THE UNRAMIFIED BRAUER GROUP OF NORM ONE
TORI

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ABSTRACT. Let $k$ be a number field and $K/k$ Galois. We transform
the construction of the unramified Brauer group of the norm one torus
$R_{K/k}^1(G_m)$ into the construction of a special abelian extension over
$K$. If $k = Q$ and $K/Q$ biquadratic, we explicitly construct the unramified
Brauer group of $R_{K/Q}^1(G_m)$.

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1. Introduction

Let $T$ be a torus over a field $k$ of characteristic zero, $X$ a principal homoge-
nous space of $T$, and $X^c$ a smooth compactification of $X$. Since the Brauer
group $\text{Br}(X^c) = H^2_{\text{ét}}(X^c, G_m)$ is a birational invariant of smooth proper vari-
ceties, which does not depend on the choice of $X^c$ but only depends on $X$;
it is called the unramified Brauer group of $X$. Let $\text{Br}_0(X^c)$ be the image
of the natural map $\text{Br}(k) \to \text{Br}(X^c)$. Formulas for $\text{Br}(X^c)/\text{Br}_0(X^c)$ can be
found in [3]. Specially, if $K/k$ is a Galois extension and $T = R_{K/k}^1(G_m)$
its norm one torus, then $\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z})$. If $K/k$ is
cyclic, then $\text{Br}(X^c) = \text{Br}_0(X^c)$.

It is well known that the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only one for $X^c$ ([9]). To compute the Brauer-Manin obstruction, one need to construct the Brauer
group. Recently, Colliot-Thélène ([2]) gave an explicit construction for a multi-norm torus of dimension 5. However, for general tori, it is still open,
even for the norm one torus $R_{K/Q}^1(G_m)$, where $K/Q$ biquadratic.

The main aim of this article is to construct the unramified Brauer group
for norm one tori $R_{K/k}^1(G_m)$ with $K/k$ Galois. In §2, we show any element
in the unramified Brauer group of $R_{K/k}^1(G_m)$ has a form from cup-product.
Furthermore, we transform the construction of the unramified Brauer group into the construction of a special abelian extension over $K$ (see Theorem 1). Some examples are also given in this section. In §3, using results of double covering of $Q^{ab}$ in [16][13], we give the explicit construction of the unramified Brauer group for the torus $R_{1/K/Q}^1(G_m)$, where $K/Q$ biquadratic.

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2. The Brauer group of $R_{1/K/k}^1(G_m)$ when $K/k$ Galois

Let $k$ be a field with $\text{char}(k) = 0$. Let $K/k$ be a field extension and $T = R_{1/K/k}^1(G_m)$ its norm one torus. Let $X$ be the affine $k$-variety defined by $N_{K/k}(\Xi) = n \in k^\times$, which is a principal homogeneous space of $T$. And $\overline{k}[X]^\times/\overline{k}^\times \cong \widehat{T}$ as a $\text{Gal}(\overline{k}/k)$-module, where $\widehat{T}$ is the character group of $T$.

Let $X^c$ be a smooth compactification of $X$. Let $\text{Br}(X)$ (resp. $\text{Br}(X^c)$) be the Brauer group of $X$ (resp. $X^c$). Let $\text{Br}_0(X)$ (resp. $\text{Br}_0(X^c)$) be the image of $\text{Br}(k)$ in $\text{Br}(X)$ (resp. $\text{Br}_0(X^c)$).

By the Hochschild-Serre Spectral sequence, we have

$$0 \to H^1(k, \overline{k}[X]^\times) \to \text{Pic}(X) \to \text{Pic}(\overline{X})^{\text{Gal}(\overline{k}/k)} \to H^2(k, \overline{k}[X]^\times)$$

$$\to \text{Ker}[\text{Br}(X) \to \text{Br}(\overline{X})] \to H^1(k, \text{Pic}(\overline{X})).$$

Since $\overline{X} \cong \mathbb{G}_m^d$ over $\overline{k}$, it implies $\text{Pic}(\overline{X}) = 0$, where $d = [K : k] - 1$. Therefore

$$H^2(k, \overline{k}[X]^\times) \cong \text{Ker}[\text{Br}(X) \to \text{Br}(\overline{X})]. \quad (1)$$

Since $X^c$ is geometrically rational, we have $\text{Br}(\overline{X^c}) = 0$. This implies the following lemma

Lemma 1. $\text{Br}(X^c)$ is contained in the image of $H^2(k, \overline{k}[X]^\times)$ in $\text{Br}(X)$.

Let $\Gamma_k = \text{Gal}(\overline{k}/k)$ and $\Gamma_K = \text{Gal}(\overline{k}/K)$. Denote $\mathbb{Z}[K/k] =: \mathbb{Z}[\Gamma_k/\Gamma_K]$.

Let $i \geq 0$, we have the cup product

$$(\cdot, \cdot) : \mathbb{Z}[K/k] \times H^i(K, \mathbb{Z}) \to H^{i+1}(K, \mathbb{Z}[K/k]).$$

Let $\text{Cor}_{K/k}^i$ be the corestriction map $H^i(K, \cdot) \to H^i(k, \cdot)$.

Lemma 2. Let $\Gamma_K \in \Gamma_k/\Gamma_K$. Then $\text{Cor}_{K/k}(\Gamma_K, \cdot) : H^i(K, \mathbb{Z}) \to H^i(k, \mathbb{Z}[K/k])$ is the inverse map of the Shapiro’s isomorphism $\text{sh} : H^i(k, \mathbb{Z}[K/k]) \to H^i(K, \mathbb{Z})$.

Proof. The case $i = 0$ is obvious. Then we only need to consider the case $i > 0$. Denote $g = \text{Cor}_{K/k}(\Gamma_K, \cdot)$. Since $f$ is an isomorphism, we only need to show $g \cdot \text{sh} = \text{id}$ or $\text{sh} \cdot g = \text{id}$.

