DEFORMED GRAPHICAL ZONOTOPAL ALGEBRAS

BORIS SHAPIRO, ILYA SMIRNOV, AND ARKADY VAINTRUB

Abstract. We study certain filtered deformations of the external zonotopal algebra of a given graph parametrized by univariate polynomials. We establish some general properties of these algebras, compute their Hilbert series for a number of graphs using Macaulay2, and formulate several conjectures.

1. Introduction

Let $G$ be a finite undirected graph. Wagner [11] and, independently, Postnikov and the first author [7] introduced a commutative graded algebra $\mathcal{C}_G$ whose dimension is equal to the number of spanning forests of $G$. They also showed that the Hilbert series of $\mathcal{C}_G$ is a specialization of the Tutte polynomial of $G$ which enumerates the spanning forests of $G$ according to their external activity. Wagner’s initial goal was to construct new algebraic invariants of graphs. Postnikov and the first author were motivated by the earlier work [9, 8], where it was shown that for the complete graph $G$, the algebra $\mathcal{C}_G$ is isomorphic to the algebra generated by the curvature forms of tautological Hermitian line bundles on the complete flag manifold. Soon it turned out that these algebras are connected to several other areas, such as the theory of power ideals, box splines, enumeration of lattice points, chip firing, etc. They have been studied under various names: circulation algebras, Postnikov-Shapiro algebras, forest-counting algebras, and (external) zonotopal algebras. We will use the latter term reflecting their connection with enumeration of lattice points in zonotopes (see e.g. [3]).

Wagner [10] and Nenashev [4] proved that the algebra $\mathcal{C}_G$ determines the graphical matroid of $G$. However, non-isomorphic graphs can have isomorphic algebras. In [6], Nenashev and the first author introduced a filtered algebra $\mathcal{K}_G$ which they called a K-theoretic analogue of $\mathcal{C}_G$. They showed that $\mathcal{K}_G$ and $\mathcal{C}_G$ are isomorphic as (non-filtered) algebras, but, unlike $\mathcal{C}_G$, the filtered algebra $\mathcal{K}_G$ is a complete invariant of $G$.

The algebra $\mathcal{K}_G$ is a deformation of $\mathcal{C}_G$ in the class of filtered algebras. It is a member of a larger family of filtered deformations $\mathcal{C}_G^f$ of $\mathcal{C}_G$ parametrized by polynomials $f \in k[u]$ which was introduced in [6]. In the current work, we...
begin to study this family of algebras trying to understand their relationship to each other and to the graph $G$.

We begin Section 2 with a review of the definition and properties of the graded algebra $\mathcal{C}_G$. Then in Section 2.2 we turn to the deformed algebras $\mathcal{C}^f_G$ and establish some general facts about them. In particular, we prove that under some mild nondegeneracy assumption on $f$, the algebra $\mathcal{C}^f_G$ is isomorphic to $\mathcal{C}_G$ as an unfiltered algebra. We introduce a natural stratification of the space of such algebras for a given graph $G$. Section 3 discusses several properties of deformed zonotopal algebras and the latter stratification. Unfortunately, unlike the graded case, we cannot explicitly find the Hilbert sequences of these algebras. In Section 4 we collect our computations of the Hilbert sequences for several examples performed in Macaulay 2. Finally, in Section 5 we present a number of conjectures and questions for further study.

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2. Preliminaries

2.1. Graded zonotopal algebras. By a graph we understand a finite undirected multigraph $G = (V, E)$, possibly with loops, with a vertex set $V$ and a (multi)set of edges $E$.

Graphs form a category $\mathcal{G}$ with morphisms $G = (V, E) \to G' = (V', E')$ defined as maps of pairs

$$\begin{align*}
(\gamma, \lambda) & : (V, E) \to (V', E'),
\end{align*}$$

injective on edges and preserving incidences between vertices and edges (i.e. if $e \in E$ is an edge in $G$ connecting $u$ and $v$, then $\lambda(e)$ is an edge in $G'$ connecting $\gamma(u)$ and $\gamma(v)$).

There are two special kinds of graph morphisms, \textit{edge deletions} $j_e : G - e \to G$ and \textit{edge contractions} $\pi_e : G \to G_e$, where $e \in E$ is an edge of $G$, $G - e$ is the subgraph of $G$ obtained by removing $e$ from $E$, and $G_e$ is the graph obtained from $G$ by identifying the endpoints $u$ and $v$ of $e$ (and thus creating a loop for each edge connecting $u$ and $v$).

All algebras in this paper are commutative unital algebras over a fixed field $k$ of characteristic 0.

\textbf{Definition 2.1.} Given a graph $G = (V, E)$, its edge algebra $\Phi_G$ is the quotient of the polynomial algebra in edge variables $\phi_e, e \in E$, by their squares,

$$\begin{align*}
\Phi_G & := k[\phi_e : e \in E]/(\phi_e^2).
\end{align*}$$
The edge algebra is a local algebra of dimension $2^{|E|}$ isomorphic to the tensor product of $|E|$ copies of the algebra of dual numbers $k[\varepsilon]/(\varepsilon^2)$, namely $\Phi_G \cong \bigotimes_{e \in E} \left( k[\phi_e]/(\phi_e^2) \right)$. It inherits a standard grading from the polynomial algebra. A basis of the $k$th graded component is given by square-free words of length $k$ in edge variables $\phi_e$.

Observe that this construction is functorial. Every graph morphism $(2.1)$ induces a natural homomorphism of algebras $\lambda^*: \Phi_{G'} \to \Phi_G$, defined on generators $\phi_{e'} \in \Phi_{G'}$ as

\begin{equation}
\lambda^*(\phi_{e'}) := \sum_{e \in \lambda^{-1}(e')} \phi_e,
\end{equation}

e.g., $\lambda^*(\phi_{\lambda(e)}) = \phi_e$ and $\lambda^*(\phi_{e'}) = 0$, if $e' \in E'$ is not in the image of $\lambda$.

**Definition 2.2.** Let $G = (V, E)$ be a graph with a linear order $<$ on its vertex set $V$. The zonotopal algebra of $G$ is the subalgebra $C_G$ of the edge algebra $\Phi_G$ generated by the elements:

\begin{equation}
X_v = \sum_{e \in G} c_{v,e} \phi_e, \quad v \in V,
\end{equation}

called vertex flows where

\begin{equation}
c_{v,e} = \begin{cases} 
1 & \text{if } e = \{v, u\}, \ v < u, \\
-1 & \text{if } e = \{v, u\}, \ v > u, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

**Remark 2.3.** Even though the coefficients $c_{v,e}$ in (2.5), and thus the elements (2.4), depend on the chosen ordering $<$ of the vertex set $V$, the subalgebras obtained from two different orderings will be identified under a graded automorphism of the algebra $\Phi_G$ given by changing signs of some generators $\phi_e$. It is clear that loop edges of $G$ do not contribute to the flow generators $X_v$. For this reason, in earlier papers [7, 4, 6] on this topic, graphs with loops were not considered. In this work we allow loops because they naturally appear when we consider graph homomorphisms involving edge contractions.

One of the important properties of algebra $C_G$ is that it is functorial with respect to graph morphisms (2.1).

**Proposition 2.4.** For every graph morphism $(\gamma, \lambda): G \to G'$, the homomorphism $\lambda^*$ (2.3) sends the subalgebra $C_{G'} \subset \Phi_{G'}$ to $C_G \subset \Phi_G$.

**Proof.** All we need to show is that for every vertex $v' \in V'$, the image of the generator $X_{v'}$ of $C_{G'}$ under the homomorphism $\lambda^*$ belongs to $C_G \subset \Phi_G$. Indeed, if $v' \notin \gamma(V)$ then $\lambda(E)$ contains no edges incident to $v'$, and so $\lambda^*(X_{v'}) = 0$. If $v' \in \gamma(V)$, then, because of the injectivity of $\lambda$, we have $\lambda^*(X_{v'}) = \sum_{v \in \gamma^{-1}(v')} X_v \in C_G$. \qed
Remark 2.5. (1) The definition of $C_G$ involves choosing a linear order $<$ on $V$. In the above proof we assume that the orders $<$ and $<'$ on $V$ and $V'$ resp. are compatible, i.e. they are chosen in such a way that the map $\gamma: V \to V'$ is monotone.

(2) Since the loop edges of $G$ do not contribute to the generators $X_v$ of the algebra $C_G$, we can remove loops from $G$ without affecting $C_G$. In particular, let $G/e$ be the graph obtained from $G$ by contracting an edge $e \in E$ by removing $e$ (without creating a new loop) and identifying the endpoints $u$ and $v$ of $e$. Then the contracting morphism $\pi_e: G \to G/e$ gives an injective homomorphism

$$\pi_e: C_{G/e} = C_{G-e} \to C_G$$

which sends the generator $X_w$ corresponding to the new vertex $w = \{u, v\}$ to $X_v + X_u \in C_G$.

