THE FIXED POINTS OF THE MULTIVARIATE SMOOTHING TRANSFORM

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ABSTRACT. Let $N, d \geq 2$ be fixed integers, let $(T_1, \ldots, T_N)$ be random $d \times d$ matrices with nonnegative entries and $Q$ a random $d$-vector with nonnegative entries. This induces a mapping (the multivariate smoothing transform) on probability laws on the nonnegative cone by

$$
S \eta := \text{Law of } \left( \sum_{i=1}^{N} T_i X_i + Q \right),
$$

where $(X_i)_{i=1}^{N}$ are iid with law $\eta$ and independent of $((T_i))_{i=1}^{N}, Q)$. Under conditions similar to those for the well-studied case $d = 1$, a complete characterization of all fixed points of $S$ is obtained.

Keywords: Smoothing transform, Markov random walks, general branching processes, multivariate stable laws, multitype branching random walk, Choquet-Deny lemma, weighted branching

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1. Introduction

Let $N \geq 2, d \geq 2$ be fixed integers. Let $\mathbb{R}_{\geq}^d = [0, \infty)^d$ be the positive cone and $\mathcal{M}_{\geq} = M(d \times d, \mathbb{R}_{\geq})$ be the set of $d \times d$-matrices with nonnegative entries. Given a random variable $(T_1, \ldots, T_N, Q)$ taking values in $\mathcal{M}_{\geq}^N \times \mathbb{R}_{\geq}^d$, define a mapping on the set $\mathcal{P}(\mathbb{R}_{\geq}^d)$ of probability measures on $\mathbb{R}_{\geq}^d$ by

$$
S : \mathcal{P}(\mathbb{R}_{\geq}^d) \to \mathcal{P}(\mathbb{R}_{\geq}^d), \quad \eta \mapsto \mathcal{L} \left( \sum_{i=1}^{N} T_i X_i + Q \right),
$$

where $\mathcal{L} (\cdot)$ denotes “law of” and $X_1, \ldots, X_N$ are independent identically distributed (iid) random variables with law $\eta$ and independent of $T := (T_1, \ldots, T_N, Q)$. Following Durrett and Liggett [23], the mapping $S$ is called the (multivariate) smoothing transform (ST).

If $S \eta = \eta$, then $\eta$ is called a fixed point (FP) of $S$. If $X, X_1, \ldots, X_N$ are iid with law $\eta$ and independent of $T$, then $X$ satisfies the stochastic fixed point equation (SFPE)

$$
X \overset{\mathcal{L}}{=} \sum_{i=1}^{N} T_i X_i + Q,
$$

here $\overset{\mathcal{L}}{=} \text{ means same law.}$
Aim of the Paper. The contribution of this paper is to completely solve the fixed point equation \( S \eta = \eta \) and to describe properties of the fixed points in the homogeneous (i.e. \( Q \equiv 0 \)) as well as in the inhomogeneous case under assumptions similar to those of Durrett and Liggett [23] for the one-dimensional case. Therefore, we have to draw on a broad variety of methods from the theory of multitype branching random walk, products of random matrices, Markov random walks, general branching processes and harmonic analysis. A tool which might be of interest in its own right is the convergence in probability of restricted versions of the intrinsic martingale in multitype branching random walk, obtained by an application of Kesten’s renewal theorem.

After introducing as little as possible notation, we will state an ad-hoc version of the main result, then we will describe applications as well as an explicit example and finish the introduction by giving an overview of the literature.

Ad-hoc Version of the Main Result. In the one-dimensional situation, the main properties of \( S^1 : \eta \to \mathcal{L} \left( \sum_{i=1}^{N} T_i X_i \right) \) are encoded in the function \( m(s) = \sum_{i=1}^{N} T_i^s \). In order to define a multivariate analogue, observe first that, using a random permutation, it may w.l.o.g. be assumed that the \( (T_i)_{i=1}^{N} \) are identically distributed. Then let \( (T_{(n)})_{n \in \mathbb{N}} \) be a sequence of iid random matrices with the same law as \( T_1, \mu \), say. Define
\[
I_\mu := \{ s \geq 0 : \mathbb{E} \| T_1 \|^s < \infty \},
\]
where \( \| \cdot \| \) denotes the operator norm w.r.t. the euclidean norm \( | \cdot | \) on \( \mathbb{R}^d \). Then
\[
m : I_\mu \to \mathbb{R}_\geq, \quad m(s) := N \lim_{n \to \infty} \left( \mathbb{E} \| T_{(n)} \cdots T_{(1)} \|^s \right)^{1/n},
\]
defines a convex function with \( m(0) = N > 1 \). Hence there are at most two values in \( I_\mu \), \( \alpha \leq \beta \), say, with \( m(\alpha) = m(\beta) = 1 \). In most of the one-dimensional studies, the existence of fixed points is proven under the assumption that \( \alpha \) exists with \( \alpha \leq 1 \). We will have to assume that the matrices \( (T_i)_{i=1}^{N} \) have no zero row or column \( \mathbb{P} \)-a.s., since otherwise the problem might be not fully \( d \)-dimensional. We call such matrices allowable. Writing \( \lambda_\alpha \) for the Perron-Frobenius eigenvalue of a positive matrix \( \alpha \), we will assume the following aperiodicity condition \((A)\): The support of \( \mu \), \( \text{supp} \mu \), contains at least two matrices \( \alpha, \beta \) with all entries strictly positive and such that \( \log \lambda_\alpha / \log \lambda_\beta \notin \mathbb{Q} \).

Now we are ready to state a simplified version of our main result. We assume \( \text{supp} \mu \) to be bounded in order to get rid of the moment assumptions (they will resemble the typical \( T \log T \) condition). Write \( S_{\geq} := \{ x \in \mathbb{R}_{\geq}^d : |x| = 1 \} \).

**Theorem 1.1.** Let \( \text{supp} \mu \) be a bounded subset of allowable matrices and let \((A)\) be satisfied. Assume that the spectral radius of \( \mathbb{E} T_1 \) is less than \( 1/N \).

Then there is \( \alpha > 0 \) such that \( m(\alpha) = 1 \), \( m'(\alpha) = 0 \) and there exists a nonzero random measurable mapping \( W : S_{\geq} \to \mathbb{R}_\geq \) such that if \( \mathcal{L}(X) \in \mathcal{P}(\mathbb{R}_{\geq}^d) \) is a fixed point of \( S \) with \( Q \equiv 0 \), then there is \( K \geq 0 \) such that for all \( u \in S_{\geq} \),
\[
\langle u, X \rangle \overset{\mathcal{L}}{=} KW(u)Z,
\]
where \( Z \) is a one-dimensional \( \alpha \)-stable r.v. having Laplace transform \( \phi(t) = \exp(-t^\alpha) \) and being independent of \( W \).

If \( 0 < \mathbb{E} Q < \infty \), then there is a random vector \( W^* \) with \( \mathbb{E} |W^*|^\alpha < \infty \), such that if \( \mathcal{L}(X) \in \mathcal{P}(\mathbb{R}_{\geq}^d) \) is a fixed point of \( S \), then there is \( K \geq 0 \) such that
\[
\langle u, X \rangle \overset{\mathcal{L}}{=} \langle u, W^* \rangle + KW(u)Z,
\]
with \( Z, W \) as above.
Observe that $W(u)$ is a random scalar which (as will be shown) depends in a nonlinear way on $u$ in contrast to the linear dependence $\langle u, W^* \rangle$.

**Application: Uniqueness of Fixed Points of the Inhomogeneous Smoothing Transform.** Mirek [43] proved the existence of the particular fixed point $L(W^*)$ of the multivariate inhomogeneous smoothing transform and showed that, if $\beta$ exists, it governs the tail behaviour of $W^*$. In particular, he proved that $W^*$ has all moments up to order $\beta$ and that $L(W^*)$ is the unique fixed point with these properties. It remained an open question—see [43, Remark 1.8], whether $W^*$ is the unique fixed point within the whole set $P(\mathbb{R}^d_{\geq})$. One-dimensional results in [2] suggested that not, and now Theorem [1.1] provides the answer: There are more fixed points, but all of these have an infinite moment of order $\alpha$. Note that existence and uniqueness of the additional fixed points described in Theorem 1.1 cannot be obtained by means of the Banach fixed point theorem.

**Application: Existence of Fixed Points in the Boundary Case.** Buraczewski, Damek and Guivarc'h [20] considered the multivariate homogeneous smoothing transform in the particular case $\alpha = 1$ and the left-hand derivative $m'(1^-) < 0$. In this situation, they proved existence of a fixed point with finite expectation and moreover, that the existence of such a fixed point implies $\alpha = 1$ and $m'(1^-) < 0$. Our result now allows to treat the general case not considered there: We prove existence and uniqueness of fixed points for general $\alpha \in (0, 1)$. Note that, in contrast to the one-dimensional case, the existence of FPs for $\alpha < 1$ cannot be deduced from the result for $\alpha = 1$ by applying the stable transformation. Moreover, our results allow to deduce the existence of a fixed point in the boundary case $\alpha = 1, m'(1) = 0$.

**Application: Generalized Multivariate Stable Laws.** A straightforward null-array argument shows that the multivariate stable laws on $P(\mathbb{R}^d_{\geq})$ are the fixed points of $S$ for the deterministic situation $T_1 = \cdots = T_N = \text{diag}(N^{-1/\alpha}, \ldots, N^{-1/\alpha})$ for $\alpha \in (0, 1)$ and $Q \equiv 0$. In this sense, FPs of (1.1) can be interpreted as a generalization of multivariate stable laws. Observe, that there is a magnitude of multivariate $\alpha$-stable laws—the spectral measure can be chosen completely arbitrary, while our theorem describes assumptions, under which there is a (up to scalar multiplication) unique solution.

Due to this uniqueness, there might be applications in statistics: If a linear combination of the empirical distribution of two iid samples “equals” the empirical distribution of the combination of both samples, one obtains an empirical fixed point equation and the solution of the stochastic fixed point equation may give a good hint for the underlying distribution of the samples.

Pursuing this aim, Eaton [24, Theorem 2] considered the inhomogeneous smoothing transform with $(T_i)$ being deterministic symmetric matrices and described conditions under which the fixed points are multivariate normal distributions. His result is generalized here in that we describe assumptions, under which similar equations have unique fixed points in $P(\mathbb{R}^d_{\geq})$.

**Application: Kinetic Models.** The Kac’ caricature of the Boltzmann equation [34] allows to describe the equilibrium distribution $V$ of particle velocity in a homogeneous Maxwell gas by the SFPE

$$V \overset{\mathcal{L}}{=} \cos \theta V_1 + \sin \theta V_2,$$

with $V, V_1, V_2$ iid and independent of the random angle $\theta$, which is uniformly distributed on $(0, 2\pi]$. Recently, there is a growing and well recognized literature on generalizations of this models, see e.g. [8, 9, 10, 49]. Though the Kac caricature itself is not in the scope
our result, the origin of this model is three-dimensional in nature, and consideration of the associated multidimensional SFPE has just started \cite{7,11}. Our hope is that the present work may promote research in the multivariate situation.

**Explicit Example.** Our result covers as well the case of deterministic \((T_1, T_2) = (a, b)\). It turns out that \(W\) is deterministic in this particular situation and we will show (in Section 18) how to explicitly compute it. To illustrate the result, consider \(d = 2, N = 2\) and

\[
a = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \frac{1}{8} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.
\]

It is easy to check that all assumptions of Theorem 1.1 are satisfied, hence if \(X, X_1, X_2\) are iid satisfying

\[
X \overset{L}{=} aX_1 + bX_2,
\]

then there is \(K \geq 0\) such that for all \(u \in Sp\),

\[
\langle X, u \rangle \overset{L}{=} KW(u)Z
\]

where \(Z\) is an \(\alpha \approx 0.84\)-stable random variable on \(\mathbb{R}_+\).

In order to describe \(W\), we identify \(S_{\geq} \approx [0, \pi/2] \) via \(u(\theta) = (\cos \theta, \sin \theta) \in S_{\geq}\). We obtain here that

\[
W(\theta) = p(\cos(\theta - \pi/4))^\alpha + q(\cos(\theta - \pi/3))^\alpha
\]

with \(p \approx 0.55, q \approx 0.45\). Plots of the functions \(W\) and \(m\) are given in Figure 1.

**References to the Literature.** Fixed points of the mapping \(S\) have been intensively studied in dimension \(d = 1\), the most recent and comprehensive results are due to Alsmeyer, Biggins and Meiners \cite{11}, classical references are Durrett and Liggett or Liu, see \cite{23,39}. Applications date back to Mandelbrot \cite{40}, who studied self-similar measures, Biggins \cite{12}, who studied martingales in the branching random walk and Durrett and Liggett, who were motivated by questions from interacting particle systems \cite{23}. Starting with Rösler \cite{51}, it was used to study limiting laws for the number of comparisons needed by divide-and-conquer algorithms. Recent applications are the analysis of the Google Page Rank algorithm \cite{54} or, as mentioned above, kinetic gas models from statistical physics.
Besides the afore-mentioned articles [11, 20, 43], multivariate smoothing transforms appear in the study of the limiting law of several functionals of divide-and-conquer algorithms, see [45], as well as in the description of functional limit theorems for generalized Pólya urns, see [32]. Several of the references mentioned even consider the smoothing transform acting on $\mathbb{R}^d$. But as in the one-dimensional case, the first step should be to investigate fixed points on $\mathbb{R}^d$—which is the aim of the present paper.

In order to avoid repetition, we will introduce first some notation in Section 2 before stating the main results in Section 3 which can then be formulated in full detail.

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2. Notation

Where necessary, we distinguish between the homogeneous ST, i.e. $Q \equiv 0$, and the inhomogeneous ST by writing $S_0$ resp. $S_Q$.

As stated before, it can w.l.o.g. be assumed that the $(T_i)_{i=1}^N$ all have the same law $\mu$, say, this will be presupposed from now on. The main condition on $\mu$ is as follows. Write $\mathring{A}$ for the topological interior of a set $A$. Then $\mathcal{M}_\geq = M(d \times d, \mathbb{R}_\geq)$ is the set of $d \times d$ positive matrices. If $a \in \mathcal{M}_\geq$, write $\lambda_a$ for its dominant (Perron-Frobenius) eigenvalue, and $v_a \in \mathring{S}_\geq$ for the corresponding eigenvector.

Definition 2.1. A matrix $a \in \mathcal{M}_\geq$ is called allowable, if it has no zero line and no zero column. A subsemigroup $\Gamma$ of $\mathcal{M}_\geq$ is said to satisfy condition $(C)$, if

1. Every $a \in \Gamma$ is allowable and
2. $\Gamma \cap \mathcal{M}_\geq \neq \emptyset$.

In addition, we say that $\Gamma$ is aperiodic, if there are $a, b \in \Gamma \cap \mathcal{M}_\geq$ such that $\frac{\log \lambda_a}{\log \lambda_b} \notin \mathbb{Q}$.

This condition is very common when studying products of nonnegative matrices, see e.g. [17, 29]. Denote by $[\text{supp } \mu]$ the smallest subsemigroup containing the support of $\mu$. We say that $\mu$ satisfies $(C)$ (is aperiodic), if $[\text{supp } \mu]$ satisfies condition $(C)$ (is aperiodic).

For an allowable matrix $a$, write

$$\iota(a) := \inf_{x \in \mathring{S}_\geq} |ax| > 0$$

and define its action on $\mathring{S}_\geq$ by

$$a \cdot x := \frac{ax}{|ax|}, \quad x \in \mathring{S}_\geq.$$  

Write $C(\mathring{S}_\geq)$ for the set of (due to compactness, bounded) continuous functions on $\mathring{S}_\geq$, equipped with the supremum norm

$$|f|_\infty = \sup_{x \in \mathring{S}_\geq} |f(x)|.$$

For a noncompact space $E$, write $C_b(E)$ for the set of bounded continuous functions, which then again is equipped with the supremum norm.
As soon as every matrix in $\operatorname{supp} \mathbf{T}_1$ is allowable, the following operator is well defined for any $s \in I_{\mu}$:

$$(2.2) \quad P^s : \mathcal{C}(\mathbb{S}_\geq) \to \mathcal{C}(\mathbb{S}_\geq), \quad P^s f(x) = \mathbb{E} [\mathbf{T}_1 x|^s f(\mathbf{T}_1 \cdot x)$$

Its adjoint operator $(P^s)'$ is a mapping on the set of bounded measures on $\mathbb{S}_\geq$. Defining $\tilde{P}^s \nu := [(P^s)' \nu(\mathbb{S}_\geq)]^{-1}(P^s)' \nu$, it induces a continuous self-map of $\mathcal{P}(\mathbb{S}_\geq)$. By the Schauder-Tychonoff theorem (see e.g. [22, Theorem V.10.5]), $\tilde{P}^s$ has a fixed point $\nu^s$, say, which is in turn an eigenmeasure of $(P^s)'$. Denote its eigenvalue by $k(s)$. Initiated by Kesten [36], properties of these operators have been studied in detail by Guivarch’h and Le Page [27, 28], see also [20, 43]. It has been shown that $\nu^s$ is unique up to scaling and that $k(s) = N^{-1} m(s)$, e.g.

$$\nu^s(P^s)' = \frac{1}{N} \nu^s.$$ 

More details will be given in Section 4.

Any $\eta \in \mathcal{P}(\mathbb{R}_+^d)$ can be uniquely identified with its Laplace transform (LT)

$$\phi_\eta(z) = \int_{\mathbb{R}_+^d} \exp \{-x, x\} \eta(dx).$$

Writing $\mathcal{S}$ for the functor $\eta \mapsto \phi_\eta$, $\mathcal{S}$ can be considered as a mapping on the set $\mathcal{L}(\mathcal{P}(\mathbb{R}_+^d))$ by the canonical definition

$$\mathcal{S} \phi_\eta = \phi_{\mathcal{S} \eta}.$$ 

If $\mathcal{S} \phi_\eta = \phi_\eta$, then we will call $\phi_\eta$ a FP of $\mathcal{S}$ as well.

In order to study iterations of $\mathcal{S}$, introduce the weighted branching process: Let $\mathfrak{T} = \bigcup_{n=0}^{\infty} \{1, \ldots, N\}^n$ be the $N$-ary tree with Ulam-Harris labelling and root $\emptyset$. For a node $v = v_1 \ldots v_n$, write $|v| = n$ for its generation, $v|k = v_1 \ldots v_k$ for its ancestor in the $k$-th generation, $k \leq n$, and $v_i$ for its $i$-th child. Attaching to each node an independent copy $T(v) = ((\mathbf{T}_i(v))_{i=1}^N, Q(v))$ of $T = ((\mathbf{T}_i)^N_{i=1}, Q)$, $T_i(v)$ can be interpreted as a weight along the path from $v$ to $v_i$. The product of the weights along the unique shortest path from $\emptyset$ to $v$ is defined recursively by $L(\emptyset) = \mathbf{1}$, the identity matrix, and

$$(2.3) \quad L(vi) = L(v)T_i(v).$$

Upon introducing an initial state $U(\emptyset) \in \mathbb{S}_\geq$ of the tree, we use the notation

$$\mathbb{P}_u (U(\emptyset) = u) = 1,$$

and for each $v \in \mathfrak{T}$, we define

$$U(v) := L(v)^T \cdot U(\emptyset), \quad S(v) := -\log |L(v)^T|.$$ 

Denote the filtration corresponding to the generations in the tree by

$$\mathfrak{B}_n := \sigma (U(\emptyset), (T(v))_{|v|<n}), \quad n \in \mathbb{N}.$$ 

With this definition, $L(v), S(v), U(v)$ are measurable w.r.t $\mathfrak{B}_{|v|}$.

Furthermore, introduce the shift operator $[\cdot]_w$: If $F$ is any function of $U(\emptyset)$ and $\mathcal{T} = (T(v))_{v \in \mathfrak{T}}$, then for $w \in \mathfrak{T}$, set

$$(2.4) \quad [F(U(\emptyset), \mathcal{T})]_w := F(U(w), (T(wv))_{v \in \mathfrak{T}}).$$

The family $[\mathcal{T}]_w$ corresponds to the subtree $[\mathfrak{T}]_w$, rooted in $w \in \mathfrak{T}$ and has the same distribution as the unshifted family $\mathcal{T}$ and is independent of $\mathfrak{B}_{|w|}$ as well as of all other subfamilies rooted at the same level. With this definition, it follows in particular that

$$L(vw) = L(v) [L(w)]_w.$$
for any $v, w \in \mathbb{T}$.

When a random variable $Y$ is given, assign to each node a copy $Y(v)$ of $Y$ such that the family $\mathcal{Y} := (Y(v))_{v \in \mathbb{T}}$ is iid and independent of $\mathcal{T}$. Then the sequence

\begin{equation}
Y_n := \sum_{|v| = n} L(v)Y(v) + \sum_{|v| < n} L(w)Q(w),
\end{equation}

$n \in \mathbb{N}_0$, is called the weighted branching process associated with $\mathcal{Y} \otimes \mathcal{T}$. It can easily be shown then that for $\mathcal{L}(Y) = \eta$,

\begin{equation}
\mathcal{L}(Y_n) = S^n \eta.
\end{equation}

This identity is the main ingredient in order to prove (see [42, Lemma 2.4]) that for $\phi \in \mathcal{L}(\mathcal{P}(\mathbb{R}_+^d))$ and all $x \in \mathbb{R}_+^d$,

\begin{equation}
S^n \phi(x) = \mathbb{E} \left[ \exp \left( -\langle x, \sum_{|v| < n} L(w)Q(w) \rangle \right) \prod_{|v| = n} \phi(L(v)^\top x) \right].
\end{equation}

For the homogeneous ST, this identity reduces to

\begin{equation}
S^n_0 \phi(x) = \mathbb{E} \left[ \prod_{|v| = n} \phi(L(v)^\top x) \right].
\end{equation}

Setting for $n \in \mathbb{N}$, $x \in \mathbb{R}_+^d$

\begin{equation}
W_n(x) = \sum_{|v| = n} \int_{\mathbb{S}_\geq} \langle L(v)^\top x, y \rangle^\alpha \nu^\alpha(dy),
\end{equation}

the sequence $(W_n)_{n \in \mathbb{N}}$ defines a nonnegative martingale w.r.t. $(\mathcal{B}_n)_{n \in \mathbb{N}_0}$:

\begin{align*}
\mathbb{E}[W_{n+1}(x)|\mathcal{B}_n] &= \mathbb{E} \left[ \sum_{|v| = n+1} \int_{\mathbb{S}_\geq} \langle L(v)^\top x, y \rangle^\alpha \nu^\alpha(dy) \right | \mathcal{B}_n] \\
&= \sum_{|v| = n} \sum_{i=1}^N \mathbb{E} \left[ \int_{\mathbb{S}_\geq} \langle L(v)^\top x, T_i(v)y \rangle^\alpha \nu^\alpha(dy) \right | \mathcal{B}_n] \\
&= \sum_{|v| = n} \sum_{i=1}^N \int_{\mathbb{S}_\geq} \mathbb{E}\left[ \langle [T_i(\theta)]_v y, L(v)^\top x \rangle^\alpha \nu^\alpha(dy) \right | \mathcal{B}_n] \nu^\alpha(dy) \\
&= \sum_{|v| = n} N \int_{\mathbb{S}_\geq} \langle L(v)^\top x, y \rangle^\alpha (N\nu^\alpha(P^\alpha)^\top)(dy) \\
&= \sum_{|v| = n} \int_{\mathbb{S}_\geq} \langle L(v)^\top x, y \rangle^\alpha (N\nu^\alpha(P^\alpha)^\top)(dy) \\
&= \sum_{|v| = n} \int_{\mathbb{S}_\geq} \langle L(v)^\top x, y \rangle^\alpha \nu^\alpha(dy) = W_n(x).
\end{align*}

Denote the $\mathbb{P}$-a.s. limit of this martingale by $W(u)$. If $\Omega$ is the underlying probability space, then $W(\cdot)$ as the limit of measurable functions is itself measurable on $\mathbb{S}_\geq \times \Omega$, and
we can set \( W := W(U(\emptyset)) \) which then is a martingale w.r.t. to each \( \mathbb{P}_u \). Introducing 
\[ H^\alpha(x) := W_0(x), \]
the martingale gets the maybe more familiar form 
\[ W_n = \sum_{v \in \mathbb{N}} H^\alpha(L(v)^T U(\emptyset)) = \sum_{v \in \mathbb{N}} |L(v)^T u|^\alpha W_0(U(v)) = \sum_{v \in \mathbb{N}} e^{-\alpha S(v)} W_0(U(v)). \]

Mean convergence of this martingale (the intrinsic martingale in multitype branching random walk) is studied in [14] (see also [4, 31, 38, 48]).

