Coupling the beam tracing code TORBEAM and the Fokker-Planck solver RELAX for fast electrons

O Maj¹, E Poli¹ and E Westerhof²
¹ Max Planck Institute for Plasma Physics, EURATOM Association, 85748 Garching bei München, Germany.
² FOM Institute DIFFER - Dutch Institute for Fundamental Energy Research, Association EURATOM-FOM, Trilateral Euregio Cluster, Nieuwegein, The Netherlands.
E-mail: omaj@ipp.mpg.de

Abstract. In this paper the interface between the beam tracing code TORBEAM [Poli, Peeters and Pereverzev, Comp. Phys. Comm. 136, 90 (2001)] and the quasi-linear Fokker-Planck solver RELAX [Westerhof, Peeters and Schippers, Rijnhuizen Report No. RR 92-211 CA, 1992] is presented together with preliminary testing results for electron cyclotron waves in ITER plasmas and their effects on the electron distribution function. The resulting numerical package allows us to account for diffraction effects in the construction of the quasi-linear wave-particle diffusion operator. The coupling of the paraxial-WKB code TORBEAM to the ray-based code RELAX requires a re-interpretation of the paraxial wave field in terms of extended rays, which are addressed in details.

1. Introduction
Electron cyclotron resonance heating, which exploits the resonant interaction of electrons with a wave beam tuned in the electron cyclotron frequency range, is a major source of fast (suprathermal) electrons in tokamak plasmas. Such suprathermal electrons are essentially generated on the high-energy tail of the distribution function, as quasi-linear effects on bulk electrons are typically very weak [1]. The detailed knowledge of such high-energy tails is essential for the understanding of soft and hard X-ray diagnostics.

A precise study of the electron distribution function requires a description of both the power carried by the wave beam and the velocity-space diffusion of the distribution function due to collisions and wave scattering.

Typically, electron cyclotron wave beams are highly collimated/focused, in order to deposit the injected power “surgically” in the plasma, and, in such regimes, diffraction effects should be properly accounted for in the description of the wave propagation. This motivates the use of the beam tracing code TORBEAM [2], which, being based on the paraxial WKB approximation developed by Pereverzev [3, 4], can describe diffracting beams in a natural way.

The electron distribution function is computed by the relativistic bounce-averaged Fokker-Planck solver RELAX [5], which, however, cannot process the TORBEAM paraxial wave field directly, as it has been optimized for a ray-based description of the wave beam.

Our strategy has been obtaining a ray-based description of the paraxial WKB solution, which fully retains the effects of diffraction: this is made possible by making use of extended rays. A novel module has been implemented in TORBEAM for the calculation of extended rays.
defined as the field lines of the wave energy flux vector. This requires the integration of an additional system of ordinary differential equations with coefficients determined in terms of the main TORBEAM output. An appropriate scheme is applied in order to avoid interpolations of TORBEAM data. Extended rays are then passed to the interface with the Fokker-Planck solver RELAX for the calculation of the electron distribution function.

The interface has been written from scratch on the model of the existing interface of RELAX with the ray tracing code TORAY-FOM [6, 7]. For this part of the package, the language of choice is Python\(^1\), which allows us to take advantage of object-oriented programming as well as of the scientific libraries NumPy and SciPy\(^2\). Time consuming procedures have been written in Cython\(^3\), a C extension for Python, in order to avoid the performance shortcomings of interpreted programming languages such as Python.

The physics ideas at the basis of TORBEAM extended rays are discussed in section 2, while preliminary numerical results are presented in section 3 for the specific case of power deposition on the flux surface \( q = 3/2 \) in ITER scenario 2, which is a very well documented test bed [7, 8].

2. Extended ray tracing and diffraction effects.

The physics basis of coupling TORBEAM to RELAX without modifications of the latter relies on the representation of diffracting paraxial wave fields in terms of a bundle of trajectories that (i) reduce to geometrical optics rays when the beam is non diffracting and (ii) represent the wave energy flow to leading order in the high frequency limit. The second condition is particularly important for building the input data to the RELAX code, in which it is assumed that rays describe the energy flow. Trajectories satisfying requirement (i) are called extended rays and have been first obtained in the framework of the complex eikonal method [9-13].

