RING EXTENSION OF ENTIRE RING WITH CONJUGATION; ARITHMETIC IN ENTIRE RINGS

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Abstract. Some basic properties of the ring of integers $\mathbb{Z}$ are extended to entire rings. In particular, arithmetic in entire principal rings is very similar than arithmetic in the ring of integers $\mathbb{Z}$. These arithmetic properties are derived from a $\star$-ring extension of the considered entire ring (ring extension with conjugation) equipped with a real function which is a multiplicative structure-preserving map between two algebras. The algebra of this ring extension is studied in detail. Some examples of such ring extension are given.

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Preface. Arithmetic on entire rings which are equipped with a relation of total order and where a relation of divisibility is defined, is very similar than arithmetic on the ring of integers \( \mathbb{Z} \). Indeed, the standard arithmetic properties in \( \mathbb{Z} \) can be extended to these algebraic structures. But, when the relation of order on elements of an entire ring, is not total, some arithmetic properties fails to be true. In particular, the greatest common divisors (gcd) and the least common divisor (lcm) of two elements in this algebraic structure, are not always defined. Then, the group of units of an entire ring may be larger than in \( \mathbb{Z} \). Moreover, an equivalence relation \( \sim \) defined on entire ring \( A \), which involves identification of elements of \( A \) which can be deduced from one to another by a global multiplicative unit, implies a quotient set \( A/\sim \). We say that two elements \( a \) and \( b \) of \( A \) are equivalent if there exists a unit \( u \) of \( A \) such that \( b = ua \). An element of the quotient set \( A/\sim \) is an equivalence class associated to an element \( a \) of \( A \) which is denoted \([a]\) and which is defined by:

\[
[a] = \{ b \in A : \exists u \in [1], \ b = ua \}
\]

where \([1]\) consists of all the units of \( A \). The element \( a \) of \( A \) is said a representative of its equivalence class.

This quotient set equipped with a suitable multiplicative law \( \cdot \), forms a group denoted \( (A/\sim, \cdot) \). Indeed, defining \([a] \cdot [b]\) as the set of all the products of any element of \([a]\) and any element of \([b]\), we have obviously:

\[
[a] \cdot [b] = [ab]
\]

Notice that the neutral element of this group is the group of units of \( A \) which is identical to the equivalence class \([1]\).

In general, we have:

\[
[a] + [b] = \bigcup_{u \in [1]} [a + ub]
\]

So, from the quotient set \( A/\sim \), regarding the product of sets and the sum set, it is not possible to form a ring. Indeed, the formula of \([a] + [b]\) above implies that we cannot establish a ring homomorphism between \( \mathbb{Z} \) and \( A/\sim \) when the product of sets and the sum set are the operations defined on \( A/\sim \). Or, the rings form a category and \( \mathbb{Z} \) is an initial object in this category\(^1\). So, \( A/\sim \) equipped with the sum set and the product of sets cannot be a ring.

The group \( (A/\sim, \cdot) \) is not necessarily isomorphic to \( (\mathbb{Z}/\sim, \cdot) \cong (\mathbb{N}, \cdot) \) when \( A \) contains at least a unit which is not equal to \( \pm 1 \). Moreover, the set of arithmetic properties in \( A \) which are invariant under \( \sim \) may be not necessarily identical to the set of arithmetic properties in \( \mathbb{Z} \) which are invariant under \( \sim \). Accordingly, when the relation of order defined on an entire ring is not total, the picture of network of elements which can be put in relation by equivalence relation \( \sim \), changes. It may be possible that an algebra in entire ring induces arithmetic which could be different than in \( \mathbb{Z} \). Nevertheless, in this paper, it is shown that when a ring extension of an entire subring \( A \) of a subfield of \( \mathbb{C} = \mathbb{R}[i] \) is equipped with a magnitude function\(^2\) which represents the size of the elements of this ring extension in a subfield of \( \mathbb{R} \), the most standard arithmetic properties can be recovered in \( A \) if \( A \) is principal, provided some conditions on the magnitude function are fulfilled.

The plan of the paper is the following one. In the section \( \text{II} \) we recall some basic facts about the group of units of an entire ring. In the section \( \text{II} \) we give some properties which characterize ideals of a principal entire ring. In the section \( \text{III} \) we define divisibility in an entire ring. In the section \( \text{IV} \) we deal with the algebraic structure of a \( \star \)-extension of an entire ring equipped with a magnitude function where the generated set of elements of this ring extension is subset of an abelian group.

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1 This function can be viewed as a partial norm. In some cases, this function is a norm.
This magnitude function is a kind of map which preserves multiplicative law between two algebras and which generalizes the concept of a norm defined on a vector space over a field. In the section 5, we give some basic arithmetic definitions relative to divisibility in an entire ring. It leads to get the generalization of the fundamental theorem of arithmetic in a principal entire ring. In the section 6, set operations on ideals of a principal entire ring are connected to divisibility. In the section 7, famous arithmetic theorems as the Bezout identity are extended to a principal entire ring. In the section 8, some arithmetic properties on the set of ideals of a principal entire ring are derived. In the section 9, a maximal ideals of a principal entire ring are obtained from prime ideals of a principal entire ring. In the section 10, examples of ring extensions of entire rings are developed. In the section 11, an algebra of entire ring generated by the generators of a Lie algebra is illustrated. It gives a generalization of the concept developed along this paper of a ring extension of an entire ring with an abelian group. But, in this algebraic structure, it turns out to be that the magnitude function is either not defined or degenerate.
1. Group of units of an entire ring

An entire ring $A[2]$ is a commutative ring which contains 1 such that $1 \neq 0$ and such that there are no zero divisors in $A$. There exist elements in $A$ which are invertible. They form a multiplicative group denoted $U(A)$ which is called the group of units of $A$ (for instance, $U(\mathbb{Z}) = \{1, -1\}$). Notice that $A$ is not necessarily a division ring. For instance $\mathbb{Z}$ is a principal entire ring but not a division ring since all the non-zero elements of $\mathbb{Z}$ are not invertible. Since the multiplicative law of $A$ is associative and defining $x^n = \underbrace{x \cdot \ldots \cdot x}_n$ for all $x \in A$ and for all $n \in \mathbb{N}$ by recurrence:

$$x^0 = 1 \quad \text{and} \quad x^{n+1} = xx^n$$

it can be shown by induction and by regarding inverse of any element $u$ of $U(A)$ that:

$$x^{m+n} = x^mx^n \quad \text{and} \quad (x^m)^n = x^{mn} \quad \forall x \in A, \forall (m,n) \in \mathbb{N}^2$$

and:

$$v^{k+l} = v^kv^l \quad \text{and} \quad (v^k)^l = v^{kl} \quad \forall v \in U(A), \forall (k,l) \in \mathbb{Z}^2$$

Notice that since $U(A)$ is a multiplicative group, $v^n$ for all $v \in U(A)$ and for all $n \in \mathbb{N}$ is invertible and its inverse is $(v^n)^{-1} = v^{-n}$ for all $v \in U(A)$ and for all $n \in \mathbb{N}$.

If $U(A)$ is a finite group, denoting $|X|$ the order of a finite subset $X$ of $A$, then from the little theorem of Lagrange, we have:

$$v^{|U(A)|} = 1 \quad \forall v \in U(A)$$

with the order $|v|$ of the element $v$ of $A$ which divides $|U(A)|$.

2. Ideals and set operations on principal ideals of an entire ring

In this section, $A$ denotes an entire ring. We recall that a (left/right) ideal of a ring $A$ is an additive subgroup of $A$ which is stable by (left/right) multiplication. Moreover, a (left/right) ideal of a ring $A$ is principal if it is generated by a singleton $\{a\}$ with $a \in A$. A principal ideal generated by a singleton $\{a\}$ with $a \in A$, is denoted $aA$. Thus, we have:

$$aA = \{ax : x \in A\}$$

Remark 2.1. The intersection of two principal ideals $a$ and $b$ is an ideal. Let prove that $a \cap b$ is really an ideal of $A$. Since $a$ and $b$ are principal ideals, there exist two elements $a, b$ of $A$ such that $a = aA$ and $b = bA$. Since $aA \cap bA$ contains $0 = a0 = b0$, since $aA \cap bA$ is stable by addition namely $\forall (ax_1 = by_1, ax_2 = by_2) \in (aA \cap bA)^2$ such that $x_i, y_j \in A$ with $i, j = 1, 2$, we have

$$ax_1 + ax_2 = by_1 + by_2$$

which implies that $a(x_1 + x_2) = b(y_1 + y_2) \in aA \cap bA$ and since $\forall c \in aA \cap bA$, $-c \in aA \cap bA$, $aA \cap bA$ is an additive subgroup of $A$. Moreover, $\forall x \in A$, $\forall c \in aA \cap bA$, we have $c \in aA$ which implies $cx \in aA$ since $aA$ is an ideal and $c \in bA$ which implies that $cx \in bA$ since $bA$ is an ideal. It follows that $\forall x \in A$, $\forall c \in aA \cap bA$, $cx \in aA \cap bA$. So, $aA \cap bA$ is an additive subgroup of $A$ which is stable by multiplication. Thus, $aA \cap bA$ is an ideal.

The sum of two principal ideals $a$ and $b$ is an ideal. Let prove that $a + b$ is really an ideal of $A$. Since $a$ and $b$ are principal ideals, there exist two elements $a, b$ of $A$ such that $a = aA$ and $b = bA$. $aA + bA$ contains $0 = a0 + b0$. Moreover, $\forall (r, s) \in A^2$, $\forall (ax_1 + by_1, ax_2 + by_2) \in (aA + bA)^2$ such that $x_i, y_j \in A$ with $i, j = 1, 2$, we have $r(ax_1 + by_1) + s(ax_2 + by_2) = a(rx_1 + sx_2) + b(ry_1 + sy_2) \in aA + bA$. Thus, $aA + bA$ is an ideal.

A principal ring is a ring such that every ideal is principal.

\(^2A\) A ring $R$ is a division ring if, and only if, its group of units $U(R)$ contains all the non-zero elements of $R$.\)
Lemma 2.2. Let $A$ be an entire principal ring. Then, any ideal which contains 1 is equal to $A$. The ideal $aA$ is equal to $A$ if, and only if, $a \in U(A)$.

Proof. $A$ is itself an ideal which contains 1. If an ideal contains 1, since it is stable by multiplication, then any element $x = 1x$ of $A$ belongs to this ideal. So, $A$ is included in this ideal which is itself included in $A$. It results that this ideal is equal to $A$.

If the ideal $aA = A$, it means that $1 \in aA$. So, there exists a non-zero element $b \in A$ such that $ab = 1$. Therefore, $a$ is invertible. Reciprocally, if $a$ is invertible, there exists a non-zero element $b$ of $A$ such that $ab = 1$. Since $aA$ is an ideal of $A$, $ab = 1 \in A$. Therefore, if $a$ is invertible, then $aA = A$.

3. Divisibility in an entire ring

In this section, we assume that the reader has a knowledge of basic concepts of the number theory. For a review, the reader can be referred to [3]. Moreover, we assume that $A$ is an entire ring which is not necessarily principal.

Definition 3.1. An element $a$ of $A$ divides an element $b$ of $A$, what it is denoted $a \mid b$, if there exists $c \in A$ such that $ac = b$. Then the element $a$ of $A$ is said to be a divisor of $b$ of $A$ and the element $b$ of $A$ is said to be a multiple of $a$ of $A$.

Definition 3.2. An element $x$ of $A$ is a divisor of zero if $x \neq 0$ and if there exists an element $y \neq 0$ of $A$ such that $xy = 0$.

Remark 3.3. The set of divisors of an element $a$ of $A$ is denoted $\mathcal{D}(a)$. We have $U(A) \subseteq \mathcal{D}(a)$ for all $a \in A$. In particular, $\mathcal{D}(1) = U(A)$. More generally, we have:

$$\mathcal{D}(a) \subseteq \bigcup_{d \mid a} dA$$

If $a \not\in U(A)$, there exists at least an element $d \neq a$ of $\mathcal{D}(a)$ which is not in $U(A)$ since $U(A)$ is a multiplicative group. Indeed, let assume absurdly that there doesn’t exist such an element $d \neq a$ of $\mathcal{D}(a)$. Accordingly, all the elements of $\mathcal{D}(a)$ except $a$ would be in $U(A)$. But, then $a \in U(A)$ since $U(A)$ is a multiplicative group. So, we reach to a contradiction meaning that if $a \not\in U(A)$, then there exists at least an element $d \neq a$ of $\mathcal{D}(a)$ which does not belong to $U(A)$.

The set of multiples of an element $a$ of $A$ is denoted $\mathcal{M}(a)$ which is equal to the ideal $aA$:

$$\mathcal{M}(a) = aA$$

In particular, we have $\mathcal{D}(0) = \mathcal{M}(0) = 0$ and $0 \in \mathcal{M}(a)$ for all $a \in A$.

The relation of divisibility which is defined on $A$ is reflexive, transitive and linear\(^3\).

Property 3.4.

$$\forall (x, y) \in A^2, x \mid y \Leftrightarrow yA \subseteq xA$$

Proof. Indeed, the ideal $xA$ is the set of multiples of $x$. Let assume $x \mid y$. Whatever $z \in yA$, we have $y \mid z$, so by transitivity of the relation of divisibility defined on $A$, $x \mid z$ and $z \in xA$. If, reciprocally, we have $yA \subseteq xA$, it comes that $y \in xA$, so $x \mid y$.

\(^3\)The property that the relation of divisibility denoted $\mid$, which is defined on $A$, is reflexive, transitive and linear (see also [3]), means that:

- reflexivity: $\forall x \in A$, we have $x \mid x$;
- transitivity: $\forall x, y, z \in A$, $x \mid y$ and $y \mid z$ imply $x \mid z$;
- linearity: for $x, y, z \in A$ such that $x \mid y$ and $x \mid z$, we have $x \mid (ay + bz)$ for all $a, b \in A$. 

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In the following, we denote by $aU(A)$, the subset of $A$ defined by:

$$aU(A) = \{ au : u \in U(A) \}$$

and for two subsets $X, Y$ of $A$, the subset $X \setminus Y$ of $A$ is the subset of all the elements of $X$ which are not in $Y$:

$$X \setminus Y = \{ x \in X : x \notin Y \}$$

**Remark 3.5.**

$$\mathcal{D}(a) = \bigcup_{d|a} dU(A)$$

### 4. Extension of an entire ring

**Definition 4.1.** Let $A$ be an entire subring with $1 \in A$ ($1 \neq 0$), of a subfield $\mathbb{F}$ of $\mathbb{C}$, which is generated by a finite number of its elements and let $G$ be a finite abelian group of $\mathbb{F}$ which is not contained in $A$ although the intersection of $A$ and $G$ is non-empty (since it contains at least 1) and such that all the elements of $G$ commute with all the elements of $A$. We denote $\mathcal{G}$ a maximal family of linearly independent elements of $G$ over $\mathbb{F}$ and $S$ the subset of $G$ which consists of all the elements in $\mathcal{G}$ which does not belong to $A$. We denote by $A[S]$ the commutative subring of $\mathbb{F}$ which is the ring extension of $A$ which includes all linear combinations of elements of $G$ with coefficients in $A$. It is understood that $A[S]$ is generated by $S$ but a basis of $A[S]$ is a set of elements which includes $S$ and a maximal family of linearly independent generators of $A$ over $A$. Moreover, we assume that there is no divisor of zero in $A[S]$. So, since $1 \neq 0$ in $A$, $A[S]$ is entire. Besides, the definition of divisibility in $A$ can be extended in $A[S]$ with the same notations. We assume that $\mathbb{F}$ is equipped with a norm $\| \|_\mathbb{F}$. We assume that any non-zero element of $A[S]$ is invertible in $\mathbb{F}$.

**Remark 4.2.** $G$ as well as $\mathcal{G}$ and so also $S$ are contained in $U(A[S])$.

**Definition 4.3.** The magnitude function $N$ is the map defined on $A[S]$ which associates the unique element $N(x)$ of the set $\mathbb{F} \cap \mathbb{R}$, to $x \in A[S]$:

$$N : A[S] \rightarrow \mathbb{F} \cap \mathbb{R}$$

$$x \mapsto N(x)$$

with properties:

$$\forall x \in A[S] \setminus (\ker N \setminus \{0\}) : N(x) \in A[S], \quad \mathcal{D}(N(x)) = \mathcal{D}(x)$$

(P.1)

$$N(x) = 0 \iff x \in \ker N$$

(P.2)

$$\forall x, y \in A[S], \quad \exists z \in A[S] : N(x) + N(y) = N(z)$$

(P.3)

$$N(x) = N(-x) \quad \forall x \in A[S]$$

(P.4)

$$\forall x \in A[S] : N(x) \in A[S], \quad N(N(x)) = N(x)$$

(P.5)

$$N(xy) = N(x)N(y) \quad \forall x, y \in A[S]$$

(P.6)

$$\|N(x)\|_\mathbb{F} = N(x), \quad \forall x \in A[S]$$

(P.7)

$$\forall x \in A[S] \setminus \ker N, \exists x' \in \mathbb{F} \cap \mathbb{R} : N(x)^2 = xx' \in A[S]$$

(P.8)

For a given element $x$ of $A[S]$, $N(x)$ is said to be the size or the magnitude of $x \in A[S]$ in $\mathbb{F} \cap \mathbb{R}$.

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4A maximal family of linearly independent elements of a set $E$ of $\mathbb{C}$ over a ring $R$ of $\mathbb{C}$ is a free family of elements of $E$ over $R$, whose cardinality is maximal. See also the definition 4.57.
Remark 4.4. For a given element $x$ of $A[S]$, the notation $N(x)^{-1}$ means the inverse of $N(x)$ in $\mathbb{F} \cap \mathbb{R}$. It is possible that $N(x)$ belongs to $U(A[S])$. In such a case, $N(x)^{-1}$ is the inverse of $N(x)$ in $A[S]$. For instance (see below for more details), when $x \in U(A[S])$, we have $N(x) \in U(A[S])$ and $N(x)^{-1} = N(x^{-1})$.

Example 4.5. For any integer $k$ of $\mathbb{Z}$, $N(k)$ is equal to the unsigned part of $k$. In other word, we have:

$$N(k) = \text{abs}(k) \quad \forall \ k \in \mathbb{Z}$$

where abs is the absolute value (or modulus) function on $\mathbb{Z}$.

Property 4.6. Let $n \in \mathbb{N}^*$ and let $x_1, \ldots, x_n$ be $n$ element(s) of $A[S]$. Then we have:

$$N(x_1) + \ldots + N(x_n) = 0 \iff x_1, \ldots, x_n \in \ker N$$

Proof. From the property (P.2) of the definition (4.3), we have $N(x_1) = 0 \iff x_1 \in \ker N$. So, the property is verified for $n = 1$. In the following, we assume that $n \geq 2$.

