Behavior of the distance exponent for $\frac{1}{|x-y|^{2d}}$ long-range percolation

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Abstract. We study independent long-range percolation on $\mathbb{Z}^d$ where the vertices $u$ and $v$ are connected with probability 1 for $\|u-v\|_\infty = 1$ and with probability $1 - e^{-\beta \int_{[u+[0,1]^d]} \int_{[u+[0,1]^d]} \frac{1}{\|x-y\|^{2d}} dx dy}$ for $\|u-v\|_\infty \geq 2$, where $\beta \geq 0$ is a parameter. There exists an exponent $\theta = \theta(\beta) \in (0,1]$ such that the graph distance between the origin 0 and $v \in \mathbb{Z}^d$ scales like $\|v\|^{\theta}$. We prove that this exponent $\theta(\beta)$ is continuous and strictly decreasing as a function in $\beta$. Furthermore, we show that $\theta(\beta) = 1 - \beta + o(\beta)$ for small $\beta$ in dimension $d = 1$.

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1 Introduction

For $\beta \geq 0$, consider independent bond percolation on $\mathbb{Z}^d$ where there is an edge between $u$ and $v$ with probability

$$p(\beta, \{u, v\}) := 1 - e^{-\beta f_{u+[0,1)^d} f_{u+[0,1)^d} \zeta^{-\frac{d}{2}}},$$

for $\|u - v\|_\infty \geq 2$ and with probability 1 for $\|u - v\|_\infty = 1$. We denote the resulting probability measure by $\mathbb{P}_\beta$ and its expectation by $\mathbb{E}_\beta$. The resulting graph is clearly connected and in [5] it is proven that the typical graph distance between the origin $0$ and a point $v \in \mathbb{Z}^d$ grows like $\|v\|^\theta$ for some $\theta = \theta(\beta) \in (0, 1]$, where $\theta = 1$ if and only if $\beta = 0$. Furthermore, the diameter of the box $\{0, \ldots, n\}^d$ also has the same asymptotic growth. More precisely, it is shown that

$$\|x\|^\theta \approx_p D(0, x) \approx_p \mathbb{E}_\beta [D(0, x)]$$

(1)

and that

$$n^\theta \approx_p \text{Diam}\left(\{0, \ldots, n\}^d\right) \approx_p \mathbb{E}_\beta \left[\text{Diam}\left(\{0, \ldots, n\}^d\right)\right]$$

(2)

where the notation $A(n) \approx_p B(n)$ means that for all $\varepsilon > 0$ there exist $0 < c < C < \infty$ such that $\mathbb{P}(cB(n) \leq A(n) \leq CB(n)) > 1 - \varepsilon$ for all $n \in \mathbb{N}$, respectively the same for all $x \in \mathbb{Z}^d$. As proven in [5, Theorem 7.1], this asymptotic scaling of the typical distance (1) also holds for many other models of long-range percolation, for example when $u$ and $v$ are connected with probability $\frac{1}{|u-v|^d} \wedge 1$ or $1 - e^{-\frac{\|u-v\|^2}{|u-v|^d}}$, which are often considered in the literature. The important feature about these connection probabilities is that they are close enough to the connection probability $p(\beta, \{u, v\})$ for large $\|u - v\|$. The results (1) and (2) were for dimension $d = 1$ already shown by Ding and Sly in [23]. The distance exponent $\theta(\beta)$ also depends on the dimension $d$, but we omit this in the notation. In [5] it is also proven how the distance exponent decays as $\beta \to \infty$. It is shown for all dimensions $d$ there exist constants $0 < c < C < \infty$ such that

$$\frac{c}{\log(\beta)} \leq \theta(\beta) \leq \frac{C}{\log(\beta)}$$

(3)

for all $\beta \geq 2$.

1.1 Main results

In this paper, we study several properties of the dependence of the distance exponent $\theta(\beta)$ on $\beta$. For $d = 1$, it is well-known that $\theta(\beta) \geq 1 - \beta$, see for example [17, 23]. In section 3, we show that for small $\beta$ this lower bound is indeed a good approximation for $\theta(\beta)$.

**Theorem 1.1.** For $d = 1$, the right-hand derivative of the distance exponent $\frac{d}{d\beta} \theta(\beta)$ exists at $\beta = 0$ and furthermore one has $\frac{d}{d\beta} \theta(\beta) \bigg|_{\beta=0} = -1$. This yields that $\theta(\beta) = 1 - \beta + o(\beta)$ as $\beta \to 0$.

It is clear that the function $\theta(\beta)$ is monotonically decreasing in $\beta$, as for $\beta_1 < \beta_2$ we can couple the respective measures in such a way that the set of edges resulting from $\mathbb{P}_{\beta_1}$ is a subset of the edge-set sampled from $\mathbb{P}_{\beta_2}$. In section 4, we show that $\theta(\beta)$ is even strictly decreasing.
Theorem 1.2. The distance exponent $\theta : \mathbb{R}_{\geq 0} \to (0, 1]$ is strictly monotonically decreasing.

The main tool in the proof of Theorems 1.1 and 1.2 is Lemma 2.1, which can be seen as a version of Russo’s formula for expectations. Finally, in section 5 we show that $\theta(\beta)$ is a continuous function.

Theorem 1.3. The distance exponent $\theta : \mathbb{R}_{\geq 0} \to (0, 1]$ is continuous in $\beta$.

So in particular, Theorem 1.3 together with the facts that $\lim_{\beta \to 0} \theta(\beta) = \theta(0) = 1$ and $\lim_{\beta \to \infty} \theta(\beta) = 0$ show that $\theta(\beta)$ interpolates continuously between 0 and 1, as $\beta$ goes from $+\infty$ to 0. The continuity of the distance exponent was also used for the comparison with different inclusion probabilities in [5, Section 7].

1.2 The continuous model

For $\beta > 0$, the described discrete percolation model has a self-similarity that comes from a coupling with the underlying continuous model, that we will now describe for any dimension. This will also explain our, at first sight complicated, choice of the connection probability. Consider the Poisson point process $\mathcal{E}$ on $\mathbb{R}^d \times \mathbb{R}^d$ with intensity $\frac{\beta}{2\|x-y\|^2}$. Define the symmetrized version $\mathcal{E}$ by $\mathcal{E} := \{(t, s) \in \mathbb{R}^d \times \mathbb{R}^d : (s, t) \in \mathcal{E}\} \cup \mathcal{E}$. We define $C = [0, 1]$. For $u, v \in \mathbb{Z}^d$ with $\|u - v\|_\infty \geq 1$ we put an edge between $u$ and $v$ if and only if $(u + C) \times (v + C) \cap \mathcal{E} \neq \emptyset$. The cardinality of $((u + C) \times (v + C)) \cap \mathcal{E}$ is a random variable with Poisson distribution and parameter $\int_{u+C} \int_{v+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt$. Thus, by the properties of Poisson processes, the probability that $u \sim v$ equals

$$\mathbb{P}\left(((u + C) \times (v + C)) \cap \mathcal{E} = \emptyset\right) = \mathbb{P}\left(((u + C) \times (v + C)) \cap \tilde{\mathcal{E}} = \emptyset\right)^2 = e^{-\int_{u+C} \int_{v+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt}\,1 - p(\beta, \{u, v\})$$

which is exactly the probability that $u \sim v$ under the measure $\mathbb{P}_\beta$. Note that for $u, v$ with $\|u - v\|_\infty = 1$ we have $\int_{u+C} \int_{v+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt = \infty$. So we really get that all edges of the form $\{u, v\}$ with $\|u - v\|_\infty = 1$ are present. The construction with the Poisson process also implies that the presence of different bonds is independent and thus the resulting measure of the random graph constructed above equals $\mathbb{P}_\beta$. The chosen connection probabilities have many advantages. First of all, the resulting model is invariant under translation and invariant under the reflection of coordinates, i.e., when we change the $i$-th component $p_i(x)$ of all $x \in \mathbb{Z}^d$ to $-p_i(x)$. Furthermore, the model has the following self-similarity: For some vector $u = (p_1(u), \ldots, p_d(u)) \in \mathbb{Z}^d$ and $n \in \mathbb{N}_{>0}$ we define the translated boxes $V_u^n := \prod_{i=1}^d \{p_i(u)n, \ldots, (p_i(u) + 1)n - 1\} = nu + \prod_{i=1}^d \{0, \ldots, n - 1\}$. Then for all points $u, v \in \mathbb{Z}^d$, and all $n \in \mathbb{N}_{>0}$ one has

$$\mathbb{P}_\beta (V_u^n \sim V_v^n) = \prod_{x \in V_u^n} \prod_{y \in V_v^n} e^{-\int_{x+C} \int_{y+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt} = e^{-\sum_{x \in V_u^n} \sum_{y \in V_v^n} \int_{x+C} \int_{y+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt} = e^{-\int_{u+C} \int_{v+C} \frac{\beta}{2\|x-y\|^2}\,ds\,dt} = \mathbb{P}_\beta (u \sim v)$$

which shows the self-similarity of the model. Also observe that for any $\alpha \in \mathbb{R}_{>0}$ the process $\alpha \tilde{\mathcal{E}} := \\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : (1/\alpha) x, (1/\alpha) y \in \tilde{\mathcal{E}}\}$ is again a Poisson point process with intensity $\frac{\beta}{2\|x-y\|^2}$.4
1.3 Notation

We use the notation $e_i$ for the $i$-th standard unit vector of $\mathbb{R}^d$ and we use the symbols $0$ and $1$ for the vector that contains only zeros, respectively ones. For a vector $y \in \mathbb{R}^d$, we write $p_i(y)$ for the $i$-th component of $y$, i.e., $p_i(y) = \langle e_i, y \rangle$. We use the symbol $C$ for the box $[0,1)^d$. For a vector $u = (p_1(u), \ldots, p_d(u)) \in \mathbb{Z}^d$ and $n \in \mathbb{N}_{>0}$, we define the box

$$V_u^n := \prod_{i=1}^d \{p_i(u)n, \ldots, (p_i(u) + 1)n - 1\} = nu + \prod_{i=1}^d \{0, \ldots, n - 1\}.$$ 

For a graph $G = (V,E)$ and $u,v \in V$ we write $D(u,v)$ for the graph distance between $u$ and $v$. For some subset $A \subset \mathbb{Z}^d$, we write $D_A(u,v)$ for the distance between $u$ and $v$ when we restrict to the edges with both endpoints inside $A$. When it is not clear, we will often write on which graph we work. When we write $\text{Diam}(A)$ we always mean the diameter inside this set, i.e., $\text{Diam}(A) = \max_{u,v \in A} D_A(u,v)$. We write $\mathbb{P}_\beta$ for the measure of independent long-range percolation with inclusion probabilities $p(\beta, \{u,v\}) = 1 - e^{-\int_{u+c}^{u+c+e} \beta x \log |x-y| dx}$, and $\mathbb{E}_\beta$ for the expectation under this measure. For some edge $e = \{u,v\}$ we write $|e| = \{u,v\} = \|u - v\|_\infty$ for the distance in the $\infty$-norm between the endpoints. We use the notation $\log(x)$ for the natural logarithm, i.e., the logarithm to the base $e$. The percolation configuration is a random element $\omega \in \{0,1\}^E$, where we say that the edge $e$ exists or is open or present if $\omega(e) = 1$. When $\omega \in \{0,1\}^E$ and $e \in E$ we define the elements $\omega^{e+}, \omega^{e-}$ by

$$\omega^{e+}(\tilde{e}) = \begin{cases} 1 & \tilde{e} = e \\ \omega(e) & \tilde{e} \neq e \end{cases} \text{ and } \omega^{e-}(\tilde{e}) = \begin{cases} 0 & \tilde{e} = e \\ \omega(e) & \tilde{e} \neq e \end{cases},$$

so this are the edge sets when we insert, respectively delete, the edge $e$. When we look at a (possibly random) subset of the edges that is defined by some $\omega \in \{0,1\}^E$ we also write $D(u,v;\omega)$ for the graph distance between $u$ and $v$ in the environment represented by $\omega$. We define the indirect distance $D^*(A,B)$ between the sets $A,B \subset \mathbb{Z}^d$ as the graph distance in the environment where we removed all edges between $A$ and $B$, which is the distance when we only consider paths between $A$ and $B$ that do not use an edge $e = \{u,v\}$ with $u \in A, v \in B$.

1.4 Related work

The scaling of the graph distance, also called chemical distance or hop-count distance, is a central characteristic of a random graph and has also been examined for many different models of percolation, see for example [1,4,8,12–15,17,21,22,24,28–30,37]. For all dimensions $d$ one can also consider the long-range percolation model with connection probability $p(\beta, \{u,v\}) = \frac{\beta}{\|u-v\|_\infty^d}$. There is a first transition of the behavior at $s = d$, which is natural as the resulting random graph is locally finite if and only if $s > d$. For $s < d$ the graph distance between two points is at most $\left\lceil \frac{d}{s-1} \right\rceil$ [10], whereas for $s = d$, the diameter of the box $\{0,\ldots,n\}^d$ is of order $\frac{\log(n)}{\log(\log(n))}$ [17, 41]. In [8, 13–15] the authors proved that for $d < s < 2d$ the typical distance between two points of Euclidean distance $n$ grows like $\log(n)^{\Delta}$, where $\Delta^{-1} = \log_2(\frac{2d}{s})$. The behavior of the typical distance for long-range percolation on $\mathbb{Z}^d$ also changes at $s = 2d$. For $s > 2d$ the graph distance grows at least linearly in the Euclidean distance of two points, as proven in [12]. For $s = 2d$, the typical distance and the diameter grow polynomially with some exponent that depends on $\beta$, as shown in [5, 23]. So in particular Theorems 1.1, 1.3 together with (3) show that, as $\beta$ goes from $+\infty$ to 0, the long-range percolation model with $s = 2d$ interpolates continuously between subpolynomial growth and linear growth.
Another line of research is to investigate what happens when one drops the assumption that \( p(\beta, \{u, v\}) = 1 \) for all nearest neighbor edges \( \{u, v\} \), but assigns i.i.d. Bernoulli random variables to the nearest neighbor edges instead. For \( d = 1 \), there is a change of behavior at \( s = 2 \). As proven by Aizenman, Newman, and Schulman in [3,38,40], an infinite cluster can not emerge for \( s > 2 \) and for \( s = 2, \beta \leq 1 \), no matter how small \( \mathbb{P}(k \sim k+1) \) is. On the other hand, an infinite cluster can emerge for \( s < 2 \) and \( s = 2, \beta > 1 \) (see [38]). See also [25] for a new proof of these results. In [3] the authors proved that there is a discontinuity in the percolation density for \( s = 2 \), contrary to the situation for \( s < 2 \), as proven in [11,32]. For models, for which an infinite cluster exists the behavior of the percolation model at and near criticality is also a well-studied question (cf. [6,7,11,16,20,32–34]). It is not known up to now how the typical distance in long-range percolation grows for \( s = 2d \) and \( p(\beta, \{u, v\}) < 1 \) for nearest-neighbor edges \( \{u, v\} \), but we conjecture also a polynomial growth in the Euclidean distance, whenever an infinite cluster exists.

For \( d = 1 \), the behavior of the mixing time is also a property that exhibits a transition at \( s = 2 \), as proven in [9]. On the line segment \( \{0, \ldots, n\} \) the mixing time grows quadratic in \( n \) for \( s > 2 \) and is of order \( n^{s-1} \) for \( 1 < s < 2 \). The behavior of the mixing time for \( s = 2 \) is still open, but we conjecture a similar behavior to that of the chemical distance, namely that the mixing time interpolates between \( n \) and \( n^2 \), as \( \beta \) goes from \( +\infty \) to 0. A better understanding of the mixing time is useful to study the heat kernel and understand the long-time behavior of the simple random walk on the cluster. In [18,19] Crawford and Sly give bounds on the heat kernel and prove a scaling limit for the case \( s \in (d, d+1) \).

Also the Ising model on the one-dimensional line with interaction energy \( J(\{x, y\}) = |x-y|^{-s} \) is a well-studied object. In [26] the author considers the case where \( s < 2 \), but there are also many results for the critical case \( s = 2 \), see for example [2,27,35]. In particular, the authors in [2] proved a discontinuity of the magnetization.

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2 Russo’s formula for expectations

In this chapter, we establish one of our main tools in the proofs of Theorem 1.1 and Theorem 1.2, which is a version of Russo’s formula. The classical Russo’s formula, also called Russo-Margulis lemma, see for example [31, Section 1.3] or [36,39] for the original papers, is a formula for i.i.d. bond percolation. It states that for any finite graph \((V, E)\) and any increasing event \( A \)

\[
\frac{d}{dp} \mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(e \text{ is pivotal for } A),
\]

where we say that an edge \( e \) is pivotal for an event \( A \) when changing the status of \( e \) also changes the occurrence of the event \( A \). Note that it does not depend on the occupation status of the edge \( e \) whether \( e \) is pivotal for \( A \). Russo’s formula (4) tells us how the probability of an event changes for i.i.d. percolation when varying the connection probability \( p \). We modify this formula in two ways. First of all, we adapt it to long-range percolation, where the inclusion probabilities of the edges are not identically distributed. Secondly, we
develop a formula that determines the derivative of the expectation of a general function rather than just the probability of a given event.

**Lemma 2.1** (Russo’s formula for expectations). Let \( G = (V, E) \) be a finite graph with a set of inclusion probabilities \((p(\beta, e))_{e \in E, \beta \geq 0}\), where \( \beta \mapsto p(\beta, e) \) is continuously differentiable on \( \mathbb{R}_{\geq 0} \) for all \( e \in E \). By \( \mathbb{P}_\beta \) we denote the Bernoulli product measure on \( \{0, 1\}^E \) with inclusion probabilities \((p(\beta, e))_{e \in E}\) and its expectation by \( \mathbb{E}_\beta \). Let \( f : \{0, 1\}^E \to \mathbb{R} \) be a function. Then

\[
\frac{d}{d\beta} \mathbb{E}_\beta [f(\omega)] = \sum_{e \in E} p'(\beta, e) \mathbb{E}_\beta \left[ f(\omega^e) - f(\omega^{-e}) \right].
\]

The lemma is stated for any set of continuously differentiable functions \( p(\beta, e) \), but one can also always think of the case where \( p(\beta, \{u, v\}) = 1 - e^{-\beta \int_{u \to v} f_{u \to v} \frac{1}{\|u-v\|^2} \, dx \, dy} \), as we only apply it to this case.

