Brane-Worlds and their Deformations

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A geometric theory of brane-worlds with large or non-compact extra dimensions is presented. It is shown that coordinate gauge independent perturbations of the brane-world correspond to the Einstein-Hilbert dynamics derived from the embeddings of the brane-world. The quantum states of a perturbation are described by Schrödinger’s equation with respect to the extra dimensions and the deformation Hamiltonian. A gauge potential with confined components is derived from the differentiable structure of the brane-world.
I. INTRODUCTION

Recent theoretical and phenomenological arguments have suggested a unification model, with large extra dimensions at TeV scale, including quantum gravity. Standard model gauge fields and ordinary matter remain confined to the four dimensional space-time, but the quantum gravitational field propagates in a region of the high dimensional space \( V_{n+1} \). Similar concepts involving trapped gauge and fermionic fields in a fluctuating space-time have appeared repeatedly in the literature, some of them based on specific models like in \( \text{(3-5)} \) while others are based on different blends of Kaluza-Klein and strings ideas \( \text{(6-11)} \). To make a theory, the brane worlds proposition requires a description of the space-time or brane-world embedded in a region of a higher dimensional manifold, capable of undergo quantum fluctuations and remain confined to the four dimensional space-time, but capable of undergo quantum fluctuations and their deformations.

We start in the next section reviewing the simpler example of hypersurface deformations. In section III it is shown that the perturbative analysis is equivalent to a dynamical process resulting from the Einstein-Hilbert principle. This is generalized to multiple parameter deformations of submanifolds in section IV. The quantum state of a deformation using Schrödinger’s principle. This is generalized to multiple parameter deformations of submanifolds in section IV. The topological changes associated with these deformations are also discussed. The paper ends showing that the differentiable structure of the space-time. The purpose of this paper is to show that these properties derive from the geometry of Riemannian submanifolds and their deformations. We start in the next section reviewing the simpler example of hypersurface deformations. In section III it is shown that the perturbative analysis is equivalent to a dynamical process resulting from the Einstein-Hilbert principle. This is generalized to multiple parameter deformations of submanifolds in section IV. The topological changes associated with these deformations are also discussed. The paper ends showing that the differentiable structure of the space-time. The purpose of this paper is to show that these properties derive from the geometry of Riemannian submanifolds and their deformations.

II. DEFORMATIONS OF A HYPERSURFACE

The following results extends Nash’s perturbation theorem of hypersurfaces in Euclidean spaces to a \( n \)-dimensional space-time \( V_n \) embedded in a \( (n+1) \)-dimensional manifold \( V_{n+1} \). It is also similar to the deformation of a 3-dimensional hypersurface in space-time within the context of canonical gravity \( \text{(8-13)} \). The reader who is already familiar with these techniques may jump to the next section where these deformations are related to a non constrained dynamics.

The isometric embedding of a background space-time \( V_n \) with metric \( \bar{g}_{ij} \) is given by the map \( \bar{X} : V_n \rightarrow V_{n+1} \) such that

\[
\bar{X}^\mu_i \bar{X}^\nu_j G_{\mu\nu} = \bar{g}_{ij}, \quad \bar{X}^\mu_i \bar{\eta}^\nu_j G_{\mu\nu} = 0, \quad \bar{\eta}^\mu_i \bar{\eta}^\nu_j G_{\mu\nu} = \varepsilon
\]

where \( G_{\mu\nu} \) denotes the metric of \( V_{n+1} \) in arbitrary coordinates and \( \bar{\eta}^\mu \) denotes the vector normal to \( V_n \) with signature \( \varepsilon/|\varepsilon| = \pm 1 \). A geometric deformation of \( V_n \) along a direction \( \zeta \) is the subset of \( V_{n+1} \) described by the coordinates \( \mathcal{Z}^\mu \) given by the perturbation of the embedding vielbein \( \bar{X}^\mu_i \):

\[
\mathcal{Z}^\mu_i(x', s) = \bar{X}^\mu_i(x) + s \mathcal{L}_\zeta \bar{X}^\mu_i \quad \eta = \bar{\eta} + s \mathcal{L}_\zeta \bar{\eta}
\]

