Poisson structures compatible with the canonical metric of $\mathbb{R}^3$

M. Boucetta

Abstract. In this Note, we will characterize the Poisson structures compatible with the canonical metric of $\mathbb{R}^3$. We will also give some relevant examples of such structures. The notion of compatibility between a Poisson structure and a Riemannian metric used in this Note was introduced and studied by the author in [1], [2], [3].

1 Introduction and main results

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman’s monograph [5].

As a continuation of the study by the author of Poisson structures compatible with Riemannian metrics in [1], [2] and [3], it is of interest to find some relevant examples of such structures in the low dimensions. So we were interested in finding all the Poisson structures compatible with the canonical metric in $\mathbb{R}^3$. The results of this search is the theme of this Note.

Let us recall some facts about the notion of compatibility between a Poisson structure and a Riemannian metric in order to motivate our investigation and to show the interest of this Note.

Let $P$ be a Poisson manifold with Poisson tensor $\pi$. A Riemannian metric on $T^*P$ is a smooth symmetric contravariant 2-form $<,>$ on $P$ such that, at each point $x \in P$, $<,>_x$ is a scalar product on $T^*_x P$. For each Riemannian metric $<,>$ on $T^*P$, we consider the contravariant connection $D$ introduced in [1] by

$$2 < D_\alpha \beta, \gamma > = \pi(\alpha) \cdot < \beta, \gamma > + \pi(\beta) \cdot < \alpha, \gamma > - \pi(\gamma) \cdot < \alpha, \beta >$$
$$+ < [\alpha, \beta]_\pi, \gamma > + < [\gamma, \alpha]_\pi, \beta > + < [\gamma, \beta]_\pi, \alpha >,$$  

(1)
where $\alpha, \beta, \gamma \in \Omega^1(P)$ and the Lie bracket $[\ , \ ]_\pi$ is given by
\[
[\alpha, \beta]_\pi = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d(\pi(\alpha, \beta));
\]
here, $\pi : T^*P \to TP$ denotes the bundle map given by
\[
\beta[\pi(\alpha)] = \pi(\alpha, \beta).
\]
The connection $D$ is the contravariant analogue of the usual Levi-Civita connection. The connection $D$ has vanishing torsion, i.e.
\[
D_\alpha \beta - D_\beta \alpha = [\alpha, \beta]_\pi.
\]
Moreover, it is compatible with the Riemannian metric $\langle , \rangle$, i.e.
\[
\pi(\alpha). \langle \beta, \gamma \rangle = \langle D_\alpha \beta, \gamma \rangle + \langle \beta, D_\alpha \gamma \rangle.
\]
The notion of contravariant connection has been introduced by Vaisman (see [5] p.55) as contravariant derivative. Recently, a geometric approach of this notion was given by Fernandes in [4].

If we put, for any $f \in C^\infty(P)$,
\[
\phi_{\langle , \rangle}(f) = \sum_{i=1}^n \langle D_{\alpha_i} df, \alpha_i \rangle
\]
where $(\alpha_1, \ldots, \alpha_n)$ is a local orthonormal basis of 1-forms, we get a derivation on $C^\infty(P)$ and hence a vector field called the modular vector field of $(P, \pi)$ with respect to the metric $\langle , \rangle$.

The couple $(\pi, \langle , \rangle)$ is compatible if, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,
\[
D\pi(\alpha, \beta, \gamma) := \pi(\alpha).\pi(\beta, \gamma) - \pi(D_\alpha \beta, \gamma) - \pi(\beta, D_\alpha \gamma) = 0.
\]

In this case, the triple $(P, \pi, \langle , \rangle)$ is called a Riemann-Poisson manifold.

Riemann-Poisson manifolds was first introduced by the author in [1]. Let us summarize some important results of Riemann-Poisson manifolds proved by the author in [2] and [3].

For a Riemann-Poisson manifold $(P, \pi, \langle , \rangle)$ the following results are true:
1. the symplectic leaves are Kählerian;
2. the symplectic foliation (when it is a regular foliation) is a Riemannian foliation;
3. \((P, \pi)\) is unimodular (see [6] for the details on the notion of unimodular Poisson manifolds) and moreover the modular vector field \(\phi_{\langle,\rangle}\) given by (2) vanishes.

With those properties in mind, we can give the main results of this Note.

**Theorem 1.1** A Poisson tensor \(\pi = \pi_{12} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_{13} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi_{23} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\) is compatible with the canonical metric \(\langle,\rangle\) of \(\mathbb{R}^3\) iff there exists a function \(f \in C^\infty(\mathbb{R}^3)\) such that

\[
\begin{align*}
\pi_{12} &= \frac{\partial f}{\partial z}, \\
\pi_{13} &= -\frac{\partial f}{\partial y}, \\
\pi_{23} &= \frac{\partial f}{\partial x},
\end{align*}
\]

and

\(d(\langle df, df \rangle) - \Delta(f)df = 0 \quad (E)\)

where \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is the usual Laplacian on \(\mathbb{R}^3\). Moreover, the function \(f\) is a Casimir function of \(\pi\).

