UNIVERSAL SIMPLICIAL COMPLEXES INSPIRED BY TORIC TOPOLOGY

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Abstract. Let $k$ be the field $\mathbb{F}_p$ or the ring $\mathbb{Z}$. We study combinatorial and topological properties of the universal complexes $X(k^n)$ and $K(k^n)$ whose simplices are certain unimodular subsets of $k^n$. We calculate their $f$-vectors, show that they are shellable but not shifted, and find their applications in toric topology and number theory.

Using discrete Morse theory, we detect that $X(k^n), K(k^n)$ and the links of their simplices are homotopy equivalent to a wedge of spheres specifying the exact number of spheres in the corresponding wedge decompositions. This is a generalisation of Davis and Januszkiewicz’s result that $K(\mathbb{Z}^n)$ and $K(\mathbb{F}_2^n)$ are $(n-2)$-connected simplicial complexes.

1. Introduction

An abstract simplicial complex on a set $S$ is a collection $K$ of subsets of $S$ closed under taking inclusions. The elements of $K$ are called simplices; in particular, the set $S$ is called the vertex set of $K$. We assume that $S \subset K$.

A simplicial complex $K$ is finite if the number of its simplices is finite, otherwise it is infinite.

The dimension of a simplex $\sigma \in K$ is $\dim \sigma = |\sigma| - 1$. The dimension of a simplicial complex $K$ is the maximal dimension of its simplices. If the dimension $K$ is not finite, we say that $K$ is infinite dimensional.

For each $I \subset S$, the full subcomplex $K_I$ is the restriction of $K$ on $I$, that is, the subcomplex $K_I = \{ \sigma \in K \mid \sigma \subset I \}$.

The link of a simplex $\sigma \in K$ is the subcomplex $\text{link}_K \sigma = \{ \tau \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset \}$.

Definition 1.1. Let $k$ be the field $\mathbb{F}_p$ or the ring $\mathbb{Z}$. A set $\{v_1, \ldots, v_m\}$ of elements from $k^n$ is called unimodular if $\text{span}\{v_1, \ldots, v_m\}$ is a direct summand of $k^n$ of dimension $m$. As a subset of a unimodular set is itself unimodular, the set of all unimodular subsets of $k^n$ is a simplicial complex. Denote this simplicial complex by $X(k^n)$.

Closely related to the complex $X(k^n)$ is the Tits building $T(V)$ for an $n$-dimensional vector space $V$. An $m$-simplex of $T(V)$ is a chain $W_0 < \ldots < W_m$ of proper subspaces. Studying finite generation of the higher algebraic $K$-groups, Quillen [18] showed that $T(V)$ has the homotopy type of a wedge of $(n-2)$ spheres. That result further prompted the study of an interesting homology group $H_{n-2}(T(V))$. This is a $\text{GL}(V)$-module called the Steinberg module. In the further study of topological properties of the Tits building $T(V)$ van der Kallen [19] proved that the link of each $i$-simplex of $T(V)$ is $(n-i-3)$-connected.

Following these results on the Tits building, we studied the homotopy type of $X(k^n)$ and its homology groups.

Another simplicial complex closely related to $X(k^n)$ is the universal simplicial complex $K(k^n)$.
Define a line through the origin to be a 1-dimensional unimodular subspace of \( \mathbb{k}^n \), that is, a submodule of rank 1 which is a direct summand.

**Definition 1.2.** Let \( P\mathbb{k}^n \), the set of lines through the origin in \( \mathbb{k}^n \), be the set of vertices of \( K(\mathbb{k}^n) \). An \((i-1)\)-simplex of \( K(\mathbb{k}^n) \) is a set of lines \( \{l_1, \ldots, l_i\} \subset P\mathbb{k}^n \) which span an \( i \)-dimensional unimodular subspace of \( \mathbb{k}^n \).

The complexes \( K(\mathbb{F}_2^n) \) and \( K(\mathbb{Z}^n) \) are important objects in toric topology since they can be seen as universal complexes classifying small covers for \( \mathbb{k} = \mathbb{F}_2 \) and quasitoric manifolds for \( \mathbb{k} = \mathbb{Z} \). Davis and Januszkiewicz \([9]\) proved that both complexes \( K(\mathbb{F}_2^n) \) and \( K(\mathbb{Z}^n) \) are \((n-2)\)-connected and that the link of an \( i \)-simplex is \((n-\i-3)\)-connected.

Quasitoric manifolds can be equipped with an additional structure called omniorientation, see \([5]\). As an analogue of the universal property of \( K(\mathbb{Z}^n) \) and \( K(\mathbb{F}_2^n) \), we shall show, as Proposition \([2.1]\) that the complex \( X(\mathbb{Z}^n) \) is universal in the sense that it classifies omnioriented quasitoric manifolds with positive orientation of the defining polytope \( P \).

The goal of this paper is combinatorial and topological study of the simplicial complexes \( X(\mathbb{k}^n) \) and \( K(\mathbb{k}^n) \) and the links of their simplices and their applications to related mathematical areas. After discussing their universal properties in Section \([2]\) we focus on determining in Theorems \([5.2]\) and \([5.6]\) the \( \mathbf{f} \)-vectors of \( X(\mathbb{F}_p^n) \), \( K(\mathbb{F}_p^n) \) and in a similar way the \( \mathbf{f} \)-vectors of the links of their simplices. These calculations will be used effectively in describing the homotopy type and the homology groups of \( X(\mathbb{k}^n) \), \( K(\mathbb{k}^n) \) for \( \mathbb{k} = \mathbb{F}_p \) or \( \mathbb{Z} \) and of the links of their simplices. We prove in Theorems \([6.2]\) and \([6.7]\) using discrete Morse theory, that all these complexes are homotopy equivalent to a wedge of spheres, and calculated the exact number of spheres in the wedge, generalising therefore Davis and Januszkiewicz, and van der Kallen’s results.

A simplicial complex \( K \) is called Cohen-Macaulay over \( \mathbb{k} \) if for all simplices \( \sigma \subset K \), \( H^i(\text{link}_K \sigma; \mathbb{k}) = 0 \) for all \( 0 \leq i < \text{dim} (\text{link}_K \sigma) \). An immediate corollary (see Corollary \([6.1]\) of Theorems \([6.2]\) and \([6.7]\)) is that both \( X(\mathbb{k}^n) \) and \( K(\mathbb{k}^n) \) are Cohen-Macaulay simplicial complexes over any coefficients. There is a well-know hierarchy between different types of pure simplicial complexes, namely, shifted \( \subset \) shellable \( \subset \) Cohen-Macaulay.

We show as Propositions \([6.2]\) and \([6.3]\) that the universal complexes \( X(\mathbb{k}^n) \) and \( K(\mathbb{k}^n) \) are not shifted but they can be endowed with a well behaved shelling making them into shellable complexes.

An important application of complexes \( K(\mathbb{F}_p^n) \) and \( K(\mathbb{Z}^n) \) is to the study of the Buchstaber invariant of simplicial complexes. The known upper and lower bound estimates of the Buchstaber invariant are obtained in \([2]\) and \([10]\) using topological and combinatorial properties of these complexes.

In Section \([7]\) we use the calculation of the \( \mathbf{f} \)-vectors of \( K(\mathbb{Z}^n) \) to give an estimate of the Buchstaber invariant of an arbitrary simplicial complex in terms of the Buchstaber invariant of \( K(\mathbb{Z}^n) \).

Beside toric topology, we relate the universal complexes \( X(\mathbb{k}^n) \) and \( K(\mathbb{k}^n) \) and their \( \mathbf{f} \)-vectors to purely number theoretical facts about divisibility. By way of application, we obtain a new result in the context of generalised factorial functions answering Bahargava’s question \([4]\) of combinatorial interpretation of certain generalised factorial function.

**2. Universality of \( X(\mathbb{Z}^n) \), \( K(\mathbb{Z}^n) \) and \( K(\mathbb{F}_2^n) \)**

Our study of the simplicial complexes \( X(\mathbb{k}^n) \) and \( K(\mathbb{k}^n) \) has been motivated by the importance \( K(\mathbb{Z}^n) \) and \( K(\mathbb{F}_2^n) \) play in toric topology.
Let \( G_d \) stand for \( \mathbb{Z}_2 \) when \( d = 1 \) and \( S^1 \) when \( d = 2 \). The spaces with actions of \( \mathbb{Z}_2^d \) and of a torus \((S^1)^n\) are objects studied in toric topology. A special attention is given to \( G_2^n \)-manifolds which are \( dn \)-manifolds with a locally standard action of \( G_2^n \) having an \( n \)-dimensional simple polytope \( P^n \) as the orbit space. Those manifolds are usually referred to as quasitoric manifolds for \( d = 2 \) and small covers for \( d = 1 \). They are topological generalisations of toric and real toric varieties. Recall that by [5, Construction 5.12] the quasitoric manifolds and small covers are determined by a simple polytope \( P^n \) and a characteristic map \( \lambda: F \to \pi_1(G_2^n) \) where \( F \) is the set of facets of \( P \). In the case of quasitoric manifolds for each facet \( F \) of \( P^n \) the vector \( \lambda(F) \) is determined up to sign. The characteristic map \( \lambda \) satisfies a certain nondegeneracy condition, that is, \( \{\lambda(F_i), \ldots, \lambda(F_\text{\textscript{ik}})\} \) is a unimodular set whenever the intersection of facets \( F_i, \ldots, F_\text{\textscript{ik}} \) is nonempty. Let \( P^* \) be a simplicial complex dual to \( P \). Then there is a simplicial map \( \lambda: P^* \to K(\pi_1(G_2^n)) \) defined on the vertices of \( P^* \) by \( \lambda \) such that a vertex \( v \) of \( P^* \) dual to the facet \( F_v \) of \( P \) is sent to the line \( l(\lambda(F_v)) \). The map \( \lambda \) is a characteristic map if and only if \( \lambda \) is a nondegenerate simplicial map. This fact implies that the set of equivalence classes of \( G_2^n \)-manifolds over \( P \) is in bijection with the set of nondegenerate simplicial maps \( f: P^* \to K(\pi_1(G_2^n)) \) modulo natural action of \( \text{Aut}(\pi_1(G_2^n)) \). Thus the universal property of \( K(\pi_1(G_2^n)) \) further motivates the combinatorial and topological study of \( K(\pi_1(G_2^n)) \).

Using omniorientation, Buchstaber, Panov and Ray [?] constructed quasitoric representative for every complex cobordism class \( \Omega^U \). A choice of omniorientation on a given quasitoric manifold \( M^{2n} \) over \( P^n \) is equivalent to a choice of orientation for \( P^n \) together with an unambiguous choice of facet vectors \( \lambda(F_i) \). More precisely, recall that a pair \((P, \Lambda)\) is called a combinatorial quasitoric pair, if \( P \) is an oriented combinatorial simple \( n \)-polytope with \( m \) facets and \( \Lambda \) is an integer \( n \times m \) matrix with the property that for each vertex \( v \) appropriate minor \( \Lambda_v \) of \( \Lambda \), consisting of columns whose facets contain \( v \), is invertible. Two combinatorial quasitoric pairs \((P, \Lambda)\) and \((P', \Lambda')\) are equivalent if \( P \cong P' \) with the same orientation and \( \Lambda \cong \Phi \Lambda' \) for some \( \Phi \) integer matrix with determinant 1.

Buchstaber, Panov and Ray [?] showed that there is a 1-1 correspondence between equivalence classes of omnioriented quasitoric manifolds and combinatorial quasitoric pairs.

We use this combinatorial description of omnioriented quasitoric manifolds to show that \( X(\mathbb{Z}^n) \) has a universal property.

