Asymptotic analysis of a GLRT test for detection with large sensor arrays

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Abstract—This paper addresses the behaviour of a classical multi-antenna GLRT test that allows to detect the presence of a known signal corrupted by a multi-path propagation channel and by an additive white Gaussian noise with unknown spatial covariance matrix. The paper is focused on the case where the number of sensors $M$ is large, and of the same order of magnitude as the sample size $N$, a context which is modeled by the large system asymptotic regime $M \to +\infty$, $N \to +\infty$ in such a way that $M/N \to \epsilon$ for $\epsilon \in (0, +\infty)$. The purpose of this paper is to study the behaviour of a GLRT test statistics in this regime, and to show that the corresponding theoretical analysis allows to accurately predict the performance of the test when $M$ and $N$ are of the same order of magnitude.

Index Terms—Multichannel detection, asymptotic analysis, large random matrices

I. INTRODUCTION

Due to the spectacular development of sensor networks and acquisition devices, it has become common to be faced with multivariate signals of high dimension. Very often, the sample size that can be used in practice in order to perform statistical inference cannot be much larger than the signal dimension. In this context, it is well established that a number of fundamental existing statistical signal processing methods fail. It is therefore of crucial importance to revisit certain classical problems in the high-dimensional signals setting. Previous works in this direction include e.g. [15] and [21] in source localization using a subspace method, or [3],[14],[16],[17] in the context of unsupervised detection.

In the present paper, we address the problem of detecting the presence of a known signal using a large array of sensors. We assume that the observations are corrupted by a temporally white, but spatially correlated (with unknown spatial covariance matrix) additive complex Gaussian noise, and study the generalized likelihood ratio test (GLRT). Although our results can be used in more general situations, we focus on the detection of a known synchronization sequence transmitted by a single transmitter in an unknown multipath propagation channel. The behaviour of the GLRT test in this context has been extensively addressed in previous works, but for the low dimensional signal case (see e.g. [1],[4],[6], [12],[13],[22], [24]). The asymptotic behaviour of the relevant statistics has thus been studied in the past, but it has been assumed that the number of samples of the training sequence $N$ converges towards $+\infty$ while the number of sensors $M$ remains fixed. This is a regime which in practice makes sense when $M \ll N$. When the number of sensors $M$ is large, this regime is however often unrealistic because in order to avoid wasting resources, the size $N$ of the training sequence is usually chosen of the same order of magnitude as $M$. Therefore, we consider in this paper the asymptotic regime in which both $M$ and $N$ converge towards $\infty$ at the same rate. We consider mainly the case where the number of paths $L$ remains fixed, but also address briefly the context where $L$ converges towards $\infty$ at the same rate as $(M,N)$. We establish that the relevant statistics converge in distribution, under hypotheses $H_0$ and $H_1$, towards non zero mean Gaussian distributions, which are then characterized in closed form.

This paper is organized as follows. In section II, we provide the signal model under hypotheses $H_0$ and $H_1$, recall the expression of the statistics $\eta_N$ corresponding to the GLRT test, and explain that, in order to study $\eta_N$, assuming that the additive noise is spatially white and that the training sequence matrix is orthogonal is not a restriction. In section III, we recall the asymptotic behaviour of $\eta_N$ in the traditional asymptotic regime $N \to +\infty$ and $M$ fixed. Under hypothesis $H_0$, $\eta_N$ behaves like a $\chi^2$ distribution while it is asymptotically Gaussian under $H_1$. The main results of this paper are presented in section IV. In subsection IV-A, we present in a comprehensive way some useful technical results. Most of them are known, but in order to study the behaviour of $\eta_N$ under hypothesis $H_1$, we also need to establish a central limit theorem for a finite sum of quadratic forms of the inverse of a Wishart matrix. In subsection IV-B, we study $\eta_N$ under hypothesis $H_0$. We prove that, in contrast with the standard asymptotic regime $M$ fixed, $\eta_N$ behaves like a Gaussian random variable with mean $L \log \frac{1}{M/N}$ and variance $\frac{L}{N} \frac{M}{M/N}$ when $L$ does not scale with $(M,N)$. When $L$ converges towards $\infty$ at the same rate than $(M,N)$, existing results of [2] and [23] concerning the asymptotic behaviour of linear statistics of large dimensional $F$-matrices allow to deduce that $\eta_N$ also exhibits a Gaussian behaviour with a different mean and variance. Performing a first order expansion of the corresponding expressions w.r.t. $L/N$ allows to recover the mean and variance obtained in the regime.
where $L$ is fixed. As shown in section $\nabla$ devoted to the numerical results, the Gaussian approximation corresponding to the regime $L \to +\infty$ appears to be more accurate, for finite values of $L, M, N$, than the approximation obtained for a fixed $L$. In subsection $\nabla\nabla$ devoted to the study of $\eta_N$ under hypothesis $H_1$, we establish that if $L$ is fixed, $\eta_N$ has a similar behaviour than in the standard asymptotic regime $N \to +\infty$ and $M$ fixed, except that the terms $L \log \frac{1}{1-M/N}$ and $L \frac{M}{N} \frac{1}{1-M/N}$ are added to the asymptotic mean and the asymptotic variance, respectively. In contrast with the context of hypothesis $H_0$, the study of $\eta_N$ in the regime $L \to +\infty$ is not covered by the existing literature. As the corresponding study requires extensive work, we do not investigate this point in the present paper. Motivated by the additive structure of the asymptotic mean and variance in the regime $L$ fixed, we propose to approximate the mean and variance in the regime $L \to +\infty$ by the expressions obtained when $L$ is fixed, but when $L \log \frac{1}{1-M/N}$ and $L \frac{M}{N} \frac{1}{1-M/N}$ are replaced by the asymptotic mean and variance of $\eta_N$ under $H_0$ deduced from [2] and [23] in the regime $L \to +\infty$. As shown in section $\nabla$ the corresponding Gaussian approximation of $\eta_N$ appears more accurate, for finite values of $L, M, N$, than the approximation obtained if $L$ does not scale with $(M, N)$. Section $\nabla$ is devoted to numerical simulations. We evaluate the accuracy of the various Gaussian approximations by comparing the asymptotic means and variances with their empirical counterparts evaluated by Monte-Carlo simulations. We also compare the ROC curves corresponding to the various approximations with the empirical ones. The numerical results show that the standard Gaussian approximation obtained when $N \to +\infty$ and $M$ fixed completely fails if $\frac{M}{N}$ is greater than $\frac{1}{8}$. The large system approximations corresponding to $L$ fixed and $L \to +\infty$ appear reliable for small values of $\frac{M}{N}$, and, of course, for larger values of $\frac{M}{N}$. For the values of $L, M, N$ that are considered, the approximations obtained in the regime $L \to +\infty$ appear the most accurate, and the corresponding ROC-curves are good approximations of the empirical ones. Therefore, the proposed Gaussian approximations allow to predict reliably the performance of the GLRT test when the number of array elements is large.

**General notations.** For a complex matrix $A$, we denote by $A^T$ and $A^*$ its transpose and its conjugate transpose, and by $\text{Tr}(A)$ and $\|A\|$ its trace and spectral norm. $I$ will represent the identity matrix and $e_n$ will refer to a vector having all its components equal to 0 except the $n$-th which is equal to 1.

The real normal distribution with mean $m$ and variance $\sigma^2$ is denoted $\mathcal{N}_R(m, \sigma^2)$. A complex random variable $Z = X + iY$ follows the distribution $\mathcal{N}_C(\alpha + i\beta, \sigma^2)$ if $X$ and $Y$ are independent with respective distributions $\mathcal{N}_R(\alpha, \frac{\sigma^2}{2})$ and $\mathcal{N}_R(\beta, \frac{\sigma^2}{2})$.

For a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ and a random variable $X$, we write

$$X_n \to X \text{ a.s. and } X_n \to_d X$$

when $X_n$ converges almost surely and in distribution, respectively, to $X$ when $n \to +\infty$. Finally, $X_n = o_P(1)$ will stand for the convergence of $X_n$ to 0 in probability, and if $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers, $X_n = O_P(a_n)$ denotes boundedness in probability (i.e. tightness) of the sequence $(X_n/a_n)_{n \in \mathbb{N}}$.

### II. Presentation of the problem.

In the following, we assume that a single transmitter sends a known synchronization sequence $(s_n)_{n=1,...,N}$ through a fixed channel with $L$ paths, and that the corresponding signal is received on a receiver with $M$ sensors. The received $M$-dimensional signal is denoted by $(y_n)_{n=1,...,N}$. When the transmitter and the receiver are perfectly synchronized, $y_n$ is assumed to be given for each $n = 1, \ldots, N$ by

$$y_n = \sum_{l=0}^{L-1} h_l s_{n-l} + v_n$$

(1)

where $(v_n)_{n \in \mathbb{Z}}$ is an additive independent identically distributed complex Gaussian noise verifying $E(v_n) = 0$, $E(v_n v_n^T) = \sigma^2 I$ and $\sigma > 0$.

When the $R > 0$ and $\frac{1}{R^2} \text{Tr}(R) = 1$. Denoting by $H$ the $M \times L$ matrix $H = (h_0, \ldots, h_{L-1})$, the received signal matrix $Y = (y_1, \ldots, y_N)$ under $H_1$ can be written as

$$H_1 : Y = HS + V$$

(2)

where $V$ is defined as $Y$ and where $S$ represents the known signal matrix. We remark that the forthcoming results are valid as soon as the matrix collecting the observations can be written as in Eq. (2). In particular, by appropriately modifying the matrices $H$ and $S$, this system model can equivalently be used for a link with multiple transmit antennas.

