SPACELIKE $B_2$-SLANT HELIX IN MINKOWSKI 4-SPACE $E^4_1$

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ABSTRACT

In this paper, we give the characterizations of spacelike $B_2$-slant helix by means of curvatures of the spacelike curve in Minkowski 4-space. Furthermore, we give the integral characterization of the spacelike $B_2$-slant helix.

Key words: Minkowski 4-space, Spacelike $B_2$-slant helix, Frenet frame.
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1. INTRODUCTION

A curve of constant slope or general helix in Euclidean 3-space $E^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix)[2]. A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [12] for details) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of the first curvature to the second curvature be constant i.e., $k_1/k_2$ is constant along the curve, where $k_1$ and $k_2$ denote the first and second curvatures of the curve, respectively. Analogue to that A. Magden has given a characterization for a curve $x(s)$ to be a helix in Euclidean 4-space $E^4$. He characterizes a helix iff the function

$$k_1^2 + \left[ \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \right]^2$$

is constant where $k_1$, $k_2$, and $k_3$ are first, second and third curvatures of Euclidean curve $x(s)$, respectively, and they are nowhere zero[7]. Corresponding characterizations of timelike helices in Minkowski 4-space $E^4_1$ were given by Kocayigit and Onder [5]. Latterly Camci and et al. have given some characterizations for a non-degenerate curve a to be a generalized helix by using its harmonic curvatures [2].

Recently, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines of the curve make a constant angle with a fixed direction and they have given a characterization of slant helix in Euclidean 3-space $E^3$ by the fact that the function

$$\frac{k_1^2}{(k_1^2 + k_2^2)^{3/2}} \left( \frac{k_2}{k_1} \right)$$

is constant [4]. After them, Kula and Yayli investigated spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix and they obtained that the spherical images are spherical helices [6]. Analogue to the definition of slant helix, Onder and et al. have defined $B_2$-slant helix in Euclidean 4-space $E^4$ by saying that the second binormal vector of the curve make a constant angle with e fixed direction and they have given some characterizations of $B_2$-slant helix in Euclidean 4-space $E^4$[8].
In this paper, we consider spacelike \( B_2 \)-slant helix in Minkowski 4-space \( E_4^1 \) and we give some characterizations and also the integral characterization of spacelike \( B_2 \)-slant helix.

2. PRELIMINARIES

Minkowski space-time \( E_4^1 \) is an Euclidean space \( E^4 \) provided with the standard flat metric given by

\[
\langle \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2
\]

where \((x_1,x_2,x_3,x_4)\) is a rectangular coordinate system in \( E_4^1 \).

Since \( \langle \cdot \rangle \) is an indefinite metric, recall that a vector \( v \in E_4^1 \) can have one of three causal characters; it can be spacelike if \( \langle v,v \rangle > 0 \) or \( v = 0 \), timelike if \( \langle v,v \rangle < 0 \) and null(lightlike) if \( \langle v,v \rangle = 0 \) and \( v \neq 0 \). Similarly, an arbitrary curve \( x(s) \) in \( E_4^1 \) can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors \( x'(s) \) are respectively spacelike, timelike or null (lightlike). Also recall that the pseudo-norm of an arbitrary vector \( v \in E_4^1 \) is given by \( ||v|| = \sqrt{\langle v,v \rangle} \). Therefore \( v \) is a unit vector if \( \langle v,v \rangle = \pm 1 \). The velocity of the curve \( x(s) \) is given by \( ||x'(s)|| \). Next, vectors \( v, w \) in \( E_4^1 \) are said to be orthogonal if \( \langle v,w \rangle = 0 \). We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively.

Denote by \( \{T(s),N(s),B_1(s),B_2(s)\} \) the moving Frenet frame along the curve \( x(s) \) in the space \( E_4^1 \). Then \( T, N, B_1, B_2 \) are the tangent, the principal normal, the first binormal and the second binormal fields, respectively. A timelike (resp. spacelike) curve \( x(s) \) is said to be parameterized by a pseudo-arclength parameter \( s \), i.e. \( \langle x'(s),x'(s) \rangle = -1 \) (resp. \( \langle x'(s),x'(s) \rangle = 1 \)).

