Finitely generated, non-artinian monolithic modules.

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Abstract

We survey noetherian rings $A$ over which the injective hull of every simple module is locally artinian. Then we give a general construction for algebras $A$ that do not have this property. In characteristic 0, we also complete the classification of down-up algebras with this property which was begun in [CLPY10] and [CM].

1 Introduction

A module $M$ is monolithic if the intersection of all nonzero submodules of $M$ is nonzero. The intersection of all nonzero submodules of a monolithic module $M$ is a simple submodule known as the lith of $M$. Thus monolithic modules have a unique lith! This terminology is due to Roseblade [Ros73], [Ros76]. It was pointed out to me by Ken Goodearl that monolithic modules are also known as subidirectly irreducible modules. We consider the following property of a noetherian ring $A$.

$(\diamond)$ Every finitely generated monolithic $A$-module is artinian.

Equivalently, the injective hull of every simple $A$-module is locally artinian. Some history concerning property $(\diamond)$ is given in the introduction to [CM]. The property is not well understood, as is shown by the following quite baffling lists of examples.

The following rings $A$ have property $(\diamond)$.

(A.0) Commutative noetherian rings, and more generally PI and FBN rings [Jat74b].

The next two examples are in fact PI rings.

(A.1) The coordinate ring of the quantum plane, that is the algebra generated by elements $a, b$ subject to the relation $ab = qba$ when $q \in K$ is a root of unity.

(A.2) The quantized Weyl algebra, that is the algebra generated by elements $a, b$ subject to the relation $ab - qba = 1$ when $q \in K$ is a root of unity.
The enveloping algebra $U(sl(2, K))$ where $K$ a field of characteristic 0, [Dah84].

The group rings $ZG$ and $KG$ where $K$ is a field which is algebraic over a finite field and $G$ is polycyclic-by-finite, [Jat74], [Ros77].

Prime noetherian rings of Krull dimension 1, [CLPY10], [Mus80].

There are simple noetherian, non-artinian rings for which any simple module is injective, and obviously these rings have property $(\diamond)$ [Coz70].

The following rings $A$ do not have property $(\diamond)$.

The coordinate ring of the quantum plane when $q \in K \setminus \{0\}$ is not a root of unity, [CM].

The quantized Weyl algebra, when $q \in K \setminus \{0\}$ is not a root of unity, [CM].

The enveloping algebra $U(b)$ over an algebraically closed field of characteristic 0, when $b$ is finite dimensional, solvable and non-nilpotent, [CH80], [Mus82].

The group algebra $KG$ where $K$ is a field which is not algebraic over a finite field and $G$ is polycyclic-by-finite which is not nilpotent-by-finite, [Mus80].

The Goodearl-Schofield example: a certain non-prime noetherian ring of Krull dimension 1, [GS86].

What has been lacking up to now is a general construction for finitely generated, non-artinian, monolithic modules. In the next section we give such a construction under fairly mild conditions on $A$. We show that examples (B.1)-(B.3) satisfy these conditions. We also apply our construction to down-up algebras in characteristic 0. Some open problems are given in the last section.

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2 The construction.

Let $K$ be a field. We make the following assumptions.

(1) $A$ is a noetherian $K$-algebra without zero divisors.

(2) $w$ is a normal element of $A$.

(3) $J$ is a maximal left ideal such that $w - \mu \in J$ for some non-zero $\mu \in K$.

From (1) and (2) it follows that there is an automorphism $\sigma$ of $A$ such that for any $x \in A$ we have

$$wx = \sigma(x)w.$$ (2.1)
Suppose that $x$ is an element of $A$ that is not a unit and set $I = Jx$. Then we have a short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0,$$

where $L = Ax/I, M = A/I$ and $N = A/Ax$.

**Lemma 2.1.** $L \cong A/J$ is a simple $A$-module.

**Proof.** The map $f$ from $A$ to $L$ sending $a$ to $ax + Jx$ is clearly surjective with kernel containing $J$. If $a \in \text{Ker } f$, then $(a - j)x = 0$ for some $j \in J$, whence $a \in J$. \hfill \Box

An interesting feature of our construction is that remaining assumptions involve only $L$ and $N$. There is a single additional assumption on $L$.