Since the space-time \( V_n \) is endowed with a general diffeomorphism group, it is always possible to find a coordinate system in which the above Lie derivative of the tangent component vanish. This undesirable coordinate gauge should be excluded from the deformations. In an example of a homogeneous elastic membrane, the tangent component of the deformation tension is canceled by the assumption that it is homogeneous and invariant under the rotation group of the tangent space. The consequence is that the fundamental modes of oscillations are described by the deformations of the membrane along the orthogonal direction. In complete analogy with this example, we take the fundamental modes of a deformations as a rules of orthogonal deformations of a hypersurface (also called pure deformations \( \text{(4)} \)). Thus, taking \( \zeta = \eta \) in \( \text{(4)} \) we obtain

\[
\mathcal{Z}^\mu_i(x, s) = \bar{X}^\mu_i(x) + s \eta^\mu_i(x).
\]

The embedding of the deformed hypersurface \( V_n \) is given by

\[
g_{ij} = Z^\mu_i Z^\nu_j G_{\mu\nu}, \quad \eta^\mu_i \eta^\nu_j G_{\mu\nu} = 0, \quad \eta^\mu_i \eta^\nu_j G_{\mu\nu} = \varepsilon
\]

Denoting by \( \bar{k}_{ij} = -X^\mu_i \bar{\eta}^\nu_j G_{\mu\nu} \) the extrinsic curvature of \( V_n \), and using \( \text{(6)} \), the deformed metric \( g_{ij} \) can be written as

\[
g_{ij} = Z^\mu_i Z^\nu_j G_{\mu\nu} = \bar{g}_{ij} - 2s \bar{k}_{ij} + s^2 \bar{g}^m n \bar{k}_{im} \bar{k}_{jn}.
\]

and the extrinsic curvature of the deformation \( V_n \) can be written as

\[\text{(5)}\]

1 All Greek indices run from 1 to \( n+1 \) in this and the next section and from 1 to \( D \) in the remaining of the paper. Small case Latin indices \( i, j, k... \) run from 1 to \( n \). An overbar denotes an object of the background space-time. The covariant derivative with respect to the metric of the higher dimensional manifold is denoted by a semicolon and \( \eta^\mu_i = \eta^\mu_i X^\mu_i \) denotes its projection over \( V_n \). The curvatures of the higher dimensional space are distinguished by a calligraphic \( \mathcal{R} \).

2
\[ k_{ij} = -Z^\nu_{ij} \eta^\mu \Gamma_{\mu \nu \rho} + \ddot{k}_{ij} - s \dot{g}^{mn} \kappa_{im} \kappa_{jn} \]  

Comparing (6) and (7), we obtain York’s relation, describing the metric evolution with respect to the deformation parameter \( s \).

\[ \frac{dg_{ij}}{ds} = g_{ij} = -2k_{ij}. \]  

(8) Notice that the inverse of (6) cannot be calculated exactly in arbitrary dimensions. To obtain the contravariant version of (8), consider the matrix notations \( \bar{g} = (\bar{g}_{mn}) \), \( g = (g_{mn}) \) and \( k = (k_{mn}) \). Then the inverse metric can be given to any order \( (k) \) of approximation as

\[ (k)^{-1} = \left( \sum_{n=0}^{k} (-1)^{n} k^{n} \right)^{2} \bar{g}^{-1}, \quad g \bar{g}^{-1} \approx 1 + 0(k^{k+1}) . \]

Using this and \( \dot{g}_{im} g_{mj} + g_{im} \dot{g}_{mj} = 0 \) and defining \( (k)^{(k)} \nabla_{ij} = \frac{1}{2} \dot{g}^{ij} \), we have

\[ (k)^{(k)} = \dot{g}^{im} \dot{g}^{jn} k_{mn} \]

Since this is true for all values of \( (k) \), it follows that

\[ k_{ij} = + \frac{1}{2} \dot{g}^{ij} \]

Consequently, the indices of the extrinsic curvature are lowered and risen \( g_{ij} \) and \( g^{ij} \) respectively, in accordance with (8):

\[ g_{im} g_{jn} k_{mn} = g_{im} g_{jn} \frac{1}{2} \dot{g}^{mn} = \frac{1}{2} \frac{d}{ds} (g_{im} g_{jn} g^{mn}) - \frac{1}{2} \dot{g}_{im} \dot{g}_{jn} g^{mn} - \frac{1}{2} g_{im} g_{jn} g^{mn} = k_{ij} . \]

