Greatest Common Divisors on the Complement of Numerically Parallel Divisors

Keping Huang and Aaron Levin

Abstract

We prove inequalities involving greatest common divisors of functions at integral points with respect to numerically parallel divisors, generalizing a result of Wang and Yasufuku (after work of Bugeaud-Corvaja-Zannier, Corvaja-Zannier, and the second author). After applying a result of Vojta on integral points on subvarieties of semiabelian varieties, we use geometry and the theory of heights to reduce to the (known) case of $\mathbb{G}_m^n$. In addition to proving results in a broader context than previously considered, we also study the exceptional set in this setting, for both the counting function and the proximity function.

1 Introduction

In 2003, Bugeaud, Corvaja, and Zannier [BCZ03] began a new line of results with their proof of the following inequality involving greatest common divisors:

**Theorem 1.1** (Bugeaud, Corvaja, Zannier [BCZ03]). Let $a, b \in \mathbb{Z}$ be multiplicatively independent integers. Then for every $\epsilon > 0$,

$$\log \gcd(a^n - 1, b^n - 1) \leq \epsilon n$$

for all but finitely many positive integers $n$.

The inequality (1.1) was subsequently generalized by Corvaja and Zannier [CZ05], allowing $a^n$ and $b^n$ to be replaced by elements $u$ and $v$, respectively, of a fixed finitely generated subgroup of $\mathbb{Q}^r$, and replacing $u - 1$ and $v - 1$ by more general pairs of polynomials in $u$ and $v$. The second author [Lev19] further generalized this inequality to multivariate polynomials, resulting in the following statement:

**Theorem 1.2.** Let $n$ be a positive integer, $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ a finitely generated group, and $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n) \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ nonconstant coprime polynomials such that not both of them vanish at $(0, 0, \ldots, 0)$. Let $h(\alpha)$ denote the (absolute logarithmic) height of an algebraic number $\alpha$. For all $\epsilon > 0$, there exists a finite union $Z$ of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that

$$\log \gcd(f(u_1, \ldots, u_n), g(u_1, \ldots, u_n)) < \epsilon \max\{h(u_1), \ldots, h(u_n)\}$$

for all $(u_1, \ldots, u_n) \in \Gamma \setminus Z$. 

1
If we identify $G^n_m$ with $\mathbb{P}^n \setminus \bigcup_{i=0}^n H_i$, where $H_0, \ldots, H_n$ are the $n + 1$ coordinate hyperplanes, and let $Y$ be the closed subscheme of $\mathbb{P}^n$ defined by the ideal generated by the (homogenizations) of $f$ and $g$, then the height $h_Y$ associated to $Y$ is closely related to the left-hand side of (1.2). In this language, the following counterpart to Theorem 1.2 was proven in [Lev19].

**Theorem 1.3.** Let $Y$ be a closed subscheme of $\mathbb{P}^n$, defined over a number field, of codimension at least 2. Suppose that $Y$ does not contain any of the points $P_0 = [1 : 0 : \cdots : 0], \ldots, P_n = [0 : 0 : \cdots : 0 : 1]$. Let $\Gamma \subset G^n_m(\overline{\mathbb{Q}})$ be a finitely generated subgroup. Then for all $\varepsilon > 0$, there exists a finite union $Z$ of translates of proper algebraic subgroups of $G^n_m$ such that

$$h_Y(P) \leq \varepsilon h(P)$$

for all $P \in \Gamma \setminus Z \subset \mathbb{P}^n(\overline{\mathbb{Q}})$.

Following Wang and Yasufuku [WY21], we study extensions of Theorem 1.3 where $\mathbb{P}^n$ is replaced by an arbitrary projective variety $X$, the coordinate hyperplanes $H_0, \ldots, H_n$ are replaced by numerically equivalent (or more generally numerically parallel) divisors $D_1, \ldots, D_{n+1}$ on $X$, and the finitely generated subgroup $\Gamma$ is replaced by a set $R$ of $(\sum_{i=1}^{n+1} D_i, S)$-integral points on $X$. More precisely, and to highlight the improvements and assumptions in our results, we work in the following general setup:

- $X$: a projective variety of dimension $n$ defined over a number field $K$.
- $\text{Div}(X)$: the group of Cartier divisors on $X$.
- $D_1, \ldots, D_{n+1}$: effective Cartier divisors on $X$, defined over $K$, that are $\mathbb{Z}$-linearly independent in $\text{Div}(X)$, and such that there exist positive integers $d_i$ such that $d_j D_i \equiv d_i D_j$ for all $1 \leq i, j \leq n+1$.
- $D$: the sum $\sum_{i=1}^{n+1} D_i$.
- $Y$: a closed subscheme of $X$, defined over $K$, of codimension at least 2.
- $M_K$: the set of (nontrivial) places of $K$.
- $S \subset M_K$: a finite set of places of $K$ containing the archimedean places.
- $\mathcal{O}_{K,S}$: the ring of $S$-integers of $K$.
- $R \subset X(K)$: a set of $(D, S)$-integral points on $X$.

Based on the powerful tools developed in [RV20], Wang and Yasufuku [WY21] proved the following generalization of Theorem 1.2.

**Theorem 1.4** (Wang, Yasufuku [WY21]). In addition to the general setup, suppose that $X$ is a Cohen–Macaulay variety, the divisors $D_1, \ldots, D_{n+1}$ are ample and in general position, and the closed subscheme $Y$ does not contain any point of the set $\bigcup_{i=1}^{n+1} \bigcap_{j \neq i} \text{Supp}D_j$. Let $\varepsilon > 0$ and let $A$ be an ample divisor on $X$. Then there exists a proper $\mathbb{Z}$ariski closed subset $Z$ of $X$ such that

$$h_Y(P) \leq \varepsilon h_A(P)$$

for all points $P \in R \setminus Z$. 

2
The main result of this paper is the following theorem.

**Theorem 1.5.** In addition to the general setup, suppose that the closed subscheme \( Y \) does not contain any point of the set \( \bigcup_{i=1}^{n+1} \bigcap_{j \neq i} \text{Supp} D_j \). Let \( \varepsilon > 0 \) and let \( A \) be a big divisor on \( X \). Then there exists a proper Zariski closed subset \( Z \) of \( X \) such that
\[
h_Y(P) < \varepsilon h_A(P)
\] (1.3)
for all \( P \in R \setminus Z \).