**Proposition 2.1.** There is a 1-1 correspondence between equivalence classes of omnioriented quasitoric manifolds with positive orientation on \( P \) and nondegenerate simplicial maps

\[
f: P^* \to X(\mathbb{Z}^n)
\]

modulo natural action of \( \text{Aut}(\mathbb{Z}^n) \).

**Proof.** For a combinatorial quasitoric pair \((P, \Lambda)\), define a map \( f: P^* \to X(\mathbb{Z}^n) \) on vertices \( v_i \) of \( P^* \) (which correspond to the facets \( F_i \) of \( P \)) to be equal to the \( i \)th column of \( \Lambda \). The condition that \( \Lambda_v \) is invertible for each vertex \( v \) of \( P \) is equivalent to \( f \) being nondegenerate.

On the other hand, for a nondegenerate simplicial map \( f: P^* \to X(\mathbb{Z}^n) \) we construct \( \Lambda \) by putting together vectors \( f(v_i) \) as appropriate columns and thus obtain a combinatorial quasitoric pair \((P, \Lambda)\). \( \square \)
3. \( f \)-vectors of \( X(\mathbb{F}_p^n) \) and \( K(\mathbb{F}_p^n) \)

The \( f \)-vector of an \((n - 1)\)-dimensional simplicial complex \( K^{n-1} \) is the integer vector

\[
\mathbf{f}(K^{n-1}) = (f_{-1}, f_0, f_1, \ldots, f_{n-1}),
\]

where \( f_{-1} = 1 \) and \( f_i = f_i(K^{n-1}) \) denotes the number of \( i \)-faces of \( K^{n-1} \) for all \( i = 1, \ldots, n-1 \).

The \( f \)-polynomial of an \((n - 1)\)-dimensional simplicial complex \( K \) is

\[
\mathbf{f}(t) = t^n + f_0 t^{n-1} + \cdots + f_{n-1}.
\]

The integral value

\[
\chi(K^{n-1}) = f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = (-1)^{n-1} \mathbf{f}(-1) + 1
\]

is called the Euler characteristic of a simplicial complex \( K^{n-1} \).

The goal of this section is to calculate the \( f \)-vector of \( X(\mathbb{F}_p^n) \) and \( K(\mathbb{F}_p^n) \) and of the links of their simplices. The \( f \)-vectors carry important combinatorial information of \( X(\mathbb{F}_p^n) \) and \( K(\mathbb{F}_p^n) \) and their calculation leads to complete determination not only of the homotopy type of the universal complexes \( X(\mathbb{F}_p^n) \) and \( K(\mathbb{F}_p^n) \) but also the homotopy types of their \( r \)-skeletons. This has applications in the study of the Buchstaber invariant of simplicial complexes. Further on, knowing the homotopy types of the skeletons of \( K(\mathbb{F}_p^n) \) and \( K(\mathbb{Z}_p^n) \) opens a possibility for new applications of \( \mathbb{F}_p \) in the study of Tor-algebras of simplicial complexes, see \( \mathbb{F}_p \).

Since the simplicial complexes \( X(\mathbb{F}_p^n) \) and \( K(\mathbb{F}_p^n) \) are given as sets of unimodular subsets of \( \mathbb{F}_p^n \), they are clearly \((n - 1)\)-dimensional and pure simplicial complexes and the group \( GL(n, \mathbb{F}_p) \) acts transitively and simplicially on them.

Let \( e_1, \ldots, e_n \) be the standard base of \( \mathbb{F}_p^n \). The vertex set of \( X(\mathbb{F}_p^n) \) are all nonzero elements of \( \mathbb{F}_p^n \). Thus,

\[
f_0(X(\mathbb{F}_p^n)) = p^n - 1.
\]

**Proposition 3.1.** Each \( i \)-simplex \( \sigma \in X(\mathbb{F}_p^n) \) is a face of exactly \( p^n - p^{i+1} (i + 1) \)-simplices of \( X(\mathbb{F}_p^n) \).

**Proof.** As \( GL(n, \mathbb{F}_p) \) acts on \( X(\mathbb{F}_p^n) \) transitively, it suffices to prove the statement for \( \sigma = \{e_1, \ldots, e_{i+1}\} \). Each of \( p^n - p^{i+1} \) elements of \( \mathbb{F}_p^n \) outside the simplex \( \sigma \) can be added to \( \sigma \) to form an \((i + 1)\)-simplex of \( X(\mathbb{F}_p^n) \) having \( \sigma \) as its face. \( \square \)

The proposition implies the following recurrence relation.

**Corollary 3.2.** For \( 0 \leq i \leq n - 2 \), there is a recurrence relation

\[
(i + 2)f_{i+1}(\mathbb{F}_p^n) = (p^n - p^{i+1})f_i(X(\mathbb{F}_p^n)).
\]

\( \square \)

Solving recurrence \( (3) \) with the starting condition \( (2) \) gives the \( f \)-vector of \( X(\mathbb{F}_p^n) \).

**Theorem 3.3.** The \( f \)-vector of \( X(\mathbb{F}_p^n) \) is given by

\[
f_i(X(\mathbb{F}_p^n)) = \frac{(p^n - p^i) \cdots (p^n - p^0)}{(i + 1)!}
\]

for \( 1 \leq i \leq n - 1 \) and \( f_{-1} = 1 \).

The vertex set of \( K(\mathbb{F}_p^n) \) are all lines through the origin in \( \mathbb{F}_p^n \). Thus

\[
f_0(K(\mathbb{F}_p^n)) = \frac{p^n - 1}{p - 1}.
\]

We give an analogue result for the \( f \)-vector of \( K(\mathbb{F}_p^n) \) to the one in Proposition \( \ref{prop:3.3} \).
Proposition 3.4. Let $\sigma \in K(F^n_p)$, be an i-simplex. Then $\sigma$ is a face of exactly $\frac{p^n-i+1}{p-1} (i+1)$-simplices of $K(F^n_p)$.

Proof. Let $l(v)$ be the line through the origin and an element $v$. As $GL(n,F_p)$ acts transitively on $K(F^n_p)$, it is sufficient to prove the statement for $\sigma = \{l(e_1), \ldots, l(e_{i+1})\}$. The lines $l(e_1), \ldots, l(e_{i+1})$ span an $(i+1)$-dimensional subspace of $F^n_p$. Thus any line determined by some element outside of this subspace can be added to $\sigma$ to form an unimodular subspace of $F^n_p$, that is, an $(i+1)$-simplex of $K(F^n_p)$ having $\sigma$ as its face. Note that the same line is determined by $p-1$ elements, so there are exactly $\frac{p^n-i+1}{p-1}$ such lines.

The proposition implies a recurrence relation which determines the $f$-vector of $K(F^n_p)$.

Corollary 3.5. For $0 \leq i \leq n-2$, we have

$$ (i+2)f_{i+1}(K(F^n_p)) = \frac{(p^n-p^{i+1})}{p-1} f_i(K(F^n_p)). $$

Theorem 3.6. The $f$-vector of $K(F^n_p)$ is given by

$$ f_i(K(F^n_p)) = \frac{(p^n-p^i)\cdots(p^n-p^0)}{(p-1)^{i+1} (i+1)!} $$

for $1 \leq i \leq n-1$ and $f_{-1} = 1$.

We proceed with calculating the $f$-vectors of link$_K(F^n_p)(\sigma)$ and link$_X(F^n_p)(\sigma)$ using the same idea.

Proposition 3.7. Let $\sigma$ be an i-simplex in $K(F^n_p)$. The $f$-vector of link$_K(F^n_p)(\sigma)$ is given by

$$ f_k(\text{link}_K(F^n_p)(\sigma)) = \frac{(p^n-p^{i+1})\cdots(p^n-p^{i+k+1})}{(p-1)^{i+k+1} (k+1)!} $$

for $1 \leq k \leq n-1$ and $f_{-1} = 1$.

Proof. Let $\sigma$ be an i-simplex in $K(F^n_p)$. A simplex $\tau = \{l_1, \ldots, l_{k+1}\}$ is a k-simplex in link$_K(\sigma)$ if and only if span($\sigma \cup \{l_1\}$) is an $(i+k+1)$-dimensional subspace of $F^n_p$.

There are $p^n-p^{i+1}$ possible choices for an element $g$ of $F^n_p$ which determines $l_1$. In fact, $g$ could be any element from $F^n_p \setminus \text{span}(\sigma)$. Since there are $p-1$ possible generators which generate the same line, there are $\frac{p^n-p^{i+1}}{p-1}$ possible ways to obtain $l_1$. Likewise, $l_{j+1}$ could be any line outside of span($\sigma \cup \{l_1, \ldots, l_j\}$) and there are $\frac{p^n-p^{i+j+1}}{p-1}$ such lines. To get the number of $k$-simplices in link$_K(\sigma)$, the product of these possibilities needs to be divided by $(k+1)!$ as the order of the generating lines in the subspace is irrelevant.

Proposition 3.8. Let $\sigma$ be an i-simplex in $X(F^n_p)$. The $f$-vector of link$_X(F^n_p)(\sigma)$ is given by

$$ f_k(\text{link}_X(F^n_p)(\sigma)) = \frac{(p^n-p^{i+1})\cdots(p^n-p^{i+k+1})}{(k+1)!} $$

for $1 \leq k \leq n-1$ and $f_{-1} = 1$.

Proof. Let $\sigma$ be an i-simplex in $X(F^n_p)$. A simplex $\tau = \{v_1, \ldots, v_{k+1}\}$ is a k-simplex in link$_X(\sigma)$ if and only if span($\sigma \cup \tau$) is an $(i+k+1)$-dimensional subspace of $F^n_p$.

We describe how to choose vertices of $\tau$. For $v_1$, we could take any element of $F^n_p \setminus \text{span}(\sigma)$ and there are $p^n-p^{i+1}$ such choices. In the same manner, there are
\[ p^n - p^j \cdot p \] possibilities for \( v_{j+1} \in \mathbb{F}_p^m \setminus \text{span}(\sigma \cup \{v_1, \ldots, v_j\}) \). By multiplying these possibilities and dividing them by \((k+1)!\) the number of \( k \)-simplices is obtained. □

Remark 3.9. Note that we have a map which sends a nonzero unimodular element of \( k^n \) into a line generated by it. This map extends to a simplicial surjection \( \phi: X(k^n) \to K(k^n) \) which is nondegenerate. The map \( \phi \) is an isomorphism for \( k = \mathbb{F}_2 \), it is \((p - 1)^k\) to 1 on \((k - 1)\)-simplices if \( k = \mathbb{F}_p \), and it is \( 2^k \) to 1 on \((k - 1)\)-simplices if \( k = \mathbb{Z} \).

Furthermore, there is a map \( \psi: K(k^n) \to X(k^n) \) defined on vertices of \( K(k^n) \) by mapping each line to an unimodular element from that line and extending it to a simplicial map. A map \( \psi \) depends on the choice of unimodular elements. It is nondegenerate and it is a section of \( \phi \), that is, \( \phi \psi = \text{Id}_{K(k^n)} \). Each such \( \psi \) embeds \( K(k^n) \) as a full subcomplex of \( X(k^n) \).

4. The homotopy type of \( K(\mathbb{F}_p^n), X(\mathbb{F}_p^n) \) and their links

The main methods in proving Theorem A come from discrete Morse theory. Discrete Morse theory, developed by Forman in [11] and [13], became in the last 20 years one of the most elegant techniques for studying homotopy types and homology groups of finite CW-complexes. It is a combinatorial analogue of the classical smooth Morse theory which origin dates back to the remarkable paper of Morse [17].