It will be shown that 
\( \lim_{n \to \infty} \sup_{|v| = n} |L(v)^T u| = 0 \) \( \mathbb{P} \)-a.s., this allows to introduce level sets \( I_t \subset \mathcal{L} \) by setting 
\[ v \in I_t \iff S(v) > t \text{ and } S(v|k) \leq t \forall k < |v|. \]

Then we can define the intrinsic martingale in continuous time by 
\[ W_t := \sum_{v \in I_t} H^\alpha(L(v)^T U(\emptyset)), \]

as well as restrictions \( W_t^\alpha \), defined in terms of a function \( f \in C_b(S_\geq \times \mathbb{R}_+) \) by 
\[ W_t^\alpha := \sum_{v \in I_t} H^\alpha(L(v)^T U(\emptyset)) f(U(v), S(v) - t) = e^{-\alpha t} \sum_{v \in I_t} e^{-\alpha(S(v) - t)} W_0(U(v)) f(U(v), S(v) - t). \]

Finally, we introduce two special classes of Laplace transforms. Write \( 1 := (1, \ldots, 1)^T \in \mathbb{R}_d \).

**Definition 2.2.** Let \( \alpha \in (0, 1) \) and \( L \) be a positive function which is slowly varying at 0. We call \( \psi \in \mathcal{L}(\mathcal{P}(\mathbb{R}_d^\perp)) \) \( \alpha \)-elementary, if it satisfies 
\[ \lim_{r \to 0} \frac{1 - \psi(r1)}{L(r)r^\alpha} = K \in (0, \infty). \]

A LT \( \psi \in \mathcal{L}(\mathcal{P}(\mathbb{R}_d^\perp)) \) is called \( \alpha \)-regular, if it satisfies 
\[ \underbar{K} \leq \liminf_{r \to 0} \frac{1 - \psi(r1)}{L(r)r^\alpha} \leq \limsup_{r \to 0} \frac{1 - \psi(r1)}{L(r)r^\alpha} \leq \overline{K} \]
for \( \underline{K}, \overline{K} \in (0, \infty). \)

If \( L \equiv 1 \), we say that the LT is \( \alpha \)-elementary (\( \alpha \)-regular).

Write 
\[ \mathcal{F}_{0/Q} := \{ \eta \in \mathcal{P}(\mathbb{R}_d^\perp) : \eta \neq \delta_0, S_{0/Q} \eta = \eta \} \]
for the set of nontrivial fixed points of the homogeneous / inhomogeneous smoothing transform in \( \mathcal{P}(\mathbb{R}_d^\perp) \), and \( \mathcal{F}_{0/Q}^\alpha \) for the subsets of \( \alpha \)-regular fixed points.

At this point, we have enough notation to formulate all main results. Some more notation will be given in Section 4 which studies implications of (C) and introduces a many-to-one / spinal tree identity which is fundamental for the rest of the paper. We refer to the appendix for a list of symbols.
3. Statement of Results

We will first state the main theorem, describing all fixed points of $S$.

**Theorem 3.1.** Let $\mu$ satisfy (C) and be aperiodic, let $\alpha \in \mathbb{I}_\mu \cap (0, 1]$ with $m(\alpha) = 1$ and $m'(\alpha) < 0$. Assume that $E \|T_1\| < \infty$ as well as

\[(T \log T) E \left( \sum_{i=1}^{N} \|T_i\|^{\alpha} \log^{+} \left( \sum_{i=1}^{N} \|T_i\| \right) \right) < \infty \]

and

\[(T \log \iota T) E \left( \|T_1\|^{\alpha} \log \iota (T_1^T) \right) < \infty. \]

Then $\mathcal{L}(\mathcal{F}_0)$ is given by the one-parameter family of LTs,

\[(r, u) \mapsto E \exp(-r^{\alpha}KW(u)), \]

indexed by $K > 0$.

If $\alpha < 1$, then every fixed point is multivariate regularly varying with index $\alpha$, more precisely, if $L(X) \in \mathcal{F}_0$, then one has the weak convergence

\[
\lim_{r \to \infty} \frac{P(|X| > sr, X/|X| \in \cdot)}{P(|X| > r)} = s^{-\alpha} \nu^\alpha(\cdot).
\]

If $\alpha = 1$, if $L(X) \in \mathcal{F}_0$, then for all $u \in S_2$,

\[
E\langle u, X \rangle = KW_0(u).
\]

Conditions $[T \log T]$ and $[T \log \iota T]$ together can be considered as the extension of the classical $T \log T$-condition to the matrix case: what we really want is to bound $|T^T u|$ for any $u \in S_2$, and the bounds are given by

\[\iota(T^T) \leq |T^T u| \leq \|T^T\| = \|T\|.\]

In Section 18, we give an example showing that without the aperiodicity condition being assumed, there might be more than just a one-parameter class of fixed points—just as in the one-dimensional arithmetic case.

The main theorem can be deduced from a couple of major results, which we are going to state in what follows. Step by step, we will give an outline of the paper by describing how which section contributes to the proof of the result at hand. At the end of this section, we will also provide a couple of results concerning the applications.

3.1. **The Intrinsic Martingale.** The mean convergence of $W_n$ is the content of our first result.

**Theorem 3.2.** Let $\mu$ satisfy (C), let $\alpha \in \mathbb{I}_\mu$ with (left hand derivative) $m'(\alpha^-) < 0$. Assume that $[T \log T]$ and $[T \log \iota T]$ hold.

Then $W_n$ converges in mean, i.e. for any $u \in S_2$,

\[
E_u W = W_0(u) = \int_{S_2} \langle u, y \rangle^{\alpha} \nu^\alpha(dy).
\]

This theorem will be proven in Section 5. Since $W_n$ is a positive martingale, it converges $\mathbb{P}_u$-a.s., hence in $L^1(\mathbb{P}_u)$ as well. The restrictions $W^f_i$ are no longer martingales, still we are able to obtain $L^1$-convergence as well as convergence in probability:
Theorem 3.3. In addition to the assumptions of Theorem 3.2 let \( \mu \) be aperiodic. Then for \( f \in C_b(S_\geq \times \mathbb{R}_\geq) \) there is \( \gamma \geq 0 \) such that for all \( u \in S_\geq \)
\[
\lim_{t \to \infty} E_u W^f_t = \gamma E_u W.
\]
(3.4)

If \( f > 0 \), then \( \gamma > 0 \). Moreover, if \( f \) is equicontinuous, then for each \( u \in S_\geq \),
\[
\lim_{t \to \infty} W^f_t = \gamma W \quad \text{in } L^1(\mathbb{P}_u).
\]

In Section 11 we introduce notation and results about \( W_t \) as a martingale in continuous time. In Section 12 we show that mean behaviour of \( W^f_t \) can be studied by using Kesten’s renewal theorem [37, Theorem 1]. Convergence in \( L^1 \) is proven in Section 13 where we refine arguments of Jagers for general branching processes [31] to yield stronger results (i.e. convergence for each \( u \in S_\geq \)) in our situation with proofs being simpler at the same time.

3.2. The Homogeneous Smoothing Transform: Existence of Fixed Points.

Theorem 3.4. Let \( \mu \) satisfy (C), let \( \alpha \in I_\mu \cap (0,1) \) with \( m'(\alpha^-) < 0 \) and let \( (T \log T), (T \log T) \) hold. Then
\[
\psi(ru) = \mathbb{E}\exp(-ru^{\alpha} W(u)), \quad (u,r) \in S_\geq \times \mathbb{R}_\geq
\]
is a nontrivial fixed point of \( S_0 \) and it holds that
\[
\lim_{r \to 0} \frac{1 - \psi(ru)}{ru^{\alpha}} = W_0(u).
\]
(3.5)

If \( \alpha \in (0,1) \), this implies the multivariate regular variation property (3.1).

This proof of this Theorem rests upon Theorem 3.2 and is given in Section 6

Remark 3.5. There is a simple sufficient condition for the existence of \( \alpha \in (0,1) \) with the above properties: It is enough that the nonnegative matrix \( NET_1 \) has the Perron-Frobenius eigenvalue smaller than 1; see [20, Lemma 3.5] resp. Proposition 4.3. Conditions \( (T \log T), (T \log T) \) are e.g. satisfied if the support of \( \mu \) is bounded.

Remark 3.6. If \( \alpha = 1 \), then with \( v^1 := \int_{S_\geq} y v^1(dy) \),
\[
W_n(u) = \langle u, \sum_{|v|=n} L(v)v^1 \rangle = \langle u, Y_n \rangle,
\]
where \( Y_n \) is the weighted branching process associated with \( Y_0 \equiv v^1 \). \( Y_n \) itself is a nonnegative martingale in this situation, hence the fixed point is given by the a.s. limit of \( Y_n \)—this corresponds to the existence result proven in [20].

If \( L(Y) \in F_0 \), then for any \( K > 0 \), \( L(KY) \in F_0 \) as well, this is an immediate consequence of (1.2) with \( Q \equiv 0 \). Choosing \( K = 0 \) yields the trivial FP \( \delta_0, K < 0 \) yields fixed points within \( \mathcal{P}(\mathbb{R}_\geq^d) \). Thus there is at least a one-parameter class of FPs. The next theorem shows that, under the additional assumption of aperiodicity, these are exactly the \( \alpha \)-regular FPs. See Section 18 for an example where the aperiodicity condition is violated.

3.3. The Homogeneous Smoothing Transform: Uniqueness of \( \alpha \)-regular Fixed Points.

We have the following result:

Theorem 3.7. Let \( \mu \) satisfy (C) and be aperiodic, let \( \alpha \in I_\mu \cap (0,1] \) with \( m'(\alpha) = 1 \) and \( m'(\alpha) < 0 \). Assume that \( (T \log T), (T \log T) \) are satisfied and that \( \mathbb{E} \| T_1 \| < \infty \).

Then \( \mathcal{L}(F_0^x) \) is given by the one-parameter family of LTs,
\[
(r,u) \mapsto \mathbb{E}\exp(-Kr^{\alpha} W(u)),
\]
(3.6)
indexed by \( K > 0 \).
The proof of this theorem is given in Section 10. As a by-product, one obtains the following necessary criterion for the existence of $L$-$\alpha$-regular FPs, proven in Section 9.

**Theorem 3.8.** Let $\mu$ satisfy (C) and be aperiodic, let $\mathbb{E} \|T_1\| < \infty$. If there is an $L$-$\alpha$-regular fixed point for some slowly varying function $L$ and some $\alpha \in (0, 1]$, then $m(\alpha) = 1$ and $m'(\alpha) \leq 0$.

The proof of these two theorems, based upon the approach of Durrett and Liggett [23] for the one-dimensional case, become very technical in our multivariate situation and need a lot of preparation. First, we show in Section 7 that an $\alpha$-elementary fixed point with LT $\phi$ is uniquely determined by the function $K(u)$ (if it exists), given by

$$
\lim_{t \to -\infty} D_t(u) := \lim_{t \to -\infty} \frac{1 - \phi(e^{\alpha t}u)}{e^{\alpha t}} = K(u).
$$

Observe, that the definition of $\alpha$-elementary only asserts the existence of $K(1)$. Based upon the Arzelá-Ascoli theorem, it is proven in Section 8 that being $\alpha$-elementary already implies that subsequences $D_{t_n}$ have a limit $K : \mathbb{S}_+ \to \mathbb{R}_+$. An argument involving a Choquet-Deny type result for products of random matrices, given in Section 9, then proves that the function $K$ is unique, in particular independent of the particular subsequence, hence the limit exists. Moreover, $K$ is proportional to $W_0$. Finally, it is shown in Section 10 that under the assumption $m'(\alpha) < 1$, every $\alpha$-regular fixed point is already $\alpha$-elementary.

### 3.4. The Homogeneous Smoothing Transform: All Fixed Points

The following theorem provides the major and final step in proving the main result.

**Theorem 3.9.** Under the assumptions of Theorem 3.7, every nontrivial fixed point of $S_0$ is $\alpha$-regular.

Its proof builds upon a technique developed by Alsmeyer, Biggins, Meiners in [11] and is given in Section 15. The technical prerequisites are in particular Theorem 3.3 and its application to particular restrictions $W_t^f$, provided in Section 14.

### 3.5. Applications

#### 3.5.1. Uniqueness of Fixed Points of the Inhomogeneous Smoothing Transform

Define the (possibly infinite) random variable

$$W^* := \sum_{n=0}^{\infty} \sum_{|v|=n} L(v)Q(v).$$

Then we have the following result:

**Theorem 3.10.** To the assumptions of Theorem 3.7 add

$$\mathbb{E}|Q| < \infty.$$ 

Then the r.v. $W^*$ is $\mathbb{P}$-a.s. finite and $L(W^*)$ constitutes the unique fixed point of $S_Q$ within the class of probability laws with a finite moment of order $s$ for some $s > \alpha$.

The set $\mathcal{L}(F_Q)$ is given by the one-parameter family of LTs,

$$\mathcal{L}(F_Q) = \{(r, u) \mapsto \mathbb{E}\exp(-r(u, W^*) - r^\alpha KW(u)), \quad (r, u) \mapsto \mathbb{E}\exp(-r(u, W^*) - r^\alpha KW(u)), \text{ indexed by } K \geq 0.\}$$
In other words, if $\mathcal{L}(Y_Q)$ is a fixed point of $S_Q$, then for all $u \in S_2$,

$$
\langle u, Y_Q \rangle \overset{\mu}{=} \langle u, W^* \rangle + (KW(u))^\frac{1}{\alpha} Z,
$$

where $K \geq 0$ and $Z$ is an $\alpha$-stable r.v. with LT $e^{-r^\alpha}$ and independent of $W^*, W(u)$. For $K = 0$, the mapping (3.7) gives the LT of the FP constructed in [43, Theorem 1.7]. The proof is given in Section 16 and builds upon the characterization result for the homogeneous equation, Theorem 3.1, the results from Section 7 and a weaker one-to-one correspondence between fixed points of $S_0$ and $S_Q$ due to Rüschendorf [52, Theorem 3.1].

3.5.2. **Existence of Fixed Points in the Boundary Case.** For the critical case $m' (\alpha) = 0$, we obtain the following existence result, which in the case $\alpha = 1$ complements [20, Theorem 2.2] where the existence of a nontrivial FP with a finite expectation has been shown under the assumption $m'(1) < 0$.

**Theorem 3.11.** Let $\mu$ satisfy (C), let $\alpha \in I_{\mu} \cap (0, 1]$ with $m'(\alpha^-) = 0$ and let $(T \log T)$ and

$$(3.8) \quad \mathbb{E}(1 + \|T_1^T\|^\alpha |\log \nu(T_1^T)|) < \infty$$

hold. Then there is a nontrivial fixed point. If $\alpha = 1$, then there is an a.s. finite fixed point with infinite expectation.

3.6. **Structure of the Paper.** A description of the contents of each section has been given above, so here we give just a brief outline.

Section 4 studies the implication of (C) and provides a many-to-one lemma by introducing the associated Markov random walk of the weighted branching process. Sections 5 and 6 contain the proofs of Theorems 3.2 resp. 3.4. Sections 7 to 9 provide the technical prerequisites for the proofs of Theorems 3.8 and 3.7 in Sections 10 resp. 11. Sections 12-13 provide the proof of Theorem 3.3 which is applied in Section 14 to deduce results needed in the proof of Theorem 3.9 which is given in Section 15. The one-to-one correspondence between homogeneous and inhomogeneous FPs is established in Section 16 and the critical case is studied in Section 17. Calculations concerning the aforementioned explicit example and the counterexample for the aperiodicity condition can be found in Section 18. See the appendix for a list of symbols and some frequently used inequalities for multivariate Laplace transforms.

4. **Markov Random Walks and a Many-To-One Identity**

This section introduces and provides detailed information about a Markov random walk $(U_n, S_n)_{n \in \mathbb{N}_0}$ generated by products of random matrices under condition (C). It is closely linked through a many-to-one identity and a change of measure with the family $(U(v), S(v))_{v \in \Xi}$ introduced before. In the following, we will cite for the reader’s convenience some results from the literature. These results are always denoted Propositions.

As identity (2.8) indicates, we often have to deal with transposes of $L(v)$. This is why we introduce a generic sequence $(M_n)_{n \in \mathbb{N}}$ of i.i.d. copies of $T_1$, and write

$$
\Pi_n := M_n \cdots M_1.
$$

Observe that with this definition, $L(v)^T \overset{\mu}{=} \Pi_n$ if $|v| = n$ and

$$
m(s) = N \lim_{n \to \infty} \left( \mathbb{E} \|\Pi_n\|^s \right)^{1/n}.
$$
For any \( u \in \mathbb{S}_> \), \((\Pi_n u)_{n \in \mathbb{N}_0}\) constitutes a Markov chain on \( \mathbb{R}_d^\ge \setminus \{0\} \). It is convenient to study this Markov chain in a polar decomposition,
\[
(U_n, S_n)_{n \in \mathbb{N}_0} := (\Pi_n \cdot u, -\log |\Pi_n u|)_{n \in \mathbb{N}_0},
\]
hence \( \Pi_n u = e^{-S_n} U_n \). Denoting its transition kernel by \( Q \), observe that for \((u, t) \in \mathbb{S}_\ge \times \mathbb{R}, A \subset \mathbb{S}_\ge, B \subset \mathbb{R}\) measurable,
\[
Q(u, t, A, B) = \bar{Q}(u, A, B - t)
\]
for a kernel \( \bar{Q} \) from \( \mathbb{S}_\ge \) to \( \mathbb{S}_\ge \times \mathbb{R} \), hence \((U_n, S_n)_{n \in \mathbb{N}_0}\) carries the additional structure of a Markov random walk (MRW). Note its initial distribution as well by the convention
\[
P_u (U_0 = u, S_0 = 0) = 1.
\]

We will use the following Choquet-Deny type result about bounded harmonic functions for the MRW.

**Proposition 4.1** ([41, Theorem 2.2]). Let \( \mu \) satisfy (C) and be aperiodic. Assume that \( H \in C_0 (\mathbb{S}_\ge \times \mathbb{R}) \) satisfies
\[
(i) \quad H(u, t) = \mathbb{E}_u H(U_1, t - S_1) \text{ for all } (u, t) \in \mathbb{S}_\ge \times \mathbb{R}, \text{ and}
\]
\[
(ii) \quad \text{for all } z \in \mathbb{S}_\ge, \lim_{y \to z} \sup_{t \in \mathbb{R}} |H(y, t) - H(z, t)| = 0.
\]
Then \( H \) is constant.

In order to study the action of \( \Pi_n \) on \( \mathbb{R}_d^\ge \) in more detail, it is useful to consider first its action on \( \mathbb{S}_\ge \); i.e. the behaviour of \((U_n)_{n \in \mathbb{N}_0}\). Besides the operator \( P_s^* \) defined above, this section is concerned with properties of another operator on \( C (\mathbb{S}_\ge) \), namely
\[
P_s f(u) := \mathbb{E}_u \left( |T_1^\top u|^s f(T_1^\top \cdot u) \right) = \mathbb{E}_u (e^{-sS_1} f(U_1)),
\]
which is well defined for all \( s \in I_\mu \) and defines a Markov transition operator for \( s = 0 \). We cite from [20] the main properties that will be used. There, a slightly different setting is considered, but the proofs are still valid. A detailed exposition will be worked out in [21].

**Proposition 4.2** ([20, Theorem 3.3]). Assume that \( \mu \) satisfies (C) and let \( s \in I_\mu \). Then the following holds:
\begin{enumerate}
\item The spectral radius and the dominant eigenvalue of \( P_s^* \) are equal to \( k(s) = N^{-1} m(s) \).
\item There is a strictly positive function \( e^s \) (unique up to scaling) and a unique probability measure \( \nu^s \) on \( \mathbb{S}_\ge \) such that
\[
P_s^* e^s = k(s)e^s, \quad (P_s^*)' \nu^s = k(s)\nu^s.
\]
\item The function \( e^s \) is \( \min\{s, 1\} \)-Hölder.
\item If \( f \in C (\mathbb{S}_\ge) \) with \( P_s^* f = k(s)f \) and \( f \) is strictly positive, then \( f = ce^s \) for some \( c > 0 \).
\item The function \( s \mapsto k(s) \) is convex on \( I_\mu \).
\item In the same way
\[
P_s^* e^*_s = k(s)e^*_s, \quad (P_s^*)' \nu^*_s = k(s)\nu^*_s,
\]
for a unique probability measure \( \nu^*_s \) and a unique strictly positive function \( e^*_s \) which satisfies the identity
\[
e^*_s(u) = \int \langle u, y \rangle^s \nu^s (du)
\]
\end{enumerate}
Proposition 4.3 (20 Lemma 3.5). Assume that $\mu$ satisfies (C) and $\mathbb{E} \|T_1\| < \infty$. Set $a = \mathbb{E}T_1$.

Then for some $n \geq 1$, $a^n \in \mathcal{M}_{\geq}$ and it holds that

$$k(1) = \lambda_a, \quad e^s(u) = \langle v_a, u \rangle, \quad \int_{\mathbb{R}_+} y \nu_y(dy) = v_a. \quad (4.5)$$

Since $k(s)$ is convex, $k(1) = \lambda_a \leq \frac{1}{N}$ is a sufficient condition for the existence of $\alpha \in (0, 1]$. Simulation algorithms for $k(s)$ and $\nu^s$ are studied in [33].

Proposition 4.3 describes as well (unbounded) nonnegative harmonic functions of $(U_n, S_n)_{n \in \mathbb{N}_0}$.

If we consider the function $H^s$ by setting

$$H^s(u, t) := e^s(u)e^{st}, \quad u \in \mathbb{R}_+, t \in \mathbb{R},$$

then it satisfies

$$k(s)H^s(u, t) = \mathbb{E}_uH^s(U_1, t - S_1), \quad (4.6)$$

i.e., if $k(s) = 1$, then $H^s(u, t)$ is a nonnegative, but in general unbounded, harmonic function. This definition of $H^s$ is consistent with the previous one $H^s(x) = W_0(x)$ by letting $u = |x|^{-1}x$, $t = \log |x|$.

It is well known (Doob h-transform), that the existence of a harmonic function allows for a change of measure. These ideas can be used for any $s \in I_\mu$ as well: Setting

$$Q^s f := \frac{1}{k(s)H^s}Q(f \cdot H^s),$$

this again defines a Markov transition kernel. Then, use the Ionescu-Tulcea theorem to define probability measures $Q^s_u$ on $(\mathbb{S}_\geq \times \mathbb{R})^{\mathbb{N}_0}$, such that with respect to $Q^s_u$, $(U_0, S_0) = (u, 0)$ a.s. and $(U_n, S_n)_{n \in \mathbb{N}_0}$ has transition kernel $Q^s$. It is shown in [20 proof of Theorem 3.3], that for any $s \in I_\mu$ and each $u \in \mathbb{S}_\geq$, the Markov chain $(U_n)_{n \in \mathbb{N}_0}$ under $Q^s_u$ has a unique stationary distribution $\pi^s_u$, which is given by

$$\pi^s_u(dx) = e^s(x)\nu^s(dx).$$

This construction captures the “Markovian” essence of $Q^s_u$, but we prefer to understand $Q^s_u$ as obtained by a change of measure applied to the sequence of matrices $(M_n)$, as we will describe now. Let $\Omega' = \mathbb{S}_\geq \times \mathcal{M}_{\geq}$ and let $(U_0, (M_n)_{n \in \mathbb{N}})$ be the coordinatewise identity. Introduce the kernel

$$q^s_n(u, a) := \frac{|au|^s e^s(a \cdot u)}{k(s)^n e^s(u)}$$

and define, for $n \in \mathbb{N}$ and $s \in I_\mu$, the probability measures $^nQ^s_u$ on $^n\Omega' = \mathbb{S}_\geq \times \mathcal{M}_{\geq}^n$ by

$$^nQ^s_u((U_0, M_1, \ldots, M_n) \in B) = \int_B q^n(u, a_1, \ldots, a_n) \mu(da_1) \cdots \mu(da_n). \quad (4.8)$$

Observe that for $s = 0$, $(M_n)_n^s$ is an iid sequence.

