Proportionate Adaptive Graph Signal Recovery

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Abstract—This paper generalizes the proportionate-type adaptive algorithm to the graph signal processing and proposes two proportionate-type adaptive graph signal recovery algorithms. The gain matrix of the proportionate algorithm leads to faster convergence than least mean squares (LMS) algorithm. In this paper, the gain matrix is obtained in a closed-form by minimizing the gradient of the mean-square deviation (GMSD). The first algorithm is the proportionate-type graph least mean squares (Pt-GELMS) algorithm which simply uses a gain matrix in the recursion process of the LMS algorithm and accelerates the convergence of the Pt-GELMS algorithm compared to the LMS algorithm. The second algorithm is the proportionate-type graph extended LMS (Pt-GLMS) algorithm, which uses the previous signal vectors alongside the signal of the current iteration. The Pt-GLMS algorithm utilizes two gain matrices to control the effect of the signal of the previous iterations. The stability analyses of the algorithms are also provided. Simulation results demonstrate the efficacy of the two proposed proportionate-type LMS algorithms.

Index Terms—Graph signal recovery, adaptive, laplacian matrix, least mean-squares, proportionate.

I. INTRODUCTION

GRAPH signal processing (GSP) [1], [2], [3] is a new research paradigm in signal processing. In GSP, the signal is defined over a graph with irregular domains. It can better represent the inherent structure of the signals defined over nodes. The potential application of GSP includes wireless sensor networks, biological network, social networks, financial networks, and vehicular networks [3], to name a few. Because of the shift of signal processing to GSP, the available tools in classical signal processing such as shifting, sampling, Fourier transform, filters, etc. is generalized to the graph domain. We can refer to graph sampling, graph signal recovery, graph topology learning, and graph spectral representation as problems within GSP framework. Graph Signal Recovery (GSR) is a basic problem in GSP which aims to recover the whole graph signal by observing signal over only a subset of graph nodes [4]. GSR uses the inherent relationship between the signal values defined over connecting nodes in the graph. This relationship is defined by matrices called weighted adjacency and Laplacian matrix of the graph which will be introduced in sequel.

There are two types of GSR algorithms. The first type is the non-adaptive GSR [4], [5], [6], [7], [8], [9], [10], [11] which often uses some optimization problems to solve GSR problems in a non-adaptive manner and in a batch-based framework. There is usually a great deal of complexity involved in these algorithms, especially in the large scale graph networks. The second type of GSR algorithms are adaptive GSR algorithms [12], [13], [14]. Similar to conventional adaptive filter algorithms, these type of GSR algorithms require lower computational complexity and have the potential to perform well in the time-varying nature of the graph signal and noise. Hence, in this paper, we focus on the adaptive GSR algorithms.

Adaptive GSR algorithms first appeared in the literature in 2016. As a pioneering work, a graph Least Mean Square (LMS) algorithm has been developed in [12], which generalizes the well-known adaptive LMS algorithm to the graph domain. In sequel, [13] suggested a distributed LMS algorithm for learning the graph signal over a network. Moreover, an LMS and a Recursive Least Square (RLS) algorithm have been developed to recover the graph signals from randomly time-varying subset of nodes [14]. Also, in [15], the LMS algorithm has been developed for the dynamic graphs in which there is a small perturbation in the Laplacian matrix. In addition, a joint graph weighted adjacency matrix learning and graph signal recovery has been suggested using a Kalman filter for auto-regressive graph signals [16]. To provide scalability and privacy in networks, an online kernel-based graph-adaptive learning algorithm has been suggested in [17]. Besides, [18] proposed a distributed adaptive learning of graph signals using in-network subspace projections. Moreover, a Normalized LMS (NLMS) graph signal estimation algorithm has been proposed in [19], which has faster convergence than LMS algorithm and has less computational complexity than RLS algorithm. Also, [20] proposed two adaptive GSR algorithms which are Extended LMS (ELMS) algorithm and Fast ELMS (FELMS) algorithm, in which the signal vectors of previous iterations are reused alongside the signal available at the current iteration. In addition, a Single-Kernel Gradraker (SKG) algorithm has been suggested for GSR which uses a Gaussian kernel and it specifies how to find a suitable variance for the kernel [21]. For adaptive estimation of the graph filter of a graph signal, a graph kernel RLS algorithm has been developed for a different but related problem [22].

As an alternative, graph-structured signals can also be defined as sparse graph signals, and compressed sensing (CS) can be applied to them. CS is applied to graph signals using the graph 2373-776X © 2023 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See https://www.ieee.org/publications/rights/index.html for more information.
Fourier transform (GFT), which consists of the eigenvectors of graph Laplacian. In [31], the authors propose to subsample the graph signal according to CS theory, assuming the graph signal is sparse in the GFT domain.

In this paper, we focus on adaptive graph LMS algorithms for GSR. We generalize the proportionate adaptive filter concept used in adaptive filtering application [23], [24], and used in distributed estimation framework [25], [26] to the graph domain. Hence, we propose adaptive proportionate GSR algorithm in which a gain matrix is used in the update of the adaptive algorithm. When we have a sparse representation of the graph signal in the domain of Graph Fourier Transforms (GFT), the proposed algorithm achieves a higher convergence rate. For determining the gain matrix, the GMSD criterion is used and a closed-form formula is obtained for the gain matrix. The stability analyses of the Graph Proportionate LMS (GPLMS) algorithm are also provided. Further, the computational complexity analysis shows that for small number of measurements, i.e., \( M \) and small memory length, i.e., \( K \), the proposed algorithm shows lower computational complexity. Finally, simulation results corroborate the efficacy of the proposed algorithm in comparison to some state-of-the-art algorithms in the literature.

This paper is organized as follows. Section II presents the essential background on GSP and problem formulation. Section III presents the proposed proportionate-type LMS algorithm for GSR. The extended proportionate-type LMS algorithm is presented in Section IV. Section V provides some theoretical aspects of the proposed algorithms. In Section VI, the simulation results are discussed. We conclude the paper in Section VII.

