Dynamics of Anisotropic Collapsing Spheres in Einstein Gauss-Bonnet Gravity

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Abstract

This paper is devoted to investigate the dynamics of the self gravitating adiabatic and anisotropic source in $5D$ Einstein Gauss-Bonnet gravity. To this end, the source has been taken as Tolman-Bondi model which preserve inhomogeneity in nature. The field equations, Misner-Sharp mass and dynamical equations have formulated in Einstein Gauss-Bonnet gravity in $5D$. The junction conditions have been explored between the anisotropic source and vacuum solution in Gauss-Bonnet gravity in detail. The Misner and Sharp approach has been applied to define the proper time and radial derivatives. Further, these helps to formulate general dynamical equations. The equations show that the mass of the collapsing system increases with the same amount as the effective radial pressure increases. The dynamical system preserve retardation which implies that system under consideration goes to gravitational collapse.

Key Words: Einstein Gauss-Bonnet Gravity; Gravitational Collapse.

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1 Introduction

The dynamics of self-gravitating objects is the topic of great interest in general theory of relativity (GR), which is modern theory of gravity. This problem becomes source of inspiration for researchers when the stellar objects remain in stable for the most of time against the perturbation caused by the self gravitational force of the massive objects. This process provides the information to study structure formation of the gravitationally collapsing objects. In the relativistic gravitational physics the dynamics of the stars studied by the Chandrasekhar [1] first time in 1964, since then there has been growing interest to study the dynamics of stars in this research direction. This work was extended by the Herrera et al. [2, 3] explicitly for spherically symmetric heat conducting, isotropic/anisotropic and viscous fluids in the framework of GR. Recently, Herrera et al. [4] have investigated the dynamics of the expansion-free fluids using first order perturbation of metric components as well as matter variables.

Several properties of the fluid play a dominant role in dynamical process of the gravitationally collapsing objects. Herrera et al. [5] have studied the expansion free condition for the collapsing sphere. Herrera and his collaborators [6, 7] discussed the dynamical process of gravitational collapse using Misner and Sharp’s formulation. They considered the matter distribution with shear-free spherically symmetric. The realistic model of heat conducting star which shows dissipation in the form of heat flux in radial direction and shear viscosity was studied by Chan [8]. Herrera et al. [9] also formulated the dynamical equations of the fluids which contains heat flux, radiation and bulk viscosity and then coupled these equations with causal transport equations. The inertia of heat flux and its significance in the dynamics of dissipative collapse was studied by Herrera [10]. The present paper is the particular case (adiabatic case) of this work in 5D Einstein Gauss-Bonnet gravity. Sharif and Azam [11]-[13] have studied the effects of electromagnetic field on the dynamical stability of the collapsing dissipative and non dissipative fluids in spherical and cylindrical geometries. This work has been further extended by Sharif and his collaborators [14]-[17] in modified theories of gravity, like $f(R)$ and $f(T)$ and $f(R,T)$. Recently [18], Abbas and Sarwar have studied the dynamical stability of collapsing star in Gauss-Bonnet gravity.

The dynamical system dealing with the dimensions greater or equal to 5 are usually discussed in the Gauss-Bonnet gravity theory. The natural
appearance of this theory occurs in the low energy effective action of the modern string theory. Boulware and Deser [19] investigated the black hole (BH) solutions in $N$ dimensional string theory with four dimensional Gauss-Bonnet invariant. This work is the extension of $N$ dimensional solutions formulated by Tangherlini [20], Merys and Perry [21]. Wheeler [22] discussed the spherically symmetric BH solutions with their physical properties in detail. The topological structure of nontrivial BHs has been explored by Cai [23]. Kobayashi [24] and Maeda [25] have formulated the structure of Vaidya BH in Gauss-Bonnet gravity. All investigations show that the presence of the Gauss-Bonnet term in the field equations would effect the final state of the gravitational collapse. Recently [26], Jhinag and Ghosh have consider the $5D$ action with the Gauss-Bonnet terms in Tolman-Bondi model and give an exact model of the gravitational collapse of a inhomogeneous dust. Motivated by these studies, we have studies, we have explored the dynamics of the gravitationally collapsing spheres in Einstein Gauss-Bonnet gravity. This paper is extension of Herrera work[10] in Einstein Gauss-Bonnet gravity. We would like to mention that the objective of this paper is to study the effects of Gauss-Bonnet term on dynamics of collapsing system. As heat flux is absent in the source equation, so transport equations and their coupling with the dynamical equations is not the objective of this paper. Hopefully it will be discussed explicitly elsewhere.