From the property (P.2) of the definition (4.3), it is obvious also that if $x_1, \ldots, x_n \in \ker N$, then $N(x_1) + \ldots + N(x_n) = 0$.

Reciprocally, if $N(x_1) + \ldots + N(x_n) = 0$ with $n \geq 2$, we have $(n \geq 2)$:

$$N(x_2) + \ldots + N(x_n) = -N(x_1)$$

From the property (P.3) of the definition (4.3), using an immediate reasoning by induction, we can find an element $z$ of $A[S]$ such that $(n \geq 2)$:

$$N(x_2) + \ldots + N(x_n) = N(z)$$

It results that:

$$N(z) = -N(x_1)$$

Using the property (P.7) of the definition (4.3), we have:

$$||N(z)||_F = N(z)$$

Using again (P.7), from the equality $N(z) = -N(x_1)$, it implies that:

$$N(z) = ||-N(x_1)||_F = ||N(x_1)||_F = N(x_1)$$

So:

$$N(x_1) = -N(x_1) \iff N(x_1) = 0 \iff x_1 \in \ker N$$

Since we can exchange $x_1$ with any $x_i$ for $i \in [1, n]$, we deduce that if $N(x_1) + \ldots + N(x_n) = 0$ with $n \geq 2$, then $x_1, \ldots, x_n \in \ker N$.

We conclude that if $x_1, \ldots, x_n$ be $n$ element(s) of $A[S]$ with $n \in \mathbb{N}^*$, $N(x_1) + \ldots + N(x_n) = 0$ is equivalent to $x_1, \ldots, x_n \in \ker N$ with $n \in \mathbb{N}^*$. \hfill $\square$

Definition 4.7. An element $r$ of a ring $R$ is said regular or simplifiable for the multiplicative law of $R$ (or multiplicatively regular or multiplicatively simplifiable) if for any couple $(x, y) \in R^2$, we have:

$$rx = ry \Rightarrow x = y$$

and

$$xr = yr \Rightarrow x = y$$
Remark 4.8. Since $A[S]$ is a commutative ring, any element $a$ is multiplicatively regular if for any couple $(x, y) \in A[S]^2$, we have:

$$ax = ay \Rightarrow x = y$$

Any invertible element of $A[S]$ is regular for the multiplicative law of $A$ (or multiplicatively regular). Indeed, let $a$ be an invertible element of $A[S]$, whose inverse is $b$. Then, we have:

$$ax = ay \Rightarrow bax = bay \Rightarrow x = y$$

Property 4.9. A non-zero element of $A[S]$ is regular for the multiplicative law of $A[S]$ (or multiplicatively regular) if, and only if, it is not a divisor of 0.

Proof. Let $a$ a non-zero element of $A[S]$ and let $x, y \in A[S]$.

If $a$ is multiplicatively regular and if $ax = 0$, then $ax = a0$. After simplification by $a$, it comes that $x = 0$. So (see the definition (3.2)), $a$ is not a divisor of 0.

Reciprocally, if $a$ is not a divisor of 0 and if $ax = ay$, then $a(x - y) = 0$. Necessarily, we have $x - y = 0$ and so $x = y$ implying that $a$ is regular.

Remark 4.10. Since $A[S]$ is an entire ring, $A[S]$ does not contain divisor of zero. So, any non-zero element of $A[S]$ is regular. In particular, any prime element of $A$ is regular.

Moreover, in a subfield of $\mathbb{C}$ as for instance $\mathbb{F}$, $\mathbb{R}$ or $\mathbb{F} \cap \mathbb{R}$, its non-zero elements are invertible and so regular.

Property 4.11.

$$N(1) = 1$$

Proof. Using the properties (P.4) and (P.6) of the definition (4.3), we have:

$$N(-x) = N(x) \quad \forall x \in A[S]$$
$$N((-1)x) = N(x) \quad \forall x \in A[S]$$
$$N(-1)N(x) = N(x) \quad \forall x \in A[S]$$
$$N(1)N(x) = N(x) \quad \forall x \in A[S]$$
$$N(x)(N(1) - 1) = 0 \quad \forall x \in A[S]$$

In particular, it is true when $x \notin \ker N$ and so when $N(x)$ is regular in $\mathbb{F} \cap \mathbb{R}$ (see the remark (4.10)). Therefore, we get:

$$N(1) - 1 = 0$$
$$N(1) = 1$$

Property 4.12.

$$0 \in \ker N$$

Proof. Using the property (P.6) of the definition (4.3), we have:

$$N(0x) = N(0)N(x) \quad \forall x \in A[S]$$

Since $0x = 0$, it comes that:

$$N(0) = N(0)N(x) \quad \forall x \in A[S]$$
$$N(0)(N(x) - 1) = 0 \quad \forall x \in A[S]$$
In particular, it is true for \( x \in A[S] \) such that \( N(x) - 1 \neq 0 \). In this case, using the remark \((4.10)\), \( N(x) - 1 \) is regular in \( \mathbb{F} \cap \mathbb{R} \) and using the property \((P.2)\) of the definition \((4.3)\), we get:

\[
N(0) = 0 \iff 0 \in \ker N
\]

\( \square \)

**Property 4.13.** Let \( v \in U(A[S]) \). Then, the inverse of \( N(v) \) denoted \( N(v)^{-1} \) in \( \mathbb{F} \cap \mathbb{R} \) is:

\[
N(v)^{-1} = N(v^{-1})
\]

**Proof.** Using the property \((P.6)\) of the definition \((4.3)\) and the property \((4.11)\), we can see that:

\[
N(vv^{-1}) = N(v)N(v^{-1}) = N(1) = 1 \quad \forall \ v \in U(A[S])
\]

What it proves that \( N(v) \in U(A[S]) \) for all \( v \in A[S] \) and the inverse of \( N(v) \) denoted \( N(v)^{-1} \) in \( \mathbb{F} \cap \mathbb{R} \) is:

\[
N(v)^{-1} = N(v^{-1}) \quad \forall \ v \in U(A[S])
\]

\( \square \)

**Remark 4.14.** Accordingly, the restriction of the application \( N \) denoted \( N \big|_{U(A[S])} \) is an endomorphism of the multiplicative group \( U(A[S]) \).

**Corollary 4.15.** If \( v \in U(A[S]) \), then \( v \notin \ker N \).

**Proof.** Since \( N(v)N(v)^{-1} = 1 \) for all \( v \in U(A[S]) \) and since \( 1 \neq 0 \), we have \( N(v) \neq 0 \) for all \( v \in U(A[S]) \). Therefore, if \( v \in U(A[S]) \), then \( v \notin \ker N \). \( \square \)

**Property 4.16.**

\[
N(x)^n = N(x^n) \quad \forall x \in A[S], \forall n \in \mathbb{N}
\]

**Proof.** This property is proved by induction by using the property \((P.6)\) of the definition \((4.3)\). \( \square \)

Then, we have:

\[
N(v)^{-n} = (N(v)^n)^{-1} = N(v^n)^{-1} = N((v^n)^{-1}) = N(v^{-n}) \quad \forall v \in U(A[S]), \forall n \in \mathbb{N}
\]

It follows the property:

**Property 4.17.**

\[
N(v)^k = N(v^k) \quad \forall v \in U(A[S]), \forall k \in \mathbb{Z}
\]

**Property 4.18.** Let \( x, y \) be two non-zero elements of \( A[S] \). Then:

\[
\mathcal{D}(x) = \mathcal{D}(y) \iff \exists \ v \in U(A[S]) : y = xv
\]

with \( v \) which is unique.

**Proof.** Let \( x, y \) be two non-zero elements of \( A[S] \).

From the remark \((3.5)\), since \( U(A[S]) \) is a multiplicative group, if there exists \( v \in U(A[S]) \) such that \( y = xv \), then \( \mathcal{D}(x) = \mathcal{D}(y) \). Indeed, we have:

\[
\mathcal{D}(y) = \bigcup_{d \mid y} dU(A[S]) = \bigcup_{d \mid xv} dU(A[S]) = \bigcup_{dv^{-1} \mid x} dU(A[S]) = \bigcup_{d' \mid x} d'U(A[S]) = \mathcal{D}(x)
\]

with \( d' = dv^{-1} \).

Reciprocally, if \( \mathcal{D}(x) = \mathcal{D}(y) \), then \( x \mid y \) and \( y \mid x \). So, there exist two elements \( d, d' \) of \( A[S] \) such that \( y = dx \) and \( x = d'y \). It gives:

\[
y = dd'y
\]
\[ y(dd' - 1) = 0 \]

Since \( y \neq 0 \), from the remark (4.10), \( y \) is regular and we deduce that:
\[ dd' = 1 \]

So, \( d \in U(A[S]) \) and its inverse is \( d^{-1} = d' \).

Let consider two elements \( v, v' \) of \( U(A[S]) \) such that \( (x, y) \neq 0 \):
\[ y = xv = xv' \]

Then we have \( (x \neq 0) \):
\[ xv - xv' = 0 \]
\[ x(v - v') = 0 \]

Since \( x \neq 0 \) of \( A[S] \) is regular (see the remark (4.10)), we have:
\[ v - v' = 0 \]
\[ v = v' \]

We conclude that \( D(x) = D(y) \) with \( x, y \neq 0 \) of \( A[S] \) if, and only if, there exists a unique element \( v \) of \( U(A[S]) \) such that \( y = xv \) with \( x, y \neq 0 \).

Corollary 4.19.

\[ \forall \ x \in A[S] \setminus \ker N : N(x) \in A[S] \ , \ \exists \ v \in U(A[S]) : x = vN(x) \]

with \( v \) which is unique.

Proof. Using the property (P.1) of the definition (4.3), the corollary (4.19) follows from the property (4.18).

Property 4.20. Let \( x \) be an element of \( A[S] \setminus (\ker N \setminus \{0\}) \) and let \( n \in \mathbb{N} \). If \( N(x) \in A[S] \) and if \( N(nx) \in A[S] \), then we have \( (x \notin \ker N \setminus \{0\}) \):
\[ N(nx) = nN(x) \]

Proof. Let \( x \) be an element of \( A[S] \setminus (\ker N \setminus \{0\}) \) and let \( n \in \mathbb{N} \).

The property is verified for \( n = 0 \) and for \( x = 0 \) since \( 0x = n0 = 0 \) and \( N(0) = 0 \). In the following, we assume that \( n \in \mathbb{N}^* \) and \( x \neq 0 \).

Notice that the element \( nx \) for \( x \in A[S] \) and \( n \in \mathbb{N}^* \) is well defined in \( A[S] \) since \( nx = x + \ldots + x \)
and \( (A[S], +) \) is an additive subgroup of \( A[S] \). Moreover, from the property (P.1) of the definition (4.3), if \( N(x) \in A[S] \), then we have \( (x \notin \ker N) \):
\[ D(x) = D(N(x)) \]

It means that \( (d \in A[S]) \):
\[ d|x \iff d|N(x) \]

We know that \( (d \in A[S]) \):
\[ d|x \Rightarrow nd|nx \]

Or, since \( n \in \mathbb{N}^* \) and so \( n1 = 1 + \ldots + 1 \) is regular (see the remark (4.10)), we have \( (d \in A[S]) \):
\[ nd|nx \Rightarrow d|x \]
It follows that \((d \in A[S] \text{ and } n \in \mathbb{N}^*)\):
\[d|x \iff nd|nx\]
and we have also \((d \in A[S] \text{ and } n \in \mathbb{N}^*)\):
\[d|N_A(x) \iff nd|nN(x)\]
So \((d \in A[S] \text{ and } n \in \mathbb{N}^*)\):
\[nd|nx \iff nd|nN(x)\]
\[d'|nx \iff d'|nN(x)\]
with \(d' = nd\). Consequently, renaming \(d'\) as \(d\), we have \((d \in A[S] \text{ and } n \in \mathbb{N}^*)\):
\[d|nx \iff d|nN(x)\]
Therefore, using the remark (3.5), we have \((x \notin \ker N \text{ and } N(x) \in A[S])\):
\[\mathcal{D}(nx) = \bigcup_{d|nx} dU(A) = \bigcup_{d|nN_A(x)} dU(A) = \mathcal{D}(nN(x))\]
Again, from the property (P.1) of the definition (4.3), if \(N(nx) \in A[S]\), then we have \((x \notin \ker N)\):
\[\mathcal{D}(nx) = \mathcal{D}(N(nx))\]
It results that \((x \notin \ker N, \ N(x) \in A[S] \text{ and } N(nx) \in A[S])\):
\[\mathcal{D}(N(nx)) = \mathcal{D}(nN(x))\]
Or, from the property (P.3) of the definition (4.3), using an immediate reasoning by induction, there exists an element \(z \in A[S]\) such that \(nN(x) = N(z)\). So, using the property (P.5) of the definition (4.3), we have \(N(nN(x)) = nN(x)\). Therefore since \(N(nx)\) is the unique element of \(\mathcal{D}(nx)\) such that \(N(N(nx)) = N(nx)\) (see the definition (4.3)) when \(N(nx) \in A[S]\), we get \(N(nx) = nN(x)\). \(\Box\)

**Corollary 4.21.** Let \(x\) be an element of \(A[S] \setminus (\ker N \setminus \{0\})\) and let \(k \in \mathbb{Z}\). Then we have:
\[N(kx) = \abs(k)N(x)\]
If \(U(A[S])\) has a finite order, since \(\forall v \in U(A[S]), \ v^{\abs(U(A[S])]} = 1\), we have \(N(v)^{\abs(U(A[S])]} = 1\). It comes that:
\[N(v)^{\abs(U(A[S])]} - 1 = 0 \ \forall \ v \in U(A[S])\]
\[(N(v) - 1) \{1 + \ldots + N(v)^{\abs(U(A[S])]} - 1\} = 0 \ \forall \ v \in U(A[S])\]
Since \(N(x_1) + \ldots + N(x_n) = 0 \iff x_1, \ldots, x_n \in \ker N\) with \(n \in \mathbb{N}^*\) (see the property (4.6) above) and since \(1 \neq 0\), the factor \(1 + \ldots + N(v)^{\abs(U(A[S])]} - 1\) is regular and so:
\[N(v) - 1 = 0 \ \forall \ v \in U(A[S])\]
\[N(v) = 1 \ \forall \ v \in U(A[S])\]
Therefore, using also the property (P.6) of the definition (4.3), we have:

**Corollary 4.22.** If \(U(A[S])\) has a finite order, then we have:
\[N(v) = 1 \ \forall \ v \in U(A[S])\]
\[N(xv) = N(x) \ \forall \ x \in A[S], \ \forall \ v \in U(A[S])\]

**Remark 4.23.** If \(U(A[S])\) has a finite order, then the endomorphism \(N\big|_{U(A[S])}\) is surjective such that \(N\big|_{U(A[S])}(U(A[S])) = \{1\}\).
**Property 4.24.** Let $x$ be an element of $A[S]$. Then we have:

$$N(x) = 1 \Rightarrow x \in U(A[S])$$

**Proof.** Let $x$ be an element of $A[S].$

If $N(x) = 1$, then from the property (P.2), $x \notin \ker N$. Using the property (P.8) of the definition (4.3), there exists an element $x' \notin \ker N$ of $A[S]$ such that $N(x)^2 = xx'$. Since $N(x) = 1$, it implies that $xx' = 1$. So, $x$ is invertible in $A[S]$ and its inverse is $x'$. Therefore, if $N(x) = 1$, then $x \in U(A[S])$. □

We saw that an element $x$ of $A[S]$ and its magnitude $N(x)$ differ from each other by an unit which is unique, when $x \notin \ker N$ such that $N(x) \in A[S]$ (see the corollary (4.19)). This result is extended in the following definition (4.25).

**Definition 4.25.** The unit function is the map defined on $A[S] \setminus \ker N$ which associates the unique element $u(x)$ of $\mathbb{F}$ to $x \notin \ker N$ of $A[S]$: 

$$u : A[S] \setminus \ker N \rightarrow \mathbb{F}$$

$$x \mapsto u(x)$$

with properties:

1. $x = u(x)N(x) \quad \forall \ x \in A[S] \setminus \ker N$ (P.9)
2. $\forall \ x \in A[S] \setminus \ker N : u(x) \in A[S] \setminus \ker N, \ u(u(x)) = u(x)$ (P.10)

For given $x \notin \ker N$ of $A[S]$, $u(x)$ is said to be the unit part of $x$.

**Remark 4.26.** For a given element $x$ of $A[S] \setminus \ker N$, the notation $u(x)^{-1}$ means the inverse of $u(x)$ in $\mathbb{F}$. It is possible that $u(x) \notin U(A[S])$. In such a case, $u(x)^{-1}$ is the inverse of $u(x)$ in $A[S]$. The theorem (4.38) allows to make the expression $u(x)^{-1}$ definite.

**Property 4.27.** Let $x$ be an element of $A[S] \setminus \ker N$. Then $(x \notin \ker N)$:

$$u(x) + u(-x) = 0$$

**Proof.** From the property (P.4) of the definition (4.3), using the definition (4.25), we have:

$$-x = -u(x)N(x) = -u(x)N(-x)$$

Or, from the property (P.9) of the definition (4.25), we have:

$$-x = u(-x)N(-x)$$

So:

$$u(-x)N(-x) = -u(x)N(-x)$$

Since $N(x) = N(-x)$ is regular for all $x \notin \ker N$ of $A[S]$ (see the remark (4.10)), it results that:

$$u(-x) = -u(x)$$

□

**Property 4.28.** Let $x, y$ be two elements of $A[S] \setminus \ker N$. Then $(x, y \notin \ker N)$:

$$u(xy) = u(x)u(y)$$
Proof. Let \( x, y \) be two elements of \( A[S] \setminus \ker N \). From the property (P.6) of the definition (4.3), we have \( (x, y \not\in \ker N) \):

\[
xy = u(xy)N(xy) = u(xy)N(x)N(y)
\]

Or, since \( x = u(x)N(x) \) and \( y = u(y)N(y) \), it comes that \( (x, y \not\in \ker N) \):

\[
xy = u(x)N(x)u(y)N(y) = u(x)u(y)N(x)N(y)
\]

So:

\[
u(xy)N(x)N(y) = u(x)u(y)N(x)N(y)
\]

Since \( x, y \) are non-zero and so regular (see the remark (4.10)), as well as \( N(x), N(y) \), it results that \( (x, y \not\in \ker N) \):

\[
u(xy) = u(x)u(y)
\]

\( \square \)

Property 4.29. Let \( x \) be an element of \( A[S] \).