(4) For all $m \geq 0$ the equation

$$\sigma^m(x)a - 1 \in J \quad (2.2)$$

has no solution for $a \in A$. For $z \in A$, denote the image of $z$ in $M = A/I$ by $\overline{z}$. Then equation (2.2) is equivalent to

$$\sigma^m(x)a\overline{z} = \overline{z} \quad (2.3)$$

and equation (2.3) always has a solution if $L$ is divisible. Since $L$ obviously cannot be injective, some condition similar to (4) must be necessary if our construction is to go through.

Finally we make the following assumptions on $N$.

(5) $N$ has a strictly descending chain of submodules

$$N \supset wN \supset \ldots \supset w^mN \supset \ldots \quad (2.4)$$

(6) Every nonzero submodule of $N$ contains $w^mN$ for some $m$.

**Theorem 2.2.** Under assumptions (1)-(6), $M$ is an essential extension of $L$.

**Proof.** Note that the assumptions are unchanged if we replace $w$ by $\mu^{-1}w$. Thus we can assume that $\mu = 1$. Suppose $U$ is a left ideal of $A$ strictly containing $I$. We need to show that $U$ contains $Ax$. It follows easily from (6) that $U$ contains an element of the form $w^m - ax$ for some $a \in A$. Set $y = \sigma^m(x)$. Then from (2.1) and (3) we have

$$y(w^m - ax) = (w^m - 1)x + (1 - ya)x$$

$$\equiv (1 - ya)x \mod Jx. \quad (2.5)$$

For $z \in A$, denote the image of $z$ in $M = A/I$ by $\overline{z}$. From (2.3) and assumption (4) we have $0 \neq (1 - ya)\overline{z} \in \overline{U} \cap A\overline{z}$, so as $L = A\overline{z}$ is simple it follows that $A\overline{z} \subseteq \overline{U}$. The result follows easily. \hfill \Box
3 Examples (B.1)-(B.3).

To check assumption (4) we use the following easy result.

**Lemma 3.1.** If for all \( m \geq 0 \), there is a subring \( B \) of \( A \) such that \( A = B \oplus J \), and \( \sigma^m(x) \in B \), then assumption (4) holds.

**Proof.** If \( \sigma^m(x)a - 1 \in J \), write \( a = b + j \) with \( b \in B \) and \( j \in J \). Then \( \sigma^m(x)b - 1 \in J \cap B = 0 \), whence \( \sigma^m(x) \) is a unit in \( A \) a contradiction, since \( x \) is assumed to be a non-unit. \( \square \)

It is not always possible to choose \( B \) to be \( \sigma \)-invariant in Lemma 3.1. From Theorem 2.2 and the next two results, we obtain the non-artinian, monolithic modules in [CM] Theorems 3.1 and 4.2.

Let \( A = K[a, b] \) be the coordinate ring of the quantum plane, as in (B.1) where \( ab = qba \) and \( q \in K \setminus \{0\} \) is a not root of unity. Let \( w = ab \) and \( J = A(ab - 1) \), \( B = K[a] \) and \( x = a - 1 \in B \). Then \( w \) is a normal element and the automorphism \( \sigma \) determined by equation (2.1) satisfies \( \sigma(a) = q^{-1}a \) and \( \sigma(b) = qb \).

**Proposition 3.2.**

(a) \( J \) is a maximal left ideal of \( A \) and assumption (4) holds.

(b) If \( N = A/Ax \), then \( N \) is non-artinian, and a complete list of non-zero submodules of \( N \) is given by equation (2.4).