This paper is organized as follow: In section 2 the Einstein Gauss-Bonnet field equations and matching conditions have been discussed. The dynamical equations have been formulated in section 3. We summaries the results of the paper in the last section.

## 2 Einstein Gauss-Bonnet Field Equations

Consider the following action in $5D$

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2k_5^2} (R + \alpha L_{GB}) \right] + S_{\text{matter}}$$

(1)

where $R$ ia the Ricci scalar in $5D$ and $k_5^2 = 8\pi G_5$ is coupling constant in $5D$. Also, the Gauss-Bonnet Lagrangian has the form

$$L_{GB} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

(2)
where coefficient $\alpha$ is coupling constant in Einstein Gauss-Bonnet gravity. Such action is derivable in the low-energy limiting case of super-string theory. Here, $\alpha$ is treated as the inverse of string tension which is positive definite and $\alpha \geq 0$ in this paper. For the 4D manifold, Gauss-Bonnet terms do not contribute to field equations. The variation of action (11) with respect to 5D metric tensor yields the following set of field equations

$$G_{\alpha\beta} = G_{\alpha\beta} + \alpha H_{\alpha\beta} = T_{\alpha\beta},$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

is the Einstein tensor and

$$H_{\alpha\beta} = 2 \left[ RR_{\alpha\beta} - 2 R_{\alpha a} R^a_b - 2 R^{ab} R_{a\beta b\gamma} + R^{ab} R_{b\alpha \gamma} - \frac{1}{2} g_{\alpha\beta} L_{GB} \right] - \frac{1}{2} g_{\alpha\beta} L_{GB},$$

is the Lanczos tensor.

A spacelike 4D hypersurface $\Sigma^{(e)}$ is taken such that it divides a 5D spacetime into two 5D manifolds, $M^{-}$ and $M^{+}$, respectively. The 5D TB spacetime is taken as an interior manifold $M^{-}$ which is inner region of a collapsing inhomogeneous and anisotropic star is given by [26]

$$ds^2 = -dt^2 + B^2 dr^2 + C^2 (d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \theta \sin^2 \phi d\psi^2),$$

where $B$ and $C$ are functions of $t$ and $r$. The energy-momentum tensor $T_{\alpha\beta}^{-\alpha}$ for anisotropic fluid has the form

$$T_{\alpha\beta}^{-\alpha} = (\mu + P_{\perp}) V_\alpha V_\beta + P_{\perp} g_{\alpha\beta} + (P_r - P_{\perp}) \chi_\alpha \chi_\beta,$$

where $\mu$ is the energy density, $P_r$ the radial pressure, $P_{\perp}$ the tangential pressure, $V^\alpha$ the four velocity of the fluid and $\chi_\alpha$ a unit four vector along the radial direction. These quantities satisfy,

$$V^\alpha V_\alpha = -1, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0$$

The expansion scalar $\Theta$ for the fluid is given by

$$\Theta = V^\alpha \chi_\alpha.$$

Since we assumed the metric (6) comoving, then

$$V^\alpha = \delta^\alpha_0, \quad \chi^\alpha = B^{-1} \delta^\alpha_1,$$

where $\delta^\alpha_0$ and $\delta^\alpha_1$ are Kronecker delta functions.
and for the expansion scalar, we get
\[ \Theta = \frac{\dot{B}}{B} + \frac{3\dot{C}}{C}. \]  

(11)

Hence, Einstein Gauss-Bonnet field equations take the form
\[ \kappa^2_{\mu} = \frac{12}{B^3 C^5} \left[ \left(C' B' + B^2 \dot{C} \dot{B} - BC''\right) \alpha \right. \\
- \left. \frac{3}{B^3 C^2} \left[ B^3 \left(1 + \dot{C}^2\right) + B^2 C \dot{C} \dot{B} + CC' B' - B(CC'' + C'^2) \right] \right] \]

(12)

\[ \kappa^2_{p_r} = -12\alpha \left( \frac{1}{C^3} - \frac{C'^2}{B^2 C^3} + \frac{\dot{C}^2}{C^5} \right) \ddot{C} + 3 \frac{C'^2}{B^2 C^2} - 3 \left( \frac{1 + \dot{C}^2 + C\ddot{C}}{C^2} \right) \]