We assume from now on that the size $N$ of the training sequence satisfies $N > M + L$. In this paper, we study the classical problem of testing the hypothesis $H_1$ characterized by Equation (2) against the hypothesis $H_0$ defined by

$$H_0 : Y = V$$

(3)

We assume from now on that $H$, $\sigma^2$ and $R$ are unknown at the receiver side. In this context, it is well established (see e.g. [4]) that the generalized maximum likelihood test consists in comparing the following statistics $\eta_N$ to a threshold:

$$\eta_N = -\log \det [I_L - T_N]$$

(4)

where $T_N$ is the $L \times L$ matrix defined by

$$T_N = \left( SS^* N^{-1/2} \right)^{-1/2} \frac{SS^*}{N} \left( YY^* N^{-1} \right) \left( SS^* N^{-1/2} \right)$$

(5)

In order to study the behaviour of the test in Eq. (4), we study the limit distribution of $\eta_N$ under each hypothesis. For this, we remark that it is possible to assume without restriction that $SS^* = I_L$ is verified and that $E(v_n v_n^*) = \sigma^2 I$, i.e. the matrix $R$ is reduced to the identity. If this is not the case, we denote by $S$ the matrix

$$S = \left( SS^* N^{-1/2} \right)$$

(6)
and by $\hat{Y}$ and $\hat{V}$ the whitened observation and noise matrices
\[ \hat{Y} = R^{-1/2}Y, \quad \hat{V} = R^{-1/2}V \]

It is clear that $\frac{SS^*}{N} = I_L$ and that $E(\hat{v}_n\hat{v}_n^*) = \sigma^2 I$. Moreover, under $H_0$, it holds that $\hat{Y} = \hat{V}$, while under $H_1$, $\hat{Y} = \hat{H}\hat{S} + \hat{V}$ where the channel matrix $\hat{H}$ is defined by
\[ \hat{H} = R^{-1/2}H(\frac{SS^*}{N})^{1/2} \]

Finally, it holds that the statistics $\eta_N$ can also be written as
\[ \eta_N = -\log \det \left[I_L - \frac{\hat{S}Y^*}{N} \left(\frac{\hat{Y}Y^*}{N}\right)^{-1} \frac{\hat{Y}S^*}{N}\right] \]

This shows that it is possible to replace $S$, $R$ and $H$ by $\hat{S}$, $I$, and $\hat{H}$ without modifying the value of statistics $\eta_N$. Therefore, without restriction, we assume from now on that
\[ \frac{SS^*}{N} = I_L, \quad R = I_M \quad (6) \]

In the following, we denote by $W$ a $(N-L) \times N$ matrix for which the matrix $\Theta = (W^T, \frac{S^*}{N})^T$ is unitary and define the $M \times (N-L)$ and $M \times L$ matrices $V_1$ and $V_2$ by
\[ (V_1, V_2) = V\Theta^* = (VW^*, V\frac{S^*}{\sqrt{N}}) \quad (7) \]

It is clear that $V_1$ and $V_2$ are complex Gaussian random matrices with independent identically distributed $\mathcal{N}_C(0, \sigma^2)$ entries, and that the entries of $V_1$ and $V_2$ are mutually independent. We notice that since $N > M + L$, the matrix $\frac{V_1V_1^*}{N}$ is invertible almost surely. We now express the statistics $\eta_N$ in terms of $V_1$ and $V_2$. We observe that
\[ \frac{VV^*}{N} = \frac{V_1V_1^*}{N} + \frac{V_2V_2^*}{N} \]

and that
\[ \frac{VS^*}{N} = \frac{1}{\sqrt{N}} (V_1, V_2) \left(\frac{W}{\sqrt{N}}\right) \frac{S^*}{\sqrt{N}} \]

coincides with $\frac{V_1V_1^*}{N}$ because $W\frac{S^*}{\sqrt{N}} = 0$. Therefore, under hypothesis $H_0$, $\eta_N$ can be written as
\[ \eta_N = -\log \det \left(I - \frac{V_2}{\sqrt{N}} \left(\frac{V_1V_1^*}{N} + \frac{V_2V_2^*}{N}\right)^{-1} \frac{V_2}{\sqrt{N}}\right) \]

Using the identity
\[ A^* (BB^* + AA^*)^{-1} = A^* (BB^*)^{-1}A (I + A^* (BB^*)^{-1}A)^{-1} \]

we obtain that, under hypothesis $H_0$, $\eta_N$ can be written as
\[ \eta_N = -\log \det \left(I_L + V_2^*/\sqrt{N} (V_1V_1^*/N)^{-1} V_2/\sqrt{N}\right) \quad (10) \]

Similarly, it is easy to check that, under $H_1$, $\eta_N$ is given by
\[ \eta_N = -\log \det \left(I_L + G_N\right) \quad (11) \]

where the matrix $G_N$ is defined by
\[ G_N = \left(H + V_2/\sqrt{N}\right)^* (V_1V_1^*/N)^{-1} \left(H + V_2/\sqrt{N}\right) \quad (12) \]

III. STANDARD ASYMPTOTIC ANALYSIS OF $\eta_N$.

In order to give a better understanding of the similarities and differences with the more complicated case where $M$ and $N$ converge towards $+\infty$ at the same rate, we first recall some standard results concerning the asymptotic distribution of $\eta_N$ under $H_0$ and $H_1$ when $N \to +\infty$ but $M$ remains fixed.

A. Hypothesis $H_0$.

A general result concerning the GLRT, known as Wilk’s theorem (see e.g. [13], [20] Chapter 8-5), implies that $\eta_N$ converges in distribution towards a $\chi^2$ distribution with $2ML$ degrees of freedom. For the reader’s convenience, we provide an informal justification of this claim. We use (10) and remark that when $N \to +\infty$ and $M$ and $L$ remain fixed, the matrices $V_1V_1^*/N$ and $\frac{1}{N}V_2^*(V_1V_1^*/N)^{-1} V_2$ converge a.s. towards $\sigma^2I$ and the zero matrix respectively. Moreover,
\[ \frac{1}{N}V_2^*(V_1V_1^*/N)^{-1} V_2 = \frac{1}{\sigma^2} V_2^* V_2/N + o_p(1/N) \]

and a standard second order expansion of $\eta_N$ leads to
\[ \eta_N = \frac{1}{\sigma^2} Tr(V_2^* V_2/N) + o_p(1/N) \]

This implies immediately that the limit distribution of $N \eta_N$ is a chi-squared distribution with $2ML$ degrees of freedom. Informally, this implies that $\mathbb{E}(\eta_N) \simeq L M N$ and $\text{Var}(\eta_N) \simeq \frac{L M N}{\sigma^2}.$

B. Hypothesis $H_1$.

Under hypothesis $H_1$, $\eta_N$ is given by (11). When $N \to +\infty$ and $M$ and $L$ remain fixed, the matrix $V_1V_1^*/N$ converges a.s. towards $\sigma^2I$ and it is easily seen that
\[ \eta_N = \log \det \left(I + \frac{HH^*}{\sigma^2}\right) + \text{Tr} \left[\left(I + \frac{HH^*}{\sigma^2}\right)^{-1} \Delta_N\right] + O_p(1/N) \]

where the matrix $\Delta_N$ is given by
\[ \Delta_N = H^* Y_N H + \frac{1}{\sigma^2} \left(\frac{V_2^*}{\sqrt{N}} H + H^* \frac{V_2}{\sqrt{N}}\right) \]

with $Y_N = (V_1V_1/N)^{-1} - I/\sigma^2.$ Standard calculations show that
\[ \sqrt{N} \left(\eta_N - \log \det \left(I + \frac{HH^*}{\sigma^2}\right)\right) \to \mathcal{N}(0, \kappa_1) \]

where $\kappa_1$ is given by
\[ \kappa_1 = \text{Tr} \left[I - \left(I + \frac{H^* H}{\sigma^2}\right)^{-2}\right] \quad (13) \]

Note that in [13] and [24], the asymptotic distribution of $\eta_N$ is studied under the assumption that the entries of the matrix $H$ are $\mathcal{O}(1/N)$ terms. In that context, $\eta_N$ behaves as a non-central $\chi^2$ distribution.
IV. ANALYSIS OF $\eta_N$ WHEN $M$ AND $N$ CONVERGE TOWARDS $\infty$ AT THE SAME RATE.

The analysis of $\eta_N$ in the asymptotic regime $M$ and $N$ converge towards $\infty$ at the same rate differs deeply from the standard regime studied in section III. In particular, it is no longer true that the empirical covariance matrix $V_1V_1^*/N$ converges in the spectral norm sense towards $\sigma^2I$. This, of course, is due to the fact that the number of entries of this $M^2$ matrix is of the same order of magnitude than the number of available scalar observations (i.e. $M(N-L) = O(MN)$).

We also note that for any deterministic $M \times M$ matrix $A$, the diagonal entries of the $L \times L$ matrix $\frac{1}{N}V_2^*AV_2$ converge towards 0 when $N \to +\infty$ and $M$ remains fixed, while this does not hold when $M$ and $N$ are of the same order of magnitude (see Proposition 1). It turns out that the asymptotic regime where $M$ and $N$ converge towards $\infty$ at the same rate is more complicated than the conventional regime of section III.

From now on, we assume that:

**Assumption 1.**

- $M$ and $N$ converge towards $+\infty$ in such a way that $c_N = \frac{M}{N} < 1$ converges towards $c$, where $0 < c < 1$.
- The number of paths $L$ remains fixed when $M$ and $N$ increase.

We note that the hypothesis $c_N < 1$ is consistent with the condition $N > M + L$. In the asymptotic regime defined by Assumption 1, $M$ can be interpreted as a function $M(N)$ of $N$. Therefore, $M$-dimensional vectors or matrices where one of the dimensions is fixed by $N$ in the following. Moreover, in order to simplify the exposition, $N \to +\infty$ should be interpreted in the following as the asymptotic regime defined by Assumption 1.

As $M$ is growing, we have to be precise with how the power of the useful signal component $HS$ is normalized. In the following, we assume that the norms of vectors $(h_l)_{l=0,\ldots,L-1}$ remain bounded when the number of antennas $M$ increases. This implies that the signal to noise ratio at the output of the matched filter $S^*H^*Y/\sqrt{N}$, i.e. $\text{Tr}((H^*H)/\sigma^2\text{Tr}(H^*H))$, is an $O(1)$ term in our asymptotic regime. We mention that the received signal to noise ratio $\text{Tr}(H^*H)/(M\sigma^2)$ converges towards 0 at rate $\frac{1}{\sqrt{N}}$ when $N$ increases.

Before studying the behaviour of $\eta_N$, we first review some useful results.

A. Useful technical results.

In this paragraph, we provide some useful technical results concerning the behaviour of certain large random matrices. In the remainder of this paragraph, $\Sigma_N$ represents a $M \times N$ matrix with $N_{\mathbb{C}}(0, \frac{I_N}{\sigma^2})$ i.i.d. elements. In the following, we give some results concerning the behaviour of the eigenvalues $\lambda_{1,N} \leq \lambda_{2,N} \leq \ldots \leq \lambda_{M,N}$ of the matrix $\Sigma_N^*\Sigma_N$ as well as on its resolvent $Q_N(z)$ defined for $z \in \mathbb{C} - \mathbb{R}^+$ by

$$Q_N(z) = (\Sigma_N^*\Sigma_N - zI_M)^{-1}.$$  

We first state the following classical result (see e.g. [2], Theorem 5.11).

**Proposition 1.** When $N \to +\infty$, $\hat{\lambda}_{1,N}$ converges almost surely towards $\sigma^2(1 - \sqrt{c})^2$ while $\hat{\lambda}_{M,N}$ converges a.s. to $\sigma^2(1 + \sqrt{c})^2$.