We denote vectors by boldface lowercase letters and matrices by boldface uppercase letters. The operators \((\cdot)^T\) and \((\cdot)^H\) represent the transpose and Hermitian transpose, respectively. \( E[\cdot] \) represents expectation. \( \text{diag}(\cdot) \) represents a diagonal matrix with its arguments. The \( m \)th element of the vector \( a \) is written as \( a_m \). The \( A_i \) denotes the \( i \)th column of the matrix \( A \), while \( \sigma_{mq} \) is the \( (m, q) \)th element of the matrix \( A \). The identity matrix of dimension \( N \) is written as \( I_N \) and the vector \( 0_N \) is a length \( N \) vector of all zeros. \( 0_{M \times N} \) is an \( M \times N \) matrix with all entries being zero. The spectral radius of the square matrix \( A \) is denoted by \( \rho(A) \triangleq \max \{|\lambda(A)|\} \).

II. PROBLEM FORMULATION AND BACKGROUND

Consider an undirected, connected, weighted graph \( G = (\mathcal{V}, \mathcal{E}) \) with \( N \) vertices indexed by \( \mathcal{V} = \{v_1, v_2, \ldots, v_N\} \) and connected together according to the set of edges \( \mathcal{E} \). The weighted adjacency matrix \( W \in \mathbb{R}^{N \times N} \) is the collection of all edge weights such that \( W_{ij} > 0 \) if \((i, j) \in \mathcal{E}, \) and \( W_{ij} = 0 \) otherwise. Let \( d_i \) be the degree of node \( i \) which is defined as \( d_i = \sum_{j=1}^{N} W_{ij} \). The degree matrix \( D \) is a diagonal matrix with the node degrees as its diagonals. Hence, graph Laplacian matrix is defined as \( L = D - W \).

For undirected graphs, the graph Laplacian is a symmetric and positive semi-definite matrix. The eigendecomposition for Laplacian matrix is

\[
L = U \Lambda U^H, \tag{1}
\]

where \( U \in \mathbb{R}^{N \times N} \) is the matrix containing all the eigenvectors of \( L \) as its columns, and \( \Lambda \in \mathbb{R}^{N \times N} \) is the diagonal matrix of eigenvalues of \( L \).

Analogous to the classical signal processing, the graph Fourier transform (GFT) of a graph signal \( x \in \mathbb{R}^{N} \) is defined as its projection onto an orthogonal set of vectors \( \{u_i\}_{i=1, \ldots, N} \), i.e.

\[
s = U^H x. \tag{2}
\]

The basis vector \( \{u_i\} \) are usually assumed to be the eigenvector set of Laplacian matrix [12], [20], or the adjacency matrix [2], [27]. Thus, the frequency-domain representation of the graph signal conveys the intrinsic information of the graph topology. In this paper, we follow the definition of GFT basis vector based on the Laplacian matrix, but the result can be extended to the approach based on the adjacency matrix. The inverse graph Fourier transform (IGFT) can be defined for reconstruction of the graph signal \( x \) from its frequency domain \( s \) as

\[
x = Us. \tag{3}
\]

In this paper, we utilize the intrinsic sparsity of bandlimited signals and we assume that the frequency domain representation of the graph signal, i.e. \( s \), is sparse. Thus, we can use the compressive sensing (CS) theory and assume the under-sampling of the \( s \) as observations. Moreover, we assume that the observation are sampled noisy versions of the graph signal. Thus, using the notations \( x^o \) and \( s^o \) for the original graph signal and its Fourier transform, respectively, we can write the observed signal at time \( n \) as

\[
y[n] = B[n]D[n]x^o + e[n] = B[n]D[n]Us^o + e[n], \tag{4}
\]

where \( B \in \mathbb{R}^{M \times N} \) is the compressed sensing matrix, where \( M \leq N \) is the number of measurements of the sampled graph signal, \( D \in \mathbb{R}^{N \times N} \) is the sampling matrix in which the row related to sampled node has just one +1 element (diagonal element) and the row related to non-sampled node is an all-zero vector, and \( e[n] \) is \( M \times 1 \) zero-mean additive measurement noise vector at time \( n \) with the covariance matrix \( C_e = \text{diag}(\sigma^2_1, \ldots, \sigma^2_M) \). The above equation can be rewritten as

\[
y[n] = A[n]s^o + e[n], \tag{5}
\]

where \( A[n] = B[n]D[n]U \) is an \( M \times N \) matrix. The aim of GSR is to estimate the graph signal \( x^o \) (or equivalently, \( s^o \)) from sampled, noisy observations \( y[n] \), where we assume that the CS sensing matrix (whose elements are random Gaussian with zero-mean and unit variance), Fourier basis functions, and the random sampling matrix are known.

III. PROPORTIONATE-TYPE GRAPH LMS ALGORITHM

The LMS algorithm is one of the most popular algorithms in adaptive filtering, and is employed in graph signal recovery [12], [13], [14], [15]. Based on the graph-LMS approach, the optimal estimation for the original graph signal can be found by the following recursive procedure

\[
s[n+1] = s[n] + \mu A^H[n](y[n] - A[n]s[n]), \tag{6}
\]

where \( \mu \) is the step-size parameter. The proportionate-type LMS (Pt-LMS) algorithm [23] was introduced as an alternative to the conventional LMS algorithm, which is proved to converge faster than the LMS algorithm by assigning a different gain to each coefficient. This gain is proportional to the magnitude of the coefficient at the current iteration. In this paper, we propose to use this Pt-LMS algorithm in graph signal recovery problem, which results in proportionate-type graph LMS (Pt-GLMS) algorithm.
Algorithm 1: Proposed Proportionate-Type Graph LMS (Pt-LMS) Algorithm.

Input Observations $y$; Fourier basis functions $U$;
Sensing matrix $B$; Sampling matrix $D$; $\mu$, $\rho$, $\delta$;
number of time instances $T$.