### III. DYNAMICS OF DEFORMATIONS

From (8) it follows that

\[ g^{ij} Z^\mu_{ij} Z^\nu_{ij} \Gamma_{\mu \nu \rho} = n \quad \text{and} \quad g^{ij} Z^\mu_{ij} Z^\nu_{ij} \eta^\alpha \Gamma_{\mu \nu \alpha} = 0 \]

Therefore, the quantity \( \xi^{\mu \nu} = g^{ij} Z^\mu_{ij} Z^\nu_{ij} \) cannot be proportional to \( \Gamma^{\mu \nu} \). Writing \( \xi^{\mu \nu} = \Gamma^{\mu \nu} + \Psi^{\mu \nu} \), then \( \Psi^{\mu \nu} \) satisfy the conditions

\[ G_{\mu \nu} \Psi^{\mu \nu} = -1 \quad \text{and} \quad \Psi^{\mu \nu} \eta_{\mu} \eta_{\nu} = -\varepsilon \]

The solution of these equations, compatible with (8) is \( \Psi^{\mu \nu} = -\varepsilon \eta^\mu \eta^\nu / \varepsilon \), so that

\[ g^{ij} Z^\mu_{ij} Z^\nu_{ij} = \Gamma^{\mu \nu - \frac{1}{\varepsilon} \eta^\mu \eta^\nu} \]

(10) The stability of the brane-world with respect to the deformation means that the integrability conditions for the embedding

\[ R_{ijkl} = \frac{2}{\varepsilon} k_{ij}(k_{jk})_{k} + R_{\alpha \beta \gamma \delta} Z^\alpha_{ij} Z^\beta_{jk} Z^\gamma_{kl} Z^\delta_{lj} \]  

(11) must be satisfied. Using (10), the contractions of (11) and (12) give

\[ R_{jk} = \frac{1}{\varepsilon}(k^i_k k^j_j - h k_{jk}) - R_{\beta} Z^\beta_{ij} Z^\gamma_{kl} \]

(13) and

\[ \kappa^k_{i,j} - h_{i} = R_{\alpha \beta} Z^\alpha_{ij} \eta^\beta \]

(15) where we have denoted by \( h = g^{ij} k_{ij} \) the mean curvature of \( V_n \), and \( \kappa^k = k^i_{ij} \).

The Einstein-Hilbert Lagrangian for \( G_{\mu \nu} \) derived directly from (14) is

\[ \mathcal{L} = R \sqrt{\bar{g}} = \left[ R - \frac{1}{\varepsilon}(k^2 - h^2) + \frac{2}{\varepsilon} R_{\alpha \beta} \eta^\alpha \eta^\beta \right] \sqrt{\bar{g}} \]

The last term in this Lagrangian may be calculated in the Gaussian normal frame of the deformation, where \( \eta^\mu = \delta^\mu_{n+1} \):

\[ R_{\alpha \beta} \eta^\alpha \eta^\beta = \Gamma^\alpha_{n+1, \alpha} - \Gamma^\alpha_{n+1, n+1, \alpha} + \Gamma^\beta_{n+1, \alpha} \Gamma^\alpha_{\beta n+1} - \Gamma^\beta_{n+1, \alpha} \Gamma^\alpha_{\beta n+1} \Gamma^\alpha_{\beta n+1} = \kappa^2 - \dot{h} \]

(16) Here the dot means derivative with respect to \( s \). Using (8) it also follows that

\[ \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial k_{ij}} = 2k^{ij} - \dot{g}^{ij} = 0 \]

Therefore, after removing the surface term, the Lagrangian reduces to the well-known expression

\[ \mathcal{L} = \left[ R + \frac{1}{\varepsilon}(k^2 + h^2) \right] \sqrt{\bar{g}} \]

(17) The sequence is similar to the standard 3 + 1 space-time decomposition, except that here we are using an internal parameter. The momentum canonically