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A crucial notion in discrete Morse theory is one of critical cells of a Morse function, or equivalently, of a Morse matching on a CW-complex, which directly relates to the topology of a CW-complex. We recall main results of discrete Morse theory, in particular those related to a Morse matching for simplicial complexes which we shall use in this paper (see [14], [16] and [12]).

A partial matching on a directed graph \( D = (V, E) \) is a subset \( A \) of the edges \( E \) such that no vertex is contained in more than one edge of \( A \). We define a new directed graph \( D_A = (V, E_A) \) by setting

\[ E_A = (E \setminus A) \cup \{\beta \to \alpha \mid \alpha \to \beta \in A\}. \]

A simplicial complex \( K \) is a partially ordered set with respect to inclusions of subsets. The Hasse diagram \( \mathcal{H}(K) \) associated with \( K \) is the directed graph whose vertices are simplices of \( K \) and there is a directed edge from \( \sigma \) to \( \tau \) if and only if \( \tau \) is a face of \( \sigma \) and \( \dim \sigma = \dim \tau + 1 \).

Definition 4.1. Let \( K \) be a simplicial complex and \( \mathcal{H}(K) \) its Hasse diagram. A partial matching \( M \) on \( \mathcal{H}(K) \) is called a Morse matching on \( K \) if \( M \) on \( \mathcal{H}(K) \) is acyclic partial matching, that is, \( \mathcal{H}_M(K) \) contains no cycles. A simplex \( \sigma \in K \) is called \( M \)-critical if it is not matched by \( M \) with any other simplex from \( K \).

Forman [11] showed that if \( K \) is a finite simplicial complex with a Morse matching \( M \) on \( \mathcal{H}(K) \) and \( C(M) \) is the set of \( M \)-critical simplices, then \( K \) is homotopy equivalent to a CW-complex that has exactly one \( n \)-dimensional cell for each \( n \)-dimensional critical simplex in \( C(M) \). This result gives a powerful tool for investigating the topology of a finite simplicial complex.

As \( K(\mathbb{F}_p^n) \) and \( X(\mathbb{F}_p^n) \) are finite simplicial complexes, we use discrete Morse theory to determine their homotopy type.

Theorem 4.2. (a) The simplicial complex \( K(\mathbb{F}_p^n) \) is homotopy equivalent to the wedge of \( A_n(p) \) spheres \( S^{n-1} \)

\[ K(\mathbb{F}_p^n) \simeq \bigvee_{i=1}^{A_n(p)} S^{n-1} \]
where

\[ A_n(p) = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i}(p^n - p^i) \cdots (p^n - p^0) \cdot (i + 1)! \cdot (p - 1)^{i+1}. \]

(b) The link of an i-simplex \( \sigma \) in \( K(\mathbb{F}_p^n) \) is homotopy equivalent to

\[ \text{link}_{K(\mathbb{F}_p^n)} \sigma^i \simeq \bigvee_{l=1}^{A_{i,n}(p)} S^{n-i-2} \]

where

\[ A_{i,n}(p) = (-1)^{n-i-1} + \sum_{k=0}^{n-i-2} (-1)^{n-i-2-k}(p^n - p^i) \cdots (p^n - p^{i+k+1}) \cdot (i + 1)! \cdot (k + 1)! \]

(c) The simplicial complex \( X(\mathbb{F}_p^n) \) is homotopy equivalent to the wedge of \( B_n(p) \) spheres \( S^{n-1} \), where

\[ B_n(p) = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i}(p^n - p^i) \cdots (p^n - p^0) \cdot (i + 1)! \]

(d) The link of i-simplex in \( X(\mathbb{F}_p^n) \) is homotopy equivalent to the wedge of \( B_{i,n}(p) \) spheres \( S^{n-i-2} \), where

\[ B_{i,n}(p) = (-1)^{n-i-1} + \sum_{k=0}^{n-i-2} (-1)^{n-i-2-k}(p^n - p^i) \cdots (p^n - p^{i+k+1}) \cdot (k + 1)! \]

Proof. Finding the homotopy type of \( X(\mathbb{F}_p^n), K(\mathbb{F}_p^n) \) and of the links of their simplices follows the same idea in all four cases.

(a) We start with \( K(\mathbb{F}_p^n) \). Denote by \( \{e_1, e_2, \ldots, e_n\} \) the standard basis of \( \mathbb{F}_p^n \) and by \( L_i := l(e_i) \). For a simplex \( \sigma = \{l_1, \ldots, l_i\} \in K(\mathbb{F}_p^n) \), define

\[ L(\sigma) := \text{span}\{l_1, \ldots, l_i\}. \]

Now consider the Hasse diagram \( \mathcal{H}(K(\mathbb{F}_p^n)) \) of \( K(\mathbb{F}_p^n) \). We define a partial matching \( U \) on \( \mathcal{H}(K(\mathbb{F}_p^n)) \) inductively in \( n \) steps.

(i) In the first step we match every simplex \( \sigma \in K(\mathbb{F}_p^n) \), such that \( L_1 \not\subseteq L(\sigma) \), with the simplex \( \sigma \cup \{L_1\} \).

Note that all simplices \( \sigma \) which contain \( L_1 \) are matched with \( \sigma \setminus \{L_1\} \) except for \( \{L_1\} \). Also all simplices \( \sigma \) with \( L_1 \not\subseteq L(\sigma) \) are matched with \( \sigma \cup \{L_1\} \), while other simplices of \( K(\mathbb{F}_p^n) \) remain unmatched in the first step.

(ii) In the \( k \)th step we match a simplex \( \sigma \in K(\mathbb{F}_p^n) \) which is not matched in the previous steps with \( \sigma \cup \{L_k\} \) if \( L_k \not\subseteq L(\sigma) \) and if \( \sigma \cup \{L_k\} \) is not matched in the previous steps.

Note that after \( k \) steps simplices from \( K(\mathbb{F}_p^n) \) which are matched are the ones which contain at least one of \( L_1, \ldots, L_k \), except for vertex \( \{L_1\} \), and simplices \( \sigma \) such that \( \text{span}\{L_1, \ldots, L_k\} \not\subseteq L(\sigma) \).

Next we prove that \( U \) is an acyclic partial matching on \( \mathcal{H}(K(\mathbb{F}_p^n)) \). Suppose there is a cycle \( C = (\sigma_1, \ldots, \sigma_t) \) in \( \mathcal{H}(K(\mathbb{F}_p^n)) \). Assume that \( \sigma_1, \sigma_3, \ldots, \sigma_{2t-1} \) are of dimension \( d \), while \( \sigma_2, \sigma_4, \ldots, \sigma_{2t} \) are of dimension \( d + 1 \) and the simplices \( \sigma_{2i-1} \) and \( \sigma_{2i} \) are matched for all \( 1 \leq i \leq t \). Let \( a(i), b(i) \in \{1, \ldots, n\} \) be such indices that

\[ \sigma_{2i} = \sigma_{2i-1} \cup \{L_{a(i)}\} \text{ and } \sigma_{2i+1} = \sigma_{2i} \setminus \{L_{b(i)}\} \text{ for } i = 1, \ldots, t. \]

Note that \( a(i) \neq b(i), a(i+1) \neq b(i) \) and \( a(t+1) = a(1) \). For all \( i \), the simplex \( \sigma_{2i-1} \) is matched with \( \sigma_{2i} = \sigma_{2i-1} \cup \{L_{a(i)}\} \) and consequently \( a(i) > b(i) \) (otherwise...
\[ \sigma_{2i-1} \text{ would be matched with } \sigma_{2i} \text{ by the definition of } U. \]

Thus, the sequence of numbers \( a(i) \) must satisfy the inequality

\[ a(1) > a(2) > \cdots > a(t) > a(1) \]

which is impossible.

Now we determine the number and the dimension of every unmatched simplex. The vertex \( \{L_1\} \) cannot be matched with any 1-simplex of \( K(\mathbb{F}_p^n) \) of type \( \{L_1, L_k\} \) in the \( k \)th step, since the vertex \( \{L_k\} \) is already matched with \( \{L_1, L_k\} \) in the first step. Moreover, \( \{L_1\} \) is the only vertex that is not in \( U \).

Furthermore, a simplex \( \sigma \in K(\mathbb{F}_p^n) \setminus \{\{L_1\}\} \) is not in \( U \) if and only if

\[ \sigma \cap \{L_1, \ldots, L_n\} = \emptyset \]

and

\[ \text{span}\{L_1, \ldots, L_n\} = \mathbb{F}_p^n \subset L(\sigma). \]

Such a simplex \( \sigma \) has to be \((n-1)\)-dimensional because \( L(\sigma) = \mathbb{F}_p^n \). Thus, besides the vertex \( \{L_1\} \), the only unmatched simplices are \((n-1)\)-simplices which do not contain any of \( L_1, \ldots, L_n \). Denote the number of unmatched simplices by \( A_n(p) \). Since all \( U \)-critical points are of the same dimension, \( K(\mathbb{F}_p^n) \) has the homotopy type of the wedge of \( A_n(p) \) spheres \( S^{n-1} \).

The number \( A_n(p) \) is equal to

\[ A_n(p) = \frac{\left| \{ \beta \mid \beta \text{ is a basis of } \mathbb{F}_p^n \text{ and } a e_i \notin \beta \text{ for } i = 1, \ldots, n; a = 1, \ldots, p-1 \} \right|}{(p-1)^n}. \]

On the other hand, using Euler characteristic \( \chi \), we get

\[ \chi(K(\mathbb{F}_p^n)) = 1 + (-1)^n A_n(p) = f_0(K(\mathbb{F}_p^n)) - \cdots + (-1)^n f_{n-1}(K(\mathbb{F}_p^n)) \]

which implies that

\[ A_n(p) = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} f_i(K(\mathbb{F}_p^n)). \]

Finally, using \( f \) we obtain

\[ A_n(p) = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} (\binom{n}{i} - \binom{n}{i+1}) \cdot (p^n - p^{n-i-1}) \cdot (p^n - p^0) \]

This proves part (a).

To prove part (b), let \( \sigma \in K := K(\mathbb{F}_p^n) \) be an \( i \)-simplex. Note that by definition of \( K(\mathbb{F}_p^n) \) and \( L(\sigma) \), a simplex \( \tau \in \text{link}_K \sigma \) if and only if \( L(\sigma) \cap L(\tau) = \emptyset \). As \( GL(n, \mathbb{F}_p) \) acts transitively on \( K(\mathbb{F}_p^n) \), we may assume that \( \sigma = \{L_1, \ldots, L_{i+1}\} \).

We define a matching \( U' \) on the Hasse diagram of \( \text{link}_K(\sigma) \) inductively in \( n-i-1 \) steps.

(i) In the first step every simplex \( \tau \in \text{link}_K(\sigma) \) such that \( L_{i+2} \not\subseteq L(\tau) \) is matched with the simplex \( \tau \cup \{L_{i+2}\} \) if the simplex \( \tau \cup \{L_{i+2}\} \in \text{link}_K(\sigma) \).

(ii) In the \( k \)th step, \( 1 \leq k \leq n-i-1 \), a simplex \( \tau \in \text{link}_K(\sigma) \) which is not matched yet is matched with \( \tau \cup \{L_{i+k+1}\} \) if \( L_{i+k+1} \not\subseteq L(\tau) \), \( \tau \cup \{L_{i+k+1}\} \in \text{link}_K(\sigma) \) and if \( \tau \cup \{L_{i+k+1}\} \) is not matched in \( U' \) in some of the previous steps.