**Proof of Lemma 2.1.** The proof is similar to the case of the classical Russo’s formula, see for example [31]. For a vector \( \beta = (\beta_e)_{e \in E} \in \mathbb{R}^E_{\geq 0} \), we define the probability measure \( \mathbb{P}_\beta \) on \( \{0, 1\}^E \) by

\[
\mathbb{P}_\beta(\omega) = \prod_{e: \omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e))
\]

so that each component \( \omega(e) \) is Bernoulli distributed with expectation \( p(\beta_e, e) \), and all components are independent. Under this measure, a function \( f : \{0, 1\}^E \to \mathbb{R} \) has the expectation

\[
\mathbb{E}_\beta [f(\omega)] = \sum_{\omega \in \{0,1\}^E} f(\omega) \prod_{e: \omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e)).
\]

For an edge \( e \in E \), differentiation with respect to \( \beta_e \) gives

\[
\frac{d}{d\beta_e} \mathbb{E}_\beta [f(\omega)] = \sum_{\omega \in \{0,1\}^E} f(\omega) \frac{d}{d\beta_e} \left( \prod_{e: \omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e)) \right)
\]

\[
= \sum_{\omega \in \{0,1\}^E} f(\omega) p'(\beta_e, f) \left( \mathbb{I}_{\omega(f)=1} - \mathbb{I}_{\omega(f)=0} \right) \prod_{e: \omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e))
\]

\[
= p'(\beta_e, f) \sum_{\omega \in \{0,1\}^E, \omega(f)=1} f(\omega) \mathbb{I}_{\omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e))
\]

\[
- p'(\beta_e, f) \sum_{\omega \in \{0,1\}^E, \omega(f)=0} f(\omega) \mathbb{I}_{\omega(e)=1} p(\beta_e, e) \prod_{e: \omega(e)=0} (1 - p(\beta_e, e))
\]

\[
= p'(\beta_e, f) \mathbb{E}_\beta \left[ f(\omega^+) \right] - p'(\beta_e, f) \mathbb{E}_\beta \left[ f(\omega^-) \right] = p'(\beta_e, f) \mathbb{E}_\beta \left[ f(\omega^+) - f(\omega^-) \right].
\]

To conclude, consider the mapping \( \phi : \mathbb{R} \mapsto \mathbb{R}^E \) defined by \( \phi(\beta) = (\beta, \ldots, \beta) \). With this and the chain rule we finally get

\[
\frac{d}{d\beta} \mathbb{E}_\beta [f(\omega)] = \frac{d}{d\beta} \mathbb{E}_{\phi(\beta)} [f(\omega)] = \sum_{e \in E} \frac{d}{d\beta_e} \mathbb{E}_\beta [f(\omega)] = \sum_{e \in E} p'(\beta, e) \mathbb{E}_\beta \left[ f(\omega^e) - f(\omega^{-e}) \right].
\]
We now consider the case where $p(\beta, \{u, v\}) = 1 - e^{-\int_{u+C} f_{v+C} \frac{\beta}{\|x-y\|^2} \, dx \, dy}$. Note that $p(\beta, e)$ decays like $\frac{\beta}{|e|^2}$ as $|e|$ tends to infinity. By the triangle inequality we have for all $x \in u + C, y \in v + C$:

$$\|u-v\| - \sqrt{d} \leq \|x-y\| \leq \|u-v\| + \sqrt{d}$$

and thus, for $\|u-v\| \geq \sqrt{d}$ we can bound the integral in the exponent from above and below by

$$\frac{1}{\|u-v\| + \sqrt{d}} \leq \int_{u+C} \int_{v+C} \frac{1}{\|x-y\|^2} \, dy \, dx \leq \frac{1}{\|u-v\| - \sqrt{d}}.$$  \hspace{1cm} (6)

Also note that we have for all edges $\{u, v\}$ with $\|u-v\| \geq 2$ that

$$1 \geq \int_{u+C} \int_{v+C} \frac{1}{\|x-y\|^2} \, dy \, dx$$

as the integrand is bounded by 1. Next, we consider the derivative of $p(\beta, e)$ for non-nearest neighbor edges $e = \{u, v\}$. By the chain rule we have

$$\frac{d}{d\beta} p(\beta, e) = \int_{u+C} \int_{v+C} \frac{1}{\|x-y\|^2} \, dy \, dx \, e^{-\int_{u+C} f_{v+C} \frac{\beta}{\|x-y\|^2} \, dx \, dy}$$

and using (6) and (7) we see that for $\|u-v\| \geq \sqrt{d}$

$$p'(\beta, \{u, v\}) = \begin{cases} \leq \frac{1}{(\|u-v\| - \sqrt{d})^2} e^{-\beta} \\ \geq \frac{1}{(\|u-v\| + \sqrt{d})^2} e^{-\beta} \end{cases}.$$ \hspace{1cm} (8)

and thus in particular for fixed $\beta > 0$ we have $p'(\beta, \{u, v\}) = \Theta(\frac{1}{\|u-v\|^2})$ for $\|u-v\| \to \infty$. For every non-nearest-neighbor edge $e$ the probability that $e$ exists is upper bounded by $1 - e^{-\beta}$. This implies that

$$|E_\beta [f(\omega^e) - f(\omega)]| \leq |E_\beta [f(\omega^e) - f(\omega^-)|] \leq \frac{1}{1 - e^{-\beta}} |E_\beta [f(\omega^e) - f(\omega)]|$$

for all functions $f : \{0, 1\}^E \to \mathbb{R}$ and all edges $e \in E$ with $|e| \geq 2$. Above we bounded the derivative $p'(\beta, e)$ from above and below. We also want to bound the connection probability $p(\beta, e)$. As $1 - e^{-s} \geq \frac{s}{2} \wedge \frac{1}{2}$ for all $s \geq 0$ one has that

$$p(\beta, \{u, v\}) \geq \int_{u+C} \int_{v+C} \frac{\beta}{2\|x-y\|^2} \, dx \, dy \wedge \frac{1}{2} e^{-\beta} \geq \frac{\beta}{2} \left(\|u-v\| + \sqrt{d}\right)^2 \wedge \frac{1}{2}.$$ \hspace{1cm} (9)

On the other hand one has $1 - e^{-x} \leq x$ and thus one can upper bound the connection probability by

$$p(\beta, \{u, v\}) \leq \int_{u+C} \int_{v+C} \frac{\beta}{\|x-y\|^2} \, dx \, dy \leq \frac{\beta}{(\|u-v\| - \sqrt{d})^2}.$$ \hspace{1cm} (10)

for $\|u-v\| \geq \sqrt{d}$. 
3 Asymptotic behavior of $\theta(\beta)$ for small $\beta$ and $d = 1$

In this section, we prove Theorem 1.1, i.e., that $\theta(\beta) = 1 - \beta + o(\beta)$ for $\beta \to 0$ for $d = 1$. Determining the asymptotic behavior of $\theta(\beta)$ for dimension two or higher for $\beta \to 0$ is more difficult for several reasons. First, there is no lower bound on $\theta(\beta)$ that arises from considering cut points or something similar. The notion of cut points and its implication on the distance exponent $\theta(\beta)$ in dimension $d = 1$ will be explained below. Secondly, it is not clear which pair of vertices $x, y \in V_0^\beta$ minimizes the expected distance $\mathbb{E}_\beta \left[ D_{V_0^\beta}(x,y) \right]$ in dimension two or higher, i.e., whether a similar statement of equation (16) holds for $d \geq 2$. However, for all dimensions $d$ there exists a constant $c > 0$ such that $\theta(\beta) \leq 1 - c\beta$. This can already be shown with the exact same technique that was used in [17].

But now let us consider dimension $d = 1$ again. Here we have $\theta(0) = 1$ and it is well known that $\theta(\beta) \geq 1 - \beta$ (see [17, 23]). So we get that $\liminf_{\beta \to 0} \frac{\theta(\beta) - \theta(0)}{\beta} \geq -1$. Thus it suffices to show that $\limsup_{\beta \to 0} \frac{\theta(\beta) - \theta(0)}{\beta} \leq -1$. For the sake of completeness, we give a short sketch of the proof of the lower bound $\theta(\beta) \geq 1 - \beta$. For this, we define the notion of a cut point. We say that the vertex $w \in \{1, \ldots, n - 2\}$ is a cut point if there exists no edge $\{u, v\}$ with $0 \leq u < w < v \leq n - 1$. We have

$$\mathbb{P}_\beta(w \text{ is a cut point}) = \prod_{0 \leq u < w} \prod_{w < v \leq n-1} e^{-\beta \int_{u+1}^{w+1} \frac{1}{|x-y|^2} \, dx \, dy} \geq e^{-\beta \int_0^w \frac{1}{|x-y|^2} \, dx \, dy} = e^{-\beta \log(w+1)} \geq n^{-\beta}. \quad (11)$$

As the distance between 0 and $n - 1$ is lower bounded by the number of cut points between 0 and $n - 1$ we get, by linearity of expectation, that

$$\mathbb{E}_\beta \left[ D_{[0,n-1]}(0,n-1) \right] \geq \mathbb{E}_\beta \left[ |\{w : w \text{ is a cut point}\}| \right] \geq (n - 2)n^{-\beta} = \Omega(n^{1-\beta}) \quad (12)$$

which shows that $\theta(\beta) \geq 1 - \beta$. As a first step towards the proof of Theorem 1.1, we remind ourselves about the submultiplicativity of the expected distance, which was proven in [5, Lemma 2.3].

Lemma 3.1. For all dimensions $d$ and all $\beta \geq 0$, the sequence

$$\Lambda(n) = \Lambda(n, \beta) := \max_{u,v \in \{0,\ldots, n-1\}^d} \mathbb{E}_\beta \left[ D_{V_0^\beta}(u,v) \right] + 1 \quad (13)$$

is submultiplicative and furthermore one has

$$\theta(\beta) = \inf_{n \geq 2} \frac{\log(\Lambda(n, \beta))}{\log(n)}. \quad (14)$$

Now we are prepared to prove Theorem 1.1. Our main tools for this are Lemma 3.1 and Russo’s formula for expectations (5).

Proof of Theorem 1.1. Note that $\Lambda(n,0) = n$ and thus $\frac{\log(\Lambda(n,0))}{\log(n)} = 1$. Using this and (14) we obtain

$$\limsup_{\beta \to 0} \frac{\theta(\beta) - \theta(0)}{\beta} = \limsup_{\beta \to 0} \inf_{n \geq 2} \frac{\log(\Lambda(n, \beta)) - \log(\Lambda(n,0))}{\beta \log(n)} \leq \inf_{n \geq 2} \limsup_{\beta \to 0} \frac{\log(\Lambda(n, \beta)) - \log(\Lambda(n,0))}{\beta \log(n)}$$

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and this works, as for fixed $n$ the function $\Lambda(n, \beta)$ is differentiable at $\beta = 0$, as the inclusion probabilities $p(\beta, \{u, v\})$ are. Now we want to calculate $\frac{d}{d\beta}\Lambda(n, \beta)\big|_{\beta=0}$. For this, let $E$ be the set of all edges of length at least 2 in the graph with vertex set $\{0, \ldots, n-1\}$. For $e \in E$, let $\omega^{e+}$ be the environment, where we added the edge $e$ (or do nothing in case it already existed before). For $\beta$ very small compared to $\frac{1}{n}$ we have that $\max_{u,v \in \{0,\ldots,n-1\}} E_\beta [D_{[0,n-1]}(u,v)] = E_\beta [D_{[0,n-1]}(0,n-1)]$. To see this, note that on the one hand for any $u, v \in \{0, \ldots, n-1\}$ we have

$$E_\beta [D_{[0,n-1]}(u,v)] \leq |u - v|,$$

whereas on the other hand we have

$$E_\beta [D_{[0,n-1]}(0,n-1)] \geq (n-1)P_\beta \left( \bigcap_{e \in E} \{e \text{ closed}\} \right).$$

As the probability of the event $\bigcap_{e \in E} \{e \text{ closed}\}$ tends to 1 for $\beta \to 0$ we see that

$$E_\beta [D_{[0,n-1]}(0,n-1)] = \max_{u,v \in \{0,\ldots,n-1\}} E_\beta [D_{[0,n-1]}(u,v)]$$

for small enough $\beta$, where small enough of course depends on $n$. Using this observation we see that

$$\frac{d}{d\beta}\Lambda(n, \beta)\big|_{\beta=0} = \lim_{\beta \to 0} \frac{\Lambda(n, \beta) - \Lambda(n, 0)}{\beta} = \lim_{\beta \to 0} E_\beta \left[ D_{[0,n-1]}(0,n-1) - E_0 \left[ D_{[0,n-1]}(0,n-1) \right] \right] = \sum_{e \in E} p'(0,e)E_0 \left[ D_{[0,n-1]}(0,n-1; \omega^{e+}) - (n-1) \right].$$

In the environment $\omega^{e+}$ sampled by $P_0$, where only the nearest-neighbor edges and the edge $e$ are present, the shortest path from 0 to $n-1$ will also take the edge $e$. By taking the edge $e$, the distance between 0 and $n-1$ decreases by $|e|-1$, and thus equals $n-1 - (|e|-1) = n - |e|$. For $d = 1$, we get from (8) that $p'(0,\{u,v\}) \geq \frac{1}{(|u-v|+1)^2}$. With this we can upper bound the derivative computed in (17) and obtain that

$$\frac{d}{d\beta}\Lambda(n, \beta)\big|_{\beta=0} = \sum_{e \in E} \frac{1}{(|e|+1)^2}(|e|-1) = \sum_{k=0}^{n-3} \sum_{j=k+2}^{n-1} \frac{1 - |j - k|}{(j - k + 1)^2} = \sum_{k=0}^{n-3} \sum_{l=2}^{n-1-k} \frac{1 - l}{(l+1)^2}. $$

For $l \in \mathbb{N}$, we have $\frac{-l}{(l+1)^2} \leq \frac{2}{l^2} - \frac{1}{l}$, as we will show now. One has

$$\frac{-l}{(l+1)^2} \leq \frac{2}{l^2} - \frac{1}{l} \iff -l^3 \leq (l+1)^2(2-l) = (l^2 + 2l + 1)(2-l) \iff l^3 \geq (l^2 + 2l + 1)(l-2) = l^3 - 2l^2 + 2l^2 - 4l + l - 2 \iff 0 \geq -3l - 2$$

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and the last line is clearly true. Using that \( \frac{1}{(l+1)^2} \leq \frac{2}{l^2} - \frac{1}{l} \) we also get that \( \frac{1-l}{(l+1)^2} \leq \frac{1}{3} + \frac{2}{l^2} - 1 \leq \frac{1}{3} \). Inserting this into (18) we get that

\[
\frac{d}{d\beta} \Lambda(n, \beta) \bigg|_{\beta=0} \leq \sum_{k=0}^{n-3} \sum_{l=2}^{n-1-k} \frac{3}{l^2} - \frac{1}{l} \leq \sum_{k=0}^{n-3} \sum_{l=2}^{n-1-k} \frac{3}{l^2} + \sum_{k=0}^{n-3} \sum_{l=2}^{k} \frac{1}{l} \\
\leq 3n + \sum_{k=0}^{n-3} \sum_{l=2}^{n-1-k} \frac{1}{l} \leq 4n + \sum_{k=0}^{n-3} \int_{1}^{n-k} \frac{1}{s} ds = 4n - \sum_{k=0}^{n-3} log(n - k) \\
= 4n - \sum_{k=3}^{n} log(k) \leq 4n - \int_{2}^{n} log(s) ds = 4n - \left[ -s + s \log(s) \right]_{2}^{n} \\
\leq 5n + 4 - n \log(n).
\]

Inserting this into (15) gives

\[
\limsup_{\beta \searrow 0} \frac{\theta(\beta) - \theta(0)}{\beta} \leq \inf_{n \geq 2} \frac{1}{\Lambda(n, 0) \log(n)} \frac{d}{d\beta} \Lambda(n, \beta) \bigg|_{\beta=0} = \inf_{n \geq 2} \frac{1}{n \log(n)} \frac{d}{d\beta} \Lambda(n, \beta) \bigg|_{\beta=0} \leq \inf_{n \geq 2} \frac{5n + 4 - n \log(n)}{n \log(n)} \leq -1
\]

where the infimum is achieved when taking \( n \to \infty \). As \( \theta(\beta) \geq 1 - \beta \), and thus \( \liminf_{\beta \searrow 0} \frac{\theta(\beta) - \theta(0)}{\beta} \geq -1 \), this finishes the proof of Theorem 1.1. \( \square \)

4 Strict monotonicity of the distance exponent

In this chapter, we prove Theorem 1.2, i.e., that the function \( \theta(\beta) \) is strictly decreasing in \( \beta \). It was known before, see [5, 17, 23], that \( \theta(\beta) \) is strictly decreasing at \( \beta = 0 \), which is equivalent to saying that \( \theta(\beta) < 1 = \theta(0) \) for all \( \beta > 0 \). With the Harris coupling (cf. [31]) one can see that the function \( \theta(\beta) \) is non-increasing. For this coupling, let \((U_{e})_{e \in E} \) be a collection of i.i.d. random variables that are uniformly distributed on \([0, 1]\) and then set \( \omega(e) := 1\{U_{e} \leq \rho(\beta, e)\} \). Then \( \omega \) is distributed according to the law of \( \mathbb{P}_\beta \) and for \( \beta \leq \beta' \) one has \( \omega(e) = 1\{U_{e} \leq \rho(\beta, e)\} \leq 1\{U_{e} \leq \rho(\beta', e)\} = \omega'(e) \). So in particular the environment defined by \( \omega' \) contains all edges defined by \( \omega \), and thus \( D(u, v; \omega') \leq D(u, v; \omega) \) for all \( u, v \in \mathbb{Z}^d \). Taking expectations on both sides of this inequality and letting \( \|u-v\| \to \infty \) already shows that \( \theta(\cdot) \) is non-increasing.