(13)

\[ \kappa^2_{p_\perp} = \frac{4\alpha}{B^4 C^2} \left[ - 2B \left(B' C' + B^2 \dot{B} \dot{C} - BC''\right) \ddot{B} + B \left(C'^2 - B^2 \left(1 + \dot{C}^2\right)\right) \right. \\
+ 2 \left(BC' - B\dot{C}'\right) - \left. \frac{1}{B^3 C^2} \left[ B^3 \left(1 + \dot{C}^2 + 2C\ddot{C}\right) + B^2 C \left(2\dot{C} \dot{B} + C \ddot{B}\right) \\
+ 2CC' B' - 2B \left(CC'' + C'^2\right) \right] \right] \]

(14)

\[ \frac{12\alpha}{B^5 C^3} \left(BC' - B\dot{C}'\right) \left(B^2 \left(1 + \dot{C}^2\right) - C'^2\right) - 3 \frac{B\dot{C}' - B\dot{C}''}{B^3 C} = 0. \]

(15)

The mass function \( m(t,r) \) analogous to Misner-Sharp mass in \( n \) manifold without \( \Lambda \) is given by [27]
\[ m(t,r) = \frac{(n-2)}{2k_n^2} V_{n-2}^k \left[ R^{n-3} \left(k - g^{ab} R_{,a} R_{,b}\right) + (n - 3)(n - 4)\alpha \left(k - g^{ab} R_{,a} R_{,b}\right)^2 \right], \]

(16)

where a comma denotes partial differentiation and \( V_{n-2}^k \) is the surface of \( (n - 2) \) dimensional unit space. For \( k = 1 \), \( V_{n-2}^1 = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \), using this relation with \( n = 5 \) and Eq. (6), the mass function (16) reduces to
\[ m(r,t) = \frac{3}{2} \left[ C^2 \left(1 - \frac{C'^2}{B^2} + \dot{C}^2\right) + 2\alpha \left(1 - \frac{C'^2}{B^2} + \dot{C}^2\right)^2 \right] \]

(17)

In the exterior region to \( \Sigma^{(e)} \), we consider Einstein Gauss-Bonnet Schwarzschild solution which is given by
\[ ds_+^2 = -F(R) dv^2 - 2dv dR + R^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \theta \sin^2 \phi d\psi^2\right). \]

(18)
where \( F(R) = 1 + \frac{R^2}{4\alpha} - \frac{R^2}{4\alpha} \sqrt{1 + \frac{16\alpha M}{\pi R^4}}. \)

The smooth matching of the 5D anisotropic fluid sphere \([6]\) to GB Schwarzschild BH solution \([18]\), across the interface at \( r = r_{\Sigma} = \text{constant} \), demands the continuity of the line elements and extrinsic curvature components (i.e., Darmonis matching conditions \([29]\)), implying

\[
\begin{align*}
\frac{dt}{\Sigma} &= \sqrt{F(R)} d\nu, \\
R &= R_{\Sigma}, \\
m(r,t) &= M, \\
\end{align*}
\]

\[
-12\alpha \left( \frac{1}{3} C^2 - \frac{C^2}{B^2 C^3} + \frac{\dot{C}^2}{C^3} \right) \ddot{C} + 3 \frac{C^2}{B^2 C^2} - 3 \left( \frac{1 + \dot{C}^2 + C\dot{C}}{C^2} \right) = 12\alpha \frac{B^2}{B^5 C^3} \left( \dot{B}C' - B\dot{C}' \right) \left( B^2 \left( 1 + \dot{C}^2 \right) - C' \right) - 3 \frac{B\dot{C}' - B\dot{C}'}{B^3 C}. \tag{22}
\]

Comparing Eq.\((22)\) with \((13)\) and \((15)\) (for detail see \([12]\)), we get

\[
p_r \Sigma^{(e)} = 0. \tag{23}
\]

Hence, the matching of the interior inhomogeneous anisotropic fluid sphere \([\Sigma]\) with the exterior vacuum Einstein Gauss-Bonnet spactime \([18]\) produces Eqs.\((6)\) and \((21)\).

3 Dynamical Equations

In this section, we formulate the equations that deal with the dynamics of collapsing process in Einstein Gauss-Bonnet gravity. Following Misner and Sharp formalism \([28]\), we discuss the dynamics of the collapsing system. We introduce proper time derivative as well as the proper radial as follows:

\[
\begin{align*}
D_T &= \frac{\partial}{\partial t}, & D_R &= \frac{1}{R'} \frac{\partial}{\partial r}, & R &= C. \tag{24}
\end{align*}
\]

The velocity of the collapsing fluid is the proper time derivative of \( R \) defined as

\[
U = D_T(R) \equiv \dot{C}. \tag{25}
\]
Using above result in the mass function given by Eq.(17)

$$m(r, t) = \frac{3}{2} \left[ C^2 \left( 1 - \frac{C'^2}{B^2} + U^2 \right) + 2\alpha \left( 1 - \frac{C'^2}{B^2} + U^2 \right)^2 \right]. \quad (26)$$

