THE MASSLESS HIGHER-LOOP TWO-POINT FUNCTION

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ABSTRACT. We introduce a new method for computing massless Feynman integrals analytically in parametric form. An analysis of the method yields a criterion for a primitive Feynman graph $G$ to evaluate to multiple zeta values. The criterion depends only on the topology of $G$, and can be checked algorithmically. As a corollary, we reprove the result, due to Bierenbaum and Weinzierl, that the massless 2-loop 2-point function is expressible in terms of multiple zeta values, and generalize this to the 3, 4, and 5-loop cases. We find that the coefficients in the Taylor expansion of planar graphs in this range evaluate to multiple zeta values, but the non-planar graphs with crossing number 1 may evaluate to multiple sums with 6th roots of unity. Our method fails for the five loop graphs with crossing number 2 obtained by breaking open the bipartite graph $K_{3,4}$ at one edge.

1. Introduction

Let $n_1, \ldots, n_r \in \mathbb{N}$ and suppose that $n_r \geq 2$. The multiple zeta value is the real number defined by the convergent nested sum:

$$\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \cdot \ldots \cdot k_r^{n_r}}. $$

An important question in perturbative quantum field theory is whether multiple zeta values, or some larger set of periods, suffice to evaluate a given class of Feynman integrals. Moreover, it is crucial for applications to find efficient methods for evaluating such Feynman integrals analytically.

In this paper, we shall consider massless Feynman integrals in $\phi_4^4$ theory with the propagators raised to arbitrary powers, and a single non-zero momentum. The simplest case is the two-loop example, pictured in figure 1 on the left. Let $a_i$ be a positive real number corresponding to each edge $1 \leq i \leq 5$. The corresponding Feynman integral is:

$$\int \int d^D k_1 d^D k_2 \left( \frac{1}{k_1^2} \right)^{a_1} \left( \frac{1}{k_2^2} \right)^{a_2} \left( \frac{1}{(k_1 - k_2)^2} \right)^{a_3} \left( \frac{1}{(k_2 - q)^2} \right)^{a_4} \left( \frac{1}{(k_1 - q)^2} \right)^{a_5},$$

where $D$ is the number of dimensions, and $q$ is the momentum entering on the left. Now suppose that $a_i = 1 + n_i \varepsilon$, where $n_i$ are positive integers, for $1 \leq i \leq 5$, and $\varepsilon$ is the parameter in dimensional regularization, i.e., $D = 4 - 2 \varepsilon$. The problem of calculating the coefficients in the Taylor expansion of (1) with respect to $\varepsilon$, is important in three and four loop calculations, and has a history spanning approximately twenty-five years (to which we refer to [3] for an account). In particular, it had been conjectured for a long time that every coefficient is a
rational linear combination of multiple zeta values. This question was finally settled in the affirmative in [3], using Mellin-Barnes techniques.

Until now, however, there seemed to be a lack of systematic methods for computing a range of Feynman integrals analytically at higher loop orders. In this paper, we introduce a new method, using iterated integration with polylogarithms, which was initiated in [8] to compute the periods of moduli spaces of curves of genus 0. Using this, we reprove the fact that the Taylor expansion of the massless two-loop two-point integral (1) evaluates to multiple zeta values, and extend the result to higher loop orders. Our method also yields results for certain examples of massive Feynman diagrams, but in the present paper we only consider massless cases.

1.1. Results. We consider three and higher-loop integrals with exactly one non-zero momentum and arbitrary powers of the propagators, generalizing the integral (1). Let $G$ be a graph in $\phi_4^4$ with two external legs, and a single momentum $q$. By Hopf algebra arguments, we can assume that $G$ is primitive, in the following sense. Let $\tilde{G}$ denote the graph formed by closing the two external legs of $G$, which now has no external edges, but gains an extra loop (see figure 1). We will suppose that $\tilde{G}$ is primitive divergent in the sense of [4], i.e., $\tilde{G}$ contains no strict divergent subgraphs and satisfies:

$$\text{#edges of } \tilde{G} = 2 \times \text{#loops of } \tilde{G}.$$ 

In this case, we say that $G$ is broken primitive divergent (bpd), and every bpd graph with a fixed number of loops can be obtained by taking the set of all primitive divergent graphs with one more loop, and breaking them open along every edge.

Now suppose that $G$ is bpd with $h$ loops and $L$ internal edges. It follows from the above that $L = 2h + 1$. For each internal edge $i$, let $a_i = 1 + n_i \varepsilon$, where $n_i$ is a positive integer, and $D = 4 - 2\varepsilon$, and consider the massless Feynman integral

$$(2) \int d^Dk_1 \ldots \int d^Dk_L \prod_{i=1}^{L} \left( \frac{1}{r_i} \right)^{a_i},$$

where $k_i$ is a momentum flowing through the $i^{th}$ edge, and $r_i$ is the propagator corresponding to the $i^{th}$ edge. The domain of integration is given by the conservation of momentum at each vertex, and there is a single external momentum entering $G$, denoted $q$. It is easy to show, using the Schwinger trick, that (2) is proportional to a certain power of $q^2$ [12]. The constant of proportionality is given by a product of

![Figure 1. Left: The two-point two-loop massless diagram. Closing up its external legs gives the wheel with three spokes (right).](image-url)
explicit gamma factors and powers of $\pi$, with the parametric integral:

$$I(G) = \int_{\alpha=1}^{\alpha_{L+1}} \frac{\prod_{i=1}^{L+1} \alpha_i^{a_i-1}}{U^{D/2}} \, d\alpha_1 \ldots \widehat{d\alpha_\lambda} \ldots d\alpha_{L+1},$$

where $U_G(\alpha_1, \ldots, \alpha_{L+1})$ is the graph polynomial (or Kirchoff polynomial) of $\tilde{G}$, $\alpha_1, \ldots, \alpha_{L+1}$ are Schwinger parameters for the $L+1$ edges in $\tilde{G}$, $\lambda$ is any index between 1 and $L+1$, and $a_{L+1}$ is a certain linear combination of $a_1, \ldots, a_L$. The problem of computing the Taylor expansion of (2) reduces to computing the expansion of (3). For example, if $\tilde{G}$ is the wheel with 3 spokes, we have:

$$U_{\tilde{G}} = \alpha_1 \alpha_2 \alpha_6 + \alpha_1 \alpha_4 \alpha_6 + \alpha_2 \alpha_5 \alpha_6 + \alpha_4 \alpha_5 \alpha_6 + \alpha_1 \alpha_3 \alpha_6 + \alpha_2 \alpha_3 \alpha_6 + \alpha_3 \alpha_4 \alpha_6 + \alpha_3 \alpha_5 \alpha_6 + \alpha_1 \alpha_3 \alpha_4 + \alpha_1 \alpha_3 \alpha_5 + \alpha_2 \alpha_3 \alpha_5 + \alpha_2 \alpha_4 \alpha_5 + \alpha_1 \alpha_4 \alpha_5 + \alpha_1 \alpha_2 \alpha_5 + \alpha_1 \alpha_2 \alpha_4,$$

and the coefficients in the Taylor expansion of (3) are given by the period integrals:

$$\int_{\alpha=1}^{\alpha_{L+1}} \frac{\log(\alpha_1)^{m_1} \ldots \log(\alpha_{L+1})^{m_{L+1}} \log(U_{\tilde{G}})^n}{U_{\tilde{G}}^{D/2}} \, d\alpha_1 \ldots \widehat{d\alpha_\lambda} \ldots d\alpha_{L+1},$$

where $m_1, \ldots, m_{L+1}, n$ are arbitrary positive integers.

We now state some results on the transcendental nature of the coefficients in the Taylor expansion of $I(G)$ with respect to $\varepsilon$, for all bpd graphs $G$ up to five loops. For every graph $G$ for which a theorem is stated below, there is also a corresponding algorithm for computing the coefficients of the Taylor expansion of $I(G)$ by integrating inside a predetermined algebra of polylogarithms.

1.1.1. Three loops. There is exactly one primitive divergent graph with four loops \cite{LS}, namely the wheel with four spokes, or cross-hairs diagram, pictured below (left). Breaking it apart along each edge gives rise to exactly two topologically distinct bpd graphs with three loops (right).

![Figure 2](image)

**Figure 2.** Left: The cross-hairs diagram is the unique primitive divergent graph with four loops. Breaking it at the edges 1, 8 gives the graphs in the middle, and on the right, respectively.

**Theorem 1.** Let $G$ be one of the two bpd 3-loop graphs depicted in figure 2 (middle and right). Then every coefficient in the Taylor expansion of $I(G)$ is a rational linear combination of multiple zeta values.
1.1.2. Four loop contributions. There are precisely three 5-loop diagrams, pictured in figure 3. The one on the left is planar and two-vertex reducible, and will be denoted $5R$. The one in the middle is planar, two-vertex irreducible, and will be denoted $5P$, and the one on the right is non-planar, and will be denoted $5N$. It turns out that there are exactly six topologically distinct ways to break the graph $5P$ open at one edge, and exactly two ways to break open $5R$, giving the eight planar topologies depicted in figure 4.

**Figure 3.** The three primitive-divergent 5-loop diagrams.

**Figure 4.** The eight planar bpd topologies with four loops.

**Theorem 2.** Let $G$ be one of the eight bpd planar 4-loop graphs depicted in figure 4. Then every coefficient in the Taylor expansion of $I(G)$ is a rational linear combination of multiple zeta values.

For the non-planar graphs, a new phenomenon occurs, and we must introduce multiple zeta values at roots of unity. Let $n_1, \ldots, n_r \in \mathbb{N}$, and consider the multiple polylogarithm function, first introduced by Goncharov [11]:

\begin{equation}
\operatorname{Li}_{n_1, \ldots, n_r}(x_1, \ldots, x_r) = \sum_{0 < k_1 < \ldots < k_r} \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}}.
\end{equation}

It converges absolutely for $|x_i| < 1$ and extends to a multivalued holomorphic function on an open subset of $\mathbb{C}^r$. For $m \geq 1$, we define $\mathcal{Z}^m$ to be the $\mathbb{Q}$-algebra generated by the values of multiple polylogarithms at $m^{\text{th}}$ roots of unity:

\[ \operatorname{Li}_{n_1, \ldots, n_r}(x_1, \ldots, x_r) \text{ such that } x_i^m = 1 \text{ for } 1 \leq i \leq r, \text{ and } (x_r, n_r) \neq (1,1). \]
The condition that \( x_r \) and \( n_r \) are not simultaneously 1 is to ensure convergence. The algebra \( Z^1 \) is the algebra of multiple zeta values, and \( Z^2 \) is known as the algebra of alternating multiple zeta values. We will call \( Z^m \) the algebra of multiple zeta values ramified at \( m^{th} \) roots of unity. Note that \( Z^a \subset Z^b \) if \( a \) divides \( b \).

**Figure 5.** The three non-planar bpd graphs on four loops obtained by breaking the graph 5\( N \) along an edge. The graph on the right has crossing number 1, but has been drawn with 2 crossings.

**Theorem 3.** Let \( G \) be one of the three non-planar bpd graphs with 4-loops as depicted in figure 5. Then every coefficient in the Taylor expansion of \( I(G) \) is a rational linear combination of multiple zeta values ramified at \( 6^{th} \) roots of unity.

Note that the theorem gives an upper bound on the set of periods which can occur \((Z^6)\), and it is an open question whether some smaller algebra \( Z^k \) for \( k = 1, 2, \) or 3 suffices to compute the Taylor expansions for the non-planar graphs, or whether there occurs a term which genuinely involves sixth roots of unity.

1.1.3. **Five-loop contributions.** Karen Yeats has computed the primitive divergent topologies up to seven loops [18]. There are nine at six loops, of which four are planar, pictured below, and five which are non-planar (figures 7,8). Although the five non-planar graphs all have genus 1 (they can be drawn on a torus without any self-crossings), it turns out that the invariant which successfully predicts the outcome of our method for computing the periods at six loops is not the genus but the crossing number. The crossing number of a graph \( G \) is defined to be the minimal number of self-crossings over all planar representations of \( G \) (which are allowed to have curved edges). It is easy to determine the crossing number for the graphs we consider here, but seems to be a difficult problem in general.

**Theorem 4.** Let \( G \) be a graph obtained by breaking open \( \tilde{G} \) at one edge, where \( \tilde{G} \) is one of the eight primitive divergent graphs at six loops pictured in figures 6 and 7. Then the coefficients in the Taylor expansion of \( I(G) \) are:

1. multiple zeta values, if \( \tilde{G} \) is planar (has crossing number 0).
2. multiple zeta values at 6\(^{th}\) roots of unity, if \( \tilde{G} \) has crossing number 1.

The one remaining primitive graph is the complete bipartite graph \( K_{3,4} \), and has crossing number exactly 2, pictured in figure 8. Our method failed for this graph.

1.2. **Discussion.** In summary, we found that for all Feynman graphs up to 6 loops, those with crossing number 0 evaluate to multiple zeta values, and those with crossing number 1 give multiple polylogarithms evaluated at 6\(^{th}\) roots of unity. There is a single example of a graph \((K_{3,4})\) with crossing number 2, and our method fails in this case. The leading term in the Taylor expansion of \( I(G) \) for \( K_{3,4} \) is actually
known to evaluate to a multiple zeta value, namely $\zeta(5, 3)$, which, interestingly, is the first occurrence of an irreducible double sum in $\phi^4$ theory.