Before going into the details of the proof of the strict monotonicity, we want to show the main idea. One of the main tools is again Russo’s formula for expectations (5). We know that \( E_{\beta} [D(0, n1)] = \Theta(n^{\theta(\beta)}) \), as proven in [5, 23], and thus \( \theta(\beta) = \lim_{n \to \infty} \frac{\log(E_{\beta}[D(0, n1)])}{\log(n)} \). But for fixed \( n \) we can calculate the derivative of \( \frac{\log(E_{\beta}[D(0, n1)])}{\log(n)} \) with Lemma 2.1 and get that

\[
\frac{d}{d\beta} \left( \frac{\log(E_{\beta}[D_{V_0^{n+1}}(0, n1)])}{\log(n)} \right) = \frac{1}{\log(n)E_{\beta}[D_{V_0^{n+1}}(0, n1)]} \frac{d}{d\beta} E_{\beta}[D_{V_0^{n+1}}(0, n1)] \\
= \frac{1}{\log(n)E_{\beta}[D_{V_0^{n+1}}(0, n1)]} \sum_{e \in E} p'(\beta, e) E_{\beta}[D_{V_0^{n+1}}(0, n1; \omega^{e+})] - D_{V_0^{n+1}}(0, n1; \omega^{e-})
\]

(19)
where $E$ is the set of edges with both endpoints in $V_0^{n+1}$. For ease of notation, we drop the subscript of $V_0^{n+1}$ in the paragraph below and will implicitly always think of this graph as the underlying graph. A fully formal proof is given in section 4.3. Our goal is to show that for each $\beta > 0$, there exists a $c(\beta) < 0$ such that $\frac{d}{d\beta} \frac{\log(E_{\beta}[D(0,n1)])}{\log(n)} < c(\beta)$ uniformly over $n$. For this, it clearly suffices to consider $n$ large enough, as the bound clearly holds for small $n$. If we prove this we get that

$$\theta(\beta + \varepsilon) - \theta(\beta) = \lim_{n \to \infty} \left\{ \frac{\log \left( E_{\beta+\varepsilon}[D(0,n1)] \right) - \log \left( E_{\beta}[D(0,n1)] \right) }{\log(n)} \right\} = \lim_{n \to \infty} \int_{\beta}^{\beta+\varepsilon} \frac{d}{ds} \frac{\log \left( E_s[D(0,n1)] \right) }{\log(n)} ds < 0$$

as we will show in the end of section 4.3 in detail. This implies strict monotonicity of the function $\beta \mapsto \theta(\beta)$. In order to show $\frac{d}{d\beta} \frac{\log(E_{\beta}[D(0,n1)])}{\log(n)} < c(\beta)$, we divide the graph into several levels, and assume $n = 2^k - 1$ for some $k \in \mathbb{N}$. The $i$-th level consists of all edges for which $c2^i < |e| \leq C2^i$ for some constants $0 < c < C < \infty$. Note that an edge can be in several levels, but at most in finitely many. By $G(0,1)$ we denote the union of all geodesics between 0 and $n1$. So the occupation status of edges outside $G(0,1)$ does not change the distance between 0 and $n1$ which implies that for all edges $e$ one has

$$\mathbb{E}_{\beta} \left[ \left[ D(0,n1;\omega) - D(0,n1;\omega^{e^-}) \right] 1_{e \not\in G(0,n1)} \right] = \mathbb{E}_{\beta} \left[ D(0,n1;\omega) - D(0,n1;\omega^{e^-}) \right].$$

For fixed $\beta > 0$, we have by (8) that $p'(\beta, e) = \Theta (\mathbb{P}_\beta(\omega(e) = 1)) = \Theta \left( \frac{1}{|e|^{d+1}} \right)$ as $|e| \to \infty$. Thus we have uniformly over all edges of length at least 2 (but not uniformly over $\beta$) that

$$p'(\beta, e) \mathbb{E}_{\beta} \left[ D(0,n1;\omega^{e^+}) - D(0,n1;\omega^{e^-}) \right] = \Theta \left( \mathbb{E}_{\beta} \left[ D(0,n1;\omega) - D(0,n1;\omega^{e^-}) \right] \right).$$

So in order to show that there exists a $c(\beta) < 0$ such that $\frac{d}{d\beta} \frac{\log(E_{\beta}[D(0,n1)])}{\log(n)} < c(\beta)$ uniformly over $n$ it suffices to show that

$$\sum_{e \in E : c2^i < |e| \leq C2^i} \mathbb{E}_{\beta} \left[ \left[ D(0,n1;\omega) - D(0,n1;\omega^{e^-}) \right] 1_{e \not\in G(0,n1)} \right] < c'(\beta) \mathbb{E}_{\beta} [D(0,n1)]$$

for some $c'(\beta) < 0$ and a positive fraction of the levels $i \in \{1, \ldots, k\}$. One needs this for a positive fraction of the levels in order to cancel the logarithm in the denominator of (19).

### 4.1 Distances of certain points

For the proof of Theorem 1.2, we import a few results from the companion paper [5]: Fix the three blocks $V_u^n$, $V_w^n$ and $V_0^n$ with $\|u\|_{\infty} \geq 2$. The next lemma deals with the distance in the infinity metric between points $x, y \in V_0^n$ with $x \sim V_u^n$, $y \sim V_w^n$ and is proven in [5, Lemma 5.1].

**Lemma 4.1.** For all $\frac{1}{n} < \varepsilon \leq \frac{1}{4}$ and $u, w \in \mathbb{Z}^d \setminus \{0\}$ with $\|u\|_{\infty} \geq 2$ one has

$$\mathbb{P}_{\beta} \left( \exists x, y \in V_0^n : \|x - y\|_{\infty} \leq \varepsilon n, x \sim V_u^n, y \sim V_w^n \mid V_0^n \sim V_u^n, V_0^n \sim V_w^n \right) \leq C'_d \varepsilon^{1/2}[\beta]^2$$

where $C'_d$ is a constant that depends only on the dimension $d$. 

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The next lemma relates long-range percolation on two boxes $V_{0}^{n'}$ and $V_{0}^{n}$ with $n' \leq n$. It is intuitive that the distances in $V_{0}^{n'}$ are bigger than those in $V_{0}^{n}$. This is shown, up to a constant, in the following lemma, cf. [5, Lemma 4.1].

**Lemma 4.2.** Let $\beta \geq 0$ and $n',n \in \mathbb{N}_{>0}$ with $n' \leq n$. For $u,v \in V_{0}^{n}$ define $u' := \lfloor \frac{u}{n} \rfloor u$, $v' := \lfloor \frac{v}{n} \rfloor v$, where the rounding operation is componentwise. There exists a coupling of the random graphs with vertex sets $V_{0}^{n}$ and $V_{0}^{n'}$ such that both are distributed according to $\mathbb{P}_{\beta}$ and
\[
D_{V_{0}^{n'}}(u',v') \leq 3D_{V_{0}^{n}}(u,v)
\] (20)
for all $u,v \in V_{0}^{n}$. The same holds true when one considers the graph $\mathbb{Z}^{d}$ instead of $V_{0}^{n}$ and this also implies that
\[
\text{Diam}(V_{0}^{n'}) \leq 3\text{Diam}(V_{0}^{n}).
\] (21)

Lemma 4.1 tells us that for a block $V_{0}^{n}$ the vertices $x,y \in V_{0}^{n}$ that are connected to different boxes $x \sim V_{u}^{n}, y \sim V_{w}^{n}$ are typically far apart in terms of Euclidean distance, whenever $\|u\|_{\infty} \geq 2$. The next lemma shows that such points $x,y$ are also typically not too near in terms of the graph distance inside $V_{0}^{n}$. It was already proven in [5, Lemma 5.2].

**Lemma 4.3.** For all dimensions $d$ and all $\beta \geq 0$, there exists a function $g_{1}(\varepsilon)$ with $g_{1}(\varepsilon) \xrightarrow{\varepsilon \to 0} 1$ such that for all $u,w \in \mathbb{Z}^{d} \setminus \{0\}$ with $\|u\|_{\infty} \geq 2$ and all large enough $n \geq n(\varepsilon)$
\[
\mathbb{P}_{\beta}(D_{V_{0}^{n}}(x,y) > \varepsilon \Lambda(n,\beta)) \text{ for all } x,y \in V_{0}^{n} \text{ with } x \sim V_{u}^{n}, y \sim V_{w}^{n} \quad \text{such that } V_{u}^{n} \sim V_{w}^{n} \geq g_{1}(\varepsilon).
\]

One of the main results of [5] is that
\[
D(0,B_{n}(0)^{C}) \approx_{P} n^{\theta(\beta)}
\]
for some $\theta(\beta) \in (0,1)$. One can ask whether the same statement is true for the distance between two sets that are separated by a euclidean distance of $n$, for example $D\left(B_{n}(0),B_{2n}(0)^{C}\right)$. However, a similar statement can never be true, as there is a uniform (in $n$) positive probability of a direct edge between the sets $B_{n}(0)$ and $B_{2n}(0)^{C}$. But if we condition on the event that there is no direct edge, then we can get such a result, as proven in [5, Lemma 4.11].

**Lemma 4.4.** Let $\mathcal{L}$ be the event that there is no direct edge between $B_{n}(0)$ and $B_{2n}(0)^{C}$. For all $\beta \geq 0$ and all $\varepsilon > 0$, there exist $0 < c < C < \infty$ such that
\[
\mathbb{P}_{\beta}\left(c\Lambda(n,\beta) \leq D\left(B_{n}(0),B_{2n}(0)^{C}\right) \leq C\Lambda(n,\beta) \mid \mathcal{L}\right) > 1 - \varepsilon
\] (22)
for all $n \in \mathbb{N}$. Let $\mathcal{L}'$ be the event that there is no direct edge between $V_{0}^{n}$ and $\bigcup_{u \in \mathbb{Z}^{d}:\|u\|_{\infty} \geq 2} V_{u}^{n}$. For all $\beta \geq 0$ and all $\varepsilon > 0$, there exist $0 < c < C < \infty$ such that
\[
\mathbb{P}_{\beta}\left(c\Lambda(n,\beta) \leq D\left(V_{0}^{n}, \bigcup_{u \in \mathbb{Z}^{d}:\|u\|_{\infty} \geq 2} V_{u}^{n}\right) \leq C\Lambda(n,\beta) \mid \mathcal{L}'\right) > 1 - \varepsilon
\] (23)
for all $n \in \mathbb{N}$. So in particular there exists a function $g$ with $\lim_{\varepsilon \to 0} g_{2}(\varepsilon) = 1$ such that
\[
\mathbb{P}_{\beta}\left(\varepsilon\Lambda(n,\beta) \leq D\left(V_{0}^{n}, \bigcup_{u \in \mathbb{Z}^{d}:\|u\|_{\infty} \geq 2} V_{u}^{n}\right) \mid \mathcal{L}'\right) \geq g_{2}(\varepsilon)
\]
for all large enough $n \geq n(\varepsilon)$. 

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4.2 The geometry inside blocks

Lemma 5.2 below implies that for all \( k \in \mathbb{N} \) and \( \beta > 0 \) there exists a constant \( C \) such that for all \( n \in \mathbb{N} \)

\[
\mathbb{E}_\beta \left[ \text{Diam} \left( V_0^n \right)^k \right] \leq C n^{k \theta(\beta)}.
\]

(24)

Let \( \delta \in (0, 1) \). We define a family of sets \( \mathcal{CO}_n^\delta \subset V_0^n \) with the following two properties:

\begin{itemize}
  \item \( \bigcup_{x \in \mathcal{CO}_n^\delta} B_{\delta n}(x) = V_0^n \), and
  \item \( |\mathcal{CO}_n^\delta| \leq C_{\mathcal{CO}} \delta^{-d} \) for all \( \delta \),
\end{itemize}

where \( C_{\mathcal{CO}} \) is a constant that depends only on the dimension \( d \), but non on \( \delta \). The abbreviation \( \mathcal{CO} \) stands for cover. Such a cover can be constructed by choosing the points in \( \mathcal{CO}_n^\delta \) at a distance of approximately \( \delta n \).

Lemma 4.5. For \( \varepsilon \in (0, 1) \), let \( \mathcal{DL}(\varepsilon) \) be the event

\[
\mathcal{DL}(\varepsilon) = \bigcap_{x \in \mathcal{CO}_n^2} \left\{ \text{Diam} \left( B_{\varepsilon^2 n}(x) \right) < \left( \varepsilon^{1.5} n \right)^{\theta} \right\}.
\]

Then there exists a function \( h_1(\varepsilon) \) with \( \lim_{\varepsilon \to 0} h_1(\varepsilon) = 1 \) such that

\[
\mathbb{P}_\beta \left( \mathcal{DL}(\varepsilon) \right) \geq h_1(\varepsilon)
\]

for all \( n \geq n(\varepsilon) \) large enough. If the event \( \mathcal{DL}(\varepsilon) \) holds, we say that \( V_0^n \) is \( \varepsilon \)-near.

Proof. By a union bound we have that

\[
\mathbb{P}_\beta \left( \mathcal{DL}(\varepsilon)^C \right) \leq \sum_{x \in \mathcal{CO}_n^2} \mathbb{P}_\beta \left( \text{Diam} \left( B_{\varepsilon^2 n}(x) \right) \geq \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right) \leq C_{\mathcal{CO}} \varepsilon^{-2d} \mathbb{P}_\beta \left( \text{Diam} \left( B_{\varepsilon^2 n}(0) \right) \geq \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right).
\]

(25)

From Markov’s inequality we know that for any \( k \in \mathbb{N} \) and \( n \geq \varepsilon^{-2} \)

\[
\mathbb{P}_\beta \left( \text{Diam} \left( B_{\varepsilon^2 n}(0) \right) \geq \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right) = \mathbb{P}_\beta \left( \text{Diam} \left( B_{\varepsilon^2 n}(0) \right)^k \geq \left( \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right)^k \right) \leq \mathbb{E}_\beta \left[ \text{Diam} \left( B_{\varepsilon^2 n}(0) \right)^k \right] \left( \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right)^{-k} \leq C \left( 2 \varepsilon^2 n + 1 \right)^{k \theta} \left( \frac{\left( \varepsilon^{1.5} n \right)^{\theta}}{3} \right)^{-k} \leq C'(k) \varepsilon^{0.5k \theta}
\]

for some constant \( C'(k) < \infty \). So using \( k = 6d[\theta^{-1}] \) and inserting this into (25) we get that

\[
\mathbb{P}_\beta \left( \mathcal{DL}(\varepsilon)^C \right) \leq \tilde{C} \varepsilon^{-2d} \min(0.5,6d)[\theta^{-1}]^{\theta} \leq \tilde{C} \varepsilon^d
\]

for some constant \( \tilde{C} < \infty \), which finishes the proof. \( \square \)
Consider long-range percolation on \( Z^d \). We split the long-range percolation graph into blocks of the form \( V^n_v \), where \( v \in Z^d \). For each \( v \in Z^d \), we contract the block \( V^n_v \subset Z^d \) into one vertex \( r(v) \). We call the graph that results from contracting all these blocks \( G' = (V', E') \). For \( r(v) \in G' \), we define the neighborhood \( N(r(v)) \) by
\[
N(r(v)) = \{ r(u) \in G' : \|v - u\|_\infty \leq 1 \},
\]
and we define the neighborhood-degree of \( r(v) \) by
\[
\text{deg}^N(r(v)) = \sum_{r(u) \in N(r(v))} \text{deg}(r(u)). \tag{26}
\]
We also define these quantities in the same way when we start with long-range percolation on the graph \( V^m_0 \), and contract the box \( V^n_v \) for all \( v \in V^m_0 \). The next lemma concerns the indirect distance between two sets, conditioned on the graph \( G' \), and is identical to \([5, \text{Lemma 5.4}]\).

**Lemma 4.6.** Let \( W(\varepsilon) \) be the event
\[
W(\varepsilon) := \left\{ D^* \left( V^n_v, \bigcup_{u \in Z^d : \|u - v\|_\infty \geq 2} V^n_u \right) > \varepsilon \Lambda(n, \beta) \right\}.
\]
For all large enough \( n \geq n(\varepsilon) \) one has
\[
\mathbb{P}_\beta (W(\varepsilon)^C \mid G') \leq 3^d \text{deg}^N(r(v))(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)),
\]
where \( g_1 \) and \( g_2 \) were defined in Lemma 4.3, respectively Lemma 4.4.

Furthermore, before going to the proof of Theorem 1.2, we introduce the following lemma that considers the structure of connected sets in the long-range percolation graph. It is proven in \([5, \text{Lemma 3.2 and (24)]}\). For a finite set \( Z \), we define its average degree by \( \text{deg}(Z) = \frac{1}{|Z|} \sum_{v \in Z} \text{deg}(v) \).

