MODULI OF TRIGONAL CURVES

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ABSTRACT. We study the moduli of trigonal curves. We establish the exact upper bound of $36(g + 1)/(5g + 1)$ for the slope of trigonal fibrations. Here, the slope of any fibration $X \to B$ of stable curves with smooth general member is the ratio $\delta_B/\lambda_B$ of the restrictions of the boundary class $\delta$ and the Hodge class $\lambda$ on the moduli space $\overline{M}_g$ to the base $B$. We associate to a trigonal family $X$ a canonical rank two vector bundle $V$, and show that for Bogomolov-semistable $V$ the slope satisfies the stronger inequality $\delta_B/\lambda_B \leq 7 + 6/g$. We further describe the rational Picard group of the trigonal locus $T_g$ in the moduli space $\overline{M}_g$ of genus $g$ curves. In the even genus case, we interpret the above Bogomolov semistability condition in terms of the so-called Maroni divisor in $\overline{F}_g$.

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1. Introduction

In this paper $\overline{M}_g$ denotes the Deligne-Mumford compactification of the moduli space of smooth curves over $\mathbb{C}$ of genus $g \geq 2$. The boundary locus $\Delta$ of $\overline{M}_g$ consists of nodal curves with finite automorphism groups, which together with the smooth curves are referred to as stable curves. The locus of hyperelliptic curves will be denoted by $I_g$, and the closure of the locus of trigonal curves will be denoted by $\overline{F}_g$.

The main objects of our study will be families of genus $g$ stable curves, whose general members are smooth. Associated to any such flat and proper family $f : X \to B$ are three basic invariants $\lambda|_B$, $\delta|_B$ and $\kappa|_B$. We define these in Section 2.1 as divisors on $B$, but for most purposes one can think of them as integers by considering their respective degrees. The invariant $\delta|_B$ counts, with appropriate multiplicities, the number of singular fibers of $X$.
The self-intersection of the relative dualizing sheaf $\omega_f$ on $X$ defines $\kappa|_B$, and its pushforward to $B$ is a rank $g$ locally free sheaf, whose determinant is $\lambda|_B$.

The basic relation $12\lambda|_B = \delta|_B + \kappa|_B$ and the positivity of the three invariants for non-isotrivial families force the slope $\frac{\delta|_B}{\lambda|_B}$ of $X/B$ to fall into the interval $[0, 12]$ (cf. Sect. 2.3).

In fact, Cornalba-Harris and Xiao establish for this slope an exact upper bound of $8 + 4/g$, which is achieved only for certain hyperelliptic families (cf. Theorem 2.2). However, if the base curve $B$ passes through a general point of $\overline{M_g}$, Mumford-Harris-Eisenbud give the better bound of $6 + o(1/g)$ (cf. Theorem 2.3). The families violating this inequality are entirely contained in the closure $\overline{\mathcal{D}}_k$ of the locus of $k$-sheeted covers of $\mathbb{P}^1$, for a suitably chosen $k$. In particular, for $k = 2$ we recover the hyperelliptic locus $\mathcal{T}_g$, for $k = 3$ - the trigonal locus $\mathcal{T}_g$, etc. Therefore, higher than the above “generic” ratio can be obtained only for families with special linear series, such as $g^1_2$, $g^1_3$, etc. These observations clearly raise the following

**Question.** According to the possession of special linear series, is there a stratification of $\overline{M_g}$ which would give successively smaller slopes $\delta/\lambda$? What would be the successive upper bounds with respect to such a stratification?

The following result, whose proof will be given in the paper, answers this question for an exact upper bound for families with linear series $g^1_3$.

**Theorem I.** If $f : X \to B$ is a trigonal nonisotrivial family with smooth general member, then the slope of $X/B$ satisfies:

$$\frac{\delta|_B}{\lambda|_B} \leq \frac{36(g + 1)}{5g + 1}.$$  

Equality is achieved if and only if all fibers are irreducible, $X$ is a triple cover of a ruled surface $Y$ over $B$, and a certain divisor class $\eta$ on $X$ is numerically zero.

To understand the importance of this result and the above question, consider Mumford’s alternative description of the basic invariants (cf. Sect. 2.2): $\lambda|_B$, $\delta|_B$ and $\kappa|_B$ are restrictions of certain rational divisor classes $\lambda, \delta, \kappa \in \text{Pic}_g \overline{M}_g$. Specifically, $\delta = \delta_0 + \cdots + \delta_{[g/2]}$, where $\delta_i$ the class of the boundary divisor $\Delta_i$ of $\overline{M}_g$, and $\text{Pic}_g \overline{M}_g$ is freely generated by the Hodge class $\lambda$ and the boundary classes $\delta_i$ for $g \geq 3$ (cf. [H2]). Thus, our question about a stratification of $\overline{M}_g$ translates into a question about the relations among the fundamental classes of various subvarieties defined by geometric conditions in $\overline{M}_g$. Moreover, such a stratification would provide a link between the global invariant $\lambda$ (the degree of the Hodge bundle on $\overline{M}_g$) and the locally defined invariant $\delta$ of the singularities of our families. In the process of estimating the ratio $\delta/\lambda$ we hope to understand the geometry of interesting loci in $\overline{M}_g$, and describe their rational Picard groups.

Such a program for the hyperelliptic locus $\mathcal{T}_g$ is completed by Cornalba-Harris (cf. Theorem 2.4), who exhibit generators and relations for $\text{Pic}_Q \mathcal{T}_g$. The typical examples of families with maximal ratio of $8 + 4/g$ are constructed as blow-ups of pencils of hyperelliptic curves, embedded in the same ruled surface.

Similar examples for trigonal families yield the slope $7 + 6/g$, but as Theorem I shows, this ratio is not an upper bound. This happens because of an “extra” Maroni locus in $\mathcal{T}_g$ (cf. Sect. 12). While a general trigonal curve embeds in $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or in the blow-up $F_1$ of $\mathbb{P}^2$ at a point, the remaining trigonal curves embed in other rational ruled surfaces and comprise a closed subset in $\mathcal{T}_g$, called the Maroni locus. The proof of Theorem II, stated below, implies that all trigonal families achieving the maximal bound lie entirely in
the Maroni locus, and moreover, their members are embedded in ruled surfaces “as far as possible from the generic” ruled surfaces $F_0$ and $F_1$.

The ratio $7 + 6/g$, though not the “correct” maximum, plays a significant role in understanding the geometry of the trigonal locus, and in describing its rational Picard group. In particular, in a linear relation is established between the Hodge class, the boundary classes on $\tau_g$, and a canonically defined vector bundle $V$ of rank 2 on a ruled surface $\tilde{Y}$ (cf. Sect. 9):

**Theorem II.** Let $\delta_0$ denote the boundary class in $\tau_g$ corresponding to irreducible singular curves, and let $\delta_{k,i}$ be the remaining boundary classes. For any trigonal non-isotrivial family with general smooth member, we have

$$(7g + 6)\lambda|_{B} = g\delta_0|_{B} + \sum_{k,i} c_{k,i}\delta_{k,i}|_{B} + \frac{g - 3}{2}(4c_2(V) - c_1^2(V)),$$

where $c_{k,i}$ is a quadratic polynomial in $i$ with linear coefficients in $g$, and it is determined by the geometry of $\delta_{k,i}$.

For example, $c_{1,i} = 3(i + 2)(g - i)/2$ corresponds to the boundary divisor $\Delta \tau_{1,i}$, whose general member is the join in three points of two trigonal curves of genera $i$ and $g - i - 2$, respectively (cf. Fig. 18).

Recall that the vector bundle $V$ is called Bogomolov semistable if its Chern classes satisfy $4c_2(V) \geq c_1^2(V)$ (cf. [Bo, Re]). We show in Section 9 the following

**Theorem III.** For any trigonal nonisotrivial family $X \to B$ with general smooth member, if $V$ is Bogomolov semistable, then

$$\frac{\delta|_B}{\lambda|_B} \leq \frac{7 + 6}{g}.$$ 

In the even genus case, the Maroni locus is in fact a divisor on $\tau_g$, whose class we denote by $\mu$. We further recognize the “Bogomolov quantity” $4c_2(V) - c_1^2(V)$ as counting roughly four times the number of Maroni fibers in $X$, and deduce

**Theorem IV.** For even $g$, $\text{Pic}_0 \tau_g$ is freely generated by all boundary divisors $\delta_0$ and $\delta_{k,i}$, and the Maroni divisor $\mu$. The class of the Hodge bundle on $\tau_g$ is expressed in terms of these generators as the following linear combination:

$$(7g + 6)\lambda|_{\tau_g} = g\delta_0 + \sum_{k,i} c_{k,i}\delta_{k,i} + 2(g - 3)\mu.$$ 

Consequently, the condition $\eta \equiv 0$ in Theorem I can be interpreted as a relation among the number of irreducible singular curves and the “Maroni” fibers in our family: $(g + 2)\delta_0|_B = -72(g + 1)\mu|_B$, and hence maximal slope families are entirely contained in the Maroni locus of $\tau_g$ (cf. Theorem 12.2). The stated theorems complete the program for the trigonal locus $\tau_g$, which was outlined earlier in this section.

An important interpretation of these results can be traced back to [EHM], where it is shown that the moduli space $\mathcal{M}_g$ is of general type. The $k$-gonal locus $\mathcal{D}_k$ is realized in terms of the generating classes as: $[\mathcal{D}_k]$ = $a\lambda - b\delta - \varepsilon$ for some $a, b > 0$, and an effective boundary combination $\varepsilon$. Restricting to a general curve $B \subset \mathcal{M}_g$, we have $\mathcal{D}_k|_B > 0$, and hence $a\lambda|_B - b\delta|_B > 0$. Because of Seshadri’s criterion for ampleness of line bundles, in effect, we are asking for all positive numbers $a$ and $b$ such that the linear combination $a\lambda - b\delta$ is ample on $\mathcal{M}_g$. The smaller the ratio $a/b$ is, the stronger result we obtain. In other words, we are aiming at a maximal bound of $\delta/\lambda$, when we think of these classes as restricted to any curve $B \subset \mathcal{M}_g$. In view of this, part of this paper can be described as
looking for all ample divisors on \( \overline{\mathbb{X}}_g \) of the form \( a\lambda - b\delta \) with \( a, b > 0 \). Theorem I then gives the necessary condition \( (5g + 1)a \geq 36(g + 1)b \) (compare with [M1, EHM, CH]). Some of the results can be applied to the study of the discriminant loci of a certain type of triple covers of surfaces.

The methods and ideas for the trigonal case are in principal extendable to more general families of \( k \)-gonal curves. We refer the reader to Sect. 13 for a general maximal bound for tetragonal curves (for \( g \) odd), and conjectures for the maximal and general bounds for any \( d \)-gonal and other families of stable curves.

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2. Preliminaries

2.1. Definition of \( \lambda|_B, \delta|_B \) and \( \kappa|_B \) in \( \text{Pic} B \). Let \( f : X \to B \) be a flat proper one-parameter family of stable curves of genus \( g \), where \( B \) is a smooth projective curve. Assume in addition that the general member of \( X \) is smooth (cf. Fig. 1).

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Figure 1. Trigonal family \( f : X \to B \)

Let \( \omega_f = \omega_X \otimes f^* \omega_B^{-1} \) be the relative dualizing sheaf of \( f \). Its pushforward \( f_*(\omega_f) \) is a locally free sheaf on \( B \) of rank \( g \), and we set

\[ \lambda|_B = \lambda_X := \wedge^g f_*(\omega_f) \in \text{Pic} B \]

to be its determinant. The sheaf \( f_*(\omega_f) \) is known as the “Hodge bundle” on \( B \), and \( \lambda|_B \) - the “Hodge class” of \( B \). In a similar way, we set \( \kappa|_B \) to be the self-intersection of \( \omega_f \):

\[ \kappa|_B = \kappa_X := f_*(c_1^2(\omega_f)) \in \text{Pic} B. \]

The definition of \( \delta|_B \), on the other hand, is local and requires some notation. Let \( q \) be any singular point of a fiber \( X_b, b \in B \). Since the general fiber of \( X \) is smooth, the total space of \( X \) near \( q \) is given locally analytically by \( xy = t^{m_q} \), where \( x \) and \( y \) are local parameters on
$X_b$, $t$ is a local parameter on $B$ near $b$, and $m_q \geq 1$. (This follows from the one-dimensional versal deformation space of the nodal singularity at $q$.) For any other point $q$ of $X$ we set $m_q = 0$. Now we can define

$$\delta|_B = \delta_X := f_*\left(\sum_{q \in X} m_q q\right) \in \text{Pic} B.$$

By abuse of notation, we shall use the same letters for the line bundles $\lambda|_B$, $\kappa|_B$ and $\delta|_B$ and for their respective degrees, e.g. $\lambda|_B = \text{deg} \lambda|_B$.

**Remark 2.1.** It is possible to define the three basic invariants for a wider variety of families. In particular, dropping the assumption of smoothness of the general fiber roughly means that the base curve $B$ is contained entirely in the boundary locus of $\overline{M}_g$. Since such families are not discussed in our paper, we shall not give here these definitions. The existence, however, of such invariants for any one-parameter family of stable curves will follow from the description of $\lambda$, $\delta$ and $\kappa$ as “global” classes in $\text{Pic}_{Q\overline{M}_g}$ (cf. Sect. 2.2).

**Remark 2.2.** It is also possible to consider families whose special members are not stable, e.g. cuspidal, tacnodal and other types of singular curves. One reduces to the above cases by applying semistable reduction (cf. [FM]).

### 2.2. The line bundles $\lambda$, $\delta$ and $\kappa$ in $\text{Pic}_{Q\overline{M}_g}$.

Another way to interpret the classes $\lambda|_B$, $\delta|_B$ and $\kappa|_B$ is to think of them as rational divisor classes on $\overline{M}_g$. In fact, Mumford shows that such invariants, defined for any proper flat family $X \to S$ and natural under base change, induce line bundles in $\text{Pic}_{Q\overline{M}_g}$. Here follows a rough sketch of the argument (cf. [M1]).

Consider $\text{Hilb}^{p(x)}$, the Hilbert scheme parametrizing all closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(x) = dx - g + 1$ for some $d = 2n(g - 1) \gg 0$ and $r = d - g$. Let $\mathcal{H} \subset \text{Hilb}^{p(x)}$ be the locally closed smooth subscheme of $n$-canonical stable curves of genus $g$. Then $\overline{M}_g$ is the GIT-quotient of $\mathcal{H}$ by $\text{PGL}_r$. Let

$$\rho : \mathcal{H} \to \overline{M}_g = \mathcal{H}/\text{PGL}_r$$

be the natural surjection, and let $(\text{Pic} \mathcal{H})^\text{PGL}_r$ be the subgroup of isomorphism classes of line bundles on $\mathcal{H}$ invariant under the action of $\text{PGL}_r$.

Consider also $\text{Pic}_{\text{fun}} \overline{M}_g$, the group of line bundles on the *moduli functor*. An element $L$ of $\text{Pic}_{\text{fun}} \overline{M}_g$ consists of the following data: for any proper flat family $f : X \to S$ of stable curves a line bundle $L_S$ on $S$ natural under base change. Two such elements are declared isomorphic under the obvious compatibility conditions.

Naturally, a line bundle on $\overline{M}_g$ gives rise by pull-back to a line bundle on the moduli functor. In fact, this map is an inclusion with a torsion cokernel, and $\text{Pic}_{\text{fun}} \overline{M}_g$ is torsion free and isomorphic to $(\text{Pic} \mathcal{H})^\text{PGL}_r$:

$$\text{Pic} \overline{M}_g \xrightarrow{\rho^*} \text{Pic}_{\text{fun}} \overline{M}_g \cong (\text{Pic} \mathcal{H})^\text{PGL}_r.$$

Hence we may regard all these groups as sublattices of $\text{Pic}_{Q\overline{M}_g}$. In particular,

$$\text{Pic}_{\text{fun}} \overline{M}_g \otimes Q \cong \text{Pic}_{Q\overline{M}_g},$$

and any line bundle on the moduli functor can be thought of as a rational class on $\overline{M}_g$.

These identifications allow us to make the following

**Definition 2.1.** In $\text{Pic}_{Q\overline{M}_g}$ we define the line bundles $\lambda$, $\kappa$ and $\delta$ by

$$\lambda = \det \pi_*(\omega_{\mathcal{C}/\mathcal{H}}), \quad \kappa = \pi_*c_1(\omega_{\mathcal{C}/\mathcal{H}})^2, \quad \delta = \mathcal{O}_\mathcal{C}(\Delta \mathcal{H}),$$
where \( \mathcal{E} \subset \mathcal{H} \times \mathbb{P}^r \) is the universal curve over \( \mathcal{H} \), \( \pi : \mathcal{E} \to \mathcal{H} \) is the projection map, \( \omega_{\mathcal{E}/\mathcal{H}} \) is the relative dualizing sheaf of \( \pi \), and \( \Delta \mathcal{H} \subset \mathcal{H} \) is the divisor of singular curves on \( \mathcal{H} \).

As defined, \( \lambda, \kappa \) and \( \delta \) lie in \( \text{Pic}_Q \overline{\mathcal{M}}_g \), and as such they are only rational Cartier divisors on \( \overline{\mathcal{M}}_g \). In [EHM] one can find examples where \( \lambda \) does not descend to a line bundle on \( \overline{\mathcal{M}}_g \). On the other hand, it is obvious from which divisor on \( \overline{\mathcal{M}}_g \) our \( \delta \) comes: \( \delta = \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta) \), where \( \Delta \) denotes the divisor on \( \overline{\mathcal{M}}_g \) comprised of all singular stable curves. Again, due to singularities of the total space of \( \overline{\mathcal{M}}_g \), \( \Delta \) is only a rational Cartier divisor. In fact, the only locus of \( \overline{\mathcal{M}}_g \) on which \( \lambda, \delta \) and \( \kappa \) are necessarily integer divisor classes is \( (\overline{\mathcal{M}}_g)_0 \) - the locus of automorphism-free curves.

We can further define the boundary classes \( \delta_0, \delta_1, \ldots, \delta_{\lfloor \frac{g}{2} \rfloor} \) in \( \text{Pic}_Q \overline{\mathcal{M}}_g \). Let \( \Delta_i \) be the \( \mathbb{Q} \)-Cartier divisor on \( \overline{\mathcal{M}}_g \) whose general member is an irreducible nodal curve with a single node (if \( i = 0 \)), or the join of two irreducible smooth curves of genera \( i \) and \( g - i \) intersecting transversally in one point (if \( i > 0 \)). Setting \( \delta_i = \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta_i) \), we have \( \Delta = \sum_i \Delta_i \) and \( \delta = \sum_i \delta_i \).

As the following result of Harer [H1, H2] suggests that, in order to describe the geometry of the moduli space \( \overline{\mathcal{M}}_g \), it will be useful to study the divisor classes defined above, and to understand the relations between them.

**Theorem 2.1** (Harer). The Hodge class \( \lambda \) and the boundary classes \( \delta_0, \delta_1, \ldots, \delta_{\lfloor \frac{g}{2} \rfloor} \) generate \( \text{Pic}_Q \overline{\mathcal{M}}_g \), and for \( g \geq 3 \) they are linearly independent.

It is easy to recognize the restrictions of \( \lambda, \delta \) and \( \kappa \) to a curve \( B \) in \( \overline{\mathcal{M}}_g \), as the previously defined \( \lambda|_B, \delta|_B \) and \( \kappa|_B \). For example, the restriction of \( \delta \) to the base curve \( B \) counts, with appropriate multiplicities, the number of intersections of \( B \) with the boundary components \( \Delta_i \) of \( \overline{\mathcal{M}}_g \).

As a final remark, applying Grothendieck Riemann-Roch Theorem (GRR) to the map \( \pi : \mathcal{E} \to \mathcal{H} \) and the sheaf \( \omega_{\mathcal{E}/\mathcal{H}} \), implies the basic relation:

\[
(2.1) \quad 12\lambda = \kappa + \delta.
\]

2.3. **Slope of non-isotrivial families.** Let \( f : X \to B \) be our family of stable curves with a smooth general member. By definition, \( \delta_B \geq 0 \). Further, all locally free quotients of the Hodge bundle \( f_* (\omega_f) \) have non-negative degrees [Fu]. If \( X \) is a non-isotrivial family, then \( \lambda|_B > 0 \), and since the relative canonical divisor \( K_{X/B} \) is nef, \( \kappa|_B > 0 \) [Be]. In particular, we can divide by \( \lambda|_B \).

**Definition 2.2.** The slope of a non-isotrivial family \( f : X \to B \) of stable curves with a smooth general member is the ratio

\[
\text{slope}(X/B) := \frac{\delta|_B}{\lambda|_B}.
\]

Suppose we make a base change \( B_1 \to B \) of degree \( d \), and set \( X_1 = X \times_B B_1 \) to be the pull-back of our family over the new base \( B_1 \) (cf. Fig. 2). Then the three invariants on \( B \) will pull-back to the corresponding invariants on \( B_1 \), and their degrees will be multiplied by \( d \), e.g. \( \lambda|_{B_1} = d\lambda|_B \), etc. In particular, the slope of \( X/B \) will be preserved.

In view of \( (2.1) \), restricting to the base curve \( B \) yields

\[
(2.2) \quad 12\lambda|_B = \kappa|_B + \delta|_B.
\]

From the positivity conditions above, we deduce that \( 0 \leq \text{slope}(X/B) < 12 \).
2.4. **Statement of the problem and what is known.** It is natural to ask whether we can find a better estimate for the slope of $X$. The first fundamental result in this direction is the following

**Theorem 2.2** (Cornalba-Harris, Xiao). Let $f : X \to B$ be a nonisotrivial family with smooth general member. Then the slope of the family satisfies:

\[
\frac{\delta|_B}{\lambda|_B} \leq 8 + \frac{4}{g}.
\]

Equality holds if and only if the general fiber of $f$ is hyperelliptic, and all singular fibers are irreducible.

Note that the upper bound is achieved only for hyperelliptic families. Such families are of very special type since the hyperelliptic locus $\overline{I}_g$ has codimension $g - 2$ in $\overline{M}_g$. On the other hand, if the base curve $B$ is general enough, a much better estimate can be shown (cf. [EHM]):

**Theorem 2.3** (Mumford-Harris-Eisenbud). If $B$ passes through a general point $[C] \in \overline{M}_g$, then

\[
\frac{\delta|_B}{\lambda|_B} \leq 6 + o\left(\frac{1}{g}\right).
\]

For example, when $g$ is odd, we can set $k = (g + 1)/2$ and define the divisor $\overline{D}_k$ in $\overline{M}_g$ as the closure of the $k$-sheeted covers of $\mathbb{P}^1$:

\[
\overline{D}_k = \{ C \in \overline{M}_g \mid C \text{ has } g^1_k \}.
\]

If our family is not entirely contained in $\overline{D}_k$, or equivalently, if $B$ passes through a point $[C] \not\in \overline{D}_k$ (cf. Fig. 3),

\[
\frac{\delta|_B}{\lambda|_B} \leq 6 + \frac{12}{g + 1}.
\]

Higher than the “generic” ratio can be obtained, therefore, only for a very special type of families: those entirely contained in $\overline{D}_k$, and hence possessing $g^1_2, g^1_3$, etc.
2.4.1. The rational Picard group of the hyperelliptic locus \( \overline{\text{Pic}}_g \). In proving the maximal bound \( 8 + 4/g \), Cornalba-Harris also describe \( \text{Pic}_Q \overline{\text{Pic}}_g \) by exhibiting generators and relations (cf. [CH]). Here we briefly discuss their result.

Recall the irreducible divisors \( \Delta_i \) on \( \overline{\text{Pic}}_g \). For \( i = 1, \ldots, [g/2] \), \( \Delta_i \) cuts out an irreducible divisor on \( \overline{\text{Pic}}_g \), while the intersection \( \Delta_0 \cap \overline{\text{Pic}}_g \) breaks up into several components:

\[
\Delta_0 \cap \overline{\text{Pic}}_g = \Xi_0 \cap \Xi_1 \cap \cdots \cap \Xi_{[g/2]}.
\]

Set \( \xi_i = 0_{\overline{\text{Pic}}_g}(\Xi_i) \) for the class of \( \Xi_i \) in \( \overline{\text{Pic}}_g \), and retain the symbols \( \lambda \) and \( \delta_i \) for their corresponding restrictions to \( \text{Pic}_Q \overline{\text{Pic}}_g \). Thus, \( \delta_i := 0_{\overline{\text{Pic}}_g}(\Delta_i \cap \overline{\text{Pic}}_g) \) for all \( i \). Finally note that the class \( \delta_0 \) is realised in \( \text{Pic}_Q \overline{\text{Pic}}_g \) as the sum

\[
\delta_0 = \xi_0 + 2\xi_1 + \cdots + 2\xi_{[g/2]}.\]

The coefficient 2 roughly means that \( \Delta_0 \) is double along \( \Xi_i \), for \( i > 0 \).

**Theorem 2.4** (Cornalba-Harris). The classes \( \xi_0, \ldots, \xi_{[g/2]} \) and \( \delta_1, \ldots, \delta_{[g/2]} \) freely generate \( \text{Pic}_Q \overline{\text{Pic}}_g \). The Hodge class \( \lambda \in \text{Pic}_Q \overline{\text{Pic}}_g \) is expressed in terms of these generators as the following linear combination:

\[
(8g + 4)\lambda = g\xi_0 + \sum_{i=1}^{[g-1]/2} 2(i+1)(g-i)\xi_i + \sum_{j=1}^{[g/2]} 4j(g-j)\delta_j.
\]

For a specific family \( f : X \to B \) of hyperelliptic stable curves this relation reads:

\[
(8g + 4)\lambda|_B = g\xi_0|_B + \sum_{i=1}^{[g-1]/2} 2(i+1)(g-i)\xi_i|_B + \sum_{j=1}^{[g/2]} 4j(g-j)\delta_j|_B.
\]

\[
\Rightarrow (8 + 4/g)\lambda|_B \geq \xi_0|_B + \sum_i 2\xi_i|_B + \sum_j 2\delta_j|_B = \delta|_B.
\]

This yields the desired \( 8 + 4/g \) inequality for the slope of a hyperelliptic family, and shows that the maximum can be obtained exactly when all \( \xi_1, \ldots, \xi_{[g/2]}; \delta_1, \ldots, \delta_{[g/2]} \) vanish on \( B \).

In other words, the singular fibers of \( X \) belong only to the boundary divisor \( \Xi_0 \), and hence are irreducible. In Appendix we review the description of the divisors \( \Xi_i \) via admissible covers, and give an alternative proof of Theorem 2.4.

2.4.2. Example of a hyperelliptic family with maximal slope. We present here a typical example in which the upper bound \( 8 + 4/g \) is achieved, and show how to calculate explicitly the basic invariants \( \lambda|_B \) and \( \delta|_B \) for this family.

**Example 2.1.** Consider a pencil \( \mathcal{P} \) of hyperelliptic curves of genus \( g \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Because of genus considerations, its members must be of type \( (2, g+1) \). Our family \( f : X \to \mathbb{P}^1 \) will be obtained by blowing-up \( \mathbb{P}^1 \times \mathbb{P}^1 \) at the \( 4g+1 \) base points of the pencil in order to separate its members (cf. Fig. 4). Hence, \( \chi(X) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) + 4(g+1) \) for the corresponding topological Euler characteristics. Riemann-Hurwitz formula for the map \( f \) gives a second relation: \( \chi(X) = \chi(\mathbb{P}^1)\chi(X_b) + \delta|_B \), where \( \mathbb{P}^1 \) is the base \( B \) and \( X_b \) is the general fiber of \( X \). Putting together, \( \delta|_B = 8g + 4 \).

The total space of \( X \) is a divisor on \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) of type \( (2, g+1, 1) \), and the map \( f : X \to \mathbb{P}^1 \) is the restriction to \( X \) of the third projection \( \pi_3 : \mathbb{P}^2 \times \mathbb{P} \times \mathbb{P}^1 \to \mathbb{P}^1 \). Using standard methods, we compute \( h^0((f_*\omega_f)(-2)) = 0 \). From the positivity of all free quotients of the Hodge
bundle on \( P^1 \), \( f_*(\omega_f) \) splits as a direct sum \( \bigoplus_{i=1}^g \mathcal{O}_{P^1}(a_i) \) for some \( a_i > 0 \). Then, for 
\[ f_*(\omega_f)(-2) = \bigoplus_i \mathcal{O}_{P^1}(a_i - 2) \] to have no sections, all \( a_i \)'s must be at most 1. Finally,

\[ f_*(\omega_f) = \bigoplus_{i=1}^g \mathcal{O}_{P^1}(+1), \quad \lambda|B = g, \quad \text{and} \quad \frac{\delta|B}{\lambda|B} = 8 + \frac{4}{g}. \]

2.4.3. The Trigonal Locus \( \mathcal{F}_g \). In a similar vein as in the above example, we consider pencils of trigonal curves on ruled surfaces, and obtain the slope \( 7+6/g \). It is somewhat reasonable to expect that this is the maximal ratio. Recall that a bundle \( \mathcal{E} \) on a curve \( B \) is \textit{semistable} if for any proper subbundle \( \mathcal{F} \), we have \( q(\mathcal{F}) \leq q(\mathcal{E}) \), where \( q \) is the quotient of the degree and the rank of the corresponding bundle. Following Xiao’s approach in the proof of Theorem 2.2, Konno shows that for non-hyperelliptic fibrations of genus \( g \) with semistable Hodge bundle \( f_*(\omega_f) \) (cf. [HN, K1]):

\begin{equation}
\frac{\delta|B}{\lambda|B} \leq 7 + \frac{6}{g}. \tag{2.7}
\end{equation}

As for any trigonal families, he establishes the inequality (cf. [K2]):

\begin{equation}
\frac{\delta|B}{\lambda|B} \leq \frac{22g + 26}{3g + 1} \sim 7 + \frac{1}{3} + o\left(\frac{1}{g}\right). \tag{2.8}
\end{equation}

Examples of trigonal families achieving this ratio were not found, which suggested that this bound might be too big. On the other hand, in trying to disprove the smaller bound \( 7 + 6/g \), we naturally arrived at counterexamples pointing to a third intermediate ratio (cf. Theorem 11.4):

\begin{equation}
\frac{36(g + 1)}{5g + 1} \sim 7 + \frac{1}{5} + o\left(\frac{1}{g}\right). \tag{2.9}
\end{equation}

The difference between the last two estimates may seem negligible, but this would not be so when both \( \lambda|B \) and \( \delta|B \) become large and we attempt to bound \( \lambda|B \) from below by \( \delta|B \). What is more important, the second ratio is in fact \textit{exact}, and we give equivalent conditions for it to be achieved (cf. Sect. 7.6, 12.4). This maximal bound confirms Chen’s result for genus \( g = 4 \) in [Ch].