Following the same arguments as in the proof of part (a), we conclude that \( U' \) is an acyclic partial matching on the Hasse diagram of \( \text{link}_K(\sigma) \). Now we determine the number and the dimension of every unmatched simplex. The vertex \( \{L_{i+2}\} \) is not matched in the first step and cannot be matched with any 1-simplex of \( \text{link}_K(\sigma) \) of type \( \{L_{i+2}, L_{i+k+1}\} \) in the \( k \)th step since the vertex \( \{L_{i+k+1}\} \) is already matched with \( \{L_{i+2}, L_{i+k+1}\} \) in the first step.
Now we exclude the vertex \( \{L_{i+2}\} \). A simplex \( \tau \in \mathrm{link}_K(\sigma) \) is not matched in \( U' \) if and only if

\[
\{L_{i+2}, \ldots, L_n\} \cap \tau = \emptyset
\]

and

\[ L_{i+k+1} \subset L(\tau) \quad \text{or} \quad L(\tau \cup \{L_{i+k+1}\}) \cap \sigma \neq \{0\} \quad \text{for all} \quad 1 \leq k \leq n-i-1. \]  

(10)

Because \( L(\tau) \cap L(\sigma) = \{0\} \), condition (10) is equivalent to

\[ L_{i+k+1} \subset L(\tau \cup \sigma) \quad \text{for all} \quad 1 \leq k \leq n-i-1. \]

Thus \( L(\tau \cup \sigma) = \mathbb{F}_p^n \), and therefore \( \tau \) is a maximal, \((n-i-2)\)-simplex, and \( \tau \cup \sigma \) does not contain \( L_{i+2}, \ldots, L_n \). Denote by \( A_{i,n}(p) \) the number of maximal unmatched simplices of \( \mathrm{link}_K(\sigma) \).

Since all \( U' \)-critical points are of the same dimension, \( \mathrm{link}_K(\sigma) \) is homotopy equivalent to a wedge of \( A_{i,n}(p) \) \((n-i-2)\)-dimensional spheres.

Similarly as in part (a), using equality (9) and Proposition 3.7, we calculate the number \( A_{i,n}(p) \) of spheres in the wedge.

This proves (b).

(c) Consider the Hasse diagram of \( X(\mathbb{F}_p^n) \) and define a partial matching \( V \) on it inductively in \( n \) steps.

(i) In the first step we match every simplex \( \sigma \in X(\mathbb{F}_p^n) \) such that \( e_1 \not\in \sigma \) with the simplex \( \sigma \cup \{e_1\} \) provided that \( \sigma \cup \{e_1\} \) is unimodular.

(ii) In the \( k \)-th step we match a simplex \( \sigma \in X(\mathbb{F}_p^n) \), such that \( \sigma \) is not matched yet with \( \sigma \cup \{e_k\} \) if \( e_k \not\in \sigma \) and if \( \sigma \cup \{e_k\} \) is unimodular, that is, if \( \sigma \cup \{e_k\} \in X(\mathbb{F}_p^n) \).

Using the analogous argument to that used in the proof of part (a), we can prove that \( V \) is an acyclic partial matching on the Hasse diagram of \( X(\mathbb{F}_p^n) \). Now we determine the number and the dimension of every unmatched simplex.

Note that all vertices \( \{v\} \) are matched with \( \{v, e_1\} \), except for vertices of the form \( \{ae_1, e_2\} \) for \( a = 1, 2, \ldots, p-1 \). However, those vertices are matched with \( \{ae_1, e_2\} \) for \( a = 2, 3, \ldots, p-1 \). Thus, the only unmatched vertex is \( \{e_1\} \) for \( n \geq 2 \).

Generally, a simplex \( \sigma \in X(\mathbb{F}_p^n) \setminus \{\{e_1\}\} \) is unmatched if and only if

\[ e_k \in \sigma \quad \text{or} \quad e_k \not\in \sigma \quad \text{for all} \quad 1 \leq k \leq n. \]  

(11)

Since \( \mathbb{F}_p \) is a field and \( \sigma \) is unimodular, the only possibility for \( \sigma \cup \{e_k\} \) not to be unimodular is if \( e_k \in \text{span}(\sigma) \). Therefore, condition (11) is equivalent to \( e_k \in \text{span}(\sigma) \), while \( e_k \not\in \sigma \) for all \( k = 1, 2, \ldots, n \). Thus \( \text{span}(\sigma) = \mathbb{F}_p^n \) and \( \sigma \) is a basis for \( \mathbb{F}_p^n \) which does not contain \( e_1, e_2, \ldots, e_n \). Denote by \( B_n(p) \) the number of such basis.

Therefore, the only unmatched simplices, aside of the vertex \( \{e_1\} \), are of dimension \( n-1 \) and there are \( B_n(p) \) such simplices. Thus, \( X(\mathbb{F}_p^n) \) has the homotopy type of the wedge of \( B_n(p) \) spheres \( S^{n-1} \). The number of spheres in the wedge can be calculated as in the previous cases using Euler characteristic (11), deduced formula (11) and Corollary 3.3. This proves (c).

(d) Let \( \sigma \) be an \( i \)-simplex in \( X := X(\mathbb{F}_p^n) \). Because \( GL(n, \mathbb{F}_p) \) acts transitively on \( X(\mathbb{F}_p^n) \), we may assume that \( \sigma = \{e_1, \ldots, e_{i+1}\} \). Define a partial matching \( V' \) on the Hasse diagram of \( X(\mathbb{F}_p^n) \) inductively in \( n-i-1 \) steps.

(i) In the first step every simplex \( \tau \in \mathrm{link}_X(\sigma) \) such that \( e_{i+2} \not\in L(\tau) \) is matched with the simplex \( \tau \cup \{e_{i+2}\} \) if the obtained simplex \( \tau \cup \{e_{i+2}\} \in \mathrm{link}_X(\sigma) \).

(ii) In the \( k \)-th step, \( 1 \leq k \leq n-i-1 \), we match a simplex \( \tau \in \mathrm{link}_X(\sigma) \) such that \( \tau \) is not matched in the previous steps with \( \tau \cup \{e_{i+k+1}\} \) if \( e_{i+k+1} \not\in L(\tau) \) and if \( \tau \cup \{e_{i+k+1}\} \in \mathrm{link}_X(\sigma) \).
Following the same arguments as in the proof of part (a), we conclude that $V'$ is an acyclic partial matching on the Hasse diagram of $\text{link}_X(\sigma)$. Now we determine the number and the dimension of every unmatched simplex.

Note further that, as in the previous cases, the vertex $\{e_{i+2}\}$ is the only vertex which is not in $V'$.

With the analysis as in part (b), a simplex $\tau \in \text{link}_X(\sigma)\setminus \{\{e_{i+2}\}\}$ is not matched if and only if $e_{i+2}, \ldots, e_n \notin \tau$ and $e_{i+2}, \ldots, e_n \in L(\sigma \cup \tau)$. Therefore, $\sigma \cup \tau$ is a basis of $\mathbb{F}_p^m$ which does not contain $e_{i+2}, \ldots, e_n$, $\tau$ has $n-i-1$ elements and is of dimension $n-i-2$.

Therefore, $\text{link}_X(\sigma)$ has the homotopy type of the wedge of $B_{i,n}(p)$ ($n-i-2$)-spheres. Using equality (10) and Proposition 3.8, the number $B_{i,n}(p)$ can be calculated.

5. The homotopy type of $K(\mathbb{Z}^n)$, $X(\mathbb{Z}^n)$ and their links

When $k = \mathbb{Z}$, the considered universal simplicial complexes and their links are infinite complexes. Therefore to find their homotopy type, the classical discrete Morse theory for finite simplicial complexes needs to be modified. The way to extend discrete Morse theory to suite infinite simplicial complexes has been a challenging problem. Recently, several authors have made significant progress in the subject, see [1] and [15].

Let $K$ be an infinite simplicial complex and let $M$ be a Morse matching on $\mathcal{H}(K)$. Let $P$ be an infinite directed simple path in $\mathcal{H}_M(K)$, called a decreasing ray. Two decreasing rays are said to be equivalent if they coincide starting at some point. The degree of a decreasing ray $P$ is the minimal dimension of a simplex in $P$. It is straightforward to see that equivalent rays have equal degrees and that there is a ray equivalent to $P$ containing only simplices of dimensions $\deg P$ and $\deg P + 1$. A Morse matching containing no decreasing rays is called rayless.

To determine the homotopy type of $X(\mathbb{Z}^n)$ and $K(\mathbb{Z}^n)$, we shall use the result of Kukiela [15] which proves that a finite dimensional, possibly infinite, simplicial complex $K$ with a rayless Morse matching $M$ on $\mathcal{H}(K)$ is homotopy equivalent to a $CW$-complex that has exactly one $n$-dimensional cell for each $n$-dimensional critical simplex in $C(M)$.

**Theorem 5.1.**

(a) The simplicial complex $K(\mathbb{Z}^n)$ is homotopy equivalent to a countable infinite wedge of $(n-1)$-spheres.

(b) The link of an $i$-simplex in $K(\mathbb{Z}^n)$ is homotopy equivalent to a countable infinite wedge of $(n-i-2)$-spheres.

(c) The simplicial complex $X(\mathbb{Z}^n)$ is homotopy equivalent to a countable infinite wedge of $(n-1)$-spheres.

(d) The link of an $i$-simplex in $X(\mathbb{Z}^n)$ is homotopy equivalent to a countable infinite wedge of $(n-i-2)$-spheres.

**Proof.** We shall prove part (a) in detail. The proof of other parts of the theorem is similar to the proof of appropriate parts of the Theorem 4.2 with modifications similar to that in part (a).

We start by defining a linear order on lines in $\mathbb{Z}^n$. For $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we say that $a < b$ if and only if

$$\sum_{i=1}^{n} |a_i| < \sum_{i=1}^{n} |b_i|$$

or if

$$\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} |b_i|, a_n = b_n, \ldots, a_{j+1} = b_{j+1}, \text{ and } a_j < b_j \text{ for some } j = n, n-1, \ldots, 1.$$
Each line $l$ in $\mathbb{Z}^n$ has two generators $g$ and $-g$. We pick the one whose first nonzero coordinate in the standard basis is positive and denote it by $g(l)$. For two lines $l$ and $s$, we say that $l < s$ if and only if $g(l) < g(s)$.

This order makes the set of all lines in $\mathbb{Z}^n$ a well ordered set, and we write its elements in an increasing order $l_1, l_2, \ldots, l_k, \ldots$. Recall that $L_i = l(e_i)$ for $1 \leq i \leq n$.

Note that $l_1 = L_1, \ldots, l_n = L_n$.

Define the matching $W$ on the Hasse diagram of $K(\mathbb{Z}^n)$ inductively.

(i) In the first step we match every simplex $\sigma \in K(\mathbb{Z}^n)$ with the simplex $\sigma \cup \{l_1\}$ provided that $\sigma \cup \{l_1\}$ is in $K(\mathbb{Z}^n)$.

(ii) In the $k$th step, we match all simplices $\sigma \in K(\mathbb{Z}^n)$ such that $\sigma$ is not matched in the previous $k - 1$ steps with $\sigma \cup \{l_k\}$, provided that $\sigma \cup \{l_k\}$ is in $K(\mathbb{Z}^n)$. Note that $\sigma \cup \{l_k\}$ is not matched in previous steps.

We split the remaining part of the proof in several steps.

(i) The simplex $\{l_1\}$ is not matched, that is, is not an element of $W$. If $\{l_1\}$ is matched, then it is matched with $\{l_1, l_i\}$, for some $i$. But the simplex $\{l_1, l_i\}$ is already matched, in the first step, with $\{l_1\}$.