Initialize $s = 0$, $G = 0$, $n = 1$.

repeat

- $A[n] = B[n]D[n]U$
- Update $\{g_i[n]\}_{n=1}^N$ using (8) or (13)
- $G[n] = \text{diag}(g_1[n], \ldots, g_N[n])$
- $s[n+1] = s[n] + \mu G[n]A^H[n](y[n] - A[n]s[n])$
- $n \leftarrow n + 1$

until $n \leq T$

The update equation for the proposed Pt-GLMS algorithm is

$$s[n + 1] = s[n] + \mu G[n]A^H[n](y[n] - A[n]s[n]),$$

where $G[n]$ is the gain matrix. This gain matrix distinguishes the Pt-LMS from conventional LMS algorithm, and is diagonal, i.e., $G[n] = \text{diag}(g_1[n], \ldots, g_N[n])$, where $g_i[n]$ is the gain factor of the signal of the $i$-th node at time $n$, and is proportional to the magnitude of the graph signal at node $i$, i.e., $g_i[n] \propto |s_i[n]|$. If $g_i[n] = 1, \forall i = 1, \ldots, N$, i.e., $G[n] = I_N$, the Pt-GLMS and GLMS algorithms are equivalent. In the literature [28], the weights $g_i[n]$ are calculated as

$$g_i[n] = \frac{\gamma_i[n]}{\sqrt{\sum_{j=1}^N \gamma_j[N]}},$$

where

$$\gamma_i[n] = \max\{\rho \gamma_{\text{min}}[n], F[|s_i[n]|]\},$$

and

$$\gamma_{\text{min}}[n] = \max\{\delta, F[|s_1[n]|], \ldots, F[|s_N[n]|]\},$$

where $\delta$ is the initialization parameter, and the parameter $\rho$ prevents the inactive coefficients from stalling. For $F[|s_i[n]|]$, different functions have been used in the literature [28].

From (7), the factor $\mu g_i[n]$ implies the effective step-size at node $i$, and indicates that in the case of large magnitude of current graph signal, i.e., $|s_i[n]|$, the value of $g_i[n]$ is also large, and, thus, the effective step-size $\mu g_i[n]$ is large, which speeds up the convergence of large coefficients. Conversely, for small magnitude of current coefficients, the effective step-size is also small. The resulting proportionate-type graph LMS algorithm is summarized in Algorithm 1.

A. Gain Matrix Calculation for Pt-GLMS

In this subsection, we calculate an optimal gain matrix in order to further speed up the convergence of the proposed Pt-GLMS algorithm. To this end, we compute the gradient mean-square deviation (GMSD), and find the optimal gains by minimizing the GMSD at time $n$. The GMSD is defined as [20]

$$\Delta[n] = E[\|s[n+1]\|_Q^2] - E[\|s[n]\|_Q^2],$$

where $\bar{s}[n] = s^o - s[n]$ is the error signal at time $n$, $Q = A^H A$, and $\|t\|_Q^2 = t^H Qt$. It is proved in Appendix A that GMSD equals

$$\Delta[n] = E \left[ \sum_{i=1}^N g_i^2[n] \sum_{j=1}^M a_{ji}^2[n] \right] + E \left[ \sum_{i=1}^N g_i[n] m_i[n] \sum_{j=1}^N a_{ji}[n] a_{jk}[n] \right] - 2 \mu E \left[ \sum_{i=1}^N g_i[n] A_i[n] C_s \bar{A}_i^H[n] \right].$$

The optimum gain for node $i$ at time instant $n$ (i.e., $g_i[n]$), by setting the derivative of $\Delta[n]$ with respect to $g_i[n]$ equal to zero, is written as

$$g_i[n] = \frac{f_3[n]}{f_4[n]},$$

where

$$\hat{e}[n] = y[n] - A[n]s[n],$$

$$f_3[n] = \mu \left[ \hat{e}^H[n] A_i[n] A_i^H[n] \hat{e}[n] - A_i[n] C_s \bar{A}_i^H[n] \right],$$

and

$$f_4[n] = m_i^2[n] \sum_{j=1}^M a_{ji}^2[n].$$

Proof: See Appendix A.

Computing the values of $g_i[n]$ using (13)–(17), and substituting $G[n] = \text{diag}(g_1[n], \ldots, g_N[n])$, the graph signal can be recovered by (7), which yields the minimum GMSD.

As an example, consider a graph with $N = 50$ nodes, as shown in Fig. 1. The spectral content of the graph signal is limited to the first 15 eigenvectors of the graph Laplacian matrix. The observation noise is drawn from a zero-mean Gaussian distribution with a diagonal covariance matrix. Fig. 2 shows the normalized MSD (NMSD), which is calculated as follows

$$\text{NMSD}[n] = \frac{\|s^o[n] - s[n]\|^2}{\|s^o[n]\|^2}.$$
in previous section. The proposed algorithm is based on the extended LMS (ELMS) algorithm proposed in [20] and speeds up the convergence of the Pt-GLMS algorithm by using the observations of previous times as well as the current observation. In this work, we propose the proportionate-type of the ELMS algorithm in GSR framework, which assigns the gain matrix to the graph signal estimation process. The resulting algorithm is proportionate-type graph ELMS (Pt-GELMS) algorithm. The update equation for the proposed Pt-GELMS algorithm is

\[ s[n + 1] = s[n] + \mu G[n]A^H[n]y[n] - A[n]s[n] \]

\[ + \mu H[n] \sum_{j=1}^{K-1} A^H[n-j] (y[n-j] - A[n-j]s[n]), \]

where \( K \) is the number of previous time instants used for estimating the graph signal at the current iteration (Memory Length), \( G[n] \) and \( H[n] \) are diagonal gain matrices assigned to the current and previous observations, respectively. Similar to the previous sections, we have \( G[n] = \text{diag}(g_1[n], \ldots, g_N[n]) \) and \( H[n] = \text{diag}(h_1[n], \ldots, h_N[n]) \). Obviously, when \( H[n] = 0_{N \times N} \), the Pt-GELMS algorithm is equivalent to the Pt-GLMS algorithm, and when \( G[n] = I_N \) and \( H[n] = 0_{N \times N} \), the Pt-GELMS algorithm reduces to the GLMS algorithm. The resulting proportionate-type graph extended LMS algorithm is summarized in Algorithm 2.

