Variational analysis of driven-dissipative bosonic fields

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We present a method to perform a variational analysis of the quantum master equation for driven-dissipative bosonic fields with arbitrary large occupation numbers. Our approach combines the P representation of the density matrix and the variational principle for open quantum systems. We benchmark the method by comparing it to wave-function Monte-Carlo simulations and the solution of the Maxwell-Bloch equation for the Jaynes-Cummings model. Furthermore, we study a model describing Rydberg polaritons in a cavity field and introduce an additional set of variational parameters to describe correlations between different modes.

I. INTRODUCTION

The theoretical analysis of driven-dissipative quantum many-body systems is a very challenging task, as many methods developed for closed quantum systems cannot be applied. While important insights have been obtained using the Wave-Function Monte-Carlo (WFMC) method [1, 2] or tensor network approaches [3–6], the numerical study of bosonic fields with large occupation numbers remains an outstanding problem [7]. Here, we show that the variational principle for open quantum systems [8] provides for a natural way to represent arbitrarily large occupation numbers in terms of a small set of variational parameters, thus providing a highly efficient description of the system.

Driven-dissipative systems of bosons arise in many different settings, ranging from semiconductor polaritonic systems [9], to cavity quantum electrodynamics arrays [10], to Rydberg polaritons in atomic quantum gases [11, 12]. Especially for the latter, the development of efficient numerical descriptions is of great interest due to the applications of Rydberg polaritons in the context of strongly correlated photon states [13] and photonic quantum computing [14–16].

In this article, we present a variational treatment of the steady state of driven-dissipative bosons. While the Lindblad formalism allows to calculate the steady state based on the solution of a quantum master equation [17], a brute-force solution becomes prohibitive for large Hilbert space dimensions. Here, we apply the variational principle for open quantum systems, which already proved to be a reliable method in the context of spin-1/2 particles [8, 18, 19]. However, a direct implementation of the variational principle for bosonic fields results in a large amount of variational parameters to describe large Hilbert space, which drastically reduces the efficiency of the method and still relies on a cut-off of the Hilbert space. Therefore, we turn to a different implementation of the variational principle based on the Glauber-Sudarshan P representation.

Phase-space representations of the density matrix have received considerable interest in recent years to classify the nonclassical properties of quantum states [20–26]. For this, the P representation by Glauber and Sudarshan [27, 28] is particularly useful, as negative values directly point to nonclassical behavior.

In this following, we want to use this representation of the density matrix to expand the variational method for open quantum systems to bosonic fields. We also show how the formalism of the variational principle translates to the Heisenberg equations of motions and how this can be used to extend the method to highly singular P distribution where an explicit representation is not feasible. We benchmark our approach by comparing to WFMC simulations [29, 30] mean-field calculations for the Jaynes-Cummings model [31, 32]. We also study a highly correlated model describing Rydberg polaritons in a cavity field. There, we introduce additional variational parameters to also implement correlation between different modes. Finally, we give an outlook how to extend our approach to incorporate other nonclassical states.

II. VARIATIONAL METHOD

Our method is based on the idea to use the P representation of the density matrix operator to express an variational state for a bosonic field for a minimazation process. First, we want to give a brief introduction to both concepts and then we show how to combine them.

In the context of open quantum systems, states are commonly described in terms of their density operator ρ. The Markovian dynamics of such open quantum systems can then be described by the Lindblad master equation

\[ \frac{d}{dt} \rho = \mathcal{L} \rho , \]

with the Liouvillian \( \mathcal{L} \) being the generator of the dynamics for the density matrix \( \rho \) [35], i.e.

\[ \frac{d}{dt} \rho(t) = \mathcal{L}(\rho) = -i[H(t), \rho(t)] + \sum_i \left( c_i \rho c_i^\dagger - \frac{1}{2} (c_i^\dagger c_i, \rho) \right) . \]  

At the same time, the equation of motions can also be expressed in the Heisenberg picture, i.e., for any given observable \( A \) as

\[ \frac{d}{dt} A(t) = i[H, A] + \sum_i \left( c_i A c_i^\dagger - \frac{1}{2} c_i^\dagger c_i, A \right) . \]  

In both equations, the jump operators \( c_i \) correspond to
incoherent processes, e.g. dephasing or dissipation, between the system and the bath.

