$, is equivalent to the requirement that

$$
\lim_{|w| \to \infty} \frac{1}{wp} \Phi^{(p)}_{N}(z, w, \bar{w})
$$

is a smooth function on $\mathbb{R}^{4|8}$. Keeping in mind this boundary condition, it is sufficient to work in the north chart only.

The operators $D^+_\alpha$ and $\bar{D}^+_\dot{\alpha}$ can be rewritten in the manner [18]

$$
D^+_\alpha = -u^{+1} \nabla_{\alpha}(w) \quad \bar{D}^+_\dot{\alpha} = -u^{+1} \bar{\nabla}_{\dot{\alpha}}(w)
$$

where $\nabla_{\alpha}(w)$ and $\bar{\nabla}_{\dot{\alpha}}(w)$ are given by eq. (2.4). Therefore, the Grassmann analyticity requirements (3.5) become

$$
\nabla_{\alpha}(w) \Phi^{(p)}_{N}(z, w, \bar{w}) = 0 \quad \bar{\nabla}_{\dot{\alpha}}(w) \Phi^{(p)}_{N}(z, w, \bar{w}) = 0.
$$

As is seen, the constraints do not specify the $\bar{w}$-dependence of $\Phi^{(p)}_{N}(z, w, \bar{w})$ at all. That is why we are in a position to truncate analytic superfields in such a way to result in projective superfields.

To represent the analytic action (3.9) in the local coordinates, we first rewrite

$$
S = \int d^4x du (D^-)^4 L^{(+4)}(z, u) | (D^-)^4 = \frac{1}{16} D^-\alpha D^-\dot{\alpha} \bar{D}^-\alpha \bar{D}^-\dot{\alpha}
$$

and then notice

$$
(D^-)^4 \Phi^{(p)} = (u^{+1})^4 \left(1 + w\bar{w}\right)^4 \frac{1}{w^2} D^4 \Phi^{(p)}
$$

for an arbitrary analytic superfield $\Phi^{(p)}$. In accordance with eq. (A.10), in the overlap of the north and south charts it is worth representing the real analytic Lagrangian $L^{(+4)}(z, u)$ as

$$
L^{(+4)}(z, u) = (u^{+1}u^{+2})^2 L^{(+4)}_{N-S}(z, w, \bar{w})
$$

where $L^{(+4)}_{N-S}$ is real with respect to the smile-conjugation (A.18). Finally, we notice the identity

$$
\int du f(u) = \frac{1}{\pi} \int \frac{d^2 w}{(1 + w\bar{w})^2} f(w, \bar{w})
$$

for any smooth function of vanishing $U(1)$ charge. As a result, the action (3.9) turns into

$$
S = \frac{1}{\pi} \int d^4x \int \frac{d^2 w}{(1 + w\bar{w})^2} D^4 L^{(+4)}_{N-S}(z, w, \bar{w})
$$

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To complete our analysis, it is also necessary to express the operator \(D^{++}\) in the local coordinates on \(S^2\). The importance of this operator consists in the fact that \(D^{++}\) moves every analytic superfield into an analytic one. If \(\Phi^{(p)}(u)\) is a function with non-negative \(U(1)\)-charge \(p\), in the north chart one readily gets

\[
D^{++}\Phi^{(p)}(u) = (u^+)^{p+2}(1 + w\bar{w})^2 \partial_{\bar{w}}\Phi^{(p)}(w, \bar{w}).
\]

(4.10)

### 4.2 From q⁺ hypermultiplet to polar hypermultiplet

Let us consider the equation

\[
D^{++}q^+(\zeta, u) = 0 .
\]

(4.11)

It defines the on-shell hypermultiplet provided \(q^+\) is required to be a global analytic superfield (a smooth superfield over the harmonic superspace). In this case the general solution to (4.11) reads

\[
q^+(z, u) = q^i(z) u^+_i \quad D^{(i)}q^j = \bar{D}^{(i)}q^j = 0 .
\]

(4.12)

But if we allow \(q^+\) to be smooth everywhere on \(S^2\) but at the north pole, the general solution of (4.11) becomes

\[
q^+(z, u) = u^+ \Upsilon(z, w) \quad \Upsilon(z, w) = \sum_{n=0}^{\infty} \Upsilon_n(z) w^n
\]

(4.13)

as a consequence of (4.10). The analytic constraints \(D^{\hat{\alpha}}q^+ = \bar{D}^{\hat{\alpha}}q^+ = 0\), to which \(q^+\) is to be subjected, tell us that \(\Upsilon(z, w)\) is nothing else but the arctic multiplet described in Section 2. Therefore, we obtain a local analytic superfield being singular at the north pole of the two-sphere.