A. Gain Matrix Calculation for Pt-GELMS

In this subsection, similar to the process in Section III-A, we find an optimal value for the gain matrices by computing and minimizing the GMSD at time \( n \). To compute GMSD, note that using the definition \( \tilde{s}[n] = s^o - s[n] \), and subtracting both sides of (19) from \( s^o \), we obtain

\[ \tilde{s}[n+1] = \tilde{s}[n] - G[n]m_1[n] - H[n]m_2[n], \]

where

\[ m_1[n] = \mu (A^H[n]A[n]\tilde{s}[n] - A^H[n]e[n]), \]

\[ m_2[n] = \mu \sum_{j=1}^{K-1} A^H[n-j]A[n-j]s[n] + \mu \sum_{j=1}^{K-1} A^H[n-j]e[n-j]. \]

Similar to Section III-A, computing the GMSD at time \( n \), i.e. \( \Delta[n] \), and setting \( \frac{\partial \Delta[n]}{\partial y[n]} = 0 \) and \( \frac{\partial \Delta[n]}{\partial s[n]} = 0 \) yields

\[ g_i[n] = r_3[n], \]

and

\[ h_i[n] = r_5[n]. \]

where

\[ r_3[n] = -m_{1,i}[n] \sum_{k=1}^{M} \sum_{j=1}^{N} h_k[n]m_{2,k}[n]a_{jk}[n]a_{ji}[n] \]

\[ -m_{1,i}[n] \sum_{k=1}^{M} \sum_{j=1}^{N} g_k[n]m_{1,k}[n]a_{jk}[n]a_{ji}[n] \]

\[ + \mu (y[n] - A[n]s[n])^T A_i[n]A_i^T[n](y[n] - A[n]s[n]) \]

\[ - \mu A_i[n]C_e A_i^T[n], \]

\[ r_4[n] = n_{1,i}[n] \sum_{j=1}^{M} a_{ji}[n], \]

\[ r_5[n] = -m_{2,i}[n] \sum_{k=1}^{M} \sum_{j=1}^{N} g_k[n]m_{1,k}[n]a_{jk}[n]a_{ji}[n] \]

\[ -m_{2,i}[n] \sum_{k=1}^{M} \sum_{j=1}^{N} h_k[n]m_{2,k}[n]a_{jk}[n]a_{ji}[n] \]

\[ - \mu A_i[n]C_e \sum_{j=1}^{K-1} A_i^T[n-j] + \mu (y[n] - A[n]s[n])^T \]

\[ \times A_i[n]A_i^T[n] \sum_{j=1}^{K-1} (y[n-j] - A[n-j]s[n]), \]

\[ r_6[n] = n_{2,i}[n] \sum_{j=1}^{M} a_{ji}[n]. \]

Proof: See Appendix B.
Algorithm 2: Proposed Proportionate-Type Graph extended LMS (Pt-GELMS) Algorithm.

Input Observations $y$: Fourier basis functions $U$; Sensing matrix $B$: Sampling matrix $D$; $\mu$, $K$; number of time instances $T$.

Initialize $s = 0$, $G = 0$, $H = 0$, $n = 1$.

repeat
  • $A[n] = B[n]D[n]U$
  • Update $\{g_i[n]\}_{i=1}^N$ and $\{h_i[n]\}_{i=1}^N$ using (23) and (24), respectively.
  • $G[n] = \text{diag}(g_1[n], \ldots, g_N[n])$ and $H[n] = \text{diag}(h_1[n], \ldots, h_N[n])$
  • $s[n + 1] = s[n] + \mu G[n]A^H[n](y[n] - A[n]s[n]) + \mu H[n] \sum_{j=1}^{K-1} A^H[n - j](y[n - j] - A[n - j])$
  • $n \leftarrow n + 1$
until $n < T$

Obviously, $g_i[n]$ and $h_i[n]$ are interdependent, and, thus, should be updated in a repetitive manner.

V. THEORETICAL ANALYSIS

In this section, some theoretical analysis of the proposed algorithms are provided which are complexity analysis, mean-square analysis, and mean performance analysis. For complexity analysis, we separately calculate the complexity of the proposed algorithms (Pt-GLMS and Pt-GELMS) as well other competing algorithms. For two other convergence analysis (mean-square and mean performance), since the Pt-GLMS is a special case of Pt-GELMS, we only discuss the theoretical analysis of Pt-GELMS, and the analysis for the Pt-GLMS can be achieved by setting $H[n] = 0_{N \times N}$.

A. Computational Complexity Analysis

This subsection presents the computational complexity analysis of the proposed algorithms. Towards that end, we calculate the required number of additions, multiplications/divisions, matrix inversions, maximum operators, and F-function. The compared results with other algorithms are shown in Table I. For the readability, we only presented the order of complexity for various algorithms. It can be seen that the complexity of the proposed Pt-GLMS and Pt-GELMS algorithms depend on the values $M$ and $K$. For instance for small values of $M$ and $K$, the complexity of the proposed algorithms are lower compared to that of other algorithms.