Let \( x(s) \) be a spacelike curve in Minkowski space-time \( E_4^1 \), parameterized by arclength function of \( s \). Then for the curve \( x(s) \) the following Frenet equation is given as follows

\[
\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix}
=
\begin{bmatrix}
0 & k_1 & 0 & 0 \\
-\varepsilon_1 k_1 & 0 & k_2 & 0 \\
0 & \varepsilon_2 k_2 & 0 & k_3 \\
0 & 0 & \varepsilon_3 k_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix},
\]

where

\[
\langle T, T \rangle = 1, \quad \langle N, N \rangle = \varepsilon_1, \quad \langle B_2, B_2 \rangle = \varepsilon_2, \quad \langle B_1, B_1 \rangle = -\varepsilon_1 \varepsilon_2, \quad \varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1
\]

and recall that the functions \( k_1 = k_1(s) \), \( k_2 = k_2(s) \) and \( k_3 = k_3(s) \) are called the first, the second and the third curvature of the spacelike curve \( x(s) \), respectively and we will assume throughout this work that all the three curvatures satisfy \( k_i(s) \neq 0, 1 \leq i \leq 3 \). Here the signs of \( \varepsilon_1 \) and \( \varepsilon_2 \) are changed by a rule. The signature rule between \( \varepsilon_1 \) and \( \varepsilon_2 \) can be given as follows
For the obvious forms of the Frenet equations in (1) we refer to the reader to see ref. [14].

**3. SPACELIKE $B_2$-SLANT HELICES IN MINKOWSKI 4-SPACE**

In this section, we give the definition and the characterizations of spacelike $B_2$-slant helix.

Let $x : I \subset IR \rightarrow E^4_1$ be a unit speed spacelike curve with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ and let $\{T, N, B_1, B_2\}$ denotes the Frenet frame of the curve $x(s)$. We call $x(s)$ as spacelike $B_2$-slant helix if its second binormal unit vector $B_2$ makes a constant angle with a fixed direction in a unit vector $U$; that is

$$\langle B_2, U \rangle = \text{constant} \quad (2)$$

along the curve. By differentiation (2) with respect to $s$ and using the Frenet formulae (1) we have

$$\langle \varepsilon_1 k_1 B_1, U \rangle = 0 .$$

Therefore $U$ is in the subspace $Sp\{T, N, B_1\}$ and can be written as follows

$$U = a_1(s)T(s) + a_2(s)N(s) + a_3(s)B_2(s) , \quad (3)$$

where

$$a_1 = \langle U, T \rangle, \quad \varepsilon_1 a_2 = \langle U, N \rangle, \quad \varepsilon_2 a_3 = \langle U, B_2 \rangle = \text{constant} .$$

Since $U$ is unit, we have

$$a_1^2 + \varepsilon_1 a_2^2 + \varepsilon_2 a_3^2 = M . \quad (4)$$

Here $M$ is +1, -1 or 0 depending if $U$ is spacelike, timelike or lightlike, respectively. The differentiation of (3) gives

$$\left( \frac{da_1}{ds} - \varepsilon_1 a_2 k_1 \right) T + \left( \frac{da_2}{ds} + a_1 k_1 \right) N + (a_2 k_2 + \varepsilon_1 a_3 k_1) B_1 = 0 ,$$

and from this equation we get

$$a_2 = -\varepsilon_1 \frac{k_3}{k_2} a_3 = \varepsilon_1 \frac{1}{k_3} \frac{da_3}{ds}, \quad \frac{da_3}{ds} = -a_1 k_1 . \quad (5)$$

Since

$$\frac{da_2}{ds} = -a_1 k_1 \quad \text{and} \quad \frac{da_2}{ds} = -\varepsilon_1 \frac{k_1'}{k_1^2} \frac{da_1}{ds} + \varepsilon_1 \frac{d^2 a_1}{ds^2} ,$$

we find the second order linear differential equation in $a_1$ given by
\[ \epsilon_1 \frac{d^2 a_1}{ds^2} - \epsilon_1 \frac{k_1'}{k_1} \frac{da_1}{ds} + a_1 k_1^2 = 0. \] (6)

If we change variables in the above equation as \( t = \int_0^s k_1(s) ds \) then we get

\[ \frac{d^2 a_1}{dt^2} + \epsilon_1 a_1 = 0. \]

This equation has two solutions: If \( \epsilon_1 = +1 \) then the solution is

\[ a_1 = A \cos \int_0^s k_1(s) ds + B \sin \int_0^s k_1(s) ds, \] (7)

and if \( \epsilon_1 = -1 \) then the solution is

\[ a_1 = A \cosh \int_0^s k_1(s) ds + B \sinh \int_0^s k_1(s) ds. \] (8)

where \( A \) and \( B \) are constant.