The sequence of measures $^nQ^s_u$ is a projective sequence, hence by Kolmogorov’s consistency theorem, it uniquely defines a probability measure $Q^s_u$ on $\Omega'$. The following many-to-one lemma then follows from the definition of $Q^s_u$. Write $v = v_1 \ldots v_n$: 
Lemma 4.4. For all \( s \in I_\mu, u \in \mathbb{S}_\geq, n \in \mathbb{N}, \) all bounded measurable \( f : \mathbb{S}_\geq \times \mathcal{M}_\geq^n \rightarrow \mathbb{R}_\geq, \)

\[
\frac{1}{H^\alpha(u)} E \left[ \sum_{\forall v = n} f \left( u, T_{v_1}^T(\emptyset), T_{v_2}^T(v_1), \ldots, T_{v_n}^T(v|n-1) \right) H^\alpha (L(v)^T u) \right] \n = \frac{N^n}{H^\alpha(u)} E \left[ f \left( u, M_1, M_2, \ldots, M_n \right) H^\alpha (\Pi_n u) \right]
\]

(4.9) \( = E_u f(u, M_1, \ldots, M_n) \)

as well as for all bounded measurable \( f : (\mathbb{S}_\geq \times \mathbb{R})^n \rightarrow \mathbb{R}_\geq, \)

\[
\frac{1}{H^\alpha(u)} E_u \left[ \sum_{\forall v = n} f \left( U(\emptyset), S(\emptyset), \ldots, U(v|n-1), S(v|n-1), U(v), S(v) \right) H^\alpha (L(v)^T u) \right] \n = \frac{N^n}{H^\alpha(u)} E_u \left[ f \left( U_0, S_0, \ldots, U_{n-1}, S_{n-1}, U_n, S_n \right) H^\alpha (\Pi_n u) \right]
\]

(4.10) \( = E_u \left[ f \left( U_0, S_0, \ldots, U_{n-1}, S_{n-1}, U_n, S_n \right) \right]. \)

Next is a strong law of large numbers for \((S_n)_{n \in \mathbb{N}_0}. \) Its remarkable feature is that it holds for all initial values \( u \in \mathbb{S}_\geq \) and that it links the drift of \( S_n \) with the (left-hand) derivative of \( k, \) which exists due to convexity. Write \( E_u^\alpha S_1 = \int_{\mathbb{S}_\geq} E_u^\alpha S_1 \pi_u^\alpha (du). \)

Proposition 4.5 (\cite{14} Theorem 3.7). Assume that \( \mu \) satisfies \((C), s \in I_\mu \) and let

\[
E \left\| T_1^\alpha \right\| \log^+ \left\| T^\top \right\| < \infty, \quad E \left\| T_1^\alpha \right\|^\alpha \log \left( \log \left( T_1 \right) \right) < \infty.
\]

Then for all \( u \in \mathbb{S}_\geq, \)

\[
\lim_{n \rightarrow \infty} \frac{S_n}{n} = E_{\pi_u^\alpha} S_1 \quad \mathbb{Q}_u^\alpha \text{-a.s.}
\]

Moreover, the (left) derivative of \( k \) exists and is continuous on \( I_\mu, \) and

\[
E_{\pi_u^\alpha} S_1 = - \frac{k'(s)}{k(s)}.
\]

In particular, \((S_n)_{n \in \mathbb{N}_0} \) has a positive drift under \( \mathbb{Q}_u^\alpha. \) This proposition will yield a \( T \log T \) condition for the mean convergence of \( W_n \) in the next section.

5. Mean Convergence of \( W_n \)

The following result, taken from \cite{14} is the main tool to prove the mean convergence of \( W_n, \) which is the topic of this section.

We use \cite{14} Theorem 1.1 (i)—this theorem was chosen over \cite{14} Proposition 8.1] to clarify the role of Proposition 4.5] Adopting the notation, the theorem reads as follows:

Proposition 5.1 (\cite{14} Theorem 1.1 (i)). Let \( u \in \mathbb{S}_\geq. \) For \( r > 0, \) let

\[
A(r) = \sum_{n=0}^\infty 1 \left( H^\alpha (U_n, -S_n) > r^{-1} \right).
\]

Suppose that there is a random variable \( Z \) such that

\[
(5.1) \quad \mathbb{P} \left( \frac{\sum_{i=1}^N H^\alpha (T_i^\top x) \left| H^\alpha (x) \right|}{H^\alpha (x)} > s \right) \leq \mathbb{P} \left( Z > s \right) \quad \forall x \in \mathbb{R}_\geq^d \setminus \{0\}, s \geq 0
\]
(stochastic domination) and a function \( L \), slowly varying at infinity, such that

\[
\sup_{r > 0} \frac{A(r)}{L(r)} < \infty \quad \mathbb{Q}_u\text{-a.s.}
\]

If \( \mathbb{E} Z L(Z) < \infty \), then \( \mathbb{E}_u W = W_0(u) \).

**Proof of Theorem 3.2** We show that under the assumptions of Theorem 3.2, Proposition 5.1 applies for any \( u \in \mathbb{S}_\geq \). First, we prove that the slowly varying function \( L(r) = 1 + \log r \) satisfies (5.2). Since \( \sup_{u \in \mathbb{S}_\geq} H_\alpha(u, 0) \leq 1 \),

\[
A(r) \leq \sum_{n=0}^{\infty} 1(e^{-\alpha S_n} > r^{-1}) \leq \sum_{n=0}^{\infty} 1(\alpha S_n < \log r) \leq \tau(\log^+ r),
\]

where

\[
\tau(s) := \sup\{n \in \mathbb{N} : \alpha S_n < s\}.
\]

The assumptions of the strong law of large numbers for \( S_n \), Proposition 4.5, hold due to \((T \log T)\) and \((T \log \log T)\). Now on the one hand, \( \sup_{r \in (0, 1)} A(r)/L(r) \) is readily bounded by \( \tau(0) \), which is finite since \( S_n \) is transient. On the other hand, it is as well a consequence of the strong law of large numbers that

\[
\lim_{s \to \infty} \frac{\tau(s)}{s} = \frac{1}{\alpha \mathbb{E}_u \pi_S^1} \quad \mathbb{P}_u\text{-a.s.}
\]

(see the argument in [19] bottom of p.219) and consequently, \( \sup_{r > 1} A(r)/L(r) \) is bounded \( \mathbb{P}_u\text{-a.s.} \), too.

As the second step, observe that, upon defining

\[
Z := C \sum_{i=1}^{\infty} \|T_i^T\|^\alpha
\]

with \( C := \sup_{u \in \mathbb{S}_\geq} H_\alpha(u, 0)^{-1} < \infty \), (5.1) is satisfied. The finiteness of \( \mathbb{E} Z L(Z) \) is then a direct consequence of assumption \((T \log T)\). \( \square \)

6. **Existence of Fixed Points**

In this section, we are going to prove that \( \phi_n(ru) = \exp(-r^\alpha W_n(u)) \) defines a sequence of Laplace transforms, which converges towards a nontrivial fixed point of \( \mathcal{S} \), thus proving Theorem 3.3.

In order to do so, observe first that

\[
\mathcal{S} : f \mapsto \left[ x \mapsto \mathbb{E} \prod_{i=1}^{N} f(T_i^T x) \right]
\]

defines a continuous mapping on \( \mathcal{C}_b(\mathbb{R}^d) \). Consequently, if \( \phi \) is the LT of a distribution on \( \mathbb{R}^d \), and \( \psi := \lim_{n \to \infty} S^n \phi \) exists and is a LT of a distribution \( \eta \) as well, then \( \eta \) is a FP of \( \mathcal{S} \), since for its LT

\[
\mathcal{S} \psi = \mathcal{S}(\lim_{n \to \infty} S^n \phi) = \lim_{n \to \infty} S^{n+1} \phi = \psi.
\]

The following lemma hints at the suitable definition of \( \phi \).

**Lemma 6.1.** Let \( \nu \) be a bounded measure on \( \mathbb{S}_\geq \) and \( \alpha \in (0, 1] \). Then

\[
\phi(x) := \exp\left( -\int_{\mathbb{S}_\geq} \langle x, y \rangle^\alpha \nu(dy) \right)
\]
is the LT of the multivariate $\alpha$-stable law on $\mathbb{R}^d_\geq$ with spectral measure $\nu$.

**Proof.** An idea of the proof is given in [47,55], see [42, Section 5.2] for a detailed account based on these works. $\square$

Denote the distribution on $\mathbb{R}^d_\geq$ having LT (6.1) by $S_\alpha(\nu)$.

**Proof of Theorem 3.4.** By the lemma above,

$$\phi_0(x) := \exp \left( -\int_{\mathbb{R}^d_\geq} \langle x, y \rangle^\alpha \nu^\alpha(dy) \right), \quad x \in \mathbb{R}^d_\geq$$

is the LT of a probability law on $\mathbb{R}^d_\geq$, namely $S_\alpha(\nu^\alpha)$. Hence $\phi_n := S^n \phi_0$ is a sequence in $\mathcal{L}(\mathcal{P}(\mathbb{R}^d_\geq))$. Moreover, recalling identity (2.8), for $(u, r) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$,

$$\phi_n(ru) = \mathbb{E} \left[ \prod_{|v|=n} \exp \left( -\int_{\mathbb{R}^d_\geq} \langle L(v) ru, y \rangle^\alpha \nu^\alpha(dy) \right) \right] = \mathbb{E} \exp(-r^\alpha W_n(u)).$$

Applying the conditional Jensen inequality and the martingale property of $W_n$

$$\phi_{n+1}(ru) = \mathbb{E} (\mathbb{E} [\exp(-r^\alpha W_{n+1}(u) | \mathcal{B}_n)])$$

$$\geq \mathbb{E} (\exp (-r^\alpha \mathbb{E} [W_{n+1} | \mathcal{B}_n])) = \mathbb{E} \exp (-r^\alpha W_n(u)) = \phi_n(ru),$$

it follows, that $(\phi_n)_{n \in \mathbb{N}_0}$ is a pointwise monotone and bounded sequence of functions, hence there exists the pointwise limit

$$\psi(x) := \lim_{n \to \infty} \phi_n(x), \quad x \in \mathbb{R}^d_\geq.$$

In particular, $\psi(0) = 1$ and hence, using the continuity theorem for multivariate LTs (see e.g. [53, Lemma 4]), $\psi$ is the LT of a probability measure $\eta$ on $\mathbb{R}^d_\geq$, which is then a FP of $S$ by the considerations above.

For all $(u, r) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$, using the bounded convergence theorem and the $\mathbb{P}$-a.s. convergence of $W_n(u)$,

$$\psi(ru) = \lim_{n \to \infty} \mathbb{E} \exp(-r^\alpha W_n(u)) = \mathbb{E} \lim_{n \to \infty} \exp(-r^\alpha W_n(u)) = \mathbb{E} \exp(-r^\alpha W(u)).$$

It remains to prove (3.5). Observe that for fixed $u \in \mathbb{S}_\geq$, the sequence of random variables $r^{-\alpha} [1 - \exp(-r^\alpha W(u))]$ is decreasing in $r \in \mathbb{R}_\geq$: Replacing $s = r^\alpha$ and fixing a realisation of $W(u)$, $s \mapsto [1 - \exp(-sW(u))] / s$ is a LT (see [25, XIII]), hence decreasing. This allows to use the monotone convergence theorem and Theorem 3.2 to infer

$$\lim_{r \to 0} \frac{1 - \psi(ru)}{r^\alpha} = \lim_{r \to 0} \mathbb{E} \left[ \frac{1 - \exp(-r^\alpha W(u))}{r^\alpha} \right]$$

$$= \mathbb{E} \left[ \lim_{r \to 0} \frac{1 - \exp(-r^\alpha W(u))}{r^\alpha} \right] = \mathbb{E} W(u) = W_0(u).$$

Finally, we turn to the multivariate regular variation. Let $X$ have Laplace transform $\psi$. Using the Tauberian theorem for LTs (25, XIII.5, (5.22)), the relation (6.4) implies that

$$\lim_{r \to \infty} r^{\alpha} \mathbb{P}(\langle u, X \rangle > r) = \frac{W_0(u)}{\Gamma(1 - \alpha)}$$
for all \( u \in \mathbb{S}_2 \). For \( \alpha \in (0,1) \), this property implies multivariate regular variation, i.e.
there is a uniquely determined probability measure \( \varrho \) such that
\[
\lim_{r \to \infty} \frac{\mathbb{P}(|X| > sr, X/|X| \in \cdot)}{\mathbb{P}(|X| > r)} = \varrho,
\]
see [5] Theorem 1.1 or [16] Corollary 2]. It remains to identify the measure \( \varrho \). Since it
is uniquely determined by \( W_0 \), it has to be the same for any probability law with LT \( \phi \)
satisfying
\[
\lim_{r \to 0} \frac{1 - \phi_ru}{r^\alpha} = W_0(u).
\]
A good choice is \( \phi = \mathcal{L}(S_\alpha(\nu^\alpha)) \), for it has been shown that for such
multivariate stable random variables, \( \varrho \) equals the spectral measure, i.e. \( \varrho = \nu^\alpha \); see [3] Corollary 3.6.20]. By
the uniqueness discussed above, (3.1) holds.

7. Behavior at 0 Identifies Fixed Points

This section provides the first ingredient for the proofs of Theorems 3.7 and 3.10; hence
we will consider the general case of the inhomogeneous ST, the corresponding results for
the homogeneous case are included by setting \( Q \equiv 0 \).

The idea is the same as in [23, 39]: Show that if \( \psi \) is a FP of \( \mathcal{S} \), and if \( \phi \in \mathcal{L}(\mathcal{P}(\mathbb{R}_{\geq 0})) \)
with
\[
\lim_{r \to 0} \frac{1 - \phi(ru)}{1 - \psi(ru)} = 1
\]
for all \( u \in \mathbb{S}_2 \), then \( \lim_{n \to \infty} S^n\phi = \psi \). Recall that a positive example is given by choosing
\( \psi \) as in Theorem 3.4 and \( \phi = \mathcal{L}(S_\alpha(\nu^\alpha)) \). The multivariate situation poses additional
problems, in particular, we have to assume that the convergence in (7.1) is uniform in \( u \in \mathbb{S}_2 \). That this
uniform convergence actually holds will be proven in Sections 8 and 9.

First is a lemma about the maximal position in multitype branching random walk. Define
\[
R_n := \max_{|v|=n} \| L(v) \|.
\]

**Lemma 7.1.** Let \( \alpha \in \mathcal{I}_\mu \) and \( m'(\alpha) < 0 \). Then \( \lim_{n \to \infty} R_n = 0 \) \( \mathbb{P}\)-a.s.

**Proof.** \( R_n \geq 0 \) for all \( n \in \mathbb{N} \), thus it suffices to show that \( \limsup_{n \to \infty} R_n = 0 \). Writing
\( R_{m,l} = \max_{|v|=ml} \| L(v) \| \), it follows that
\[
\limsup_{n \to \infty} R_n \leq \sum_{k=0}^{l-1} \sum_{|v|=k} \| L(v) \| \limsup_{m \to \infty} [R_{m,l}]_v.
\]

Thus it is enough to consider \( \limsup_{m \to \infty} [R_{m,l}]_v \), some \( l \in \mathbb{N} \) and \( |v| \leq l \). By
the assumption \( \alpha \in \mathcal{I}_\mu \), there is \( s > \alpha \in \mathcal{I}_\mu \), such that \( m(s) < 1 \). Referring to the definition
of \( m(s) \), there is \( l \in \mathbb{N} \) such that
\[
\varrho(s) := \mathbb{E} \sum_{|v|=l} \| L(v) \|^s = N^l \mathbb{E} \| \Pi_l \|^s < 1.
\]

Fix this \( l \). Define \( Z_0 = 1 \) and
\[
Z_m = \sum_{|v|=l} \| L(v) \|^s [Z_{m-1}]_v = \sum_{|v|=ml} \mathbb{I} \| [L(v)[kl]]_{v(l-1)} \|^s
\]
as the sum over the norms of the weights, taken in blocks of \( l \) generations. Hence \( \mathbb{E}Z_1 = \varrho(s) \) and \( ([R_{m,l}]_v)^s \leq [Z_m]_v \) for all \( m \in \mathbb{N} \), \( v \in \mathcal{I} \).
Considering the filtration $\mathcal{F}_m := \mathcal{B}_m$ and using the independence of $\{T\}_v$ and $\mathcal{B}_v$, it can easily be seen that $U_m := \phi(s)^{-m} Z_m$ is a nonnegative $\mathcal{F}_m$ martingale. Thus it converges to a random variable $\bar{U}$ and by Fatou’s lemma, $\mathbb{E} \, \bar{U} \leq \mathbb{E} \, \phi(s) Z_1 = 1$. In particular, $\bar{U}$ is almost sure finite, and this gives the final estimate

$$\limsup_{m \to \infty} (\{R_m, l\}_v)^s \leq \limsup_{m \to \infty} \phi(s)^m [U_m]_v = 0 \quad \mathbb{P} \text{-a.s.}$$

for all $|v| \leq l$. \hfill $\square$

The next lemma is the extension of [39, Lemma 7.3] to the multivariate and inhomogeneous case.

**Lemma 7.2.** Let $\alpha \in \tilde{I}_\mu$ with $m'(\alpha) < 0$. If $\phi, \varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}^d_+))$ and there is $r_0 \in \mathbb{R}_+$ such that for all $(y, s) \in \mathbb{S}_\times [0, r_0]$,

$$\phi(sy) \leq \varphi(sy),$$

then for all $(u, r) \in \mathbb{S}_\times \mathbb{R}_+$

$$\liminf_{n \to \infty} S^n_Q \phi(ru) \leq \liminf_{n \to \infty} S^n_Q \varphi(ru) \quad \text{and} \quad \limsup_{n \to \infty} S^n_Q \phi(ru) \leq \limsup_{n \to \infty} S^n_Q \varphi(ru).$$

**Proof.** Fix $(u, r) \in \mathbb{S}_\times \mathbb{R}_+$. Let $\varepsilon > 0$. By Lemma 7.1 there is $n_0 \in \mathbb{N}$ such that $\mathbb{P} \{r R_n > r_0\} < \varepsilon$ for all $n \geq n_0$. On the set $r R_n \leq r_0$, by assumption $\phi(r L(v)^T u) \leq \varphi(r L(v)^T u)$ for all $v$ with $|v| = n$ and the same holds true when multiplying both sides with $\exp(-r(u, \sum_{|w| < n} L(w) Q(w)))$. Therefore, for all $n \geq n_0$

$$S^n_Q \phi(ru) = \mathbb{E} \exp(-r(u, \sum_{|w| < n} L(w) Q(w))) \prod_{|v| = n} \phi(r L(v)^T u)$$

$$= \mathbb{E} \mathbb{1}_{\{r R_n \leq r_0\}} \exp(-r(u, \sum_{|w| < n} L(w) Q(w))) \prod_{|v| = n} \phi(r L(v)^T u)$$

$$+ \mathbb{E} \mathbb{1}_{\{r R_n > r_0\}} \exp(-r(u, \sum_{|w| < n} L(w) Q(w))) \prod_{|v| = n} \phi(r L(v)^T u)$$

$$\leq \mathbb{E} \mathbb{1}_{\{r R_n \leq r_0\}} \exp(-r(u, \sum_{|w| < n} L(w) Q(w))) \prod_{|v| = n} \varphi(r L(v)^T u) + \mathbb{P} \{r R_n > r_0\} \mathbb{E} \exp(-r(u, \sum_{|w| < n} L(w) Q(w))) \prod_{|v| = n} \varphi(r L(v)^T u) + \varepsilon \leq S^n_Q \phi(ru) + \varepsilon.$$

Now take the infimum resp. supremum, and then let $\varepsilon \to 0$ to infer the assertion. \hfill $\square$

**Lemma 7.3.** Assume that $\alpha \in \tilde{I}_\mu$ and $m'(\alpha) < 0$. Let $\psi, \phi \in \mathcal{L}(\mathcal{P}(\mathbb{R}^d_+))$ with

$$(7.2) \quad \lim_{r \to 0} \left[ \frac{1 - \psi(r \cdot)}{r^\alpha L(r)} - h \right]_\infty = \lim_{r \to 0} \left[ \frac{1 - \phi(r \cdot)}{r^\alpha L(r)} - h \right]_\infty = 0$$

for a positive function $L$, slowly varying at 0 and a continuous function $h : \mathbb{S}_\times \to (0, \infty)$. Assume that $S \psi = \psi$. Then $\lim_{n \to \infty} S^n \phi = \psi$.

**Proof.** Since $L$ is slowly varying at zero, it holds for all $p > 0$ that

$$\lim_{r \to 0} \left| \frac{1}{h} \frac{1 - \psi(pr \cdot)}{h^\alpha L(pr)} \cdot \frac{L(r)}{L(pr)} - 1 \right|_\infty = \lim_{r \to 0} \left| \frac{1}{h} \frac{1 - \psi(pr \cdot)}{h^\alpha L(r)} - p^\alpha \right|_\infty = 0.$$
It follows that

\[\lim_{r \to 0} \left| \frac{1 - \psi(p r)}{1 - \psi(r)} - p^a \right|_\infty = 0.\]  

(7.3)

For \( p > 1 \) arbitrary but fixed, set

\[\psi(r u) := \psi(p ru), \quad \text{and} \quad \overline{\psi}(r u) := \psi(p^{-1} ru).\]

Note that both \( \psi, \overline{\psi} \) are just scaled versions of \( \psi \), consequently, they are FPs themselves. Combining (7.2) with (7.3), it follows that

\[\lim_{r \to 0} \left| \frac{1 - \phi(r)}{1 - \overline{\psi}(r)} - p^a \right|_\infty = 0,\]

(7.4)

where \( p^a > 1 > p^{-a} \). Hence there is \( r_0 > 0 \) such that for all \((y, s) \in \mathbb{S}_\geq \times [0, r_0]\)

\[\psi(sy) \leq \phi(sy) \leq \overline{\psi}(sy).\]

Considering Lemma 7.2,

\[\overline{\psi}(ru) = \liminf_{n \to \infty} S^n \psi(ru) \leq \liminf_{n \to \infty} S^n \phi(ru) \]

\[\leq \limsup_{n \to \infty} S^n \phi(ru) \leq \limsup_{n \to \infty} S^n \overline{\psi}(ru) = \overline{\psi}(ru).\]

Since \( p \) was arbitrary, \( \overline{\psi} \) and \( \psi \) can be brought arbitrarily close to infer first the convergence of \( S^n \phi(ru) \) and next that \( \lim_{n \to \infty} S^n \phi(ru) = \psi(ru) \) for all \((u, r) \in \mathbb{S}_\geq \times \mathbb{R}_\geq.\) \(\square\)

8. Uniform Behaviour at Zero via the Arzelá-Ascoli-Theorem

In the previous section we proved that fixed points can uniquely be identified by the behaviour of

\[(r, u) \mapsto \frac{1 - \psi(r u)}{r^a L(r)}\]

at zero, though it is far from being obvious, whether a limit at zero exists. This section will provide a first positive result in that direction, by showing that for \( L^-\alpha \)-regular LTs, at least subsequential limits in (8.1) exists for \( r \to 0 \). In due course, we will obtain as well that this convergence is uniform on \( \mathbb{S}_\geq \). This will be done by an application of the Arzelá-Ascoli theorem; and the first lemma will provide some results needed for the proof of uniform continuity on \( \mathbb{S}_\geq \).

