EXTRINSIC CURVATURE OF CODIMENSION ONE ISOMETRIC IMMERSIONS WITH HÖLDER CONTINUOUS DERIVATIVES

SÖREN BEHR AND HEINER OLBERMANN

Abstract. We prove that if \( n \) is even, \((M, g)\) is a compact \( n \)-dimensional Riemannian manifold whose Pfaffian form is a positive multiple of the volume form, and \( y \in C^{1,\alpha}(M; \mathbb{R}^{n+1}) \) is an isometric immersion with \( n/(n+1) < \alpha \leq 1 \), then \( y(M) \) is a surface of bounded extrinsic curvature. This is proved by showing that extrinsic curvature, defined by a suitable pull-back of the volume form on the \( n \)-sphere via the Gauss map, is identical to intrinsic curvature, defined by the Pfaffian form. This latter fact is stated in form of an integral identity for the Brouwer degree of the Gauss map, that is classical for \( C^2 \) functions, but new for \( n > 2 \) in the present context of low regularity.

1. Introduction

1.1. Statement of results. Let \( M \) be a compact \( n \)-dimensional Riemannian manifold, where \( n \) is even. We adapt Cartan’s method of moving frames. Let \( X_i, i = 1, \ldots, n \) be an orthonormal frame on \( M \), and let \( \theta^i, \omega^j_i, \Omega^j_i, i, j = 1, \ldots, n \) be the associated dual forms, connection one-forms and curvature two-forms respectively, defined by the equations

\[
\begin{align*}
\theta^i(X_j) &= \delta^i_j \\
\text{d}\theta^i &= \sum_{j=1}^{n} \omega^j_i \wedge \theta^j \\
\Omega^j_i &= \text{d}\omega^j_i + \sum_{k=1}^{n} \omega^k_i \wedge \omega^j_k,
\end{align*}
\]

(1)

where \( \delta^i_j \) denotes the Kronecker delta, and \( i, j \in \{1, \ldots, n\} \). We define the Pfaffian of \((M, g)\) by

\[
Pf(\Omega) = \frac{1}{n(n/2)!} \sum_{\zeta \in \text{Sym}(n)} \Omega^{(2)}_{\zeta(1)} \wedge \cdots \wedge \Omega^{(n)}_{\zeta(n-1)},
\]

where \( \text{Sym}(n) \) denotes the group of permutations of \( \{1, \ldots, n\} \). It turns out (see [28]) that this formula is independent of the chosen orthonormal frame, and thus makes \( Pf(\Omega) \) defined on all of \( M \). Additionally, for every isometric immersion \( y \in C^2(M; \mathbb{R}^{n+1}) \) with normal \( \nu : M \to S^n \), we have by Gauss’ equation

\[
Pf(\Omega) = \nu^*\sigma_{S^n},
\]

(2)

where \( \sigma_{S^n} \) denotes the canonical volume element on \( S^n \), and \( \nu^* \) the pull-back by \( \nu \). (Such a relation only exists for even \( n \), which is the reason why our analysis is limited to this case.) Let us consider the isometric immersion \( y \), its normal \( \nu \) and the Pfaffian \( Pf(\Omega) \) in

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a chart $U \subset \mathbb{R}^n$. In this chart, the metric $g$ is given by the $n \times n$ matrix-valued function $Dy^T Dy$. As a direct consequence of \((2)\), we have the change of variables type formula
\[
\int_U \varphi \circ \nu \text{Pf}(\Omega) = \int_{S^n} \varphi(z) \deg(\nu, U, z) d\mathcal{H}^n(z) \quad \text{for every } \varphi \in L^\infty(S^n \setminus \nu(\partial U)),
\]
where $\deg(\nu, U, \cdot)$ denotes the Brouwer degree of $\nu : U \to S^n$. Our first result is the validity of this formula for every isometric immersion $y \in C^{1,\alpha}(M; \mathbb{R}^{n+1})$ with $\alpha > n/(n + 1)$:

**Theorem 1.** Let $n$ be even, $U \subset \mathbb{R}^n$ open and bounded, and $n/(n + 1) < \alpha \leq 1$. Furthermore, let $y \in C^{1,\alpha}(U; \mathbb{R}^{n+1})$ with $g = Dy^T Dy \in C^\infty(U; \text{Sym}^+_n)$, and let $\Omega$ be the curvature form associated to the metric $g$. Then
\[
\int_U \varphi \circ \nu \text{Pf}(\Omega) = \int_{S^n} \varphi(z) \deg(\nu, U, z) d\mathcal{H}^n(z) \quad \text{for every } \varphi \in L^\infty(S^n \setminus \nu(\partial U)).
\]

Now let us consider Riemannian manifolds $(M, g)$ whose Pfaffian is a positive multiple of the volume form. Then for any smooth immersion $y$, the index of the normal map $\nu$ is positive everywhere. Hence, the Brouwer degree $\deg(\nu, M, \cdot)$ is positive everywhere on $\nu(M) \setminus \nu(\partial M)$, and one can estimate the $n$-dimensional Hausdorff measure of images of the normal map by the formula \((3)\). This allows for estimates on extrinsic curvature, that we define following Pogorelov \[27]:

**Definition 1.** Let $y : M \to \mathbb{R}^{n+1}$ be an immersed manifold of class $C^1$. Denote the surface normal by $\nu : M \to S^n$. The extrinsic curvature of $M$ is given by
\[
\sup \left\{ \sum_{i=1}^N \mathcal{H}^n(\nu(E_i)) : N \in \mathbb{N}, \{E_i\}_{i=1,\ldots,N} \right. 

a collection of closed disjoint subsets of $M$ \}. \in [0, \infty].
\]

If this quantity is finite, we say $y(M)$ is of bounded extrinsic curvature.

Here, $\mathcal{H}^n$ denotes $n$-dimensional Hausdorff measure. Using Theorem \[1\], we can show that the reasoning above still applies for isometric immersions $y \in C^{1,\alpha}(U; \mathbb{R}^{n+1})$ if $\alpha > n/(n + 1)$:

**Theorem 2.** Let $n$ be even, and let $(M, g)$ be a precompact $n$-dimensional Riemannian manifold with smooth metric and positive Pfaffian. Furthermore, let $n/(n + 1) < \alpha \leq 1$, and let $y \in C^{1,\alpha}(M; \mathbb{R}^{n+1})$ be an isometric immersion. Then the immersed surface $y(M)$ has bounded extrinsic curvature.

1.2. **Scientific context.** The Weyl problem is the task of finding an isometric immersion $y \in C^2(S^2; \mathbb{R}^3)$ for a manifold $(S^2, g)$ with positive Gauss curvature (where for simplicity, we assume $g \in C^\infty$). Existence of such an immersion has been proved independently by Pogorelov and Nirenberg. If a solution of this problem is unique up to rigid motions, then it is called rigid. The proof of rigidity for the case of analytic immersions is due to Cohn-Vossen \[13], and for $C^2$ immersions, it has been given by Pogorelov.

The regularity assumption in the Weyl problem is crucial for uniqueness questions. Recall that an immersion $y \in (S^2; \mathbb{R}^3)$ is called short if every (Lipschitz) path $\gamma \subset M$ gets mapped to a shorter path $y(\gamma) \subset \mathbb{R}^3$. The famous Nash-Kuiper Theorem \[22, 25] states
that any short immersion can be approximated in $C^0$ by isometric immersions of regularity $C^1$. (To avoid confusion, we remark that the Nash-Kuiper Theorem is not limited to $M = S^2$ and immersions $M \to \mathbb{R}^3$, but it holds for any short immersion of codimension at least one.) Hence, there exists a vast set of solutions to the Weyl problem in the class of $C^1$ immersions. Historically, this was the first instance of the so-called "$h$-principle", nowadays associated with Gromov [19].

Comparing these two cases, it immediately arises the question of what can be said about the intermediate ones: For which range of $\alpha \in (0,1)$ are isometric immersions $y \in C^{1,\alpha}(S^2;\mathbb{R}^3)$ rigid? For which range does the $h$-principle hold? This is a matter of some interest – the question of existence of a critical $\alpha$ can be found as problem 27 in Yau’s list of open problems in geometry [31]. For large codimension, a solution has been given by Källén in [21].

In codimension one, a partial answer can be found in a series of articles by Borisov [2, 3, 4, 5, 6, 7, 8]. He proved that the $h$-principle holds locally for $\alpha < \frac{1}{n^2+n+1}$, where $n$ is the dimension of the manifold (provided the metric $g$ is analytic), while for $\alpha > \frac{2}{3}$, $C^{1,\alpha}$-isometric immersions of manifolds $(S^2,g)$ with positive Gauss curvature are rigid.