**Lemma 4.7.** Let \( CS_k = CS_k(Z^d) \) be all connected subsets of the long-range percolation graph with vertex set \( Z^d \) of size \( k \) that contain the origin \( 0 \). We write \( \mu_\beta \) for \( \mathbb{E}_\beta [\text{deg}(0)] \). Then for all \( \beta > 0 \)
\[
\mathbb{P}_\beta (\exists Z \in CS_k : \text{deg}(Z) \geq 20\mu_\beta) \leq e^{-4k\mu_\beta}
\]
and
\[
\mathbb{E}_\beta \left[ |CS_k(Z^d)| \right] \leq 4^k\mu_\beta^k.
\]

### 4.3 The proof of Theorem 1.2

With the knowledge from the previous subsections, we are now ready to go to the proof of Theorem 1.2. The proof consists out of three main parts: First, we define what good paths in a renormalized graph. Then we show that every long enough path is good, with high probability. Finally, we argue how this implies strict monotonicity of the distance exponent.
Proof of Theorem 1.2. Consider the graph $V_0^{2n}$. For $k \leq n$, define the graph $G'$ by contracting all blocks of the form $V_u^{2k}$. We define $r(u) \in G'$ as the vertex that results from contracting $V_u^{2k}$. In analogy to $Z^d$, we call the vertex $r(0)$ the origin of $G'$. We define a metric on $G'$ by $\|r(u) - r(v)\|_{\infty} = \|u - v\|_{\infty}$. Now consider a self-avoiding path $P = (r(u_0), r(u_1), \ldots, r(u_t)) \subset G'$, where $u_0 = 0$, and $t$ is very large (depending on $d$ and $\beta$). We divide the path into blocks of length $K = 3^d + 1$: For $j \leq \lfloor \frac{t}{K} \rfloor - 1$, we define $R_j = (r(u_jK), \ldots, r(u_{jK+3^d})).$ For each such $j$ and $R_j$, we define a set $\tilde{R}_j$ as follows: If there exist $(r(u_i), r(u_{i+1})) \in R_j$ with $\|u_i - u_{i+1}\|_{\infty} \geq 2$, we simply set $\tilde{R}_j = R_j$. If $\|r(u_i) - r(u_{i+1})\|_{\infty} = 1$ for all $i \in \{jK, \ldots, jK + 3^d\}$, then we set $\tilde{R}_j = R_j \cup \mathcal{N}(r(u_{jK})).$ The set $\bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j$ is a connected set and its cardinality is bounded from below by

$$\left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \geq \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} R_j \right| \geq K \left| \frac{t}{K} \right| \geq \frac{t}{2},$$

and bounded from above by

$$\left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \leq 3^d \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} R_j \right| \leq 3^d t.$$

From now on will always work on the event

$$\mathcal{H}_t := \{ \text{deg}(Z) < 20\mu_\beta \text{ for all } Z \in CS_{\geq t/2} (G') \}.$$

Note that

$$\mathbb{P}_\beta (\mathcal{H}_t^C) \leq e^{-2t\mu_\beta} \leq 2^{-t}$$

by Lemma 4.7. We define the degree of $\tilde{R}_j$ by

$$\text{deg} (\tilde{R}_j) = \sum_{r(u) \in \tilde{R}_j} \text{deg}(r(u)).$$

Note that we do not necessarily have

$$\sum_{i=0}^{\lfloor \frac{t}{K} \rfloor - 1} \text{deg} (\tilde{R}_j) = \sum_{r(u) \in \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j} \text{deg}(r(u)),$$

as some vertices $r(u_i)$ might be included in more than one of the sets $\tilde{R}_j$. However, each vertex $r(u_i)$ can be included in at most $3^d$ sets $\tilde{R}_j$ and thus we have

$$\sum_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \text{deg} (\tilde{R}_j) \leq 3^d \sum_{r(u) \in \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j} \text{deg}(r(u)) \leq 3^d 20\mu_\beta \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \leq 9^d 20\mu_\beta t.$$

There are $\lfloor \frac{t}{K} \rfloor \geq \frac{t}{2K} \geq \frac{t}{8^d}$ indices $j$, and thus we have

$$\frac{1}{\lfloor t/K \rfloor} \sum_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \text{deg} (\tilde{R}_j) \leq \frac{8^d}{t} 9^d 20\mu_\beta t = 72^d 20\mu_\beta.$$
which implies that there are at least \( \left\lceil \frac{t}{20d} \right\rceil \) many indices \( j \in \{0, \ldots, \lfloor t/K \rfloor - 1 \} \) with
\[
\deg(\tilde{R}_j) \leq 80^{d+1} \mu_\beta.
\]
Say that \( j_1, \ldots, j_{\left\lceil \frac{t}{20d} \right\rceil} \) are the first such indices. We define a further subset \( \mathcal{IND} = \mathcal{IND}(P) \) of these indices by starting with \( \mathcal{IND}_0 = \emptyset \) and then iteratively define \( \mathcal{IND}_i^{\left\lceil \frac{t}{20d} \right\rceil} \) by
\[
\mathcal{IND}_i := \left\{ \mathcal{IND}_{i-1} \cup \{j_i\} \right. \text{ if } \tilde{R}_{j_i} \sim \bigcup_{u \in \mathcal{IND}_{i-1}} \tilde{R}_u \bigcup \left. \mathcal{IND}_{i-1} \right\}
\]
So in particular there is no edge between \( \tilde{R}_j \) and \( \tilde{R}_{j'} \) for different \( j, j' \in \mathcal{IND} := \mathcal{IND}_{\left\lceil \frac{t}{20d} \right\rceil} \). The set \( \mathcal{IND} \) has a cardinality of at least \( \frac{1}{80^{d+1} \mu_\beta + 1} \left\lceil \frac{t}{20d} \right\rceil \), as for \( j \in \mathcal{IND} \), the set \( \tilde{R}_j \) has a degree of at most \( 80^{d+1} \mu_\beta \) and can thus block at most \( 80^{d+1} \mu_\beta \) many other elements from getting included. So in particular we also have
\[
|\mathcal{IND}| \geq \frac{1}{80^{d+1} \mu_\beta + 1} \left\lceil \frac{t}{20d} \right\rceil \geq \frac{t}{2000^{d+1} \mu_\beta}.
\]
(28)
If \( R_i = (r(u_{iK}), \ldots, r(u_{iK+3d})) \) is a path of length \( K = 3^d + 1 \) in the graph \( G' \) with \( \deg(\tilde{R}_i) \leq 80^{d+1} \mu_\beta \), we want to investigate the typical minimal length of a path in the original model that goes through the blocks \( \{V_{u_{iK}}^{2^k}, \ldots, V_{u_{iK+3d}}^{2^k}\} \). If there exists \( j \in \{iK, \ldots, iK+3^d-1\} \) with \( \|u_{j+1} - u_j\|_\infty \geq 2 \), let \( j \) be the smallest such index. The probability that there exist \( x, y \in V_{u_j} \) such that \( x \sim V_{u_{j+1}}^{2^k}, y \sim V_{w}^{2^k} \), where \( w \notin \{u_j, u_{j+1}\} \), and \( D_{V_{u_j}^{2^k}}(x, y) \leq \varepsilon \Lambda(2^k, \beta) \) is bounded by \( \deg(V_{u_j}^{2^k})(1 - g_1(\varepsilon)) \leq 80^{d+1} \mu_\beta (1 - g_1(\varepsilon)) \), by Lemma 4.3. If \( D_{V_{u_j}^{2^k}}(x, y) > \varepsilon \Lambda(2^k, \beta) \) for all \( x, y \in V_{u_j}^{2^k} \) such that \( x \sim V_{u_{j+1}}^{2^k}, y \sim V_{w}^{2^k} \), where \( w \notin \{u_j, u_{j+1}\} \), we say that the block \( R_i \) is \( \varepsilon \)-separated.

Now suppose that \( \|u_j - u_{j+1}\|_\infty = 1 \) for all \( j \in \{iK, \ldots, iK+3^d-1\} \). There exists an index \( j \in \{iK+1, \ldots, iK+3^d\} \) with \( \|u_{iK} - u_j\|_\infty \geq 2 \), as there are only \( 3^d - 1 \) many points \( w \in \mathbb{Z}^d \) with \( \|u_{iK} - w\|_\infty = 1 \). When the path exits the cube \( V_{u_{iK}}^{2^k} \) for the last time, it goes to \( V_{u_{iK+1}}^{2^k} \), so in particular the walk does not use a long edge from \( V_{u_{iK}}^{2^k} \) to \( \bigcup_{w: \|w-u_{iK}\|_\infty \geq 2} V_{u_{iK+1}}^{2^k} \) for the last exit. If the indirect distance between the sets \( V_{u_{iK}}^{2^k} \) and the set \( \bigcup_{r(w) \in G', \|w-u_{iK}\|_\infty \geq 2} V_{u_{iK+1}}^{2^k} \) is at least \( \varepsilon \Lambda(2^k, \beta) \), i.e, if
\[
D_{V_{u_{iK}}^{2^k}}^*(\bigcup_{w: \|w-u_{iK}\|_\infty \geq 2} V_{u_{iK+1}}^{2^k}) \geq \varepsilon \Lambda(2^k, \beta),
\]
we also say that the subpath \( R_i \) and the set \( \tilde{R}_i \) are \( \varepsilon \)-separated. As \( \deg'(r(u_i)) \leq 80^{d+1} \mu_\beta \), the probability that there is a path of length at most \( \varepsilon \Lambda(2^k, \beta) \) that goes through \( V_{u_{iK}}^{2^k}, \ldots, V_{u_{iK+3d}}^{2^k} \) is bounded by \( 3^d 80^{d+1}(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)) \), by Lemma 4.6. So we see that in all cases, with probability at least
\[
1 - \left(3^d 80^{d+1}(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon))\right)
\]
the original path needs to walk a distance of at least $\varepsilon \Lambda(2^k, \beta)$ inside the sets $V_{u_{i,k+1}}^{2^k}$, $V_{u_{i,k+2}}^{2^k}, \ldots, V_{u_{i,k+3^d-1}}^{2^k}$, and this distance can be witnessed from the set of edges with at least one end in $\tilde{R}_t$. Note that we have two notions of $\varepsilon$-separated: one for subpaths that make a jump of size at least 2 and one for subpaths that do not make such a jump. However, the idea is in both cases that a path that walks through the set $R_t$ needs to walk a distance of at least $\varepsilon \Lambda(2^k, \beta)$ in the original model.

We say that a sequence $R_t$ is $\varepsilon$-influential, if $\tilde{R}_t$ is $\varepsilon$-separated and all boxes $V_{u_{i,k}}^{2^k}, \ldots, V_{u_{i,k+3^d}}^{2^k}$ are $\varepsilon^{1/\theta}$-near (see Lemma 4.5 for the definition of $\varepsilon$-near). For a block $R_t$ with $i \in \mathcal{ND}$, we can bound the probability that a sequence $R_t$ is not $\varepsilon$-influential by

$$P_\beta (R_t \text{ is not } \varepsilon\text{-influential}) \leq 3^d 80^{d+1} (1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)) + (3^d + 1) \left(1 - h_1 \left(\varepsilon^{1/\theta}\right)\right).$$

Note that it only depends on edges with at least one endpoint inside $\tilde{R}_t$, whether $R_t$ is $\varepsilon$-influential. For different values of different $j_1, \ldots, j_l \in \mathcal{ND}$, the sets $\left(\tilde{R}_{j_i}\right)_{i \in \{1, \ldots, l\}}$ are not connected, and thus it is independent whether these blocks are $\varepsilon$-influential. Next, let $\varepsilon$ be small enough such that

$$\left(3^d 80^{d+1} (1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)) + (3^d + 1) \left(1 - h_1 \left(\varepsilon^{1/\theta}\right)\right)\right)^{\frac{1}{2 \cdot 2000^{d+1} \mu_\beta}} \leq \frac{1}{100 \mu_\beta^2}. \quad (29)$$

Let $\mathcal{NF} = \mathcal{NF}(P) \subset \mathcal{ND}(P)$ be all indices $i \in \mathcal{ND}$, for which $R_t$ is $\varepsilon$-influential. If $\mathcal{H}_t$ holds we want to get bounds on the cardinality of the set $\mathcal{NF}$ for a fixed path $P \subset G'$ of length $t$. Remember that we have

$$|\mathcal{ND}| \geq \frac{t}{2000^{d+1} \mu_\beta},$$

as shown in (28). For a path $P = (r(u_0), r(u_1), \ldots, r(u_t)) \subset G'$ one thus has

$$P_\beta \left(\mathcal{NF} \leq \frac{t}{2 \cdot 2000^{d+1} \mu_\beta} \mid G'\right) = P_\beta \left(\bigcup_{U \subset \mathcal{ND} \mid |U| \geq \frac{t}{2 \cdot 2000^{d+1} \mu_\beta}} \{R_t \text{ not } \varepsilon\text{-influential } \forall i \in U\} \mid G'\right) \leq 2^{\mathcal{ND}} \left(3^d 80^{d+1} (1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)) + (3^d + 1) \left(1 - h_1 \left(\varepsilon^{1/\theta}\right)\right)\right)^{\frac{1}{2 \cdot 2000^{d+1} \mu_\beta}} \leq 2^t \left(\frac{1}{100 \mu_\beta^2}\right)^t \leq \frac{1}{50^t \mu_\beta^2}$$

where used the assumption on $\varepsilon$ (29) for the last inequality. This shows that a specific path $P$ is satisfies $|\mathcal{NF}(P)| \geq \frac{t}{2 \cdot 2000^{d+1} \mu_\beta}$ with high probability. Next, we want to show that all paths $P \subset G'$ of length $t$ starting at the origin $r(0)$ satisfy $|\mathcal{NF}(P)| \geq \frac{t}{2 \cdot 2000^{d+1} \mu_\beta}$ with high probability in $t$. Let $\mathcal{P}_t$ be the set of all paths in $G'$ of length $t$ starting at $r(0)$. We call the previously mentioned event $\mathcal{G}_t$, i.e.,

$$\mathcal{G}_t = \left\{ |\mathcal{NF}(P)| \geq \frac{t}{2 \cdot 2000^{d+1} \mu_\beta} \text{ for all } P \in \mathcal{P}_t \right\}.$$

By a comparison with a Galton-Watson tree we get that $E_\beta \left(|\mathcal{P}_t|\right) \leq \mu_\beta^t$. Thus we have, by a union bound

$$P_\beta (\mathcal{G}_t^C) \leq P_\beta (\mathcal{H}_t^C) + P_\beta (|\mathcal{P}_t| > 2^t \mu_\beta^t) + P_\beta (\mathcal{G}_t^C \mid \mathcal{H}_t, |\mathcal{P}_t| \leq 2^t \mu_\beta^t)$$

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\[
\begin{align*}
&\leq \Pr(\mathcal{H}_t^c) + \Pr(\{|P_t| > 2^t\mu_2^t\} + 2^t\mu_2^t \frac{1}{50\mu_2^t} \leq 2^{-t} + \frac{\E[|P_t|]}{2^t\mu_2^t} + \frac{1}{25^t} \leq 3 \cdot 2^{-t}.
\end{align*}
\]

and with another union bound we get for the event \(\mathcal{G}_{\geq t} := \bigcap_{\nu=1}^{\infty} \mathcal{G}_t\) that
\[
\Pr(\mathcal{G}_{\geq t}^c) \leq 6 \cdot 2^{-t}.
\]

So we see that all paths of length at least \(t\) contain at least \(\frac{t}{2\cdot 2000\mu_2^t + \mu_3}\) many \(\varepsilon\)-influential subpaths \(R_i\) with very high probability in \(t\), for \(\varepsilon\) small enough as in (29). Now, let \(\hat{P}\) be a geodesic between \(0\) and \((2^n - 1)1\) in the graph with vertex set \(V_0^{2^n}\). Let \(\hat{P}\) be the projection of this path onto \(G'\), and let \(P\) be the loop-erased version of it. Whenever the path \(P\) crosses an influential subset \(R_i = (r(u_{iK}), \ldots, r(u_{iK+3\varepsilon})) \subset P\), let \(l = l(i) \in \{iK, \ldots, iK + 3\varepsilon\}\) be the first index for which \(\|r(u_l) - r(u_{iK})\|_\infty \geq 2\) if such an index exists. Respectively let \(l = l(i) \in \{iK, \ldots, iK + 3\varepsilon\}\) be the first index for which \(\|r(u_l) - r(u_{iK})\|_\infty \geq 2\), if there does not exist such an index with \(\|r(u_l) - r(u_{iK})\|_\infty \geq 2\). Whenever the path \(P\) crosses the set \(R_i\), it enters \(V_{u_{iK}}^{2^k}\) through some vertex \(x_0\) and it leaves \(V_{u_{i+1}}^{2^k}\) to \(V_{u_{i+1}}^{2^k}\) through some vertex \(x_R\). As the boxes \(V_{u_{iK}}^{2^k}\) and \(V_{u_l}^{2^k}\) are \(1/\theta\)-near, there exist cubes \(B_L, B_R\) of side length at least \(2\varepsilon/\theta 2^k\) such that
\[
x_L \in B_L \subset V_{u_{iK}}^{2^k}, x_R \in B_R \subset V_{u_{i+1}}^{2^k}, \text{ and }

\text{Diam}(B_L), \text{Diam}(B_R) < \left(\varepsilon \frac{1.5 \cdot 2^k}{\theta}\right) \leq \varepsilon \frac{1.5 \cdot 3}{2^{k\theta}} .
\]

The graph distance between \(x_L\) and \(x_R\) is at least \(\varepsilon \Lambda (2^k, \beta)\), as we will argue now. If there exists an index \(l = l(i) \in \{iK, \ldots, iK + 3\varepsilon\}\) for which \(\|r(u_l) - r(u_{iK})\|_\infty \geq 2\), then we know that the box \(V_{u_l}^{2^k}\) is \(\varepsilon\)-separated. At the last visit of the box \(V_{u_l}^{2^k}\), the geodesic \(\hat{P}\) enters the box \(V_{u_l}^{2^k}\) through some point \(z \in V_{u_l}^{2^k}\) with \(z \sim V_w^{2^k}\), for some \(w \in V_0^{2^n-k} \setminus \{u_l, u_{i+1}\}\). We have \(w \neq u_{i+1}\), as the loop-erased projection \(P\) is path-avoiding. As \(D_{V_{u_l}^{2^k}}(x_R, z) \geq \varepsilon \Lambda (2^k, \beta)\) for all \(z \in V_{u_l}^{2^k}\) with \(z \sim V_w^{2^k}\) for \(w \neq \{u_l, u_{i+1}\}\), we automatically get that
\[
D(x_L, x_R) \geq \varepsilon \Lambda (2^k, \beta) , \text{ as either } x_L \in V_{u_{iK}}^{2^k} \text{ with } x_L \sim V_{u_{iK}}^{2^k}, \text{ or } x_L \notin V_{u_{iK}}^{2^k} . \text{ If there does not exist an index } l \in \{iK, \ldots, iK + 3\varepsilon\}\) for which \(\|r(u_l) - r(u_{iK})\|_\infty \geq 2\), then we know that \(\|u_l - u_{iK}\|_\infty \leq 2\) and the geodesic between \(x_L\) and \(x_R\) walks through the set \(\bigcup_{\{w \in G' \mid \|w - u_{i,K}\|_\infty \geq 2\}} V_w^{2^k}\), and thus its length is at least
\[
D_{V_w^{2^k}}(x_L, x_R) \geq D_{V_w^{2^k}}\left(V_{u_{iK}}^{2^k}, \bigcup_{r(u) \in G' \cap \{w - u_{i,K}\|_\infty \geq 2\}} V_w^{2^k}\right) \geq \varepsilon \Lambda (2^k, \beta) ,
\]
where the last inequality holds, as the subpath \(R_i\) was assumed to be \(\varepsilon\)-separated.