As was proved in [10, 8], the algebra $C_G$ is a power algebra, i.e. it is isomorphic to a quotient of the polynomial algebra by an ideal generated by powers of linear forms. Namely, for a subset $I \subset V$ of vertices denote by $D_I$ the number of edges $e \in E$ connecting a vertex from $I$ with one in the complementary subset $I^c = V - I$.

Theorem 2.6 ([10, 8]). The zonotopal algebra $C_G$ is isomorphic to the quotient of the polynomial algebra $k[x_v : v \in V]$ by the ideal $(p_I)_{I \subseteq V}$ generated by the polynomials

$$p_I = \left( \sum_{v \in I} x_v \right)^{D_I+1}.$$  

The dimension and the Hilbert function of the algebra $C_G$ were also found in [10, 8].

Theorem 2.7. (i) The dimension of the algebra $C_G$ is equal to the number of spanning subforests in $G$ (which is the same as the number of acyclic subsets of edges $S \subseteq E$).

(ii) The dimension of the $k$th graded component of $C_G$ is equal to the number of subforests $S \subseteq E$ with the external activity $|E - S| - k$.

Proof. Functoriality of the algebra $C_G$ with respect to graph homomorphisms leads to simple proofs of Theorems 2.6 and 2.7.

Indeed, let $e \in E$ be a non-loop edge of a graph $G$. Consider two algebra homomorphisms, a projection $j_e^*: \mathcal{G}_G \to \mathcal{G}_{G-e}$ (which corresponds to sending $\phi_e \in \Phi_G$ to 0) and an embedding $\pi_e^*: \mathcal{G}_{G/e} = \mathcal{G}_{G-e} \to C_G$ (which maps $X_w$ to $X_u + X_v$).

Denote by $\delta_e := \frac{\partial}{\partial \phi_e}: \Phi_G \to \Phi_{G-e}$, the partial derivative with respect to the edge variable $\phi_e \in \Phi_G$. Modulo $(\phi_e)$ the map $\delta_e$ is indeed a derivation of the edge algebra $\Phi_G$ which sends the subalgebra $C_G$ onto $C_{G-e}$ and generates
an exact sequence of graded spaces

\[ 0 \to \mathcal{E}_{G/e} \to \mathcal{E}_G \to \mathcal{E}_{G-e}[1] \to 0, \]

where the rightmost map (induced by \( \delta \)) decreases the grading by 1. This exact sequence implies the relation

\[ (2.8) \quad H_{\mathcal{E}_G}(t) = H_{\mathcal{E}/e}(t) + tH_{\mathcal{E}_{G-e}}(t), \]

for the Hilbert series which proves Theorem 2.7 by induction on \(|E|\).

To settle Theorem 2.6 we can use similar maps and an exact sequence

\[ 0 \to \mathcal{B}_{G/e} \to \mathcal{B}_G \to \mathcal{B}_{G-e}[1] \to 0, \]

for quotient algebras \( \mathcal{B}_G := k[x_v : v \in V]/(p_I)_{I \subseteq V} \), where the polynomial \( p_I \) is given by (2.7). \( \square \)

In [5] G. Nenashev has shown that \( \mathcal{E}_G \) contains all information about the graphical matroid of \( G \) and only it.

**Proposition 2.8** (Theorem 5 of [5]). Given two undirected (multi)graphs \( G_1 \) and \( G_2 \), algebras \( \mathcal{E}_{G_1} \) and \( \mathcal{E}_{G_2} \) are isomorphic if and only if the graphical matroids of \( G_1 \) and \( G_2 \) coincide. (The latter isomorphism can be thought of either as graded or as non-graded, the statement holds in both cases.)

### 2.2. Deformed algebras \( \mathcal{E}^f_G \).

#### 2.2.1. Definition. The main object of our study is a certain family of filtered algebras, introduced in [6], which we call **deformed zonotopal algebras**.

**Definition 2.9.** A formal power series \( f = a_0 + a_1 u + a_2 u^2 + \ldots \in k[[u]] \) is called **nondegenerate** if \( a_1 \neq 0 \), i.e. when \( f'(0) \neq 0 \).

**Definition 2.10.** For a graph \( G = (V,E) \) and a nondegenerate power series \( f \in k[[u]] \), we define the **deformed zonotopal algebra** of \( G \) associated to \( f \) as the subalgebra of the edge algebra \( \Phi_G \) generated by the elements

\[ Y_v := f(X_v) = f(\sum_{e \in E} c_{v,e} \phi_e), \quad v \in V, \]

where \( c_{v,e} \) are given by (2.5).

In particular, for \( f = u \), the algebra \( \mathcal{E}^f_G \) is the usual zonotopal algebra \( \mathcal{E}_G \) discussed above and for \( f = e^u \), this algebra coincides with the \( K \)-theoretic analog \( \mathcal{X}_G \) of \( \mathcal{E}_G \) studied in [6].

**Remark 2.11.** Since \( \phi_e^2 = 0 \), the element \( X_v \) is nilpotent with \( X_v^n = 0 \) for \( n > \deg v \), and so plugging it into a power series is well-defined. Moreover, this argument also shows that the terms of \( f \) of degree higher than

\[ \text{md}_G := \max_{v \in V} \deg v, \]
the maximal degree of a vertex in \( G \), do not affect any of the generators \( Y_v \) of \( \mathcal{C}_G^f \). Therefore, we can restrict our attention to those \( f \in k[[u]] \) which are polynomials in \( u \) of degree at most \( m \deg G \).

The algebra \( \mathcal{C}_G^f \) is endowed with an increasing filtration

\[(2.9) \quad k = \mathcal{C}_G^{f,0} \subset \mathcal{C}_G^{f,1} \subset \mathcal{C}_G^{f,2} \subset \ldots ,\]

where the subspace \( \mathcal{C}_G^{f,k} \) is spanned by the monomials of degree at most \( k \) in the generators \( Y_v = f(X_v) \).

**Remark 2.12.** Notice that neither the algebra \( \mathcal{C}_G^f \) nor this filtration depend on the constant term \( a_0 = f(0) \) of \( f \), since changing \( f \) by a constant modifies \( Y_v \) by this constant which is an element of \( k = \mathcal{C}_G^{f,0} \). It is also clear that multiplying \( f \) by a nonzero constant does not change the filtration (2.9).

For this reason, from now on, we will assume that \( f \) has no constant term i.e.

\[ f - u^k \in u^{k+1}k[[u]] \text{ for } k \geq 1. \]

When \( f = u \), this filtration coincides with the filtration induced by the grading on \( \mathcal{C}_G \). Let us now present some basic properties of \( \mathcal{C}_G^f \) proven in [6].

**Proposition 2.13 ([6, Proposition 2]).** If \( f \) is a nondegenerate series, then the algebra \( \mathcal{C}_G^f \) coincides, as a subalgebras of \( \Phi_G \), with the usual zonotopal algebra \( \mathcal{C}_G \). In other words, in this case the only difference between the graded algebra \( \mathcal{C}_G \) and \( \mathcal{C}_G^f \) is in their filtrations.

**Proof.** Firstly, by Remark 2.11 we can assume that \( f \) is a polynomial. Secondly, if \( f \) is a polynomial we have the inclusion \( \mathcal{C}_G^f \subseteq \mathcal{C}_G \) because \( f(X_i) \in \mathcal{C}_G \) for every \( i \). Finally, there exists a polynomial \( g(u) \) such that \( X_i = g(f(X_i)) \) for every \( i \) which finishes the proof. Indeed, the formal power series \( g(u) = \sum_{i \geq 1} a_i u^i \) can be found by requiring that

\[ u = f(\sum_{i \geq 1} a_i u^i) = c_1 a_1 u + (c_1 a_2 + c_2 a_1^2) u^2 + (c_1 a_3 + 2c_2 a_1 a_2 + c_3 a_1^3) u^3 + \cdots . \]

Under the assumption that \( c_1 \neq 0 \), this system of equations can solved for each \( a_j \) consecutively by induction. The resulting power series \( g(u) \) can be truncated to a polynomial in view of Remark 2.11.

This proposition together and the above remark explain why we focus our attention on (nondegenerate) polynomials \( f = u + \ldots \).