**Proof.** Since \( A = B \oplus J \) and \( \sigma \) preserves \( B \), the result follows from Steps 1 and 2 in the proof of [CM] Theorem 3.1. \( \square \)

Let \( A = K[a, b] \) be the quantized Weyl algebra, as in (B.2) where \( ab - qba = 1 \) and \( q \in K \setminus \{0\} \) is a not root of unity. If \( w = ab - ba \), then \( w \) is a normal element of \( A \) and \( w - 1 = (q - 1)ba \in J = Aa \). The automorphism \( \sigma \) determined by equation (2.1) satisfies \( \sigma(a) = q^{-1}a \) and \( \sigma(b) = qb \). We have \( A = B \oplus J \) with \( B = K[b] \), and \( \sigma(B) = B \). Let \( x = (1 - q)b - 1 \in B \).

**Proposition 3.3.**

(a) \( J \) is a maximal left ideal of \( A \) and assumption (4) holds.

(b) If \( N = A/Ax \), then \( N \) is non-artinian, and a complete list of non-zero submodules of \( N \) is given by equation (2.4).

**Proof.** By [CM] Lemma 4.1, \( J \) is a maximal left ideal of \( A \), and (4) follows as before. Note that \( N \cong K[a] \) as a \( K[a] \)-module. Let \( u_0 = 1 + Ax \), and define inductively \( u_{n+1} = (q^{-n}a - 1)u_n \). Then

\[
au_n = q^n(u_n + u_{n+1}) \quad \text{and} \quad bu_n = \frac{q^{-n}}{1 - q}u_n.
\]

Thus (b) follows as in the proof of [CM] Theorem 2.4 (b). \( \square \)
Next we show that certain Ore extensions with Gelfand-Kirillov dimension 2 do not have property \((\circ)\). Assume that \(K\) has characteristic zero, and let \(d\) be the derivation of the polynomial algebra \(K[a]\) determined by \(d(a) = a^r\) where \(r \geq 1\). Let \(A = K[a, b]\) be the resulting Ore extension, where for \(p \in K[a]\),
\[
pb = bp + d(p).
\]
(3.1)
In particular
\[
ab = ba + a^r.
\]
Thus if \(w = a\), then \(w\) is a normal element and the automorphism \(\sigma\) determined by equation (2.1) satisfies \(\sigma(a) = a\) and \(\sigma(b) = b + a^{r-1}\). We show below that \(A\) does not have property \((\circ)\). When \(r = 1\), \(A\) is isomorphic to the enveloping algebra \(U(b)\) where, \(b\) is a Borel subalgebra of \(\mathfrak{sl}(2, K)\). Now by [BGR73] Lemma 6.12, if \(K\) is algebraically closed, then any finite dimensional solvable Lie algebra which is non-nilpotent has \(b\) as an image, and thus we recover the result in (B.3).

Lemma 3.4. Any ideal invariant under \(d\) is generated by a power of \(a\).

Proof. This follows from the well known fact that if an ideal \(Q\) is invariant under a derivation, then so too are all the prime ideals that are minimal over \(Q\), see for example [BGR73] Lemma 4.1.

Let \(J = A(a - 1)\) and \(x = b - 1\).

Proposition 3.5.
(a) \(J\) is a maximal left ideal of \(A\) and assumption (4) holds.
(b) If \(N = A/Ax\), then \(N\) is non-artinian, and a complete list of non-zero submodules of \(N\) is given by equation (2.4).
(c) The submodules of \(N\) are pairwise non-isomorphic.

Proof. (a) Set \(v_n = b^n + J\). The elements \(\{v_n\}_{n \geq 0}\) form a basis for \(A/J\), and \(av_0 = v_0\). Assume by induction that
\[
(a - 1)^n v_n = n!v_0.
\]
(3.2)
Then by equation (3.1), we have
\[
(a - 1)^{n+1} v_{n+1} = (a - 1)^{n+1} bv_n = b[(a - 1)^{n+1} + (n + 1)a^r(a - 1)^n]v_n = (n + 1)!v_0.
\]
It follows easily from equation (3.2) that \(A/J\) is simple. Since \(\sigma^m(x) = b - 1 + ma^{r-1}\) we have \(A = B \oplus J\) where \(B = K[\sigma^m(x)]\), thus (4) holds.