When we insert an edge between the boxes \(B_L\) and \(B_R\), the distance between \(x_L\) and \(x_R\) is at most \(2^k \varepsilon^{1.5} \cdot 2^{k\theta} + 1\). Remember that \(\Lambda (2^k, \beta) \geq 2^{k\theta}\). Thus we have for all edges \(e \in B_L \times B_R\)
\[
D_{V_0^{2^n}}(x_L, x_R; w) - D_{V_0^{2^n}}(x_L, x_R; w^{e+}) \geq \varepsilon \Lambda (2^k, \beta) - 2 \varepsilon^{1.5} \cdot 2^{k\theta} - 1 \geq \varepsilon \Lambda (2^k, \beta) ,
\]
where the last inequality holds for \(k\) large enough. The boxes \(B_L\) and \(B_R\) are of side length at least \(2\varepsilon^{2/\theta} 2^k\) and are disjoint, as \(D_{V_0^{2^n}}(x_L, x_R) > \text{Diam} (B_L) + \text{Diam} (B_R)\). Thus there are at least \((\varepsilon^{2/\theta} 2^k)^d \cdot (\varepsilon^{2/\theta} 2^k)^d\) pairs of vertices \((a, b) \in B_L \times B_R\) for which \(|\{a, b\}| \geq \varepsilon^{2/\theta} 2^k\).
On the other hand, we also have $|\{a, b\}| \leq (3^d + 1)2^k \leq 6d2^k$ for all pairs $(a, b) \in B_L \times B_R$, as $\|r(\kappa) - r(\kappa)\|_\infty \leq 3^d$. So in particular we have

$$
\sum_{e \in B_L \times B_R; \quad \varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} p'(\beta, e)\left(D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega) - D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega^{e^+})\right) \\
\geq \sum_{e \in B_L \times B_R; \quad \varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} p'(\beta, e)\varepsilon \Lambda(2^k, \beta) / 4 \geq \sum_{e \in B_L \times B_R; \quad \varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} e^{-\beta} \varepsilon \Lambda(2^k, \beta) / 4 \geq c \Lambda(2^k, \beta)
$$

with a constant $c > 0$, that depends on $\varepsilon, \theta$, and $d$, and for $k$ large enough. For the two points $0$ and $(2^n - 1)1$, and points $x_L$ and $x_R$ which are in a geodesic between $0$ and $(2^n - 1)1$ in this order, and any edge $e$ we have

$$D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) = D_{\mathcal{V}_0^{2n}}(0, x_L; \omega) + D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega) + D_{\mathcal{V}_0^{2n}}(x_R, (2^n - 1)1; \omega),$$

$$D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+}) = D_{\mathcal{V}_0^{2n}}(0, x_L; \omega^{e^+}) + D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega^{e^+}) + D_{\mathcal{V}_0^{2n}}(x_R, (2^n - 1)1; \omega).$$

Subtracting these two (in)equalities already yields that

$$D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) - D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+}) \geq D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega) - D_{\mathcal{V}_0^{2n}}(x_L, x_R; \omega^{e^+}),$$

so in particular we also have

$$\sum_{e \in B_L \times B_R; \quad \varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} p'(\beta, e)\left(D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) - D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+})\right) \geq c \Lambda(2^k, \beta).$$

The above inequality holds for fixed $B_L \subset V_{u,k}^{2k}, B_R \subset V_{u,t}^{2k}$. However, such boxes exist for all indices $i \in \mathcal{V}_L P$. Thus, assuming that $D_{\mathcal{G}'}(0, (2^{n-k} - 1)1) = t$ and $\mathcal{G}_{\geq t}$ holds for large enough $t \geq T$, we have for large enough $k$

$$\sum_{\varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} p'(\beta, e)\left(D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) - D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+})\right) \geq |\mathcal{V}_L P| c \Lambda(2^k, \beta) \geq 2 \cdot 2000^{d+1} / \mu_\beta \cdot c \Lambda(2^k, \beta) =: c't \Lambda(2^k, \beta).$$

So far, we always worked on the event $\mathcal{G}_{\geq t}$. Now, we want to get a similar bounds in expectation, not conditioning on $\mathcal{G}_{\geq t}$. Writing $E_k$ for the set of edges $e$ with $\varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k$ we get that there exists a large enough $T < \infty$

$$\sum_{\varepsilon^{2/\theta 2^k} \leq |e| \leq 6^d2^k} p'(\beta, e) E_\beta \left[D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) - D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+})\right]$$

$$\geq \sum_{t = T}^\infty \sum_{e \in E_k} p'(\beta, e) E_\beta \left[D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega) - D_{\mathcal{V}_0^{2n}}(0, (2^n - 1)1; \omega^{e^+})\right]$$

$$\geq \sum_{t = T}^\infty c't \Lambda(2^k, \beta) E_\beta \left[D_{\mathcal{G}'}(0, \{(2^{n-k} - 1)1\}) = t\right] \mathbb{1}\{\mathcal{G}_{\geq t}\}$$

$$\geq \sum_{t = T}^\infty c't \Lambda(2^k, \beta) \sum_{t = T}^\infty \left(\mathbb{P}_\beta\left[D_{\mathcal{G}'}(0, \{(2^{n-k} - 1)1\}) = t\right] - \mathbb{P}_\beta\left[\mathcal{G}_{\geq t}^C\right]\right)$$
\[
\sum_{e} p'(\beta, e) \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1; \omega) - D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^-}) \right] \\
\geq c_1 \sum_{k=1}^{n} \sum_{e \in E_k} p'(\beta, e) \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1; \omega) - D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^-}) \right] \\
\geq c_2 \sum_{k=1}^{n} \Lambda(2^n, \beta) \geq c_3 \log(2^n) \Lambda(2^n, \beta)
\]

for constants \(c_1, c_2, c_3 > 0\) and \(n\) large enough. This already implies that

\[
\sum_{e} p'(\beta, e) \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^-}) - D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^+}) \right] \\
\geq \sum_{e} p'(\beta, e) \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1; \omega) - D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^+}) \right] \geq c_3 \log(2^n) \Lambda(2^n, \beta).
\]

(31)

Now, let us see how this bound implies strict monotonicity of the distance exponent \(\theta(\beta)\). We know that

\[
\theta(\beta) = \lim_{n \to \infty} \log \left( \frac{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right]}{\log(2^n)} \right)
\]

and that for fixed \(n\) the function

\[
\beta \mapsto \log \left( \frac{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right]}{\log(2^n)} \right)
\]

is, by Russo’s formula for expectations (5), differentiable. So we can calculate the derivative and bound it from above by

\[
\begin{align*}
\frac{d}{d\beta} \log \left( \frac{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right]}{\log(2^n)} \right) \\
= \frac{1}{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right]} \log(2^n) \frac{d}{d\beta} \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right] \\
= \frac{\sum_{e \in E} p'(\beta, e) \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^+}) - D_{V_0^{2n}}(0, (2^n-1)1; \omega^{e^-}) \right]}{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right]} \log(2^n)
\end{align*}
\]

(31)

\[
\leq \frac{-c_3 \Lambda(2^n, \beta) \log(2^n)}{\mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n-1)1) \right] \log(2^n)} \leq -c_3 =: c(\beta)
\]

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for some $c(\beta) < 0$ and this holds for all $n \in \mathbb{N}_{>0}$ large enough. Now fix $0 < \beta_1 < \beta_2 < \infty$. We want to show that $\theta(\beta_1) > \theta(\beta_2)$. For each fixed $\beta \in [\beta_1, \beta_2]$ there exists $n(\beta) < \infty$ such that for all $n \geq n(\beta)$

$$\frac{d}{d\beta} \log \left( \frac{\mathbb{E}_\beta \left[ D_{V_0^{2^n}} (0, (2^n - 1) \mathbf{1}) \right]}{\log (2^n)} \right) \leq \frac{c(\beta)}{2}$$

(32)

holds. So in particular we can take $N$ large enough, and $c < 0$ with $|c|$ small enough so that the set of $\beta \in [\beta_1, \beta_2]$ which satisfy $\frac{c(\beta)}{2} < c$, and which satisfy (32) for all $n \geq N$, has Lebesgue measure of at least $\frac{\beta_2 - \beta_1}{2}$. Thus we get

$$\theta(\beta_2) - \theta(\beta_1) = \lim_{n \to \infty} \left( \frac{\log \left( \mathbb{E}_{\beta_2} \left[ D_{V_0^{2^n}} (0, (2^n - 1) \mathbf{1}) \right] \right)}{\log (2^n)} - \frac{\log \left( \mathbb{E}_{\beta_1} \left[ D_{V_0^{2^n}} (0, (2^n - 1) \mathbf{1}) \right] \right)}{\log (2^n)} \right)$$

$$= \lim_{n \to \infty} \int_{\beta_1}^{\beta_2} \frac{d}{d\beta} \log \left( \frac{\mathbb{E}_\beta \left[ D_{V_0^{2^n}} (0, (2^n - 1) \mathbf{1}) \right]}{\log (2^n)} \right) d\beta$$

$$\leq \frac{\beta_2 - \beta_1}{2} c < 0,$$

which finishes the proof of the strict monotonicity.

5 Continuity of the distance exponent

In this section, we show that the distance exponent is continuous in $\beta$. This result is also useful for comparing different percolation models with each other, as shown in [5, Section 7]. With the tools that we have developed so far, we can already prove continuity from the left:

Lemma 5.1. The distance exponent $\theta(\beta)$ is continuous from the left.

Proof. Remember that

$$\theta(\beta) = \inf_{n \geq 2} \frac{\log (\Lambda(n, \beta))}{\log (n)}$$

which is stated in Lemma 3.1. For fixed $n$, the function $\beta \mapsto \Lambda(n, \beta)$ is continuous and decreasing in $\beta$. The continuity holds, as the inclusion probabilities $p(\beta, e)$ are continuous in $\beta$ for all edges $e$, and we only consider the finitely many edges with both endpoints in $V_0^n$. As the functions $p(\beta, e)$ are also increasing in $\beta$ for all edges $e$, one can see with the Harris coupling that the function $\beta \mapsto \Lambda(n, \beta)$ is also decreasing. So we get that for all $\beta > 0$

$$\lim_{\varepsilon \downarrow 0} \theta(\beta - \varepsilon) = \inf_{\varepsilon > 0} \theta(\beta - \varepsilon) = \inf_{\varepsilon > 0, n \geq 2} \frac{\log (\Lambda(n, \beta - \varepsilon))}{\log (n)} = \inf_{n \geq 2} \inf_{\varepsilon > 0} \frac{\log (\Lambda(n, \beta - \varepsilon))}{\log (n)} = \inf_{n \geq 2} \frac{\log (\Lambda(n, \beta))}{\log (n)} = \theta(\beta),$$

and this shows continuity from the left.

The proof of continuity from the right is more difficult. We consider independent bond percolation on the complete graph with vertex set $V = V_0^{2^n}$ and edge set $E = \{ \{x, y\} :$
For $k \in \{1, \ldots, n\}$ and $\beta_1, \beta_2 > 0$, we denote by $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$ the product probability measure on the $\{0,1\}^E$ where edges $e = \{u,v\}$ are open with the following probabilities:

$$
\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}(\omega(\{u,v\}) = 1) = \begin{cases} 
1 - e^{-\beta_1 \int_{u+c} f_{u+c} x \int_{u+y} e^{dy} } & \text{if } 1 < |\{u,v\}| \leq 2^k - 1 \\
1 - e^{-\beta_2 \int_{u+c} f_{u+c} x \int_{u+y} e^{dy} } & \text{if } |\{u,v\}| \geq 2^k \\
1 & \text{if } |\{u,v\}| = 1
\end{cases},
$$

so in particular the measure $\mathbb{P}_{\beta_1 \leq 1}^{\beta_2 > 1}$ is identical to the measure $\mathbb{P}_{\beta_2}$, and the measure $\mathbb{P}_{\beta_1 \leq n}^{\beta_2 > n}$ on the graph with vertex set $V_0^{2n}$ is identical to the measure $\mathbb{P}_{\beta_1}$. For $k \in \{2, \ldots, n-1\}$, we think of the measure $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$ as an interpolation between the probability measures $\mathbb{P}_{\beta_1}$ and $\mathbb{P}_{\beta_2}$ on the graph with vertex set $V_0^{2n}$. We will mostly work on this graph in this chapter and the distances should be considered as the graph distances inside this graph. We denote by $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$ the expectation under $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$. Our main strategy of the proof of Theorem 1.3 is as follows: We know that

$$
\theta(\beta) = \lim_{n \to \infty} \frac{\log \left( \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right)}{\log(2n)} = \lim_{n \to \infty} \frac{\log \left( \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right)}{\log(2n)}
$$

and thus we also have

$$
\theta(\beta) - \theta(\beta + \varepsilon) = \lim_{n \to \infty} \frac{1}{\log(2)} \left( \log \left( \mathbb{E}_\beta \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right) - \log \left( \mathbb{E}_{\beta + \varepsilon} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right) \right) = \lim_{n \to \infty} \frac{1}{\log(2)} \left( \log \left( \mathbb{E}_{\beta + \varepsilon}^{\beta > k} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right) - \log \left( \mathbb{E}_{\beta \leq k - 1}^{\beta + \varepsilon > k - 1} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right] \right) \right)
$$

$$
= \frac{1}{\log(2)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \varepsilon > k - 1} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]} \right). \tag{33}
$$

So in order to show that $\lim_{\varepsilon \to 0} \theta(\beta + \varepsilon) = \theta(\beta)$ it suffices to show that

$$
\lim_{\varepsilon \to 0} \frac{1}{n} \sum_{k=2}^{n} \log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \varepsilon > k - 1} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]} \right) = 0, \tag{34}
$$

and in order to show this, it is sufficient to show that the terms

$$
\log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \varepsilon > k - 1} \left[ D_{V_0^{2n}}(0, (2^n - 1)1) \right]} \right), \quad k \in \{2, \ldots, n\},
$$

are bounded uniformly and converge to 0, as $\varepsilon \to 0, k, n - k \to \infty$. Before going to the proof, we need to prove several technical results. In Lemma 5.2, we investigate the exponential moments of $\frac{\text{Diam}(V_0^{2n})}{m^\theta}$, uniformly over $m$. In subsection 5.1, we derive several inequalities for the mixed measure $\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k}$ that we need later in the proof. Then, in subsection 5.2 we show how this implies (34) and thus continuity of the distance exponent $\theta$.  

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Lemma 5.2. For all $\beta \geq 0$ there exists a constant $C_1 < \infty$ such that for all $s \geq 1$, and all $m \in \mathbb{N}_{>0}$

$$E_\beta \left[e^{s \frac{\text{Diam}(V^m_\beta)}{m^{1/2}}} \right] \leq e^{C_1 s^2}.$$  \hfill (35)

Proof. Define $Y_m := \frac{\text{Diam}(V^m_\beta)}{m^{1/2}}$. In [5, Theorem 6.1], we proved that for each $\beta \geq 0$ there exists some $\eta > 1$, and a $C < \infty$ such that

$$E_\beta \left[e^{\eta Y_m} \right] \leq C$$  \hfill (36)

for all $m \in \mathbb{N}$. For all $y, s > 0$ one has

$$sy \leq sy \mathbb{1}_{\{s < y^{-1}\}} + sy \mathbb{1}_{\{s \geq y^{-1}\}} \leq y^{1 - 1} y \mathbb{1}_{\{s < y^{-1}\}} + sy \mathbb{1}_{\{s \geq y^{-1}\}} \leq y + s^{1 - 1}.$$  

Inserting this into (36), we get that for all $s \geq 1$

$$E_\beta \left[e^{s Y_m} \right] \leq E_\beta \left[e^{s \frac{\text{Diam}(V^m_\beta)}{m^{1/2}}} \right] \leq C e^{s^{1 - 1}} \leq e^{C_1 s^2}$$

for some $C_1 < \infty$. \hfill $\square$

5.1 Uniform bounds for the mixed measure

In this chapter, we give several bounds for the measure $\mathbb{P}^{\beta + \varepsilon > k}$ that hold uniformly over $\varepsilon \in [0, 1]$ and $k \leq n$. These bounds were partially already proven in the previous chapters or in [5, 23] for fixed $\beta$ and $\varepsilon = 0$. One can couple the measures $\mathbb{P}^{\beta + \varepsilon > k}$ for different $\varepsilon$ with the Harris coupling. For some set $V \subset \mathbb{Z}^d$ and $E = \{\{u, v\} : u, v \in V, u \neq v\}$, let $(U_\varepsilon)_{\varepsilon \in E}$ be independent random variables with uniform distribution on the interval $[0, 1]$. Define the function $p(\beta, \beta + \varepsilon, k, \{u, v\}) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \times E \to [0, 1]$ by

$$p(\beta, \beta + \varepsilon, k, \{u, v\}) = \begin{cases} 1 - e^{-\int_{\beta + \varepsilon}^{\beta + k} \frac{1}{\|x - y\|^2} dx dy} & 1 < \{|u, v\}| \leq 2^k - 1 \\ 1 - e^{-\int_{\beta + \varepsilon}^{\beta + k} \frac{1}{\|x - y\|^2} dx dy} & |\{u, v\}| \geq 2^k \\ 1 & \{|u, v\}| = 1 \end{cases}.$$  

Define the environment $\omega^\varepsilon \in \{0, 1\}^E$ by $\omega(\varepsilon) = \mathbb{1}_{\{U_{\varepsilon} \leq p(\beta, \beta + \varepsilon, k, \{u, v\})\}}$. Then $\omega^\varepsilon$ is distributed according to the measure $\mathbb{P}^{\beta + \varepsilon > k}$. For $0 \leq \varepsilon_1 < \varepsilon_2$, this construction couples the measures $\mathbb{P}^{\beta + \varepsilon_1 > k}$ and $\mathbb{P}^{\beta + \varepsilon_2 > k}$ in such a way that all edges contained in the environment defined by $\omega^\varepsilon_1$ are also contained in the environment defined by $\omega^\varepsilon_2$. The next two lemmas deal with the graph distance of certain points in boxes, that have direct edges to other far away blocks.

