Linear perturbations in Eddington-inspired Born-Infeld gravity

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Based on: Ke Yang, Xiao-Long Du, YXL, arXiv:1307.2969.

ICTS, USTC Sep 5, 2013
Outline

1. Introduction to EiBI gravity
2. Linear Perturbations
3. The stability of the perturbations
4. Conclusion and discussion
1. Introduction to EiBI gravity

1915: General Relativity

- It provides precise descriptions to a variety of phenomena in our Universe for almost a century.
- It also suffers various troublesome theoretical problems: dark matter/energy, nonrenormalization, singularity...

The the Einstein-Hilbert action is

\[ S_{\text{EH}}[g] = \int d^4x \sqrt{-g} \left[ R(g) - 2\Lambda \right]. \]  

(1)
1. Introduction to EiBI gravity

**Modified Gravity**

- Scalar-tensor (Brans-Dicke) gravity
- Einstein-Aether gravity
- $F(R)$ gravity and general higher-order theories,
- Horava-Lifschitz gravity
- Galileons
- Ghost Condensates
- Models of extra dimensions: KK, ADD, RS, DGP
- Born-Infeld Gravity
- Bimetric theories
- ...
1924, Eddington proposed a purely affine gravity.

Eddington gravity

\[ S_{\text{Edd}}(\Gamma) = \frac{1}{8\pi G} \frac{2}{\kappa} \int d^4x \sqrt{-|\kappa R_{\mu\nu}(\Gamma)|}, \]  

(2)

\[ \Downarrow (g_{\mu\nu} = \kappa R_{\mu\nu}) \]

\[ g_{\mu\nu;\lambda} = 0, \text{ i.e., } \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \]

\[ \Downarrow (\Lambda = \frac{1}{\kappa}) \]

\[ R_{\mu\nu} = \Lambda g_{\mu\nu} \]  

(3)

Eddington’s theory is totally equivalent to the GR with \( \Lambda \).

HOWEVER, it is incomplete because matter is not included.
Consider the Palatini action for gravity with $\Lambda$

$$S_P[g, \Gamma] = \int d^4x \left( \sqrt{-g} \ g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda \right),$$  \hspace{1cm} (4)$$

Eliminating the connection using its own EoM gives

$$S_{EH}[g] = \int d^4x \sqrt{-g} \left[ R(g) - 2\Lambda \right],$$  \hspace{1cm} (5)$$

If $\Lambda \neq 0$, eliminating the metric yields [Annals Phys. 162(1985)31]

$$S_{Edd}[\Gamma] = \frac{2}{\Lambda} \int d^4x \sqrt{-|R_{\mu\nu}(\Gamma)|}.$$  \hspace{1cm} (6)$$

$S_P[g, \Gamma]$ is called the Parent action, while the Einstein-Hilbert action and Eddington’s action are its daughters.

$S_{EH}[g]$ and $S_{Edd}[\Gamma]$ are said to be dual to each other, and in many respects they are equivalent.
The vector Born-Infeld theory (1934 Born, Infeld)

\[ S_{\text{VBI}} = \int d^4x \sqrt{-|g_{\mu\nu} + F_{\mu\nu}|}. \]  

(7)

The Born-Infeld gravity [Deser and Gibbons, CQG 15, L35 (1998)]

\[ S_{\text{BI}}[g_{\mu\nu}] = \int d^4x \sqrt{-|g_{\mu\nu} - l^2 R_{\mu\nu} + X_{\mu\nu}(R)|}. \]  

(8)

- \( X_{\mu\nu}(R) \) must be chosen such that the action is free of ghost.
- For the vector BI theory, the EoM is of second order.
- For the spin two theory this is not automatic and requires the addition of \( X_{\mu\nu}(R) \).
Inspired by the Eddington and BI gravity theories, a new theory was put forward by Bañados and Ferreira [PRL 105 (2010) 011101].

Eddington Inspired Born-Infeld (EiBI) gravity

\[
S_{\text{EiBI}}[g, \Gamma, \Phi] = \frac{2}{\kappa} \int d^4x \left( \sqrt{-\left| g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma) \right|} - \lambda \sqrt{-\left| g_{\mu\nu} \right|} \right) + S_M(g, \Gamma, \Phi),
\]

(9)

where the dimensionless parameter \( \lambda \) must be nonvanishing.
By varying the action with respect to the metric simply gives

\[ \sqrt{-|g_{\mu\nu} + \kappa R_{\mu\nu}| \left[ (g_{\mu\nu} + \kappa R_{\mu\nu})^{-1} \right]^{\alpha\beta} = \sqrt{-|g_{\mu\nu}|} \left( \lambda g^{\alpha\beta} - \kappa T^{\alpha\beta} \right), \]  

(10)

By introducing an auxiliary metric

\[ q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}, \]  

(11)

the variation to the connection simply gives us \( q_{\mu\nu;\sigma} = 0 \), i.e., \( \Gamma \) is just the Christoffel symbol of the auxiliary metric.