In the following, we will show that $g \cdot \text{sh} = \text{id}$. Let $C^i(\Gamma_k, \mathbb{Z}) \to \mathbb{Z}$ (resp. $C^i(\Gamma_k, \mathbb{Z}[K/k]) \to \mathbb{Z}[K/k]$) be a continuous $\Gamma_k$-module resolution of $\mathbb{Z}$ (resp. $\mathbb{Z}[K/k]$). Suppose $x \in H^i(k, \mathbb{Z}[K/k])$, choose an $i$-cocycle $u \in C^i(\Gamma_k, \mathbb{Z}[K/k])$ which represents $x$. Hence

$$\langle g \cdot \text{sh}(u)(\sigma) \rangle = g(\text{id}_{\Gamma_K}(u(\sigma))) = \text{Cor}_{K/k}(\text{id}_{\Gamma_K}(u(\sigma)))_{\Gamma_K}$$

$$= \sum_{\gamma \in \Gamma_K} \text{id}_{\Gamma_K}(u(\gamma^{-1}\sigma)) \gamma_{\Gamma_K},$$

for all $\gamma \in \Gamma_K$. Therefore

$$g \cdot \text{sh}(u)(\sigma) = \text{id}(u(\sigma)) \gamma_{\Gamma_K},$$

for all $\gamma \in \Gamma_K$. This implies $g \cdot \text{sh} = \text{id}$.
where \( j_{\Gamma_K} \) is the projection \( \mathbb{Z}[K/k] \rightarrow \mathbb{Z} \) by \( \Gamma_K \mapsto 1, \gamma \Gamma_K \mapsto 0 \) if \( \gamma \notin \Gamma_K \). Similarly we can define \( j_{\gamma \Gamma_K} \) for any \( \gamma \Gamma_K \in \Gamma_k/\Gamma_K \). Since \( u \) is \( \Gamma_k \)-linear, we have

\[
(g \cdot sh(u))(\sigma) = \sum_{\gamma \Gamma_K \in \Gamma_k/\Gamma_K} j_{\Gamma_K}(\gamma^{-1}u(\sigma))\gamma \Gamma_K
\]

\[
\quad = \sum_{\gamma \Gamma_K \in \Gamma_k/\Gamma_K} j_{\gamma \Gamma_K}(u(\sigma))\gamma \Gamma_K = u(\sigma).
\]

\[\square\]

In the reminder of this paper we always assume that \( k \) is a field of characteristic zero and \( H^3(k, \mathbb{Z}) = 0 \), eg. \( k \) is a number field or \( p \)-adic number field, see [7] Corollary 4.7. The following lemma will show any element of \( \text{Br}(X) \) has the form \( \text{Cor}_{K/k}(\Xi, \chi) \).

**Lemma 3.** Each element of \( \text{Br}(X)/\text{Br}_0(X) \) in the image of \( H^2(k, \bar{k}[X]^\times) \) is of the form \( \text{Cor}_{K/k}(\Xi, \chi) \), where \( \chi \) is a character of \( \Gamma_K \) and \( \Xi \in \{k[X]^\times \} \) is a \( K \)-‘variable’. And \( \text{Cor}_{K/k}(\Xi, \chi) = 0 \in \text{Br}(X)/\text{Br}_0(X) \) if and only if \( \chi \) is the restriction of a character of \( \text{Gal}(\bar{k}/k) \).

**Proof.** Using the natural exact sequence

\[0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[K/k] \rightarrow \hat{T} \rightarrow 0,\]

and the assumption \( H^3(k, \mathbb{Z}) = 0 \), we deduce

\[H^2(k, \mathbb{Z}) \rightarrow H^2(k, \mathbb{Z}[K/k]) \,\xrightarrow{g}\, H^2(k, \hat{T}) \rightarrow H^3(k, \mathbb{Z}) = 0.\]

(2)

Define the \( \text{Gal}(\bar{k}/k) \)-morphism \( j : \mathbb{Z}[K/k] \rightarrow \bar{k}[X]^\times \) by \( \Gamma_K \mapsto \Xi \). Then we have the maps \( \mathbb{Z}[K/k] \rightarrow \bar{k}[X]^\times \rightarrow \bar{k}[X]^\times / \bar{k}^\times \) (\( \cong \hat{T} \)). Hence the following diagram is commutative

\[\begin{array}{ccc}
H^2(k, \mathbb{Z}[K/k]) & \xrightarrow{g} & H^2(k, \bar{k}[X]^\times) \\
\downarrow{f} & & \downarrow{f} \\
H^2(k, \hat{T}) & & \\
\end{array}\]

By the sequence (2), \( g \) is surjective. Hence \( f \) is also surjective. Write \( \text{Ima}[\text{Br}(k)] \) to be the image of \( \text{Br}(k) \) in \( H^2(k, \bar{k}[X]^\times) \). There is the following isomorphism

\[H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)] \cong H^2(k, \hat{T}).\]

(4)

By Lemma 2 and the functoriality of cup product, we have the following commutative diagram

\[\begin{array}{ccc}
H^2(k, \mathbb{Z}) & \xrightarrow{g_1} & H^2(K, \mathbb{Z}) \\
\downarrow{\text{Res}_{k/K}} & & \downarrow{\text{Res}_{k/K}} \\
H^2(k, \mathbb{Z}) & \xrightarrow{g_2} & H^2(k, \bar{k}[X]^\times) \\
\end{array}\]

where \( g_1 = \text{Cor}_{K/k}(\Gamma_K, \cdot) \) is the inverse of the Shapiro’s isomorphism (by Lemma 2) and \( g_2 = \text{Cor}_{K/k}(\Xi, \cdot) \). Then the above exact sequence (2) identifies with

\[H^2(k, \mathbb{Z}) \xrightarrow{\text{Res}_{k/K}} H^2(K, \mathbb{Z}) \xrightarrow{h} H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)] \rightarrow 0,\]

(5)
where \( h = \rho \cdot \text{Cor}_{K/k}(\Xi, \cdot) \) and \( \rho \) is the quotient map

\[
H^2(k, \bar{k}[X]^\times) \rightarrow H^2(k, \bar{k}[X]^\times) / \text{Ima}[\text{Br}(k)].
\]

Using the canonical isomorphism \( H^1(K, \mathbb{Q}/\mathbb{Z}) \cong H^2(K, \mathbb{Z}) \), each element of \( H^2(k, \bar{k}[X]^\times) / \text{Ima}[\text{Br}(k)] \) is of the form \( \text{Cor}_{K/k}(\Xi, \chi) \) by \((9)\), where \( \chi \) is a character of \( \Gamma_K \). Then each element of \( \text{Br}(X)/\text{Br}_0(X) \) in the image of \( H^2(k, \bar{k}[X]^\times) \) is of the form \( \text{Cor}_{K/k}(\Xi, \chi) \).