2 For generality of signature, we denote \( \sqrt{\bar{g}} \) as meaning \( \sqrt{-g_{\mu \nu}} \), where \( g = \det(g_{ij}) \), when \( V_n \) has Euclidean signature and meaning \( \sqrt{-g_{\mu \nu}} \) when \( V_n \) has Minkowski signature.
conjugated to $G_{\alpha \beta}$, with respect to the deformation parameter $s$ is defined by

$$p^{\alpha \beta} = \frac{\partial \mathcal{L}}{\partial (G_{\alpha \beta})}$$

Using (13), the momentum components corresponding to $g_{ij}$ are

$$p^{ij} = \frac{1}{\varepsilon} (k^{ij} + hg^{ij}) \sqrt{G}$$

Notice that the compatibility between (8) and (9) and the tensor algebra of $V_n$ requires that $p_{ij} = -\partial \mathcal{L}/\partial \dot{g}^{ij}$. Since the deformation does not prescribe the evolution of $G_{i,n+1}$ and $G_{n+1,n+1}$, the corresponding momenta do not follow from York’s relation and their values are set as constraints to the deformations:

$$p^{i,n+1} = -2 \frac{\partial R_{\alpha \beta}^i \eta^{\alpha \beta}}{\partial G_{i,n+1}} \sqrt{G} = 0$$

$$p^{n+1,n+1} = -2 \frac{\partial R_{\alpha \beta}^{n+1} \eta^{\alpha \beta}}{\partial G_{n+1,n+1}} \sqrt{G} = 0$$

The constraint (14) is equivalent to the fact that the deformation does not have tangent components, a condition already imposed to guarantee the coordinate gauge independence. Equation (19) corresponds to the fact that the evolution of the system is measured by the parameter $s$ alone, not depending on the variations of $g_{n+1,n+1}$. Using the definition $p = G_{\alpha \beta} p^{\alpha \beta}$, (13) and (20), we obtain the expressions

$$k_{ij} = \frac{\varepsilon}{\sqrt{G}} \left( \frac{pg_{ij}}{n+1} - p_{ij} \right), \quad h = \frac{-\varepsilon p}{(n+1)\sqrt{G}}$$

In addition, $k^2 + h^2 = -(\eta^{2i} + p^{ij}p_{ij})/G$. The deformation Hamiltonian is obtained from the usual Legendre transformation

$$\mathcal{H} = p^{\alpha \beta} \dot{G}_{\alpha \beta} - \mathcal{L} = -R \sqrt{G} - \frac{\varepsilon}{G} \left( \frac{p^2}{n+1} - p_{ij}p^{ij} \right) \sqrt{G}$$

and Hamilton’s equations with respect to $s$ are

$$\dot{g}^{ij} = \frac{2\varepsilon}{\sqrt{G}} \left( \frac{p}{n+1} g^{ij} - p^{ij} \right) \sqrt{G},$$

$$\dot{p}^{ij} = -(R^{ij} - \frac{1}{2} Rg^{ij}) \sqrt{G}$$

$$+ \frac{1}{\sqrt{G}} \left[ pp^{ij} - 2p^{im}p_{mj} + \frac{1}{2} (p^2 - p_{mn}p_{mn}) g^{ij} \right]$$

The equation in $g_{ij}$ coincides with (8) expressed in terms of $p_{ij}$. It follows that the deformation originally described as a perturbative process is an exact dynamical process governed by the Einstein-Hilbert principle.

Hypersurface deformations do not depend on the compactness of the extra dimension. In principle we could apply it to implement the Randall-Sundrum model [1], taking $V_{n+1}$ to be a 5-dimensional AdS manifold. In this case the extra dimensional slice is determined by the limits in $s$ imposed by the regularity of (6). This will be discussed in the next sections together with the quantum deformations.

**IV. BRANE-WORLD DEFORMATIONS**

A single extra degree of freedom is not compatible with the full development of brane-world theory, especially in what concerns quantum gravity and the spin-statistics theorem [2], and with the standard model of gauge interactions. In this section we will see that the geometric geometry of gauge field requires at least two extra dimensions. A given background space-time $V_n$ may be locally embedded into a manifold $V_D$, with sufficiently large dimension $D$ and with metric signature $(P,Q)$. The number of extra dimensions $N = D - n$ depends on the geometries of $V_n$ and of $V_D$, and of course of the embedding map. The best known examples are those of space-times isometrically embedded in a flat space $M_D$, where the embedding is given by analytic functions $\tilde{A}$. The analyticity simplifies some embedding results but it imposes a maximum embedding dimension for all four-dimensional space-times to be $D = 10$. Since it is not obvious that the analyticity will hold at TeV scale excitations, we may assume that the deformed manifolds remain (at least) differentiable. In this case, the limit dimension for flat embeddings rises to $D = 14$, with a wide range of compatible signatures [3]. However, in general $V_D$ is not flat and its actual dimension and signature depends on the geometry imposed on $V_D$. In agreement with the Kaluza-Klein principle, $V_D$ is taken to be a solution of the higher dimensional Einstein’s equations, with or without a positive or negative cosmological term. The multiparameter deformation is a straightforward generalization of (3) and (4), extended to the $N = D - n$ directions orthogonal to the background space-time:

$$Z^\mu = \tilde{X}^\mu + s^A \eta_A^\mu$$

$$\eta_A^\mu = \tilde{\eta}_A^\mu + s^{ij}[\tilde{\eta}_A, \tilde{\eta}_B]$$

With the condition that each $\eta_A$ is independent of the other, the Lie bracket in the last expression vanishes so that it simplifies to $\eta_A = \tilde{\eta}_A$. Notice however that there is a degree of freedom in the choice of the orthogonal basis $\{\tilde{\eta}_A\}$, given by the isometry group of the internal
space $B_N$ tangent to the normal vectors $\bar{\eta}_A$. This symmetry will impact on the brane-worlds gauge structure.

The embedding of the deformed submanifold is now given by

$$Z^\mu_i Z^\nu_j \eta_{ij} = g_{ij}, \quad Z^\mu_i \eta_{ij} \eta^\nu = g_{ij}, \quad \eta^\mu_i B \eta^\nu = g_{AB}$$

(26)

where $g_{AB}$ denotes the metric of the internal space $B_N$ with tangent vectors $\eta_A$. The cross terms are

$$\eta_{ijA} = s^M A_{iMA},$$

(27)

which appear only when we have two or more extra dimensions.

The extrinsic curvatures are defined for each direction $\eta_A$ as

$$\kappa_{ij} = -Z^\mu_i \eta^\nu \eta_{ij} \eta_{ij} \eta_{ij}$$

(28)

From (24) we obtain

$$g_{ij} = \eta_{ij} - 2s^A \kappa_{ij} + s^A S^B (g_{mn} \kappa_{im} A_{jnB} + g_{MN} A_{iMA} A_{jNB})$$

and

$$\kappa_{ij} = \kappa_{ij} - s^B (g_{mn} \kappa_{im} A_{jnB} + g_{MN} A_{iMA} A_{jNB})$$

(29)

The derivative of $g_{ij}$ with respect to $s^A$ gives a generalization of (30), describing the brane metric evolution along the extra dimension $\eta_A$

$$\partial g_{ij} = -2\kappa_{ij}$$

(30)

The mean curvature is also defined for each normal direction as $h_A = g^{ij} \kappa_{ij} A$ and its norm is

$$h^2 = g^{AB} h_A h_B.$$

As in the case of hypersurfaces, we apply the integrability conditions for the embedding, required to guarantee that the deformation is again an embedded submanifold:

$$\kappa_{ijkl} = 2g^{MN} \kappa_{[kMA} A_{jNB]} + \nabla_{\mu} R_{ijkl} + \kappa_{ijkl}$$

where we have denoted $\kappa_{ij} = \kappa_{ij}$. Using the extended Riemann normal coordinates defined by the deformed submanifold and $\eta_A = \delta^\mu_i$, it follows that the last term vanishes and that

$$g^{AB} R_{\mu \nu} \eta_A^\mu \eta_B^\nu = -g^{AB} \frac{\partial h_A}{\partial s^B} + \kappa^2$$

Therefore,

$$R = R - (\kappa^2 + h^2) - 2g^{AB} \frac{\partial h_A}{\partial s^B}$$

After discarding the hypersurface terms, the Einstein Hilbert Lagrangian for $V_D$ becomes remarkably simple

$$\mathcal{L}(g, g_A) = \left[ R + (\kappa^2 + h^2) \right] \sqrt{g}$$

(31)