In the following, we denote by $I_c$ the interval defined by

$$I_c = [\sigma^2(1 - \sqrt{c})^2 - \epsilon, \sigma^2(1 + \sqrt{c})^2 + \epsilon]$$

and by $E_N$ the event defined by

$$E_N = \{\text{one of the } (\hat{\lambda}_{k,N})_{k=1,\ldots,M} \text{ escapes from } I_c\}$$

and remark that the almost sure convergence of $\hat{\lambda}_{1,N}$ and $\hat{\lambda}_{M,N}$ implies that

$$\Pr(E_N) = 1$$

almost surely for $N$ large enough.

**Proposition 1.** implies that the resolvent $Q_N(z)$ is almost surely defined on $\mathbb{C} - I_c$ for $N$ large enough, and in particular for $z = 0$.

Another important property is the almost sure convergence of the eigenvalue empirical distribution $\hat{\mu}_N = \frac{1}{M}\sum_{k=1}^{M}\delta_{\hat{\lambda}_{k,N}}$ of $\Sigma_N^*\Sigma_N$ towards the Marcenko-Pastur distribution (see e.g. [2] and [19] and the references therein). Formally, this means that the Stieltjes transform $m_N(z)$ of $\hat{\mu}_N$ defined by

$$m_N(z) = \int_{\mathbb{R}} \frac{d\hat{\mu}_N(\lambda)}{\lambda - z} = \frac{1}{M} \text{Tr}(Q_N(z))$$

satisfies

$$\lim_{N \to +\infty} (m_N(z) - m_{c,N}(z)) = 0$$

almost surely for each $z \in \mathbb{C} - I_c$ (and uniformly on each compact subset of $\mathbb{C} - \mathbb{R}^+$), where $m_{c,N}(z)$ represents the Stieltjes transform of the Marcenko-Pastur distribution of parameter $c_N$, denoted by $\mu_{c,N}$ in the following, satisfies the following fundamental equation

$$m_{c,N}(z) = \frac{1}{N} - (1 + \sigma^2c_Nm_{c,N}(z)) + \sigma^2(1 - c_N)$$

for each $z \in \mathbb{C}$. $\mu_{c,N}$ is known to be absolutely continuous, its support is the interval $[\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$, and its density is given by

$$\sqrt{\left(x - x_{c,N}\right)\left(x_{c,N} - x\right)}$$

$$2\sigma^2c_N\pi x^4$$

$$1$$

$$\left|x - x_{c,N}\right|$$

with $x_{c,N} = \sigma^2(1 - \sqrt{c_N})^2$ and $x_{c,N} = \sigma^2(1 + \sqrt{c_N})^2$. As $\mu_{c,N}$ is supported by $[\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$, the almost sure convergence (19) holds not only on $\mathbb{C} - \mathbb{R}^+$, but also for each $z \in \mathbb{C} - [\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$. In particular, (19) is valid for $z = 0$. Solving the equation (20) for $z = 0$ leads immediately to $m_{c,N}(0) = \frac{1}{\sigma^2(1 - c_N)}$, and to

$$\lim_{N \to +\infty} \frac{1}{M} \text{Tr}(\Sigma_N^*\Sigma_N)^{-1} - \frac{1}{\sigma^2(1 - c_N)} = 0$$

almost surely. Taking the derivative of (19) w.r.t. $z$ at $z = 0$, and using that $m_{c,N}'(0) = \frac{1}{\sigma^2(1 - c_N)}$, we also obtain that

$$\lim_{N \to +\infty} \frac{1}{M} \text{Tr}(\Sigma_N^*\Sigma_N)^{-2} - \frac{1}{\sigma^4(1 - c_N)^3} = 0$$

(22)
almost surely. Moreover, it is possible to specify the convergence speed in (21) and (22). The following proposition is a direct consequence of Theorem 9.10 in [2].

**Proposition 2.** It holds that

\[
\frac{1}{M} \text{Tr} \left( \Sigma_N \Sigma_N^* \right)^{-1} - \frac{1}{\sigma^2(1-c_N)} = O_p \left( \frac{1}{N} \right) \quad (23)
\]

\[
\frac{1}{M} \text{Tr} \left( \Sigma_N \Sigma_N^* \right)^{-2} - \frac{1}{\sigma^4(1-c_N)^3} = O_p \left( \frac{1}{N} \right) \quad (24)
\]

Theorem 9.10 in [2] implies that the left hand side of (23), renormalized by \(N\), converges in distribution towards a Gaussian distribution, which, in turn, leads to (23). (24) holds for the same reason.

**Remark 1.** As \(c_N \to c\), the previous results of course imply that \(\frac{1}{M} \text{Tr} \left( \Sigma_N \Sigma_N^* \right)^{-1}\) (resp. \(\frac{1}{M} \text{Tr} \left( \Sigma_N \Sigma_N^* \right)^{-2}\)) converge towards \(\frac{1}{\sigma(1-c)}\) (resp. \(\frac{1}{\sigma^2(1-c)^3}\)). However, the rate of convergence is not a \(O_p(\frac{1}{N})\) term if the convergence speed of \(c_N\) towards \(c\) is less than \(O(\frac{1}{N})\). Therefore, it is more relevant to approximate the left hand sides of (23) and (24) by \(\frac{1}{\sigma^2(1-c_N)}\) and \(\frac{1}{\sigma^4(1-c_N)^3}\), respectively.

The above results allow to characterize the asymptotic behaviour of the normalized trace of \((\Sigma_N \Sigma_N^*)^{-1}\) and \((\Sigma_N \Sigma_N^*)^{-2}\). However, it is also useful to obtain similar results on the bilinear forms of these matrices.

**Proposition 3.** We consider two deterministic \(M\)-dimensional unit norm vectors \(u_N\) and \(v_N\). Then, it holds that

\[
\lim_{N \to +\infty} u_N^* (\Sigma_N \Sigma_N^*)^{-1} v_N - \frac{u_N^* v_N}{\sigma^2(1-c_N)} = 0 \quad (25)
\]

and that

\[
\lim_{N \to +\infty} u_N^* (\Sigma_N \Sigma_N^*)^{-2} v_N - \frac{u_N^* v_N}{\sigma^4(1-c_N)^3} = 0 \quad (26)
\]

almost surely. Moreover,

\[
u_N^* (\Sigma_N \Sigma_N^*)^{-1} v_N - \frac{u_N^* v_N}{\sigma^2(1-c_N)} = O_p \left( \frac{1}{\sqrt{N}} \right) \quad (27)
\]

Finally, if \(C_N\) is a positive \(M \times M\) matrix such that \(\text{Rank}(C_N) = K\) is independent of \(N\), and satisfying for each \(N' < d_1 \leq \text{Tr}(C_N^2) < d_2 < \infty\) for some constants \(d_1\) and \(d_2\), then, we consider the sequence of random variables \((\kappa_N)_{N \geq 1}\) defined by

\[\kappa_N = \text{Tr} \left( C_N (\Sigma_N \Sigma_N^*)^{-1} \right) \quad (28)\]

Define by \(\theta_N\) the term

\[\theta_N = \frac{\text{Tr}(C_N^2)}{\sigma^4(1-c_N)^3} \quad (29)\]

Then, it holds that

\[
E \left[ \exp \left( iu \sqrt{\frac{N}{\theta_N}} \left( \kappa_N - \frac{\text{Tr}(C_N)}{\sigma^2(1-c_N)} \right) \right) \right] - \exp \left( - \frac{\theta_N u^2}{2} \right) \to 0 \quad (30)
\]

for each \(u \in \mathbb{R}\), and that

\[
\sqrt{\frac{N}{\theta_N}} \left( \kappa_N - \frac{\text{Tr}(C_N)}{\sigma^2(1-c_N)} \right) \to_{D} \mathcal{N}(0, 1) \quad (31)
\]

The almost sure convergence result (25) is well known (see e.g. [11] in the context of a more general matrix model), while (26) can be established by differentiating the behaviour of the bilinear forms of \(Q_N(z)\) w.r.t. \(z\). Moreover, (27) is a consequence of (31) used for the rank 1 matrix \(C_N = v_N u_N^\star\). (30) and (31) are new and need to be established. The technical arguments leading to (30) and (31) are presented in the appendix. The proof is based on Gaussian tools (integration by parts and Poincaré–Nash inequality, see [19] for an exhaustive presentation, and section III in [9] for a presentation focused on the models considered here) classically used to evaluate the behaviour of functionals of the resolvent of large random matrices with Gaussian entries (see [18] where this approach was first introduced, and [19] for more details). However, a technical difficulty appears in the present context because we consider the resolvent of the matrix \(\Sigma_N \Sigma_N^*\) at \(z = 0\) while in previous works, \(z\) is supposed to be belong to \(\mathbb{C} - \mathbb{R}^+\). For \(z \in \mathbb{C} - \mathbb{R}^+\), the matrix \(Q_N(z)\) is uniformly bounded, because it holds that

\[
\|Q_N(z)\| \leq \frac{1}{d(z, \mathbb{R}^+)} \quad (32)
\]

for each \(N\). This differs from the context of Proposition 3 because \(Q_N(0) = (\Sigma_N \Sigma_N^*)^{-1}\) is no longer uniformly bounded, in the sense that, despite Proposition 1, there does not necessarily exist a deterministic constant \(\alpha\) such that \(\|\Sigma_N \Sigma_N^*\|^{-1} = \frac{1}{\alpha \sqrt{N}} < \alpha\) for each \(N\) greater than a non random integer. In order to solve this issue, we use in the appendix the regularization technique introduced in a more general context in [10].

We finish this paragraph by a standard result whose proof is omitted.

**Proposition 4.** We consider a \(M \times L\) random matrix \(\Gamma_N\) with \(N_{G}(0, \frac{1}{N})\) i.i.d. entries, as well as the following deterministic matrices: \(A_N\) is \(M \times M\) and hermitian, \(B_N\) is \(M \times L\) and satisfies \(\sup_N \|B_N\| < +\infty\) while \(D_N\) is a positive \(L \times L\) matrix and also verifies \(\sup_N \|D_N\| < +\infty\). Then, if \((\omega_N)_{N \geq 1}\) represents the sequence of random variables defined by

\[
\omega_N = \text{Tr} \left[ D_N \left( \Gamma_N^A A_N \Gamma_N + \Gamma_N^B B_N + \Gamma_N^B \Gamma_N \right) \right] \quad (33)
\]

it holds that

\[
E(\omega_N) = \sigma^2 \frac{1}{N} \text{Tr}(A_N) \text{Tr}(D_N), \quad (34)
\]

\[
\text{Var}(\omega_N) = \frac{1}{N} \zeta_N \quad (35)
\]

where \(\zeta_N\) is defined by

\[
\zeta_N = \sigma^4 \frac{1}{N} \text{Tr}(A_N^2) \text{Tr}(D_N^2) + 2 \sigma^2 \frac{1}{N} \text{Tr}(D_N^2 B_N^2 B_N^\star) + \frac{1}{N^3} \text{Tr}(A_N^8) \quad (36)
\]

Moreover,

\[
E |\omega_N - E(\omega_N)|^4 \leq \frac{a_1}{N^2} + \frac{a_2}{N^2} \left( \frac{1}{N} \text{Tr}(A_N^2) \right)^2 + \frac{a_3}{N^3} \frac{1}{N^2} \text{Tr}(A_N^8) \quad (37)
\]
where $a_1, a_2, a_3$ are constant terms depending on $L, \sup \|B_N\|$, and $\sup \|D_N\|$. Finally, if $\lim \sup \zeta_n < +\infty$, it holds that
\[
\mathbb{E} \left( \exp i u \sqrt{N} (\omega_N - \mathbb{E}(\omega_N)) \right) - e^{-\frac{u^2 \zeta_n}{2}} \to 0 \quad (37)
\]
for each $u \in \mathbb{R}$.