(ii) All simplices from $K(\mathbb{Z}^n)$ which are not in $W$ are $\{l_1\}$ and only $(n - 1)$-dimensional simplices.

Let $\sigma \in K(\mathbb{Z}^n) \setminus \{\{l_1\}\}$ be a simplex which is not matched and assume $\dim(\sigma) < n - 1$. Then $L(\sigma)$ is a direct summand of $\mathbb{Z}^n$, that is, $L(\sigma) \oplus A \cong \mathbb{Z}^n$. Let $\{l_1, \ldots, l_k\}$ be a basis of $A$ with $i_1 < \ldots < i_k$. Then $\sigma$ is matched with $\sigma \cup \{l_{i_1}\}$ in $i_1$th step if not before.

(iii) The matching $W$ is an acyclic partial matching on the Hasse diagram of $K(\mathbb{Z}^n)$.

Assume that $W$ is not an acyclic matching and let $C = (\sigma_1, \ldots, \sigma_{2t})$ be a cycle in $\mathcal{H}_M(K(\mathbb{Z}^n))$. Assume that $\sigma_1$ is a simplex in $C$ with the smallest dimension $d$. The cycle $C$ implies that the pairs $(\sigma_1, \sigma_2), \ldots, (\sigma_{2t-1}, \sigma_2)$ are matched. Moreover, $\sigma_1, \sigma_3, \ldots, \sigma_{2t-1}$ are of dimension $d$, while $\sigma_2, \sigma_4, \ldots, \sigma_{2t}$ are of dimension $d + 1$, and $\sigma_3 < \sigma_2, \ldots, \sigma_2t-1 < \sigma_2t-2$ and $\sigma_3 < \sigma_2t$. Define $\sigma_{2t+1} := \sigma_1$ and $\sigma_{2t+2} := \sigma_2$.

More precisely, let $a(i), b(i) \in \mathbb{N}$ be such indices that

$$\sigma_{2i} = \sigma_{2i-1} \cup \{a(i)\} \qquad \text{and} \qquad \sigma_{2i+1} = \sigma_{2i} \setminus \{b(i)\} \quad \text{for } i = 1, \ldots, t.$$ 

Note that $a(i) \neq b(i), a(i + 1) \neq b(i)$ and $a(t + 1) = a(1)$.

Define $A := \{a(1), \ldots, a(t)\}$ and $B := \{b(1), \ldots, b(t)\}, \varepsilon_i := \eta_i \cap A$ and $\eta_i := \sigma_i \setminus \varepsilon_i$. Since we are adding one of $\eta_i$’s and subtracting possibly some other lines, we conclude that $|\varepsilon_{i+2}| \geq |\varepsilon_i|$ for all $i = 1, 2, \ldots, 2t$. Therefore,

$$|\varepsilon_1| \leq |\varepsilon_2| \leq \ldots \leq |\varepsilon_{2t-1}| \leq |\varepsilon_1|$$

implying that $|\varepsilon_1| = |\varepsilon_2| = \ldots = |\varepsilon_{2t-1}|$, that is, we always subtract one of $\eta_i$’s when coming from $\sigma_{2k}$ to $\sigma_{2k+1}$. Therefore, $\eta_1 = \eta_2 = \ldots = \eta_{2t}$.

Since $C$ is a cycle, we may assume that $a(1) = \min\{a(1), \ldots, a(t)\}$.

Consider $\sigma_{2i+1}$. Since $\sigma_{2i+2} = \sigma_{2i+1} \cup \{a(i+1)\}$, the simplex $\sigma_{2i+1}$ is not matched with $l_1, l_2, \ldots, l_{a(i+1)-1}$. But $\sigma_{2i+1}$ could be matched with $l_{b(i)}$ and therefore, since $\sigma_{a(i+1)} \neq l_{b(i)}$ we get

$$a(i + 1) < b(i) \quad \text{for all } i = 1, 2, \ldots, t.$$ 

Since $\{a(1), \ldots, a(t)\} = \{b(1), \ldots, b(t)\}$, in order to get back to $\sigma_1$ there is some $k$ such that $b(k) = a(1)$. Then $a(k + 1) < b(k) = a(1)$, which is not possible since $a(1)$ is minimal. This contradicts the assumption that there is a cycle $C$.

(iv) The matching $W$ does not contain any rays.

Assume that there is a ray $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ of degree $k$. We may assume that $\dim \sigma_1 = k$. Then $\dim \sigma_{2t-1} = k$ and $\dim \sigma_{2t} = k + 1$ for all $t \geq 1$. 


Let \( a(i), b(i) \in \mathbb{N} \) be such indices that
\[
\sigma_{2i} = \sigma_{2i-1} \cup \{a(i)\} \quad \text{and} \quad \sigma_{2i+1} = \sigma_{2i} \setminus \{b(i)\} \quad \text{for} \quad i = 1, \ldots, t, \ldots
\]
Note that \( a(i) \neq b(i), a(i+1) \neq b(i) \) and \( a(i+1) < b(i) \) for all \( i = 1, 2, \ldots \).

Denote by \( w(\sigma) := \sum_{i \in \sigma} i \). Then
\[
w(\sigma_1) > w(\sigma_3) > \ldots > w(\sigma_{2k+1}) > \ldots \]
There is only a finite number of simplices \( \tau \) with \( w(\sigma_1) > w(\tau) \), so we arrive at contradiction.

(v) The simplicial complex \( K(\mathbb{Z}^n) \) is homotopy equivalent to an infinite countable wedge of spheres.

Since \( H_n(K(\mathbb{Z}^n)) \) does not contain any rays and all the critical simplices of \( K(\mathbb{Z}^n) \) are in dimension \( n - 1 \) but one which is in dimension zero, the simplicial complex \( K(\mathbb{Z}^n) \) is homotopy equivalent to a wedge of \( (n - 1) \)-spheres which are in bijection with the critical simplices. The number of spheres in the wedge is countable as there are only countably many simplices in \( K(\mathbb{Z}^n) \). We prove that for \( n \geq 2 \) this number cannot be finite by finding an explicit infinite family of critical simplices.

For \( n \geq 2 \) and \( k \geq 1 \), consider the simplex
\[
\sigma_k := \{ L(e_1 + ke_2), L(2e_1 + (2k - 1)e_2), L(e_1 + e_3), \ldots, L(e_1 + e_n) \}.
\]
Note that \( \sigma_k \) is unimodular and of dimension \( n - 1 \). We shall show that \( \sigma_k \) is not matched with any of its subsimplices because they are already matched with some other simplex.

Consider the subsimplex
\[
\{ L(2e_1 + (2k - 1)e_2), L(e_1 + e_3), \ldots, L(e_1 + e_n) \}.
\]
It cannot be matched with \( \sigma_k \) in the step corresponding to the line \( L(e_1 + ke_2) \), because \( L(e_1 + ke_2) \) is already matched before, in the step corresponding to \( L(e_1 + (k - 1)e_2) \) with the simplex
\[
\{ L(e_1 + (k - 1)e_2), L(2e_1 + (2k - 1)e_2), L(e_1 + e_3), \ldots, L(e_1 + e_n) \}
\]
or maybe in some earlier step.

The subsimplex
\[
\{ L(e_1 + ke_2), L(e_1 + e_3), \ldots, L(e_1 + e_n) \}
\]
is matched in the second step with
\[
\{ L(e_2), L(e_1 + ke_2), L(e_1 + e_3), \ldots, L(e_1 + e_n) \}.
\]
Finally, a subsimplex \( \tau \) of \( \sigma_k \) of the form \( \tau = \sigma_k \setminus \{ L(e_1 + e_i) \} \) for some \( i \geq 3 \) is matched in the step corresponding to \( L(e_i) \) with \( \tau \cup \{ L(e_i) \} \). Therefore \( \{ \sigma_k \mid k \geq 1 \} \) is an infinite family of critical simplices.

\( \square \)

### 6. Shellability of Universal Complexes

In this section the universal complexes \( X(k^n) \) and \( K(k^n) \) will be studied from the viewpoint of shifted, shellable and Cohen-Macaulay complexes.

In commutative algebra Cohen-Macaulay rings and modules have been one of the most studied objects in the last three decades. A **Cohen-Macaulay ring** is a Noetherian commutative unit ring \( R \) in which any proper ideal \( I \) of height \( n \) contains a sequence \( x_1, \ldots, x_n \) of elements (called a regular sequence) such that for all \( i = 1, \ldots, n \), the residue class of \( x_i \) in the quotient ring \( R/(x_1, \ldots, x_{i-1}) \) is a non-zero divisor.

Given an abstract simplicial complex \( K \) on the vertex set \( \{ x_1, \ldots, x_m \} \) and a ring \( k \), the corresponding **Stanley-Reisner ring**, or face ring, of \( K \), denoted \( k[K] \),
is the polynomial ring \( k[x_1, \ldots, x_n] \) quotiented by the ideal \( I_K \) generated by the square-free monomials corresponding to the non-simplices of \( K \):

\[
I_K = \langle x_i \ldots x_k | \{x_{i_1}, \ldots, x_{i_t}\} \notin K \rangle.
\]

A simplicial complex is said to be a Cohen-Macaulay complex if its Stanley-Reisner ring is a Cohen-Macaulay ring.

Reisner’s criterion \([?]\) asserts that a simplicial complex \( K \) is Cohen-Macaulay over \( k \) if and only if for all simplices \( \sigma \subseteq K \), all reduced simplicial homology groups of the link of \( \sigma \) in \( K \) with coefficients in \( k \) are zero except the top dimensional one.

**Proposition 6.1.** The universal complexes \( X(k^n) \) and \( K(k^n) \) are Cohen-Macaulay for \( k = \mathbb{Z} \) or \( \mathbb{F}_p \).

**Proof.** By Reisner’s criteria, it is sufficient to check that for \( L = X(k^n) \) or \( K(k^n) \), if for all simplices \( \sigma \subseteq L, H^i(\text{link}_L(\sigma); k) = 0 \) for all \( 0 \leq i < \dim(\text{link}_L(\sigma)) \). Since by Theorems 4.2 and 5.1, the corresponding links are \((n - i - 3)\)-connected and \((n - i - 2)\)-dimensional CW-complexes the statement follows. \( \square \)

In combinatorial algebra two other families of complexes stand out in relation to Cohen-Macaulay complexes, these are shifted and shellable complexes.

A simplicial complex \( K \) is shifted if there is a labelling of its vertices by some set of positive integers such that different vertices have distinct labels and if in a simplex of \( K \) any vertex is replaced by a vertex with smaller label, then the resulting set of vertices is a simplex of \( K \).

A simplicial complex is shellable if its facets can be arranged in linear order \( F_1, F_2, \ldots, F_q \) in such a way that \( (\bigcup_{i=1}^{q-1} F_i) \cap F_q \) is pure and \((\dim F_k - 1)\)-dimensional for all \( k = 2, \ldots, q \). Such an ordering of facets is called a shelling order or shelling.

It is well known that

\[
\text{shifted} \subset \text{shellable} \subset \text{Cohen-Macaulay}.
\]

Motivated by Björner’s result \([?, \text{Theorem 1.3}]\) that pure shellable complexes have the homotopy type of a wedge of spheres and the results of our Theorems 4.2 and 5.1, we investigate whether the universal complexes \( X(k^n) \) and \( K(k^n) \) are shellable and shifted. We first observe that the universal complexes are not shifted except if they are of dimension zero or one. More precisely, the only universal complexes that are shifted are \( X(k), X(F_2^2), K(k) \) and \( K(k^2) \).