The momentum canonically conjugated to the metric with respect to the normal direction $\eta_A$ is

$$p_{(A)}^\alpha = \frac{\partial \mathcal{L}}{\partial \partial \eta^\alpha}$$

In particular, using (24) we obtain the components

$$p_{ij}^{(A)} = -(\kappa^2 + h A g^{ij}) \sqrt{g}$$

As before, the remaining components are taken as momentum constraints

$$p_{i}^{(A)} = 0$$

(35)

$$p_{i}^{(C)} = 0$$

(36)

We notice also that the compatibility of (24) with the tensor algebra of $V_n$ requires the definition of the contravariant momentum components to be

$$p_{ij}^{(A)} = -\partial \mathcal{L}/\partial \eta^i_j A = -\left(\kappa_{ij} + h A g^{ij} \right) \sqrt{g}$$

Denoting $p_A = g^{ij} p_{ij}^{(A)}$, $p^A = g^{AB} p_B$ we obtain $h_A = -p_A / (n + 1) \sqrt{g}$ and defining the orthogonal momentum norm $p^2 = g^{AB} p_A p_B$, it follows that

$$h^2 = \frac{p^2}{(n + 1)^2 g}, \quad \kappa^2 = \frac{1}{g} \left[ \frac{(n + 2)}{(n + 1)^2} p^2 + p_{ij}^{(A)} p_{ij}^{(A)} \right]$$

The Hamiltonian of the deformation is defined by the Legendre transformation

$$\mathcal{H} = \sum_{A=n+1}^{D} p_{ij}^{(A)} g_{ij} - \mathcal{L} =$$

$$-R \sqrt{g} - \frac{1}{g} \left( \frac{p^2}{n + 1} - \sum_{A=n+1}^{D} p_{ij}^{(A)} p_{ij}^{(A)} \right)$$

(37)

Except for the signature and the number of dimensions this is the same Hamiltonian (21), leading to similar Hamilton’s equations, one for each internal index ($A$)
\[
\frac{d\delta g_{ij}}{ds^A} = \frac{\delta H}{\delta p_{(A)}^{ij}}, \quad \frac{d\delta p_{(A)}^{ij}}{ds^A} = \frac{\delta H}{\delta g_{ij}}
\]

The first of these equations coincide with (24) and the second equation expresses the variation of the extrinsic curvature in terms of the momentum. The conclusion is the same as before: The deformation of a brane-world can be derived from the Einstein-Hilbert action for the metric of \( V_D \). The difference is that now the internal symmetry has to be taken into account.

V. QUANTUM BRANE-WORLD DEFORMATIONS

One of the basic requirements of brane-world is that the extra dimensions are accessed by quantum fluctuations of the space-time geometry. The meaning of quantum geometry is not at all clear as it requires the adaptation of concepts that are typical of physics to geometry. For example, in such discipline we could talk about an atom (or quantum) of length, area, connection and curvature. Going further we may speculate on the quantum topology of a manifold and its consequences, regardless if this is a metric topology or not.

In brane-worlds the metric is also a physical field and the quantum geometry comes after the quantization of the metric. However, the quantization of the relativistic gravitational field as an isolated system has proven to be a difficult subject. In fact, the gravitational field is a prime example of a constrained canonical system, to which Dirac’s standard procedure for such systems could be applied. As it happens, the requirements of diffeomorphism invariance of the theory implies that the propagation of the Poisson’s bracket structure does not produce the expected results, except possibly using preferred frames.

However, with brane-worlds the basic requirement is that of a quantum fluctuation of the space-time, prior to any consideration on the second quantization of the metric field. The suggestion comes from the compactification hypothesis introduced by O. Klein, to make Kaluza’s theory consistent with quantum mechanics. Accordingly, all functions are harmonically expanded in terms of the internal parameters. Therefore, these discrete internal modes would correspond to the quantum modes with respect to the internal parameters, as if they were internal times. In brane-worlds such discrete modes are more difficult to realize, firstly because the internal space is not necessarily compact and secondly because of the hierarchical distinction between the gravitational and gauge interactions. Nonetheless, it is possible to quantize the brane embedding map with respect to the internal parameters. In fact, we have seen that the deformations of the space-time submanifold geometry with respect to the internal parameters correspond to the deformation Hamiltonian. Since the internal parameters do not share the same diffeomorphism of the space-time coordinates, we may use Schrödinger’s picture to describe the quantum wave function of a deformation along a direction \( \eta_A \) as a solution of