B. Asymptotic behaviour of $\eta_N$ under hypothesis $H_0$.

In order to study $\eta_N$ under $H_0$, we use (40) and denote by $F_N$ the $L \times L$ matrix
\[
F_N = V_2^* / \sqrt{N} \left( V_1 V_1^* / N \right)^{-1} V_2 / \sqrt{N} \quad (38)
\]
We now specify the asymptotic behaviour of $\eta_N$ in our asymptotic regime.

**Theorem 1.** It holds that
\[
\eta_N - L \log \left( \frac{1}{1 - c_N} \right) \to 0 \text{ a.s.} \quad (39)
\]
and that
\[
\frac{\sqrt{N}}{\sqrt{1 - \frac{c_N}{1 - c_N}}} \left( \eta_N - L \log \left( \frac{1}{1 - c_N} \right) \right) \to_D \mathcal{N}(0,1) \quad (40)
\]

**Remark 2.** Asymptotic distribution of $\eta_N$ in the regime $L, M, N \to \infty$ at the same rate. We remark that the eigenvalues of $F_N$ coincide with the non-zero eigenvalues of $(V_2 V_2^*/N)(V_1 V_1^*/N)^{-1}$, and that $\eta_N$ appears as a linear statistics of the eigenvalues of this matrix. $(V_2 V_2^*/N)(V_1 V_1^*/N)^{-1}$ is a multivariate $F$-matrix. The asymptotic behaviour of the empirical eigenvalue distribution of this kind of random matrices as well as the corresponding central limit theorems are well established (see e.g. Theorem 4-10 and Theorem 9-14 in [2]) when the dimensions of $V_1$ and $V_2$ converge towards $+\infty$ at the same rate. In our particular context, this is not the case, because the number of columns $L$ of $V_2$ is fixed while its number of columns $M$ converges towards $+\infty$. Therefore, the results of [2] cannot be used to formally prove Theorem 7. We note in particular that when $L, M, N$ are of the same order of magnitude, the asymptotic behaviour of $\eta_N$ deeply differs from Theorem 7 because $\mathbb{E}(\eta_N) \to +\infty$ and does not behave as $-L \log(1 - c_N)$ and that $(\mathbb{E}(\eta_N) - \mathbb{E}(\eta_N))$ converges towards a Gaussian distribution (instead of $\sqrt{N}(\eta_N - \mathbb{E}(\eta_N))$ if $L$ is fixed). If we denote by $d_N$ the term $\frac{d_N}{N}$ assumed to converge towards $d \in (0, +\infty)$, $\mathbb{E}(\eta_N)$ can be written as $\mathbb{E}(\eta_N) = \eta_N + O\left( \frac{1}{N} \right)$ where $\eta_N$ is given by
\[
\tilde{\eta}_N = - N ((1 - c_N) \log(1 - c_N) + (1 - d_N) \log(1 - d_N)) + N(1 - c_N - d_N) \log(1 - c_N - d_N) \quad (41)
\]
The asymptotic variance $\tilde{\delta}_N$ of $\eta_N$ is equal to
\[
\tilde{\delta}_N = - \log \left( \frac{2 \sqrt{a_N^2 + b_N^2}}{a_N + \sqrt{a_N^2 + b_N^2}} \right) \quad (42)
\]
where
\[
a_N = \left( 1 - d_N \right)^2 + d_N \left( 1 + c_N(1 - c_N) \right) \quad (43)
\]
\[
b_N = 2 \frac{d_N}{1 - d_N} \left( c_N(1 - c_N) \right) \quad (44)
\]
It is possible, from [27] and [42], to informally obtain the expressions of the asymptotic mean and variance of $\eta_N$ in Theorem 7. For this, we remark that a first order expansion w.r.t. $\frac{1}{N}$ of $\eta_N$ and $\tilde{\delta}_N$ leads to
\[
\tilde{\eta}_N = L \left( \log \left( \frac{1}{1 - c_N} \right) + O(L/N) \right) \quad (45)
\]
and to
\[
\tilde{\delta}_N = \frac{L}{N} c_N \quad (46)
\]
which, of course, is in accordance with Theorem 7.

**Proof of Theorem 1** In order to establish Theorem 1, we use the results of subsection 4-V-A for the matrix $\Sigma_N = \frac{1}{N} V_1 V_1^*$. We note that $\frac{1}{N} V_1 V_1^*$ is a $M \times (N - L)$ matrix while the results of subsection IV-A have been presented in the context of a $M \times N$ matrix. In principle, it should be necessary to exchange $N$ by $N - L$ in Propositions 1 to 3. However, $c_N - \frac{M}{N - L} = O\left( \frac{1}{N} \right)$, so that it is possible to use the results of the above propositions without exchanging $N$ by $N - L$.

We first verify (39). For this, we introduce the event $\mathcal{E}_N$ defined by (40). We first remark that $\eta_N - \mathbb{E}(\xi_N) \mathbb{P}_N \to 0$, a.s. It is thus sufficient to study the behaviour of $\mathbb{E}(\xi_N)$ which is also equal to
\[
\mathbb{E}(\xi_N) = \log \det \left( I + F_N \xi_N \right) \quad (43)
\]
We now study the behaviour of each entry $(k,l)$ of matrix $\mathbb{E}(\xi_N) F_N$. For this, we use Proposition 4 for $D_N = e_k e_l^T$, $\Gamma_N = \frac{N^2}{N}$ and $A_N = \mathbb{E}(\xi_N) \left( V_1 V_1^* \right)$. $A_N$ is of course not deterministic, but as $V_2$ and $V_1$ are independent, it is possible to use the results of Proposition 4 by replacing the mathematical expectation operator by the mathematical expectation operator $\mathbb{E}_{V_2}$ w.r.t. $V_2$. We note that the present matrix $A_N$ verifies
\[
A_N \leq \frac{I}{\sigma^2(1 - \sqrt{\epsilon})^2 - \epsilon} \quad (44)
\]
because $\xi_N \neq 0$ implies that all the eigenvalues of $\frac{V_1 V_1^*}{N}$ belong to $L = [\sigma^2(1 - \sqrt{\epsilon})^2 - \epsilon, \sigma^2(1 + \sqrt{\epsilon})^2 + \epsilon]$. Therefore, (46) immediately implies that
\[
\mathbb{E}_{V_2} \left| F_{N,k,l} \xi_N - \mathbb{E}_{V_2} \left( F_{N,k,l} \xi_N \right) \right|^4 \leq \frac{a}{N^2} \quad (45)
\]
where $a$ is a deterministic constant. Taking the mathematical expectation of the above inequality w.r.t. $V_1$, and using the Borel-Cantelli Lemma lead to
\[
F_{N,k,l} \xi_N - \mathbb{E}_{V_2} \left( F_{N,k,l} \xi_N \right) \to 0 \text{ a.s.} \quad (46)
\]
or equivalently, to
\[
F_{N,k,l} \xi_N - \delta(k - l) \sigma^2 c_N \frac{1}{M} \text{Tr} \left( V_1 V_1^* \right) \to 0 \text{ a.s.} \quad (47)
\]
(21) implies that $F_{N,k,l} \xi_{E_N} - \delta(k-l) \frac{c_N}{1-c_N} \to 0$ almost surely, or equivalently that

$$F_{N} - \frac{c_N}{1-c_N} I \to 0 \quad a.s.$$  

This eventually leads to (39).

We now establish (40). For this, we first remark that (17) implies that $\eta_N = \eta_N \xi_{E_N} + O_p(\frac{1}{N})$ for each integer $p$. Therefore, the asymptotic behaviour of the distribution of the left hand side of (40) is not modified if $\eta_N$ is replaced by $\eta_N \xi_{E_N}$ given by (43). We denote by $\Delta_N$ the matrix defined by

$$\Delta_N = F_{N} \xi_{E_N} - \frac{c_N}{1-c_N} I$$  

We first prove that $\Delta_N = O_p(\frac{1}{\sqrt{N}})$. For this, we express $\Delta_N$ as

$$\Delta_N = \left( F_{N} \xi_{E_N} - \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-1} \xi_{E_N} I \right) + \frac{\sigma^2 c_N}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-1} \xi_{E_N} I - \frac{c_N}{1-c_N} I \quad (45)$$

The first term of the right hand side of (45) is $O_p(\frac{1}{\sqrt{N}})$ because the fourth-order moments of its entries are $O(\frac{1}{N})$ terms. As for the second term, (23) implies that it is a $O_p(\frac{1}{N})$. A standard second order expansion of $\log det(I + F_{N} \xi_{E_N})$ leads to

$$\eta_N \xi_{E_N} = L \log \frac{1}{1-c_N} + (1-c_N) \text{Tr}(\Delta_N) + O_p(\frac{1}{N})$$

Therefore, it holds that

$$\sqrt{N} \left( \eta_N \xi_{E_N} - L \log \frac{1}{1-c_N} \right) = \sqrt{N}(1-c_N) \text{Tr}(\Delta_N) + O_p\left(\frac{1}{\sqrt{N}}\right),$$

or, using (45), that

$$\sqrt{N} \left( \eta_N \xi_{E_N} - L \log \frac{1}{1-c_N} \right) = \sqrt{N}(1-c_N) \text{Tr} \left( F_{N} \xi_{E_N} - \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-1} \xi_{E_N} \right)$$

As

$$\mathbb{E} \left( \text{Tr} \left( F_{N} \xi_{E_N} \right) \right) = \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-1} \xi_{E_N} ,$$

Proposition 4 used for $A_N = (V_1 V_1^* - \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-1} \xi_{E_N}$, $B_N = 0$ and $D_N = (1-c_N) I$ leads to

$$\mathbb{E} \left( \exp \frac{L \log \frac{1}{1-c_N}}{2} \right) \times \exp \left\{ \frac{-u^2}{2} \sigma^4 L^2 (1-c_N)^2 c_N \frac{1}{M} \text{Tr} \left( \frac{V_1 V_1^*}{N} \right)^{-2} \xi_{E_N} \right\} \to 0$$

a.s. for each $u \in \mathbb{R}$. (22) and the dominated convergence theorem finally implies that

$$\mathbb{E} \left( \exp iu \sqrt{N} \left( \eta_N - L \log \frac{1}{1-c_N} \right) \right) = \mathbb{E} \left( \exp \left\{ - \frac{u^2}{2} \frac{L c_N}{1-c_N} \right\} \right) \to 0$$

This establishes (40).