**Proposition 6.2.** The following universal complexes are not shifted:

- (i) \( X(k^n) \) for \( n \geq 2 \) and \( k \neq \mathbb{F}_2 \),
- (ii) \( X(F_2^2) \) for \( n \geq 3 \),
- (iii) \( K(k^n) \) for \( n \geq 3 \).

**Proof.** We prove the statement for \( X(k^n) \) leaving out the proof for \( K(k^n) \) as it is analogous to part (ii) as the 1-skeleton of \( K(k^n) \) is a complete graph.

Assume that \( X(k^n) \) is shifted and denote by \( m_1 \) the smallest vertex in a vertex set ordering of \( X(k^n) \). If \( \{v_i, v_j\} \) is a 1-simplex of \( X(k^n) \), then \( \{m_1, v_i\} \) is also a simplex of \( X(k^n) \). Therefore, \( \{m_1, 2m_1\} \) should be a simplex of \( X(k^n) \), which is not true unless \( k = \mathbb{F}_2 \). This proves that \( X(k^n) \) is not shifted for \( n \geq 2 \) and \( k \neq \mathbb{F}_2 \).

As \( X(F_2^2) = \partial \Delta^2 \), it is shifted. To prove that \( X(F_2^2) \) for \( n \geq 3 \) is not shifted assume that for any given ordering of vertices of \( X(F_2^2) \), the vertices \( m_1 \) and \( m_2 \) are minimal. Assuming that \( X(F_2^2) \) is shifted, if \( \{v_i, v_j, v_\ell\} \) is a 2-simplex of \( X(F_2^2) \), then \( \{m_1, m_2, v_\ell\} \) is also a simplex of \( X(F_2^2) \). Therefore, \( \{m_1, m_2, m_1 + m_2\} \) should be a simplex of \( X(F_2^2) \), which is not true. \( \square \)

**Proposition 6.3.** The universal complexes \( X(k^n) \) and \( K(k^n) \) are shellable.
Proof. First we consider the case $k = \mathbb{F}_p$.

We prove that $K(\mathbb{F}_p^n)$ is shellable omitting the proof for $X(\mathbb{F}_p^n)$ which follows the same lines.

The proof is by induction on $n$. For $n = 1$, $K(\mathbb{F}_p^n)$ is a vertex and thus shellable. Suppose $K(\mathbb{F}_p^n)$ is shellable for some $n$. The standard inclusion $\mathbb{F}_p^n \hookrightarrow \mathbb{F}_p^{n+1}$ induces the canonical inclusion $K(\mathbb{F}_p^n) \hookrightarrow K(\mathbb{F}_p^{n+1})$ and we identify $K(\mathbb{F}_p^n)$ with its image in $K(\mathbb{F}_p^{n+1})$. Denote by $V_0$ the set of vertices of $K(\mathbb{F}_p^n)$ and by $V_1$ the set of the remaining vertices of $K(\mathbb{F}_p^{n+1})$. Assume that the vertices of $V_1$ have total linear ordering as in the proof of Theorem 5.1. This ordering induces total ordering on the sets of all $\ell$ sized subsets of $V_1$ and therefore on $\ell - 1$ simplices spanned by those vertices. Our construction of a shelling will rely on two properties of shellable complexes: any $\ell$-skeleton of a shellable complex is shellable and the link of any simplex of a shellable complex is also shellable.

To construct a shelling on $K(\mathbb{F}_p^{n+1})$ notice that for any vertex $L(v) \in V_1$, due to being linearly independent with simplices of $K(\mathbb{F}_p^n)$, the complex $L(v) \ast K(\mathbb{F}_p^n)$ is a subcomplex of $K(\mathbb{F}_p^{n+1})$. Denote by $\{F_i\}_{i=1}^m$ the shelling of $K(\mathbb{F}_p^n)$. We start building a shelling $\{F_i\}$ of $K(\mathbb{F}_p^{n+1})$ by setting $F_1 = F_0 \ast L(v_1)$. The two defining conditions are satisfied as $(\cup_{i=1}^{j-1} F_i) \cap F_k = (\cup_{i=1}^{j-1} F_i \cap \bar{F}_k) \ast L(v)$ is pure and of dimension $\dim((\cup_{i=1}^{j-1} F_i) \cap \bar{F}_k) + 1 = n - 1$. Using ordering of $V_1$, we continue ordering facets of $K(\mathbb{F}_p^{n+1})$ as $F_{(j-i)m+1} = F_i \ast L(v_j)$. Again it is easy to check that the conditions are satisfied as $(\cup_{i=1}^{j-1} F_i) \cap F_{(j-i)m+k+1} = (\cup_{i=1}^{j-1} m+k F_i) \cap \bar{F}_k \ast L(v_j) = F_{k+1} + 1$.

Next, we observe that for the edge $\{L(v_1), L(v_j)\}$, where $L(v_1)$, $L(v_j) \in V_1$, there is a unique $L(z) \in K(\mathbb{F}_p^n)$ lying in span $\{L(v_1), L(v_j)\}$. Therefore link$_K(\mathbb{F}_p^n)(L(z)) = \{L(v_1), L(v_j)\}$ is a subcomplex of $K(\mathbb{F}_p^{n+1})$ and we continue making the shelling by ordering facets $F_{n-2}^p \ast \{L(v_1), L(v_j)\}$ respecting first ordering on the shelling $\{F_{n-2}^p\}$ of link$_K(\mathbb{F}_p^n)(L(z))$ and the ordering of $\{L(v_1), L(v_j)\}$ for every pair $L(v_1), L(v_j) \in V_1$. To verify that our ordering still satisfies the shelling conditions, notice that the necessary intersection is always of the form $\tau \ast L(v_1) \cup \tau \ast L(v_j)$ for some $(n-2)$-facet $\tau$ of link$_K(\mathbb{F}_p^n)(L(z))$ and therefore $(n-1)$-dimensional pure complex.

Continuing in the same manner, for any simplex $\{L(v_1), \ldots, L(v_j)\} \subset V_1$ there is a unique $(i-1)$-simplex $\sigma \in K(\mathbb{F}_p^n)$ lying in its span. Thus, link$_K(\mathbb{F}_p^n)\sigma = \{L(v_1), \ldots, L(v_j)\}$ is a subcomplex of $K(\mathbb{F}_p^{n+1})$. Using the shelling on link$_K(\mathbb{F}_p^n)\sigma$ and respecting the ordering of $\{L(v_1), \ldots, L(v_j)\} \subset V_1$, we can order facets of $K(\mathbb{F}_p^{n+1})$ of the form $F_{n-i}^p \ast \{L(v_1), \ldots, L(v_j)\}$. After performing these steps, we add the simplices $\{L(v_1), \ldots, L(v_{n+1})\} \subset V_1$. This constructs a shelling on $K(\mathbb{F}_p^{n+1})$ and finishes the proof.

Next we prove that $K(Z^n)$ is a shellable simplicial complex following the same idea as before, that is, by induction on $n$ and using the fact the links of simplices in a shellable simplicial complex are also shellable. Technically the proof is more complicated as $K(Z^n)$ is an infinite complex.

Using the linear order on vertices of $K(Z^n)$ (that is lines in $Z^n$) defined in the proof of Theorem 5.1, we define the linear order on simplices of $K(Z^n)$ in the following way.

Let $l_1(\sigma) > l_2(\sigma) > \ldots > l_k(\sigma)$ and $l_1(\tau) > l_2(\tau) > \ldots > l_s(\tau)$ be vertices of simplices $\sigma$ and $\tau$, respectively. Then $\sigma < \tau$ if

$$l_1(\sigma) = l_1(\tau), \ldots, l_{t-1}(\sigma) = l_{t-1}(\tau), l_t(\sigma) < l_t(\tau) \text{ for some } t = 1, 2, \ldots, n + 1.$$
Note that this ordering makes the set of simplices of $K(\mathbb{Z}^n)$ a well ordered set and that we have if $\sigma$ is a simplex of $\tau$, then $\sigma < \tau$.

For $n = 1$, $K(\mathbb{Z})$ is discrete so it is shellable.

Note that link $K(\mathbb{Z}^n)(\sigma) = link_{K(\mathbb{Z}^n)}(\tau)$ if and only if $L(\sigma) = L(\tau)$, that is, link $K(\mathbb{Z}^n)(\sigma)$ only depends on $L(\sigma)$. We write link $K(\mathbb{Z}^n)(L(\sigma))$ for link $K(\mathbb{Z}^n)(\sigma)$.

Assume, as inductive hypothesis, that $K(\mathbb{Z}^n)$ is shellable and fix the shelling $\{F_i\}_{i=1}^\infty$ of $K(\mathbb{Z}^n)$ as well as all the shellings $\{F_i(L(\sigma))\}_{i=1}^\infty$ of links of $\sigma$ for all simplices $\sigma$ of $K(\mathbb{Z}^n)$.

Note that $K(\mathbb{Z}^n)$ is naturally embedded in $K(\mathbb{Z}^{n+1})$. Let $V_0$ denote the vertices of $K(\mathbb{Z}^{n+1})$ which are in $K(\mathbb{Z}^n)$ and $V_1$ the set of remaining vertices of $K(\mathbb{Z}^{n+1})$.

For a facet $C$ of $K(\mathbb{Z}^{n+1})$, let $\tau(C)$ be the face of $C$ consisting of all vertices of $C$ which are in $V_1$. Note that $\tau(C)$ cannot be empty. Then the sequence

$$F_1(L(\sigma) \cap \mathbb{Z}^n) * \sigma, F_2(L(\sigma) \cap \mathbb{Z}^n) * \sigma, \ldots$$

represents all facets $C$ of $K(\mathbb{Z}^{n+1})$ for which $\tau(C) = \sigma$, where $\sigma$ is a nonempty simplex in $K(\mathbb{Z}^{n+1})$ with vertices from $V_1$. If $\sigma$ is a maximal simplex, then the sequence has only one term: $\sigma$. We write shortly

$$F_i(\sigma) := F_i(L(\sigma) \cap \mathbb{Z}^n).$$

Let $\sigma_1 < \sigma_2 < \sigma_3 < \ldots$ be all simplices from $K(\mathbb{Z}^{n+1})$ with vertices from $V_1$. Then all facets from $K(\mathbb{Z}^{n+1})$ are of the form

$$F_i(\sigma_k) * \sigma_k, \text{ for } i, k = 1, 2, \ldots$$

If $\sigma$ is a vertex from $V_1$ we use the shelling of $K(\mathbb{Z}^n)$. Note that $\sigma_1 = \{l(e_{n+1})\}$.

Now we define linear order on facets inductively on $k$ and $i$ and prove that it is a shelling.

We need each facet to have a finitely many smaller facets. For that we require the following condition to hold

$$F_i(\sigma_k) * \sigma_k < F_j(\sigma_s) * \sigma_s \text{ for } k < s \text{ and } k + i < j + s. \quad (12)$$

We also need that order we are defining respects orders from the shellings of the links, that is, we require

$$F_i(\sigma_k) * \sigma_k < F_j(\sigma_k) * \sigma_k \text{ for } i < j. \quad (13)$$

Thus to get a shelling we need the following condition.

For all $\tau \not\subset \sigma$ and $k \geq 1$, there exists $j \geq 1$ such that

$$F_j(\tau) * \tau < F_k(\sigma) * \sigma \text{ and } F_k(\sigma) < F_j(\tau). \quad (14)$$

For $k = 1$, we use the order obtained from the shelling of appropriate link, that is,

$$F_1(\sigma_1) * \sigma_1 < F_2(\sigma_1) * \sigma_1 < F_3(\sigma_1) * \sigma_1 < \ldots$$

Assume, as inductive hypothesis, that all $F_i(\sigma_k) * \sigma_k$ for $i = 1, 2, \ldots$ and $1 \leq k \leq t - 1$ are linearly ordered so that conditions $\{12\}$, $\{13\}$ and $\{14\}$ hold.