Lemma 5.3. Let $V^k_u$ be a block with side length $2^k$ that is connected to $V^k_0$ and let $\|u\|_\infty \geq 2$. Let $B(u)(\delta)$ be the following event:

$$B_u(\delta) = \bigcap_{x, y \in V^k_u, x, y \sim V^k_u, x \neq y} \left\{ D_{V^k_0}(x, y) \geq 2^{k(\delta)} \right\}.$$  

For every $\beta > 0$, there exists a function $f_1(\delta)$ with $f_1(\delta) \xrightarrow{\delta \to 0} 1$ such that for all large enough $k \geq k(\delta)$, all $u \in \mathbb{Z}^d$ with $\|u\|_\infty \geq 2$, and all $\epsilon \in [0, 1]$

$$\mathbb{P}^{\beta + \varepsilon > k} \left( B_u(\delta) \mid V^k_0 \sim V^k_u \right) \geq f_1(\delta).$$
Lemma 5.4. Let $V_{u}^{2k}, V_{v}^{2k}$ be two blocks of side length $2^k$ that are connected to $V_{0}^{2k}$, with $u \neq v \neq 0$ and $\|u\|_{\infty} \geq 2$. Let $A_{u,v}(\delta)$ be the following event:

$$A_{u,v}(\delta) = \bigcap_{x \in V_{0}^{2k}} \bigcap_{y \in V_{0}^{2k}} \left\{ D_{\delta}^{2k} (x, y) \geq \delta 2^{k\theta(\beta)} \right\}. $$

For every $\beta > 0$, there exists a function $f_{2}(\delta)$ with $f_{2}(\delta) \xrightarrow{\delta \to 0} 1$ such that for all large enough $k \geq k(\delta)$, all $u, v \in \mathbb{Z}^{d} \setminus \{0\}$ with $\|u\|_{\infty} \geq 2$, and all $\varepsilon \in [0, 1]$

$$\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( A_{u,v}(\delta) \mid V_{u}^{2k} \sim V_{0}^{2k} \sim V_{v}^{2k} \right) \geq f_{2}(\delta).$$

Proof of Lemma 5.4 and Lemma 5.3. By the Harris coupling, it suffices to consider the case $\varepsilon = 1$. From here on, the proof is analogous to the proofs of [5, Lemma 5.1] and [5, Lemma 5.2]. The spacing in terms of infinity distance between distinct points $x, y \in V_{0}^{2k}$ with $x \sim V_{u}^{2k}, y \sim V_{v}^{2k}$ can be bounded in the same way as in Lemma 5.1. As the structure inside $V_{0}^{2k}$ is not affected by any change of $\varepsilon$, the graph distance between such points $x, y$ can be bounded as in Lemma 5.2. \hfill \Box

In the following lemma, we define the graph $G'$ as the graph, in which we contract boxes of the form $V_{u}^{2k}$ for $u \in \mathbb{Z}^{d}$. The vertex that results from contracting the box $V_{u}^{2k}$ is called $r(u)$.

Lemma 5.5. Let $B(\delta)$ be the event

$$B(\delta) := \left\{ D^{*} \left( V_{0}^{2k}, \bigcup_{u \in \mathbb{Z}^{d} | \|u\|_{\infty} \geq 2} V_{u}^{2k} \right) \geq \delta 2^{k\theta(\beta)} \right\}. $$

For every $\beta > 0$ there exists a function $f_{3}(\delta)$ with $f_{3}(\delta) \xrightarrow{\delta \to 0} 1$ such that for all large enough $k \geq k(\delta)$, all $\varepsilon \in [0, 1]$, and all realizations of $G'$ with $\text{deg}^{N}(r(0)) \leq 9d 100 \mu_{\beta}$

$$\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} (B(\delta) \mid G') \geq f_{3}(\delta).$$

The proof of this lemma is similar to the proof of Lemma 4.6, and we omit it. From here on we also use the notation $f(\delta) = \min \{f_{1}(\delta), f_{2}(\delta), f_{3}(\delta)\}$.

Lemma 5.6. For all $\beta > 0$, there exist constants $0 < c_{\beta} < C_{\beta} < \infty$ such that uniformly over all $\varepsilon \in [0, 1]$

$$c_{\beta} A \left( 2^{k}, \beta \right) A \left( 2^{n-k}, \beta + \varepsilon \right) \leq \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_{0}^{2n}} (0, (2^{n} - 1)e_{1}) \right] \leq C_{\beta} A \left( 2^{k}, \beta \right) A \left( 2^{n-k}, \beta + \varepsilon \right)$$

(37)

The proof is completely analogous to the proofs of [5, Lemma 2.3 and Lemma 5.5], so we omit it here. The proof of the first inequality is analogous as the proof of [5, Lemma 5.5], and the proof of the second inequality is analogous to the proof of [5, Lemma 2.3]. We want to get similar bounds on the second moment of distances $D_{V_{0}^{2n}}$ under the measure $\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k}$. For this, we introduce the following lemma, which was already proven in [5, Lemma 4.5].

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Lemma 5.7. For all $\beta \geq 0$, there exists a constant $C_\beta < \infty$ such that for all $n \in \mathbb{N}$, all $\varepsilon \in [0,1]$ and all $x,y \in V_0^n$

$$E_{\beta+\varepsilon} \left[ D_{V_0^n}(x,y)^2 \right] \leq C_\beta \Lambda(n, \beta + \varepsilon)^2. \quad (38)$$

Having this lemma allows us to prove a uniform bound on the second moment of distances under the measure $P_{\beta+\varepsilon}^{\geq k}$.

Lemma 5.8. For all $\beta \geq 0$, there exists a constant $C_\beta < \infty$ such that uniformly over all $\varepsilon \in [0,1]$, all $k \leq n$, and all $x,y \in V_0^{2n}$

$$E_{\beta \leq k}^{\geq k} \left[ D_{V_0^{2n}}(x,y)^2 \right] \leq C_\beta \Lambda(2^k, \beta)^2 \Lambda(2^{n-k}, \beta + \varepsilon)^2. \quad (39)$$

Proof. We use a renormalization structure for this proof. We first define the renormalized graph $G'$ where we contract all vertices of the set $V_u^{2k}$ to one vertex $r(u)$ and do this for all $u \in V_0^{2n-k}$. In the graph $G'$, there is an edge between $r(u)$ and $r(v)$ if and only if there is an edge between $V_u^{2k}$ and $V_v^{2k}$. Now, let $x, y \in V_0^{2n}$ be arbitrary, say with $x \in V_u^{2k}$ and $y \in V_v^{2k}$. The claim is clear in the case where $u = v$, so we will assume $u \neq v$ from here on. Consider the shortest path between $r(u)$ and $r(v)$. Say that $(r(u_0), \ldots, r(u_K))$ is this path, where $K = D_{G'}(u,v)$, $u_0 = u$, and $u_K = v$. There is a path between $x$ and $y$ that uses only edges in or between the sets $V_{u_i}^{2k}$ for $i = 0, \ldots, K$. Thus we have an upper bound on the graph distance between $x$ and $y$ given by

$$D_{V_0^{2n}}(x,y) \leq \sum_{i=0}^K \left( \text{Diam} \left( V_{u_i}^{2k} \right) + 1 \right). \quad (40)$$

The random variables $\text{Diam} \left( V_{u_i}^{2k} \right)$ and $K = D_{G'}(u,v)$ are independent, as the diameters $\text{Diam} \left( V_{u_i}^{2k} \right)$ depend only on edges with both endpoints inside $V_{u_i}^{2k}$, whereas the distance $K = D_{G'}(u,v)$ depends only on edges that are between two different boxes. For $(X_i)_{i \in \mathbb{N}}$ i.i.d. random variables that are furthermore independent of an integer-valued random variable $K$ one has

$$E \left[ \left( \sum_{i=1}^K X_i \right)^2 \right] \leq E \left[ \left( \sum_{i=1}^\infty \mathbb{1}_{\{i \leq K\}} X_i \right)^2 \right] = E \left[ \sum_{i=1}^\infty \sum_{j=1}^\infty \mathbb{1}_{\{i \leq K\}} \mathbb{1}_{\{j \leq K\}} X_i X_j \right]$$

$$= \sum_{i=1}^\infty \sum_{j=1}^\infty E \left[ \mathbb{1}_{\{i \leq K\}} \mathbb{1}_{\{j \leq K\}} \right] E [X_i X_j] \leq E \left[ \sum_{i=1}^\infty \sum_{j=1}^\infty \mathbb{1}_{\{i \leq K\}} \mathbb{1}_{\{j \leq K\}} \right] E [X_1^2]$$

$$= E \left[ K^2 \right] E [X_1^2].$$

We know that $E_\beta \left[ \text{Diam(} V_{u_i}^{2k} \right)^2 \leq C_\beta' \Lambda(2^k, \beta)^2$ for some $C_\beta' < \infty$, which follows for example from Lemma 5.2. The distance $D_{G'}(r(u), r(v))$ only depends on the occupation status of edges with both ends in $V_0^{2n}$ that have a length of at least $2^k$. Thus $D_{G'}(r(u), r(v))$ has exactly the same distribution as $D_{V_0^{2n-k}}(u, v)$ under the measure $P_{\beta+\varepsilon}$. The previous observations together with (40) imply that

$$E_{\beta \leq k}^{\geq k} \left[ D_{V_0^{2n}}(x,y)^2 \right] \leq E_{\beta \leq k}^{\geq k} \left[ \left( \sum_{i=0}^K \left( \text{Diam} \left( V_{u_i}^{2k} \right) + 1 \right) \right)^2 \right]$$
\[ \leq \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ (D_{G'} (r(u), r(v)) + 1)^2 \right] \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ \left( \text{Diam} \left( V_0^{2k} \right) + 1 \right)^2 \right] \]

\[ \leq \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ (2D_{G'} (r(u), r(v)))^2 \right] \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ \left( 2\text{Diam} \left( V_0^{2k} \right) \right)^2 \right] \]

\[ \leq 4E_{\beta \leq k} \left[ D_{V_0^{2n-k}} (u, v)^2 \right] 4C_\beta' \Lambda(2^k, \beta)^2 \leq 16C_\beta' \Lambda(2^{n-k}, \beta + \varepsilon)^2 \Lambda(2^k, \beta)^2, \]

where we used Lemma 5.7 for the last inequality.

\subsection{The proof of Theorem 1.3}

In order to prove Theorem 1.3, we use a coupling between the measures $\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k}$ and $\mathbb{P}_{\beta \leq k-1}^{\beta + \varepsilon > k-1}$. Let $\omega \in \{0,1\}^E$ be distributed according to $\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k}$. Let $\chi \in \{0,1\}^E$ be a random vector that is independent of $\omega$ and has independent coordinates such that

\[ \mathbb{P} (\chi (\{u, v\}) = 1) = \begin{cases} 1 - e^{-\varepsilon f_{u+c} f_{v+c} \frac{1}{||x-y||^2} dxdy} \quad &2k-1 \leq ||\{u, v\}|| \leq 2k - 1, \\ 0 \quad &\text{else} \end{cases} \quad \text{(41)} \]

Then set $\omega' (e) = \omega (e) \lor \chi (e) = \max \{\omega (e) , \chi (e)\}$ for all edges $e \in E$. The coordinates of $\omega'$ are independent and for $e = \{u, v\} \in E$ with $2k-1 \leq |e| \leq 2k - 1$ we have

\[ \mathbb{P} (\omega' (e) = 0) = \mathbb{P} (\omega (e) = 0) \mathbb{P} (\chi (e) = 0) = e^{-f_{0+c} f_{e+c} \frac{1}{||x-y||^2} dxdy} e^{-f_{u+c} f_{v+c} \frac{1}{||x-y||^2} dxdy} = e^{-f_{u+c} f_{v+c} \frac{1}{||x-y||^2} dxdy} - p(\beta + \varepsilon, \{u, v\}) \]

and thus $\omega'$ is distributed according to the measure $\mathbb{P}_{\beta \leq k-1}^{\beta + \varepsilon > k-1}$. For a block $V_0^{2k} = \prod_{i=1}^d \{p_i(u) 2^k \ldots , (p_i(u) + 1) 2^k - 1 \}$ of side length $2^k$ and every vertex $v \in V_0^{2k}$, there are at most $(2(2k-1) + 1)^d \leq 2^{(k+1)d}$ vertices $w$ with $2k-1 \leq ||\{v, w\}|| \leq 2k - 1$. As $\chi$ can only be +1 on edges of $2k-1 \leq |e| \leq 2k - 1$, we have

\[ \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( \exists v \in V_0^{2k}, w \in \mathbb{Z}^d \text{ with } \chi (\{v, w\}) = 1 \right) \leq 2^{kd} \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( \exists w \in \mathbb{Z}^d \text{ with } \chi (\{0, w\}) = 1 \right) \leq 2^{kd} \sum_{w \in \mathbb{Z}^d : ||w||_\infty \in [2^{k-1}, 2^k-1]} \left( 1 - e^{-f_{0+c} f_{e+c} \frac{1}{||x-y||^2} dxdy} \right) \leq 2^{kd} \sum_{w \in \mathbb{Z}^d : ||w||_\infty \in [2^{k-1}, 2^k-1]} \frac{2^{2d} \varepsilon}{2^{2kd}} \leq 2^{kd} 2^{(k+1)d} \frac{2^{2d} \varepsilon}{2^{2kd}} = 2^{5d} \varepsilon, \quad \text{(42)} \]

where we used that

\[ 1 - e^{-\varepsilon f_{u+c} f_{v+c} \frac{1}{||x-y||^2} dxdy} = \mathbb{P}_\varepsilon (u \sim v) \leq \frac{2^{2d} \varepsilon}{||u - v||_\infty^{2d}} \quad \text{(43)} \]

for all $\varepsilon \geq 0$, all $n \in \mathbb{N}$, and all $u, v \in \mathbb{Z}^d$ with $||u - v||_\infty \geq 2$. This was proven in [5, Lemma 2.1].

Next, we define a notion of good sets inside the graph with vertex set $V_0^{2n}$. For $w \in V_0^{2n-k}$, we contract the box $V_0^{2k} \subset V_0^{2n}$ to vertices $r(w)$ and call the resulting graph $G'$. Remember the definition of the events $\mathcal{B}(\delta), \mathcal{B}_{u}(\delta)$, and $\mathcal{A}_{u,v}(\delta)$ from Lemmas 5.4, 5.3, and 5.5. For a small $\delta > 0$ (that will be defined in (48) below), we call a vertex $r(w)$ and
the underlying block $V_{w}^{2k}$ δ-good, if all the translated events of $B(\delta), B_{u}(\delta)$, and $A_{u,v}(\delta)$ occur, i.e., if

$$\bigcap_{x \in V_{w}^{2k}} \bigcap_{y \in V_{w}^{2k}} \left\{ D_{V_{w}^{2k}}(x, y) \geq \delta 2^{k\theta(\beta)} \right\}$$  \hspace{1cm} (44)$$

for all $u \neq v$ for which $w \neq v$, $V_{u}^{2k} \sim V_{w}^{2k} \sim V_{u}^{2k}$ and $\|u - w\|_{\infty} \geq 2$, and if

$$\bigcap_{x, y \in V_{w}^{2k}} \left\{ D_{V_{w}^{2k}}(x, y) \geq \delta 2^{k\theta(\beta)} \right\}$$  \hspace{1cm} (45)$$

for all $u$ with $\|u - w\|_{\infty} \geq 2$ and $V_{u}^{2k} \sim V_{w}^{2k}$, and if

$$D^{*} \left( V_{w}^{2k}, \bigcup_{u \in \mathbb{Z}^d : \|u - w\|_{\infty} \geq 2} V_{u}^{2k} \right) \geq \delta 2^{k\theta(\beta)}.$$  \hspace{1cm} (46)$$

Suppose that a path crosses a good set $V_{w}^{2k}$, in the sense that it starts somewhere outside of the set $\bigcup_{u \in \mathbb{Z}^d : \|u - w\|_{\infty} \leq 1} V_{u}^{2k}$, then goes to the set $V_{w}^{2k}$, and then leaves the set $\bigcup_{u \in \mathbb{Z}^d : \|u - w\|_{\infty} \leq 1} V_{u}^{2k}$ again. When the path enters the set $V_{w}^{2k}$ at the vertex $x$, coming from some a block $V_{w}^{2k}$ with $\|u - w\| \geq 2$, the path needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ to reach a vertex $y \in V_{w}^{2k}$ that is connected to the complement of $V_{w}^{2k}$, because of (44) and (45). When the path enters the set $V_{w}^{2k}$ from a block $V_{w}^{2k}$ with $\|u - w\|_{\infty} = 1$, then the path crosses the annulus between $V_{w}^{2k}$ and $\bigcup_{u \in \mathbb{Z}^d : \|u - w\|_{\infty} \geq 2} V_{u}^{2k}$. So in particular it needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ in order to cross this annulus, because of (46). Overall, we see that the path needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ within the set $\bigcup_{u \in \mathbb{Z}^d : \|u - w\|_{\infty} \leq 1} V_{u}^{2k}$ in order to cross the set $V_{w}^{2k}$. Let δ be small enough such that

$$9^{2d} 5000 \mu_{\beta+1}^2 (1 - f(\delta)) \leq (32 \mu_{\beta+1})^{-9^{d}400 \mu_{\beta+1}}.$$  \hspace{1cm} (47)$$

Such a δ > 0 exists, as $f(\delta)$ tends to 1 for $\delta \to 0$. From here on we call a block $V_{w}^{2k}$ good if it is δ-good for this specific choice of δ, and we call a vertex $r(u) \in G'$ good if the underlying block $V_{w}^{2k}$ is good. For a connected set $Z \subset G'$, we are interested in the number of separated good vertices inside this set, that are good vertices $r(u)$ such that the sets $\mathcal{N}(r(u))$ are not connected by a direct edge.