Let us introduce an isospinor \(s^i\) and its conjugate \(\bar{s}_i\)

\[
s^i = (1, 0) \quad s_i = (0, 1) \\
\bar{s}_i = (1, 0) \quad \bar{s}^i = (0, -1)
\]

(4.14)

which corresponds to the south and north poles of \(S^2\), respectively. Then we can rewrite eq. (4.13) as follows

\[
q^+(z, u) = u^+ \sum_{n=0}^{\infty} \Upsilon_n(z) \frac{(u^+s)^n}{(u^+\bar{s})^n}
\]

(4.15)
where \((u^+ s) = u^+ s_i\) and completely similar for \((u^+ \bar{s})\). In accordance with (3.8), the
smile-conjugate of this superfield reads

\[
\tilde{q}^+(z, u) = u^{+2} \sum_{n=0}^{\infty} (-1)^n \tilde{\Upsilon}_n(z) \frac{(u^+ \bar{s})^n}{u^+ s}\n\]  
(4.16)
or, equivalently,

\[
\tilde{q}^+(z, u) = u^{+2} \tilde{\Upsilon}(z, w) \quad \tilde{\Upsilon}(z, w) = \sum_{n=0}^{\infty} (-1)^n \tilde{\Upsilon}_n(z) \frac{1}{w^n} .
\]  
(4.17)

This superfield satisfies the constraints \(D_\alpha^+ \tilde{q}^+ = \bar{D}_\dot{\alpha}^+ \tilde{q}^+ = 0\), since the smile-conjugation
is analyticity-preserving. Our consideration shows that \(q^+\) and \(\tilde{q}^+\) possess singularities
at the north and south poles of \(S^2\), respectively. Therefore, this multiplet lives in the
two-point-punctured version \(\mathbb{R}^{4\times} \times \mathbb{C}^\ast\) of harmonic superspace. It is also obvious that the
superfield \(\tilde{\Upsilon}(z, w)\) just introduced is exactly the antarctic multiplet described in Section 2.

Since \(q^+\) and \(\tilde{q}^+\) are holomorphic superfields, the \(q^+\) hypermultiplet action (3.11)
vanishes for such superfields. But now we can construct another analytic Lagrangian

\[
L^{(+4)}[\Upsilon] = (u^+ \bar{s})(u^+ s) \tilde{q}^+ q^+ \quad \text{holomorphic on } \mathbb{C}^\ast ,
\]  
(4.18)

which is holomorphic on \(\mathbb{C}^\ast\),

\[
D^{++} L^{(+4)} = 0 .
\]  
(4.19)

It can be used to construct the following supersymmetric action

\[
S[\Upsilon] = \frac{1}{2\pi i} \int \frac{dw}{w} \int d\zeta^{(-4)} L^{(+4)}[\Upsilon] \quad \text{coinciding with the polar hypermultiplet action discussed in Section 2.}
\]  
(4.20)

The polar hypermultiplet can be obtained as the limit of a sequence of global analytic
superfields. Let us introduce an auxiliary smooth function \(f_{R, \epsilon}(x)\) of a real variable
\(x \in [0, \infty)\):

\[
f_{R, \epsilon}(x) = \begin{cases} \exp \left( \frac{1}{x-R-\epsilon} - \frac{1}{x-R} \right) & R < x < R + \epsilon \\ 0 & x \in [0, R] \cup [R + \epsilon, \infty) \end{cases}
\]  
(4.21)

which we apply to construct another function

\[
F_{R, \epsilon}(x) = \int_x^{R+\epsilon} dt \, f_{R, \epsilon}(t) / \int_R^{R+\epsilon} dt \, f_{R, \epsilon}(t) \quad \text{coinciding with the polar hypermultiplet action discussed in Section 2.}
\]  
(4.22)
to be used in what follows. Here $R$ and $\epsilon$ are ‘large’ and ‘small’ positive parameters, respectively. The function $F_{R,\epsilon}(x)$ is equal to one when $0 \leq x \leq R$, decreases from one to zero when $R < x < R + \epsilon$, and is equal to zero when $x \geq R + \epsilon$.

Now, we define global analytic superfields $q^+_R,\epsilon$ and $\tilde{q}^+_R,\epsilon$ given in the north chart as follows:

$$q^+_R,\epsilon(z, u) = q^+(z, u) F_{R,\epsilon}(|w|) = u^{+1} \Upsilon(z, w) F_{R,\epsilon}(|w|)$$
$$\tilde{q}^+_R,\epsilon(z, u) = \tilde{q}^+(z, u) F_{R,\epsilon}(|w|) = u^{+2} \tilde{\Upsilon}(z, w) F_{R,\epsilon}(|w|).$$

(4.23)