Solving Eq. (1) is often very challenging [7], i.e., typically some approximations have to be made. Within the variational principle for open quantum systems [8], the density matrix is approximated by the usage of a variational ansatz. In case we want to solve for the steady state of the system \( \rho = 0 \) we need to minimize

\[
\|\hat{\rho}_{\text{var}}\| = \|\mathcal{L}(\hat{\rho}_{\text{var}})\| \to \min
\]

with \( \|\hat{\rho}\| = \text{Tr} \{ |\hat{\rho}| \} \) being the trace norm. Here, we want to define a similar approach in the Heisenberg picture, where the steady state is defined as \( \frac{d}{dt} \langle \hat{A} \rangle (t) = 0 \) for all operators \( \hat{A} \). Therefore, we determine the steady state by minimizing a suitable norm for a small subset of operators given by

\[
\sum_n \frac{d}{dt} \langle \hat{A}_n \rangle_{\text{var}}(t) = \sum_n |\mathcal{L}(\langle \hat{A}_n \rangle_{\text{var}})| \to \min.
\]

with \( \langle \hat{A}_n \rangle_{\text{var}} \) being the \( n \)th variational expectation value \( \text{Tr} \{ \hat{\rho}_{\text{var}} \hat{A}_n \} = \langle \hat{A}_n \rangle_{\text{var}} \). In the next paragraph, we will discuss how the P representation can be used to construct \( \hat{\rho}_{\text{var}} \) for bosonic fields.

The most straightforward implementation of a variational density matrix is to use each entry of the matrix as a variational parameter. This is, however, not feasible in most cases. In the case of an infinite Hilbert space the number of variational parameters also goes to infinity. A solution for this problem can be found in the P representation of the density matrix [27].

\[
\rho = \int d^2 \alpha \ P(\alpha) \ |\alpha\rangle \langle \alpha|.
\]

The non-orthogonality of coherent states forms an overcomplete basis set of states which we can use to represent the density matrix if we combine them with an appropriate choice of quasiprobability distribution \( P(\alpha) \). Such a distribution can theoretically be found for any kind of density matrix [30] if we allow the class of generalized function in our distribution. This excludes any interpretation as an analogue to classical distribution functions because \( P(\alpha) \) can become negative or more singular than a Dirac delta function \( \delta(x) \).

A useful property of this specific representation is the way how expectation values of annihilation (creation) operators \( a \ (a^\dagger) \) are calculated through c-number integrals

\[
\langle a^\dagger p a^q : \rangle = \text{Tr} \{ \rho a^\dagger p a^q \} = \int d^2 \alpha P(\alpha) \alpha^p \alpha^q ,
\]

where \( \langle a^\dagger p a^q : \rangle \) indicates normal ordering of the operators. We will drop the indicator for normal ordering and always assume that the expectation values are in that order for the rest of the paper.

We also want to consider the combination of different quantum states to expand the variational manifold. The construction of the corresponding P distribution is done by a convolution

\[
P(\alpha) = (P_1 \ast P_2)(\alpha) = \int d\alpha' P_1(\alpha') P_2(\alpha - \alpha')
\]

of the original distributions \( P_1 \) and \( P_2 \) [27]. If we insert \( P_7 \) into \( P_6 \) we obtain

\[
\langle a^\dagger p a^q \rangle = \sum_{n,m} \xi_{p,q} \langle (a^\dagger)^n a^m \rangle P_i \langle (a)^p (a^\dagger)^n (a^\dagger)^m \rangle P_j \]

\[
\]

to compute expectation values of a convoluted P distribution with \( \xi_{n,m} \) as the number of possible combinations of the given expectation values from \( \langle a^\dagger n a^m \rangle \). For example, if we assume that one of the distribution are for the thermal state we regain the same result as in [35]. We see that the calculation depends on all expectation values up to the orders \( p, q \) of \( a, a^\dagger \) of the original expectation value but are calculated for the single P distributions \( P_i \) and \( P_j \). This process can then be repeated multiple times to combine multiple distributions. Table I in App. A shows examples of the convolution of two different states.