\[
-\imath \hbar \frac{d\Psi_A}{ds^A} = \hat{H}\Psi_A
\]

where the operator \( \hat{H} \) is constructed with (27). The probability of a deformation being in a state \( \Psi_A \) is given by \( |\Psi_A|^2 = \int \Psi_A^\dagger \Psi_A dv \), with the integral extending over the volume of the deformed region in \( V_D \). The superposition of quantum deformations states defined in the same region of \( V_n \) is given by \( \Psi = \sum \Psi_A \). Classically, this superposition correspond to a deformed submanifold along a direction \( \zeta = \sum A s^A \eta_A \), with norm \( |\zeta|^2 = \sum g_{AB} s^A s^B = s^2 \). The classical deformation along the unit direction \( \eta = \zeta/s \) is given by

\[
Z^\mu_{\bar{s}} = \bar{X}^\mu_{\bar{s}} + s\eta^\mu = \bar{X}^\mu_{\bar{s}} + \sum A s^A \eta^\mu_{A,\bar{s}}
\]

(39)

Given two deformations \( \Psi_A \) and \( \Psi_B \), the transition probability between them is given by the Hilbert product \( \langle \Psi_A, \Psi_B \rangle \). The interpretations of the classical limit of the manifold quantization depends on the signature of \( V_D \). For example, if \( \eta_A \) and \( \eta_B \) are both space-like, then \( \langle \Psi_A, \Psi_B \rangle \) corresponds in the classical limit to a space-like handle. On the other hand, if \( \eta_A \) and \( \eta_B \) have both time-like signatures the classical limit of \( \langle \Psi_A, \Psi_B \rangle \) corresponds to a classical loop involving two internal time parameters. In this case, a deformation along \( \eta_A \) with evolution scale \( s^A \) has a transition to \( \eta_B \) with different scale \( s^B \) as if an internal time machine. Finally, if \( \eta_A \) and \( \eta_B \) have different signatures, then the transition probability \( \langle \Psi_A, \Psi_B \rangle \) corresponds to a signature change. An example is given by the Kruskal space-time regarded as a deformation of the Schwarzschild space-time, in such a way that the latter becomes geodesically complete. These spaces are both minimally embedded in six dimensional spaces but with different signatures \((5, 1)\) and \((4, 2)\) respectively. Since the Schwarzschild space-time is a subset of Kruskal space-time, we cannot have both space-times in the same fixed embedding space. However, they can be considered as classical limits of a quantum deformation of the Schwarzschild space-time with a signature transition at the horizon. This agrees with the known fact that those solutions have different topologies.

Equation (35) is constructed with the Einstein-Hilbert action and therefore it may be referred to as a first quantization of the space-time geometry where the expectation value of the metric is given by \( \langle \Psi | g_{ij} | \Psi \rangle \). The second quantization of the metric as a field can be adapted from many of the current ideas to the brane-world configuration. In one example we may regard the gravitational field as an effective field theory in \( D \) dimensions, where \( V_D \) is seen as the space of all
deformed metrics, in which the effective Planck mass is taken as the regularization mass \[23\].

VI. CONFINED GAUGE FIELDS

As we have seen, equations \[30\] are responsible for the stability of submanifold structure under classical deformations. Therefore, these equations should also say about the confinement of the gauge interactions. In fact, the basic field variables in \[30\] are \(g_{ij}, \kappa_{ijA}\) and \(A_{iAB}\). The first two vary with the deformation and they are related by \[29\]. On the other hand, \(A_{iAB} = -A_{iBA}\) and

\[
A_{iAB} = \tilde{\eta}^\mu_{iB}\tilde{\eta}^\nu_A g_{\mu\nu} = \tilde{\eta}^\mu_{iB}\tilde{\eta}^\nu_A g_{\mu\nu} = \tilde{A}_{iAB} \tag{40}
\]

so that \(A_{iAB}\) does not propagate with \(s^A\). It was shown elsewhere the relevant property that \(A_{iAB}\) transforms like the components of a gauge field with respect to the group of isometries of \(B_N\) \[21,24\]. Actually, \(A_{iAB}\) is a Yang-Mills potential in the Einstein-Yang-Mills equations for a deformed metric with respect to the mentioned group of isometries. To see this, consider the metric of \(V_D\) written in the Gaussian normal frame of the deformed submanifold, with separate components