One possible reason for this is that our method concerns all terms in the Taylor expansion of the integral $I(G)$, with arbitrary powers of logarithms in the numerator. In the particular case when there are no logarithmic terms in the numerator, the set of singularities of the integrand are slightly reduced, and it is not impossible that our method might work for $K_{3,4}$ with this restriction, although we have not checked this. Another possible reason for this could be because our algorithm has some room for improvement, in at least two ways. First, the algorithm involves the repeated factorization of polynomials derived from the graph polynomial. In our computations, we only considered factorizations that occurred over the field of rationals $\mathbb{Q}$, although it is conceivable that some of the polynomials which occur are irreducible over $\mathbb{Q}$, but factorize over an algebraic extension of $\mathbb{Q}$. 
A second, and more promising, possibility is to extend our algorithm to deal with quadratic terms, and this will be explained in further detail in §4.5. It is possible, by extending our method in this way, that a larger class of graphs will become tractable. However, it is more likely than not that eventually one will find periods of motives which are not mixed Tate (c.f. [1]), and this will pose a genuine obstruction to the present method. We expect that our method of polylogarithmic integration should also help to exhibit the first example of such a period in massless $\phi^4_4$, if and when it occurs. The idea would be to strip away from a candidate Feynman integral terms which evaluate to multiple zeta values, until one is left with a totally irreducible period integral which is verifiably not of mixed Tate type. Thus, the eventual failure of our method should at the same time exhibit the first non-MZV-type period, when it occurs.

1.3. Plan of the paper. The paper is divided into two halves. The first half (§2–4) gives an overview of our method, and can be read linearly, as a long introduction. In §2 we briefly recall how to rewrite the Feynman integrals $I(G)$ in Schwinger-parametric form using graph polynomials. In §3, we outline the main idea of our method, and in §4, we translate this into an elementary reduction algorithm on graph polynomials. The main theorem 18 in §4.4 states a sufficient condition for a bpd graph $G$ to evaluate to multiple zeta values in its Taylor expansion.

The purpose of the second half is to give a complete worked example of our method, in the case of the wheel with three spokes. In particular, in §7, we give a new proof of the well-known result that the leading term of $I(G)$ is $6\zeta(3)$. This is, to our knowledge, the first time that such a computation has appeared in print using a parametric representation for the Feynman integral. The purpose of section 5 is to provide sufficiently many details to understand the intricacies of the method in general, and section 6 provides worked examples of taking primitives and limits of polylogarithms in two variables, which are then applied directly in the $6\zeta(3)$ computation. It is perhaps advisable, after having read §2–4, to refer directly to the example in §7, and then read §5 and §6 bearing the example in mind.

The proofs of our results will be written up in full detail in [9], including a study of the underlying algebraic geometry, which is completely absent from this paper, and a generalization of the above results to some infinite families of graphs. Much of the background on iterated integrals, and also our method of integration with polylogarithms, is explained in detail in [7, 8] in the case of moduli spaces of genus 0 curves. These may serve as a useful introduction to this paper, since there are many similarities with the integrals we study here, even though the geometry underlying higher loop Feynman integrals is considerably more complex.

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2. Parametric representations

Let $G$ be a Feynman graph with $h$ loops, $L$ internal edges, and $E$ external legs. The graph polynomial of $G$ is a homogeneous polynomial of degree $h$ in variables $\alpha_1, \ldots, \alpha_L$ indexed by the set of internal edges of $G$. It is defined by the formula:

$$U_G = \sum_T \prod_{\ell \in T} \alpha_\ell .$$

The sum is over all spanning trees $T$ of $G$, i.e., subgraphs $T$ of $G$ which pass through every vertex of $G$ but which contain no loop. Next consider the homogeneous polynomial of degree $h + 1$ defined by:

$$V_G = \sum_S \prod_{\ell \in S} \alpha_\ell (q^S)^2 .$$

The sum is over graphs $S \subset G$ with exactly two connected components $S = T_1 \cup T_2$ where both $T_1$ and $T_2$ are trees, such that $S$ is obtained by cutting a spanning tree $S'$ along an edge $e$, and $q^S$ is the momentum flowing through $e$ in $S'$.

**Example 5.** Let $G$ denote the two-point two-loop diagram depicted in Fig 9. Then

$$U_G = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) + \alpha_3(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) ,$$

$$V_G = (\alpha_3(\alpha_1 + \alpha_2)(\alpha_4 + \alpha_5) + (\alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_2\alpha_4)) q^2 .$$

![Figure 9](image)

*Figure 9.* On the left are shown the eight spanning trees for $G$, corresponding to the eight terms in $U_G$. On the right are the eight pairs of trees $T_1 \cup T_2$ which correspond to the eight terms in $V_G$.

To each internal edge $\ell$ of $G$ we associate a decoration $a_\ell$, which is a positive real number (the power to which the corresponding propagator is raised) and set

$$a = \sum_{1 \leq \ell \leq L} a_\ell .$$

The unregularised massless Feynman integral $I_G(a, q, D)$ in dimension $D$, expressed in Schwinger coordinates (12], (2.36)) , is :

$$(-1)^a \frac{e^{\pi |a+h(1-D/2)|/2} \pi^{D/2}}{\prod_{\ell} \Gamma(a_\ell)} \int_0^\infty \! \! d\alpha_1 \ldots \int_0^\infty \! \! d\alpha_L \frac{\prod_{\ell} \alpha_\ell^{a_\ell-1} e^{-V_G/U_G}}{U_G^{D/2}} .$$

Now let $\lambda$ denote a non-empty set of internal edges of $G$. By making the change of variables $\alpha_i = \alpha_i' t$, for $1 \leq i \leq L$, where $t = \sum_{\ell \in \lambda} \alpha_\ell$, integrating with respect
to $t$ from 0 to $\infty$, and finally replacing $\alpha_i'$ with $\alpha_i$ once again, this integral can be rewritten (\cite{18}, (3.32)):

$$
\int_{H_\lambda} \prod_{\ell=1}^L \alpha^{a_{\ell}-1}_\ell \left( \frac{U_G}{V_G} \right)^{a-hD/2} \Omega_{L}.
$$

Here, $\Omega_L = \sum_{i=1}^L (-1)^i \alpha_i d\alpha_1 \ldots d\alpha_L$, and $H_\lambda = \{ \alpha_i : \sum_{i \in \lambda} \alpha_i = 1 \}$. The fact that the integral does not depend on the choice of subset $\lambda$ follows from the fact that it is really a projective integral, but is known in the physics literature as the Cheng-Wu theorem. In this paper, we shall always take $\lambda$ to be a single edge, and set $D = 4 - 2\varepsilon$.

2.1. Primitive divergent graphs. We say that $G$ is broken primitive divergent if it has exactly 2 external legs, contains no divergent subgraphs, and satisfies:

$$
L = 2h + 1.
$$

In this case, let $q$ denote the momentum entering or leaving each external leg. By (7), we have $V_G = V_G q^2$, where $V_G$ is a polynomial in the $\alpha_i$ and does not depend on $q$. Up to gamma-factors, we can therefore rewrite (11) as the integral:

$$
I(a_1, \ldots, a_L, \varepsilon) = q^{h(2-\varepsilon)-a} \int_{H_\lambda} \frac{\Omega_L}{U_G V_G} \prod_{\ell=1}^L \left( \frac{\alpha_\ell U_G}{V_G} \right)^{a_\ell-1} \left( \frac{U_G^{h+1}}{V_G^{h+1}} \right)^{\varepsilon},
$$

which converges for all Re $a_\ell > 0$. As is customary, we set

$$
a_\ell = 1 + n_\ell \varepsilon, \quad \text{for } 1 \leq \ell \leq L,
$$

where $n_\ell$ are positive integers. Before taking the Taylor expansion with respect to $\varepsilon$, we first express the integrand of (13) as an integral of a simpler function.

When $G$ is broken primitive divergent, we can close up the two external legs of $G$ to form a graph $\tilde{G}$ with $h+1$ loops, and $L+1 = 2(h+1)$ edges (see figure 1, right). In \cite{4}, such graphs were called primitive divergent, and one verifies that

$$
\tilde{U}_G = V_G + \alpha_{L+1} U_G,
$$

where $\alpha_{L+1}$ is the parameter attached to the edge of $\tilde{G}$ obtained by gluing the external legs of $G$ together (see §2.2 below). The following lemma is nothing other than the definition of Euler’s beta function.

**Lemma 6.** Let $0 < r < s$, and $u, v \neq 0$. Then

$$
\int_0^\infty \frac{x^{r-1}}{(u x + v)^s} dx = \frac{1}{u^r v^{s-r}} \frac{\Gamma(r) \Gamma(s-r)}{\Gamma(s)}.
$$

On comparison with (11), we set $r = (h+1)\frac{D}{2} - a$, and $s = \frac{D}{2}$. Substituting (12) and (14) into the expression for $r$ leads us to define:

$$
a_{L+1} = 1 - (h + 1 + \sum_{i=1}^L n_i) \varepsilon.
$$

Therefore, using the previous lemma, we can rewrite the integral (13) as the product of certain explicit gamma factors with the following integral:

$$
I(a_1, \ldots, a_{L+1}, \varepsilon) = \int_{H_\lambda} \prod_{\ell=1}^{L+1} \frac{\alpha^{a_{\ell}-1}_\ell}{U_G^{2-\varepsilon}} \Omega_{L+1},
$$

where $\Omega_L = \sum_{i=1}^L (-1)^i \alpha_i d\alpha_1 \ldots d\alpha_L$, and $H_\lambda = \{ \alpha_i : \sum_{i \in \lambda} \alpha_i = 1 \}$. The fact that the integral does not depend on the choice of subset $\lambda$ follows from the fact that it is really a projective integral, but is known in the physics literature as the Cheng-Wu theorem. In this paper, we shall always take $\lambda$ to be a single edge, and set $D = 4 - 2\varepsilon$. **Lemma 6.** Let $0 < r < s$, and $u, v \neq 0$. Then

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$$

where $\Omega_L = \sum_{i=1}^L (-1)^i \alpha_i d\alpha_1 \ldots d\alpha_L$, and $H_\lambda = \{ \alpha_i : \sum_{i \in \lambda} \alpha_i = 1 \}$. The fact that the integral does not depend on the choice of subset $\lambda$ follows from the fact that it is really a projective integral, but is known in the physics literature as the Cheng-Wu theorem. In this paper, we shall always take $\lambda$ to be a single edge, and set $D = 4 - 2\varepsilon$. **Lemma 6.** Let $0 < r < s$, and $u, v \neq 0$. Then

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$$
a_{L+1} = 1 - (h + 1 + \sum_{i=1}^L n_i) \varepsilon.
$$

Therefore, using the previous lemma, we can rewrite the integral (13) as the product of certain explicit gamma factors with the following integral:
where \( \Omega_{L+1} = \Omega_L d\alpha_{L+1} \), and \( H_\lambda \) is the hyperplane \( \alpha_\lambda = 1 \) in \( [0, \infty]^{L+1} \). The Taylor expansion of the original Feynman integral (11) can be retrieved from the following well-known formula for the Taylor expansion of the gamma function:

\[
\Gamma(s + 1) = \exp(-\gamma s) \exp \left( \sum_{k=2}^{\infty} (-1)^k \zeta(k) s^k \right).
\]

The coefficients in the Taylor expansion of (17) are given by:

\[
\int_{H_\lambda} \frac{\Omega_{L+1}}{U_G^2} P(\alpha_i, \log(\alpha_i), \log(U_G)) ,
\]

where \( m \) is an integer, and \( P \) is a polynomial in \( \alpha_i, \log(\alpha_i) \) for \( 1 \leq i \leq L + 1 \) and \( \log U_G \) with rational coefficients. This is a period integral in the sense of [13].

In the first half of this paper, we will state a criterion on \( \tilde{G} \) for (18) to evaluate to multiple zeta values, and in the second half, we will outline an algorithm for computing these values.