For fixed parameters $R$ and $\epsilon$, such superfields form an off-shell multiplet with respect to the $N = 2$ supersymmetric transformations. Let us assume that $R \gg 1$, $\epsilon \ll 1$ and evaluate the $q^+$ hypermultiplet action (3.11) for the superfields (4.23). Since $\Upsilon$ is holomorphic, the operator $D^{++}$ in (3.11) acts on $F_{R,\epsilon}(|w|)$ only. Accounting the properties of $F_{R,\epsilon}$, one observes that the integration over $S^2$ produces a non-vanishing contribution only in a small region enclosed between the two circles of radii $R$ and $R + \epsilon$. If one introduces real variables $\rho$ and $\varphi$ defined by $w = \rho e^{i\varphi}$, the action can be brought to the form

$$S[q^+_R,\epsilon] = -\frac{1}{2\pi} \int d^4x D^4 \int_{0}^{2\pi} d\varphi \int_{R}^{R+\epsilon} d\rho \, F_{R,\epsilon}(\rho^{-1}) \tilde{\Upsilon}(z, w) \Upsilon(z, w) \partial_\rho F_{R,\epsilon}(\rho).$$

(4.24)

From here one readily gets

$$\lim_{\epsilon \to 0} S[q^+_R,\epsilon] = S[\Upsilon] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^4x D^4(\tilde{\Upsilon}\Upsilon).$$

(4.25)

Therefore, the polar multiplet action has its origin in harmonic superspace.

### 4.3 Projective action rule

It is easy to derive the projective action rule (2.14) from harmonic superspace. First of all, one should define a global analytic real superfield $\mathcal{L}_{R,\epsilon}^{(+4)}(z, u)$ of $U(1)$-charge +4, which looks like

$$\mathcal{L}_{R,\epsilon}^{(+4)}(z, u) = (u^{+1}u^{+2})^2 F_{R,\epsilon}(|w|) \mathcal{L}(z, w) F_{R,\epsilon}(|w|)$$
$$\equiv (u^{+1}u^{+2})^2 L_{R,\epsilon}(z, w, \bar{w}).$$

(4.26)

Associated with the analytic Lagrangian $\mathcal{L}_{R,\epsilon}^{(+4)}$ is the supersymmetric action

$$S_{R,\epsilon} = \int du \, d\zeta^{(-4)} \mathcal{L}_{R,\epsilon}^{(+4)}(\zeta, u)$$
$$= \frac{1}{\pi} \int d^4x \int \frac{d^2w}{(1 + w\bar{w})^2} D^4 \mathcal{L}_{R,\epsilon}(z, w, \bar{w}).$$

(4.27)
We then can represent
\[
\frac{1}{(1 + w\bar{w})^2} D^4 \mathcal{L}_{R,\epsilon}(z, w, \bar{w}) = - \left( \partial_{\bar{w}} \frac{1}{1 + w\bar{w}} \right) \frac{1}{w} D^4 \mathcal{L}_{R,\epsilon}(z, w, \bar{w}) \ . \tag{4.28}
\]
Finally, it remains to make the following steps: (i) integrate by parts in (4.27) (this is possible since the function \( w^{-1} D^4 \mathcal{L}_{R,\epsilon}(z, w, \bar{w}) \) is regular at \( w = 0 \)); (ii) account that \( \mathcal{L}(z, w) \) is holomorphic; (iii) introduce the real variables \( \rho \) and \( \varphi \) defined by \( w = \rho e^{i\varphi} \). As a result, one observes
\[
\lim_{\epsilon \to 0} \int du \, d\zeta \, (-4) L^{(+4)}_{R,\epsilon}(\zeta, u) = \frac{R^2 - 1}{R^2 + 1} \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^4x \, D^4 \mathcal{L} \ . \tag{4.29}
\]

### 4.4 Hypermultiplet propagators

We have seen that the \( q^+ \) hypermultiplet and the \( \Upsilon \) hypermultiplet are closely related to each other. Therefore, there should exist a relationship between their propagators.

The \( q^+ \) hypermultiplet propagator \( \square \) reads
\[
< q^+(z_1, u_1) \, \bar{q}^+(z_2, u_2) > = i \frac{(D^+_1)^4(D^+_2)^4}{\Box} \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \tag{4.30}
\]
where
\[
(D^+_\alpha) = \frac{1}{16} D^+ \alpha D^+ \bar{\alpha} \bar{D}^+ \bar{\alpha}
\]
and
\[
(u_1^+ u_2^+) = u_1^+ i u_2^+ . \tag{4.31}
\]
To compare the above propagator with that of the \( \Upsilon \) hypermultiplet, we are to express \( (4.30) \) in the local coordinates on \( S^2 \). For this purpose we represent
\[
q^+(z, u) = u^+ w^1 q^+_N(z, w, \bar{w})
\]
\[
\bar{q}^+(z, u) = u^{-2} \tilde{q}^+_S(z, y(w), \bar{y}(\bar{w})) \equiv w u^+ \tilde{q}^+_S(z, w, \bar{w}) . \tag{4.33}
\]
Further, we have to express the operators \( D^+_\alpha \) and \( \bar{D}^+_\alpha \) via \( \nabla_\alpha \) and \( \bar{\nabla}_{\bar{\alpha}} \) by the rule \( (4.3) \). Finally, we should make use of the identity
\[
(u_1^+ u_2^+) = u_1^+ u_2^+ (w_1 - w_2) . \tag{4.34}
\]
Therefore, we result with
\[
< q^+_N(z_1, w_1, \bar{w}_1) \, \tilde{q}^+_S(z_2, w_2, \bar{w}_2) > = i \frac{(\nabla_1)^4(\nabla_2)^4}{\Box} \frac{\delta^{12}(z_1 - z_2)}{w_2 (w_1 - w_2)^3} . \tag{4.35}
\]
This expression coincides in form with the polar hypermultiplet propagator \([13]\). Of course, the distributions \((w_1-w_2)^{-3}\) which enter the \(q^+\) and \(\Upsilon\) hypermultiplet propagators are defined on different functional spaces. But since we know how to truncate the \(q^+\) hypermultiplet to result with the \(\Upsilon\) hypermultiplet, one can immediately reproduce the \(\Upsilon\) hypermultiplet propagator without tedious calculations.

