H-inf and State Jumps Control for Switched Singular Systems

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Abstract. In this paper, the problem of switched singular systems with exogenous disturbance aiming to minimize $H_\infty$ norm bound and inconsistent state jumps when switch occurs between two different singular subsystems is studied. By designing static output feedback controllers, under which the closed-loop system can be admissible (regular, impulse free and stable) as well as minimizing $H_\infty$ norm bound and instantaneous state jumps at arbitrary switching point. The validity of the main result is illustrated by an example.

1. Introduction
Recently, there has been increasing interest in analysis and synthesis for singular switched systems [1-6]. For the switched singular system, state jumps occur not only at initial starting point, but also at switching points with inconsistent initial conditions. Such instantaneous jumps should be avoided since the systems will be destroyed when the jumps strength is too large. This problem has attracted much attention [7, 8].

In this paper, we considers the problem of designing static output feedback controllers that stabilize singular switch systems with exogenous disturbance and minimize $H_\infty$ norm bound and the possible state jumps. Firstly, an explicit formula of state jumps vector for switched singular systems with exogenous disturbance is presented. Secondly, the resulting closed-loop system guarantees the admissibility as well as a desire $H_\infty$ performance level via static output feedback. Finally, two convex optimization problems are presented to minimize $H_\infty$ norm bound and state jumps at arbitrary switching point, respectively.

The major contribution of the paper is guaranteeing admissibility as well as a small $H_\infty$ performance level and compressing state jumps of switched singular systems with exogenous disturbance at arbitrary switching point. Furthermore, the designed static output feedback gain need satisfying both some inequalities and equalities, which is a numerically hard problem. In order to tackle such a problem, the Projection lemma and Finsler's lemma combined with matrix inequality transformation technique, sufficient conditions are derived, under which the resulting closed-loop system guarantees the admissibility as well as a small $H_\infty$ performance level and state jumps via static output feedback, respectively.

2. Problem Statement and Preliminaries
Consider the following switched singular system:
\[
E \dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u_{\sigma(t)} + G_{\sigma(t)} \omega \\
z = C_{z\sigma(t)} x \\
y = C_{y\sigma(t)} x
\]  

(1)

Where \( \sigma(t) : [0, +\infty) \to \chi = \{1, 2, \cdots, m\} \) is the switching signal. \( x \in R^n \) is the state, \( u_{\sigma(t)} \in R^m \) is the control input, \( \omega \in R^p \) is the exogenous input and \( z \in R^q \) is the controlled output. \( E \in R^{r \times m} \) And \( \text{rank} E = r \leq n \), \( A_{\sigma(\cdot)}, B_{\sigma(\cdot)}, G_{\sigma(\cdot)}, C_{z\sigma(\cdot)} \) are known constant matrices.

**Assumption 1.** Each singular system \((E, A_i)\) is regular, impulse-free and stabilizable.

In this paper, the following static output feedback controller

\[
u_{\sigma(t)} = K_{\sigma(t)} y
\]

(2)

Is designed. Under the controller (2), the corresponding closed-loop system is

\[
E \dot{x} = (A_{\sigma(t)} + B_{\sigma(t)} K_{\sigma(t)} C_{y\sigma(t)}) x + G_{\sigma(t)} \omega
\]

(3)

**Lemma 1.** [9] The closed-loop system (3) is admissible and with \( H_\infty \) norm bound \( \gamma \) if and only if

\[
E^T X = X^T E \geq 0
\]

(4)

\[
(A_i + B_i K_i C_{yi})^T X + X^T (A_i + B_i K_i C_{yi}) + C_{yi}^T C_{yi} + \frac{1}{\gamma^2} X^T G_i G_i^T X < 0
\]

(5)

**Lemma 2.** [10] Let matrices \( G = G^T \in R^{n \times m}, U \in R^{k \times n} \) and \( V \in R^{m \times n} \) be given. Then, there exists \( X \) such that if and only if \( U_i^T G U_i < 0 \) and \( V_i^T G V_i < 0 \) where \( U_i \) and \( V_i \) are bases of the 100 null spaces of \( U \) and \( V \) respectively.

\[
G + U_i^T X V + V_i^T X_i U < 0
\]

(6)

**Lemma 3.** [10] Let \( Q = Q^T \in R^{n \times n} \) and \( M \in R^{n \times n} \) such that \( \text{rank}(M) < n \). The following statements are equivalent.

i) \( M_i^T Q M \_ > 0 \) Where \( M \_ \) is a base of the null spaces of \( M \), that is, \( MM \_ = 0 \).

ii) There exists a scalar \( \sigma \in R \) such that \( Q - \sigma MM \_ > 0 \).