The following proposition and the theorem above give all the linear Poisson structures on \(\mathbb{R}^3\) compatible with the canonical metric.

**Proposition 1.1** The polynomial functions of degree 2 solutions of (E) are

\[
f(x, y, z) = (a + c)x^2 + (a + b)y^2 + (b + c)z^2 - 2\sqrt{bc}xy + 2\sqrt{ab}xz + 2\sqrt{ac}yz,
\]

where \(a, b, c \in \mathbb{R}\) and \(ab, ac, bc \in \mathbb{R}_+\).

Let \(\pi_{so(3)} = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\) be the linear Poisson structure on \(\mathbb{R}^3\) corresponding to the Lie algebra \(so(3)\). In [2], we have shown that there isn’t any Riemannian metric on \(\mathbb{R}^3\) compatible with \(\pi_{so(3)}\). However, we have the following proposition.

**Proposition 1.2** The function \(f(x, y, z) = (x^2 + y^2 + z^2)^{\frac{1}{2}}\) is a solution of (E) and then \((x^2 + y^2 + z^2)^{\frac{1}{2}}\pi_{so(3)}\) is compatible with the canonical metric of \(\mathbb{R}^3\).

**Remarks.** 1. The fact that there isn’t any metric compatible with \(\pi_{so(3)}\) and, however, the Poisson structure \((x^2 + y^2 + z^2)^{\frac{1}{2}}\pi_{so(3)}\) is compatible with the canonical metric seems curious. But it can be explained easily. In fact, let
\((P, \pi, <, >)\) be a Poisson manifold with a contravariant Riemannian metric. If we change the Poisson structure by \(f\pi\) where \(f \in C^\infty(P)\), the contravariant Levi-Civita connection given by (1) become more complicated and is given by

\[
D^{f\pi}_\alpha \beta = fD^\pi_\alpha \beta + \frac{1}{2} \pi(\alpha, \beta)df - \frac{1}{2} < df, \beta > J\alpha - \frac{1}{2} < df, \alpha > J\beta
\]

where \(J\) is the field of homomorphisms given by \(\pi(\alpha, \beta) = < J\alpha, \beta >\).

2. The Poisson structures \(\pi_{so(3)}\) and \((x^2 + y^2 + z^2)^{1/2} \pi_{so(3)}\) have the same symplectic foliation and, in restriction to a symplectic leaf, the two symplectic structures differ by a constant.

3. It is possible that the compatibility of \((x^2 + y^2 + z^2)^{1/2} \pi_{so(3)}\) with the canonical metric has some physical signification.

2 Proof of Theorem 1.1

Let \(\pi\) be a bivectors field on \(\mathbb{R}^3\) given by

\[
\pi = \pi_{12} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_{13} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi_{23} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.
\]

We consider the canonical metric \(<, >\) on \(\mathbb{R}^3\) as contravariant metric given by

\[
< dx, dx > = < dy, dy > = < dz, dz > = 1, < dx, dy > = < dx, dz > = < dy, dz > = 0.
\]

We denote by \(D\) the Levi-Civita contravariant connection associated with \((\pi, <, >)\).

Firstly, remark that the compatibility between \(\pi\) and \(<, >\) implies the vanishing of the modular vector field given by

\[
\phi_{<,>}(f) = < D_{dx} df, dx > + < D_{dy} df, dy > + < D_{dz} df, dz >.
\]

A straightforward calculation shows that the vanishing of \(\phi_{<,>}\) is equivalent to

\[
\frac{\partial \pi_{12}}{\partial y} + \frac{\partial \pi_{13}}{\partial z} = 0, \quad \frac{\partial \pi_{12}}{\partial x} - \frac{\partial \pi_{23}}{\partial z} = 0, \quad \frac{\partial \pi_{13}}{\partial x} + \frac{\partial \pi_{23}}{\partial y} = 0.
\]
Now, it is easy to see that (4) is equivalent to the fact that $\pi_{23}dx - \pi_{13}dy + \pi_{12}dz$ is a closed 1-form and hence is exact. So, there exists a function $f \in C^\infty(\mathbb{R}^3)$ such that
\[
\pi_{12} = \frac{\partial f}{\partial z}, \quad \pi_{13} = -\frac{\partial f}{\partial y}, \quad \pi_{23} = \frac{\partial f}{\partial x}.
\] (5)