If \( x \not\in \ker N \) and \( u(x) \in A[S] \), then we have \( (x \not\in \ker N) \):

\[
N(u(x)) = 1
\]

If \( N(x) \in A[S] \setminus \ker N \), then we have \( (N(x) \not\in \ker N) \):

\[
u(N(x)) = 1
\]

Proof. Let \( x \) be an element of \( A[S] \).

If \( x \not\in \ker N \) and \( u(x) \in A[S] \), then from the properties (P.5), (P.6) of the definition (4.3) and from the property (P.9) of the definition (4.25), we have \( (x \not\in \ker N \) and \( u(x) \in A[S] \):

\[
N(x) = N(u(x)N(x)) = N(u(x))N(N(x)) = N(u(x))N(x)
\]

\[
N(x)(N(u(x)) - 1) = 0
\]

Since \( x \not\in \ker N \), from the property (P.2) of the definition (4.3), \( N(x) \neq 0 \). So, since the set \( A[S] \) does not contain divisor of 0, we get \( (x \not\in \ker N \) and \( u(x) \in A[S] \):

\[
N(u(x)) - 1 = 0
\]

\[
N(u(x)) = 1
\]

Moreover, if \( N(x) \in A[S] \setminus \ker N \), then from the properties (P.9) and (P.10) of the definition (4.25) of the unit part of an element \( x \not\in \ker N \) of \( A \), using the property (4.28), we have \( (N(x) \in A[S] \setminus \ker N) \):

\[
u(x) = u(u(x)N(x)) = u(u(x))u(N(x))
\]

\[
u(x)u(x)u(N(x))
\]

\[
N(x)(u(N(x)) - 1) = 0
\]

Since \( u(x) \) for \( x \not\in \ker N \) is well defined and is regular (\( u(x) \) is invertible in \( \mathbb{F} \)), we have \( (N(x) \in A[S] \setminus \ker N) \):

\[
u(N(x)) - 1 = 0
\]

\[
u(N(x)) = 1
\]

Therefore, we conclude that if \( u(x) \in A[S] \) then \( (x \not\in \ker N) \):

\[
N(u(x)) = 1
\]

and if \( N(x) \in A[S] \setminus \ker N \) then \( (N(x) \not\in \ker N) \):

\[
u(N(x)) = 1
\]

\( \square \)
Remark 4.30. The equality $N(u(x)) = 1$ for all $x \in A[S] \setminus \ker N$ such that $u(x) \in A[S]$, is true even if $U(A[S])$ has not finite order.

Property 4.31. Let $x$ be an element of $A[S] \setminus \ker N$. Then we have ($x \not\in \ker N$):

$$N(x) = x \iff u(x) = 1$$

Proof. Let $x$ be an element of $A[S] \setminus \ker N$.

If $u_A(x) = 1$, then from the property (P.9) of the definition (4.25) of the unit part of an element $x \not\in \ker N$ of $A[S]$, it is obvious that $x = N(x)$.

Reciprocally, if $N(x) = x$, from the property (4.29), we have ($x \not\in \ker N$):

$$u(N(x)) = u(x) = 1$$

Therefore, we conclude that for any $x \not\in \ker N$ of $A[S]$, we have $N(x) = x \iff u(x) = 1$. □

Example 4.32. We know that $N(1) = 1$. So, we have, $u(1) = 1$.

Property 4.33. Let $x$ be a non-zero element of $A[S] \setminus \ker N$. Then we have ($x \not\in \ker N$):

$$N(x) = -x \iff u(x) = -1$$

Proof. Let $x$ be an element of $A[S] \setminus \ker N$.

If $u(x) = -1$, then from the property (P.9) of the definition (4.25) of the unit part of an element $x \not\in \ker N$ of $A[S]$, it is obvious that $x = N(x)$.

Reciprocally, if $N(x) = -x$, from the properties (4.27) and (4.29), we have ($x \not\in \ker N$):

$$u(N(x)) = u(-x) = -u(x) = 1$$

Therefore, we conclude that for any $x \not\in \ker N$ of $A[S]$, we have $N(x) = -x \iff u(x) = -1$. □

Example 4.34. We know that $N(-1) = N(1) = -(-1)$. So, we have, $u(-1) = -1$.

Remark 4.35. If $x \in U(A[S])$ (as for instance $x \in G \supset S$), then $x^{-1}$ exists and is unique in $A[S]$ and since $x \not\in \ker N$ in this case, from the property (P.8) of the definition (4.3), there exists an element $x'$ of $F \cap \mathbb{R}$ such that $N(x')^2 = xx' \in A[S]$. So, if $x \in U(A[S])$, since $A[S]$ is a ring, then $x' = x^{-1}N(x)^2$ exists and belongs to $A[S]$. It follows that ($x \in U(A[S])$ and so $x' \in A[S]$):

$$N(x') = N(x) \neq 0$$

Thus, if $x \in U(A[S])$, then $x' \not\in \ker N$.

Notice that $x'$ is unique. Indeed, let $x'' \in A[S] \setminus \ker N$ such that ($x, x', x'' \not\in \ker N$):

$$xx' = xx''$$

Then since $x \not\in \ker N$ and so since $x$ is regular, we get ($x', x'' \not\in \ker N$):

$$x' = x''$$

Moreover, since $x, x' \in A[S] \setminus \ker N$, we have ($x \in U(A[S])$ and so $x, x' \not\in \ker N$):

$$N(x)^2 = u(x)N(x)u(x')N(x') = N(x)u(x)u(x')N(x')$$

$$N(x)(N(x) - u(x)u(x')N(x')) = 0$$

Since $N(x)$ is regular for $x \not\in \ker N$ of $A[S]$ (see the remark (4.10)), it comes that ($x \in U(A[S])$ and $x, x' \not\in \ker N$):

$$N(x) - u(x)u(x')N(x') = 0$$
\[ N(x) = u(x)u(x')N(x') \]
or equivalently \((x \in U(A[S])\) and so \(x, x' \not\in \ker N)\):
\[ u(x)^{-1}N(x) = u(x')N(x') = x' \]

We shall identify \(x'\) when \(x \in U(A[S])\), with what we call \(x^*\) (see below the definition (4.36)).

**Definition 4.36.** The operation \(\ast\) is the involution defined on \(A[S]\), which maps element \(x\) of \(A[S]\) to element \(x^*\) of \(A[S]\):

\[
\ast : A[S] \longrightarrow A[S] \\
x \mapsto x^* = \begin{cases} 
  x & \text{if } x \not\in U(A[S]) \\
  x' & \text{if } x \in U(A[S])
\end{cases}
\]
with \(x'\) defined in the remark (4.35) and properties:

\[
\begin{align*}
(x^*)^* &= x \quad \forall \ s \in A[S] & (P.11) \\
(x + y)^* &= x^* + y^* \quad \forall \ x, y \in A[S] & (P.12) \\
(xy)^* &= y^*x^* \quad \forall \ x, y \in A[S] & (P.13)
\end{align*}
\]

The element \(x^*\) of \(A[S]\) is said to be the conjugate of the element \(x\) of \(A[S]\).

**Remark 4.37.**

\[
\begin{align*}
0^* &= 0 \\
1^* &= 1 \\
(-1)^* &= -1
\end{align*}
\]

For instance, let prove that \(0^* = 0\). Indeed, using the property (P.12) of the definition (4.36), we have \((x \in A[S])\):
\[
(x + 0)^* = x^* + 0^*
\]

Or, \(x + 0 = x\). It comes that \((x \in A[S])\):
\[
\begin{align*}
x^* &= x^* + 0^* \\
x^* + 0 &= x^* + 0^* \\
0 &= 0^*
\end{align*}
\]
Moreover, let prove that \(1^* = 1\). Indeed, using the property (P.13) of the definition (4.36), we have \((x \in A[S])\):
\[
(x1)^* = 1^*x^*
\]
Since \(x1 = 1x = x\), we obtain \((x \in A[S])\):
\[
\begin{align*}
x^* &= 1^*x^* \\
1x^* &= 1^*x^*
\end{align*}
\]
In particular, it is true when \(x\) is regular meaning that:
\[
1 = 1^*
\]

**Theorem 4.38.** Let \(x\) be an element of \(A[S]\). Then we have \((x \in A[S])\):
\[
N(x^*) = N(x)
\]
Moreover, if \(x \in U(A[S])\), then we have:
\[
u(x^*) = u(x)^{-1}
\]

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Proof. If \( x \notin U(A[S]) \), from the definition \((4.36)\), we have \( x = x^* \). It implies that \( (x \notin U(A[S])): \)

\[ N(x) = N(x^*) \]

If \( x \in U(A[S]) \), then from the definition \((4.36)\) we have \( x^* = x' \) and from the remark \((4.35)\), we know that \( (x \in U(A[S])): \)

\[ N(x) = N(x') = N(x^*) \]

Moreover, if \( x \in U(A[S]) \), then from the remark \((4.35)\), we know that:

\[ u(x)^{-1}N(x) = u(x')N(x') = x' \]

Using \( N(x) = N(x') \), it comes that:

\[ u(x)^{-1}N(x) = u(x')N(x) \]

Since \( x' \notin \ker N \), \( N(x) \) is regular. It results that \( (x \in U(A[S])): \)

\[ u(x') = u(x)^{-1} \]

Therefore, if \( x \in U(A[S]) \), then we have:

\[ u(x^*) = u(x)^{-1} \]

\[ \square \]

**Property 4.39.** Let \( x \) be an element of \( A[S] \setminus \ker N \). Then we have \( (x \notin \ker N) : \)

\[ x^* = x \iff u(x) \in \{1,-1\} \]

Proof. Let \( x \) be an element of \( A[S] \setminus \ker N \).

Using the definition \((4.26)\) of the unit part of a non-zero element \( x \) of \( A[S] \), we have \( (x \notin \ker N_A) : \)

\[ x^* = x \iff u(x^*)N(x^*) = u(x)N(x) \iff u(x^*)N(x) = u(x)N(x) \]

Since for \( x \notin \ker N \), \( N(x) \) is regular (see the remark \((4.10)\)), we have \( (x \notin \ker N) : \)

\[ x^* = x \iff u(x^*) = u(x) \iff u(x)^{-1} = u(x) \iff u(x)^2 = 1 \]

\[ x^* = x \iff u(x)^2 - 1 = 0 \iff (u(x) - 1)(u(x) + 1) = 0 \]

Since there doesn’t exist divisor of zero in \( A[S] \), we get \( (x \notin \ker N) : \)

\[ x^* = x \iff u(x) - 1 = 0 \text{ or } u(x) + 1 = 0 \]

\[ x^* = x \iff u(x) = 1 \text{ or } u(x) = -1 \]

\[ \square \]

**Property 4.40.** Let \( x \) be a non-zero element of \( A[S] \setminus \ker N \). If there exists an units \( i \) of \( U(A[S]) \) such that \( i^2 + 1 = 0 \), then we have \( (x \notin \ker N) : \)

\[ x^* = -x \iff u(x) \in \{i,-i\} \]

Proof. Let \( x \) be an element of \( A[S] \setminus \ker N \).

Using the property \((12.9)\) of the definition \((4.25)\) of the unit part of an element \( x \) of \( A[S] \setminus \ker N \), we have \( (x \notin \ker N) : \)

\[ x^* = -x \iff u(x^*)N(x^*) = -u(x)N(x) \iff u(x^*)N(x) = -u(x)N(x) \]

Since for \( x \notin \ker N \), \( N(x) \) is regular (see the remark \((4.10)\)), we have \( (x \notin \ker N) : \)

\[ x^* = -x \iff u(x^*) = -u(x) \iff u(x)^{-1} = -u(x) \iff u(x)^2 = -1 \]

\[ x^* = -x \iff u(x)^2 + 1 = 0 \]

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Assuming that there exists an unit $i$ in $U(A[S])$ such that $i^2 + 1 = 0$, we have $(x \not\in \ker N)$:

$$(u(x) - i)(u(x) + i) = 0$$

Since there doesn’t exist divisor of zero in $A[S]$, we get $(x \not\in \ker N)$:

$$u(x) - i = 0 \text{ or } u(x) + i = 0$$
$$u(x) = i \text{ or } u(x) = -i$$

It results that $(x \not\in \ker N)$:

$$x^* = -x \iff u(x) \in \{i, -i\} \square$$

**Remark 4.41.** Assuming that there exists an unit $i$ in $U(A[S])$ such that $i^2 + 1 = 0$, we have:

$$N(i^2) = N(-1) = N(1) = 1$$

Since $N(i^2) = N(i)^2$, it follows that:

$$N(i)^2 = 1 \iff N(i)^2 - 1 = 0 \iff (N(i) - 1)(N(i) + 1) = 0$$

Since there doesn’t exist divisor of zero in $A[S]$, we get:

$$N(i)^2 = 1 \iff N(i) - 1 = 0 \text{ or } N(i) + 1 = 0$$
$$N(i)^2 = 1 \iff N(i) = 1 \text{ or } N(i) = -1$$

But, if $N(i) = -1$, then $N(N(i)) = N(i) = N(-1) = 1$. We reach to a contradiction meaning that:

$$N(i) = 1$$

It results that:

$$u(i) = i$$

From the property (4.40), we deduce that:

$$i^* = -i$$

**Property 4.42.** Let $x$ be an element of $A[S] \setminus \ker N$. Then in $F$, we have $(x \not\in A[S] \setminus \ker N)$:

$$xu(x^*) = x^*u(x)$$

**Proof.** From the property (P.2) of the definition (4.25), using the theorem (4.38), we have in $F$ $(x \not\in \ker N)$:

$$u(x)x^* = u(x)N(x^*)u(x^*) = u(x)N(x)u(x^*) = xu(x^*) \square$$

**Example 4.43.** Let $A = \mathbb{Z}$ be the ring of integers, let $S = \{i\}$ such that $i^2 + 1 = 0$ and let $F = \mathbb{C}$ equipped with the usual modulus norm $|| \cdot ||_c$. So, $A[S]$ is the subring $\mathbb{Z}[i]$ of gaussian integers in $\mathbb{C}$. Let calculate the magnitude and the unit part of the element $1 + i$. Using the property (4.42), we have:

$$(1 + i)u(1 - i) = (1 - i)u(1 + i)$$

Since:

$$1 - i = -i(1 + i)$$

and since $-i = i^* = i^{-1} \in U(A[S])$, we can notice that:

$$\mathcal{D}(1 - i) = \mathcal{D}(-i(1 + i)) = \mathcal{D}(1 + i)$$

From the property (4.18), there exists a unique element $u$ of $U(A[S])$ such that:

$$1 - i = u(1 + i)$$
Since $1 - i = -i(1 + i)$, we have $u = -i$. From the equality $(1 + i)u(1 - i) = (1 - i)u(1 + i)$, since the elements $1 \pm i \neq 0$ are regulars (see the remark (4.10), we have:

\[ u(1 - i) = -iu(1 + i) \]
\[ u(1 - i)u^{-1}(1 + i) = -i \]
\[ u(1 - i)^2 = -i = \left( \frac{1 - i}{\sqrt{2}} \right)^2 \]
\[ u(1 - i)^2 - \left( \frac{1 - i}{\sqrt{2}} \right)^2 = 0 \]
\[ \left( u(1 - i) - \frac{1 - i}{\sqrt{2}} \right) \left( u(1 - i) + \frac{1 - i}{\sqrt{2}} \right) = 0 \]

Since $A[S]$ is an entire ring, we get:

\[ u(1 - i) - \frac{1 - i}{\sqrt{2}} = 0 \quad \text{or} \quad u(1 - i) + \frac{1 - i}{\sqrt{2}} = 0 \]
\[ u(1 - i) = \frac{1 - i}{\sqrt{2}} \quad \text{or} \quad u(1 - i) = -\frac{1 - i}{\sqrt{2}} \]

If absurdly $u(1 - i) = -\frac{1 - i}{\sqrt{2}}$, then we would have:

\[ 1 - i = -\frac{1 - i}{\sqrt{2}} N(1 - i) \]
\[ (1 - i) \left\{ 1 + \frac{N(1 - i)}{\sqrt{2}} \right\} = 0 \]

Since $1 - i \neq 0$ is regular, it would result that:

\[ 1 + \frac{N(1 - i)}{\sqrt{2}} = 0 \]

Since $\sqrt{2} \in \mathbb{R}$ is invertible in $\mathbb{R}$, it would imply that:

\[ N(1 - i) = -\sqrt{2} \]

and so:

\[ ||N(1 - i)||_c = ||-\sqrt{2}||_c = \sqrt{2} \]

But it would contradict the fact that (see the property (P.7) of the definition (4.3)):

\[ ||N(1 - i)||_c = N(1 - i) \]

It means that:

\[ u(1 - i) = \frac{1 - i}{\sqrt{2}} \]

which implies that:

\[ N(1 - i) = \sqrt{2} \]

Therefore:

\[ u(1 + i) = \frac{1 + i}{\sqrt{2}} \]

and:

\[ N(1 + i) = \sqrt{2} \]
Remark 4.44. Using the properties (P.11) and (P.12), we can notice that:
\[(x + x^*)^* = x^* + x\]
From the property (4.39), it means that:
\[u(x + x^*) \in \{-1; 1\}\]
Using the property (P.9) of the definition (4.25) and using the definition (4.3), it gives:
\[x + x^* = \pm N(x + x^*) \in \mathbb{F} \cap \mathbb{R}\]
Thus, we can set the following definition.

Definition 4.45. The real function \(\text{Re}\) is the map defined on \(A[S]\) which associates the element \(\text{Re}(x) = \frac{x + x^*}{2}\) of the set \(\mathbb{F} \cap \mathbb{R}\), to \(x \in A[S]\):
\[
\text{Re} : A[S] \rightarrow \mathbb{F} \cap \mathbb{R} \\
x \mapsto \text{Re}(x) = \frac{x + x^*}{2}
\]
The number \(\text{Re}(x)\) is called the real part of the element \(x\) of \(A[S]\).

Property 4.46. Let \(x\) be an element of \(A[S]\). Then, we have:
\[\text{Re}(x^*) = \text{Re}(x)\]
Proof. The property (4.46) follows from the definition (4.45) of the real function \(\text{Re}\).

Property 4.47. Let \(x, y\) be two elements of \(A[S]\). Then we have:
\[\text{Re}(x + y) = \text{Re}(x) + \text{Re}(y)\]
Proof. The property (4.47) follows from the definition (4.45) of the real function \(\text{Re}\).

