GEGENBAUER-CHEBYSHEV INTEGRALS AND RADON TRANSFORMS

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Abstract. We suggest new versions of Helgason’s support theorems and related characterizations of the kernel (the null space) for the classical hyperplane Radon transform and its dual, the totally geodesic transforms on the sphere and the hyperbolic space, the spherical slice transform, and the spherical mean transform for spheres through the origin. The assumptions for functions are close to minimal and formulated in integral terms. The proofs rely on projective equivalence of these transforms and new facts for the Gegenbauer-Chebyshev fractional integrals.

1. Introduction

The Gegenbauer-Chebyshev integrals generalize Abel type operators of fractional integration in analysis. The history and basic properties of these integrals are described in van Berkel [7], van Berkel and van Eijndhoven [8], Samko, Kilbas, and Marichev [51, Chapter 7].

According to Ludwig [34, p. 50], who referred to the private communication by L. Sarason, the connection between the Gegenbauer-Chebyshev integrals and the Radon transform on $\mathbb{R}^n$ was known to Lax and Phillips [29]. In the case $n = 2$, this connection was independently discovered by Cormack [16]; see also the later publications by Cormack and Quinto [17], Deans [19, Chapter 7], Helgason [26, Chapter I, Section 2], Natterer [37, p. 25].

Apart of explicit inversion formulas on the corresponding subspaces spanned by spherical harmonics, the Gegenbauer-Chebyshev integrals encode information about the kernel (the null space) of the Radon transform and its dual. Moreover, a simple unilateral structure of the Gegenbauer-Chebyshev integrals and their inverses can be used to retrieve information about the support of a function from the information about the support of the Radon transform of that function. The last observation is intimately related to the celebrated Helgason’s support

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theorem which states that if the Radon transform of a function \( f \) on \( \mathbb{R}^n \) vanishes on all hyperplanes at distance \( > A \) from the origin, then \( f(x) = 0 \) for all \( |x| > A \) provided that \( f \in C(\mathbb{R}^n) \) and \( |x|^m |f(x)| \) is bounded for every integer \( m > 0 \). This remarkable result, which extends to arbitrary convex sets, dates back to Helgason’s 1963 address [23]; it was mentioned in [24, p. 438] and presented with detailed proof in [25]; see also [27, p. 10]. The original Helgason’s proof was beautiful and did not use Gegenbauer-Chebyshev integrals.

An important question is whether it is possible to weaken the assumptions in Helgason’s support theorem.

This question and related problems for some other important Radon-like transforms are studied in the present paper. We establish new facts for the Gegenbauer-Chebyshev integrals and use these facts to prove new kernel and support theorems for several Radon-like transforms in integral geometry. The latter include the classical hyperplane Radon transform and its dual, the totally geodesic transforms on the sphere and the hyperbolic space, the spherical slice transform, and the spherical mean transform for spheres through the origin. The basic idea is to fully use the unilateral structure of the Radon transform which is especially transparent on subspaces generated by spherical harmonics.

The distinctive feature of our treatment is that unlike the aforementioned publications, where the functions are smooth and rapidly decreasing, we deal with arbitrary locally integrable functions satisfying certain minimal integral conditions.

Plan of the paper, main results, and comments.

Section 2 contains necessary preliminaries. Section 3 is devoted to the Gegenbauer-Chebyshev integrals. In Section 4 we revise known facts related to the action of the Radon transform and its dual on the subspaces generated by spherical harmonics and prove the corresponding kernel and support theorems. The main results are stated in Theorems 4.5, 4.11, 4.13 and 4.14.

Some comments are in order. The results of Section 4 trace back to the aforementioned works by Helgason and Ludwig; see also Helgason [26, 27]. Regrettfully, some important justifications in [34] are skipped. For example, it is not explained how to choose the functions \( \psi_{j,\ell}(s) \) in the proof of Theorem 4.2; see [34, p. 65]. This theorem plays a crucial role in the subsequent reasoning in that paper. We circumvent this difficulty and suggest a completely different approach which invokes the Semyanistyi-Lizorkin spaces of Schwartz functions orthogonal to all polynomials.

One should also mention the papers by Perry [40] and Quinto [43] about the kernel of the exterior Radon transform and related singular
value decompositions. A nice discussion of Helgason’s support theorem is presented by Quinto [44]; see also Boman [9, 10, 11, 12], Boman and Quinto [13].