B. Mean-Square Analysis

In this subsection, we study the mean-square behavior of the proposed algorithms. As described in Section IV, the proposed Pt-GELMS algorithm is equivalent to the Pt-GLMS if $H[n] = 0_{N \times N}$, and is equivalent to LMS if $G[n] = I_N$ and $H[n] = 0_{N \times N}$, respectively. Thus, we write a general equation for all these algorithms as

$$s[n + 1] = [I - \mu B_1[n]]s[n] - \mu B_2[n] - \mu B_3[n],$$

where

$$B_1[n] = G[n]A^H[n]A[n] + H[n] \sum_{j=1}^{K-1} A^H[n - j]A[n - j],$$

$$B_2[n] = G[n]A^H[n]e[n],$$

$$B_3[n] = H[n] \sum_{j=1}^{K-1} A^H[n - j]e[n - j].$$

In this paper, similar to [12], we consider a general weighted squared error sequence $||\tilde{s}[n]||_2^2$, where $\Phi \in \mathbb{R}^{N \times N}$ is an arbitrary matrix. Then, $\Phi$—weighted norm of both sides of (29) yields

$$E[||\tilde{s}[n + 1]||_2^2] = E[||\tilde{s}[n]||_2^2]$$
$$+ \mu^2 E[e^T[n]A[n]G^T[n]\Phi G[n]A^H[n]e^T[n]]$$
$$+ \mu^2 E[\sum_{j=1}^{K-1} e^T[n - j]A[n - j]H^T[n]\Phi]$$
$$\times H[n] \sum_{j=1}^{K-1} A^H[n - j]e[n - j],$$

where

$$\Phi' = (I - \mu B_1[n])^T \Phi (I - \mu B_1[n]).$$

Let $\phi = \text{vec}(\Phi)$, where the operator vec(.) aggregates the columns of the matrix therein on top of each other; then, we have

$$E[||\tilde{s}[n + 1]||_2^2] = E[||\tilde{s}[n]||_2^2]Q \phi$$
$$+ \mu^2 \text{Tr}[\Phi G[n]A^H[n]C_eA[n]G^T[n]]$$
$$+ \mu^2 \text{Tr}[\Phi H[n] \sum_{j=1}^{K-1} A^H[n - j]C_eA[n - j]H^T[n]],$$

where $\text{Tr}(\cdot)$ is the trace operator, and we have

$$Q = (I - \mu B_1[n]) \odot (I - \mu B_1[n]).$$

TABLE I

| Algorithm          | A | M/D | MI | max | F-function |
|--------------------|---|-----|----|-----|------------|
| GLMS               | $O(N^2)$ | $O(N^2)$ | -  | -  | -          |
| GNGLMS             | $O(N^2 F)$ | $O(N^2 F)$ | -  | -  | -          |
| Pt-GLMS            | $O(N^2 K)$ | $O(N^2 K)$ | -  | -  | -          |
| Pt-GELMS (proposed)| $O(NL)$+ | $O(NL)$+ | -  | -  | -          |
| Pt-GELMS           | $O(KMN)$ | $O(KMN)$ | -  | -  | -          |

A: Addition, M: Multiplication, D: Division, max: maximum operator, MI: Matrix Inversion M: Number of measurements, F: sparsity of GFT; N: Number of nodes, K: Memory Length, and $L = N + M$.
Assume that the model holds. Then, for a zero-mean Gaussian distribution with a diagonal covariance matrix $\mathbf{C}_e = \sigma_e^2 \mathbf{I}$ with $\sigma_e^2 = 0.01$. We compare the results of our proposed algorithms with that of the graph LMS algorithm [12], and extended proportionate-type LMS [20]. In all simulations, we have compared the results of the Pt-GLMS algorithm with two settings: (i) in the first case, we set the weights $g_i[n]$ according to (8), where in (10), we have calculated the function $F(s_i[n])$ as presented in [28], such that $F(s_i[n]) = \log(1 + ||s_i[n]||^2)$. Then, using (9) and (10), the weights are obtained. (ii) In the second case, the weights are calculated using the proposed algorithm (13).

Fig. 3 shows the transient behavior of the NMSD in (18) versus the iteration index for the number of previous time instants $K = 8$, number of CS measurements $M = 30$, number of samples $|S| = 20$, and step-size $\mu = 0.01$. Each point in the curves is the result of ensemble average over 50 independent simulations. It can be seen that the proposed proportionate-type LMS algorithms converge faster than the conventional LMS. Moreover, using the proposed algorithm for estimation.

**VI. SIMULATION RESULTS**

In this section, we evaluate the performance of the proposed algorithms via some simulations on synthetic and real-world data.

In the first scenario, we consider a synthetic graph with $N = 50$ nodes, as shown in Fig. 1. Similar to [29], the edge weights are drawn randomly from a uniform distribution $W_{ij} \sim U(0, 1)$, and the weighted adjacency matrix is derived as $\mathbf{W} = (\mathbf{W} + \mathbf{W}^T)/2$. The graph Laplacian matrix is calculated as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where $\mathbf{D}$ is the diagonal degree matrix with the diagonal elements as $d_i = \sum_{j=1}^{N} W_{ij}$. The spectral content of the graph is limited to the first 15 eigenvectors of the graph Laplacian matrix. The observation noise in (5) is drawn from a zero-mean Gaussian distribution with a diagonal covariance matrix $\mathbf{C}_e = \sigma_e^2 \mathbf{I}$ with $\sigma_e^2 = 0.01$. We compare the results of our proposed algorithms with that of the graph LMS algorithm [12], and extended proportionate-type LMS [20]. In all simulations, we have compared the results of the Pt-GLMS algorithm with two settings: (i) in the first case, we set the weights $g_i[n]$ according to (8), where in (10), we have calculated the function $F(s_i[n])$ as presented in [28], such that $F(s_i[n]) = \log(1 + ||s_i[n]||^2)$. Then, using (9) and (10), the weights are obtained. (ii) In the second case, the weights are calculated using the proposed algorithm (13).

Fig. 3 shows the transient behavior of the NMSD in (18) versus the iteration index for the number of previous time instants $K = 8$, number of CS measurements $M = 30$, number of samples $|S| = 20$, and step-size $\mu = 0.01$. Each point in the curves is the result of ensemble average over 50 independent simulations. It can be seen that the proposed proportionate-type LMS algorithms converge faster than the conventional LMS. Moreover, using the proposed algorithm for estimation.
Fig. 5. Transient NMSD behaviour for $K = 8$, $|S| = 20$, and different values of $M$.