2.1. Extended rays

In order to fix the notation, let us begin from the standard definition of geometrical optics rays [14] for the construction of asymptotic solutions of the Maxwell’s wave equation, which, in normalized form, reads [4],

\[
\nabla \times (\nabla \times E(\kappa, x)) - \kappa^2 \varepsilon(\kappa, x)E(\kappa, x) = 0, \quad \kappa = \omega L/c \to +\infty, \tag{1}
\]

where \( L \) is the typical scale length of the medium inhomogeneity, \( x = (x^1, x^2, x^3) \) is the spatial position normalized to \( L \), \( E(\kappa, x) \) is the wave electric field of a beam of frequency \( \omega \) in a stationary spatially non-dispersive medium with dielectric tensor \( \varepsilon(\kappa, x) = \varepsilon_0(x) + \kappa^{-1}\varepsilon_1(x) \) and with \( \varepsilon_0(x) \) being Hermitian (weakly dissipative medium).

Geometrical optics rays are the spatial projection \( x = x(t) \) of orbits \( (x(t), N(t)) \) of the Hamiltonian system

\[
dx/dt = \nabla_N H(x, N), \quad dN/dt = -\nabla_x H(x, N), \tag{2}
\]

with the constraint \( H(x, N) = 0 \), and \( N = (N_1, N_2, N_3) \) has the physical meaning of the refractive index vector carried by the ray. The Hamiltonian function \( H(x, N) \) is in the wave-\( x \)-\( N \) phase-space is obtained as follows. First, one considers the dispersion tensor \( D_0(x, N) \), given in components by

\[
D_{0,ij}(x, N) = N^2\delta_{ij} - N_iN_j - \varepsilon_{0,ij}(x), \tag{3}
\]

which is the leading order term in the plane-wave response, namely,

\[
e^{-i\kappa N \cdot x} \mathcal{L}(\kappa, x, \nabla)e^{i\kappa N \cdot x} = \kappa^2 D_0(x, N) + O(\kappa),
\]

\(^1\) http://www.python.org/
\(^2\) http://numpy.scipy.org/
\(^3\) http://www.cython.org/
\( L \) being the operator on the left-hand side of (1). With our assumptions, \( D_0 \) is Hermitian, and its eigenvectors determine the polarization of wave modes supported by the medium. Let \( \epsilon(x, N) \) be the specific eigenvector corresponding to the considered wave mode (mode conversion is not discussed here), then the eigenvalue \( H(x, N) = \epsilon^*(x, N) \cdot D_0(x, N) \epsilon(x, N) \) is the Hamiltonian in equation (2).

Upon computing (when possible) the integral of \( ds = N \cdot dx \) along solutions of (2) one obtains the action function \( s(x) \) which solves the Hamilton-Jacobi equation

\[
H(x, \nabla s) = 0. 
\] (4)

We can now define the normalized group velocity field

\[
V(x) = \nabla_N H(x, \nabla s(x)), 
\] (5)

which has the property of being tangent to geometrical optics rays, i.e., \( dx(t)/dt = V(x(t)) \).

In addition, the vector field \( V \) determines the wave energy transport, which is governed by the amplitude transport equation

\[
\nabla \cdot [V(x)|A(x)|^2] = -2\gamma(x)|A(x)|^2, 
\] (6)

where \( A(x) \) is the complex scalar amplitude of the wave field, and \( \gamma(x) = \epsilon^*(x, \nabla s) \epsilon^*_i(x) e(x, \nabla s) \), with \( \epsilon^*_i \) being the anti-Hermitian part of \( \epsilon_1 \). The wave electric field is then obtained in the form

\[
E(\kappa, x) = e(x, \nabla s) A(x) e^{i\kappa s(x)}, 
\] (7)

and one can check that this is an asymptotic solution of (1) in the limit \( \kappa \to +\infty \).

Summarizing, in the standard geometrical optics theory, the wave energy density flux vector is proportional to the vector \( V(x) \) defined in equation (5), and thus to geometrical optics rays.