Put $F_1(\sigma_1) * \sigma_1$ in-between any two facets so that conditions $\{12\}$ and $\{14\}$ hold.

For condition $\{12\}$ to hold, $F_1(\sigma_1) * \sigma_1$ has to be bigger than all $F_i(\sigma_k) * \sigma_k$ for $k + i < t + 1$, and there are only finitely many such facets, so it could be done.

For condition $\{14\}$ to hold, we pick for each proper subsimplex $\tau \not\subset \sigma_1$ a facet $F_j(\tau)$ from its shelling, so that $F_1(\sigma_1) \subset F_j(\tau)$. We require $F_j(\tau) * \tau < F_1(\sigma_1) * \sigma_1$. Since there are only finitely many subsimplices of $\sigma_1$ we obtain condition $\{14\}$.

Suppose that we ordered $F_1(\sigma_1) * \sigma_1, \ldots, F_s(\sigma_1) * \sigma_1$ and put $F_{s+1}(\sigma_1) * \sigma_1$ in-between any two facets so that conditions $\{12\}$ and $\{14\}$ hold (this is possible with the same explanation as for $F_1(\sigma_1) * \sigma_1$) and that

$$F_s(\sigma_1) * \sigma_1 < F_{s+1}(\sigma_1) * \sigma_1.$$
This way we defined inductively order on facets from $K(\mathbb{Z}^{n+1})$ which is linear. Condition \ref{cond:linear} guarantees that it is a well ordering.

Fix a facet $F_*(\sigma_k) * \sigma_k$. We claim that
\begin{equation}
F_*(\sigma_k) * \sigma_k \cap (\bigcup \{\text{smaller facets}\}) = (F_*(\sigma_k) \cap (\bigcup_{j=1}^{n-1} F_*(\sigma_k))) * \sigma_k \cup F_*(\sigma_k) * \partial \sigma_k,
\end{equation}
where $\partial \sigma$ is the boundary of a simplex $\sigma$.

Condition \ref{cond:boundary} gives that the first part of the right hand side of \ref{eq:boundary} is a subset of the left hand side, while condition \ref{cond:subset} gives that $F_*(\sigma_k) * \tau$ is a subset of the left hand side for each $\tau \not\subseteq \sigma_k$. Therefore, the right hand side of \ref{eq:boundary} is a subset of the left hand side.

Now we prove the opposite. Let $F_*(\tau) * \tau < F_*(\sigma_k) * \sigma_k$.
If $\tau \cap \sigma_k \neq \sigma_k$, then
$$F_*(\sigma_k) * \sigma_k \cap F_*(\tau) * \tau = (F_*(\sigma_k) \cap F_*(\tau)) * (\sigma_k \cap \tau) \subset F_*(\sigma_k) * \partial \sigma_k.$$  

If $\tau = \sigma_k$ and $F_*(\tau) * \tau < F_*(\sigma_k) * \sigma_k$, then $j < i$ and
$$F_*(\sigma_k) * \sigma_k \cap F_*(\tau) * \tau$$
is obviously a subset of the first part of the right hand side of \ref{eq:boundary}.
If $\sigma_k \not\subseteq \tau$, condition \ref{cond:subset} gives us that there exists $s \geq 1$ such that
$$F_*(\sigma_k) * \sigma_k < F_*(\tau) * \tau < F_*(\sigma_k) * \sigma_k.$$  

We get that $F_*(\sigma_k) * \sigma_k < F_*(\tau) * \tau$ and therefore $s < i$.

We get
$$F_*(\sigma_k) * \sigma_k \cap F_*(\tau) * \tau \subset (F_*(\sigma_k) \cap F_*(\sigma_k)) * \sigma_k$$
which is obviously subset of the first part of the right hand side of \ref{eq:boundary}. This proves the claim \ref{eq:boundary}.

Note that the complex on the right hand side of \ref{eq:boundary} is pure and of dimension $\dim(F_*(\sigma_k) * \sigma_k) - 1$, so we obtained a shelling. 

\[\square\]

7. Application: Buchstaber Invariant

Let $K$ be a finite simplicial complex on vertex set $[m]$ and $(X, A)$, $A \subset X$ be a pair of topological spaces. For every $\sigma \in K$ we define
$$(X, A)^{\sigma} := \{x_1, x_2, \ldots, x_m \in X^m | x_i \in A \text{ if } i \notin \sigma\}.$$

**Definition 7.1.** The polyhedral $K$ product of $(X, A)$ is a topological space

$$(X, A)^K := \bigcup_{\sigma \in K} (X, A)^{\sigma}.$$  

The polyhedral products $(D^2, S^1)^K$ and $(I, S^0)^K$ are of special interests in toric topology and they are called moment-angle complex $\mathbb{Z}_K$ and real moment-angle complex $\mathbb{R}\mathbb{Z}_K$. $\mathbb{Z}_K$ has the natural coordinatewise action of torus $T^m$ while $\mathbb{R}\mathbb{Z}_K$ has the natural coordinatewise action of real torus $\mathbb{R}^m$.

**Definition 7.2.** A complex Buchstaber invariant $s(K)$ of $K$ is a maximal dimension of a toric subgroup of $T^m$ acting freely on $\mathbb{Z}_K$.

A real Buchstaber invariant $s_{\mathbb{R}}(K)$ of $K$ is a maximal rank of a subgroup of $\mathbb{F}_2^m$ acting freely on $\mathbb{R}\mathbb{Z}_K$.

It is a tractable problem in toric topology to find a combinatorial description of $s(K)$ and $s_{\mathbb{R}}(K)$. Buchstaber’s numbers are closely related with universal simplicial complexes $K(\mathbb{Z}^n)$ and $K(\mathbb{Z}^n_2)$. We recall few their properties needed for our work following [3, Section 2].
Proposition 7.5. Let \( K \) be a simplicial complex and let \( K \) be a simplicial complex. Suppose that there is a nondegenerate simplicial map \( f : K \to K(\mathbb{Z}^r) \).

A real Buchstaber invariant of \( K \) is the number \( s_{\mathbb{R}}(K) = m - r \), where \( r \) is the least integer such that there is a nondegenerate simplicial map

\[
f : K \to K(\mathbb{F}_2^r).
\]

Because there are nondegenerate maps \( \phi : X(k^n) \to K(k^n) \) and \( \psi : K(k^n) \to X(\mathbb{Z}^n) \) (see Remark 3.9), we may use \( X(\mathbb{Z}^n) \) instead of \( K(\mathbb{Z}^r) \) in Proposition 7.3 while for the real case \( X(\mathbb{F}_2^r) = K(\mathbb{F}_2^r) \).

Finding explicit values of Buchstaber numbers for a given simplicial complex \( K \) is an important open problem in toric topology and for a general case we know only some estimates of these numbers. A sequence \( \{L^i, i = 1, 2, \ldots \} \) of simplicial complexes is called an increasing sequence if for every \( i \) and \( j \) such that \( i < j \) there is a nondegenerate simplicial map \( f : L^i \to L^j \). Here we observe that we can use Proposition 7.3 to define Buchstaber number \( s_{\mathbb{R}}(L^i) \) for any increasing sequence of simplicial complexes \( \{L^i\} \) instead of \( \{K(\mathbb{F}_2^r)\} \) or \( \{K(\mathbb{Z}^n)\} \). For the sequence of standard simplicial complexes \( \{\Delta^n\} \), we obtain \( m - \gamma(K) \) as its Buchstaber number, where \( \gamma(K) \) is the classical chromatic number of a simplicial complex. Specially, for a prime number \( p \) we can consider the family \( \{K(\mathbb{F}_p^n)\} \) and the corresponding Buchstaber invariant of \( K \) is denoted by \( s_{\mathbb{F}_p}(K) \). Notice that \( s_{\mathbb{R}}(K) \) is \( s_{\mathbb{F}_p}(K) \) in this notation. However, we do not know if there is some other meaning for numbers \( s_{\mathbb{F}_p}(K) \) when \( p \geq 3 \) is a prime number.

Problem 7.4. Is there a topological description of \( s_{\mathbb{F}_p}(K) \) via maximal rank of subgroup of \( \Gamma^m \) acting freely on the polyhedral product \( (X, A)^K \) where \( G \) is a group and \( (X, A) \) a pair of topological \( G \)-spaces?

Ayzenberg and Erokhovets in [2] and [10] studied nondegenerate maps between various increasing sequences of simplicial complexes to establish estimation of Buchstaber’s numbers in terms of known invariants such as the dimension and the chromatic number of simplicial complexes. Recall that the chromatic number of a simplicial complex \( K \) is the minimal number \( \gamma(K) \) such that there is a nondegenerate map \( f : K \to \Delta^{\gamma(K)-1} \). They obtained two interesting results.

Proposition 7.5 (3, Proposition 2). Let \( \{L^i\} \) and \( \{M^i\} \) be two increasing sequences of simplicial complexes such that for each \( i \) there is a nondegenerate map \( f_i : L^i \to M^i \). It holds

\[
s_{\{L^i\}}(K) \geq s_{\{M^i\}}(K).
\]

\[\square\]

Proposition 7.6 (3, Proposition 4). Let \( \{L^i\} \) be an increasing sequence of simplicial complexes and let \( K_1 \) and \( K_2 \) be simplicial complexes on vertex sets \( [m_1] \) and \( [m_2] \), respectively. Suppose that there exists a nondegenerate map \( f : K_1 \to K_2 \). It holds

\[
s_{\{L^i\}}(K_1) - s_{\{L^i\}}(K_2) \geq m_1 - m_2.
\]

\[\square\]

There is an inclusion \( \Delta^{n-1} \) into \( K(\mathbb{Z}^n) \) and a modulo \( p \) reduction map from \( K(\mathbb{Z}^n) \) to \( K(\mathbb{F}_2^r) \) which is nondegenerate for any prime number \( p \). Therefore, by Proposition 7.5 holds

\[
m - \gamma(K) \leq s(K) \leq s_{\mathbb{F}_p}(K) \leq m - \dim K - 1
\]

where \( m \) is the number of vertices of \( K \).
Proposition 7.7. For every simplicial complex $K$, there is an infinite set of prime numbers denoted by $P(K)$ such that for all primes $p, q \in P(K)$

$$s_{P_p}(K) = s_{P_q}(K).$$

Proof. Using the Pigeonhole principle, relation (16) implies the claim since the set of values of $s_{P_p}(K)$ is a finite set, which contradicts the fact that the set of prime numbers is a countable infinite set. □

As an immediate corollary of Propositions 7.5 and 7.6 we get the following result.

Corollary 7.8. Let $M$ be a simplicial complex on vertex set $[m]$ and $p$ a prime number. Then the following inequality holds

$$(17) \quad s(M) \leq s_{P_p}(M) \leq m - \lceil \log_p((p-1)\gamma(M) + 1) \rceil.$$

Proof. We only need to prove the second inequality. Let $r$ be the minimal integer such that there is a nondegenerate map $f: M \to K(\mathbb{F}_p^r)$. There is a nondegenerate map $e: K(\mathbb{F}_p^r) \to \Delta^{|\mathbb{F}_p^r| - 1}$ arising from a bijection from the set of $\frac{k-1}{p-1}$ vertices of $K(\mathbb{F}_p^r)$ onto the vertices of $\Delta^{|\mathbb{F}_p^r| - 1}$. Thus, the composition $e \circ f$ is a nondegenerate map from $M$ to $\Delta^{|\mathbb{F}_p^r| - 1}$. However, $\gamma(M) - 1$ is the minimal dimension of a simplex into which $M$ can be mapped nondegeneratively so

$$\gamma(M) \leq \frac{p^r - 1}{p - 1}.$$

The desired inequality easily follows from the inequality above. □

Using the same argument with appropriate modifications as in the Ayz enberg’s proof of [3, Theorem 1], we deduce the following result.

