Asymptotic properties of gravitational and electromagnetic fields in higher dimensions

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Abstract.
We summarize the fall-off of electromagnetic and gravitational fields in n > 5 dimensional Ricci-flat spacetimes along an asymptotically expanding non-singular geodesic null congruence.

1. Introduction
Under suitable assumptions, the well-known peeling-off property characterizes the behavior of the gravitational and electromagnetic fields at null infinity (see, e.g., [1, 2] and references therein). It has been observed [3] that the Weyl tensor peels off differently in n > 4 dimensions. Here, we summarize our recent results [4, 5] on the leading-order behavior of gravitational and electromagnetic fields in higher dimensions. Ref. [4] partly recovers the results of [3] but uses a different method and different assumptions. We restrict to Ricci-flat spacetimes with suitable properties at null infinity (a cosmological constant can be included [4, 5]), formulated in terms of a geodesic null vector field $\ell = \partial_r$ ($r$ is an affine parameter) and of the Weyl tensor, using a “null” frame [6] based on two null vectors $m_{(0)} = \ell$, $m_{(1)} = n$ and $n - 2$ orthonormal spacelike vectors $m_{(i)}$ ($i, j, \cdots = 2, \ldots, n - 1$). First, we assume that the optical matrix $\rho_{ij} = \ell_a m^a_{(i)} \ell^b m^b_{(j)}$ is asymptotically non-singular and expanding [4, 5] (this includes asymptotically flat spacetimes [3] but also holds more generally – see [7] in four dimensions). Furthermore, we assume that the boost-weight (b.w.) $+2$ Weyl components $\Omega_{ij} \equiv C_{00ij} = C_{abcd} \ell^a m^b_{(i)} \ell^c m^d_{(j)}$ fall off as

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2). \quad (1)$$

Again, this is satisfied in asymptotically flat spacetimes [3] (e.g., $\Omega_{ij} = O(r^{-5})$ in the 4D spacetimes of [7]). Under the above conditions, one is able to determine how the Maxwell and Weyl tensors fall off as $r \to \infty$, as we summarize in sections 2 and 3. However, as an intermediate step, one also needs the $r$-dependence of the Ricci rotation coefficients and of the derivative operators [6], which is given in [4] (it follows from the Ricci identities [8], also using the commutators [9] and the Bianchi identities [10]). For example, $\rho_{ij} = \delta_{ij} + \ldots$. For brevity, in this paper, we discuss only results in $n > 5$ dimensions – the case $n \geq 5$ is studied in [4, 5].

2. Electromagnetic field
We start from the simpler case of test Maxwell fields in the background of an $n$-dimensional Ricci-flat spacetime satisfying the assumptions of section [1, 5]. The gravitational field (Weyl tensor)
can be treated similarly, however, resulting in a larger number of possible cases (section 3).

In the frame of section 1, we assume that for $r \to \infty$ the Maxwell components have a power-like behavior described by

$$
F_{0i} = O(r^\alpha), \quad F_{01} = O(r^\beta), \quad F_{ij} = O(r^\gamma), \quad F_{1i} = O(r^\delta). \quad (2)
$$

The empty-space Maxwell equations $F^a_{\ b;a} = 0 = F_{[abc]}$ (see [5][11] for their GHP and NP form) determine the possible values of $\alpha$, $\beta$, $\gamma$ and $\delta$. We assume that if a generic component $f$ behaves as $f = O(r^{-\zeta})$ then $\partial_r f = O(r^{-\zeta - 1})$ and $\partial_A f = O(r^{-\zeta})$. As it turns out, $\alpha$ can be chosen arbitrarily, giving raise to two main cases, $\alpha \geq -2$ or $\alpha < -2$. In the latter, one needs to choose whether $\gamma \geq -2$ or $\gamma < -2$, and then specify more precisely the value of $\alpha$, as we detail.

### 2.1. Case $\alpha \geq -2$.

In this case, all components fall off at the same speed, i.e.,

$$
F_{0i} = O(r^\alpha), \quad F_{01} = O(r^\alpha), \quad F_{ij} = O(r^\alpha), \quad F_{1i} = O(r^\alpha). \quad (3)
$$

The electromagnetic field does not peel. This describes, e.g., a uniform magnetic field permeating asymptotically flat black holes [12] (or black rings [13] if $n = 5$ is included, cf. [5]).