The standard geometrical optics theory, however, does not allow us to describe diffractive beams, which is a severe limitation when it comes to the electron cyclotron beams currently employed. The complex eikonal theory [9-13] is an extension of geometrical optics which includes diffractive beams. The solution (7) is replaced by

\[
E(\kappa, x) = e(x, \nabla s) A(x) e^{i\kappa s(x) - \kappa \phi(x)}, 
\] (8)

where \( e(x, N) \) is the same as in (7), but the amplitude \( A(x) \) and the phase \( s(x) \) are modified by the introduction of a new non-negative function \( \phi \geq 0 \), which is regarded as the imaginary part of the complex phase (eikonal) \( \psi = s + i\phi \). As a consequence the Hamilton-Jacobi equation (4) is replaced by

\[
H(x, \nabla s) - \frac{1}{2} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \frac{\partial^2 H}{\partial N_i \partial N_j}(x, \nabla s) = 0, 
\] (9)

together with the constraint \( \nabla \phi \cdot \nabla_N H(x, \nabla s) = 0 \). We note from (8) that the envelope of the wave field \( e^{-\kappa \phi} \) has a maximum where \( |\nabla \phi| = 0 \), which is where the wave field is localized. We are therefore interested in a region of space where \( |\nabla \phi| \) is actually small so that the second term in (9) acts as a perturbation of the standard Hamilton-Jacobi equation. The corresponding Hamiltonian orbits are the extended rays of complex geometrical optics. It is worth noting that Hamilton’s equations for extended rays depend on \( \nabla \phi \), which, in turn, satisfies a partial differential constraint.

We can still define the vector field \( V(x) \) by equation (5), \( s \) being now a solution of (9), but, differently from geometrical optics rays, for extended rays we have,

\[
\frac{dx}{dt} = V(x) - \frac{1}{2} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \frac{\partial^3 H}{\partial N_i \partial N_j \partial N}, 
\] (10)
i.e., \( V(x) \) is no longer exactly tangent to ray trajectories. It is however approximately tangent to ray trajectories in the region of interest, where \(|\nabla \phi|\) is small enough.

As for the wave energy flux it can be shown [15] that equation (6) holds in complex geometrical optics as well, with the only difference that the function \( s \) is now a solution of the modified Hamilton-Jacobi equation (9).

It follows that, according to the standard definition, extended rays satisfy the first requirement stated at the beginning of this section, i.e., (i) they reduce to geometrical optics rays when diffraction is negligible. The second requirement on the energy flux, however, is not exactly satisfied, cf. equation (10), though it is approximately satisfied.

One could define a second set of extended rays as the field lines of the vector field \( V(x) \), which is proportional to the wave energy flux. Such rays satisfy condition (ii) on the energy flow by construction and, in addition, satisfy condition (i) as well, since the function \( s \) reduces to the corresponding geometrical optics solution for non diffracting beams.

Quantitatively, the differences between extended rays computed according to the foregoing two definitions is found to be small in the neighborhood of the beam trajectory [15], so that the distinction of the two different bundle of extended rays has no practical consequence.

In addition, one should note that the calculation of the field lines of the vector \( V(x) \) requires the knowledge of the phase function \( s(x) \), which, in turn, requires the calculation of standard extended rays.

### 2.2. The case of paraxial WKB wave fields and extended rays from \textsc{torbeam}

The paraxial WKB solution for the lowest order Gauss-Hermite mode [3, 4] is formally the same as equation (8), but with a different representation of the amplitude \( A \), the phase \( s \), and the field envelope \( \phi \), which we briefly recall. The trajectory of the beam is identified with the locus of points \( \phi(x) = 0 \) and it is found to be a particular geometrical optics ray \( x = \bar{x}(t) \), called reference ray and carrying the refractive index \( N = N(t) \). In a neighborhood of the reference ray, one extends the parameter \( t \) to a curvilinear coordinate \( t(x) \), and consider the Taylor expansion

\[
A(x) \approx A(t(x)), \quad s(x) \approx s_0(t(x)) + \bar{N}(t(x)) \cdot \ddot{x} + \frac{1}{2} \ddot{x} \cdot S(t(x)) \ddot{x}, \quad \phi(x) = \frac{1}{2} \ddot{x} \cdot \Phi(t(x)) \ddot{x},
\]

where \( \ddot{x} = x - \ddot{x}(t(x)) \), while the amplitude \( A(t) \) and the two beam tracing matrices \( S(t) \) and \( \Phi(t) \) satisfy ordinary differential equations describing their evolution along the reference ray. (The first two terms in the expansion of \( \phi \) vanish because of the definition of reference rays and the condition \( \phi \geq 0 \).) The physical meaning of the two symmetric matrices \( S(t) \) and \( \Phi(t) \) has been extensively discussed [2-4].