The reader may ask why the “generic” examples for the maximum in the hyperelliptic case fail to provide also the maximum in the trigonal case. As we noted in the Introduction, the answer is closely related to the so-called \textit{Maroni} locus in \( \mathcal{F}_g \). More precisely, if \( F_k = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(k)) \) denotes the corresponding rational ruled surface, a general curve \( C \) embeds in \( F_0 \) if \( g \) is even, and in \( F_1 \) if \( g \) is odd. The Maroni locus consists of those curves that embed in \( F_k \) with \( k \geq 2 \). The number \( k/2 \) is referred to as the \textit{Maroni invariant} of \( C \). In these terms, the examples of pencils of trigonal curves on \( F_0 \) and \( F_1 \) have the lowest possible constant Maroni invariant, and we shall see that the maximum bound can be obtained only for families entirely contained in the Maroni locus, and having very high Maroni invariants.
The "semistable" bound $7 + 6/g$ appears in Theorem 11.2, where we give instead a sufficient \textit{Bogomolov-semistability} condition $4c_2(V) - c_1^2(V) \geq 0$ for a canonically associated to $X$ vector bundle $V$ of rank 2. The rational Picard group of $\Gamma_g$ is described in terms of generators and relations in Section 12.2, providing thus in the trigonal case an analog of Theorem 2.4. Note the apparent similarity of the coefficients $\tilde{c}_{k,i}$ of the trigonal boundary classes and the coefficients of the hyperelliptic boundary classes. This is not coincidental. In fact, the $\tilde{c}_{k,i}$'s are in a sense the "smallest" coefficients that could have been associated to the corresponding classes $\delta_{k,i}$ (cf. Fig. 18): they are symmetric with respect to the two genera of the components in the general member of $\delta_{k,i}$. A crucial role in the proof of Theorem 12.1 is played by the interpretation of the above Bogomolov semistability condition in terms of the Maroni locus it $\Gamma_g$ (cf. Sect. 12.3).

2.5. The idea of the proof. Let $f : X \to B$ be a family of stable curves, whose general member $X_b$ is a smooth trigonal curve. By definition, $X_b$ is a triple cover of $\mathbb{P}^1$. We would like to study how this triple cover varies as $X_b$ moves in the family $X$. Thus, it would be desirable to represent $X$, by analogy with $X_b$, as a triple cover of a ruled surface $Y$, comprised by the image lines $\mathbb{P}^1$. Unfortunately, due to existence of hyperelliptic and other special singular fibers, this is not always possible.

![Figure 5. Basic construction](image)

2.5.1. The basic construction. The "closest" model of such a triple cover can be obtained after a finite number of birational transformations on $X$, and a possible base change over the base $B$. This way we construct a \textit{quasi-admissible} cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$ over a new base $\tilde{B}$ (cf. Prop. 3.1). Here $\tilde{Y}$ is a \textit{birationally} ruled surface over $\tilde{B}$ with reduced, but non necessarily irreducible, special fibers: $\tilde{Y}$ allows for \textit{pointed stable} rational fibers, i.e. trees of $\mathbb{P}^1$'s with points marked in a certain (stable) way.

The map $\tilde{\phi}$ expresses any fiber $\tilde{X}_b$ as a triple quasi-admissible cover of the corresponding \textit{pointed stable} rational curve $\tilde{Y}_b$. To calculate effectively our invariants $\lambda, \delta$ and $\kappa$, we need that $\tilde{\phi}$ be \textit{flat}, which could force a few additional blow-ups on $\tilde{X}$ and $\tilde{Y}$. We end up with a flat proper triple cover $\hat{\phi} : \hat{X} \to \hat{Y}$, where certain fibers of $\hat{X}$ and $\hat{Y}$ are allowed to be \textit{non-reduced}: these are the scheme-theoretic preimages under the blow-ups on $\hat{X}$ and $\hat{Y}$. We call such covers $\hat{\phi}$ \textit{effective}.

We observe next that any smooth trigonal curve $C$ can be naturally embedded in a ruled surface $\mathbf{F}_k$ over $B$. If $\alpha : C \to \mathbb{P}^1$ is the corresponding triple cover, there is an exact sequence of locally free sheaves on $\mathbb{P}^1$:

$$0 \to V \to \alpha_* \mathcal{O}_C \xrightarrow{\text{tr}} \mathcal{O}_{\mathbb{P}^1} \to 0.$$
The projectivization $PV$ of the rank 2 vector bundle $V$ is the ruled surface $F_k$.

This construction can be extended as $C$ moves in the effective cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$. The flatness of $\tilde{\phi}$ forces the pushforward $\phi_*\mathcal{O}_{\tilde{X}}$ to be a locally free sheaf of rank 3 on $\tilde{Y}$, and the finiteness of $\tilde{\phi}$ ensures the existence of a trace map $\text{tr} : \phi_*\mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Y}}$. Again, the kernel $V$ of $\text{tr}$ is the desired rank 2 vector bundle on $\tilde{Y}$, in whose projectivization, $PV$, we embed $\tilde{X}$ (cf. Fig. 3).

2.5.2. Chow Rings Calculations. We can now use the relations in the Chow rings of $\mathbb{A}(PV)$, $\mathbb{A}\tilde{Y}$ and $\mathbb{A}\tilde{X}$ to calculate the invariants $\lambda_{\tilde{X}}$ and $\delta_{\tilde{X}}$, appropriately defined for the new family $\tilde{X} \to \tilde{B}$ of semistable and occasionally non-reduced fibers. Then, of course, we translate $\lambda_{\tilde{X}}$ and $\delta_{\tilde{X}}$ into $\lambda_X$ and $\delta_X$ with the necessary adjustments from the birational transformations on $X$ and the base change on $B$. We compare the resulting expressions to obtain a relation among $\lambda_X$ and $\delta_X$.

2.5.3. Boundary of the Trigonal Locus. As we vary the base curve $B$ inside $\overline{\Sigma}_g$, we actually obtain a relation among the restrictions of $\lambda$ and $\delta$ in $\text{Pic}_Q \overline{\Sigma}_g$, rather than just among $\lambda|_B = \lambda_X$ and $\delta|_B = \delta_X$ in $\text{Pic} B$.

In terms of what have we thus represented and linked $\lambda|_{\overline{\Sigma}_g}$ and $\delta|_{\overline{\Sigma}_g}$? To answer this question, we need first to understand the boundary divisors of the trigonal locus $\overline{\Sigma}_g$. As we shall see, there are seven types of such divisors, denoted by $\Delta_{\overline{\Sigma}_0}$ and $\Delta_{\overline{\Sigma}_{k,i}}$ for $k = 1, ..., 6$. Each type is determined by the specific geometry of its general member. For example, $\Delta_{\overline{\Sigma}_0}$ is the closure of all irreducible trigonal curves with one node, while $\Delta_{\overline{\Sigma}_{k,i}}$ corresponds to joins in two points of a trigonal and a hyperelliptic curve with genera $i$ and $g - 1 - i$, respectively (cf. Fig. 13). Naturally, we derive an expression for the restriction of the divisor class $\delta \in \text{Pic}_Q \overline{\Sigma}_g$ to $\overline{\Sigma}_g$:

$$\delta|_{\overline{\Sigma}_g} = \delta_0 + \sum_{i=1}^{[\frac{(g-2)/2]}{2}} 2\delta_{1,i} + \sum_{i=1}^{g-2} 3\delta_{2,i} + \sum_{i=1}^{[\frac{g/2}]}/2} \delta_{3,i} + \sum_{i=1}^{[\frac{(g-1)/2}]}/2} \delta_{4,i} + \sum_{i=1}^{g-1} \delta_{5,i} + \sum_{i=1}^{[\frac{g/2}]}/2} \delta_{6,i}.$$

Here $\delta_0$ and $\delta_{k,i}$ are the divisor classes of $\Delta_{\overline{\Sigma}_0}$ and $\Delta_{\overline{\Sigma}_{k,i}}$ in $\text{Pic}_Q \overline{\Sigma}_g$.

2.5.4. Relations among $\lambda$ and $\delta$. For a fixed family $X \to B$ with a smooth trigonal general member, we establish a relation among the Hodge class $\lambda|_B$, the boundary classes $\delta_{k,i}|_B$, and the Bogomolov quantity $4c_2(V) - c_1^2(V)$ for the associated vector bundle $V$:

$$(7g + 6)\lambda|_B = g\delta_0|_B + \sum_{k,i} \tilde{c}_{k,i} \delta_{k,i}|_B + \frac{g-3}{2} (4c_2(V) - c_1^2(V)).$$

The polynomial coefficients $\tilde{c}_{k,i}$ are comparatively larger than the corresponding coefficients of the boundary divisors in the expression for $\delta|_{\overline{\Sigma}_g}$. As a result, we rewrite (2.10) as

$$(7g + 6)\lambda|_B = g\delta|_B + \epsilon|_B + \frac{g-3}{2} (4c_2(V) - c_1^2(V)), $$

where $\epsilon$ is an effective combination of the boundary classes on $\overline{\Sigma}_g$. In particular, if $V$ is Bogomolov semistable, the slope satisfies (cf. Theorem 1.3):

$$\text{slope}(X/B) \leq 7 + \frac{6}{g}.$$  

Further, we describe $\text{Pic}_Q \overline{\Sigma}_g$ as generated freely by the restriction $\lambda|_{\overline{\Sigma}_g}$ and the boundary classes of $\overline{\Sigma}_g$. In the even genus $g$ case, we can replace $\lambda|_{\overline{\Sigma}_g}$ by a geometrically defined class $\mu$, corresponding to the so-called Maroni divisor in $\overline{\Sigma}_g$. This, of course, means that
the Hodge class $\lambda|_{\mathcal{T}_g}$ must be some linear combination of the boundary classes and $\mu$. The Bogomolov quantity is interpreted as
\[
4c_2(V) - c_1^2(V) = 4\mu|_B + 0 \cdot \delta_0|_B + \sum_{k,i} \alpha_{k,i}\delta_{k,i}|_B,
\]
which in turn “lifts” (2.10) to the wanted relation in $\text{Pic}_Q\mathcal{X}_g$:
\[
(7g + 6)\lambda|_{\mathcal{T}_g} = g\delta_0 + \sum_{k,i} \hat{c}_{k,i}\delta_{k,i} + 2(g - 3)\mu.
\]
We have not yet computed explicitly all coefficients $\hat{c}_{k,i}$. In the cases which we have completed ($\Delta_0\mathcal{X}_g$ and $\Delta_{1,i}\mathcal{X}_g$), these coefficients turn out again sufficiently large so that we can repeat the argument in (2.11). Thus, if $X$ has at least one non-Maroni fiber, and its singular fibers belong to $\Delta_0\mathcal{X}_g \cup \Delta_{1,i}\mathcal{X}_g$, then $\mu|_B \geq 0$, and hence the stronger bound of (2.12) holds (cf. Prop. 12.3 and Conj. 12.1).

2.5.5. Maximal Bound. Since the Bogomolov semistability condition $4c_1^2(V) - c_2(V) \geq 0$ is not always satisfied, the above discussion shows that $7 + 6/g$ is not the maximal bound for the slope of trigonal families. Therefore, we need another, more subtle, estimate. The expressions for $\lambda|_B$ and $\delta|_B$ suggest that any maximal bound would be equivalent to an inequality involving $c_1^2(V)$, $c_2(V)$, and possibly some other invariants. We construct a specific divisor class $\eta$ on $\tilde{X}$, for which the Hodge Index theorem implies $\eta^2 \leq 0$, and we translate this into $9c_1^2(V) - 2c_2^2(V) \geq 0$ (cf. Prop. 10.1). We notice that the only reasonable way to replace Bogomolov’s condition $4c_1^2(V) - c_2(V) \geq 0$ by the newly found inequality is by subtracting the following quantities:
\[
36(g + 1)\lambda|_B - (5g + 1)\delta|_B = \mathcal{E}'|_B + (g - 3)(9c_2(V) - 2c_1^2(V)),
\]
so that the “left-over” linear combination of boundary divisors $\mathcal{E}'$ is again effective (cf. Theorem 11.3). Hence, we conclude that for all trigonal families:
\[
slope(X/B) \leq \frac{36(g + 1)}{5g + 1}.
\]

2.6. The organization of the paper. The presentation of the Basic Construction is done in several stages. Fig. 6 shows schematically the connection between the three types of covers, admissible, quasi-admissible and effective, in relation to the original family $X \to B$ of stable curves.

![Figure 6. Types of covers](image-url)

We start in Section 3.1 by introducing a compactification $\mathcal{H}_{d,g}$ of the Hurwitz scheme, parametrizing admissible $d$-uple covers of stable pointed rational curves. Using its coarse moduli properties, we show in Section 3.3 the existence of admissible covers of surfaces.
$X^a \to Y^a$ associated to the original family $f : X \to B$. Next we modify these covers to quasi-admissible covers $\phi : \tilde{X} \to \tilde{Y}$ (cf. Prop. 3.1), and further to effective covers $\hat{\phi} : \hat{X} \to \hat{Y}$ in order to resolve the technical difficulties arising from the non-flatness of $\tilde{\phi}$ (cf. Sect. 4.4).

We devote Section 4 to the study of the boundary components of the trigonal locus $\overline{T}_g$ inside the moduli space $\overline{M}_g$, and express the restriction $\Delta|_{\overline{T}_g}$ as a linear combination of the boundary divisors (cf. Prop. 4.1). In Section 6 we complete the Basic Construction by embedding the effective cover $\hat{X}$ in a rank 1 projective bundle $\mathbb{P}V$ over $\hat{Y}$.

For convenience of the reader, the proofs of the maximal $36(g+1)/(5g+1)$ and the semistable $7 + 6/g$ bounds are presented first in the special, but fundamental case when the original family $f : X \to B$ is already an effective triple cover of a ruled surface $Y$ (cf. Sect. 7).

The discussion results in finding the coefficients of $\delta_0$ in two different expressions of $\lambda|_{\overline{T}_g}$, but, as it turns out, the knowledge of these coefficients is enough to determine the desired two bounds. We refer to this as the global calculation. The Hodge Index Theorem and Nakai-Moishezon criterion on $X$ complete the global calculation in Sect. 7.5. A discussion of maximal bound examples can be found in Section 7.6.

The local calculations in Sections 8-10 compute the contributions of the other boundary classes $\delta_{k,i}$, and express $\lambda|_{\overline{T}_g}$ in terms of these contributions and the Chern classes of the rank 2 vector bundle $V$ on $Y$. For clearer exposition, the proofs of the two bounds are shown first for a general base curve $B$ (i.e. $B$ intersects transversally the boundary components in general points), and then in Section 11 the results are extended to any base curve $B$. We develop the necessary notation and techniques for the local calculations in Section 8.1.1.

Section 12 discusses the relation between the Bogomolov semistability condition and the Maroni locus, and describes the structure of $\text{Pic}_Q \overline{T}_g$. In Section 12.4 we give another interpretation of the conditions for the maximal bound.

We present further results and conjectures for $d$-gonal families in Section 13. In the Appendix, we give another proof of the $8 + 4/g$ bound in the hyperelliptic case and show an application of the maximal trigonal bound to the study of the discriminant locus of certain triple covers.

### 3. Quasi-Admissible Covers of Surfaces

We first review briefly the theory of admissible covers. For more details, we refer the reader to [EHM], [HM].

#### 3.1. The Hurwitz scheme $\overline{H}_{d,g}$

Let $H_{d,g}$ be the small Hurwitz scheme parametrizing the pairs $(C, \phi)$, where $C$ is a smooth curve of genus $g$ and $\phi : C \to \mathbb{P}^1$ is a cover of degree $d$, simply branched over $b = 2d + 2g - 2$ distinct points. Since $C \in M_g$, there is a natural map $H_{d,g} \to M_g$, whose image contains an open dense subset of $M_g$. The theory of admissible covers provides the commutative diagram in Fig. 7.

There $\mathcal{M}_{0,b}$ (resp. $\overline{\mathcal{M}}_{0,b}$) is the moduli space of $m$-pointed $\mathbb{P}^1$'s (resp. of stable $m$-pointed rational curves), and $\overline{H}_{d,g}$ is a compactification of the Hurwitz scheme. The points of $\overline{H}_{d,g}$ correspond to triples $(C, (P; p_1, ..., p_m), \phi)$, where $C$ is a connected reduced nodal curve of genus $g$, $(P; p_1, ..., p_m)$ is a stable $m$-pointed rational curve, and $\phi : C \to P$ is a so-called admissible cover.

**Definition 3.1.** Given the curves $C$ and $P$ as above, an admissible cover $\phi : C \to P$ is a regular map satisfying the following conditions:
(A1) $\phi^{-1}(P_{sm}) = C_{sm}$ and $\phi : C_{sm} \to P_{sm}$ is simply branched over the distinct points $p_1, \ldots, p_b \in P_{sm}$.

(A2) for every $q \in C_{sing}$ lying over a node $p \in P$, the two branches through $q$ map with the same ramification index to the two branches through $p$.

Note that $C$ is not necessarily a stable curve, but contracting its destabilizing rational chains yields the corresponding stable curve $pr_1(C) \in \overline{\mathcal{M}}_g$. In such a case, we say that $C$ is the “admissible model” for $pr_1(C)$ (cf. Fig. 8). Harris-Mumford have shown that the compactification $\overline{\mathcal{H}}_{d,g}$ is in fact a coarse moduli space for the admissible covers $\phi : C \to P$.

3.2. Local properties of admissible covers. When we vary the admissible covers of curves in families, the local structure of the corresponding total spaces becomes apparent.

Let $\phi : \mathcal{C} \to \mathcal{P}$ be a proper flat family (over a scheme $\mathcal{B}$) of admissible covers of curves (cf. Fig. 9). Assume that $\phi$ is étale everywhere except over the nodes of the fibers of $\mathcal{P}/\mathcal{B}$, and except over some sections $\sigma_i : \mathcal{B} \to \mathcal{C}$ and their images $\omega_i : \mathcal{B} \to \mathcal{P}$: there $\phi$ is simply branched along $\sigma_i$ over $\omega_i$ for all $i$. If $q \in \mathcal{C}_b$ is a point lying above a node $p \in \mathcal{P}_b$ for some $b \in \mathcal{B}$, then $\mathcal{C}_b$ has a node at $q$, and locally analytically we can describe $\mathcal{C}, \mathcal{P}$ and $\phi$ near $q$ and $p$ by:
admissible covers \( \phi \) map. From the weak valuative criterion for properness, this means that given a family of \( \geq \) families \( \phi \) for which when the base \( B \) singularities on the total spaces of \( C \) can always pick a generator \( \phi \). Example 3.1. Let the triple admissible cover of curves. From now on, by admissible covers we mean, more generally, families \( \mathcal{C} \rightarrow \mathcal{P} \) over \( B \) with the above description.

The local properties of the admissible cover \( \phi : \mathcal{C} \rightarrow \mathcal{P} \) over the nodes in \( \mathcal{P}_b \) forces singularities on the total spaces of \( \mathcal{C} \) and \( \mathcal{P} \). Since we will be interested only in the cases when the base \( B \) is a smooth projective curve \( B \) and the general fiber of \( \mathcal{C} \) is smooth, we can always pick a generator \( t \) for \( \hat{\mathcal{O}}_{b,B} \), and express \( a = t^l \) for some \( l \in \mathbb{N} \).

\[
\begin{align*}
\mathcal{C} : & \quad xy = a, \quad x, y \text{ generate } \mathfrak{m}_{a,c_b}, \quad a \in \hat{\mathcal{O}}_{b,B}, \\
\mathcal{P} : & \quad uv = a^n, \quad u, v \text{ generate } \mathfrak{m}_{p,p_b}, \\
\phi : & \quad u = x^n, v = u^n.
\end{align*}
\]

One can see that \( n \) is the index of ramification of \( \phi \) at \( q \), and that fiberwise \( \mathcal{C}_b \rightarrow \mathcal{P}_b \) is an admissible cover (of curves). From now on, by admissible covers we mean, more generally, families \( \mathcal{C} \rightarrow \mathcal{P} \) over \( B \) with the above description.

The local properties of the admissible cover \( \phi : \mathcal{C} \rightarrow \mathcal{P} \) over the nodes in \( \mathcal{P}_b \) forces singularities on the total spaces of \( \mathcal{C} \) and \( \mathcal{P} \). Since we will be interested only in the cases when the base \( B \) is a smooth projective curve \( B \) and the general fiber of \( \mathcal{C} \) is smooth, we can always pick a generator \( t \) for \( \hat{\mathcal{O}}_{b,B} \), and express \( a = t^l \) for some \( l \in \mathbb{N} \).

\[
\begin{align*}
\mathcal{C} : & \quad xy = t^l, \quad \text{At } q \text{ is given by } xy = t^l, \text{ and at } p, \mathcal{P} \text{ is given by } uv = t^{2l}, \text{ where } u = x^2, v = y^2. \text{ This forces at } r \text{ the local equation } xy = t^{2l} (u = x, v = y). \text{ Even if } \mathcal{C} \text{ is smooth at } q (l = 1), \mathcal{C} \text{ and } \mathcal{P} \text{ will be singular at } r \text{ and } p, \text{ respectively } (xy = t^2, uv = t^2). \text{ Compare this with the non-flat cover of ramification index 1 in Fig. 24.}
\end{align*}
\]

Recall that a rational double point \( s \) on a surface \( S \) is of type \( A_{l-1} \) if locally analytically \( S \) is given at \( s \) by the equation \( xy = t^l \). Thus, \( r \) and \( p \) above are rational double points on \( \mathcal{C} \) and \( \mathcal{P} \), respectively, of type \( A_{l-1} \).

Remark 3.1. In the sequel, we use the fact that the projection \( pr_1 : \overline{\mathcal{M}}_{d,g} \rightarrow \overline{\mathcal{M}}_{0,b} \) is a finite map. From the weak valuative criterion for properness, this means that given a family of admissible covers \( \phi : \mathcal{C}^* \rightarrow \mathcal{P}^* \) over the punctured disc \( \text{Spec } \mathbb{C}((t)) \), there is some \( n \in \mathbb{N} \) for which \( \phi \) extends to a family \( \phi_n : \mathcal{C}_n \rightarrow \mathcal{P}_n \) of admissible covers over \( \text{Spec } \mathbb{C}[[t^{1/n}]] \). In particular, if the base for the admissible cover \( \phi : X^* \rightarrow Y^* \) is an open set \( B^* \) of a smooth projective curve \( B \), modulo a finite base change, we can extend this to a family of admissible covers \( X^a \rightarrow Y^a \) over the whole curve \( B \).

3.3. Admissible covers of surfaces. Consider a family \( f : X \rightarrow B \) of stable curves of genus \( g \), whose general member is smooth and \( d \)-gonal. Let \( \psi : B \rightarrow \overline{\mathcal{M}}_g \) be the canonical map, and let \( \overline{B} \) denote the fiber product \( B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{d,g} \).

If the general member of \( X \) has infinitely many \( g_1 \)'s, the variety \( \overline{B} \) will have dimension \( \geq 2 \). We can resolve this by considering an intersection of the appropriate number of hyperplane sections of \( \overline{B} \), and picking a one-dimensional component \( \overline{B}_0 \) dominating \( B \). The curve \( \overline{B}_0 \) might be singular, but by normalizing it and pulling \( X \) over it, we get another family of stable curves (cf. Fig. 24):

\[
\overline{X} = X \times_B (\overline{B}_0)^{\text{norm}} \rightarrow (\overline{B}_0)^{\text{norm}}.
\]
Since the two families have the same basic invariants, we can replace the original with the new one, and assume the existence of a map \( \eta : B \to \mathcal{H}_{d,g} \) compatible with \( \psi : B \to \mathcal{M}_g \).

In other words, \( \eta \) associates to every fiber \( C \) of \( X \) a specific \( g_1^d \) on \( C \) or, possibly, a \( g_1^d \) on an admissible model \( C^a \) of \( C \).

Let \( B^* \) be the open subset of \( B \) over which all fibers are smooth and \( d \)-gonal. For simplicity, assume for now that all the fibers over \( B^* \) can be represented as admissible covers of \( \mathbb{P}^1 \) via the chosen \( g_1^d \)'s, i.e. they are simply branched covers of \( \mathbb{P}^1 \) over \( m \) distinct points of \( \mathbb{P}^1 \). Denote by \( X^* \) the restriction of \( X \) over \( B^* \) (cf. Fig. 12).

The map \( \eta : B^* \to \mathcal{H}_{d,g} \) induces a section

\[
\sigma : B^* \to \text{Pic}^d(X^*/B^*),
\]
where \( \text{Pic}^d(X^*/B^*) \) is the relative degree \( d \) Picard variety of \( X^* \) over \( B^* \). \( \text{Pic}^d(X^*/B^*) \) parametrizes the line bundles on \( X^* \) of relative degree \( d \). The image \( \sigma(B^*) \subset \text{Pic}^d(X^*/B^*) \) is a class of line bundles on \( X^* \) whose fiberwise restrictions are the chosen \( g_1^d \)'s. Let \( \mathcal{L} \) be a representative of this class, and let \( Y^* \) be the ruled surface \( \mathbb{P}((f_*\mathcal{L})^\wedge) \) over \( B^* \). The map \( \phi : X^* \to Y^* \) induced by \( \mathcal{L} \) defines an admissible cover over \( B^* \), as shown in Fig. 13.

\[
X^* \xrightarrow{\phi} Y^* = \mathbb{P}((f_*\mathcal{L})^\wedge)
\]

**Figure 13.** Construction of \( Y^* \)

From Remark 3.1, \( \phi \) extends to a family of admissible covers \( \phi^a : X^a \to Y^a \) over the whole base \( B \). Since \( X^a \) and \( X \) are isomorphic over \( B^* \), they are birational to each other. In other words, the fibers \( C \) of \( X \), over which \( \mathcal{L} \) does not extend to the base-point free linear series \( g_1^d = \sigma_1(b) \), are modified by blow-ups and blow-downs so as to arrive at their admissible models in \( X^a \). We have thus proved the following

**Lemma 3.1.** Let \( f : X \to B \) be a family of stable curves, whose general member over an open subset \( B^* \subset B \) is a smooth \( d \)-uple admissible cover of \( \mathbb{P}^1 \). Then, modulo a finite base change, there exists an admissible cover of surfaces \( X^a \to Y^a \) over \( B \) such that \( X^a \) is obtained from \( X \) by a finite number of birational transformations performed on the fibers over \( B - B^* \).
3.4. **Quasi-admissible covers.** In case the general member of \( X \) is *not* an admissible cover of \( \mathbb{P}^1 \), e.g. it is trigonal with a total point of ramification, we have to modify the above construction. To start with, we cannot expect to obtain an *admissible cover* \( X^* \to Y^* \), even modulo a finite base change. This leads us to consider a different kind of covers, which we call *quasi-admissible*.

**Definition 3.2.** A *quasi-admissible cover* \( \tilde{\phi} : C \to P \) of a nodal curve \( C \) over a semistable pointed rational curve \( P \) is a regular map which behaves like an admissible cover over the singular locus of \( P \), i.e. for any \( q \in C \) lying over a node \( p \in P \) the two branches through \( q \) map with the same ramification index to the two branches through \( p \).

![Figure 14. Quasi-admissible covers over \( \mathbb{P}^1 \)](image)

Quasi-admissible covers differ from admissible covers in allowing more diverse behavior of \( C \) over \( P_{\text{sm}} \), e.g. having singularities, higher ramification points and multiple simple ramification points. Fig. 14 displays several degree 3 quasi-admissible covers over \( \mathbb{P}^1 \):

However, any quasi-admissible cover can be obtained from an admissible cover \( \phi^a : C^a \to P^a \) by simultaneous contractions of components in \( P^a \) and their (rational) inverses on \( C^a \).

**Definition 3.3.** A *minimal* quasi-admissible cover \( \tilde{\phi} : C \to P \) is minimal with respect to the number of components of \( P \). In other words, one cannot apply more simultaneous contractions on \( C \to P \) and end with another quasi-admissible cover.

**Example 3.2.** A smooth trigonal curve \( C \) with a total point of ramification \( q \) is a minimal quasi-admissible cover of \( P = \mathbb{P}^1 \). Blowing up \( q \) on \( C \) and \( p = \tilde{\phi}(q) \in P \), gives an admissible cover \( C^a = C \cup C_1 \to P \cup P_1 \), where \( C_1 \cong \mathbb{P}^1 \) maps three-to-one onto \( P_1 \cong \mathbb{P}^1 \) with a total point of ramification \( q = C_1 \cap C \) (cf. Fig. 15).

![Figure 15. Quasi versus admissible covers](image)

The motivation for using *minimal* quasi-admissible covers, instead of just admissible or quasi-admissible covers, is that the former are the closest covers to the original families \( X \to B \) of stable curves, and calculations on them will yield the best possible estimate for the ratio \( \delta_X/\lambda_X \) (cf. Fig. 6).

### 3.4.1. Quasi-admissible covers for families with higher ramification sections.

Now let us consider the remaining case of a family \( X \to B \), whose general member over \( B^* \) is smooth and \( d \)-gonal, but *not* an admissible cover of \( \mathbb{P}^1 \). After a possible base change, we still have the map (cf. Fig. 12)

\[ \eta : B \to \mathcal{M}_{d,g}. \]
It associates to every fiber $C$ a $g^1_d$ on its admissible model $C^a$. Let $C^a \to P^a$ be the corresponding admissible cover. Since $C$ itself is $d$-gonal, and by assumption it does not possess a $g^1_d$ with $e < d$, $C$ must be a $d$-uple cover of some component of $P^a$. In particular, the $g^1_d$ on $C^a$ restricts to a $g^1_d$ on $C$. Thus, in effect, $\eta$ gives again a section $\sigma : B^* \to \text{Pic}^d(X^*/B^*)$. As before, we obtain a degree $d$ finite map $\phi : X^* \to Y^*$ to the ruled surface $\mathbb{P}(f_\mathcal{L}^*)$ over $B^*$. Note that this is a family of minimal quasi-admissible covers.

We extend $\phi$ over the curve $B$ as follows. For simplicity, assume that $d = 3$. Let $R$ be the ramification divisor of $\phi$ in $X^*$. By hypothesis, there is a component $R_0$ of $R$ which passes through total ramification points and dominates $B^*$. Letting $\overline{R}_0$ be the closure of $R_0$ in $X$, we can normalize it and pull the family $\eta$ over it. So we may assume that $\overline{R}_0$ is a section of $X \to B$. If there are some other components $R_1, R_2, \ldots, R_i$ of the ramification divisor $R$ passing through higher ramification points, we repeat the same procedure for them, until we “straighten out” all $\overline{R}_i$’s into sections of $X \to B$. Let $E_i = \phi(R_i)$ be the corresponding sections of $Y^*$ over $B^*$. We can shrink $B^*$ in order to exclude any fibers with isolated higher ramification points.

Consider a fiber $C$ in $X^*$. Let $\{r_i = C \cap R_i\}$ be its total ramification points, and let $\{p_i = \phi(r_i)\}$ be their images on $P = \phi(C)$ in $Y^*$. It is clear that blowing-up all $r_i$’s and $p_i$’s will give an admissible triple cover $C^a = \text{Bl}_{\{r_i\}}(C) \to P^a = \text{Bl}_{\{p_i\}}(P)$. The $g^1_d$ giving this cover, is the original one assigned by $\eta : B^* \to \overline{R}_{d,g}$. We globalize this construction by blowing-up the sections $R_i$ on $X^*$ and $E_i$ on $Y^*$. Similarly as above, we obtain a triple admissible cover of surfaces $\phi^a : \text{Bl}_{\cup R_i}(X^*) \to \text{Bl}_{\cup E_i}(Y^*)$ over $B^*$. The properness of $pr_1 : \overline{R}_{d,g} \to \overline{R}_g$ allows us to extend this to an admissible cover $\phi^a : \text{Bl}_{\cup R_i}(X^*) \to \text{Bl}_{\cup E_i}(Y^*)$ over $B$ (cf. Fig. 16).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{Blowing up $R_i$ and $E_i$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{Over $B^*$}
\end{figure}

Denote by $\overline{R}_i$ the component of $\text{Bl}_{\cup R_i}(X^*)$, obtained by blowing up $R_i \subset X^*$, and let $\overline{R}_i$ be its closure in $\text{Bl}_{\cup R_i}(X^*)$. Define similarly $\overline{E}_i \subset \text{Bl}_{\cup E_i}(Y^*)$ and $\overline{E}_i \subset \text{Bl}_{\cup E_i}(Y^*)$. The admissible cover $\phi^a$ maps $\overline{R}_i$ to $\overline{E}_i$, so that after removing all the $\overline{R}_i$’s and $\overline{E}_i$’s we still have a triple cover $\phi^a : X^q = \text{Bl}_{\cup R_i}(X^*) - \cup \overline{R}_i \to Y^q = \text{Bl}_{\cup E_i}(Y^*) - \cup \overline{E}_i$.