3. Jumps Analysis

There exist nonsingular matrices \( M, N \) such that \( MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, MG_i = \begin{bmatrix} G_{i1} \\ G_{i2} \end{bmatrix}, M(A_i + B_i K_i) \cdot \)

\[
C_{yi} N = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

Then the system (3) is equivalent to
\begin{align}
\dot{z}_{i1} &= \bar{A}_{i11} z_{i1} + \bar{A}_{i12} z_{i2} + G_{i1} \omega_s, \\
0 &= \bar{A}_{i21} z_{i1} + \bar{A}_{i22} z_{i2} + G_{i2} \omega_s,
\end{align}
(7)

Where \( N^{-1} \chi = z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, z_1 \in \mathbb{R}^r, z_2 \in \mathbb{R}^{n-r}. \) By the above section, there exist a static output feedback controller such that the closed-loop system (3) is admissible. Thus, one gets that \( \bar{A}_{i22} \) is invertible. When \( t \in [t_k, t_{k+1}) \), if follows that

\begin{align}
z_{i1}(t) &= e^{\bar{A}_{i11} (t-t_k)} z_{i1}(t_k) + \int_{t_k}^t e^{\bar{A}_{i11}(t-\tau)} \tilde{G}_i \omega(\tau) d\tau, \\
z_{i2}(t) &= -\bar{A}_{i22}^{-1} \bar{A}_{i21} z_{i1}(t) - \bar{A}_{i22}^{-1} \tilde{G}_{i2} \omega(t),
\end{align}
(8)

(9)

Where \( \tilde{A}_i = \bar{A}_{i11} - \bar{A}_{i12} \bar{A}_{i22}^{-1} \bar{A}_{i21}, \tilde{G}_i = G_{i1} - \bar{A}_{i12} \bar{A}_{i22}^{-1} G_{i2}. \) When \( t \to t_k^+ \), one has

\begin{align}
z_{i1}(t_k) &= z_{i1}(t_k^-), \\
z_{i2}(t_k^+) &= -\bar{A}_{i22}^{-1} \bar{A}_{i21} z_{i1}(t_k) - \bar{A}_{i22}^{-1} \tilde{G}_{i2} \omega(t_k^+).
\end{align}
(10)

(11)

**Proposition 1.** If there exists a static output feedback gain \( K_i \) such that the system (3) is no impulses at arbitrary switching sequence, the instantaneous state jump vector \( e(t_i) \) can be denoted by

\[ e(t_k) = \Gamma_i \xi(t_k), \]
(12)

Where \( \Gamma_i = N \begin{bmatrix} 0 & 0 & 0 \\ -\bar{A}_{i22}^{-1} \bar{A}_{i21} & -\bar{A}_{i22}^{-1} \tilde{G}_{i2} & N^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ N^{-1} \end{bmatrix}, \xi(t_k) = \begin{bmatrix} x(t_k) \\ \omega(t_k) \end{bmatrix}. \)

4. Admissible control and \( H_\infty \) control

**Theorem 1.** If there exist matrices \( Z > 0 \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{(n-r) \times n}, W \in \mathbb{R}^{m \times (n-r)} \) of full-column rank satisfying \( E^T W = 0 \) and scalars \( \sigma_i < 0, \gamma > 0 \) for a given \( \delta_i > 0, \epsilon_i \), such that the following inequalities

\[
\begin{bmatrix}
\Phi_i & (ZE + WU)^T G_i & ZE + WU \\
* & -\gamma^2 I & 0 \\
* & * & \sigma_i^{-1} \delta_i^{-1} I
\end{bmatrix} < 0
\]
(13)
\[ Q_{i\perp}^T \begin{bmatrix} \Omega_i & (ZE + WU)^T G_i \\ * & -\gamma^2 I \end{bmatrix} Q_{i\perp} < 0 \] (14)

Where \( \Phi_i = A_i^T (ZE + WU) + (ZE + WU)^T A_i + C_{zi}^T C_{zi} - \varepsilon_i (ZE + WU)^T - \varepsilon_i (ZE + WU) \\
- \varepsilon_i^2 \sigma_i^{-1} (B_i B_i^T + \delta_i I)^{-1}, \Omega_i = A_i^T (ZE + WU) + (ZE + WU)^T A_i + C_{zi}^T C_{zi} \) for arbitrary \( i \in \mathcal{X} \), then the system (3) is admissible and has a desire H1 performance level via static output feedback.