Then Eq. (10) is rewritten as

\[ \sqrt{-|q_{\mu\nu}|} q^{\alpha\beta} = \lambda \sqrt{-|g_{\mu\nu}|} g^{\alpha\beta} - \kappa \sqrt{-|g_{\mu\nu}|} T^{\alpha\beta}, \]  

(12)

Eqs. (11) and (12) and matter field equations form a complete set of equations of the theory.
Some properties of the EiBI gravity:

- When $\kappa R \gg g$, $S_{EiBI} \rightarrow S_{Edd}$.
- When $\kappa R \ll g$, the EiBI action reproduces the GR with $\Lambda_{\text{eff}}$ in lowest order approximation:

$$S_{EiBI} \approx \frac{1}{2\kappa} \int d^n x \sqrt{-g_{\mu\nu}} \left( R - 2\Lambda_{\text{eff}} + \frac{\kappa}{4} R R - \frac{\kappa}{2} R_{\mu \nu} R^{\mu \nu} \right) + S_M[g, \Gamma, \Phi],$$

where $\Lambda_{\text{eff}} \equiv (\lambda - 1)/\kappa$.

- In the nonrelativistic limit, the EiBI theory gives the modified Poisson equation

$$\nabla^2 \Phi = -\frac{1}{2} \rho - \frac{\kappa}{4} \nabla^2 \rho.$$  \hspace{1cm} (14)

- So, it reproduces Einstein gravity precisely within the vacuum but deviates from it in the presence of source.
Homogeneous and isotropic Universe

\[ ds^2 = -dt^2 + a^2(t) \mathbf{d}x \cdot \mathbf{d}x \]  \hspace{1cm} (15)

With coupling to an ideal fluid \( T^{\mu\nu} = (\rho + \rho)u^\mu u^\nu + pg^{\mu\nu} \), the
Friedmann equation is given by

\[ H^2 = \frac{2}{3} \frac{G}{F^2}, \]  \hspace{1cm} (16)

\[ F = 1 - \frac{3\kappa(\rho_T + p_T)(1 - \omega - b\kappa \rho_T - \kappa p_T)}{(1 + \kappa \rho_T)(1 - \kappa p_T)}, \]

\[ G = \frac{1}{\kappa} [1 + 2U - 3 \frac{U}{V}], \]

\[ U = (1 + \kappa \rho_T)^{-1/2}(1 - \kappa p_T)^{3/2}, \]

\[ V = a^2(1 + \kappa \rho_T)^{1/2}(1 - \kappa p_T)^{1/2}, \]  \hspace{1cm} (17)

with \( \rho_T = \rho + \frac{A}{\kappa} \) and \( p_T = p - \frac{A}{\kappa} \).
Focus on the evolution of the scale factor at early times.

Assume radiation domination: $\rho_T = \rho$, $p_t = p = \frac{1}{3}\rho$.

Define $\bar{\rho} = \kappa \rho$, the Friedmann equation becomes

$$3H^2(\bar{\rho}) = \left[ \bar{\rho} - 1 + \frac{\sqrt{(1 + \bar{\rho})(3 - \bar{\rho})^3}}{3\sqrt{3}} \right] \frac{(1 + \bar{\rho})(3 - \bar{\rho})^2}{\kappa(3 + \bar{\rho}^2)^2}.$$

For small $\bar{\rho}$ we recover the conventional Friedmann universe, $H^2 \approx \rho/3$.

But at high density there is a stationary point $H^2 = 0$

at $\bar{\rho} = 3$ ($\rho = 3/\kappa$) for $\kappa > 0$, and

at $\bar{\rho} = -1$ ($\rho = -1/\kappa$) for $\kappa < 0$. 
The Hubble rate $H^2$ as a function of energy density $\rho$

- The new stationary points correspond to a maximum density $\rho_B$ ($\rho_B = 3/\kappa$ for $\kappa > 0$ and $\rho_B = -1/\kappa$ for $\kappa < 0$)
The scale factor $a$ as a function of time $t$

- The maximum density $\rho_B$ corresponds to a minimum length $a_B$ in cosmology.
- Thus the universe may be entirely singularity free.
Eddington-Born-Infeld action for dark matter and dark energy

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(Received 10 March 2008; published 24 June 2008)

We argue that Einstein gravity coupled to a Born-Infeld theory provides an attractive candidate to represent dark matter and dark energy. For cosmological models, the Born-Infeld field has an equation of state which interpolates between matter, $p = 0$ (small times), and a cosmological constant $p = -\rho$ (large times). On galactic scales, the Born-Infeld theory predicts asymptotically flat rotation curves.

DOI: 10.1103/PhysRevD.77.123534

PACS numbers: 95.35.+d, 95.36.+x, 98.80.-k, 04.50.Kd

FIG. 2. Evolution of the equation of state.

FIG. 5. Rotation curves for $k_0 = 1.5, 1.03, 1.003, 1.0005$. 
Tensor instability in the Eddington-inspired Born-Infeld theory of gravity

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(Received 7 April 2012; published 24 April 2012)

In this paper, we consider an extension to Eddington’s proposal for the gravitational action. We study tensor perturbations of a homogeneous and isotropic space-time in the Eddington regime, where modifications to Einstein gravity are strong. We find that the tensor mode is linearly unstable deep in the Eddington regime and discuss its cosmological implications.