Note that $X^q \cong X$ and $Y^q \cong Y$ over the open set $B^*$, and that $Y^q$ is a birationally ruled surface over $B$ (cf. Fig. 17). Finally, note that from the quasi-admissible cover $\phi^a : X^q \to Y^q$ we obtain a family $\phi : \tilde{X} \to \tilde{Y}$ of minimal quasi-admissible covers: simply contract the unnecessary rational components in the fibers of $X^q$ and $Y^q$, and observe that the triple map $\phi^a$ restricts to the corresponding triple map $\phi$. 
This completes the construction of minimal quasi-admissible covers for any family $X \to B$ with general smooth trigonal member. The cases $d > 3$ are only notationally more difficult. One has to keep track of the possibly different higher multiplicities in $C$ and multiple double points in $C$ over the same $p \in P$. The construction of an admissible cover $X^a \to Y^a$ goes through with minimal modifications. We combine the results of this section in the following

**Proposition 3.1.** Let $f : X \to B$ be a family of stable curves, whose general member over an open subset $B^* \subset B$ is smooth and $d$-gonal. Then, modulo a finite base change, there exists a minimal quasi-admissible cover of surfaces $\tilde{X} \to \tilde{Y}$ over $B$ such that $\tilde{X}$ is obtained from $X$ by a finite number of birational transformations performed on the fibers over $B - B^*$.

4. The Boundary $\Delta \mathfrak{T}_g$ of the Trigonal Locus $\mathfrak{T}_g$

4.1. Description and notation for the boundary of $\mathfrak{T}_g$. In this section we shall see that there are seven types of boundary divisors of $\mathfrak{T}_g$, each denoted by $\Delta \mathfrak{T}_{k,i}$ for $k = 0, 1, ..., 6$. The second index $i$ is determined in the following way. Let $C = C_1 \cup C_2$ be the general member of $\Delta \mathfrak{T}_{k,i}$, where $C_1$ and $C_2$ are smooth curves. If $C_1$ and $C_2$ are both trigonal or both hyperelliptic, then we set $i$ to be the smaller of the two genera $p(C_1)$ or $p(C_2)$. If, say, $C_1$ is a trigonal, but $C_2$ is hyperelliptic, then we set $i$ to be genus of the trigonal component $C_1$. The only exception to this rule occurs when $C$ is irreducible (and hence of genus $g$ with exactly one node). We denote this boundary component by $\Delta \mathfrak{T}_0$.

When we view a general member $C$ roughly as a triple cover of $\mathbb{P}^1$’s in the Hurwitz scheme (consider the pull-back $p^i_1[C] \in \mathfrak{F}_3$), then it may or may not be ramified. If there is no ramification, then $C$ lies in one of the first four types of trigonal boundary divisors $\Delta \mathfrak{T}_{k,i}$, $k = 0, 1, 2, 3$. Ramification index 1 characterizes the general members of $\Delta \mathfrak{T}_{4,i}$ and $\Delta \mathfrak{T}_{5,i}$, and in case of $\Delta \mathfrak{T}_{6,i}$ the ramification index is 2 (cf. Fig. 18).

There is an alternative description of the boundary components $\Delta \mathfrak{T}_{k,i}$’s of $\mathfrak{T}_g$. If one such $\Delta \mathfrak{T}_{k,i}$ lies in the restriction $\Delta_0|_{\mathfrak{T}_g}$ of the divisor $\Delta_0$ in $\mathfrak{M}_g$, then $\Delta \mathfrak{T}_{k,i}$ is one of $\Delta \mathfrak{T}_0$, $\Delta \mathfrak{T}_{1,i}$, $\Delta \mathfrak{T}_{2,i}$, or $\Delta \mathfrak{T}_{4,i}$. The partial normalization of their general members $C$ is still connected, i.e. $C$ is either irreducible, or the join of two smooth curves meeting in at least two points. Correspondingly, for the general member $C$ of the remaining three types of boundary components, $\Delta \mathfrak{T}_{3,i}$, $\Delta \mathfrak{T}_{5,i}$ and $\Delta \mathfrak{T}_{6,i}$, the irreducible components of $C$ intersect transversely in exactly one point, so that the normalization of $C$ is disconnected.

![Figure 18. Boundary Components $\Delta \mathfrak{T}_{k,i}$ of $\mathfrak{T}_g$](image)

**Proposition 4.1.** The boundary divisors of $\mathfrak{T}_g$ can be grouped in seven types: $\Delta \mathfrak{T}_0$ and $\Delta \mathfrak{T}_{k,i}$ for $k = 1, ..., 6$. Their general members and range of index $i$ are shown in Fig. 18. The boundary of $\mathfrak{T}_g$ consists of $\Delta \mathfrak{T}_0$, $\Delta \mathfrak{T}_{k,i}$, and the codimension 2 component $\mathfrak{T}_g$ of hyperelliptic curves.
Consider the projection map $pr_1 : \mathcal{H}_{3,g} \rightarrow \mathcal{M}_g$, whose image is the trigonal locus $\mathcal{F}_g$. Thus, the inverse image of each boundary divisor $\Delta \mathcal{F}_{k,i}$ will be a boundary divisor $\Delta \mathcal{H}_{k,i}$ in $\mathcal{H}_{3,g}$. The converse, however, is not always true, i.e. certain boundary divisors of $\mathcal{H}_{3,g}$ contract under $pr_1$ to smaller subschemes of $\mathcal{F}_g$, e.g. the hyperelliptic locus $\mathcal{F}_g$. With the description of the Hurwitz scheme $\mathcal{H}_{3,g}$, given in Section 3, it is easier to determine first $\mathcal{H}_{3,g}$’s boundary divisors. Thus, we postpone the proof of Proposition 4.1 until the end of the next subsection.

### 4.1.1. The Boundary of $\mathcal{H}_{3,g}$

**Proposition 4.2.** The boundary divisors of $\mathcal{H}_{3,g}$ can be grouped in six types: $\Delta \mathcal{H}_{k,i}$ for $k = 1, ..., 6$. Their general members and range of index $i$ are shown in Fig. 19.

![Figure 19. Boundary Components of $\mathcal{H}_{3,g}$](image)

**Proof.** A general member $A$ of the boundary $\Delta \mathcal{H}$ is a triple admissible cover of a chain of two $\mathbb{P}^1$. (From the dimension calculations that follow it will become clear that an admissible cover of a chain of three or more $\mathbb{P}^1$’s will generate a subscheme in $\mathcal{H}_{3,g}$ of codimension $\geq 2$.) Note that *three* connected components of $A$ over one $\mathbb{P}^1$ means that they are all smooth $\mathbb{P}^1$’s themselves, and hence they can all be contracted simultaneously, leaving us with a smooth trigonal curve, or with a hyperelliptic curve with an attached $\mathbb{P}^1$, neither of which cases by dimension count corresponds to a general member of a boundary component $\Delta \mathcal{H}_{k,i}$. Considering all combinations of one or two connected components of $A$ over each $\mathbb{P}^1$, we generate a list of the possible general members of the boundary divisors $\Delta \mathcal{H}_{k,i}$.

To see which of these are indeed of codimension 1 in $\mathcal{H}_{3,g}$, we do the following calculation. First we note that, for a fixed set of $2i + 4$ ramification points in $\mathbb{P}^1$, there are finitely many covers of degree 3 and genus $i$, that is,

$$\dim \mathcal{F}_i = 2i + 4 - 3 = 2i + 1.$$  

Subtracting 3 takes into account the projectively equivalent triples of points on $\mathbb{P}^1$. In particular, $\dim \mathcal{F}_g = 2g + 1$. A similar argument (with $2i + 2$ ramification points) shows that for the hyperelliptic locus:

$$\dim \mathcal{J}_i = 2i + 2 - 3 = 2i - 1.$$  

These computations are valid for $i > 0$, whereas $0 = \dim \mathcal{J}_i = \dim \mathcal{J}_i$.

Thus, to compute the dimensions of the six types of subschemes of $\mathcal{H}_{3,g}$, one adds the corresponding dimensions of $\mathcal{F}_i$ and $\mathcal{J}_j$, making the necessary adjustments for the choice of intersection points on the components of each curve $A$. For example, when $i > 0$ the dimension of the subscheme with general member $A$, shown in Fig. 19, is

$$\dim \mathcal{F}_i + \dim \mathcal{F}_{g-i-2} + 1 + 1 = 2g.$$
The final 1’s account for the choice of triples of points in the $g_3^1$’s on each component. We conclude that for $i = 1, 2, \ldots, [(g-2)/2]$ the join at three points of two trigonal curves, one of genus $i$ and the other of genus $g-i-2$, is the general member of a boundary component of $\overline{\mathcal{H}}_{3,g}$. We denote it by $\Delta \mathcal{H}_{1,i}$. The range of $i$ stops at $[(g-2)/2]$ for symmetry considerations. When $i = 0$, the corresponding subscheme has a smaller dimension of $2g-2$ and hence no boundary divisor is generated by such curves.

As another example, consider the fifth sketch in Fig. 19. It corresponds to the join at one point of a trigonal curve $C_1$ of genus $i$, a hyperelliptic curve $C_2$ of genus $g-i$, and an attached $\mathbb{P}^1$ to $C_2$ to make the whole curve a triple cover. Note that $C_1$ and $C_2$ intersect transversally at a point $q$, but when presented as covers of $\mathbb{P}^1$ they both have ramifications at $q$ of index 1. On all such curves $C_1$ and $C_2$ the total number of ramification points over $\mathbb{P}^1$ is finite, and hence their choice does not affect the dimension of our subscheme. Thus,

$$\dim \mathcal{F}_i + \dim \mathcal{F}_{g-i} = 2i + 1 + 2(g-i) = 2g.$$ 

Therefore, this subscheme is in fact a divisor in $\overline{\mathcal{H}}_{3,g}$, which we denote by $\Delta \mathcal{H}_{5,i}$. The cases of $i = 0$ or $i = g$ lead to contractions of unstable rational components ($C_1$ or $C_2$), and do not yield the necessary dimension of $2g$. Hence, $i = 1, 2, \ldots, g-1$.

In the case of $\Delta \mathcal{H}_{6,i}$, the two components $C_1$ and $C_2$ meet transversally in one point $q$, but both have ramification of index 2 at $q$ as triple covers of $\mathbb{P}^1$. Smooth trigonal curves of genus $i$ with such high ramification form a codimension 1 subscheme of the trigonal locus $\mathcal{X}_1$, hence the dimension of $\Delta \mathcal{H}_{6,i}$ is

$$\dim \mathcal{F}_i - 1 + \dim \mathcal{F}_{g-i} - 1 = (2i+1)-(2(g-i)+1) = 2g.$$ 

Thus, $\Delta \mathcal{H}_{6,i}$ is a boundary divisor in $\overline{\mathcal{H}}_{3,g}$ for $i = 1, 2, \ldots, [g/2]$. The case of $i = 0$ yields dimension $2g-1$, and hence we disregard it.

The remaining cases are treated similarly. We conclude that $\overline{\mathcal{H}}_{3,g}$ has six types of boundary divisors, $\Delta \mathcal{H}_{k,i}$, whose general members and range of indices are indicated in Fig. 19. 

4.1.2. Boundary of $\mathcal{X}_g$. Proof of Proposition 4.1. Having described the boundary of $\overline{\mathcal{H}}_{3,g}$, it remains to check which of the divisors $\Delta \mathcal{H}_{k,i}$ preserve their dimension under the map $pr_1$ and hence map into divisors of $\mathcal{X}_g$. The only “surprises” can be expected where $pr_1$ contracts unstable $\mathbb{P}^1$, such as in $\Delta \mathcal{H}_{2,i}$, $\Delta \mathcal{H}_{3,i}$, and $\Delta \mathcal{H}_{5,i}$. In fact, only $\Delta \mathcal{H}_{2,g-1}$ and $\Delta \mathcal{H}_{3,0}$ diverge from the common pattern; in all other cases, we set $\Delta \mathcal{F}_{k,i} := pr_1(\Delta \mathcal{H}_{k,i})$ to be the corresponding boundary divisor in $\mathcal{X}_g$.

The map $pr_1$ contracts the three rational components of the general member of $\Delta \mathcal{H}_{3,0}$, leaving only a smooth hyperelliptic curve of genus $g$. Thus, the image $pr_1(\Delta \mathcal{H}_{3,0})$ is the hyperelliptic locus $\mathcal{F}_g$, which is of dimension $2g-1$. Hence $\Delta \mathcal{H}_{3,0}$ does not yield a divisor in $\mathcal{X}_g$, but a boundary component of codimension 2.

Finally we consider $\Delta \mathcal{H}_{2,g-1}$. After we contract its two rational components, we arrive at an irreducible nodal trigonal curve with exactly one node. The dimension of the subscheme of such curves is

$$\dim \mathcal{F}_{g-1} + 1 = 2(g-1) + 1 + 1 = 2g,$$

where the final 1 indicates the choice of a triple of points on a smooth trigonal curve (belonging to the $g_3^1$), two of which will be identified as a node. Correspondingly, we obtain another divisor in $\mathcal{X}_g$, which we denote by $\Delta \mathcal{F}_0$. 

4.2. Multiplicities of the boundary divisors $\Delta \mathfrak{T}_{k,i}$ in the restriction $\delta|_{\mathfrak{T}_g}$. By abuse of notation, we will denote by $\delta_0$ and $\delta_{k,i}$ the classes in $\text{Pic}_Q \mathfrak{T}_g$ of $\Delta \mathfrak{T}_0$ and $\Delta \mathfrak{T}_{k,i}$, respectively.

**Proposition 4.3.** The divisor class $\delta \in \text{Pic}_Q \mathfrak{T}_g$ restricts to $\mathfrak{T}_g$ as the following linear combination of the boundary classes in $\mathfrak{T}_g$:

\[
(4.1) \quad \delta|_{\mathfrak{T}_g} = \delta_0 + \sum_{i=1}^{[g/2]} 3\delta_{1,i} + \sum_{i=1}^{[g/2]} 2\delta_{2,i} + \sum_{i=1}^{[g/2]} \delta_{3,i} + \sum_{i=1}^{[g/2]} 3\delta_{4,i} + \sum_{i=1}^{[g/2]} \delta_{5,i} + \sum_{i=1}^{[g/2]} \delta_{6,i}.
\]

**Proof.** Let us rewrite equation (4.1) in the form

\[
\delta|_{\mathfrak{T}_g} = (\text{mult}_\delta \delta_0)\delta_0 + \sum_{k,i} (\text{mult}_\delta \delta_{k,i})\delta_{k,i},
\]

and call $\text{mult}_\delta \delta_{k,i}$ the multiplicity of $\delta_{k,i}$ in $\delta|_{\mathfrak{T}_g}$. This linear relation simply counts the contribution of each singular curve of a specific boundary type in $\Delta \mathfrak{T}_g$ to the degree of $\delta$.

Recall that for any trigonal family $f : X \to B$:

\[
\deg \delta|_B = \sum_{q \in X} m_q.
\]

Here $m_q$ denotes the local analytic multiplicity of the total space of $X$ nearby $q$ measured by the equation $xy = t^{m_q}$, where $x$ and $y$ are local parameters on the singular fiber $X_b$, and $t$ is a local parameter on $B$ near $b = f(q)$.

For each boundary class $\Delta \mathfrak{T}_{k,i}$ of $\mathfrak{T}_g$, we consider its general member $C = C_1 \cup C_2$, and a base curve $B$ in $\mathfrak{T}_g$ which intersects transversally $\Delta \mathfrak{T}_{k,i}$ in $[C]$. In the corresponding one-parameter trigonal family $f : X \to B$, we must find the sum of the multiplicities $m_q$ of the singularities of $C$. Thus,

\[
\text{mult}_\delta \delta_{k,i} = \sum_{q \in C_{\text{sing}}} m_q.
\]

For most of the divisors classes, this sum is actually quite straightforward. For example, the general member $[C] \in \Delta \mathfrak{T}_{3,i}$ is the join of two smooth hyperelliptic curves $C_1$ and $C_2$, which intersect transversally in one point $q$. The family $X$, constructed as above, will be given locally analytically nearby $q$ by $xy = t$, and hence $\text{mult}_\delta \delta_{k,i} = m_q = 1$. A similar situation occurs in the cases of $\Delta \mathfrak{T}_0$, $\Delta \mathfrak{T}_{5,i}$ and $\Delta \mathfrak{T}_{6,i}$: there is one point of transversal intersection (or one node) forcing $\text{mult}_\delta \delta_0 = \text{mult}_\delta \delta_{k,i} = 1$ for $k = 3, 5, 6$.

In the cases of $\Delta \mathfrak{T}_{2,i}$ and $\Delta \mathfrak{T}_{1,i}$ there are correspondingly two or three points of transversal intersection, forcing

\[
\text{mult}_\delta \delta_{2,i} = 2 \quad \text{and} \quad \text{mult}_\delta \delta_{1,i} = 3.
\]

This can be also interpreted by the fact that $\Delta \mathfrak{T}_{2,i}$ and $\Delta \mathfrak{T}_{1,i}$ lie entirely in the divisor $\Delta_0$ in $\mathfrak{T}_g$ with, $\Delta_0$ being double along $\Delta \mathfrak{T}_{2,i}$, and triple along $\Delta \mathfrak{T}_{1,i}$.

A slightly more complex situation occurs in the case of $\Delta \mathfrak{T}_{4,i}$. The general member $C$ consists of two curves $C_1$ and $C_2$, meeting transversally in two points $q$ and $r$ (see Fig. 20). But, as in an admissible triple cover of two $\mathbb{P}^1$s, the points $q$ and $r$ behave differently: at one of them, say $r$, the triple cover is not ramified, while at $q$ there is ramification of index 1. In the local analytic rings of $p, q$ and $r$ the generators of $\mathfrak{O}_{Y, p}$ map into the squares of the generators of $\mathfrak{O}_{X, q}$: $u \mapsto x^2, v \mapsto y^2$, and of course, $t \mapsto t$, so that the local equation of $Y$ near $p$ is $uv = t^2$, and that of $X$ near $q$ is $xy = t$. But since the triple cover is a local...
Figure 20. The multiplicity $\text{mult}_{\delta_0} \delta_{4,i}$

isomorphism of $\hat{\mathcal{O}}_{Y,p}$ into $\hat{\mathcal{O}}_{X,r}$, the total space of $X$ near $r$ is given locally analytically by $zw = t^2$ ($u \mapsto z, v \mapsto w, t \mapsto t$). Therefore, $m_q = 1$, but $m_r = 2$, and

$$\text{mult}_{\delta_0} \delta_{4,i} = m_q + m_r = 3.$$  

4.3. The hyperelliptic locus $\mathcal{I}_g$ inside $\mathbb{T}_g$. Although the relations proved in this paper will be valid on the Picard group $\text{Pic}_Q \mathbb{T}_g$, it will be interesting to check what happens with the hyperelliptic curves inside the trigonal locus $\mathbb{T}_g$.

We noted that $\mathcal{I}_g$ is the only boundary component of $\mathbb{T}_g$ of codimension 2. It is obtained as the image $\text{pr}_1(\Delta \mathcal{H}_{3,0})$. By blowing up a point on a smooth hyperelliptic curve $C$, we add a $\mathbf{P}^1$-component to $C$ to make it a triple cover $C'$ (cf. Fig. 45). It terms of the quasi-admissible covers, such $C'$ behaves exactly as an irreducible singular trigonal curve in $\Delta \mathcal{H}_{0}$. However, $C$ does not contribute to the invariant $\delta_B$, as defined in Section 2.1. In fact, in a certain sense, it even decreases $\delta_B$.

To simplify the exposition, we shall postpone the discussion of families with hyperelliptic fibers until Section 11, where we will explain the behavior of trigonal families with finitely many hyperelliptic fibers in terms of the exceptional divisor $\Delta \mathcal{H}_{1,0}$ of the projection $\text{pr}_1$.

A similar phenomenon occurs with the boundary component $\Delta \mathcal{I}_{1,0} = \text{pr}_1(\Delta \mathcal{H}_{1,0})$, but it does not make sense to exclude its members from our discussion, since they behave exactly as members of the boundary divisor $\Delta \mathcal{I}_{1,i}$ for $i \geq 1$.

4.4. The invariants $\mu(C)$. In the transition from the original family $X \to B$ to the minimal quasi-admissible family $\tilde{X} \to \tilde{Y}$ over $\tilde{B}$, certain changes occur in the calculation of the basic invariants. To start with, it is easy to redefine $\lambda_{\tilde{X}}, \kappa_{\tilde{X}}$ and $\delta_{\tilde{X}}$ for $\tilde{X} \to \tilde{B}$: simply use the corresponding definitions from Section 2.1. Since we are interested in the slope of the family, which is invariant under base change, we may assume that $\tilde{B} := B$ and that $X$ is the pull-back over the new base $\tilde{B}$. Now the difference between $X$ and $\tilde{X}$ is reduced to several “quasi-admissible” blow-ups on $X$.

Blowing up smooth or rational double points on a surface does not affect its structure sheaf. Therefore, the degrees of the Hodge bundles on the two surfaces $X$ and $\tilde{X}$ will be the same: $\lambda_{\tilde{X}} = \lambda_X$. On the other hand, blowing up a smooth point on a surface decreases the square of its dualizing sheaf by 1, while there is no effect when blowing up a rational double point. Each type of singular fibers $C$ in $X$ requires apriori different quasi-admissible modifications (or no modifications at all), and thus decreases $\kappa_X$ by some nonnegative integer, denoted by $\mu(C)$:

$$\kappa_X = \kappa_{\tilde{X}} + \sum_C \mu(C).$$
Thus, $\mu(C)$ counts the number of “smooth blow-ups” on $C$, which are needed to obtain the minimal quasi-admissible cover $\tilde{C} \to C$ within the surface quasi-admissible cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$.

In the following Lemma, we compute the invariants $\mu(C)$ only for the general members of the boundary $\Delta \Sigma_g$ (cf. Fig. 18). The remaining, more special, singular curves in $\Delta \Sigma_g$ will be linear combinations of these $\mu(C)$’s (cf. Sect. 11).

**Lemma 4.1.** If $\mu_{k,i}$ denotes the invariant $\mu(C)$ for a general curve $C \in \Delta \Sigma_{k,i}$, then

(a) $\mu_0 = \mu_1,i = \mu_4,i = \mu_6,i = 0$;
(b) $\mu_2,i = 1$;
(c) $\mu_3,i = \mu_5,i = 2$.

**Proof.** The general members of the boundary $\Delta \mathcal{H}$ are in fact the minimal quasi-admissible covers associated to the general members of the boundary $\Delta \Sigma$, except for $\Delta_0$ which has $\mu_0 = 0$. Thus, we trace the blow-ups necessary to transform the curves in Fig. 18 to the curves in Fig. 19. For example, no blow-ups are needed in the case of $\Delta_{1,i}$, so that $\mu_{1,i} = 0$, while we need 2 blow-ups in the case of $\Delta_{3,i}$, and hence $\mu_{3,i} = 2$.

The only interesting situation occurs for $\Delta_{5,i}$. Apparently, there is only one added component $\mathbf{P}^1$ to the original $C \in \Delta_{3,i}$, but the lemma states that $\mu_{3,i} = 2$. The difference comes from the fact that near the intersection $r = C \cup \mathbf{P}^1$ the surface $\tilde{X}$ has equation $xy = t^2$, i.e. $r$ is a rational double point on $\tilde{X}$ of type $A_1$ (a similar situation occurred in Fig. 20). To obtain such a point $r$ in place of a smooth point $r_1$ on $X$, we first blow up $r_1$, and then on the obtained exceptional divisor we blow up another point $r_2$, so as to end with a chain of two $\mathbf{P}^1$’s (cf. Fig. 21). Finally, we blow down the first $\mathbf{P}^1$, and develop the required rational double point $r$. As a result, we have two “smooth” and one “singular” blow-ups, which implies $\mu_{5,i} = 2$.

**Figure 21.** Quasi-admissible blow-ups on $\Delta_{5,i}$

5. Effective Covers

In this section we construct the final type of triple covers in the Basic Construction. These will not be necessary for the global calculation in Section 7, so the reader may wish to skip this more technical part on a first reading, and assume in Section 6 that all covers are flat.

5.1. Construction of effective covers $\tilde{X} \to \tilde{Y}$. Consider a quasi-admissible cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$, as given in Prop. 3.1. In order to use the fact that the pushforward $\tilde{\phi}_* \mathcal{O}_{\tilde{X}}$ is locally free on $\tilde{Y}$, we need to assure that the map $\tilde{\phi}$ is flat. Unfortunately, there are certain points on $\tilde{X}$ where this fails to be true: exactly where the fibers of $\tilde{X}$ are ramified as triple covers of the corresponding fibers of $\tilde{Y}$. The situation can be resolved by several further blow-ups.

Namely, we work locally analytically near the points in $\tilde{X}$ of ramifications index 1 or 2, and consider correspondingly two cases.
5.1.1. Case of ramification index 1. This case involves members of the boundary divisors $\Delta_{T,4,i}$ and $\Delta_{T,5,i}$. Let $q$ be the point of ramification in the fiber of $\widetilde{X}$ over the point $p$ in the fiber of $\widetilde{Y}$ (cf. Fig. 22).

We use the pull-back of the map $\tilde{\phi}$ to study the embedding of the completion of the local ring of $p$ into that of $q$:

$$\hat{\mathcal{O}}_{\widetilde{Y},p} = \mathbb{C}[u,v,t]/(uv - t^2) \xrightarrow{\tilde{\phi}^\#} \hat{\mathcal{O}}_{\widetilde{X},q} = \mathbb{C}[x,y,t]/(xy - t).$$

Therefore, as an $\hat{\mathcal{O}}_{\widetilde{Y},p}$-module,

$$\hat{\mathcal{O}}_{\widetilde{X},q} = \hat{\mathcal{O}}_{\widetilde{Y},p} + \hat{\mathcal{O}}_{\widetilde{Y},p}x + \hat{\mathcal{O}}_{\widetilde{Y},p}y.$$ 

However, this is not a locally-free $\hat{\mathcal{O}}_{\widetilde{Y},p}$-module: for instance, one relation among the generators is $(v - t)x + (u - t)y = 0$.

Alternatively, the fiber of $\phi$ over $p$ is supported at $q$, but it is of degree 3 rather than 2, which would have been necessary for the flatness of a degree 2 map. Indeed, as $\mathbb{C}$-vector spaces:

$$\hat{\mathcal{O}}_{\widetilde{X},q} \otimes \hat{\mathcal{O}}_{\widetilde{Y},p} \text{Spec } k(p) \cong \hat{\mathcal{O}}_{\widetilde{X},q} / \hat{\mathcal{O}}_{\widetilde{Y},p} \cong \mathbb{C}[[x,y]]/(x^2, y^2, xy) = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y.$$

In Fig. 22 one can visually observe the two distinct tangent directions at $q$ making it a fat point of degree 3.

We conclude that $\tilde{\phi}$ is indeed non-flat at $q$. To resolve this, we blow-up $\widetilde{Y}$ at $p$ and $\widetilde{X}$ at $q$, denoting the new surfaces by $\hat{\widetilde{Y}}$ and $\hat{\widetilde{X}}$. It is easy to see that they fit into the following coming diagram:

In order to keep the map $\hat{\tilde{\phi}} : \hat{\widetilde{X}} \to \hat{\widetilde{Y}}$ of degree 3, we need to blow-up one further point on $\hat{\widetilde{X}}$: if the inverse image of $p$ is $\{q, r\}$ we blow-up $r$, and thus we add the necessary component to $\hat{\widetilde{X}}$ to make it a triple cover of $\hat{\widetilde{Y}}$ (cf. Fig. 39).
5.1.2. Case of ramification index 2. The only boundary component, where ramification index 2 occurs, is $\Delta \Sigma_{k,i}$. Similarly as above, $\phi : \tilde{X} \to \tilde{Y}$ is non-flat at $q$. Indeed, $\tilde{\mathcal{O}}_{\tilde{X},q}$ is generated as an $\tilde{\mathcal{O}}_{\tilde{Y},p}$-module by $1, x, y, x^2, y^2$, but not-freely due to the relation $u \cdot x + v \cdot y - t \cdot x^2 - t \cdot y^2 = 0$. To resolve the apparent non-flatness of $\tilde{\phi}$, we can blow-up once $\tilde{X}$ and $\tilde{Y}$ at $q$ and $p$, but this would not be sufficient. In fact, we must make further blows-ups on each surface, as Fig. 24 suggests: two more on $\tilde{X}$ and one more on $\tilde{Y}$.

![Figure 24. Resolving the case of ram. index 2](image)

In both cases of ramification index 1 or 2, the new map $\hat{\phi} : \hat{X} \to \hat{Y}$ is obtained from $\tilde{\phi}$ by a base change, and hence $\hat{\phi}$ is proper and finite, and by construction, also a flat morphism. We call such covers effective.

The above considerations combined with Prop. 3.1 imply the existence of effective covers for our families of trigonal curves:

**Proposition 5.1.** Let $X/B$ be a family of trigonal curves with smooth general member. After several blow-ups (and possibly modulo a base change) we can associate to it an effective cover $\hat{\phi} : \hat{X} \to \hat{Y}$.

Here $\hat{Y}$ is a birationally ruled surface over $B$. If the base curve $B$ is not tangent to the boundary divisors $\Delta \Sigma_{k,i}$, then the resulting surfaces $\hat{X}$ and $\hat{Y}$ will have smooth total spaces. If, moreover, $B$ intersects the $\Delta_{k,i}$’s only in their general points (as given in Fig. 18), then the special fibers of $\hat{Y}$ and $\hat{X}$ are easy to describe (cf. Fig. 38-40). For example, $\hat{Y}$’s special fibers are either chains of two or three reduced projective lines, or chains of five smooth rational curves with non-reduced middle component of multiplicity two. The special fibers of $\hat{X}$ can also contain nonreduced components (of multiplicity 2 or 3), and this occurs only in the ramification cases discussed above ($\Delta \Sigma_{k,i}$ for $k = 4, 5, 6$).

5.2. Change of $\lambda_X, \kappa_X$ and $\delta_X$ in the effective covers. This is an analog to the discussion in Section 4.4. After the necessary base changes we again identify, without loss of generality, the new base curve $\hat{B}$ with $B$, and the pull-back of $X$ over $\hat{B}$ with $X$, and we redefine the basic invariants $\lambda_{\hat{X}}$ and $\kappa_{\hat{X}}$ for the effective family $\hat{X}$ over $\hat{B}$. (It doesn’t make sense to define directly $\delta_{\hat{X}}$, because of the nonreduced fiber components in $\hat{X}$. We could, of course, set $\delta_{\hat{X}} = 12\lambda_{\hat{X}} - \kappa_{\hat{X}}$, but we will not need this in the sequel.) Now the original $X$ and the effective $\hat{X}$ differ by “quasi-admissible” and “effective” blow-ups. The connections between the invariants of $X$, $\tilde{X}$ and $\hat{X}$ are expressed by the following
Lemma 5.1. With the above notation,

(a) $\lambda_X = \lambda_{\tilde{X}} = \hat{\lambda}_X$;
(b) $\kappa_X = \kappa_{\tilde{X}} + \sum C \mu(C)$;
(c) $\kappa_{\tilde{X}} = \kappa_{\hat{X}} + \sum 1 + \sum 3$.