The above consideration was restricted to the case of massless hypermultiplet, but it can be readily generalized to the massive case. A general feature of \(N = 2\) off-shell hypermultiplets is that the presence of a non-vanishing mass is equivalent to the coupling to a background \(N = 2\) \(U(1)\) vector multiplet with constant strength [4, 3, 7, 22]. It is the mechanism which was used for constructing the massive hypermultiplet propagators in harmonic superspace [7, 22] and projective superspace [14]. Similarly to the massless case, the massive propagators coincide in form.

### 4.5 From \(V^{++}\) multiplet to tropical multiplet

Let us consider the equation

\[
D^{++} V^{++} = 0. \tag{4.36}
\]

It describes the free \(N = 2\) tensor multiplet provided \(V^{++}\) is required to be an analytic real superfield,

\[
D^{\alpha}_a V^{++} = \bar{D}^{\dot{\alpha}}_a V^{++} = 0 \quad \dot{V}^{++} = V^{++}, \tag{4.37}
\]

globally defined on harmonic superspace. In this case \(V^{++}\) looks like

\[
V^{++}(z, u) = V^{(ij)}(z)u_i^+ u_j^+ \quad D^{(i} V^{jk)} = \bar{D}^{(i} V^{jk)} = 0 \tag{4.38}
\]

with \(V^{(ij)}(z)\) being a real isovector superfield. However, if one allows \(V^{++}\) to be singular at the north and south poles, but keeps intact the basic constraints (4.37), the general solution becomes

\[
V^{++}(z, u) = iu_1^+ u_2^+ V(z, w) \tag{4.39}
\]

where \(V(z, w)\) is now the tropical multiplet described in Section 2. Because of the reality condition, \(V^{++}\) cannot be singular only at a single point.

The tropical multiplet is closely related to the analytic gauge superfield which we briefly discussed in Section 3. To establish such a relationship, let us introduce a special global analytic superfield \(V^{++}_{R,\epsilon}\) defined with help of the infinitely differentiable function (4.22):

\[
V^{++}_{R,\epsilon}(z, u) = F_{R,\epsilon}(|w|^{-1}) V^{++}(z, u) F_{R,\epsilon}(|w|). \tag{4.40}
\]
In the limit $R \to \infty$, $V_{R,\epsilon}^{++}$ turns into $V^{++}$ \((1.39)\) defining the tropical multiplet. The tropical action \((2.22)\) can be derived from that corresponding to the analytic gauge superfield \((3.14)\). It is a simple exercise to prove the relation

$$
\lim_{\epsilon \to 0} S[V_{R,\epsilon}^{++}] = \left(\frac{R^2 - 1}{R^2 + 1}\right)^2 S[V].
$$

We see that the tropical multiplet action \((2.22)\) has its origin in the harmonic superspace approach. To derive the tropical gauge transformation \((2.21)\) from that corresponding to the analytic gauge superfield \((3.13)\), let us choose $V_{R,\epsilon}^{++}$ in the role of $V^{++}$ and consider the following variation

$$
\delta V_{R,\epsilon}^{++} \equiv F_{R,\epsilon}(|w|^{-1}) \delta V^{++} F_{R,\epsilon}(|w|) = -F_{R,\epsilon}(|w|^{-1})(D^{++}\Lambda) F_{R,\epsilon}(|w|)
$$

$$
= -D^{++}\left\{F_{R,\epsilon}(|w|^{-1}) \Lambda F_{R,\epsilon}(|w|)\right\} + \Lambda D^{++}\left\{F_{R,\epsilon}(|w|^{-1}) F_{R,\epsilon}(|w|)\right\}
$$

with the parameter $\Lambda$ being defined as follows

$$
\Lambda = \tilde{\Lambda} = \left(\frac{1}{2} + u^{+1}u^{-2}\right)(\tilde{\Sigma} - \Sigma)
$$

where $\Sigma(z, w)$ is required, for a moment, to be a projective superfield only. In the limit $R \to \infty$, the variation \((1.42)\) formally turns into the transformation law \((2.21)\). The fact that $\Sigma$ must be arctic is quite understandable. As is obvious, the first term in the second line of \((1.42)\) does not contribute to the corresponding variation of $S[V_{R,\epsilon}^{++}]$. Keeping in mind this observation, one then finds

$$
\lim_{\epsilon \to 0} \delta S[V_{R,\epsilon}^{++}] = \left(\frac{R^2 - 1}{R^2 + 1}\right)^2 \delta S[V].
$$