### 2.2. Case $\alpha < -2$.

Generically, we have

$$
F_{0i} = O(r^\alpha), \quad (4)
$$

$$
F_{01} = O(r^{-2}), \quad F_{ij} = O(r^{-2}), \quad (5)
$$

$$
F_{1i} = O(r^{-2}). \quad (6)
$$

The above behavior includes the special case when $\ell$ is an aligned null direction of the Maxwell field, i.e., $F_{0i} = 0$ (in the formal limit $\alpha \to -\infty$). The leading term is of type II. Examples can be obtained as a “linearized” Maxwell field limit of certain full Einstein-Maxwell solutions given in [14] for even $n$. Several subcases are possible when $\gamma < -2$.

#### 2.2.1. Subcase (a): $\gamma < -2$ with $1 - \frac{n}{2} \leq \alpha < -2$.

In this case, one has the same results as in section 2.1 above. This subcase does not exist for $n = 6$.

#### 2.2.2. Subcase (b): $\gamma < -2$ with $-\frac{n}{2} \leq \alpha < 1 - \frac{n}{2}$.

Here, we have

$$
F_{0i} = O(r^\alpha), \quad (7)
$$

$$
F_{01} = O(r^\alpha), \quad F_{ij} = O(r^\alpha), \quad (8)
$$

$$
F_{1i} = O(r^{1-n/2}). \quad (9)
$$

The leading term falls off as $1/r^{\frac{n}{2} - 1}$ and is of type N. This is characteristic of radiative fields (note that $T_{11} \propto F_{11}F_{1i} \sim 1/r^{n-2}$ and the energy flux along $\ell$ can be directly related to the energy loss, at least in the case of asymptotically flat spacetimes – cf. [15][17] for $n = 4$). As opposed to the well-known four-dimensional case, here, $\ell$ cannot be aligned with $F_{ab}$ if radiation is present (since $\alpha \geq -\frac{n}{2}$). In the case $\alpha = -\frac{n}{2}$, if one assumes that $F_{1i}$ has a power-like behavior also at the subleading order, from the Maxwell equations, one finds $F_{1i} = F_{1i}^{(0)}r^{1-n/2} + O(r^{-n/2})$, which gives the peeling-off behavior

$$
F_{ab} = \frac{N_{ab}}{r^{\frac{n}{2} - 1}} + \frac{G_{ab}}{r^{\frac{n}{2}}} + \ldots \quad (\alpha = -\frac{n}{2}). \quad (10)
$$

The subleading term is algebraically general, which is qualitatively different from the 4D case [12][16][17]. This resembles the behavior of the Weyl tensor of higher dimensional asymptotically flat spacetimes [3]. See [3] for a possible different peeling-off in five dimensions.
2.2.3. Subcase (c): $\gamma < -2 \text{ with } 2 - n \leq \alpha < -\frac{n}{2}$. The same results as in section 2.1 apply.

2.2.4. Subcase (d): $\gamma < -2 \text{ with } \alpha < 2 - n$. We have

\begin{align*}
F_{0i} &= O(r^\alpha), \\
F_{01} &= O(r^{2-n}), \\
F_{ij} &= o(r^{2-n}),
\end{align*}

The leading term is of type II and falls off as $1/r^{n-2}$ (it is purely electric in the subcase $F_{1i} = o(r^{2-n})$). This behavior includes the Coulomb field of a weakly charged asymptotically flat black hole [12, 18] (or black ring [13] if $n = 5$ is included [5]). In the special subcase $F_{01} = o(r^{2-n})$, the same results as in section 2.1 again apply (for example, for $n = 5$ and $\alpha = -4$, this is the case of the weak-field limit of the 5D dipole black rings of [19]).

Let us observe that in all cases, type N fields for which $\ell$ is aligned are not permitted [11, 20].

2.3. The case of $p$-forms

The above results for a 2-form $F_{ab}$ can be extended easily [5] to $p$-form fields satisfying the generalized Maxwell equations (given in [11] in the GHP notation). In even dimensions, the special case $p = n/2$ (including $n = 4, p = 2$) has unique properties. It peels off as

\begin{equation}
F_{a_1...a_p} = \frac{N_{a_1...a_p}}{r^{\frac{n}{2} - 1}} + \frac{I_{a_1...a_p}}{r^\nu} + \ldots \quad (p = \frac{n}{2}).
\end{equation}

The (radiative) leading term is of type N and falls off as $1/r^{n/2 - 1}$. In contrast to the case $p = 2$ discussed above (or, in fact, any other $p \neq n/2$), Maxwell fields of type N aligned with $\ell$ are now permitted [5] and the peeling (14) applies also in the presence of a cosmological constant [5]. Corresponding solutions of the full Einstein-Maxwell equations have recently been obtained [21].