**Lemma 5.9.** Let $\varepsilon \in [0, 1]$, let $G = (V_{0}^{2\varepsilon}, E)$ be sampled according to the measure $\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}$, and let $G'$ be the graph that results from contracting boxes of the form $V_{w}^{2k}$. Then for large enough K one has

$$\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k} \left( \exists Z \in \mathcal{CS}_{K}(G') \text{ with less than } \frac{K}{9^{d}400 \mu_{\beta+1}} \text{ separated good vertices} \right) \leq 3 \cdot 2^{-K}. \hspace{1cm} (47)$$

**Proof.** Let $\hat{Z} = \{r(v_{1}), \ldots, r(v_{K})\}$ be a connected set in $G'$. Let $\preceq$ be a fixed total ordering of $\mathbb{Z}^d$, where we write $\prec$ for strict inequalities. Such an ordering can be obtained by considering a bijection $f : \mathbb{N} \to \mathbb{Z}^d$ and defining $u \preceq v \iff f^{-1}(u) \leq f^{-1}(v)$. So we can
assume that $\hat{Z} = \{r(v_1), \ldots, r(v_K)\}$, where $v_1 \prec v_2 \prec \ldots \prec v_K$. For such a set, we add the nearest neighbors to it. Formally, we define the set

$$\hat{Z}^N = \bigcup_{r(v) \in \hat{Z}} \mathcal{N}(r(v))$$

which is still a connected set and satisfies $K \leq |\hat{Z}^N| \leq 3^d K$. A vertex $r(u) \in G'$ can be included into the set $\hat{Z}^N$ in more than one way, meaning that there can be different vertices $r(v), r(\tilde{v}) \in \hat{Z}$ such that $r(u) \in \mathcal{N}(r(v))$ and $r(u) \in \mathcal{N}(r(\tilde{v}))$. However, each vertex $r(u) \in G'$ can be included into the set $\hat{Z}^N$ in at most $3^d$ many different ways. So in particular we have

$$\sum_{i=1}^{K} \deg^N(r(v_i)) \leq 3^d \sum_{r(v) \in \hat{Z}^N} \deg(r(v)),$$

where the neighborhood-degree of a vertex $\deg^N(r(u))$ was defined in (26). Next, we iteratively define a set $\mathbb{L}_I = \mathbb{L}_I(\hat{Z}) = \mathbb{L}_I K \subset \hat{Z}$ as follows:

0. Start with $\mathbb{L}_I_0 = \emptyset$.

1. For $i = 1, \ldots, K$: If $\deg^N(r(v_i)) \leq 9^{d50}\mu_{\beta+1}$ and $\mathcal{N}(r(v_i)) \sim \mathbb{L}_{i-1}$, then set $\mathbb{L}_{i} = \mathbb{L}_{i-1} \cup r(v_i)$; else set $\mathbb{L}_{i} = \mathbb{L}_{i-1}$.

On the event where $\overline{\deg}(Z) \leq 20\mu_{\beta+1}$, for all $Z \in \mathcal{C}_S_{\geq K} (G')$, we have

$$\sum_{i=1}^{K} \deg^N(r(v_i)) \leq 3^d \sum_{r(v) \in \hat{Z}^N} \deg(r(v)) \leq 3^d 20\mu_{\beta+1} \left| \hat{Z}^N \right| \leq 9^d 20^d \mu_{\beta+1} K$$

and thus there can be at most $\frac{K}{2^d}$ many vertices $r(v_i)$ with $\deg^N(r(v_i)) > 9^{d50}\mu_{\beta+1}$, which implies that there are at least $\frac{K}{2^d}$ many vertices with $\deg^N(r(v_i)) \leq 9^{d50}\mu_{\beta}$. Whenever we include such a vertex in the set $\mathbb{L}_I$, we can block at most $9^{d50}\mu_{\beta+1}$ different vertices, which already implies

$$|\mathbb{L}_I| \geq \frac{K}{2(9^{d50}\mu_{\beta+1} + 1)} \geq \frac{K}{9^d 200\mu_{\beta+1}}.$$

The event where $\overline{\deg}(Z) \leq 20\mu_{\beta+1}$ for all $Z \in \mathcal{C}_S_{\geq K} (G')$ is very likely for large $K$, by Lemma 4.7.

Conditioned on the degree of the block $V^2_w$, and assuming that $\deg^N(r(w)) \leq 9^{d50}\mu_{\beta+1}$, the probability that the block $V^2_w$ is not $\delta$-good is bounded by

$$\deg(r(w))^2 (1 - f_\delta(\delta)) + \deg^N(r(w))(1 - f_\delta(\delta)) \leq 9^{2d} 5000 \mu_{\beta+1}^2 (1 - f(\delta)),$$

where $f$ was defined by $f(\delta) = \min\{f_\delta(\delta), f_\delta(\delta), f_\delta(\delta)\}$. Remember that we chose $\delta > 0$ small enough so that

$$\delta(\delta) := 9^{2d} 5000 \mu_{\beta+1}^2 (1 - f(\delta)) \leq (32\mu_{\beta+1})^{-9^{d} 400\mu_{\beta+1}}.$$ (48)

We now claim that the set $\mathbb{L}_I$ contains at least $\frac{K}{9^{d} 400\mu_{\beta+1}}$ many separated good vertices with high probability. Given the graph $G'$, it is independent whether different vertices in $\mathbb{L}_I$ are good or not, as we will argue now. For a vertex $r(u)$, it depends only on edges with
at least one end in the set \( \bigcup_{r(v) \in \mathcal{N}(r(u))} V_v^2 \) whether the vertex \( r(u) \) is good or not. But for different vertices \( r(u), r(u') \in \mathbb{L} \) there are no edges with one end in \( \bigcup_{r(v) \in \mathcal{N}(r(u))} V_v^2 \) and the other end in \( \bigcup_{r(v) \in \mathcal{N}(r(u'))} V_v^2 \), as \( \mathcal{N}(r(u)) \sim \mathcal{N}(r(u')) \). Thus, it is independent whether different vertices in \( \mathbb{L} \) are good. So in particular, the probability that there are \( \frac{|\mathbb{L}|}{2} \) or more vertices in the set \( \mathbb{L} \) that are not good is bounded by

\[
2^{\frac{|\mathbb{L}|}{2}} \overline{f}(\delta) \frac{|\mathbb{L}|}{2} \leq 2^K \overline{f}(\delta) \frac{K}{\beta + 1} \leq 2^K (32\mu_{\beta+1})^{-K} = (16\mu_{\beta+1})^{-K}
\]

and thus the set \( \mathbb{L} \) (and also the set \( \hat{Z} \)) contains at least \( \frac{K}{\beta + 1} \) good vertices with very high probability. Furthermore, the separation property directly follows from the construction. Next, we want to translate such a bound from one connected set to all connected sets simultaneously. Using Lemma 4.7, we get that

\[
\mathbb{P}_{\beta \leq k}^{\beta + \epsilon \geq k} \left( \exists Z \in \mathcal{CS}_K (G') \text{ with less than } \frac{K}{\beta + 1} \text{ separated good vertices} \right)
\leq \mathbb{P}_{\beta \leq k}^{\beta + \epsilon \geq k} \left( \exists Z \in \mathcal{CS}_K (G') : |\{r(v) \in \mathbb{L}(Z) : r(v) \text{ good}\}| \leq \frac{K}{\beta + 1} \right)
\leq \mathbb{P}_{\beta + \epsilon \geq k} \left( \exists Z \in \mathcal{CS}_K (G') : \overline{\text{deg}}(Z) > 20\mu_{\beta+1} \right) + \mathbb{P}_{\beta \leq k}^{\beta + \epsilon \geq k} \left( |\mathcal{CS}_K (G')| > 8^K \mu_{\beta+1} \right) + 8^K \mu_{\beta+1}^{-K} (16\mu_{\beta+1})^{-K} \leq 3 \cdot 2^{-K}
\]

which finishes the proof. \( \square \)

With this we can now go to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We want to show that for all \( \beta \geq 0 \) the difference \( \theta(\beta) - \theta(\beta + \epsilon) \) converges to 0 as \( \epsilon \to 0 \). At the beginning of section 5, we already showed that the function \( \theta(\cdot) \) is continuous from the left, so it suffices to consider \( \epsilon > 0 \) now. We have also seen in (33) that

\[
\theta(\beta) - \theta(\beta + \epsilon) = \frac{1}{\log(2)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon \geq k} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \epsilon \geq k - 1} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]} \right). \tag{49}
\]

Each of the summands in (49) is bounded, which follows directly from the results of Lemma 5.6. So in order to show that \( \theta(\beta) = \lim_{\epsilon \to 0} \theta(\beta + \epsilon) \), it suffices to show that the summands converge to 0, for large \( k, n - k \), as \( \epsilon \to 0 \). Showing the convergence of a summand in (49) to 0 is equivalent to proving that the expression inside the logarithm converges to 1, which is equivalent to showing that

\[
\frac{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon \geq k} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \epsilon \geq k - 1} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]} - \frac{\mathbb{E}_{\beta \leq k - 1}^{\beta + \epsilon \geq k - 1} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]}{\mathbb{E}_{\beta \leq k - 1}^{\beta + \epsilon \geq k - 1} \left[ D_{V_0^{2^n}} (0, (2^n - 1) e_1) \right]}
\]

converges to 0 \( , \) as \( \epsilon \to 0 \). Again, we write \( G' \) for the graph where we contracted boxes of the form \( V_v^2 \) into vertices \( r(v) \). So each vertex \( r(v) \) in \( G' \) corresponds to the set \( V_v^2 \). We write \( \text{Diam}(r(v)) \) for \( \text{Diam}(V_v^2) \). Next, we want to investigate the sum of diameters in connected sets. We claim that there exists a constant \( 1 < C' < \infty \) such that \( \sum_{r(v) \in Z} \text{Diam}(r(v)) \leq C' |Z| \beta \) for all connected sets \( Z \) of some size with high probability. Let \( Z \) be a fixed set in \( G' \). Under the measure \( \mathbb{P}_{\beta \leq k}^{\beta + \epsilon \geq k} \), the diameter of
the box corresponding to some vertex \( r(v) \in G' \) always has the same distribution, not depending on \( \varepsilon \). By Markov’s inequality we have

\[
P_\beta \left( \sum_{r(v) \in Z} \text{Diam}(r(v)) > C'|Z|2^{k^\varepsilon(\beta)} \right) = P_\beta \left( \sum_{r(v) \in Z} \varepsilon \cdot \frac{\text{Diam}(r(v))}{2^{k^\varepsilon(\beta)}} > e^{C'|Z|} \right)
\]

\[
\leq \mathbb{E}_\beta \left[ e^{e^{C'|Z|}} |Z| e^{-C'|Z|} \right] \leq (8\mu_{\beta+1})^{-|Z|}
\]

for \( C' \) large enough, as \( \frac{\text{Diam}(r(v))}{2^{k^\varepsilon(\beta)}} \) has uniform exponential moments (see for example Lemma 5.2). As the diameter of some vertex \( v \) is independent of the edges in \( G' \) we get by a union bound that

\[
P_{\beta+\varepsilon} \left( \exists Z \in \mathcal{CS}_K \left( G' \right) : \sum_{r(v) \in Z} \text{Diam}(r(v)) > C'|Z|2^{k^\varepsilon(\beta)} \right)
\]

\[
\leq \mathbb{E}_{\beta+\varepsilon} \left[ |\mathcal{CS}_K \left( G' \right)| \right] (8\mu_{\beta+1})^{-K} \leq 2^{-K}. \quad (50)
\]

Let \( Z_K \) be the event that every connected set \( Z \in \mathcal{CS}_K \left( G' \right) \) satisfies \( \sum_{v \in Z} \text{Diam}(v) \leq C'K2^{k^\varepsilon(\beta)} \) and that every connected set \( Z \in \mathcal{CS}_K \left( G' \right) \) contains at least \( K \) separated good vertices. We also define \( Z_{\geq K} := \bigcap_{t=K}^{\infty} Z_t \). By (47), (50) and a union bound over all \( t \geq K \) we know that

\[
P_{\beta+\varepsilon}^{\beta+\varepsilon > k} \left( Z_{\geq K}^C \right) \leq \sum_{t=K}^{\infty} P_{\beta+\varepsilon}^{\beta+\varepsilon > k} \left( Z_t^C \right) \leq 10 \cdot 2^{-K} \quad (51)
\]

for all large enough \( K \). Now assume that the event \( Z_{\geq K} \) holds and that \( D_{G'} \left( r(0), r((2^n-k-1)e_1) \right) = K \). So it is possible to walk from \( 0 \) to \( (2^n-1)e_1 \) and to touch only \( K+1 \) boxes of the form \( V_{u^k}^{\delta_k} \), by going along the shortest path between \( r(0) \) and \( r((2^n-k-1)e_1) \) in \( G' \). This path is also a connected set in \( G' \). Between these boxes, one needs to take on additional step. Thus we have that

\[
D_{V_{0}^{\delta}} \left( 0, (2^n-1)e_1 \right) \leq C'(K+1)2^{k^\varepsilon(\beta)} + K \leq 2C'K2^{k^\varepsilon(\beta)}. \quad (52)
\]

On the other hand, let \( P \) be a path from \( 0 \) to \( (2^n-1)e_1 \) and let \( \hat{P} \) be its projection onto \( G' \). Then the projection \( \hat{P} \) goes through at least \( K \) blocks of the form \( V_{u^k}^{\delta_k} \) and the projection \( \hat{P} \) is a connected set in \( G' \). Thus, the set \( \hat{P} \) contains at least \( K \) \( \frac{K}{2^{400\mu_{\beta+1}}} \) separated good vertices. Now consider the situation where the path \( P \) crosses a good block \( V_{u^k}^{\delta_k} \). In this case, the path \( P \) already needs to make at least \( \delta 2^{k^\varepsilon(\beta)} \) steps inside the set \( \bigcup_{u \in Z^k, ||u-w|| = 1} V_{u^k}^{\delta_k} \). The sets \( \bigcup_{u \in Z^k, ||u-w|| = 1} V_{u^k}^{\delta_k} \) are not directly connected for different separated good vertices \( r(u) \) inside \( \hat{P} \). The path \( P \) crosses at least \( K \) \( \frac{K}{2^{400\mu_{\beta+1}}} - 2 \) separated good boxes, where the subtraction of two is necessary because the path touches boxes at the beginning/end without crossing them. This already implies that

\[
\text{length}(P) \geq \left( \frac{K}{2^{400\mu_{\beta+1}}} - 2 \right) \delta 2^{k^\varepsilon(\beta)} \quad (53)
\]