Assume that \( \epsilon_1 = -1 \) and consider the solution (8). From (5) and (8) we have

\[ a_2 = \frac{k_3}{k_2} a_3 = -A \sinh \int_0^s k_1(s) ds - B \cosh \int_0^s k_1(s) ds, \]

\[ a_3 = -\frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' a_3 = A \cosh \int_0^s k_1(s) ds + B \sinh \int_0^s k_1(s) ds. \]

From these equations it follows that

\[ A = a_3 \left( \frac{k_3}{k_2} \sinh \int_0^s k_1(s) ds - \frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' \cosh \int_0^s k_1(s) ds \right), \] (9)

\[ B = a_3 \left( \frac{k_3}{k_2} \cosh \int_0^s k_1(s) ds + \frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' \sinh \int_0^s k_1(s) ds \right). \] (10)

Hence, using (9) and (10) we get

\[ B^2 - A^2 = \left[ \left( \frac{k_3}{k_2} \right)^2 - \frac{1}{k_1^2} \left( \frac{k_3}{k_2} \right)'^2 \right] a_3^2 = \text{constant}, \]

so that

\[ \left( \frac{k_3}{k_2} \right)^2 - \frac{1}{k_1^2} \left( \frac{k_3}{k_2} \right)'^2 = \text{constant} := m. \] (11)

From (4), (9), (10) and (11) we have

\[ B^2 - A^2 = a_3^2 m = M - 1. \]

Thus, the sign of the constant \( m \) agrees with the sign of \( B^2 - A^2 \). So, if \( U \) is timelike or lightlike then \( m \) is negative. If \( U \) is spacelike then \( m = 0 \). Then we can give the following corollary.
Corollary 3.1. Let $x(s)$ be a spacelike $B_2$-slant helix with timelike principal normal $N$ in Minkowski 4-space $E^4_1$ and $U$ be a unit constant vector which makes a constant angle with the second binormal $B_2$. Then the vector $U$ is spacelike if and only if there exist a constant $K$ such that

$$
\frac{k_3}{k_2}(s) = K \exp \left( \int_0^s k_i(t) dt \right).
$$

When $\epsilon_i = +1$, by using (7) with similar calculations as above we get that the spacelike curve $x(s)$ is a spacelike $B_2$-slant helix if and only if

$$
\left( \frac{k_3}{k_2} \right)^2 + \frac{1}{k_i^2} \left( \frac{k_3}{k_2} \right)'^2 = \text{constant}.
$$

(12)

Thus, using (11) and (12), we can characterize the spacelike $B_2$-slant helix $x(s)$ by the fact that

$$
\left( \frac{k_3}{k_2} \right)^2 + \epsilon_i \frac{1}{k_i^2} \left( \frac{k_3}{k_2} \right)'^2 = \text{constant}
$$

(13)

Conversely, if the condition (13) is satisfied for a regular spacelike curve we can always find a constant vector $U$ which makes a constant angle with the second binormal $B_2$ of the curve.

Consider the unit vector $U$ defined by

$$
U = \left[ \epsilon_i \frac{1}{k_i} \left( \frac{k_3}{k_2} \right)' T - \epsilon_i \frac{k_3}{k_2} N + B_2 \right].
$$

By taking account of the differentiation of (13), differentiation of $U$ gives that $\frac{dU}{ds} = 0$, this means that $U$ is a constant vector. So that, we can give the following theorem:

Theorem 3.1. A unit speed spacelike curve $x : I \subset \mathbb{R} \to E^4_1$ with nonzero curvatures $k_i(s)$, $k_3(s)$ and $k_3(s)$ is a spacelike $B_2$-slant helix if and only if the following condition is satisfied,

$$
\left( \frac{k_3}{k_2} \right)^2 + \epsilon_i \frac{1}{k_i^2} \left( \frac{k_3}{k_2} \right)'^2 = \text{constant}.
$$

From Theorem 3.1 one can easily see that the constant function in Theorem 3.1, is independent of $\epsilon_2$. So, we can give the following corollary.

Corollary 3.2. The characterizations of the spacelike $B_2$-slant helix are independent of the Lorentzian causal character of the second binormal vector $B_2$. It is only related to the Lorentzian causal character of the unit principal normal vector $N$. 
Now, we give another characterization of spacelike \( B_2 \)-slant helix in Minkowski 4-space. Let assume that \( x: I \subset \mathbb{R} \rightarrow E^4_1 \) is a spacelike \( B_2 \)-slant helix. Then, Theorem 3.1 is satisfied. By differentiating (13) with respect to \( s \) we get

\[
\left( \frac{k_1}{k_2} \right) \frac{d}{ds} \left( \frac{k_3}{k_2} \right) + \varepsilon \frac{d}{ds} \left( \frac{k_3}{k_2} \right) \frac{d}{ds} \left[ \frac{1}{k_1} \frac{d}{ds} \left( \frac{k_3}{k_2} \right) \right] = 0 ,
\]

and hence

\[
\varepsilon \frac{d}{ds} \left( \frac{k_3}{k_2} \right) = - \frac{\left( \frac{k_3}{k_2} \right) \left( \frac{k_3}{k_2} \right)'}{\left[ \frac{1}{k_1} \left( \frac{k_3}{k_2} \right) \right]'}.
\]