Proof. In view of Lemma 4.1, the first and the second statements are obvious. Obtaining a flat cover $\hat{X} \to \hat{Y}$ requires blowing up on $\tilde{X}$ one smooth point for each ramification index 1, and three smooth points for each ramification index 2. Hence the relation between $\kappa_{\hat{X}}$ and $\kappa_{\tilde{X}}$.

6. Embedding $\hat{X}$ in a Projective Bundle over $\hat{Y}$

Given the effective degree 3 map $\hat{\phi} : \hat{X} \to \hat{Y}$, our next step is to embed $\hat{X}$ into a projective bundle $P^V$ of rank 1 over the birationally ruled surface $\hat{Y}$. We shall consider a degree 3 morphism $\hat{\phi}$, but the same discussion is valid for any degree $d$.

6.1. Trace map. Since $\hat{\phi}$ is flat and finite, the pushforward $\hat{\phi}_* (O_{\hat{X}})$ is a locally free sheaf on $\hat{Y}$ of rank 3. Define the trace map

$$\text{tr} : \hat{\phi}_* (O_{\hat{X}}) \to O_{\hat{Y}}$$

as follows. The finite field extension $K(\hat{X})$ of $K(\hat{Y})$ induces the algebraic trace map $\text{tr}^# : K(\hat{X}) \to K(\hat{Y})$, defined by $\text{tr}^# (a) = \frac{1}{3}(a_1 + a_2 + a_3)$. Here the $a_i$'s are the conjugates of $a$ over $K(\hat{Y})$ in an algebraic closure of $K(\hat{X})$. The restriction $\text{tr}^# |_{K(\hat{Y})} = \text{id}_{K(\hat{Y})}$. Over an affine open $U = \text{Spec } A \subset \hat{Y}$ and its affine inverse $\hat{\phi}^{-1}(U) = \text{Spec } B \subset \hat{X}$, $B$ is the integral closure of $A$ in its field of fractions $K(\hat{X})$. Therefore, the trace map restricts to the $A$-module homomorphism $\text{tr}^# : B \to A$. We have a commutative diagram:

\[
\begin{array}{ccc}
\hat{\phi}^{-1}(U) & \xrightarrow{\hat{\phi}} & \hat{X} \\
\downarrow & & \downarrow \text{tr}^# \\
U = \text{Spec } A & \xrightarrow{A} & K(\hat{X})
\end{array}
\]

**Figure 25.** The trace map

The so-defined local maps $\text{tr} : \hat{\phi}_* O_{\text{Spec } B} \to O_{\text{Spec } A}$ patch up to give a global trace map $\text{tr} : \hat{\phi}_* O_{\hat{X}} \to O_{\hat{Y}}$. Let $\mathcal{V}$ be the kernel of $\text{tr}$:

\[
(6.1) \quad 0 \to \mathcal{V} \to \hat{\phi}_* O_{\hat{X}} \xrightarrow{\text{tr}} O_{\hat{Y}} \to 0.
\]

Note that $\mathcal{V}$ is locally free of rank 2. The natural inclusion $O_{\hat{Y}} \to \hat{\phi}_* O_{\hat{X}}$, composed with $\text{tr}$, is the identity on $O_{\hat{Y}}$, hence the exact sequence splits:

\[
(6.2) \quad \hat{\phi}_* O_{\hat{X}} = O_{\hat{Y}} \oplus \mathcal{V}.
\]
6.1.1. Geometric interpretation of the trace map. It is useful to interpret the trace map geometrically in terms of the corresponding vector bundles $\hat{\phi}_*O_{\hat{X}}$, $O_{\hat{Y}}$ and $V$ associated to the sheaves $\hat{\phi}_*O_{\hat{X}}$, $O_{\hat{Y}}$ and $V$. We again localize over affine opens, and if necessary, we shrink $U = \text{Spec } A$ so that $\hat{\phi}_*O_{\hat{X}}$ becomes a free $O_{\hat{Y}}$-module.

Let $p$ be a closed point in $\text{Spec } A$ with maximal ideal $p \subset A$, having three distinct preimages $q, r, s \in \text{Spec } B$ with maximal ideals $q, r, s \subset B$. Since $B$ is a free $A$-module, the quotient $B/pB$ is a 3-dim'l algebra over the ground field $k(p) = A/p$, i.e. a 3-dim'l vector space over $C$. On the other hand, from $\mathfrak{qrs} = q\cap r\cap s$ and the Chinese Remainder Theorem, it is clear that $B/pB \cong B/q \oplus B/r \oplus B/s \cong C\hat{f}_q \oplus C\hat{f}_r \oplus C\hat{f}_s$. The generators $\hat{f}_q, \hat{f}_r, \hat{f}_s$ are chosen as usual: $\hat{f}_q$, for instance, is the residue in $k(q)$ of a function $f_q \in B$ such that $f_q \equiv 1(\text{mod } q)$, $f_q \equiv 0(\text{mod } r, s)$.

In the Grothendieck style, the vector bundle over $\hat{Y}$ associated to $\hat{\phi}_*O_{\hat{X}}$ is $\text{Spec } S(B_A)$, where $S(B_A)$ is the symmetric algebra of $B$ over $A$. Its fiber over $p$ is the pull-back $\text{Spec } S(B_A) \times_{\text{Spec }A} \text{Spec } k(p) = \text{Spec } (S(B_A) \times_A A/p) = \text{Spec } S(B/pB)$. We prefer to work with the dual $\hat{\phi}_*O_{\hat{X}}$ of this bundle, and the same goes for projectivizations: we projectivize the 1-dim'l subspaces of $\hat{\phi}_*O_{\hat{X}}$ rather than its 1-dim'l quotients.

In view of this convention, the fiber of the bundle $\hat{\phi}_*O_{\hat{X}}$ is canonically identified as

$$(\hat{\phi}_*O_{\hat{X}})_p = B/pB \cong C\hat{f}_q \oplus C\hat{f}_r \oplus C\hat{f}_s \cong A^3_C.$$ 

The generators $\hat{f}_q, \hat{f}_r, \hat{f}_s$ span three lines in $A^3_C$, which can be canonically described: the line $\Lambda_q = C\hat{f}_q$, for example, corresponds to all functions regular at $q, r$ and $s$, and vanishing at $r$ and $s$.

Similarly, the vector bundle $O_{\hat{Y}}$ associated to $O_{\hat{Y}}$ has fiber $(O_{\hat{Y}})_p = A/p \cong C\hat{h}_p$, where $h_p$ is a function near $p$ having residue $h_p(p) = 1$ in $k(p)$. The trace map $\text{tr} : \hat{\phi}_*O_{\hat{X}} \rightarrow O_{\hat{Y}}$ translates fiberwise in terms of the vector bundles $\hat{\phi}_*O_{\hat{X}}$ and $O_{\hat{Y}}$ as:

$$\text{tr}_p : C\hat{f}_q \oplus C\hat{f}_r \oplus C\hat{f}_s \rightarrow C\hat{h}_p, \hat{f} \mapsto \frac{1}{3}(f(q) + f(r) + f(s)).$$

Finally, the locally free subsheaf $V = \ker(\text{tr}) \subset \hat{\phi}_*O_{\hat{X}}$ is associated to a vector bundle $V$ with fiber $V_p = \{\hat{f} \mid f(q) + f(r) + f(s) = 0\} \subset (\hat{\phi}_*O_{\hat{X}})_q$. Equivalently, from the direct sum (6.2), $V_p = (\hat{\phi}_*O_{\hat{X}})_p/\Lambda$, where the line $\Lambda = \{\hat{f} \mid f(q) = f(r) = f(s)\}$ corresponds to pull-backs of functions regular at $p$.

![Figure 26. Geometric interpretation of tr](image-url)
Since the four lines $\Lambda_q, \Lambda_r, \Lambda_s$ and $\Lambda$ are in general position in the fiber $(\widehat{\phi}_s O_\widehat{X})_p$, modding out by $\Lambda$ yields three distinct lines in the fiber $V_p$ (cf. Fig. 26). Therefore, projectivizing $V_p$ naturally induces three distinct points $Q, R, S$ in the fiber $\mathbb{P}^1$ of $PV$. Going the other way around the diagram, we first projectivize $(\widehat{\phi}_s O_\widehat{X})_q \cong \mathbb{A}^3$, and then we project from the point $[\Lambda]$ onto the fiber of $PV$. In other words, $\pi[\Lambda] : \mathbb{P}^2 \rightarrow \mathbb{P}^1$ is well-defined at $[\Lambda_q], [\Lambda_r]$ and $[\Lambda_s]$.

This completes the interpretation of the trace map in the case of three distinct preimages $q, r, s$ in $\widehat{X}$. In case of only two or one preimage of $p$ in $\widehat{X}$, one modifies correspondingly the above interpretation.

6.2. $\widehat{X}$ embeds naturally in $PV$ over $\widehat{Y}$. We construct the map $i : \widehat{X} \hookrightarrow PV$ via the use of an invertible relative dualizing sheaf $\omega_{\widehat{X}/\widehat{Y}}$. Its existence imposes a mild technical condition on the schemes $\widehat{X}$ and $\widehat{Y}$: we assume that they are Gorenstein, i.e. Cohen-Macaulay with invertible dualizing sheaves $\omega_{\widehat{X}/\mathbb{C}}$ and $\omega_{\widehat{Y}/\mathbb{C}}$. In our situation this will be sufficient. As we noted in Section 5.1, when the base curve $B$ is not tangent to the boundary divisors $\Delta \Sigma_{k,i}$, then $\widehat{X}$ and $\widehat{Y}$ are smooth surfaces. The remaining cases are “local” base changes of these, and the construction carries over.

**Proposition 6.1.** Let $\widehat{\phi} : \widehat{X} \rightarrow \widehat{Y}$ be a flat and finite degree $d$ morphism of Gorenstein schemes, with $\widehat{Y}$ integral. Then $\widehat{\phi}$ factors through a natural embedding of $\widehat{X}$ into the projective bundle $PV$, followed by the projection $\pi : PV \rightarrow \widehat{Y}$ (cf. Fig. 29).

For easier referencing in the sequel, we have kept the notation $\widehat{X}$ and $\widehat{Y}$, but the statement is true for any schemes satisfying the Gorenstein condition. For another proof of Prop. 5.1, see [CE].

**Proof of Prop. 6.1.** Here we construct the map $i : \widehat{X} \rightarrow PV$, give the proof of its regularity, and point out how to show its injectivity.

$$
\begin{array}{ccc}
P(O_{\widehat{Y}}) & \xrightarrow{\psi} & P(\widehat{\phi}_s O_\widehat{X}) \\
\downarrow{i} & \downarrow{\rho} & \\
\widehat{X} & \rightarrow & PV
\end{array}
$$

**Figure 27.** Embedding $i : \widehat{X} \hookrightarrow PV$

6.2.1. **Construction of the embedding map.** According to Prop. II.7.12 [H], to give a morphism $\psi : \widehat{X} \rightarrow P(\widehat{\phi}_s (O_\widehat{X}))$ over $\widehat{Y}$ is equivalent to give an invertible sheaf $L$ on $\widehat{Y}$ and a surjective map of sheaves $\widehat{\phi}^* (\widehat{\phi}_s (O_\widehat{X})) \rightarrow L$. Recall from relative Serre-duality that $(\widehat{\phi}_s O_\widehat{X})^* \cong \widehat{\phi}_s \omega_{\widehat{X}/\widehat{Y}}$, and let $L = \omega_{\widehat{X}/\widehat{Y}}$. The natural morphism

$$
\sigma : \widehat{\phi}^* \widehat{\phi}_s \omega_{\widehat{X}/\widehat{Y}} \rightarrow \omega_{\widehat{X}/\widehat{Y}}
$$

is surjective. This is in fact true for any quasicoherent sheaf $F$ on $\widehat{X}$. Indeed, restricting to the affine open sets $\widehat{\phi} : \text{Spec } B \rightarrow \text{Spec } A$, we have $F = \widehat{F}$ for some finitely generated $B$-module $M$, and $\widehat{\phi}^* \widehat{\phi}_s F = \widehat{\phi}^* (\widehat{M}) = (\widehat{M} \otimes_A B)$. The surjective $B$-module homomorphism $M_A \otimes_B B \rightarrow M$, given by $m \otimes b \mapsto b \circ m$, induces $\widehat{\phi}^* \widehat{\phi}_s F \rightarrow F$. 

Thus, the above map $\sigma$ naturally defines a morphism $\psi : \hat{X} \to \mathbf{P}(\hat{\phi}_*(\mathcal{O}_X))$ over $\hat{Y}$. Projectivizing $0 \to \mathcal{O}_\hat{Y} \to \hat{\phi}_*\mathcal{O}_X \to \mathcal{V} \to 0$ gives a sequence of projective bundles, as in Fig. 27. The map $\rho$ is undefined exactly on the image of $\text{tr}^\#$. Composing $\rho$ with the map $\psi$ yields the map $i : \hat{X} \to \mathbf{P}V$, which we claim is a regular map.

6.2.2. Regularity and injectivity of $i$. To see regularity, we show that the restriction of $\sigma_{\hat{\phi}_*(\mathcal{V}^*)} : \hat{\phi}^*(\mathcal{V}^*) \to \omega_{\hat{X}/\hat{Y}}$ is also surjective. Indeed, we again work locally, and let $B = A \oplus C$ be the decomposition of $B$ via the trace map as a free $A$-module, where $C = A \cdot b_1 \oplus A \cdot b_2$ with $\text{tr} b_1 = \text{tr} b_2 = 0$. Let $\omega_{\hat{X}/\hat{Y}} = M^\sim$ for some finitely generated $B$-module $M$. Recall that $\hat{\phi}_*\omega_{\hat{X}/\hat{Y}} \cong (\hat{\phi}_*\mathcal{O}_X)^\sim$, so that as $A$-modules: $M \cong (B_A)^\sim = \text{Hom}_A(B, A)$, and $B$ acts on $M$ by

$$(b \circ f)(x) = f(bx) \text{ for } f \in \text{Hom}_A(B, A), x \in B.$$ Naturally, the sheaf $\mathcal{V} = C^\sim$, and $\hat{\phi}^*(\mathcal{V}^*) = (\text{Hom}_A(C, A) \otimes_A B)^\sim$, where we think of $f \in \text{Hom}_A(C, A)$ as an element of $\text{Hom}_A(B, A)$ by extending it via $f(1) = 0$. Our statement is equivalent to showing that the $B$-module homomorphism

$$\sigma : \text{Hom}_A(C, A) \otimes_A B \to (\text{Hom}_A(B, A))_B, \ f \otimes b \mapsto b \circ f,$$ is surjective. In fact, it suffices to show that the trace map is in the image of $\sigma$, i.e. to find $c_1, c_2 \in B$ such that

$$\text{tr} \equiv c_1 \circ \pi_1 + c_2 \circ \pi_2.$$ Here $\pi_j : B \to A$ gives the $b_j$-coordinate of $b \in B$, $j = 1, 2$. Set $c_1 = b_1 - \pi_1(b_1^2)$ and $c_2 = -\pi_1(b_1 b_2)$. Evaluating both sides of (6.3) at $1, b_1$ and $b_2$ yields the same result, hence the identity is established, and $\sigma|_{\hat{\phi}_*(\mathcal{V}^*)} : \hat{\phi}^*(\mathcal{V}^*) \to \omega_{\hat{X}/\hat{Y}}$ is surjective.

We have shown that the composition $\rho \circ \psi = i : \hat{X} \to \mathbf{P}V$ is a regular map on $\hat{X}$. Alternatively, one could employ the geometric interpretation of the trace map. A general point $p \in \hat{Y}$ has three preimages $q, r, s$ in $\hat{X}$, each of which defines canonically a distinct point $[\Lambda_p]$, $[\Lambda_q]$ or $[\Lambda_s]$ in the fiber of $\mathbf{P}(\hat{\phi}_*\mathcal{O}_X)$. As we pointed above, the projection $\pi|_{[\Lambda]} : \mathbf{P}^2 \to \mathbf{P}^1$ is well-defined at $[\Lambda_q]$, $[\Lambda_q]$ and $[\Lambda_s]$. But $\pi|_{[\Lambda]}$ is precisely the fiberwise restriction of $\mathbf{P}(\hat{\phi}_*\mathcal{O}_X) \xrightarrow{\rho} \mathbf{P}V$, which shows again that the composition $i = \rho \circ \psi : \hat{X} \to \mathbf{P}V$ is regular on an open set of $\hat{X}$. One makes the necessary modifications in the cases of fewer preimages of $p$ in $\hat{X}$. Finally, one can show, using similar methods (either algebraically or geometrically), that the map $i$ is also an embedding. \hfill $\square$

**Remark 6.1.** Since the general fiber $C$ of $\hat{X}$ is a smooth trigonal curve, the restriction $i|_C$ embeds $C$ in a ruled surface $F_k$ over the corresponding fiber $F_{\hat{Y}} = \mathbf{P}^1$ of $\hat{Y}$, where $F_k = \mathbf{P}(V|_{F_{\hat{Y}}})$. In Section 12.1, we will describe how the ruled surface $F_k$ varies as the fiber $C$ varies in $\hat{X}$.

7. Global Calculation on a Triple Cover $X \to Y$

In this section we consider the simplest case of effective covers, namely, when the original family $X$ is itself a triple cover of a ruled surface $Y$ over the base curve $B$. This happens exactly when all fibers of $X$ are irreducible, and the linear system of $g^3_2$’s on the smooth fibers extends over the singular fibers to base-point free line bundles of degree 3 with two linearly independent sections. As we saw in Section 2.6, we can patch together all these $g^3_2$’s in a line bundle $\mathcal{L}$ on the total space of $X$. Thus, $X$ will map to $\mathbf{P}(H^0(X, \mathcal{L})^\sim)$ via $\phi_\mathcal{L}$, and this map will factor through our ruled surface $Y$ over $B$: 
Equivalently, we can describe such a family $X \to B$ via the classification of the boundary components of the trigonal locus in Section 4: in $\overline{\Sigma}_g$ the base curve $B$ meets only the boundary component $\Delta \overline{\Sigma}_0$ of irreducible curves ($\delta_0|_B > 0$), and there are no hyperelliptic fibers in $X$ ($B \cap \overline{\Sigma}_g = \emptyset$).

7.1. Global versus local calculation. As it will turn out, the calculation of the slope $\delta_X/\lambda_X$ in this basic case yields the actual maximal bound $\frac{36(g+1)}{9g+1}$: any addition of singular fibers belonging to other boundary components of $\overline{\Sigma}_g$ will only decrease the ratio. Henceforth, we distinguish among two types of calculation: global and local. The global calculation refers to finding the coefficients of $\delta_0$ and the Hodge bundle $\lambda|_{\overline{\Sigma}_g}$ in a relation in $\text{Pic}_Q \overline{\Sigma}_g$ involving all boundary classes. The local calculation provides the remaining coefficients by considering local invariants of each individual boundary class (cf. Sect. 8).

7.2. The basic construction. For the remainder of this section, we consider a family $X \to B$ such that, as above, $X$ is a triple cover of the corresponding ruled surface $Y$, and we carry out the proposed global calculation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure29.png}
\caption{Triple Case}
\end{figure}

Recall that the pushforward of the structure sheaf $\mathcal{O}_X$ to $Y$ is a locally free sheaf of rank 3. In the exact sequence:

$$0 \to E \to \phi_* \mathcal{O}_X \xrightarrow{\text{tr}} \mathcal{O}_Y \to 0,$$

the kernel of the trace map $\text{tr}$ is a vector bundle $E$ on $Y$ of rank 2, and $X$ naturally embeds in the rank 1 projective bundle $PV$ over $Y$, where $V = E^\perp$. Any rank 2 vector bundle $E$ has the same projectivizations as its dual bundle $V$ since $E \cong \bigwedge^2 E \otimes V$, where $\bigwedge^2 E$ is a line bundle. For easier notation, in the trigonal case we use the dual $V$ instead of $E$ from Section 5.2.

A basis for $\text{Pic} Y$ can be chosen by letting $F_Y$ be the fiber of $Y$, and $B'$ be any section of $Y \to B$. Hence $\text{Pic} Y = \mathbb{Z}B' \oplus \mathbb{Z}F_Y$. We normalize by replacing $B'$ with the $\mathbb{Q}$-linear combination $B_0 = B' - \frac{(B')^2}{2} F_Y$, and provide a $\mathbb{Q}$-basis for $\text{Pic}_Q Y$:

$$\text{Pic}_Q Y = \mathbb{Q}B_0 \oplus \mathbb{Q}F_Y$$

with $B_0^2 = F_Y^2 = 0$ and $B_0 \cdot F_Y = 1$. 

\begin{align}
(7.1) \quad \text{Pic}_Q Y = \mathbb{Q}B_0 \oplus \mathbb{Q}F_Y \quad \text{with} \quad B_0^2 = F_Y^2 = 0 \quad \text{and} \quad B_0 \cdot F_Y = 1.
\end{align}
Let \( \zeta \) denote the class of the hyperplane line bundle \( \mathcal{O}_{\mathbb{P}^V}(+1) \) on \( \mathbb{P}^V \), and let \( c_1(V) \) and \( c_2(V) \) be the Chern classes of \( V \) on \( Y \). The Chow ring \( A(\mathbb{P}^V) \) is generated as a \( \pi^*(A(Y)) \)-module by \( \zeta \) with the only relation:

\[
\zeta^2 + \pi^*c_1(V)\zeta + \pi^*c_2(V) = 0.
\]

In particular, for the Picard groups:

\[
\text{Pic}_Q(\mathbb{P}^V) = \pi^*(\text{Pic}_Q Y) \oplus \mathbb{Q}\zeta.
\]

As a divisor on \( \mathbb{P}^V \), the surface \( X \) meets the fiber \( F_\pi \) of \( \pi \) generically in three points (\( X \) maps three-to-one onto \( Y \)). Thus in the Chow ring \( A(\mathbb{P}^V) \) we have \([X] \cdot [F_\pi] = 3\), which simply means that \( X \) can be expressed as

\[
X \sim 3\zeta + \pi^*D
\]

for some divisor \( D \) on \( Y \) (see (7.3)). With respect to the chosen basis for \( \text{Pic}_Q Y \):

\[
D \sim aB_0 + bF_Y \quad \text{and} \quad c_1(V) \sim cB_0 + dF_Y
\]

for some \( a, b, c, d \in \mathbb{Z} \). Note that \( \deg(D|B_0) = b \) and \( \deg(c_1(V)|B_0) = d \).

### 7.3. Relation among the divisor classes \( D \) and \( c_1(V) \)

It is evident that the divisors \( D \) and \( c_1(V) \) cannot be independent on ruled surface \( Y \) since both are canonically determined by the surface \( X \). The relation is in fact quite straightforward.

**Lemma 7.1.** With the above notation for the triple cover \( \phi : X \to Y \), we have \( D = 2c_1(V) \) in \( \text{Pic} Y \).

**Proof.** We start with the standard exact sequence for the divisor \( X \) on \( \mathbb{P}^V \):

\[
0 \to \mathcal{O}_{\mathbb{P}^V}(-X) \to \mathcal{O}_{\mathbb{P}^V} \to \mathcal{O}_X \to 0.
\]

Pushing to \( Y \) yields:

\[
0 \to \pi_*\mathcal{O}_{\mathbb{P}^V}(-X) \to \pi_*\mathcal{O}_{\mathbb{P}^V} \to \pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_{\mathbb{P}^V}(-X) \to R^1\pi_*\mathcal{O}_{\mathbb{P}^V} \to \cdots
\]

It is easy to show that \( R^1\pi_*\mathcal{O}_{\mathbb{P}^V} = 0 \) and \( \pi_*\mathcal{O}_{\mathbb{P}^V}(-X) = 0 \). This follows from Grauert’s theorem \([11]: h^1(\mathcal{O}_{\mathbb{P}^V}|_{F_\pi}) = h^1(\mathcal{O}_{\mathbb{P}^1}) = 0 \), and

\[
h^0(\mathcal{O}_{\mathbb{P}^V}(-X)|_{F_\pi}) = h^0(\mathcal{O}_{\mathbb{P}^V}(-3\zeta - \pi^*D)|_{F_\pi}) = h^0(\mathcal{O}_{\mathbb{P}^1}(-3)) = 0.
\]

Furthermore, \( \pi_*\mathcal{O}_{\mathbb{P}^V} = \mathcal{O}_Y \) and \( \pi_*\mathcal{O}_X = \phi_*\mathcal{O}_X \), so that (7.6) becomes

\[
0 \to \mathcal{O}_Y \to \phi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_{\mathbb{P}^V}(-X) \to 0.
\]

From relative Serre-duality, and using the first Chern class of the relative dualizing sheaf, \( c_1(\omega_\pi) \) (cf. \([BPV]\)):

\[
R^1\pi_*\mathcal{O}_{\mathbb{P}^V}(-X) \cong (\pi_*\omega_\pi \otimes \mathcal{O}_{\mathbb{P}^V}(X)) = (\pi_*\mathcal{O}_{\mathbb{P}^V}(\zeta + \pi^*D - \pi^*c_1(V)))^\sim.
\]

Since \( \pi_*\mathcal{O}_{\mathbb{P}^V}(\zeta) = V^\sim \) (cf. \([BPV]\)),

\[
R^1\pi_*\mathcal{O}_{\mathbb{P}^V}(-X) \cong [V^\sim \otimes \mathcal{O}_Y(D - c_1(V))]^\sim.
\]

Finally, combining (7.8) with (7.7) and \( \phi_*\mathcal{O}_X/\mathcal{O}_Y = V^\sim \), we arrive at

\[
V \cong V^\sim \otimes \mathcal{O}_Y(D - c_1(V)) \Rightarrow \mathcal{O}_Y(D - c_1(V)) \cong \bigwedge^2 V \cong \mathcal{O}_Y(c_1(V)),
\]

and hence \( D = 2c_1(V) \) in \( \text{Pic} Y \). \( \square \)
7.4. Global calculation of $\lambda_X$ and $\kappa_X$. In the following proposition we express $\lambda_X$ and $\kappa_X$ in terms of $\deg(c_1(V)|_{B_0}) = d$ and the Chern polynomial $c_1^2(V) - 4c_2(V)$, both of which are independent of the choice of the vector bundle $V$. Indeed, if we twist $V$ by a line bundle $M$ on $Y$ and set $V' = V \otimes M$, then

$$c_1(V') = c_1(V) + 2c_1(M), \quad c_2(V') = c_2(V) + c_1(V)c_1(M) + c_1(M)^2,$$

$$\implies c_1(V')^2 - 4c_2(V') = c_1(V)^2 - 4c_2(V).$$

Recall the notation of (7.4). In order to make $d$ also invariant, we use $b = 2d$ from Lemma 7.3 and write $d = 2b - 3d$. Now, if we replace $PV$ with its isomorphic $PV'$ (cf. Fig. 29), and set $\zeta' = i^*\zeta \otimes (\pi')^*M^{-1}$ to be the new hyperplane bundle, then in $\text{Pic}(PV)$: $X \sim 3\zeta' + (\pi')^*D'$ with $D' \sim D + 3c_1(M)$. Hence

$$2D' - 3c_1(V') = 2D + 6c_1(M) - 3c_1(V) - 6c_1(M) = 2D - 3c_1(V),$$

and equating their degrees on $B_0$, we obtain $2b' - 3d' = 2b - 3d$.

In other words, the following formulas for $\lambda_X$ and $\kappa_X$ would be valid for any vector bundle $V'$ in place of the canonically defined $V$, as long as the diagram of the basic construction (cf. Fig. 29) is satisfied, and as long as we adjust the degree $d = \deg(c_1(V)|_{B_0})$ by its invariant form $2b - 3d = 2\deg(D|_{B_0}) - 3\deg(c_1(V)|_{B_0})$.

**Proposition 7.1.** Let $\phi : X \to Y$ be a degree 3 map from the original family $X$ of trigonal curves to the ruled surface $Y$ over $B$. The invariants $\lambda_X$ and $\kappa_X$ are given by the formulas:

$$\lambda_X = \frac{g}{2} \deg(c_1(V)|_{B_0}) + \frac{1}{4}(c_1(V)^2 - 4c_2(V)),$$

$$\kappa_X = \frac{5g - 6}{2} \deg(c_1(V)|_{B_0}) + \frac{3}{4}(c_1(V)^2 - 4c_2(V)).$$

We defer the proof of Prop. 7.1 to Subsections 7.4.2-3.

7.4.1. *Notation and Basic Tools.* The proof of Prop. 7.1 consists of two calculations in the Chow ring of $PV$; one uses versions of Riemann-Roch theorem on $X$ and $PV$, and the other uses the adjunction formula on $PV$ for the divisor $X$. Here we discuss these statements and set up the necessary notation.

In order to work in $\mathbb{A}(PV)$, we express the Chern classes of $PV$ in terms of the hyperplane class $\zeta$ and the Chern classes of $Y$. We first recall that $\pi_*\mathcal{O}_{PV}(+1) \cong \mathcal{V}$. In the Euler sequence on $PV$:

$$0 \to \mathcal{O}_{PV} \to \mathcal{O}_{PV}(+1) \otimes \pi^*V \to \mathcal{F}_{\pi} \to 0,$$

we compare the Chern polynomials $c_t(\mathcal{O}_{PV}(+1) \otimes \pi^*V) = c_t(\mathcal{F}_{\pi})$, and obtain:

$$K_{PV} = -2\zeta - \pi^*c_1(V) + \pi^*K_Y,$$

$$c_1(\mathcal{F}_{\pi}) = -2\zeta - \pi^*c_1(V),$$

$$c_2(PV) = -2\zeta \pi^*K_Y + \pi^*(c_1(V)K_Y + c_2(Y)).$$
Here $\mathcal{T}_\pi$ and $\omega_\pi$ are correspondingly the relative tangent and the relative dualizing sheaves of $\pi$, while $K_{PV}$ is the class of the canonical sheaf on $PV$. On the ruled surface $Y$ over the curve $B$ of genus $g_B$ we similarly have

$$K_Y = -2B_0 + h^*(K_B) \equiv -2B_0 + (2g_B - 2)F_Y,$$

(7.13)

$$c_2(Y) = 4(1 - g_B).$$

(7.14)

Now let $C$ be the general fiber of $X$, i.e. a smooth trigonal curve of genus $g$. Assuming the Basic construction for the triple cover $X \to Y$ (cf. Fig. [23]), we have the following lemmas.

**Lemma 7.2.** If $\chi(\mathcal{E})$ denotes the holomorphic Euler characteristic of any sheaf $\mathcal{E}$, then the invariant $\lambda_X$ is expressible as $\lambda_X = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_B) \cdot \chi(\mathcal{O}_C)$.

**Proof.** From Grothendieck-Riemann-Roch theorem for the map $\pi: Y \to B$.

$$\chi(f_!\mathcal{O}_X) \cdot \text{td} \mathcal{E}_B = f_*(\chi \mathcal{O}_X \cdot \text{td} \mathcal{E}_B),$$

where $\mathcal{E}_X$ and $\mathcal{E}_B$ are the corresponding tangent sheaves. Since the fibers of $f$ are one-dimensional, $f_!\mathcal{O}_X = f_\ast \mathcal{O}_X - R^1f_\ast \mathcal{O}_X = \mathcal{O}_B - (f_\ast \omega_f)$. Substituting:

$$\chi(f_!\mathcal{O}_X) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_B) \cdot \chi(\mathcal{O}_C).$$

Note the similarity between this formula and the formula for $\delta_B$ in Example 2.1. Both quantities are expressed as differences of the Euler characteristic (holomorphic or topological) on the total space of $X$ and the product of the corresponding Euler characteristics on the base $B$ and the general fiber $C$. Lemma 7.2 suggests that in order to calculate $\lambda_X$, we must have control over $\chi(\mathcal{O}_X)$.