**Lemma 8.1.** Let \( \phi \) be \( L^-\alpha \)-regular. Then there is \( s_0 > 0 \) and \( K > \overline{K} \) such that for all \( s \in [0, s_0], u, w \in \mathbb{S}_\geq \),

\[\left| \frac{1 - \phi(s ru)}{s^a L(s)} - \frac{1 - \phi(s rw)}{s^a L(s)} \right| \leq K(1 \lor r) |u - w|^\alpha.\]

(8.2)

Moreover, let \( C \subset \mathbb{S}_\geq \) compact. Then with

\[K_C := \left( \min_{y \in C} \min_i y_i \right),\]

it holds that for each \( r \in \mathbb{R}_\geq \) there is \( s_1 = s_1(r) \leq s_0 \) such that for all \( u, w \in C, s \in [0, s_1], \)

\[\left| \frac{1 - \phi(s ru)}{1 - \phi(s u)} - \frac{1 - \phi(s rw)}{1 - \phi(s w)} \right| \leq 4K K_C (1 \lor r^2) |u - w|^\alpha.\]

(8.3)
Due to symmetry, it is enough to consider \( u \wedge w \). Then \( u - u \wedge w, w - u \wedge w \in \mathbb{R}^d_+ \). Let \( X \) be a r.v. with LT \( \phi \). Consider

\[
|1 - \phi(sr u) - 1 - \phi(sr w)| 
\leq E |\exp(-sr \langle u, X \rangle) - \exp(-sr \langle w, X \rangle)| 
\leq E |\exp(-sr \langle u \wedge w, X \rangle) (1 - \exp(-sr \langle u - u \wedge w, X \rangle))| 
+ E |\exp(-sr \langle u \wedge w, X \rangle) (1 - \exp(-sr \langle w - u \wedge w, X \rangle))| 
\leq E |1 - \exp(-sr \langle u - u \wedge w, X \rangle)| + E |1 - \exp(-sr \langle w - u \wedge w, X \rangle)| 
= 1 - \phi(sr[u - u \wedge w]) + 1 - \phi(sr[w - u \wedge w]) 
\]

Due to symmetry, it is enough to consider \( 1 - \phi(sr[u - u \wedge w]) \). Using inequality \((\mathcal{A.3})\) and then \((\mathcal{A.3})\) resp. \((\mathcal{A.4})\), we infer

\[
1 - \phi(sr[u - u \wedge w]) \leq 1 - \phi(sr|u - u \wedge w| 1) 
\leq \begin{cases} 
1 - \phi(s|u - u \wedge w| 1) & r < 1 \\
(r - 1 - \phi(s|u - u \wedge w| 1)) & r \geq 1 
\end{cases} 
\]

Since by assumption,

\[
\limsup_{s \to 0} \frac{1 - \phi(s|u - u \wedge w| 1)}{s^\alpha|u - u \wedge w|^\alpha L(s|u - u \wedge w|)} \leq \overline{K} 
\]

with \( L \) slowly varying at \( 0 \), there is \( s_0 > 0 \) and \( K' > \overline{K} \) such that

\[
\frac{1 - \phi(s|u - u \wedge w| 1)}{s^\alpha L(s)} \leq K'|u - u \wedge w|^\alpha \leq K'(1 \vee r)|u - w|^\alpha 
\]

for all \( s \in [0, s_0] \). This proves the first assertion.

Turning now to the second assertion, write \( F(x) = 1 - \phi(x) \). Then for all \( s \leq s_0 \),

\[
\leq \left| \frac{1 - \phi(sr u)}{1 - \phi(s u)} \right| \leq \left| \frac{F(sr u)}{F(s u)} \right| \left| \frac{F(s u) - F(u)}{F(s u)} \right| + \left| \frac{F(sr u) - F(s u)}{F(s u)} \right| 
\leq (1 \vee r) \left| \frac{F(s u) - F(u)}{s^\alpha L(s)} \right| s^\alpha L(s) + (1 \vee r) \left| \frac{F(sr u) - F(s u)}{s^\alpha L(s)} \right| s^\alpha L(s) 
\leq (1 \vee r)K'|u - u|^\alpha \frac{s^\alpha L(s)}{F(s u)} + (1 \vee r)^2 K'|u - u|^\alpha \frac{s^\alpha L(s)}{F(s u)} 
\]

To estimate further, observe that by \((\mathcal{A.3})\)

\[
\frac{F(s u)}{F(s 1)} \geq \min_i u_i, 
\]

hence

\[
\ldots \leq (1 \vee r)^2 K'|u - u|^\alpha (\min_i u_i)^{-1} \left( \frac{s^\alpha L(s)}{F(s 1)} + \frac{s^\alpha L(s)}{F(s u)} \right) 
\]

The term in the bracket is bounded by \( 2/K \) for \( s \to 0 \), hence there is \( s_1 \), depending on \( r \), such that the expression is bounded by \( 4/K \) for all \( s \leq s_1 \). To make the bound independent of \( u \), replace \((\min_i u_i)^{-1} \) by \( K' \). Finally, choose \( K = \max\{K', K'/K\} \). \( \square \)
Given an $L$-$\alpha$-regular LT $\phi$, define the family of functions $(D_{L,s})_{s \in \mathbb{R}}$ by

$$D_{L,s}(u,t) := \frac{1 - \phi(e^{s+t}u)}{H^\alpha(u,t)e^{\alpha L(e^s)}} = \frac{1 - \phi(e^{s+t}u)}{e^s(u)e^{\alpha(s+t)}L(e^s)}.$$  

In particular, if $\phi$ is the FP constructed in Theorem 3.4 then with $L \equiv 1$

$$\lim_{s \to -\infty} D_{L,s}(u,t) = 1$$

for all $u \in \mathbb{S}_\geq$. As said before, we want to prove that this convergence is uniform in $u$ (for fixed $t$). The idea is to show that $(D_{L,s})$ belongs to a compact subset of $C(\mathbb{S}_\geq \times \mathbb{R})$ w.r.t. to the topology of compact uniform convergence. Such a set is defined below. When reading its definition, the reader should keep the sequence $(D_{L,s})$ in mind as the prototype example.

**Definition and Lemma 8.2.** For $\alpha \in (0,1)$, $K > 0$ let $J^K_\alpha$ be the set of continuous functions

$$g : \mathbb{S}_\geq \times \mathbb{R} \to [0,\infty)$$

satisfying

1. $\sup_{u \in \mathbb{S}_\geq} g(u,0) H^\alpha(u,0) \leq K$,
2. $t \mapsto g(u,t)e^{\alpha t}$ is increasing for all $u \in \mathbb{S}_\geq$,
3. $t \mapsto g(u,t)e^{(1-\alpha)t}$ is decreasing for all $u \in \mathbb{S}_\geq$,
4. $u \mapsto g(u,t)H^\alpha(u,t)$ is $\alpha$-Hölder with constant $(1 \vee e^t)K$ for each $t \in \mathbb{R}$

Then $J^K_\alpha$ is a compact subset of $C(\mathbb{S}_\geq \times \mathbb{R})$ w.r.t. to the topology of uniform convergence on compact sets.

**Proof.** The assertion will follow from the general Arzelà-Ascoli theorem for locally compact metric spaces, see e.g. [35] Theorem 7.18. Properties (1)-(4) together imply the uniform bounds, valid for all $g \in J^K_\alpha$

$$g(u,t) \leq \begin{cases} KH^\alpha(u,0)^{-1}e^{(1-\alpha)t} & t \geq 0, \\ KH^\alpha(u,0)^{-1}e^{-\alpha t} & t \leq 0. \end{cases}$$

Properties (1)-(4) are closed even under pointwise convergence of functions, thus $J^K_\alpha$ is particularly closed under compact uniform convergence. Turning to equicontinuity, fix $(u_0,t_0) \in \mathbb{S}_\geq \times \mathbb{R}$ and $\varepsilon > 0$ and consider first the variation in $t$. Let $\delta > 0$. Then for any $g \in J^K_\alpha$, it follows from property (2) that for all $u \in \mathbb{S}_\geq$ and $t \in [t_0 - \delta, t_0 + \delta]$

$$g(u,t)e^{\alpha(t_0-\delta)} \leq g(u,t)e^{\alpha t} \leq g(u,t_0 + \delta)e^{\alpha(t_0+\delta)},$$

thus $g(u,t) \leq g(u,t_0 + \delta)e^{2\alpha \delta}$. Similarly, from property (3), $g(u,t) \geq g(u,t_0 + \delta)e^{2(\alpha-1)\delta}$ and consequently

$$|g(u,t) - g(u,t_0)| \leq g(u,t_0 + \delta)e^{2\alpha \delta} - g(u,t_0 + \delta)e^{2(\alpha-1)\delta} \leq M \left(e^{2\alpha \delta} - e^{2(\alpha-1)\delta}\right),$$

where the uniform bound $M$ exists due to (8.5). Hence there is $\delta_1 > 0$ such that

$$|g(u,t) - g(u,t_0)| < \frac{\varepsilon}{2}$$

for all $t \in B_{\delta_1}(t_0)$ and all $u \in \mathbb{S}_\geq$. Considering the variation in $u$, it follows again from (8.5) that for $h(u,t) := g(u,t)H^\alpha(u,t)$,

$$L := \sup\{h(u,t) : g \in J^K_\alpha, (u,t) \in \mathbb{S}_\geq \times [t_0 - \delta_1, t_0 + \delta_1]\} < \infty.$$
Using property (4), we infer that for all $u \in \mathcal{S}$,
\[
|g(u, t) - g(u_0, t_0)| \leq \frac{|h(u, t_0) - h(u_0, t_0)|}{H^\alpha(u_0, t_0)} + h(u, t_0) \left( \frac{1}{H^\alpha(u, t_0)} - \frac{1}{H^\alpha(u_0, t_0)} \right) \leq K \left( 1 + \varepsilon \right) \left( |u - u_0| \right) + L \left( \frac{1}{H^\alpha(u, t_0)} - \frac{1}{H^\alpha(u_0, t_0)} \right).
\]
Hence there is $\delta_2 > 0$ such that
\[
(8.7) \quad |g(u, t) - g(u_0, t_0)| \leq \varepsilon/2
\]
for all $u \in B_{\delta_2}(u_0)$. Combining (8.6) and (8.7), it holds that for all $(u, t) \in B_{\delta_2}(u_0) \times B_{\delta_1}(t_0)$,
\[
|g(u, t) - g(u_0, t_0)| \leq |g(u, t) - g(u_0, t_0)| + |g(u_0, t_0) - g(u, t)| \leq \varepsilon.
\]
This proves the equicontinuity, hence Arzelà-Ascoli applies and yields the assertion. □

Now we come back to the sequence $D_{L,s}$ defined above.

**Lemma 8.3.** Let $\phi$ be $L$-$\alpha$-regular. Then there is $s_0 \in \mathbb{R}$ and $K > 0$ such that $(D_{L,s}(u, t))_{s \leq s_0} \subset J^K$.

**Proof.** We have to check properties (1)-(4):

1. $\sup_{u \in \mathcal{S}} D_{L,s}(u, 0)H^\alpha(u, 0) \leq \frac{1 - \phi(e^{s_1}t)}{e^{s_1}H^\alpha(u, 0)} \leq K_s$, with $K$ bounded by $C_2$ asymptotically.

2. Just observe that $D_{L,s}(u, t)e^{\alpha t} = \frac{1 - \phi(e^{s_1}t)}{H^\alpha(u, 0)e^{s_1}L(s)}$ is increasing as a function of $t$.

3. Recall that $(e^{-s})e^{-t}(1 - \phi(e^{s_1}t))$ is a LT, hence decreasing. Consequently, $D_{L,s}(u, t)e^{(\alpha - 1)t} = e^{-t}1 - \phi(e^{s_1}t)$ is decreasing as a function of $t$ as well.

4. This is the content of Lemma 8.1. It gives $e^{s_0} > 0$ and $K > 0$ such that for all $s < s_0$,
\[
|D_{L,s}(u, t)H^\alpha(u, t) - D_{L,s}(w, t)H^\alpha(w, t)| \leq K(1 + e^t)|u - w|\alpha.
\]
 Possibly by making $s_0$ smaller, $K_s \leq K$ for all $s \leq s_0$, i.e. property (1) holds with this $K$ as well. □

**Corollary 8.4.** The convergence in Theorem 3.4 (3.5) is uniform on $\mathcal{S}$.

**Proof.** We can rewrite (3.5) as follows: For all $t \in \mathbb{R}$,
\[
\lim_{s \to -\infty} \frac{1 - \psi(e^{s_1}u)}{e^{\alpha s}} = W_0(u)e^{\alpha t} = H^\alpha(u, t),
\]
i.e., for all $u \in \mathcal{S}$,
\[
(8.8) \quad \lim_{s \to -\infty} D_{L,s}(t, u) = 1.
\]
Equation (3.5) states in particular, that $\phi$ is $\alpha$-regular, hence Lemma 8.3 applies. Since $J^K_s$ is compact by Lemma 8.7, any sequence $(D_{L,s_n})_{n \in \mathbb{N}}$ with $s_n \leq s_0$ and $\lim_{n \to \infty} s_n = -\infty$ has a convergent subsequence, and this convergence is uniform on compact subsets of $\mathcal{S} \times \mathbb{R}$. But due to (8.8), the limit is always the same, hence $\lim_{s \to -\infty} D_{L,s} = 1$ uniformly on compact subsets of $\mathcal{S} \times \mathbb{R}$, in particular on $\mathcal{S} \times \{0\}$, i.e.
\[
\lim_{s \to -\infty} \sup_{u \in \mathcal{S}} \left| \frac{1 - \psi(e^{s_1}u)}{e^{s_1}e^{\alpha s}} - 1 \right| = 0.
\] □
9. Uniqueness of $\alpha$-elementary Fixed Points

This section is the technical cornerstone in proving Theorem 3.7. We are going to prove that for any $L$-$\alpha$-regular FP $\psi$, its associated sequence $D_{L_n}$ has convergent subsequences (as $s \to -\infty$), and the limits can be identified with bounded harmonic functions for $(U_n, S_n)_{n \in \mathbb{N}}$, which are unique up to scaling by the Choquet-Deny type result, Proposition 4.1. The main results of this section are collected in Theorem 9.5, which is essential for the proof of Theorem 3.7 in Section 10. A direct consequence of Theorem 9.5 and Lemma 7.3 will be the uniqueness of $\alpha$-elementary FPs.

We start by introducing a subset $H_{\alpha,c}^K$ of $J_{\alpha}^K$, which will contain the subsequential limits of $D_{L_n}$ for $L$-$\alpha$-regular fixed points.

Definition and Lemma 9.1. Let $\mu$ satisfy (C). For $\alpha, c \in (0, 1]$, define the subset $H_{\alpha,c}^K \subset J_{\alpha}^K$ as follows: A function $g \in J_{\alpha}^K$ is in $H_{\alpha,c}^K$, if it satisfies the following additional properties:

1. $\sup_{u \in S_2} g(u, 0) H^\alpha(u, 0) = c$ and $g(u, 0) H^\alpha(u, 0) \geq \min_i u_i$ for all $u \in S_2$.
2. For all $(u, t) \in S_2 \times \mathbb{R}$,
   \[ g(u, t) = m(\alpha) E^\alpha_u g(U_1, t - S_1). \]
3. Introducing
   \[ L_t : S_2 \times \mathbb{R} \to \mathbb{R}_+, \quad (u, z) \mapsto \frac{g(u, t + z)}{g(u, z)}, \]
   the following holds: For all $t \in \mathbb{R}$, all compact $C \subset S_2$, all $u, w \in C$:
   \[ \sup_{z \in \mathbb{R}} e^{-\alpha t} |L_t(u, z) - L_t(w, z)| \leq 4K_K C (1 + e^{2t}) |u - w|^{\alpha}, \]
   with $K_K := \left( \min \{ y_i : y \in C, i = 1, \ldots, d \} \right)^{-1}$.

Here, validity of (1') and (5) implies that the function $L_t$ is well defined and continuous on $S_2 \times \mathbb{R}$.

If $\mathbb{E} \| T_1 \| < \infty$, then the set $H_{\alpha,c}^K$ is a compact subset of $C(S_2 \times \mathbb{R})$ w.r.t the compact uniform convergence.

Property (6) will provide the uniform continuity needed in the Choquet-Deny-Lemma 4.1.

Proof. The function $L_t$ is well defined and continuous as soon as $g(u, z) > 0$ for all $(u, z) \in S_2 \times \mathbb{R}$. For $u \in S_2$, this is a direct consequence of (1'), combined with the lower bounds
\[ g(u, t) \geq \begin{cases} g(u, 0) e^{-\alpha t} & t \geq 0, \\ g(u, 0) e^{(1-\alpha)t} & t \leq 0. \end{cases} \]

But due to property (2) of condition (C), $Q^\alpha_u \{ U_n \in S_2 \} > 0$ for all $u \in S_2$, hence using property (5), $g(u, t) > 0$ everywhere.

Since $J_{\alpha}^K$ is compact, it suffices to show that the subset $H_{\alpha,c}^K$ is closed. It is readily checked that properties (1'), (6) persist to hold even under pointwise convergence of functions $g_n \to g$. In order to show the closedness of property (5), uniform integrability of the sequence $g_n(U_1, t - S_1)$ w.r.t the measures $Q^\alpha_u$ is needed. This is the content of the subsequent lemma.

Lemma 9.2. Let $\mu$ satisfy (C), $\alpha \in (0, 1]$ and let $\mathbb{E} \| T_1 \| < \infty$. Then for all $(u, t) \in S_2 \times \mathbb{R}$, the family $\{ g(U_1, t - S_1) : g \in J_{\alpha}^K \}$ is uniformly integrable w.r.t $Q^\alpha_u$. 

\[ \square \]
Proof. Recalling the uniform bounds (8.5), valid for all $g \in J^K_{\alpha}$ and the finiteness of
\[ C := \sup_{y \in S_2} H^{\alpha}(y, 0)^{-1} = \frac{1}{\inf_{y \in S_2} c^\alpha_*(y)} \]
due to Proposition 4.2, it is sufficient to show that
\[ e^{(1-\alpha)(t-S_1)} \mathbf{1}_{t \geq S_1} + e^{-\alpha(t-S_1)} \mathbf{1}_{t \leq S_1} \]
is integrable w.r.t. $\mathbb{Q}^{\alpha}_u$. Using the definition of $\mathbb{Q}^{\alpha}_u$,
\[ \mathbb{E}_u e^{(1-\alpha)(t-S_1)} = \frac{1}{e^\alpha_*(u)} \mathbb{E}_u e^\alpha_*(U_1) e^{(1-\alpha)(t-S_1)} e^{\alpha(t-S_1)} \leq C e^\alpha \mathbb{E} \left| T^T_1 u \right| \leq C e^\alpha \mathbb{E} \left| T_1 \right|, \]
\[ \mathbb{E}_u e^{-\alpha(t-S_1)} = \frac{1}{e^\alpha_*(u)} \mathbb{E}_u e^\alpha_*(U_1) e^{-\alpha(t-S_1)} e^{-\alpha(t-S_1)} \leq C, \]
hence it is integrable due to the assumption $\mathbb{E} \left| T_1 \right| < \infty$. \(\square\)

As a compact subset of a locally convex topological space, namely $\mathbb{C}(S_2 \times \mathbb{R})$, the set $H^K_{\alpha,c}$ (if non-void) is contained in the convex hull of its extremal points due to the Krein-Milman theorem [22, Theorem V.8.4]. Using Proposition 4.1 we will compute all possible extremal points.

Lemma 9.3. Let $\mu$ be aperiodic and satisfy (C), and let $\mathbb{E} \left| T_1 \right| < \infty$. The extremal points of $H^K_{\alpha,c}$ are contained in the set
\[ E_{\alpha,c} := \left\{ (u, t) \mapsto \frac{H^\chi(u, t)}{H^\alpha(u, t)} : \chi \in (0, 1], \ m(\chi) = 1 \right\} \]

Proof. Let $g \in H^K_{\alpha,c}$ be extremal.

Step 1: Use property (5) to compute for all $u \in S_2$, $s, t \in \mathbb{R}$
\[ g(u, t + s) = m(\alpha) \mathbb{E}_u g(U_1, t + s - S_1) \]
\[ = m(\alpha) \int g(y, t + s - z) \mathbb{Q}^\alpha_u(U_1 \in dy, S_1 \in dz) \]
\[ = m(\alpha) \int \frac{g(y, t + s - z)}{g(y, s - z)} \frac{g(y, s - z)}{g(u, s)} \mathbb{Q}^\alpha_u(U_1 \in dy, S_1 \in dz) \]
Recall that by Lemma 9.1, $g > 0$, thus the denominators are positive. Using (9.2) with $t = 0$, it follows that
\[ m(\alpha) \int \frac{g(y, s - z)}{g(u, s)} \mathbb{Q}^\alpha_u(U_1 \in dy, S_1 \in dz) = 1. \]
Hence (9.3) is a convex combination of functions
\[ g_{y,z}(u, s, t) = \frac{g(y, t + s - z)}{g(y, s - z)} g(u, s), \]
Consequently, since $g$ is extremal,
\[ \frac{g(u, t + s)}{g(u, s)} = \frac{g(y, t + s - z)}{g(y, s - z)} \]
for all $u \in S_2$, $t, s \in \mathbb{R}$ and all $(y, z) \in \text{supp} \mathbb{Q}^\alpha_u((U_1, S_1) \in \cdot)$. But this support is the same as $\text{supp} \mathbb{Q}_u((U_1, S_1) \in \cdot)$. This yields that $L_t(u, s) := \frac{g(u, t + s)}{g(u, s)}$ satisfies
\[ L_t(u, s) = \mathbb{E}_u L_t(U_1, s - S_1). \]

Step 2: Proposition 4.1 will be applied in order to show that $L_t$ is constant on $S_2 \times \mathbb{R}$, i.e. equation (9.4) holds for all $u, y \in S_2$, $s, t \in \mathbb{R}$. The aperiodicity of $\mu$ enters here. Property (6) yields condition (ii) of the proposition, while (9.5) is its condition (i).
It remains to show that $L_t$ is bounded (for fixed $t$). If $t \leq 0$, by property (2), $g(u, t + s)e^{\alpha(t+s)} \leq g(u, s)e^{\alpha s}$, thus

$$0 < L_t(u, s) = e^{-\alpha t} \frac{g(u, t + s)e^{\alpha(t+s)}}{g(u, s)e^{\alpha s}} \leq e^{-\alpha t}.$$  

For $t \geq 0$, use property (3) for an analogue argument. Referring also to Lemma 9.1, $L_t \in C_b(S_{\geq} \times \mathbb{R})$, hence as a bounded harmonic function, it is constant.