Solving above equation for $\frac{C'}{B}$, we get the positive and negative roots, the positive roots are given by

$$E = \frac{C'}{B} = \sqrt{1 + U^2 + \frac{R^2}{4\alpha} \pm \sqrt{3R^2 + 16m\alpha}}. \quad (27)$$

The rate of change of mass (Eq.(17)) with respect proper time is given by

$$D_Tm(t, r) = -\kappa_5^2 P_r U R^3, \quad (28)$$

where we have used Einstein Gauss-Bonnet field equations Eqs.(13) and (15). The right hand side of this equation has a single term. The this term is due to effective pressure (means the pressure is effected by the Gauss-Bonnet term) in $r$-direction. This term is positive in case of collapse ($U < 0$). This implies that as effective pressure in $r$-direction increases, mass (energy) also increases with the same amount. Similarly, we can calculate

$$D_Rm(t, r) = \frac{2}{3} k_5^2 \mu R^3, \quad (29)$$

where we have used Einstein Gauss-Bonnet field equations Eqs.(12) and (15). This equation explain how effective energy density affects the mass between neighboring hypersurfaces in the interior fluid distribution. Integration of Eq.(29) yields

$$m(t, r) = \frac{2}{3} k_5^2 \int_0^R (R^3 \mu) dR. \quad (30)$$

The dynamical equations can be obtained from the contracted Bianchi identities $T^{a\beta}_{\,\,;b} = 0$. Consider the following two equations

$$T^{\alpha\beta}_{\,\,;\beta} V_{\alpha} = \left[ \dot{\mu} + (\mu + P_r) \frac{\dot{B}}{B} + 3 (\mu + P_{\perp}) \frac{\dot{C}}{C} \right] = 0, \quad (31)$$

$$T^{\alpha\beta}_{\,\,;\beta} \chi_{\alpha} = \frac{1}{B} \left[ P'_r + 3 (P_r - P_{\perp}) \frac{C'}{C} \right] = 0. \quad (32)$$
The acceleration of the collapsing fluid is defined as

\[ D_T U = \ddot{C}. \] (33)

Using Eqs. (32), (33) and (13), we get

\[
\begin{align*}
12\alpha C & \left( \frac{9(p_r - p_\perp)^2}{C^3} (1 + U^2) + \left( \frac{p_r'}{B^2} + 1 \right) \right) D_T U \\
& = -9(p_r - p_\perp)^2 \left[ \kappa^2 p_r C + 3 \left( \frac{1}{C} + \frac{U^2}{C} \right) \right].
\end{align*}
\] (34)

This equation yields the effect of different forces on the collapsing process. It can be interpreted in the form of Newton’s second law of motion i.e., Force = mass density × acceleration. The term within square bracket on left side of above equation represent the inertial or passive gravitational mass. All the quantities on right side in square bracket are positive, hence this side is consequently negative and implies the retardation of dynamical system giving rise to the collapse of the system.

4  Outlook

This paper investigates the effects of the Gauss-Bonnet term on the dynamics of anisotropic fluid collapse in the 5D Einstein Gauss-Bonnet gravity. We have extended the work of Herrera [10] to 5D Einstein Gauss-Bonnet gravity. To this end, the non-conducting anisotropic fluid with 5D spherical symmetry has been taken as the source of gravitation in Einstein Gauss-Bonnet gravity. The Misner-Sharp mass has been calculated in the present scenario. The smooth matching of the interior source has been carried out with 5D Schwarzschild BH solution in Einstein Gauss-Bonnet gravity by using the Darmois [29] junction conditions. The matching of the two regions implies the vanishing of radial pressure over the boundary of the star and continuity of the gravitational masses in the interior and exterior regions. By using the Misner and Sharp approach for the proper time and radial derivatives, we have formulated the velocity as well as acceleration of the system. These definitions have been also applied to formulate the general dynamical equations in Gauss-Bonnet gravity.

The analysis of the dynamical equations predicts the following consequences:
• Mass of the collapsing spheres increases with the passage of time.

• Effective energy density of the system would effects the mass of the system during the different stages of the collapse

• The system under consideration goes to retardation implying the gravitational collapse.

We would like to mention that transport equations and their coupling with dynamical equations are not the objective of this paper as heat flux is absent in anisotropic fluid. This will be done in an other investigation with the inclusion of charge term in the interior source in future.

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