Fig. 6. Transient NMSD of the adaptive algorithms for $K = 8$, $|S| = 20$, $M = 30$, and different values of $|F|$.

Fig. 7. Transient NMSD of the adaptive algorithms versus iteration index for $K = 8$, $|F| = 15$, $M = 30$, and different values of $|S|$.

Fig. 8. The NMSD versus iteration index for $M = 30$, $|S| = 30$, and different values of $\mu$.

Fig. 9. The simulated and theoretical steady-state NMSD learning curves of the proposed Pt-GELMS algorithm for $M = 30$, and $|S| = 20$.

of gain matrices lead to a further increase in the convergence rate.

In Fig. 4, the transient behaviour of the NMSD versus the iteration index are shown for $M = 30$, step-size $\mu = 0.01$, and for different values of $K$. The curves are averaged over 50 independent trials. It can be seen that increasing the value of the parameter $K$, which incorporates further previous signals in estimating the current signal, results in higher convergence speed of the algorithms.

Fig. 5 depicts the transient behaviour of the NMSD for LMS, FELMS, Pt-GLMS (with the proposed gain matrix), and Pt-GELMS algorithms, considering different number of CS measurements, i.e., different values of $M$, and $K = 8$, $|S| = 20$, and the step-size $\mu = 0.01$. The results are averaged over 50 independent trials. As can be seen from the curves, increasing the number of CS measurements, which means increasing the number of linear combination of signal ensembles in hand, leads to an increase in convergence rate and decrease in NMSD. Moreover, with the same number of measurements, the proposed algorithms perform better than their peer algorithms, i.e., Pt-GLMS outperforms LMS and Pt-GELMS outperforms FELMS.

In Fig. 6, the transient behaviour of the NMSD for LMS, FELMS, Pt-GLMS (with the proposed gain matrix), and Pt-GELMS algorithms are presented, considering three
different values of bandwidth, $|F|$, and $K = 8$, $|S| = 20$, $M = 30$, and the step-size $\mu = 0.01$. The results are averaged over 50 independent trials. As can be seen from the curves, increasing the bandwidth of the signal spectrum, leads to an increase in convergence rate and decrease in NMSD.

In Fig. 7, the transient behaviour of the NMSD for LMS, FELMS, Pt-GLMS (with the proposed gain matrix), and Pt-GELMS algorithms are presented, considering three different number of samples selected, $|S|$, and $K = 8$, $|F| = 15$, $M = 30$, and the step-size $\mu = 0.01$. The results are averaged over 50 independent trials. As can be seen from the curves, increasing number of nodes selected in the sampling procedure, leads to an increase in convergence rate and decrease in NMSD. As seen, both the proposed Pt-GELMS and Pt-GLMS algorithms outperforms their non-proportionate type peers, i.e., FELMS and LMS, respectively. Furthermore, in all cases, the Pt-GLMS with the proposed gain matrix perform better than the Pt-GLMS with conventional gain matrix used in the literature.

To verify our theoretical results, we compare the NMSD for the proposed Pt-GELMS algorithm with $K = 8$, $|S| = 30$, $|F| = 15$, and $M = 30$, and for different values of $\mu$, as shown in Fig. 8. This figure shows that when $\mu$ is larger than $\frac{2}{\max_{n,m}|A(n,m)|}$, the convergence does not occur. In Fig. 9, the theoretical and simulated steady-state NMSD for the proposed Pt-GELMS algorithm with $K = 4, 8$ are depicted for $M = 30$, and $|S| = 20$. This figure shows a good match between theoretical and simulated steady-state NMSD values.

One of the applications of graph signal processing is in the analysis of temperature signals. For example, if a number of sensors are out of order, it is possible to estimate the temperature of those sensors by using GSR methods. The GSR is also used to predict the temperature. Thus, in the second scenario, for assessing the performance of the adaptive algorithms with real-world data, we consider the temperature graph signal. In this case, we use the dataset downloaded from the Intel Berkeley Research lab (refer to [10], [11], [30]), in which the temperature values from a total of $N = 54$ sensors are acquired. We aim to estimate the temperature values using the competing adaptive methods. Similar to the first case, we add an observation noise which is a zero-mean Gaussian noise with a diagonal covariance matrix $C_e = \sigma_e^2 I$ with $\sigma_e^2 = 3$ (which yields $SNR \sim 25$ dB). The transient NMSD versus the iteration index is depicted in Fig. 10 for three different numbers of selected nodes ($M = 30, 40, 54$), where we have used $K = 6$. The figures show the superiority of the proposed algorithm in comparison to the other adaptive algorithms.

For an intuitive comparison, in Fig. 11, the results about the temperature over nodes are given for $M = 40$ and $|S| = 30$. It can be observed that the proposed algorithms can estimate the true signal very well.

VII. CONCLUSION

In this paper, we formulated the proportionate adaptive graph signal recovery algorithm. Hence, the proportionate adaptive filtering algorithm in the classical signal processing is generalized to proportionate adaptive GSR algorithm in GSP. Two proportionate-type algorithms are proposed for adaptive GSR, of which the first is the Pt-GLMS algorithm and the second is the Pt-GELMS algorithm. In two algorithms, the gain matrix (or matrices) are obtained optimally in a closed form via minimizing the GMSD. Some theoretical analysis of the proposed algorithms such as mean-square convergence analysis, mean performance analysis, and steady-state performance analysis are also presented. Simulation results in both synthetic and real data, demonstrate the faster convergence of the proposed proportionate-type algorithms in comparison to non-proportionate-type counterparts.