These ingredients are all that is required to formulate the variational principle in terms of the P distribution. The equations of motion created by Eq. (2) for the expectation value \( \hat{A} \) depend only on expectation values like

\[
\frac{d}{dt} \langle \hat{A} \rangle (t) = F(\{ \langle a^\dagger p a^q \rangle \}_{p,q}).
\]

This means that we can write Eq. (4) as

\[
D = \sum_i F_i (\{ \langle a^\dagger p a^q \rangle \}_{p,q}) \to \min
\]

with \( F_i \) describing the right hand side of Eq. (2), which depends on the set of expectation values \( \{ \langle a^\dagger p a^q \rangle \}_{p,q} \). To see how the P representation can be used to describe these expectation values in terms of variational parameters, it is instructive to have a look at some well known cases for \( P(\alpha) \). First, we consider two classical states, a coherent and a thermal state, represented by

\[
P_{\text{coherent}}(\alpha) = \delta(\alpha - \alpha_0)
\]

\[
P_{\text{thermal}}(\alpha) = \frac{1}{\pi n_0} \exp \left( -\frac{|\alpha|^2}{n_0} \right).
\]

In addition, we can find a expression for a highly non-classical state in form of the Fock states that look like

\[
P_{\text{Fock}} = \frac{1}{R} e^{\alpha^2} \frac{\partial^2}{\partial \alpha \partial \alpha^*} \delta^{(2)}(\alpha)
\]

The distribution includes derivatives of the delta distribution which are defined as \( \int dx \delta^{(n)}(x) \psi(x) = (-1)^n \psi^{(n)}(0) \). We can immediately see that each distribution has one defining parameter, i.e., \( \alpha_0 \in \mathbb{C} \), \( n_0 \in \mathbb{R} \),
or \( l \in \mathbb{N} \). The convolution of two \( P \) distributions results in a new \( P \)-distribution that depends on the set \( \{ \beta \} = \alpha_0, n_0, l \). This allows us to formulate Eq. (10) as

\[
D = \sum_i F_i(\{ \beta \}) \to \min
\]

Upon inspection of Eq. (8) we can also see that we do not need to know the complete form of the \( P \) distribution that corresponds to a specific state. Instead, it is enough to know how all expectation values depend on the variational parameters \( \beta \). This is for example useful if it is difficult to find a complete expression of the \( P \) distribution like in the case of the squeezed coherent state. The state can be obtained by convolution of the coherent state and the squeezed vacuum state where the distribution is known or by directly evaluating the expectation values for this particular state. In this case we know that the squeezing operator \( S(r, \Phi) \) with squeezing parameter \( r \) and angle \( \Phi \) changes the annihilation operator \( \hat{a} \) like

\[
S^\dagger \hat{a} S = \hat{a} \cosh(r) - e^{i\Phi} \hat{a}^\dagger \sinh(r)
\]

which allows us to directly calculate how the expectation values in Eq. (10) depend on the parameters \( r \) and \( \Phi \).

III. JAYNES-CUMMINGS MODEL

In order to benchmark our variational approach, let us turn to a driven-dissipative variant of the Jaynes-Cummings model, where we compare the variational method to WFMC simulations using the QuTiP package. The Jaynes-Cummings model describes a atom interacting with a light field that is trapped inside a cavity. The Hamiltonian is of the form

\[
H = \Delta_c \hat{a}^\dagger \hat{a} + \Delta_a \sigma^+ \sigma^- + g (\hat{a} \sigma^+ + \hat{a}^\dagger \sigma^-) + p (\hat{a}^\dagger + \hat{a}).
\]

The first two terms describe the detunings \( \Delta_c, \Delta_a \) for the cavity respectively the atoms from the driving frequency. The atom and the cavity are coupled with a strength \( g \) and we pump the cavity with an driving amplitude \( p \). Additionally, we include cavity losses and spontaneous emission of the atoms into other modes than the cavity via the jump operators \( \hat{c}_i = \sqrt{\gamma} \hat{a} \) and \( \hat{c}_n = \sqrt{\kappa} \hat{a}^\dagger \) with decay rate \( \gamma \) for the cavity mode and \( \kappa \) for the atom, respectively.