Since the map \( H^2(k, \bar{k}[X]^\times) \rightarrow \text{Br}(X) \) is injective by \((1)\), the map

\[
H^2(k, \bar{k}[X]^\times) / \text{Ima}[\text{Br}(k)] \rightarrow \text{Br}(X)/\text{Br}_0(X)
\]

is also injective. By the exact sequence \((3)\), we immediately have \( \text{Cor}_{K/k}(\Xi, \chi) \) is zero in \( \text{Br}(X)/\text{Br}_0(X) \) if and only if \( \chi \) is the restriction of a character of \( \text{Gal}(\bar{k}/k) \).

\(\square\)

Suppose \( K/k \) is Galois. We say a field extension \( L/K \) satisfies the condition \((*)\) if:

\( L/F \) is abelian for any subfield \( F \subset K \) satisfying \( K/F \) is cyclic.

Obviously \( L/K \) is abelian if \( L/K \) satisfies the condition \((*)\).

**Lemma 4.** Suppose \( K/k \) is Galois and \( L/K \) satisfies the condition \((*)\). Then \( L/k \) is Galois and \( \text{Gal}(L/K) \) is contained in the center of \( \text{Gal}(L/k) \).

**Proof.** We will show \( \sigma(L) = L \) for any \( \sigma \in \text{Gal}(\bar{k}/k) \). Let \( K^\sigma \) be the fixed subfield of \( \sigma \) in \( K \). Obviously \( K/K^\sigma \) is cyclic, then we have \( L/K^\sigma \) is abelian by the condition \((*)\). Therefore \( \sigma(L) = L \). So \( L/k \) is Galois.

Let \( \sigma \in \text{Gal}(L/K) \). For any \( g \in \text{Gal}(L/k) \), let \( K^g \) be the subfield of \( K \) fixed by \( g \). Hence \( K/K^g \) is cyclic. By the condition \((*)\), \( \text{Gal}(L/K^g) \) is abelian. This implies \( \sigma g = g \sigma \) since \( \sigma, g \in \text{Gal}(L/K^g) \). So \( \sigma \) is contained in the center of \( \text{Gal}(L/k) \).

\(\square\)

**Theorem 1.** Suppose \( K/k \) is Galois. Let \( \chi \) be a character of \( \text{Gal}(\bar{k}/K) \).

Then:

1. All elements of \( \text{Br}(X^c)/\text{Br}_0(X^c) \) are of the form \( \text{Cor}_{K/k}(\Xi, \chi) \).
2. \( \text{Cor}_{K/k}(\Xi, \chi) \in \text{Br}(X^c) \) if and only if \( \chi \) can factor through an abelian extension \( L/K \) which satisfies the condition \((*)\).

**Proof.** The part (1) follows from Lemma\((1)\) and \((3)\). So we only need to prove part (2)

Let \( k(X) \) be the function field of \( X \). Let \( A \) be a discrete valuation ring containing \( k \) with fraction field \( k(X) \) and residue field \( \kappa_A \). There is a residue map

\[
\partial_A : \text{Br}(k(X)) \rightarrow H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).
\]

By Grothendieck’s purity theorem, we have

\[
\text{Br}(X^c) = \bigcap_A \text{Ker}(\partial_A) \subset \text{Br}(k(X)),
\]

where \( A \) runs through all above discrete valuation rings.

Suppose \( L/K \) satisfies the condition \((*)\). Let \( \chi \) be a character of \( \text{Gal}(\bar{k}/K) \) which factors through \( \text{Gal}(L/K) \). Let \( B = \text{Cor}_{K/k}(\Xi, \chi) \in \text{Br}(X) \), we will
prove $B \in \text{Br}(X^c)$. Assume it is not, then there is a discrete valuation ring $A$ such that
$$\partial_A(B) \neq 0 \in H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).$$
Then there is an element $g \in \text{Gal}(\bar{\kappa}_A/\kappa_A)$ such that
$$\partial_A(B)(g) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$  
Since $k \subset \kappa_A$, we can fix an embedding $\bar{k} \hookrightarrow \bar{\kappa}_A$. Let $K^g$ be the fixed field of $g$ in $K$. Let $f$ be the natural map $\text{Br}(X) \to \text{Br}(X_{K^g})$. Since $K^g(X) = K^g.k(X)$ is a finite unramified extension over $k(X)$, there is a discrete valuation ring $A_g \in K^g(X)$ which extends $A$. Then we have $\kappa_{A_g} = \kappa_A.K^g$ and $g \in \text{Gal}(\kappa_A/\kappa_{A_g})$. By Proposition 1.1.1 in [5], we have
$$\partial_A(B)(g) = \partial_{A_g}(f(B))(g).$$
Therefore $\partial_{A_g}(f(B)) \neq 0$.
Let
$$G = \text{Gal}(\bar{k}/k), U = \text{Gal}(\bar{k}/K^g) \text{ and } H = \text{Gal}(\bar{k}/K).$$
Choose a representation of double coset of $G$
$$G = \bigsqcup_{\sigma} U\sigma H.$$  
(6)
By [8, Proposition 1.5.6], we have
$$\text{Res}_{G/U} \cdot \text{Cor}_{H/G} = \sum_{\sigma} \text{Cor}_{U\cap_\sigma H\sigma^{-1}/U} \cdot \sigma_* \cdot \text{Res}_{H/H\cap_\sigma U\sigma^{-1}},$$
where $\sigma$ runs through all elements in (6). Note that $H$ is a normal subgroup of $G$ and contained in $U$, then we have
$$\text{Res}_{G/U} \cdot \text{Cor}_{H/G} = \sum_{\sigma} \text{Cor}_{H/U} \cdot \sigma_*.$$  
Therefore
$$f(B) = \text{Res}_{G/U} \cdot \text{Cor}_{H/G}(\Xi, \chi) = \sum_{\sigma} \text{Cor}_{H/U} \cdot \sigma_*(\Xi, \chi) = \sum_{\sigma} \text{Cor}_{H/U}(\Xi^\sigma, \chi^\sigma).$$
Note that $K/K^g$ is cyclic, we have $\text{Gal}(L/K^g)$ is abelian by the condition $(\ast)$. Then we can choose a character $\hat{\chi}$ of $\text{Gal}(\bar{k}/K^g)$ which factors through $\text{Gal}(L/K^g)$ and lifts $\chi$. Then
$$f(B) = \sum_{\sigma} \text{Cor}_{H/U}(\Xi^\sigma, \text{Res}_{U/H}(\hat{\chi})) = \sum_{\sigma}(N_{K/K^g}(\Xi^\sigma), \hat{\chi}) = (N_{K/k}(\Xi), \hat{\chi}) = (n, \hat{\chi}).$$
Obviously
$$\partial_{A_g}(f(B)) = v_{A_g}(n)\hat{\chi} = 0,$$
where \( g \) 