Note that the conditions on the variety \( X \) and the divisors \( D_i \) are weaker than those in Theorem 1.4. In particular, we do not require the divisors \( D_i \) to be in general position, nor do we require the divisors \( D_i \) to be ample.

The geometric condition on \( Y \) in Theorems 1.3–1.5 (or the nonvanishing condition in Theorem 1.2) cannot be dropped in general. However, going back to the work of Corvaja and Zannier [CZ05, Prop. 4], it has been observed that this geometric condition may be dropped if one avoids a finite number of local contributions to \( h_Y \) (or the logarithmic gcd).

More precisely, outside of the support of \( Y \), we can decompose the height \( h_Y \) into a sum of local heights
\[
h_Y(P) = \sum_{v \in M_K} \lambda_{Y,v}(P),
\]
and write \( h_Y = m_{Y,S} + N_{Y,S} \), where \( m_{Y,S} = \sum_{v \in S} \lambda_{Y,v} \) and \( N_{Y,S} = \sum_{v \in M_K \setminus S} \lambda_{Y,v} \) are known as the proximity function and counting function, respectively, associated to \( Y \) and the finite set of places \( S \subset M_K \). For the counting function, the following result was proved in [Lev19].

**Theorem 1.6.** Let \( Y \) be a closed subscheme of \( \mathbb{P}^n \) of codimension at least 2, defined over a number field \( K \). Let \( S \) be a finite set of places of \( K \) containing the archimedean places. For all \( \varepsilon > 0 \), there exists a finite union \( Z \) of translates of proper algebraic subgroups of \( \mathbb{G}_m^n \) such that
\[
N_{Y,S}(P) \leq \varepsilon h(P)
\]
for all \( P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus Z \subset \mathbb{P}^n(K) \).

Theorem 1.6 was generalized by Wang and Yasufuku under the assumption that \( Y \) consists of a finite number of points:

**Theorem 1.7.** ([WY21]) In addition to the general setup, suppose that \( X \) is a Cohen–Macaulay variety, the divisors \( D_1, \ldots, D_{n+1} \) are ample and in general position, and \( \dim Y = 0 \). Let \( \varepsilon > 0 \) and let \( A \) be an ample divisor on \( X \). Then there exists a proper Zariski closed subset \( Z \) of \( X \) such that
\[
N_{Y,S}(P) \leq \varepsilon h_A(P)
\]
for all points \( P \in R \setminus Z \).
We give a complete generalization of Theorem 1.6 (and Theorem 1.7) under the hypotheses of the general setup.

**Theorem 1.8.** Assume the general setup. Let $\varepsilon > 0$ and let $A$ be a big divisor on $X$. Then there exists a proper Zariski closed subset $Z$ of $X$ such that

$$N_{Y,S}(P) < \varepsilon h_A(P)$$

(1.4)

for all $P \in R \setminus Z$.

In the case when the divisors $D_1, \ldots, D_{n+1}$ are ample, we can say more about the exceptional $Z$ in Theorem 1.8. In some sense, the only obstruction to $N_{Y,S}$ being small is that the subscheme $Y$ becomes an effective divisor when restricted to some subvariety $Z$ of $Y$, in which case the counting function $N_{Y,S}$ may indeed be large on $Z$.

**Theorem 1.9.** In addition to the general setup, assume further that $\bigcap_{i=1}^{n+1} D_i = \emptyset$ and $D_1, \ldots, D_{n+1}$ are ample. Let $\varepsilon > 0$ and let $A$ be an ample divisor on $X$. Then there exists a proper Zariski closed subset $Z$ of $X$ such that for each irreducible component $Z'$ of $Z$, the subscheme $Z' \cap Y$ has codimension at most 1 in $Z'$, and

$$N_{Y,S}(P) < \varepsilon h_A(P)$$

(1.5)

for all $P \in R \setminus Z$.

As a complement to Theorem 1.9, we similarly study the exceptional set for analogous inequalities where the counting function $N_{Y,S}$ is replaced by the proximity function $m_{Y,S}$ (Theorem 4.4).

The proofs of our main results proceed in roughly two steps. First, we use a result of Vojta on integral points on subvarieties of semiabelian varieties to reduce to considering the case where the divisors $D_i$ are, up to some positive integer multiple, linearly equivalent (instead of just numerically equivalent). Second, we use the linear equivalence to construct a suitable morphism to projective space, where we apply functoriality of heights and results of the second author on $G^m_n$ to derive the desired inequalities. In the second step, in the case of the proximity function, we prove the requisite result on $G^m_n$ in Theorem 3.2, and provide a more refined analysis in Section 4.

Finally, we discuss analogues of our main results in Nevanlinna theory. The proofs of our arithmetic results rely on gcd inequalities of the second author in [Lev19]. Analogous inequalities in Nevanlinna theory, proved more generally for semiabelian varieties, are due to Noguchi, Winkelmann, and Yamanoi:

**Theorem 1.10** (Noguchi, Winkelmann, Yamanoi [NWY02], Theorem 5.1). Let $f : \mathbb{C} \to A$ be a holomorphic map to a semiabelian variety $A$ with Zariski-dense image. Let $Y$ be a closed subscheme of $A$ with $\text{codim} \ Y \geq 2$ and let $\varepsilon > 0$.

1. Then

$$N_f(Y, r) \leq \varepsilon T_f(r).$$

2. There exists a compactification $\bar{A}$ of $A$, independent of $\varepsilon$, such that for the Zariski closure $\bar{Y}$ of $Y$ in $\bar{A}$,

$$T_{\bar{Y},f}(r) \leq \varepsilon T_f(r).$$
Here \( N_f(Y, r) \) is a counting function associated to \( f \) and \( Y \), \( T_{\nabla, f}(r) \) is a Nevanlinna characteristic function associated to \( f \) and \( \nabla \), and \( T_f(r) \) is any characteristic function associated to an appropriate ample line bundle. The notation \( \leq_{\text{exc}} \) means that the estimate holds for all \( r \) outside of a set of finite Lebesgue measure, possibly depending on \( \varepsilon \). See also [NW14], Section 6.5 and [LW20] for more discussion.