We use a product ansatz for the atom and the cavity

\[
\rho = \rho_{\text{cavity}} \otimes \rho_{\text{atom}}
\]

in the variational approach and use the variational parameter \( \alpha_i \) in \( \sum_{i=x,y,z} \alpha_i \sigma_i \) to describe the atomic part, while we use the \( P \) representation to account for the cavity mode. As our variational parameter set we use a convolution of coherent, thermal, fock and squeezed states.

To show an immediate advantage of the variational approach, we also want to analyze the Maxwell-Bloch equations of the Jaynes-Cummings model. This set of equation describe the time evolution of the lowest order of expectation values. The atom and cavity decouples in a similar fashion like in Eq. (17), but it also decouples the equation from higher order terms of the cavity field through the neglect of any correlation term of the second or higher order. The Maxwell-Bloch equations for the Jaynes-Cummings model are given by

\[
\frac{d}{dt} \langle a \rangle = -\left( \kappa + i\delta_c \right) \langle a \rangle - ig \langle \sigma^- \rangle - i p
\]

\[
\frac{d}{dt} \langle \sigma^- \rangle = -\frac{\gamma}{2} - i \Delta_a \langle \sigma^- \rangle + ig \langle a \rangle \langle \sigma^+ \rangle
\]

\[
\frac{d}{dt} \langle \sigma^+ \rangle = -\gamma (\langle \sigma^+ \rangle + 1) + 2ig (\langle a \rangle \langle \sigma^+ \rangle - \langle a \rangle \langle \sigma^+ \rangle).
\]

Fig. 1 shows a comparison between the solution of the Maxwell-Bloch equations, the Monte-Carlo wave function solution and the variational expectation value approach for the cavity field \( \langle a^\dagger a \rangle \). The mean-field solution (orange) shows a large area of bistability between two solutions. A comparison of the norms of the two solutions inserted in a set of first-order equations of motion resolves the bistability and indicates a jump between the solutions at the grey line. The third line (blue) shows the solution of expanding the equations of motion up to second order in the variational approach, resulting in a clear improvement. Fig. 2 shows a reconstructed \( P \) distribution from the variational expectation values through
the usage of the characteristic function

\[ \chi(z) = \sum_{k,l=0}^{\infty} \frac{z^k}{k!} \frac{(-z^*)^l}{l!} (a^k a^l) \quad (21) \]

and

\[ P(\alpha, \alpha^*) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} d^2 z \chi(z) e^{-iz^* a^l e^{-iza}}. \quad (22) \]

The non-classicality of the steady state is clearly shown by the negative values of \( P(\alpha) \). The remaining difference with the WFMC simulations can be attributed to the neglect of correlations between the atom and the cavity mode due to our product ansatz in Eq. (17).

IV. RYDBERG CAVITY POLARITONS

Let us now turn to a model where correlations beyond a single mode are particularly important. For this, we investigate an effective three-boson model to describe strongly interacting Rydberg atoms inside a cavity [11][12], which describes nonlinear effects that arises from the interaction of the Rydberg atoms. Before turning to the variational analysis, we briefly want to recapture the key pieces of the model.

Consider a cavity filled with \( N \) three-level atoms with energy level \( g, e, r \) as the ground state \( |g\rangle \), an intermediate level \( |e\rangle \) and an excited state which we denote as the Rydberg level \( |r\rangle \). The key idea is to restrict the dynamics to three bosonic modes that describe the cavity mode and the symmetric subspaces of the atomic excitations. This restriction of the atoms to their symmetric subspace is valid as long as the total number of atomic excitations is small compared to \( N \).