Here $\delta S[V]$ is the variation of the tropical multiplet action with respect to \((2.21)\). The variation $\delta S[V]$ vanishes only if $\Sigma$ is an arctic superfield \([13]\).

Eq. \((2.22)\) defines the linearized action of the pure $N = 2$ super Yang-Mills theory in the projective superspace approach. By now, the full nonlinear action has not derived in terms of the tropical prepotential $V(z, w)$. In principle, it can be deduced from the well-known harmonic action for the $N = 2$ super Yang-Mills theory \([23]\), but with use of a more delicate truncation than that considered above.

4.6 Hypermultiplet coupled to abelian vector multiplet

It is interesting to compare the harmonic and projective off-shell realizations for a charged massless hypermultiplet coupled to an abelian vector multiplet. In the harmonic super-
space approach, the action reads [1]

$$S[q^+, V^{++}] = -\int du\, d\zeta (-4) \dot{q}^+ (D^{++} + iV^{++}) q^+$$  \hspace{1cm} (4.45)$$

and is invariant under the gauge transformations

$$\delta q^+ = i\lambda q^+ \quad \delta V^{++} = -D^{++} \lambda$$  \hspace{1cm} (4.46)$$

with an arbitrary real scalar analytic parameter \(\lambda\). In the projective superspace approach, the action reads [17]

$$S[\Upsilon, V] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^4x\, D^4(\check{\Upsilon} e^V \Upsilon)$$  \hspace{1cm} (4.47)$$

and the corresponding gauge invariance is

$$\delta \Upsilon = i\Sigma \Upsilon \quad \delta V = i(\check{\Sigma} - \Sigma)$$  \hspace{1cm} (4.48)$$

where the gauge parameter \(\Sigma\) is an arctic superfield.

To link the two descriptions, it is worth replacing \(\Upsilon\) by a local analytic superfield

$$Q^+(z, u) = u^{+1} \exp \left( u^{-1} u^{+2} V(z, w) \right) \Upsilon(z, w)$$  \hspace{1cm} (4.49)$$

which, in contrast to \(q^+\) (4.13), possesses singularities at both poles and is covariantly holomorphic,

$$\left( D^{++} + iV^{++} \right) Q^+ = 0$$  \hspace{1cm} (4.50)$$

with \(V^{++}\) defined as in eq. (4.39). The matter (\(Q^+\)) and gauge (\(V^{++}\)) superfields transform similar to eq. (4.46),

$$\delta Q^+ = i\Lambda Q^+ \quad \delta V^{++} = -D^{++} \Lambda$$  

$$\Lambda = u^{-1}u^{+2}\check{\Sigma} - u^{+1}u^{-2}\Sigma$$  \hspace{1cm} (4.51)$$

but now the gauge parameter \(\Lambda\) becomes singular at the north and south poles. To reproduce the action (4.47), it is sufficient to evaluate \(S[Q^+_{R,e}, V^{++}]\) in the limit \(e \to 0\), where

$$Q^+_{R,e}(z, u) = \exp \left( u^{-1} u^{+2} V(z, w) \right) q^+_{R,e}(z, u)$$  \hspace{1cm} (4.52)$$

with \(q^+_{R,e}\) defined as in (4.23). Both \(Q^+_{R,e}\) and \(V^{++}\) are not globally defined on the harmonic superspace. However, the (manifestly gauge invariant) action \(S[Q^+_{R,e}, V^{++}]\) appears to be well defined, since either \(Q^+_{R,e}\) or its conjugate \(\check{Q}^+_{R,e}\) vanishes just in a small region where \(V^{++}\) and the other matter superfield become singular.
5 Low-energy hypermultiplet action

In quantum $N = 2$ super Yang-Mills theories, the effective dynamics of hypermultiplets are described by a low-energy action of the general form [4]

$$S_{\text{eff}}[q^+] = \int du \, d\zeta^{(-4)} K^{(4)}_{\text{eff}}(q^+, \bar{q}^+, D^{++}q^+, D^{++}\bar{q}^+, \cdots, u).$$  (5.1)