APPENDIX A

PROOF OF GAIN MATRIX FOR Pt-GLMS

The GMSD at time $n$ can be written as

$$\Delta[n] = E[\|\tilde{s}[n+1]\|^2_Q - E[\|\tilde{s}[n]\|^2_Q].$$

Moreover, from (7), we have

$$\tilde{s}[n+1] = \tilde{s}[n] - G[n]m[n],$$

where $m[n] = \mu A^H[n](y[n] - A[n]\tilde{s}[n])$. Thus, we have

Fig. 10. Adaptive algorithms performance when estimating the temperature data from [30].

Fig. 11. The result about the temperature over nodes for $M = 40$ and $|S| = 30$. 

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∥\tilde{s}[n + 1]\|^2_Q = ∥\tilde{s}[n] - G[n]m[n]∥^2_Q
= \tilde{s}^T[n]A^H[n]A[n]\tilde{s}[n]
- 2m^T[n]G^T[n]A^H[n]A[n]\tilde{s}[n]
+ m^T[n]G^T[n]A^H[n]A[n]G[n]m[n] \quad (44)

It can be seen that the first term in the above equation is ∥\tilde{s}[n]\|^2_Q. Thus, after some algebraic effort, we obtain

∥\tilde{s}[n + 1]\|^2_Q = ∥\tilde{s}[n]\|^2_Q
- 2\sum_{k=1}^{N} \tilde{s}_k[n] \sum_{j=1}^{M} \sum_{i=1}^{N} g_i[n]m_i[n]a_{ji}[n]a_{jk}[n]
+ M \left( \sum_{i=1}^{N} g_i[n]m_i[n]a_{ji}[n] \right)^2. \quad (45)

where

\[ m_i[n] = \mu A_i^T[n](y[n] - A[n]s[n]). \]

Thus, we have

\[ \Delta[n] = f_1[n] - 2f_2[n], \]

(47)

where

\[ f_1[n] = E \left[ \sum_{j=1}^{M} \left( \sum_{i=1}^{N} g_i[n]m_i[n]a_{ji}[n] \right)^2 \right] \]

\[ = E \left[ \sum_{i=1}^{N} g_i^2[n]m_i^2[n] \sum_{j=1}^{M} a_{ji}^2[n] \right] + \sum_{i=1}^{N} g_i[n]m_i[n] \sum_{j=1}^{N} g_k[n]m_k[n]a_{ji}[n]a_{jk}[n]. \]

(48)

\[ f_2[n] = \sum_{k=1}^{N} \tilde{s}_k[n] \sum_{j=1}^{M} \sum_{i=1}^{N} g_i[n]m_i[n]a_{ji}[n]a_{jk}[n]
= E \left[ \sum_{i=1}^{N} g_i[n]m_i[n]\tilde{s}_i^T[n]A_i^T[n]A_i[n] \right]. \]

(49)

It can be seen that we can calculate the GMSD if \( f_1[n] \) and \( f_2[n] \) are given. On the other hand, since the signal error vector \( \tilde{s} \) is not available, we can not compute \( f_2[n] \) directly. To solve this problem, we rewrite \( f_2[n] \) as

\[ f_2[n] = E \left[ \sum_{i=1}^{N} g_i[n]m_i[n](A[n]\tilde{s}[n])^T A_i[n] \right] \]

\[ = E \left[ \sum_{i=1}^{N} g_i[n]m_i[n](A[n](s^o - s[n]))^T A_i[n] \right] \]

\[ = E \left[ \sum_{i=1}^{N} g_i[n]m_i[n](y[n] - e[n] - A[n]s[n])^T A_i[n] \right]. \]

(50)

Assuming that the noise \( e[n] \) is independent of the other processes, we have

\[ f_2[n] = \mu E \left[ \sum_{i=1}^{N} g_i[n](y[n] - A[n]s[n])^T A_i[n]A_i^H[n](y[n] - A[n]s[n]) \right] - A[n]s[n] \]

\[ - \mu E \left[ \sum_{i=1}^{N} g_i[n]A_i[n]C \right]. \]

(51)

Thus, the GMSD becomes as

\[ \Delta[n] = E \left[ \sum_{i=1}^{N} g_i^2[n]m_i^2[n] \sum_{j=1}^{M} a_{ji}^2[n] \right] + \sum_{i=1}^{N} g_i[n]m_i[n] \sum_{j=1}^{N} g_k[n]m_k[n]a_{ji}[n]a_{jk}[n]. \]

\[ - 2\mu E \left[ \sum_{i=1}^{N} g_i[n](y[n] - A[n]s[n])^T A_i[n]A_i^H[n](y[n] - A[n]s[n]) \right] + 2\mu E \left[ \sum_{i=1}^{N} g_i[n]A_i[n]C \right]. \]

(52)

The optimum gain for node \( i \) at time instance \( n \) (i.e., \( g_i[n] \)) can be found by setting the derivative of \( \Delta[n] \) with respect to \( g_i[n] \) to zero and obtain

\[ g_i[n] = \frac{f_4[n]}{f_4[n]} \]

(53)

where

\[ \hat{e}[n] = y[n] - A[n]s[n] \]

\[ f_3[n] = \mu \left[ \hat{e}^T[n]A_i[n]A_i^H[n]\hat{e}[n] - A_i[n]C \right]. \]

(54)

\[ f_4[n] = m_i^2[n] \sum_{j=1}^{M} a_{ji}^2[n]. \]

(56)

**APPENDIX B**

**PROOF OF GAIN MATRIX FOR Pt-GELMS**

Using (20), we have

\[ ∥\tilde{s}[n + 1]\|^2_Q = ∥\tilde{s}[n] - G[n]m_1[n] - Hm_2[n]\|^2_Q \]

\[ = \tilde{s}^T[n]A^H[n]A[n]\tilde{s}[n]
- 2m_1^T[n]G^T[n]A^H[n]A[n]\tilde{s}[n]
- 2m_2^T[n]H^T[n]A^H[n]A[n]m_2[n]
+ m_1^T[n]G^T[n]A^H[n]A[n]G[n]m_1[n]
+ m_2^T[n]H^T[n]A^H[n]A[n]H[n]m_2[n]
+ 2m_1^T[n]G^T[n]A^H[n]A[n]H[n]m_2[n]. \]