Property 4.48. Let \(a\) be an element of \(A[S] \setminus U(A[S])\). Then for all \(x \in A[S]\), we have (\(a \not\in U(A[S])\)):
\[\text{Re}(ax) = a\text{Re}(x)\]
Proof. Let \(a\) be an element of \(A[S] \setminus U(A[S])\). From the definition (4.36), we have \(a^* = a\). Using the definition (4.45) of the real function, we have (\(x \in A[S]\) and \(a \not\in U(A[S])\)):
\[
\text{Re}(ax) = \frac{ax + x^*a^*}{2} = \frac{ax + x^*a}{2} \\
\text{Re}(ax) = \frac{ax + ax^*}{2} = a\frac{x + x^*}{2} = a\text{Re}(x)
\]

Remark 4.49. Using the properties (P.11) and (P.12) with the remark (4.41), we can notice that:
\[(-i(x - x^*))^* = (x^* - x)(-i) = -i(x - x^*)\]
From the property (4.39), it means that:
\[u(x + x^*) \in \{-1; 1\}\]
Using the property (P.9) of the definition (4.25) and using the definition (4.3), it gives:
\[-i(x - x^*) = \pm N(x + x^*) \in \mathbb{F} \cap \mathbb{R}\]
Thus, we can set the following definition.
**Definition 4.50.** The imaginary function $\text{Im}$ is the map defined on $A[S]$ which associates the element $\text{Im}(x) = -i\frac{x-x^*}{2}$ of the set $\mathbb{F} \cap \mathbb{R}$, to $x \in A[S]$:  
\[
\text{Im} : A[S] \rightarrow \mathbb{F} \cap \mathbb{R} \\
x \mapsto \text{Im}(x) = -i\frac{x-x^*}{2}
\]
The number $\text{Im}(x)$ is called the imaginary part of the element $x$ of $A[S]$.

**Property 4.51.** Let $x$ be an element of $A[S]$. Then, we have:  
$\text{Im}(x^*) = -\text{Im}(x)$

**Proof.** The property (4.51) follows from the definition (4.50) of the real function $\text{Im}$. □

**Property 4.52.** Let $x, y$ be two elements of $A[S]$. Then we have:  
$\text{Im}(x + y) = \text{Im}(x) + \text{Im}(y)$

**Proof.** The property (4.52) follows from the definition (4.50) of the imaginary function $\text{Im}$. □

**Property 4.53.** Let $a$ be an element of $A[S] \setminus U(A[S])$. Then for all $x \in A[S]$, we have ($a \not\in U(A[S])$):  
$\text{Im}(ax) = a\text{Im}(x)$

**Proof.** Let $a$ be an element of $A[S] \setminus U(A[S])$. So, we have $a^* = a$. Using the definition (4.50) of the imaginary function, we have ($x \in A[S]$ and $a \not\in U(A[S])$):
\[
\text{Im}(ax) = -i\frac{ax - x^*a^*}{2} = -i\frac{ax - x^*a}{2} \\
\text{Im}(ax) = -i\frac{ax - ax^*}{2} = a\left(-i\frac{x-x^*}{2}\right) = a\text{Im}(x)
\]
□

**Theorem 4.54.** For all $x \in A[S]$, we have:  
$x = \text{Re}(x) + i\text{Im}(x)$

**Proof.** Using the definitions (4.45) and (4.50), for all $x \in A[S]$, we have:  
$\text{Re}(x) + i\text{Im}(x) = \frac{x + x^*}{2} + i(-i)\frac{x - x^*}{2}$

Since $i^2 + 1 = 0$, it gives:  
$\text{Re}(x) + i\text{Im}(x) = \frac{x + x^*}{2} + \frac{x - x^*}{2} = \frac{x + x^* + x - x^*}{2} = \frac{2x}{2} = x$

So, we deduce that:  
$\text{Re}(x) + i\text{Im}(x) = x$

□

**Corollary 4.55.** Let $x$ be an element of $A[S]$. Then, we have:  
$x^* = \text{Re}(x) - i\text{Im}(x)$
Proof. From the theorem (4.54), we have:
\[ x^* = \Re(x^*) + i\Im(x^*) \]

Using the properties, it results that:
\[ x^* = \Re(x) - i\Im(x) \]

□

Remark 4.56. Since \(0^* = 0\), using also the theorem (4.54) and the corollary (4.55), we have:
\[ x = 0 \Rightarrow x^* = 0^* = 0 \]

and:
\[ \Re(x) + i\Im(x) = 0 \]
\[ \Re(x) - i\Im(x) = 0 \]

Taking the sum and the difference side by side of these two equations, since \(2 \neq 0\) is regular, we deduce that:
\[ x = 0 \iff \begin{cases} 
\Re(x) = 0 \\
\text{and} \\
\Im(x) = 0 
\end{cases} \]

Definition 4.57. Let \(k \in \mathbb{N}^*\) and \(\{f_1, \ldots, f_k\}\) be a family of a subfield \(\mathbb{F}\) of \(\mathbb{C}\). We said that the family \(\{f_1, \ldots, f_k\}\) is free over a subring \(A\) of \(\mathbb{C}\), if for \(a_1, \ldots, a_k \in A\):
\[ \sum_{i=1}^{k} a_i f_i = 0 \Rightarrow a_1 = \ldots = a_k = 0 \]

In other words, when a family \(\{f_1, \ldots, f_k\}\) of a subfield \(\mathbb{F}\) of \(\mathbb{C}\) is free over a subring \(A\) of \(\mathbb{C}\), it means that the elements \(f_1, \ldots, f_k\) are linearly independent over \(A\).

Theorem 4.58. Let \(e = \{e_0, e_1, \ldots, e_{n-1}\}\) with \(n \in \mathbb{N}^*\) be a maximal free family of elements of \(G\) over \(\mathbb{C}\) with:
\[ e_0 = e_n = 1 \]

such that \((i = 1, \ldots, n-1)\):
\[ \Im(e_i) \neq 0 \]

If for all \(a \in A\), \(a^* = a\), then the image of the free family \(e \setminus \{e_0\} = \{e_1, \ldots, e_{n-1}\}\) with \(n \in \mathbb{N}^*\), of elements of \(G\) under \(\Im\), is a free family of elements of \(\mathbb{F} \cap \mathbb{R}\) over \(A\).

Proof. Let \(e = \{e_0, e_1, \ldots, e_{n-1}\}\) with \(n \in \mathbb{N}^*\) be a maximal free family of elements of \(G\) over \(\mathbb{C}\) with:
\[ e_0 = e_n = 1 \]

such that \((i = 1, \ldots, n-1)\):
\[ \Im(e_i) \neq 0 \]

If for elements \(a_1, \ldots, a_{n-1}\) of \(A\) with \(n \in \mathbb{N}^*\):
\[ \sum_{i=1}^{n-1} a_i \Im(e_i) = 0 \]
then since \( a_i^* = a_i \) for all \( i \in \{1, \ldots, n-1\} \), using the properties (4.32) and (4.33), we have \( (n \in \mathbb{N}^*) \):

\[
\text{Im} \left( \sum_{i=1}^{n-1} a_i e_i \right) = 0
\]

which implies that (see also the properties (4.47) and (4.48), \( n \in \mathbb{N}^* \)):

\[
\sum_{i=1}^{n-1} a_i e_i = \text{Re} \left( \sum_{i=1}^{n-1} a_i e_i \right) = \sum_{i=1}^{n-1} a_i \text{Re}(e_i)
\]

Since \( \text{Re}(e_i) \in \mathbb{F} \cap \mathbb{R} \), then there exists an element \( c_0 \) of \( \mathbb{F} \cap \mathbb{R} \) such that \( (n \in \mathbb{N}^*) \):

\[
c_0 e_0 = \sum_{i=1}^{n-1} a_i \text{Re}(e_i)
\]

It gives \( (n \in \mathbb{N}^*) \):

\[
\sum_{i=1}^{n-1} a_i e_i = c_0 e_0
\]

Since \( e = \{e_0, e_1, \ldots, e_{n-1}\} \) is a free family over \( \mathbb{C} \), it implies that \( (n \in \mathbb{N}^*) \):

\[
c_0 = a_1 = \ldots = a_{n-1} = 0
\]

So, we get \( (n \in \mathbb{N}^* \) and \( a_1, \ldots, a_{n-1} \in A) \):

\[
\sum_{i=1}^{n-1} a_i \text{Im}(e_i) = 0 \Rightarrow a_1 = \ldots = a_{n-1} = 0
\]

It means that \( \{\text{Im}(e_1), \ldots, \text{Im}(e_{n-1})\} \) with \( n \in \mathbb{N}^* \) is a free family of elements of \( \mathbb{F} \cap \mathbb{R} \) over \( A \). \( \square 

**Theorem 4.59.** Let \( e = \{e_0, e_1, \ldots, e_{n-1}\} \) with \( n \in \mathbb{N}^* \) be a maximal free family of elements of \( G \) over \( \mathbb{C} \) with:

\[
e_0 = e_n = 1
\]

such that \( (i = 1, \ldots, n-1) \):

\[
\text{Re}(e_i) \neq 0
\]

If for all \( a \in A \), \( a^* = a \), then the image of the free family \( e \setminus \{e_0\} = \{e_1, \ldots, e_{n-1}\} \) with \( n \in \mathbb{N}^* \), of elements of \( G \) under \( \text{Re} \), is a free family of elements of \( \mathbb{F} \cap \mathbb{R} \) over \( A \).

**Proof.** Let \( e = \{e_0, e_1, \ldots, e_{n-1}\} \) with \( n \in \mathbb{N}^* \) be a maximal free family of elements of \( G \) with:

\[
e_0 = e_n = 1
\]

such that \( (i = 1, \ldots, n-1) \):

\[
\text{Re}(e_i) \neq 0
\]

If for elements \( a_1, \ldots, a_{n-1} \) of \( A \) with \( n \in \mathbb{N}^* \):

\[
\sum_{i=1}^{n-1} a_i \text{Re}(e_i) = 0
\]

then since \( a_i^* = a_i \) for all \( i \in \{1, \ldots, n-1\} \), using the properties (4.37) and (4.38), we have \( (n \in \mathbb{N}^*) \):

\[
\text{Re} \left( \sum_{i=1}^{n-1} a_i e_i \right) = 0
\]
which implies that (see also the properties (4.52) and (4.53), \( n \in \mathbb{N}^* \)):

\[
\sum_{i=1}^{n-1} a_i e_i = i \text{Im} \left( \sum_{i=1}^{n-1} a_i e_i \right) = i \sum_{i=1}^{n-1} a_i \text{Im}(e_i)
\]

Since \( \text{Im}(e_i) \in \mathbb{F} \cap \mathbb{R} \), then there exists an element \( c_0 \) of \( \mathbb{F} \cap \mathbb{R} \) such that \( (n \in \mathbb{N}^*) \):

\[
c_0 e_0 = \sum_{i=1}^{n-1} a_i \text{Im}(e_i)
\]

It gives \( (n \in \mathbb{N}^*) \):

\[
\sum_{i=1}^{n-1} a_i e_i = ic_0 e_0
\]

Since \( e = \{e_0, e_1, \ldots, e_{n-1}\} \) is a free family over \( \mathbb{C} \), it implies that \( (n \in \mathbb{N}^*) \):

\[
c_0 = a_1 = \ldots = a_{n-1} = 0
\]

So, we get \( (n \in \mathbb{N}^* \text{ and } a_1, \ldots, a_{n-1} \in A) \):

\[
\sum_{i=1}^{n-1} a_i \text{Im}(e_i) = 0 \Rightarrow a_1 = \ldots = a_{n-1} = 0
\]

It means that \( \{\text{Re}(e_1), \ldots, \text{Re}(e_{n-1})\} \) with \( n \in \mathbb{N}^* \) is a free family of elements of \( \mathbb{F} \cap \mathbb{R} \) over \( A \). \( \square \)

**Property 4.60.** Let \( x \) be an element of \( A[S] \). Then we have:

\[
x x^* = N(x)^2
\]

**Proof.** Let \( x \) be an element of \( A[S] \). If \( x \in \ker N \), then since \( 0^* = 0 \) (see the remark (4.37)), using the property (P.2) of the definition (4.3), we have \( (x \in \ker N) \):

\[
00^* = 00 = 0 = N(x)
\]

In the following, we assume that \( x \not\in \ker N \). From the property (P.9) of the definition (4.25), using the theorem (4.38), we have \( (x \not\in \ker N) \):

\[
x x^* = u(x) N(x) u(x^*) N(x^*) = u(x) N(x) u(x)^{-1} N(x)
\]

\[
x x^* = u(x) u(x)^{-1} N(x) N(x) = N(x)^2
\]

\( \square \)

**Remark 4.61.** Let \( x, y \) be two elements of \( A[S] \). Then we have:

\[
N(x + y)^2 = (x + y)(x + y)^* = (x + y)(x^* + y^*)
\]

\[
N(x + y)^2 = xx^* + xy^* + yx^* + yy^*
\]

\[
N(x + y)^2 = N(x)^2 + 2 \text{Re}(xy^*) + N(y)^2
\]

So, \( N \) satisfies a triangular inequality if, and only if \( \text{Re}(xy^*) \leq N(xy) \). In this case, \( N \) behaves as a norm on \( A[S] \).
**Definition 4.62.** The radius function is the function defined on $A[S]$ which associates the unique element $\sqrt{xx^*}$ of $\mathbb{R}_+$ to $x \in A[S]:$

$$r : A[S] \rightarrow \mathbb{R}_+$$

$$x \mapsto r(x) = \sqrt{xx^*}$$

with property:

$$||r(x)||_F = r(x)$$ (P.14)

**Corollary 4.63.** Let $x$ be an element of $A[S]$. Then, we have:

$$N(x) = r(x)$$

**Proof.** Let $x$ be an element of $A[S]$. Using the property (4.60), we have ($x \in A[S]$):

$$N(x)^2 = xx^* = r^2(x)$$

So, either $N(x) = -r(x)$ or $N(x) = r(x)$. If absurdly, $N(x) = -r(x)$, from the property (P.7) of the definition (4.3), then we have:

$$||N(x)||_F = N(x) = -r(x)$$

Using the property (P.14) of the definition (4.62), it gives:

$$||N(x)||_F = ||r(x)||_F = r(x)$$

It results that $||N(x)||_F = r(x)$ which contradicts the assumption. Therefore, we have:

$$||N(x)||_F = N(x) = r(x)$$

□

Recall that any non-zero element of $A[S]$ is invertible in $\mathbb{F}$. Thus, for any non-zero element $x$ of $A[S]$, the fraction $\frac{1}{x}$ defined on $\mathbb{F}$ means also the inverse $x^{-1}$ in $\mathbb{F}$.

**Property 4.64.** Let $x$ be an element of $A[S] \setminus \ker N$. Then in $\mathbb{F}$, we have ($x \notin \ker N$):

$$\frac{1}{x} = \frac{1}{N(x)^2} x^*$$

**Proof.** Let $x$ be an element of $A[S] \setminus \ker N$. We have obviously:

$$xx^* = xx^*$$

Since $x \neq 0$ is assumed to be invertible in $\mathbb{F}$, then in $\mathbb{F}$, we have ($x \notin \ker N$):

$$\frac{1}{x} xx^* = x^*$$

Since $xx^* \neq 0$ is invertible in $\mathbb{F}$, then from the property (4.60), in $\mathbb{F}$, we have ($x \notin \ker N$):

$$\frac{1}{x} = x^* \frac{1}{xx^*} = \frac{1}{N(x)^2} x^*$$

□

**Property 4.65.** Let $x$ be an element of $A[S] \setminus \ker N$. Then in $\mathbb{F}$, we have ($x \notin \ker N$):

$$u(x) = \frac{1}{N(x)} x$$

□
Proof. Let \( x \) be an element of \( A[S] \setminus \ker N \). Using the definition (4.25), we have \( x \notin \ker N \):

\[
x = N(x)u(x)
\]

Then in \( \mathbb{F} \), we have \( x \notin \ker N \):

\[
\frac{1}{N(x)}x = u(x)
\]

\( \square \)

**Property 4.66.** If \( || \cdot ||_F \) is an extension of \( r \) to \( \mathbb{F} \), then for all \( x \notin \ker N \) of \( A[S] \), in \( \mathbb{F} \), we have \( x \notin \ker N \):

\[
||u(x)||_F = 1
\]

**Proof.** Let \( x \) be an element of \( A[S] \setminus \ker N \). Then in \( \mathbb{F} \), we have \( x \notin \ker N \):

\[
||u(x)||_F = \left|\left| \frac{1}{N(x)}x \right|\right|_F = \left|\left| \frac{1}{N(x)} \right|\right|_F ||x||_F
\]

\[
||u(x)||_F = \frac{1}{||N(x)||_F} ||x||_F = \frac{1}{r(x)}r(x) = 1
\]

\( \square \)

**Property 4.67.** Let \( x \) be an element of \( A[S] \setminus \ker N \). If \( || \cdot ||_F \) is an extension of \( r \) to \( \mathbb{F} \) and if \( \cdot \) is an extension of \( * \) operation to \( \mathbb{F} \) such that in \( \mathbb{F} \) \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{||x||_F^2}x
\]

then in \( \mathbb{F} \) \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{x}
\]

**Proof.** Let \( x \) be an element of \( A[S] \setminus \ker N \). From the property (4.64), we have \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{N(x)^2}x
\]

Using the property (4.11) of the definition (4.36) and the theorem (4.38), it comes that \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{N(x)^2}x
\]

Since \( || \cdot ||_F \) is an extension of \( r \) to \( \mathbb{F} \), using the corollary (4.63), it gives \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{||x||_F^2}x = \frac{1}{x}
\]

Or, since \( \cdot \) is an extension of \( * \) operation to \( \mathbb{F} \), we obtain \( x \notin \ker N \):

\[
\frac{1}{x} = \frac{1}{x}
\]

\( \square \)
5. The fundamental theorem of arithmetic in an entire ring

In this section, we shall consider an entire subring $A$ of a subfield $\mathbb{F}$ of $\mathbb{C}$ such that:

\[ \forall a \in A \setminus \{0\}, \ a \notin \ker N \]
\[ \forall a \in A, \ N(a) \in A \]

**Definition 5.1.** Let $a, b$ be two elements of $A$.

Let $\mathcal{D}(a, b)$ be the set of common divisors of $a, b$. If there exists an element $g$ of $\mathcal{D}(a, b)$ such that whatever $d \in \mathcal{D}(a, b)$, $d|g$ and if $N(g)$ belongs to $A \setminus (U(A) \setminus \{1\})$, then $N(g)$ is defined as the greatest common divisor of $a, b$. The element $N(g)$ of $\mathcal{D}(a, b)$ is denoted $\gcd(a, b)$. Of course, if $\gcd(a, b)$ exists, then $\gcd(a, b) = \gcd(b, a)$.