We also need the analogue for holomorphic curves of Vojta’s results on integral points on subvarieties of semiabelian varieties due to Noguchi [Nog81] (as formulated by Noguchi-Winkelmann [NW14, Th. 4.9.7]).

Using these results, one can derive the following Nevanlinna-theoretic inequalities, analogous to Theorems 1.5 and 1.8, and generalizing a result of Yasufuku-Wang [WY21, Th. 1.8]. We omit the details here.

**Theorem 1.11.** Assume the general setup, with the number field \( K \) replaced by \( K = \mathbb{C} \). Let \( \varepsilon > 0 \) and let \( A \) be a big divisor on \( X \). Let \( f : \mathbb{C} \to X \setminus \bigcup_{i=1}^{n+1} D_i \) be a holomorphic map with Zariski-dense image. Then we have

\[
N_f(Y, r) \leq_{\text{exc}} \varepsilon T_{A, f}(r).
\]

Suppose furthermore that \( Y \) does not contain any point of the set \( \bigcup_{i=1}^{n+1} \bigcap_{j \neq i} D_j \). Then we have

\[
T_Y(f)(r) \leq_{\text{exc}} \varepsilon T_{A, f}(r).
\]

The background and notation for our results will be given in Section 2. We prove Theorems 1.5 and 1.8 in Section 3, ending with a concrete example on Hirzebruch surfaces. Finally, Section 4 is devoted to inequalities on toric varieties and their applications to the exceptional set for the proximity function under our general setup.

## 2 Background and Notation

In [Sil87], Silverman generalized the Weil height machine for Cartier divisors to height functions on projective varieties with respect to closed subschemes. More precisely, let \( X \) be a projective variety over a number field \( K \), and let \( Z(X) \) denote the set of closed subschemes of \( X \). Note that the closed subschemes \( Y \in Z(X) \) are in one-to-one correspondence with quasi-coherent ideal sheaves \( \mathcal{I}_Y \subseteq \mathcal{O}_X \), and we identify a closed subscheme \( Y \) with its ideal sheaf \( \mathcal{I}_Y \). Silverman assigned to each \( Y \in Z(X) \) and each place \( v \in M_K \) a local height function \( \lambda_{Y,v} \), and to each \( Y \in Z(X) \) a global height function \( h_Y = \sum_{v \in M_K} \lambda_{Y,v} \) that generalizes the Weil height machine for Cartier divisors.

**Theorem 2.1.** ([Sil87]) Let \( X \) be a projective variety over a number field \( K \). There are maps

\[
\lambda : Z(X) \times M_K \to \{ \text{functions } X(K) \to [0, +\infty] \}, \\
h : Z(X) \to \{ \text{functions } X(K) \to [0, +\infty] \},
\]

satisfying the following properties:
1. If \( W, Y \in Z(X) \) satisfy \( W \subseteq Y \), then \( h_W \leq h_Y + O(1) \) and \( \lambda_{W,v} \leq \lambda_{Y,v} + O(1) \) for all \( v \in M_k \).

2. If \( W, Y \in Z(X) \) satisfy \( \text{Supp}(W) \subseteq \text{Supp}(Y) \), then there exists a constant \( C \) such that \( h_W \leq C \cdot h_Y + O(1) \) and \( \lambda_{W,v} \leq C \cdot \lambda_{Y,v} + O(1) \) for all \( v \in M_k \).

3. For all \( W, Y \in Z(X) \), we have \( h_{W+Y} = h_W + h_Y + O(1) \) and \( \lambda_{W+Y,v} = \lambda_{W,v} + \lambda_{Y,v} + O(1) \) for all \( v \in M_k \).

4. Let \( \phi : X' \to X \) be a morphism of projective varieties over \( K \), and let \( Y \in Z(X) \). Then
   \[
   h_{X',\phi^*Y} = h_{X,Y} \circ \phi + O(1),
   \]
   \[
   \lambda_{X',\phi^*Y,v} = \lambda_{X,Y,v} \circ \phi + O(1),
   \]
   for all \( v \in M_k \).

Here, \( Y \subset Z \), \( Y + Z \), and \( \phi^*Y \) are all defined in terms of the associated ideal sheaves (see [Sil87]).

**Remark.** Note that \( \phi^*Y \) is the closed subscheme corresponding to the inverse image ideal sheaf
\[
\phi^{-1}I_Y \cdot \mathcal{O}_{X'},
\]
instead of the pullback of the ideal sheaf. See Caution 7.12.2 of [Har77] for more details.

We shall reduce the estimates of counting functions to those over a compactification \( \mathbb{P}^n \) of \( G_m \). For this purpose we need the following elementary fact (see, e.g., [GW20, p. 112]).

**Lemma 2.2.** Let \( \phi : X \to V \) be a morphism of projective varieties and let \( Y \in Z(V) \). Then \( \text{Supp} \phi^*Y = \phi^{-1}(\text{Supp} Y) \).

We use the following definition of integrality (see [Voj87] for more details).

**Definition 2.3.** Let \( D \) be an effective Cartier divisor on \( X \), let \( R \) be a subset of \( X(K) \setminus \text{Supp} D \), and let \( S \) be a finite set of places of \( K \) containing all the archimedean places. We say that \( R \) is a set of \( (D,S) \)-integral points on \( X \) if there exists a global Weil function \( \lambda_{D,v} \) and constants \( c_v, v \in M_K \), such that

- \( c_v = 0 \) for all but finitely many \( v \), and
- for all \( v \in M_K \setminus S \), we have \( \lambda_{D,v}(P) \leq c_v \) for all \( P \in R \).

We denote numerical and linear equivalence of divisors by \( \equiv \) and \( \sim \), respectively.
3 Proofs of Theorems 1.5, 1.8, and 1.9

We need the following lemma, whose essential ingredient is a result of Vojta [Voj96] on integral points on subvarieties of semiabelian varieties.