Step 3: Validity of (9.4) for any $u, y \in S_{\geq}$, $t, s, z \in \mathbb{R}$ implies that for some $\tilde{f} : S_{\geq} \to (0, \infty)$, $a \in \mathbb{R}_{\geq}$, $b \in \mathbb{R}$,

$$g(u, t) = \tilde{f}(u)e^{bt}.$$  

Considering properties (2) and (3) it follows that $b \in [\alpha - 1, \alpha]$, i.e. $b = \chi - \alpha$ for some $\chi \in [0, 1]$. Rewriting $af(u) = c^+ \chi^+(u) = f(u)$, it follows that

$$g(u, t) = \frac{f(u)}{\chi(u)}e^{(\chi - \alpha)t} = f(u)e^{\chi t}H^{\alpha}(u, t).$$

It remains to compute the possible values of $f$ and $\chi$. Therefore, use property (5) which gives

$$f(u) = e^{-\chi t}H^{\alpha}(u, t)\mathbb{E}_u,^\alpha\left(\frac{f(U_1)}{H^{\alpha}(U_1, t - S_1)}e^{\chi(t - S_1)}\right) = H^{\alpha}(u, t)\mathbb{E}_u\left(\frac{H^{\alpha}(U_1, t - S_1)}{H^{\alpha}(U_1, t - S_1)}f(U_1)e^{-\chi S_1}\right) = N\mathbb{E}_u f(X_1)e^{-\chi S_1} = N\mathbb{E}\left(f(T_1^+ \cdot u) \mid T_1^+ u \chi\right) = NP^\infty f(u).$$

This means that $f$ is an eigenfunction of $P^\infty$ with eigenvalue $\frac{1}{N}$. Referring to the definition of $H^{K,c}_ \alpha$, $f > 0$. By (4), scalar multiples of $\chi^+$ are the only strictly positive eigenfunctions of $P^\infty$. Thus $f = c\chi^+$ where $c$ is given by property (1). The eigenvalue of $P^\infty$ corresponding to $\chi^+$ is $k(\chi)$. If now $k(\chi) = \frac{1}{N}$, then $m(\chi) = Nk(\chi) = 1$, which shows that all extremal points of $H^{K,c}_ \alpha$ are in $E^{K,c}_ \alpha$. \(\square\)

Together with the following lemma, this result allows to prove that if there is a $L\alpha$-regular FP, then there is $\chi \in (0, 1]$ with $m(\chi) = 1$, $m'(\chi) \leq 0$: Because if the compact set $H^{K,c}_ \alpha$ is non-void, then it has extremal points (see Lemma 8.2], thus $E^{K,c}_ \alpha \neq \emptyset$, which gives the existence of $\chi$ when taking additionally into account that $m(0) = N > 1$ and that $m$ is convex.

**Lemma 9.4.** Let $\psi$ be a $L\alpha$-regular FP of $S$, with associated sequence $D_{L,s}$. Then for any sequence $(s_k)_{k \in \mathbb{N}}$ with $s_k \to -\infty$, there is a subsequence $(s_n)_{n \in \mathbb{N}}$ such that $D_{\infty} = \lim_{n \to \infty} D_{L,s_n}$ exists and is an element of $H^{K,c}_ \alpha$ for some $c > 0$. The convergence is uniform on compact subsets of $S_{\geq} \times \mathbb{R}$.

**Proof.** By Lemma 8.3, $(D_{L,s})_{s \leq s_0} \in J^K_{\alpha}$ for some $K' > 0$ and $s_0 \in \mathbb{R}$. Hence for any sequence $s_k \to -\infty$ there is a convergent subsequence $(s_n)$, and the limit $D_{\infty}$ of $D_{L,s_n}$ is again an element of $J^K_{\alpha}$. By Lemma 8.2, the convergence is uniform on compact sets. Thus the burden of the proof is to show that the additional properties (1'), (5) and (6) hold for the limit $h$.

Step 1, Property (1'): Using (A.8), we infer that

$$1 - \psi(e^\alpha u) \geq 1 - \psi(e^\alpha \min_i u_i 1) \geq \min_i u_i (1 - \psi(e^\alpha 1)).$$
Thus for any $s$,

$$D_{L,s}(u, 0)H^s(u, 0) = \frac{1 - \psi(e^s u)}{e^{as} L(e^s)} \geq \min_i u_i \frac{1 - \psi(e^s 1)}{e^{as} L(e^s)},$$

and this is bounded from below by $C_1$ for $s \to -\infty$, since $\psi$ is $L$-regular. This proves the lower bound for $D_\infty(u, 0)$. Due to property (1), it is also bounded from above, thus $c := \sup_{u \in S_\geq} D_\infty(u, 0)H^\alpha(u, 0)$ exists.

**Step 2, Property (5):** Write

$$G(x) := \mathbb{E} \left[ \prod_{i=1}^{N} \psi(T_i^\top x) + \sum_{i=1}^{N} (1 - \psi(T_i^\top x)) - 1 \right].$$

$G(x) \geq 0$ by a simple translation of the arguments in [23] Lemma 2.4. We use that $S\psi = \psi$ and a linearization to derive the following:

$$D_{L,s}(u, t) = \frac{1 - \psi(e^{s+t}u)}{H^\alpha(u, t)L(e^s)e^{as}} = \frac{1 - \mathbb{E}\prod_{i=1}^{N} \psi(e^{s+t}T_i^\top u)}{H^\alpha(u, t)L(e^s)e^{as}}$$

$$= \frac{\mathbb{E}\sum_{i=1}^{N} (1 - \psi(e^{s+t}T_i^\top u))}{H^\alpha(u, t)L(e^s)e^{as}} - \frac{G(e^{s+t}u)}{H^\alpha(u, t)L(e^s)e^{as}}$$

$$= \frac{m(\alpha)}{k(\alpha)H^\alpha(u, t)} \mathbb{E} u [D_{L,s}(U_1, t - S_1)H^\alpha(U_1, t - S_1)] - \frac{G(e^{s+t}u)}{H^\alpha(u, t)L(e^s)e^{as}}$$

$$= m(\alpha)\mathbb{E} u D_{L,s}(U_1, t - S_1) - \frac{L(e^{s+t})}{L(e^s)e^a(u)} \frac{G(e^{s+t}u)}{L(e^{s+t})e^a(s+t)}$$

Due to Lemma 9.2 the sequence $D[L, s_n](U_1, t - S_1)$ is uniformly integrable w.r.t. $Q^n_s$, hence it remains to show that the second part tends to zero for $s \to -\infty$. The following argument follows closely the ideas of [23] Lemma 2.6. Since $L$ is slowly varying at 0, the quotient $L(e^{s+t})/L(e^s)$ is bounded when $s$ tends to $-\infty$.

Consider $G(ru), r \in \mathbb{R}_>, u \in S_\geq$. Defining the bounded and increasing function

$$f : \mathbb{R}_+ \to \mathbb{R}_+, \quad f(s) = \begin{cases} e^{-s} + s - 1 & s \leq N \\ e^{-N} + N - 1 & s \geq N \end{cases},$$

and using the inequality $s \leq e^{-(1-s)}$ as well as (4.7), we calculate

$$G(ru) \leq \mathbb{E} \left[ \exp (-\sum_{i=1}^{N} (1 - \psi(T_i^\top ru))) + \sum_{i=1}^{N} (1 - \psi(T_i^\top ru)) - 1 \right]$$

$$= \mathbb{E} f \left( \sum_{i=1}^{N} (1 - \psi(T_i^\top ru)) \right) \leq \mathbb{E} f \left( \sum_{i=1}^{N} (\|T_i\| \vee 1)(1 - \psi(r1)) \right)$$

Writing $C(T) = \sum_{i=1}^{N}(\|T_i\| \vee 1)$, use that $\mathbb{E}C(T) \leq N(1 + \mathbb{E} \|T_i\|) < \infty$, the regular variation of $\psi$ and $\lim_{s \to 0} f(s)/s$ to deduce that

$$0 \leq \limsup_{r \to 0} \sup_{u \in S_\geq} \frac{G(ru)}{L(r)^{\alpha}} \leq \limsup_{r \to 0} \mathbb{E} \left[ \frac{f \left( C(T)(1 - \psi(r1)) \right) }{C(T)(1 - \psi(r1))} \right] \frac{1 - \psi(r1)}{L(r)^{\alpha}} = 0$$

Consequently, for the limit $D_\infty(u, t) = m(\alpha)\mathbb{E} u D_\infty(U_1, t - S_1)$. 


Step 3. Property (6): Fix \( t \in \mathbb{R}, C \subset \mathbb{S}_\geq \) compact and compute for \( u, w \in C, z \in \mathbb{R}: \)
\[
e^{-\alpha t} \left| \frac{D_{L,s}(u, t + z)}{D_{L,s}(u, z)} - \frac{D_{L,s}(w, t + z)}{D_{L,s}(w, z)} \right| = \left| \frac{1 - \varphi(e^{s\alpha}u)}{1 - \varphi(e^{s\alpha}w)} - \frac{1 - \psi(e^{s\alpha}u)}{1 - \psi(e^{s\alpha}w)} \right|.
\]
Using Lemma 9.1, there is \( s_1 \in \mathbb{R} \) such that the right hand side is bounded by \( 4K_1\|C\| (1 + e^{\alpha t}) |u - w|^\alpha \) as soon as \( e^{s\alpha} \leq s_1 \). For any fixed \( z \), this condition is satisfied eventually when taking the limit \( s_n \to -\infty \). Hence in the limit,
\[
e^{-\alpha t} \left| \frac{D_{L,s}(u, t + z)}{D_{L,s}(u, z)} - \frac{D_{L,s}(w, t + z)}{D_{L,s}(w, z)} \right| \leq 4K_1\|C\| (1 + e^{\alpha t}) |u - w|^\alpha
\]
for all \( r \in \mathbb{R} \).

Now we turn to the main result of this section; which implies, together with Lemma 7.3, the uniqueness of \( \alpha \)-elementary fixed points.

**Theorem 9.5.** Let \( \mu \) be aperiodic and satisfy (C), let \( \mathbb{E} \|T_1\| < \infty. \)

1. If there is an \( L\alpha \)-regular FP of \( \mathcal{S} \), then \( m(\alpha) = 1, m'(\alpha) \leq 0. \)
2. If \( \varphi \) is an \( L\alpha \)-regular FP of \( \mathcal{S} \), then \( \varphi \) is regularly varying at 0, i.e. for each fixed \( s, u, v \in \mathbb{S}_\geq, \)
\[
\lim_{r \to 0} \left| \frac{1 - \varphi(sru)}{1 - \varphi(rv)} - s^\alpha \frac{H^\alpha(u)}{H^\alpha(v)} \right|_\infty = 0.
\]
3. If \( \varphi \) is an \( L\alpha \)-regular FP of \( \mathcal{S} \), then for any sequences \( \{u_n\} \subset \mathbb{S}_\geq \) and \( r_n \to 0 \)
and all \( s \in \mathbb{R}_+, \)
\[
\lim_{n \to \infty} \left| \frac{1 - \varphi(sru_n)}{1 - \varphi(rv_n)} \right| = s^\alpha.
\]
4. If \( \psi \) is a \( L\alpha \)-elementary FP of \( \mathcal{S} \), then there is \( K > 0 \) such that
\[
\lim_{r \to 0} \left| \frac{1 - \psi(r \cdot \cdot \cdot)}{L(r)^{\alpha \cdot \cdot \cdot}} - Ke^\alpha \right|_\infty = 0.
\]

**Proof.** Step 1: Considering Lemma 9.3, there are at most two values \( \chi_1, \chi_2 \in (0, 1], \)
\( m(\chi_1) = m(\chi_2) = 1, \) such that every function in \( H^K_{\alpha, c} \) can be written as a convex combination
\[
(u, t) \mapsto \frac{c}{e^{\alpha c}(u)} \left( \lambda e^{\chi_1}(u)e^{(1-\chi_1-\alpha)t} + (1 - \lambda)e^{\chi_2}(u)e^{(\chi_2-\alpha)t} \right)
\]
for \( \lambda \in [0, 1]. \) Observe that unless \( \alpha \in \{\chi_1, \chi_2\}, \) none of this convex combinations is a bounded function in \( t \) for fixed \( u. \)

Let \( \varphi \) be an \( L\alpha \)-regular FP. Write \( \mathbf{1} := \sqrt{d-1} \mathbf{1} \in \mathbb{S}_\geq. \) By Lemma 9.4, there is a subsequence \( s_n \to -\infty, \) such that
\[
D_{\infty}(\mathbf{1}, t) = \lim_{n \to \infty} D_{L,s_n}(\mathbf{1}, t) = \lim_{n \to \infty} \frac{1 - \varphi(e^{s_n+t}\mathbf{1})}{H^\alpha(\mathbf{1}, t)L(e^{s_n})e^{\alpha s_n}}.
\]
Considering the definition of \( L\alpha \)-regularity, the function \( D_{\infty}(\mathbf{1}, \cdot) \) is bounded from below and above by \( C_1 \) resp. \( C_2. \) Hence by the above, \( \alpha \in \{\chi_1, \chi_2\}. \)

Supposing that \( \alpha = \chi_2, \) the upper bound
\[
\lim_{r \to 0} \frac{1 - \varphi(r \cdot \cdot \cdot)}{L(r)^{\alpha \cdot \cdot \cdot}} \leq C_2 < \infty
\]
still implies, using the Tauberian theorem for LTs \cite[XIII.5]{Feller}, that if $Y$ is a random variable with LT $\varphi$, then for any $\varepsilon > 0$ there is $C$ such that

$$\Pr \left( Y, 1 > r \right) \leq C r^{-\chi_2 + \varepsilon}.$$ 

Thus there is $\chi_1 < s < \chi_2 \leq 1$ with $m(s) < 1$ and $\mathbb{E}[Y]^s \leq \mathbb{E}(Y, 1)^s < \infty$. But it can be deduced from \cite[Lemma 3.3]{Koeckler} (see \cite[Section 4]{Koeckler} for details), that the unique FP of $S$ with finite $s$-moment for $m(s) < 1$ and $s \leq 1$ is $\delta$. Hence, $\alpha = \chi_1$. This proves $m(s) = 1, m'(s) \leq 0$.

**Step 2:** Moreover, the formula (9.8) for functions in $H^K_{\alpha,s}$, in particular for the limit $D_\infty$, simplifies to

$$D_\infty(u,t) = e \left( \lambda + \left( 1 - \lambda \right) e^{\chi_2} \left( u \right) e^{(\chi_2 - \alpha) t} \right).$$

Reasoning as before, the only possible choice is $\lambda = 1$, since otherwise $D_\infty$ would be unbounded. This proves that any subsequential limit of $D_{L_s, u}$ is a positive constant function, nevertheless, the value of the constant may depend on the subsequence. But this suffices to prove regular variation, since for any subsequence $t_n$ such that $D_{L_{t_n}}$ converges,

$$\lim_{n \to \infty} \frac{1 - \varphi(e^{s+t_n} u)}{1 - \varphi(e^{s} v)} = \lim_{n \to \infty} \frac{D[L, t_n] (s, u)}{D[L, t_n] (0, v)} \frac{H^\alpha (e^{s} u)}{H^\alpha (v)} = \frac{H^\alpha (e^{s} u)}{H^\alpha (v)},$$

i.e. the limit is independent of the particular subsequence. Since every subsequential limit is the same, the asserted limit for $t \to 0$ exists.

The convergence is uniform, since $D_{L_{t_n}} \to D_\infty$ uniform on the compact set $\mathbb{S}_\geq \times [0, s]$ by Lemma 9.4.

**Step 3:** This again follows from the uniform convergence: For all $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that

$$\left| \frac{1 - \varphi(s r_n u_n)}{1 - \varphi(r_n u_n)} - s^\alpha \right| = \left| \frac{1 - \varphi(s r_n u_n)}{1 - \varphi(r_n u_n)} - \frac{s^\alpha H^\alpha (u_n)}{H^\alpha (u_n)} \right| \leq \varepsilon,$$

independent of the particular $u_n$.

**Step 4:** If now $\psi$ is $L_{\alpha}$-elementary, then $t \mapsto h(1, t) \equiv K$, hence $\lambda = 1$, and $h \equiv K$ on $\mathbb{S}_\geq \times \mathbb{R}$. This gives that any subsequence $D_{L_{s_n}}$ with $s_n \to -\infty$ has the same limit $K$, hence the compact uniform convergence $h_s \to K$. In particular,

$$\lim_{s \to -\infty} \frac{1 - \varphi(e^{s+0} u)}{H^\alpha (u) L(e^s) e^{\alpha s}} = K$$

uniformly on the compact set $\mathbb{S}_\geq \times \{0\}$. Replacing $t = e^s$, this gives the assertion. \hfill \Box

**Proof of Theorem 3.3** This is a direct consequence of Theorem 9.5. \hfill \Box

**10. Disintegration and Uniqueness of \(\alpha\)-regular Fixed Points**

In this section, we will finish the characterization of \(\alpha\)-regular fixed points by first showing that, for $m'(s) < 0$, every \(\alpha\)-regular fixed point is already \(\alpha\)-elementary; which we then can identify by using the results from Section 7. The main tool used therefore is the disintegration of fixed points (also known as multiplicative martingale), which we are going to introduce now.

**Definition and Lemma 10.1.** Let $\phi$ be a FP of $S$. For each $u \in \mathbb{S}_\geq$, $r \in \mathbb{R}_\geq$, the sequence

$$M_n(ru) := \prod_{|v|=n} \phi(L(v)^\top ru)$$

is a bounded martingale, and its limit $M$ is called the disintegration of $\phi$. It holds that
Let \( \phi \) be an \( \alpha \)-regular FP with disintegration \( M \).

1. For all \( r \in \mathbb{R}_+ \), \( M(\mu_t) = e^{-r\alpha Z(u)} \), where \( Z(u) := -\log M(u) \).
2. For all \( u \in \mathbb{R}_+ \), \( \mathbb{E}Z(u) < \infty \) (as soon as \( \mathbb{E}W(u) > 0 \)).
3. \( \phi \) is \( \alpha \)-elementary.

The idea of proof is borrowed from Lemma 7.6 & Theorem 10.2:

Proof. 1 Using property 2 from Lemma 10.1 as well as property 3 from Theorem 9.5:

\[
-\log M(\mu_t) = \lim_{n \to \infty} \sum_{|v|=n} 1 - \phi_r(u_v)\left|v\right| = \lim_{n \to \infty} \sum_{|v|=n} (1 - \phi_r(u_v)) = \mathbb{E}W(u) = KW(u) \]

Since \( \phi \) is the LT of a probability measure, \( Z(u) < \infty \) \( \mathbb{P} \)-a.s.

2 The finiteness of \( Z(u) \) can then be used to infer its integrability: We have the monotone convergence \( \lim_{t \to 0} t^{-\alpha}(1 - \exp(-r\alpha Z(u))) = Z(u) \), thus

\[
\lim_{t \to 0} \frac{1 - \phi_r(u_v)}{r^\alpha} = \lim_{t \to 0} \mathbb{E}\left[1 - e^{-r\alpha Z(u)}\right] = \mathbb{E}Z(u),
\]

being finite or not. Supposing that \( \mathbb{E}Z(u) = \infty \), then for all \( K > 0 \),

\[
Z(u) = \sum_{|v|=n} (1 - \phi_r(u_v)) \geq \sum_{|v|=n} K \mathbb{E}Z(u) = \lim_{n \to \infty} K W_n(u) = KW(u) \]

Hence, \( Z(u) = \infty \) on the set \( \{W(u) > 0\} \). But this is a contradiction with \( \mathbb{E}W(u) = e^{\alpha} \) and the fact \( Z(u) < \infty \) \( \mathbb{P} \)-a.s. Hence \( \mathbb{E}Z(u) < \infty \).

3 This follows from (10.3) as soon as \( \mathbb{E}Z(u) > 0 \). But \( Z(u) \) is nonnegative, and due to (10.2), it is non-zero as well, for otherwise \( \phi \equiv 1 \), which cannot be the case since \( \phi \) was assumed to be \( \alpha \)-regular. \( \Box \)
Now we are ready to prove the characterization result for \( \alpha \)-regular fixed points:

**Theorem 10.3.** If \( \varphi \) is an \( \alpha \)-regular FP of \( S \) then there is \( K > 0 \) such that
\[
\varphi(ru) = E \exp(-r^\alpha KW(u)).
\]

**Proof.** By Lemma 10.2 assertion (3), \( \varphi \) is in fact an \( \alpha \)-elementary FP. Then Theorem 9.5 assertion (4) yields that there is \( K > 0 \) such that
\[
\lim_{r \to 0} \left| \frac{1 - \varphi(r \cdot)}{r^\alpha} - KW_0 \right|_\infty = 0.
\]

Considering Theorem 3.4 together with Corollary 8.4, we have for the FP \( \psi(ru) = \exp(-r^\alpha KW(u)) \) that
\[
\lim_{r \to 0} \left| \frac{1 - \psi(r \cdot)}{r^\alpha} - KW_0 \right|_\infty = 0.
\]

But then \( \psi = \varphi \) by Lemma 7.3. \( \square \)

**Proof of Theorem 3.7.** That the mapping is well defined and injective, follows from Theorem 3.4. Theorem 10.3 shows that it is surjective as well. \( \square \)

### 11. Stopping Lines

In this section, we will first introduce the concept of stopping lines (which formalizes the study of the tree with continuous time level sets) together with corresponding filtrations. Subsequently, we will study the many-to-one identity and the martingale \( W_t \) in this new structure.

Let \( \sigma = \sigma((u, (m_k)_{k \in \mathbb{N}})) \) be a stopping time of the form
\[
\sigma((u, (m_k)_{k \in \mathbb{N}})) = \inf \{ n \geq 0 : (u, (m_k)_{k=1}^n) \in A_n \}
\]
for some sets \( A_n \). This gives rise to the homogeneous stopping line \( I_\sigma \) by the definition
\[
I_\sigma := \{ v | \sigma(u, (T(v \upharpoonright k - 1))_{k \in \mathbb{N}}) : v \in \{1, \ldots, N\}^\mathbb{N} \}
\]

If there is \( k \leq |v| \) such that \( w = v \upharpoonright k \), then \( w \) is called an ancestor of \( v \). The pre-\( I_\sigma \) \( \sigma \)-algebra \( \mathcal{B}_I \) associated with the stopping line \( I \) is defined as
\[
\mathcal{B}_I := \sigma(U(\emptyset), \{ T(v) : v \text{ has no ancestor in } I \})
\]

A HSL \( I \) is called *anticipating*, if \( \{ v \in I \} \in \mathcal{B}_I \) for all \( v \in I \). It is called a.s. *dissecting*, if \( \max\{|v| : v \in I\} \) is finite a.s. (see [1, Section 7]).

Observe that for each \( t \in \mathbb{R}_+ \), the HSL
\[
I_t := I_{\sigma_t} = \{ v \in I : S(v) > t, S(v \upharpoonright k) \leq t \forall k < |v| \}
\]
is dissecting by Lemma 7.1 and anticipating as well, since \( S(v) \) only depends on \( U(\emptyset) \) and \( (T(v \upharpoonright k))_{k < |v|} \).

In what follows, we will focus on the particular stopping lines \( I_t \), and write \( \mathcal{B}_t \) for \( \mathcal{B}_{I_t} \). Note, that \( \mathcal{B}_t \) is a filtration with
\[
\mathcal{B}_\infty = \lim_{t \to \infty} \sigma(U(\emptyset), (\mathcal{B}_s, s \leq t)) = \sigma(U(\emptyset), (T(v), v \in I))
\]
see the proof of [1, Lemma 8.7] for details.

The many-to-one identity remains valid under the application of dissecting HSL:
Lemma 11.1. For any a.s. dissecting HSL defined as above, any bounded measurable $f$,
\[
\frac{1}{c_\sigma(u)} \mathbb{E} \left( \sum_{v \in I_\sigma} f(L(v)^\top, L(v)^\top u) H^\alpha(L(v)^\top u) \right) = \mathbb{P}_u^\alpha f(\Pi_\sigma, e^{S_\sigma X_\sigma}).
\]

Proof. Just by summing the many-to-one identity (4.9) over $n \in \mathbb{N}$ when considering the sets $\{\sigma = n\}$.

The following lemma is then a direct consequence of Lemma 11.1, applied with $f \equiv 1$.

Lemma 11.2. For each $u \in S_\geq$, the sequence
\[
W_t := W_{I_t} = \int_{S_\geq} \sum_{v \in I_t} (L(v)^\top U(\emptyset), y) \nu^\alpha(dy) = \sum_{v \in I_t} H^\alpha(L(v)^\top U(\emptyset))
\]
is a $\mathbb{P}_u$-martingale with respect to the filtration $\mathcal{B}_{I_t}$.

The next step is to prove that the limit of $W_t$ is still given by $W$.