If $\mathcal{D}(a, b) = U(A)$, by convention, we set $\gcd(a, b) = 1$. In such a case, the two elements $a, b$ are said to be relatively primes. In particular, $\gcd(a, b) = 1$ for all $a \in U(A)$ and for all $b \in A$. We have also $\gcd(1, a) = 1$ for all $a \in A$. Moreover, for any $v, v' \in U(A)$, $\gcd(v, v') = U(A)$ and $\gcd(v, v') = 1$.

A non-zero element $p$ of $A$ is said to be irreducible if, and only if, $p \notin U(A) \cup \{0\}$ and we have $\mathcal{D}(p) = U(A) \cup pU(A)$. Moreover, a non-zero irreducible element $p$ of $A$ is said to be a prime if, and only if, $N(p) = p$. By convention, the elements of $U(A)$ are not irreducible. So, any irreducible element of $A$ is not invertible. Moreover, when $p$ of $A$ is prime, we have $\gcd(a, p) = 1$ if $p \nmid a$ and $\gcd(a, p) = p$ if $p \mid a$.

Let $\mathcal{M}(a, b)$ be the set of common multiples of $a, b$. If there exists an element $\ell$ of $\mathcal{M}(a, b)$ such that whatever $m \in \mathcal{M}(a, b)$, $\ell|m$ and if $N(\ell)$ belongs to $A \setminus (U(A) \setminus \{1\})$, then $N(\ell)$ is defined as the least common multiple of $a, b$. The element $N(\ell)$ of $\mathcal{M}(a, b)$ is denoted $\lcm(a, b)$. Of course, if $\lcm(a, b)$ exists, then $\lcm(a, b) = \lcm(b, a)$.

**Remark 5.2.**

\[ \mathcal{D}(a, b) = \mathcal{D}(a) \cap \mathcal{D}(b) \]
\[ U(A) \subseteq \mathcal{D}(a, b) \]
\[ \mathcal{M}(a, b) = \mathcal{M}(a) \cap \mathcal{M}(b) = aA \cap bA \]

**Property 5.3.** If $A$ contains at least a prime element and if $U(A)$ has finite order, then any non-zero element of $A$ which doesn’t belong to $U(A) \cup \{0\}$ such that $|\mathcal{D}(x)|$ is finite, has a prime divisor.

**Proof.** We assume that $A$ contains at least a prime element.

Let $x$ be a non-zero element of $A$ which does not belong to $U(A) \cup \{0\}$ such that $|\mathcal{D}(x)|$ is finite. Since $x \in \mathcal{D}(x)$, $\mathcal{D}(x)$ is non-empty. If $x$ is prime, we find a prime divisor of $x$, namely $x$ itself.

Using the remark $\square_3$, there exists a non-zero element $d_1$ of $\mathcal{D}(x)$ which does not belong to $U(A)$ such that $\mathcal{D}(d_1) \subseteq \mathcal{D}(x)$ and:

\[ x = a_1d_1 \]

with $a_1 \in A \setminus \{0\}$. Notice that $\mathcal{D}(d_1)$ is non-empty since $d_1 \in \mathcal{D}(d_1)$ and $\gcd(a_1, d_1)$ is not necessarily equal to 1. Notice also that if we cannot find $d_1$ in $\mathcal{D}(x)$ such that $a_1 \notin U(A)$, then $\mathcal{D}(x) = U(A) \cup xU(A)$ and so $N(x)$ is prime. If $d_1$ is not a prime element of $A$ and if $a_1 \notin U(A)$, since $d_1$ is a non-zero element of $A$ which does not belong to $U(A)$, there exists a non-zero element $d_2$ of $\mathcal{D}(d_1)$ which does not belong to $U(A)$ such that $\mathcal{D}(d_2) \subseteq \mathcal{D}(d_1)$ with $\mathcal{D}(d_2) \neq \emptyset$ and:

\[ d_1 = a_2d_2 \]
So:

\[ x = a_1a_2d_2 \]

with \( a_1 \in A \setminus (U(A) \cup \{0\}) \) and \( a_2 \in A \setminus \{0\} \). Notice that \( \text{gcd}(a_2, d_2) \) as well as \( \text{gcd}(a_1a_2, d_2) \) is not necessarily equal to 1. Notice also that if we cannot find \( d_2 \) in \( D(d_1) \) such that \( a_2 \notin U(A) \), then \( D(d_1) = U(A) \cup d_1U(A) \) and so \( N(d_1) \) is prime. If \( d_2 \) is not a prime element of \( A \) and \( a_2 \notin U(A) \), we follow the same steps than above. Thus, we get a sequence \((D(d_i))\) of nested non-empty subsets of \( A \) such that \( (x \notin U(A) \cup \{0\}) \):

\[
D(d_0) = D(x) \\
D(d_{i+1}) \subseteq D(d_i) \quad \text{with} \quad d_i, d_{i+1} \in D(x) \setminus U(A) \\
D(d_i) \neq \emptyset \quad \text{with} \quad d_i \in D(x) \setminus U(A)
\]

and \( (i \geq 2, a_1, a_2, \ldots, a_{i-1} \in A \setminus (U(A) \cup \{0\}) \) and \( a_i \in A \setminus \{0\} \):

\[ x = a_1 \ldots a_{i-1}a_id_i \]

Since \( D(x) \) has a finite order by assumption, the sequence \((D(d_i))\) is finite. It follows that there exists \( n \in \mathbb{N} \) such that \( D(d_{n+1}) = D(d_n) \). So, using the property (4.18), \( d_{n+1}|d_n \) and \( d_n|d_{n+1} \) meaning that \( d_n \) and \( d_{n+1} \) differ from a multiplicative unit namely \((a_{n+1} = v \in U(A))\):

\[ d_n = ud_{n+1} \]

It comes that \((a_1, a_2, \ldots, a_n \in A \setminus (U(A) \cup \{0\}))\):

\[ x = a_1 \ldots a_nd_n \]

and \( (d_n \notin U(A) \cup \{0\}) \) exists so \( D(d_n) \neq \emptyset \):

\[ D(d_n) \subseteq \ldots \subseteq D(d_0) \]

The natural number \( n \) is equal to the greatest integer for which \( d_n|x\):

\[ n = \max\{i \in \mathbb{N} : x = a_1 \ldots a_id_i \text{ with } a_1, \ldots, a_i \in A \setminus \{0\} \text{ and } d_i \in A \setminus (U(A) \cup \{0\})\} \]

So, only \( d_n \) and \( d_{n+1} \) with their symmetric opposites \(-d_n \) and \(-d_{n+1} \) in \( D(d_n) \) do not belong to \( U(A) \). Otherwise, there exists a non-zero element \( b \) in \( D(d_n) \) such that \( b \notin v \) and \( b \neq vd_n \) with \( v \in U(A) \), which divides \( d_n, d_{n+1} \) and so \( x \) by transitivity of the relation of divisibility defined on \( A \). But, then \( d_n = bc \) with \( c \in A \setminus (U(A) \cup \{0\}) \) and \( n \) would not be the greatest integer such that \( d_n|x\). We reach to a contradiction meaning that \( b \) doesn’t exist. It results that \( D(d_n) = U(A) \cup d_nU(A) \) with \( d_n \notin U(A) \cup \{0\} \). So, \( N(d_n) \) is a prime element of \( A \) which divides \( x \). It proved that \( x \) has at least a prime divisor.

\[ \square \]

**Corollary 5.4.** If \( A \) contains at least a prime element and if \( U(A) \) has finite order, then for any non-zero element \( x \) of \( A \) which doesn’t belong to \( U(A) \cup \{0\} \) such that \(|D(x)|\) is finite, there exists a prime element \( p \) of \( A \) and a non-zero natural number \( n \) such that:

\[ x = ap^n \text{ with } a \in A \setminus \{0\} \text{ such that } \text{gcd}(a, p) = 1 \]

**Proof.** We assume that \( A \) contains at least a prime element.

Let \( x \in A \setminus (U(A) \cup \{0\}) \). From the property (5.4.), there exist a prime element \( p \) and an element \( b \) of \( A \setminus \{0\} \) such that \( x = b_1p \). If \( \text{gcd}(b_1, p) = 1 \), then the property is verified with \( n = 1 \). If \( p|b_1 \), then there exists a non zero element \( b_2 \) such that \( b_1 = b_2p \) and so \( x = b_2p^2 \). If \( \text{gcd}(b_2, p) = 1 \), then the property is verified with \( n = 2 \). If \( p|b_2 \), we follow the steps above. Thus, we get a sequence \((b_i)\) of non-zero elements of \( A \) such that:

\[ b_{i+1} = b_ip \]

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and:

\[ x = b_i p^i \]

with \( i \in \mathbb{N}^* \) and \( b_i \in A \setminus (U(A) \cup \{0\}) \). Since \(|D(x)|\) is finite, there exists a non-zero natural number \( n \) such that \( x = b_n p^n \) and \( p \not\mid b_n \). So, \( b_n \) and \( p \) are relatively primes which implies that \( \gcd(b_n, p) = 1 \). Setting \( a = b_n \), we obtain \( x = ap^n \) with \( \gcd(a, p) = 1 \) and \( a \in A \setminus (U(A) \cup \{0\}) \).

\[ \square \]

**Theorem 5.5** (The fundamental theorem of arithmetic in \( A \)). Let \( k \in \mathbb{N}^* \).

If \( A \) contains at least a prime element and if \( U(A) \) has finite order, any non-zero element \( x \) of \( A \) which does not belong to \( U(A) \cup \{0\} \) such that \(|D(x)|\) is finite, has a decomposition into prime factors up to a multiplicative unit \( v \in U(A) \) as:

\[ x = v p_1^{n_1} \ldots p_k^{n_k} \]

where \( p_1, \ldots, p_k \) which are primes such that \( p_i \neq p_j \) for \( i \neq j \) with \( i, j \in [1, k] \) and \( n_1, \ldots, n_k \in \mathbb{N}^* \).

This decomposition is unique up to the order of factors.

**Proof.** Let \( k \in \mathbb{N}^* \).

We assume that \( A \) contains at least a prime element.

Let \( x \) be an element of \( A \setminus (U(A) \cup \{0\}) \). From the corollary \( \square \), we know that there exists a prime element \( p_1 \) in \( D(x) \) and a non-zero natural number \( n_1 \) such that:

\[ x = p_1^{n_1} a_1 \]

with \( a_1 \in A \setminus \{0\} \) and \( \gcd(a_1, p_1) = 1 \). If \( a_1 = v_1 \in U(A) \), then \( x = v_1 p_1^{n_1} \) and the property is verified. If \( a_1 \not\in U(A) \), then from the corollary \( \square \), we can find a prime element \( p_2 \) in \( D(a_1) \) and a non-zero natural number \( n_2 \) such that:

\[ a_1 = p_2^{n_2} a_2 \]

and so:

\[ x = p_1^{n_1} p_2^{n_2} a_2 \]

with \( a_2 \in A \setminus \{0\} \), \( \gcd(a_2, p_2) = 1 \) and \( p_1 \neq p_2 \). If \( a_2 = v_2 \in U(A) \), then \( x = v_2 p_1^{n_1} p_2^{n_2} \) and the property is verified. If \( a_2 \not\in U(A) \), we follow the same steps than above. Thus, we get a sequence \( (a_i) \) of elements of \( A \setminus \{0\} \) such that \( (i \in \mathbb{N}^*, p_i \) which is prime of \( A \) and \( n_i \in \mathbb{N}^* \): \n
\[ a_{i+1} = p_i^{n_i} a_i \]

and so:

\[ x = p_1^{n_1} \ldots p_i^{n_i} a_i \]

with \( p_m \neq p_j \) for \( m \neq j \) (\( m, j \in [1, i] \)). Since \(|D(x)|\) is finite, the sequence \( (p_i^{n_i}) \) is finite. Or, the decomposition of \( x \) as \( p_1^{n_1} \ldots p_i^{n_i} a_i \) is achieved when \( a_i \in U(A) \). It follows that there exists \( k \in \mathbb{N} \) such that \( a_k \in U(A) \). Setting \( a_k = v \in U(A) \), it results that:

\[ x = vp_1^{n_1} \ldots p_k^{n_k} \]

Afterwards, let prove that the decomposition of \( x \) as \( vp_1^{n_1} \ldots p_k^{n_k} \) with \( u \in U(A) \), \( p_i \) which is prime for all \( i \in [1, k] \) and \( n_1, \ldots, n_k \in \mathbb{N}^* \), is unique. Let consider two decompositions of \( x \):

\[ x = vp_1^{n_1} \ldots p_k^{n_k} = wp_1^{m_1} \ldots p_k^{m_k} \]

with \( v, w \in U(A) \), \( p_i \) which is prime for all \( i \in [1, k] \) and \( n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}^* \).
Since $v \in U(A)$, we have:

$$p_1^{n_1} \ldots p_k^{n_k} = w v^{-1} p_1^{m_1} \ldots p_k^{m_k}$$

Since $p_i$ for all $i \in [1, k]$ cannot divide $w v^{-1} \in U(A)$, it remains only one possibility that is to say $w v^{-1} = 1$ and so $w = v$. It implies that:

$$p_1^{n_1} \ldots p_k^{n_k} = p_1^{m_1} \ldots p_k^{m_k}$$

Let assume absurdly that $n_1 \neq m_1$ say $n_1 < m_1$. Since $p_1$ is regular (see the remark [4.10]), we have $(n_1 < m_1)$:

$$\prod_{i \neq 1}^k p_i^{n_i} = p_1^{m_1-n_1} \prod_{i \neq 1}^k p_i^{n_i}$$

Since $\gcd(p_i, p_j) = 1$ for $i \neq j$ with $i, j \in [1, k]$, no factor of $\prod_{i \neq 1}^k p_i^{n_i}$ divides $p_1^{m_1-n_1}$. It remains only one possibility that is to say $p_1^{m_1-n_1} = 1$ and so $m_1 = n_1$. Following this reasoning for every $i \in [1, k]$, it can be shown that $m_i = n_i$ for all $i \in [1, k]$. Therefore, the decomposition of $x$ as $u p_1^{n_1} \ldots p_k^{n_k}$ with $u \in U(A)$, $p_1, \ldots, p_k$ which are prime such that $p_i \neq p_j$ for $i \neq j$ such that $i, j \in [1, k]$ and $n_1, \ldots, n_k \in \mathbb{N}^*$, is unique. 

\[ \square \]

6. Set operations on ideals of a principal entire ring and divisibility

**Theorem 6.1.** Let $a, b$ be two non-zero elements of the entire principal ring $A$. If $\text{lcm}(a, b)$ exists, then:

$$aA \cap bA = \text{lcm}(a, b)A$$

**Proof.** We assume that $\text{lcm}(a, b)$ exists.

Notice that since $a | \text{lcm}(a, b)$ and $b | \text{lcm}(a, b)$, we have $\text{lcm}(a, b)A \subseteq aA$ and $\text{lcm}(a, b)A \subseteq bA$.

So, $\text{lcm}(a, b)A \subseteq aA \cap bA$.

Since $A$ is a principal ring and since $aA \cap bA$ is an ideal, there exists an element $m$ of $A$ such that $aA \cap bA = mA$. The element $m$ is a generator of $aA \cap bA$ (notice that it is not unique since $-m$ is also a generator of $aA \cap bA$). Let $n \in A$ be a common multiple of $a, b$. Then, $a | n$ which implies that $nA \subseteq aA$ and $b | n$ which implies that $nA \subseteq bA$. So, $nA \subseteq aA \cap bA$ or equivalently $nA \subseteq mA$. It means that $m | n$. Since $n$ is arbitrary common multiple of $a, b$ and since any ideal of $A$ is stable by multiplication by $-1$, we deduce that $mA = \text{lcm}(a, b)A$. It results that $aA \cap bA = \text{lcm}(a, b)A$. \[ \square \]

**Remark 6.2.** If $\text{lcm}(a, b)$ exists such that $\text{lcm}(a, b) \neq a$ and if $\text{lcm}(a, b) \neq b$, the intersection $aA \cap bA$ of the two ideals $aA$ and $bA$ with $a, b \in A$ does not contain $a$ and $b$ since the generator of $aA \cap bA$ is $\text{lcm}(a, b)$. Therefore, $aA \cap bA$ cannot be the smallest ideal which is generated by the subset $\{a, b\}$.

**Theorem 6.3.** Let $a, b$ be two non-zero elements of the entire principal ring $A$. If $\text{gcd}(a, b)$ exists, then:

$$aA + bA = \text{gcd}(a, b)A$$

**Proof.** We assume that $\text{gcd}(a, b)$ exists.

Before giving the details of the proof, notice that since $\text{gcd}(a, b) | a$ and $\text{gcd}(a, b) | b$, by linearity of the relation of divisibility defined on $A$, we have $\text{gcd}(a, b) | ax + by$ for any $x, y \in A$. Since $aA + bA$ is an ideal and so since $ax + by \in aA + bA$ for any $x, y \in A$, we deduce that $aA + bA \subseteq \text{gcd}(a, b)A$.

Since $A$ is a principal ring and $aA + bA$ is an ideal, there exists an element $g$ of $A$ such that $aA + bA = gA$. The element $g$ of $A$ is a generator of $aA + bA$ (notice that $g$ is not unique since its symmetric $-g$ is also a generator of $aA + bA$ and notice that $g \in D(a, b)$). Since $g \in aA + bA$,
there exist two elements \(x, y\) of \(A\) such that \(g = ax + by\). Let \(d \in A\) be a common divisor of \(a, b\).

Then, by linearity of the relation of divisibility defined on \(A\), \(d | g\). Since \(d\) is any common divisor of \(a, b\) and since any ideal of \(A\) is stable by multiplication by \(-1\), we deduce that \(gA = \gcd(a, b)A\). It results that \(aA + bA = \gcd(a, b)A\).

In particular, if \(\mathcal{D}(a, b) = U(A)\), then by convention \(\gcd(a, b) = 1\) and the ideal \(aA + bA\) of \(A\) is generated by an element \(g \in U(A)\) since \(g \in \mathcal{D}(a, b) = U(A)\). From the lemma \((2.2)\), since \(\gcd(a, b) = 1\), it means that \(aA + bA = A = \gcd(a, b)A\).

**Remark 6.4.** Provided \(\gcd(a, b)\) and \(\text{lcm}(a, b)\) exist, since \(\gcd(a, b) | \text{lcm}(a, b)\), we have the inclusion \(
\text{lcm}(a, b)A \subseteq \gcd(a, b)A.
\)

It is compatible with the fact that:

\[
aA \cap bA \subseteq aA \cup bA \subseteq aA + bA
\]

Notice that \(aA \cup bA\) is not always an ideal.