DOI: 10.1103/PhysRevD.85.087302

PACS numbers: 98.80.–k, 04.30.–w, 14.70.Kv
Motivation:

Do the linear scalar and vector perturbations stable in Eddington regime?

So, we study the full linear perturbations of a homogeneous and isotropic spacetime in the EiBI gravity.
2. Linear Perturbations of EiBI cosmology

2.1 The perturbed metrics

The background space-time metric is

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \]  

(18)

The background auxiliary metric is

\[ ds'^2 = q_{\mu\nu} dx^\mu dx^\nu = -X^2(t) dt^2 + a^2(t) Y^2(t) \delta_{ij} dx^i dx^j. \]  

(19)
The perturbed space-time metric is

\[ d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu} + H_{\mu\nu})dx^\mu dx^\nu \]

\[ = (-1 + h_{00}(x))dt^2 + a^2(t)(\delta_{ij} + h_{ij}(x))dx^i dx^j + 2h_{0i}(x)dtdx^i. \] (20)

The perturbed auxiliary metric is

\[ d\tilde{s}'^2 = \tilde{q}_{\mu\nu}dx^\mu dx^\nu = (q_{\mu\nu} + \Pi_{\mu\nu})dx^\mu dx^\nu \]

\[ = X^2(t)(-1 + \gamma_{00}(x))dt^2 + a^2(t)Y^2(t)(\delta_{ij} + \gamma_{ij}(x))dx^i dx^j + 2Y^2(t)\gamma_{0i}(x)dtdx^i. \] (21)

\[ h \equiv \eta^{\mu\nu}h_{\mu\nu}, \quad \gamma \equiv \eta^{\mu\nu}h_{\mu\nu}. \] (22)
2.2 The background field equation

The 1st field equation reads

$$\sqrt{-|q_{\mu\nu}|q^{\mu\nu}} = \lambda \sqrt{-|g_{\mu\nu}|g^{\mu\nu} - \kappa \sqrt{-|g_{\mu\nu}|T^{\mu\nu}},} \quad (23)$$

$$\frac{Y^3}{X} = \lambda + \kappa \rho, \quad (24)$$

$$XY = \lambda - \kappa P. \quad (25)$$

The 2nd field equation

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}, \quad (26)$$

$$X^2 = 1 + 3\kappa \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 2 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right], \quad (27)$$

$$Y^2 = 1 + \kappa \frac{Y^2}{X^2} \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 6 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y} + 2 \frac{\dot{Y}^2}{Y^2} \right). \quad (27)$$
2.3 The energy-momentum tensor

Here the matter is the perfect fluid,

\[ T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho)u^\mu u^\nu, \]  

(28)

and the static observer is \( u^\mu = (1, 0, 0, 0) \) with \( g_{\mu\nu} u^\mu u^\nu = -1 \).

The 1st order perturbation of the energy-momentum tensor is

\[
\begin{align*}
\delta T^{00} &= \delta \rho + \rho h_{00}, \\
\delta T^{i0} &= -a^{-2} \rho h_{0i} + a^{-2} (P + \rho) \delta u_i, \\
\delta T^{ij} &= a^{-2} \delta P \delta_{ij} - a^{-2} P h_{ij}.
\end{align*}
\]  

(29) (30) (31)
2.4 The first perturbed field equation

Perturbing the first field equation

\[ \sqrt{-|q_{\mu\nu}|q^{\mu\nu}} = \lambda \sqrt{-|g_{\mu\nu}|g^{\mu\nu}} - \kappa \sqrt{-|g_{\mu\nu}|T^{\mu\nu}}, \quad (32) \]

we can get

\[
\gamma_{00} = h_{00} + \frac{\kappa \delta \rho}{2(\lambda + \kappa \rho)} + \frac{3\kappa \delta P}{2(\lambda - \kappa P)},
\]

\[
\gamma_{0i} = h_{0i} - \frac{(P + \rho)}{2(\lambda + \kappa \rho)} \kappa \delta u_i,
\]

\[
\gamma_{ij} = h_{ij} + \left[ \frac{\kappa \delta \rho}{2(\lambda + \kappa \rho)} - \frac{\kappa \delta P}{2(\lambda - \kappa P)} \right] \delta_{ij}. \quad (33)
\]
2.5 The perturbed Ricci tensor

Perturbing the Ricci tensor

\[
R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\alpha} \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu}.
\] (34)

we have

\[
\delta R_{00} = -\frac{1}{2} \frac{X^2}{Y^2} a^{-2} \partial_i \partial_i \gamma_{00} - \frac{3}{2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) \partial_0 \gamma_{00} - \frac{1}{2} \partial_0 \partial_0 \gamma_{ii}
\]
\[
- \left( \frac{\dot{a}}{a} - \frac{1}{2} \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) \partial_0 \gamma_{ii} + a^{-2} \partial_0 \partial_i \gamma_{i0}
\]
\[
+ a^{-2} \left( 2 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \partial_i \gamma_{i0},
\]

\[
\delta R_{0i} = \frac{Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 6 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y} + 2 \frac{\dot{Y}^2}{Y^2} - \frac{\dot{X} \dot{Y}}{X Y} \right] \gamma_{i0}
\]
\[
- \left[ \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right] \partial_i \gamma_{00} + \frac{1}{2} \partial_0 \partial_j \gamma_{i0} - \frac{1}{2} \partial_0 \partial_i \gamma_{j0} - \frac{1}{2} a^{-2} \partial_j \partial_j \gamma_{i0}
\]
\[
+ \frac{1}{2} a^{-2} \partial_i \partial_j \gamma_{j0},
\]