**Theorem 2.14 ([6, Theorem 6]).** Let \( f \) be a polynomial with non-vanishing linear and quadratic terms and let \( G_1 \) and \( G_2 \) be two simple graphs without isolated vertices. Then \( \mathcal{C}_G^{f_1} \) and \( \mathcal{C}_G^{f_2} \) are isomorphic as filtered algebras if and only if the graphs \( G_1 \) and \( G_2 \) are isomorphic.
Definition 2.15. Given a graph $G$, we will call the affine space
\begin{equation}
A_G = \{ f \in k[u] \mid f(0) = 0, f'(0) = 1, \deg f \leq \text{md}_G \}
\end{equation}
the space of parameters of deformed zonotopal algebras of $G$.

For $f \in A_G$, we will be interested in the Hilbert sequence
\[ \mathcal{H}_G^f := (\dim_k \mathcal{C}_G^f / \mathcal{C}_G^{f-j})_{j \geq 0} \]
of the filtered algebra $\mathcal{C}_G^f$ (where, by convention, $\mathcal{C}_G^{f-j} = 0$).

Proposition 2.16 ([6, Proposition 3]; see also Theorem 3.8.). There exists a non-empty Zariski open subset $U \subset A_G$ such that the Hilbert sequences $\mathcal{H}_G^f$, for all $f \in U$, are the same and maximal among all possible Hilbert sequences $\mathcal{H}_G^f$, $f \in A_G$ in the lexicographic order.

We will call the above maximal Hilbert sequence the general Hilbert sequence and will denote it simply by $\mathcal{H}_G$.

3. Algebraic properties of deformed zonotopal algebras

3.1. Generators and relations.

Proposition 3.1. Let $A$ be a finite dimensional local algebra over $k$ with maximal ideal $m$ and with a set of algebra generators $x_1, \ldots, x_n \in m$. Let $f \in k[[u]]$ be a nondegenerate series with $f(0) = 0$. Then

1. the map $f : A \to A$, $a \mapsto f(a)$ is well-defined and invertible;
2. the elements $y_1 = f(x_1), \ldots, y_n = f(x_n)$ generate $A$;
3. for $L \in k[t_1, \ldots, t_n]$, the relation $L(x_1, \ldots, x_n) = 0$ holds in $A$ if and only if the relation $L(f^{-1}(y_1), \ldots, f^{-1}(y_n)) = 0$ holds in $A$.

Proof. To settle (1) notice that since $A$ is an Artinian algebra, any $a \in A$ is nilpotent which implies that $f(a)$ is a finite sum. Therefore $f(a)$ is well-defined for any $a$. The inverse of $f$ has been already constructed in the proof of Proposition 2.13.

Now, (2) is obvious since $f$ is invertible as a map of $A$ and therefore $x_i = f^{-1}(y_i)$ for every $i$. The first part of (3), claiming that the relation
\[ L(f^{-1}(y_1), \ldots, f^{-1}(y_n)) = 0 \]
holds in $A$, is obvious. Conversely, assume that a relation $R(y_1, \ldots, y_n) = 0$ holds in $A$. Then the equation $R(f(x_1), \ldots, f(x_n)) = 0$ gives a relation in $A$ in terms of the original generating set $x_1, \ldots, x_n$ which then satisfies the claim because of invertibility of $f$. \hfill \Box

Corollary 3.2. For a graph $G = (V, E)$ and a non-degenerate $f \in k[[u]]$, we have the isomorphism of $k$-algebras $\mathcal{C}_G^f \cong k[y_v : v \in V] / I_G^f$; the latter is the quotient of the polynomial algebra by the ideal $I_G^f$ generated by two sets $\{ y_v^{\deg f+1} \mid v \in V \}$ and $\{ (\sum_{v \in I} f^{-1}(y_v))_{D_I+1} \mid I \subset V, |I| \geq 2 \}$ where $D_I$ has been defined before Theorem 2.6.
Proof. As we have already seen, $X_v^{|v|+1} = 0$ which implies that $Y_v^{-|v|} = (f(X_v))^{-|v|+1} = 0$ by plugging directly into the formula for $f(u)$. By Proposition 3.1 and Theorem 2.6, $(\sum_{v \in I} f^{-1}(Y_v))^{D_{I^{\pm 1}}} = 0$ are still relations in $\mathcal{C}_G$. Thus by mapping $y_i \mapsto Y_i$ we see that $\mathcal{C}_{G_f}$ is a homomorphic image of $k[y_v : v \in V]/I_{G_f}$. Due to invertibility of $f(u)$, there are no other relations (i.e., the kernel of the homomorphism is zero) as this would contradict Theorem 2.6.

In view of Proposition 2.13, we can view each $f$ as a certain choice of filtration on a fixed algebra $\mathcal{C}_G$. We can always recover the original algebra by taking the associated graded ring with respect to the distinguished ideal.

Corollary 3.3. Let $f$ be a nondegenerate polynomial and let $I$ be the ideal of the algebra $\mathcal{C}_G$ generated by the elements $Y_v = f(X_v)$, $v \in V$. Then the associated graded algebra $Gr_{J^i}(\mathcal{C}_G)$ for the $J$-adic filtration is isomorphic to the graded algebra $\mathcal{C}_G$. In particular, the Hilbert series of $\mathcal{C}_G$ and $\mathcal{C}_G$ coincide.

Proof. In the notation of Corollary 3.2, the associated graded algebra $\mathcal{C}_{G_f}$ is the quotient of $k[Y_v]$ by the initial ideal of $I_{G_f}$, i.e., the ideal generated by the lowest degree homogeneous forms of elements $f \in I_{G_f}$. Clearly, $Y_v^{-|v|}, v \in V$, are contained in the initial ideal and the expression $(\sum_{v \in I} f^{-1}(Y_v))^{D_{I^{\pm 1}}}$ will contribute the homogeneous form arising from the linear part of $f^{-1}$, i.e., $(\sum_{v \in I} Y_v)^{D_{I^{+1}}}$. These forms are the defining relations of the algebra $\mathcal{C}_G$ and there are no other relations because $\dim \text{gr}(\mathcal{C}_{G_f}) = \dim \mathcal{C}_{G_f} = \dim \mathcal{C}_G$ by Proposition 2.13 and properties of associated graded algebras. Thus the graded algebras $Gr_{J^i}(\mathcal{C}_{G_f})$ and $\mathcal{C}_G$ are isomorphic.

Several concrete examples of such relations can be found later in the text.

Remark 3.4. The relation $\sum_{v \in V} f^{-1}(Y_v) = 0$ corresponding to $I = V$ in Corollary 3.2 can be used to remove half of the generators of the ideal $I_{G_f}$. Namely, for a subset $I \subset V$ with $1 < |I| < |V|$, consider the complementary subset $I^c = V \setminus I$. If $|I| < |I^c|$, then keep the generator $(\sum_{v \in I} f^{-1}(Y_v))^{D_{I^{\pm 1}}}$ corresponding to $I$ and remove the one corresponding to $I^c$. If $|I^c| > |I|$, then keep the generator corresponding to $I^c$ and remove the one corresponding to $I$. Finally, if $|I| = |I^c|$ (which can happen only if $|V|$ is even), then keep any of these two generators and remove the other one. The generator corresponding to $I = V$ should not be removed.

3.2. Stratification of the space of deformed zonotopal algebras of a graph. Let $A_G$ be the affine space (2.10) of parameters of algebras $\mathcal{C}_{G_f}$ for a graph $G$. Our ambition is to study the stratification of $A_G$ according to the Hilbert sequences $H^i_G$ of $\mathcal{C}_{G_f}$. 


Since the generators $Y_v$, $v \in V$, of the algebra $C_f^G$ are nilpotent, the corresponding Hilbert sequence $\mathcal{H}_f^G$ has finitely many non-zero terms. We denote by $\overline{\mathcal{H}}_f^G$ the finite sequence obtained by removing all zero terms of $\mathcal{H}_f^G$.

**Definition 3.5.** Given a graph $G$ and a sequence $M = (1, m_1, m_2, \ldots, m_n)$ of positive integers, the *associated Hilbert stratum* is the subset $S^M_G$ of the parameter space $A_G$ consisting of all $f$ such that $\overline{\mathcal{H}}_f^G = M$.

We will show that each $S^M_G$ is a constructible algebraic set and we will be interested in the decomposition of its closure into irreducible closed algebraic components. (Observe that $S^M_G$ might be empty.) Let us describe the adjacency of these strata.

### 3.3. Semicontinuity

First recall the following notion.

**Definition 3.6.** Let $X$ be a topological space and let $(\Lambda, \prec)$ be a partially ordered set. We say that a function $g: X \to \Lambda$ is upper semicontinuous if for every $\lambda \in \Lambda$, the set $g^{-1}(\prec \lambda) := \{x \in X \mid g(x) \prec \lambda\}$ is open.