Also in the paraxial WKB method [3, 4], the wave energy flux is proportional to the wave field \( V(x) \) defined in equation (5) with \( s(x) \) given in the form (12). In order to see this, let us note that equation (1) implies the continuity equation \( \nabla \cdot F = -2E^* \cdot \varepsilon_1^E \), where \( \varepsilon_1^E \) is defined after equation (6) and

\[
F(\kappa, x) = \frac{2}{\kappa} \text{Im}[E^*(\kappa, x) \times (\nabla \times E(\kappa, x))],
\]

is the Poynting flux properly normalized. We substitute the wave object (8) for \( E(\kappa, x) \) and, retaining the lowest order term only in the limit \( \kappa \to +\infty \), we obtain

\[
F(\kappa, x) = [2\nabla s - (e^* \cdot \nabla s)e - (e \cdot \nabla s)e^*] e^{-2\kappa \phi} |A|^2 + R_1(\nabla \phi) + O(1/\kappa),
\]

the remainder \( R_1(\nabla \phi) \) being proportional to \( \nabla \phi \). The term in square brackets on the right-hand side reads

\[
[2\nabla s - (e^* \cdot \nabla s)e - (e \cdot \nabla s)e^*] = e^*(x, \nabla s) \cdot \frac{\partial D_0(x, \nabla s)}{\partial N}e(x, \nabla s),
\]
and, again, upon differentiating the eigenvalue equation $D_0e = He$ with respect to the refractive index and making use of (9), one has

$$e'(x, \nabla s) - \frac{\partial D_0(x, \nabla s)}{\partial N}e(x, \nabla s) = \nabla_N H(x, \nabla s) + R_2(\nabla \phi),$$

where the remainder $R_2(\nabla \phi)$ is quadratic in $\nabla \phi$. In both the complex eikonal and the paraxial WKB theories, the estimate $|\nabla \phi|^n e^{-\kappa \phi} = O(\kappa^{-n/2})$ is proven for every non-negative integer $n \geq 0$, [4, 15], and that implies

$$\bar{F}(\kappa, x) = V(x)e^{-2\kappa \phi(x)}|A(x)|^2 + O(1/\sqrt{\kappa}),$$

showing that the leading order term in the (normalized) Poynting flux is proportional to the vector field $V(x) = \nabla_N H(x, \nabla s)$. This asymptotic form of the Poynting flux vector is obtained regardless for the specific representation of the functions $A$, $s$, and $\phi$, and, thus, it holds for both the complex eikonal method and the lowest-order Hermite-Gaussian mode of the paraxial WKB solution.

In the latter case, one can make use of equation (12) in order to compute $\nabla s(x)$ up to an $O(\tilde{x}^2)$ remainder, namely, $\nabla s(x) = \tilde{N}(t(x)) + S(t(x))\tilde{x} + O(\tilde{x}^2)$ and this approximation can be used in the calculation of the vector field $V(x) = \nabla_N H(x, \nabla s(x))$ in a neighborhood of the reference ray. This leads to the following system of ordinary differential equations for extended rays,

$$\frac{dx}{dt} = \nabla_N H(x, \nabla s(x)), \quad \nabla s(x) = \tilde{N}(t(x)) + S(t(x))[x - \tilde{x}(t(x))],$$

(16)

where the vector $\tilde{N}(t)$ and the symmetric matrix $S(t)$ are obtained from the solution of the standard equations of paraxial WKB [2-4].

Equations (16) are used to compute the bundle of extended rays in TORBEAM. With this aim an appropriate numerical scheme has been developed on the basis of the second-order Heun’s method for the solution of ordinary differential equation, which belongs to the family of Runge-Kutta methods. This is, indeed, a rather unconventional choice of Runge-Kutta method, but it allows us to combine nicely the calculation of extended rays with the main TORBEAM calculation, without making use of interpolations for the refractive index vector $\tilde{N}(t)$ and the matrix $S(t)$.