Lemma 11.3. For all $t \in \mathbb{R}_\geq$, $u \in S_\geq$
\[
W_t = \mathbb{E}(W|\mathcal{B}_t) \quad \mathbb{P}_u$-a.s.,
\]
moreover, $\mathbb{P}_u$-a.s. and in $L^1(\mathbb{P}_u)$,
\[
\lim_{t \to \infty} W_t = W.
\]

Here and in what follows, $\mathbb{P}$-a.s.-convergence for $t \to \infty$ means that for every sequence $t_n \to \infty$, there is a set of full measure, on which the convergence takes place. This will be enough for our purposes.

Proof. The first part of the proof is valid for any anticipating and $\mathbb{P}_u$-a.s. dissecting HSL $\mathcal{I}$. Following the lines of the proof of [13, Lemma 6.1], write
\[
E_\mathcal{I}(j) = \{v \in \mathcal{I} : |v| = j, v \in \mathcal{I}\}
\]
and
\[
A_\mathcal{I}(j) = \{v \in \mathcal{I} : |v| = j, v \text{ has no ancestor in } \mathcal{I}\}.
\]
Consequently, if $v \in A_\mathcal{I}(j)$, then $\sigma((T(v|k))_{k=0}^{|v|}) \subset \mathcal{B}_\mathcal{I}$ as well as $[T]_v$ and $B_{A_\mathcal{I}(j)}$ are independent for $v \in E_\mathcal{I}(j)$. Then for $m \in \mathbb{N}$, under $\mathbb{P}_u$,
\[
\mathbb{E}[W_m|\mathcal{B}_\mathcal{I}] = \mathbb{E}\left[ \sum_{j=1}^{m+1} \mathbb{E}\left[ \sum_{v \in E_\mathcal{I}(j)} H^\alpha(L(v)^\top u) + \sum_{v \in A_\mathcal{I}(m)} H^\alpha(L(v)^\top u) | \mathcal{B}_\mathcal{I} \right] \right]
\]
\[
= \sum_{j=1}^{m+1} \mathbb{E}\left[ \sum_{v \in E_\mathcal{I}(j)} H^\alpha([L(u)]_v L(v)^\top u) | \mathcal{B}_\mathcal{I} \right] + \sum_{v \in A_\mathcal{I}(m)} H^\alpha(L(v)^\top u)
\]
\[
= \sum_{j=1}^{m+1} \mathbb{E}\left[ H^\alpha(L(v)^\top u) | \mathcal{B}_\mathcal{I} \right] + \sum_{v \in A_\mathcal{I}(m)} H^\alpha(L(v)^\top u) \quad \mathbb{P}_u$-a.s.
\]
Since $\mathcal{I}$ is $\mathbb{P}_u$-a.s. dissecting, in the limit $m \to \infty$,
\[
\mathbb{E}[W|\mathcal{B}_\mathcal{I}] = \sum_{v \in \mathcal{I}} H^\alpha(L(v)^\top u) = W_{\mathcal{I}} \quad \mathbb{P}_u$-a.s.
Lemma 11.4. Let \( \phi \) be a fixed point of \( \mathcal{S} \) with disintegration \( M \). Then for all \( u \in \mathbb{S}_2 \),
\begin{align*}
(1) \quad & \lim_{t \to \infty} \prod_{v \in \mathcal{I}_t} \phi(L(v)^\top u) = M(u) \mathbb{P}\text{-a.s., and} \\
(2) \quad & \lim_{t \to \infty} \sum_{v \in \mathcal{I}_t} (1 - \phi(L(v)^\top u)) = -\log M(u) \mathbb{P}\text{-a.s.}
\end{align*}

Proof. (1) The same proof as for Lemma 11.3 yields that
\[ \lim_{t \to \infty} \prod_{v \in \mathcal{I}_t} \phi(L(v)^\top u) = \mathbb{E}[M(u)|\mathcal{B}_t], \]
and the latter one converges to \( M(u) \mathbb{P}\text{-a.s.} \).
(2) This is the same proof as for \( \{10.1\}_2 \), using now that for \( v \in \mathcal{I}_t \), \( |L(v)^\top u| \leq e^{-t}. \)

12. Kesten’s Renewal Theorem

This section provides us with a first result about the convergence of \( \mathbb{E}_n W_t^f \) by an application of Kesten’s renewal theorem. We start with a variant of the many-to-one lemma:

Under \( \mathbb{Q}_0^\alpha \), the MRW \( (U_n, S_n)_{n \in \mathbb{N}_0} \) is transient with \( \lim_{n \to \infty} S_n = \infty \) a.s. Thus \( \sigma_1 := \inf\{n : S_n > t\} \) is a.s. finite, and one can define a semi-Markov process by
\[ U(t) := U_{\sigma_1}, \quad R(t) := S_{\sigma_1} - t \]
for all \( t \in \mathbb{R}_2 \). With this definition, Lemma 11.1 yields the following very helpful identity:

Corollary 12.1. For all \( t \in \mathbb{R}_2 \), all bounded measurable \( f \)
\begin{equation}
(12.1) \quad \frac{1}{c_\alpha^t(u)} \mathbb{E}_u \left( \sum_{v \in \mathcal{I}_t} f(U(v), S(v) - t) H^\alpha(L(v)^\top u) \right) = \mathbb{E}_u^\alpha f(U(t), R(t)).
\end{equation}

This identity is important because Kesten’s renewal theorem \[37\] Theorem 1] describes the asymptotic behaviour of the right hand side. Therefore, consider the ladder MRW \( (U_\sigma^\geq, S_\sigma^\geq) = (U_{\sigma_\geq}, S_{\sigma_\geq}) \) generated by the sequence of stopping times \( \sigma_n^\geq \) with
\[ \sigma_0^\geq = 0, \quad \sigma_n^\geq := \inf\{n > \sigma_{n-1} : S_n > S_{\sigma_{n-1}}\}. \]

Proposition 12.2. Let \( \mu \) satisfy (C) and be aperiodic. Let \( \alpha \in \mathcal{I}_\mu \) with \( m'(\alpha) < 0 \). Let \( \{T \log T\} \) and \( \{\log \log T\} \) hold. Then there is a finite stationary measure \( \nu^\geq \) for \( (U_n)^\geq \) w.r.t. \( \mathbb{Q}_0^\alpha \), and for all \( f \in \mathcal{C}^\alpha(S_\geq \times \mathbb{R}_2) \),
\[ \lim_{t \to \infty} \mathbb{E}_u^\alpha f(U(t), R(t)) = \int_{S_\geq} \int_{S_\geq \times \mathbb{R}_2} \int_0^\tau f(y, s) ds \mathbb{Q}_0^\alpha(U_1^\geq \in dy, S_1^\geq \in dr) \nu^\geq(du). \]
Proof. We have to check that the Markov Random Walk \((U_n, S_n)\) under \(Q_u^\alpha\) satisfies the conditions I.1 - I.4 on [37] page 359. Then the assertions follow from [37] Theorem 1 resp. [37] Lemma 2 (stationarity of \(\nu^>\)). Note that Kesten used a different norming for the measure \(\nu^>\), which results in an additional factor in his statement of the theorem.

Observe, that our assumptions cover those of [36], where validity of conditions I.1-I.4 is proven for the particular measure \(Q_u^\alpha\), where \(\alpha > 0\) with \(\kappa(\alpha) = 1\). Mutatis mutandis, the proof carries over to \(Q_u^\alpha\), this is why we will only indicate ideas of the (quite long) proof. For details, we refer as well to [20] Proposition 5.5 and [43] Section 4.

1.1 The kernel \(Q^\alpha\) has the unique stationary probability measure \(\pi^\alpha\), and open sets are recurrent by Breiman’s strong law of large numbers for Markov chains [18].

1.2 This is a direct consequence of Proposition 4.5.

1.3 This is a consequence of the aperiodicity condition, the basic idea is to choose the \((\zeta,\nu)\) in I.3 as \((\log \lambda, \log \lambda)\) with \(\lambda \notin Q\); see [20] Proposition 5.5 for details.

1.4 In [36] p. 218-219, it is proven that this condition results from

\[
Q_u^\alpha \left( \{ \inf_{n \in \mathbb{N}} \frac{\|\Pi_n u\|}{\|\Pi_n\|} > 0 \} \right) = 1,
\]

which in turn is mainly a consequence of \(\text{supp } \mu \cap \bar{M} \neq \emptyset\), see [36] p. 225 for details.

\[\square\]

Remark 12.3. The expression for the limit above obviously defines a probability measure \(\varrho\) on \(S_\geq \times \mathbb{R}_\geq\), with \(Q_u^\alpha(U(t), R(t) \in \cdot) \to \varrho\) weakly. It is sometimes referred to as the stationary Markov delay distribution.

Lemma 12.4. It holds that \(\varrho(S_\geq \times \mathbb{R}_\geq) = 1\).

Proof. Observe that \(\text{supp } \mu \cap \bar{M} \neq \emptyset\) implies that \(T := \inf \{ n : \Pi_n \in \bar{M} \}\) is finite a.s. by a geometric trials argument, and moreover, \(\Pi_{T+k} \in \bar{M}_\geq\) for all \(k \geq 0\). Consequently, \((U_{T+k})_{k \in \mathbb{N}_0} \in S_\geq\) independent of the starting point \(U(0)\). Since \(\nu^>\) is stationary for the subsequence \((U_n^>)\) of \((U_n)\), it necessarily satisfies \(\nu^>(S_\geq) = 1\), which then implies the assertion for \(\varrho\).

\[\square\]

We will need the following stronger version of Proposition 12.2, the proof of which can be found in [21].

Proposition 12.5. Under the assumptions of Proposition 12.2 let \(f \in C_b(S_\geq \times \mathbb{R}_\geq)\) be uniformly continuous. Then

\[
\lim_{t \to \infty} \sup_{u \in S_\geq} \left| \mathbb{E}_u^\alpha \{ f(U(t), R(t)) \} - \int_{S_\geq \times \mathbb{R}_\geq} f(y, s) \varrho(dy, ds) \right| = 0.
\]

13. General Branching Processes

This section is concerned with results about the convergence in probability of restricted versions of \(W_t\), namely the behaviour for \(t \to \infty\) of

\[
W_t^f = \sum_{\nu \in \mathcal{X}_f} H^\alpha(L(\nu)^T U(0)) f(U(\nu), S(\nu) - t)
\]

for \(f \in C_b(S_\geq \times \mathbb{R}_\geq)\); a typical choice for \(f\) will be a smoothed version of \(1_{\{S(\nu) > t+c\}}\), for example.
Such random variables are particular cases of so called \( \chi \)-counted populations, appearing in the study of general branching processes, see [30, 31, 46, 48]. Our approach follows [31], but due to our special situation, it is possible to obtain stronger results with still simpler proofs.

A first convergence result for the means is obtained from the renewal theorem. Write

\[
\gamma := \int_{\mathbb{S}_\geq \times \mathbb{R}_+} f(y, s) \varrho(dy, ds).
\]

**Lemma 13.1.** Let \( \mu \) satisfy (C) and be aperiodic. Let \( \alpha \in \bar{I}_\mu \) with \( m'(\alpha) < 0 \) and let (1 \( \log t \)) and (1 \( \log^\ell t \)) hold. Then for \( f \in C_b (\mathbb{S}_\geq \times \mathbb{R}_+) \) and all \( u \in \mathbb{S}_\geq \),

\[
\lim_{t \to \infty} \mathbb{E}_u W_t^f = \gamma \mathbb{E}_u W.
\]

The convergence is uniform in \( u \in \mathbb{S}_\geq \) as soon as \( f \) is uniformly continuous.

**Proof.** By Corollary [12.1]

\[
\mathbb{E}_u W_t^f = e^\alpha(u) \mathbb{E}_u^\alpha f(U(t), R(t)) = \mathbb{E}_u W \cdot \mathbb{E}_u^\alpha f(U(t), R(t)),
\]

and the second expectation converges to \( \int f \, d\varrho \) by Proposition [12.2]. The uniform convergence follows from Proposition [12.5]. \( \square \)

The main result of this section is now that the same convergence, namely \( W_t^f \to \gamma W \) is also valid in probability, not only in mean. We will even prove that it holds in \( L^1 \):

**Theorem 13.2.** In addition to the assumptions of Lemma [13.1] let \( f \) be uniformly continuous. Then for all \( u \in \mathbb{S}_\geq \),

\[
\lim_{t \to \infty} W_t^f = \gamma W
\]

in \( \mathbb{P}_u \)-probability and in \( L^1 (\mathbb{P}_u) \).

The proof will rely on several lemmata. The strategy is the same as in [31]: After two technical lemmata, we first prove the weak \( L^1 (\mathbb{P}_u, \mathcal{B}_\infty) \)-convergence \( W_t^f \to \gamma W \), then we show that in the decomposition

\[
W_t^f - \gamma W = (W_t^f - \mathbb{E}[W_t^f | \mathcal{B}_{t/2}]) + (\mathbb{E}[W_t^f | \mathcal{B}_{t/2}] - \gamma W_{t/2}) + (\gamma W_{t/2} - W),
\]

each bracket converges to 0 in \( \mathbb{P}_u \)-probability; finally, weak \( L^1 \)-convergence and convergence in probability together imply the (strong) \( L^1 \)-convergence.

Here, a sequence of random variables \( X_n \) is said to converge weakly in \( L^1 (\mathbb{P}, \mathcal{B}) \) to a random variable \( X \), if for all \( A \in \mathcal{B} \),

\[
\lim_{n \to \infty} \mathbb{E} 1_A X_n = \mathbb{E} 1_A X.
\]

**Two Lemmata.** Here and in the following subsections, the assumptions of Theorem [13.2] will be in force throughout.

**Lemma 13.3.** We have the uniform integrability

\[
\lim_{q \to \infty} \sup_{u \in \mathbb{S}_\geq} \sup_{i \in \mathbb{R}_+} \mathbb{E}_u \left( W_t^f 1_{\{W_t^f > q_i\}} \right) = \lim_{q \to \infty} \sup_{u \in \mathbb{S}_\geq} \sup_{i \in \mathbb{R}_+} \mathbb{E}_u \left( W_t 1_{\{W_t > q_i\}} \right) = 0.
\]

**Proof.** Since \( 0 \leq W_t^f \leq |f|_\infty W_t \) and \( f \) is assumed to be bounded, it satisfies to prove the second equality.
Let \( u_1, \ldots, u_d \in S_\geq \) be the standard basis of \( \mathbb{R}^d \), then for all \( u \in S_\geq, n \in \mathbb{N} \) it follows right from the definition \(^2.9\) of \( W_n \) that

\[
W_n(u) \leq \sum_{i=1}^{d} W_n(u_i),
\]

and consequently, due to the \( \mathbb{P}\) a.s.-convergence, \( W(u) \leq \sum_{i=1}^{d} W(u_i) \). Now for fixed \( g \), the function \( g(t) := t 1_{(q, \infty)}(t) \) is convex. Using the estimate above, the conditional Jensen inequality and Lemma \(^11.3\) we compute that for all \( u \in S_\geq, t \in \mathbb{R}_> \)

\[
\mathbb{E}_u g(W_t) = \mathbb{E}_u g(\mathbb{E}[W|\mathcal{B}_t]) \leq \mathbb{E}_u \mathbb{E}[g(W)|\mathcal{B}_t]
\]

\[
= \mathbb{E}_u g(W) = \mathbb{E} g(W(u)) \leq \mathbb{E} g \left( \sum_{i=1}^{d} W(u_i) \right).
\]

Since \( \sum_{i=1}^{d} W(u_i) \) is \( \mathbb{P}\)-integrable, the last expression tends to zero for \( q \to \infty \). \( \square \)

Set

\[
F(u, t) := \mathbb{E}_u W_t^f = e_\alpha^n(u) \mathbb{E}_u f\left(U(t), R(t)\right),
\]

as well as

\[
\eta_t(u) := W_t^f(u) - \mathbb{E}_u W_t^f = W_t^f(u) - F(u, t); \quad \eta_t := \eta_t(U(\emptyset)).
\]

Then the assertion of uniform convergence in Lemma \(^13.1\) together with Lemma \(^13.3\) above imply the following corollary:

**Corollary 13.4.** We have the uniform integrability

\[
\lim_{q \to \infty} \sup_{u \in S_\geq} \sup_{t \in \mathbb{R}_>} \mathbb{E}_u \left( \eta_t \mathbf{1}_{\{|\eta_t|>q\}} \right) = 0.
\]

**Lemma 13.5.** If \( r < t \), then for all \( u \in S_\geq \),

\[
\mathbb{E}[W_t^f|\mathcal{B}_r] = \sum_{v \in \mathcal{I}_r} e^{-\alpha S(v)} F(U(v), t - S(v)) \quad \mathbb{P}_u\text{-a.s.}
\]

**Proof.** If \( r < t \), then \( \mathcal{I}_r \prec \mathcal{I}_t \) in the sense that for every \( x \in \mathcal{I}_r \) there is \( v \in \mathcal{I}_r \) with \( v \prec x \), i.e. there is \( w \in \mathcal{I} \) s.t. \( x = vw \). Hence, we have the general decomposition

\[
W_t^f = \sum_{v \in \mathcal{I}_r} \sum_{w \in \mathcal{I}} H^\alpha(L(vw)^T \cdot U(\emptyset)) f(U(vw), S(vw) - t) 1_{\{S(vw) > t, S(vw) \leq t \forall k \leq |vw|\}}
\]

\[
= \sum_{v \in \mathcal{I}_r} \sum_{w \in \mathcal{T}} e^{-\alpha(S(v) + |S(w)|_v)} e_\alpha^n \left([L(w)]_v \cdot U(v)\right) \times
f \left([L(w)]_v \cdot U(v), S(v) + |S(w)|_v - t\right) \times
1_{\{|S(w)|_v > t - S(v), |S(w)|_v \leq t - S(v) \forall k \leq |w|\}}
\]

\[
= \sum_{v \in \mathcal{I}_r} \sum_{w \in \mathcal{I}_{t-S(v)}} \left[e^{-\alpha S(w)} e_\alpha^n(U(w) f\left(U(w), S(w) - (t - S(v))\right))\right]_v
\]

\[
= \sum_{v \in \mathcal{I}_r} e^{-\alpha S(v)} \left[W_{t-S(v)}^f\right]_v
\]
Note that in this notation, the shift $[\cdot]_v$ does not affect $t - S(v)$; and that $W^f_t := f(U(0), -s)$ for $s < 0$. Recalling the definition of $\mathcal{B}_r$, we obtain for $r < t$,

$$
\mathbb{E}[W^f_t|\mathcal{B}_r] = \sum_{v \in \mathcal{I}_r} e^{-\alpha S(v)} F(U(v), t - S(v)).
$$

\[\square\]

**Weak $L^1$-convergence.**

**Lemma 13.6.** $W^f_t \to \gamma W$ weakly in $L^1(\mathbb{P}_u, \mathcal{B}_\infty)$ for all $u \in \mathcal{S}_\geq$.

**Proof.** Consider $\mathbb{E}[W^f_t|\mathcal{B}_r]$ for any fixed $r$. By (13.2),

$$
\mathbb{E}[W^f_t|\mathcal{B}_r] = \sum_{v \in \mathcal{I}_r} e^{-\alpha S(v)} F(U(v), t - S(v)),
$$

with $\#\mathcal{I}_r < \infty$ a.s., since $\mathcal{I}_r$ is dissecting and each generation has finite size. Thus we can take the limit $t \to \infty$ inside the sum to obtain, recalling Lemma [13.1],

$$
\lim_{t \to \infty} \mathbb{E}[W^f_t|\mathcal{B}_r] = \gamma \sum_{v \in \mathcal{I}_r} e^{-\alpha S(v)} e^\alpha(U(v)) = \gamma W = \gamma \mathbb{E}[W|\mathcal{B}_r] \text{ } \mathbb{P}_u\text{-a.s.}
$$

The last identity holds due to Lemma [11.3]. By Lemma [13.3], the family $(W^f_t)_t$ is uniformly integrable. Then for all $A \in \mathcal{B}_r$, $(\mathbb{E}[W^f_t|\mathcal{B}_r]1_A)_t$ is uniformly integrable as well, and we obtain

$$
\lim_{t \to \infty} \mathbb{E}_u 1_A W^f_t = \mathbb{E}_u 1_A \lim_{t \to \infty} \mathbb{E}[W^f_t|\mathcal{B}_r] = \mathbb{E}_u 1_A \mathbb{E}[W|\mathcal{B}_r] = \mathbb{E}_u 1_A W.
$$

But for any set $A \in \sigma(\mathcal{T})$, there is $r > 0$ with $A \in \mathcal{B}_r$, thus this proves the weak $L^1$-convergence $W^f_t \to W$. \[\square\]

**Convergence in Probability.**

**Lemma 13.7.** For all $u \in \mathcal{S}_\geq$, $W^f_t - \mathbb{E}[W^f_{t/2}|\mathcal{B}_t]$ $\to 0$ in $\mathbb{P}_u$-probability.

**Proof.** Write $\eta_u(u) := W^f_t(u) - \mathbb{E}_u W^f_t$. Referring to Lemma [13.5],

$$
W^f_t - \mathbb{E}[W^f_{t/2}|\mathcal{B}_t] = \sum_{v \in \mathcal{I}_{t/2}} e^{-\alpha S(v)} [\eta_u - S(v)]_v \text{ } \mathbb{P}_u\text{-a.s.}
$$

For $\mathbb{P}_u$-a.e. $\omega \in \Omega$, the right hand side is a sum of independent centered random variables with respect to the conditional law $\mathbb{P}_u(\cdot|\mathcal{B}_{t/2})$, the data $S(v), U(v)$ can considered to be constant while $[\eta_u - S(v)]_v$ is centered by its definition and its randomness stems only from $[\mathcal{T}]_w$, which is independent of $\mathcal{B}_{t/2}$ as well as of any other $[\mathcal{T}]_w$ with $v \neq w \in \mathcal{I}_{t/2}$. Considering a sequence $t_n \to \infty$, we obtain (for fixed $\omega$) a triangular array with the law in every row given by $\mathbb{P}_u(\cdot|\mathcal{B}_{t_n/2})$. Using [25] Theorem XVII.7, we will show that it convergence in distribution to $\delta_0$, i.e. to the infinitely divisible law with vanishing canonical measure $M$. We proceed as in [31] p.209]

Write $G(u, t, q) := e^\alpha(u) - \mathbb{E}_u(\eta_u 1_{|\eta_u| > q})$. By Corollary [13.4],

$$
\lim_{q \to \infty} \sup_{u \in \mathcal{S}_\geq} \sup_{t \in \mathbb{R}_>} G(u, t, q) = 0.
$$

We infer first that for any $r > 0$,

$$
\sum_{v \in \mathcal{I}_{t_n/2}} \mathbb{P}_u \left( |e^{-\alpha S(v)} [\eta_u - S(v)]_v | > r \right| \mathcal{B}_{t_n/2})
$$
Abbreviating

So for all\( (13.3) \)

all

for some fixed

limiting distribution (if such exists) vanishes outside zero. This proves that the triangular array is infinitesimal (is a null array). Considering

Lemma 13.8.

For all

well in probability. This holding for any sequence

We are going to prove that for all

Proof. Thesecondtermgoestozeroby\((13.3)\); thefirsttermconvergesalmostsurelyto\(\epsilon\)
henceitcanbemadearbitrarilysmallaswell. Thuswehaveproven\((13.4)\). HenceTheorem

\[\sum_{v \in I_{t_n/2}} \mathbb{E}_u \left( \left| e^{-\alpha S(v)} [\eta_{n-S(v)}]_v \right|^2 \mathbf{1}_{\left\{ |e^{-\alpha S(v)} [\eta_{n-S(v)}]_v| \leq r \right\}} \right) \mathcal{B}_{t_n/2} \] (\(\omega\)) \rightarrow 0.