In [15], Conti, De Lellis and Székelyhidi have given simplified versions of Borisov’s proofs of these facts. For the case $\alpha < \frac{1}{n^2+n+1}$, it has been shown there that the $h$-principle also holds for non-analytic $g$. In the recent paper [26], it has been proved that in dimension $n = 2$, it holds in the (larger) range $\alpha > \frac{1}{3}$. Concerning rigidity, [15] contains the statements of our Theorems 1 and 2 for the case $n = 2$. Combining the latter with classical results by Pogorelov on surfaces of bounded extrinsic curvature [27], the rigidity result for isometric immersions $y \in C^{1,\alpha}(S^2;\mathbb{R}^3)$ with $\alpha > \frac{2}{3}$ follows (where, of course, Gauss curvature is assumed to be positive).

Our Theorems 1 and 2 generalize the results on extrinsic curvature from [15] to even dimension $n > 2$. However, the results by Pogorelov from [27] that allow to conclude that isometric immersions of bounded extrinsic curvature are rigid have only been proved for the case $n = 2$. The question whether or not their analogues in higher dimension are valid will not be addressed here. Thus, the question whether codimension one $C^{1,\alpha}$-isometric embeddings of $n$-dimensional manifolds whose Pfaffian is positive for $\alpha > n/(n+1)$ and even $n$ are rigid, remains open too.

As in two dimensions, we require the Pfaffian form to be a positive multiple of the volume form. In other words, we require the Gauss-Bonnet integrand (also known as the Lipschitz-Killing curvature, or Gauss-Kronecker curvature) to be positive. In passing, we mention that it is a known fact that for smooth immersions, positivity of the Gauss-Bonnet integrand implies that the immersed surface is the boundary of a convex body also in even dimensions $n > 2$, see [15].

The present paper builds on the recent results by the second author from [26]. The main focus of that work are the integrability properties of the Brouwer degree. The results there are achieved by a suitable definition of the distributional Jacobian $\det Du$ for
$u \in C^{0,\alpha}(U; \mathbb{R}^n)$ through real interpolation. In a similar way, the distributional Jacobian had been defined by Brezis and Nguyen in [11], building on an idea by Bourgain, Brezis and Mironescu [9, 10]. Here, we will adapt these techniques to the Pfaffian form. Another ingredient that has been used in [26] and will also be used here, is a well defined notion of integration of H"older continuous forms over fractals. This follows closely the definitions from the paper [20] by Harrison and Norton. The main ingredients of our proofs will be suitable generalizations of the results from [26]; for the convenience of the reader, we will give full proofs of these statements, and not refer to that work.

1.3. Plan of the paper. In Section 2, we will recall some well known facts: The Gauss-Bonnet-Chern Theorem, real interpolation by the trace method, the definition of box dimension, the Whitney decomposition of an open subset of $\mathbb{R}^n$, properties of the level sets of H"older functions and the approximation of $C^{1,\alpha}$ isometric immersions by mollification. These results will be used in Section 3 to give a suitable definition of the distributional Jacobian, the distributional Pfaffian and well defined notions of their integrals over sets with fractal boundary. This section parallels most of the ideas from [26] and adapts them to the current setting. In Section 4, we combine the results from Section 3 to prove Theorem 1. Theorem 2 will then be obtained from Theorem 1 by arguing in exactly the same way as in the proof of the case $n=2$ in [15].

1.4. Notation. Except for the proof of Theorem 2, our investigations will take place in a single chart $U \subset \mathbb{R}^n$ of an $n$-dimensional Riemannian manifold $(M, g)$. For $k = 0, 1, 2, \ldots$, the $C^k$-norms $\| \cdot \|_{C^k(U)}$ are defined by

$$
\|u\|_{C^k(U)} = \sum_{0 \leq j \leq k} \sup_{x \in U} |D^j u(x)|.
$$

For $\alpha \in (0, 1]$, the H"older semi-norms $[\cdot]_\alpha$ are defined by

$$
[u]_\alpha = \sup_{x, x' \in U, x \neq y} \frac{|u(x) - u(x')|}{|x - x'|}.
$$

Finally, the $C^{k,\alpha}$ norms $\| \cdot \|_{C^{k,\alpha}(U)}$ are defined by

$$
\|u\|_{C^{k,\alpha}(U)} = \|u\|_{C^k(U)} + [D^k u]_\alpha.
$$

In the chart $U$, the metric $g$ is a smooth function on $U$ with values in the positive definite $n \times n$ matrices $\text{Sym}^+_n$, $g \in C^\infty(U; \text{Sym}^+_n)$. We make $C^{k,\alpha}(U; \text{Sym}^+_n)$ a normed space by setting

$$
\|g\|_{C^{k,\alpha}(U; \text{Sym}^+_n)} = \sum_{i,j=1,\ldots,n} \|g_{ij}\|_{C^{k,\alpha}(U)}.
$$

An immersion $y \in C^1(U; \mathbb{R}^{n+1})$ is an isometric immersion (w.r.t. $g$) if and only if $Dy^T Dy = g$. The tangent space at every $x \in U$ will be identified with $\mathbb{R}^n$, and hence vector fields are identified with $\mathbb{R}^n$ valued functions on $U$. Let $\Lambda^p \mathbb{R}^n$ denote the set of rank $p$ multi-vectors in $\mathbb{R}^n$, i.e., the linear space

$$
\Lambda^p \mathbb{R}^n = \left\{ \sum_{i_1, \ldots, i_p \in \{1, \ldots, n\}} a_{i_1, \ldots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} : a_{i_1, \ldots, i_p} \in \mathbb{R} \right\},
$$

for $0 \leq p \leq n$. The H"older $C^{0,\alpha}(U; \mathbb{R}^{n+1})$ valued functions on $U$ are identified with $\Lambda^p \mathbb{R}^n$ valued functions on $U$. This space is normed by setting

$$
\|\varphi\|_{C^{0,\alpha}(U; \Lambda^p \mathbb{R}^n)} = \sum_{i_1, \ldots, i_p \in \{1, \ldots, n\}} \|a_{i_1, \ldots, i_p}\|_{C^{0,\alpha}(U)}.
$$

The operator $\mathbf{d}$ is defined to be the exterior derivative, i.e., the linear operator on $\Lambda^p \mathbb{R}^n$ valued functions on $U$ that sends a function $\varphi$ to $\mathbf{d}\varphi = \sum_{i_1, \ldots, i_p \in \{1, \ldots, n\}} a_{i_1, \ldots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $a_{i_1, \ldots, i_p}$ is the coefficient of $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ in $\varphi$.

For $p = 0$, we set $\Lambda^0 \mathbb{R}^n = \mathbb{R}$. For $a = (a_{i_1, \ldots, i_p}) \in \Lambda^p \mathbb{R}^n$, we define $a(x) = a_{i_1, \ldots, i_p}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then, for $\varphi \in C^{0,\alpha}(U; \mathbb{R}^{n+1})$, $\mathbf{d}\varphi$ is a well defined $C^{0,\alpha}(U; \Lambda^p \mathbb{R}^n)$ valued function on $U$. This space is normed by setting

$$
\|\varphi\|_{C^{0,\alpha}(U; \Lambda^p \mathbb{R}^n)} = \sum_{i_1, \ldots, i_p \in \{1, \ldots, n\}} \|a_{i_1, \ldots, i_p}\|_{C^{0,\alpha}(U)}.
$$

Finally, for $p = 0$, we set $\Lambda^0 \mathbb{R}^n = \mathbb{R}$. For $a = (a_{i_1, \ldots, i_p}) \in \Lambda^p \mathbb{R}^n$, we define $a(x) = a_{i_1, \ldots, i_p}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then, for $\varphi \in C^{0,\alpha}(U; \mathbb{R}^{n+1})$, $\mathbf{d}\varphi$ is a well defined $C^{0,\alpha}(U; \Lambda^p \mathbb{R}^n)$ valued function on $U$. This space is normed by setting

$$
\|\varphi\|_{C^{0,\alpha}(U; \Lambda^p \mathbb{R}^n)} = \sum_{i_1, \ldots, i_p \in \{1, \ldots, n\}} \|a_{i_1, \ldots, i_p}\|_{C^{0,\alpha}(U)}.
$$
Hence, $p$-forms will be functions on $U$ with values in $\Lambda^p \mathbb{R}^n$. We make $C^{k,\alpha}(U; \Lambda^p \mathbb{R}^n)$ a normed space by setting

$$\|a\|_{C^{k,\alpha}(U; \Lambda^p \mathbb{R}^n)} = \sum_{i_1, \ldots, i_p} \|a_{i_1, \ldots, i_p}\|_{C^{k,\alpha}(U)}$$

for $a = \sum_{i_1, \ldots, i_p} a_{i_1, \ldots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

The symbol “$C$” will be used as follows: A statement such as “$a \leq C(\alpha)b$” is to be understood as “there exists a numerical constant $C$ only depending on $\alpha$ such that $a \leq Cg$”. Whenever the dependence of $C$ on other parameters is clear, we also write “$a \lesssim b$” in such a situation.