ITP,LZU

Linear perturbations of EiBI Gravity
\[
\delta R_{ij} = \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} + 2 \frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 6 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} + 2 \frac{\dot{Y}^2}{Y^2} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right]
\times \left[ \gamma_{00} \delta_{ij} + \gamma_{ij} \right] + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \partial_0 \gamma_{00} \delta_{ij}
\]

\[
+ \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ 3 \frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y} \right] \partial_0 \gamma_{ij} - \frac{1}{2} \frac{Y^2}{X^2} \left[ \frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y} \right] \partial_{i} \gamma_{j0}
\]

\[
- \frac{1}{2} \frac{Y^2}{X^2} \left[ \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right] \partial_{j} \gamma_{i0} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \partial_{0} \gamma_{kk} \delta_{ij}
\]

\[
- \frac{Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \partial_{k} \gamma_{0i} + \frac{1}{2} \partial_{i} \partial_{j} \gamma_{00} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_{0} \partial_{0} \gamma_{ij}
\]

\[
+ \frac{1}{2} \partial_{k} \partial_{j} \gamma_{ki} + \frac{1}{2} \partial_{k} \partial_{i} \gamma_{kj} - \frac{1}{2} \partial_{k} \partial_{k} \gamma_{ij} - \frac{1}{2} \partial_{i} \partial_{j} \gamma_{kk}
\]

\[
- \frac{1}{2} \frac{Y^2}{X^2} \partial_{0} \partial_{i} \gamma_{j0} - \frac{1}{2} \frac{Y^2}{X^2} \partial_{0} \partial_{j} \gamma_{i0}.
\]

(35)
2.6 The perturbed field equation

With the scalar-vector-tensor decomposition of the perturbed metric $h_{\mu\nu}$ and $\delta u_i$

$$h_{00} = -E, \quad h_{i0} = \partial_i F + G_i,$$
$$h_{ij} = A\delta_{ij} + \partial_i \partial_j B + \partial_j C_i + \partial_i C_j + D_{ij},$$
$$\delta u_i = \partial_i \delta u + \delta U_i,$$

where $\partial_i C_i = \partial_i G_i = \partial_i \delta U_i = 0$, $\partial_i D_{ij} = 0$, and $D_{ii} = 0$, the 2nd perturbed field equation

$$\delta q_{\mu\nu} = \delta g_{\mu\nu} + \kappa \delta R_{\mu\nu} \quad (36)$$

can be calculated as
00-component:

\[
\frac{1}{2} \frac{X^2}{Y^2} a^{-2} \nabla^2 E + 3 \left( \frac{a}{a} + \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 2 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right) E \\
+ \frac{3}{2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) \dot{E} - \frac{1}{2} \left[ 3 \dot{A} + \nabla^2 \dot{B} \right] - \left( \frac{\dot{a}}{a} - \frac{1}{2} \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) \left[ 3 \dot{A} + \nabla^2 \dot{B} \right] \\
+ a^{-2} \nabla^2 \dot{F} + a^{-2} \left( 2 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \nabla^2 F - \frac{\kappa}{4} a^{-2} \frac{X^2}{Y^2} \frac{\nabla^2 \delta \rho}{\lambda + \kappa \rho} \\
- \frac{3\kappa}{4} a^{-2} \frac{X^2}{Y^2} \frac{\nabla^2 \delta P}{\lambda - \kappa P} - \frac{3\kappa}{4} \partial_0 \partial_0 \left[ \frac{\delta \rho}{\lambda + \kappa \rho} \right] + \frac{3\kappa}{4} \partial_0 \partial_0 \left[ \frac{\delta P}{\lambda - \kappa P} \right] \\
- \frac{3\kappa}{4} \left( 3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \partial_0 \left[ \frac{\delta \rho}{\lambda + \kappa \rho} \right] - \frac{3\kappa}{4} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} + \frac{\dot{X}}{X} \right) \partial_0 \left[ \frac{\delta P}{\lambda - \kappa P} \right] \\
- \frac{1}{2} \left[ 1 + 3\kappa \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 2 \frac{\dot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right) \right] \left[ \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3\delta P}{\lambda - \kappa P} \right] \\
- \kappa a^{-2} \partial_0 \left[ \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u \right] - \kappa a^{-2} \left( 2 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u = 0. \tag{37}
\]
\( \partial_i \left\{ \left[ \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right] E - \partial_i \dot{A} - \frac{\kappa}{2} \partial_0 \left[ \frac{\partial_i \delta \rho}{\lambda + \kappa \rho} \right] + \frac{\kappa}{2} \partial_0 \left[ \frac{\partial_i \delta P}{\lambda - \kappa P} \right] \right. \\
- \frac{\kappa}{2} \left[ \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right] \left( \frac{\partial_i \delta \rho}{\lambda + \kappa \rho} + \frac{3 \partial_i \delta P}{\lambda - \kappa P} \right) \left. + \frac{P + \rho}{\lambda + \kappa \rho} \partial_i \delta u \right\} \right. \\
+ \frac{1}{2} \nabla^2 \dot{C}_i - \frac{1}{2} a^{-2} \nabla^2 G_i + \frac{\kappa}{2} \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta U_i + \frac{P + \rho}{\lambda + \kappa \rho} \delta U_i \\
= 0. \right) (38)
\[
\begin{align*}
\left\{ \begin{array}{l}
- \frac{a^2 Y^2}{X^2} \left[ \frac{\dddot{a}}{a} + \frac{\dddot{Y}}{Y} + 2 \frac{\dddot{a}^2}{a^2} - \frac{\dddot{a} \dddot{X}}{a X} + 6 \frac{\dddot{a} \dddot{Y}}{a Y} + 2 \dddot{Y^2} - \frac{\dddot{X} \dddot{Y}}{X Y} \right] \dot{E} \\
- \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \dot{E} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \frac{\dot{a}}{a} \dot{X} - \frac{1}{2} \nabla^2 A \\
- \frac{Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \nabla^2 F + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{3 \ddot{a}}{a} + \frac{3 \dddot{Y}}{Y} - \frac{\dddot{X}}{X} \right] \dot{A} \\
+ \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \left[ 3 \dot{A} + \nabla^2 \dot{B} \right] + \frac{\kappa a^2 Y^2}{4 X^2} \partial_0 \partial_0 \left[ \frac{\delta \rho}{\lambda + \kappa \rho} \right] \\
- \frac{\kappa a^2 Y^2}{4 X^2} \partial_0 \partial_0 \left[ \frac{\delta P}{(\lambda - \kappa P)} \right] - \frac{\kappa}{4} \left[ \frac{\nabla^2 \delta \rho}{\lambda + \kappa \rho} - \frac{\nabla^2 \delta P}{\lambda - \kappa P} \right] \\
+ \frac{\kappa a^2 Y^2}{2 X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} + 2 \frac{\dddot{a}^2}{a^2} - \frac{\dddot{a} \dddot{X}}{a X} + 6 \frac{\dddot{a} \dddot{Y}}{a Y} + 2 \dddot{Y^2} - \frac{\dddot{X} \dddot{Y}}{X Y} \right] \\
\times \left[ \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3 \delta P}{\lambda - \kappa P} \right] + \frac{\kappa a^2 Y^2}{4 X^2} \left[ \frac{7 \ddot{a}}{a} + \frac{7 \dddot{Y}}{Y} - \frac{\dddot{X}}{X} \right] \partial_0 \left[ \frac{\delta \rho}{\lambda + \kappa \rho} \right] \\
- \frac{1}{2} a^2 \left[ \frac{\delta \rho}{\lambda + \kappa \rho} - \frac{\delta P}{\lambda - \kappa P} \right] + \frac{\kappa}{X^2} \left[ \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} \right] \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u \\
- \frac{\kappa a^2 Y^2}{4 X^2} \left[ \frac{3 \ddot{a}}{a} + \frac{3 \dddot{Y}}{Y} - \frac{\dddot{X}}{X} \right] \partial_0 \left[ \frac{\delta P}{\lambda - \kappa P} \right] \right\} \delta_{ij}
\end{array} \right.
\]
\[ \partial_i \partial_j \left\{ -\frac{1}{2} E - \frac{1}{2} A + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \right\} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ 3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ 3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left[ \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right] \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \kappa \frac{Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} - \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ - \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} \]

\[ + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial_j D_{ij} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \partial_i \partial_j \partial_i \partial j
A. Scalar modes:

The 00-component of the perturbed equation (36) gives

\[
\frac{1}{2} \frac{X^2}{Y^2} a^{-2} \nabla^2 E + 3 \left( \frac{\ddot{a}}{a} + \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 2 \frac{\ddot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\ddot{X}}{X} \frac{\dot{Y}}{Y} \right) E + \frac{3}{2} \left( \frac{\ddot{a}}{a} + \frac{\dot{Y}}{Y} \right) \dot{E}
\]

\[-\frac{1}{2} (3\ddot{A} + \nabla^2 \ddot{B}) - \left( \frac{\ddot{a}}{a} - \frac{1}{2} \frac{\dot{X}}{X} + \frac{\ddot{Y}}{Y} \right) (3\dot{A} + \nabla^2 \dot{B}) + a^{-2} \nabla^2 \dot{F}
\]

\[+ a^{-2} \left( \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \nabla^2 F - \frac{\kappa}{4} a^{-2} \frac{X^2}{Y^2} \frac{\nabla^2 \delta \rho}{\lambda + \kappa \rho} - \frac{3\kappa}{4} a^{-2} \frac{X^2}{Y^2} \frac{\nabla^2 \delta P}{\lambda - \kappa P}
\]

\[-\frac{3\kappa}{4} \partial_0 \partial_0 \frac{\delta \rho}{\lambda + \kappa \rho} - \frac{3\kappa}{4} \left( \frac{3\ddot{a}}{a} + 3 \frac{\ddot{Y}}{Y} - \frac{\ddot{X}}{X} \right) \partial_0 \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3\kappa}{4} \partial_0 \partial_0 \frac{\delta P}{\lambda - \kappa P}
\]