**Lemma 7.3.** In the Chow ring of $PV$:

$$\chi(\mathcal{O}_X) = \frac{1}{12} X[(X + K_{PV})(2X + K_{PV}) + c_2(\mathcal{O}_X)].$$

**Proof.** From the standard exact sequence (7.3) for the divisor $X$ on $PV$ we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{PV}) - \chi(\mathcal{O}_{PV}(-X))$. On the other hand, Hirzebruch-Riemann-Roch claims that for any sheaf $\mathcal{E}$ on $PV$: $\chi(\mathcal{E}) = \deg(\chi(\mathcal{E}) \cdot \text{td} \mathcal{E}_X)$. Applying this to the line bundles $\mathcal{O}_{PV}$ and $\mathcal{O}_{PV}(\cdot)$, and subtracting the results completes the proof of the lemma.

The reader may have noticed that all quantities discussed in the above lemmas are elements of the third graded piece $\mathbb{A}^3(\mathbb{P}^V) \otimes \mathbb{Q}$ of the Chow ring $\mathbb{A}(\mathbb{P}^V) \otimes \mathbb{Q}$. Hence they are cubic polynomials in the class $\zeta$, whose coefficients are appropriate products of pull-backs from $\mathbb{A}(Y) \otimes \mathbb{Q}$. The higher degrees $\zeta^3$ and $\zeta^2$ can be decreased using the basic relation (7.2), while $\zeta$ itself can be altogether eliminated by noticing that for any $\vartheta \in \mathbb{A}^2(Y)$:

$$\zeta, \pi^*(\text{point}) = \zeta, F_{\pi} = 1 \Rightarrow \zeta, \pi^*\vartheta = \deg \vartheta.$$

It is also useful to remember the trivial fact that for any divisors $D_i$ on $Y$, $\dim Y = 2$ implies $D_1, D_2, D_3 = 0 = D_1, c_2(V)$.

**Lemma 7.4** (Adjunction formula). The canonical bundle $\omega_Z$ of a smooth divisor $Z$ on the smooth variety $T$ can be expressed as $\omega_Z \cong \omega_T \otimes \mathcal{O}_T(Z) \otimes \mathcal{O}_Z$. Consequently,

$$K_X^2 = (K_{PV} + X)^2 X \text{ and } g + 2 = \deg c_1(V)|F_Y.'
Proof. For the general statement of the adjunction formula see [Hr]. The expression for $K_X^2$ is a straightforward application to the divisor $X$ on $\mathbf{P}V$: $K_X = (K_{\mathbf{P}V} + X)|_X$ is being squared in $\mathcal{A}(\mathbf{P}V)$. As for the genus $g$ of the general member $C$ of our family, we consider a general fiber $F_Y$ of $Y$ (cf. Fig. 5). Its pullback $\pi^*F_Y$ is a rational ruled surface $\mathbf{F}$ over $F_Y$, embedded in the 3-fold $\mathbf{P}V$. The intersection of $\mathbf{F}$ with the surface $X$ is the trigonal fiber $C = X: \pi^*F_Y = (3\zeta + 2\pi^*c_1(V))|_{\pi^*F_Y}$.

From the adjunction formulas for $C \subset \pi^*F_Y$ and $\pi^*F_Y \subset \mathbf{P}V$:

$$2g - 2 = (K_{\pi^*F_Y} + C) \cdot C = ((K_{\mathbf{P}V} + \pi^*F_Y)|_{\pi^*F_Y} + X|_{\pi^*F_Y}) \cdot X|_{\pi^*F_Y}$$

$$= (\zeta + \pi^*c_1(V) + \pi^*K_Y + \pi^*F_Y) (3\zeta + 2\pi^*c_1(V)) \cdot \pi^*F_Y$$

$$= (2c_1(V) + 3K_Y) \cdot F_Y = 2 \deg c_1(V)|_{F_Y} - 6. \quad \square$$

7.4.2. Global Calculation of $\lambda_X$. We substitute in Lemma 7.8 the expressions (7.10–7.12) for $X$, $K_{\mathbf{P}V}$ and $c_2(V)$, as well as the identity $D = 2c_1(V)$:

$$\chi(\mathcal{O}_X) = \frac{3\xi + 2\pi^*c_1(V)}{12} \left[ \left( \xi + \pi^*c_1(V) + \pi^*K_Y \right) \left( 4\xi + 3\pi^*c_1(V) + \pi^*K_Y \right) \right]$$

$$- 2\xi \pi^*K_Y + \pi^*c_1(V)\pi^*K_Y + \pi^*c_2(Y) \right].$$

Applying the necessary reductions, we arrive at:

$$\chi(\mathcal{O}_X) = \frac{1}{2} (c_1^2(V) - 2c_2(V)) + \frac{1}{2} c_1(V)K_Y + \frac{1}{4} (K_Y^2 + c_2(Y)).$$

We expect our formula for $\lambda_X$ to be independent of the base curve $B$. The contribution of $g_B$ in $\chi(\mathcal{O}_X)$ can be written as: $(g_B - 1) \deg c_1(V)|_{F_Y} + \chi(\mathcal{O}_Y) = (g_B - 1)(g - 1)$, but this is precisely the adjustment $\chi(\mathcal{O}_B)\chi(\mathcal{O}_C)$ given in Lemma 7.2. Thus,

$$\lambda_X = \frac{1}{2} (c_1^2(V) - 2c_2(V)) - \deg c_1(V)|_{B_0}.$$

It remains to notice that $c_1^2(V) = 2 \deg c_1(V)|_{F_Y} \deg c_1(V)|_{B_0} = 2(g + 2) \deg c_1(V)|_{B_0}$ and rewrite $\lambda_X$ in the form

$$\lambda_X = \frac{1}{4} (c_1^2(V) - 4c_2(V)) + \frac{g}{2} \deg c_1(V)|_{B_0}. \quad \square$$
7.4.3. Global Calculation of $\kappa_X$. Since $\omega_f = \omega_X \otimes \omega_B^{-1}$,
\begin{equation}
\kappa_X = (K_X - \pi^*K_B)^2 = K_X^2 - 8(g_B - 1)(g - 1).
\end{equation}
From Lemma 7.4, we calculate
\[ K_X^2 = (K_{PV} + X)^2 = (\xi + \pi^*c_1(V) + \pi^*K_Y)^2(3\xi + 2\pi^*c_1(V)) = 2c_1^2(V) - 3c_2(V) + 4c_1(V)K_Y + 3K_Y^2. \]
We calculate the contribution of $g_B$ in $K_X^2$: $8(g_B - 1)\deg c_1(V)|_{F_Y} + 24(1 - g_B) = 8(g_B - 1)(g - 1)$, which is exactly the necessary adjustment for $\kappa_X$ in (7.16). Therefore,
\[ \kappa_X = 2c_1^2(V) - 3c_2(V) - 8\deg c_1(V)|_{B_0} = \frac{3}{4}(c_1^2(V) - 4c_2(V)) + \frac{5}{2} \deg c_1(V)|_{B_0} \deg c_1(V)|_{F_Y} - 8 \deg c_1(V)|_{B_0} = \frac{3}{4}(c_1^2(V) - 4c_2(V)) + \frac{5g - 6}{2} \deg c_1(V)|_{B_0}. \quad \Box \]

7.5. Index theorem on the surface $X$. Now that we have completed the proof of Prop. 7.1, we notice that any bound on the ratio $\delta_X/\lambda X$ would be equivalent to some inequality involving the genus $g$ and the two invariants discussed earlier: $\deg c_1(V)|_{B_0}$ and the quantity $c_1(V)^2 - 4c_2(V)$. This inequality should be a fairly general one, since the only relevant information in our situation is that $X$ is a triple cover of a ruled surface $Y$. One way of obtaining such general inequalities in $\mathbb{A}^2(X) \otimes \mathbb{Q}$ via

**Theorem 7.1** (Hodge Index). Let $H$ be an ample divisor on the smooth surface $X$, and let $\eta$ be a divisor on $X$, numerically not equivalent to 0. If $\eta \cdot H = 0$, then $\eta^2 < 0$.

The question here, of course, is how to find suitable divisors $H$ and $\eta$ that would yield our result for the maximal slope bound. For that, we make use of the triple cover $\phi : X \to Y$. If $H$ is any ample divisor on $Y$, then its pullback $\phi^*H$ to $X$ is also ample. This follows from

**Theorem 7.2** (Nakai-Moishezon Criterion). A divisor $A$ on the smooth surface $X$ is ample if and only if $A^2 > 0$ and $A \cdot C > 0$ for all irreducible curves $C$ in $X$.

Since $H$ is ample itself, $(\phi^*H)^2 = 3H^2 > 0$ and $(\phi^*H) \cdot C = H \cdot \phi_*(C) > 0$ for any curve $C$ on $X$, so that $\phi^*H$ is also ample on $X$. Now, if we find a divisor $\eta$ on $X$ such that $\eta \cdot \phi^*\Pic Y = 0$, we will have assured that $\eta \cdot \phi^*H = 0$, and then the Index theorem will assert $\eta^2 \leq 0$. As $X$ is a divisor itself on $PV$, its Picard group naturally contains the restriction of $\Pic PV$ to $X$. We look for $\eta$ inside this subgroup, and for our purposes we may write it in the form $\eta = (\zeta + \pi^*C_1)|_X$ for some $C_1 \in \Pic Q Y$. Let $C$ be any divisor class $\Pic Q Y$. We compute
\[ \eta \cdot \phi^*C = (\zeta + \pi^*C_1)(3\zeta + 2\pi^*c_1(V))\pi^*C = C(3C_1 - c_1(V)). \]
We want this to be zero for all $C$, so we naturally take $C_1 = \frac{1}{3}c_1(V) \in \Pic Q Y$. We summarize the above discussion in

**Lemma 7.5** (Index Theorem on $X$). The divisor class $\eta = (\zeta + \frac{1}{3}\pi^*c_1(V))|_X$ on $X$ satisfies $\eta \cdot \phi^*\Pic Y = 0$. In particular, for an ample divisor $H$ on $Y$, the pullback $\phi^*H$ is also ample on $X$ and $\eta \cdot \phi^*H = 0$. Consequently, $\eta^2 \leq 0$ with equality if and only if $\eta$ is numerically equivalent to 0 on $X$.

We have shown that
\begin{equation}
0 \geq 3\eta^2 = 3(\zeta + \frac{1}{3}\pi^*c_1(V))^2(3\zeta + 2\pi^*c_1(V)) = 2c_1^2(V) - 9c_2(V),
\end{equation}
or equivalently,

\[(7.18) \quad 2(g + 2) \deg c_1(V)|_{B_0} - 9(c_1^2(V) - 4c_2(V)) \geq 0.\]

We are now ready to find a maximal bound for the slope of \(X\). Recall the formulas for \(\lambda_X\) and \(\kappa_X\) (cf. Prop. 7.1), and write

\[
\delta_X = 12\lambda_X - \kappa_X = \frac{7g + 6}{2} \deg c_1(V)|_{B_0} + \frac{9}{4}(c_1^2(V) - 4c_2(V)).
\]

In view of the type of bound for the ratio \(\delta_X/\lambda_X\), which we aim to achieve, we have to eliminate any extra terms and use inequality \((7.17)\). Our only choice is to subtract

\[
36(g + 1)\lambda_X - (5g + 1)\delta_X = \frac{1}{2}(36(g + 1)g - (5g + 1)(7g + 6)) \deg c_1(V)|_{B_0} + \\
+ \frac{1}{4}(9(g + 1) - 9(5g + 1))(c_1^2(V) - 4c_2(V))
\]

\[
= \frac{1}{2}(g^2 - g - 6) \deg c_1(V)|_{B_0} - \frac{9}{4}(g - 3)(c_1^2(V) - 4c_2(V))
\]

\[
= \frac{g - 3}{4}[2(g + 2) \deg c_1(V)|_{B_0} - 9(c_1^2(V) - 4c_2(V))]
\]

\[
= (g - 3)(9c_2(V) - 2c_1^2(V)) \geq 0.
\]

As a result, we establish an exact maximal bound for the slopes of our triple covers:

**Theorem 7.3 (Main Theorem in Triple Cover Case).** Given a triple cover \(\phi: X \to Y\) satisfying in the Basic construction, the slope of \(X\) satisfies

\[
\frac{\delta_X}{\lambda_X} \leq \frac{36(g + 1)}{5g + 1}.
\]

Equality is achieved if and only if \(g = 3\), or \(g > 3\) and \(\eta \equiv 0\) on \(X\).

### 7.6. When is the maximal bound achieved?

#### 7.6.1. The branch divisor of \(\phi\).

From GRR, applied to \(\phi: X \to Y\) and the sheaf \(\mathcal{O}_X\), we obtain a description of \(c_1(V)\):

\[
\text{ch}(\phi_!\mathcal{O}_X). \text{td} \mathcal{T}_Y = \phi_*(\text{ch} \mathcal{O}_X. \text{td} \mathcal{T}_X),
\]

\[
\text{ch}(\phi_*\mathcal{O}_X)(1 - \frac{1}{2}K_Y + \frac{1}{12}(K_Y^2 + c_2(Y))) = \phi_*(1 - \frac{1}{2}K_X + \frac{1}{12}(K_X^2 + c_2(X))
\]

\[
\Rightarrow c_1(\phi_*\mathcal{O}_X) = -\frac{1}{2}(\phi_*K_X - 3K_Y).
\]

For the ramification divisor \(R\) on \(X\) we know \(K_X = \phi^*K_Y + R\), so that \(\phi_*K_X = 3K_Y + \phi_*R\). Hence \(c_1(V) = -c_1(\phi_*\mathcal{O}_X) = \frac{1}{2}\phi_*R\). In other words, from Lemma 7.1 we conclude that \(c_1(V)\) is half of the branch divisor \(D\) on \(Y\). On the other hand, we can rewrite the condition \(\eta \equiv 0\) in the following way:

\[
0 \equiv 3\eta = (3\zeta + \pi^*c_1(V))|_X = (X - \pi^*c_1(V))|_X = c_1(\mathcal{O}_\mathcal{P}^V(X)|_X) - \pi^*c_1(V)|_X
\]

\[
\Leftrightarrow c_1(\mathcal{O}_\mathcal{P}^V(X)|_X) \equiv \frac{1}{2}\phi^*D.
\]

The self-intersection of \(X\) on \(\mathcal{P}^V\) satisfies (cf. [LMS])

\[
i^*i_*1_X = c_1(N_{X/\mathcal{P}^V}) \Rightarrow X \cdot X = i_*c_1(N_{X/\mathcal{P}^V}).
\]

In particular, our condition \(\eta \equiv 0\) can be expressed as \(c_1(N_{X/\mathcal{P}^V}) \equiv \frac{1}{2}\phi^*D\).
7.6.2. Examples of the maximal bound. Constructing examples of families achieving the maximal bound is not so easy, considering that the condition $\eta \equiv 0$ is not useful in practice. Instead, we start from the Basic construction and attempt to find a ruled surface $Y$ and a rank 2 vector bundle $V$ on it satisfying the equality in (7.15), as well as the “genus condition” given in Lemma 7.4. The former will ensure the maximal ratio $\delta/\lambda = 36(g+1)/(5g+1)$, while the latter will imply that the fibers of our family are indeed of genus $g$. The remaining question is what linear series $3\zeta + \phi^*D$ has an irreducible member with at most rational double points as singularities, which would serve as the total space of our family $X$.

It is hard to work with the canonically defined bundle $V = \phi_* (\mathcal{O}_X)/\mathcal{O}_Y$, since not every vector bundle $W$ of rank 2 on $Y$ is of this form for some surface $X$. But any $W$ is congruent to some $V$ after a twist by an appropriate line bundle $M$: $V = W \otimes M$, and $PV \cong PW$. So, it seems reasonable to start with $W$ rather than $V$, and use the invariant forms of our required equalities (cf. Sect. 7.4). This means replacing the degrees of $c_1(V)$ on $B_0$ and $F_Y$ by the corresponding invariant degrees of $2D – 3c_1(V)$. Thus, we need for some divisor $\widehat{D}$ on $Y$:

\[(7.19) \quad 2(g+2)(2\deg \widehat{D}|_{B_0} - 3\deg c_1(W)|_{B_0}) = 9(\zeta^2(W) - 4c_2(W)),\]

\[(7.20) \quad g + 2 = 2\deg \widehat{D}|_{F_Y} - 3\deg c_1(W)|_{F_Y}.

For a general fiber $F_Y$ of $Y$ consider the rational ruled surface (cf. Fig. 31):

\[F_e = \pi^*F_Y = P(W|_{F_Y}) = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(e)), \text{ with } e \geq 0,
\]

Let $S'$ be the section in $F_e$ with self-intersection $(S')^2 = -e$, and let $F_\pi$ be the fiber of $F_e$ (in terms of the map $\pi : PV \to Y$, $F_\pi = \pi^*(pt)$). Since a general fiber $C$ of our family is embedded in $F_e$, the linear system

\[|C| = |3S' + \frac{g+2+3e}{2}F_\pi|
\]

has an irreducible nonsingular member. Equivalently, $C \cdot S' \geq 0$, i.e.

\[(7.21) \quad e \leq (g+2)/3 \quad \text{and} \quad e \equiv g(\text{mod } 2),
\]

(compare with Lemma 12.1). This forces three types of extremal examples according to the remainders $g(\text{mod } 3)$.

**Example 7.1** $(g \equiv 0(\text{mod } 3))$. Let $g = 3e$ for some $e \in \mathbb{N}$. Set the base curve $B = P^1$, and the ruled surface

\[Y = P(\mathcal{O}_B \oplus \mathcal{O}_B(6)) = F_6.\]

Let $B'$ be the section in $Y$ with smallest self-intersection: $(B')^2 = -6$, thus $B_0 = B' + 3F_Y$ with $B_0^2 = 0$. Let $Q = B' + 6F_Y$, and we choose two divisors $\widehat{D}$ and $E$ on $Y$ as follows:

\[\widehat{D} = (g+1)Q \quad \text{and} \quad E = eB' + 2(g+1)F_Y.
\]

For the vector bundle $W$ on $Y$ we set $W = \mathcal{O}_Y \oplus \mathcal{O}_Y(E)$ so that $c_1(W) = E$ and $c_2(W) = 0$. We claim that the linear system $L = |3\zeta + \phi^*\widehat{D}|$ on the 3-fold $PW$ contains an irreducible smooth member, which we set to be our surface $X$ with maximal ratio $\delta/\lambda$. Indeed, it is trivial to check conditions (7.19–20). Further, for *any* fiber $F_Y$ of $Y$:

\[\pi^*F_Y = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(E \cdot F_Y)) = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(e)) = F_e,
\]

so that $e = g/3$ satisfies the required conditions (7.21).
The only nontrivial fact is the existence of the wanted member $X$ in the linear system $L$ on $\mathbb{P}W$. Consider two sections $\Sigma_0$ and $\Sigma_1$ of $\mathbb{P}W$ corresponding to the subbundles $\mathcal{O}_Y$ and $\mathcal{O}_Y(E)$ of $W$, respectively: $\Sigma_0 \in |\zeta|$, $\Sigma_1 \in |\zeta + \pi^*(E)|$, so that $\Sigma_1 \sim \Sigma_0 + E$ (cf. Fig. 32).

Note that $\Sigma_0 \cdot \Sigma_1 = 0$ and $\Sigma_0 \cdot L = \Sigma_0 \cdot \pi^*B'$. In other words, if $G = \pi^*B'$ is the ruled surface over $B'$, then $\Sigma_0$ intersects every irreducible member of $L$ in the curve $R = \Sigma_0 \cap G$. On the other hand, if a member of $L$ meets $\Sigma_0$ in a point outside $R$, then this member contains entirely $\Sigma_0$. Thus, $L$ does not distinguish the points on $\Sigma_0$, and $R$ is in the base locus of $L$. Similarly, the restriction $L|_G = |3\Sigma_0|_G = |3R|$ has exactly one section on $G$, namely, $3R$. Again it follows that $L$ does not distinguish the points on $G$.

Away from the closed subset $Z = \Sigma_0 \cup G$, the linear system $L$ is in fact very ample. This can be checked by showing directly that $L$ separates points and tangent vectors on $\mathbb{P}W - Z$. Therefore, $L$ defines a rational map

$$\phi_L : \mathbb{P}W \to \mathbb{P}(H^0(L)^+) = \mathbb{P}^N.$$

The map $\phi_L$ is regular on $\mathbb{P}W - R$, embeds $\mathbb{P}W - Z$, and contracts $\Sigma_0 - R$ and $G - R$ to two distinct points $p$ and $q$ in $\mathbb{P}^N$. By Bertini’s theorem (cf. [11]), the general member of $L$ is smooth away from the base locus $R$. Let $H$ be a general hyperplane in $\mathbb{P}^N$ not passing through $p$ and $q$. Pulling $H$ back to $\mathbb{P}W$ yields a member $X$ of $L$ not containing $\Sigma_0$ or $G$, and hence irreducible.

It remains to show that the total space of $X$ has at most finitely many double point singularities along the curve $R$. Since the member $3\Sigma_1 + G \in |L|$ is smooth along $R$, then the general member of $|L|$ must be smooth along $R$. Hence our surface $X$ has, in fact, a smooth total space. This concludes the construction of our maximal bound family of trigonal curves.

**Example 7.2** ($g \equiv 1(\text{mod}3)$). Set $g = 3e - 2$ for $e \in \mathbb{N}$. Then $e$ satisfies the requirements of our construction: $e = (g + 2)/3$ and $e \equiv g(\text{mod}2)$. For the ruled surface $Y$ we choose $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Let $E$ and $\tilde{D}$ be the following divisors on $Y$: $E = eB_0 + fF_Y$ and $\tilde{D} = 3E$, where $f \in \mathbb{N}$. The vector bundle $W$ on $Y$ is then defined by $W = \mathcal{O}_Y \oplus \mathcal{O}_Y(E)$. Finally, we indentify the total space of the surface $X$ with an irreducible smooth member of the linear system $L = |3\zeta + \pi^*\tilde{D}|$ on the $3$-fold $\mathbb{P}W$.

The verification of this construction is similar to the previous example. Here $L$ is very ample everywhere on $\mathbb{P}W$ except on the section $\Sigma_0$, which is contracted to a point under the map $\phi_L$. This example, in somewhat different context, is shown in [K3].

**Remark 7.1** The case of $g \equiv 2(\text{mod}3)$ is complicated by the fact that we cannot take $e = (g + 1)/3$, for then $e \not\equiv g(\text{mod}2)$. For example, if $g = 5$, then the only possibility is $e = 1$. In the notation of Section 12, all trigonal curves have lowest Maroni invariant of 1, and there is no Maroni locus. For now, in this case we have not been able to construct a trigonal family with singular general member, whose ratio is $36(g + 1)/(5g + 1)$.
8. Local Calculation of $\lambda, \delta$ and $\kappa$ in the General Case

8.1. Notation and conventions. In this section we consider the general case of a trigonal family $X \to B$. For convenience of notation, we shall assume that the base curve $B$ intersects transversally and in general points the boundary divisors of $\mathfrak{M}_g$ (cf. Fig. 8). We will call such a base curve general, and use this definition throughout Section 8-10. Since we work in the rational Picard group of $\mathfrak{M}_g$, all arguments and statements in the remaining cases are shown similarly in Sect. 11. From Prop. 6.1, we may assume that modulo a base change, our family $X \to B$ fits in the following commutative diagram:

\[ \begin{array}{ccc}
\hat{X} & \xrightarrow{\phi} & \hat{Y} \\
\downarrow & & \downarrow \\
\hat{X} & \xrightarrow{\phi} & \hat{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & B \\
\end{array} \]

Figure 33. General base $B$

8.1.1. Relations in $\text{Pic}_Q \hat{Y}$ and $\text{Pic}_Q P V$. The special fibers of of $\hat{X}$ and of the birationally ruled surface $\hat{Y}$ over $B$ are described in Fig. 38-40. Since each such fiber in $\hat{Y}$ is a chain $T$ of rational components, we can fix one of the end components to be the root $R$. We keep the notation $E^- (E^+, \text{ respectively})$ for the ancestor (descendants, respectively) of a component $E$ in $T$. We also fix a general fiber $F_{\hat{Y}} \cong P^1$ of $\hat{Y}$, and a section $B_{\hat{Y}}$, which is the pullback of the corresponding section $B_0$ in $\hat{Y}$ (cf. (7.1)). The rational Picard group of $\hat{Y}$ is generated by $F_{\hat{Y}}, B_{\hat{Y}}$ and all non-root components $E$ of the special fibers of $\hat{Y}$:

\[ \text{Pic}_Q \hat{Y} = \mathbb{Q}B_{\hat{Y}} \bigoplus \mathbb{Q}F_{\hat{Y}} \bigoplus \mathbb{Q}E_{\text{not root}} \]

The intersection numbers of these generators are as follows: $B_{\hat{Y}}^2 = 0 = F_{\hat{Y}}^2$, $B_{\hat{Y}} \cdot F_{\hat{Y}} = 1$, and $E \cdot B_{\hat{Y}} = E \cdot F_{\hat{Y}} = 0$.

\[ \begin{array}{c}
E^- \quad E^+ \\
E^- \quad E \\
E^- \quad E^+ \quad E^+ \\
R \quad E^- \\
\end{array} \]

Figure 34. $m_E$

Figure 35. $E^2$

Figure 36. $\theta_E$

We also set $m_E = E \cdot E^-$ (cf. Fig. 34):

\[ m_E = \begin{cases} 
0 & \text{if } E = R \text{ root}, \\
1 & \text{if } E \text{ and } E^- \text{ reduced}, \\
2 & \text{if } E \text{ or } E^- \text{ nonreduced}.
\end{cases} \]
In this notation, due to the fact that \( E \cdot T = E \cdot F_{\hat{Y}} = 0 \), the self-intersection of any \( E \) is computed by (cf. Fig. 35):

\[
E^2 = - \sum_{E' \neq E} E \cdot E' = - \sum_{E' = E^+} m(E')
\]

In order to express the dualizing sheaf \( K_{\hat{Y}} \) in terms of the above generators of \( \text{Pic}_Q \hat{Y} \), for each component \( E \) in \( \hat{Y} \) we denote by \( \theta_E \) the length of the path \( RE \), omitting any nonreduced components except for \( E \) itself. For example, in the two cases in Fig. 36 we have \( \theta_E = 1 \) and \( \theta_E = 2 \). Note that \( \theta_R = 0 \).

Considering the “effective” blow-ups on \( \hat{Y} \), necessary to construct \( \hat{Y} \), we immediately obtain the following identities (compare with (7.10) and (7.13)).

**Lemma 8.1.** In \( \text{Pic}_Q \hat{Y} \) and \( \text{Pic}_Q PV \):

(a) \( K_{\hat{Y}} \equiv -2B_{\hat{Y}} + (2g_B - 2)F_{\hat{Y}} + \sum_E \theta_E E \),

(b) \( K_{PV} \equiv -2\zeta - \pi^*c_1(V) + \pi^*K_{\hat{Y}} \),

(c) \( K_{PV/B} \equiv -2\zeta - \pi^*c_1(V) - 2\pi^*B_{\hat{Y}} + \sum \theta_E \pi^*E \).

The hyperplane section \( \zeta \) of \( PV \) and the rank 2 vector bundle \( PV \) on \( \hat{Y} \) are defined similarly as in Section 7. Thus, in \( \text{Pic}_Q PV \) we have \( \hat{X} \sim 3\zeta + \pi^*D \) for a certain divisor \( D \) on \( \hat{Y} \). By analogy with Lemma 7.1, one shows that \( D \equiv 2c_1(V) \) in \( \text{Pic}_Q Y \), so that

\[
\hat{X} \equiv 3\zeta + 2\pi^*c_1(V).
\]

Using the above notation for \( \text{Pic}_Q \hat{Y} \) we can write for some half-integers \( c, d, \gamma_E \):

\[
c_1(V) \equiv cB_{\hat{Y}} + dF_{\hat{Y}} + \sum_E \gamma_E E.
\]

Here we can assume that \( \gamma_R = 0 \) by replacing \( R \) with a linear combination of the remaining components \( E \) in its chain \( T \) (compare with (7.3)).

Finally, we need the top Chern classes of \( Y \) and \( PV \) in terms of intersections of known divisors and other known invariants of the two surfaces (compare with (7.12) and (7.14)).

**Lemma 8.2.** In the Chow rings \( \mathbb{A}(\hat{Y}) \) and \( \mathbb{A}(PV) \) the following equalities are true:

(a) \( c_2(\hat{Y}) = c_2(Y) + \sum_{E \neq R} 1 = 4(1 - g_B) + \sum_{E \neq R} 1 \),

(b) \( c_2(PV) = c_2(\hat{Y}) - \pi^*K_{\hat{Y}}(2\zeta + \pi^*c_1(V)) \),

(c) \( c_2(PV/B) = -\pi^*K_{\hat{Y}/B}(2\zeta + \pi^*c_1(V)) + \sum_{E \neq R} 1 \).

8.1.2. A technical lemma. In the sequel, we will work with several functions defined on the set of components \( \{ E \} \) in \( \hat{Y} \). For easier calculations, to any such function \( f \) we associate the difference function \( F \) by setting \( F_E := f_E - f_{\overline{E}} \) for all \( E \). Since \( R^- \) does not exist, we define \( f_{R^-} = 0 \) for all roots \( R \) in \( \hat{Y} \).

**Lemma 8.3.** For any functions \( f \) and \( h \) defined on the set of components \( \{ E \} \) in \( \hat{Y} \), the following identity holds true:

\[
\sum_E f_E \cdot \sum_E h_E E = - \sum_E (m \cdot F \cdot H)_E.
\]
Proof. We rewrite the lefthand side as \[ \sum_{E_1 \neq E_2} f_{E_1} h_{E_1} E_1 E_2 + \sum_{E} f_{E} h_{E} E^2 = \]

\[ = \sum_{E} \left( f_{E} h_{E} + f_{E} h_{E^-} \right) m_E - \sum_{E} \left( f_{E} h_{E} + f_{E} h_{E^-} \right) m_E = \]

\[ = \sum_{E} \left( f_{E^-} - f_{E} \right) \left( h_{E} - h_{E^-} \right) m_E = \sum_{E} (m \cdot F \cdot H)_E . \]

We have noted that all three functions \( m, \theta \) and \( \gamma \) are zero on the roots \( R \) in \( \hat{Y} \). Since we shall be working specifically with these three functions, it makes sense to restrict from now on all sums \( \sum_{E} \) only to the non-roots \( E \) in \( \hat{Y} \). With this in mind, in every application of Lemma 8.3 one must check that the corresponding functions \( f \) and \( h \) have the same property: \( f_R = 0 = h_R \), so that we can restrict the sums in Lemma 8.3 also to all non-roots \( E \) in \( \hat{Y} \). In fact, in all cases this verification will be obvious as \( f \) and \( h \) will be, for the most part, linear combinations of \( \theta \) and \( \gamma \).

Example 8.1. From expression \( \text{(8.3)} \) for \( c_1(V) \) as a divisor on \( \hat{Y} \), and Lemma 8.3

\[ c_1^2 (V) = 2cd + \sum_{E} \gamma_E E \cdot \sum_{E} \gamma_E E = 2cd - \sum_{E} m_E \Gamma^2 . \]

8.2. Computation of the invariants \( \lambda_{\hat{X}}, \kappa_{\hat{X}} \) and \( \delta \). The following proposition 8.1 is a generalization of the corresponding statement in Section 7 (cf. Prop. 7.1). We set \( \Gamma_E = \gamma_E - \gamma_E^- \) and \( \Theta_E = \theta_E - \theta_E^- \) to be the difference functions of \( \gamma \) and \( \theta \).

