Gauge conditions in combined dark energy and dark matter systems

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When analysing a system consisting of both dark matter and dark energy, an often used practice in the literature is to neglect the perturbations in the dark energy component. However, it has recently been argued, through the use of numerical simulations, that one cannot do so. In this work we show that by neglecting such perturbations one is implicitly making a choice of gauge. As such, one no longer has the freedom to choose, for example, a gauge comoving with the dark matter – in fact doing so will give erroneous, gauge dependent results. We obtain results consistent with the numerical simulations by using the formalism of cosmological perturbation theory, and thus without resorting to involved numerical calculations.

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I. INTRODUCTION

Cosmological data sets have increased in both their number and quality in recent years and we now hold a wealth of data about our universe. This allows us to confront the theory with observations and thus obtain a knowledge of the intricacies of the universe and its evolution better than ever before. Current data allows us to conclude that the majority of the energy density of the universe today is in the so called ‘dark sector’. That is, the universe is currently predominantly made up of dark energy – the mysterious force responsible for the accelerated expansion of the universe at the current epoch – and dark matter.\(^\dagger\)\(^\ddagger\)

The nature of dark energy is arguably one of the most important open questions in modern cosmology. An obvious ‘candidate’ for the dark energy component is the cosmological constant, \(\Lambda\). In fact, current observations are consistent with \(\Lambda CDM\), the standard cosmological model in which the dark energy is a cosmological constant.\(^\ddagger\)\(^\ddagger\) However, in light of the ‘cosmological constant problem,’ that the observed value of \(\Lambda\) differs from that predicted from fundamental theories by over a hundred orders of magnitude, the cosmological constant is perhaps a less favourable candidate for dark energy. Thus, much recent work has been focussed on a dynamical agent for the dark energy with a negative pressure which will cause the universe to accelerate in its expansion (see Ref. \[\ddagger\] and references contained within).

One of the most useful tools that theoretical cosmologists have at their disposal is cosmological perturbation theory. The basic premise is simple: since the universe appears homogeneous and isotropic on large scales, one starts with a Friedmann–Robertson–Walker spacetime as a background solution, and then adds small inhomogeneous perturbations on top. (For a selection of papers and reviews on the topic see, e.g., Ref. \[\ddagger\] and references therein). Since one perturbs the geometry of the background spacetime, through Einstein’s field equations, this invokes perturbations on its matter content. For a universe whose dark energy component is pure cosmological constant only the dark matter will be perturbed, but in the case of dynamical dark energy there will be perturbations to both the dark matter and the dark energy components. However, an often used practice in the literature (e.g. Ref. \[\ddagger\] and see references in \[\ddagger\]) is to ignore the perturbations in the dark energy and simply use the well-known evolution equation given by

\[
\ddot{\delta_c} + 2H\dot{\delta_c} - 4\pi G\rho_c\delta_c = 0, \tag{1.1}
\]

where \(\delta_c \equiv \delta\rho_c/\rho_c\) is the dark matter density contrast and an overdot denotes a time derivative. This issue was addressed in Ref. \[\ddagger\], where Park et. al showed, using numerical methods, that dark energy perturbations are not negligible and so, in general, cannot be ignored. In this paper we address a similar question. Using the formalism of cosmological perturbation theory, we show that ignoring the dark energy perturbations is in fact a gauge choice, and so if we choose to set the dark energy perturbations to zero then we no longer have the freedom to choose a gauge in which the dark matter evolution equation takes the form of Eq. \(\ddagger\) – we are still left with a system of coupled differential equations. Since this work focuses on the formalism of cosmological perturbation theory, and mathematics, and not involved numerical calculations, we make no numerical approximations in obtaining this result.

The paper is structured as follows. In the next section we derive and present the governing equations for the dark energy and dark matter system under consideration. This is followed, in Section \[\ddagger\] by a brief recap of the gauge transformations of matter and metric perturbations. Finally, in Section \[\ddagger\] we present our results and conclude.
II. GOVERNING EQUATIONS

We consider here only scalar perturbations to a flat ($K = 0$) Friedmann-Robertson-Walker spacetime with the line element
\[
d s^2 = -(1 + 2\phi) dt^2 + 2a^2 \delta_{ij} dx^i dx^j + a^2 (1 - 2\psi) \delta_{ij} + 2E_{ij} dx^i dx^j ,
\]  
(2.1)
where $a = a(t)$ is the scale factor, $\phi$ is the lapse function, $\psi$ is the dimensionless curvature perturbation and $E$ and $B$ make up the scalar shear, $\sigma$, as
\[
\sigma \equiv a^2 \dot{E} - aB.
\]  
(2.2)
All perturbations are functions of space and time (e.g. $\phi \equiv \phi(x^i, t)$). Throughout this paper, Latin indices $i, j, k$, take the value 1, 2 or 3, Greek indices $\mu, \nu, \lambda$, cover the full spacetime indices and we denote a partial derivative with a subscript comma.