**Lemma 3.1.** Assume the general setup. Suppose that the set of \((D, S)\)-integral points \(R\) is Zariski dense in \(X\). Then \(q = h^1(X, \mathcal{O}_X) = 0\) and \(d_jD_i \sim d_iD_j \) (linear equivalence) for all \(i\) and \(j\). If \(\text{div}(f_i) = d_{n+1}D_i - d_iD_{n+1}\), \(i = 1, \ldots, n\), then the morphism

\[
\psi : X \setminus D \to \mathbb{G}_m^n,
\psi(P) = (f_1(P), \ldots, f_n(P))
\]

is dominant and there exists a finite set of places \(S' \supset S\) such that \(\psi(R) \subset \mathbb{G}_m^n(\mathcal{O}_{K, S'})\).

**Proof.** Suppose first that \(q > 0\). Then \(D\) contains the \(n+1\) linearly independent divisors \(D_i\) and \(n + 1 \geq \dim X - q + \text{rk}\{D_i\}_{i=1}^{n+1} + 1 = n + 2 - q\), where \(\text{rk}\{D_i\}_{i=1}^{n+1}\) denotes the rank of the subgroup generated by \(D_1, \ldots, D_{n+1}\) in the Néron-Severi group \(\text{NS}(X)\). Then \(R\) is not Zariski dense by a result of Vojta [Voj96, Corollary 0.3] (noting that the Picard number \(\rho\) in Vojta’s statement may clearly be replaced by \(\text{rk}\{D_i\}_{i=1}^{n+1}\) in the proof).

Suppose now that \(q = 0\). In this case the Albanese variety \(\text{Alb}(X)\) is trivial. Since the Picard variety and Albanese variety are dual, the Picard variety \(\text{Pic}^0(X)\) is also trivial. From the exact sequence (of groups)

\[
0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \text{NS}(X) \to 0,
\]

we have \(\text{Pic}(X) \cong \text{NS}(X)\). Since by assumption \(d_jD_i \equiv d_iD_j\) for all \(i\) and \(j\), it follows that \(d_jD_i \sim d_iD_j\) for all \(i\) and \(j\). Let \(f_1, \ldots, f_n \in K(X)\) be rational functions such that \(\text{div}(f_i) = d_{n+1}D_i - d_iD_{n+1}\), \(i = 1, \ldots, n\). Define a morphism \(\psi : X \setminus D \to \mathbb{G}_m^n\) by \(\psi(P) = (f_1(P), \ldots, f_n(P))\). Since \(R \subset X(K)\) is a set of \((D, S)\)-integral points and the functions \(f_i\) have no zeros or poles outside \(D\), by [Voj87, Lemma 1.4.6] there exists a constant \(c \in K^*\) such that \(cf_i(P), c/f_i(P) \in \mathcal{O}_{K,S}\) for all \(P \in R\). Then for some finite set of places \(S' \supset S\), we have

\[
f_i(P) \in \mathcal{O}_{K,S'}^k, \quad (3.1)
\]

for all \(P \in R\), where \(\mathcal{O}_{K,S'}^k\) is the group of \(S'\)-units of \(K\). It follows that \(\psi(R) \subset \mathbb{G}_m^n(\mathcal{O}_{K,S'})\).

If \(\psi\) is not dominant, then \(\psi(R)\) is not Zariski dense in \(\mathbb{G}_m^n\), and by Laurent’s theorem, \(\psi(R)\) is contained in a finite union \(T\) of translates of proper algebraic subgroups of \(\mathbb{G}_m^n\). Since \(R\) is Zariski dense in \(X\), it follows that \(\psi(X) \subset T\). But \(X\) is irreducible, and so \(\psi(X)\) is in fact contained in an irreducible component of \(T\). It follows that there is a nontrivial multiplicative relation \(f_1^{e_1}\ldots f_n^{e_n} = c\) with \(c \in \bar{K}^\times\). Since \(\text{div}(f_i) = d_{n+1}D_i - d_iD_{n+1}\), this yields a nontrivial linear dependence relation for \(D_1, \ldots, D_{n+1}\), contradicting our **general setup**. It follows that \(\psi\) is dominant.

We now give the proof of Theorem 1.8.
Proof of Theorem 1.8. First, by analyzing the irreducible components of $Y$ one by one, we may assume that $Y$ is itself irreducible. By part 2 of Theorem 2.1 we may assume further that $Y$ is reduced. Finally, we may assume that $R$ is Zariski dense in $X$.

Let $\psi : X \dasharrow \mathbb{P}^n$ be the rational map defined in Lemma 3.1. By blowing up the indeterminacy locus of $\psi$, we may find a projective variety $X'$ and morphisms $\pi : X' \to X$ and $\phi : X' \to \mathbb{P}^n$ such that $\pi$ is birational and $\psi \circ \pi = \phi$ on $\pi^{-1}(X \setminus D)$.

By Definition 2.3, for all $P \in R$ and $P' = \pi^{-1}(P)$,

$$N_{D,S}(P) = N_{\pi^*D,S}(P') + O(1) \leq O(1). \quad (3.2)$$

Suppose first that $Y \subseteq D$. In this case, by (3.2), for all $P \in R$, we have

$$N_{Y,S}(P) \leq N_{D,S}(P) < C,$$

where $C$ depends only on $R$. The result now follows from an appropriate version of Northcott’s theorem [Voj87, Prop. 1.2.9(h)].

Now assume that $Y \not\subseteq D$. Therefore $\text{Supp}(Y \cap D)$ is a proper Zariski closed subset of $Y$. Let $W$ be the closed subscheme with the reduced induced structure on the Zariski closure of $\phi(\pi^{-1}(Y \setminus D)) = \psi(Y \setminus D)$. Then $\dim W = \dim(\psi(Y \setminus D)) \leq \dim(Y \setminus D) = \dim Y$. Since $\dim X = n$ and $\text{codim} Y \geq 2$, it follows that $W$ is of codimension at least 2 in $\mathbb{P}^n$.