Abbreviating \(N(v) := e^{-\alpha S(v)} [\eta_{n-S(v)}]_v\), we get for any \(0 < \epsilon < r\) that

\[\sum_{v \in I_{t_n/2}} \mathbb{E}_u \left( |N(v)|^2 \mathbf{1}_{\{|N(v)| \leq r\}} \right) \mathcal{B}_{t_n/2} \]

\[\leq \sum_{v \in I_{t_n/2}} \varepsilon \mathbb{E}_u \left( |N(v)| \mathbf{1}_{\{|N(v)| \leq \varepsilon\}} \right) \mathcal{B}_{t_n/2} + \sum_{v \in I_{t_n/2}} r^2 \mathbb{E}_u \left( \mathbf{1}_{\{\varepsilon < |N(v)| \leq r\}} \right) \mathcal{B}_{t_n/2} \]

\[\leq \varepsilon \{sup G(y,s,0) \} \sum_{v \in I_{t_n/2}} e^{-\alpha S(v)} e_{s}^{\alpha} (U(v)) + r^2 \sum_{v \in I_{t_n/2}} \mathbb{P}_u \left( |N(v)| > \varepsilon \right) \mathcal{B}_{t_n/2} \]

The second term goes to zero by\((13.3)\); the first term converges almost surely to \(\varepsilon \{sup G(y,s,0) \} W(\omega)\), hence it can be made arbitrarily small as well. Thus we have proven\((13.4)\). Hence Theorem

\[\text{THEOREM XVII.7} \] yields that

\[\lim_{n \rightarrow \infty} \mathbb{P}_u \left( W_{t_n}^f - \mathbb{E}[W_{t_n}^f | \mathcal{B}_{t_n/2}] \in \cdot \big| \mathcal{B}_{t_n/2} \right) (\omega) = \delta_0 \]

for any \(\omega \in \Omega\), any sequence \(t_n \rightarrow \infty\). Consequently, for all \(r \in \mathbb{R}, u \in \mathbb{S}_2\)

\[\lim_{n \rightarrow \infty} \mathbb{P}_u \left( W_{t_n}^f - \mathbb{E}[W_{t_n}^f | \mathcal{B}_{t_n/2}] \leq r \right) = \mathbb{E}_u \left( \lim_{n \rightarrow \infty} \mathbb{P}_u \left( W_{t_n}^f - \mathbb{E}[W_{t_n}^f | \mathcal{B}_{t_n/2}] \leq r \right) \right) = \mathbb{1}_{\mathbb{R}_2}(u), \]

i.e. \(W_{t_n}^f - \mathbb{E}[W_{t_n}^f | \mathcal{B}_{t_n/2}] \rightarrow 0\) in distribution, and thus, since the limit is constant, as well in probability. This holding for any sequence \(t_n \rightarrow \infty\), we infer the assertion. \(\square\)

Lemma 13.8. For all \(u \in \mathbb{S}_2, \mathbb{E}[W_{t_n}^f | \mathcal{B}_{t_n/2}] - \gamma W_{t_n/2} \rightarrow 0\) in \(\mathbb{P}_u\)-probability.

Proof. We are going to prove that for all \(\varepsilon > 0\), there is \(c_0\) such that

\[\mathbb{E}_u \left| \mathbb{E}_u \left[ W_{t_n}^f | \mathcal{B}_{t_n/2} \right] - \gamma W_{t_n/2} \right| < \varepsilon \]
for all $t > c \geq c_0$. This $L^1$-convergence then implies the asserted convergence in probability.

Using (13.2) and Corollary 12.1, we obtain
\[
E_u \left| E \left[ W_t^f | B_{t-c} \right] - \gamma W_{t-c} \right| 
\leq E_u \sum_{v \in I_{t-c}} H^\alpha(L(v)^\top u) \left| \frac{F(U(v), t - c - S(v) + c)}{e^\alpha(U(v))} - \gamma \right|
\]
\[
= e^\alpha(u) E_u \left| \frac{F(U(t - c), c - R(t-c))}{e^\alpha(U(t - c))} - \gamma \right|
\]
\[
(13.5) = e^\alpha(u) \int \frac{F(y, c - r)}{e^\alpha(y)} - \gamma \left| Q_u^\alpha(U(t-c) \in dy, R(t-c) \in dr) \right|
\]

By Lemma 13.1, $\lim_{s \to \infty} |e^\alpha(\cdot)^{-1}F(\cdot, s) - \gamma| = 0$, in particular, this expression is bounded, by $F_0$, say. By Remark 12.3, $Q_u^\alpha(U(t), R(t)) \in \cdot$ converges weakly, in particular, it is tight. Thus for each $\varepsilon > 0$, we may choose a compact set $C = S_2 \times [a, b] \subset S_2 \times \mathbb{R}$, with
\[
Q_u^\alpha(U(t), R(t)) \in C^c \leq \frac{\varepsilon}{2F_0 e^\alpha(u)} \quad \forall t \geq 0.
\]

If $(y, r) \in C$, then $c - r \geq c - b$, and we can choose $c_0$ such that
\[
\sup_{s \geq c_0 - b, y \in S_2} \left| \frac{F(s, l)}{e^\alpha(y)} - \gamma \right| < \frac{\varepsilon}{2e^\alpha(u)}.
\]

Then we have the estimate, valid for all $t > c \geq c_0$,
\[
(13.5) \leq e^\alpha(u) \int_C \sup_{t \geq c_0 - b, y \in S_2} \left| \frac{F(y, l)}{e^\alpha(y)} - \gamma \right| Q_u^\alpha(U(t-c) \in dy, R(t-c) \in dr)
\]
\[
+ F_0 Q_u^\alpha(U(t), R(t)) \in C^c
\]
\[
< \varepsilon
\]

□

**Proof of Theorem 13.2** Combining the lemmata 11.3, 13.7 and 13.8 shows that $W_t^f \to \gamma W$ in $\mathbb{P}_u$-probability for all $u \in S_2$. By Theorem IV.8.12, convergence in probability together with the weak $L^1$-convergence, proven in Lemma 13.6, implies the (strong) $L^1$-convergence.

□

**14. Applications of General Branching Processes**

This section prepares the final proof of Theorem 3.9 in the next section by applying the results of Section 13 to particular functions $f$.

**Lemma 14.1.** For all $\varepsilon > 0$ there is a $c > 0$ and an equicontinuous function $f_c$ such that for all $u \in S_2$
\[
(14.1) \sum_{v \in I_t} H^\alpha(L(v)^\top u) 1_{\{S(v) - c > c\}} \leq W_t^{f_c} \quad \mathbb{P}_u\text{-a.s.,}
\]
and
\[
\lim_{t \to \infty} W_t^{f_c} = \varepsilon W(u)
\]
in $\mathbb{P}_u$-probability.
Proof. Fix $\delta > 0$. For any $c > \delta$, just choose $f_c$ as an equicontinuous function such that $1_{(c, \infty)} \leq f_c \leq 1_{(c-\delta, \infty)}$. Then under $\mathbb{P}_u$,

$$
\sum_{v \in I_t} H^\alpha(L(v)^\top u) 1_{(c, \infty)}(S(v) - t) \leq \sum_{v \in I_t} H^\alpha(L(v)^\top u) f(S(v) - t) = W_t^{f_c}.
$$

The function $f$ satisfies the assumption of Theorem 13.2 hence

$$
\lim_{t \to \infty} W_t^{f_c} = W(u) \int f_c(r) \varrho(S \times \mathbb{R}_+ \times dr)
$$

in $\mathbb{P}_u$-probability. Now $\int f_c d\varrho \leq \varrho(S \times (c-\delta, \infty))$, thus for $c$ large, this quantity becomes arbitrary small. □

Lemma 14.2. For all sufficiently large $c > 0$ there is a equicontinuous function $g_c$ and a $C > 0$ such that for all $u \in S\geq$

$$
\sum_{v \in I_t} H^\alpha(L(v)^\top u) \min_j U(v)_j \, 1_{\{S(v) - t \leq c\}} \geq W_t^{\beta_c} \quad \mathbb{P}_u\text{-a.s.,}
$$

and

$$
\lim_{t \to \infty} W_t^{\beta_c} = CW(u)
$$

in $\mathbb{P}_u$-probability.

Proof. First we are going to prove that there is $C' > 0$ such that

$$
\lim_{t \to \infty} W_t^{\min} := \lim_{t \to \infty} \sum_{v \in I_t} H^\alpha(L(v)^\top u) \min_j U(v)_j = C' W(u)
$$

in probability. The function $u \mapsto \min_j u_j$ is continuous on the compact set $S\geq$, hence equicontinuous; thus Theorem 13.2 applies and gives the convergence with

$$
C' = \int_{S\geq} \min_j u_j \varrho(du \times \mathbb{R}_+).
$$

By Lemma 12.4, $\varrho(\cdot \times \mathbb{R}_+)$ is concentrated on $S'\geq$, hence $\min_j u_j > 0$ $\varrho$-a.s., and thus $C' > 0$.

Now, for fixed $\delta > 0$ and any $c > \delta$, choose $g'_c$ as an equicontinuous function on $\mathbb{R}_+$, satisfying $1_{(c, \infty)} \leq g'_c \leq 1_{(c-\delta, \infty)}$. Then the function $((u, r) \mapsto g'_c(r)(\min_j u_j)$ is equicontinuous, and another application of Theorem 13.2 yields that

$$
\lim_{t \to \infty} W_t^{\min \times g'_c} := \lim_{t \to \infty} \sum_{v \in I_t} H^\alpha(L(v)^\top u) \left( \min_j U(v)_j \right) g'_c(S(v) - t)
$$

$$
= W(u) \int (\min_j y_j) g'_c(r) \varrho(dy \times dr).
$$

The factor on the right hand side is smaller than $\int (\min_j y_j) 1_{(c-\delta, \infty)}(r) \varrho(dr \times du)$, hence for choosing $c$ large, this becomes less than $\varepsilon C'$ for some $\varepsilon < 1$.

Finally, for this choice of $c$, set $g_c(u, r) := (\min_j u_j) \times (1-g'_c(r)) \leq (\min_j u_j) 1_{[0, c]}(r)$. With this choice, (14.2) holds. Moreover, in $\mathbb{P}_u$-probability,

$$
\lim_{t \to \infty} W_t^{\beta_c} = \lim_{t \to \infty} (W_t^{\min} - W_t^{\min \times g'_c}) = (1-\varepsilon)C'W(u) =: CW(u).
$$

□
15. Every Fixed Point is $\alpha$-regular

In this section, we finally give the proof of Theorem 3.9, showing that any nontrivial and nondegenerate (i.e., almost sure finite) fixed point is $\alpha$-regular.

In what follows, let $\phi$ be (the LT of) such a fixed point of $S$, and write

$$D(ru) := \frac{1 - \phi(ru)}{H^\alpha(ru)} = \frac{1 - \phi(ru)}{r^\alpha e^\alpha(u)}.$$  

Thus we are going to show that

$$0 < K := \liminf_{r \to 0} D(r1) \leq \limsup_{r \to 0} D(r1) =: \overline{K} < \infty.$$  

Therefore, we start with a Lemma similar to [1, Lemma 11.4]:

**Lemma 15.1.** For all $u \in S_\geq$, $t \in \mathbb{R}$ it holds that

$$\frac{D(e^{-t}u)}{D(e^{-t}1)} \leq \frac{H^\alpha(1)}{H^\alpha(u)} := R \quad \text{and} \quad \frac{D(e^{-t}u)}{D(e^{-t}1) \min_j u_j} \geq R.$$  

Moreover, for all $c > 0$ there is $\delta > 0$ such that for all $u \in S_\geq$ and all $0 \leq a \leq c$,

$$\frac{D(e^{-(t+a)}u)}{D(e^{-t}1)} \leq Re^\delta \quad \text{and} \quad \frac{D(e^{-(t-a)}u)}{D(e^{-t}1) \min_j u_j} \geq Re^{-\delta}. \tag{15.1}$$  

**Proof.** The first inequality results from Inequality (A.5):

$$\frac{D(e^{-t}u)}{D(e^{-t}1)} = \frac{1 - \phi(e^{-t}u) H^\alpha(e^{-t}1)}{1 - \phi(e^{-t}1) H^\alpha(e^{-t}u)} \leq 1 \cdot \frac{e^{-a\alpha} H^\alpha(1)}{e^{-a\alpha} H^\alpha(u)} = R.$$  

For the second inequality, we use (A.8),

$$\frac{D(e^{-t}u)}{D(e^{-t}1) \min_j u_j} = \frac{1 - \phi(e^{-t}u)}{(\min_j u_j)(1 - \phi(e^{-t}1))} \frac{e^{-a\alpha} H^\alpha(1)}{e^{-a\alpha} H^\alpha(u)} \geq R.$$  

To derive the remaining inequalities, use that $e^{-a\alpha}D(e^{-t}u)$ is decreasing in $t$:

$$\frac{D(e^{-(t+a)}u)}{D(e^{-t}u)} = \frac{e^{-a(t+a)}D(e^{-(t+a)}u)}{e^{-a(t+a)}D(e^{-t}u)} \leq \frac{e^{-a\alpha}D(e^{-t}u)}{e^{-a(t+a)}D(e^{-t}u)} \leq e^{a\alpha}.$$  

Setting $\delta := a\alpha$, one obtains by taking the reciprocal that

$$\frac{D(e^{-(t-a)}u)}{D(e^{-t}u)} \geq e^{-\delta}.$$  

Now plug in the first and second inequality to derive the third resp. fourth one. $\square$

Let $\phi$ be a fixed point of $S$ with disintegration $M$. Set $Z(u) := -\log M(u)$ for $u \in S_\geq$. Then we have the following lemma:

**Lemma 15.2.** For all $u \in S_\geq$,

$$\mathbb{P}(Z(u) < \infty, W(u) > 0) > 0$$  

and there is $u_0 \in S_\geq$ such that

$$\mathbb{P}(Z(u_0) > 0, W(u_0) < \infty) > 0.$$
Proof. Considering the first probability, Theorem 3.2 gives that \(\mathbb{E} W(u) = e^\alpha(u) > 0\). Fix any \(u \in \mathbb{S}_2\), then \(\mathbb{P}(W(u) > 0) := p > 0\).

If \(Y\) is a r.v. with LT \(\phi\), then \(\phi\) is \(\alpha\)-regular if and only if this holds for the LT of \(rY\), any \(r > 0\). Since \(Y < \infty\) \(\mathbb{P}\)-a.s., \(\lim_{r \to 0} \phi(ru) = 1\), hence by replacing \(Y\) with \(rY\) for suitable \(r\), we may w.l.o.g. assume that \(\phi(u) > 1 - p\). Since \(\phi(u) = \mathbb{E} e^{-Z(u)}\), necessarily \(\mathbb{P}(Z(u) < \infty) > 1 - p\). Consequently, \(\mathbb{P}(Z(u) < \infty, W(u) > 0) > 0\).

Turning to the second probability, we have that \(\mathbb{P}(W(u) < \infty) = 1\) for all \(u \in \mathbb{S}_2\). Since \(\phi\) is not trivial, there is \(u_0 \in \mathbb{S}_2\) such that
\[
1 > \phi(u_0) = e^{-Z(u_0)}.
\]
thus \(\mathbb{P}(Z(u_0) > 0) > 0\) and the assertion follows. \(\square\)

The following two lemmata give proof to Theorem 3.9.

**Lemma 15.3.** The value \(K\) is finite.

**Proof.** Fix \(u \in \mathbb{S}_2\). Then, under \(\mathbb{P}_u\),
\[
\sum_{v \in I_t} (1 - \phi(L(v)^\top u)) = \sum_{v \in I_t} H^\alpha(L(v)^\top u) \frac{1 - \phi(L(v)^\top u)}{H(\phi(L(v)^\top u))} \geq \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(\phi(L(v)^\top u)) 1_{\{S(v) \leq t+c\}}
\]
\[
= \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(e^{-S(v)}U(v)) 1_{\{t < S(v) \leq t+c\}}
\]
\[
\geq K e^{-\delta} D(e^{-(t+c)} 1) \sum_{v \in I_t} (\min(U(v))_j) H^\alpha(L(v)^\top u) 1_{\{S(v) \leq t+c\}}
\]
Referring to Lemma 14.2, we have for \(c\) large enough and letting \(t \to \infty\) along a suitable subsequence, that
\[
Z(u) \geq K e^{-\delta} K W(u) \quad \mathbb{P}\text{-a.s.}
\]
By Lemma 15.2, for any \(u \in \mathbb{S}_2\), \(\mathbb{P}(Z(u) < \infty, W(u) > 0) > 0\), thus necessarily, \(K < \infty\). \(\square\)

**Lemma 15.4.** The value \(K\) is positive.

**Proof.** Again using Lemma 15.1, we estimate under \(\mathbb{P}_u\)
\[
\sum_{v \in I_t} (1 - \phi(L(v)^\top u))
\]
\[
= \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(L(v)^\top u) 1_{\{S(v) \leq t+c\}} + \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(L(v)^\top u) 1_{\{S(v) > t+c\}}
\]
\[
= \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(e^{-S(v)}U(v)) 1_{\{t < S(v) \leq t+c\}}
\]
\[
+ \sum_{v \in I_t} H^\alpha(L(v)^\top u) D(e^{-S(v)}U(v)) 1_{\{S(v) > t+c\}}
\]
\[
\leq e^\delta D(e^{-t} 1) \sum_{v \in I_t} H^\alpha(L(v)^\top u) + R \left( \sup_{s \geq t+c} D(e^{-s} 1) \right) \sum_{v \in I_t} H^\alpha(L(v)^\top u) 1_{\{S(v) > t+c\}}.
\]
Considering Lemma 14.1, we have for \(t \to \infty\) along a suitable subsequence, that
\[
Z(u) \leq e^\delta K W(u) + R K e^\delta W(u) \quad \mathbb{P}\text{-a.s.}
\]
By Lemma 15.2 there is \( u_0 \in S_\geq \) with \( \mathbb{P} (Z(u_0) > 0, W(u_0) < \infty) > 0 \) for some \( u \). Since \( \varepsilon \) can be made arbitrarily small for \( c \) large, we have that necessarily, \( K > 0 \). \( \square \)

16. The Inhomogeneous Smoothing Transform

In this section, we study FPs of the inhomogeneous multivariate ST. We prove that they are of the form “homogeneous FP + particular inhomogeneous FP”. Using the results for the homogeneous ST, we describe the set of all FPs. This allows inter alia to explain why the particular FP of \( S_Q \), constructed by Mirek in [43], is not unique.

**Lemma 16.1.** Let \( m(s) < 1 \) for some \( s \in (0, 1] \) and let \( \mathbb{E}|Q|^s < \infty \). Then the series

\[
W^* = \sum_{n=0}^{\infty} \sum_{|v|=n} L(v)Q(v)
\]

converges a.s. and \( L(W^*) \) is the unique FP of \( S_Q \) with a finite moment of order \( s \).

**Proof.** Since the sum is taken over vectors with nonnegative entries, it suffices to show that \( \mathbb{E}|W^*|^s < \infty \). This follows from the simple estimate (recall \( s \leq 1 \))

\[
\mathbb{E}|W^*|^s \leq \sum_{n=0}^{\infty} \sum_{|v|=n} \mathbb{E}|L(v)Q(v)|^s
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{|v|=n} |L(v)||Q(v)|^s
\]

\[
= \sum_{n=0}^{\infty} n^n \mathbb{E}||H_n||^s \mathbb{E}|Q|^s
\]

where we also used that \( Q(v) \) is independent of \( L(v) \), since the latter only depends on the weights up to level \( |v| - 1 \). The final expression is convergent since \( m(s) < 1 \).

The law of \( W^* \) is a FP of \( S_Q \) by the simple expansion

\[
W^* = \sum_{i=1}^{N} T_i(\emptyset) \sum_{n=0}^{\infty} \sum_{|v|=n} [L(v)Q(v)]_i + Q(\emptyset) = \sum_{i=1}^{N} T_i(\emptyset) [W^*_i] + Q(\emptyset).
\]

Its uniqueness within the specified class follows from standard results using the contraction method, see e.g. [44, Lemma 3.3]. \( \square \)

Now we want to study \( \alpha \)-elementary FPs of \( S_Q \). Therefore, we exploit a one-to-one correspondence between FPs of \( S_0 \) and \( S_Q \).

Let \( s \in (0, 1] \). The minimal \( L_s \) metric is defined by

\[
l_s(\varrho, \eta) = \inf\{\mathbb{E}|X - Y|^s : \mathcal{L}(X) = \varrho, \mathcal{L}(Y) = \eta\}
\]

for probability laws \( \varrho, \eta \). It is a particular case of a Wasserstein distance. The following result can easily be obtained from [52, Theorem 3.1], where the one-dimensional situation is covered:

**Lemma 16.2.** Let \( s \in (0, 1] \) and \( \mathbb{E}\|T_1\|^s + |Q|^s < \infty \). Suppose that \( m(s) < 1 \). Then the following holds:

1. For any \( \eta_0 \in \mathcal{P}(\mathbb{R}_+^d) \) such that \( S_0\eta_0 = \eta_0 \), there exists exactly one \( \eta_Q \in \mathcal{P}(\mathbb{R}_+^d) \) such that

\[
S_Q\eta_Q = \eta_Q \quad \text{and} \quad l_s(\eta_0, \eta_Q) < \infty.
\]
(2) For any \( \eta_Q \in \mathcal{P}(\mathbb{R}_+^d) \) such that \( S_Q \eta_Q = \eta_Q \), there exists exactly one \( \eta_0 \in \mathcal{P}(\mathbb{R}_+^d) \) such that
\[
S_0 \eta_0 = \eta_0 \quad \text{and} \quad l_s(\eta_0, \eta_Q) < \infty.
\]

Observe that \( \mathcal{L}(W^*) \) is the FP of \( S_Q \) corresponding to the trivial FP \( \delta_0 \) of \( S_0 \). This one-to-one-correspondence can be made more explicit:

**Lemma 16.3.** Let \( \mu \) satisfy (C) and be aperiodic, let \( \alpha \in I_\mu \cap (0, 1) \) with \( m(\alpha) = 1 \) and \( m'(\alpha) < 0 \). Assume that \( \log(1) \), \( \log(1) \) are satisfied and that \( \mathbb{E} \|T_1\| + |Q| < \infty \).

Then there is \( s \in (\alpha, 1] \) such that the assumptions of Lemma 16.2 hold. Let \( \eta_0 \) and \( \eta_Q \) be corresponding FPs of \( S_0 \) resp. \( S_Q \) as given by that Lemma, with \( \mathcal{L} \)\( \psi_0 \) resp. \( \psi_Q \). Then either

(1) \( \eta_0 = \delta_0 \) and \( \eta_Q = \mathcal{L}(W^*) \), or
(2) both FPs are \( \alpha \)-elementary and there is \( K > 0 \) such that for all \( u \in \mathbb{S}_+ \)
\[
\lim_{r \downarrow 0} \left| \frac{1 - \psi_Q(ru)}{r^\alpha} - Ke^\alpha(u) \right| = 0.
\]

**Proof.** Since \( \alpha \in I_\mu \cap (0, 1) \) with \( m'(\alpha) < 0 \), there is \( s > \alpha \) with \( m(s) < 1 \). Additionally, \( \mathbb{E} \|T\| + |Q| < 2 + \mathbb{E} \|T_1\| + |Q| \), hence Lemma 16.2 applies.

Suppose that \( \eta_0 \neq \delta_0 \). Then necessarily, \( \psi_0 \) is \( \alpha \)-elementary by Theorem 1.9.