Informally, Theorem 1 leads to $\mathbb{E}(\eta_N) \approx -L \log(1-c_N)$ and $\text{Var}(\eta_N) \approx \frac{L c_N}{N(1-c_N)}$. We recall that if $M$ is fixed, $N \eta_N$ behaves like a $\chi^2$ distribution with $2ML$ degrees of freedom. In that context, $\mathbb{E}(\eta_N) \approx L c_N$ and $\text{Var}(\eta_N) \approx \frac{L c_N}{N}$. Therefore, the behaviour of $\eta_N$ in the two asymptotic regimes deeply differ. However, if $c_N \to 0$, $-\log(1-c_N) \approx c_N$, and the asymptotic means and variances of $\eta_N$ tend to coincide.

C. Asymptotic behaviour of $\eta_N$ under hypothesis H1.

The behaviour of $\eta_N$ under hypothesis H1 is given by the following result.

**Theorem 2.** It holds that

$$\eta_N - \eta_{N,1} \to 0 \quad a.s.$$

where $\eta_{N,1}$ is defined by

$$\eta_{N,1} = L \log \frac{1}{1-c_N} + \text{log det} \left( I + H^* H / \sigma^2 \right)$$

Moreover,

$$\sqrt{N} \left( \frac{L c_N}{1-c_N} + \kappa_1 \right)^{1/2} (\eta_N - \eta_{N,1}) \to D N_0(0, 1)$$

where $\kappa_1$ is defined by (7).

**Remark 3.** Interestingly, it is seen that the asymptotic mean and variance of $\eta_N$ are equal to the sum of the asymptotic mean and variance of $\eta_N$ in the standard regime $N \to +\infty$ and $M$ fixed, with the extra terms $L \log \frac{1}{1-c_N}$ and $\frac{L c_N}{N(1-c_N)}$, which coincide with the asymptotic mean and variance of $\eta_N$ under $H_0$.

**Remark 4.** Asymptotic distribution of $\eta_N$ under $H_1$ in the regime $L, M, N \to \infty$ at the same rate. The asymptotic distribution under $H_0$ for the regime $L, M, N \to \infty$ is known, as we noted in Remark 2 when $M, N \to \infty$.
and $L$ fixed, under $H_1$, the asymptotic mean $\eta_{N,1}$ is the sum of the asymptotic mean under $H_0$ given by (39) and the second term $\log \det (I + H^*H/\sigma^2)$. Thus, in the regime where $N, M, L \to \infty$, it is reasonable to approximate the asymptotic mean of $\eta_N$ by the sum of the expected value in Remark 2 and this term. We can reason similarly with the variance. The asymptotic variance under $H_1$, (48), is the sum of the asymptotic variance under $H_0$, outlined in theorem 7 and the extra term $\frac{\pi}{\sqrt{2}}$. Therefore, the asymptotic variance under $H_1$ in the regime where $N, M, L \to \infty$ can be approximated by the asymptotic variance under $H_0$ for the same regime, plus the extra term $\frac{\pi}{\sqrt{2}}$. The results provided by this approximation are evaluated numerically in section $\text{V}$.

**Proof.** We recall that, under $H_1$, $\eta_N$ is given in (11). As in subsection $\text{IV-B}$, it is sufficient to study the regularized statistics $\eta_N I_{\xi_N}$, which is also equal to

$$\eta_N I_{\xi_N} = \log \det (I + I_{\xi_N} G_N).$$

In order to evaluate the almost sure behaviour of $\eta_N I_{\xi_N}$, we expand $G_N I_{\xi_N}$ as

$$G_N I_{\xi_N} = H^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} + F_N I_{\xi_N} + (V_2/\sqrt{N})^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} + H^* (V_1 V_1^*/N)^{-1} (V_2/\sqrt{N}) I_{\xi_N}.$$  \hfill (49)

By (25), the first term of the right hand side of (49) behaves almost surely as $\frac{H^* H}{\sigma^2 (1 - c_N)}$, while it has been shown in subsection $\text{IV-B}$ that the second term converges a.s. towards $\frac{c_N}{\sigma^2 (1 - c_N)} I$. To address the behaviour of entry $(k, l)$ of the sum of the third and the fourth terms, we use Proposition 4 for $\Gamma_N = V_2$, $A_N = 0$, $B_N = (V_1 V_1^*/N)^{-1} H I_{\xi_N}$ and $D_N = e_k e_l$. \hfill (36)

implies that entry $(k, l)$ converges almost surely towards 0.

Therefore, we have proved that

$$G_N = \left( \frac{H^* H}{\sigma^2 (1 - c_N)} + \frac{c_N}{1 - c_N} I \right) \to 0 \ a.s.$$ from which (46) follows immediately.

The proof of (48) is similar to the proof of (40), thus we do not provide all the details. We replace $\eta_N$ by $\eta_N I_{\xi_N}$, and remark that the matrix $\Delta_N$, given by

$$\Delta_N = G_N I_{\xi_N} - \left( \frac{H^* H}{\sigma^2 (1 - c_N)} + \frac{c_N}{1 - c_N} I \right)$$ verifies $\Delta_N = O_P \left( \frac{1}{\sqrt{N}} \right)$. To check this, it is sufficient to use the expansion (49), and to recognize that:

- by (27),

$$H^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} - \frac{H^* H}{\sigma^2 (1 - c_N)} = O_P \left( \frac{1}{\sqrt{N}} \right),$$

- by Proposition 4 and (36),

$$(V_2/\sqrt{N})^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} + H^* (V_1 V_1^*/N)^{-1} (V_2/\sqrt{N}) I_{\xi_N} = O_P \left( \frac{1}{\sqrt{N}} \right),$$

- it has been shown in subsection $\text{IV-B}$ that

$$F_N I_{\xi_N} - \frac{c_N}{1 - c_N} I = O_P \left( \frac{1}{\sqrt{N}} \right),$$

This implies that

$$\eta_N I_{\xi_N} - \eta_{N,1} = \text{Tr} \left( (1 - c_N) (I_L + H^*H/\sigma^2)^{-1} \Delta_N \right) + O_P (1/N)$$

We denote by $D_N$ the $L \times L$ matrix given by

$$D_N = (1 - c_N) (I_L + H^*H/\sigma^2)^{-1}$$

and by $C_N$ the $M \times M$ matrix defined by

$$C_N = (1 - c_N) H (I_L + H^*H/\sigma^2)^{-1} H^*$$

and express $\eta_N I_{\xi_N}$ as

$$\eta_N I_{\xi_N} = \kappa_N + \omega_N + O \left( \frac{1}{N^p} \right)$$

where $\kappa_N$ and $\omega_N$ are defined by

$$\kappa_N = \text{Tr} \left( C_N (V_1 V_1^*/N)^{-1} \right)$$

and

$$\omega_N = \text{Tr} \left[ D_N F_N I_{\xi_N} \right] + \text{Tr} \left[ D_N (V_2/\sqrt{N})^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} \right] + \text{Tr} \left[ D_N H^* (V_1 V_1^*/N)^{-1} (V_2/\sqrt{N}) I_{\xi_N} \right].$$

Using (45), we obtain that

$$\omega_N = \text{E} \text{V}_2 (\omega_N) = \omega_N - \frac{c_N}{1 - c_N} \text{Tr} (D_N) + O \left( \frac{1}{N} \right)$$

Therefore, it holds that

$$\sqrt{\text{N}} \left( \eta_N I_{\xi_N} - \eta_{N,1} \right) = \sqrt{\text{N}} \left( \text{Tr} (D_N \Delta_N) \right)$$

can be written as

$$\sqrt{\text{N}} (\eta_N I_{\xi_N} - \eta_{N,1}) = \sqrt{\text{N}} \left( \kappa_N - \frac{\text{Tr} (C_N)}{\sigma^2 (1 - c_N)} \right) + \sqrt{\text{N}} (\omega_N - \text{E} \text{V}_2 (\omega_N)) + O \left( \frac{1}{\sqrt{N}} \right)$$

We denote by $\zeta_N$ the term

$$\zeta_N = \frac{\sigma^4}{N} \text{Tr} \left( (V_1 V_1^*/N)^{-2} I_{\xi_N} \right) \text{Tr} (D_N^2) + 2 \sigma^2 \frac{1}{N} \text{Tr} \left( D_N^2 H^* (V_1 V_1^*/N)^{-1} H I_{\xi_N} \right)$$

We use Proposition 4 and (37) for $\Gamma_N = V_2/\sqrt{N}$, $A_N = (V_1 V_1^*/N)^{-1} I_{\xi_N}$ and $B_N = (V_1 V_1^*/N)^{-1} H I_{\xi_N}$, and obtain that

$$\text{E} \text{V}_2 \left[ \exp \left( i u \sqrt{\text{N}} (\eta_N I_{\xi_N} - \eta_{N,1}) \right) \right] - \exp \left( - \frac{u^2}{2} \zeta_N \right) \to 0$$\hfill (50)

a.s. $\zeta_N$ has almost surely the same behaviour as $\zeta$ given by

$$\zeta = \frac{c_N}{(1 - c_N)^3} \text{Tr} (D_N^2) + 2 \frac{c_N}{(1 - c_N)} \text{Tr} (D_N^2 H^* H)$$

which implies that

$$\exp \left( - \frac{u^2}{2} \zeta_N \right) - \exp \left( - \frac{u^2}{2} \zeta \right) \to 0 \ a.s.$$
Therefore, taking the mathematical expectation of (50) w.r.t \( V_i \) and using the dominated convergence theorem as well as (30), lead, after some calculations, to

\[
\mathbb{E} \left[ \exp \left( i u \sqrt{N} (\eta_N - \eta_{N,1}) \right) \right] - \exp \left[ - \frac{u^2}{2} \left( \frac{Lc_N}{1 - c_N} + \kappa_1 \right) \right] \to 0 \quad (51)
\]

for each \( u \). As \( \inf_N \left( \frac{Lc_N}{1 - c_N} + \kappa_1 \right) > 0 \), (48) follows from (51) (see Proposition 6 in [9]).

**Remark 5.** It is useful to recall that the expression of the asymptotic mean and variance of \( \eta_N \) provided in Theorem 2 assumes that \( R = I \) and that \( \frac{SS^*}{N} = I \). If this is not the case, we have to replace \( H \) by \( R^{-1/2}H (SS^*/N)^{1/2} \) in Theorem 2.