2.2. Contraction and deletion of edges. For any graph \( G \), its graph polynomial \( U_G \) is linear with respect to each variable \( \alpha_i \). We can write

\[
U_G = U_{G/\{i\}} + U_{G\setminus\{i\}} \alpha_i ,
\]

where \( G/\{i\} \) is the graph obtained from \( G \) by contracting the edge labelled \( i \), and \( G\setminus\{i\} \) is the graph obtained from \( G \) by deleting the edge labelled \( i \). Using (15), this implies that

\[
U_G = U_{\tilde{G}\setminus\{L+1\}} \quad \text{and} \quad V_G = U_{\tilde{G}/\{L+1\}} .
\]

From (19) and (20), one deduces a similar formula for \( V_G \), namely:

\[
V_G = V_{G/\{i\}} + V_{G\setminus\{i\}} \alpha_i .
\]

Notation 7. When the graph \( G \) is implicit, we will often write \( U, V \) instead of \( U_G, V_G \). In order to lighten the notation, we adopt the useful convention from [4], which consists in writing, for any polynomial \( \Psi \) which is linear in the variable \( \alpha_i \):

\[
\Psi^{(i)} = \frac{\partial}{\partial \alpha_i} \Psi , \quad \text{and} \quad \Psi_i = \Psi \bigg|_{\alpha_i = 0} .
\]

We therefore have \( U_i = U_{G/\{i\}}, U^{(i)} = U_{G\setminus\{i\}} \), and (19) and (21) become

\[
U = U_i + U^{(i)} \alpha_i \quad \text{and} \quad V = V_i + V^{(i)} \alpha_i .
\]

Since the operations of contracting and deleting distinct edges commute, we can write \( U_{12} = U_{G/\{1,2\}}, U_{12}^{(1)} = U_{G/\{2\}\setminus\{1\}}, U_{12}^{(12)} = U_{G\setminus\{1,2\}} \), and so on, where indices in the superscript (subscript) correspond to deleted (resp. contracted) edges.
3. Symbolic integration using polylogarithms

Let $G$ be a broken primitive divergent graph, and let $\tilde{G}$ be the graph obtained by closing its external legs. To illustrate our integration method, let us begin to compute the integral (13) in the case where all decorations $a_\ell = 1$, and $\varepsilon = 0$. Therefore consider the convergent integral

$$I_G = \int_{H_\lambda} \frac{\Omega_{L+1}}{U_G^2}, \quad \text{(Step 0)}$$

which was studied in [4]. We will assume that $\lambda$ is a single index, say $L$. In this case we can write $\Omega_L = d_0 d_1 d_2 \ldots d_{L-1}$ and $\Omega_{L+1} = \Omega_L d_0 a_{L+1}$. The domain of integration is simply $H_\lambda = \{a_L = 1, 0 \leq a_i \leq \infty, i \neq L\}$. Using equation (15) to replace $U_G$ with $V_G + a_{L+1} U_G$, we can perform one integration with respect to $a_L$ from 0 to $\infty$, to obtain:

$$I_G = \int_{a_L=1}^{\infty} \frac{d_0 d_1 d_2 \ldots d_{L-1}}{(V_G + a_{L+1} U_G)^2} \Omega_L = \int_{H_\lambda} \frac{\Omega_L}{U_G V_G} \quad \text{(Step 1)}.$$

From here on, we write $U, V$ instead of $U_G, V_G$ and use the notations of §2.2. Now choose any variable in the integrand, say $a_1$, and write $I_G$ in terms of $a_1$:

$$I_G = \int_{a_L=1} \frac{\Omega_L}{(U_1 + U^{(1)} a_1)(V_1 + V^{(1)} a_1)}.$$

By decomposing into partial fractions with respect to $a_1$, this gives

$$I_G = \frac{\Omega_L}{(U^{(1)} V_1 - U_1 V^{(1)}) (U^{(1)} V_1 - U_1 V^{(1)})} \left( \frac{U^{(1)}}{U^{(1)} V_1 - U_1 V^{(1)}} - \frac{V^{(1)}}{V_1 + V^{(1)} a_1} \right).$$

Now we can perform an integration with respect to $a_1$ from 0 to $\infty$ at the expense of introducing a new function, namely the logarithm.

$$I_G = \int_{a_L=1} \log U^{(1)} - \log U - \log V^{(1)} + \log V_1 \frac{d_0 d_1 d_2 \ldots d_{L-1}}{U^{(1)} V_1 - U_1 V^{(1)}} \quad \text{(Step 2)}.$$

The logarithm $\log f$ should be regarded as a symbol which satisfies the formal rule $d \log f = f^{-1} df$, since changing the constant of integration does not affect the integrand of (24). At this stage, something remarkable occurs. The denominator factorizes as the square of a polynomial $D$ which is linear in each variable $a_2, \ldots, a_L$:

$$D^2 = U^{(1)} V_1 - U_1 V^{(1)}$$

As observed in [4], this phenomenon is quite general and follows from a result due to Dodgson on determinants of matrices. One can also give a formula for $D$ in terms of trees in $G$, but this will not be required here.

We can repeat this argument. Choose a variable, say $a_2$, and write

$$D = D_2 a_2 + D^{(2)}.$$

By decomposing the integrand of (24) into partial fractions with respect to the variable $a_2$, we can integrate with respect to the variable $a_2$. To simplify the notation, we define

$$\{p, q|r, s\} = \frac{p \log(p) + \log(s) - \log(q) - \log(r)}{r(ps - qr)} + \frac{\log(q)}{rs}.$$
One verifies that
\[ \int_0^\infty \frac{\log(px + q)}{(rx + s)^2} \, dx = \{ p, q \mid r, s \} . \]

**Corollary 8.** The integral \( I_G \) is equal to the \( L - 2 \) dimensional integral:

\[
I_G = \int_{\alpha_{L-1}} \left( \{ U_2^{(1)}, U_1^{(2)} \mid D_2, D^{(2)} \} - \{ U_{12}, U_1^{(2)} \mid D_2, D^{(2)} \} - \{ V_2^{(1)}, V_1^{(2)} \mid D_2, D^{(2)} \} \\
+ \{ V_{12}, V_1^{(2)} \mid D_2, D^{(2)} \} \right) d\alpha_3 \ldots d\alpha_{L-1} \quad \text{(Step 3)} .
\]

As long as there exists a variable with respect to which all polynomials which occur in the integrand are linear, this can be repeated.

At the next stage of the integration process, one has to introduce the dilogarithm \( \text{Li}_2 \), which is formally defined by the differential equation

\[
(27) \quad d\text{Li}_2(f) = -\frac{\log(1 - f)}{f} \, df .
\]

In this manner, we obtain a conditional algorithm for computing integrals such as \( [18] \), which can be approximately formalised as follows:

1. Choose a variable in which all terms of the integrand are linear.
2. Formally take a primitive of the integrand with respect to this variable.
3. Evaluate this primitive at 0 and \( \infty \), and repeat.

The algorithm fails exactly when in (1), we can no longer find a variable with respect to which all terms in the integrand are linear. This will be formulated precisely in the following section. It can be checked in advance whether the algorithm will terminate, as this depends only on the topology of the graph \( G \). When the algorithm does terminate, any integral \( [13] \) with arbitrary decorations can always be computed in a finite number of steps in terms of a fixed differential algebra of polylogarithms determined in advance by the topology of \( G \).

For example, in the above, we can keep track of the singularities of the integrand (i.e., terms occurring in the denominator, or arguments of the logarithm functions) at each step. At the first step, we represent the singularities as the set:

\[ \{ U, V \} . \]

After the second integration, the singularities are given by:

\[ \{ U^{(1)}, U_1, V^{(1)}, V_1, D \} . \]

At the third step (corollary \( [5] \), we have in the same manner

\[ \{ U^{(12)}, U_2^{(1)}, U_1^{(2)}, U_{12}, V^{(12)}, V_2^{(1)}, V_1^{(2)}, V_{12}, D^{(2)}, D_2 \} , \]

along with the set of irreducible factors of the four denominator terms:

\[ D^{(2)}U_2^{(1)} - D_2U^{(12)}, D^{(2)}U_{12} - D_2U_1^{(2)}, D^{(2)}V_2^{(1)} - D_2V^{(12)}, D^{(2)}V_{12} - D_2V_1^{(2)} . \]

We are therefore led to consider an algorithm for the reduction of sets of polynomials with respect to their variables. This will ultimately lead to a criterion for the computability of broken primitive divergent Feynman graphs (theorem \( [18] \)).

---

\(^1\)As we shall see later, it is more convenient to use the function satisfying the differential equation \( dL(f) = f^{-1} \log(f + 1) \, df \), which is a close relative of the dilogarithm.
4. Reduction of Polynomials

Keeping track of the polynomials, or singular loci, which can occur in this integration process gives rise to a reduction algorithm which can be used to check the outcome of the integration process without actually doing it.

4.1. The simple reduction algorithm. Let \( S = \{ f_1, \ldots, f_N \} \), where \( f_1, \ldots, f_N \) are polynomials in the variables \( \alpha_1, \ldots, \alpha_m \), with rational coefficients.

1. Suppose that there exists an index \( 1 \leq r \leq m \) with respect to which every polynomial \( f_1, \ldots, f_N \) is linear in the variable \( \alpha_r \). Then we can write:

\[
    f_i = g_i \alpha_r + h_i \\
    \text{for } \quad 1 \leq i \leq N,
\]

where \( g_i = \partial f_i / \partial \alpha_r \), and \( h_i = f_i|_{\alpha_r=0} \). Define a new set of polynomials:

\[
    \overline{S}(r) = \{ (g_i)_{1 \leq i \leq N}, (h_i)_{1 \leq i \leq N}, (h_i g_j - g_i h_j)_{1 \leq i < j \leq N} \}.
\]

2. Let \( S(r) \) be the set of irreducible factors of polynomials in \( \overline{S}(r) \).

The polynomials now occurring in \( S(r) \) are functions of one fewer variables, namely \( \alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_m \). This process can be repeated. If at each stage there exists a variable in which all polynomials are linear, we can proceed to the next stage. This gives a sequence of variables \( \alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_n} \), and a sequence of sets

\[
    S_{(r_1)}, S_{(r_1, r_2)}, \ldots, S_{(r_1, \ldots, r_n)}.
\]

If there exists a sequence \( (r_1, \ldots, r_m) \) such that every variable is eventually eliminated, then we say that the reduction terminates. When this happens, we say that the set \( S \) is simply reducible.

Remark 9. We can remove any constants, and any monomials of the form \( \alpha_i \) which occur as elements in \( S(r) \), as this does not affect the outcome of the algorithm.

Definition 10. Let \( G \) be a broken primitive divergent graph, and let

\[
    S_G = \{ U_G \}.
\]

We say that \( G \) is simply reducible if \( S_G \) is simply reducible.

Observe that \( \{ U_G \}_{L+1} = \{ U_G, V_G \} \), where \( \alpha_{L+1} \) is the edge variable of \( \bar{G} \) obtained by closing the external legs of \( G \), by equation (4).

Example 11. Consider the 2-loop 2-point graph \( G \) depicted in fig. 1. We write \( S = S_G = \{ U_G \} \). After reducing with respect to \( \alpha_6 \), we have \( S_{(6)} = \{ U, V \} \), where \( U = U_G, V = V_G \) are given by (8):

\[
    S_{(6)} = \{ \alpha_1 \alpha_2 + \alpha_1 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 \},
\]

\[
    \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_3 \alpha_5 + \alpha_3 \alpha_5 \}.
\]

Since both polynomials are linear in \( \alpha_1 \), we can reduce with respect to \( \alpha_1 \) to obtain

\[
    \overline{S}_{(6, 1)} = \{ U^{(1)}, U_1, V^{(1)}, V_1, U^{(1)} V_1 - U_1 V^{(1)} \}.
\]

But \( U^{(1)} V_1 - U_1 V^{(1)} \) factorizes, by the Dodgson identity, and so at the second stage we have \( S_{(6, 1)} = \{ U^{(1)}, U_1, V^{(1)}, V_1, D \} \), where \( D^2 = (U^{(1)} V_1 - U_1 V^{(1)} ) \):

\[
    S_{(6, 1)} = \{ \alpha_2 \alpha_3 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_4 \alpha_5 + \alpha_4 \alpha_5 + \alpha_4 \alpha_5 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 + \alpha_5 \alpha_6 \}.
\]
Since each polynomial is linear, we can reduce with respect to the variable α₂:

\[ S_{(6,1,2)} = \{ α_4 + α_5 , α_3α_4 + α_3α_5 + α_4α_5 , α_3 + α_4 , α_3 + α_5 , α_3 - α_4 \} . \]

Next, reducing with respect to the variable α₅, gives

\[ S_{(6,1,2,5)} = \{ α_3 + α_4 , α_3 - α_4 \} , \]

and finally, reducing with respect to α₄ gives

\[ S_{(6,1,2,5,3)} = \emptyset . \]

Here, and from now on, we will adopt the convention (remark 9) that we remove all constant terms and variables αᵢ from each stage of the reduction.

In §3, the domain of integration for \( I_G \) was a hyperplane \( α₃ = 1 \) for some index λ. If \( S_G \) is simply reducible with respect to some order \((α_1, \ldots, α_m)\) of the variables, we can choose λ to be the index of one of the two final variables \( α_{r_n-1}, α_{r_n} \). In the previous example, we can take \( λ = 4 \), and set \( α_4 = 1 \), \( S' = S_G|_{α_4 = 1} \). This gives

\[ S'_{(6,1,2)} = \{ α_5 + 1 , α_3 + α_5 + α_3α_5 , α_3 + 1 , α_3 + α_5 , α_3 - 1 \} , \]

and

\[ S'_{(6,1,2,5)} = \{ α_3 + 1 , α_3 - 1 \} . \]

The reduction algorithm reflects the set of singularities which occur at each stage of the integration process. Roughly speaking, the integrand at the \( k^{th} \) stage, after integrating with respect to \( α_{r₁}, \ldots, α_{rₖ} \), will have singularities along the zero locus of polynomials in the sets \( S_{(r₁, \ldots, rₖ)} \), for \( m ≥ k \), and along the axes \( α_i = 0 \).

**Remark 12.** The algorithm described above is similar to an algorithm defined by Stembridge to study the zeros of graph polynomials over finite fields. A similar argument was also used in [4] to obtain a Tate filtration on graph hypersurfaces.