\[-\frac{1}{2} \left[ 1 + 3\kappa \left( \frac{\ddot{a}}{a} + \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 2 \frac{\ddot{a}}{a} \frac{\dot{Y}}{Y} - \frac{\ddot{X}}{X} \frac{\dot{Y}}{Y} \right) \right] \left( \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3\delta P}{\lambda - \kappa P} \right)
\]

\[-\frac{3\kappa}{4} \left( \frac{\ddot{a}}{a} + \frac{\dot{Y}}{Y} + \frac{\dot{X}}{X} \right) \partial_0 \frac{\delta P}{\lambda - \kappa P} - \kappa a^{-2} \partial_0 \left[ \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u \right]
\]

\[-\kappa a^{-2} \left( \frac{2\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u = 0. \quad (39)
\]
The part of $ij$-component of (36) proportional to $\delta_{ij}$ gives

\[
-\frac{a^2 Y^2}{X^2} \left( \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} + 2\frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 6\frac{\dot{a}}{a} \frac{\dot{Y}}{Y} + 2\frac{\dot{Y}^2}{Y^2} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right) E + \frac{1}{2} \frac{a^2 Y^2}{X^2} \ddot{A} \\
- \frac{1}{2} \frac{a^2 Y^2}{X^2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) \dot{E} - \frac{1}{2} \nabla^2 A - \frac{Y^2}{X^2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) \nabla^2 F \\
+ \frac{1}{2} \frac{a^2 Y^2}{X^2} \left( 3\frac{\dot{a}}{a} + 3\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \dot{A} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) (3\dot{A} + \nabla^2 \dot{B}) \\
+ \frac{\kappa a^2 Y^2}{4 X^2} \partial_0 \partial_0 \frac{\delta \rho}{\lambda + \kappa \rho} - \frac{\kappa a^2 Y^2}{4 X^2} \partial_0 \partial_0 \frac{\delta P}{\lambda - \kappa P} \\
- \frac{\kappa}{4} \left( \frac{\nabla^2 \delta \rho}{\lambda + \kappa \rho} - \frac{\nabla^2 \delta P}{\lambda - \kappa P} \right) - \frac{1}{2} \frac{a^2}{a^2} \left( \frac{\delta \rho}{\lambda + \kappa \rho} - \frac{\delta P}{\lambda - \kappa P} \right) \\
+ \frac{\kappa a^2 Y^2}{2 X^2} \left( \frac{\ddot{a}}{a} + \frac{\ddot{Y}}{Y} + 2\frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a} \frac{\dot{X}}{X} + 6\frac{\dot{a}}{a} \frac{\dot{Y}}{Y} + 2\frac{\dot{Y}^2}{Y^2} - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} \right) \\
\times \left( \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3\delta P}{\lambda - \kappa P} \right) + \frac{\kappa a^2 Y^2}{4 X^2} \left( 7\frac{\dot{a}}{a} + 7\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \partial_0 \frac{\delta \rho}{\lambda + \kappa \rho} \\
- \frac{\kappa a^2 Y^2}{4 X^2} \left( 3\frac{\dot{a}}{a} + 3\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \right) \partial_0 \frac{\delta P}{\lambda - \kappa P} + \frac{\kappa Y^2}{X^2} \left( \frac{\dot{a}}{a} + \frac{\dot{Y}}{Y} \right) \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta u \\
= 0.
\]
The part of $i0$-component of (36) with the form $\partial_i S$ (where $S$ is any scalar) gives

\[
\left(\frac{\dot{a}}{a} + \frac{\dot{Y}}{Y}\right)E - \dot{A} - \frac{\kappa}{2} \partial_0 \frac{\delta \rho}{\lambda + \kappa \rho} + \frac{\kappa}{2} \partial_0 \frac{\delta P}{\lambda - \kappa P} + \frac{P + \rho}{\lambda + \kappa \rho} \delta u
\]

\[-\frac{\kappa}{2} \left(\frac{\dot{a}}{a} + \frac{\dot{Y}}{Y}\right) \left(\frac{\delta \rho}{\lambda + \kappa \rho} + \frac{3 \delta P}{\lambda - \kappa P}\right) = 0.
\]  

(41)

The part of $ij$-component of (36) of the form $\partial_i \partial_j S$ gives

\[
-\frac{1}{2} E - \frac{1}{2} A + \frac{1}{2} \frac{a^2 Y^2}{X^2} \ddot{B} + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left(3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right) \ddot{B} - \frac{Y^2}{X^2} \ddot{F}
\]

\[-\frac{Y^2}{X^2} \left(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y}\right) F + \kappa \frac{\delta P}{\lambda - \kappa P} + \kappa \frac{Y^2}{X^2} \partial_0 \left(\frac{P + \rho}{\lambda + \kappa \rho}\delta u\right)
\]

\[+ \kappa \frac{Y^2}{X^2} \left(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y}\right) \frac{P + \rho}{\lambda + \kappa \rho} \delta u = 0.
\]  