Since the ideal \(aA + bA\) contains \(a\) and \(b\) \((a = a1 + b0\) and \(b = a0 + b1\)), then the ideal \(aA + bA\) is contained in any ideal which contains \(a\) and \(b\). Therefore, \(aA + bA\) is the smallest ideal which is generated by the subset \(\{a, b\}\) of \(A\).

**Corollary 6.5.** If \(\gcd(a, b)\) exists, then there exist two elements \(x, y\) of \(A\) such that:

\[
\gcd(a, b) = ax + by
\]

**Proof.** We assume that \(\gcd(a, b)\) exists.

We know that \(\gcd(a, b) \in aA + bA\). It results that there exist two elements \(x, y\) of \(A\) such that \(\gcd(a, b) = ax + by\). □

**Remark 6.6.** Thus, if \(a\) and \(b\) of \(A\) are relatively primes, then there exists \((x, y) \in A^2\) such that \(1 = ax + by\).

### 7. The Bezout identity and the Euclid’s lemma in a principal entire ring

**Theorem 7.1** (Generalization of the Bezout theorem). Let \(a, b\) be two elements of \(A\).

\(\mathcal{D}(a, b) = U(A)\) if, and only if, \(aA + bA = A\).

**Proof.** If \(\mathcal{D}(a, b) = U(A)\), then \(\gcd(a, b) = 1\) which implies that \(aA + bA = A\). Reciprocally, if \(aA + bA = A\), since \(1 \in A\), there exist two elements \(x, y\) of \(A\) such that \(1 = ax + by\). Let \(d \in \mathcal{D}(a, b)\). By linearity of the relation of divisibility defined on \(A\), \(d | ax + by\) and so \(d \mid 1\). Accordingly, \(d \in U(A)\). Therefore, since \(d\) is any element of \(\mathcal{D}(a, b)\), we have \(\mathcal{D}(a, b) \subseteq U(A)\) meaning that \(\mathcal{D}(a, b) = U(A)\). It completes the proof that \(\mathcal{D}(a, b) = U(A)\) if, and only if, \(aA + bA = A\). □

**Lemma 7.2** (Euclid’s lemma). Let \(a, b, c\) be three elements of \(A\). If \(\gcd(a, b) = 1\) and if \(a\) divides \(bc\), then \(a\) divides \(c\).

**Proof.** If \(\gcd(a, b) = 1\), we know that there exist two elements \(x, y\) of \(A\) such that \(ax + by = 1\). So, for \(c \in A\), we have \(c = axc + byc\) which gives \(c = acx + bcy\). The element \(a\) divides \(acx\). Besides, we assume that the element \(a\) of \(A\) divides \(bc\) and so \(a\) divides \(bey\). Accordingly, by linearity of the relation of divisibility, \(a | c\).

**Remark 7.3.** If \(\gcd(a, b) = 1\) and if \(a | bc\) with \(c \in U(A)\), then from the generalization of the Euclid’s lemma \((7.2)\), \(a | c\) and so \(a\) should belong to \(U(A)\). In such a case, we have also \(a | b\) since \(a \in U(A)\).
8. Some arithmetic properties on the set of ideals of a principal entire ring

Property 8.1. If \(a, b\) are two elements of \(A\) such that \(\gcd(a, b) = 1\), then the common multiples of \(a, b\) are multiples of \(ab\). In particular, provided \(\gcd(a, b) = 1\) and provided \(\mathop{lcm}\(a, b)\) exists, we have:
\[
\mathop{lcm}(a, b)A = abA
\]

Proof. We assume that \(\mathop{lcm}(a, b)\) exists.

It is obvious that the multiples of \(ab\) are also common multiples of \(a, b\). Then, \(abA \subseteq aA \cap bA\) or equivalently \(abA \subseteq \mathop{lcm}(a, b)A\).

Reciprocally, let assume that \(m \in aA \cap bA\). Then, we have \(m = bc\) with \(c \in A\). As \(a \mid m\) (namely, \(a \mid bc\)) and \(\gcd(a, b) = 1\), from the Euclid’s lemma (7.2), \(a \mid c\). So, there exists an element \(x\) of \(A\) such that \(c = ax\). Therefore, \(m = abx\) with \(x \in A\). Accordingly, \(ab\mid m\). Since \(m\) is any element of the subset \(aA \cap bA\) of \(A\), it implies that \(aA \cap bA \subseteq abA\) or equivalently \(\mathop{lcm}(a, b)A \subseteq abA\).

Therefore, we conclude that \(\mathop{lcm}(a, b)A = abA\).

\(\square\)

Corollary 8.2. Let \(n \in \mathbb{N}^*\). If elements \(a_1, \ldots, a_n\) of \(A\) whose \(\gcd\) is equal to 1, divides an element \(m\) of \(A\), then their product \(a_1 \ldots a_n\) divides \(m\).

Corollary 8.3. Let \(a, b\) be two non-zero elements of \(A\) such that their \(\gcd\) exists, \(a = ga', b = gb'\) with \(g = \gcd(a, b)\) which is multiplicatively regular. We assume that \(\mathop{lcm}(a, b)\) exists. Then, the common multiples of \(a, b\) are multiples of \(ga'b'\). In particular, we have:
\[
\mathop{lcm}(a, b) \gcd(a, b)A = abA
\]

Proof. We assume that \(\gcd(a, b)\) exists and is regular. We assume also that \(\mathop{lcm}(a, b)\) exists.

Notice that \(\gcd(a', b') = 1\). Indeed, denoting \(\gcd(a, b) = g\), there exist two elements \(x, y\) of \(A\) such that \(g = ax + by\). Since \(a = ga'\) and \(b = gb'\), it comes that \(g = ga'x + gb'y\). Since \(g\) is multiplicatively regular, after simplification, we get \(1 = a'x + b'y\). So, the ideal \(a'A + b'A\) contains 1 meaning that \(a'A + b'A = A\) (see the lemma (7.2)). From the generalization of the Bezout theorem (7.1), we have \(\mathcal{D}(a', b') = U(A)\). It means that \(\gcd(a', b') = 1\).

It is obvious that \(ga'b' = ab' = a'b\) is a multiple of \(a, b\). Then \(ga'b'A \subseteq aA \cap bA\) and so \(ga'b'A \subseteq \mathop{lcm}(a, b)A\).

Reciprocally, let \(m\) be a common multiple of \(a, b\). Then by transitivity of the relation of divisibility, \(m\) is a multiple of \(g\). So, we have \(m = gm', m = ac\) and \(m = bd\) with \(m', c, d \in A\). Since \(a = ga'\) and \(b = gb'\), it gives \(gm' = ga'c\) and \(gm' = gb'd\). Whence since \(g\) is multiplicatively regular, we obtain \(m' = a'c\) and \(m' = b'd\) with \(c, d \in A\). We deduce that \(a'|m'\) and \(b'|m'\). Since \(a'|m', b'|m'\) and since \(\gcd(a', b') = 1\), it results that \(a'b'|m'\). Therefore, \(ga'b'|gm'\) and so \(ga'b'|m\). Since \(m\) is any multiple of \(a, b\), we deduce that \(aA \cap bA \subseteq ga'b'A\) and so \(\mathop{lcm}(a, b)A \subseteq ga'b'A\).

Therefore, we conclude that \(\mathop{lcm}(a, b)A = ga'b'A\) or equivalently \(\mathop{lcm}(a, b) = ab'A = a'bA\). It results that \(\mathop{lcm}(a, b) \gcd(a, b)A = abA\).

\(\square\)

9. Maximal ideals in a principal entire ring

A prime ideal \([2]\) in \(A\) is an ideal \(p \not \subseteq A\) such that \(A/p\) is entire. Equivalently, we could say that it is an ideal \(p \not \subseteq A\) such that, whenever \(x, y \in A\) and \(xy \in p\), then \(x \in p\) or \(y \in p\).

Property 9.1. Let \(z\) be an element of \(A\) such that \(N(z) = z\).

\(A/zA\) is entire if, and only if, \(z\) is prime or null.
Proof. If $z = 0$, then $A/zA$ is equal to $A$ which is entire. In the following, we assume that $z$ is not null.

Let assume that $z = N(z) \neq 0$ is a prime element of $A$ (see above for a definition). Then, $zA$ is an ideal of $A$ which is not equal to $A$ since $z$ is not invertible. Let $x, y$ be two elements of $A$ such that $z|xy$. Then either $z$ divides both $x, y$ or else $z$ and one of the elements $x, y$ say $x$ are relatively primes. In the second case, from the Euclid’s lemma, it follows that $z$ divides the other element namely $y$ among the elements $x, y$ of $A$. To sum up, if $z$ is a prime element of $A$ and if $z|xy$ with $x, y \in A$, then $z|x$ or $z|y$. It is equivalent to say that $zA \neq A$ is an ideal such that whenever $x, y \in A$ and $xy \in zA$, then $x \in zA$ or $y \in zA$. From the definition of a prime ideal, it results that $A/zA$ is entire.

Afterwards, let assume that $z \neq 0$ is not a prime element of $A$. If $z \in U(A)$, then $A/zA$ is reduced to one residue class of elements of $A$ which is equal to the zero class. In this case, $A/zA$ is clearly not entire. Let assume that $z \notin U(A)$. Since $z$ is not prime and $z \notin U(A)$, $D(z)$ is not reduced to $U(A)$. So, there exist two elements $x, y \in A$ such that $z = xy$ (with at least one of the elements $x, y$, which does not belong to $U(A)$). If one of the elements $x, y$ belongs to $U(A)$, then $D(z) = U(A) \cup zU(A)$.

Since $z \notin U(A)$ and $z \neq 0$, from the definition of a prime element of $A$, it would mean that $z$ is prime. What it is impossible. So, we have necessarily $z = xy$ with $x, y \in A$ such that $x, y \notin U(A)$. Then, the residue class $\hat{x}, \hat{y}$ of $x, y$ in $A/zA$ are non-zero and their product is zero. It means that $A/zA$ has divisors of zero and so $A/zA$ is not entire. Thus, we proved that if $z$ is not a prime element of $A$, then $A/zA$ is not entire. It is equivalent to say that if $A/zA$ is entire, then $N(z)$ is prime.

We conclude that $A/zA$ is entire if, and only if, $z$ is prime or null. □

Property 9.2. Let $z$ be an element of $A$.

If $z$ is prime, then $zA$ is maximal.

Proof. Let $z$ be a prime element of $A$. Let assume that there exists $w \in A$ such that $zA \subseteq wA$. Then $w|z$. Since $z$ is prime (see the definition of a prime element of $A$), then there exists $u \in U(A)$ such that $z = wu$. It results that $zA = wuA = wA$. □

Corollary 9.3. Let $z$ be an element of $A$ such that $N(z) = z$.

$A/zA$ is a field if, and only if, $z$ is prime.

Proof. Let $z$ be an element of $A$ such that $N(z) = z$.

We know that the ideal $zA$ is maximal if, and only if, $A/zA$ is a field (see p. 93 of [2]). So, if $z$ is prime, then from the property above, the ideal $zA$ is maximal. It results that if $z$ is prime, then $A/zA$ is a field.

Reciprocally, if $A/zA$ is a field with $z = N(z)$, then $z \neq 0$ and $z \notin U(A)$ ($A$ is not a field by assumption and $A/A = \{0\}$ is not also a field). Let consider a non-zero element $x$ whose residue class $\hat{x}$ in $A/zA$ is invertible in $A/zA$. Notice that $x \neq zq$ with $q \in A$ and in particular $x \neq zu$ with $u \in U(A)$. Since $A/zA$ is a field, there exists an element $y \in A$ such that $\hat{x} \hat{y} = \hat{1}$. It means that $1 = xy + zw$ with $w \in A$. Then, it comes that for any $a \in A$, we have $a = xya + zwa$. Accordingly, since $a$ is arbitrary in $A$, it results that $A \subseteq xA + zA$ which gives $xA + zA = A$. Or, from the generalization of the Bezout theorem (7.1), $xA + zA = A$ is equivalent to $D(z, x) = U(A)$. Since $x$ is any non-zero element of $A$ whose residue class $\hat{x}$ in $A/zA$ is invertible in $A/zA$ (namely $x \neq zq$ with $q \in A$ and in particular $x \neq zu$ with $u \in U(A)$), only elements of $U(A)$ and elements of $zU(A)$
in \( A \), divide \( z \). So, we get \( \mathcal{D}(z) = U(A) \cup zU(A) \). Since \( z \neq 0 \) and \( z \not\in U(A) \) if \( A/zA \) is a field with \( z = N(z) \), we deduce that if \( A/zA \) is a field, then \( z \) is prime.

We conclude that when \( z = N(z) \), \( A/zA \) is a field if, and only if, \( z \) is prime. \( \Box \)

10. Examples of ring extensions of entire rings

In this section, the element \( i \) which verifies the polynomial equation \( z^2 + 1 = 0 \) in \( C \), is also written in its exponential form as:

\[
i = e^{i\pi/2}
\]

Thus, the equation \( i^2 + 1 = 0 \) can be rewritten as:

\[
e^{i\pi} + 1 = 0
\]

Since \( i^* = -i \) (see the remark [1.41]), using the definition [4.25], we have necessarily \( i = i' \) (recall that \( i \) is defined as a unit in \( A[S] \) (see the property [1.30]) and so is invertible). From the remark [1.35], using again the remark [4.41], it follows that:

\[
i^* = i^{-1}N(i)^2 = i^{-1}
\]

Therefore:

\[
(e^{i\pi/2})^* = (e^{i\pi/2})^{-1} \\
(e^{i\pi})^* = e^{-i\pi} = -i
\]

Since \( i^* = -i \), it gives:

\[
(e^{i\pi})^* = e^{-i\pi} = -i
\]

In the following, for all \( n \in \mathbb{N}^* \), we set:

\[
e_{n,0} = e_{n,n} = -i^2 = -e^{i\pi} = 1
\]

It comes that:

\[
e_{n,0}^2 = e_{n,n}^2 = i^4 = e^{2i\pi} = 1
\]

Let \( n \in \mathbb{N}^* \) and let \( U_n \) the subset of the \( n \)th-root of unity in \( C \):

\[
U_n = \{e_{n,k} : e_{n,k}^n = e_{n,0}, k = 0, 1, \ldots, n - 1\}
\]

with \( (k = 0, 1, \ldots, n - 1 \text{ and } n \in \mathbb{N}^*) \):

\[
e_{n,k} = (-e_{n,0})^{\frac{2k}{n}} = (e^{i\pi})^{\frac{2k}{n}} = e^{\frac{2ik\pi}{n}}
\]

and for \( k, l \in \{0, 1, \ldots, n - 1\} \):

\[
e_{n,k} = e_{n,l} \iff k = l
\]

Notice that the formula of \( e_{n,k} \) works also for \( k = n \):

\[
e_{n,n} = e^{2i\pi} = 1 = e_{n,0}
\]

Notice also that the square root function defined on \( C \) is the function which associates at least one complex number \( w \) to any complex number \( z \) such that \( w^2 = z \). It maps \( U_n \) onto \( U_{2n} \cup e^{i\pi}U_{2n} \) with \( n \in \mathbb{N}^* \) since if \( (k = 0, 1, \ldots, n - 1 \text{ with } n \in \mathbb{N}^*) \):

\[
w^2 = e^{\frac{2ik\pi}{n}} = e^{\frac{2ik\pi}{n}}
\]

then \( (k = 0, 1, \ldots, n - 1 \text{ with } n \in \mathbb{N}^*) \):

\[
w = \pm e^{\frac{2ik\pi}{2n}} = \pm e_{2n,k}
\]

The ambiguity of sign comes from the fact that we can write \( z = ze^{2i\pi} \). So, we can replace \( z \) by \( ze^{2i\pi} \) in the equality \( w^2 = z \). It may change \( w \) into \(-w \). In order to have a single value, we cut the
complex plane where the square root function is multi-valued. The set of points (images of complex numbers in the complex plane) where the square root function keeps a constant sign is called a branch (or a sheet). The result is to make the square root function uniform (within a branch). The corresponding value of the square root function at a complex number \( z \) in the cut complex plane is the chosen determination of the square root of a complex number \( z \). When the argument of \( z \) (namely the angle coordinate of \( z \) when \( z \) is described by its polar coordinates in the complex plane) belongs to the interval \( [−π; π] \), the chosen determination of the square root of a complex number \( z \) is called its principal square root denoted \( \sqrt{z} = z^{\frac{1}{2}} \). It is usual to take the branch cut in the complex plane as the non-positive part of the real axis in the complex plane. It stems from the fact that we can associate two purely imaginary complex numbers which are the solutions of a polynomial equation as \( z^2 + x = 0 \) of unknown \( z \) in \( \mathbb{C} \), to any strictly negative real number \(-x\) with \( x > 0 \). Each time we go through the branch cut, the square root function takes a multiplicative global sign \(-1\). So, the square root function has a discontinuity near the branch cut.

For instance, the principal square root of \( e_{2,1} = e^{i\pi} = -e_{2,0} = -1 \) is given by:

\[
i = e_{4,1} = \sqrt{e_{2,1}} = \sqrt{-e_{2,0}} = e_{2,0}\sqrt{-1}
\]

where we used the fact that \( \sqrt{e_{n,0}} = e_{n,0} = 1 \) for all \( n \in \mathbb{N}^* \).

More generally, the principal square root of \( e_{n,k} \) for \( 0 \leq k \leq \frac{n}{2} \) with \( n \in \mathbb{N}^* \) is given by:

\[
\sqrt{e_{n,k}} = e_{2n,k}
\]

Therefore, the principal square root function defined on the cut complex plane, maps \( U_n \) onto \( U_{2n} \) with \( n \in \mathbb{N}^* \).