(b) Since \(A = K[a] \oplus Ax\), we can identify \(N\) with \(K[a]\) as a \(K[a]\)-module. Suppose \(N'\) is a submodule of \(N\), and \(N' = pK[a]\) for some \(p \in K[a]\). Then
\[
bp = pb - d(p) = p - d(p) \mod Ax,
\]
and hence \(d(p) \in pK[a]\). Thus (b) follows from Lemma 3.4.

(c) As above we identify \(N\) with \(K[a]\). If \(\phi : a^m N \to a^{m_1}N\) is an isomorphism, then \(\phi(a^m) = a^{m_1}q(a)\) for some polynomial \(q\) with \(q(0) \neq 0\). Thus

\[
\phi(ba^m) = \phi(a^m - ma^{m+r-1}) = (1 - ma^{r-1})a^{m_1}q(a).
\]

and

\[
 b\phi(a^m) = b(a^{m_1}q(a)) = a^{m_1}q(a) - a^r(a^{m_1}q(a))'.
\]

This easily gives

\[
(m_1 - m)a^{m_1+r-1}q(a) = a^{m_1+r}q'(a).
\]

Now we must have \(m = m_1\) since otherwise the left side has 0 as a root of multiplicity at most \(m - r + 1\), whereas the right side has 0 as a root of multiplicity at least \(m - r\).

\[\square\]

4 Down-up Algebras.

Given a field \(K\) and \(\alpha, \beta, \gamma\) elements of \(K\), the associative algebra \(A = A(\alpha, \beta, \gamma)\) over \(K\) with generators \(d, u\) and defining relations

\[(R1) \quad d^2u = \alpha ud + \beta ud^2 + \gamma d\]
\[(R2) \quad du^2 = \alpha ud + \beta d^2u + \gamma u\]

is called a down-up algebra. Down-up algebras were introduced by G. Benkart and T. Roby [BR98], [BR99]. In [AMP99] it is shown that \(A = A(\alpha, \beta, \gamma)\) is noetherian if and only if \(\beta \neq 0\), and that these conditions are equivalent to \(A\) being a domain. The main result of this section is as follows.

**Theorem 4.1.** If \(A(\alpha, \beta, \gamma)\) is a noetherian down-up algebra over a field \(K\) of characteristic zero, then any finitely generated monolithic \(A(\alpha, \beta, \gamma)\)-module is artinian if and only if the roots of \(X^2 - \alpha X - \beta\) are roots of unity.

From now on we assume that \(X^2 - \alpha X - \beta = (X - 1)(X - \eta)\) where \(\eta = -\beta\) is not a root of 1, and that \(\beta \neq 0\). Thus \(A(\alpha, \beta, \gamma)\) is a Noetherian domain by the above remarks, and \(\alpha + \beta = 1\). In addition we assume that \(\gamma \neq 0\). Hence \(A(\alpha, \beta, \gamma)\) is isomorphic to a down-up algebra

\[A_\eta = A(1 + \eta, -\eta, 1).
\]

To prove Theorem 4.1 it is enough to prove the result below, as noted in [CM].

**Theorem 4.2.** If \(\eta\) is not a root of unity, then \(A_\eta\) does not have property (\(\diamond\)).
For the remainder of this section we assume that $A = A_\eta$ as in Theorem 4.2. We begin with some consequences of (R1) and (R2). Since $\eta \neq 1$, we have $\alpha \neq 2$. Set $\epsilon = (\alpha - 2)^{-1},$ and $\phi = 1 - \alpha \epsilon = -2(\alpha - 2)^{-1}$. As noted in [CM00] Section 1.4 Case 2, the element $w = -ud + du + \epsilon$ satisfies

$$dw = \eta wd, \quad uw = \eta^{-1}wu,$$

and hence $wA = Aw$. We remark that $A/Aw$ is isomorphic to the first Weyl algebra (this fact is not used below).