**Proof:** Choose the Lyapunov function

\[ V(x(t)) = x^T(t)E^T X x(t) \] (15)

By Lemma 1, the system (3) is admissible and with norm bound \( \gamma \) if and only if the (4) and (5) hold. (5) Can be written as, by the Schur complement,

\[ \Psi_i + S_i^T K_i Q_i + Q_i^T K_i^T S_i < 0 \] (16)

\[ \Psi_i = \begin{bmatrix} A_i^T X + X^T A_i + C_{zi}^T C_{zi} & X^T G_i \\ G_i^T X & -\gamma^2 I \end{bmatrix}, S_i^T = \begin{bmatrix} X^T B_i \\ 0 \end{bmatrix}, Q_i = \begin{bmatrix} C_{zi} & 0 \end{bmatrix}. \] By Lemma 2, (16) is turned into

\[ S_{i\perp}^T \Psi_i S_{i\perp} < 0, \] (17)

\[ Q_{i\perp}^T \Psi_i Q_{i\perp} < 0. \] (18)

Where \( S_{i\perp} \) and \( Q_{i\perp} \) are bases of the null spaces of \( S_i \) and \( Q_i \) respectively. Obviously, (18) is a linear matrix inequality and (17) is not linear due to containing unknown Lyapunov matrix \( X \). According to Lemma 3 and the Schur complement, (17) is equivalent

\[ \begin{bmatrix} \Theta_i & X^T G_i \\ G_i^T X & -\gamma^2 I \end{bmatrix} < 0 \] (19)

Where \( \Theta_i = A_i^T X + X^T A_i + C_{zi}^T C_{zi} + \sigma_i X^T B_i B_i^T X \). Owning to \( B_i B_i^T \geq 0 \), we must find a \( \delta_i > 0 \) to satisfy \( B_i B_i^T + \delta_i I > 0 \). Then the (19) can be rewritten in the form

\[ \begin{bmatrix} \Pi_i & X^T G_i \\ G_i^T X & -\gamma^2 I \end{bmatrix} < 0 \] (20)

Where \( \Pi_i = A_i^T X + X^T A_i + C_{zi}^T C_{zi} + \sigma_i X^T (B_i B_i^T + \delta_i I) X - \sigma_i \delta_i X^T X \). According to

\[ \left[ \varepsilon_i X + \sigma_i^{-1} (B_i B_i^T + \delta_i I)^{-1} \right]^T \sigma_i (B_i B_i^T + \delta_i I) \left[ \varepsilon_i X + \sigma_i^{-1} (B_i B_i^T + \delta_i I)^{-1} \right] \leq 0, \] one has
\(e_i X^T \sigma_i (B_i B_i^T + \delta_i I) e_i X \leq -\varepsilon_i X^T - \varepsilon_i X - \sigma_i^{-1} (B_i B_i^T + \delta_i I)^{-1}.\) By the Schur complement, one deduces that (20) holds if the following inequality holds, where

\[
\Xi_i = A_i^T X + X^T A_i + C_{zi} C_{zi} - e_i X^T - e_i X - \sigma_i^{-1} e_i^2 (B_i B_i^T + \delta_i I)^{-1}.
\]

Noting that (4) is not strict inequality. By making use of the method proposed in [11], let \(X = ZE + WU\) with \(W \in R^{\alpha(n-r)}\) and \(\text{rank}(W) = n - r\) satisfying \(E^T W = 0.\) (4) And (5) are equivalent to (13) and (14)

\[
\begin{bmatrix}
\Xi_i & X^T G_i & X^T \\
G_i^T X & -\gamma^2 I & 0 \\
X & 0 & \sigma_i^{-1} \delta_i^{-1} I
\end{bmatrix} < 0
\]

(21)

5. Minimum \(H_\infty\) Norm Bound and Inconsistent State Jumps

Firstly, we minimize the \(H_\infty\) norm bound through the following optimization problem.