Proposition 7.9. For any simple graph $\Gamma$, it holds

$$s_{P_p}(\Gamma) = m - \lceil \log_p((p-1)\gamma(\Gamma) + 1) \rceil.$$

Corollary 7.10. Let $\Gamma$ be a simple graph on $m$ vertices and $p$ a prime number. Let $k$ be such natural number so that

$$p^{k-2} + p^{k-3} + \ldots + 1 \leq \gamma(\Gamma) \leq p^{k-1} + p^{k-2} + \ldots + 1.$$

Then

$$s_{P_p}(\Gamma) = m - k.$$

Specially, if $\Gamma$ is not a discrete graph and $\gamma(\Gamma) \leq p + 1$ then $s_{P_p}(\Gamma) = m - 2$.

Proof. For assumed $k$, we have

$$p^{k-1} < (p-1)\gamma(\Gamma) + 1 \leq p^k.$$

We apply $\log_p$ to this inequality and obtain

$$\lceil \log_p((p-1)\gamma(\Gamma) + 1) \rceil = k.$$

Using Proposition 7.9 finishes the proof. □

Remark 7.11. Corollary 7.10 implies that a simple graph $\Gamma$ can be mapped by a nondegenerate map in $K(\mathbb{F}_p^k)$ when $k$ is such that

$$p^{k-2} + p^{k-3} + \ldots + 1 \leq \gamma(\Gamma) \leq p^{k-1} + p^{k-2} + \ldots + 1.$$

It is easy to realise a map like that. Let $f: V(\Gamma) \to [\gamma(\Gamma)]$ be a proper colouring of the vertices of $\Gamma$. Since the number of vertices of $K(\mathbb{F}_p^k)$ is $p^{k-1} + \ldots + 1$, we have an injection $i: [\gamma(\Gamma)] \to K(\mathbb{F}_p^k)$. The composition $i \circ f: V(\Gamma) \to K(\mathbb{F}_p^k)$ has an
Theorem 7.16. Universal complexes $K(\mathbb{F}_p^n)$ are closely related with the projective spaces $\mathbb{P}_p^n$ which are exceedingly studied in finite geometry.

**Definition 7.12.** A finite projective space is a finite set of the points $P$, together with a set of lines $L \subset \mathcal{P}(P)$ satisfying the following axioms:

(i) Every two points $p, q \in P$ lie on the unique line $l(p, q) \in L$;
(ii) Every line from $L$ contains at least 3 points from $P$;
(iii) If $a, b, c, d$ are distinct points and the lines $l(a, b)$ and $l(c, d)$ meet, then so do the lines $l(a, c)$ and $l(b, d)$.

The motivation for studying finite projective spaces came from the theory of combinatorial design. More details about finite projective planes may be found in [1]. A finite projective spaces $\mathbb{P}_p^n$ are projectivization of the vector space $\mathbb{F}_p^{n+1}$ over a finite field $\mathbb{F}_p$. Having a structure of pure simplicial complexes, the universal simplicial complexes $K(\mathbb{F}_p^n)$ from topological point of view are the objects providing concept where classical topics of embeddings and nondegenerate maps naturally appear. Embeddings of graphs into finite projective planes has become an active research area in the last years, see [2]. In the same spirit, we study nondegenerate maps among different universal simplicial complexes $K(\mathbb{k}^n)$.

**Proposition 7.13.** Let $p, q$ be prime numbers and $m$ and $n$ be positive integers. Let $f$ be a nondegenerate map $f : K(\mathbb{F}_p^m) \to K(\mathbb{F}_q^n)$. Then $f$ is an injection.

**Proof.** By dimensional reasons, if $f$ is a nondegenerate map $m$ must be less than $n$. Observe that the 1-skeletons $K(\mathbb{F}_p^m)$ and $K(\mathbb{F}_q^n)$ are complete graphs, so nondegeneracy of $f$ implies $f$ is an injection on the set of vertices of $K(\mathbb{F}_p^m)$. It follows that $f$ is an injection since it is a simplicial map. 

An instant consequence of Proposition [7.13] is the following fact.

**Corollary 7.14.** If there is a nondegenerate map $f : K(\mathbb{F}_p^m) \to K(\mathbb{F}_q^n)$, then it holds

\begin{align*}
(18) & \quad m \leq n, \\
(19) & \quad f_i(K(\mathbb{F}_p^m)) \leq f_i(K(\mathbb{F}_q^n)) \text{ for all } i. 
\end{align*}

For any two prime numbers $p$ and $q$, define a function $\varsigma_{p,q}(n)$ that assigns to each positive integer $n$ the minimal integer $r$ such that there is a nondegenerate map from $K(\mathbb{F}_p^n)$ to $K(\mathbb{F}_q^r)$. Similarly, for a prime number $p$ define a function $\vartheta_p(n)$ such that $\vartheta_p(n)$ is the minimal integer $r$ such that there is a nondegenerate map from $K(\mathbb{F}_p^n)$ to $K(\mathbb{Z}^r)$.

**Proposition 7.15.** The functions $\varsigma_{p,q}(n)$ and $\vartheta_p(n)$ are increasing.

**Proof.** The statement follows from the inclusion of $K(\mathbb{k}^n)$ into $K(\mathbb{k}^{n+1})$ and Proposition [7.13]. 

**Theorem 7.16.**

\[ \log_q \left( \frac{(q-1)(p^n-1)}{p-1} + 1 \right) \leq \varsigma_{p,q}(n) \leq \frac{p^n-1}{p-1}. \]

**Proof.** Corollary [7.14] implies the following relation

\[ \frac{p^n-1}{p-1} \leq q^{\varsigma_{p,q}(n)} - 1 \quad \frac{q^n-1}{q-1}. \]
On the other hand, there are inclusions
\[(21) \quad K(\mathbb{F}_p^n) \hookrightarrow \Delta \frac{p^n - 1}{p} - 1 \hookrightarrow K(\mathbb{F}_q^n).
\]
From (20) and (21), we deduce the desired inequalities. \hfill \Box

By Theorem 7.16, we get that \(\varsigma_{p,q}(n)\) is bounded above with \(\frac{p^n - 1}{p} - 1\), implying that it is well defined.

For any positive integer \(n\), there is a composition of nondegenerate maps
\[K(\mathbb{F}_p^n) \to K(\mathbb{Z}^{\vartheta_p(n)}) \to K(\mathbb{F}_q^{\vartheta_q(n)})\]
implying the following statement.

**Proposition 7.17.** For every positive integer \(n\) and any two prime numbers \(p\) and \(q\), it holds
\[\vartheta_p(n) \geq \varsigma_{p,q}(n).\]
\hfill \Box

The sharpest lower bound for \(\vartheta_p(n)\) in Proposition 7.17 is achieved for \(q = 2\) so
\[\vartheta_p(n) \geq \left\lceil \log_2 \left( \frac{p^n - 1}{p - 1} + 1 \right) \right\rceil.\]

The similar reasoning yielding (21) implies that
\[\vartheta_p(n) \leq \frac{p^n - 1}{p - 1}.
\]

8. **Application: Bhargava’s generalized factorial function**

A fundamental interpretation of the factorial function is the number of all permutations of a set with \(n\) elements. Its countless appearance in numerous combinatorial formulas such as binomial coefficients, Stirling numbers, Catalan numbers, etc. makes the factorial function one of the principal functions in combinatorics. However, the factorial function has important appearances in number theory and mathematical analysis as well, that motivated investigation of series of its generalisations that are applied in variety of number-theoretic, ring-theoretic and combinatorial problems.

Bhargava [4] considered one such generalisation of the classical factorial function on \(\mathbb{Z}\) that satisfies the corresponding analogues of classical theorems of elementary number theory: the binomial coefficient theorem, the theorem about the greatest common factor of the set of all integer values of a primitive polynomial (that is, a polynomial whose coefficients are relatively prime numbers), the theorem on the number of polynomial functions from \(\mathbb{Z}\) to \(\mathbb{Z}_n\), and the theorem about the product of pairwise differences of any \(n + 1\) integers.

Let \(S\) be a subset of \(\mathbb{Z}\) and \(p\) be a prime number.

**Definition 8.1.** A \(p\)-ordering on \(S\) is a sequence of elements \((a_i)_{i=0}^{\infty}\) formed by induction according to the following rules:

(i) Choose an element \(a_0 \in S\) arbitrary.

(ii) For any positive integer \(n\), the integer \(a_n\) is chosen so that it minimises the highest powers of \(p\) dividing \((a_n - a_{n-1}) \cdots (a_n - a_1)(a_n - a_0)).

For a \(p\)-ordering on \(S\) and \(k\) a positive integer, the number \(\nu_k(S,p)\) is defined to be the highest power of \(p\) dividing \((a_k - a_{k-1}) \cdots (a_k - a_1)(a_k - a_0).\) There are many distinct \(p\)-orderings on \(S\), but the numbers \(\nu_k(S,p)\) are independent of a choice of \(p\)-ordering. Therefore, \(\nu_k(S,p)\) are invariant of the subset \(S\), as it was proved by
Bhargava in [4]. Using this observation, Bhargava defined the generalised factorial function for any subset $S$ as

$$k!_S = \prod_p \nu_p(S, p)$$

where the product is taken over all prime numbers.

For more details about the generalised factorial function and its number-theoretical properties see the aforementioned Bhargava’s paper. In the same paper the author motivated by the classical combinatorial description of the standard factorial function formulated the following problem.

**Problem 8.2 ([4], Question 27).** For a subset of $S$, is there a natural combinatorial interpretation of the number $k!_S$.

We solve Problem 8.2 when $S$ is the set of powers of a prime number $p$. Bhargava [4] that if $S$ is a geometric progression in $\mathbb{Z}$ with common ratio $q$ and first term $a$ then

$$(22) \quad k!_S = a^k(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}).$$

**Proposition 8.3.** For the set $S$ of powers of a prime number $p$, the generalised factorial function $i!_S$ is $i!$ multiple of the number of $f_{i-1}(X(\mathbb{F}_p^i))$.

**Proof.** It is easy to verify that $1, p, p^2, p^3, \ldots$ forms a $l$-ordering of $S$ for all primes $l$. Using identity (22) and Theorem 3.3,

$$i!_S = (p^i - p^{i-1}) \cdots (p^i - p^0) = i!f_{i-1}(X(\mathbb{F}_p^i)).$$

We remark that in this case there is another interpretation for $i!_S$ as the number of $(i-1)$-simplices of the Tits building $T(\mathbb{F}_p^i)$, that is, the number of unimodular sequences in $\mathbb{F}_p^i$ of length $i$, see [19, Definition 2.3].

Bhargava in [4] established that for $T \subset S \subset \mathbb{Z}$, generalised factorial function $k!_S$ divides $k!_T$ for all positive integers $k$.

**Lemma 8.4.** There is a combinatorial interpretation of divisibility of $k!_T$ by $k!_S$ when $S = \mathbb{Z}$ and $T = \{1, p, \ldots, p^k, \ldots\}$.

**Proof.** By Theorem 3.3

$$f_{k-1}(X(\mathbb{F}_p^k)) = \frac{(p^k - p^{k-1}) \cdots (p^k - p^0)}{k!} = \frac{k!_T}{k!_S}.$$

It would be interesting to find a purely combinatorial proof of Bhargava’s result on divisibility in general.

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