Hence

**Proof.** We know \( Br(L/k) \)

Example 1. Let \( K = \mathbb{Q} (\sqrt[3]{-1}, \sqrt[3]{7}) \). Then the equation \( N_{K/Q}(\Xi) = n \) is solvable over \( \mathbb{Q} \) if and only if the following conditions hold:
Proposition 2. We immediately have the following result:

\[
1 \quad \text{for a rational solution } (\Xi, \chi) \text{ of } \chi_{\bar{n}} = (\Xi, \chi)_{\bar{n}}.
\]

Example 2. Let \( K = k(\sqrt{d_1}, \sqrt{d_2}) \) be as above. Then \( \text{Cor}_{K/k}(\Xi, \chi) \) is the unique generator of \( \text{Br}(X^c)/\text{Br}_0(X^c) \).

Finally we will use Proposition 2 to give an explicit example. Write \( n = (-1)^{s_0}2^{s_1}13^{s_2}17^{s_3}p_1^{e_1} \cdots p_y^{e_y} \in \mathbb{Q}^\times \). Let \( D(n) = \{p_1, \ldots, p_y\} \) and \( n_1 = p_1^{e_1} \cdots p_y^{e_y} \). Let \( K = \mathbb{Q}(\sqrt{13}, \sqrt{17}) \). Obviously \( 15^2 - 13 \cdot 4^2 = 17 \). Denote

\[
D_1 = \{p \in D(n) : \left(\frac{13}{p}\right) = -1\}
\]

\[
D_2 = \{p \in D(n) : \left(\frac{17}{p}\right) = 1 \text{ and } \left(\frac{15 + 4\sqrt{13}}{p}\right) = -1\}.
\]

Example 2. Let \( K = \mathbb{Q}(\sqrt{13}, \sqrt{17}) \). Then the equation \( N_{K/\mathbb{Q}}(\Xi) = n \) is solvable over \( \mathbb{Q} \) if and only if the following conditions hold:

(1) \( s_0 \) is even; \( \left(\frac{n}{13}\right) \cdot (-1)^{s_1} = \left(\frac{n}{17}\right) = 1 \); and \( e_i \) is even when \( K/k \) is not totally split over \( p_i \).

(2) \( (-1)^{\sum p_i e_i/2 + \sum p_i e_i} = (-1)^{s_1 + s_2} \cdot \left(\frac{n}{13}\right) \).

Remark 1. Let \( k \) be a number field and \( K = k(\sqrt{a}, \sqrt{b}) \) a biquadratic field. Sansuc (\[10, Proposition 6\]) gave a method to determine the existence of rational points for \( X \) or not. For each \( n \in \mathbb{Q}^\times \), his method need to look for a rational solution \( (\Xi_1, \Xi_2) \in k(\sqrt{a})^\times \times k(\sqrt{b})^\times \) of the equation

\[
N_{k(\sqrt{a})/k}(\Xi_1) \cdot N_{k(\sqrt{b})/k}(\Xi_2) = n.
\]

3. The case that \( k = \mathbb{Q} \) and \( K/\mathbb{Q} \) biquadratic

In \[3,7\], we will recall some results of double covering of \( \mathbb{Q}^{ab}/\mathbb{Q} \). In \[3,2\], the explicit construction for the biquadratic case will be given using Theorem 1 in \[2\] and double coverings in \[3,7\].

3.1. Double covering of \( \mathbb{Q}^{ab} \). Suppose \( K/F \) is Galois. A double covering of \( K/F \) (defined in \[6\]) is an extension \( \hat{K}/K \) of degree \( 2 \) such that \( K/F \) is Galois. Let \( \mathbb{Q}^{ab} \) be the maximal abelian extension of \( \mathbb{Q} \). In the following we will describe all double coverings of \( \mathbb{Q}^{ab} \) (see \[1,6\]) and of the cyclotomic field \( \mathbb{Q}(\zeta_n) \) (see \[13\]).