Next, we want to investigate how this helps us to bound the difference

\[
E_{\beta+\varepsilon}^{\beta+\varepsilon > k} \left[ D_{V_{0}^{\delta}} \left( 0, (2^n-1)e_1 \right) \right] - E_{\beta+\varepsilon}^{\beta+\varepsilon > k-1} \left[ D_{V_{0}^{\delta}} \left( 0, (2^n-1)e_1 \right) \right].
\]
We use the same notation as in the beginning of this chapter, i.e., we assume that \( \omega \) is distributed according to \( \mathbb{P}^{\beta + \varepsilon > k} \) and \( \chi \) is independent of \( \omega \) and distributed as described in (41). Then \( \omega' := \omega \lor \chi \) has law \( \mathbb{P}^{\beta + \varepsilon > k - 1} \). The structure of the graph \( G' \), in which we contracted blocks of side length \( 2^k \), does not change, as the edges inserted are either inside the blocks \( V_u^{2k} \) or between neighboring blocks. The probability that a block \( V_u^{2k} \) is adjacent to a bond in \( \omega' \) that did not exist in \( \omega \) is bounded by \( 2^{5d \varepsilon} \), see (42). For a connected set \( Z \subseteq G' \), we write \( Z_\chi \) for the set of vertices \( r(w) \in Z \) for which there exists an edge \( e \) that is adjacent to \( V_u^{2k} \) and satisfies \( \chi(e) = 1 \). For a fixed set \( Z \) we expect that \( |Z_\chi| \) is of order \( \varepsilon |Z| \) but it is not clear how to show such a statement for all connected sets of some size. Instead, we show that with high probability for all connected sets \( Z \) of size \( K \) the set \( Z_\chi \) is not larger than \( \frac{16 \mu_{\beta + 1} K}{\log(1/\varepsilon)} \). For a fixed connected set \( Z \in \mathcal{CS}_K(G') \), define the set \( Z'_\chi \) by the vertices \( r(w) \in Z_\chi \) for which an edge \( e \) with \( \chi(e) = 1 \) exists, so that \( e \) has both endpoints in \( V_u^{2k} \), or one endpoint is in \( V_u^{2k} \) and one endpoint is in \( V_w^{2k} \) with \( r(u) \notin Z \), or one endpoint in \( V_w^{2k} \) and one in \( V_u^{2k} \) with \( w < u \) and \( r(u) \in Z \). For different vertices \( r(u) \in Z \), it is independent whether they are in the set \( Z'_\chi \) or not. Hence the size of the set \( Z'_\chi \) is stochastically dominated by \( \sum_{i=1}^K X_i \), where \( X_i \) are independent Bernoulli-distributed random variables with parameter \( 2^{5d \varepsilon} \). Furthermore, one has \( |Z_\chi| \leq 2 |Z'_\chi| \), as each edge \( e \) with \( 1 - \omega(e) = \omega'(e) = 1 \), that creates a vertex in \( Z'_\chi \), can add at most two vertices to \( Z_\chi \). As the structure of the graph \( G' \) and the sets \( Z_\chi \) are independent, we get for small enough \( \varepsilon > 0 \) that

\[
\begin{align*}
\mathbb{P}^{\beta + \varepsilon > k}_{\beta \leq k} \left( \exists Z \in \mathcal{CS}_K(G') : |Z_\chi| > \frac{\mu_{\beta + 1} 16 K}{\log(1/\varepsilon)} \right) \\
\leq \mathbb{E}^{\beta + \varepsilon > k}_{\beta \leq k} \left[ |\mathcal{CS}_K(G')| \right] \mathbb{P} \left( 2 \sum_{i=1}^K X_i > \frac{\mu_{\beta + 1} 16 K}{\log(1/\varepsilon)} \right) \\
\leq 4^K \mu_{\beta + 1} K \mathbb{P} \left( \sum_{i=1}^K X_i > \frac{\mu_{\beta + 1} 18 K}{\log(1/\varepsilon)} \right) \leq 4^K \mu_{\beta + 1} K \left( 2^{5d \varepsilon} \right)^{\frac{\mu_{\beta + 1} 18}{\log(1/\varepsilon)} K} \\
= \left( 2^{5d} \right)^{\frac{\mu_{\beta + 1} 18}{\log(1/\varepsilon)} K} 8 K \mu_{\beta + 1} e^{-\mu_{\beta + 1} 8 K} \leq 2^{-K}
\end{align*}
\]

where the last inequality holds for small enough \( \varepsilon \). Next, let us see how the sums of the inside diameters of the sets \( Z_\chi \) grow. Let \( C_1 \in (0, \infty) \) be a constant such that

\[
\mathbb{E} \left[ e^{\text{Diam} \left( \frac{V_u^{2k}}{2^{k\theta(\beta)}} \right)} \right] < e^{C_1 s} \quad \text{for all } s \geq 1 \text{ and } k \in \mathbb{N}.
\]

Such a constant exists by Lemma 5.2.

We define the functions \( r(\varepsilon) = \log(1/\varepsilon)^{-\frac{1}{2k\theta(\beta)}} \) and \( s(\varepsilon) = \frac{\log(1/\varepsilon)}{\mu_{\beta + 1} 16 C_1} \). Let \( Z' \subseteq G' \) be a fixed set of size at most \( \frac{\mu_{\beta + 1} 16 K}{\log(1/\varepsilon)} K \). Then we have for all small enough \( \varepsilon \) that

\[
\begin{align*}
\mathbb{P}^{\beta + \varepsilon > k}_{\beta \leq k} \left( \sum_{r(v) \in Z'} \text{Diam} \left( r(v) \right) > r(\varepsilon) 2^{k\theta(\beta)} K \right) \\
\leq \mathbb{E} \left[ \exp \left( s(\varepsilon) \text{Diam} \left( \frac{V_u^{2k}}{2^{k\theta(\beta)}} \right) \right)^{|Z'|} \right] e^{-s(\varepsilon) r(\varepsilon) K} \leq e^{\frac{\mu_{\beta + 1} 16 K}{\log(1/\varepsilon)} C_1 s(\varepsilon) C_1} e^{-s(\varepsilon) r(\varepsilon) K} \\
= e^K e^{\left( \frac{\mu_{\beta + 1} 16 C_1}{\log(1/\varepsilon)} \right)} \leq 16^{-K} \mu_{\beta + 1} K
\end{align*}
\]

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where the last inequality holds for \( \varepsilon \) small enough. As the inside structure of blocks of the form \( V^{2k}_V \) in the graph defined by \( \omega \), the sets \( Z_\chi \), and the connections inside the graph \( G' \) are independent, we get that for \( \varepsilon > 0 \) small enough

\[
\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( \exists Z \in \mathcal{CS}_K(G') : \sum_{r(v) \in Z_\chi} \text{Diam} (r(v)) > r(\varepsilon)2^{k\theta(\beta)}K \right)
\leq \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( |\mathcal{CS}_K(G')| > 8^K\mu_{\beta+1}K \right) + \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( \exists Z \in \mathcal{CS}_K(G') : |Z_\chi| > \frac{\mu_{\beta+1}+16}{\log(1/\varepsilon)}K \right)
+ 8^K\mu_{\beta+1}16^{-K}\mu_{\beta+1} \leq 3 \cdot 2^{-K}.
\] (54)

Let \( D_K \) be the event that \( \sum_{v \in \hat{\mathcal{Z}}} \text{Diam} (r(v)) \leq r(\varepsilon)2^{k\theta(\beta)}K \) for all \( Z \in \mathcal{CS}_K(G') \), and let \( D_{\geq K} = \bigcap_{t=K}^{\infty} D_t \). From (51) and (54) we get that for \( K \) large enough

\[
\mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( (Z_{\geq K} \cap D_{\geq K})^C \right) \leq \sum_{t=K}^{\infty} \left( \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( Z_t^C \right) + \mathbb{P}_{\beta \leq k}^{\beta + \varepsilon > k} \left( D_t^C \right) \right) \leq 20 \cdot 2^{-K}.
\]

Now assume that \( D_{G'} (r(0), r((2^{n-k}-1)e_1)) = K \) and the events \( Z_{\geq K} \) and \( D_{\geq K} \) both hold; Consider a path \( P \) between \( 0 \) and \((2^{n}-1)e_1\) in the environment \( \omega' = \omega \lor \chi \) and its projection \( \hat{P} \) on \( G' \). Assume that the events \( D_{\geq K} \) and \( Z_{\geq K} \) both hold, and that \( K = D_{G'} (r(0), r((2^{n-k}-1)e_1)) \) is large enough. The path \( \hat{P} \) is a (not necessarily self-avoiding) walk on \( G' \) between \( r(0) \) and \( r((2^{n-k}-1)e_1) \). In the environment \( \omega \) for \( |\hat{P}| \) large enough, every path that touches \( |\hat{P}| \) distinct \( 2^k \)-blocks has length at least

\[
\left( \frac{|\hat{P}|}{2^{d400\mu_{\beta+1}}} - 2 \right) \delta 2^{k\theta(\beta)} \quad \text{by (53).}
\]

In the environment \( \omega \lor \chi \) such a path may be shorter, but by at most \( \sum_{r(v) \in \hat{P}_\chi} \text{Diam}(r(v)) \). So we get that

\[
\text{length}(P) \geq \left( \frac{|\hat{P}|}{2^{d400\mu_{\beta+1}}} - 2 \right) \delta 2^{k\theta(\beta)} - \sum_{r(v) \in \hat{P}_\chi} \text{Diam}(r(v))
\geq \left( \frac{|\hat{P}|}{2^{d400\mu_{\beta+1}}} - 2 \right) \delta 2^{k\theta(\beta)} - r(\varepsilon) |\hat{P}| 2^{k\theta(\beta)} \geq c_12^{k\theta(\beta)} |\hat{P}|
\] (55)

for some small \( c_1 > 0 \), \( \varepsilon \) small enough, and \( |\hat{P}| \) large enough. Now consider the shortest path \( P \) between \( 0 \) and \((2^{n}-1)e_1\) in the environment \( \omega' \). Combining the inequalities (52) and (55) we get that for \( K = D_{G'} (r(0), r((2^{n-k}-1)e_1)) \)

\[
2C'K2^{k\theta(\beta)} \geq D_{V^{2^{2k}}} (0, (2^n-1)e_1) = \text{length}(P) \geq c_1 |\hat{P}| 2^{k\theta(\beta)}
\]

and thus

\[
|\hat{P}| \leq \frac{2C'}{c_1} D_{G'} (r(0), r((2^{n-k}-1)e_1)) =: C_w D_{G'} (r(0), r((2^{n-k}-1)e_1)).
\]

So the shortest path \( P \) between \( 0 \) and \((2^n-1)e_1\) in the environment \( \omega' = \omega \lor \chi \) does not touch more than \( C_w D_{G'} (r(0), r((2^{n-k}-1)e_1)) \) blocks with side length \( 2^k \). This is an
interesting observation, as the path also needs to touch at least $D_{G'}(r(0), r((2^{n-k} - 1)e_1))$ many blocks with side length $2^k$.

Now let us bound the difference $D(0, (2^n - 1)e_1; \omega) - D(0, (2^n - 1)e_1; \omega')$. Let $(x_0, \ldots, x_s)$ be the shortest path between $x_0 = 0$ and $x_s = (2^n - 1)e_1$ in the environment $\omega'$. Then we build a path $(y_0, \ldots, y_s)$ between $0$ and $(2^n - 1)e_1$ in the environment $\omega$ as follows. As long as $\omega (\{x_i, x_{i+1}\}) = 1$, we follow the path $P$. If $\omega (\{x_i, x_{i+1}\}) = 0$, say with $x_i \in V_w^{2^k}$ and $x_{i+1} \in V_w^{2^k}$, we take the shortest path from $x_i$ to $x_{i'}$ where $i' = \max \{s' : x_{s'} \in V_w^{2^k}\}$.

That means, we go to the point $x_{i'}$ where the path $(x_0, \ldots, x_s)$ leaves the box $V_w^{2^k}$ for the last time. As $\omega$ and $\omega'$ can only differ at edges with length in $[2^{k-1}, 2^k]$, we already have $|u - w|_{\infty} \leq 1$, and thus the distance between $x_i$ and $x_{i'}$ is at most $\text{Diam}(r(u)) + \text{Diam}(r(w)) + 1$. So the length of the path constructed by this procedure is at most $\text{Diam}(r(u)) + \text{Diam}(r(w))$ longer compared to the original path. When at $x_{i'}$, we follow the path $P$ again until there appears again an edge $e = \{x_j, x_{j+1}\}$ with $\omega'(e) = 1 - \omega(e)$ and do the same procedure as before. This construction gives a path in the environment of $\omega$ of length at most $s + 2 \sum_{e \in P} \text{Diam}(v)$ and this already implies

$$D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1; \omega) - D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1; \omega') = D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1; \omega) - s \leq 2 \sum_{r(v) \in P} \text{Diam}(v) \leq 2C_w r(\epsilon) D_{G'}(r(0), r((2^{n-k} - 1)e_1)) 2^{k \theta(\beta)}$$

for small enough $\epsilon$ and when $D_{\geq K} \cap \mathcal{Z}_{\geq K}$ and $D_{G'}(r(0), r((2^{n-k} - 1)e_1)) \geq K$ hold for large enough $K$. This gives us a bound on the difference of $D(0, (2^n - 1)e_1; \omega)$ and $D(0, (2^n - 1)e_1; \omega')$ that goes to 0, as $\epsilon \to 0$. This bound only holds on the previously mentioned event, but we can also choose $K$, depending on $n - k$, in such a way such that the probability of this event goes to 1 as $n - k \to \infty$. The residual terms, where the previously mentioned events do not hold, can be estimated with the Cauchy-Schwarz inequality. In the following equations we simply write $D(\cdot, \cdot)$ for the distance $D_{V_{\omega}^{2^n}}(\cdot, \cdot)$.

For small enough $\epsilon > 0$ and large enough $k$, $n - k$ we have

$$\mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1) - \mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k - 1} [D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1)] \right]$$

$$= \mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1; \omega) - D_{V_{\omega}^{2^n}}(0, (2^n - 1)e_1; \omega') \right]$$

$$= \mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ (D(0, (2^n - 1)e_1; \omega) - D(0, (2^n - 1)e_1; \omega')) \mathbb{1}_{\{|D_{\geq n-k} \cap \mathcal{Z}_{\geq n-k}\}} \mathbb{1}_{\{|D_{G'}(r(0), r((2^{n-k} - 1)e_1))| < n-k\}} \right]$$

$$+ \mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ (D(0, (2^n - 1)e_1; \omega) - D(0, (2^n - 1)e_1; \omega')) \mathbb{1}_{\{|D_{\geq n-k} \cap \mathcal{Z}_{\geq n-k}\}^C} \mathbb{1}_{\{|D_{G'}(r(0), r((2^{n-k} - 1)e_1))| < n-k\}} \right]$$

$$\leq \mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ D_{G'}(r(0), r((2^{n-k} - 1)e_1)) 2C_w r(\epsilon) 2^{k \theta(\beta)} \right]$$

$$+ \sqrt{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ D(0, (2^n - 1)e_1)^2 \right]} \sqrt{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ \mathbb{1}_{\{|D_{G'}(r(0), r((2^{n-k} - 1)e_1))| < n-k\}} \right]^2}$$

$$+ \sqrt{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ D(0, (2^n - 1)e_1)^2 \right]} \sqrt{\mathbb{E}_{\beta \leq k}^{\beta + \epsilon + k} \left[ \mathbb{1}_{\{|D_{\geq n-k} \cap \mathcal{Z}_{\geq n-k}\}}^C \right]^2}$$

$$\leq 2C_w r(\epsilon) 2^{k \theta(\beta)} \mathbb{E}_{\beta + \epsilon} \left[ D_{V_{\omega}^{2^n-k}}(0, (2^{n-k} - 1)e_1) \right] + \left( \sqrt{C_{\beta} A(2^k, \beta)} \cdot A(2^{n-k}, \beta + \epsilon) \mathbb{E}_{\beta + 1} \left[ D_{V_{\omega}^{2^n-k}}(0, (2^{n-k} - 1)e_1) < n-k \right]^{1/2} \right)$$
\[ + \sqrt{C_\beta \Lambda(2^k, \beta) \Lambda(2^{n-k}, \beta + \varepsilon) 20 \cdot 2^{\frac{n-k}{2}}} \]
\[ \leq \Lambda(2^{n-k}, \beta + \varepsilon) \left( 2C_{w}(\varepsilon) 2^{k\theta(\beta)} + \sqrt{C_\beta \Lambda(2^k, \beta) \mathbb{P}_{\beta+1} \left( D_{V_0^2} n^{-k} (0, (2^{n-k} - 1) e_1) < n - k \right)^{1/2}} \right) + \sqrt{C_\beta \Lambda(2^k, \beta) 20 \cdot 2^{\frac{n-k}{2}}} \]

where we used Lemma 5.8 for the second inequality and the Cauchy-Schwarz inequality and (56) for the first inequality. Using Lemma 5.6, we get that

\[ \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right] - \mathbb{E}_{\beta \leq k-1}^{\beta + \varepsilon > k-1} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right] \]
\[ \leq \frac{\Lambda(2^{n-k}, \beta + \varepsilon) \Lambda(2^k, \beta)}{c_\beta \Lambda(2^{n-k+1}, \beta + \varepsilon) \Lambda(2^{k-1}, \beta)} \left( 2C_{w}(\varepsilon) 2^{k\theta(\beta)} \right) \]
\[ + \mathbb{P}_{\beta+1} \left( D_{V_0^2} n^{-k} (0, (2^{n-k} - 1) e_1) < n - k \right)^{1/2} \]
\[ \leq C_f \left( r(\varepsilon) + 20 \cdot 2^{\frac{n-k}{2}} + \mathbb{P}_{\beta+1} \left( D_{V_0^2} n^{-k} (0, (2^{n-k} - 1) e_1) < n - k \right)^{1/2} \right) \]

for some finite constant \( C_f < \infty \). This is true, as both fractions

\[ \frac{\Lambda(2^{n-k}, \beta + \varepsilon) \Lambda(2^k, \beta)}{c_\beta \Lambda(2^{n-k+1}, \beta + \varepsilon) \Lambda(2^{k-1}, \beta)} \]

and

\[ \frac{\Lambda(2^{n-k}, \beta + \varepsilon) 2^{k\theta(\beta)}}{c_\beta \Lambda(2^{n-k+1}, \beta + \varepsilon) \Lambda(2^{k-1}, \beta)} \]

are bounded uniformly over all \( \varepsilon \in [0, 1], k \leq n \in \mathbb{N} \). The last term in the above calculation is the probability \( \mathbb{P}_{\beta+1} \left( D_{V_0^2} n^{-k} (0, (2^{n-k} - 1) e_1) < n - k \right) \), which tends to 0 as \( n - k \) goes to infinity, as the graph distance between 0 and \((2^{n-k} - 1)e_1\) is of order \(2^{(n-k)}\theta(\beta+1) \gg n-k\) under the measure \( \mathbb{P}_{\beta+1} \). In particular this implies that for large enough \( k \)

\[ \mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right] - \mathbb{E}_{\beta \leq k-1}^{\beta + \varepsilon > k-1} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right] \]
\[ \leq \log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right]}{\mathbb{E}_{\beta \leq k-1}^{\beta + \varepsilon > k-1} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right]} \right) \]

converges to zero, as \( \varepsilon \to 0 \) and \( n - k \to \infty \). Thus

\[ \lim_{\varepsilon \to 0} \theta(\beta) - \theta(\beta + \varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{\log(2)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \log \left( \frac{\mathbb{E}_{\beta \leq k}^{\beta + \varepsilon > k} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right]}{\mathbb{E}_{\beta \leq k-1}^{\beta + \varepsilon > k-1} \left[ D_{V_0^2} (0, (2^n - 1) e_1) \right]} \right) = 0 \]

which shows continuity from the right of the distance exponent \( \theta(\cdot) \) and thus finishes the proof of Theorem 1.3. \( \square \)
6 References

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