Proposition 8.1. The degrees of the invariants \( \lambda_{\hat{X}}, \kappa_{\hat{X}} \) on \( \hat{X} \) are given by

\[ \lambda_{\hat{X}} = d(g + 1) - c_2(V) - \frac{1}{4} \sum_{E} \left\{ m_E \cdot (2 \Gamma^2 + 2 \Gamma \cdot \Theta + \Theta^2)_E - 1 \right\} , \]

\[ \kappa_{\hat{X}} = 4dg - 3c_2(V) - \sum_{E} m_E (2 \Gamma^2 + 4 \Gamma \Theta + 3 \Theta^2)_E . \]

Proof. One starts with the Euler characteristic formula \( \lambda_{\hat{X}} = \chi(\mathcal{O}_{\hat{X}}) - \chi(\mathcal{O}_C) \cdot \chi(\mathcal{O}_B) \), or the adjunction formula \( \kappa_{\hat{X}} = (\hat{X} + K_{\hat{X} / \hat{C}}^2)^2 \hat{X} \). The rest of the proof is a straightforward calculation, which uses the equalities given in (8.1), and is substantially simplified by Lemma 8.3.

Corollary 8.1. The degree \( \delta \) on the original family \( X \) is given by

\[ \delta = 4d(2g + 3) - 9c_2(V) - \sum_{T} \mu(T) - \sum_{\text{ram } 1} 1 - \sum_{\text{ram } 2} 3 - \sum_{E} \left\{ m_E (4 \Gamma^2 + 2 \Gamma \Theta)_E - 3 \right\} . \]

Here \( \mu(T) \) stands for the quasi-admissible contribution to \( \kappa_{\hat{X}} \) of the preimage \( C = \hat{\phi}^* T \) in \( \hat{X} \), as defined in Lemma 4.4.

Proof. Since \( \lambda = \lambda_{\hat{X}}, \kappa = \kappa_{\hat{X}} + \sum_{T} \mu(T) + \sum_{\text{ram } 1} 1 + \sum_{\text{ram } 2} 3, \) and \( \delta = 12 \lambda - \kappa \), the statement immediately follows from Prop. 8.1.
8.3. The arithmetic genus $p_E$, and the invariants $\Gamma_{E'}$ and $\Theta_{E'}$. For a component $E$ in a special fiber $T$ of $\tilde{Y}$, we define $T(E)$ to be the subtree of $T$ generated by the component $E$. In other words, $T(E)$ is the union of all components $E' \in T$ such that $E' \geq E$ (cf. Fig. 37). For simplicity, we set $p_E := p_a(\phi^*(T(E)))$ to be the arithmetic genus of the inverse image $\phi^*(T(E))$ in $\tilde{X}$. It can be easily computed via the following analog of Lemma 7.4, where $T$ consisted of a single component $E = R$.

**Lemma 8.4.** For a general base curve $B$ and for any non-root component $E \in T$:

\[
(8.4) \quad p_E = -m_E \left( \Gamma_E + \frac{3(\Theta_E + 1)}{2} \right) + 1.
\]

**Proof.** From the adjunction formula for the divisor $\phi^*(T(E))$ in $\tilde{X}$:

\[
2p_E - 2 = (K_{\tilde{X}} + \phi^*(T(E)))|_E + \sum_{E'} \delta_{E'}\phi^*E'.
\]

Here $\delta_{E'} = 0$ if $E' < E$, and $\delta_{E'} = 1$ otherwise. Thus, the sums above are effectively taken over all $E' \geq E$. Substituting the expressions for $K_{\text{P}V}$ and $\tilde{X}$ as divisors in $\text{P}V$ from Lemma 8.1 and (8.1), we arrive at

\[
2p_E - 2 = \sum_{E'} \left( 2\gamma_{E'} + 3\theta_{E'} + 3\delta_{E'} \right) E' \sum_{E'} \delta_{E'} E'.
\]

Set $\Delta_E = \delta_E - \delta_{E'}$, i.e. $\Delta_{E'} = 1$ only if $E' = E$; otherwise, $\Delta_{E'} = 0$. By Lemma 8.3,

\[
2p_E - 2 = -\sum_{E'} m_{E'} \left( 2\Gamma_{E'} + 3\Theta_{E'} + 3\Delta_{E'} \right) \Delta_{E'} \Rightarrow 2p_E - 2 = -m_E \left( 2\Gamma_E + 3\Theta_E + 3 \right).
\]

Now we can easily compute the invariants $m_{E'}$, $\Theta_{E'}$, and $\Gamma_{E'}$, appearing in the formulas for $\lambda_X$ and $\kappa_X$.

**Corollary 8.2.** There are three possibilities for the triple $(m_{E'}, \Theta_{E'}, \Gamma_{E'})$, depending on whether the components $E$ and $E^-$ of $T$ are reduced:

(a) if $E, E^-$ reduced, then $m_E = 1$, $\Theta_E = 1$, $\Gamma_E = -(p_E + 2)$.
(b) if $E$ nonreduced, then $m_E = 2$, $\Theta_E = 1$, $\Gamma_E = -(p_E + 5)/2$.
(c) if $E^-$ nonreduced, then $m_E = 2$, $\Theta_E = 0$, $\Gamma_E = -(p_E + 2)/2$.

**Proof.** Note that for the list all possible special fibers $T$ of $\tilde{Y}$, each component $E$ fits in exactly one of the three cases above (cf. Fig. 8.91). The proof of the statement is immediate from the definitions of $m_E$ and $\Theta_{E'}$, and from Lemma 8.4.
9. The Bogomolov Condition $4c_2 - c_1^2$ and the $7 + 6/g$ Bound in $\overline{\Sigma}_g$

With the conventions of Section 8, we state the main proposition of the section.

**Proposition 9.1.** There exists an effective $\mathbb{Q}$-linear combination $E$ of boundary divisors $\Delta \Sigma_{k,i}$, not containing $\Delta \Sigma_0$, such that for a general base curve $B$ in $\overline{\Sigma}_g$:

$$(7g + 6)\lambda_B = g\delta_B + E|_B + \frac{g - 3}{2} (4c_2(V) - c_1^2(V)).$$

For a shorthand notation, we denote by $\mathcal{G}$ the difference

$$\mathcal{G} := (7g + 6)\lambda_B - g\delta_B - \frac{g - 3}{2} (4c_2(V) - c_1^2(V)).$$

Using the results of the previous section, we can write:

$$\mathcal{G} = -\frac{1}{4} \sum_E \left\{ m_E (6\Gamma^2 + 6(g + 2)\Gamma \Theta + (7g + 6)\Theta_E^2) + 5g - 6 \right\}$$

$$+ \sum_T g\mu(T) + \sum_{\text{ram} 1} g + \sum_{\text{ram} 3} 3g.$$  

We defer the proof of Prop. 9.1 until the end of this section, when all of the terms in this sum will be computed.

9.1. **Grouping the contributions of each $\Delta \Sigma_{k,i}$ in $\mathcal{G}$**. Substituting the results of Corollary 8.2 in the expression for $\mathcal{G}$, we eliminate $m_E$, $\Theta_E$, and $\Gamma_{E^c}$:

$$\mathcal{G} = \sum_T g\mu(T) + \sum_{\text{ram} 1} g + \sum_{\text{ram} 2} 3g + \frac{1}{4} \sum_{E,E^c \text{ red}} \left( 6(2 + p_E)(g - p_E) - 12g \right)$$

$$- \frac{1}{4} \sum_{E^c \text{ nonred}} \left( 3(p_E + 2)^2 + 5g - 6 \right) + \frac{1}{4} \sum_{E^{\text{nonred}}} \left( 3(p_E + 5)(2g - p_E - 1) - 19g + 6 \right).$$

For each chain $T$ in $\tilde{Y}$, the inverse image $\tilde{\phi}^*(T)$ in $\tilde{X}$ is a member (or a blow-up of a member) of exactly one boundary divisor $\Delta \Sigma_{k,i}$. Consequently, to find the contribution to $\mathcal{G}$ of a specific $\Delta \Sigma_{k,i}$, we calculate the sum in $\mathcal{G}$ corresponding to all types of special fibers $\tilde{\phi}^*(T)$.

**Figure 38.** Coefficients with no ramification

9.1.1. **Contributions of $\Delta \Sigma_{1,i}, \Delta \Sigma_{2,i}$ and $\Delta \Sigma_{3,i}$**. Fig. 38 presents the special fibers corresponding to the boundary divisors $\Delta \Sigma_{1,i}$, $\Delta \Sigma_{2,i}$, $\Delta \Sigma_{3,i}$. In each of these cases, there is only one component $E$ in $T$ besides the root $R = E^-$. Thus, the subchain $T(E)$ in $T$ is trivial – it consists only of $E$. Its inverse image $\tilde{\phi}^*(E)$ is connected for $\Delta \Sigma_{1,i}$, and consists of two connected curves for $\Delta \Sigma_{2,i}$, and $\Delta \Sigma_{3,i}$. Setting the genus of the inverse image of $R$ to be $i$, it is easy to see that the genus $p_E$ of $\phi^*(E)$ is $g - i - 2$ in the first two cases, and $g - i - 1$ in the third case. (The total genus of the original fiber of $X$, drawn in full lines, must be $g$.) Finally, counting the number of “quasi-admissible” blow-ups (drawn by dashed lines), we
see that $\mu(T) = 0$ for $\Delta \mathfrak{S}_{1,i}$, $\mu(T) = 1$ for $\Delta \mathfrak{S}_{2,i}$, and $\mu(T) = 2$ for $\Delta \mathfrak{S}_{3,i}$ (cf. Lemma 4.1). Note that there are no ramification modifications.

The contribution of each such fiber $\hat{\varphi}^*T$ to the sum $\mathfrak{S}$ is only one summand of the first type $(E, E^-)$ reduced, plus the quasi-admissible adjustment $g_\mu(T)$. If $\hat{\varphi}^*T$ corresponds to the boundary divisor $\Delta \mathfrak{S}_{k,i}$, we denote this contribution by $c_{k,i}$. In conclusion,

$$c_{k,i} = \frac{1}{4}(6(2 + p_{E_j})(g - p_{E_j}) - 12g) + g_\mu(T) \Rightarrow c_{k,i} = \frac{3}{2}(i + 2)(g - i) - (4 - k)g, \ k = 1, 2, 3.$$

9.1.2. Contributions of $\Delta \mathfrak{S}_{4,i}$ and $\Delta \mathfrak{S}_{5,i}$: ramification index 1. In each of these cases, the fiber $T$ of $\hat{\varphi}$ consists of two rational curves $E_1$ and $E_2$, and the root $R = E_1^-$ (cf. Fig. 39). There are no nonreduced components in $T$, so the contribution to $\mathfrak{S}$ consists of two summands of the first type $(E, E^-)$ nonreduced, plus a quasi-admissible adjustment of $\mu(T) = 1$ for $\Delta \mathfrak{S}_{5,i}$, and a ramification adjustment of $g$ in both cases:

$$c_{k,i} = \frac{1}{4} \sum_{j=1,2} (6(2 + p_{E_j})(g - p_{E_j}) - 12g) + g_\mu(T) + g \text{ for } k = 4, 5.$$

![Figure 39. Coefficients for ramification index 1](image)

The arithmetic genus of the nonreduced component of $\hat{\varphi}$ is $-2$, and its intersection number with each of the neighboring components is 2. Setting $p_{E_j}^\mu(\hat{\varphi}^*R) = i$ forces $p_{E_j}^\mu(\hat{\varphi}^*E_2) = g - i - 1$. Hence, $p_{E_1} = g - i - 1$ and $p_{E_2} = g - i - 2$. Substituting:

$$c_{k,i} = 3(g - i)(i + 1) - \frac{7g - 3}{2} + g_\mu(T),$$

$$c_{4,i} = 3(i + 1)(g - i) - \frac{7g - 3}{2}, \ c_{5,i} = 3(i + 1)(g - i) - \frac{7g - 3}{2} + 2g.$$

9.1.3. Contribution of $\Delta \mathfrak{S}_{6,i}$: ramification index 2. It remains to consider the case of ramification index 2. Here there are four components $E$ besides the root $R$ in the special fiber $T \subset \hat{\varphi}$. Consequently, there are four summands in $\mathfrak{S}$ corresponding to the $E_i$'s: $E_1$ and $E_4$ yield summands of the first type $(E, E^-)$ reduced, $E_2$ yields a summand of the second type $(E$ nonreduced), and $E_3$ yields a summand of the third type $(E^-)$ nonreduced).

Since $\mu(T) = 0$, and the ramification adjustment is $3g$, we obtain for the contribution of $\Delta \mathfrak{S}_{6,i}$ to $\mathfrak{S}$ the following expression:

$$c_{6,i} = \frac{1}{4}(6(2 + p_{E_1})(g - p_{E_1}) - 12g) + \frac{1}{4}(6(2 + p_{E_4})(g - p_{E_4}) - 12g) +$$

$$+ \frac{1}{4}(3(p_{E_2} + 5)(2g - p_{E_2} - 1) - 19g + 6) - \frac{1}{4}(3(p_{E_3} + 2)^2 + 5g - 6) + 3g.$$
The arithmetic genera of the components in $\widehat{X}$ are denoted in the Fig. [40]. It is easy to see that $p_{E_4} = i$, $p_{E_3} = i - 3$, $p_{E_2} = i - 2$, $p_{E_1} = i - 2$. Finally,

$$c_{6,i} = \frac{9}{2} i (g - i) - \frac{3}{2} (g - 1).$$

9.2. **Proof of Proposition 9.1** In the above discussion we calculated the contributions of the boundary divisors $\Delta \mathfrak{T}_{k,i}$ to the sum $\mathfrak{S}$, so that $\mathfrak{S} = \sum_{k,i} c_{k,i} \Delta \mathfrak{T}_{k,i}$ with $k = 1, \ldots, 6$, and the corresponding limits for the index $i$ (cf. Prop. [4.1]). It is now clear what the divisor $\mathcal{E}$ should be. We set $\mathcal{E} := \sum_{k,i} c_{k,i} \Delta \mathfrak{T}_{k,i}$, and thus, $\mathfrak{S} = \mathcal{E}|_B$,

$$\Rightarrow (7g + 6) \lambda|_B = g \delta|_B + \mathcal{E}|_B + g - \frac{3}{2} (4c_2(V) - c_1^2(V)).$$

Using the restrictions on the index $i$ for each type of boundary divisor $\Delta \mathfrak{T}_{k,i}$, one can easily deduce that all coefficients $c_{k,i} > 0$. For instance, when $i = 1, \ldots, \lfloor g/2 \rfloor$:

$$c_{6,i} = \frac{9}{2} i (g - i) - \frac{3}{2} (g - 1) > \frac{9}{2} \cdot (g - 1) - \frac{3}{2} (g - 1) = 3(g - 1) > 0.$$  

In other words, $\mathcal{E}$ is an effective rational linear combination of boundary divisors in $\mathfrak{T}_g$, which by construction does not contain $\Delta \mathfrak{T}_0$. [ ]

9.3. **The slope bound $7 + 6/g$ and a relation restricted to the base curve $B$.** Recall that a vector bundle $V$ of rank 2 is Bogomolov semistable if $4c_2(V) \geq c_1^2(V)$.

**Proposition 9.2 ($7 + 6/g$ bound).** For a general base curve $B$, if the canonically associated vector bundle $V$ is Bogomolov semistable, then the slope of $X/B$ is bounded by

$$\frac{\delta|_B}{\lambda|_B} \leq 7 + \frac{6}{g}.$$  

**Proof.** The statement follows directly from Prop. [9.1]. Indeed, since $\mathcal{E}$ is effective, then $\mathcal{E}|_B \geq 0$. By hypothesis, $4c_2(V) - c_1^2(V) \geq 0$, and $g \geq 3$. Hence, $(7g + 6) \lambda|_B \geq g \delta|_B$. [ ]
Corollary 9.1. For a general base curve $B$ the following relation holds true:

$$(7g + 6)|B| = g\delta_0|B| + \frac{g - 3}{2} (4c_2(V) - c_1^2(V))$$

$$+ \sum_{i=1}^{[(g-2)/2]} \frac{3}{2}(i+2)(g-i)\delta_{1,i}|B| + \sum_{i=1}^{g-2} \frac{3}{2}(i+2)(g-i)\delta_{2,i}|B|$$

$$+ \sum_{i=1}^{[g/2]} \frac{3}{2}(i+1)(g-i+1)\delta_{3,i}|B| + \sum_{i=1}^{[(g-1)/2]} (3(i+1)(g-i) - \frac{g-3}{2})\delta_{4,i}|B|$$

$$+ \sum_{i=1}^{g-1} (3(i+1)(g-i) - \frac{g-3}{2})\delta_{5,i}|B| + \sum_{i=1}^{[g/2]} \frac{9}{2}(g-i) - \frac{g-3}{2})\delta_{6,i}|B|.$$

Proof. This is an immediate consequence of the established relation in Prop. 9.1. We replace $\delta$ by the linear combination (4.1) of the boundary classes of $\overline{\mathcal{X}}_g$, and write

$$(7g + 6)\lambda = g\delta_0|B| + \sum_{k,i} \tilde{c}_{k,i}\delta_{k,i}|B| + \frac{g - 3}{2} (4c_2(V) - c_1^2(V)),$$

for some new coefficients $\tilde{c}_{k,i}$. Recall that $\text{mult}_\delta(\delta_{k,i})$ denotes the multiplicity of $\delta_{k,i}$ in $\delta$, so that $\tilde{c}_{k,i} = c_{k,i} + \text{mult}_\delta(\delta_{k,i})g$. For example, the coefficient of $\delta_{1,i}$ is

$$\tilde{c}_{1,i} = \left\{ \frac{3}{2}(i+2)(g-i) - 3g \right\} + 3g = \frac{3}{2}(i+2)(g-i),$$

or the coefficient of $\delta_{5,i}$ is

$$\tilde{c}_{5,i} = \left\{ 3(i+2)(g-i) - \frac{7g-3}{2} + 2g \right\} + g = 3(i+1)(g-i) - \frac{g-3}{2}. \qed$$

10. Generalized Index Theorem and Upper Bound

Proposition 10.1 (Index Theorem on $\hat{X}$). For a general base curve $B$ and for the rank 2 vector bundle $V$ on $\hat{Y}$, we have $9c_2(V) - 2c_1^2(V) \geq 0$.

Proof. The proof is identical to that of Theorem 7.3. One considers the divisor $\eta$ on $\hat{X}$ defined by

$$\eta := \left( \zeta + \frac{1}{3}\pi^*c_1(V) \right) |_{\hat{X}},$$

and shows that $\eta$ kills the pullback of any divisor on $\hat{Y}$. In particular, $\eta$ kills an ample divisor on $\hat{X}$. By the index theorem on $\hat{X}$, $\eta^2 \leq 0$. From expression (8.1), this can be also written as $9c_2(V) - 2c_1^2(V) \geq 0. \quad \square$

As in Section 7, the index theorem on $\hat{X}$ suggests to replace the Bogomolov difference $4c_2(V) - c_1^2(V)$ by another linear combination of $c_2(V)$ and $c_1^2(V)$, which would behave in a more “predictable” way, namely, by $9c_2(V) - 2c_1^2(V)$. In the process of doing so, the only way to eliminate the unnecessary global terms $d$ and $e$ from a relation among $\lambda|B$ and $\delta|B$ is to subtract: $36(g+1)\lambda|B - (5g+1)\delta|B$.

Proposition 10.2. For a general base curve $B$ and an effective rational combination $\mathcal{E}'$ of the boundary divisors $\Delta\mathcal{X}_{k,i}$, not containing $\Delta\mathcal{X}_0$, we have:

$$36(g+1)\lambda|B = (5g+1)\delta|B + \mathcal{E}'|B + (g - 3)(9c_2(V) - 2c_1^2(V)).$$
Note the apparent similarity between this relation and Prop. 9.1. One may use the latter to prove the former, but the calculations are not simpler than if one starts from scratch. We will show a sketch of this proof, leaving the details to the reader, and referring to the proof of Prop. 9.1 for comparison.

Proof. We denote by $\mathcal{S}'$ the difference
\[ \mathcal{S}' := 36(g + 1)|B| - (5g + 1)\delta|B| - (g - 3)(9c_2(V) - 2c_1^2(V)). \]
Substituting for $\delta|B|$, $\lambda|B|$ and $c_1^2(V)$ the corresponding identities from Prop. 8.1 and Example 8.1, and recalling that $c = g + 2$ (cf. Lemma 7.4), we write $\mathcal{S}'$ as
\[ \mathcal{S}' = -\sum_E \left\{ m_E \left( 8\Gamma^2 + 8(g + 2)\Gamma\Theta + 9(g + 1)\Theta^2 \right)_E + 6(g - 1) \right\} \]
\[ + (5g + 1) \left( \sum_T \mu(T) + \sum_{\text{ram 1}} 1 + \sum_{\text{ram 3}} 3 \right). \]
As in Lemma 9.1, we group the above summands for every special fiber in $\hat{X}$, and correspondingly, for every chain $T$ in $\hat{Y}$. Recall Corollary 8.2 and the computations of the arithmetic genera $p_E$ in the previous section:
\[ \mathcal{S}' = (5g + 1) \left( \sum_T \mu(T) + \sum_{\text{ram 1}} 1 + \sum_{\text{ram 2}} 3 \right) + \sum_{E, E \text{ ram } 3} \left( 8(p_E + 2)(g - p_E) - 3(5g + 1) \right) \]
\[ - \sum_{E \text{ nonred}} \left( 4(p_E + 2)^2 + 6(g - 1) \right) + \sum_{E \text{ nonred}} \left( 4(p_E + 5)(2g - 1 - p_E) - 12(g - 1) \right). \]
With this at hand, it is not hard to calculate the contributions $d_{k,i}$ of each boundary component $\Delta \Xi_{k,i}$ to the sum $\mathcal{S}'$:

| $d_{1,i}$ | $d_{2,i}$ | $d_{3,i}$ | $d_{4,i}$ | $d_{5,i}$ | $d_{6,i}$ |
|----------|----------|----------|----------|----------|----------|
| $8(i + 2)(g - i) - 3(5g + 1)$ | $8(i + 2)(g - i) - 2(5g + 1)$ | $8(i + 1)(g - i + 1) - (5g + 1)$ | $16(i + 1)(g - i) - 2(g - 3) - 3(5g + 1)$ | $16(i + 1)(g - i) - 2(g - 3) - (5g + 1)$ | $24(i)(g - i) - (5g + 1)$. |

Let $\mathcal{E}' = \sum_i d_{k,i} \Delta \Xi_{k,i}$. Then $\mathcal{S}' = \mathcal{E}'|_B$, and the desired relation would be established if $\mathcal{E}'$ is effective. Given the restrictions on the indices $i$ of the coefficients $d_{k,i}$ in Prop. 10.1, one easily shows that all $d_{k,i} > 0$.

Proposition 10.3 (Maximal Bound). For a general base curve $B$, the slope satisfies:
\[ \frac{\delta}{\lambda} \leq \frac{36(g + 1)}{5g + 1}, \]
with equality if and only if all fibers of $X$ are irreducible curves, and either $g = 3$ or the divisor $\eta$ on the total space of $X$ is numerically zero.

Proof. From the Index Theorem on $\hat{X}$, it follows that $9c_2(V) - 2c_1^2(V) \geq 0$. Since $\mathcal{E}'$ is effective, $\mathcal{E}'|_B \geq 0$. Then Prop. 10.2 implies $36(g + 1)|_B \geq (5g + 1)\delta|_B$, with equality exactly when $9c_2(V) - 2c_1^2(V) = 0$ and $\mathcal{E}'|_B = 0$. The latter means that $B \cap \Delta \Xi_{k,i} = \emptyset$ because all coefficients $d_{k,i}$ of $\mathcal{E}'$ are strictly positive. In other words, the family $\hat{X}$ has only irreducible fibers ($B \cap \Delta \Xi_{0} = \emptyset$). This takes us back to Section 7, where we presented the global calculation on the triple cover $X \to Y$. There we concluded that the index condition $9c_2(V) - 2c_1^2(V) = 0$ was equivalent to $\eta \equiv 0$ on $X(= \hat{X})$, or the genus $g = 3$. \qed
Corollary 10.1. For a general base curve $B$, 
\[ 36(g + 1)\lambda|_B = (5g + 1)\delta_0|_B + (g - 3) \left( 9c_2(V) - 2c_1^2(V) \right) + \sum_{[g/2]} 8(i + 2)(g - i)\delta_1,i|_B + \sum_{[g-1]/2} 8(i + 2)(g - i)\delta_2,i|_B + \sum_{i=1}^{g-1} 16(i + 1)(g - i)\delta_3,i|_B + \sum_{i=1}^{[g/2]} (16(i + 1)(g - i) - 2(g - 3))\delta_4,i|_B + \sum_{i=1}^{g-1} 24i(g - i)\delta_5,i|_B. \]

Proof. We only need to substitute the known expressions for the divisors $\mathcal{E}'$ and $\delta$ into Prop. 10.2:
\[ 36(g + 1)\lambda|_B = (5g + 1)\delta_0|_B + \sum_{k,i} \left( (5g + 1)\text{mult}_\delta(\delta_{k,i}) + d_{k,i} \right) + (g - 3)(9c_2(V) - 2c_1^2(V)). \]

The rest is a simple calculation. For example, the total coefficient $d_{3,i}$ of $\delta_{3,i}$ equals 
\[ d_{3,i} + (5g + 1)\text{mult}_\delta(\delta_{3,i}) = \{8(i + 1)(g - i + 1) - (5g + 1)\} + (5g + 1) \cdot 1 = 8(i + 1)(g - i + 1). \]

11. Extension to an Arbitrary Base $B$

We extend now the results of Sect. 8–10 to arbitrary nonisotrivial families $X \to B$ with smooth trigonal general member. The essential case is when $B$ is not tangent to the boundary $\Delta \mathcal{F}$, from which the remaining cases easily follows.

11.1. The base curve $B$ not tangent to $\Delta \mathcal{F}$. We now drop the hypothesis of the base curve $B$ intersecting the boundary divisors in general points. Instead, for now we only assume that the base curve $B$ is not tangent to the boundary $\Delta \mathcal{F}$. This means that all special fibers of $X$ locally look like the general ones (cf. Fig. 38–39). Therefore, from the quasiadmissible cover $\tilde{X} \to \tilde{Y}$ we can construct an effective cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$ of smooth surfaces $\tilde{X}$ and $\tilde{Y}$. (The smoothness indicates that $B$ is not tangent to any $\Delta \mathcal{F}_{k,i}$). Otherwise, there would be a higher local multiplicity $xy = t^n$ near a node of a special fiber $C_X$, $n > 1$. Hence $\tilde{X}$ would be obtained locally via a base change from a smooth surface, but $\tilde{X}$ would have a singular total space.)

Now the special fibers of $\tilde{Y}$ are trees $T$ (rather than just chains) of reduced smooth rational curves with occasional nonreduced rational components of multiplicity 2. The latter occur again exactly for each singular point in $\tilde{C}_X$ of ramification index 2 under the quasiadmissible cover $\tilde{\phi} : \tilde{X} \to \tilde{Y}$ (cf. Fig. 40).

The notation and conventions from Sections 8.1.1 are also valid here. In particular, for any tree $T$, we fix one of its end (nonreduced) components to be its root $R$, and we define as before the functions $m, \theta, \gamma$ on the components $E$ of $T$. Moreover, since Lemma 8.3 can be applied also for any tree $T$, the calculations of $\lambda_{\tilde{X}}, \kappa_{\tilde{X}}$ and $\delta$ in Prop. 8.1 and Cor. 8.1 go through without any modifications.

Finally, we wish to extend all results of Sections 8–10 over the new base $B$. The only difference arises in the final calculation of the coefficients $c_{k,i}$ and $d_{k,i}$. The fiber $C_X$ in $X$, corresponding to a tree $T$, may now lie in the intersection of several boundary divisors $\Delta \mathcal{F}_{k,i}$. 
Such a trigonal curve $C_X$ is called a special boundary curve. Accordingly, its contribution $c_T$ to $\mathcal{S}$ (or $d_T$ to $\mathcal{S}'$) will be distributed among these divisors $\Delta \Xi_{k,i}$'s, rather than just yielding a single coefficient $c_{k,i}$ (or $d_{k,i}$) as before.

This can be easily resolved. The idea is to replace any special singular fiber in $\hat{X}$ by a suitable combination of general fibers, without changing the sums $\mathcal{S}$ and $\mathcal{S}'$. We can imagine this as “moving” the base curve $B$ in $\hat{X}$ away from the special singular locus of $\Xi_g$, and replacing it with a general base curve $B'$, as defined in Section 8. For example, in Fig. 41 the base $B$ passes through a point $p$ in the intersection of two boundary divisors $\Delta \Xi_{k,i}$. Two new general points $p_1$ and $p_2$, each lying on a single $\Delta \Xi_{k,i}$, replace the special point $p$, and thus $B$ moves to a general curve $B'$.

**Lemma 11.1.** Let $C_X$ be a special boundary curve in $\Xi_g$. Denote by $\alpha_{k,i}$ the degree of the point $[C_X]$ in the intersection $\Delta \Xi_{k,i} \cdot B$. Then the contributions of $T = \hat{\phi}(C_X)$ to $\mathcal{S}$ and to $\mathcal{S}'$ are $c_T = \sum_{k,i} \alpha_{k,i} c_{k,i}$ and $d_T = \sum_{k,i} \alpha_{k,i} d_{k,i}$, respectively.

**Proof:** Rewrite $\mathcal{S}$ and $\mathcal{S}'$ as sums over the non-root components $E$ of the special trees $T$:

$$\mathcal{S} = \sum_{E, E^- \text{ red}} F_1(p_E) + \sum_{E^- \text{ non red}} F_2(p_E) + \sum_{E \text{ non red}} F_3(p_E) + gH,$$

$$\mathcal{S}' = \sum_{E, E^- \text{ red}} G_1(p_E) + \sum_{E^- \text{ non red}} G_2(p_E) + \sum_{E \text{ non red}} G_3(p_E) + (5g+1)H,$$

where $H = \sum_T \mu(T) + \sum_{\text{ram 1}} 1 + \sum_{\text{ram 2}} 3$ is the quasi-admissible and effective adjustment, and the functions $F_i$ and $G_j$ are quadratic polynomials in $p_E$ with linear coefficients in $g$.

There is a simple way to recognize the boundary divisors $\Delta \Xi_{k,i}$ in which a special trigonal fiber $C_X$ lies. Consider the corresponding “effective” fiber $C_X = \hat{\phi}^*T$ in $\hat{X}$. For any non-root component $E$ in $T$ there are two possibilities: either $\hat{\phi}^*E$ and $\hat{\phi}^*E^−$ are both reduced, or $E$ is part of a chain of length 3 or 5, constructed to resolve ramifications in the quasi-admissible fiber $C_X$.