In the previous example, we see that at the penultimate stage, we expect to have polylogarithms in \( α₃ \), with singularities along \( α₃ = 0, α₃ = -1, α₃ = 1 \). This would imply that at the final stage, the integral \( I_G \) is a rational linear combination of alternating multiple sums \( Z^2 \), i.e., periods of \( \mathbb{P}^1 \setminus \{0, -1, 1, \infty\} \). This is not good enough, since it is known that the integral \( I_G \) in fact gives multiple zeta values. To rectify this problem, we introduce the Fubini reduction algorithm.

### 4.2. The Fubini reduction algorithm

In the simple reduction algorithm, the sets \( S_{(r₁, \ldots, rₖ)} \) which control the singularities of the integration process, depend in an essential way on the order of the variables \( r₁, \ldots, rₖ \) which was chosen. However, it is an obvious consequence of Fubini’s theorem that the final integral does not depend on the particular order of integration. More precisely, if \( F \) is the integrand obtained at the \( k^{th} \) stage, we clearly have

\[ \int_0^∞ dα_{r₁} \int_0^∞ dα_{r₂} F = \int_0^∞ dα_{r₃} \int_0^∞ dα_{r₄} F . \]

The left-hand integral has singularities contained in the zero locus of polynomials in \( S_{(r₁, \ldots, r₄)} \); the right-hand integral has singularities contained in the zero locus of polynomials in \( S_{(r₁, \ldots, r₃, r₄)} \). It follows that both have singularities contained in the zero locus of polynomials in the intersection \( S_{(r₁, \ldots, r₃, r₄)} \cap S_{(r₁, \ldots, r₃, r₄)} \).
We compute the Fubini reduction algorithm for the two-loop two-point function, as in example 11. Let
\[
\alpha = \alpha_3 + \alpha_4 + \alpha_1 + \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4, \quad \alpha_1 + \alpha_3, \quad 2 \alpha_3 \alpha_4 + \alpha_3^2 + \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4 + \alpha_3^2
\]

and \(S_{[2,6][5]}\) by
\[
\{\alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_1, \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4, \alpha_1 + \alpha_3, \alpha_1 \alpha_3 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4, \alpha_1 + \alpha_3, \alpha_1 \alpha_4, \alpha_1 + \alpha_4, \alpha_1 + \alpha_4, \alpha_1 - \alpha_4\}
\]

Taking the intersection of all three sets gives
\[
S_{[2,5][6]} = \{\alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_1, \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_1 + \alpha_4\}
\]

By performing an ordinary reduction with respect to the variable \(\alpha_1\), one obtains:
\[
S_{[2,5][6][1]} = \{\alpha_3 + \alpha_4, \alpha_3 \alpha_4 + \alpha_3^2 + \alpha_4^2\}
\]

By computing \(S_{[6,1,2][5]}, S_{[6,1,5][2]}, S_{[1,2,5][6]}\) and intersecting all four (actually it suffices to intersect with \(S_{[6,1,2,5]}\) from example 11 in this case), one verifies that
\[
S_{[6,1,2,5]} = \{\alpha_3 + \alpha_4\}.
\]
As before, we can take our domain of integration to be the hyperplane $\alpha_4 = 1$. At the final stage of the integration process, the Fubini reduction algorithm will predict (see theorem 18) that the integrand has singularities in $\alpha_3 = 0$, $\alpha_3 = 1$. Therefore we expect to obtain multiple polylogarithms with singularities in $\{0, 1, \infty\}$ at the penultimate stage, and hence multiple zeta values as the final answer. In this way, the problem raised at the end of the previous section has apparently been overcome.

However, constants can appear at every stage during the integration process (see §7), and one needs to verify that we only obtain multiple zeta values every time. This leads to a further ramification condition to be verified, for each set $S_{[r_1, \ldots, r_k]}$.

### 4.3. The ramification condition

Let $S$ be Fubini reducible for some order $(r_1, \ldots, r_m)$ of the variables. After setting the final variable $\alpha_{r_m} = 1$ in the sequence (30), we obtain a new sequence of sets (the reductions of $S' = S|_{\alpha_{r_m}=1}$):

$$(31) \quad S'_{[r_1]} \ , \ S'_{[r_1,r_2]} \ , \ \ldots \ , \ S'_{[r_1,\ldots,r_{m-1}]} \ ,$$

where every polynomial in $S'_{[r_1,\ldots,r_k]}$ is linear in the variable $\alpha_{r_{k+1}}$. Therefore, if we write $S'_{[r_1,\ldots,r_k]} = \{f_1, \ldots, f_{M_k}\}$, then we have

$$f_i = a_i \alpha_{r_{k+1}} + b_i \ , \ \text{for} \ 1 \leq i \leq M_k \ ,$$

where $a_i, b_i$ are polynomials in $\alpha_{r_{k+2}}, \ldots, \alpha_{r_m}$. We define $\Sigma_{\alpha_k}$ to be

$$(32) \quad \Sigma_{\alpha_k} = \{-\frac{b_i}{a_i} \ \text{such that} \ a_i \neq 0\} \ .$$

The set $\Sigma_{\alpha_k}$ clearly depends on the ordering of the variables $(r_1, \ldots, r_m)$.

**Definition 16.** We say that $\Sigma_{\alpha_k}$ is unramified if:

$$(33) \quad \lim_{\alpha_{r_m} \to 0} \left( \lim_{\alpha_{r_{m-1}} \to 0} \left( \ldots \left( \lim_{\alpha_{r_{k+2}} \to 0} \Sigma_{\alpha_k} \right) \ldots \right) \right) \subseteq \{0, -1, \infty\} \ .$$

We say that the sequence of sets (30) is unramified if the corresponding $\Sigma_{\alpha_k}$ are unramified for all $1 \leq k \leq m-1$. Finally, we say that $S$ is unramified if it is Fubini reducible, and if there exists a sequence $(r_1, \ldots, r_m)$ such that (31) is unramified.

**Example 17.** One can check that the sequence of sets given in example 15 is unramified. We check the last two terms only. Setting $\alpha_4 = 1$, we have:

$$S'_{[6,2,5]} = \{1 + \alpha_3 , \ 1 + \alpha_1 + \alpha_3 \ , \ \alpha_1 + \alpha_3 + \alpha_1 \alpha_3 \ , \ \alpha_1 + \alpha_3\} \ , \ S'_{[6,1,2,5]} = \{1 + \alpha_3\} \ .$$

Thus

$$\Sigma_{\alpha_1} = \{-\alpha_3 + 1\} , \ -\frac{\alpha_3}{1 + \alpha_3} , \ -\alpha_3 \} , \ \text{and} \ \Sigma_{\alpha_3} = \{-1\} \ .$$

The ramification condition (33) is satisfied in both cases, since $\Sigma_{\alpha_3} \subset \{0, -1, \infty\}$, and $\lim_{\alpha_{r_m} \to 0} \Sigma_{\alpha_1} = \{0, -1\}$.

### 4.4. The main theorem

The main theorem gives a simple criterion for a master Feynman integral to evaluate to multiple zeta values.

**Theorem 18.** Let $G$ be a broken primitive divergent Feynman diagram and let $S_G = \{U_G\}$. If $S_G$ is Fubini reducible and unramified, then the coefficients in the Taylor expansion of the integral $I(G)$ (17) are rational linear combinations of multiple zeta values.
We also require a variant to allow for multiple zeta values ramified at roots of unity. Let $S$ be Fubini reducible, as above. We say that $S$ is ramified at $p^\text{th}$ roots of unity if there exists a sequence $(r_1, \ldots, r_m)$ such that the corresponding sets $\Sigma_{\alpha_k}$ satisfy

$$\Sigma_{\alpha_k} \subseteq \{0, \infty\} \cup \{-\omega : \omega^p = 1\}.$$ 

If $p = 1$, then this coincides with the definition 16 above.

**Theorem 19.** Let $G$ be a broken primitive divergent Feynman diagram and let $S_G = \{U_G\}$. If $S_G$ is Fubini reducible and ramified at $p^\text{th}$ roots of unity, then the coefficients in the Taylor expansion of the integral $I(G)$ are rational linear combinations of multiple zeta values ramified at $p^\text{th}$ roots of unity, i.e., in $\mathbb{Z}_p$.

The results stated in the introduction are the result of applying these two theorems to the set of all primitive divergent graphs up to 5 loops. We computed the Fubini reduction algorithm for the cross-hairs diagram (for which the graph polynomial has 45 terms), the graphs $5R$, $5P$ and $5N$ (for which it has 128, 130 and 135 terms, respectively), and all eight primitive divergent graphs with 6 loops and crossing number 0 and 1, as depicted in figures 6 and 7 (for which the graph polynomials have around 400 terms). We found that all planar bpd graphs up to 5 loops are Fubini reducible and unramified, and all non-planar bpd graphs with crossing number 1 are Fubini reducible but ramified at $6^\text{th}$ roots of unity. The bipartite graph $K_{3,4}$ has crossing number 2 (figure 8), and is not Fubini reducible. These calculations were done using maple.

The outcome of the Fubini reduction algorithm may depend on the particular choice of two final variables. Typically, the ramification is better if one chooses the two final variables to correspond to edges meeting at a four-valent vertex. In each example, we computed the algorithm for all possible pairs of final variables, and chose a pair giving the least ramification. In an ideal world, one could simply take the intersection of the period rings obtained for every such choice, and the algorithm would in that case not depend on any choices. In general, however, taking the intersection of rings of periods requires some powerful diophantine results, which are at present totally out of reach.

**4.5. Extensions.** There are a number of ways in which the reduction algorithm might be improved. First of all, in our computer calculations, we factorized our polynomials over $\mathbb{Q}$, rather than over the algebraic closure $\overline{\mathbb{Q}}$, which may or may not have made a significant difference.

More interestingly, the Fubini reduction method stops as soon as it finds that, for every variable $\alpha_i$, there is a polynomial which is quadratic in that variable. It is possible, by introducing new variables which are the square roots of discriminants (by passing to a ramified cover), one could generalize the method to deal with some of the quadratic terms. The basic idea is that, in the case of a plane curve for example, a polynomial need not necessarily be linear to define a curve of genus 0. A conic, of degree 2, also defines a curve of genus 0, and our method should also extend to this case. A genuine obstruction should occur at the degree 3 level, since one expects to find elliptic components in this case. In any case, by extending our method to deal with quadratic terms, a larger class of graphs may then become reducible, and amenable to computation.

Finally, if one is interested only in the leading term of the Taylor expansion of the graph $I(G)$, then one can simplify the reduction algorithm. Recall that, for the
general integral (with arbitrary logarithms in the numerator), we obtained at the second step, the set of singularities:

$$S_{[L+1,1]} = \{U^{(1)}, U_1, V^{(1)}, V_1, D\}.$$  

Notice that in (24), where the numerator has no logarithmic terms, not all pairs of singularities can occur. Thus one should perform a Fubini reduction separately on the four sets

$$\{U^{(1)}, D\}, \{U_1, D\}, \{V^{(1)}, D\}, \{V_1, D\},$$

which may improve the ramification. There may also exist graphs which are not Fubini reducible, but for which each of the four sets above is Fubini reducible. This would prove that the first term only in the Taylor expansion is a multiple zeta value.

In the remainder of the paper we explain how to compute the coefficients in the Taylor expansion when the conditions of theorems 18 and 19 hold.
5. Hyperlogarithms, polylogarithms, and primitives

In this section, we outline the function theory underlying the iterated integration procedure. For a more detailed introduction, see [6, 7, 8].

5.1. Hyperlogarithms. Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$, where $\sigma_i$ are distinct points of $\mathbb{C}$. We will always assume that $\sigma_0 = 0$. For now we will consider the points $\sigma_i$ to be stationary, but later we will allow them to be variables in their own right.

Let $A = \{a_0, a_1, \ldots, a_N\}$ denote an alphabet on $N+1$ letters, where each symbol $a_i$ corresponds to the point $\sigma_i$. Let $\mathbb{Q}(A)$ denote the vector space generated by all words $w$ in the alphabet $A$, along with the empty word which we denote by $e$. To each such word $w$, we associate a hyperlogarithm function:

$$L_w(z) : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}$$

which is multivalued, i.e., it is a meromorphic function on the universal covering space of $\mathbb{C} \setminus \Sigma$. Let $\log(z)$ denote the principal branch of the logarithm.

**Definition 20.** Let $A^\times$ denote the set of all words $w$ in $A$, including $e$. The family of functions $L_w(z)$ is uniquely determined by the following properties:

1. $L_e(z) = 1$, and $L_{a^N}(z) = \frac{1}{N!} \log^n(z)$, for all $n \geq 1$.
2. For all words $w \in A^\times$, and all $0 \leq i \leq N$,

$$\frac{\partial}{\partial z} L_{a_i w}(z) = \frac{1}{z-\sigma_i} L_w(z), \quad \text{for } z \in \mathbb{C} \setminus \Sigma.$$  

3. For all words $w \in A^\times$ not of the form $w = a_0^n$,

$$\lim_{z \to 0} L_w(z) = 0.$$  

The weight of the function $L_w(z)$ is defined to be the number of letters which occur in $w$. The functions $L_w(z)$ are defined inductively by the weight: if $L_w(z)$ has already been defined, then $L_{a_i w}(z)$ is uniquely determined by the differential equation (2) since the constant of integration is given by (3).