**Theorem 2.** If the following optimization problem with variable \(Z > 0 \in R^{\alpha n}, U \in R^{(n-r)\alpha n},\)

\(W \in R^{\alpha(n-r)}\) of full-column rank satisfying \(E^T W = 0\) and scalars \(\delta_i < 0,\) \(\bar{\nu}\) has solutions

\[
\min_{Z,U,\sigma_i,\bar{\nu}} \bar{\nu}
\]

s.t. \((i)(13)\)

\((ii)(14)\)

Where \(\bar{\nu} = \gamma^2,\) then the system (4) is admissible and has small norm bound.

Matrix \(X\) can be solved from the above optimization problem. Substituting \(X\) into (16), (16) can be turned into a linear matrix inequality with respect to only \(K_i.\)

In the following, we try to reduce the jump strength as small as possible, which can be carried out by minimizing the Fresenius norm of \(\Gamma_i\) since \(x(t_k)\) and \(\omega(t_k)\) are unknown at arbitrary switching point \(t_k.\) The Fresenius norm of \(\Gamma_i\) denoted by \(||\Gamma_i||\), is defined as \(||\Gamma_i|| = \left(\text{trace}(\Gamma_i^T \Gamma_i)\right)^{\frac{1}{2}}\). The minimization problem of \(||\Gamma_i||^2\) will be transformed into the following optimization problem. \(\Gamma_i\) In (12) can be rewritten as

\[
\Gamma_i = N \begin{bmatrix}
0 & 0 \\
0 & -A_{i2} \end{bmatrix} \begin{bmatrix}
0 & A_{i2} \\
A_{i2} & G_{i2}
\end{bmatrix} \begin{bmatrix}
N^{-1} & 0 \\
0 & I
\end{bmatrix} N^{-1} = N \begin{bmatrix}
0 & 0 \\
0 & -A_{i2}
\end{bmatrix} \begin{bmatrix}
A_{i2} & 0 \\
A_{i2} & G_{i2}
\end{bmatrix} N^{-1} G_{i2}
\]

(23)

Let \(\Lambda_i = \begin{bmatrix}
A_{i2} & 0 \\
A_{i2} & G_{i2}
\end{bmatrix} N^{-1} G_{i2},\) then \(||\Gamma_i|| \leq ||N|| \cdot ||0 & 0 \begin{bmatrix}
0 & 0 \\
0 & -A_{i2}
\end{bmatrix} \| ||\Lambda_i|| = ||N|| \cdot ||A_{i2}^{-1}|| ||\Lambda_i||\)

Then the minimization problem of \(||\Gamma_i||^2\) can be converted into an optimization problem below.
Theorem 3. If the optimization problem with respect to $K_i$, $F_i$, $\Upsilon_i$ has solutions, for arbitrary $i, j \in \mathcal{X}$, then the system (3) is admissible and has smaller state jump strength via $u = K_i y$.

\[
\min_{K_i, F_i, \Upsilon_i} \text{trace}(F_i) + \text{trace}(\Upsilon_i)
\]

\[\begin{align*}
(i) & \quad \left[ -\Upsilon_i \quad \Lambda_i^T \right] < 0 \\
(ii) & \quad \left[ -F_i \quad \bar{A}_{i22}^T \right] < 0 \\
(iii) & \quad (18)
\end{align*}\]

Proof: Actually, the condition (i) is equivalent to $\Lambda_i^T \Lambda_i < \Upsilon_i$. Thus, minimizing $\text{trace}(\Upsilon_i)$ can ensure the minimization of $\text{trace}(\Lambda_i^T \Lambda_i)$. And according to the above analysis and Theorem 1, the proof is easily completed and omitted.

6. Example
Consider the singular switched system composed of two subsystems

\[
\begin{align*}
E \dot{x}(t) &= A_i x(t) + B_i u(t) + G_i w(t) \\
z(t) &= C_{zi} x(t) \\
y(t) &= C_{yi} x(t)
\end{align*}
\]

where

\[
E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 1.6 \\ 0 & -2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5 & -3 \\ 0 & -6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -10 \\ -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad C_{z1} = [-3.5 \ -0.8],
\]

\[
C_{z2} = [-1 \ -2.2], \quad C_1 = [-1 \ 0], \quad C_2 = [0 \ 2], \quad G_1 = \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1.4 \\ 0 \end{bmatrix}.
\]

Choose $\delta_1 = 1/2$, $\delta_2 = 1/3$, satisfy $B_1 B_1^T + \delta_1 I > 0$, $B_2 B_2^T + \delta_2 I > 0$, $\epsilon_1 = 10$, $\epsilon_2 = 8$, $W = [0 \ 1]^T$.