**Lemma 4.3.** For $n \geq 1$, we have

$$du^{2n} = u^{2n}d + n\phi u^{2n-1} + \alpha \sum_{i=0}^{n-1} \eta^{-2i-1}wu^{2n-1} \quad (4.1)$$

and for $n \geq 0$,

$$du^{2n+1} = u^{2n+1}d + u^{2n}w + (n\phi - \epsilon)u^{2n} + \alpha \sum_{i=0}^{n-1} \eta^{-1-2i}wu^{2n} \quad (4.2)$$

**Proof.** We have

$$du = w + ud - \epsilon. \quad (4.3)$$

Using (R2), then (4.3) and the fact that $\alpha + \beta = 1$, we see that for $j \geq 2$,

$$dw^j = [\alpha ud + \beta u^2d + u]w^{j-2}$$

$$= [\alpha u(w + ud - \epsilon) + \beta u^2d + u]w^{j-2}$$

$$= [(\alpha + \beta)u^2d + \alpha uw + (1 - \alpha \epsilon)u]w^{j-2}$$

$$= u^2dw^{j-2} + \alpha uw^{j-2} + \phi w^{j-1}.$$

The result follows easily by induction. \hfill \Box

Consider the module $N = A/A(d - 1)$, and if $a \in A$, denote the image of $a$ in $N$ by $\pi$. Then $N$ has a basis $w^i\pi^j$ with $i, j \geq 0$. Thus if $B = K[u, w]$, then $N \cong B$ as a left $B$-module. Since $dw^m = \eta^m w^md$, $N$ has a strictly descending chain of submodules as in Assumption (5). Next we define a filtration on $N$ by setting

$$N_n = \sum_{i=0}^{n} u^iK[\pi] = \sum_{i=0}^{n} K[w]\pi^i.$$

It follows from (4.1) and (4.2) that $dN_n \subseteq N_n$. Also for $f \in K[w]$, we have

$$df(w)\pi^m \equiv f(\eta w)\pi^m \mod N_{n-1}. \quad (4.4)$$

**Lemma 4.4.** If $U$ is a non-zero submodule of $N$, then $U$ contains $\pi^m$ for some $m$. 


Proof. Suppose that $n$ is minimal such that $U \cap N_n \neq 0$. We claim that $n = 0$. If this is not the case then $U + N_{n-1}$ contains an element of the form $x = f(w)\overline{w}^n$ for some non-zero polynomial $f$. Write $f(w) = \sum_{i=r} a_i w^i$, where $a_r \neq 0 \neq a_s$. If $r < s$, then $U + N_{n-1}$ contains an element of the form $y = w^r\overline{w}^n$, because $\prod_{i=r+1}^s (d_i - \eta^i)x \in U + N_{n-1}$. Thus if $n = 2m$ is even, we can assume that
\[ y = w^r\overline{w}^{2m} + \sum_{i=0}^{2m-1} g_i(w)\overline{w}^i \in U. \]

Then
\[(d - \eta^r)y \equiv [\eta^r w^r (m\phi + \alpha \sum_{i=0}^{n-1} \eta^{-2i}w) + g_{2m-1}(\eta w) - \eta^r g_{2m-1}(w)]\overline{w}^{2m-1} \mod N_{n-2}. \]

By the choice of $n$, $(d - \eta^r)y$ must be zero mod $N_{n-2}$. Note that the coefficient of $w^r$ in $g_{2m-1}(\eta w) - \eta^r g_{2m-1}(w)$ is zero. Thus looking at the coefficient of $w^r\overline{w}^{2m-1}$ on the right side above yields $m\phi = 0$, which is a contradiction. Thus $n = 2m + 1$ is odd, and we can assume that
\[ y = w^r\overline{w}^m + \sum_{i=0}^{2m} f_i(w)\overline{w}^i \in U + N_{n-2}. \]