**Lemma 3.7.** Let $A$ be a commutative ring and $M$ be a matrix with entries in $A$. For a prime $p \in A$, let $M(p)$ be the matrix obtained from $M$ by replacing the entries by their images in $k(p)$. Then the real-valued function $p \mapsto \text{rk} \ M(p)$ (respectively, $p \mapsto \text{dim ker} \ M(p)$) is lower (resp., upper) semicontinuous.

**Proof.** We use the fact that non-vanishing of a minor is an open condition. Namely, if $B$ is a square matrix then $\det B(p) \neq 0$ if and only if $\det B \notin p$. Hence, for any $p$ such that $\text{rk} \ M(p) \geq r$, there is an open neighborhood where the same condition holds. \qed

**Theorem 3.8.** Let $A$ be a commutative ring and $R = A[X_1, \ldots, X_N]/I$ be a finite $A$-module. Then

1. the function $p \mapsto \text{dim}_{k(p)} R \otimes_A k(p)$ is upper semicontinuous on $\text{Spec} \, R$;
2. for any positive integer $m$, $H_m: p \mapsto \text{dim}_{k(p)} \langle X_1^{\alpha_1} \cdots X_N^{\alpha_N} \mid \alpha_1 + \cdots + \alpha_n \leq m \rangle$, where the latter is a submodule of $R \otimes_A k(p)$, is also a lower semicontinuous function on $\text{Spec} \, A$;
3. for $p \subset q$, we have the equality $H_m(p) = H_m(q)$ if and only if $\langle X_1^{\alpha_1} \cdots X_N^{\alpha_N} \mid \alpha_1 + \cdots + \alpha_n \leq m \rangle \otimes_A A/p_q$ is a free $\otimes_A A/p_q$-module.

**Proof.** Since $R$ is a finite $A$-module, it can be generated (as a module) by monomials of bounded degree. We fix such system of generators of $R$ to define its presentation as an $A$-module in the form:

$$A^\oplus m \to A^\oplus n \to R \to 0.$$
It is now clear that \( \dim_{k(p)} R \otimes_A k(p) = n - \text{rk}(p) \), so the first claim follows from Lemma 3.7.

For the second claim, we take \( V_k \) to be a free submodule of \( A^{\oplus n} \) corresponding to monomials of degree at most \( k \). Then

\[
\dim_{k(p)}(X_1^{\alpha_1} \cdots X_N^{\alpha_n} | \alpha_1 + \cdots + \alpha_n \leq k) = \text{rk}(V_k) - \dim_{k(p)} \text{Im}(M(p) \cap V_k(p)).
\]

By a standard linear algebra argument

\[
\dim_{k(p)}(\text{Im}(M(p)) \cap V_k(p)) = \dim_{k(p)} \ker[M | -V_k](p)
\]

which is an upper semicontinuous function by Lemma 3.7.

Last, observe that for any finite \( A \)-module \( M \), the equality \( \dim_{k(p)} M \otimes_A k(p) = \dim_{k(q)} M \otimes_A k(q) \) is equivalent to \( M \otimes_A A/p_q \) being \( A/p_q \)-free since the generic rank and the minimal number of generators of this module have to be equal. \( \Box \)

**Corollary 3.9.** Let \( A \) be a commutative ring and \( R = A[X_1, \ldots, X_N]/I \) be a finite \( A \)-module. Suppose that \( R \) is generated (as a module) by monomials of degree at most \( D \) and set \( \Lambda = \mathbb{Z}^{D+1} \) with the lexicographic order. In the notation of Theorem 3.8, define the function \( H : \text{Spec} \ A \to \Lambda, p \mapsto (H_0, H_1, \ldots, H_D) \). Then \( H \) is lower semicontinuous. Moreover, for any vector \( \lambda = (\lambda_0, \lambda_1, \ldots) \), the set

\[
H^{-1}(\lambda) = \{ p \in \text{Spec} \ A | H(p) = \lambda \}
\]

is the intersection of an open set \( H^{-1}(\succeq_{\text{lex}} \lambda) \) and a closed set \( H^{-1}(\preceq_{\text{lex}} \lambda) \).

In particular, the closure of \( H^{-1}(\lambda) \) coincides with \( H^{-1}(\succeq_{\text{lex}} \lambda) \).

**Proof.** For the next claim, we first note that we may stop at \( H_D \). Since each function \( H_k \) is discrete, the set \( H_k^{-1} (\succeq a) = H_k^{-1} (a - 1) \) is open. Thus if we set \( \lambda = (\lambda_0, \ldots, \lambda_D) \), we may decompose

\[
H^{-1}(\succeq_{\text{lex}} \lambda) = H_0^{-1}(\succeq \lambda_0) \cup H_0^{-1}(\succeq \lambda_0) \cap H_1^{-1}(\succeq \lambda_1) \cup \cdots \cup H_0^{-1}(\succeq \lambda_0) \cap \cdots \cap H_D^{-1}(\succeq \lambda_D)
\]

and see that \( H^{-1}(\succeq_{\text{lex}} \lambda) \) is open. Similarly, replacing in the last line \( H_D^{-1}(\succeq \lambda_D) \) with \( H_D^{-1}(\succeq \lambda_D) \) we see that \( H^{-1}(\preceq_{\text{lex}} \lambda) \) is also open. It remains to note that \( H^{-1}(\preceq_{\text{lex}} \lambda) \) is closed because it can be decomposed as:

\[
H_{-1}(\preceq_{\text{lex}} \lambda) = H_0^{-1}(\preceq \lambda_0) \cup H_0^{-1}(\preceq \lambda_0) \cap H_1^{-1}(\preceq \lambda_1) \cup \cdots \cup H_0^{-1}(\preceq \lambda_0) \cap \cdots \cap H_D^{-1}(\preceq \lambda_D - 1) \cap H_D^{-1}(\preceq \lambda_D).
\]

\( \Box \)

**Corollary 3.10.** The stratum \( S^M_G \) is constructive and, in the lexicographic partial order, we have \( \overline{S^M_G} = S_G^{\leq M} \).
Proof. By Remark 2.11, polynomial in $A_G$ is a specialization of the generic polynomial $f = u + T_2u^2 + \cdots + T_du^d$, where $d = \text{md}_G$ is the maximal degree of $G$. For this $f$, we define the algebra $R = \mathcal{C}_G^{[f]}$ as the quotient of $k[X_v, T_2, \ldots, T_d], v \in V$, by the relations from Corollary 3.2. Note that the obvious relation $f^{-1}(f(u)) = u$ gives explicit polynomial formulas for the coefficients of $f^{-1}$ in terms of $T_i$. Since $R$ is a finitely generated module over $A = k[T_2, \ldots, T_d]$, the theorem applies and we may use the fact that $A_G$ can be identified with the set of $k$-rational points of Spec $A$.

Next we introduce a natural $k^*$-action on $A_G$.

Lemma 3.11. For any graph $G$, the natural $k^*$-action on $A_G$ given by $f(u) \mapsto \frac{1}{\epsilon}f(\epsilon u)$ with $\epsilon \in k \setminus \{0\}$, preserves the Hilbert stratification. This action is free on $A_G \setminus \{u\}$ and the point $u \in A_G$ belongs to the closure of every Hilbert stratum.

Proof. We substitute $Y_v = X_v/\epsilon$ to get an isomorphism of algebras

$$k[X_v : v \in V]/I_G^{f(u)} \cong k[Y_v : v \in V]/I_G^{f(\epsilon u)}.$$ 

The second claim follows from the fact that if $f = u + \cdots \neq u$, then $f(u) \neq \frac{1}{\epsilon}f(\epsilon u)$ for $\epsilon \neq 1$. But, by Borel’s fixed point theorem, the closure of each stratum should contain a fixed point.

Denote by $P A_G = (A_G \setminus \{u\}) / k^*$ the weighted projective space obtained as the latter quotient. Taking the quotient we obtain the induced Hilbert stratification of $P A_G$ which we will be interested in. Consider the (finite) poset of Hilbert strata. This poset has the minimal element corresponding to $\{u\}$ and the maximal element corresponding to the stratum with a generic Hilbert sequence.

Corollary 3.12. The Hilbert sequence of the graded algebra $\mathcal{C}_G$ is minimal. The maximal length of the Hilbert sequence of a generalized zonotopal algebras in $A_G$ is attained for $f(u) = u$ and equals $|G|$ which is the total number of edges in $G$.

Example 3.13. For $G = K_4$, $P A_{K_4}$ is a weighted projective line (topologically $S^2$). Its Hilbert stratification consists of two points corresponding to $f(u) = u + u^2$ and $f(u) = u + u^3$ and a complex 1-dimensional stratum (coinciding with $S^2$ minus two points) which is the factor of $u + au^2 + cu^3$ with $b \neq 0$ and $c \neq 0$ mod the above $k^*$-action, see subsection 4.4.2 below.