3. Gravitational field

The method to be used for the Weyl tensor [4] is essentially similar, now $-\nu$ playing the role that $\alpha$ played above. Instead of the Maxwell equations, one has to integrate the system “Bianchi-Ricci-commutators”. However, there is now extra freedom in the choice of possible boundary conditions. In particular, three possible choices for the behavior of b.w. +1 components $\Psi_{ijkl}$ are possible (cases (i), (ii) and (iii) below). Once the fall-off of $\Omega_{ij}$ and $\Psi_{ijkl}$ has been specified, the next step is to determine the fall-off of the b.w. 0 components $\Phi_{ijkl}$

\begin{equation}
\Phi_{ijkl} = O(r^{\beta_\nu}).
\end{equation}

The parameter $\beta_\nu$ can then be used to label various possible subcases, which we now present.

3.1. Case (i): $\Omega_{ij} = O(r^{-\nu}), \Psi_{ijkl} = O(r^{-\nu})$

In all cases given here, we have (this will not be repeated every time below)

\begin{equation}
\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2), \quad \Psi_{ijkl} = O(r^{-\nu}).
\end{equation}

3.1.1. Subcase (A): $\beta_\nu = -2$. In this case, necessarily $\beta_\nu > -\nu$ and we have the following possible behaviors, depending on how $\nu$ is chosen (cf. [2] for a few further special subcases):
A1: 
\[
\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = o(r^{-2}), \quad \Phi_{ij}^A = o(r^{-2}) \quad (2 < \nu \leq 3), \\
\Psi'_{ijk} = O(r^{-2}), \\
\Omega'_{ij} = O(r^{\sigma}) \quad (-2 \leq \sigma < -1);
\]

A2: 
\[
\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{-3}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-3}) \quad (3 < \nu < 4), \\
\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{-3}), \\
\Omega'_{ij} = O(r^{-2});
\]

A3: 
\[
\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{-3}), \quad \Phi = O(r^{-4}), \quad \Phi_{ij}^A = O(r^{-3}) \quad (\nu \geq 4), \\
\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{-3}), \\
\Omega'_{ij} = O(r^{-2}),
\]

with the further restrictions \(\Phi_{ij}^S = O(r^{1-\nu})\) for \(4 \leq \nu < 5\) and \(\Phi_{ij}^S = O(r^{-4})\) for \(\nu \geq 5\):

A4: 
\[
\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{1-\nu}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-\nu}) \quad (\nu \geq 4, \nu \neq n), \\
\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{1-\nu}), \\
\Omega'_{ij} = O(r^{-2});
\]

A5: 
\[
\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{1-n}), \quad \Phi_{ij}^A = O(r^{-n}) \quad (\nu \geq n), \\
\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{1-n}), \\
\Omega'_{ij} = O(r^{-2}).
\]

None of the above five cases can describe asymptotically flat spacetimes, cf. [3]. In cases A2–A5, the leading term falls off as \(1/r^2\) at infinity and it is of type II(abd). In cases A3–A5, \(\ell\) can be a multiple WAND. Examples in case A5 are Robinson-Trautman spacetime [22].

When \(\beta_c < -2\), its precise value depends on the value of \(\nu\) so that we have to consider the following possible cases.

3.1.2. Subcase (B): \(\beta_c < -2\) with \(\frac{n}{2} < \nu \leq 1 + \frac{n}{2}\). In this case, \(\beta_c = -\frac{n}{2}\) and we have
\[
\Phi_{ijkl} = O(r^{-n/2}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-\nu}) \quad \left(\frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right), \\
\Psi'_{ijk} = O(r^{-n/2}), \\
\Omega'_{ij} = O(r^{1-n/2}).
\]

Here, \(\ell\) cannot be a WAND. The leading term at infinity falls off as \(1/r^{n/2-1}\) and it is of type N. This includes radiative spacetimes [3] that are asymptotically flat in the Bondi
definition \[23,24\]. If one takes for b.w. +2 components \( \nu = 1 + \frac{n}{2} \) and additionally assumes that 
\( \Omega_{ij} = \Omega_{ij}^{(0)} r^{-n/2-1} + \Omega_{ijkl}^{(1)} r^{-n/2-2} + o(r^{-n/2-2}) \), then one finds \[4\] the peeling-off behavior
\[
C_{abcd} = \frac{N_{abcd}}{r^{n/2-1}} + \frac{H_{abcd}}{r^{n/2}} + o(r^{-n/2}).
\] (23)

This agrees with \[3\] for asymptotically flat spacetimes. See \[3, 4\] for special properties of the case
\( n = 5 \). When \( \beta_c < -2 \) but \( \nu \) is not in the range \( \frac{n}{2} < \nu < 1 + \frac{n}{2} \) one has the following
subcases \((B^*)\) and \((C)\).