Otherwise, since \((k = 0, 1, \ldots, n - 1)\):

\[
e^{2i\pi} = 1 \Rightarrow (e^{2i\pi})^k = e^{2ik\pi} = 1
\]

we can observe that \((k = 0, 1, \ldots, n - 1)\):

\[
(e^{\frac{2ik\pi}{n}})^{n-1} = e^{\frac{2ik(n-1)\pi}{n}} = e^{-\frac{2ik\pi}{n}} = e^{\frac{2i(k-1)\pi}{n}}
\]

So \((k = 0, 1, \ldots, n - 1)\):

\[
e^n_{n,k} = e_{n,0} \iff e_{n,k}e_{n,k}^{-1} = e_{n,0} \iff e_{n,k}e_{n,n-k} = e_{n,0}
\]

and \((k = 0, 1, \ldots, n - 1)\):

\[
e^n_{n,k} = e_{n,0} \iff e_{n,0}e_{n,k}^{-1} = e_{n,0} \iff e_{n,n-k}e_{n,k} = e_{n,0}
\]

Therefore \((k = 0, 1, \ldots, n - 1)\):

\[
e_{n,k}e_{n,n-k} = e_{n,n-k}e_{n,k} = e_{n,0}
\]

It results that each \( e_{n,k} \) with \( n \in \mathbb{N}^* \) and \( k = 0, 1, \ldots, n - 1 \) is invertible. Consequently, \( N(e_{n,k}) \neq 0 \) for all \( k \in \{0, 1, \ldots, n - 1\} \) with \( n \in \mathbb{N}^* \). Regarding the Euclid division of the product \( km \) of two integers \( k, m \in \{0, 1, \ldots, n - 1\} \) by \( n \):

\[
km = nq + r \text{ with } 0 \leq r < n
\]

where \( q = \lfloor \frac{km}{n} \rfloor \), we can remark that \((k, m = 0, 1, \ldots, n - 1 \text{ and } r \text{ the remainder of the Euclid division of } km \text{ by } n)\):

\[
(e^{\frac{2ik\pi}{n}})^m = e^{\frac{2im\pi}{n}} = e^{2i\pi q + 2i\pi r} = e^{2i\pi r}
\]

and so we have \((k, m = 0, 1, \ldots, n - 1 \text{ and } r \text{ the remainder of the Euclid division of } km \text{ by } n)\):

\[
e^n_{m,k} = e_{n,r}
\]
Or \((k = 0, 1, \ldots, n - 1)\):

\[ N(e_{n,k}^n) = N(e_{n,0}) = N(1) = 1 \]

Using the property \([4.16]\), it comes that:

\[ N(e_{n,k})^n = 1 \]

\[ N(e_{n,k})^n - 1 = 0 \]

\[ (N(e_{n,k}) - 1)(1 + \ldots + N(e_{n,k})^{n-1}) = 0 \]

Since in the sum \(1 + \ldots + N(e_{n,k})^{n-1}\), each term can be written as \(N(e_{n,r}) \neq 0\) with \(0 \leq r < n\), it implies that \((k = 0, 1, \ldots, n - 1)\):

\[ N(e_{n,k}) = 1 \]

It results also that \((k = 0, 1, \ldots, n - 1)\):

\[ u(e_{n,k}) = e_{n,k} \]

Since \(e_{n,k}\) is invertible, using the definition \([4.25]\), we have \((k = 0, 1, \ldots, n - 1)\):

\[ (e_{n,k})^* = e_{n,k}' = e_{n,k}^{-1}N(e_{n,k}) = e_{n,k}^{-1} = e_{n,n-k} \]

or equivalently \((k = 0, 1, \ldots, n - 1)\):

\[ (e^{2ik\pi/n})^* = (e^{-2ik\pi/n})^{-1} = e^{-2ik\pi/n} = e^{2\pi(k-n)\pi/(n)} \]

Thus, we have \((k = 0, 1, \ldots, n - 1)\):

\[ N(e_{n,k})^2 = e_{n,k}(e_{n,k})^* = (e_{n,k})^*e_{n,k} = e_{n,0} = 1 \]

Besides, since \(n - k \in [1, n - 1]\) for \(k = 1, \ldots, n - 1\) and since \(e_{n,0} = e_{n,0}\), \(e_{n,k}^* \in U_n\) for all \(k \in [0, n - 1]\) and the star operation which is well defined on \(U_n\), maps \(U_n\) onto \(U_n\).

It is straightforward to verify that \(U_n\) is a multiplicative cyclic group of order \(n\). In particular, a generator of \(U_n\) is given by:

\[ e_{n,1} = e^{2\pi i/n} \]

and we have \((k = 0, 1, \ldots, n - 1)\):

\[ e_{n,k} = e_{n,1}^k \]

Notice that if \(n\) is even (but not zero), then \(n/2\) is a non-zero natural number and we have \((n \in 2\mathbb{N}^*)\):

\[ e_{n,n/2} = -1 = -e_{n,0} \]

For \(n\) odd, it is impossible that an element among the elements \(e_{n,k}\)s with \(k \in [0, n - 1]\), of \(U_n\), be equal to \(-1\).

Moreover, we can define elements \(e_{n,m}\) for all \(m \in \mathbb{Z}\). Indeed, regarding the Euclid division of \(\text{abs}(m)\) by \(n\):

\[ \text{abs}(m) = an + k \text{ with } 0 \leq k < n \]

with \(a = \left[\frac{\text{abs}(m)}{n}\right]\), we have:

\[ e_{n,\text{abs}(m)} = e^{2i\pi m / n} = e^{2ia\pi} e^{2ik\pi/n} = e^{2ik\pi/n} = e_{n,k} \]

and we have:

\[ e_{n,-\text{abs}(m)} = e_{n,\text{abs}(m)}^{-1} = e_{n,k}^{-1} = e_{n,n-k} \]

In particular, for \(m, l \in \mathbb{Z}\):

\[ e_{n,\text{abs}(m)} = e_{n,\text{abs}(l)} \iff \text{abs}(m) = \text{abs}(l) + qn \text{ with } q \in \mathbb{Z} \]
Or, we have \((k = 0, 1, \ldots, n - 1 \text{ and } n \in \mathbb{N}^*)\):
\[
e_{n,k} = (e_{n,k})^* \iff e^{\frac{2i\pi k}{n}} = e^{-\frac{2i\pi k}{n}} \iff e^{\frac{4i\pi k}{n}} = 1 \iff e_{n,2k} = e_{n,0} \text{ or } e_{n,2k} = e_{n,n}
\]
So, since \(2k \in \{0, 2, \ldots, 2n - 2\}\) for \(k \in \{0, 1, \ldots, n - 1\} \text{ with } n \in \mathbb{N}^*\), we have either \(2k = 0\) which gives \(k = 0\) or \(2k = n\) which gives \(k = n/2\). The case \(k = n/2\) is only possible if \(n\) is even (but not zero). So, provided \(n\) is even, the set \(U_n\) contains two real numbers namely \(e_{n,0} = 1\) and \(e_{n,n/2} = -e_{n,0} = -1\).

In conclusion, we get:
\[
U_n \cap \mathbb{R} = \begin{cases} 
\{1\} & \text{if } n \equiv 1 \pmod{2} \\
\{1,-1\} & \text{if } n \equiv 0 \pmod{2}
\end{cases}
\]

In the following, we denote \(I_n\) the integer interval:
\[
I_n = [0, n - 1]
\]

Let \(\mathbb{Z}[S_n]\) the subring of \(\mathbb{C}\) generated by \(S_n\) over the subring \(\mathbb{Z}\) of integers of \(\mathbb{C}\) with \(S_n = G_n \setminus \{e \in G_n : e \in \mathbb{Z}\}\) where \(G_n\) is a maximal family of linearly independent elements of \(U_n\) over \(\mathbb{C}\):
\[
\mathbb{Z}[S_n] = \{a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_{n-1} e_{n,n-1} : a_k \in \mathbb{Z}, k = 0, 1, \ldots, n - 1\}
\]

We can notice that for \(n = 1, 2\), \(U_n \subseteq \{1,-1\}\) and so \(\mathbb{Z}[S_n] = \mathbb{Z}\). For \(n \geq 3\), since \((k \in I_n)\):
\[
e_{n,k} = e_{n,0} = 1
\]
and:
\[
e_{n,0} + e_{n,1} + \ldots + e_{n,n-1} = 0
\]
the family \(\{e_{n,k}\}_{k \in I_n}\) with \(n \in \mathbb{N}^*\) which generates \(\mathbb{Z}[S_n]\), is not free. It results that for \(n \geq 3\), \(\mathbb{Z}[S_n]\) is generated by the family \(\{e_{n,0}, e_{n,1}, \ldots, e_{n,n-2}\}\) if \(n \equiv 1 \pmod{2}\) and \(\mathbb{Z}[S_n]\) is generated by the family \(\{e_{n,0}, e_{n,1}, \ldots, e_{n,n-2}\} \setminus \{e_{n,n/2}\}\) if \(n \equiv 0 \pmod{2}\).

For instance, for \(n = 3\), setting \(j = e_{3,1} = e^{\frac{2i\pi}{3}}\), we have \((j^3 = 1)\):
\[
1 + j + j^2 = 0
\]
and:
\[
j^2 = j^* = -e_{3,0} - j
\]
Notice that:
\[
jj^* = j^j = 1
\]
The subring \(\mathbb{Z}[S_3]\) of \(\mathbb{C}\) is generated by \(S_3 = G_3 \setminus \{e_{3,0}\} = \{j\}\) where \(G_3 = \{e_{3,0}, j\}\) with \(e_{3,0} = 1\):
\[
\mathbb{Z}[S_3] = \mathbb{Z}[j] = \{ae_{3,0} + bj : a, b \in \mathbb{Z}\}
\]
Since \(\{e_{3,0}, j\}\) is free and is maximal in \(U_3\), \(\mathbb{Z}[S_3]\) has a basis namely \(\{e_{3,0}, j\}\).

In this case, we have \((e^{i\pi} = -1)\):
\[
U(\mathbb{Z}[j]) = U_3 \cup e^{i\pi} U_3 = \{e_{3,0}, -e_{3,0}, j, -j, j^*, -j^*\}
\]
Indeed, let \(ae_{3,0} + bj \in U(\mathbb{Z}[j])\). Then, there exists \(a', b' \in \mathbb{Z}\) such that:
\[
(aa'e_{3,0} + (a'b + a'b)j + bb'j^2 = e_{3,0}
\]
\[
(aa'e_{3,0} + (ab' + a'b)j - bb'(e_{3,0} + j) = e_{3,0}
\]
\[
(aa' - bb')e_{3,0} + (ab' + a'b - bb')j = e_{3,0}
\]
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So, since \( \{e_{3,0}, j\} \) is a basis of \( \mathbb{Z}[S_3] \), we have:

\[
\begin{cases}
   aa' - bb' = 1 \\
   ab' + ba' - bb' = 0
\end{cases}
\]

The equality \( ab' + ba' - bb' = 0 \) can be rewritten as:

\[ bb' = ab' + ba' \]

The equation \( aa' - bb' = 1 \) means that \( \gcd(a, b) = 1 \). Moreover, the equation \( aa' - bb' = 1 \) can be rewritten like \( a(a' - b') - (b - a)b' = 1 \) meaning that also \( \gcd(a, \text{abs}(b - a)) = 1 \). Besides the equation \( ab' + ba' - bb' = 0 \) can be rewritten as:

\[ b'(b - a) = a'b \]

Since \( \gcd(a, \text{abs}(b - a)) = 1 \), from the Euclid’s lemma, \( b - a|a' \) and \( b|b' \). So, there exists an integer \( k \) such that:

\[
\begin{align*}
   a' &= (b - a)k \\
   b' &= bk
\end{align*}
\]

Using the equation \( aa' - bb' = 1 \), it implies that:

\[ (ab - (a^2 + b^2))k = 1 \]

So, either \( k = 1 \) or \( k = -1 \). If \( k = 1 \), then \( ab - (a^2 + b^2) = 1 \) which is equivalent to \( a^2 - ab + b^2 + 1 = 0 \). This equation of degree 2 in variable \( a \) has no solution in \( \mathbb{R} \) and so in \( \mathbb{Z} \) since its discriminant is \( \Delta = -3b^2 - 4 < 0 \). So, the case \( k = 1 \) is not possible. If \( k = -1 \), then \( ab - (a^2 + b^2) = -1 \) which is equivalent to \( a^2 - ab + b^2 - 1 = 0 \). The discriminant of this equation of degree 2 in variable \( a \) is \( \Delta = 4 - 3b^2 \). Since \( b \) is an integer, either \( b = -1 \), either \( b = 0 \) or \( b = 1 \). Notice that for \( b = 0 \) or \( b = \pm 1 \), \( \Delta \) is strictly positive and is a perfect square. It gives rise to the possible values for \( a \):

\[
\begin{align*}
   b = -1 & \Rightarrow a = 0 \text{ or } a = -1 \\
   b = 0 & \Rightarrow a = -1 \text{ or } a = 1 \\
   b = 1 & \Rightarrow a = 0 \text{ or } a = 1
\end{align*}
\]

Therefore, the elements of \( U(\mathbb{Z}[j]) \) are:

\[ e_{3,0}, -e_{3,0}, j, -j, j^*, -j^* \]

We can notice that \( \mathbb{Z}[j] = \mathbb{Z} \oplus j\mathbb{Z} \). It is because any element \( x \) of \( \mathbb{Z}[j] \) is written in an unique way as \( x = au_{3,0} + bj \) with \( a, b \in \mathbb{Z} \) and because \( \mathbb{Z} \cap j\mathbb{Z} = \{0\} \). Indeed, the existence of \( a, b \in \mathbb{Z} \) stems from the algebraic structure of \( \mathbb{Z}[j] \). For proving the uniqueness of \( a, b \in \mathbb{Z} \) such that \( x = au_{3,0} + bj \), let consider two other integers \( a', b' \) which verify:

\[ ae_{3,0} + bj = a'e_{3,0} + b'j \]

Since \( \{e_{3,0}, j\} \) is a basis of \( \mathbb{Z}[S_3] \), it gives \( a = a' \) and \( b = b' \) meaning that \( x \) is written uniquely as \( x = a + bj \) with \( a, b \in \mathbb{Z} \). Moreover, let \( a, b \in \mathbb{Z} \) such that:

\[ ae_{3,0} = bj \]

Then, since \( \{e_{3,0}, j\} \) is a basis of \( \mathbb{Z}[S_3] \), \( a = b = 0 \) meaning that \( \mathbb{Z} \cap j\mathbb{Z} = \{0\} \). So, for any subset \( \mathcal{W} \) of \( \mathbb{Z}[j] \), there exist two subsets \( \mathcal{X}, \mathcal{Y} \) of \( \mathbb{Z} \) such that \( \mathcal{W} = \mathcal{X} \oplus j\mathcal{Y} \). Notice that \( \mathcal{X}, \mathcal{Y} \) are unique since if \( \mathcal{X}' \oplus j\mathcal{Y}' = \mathcal{X} \oplus j\mathcal{Y} \), then \( \mathcal{X}' = \mathcal{X} \) and \( \mathcal{Y}' = \mathcal{Y} \). Let \( \mathcal{L} \) an ideal of \( \mathbb{Z}[j] \). Since \( \mathcal{L} \) is a subset of \( \mathbb{Z}[j] \), there exist two subsets \( \mathcal{I}, \mathcal{J} \) of \( \mathbb{Z} \) such that \( \mathcal{L} = \mathcal{I} \oplus j\mathcal{J} \). Since \( \mathcal{L} \) is an additive subgroup of \( \mathbb{Z}[j] \) which is stable by multiplication, then \( \mathcal{I}, \mathcal{J} \) should be additive subgroups of \( \mathbb{Z} \) which are stable by multiplication. So, \( \mathcal{I}, \mathcal{J} \) are ideals of \( \mathbb{Z} \). Reciprocally, if \( \mathcal{L} = \mathcal{I} \oplus j\mathcal{J} \) where \( \mathcal{I}, \mathcal{J} \) are ideals of \( \mathbb{Z} \), it is obvious that \( \mathcal{L} \) is also an ideal of \( \mathbb{Z}[j] \).
Since in \( \mathbb{Z} \), any ideal is principal, any ideal of \( \mathbb{Z}[j] \) has the form \((a, b) \in \mathbb{Z} \):

\[
\mathcal{L}_{a,b} = \{ae_{3,0}x + byj : x, y \in \mathbb{Z} \}
\]

Moreover, we know that for \( a, b \) in \( \mathbb{Z} \), there exists two integers \( s, t \) such that:

\[
a = s \gcd(a, b)
\]

and:

\[
b = t \gcd(a, b)
\]

Therefore, any element of \( \mathcal{L}_{a,b} \) can be expressed as \((x, y) \in \mathbb{Z}\):

\[
ae_{3,0}x + byj = \gcd(a, b)(ae_{3,0}s + btj)
\]

It results that \( \mathcal{L}_{a,b} \) is generated by \( \gcd(a, b) \) and is so principal. We conclude that \( \mathbb{Z}[j] \) is a principal entire subring of \( \mathbb{C} \).