The non-negative real line will be denoted by $\mathbb{R}^+ = [0, \infty)$. On $\mathbb{R}^+$, we write $dt/t$ for the measure defined by $A \mapsto \int_{\mathbb{R}^+} \chi_A(t) dt$, where $\chi_A$ is the characteristic function of the measurable set $A \subset \mathbb{R}^+$.

Acknowledgments. This paper presents the main results of the first author’s Masters thesis \[1\].

2. Results from the literature

2.1. The Gauss-Bonnet-Chern Theorem. The Gauss-Bonnet-Chern Theorem states in particular that the Pfaffian form is an exact form, with an explicit formula for a primitive that can be written as a polynomial in the connection and curvature forms. Let $(M, g)$ be a Riemannian manifold of even dimension $n$. For a given orthonormal frame $\{X_i\}_{i=1, \ldots, n}$ and associated connection and curvature forms $\omega^1_i$, $\Omega^j_i$, let the forms $\Phi_i$, $i = 1, \ldots, n/2 - 1$ be defined by

$$\Phi_i = \sum_{\zeta \in \text{Sym}(\mathbb{N})} \text{sgn} \omega^1_{\zeta(2)} \wedge \omega^1_{\zeta(3)} \wedge \cdots \wedge \omega^1_{\zeta(n-2i)} \wedge \Omega^j_{\zeta(n-2i+2)} \wedge \cdots \wedge \Omega^j_{\zeta(n)}. \quad (4)$$

**Theorem 3** (Gauss-Bonnet-Chern, \[12\]). We have $\text{Pf}(\Omega) = d\Pi(\omega)$, where

$$\Pi(\omega) = \frac{1}{\pi^n} \sum_{i=0}^{n-1} \frac{(-1)^i}{1 \cdot 3 \cdots (2n-2i-1)} \frac{1}{2^{n+1}} \Phi_i, \quad (5)$$

with $\Phi_i$, $i = 1, \ldots, n/2 - 1$, as defined in (4), and $\Omega^j_i$ defined as a function of the $\omega^k_i$ through (1).

2.2. Real interpolation via the trace method. We are going to use some standard constructions from real interpolation theory. The following way to introduce the real interpolation spaces is due to Lions \[23\]. Let $E_0$ and $E_1$ be two Banach spaces that are continuously embedded in a Hausdorff topological vector space. This is only necessary to guarantee that $E_0 \cap E_1$ and $E_0 + E_1 = \{c_0 + c_1 \mid c_0 \in E_0, \ c_1 \in E_1\}$ are also Banach spaces with the following norms:

$$\|x\|_{E_0 \cap E_1} := \max\{\|x\|_{E_0}, \|x\|_{E_1}\}$$

$$\|x\|_{E_0 + E_1} := \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x_0 \in E_0, \ x_1 \in E_1, \ x_0 + x_1 = x\}$$
Definition 2. For \( \theta \in (0, 1) \) and \( 1 \leq p \leq \infty \) we denote by \( V(\theta, \theta, E_0) \) the set of all functions \( u \in W_{loc}^1(\mathbb{R}^+, E_0 \cap E_1; dt/t) \) such that, with \( u_{*, \theta}(t) := t^\theta u(t) \) and \( u'_{*, \theta} := t^\theta u'(t) \), we have

\[
u_{*, \theta} \in L^p(\mathbb{R}^+, E_1; dt/t), \quad u'_{*, \theta} \in L^p(\mathbb{R}^+, E_0; dt/t),
\]

and we define a norm on \( V = V(p, \theta, E_1, E_0) \) by

\[
\|u\|_V := \|u_{*, \theta}\|_{L^p(\mathbb{R}^+, E_1; dt/t)} + \|u'_{*, \theta}\|_{L^p(\mathbb{R}^+, E_0; dt/t)}.
\]

It can be shown that those functions are continuous in \( t = 0 \). We define the real interpolation spaces as follows:

Definition 3. The real interpolation space \((E_0, E_1)_{\theta, p}\) is defined as set of traces of functions belonging to \( V(p, 1 - \theta, E_1, E_0) \) at \( t = 0 \) together with the norm:

\[
\|x\|_{T_r} = \inf \{\|u\|_V \mid u \in V(p, 1 - \theta, E_1, E_0), \ u(0) = x\}.
\]

We conclude with two estimates for \( x \in E_0 \cap E_1 \).

Lemma 1. Let \( x \in E_0 \cap E_1 \). Then

\[
\|x\|_{T_r} \leq C\|x\|^{1-\theta}_{E_0}\|x\|^\theta_{E_1}, \quad (6)
\]

and for \( u \in V(\infty, 1 - \theta, E_1, E_0) \) with \( u(0) = x \):

\[
\|x - u(t)\|_{E_0} \leq C(\theta)t^\theta\|u\|_V. \quad (7)
\]

Proof. For \( r > 0 \), we set

\[
u(t) := \begin{cases} (1 - t/r)x & \text{if } t < r \\ 0 & \text{else} \end{cases}.
\]

Note that

\[
\|t^{1-\theta}u(t)\|_{E_1} \leq r^{1-\theta}\|x\|_{E_1}
\]

and

\[
\|t^{1-\theta}u'(t)\|_{E_0} \leq r^{-\theta}\|x\|_{E_0}
\]

and choose \( r = \frac{\|x\|_{E_1}}{\|x\|_{E_0}} \). This yields \( (6) \).

For the second estimate, we write \( u(0) - u(t) = \int_0^t u'(s)ds \) and conclude

\[
\|x - u(t)\|_{E_0} \leq \int_0^t \|u'(s)\|_{E_0}ds
\]

\[
\leq \int_0^t s^{\theta-1}\|s^{1-\theta}u'(s)\|_{E_0}ds
\]

\[
\leq \int_0^t s^{\theta-1}\|u\|_Vds
\]

\[
\leq C(\theta)t^\theta\|u\|_V. \quad \square
\]
2.3. Box dimension, Whitney decomposition and level sets of Hölder functions. We recall the following decomposition of an open set into cubes, due to Whitney [30], and a bound of the number of cubes of a certain size.

**Lemma 2.** Let \( U \subset \mathbb{R}^n \) be open. Then there is a countable collection of mutually disjoint axis-aligned cubes \( W \) whose diameters are comparable to their distance from \( \partial U \), i.e.

(i) \( U = \bigcup_{Q \in W} Q \)

(ii) For all \( Q \in W \) there is \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z}^n \) such that \( Q = 2^{-k}(l + (0,1)^n) \). We denote the sub-collection of all cubes of size \( k \) by \( W_k \).

(iii) \( Q \cap Q' = \emptyset \) if \( Q \neq Q' \)

(iv) \( \text{diam } Q \leq \text{dist}(Q,\partial U) \leq 4 \text{ diam } Q \)

For a proof we refer to [29].

It turns out that \( |W_k| \) is related to the box dimension of the boundary of the decomposed set – this is made precise in Lemma 3 below. First, we give the definition of box dimension (cf. e.g. [17]):

**Definition 4.** Let \( E \subset \mathbb{R}^n \) be bounded and \( \beta > 0 \). We introduce the (upper) Hausdorff-type content

\[
\overline{H}^\beta(E) = \limsup_{\varepsilon \to 0} \inf \{ m\varepsilon^\beta \mid E \subset \bigcup_{i=1}^m B_{\varepsilon}(x_i) \}
\]

and define the (upper) box dimension of \( E \) to be

\[
\dim_{\text{box}} E = \sup \{ s \mid \overline{H}^s(E) = \infty \}.
\]

**Lemma 3 ([24], Theorem 3.12).** Let \( U \subset \mathbb{R}^n \) be open and bounded such that \( \dim_{\text{box}} \partial U < d \). Then there is \( M > 0 \) such that

\[
|W_k| \leq M 2^{dk}.
\]

We conclude this subsection with a lemma regarding the box dimension of pre-images of Hölder continuous functions.