We assume that the dark matter and dark energy are non-interacting, and so energy momentum conservation for each fluid gives
\[
\nabla_{\mu} T^\mu_{(\alpha)\nu} = 0 .
\]  
(2.4)
Then, we obtain an evolution equation for each fluid from the energy (temporal) component of Eq. (2.4),
\[
\delta \rho_\alpha + 3H(\delta \rho_\alpha + \delta P_\alpha) + (\rho_\alpha + P_\alpha) \frac{\nabla^2}{a^2}(V_\alpha + \sigma) = 3(\rho_\alpha + P_\alpha)\dot{\psi} ,
\]  
(2.5)
where the covariant velocity potential of each fluid is defined as
\[
V_\alpha = a(v_\alpha + B) ,
\]  
(2.6)
$v_\alpha$ is the scalar velocity potential of the $\alpha$th fluid and $H = \dot{a}/a$ is the Hubble parameter.

Considering the dark matter fluid and the dark energy scalar field, respectively, Eq. (2.5) then gives
\[
\dot{\delta} + 3\dot{H} \delta + \left( U_{\phi} - \frac{\nabla^2}{a^2}\right) \delta = 0 ,
\]  
(2.7)
\[
\dot{\phi} + 3H\dot{\phi} + \left( U_{\phi} - \frac{\nabla^2}{a^2}\right) \phi = 3\psi - \frac{\nabla^2}{a^2}\phi + 2U_{\phi}\dot{\phi} ,
\]  
(2.8)
where we have treated the field as a perfect fluid with energy density and pressure
\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + U_{\phi} ,
\]  
(2.9)
\[
P_\phi = \frac{1}{2} \dot{\phi}^2 - U_{\phi} .
\]  
(2.10)
We have also used the background evolution equation for the dark energy scalar field
\[
\ddot{\phi} + 3H\dot{\phi} + U_{\phi} = 0 ,
\]  
(2.11)
and the fact that
\[
V_\phi = -\frac{\delta \phi}{\dot{\phi}} .
\]  
(2.12)
There is also a momentum conservation equation, coming from the spatial component of Eq. (2.4), corresponding to each fluid
\[
\dot{V}_\alpha - 3Hc^2_\alpha V_\alpha + \phi + \frac{\delta P_\alpha}{\rho_\alpha + P_\alpha} = 0 ,
\]  
(2.13)
where $c^2_\alpha = \dot{P}_\alpha/\rho_\alpha$.

The Einstein field equations give
\[
3H(\ddot{\psi} + H\dot{\psi}) - \frac{\nabla^2}{a^2}(\psi + H\sigma) + 4\pi G\delta \rho = 0 ,
\]  
(2.14)
\[
\ddot{\psi} + H\dot{\psi} + 4\pi G(\rho + P)V = 0 ,
\]  
(2.15)
\[
\ddot{\sigma} + H\dot{\sigma} - \dot{\psi} + \psi = 0 ,
\]  
(2.16)
\[
\ddot{\phi} + 3H\dot{\phi} + (3H^2 + 2\dot{H})\phi - 4\pi G\delta \rho P = 0 ,
\]  
(2.17)
where the total matter quantities are defined as the sum of the quantity for each fluid/field, i.e.
\[
\delta \rho = \delta \rho_\phi + \delta \rho_c ,
\]  
(2.18)
\[
\delta P = \delta P_\phi ,
\]  
(2.19)
\[
(\rho + P)V = (\rho_\phi + P_\phi)V_\phi + \rho_c V_c ,
\]  
(2.20)
and we have used the fact that the dark matter is pressureless, i.e. $P_c = 0 = \delta P_c$. Note that, by treating the scalar field as a fluid, we can write the energy density and pressure perturbation for the dark energy, respectively, as
\[
\delta \rho_\phi = \phi \delta \phi - \phi \dot{\phi}^2 + U_{\phi}\delta \phi ,
\]  
(2.21)
\[
\delta P_\phi = \phi \delta \phi - \phi \dot{\phi}^2 - U_{\phi}\delta \phi .
\]  
(2.22)
Introducing a new variable $Z$, both for notational convenience and to assist with the following calculations, defined as
\[
Z \equiv 3(\ddot{\psi} + H\dot{\psi}) - \frac{\nabla^2}{a^2}\phi ,
\]  
(2.23)
we can rewrite Eq. (2.17), using Eqs. (2.14) and (2.10), as
\[
\ddot{Z} + 2H\dot{Z} + \left( 3\ddot{H} + \frac{\nabla^2}{a^2}\right) \phi = 4\pi G(\delta \rho + 3\delta P) .
\]  
(2.24)