\[
g_{ij} = Z_i^\mu Z_j^\nu g_{\mu\nu} = \bar{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B (\bar{g}^{mn}\bar{k}_{imA}\bar{k}_{jnB} + \bar{g}^{MN}A_{iMA}A_{jNB})
\]

\[
g_{ij} = Z_i^\mu Z_j^\nu g_{\mu\nu} = \bar{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B \bar{g}^{MN}A_{iMA}A_{jNB}
\]

\[
g_{ij} = Z_i^\mu Z_j^\nu g_{\mu\nu} = \bar{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B \bar{g}^{MN}A_{iMA}A_{jNB}
\]

or, after denoting

\[
\bar{g}_{ij} = \bar{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A s^B \bar{g}^{MN}A_{iMA}A_{jNB} \tag{41}
\]

\[
A_{iA} = s^M A_{iMA} \tag{42}
\]

we may write the metric of \(V_D\) as

\[
\bar{g}_{\alpha\beta} = \begin{pmatrix}
\bar{g}_{ij} + g^{MN}A_{iM}A_{jN} & A_{iA} \\
A_{iB} & g_{AB}
\end{pmatrix}
\]

This has the same appearance as the Kaluza-Klein metric ansatz, with the exception that the cross terms are determined by \(A_{iAB}\). As with Kaluza-Klein theory, the Einstein-Hilbert Lagrangian derived from \[33\] is

\[
\mathcal{L} = R\sqrt{\bar{g}} = R\sqrt{\bar{g}} + \frac{1}{4} trF^2\sqrt{\bar{g}} + \text{eventual extra terms} \tag{44}
\]

where we have denoted \(\epsilon = det(g_{AB})\), \(F^2 = F_{ij}F^{ij}\) and \(F_{ij} = [D_i, D_j]\) where \(D_i = \partial_i + A_i\), with the connection \(A_i\) given by

\[
A_i = s^M A_{iMA}K^A
\]

Here \(\{K^A\}\) denotes the Killing basis of the Lie algebra of the group of isometries of the metric \(g_{AB}\).

The Einstein-Yang-Mills equations derived from the above Lagrangian provide the dynamics of the confined gauge fields. In addition it provides the necessary relationship between the internal parameters and the space-time geometry.

The range of the internal parameters \(s^A\) depends on the curvature of \(V_n\). This can be seen after writing \[11\] as

\[
\bar{g}_{ij} = \bar{g}^{mn}(\bar{g}_{im} - s^A\bar{k}_{imA})(\bar{g}_{jn} - s^B\bar{k}_{jnB})
\]

Since the determinant equation \(det(\bar{g}_{ij} - s^A\bar{k}_{ij}) = 0\) defines the manifold of curvature centers of \(\bar{V}_n\), the values of \(s^A\) are limited by this manifold, which is a singular and forbidden region for \[13\]. Denoting by \(\bar{V}_n\) the closest regular manifold to that manifold, the parameters \(s^A\) are restricted to the region of the higher dimensional space sandwiched between the background \(V_n\) and \(\bar{V}_n\). This provides an explanation for the thickness hypothesis in the 5-dimensional example in \[2\].

We would like to end with a comment on the dimension of the brane-world. The higher dimensional space \(V_D\) may be thought of as composed of substructures of different degrees of complexity, which are submanifolds of dimension \(n\), where the value of \(n\) can be anything from 0 to \(D\). The simplest case \(n = 0\) corresponds to point particles. With \(n = 1\), we may formulate a classical string theory. For \(n \geq 2\) we obtain surfaces and in general \(n\)-branes. In this last case the internal space play a more significant role. In fact, \[34\] are differentiable equations, which can be solved without appeal to the analyticity of the embedding functions. If so, then the limiting dimension changes to \(D = n(n + 3)/2\), so that \(n^2 + n - 2N \geq 0\). Taking the Standard model gauge group \(SU(3) \times SU(2) \times U(1)\) acting on a seven dimensional projective space, the nearest integer dimension is \(n = 4\). More appropriately, we should look for the GUT group which also acts as the group of isometries of \(B_N\). The \(SO(10)\) model gives exactly the value \(n = 4\), suggesting a particular fourteen dimensional model with signature \((11,3)\) and with \(SO(10)\) as the gauge group.

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