Then mod $N_{n-2}$,
\[(d - \eta^r)y \equiv \eta^r w^r [u^{2m}\overline{w} + (m\phi - \epsilon)\overline{w}^{2m} + \alpha \sum_{i=0}^{m-1} \eta^{-1-2i}w\overline{w}^{2m}] + [f_{2m}(\eta w) - \eta^r f_{2m}(w)]\overline{w}^{2m}. \]

By the choice of $n$, $(d - \eta^r)y$ must be zero mod $N_{n-2}$. Then looking at the coefficient of $w^r\overline{w}^{2m}$ we obtain $m\phi = \epsilon$ which leads to the contradiction $2m + 1 = 0$. Thus $U$ contains an element of the form $f(\overline{w})$ with $f \neq 0$, and the result follows easily. \hfill \Box

We have verified assumptions (5) and (6) for the module $N$, and we now turn our attention to the simple module $L$.

Following [BR98] Proposition 2.2, we define the Verma module $V(\lambda)$ with highest weight $\lambda \in K$. Let $\lambda_{-1} = 0, \lambda_0 = \lambda$ and for each $n > 0$ set,
\[ \lambda_n = \alpha \lambda_{n-1} + \beta \lambda_{n-2} + 1. \]

The Verma module $V(\lambda)$ has basis $\{v_n | n \in \mathbb{N}\}$. The action of $A$ is defined by
\[ d.v_0 = 0, \text{ and } d.v_n = \lambda_{n-1} v_{n-1}, \text{ for all } n \geq 1 \]
\[ u.v_n = v_{n+1}. \]

In [BR98] Proposition 2.4 it is shown that $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for all $n \geq 0$. Furthermore, by [CM00] Lemma 2.5, $\lambda_{n-1} = 0$ if and only if
\[ \lambda(\eta - 1) = -(1 - n(\sum_{i=0}^n \eta^i)^{-1}). \]

(4.6)
Lemma 4.5. The algebra $A$ has infinitely many pairwise non-isomorphic simple Verma modules.

Proof. The result is evident if $K$ is uncountable, because then we simply require that the highest weight $\lambda$ does not satisfy the condition in (4.6) for any $n$. In general we argue as follows. By [CM00] Proposition 5.5, any Verma module has length at most 3, so by [BR98] Proposition 2.23, any Verma module has a simple Verma submodule. Also if $V(\lambda)$ is not simple this submodule is generated by $v_n$ where $n$ is the largest integer such that $\lambda_{n-1} = 0$. This submodule is isomorphic to $V(\lambda_n)$. Note that the case covered by [BR99] does not arise here. Now if $\mu = \lambda_n$ and $V(\mu)$ is simple, we can solve the recurrence (4.5) in reverse to find all Verma modules $V(\lambda)$ containing as a $V(\mu)$ simple submodule. Since there can be at most 3 such $\lambda$ and $K$ is infinite, the result follows.

Unfortunately it does not seem possible to verify assumption (4) for a simple Verma module. Instead we consider the universal lowest weight modules $W(\kappa)$ defined in [BR98] Proposition 2.30 (a). For $\kappa \in K$, set $\kappa_0 = \kappa$ and define for each $n > 0$,

$$\kappa_n = \eta^{-1}(\alpha \kappa_{n-1} - \kappa_{n-2} + 1).$$

Then $W(\kappa)$ has basis $\{a_n|n \in \mathbb{N}\}$. The action of $A$ is defined as follows,

- $u.a_0 = 0$, and $u.a_n = \kappa_{n-1}a_{n-1}$, for all $n \geq 1$
- $d.a_n = a_{n+1}$.

Corollary 4.6. The algebra $A$ has infinitely many pairwise non-isomorphic simple lowest weight modules $W(\kappa)$.

Proof. By [CM00] Lemma 4.1, there is an isomorphism from $A$ onto $A' = A_{\eta^{-1}}$ which interchanges the generators $u$ and $d$. Under this isomorphism, any Verma module for $A'$ becomes a module of the form $W(\kappa)$ for $A$, so the result follows.