Let $x^n \approx x(t_n)$ be an approximation of the position of a ray at the discrete time-like parameter $t_n$, and, analogously, $(\nabla s)^n \approx \nabla s(x(t_n))$; then, the second-order Heun’s integration step for the position of the rays at the next time-like point $t_{n+1} = t_n + \tau_n$, $\tau_n$ being the local step size, can be written as a predictor-corrector scheme,

$$x^{n+1} = x^n + \frac{1}{2}[\nabla_N H(x^n, (\nabla s)^n) + \nabla_N H(x^*, (\nabla s)^*)]\tau_n,$$

(17)

where $x^*$ is the (explicit Euler) predictor for the advanced position of the ray, $(\nabla s)^* \approx \nabla s(x^*)$, and $x^{n+1}$ is the corrected advanced position of the ray. Under the assumption that the curvilinear coordinate $t(x)$ along the propagation of the beam satisfies the condition $t(x^n) = t_n$, at least approximately, the gradients of the phase are computed by

$$(\nabla s)^n = \tilde{N}(t_n) + S(t_n)[x^n - \tilde{x}(t_n)], \quad (\nabla s)^* = \tilde{N}(t_{n+1}) + S(t_{n+1})[x^* - \tilde{x}(t_{n+1})].$$

(18)

All coefficients in the integration step (17) are then known from the standard TORBEAM output at the time-like points $t_n$ and $t_{n+1}$, and the bundle of extended rays can be advanced in parallel to and without interfering with the main TORBEAM calculation. It is worth noting that, in contrast to the complex eikonal method, rays here are not used to compute the phase $s$ and the beam.
**Figure 1.** Three-dimensional representation of the extended rays obtained from the beam tracing code TORBEAM for the specific case of the ITER upper launcher, for the standard scenario 2 with power deposition on the flux surface $q = 3/2$. Rays are color-coded with the fraction of carried power (from red for high power to blue for low power) and the position of the beam with respect to the ITER separatrix is shown.

Envelope $\phi$, as those are already provided by the TORBEAM code; thus, reasonable results can be obtained with a limited number of extended rays.

Figure 1 shows an array of $9 \times 9$ extended rays computed by a new module implemented in the beam tracing code TORBEAM on the basis of the foregoing numerical scheme. Launching conditions are those relevant to the ITER upper launcher for power deposition on the flux surface $q = 3/2$ in the standard scenario 2; specifically the beam is assumed to be stigmatic and to have an initially circular cross section of width $w = 4.599\,\text{cm}$ and phase-front radius of curvature $R = 193.4\,\text{cm}$; poloidal and toroidal injection angles are $\alpha = 53^\circ$ and $\beta = 20^\circ$, respectively; the coordinates of the launching position are $(688., 0., 418.)$ in centimeters. One can note that extended rays do not cross each other: diffraction effects appear to be properly accounted for.

### 3. Numerical results

In this section, preliminary numerical results of the Fokker-Planck calculation are presented, with the aim of testing the TORBEAM-RELAX package.

For ITER parameters, quasi-linear effects are completely negligible [1, 7] and thus linear power-deposition and current drive profiles, obtained from TORBEAM, should agree with the corresponding quasi-linear profiles obtained from RELAX.

In figure 2 the results for the ITER upper launcher scenario 2, with power deposition on the $q = 3/2$ flux surface, are reported; beam parameters are the same as in figure 1, the launched power is $1\,\text{MW}$, and the array of $9 \times 9$ extended rays of figure 1 is used. (Qualitatively similar results can also be obtained with a $3 \times 3$ array.) The non-linearity parameter of Harvey et al. is $(dP/dV[\text{W/cm}^3])/N[10^{13}\text{cm}^{-3}]^2 = 0.0004$ [1], whereas quasi-linear effects start to be relevant for $(dP/dV[\text{W/cm}^3])/N[10^{13}\text{cm}^{-3}]^2 \approx 0.5$, which, for the considered case, is achieved for $\approx 1.1\,\text{GW}$ of injected power. One can note that the temperature profile is practically unchanged by the Fokker-Planck calculation and the anisotropy induced by the wave-particle interaction, quantified by the ratio of perpendicular and parallel energies, is extremely low. Both these results confirm that quasi-linear effects are totally negligible and, therefore, the current drive
and the power deposition profile computed by TORBEAM and RELAX should be the same. The agreement shown in figure 2, lower panels, is qualitatively the same as that observed with the TORAY-RELAX package [7]; the total absorbed power in the two codes agrees within an error of \( \approx 1.1\% \). Let us remark that both the linear code TORBEAM and the quasi-linear code RELAX compute the current drive and power deposition profiles independently and without any ad hoc renormalization of the absorbed power. Furthermore, a momentum conserving electron-electron collision operator is used in both TORBEAM and RELAX current drive calculations.