**11.1.1. Contributions to the degrees $\alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i}$**. Consider the first situation, and denote by $C'$ the preimage $\hat{\phi}^*E \cup \hat{\phi}^*E^−$ in $\hat{X}$. Thus, $C'$ corresponds to a general member of $\Delta \Xi_{1,i}, \Delta \Xi_{2,i}, \Delta \Xi_{3,i}$, possibly of lower genus (cf. Fig. 42). As part of the fiber $C_X$, the curve $C'$ is represented for simplicity by the triple intersection of two smooth trigonal curves (in $\Delta \Xi_{1,i}$), but it could have corresponded to any general member of $\Delta \Xi_{2,i}$ or $\Delta \Xi_{3,i}$. The solid box encompasses the preimage $\hat{\phi}^*T(E)$, and the dashed box encompasses the preimage of

![Figure 41. Moving B](image-url)
the rest, \( \hat{\phi}^*(T - T(E)) \). Each of these boxes represents a limit of a quasi-admissible curve,\( C_1 \) or \( C_2 \), which is naturally a triple cover of\( \mathbb{P}^1 \). Thus, we can “smoothen” each box to such a curve \( C_i \). As a result we obtain a quasi-admissible curve \( C_1 \cup C_2 \) of total genus \( g \), which corresponds to a general member of \( \Delta_T^1,i, \Delta_T^2,i, \) or \( \Delta_T^3,i \). Depending on which divisor \( \Delta_T^k,i \) is evoked, there is a corresponding contribution of 1 to the coefficient \( \alpha_{k,i}^* \): 

\[ [C_X] \in \Delta \mathcal{S}_{k,i}. \]

Note that the arithmetic genus of \( C_2 \) is the previously defined \( p_E \). The contribution of \( E \) to \( \mathcal{S} \) is \( F_1(p_E) \) plus the possible quasi-admissible adjustment in \( \mu(T) \) needed to obtain \( \hat{\phi}^*(E \cup E^-) \). In view of the above “smoothening”, this can be thought of as the contribution of \( C_2 \) in the effective curve \( C_1 \cup C_2 \), and by Prop. 9.1 this is exactly the coefficient \( c_{k,i} \). The same argument works in the case of \( \mathcal{S}' \) from Prop. 10.2. We conclude that \( \alpha_{k,i}^* \) (for \( k = 1, 2, 3 \)) equals the number of \( c_{k,i} \)'s and \( d_{k,i} \)'s in \( \mathcal{S} \) and \( \mathcal{S}' \), respectively.

### 11.1.2. Contributions to the degrees \( \alpha_{4,i}, \alpha_{5,i}, \alpha_{6,i} \). We treat analogously the remaining case when the component \( E \) is part of a chain of length 3 or 5. Here, however, one must consider simultaneously all the components \( E \) of \( T \) participating in such a chain, and take a quasi-admissible limit only over the reduced curves in \( \hat{X} \). In Fig. 43 one can see all three ramification cases, or equivalently, the boundary divisors \( \Delta T_{4,i}, \Delta T_{5,i}, \) and \( \Delta T_{6,i} \). For simplicity, we have again depicted the reduced components in \( \hat{X} \) by smooth trigonal curves, which may not always be true for every tree \( T \); they could, for instance, be singular or reducible, but they will keep the ramification index 1 or 2 at the appropriate points.

Figure 42. \( E \nsubseteq \) chain \( \rightarrow \) \( \alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i} \)

Figure 43. \( E \subseteq \) chain \( \rightarrow \) \( \alpha_{4,i}, \alpha_{5,i}, \alpha_{6,i} \)
To see how \( c_{k,i} \) and \( d_{k,i} \) are obtained, let us calculate, for example, the contributions of \( E_1, E_2, E_3 \) and \( E_4 \) in the case of \( \Delta \Xi_{6,i} \). The inverse images in \( \tilde{X} \) of \( T - T(E_1) \) and \( T(E_4) \) are marked by dashed and solid boxes, respectively. We “smoothen” each box by a smooth trigonal curve, \( C_1 \) or \( C_2 \), and keep the inverse images of \( E_1, E_2 \) and \( E_3 \). Thus, we obtain a general member \( C'' \) of \( \Delta \Xi_{6,i} \). The arithmetic genera, necessary to calculate the contribution of \( C'' \) to \( \mathcal{G} \), are given from right to left by:

\[
p_a(C_2) = p_{E_4}, \quad p_{E_3}^c = p_{E_4} - 3, \quad p_{E_2} = p_{E_4} - 2, \quad p_{E_1} = p_{E_4} - 2.
\]

As in the proof of Prop. 9.1, we substitute these in the sum \( \mathcal{G} \), and for \( i = p_{E_4} \) we obtain

\[
F_1(E_1) + F_1(E_4) + F_2(E_2) + F_3(E_3) + 3g = \frac{9}{2} p_{E_4} (g - p_{E_4}) - \frac{3}{2} (g - 1) = e_{6,i}.
\]

Combining all of the above results, we conclude that the contributions of any tree \( T \) to the sums \( \mathcal{G} \) and \( \mathcal{G}' \) are \( c_T = \sum_{k,i} \alpha_{k,i} c_{k,i} \) and \( d_T = \sum_{k,i} \alpha_{k,i} d_{k,i} \).

This allows us to extend all results of Sect. 8.3, 9.3 to the case of a base curve \( B \) meeting transversally the boundary \( \Delta \Xi_g \).

11.2. **Extension to an arbitrary base \( B \), not contained in \( \Delta \Xi_g \).** If the base curve \( B \) happens to be tangent to a boundary divisor \( \Delta \Xi_{k,i} \) at a point \( [C_X] \), then over some node \( p \) of the corresponding tree \( T = \phi(C_X) \) all local analytic multiplicities \( m_q \) (cf. Sect. 2.1) will be multiplied by the degree of tangency of \( B \) and \( \Delta \Xi_{k,i} \). Fig. 44 presents a few examples of possible fibers in \( \tilde{X} \):

\[
\begin{align*}
\text{Fig. 44. Local multiplicities}
\end{align*}
\]

In the nonramification cases of \( \Delta \Xi_{1,i}, \Delta \Xi_{2,i} \) and \( \Delta \Xi_{3,i} \), this would force rational double points as singularities on the total spaces of \( \tilde{X} \) and \( \tilde{Y} \), whereas in the ramification cases of \( \Delta \Xi_{4,i}, \Delta \Xi_{5,i} \) and \( \Delta \Xi_{6,i} \), one may arrive at surfaces \( \tilde{X} \) and \( \tilde{Y} \), nonnormal over some nonreduced fibers. But in both cases, one can roughly view the corresponding fibers as being obtained by a base change from the general or special fibers of Sect. 8 and Sect. 11.1. Alternatively, one can go through the arguments of the paper for the new surfaces \( \tilde{X} \) and \( \tilde{Y} \) (normalizing, if necessary), and notice that all formulas (e.g. Euler characteristic formula for \( \lambda \), Index theorem on \( \tilde{X} \), adjunction formula in \( \mathbb{P} V \), etc.) hold for surfaces with double point singularities.

Thus, in effect, one may replace a given singular fiber \( C_X \) by a bunch of general boundary curves \( C \), following the procedure described in Section 11.1. Furthermore, if some of these general curves \( C \) are “multiple” (i.e. \( B \) is tangent to \( \Delta \Xi_{k,i} \) at \( [C] \)), one may in turn replace each \( C \) by several “transversal” general boundary curves, and refer to the statements in Sections 8.3 and 9.3. The only notational difference in this approach will appear in the definition of the invariants \( m, \theta \) and \( \gamma \) from Sect. 8: now we have to allow for them to be rational, rather than integral, due to possible rational intersections \( E \cdot E' \). This will be “compensated” in the final calculations, which will take into account the multiplicity of the
corresponding fibers, and roughly speaking, will “multiply back” our invariants \( \delta, \lambda \) and \( \kappa \) by what they were divided by in the beginning of the calculations.

This concludes the proof of our results for all families of stable curves \( X \to B \) with general smooth trigonal members.

11.3. **Statements of the results for any family** \( X \to B \). In the following list of results, Theorems 11.1 and 11.3 can be viewed as local trigonal analogs of Cornalba-Harris’s relation in the Picard group of the hyperelliptic locus \( \overline{\mathcal{M}}_g \) (cf. Theorem 2.6). Similarly, Theorem 11.4 is the analog of the \( 8 + 4/g \) maximal bound in the hyperelliptic case (cf. Theorem 2.2).

**Theorem 11.1** (7 + \( 6/g \) relation). For any family \( X \to B \) of stable curves with smooth trigonal general member, if \( V \) is the canonically associated to \( X \) vector bundle of rank 2, then the following relation holds true

\[
(7g + 6)\lambda|_B = g\delta|_B + \mathcal{E}|_B + \frac{g - 3}{2} (4c_2(V) - c_1^2(V)),
\]

where \( \mathcal{E} \) is an effective rational linear combination of boundary components of \( \overline{\mathcal{M}}_g \), not containing \( \Delta\overline{\mathcal{S}}_0 \). In particular,

\[
(7g + 6)\lambda|_B = g\delta_0|_B + \sum_{k,i} \tilde{c}_{k,i} \delta_{k,i}|_B + \frac{g - 3}{2} (4c_2(V) - c_1^2(V)),
\]

where \( \tilde{c}_{k,i} \) is quadratic polynomial in \( i \) with linear coefficients in \( g \), and it is determined by the geometry of \( \delta_{k,i} \) (cf. p. 44).

**Theorem 11.2** (7 + \( 6/g \) bound). For any nonisotrivial family \( X \to B \) of stable curves with smooth trigonal general member, if the canonically associated to \( X \) vector bundle \( V \) is Bogomolov semistable, then the slope of \( X/B \) is bounded from above by

\[
\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}.
\]

**Theorem 11.3** (Index relation). For any family \( X \to B \) of stable curves with smooth trigonal general member, if \( V \) is the canonically associated to \( X \) vector bundle of rank 2, then the following relation holds true

\[
36(g + 1)\lambda|_B = (5g + 1)\delta|_B + \mathcal{E}'|_B + (g - 3) (9c_2(V) - 2c_1^2(V)),
\]

where \( \mathcal{E}' \) is an effective rational combination of the boundary divisors \( \Delta\overline{\mathcal{S}}_{k,i} \), not containing \( \Delta\overline{\mathcal{S}}_0 \). In particular,

\[
36(g + 1)\lambda|_B = (5g + 1)\delta_0|_B + \sum_{k,i} \tilde{d}_{k,i} \delta_{k,i}|_B + (g - 3) (9c_2(V) - 2c_1^2(V)),
\]

where \( \tilde{d}_{k,i} \) is quadratic polynomial in \( i \) with linear coefficients in \( g \), and it is determined by the geometry of \( \delta_{k,i} \) (cf. p. 44).

**Theorem 11.4** (Maximal bound). For any nonisotrivial family \( X \to B \) of stable curves with smooth trigonal general member, the slope of \( X/B \) satisfies:

\[
\frac{\delta}{\lambda} \leq \frac{36(g + 1)}{5g + 1},
\]

with equality if and only all fibers of \( X \) are irreducible curves, and either \( g = 3 \) or the divisor \( \eta \) on the total space of \( X \) is numerically zero.
11.4. **What happens with the hyperelliptic locus** $\mathcal{F}_g$. As we promised in Section 11.3, we consider the contribution of the hyperelliptic locus to the above theorems. For any hyperelliptic curve $C$, we need to blow up a point on $C$ before it starts “behaving” like a trigonal curve in the quasi-admissible and effective covers. Below we have shown what happens to a smooth or general singular hyperelliptic curve (cf. Fig. 52 for the admissible classification of the boundary locus $\Delta \mathcal{F}_g$).

![Figure 45. $\mathcal{F}_g \cap \Delta \mathcal{F}_0$](image)

**11.4.1. Smooth hyperelliptic curves.** We blow up $C$ at a point, and thus add a smooth rational component $\mathbb{P}^1$ to make it a triple cover $C'$ (cf. Fig. 45). The quasi-admissible adjustment of $C$ is $\mu(C') = 1$. From here on, $C$ will behave essentially like a smooth trigonal curve. Therefore, in all relations $C$ is going to contribute $g$ or $(5g + 1)$, depending on what $\delta$ is multiplied by.

**11.4.2. Singular hyperelliptic curves in $\Delta \mathcal{F}_{2,i}$ and $\Delta \mathcal{F}_{5,i}$.** The necessary effective and quasi-admissible modifications are shown in Fig. 46–47.

In the first case, there are two hyperelliptic components intersecting transversally in two points. For the quasi-admissible cover, we need two “smooth” blow-ups, which makes $\mu = 2$. From now on, this curve will behave like a element of $\Delta \mathcal{F}_{2,i}$, where $\mu_{2,i} = 1$. Thus, the coefficient in, say, the maximal bound relation will be: $\tilde{d}_{2,i} + (5g + 1)$, due to the extra blow-up in $\mu$.

![Figure 46. $\mathcal{F}_g \cap \Delta \mathcal{F}_{2,i}$](image) ![Figure 47. $\mathcal{F}_g \cap \Delta \mathcal{F}_{5,i}$](image)

In the second case, two hyperelliptic components meet transversally in one point, but have a ramification index 1 at this point when viewed as double covers. Fig. 46 presents first the quasi-admissible modification: as in the case of $\Delta \mathcal{F}_{5,i}$, the local analytic multiplicity between the two rational components is 2, which means that we must have made three “smooth” blow-ups and one “singular” blow-down. As a result, $\mu = 3$. From here on, this curve behaves exactly as a general member of $\Delta \mathcal{F}_{5,i}$. Recall that $\mu_{5,i} = 2$, and the extra 1 in the hyperelliptic case accounts for the one extra blow-up. Therefore, the coefficient of this fiber $C$, say, in the maximal bound relation, will be $\tilde{d}_{5,i} + (5g + 1)$.

We conclude that a base curve $B$, passing through the hyperelliptic locus, will contribute in the results listed in Section 11.3 roughly $g$, or $(5g + 1)$, times the number of elements in
We cannot write the latter in the form of a scheme-theoretic intersection, since $\overline{\mathcal{I}}_g$ is of a larger codimension in $\overline{\mathcal{M}}_g$.

One can explain these extra summands in the following way. Recall the projection map $pr_1: \overline{\mathcal{M}}_{3,0} \to \overline{\mathcal{M}}_g$. The exceptional locus of $pr_1$ is the admissible boundary divisor $\Delta_{3,0}$, which is blown down to the codimension 2 hyperelliptic locus $\overline{\mathcal{I}}_g$ inside $\overline{\mathcal{M}}_g$. For calculation purposes, it will be easier to work instead with the space of minimal quasi-admissible covers $\overline{\mathcal{Q}}_{3,0}$, which replaces $\overline{\mathcal{M}}_{3,0}$. The same situation of a blow-down occurs, where the exceptional divisor in $\overline{\mathcal{Q}}_{3,0}$ consists of reducible curves $C'$, as shown in Fig. 45.

Let $D$ be the linear combination of divisors in $\overline{\mathcal{M}}_g$ given by the restriction $\Delta|_{\overline{\mathcal{M}}_g}$, and consider a curve $B \subset \overline{\mathcal{M}}_g$, intersecting the hyperelliptic locus in finitely many points. By abuse of notation, we denote by $pr_1$ the projection from $\overline{\mathcal{Q}}_{3,0}$ to $\overline{\mathcal{M}}_g$. Then for the intersection $D \cdot B$ we have:

$$D \cdot B = pr_1^*(D) \cdot pr_1^*(B) = pr_1^*(D) \cdot (\overline{B} + \sum E_j),$$

where $\overline{B}$ is the proper transform of $B$, and the $E_j$'s are the corresponding exceptional curves above $B$. Note that each $E_j$ is in fact a line $\mathbb{P}^1$ representing all possible quasi-admissible covers, arising from a hyperelliptic curve $[C] \in B \cap \overline{\mathcal{I}}_g$. From Fig. 45, these are the blow-ups of $C$ at a point, one for each involution pair $\{p_1, p_2\} \in \mathfrak{g}_2$, and that is why $E_j \cong \mathbb{P}^1$.

The extra summands on p. 53, induced by the base curve $B$, are result of the extra intersections $pr_1^*(D) \cdot E_j$ from above. Indeed, the relations, as they stand, compute only $pr_1^*(D) \cdot \overline{B}$, the component corresponding to families with general smooth members. From the calculations on p. 53, we expect that each $pr_1^*(D) \cdot E_j = 1$, and this will account for the extra 1 appearing in all $\mu$'s.

To verify this, we only needs to show $\delta|_{E_j} = 1$. Since we cannot pick out canonically one point $p_i$ from each hyperelliptic pair $\{p_1, p_2\}$ on $C$, and thus construct a family of blow-ups at $p_i$ of $C$ over $E_j \cong \mathbb{P}^1$, we make a base change of degree two $C \to E_j$.

We construct a family over $C$, corresponding to all blow-ups of $C$ at point $p \in C$. This is simply the products $C \times C$ and $\mathbb{P}^1 \times C$, identified at two sections $S_1$: $S_1$ is the diagonal on $C \times C$, and $S_2$ is a trivial section of $\mathbb{P}^1 \times C$ over $C$ (cf. Fig. 49). From \cite{CH}, for the base curve $C$ of this family, the degree $\delta|_C$ is computed as

$$\delta|_C = \delta_{C \times C} + \delta_{\mathbb{P}^1 \times C} + S_1^2 + S_2^2 = 0 + 0 + 2 + 0 = 2.$$
Taking into account the base change \( C \rightarrow E_j \), \( \delta|_{E_j} = 1 \).

Finally, if we allow for our families to have finitely many hyperelliptic fibers, we adjust the relation in 11.1 by \( g \Delta H_{3,0} \cdot B \), and the relation in 11.3 by \( (5g + 1) \Delta H_{3,0} \cdot B \). The two bounds in Theorems 11.2-11.4 are unaffected by the above discussion.

12. Interpretation of the Bogomolov Index \( 4c_2 - c_1^2 \) via the Maroni Divisor

12.1. The Maroni invariant of trigonal curves. For any smooth trigonal curve \( C \), consider the triple cover \( f: C \rightarrow \mathbb{P}^1 \). The pushforward \( f_* (\mathcal{O}_C) \), as we noted before, is a locally free sheaf of rank 3 on \( \mathbb{P}^1 \), and hence decomposes into a direct sum of three invertible sheaves on \( \mathbb{P}^1 \):

\[
f_* (\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b).
\]

The first summand is trivial due to the split exact sequence

\[
0 \rightarrow V \rightarrow \alpha_* \mathcal{O}_C \overset{\text{tr}}{\rightarrow} \mathcal{O}_{\mathbb{P}^1} \rightarrow 0,
\]

where \( V = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \). From GRR, \( a + b = g + 2 \). We have observed in Section 6 that \( C \) embeds in the rational ruled surface \( \mathbb{P}V = F_k \), for \( k = \frac{|b - a|}{2} \).

Definition 12.1. The Maroni invariant of an irreducible trigonal curve \( C \) is the difference \( \frac{|b - a|}{2} \). The Maroni locus in \( \mathbb{P}_g \) is the closure of the set of curves with Maroni invariants \( \geq 2 \) (cf. [Ma]).

Lemma 12.1. For a general trigonal curve \( C \) the vector bundle \( V \) is balanced, i.e. the integers \( a \) and \( b \) are equal or 1 apart according to \( g \pmod{2} \).

Proof. Let \( a \leq b \). The statement follows from a dimension count of the linear system \( L = |3B_0 + \frac{2+2}{2} F| \) on the ruled surface \( \mathbb{P}F_{b-a} = F_k \). Indeed, all trigonal curves with Maroni invariant \( \frac{b-a}{2} \) are elements of \( L \). If \( p: F_k \rightarrow \mathbb{P}^1 \) is the projection map, the projective dimension of \( L \) equals

\[
r(L) = h^0 (p_* \mathcal{O}_{F_k} (3B_0 + \frac{2+2}{2} F)) - 1.
\]

Denoting by \( \tilde{B} = B_0 - \frac{k}{2} F \) the section of \( F_k \) with smallest self-intersection of \(-k\), we have \( p_* \mathcal{O}_{F_k} (\tilde{B}) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-k) \). The necessary pushforward from above is:

\[
p_* \mathcal{O}_{F_k} (3\tilde{B} + \frac{2+2}{2} + 3k F) = \text{Sym}^3 (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-k)) \otimes \mathcal{O}_{\mathbb{P}^1} (\frac{3+2+3k}{2}) = \bigoplus_{j=\pm 1, \pm 3} \mathcal{O}_{\mathbb{P}^1} (\frac{3+2+3k}{2}).
\]

Since an irreducible trigonal curve \( C \) lies in \( L \), we have \( C \cdot \tilde{B} \geq 0 \), hence \( g + 2 - 3k \geq 0 \) and \( g \equiv k \pmod{2} \). Evaluating the sections of this sum of sheaves, we obtain \( r(L) = 2g + 7 \).
The ruled surface $F_k$ has automorphisms, inducing automorphisms of the linear system $L$. We need to mod out these in order to obtain the dimension of the space of trigonal curves embedded in $F_k$. The group $\text{Aut} F_k$ is a product (not necessarily direct) of the base automorphisms $\text{Aut} P^1 = \text{PGL}_2$, and the projective automorphisms of the vector bundle $V$. The latter is an open set (up to projectivity) of the homomorphisms of $V$ into $V$, and hence has the same dimension as:

$$\text{Hom}(V,V) \cong H^0(V \otimes V^{-}) = H^0(\mathcal{O}_{P^1}(-k) \oplus \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(k)).$$

For $k > 0$, $\dim \text{Aut} V = k + 3$, while for $k = 0$, $\dim \text{Aut} V = 4$. We conclude that the dimension of the set of trigonal curves with Maroni invariant $k/2$ is

$$r(L) - \dim \text{Aut} F_k = \begin{cases} 2g + 1 & \text{if } k = 0, \\ 2g + 2 - k & \text{if } k > 0. \end{cases}$$

When $k = 0$ or $k = 1$, this space corresponds to an open dense set of $\overline{\Sigma}_g$. For an even $g$ a general trigonal curve has Maroni invariant 0 and therefore embeds in $F_0 = P^1 \times P^1$, while for an odd $g$ a general trigonal curve has Maroni invariant 1 and embeds in $F_1 = \text{Bl}_{pt}(P^2)$. In both cases, the vector bundle $V$ is balanced.

**Corollary 12.1.** For $g$ even, the Maroni locus is a divisor in $\overline{\Sigma}_g$ whose general member embeds in $F_2$. For $g$ odd, the Maroni locus has codimension 2 in $\overline{\Sigma}_g$ and its general member embeds in $F_3$.

**Remark 12.1.** It will be useful to identify precisely the group of automorphisms of the linear system $L$ for $k = 0, 1$. We have $\text{Aut}(P^1 \times P^1) \cong \text{PGL}_2 \times \text{PGL}_2 \times \mathbb{Z}/2\mathbb{Z}$. The last factor comes from the exchange of the fiber and the base of $P^1 \times P^1$ and it is relevant only for $g = 4$: then $L = |3B_0 + 3F|$. Otherwise, for any even $g > 4$:

$$\text{Aut} L \cong \text{PGL}_2 \times \text{PGL}_2.$$

When $g$ is odd, the ruled surface $F_1$ can be thought of as the blow-up of $P^2$ at the point $q = [0, 0, 1]$. Any automorphism of $\text{Bl}_q P^2$ carries the exceptional divisor $E_q$ of $F_1$ to itself, and hence is induced by an automorphism of the plane preserving the point $q$. The group of such automorphisms of $P^2$ is the subgroup of $\text{PGL}_3$ corresponding to matrices:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$ 

Taking into account the discriminant of these matrices, we easily identify for odd $g$:

$$\text{Aut} L \cong A^2 \times \text{GL}_2.$$ 

Note that all of the above groups $\text{Aut} L$ have dimension 6, which was claimed already in Lemma 12.1.

### 12.2. Generators of $\text{Pic}_Q \overline{\Sigma}_g$.

**Proposition 12.1.** The rational Picard group of $\overline{\Sigma}_g$, $\text{Pic}_Q \overline{\Sigma}_g$, is freely generated by the boundary classes $\delta_0$, $\delta_k$, and one additional class, which for even genus $g$ coincides with the Maroni class $\mu$.

**Proof.** Since a general trigonal curve $C$ embeds in the ruled surface $F_k$ ($k = 0, 1$), $C$ is a member of the linear system $L = |3B_0 + \frac{g+2}{2}F|$ on $F_k$. Let $U$ be the open set inside $PL \cong P^{2g+7}$ corresponding to the smooth trigonal members of $L$. The surjection

$$\mathbb{Z} = \text{Pic} P^{2g+7} \twoheadrightarrow \text{Pic} U$$
has a nontrivial kernel, because the set of singular trigonal curves in $F_k$ is a divisor in $PL$. Hence Pic $U = \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 0$, and Pic$_Q U = 0$.

The image of the natural projection map $p : U \to Y_{g}$ is the open dense set $W$ of smooth trigonal curves with lowest Marone invariant of 0 or 1. Let $F$ denote the fiber of $p$. From Remark 12.1,

$$F \cong \begin{cases} 
PGL_2 \times PGL_2 & \text{if } g - \text{even}, g > 4; \\
PGL_2 \times PGL_2 \times \mathbb{Z}/2\mathbb{Z} & \text{if } g = 4; \\
\text{Aut } L \cong \mathbb{A}^2 \times GL_2 & \text{if } g - \text{odd}.
\end{cases}$$

Leray spectral sequence or other methods (cf. [GH, Mi]) yield:

$$H^1(W, f_*\mathcal{O}_Y^\ast) \hookrightarrow H^1(U, \mathcal{O}_U^\ast).$$

Pushing the exponential sequence on fibers, i.e. the base curve $B$, (cf. p. 53), the boundary divisors are not since the class of the Hodge bundle $U \to B$ is a linear combination of the boundary classes, which is not true. Hence, ker $p^* \subset H^0(W, R^1f_*\mathcal{O}_Z) \subset H^1(F, \mathcal{O}_Z)$. For even $g$, $H^1(F, \mathcal{O}_Z)$ is torsion (a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$), but for odd $g$ it is isomorphic to $\mathbb{Z}$.

Hence, for even $g$ we have the natural embedding $p^* : \text{Pic}_Q W \hookrightarrow \text{Pic}_Q U$, and in view of Pic$_Q U = 0$, it follows that Pic$_Q W = 0$. The complement of $W$ in $\overline{Y}_g$ is the union of the boundary of $\overline{Y}_g$ and the Maroni divisor. Therefore, $\delta_0, \delta_{k,i}$ and $\mu$ generate Pic$_Q \overline{Y}_g$.

Since the class of the Hodge bundle $\lambda$ is not a linear combination of the boundary classes (cf. p. 53), the boundary divisors are not sufficient to generate the rational Picard group of $\overline{Y}_g$, and $\mu$ must be linearly independent of them. We conclude that $\delta_0, \delta_{k,i}$, and $\mu$ generate freely Pic$_Q \overline{Y}_g$ for even genus $g$.

For $g$-odd, $p^* : \text{Pic}_Q W \to \text{Pic}_Q U$ is either an inclusion, or has a kernel with one generator. Since the Maroni locus for $g$-odd is not a divisor, an inclusion would imply as above that $\lambda$ is a linear combination of the boundary classes, which is not true. Hence, ker $p^* = \mathbb{Q}$ and Pic$_Q W$ is generated freely by the boundary classes $\delta_0$ and $\delta_{k,i}$, and one additional class. □

12.3. The Bogomolov condition and the Maroni divisor.

**Proposition 12.2.** For even genus $g$ and a base curve $B$, not contained in $\Delta \overline{Y}_g$:

$$(7g + 6)\lambda = g\delta_0 + \sum_{k,i} \tilde{c}_{k,i} \delta_{k,i} + 2(g - 3)\mu,$$

where $\tilde{c}_{k,i}$ are certain polynomial coefficients computed similarly as $\tilde{c}_{k,i}$. (cf. p. 53)

**Proof.** We set $g = 2(m - 1)$. Let us consider for now only families with irreducible trigonal fibers, i.e. the base curve $B$ intersects only the boundary component $\Delta \overline{Y}_0$.

**Case 1.** If $B$ does not intersect the Maroni divisor $\mu$, then the Maroni invariant of the fibers in $X$ is constant, and equal to 0. The fibers $C$ of $X$ embed in the projectivization $P(V|_{F_Y}) \cong P^1 \times P^1$. Since deg $V|_{F_Y} = g + 2$ and $V$ is balanced, the restriction of $V$ to the fiber $F_Y$ on the ruled surface $Y$ is

$$V|_{F_Y} = \mathcal{O}_{P^1}(m) \oplus \mathcal{O}_{P^1}(m).$$

Moreover, $V|_{F_Y}$ does not jump as $F_Y$ moves, so that $V$ can be written as:

$$V \cong h^* M \otimes \mathcal{O}_Y(mB_0).$$
for some vector bundle $M$ of rank 2 on $B$. But the Bogomolov index $4c_2(V) - c_1^2(V)$ is independent of twisting $V$ by line bundles, in particular, by $O_Y(mB_0)$, so that

$$4c_2(V) - c_1^2(V) = 4c_2(h^*M) - c_1^2(h^*M) = 4c_2(M) - c_1^2(M) = 0.$$  

The last equality follows from $c_2(M) = 0 = c_1^2(M)$ for any bundle on the curve $B$. We conclude that $4c_2(V) - c_1^2(V) = 4\mu|_B = 0$.

**Case 2.** Now let $B$ intersect the Maroni divisor $\mu$ in finitely many points. Assume also that these points are general in $\mu$, i.e. they correspond to trigonal curves $C$ embeddable in the ruled surface $F_2$. We twist $V$ by a line bundle $M = O_Y(mB_0)$, and set $\tilde{V} = V \otimes M$, so that $\deg \tilde{V}|_{F_Y} = 0$ and

$$\begin{cases} 
\tilde{V}|_{F_Y} = O_{P_1} \oplus O_{P_1} & \text{when } F_Y \text{ is generic,} \\
\tilde{V}|_{F_Y} = O_{P_1}(-1) \oplus O_{P_1}(1) & \text{when } F_Y \text{ is special.}
\end{cases}$$

Then $\tilde{V}$ is the middle term of a short exact sequence on $Y$

$$0 \to \tilde{V}' \to \tilde{V} \to \mathcal{I} \to 0,$$

where $\tilde{V}' = h^*(h_\ast \tilde{V})$ is a vector bundle of rank 2 on $Y$. In the notation of [Br], let $W$ be the sum of the special fibers of $Y$, and let $Z$ be the union of certain isolated points on each member of $W$, so that $\mathcal{I} = \mathcal{I}_{Z,W}$ is the ideal sheaf of $Z$ inside $W$. Note that the number of the special fibers, which comprise $W$, equals $\deg Z = \mu|_B$.

We can now compute the Chern classes of $\tilde{V}$:

$$\begin{cases} 
c_1(\tilde{V}) = c_1(\tilde{V}') + W = \text{a sum of fibers of } Y, \\
c_2(\tilde{V}) = c_2(\tilde{V}') + \deg Z = \deg Z.
\end{cases}$$

The last equality follows from the fact that $\tilde{V}'$ is the pull-back of a bundle on the curve $B$, hence of zero higher Chern classes. We conclude that $c_1^2(\tilde{V}) = 0$, and

$$4c_2(\tilde{V}) - c_1^2(\tilde{V}) = 4c_2(\tilde{V}) - c_1^2(\tilde{V}) = 4\deg Z = 4\mu|_B.$$  

Putting the above two cases together, we have for any family with irreducible trigonal members, not entirely contained in the Maroni locus:

$$4c_2(V) - c_1^2(V) = 4\mu|_B. \quad (12.1)$$

Prop. [12] then implies that $\lambda$ is a linear combination of the boundary and the Maroni class:

$$(7g + 6)\lambda|_{\overline{\mathcal{M}}_g} = g\delta_0 + \sum_{k,i} \widehat{c}_{k,i} \delta_{k,i} + 2(g-3)\mu,$$

where the coefficients $\widehat{c}_{k,i}$ are computed in a similar way, or by direct computation with families of singular trigonal curves (cf. [CH]). \qed
Remark 12.2. Note that the coefficients $\tilde{c}_{k,i}$ depend on the specific descriptions of the Maroni curves that appear in the boundary divisors $\Delta \mathcal{F}_{k,i}$, and they are not always equal to the corresponding coefficients $c_{k,i}$ in Theorem 11.1. Indeed, in the above Proposition, we have shown that

\begin{equation}
(12.2) \quad 4c_2(V) - c_1^2(V) = 4\mu|_B + \sum_{k,i} \alpha_{k,i} \delta_{k,i},
\end{equation}

for some $\alpha_{k,i}$, which may be non-zero. Hence, $\tilde{c}_{k,i} = \tilde{c}_{k,i} + \frac{g-3}{2}\alpha_{k,i}$.