Under reasonable assumptions on the structure of $K^{(4)}_{\text{eff}}$, this action can be readily truncated to projective superspace to result with [17]

$$S_{\text{eff}}[\Upsilon] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^4x \, D^4 K_{\text{eff}}(\Upsilon, \bar{\Upsilon}, w)|.$$  (5.2)

In the simplest case when $K_{\text{eff}}$ is $w$-independent, one can immediately evaluate leading contributions to the low-energy action which come from the physical $N = 1$ chiral ($\Phi^I$) and complex linear ($\Gamma^I$) superfields contained in $\Upsilon^I$,

$$\Upsilon^I(w) = \Phi^I + w\Gamma^I + \text{auxiliary superfields}$$

$$\bar{D}_\alpha \Phi^I = 0 \quad \bar{D}^2 \Gamma^I = 0.$$  (5.3)

One gets

$$S_{\text{eff}}[\Upsilon] = \frac{1}{2\pi i} \oint \frac{dw}{w} \int d^4x \, D^4 K_{\text{eff}}(\Upsilon, \bar{\Upsilon})|$$

$$= \int d^8z \left\{ K_{\text{eff}}(\Phi, \bar{\Phi}) - \Gamma^I \bar{\Gamma}^J \frac{\partial^2}{\partial \Phi^I \partial \bar{\Phi}^J} K_{\text{eff}}(\Phi, \bar{\Phi}) \right\} + \ldots$$  (5.4)

This manifestly $N = 2$ supersymmetric sigma-model possesses a Kähler invariance of the form

$$K_{\text{eff}}(\Upsilon, \bar{\Upsilon}) \rightarrow K_{\text{eff}}(\Upsilon, \bar{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon})$$  (5.5)

with an arbitrary holomorphic function $\Lambda$. Hence $K_{\text{eff}}(\Phi, \bar{\Phi})$ is a Kähler potential. As is well-known, the target spaces of $N = 2$ supersymmetric sigma-models are hyper-Kähler manifolds [24]. The case under consideration turns out to be very specific. The physical scalars $\Phi^I$ parametrize a Kähler manifold, while $\Gamma^I$ the tangent space at the point $\{\Phi^I\}$ of the Kähler manifold. This follows from the fact that a holomorphic reparametrization

$$\Upsilon^I \rightarrow \Upsilon''^I = f^I(\Upsilon)$$  (5.6)

implies

$$\Phi^I \rightarrow \Phi''^I = f^I(\Phi)$$

$$\Gamma^I \rightarrow \Gamma''^I = \frac{\partial f^I}{\partial \Phi^J} \Gamma^J.$$  (5.7)
Therefore, the whole set of physical scalars \{\Phi^I, \Gamma^J\} parametrizes the tangent bundle of some Kähler manifold.

The action presented in the second line of (5.4) is the so-called chiral–non-minimal nonlinear sigma models [19] which was proposed to describe a supersymmetric low-energy QCD action.

It is worth pointing out that the computation of the low-energy hypermultiplet action (5.4) is rather simple. To determine it, we should in fact evaluate the effective action for non-vanishing values of the matter chiral superfields \Phi^I only.

6 Conclusion

Some years ago it was shown [4] how to realize the \( N = 2 \) off-shell matter multiplets with finitely many components fields (the tensor multiplets [25], [26], [27], [28], the relaxed hypermultiplet [23] and its higher relaxations [31], [4], the generalized tensor or \( O(2k) \) multiplets [21], [4]) in the harmonic superspace approach. The polar and tropical multiplets, which are the most interesting, for applications, multiplets in projective superspace, possess infinitely many auxiliary or purely gauge components. We have shown in the present paper that these projective multiplets naturally originate in harmonic superspace as well.

In our opinion, the importance of the projective superspace approach is that it defines a minimal truncation of unconstrained analytic superfields, which preserves several fundamental properties of multiplets and is most suitable for reduction to \( N = 1 \) superfields. The \( q^+ \) hypermultiplet cannot be truncated to a multiplet with finitely many components, since the charged hypermultiplet must possess infinitely many auxiliary fields [20]. But it can be decomposed into a sum of two submultiplets, one of which is just the polar multiplet and the other is purely auxiliary.

The polar and \( O(2k) \) multiplets involve, as one of their \( N = 1 \) components, a remarkable representation of \( N = 1 \) supersymmetry – the non-minimal scalar multiplet being described by a complex linear superfield [31], [32]. This multiplet has remained for a long time in shadow of the chiral scalar which is traditionally used to describe supersymmetric matter. In conventional \( N = 2 \) supersymmetry, the non-minimal scalar multiplet is seen to be unavoidable. It is worth also remarking that \( N = 2 \) supersymmetry provides us with an explanation of the magic \( N = 1 \) mechanism of generating masses for non-minimal scalars in tandem with chiral superfields [32] (see [33] for a recent review). Coupling of the polar multiplet to an external \( N = 2 \) vector multiplet is achieved by deforming the
polar multiplet constraints via the covariantization of the $N = 2$ covariant derivatives. If we now choose the background $N = 2$ vector multiplet to possess a constant strength, we result in a massive $N = 1$ non-minimal scalar multiplet.