By (22), a minimal $H_\infty$ performance level can be solved $\Upsilon = 1.4998$ and the following matrices and scalars can be solve $\sigma_1^{-1} = -0.6186$, $\sigma_2^{-1} = -1.1653$, $Z = \begin{bmatrix} 1.400 & -0.0000 \\ -0.0000 & 265.36 \end{bmatrix}$.

\[
U = \begin{bmatrix} 1.0583 & 18.8946 \end{bmatrix}. \quad \text{By simply calculation, we get Lyapunov matrix } X = \begin{bmatrix} 2.8851 & 0 \\ 1.0583 & 18.8946 \end{bmatrix}.
\]

By (24), static output feedback control gain can be solved $K_1 = -1.9567$, $K_2 = -0.2857$.

Now we check the performance for the obtained controller. With the designed controller, the system (25) is admissible. Let the switching interval be 0.5s and the switching sequence be $\{1, 2, 1, 2, \cdots\}$. When the system 1 is activated at $t_k = 0, 1, 2, \cdots$, one gets $\|e(t_k)\| \leq 1.8575 \cdot \|x(t_k)\|$; when the subsystem 2 is activated at $t_k = 0.5, 1.5, 2.5, \cdots$, one gets $\|e(t_k)\| \leq 0.3844 \cdot \|x(t_k)\|$. 

6
7. Conclusion
In the study, we investigates the admissible control and minimum of $H_\infty$ norm bound and state jumps for switched singular systems with exogenous disturbance by designing static output feedback controllers. These can be solved through two convex optimization problems by means of special matrix inequality transformation. Some sufficient conditions are obtained under which the closed-loop system can be admissible and has minimum $H_\infty$ norm bound and state jumps at switching instants.

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References
[1] J. X. Xi, H. Liu, Z. C. Yao, X. G. Yang, G. B. Liu, Distributed admissible consensus control for singular swarm systems with switching topologies, International Journal of Robust and Nonlinear Control, 25 (2015) 2816 - 2828.
[2] X. N. Song, Z. Wang, H. Shen, F. Li, B. Chen, J. W. Lu, A unified method to energy-to-peak filter design for networked Markov switched singular systems over a finite-time interval, Journal of the Franklin Institute, 354 (2017) 7899 – 7916.
[3] X. Liu, S. M. Zhong, X. Y. Ding, A Razumikhin approach to exponential admissibility of switched descriptor delayed systems, Applied Mathematical Modelling, 38 (2014) 1647 - 1659.
[4] X. Q. Xiao, J. H. Park, L. Zhou, G. P. Lu, New Results on Stability Analysis of Markovian Switching Singular Systems, IEEE Transactions on Automatic Control, 64(2018) 2084 – 2091.
[5] J. M. Zhao, L. J. Zhang and X. Qi, A Necessary and Sufficient Condition for Stabilization of Switched Descriptor Time-Delay Systems Under Arbitrary Switching, Asian Journal of Control, 18 (2016) 266 - 272.
[6] M. P. Xing, J. W. Xia, X. Huang, H. Shen, On dissipativity-based filtering for discrete-time Switched singular systems with sensor failures: a persistent dwell-time scheme, IET Control Theory & Applications, 13(2019) 1814 – 1822.
[7] Y. J. Yin, J. Zhao and Y. Z. Liu, H-infinity control for switched and impulsive singular systems, J. Control Theory Appl., 6 (2008) 86 - 92.
[8] D. Liberzon and S. Trenn, On stability of linear switched differential algebraic equations, Proceedings of the 48th IEEE Conference on Decision and Control, 2009, pp.2156 - 2161.
[9] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda, $H_\infty$ control for descriptor systems:a matrix in-equalities approach, Automatica, 33 (1997) 669 - 673.
[10] K. C. Goh, M. G. Safonov and G. P. Papavassilopoulos, A global optimization approach for the B BMI problem, in Proc. 33rd IEEE Conf. Decision and Control, Lake Buena Vista, FL, Dec., 1994, pp.2009 - 2014.
[11] L. Q. Zhang, B. Huang and J. Lam, LMI synthesis of $H_2$ and Mixed $H_2/H_\infty$ controllers for singular systems, IEEE Trans. Circuits Syst. II, Analog Digit.Signal Process, 50 (2003) 615 - 626.