Let \( (Y_0, Y_Q) \) be a coupling of \( \eta_0 \) and \( \eta_Q \) with \( \mathbb{E}|Y_0 - Y_Q|^s < \infty \). Such a coupling exists since \( l_s(\eta_0, \eta_Q) < \infty \). Using the inequality \( |a^s - b^s| \leq |a - b|^s \) which is valid for \( s \in [0, 1] \) and \( a, b \in \mathbb{R}_+ \), it follows that for all \( u \in \mathbb{R}_+^d \)
\[
\mathbb{E}|\langle u, Y_Q \rangle^s - \langle u, Y_0 \rangle^s| \leq \mathbb{E}|\langle u, Y_Q - Y_0 \rangle| \leq |u|^s \mathbb{E}|Y_Q - Y_0|^s < \infty.
\]

Referring to [26 Lemma 9.4],
\[
\int_0^\infty \frac{1}{r} \left( r^s \mathbb{P}(\langle u, Y_Q \rangle > r) - \mathbb{P}(\langle u, Y_0 \rangle > r) \right) dr < \infty.
\]

From the fact that \( \int_1^\infty \frac{1}{r} dr \) diverges, it follows that necessarily
\[
\limsup_{r \to \infty} r^s \mathbb{P}(\langle u, Y_Q \rangle > r) - \mathbb{P}(\langle u, Y_0 \rangle > r) = 0.
\]

Since \( s > \alpha \), in particular
\[
\lim_{r \to \infty} |r^\alpha \mathbb{P}(\langle u, Y_Q \rangle > r) - r^\alpha \mathbb{P}(\langle u, Y_0 \rangle > r)| = 0.
\]

Now, combining the fact that \( \psi_0 \) is \( \alpha \)-elementary with the Tauberian theorem for LTs [25 XIII.(5.22)] gives that
\[
\lim_{r \to \infty} r^\alpha \mathbb{P}(\langle u, Y_0 \rangle > r) = K'e^\alpha(u) > 0
\]
for some \( K' \). It follows that
\[
\lim_{r \to \infty} \frac{\mathbb{P}(\langle u, Y_Q \rangle > r)}{\mathbb{P}(\langle u, Y_0 \rangle > r)} = 1
\]
\[
\lim_{r \to \infty} (r^\alpha \mathbb{P}(\langle u, Y_0 \rangle > r))^{-1} r^\alpha \mathbb{P}(\langle u, Y_Q \rangle > r) = K'e^\alpha(u) > 0.
\]

First this gives
\[
\lim_{r \to \infty} r^\alpha \mathbb{P}(\langle u, Y_Q \rangle > r) = K'e^\alpha(u)
\]
for all \(u \in \mathbb{S}_2\), in other words, \(Y_0\) and \(Y_Q\) have the same tail behaviour. From this, the asserted result \([16.1]\) for the LTs follows by using again the Tauberian theorem.

Proof of Theorem 3.10 The first assertion concerning \(W^*\) is the content of Lemma \([16.1]\).

Considering the second assertion, let \(\psi_Q \in \mathcal{L}(\mathcal{F}_Q)\). Referring to Lemma \([16.3]\) either \(\psi_Q(ru) = E e^{-r(u,W^*)}\), which corresponds to \(K = 0\), or there is \(K > 0\) such that, using Lemma \([7.3]\), \(\psi = \lim_{n \to \infty} S_Q^n \phi\) with
\[
\phi(ru) = \exp \left( -Kr^\alpha \int_{S_2} \langle u, y \rangle^\alpha \nu^\alpha(dy) \right).
\]

Taking \((2.7)\) into account,
\[
S_Q^n \phi(ru) = E \left[ \exp \left( -r \langle u, \sum_{|w| < n} L(w)Q(w) \rangle \prod_{|v| = n} \phi(rL(v)^T u) \right) \right]
= E \left[ \exp \left( -r \langle u, \sum_{|w| < n} L(w)Q(w) \rangle - \sum_{|v| = n} Kr^\alpha \int_{S_2} \langle L(v)^T u, y \rangle^\alpha \nu^\alpha(dy) \right) \right]
= E \exp \left( -r \langle u, W_n^* \rangle - Kr^\alpha W_n(u) \right)
\]

Using the \(\mathbb{P}\)-a.s.-convergence of \(W_n^*\), \(W_n(u)\) (Lemma \([16.1]\) resp. Theorem \(3.4\)) and the bounded convergence theorem,
\[
\psi_Q(ru) = \lim_{n \to \infty} S_Q^n \phi(ru) = E \exp(-r \langle u, W^* \rangle + Kr^\alpha W(u)).
\]

\[\square\]

17. Existence of Fixed Points in the Boundary Case

In this section, we prove that in the situation \(m(\alpha) = 1\) with \(m'(\alpha) = 0\) for \(\alpha \in (0,1]\), there still exists a nontrivial fixed point, thus extending the results of \([20]\) to the situation \(m(1) = 1\), \(m'(1) = 0\). The existence of a fixed point in the boundary case is proven by the same approximation argument as in \([23,\text{Theorem 3.5}]\). This is why we just sketch the main ideas and refer the interested reader to \([42,\text{Section 10}]\) for details.

For \(\chi \in (0, \alpha]\), define a biased version of \(S\) by
\[
(17.1)\quad S_S : \nu \mapsto \mathcal{L} \left( \sum_{i=1}^N m(\chi)^{-1/\chi} T_i X_i \right),
\]
where \(X_i\) are iid with law \(\nu\) and independent of \(T\). Writing
\[
T_S = (T_{\chi,i})_{i=1}^N = (m(\chi)^{-1/\chi} T_i)_{i=1}^N,
\]
define \(m_\chi\) as the spectral function associated with \(T_S\), and \(\mu_\chi\) as the distribution of \(T_{\chi,1}\).

Then it is readily checked that \(\mu_\chi\) satisfies \((C)\) (see \([42,\text{Lemma 10.3}]\)), that \(m_\chi(\chi) = 1\) and \(m_\chi'(\chi) < 0\) and that \(T \log T\) and \((3.8)\) for \(T\) imply the validity of \((T \log T), (T \log^2 T)\) for \(T_S\) and with \(\alpha\) replaced by \(\chi\).

Hence Theorem \(3.4\) applied to \(S_S\) gives the existence of a nontrivial FP with LT \(\psi_\chi\), say. Fix \(u_0 \in \mathbb{S}_2\), then, possibly after rescaling, \(\psi_\chi(u_0) = 1/2\). In this manner, construct a family \((\psi_\chi)_{\chi \in (0,\alpha]}\), such that \(\psi_\chi(u_0) = 1/2\) and \(S_S \psi_\chi = \psi_\chi\) for all \(\chi \in (0, \alpha]\).

For any sequence \(\chi_n \to \alpha\), there is a convergent subsequence \(\psi_{\chi_n}^\prime\) with limit \(\psi_\chi\), which is again a LT of a (sub-)probability measure, with \(\psi(u_0) = 1/2\). It can be checked that
Since this holds for arbitrary \( k \), this is sufficient for the existence of \( E \). Hence, all the assumptions of Theorem 3.1 are satisfied.

In the particular case \( \alpha = 1 \), it is shown in [20] Theorem 2.2 that the existence of a nontrivial FP with finite expectation is equivalent to \( m' (1) < 0 \). Thus if \( m' (1) = 0 \), then the nontrivial FP constructed above necessarily has infinite expectation. It is a.s. finite since \( \psi (0^+) = 1 \).

18. An Explicit Example

This section contains the calculations related to the example given in the introduction, as well as a second, similar example, which shows that the aperiodicity condition is necessary for the uniqueness (up to scalar multiplication) of fixed points.

In order to get a better feeling for the results, we will give explicit formulas for \( m (s), \nu^s \) and \( W_0 \) in a somewhat minimal example, which was also considered in the Introduction.

Let \( d = 2 \). Let \( a, b \in \mathcal{M}_d \) be two rank-one projections (i.e. \( a \cdot x = v_a \) for all \( x \in S_d \)) such that \( \log \frac{\lambda_a}{\lambda_b} \notin \mathbb{Q} \) and \( \lambda_a + \lambda_b < 1 \). Let \( N = 2, T_1, T_2 \) be iid with law

\[
\mu = \frac{1}{2} \delta_a + \frac{1}{2} \delta_b.
\]

Then \( ET_1 = \frac{1}{2} (a + b) \) has spectral radius less or equal to \( \frac{1}{2} (\lambda_a + \lambda_b) < 1/2 \). Referring to Lemma 4.3, this is sufficient for the existence of \( \alpha \in (0, 1) \) with \( m (\alpha) = 1, m' (\alpha) < 0 \). Hence, all the assumptions of Theorem 3.1 are satisfied.

By the special form of \( \mu \), we deduce that for all \( s \geq 0 \),

\[

\nu^s = p_s \delta_{v_a} + q_s \delta_{v_b}.
\]

Hence, applying \( \nu^s (P^s)' = k (s) \nu^s \), we have for all \( f \in C (S_d) \),

\[

\int P^s f(x) \nu^s (dx) = \frac{1}{2} f (v_a) (p_s |a v_a|^s + q_s |a v_b|^s) + \frac{1}{2} f (v_b) (p_s |b v_a|^s + q_s |b v_b|^s)
\]

\[
= k (s) (p_s f (v_a) + q_s f (v_b)).
\]

Since this holds for arbitrary \( f \), we infer that

\[

k (s) \begin{pmatrix} p_s \\ q_s \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \lambda_a^s |a v_b|^s \\ \lambda_b^s |b v_a|^s \end{pmatrix} \begin{pmatrix} p_s \\ q_s \end{pmatrix} = L_s \begin{pmatrix} p_s \\ q_s \end{pmatrix},
\]

i.e. \( k (s) \) is the Perron-Frobenius eigenvalue of the positive matrix \( L_s \), while the measure \( \nu^s \) is determined by the corresponding eigenvector. In particular, with \( t_s := \text{trace} (L_s), d_s := \det (L_s) \), we have

\[
k (s) = t_s / 2 + \sqrt{t_s^2 / 4 - d_s}.
\]

Thus, \( \alpha \) can be at least computed numerically. Furthermore, we obtain by its very definition that

\[

W_0 (u) = \int \langle u, x \rangle^\alpha \nu^\alpha (dx) = p_\alpha \langle u, v_a \rangle^\alpha + q_\alpha \langle u, v_b \rangle^\alpha.
\]

In fact, our Theorem does not exclude the completely deterministic case \( (T_1, T_2) = (a, b) \). In this case, \( W_0 (u) \) is deterministic as well, thus, due to the martingale property, \( W (u) = W_0 (u) \). We deduce that any solution to the SFPE

\[

X \overset{\mathcal{L}}{=} a X_1 + b X_2
\]

(18.1)
satisfies for all $u \in S_2$

$$
\langle u, X \rangle \overset{\xi}{=} K^{1/\alpha} \left( p_s^{1/\alpha} \langle u, v_a \rangle Z_1 + q_s^{1/\alpha} \langle u, v_b \rangle Z_2 \right),
$$

for some $K \in \mathbb{R}$, where $Z_1, Z_2$ are independent one-dimensional $\alpha$-stable r.v.'s with LT $e^{-\rho_s^\alpha}$.

**On the Aperiodicity Condition.** Finally, we will give an example where the aperiodicity condition is violated, and more than a one-parameter class of fixed point appears. Consider

$$
a := \frac{1}{10} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad b := \frac{1}{10} \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
$$

Direct calculations show that

$$
L_s = \frac{1}{2} \left( \frac{3}{10}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} |a v_a|^s & |a v_b|^s \\ |b v_a|^s & |b v_b|^s \end{pmatrix} \right),
$$

in particular, the aperiodicity condition does not hold. One obtains that $k(s) = (0.3)^s$, consequently $\alpha = -\log 2/\log(0.3)$ and that $p_s = q_s = 0.5$.

But in addition to $\psi(x) = \exp(-KW(x))$, there are more fixed points: Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be multiplicatively $r := 3/10$-periodic, i.e. $H(rt) = H(t)$ for all $t$ and such that $F(t) := H(t)t^\alpha$ is completely monotone. It follows from [23, Section 5] that this set is non-trivial. Consider

$$
W^H(x) := \int H(\langle x, y \rangle) \langle x, y \rangle^\alpha \nu(\langle x, v_a \rangle + F(\langle x, v_b \rangle))
$$

The multivariate Bernstein theorem [15, Theorem 4.2.1] allows to prove that $\phi(x) := \exp(-W^H(x))$ is the Laplace transform of a probability distribution on $\mathbb{R}_+^d$.

Still considering the situation $(T_1, T_2) = (a, b)$, we obtain that $S\phi(x) = \exp(-W^H_1(x))$, with

$$
W^H_1(x) = W^H(\begin{pmatrix} a^\top & x \end{pmatrix}) + W^H(\begin{pmatrix} b^\top & x \end{pmatrix})
$$

$$
= \frac{1}{2} \left[ 2F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) + F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) \right]
$$

$$
= \frac{1}{2} \left[ 2F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) \right]
$$

$$
= \frac{1}{2} \left[ 2F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) + F(\frac{3}{10} \langle x, v_a \rangle) + F(\frac{3}{10} \langle x, v_b \rangle) \right]
$$

$$
= W^H(x),
$$

thus $\phi(x)$ is a fixed point of $S$, too.

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List of Symbols

1, \overline{1} \quad 1 = (1, \ldots, 1)^T \in \mathbb{R}_+^d, \overline{1} = d^{-1/2} \mathbf{1} \in \mathcal{S}_{\geq} \\
\subseteq \quad \text{same law} \\
\hat{\mathcal{M}} \quad \text{topological interior of the set } \mathcal{M} \\
[\cdot, \cdot] \quad \text{shift operator in } \mathfrak{T}, \text{ see \eqref{eq:shift-operator}} \\
aperiodic \quad \exists \, \mathbf{a}, \mathbf{b} \in \mathcal{M}_{\geq} \text{ s.t. } \log \lambda_a / \log \lambda_b \notin \mathbb{Q} \\
\alpha\text{-regular} \quad \text{see Definition \ref{def:alpha-regular}} \\
\mathfrak{B}_n \quad \text{filtration, } \mathfrak{B}_n = \sigma \left( U(\emptyset), \{ T(v) : v \text{ has no ancestor in } \mathcal{I} \} \right) \\
\mathfrak{B}_t = B_{\mathcal{I}_t} \quad \mathfrak{B}_2 := \sigma \left( U(\emptyset), \{ T(v) : v \text{ has no ancestor in } \mathcal{I} \} \right) \\
(C) \quad \text{condition imposed on the supp of } \mathcal{L} \left( \mathcal{T}_1 \right), \text{ see Definition \ref{def:condition-C}} \\
\mathcal{D}_{s,L} \quad \mathcal{D}_{s,L}(u, t) = \frac{1 - \phi(e^{-u} t)}{H^s(u, t) - L(e^{-u} t)} \\
\mathcal{E}_{u} \quad \text{expectation symbol of } \mathbb{Q}^u_\sigma, \text{ see \eqref{eq:expectation-symbol}} \\
\mathcal{e}^s, \mathcal{e}^w \quad \text{positive functions on } \mathcal{S}_{\geq}, \text{ satisfies } \mathcal{P}^w \mathcal{e}^s = k(s) \mathcal{e}^s \text{ resp. } \mathcal{P}^w \mathcal{e}^w = k(s) \mathcal{e}^w \\
\mathcal{E}_{\alpha, c} \quad \text{extremal points of } H^K_{\alpha, c}, \text{ see Lemma \ref{lem:extremal-points}} \\
\Gamma \quad \text{smallest subsemigroup of } \mathcal{M}_{\geq} \text{ containing supp } \mu \\
H^s \quad \text{harmonic function, } H^s(u, t) = e^s(t) e^{st}; H^s(x) = H^s \left( \frac{x}{|x|}, \log |x| \right) \\
H^K_{\alpha, c} \quad \text{see Definition \ref{def:harmonic-functions}} \\
I_\mu \quad I_\mu = \{ s \geq 0 : \mathbb{E} \| T_1 \|^s < \infty \} \\
\iota(\mathbf{a}) \quad \iota(\mathbf{a}) := \inf \{ x \in \mathbb{R}^d : |x| \} \\
\mathfrak{T}_t \quad \text{stopping line, } \mathfrak{T}_t := \left\{ v \in \mathfrak{T} : S(v) > t \text{ and } S(v) | k \leq t \forall k < |v| \} \right. \\
J^K_\mu \quad \text{see Definition \ref{def:dominant-eigenvalue}} \\
k(s) \quad \text{dominant eigenvalue of } \mathcal{P}^w \text{ and } \mathcal{P}^w \text{ satisfies } m(s) = N k(s) \\
K_C \quad K_C = (\min_{y \in C} \min_{i} y_i) \text{ for compact } C \subset \mathcal{S}_{\geq} \\
\mathcal{K}_C, \mathcal{K}_u \quad \text{see Definition \ref{def:dominant-eigenvalue}} \\
\mathcal{L} \quad \text{functor, mapping probability laws on } \mathbb{R}^d_{\geq} \text{ to their Laplace transforms} \\
L(v) \quad \text{recursively defined by } L(\emptyset) = \mathbf{1}d \text{ and } L(v) = L(v) \mathcal{T}_1(v) \\
L-\alpha\text{-elementary} \quad \text{see Definition \ref{def:L-alpha}} \\
\lambda_a \quad \text{Perron-Frobenius eigenvalue of } \mathbf{a} \in \hat{\mathcal{M}}_{\geq} \\
M \quad \text{Disintegration, see Definition \ref{def:disintegration}} \\
\mathcal{M}_{\geq} \quad \text{set } M(d \times d, \mathbb{R}^d_\geq) \text{ of nonnegative } d \times d \text{ matrices} \\
m(s) \quad m(s) = N \lim_{n \to \infty} \mathbb{E} \| T_1 \cdots T_n \|^s / n, \text{ where } (T_i)_{i \in \mathbb{N}} \text{ are iid with law } \mu \\
(M_n)_{n \in \mathbb{N}} \quad \text{sequence of iid copies of } T_1 \text{ under } \mathbb{P} \\
\mu \quad \text{law of } T_1 = \ldots = \text{law of } T_N \\
\nu^s, \nu^w \quad \text{prob. meas. on } \mathcal{S}_{\geq}, \text{ satisfy } (\mathcal{P}^w)^s \nu^s = k(s) \nu^s \text{ resp. } (\mathcal{P}^w)^s \nu^w = k(s) \nu^w \\
\mathcal{F}_u \quad \text{operators on } \mathcal{C} \left( \mathcal{S}_{\geq} \right) \text{ defined in \eqref{eq:linear-operators} resp. \eqref{eq:nonlinear-operators}} \\
\Pi_n \quad \Pi_n := \mathcal{M}_n \cdots \mathcal{M}_1 \\
\pi^s \quad \text{stationary law for } (U_n) \text{ under } \mathbb{Q}_u^s; \pi^s(dx) = e^s(x) \nu^s(dx) \\
\mathbb{Q}_u^c \quad \text{exponentially changed measure, see \eqref{eq:exponential-change}} \\
R(t) \quad R(t) := S_{\alpha} - t \\
\theta \quad \text{prob. on } \mathcal{S}_{\geq} \times \mathbb{R}_+, \text{ see Remark \ref{rem:theta}} \\
S(v), S_n, S(t) \quad S(v) = - \log \left[ L(v)^T U(\emptyset) \right] \text{, } S_n := - \log \left[ \mathcal{M}_n \cdots \mathcal{M}_1 U_0 \right] \text{, } S(t) = S_{\sigma_t} \\
S_\alpha(v) \quad \text{multivariate } \alpha\text{-stable law with LT } \left( \mathcal{T}_1 \right) \text{ of } \mathbb{P} \\
\mathcal{S}_{\geq} \quad \mathcal{S}_{\geq} = \mathcal{S} \cap \mathbb{R}^d_{\geq} \text{ intersection of the unit sphere and the nonnegative cone in } \mathbb{R}^d \\
\sigma_t \quad \sigma_t := \inf \{ n : S_n > t \} \\
\mathcal{T} \quad \mathcal{T} = \bigcup_{n \in \mathbb{N}} \{ 1, \ldots, N \}^n \\
\mathbb{U}(v), U_n, U(t) \quad \mathbb{U}(v) = L(v)^T U(\emptyset) \text{, } U_n = \mathcal{M}_n \cdots \mathcal{M}_1 \cdot U_0 \text{, } U(t) = U_{\sigma_t} \\
\nu_a \quad \text{Perron-Frobenius eigenvalue of } \mathbf{a} \in \hat{\mathcal{M}}_{\geq} \\
W_n, W \quad \text{martingale, see \eqref{eq:W-n}, } W_n \to W, W_0(u) = e^w(u) = H^w(u, 0) \\
W^\star \quad \text{particular fixed point of } \mathcal{S}_Q, \text{ see Lemma \ref{lem:fixed-point}} \\
W^f_k \quad W^f_k = \sum_{v \in \mathfrak{T}_k} H^s(L(v)^T U(v)) f(U(v), S(v) - t), \\
Y_n \quad \text{weighted branching process, see \eqref{eq:weighted-branching-process}} \\
Z(u) \quad \text{in Sections \ref{sec:fixed-points} and \ref{sec:branching}}, \quad Z(u) := - \log \mathcal{M}(u)
Appendix A. Inequalities for Laplace Transforms

If $\phi$ is the LT of a r.v. $R \geq 0$, then $t^{-1}(1 - \phi(t))$ is again a LT of a measure on $R \geq 0$ (see [25, XIII (2.7)]). Consequently, it is decreasing and thus for all $t \in \mathbb{R} \geq 0$, $0 < a < 1$:

\[
\frac{1 - \phi(at)}{at} \geq \frac{1 - \phi(t)}{t} \Rightarrow 1 - \phi(at) \geq a(1 - \phi(t)),
\]

as well as, for $b \geq 1$,

\[
1 - \phi(bt) \leq b(1 - \phi(t)).
\]

This proves the first four inequalities in the subsequent lemma:

Lemma A.1. Let $\phi$ be the Laplace transform of a distribution on $\mathbb{R}^d_\geq$, $u \in S_\geq$, $t \in \mathbb{R} \geq$ and $A \in M(d \times d, \mathbb{R}_\geq)$. Then

\begin{align*}
(A.1) & \quad 1 - \phi(atu) \leq 1 - \phi(tu) \quad \text{for } a < 1, \\
(A.2) & \quad 1 - \phi(atu) \geq a(1 - \phi(tu)) \quad \text{for } a < 1, \\
(A.3) & \quad 1 - \phi(btu) \geq 1 - \phi(tu) \quad \text{for } b > 1, \\
(A.4) & \quad 1 - \phi(btu) \leq b(1 - \phi(tu)) \quad \text{for } b > 1, \\
(A.5) & \quad 1 - \phi(tu) \leq 1 - \phi(t1) \\
(A.6) & \quad 1 - \phi(tAu) \leq 1 - \phi(t|Au|1) \leq 1 - \phi(t\|A\|1) \\
(A.7) & \quad 1 - \phi(tAu) \leq (\|A\| \vee 1)(1 - \phi(t1)) \\
(A.8) & \quad 1 - \phi(tu) \geq 1 - \phi(t\min_i u_i1) \geq (\min_i u_i)(1 - \phi(t1))
\end{align*}

Proof. Let $Z$ be a r.v. with LT $\phi$. For all $u \in S_\geq$, $\langle u, Z \rangle \leq \langle 1, Z \rangle$. Thus

\[
1 - \phi(tu) = \mathbb{E} \left( 1 - e^{-t\langle u, Z \rangle} \right) = \int_0^\infty te^{-tr} \mathbb{P}(\langle u, Z \rangle > r) \, dt \\
\leq \int_0^\infty te^{-t\langle 1, Z \rangle} \mathbb{P}(\langle 1, Z \rangle > r) \, dt = 1 - \phi(t1).
\]

From (A.5) and (A.1) now (A.6) follows:

\[
1 - \phi(tAu) = 1 - \phi(t|Au|1) \leq 1 - \phi(t\|A\|1) \leq 1 - \phi(t\langle 1, Z \rangle).
\]

Then (A.7) follows by applying (A.1) resp. (A.4) in (A.6).

In order to prove (A.8), observe that

\[
\langle u, Z \rangle = \sum_{i=1}^d u_i Z_i \geq \min_i u_i \sum_{i=1}^d Z_i = \min_i u_i (1, Z).
\]

Then the argument is the same as given for (A.5), with an additional use of (A.2). \qed