Let \( \mathcal{A} \) be the free abelian group on the symbols of the form \( [a] \) \( (a \in \mathbb{Q}) \) modulo the identifications

\[
[a] = [b] \iff a - b \in \mathbb{Z}.
\]
For all odd primes \( p < q \), put
\[
a_{pq} = \sum_{i=1}^{\frac{q-1}{2}} \left( \left[ \frac{i}{p} \right] - \sum_{k=0}^{\frac{p-1}{2}} \left[ \frac{i}{pq} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left( \left[ \frac{j}{q} \right] - \sum_{l=0}^{\frac{q-1}{2}} \left[ \frac{j}{pq} + \frac{l}{p} \right] \right)
\]
and for prime \( q > 2 \), put
\[
a_{2q} = \left( \left[ \frac{1}{4} \right] - \sum_{k=0}^{\frac{q-3}{2}} \left[ \frac{k}{q} + \frac{1}{4q} \right] \right) - \frac{q-1}{2} \left( \left[ \frac{2}{q} \right] + \left[ \frac{1}{2q} \right] - \left[ \frac{3}{2q} \right] - \left[ \frac{1}{4q} \right] \right).
\]

Let
\[
\sin : A \to \mathbb{Q}^{ab \times}
\]
be the unique homomorphism such that
\[
\sin[a] = \begin{cases} 2\sin(\pi a) (= 1 - e^{2\pi i a}) & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0 \end{cases} \quad (a \in \mathbb{Q} \cap [0, 1)).
\]

The composition field of all double coverings of \( \mathbb{Q}^{ab} \) is (see [1] main theorem)
\[
\mathbb{Q}^{ab}\left( \left\{ \sqrt{l} \right\}_{l \text{ prime}} \cup \{ \sin a_{pq} \}_{p, q \text{ prime } p < q} \right).
\]

Let \( n \not\equiv 2 \mod 4 \), we define a subset \( S_n \) of \( \mathbb{Z} \) associated to \( n \) as following:

i) if \( 2 \mid n \), set \( S_n := \{ \text{odd prime factors of } n \} \);

ii) if \( 4 \mid n \) and \( 8 \nmid n \), set \( S_n := \{-1\} \cup \{ \text{odd prime factors of } n \} \);

iii) if \( 8 \mid n \), set \( S_n := \{-1, 2\} \cup \{ \text{odd prime factors of } n \} \).

If \( 4 \mid n \), then for all \( p, q \in S_n \) and \( p < q \), we set
\[
u_{pq} := \begin{cases} \sqrt{q} & \text{if } p = -1 \\ \sin a_{pq} & \text{otherwise.} \end{cases}
\]

If \( 2 \nmid n \), then for primes \( p, q \in S_n \) and \( p < q \), we set
\[
u_{pq} := \begin{cases} \sin a_{pq} & \text{if } p \equiv q \equiv 1 \mod 4 \\ \sqrt{p} \cdot \sin a_{pq} & \text{if } p \equiv 1, q \equiv 3 \mod 4 \\ \sqrt{q} \cdot \sin a_{pq} & \text{if } p \equiv 3, q \equiv 1 \mod 4 \\ \sqrt{pq} \cdot \sin a_{pq} & \text{if } p \equiv q \equiv 3 \mod 4 \end{cases}
\]
(8)

Let \( F = \mathbb{Q}(\xi_n) \), where \( \xi_n \) is a primitive root of unity. Then the composition field of all double coverings of \( F/\mathbb{Q} \) is (see [13] Thoerem 1)
\[
F(\{ \sqrt{v_{pq}} \}_{p, q \in S_n}).F',
\]
where \( F' = F(\{ \sqrt{-1} \} \cup \{ \sqrt{l} \}_{l \text{ prime}}) \).

3.2. Construction of the Brauer group. Let \( K = \mathbb{Q}(\sqrt{d_1 d_2}, \sqrt{d_1 d_3}) \) with \( d_1, d_2, d_3 \in \mathbb{Z} \) are square-free and relatively prime each other. Without loss generality, we can assume \( d_1 d_2 > 0 \). In this section, we will explicitly construct the unramified Brauer group of the affine variety \( X \) over \( \mathbb{Q} \) defined by \( N_{K/\mathbb{Q}}(\Xi) = n \), where \( n \in \mathbb{Q}^{\times} \).
Denote $$S_i = \{ p \text{ rational prime} : p \mid d_i \}$$ for $$1 \leq i \leq 3$$,
\[
R = \bigcup_{i < j} S_i \times S_j, \quad \text{where} \quad 1 \leq i, j \leq 3
\]
\[
N = \begin{cases} 
|d_1d_2d_3| & \text{if } d_1d_2 \equiv d_1d_3 \equiv 1 \mod 4 \\
4|d_1d_2d_3| & \text{otherwise}.
\end{cases}
\]

Let $$F = \mathbb{Q}(\xi_N)$$. It’s clear that $$K$$ is contained in the cyclotomic field $$F$$. For simplicity of the notation, we extend the definition of $$a_{pq}$$ and $$u_{pq}$$ for $$p > q$$ by
\[
a_{pq} = a_{qp} \text{ and } u_{pq} = u_{qp}.
\]

Let
\[
\Delta = \left\{ \prod_{(p,q) \in R} \sin a_{pq} \right\} \sqrt{d_1d_2} \prod_{(p,q) \in R} \sin a_{pq} \cdot \sqrt{d_1d_2},
\]
\[
L = F(\sqrt{\Delta}).
\]

**Lemma 5.** The field extension $$L/\mathbb{Q}$$ is Galois and $$L \not\subset \mathbb{Q}^{ab}$$.

**Proof.** If $$4 \mid N$$, it follows from Theorem 11 and 12 in [6]. So we only need to consider the case $$4 \nmid N$$, i.e., $$d_1d_2 \equiv d_1d_3 \equiv 1 \mod 4$$.

Let $$\Delta' = \prod_{(p,q) \in R} u_{pq}$$. By an easy computation, we have
\[
\Delta' = \prod_{(p,q) \in R} \sin a_{pq} \prod_{p \mid d_1d_2d_3} \sqrt{p^{e_p}},
\]
where
\[
e_p = \begin{cases} 
\# \{ p \mid d_2d_3 : p \equiv 3 \mod 4 \} & \text{if } p \mid d_1 \\
\# \{ p \mid d_1d_3 : p \equiv 3 \mod 4 \} & \text{if } p \mid d_2 \\
\# \{ p \mid d_1d_2 : p \equiv 3 \mod 4 \} & \text{if } p \mid d_3.
\end{cases}
\]

Since $$d_1d_2 > 0$$, it implies $$e_p$$ is even when $$p \mid d_2$$.