3.4. Specialization. If $H$ is a subgraph of $G$ and $v$ is a vertex of $H$, then the embedding $H \hookrightarrow G$ sends the generator $X^G_v$ of $\mathcal{C}_G^f$ to $X^H_v$. Therefore, we obtain a surjective homomorphism $\pi_H : \mathcal{C}_G^f \to \mathcal{C}_H^f$.

We will now study the effect of this map on stratifications.

Lemma 3.14. Let $A$ be a commutative ring and $M$ be an $A$-module such that $M = M_1 \oplus M_2$. Suppose that $0 \neq N \subseteq M$ is a submodule that satisfies
a commutative diagram

\[
\begin{array}{ccc}
N & \xrightarrow{f} & N_2 \\
\downarrow & & \downarrow \\
M & \xrightarrow{\pi} & M_2
\end{array}
\]

for some $A$-module $N_2 \neq 0, N$. Then $N = N_1 \cap N \oplus M_2 \cap N$.

**Proof.** It is easy to check that the diagram implies that $\ker f = N \cap M_1 \neq 0$. Then the inclusion $N_2 \subseteq M_2$ given in the diagram implies that $N \subseteq N \cap M_1 + M_2$ as submodules of $M$. Since $N$ is a submodule, this forces the containment $N \subseteq N \cap M_1 + M_2$ and the claim easily follows. \qed

As in the proof of Corollary 3.10, we can define $\mathcal{C}_G^f(A)$ for any ground ring $A$ and $f \in A[u]$. As above, we have a surjective homomorphism of $A$-algebras, $\mathcal{C}_G^f(A) \to \mathcal{C}_H^f(A)$.

**Theorem 3.15.** Let $A$ be a commutative ring and $f \in A[u]$ such that $f \in (u) \setminus (u^2)$. Let $H \subset G$ be graphs. Then the natural projection map splits and allows to identify $\mathcal{C}_H^f(A)$ with a direct summand of $\mathcal{C}_G^f(A)$ in the category of filtered $A$-modules.

**Proof.** It suffices to assume that $H$ is obtained by removing a single edge. We can similarly define the algebras $\Phi_G(A), \Phi_H(A)$ as square-free algebras on the sets of edges with coefficients in $A$. It is clear from the relations that $\mathcal{C}_G^f(A) \subseteq \Phi_G(A)$.

We note that the map $\pi_H: \mathcal{C}_G^f(A) \to \mathcal{C}_H^f(A)$, sending $X_i^G \mapsto X_i^H$, is induced by the natural projection map $\pi_H: \Phi_G(A) \to \Phi_H(A)$. By the definition of $\Phi_G$ as a square-free algebra on the set of edges, the map $\pi_H$ splits, i.e., $\Phi_G(A) = \Phi_H(A) \oplus e\Phi_H(A)$. The functoriality of the definitions gives a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_G^f & \xrightarrow{\pi_H} & \mathcal{C}_H^f \\
\downarrow & & \downarrow \\
\Phi_G & \xrightarrow{\pi_H} & \Phi_H
\end{array}
\]

Therefore, Lemma 3.14 asserts that $\mathcal{C}_G^f(A) = e\Phi_H(A) \cap \mathcal{C}_G^f(A) \oplus \Phi_H(A) \cap \mathcal{C}_G^f(A)$. Using Lemma 3.14 again, we extend the splitting to the filtrations; note that $\pi_H$ respects the filtration, i.e.,

\[
\pi_H \left( \prod f(X_i)^{a_i} \mid \sum a_i \leq n \right) \subseteq \left( \prod (\pi_H f(X_i))^{a_i} \mid \sum a_i \leq n \right) \subseteq \mathcal{C}_H^f(A).
\]

\[
\boxdot
\]

**Theorem 3.16.** Suppose that $k$ is algebraically closed. Then the projection map $\pi_H$ induces the surjective map $A_G \to A_H$ which preserves the Hilbert
stratifications, i.e., the image of a connected component of a Hilbert stratum is still contained in one stratum.

Proof. As in the proof of Corollary 3.10 we will parametrize $A_G$ as the fibers of $A := k[T_2, \ldots, T_v] \to \mathcal{O}_G^f(A)$.

By Corollary 3.10, it suffices to show that if $p \in \text{Spec } A$ and $s \notin p$ are such that $V_k(p) \cap D(s)$ is contained in a single Hilbert stratum of $\mathcal{O}_G^f(A)$, then it is contained in a single Hilbert stratum of $\mathcal{O}_H^f(A)$. Here, $V_k(p)$ denotes the set of $k$-rational points containing $p$ and $D(s)$ denotes the distinguished open set of points not containing $s$. By the construction we can pass to $B := A_s/pA_s$ and consider the stratifications induced by $B$-algebras $\mathcal{O}_G^f(B)$ and its filtered direct summand $\mathcal{O}_H^f(B)$, see Theorem 3.15.

Because $k$ is algebraically closed, the set of $k$-rational points is dense, so the Hilbert stratification of $G$ is constant on the entire Spec $B$. Thus, by Theorem 3.8 and Corollary 3.10 the condition on the stratification can be restated as freeness of $\mathcal{O}_G^f(B) \otimes_B B_q$ (as a filtered $B_q$-module) for every $q$-rational point of Spec $B_q$. But a direct summand of a free module is projective and projective modules over a local ring are free. □

Remark 3.17. Like other similar results, Theorem 3.16 shows that when $k$ is not algebraically closed, it is better to consider the stratification of the entire space Spec $A$ rather than only of the set of its $k$-points.

Corollary 3.18. For any usual graph $G$ on $n$ vertices, the Hilbert stratification of $A_G$ is a coarsening of the Hilbert stratification of $A_{K_n}$, where $K_n$ denotes a complete graph on $n$ vertices.

4. Experimental results

In this section, we present the results of computations of the Hilbert sequences $\mathcal{H}_G^f$ of deformed zonotopal algebras $\mathcal{O}_G^f$ for various graphs $G$ and nondegenerate polynomials $f$ using Macaulay2 ([2]). Notice that, when $f = u$, i.e. in the graded case, the answer is provided by Theorem 2.7. In particular, if $G$ is a tree with $n$ edges, then the Hilbert series of $\mathcal{O}_G^f$ is equal to $(1 + t)^n$. For this reason, listing results of our computations below, we usually exclude the graded case $f = u$. Also we do not include in $f$ monomials of degree higher than $\text{md}_G$, since as explained in Remark 2.11, they do not affect $\mathcal{C}_G$.

4.1. Special families of graphs. In the tables below we present the Hilbert sequences for several families of graphs with $n = |V|$ vertices. Each row always starts with 1, and the second entry for all $f \neq u$ equals $n$.

Example 4.1. For the chain graph with $n \geq 3$ vertices, $A_n = \cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot$, besides the graded case, $f = u$, we only need to consider $f = u + u^2$. The corresponding Hilbert sequences for $3 \leq n \leq 17$, are listed below.
Conjectures:

- for $n \geq 5$, the 3rd entry equals $\frac{n^2 + n - 10}{2}$;
- for $n \geq 7$, the 4th entry equals $\frac{n^3 + 3n^2 - 46n + 30}{6}$;
- for $n \geq 9$, the 5th entry equals $\frac{n^4 + 6n^3 - 121n^2 + 42n + 1080}{24}$;
- for $n \geq 11$, the 6th entry equals $\frac{n^5 + 10n^4 - 245n^3 - 250n^2 + 9364n - 12000}{120}$;
- for $n \geq 13$, the 7th entry equals $\frac{n^6 + 15n^5 + 3325n^4 + 1785n^3 + 11874n^2 - 201960n + 93600}{6!}$;
- for $n = 2k$, the $(k+1)$st entry equals $3^{k-1}$.

Example 4.2. For the cycle graph with $n$ vertices, $\hat{A}_n = \begin{array}{c}
\vdots
\end{array}$

besides the graded case, $f = u$, we only need to consider $f = u + u^2$. The corresponding Hilbert sequences, for $3 \leq n \leq 13$ are listed below.
1 3 2 1
1 4 7 3
1 5 14 10 1
1 6 20 31 5
1 7 27 63 28 1
1 8 35 96 106 9
1 10 54 190 405 346 17
1 11 65 253 627 891 198 1
1 12 77 328 921 1620 1103 33
1 13 90 416 1300 2691 3159 520 1

Conjectures:
- for \( n \geq 5 \), the 3rd entry equals \( \frac{n^2 + (n-2)}{2} \).
- for \( n \geq 7 \), the 4th entry equals \( \frac{n^3 + 3n^2 - 16n}{6} \).