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1 This assumption is for brevity, and to allow us to deal with more manageable equations. The results highlighted in this work will still hold in the case of interacting fluids for which the overall energy-momentum tensor is covariantly conserved, but the components obey
\[
\nabla_\mu T^\mu_{(\alpha)\nu} = Q_{(\alpha)\nu} ,
\]  
(2.3)
where $Q_{(\alpha)\nu}$ is the energy-momentum transfer to the $\alpha$th fluid.

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\[ Z = \dot{\xi} + 3H\phi + \frac{\nabla^2}{a^2}V_c, \quad (2.25) \]
and from Eq. (2.13) for the dark matter fluid (for which \( c_s^2 = 0 \)),
\[ \phi = -\dot{V}_c. \quad (2.26) \]

Differentiating Eq. (2.25) gives
\[ \dot{Z} = \dot{\xi} + 3\dot{H}\phi + \frac{\nabla^2}{a^2}\left(\dot{V}_c - 2HV_c\right). \quad (2.27) \]

Substituting this into Eq. (2.25) gives
\[ \dot{\xi} + 2H\dot{\xi} - 4\pi G\rho_\xi\dot{\xi} = 8\pi G(2\dot{\phi}\delta\phi - U_{,\phi}\delta\phi) \]
\[ + \dot{V}_c\left(6\dot{H} + H^2 + 16\pi G\dot{\phi}^2\right) + 3H\dot{V}_c. \quad (2.28) \]

The evolution equation for the field is then obtained solely in terms of matter perturbations from Eq. (2.28) by using Eqs. (2.26) and (2.27):
\[ \ddot{\phi} + 3H\dot{\phi} + \left(U_{,\phi}\phi - \frac{\nabla^2}{a^2}\right)\phi \quad (2.29) \]
\[ = \dot{\phi}\left(\dot{\xi} + \frac{\nabla^2}{a^2}V_c - \dot{V}_c\right) + 2U_{,\phi}\dot{V}_c. \]

It is worth restating that we have derived these equations in a general form without fixing a gauge. If we set the dark matter velocity to zero, i.e. \( V_c = 0 \), then they reduce to those presented in, for example, Refs. [14, 17] (for the case of a zero energy-momentum transfer).

### III. GAUGE TRANSFORMATIONS

As mentioned above, so far the equations have been presented without choosing a gauge. In order to choose a gauge, we need to first consider how the variables change under a gauge transformation, that is, a change of coordinates in the physical spacetime while holding the background coordinates fixed. Recall that, in general the gauge transformation of an arbitrary tensor field \( T \) is given by the exponential map [12, 18]
\[ \tilde{T} = e^{L_\xi}T. \quad (3.1) \]

This can then be expanded to give the gauge transformation at linear order in perturbation theory as
\[ \delta\tilde{T} = \delta T + \mathcal{L}_\xi T_0, \quad (3.2) \]
where \( T_0 \) is the value of the tensor field \( T \) in the background and \( \mathcal{L}_\xi \) denotes the Lie derivative with respect to the gauge transformation generating vector, \( \xi^\mu \). This can be split up as
\[ \xi^\mu = (\alpha, \beta^i), \quad (3.3) \]
such that the coordinates transform as
\[ \tilde{t} = t - \alpha, \quad (3.4) \]
\[ \tilde{x}^i = x^i - \beta^i. \quad (3.5) \]

Then, one can show that scalar quantities such as the field perturbation transform as
\[ \tilde{\phi} = \phi + \dot{\phi}, \quad (3.6) \]
the density contrast transforms as
\[ \tilde{\rho} = \rho + \dot{\rho}, \quad (3.7) \]
and the components of the velocity potential as
\[ \tilde{v}_\alpha = v_\alpha - \alpha. \quad (3.8) \]

Furthermore, by considering the transformation behaviour of the metric tensor gives us the following transformation rules for the scalar metric perturbations
\[ \tilde{\psi} = \psi - H\alpha, \quad (3.9) \]
\[ \tilde{B} = B - \frac{1}{a}\alpha + a\beta, \quad (3.10) \]
\[ \tilde{E} = E + \beta, \quad (3.11) \]
and so the scalar shear transforms as
\[ \tilde{\sigma} = \sigma + \alpha. \quad (3.12) \]