If $$d_1d_3 > 0$$, we have $$d_2d_3 > 0$$ too. Then $$e_p$$ is even when $$p \mid d_1d_2$$. So $$\Delta' = \pm \Delta \cdot u^2$$ with $$u \in F^\times$$. Therefore $$L = F(\sqrt{\Delta}) = F(\sqrt{\Delta'})$$ or $$F(\sqrt{-\Delta})$$.

Then $$L/\mathbb{Q}$$ is Galois and $$L \not\subset \mathbb{Q}^{ab}$$ by Theorem 1 in [13].

If $$d_1d_3 < 0$$, we have $$d_2d_3 < 0$$ too. Then $$e_p$$ is odd when $$p \mid d_1d_2$$. So $$\Delta' = \pm \Delta \cdot u^2$$ with $$u \in F^\times$$. Therefore $$L = F(\sqrt{\Delta}) = F(\sqrt{\Delta'})$$ or $$F(\sqrt{-\Delta})$$.

Then $$L/\mathbb{Q}$$ is Galois and $$L \not\subset \mathbb{Q}^{ab}$$ by Theorem 1 in [13].

**Theorem 2.** Let $$X$$ be the affine variety over $$\mathbb{Q}$$ defined by $$N_{K/\mathbb{Q}}(\Xi) = n \in \mathbb{Q}^\times$$. Then $$\text{Cor}_{K/\mathbb{Q}}(\Xi, \chi)$$ generates $$Br(X^c)/Br_0(X^c)$$, where $$\chi$$ is a character of $$\text{Gal}(\mathbb{Q}/K)$$ which factors through $$\text{Gal}(L/K)$$ and nontrivial on $$\text{Gal}(L/F)$$.

**Proof.** First we show $$\text{Cor}_{K/\mathbb{Q}}(\Xi, \chi) \in \text{Br}(X^c)$$. Using Theorem 1 we only need to show $$L/\mathbb{Q}$$ satisfies the condition ($*$).

Let $$K'$$ be a subfield of $$K$$ such that $$K/K'$$ is cyclic. We want to show $$\text{Gal}(L/K')$$ is abelian. Since $$\text{Gal}(L/K')$$ is a quotient of $$\text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K')$$, we only need to show $$\text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K')$$ is abelian. The extension $$\mathbb{Q}^{ab}(\sqrt{\Delta})/K'$$ is Galois by the main theorem in [11]. Let $$G^{ab} = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$. There is the central extension
\[
\Sigma : 0 \to \mathbb{Z}/2 \to \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/\mathbb{Q}) \to G^{ab} \to 0
\]
Let
\[ \vartheta : \mathbb{Q}^{ab} \to \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/\mathbb{Q}) \]
be any set-theoretic splitting of the central extension \( \Sigma \). Then \( \Sigma \) gives a 2-cocycle
\[ a \in Z^2(\mathbb{G}^{ab}, \mathbb{Z}/2) \]
defined by the formula
\[ a_{\sigma,\tau} = \vartheta(\sigma)\vartheta(\tau)\vartheta(\sigma \tau)^{-1} \quad (\sigma, \tau \in \mathbb{G}^{ab}). \]
For each odd prime \( p \), let \( G_p \subset \mathbb{G}^{ab} \) be the inertia subgroup at \( p \). Let \( G_{-1} \subset \mathbb{G}^{ab} \) be the subgroup generated by the restriction of complex conjugation to \( \mathbb{Q}^{ab} \). Let \( G_2 \subset \mathbb{G}^{ab} \) be the subgroup of the inertial subgroup at 2 fixing \( \sqrt{-1} \). Let \( S = \{-1\} \cup \{ p \mid p \text{ is rational prime} \} \). We have
\[ \mathbb{G}^{ab} = \coprod_{p \in S} G_p. \]

For \( p \in S \) the profinite group \( G_p \) is procyclic. Let \( \sigma_p \in \mathbb{G}^{ab} \) such that \( \sigma_p \) projects to a topological generator of \( G_p \) and projects to 1 in \( G_q \) for \( q \neq p \). Let \( \alpha \in Z^2(\mathbb{G}^{ab}, \mathbb{Z}/2) \) defined by \( \alpha_{\sigma,\tau} = \alpha_{\sigma,-} - \alpha_{-,\sigma} \) (\( \sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K') \)). It’s easy to check \( \alpha \) is a skew-symmetric (symmetric) bilinear map.

1) Suppose \( d_1 d_3 > 0 \). Then \( \Delta = \prod_{(p,q) \in T} \sin \alpha_{pq} \) by our definition. By the Log wedge Formula in §3.4 and §4.3.4 in [1], we have
\[ \alpha = \sum_{(p,q) \in R} \delta_{p,q} \in Z^2(\mathbb{G}^{ab}, \mathbb{Z}/2), \]
where \( \delta_{p,q} = \delta_{q,p} : \mathbb{G}^{ab} \times \mathbb{G}^{ab} \to \mathbb{Z}/2 \) is defined by
\[ ((\sigma_i^{i_1})_{i \in S}, (\sigma_i^{i_2})_{i \in S}) \mapsto i_{p,1}q_2 + i_{p,2}^q, \]
and see [4] for the definition of \( R \).