**Lemma 4.** Let \( U \subset \mathbb{R}^n \) be open and bounded, \( f \in C^{0,\alpha}(U) \). If \( \beta \geq n - \alpha \) then we have

\[
\dim_{\text{box}} f^{-1}(r) \leq \beta \quad \text{for a.e. } r \in \mathbb{R}.
\]

**Proof.** For the sake of contradiction, assume that there exists a set \( A \subset \mathbb{R} \) of positive measure such that for all \( r \in A \),

\[
\dim_{\text{box}} f^{-1}(r) > \beta.
\]

Let \( k \in \mathbb{N} \) be arbitrary. Then, by assumption, there is \( 0 < \varepsilon_r < 1 \) for every \( r \in A \) such that

\[
\inf \{ m\varepsilon^\beta \mid f^{-1}(r) \subset \bigcup_{i=1}^m B_{\varepsilon_r}(x_i) \} > k.
\]

We conclude that at least \( k\varepsilon_r^{-\beta} \) balls of radius \( \varepsilon_r \) are necessary to cover \( f^{-1}(r) \) for \( r \in A \). Obviously,

\[
A \subset \bigcup_{r \in A} B_{C\varepsilon_r}(r).
\]

Using the Vitali covering lemma, we obtain \( J \subset A \) countable such that the balls \( B_{C\varepsilon_r}(j) \) (for \( j \in J \)) are pairwise disjoint and \( A \subset \bigcup_{j \in J} B_{3C\varepsilon_r}(j) \). Therefore, we have that

\[
\sum_{j \in J} 2C\varepsilon_r^\beta \geq \frac{1}{5} \mathcal{L}^1(A).
\]
Again using the Vitali covering lemma on the families \( \{ B_{\frac{1}{k}x_0}(x) \mid x \in f^{-1}(j) \} \), we obtain countable collections \( \mathcal{I}_j \) of pairwise disjoint balls, such that
\[
f^{-1}(j) \subset \bigcup_{B \in \mathcal{I}_j} B^* ,
\]
where \( B^* \) denotes the concentric ball with five times the radius. Note that by \( \mathcal{I}_j \), we have \(| \mathcal{I}_j | > k\varepsilon^{-\beta} \) for every \( j \in J \). Moreover, for \( B = \overline{B}_{\frac{1}{k}x_0}(x) \in \mathcal{I}_j \) and \( B' = \overline{B}_{\frac{1}{k}x_0}(x') \in \mathcal{I}_j' \) we observe that if \( B \cap B' \neq \emptyset \):
\[
| j - j' | = | f(x) - f(x') | \leq \| f \|_{C^{0,\alpha}} | x - x' |^{\alpha} < \| f \|_{C^{0,\alpha}} \varepsilon_j^{\alpha}
\]
and hence \( j = j' \). We conclude that
\[
\mathcal{L}^n(f^{-1}(A)) \geq \sum_{j \in J} \sum_{B \in \mathcal{I}_j} \mathcal{L}^n(B) \geq \sum_{j \in J} k\varepsilon_j^{n-\beta} \geq \sum_{j \in J} k\varepsilon_j^{\alpha} \geq \frac{k}{10\| f \|_{C^{0,\alpha}}} \mathcal{L}^1(A) ,
\]
where we used that by assumption \( n - \beta \leq \alpha \). This is a contradiction since \( k \) was arbitrary.

2.4. Approximating \( C^{1,\alpha} \) isometric immersions by smooth ones. Let \( U \subset \mathbb{R}^n \) be open and bounded, and \( \alpha \in (0,1] \). We consider an immersion \( y \in C^{1,\alpha}(U;\mathbb{R}^{n+1}) \), and write
\[
Dy^T Dy = g .
\]
We will assume that \( g \) is a smooth function \( U \rightarrow \mathbb{R}^{n \times n} \), with values in the positive definite \( n \times n \) matrices \( \text{Sym}^+_n \). I.e., \( g \in C^{\infty}(U;\text{Sym}^+_n) \).

Let \( \varphi \) be a standard symmetric mollifier; i.e., \( \varphi \in C^\infty_c(\mathbb{R}^n) \), \( \varphi(x) = \varphi(-x) \) for \( x \in \mathbb{R}^n \) and \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \). We set \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\frac{x}{\varepsilon}) \) and define the mollifications \( y_\varepsilon \) as well as their induced metrics \( g_\varepsilon \) by setting
\[
y_\varepsilon := \varphi_\varepsilon * (y\chi_U) \quad \text{and} \quad g_\varepsilon := Dy_\varepsilon^T Dy_\varepsilon .
\]
(10)

The following estimate from \cite{15} is crucial for our analysis:

**Lemma 5** (Proposition 1 in \cite{15}). If \( y \in C^{1,\alpha} \), then we have
\[
\|Dy_\varepsilon^T Dy_\varepsilon - Dy^T Dy\|_{C^r(U;\text{Sym}^+_n)} \leq C\varepsilon^{2\alpha-r} .
\]
For the reader who is unfamiliar with this estimate, we mention that its proof is based on the commutator estimate
\[
\|(fg) * \varphi_\varepsilon - (f * \varphi_\varepsilon)(g * \varphi_\varepsilon)\|_{C^r} \leq C_r \varepsilon^{2\alpha-r} [f]_{C^{0,\alpha}} [g]_{C^{0,\alpha}} .
\]
which appeared first in context of the Onsager conjecture on Energy Conservation for Euler’s equation (see \cite{14}).

From Lemma 5 and the interpolation inequality \( \| u \|_{C^{1,\beta}} \leq C \| u \|_{C^{1,\beta}}^{\frac{1-\beta}{\gamma}} \| u \|_{C^2}^{\frac{\beta}{\gamma}} \) (see Lemma 1), we obtain
Lemma 6. If \( y \in C^{1,\alpha}(U, \mathbb{R}^n) \) and \( g_\varepsilon \) is defined by \([\text{III}]\), then
\[
ge_{\varepsilon} \to g \quad \text{in } C^{1,\beta}(U; \text{Sym}^n_+) \quad \text{for all } \beta < 2\alpha - 1.
\]

For the rest of this section, let \( 0 < \beta < 2\alpha - 1 \). Let \( \mathcal{U} \) be a small neighborhood of \( g \) in \( C^{1,\beta}(U, \text{Sym}^n_+) \). At every point \( x \in U \), an ordered basis of the tangent space is given by \((\partial_1, \ldots, \partial_n)\). To this ordered basis, and for every \( \tilde{g} \in \mathcal{U} \), we may apply the Gram-Schmidt process with respect to \( \tilde{g}(x) \), and obtain an orthonormal frame \((X_1, \ldots, X_n)\). Let \( U_1 \) be a suitably chosen neighborhood of \( g(x) \) in \( \text{Sym}^n_+ \). Note that the map
\[
U_1 \to \mathbb{R}^{n \times n}
\]
\[
\tilde{g}(x) \mapsto (X_1(x), \ldots, X_n(x))
\]
is in \( C^\infty(U_1; \mathbb{R}^{n \times n}) \). Hence, the map
\[
\mathcal{U} \to C^{1,\beta}(U; \mathbb{R}^{n \times n})
\]
\[
\tilde{g} \mapsto (X_1, \ldots, X_n)
\]
is continuous. (It is in order to have this continuity why we choose a particular orthonormal frame, instead of choosing an arbitrary one.) We define dual one-forms \( \theta^i \in C^\infty(U; \Lambda^1 \mathbb{R}^n) \) by requiring \( \theta^i(X_j) = \delta^i_j \), where \( \delta^i_j \) is the Kronecker delta. By this definition, we have
\[
(\theta^i(\partial_j))_{i,j=1,\ldots,n} = ((dx_j(X_i))_{i,j=1,\ldots,n})^{-1}
\]
which is again smooth as a function of the \( X_i \) (in some uniform neighborhood of the \( X_i(x) \) given by \( \tilde{g}(x) \mapsto (X_1(x), \ldots, X_n(x)) \), \( x \in U \), \( \tilde{g} \in \mathcal{U} \)), and hence the map \( \tilde{g} \mapsto (\theta^1, \ldots, \theta^n) \) is continuous from \( \mathcal{U} \) to \( C^{1,\beta}(U; \Lambda^1 \mathbb{R}^n)^n \). The connection one forms associated to the frame \( X \) are defined by the first structural equation,
\[
d\theta^i = \sum_j \omega^i_j \wedge \theta^j, \quad \omega^i_j = -\omega^j_i.
\]
This defines \( (\omega^i_j(x))_{i,j=1,\ldots,n} \) as a smooth function of \( (d\theta^1(x), \ldots, d\theta(x)) \) and \( (\theta^1(x), \ldots, \theta(x)) \). This in turn implies our final conclusion, which we state as a lemma:

**Lemma 7.** The map \( \tilde{g} \mapsto (\omega^i_j)_{i,j=1,\ldots,n} \) defined above, associating to a metric \( \tilde{g} \) a connection-one form satisfying \([\text{III}]\), is continuous from some neighborhood \( \mathcal{U} \subset C^{1,\beta}(U; \text{Sym}^n_+) \) of \( g \) to \( C^{0,\beta}(U; \Lambda^1 \mathbb{R}^n)^{n \times n} \).