**Remark 6.** We note that Theorem 2 allows to quantify the influence of an overdetermination of \( L \) on the asymptotic distribution of \( \eta_N \) under \( H_1 \). This analysis is interesting from a practical point of view, since it is not always possible to know the exact number of paths and their delays. If \( L \) is overestimated, i.e. if the true number of paths is \( L_1 < L \), then, matrix \( H \) can be written as \( H = (H_1, 0) \). We also denote by \( S_1 \) and \( S_2 \) the \( L_1 \times N \) and \( (L - L_1) \times N \) matrices such that \( S = (S_1^T, S_2^T)^T \). It is easy to check that the second term of \( \eta_{N,1} \), i.e.

\[
\log \det \left( I_L + (SS^*/N)^{1/2}H^*R^{-1}H(SS^*/N)^{1/2} \right)
\]

coincides with

\[
\log \det \left( I_{L_1} + (S_1S_1^*/N)^{1/2}H_1^*R^{-1}H_1(S_1S_1^*/N)^{1/2} \right)
\]

and is thus non affected by the over-determination of \( L \). Therefore, choosing \( L > L_1 \) increases \( \eta_{N,1} \) by the factor \( (L - L_1) \log \left( \frac{1}{1 - c_N} \right) \). As for the asymptotic variance, it is also easy to verify that \( \kappa_1 \) is not affected by the over-determination of the number of paths, so that the asymptotic variance is increased by the factor \( (L - L_1) \frac{c_N}{1 - c_N} \). It is interesting to notice that the standard asymptotic analysis of subsection IV-B does not allow to predict any influence of the over-determination of \( L \) on the asymptotic distribution of \( \eta_N \).

**V. Numerical Results.**

In this section, we validate the relevance of the Gaussian approximations of section IV. In our numerical experiments, we have calculated the asymptotic expected values and variances as well as their empirical counterparts, evaluated by Monte Carlo simulations with 100,000 trials.

In our simulations we calculate the results for three regimes:

The first is the classical asymptotic analysis, where we assume that \( M, L \) are fixed and small while \( N \to \infty \), denoted by small \( M, L \) in the following. The regime where \( L \) is small and fixed but where \( M, N \to \infty \) is denoted small \( L \). Finally, the regime mentioned in remarks 2 and 4, where \( L, M, N \to \infty \), has no additional notation.

The fixed channel \( H \) is equal to \( H = \frac{1}{(\text{Tr}(HH^*)))^{1/2}} \tilde{H} \) where \( \tilde{H} \) is a realization of a \( M \times L \) Gaussian random matrix with i.i.d. \( N_c(0, \frac{1}{M}) \) entries. We remark that \( \text{Tr}(HH^*) = 1 \).

The rows of the training sequence matrix \( S \) are chosen as cyclic shifts of a Zadoff-Chu sequence of length \( N \). Due to the autocorrelation properties of Zadoff-Chu sequences, designed so that the correlation between any shift of the sequence with itself is zero, we have that \( SS^*/N = I_L \). We first evaluate the behaviour of the means and variances of the three Gaussian approximations in terms of \( c_N = \frac{M}{N} \). We only show the results for the asymptotic variance under \( H_1 \), but note that the results are similar for the expected values and under hypothesis \( H_0 \). Figure 1 compares the theoretical variances with the empirical variance obtained by simulation, under hypothesis \( H_1 \), as a function of \( c_N \), the ratio between \( M \) and \( N \). In this simulation, \( M = 10 \), \( L = 5 \) and \( N = 20, 40, 60, 80, 160, 320 \). When \( c_N \) is small, the three approximations give the same variance, as expected, and are very close to the empirical variance. When \( c_N \geq \frac{1}{4} \), the assumption that \( M \) is small compared to \( N \) is no longer valid, and the classical asymptotic analysis fails. The two large system approximations provide similar results when \( c_N \leq \frac{1}{8} \), i.e. when \( N = 40 \), or equivalently when \( \frac{L}{N} \leq \frac{1}{8} \). However, when \( N = 20 \), i.e. \( \frac{L}{N} = \frac{1}{4} \), the approximation corresponding to the regime where \( L, M, N \) converge towards \( \infty \) leads to a much more accurate prediction of the empirical variance. We remark that the above approximation is reliable for rather small values of \( L, M, N \), i.e. \( L = 5, M = 10, N = 20 \). We also remark that the regimes where \( M, N \) are of the same order of magnitude capture the actual performance even when \( c_N \) is small, which, by extension, implies that the standard asymptotic analysis always performs worse compared to the two large system approximations.

If \( N, M \) increase while \( c_N \) stays the same, the results will be even closer to the theoretical values, since the number of samples is larger. In the simulations that follow, we will use \( c_N = 1/2 \) with \( N = 300, M = 150 \) and \( L = 10 \), if not otherwise stated.

![Fig. 1. Proposed asymptotic analysis with standard asymptotic analysis](image-url)
B. Comparison of the asymptotic means and variances of the approximations of $\eta_N$ under $H_0$

We first compare in figures 2 and 3 the asymptotic expected values and variances with the empirical ones when $L$ increases from $L = 1$ to $L = 30$ while $M = 150$ and $N = 300$, i.e. $c_N = 1/2$. The figures show that the standard asymptotic analysis of section III completely fails for all values of $L$. This is expected, given the value of $\frac{M}{N}$. As $L$ increases, the assumption that $L$ is small becomes increasingly invalid, and the only model that is correct in this regime is the model from remark 2. This is valid both for the expected value and variance, and the theoretical values are very close to their empirical counterparts. We remark that the approximation of remark 2 valid when $L \to +\infty$, also allows to capture the actual empirical performance when $L$ is small.

Fig. 2. $H_0$: Asymptotic expected values as a function of $L$

![Expected value under $H_0$](image)

Fig. 3. $H_0$: Asymptotic variances as a function of $L$

![Variance under $H_0$](image)

C. Validation of asymptotic distribution under $H_0$

Although the expected values and variances can be very accurate, this does not necessarily mean that the empirical distribution is Gaussian. Therefore, we need to validate also the distribution under $H_0$. The asymptotic distribution under $H_0$ can be validated by analyzing its accuracy when calculating a threshold used to obtain ROC-curves. Note that this analysis also shows the applicability of the results for a practical case of timing synchronization.

We calculate the ROC curves in two different ways. The first is the ROC curve calculated empirically. We determine a threshold $s$ from the empirical distribution under $H_0$ which gives a given probability of false alarm as $P_{fa} = P(\eta_N > s)$. Its corresponding probability of non-detection, $P_{nd}$, is then obtained as the probability that the empirical values of the synchronization statistics under $H_1$ pass this threshold. The other ROC-curves are obtained by calculating the threshold $s$ from the asymptotic Gaussian distributions under $H_0$, and using this theoretical threshold to calculate the $P_{nd}$ from the empirical distribution under $H_1$.

Figure 4 shows the ROC-curves obtained with the approaches mentioned above when $L = 10, M = 150, N = 300$. Since the standard asymptotic analysis small $M,L$ gives very bad results, its results are omitted. It is clear that ROC-curve obtained by using the asymptotic distribution obtained with the assumption that $L$ is small differs greatly from the results from the regime of Remark 2 even for this relatively small value of $L$. This is because the theoretical threshold depends greatly on the expected value, and if it is not precisely evaluated, it gives erroneous results. In the model where $N,M,L \to \infty$, the expected value and variance are very close to their empirical counterparts, and the resulting threshold can be used to precisely predict the synchronization performance for the set of parameters used when $P_{fa} \geq 10^{-3}$ and $P_{nd} > 10^{-3}$. Figure 5 shows, for the regime $N,M,L \to \infty$, the ROC curves obtained with the theoretical threshold, together with the empirical results. In the figure, $L$ goes from 1 to 20, while $M = 15L$ goes from 15 to 300 and $N = 30L$ goes from 30 to 600. It is seen that when the three parameters grow, the distance between the theoretical and empirical ROC curves decreases.

![ROC curve obtained with theoretical threshold](image)
D. Comparison of the asymptotic means and variances of the approximations of $\eta_N$ under $H_1$

In this section, we will proceed to validate the expected value and variance under $H_1$.

Figures 6 and 7 validate the asymptotic expected values and variances under $H_1$. Similarly to hypothesis $H_0$, the theoretical expected values and variances are poorly evaluated using the standard asymptotic analysis. We note that the asymptotic expected values deduced for the regime $N, M, L \to \infty$, see remark 2, are very close to the empirical expected values and variances. For an $L$ sufficiently small, however, the regime $N, M \to \infty$ with small $L$ give asymptotic expected values and variances that are close to their empirical counterparts.

E. Validation of asymptotic distribution under $H_1$

To validate the asymptotic distributions under $H_1$, we calculate theoretical ROC-curves using both asymptotic distributions. For each $P_{fa}$, a threshold $s$ is calculated from the theoretical Gaussian distribution under $H_0$. This threshold is then used to calculate the $P_{nd}$ from the theoretical Gaussian distribution under $H_1$, using $P_{nd} = 1 - \Phi_{H_1}(\eta_N > s)$. Figure 8 shows these theoretical ROC curves plotted together with the empirical ROC curve. Here, $L = 10, M = 150$ and $N = 300$. It is seen that the approximation corresponding to the regime $N, M, L \to \infty$ provides, as in the context of hypothesis $H_0$, a more accurate theoretical ROC curve. It is seen that the ROC curve associated with the regime small $L$ is closer from the empirical ROC curve than in the context of hypothesis $H_0$. This is because the corresponding asymptotic means are, for both $H_0$ and $H_1$, less than the actual empirical means. These two errors tend to compensate in the theoretical ROC curves, which explains why the theoretical ROC curve of figure 8 is more accurate than the corresponding ROC curve of figure 4, for small $L$.
more accurately the asymptotic behaviour of $\eta_N$ under $H_1$ in
the regime $L \to +\infty$, and to check if the residual error tends
to diminish. However, as mentioned in Remark 4 this needs
to establish a central limit theorem for linear statistics of the
eigenvalues of non zero mean large F-matrices, which is a non
trivial task.