Example 4.3. For the graph \( D_n = \ldots \) with \( n \geq 4 \) vertices and \( f = u + u^3 \) (for \( f = u + u^2 \) and \( f = u + u^2 + u^3 \) the Hilbert sequences are the same as for the graph \( A_n \), see Example 4.1), we have the following Hilbert sequences

1 4 3
1 5 7 3
1 6 12 10 3
1 7 18 22 13 3
1 8 25 40 35 16 3
1 9 33 65 75 51 19 3
1 10 42 98 140 126 70 22 3
1 11 52 140 238 266 196 92 25 3

Conjecture. These numbers satisfy a Pascal-type recursion relation.

Example 4.4. For the graph \( G = \ldots \) with \( n \geq 5 \) vertices and \( f = u + u^2 \) (for \( f = u + u^3 \) the Hilbert sequences are the same as for the graph \( D_n \) from Example 4.3), we have the following Hilbert sequences

1 5 10
1 6 15 10
1 7 22 34
1 8 30 59 30
1 9 39 95 112
1 10 49 141 221 90
1 11 60 198 388 366

Example 4.5. For \( G = \ldots \) (\( A_{n-1} \) with a leg on the 4th place), with \( n \geq 7 \) vertices and \( f = u + u^2 \) (for \( f = u + u^3 \) and \( f = u + u^2 + u^3 \),
the results are the same as for the graphs $D_n$ from Example 4.3 and $A_n$, from Example 4.1, respectively), we have the Hilbert sequences

\[
\begin{array}{cccccc}
1 & 7 & 22 & 34 \\
1 & 8 & 30 & 62 & 27 \\
1 & 9 & 39 & 96 & 111 \\
1 & 10 & 49 & 142 & 229 & 81 \\
1 & 11 & 60 & 199 & 393 & 360 \\
\end{array}
\]

**Example 4.6.** For $G = \cdots$ (with a leg on 5th place) with $n \geq 6$ vertices and $f = u + u^2$ (for $f = u + u^3$ and $f(u) = u + u^2 + u^3$, the results are the same as for graphs $D_n$ and $A_n$, respectively), we have the Hilbert sequences

\[
\begin{array}{cccccc}
1 & 6 & 16 & 9 \\
1 & 7 & 23 & 33 \\
1 & 8 & 30 & 59 & 30 \\
1 & 9 & 39 & 96 & 111 \\
1 & 10 & 49 & 140 & 222 & 90 \\
1 & 11 & 60 & 197 & 392 & 363 \\
\end{array}
\]

**Example 4.7.** For graph $\hat{D}_n = \cdots$ and $f = u + u^3$ (for $f = u + u^2$ and $f = u + u^2 + u^3$, the results are the same as for $A_n$), we have the Hilbert sequences

\[
\begin{array}{cccccc}
1 & 5 & 7 & 3 \\
1 & 6 & 14 & 10 & 1 \\
1 & 7 & 22 & 25 & 9 \\
1 & 8 & 30 & 47 & 33 & 9 \\
1 & 9 & 39 & 77 & 79 & 42 & 9 \\
1 & 10 & 49 & 116 & 155 & 121 & 51 & 9 \\
1 & 11 & 60 & 165 & 270 & 276 & 172 & 60 & 9 \\
1 & 12 & 72 & 225 & 434 & 546 & 448 & 232 & 69 & 9 \\
\end{array}
\]

Conjecture. These numbers exhibit a Pascal-type behavior.

**Example 4.8.** For $G = \cdots$ and $f = u + u^3$ (for $f = u + u^2$ and $f = u + u^2 + u^3$ the answers are the same as in Example 4.4 and Example 4.1, respectively), we have the Hilbert sequences

\[
\begin{array}{cccccc}
1 & 7 & 20 & 24 & 11 & 1 \\
1 & 8 & 29 & 47 & 34 & 9 \\
1 & 9 & 38 & 77 & 80 & 42 & 9 \\
\end{array}
\]

**Example 4.9.** For $G = \cdots$ and $f = u + u^3$ (for $f = u + u^2$ and $f = u + u^2 + u^3$ the results are the same as in Example 4.5 and Example 4.1, respectively), we have the Hilbert sequences
Conjecture: Pascal-type behavior except for the entry 14.

**Example 4.10.** For $G = \cdots$ and $f = u + u^3$, we have the Hilbert sequences

| 1 | 5   | 7   | 3   |
|---|------|------|-----|
| 1 | 6   | 14  | 10  |
| 1 | 7   | 20  | 24  |
| 1 | 8   | 27  | 44  |
| 1 | 9   | 35  | 79  |
| 1 | 10  | 44  | 126 |

Conjecture: A Pascal-type behavior.

**Example 4.11.** For $G = \cdots$ and $f = u + u^3$, we have the Hilbert sequences

| 1 | 8   | 30  | 47  | 33  | 9   |
|---|------|------|-----|-----|-----|
| 1 | 9   | 38  | 77  | 80  | 42  | 9   |
| 1 | 10  | 47  | 115 | 157 | 122 | 51  | 9   |

Conjecture: A Pascal-type behavior.

**4.2. Trees of maximal degree at most three.**

**4.2.1. 4 vertices.** The homogeneous (i.e. corresponding to $f(u) = u$) Hilbert sequence is $(1, 3, 3, 1)$.

- $\odot$ has the Hilbert sequence $(1, 4, 3)$ for all $f(u) = u + au^2$ with $a \neq 0$.
- $\odot$ has the Hilbert sequence $(1, 4, 3)$ for all $f(u) = u + au^2 + bu^3$ unless $a = b = 0$.

**4.2.2. 5 vertices.** The homogeneous Hilbert sequence is $(1, 4, 6, 4, 1)$.

- $\odot$ has the Hilbert sequence $(1, 5, 10)$ for all $f(u) = u + au^2$ with $a \neq 0$.
- $\odot$ has the Hilbert sequence $(1, 5, 10)$ for all $f(u) = u + au^2 + bu^3$ unless $a = 0$. For $f(u) = u + bu^3$ with $b \neq 0$, the Hilbert sequence is $(1, 5, 7, 3)$.

**4.2.3. 6 vertices.** The homogeneous Hilbert sequence is $(1, 5, 10, 10, 5, 1)$. Surprisingly all 4 non-isomorphic trees on 6 vertices have the same general Hilbert sequence $(1, 6, 16, 9)$ for $f(u) = u + au^2 + bu^3$ with $a, b \neq 0$.

- $\odot$ has the Hilbert sequence $(1, 6, 16, 9)$ for all $f(u) = u + au^2$ with $a \neq 0$. 
\[ f(u) = u + bu^3, \ b \neq 0 \] gives the Hilbert sequence \((1, 6, 12, 10, 3)\).

\[ \square \] also has special value at \(a = 0\): the family \(f(u) = u + bu^3, \ b \neq 0\) gives the Hilbert sequence \((1, 6, 14, 10, 1)\).

\[ \square \] has two special values: \(a = 0, b \neq 0\) gives the Hilbert sequence \((1, 6, 12, 10, 3)\) while \(a \neq 0, b = 0\) gives \((1, 6, 15, 10)\).

\subsection*{4.2.4. 7 vertices.} The homogeneous Hilbert sequence is \((1, 6, 15, 20, 15, 6, 1)\).

\[ \square \] has the Hilbert sequence \((1, 7, 23, 33)\) for all \(f(u) = u + au^2\) with \(a \neq 0\).

\[ \square \] has the general Hilbert sequence \((1, 7, 23, 33)\) and one special value \(a = 0\) which for \(f(u) = u + bu^3, \ b \neq 0\) gives the Hilbert sequence \((1, 7, 22, 25, 9)\).

\[ \square \] has the general Hilbert sequence \((1, 7, 23, 33)\) and one special value \(a = 0\) which for \(f(u) = u + bu^3, b \neq 0\) gives the Hilbert sequence \((1, 7, 22, 25, 9)\).

\[ \square \] has a different general Hilbert sequence \((1, 7, 22, 34)\) and at least two special values \(a = 0\) and \(b = 0\). In particular, for \(f(u) = u + u^2\), the Hilbert sequence equals \((1, 7, 21, 35)\) and for \(f(u) = u + u^3\), the Hilbert sequence equals \((1, 7, 18, 22, 13, 3)\).

\[ \square \] has the general Hilbert sequence \((1, 7, 23, 33)\) and at least two special values \(a = 0\) and \(b = 0\). In particular, for \(f(u) = u + u^2\), the Hilbert sequence equals \((1, 7, 22, 34)\) and for \(f(u) = u + u^3\), the Hilbert sequence equals \((1, 7, 20, 24, 11, 1)\).

\[ \square \] has the general Hilbert sequence \((1, 7, 23, 33)\) and at least two special values \(a = 0\) and \(b = 0\). For \(f(u) = u + u^2\) the Hilbert sequence equals \((1, 7, 22, 34)\) and for \(f(u) = u + u^3\) the Hilbert sequence equals \((1, 7, 18, 22, 13, 3)\).