3. Distributional Pfaffians, Jacobians, and their integrals over sets with fractal boundary

3.1. Distributional Jacobians and Pfaffians for Hölder functions. The aim of the present subsection is to give a definition of the Jacobian determinant for Hölder continuous functions, as well as a definition of the Pfaffian form for metrics with Hölder continuous derivatives. In both cases, the Hölder exponent has to be large enough for this to be possible. The main technical ingredient is real interpolation, and the core of the argument is contained in the proof of Proposition \([\text{IV}]\) below.

Recall that for an open and bounded set \( U \subset \mathbb{R}^n \), we denote the space of smooth \( k \)-forms on \( U \) by \( C^\infty(U; \Lambda^k \mathbb{R}^n) \). Furthermore, we set
\[
C^\infty_{cd}(U; \Lambda^k \mathbb{R}^n) = \{ \omega \in C^\infty(U; \Lambda^k \mathbb{R}^n) \mid d\omega = 0 \}.
\]
In the following, we will only be interested in the case $k = n - 1$. We introduce two norms on $C^\infty(U; \Lambda^{n-1}\mathbb{R}^n)/C^\infty_1(U; \Lambda^{n-1}\mathbb{R}^n)$,

$$
\|\omega\|_{X_0^{n-1}} := \inf\{\|\omega + a\|_{C^0(U; \Lambda^{n-1}\mathbb{R}^n)} \mid a \in C^\infty_1(U; \Lambda^{n-1}\mathbb{R}^n)\}
$$

and denote by $X_0^{n-1}$ and $X_1^{n-1}$ the completion of $C^\infty(U; \Lambda^{n-1}\mathbb{R}^n)/C^\infty_1(U; \Lambda^{n-1}\mathbb{R}^n)$ with respect to these norms. We introduce the shorthand notation

$$
X_\theta = (X_0^{n-1}, X_1^{n-1})_{\theta, \infty}.
$$

**Proposition 1.** Let $k_1, \ldots, k_J \in \mathbb{N}$ and $0 \leq I < J$ with $\sum_{i=1}^J k_i + I = n - 1$. For $\omega_i \in C^\infty(U; \Lambda^{k_i}\mathbb{R}^n)$ with $i = 1, \ldots, J$, set

$$
M_{k,I}(\omega_1, \ldots, \omega_J) = \omega_1 \wedge \cdots \wedge \omega_{J-I} \wedge d\omega_{J-I+1} \wedge \cdots \wedge d\omega_J.
$$

Furthermore, let $\beta \in (0, 1]^J$ such that $\theta := \min_{i=1, \ldots, J-I} \beta_i + \sum_{i=J-I+1}^J \beta_i - I \in (0, 1)$, and $X_\theta = (X_0^{n-1}, X_1^{n-1})_{\theta, \infty}$. Then

$$
\|M_{k,I}(\omega_1, \ldots, \omega_J)\|_{X_\theta} \leq \prod_{i=1}^J \|\omega_i\|_{C^{0, \beta_i}}. \tag{11}
$$

Moreover, for $\bar{\theta} < \theta$, $M_{k,I}$ extends to a multi-linear continuous operator

$$
C^{0, \beta_1}(U; \Lambda^{k_1}\mathbb{R}^n) \times \cdots \times C^{0, \beta_J}(U; \Lambda^{k_J}\mathbb{R}^n) \to X_{\bar{\theta}}.
$$

**Proof.** We use that $C^{0, \beta_i} = (C^0, C^1)_{\beta_i, \infty}$, write

$$
\omega_i = \sum_{1 \leq i_1 < \cdots < i_{k_i} \leq n} v_{i_1, \ldots, i_{k_i}}^i(0) dx_{i_1} \wedge \cdots \wedge dx_{i_{k_i}}
$$

for some $v_{i_1, \ldots, i_{k_i}}^i \in V(\infty, 1 - \beta_i, C^0, C^0)$. Without loss of generality, we may assume that $\text{supp} v_{i_1, \ldots, i_{k_i}}^i \subset [0, 1]$ and set

$$
v^i(t) = \sum_{1 \leq i_1 < \cdots < i_{k_i} \leq n} v_{i_1, \ldots, i_{k_i}}^i(t) dx_{i_1} \wedge \cdots \wedge dx_{i_{k_i}}.
$$

The main idea is to write $dM_{k,I}(v_1(t), \ldots, v_J(t))$ and $(M_{k,I}(v_1(t), \ldots, v_J(t)))'$ as wedge products of $(I + 1)$ derivatives $dv^i$ or $(v^i)'$ and $J - I - 1$ factors of the form $v^i$. Observe that

$$
dM_{k,I}(\omega_1, \ldots, \omega_J) = \sum_{i=1}^J (-1)^{J-I} \sum_{j=1}^{I-1} \omega_i \wedge \cdots \wedge \omega_{i-1} \wedge d\omega_i \wedge \cdots \wedge d\omega_{J-I} \wedge d\omega_{J-I+1} \cdots \wedge d\omega_J
$$

and hence, for all $t \in \mathbb{R}^+$,

$$
\|M_{k,I}(v^1(t), \ldots, v^J(t))\|_{X_\theta^{n-1}} \leq \sum_{i=1}^{J-I} \|v^i(t)\|_{C^1} \prod_{j=1}^{J-I} \|v^j(t)\|_{C^0} \prod_{l=J-I+1}^{J} \|v^l(t)\|_{C^1}.
$$
For $(M_{k,l}(v^1, \ldots, v^j))'$, we use that the $X_0^{n-1}$-norm is only defined up to a closed form to avoid factors involving two derivatives. Note that

$$v^1 \wedge \cdots \wedge v^{i-1} \wedge dv^{i-1} \wedge \cdots \wedge dv^j = \pm d \left( v^1 \wedge \cdots \wedge v^{i-1} \wedge dv^i \wedge dv^{i+1} \wedge \cdots \wedge dv^j \right)$$

$$\pm d \left( v^1 \wedge \cdots \wedge v^j \wedge (v^i)' \wedge dv^{i+1} \wedge \cdots \wedge dv^j \right).$$

Note that the $n - 1$ form in the last line above is closed. When computing the $X_0^{n-1}$-norm of $(M_{k,l}(v^1, \ldots, v^j))'$, we therefore may replace every term involving $(dv)^i$ with a sum of terms involving $I$ exterior and one time derivative. Thus we have

$$\| (M_{k,l}(v^1(t), \ldots, v^j(t)))' \|_{X_0^{n-1}} \leq \sum_{i=1}^J \| (v^i)'(t) \|_{C^0} \prod_{j=1, \ldots, J-I, j \neq i} \| v^j(t) \|_{C^0} \prod_{l=J-I+1, \ldots, J} \| v^l(t) \|_{C^1}$$

for all $t \in \mathbb{R}^+$. By the definition of $\theta$, we have that

$$t^{1-\theta} = t^{1-\min_{i=1, \ldots, J-I} \beta_i} \prod_{i=J-I+1}^J t^{1-\beta_i}.$$

Recall that

$$t^{1-\beta_i} \left( \| v^i(t) \|_{C^1} + \| (v^i)'(t) \|_{C^0} \right) \leq \| v^i \|_V$$

for all $t \in \mathbb{R}^+$, and observe that by the second estimate in Lemma 1, we have

$$\| v^i(t) \|_{C^0} \leq \| v^i(0) \|_{C^0} + \| v^i(t) - v^i(0) \|_{C^0} \leq \| v^i(0) \|_{C^0} + t^{\beta_i} \| v^i \|_V \lesssim \| v^i \|_V.$$

We conclude that for all $t \in \mathbb{R}^+$, we have

$$\| t^{1-\theta} M_{k,l}(v^1(t), \ldots, v^j(t))' \|_{X_0^{n-1}} \lesssim \prod_{j=1, \ldots, J} \| v^j \|_{V(\infty, 1-\beta_j, C^1, C^0)}$$

$$\| t^{1-\theta} (M_{k,l}(v^1(t), \ldots, v^j(t)))' \|_{X_0^{n-1}} \lesssim \prod_{j=1, \ldots, J} \| v^j \|_{V(\infty, 1-\beta_j, C^1, C^0)}.$$

Taking appropriate infima on both sides of these estimates completes the proof of (11).