VI. CONCLUSION.

In this paper, we have studied the behaviour of the multi-
antenna GLRT detection test of a known signal corrupted
by a multi-path deterministic channel and an additive white
Gaussian noise with unknown spatial covariance. We have
addressed the case where the number of sensors and the
number of samples of the training sequence are large and of
the same order of magnitude. Under hypothesis $H_0$, we have
recalled that in the standard asymptotic regime $N \to +\infty$ and
$M$ fixed, the GLRT test statistics $\eta_N$ converges towards a $\chi^2$
distribution. If $M$ and $N$ converge towards $\infty$ at the same rate
and that $L$ does not scale with $(M,N)$, we have established that
$\eta_N$ has a Gaussian behaviour with asymptotic mean
$L \log \frac{1}{1-M/N}$ and variance $\sqrt{\frac{L}{N}} \frac{M/N}{1-M/N}$. Using known results
of [2] and [23], concerning the behaviour of linear statistics
of the eigenvalues of large F-matrices, we have deduced that
in the regime where $L,M,N$ converge to $\infty$ at the same rate,
$\eta_N$ still has a Gaussian behaviour, but with a different mean
and variance. Under hypothesis $H_1$, we have shown that $\eta_N$
has a Gaussian behaviour when $M$ and $N$ converge towards
$\infty$ at the same rate and $L$ remains fixed. The corresponding
asymptotic mean and variance are obtained as the sum of the
asymptotic mean and variance in the standard regime
$N \to +\infty$ and $M$ fixed, and $L \log \frac{1}{1-M/N}$ and $\sqrt{\frac{L}{N}} \frac{M/N}{1-M/N}$
respectively, i.e. the asymptotic mean and variance under $H_0$.
The analysis of $\eta_N$ under $H_1$ when $L,M,N$ converge to $\infty$
needs to establish a central limit theorem for linear statistics of
the eigenvalues of large non zero-mean F-matrices, a difficult
task that we will address in a future work. Motivated by the
results obtained in the case where $L$ remains finite, we have
proposed to approximate the asymptotic mean and variance
when $L \to +\infty$ by the sum of the asymptotic mean and
variance under $H_0$ when $L \to +\infty$ with the asymptotic
mean and variance under $H_1$ in the standard regime $N \to
+\infty$ and $M$ fixed. Numerical experiments have shown that
the Gaussian approximation corresponding to the standard
regime $N \to +\infty$ and $M$ fixed completely fails as soon as $\frac{M}{N}$
is not small enough. The large system approximations
provide better results when $\frac{M}{N}$ increases, while also allowing
to capture the actual performance for small values of $\frac{M}{N}$.
We have also observed that, for finite values of $L,M,N$,
the Gaussian approximation obtained in the regime $L,M,N$
converge towards $\infty$ is more accurate than the approximation
in which $L$ is fixed. In particular, the ROC curves that are
obtained using the former large system approximation are
accurate approximations of the empirical ones in a reasonable
range of $P_{fa}, P_{nd}$. We therefore believe that our results can
be used to reliably predict the performance of the GLRT test,
and that the tools that are developed in this paper are useful
in the context of large antenna arrays.

APPENDIX

To establish (51), we follow the approach of [9] which is
based on the joint use of the integration by parts formula
and of the Poincaré-Nash inequality (see section III-B of [9]).
However, the approach of [9] allows to manage functionals
of the resolvent $Q_N(z)$ for $z \in \mathbb{C} - \mathbb{R}^+$. For this, the inequality
(32) plays a fundamental role. For $z = 0$, $\|Q_N(0)\|$ coincides
with $\frac{1}{\lambda_{1,N}}$ which is not upper-bounded by a deterministic
positive constant for $N$ greater than a non random integer.
However, Proposition 1 strongly suggests that it is possible to
replace matrix $(\Sigma_N \Sigma_N^*)^{-1}$ by $(\Sigma_N \Sigma_N^*)^{-1} \chi_N$
where $\chi_N$ is a scalar regularization term depending on the entries of $\Sigma_N$
which vanishes when the smallest eigenvalue $\lambda_{1,N}$ deviates
significantly from its almost sure limit $\sigma^2(1-\sqrt{\epsilon})^2$. As the
integration by parts formula and the Poincaré-Nash inequality
need to consider smooth enough functions of $\Sigma_N$, the regu-
larization term $\chi_N$, considered as a function of $\Sigma_N$, should
be itself smooth enough. Motivated by [10], we consider the
regularization term $\chi_N$ defined by

$$
\chi_N = \det \left[ \phi \left( \Sigma_N \Sigma_N^* \right) \right]
$$

(52)

where $\phi$ is a smooth function such that

$$
\phi(\lambda) = 1 \text{ if } \lambda \in \mathcal{I}_c = [\sigma^2(1-\sqrt{\epsilon})^2 - \epsilon, \sigma^2(1+\sqrt{\epsilon})^2 + \epsilon]
$$
$$
\phi(\lambda) = 0 \text{ if } \lambda \in [\sigma^2(1-\sqrt{\epsilon})^2 - 2\epsilon, \sigma^2(1+\sqrt{\epsilon})^2 + 2\epsilon]\n$$
$$
\phi \in [0,1] \text{ elsewhere}
$$

In the following, we need to use the following property: for
each $\epsilon > 0$, it holds that

$$
P \left( E_N \right) = O \left( \frac{1}{NP} \right)
$$

(53)

where $E_N$ is defined by (16). Property (53) is not mentioned in
Theorem 5.11 of [2] which addresses the non Gaussian case.
However, (53) follows directly from Gaussian concentration
arguments.

It is clear that

$$
(\Sigma_N \Sigma_N^*)^{-1} \chi_N \leq \frac{1}{\sigma^2((1-\sqrt{\epsilon})^2 - 2\epsilon)}
$$

(54)
Lemma 3-9 of [10] also implies that, considered as a function of the entries of $\Sigma_N$, $\chi_N$ is continuously differentiable. Moreover, it follows from Proposition 1 that almost surely, for $N$ large enough, $\chi_N = 1$ and $\kappa_N = \kappa_N \chi_N$. Therefore, it holds that $\kappa_N \chi_N = \kappa_N + O_p(\frac{1}{N^p})$, and that

$$\sqrt{N} \left( \kappa_N - \frac{\text{Tr}(C_N)}{\sigma^2(1 - c_N)} \right) = \sqrt{N} \left( \kappa_N \chi_N - \frac{\text{Tr}(C_N)}{\sigma^2(1 - c_N)} \right) + O_p(\frac{1}{N^p}) \quad (55)$$

for each $p \in \mathbb{N}$. In order to establish (50), it is thus sufficient to prove that

$$\mathbb{E} \left[ \exp \left( i u \sqrt{N} \left( \kappa_N \chi_N - \frac{\text{Tr}(C_N)}{\sigma^2(1 - c_N)} \right) \right) \right] - \exp \left( - \frac{\theta_N u^2}{2} \right) \to 0 \quad (56)$$

for each $u$. To obtain (51), we remark that, as $\inf_N \theta_N > 0$, it follows from (56) that

$$\sqrt{N} \mathbb{E} \left( \chi \right) = \frac{\text{Tr}(C_N)}{\sigma^2(1 - c_N)} \to \mathcal{N}(0, 1)$$

(see Proposition 6 in [9]). (51) eventually appears as a consequence of (55).

The above regularization trick thus allows to replace the matrix $(\Sigma_N \Sigma_N)^{-1}$ by $(\Sigma_N \Sigma_N)^{-1} \chi_N$, which verifies (54). In order to establish (56), it is sufficient to prove that

$$\mathbb{E} \left( \kappa_N \chi_N - \frac{\text{Tr}(C_N)}{\sigma^2(1 - c_N)} \right) = o\left( \frac{1}{\sqrt{N}} \right) \quad (57)$$

and that

$$\mathbb{E} \left[ \exp \left( i u \sqrt{N} \left( \kappa_N \chi_N - \mathbb{E}(\kappa_N \chi_N) \right) \right) \right] - \exp \left( - \frac{\theta_N u^2}{2} \right) \to 0 \quad (58)$$

for each $u$.

In the rest of this section, to simplify the notations, we omit to write the dependance on $N$ of the various terms $\Sigma_N$, $Q_N(0)$, $\chi_N$, ..., and denote them by $\Sigma$, $Q(0)$, $\chi$, ... However, we keep the notation $c_N$, in order to avoid confusion between $c_N$ and $c$. Furthermore, the matrix $Q(0)$ is denoted by $Q$. If $x$ is a random variable, $x^\circ$ represents the zero mean variable $x^\circ = x - \mathbb{E}(x)$. In the following, we denote by $\delta$ the random variable defined by

$$\delta = \sqrt{N} \kappa \chi$$

and by $\psi(\delta)$ the characteristic function of $\delta$, defined by

$$\psi(\delta) = \mathbb{E}(\exp \{ i u \delta \})$$

We first establish the following Proposition.

**Proposition 5.** It holds that

$$(\psi(\delta))' = -u \mathbb{E} \left( \text{Tr}(C^2 Q^2 \chi) \right) \psi(\delta) + O\left( \frac{1}{\sqrt{N}} \right) \quad (59)$$

where $'$ represents the derivative w.r.t. the variable $u$.

**Proof.** We consider the characteristic function $\psi(\delta)$ of $\delta$, and evaluate

$$\psi'(\delta) = i u \mathbb{E} \left( \text{Tr}(Q \Sigma \Sigma^\ast \chi e^{i u \delta}) \right)$$

We remark that $Q \Sigma \Sigma^\ast = I$ so that

$$\mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) = \mathbb{E} \left( e^{i u \delta} \right) I$$

We claim that

$$\mathbb{E} \left( e^{i u \delta} \right) = \psi(\delta) + O\left( \frac{1}{N^p} \right) \quad (60)$$

for each $p$. We remark that

$$\left| \mathbb{E} \left( e^{i u \delta} \right) \right| \leq 1 - \mathbb{E}(\chi)$$

We recall that the event $\mathcal{E}$ is defined by $\text{Prob}(\mathcal{E}) = \mathcal{O}(\frac{1}{N^p})$ for each $p$ (see (53)). $\mathcal{E}$ leads to $1 - \mathbb{E}(\chi) \leq P(\mathcal{E})$. This justifies (60). Therefore, it holds that

$$\mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) = \left( \psi(\delta) + O\left( \frac{1}{N^p} \right) \right) I \quad (61)$$

and that

$$\mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j$$

The integration by parts formula leads to

$$\mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j$$

After some algebra, we obtain that

$$\mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j = \mathbb{E} \left( Q \Sigma \Sigma^\ast \chi e^{i u \delta} \right) \Sigma_t j$$

We now need to study more precisely the properties of the derivative of $\chi$ w.r.t. $\Sigma_t j$. For this, we give the following Lemma.

**Lemma 1.** We denote by $A$ the event:

$$A = \{ \text{one of the } \lambda_{N, k} \text{ escapes from } \mathcal{I}_k \} \cap \{ (\lambda_{N, k})_{k=1,...,M} \in \text{supp}(\phi) \} \quad (63)$$

Then, it holds that

$$\frac{\partial \chi}{\partial \Sigma_t j} = 0 \text{ on } A^c \quad (64)$$
and that
\[ \mathbb{E} \left| \frac{\partial \chi}{\partial \sum_{i,j}} \right|^2 = \mathcal{O} \left( \frac{1}{N^p} \right) \] (65)
for each \( p \).

**Proof.** Lemma 1 follows directly from Lemma 3.9 of [10] and from the calculations in the proof of Proposition 3.3 of [10].