Proof of Theorem 4.2. Let $L = W(\kappa)$ be a simple lowest weight module, and let $J$ be the annihilator of the lowest weight vector $a_0$ in $A$. Then $J = Au + A(ud - \kappa)$. The normal element $w = -ud + du + \epsilon$ satisfies $w - \mu \in J$ where $\mu = -\kappa + \epsilon$. By Corollary 4.6 we can arrange that $\mu$ is non-zero. Set $x = d - 1$. It only remains to check assumption (4). This holds because $A = B \oplus J$ with $x \in B = K[d]$, and $B$ is $\sigma$-invariant.

5 Remarks and Problems.

(a) We call a finitely generated module $E$ over a left noetherian ring uniserial if the submodules of $E$ are totally ordered by inclusion. For $E$ uniserial define a descending chain of submodules $\{E_\alpha\}$ as follows. For any ordinal $\alpha$, if $E_\alpha \neq 0$ let $E_{\alpha+1}$ be the unique maximal submodule of $E_\alpha$. For a limit ordinal $\beta$ such that $E_\alpha \neq 0$ for $\alpha < \beta$, set $E_\beta = \bigcap_{\alpha < \beta} E_\alpha$. There is a smallest ordinal $\tau$ such that $E_\tau = 0$, and we call $\tau$ the depth of $E$. As noted in the introduction to
it follows from \[\text{[Jat69]}\] Theorem 4.6, that for any ordinal \(\tau\) there is a left noetherian ring \(A\) such that the left regular module is uniserial with depth \(\tau\). The modules \(M\) constructed using Theorem 2.2 with the aid of the results in Section 3 are all uniserial with depth \(\omega + 1\) where \(\omega\) is the first infinite ordinal. What other module depths are possible for uniserial modules over (two-sided) noetherian rings?

(b) If \(N\) is as in Propositions 3.2 and 3.3 (resp. 3.5), then \(N\) is incompressible and critical by \[\text{[CM]}\] Theorems 3.1 and 4.2, (resp. Proposition 3.3 (c)). The first example of an a incompressible and critical module was found by Ken Goodearl, see \[\text{[Goo80]}\], to which we refer for the definitions. In general is there a connection between rings that do not have \(A\) property \((\diamondsuit)\), and incompressible critical modules?

(c) Suppose that \(A\) is a Noetherian ring, and \(P\) an ideal of \(A\) such that \(A/P\) is simple artinian with simple module \(S\). Is the injective hull of a \(S\) as an \(A\)-module locally artinian?

(d) Define a noetherian ring \(A\) to be \((\diamondsuit)\) extremal if it does not have property \((\diamondsuit)\), but every proper homomorphic image has property \((\diamondsuit)\). What can be said about \((\diamondsuit)\) extremal rings? If \(A\) is an algebra over a field having finite Gelfand-Kirillov dimension and \(A\) is \((\diamondsuit)\) extremal, must \(A\) be prime? The Goodearl-Schofield example shows that this is not true without the GK dimension hypothesis. It seems likely that the algebra \(A_\eta\) in Theorem 4.2 is \((\diamondsuit)\) extremal. We note the following result.

**Proposition 5.1.** Suppose that \(A\) is a \(K\)-algebra such that the endomorphism ring of every simple \(A\)-module is algebraic over \(K\). If \(A\) is \((\diamondsuit)\) extremal the center \(Z\) of \(A\) is algebraic over \(K\).

**Proof.** If \(Z\) is not algebraic over \(K\) then, for every simple module \(L\), the natural map \(Z \rightarrow \text{End}_A L\) has non-zero kernel \(m\). Then if the injective hull of \(L\) as an \(A/mA\) is locally artinian, then so too is its injective hull over \(A\), see \[\text{[CLPY10]}\] Proposition 1.6. Thus \(A\) cannot be \((\diamondsuit)\) extremal.

The hypothesis that the endomorphism ring of every simple \(A\)-module is algebraic over \(K\) is known to hold for many algebras, for example it holds for almost commutative algebras (Quillen’s Lemma) and for an algebra of countable dimension over an uncountable field.
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