By Theorem 1.6, for any given $\varepsilon > 0$, there exists a proper Zariski closed subset $Z_0 \subset \mathbb{P}^n$ such that for all $\psi(P) \in \psi(R) \setminus Z_0$, we have

$$N_{W,S}(\psi(P)) \leq \varepsilon \cdot h(\psi(P)) + O(1).$$

Equivalently, if $P' = \pi^{-1}(P)$, then

$$N_{W,S}(\phi(P')) \leq \varepsilon \cdot h(\phi(P')) + O(1).$$

If $A' = \phi^*H$, where $H$ is a hyperplane of $\mathbb{P}^n$, then $A'$ is a big divisor and by Theorem 2.1, for all $P' \in \pi^{-1}(R)$ such that $\phi(P') \notin Z_0$, we have

$$N_{\phi^*W,S}(P') \leq \varepsilon h_{A'}(P') + O(1). \quad (3.3)$$

Now we note that

$$\pi^{-1}(Y) \subseteq \pi^{-1}(Y \setminus D) \cup \pi^{-1}(D) \subseteq \phi^{-1}(\psi(Y \setminus D)) \cup \pi^{-1}(D) \subseteq \phi^{-1}(W) \cup \pi^{-1}(D),$$

and it follows from Lemma 2.2 that $\text{Supp} \pi^*Y \subseteq \text{Supp}(\phi^*W + \pi^*D)$. By Theorem 2.1, there exists a constant $C$ such that

$$N_{\pi^*Y,S}(P') \leq C(N_{\phi^*W,S}(P') + N_{\pi^*D,S}(P')) + O(1).$$

Figure 1: The Diagram
Then by (3.2) and (3.3), for all $P' \in \pi^{-1}(R) \setminus \phi^{-1}(Z_0)$, we find that

$$N_{\pi^*Y,S}(P') \leq \varepsilon h_A(P') + O(1).$$

Recalling that $\pi$ is birational and $\psi$ is dominant, combining this with Lemma 2.2 of [Lev19], there exists a constant $C' > 0$ and a proper Zariski closed subset $Z \supset \pi(\phi^{-1}(Z_0))$ of $X$ such that

$$N_{Y,S}(P) \leq \varepsilon C'h_A(P) + O(1)$$

for all $P \in R \setminus Z$, finishing the proof. \qed

Under an ampleness assumption, we repeatedly apply Theorem 1.8 to derive the stronger conclusion on the exceptional set in Theorem 1.9.

**Proof of Theorem 1.9.** Let $Z$ be the Zariski closure of the set of points $P \in R$ such that

$$N_{Y,S}(P) \geq \varepsilon h_A(P).$$

Since all of the assumptions of Theorem 1.8 are verified, $Z$ is a proper Zariski closed subset of $X$. After replacing $K$ by a finite extension, we may assume that every irreducible component of $Z$ is geometrically irreducible. Let $Z'$ be such an irreducible component and suppose that $Y|_{Z'}$ has codimension $\geq 2$ in $Z'$.

Let $\iota : Z' \hookrightarrow X$ be the embedding of $Z'$ in $X$ (identifying $Z'$ with its image where convenient). Since $R$ is disjoint from the support of $D$, $Z'$ is not contained in the support of any of the divisors $D_i$, and we can pull back the divisors $D_1, \ldots, D_{n+1}$ to $Z'$. The numerical equivalences $d_iD_i|_{Z'} \equiv d_iD_j|_{Z'}$ are preserved and the divisors $D_i|_{Z'}$ remain ample for $i = 1, \ldots, n+1$. By Theorem 2.1, the set $R \cap Z'(K)$ is a $(D|_{Z'}, S)$-integral set of points. Set $n' = \dim Z'$. Since $\bigcap_{i=1}^{n+1} D_i = \emptyset$, by Lemma 4.6 of [Lev16], the set of divisors $\{D_1|_{Z'}, \ldots, D_{n+1}|_{Z'}\} \subseteq \text{Div}(Z')$ generate a subgroup of rank at least $n' + 1$ in $\text{Div}(Z')$. Rearranging, we may assume that $D_1|_{Z'}, \ldots, D_{n'+1}|_{Z'}$ are linearly independent in $\text{Div}(Z')$. Then by Theorem 1.8 applied to the variety $Z'$, the divisors $D_1|_{Z'}, \ldots, D_{n'+1}|_{Z'}$, and the subscheme $\iota^*Y = Y|_{Z'}$, for any $\varepsilon' > 0$, there exists a proper Zariski closed subset $Z'' \subset Z'$ such that $N_{\iota^*Y,S}(P) < \varepsilon' h_{\iota^*A}(P)$ for all $P \in R \cap (Z' \setminus Z'')(K)$. By Theorem 2.1,

$$N_{Y,S}(\iota(P)) < \varepsilon h_A(\iota(P)) + C' \quad (3.4)$$

for some constant $C'$ and all $P \in R \cap (Z' \setminus Z'')(K)$. Since $A$ is ample, we may eliminate the term $C'$ by adding finitely many points to $Z''$, and the resulting inequality contradicts the definition of $Z'$. Thus, $Y|_{Z'}$ can have codimension at most one in $Z'$. \qed

We now give an estimate for the proximity function. Compared with earlier results (e.g., [WY21, Th. 3.1], [HL21, Cor. 1.5]) we do not require the divisors $D_i$ to be ample, nor do we require them to be in general position.
for all \( P \in R \setminus Z \).

**Proof.** We may assume that \( R \) is Zariski dense in \( X \) (otherwise the conclusion of the theorem is trivial). Then by Lemma 3.1, after replacing the divisors \( D_i \) by suitable integer multiples, we assume that the divisors \( D_i \) are linearly equivalent. We define maps \( \psi, \phi, \pi \) as in the proof of Theorem 1.8, so that \( \psi \circ \pi = \phi \) and the rational map \( \psi: X \rightarrow \mathbb{P}^n \) is given by

\[
\psi(P) = [f_1(P) : \cdots : f_n(P) : f_{n+1}(P)],
\]

where \( \text{div}(f_i) = D_i - D_{n+1}, i = 1, \ldots, n + 1 \), and \( f_{n+1} = 1 \). Noting that \( \psi_i := [f_1/f_i : \cdots : f_{n+1}/f_i] \) is defined for points outside \( D_i \), \( i = 1, \ldots, n + 1 \), we see that \( \psi \) may be extended to a morphism on \( X \setminus \bigcap_{i=1}^{n+1} D_i \).