Since projective superspace admits only the $U(1)$ subgroup of the automorphism group $SU(2)_A$, it seems to be perfectly suited for formulating $N = 2$ anti-de Sitter supersymmetry as well as for realizing the $N = 2$ higher-superspin massless multiplets \cite{34} in a manifestly supersymmetric form. As concerns $N = 2$ anti-de Sitter supersymmetry, it can be most likely realized in harmonic superspace by choosing a $u$-dependent vacuum solution for a compensator of $N = 2$ supergravity.

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A Tensor fields on the two-sphere

In this appendix we describe, for completeness, the well-known one-to-one correspondence between smooth tensor fields on $S^2 = SU(2)/U(1)$ and smooth scalar functions over $SU(2)$ with definite $U(1)$ charges. The two-sphere is obtained from $SU(2)$ by factorization with respect to the equivalence relation

$$u^i \sim e^{i\varphi} u^i \quad \varphi \in \mathbb{R}. \quad (A.1)$$

We start by introducing two open charts forming an atlas on $SU(2)$ which, upon identification on (A.1), provides us with a useful atlas on $S^2$. The north patch is defined by

$$u^1 \neq 0 \quad (A.2)$$

and here we can represent

$$u^+ i = u^1 w^i \quad w^i = (1, u^{+2}/u^1) = (1, w)$$

$$u^- i = u^1 \bar{w}_i \quad \bar{w}_i = (1, \bar{w}) \quad |u^1|^2 = (1 + w\bar{w})^{-1}. \quad (A.3)$$
The south patch is defined by

\[ u^+ \neq 0 \]  

(A.4)

and here we have

\[ u^+ = u^+ y^i \quad y^i = (u^+/u^+, 1) = (y, 1) \]

\[ u^- = u^+ \bar{y}_i \quad \bar{y}_i = (\bar{y}, 1) \quad |u^+|^2 = (1 + y\bar{y})^{-1}. \]  

(A.5)

In the overlap of the two charts we have

\[ u^+ i = e^{i\alpha} \sqrt{1 + w\bar{w}} \quad w^i = e^{i\beta} \sqrt{1 + y\bar{y}} \quad y^i \]

(A.6)

where

\[ y = \frac{1}{w} \quad e^{i\beta} = \sqrt{\frac{w}{\bar{w}}} e^{i\alpha}. \]  

(A.7)

The variables \( w \) and \( y \) are seen to be local complex coordinates on \( S^2 \) considered as the Riemann sphere, \( S^2 = \mathbb{C} \cup \{\infty\} \); the north chart \( U_N = \mathbb{C} \) is parametrized by \( w \) and the south patch \( U_S = \mathbb{C}^* \cup \{\infty\} \) is parametrized by \( y \).

Along with \( w^i \) and \( \bar{w}_i \), we often use their counterparts with lower (upper) indices

\[ w_i = \varepsilon_{ij} w^j = (-w, 1) \quad \bar{w}_i = \varepsilon^{ij} \bar{w}_j = (\bar{w}, -1) \quad \bar{w}_i = -\bar{w}^i \]  

(A.8)

and similar for \( y_i \) and \( \bar{y}_i \).

Let \( \Phi^{(p)}(u) \) be a smooth function on \( SU(2) \) with \( U(1) \)-charge \( p \) which we choose, for definiteness, to be non-negative, \( p \geq 0 \). Such a function possesses a convergent Fourier series of the form

\[ \Phi^{(p)}(u) = \sum_{n=0}^{\infty} \Phi^{(i_1 \cdots i_{n+p} j_1 \cdots j_n)}_{u_{i_1}^+ \cdots u_{i_{n+p}}^+ u_{j_1}^- \cdots u_{j_n}^-}. \]  

(A.9)

In the north patch we can write

\[ \Phi^{(p)}(u) = (u^+)^p \Phi^{(p)}_N (w, \bar{w}) \]

\[ \Phi^{(p)}_N (w, \bar{w}) = \sum_{n=0}^{\infty} \Phi^{(i_1 \cdots i_{n+p} j_1 \cdots j_n)}_{w_{i_1}^+ \cdots w_{i_{n+p}}^+ \bar{w}_{j_1}^- \cdots \bar{w}_{j_n}^-} \frac{1}{(1 + w\bar{w})^n}. \]  

(A.10)

In the south patch we have

\[ \Phi^{(p)}(u) = (u^+)^p \Phi^{(p)}_S (y, \bar{y}) \]

\[ \Phi^{(p)}_S (y, \bar{y}) = \sum_{n=0}^{\infty} \Phi^{(i_1 \cdots i_{n+p} j_1 \cdots j_n)}_{y_{i_1}^+ \cdots y_{i_{n+p}}^+ \bar{y}_{j_1}^- \cdots \bar{y}_{j_n}^-} \frac{1}{(1 + y\bar{y})^n}. \]  