CAUTION: Parametric Gröbner bases were not computed which means that there could be missing additional special values of parameters which we have not found.

\subsection*{4.2.5. 8 vertices.} The homogeneous Hilbert sequence is \((1, 7, 21, 35, 35, 21, 7, 1)\).

\[ \square \] For \(f(u) = u + au^2, \ a \neq 0\) the Hilbert sequence equals \((1, 8, 31, 61, 27)\).
For $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$, the Hilbert sequence equals $(1, 8, 31, 61, 27)$. For $f(u) = u + u^3$, it equals $(1, 8, 25, 40, 35, 16, 3)$.

For $f(u) = u + u^2$, the Hilbert sequence equals $(1, 8, 30, 62, 27)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 8, 31, 61, 27)$, and for $f(u) = u + u^3$, it equals $(1, 8, 27, 44, 35, 12, 1)$.

For $f(u) = u + u^2$, the Hilbert sequence equals $(1, 8, 30, 59, 30)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 8, 31, 61, 27)$, and for $f(u) = u + u^3$, it equals $(1, 8, 29, 47, 34, 9)$.

For $f(u) = u + u^2$ and $f(u) = u + u^2 + u^3$, the Hilbert sequence equals $(1, 8, 31, 61, 27)$; for $f(u) = u + u^3$, it equals $(1, 8, 30, 47, 33, 9)$.

For $f(u) = u + u^2$, the Hilbert sequence equals $(1, 8, 30, 62, 27)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 8, 31, 61, 27)$, and for $f(u) = u + u^3$, it equals $(1, 8, 25, 51, 34, 5)$.

For $f(u) = u + u^2$, the Hilbert sequence equals $(1, 8, 30, 59, 30)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 8, 31, 61, 27)$, and for $f(u) = u + u^3$, it equals $(1, 8, 25, 40, 35, 16, 3)$.

For $f(u) = u + u^2$, the Hilbert sequence equals $(1, 8, 30, 62, 27)$; for $f(u) = u + u^2 + u^3$, it equals $(1, 8, 31, 61, 27)$, and for $f(u) = u + u^3$, it equals $(1, 8, 25, 40, 35, 16, 3)$.

CAUTION: Parametric Gröbner bases were not computed which means that there could be missing additional special values of parameters which we have not found.

4.3. **Sporadic examples.** Here we present the Hilbert sequences for several special graphs.

(A) We start with the complete graph $K_5$. 
| \( f(u) \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|---|
| \( u \) | 1 | 4 | 10 | 20 | 35 | 64 | 60 | 35 | 10 | 1 |
| \( u + u^2 \) | 1 | 5 | 14 | 30 | 55 | 80 | 77 | 15 | 9 | 4 | 1 |
| \( u + u^3 \) | 1 | 5 | 15 | 33 | 60 | 76 | 60 | 27 | 9 | 4 | 1 |
| \( u + u^4 \) | 1 | 5 | 14 | 30 | 53 | 73 | 60 | 41 | 9 | 4 | 1 |
| \( u + u^2 + u^3 \) | 1 | 5 | 15 | 34 | 64 | 90 | 53 | 15 | 9 | 4 | 1 |
| \( u + u^2 + u^4 \) | 1 | 5 | 15 | 35 | 67 | 91 | 48 | 15 | 9 | 4 | 1 |
| \( u + u^3 + u^4 \) | 1 | 5 | 15 | 33 | 63 | 82 | 56 | 21 | 9 | 4 | 1 |
| \( u + u^2 + u^3 + u^4 \) | 1 | 5 | 15 | 35 | 67 | 91 | 48 | 15 | 9 | 4 | 1 |
| \( u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} \) | 1 | 5 | 15 | 35 | 63 | 84 | 59 | 15 | 9 | 4 | 1 |

(B) \( K_5 \setminus e \), where \( e \) is any edge.

| \( f(u) \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| \( u \) | 1 | 4 | 10 | 20 | 33 | 45 | 46 | 29 | 9 | 1 |
| \( u + u^2 \) | 1 | 5 | 14 | 30 | 53 | 62 | 33 | |
| \( u + u^3 \) | 1 | 5 | 15 | 33 | 55 | 59 | 28 | 2 |
| \( u + u^4 \) | 1 | 5 | 14 | 30 | 48 | 50 | 37 | 13 |
| \( u + u^2 + u^3 \) | 1 | 5 | 15 | 34 | 62 | 68 | 13 |
| \( u + u^2 + u^4 \) | 1 | 5 | 15 | 35 | 65 | 64 | 13 |
| \( u + u^3 + u^4 \) | 1 | 5 | 15 | 35 | 65 | 64 | 13 |
| \( u + u^2 + u^3 + u^4 \) | 1 & 5 & 15 & 35 & 65 & 64 & 13 |
| \( u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} \) | 1 & 5 & 15 & 35 & 60 & 59 & 23 |

(C) \( K_5 \setminus (e_1 \cup e_2) \), where \( e_1 \) and \( e_2 \) are any two disjoint edges of \( K_5 \).

| \( f(u) \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| \( u \) | 1 | 4 | 10 | 20 | 31 | 35 | 24 | 8 | 1 |
| \( u + u^2 \) | 1 | 5 | 14 | 30 | 51 | 26 | 7 |
| \( u + u^3 \) | 1 | 5 | 15 | 33 | 45 | 29 | 6 |
| \( u + u^4 \) | 1 | 5 | 15 | 33 | 34 | 33 | 18 | 4 |
| \( u + u^2 + u^3 \) | 1 | 5 | 15 | 34 | 56 | 19 | 4 |
| \( u + u^2 + u^4 \) | 1 & 5 & 15 & 34 & 56 & 19 & 4 |
| \( u + u^3 + u^4 \) | 1 & 5 & 15 & 34 & 47 & 28 & 4 |
| \( u + u^2 + u^3 + u^4 \) | 1 & 5 & 15 & 35 & 56 & 18 & 4 |
| \( u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} \) | 1 & 5 & 15 & 35 & 51 & 20 | 7 |

(D) A square with parallel double edges.

| \( f(u) \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| \( u \) | 1 | 3 | 6 | 9 | 8 | 4 | 1 |
| \( u + u^2 \) | 1 | 4 | 9 | 15 | 3 |
| \( u + u^3 \) | 1 | 4 | 10 | 11 | 5 | 1 |
| \( u + u^2 + u^3 \) | 1 | 4 | 10 | 15 | 2 |
| \( u - \frac{u^2}{2} + \frac{u^3}{3} \) | 1 & 4 & 10 | 14 & 3 |

(E) A square with adjacent double edges.

The matroid is the same as in the previous example, so the original, undeformed algebra does not distinguish the two [6].
| \(f(u)\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \(u\) | 1 | 3 | 6 | 9 | 8 | 4 | 1 |
| \(u + u^2\) | 1 | 4 | 9 | 15 | 3 |
| \(u + u^3\) | 1 | 4 | 10 | 12 | 5 |
| \(u + u^4\) | 1 | 4 | 8 | 10 | 7 | 2 |
| \(u + u^2 + u^3\) | 1 | 4 | 10 | 16 | 1 |
| \(u + u^2 + u^4\) | 1 | 4 | 10 | 16 | 1 |
| \(u + u^3 + u^4\) | 1 | 4 | 10 | 13 | 4 |
| \(u + u^2 + u^3 + u^4\) | 1 | 4 | 10 | 16 | 1 |
| \(u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4}\) | 1 | 4 | 10 | 15 | 2 |

4.4. Two examples with analysis of relations. Here we present two small examples for which we can provide complete analysis of their weighted projective spaces \(PA_G\) of deformed zonotopal algebras.

4.4.1. Multigraph \(K_3 + e\) with a double edge.

**Proposition 4.12.** For \(G = K_3 + e\), the space \(PA_G\) is stratified by the following functions over a field \(k\) of characteristic 0.

| \(f(u)\) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \(u\) | 1 | 2 | 3 | 3 | 1 |
| \(u + bu^2 + cu^3\) | 1 | 3 | 5 | 1 |
| * | 1 | 3 | 4 | 2 |

In the last row either \(b = 0\), or \(c = 0\), or \(3c = 4b^2\).