(42)
The 0-component of the perturbed conservation equation gives

\[
\delta \dot{\rho} + 3 \frac{\dot{a}}{a} (\delta \rho + \delta P) + \frac{1}{2} (P + \rho) (3 \dot{A} + \nabla^2 \dot{B}) \\
- a^{-2} (P + \rho) \nabla^2 (F - \delta u) = 0. \tag{43}
\]

The part of \(i\)-component of the perturbed conservation equation of the form \(\partial_i S\) (where \(S\) is any scalar) gives

\[
\delta P + \frac{1}{2} (P + \rho) E + (P + \rho) \delta u + (\dot{P} + \dot{\rho}) \delta u + 3 \frac{\dot{a}}{a} (P + \rho) \delta u = 0. \tag{44}
\]
B. Vector modes:

The part of $i0$-component of the perturbed equation (36) with the form $V_i$ (where $V_i$ is any vector satisfying $\partial_i V_i = 0$) gives

$$\frac{1}{2} \nabla^2 \dot{C}_i - \frac{1}{2} a^{-2} \nabla^2 G_i + \frac{\kappa}{2} a^{-2} \frac{P + \rho}{\lambda + \kappa \rho} \nabla^2 \delta U_i + \frac{P + \rho}{\lambda + \kappa \rho} \delta U_i = 0. \quad (45)$$

The part of $i$-component of the perturbed conservation equation with the form $V_i$ (where $V_i$ is any vector satisfying $\partial_i V_i = 0$) gives

$$(P + \rho) \delta \dot{U}_i + (\dot{P} + \dot{\rho}) \delta U_i + 3 \frac{\dot{a}}{a} (P + \rho) \delta U_i = 0. \quad (46)$$
The part of $ij$-component of (36) with the form $\partial_i V_j$ (where $V_j$ is any vector satisfying $\partial_j V_j = 0$) gives

\[
\frac{1}{2} \frac{a^2 Y^2}{X^2} \ddot{C}_j + \frac{1}{2} \frac{a^2 Y^2}{X^2} \left(3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right) \dot{C}_j - \frac{1}{2} \frac{Y^2}{X^2} \dot{G}_j \\
- \frac{1}{2} \frac{Y^2}{X^2} \left(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y}\right) G_j + \frac{\kappa Y^2}{2 X^2} \partial_0 \left(\frac{P + \rho}{\lambda + \kappa \rho} \delta U_j\right) \\
+ \frac{\kappa Y^2}{2 X^2} \left(\frac{\dot{a}}{a} - \frac{\dot{X}}{X} + 3 \frac{\dot{Y}}{Y}\right) \frac{P + \rho}{\lambda + \kappa \rho} \delta U_j = 0. \tag{47}
\]
B. Tensor modes:

The part of $ij$-component of (36) with the form of a transverse-traceless tensor is

$$-
abla^2 D_{ij} + \frac{a^2 Y^2}{X^2} \ddot{D}_{ij} + \frac{a^2 Y^2}{X^2} \left(3 \frac{\dot{a}}{a} + 3 \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right) \ddot{D}_{ij} = 0. \quad (48)$$
3. The stability of the perturbations

- The perturbed equations involve 7 scalars modes $E, F, A, B, \delta \rho, \delta P, \delta u$, 3 transverse vectors modes $C_i, G_i, \delta U_i$, and 1 transverse-traceless tensor mode $D_{ij}$.

- For scalar perturbed modes we work in the Newtonian gauge, i.e., we set $B = F = 0$ in the linear perturbed equations.

- For vector mode, we fix the gauge freedom to eliminate $C_i$.

- Finally, with the state equation $\tilde{P} = \omega \tilde{\rho}$, then $\delta P = \omega \delta \rho$, all the remaining perturbed modes are solvable.

- The dominant component in Eddington regime is the highly relativistic ideal gas ($\omega = 1/3$, $\delta P = \frac{1}{3} \delta \rho$) and the cosmological constant can be neglected ($\lambda = 1$).
3.1 The case $\kappa > 0$

For $\kappa > 0$, the approximate background solution near the maximum density ($t \to -\infty$) is given by

[Escamilla-Rivera, Banados, and Ferreira, PRD 85(2012)087302],

[Scargill, Banados, and Ferreira, PRD 86 (2012) 103533]

\[ a = a_B \left[ 1 + e^{\sqrt{\frac{8}{3\kappa}}(t-t_0)} \right], \]

\[ X = U^{\frac{1}{2}} = 2e^{\frac{3}{4} \sqrt{\frac{8}{3\kappa}}(t-t_0)}, \]

\[ Y = V^{\frac{1}{2}} = 2e^{\frac{1}{4} \sqrt{\frac{8}{3\kappa}}(t-t_0)}. \]  

(49)
A. Scalar perturbations

For scalar perturbations, the solution \((t \to -\infty)\) is given by

\[
A \simeq 2c_1(x_i)e^{\frac{7}{4}b(t-t_0)},
\]
\[
E \simeq -2c_1(x_i)e^{\frac{11}{4}b(t-t_0)},
\]
\[
\delta\rho \simeq -\frac{12}{\kappa}c_1(x_i)e^{\frac{7}{4}b(t-t_0)},
\]
\[
\delta u \simeq \frac{4}{7b}c_1(x_i)e^{\frac{7}{4}b(t-t_0)} + c_2(x_i).
\]