(A.11)

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Finally, in the overlap of the two charts $\Phi^{(p)}_N$ and $\Phi^{(p)}_S$ are simply related to each other

$$\Phi^{(p)}_S(y, \bar{y}) = \frac{1}{w^p} \Phi^{(p)}_N(w, \bar{w}). \tag{A.12}$$

If we redefine

$$\hat{\Phi}^{(p)}_N(w, \bar{w}) = e^{i\nu \pi/4} \Phi^{(p)}_N(w, \bar{w}) \quad \hat{\Phi}^{(p)}_S(y, \bar{y}) = e^{-i\nu \pi/4} \Phi^{(p)}_S(y, \bar{y})$$

the above relation takes the form

$$\hat{\Phi}^{(p)}_S(y, \bar{y}) = \left( \frac{\partial y}{\partial w} \right)^{p/2} \hat{\Phi}^{(p)}_N(w, \bar{w}) \tag{A.13}$$

and thus defines a smooth tensor field on $S^2$.

In accordance with eq. (3.8), the smile-conjugate of function (A.9) reads

$$\check{\Phi}^{(p)}(u) = (-1)^p \sum_{n=0}^{\infty} \hat{\Phi}^{(i_1 \cdots i_n \nu \xi_1 \cdots \xi_n)} u_{i_1}^{+} \cdots u_{i_n}^{+} u_{j_1}^{-} \cdots u_{j_n}^{-}. \tag{A.14}$$

It is easy to check that $\check{\Phi}^{(p)}_S(y, \bar{y})$ is obtained from $\Phi^{(p)}_N(w, \bar{w})$ by composing the complex conjugation with replacement $w \rightarrow -\bar{y}$,

$$\check{\Phi}^{(p)}_S(y, \bar{y}) = \left. \check{\Phi}^{(p)}_N(w, \bar{w}) \right|_{w \rightarrow -\bar{y}}. \tag{A.15}$$

If $p$ is even, in the overlap of the north and south charts we can represent

$$\Phi^{(2k)}(u) = (iu^{+1}u^{+2})^k \Phi^{(p)}_{N-S}(w, \bar{w}). \tag{A.16}$$

Then

$$\check{\Phi}^{(2k)}(u) = (iu^{+1}u^{+2})^k \check{\Phi}^{(2k)}_{N-S}(w, \bar{w}) \tag{A.17}$$

where $\check{\Phi}^{(2k)}_{N-S}(w, \bar{w})$ is obtained from $\Phi^{(2k)}_{N-S}(w, \bar{w})$ by composing the complex conjugation with replacement $w \rightarrow -\frac{1}{w}$,

$$\check{\Phi}^{(2k)}_{N-S}(w, \bar{w}) = \left. \check{\Phi}^{(2k)}_{N-S}(w, \bar{w}) \right|_{w \rightarrow -\frac{1}{w}}. \tag{A.18}$$

From here we recover the projective superspace conjugation (2.10).
B  \(O(2k)\) multiplet in harmonic superspace

The \(O(2k)\) multiplet is described in harmonic superspace \([4]\) by an analytic real superfield \(\Omega^{(2k)}(z, u)\),

\[
D^\alpha \Omega^{(2k)} = \bar{D}^\alpha \Omega^{(2k)} = 0 \quad \bar{\Omega}^{(2k)} = \Omega^{(2k)},
\]

which, in addition, is constrained by

\[
D^{++} \Omega^{(2k)} = 0.
\]

This constraint along with the analyticity conditions imply

\[
\Omega^{(2k)}(z, u) = \Omega^{i_1 \cdots i_{2k}} u_{i_1}^+ \cdots u_{i_{2k}}^+ \quad D^j \Omega^{i_1 \cdots i_{2k}} = \bar{D}^j \Omega^{i_1 \cdots i_{2k}} = 0.
\]

In the north chart we can represent

\[
\Omega^{(2k)}(z, u) = \frac{1}{(2k-2)!} (i u^+ u^2)^k \Omega^{[2k]}(z, w)
\]

where \(\Omega^{[2k]}\) is given by eq. (2.23). The action reads

\[
S = \frac{1}{2} (4k-3)! \int du d\zeta (-4) [(u^- s)(u^- \bar{s})]^{2k-2} (\Omega^{(2k)})^2.
\]

Here \(s\) and \(\bar{s}\) are the constant isospinors \([4.14]\) defining the south and north poles. In spite of the fact that the constraints \([B.1]\) and \([B.2]\) are \(SU(2)_A\) covariant, for \(k > 1\) the action is invariant only with respect to the \(U(1)\) subgroup of \(SU(2)_A\).

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