Finally Eq. (3.8) along with Eq. (2.6), gives the transformation behaviour of the components of the covariant velocity potential
\[ \tilde{V}_\alpha = V_\alpha - \alpha. \quad (3.14) \]

### IV. DISCUSSION AND CONCLUSIONS

When splitting the spacetime into a background and a perturbation, as is done in cosmological perturbation theory, one introduces a gauge problem. That is, there exist spurious gauge modes which must be removed in order to draw physically meaningful conclusions. This problem arises because, while general relativity is a fully covariant theory, this splitting is not a covariant process (see, e.g., Ref. [18]). In order to resolve this issue Bardeen proposed a systematic method of dealing with the gauge modes, by considering the transformation behaviour of quantities and constructing gauge invariant variables [19]. We follow that approach here, by considering the gauge transformations of perturbations presented above in Section III and making gauge choices such that we can remove the components of the generating vector \( \alpha \) and \( \beta \). Any remaining quantities will then be gauge invariant.
Turning now to the case at hand, choosing the gauge in which the perturbation in the dark energy field is zero, \( \delta \phi = 0 \), fixes \( \alpha \) as

\[
\alpha = -\frac{\delta \varphi}{\varphi}. \tag{4.1}
\]

Since none of the gauge transformations of the quantities involved in the governing equations depend upon \( \beta \), we do not need to consider fixing this explicitly here. (Of course, one can rigorously fix \( \beta \) by choosing a suitable gauge condition, for example, by setting \( \tilde{E} = 0 \).) The governing equations in this gauge are then

\[
\ddot{\delta}_c + 2H \dot{\delta}_c - 4\pi G \rho_c \delta_c = \dot{\hat{V}}_c \left( 6(\dot{H} + H^2) + 16\pi G \rho^2 \right) + 3H \ddot{\delta}_c, \tag{4.2}
\]

\[
\ddot{\hat{V}}_c - \frac{2U_c}{\varphi} \dot{\hat{V}}_c - \frac{\nabla^2}{a^2} \hat{V}_c = \dot{\delta}_c, \tag{4.3}
\]

where the hat denotes that the variables are evaluated in the uniform field fluctuation gauge. That is, in this gauge, \( \hat{\delta}_c \) and \( \hat{V}_c \) are gauge invariant variables defined as

\[
\hat{\delta}_c = \delta_c - \frac{\rho_c}{\varphi} \delta \varphi, \quad \hat{V}_c = V_c + \frac{\delta \varphi}{\varphi}. \tag{4.4}
\]

Alternatively, choosing a gauge comoving with the dark matter, in which \( \tilde{V}_c = 0 \) fixes the generating vector as

\[
\alpha = V_c, \tag{4.5}
\]

and reduces the governing equations to

\[
\ddot{\delta}_c + 2H \dot{\delta}_c - 4\pi G \rho_c \delta_c = 8\pi G(2\dot{\varphi} \delta \varphi - U_c \delta \varphi), \tag{4.6}
\]

\[
\ddot{\varphi} + 3H \dot{\varphi} + \left( \frac{U_c \varphi}{2a^2} \right) \delta \varphi = \varphi \delta_c, \tag{4.7}
\]

where the bar denotes variables in the comoving gauge and we have

\[
\bar{\delta}_c = \delta_c + \frac{\rho_c}{\bar{\rho}_c} V_c, \quad \bar{\delta} \varphi = \delta \varphi + \varphi V_c. \tag{4.8}
\]

By studying the above systems of equations, it is evident that choosing the dark energy field perturbation to be zero is a well defined choice of gauge, reducing the governing equations to Eqs. (4.2) and (4.3). Then, having done so, we are no longer allowed the freedom to make another choice of gauge. Alternatively, choosing a gauge comoving with the dark matter uses up the gauge freedom, and so we are not permitted to neglect the perturbation in the dark energy field. In fact, doing so will result in erroneous gauge dependent results. It is clearest to see why this is the case by considering the set of governing equations. By making our choice of gauge we are left with a set of equations which is gauge invariant: that is, performing a gauge transformation will leave the set of equations unchanged. However, by neglecting the perturbation in the dark energy field after having chosen the gauge comoving with the dark matter amounts to setting the right hand side of Eq. (4.6) to zero. This resulting equation will, in general, then no longer be gauge invariant.

Thus, we conclude that the dark energy perturbation must be considered in a system containing a mixture of dark matter and dark energy. Our result is consistent with that of Ref. [10], though we have shown this by simply using the formalism of cosmological perturbation theory instead of relying on more involved numerical calculations.

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