\[3.1.3. \text{Subcase } (B^*): \beta_c < -2 \text{ with } 2 < \nu \leq \frac{n}{2} \text{ or } 1 + \frac{n}{2} < \nu \leq n - 1. \]
In this case, \( \beta_c = -\nu \) and we have (cf. section IV A 5 of \[4\])
\[
\begin{align*}
\Phi_{ijkl} &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}), \\
\Psi'_{ijk} &= O(r^{-2}) \quad \text{if } 2 < \nu \leq 3, & \Psi'_{ijk} &= O(r^{-\nu}) \quad \text{if } \nu > 3, \\
\Omega'_{ij} &= o(r^{1-\nu}) \quad \text{if } \nu \neq \frac{n}{2}, & \Omega'_{ij} &= O(r^{1-n/2}) \quad \text{if } \nu = \frac{n}{2}.
\end{align*}
\] (24)

Here, \( \ell \) cannot be a WAND.

\[3.1.4. \text{Subcase } (C): \beta_c < -2 \text{ with } \nu > n - 1. \]
In this case, \( \beta_c = 1 - n \) and we have
\[
\begin{align*}
\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= o(r^{1-n}) \quad (\nu > n - 1), \\
\Psi'_{ijk} &= O(r^{1-n}), \\
\Omega'_{ij} &= o(r^{2-n}),
\end{align*}
\] (25)

with \( \Phi_{ij}^A = O(r^{-\nu}) \) for \( n - 1 < \nu < n \) and \( \Phi_{ij}^A = O(r^{-n}) \) for \( \nu \geq n \). Here, \( \ell \) can become a multiple WAND, cf. \[25\]. This includes asymptotically flat spacetimes in the case of vanishing radiation \[3\], such as those for which \( \ell \) is a multiple WAND \[25\], e.g., the Schwarzschild-
Tangherlini metric and Kerr-Schild spacetimes \[20\] with a non-degenerate Kerr-Schild vector.

\[3.2. \text{Case (ii): } \Omega_{ij} = o(r^{-n}), \quad \Psi_{ijk} = O(r^{-n}) \]
\[3.2.1. \text{Subcase } \beta_c = -2. \]
Generically, one has
\[
\begin{align*}
\Omega_{ij} &= o(r^{-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_{j} &= O(r^{-3}), \\
\Omega'_{ij} &= O(r^{-2}).
\end{align*}
\] (26)

For \( \Psi_{ijk}^{(n)} = 0 \), this case reduces to \[19\] (with \( \nu > n \)). See \[4\] for possible subcases.

\[3.2.2. \text{Subcase } \beta_c = 1 - n. \]
When \( \beta_c < -2 \) then necessarily \( \beta_c = 1 - n \) and generically, one has
\[
\begin{align*}
\Omega_{ij} &= o(r^{-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{1-n}), & \Psi'_{j} &= O(r^{1-n}), \\
\Omega'_{ij} &= o(r^{2-n}).
\end{align*}
\] (27)
This includes asymptotically flat spacetimes in the case of vanishing radiation \cite{3}. For $\Psi_{ijk}^{(n)} = 0$, this case reduces to (25) (with $\nu > n$).

3.3. Case (iii): $\Omega_{ij} = o(r^{-3})$, $\Psi_{ijk} = O(r^{-3})$

This case cannot represent asymptotically flat spacetimes \cite{3}. Generically, $\beta_c = -2$ and

$$
\begin{align*}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu > 3), \\
\Psi_{ijk} &= O(r^{-3}), \quad \Psi_i = o(r^{-3}), \\
\Phi_{ijk} &= O(r^{-2}), \quad \Phi_i^O = O(r^{-3}), \quad \Phi = o(r^{-3}), \quad \Phi_i^O = O(r^{-3}), \\
\Psi_{ijk}' &= O(r^{-2}), \quad \Psi_i' = O(r^{-3}), \\
\Omega_{ij}' &= O(r^{-2}),
\end{align*}
\tag{28}
$$

where $\Psi_i = O(r^{-\nu}), \Phi = O(r^{-\nu})$ for $3 < \nu \leq 4$ while $\Psi_i = O(r^{-4}), \Phi = O(r^{-4})$ for $\nu > 4$. Here, $\ell$ can be a single WAND and the asymptotically leading term is of type II(abd). For $\Psi_{ijk}^{(3)} = 0$, this case reduces for $3 < \nu < 4$ to \cite{15} (with $\nu > n$), for $4 \leq \nu \leq n$ to \cite{19} and for $\nu > n$ to \cite{20}. If $\beta_c < -2$ then $\Phi_{ijkl} = O(r^{-3})$ and the leading term at infinity becomes of type III(a).

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