It follows from the definitions that

$$L_{a_i}(z) = \log(z - \sigma_i) - \log(\sigma_i).$$

Later, we will only consider linear combinations of functions $L_w(z)$ which are single-valued on the real interval $(0, \infty)$. Any such function is uniquely defined on $(0, \infty)$ after having fixed the branch of the logarithm function $L_{a_0}(z) = \log(z)$.

5.1.1. The shuffle product. The shuffle product, denoted $\shuffle$, is a commutative multiplication law on $\mathbb{Q}(A)$ defined inductively by the formulae:

$$w \shuffle e = e \shuffle w = w \quad \text{for all } w \in A^\times,$$

$$a_i w_1 \shuffle a_j w_2 = a_i (w_1 \shuffle a_j w_2) + a_j (a_i w_1 \shuffle w_2) \quad \text{for all } w_1, w_2 \in A^\times, a_i, a_j \in A.$$

We extend the definition of the functions $L_w(z)$ to $\mathbb{Q}(A)$ by linearity:

$$L_w(z) = \sum_{i=1}^m q_i L_{w_i}(z) \quad \text{where } w = \sum_{i=1}^m q_i w_i, q_i \in \mathbb{Q}, \ w_i \in A^\times.$$

The following lemma is well-known.

**Lemma 21.** The functions $L_w(z)$ satisfy the shuffle relations:

$$L_{w_1}(z)L_{w_2}(z) = L_{w_1 \shuffle w_2}(z), \quad \text{for all } w_1, w_2 \in \mathbb{Q}(A), \quad \text{and } z \in \mathbb{C} \setminus \Sigma.$$
Definition 22. Now let us define
\[ \mathcal{O}_\Sigma = \mathbb{Q}\left[\frac{1}{z - \sigma_0}, \ldots, \frac{1}{z - \sigma_N}\right], \]
and let \( L(\Sigma) \) be the \( \mathcal{O}_\Sigma \)-module generated by the functions \( L_w(z) \), for \( w \in A^\times \).

The shuffle product makes \( L(\Sigma) \) into a commutative algebra. It is a differential algebra for the operator \( \partial / \partial z \), and is graded by the weight.

Theorem 23. The functions \( \{L_w(z)\}_{w \in A^\times} \), are linearly independent over \( \mathbb{C} \otimes \mathbb{Q} \mathcal{O}_\Sigma \).

It follows from the theorem that \( L(\Sigma) \) is a polynomial ring, and a convenient polynomial basis is given by the functions \( L_w(z) \), where \( w \) are Lyndon words \([14]\).

In order to take primitives in \( L(\Sigma) \), we have to enlarge the ring of coefficients slightly. Therefore we must consider
\[ \mathcal{O}_\Sigma^+ = \mathcal{O}_\Sigma\left[\sigma_i, \frac{1}{\sigma_i - \sigma_j}, 0 \leq i < j \leq N\right], \]
and let \( L^+(\Sigma) = \mathcal{O}_\Sigma^+ \otimes \mathcal{O}_\Sigma L(\Sigma) \) be \( \mathcal{O}_\Sigma^+ \)-module spanned by \( L_w(z) \), for \( w \in A^\times \).

Theorem 24. Every element \( f \) in \( L(\Sigma) \) of weight \( n \) has a primitive in \( L^+(\Sigma) \) of weight at most \( n + 1 \), i.e., an element \( F \in L^+(\Sigma) \) such that
\[ \partial F / \partial z = f. \]

The primitive of any generator \( f(z)L_w(z) \in L(\Sigma) \), where \( f(z) \in \mathcal{O}_\Sigma \), can be found explicitly by decomposing \( f(z) \) into partial fractions in \( \mathcal{O}_\Sigma^+ \). The fact that one must enlarge \( \mathcal{O}_\Sigma \) to \( \mathcal{O}_\Sigma^+ \) is clear from the identity:
\[ \frac{1}{(z - \sigma_i)(z - \sigma_j)} = \frac{1}{\sigma_i - \sigma_j}\left(\frac{1}{z - \sigma_i} - \frac{1}{z - \sigma_j}\right). \]

One is thereby reduced to the case of finding a primitive of functions of the form \( (z - \sigma_i)^nL_w(z) \), where \( n \in \mathbb{Z} \).

In the case \( n = -1 \), a primitive is given by \( L_{\alpha_i}(z) \) by definition (2) above. In all other cases, integration by parts enables one to reduce the case of finding a primitive of a function of lower weight. Thus a primitive of \( f(z)L_w(z) \) can be computed algorithmically in at most \( n \) steps, where \( n \) is the weight of \( L_w(z) \).

5.1.2. Logarithmic regularization at infinity. Having fixed a branch of the logarithm \( L_{\alpha_i}(z) \) we can define the regularization of hyperlogarithms at infinity.

Proposition 25. Every function \( f(z) \in L(\Sigma) \) can be uniquely written in the form
\[ f(z) = \sum_{i=0}^{m} f_i(z) \log^i(z), \]
where \( f_i(z) \) is holomorphic at \( z = \infty \), for \( 0 \leq i \leq m \).

We can therefore define the regularized value of \( f \) at infinity to be:
\[ \text{Reg}_{z=\infty} f(z) = f_0(\infty). \]

Clearly, \( \text{Reg}_{z=\infty} L_{\alpha_i}(z) = 0 \), and for all \( 1 \leq i \leq N, \text{Reg}_{z=\infty} L_{\alpha_i}(z) = -\log(\sigma_i). \)

Note that the regularization operator \( \text{Reg}_{z=\infty} \) respects multiplication.
There is an analogous notion of regularization at \( z = 0 \). Note that by the definition of the functions \( L_w(z) \) (properties (1) and (3)), we always have
\[
\text{Reg}_{z=0} L_w(z) = 0, \quad \text{for all } w \in A^\times, \: w \neq e.
\]

**Definition 26.** Suppose that \( f(z) \in L(\Sigma) \) is holomorphic on the real interval \((0, \infty)\), and that \( f(z) \, dz \) has at most logarithmic singularities at \( z = 0, \infty \). We define the regularized integral of \( f(z) \, dz \) along \([0, \infty]\) to be:
\[
\int_0^\infty f(z) \, dz = \text{Reg}_{z=\infty} F(z) - \text{Reg}_{z=0} F(z),
\]
where \( F(z) \in L(\Sigma) \) is a primitive of \( f(z) \). The integral converges. In practice, we write \( F(z) \) in terms of the basis of functions \( L_w(z) \), and choose the constant term to be zero. It will follow that \( \text{Reg}_{z=0} F(z) \) vanishes, and the integral is simply given by \( \text{Reg}_{z=\infty} F(z) \) in this case.

It is clear that the regularized integral is additive. The point is that, in order to compute an integral which is convergent, one is allowed to break it into a sum of logarithmically divergent pieces, compute the regularized integral of each, and add the answers together. This is illustrated in \[17a2\]

5.1.3. **Multiple Zeta Values.** Consider the case \( N = 1 \), and \( \sigma_0 = 0, \: \sigma_1 = -1 \). After a change of variables \( z \mapsto -z \), we retrieve the classical situation on \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \).

The functions \( L_w(-z) \) are therefore multiple polylogarithms in one variable \[7\], and we can compute the regularized values at infinity in terms of multiple zeta values using standard methods. To do this, let \( X = \{x_0, x_1\} \) denote an alphabet with two letters, and let \( Z(x_0, x_1) \) be Drinfeld’s associator:
\[
Z(x_0, x_1) = \sum_{w \in X^\times} \zeta_{\Xi}(w) \cdot w,
\]
where the numbers \( \zeta_{\Xi}(w) \in \mathbb{R} \) satisfy:
\[
\zeta_{\Xi}(x_0) = \zeta_{\Xi}(x_1) = 0, \quad \text{and} \quad \zeta_{\Xi}(x_0^{n_2-1} x_1 \ldots x_0^{n_{k-1}} x_1) = \zeta(n_1, \ldots, n_k),
\]
for all integers \( n_1, \ldots, n_k \geq 1 \) such that \( n_k \geq 2 \). Extending \( \zeta_{\Xi}(w) \) by linearity to \( \mathbb{Q}\langle X \rangle \), we have the shuffle relations
\[
\zeta_{\Xi}(w_1) \zeta_{\Xi}(w_2) = \zeta_{\Xi}(w_1 \shuffle w_2) \quad \text{for all } w_1, w_2 \in X^\times.
\]
The numbers \( \zeta_{\Xi}(w) \) are uniquely determined by these properties. Using the fact that \( \zeta(1, 2) = \zeta(3) \), one can verify that, up to weight three:
\[
Z(x_0, x_1) = 1 + \zeta(2)(x_0 x_1 - x_1 x_0) + \zeta(3)(x_0^2 x_1 + x_0 x_1^2 + x_1 x_0^2 + x_1^2 x_0 - 2x_0 x_1 x_0 - 2x_1 x_0 x_1) + \ldots
\]

**Lemma 27.** Let \( \Sigma = \{0, -1, \infty\} \), and let \( w \in X^\times \). Set
\[
\zeta_{\infty}(w) = \text{Reg}_{z=\infty} L_w(z).
\]
Then the generating series of regularized values at infinity is given by:
\[
\sum_{w \in X^\times} \zeta_{\infty}(w) \cdot w = Z^{-1}(x_1 - x_0, -x_0).
\]

A simple computation shows that:
\[
Z^{-1}(x_1 - x_0, -x_0) = 1 + \zeta(2)(x_1 x_0 - x_0 x_1) + \zeta(3)(x_0 x_1^2 + x_1 x_0^2 - 2x_1 x_0 x_1) + \ldots
\]
which allows us to read off the values of \( \zeta_{\infty}(w) \) for \( w \) up to weight 3.
5.2. An algebra of polylogarithms for reducible graphs. The integration process takes place in an algebra of polylogarithm functions in several variables. A basis of such functions will be products of hyperlogarithms \( L_{w_1}(z_1) \ldots L_{w_k}(z_k) \), each of which is viewed as a function of a single variable \( z_i \), where \( z_i \) is some Feynman parameter. Now, however, the singularities \( \sigma_0, \ldots, \sigma_N \) will themselves be rational functions in the remaining Feynman parameters \( \alpha_1, \ldots, \alpha_k \).

Let \( S = \{ f_1, \ldots, f_M \} \) denote a Fubini reducible set of polynomials in the variables \( \alpha_1, \ldots, \alpha_N \). We can assume that the algorithm of \( \text{[4.2]} \) terminates when we reduce \( S \) with respect to the variables \( \alpha_1, \ldots, \alpha_N \) in that order. We define a nested sequence of rings \( R_N \subset R_{N-1} \subset \ldots \subset R_1 \) recursively, as follows. Set \( R_{N+1} = \mathbb{Q} \) and define:

\[
R_k = R_{k+1} \left[ \alpha_k, \frac{1}{\alpha_k}, \frac{1}{f} : f \in S_{[\alpha_1, \alpha_2, \ldots, \alpha_k]} \right], \quad \text{for } 2 \leq k \leq N.
\]

The fibration algorithm ensures that every \( f \in S_{[\alpha_1, \alpha_2, \ldots, \alpha_k]} \) is linear in \( \alpha_k \). Thus, if \( \{ f_1, \ldots, f_{m_k} \} \) is the set of elements in \( S_{[\alpha_1, \alpha_2, \ldots, \alpha_k]} \) whose coefficient of \( \alpha_k \) is non-zero, we can write:

\[
 f_i = a_i \alpha_k + b_i, \quad \text{where } a_i, b_i \text{ are invertible in } R_{k+1},
\]

for all \( 1 \leq i \leq m_k \). As in \( \text{[4.2]} \) we set

\[
 \Sigma_k = \{ \sigma_1, \ldots, \sigma_{m_k} \}, \quad \text{where } \sigma_i = -\frac{b_i}{a_i},
\]

and with these notations, we can write:

\[
 R_k = R_{k+1} \left[ \alpha_k, \frac{1}{\alpha_k - \sigma_i}, \sigma_i \in \Sigma_k \right]
\]

We refer the reader to \( \text{[7.1]} \) for a worked example in the case of the wheel with 3 spokes diagram.

**Definition 28.** Let \( L(R_N) = R_N \), and define inductively

\[
 L(R_k) = L(\Sigma_k) \otimes L(R_{k+1}) \quad \text{for } 1 \leq k \leq N.
\]

Every element of \( L(R_k) \) can be represented as a sum of terms of the form:

\[
 \phi = f(\alpha_k, \ldots, \alpha_N) L_{w_1}(\alpha_k) L_{w_2}(\alpha_{k+1}) \ldots L_{w_{N-k+1}}(\alpha_N),
\]

where \( f \in R_k \) is a rational function of \( \alpha_k, \ldots, \alpha_N \), and \( L_{w_i}(\alpha_{k+i-1}) \in L(\Sigma_{k-i+1}) \) is a hyperlogarithm in \( \alpha_{k+i-1} \), whose set of singularities \( \Sigma_{k-i+1} \) are rational functions of the higher variables \( \alpha_{k+i}, \ldots, \alpha_N \).

**Definition 29.** The weight of the element \( \phi \) is \( |w_1| + \ldots + |w_N| \), where \( |w| \) denotes the number of letters in a word \( w \).