2) Suppose \( d_1 d_3 < 0 \). Then \( \Delta = \sqrt{d_1 d_3} \prod_{(p,q) \in T} \sin \alpha_{pq} \). By the Log wedge Formula in §3.4 and §4.3.4 in [1], we have
\[ \alpha = \sum_{(p,q) \in R} \delta_{p,q} + \sum_{p|d_1 d_2} \delta_{-1,p}, \]
where \( \delta_{-1,p} : \mathbb{G}^{ab} \times \mathbb{G}^{ab} \to \mathbb{Z}/2 \) is defined by
\[ ((\sigma_i^{i_1})_{i \in S}, (\sigma_i^{i_2})_{i \in S}) \mapsto i_{-1,1}p_2 + i_{-1,2}q_1. \]

We have the central extension
\[ \Sigma_{K'} : 0 \to \mathbb{Z}/2 \to \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K') \to \text{Gal}(\mathbb{Q}^{ab}/K') \to 0 \]
Then \( \Sigma_{K'} \) gives a 2-cocycle \( a' = \text{Res}_{\mathbb{Q}/K'}(a) \in Z^2(\text{Gal}(\mathbb{Q}^{ab}/K'), \mathbb{Z}/2) \). Let
\[ a'_{\sigma,\tau} = a'_{\sigma,-} - a'_{-,\sigma} \quad (\sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K')). \]
We can verify (see [1] Lemma 2.8) that \( \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K') \) is abelian if and only if
\[ a'_{\sigma,\tau} = 0 \text{ for any } \sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K') \]
(i) Suppose \( d_1 d_3 > 0 \). Without loss generality, we can assume \( K' = \mathbb{Q}(\sqrt{d_1 d_2}) \). Let
\[ g_1 = (\sigma_p^{i_{p,1}})_{p \in S}, g_2 = (\sigma_p^{i_{p,2}})_{p \in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset \mathbb{G}^{ab}. \]
Since \( g_1, g_2 \) fix \( K' \), we have
\[
\sum_{p|d_1 d_2} i_{p,j} = \sum_{p|d_1} i_{p,j} + \sum_{p|d_2} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.
\]

Then
\[
\alpha'_{g_1, g_2} = \alpha_{g_1, g_2} = \sum_{(p,q)\in R} (i_{p,1}i_{q,2} + i_{p,2}i_{q,1})
\]
\[
= \sum_{(p,q)\in S_1 \times S_2} (i_{p,1}i_{q,2} + i_{p,2}i_{q,1}) + \sum_{(p,q)\in S_1 \times S_3} (i_{p,1}i_{q,2} + i_{p,2}i_{q,1})
\]
\[
= \sum_{p|d_1} i_{p,1} \sum_{p|d_2} i_{p,2} + \sum_{p|d_1} i_{p,1} \sum_{p|d_2} i_{p,2} + \sum_{p|d_2} i_{p,2} \sum_{p|d_1} i_{p,1}
\]
\[
\equiv 0 \pmod{2}.
\]

(ii) Suppose \( d_1 d_3 < 0 \).
(a) Suppose \( K' = \mathbb{Q}(\sqrt{d_1 d_2}) \). Let
\[
g_1 = (\sigma_{p,i_{p,1}})_{p\in S}, g_2 = (\sigma_{p,i_{p,2}})_{p\in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset G^{ab}.
\]
Then we have
\[
\sum_{p|d_1 d_2} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.
\]

Similar as above one has
\[
\alpha'_{g_1, g_2} = \alpha_{g_1, g_2} = \sum_{(p,q)\in R} (i_{p,1}i_{q,2} + i_{p,2}i_{q,1}) + \sum_{p|d_1 d_2} (i_{-1,1}i_{p,2} + i_{-1,2}i_{p,1})
\]
\[
\equiv 0 + i_{-1,1} \sum_{p|d_1 d_2} i_{p,2} + i_{-1,2} \sum_{p|d_1 d_2} i_{p,1}
\]
\[
\equiv 0 \pmod{2}.
\]

(b) Suppose \( K' = \mathbb{Q}(\sqrt{d_1 d_3}) \) (similar proof for \( K' = \mathbb{Q}(\sqrt{d_2 d_3}) \)). Let
\[
g_1 = (\sigma_{p,i_{p,1}})_{p\in S}, g_2 = (\sigma_{p,i_{p,2}})_{p\in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset G^{ab}.
\]
Then we have
\[
i_{-1,j} + \sum_{p|d_1 d_3} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.
\]
Then
\[ \alpha'_{g_1, g_2} = \sum_{(p, q) \in R} (i_{p,1}i_{q,2} + i_{p,2}i_{q,1}) + \sum_{p \mid d_1d_2} (i_{-1,1}i_{p,2} + i_{-1,2}i_{p,1}) \]
\[ = \sum_{p \mid d_1} i_{p,1} \sum_{p \mid d_2} i_{p,2} + \sum_{p \mid d_1} i_{p,2} \sum_{p \mid d_3} i_{p,1} + \sum_{p \mid d_2} i_{p,1} \sum_{p \mid d_3} i_{p,2} \]
\[ + \sum_{p \mid d_1d_2} (i_{-1,1}i_{p,2} + i_{-1,2}i_{p,1}) \]
\[ \equiv - (i_{-1,1} + \sum_{p \mid d_3} i_{p,2} - (i_{-1,1} + \sum_{p \mid d_3} i_{p,1}) \sum_{p \mid d_2} i_{p,2} + \sum_{p \mid d_3} i_{p,1} \sum_{p \mid d_2} i_{p,2} \]
\[ - (i_{-1,2} + \sum_{p \mid d_2} i_{p,2}) \sum_{p \mid d_3} i_{p,1} - (i_{-1,2} + \sum_{p \mid d_2} i_{p,2}) \sum_{p \mid d_3} i_{p,1} + \sum_{p \mid d_3} i_{p,2} \sum_{p \mid d_3} i_{p,1} \]
\[ + \sum_{p \mid d_1d_2} (i_{-1,1}i_{p,2} + i_{-1,2}i_{p,1}) \]
\[ \equiv - i_{-1,1} \sum_{p \mid d_2d_3} i_{p,2} - i_{-1,2} \sum_{p \mid d_2d_3} i_{p,1} + \sum_{p \mid d_1d_2} (i_{-1,1}i_{p,2} + i_{-1,2}i_{p,1}) \]
\[ \equiv - i_{-1,1} i_{-1,2} - i_{-1,2} i_{-1,1} \equiv 0 \pmod{2}. \]

Therefore \( L/K \) satisfies the condition (*).