Now suppose \( Y \) verifies the assumptions of the theorem. Since \( Y \) does not pass through any point in \( \bigcap_{i=1}^{n+1} D_i \), we have \( \psi(Y) = \phi(\pi^{-1}(Y)) \), which is closed as \( \phi \) is proper. Let \( P_0 = [1 : 0 : \cdots : 0], \ldots, P_n = [0 : 0 : \cdots : 0 : 1] \) in \( \mathbb{P}^n(K) \). If \( P \notin \text{Supp} \, D_i \) and \( \psi_i(P) \) has only one nonzero coordinate, then clearly \( P \in \bigcap_{j \neq i} D_i \). It follows that the image \( \psi(Y) \) does not contain any of the points \( P_i, i = 0, \ldots, n \).

Then we may find an effective divisor \( E \) on \( \mathbb{P}^n \) such that \( \psi(Y) \subseteq \text{Supp} \, E \) and \( P_0, \ldots, P_n \notin \text{Supp} \, E \). By Theorem 4.1 of [Lev19], there exists a finite union \( Z' \) of translates of proper algebraic subgroups of \( \mathbb{G}_m^n \) such that \( m_{E,S}(P') \leq \varepsilon \cdot h(P') \) for all points \( P' \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus Z' \subseteq \mathbb{P}^n(K) \). This implies that \( m_{\pi^*Y,S}(P) \ll m_{\phi^*E,S}(P) \leq \epsilon h_{E,H}(P) \) for all \( P \in \pi^{-1}(R) \) outside some proper Zariski closed subset of \( X' \), where \( H \) is a hyperplane in \( \mathbb{P}^n \). As in the proof of Theorem 1.8, this in turn implies that there exists a proper Zariski closed subset \( Z \subseteq X \) such that

\[
m_{Y,S}(P) < \varepsilon h_A(P)
\]

for all \( P \in R \setminus Z \).

**Proof of Theorem 1.5.** This is a straightforward combination of Theorem 1.8 and Theorem 3.2.

We end this section by describing a family of surfaces \( X \setminus D \) where: Theorem 1.8 applies, the surfaces \( X \setminus D \) contain dense sets of integral points (so that the inequalities are not vacuous), and previous results do not naively apply (of course, our results are ultimately derived from [Lev19]).

**Example 3.3.** Consider the \( n \)th Hirzebruch surface \( X := \mathbb{F}_n, n \geq 1 \). Let \( C_0 \) denote a section of \( \pi: X \rightarrow \mathbb{P}^1 \), and let \( f \) be a fiber of \( \pi \). We say that a divisor \( D \) on \( X \) is of type \((a, b)\) if \( D \sim aC_0 + bf \). Let \( b \geq n \) be an integer and let \( B_1, B_2, B_3, B_4 \) be distinct
irreducible curves of types \((0, 1), (0, 1), (1, b), \text{ and } (1, b+1)\), respectively, all defined over a number field \(L\). Then we have three linearly independent (over \(\text{Div}(X)\)) divisors

\[
\begin{align*}
D_1 &:= B_1 + B_3, \\ D_2 &:= B_2 + B_3, \\ D_3 &:= B_4,
\end{align*}
\]

all of type \((1, b+1)\). Let \(D := D_1 + D_2 + D_3\), and let \(B := B_1 + B_2 + B_3 + B_4\), a reduced divisor of type \((2, 2b+3)\). By Theorem 5.5.4 of [Lev05], there exist choices of the curves \(B_1, B_2, B_3, B_4\) such that \(B\) is a normal crossings divisor and there exist dense sets of \(D\)-integral points over sufficiently large number fields \(M \supset L\). Note that the canonical divisor \(K_X\) is of type \((-2, -n - 2)\), and in this case (when \(B\) is a normal crossings divisor) the log-canonical divisor \(K_X + B\) is of type \((0, 2b - n + 1)\). In particular, since \(2b - n + 1 \geq 1\), \(X \setminus D\) has positive log Kodaira dimension, and \(X \setminus D \not\cong \mathbb{G}_m\). Note also that the divisors \(D_1, D_2, D_3\) are not in general position. Let \(S\) be a finite set of places of \(L\) containing all the archimedean places, let \(R \subseteq X(L)\) be a set of \((D, S)\)-integral points, and let \(A\) be a big divisor on \(X\). Let \(Y \subseteq X\) be a closed subscheme of codimension 2.

By Theorem 1.8, there exists a proper Zariski closed subset \(Z \subseteq X\) such that for all \(P \in R \setminus Z\) we have \(N_{Y, S}(P) \leq \varepsilon h_A(P)\).

4 Proximity Functions on Toric Varieties with Applications

In the final section, we study the proximity function in our setting. We prove a result on the dimension of the exceptional set for the proximity function, first in the case of projective space, and then under our general setup. The proof for projective space exploits the toric structure on \(\mathbb{P}^n\). A toric variety \(X\) is a variety over \(K\) containing \(\mathbb{G}_m^n\) as a dense open subvariety such that the action of \(\mathbb{G}_m^n\) on itself extends to an algebraic action of \(\mathbb{G}_m^n\) on \(X\). We refer to [CLS11] for more details on toric varieties.

**Theorem 4.1.** Let \(X\) be a projective toric variety of dimension \(n\) defined over a number field \(K\). Let \(A\) be an ample divisor on \(X\). Let \(Y\) be a closed subscheme of \(X\), defined over \(K\), whose support does not contain a torus fixed point. Let \(S\) be a finite set of places of \(K\) including the archimedean ones and let \(\varepsilon > 0\). Then there exists a finite union \(Z\) of translates of proper algebraic subgroups of \(\mathbb{G}_m^n\) such that

\[
m_{Y, S}(P) \leq \varepsilon h_A(P) + O(1)
\]

for all points \(P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus Z \subset X(K)\).