To prove the statement about the extension of $M_{k,l}$, we only need to choose $\tilde{\beta}_i < \beta_i$ for $i = 1, \ldots, J$ such that

$$\tilde{\theta} = \min_{i=1, \ldots, J-I} \tilde{\beta}_i + \sum_{i=J-I+1}^J \tilde{\beta}_i - I$$

and note that $\omega_i$ can be approximated in $C^0, \tilde{\beta}_i(U; \Lambda^k \mathbb{R}^n)$ by smooth functions $\omega_i, \delta \in C^\infty(U; \Lambda^k \mathbb{R}^n)$,

$$\omega_i, \delta \to \omega_i \quad \text{in} \; C^0, \tilde{\beta}_i(U; \Lambda^k \mathbb{R}^n) \quad \text{as} \; \delta \to 0.$$

By the estimate (11) applied with $\tilde{\theta}$ and $\tilde{\beta}_i$, $i = 1, \ldots, J$,

$$\delta \mapsto M(\omega_1, \delta, \ldots, \omega_J, \delta)$$

is a continuous function with values in $X_{\theta}$, whose limit does not depend on the choice of the approximations $\omega_i, \delta$. This proves the proposition. \qed
Proof. Let us fix \( Q \) and furthermore, the map \( \partial Q \) and \( X \) is continuous on \( \partial Q \) and \( X \) (not including), such that elements in this space can be integrated over fractals of dimension up to (but not including) \( n \). Then, we have the following:

\[
\Pi(\omega) = \sum_{I=1, \ldots, n/2-1}^{i_1, \ldots, i_{2(n-I-1)} \in \{1, \ldots, n\}} e_I^{i_1, \ldots, i_{2(n-I-1)}} M(1, \ldots, 1) \prod_{j \text{ times } I} \omega^{i_1, \ldots, i_{2(n-I-1)}}(1, \ldots, 1),
\]

where \( J = n - 1 - I \). By Proposition 7, it follows that the map

\[
\omega \mapsto \Pi(\omega)
\]

is continuous from \( C^{0,\beta}(\mathbb{R}^n) \) to \( X_\theta \) for every \( \theta < n/2 - (n/2 - 1) \).

3.2. Integrating distributional Pfaffians and Jacobians over sets with fractal boundary. The interpolation space \( X_\theta = (X^{n-1}_0, X^{n-1}_1)_{\theta, \infty} \) has been chosen in a way such that elements in this space can be integrated over fractals of dimension up to (but not including) \( n - 1 + \theta \); it will be shown now how this works. We adapt the arguments from [24], and give a well defined meaning to integrals over differentials \( dM \) with \( M \in X_\theta \).

We fix some \( U \subset \mathbb{R}^n \) with \( d := \dim_{\text{box}} U < n - 1 + \theta \). Let \( W \) denote the Whitney decomposition of \( U \). Recalling the properties of trace spaces from Section 2.2, we have that for \( M \in X_\theta \) there exists \( M(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^n; C^1(U; \mathbb{R}^n)) \) such that

\[
t^{1-\theta} \left( \|M(t)\|_{C^1} + \|M'(t)\|_{C^0} \right) \leq \|M\|_{X_\theta} \quad \text{for all } t \in \mathbb{R}^+,
\]

and

\[
\lim_{t \to 0} \|M - M(t)\|_{C^0} = 0.
\]

Definition 5. For \( M \in (X^{n-1}_0, X^{n-1}_1) \), we define the integral \( \int_U dM \) by

\[
\int_U dM := \sum_{Q \in W} \int_Q dM(\text{diam } Q) + \int_{\partial Q} (M - M(\text{diam } Q)).
\]

Lemma 8. The above definition makes \( \int_U dM \) well defined for \( M \in X_\theta \) for \( n - 1 + \theta > d \). Furthermore, the map

\[
M \mapsto \int_U dM
\]

is continuous on \( X_\theta \).

Proof. Let us fix \( M \in X_\theta \) and choose \( M(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^+; C^1(U; \mathbb{R}^n)) \) as above. Let \( Q \in W \). We estimate

\[
\left| \int_Q dM(\text{diam } Q) \right| \leq \mathcal{L}^n(Q) \|M(\text{diam } Q)\|_{X^{n-1}_1} \leq \mathcal{L}^n(Q)(\text{diam } Q)^{\theta - 1} \|M\|_{X_\theta}
\]

and

\[
\left| \int_{\partial Q} (M - M(\text{diam } Q)) \right| \leq \mathcal{H}^{n-1}(Q) \|M - M(\text{diam } Q)\|_{X^{n-1}_0} \leq \mathcal{H}^{n-1}(Q)(\text{diam } Q)^{\theta} \|M\|_{X_\theta}.
\]
where we have used Lemma 1 in the second estimate. By Lemma 3, the number of cubes in $W$ of sidelength $2^{-k}$ can be estimated by $C2^{kd}$, where the constant $C$ may depend on the domain $U$, and $d = \dim_{\text{box}} \partial U$. Hence we may estimate

$$\left| \int_U dM \right| \leq C 2^{kd} \| M \|_{X_\theta} + \sum_{Q \in W} H^{n-1}(Q)(\text{diam } Q)^{\theta} \| M \|_{X_\theta}$$

The sum on the right hand side is absolutely convergent, by the assumption $d < n - 1 + \theta$. This implies that $\int_U dM$ exists and is independent of the choice of $M(\cdot)$ (which makes $\int_U dM$ well defined). Moreover the map $M \mapsto \int_U dM$ is linear and thus continuous. □

### 3.3. Weak convergence of the Brouwer degree.

Let $U \subset \mathbb{R}^n$ with $\dim_{\text{box}} \partial U = d \in [n - 1, n)$, and $\alpha \in (0, 1)$ such that $n\alpha - d > 0$. The following lemma and proposition are taken from [26]; we repeat the proofs for the convenience of the reader. In the lemma, we use the notation $(A)_\varepsilon := \{ x \in \mathbb{R}^n : \text{dist}(x, A) < \varepsilon \}$ for $A \subset \mathbb{R}^n$.

**Lemma 9.** Let $V \subset \mathbb{R}^n$ be open and bounded, $U \subset V$, $n - 1 < \dim_{\text{box}} \partial U = d < n$, $0 < \alpha < 1$ such that $n\alpha > d$, and $u \in C^{0,\alpha}(V; \mathbb{R}^n)$. Then

$$\mathcal{L}^n((\partial U)_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$  

**Proof.** We choose $\delta := \varepsilon^{\alpha - 1}$. Let $x_i \in \partial U$, $i = 1, \ldots, k$, be a finite collection of points in the boundary such that

$$\partial U \subset \bigcup_{i=1}^k B(x_i, \delta)$$

$$B(x_i, \delta/5) \cap B(x_j, \delta/5) = \emptyset \quad \text{for } i, j \in \{1, \ldots, k\}, i \neq j.$$  

The existence of the collection $\{x_i\}$ is assured by the Vitali Covering Lemma. Now let $d < \tilde{d} < n\alpha$. This implies $H^{\tilde{d}}(\partial U) = 0$ (with $H$ defined in [3]). Choosing $\varepsilon$ small enough, we may assume that

$$k\delta^{\tilde{d}} \leq 1,$$

We observe that the image of the boundary is covered by the collection of balls with centers $u(x_i)$ and radius $\| u \|_{C^{0,\alpha}} \delta^{\alpha}$,

$$u(\partial U) \subset \bigcup_{i=1}^k B(u(x_i), \| u \|_{C^{0,\alpha}} \delta^{\alpha})$$

$$= \bigcup_{i=1}^k B(u(x_i), \| u \|_{C^{0,\alpha}} \varepsilon).$$

Next we define $c_0 = \| u \|_{C^{0,\alpha}} + 1$ and obtain

$$(u(\partial U))_\varepsilon \subset \bigcup_{i=1}^k B(u(x_i), c_0 \varepsilon) .$$
Putting it all together, we have the chain of inequalities

\[
L^n \left( (u(\partial U))_\varepsilon \right) \leq k L^n(B(0,1))(\varepsilon_0 \varepsilon)^n
\]
\[
\leq C(u, n)(\varepsilon_0^{-d}) k \varepsilon^d
\]
\[
\leq C(u, n)(\delta_{\alpha})
\]
\[
\rightarrow 0 \text{ as } \varepsilon \rightarrow 0 ,
\]

which proves the lemma. □

**Proposition 2.** Let \( u^j \in C^\infty(U; \mathbb{R}^n) \) with \( u^j \rightarrow u \) in \( C^{0,\alpha}(U; \mathbb{R}^n) \), and \( 1 < p < n\alpha/d \). Then

\[
\deg(u^j, U, \cdot) \rightarrow \deg(u, U, \cdot) \text{ in } L^p(\mathbb{R}^n).
\]