On going back to the anisotropy index, figure 2 upper right panel, we note that the perpendicular energy is actually reduced with respect to the parallel energy; this result might appear counter-intuitive as one would expect cyclotron wave to increase the perpendicular energy. In order to understand this point, we can increase the injected power holding constant the other parameters up to the level where quasi-linear effects are strong enough to be appreciated in the electron distribution function. For the case of the ITER tokamak, this takes an unrealistic value of the injected power (8GW, corresponding to the non-linearity parameter of Harvey et al. (\(dP/dV\)[W/cm\(^3\)]/\(N[10^{13}\text{cm}^{-3}]^2 = 3.76\)).

Figure 3 shows the contour plot of the electron distribution function in the \((p_\parallel, p_\perp)\)-plane in logarithmic scale, at the crossing point of the beam on the surface where the maximum absorbed power density is achieved, i.e., \(\rho = \sqrt{\psi} = 0.745\) (this value is significantly shifted with respect to the maximum of the deposition profile for the 1MW case). The black circle denotes the locus of points where the first-harmonic wave-particle resonance condition

\[
\gamma - \frac{\omega_c}{\omega} N_{\parallel} \frac{P_\parallel}{m_e c} = 0,
\]  

(19)
is satisfied, with the electron cyclotron frequency $\omega_c$ and the parallel refractive index $N_\parallel$ being estimated by their average values over crossing points of rays through the considered magnetic surface. We can note that the deformation of the electron distribution function due to the interaction with the electron cyclotron beam occurs in the region of the velocity space delimited by the resonance condition as expected. For the current drive scenario under consideration such region is localized in the part of the velocity space for which $p_\parallel < 0$ and $p_\perp$ is small. The section of the distribution function at fixed velocity pitch-angle $\vartheta$ also shows a significant deformation for $\vartheta = \pi$ only, i.e., in the direction anti-parallel to the equilibrium magnetic field. This explains the anisotropy profile of figure 2.

Acknowledgments
We wish to thank Michael Kraus for his assistance with the Cython modules, and Nicola Bertelli for sharing with us his previous work on TORBEAM and RELAX. The output routines of RELAX have greatly benefited by the formatted output library fmtout by Roberto Bilato.

References
[1] R. W. Harvey, M. G. McCoy and G . D. Kerbel, Phys. Rev. Lett. 62, 426 (1989).
[2] E. Poli, A. G. Peeters and G. V. Pereverzev, Comp. Phys. Comm. 136, 90 (2001).
[3] G. V. Pereverzev, in Reviews of Plasma Physics, Vol. 19, edited by B. B. Kadomtsev, Consultants Bureau (New York, 1996).
[4] G. V. Pereverzev, Phys. Plasmas 5, 3529 (1998).
[5] E. Westerhof, A. G. Peeters and W. L. Schippers, RELAX, a computer code for the study of collisional and wave driven relaxation of the electron distribution function in toroidal geometry, Rijnhuizen Report No. RR 92-211 CA, 1992.
[6] E. Westerhof, Implementation of TORAY at JET National Technical Information Service Document no. PB91-114819INZ Rijnhuizen Report 89-183, 1989.
[7] N. Bertelli and E. Westerhof, Nucl. Fusion 49, 095018 (2009).
[8] R. Prater, Nucl. Fusion 48, 035006 (2008).
[9] S. Choudhary and L. B. Felsen, IEEE Trans. Antennas Propag. AP-21, 827 (1973).
[10] E. Mazzucato, Phys. Fluids B 1, 1855 (1989).
[11] S. Nowak and A. Orefice, Phys. Fluids B 5, 1945 (1993).
[12] A. G. Peeters, Phys. Plasmas 3, 4386 (1996).
[13] D. Farina, Fusion Sci. Technol. 52, 154 (2007).
[14] Yu. A. Kravtsov and Yu. I. Orlov, Geometrical Optics of Inhomogeneous Media, Springer (Berlin 1990).
[15] O. Maj, A. Mariani, E. Poli and D. Farina, Extended rays and wave energy flux in complex geometrical optics, to be submitted 2012.