**Proof.** We may find an effective Cartier divisor \(D\) on \(X\) such that \(D\) does not contain any torus fixed point and \(Y \subset D\) as closed subschemes. In this case, \(m_{Y, S}(P) \leq m_{D, S}(P) + O(1)\), and we may reduce to the case that \(Y = D\) is a divisor. By Proposition 3.9.1 of [Kol07], there exists an equivariant resolution of singularities \(\pi : \tilde{X} \to X\). Since \(\pi\) is equivariant, the pullback \(\pi^*D\) does not contain a torus fixed point either.
Now we apply the proof of Theorem 4.4 of [Lev19] to the divisor $\pi^*D$ and the big divisor $\pi^*A$ to obtain
\[
m_{\pi^*D,S}(P) \leq \varepsilon h_{\pi^*A}(P) + O(1) \tag{4.1}
\]
for all points $P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus \bar{Z} \subset \bar{X}(K)$, where $\bar{Z}$ is a finite union of translates of proper algebraic subgroups of $\mathbb{G}_m^n$.

The desired inequality then follows from functoriality. \qed

We give a more refined characterization of the exceptional set in the following theorem.

**Theorem 4.2.** Let $X$ be a projective toric variety of dimension $n$ defined over a number field $K$. Let $A$ be an ample divisor on $X$. Let $Y$ be a closed subscheme of $X$ defined over $K$. Let $S$ be a finite set of places of $K$ and let $\varepsilon > 0$. Let $\ell$ be the maximum non-negative integer such that there exists an orbit $O(P), P \in X(K)$, of codimension $\ell$ that intersects $Y$. Then there exists a finite union $Z$ of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ of dimension at most $\ell$ such that
\[
m_{Y,S}(P) \leq \varepsilon h_A(P) + O(1)
\]
for all points $P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus (Y \cup Z) \subset X(K)$.

**Proof.** If $\ell \geq n$, then the result is vacuous. So we may assume that $\ell < n$. This implies that $Y$ does not contain any torus fixed point of $X$. By Theorem 4.1, there exists a finite union $Z'$ of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that
\[
m_{Y,S}(P) \leq \varepsilon h_A(P) + O(1)
\]
holds for all points $P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus Z' \subset X(K)$.

Let $W = gT$ be such a translate in $Z'$. After possibly replacing $K$ by a finite extension (and replacing $S$ by the set of places lying above places of $S$), we may assume that $T$ is geometrically irreducible. Clearly, by induction, it suffices to show that if $\dim W > \ell$ and $W \not\subset Y$, then there exists a proper subset $Z' \subset W$ such that $Z'$ is a finite union of translates of proper algebraic subgroup of $\mathbb{G}_m^n$ and
\[
m_{Y,S}(P) \leq \varepsilon h_A(P) + O(1)
\]
for all points $P \in W(K) \cap \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus Z'' \subset X(K)$.

By enlarging $S$, we may assume that $g \in \mathbb{G}_m^n(\mathcal{O}_{K,S})$. Then $gP \in (gT)(K) \cap \mathbb{G}_m^n(\mathcal{O}_{K,S})$ if and only if $P \in T'(K) \cap \mathbb{G}_m^n(\mathcal{O}_{K,S})$. We also have
\[
m_{Y,S}(gP) = m_{g^*Y,S}(P) + O(1), \quad h_A(gP) = h_{g^*A}(P) + O(1).
\]

An orbit $O(P)$ intersects $Y$ if and only if it intersects $g^*Y$. Then after replacing $Y$ by $g^*Y$ and $W = gT$ by $T$, it suffices to prove the desired result when $W = T \cong \mathbb{G}_m^{\dim T}$ is a subtorus of $\mathbb{G}_m^n$. Let $\overline{T}$ be the Zariski closure of $T$ in $X$. Then $T$ acts naturally on $\overline{T}$, and $\overline{T}$ has an induced structure of a toric variety. We claim that $Y|_{\overline{T}}$ does not contain any $T$-invariant point. Indeed, suppose that $P \in \overline{T}(K)$ is such a $T$-invariant point. Consider the orbit $O(P)$ under the action of $\mathbb{G}_m^n$. Let $\text{stab}(P)$ be the stabilizer of $P$. Since $\text{stab}(P) \supset T$, we have that $\text{stab}(P)$ has dimension at least $\dim T$. Hence the orbit $O(P)$ under the action of $\mathbb{G}_m^n$ has codimension (in $X$) at least $\dim T > \ell$, contradicting the definition of $\ell$. Then functoriality and Theorem 4.1 applied to $\overline{T}$ and $Y|_{\overline{T}}$ give the desired result. \qed

12
We have the following important immediate corollary for the case of \( \mathbb{P}^n \).

**Theorem 4.3.** Let \( Y \) be a closed subscheme of \( \mathbb{P}^n \) defined over a number field \( K \). Let \( H_1, \ldots, H_{n+1} \) be the \( n+1 \) coordinate hyperplanes in \( \mathbb{P}^n \). Suppose that \( \bigcap_{i \in I} H_i \cap Y = \emptyset \) for every subset \( I \subset \{1, \ldots, n+1\} \) with \( |I| > \ell \). Let \( S \) be a finite set of places of \( K \) containing the archimedean places. For any \( \varepsilon > 0 \), there exists a finite union \( Z \) of translates of proper algebraic subgroups of \( \mathbb{G}_m^n \) of dimension at most \( \ell \) such that

\[
m_{Y,S}(P) \leq \varepsilon h(P) + O(1)
\]

for all points \( P \in \mathbb{G}_m^n(\mathcal{O}_{K,S}) \setminus (Y \cup Z) \subset \mathbb{P}^n(K) \). \( \square \)

More generally, we consider the exceptional set for the proximity function in the general case.

**Theorem 4.4.** In addition to the general setup, let \( \varepsilon > 0 \) and let \( A \) be an ample divisor on \( X \). Assume that \( \bigcap_{i=1}^{n+1} D_i = \emptyset \) and that \( D_1, \ldots, D_{n+1} \) are ample. Suppose that \( \bigcap_{i \in I} D_i \cap Y = \emptyset \) for every subset \( I \subset \{1, \ldots, n+1\} \) with \( |I| > \ell \). Then there exists a proper Zariski closed subset \( Z \) of \( X \) of dimension at most \( \ell \) such that

\[
m_{Y,S}(P) < \varepsilon h_A(P)
\]

for all \( P \in R \setminus (Y \cup Z) \).