Every element \( \phi \in L(R_k) \) is a multivalued function on a certain open subset of \( \mathbb{C}^{N-k} \). When we integrate, we will only consider elements \( \phi \) which are holomorphic, and hence single-valued, on the real hypercube \( (0, \infty)^{N-k} \subset \mathbb{C}^{N-k} \).

---

2 We use the word hyperlogarithm to denote a function \( L_w(z) \), considered as a function of the single variable \( z \), with constant singularities \( \sigma_i \in \mathbb{C} \). In contrast, we use the word polylogarithm to denote \( L_w(z) \) also, but this time viewed as a function of several variables \( z \) and \( \alpha_k, \ldots, \alpha_N \) where some of the singularities \( \sigma_i \) depend on the \( \alpha_k, \ldots, \alpha_N \).
In this manner, we have defined an explicit algebra of polylogarithm functions associated to a reducible Feynman graph \( G \). Its elements can be represented symbolically as a linear combination of products of words. We next show how to compute any integral of the type \( (4) \) associated to \( G \) by working inside this algebra.

5.3. Existence of primitives. The integration process requires finding primitives at each stage. As explained earlier, we need to enlarge the coefficients of the algebra \( L(R_k) \) slightly in order to do this. In the above notation, we set

\[
R_k^+ = R_k \left[ \frac{1}{\sigma_i - \sigma_j}, \text{for } \sigma_i, \sigma_j \in \Sigma_k \text{ distinct} \right].
\]

The following theorem establishes the existence of primitives.

**Theorem 30.** Let \( f \in L(R_k) \) denote a function of weight \( n \) in the variables \( \alpha_k, \ldots, \alpha_N \). Then there exists a unique function \( F \in R_k^+ \otimes R_k L(R_k) \) of weight at most \( n+1 \), which is a primitive of \( f \) with respect to \( \alpha_k \), and is regularized at 0:

\[
\frac{\partial F}{\partial \alpha_k} = f, \quad \text{and} \quad \text{Reg}_{\alpha_k=0} F = 0.
\]

The construction of the primitive follows immediately from theorem 24. We can assume \( f \) is a generator of \( L(\Sigma_k) \) of the form \( f = f_1 g \), where \( f_1 \in L(\Sigma_k) \) and \( g \in L(R_{k+1}) \) is a function of \( \alpha_{k+1}, \ldots, \alpha_N \) only. Theorem 24 provides a primitive \( F_1 \in R_k^+ \otimes R_k L(R_k) \) for \( f_1 \). The required primitive of \( f \) is therefore \( F = F_1 g \).

5.4. Restricted Regularization. In the notations of theorem 30, we must next consider the regularized limit of such a primitive \( F \) at \( \alpha_k = \infty \),

\[
\int_0^\infty f \, d\alpha_k = \text{Reg}_{\alpha_k=\infty} F.
\]

This is a function of \( \alpha_{k+1}, \ldots, \alpha_N \), but will not lie in \( \mathbb{C}(\alpha_{k+1}, \ldots, \alpha_N) \otimes_\mathbb{Q} L(R_{k+1}) \), since it will have extra singularities corresponding to the loci \( \sigma_i = \sigma_j \), where \( \sigma_i, \sigma_j \in \Sigma_k \). The main technical point is that these extra singularities will cancel during the iterated integration process. Therefore, by mapping superfluous singular terms to zero, we can compute the integrals by using a more economical space of functions.

In this way, we will define a restricted regularization map:

\[
(42) \quad \text{RReg}_{\alpha_k=\infty} : L(R_k) \rightarrow \mathbb{C}(\alpha_{k+1}, \ldots, \alpha_N) \otimes_\mathbb{Q} L(R_{k+1}),
\]

which consists of first taking the regularization at infinity as defined in \( (5.1.2) \) and then projecting onto the algebra \( \mathbb{C}(\alpha_{k+1}, \ldots, \alpha_N) \otimes_\mathbb{Q} L(R_{k+1}) \).

The projection map is easily illustrated for functions of a single variable, i.e., in the case of hyperlogarithms. Consider two sets of distinct points in \( \mathbb{C} \):

\[
\Sigma = \{ \sigma_0, \sigma_1, \ldots, \sigma_n \} \subset \Sigma' = \{ \sigma_0, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_m \},
\]

and let \( L(\Sigma), L(\Sigma') \) denote the corresponding hyperlogarithm algebras, indexed by the set of words in the alphabets \( A = \{ a_0, \ldots, a_n \} \subset A' = \{ a_0, \ldots, a_m \} \), respectively. There is a projection map

\[
(43) \quad \pi_\Sigma : L(\Sigma') \rightarrow \mathcal{O}_{\Sigma'} \otimes_\mathbb{Q} L(\Sigma)
\]

\[
L_w(z) \mapsto L_{\pi_A(w)}(z),
\]

where \( \pi_A(w) = w \) if \( w \in A^2 \), but \( \pi_A(w) = 0 \) if \( w \) contains a letter in \( A' \setminus A \). The map \( \pi_\Sigma \) is a homomorphism for the shuffle product. Thus, for any set \( \Sigma' \) containing \( \Sigma \), we can in this way project out any singularities in \( \Sigma' \) which are not in \( \Sigma \).
The idea is to apply this argument to \( \text{Reg}_{\alpha_k=\infty} F \). If one considers \( \alpha_{k+2}, \ldots, \alpha_N \) to be constant, then one can verify that \( \text{Reg}_{\alpha_k=\infty} F \) is a hyperlogarithm function in the variable \( \alpha_{k+1} \). We can therefore write it in the form:

\[
\text{Reg}_{\alpha_k=\infty} F = \sum_i q_i L_{w_i}(\alpha_{k+1}) \text{, where } L_{w_i}(\alpha_{k+1}) \in L(\Sigma')
\]

where \( L_{w_i}(\alpha_{k+1}) \) has singularities in some set \( \Sigma' \) which may be larger than \( \Sigma_{k+1} \). After applying the projection map \( \pi_{\Sigma_{k+1}} \), we have

\[
\pi_{\Sigma_{k+1}}(\text{Reg}_{\alpha_k=\infty} F) = \sum_i q_i L_{w_i}(\alpha_{k+1}) \text{, where } L_{w_i}(\alpha_{k+1}) \in L(\Sigma_{k+1})
\]

The coefficients \( q_i \) are functions of \( \alpha_{k+2}, \ldots, \alpha_N \), and can themselves be projected down to \( L(\Sigma_{k+2}), \ldots, L(\Sigma_N) \) in turn. Thus we can define:

\[
(44) \quad \text{RReg}_{\alpha_k=\infty} F = \pi_{\Sigma_N} \circ \cdots \circ \pi_{\Sigma_{k+2}} \circ \pi_{\Sigma_{k+1}}(\text{Reg}_{\alpha_k=\infty} F)
\]

Therefore \( \text{RReg}_{\alpha_k=\infty} F \) lies in \( \mathbb{C}(\alpha_{k+2}, \ldots, \alpha_N) \otimes L(R_{k+1}) \).

**Theorem 31.** If \( S \) is unramified (definition 16), then the coefficients of \( \text{RReg}_{\alpha_k=\infty} F \) are multiple zeta values.

The restricted regularization \( \text{RReg}_{\alpha_k=\infty} f \) for any element \( f \in L(R_k) \) can be computed algorithmically by successive differentiation with respect to \( \alpha_{k+1}, \ldots, \alpha_N \). This uses an induction on the weight which is illustrated on some examples in §6.1.1 below. The only difficulty in practice is to compute the constant terms

\[
\text{Reg}_{\alpha_N=0} \circ \cdots \circ \text{Reg}_{\alpha_{k+1}=0}(\text{Reg}_{\alpha_k=\infty} F) \in \mathbb{C}.
\]

The statement of the theorem is that this number lies in \( \mathcal{Z} \) under suitable conditions. The proof uses associators in higher dimensions generalizing the ideas behind §5.1.1.

### 5.5. The integration algorithm.

We wish to compute

\[
I = \int_0^\infty \cdots \int_0^\infty f_1 \, d\alpha_1 \ldots d\alpha_N \text{, where } f_1 \in R_1.
\]

The idea is to integrate, one variable at a time, using the polylogarithm functions in \( L(R_1) \). At the \( k \)-th stage of the integration process, we will have:

\[
I = \int_0^\infty \cdots \int_0^\infty f_k \, d\alpha_k \ldots d\alpha_N \text{, where } f_k \in \mathcal{Z} \otimes \mathbb{Q} L(R_k),
\]

and \( f_k \) has weight at most \( k \). The integrands \( f_k \) are calculated recursively as follows:

1. The function \( f_k \) has a primitive \( F_k \in R_k^+ \otimes \mathbb{R} L(R_k) \) of weight at most \( k+1 \).
2. Since we can assume that \( \text{Reg}_{\alpha_k=0} F_k = 0 \), we have

\[
\int_0^\infty f_k \, d\alpha_k = \text{Reg}_{\alpha_k=\infty} F_k.
\]

In general, however, \( \text{Reg}_{\alpha_k=\infty} F_k \) is not an element of \( \mathcal{Z} \otimes \mathbb{Q} L(R_{k+1}) \), so we must use restricted regularization instead.

3. Therefore, we define

\[
f_{k+1} = \text{RReg}_{\alpha_k=\infty} F_k \in \mathcal{Z} \otimes \mathbb{Q} L(R_{k+1})
\]

Although \( \int_0^\infty f_k \, d\alpha_k \) is not in general equal to \( f_{k+1} \), one can prove that

\[
\int_0^\infty \cdots \int_0^\infty f_k \, d\alpha_k \ldots d\alpha_N = \int_0^\infty \cdots \int_0^\infty f_{k+1} \, d\alpha_{k+1} \ldots d\alpha_N.
\]
and the induction goes through.

At the last stage of the integration, we have

$$ I = \int_0^\infty f_N \, d\alpha_N = \text{Reg}_{\alpha_N=\infty} F_N, $$

where $F_N$ is the primitive of $f_N$ defined in (2) above. Since $F_N \in \mathbb{Z} \otimes L(R_N)$, its regularized value at $\infty$ is a $\mathbb{Q}$-linear combination of multiple zeta values by §5.1.3.

This proves that $I \in \mathbb{Z}$.

Since the weight of the integrand increases by at most one at each integration, we can use the above argument to bound the weights of the periods we obtain.

Remark 32. There is never any need to choose branches of any hyperlogarithm functions, because one can prove that every function $f_k$ which occurs in the above process is in fact holomorphic on the open hypercube

$$ \{ (\alpha_k, \dots, \alpha_N) \in \mathbb{R}^{N-k} : 0 < \alpha_k, \dots, \alpha_N < \infty \}. $$

The functions $f_k$ can, however, have logarithmic singularities along the codimension 1 faces of this hypercube. A detailed study of the locus of singularities in the Feynman case, and the corresponding compactifications, will be given in [9].

Note that the entire integration process is algorithmic, from the calculation of primitives to the regularizations at infinity, and can be reduced to a sequence of elementary manipulations on symbols.
6. EXAMPLES OF HYPERLOGARITHMS AND REGULARIZATION

We consider some one-dimensional examples which will occur in the calculation of the wheel with three spokes diagram. First let \( N = 1, \sigma_0 = 0, \) and \( \sigma_1 = -1, \) and denote the corresponding alphabet by \( X = \{ x_0, x_1 \}. \) We set:

\[
\Sigma_2 = \{ 0, -1 \}, \quad \text{and} \quad \mathcal{O}_{\Sigma_2} = \mathbb{Q}\left[ x, \frac{1}{x-1}, \frac{1}{x+1} \right],
\]

and \( L(\Sigma_2) \) is the \( \mathcal{O}_{\Sigma_2} \)-module spanned by the hyperlogarithms \( \text{Li}_w(x) \), where \( w \) is any word in \( X \). Thus, in \( L(\Sigma_2) \), there are exactly two hyperlogarithms of weight 1:

\[
\text{Li}_{x_0} = \log(x) \quad \text{and} \quad \text{Li}_{x_1} = \log(x+1).
\]

Note the absence of a minus sign in front of \( \text{Li}_{x_1} \), which differs from the usual convention \( \text{Li}_z(\zeta) = -\log(1-z) \). In weight two there are precisely four functions:

\[
\text{Li}_{x_0^2}(x), \text{Li}_{x_0 x_1}(x), \text{Li}_{x_1 x_0}(x), \text{Li}_{x_1^2}(x).
\]

Using definition \(20\) one verifies that these can also be expressed as:

\[
\frac{1}{2} \log^2(x), -\text{Li}_2(-x), -\text{Li}_2(x+1), \frac{1}{2} \log^2(x+1),
\]

respectively. They are related by a single shuffle product:

\[
\text{Li}_{x_0}(x)\text{Li}_{x_1}(x) = \text{Li}_{x_0 x_1}(x) + \text{Li}_{x_1 x_0}(x).
\]

Finally, in weight three there are exactly 8 linearly independent hyperlogarithms. One can check, for example, that \( \text{Li}_{x_1^3}(x) = -\text{Li}_3(-x) \). If we denote the regularized values at infinity by \( \zeta_{\infty}(w) = \operatorname{Reg}_x=\infty L_w(x) \), we have from \(5.1.3\):

\[
\zeta_{\infty}(x_0^3) = 0, \quad \zeta_\infty(x_0 x_1) = -\zeta(2), \quad \zeta_\infty(x_1 x_0) = \zeta(2), \quad \zeta_{\infty}(x_0 x_1) = 0.
\]