We then can describe the system in terms of collective operators describing the symmetric subspace with \( a \) being the annihilation operator for the cavity mode and \( b = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \sigma_{ge} \) and \( c = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \sigma_{gr} \) as the collective operators for the atomic transition modes \( \sigma_{ge} \) and \( \sigma_{gr} \). With that the Hamiltonian reads as

\[ H = -\Delta_c a^\dagger a + p(a + a^\dagger) - \Delta_e b^\dagger b - \Delta_r c^\dagger c \]

\[ + g\sqrt{N}(ab^\dagger + a^\dagger b) + \frac{\omega}{2}(bc^\dagger + b^\dagger c) + \frac{\kappa_r}{2} c^\dagger c \quad (23) \]

and the the jump operators are given by \( \Delta_c = \sqrt{\kappa_c} b, \Delta_e = \sqrt{\kappa_e} e \) and \( \Delta_r = \sqrt{\kappa_r} r \) for the intermediate and Rydberg state and also for the cavity. The nonlinear terms in the Hamiltonian arise from the van-der-Waals-force between atoms in the Rydberg state. The interaction also couples the symmetric subspace to the antisymmetric subspace which leads to an additional nonlinear dissipation term \( \Delta_n = \sqrt{\kappa_n} c \).

We now study the model by working in the eigenbasis of the non-interacting Hamiltonian at \( \kappa_r = 0 \). The diagonalisation of Eq. (23) results in \( H = \sum_q \frac{\omega_q}{2} \psi_q \Psi_q \Psi_q \).

The new states \( \psi_q \) form polariton states. There are defined as a quasi particle consisting of both light and matter. For a three level atomic system we get two different types of polaritons. The ones with \( q \in \pm \) are bright state polaritons, while \( q = 0 \) is the dark state polariton. The dark state polariton shows very different behavior as it is decoupled from the intermediate atomic level, which leads to long lifetimes in the cavity.

The previously neglected interaction between the polariton leads to a strongly correlated many-body system which provides a difficult task for numerical calculation especially for large atom numbers [13].

To also be able to capture correlations between the modes we need additional variational parameters. If we
parameters of the effective three boson model in the polariton picture for FIG. 4. Intensity(Top) and squeezing parameter \( r \) (bottom) of the resonant parameters we have chosen here an uneven population. All modes reach a saturation around \( p \approx 5 \gamma_c \). However, only the dark state polariton mode displays squeezing. For the bright state polaritons, the squeezing parameter essentially vanishes.

look at the lowest order expectation values between different modes we get

\[
\langle ab \rangle = \langle a \rangle \langle b \rangle + \delta(ab)
\]

with \( \delta(ab) \) as the correlation function between mode \( a \) and \( b \). These kind of factorizations for expectation values can be done for all orders and provides us with the needed variational parameter in form of the correlation functions \( \delta(a^n b^m) \). [44–46]

**V. POSSIBLE EXTENSIONS**

In all of our previous calculations, we worked with only a handful of different convoluted states to successfully construct our variational manifold. However, we would like to point out that it is possible to extend our approach to even broader classes of quantum states. As already mentioned, it is not necessary to know the full P distribution function as it is sufficient to be able to calculate expectation values of the given state, which gives us access to a great variety of non-classical states.

In the previous chapters we already discussed the coherent squeezed states as the most prominent candidate for squeezing but there are similar definitions for squeezed Fock state \( |l\rangle_{sf} \) and squeezed thermal states \( |n\rangle_{sth} \)[47–49]

\[
|l\rangle_{sf} = S(r, \phi) |l\rangle
\]

\[
|n\rangle_{sth} = S(r, \phi) |n\rangle.
\]

Both classes of states have already been investigated in some detail, with explicit expression for expectation values of all orders being known [35–50]. Hence, these squeezed Fock states can also be readily integrated into our variational approach.

Furthermore, it is also possible to employ highly entangled Schrödinger cat states given by

\[
|\psi\rangle = A(|\alpha_1 \rangle + \Theta |\alpha_2 \rangle),
\]

with \( |\alpha_1 \rangle \) and \( |\alpha_2 \rangle \) being two different coherent states. The expectation values for this state can be calculated via the explicit P distribution [51].