For example, consider the case of $\Delta \mathcal{F}_{1,i}$, and let $C = C_1 \cup C_2$ be a general member of it. If $C$ is also Maroni, then there exists a family $X \to B$, whose general fiber is an irreducible Maroni curve, and one of whose special fibers is our $C$. We can assume, modulo a base change and certain blow-ups not affecting $C$, that this family fits in the basic construction diagram (cf. Fig. 33). Let $R_1$ and $R_2$ be the two ruled surfaces in which $C_1$ and $C_2$ are embedded, and let $E_1$ and $E_2$ be the projections of $C_1$ and $C_2$ in the birationally ruled surface $\tilde{Y}$. Then $F = E_1 + E_2$ is a special fiber of $\tilde{Y}$, with self-intersections $E_1^2 = E_2^2 = -1$.

Now, the general member of $X$, being Maroni, is embedded in a ruled surface $F_2$ with a section $L$ of self-intersection $-2$. The union of such $L$'s forms a surface in the 3-fold $PV$, whose closure we denote by $S$. Evidently, $S \cong \tilde{Y}$, at least outside their special fibers. Let $S$ intersect $R_1$ and $R_2$ in curves $L_1$ and $L_2$ (over $E_1$ and $E_2$). We claim that at least one of $R_1$ and $R_2$ is not isomorphic to $F_0 = P^1 \times P^1$. It will suffice to show that $L_1$ or $L_2$ has negative self-intersection.

Indeed, suppose to the contrary that $L_m^2 \geq 0$ in $R_m$ ($m = 1, 2$). Note that $S \cdot R_m = L_m$ in $PV$, so that

\[ L_m^2 = S|_{R_m} \cdot S|_{R_m} = S \cdot R_m \Rightarrow S^2(R_1 + R_2) \geq 0. \]

On the other hand, $R_1 + R_2$ is the fiber of the projection $PV \to \tilde{Y}$, and as such it is linearly equivalent to the general fiber $F_2$. Hence

\[ 0 \leq S^2 \cdot F_2 = S|_{F_2} \cdot S|_{F_2} = L^2 = -2, \]

a contradiction. We conclude that if $C = C_1 \cup C_2$ is a Maroni curve of boundary type $\Delta \mathcal{F}_{1,i}$, then either $C_1$ or $C_2$ (or both) is embedded in a ruled surface $F_k$ with $k \geq 1$. This already distinguishes the cases of odd and even genus $i$.

When $i = g(C_1)$ is even (and hence $j = g(C_2) = g - j - 2$ is also even), the general member of $\Delta \mathcal{F}_{1,i}$ is embedded in a join of two $F_0$'s (each $C_m \subset F_0$), and hence it is not Maroni. Based on this observation, one can easily find the coefficient $\alpha_{1,i}$ for $i$-even. To do this, consider the birationally ruled surface $Y$ which is the blow-up of $F_0$ at one point. Let again the two components of the special fiber of $Y$ be $E_1$ and $E_2$, and projectivize the trivial vector bundle $V = O_Y \oplus O_Y$: $PV = Y \times P^1$. By taking an appropriate linear system in $PV$, one obtains a family of trigonal curves $X$, whose fibers are all irreducible and embedded in $F_0$, except for a special reducible curve $C$ over $E_1 \cup E_2$ of the specified above type. Hence none of $X$'s members are Maroni, and so $\mu|_B = 0$. Further, $4c_2(V) - c_1^2(V) = 0$, and $\delta_{1,i}|_B = 1$, so that equation (12.2) implies $\alpha_{1,i} = 0$, and hence $\tilde{c}_{k,i} = \tilde{c}_{k,i}$ for $i$-even.
The situation is quite different when the genus $i$ is odd. Then both components of the general member $C$ of $\Delta_{1,i}T_g$ are embedded in $F_1$'s, and hence $C$ is potentially Maroni. One can take further the above general argument of intersection theory on $PV$, and show that the curves $L_1$ and $L_2$ are in fact both sections of negative self-intersection $-1$ in these $F_1$'s: consider the product $S \cdot X \cdot F_2$ and its variation over the special fiber of $\tilde{Y}$. But we know that $L_1$ and $L_2$ intersect, as the fiber of $S$ over $\tilde{Y}$ is connected.

Thus, the curve $C$ would be Maroni if and only if the two corresponding ruled surfaces $F_1$ are glued along one of their fibers so that their negative sections intersect on that fiber. (This description can be alternatively derived by considering the degenerations of the $g_3$'s on the irreducible Maroni curves.) To find $\alpha_{1,i}$ in this case, we construct a similar example as above, only changing $V$ to $O_Y \oplus O_Y(E_1)$. This, while keeping the general fiber embedded in $F_0$, has the effect of embedding the special one in a “Maroni” gluing of two $F_1$’s. We have $4c_2(V) - c_1^2(V) = -E_1^2 = 1$, $\mu|_B = 1$, and $\delta_{1,i}|_B = 1$, so that equation (12.2) implies $\alpha_{1,i} = -3$ for $i$-odd, and hence $\widehat{\epsilon}_{k,i} = \check{\epsilon}_{k,i} - 3/(g-3)$.

One can similarly compute the remaining coefficients $\alpha_{k,i}$, by first figuring out which boundary curves in $\Delta_{k,i}T_g$ are Maroni, then constructing an appropriate vector bundle $V$, and finally using equation (12.2) to compute $\alpha_{k,i}$, and hence $\widehat{\epsilon}_{k,i}$.

**Proposition 12.3.** For $g$-even, if the base curve $B$ is not entirely contained in the Maroni divisor, and the singular members of $X$ belong only to $\Delta_0T_g \cup \Delta_{1,i}T_g$, then the slope of the family $X/B$ satisfies:

$$\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}.$$ 

**Conjecture 12.1.** For $g$-even, if the base curve $B$ is not entirely contained in the Maroni divisor, then the slope of the family $X/B$ satisfies:

$$\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}.$$ 

12.4. The Maroni divisor and the maximal bound. Even though for odd genus $g$ the Maroni locus is not large enough to be a divisor in $\Xi_g$, we can define a generalized Maroni divisor class by extending the relation from the $g$-even case.

**Definition 12.2.** For any genus, we define the generalized Maroni class $\mu$ in $\text{Pic}_Q \Xi_g$ by

$$\mu := \frac{1}{2(g-3)} \left\{ (7g + 6)\lambda - g\delta_0 - \sum_{k,i} \widehat{\epsilon}_{k,i}\delta_{3,i} \right\}.$$
Theorem 12.2. The maximal bound $36(g + 1)/(5g + 1)$ is attained for a trigonal family of curves $X \to B$ if and only if all fibers of $X$ are irreducible and

$$
\delta_0|_B = -\frac{72(g + 1)}{g + 2} \mu|_B.
$$

Proof. The fact that $X$ must have only irreducible fibers in order to attain the maximum bound is already known from Theorem 10.3. This means $\delta_{k,i}|_B = 0$ for all $k, i$. Then, Theorem 9.1 implies:

$$(7g + 6)\lambda|_B = g\delta_0|_B + \frac{g - 3}{2} \mu|_B.$$  (12.3)

Assume that the maximal bound is attained, i.e. $36(g + 1)\lambda|_B = (5g + 1)\delta_0|_B$. Substituting for $\lambda|_B$ in the above equation, yields the desired equality. The converse follows similarly. \[\square\]

Remark 12.3. In the $g$-even case, this equality has a specific meaning. Since the Maroni class $\mu$ corresponds to an effective divisor on $\overline{T}_g$, the equality (and hence the maximal bound) is achieved only for base curves $B$ entirely contained in the Maroni divisor, so that the restriction $\mu|_B$ can be negative. In fact, in all found examples, the base $B$ is contained in a very small subloci of the Maroni loci, defined by the highest possible Maroni invariant.

Remark 12.4. Theorem 12.1 and Prop. 12.2 do not have analogs in the hyperelliptic case: there is no additional Maroni divisor to generate $\text{Pic}_Q \overline{T}_g$ together with the boundary $\Delta \overline{T}_g$.

Remark 12.5. When $g = 3$, there is no Maroni locus in $\overline{T}_3$ either. Indeed, since an irreducible trigonal curve of genus 3 embeds only in ruled surfaces $F_k$ with $k$-odd and $k \leq (g + 2)/3 = 5/3$, then all irreducible trigonal curves embed in $F_1$, and correspondingly they all have the lowest possible Maroni invariant $k = 1$. However, $\text{Pic}_Q \overline{T}_3$ is not generated by the boundary classes of $\overline{T}_3$: as Prop. 12.1 asserts, in the odd genus case there is always one additional generating class.

On the other hand, the results on p. 53 yield a priori two relations among $\lambda$ and the $\delta_{k,i}$'s. This would have been a contradiction to the freeness of the generators above, unless these two relations are the same. This is in fact what happens:

$$9\lambda = \delta_0 + 3\delta_{2,1} + 3\delta_{3,1} + 4\delta_{4,1} + 4\delta_{5,1} + 3\delta_{5,2} + 3\delta_{6,1},$$

as restricted to any base curve $B \not\subseteq \Delta \overline{T}_3$. Note the convenient disappearance of the “extra” $(g - 3)$--summands in the coefficients of $\delta_{4,i}, \delta_{5,i}, \delta_{6,i}$). Then the maximal and the semistable ratios both equal 9, and are attained for families with irreducible trigonal members.

13. Further Results and Conjectures

13.1. Results and conjectures for $d$-gonal families, $d \geq 4$. I have carried out some preliminary research in the $d$-gonal case, and while the methods and ideas for the trigonal case are in principle extendable, this appears to be a substantially more subtle and complex problem. More precisely, let $\overline{D}_d$ be the closure in $\overline{M}_g$ of the stable curves expressible as $d$-sheeted covers of $\mathbb{P}^1$. One possible goal is to complete the program of describing generators and relations for the rational Picard groups $\text{Pic}_Q \overline{D}_d$, and to find the exact maximal bounds for the slopes of $d$-gonal families.

For example, I have obtained the following bound for the slope of a general tetragonal family with smooth general member (for odd genus $g$):

$$\frac{\delta}{\lambda} \leq \frac{2}{3} + \frac{64}{3(3g + 1)} = \frac{4(5g + 7)}{3g + 1}.$$
I have also conjectured formulas for the maximal and general bounds for any \( d \)-gonal and other families of stable curves. Entering these formulas are the Clifford index of curves, Bogomolov semistability conditions for higher rank bundles, and some new geometrically described loci in \( \overline{D}_d \). Generalizing the idea of the Maroni locus in the trigonal case, these loci are characterized, for example, in the tetragonal case by the dimensions of the multiples of the \( g^1_4 \)-series. In particular, there will be another generator of Pic\(_{\mathbb{Q}}\overline{\mathcal{T}}_4\) besides the boundary and Maroni divisors.

In the following I present some of these conjectures on the upper bounds for \( \overline{M}_d \). We start by comparing all known maximal and general bounds functions of the genus \( g \):

| locus in \( \overline{M}_g \) | bound \( g = 1 \) | \( g = 2 \) | \( g = 3 \) | \( g = 5 \) |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| general \( \overline{M}_g \) | \( 6 + \frac{12}{g + 1} \) | 12              | 10              | 9               |
| hyperelliptic \( \overline{T}_g = \overline{D}_2 \) | \( 8 + \frac{4}{g} \) | 12              | 10              | –               |
| trigonal \( \overline{T}_g = \overline{D}_3 \) | \( \frac{36(g + 1)}{5g + 1} \) | 12              | –               | 9               |
| gen. tetragonal = \( \overline{T}_4 \) | \( \frac{4(5g + 7)}{3g + 1} \) | 12              | –               | –               |

The pattern appearing in this table is clear: the general bound \( 6 + 12/(g + 1) \) coincides with each of the other bounds exactly twice for some special values of the genus \( g \). Evidently, \( g = 1 \) is one of these special values, yielding 12 everywhere. (I owe this observation to Benedict Gross.) Let \( g_d \) be the other genus \( g \) for which the general formula in \( \overline{M}_g \) and the maximal formula for \( \overline{D}_d \) coincide, i.e. \( g_2 = 2, g_3 = 3, g_5 = 5 \). We notice that for these genera \( g_d \) the moduli spaces \( \overline{M}_2, \overline{M}_3 \) and \( \overline{M}_5 \) consist only of hyperelliptic, trigonal or tetragonal curves, respectively. In general, Brill-Noether theory (cf. [ACGH]) asserts that for complete linear series \( g^1_r \)'s on a smooth curve of genus \( g \) is \( \rho = g - (r + 1)(g - d + r) = 2(d - 1) - g \), and hence the smallest genus \( g \) for which \( \overline{M}_g = \overline{D}_d \supset \overline{D}_{d - 1} \) is \( g = 2d - 3 \). Thus we set \( g_d = 2d - 3 \) for \( d \geq 3 \) and \( g_2 = 2 \). Note that this coincides with the previously found \( g_3 = 3 \) and \( g_5 = 5 \).

**Conjecture 13.1.** If \( \mathcal{F}_d(g) \) is an exact upper bound for the slopes of families of stable curves with smooth \( d \)-gonal general member (locus \( \overline{D}_d \)), then

\[
(a) \quad \mathcal{F}_d(1) = 12.
\]

\[
(b) \quad \mathcal{F}_d(g) = 6 + \frac{12}{g_d + 1}.
\]

It is reasonable to expect that the upper bounds for \( \overline{D}_d \) will be ratios of linear functions of the genus \( g \): \( \mathcal{F}_d(g) = (Ag + B)/(Cg + D) \). Conjecture [13.1] then estimates the difference between \( \mathcal{F}_d(g) \) and the general bound for \( \overline{M}_g \) up to a factor \( f_d = D/C \).

**Conjecture 13.2.** The exact upper bounds \( \mathcal{F}_d(g) \) are given by

\[
\mathcal{F}_d(g) = 6 + \frac{12}{g + 1} + 6 \frac{(1 - f_d)(g - g_d)(g - 1)}{(g + f_d)(g_d + 1)(g + 1)},
\]

or equivalently,

\[
\mathcal{F}_d(g) = 6 + \frac{6}{g + f_d} \left( 1 + f_d + \frac{1 - f_d}{g_d + 1}(g - 1) \right).
\]
I have a conjecture on how to determine the remaining factor \( f_d \), which seems to be closely related to the coefficients of the linear expression in [EMH] for the divisor \( \overline{D}_{g+1} \) in terms of the Hodge bundle \( \lambda \) and the boundary classes \( \delta_i \) on \( \overline{M}_g \). These conjectures are supported by the work of Cornalba-Harris on the hyperelliptic locus \( \overline{H}_g = \overline{D}_2 \), by the results of this paper on the trigonal locus \( \overline{T}_g = \overline{D}_3 \), and by partial results on the tetragonal locus \( \overline{D}_4 \).

In view of Remark 12.5, the equality between the maximal and semistable trigonal bounds for \( g = 3 \) suggests that a similar situation might occur for other \( d \)-gonal families. It is reasonable to expect two or more “semistable” bounds, depending on the number of extra generators in \( \text{Pic} \overline{D}_d \).

One of these “semistable” bounds relates to families obtained as blow-ups of pencils of \( d \)-gonal curves on a ruled surface \( F_k \). Example 2.1 yields the maximal bound \( 8 + 4/g \) for hyperelliptic families (no extra generator besides the boundary classes), and a similar example in the trigonal case yields the \( 7 + 6/g \) semistable bound (one extra generator, the Maroni locus). We generalize this to any \( d \)-gonal family of curves embedded in an arbitrary ruled surface \( F_k \). Invariably, the slope of \( X/B \) is:

\[
\frac{\delta_{|B}}{\lambda_{|B}} = \left( 6 + \frac{2}{d-1} \right) + \frac{2d}{g}.
\]

**Conjecture 13.3.** Let \( X \) be a family of \( d \)-gonal curves of genus \( g \) whose base \( B \) is not contained in a certain codimension 1 closed subset of \( \overline{D}_d \). Then the slope of \( X/B \) satisfies:

\[
\frac{\delta_{|B}}{\lambda_{|B}} \leq \left( 6 + \frac{2}{d-1} \right) + \frac{2d}{g}.
\]

Conjectures 13.3–4 are modifications of earlier conjectures of Joe Harris.

**13.2. A look at families with special \( g_r^d \)'s, \( r \geq 2 \).** The discussion so far was primarily concerned with the loci \( \overline{D}_d \subset \overline{M}_g \) corresponding to linear series \( g_1^d \). But all of our problems are well-defined and quite interesting to solve for curves with series \( g_r^d \) of dimension \( r > 1 \). Equivalently, we consider the loci \( \overline{D}_r^d \) of curves mapping with degree \( d \) to \( P^r \), \( r \geq 1 \).

**Definition 13.1.** The Clifford index \( \epsilon \) of a smooth curve \( C \) is defined as

\[
\epsilon = \min_L \{ \deg L - 2 \dim L \}
\]

where \( L \) runs over all effective special linear series \( L \) on \( C \).

Clifford’s theorem implies \( \epsilon \geq 0 \), with equality if and only if \( C \) is hyperelliptic, i.e. \( L = g_2^1 \) (cf. [ACGH]). On the other hand, \( \epsilon = 1 \) means that there exists a \( g_r^d \) on \( C \) with \( d - 2r = 1 \). From Martin’s Theorem, \( \dim W_d^r (C) \leq d - 2r - 1 = 0 \), where \( W_d^r \) is the variety parametrizing complete linear series on \( C \) of degree \( d \) and dimension at least \( r \). Therefore, we must have \( \dim W_d^r = 0 \). But then Mumford’s theorem asserts that \( C \) is either trigonal, or bi-elliptic, or a smooth plane quintic. The bi-elliptic case would mean that \( W_d^r \) consists of \( g_6^2 \)'s, which contradicts the dimension of \( \dim W_d^r \). In short, \( \epsilon = 1 \) if and only if \( C \) is not hyperelliptic and possesses a \( g_3^1 \) or a \( g_2^2 \).

Thus, according to the Clifford index, the first case with \( r \geq 2 \) is the space of plane quintics. Consider a general pencil of such, and blow up the plane at its 25 base points. The resulting family \( X = Bl_{25} P^2 \to P^1 \) is easily seen to have slope \( 8 = 7 + 6/g \), which corresponds to the bound in Conjecture 13.3 with \( d - 2 \) replaced by the Clifford index \( \epsilon = 1 \). Finally, note that for a \( d \)-gonal curve \( C \) of genus \( g \), by definition \( \epsilon \leq d - 2 \), so that when \( g \gg d \) we may generalize to:
Conjecture 13.4. For a general family $X \to B$ of genus $g$ stable curves whose general member has Clifford index $c$ and whose base $B$ is a general curve in $\mathcal{D}_d$, the slope of $X/B$ satisfies:

$$\frac{\delta_X}{\lambda_X} \leq \left(6 + \frac{2}{c+1}\right) + \frac{2c+4}{g} \quad \text{for} \quad c \ll g.$$ 

Remark 13.1. It is worth noting that the stratification of $\overline{M}_g$, for which we asked in the Introduction, is not obtained via the Clifford index $c$. For example, Xiao constructs families of bi–elliptic curves $C$ with slope 8 (cf. [Xi]), which is between the hyperelliptic and the trigonal maximal bounds. Since $C$ has a $g_1^4$ as bi–elliptic, this already exceeds the conjectured maximal bounds for the tetragonal case. This shows that in some of the above conjectures we have to exclude the subset of bi–elliptic curves from the tetragonal locus $\mathcal{D}_4$, and that similar modifications might be necessary for the other loci $\mathcal{D}_d$. More precisely, it seems plausible that the stratification of $M_g$ according to successively lower slope bounds is related not just to the existence of a specific linear series $g^r_d$, but also to the number, dimension and description of the irreducible components of corresponding varieties $W^r_d$.

13.3. Other methods via the moduli space $\overline{M}_{g,n}(P^r, d)$. The approach in the $g_1^1$-cases is based on a modification of the Harris-Mumford’s [EHM] Hurwitz scheme of admissible covers, which parametrized the $d$-uple covers of stable pointed rational curves. However, in the more general situation for linear series with larger dimensions $r > 1$, such a compactification via admissible covers does not exist, so we have to look for a different solution.

Consider moduli spaces of stable maps $\overline{M}_{g,n}(P^r, d)$. They parametrize stable maps $(C, p_1, p_2, \ldots, p_n; \mu)$, where $C$ is a projective, connected nodal curve of arithmetic genus $g$, the $p_i$'s are marked distinct nonsingular points on $C$, and the map $\mu : C \to P^r$ has image $\mu_*([C]) = d[\text{line}]$ and satisfies certain stability conditions (cf. [K, KM]). The space $\overline{M}_{g,n}(P^r, d)$ seems to be the right compactification which we need in order to extend our results to families with $g^r_d$-series on the fibers: the moduli space of stable maps is somewhat more “sensitive” in describing our loci $\mathcal{D}_d$ in terms of their geometry.

Going back to the $g_1^1$-problems, one can also see the combinatorial flavor that stands in the background of these questions. It is probably not coincidental that the spaces $\overline{M}_{g,n}(P^r, d)$ are also combinatorially defined and give rise to many enumerative problems. It will be useful to understand better the loci $\overline{D}_d$ via their connection with the Kontsevich spaces $\overline{M}_{g,n}(P^r, d)$, and ultimately to solve the remaining questions on $\text{Pic}_Q \overline{D}_d$ for any $d, r$, as well as related interesting enumerative problems that will inevitably arise from such considerations.

14. Appendix: The Hyperelliptic Locus $\mathcal{I}_g$

In this section we give a proof of Theorems 2.4 and 2.2, following the same ideas and methods as in the trigonal case. We refer the reader to previous sections for a detailed proof of certain statements.

14.1. Boundary locus of $\mathcal{I}_g$. Cornalba-Harris describe the boundary of $\mathcal{I}_g$ as consisting of several boundary components, whose general members and indexing are shown in Fig. 52 (cf. [CH]). The restriction of the divisor class $\delta$ to $\mathcal{I}_g$ is the following linear combination:

$$\delta|_{\mathcal{I}_g} = \delta_0 + 2 \sum_{i=1}^{[g-1]/2} \xi_i + \sum_{j=1}^{[g/2]} \delta_j,$$

where $\xi_i$ and $\delta_i$ are the classes in $\text{Pic}_Q \mathcal{I}_g$ of the boundary divisors $\Xi_i$ and $\Delta_j$. 
**Figure 52. Boundary of the hyperelliptic locus \( \overline{\mathcal{F}}_g \)**

\[ \Xi_0; \Xi_i, i = 1, \ldots, [(g-1)/2]; \Delta_j, j = 1, \ldots, [g/2] \]

14.2. **Effective covers and embedding for hyperelliptic families.** In the case of a hyperelliptic family \( f : X \to B \), a minimal quasi-admissible cover coincides with the original family \( X \), because no blow-ups are necessary to perform on the fibers of \( X \): these are already quasi-admissible double covers. Thus, we have a degree 2 map \( \phi = \tilde{\phi} : X \to Y \) for some birationally ruled surface \( Y \) over \( B \). As for an effective cover \( \tilde{\phi} : \tilde{X} \to \tilde{Y} \), only the boundary divisors \( \Delta_j \) require blow-ups (cf. Fig. 52). This is analogous to the “ramification index 1” discussion in Fig. 22–23. Thus, while in \( \tilde{X} \) the special fibers may have occasional nonreduced rational components of multiplicity 2, the fibers of \( \tilde{Y} \) are always trees of reduced smooth \( \mathbb{P}^1 \)'s.

In the case of a smooth hyperelliptic curve \( C \), we consider the natural double sheeted map \( f : C \to \mathbb{P}^1 \). The pushforward \( f_* \mathcal{O}_C \) is a rank 2 vector bundle on \( \mathbb{P}^1 \), which fits into the short exact sequence

\[ 0 \to \mathcal{O}_{\mathbb{P}^1}(g + 1) \to f_* \mathcal{O}_C \to \mathcal{O}_{\mathbb{P}^1} \to 0. \]

We can embed \( C \) in the rational ruled surface \( \mathbb{P}((f_* \mathcal{O}_C)) \). We generalize this construction to the effective cover \( \tilde{\phi} : \tilde{X} \to \tilde{Y} \) by setting \( V := (\phi_* \mathcal{O}_X)^\vee \). For some line bundle \( E \) on \( \tilde{Y} \):

\[ 0 \to E \to \tilde{\phi}_* \mathcal{O}_\tilde{X} \to \mathcal{O}_{\tilde{Y}} \to 0. \]

Then \( \tilde{X} \) naturally embeds in the threefold \( \mathbb{P}V \). Let \( \pi : \mathbb{P}V \to \tilde{Y} \) be the corresponding projection map.

14.3. **The invariants \( \lambda, \delta \) and \( \kappa \).** As a divisor in \( \mathbb{P}V \), \( \tilde{X} \equiv 2\zeta + \pi^*D \), for some divisor \( D \) on \( \tilde{Y} \). From the adjunction formula, \( g = \deg c_1(V)|_{F_Y} - 1 = c - 1 \), where \( c_1(V) = cB_0 + dF_Y \).

The arithmetic genus of the inverse image \( \tilde{\phi}^*T(E) \) is given by

\[ p_E = -m_E \left( \Gamma_E + \Theta_E \right). \]

It turns out that these are the only differences between the set-up of the hyperelliptic and the trigonal case. The definitions of the functions \( m, \theta \) and \( \gamma \), as well as the formulas for \( c_1(V), K_{\mathbb{P}V}, c_2(\mathbb{P}V) \) and the congruence \( D \equiv 2c_1(V) \) are valid without any modifications.

As in the trigonal case, it will be sufficient to consider only the cases when the base curve \( B \) intersects transversally the boundary divisors of \( \overline{\mathcal{F}}_g \). But then for all non-root components \( E \) in \( \tilde{Y} \):

\[ m_E = 1 = \Theta_E \quad \text{and} \quad \Gamma_E = -(p_E + 1). \]

We can now easily calculate the invariants on \( X \).
Proposition 14.1. For any family \( f : X \to B \) of hyperelliptic curves with smooth general member and a base curve \( B \) intersecting transversally the boundary of \( \overline{\mathcal{M}}_g \):

\[
\lambda_X = dg + \frac{1}{2} \sum_{E \neq R} \Gamma_E (\Gamma_E + 1),
\]

\[
\kappa_X = 4d(g - 1) - 2 \sum_{E \neq R} (\Gamma_E + 1)^2 + \sum_{\text{ram} 1} 1,
\]

\[
\delta_X = 4d(2g + 1) + 2 \sum_{E \neq R} (\Gamma_E + 1)(1 - 2\Gamma_E) + \sum_{\text{ram} 1} 1.
\]

With this, we are ready to show the linear relations among \( \lambda|_B \) and the boundary restrictions \( \delta|_B \) and \( \xi|_B \). It is evident that in order to cancel the “global” term \( d \), one must subtract \( (8g + 4)\lambda|_B - g\delta|_B \), which is the main idea of the next theorem.

Theorem 14.1. There exists an effective linear combination \( \mathcal{E}_h \) of the boundary divisors of \( \overline{\mathcal{M}}_g \), not containing \( \Xi_0 \), such that for any family \( f : X \to B \) of hyperelliptic curves with smooth general member:

\[
(8g + 4)\lambda_X|_B = g\delta|_B + \mathcal{E}_h|_B
\]

Proof. We consider the difference

\[
\mathcal{G}_h = (8g + 4)\lambda|_B - g\delta|_B = 2 \sum_{E \neq R} (1 + \Gamma_E)(g + \Gamma_E) + \sum_{\text{ram} 1} g.
\]

In the hyperelliptic case, as opposed to the trigonal case, there is only one type of summands in \( \mathcal{G}_h \).

As in Section 9.3, it is sufficient to calculate the above sum for general members of \( \Xi_i \) and \( \Delta_i \), as described in Prop. 18, i.e. for a general base curve \( B \).

14.3.1. Contribution of the boundary divisors \( \Xi_i \). This case is analogous to the case of \( \Delta_{3;i} \) (cf. Subsection 9.1.1). The arithmetic genus \( p_E = g - i - 1 \), and the corresponding summand in \( \mathcal{G}_h \) is

\[
e_i = 2p_E (g - 1 + p_E) = 2i(g - i - 1) > 0,
\]

where \( i = 1, ..., [(g - 1)/2] \).

14.3.2. Contribution of the boundary divisors \( \Delta_j \). Compare this with the contribution of \( \Delta_{5;j} \) (subsection 9.1.2). There are two non-root components \( E_1 \) and \( E_2 \) in the special fiber of \( \hat{Y} \) \( (E_1^- = R) \), whose invariants are \( p_{E_1} = g - j - 1 \) and \( p_{E_2} = g - j \). With the ramification adjustment of \( g \), the contribution of \( \Delta_j \) to the sum \( \mathcal{G}_h \) is

\[
f_j = 2p_{E_1} (g - 1 + p_{E_1}) + p_{E_2} (g - 1 + p_{E_2}) + g = 4j(g - j) - g > 0,
\]

where \( j = 1, ..., [g/2] \).

Finally, for the appropriate indices \( i \) and \( j \) we set \( \mathcal{E}_h := \sum_{i > 0} e_i \Xi_i + \sum_{j > 0} f_j \Delta_j \). This is an effective combination of boundary divisors in \( \overline{\mathcal{M}}_g \), not containing \( \Delta_0 \) by construction, and satisfying \( \mathcal{G}_h = \mathcal{E}_h|_B \). \( \square \)
Theorem 14.1 implies immediately the following

**Corollary 14.1.** Let $f : X \to B$ be a nonisotrivial family with smooth general member. Then the slope of the family satisfies:

$$\frac{\delta|_B}{\lambda|_B} \leq 8 + \frac{4}{g}.$$  

(14.2)

Equality holds if and only if the general fiber of $f$ is hyperelliptic, and all singular fibers are irreducible.

It is now straightforward to prove the fundamental relation in $\text{Pic}_{\mathbb{Q}} \overline{I}_g$, shown first in [CH]. In Theorem 14.1, we add to the coefficients $e_i$ and $f_j$ the corresponding multiplicities $\text{mult}_\delta \xi_i$ and $\text{mult}_\delta \delta_j$:

$$\tilde{e}_i = e_i + 2 \cdot g = 2(i + 1)(g - i), \quad \tilde{f}_j = f_j + 1 \cdot g = 4j(g - j).$$

Using the fact that $\text{Pic}_{\mathbb{Q}} \overline{I}_g$ is generated freely by the boundary classes $\xi_i$ and $\delta_j$ (see [CH]), we obtain

$$(8g + 4)\lambda = g\delta_0 + \sum_{i > 0} \tilde{e}_i \xi_i + \sum_{j > 0} \tilde{f}_j \delta_j.$$  

**Theorem 14.2.** In the Picard group of the hyperelliptic locus, $\text{Pic}_{\mathbb{Q}} \overline{I}_g$, the class of the Hodge bundle $\lambda$ is expressible in terms of the boundary divisor classes of $\overline{I}_g$ as:

$$(8g + 4)\lambda = g\xi_0 + \sum_{i=1}^{(g-1)/2} 2(i + 1)(g - i)\xi_i + \sum_{j=1}^{[g/2]} 4j(g - j)\delta_j.$$  

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