**Proof.** Let \( \overline{R} \) be the Zariski of \( R \) in \( X \). After possibly extending \( K \), we may assume that every irreducible component of \( \overline{R} \) (over \( K \)) is geometrically irreducible. After replacing \( D_i \) by an appropriate multiple, we may further assume that \( d_i = 1 \) for all \( i \). Let \( X' \not\subseteq Y \) be an irreducible component of \( \overline{R} \), and let \( R' = R \cap X', Y' = Y|_{X'}, D'_i = D_i|_{X'}, A' = A|_{X'} \). By functoriality, it suffices to show that there exists a proper Zariski closed subset \( Z \) of \( X' \) of dimension at most \( \ell \) such that

\[
m_{Y',S}(P) < \varepsilon h_{A'}(P)
\]

for all \( P \in R' \setminus Z \). Let \( n' = \dim X' \). Since \( \bigcap_{i=1}^{n+1} D_i = \emptyset \), it follows from [Lev16, Lemma 4.6] that \( D'_1, \ldots, D'_n \) generate a subgroup of rank at least \( n'+1 \) in \( \text{Div}(X') \). Since \( R' \) is a Zariski dense set of \( (D',S) \)-integral points on \( X' \), where \( D' = \sum_{i=1}^{n+1} D'_i \), it follows from the proof of Lemma 3.1 that \( D'_i \sim D'_j \) for all \( i \) and \( j \). As \( \bigcap_{i=1}^{n+1} D'_i = \emptyset \) as well, we can construct a morphism \( \phi : X' \to \mathbb{P}^n \) such that if \( H_1, \ldots, H_{n+1} \) are the \( n+1 \) coordinate hyperplanes of \( \mathbb{P}^n \), then \( \phi^*H_i = D_i \) for all \( i \). It follows that \( \bigcap_{i \in I} H_i \cap \phi^*(Y) = \emptyset \) for every subset \( I \subset \{1, \ldots, n+1\} \) with \( |I| > \ell \), Furthermore, after possibly enlarging \( S \), \( \phi(R') \subset \mathbb{G}_m^n(\mathcal{O}_{K,S}) \). Then by Theorem 4.3 applied to the closed subscheme \( \phi(Y) \), Lemma 2.2, and Theorem 2.1, there exists a closed subset \( W \subset \mathbb{G}_m^n, \dim W \leq \ell \), such that

\[
m_{Y',S}(P) < \varepsilon h_{A'}(P)
\]

for all \( P \in R' \setminus \phi^{-1}(W) \). Finally, since \( D'_1, \ldots, D'_n \) are ample, \( \phi \) is a finite morphism [Laz04, Cor. 1.2.15]. Then \( \dim \phi^{-1}(W) = \dim W \), completing the proof. \( \square \)
Acknowledgment

We would like to thank Zheng Xiao for many helpful discussions. We also thank Julie Wang for useful discussions, especially on Nevanlinna theory.

References

[BCZ03] Yann Bugeaud, Pietro Corvaja, and Umberto Zannier, *An upper bound for the G.C.D. of $a^n - 1$ and $b^n - 1* Math. Z. 243 (2003), no. 1, 79–84. MR 1953049

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[CZ05] Pietro Corvaja and Umberto Zannier, *A lower bound for the height of a rational function at $S$-unit points*, Monatsh. Math. 144 (2005), no. 3, 203–224. MR 2130274

[GW20] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I. Schemes—with examples and exercises*, Springer Studium Mathematik—Master, Springer Spektrum, Wiesbaden, [2020] ©2020, Second edition [of 2675155]. MR 4225278

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[HL21] Gordon Heier and Aaron Levin, *A generalized schmidt subspace theorem for closed subschemes*, American Journal of Mathematics 143 (2021), no. 1, 213–226.

[Kol07] János Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. MR 2289519

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR 2095471

[Lev05] Aaron Levin, *Generalizations of Siegel’s and Picard’s theorems*, PhD thesis, University of California, Berkeley (2005).

[Lev16] Aaron Levin, *Integral points of bounded degree on affine curves*, Compos. Math. 152 (2016), no. 4, 754–768. MR 3484113
[Lev19] _, Greatest common divisors and Vojta’s conjecture for blowups of algebraic tori, Invent. Math. 215 (2019), no. 2, 493–533. MR 3910069

[LW20] Aaron Levin and Julie Tzu-Yueh Wang, Greatest common divisors of analytic functions and Nevanlinna theory on algebraic tori, J. Reine Angew. Math. 767 (2020), 77–107. MR 4160303

[Nog81] Junjiro Noguchi, Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. 83 (1981), 213–233. MR 632655

[NW14] Junjiro Noguchi and Jörg Winkelmann, Nevanlinna theory in several complex variables and Diophantine approximation, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 350, Springer, Tokyo, 2014. MR 3156076

[NWY02] Junjiro Noguchi, Jörg Winkelmann, and Katsutoshi Yamanoi, The second main theorem for holomorphic curves into semi-abelian varieties, Acta Math. 188 (2002), no. 1, 129–161. MR 1947460

[RV20] Min Ru and Paul Vojta, A birational Nevanlinna constant and its consequences, Amer. J. Math. 142 (2020), no. 3, 957–991. MR 4101336

[Sil87] Joseph H. Silverman, Arithmetic distance functions and height functions in Diophantine geometry, Math. Ann. 279 (1987), no. 2, 193–216. MR 919501

[Voj87] Paul Vojta, Diophantine approximations and value distribution theory, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987. MR 883451

[Voj96] _, Integral points on subvarieties of semiabelian varieties. I, Invent. Math. 126 (1996), no. 1, 133–181. MR 1408559

[WY21] Julie Tzu-Yueh Wang and Yu Yasufuku, Greatest common divisors of integral points of numerically equivalent divisors, Algebra Number Theory 15 (2021), no. 1, 287–305. MR 4226990