Since \( Br(X^c)/Br_0(X^c) \cong \mathbb{Z}/2 \), we only need to show \( Cor_{K/Q}(\Xi, \chi) \) is nontrivial. Recall
\[ F = \mathbb{Q}(\xi_N), K \subset F \subset \mathbb{Q}^{ab} \] and \( L = F(\sqrt{\Delta}) \).

By Lemma 3 we only need to show \( \chi \) is not the restriction of a character of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Otherwise we assume \( \chi \) is trivial on \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \). On the other hand, \( \chi \) factors through \( \text{Gal}(L/K) \). Therefore we have \( \chi \) is trivial on \( \text{Gal}(\overline{\mathbb{Q}}/L \cap \mathbb{Q}^{ab}) \). Note that \( L/Q \) is non-abelian (see Lemma 5), hence \( F = L \cap \mathbb{Q}^{ab} \). Therefore \( \chi \) is trivial on \( \text{Gal}(\overline{\mathbb{Q}}/F) \), this is a contradiction to that \( \chi \) factors through \( \text{Gal}(L/K) \) and nontrivial on \( \text{Gal}(L/F) \).

\[ \square \]

**Theorem 3.** Let \( m \neq 2 \pmod{4} \). Let \( K = \mathbb{Q}(\xi_m) \) be a cyclotomic field and \( L = K(\sqrt[4]{\mu_p})_{p < q \in S_m} \). Then the 2-torsion subgroup of \( Br(X^c)/Br_0(X^c) \) is generated by all \( Cor_{K/Q}(\Xi, \chi) \), where \( \chi \) runs through all characters in the image by the natural map \( \text{Hom}(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\text{Gal}(\overline{\mathbb{Q}}/K), \mathbb{Q}/\mathbb{Z}) \).

**Proof.** Let \( \chi \) be such a nontrivial character of \( \text{Gal}(\overline{\mathbb{Q}}/K) \), then there is a subfield \( L' \supset K \) of \( L \) with \( [L' : K] = 2 \) such that \( \chi \) factors through \( \text{Gal}(L'/K) \). And \( L'/\mathbb{Q} \) is Galois (non-abelian) by [13 Theorem 1]. That \( L'/K \) satisfies the condition (*) follows from the fact:

- if the abelian group \( M \) is cyclic and \( G \) satisfies the central extension \( 0 \to \mathbb{Z}/2 \to G \to M \to 0 \),
then $G$ is abelian.

Therefore $\text{Cor}_{K/Q}(\Xi, \chi) \in \text{Br}(X^c)$ by Theorem 1.

Let $d = \# S_m$. On the other hand, the Galois group $\text{Gal}(L/K) \cong (\mathbb{Z}/2)^{d(d-1)/2}$ by the linear independent $u_{pq}$ in $K^\times/K^\times^2$ (see [13] Lemma 4). Therefore the subgroup of $\text{Br}(X^c)/\text{Br}_0(X^c)$ generated by all $\text{Cor}_{K/Q}(\Xi, \chi)$ is of 2-rank $d(d-1)/2$. Using the Künneth formula (p. 96 in [8]), we can calculate that the 2-rank of $H^3(\text{Gal}(K/k), \mathbb{Z})$ is also $d(d-1)/2$. Since $\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z})$, the 2-rank of $\text{Br}(X^c)/\text{Br}_0(X^c)$ is $d(d-1)/2$. Therefore all $\text{Cor}_{K/Q}(\Xi, \chi)$ generate the 2-torsion subgroup of $\text{Br}(X^c)/\text{Br}_0(X^c)$.

Finally we will use Theorem 3 to give an explicit example associated to a cyclotomic field. Write $n = 2^{3} \cdot 3 \cdot 7 \cdot 53 \cdot p_1^{e_1} \cdots p_g^{e_g} \in \mathbb{Q}^\times$. Let $D(n) = \{p_1, \ldots, p_g\}$. Denote

$$D_1 = \{p \in D(n) : \left(\frac{-7}{p}\right) = \left(\frac{53}{p}\right) = -1\}$$

$$D_2 = \{p \in D(n) : \left(\frac{-7}{p}\right) = \left(\frac{53}{p}\right) = 1 \text{ and } \left(\frac{5 + 2\sqrt{-7}}{p}\right) = -1\}.$$

**Example 3.** Let $K = \mathbb{Q}(\sqrt[7]{2}, \sqrt[53]{7})$. Then the equation $N_{K/Q}(\Xi) = n$ is solvable over $\mathbb{Q}$ if and only if the following conditions hold:

1. The equation $N_{K/Q}(\Xi) = n$ is solvable over $\mathbb{Q}_p$ for each $p$.
2. $(-1)^{\sum_{p_i \in D_1} e_i/2 + \sum_{p_i \in D_2} e_i} = (-1)^{e_2} \cdot \left(\frac{-1}{2}\right)$.

**Proof.** Let $X$ be the affine variety defined by $N_{K/Q}(\Xi) = n \in \mathbb{Q}^\times$. We can see

$$\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z}) \cong \mathbb{Z}/2.$$ 

Let $L = K(\sqrt{\alpha_{pq}})$. It is easy to verify that $K(\sqrt{5 + 2\sqrt{-7}})/Q$ is Galois. Then we have $L = K(\sqrt{5 + 2\sqrt{-7}})$ by the fact that $L$ is the composition field of all double coverings of $K/Q$. Let $\chi$ be the unique nontrivial character of $\text{Gal}(\overline{Q}/K)$ which factors through $\text{Gal}(L/K)$. Then $\text{Cor}_{K/Q}(\Xi, \chi)$ is the unique generator of $\text{Br}(X^c)/\text{Br}_0(X^c)$ by Theorem 3.

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