Fig. [5] shows the Wigner distribution of all three states. The Wigner distribution is more suitable for a visual representation because it does not have singularities for highly non-classical states that are found in the P distribution. The transformation

\[
W(\alpha) = \frac{2}{\pi} \int d^2 \alpha e^{-2|\alpha - \alpha'|^2} P(\alpha')
\]

connects both distributions.

We also want to make a clear distinction between the squeezed thermal (Fock) state and the convoluted distribution of a squeezed coherent state with a thermal (Fock) state. Especially in the case of the thermal state it is not straightforward to see from their Wigner functions that the two results are actually different. Therefore, it is instructive to look at the difference of their intensities, which is given by

\[
\langle a^\dagger a \rangle_{sth} - \langle a^\dagger a \rangle_{s+th} = 2n_0 \sinh^2 r.
\]

The difference is even enhanced for higher-order expectation values, which can significantly change the result of the minimization in Eq. (14).

In case of the cat state the situation is reversed. Although the visual representation in [5] is clearly distinguishable from a simple coherent state, the difference enters only in higher orders, as the lowest order is given by

\[
\langle a \rangle_{cat} = \alpha_1 + \alpha_2 = \bar{\alpha}.
\]

Only the scaling with higher order expectation values can reveal the true nature of this
state and shows the importance of incorporating as many orders as possible for the equations of motion.

Finally, we would like to mention two additional classes of states that could be included in the variational analysis. Both the single-variable Hermite polynomial states and the photon-added (subtracted) coherent states appear to be good candidates for a further expansion of the variational approach.

VI. SUMMARY

In summary, we have extended the variational principle for open quantum systems through the usage of the P distribution of the density matrix. Despite its simplicity, we find that our method yields even quantitatively reasonable results for the driven-dissipative Jaynes-Cummings model. Furthermore, we have successfully applied our approach to an effective model describing a many-body system of Rydberg atoms in a cavity, where we can identify strong squeezing of a dark state polariton mode. Our approach could be especially fruitful for applications where strong nonclassical correlations play an important role, such as gravitational wave detection using squeezed light or the preparation of nonclassical states of light in photonic condensates. Finally, we have presented several directions how the class of variational states could be extended further.

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Appendix A: Convoluted P distributions

Tab. I shows examples of convolutions of two P distributions used in the main text.

|                      | Coherent $\alpha_1 = i$ | Squeezed coherent state $r_1 = 0.5, \phi_1 = 0$ | Thermal state $n_1 = 0.1$ | Fock state $l_1 = 1$ |
|----------------------|-------------------------|-----------------------------------------------|--------------------------|---------------------|
| **Identity Matrix**  | ![Image](image1.png)    | ![Image](image2.png)                         | ![Image](image3.png)     | ![Image](image4.png) |
| **Coherent state**   | ![Image](image5.png)    | ![Image](image6.png)                         | ![Image](image7.png)     | ![Image](image8.png) |
| $\alpha_2 = 1$       | ![Image](image9.png)    | ![Image](image10.png)                        | ![Image](image11.png)    | ![Image](image12.png) |
| **Squeezed coherent state** $r_2 = 1$, $\phi_2 = -\pi/2$ | ![Image](image13.png) | ![Image](image14.png)                       | ![Image](image15.png)    | ![Image](image16.png) |
| **Thermal state**    | ![Image](image17.png)   | ![Image](image18.png)                         | ![Image](image19.png)    | ![Image](image20.png) |
| $n_2 = 10^{-3}$       | ![Image](image21.png)   | ![Image](image22.png)                         | ![Image](image23.png)    | ![Image](image24.png) |
| **Fock state**       | ![Image](image25.png)   | ![Image](image26.png)                         | ![Image](image27.png)    | ![Image](image28.png) |
| $l_1 = 2$            | ![Image](image29.png)   | ![Image](image30.png)                         | ![Image](image31.png)    | ![Image](image32.png) |

**TABLE I.** Convoluted Wigner functions of the coherent, squeezed coherent, thermal and Fock state with different parameters.