(57)
where
\[
m_1[n] = \mu (A^H[n]A[n]s[n] - A^H[n]e[n]), \tag{58}
\]
\[
m_2[n] = \mu \sum_{j=1}^{K-1} A^H[n-j]A[n-j]s[n] + \mu \sum_{j=1}^{K-1} A^H[n-j]e[n-j]. \tag{59}
\]

It is evident that the first term in (57) is \(\|s[n]\|^2_Q\). Thus, after some algebraic effort, the GMSD is obtained as
\[
\Delta[n] = E[\|s[n]\|^2_Q - E[\|s[n]\|^2_Q]
\]
\[
= -2E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} \hat{s}_k[n]g_i[n]m_{1,i}[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
- 2E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} \hat{s}_k[n]h_i[n]m_{2,i}[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
+ 2E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} m_{2,k}[n]h_k[n]m_{1,i}[n]g_i[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
+ E \left( \sum_{i=1}^{N} g_i[n]m_{1,i}[n]a_{j,i}[n] \right)^2
\]
\[
+ E \left( \sum_{i=1}^{N} h_i[n]m_{2,i}[n]a_{j,i}[n] \right)^2
\]
\[
= r_1[n] - 2r_2[n]. \tag{60}
\]
where \(m_{1,i}[n]\) and \(m_{2,i}[n]\) denote the \(i\)th element of \(m_1[n]\) and \(m_2[n]\), respectively, and
\[
r_1[n] = 2E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} m_{2,k}[n]h_k[n]m_{1,i}[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
+ E \left( \sum_{i=1}^{N} g_i[n]m_{1,i}[n]a_{j,i}[n] \right)^2
\]
\[
+ E \left( \sum_{i=1}^{N} h_i[n]m_{2,i}[n]a_{j,i}[n] \right)^2,
\]
\[
r_2[n] = E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} \hat{s}_k[n]g_i[n]m_{1,i}[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
+ E \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{N} \hat{s}_k[n]h_i[n]m_{2,i}[n]a_{j,k}[n]a_{j,i}[n]
\]
\[
= E \left( \sum_{i=1}^{N} g_i[n]m_{1}[n] + h_i[n]m_{2}[n] \right) s^T[n] A^H[n] A_i[n]. \tag{61}
\]

Similar to Section III-A, the signal error vector \(\hat{s}[n]\) is not available, and thus, \(r_2[n]\) is not computable. Thus, we solve this problem using the following relations
\[
\hat{A}[n] = A(s^n - s[n]) = y[n] - e[n] - A[n]s[n], \tag{62}
\]
which yields
\[
r_2[n] = E \left( \sum_{i=1}^{N} (A[n]s[n])^T A_i[n] \right)
\]
\[
= \mu E \left[ \sum_{i=1}^{N} g_i[n] (y[n] - A[n]s[n])^T A_i[n] \right]
\]
\[
+ \mu E \left[ \sum_{i=1}^{N} h_i[n] (y[n] - A[n]s[n])^T A_i[n] \right]
\]
\[
- \mu E \left[ \sum_{i=1}^{N} g_i[n] A_i[n] C e A_i^H[n] \right]
\]
\[
- \mu E \left[ \sum_{i=1}^{N} h_i[n] A_i[n] C e \sum_{j=1}^{K-1} A_j^H[n-j] \right]. \tag{63}
\]

Substituting the above equation in (61) and setting \(\frac{\partial \Delta[n]}{\partial g_i[n]} = 0\) and \(\frac{\partial \Delta[n]}{\partial h_i[n]} = 0\) yields
\[
g_i[n] = \frac{r_3[n]}{r_4[n]}, \tag{64}
\]
and
\[
h_i[n] = \frac{r_5[n]}{r_6[n]}, \tag{65}
\]
where
\[
r_3[n] = - m_{1,i}[n] \sum_{k=1}^{N} h_k[n] m_{2,k}[n] a_{j,k}[n] a_{j,i}[n]
\]
\[
- m_{1,i}[n] \sum_{j=1}^{M} g_k[n] m_{1,k}[n] a_{j,k}[n] a_{j,i}[n]
\]
\[
+ \mu (y[n] - A[n]s[n])^T A_i[n] (y[n] - A[n]s[n])
\]
\[
- \mu A_i[n] C e A_i^H[n], \tag{66}
\]
\[
r_4[n] = m_{2,i}[n] \sum_{j=1}^{N} a_{j,i}[n], \tag{67}
\]
\[
r_5[n] = - m_{2,i}[n] \sum_{k=1}^{N} g_k[n] m_{1,k}[n] a_{j,k}[n] a_{j,i}[n]
\]
\[
- m_{2,i}[n] \sum_{j=1}^{M} h_k[n] m_{2,k}[n] a_{j,k}[n] a_{j,i}[n]
\]
which happens if the condition holds, which holds true for any step-sizes $\bar{C}$ is equivalent to $\mu = (\text{conv})$ converges to a finite value, which results is stable, $\|s[n]\|$ is the initial condition. First, it

that the matrix $Q$ tends to zero asymptotically, and thus, the convergence of the second term to a finite value relies on the term $\sum_{l=0}^{n-1} \bar{Q}^l$, which happens if the matrix $Q$ is stable [12]. Indeed, for a stable $Q$, the right hand side of (70) converges to a finite value, which results in the convergence of $E[\|s[n+1]\|_p^2]$ to a steady-state value. On the other hand, it is clear from (36) that the matrix $Q$ is stable if $I - \mu B_4[n]$ is stable, which happens if the condition $\|I - \mu B_4[n]\|_2 < 1$ holds, which holds true for any step-sizes satisfying (39).

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1Since from (36) we have $Q = (I - \mu B_4[n]) \otimes (I - \mu B_4[n])$