Therefore, the scalar perturbations vanish when the universe approaches the maximum density and we conclude that the scalar perturbations are stable in the Eddington regime.
3.1 The case $\kappa > 0$

B. Transverse vector modes

For transverse vector modes, from Eqs. (46) and (47), we have

$$G_i \simeq c_3(x_i), \quad (54)$$
$$\delta U_i \simeq c_4(x_i). \quad (55)$$

The vector perturbations are also stable in the Eddington regime.
3.1 The case $\kappa > 0$

C. Transverse-traceless tensor mode

The transverse-traceless tensor mode is given by Eq. (48), with the approximate solution (49), the dominant part simply gives

$$\ddot{D}_{ij} = 0.$$  \hfill (56)

So it gives that

$$D_{ij} \propto m(x)t + n(x).$$  \hfill (57)

When the universe approaches the maximum density ($t \to -\infty$), the tensor perturbation is divergent, it causes an instability in the Eddington regime.
For $\kappa < 0$, the approximate background solution near the maximum density ($t \to 0$) is given by

[Escamilla-Rivera, Banados, and Ferreira, PRD 85(2012)087302],
[Scargill, Banados, and Ferreira, PRD 86 (2012) 103533]

\begin{align*}
    a &= a_B \left( 1 - \frac{2}{3\kappa} |t|^2 \right), \quad (58) \\
    X &= \frac{2}{\sqrt{3}} (-\kappa/2)^{1/4} |t|^{-\frac{1}{2}}, \quad (59) \\
    Y &= \frac{2}{\sqrt{3}} (-2/\kappa)^{1/4} |t|^{\frac{1}{2}}. \quad (60)
\end{align*}
3.2 The case $\kappa < 0$

The perturbations are approximately given by

$$A \simeq \frac{\kappa}{2} C_1(x_i) \mid t \mid^{\frac{3}{2}}, \quad E \simeq \frac{\kappa^2}{16} C_1(x_i) \mid t \mid^{-\frac{1}{2}},$$

$$\delta \rho \simeq C_1(x_i) \mid t \mid^{\frac{3}{2}}, \quad \delta u \simeq -\frac{\kappa^2}{16} C_1(x_i) \mid t \mid^{\frac{1}{2}},$$

$$\delta U \simeq C_2(x_i), \quad G_i \simeq C_3(x_i) \mid t \mid^{-2} + \frac{2}{3} C_2(x_i),$$

$$D_{ij} \simeq C_4(x_i) \mid t \mid^{-1} + C_5(x_i).$$

Therefore, scalar, vector and tensor modes will all cause instabilities in the Eddington regime in this case.
We studied the full linear perturbations in the radiation era of a homogeneous and isotropic spacetime in EiBI theory.

- For $\kappa > 0$, the scalar and transverse vector modes are stable, while the tensor mode is unstable in Eddington regime.
- For $\kappa < 0$, all the scalar, vector and tensor modes cause instabilities in Eddington regime.
- It may necessary to consider the nonlinear perturbations.
In Einstein theory, in the radiation era, \((\frac{\dot{a}}{a})^2 = \frac{\rho}{3}\), \(a = a_0 \sqrt{t}\).

The linear perturbed modes are

\[
E = -A \simeq d_1 t^{-\frac{3}{2}} + d_2,
\]

\[
\delta u \simeq d_1 t^{-\frac{1}{2}} - \frac{d_2}{2} t,
\]

\[
\delta \rho \simeq \frac{3}{2} d_1 t^{-\frac{7}{2}} - \frac{3}{4} d_2 t^{-2},
\]

\[
G_i \simeq d_3 t^{-\frac{1}{2}}, \quad \delta U_i \simeq d_4 t^{\frac{1}{2}},
\]

\[
D_{ij} \simeq d_5 t^{-\frac{1}{2}} + d_6.
\]

All modes are all unstable in early universe.
Thus the EiBI cosmology with $\kappa > 0$ presents as an interesting theory with **stable scalar and vector modes**.

This was also demonstrated in the following references:

- Pani, Cardoso, Delsate, *Compact stars in eddington inspired gravity*, PRL **107** (2011) 031101;
- P. Avelino and R. Ferreira, *Bouncing eddington-inspired born-infeld cosmologies: an alternative to inflation?*, PRD **86** (2012) 041501;
- P. Pani, T. Delsate, and V. Cardoso, *Eddington-inspired born-infeld gravity. phenomenology of non-linear gravity-matter coupling*, PRD **85** (2012) 084020,

where the EiBI theory with positive $\kappa$ shows more well properties than negative one.
Conclusion and discussion

- For a 5D **braneworld model** with

\[
S_{\text{EiBI}} = \frac{1}{\kappa} \int d^5 x \left( \sqrt{-g_{PQ} + \kappa R_{PQ}(\Gamma)} - \lambda \sqrt{-\mathcal{g}_{PQ}} \right)
- \int d^5 x \sqrt{-\mathcal{g}_{PQ}} \left( \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi + V(\phi) \right),
\]

\[
ds^2 = a^2(x) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]

- We found in [PRD 85(2012)124053] a brane solution for \( \kappa > 0 \) and the TT tensor perturbation is stable.
Thank you!