**Proof.** Since \( u^j \) is smooth, we have the classical change of variables type formula

\[
\int_U \varphi(u^j(x)) \det Du^j dx = \int_{\mathbb{R}^n} \varphi(z) \deg(u^j, U, z) dz
\]

for any \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \). Let \( p' \) be given by \( p^{-1} + (p')^{-1} = 1 \). We will show

\[
\sup_{j \rightarrow \infty} \sup \left\{ \int_U \varphi(u^j(x)) \det Du^j dx : \varphi \in L^{p'}(\mathbb{R}^n), \| \varphi \|_{L^{p'}} \leq 1 \right\} < \infty. \tag{12}
\]

This implies that \( \deg(u^j, U, \cdot) \) is bounded in \( L^p \) and hence there exists a weakly convergent subsequence. By the convergence \( u^j \rightarrow u \) in \( C^{0,\alpha}(U; \mathbb{R}^n) \), we have that \( \deg(u^j, U, \cdot) \rightarrow \deg(u, U, \cdot) \) pointwise in \( S^2 \setminus u(\partial U) \). By Lemma \([9]\), \( u(\partial U) \) has measure 0, which implies

\[
\deg(u^j, U, \cdot) \rightarrow \deg(u, U, \cdot) \text{ pointwise a. e.,}
\]

and hence we conclude

\[
\deg(u^j, U, \cdot) \rightarrow \deg(u, U, \cdot) \text{ in } L^p(S^n).
\]

Since we would have obtained the same starting from any subsequence of \( u^j \), we get the claim of the proposition. It remains to show (12).

Let us fix \( \varphi \in L^{p'}(\mathbb{R}^n) \). We define \( \zeta \in W^{2,p}(\mathbb{R}^n) \) by

\[
\Delta \zeta = \varphi,
\]

and set \( \psi(x) = D\zeta(x) - D\zeta(0) \). By standard elliptic regularity, we have \( D\psi \in L^{p'}(\mathbb{R}^n; \mathbb{R}^{n \times n}) \) with

\[
\| D\psi \|_{L^{p'}} \leq C \| \varphi \|_{L^{p'}}.
\]

Since \( p < n\alpha/d < n/(n - 1) \), we have \( p' > n \) and hence, Morrey’s inequality implies

\[
[\psi]_{C^{0,1-n/p'}} \leq C \| \varphi \|_{L^{p'}}. \tag{13}
\]

Let \( \tilde{\alpha} := (1 - n/p')\alpha \). We claim that \( \psi \circ u^j \in C^{0,\tilde{\alpha}} \) with

\[
\| \psi \circ u^j \|_{C^{0,\tilde{\alpha}}} \leq C \| \varphi \|_{L^{p'}} \| u^j \|_{C^{0,\tilde{\alpha}} \alpha}^{1-n/p'}. \tag{14}
\]
Indeed we have
\[
\sup_x |\psi \circ u^j(x)| = \sup_x |\psi(u^j(x)) - \psi(0)|
\leq [\psi]_{C^{0,1-n/p'}} \sup_x |u^j(x)| - 0 \left|^{1-n/p'}\right.
\leq [\psi]_{C^{0,1-n/p'}} \|u^j\|_{C^{0,\alpha}},
\]
and furthermore
\[
|\psi \circ u^j(x) - \psi \circ u^j(x')| \leq [\psi]_{C^{0,1-n/p'}} \left|u^j(x) - u^j(x')\right|^{1-n/p'}
\leq [\psi]_{C^{0,1-n/p'}} \|u^j\|_{C^{0,\alpha}} \left|x - x'\right|^{(1-n/p')}.
\]
This proves the claim (14). Next, we recall the identity
\[
\varphi \circ u^j \det Du^j = \text{div} \left( \psi \circ u^j \text{cof} Du^j \right),
\]
which can be verified easily by noting \(\text{div} \text{cof} Du^j = 0\) and \(Du^j \text{cof} Du^j = \det Du^j \text{Id}_{n \times n}\).

For the rest of this proof, we write
\[
M \equiv M_{(0,\ldots,0),n-1},
\]
where the right hand side has been defined in Section 3.1. We recall
\[
M(u^1, \ldots, u^n) = u^1 du^2 \wedge \cdots \wedge du^n \quad \text{up to a closed (n-1)-form},
\]
and express the identity (15) using this notation:
\[
\varphi \circ u^j dM(u^1, \ldots, u^n) = \sum_{i=1}^n dM(u^1, \ldots, u^i_{i-1}, \psi_i \circ u^i, u^i_{i+1}, \ldots, u^n).
\]
Note that
\[
(n-1)(\alpha - 1) + \tilde{\alpha} = \frac{n\alpha}{p'}
= n\alpha - \frac{n\alpha}{p}
< n\alpha - d.
\]
Hence we may choose \(\theta \in (n\alpha/p', n\alpha - d)\), and we may estimate as follows:
\[
\left| \int_U \varphi(u^j(x)) \det Du^j(x) dx \right| \leq \left| \int_U \sum_{i=1}^n dM(u^1, \ldots, u^i_{i-1}, \psi_i \circ u^i, u^i_{i+1}, \ldots, u^n) \right|
\leq \sum_i \|M(u^1, \ldots, u^i_{i-1}, \psi_i \circ u^i, u^i_{i+1}, \ldots, u^n)\|_{X_{\theta}}
\leq \sum_i \|u^i\|_{C^{0,\alpha}} \|\psi_i \circ u^i\|_{C^{0,\alpha}}
\leq \sum_i \|u^i\|_{C^{0,\alpha}} \|\varphi\|_{L^{p'}}.
\]
This proves (12) and hence the proposition. \(\square\)
Remark 2. Let \( \nu \) be as in Section 2.4. By using a smooth atlas on \( S^n \), and considering the situation in coordinate charts, we get as an immediate consequence of Proposition 2 that

\[
\deg(\nu, U, \cdot) \to \deg(\nu, U, \cdot) \quad \text{in } L^p(S^n) \quad \text{for } 1 < p < \frac{n\alpha}{d}.
\]

4. Proof of Theorems 1 and 2

For the proof of Theorem 1, we first consider the case \( \varphi = \chi_U \).

Proposition 3. Let \( U \subset \mathbb{R}^n \) be open and bounded with \( \dim \text{box } \partial U = d \in [n-1, n) \). Let \( y \in C^{1,\alpha}(U; \mathbb{R}^{n+1}) \) be an immersion with \( n\alpha > d \), let \( \nu \in C^{0,\alpha}(U; S^n) \) be the unit normal, and let \( \Pi(\Omega) \) be the Pfaffian form obtained from the metric \( g = D_y T D_y \). Then

\[
\hat{\Omega} = \int_{S^n} \deg(\nu, U, z) d\mathcal{H}^n(z).
\]

Proof. Let \( y_\varepsilon, g_\varepsilon \) be as in (10), and let \( \omega, \omega_\varepsilon \) be the connection one-forms associated to \( g, g_\varepsilon \) respectively, as in Section 2.4. Furthermore, let \( \Omega, \Omega_\varepsilon \) be the curvature forms associated to \( g, g_\varepsilon \) respectively. Since \( y_\varepsilon \) is smooth, we have

\[
\int_U d\Pi(\omega_\varepsilon) = \int_{S^n} \deg(\nu_{y_\varepsilon}, U, z) d\mathcal{H}^n.
\]

We are going to pass to the limit \( \varepsilon \to 0 \) on both sides. On the right hand side, the limit is \( \int_{S^n} \deg(\nu, U, z) d\mathcal{H}^n \) by Remark 2. It remains to show that the limit on the right hand side is \( \int_U d\Pi(\omega) \). By Lemma 6 and Lemma 7, we have

\[
\left( \omega_i^j(g_\varepsilon) \right)_{i,j=1,...,n} \to \left( \omega_i^j(g) \right)_{i,j=1,...,n} \quad \text{in } C^{0,\beta}(U; \Lambda^1 \mathbb{R}^n)
\]

for all \( \beta < 2\alpha - 1 \). Remark 1 implies that for

\[
\theta < \frac{n\alpha - 1}{2} - \left( \frac{n}{2} - 1 \right) = n\alpha - (n - 1),
\]

we have

\[
\Pi(\omega(g_\varepsilon)) \to \Pi(\omega(g)) \quad \text{in } X_\theta \text{ as } \varepsilon \to 0.
\]

By our assumptions on \( d, \alpha \), we may choose \( \theta \) such that it fulfills (18) and additionally \( \theta > d - (n - 1) \). By Lemma 8 we get

\[
\int_U d\Pi(\omega(g_\varepsilon)) \to \int_U d\Pi(\omega(g)) \quad \text{as } \varepsilon \to 0.
\]

This proves the proposition. 