If we write

\[
f(s) = - \frac{\left( \frac{k_3}{k_2} \right)'}{\left[ \frac{1}{k_1} \left( \frac{k_3}{k_2} \right) \right]'},
\]

then

\[
f(s)k_1 = \varepsilon \frac{d}{ds} \left( \frac{k_3}{k_2} \right)'.
\]

From (14) it can be written

\[
\left[ \varepsilon \frac{d}{ds} \left( \frac{k_3}{k_2} \right) \right]' = - k_1 \frac{k_3}{k_2}.
\]

By using (17) and (18) we have

\[
\frac{d}{ds} f(s) = - k_1 \frac{k_3}{k_2}.
\]

Conversely, let \( f(s)k_1 = \varepsilon \frac{d}{ds} \left( \frac{k_3}{k_2} \right) \) and \( \frac{d}{ds} f(s) = - k_1 \frac{k_3}{k_2} \). If we define a unit vector \( U \) by

\[
U = - f(s)T + \varepsilon \frac{k_3}{k_2} N - B_2
\]

we have that \( U \) and \( \langle B_2, U \rangle \) are constants. So, we have the following theorem:

**Theorem 3.2.** A unit speed spacelike curve \( x: I \subset \mathbb{R} \rightarrow E^4_1 \) with nonzero curvatures \( k_1(s), k_2(s) \) and \( k_3(s) \) is a \( B_2 \)-slant helix if and only if there exists a \( C^2 \)-function \( f \) such that
Now, we give the integral characterization of the spacelike $B_2$-slant helix.

Suppose that, the unit speed spacelike curve $x: I \subset IR \rightarrow E^4_1$ with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ is a spacelike $B_2$-slant helix. Then theorem 3.2 is satisfied. Let us define $C^2$-function $\varphi$ and $C^1$-functions $m(s)$ and $n(s)$ by

$$\varphi = \varphi(s) = \int_{0}^{s} k_1(s)ds,$$

$$m(s) = \frac{k_3}{k_2} \eta(\varphi) + f(s) \mu(\varphi),$$

$$n(s) = \frac{k_3}{k_2} \mu(\varphi) - \epsilon_1 f(s) \eta(\varphi),$$

where $\eta(\varphi) = \cosh(\varphi)$, $\mu(\varphi) = \sinh(\varphi)$ if $\epsilon_1 = -1$; and $\eta(\varphi) = \cos(\varphi)$, $\mu(\varphi) = \sin(\varphi)$ if $\epsilon_1 = +1$. If we differentiate equations (23) with respect to $s$ and take account of (22) and (21) we find that $m' = 0$ and $n' = 0$. Therefore, $m(s) = C$, $n(s) = D$ are constants. Now substituting these in (23) and solving the resulting equations for $\frac{k_3}{k_2}$, we get

$$\frac{k_3}{k_2} = C\eta(\varphi) + D\mu(\varphi).$$

Conversely if (24) holds then from the equations in (23) we get

$$f = \epsilon_1(C\mu(\varphi) - D\eta(\varphi)),$$

which satisfies the conditions of Theorem 3.2. So, we have the following theorem:

**Theorem 3.3.** A unit speed spacelike curve $x: I \subset IR \rightarrow E^4_1$ with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ is a spacelike $B_2$-slant helix if and only if the following condition is satisfied

$$\frac{k_3}{k_2} = C\eta(\varphi) + D\mu(\varphi),$$

where $C$ and $D$ are constants, $\eta(\varphi) = \cosh(\varphi)$, $\mu(\varphi) = \sinh(\varphi)$ if $\epsilon_1 = -1$; and $\eta(\varphi) = \cos(\varphi)$, $\mu(\varphi) = \sin(\varphi)$ if $\epsilon_1 = +1$.

**Conclusions**

In this paper, the spacelike $B_2$-slant helix is defined and the characterizations of the spacelike $B_2$-slant helix are given in Minkowski 4- space $E^4_1$. It is shown that a spacelike curve $x: I \subset IR \rightarrow E^4_1$ is a $B_2$-slant helix if an equation holds between the first, second and third curvatures of the curve. Furthermore, the integral characterization of the spacelike $B_2$-slant helix is given.
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