Lemma 1 implies that the last term of (62) is \( \mathcal{O}(\frac{1}{N^p}) \) for each \( p \). To check this, we remark that
\[ \mathbb{E} \left( \mathcal{Q}_{r,t} \sum_{s,j} e^{i \alpha \delta} \frac{\partial \chi}{\partial \sum_{t,j}} \right) = \mathbb{E} \left( \mathcal{Q}_{r,t} \sum_{s,j} e^{i \alpha \delta} A \frac{\partial \chi}{\partial \sum_{t,j}} \right) \]
The Schwartz inequality leads to
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{t,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{t,j}} \right|^2 \right)^2 \]
On each \( A \), all the eigenvalues of \( \Sigma \Sigma^* \) belong to \( [\sigma^2(1 - \sqrt{c})^2 - 2\epsilon, \sigma^2(1 + \sqrt{c})^2 + 2\epsilon] \). Therefore, \( \mathcal{Q}_{r,t} \sum_{t,j} \) is bounded and (65) implies that the last term of (62) is \( \mathcal{O}(\frac{1}{N^p}) \) for each \( p \). Summing (62) over \( t \), we obtain that
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \]
where we recall that \( \hat{m}(0) = \frac{1}{N} \text{Tr}(Q) \) represents the Stieltjes transform of the empirical eigenvalue distribution of \( \sum \) at \( z = 0 \). Using that \( 1 - \chi \leq 1 \), it is easy to check that for each \( p \), it holds that
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \]
We denote by \( \beta \) the term \( \hat{m}(0) \chi \), and express \( \beta \) in terms of \( \beta = \alpha + \beta^* \). Replacing \( \chi \) by \( \chi^2 \) in the second term of the right-hand side of (66) and plugging \( \beta = \alpha + \beta^* \) into (66), we obtain that immediately that
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \]
Summing over \( j \), we get that
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \]

or, using that \( \mathcal{Q} \Sigma \Sigma^* = I \),
\[ \mathbb{E} \left( e^{i \alpha \delta} \right) \delta(r = s) = \frac{\sigma^2}{1 + \sigma^2 c_N \alpha} \mathbb{E} \left( \mathcal{Q}_{r,s} \chi e^{i \alpha \delta} \right) \]
\[ - \frac{i \sigma^2 u}{\sqrt{N}(1 + \sigma^2 c_N \alpha)} \mathbb{E} \left( \left( \mathcal{Q}^2 \mathcal{C} \right)_{r,s} \chi e^{i \alpha \delta} \right) \]
\[ - \frac{\sigma^2 c_N}{1 + \sigma^2 c_N \alpha} \mathbb{E} \left( \beta^\alpha \chi e^{i \alpha \delta} \right) \delta(r = s) + \mathcal{O}(\frac{1}{N^p}) \] (69)
In order to evaluate \( \alpha \), we take \( u = 0 \) and sum over \( r = s \) in (69), and obtain that
\[ \alpha = \frac{1}{\sigma^2(1 - c_N)} + \frac{1}{1 - c_N} \mathbb{E} (\beta^\alpha \chi) + \mathcal{O}(\frac{1}{N^p}) \]
Therefore, we get that
\[ \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \leq \mathbb{E} \left( \left| \frac{\partial \chi}{\partial \sum_{s,j}} \right|^2 \right)^2 \]
We now use (69) in order to evaluate \( \mathbb{E} \left( \left( \mathcal{Q}_{r,s} \chi \right)^2 \right) e^{i \alpha \delta} \right) \). For this, we first establish that the use of (54) and of the Poincaré-Nash inequality implies that
\[ \mathbb{E} \left( \left( \mathcal{Q}_{r,s} \chi \right)^2 \right) e^{i \alpha \delta} \right) \]
We just evaluate the terms corresponding to the deriviates with respect to the terms \( (\sum_{i,j})_{i=1,...,N,j=1,...,N} \). It is easily seen that
\[ \frac{\partial \beta}{\partial (\sum_{i,j})} = -\frac{1}{M} \left( e^{i \alpha \delta} \mathcal{Q}_{i,j} \right) + \frac{1}{M} \mathbb{E} (\mathcal{Q}_{i,j}) \frac{\partial \chi}{\partial (\sum_{i,j})} \] (see (64)), we obtain that
\[ \frac{\partial \beta}{\partial (\sum_{i,j})} \leq 2 \frac{1}{M} \mathbb{E} \left( \left( \mathcal{Q}_{i,j} \right)^2 \right) + 2 \frac{1}{M} \mathbb{E} (\mathcal{Q}) \left| \frac{\partial \chi}{\partial (\sum_{i,j})} \right|^2 \]
Therefore, it holds that
\[ \left| \frac{\partial \beta}{\partial (\sum_{i,j})} \right|^2 \leq 2 \frac{1}{M} \mathbb{E} \left( \left( \mathcal{Q}_{i,j} \right)^2 \right) + 2 \frac{1}{M} \mathbb{E} (\mathcal{Q}) \left| \frac{\partial \chi}{\partial (\sum_{i,j})} \right|^2 \]
Using the identity \( \mathcal{Q} \Sigma \Sigma^* = I \) as well that \( \frac{\partial \chi}{\partial (\sum_{i,j})} = \mathbb{E} (\mathcal{Q}) \left| \frac{\partial \chi}{\partial (\sum_{i,j})} \right|^2 \)
see (64), we obtain that
\[ \frac{\partial \beta}{\partial (\sum_{i,j})} \leq 2 \frac{1}{M} \mathbb{E} \left( \left( \mathcal{Q}_{i,j} \right)^2 \right) + 2 \frac{1}{M} \mathbb{E} (\mathcal{Q}) \left| \frac{\partial \chi}{\partial (\sum_{i,j})} \right|^2 \]
On the set \( A \), the eigenvalues of \( \Sigma \Sigma^* \) are located into \( [\sigma^2(1 - \sqrt{c})^2 - 2\epsilon, \sigma^2(1 + \sqrt{c})^2 + 2\epsilon] \). Therefore, we get that
\[ \frac{1}{M} \mathbb{E} (\mathcal{Q}) \leq \frac{1}{\sigma^2(1 - \sqrt{c})^2 - 2\epsilon} \]
Using (65), we obtain that
\[
\frac{2 \sigma^2}{N} \mathbb{E} \left( \frac{1}{M} \text{Tr}(Q) \sum_{i,j} \left| \frac{\partial \chi}{\partial \Sigma_{i,j}} \right|^2 \right) = \mathcal{O} \left( \frac{1}{N^p} \right)
\]
for each \( p \). Moreover, (54) implies that
\[
\frac{1}{M} \text{Tr}(Q^3) \chi < \frac{1}{(\sigma^2(1 - \sqrt{\epsilon})^2 - 2\epsilon)^3}
\]
and that
\[
2 \sigma^2 \frac{1}{MN} \mathbb{E} \left( \frac{1}{M} \text{Tr}(Q^3) \chi \right) = \mathcal{O} \left( \frac{1}{N^2} \right)
\]
This establishes (72).

Therefore, the Schwartz inequality leads to \( \mathbb{E} (\beta^o \chi e^{iu\delta}) = \mathcal{O}(\frac{1}{N}) \). Writing \( \mathbb{E} (Q_{r,s} \chi e^{iu\delta}) \) as
\[
\mathbb{E} (Q_{r,s} \chi e^{iu\delta}) = \mathbb{E}(Q_{r,s} \chi^2 e^{2iu\delta}) + \mathcal{O}(\frac{1}{N})
\]
\[
\mathbb{E}(Q_{r,s} \chi e^{iu\delta}) = \mathbb{E}(Q_{r,s} \chi) + \mathcal{O}(\frac{1}{N})
\]
and that
\[
\mathbb{E}(Q_{r,s} \chi) = \mathbb{E}(Q_{r,s} \chi^2) + \mathcal{O}(\frac{1}{N})
\]
\[
\mathbb{E}(Q_{r,s} \chi^2) = \mathbb{E}(Q_{r,s} \chi^2) + \mathcal{O}(\frac{1}{N})
\]
\( (70) \), (71) and (69) lead to
\[
\mathbb{E}(Q_{r,s} \chi e^{iu\delta}) = \frac{iu}{\sqrt{N}} \mathbb{E}(Q_{r,s} \chi)^2 + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
or equivalently to
\[
\mathbb{E}(\delta^o e^{iu\delta}) = \frac{iu}{\sqrt{N}} \mathbb{E}(Q_{r,s} \chi)^2 e^{iu\delta} + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
Using the Nash-Poincaré inequality, it can be checked that
\[
\text{Var} (\text{Tr}(Q^2 \chi)) = \mathcal{O}(\frac{1}{N})
\]
Therefore, the Schwartz inequality leads to
\[
\mathbb{E}(\text{Tr}(Q^2 \chi) e^{iu\delta}) = \mathbb{E}(\text{Tr}(Q^2 \chi) e^{iu\delta}) + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
and we get that
\[
\mathbb{E}(\delta^o e^{iu\delta}) = \frac{iu}{\sqrt{N}} \mathbb{E}(\text{Tr}(Q^2 \chi) e^{iu\delta}) + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
Plugging \( \delta = \delta^o + \mathbb{E}(\delta) \) into (74) eventually leads to
\[
\mathbb{E}(\delta^o e^{iu\delta^o}) = \frac{iu}{\sqrt{N}} \mathbb{E}(\text{Tr}(Q^2 \chi) e^{iu\delta^o}) + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
which is equivalent to (59). This, in turn, establishes Proposition 5.

We now complete the proof of (58). We integrate (59), and obtain that
\[
\psi^o(u) = \exp \left[ - \frac{u^2}{2} \mathbb{E}(\text{Tr}(Q^2 \chi)) \right] + \mathcal{O}(\frac{1}{\sqrt{N}})
\]
(see section V-C of [9] for more details). 26 implies that
\[
\text{Tr}(Q^2 \chi^2) - \frac{\text{Tr}(C^2)}{\sigma^4(1 - \epsilon N)^3} \rightarrow 0 \text{ a.s.}
\]
As \( \text{Tr}(Q^2 \chi^2) \) also converges to 0 almost surely, we obtain that
\[
\text{Tr}(Q^2 \chi) - \frac{\text{Tr}(C^2)}{\sigma^4(1 - \epsilon N)^3} \rightarrow 0 \text{ a.s.}
\]
As matrix \( Q^2 \chi \) is bounded and \( \sup_N \text{Tr}(C^2) < +\infty \), it is possible to use the Lebesgue dominated convergence theorem and to conclude that
\[
\mathbb{E}(\text{Tr}(Q^2 \chi^2)) - \frac{\text{Tr}(C^2)}{\sigma^4(1 - \epsilon N)^3} \rightarrow 0
\]
This proves (58). It remains to establish (57). For this, we use (71), and obtain that
\[
\mathbb{E}(\text{Tr}(Q \chi^2)) - \frac{\text{Tr}(C)}{\sigma^2(1 - \epsilon N)^3} = \mathcal{O}(\frac{1}{N^p})
\]
for each \( p \). This, of course, implies (57).

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