Likewise, in weight three, we deduce from lemma \(27\) that:

\[
zeta_{\infty}(x_1 x_0 x_1) = -2 \zeta(3), \quad \zeta_{\infty}(x_0 x_1^2) = \zeta(3), \quad \zeta_{\infty}(x_1^2 x_0) = \zeta(3),
\]

\[
zeta_{\infty}(x_0^2 x_1) = 0, \quad \zeta_{\infty}(x_0 x_1 x_0) = 0, \quad \zeta_{\infty}(x_1 x_0^2) = 0, \quad \zeta_{\infty}(x_1^3) = 0, \quad \zeta_{\infty}(x_0^3) = 0.
\]

6.0.1. The enlarged case \( \{ 0, \pm 1 \} \). Now consider the case when \( N = 2, \sigma_0 = 0, \sigma_1 = -1, \sigma_2 = 1 \). We set

\[
\Sigma_2^+ = \{ 0, -1, 1 \}, \quad \text{and} \quad \mathcal{O}_{\Sigma_2^+} = \mathbb{Q}\left[ x, \frac{1}{x-1}, \frac{1}{x+1} \right].
\]

We denote the corresponding alphabet by \( X^+ = \{ x_0, x_1, x_2 \} \). The algebra \( L(\Sigma_2^+) \) now contains 3 hyperlogarithms in weight 1, namely:

\[
\text{Li}_{x_0}(x) = \log(x) \quad \text{Li}_{x_1}(x) = \log(x+1) \quad \text{Li}_{x_2}(x) = \log(x-1),
\]

and 3^n hyperlogarithms in weight \( n \). In this case, the regularized values at infinity are no longer multiple zeta values \( (Z^1) \), but the larger set of alternating multiple zeta values \( (Z^2) \). Via the inclusion \( L(\Sigma_2) \to L(\Sigma_2^+) \), we can identify \( L(\Sigma_2) \) as a sub-algebra of \( L(\Sigma_2^+) \). In \(5.2.4\) we defined a projection map:

\[
\pi_{\Sigma_2} : L(\Sigma_2^+) \longrightarrow \mathcal{O}_{\Sigma_2^+} \otimes \mathcal{O}_{\Sigma_2} L(\Sigma_2)
\]

\[
f(x) L_w(x) \quad \mapsto \quad f(x) L_w(x) \quad \text{if} \; w \in X^*,
\]

\[
f(x) L_w(x) \quad \mapsto \quad 0 \quad \text{if} \; w \text{ contains the letter } x_2.
\]
6.1. Second example. We now consider an example in two variables. Let
\( \Sigma_y = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3 \} \), where \( \sigma_0 = 0 \), \( \sigma_1 = -1 \), \( \sigma_2 = -x \), \( \sigma_3 = -\frac{x}{x+1} \).

We set
\[
O_{\Sigma_y} = \mathbb{Q}\left[ y, \frac{1}{y}, \frac{1}{y+y+1}, \frac{1}{x+y}, \frac{1}{y+x+y+1} \right].
\]

Let us denote the corresponding alphabet by \( Y = \{ y_0, y_1, y_2, y_3 \} \), and thus
\[
L_{y_0}(y) = \log(y), \quad L_{y_1}(y) = \log(y+1), \quad L_{y_2}(y) = \log(x+y) - \log(x), \quad L_{y_3}(y) = \log(xy+x+y) - \log(x).
\]

Thus \( L(\Sigma_y) \) is spanned (as a vector space) in weight \( n \) by exactly \( 4^n \) hyperlogarithm functions \( L^w \), which can be considered as functions of the single variable \( y \), for constant \( x \). If we wish to consider the dependence on \( x \), we must remove the singular locus where the \( \sigma_i \) collide.

Recall that \( O^+_{\Sigma_y} \) was defined to be \( O_{\Sigma_y}((\sigma_i, (\sigma_i-\sigma_j)^{-1}) \), and so

\[
O^+_{\Sigma_y} = \mathbb{Q}\left[ x, \frac{1}{x}, \frac{1}{x+1}, \frac{1}{x-1} \right][y, \frac{1}{y}, \frac{1}{y+1}, \frac{1}{x+y}, \frac{1}{y+x+y+1}].
\]

This is geometrically a linear fibration over \( O^+_{\Sigma_y} \) and is pictured below:

Let \( U = \mathbb{C}^2 \setminus \{x = 0, x = \pm 1, y = 0, y = -1, x + y = 0, xy + x + y = 0 \} \), as shown. The elements of \( L(\Sigma_y) \) can be viewed as multi-valued functions on \( U \). The full space of polylogarithms is a product of two hyperlogarithm algebras:
\[
L(O^+_{\Sigma_y}) = L(\Sigma_y) L(\Sigma_y).
\]

It is generated by elements \( f(x, y) L_{w_1}(x)L_{w_2}(y) \) where \( w_1 \in X^x \), \( w_2 \in Y^x \), and \( f(x, y) \in O^+_{\Sigma_y} \). The set of elements of weight two are:
\[
Li_{x_i,x_j}(x) , \quad Li_{x_i}(x)Li_{y_i}(y) , \quad Li_{y_i,x_j}(y),
\]
where the \( x \)'s are in \( X^+ \), and the \( y \)'s are in \( Y \). There are, respectively, 9, 12, and 16 such elements, giving a total of 37 polylogarithms of weight 2.
6.1.1. Regularized values. Let \( \mathcal{Z} \) denote the \( \mathbb{Q} \)-algebra spanned by all multiple zeta values. We defined a regularization map

\[
\text{Reg}_{y=\infty} : L(\Sigma_y) \rightarrow \mathcal{Z} \otimes \mathbb{Q} L(\mathcal{O}^+_{\Sigma_y}) .
\]

The regularized values at \( y = 0 \) of all functions \( L_w(y) \), \( w \in Y^\times \) vanish. At \( y = \infty \),

\[
(50) \quad \text{Reg}_{y=\infty} L_y(y) = 0 \text{ , } \text{Reg}_{y=\infty} L_y(1) = 0 .
\]

\[
\text{Reg}_{y=\infty} L_y(y) = - \log(x) \text{ , } \text{Reg}_{y=\infty} L_y(1) = \log(x+1) - \log(x) .
\]

We now compute the regularized values at infinity of some functions of weight 2.

To compute \( \text{Reg}_{y=\infty} L_w(y) \), we can differentiate with respect to \( x \), which gives a function of lower weight, compute the regularized value at infinity of this function by induction, and take the primitive with respect to \( x \). In other words,

\[
\text{Reg}_{y=\infty} L_w(y) = \int \text{Reg}_{y=\infty} \left( \frac{\partial}{\partial x} L_w(y) \right) dx .
\]

The constant of integration is determined from the regularized values:

\[
\text{Reg}_{y=\infty} \text{Reg}_{x=0} L_w(y) \in \mathcal{Z} ,
\]

which can be calculated by a generalization of the associator argument of \( \text{Reg}_{y=\infty} \).

6.1.2. First example. Let \( w = y_0y_2 \). To compute \( \frac{\partial}{\partial x} L_w(y) \), we have:

\[
\frac{\partial}{\partial y} \frac{\partial}{\partial x} L_{y_0y_2}(y) = \frac{\partial}{\partial x} L_{y_0y_2}(y) = \frac{\partial}{\partial x} L_{y_2}(y) = \frac{1}{y} \left( \frac{1}{x+y} - \frac{1}{x} \right) .
\]

Thus

\[
\frac{\partial}{\partial x} L_{y_0y_2}(y) = \int \frac{-1}{x(x+y)} dy = -\frac{1}{x} L_{y_2}(y) .
\]

The constant of integration is determined by the fact that \( L_{y_0y_2}(y) \), and hence \( \frac{\partial}{\partial x} L_{y_0y_2}(y) \), vanishes along \( y = 0 \). Taking the regularized value at \( y = \infty \) gives

\[
\text{Reg}_{y=\infty} \left( \frac{\partial}{\partial x} L_{y_0y_2}(y) \right) = -\frac{1}{x} \text{Reg}_{y=\infty} L_{y_2}(y) = \frac{\log(x)}{x} .
\]

By taking a primitive with respect to \( x \) (working now in \( L(\Sigma_y^+) \)), we deduce that

\[
\text{Reg}_{y=\infty} L_{y_0y_2}(y) = \int \frac{\log x}{x} dx = L_{y_2}(x) + c = \frac{1}{2} \log(x)^2 + c ,
\]

where \( c \) satisfies \( c = \text{Reg}_{x=0} \text{Reg}_{y=\infty} L_{y_0y_2}(y) = 0 \). It follows that

\[
(51) \quad \text{Reg}_{y=\infty} L_{y_0y_2}(y) = L_{y_2}(x) .
\]

6.1.3. Second example. Let \( w = y_1y_1 \). By a similar calculation, we have

\[
\frac{\partial}{\partial x} L_{y_1y_1}(y) = \frac{y+1}{xy+x+y} L_{y_1}(y) - \frac{1}{x+1} L_{y_1}(y) .
\]

Taking the regularized limit at \( y = \infty \) gives

\[
\frac{\partial}{\partial x} \text{Reg}_{y=\infty} L_{y_1y_1}(y) = -\frac{1}{x+1} (\log(x) - \log(x+1)) .
\]

To determine the constant of integration, we use the fact that \( \text{Reg}_{y=\infty} \text{Reg}_{x=0} \text{Reg}_{y=\infty} L_{y_1y_1}(y) = \zeta(2) \). Taking a primitive with respect to \( x \) in the ring \( L(\Sigma_y^+) \), we deduce that

\[
\text{Reg}_{y=\infty} L_{y_1y_1}(y) = L_{y_1y_0}(x) - L_{y_1}(x) + \zeta(2) .
\]
6.1.4. Further examples. In a similar way, one can verify the following:

\begin{equation}
\text{Reg}_{y=\infty} L_{y_1y_2}(y) = L_{x_0}(x) + L_{x_1}(x) - L_{x_0x_1}(x) - L_{x_1x_0}(x),
\end{equation}

\begin{equation}
\text{Reg}_{y=\infty} L_{y_1y_3}(y) = L_{x_0x_1}(x) - L_{x_1}(x) + L_{x_2}(x),
\end{equation}

which will be used in the calculation of the wheel with three spokes diagram overleaf.

6.1.5. Restricted regularization. All the previous examples happen to lie in \( L(S_x) \). In the general case, one obtains an answer with an extra singularity at \( x = 1 \), i.e., a hyperlogarithm in \( L(S^+_x) \). For example, one can check that:

\begin{equation}
\frac{\partial}{\partial x} L_{y_1y_2}(y) = \frac{L_{y_1}(y)}{x(x-1)} - \frac{L_{y_2}(y)}{x-1},
\end{equation}

\begin{equation}
\text{Reg}_{y=\infty} L_{y_1y_2}(y) = L_{x_2x_0}(x).
\end{equation}

The reason this happens is because \( L_{y_1y_2}(y) \) has a singularity at \( \sigma_1 = \sigma_2 \). In this case, the restricted regularization map of \( \{\sigma_1, \sigma_2\} \) gives

\[ \text{RReg}_{y=\infty} L_{y_1y_2}(y) = \pi_{\Sigma_x} L_{x_2x_0}(x) = 0. \]

Since the regularized value of every hyperlogarithm occurring in the calculation of the wheel with three spokes diagram overleaf already lies in \( L(S_x) \), we do not in fact need to use the regularized restriction in the calculation.