Remark 3. We note that Proposition 3 could also have been deduced using the techniques from [32]. It suffices to note that by the Gauss-Bonnet-Chern Theorem, the Pfaffian form has the right structure to apply Theorem 3.2 from [32], and hence one can pass to the limit \( \varepsilon \to 0 \) on the left hand side.

Proof of Theorem 1. Let \( \varphi_k \in C^1(S^n \setminus \nu(\partial U)) \) be a sequence that is bounded uniformly in \( L^\infty \) and converges pointwise to \( \varphi \). It is sufficient to prove the claim for \( \varphi_k \), and then apply the dominated convergence theorem to obtain it for \( \varphi \). Hence, from now on, we may assume \( \varphi \in C^1(S^n \setminus \nu(\partial U)) \).

We set

\[
A_r := \{ x \in U : \varphi \circ \nu(x) > r \}.
\]
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Note that \( \varphi \circ \nu \in C^{0,\alpha}(S^n) \), and hence by Lemma 4, we have

\[
\dim_{\text{box}} \partial A_r = \dim_{\text{box}}(\varphi \circ \nu)^{-1}(r) \leq n - \alpha \quad \text{for a.e. } r \in \mathbb{R}.
\]

(19)

Denoting the characteristic function of \( A_r \) by \( \chi_{A_r} \), we have for every \( x \in U \),

\[
\varphi \circ \nu(x) = \int_0^\infty \chi_{A_r}(x)dr - \int_{-\infty}^0 (1 - \chi_{A_r}(x))dr.
\]

By Fubini’s Theorem, we get

\[
\int_U \varphi \circ \nu Pf(\Omega) = \int_0^\infty \int_{A_r} Pf(\Omega)dr - \int_{-\infty}^0 \int_{U \setminus A_r} Pf(\Omega)dr.
\]

(20)

Now let \( \hat{A}_r := \{ z \in S^n : \varphi(z) > r \} \). Obviously, \( A_r = \nu^{-1}(\hat{A}_r) \), and hence for every \( z \in S^n \setminus \nu(\partial A_r) \),

\[
\deg(\nu, A_r, z) = \chi_{\hat{A}_r}(z) \deg(\nu, U, z),
\]

\[
\deg(\nu, U \setminus A_r, z) = (1 - \chi_{\hat{A}_r}(z)) \deg(\nu, U, z).
\]

(21)

Finally, for every \( z \in S^n \), we have

\[
\varphi(z) = \int_0^\infty \chi_{\hat{A}_r}(z)dr - \int_{-\infty}^0 (1 - \chi_{A_r}(z))dr.
\]

(22)

Combining (20), (21), (22) and Fubini’s Theorem, we obtain

\[
\int_U \varphi \circ \nu Pf(\Omega) = \int_{S^n} \varphi(z) \deg(\nu, U, z)d\mathcal{H}^n(z).
\]

This proves the theorem. \( \square \)

**Proof of Theorem 2.** The proof works as in [15]. We claim that for all \( V \subset M \) open with smooth boundary, we have that

\[
\deg(\nu, V, \cdot) \geq \chi_{\nu(V) \setminus \nu(\partial V)}.
\]

(23)

Without loss of generality, we may assume that \( V \) is diffeomorphic to an open subset of \( \mathbb{R}^n \). If not, cover \( V \) by finitely many open sets \( V_1, \ldots, V_r \) with smooth boundary that are diffeomorphic to an open subset of \( \mathbb{R}^n \). Set \( \tilde{V}_i = V \cap (V_i \setminus \cup_{j<i} V_j)^\circ \), where we have used
the notation $A^\circ$ to denote the interior of a set $A \subset M$. Using additivity of the mapping degree we obtain for $z \notin \nu(\bigcup_{i=1}^r \tilde{V}_i)$,

$$
\deg(\nu, V, z) = \deg(\nu, \bigcup_{i=1}^r \tilde{V}_i, z) = \sum_{i=1}^r \deg(\nu, \tilde{V}_i, z) \\
\geq \sum_{i=1}^r \chi_{\nu(\tilde{V}_i) \setminus \nu(\partial \tilde{V}_i)}(z) \\
\geq \chi_{\nu(V) \setminus \nu(\partial V)}(z)
$$

But $\mathcal{H}^n(\nu(\partial \tilde{V}_i)) = 0$ for $i = 1, \ldots, r$, and since $\deg(\nu, V, \cdot)$ is locally constant we obtain the inequality for all $z \in S^n \setminus \nu(\partial V)$.

By definition, we have $\deg(\nu, V, z) = 0$ if $z \notin \nu(V)$. For the sake of contradiction, assume that there is $z_0 \in \nu(V)$ such that $\deg(\nu, V, z_0) \leq 0$. We consider a small disk $D$ around $z_0$ with

$$
D \cap \nu(\partial V) = \emptyset
$$

and set $W = \nu^{-1}(D)$. Note that $\nu(W) \subset D$ and by continuity of $\nu, \nu(\partial W) \subset \partial D$. Hence, $\deg(\nu, W, z) = 0$ for $z \in S^n \setminus D$ and $\deg(\nu, W, z) = k$ for $z \in D$, where $k$ is some integer.

Let $\varphi \in C^1(S^n)$ with $\varphi \geq 0$, $\varphi(z_0) > 0$ and $\text{supp} \varphi \subset D$. By Theorem 1 we have

$$
\int_{\partial D} \varphi(z) \deg(\nu, W, z) \, dz = \int_W \varphi \circ \nu \text{Pf}(\Omega) > 0.
$$

This implies that $k > 0$. By additivity of the degree we have

$$
0 < \deg(\nu, W, z_0) = \deg(\nu, V, z_0) - \deg(\nu, V \setminus W, z_0) = \deg(\nu, V, z_0) \leq 0
$$

since, by construction $z_0 \notin \nu(V \setminus W)$. But this is a contradiction.

Now let $F_1, \ldots, F_r \subset M$ be closed and pairwise disjoint. We can cover them with disjoint open sets $V_1, \ldots, V_r \subset M$ with smooth boundary and use (23) and Proposition 3 to obtain

$$
\sum_{i=1}^r \mathcal{H}^n(\nu(F_i) \setminus \nu(\partial V_i)) \leq \sum_{i=1}^r \mathcal{H}^n(\nu(V_i) \setminus \nu(\partial V_i)) \\
\leq \sum_{i=1}^r \int_{S^n} \deg(\nu, V_i, z) \, dz \\
= \sum_{i=1}^r \int_{V_i} \text{Pf}(\Omega) \\
\leq \int_M \text{Pf}(\Omega),
$$

which is finite. By our choice of the $V_i$, we have $\mathcal{H}^n(\nu(\partial V_i)) = 0$ for $i = 1, \ldots, r$, and hence the theorem is proved.

\[\square\]

**Remark 4.** As is easily seen from the proof, we could have deduced Theorem 2 directly from Proposition 3 (without using Theorem 1) by choosing $\varphi \equiv \chi_D$ in (24).
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(Sören Behr) Hausdorff Center for Mathematics & Institute for Applied Mathematics, Bonn, Germany

E-mail address: s6sobehr@uni-bonn.de

(Heiner Olbermann) Hausdorff Center for Mathematics & Institute for Applied Mathematics, Bonn, Germany

E-mail address: heiner.olbermann@hcm.uni-bonn.de