The expression for the magnitude function is given by:

\[
N(ae_{3,0} + bj) = \sqrt{(ae_{3,0} + bj)(ae_{3,0} + bj^*)} = \sqrt{a^2e_{3,0}^2 + ab(j + j^*) + b^2jj^*}
\]

\[
N(ae_{3,0} + bj) = \sqrt{(a^2 - 2ab + b^2)e_{3,0}} = \sqrt{(a - b)^2e_{3,0}}
\]

\[
N(ae_{3,0} + bj) = \abs(a - b)\sqrt{e_{3,0}} = \abs(a - b)e_{3,0} = \abs(a - b)
\]

Thus, we have:

\[
\ker N = \{a(u_{3,0} + j) : a \in \mathbb{Z} \}
\]

It follows that the expression for the unit function is given by \((a \neq b)\):

\[
u(ae_{3,0} + bj) = \frac{1}{\abs(a - b)}(a + bj)
\]

For \( n = 4 \), we know that \( i = e_{4,1} = e^{\frac{i\pi}{4}} \). The subring \( \mathbb{Z}[S_4] \) of \( \mathbb{C} \) is generated by \( S_4 = \mathcal{G}_4 \setminus \{e_{4,0} \} = \{i\} \)

where \( \mathcal{G}_4 = \{e_{4,0}, i, e_{4,2}\} \setminus \{e_{4,2}\} \) namely \( \mathcal{G}_4 = \{e_{4,0}, i\} \) with \( e_{4,0} = 1 \). Since \( \{u_{4,0}, i\} \) is free and is maximal in \( U_4 \), \( \mathbb{Z}[S_4] \) has a basis namely \( \mathcal{G}_4 = \{e_{4,0}, i\} \). So, \( \mathbb{Z}[S_4] \) is the subring of Gaussian integers:

\[
\mathbb{Z}[S_4] = \mathbb{Z}[i] = \{ae_{4,0} + bi : a, b \in \mathbb{Z} \}
\]

In this case, we have:

\[
U(\mathbb{Z}[i]) = U_4 = \{e_{4,0}, -e_{4,0}, i, -i\}
\]

Indeed, let \( ae_{4,0} + bi \in U(\mathbb{Z}[i]) \). Then, there exists \( a', b' \in \mathbb{Z} \) such that:

\[
(ae_{4,0} + bi)(a' e_{4,0} + b'i) = e_{4,0}
\]

\[
(aa' - bb')e_{4,0} + (ab' + ba')i = e_{4,0}
\]

So:

\[
\begin{cases}
  aa' - bb' = 1 \\
  ab' + ba' = 0
\end{cases}
\]

The first equation means that \( \gcd(a, b) = 1 \) and so \( a \) and \( b \) are relatively primes. Using this fact, the second equation which can be rewritten \( ab' = -ba' \), implies from the Euclid’s lemma that \( a|a' \) and \( b|b' \). So, there exists an integer \( k \) in \( \mathbb{Z} \) such that:

\[
a' = ak
\]

and:

\[
b' = bk
\]
From the equation \( aa' - bb' = 1 \), it gives:
\[
k(a^2 + b^2) = 1
\]
So, either \( k = 1 \) or \( k = -1 \). If \( k = -1 \), then \( a^2 + b^2 = -1 \) which is not possible. It remains \( k = 1 \).
It gives \( (k = 1) \):
\[
a = \pm 1 \quad \text{and} \quad b = 0
\]
or
\[
a = 0 \quad \text{and} \quad b = \pm 1
\]
Therefore, the elements of \( U(\mathbb{Z}[i]) \) are:
\[
e_{4,0}, -e_{4,0}, i, -i
\]
Since \( \mathbb{Z}[i] = \mathbb{Z} \oplus i\mathbb{Z} \) (it is because any element \( x \) of \( \mathbb{Z}[i] \) is written in an unique way as \( x = a + bi \) with \( a, b \in \mathbb{Z} \) and because \( \mathbb{Z} \cap i\mathbb{Z} = \{0\} \)), for any subset \( \mathcal{W} \) of \( \mathbb{Z}[i] \), there exist two subsets \( \mathcal{X}, \mathcal{Y} \) of \( \mathbb{Z} \) such that \( \mathcal{W} = \mathcal{X} \oplus i\mathcal{Y} \). Notice that \( \mathcal{X}, \mathcal{Y} \) are unique since if \( \mathcal{X}' \oplus i\mathcal{Y}' = \mathcal{X} \oplus i\mathcal{Y} \), then \( \mathcal{X}' = \mathcal{X} \) and \( \mathcal{Y}' = \mathcal{Y} \). Let \( \mathcal{L} \) an ideal of \( \mathbb{Z}[i] \). Since \( \mathcal{L} \) is a subset of \( \mathbb{Z}[i] \), there exist two subsets \( \mathcal{I}, \mathcal{J} \) of \( \mathbb{Z} \) such that \( \mathcal{L} = \mathcal{I} \oplus i\mathcal{J} \). Since \( \mathcal{L} \) is an additive subgroup of \( \mathbb{Z}[i] \) which is stable by multiplication, then \( \mathcal{I}, \mathcal{J} \) should be additive subgroups of \( \mathbb{Z} \) which are stable by multiplication. So, \( \mathcal{I}, \mathcal{J} \) are ideals of \( \mathbb{Z} \). Reciprocally, if \( \mathcal{L} = \mathcal{I} \oplus i\mathcal{J} \) where \( \mathcal{I}, \mathcal{J} \) are ideals of \( \mathbb{Z} \), it is obvious that \( \mathcal{L} \) is also an ideal of \( \mathbb{Z}[i] \).
Since in \( \mathbb{Z} \), any ideal is principal, any ideal of \( \mathbb{Z}[i] \) has the form \( (a, b \in \mathbb{Z}) \):
\[
\mathcal{L}_{a,b} = \{ ae_{4,0}x + byi : x, y \in \mathbb{Z} \}
\]
Moreover, we know that for \( a, b \) in \( \mathbb{Z} \), there exists two integers \( s, t \) such that:
\[
a = s \gcd(a, b)
\]
and:
\[
b = t \gcd(a, b)
\]
Therefore, any element of \( \mathcal{L}_{a,b} \) can be expressed as \( (x, y \in \mathbb{Z}) \):
\[
ae_{4,0}x + byi = \gcd(a, b)(ae_{4,0}s + bti)
\]
It results that \( \mathcal{L}_{a,b} \) is generated by \( \gcd(a, b) \) and is so principal. We conclude that \( \mathbb{Z}[i] \) is a principal entire subring of \( \mathbb{C} \).

The expression for the magnitude function is given by:
\[
N(ae_{4,0} + bi) = \sqrt{(ae_{4,0} + bi)(ae_{4,0} - bi)} = \sqrt{a^2 - abi + bai - b^2i^2}
\]
\[
N(ae_{4,0} + bi) = \sqrt{a^2 + b^2}
\]
Thus, we have:
\[
\ker N = \{0\}
\]
It follows that the expression for the unit function is given by \( (a, b \neq 0) \):
\[
u(ae_{4,0} + bi) = \frac{1}{\sqrt{a^2 + b^2}}(a + bi)
\]
Let consider a natural number \( n \geq 3 \) which is odd. Let prove by induction that any family \( \{e_{n,0}, e_{n,1}, \ldots, e_{n,k}\} \) for \( k \in \llbracket 0, n - 2 \rrbracket \) is free. It will prove that the family \( \{e_{n,0}, e_{n,1}, \ldots, e_{n,n-2}\} \) forms a basis of \( \mathbb{Z}[S_n] \) when \( n \) is an odd positive integer. The case where \( n \) is even can be done in a similar way.

When \( k = 0 \), we have \( e_{n,0} = 1 \) and it is obvious that \( \{e_{n,0}\} \) is free. Let assume that for an integer
If an odd positive integer. A similar reasoning implies that the family \( Z \) is free. Therefore, the family \( \{ e_n, e_{n,1}, \ldots, e_{n,k} \} \) is free. If \( a_0, a_1, \ldots, a_k, a_{k+1} \) are integers such that:

\[
a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_k e_{n,k} + a_{k+1} e_{n,k+1} = 0
\]

then since \( e_{n,i} = e_{n,1}^i \) for \( i \in \mathbb{Z} \), it comes that:

\[
a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_k e_{n,k} + a_{k+1} e_{n,k+1} = 0
\]

Multiplying the left and the right hand sides of this equation by \( e_{n,1}^{n-k-1} \), it gives \( (e_{n,1}^n = 1) \):

\[
e_{n,1}^{n-k-1} \{ a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_k e_{n,k} + a_{k+1} e_{n,k+1} \} = 0
\]

\[
e_{n,1}^{n-k-1} \{ a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_k e_{n,k} \} = -a_{k+1}
\]

By conjugation, we have also:

\[
(e_{n,1}^{n-k-1})^* \{ a_0 e_{n,0} + a_1 e_{n,1}^* + \ldots + a_k (e_{n,1}^k)^* \} = -a_{k+1}
\]

It results that:

\[
a_0 \text{Im}(e_{n,1}^{n-k-1}) + a_1 \text{Im}(e_{n,1}^{n-k}) + \ldots + a_k \text{Im}(e_{n,1}^{n-1}) = 0
\]

\[
a_0 \text{Im}(e_{n,n-k-1}) + a_1 \text{Im}(e_{n,n-k}) + \ldots + a_k \text{Im}(e_{n,n-1}) = 0
\]

Since the family \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,k} \} \) is free from the assumption, from the theorem \([4,5,8]\), the family \( \{ \text{Im}(e_{n,n-k-1}), \text{Im}(e_{n,n-k}), \ldots, \text{Im}(e_{n,n-1}) \} \) is also free. It implies that:

\[
a_0 = a_1 = \ldots = a_k = 0
\]

and so:

\[
a_{k+1} = 0
\]

We deduce that if \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,k} \} \) is free for \( k \in [0, n-3] \), then \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,k}, e_{n,k+1} \} \) is also free. It achieved the proof by induction of the property that any family \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,k} \} \) for \( k \in [0, n-2] \) is free. Therefore, the family \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,n-2} \} \) forms a basis of \( \mathbb{Z}[S_n] \) when \( n \) is an odd positive integer. A similar reasoning implies that the family \( \{ e_{n,0}, e_{n,1}, \ldots, e_{n,n-2} \} \setminus \{ e_{n,n/2} \} \) forms a basis of \( \mathbb{Z}[S_n] \) when \( n \) is a non-zero even positive integer.

Thus, \( \mathbb{Z}[S_n] \) with \( n \geq 3 \) is a free module such that \( (n \geq 3) \):

\[
\mathbb{Z}[S_n] = \begin{cases} 
\mathbb{Z}[e_{n,1}, \ldots, e_{n,n/2-1}, e_{n,n/2+1}, \ldots, e_{n,n-2}] & \text{if } n \equiv 0 \pmod{2} \\
\mathbb{Z}[e_{n,1}, \ldots, e_{n,n-2}] & \text{if } n \equiv 1 \pmod{2}
\end{cases}
\]

If \( n \geq 3 \) is odd, then the expression for the magnitude function is given by:

\[
N(a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_{n-2} e_{n,n-2}) = \sqrt{\sum_{i=0}^{n-2} a_i^2 e_{n,0} + \sum_{i=0}^{n-2} \sum_{j \neq i} a_i a_j (e_{n,1}^i (e_{n,1}^j)^* + (e_{n,1}^i)^* e_{n,1}^j)}
\]

If \( n \geq 3 \) is even, then the expression for the magnitude function is given by:

\[
N(a_0 e_{n,0} + a_1 e_{n,1} + \ldots + a_{n/2-1} e_{n,n/2-1} + a_{n/2+1} e_{n,n/2+1} + \ldots + a_{n-2} e_{n,n-2})
\]

\[
= \sqrt{\sum_{i \neq n/2} a_i^2 e_{n,0} + \sum_{i \neq n/2} \sum_{j \neq i, n/2} a_i a_j (e_{n,1}^i (e_{n,1}^j)^* + (e_{n,1}^i)^* e_{n,1}^j)}
\]

More generally, let \( n \in \mathbb{N}^* \) and let \( \{ e_1, \ldots, e_n \} \) a finite family of elements of \( \mathbb{C} \) such that \( (i, j = 1, \ldots, n) \):

\[
e_n = e_0 = 1
\]
We assume that the elements $c_i$ are understood. The set $e_i$ such that $(i, j)$ where the Einstein summation over repeated indices $n$ from 1 to an entire subring $A$ is associated to a finite-dimensional complex Lie group $G$ and let $e_i$ satisfy the 

$\{e_0, e_1, \ldots, e_n\}$ with:

$$|e_i^n = e_i \ldots e_i = e_{ni} = e^n_i = e^{ni}_0 = e_0$$

So, $G$ is a cyclic finite subgroup of $A[S]$ of order $n$.

Moreover, we have:

$$e_i = e^i_1$$

So, a generator of $\{e_0, \ldots, e_n\}$ is $e_1$. Besides, we have $(i = 0, 1, \ldots, n - 1)$:

$$N(e^n_i) = N(e_i)^n = N(e_0) = N(1) = 1$$

Since $||N(e_i)||_C = N(e_i)$, it results that $(i = 0, 1, \ldots, n - 1)$:

$$N(e_i) = 1$$

Therefore, we have $(i = 0, 1, \ldots, n - 1)$:

$$u(e_i) = e_i$$

11. An algebra of entire ring generated by the generators of a Lie algebra

A more general framework is to consider a finite maximal free family $\mathfrak{c} = \{e_1, \ldots, e_n\}$ with $n \in \mathbb{N}^*$ of generators of a Lie algebra $\mathfrak{g}$ associated to a finite-dimensional complex Lie group $G$ such that $(i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$e_n = e_0$$

$$e_i e_j = c_{ij}^k e_k$$

where the Einstein summation over repeated indices $k$ from $k = 0$ to $n - 1$ with $n \in \mathbb{N}^*$, is understood. The set $\mathfrak{c}$ is a basis of the associated Lie algebra $\mathfrak{g}$ of the Lie group $G$.

We assume that the elements $c_{ij}^k$ are called the structure constants of $\mathfrak{g}$ with respect to basis $\mathfrak{c}$, belong to an entire subring $A$ of $\mathbb{C}$ such that $A = \text{Re}(A)$. Denoting $s = \mathfrak{c} \setminus \{e_n\} = \{e_1, \ldots, e_{n-1}\}$, the
set $A[s]$ is the extension of ring $A$ which includes all linear combinations of elements of $g$ with coefficients in $A$. It is understood that $A[s]$ is generated by $s$. Similarly as in the case where the Lie algebra $g$ associated to the Lie group $G$ is replaced by an abelian group (see above), a star operation is defined on $A[s]$. Since the structure constants $c_{ij}^k$ with $i, j, k = 0, 1, \ldots, n - 1$, belong to $A = \text{Re}(A)$, we have $(i, j, k = 0, 1, \ldots, n - 1$ with $n \in \mathbb{N}^*$):

$$(c_{ij}^k)^* = c_{ij}^k$$

We define another finite family $\{e'_1, \ldots, e'_n\}$ with $n \in \mathbb{N}^*$ by $(i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$e'_n = e'_0 = e_0$$

$$e'_i - v_i e_0 = (\delta_i^j - v_i \delta_0^j)e_j$$

$$(e'_i)^* = -e'_i$$

with $v_1, \ldots, v_{n-1} \in \mathbb{C}$ and $v_n = 0$ with $n \in \mathbb{N}^*$.

The family $\{e'_1, \ldots, e'_n\}$ is free. Indeed, if $(a'_0, a'_1, \ldots, a'_{n-1} \in \mathbb{C}$ and $n \in \mathbb{N}^*)$:

$$a_0 e_0 + a'_1 e'_1 + \ldots + a'_{n-1} e'_{n-1} = 0$$

then $(a'_0, a'_1, \ldots, a'_{n-1} \in \mathbb{C}$ and $n \in \mathbb{N}^*)$:

$$a_0 e_0 + a'_1 e'_1 + \ldots + a'_{n-1} e'_{n-1} = 0$$

with $(a'_0, a'_1, \ldots, a'_{n-1} \in \mathbb{C}$ and $n \in \mathbb{N}^*)$:

$$a_0 = a'_0 - (v_1 a'_1 + \ldots + v_{n-1} a'_{n-1})$$

and $(i = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$a'_i = a_i$$

Since the family $\{e_1, \ldots, e_n\}$ is free, it implies that $(n \in \mathbb{N}^*)$:

$$a_0 = a_1 = \ldots = a_{n-1} = 0$$

It gives $(i = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$a_i = 0$$

and:

$$a'_0 = a_0 = 0$$

Therefore the family $\{e'_1, \ldots, e'_n\}$ is free.

Moreover, it comes that $(i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$e_i = e'_i + v_i e_0 = (\delta_i^j + v_i \delta_0^j)e'_j$$

Then, we have $(i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$e_i e_j = (\delta_i^l + v_i \delta_0^l)(\delta_j^e + v_j \delta_0^e)e'_m$$

$$e_i e_j = \delta_i^l \delta_j^m e'_l e'_m + v_i v_j \delta_0^l \delta_0^m e'_l e'_m + v_i \delta_0^l v_j \delta_0^m e'_l e'_m + v_i v_j \delta_0^l \delta_0^m \delta_0^m e'_l e'_m$$

$$e_i e_j = e'_i e'_j + v_i e'_i + v_j e'_i + v_i v_j e_0^2 = e'_i e'_j + (v_i \delta_j^i + v_j \delta_i^j)e'_k$$

Since $e_i e_j = c_{ij}^k e_k$ for $i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*$, we have also $(i, j = 1, \ldots, n$ and $n \in \mathbb{N}^*)$:

$$e_i e_j = c_{ij}^k (\delta_k^l + v_k \delta_0^l)e'_l$$

$$e_i e_j = c_{ij}^k \delta_k^l e'_l + c_{ij}^k v_k \delta_0^l e'_l$$

$$e_i e_j = c_{ij}^k e'_k + c_{ij}^l v_k \delta_0^l e'_k$$
So \((i,j = 1,\ldots,n\) and \(n \in \mathbb{N}^*\):
\[
e'_i e'_j = d_{ij}^k e'_k
\]
with \((i,j,k = 1,\ldots,n\) and \(n \in \mathbb{N}^*\):
\[
d_{ij}^k = c_{ij}^k + c_{ij}^l v_l \delta^k_0 - (v_i \delta^k_j + v_j \delta^k_i)
\]
Notice that \((i,j = 0,1,\ldots,n-1)\):
\[
(e'_i e'_j)^* = (d_{ij}^k e'_k)^* = -(d_{ij}^k)^* e'_k
\]
\[
(e'_i e'_j)^* = (e'_j)^* (e'_i)^* = e'_j e'_i = d_{ji}^k e'_k
\]
It results that \((i,j,k = 0,1,\ldots,n-1)\):
\[
(d_{ij}^k)^* = -d_{ji}^k
\]
The conjugate of \(e_i\) for \(i = 0,1,\ldots,n-1\) with \(n \in \mathbb{N}^*\) is given by \((i = 0,1,\ldots,n-1\) and \(n \in \mathbb{N}^*\):
\[
e_i^* = v_i^* e_0 - e'_i
\]
\[
e_i^* = v_i^* e_0 - (e_i - v_i e_0)
\]
\[
e_i^* = -e_i + (v_i + v_i^*) e_0
\]
It follows that \((i = 0,1,\ldots,n-1\) with \(n \in \mathbb{N}^*\):
\[
e_i e_i^* = -e_i^2 + (v_i + v_i^*) e_0 e_0 = -e_i^2 + (v_i + v_i^*) c_{00}^k e_k
\]
and \((i = 0,1,\ldots,n-1\) with \(n \in \mathbb{N}^*\):
\[
e_i^* e_i = -e_i^2 + (v_i + v_i^*) e_0 e_i = -e_i^2 + (v_i + v_i^*) c_{0i}^k e_k
\]
So, it comes that:
\[
e_i e_i^* = e_i^* e_i + (v_i + v_i^*) (e_{i0}^k - c_{0i}^k) e_k
\]
It results that the function \(N\) is not well defined on \(A[s]\) unless \((i = 0,1,\ldots,n-1; k = 0,\ldots,n-1\) with \(n \in \mathbb{N}^*\):
\[
e_{i0}^k = c_{0i}^k
\]
Or, since \(e_i e_0 = -e_0 e_i\), we have \(c_{i0}^k = -c_{0i}^k\). Therefore, the function \(N\) is well defined on \(A[s]\) if \((i = 0,1,\ldots,n-1; k = 0,1,\ldots,n-1\) with \(n \in \mathbb{N}^*\):
\[
e_{i0}^k = e_{0i}^k = 0
\]
But, in this case, since \(g\) is Lie algebra, we have:
\[
e_i e_i^* = e_i^* e_i = -e_i e_i
\]
meaning that:
\[
e_i e_i^* = e_i^* e_i = 0
\]
It involves that the function \(N\) which can be defined on any element of \(A[s]\) which commutes with its conjugate cancels. We conclude that \(N\) is either degenerate on a subset of \(A[s]\) or is not defined on \(A[s]\).

12. Acknowledgements

The author would like to thank Aleks Kleyn for helpful comments.
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