Remark 33. In order to compute the values \( \text{Reg}_{x=\epsilon_0} \text{Reg}_{y=\infty} L_w(y) \), one can set

\[ t_1 = -\sigma_2, \quad t_2 = -\sigma_3, \quad \text{and} \quad t_3 = -y. \]

The hyperlogarithm \( L_w(y) \), viewed as a function of the independent variables \( t_1, t_2, t_3 \), is then a unipotent function on the moduli space

\[ \mathcal{M}_{0,6} = \text{Spec} \mathbb{Z} \left[ t_1, t_2, t_3, \frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3} \right]. \]

Setting \( t_1 = x, t_2 = \frac{x}{t_1}, t_3 = -y \) defines a surface inside \( \mathcal{M}_{0,6} \), and taking the limit first as \( y \to 0 \) and then \( x \to 0 \), corresponds to taking the regularized limit of \( L_w(y) \) at the tangential base-point at \( t_1 = 0, t_2 = 0, t_3 = 0 \) defined by the sector

\[ 0 \leq t_3 \leq t_2 \leq t_1 \leq 1. \]

Equivalently, this corresponds to a unique point on the compactification \( \overline{\mathcal{M}}_{0,6} \) which lies in the deepest stratum, and defined over the integers. Likewise, taking the regularized limit at \( x = 0, y = \infty \) corresponds to taking a limit at a different point in \( \overline{\mathcal{M}}_{0,6} \). By the results of [8], this limit can be described by a generalized associator and is expressible in terms of multiple zeta values. The general case is similar and follows from the ramification condition [33] (this will be discussed in [9]).
7. The wheel with three spokes

7.1. The Fubini reduction algorithm. Let \( G \) be the wheel with three spokes depicted in fig. 1. We take \( \lambda = 5 \), and set \( \alpha_5 = 1 \) from the outset. We therefore have \( S^\prime = \{U_{G}^\prime | \alpha_5 = 1 \} \), where \( U_{G}^\prime = U_G \alpha_6 + V_G \), and \( U_G, V_G \) are given by \([8]\). We wish to compute integrals of the form:

\[
I_G = \int \prod_{i=1}^{5} \frac{\log m_i(\alpha_i) \log m(\alpha)}{U_G^2} d\alpha_6 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 , \quad m_i, m \in \mathbb{N} ,
\]

by integrating with respect to the variables \( \alpha_6, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), in that order. The Fubini reduction algorithm gives:

\[
S^\prime_{[6]} = \{ \alpha_1 \alpha_2 + \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4 , \alpha_1 \alpha_3 \alpha_4 + \alpha_1 \alpha_2 \alpha_4 + \alpha_3 \alpha_2 \alpha_4 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 + \alpha_2 \alpha_4 + \alpha_1 \alpha_4 + \alpha_1 \alpha_2 \} .
\]

After reducing with respect to \( \alpha_1 \), we have:

\[
S^\prime_{[6,1]} = \{ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_2 \alpha_3 + \alpha_3 \alpha_4 , \alpha_2 + \alpha_3 + \alpha_4 , \alpha_3 \alpha_4 + \alpha_3 + \alpha_4 , \alpha_2 \alpha_4 + \alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4 , \alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4 \}
\]

Reducing with respect to the variable \( \alpha_2 \) gives:

\[
S^\prime_{[6,1,2]} = \{ \alpha_4 + 1 , \alpha_3 + 1 , \alpha_3 \alpha_4 + \alpha_3 + \alpha_4 \}
\]

Finally, by reducing with respect to the variable \( \alpha_3 \), we obtain:

\[
S^\prime_{[6,1,2,3]} = \{ \alpha_4 + 1 \}
\]

Note that at each stage, every polynomial is linear with respect to every variable. The corresponding sets of singularities \( \Sigma_1, \ldots, \Sigma_4 \) are therefore:

\[
\Sigma_1 = \{0, -\frac{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_2 \alpha_3 + \alpha_3 \alpha_4}{\alpha_2 + \alpha_3 + \alpha_4} , -\frac{\alpha_2 \alpha_3 + \alpha_2 \alpha_4 + \alpha_2 \alpha_3 \alpha_4}{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_2 \alpha_4 + \alpha_3 \alpha_4} \},
\]

\[
\Sigma_2 = \{0,-(\alpha_3 + \alpha_4), -(\alpha_3 + \alpha_4 + \alpha_3 \alpha_4), -\frac{\alpha_3 + \alpha_4 + \alpha_3 \alpha_4}{\alpha_3 + 1} , -\frac{\alpha_3 + \alpha_4 + \alpha_3 \alpha_4}{\alpha_4 + 1} \},
\]

\[
\Sigma_3 = \{0,-1,-(\alpha_4), -\frac{\alpha_4}{\alpha_4 + 1} \} , \quad \Sigma_4 = \{0,-1\} .
\]

Taking the limits as \( \alpha_2 \to 0, \alpha_3 \to 0, \alpha_4 \to 0 \), in that order, we obtain singularities in \( 0,-1 \) only. Therefore the conditions of theorem \([8]\) are satisfied.

**Corollary 34.** Every integral of the form \([54]\) lies in \( \mathbb{Z} \).

We have a nested sequence of rings \( \mathbb{Q} \subset R_4 \subset R_3 \subset R_2 \subset R_1 \), defined as follows:

\[
R_4 = \mathbb{Q}[\alpha_4, \alpha_4^{-1}, (\alpha_4 + 1)^{-1}] , \\
R_3 = R_4[\alpha_3, \alpha_3^{-1}, (\alpha_3 + 1)^{-1}, (\alpha_3 + \alpha_4)^{-1}, (\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)^{-1}] , \\
R_2 = R_3[\alpha_2, \alpha_2^{-1}, (\alpha_2 + \alpha_3 + \alpha_4)^{-1}, (\alpha_2 \alpha_3 + \alpha_2 + \alpha_3 + \alpha_4)^{-1}, (\alpha_2 \alpha_4 + \alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4)^{-1}, (\alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4)^{-1}] , \\
R_1 = R_2[\alpha_1, \alpha_1^{-1}, (\alpha_1 \alpha_2 + \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_3 \alpha_2 + \alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4)^{-1}, (\alpha_3 \alpha_4 + \alpha_2 + \alpha_3 + \alpha_4)^{-1}] .
\]

One can compute any particular integral \([54]\) by working in the algebra of polylogarithms \( L(R_1) = L(\Sigma_1) L(\Sigma_2) L(\Sigma_3) L(\Sigma_4) \), where \( L(\Sigma_1), \ldots, L(\Sigma_4) \) are hyperlogarithm algebras on 3, 5, 4, 2 letters respectively.
7.2. Calculation of the leading term for the wheel with 3 spokes. We illustrate the method by calculating in complete detail the Feynman amplitude of the wheel with three spokes \( G \), and reprove the result, due originally to Broadhurst and Kreimer, that it evaluates to \( 6 \zeta(3) \).

Let \( U = U_G, V = V_G \) be as in (56). We set \( \lambda = \{ 5 \} \), and wish to compute (57):

\[
I_G = \int_{[\alpha_5=1]}^{\infty} \frac{1}{UV} d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 .
\]

From equation (25), we have \( D = \alpha_2 \alpha_5 + \alpha_4 \alpha_5 + \alpha_3 \alpha_4 + \alpha_3 \alpha_5 \), and

\[
U^{(1,2)} = \alpha_3 \alpha_4 + \alpha_3 \alpha_5 + \alpha_4 \alpha_5 , \quad U_2^{(1)} = \alpha_4 + \alpha_5 , \quad U_1^{(2)} = \alpha_3 + \alpha_4 , \quad U_{12} = 1 .
\]

Let us set \( \alpha_5 = 1 \) once and for all. Using the notation (26) we have:

\[
\{ U_2^{(1)}, U^{(1,2)}|D_2, D^{(2)} \} = \frac{(\alpha_3 + 1) \log(\alpha_3 + 1)}{\alpha_3 \alpha_4 + \alpha_3 + \alpha_4} + \frac{\log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)}{\alpha_3 \alpha_4 + \alpha_3 + \alpha_4} ,
\]

\[
\{ U_{12}, U^{(2)}|D_2, D^{(2)} \} = \frac{\log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)}{\alpha_3 \alpha_4} ,
\]

\[
\{ V_2^{(1)}, V^{(1,2)}|D_2, D^{(2)} \} = 2 \frac{\log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)}{(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)} ,
\]

\[
\{ V_{12}, V^{(2)}|D_2, D^{(2)} \} = \frac{(\alpha_4 + 1) \log(\alpha_4 + 1)}{\alpha_4 (\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)} + \frac{\log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)}{\alpha_3 \alpha_4 + \alpha_3 + \alpha_4} .
\]

By corollary 8 we can skip the first two integration steps and go straight to

\[
I_G = \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{\alpha_3 \alpha_4} \left( \log(\alpha_3 + 1) + \log(\alpha_3 + \alpha_4) - \log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4) \right)
\]

\[
+ \frac{1}{\alpha_4 (\alpha_3 \alpha_4 + \alpha_3 + \alpha_4)} \left( (\alpha_4 + 1) \log(\alpha_4 + 1) - \log(\alpha_3 + 1) - \alpha_4 \log(\alpha_3 \alpha_4 + \alpha_3 + \alpha_4) \right) d\alpha_3 d\alpha_4 .
\]

One can verify that the integrand has no polar singularities along the faces of the domain of integration \( X = [0, \infty] \times [0, \infty] \). It has a pole of order exactly one along the hypersurface \( \alpha_3 \alpha_4 + \alpha_3 + \alpha_4 = 0 \), which meets the boundary of \( X \) in codimension two. Even though the integral converges, the simplest way to compute it is to calculate the regularised integral of each individual term, which may be at most logarithmically divergent, and add the contributions together.

We set \( \alpha_3 = y, \alpha_4 = x \). In [40] \( R_3 \) and \( R_4 \) were called \( \mathcal{O}_{\Sigma_3} \) and \( \mathcal{O}_{\Sigma_4} \), respectively. In the hyperlogarithm notation of [5, 1] the integrand of \( I_G \) is an element of \( L(R_4) = L(\mathcal{O}_{\Sigma_4}) \) of weight 1. We rewrite \( I_G \) as follows:

\[
I_G = \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{xy} \left( L_{y_1}(y) + L_{y_2}(y) - L_{y_3}(y) \right) + \frac{1}{x} L_{x_1}(x) \left( \frac{1}{y + 1} \right)
\]

\[
- \frac{1}{x(x+1)} \frac{1}{y + \frac{1}{x+1}} \left( L_{y_1}(y) + x L_{y_2}(y) + x L_{y_0}(x) \right) dxdy .
\]

Using theorem 20 we can take a primitive with respect to \( y \), which gives:

\[
I_G = \int_{0}^{\infty} \frac{1}{x} \text{Reg}_{y=\infty} \left[ L_{y_0 y_1}(y) + L_{y_0 y_2}(y) - L_{y_0 y_3}(y) + L_{x_1}(x) L_{y_3}(y) \right]
\]

\[
- \frac{1}{x(x+1)} \text{Reg}_{y=\infty} \left[ L_{y y_3}(y) + x L_{y y_3}(y) + x L_{y_0}(x) L_{y_3}(y) \right] dx .
\]
Using the shuffle relations, and equation (50), observe that
\[
\text{Reg}_{y=\infty} L_{x_1}(x)L_{y_2}(y) = L_{x_1}(x)\left(L_{x_1}(x) - L_{x_0}(x)\right) = 2 L_{x_1^2}(x) - L_{x_1x_0}(x) - L_{x_0x_1}(x)
\]
and similarly,
\[
\text{Reg}_{y=\infty} L_{x_0}(x)L_{y_2}(y) = L_{x_0x_1}(x) + L_{x_1x_0}(x) - 2 L_{x_2}(x).
\]
All the remaining regularized limits were given in (57). Substituting them in gives:
\[
I_G = \int_0^\infty \frac{1}{x} \left[ \zeta(2) + L_{x_0x_1}(x) + L_{x_1x_0}(x) - L_{x_1^2}(x) + L_{x_1}(x)\left(L_{x_1}(x) - L_{x_0}(x)\right) \right]
\]
\[
- \frac{1}{x(x+1)} \left[ \zeta(2) + L_{x_1x_0}(x) - L_{x_1^2}(x) + x\left(L_{x_0x_1}(x) - L_{x_1^2}(x) + L_{x_0}(x)(L_{x_1}(x) - L_{x_0}(x))\right) \right].
\]
After decomposing into partial fractions, and expanding out the shuffle products, we obtain the fourth step:
\[
(57) \quad I_G = \int_0^\infty \frac{1}{x} \left[ 2 L_{x_1^2}(x) - L_{x_1x_0}(x) \right] + \frac{1}{x+1} \left[ \zeta(2) + L_{x_0}(x) - 2 L_{x_0x_1}(x) \right] dx.
\]
This is an integral of functions of weight at most two in a single variable \(x\), in the hyperlogarithm algebra \(Z \otimes Q L(R_4) = Z \otimes Q L(\Sigma_2)\). A priori, the regularized limits at \(y = \infty\) which were substituted in at the previous stage could also have had singularities in \(\Sigma_2^+\), i.e., at \(x = 1\) also, but the method (the Fubini argument) predicts that the integrand at the fourth stage (57) cannot, as is indeed the case.

To complete the calculation, we work in \(L(R_4)\). A further integration gives:
\[
I_G = \text{Reg}_{x=\infty} \left( 2 L_{x_0x_1^2}(x) - L_{x_0x_1x_0}(x) + \zeta(2)L_{x_1}(x) + L_{x_1x_0^2}(x) - 2 L_{x_1x_0x_1}(x) \right),
\]
Therefore at the fifth and final step, we obtain
\[
I_G = 2 \zeta_\infty(x_0x_1^2) - \zeta_\infty(x_0x_1x_0) + \zeta_\infty(x_1x_0^2) - 2 \zeta_\infty(x_1x_0x_1).
\]
Substituting the values given in (40), we conclude that
\[
I_G = 6 \zeta(3).
\]

Remark 35. One can rewrite (57) using only dilogarithms and logarithms:
\[
(58) \quad I_G = \int_0^\infty \frac{1}{\alpha_4} \left( - \text{Li}_2(-\alpha_4) + \log^2(\alpha_4) + \log(\alpha_4) \log(\alpha_4 + 1) \right)
\]
\[
+ \frac{1}{\alpha_4 + 1} \left( 2 \text{Li}_2(-\alpha_4) + \frac{1}{2} \log^2(\alpha_4) + \zeta(2) \right) d\alpha_4,
\]
where each term in the integrand has singularities contained in \(\{0, -1, \infty\}\). One can obtain (58) directly from (56) by integrating using the dilogarithm function, regularizing at infinity, and applying the inversion relation for Li_2 to arrive at (58). Although this gives a substantial shortcut, such a method is ad hoc, and will not work in a more general setting.
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