Now, let us compute the contravariant connection $D$. We will use the Christoffel symbols $\Gamma^k_{ij}$. For example, $D_{dx}dx = \Gamma^1_{11}dx + \Gamma^2_{11}dy + \Gamma^3_{11}dz$. From (1), we get:

\[
\Gamma^1_{11} = 0, \quad \Gamma^2_{11} = -\frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma^3_{11} = \frac{\partial^2 f}{\partial x \partial y},
\]

\[
\Gamma^1_{12} = \frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma^2_{12} = 0, \quad \Gamma^3_{12} = \frac{1}{2} \left( -\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right),
\]

\[
\Gamma^1_{21} = 0, \quad \Gamma^2_{21} = -\frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma^3_{21} = \frac{1}{2} \left( -\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right),
\]

\[
\Gamma^1_{13} = -\frac{\partial^2 f}{\partial x \partial y}, \quad \Gamma^2_{13} = \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right), \quad \Gamma^3_{13} = 0,
\]

\[
\Gamma^1_{31} = 0, \quad \Gamma^2_{31} = \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right), \quad \Gamma^3_{31} = \frac{\partial^2 f}{\partial y \partial z},
\]

\[
\Gamma^1_{22} = \frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma^2_{22} = 0, \quad \Gamma^3_{22} = -\frac{\partial^2 f}{\partial x \partial y},
\]

\[
\Gamma^1_{23} = \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right), \quad \Gamma^2_{23} = \frac{\partial^2 f}{\partial x \partial y}, \quad \Gamma^3_{23} = 0,
\]

\[
\Gamma^1_{32} = \frac{1}{2} \left( -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right), \quad \Gamma^2_{32} = 0, \quad \Gamma^3_{32} = -\frac{\partial^2 f}{\partial x \partial z},
\]

\[
\Gamma^1_{33} = -\frac{\partial^2 f}{\partial y \partial z}, \quad \Gamma^2_{33} = \frac{\partial^2 f}{\partial x \partial z}, \quad \Gamma^3_{33} = 0.
\]

Now, we will compute $D_{dx}\pi$, $D_{dy}\pi$ and $D_{dz}\pi$. We have
\[
D_{dx}\pi = \pi(dx) \left( \frac{\partial f}{\partial z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \pi(dx) \left( \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \pi(dx) \left( \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} 
\right.ight.ight.
\]

\[
+ \frac{\partial f}{\partial z} \left( (D_{dx} \frac{\partial}{\partial x}) \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge (D_{dz} \frac{\partial}{\partial y}) \right)
\]
\[- \frac{\partial f}{\partial y} \left( (D_{dx} \frac{\partial}{\partial x}) \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \wedge (D_{dx} \frac{\partial}{\partial z}) \right) \]
\[+ \frac{\partial f}{\partial x} \left( (D_{dx} \frac{\partial}{\partial y}) \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \wedge (D_{dx} \frac{\partial}{\partial z}) \right) . \]

On other hand, we have
\[\pi(dx) = \frac{\partial f}{\partial z} \frac{\partial}{\partial y} \frac{\partial}{\partial z} - \frac{\partial f}{\partial y} \frac{\partial}{\partial z} , \]
\[D_{dx} \frac{\partial}{\partial x} = - \frac{\partial^2 f}{\partial x \partial z} \frac{\partial}{\partial y} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial z} , \]
\[D_{dx} \frac{\partial}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \frac{\partial}{\partial x} , \]
\[D_{dx} \frac{\partial}{\partial z} = - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right) \frac{\partial}{\partial y} . \]

Substituting those expressions into the expression of $D_{dx} \pi$, we get
\[D_{dx} \pi = \left( \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \frac{\partial f}{\partial y} \left( - \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left( \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \frac{\partial f}{\partial z} \left( - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} . \]

In the same manner we can get
\[D_{dy} \pi = \left( - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial z} + \frac{1}{2} \frac{\partial f}{\partial x} \left( - \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left( - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial z} + \frac{1}{2} \frac{\partial f}{\partial z} \left( - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} . \]
\[D_{dz} \pi = \left( - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial z} + \frac{1}{2} \frac{\partial f}{\partial x} \left( - \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \left( - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial z} + \frac{1}{2} \frac{\partial f}{\partial y} \left( - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} . \]

Now, it is easy to show that $D \pi = 0$ iff $f$ satisfies $(E)$. It is also easy to show that $f$ is a Casimir function. Remark that $D \pi = 0$ implies that the bracket of Schouten $[\pi, \pi]$ vanishes which finish the proof of Theorem 1.1. □
References

[1] M. Boucetta, Compatibilité des structures pseudo-riemanniennes et des structures de Poisson, C. R. Acad. Sci. Paris, t. 333, Série I, (2001) 763-768.
[2] M. Boucetta, Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras, Preprint math.DG/0206102. To appear in Differential Geometry and its Applications.
[3] M. Boucetta, Riemann-Poisson manifolds and Kähler-Riemann foliations, C. R. Acad. Sci. Paris, Ser. I 336 (2003) 423-428.
[4] R. L. Fernandes, Connections in Poisson Geometry 1: holonomy and invariants, J. Diff. Geom. 54 , (2000) 303-366.
[5] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Progress in Mathematics, vol. 118, Birkhäuser, Berlin, 1994.
[6] A. Weinstein, The Modular Automorphism Group of a Poisson Manifold, J. Geom. Phys. 23, (1997) 379-394.

Mohamed Boucetta
Faculté des Sciences et Techniques Gueliz
BP 549 Marrakech Morocco
E-mail: boucetta@fstg-marrakech.ac.ma