**Proof.** By Corollary 3.2 our relations have the form

\[
\begin{align*}
\lambda^4 &= x_1^4 = x_2^4 = x_3^3 = 0, \\
(f(x_1) + f(x_2) + f(x_3)) &= 0, \\
((f(x_1) + f(x_2))^3 - (f(x_1) + f(x_3))^4) &= 0.
\end{align*}
\]

Note that, the first group of equations implies that \(f(x_1)^4 = f(x_2)^4 = f(x_3)^3 = 0\), so the third group of equation is redundant.

When \(b \neq 0\) we may substitute \(u \mapsto u/b\) to transform \(f(u)\) to the function of the form \(f(u) = u + u^2 + \lambda u^3\). Using \(\lambda\) as a variable, we can verify using Macaulay2 ([1]) that \((x_1 + x_2 + x_3)^2\) is always a relation. Note that since the sums of consecutive entries in any Hilbert sequence are increasing, the sequence is determined if there is no further relation in degree 2. We want to show that \(\lambda = 0, 4/3\) are the only two exceptional cases where we get an additional quadratic relation – this will separate the two rows of the table. It can be verified by hand or using Macaulay2, that Hilbert sequences of the exceptional values of \(\lambda\) coincide.

By symmetry (if we had two relations which are switched by interchanging \(x_1\) and \(x_2\), then we add them), we must have a quadratic relation of the form

\[
a_1(x_1^2 + x_2^2) + a_2x_1x_2 + a_3(x_1 + x_2) + bx_3(x_1 + x_2) + c_2x_3^2 + c_1x_3 = 0.
\]
By subtracting the existing relation \((x_1 + x_2 + x_3)^2 = 0\) we may assume that \(b = 0\). Since \(f(x_3)^2 = x_3^2\) we may now rewrite the relation as

\[
a_1(x_1^2 + x_2^2) + a_2 x_1 x_2 + a_3 (x_1 + x_2) = Q(-f(x_3)),
\]

where \(Q\) is a quadratic polynomial such that \(Q(0) = 0\). Since, \(Q(-f(x_3)) = Q(f(x_1) + f(x_2))\) we need to find when there is \(Q\) such that \(\deg Q(f(x_1) + f(x_2)) \leq 2\). We may assume that \(Q(T) = T^2 + aT\), because \(Q(T) = T\) cannot work.

One can check that modulo existing relations \(x_1^4 = x_4^2 = (f(x_1) + f(x_2))^3 = 0\), the polynomial \(Q(f(x_1) + f(x_2))\) has degree at most 3 and its cubic term is

\[
(-a\lambda + 4/3a)(x_1 + x_2)^3 + (\lambda + 4/3a)(x_1^3 + x_2^3).
\]

Both coefficients vanish if and only if either \(\lambda = 4/3, a = -1\) or \(a = \lambda = 0\). \(\square\)

4.4.2. Graph \(K_4\).

**Proposition 4.13.** For \(G = K_4\), the space \(A_G\) of parameters is stratified by the following functions with \(b, c \neq 0\) over a field \(k\) of characteristic 0.

| \(f(u)\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|----------|---|---|---|---|---|---|---|
| \(u\)    | 1 | 3 | 6 | 10| 11| 6 | 1 |
| \(u + bu^2\) | 1 | 4 | 9 | 15| 5 | 3 | 1 |
| \(u + cu^3\) | 1 | 4 | 9 | 12| 8 | 3 | 1 |
| \(u + bu^2 + cu^3\) | 1 | 4 | 10| 14| 5 | 3 | 1 |

Thus we have a linear order of strata of \(A_G\) given by \(u < u + cu^3 < u + bu^2 < u + bu^2 + cu^3\).

**Proof.** We may use substitution \(u \mapsto u/b\) to transform the last equation into \(u + u^2 + cu^3\). Let us represent our algebra as a quotient of \(K[x_1, x_2, x_3, x_4]\).

By Corollary 3.2 we have relations of the form

\[
\begin{aligned}
&x_i^4 = 0 \\
&(x_i + x_j + x_i^2 + x_j^2 + x_i^3 + x_j^3))^5 = 0 \quad \text{for } i \neq j \quad \text{for } i \neq j \neq k \\
&(x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + c(x_1^3 + x_2^3 + x_3^3 + x_4^3)) = 0.
\end{aligned}
\]

It is then easy to see that

\[
(x_i + x_j + x_i^2 + x_j^2 + c(x_i^3 + x_j^3))^5 \equiv (x_i + x_j + x_i^2 + x_j^2)^5 \mod (x_1, x_2, x_3, x_4),
\]

In fact, this relation reduces to \(40x_i^3 x_j^3 + 10x_i^3 x_j^2 + 10x_i^2 x_j^3 \equiv 0\). After multiplication by \(x_i\) we get that \(x_i^3 x_j^3 \equiv 0 \mod (x_1, x_2, x_3, x_4)\). So the ideal generated by the first two classes of relations is generated by \(x_1, x_2, x_3, x_4\) for \(i \neq j\). It is now easy to verify that

\[
(x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2 + c(x_i^3 + x_j^3 + x_k^3))^4 \equiv (x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2)^4
\]
modulo the ideal generated by the previous relations. We conclude that the relations are, in fact, less dependent on $c$:

$$
\begin{align*}
&x_i^4 = 0 \quad \text{for } i = 1, \ldots, 4 \\
&(x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2 + x_i^2 + x_j^2 + x_k^2)^5 = 0 \quad \text{for } i \neq j \\
&(x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2)^4 = 0 \quad \text{for } i \neq j \neq k \\
x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + (x_1^3 + x_2^3 + x_3^3 + x_4^3) = 0.
\end{align*}
$$

It is immediate that $c = 0$ gives a quadratic relation. It is also easy to see that no other value of $c$ can give one. Namely, since the ideal $I$ generated by the first three groups of relations, which are independent of $c$, does not contain any element of order less than 4, then for any $f \in I$, the polynomial

$$
f + x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + c(x_1^3 + x_2^3 + x_3^3 + x_4^3)
$$

has a cubic term.

The case $f(u) = u + cu^3$ is similar. \hfill \square

5. Outlook

Here we present a small sample of open problems about deformed zonotopal algebras $C^f_G$ for future investigation.

Our experiments with Maculay2 show that for many graphs $G$ and functions $f$, the Hilbert sequence of the algebra $C^f_G$ is logarithmically concave. In an earlier preprint version of our paper we conjectured that this was true for all graphs $G$ and all non-degenerate functions $f$. However, in this generality the conjecture does not hold. A counterexample is given by the complete graph $G = K_5$ and $f = u + u^4$, for which the Hilbert sequence is given by $H^f_G = (1, 5, 14, 30, 53, 73, 60, 41, 9, 4, 1)$ which is not log-concave, since $9^2 < 41 \cdot 4 = 164$.

However, recently a truly remarkable proof of this fact in the graded case (i.e., $f = u$) was found in [1]. This circumstance gives hope that the log-concavity might hold for a larger class of graphs and functions.

**Problem 5.1.** Find families of graphs $G$ and nondegenerate functions $f \neq u$ for which the Hilbert sequence $H^f_G$ is log-concave. In particular, prove log-concavity for chain and cycle graphs and $f = u + u^2$.

**Problem 5.2.** When $G$ is a tree, is it true that the Hilbert stratification of $A_G$ consists of coordinate subspaces? In particular, does $f = e^u$ give a general Hilbert sequence?

**Problem 5.3.** Find a graph-theoretical interpretation of the general Hilbert sequence $H^f_G$? Find a graph-theoretical interpretation of $H^f_G$ in some special cases, for example, for $f = e^u$. 
Problem 5.4. Is it true that if $H_{G_1}^f = H_{G_2}^f$ for every function $f$, then $G_1$ is isomorphic to $G_2$?

Problem 5.5. Determine $G$ and $f$ for which the algebra $C_G^f$ is Gorenstein/quadratic/Koszul.

When $f = u$, i.e. in the graded case, the Hilbert sequences of $C_G$ satisfy the deletion-contraction relation (2.8) which allows to compute them recursively. However, for $f \neq u$ the relation (2.8) does not hold.

Problem 5.6. Study the behavior of $H_{G}^f$ under standard graph operations on graphs, for example under deletions and contractions of edges.

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Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: shapiro@math.su.se

BCAM – BÁSQUE CENTER FOR APPLIED MATHEMATICS, MAZARREDO 14, 48009 BILBAO, SPAIN AND IKERBASQUE, BÁSQUE FOUNDATION FOR SCIENCE, PLAZA EUSKADI 5, 48009 BILBAO, SPAIN
Email address: ismirnov@bcamath.org

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
Email address: vaintrob@uoregon.edu