The question about the kernel of the Radon transform \( f \rightarrow Rf \) is closely connected with non-injectivity of this operator when \( f \) does not belong to \( L^p(\mathbb{R}^n) \) with \( 1 \leq p < n/(n-1) \) (otherwise, \( R \) is injective). According to Armitage and Goldstein [3], there exists a nonconstant harmonic function \( h \) on \( \mathbb{R}^n, n \geq 2 \), such that \( \int_P |h| < \infty \) and \( \int_P h = 0 \) for every \((n-1)\)-dimensional hyperplane \( P \). This result amounts to Zalcman [56] for \( n = 2 \). Another construction, similar to Zalcman’s, but more elementary, was suggested by Armitage [2]; see also Helgason [27, p. 19]. Our results give more non-injectivity examples and describe the kernel of \( R \) explicitly in the wide class of functions satisfying minimal integrability conditions.

The description of the kernel of our Radon-like operators is given in terms of the Fourier-Laplace coefficients. This approach seems to be universal. Equivalent statements in any topological space containing finite linear combinations of spherical harmonics can be obtained by taking the closure in the corresponding topology.

Section 5 is devoted to the spherical mean transform that assigns to a function \( f \) on \( \mathbb{R}^n \) the integrals of \( f \) over spheres passing through the origin. In the case \( n = 2 \), this transform was introduced by Cormack [16] who obtained a formal inversion formula in terms of the Fourier series. A similar inversion problem for spheres through the origin in \( \mathbb{R}^n, n \geq 3 \) odd, was studied by Chen [14, 15] and Rhee [45, 46] in connection with the Darboux equation. Their consideration relies on certain paraboloidal means. The case of all \( n \geq 2 \) was investigated by Cormack and Quinto [17]. These authors used spherical harmonic expansions, the link with the dual Radon transform in \( \mathbb{R}^n \), and the relevant results of Ludwig [34]; see also Quinto [42, 43, 44] and Solmon [54, p. 340]. Our treatment of this class of operators (see Theorems 5.4, 5.5) also relies on the connection with the Radon transform, but the method is different, the exposition is self-contained, and functions under consideration may not be smooth.

A similar work has been done in Sections 6, 7, and 8 for the Funk transform on the unit sphere \( S^n \), the corresponding spherical slice transform for geodesic spheres through the north pole, and the totally geodesic Radon transform on the \( n \)-dimensional real hyperbolic space.

The name spherical slice transform was adopted by Helgason for the Funk-type transform previously studied by Abouelaz and Daher [1] on zonal functions. In the case of the 2-sphere in \( \mathbb{R}^3 \) it was proved
(see Helgason [27, p. 145]) that there is a link between the spherical slice transform and the Radon transform over lines in the 2-plane. We extend Helgason’s idea to the \( n \)-dimensional case for all \( n \geq 2 \) and combine it with the corresponding statements from Section 4. As a result, the smoothness conditions in [27] are replaced by the less restrictive integral conditions and the kernel of the spherical slice transform is described; see Theorems 7.10 and 7.10.

We conclude this discussion by noting that the idea of the projective equivalence of some Radon-like transforms is not new; cf. [21]. The link between the Radon transform and the corresponding transforms on other constant curvature spaces was used by Kurusa [28] to transfer Helgason’s support theorem from \( \mathbb{R}^n \) to the sphere and the hyperbolic space; see also [4, 5]. Our formulas are different and the functions may not be smooth. Moreover, we describe the kernel of the corresponding operators and give examples of their non-injectivity.

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2. Preliminaries

2.1. Notation. In the following \( \mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C} \) are the sets of all integers, positive integers, real numbers, and complex numbers, respectively; \( \mathbb{Z}_+ = \{ j \in \mathbb{Z} : j \geq 0 \} \); \( \mathbb{R}_+ = \{ a \in \mathbb{R} : a > 0 \} \); \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) is the unit sphere in \( \mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n \), where \( e_1, \ldots, e_n \) are the coordinate unit vectors. For \( \theta \in S^{n-1} \), \( d\theta \) denotes the surface element on \( S^{n-1} \), \( \sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \) - the surface area of \( S^{n-1} \). We set \( d\star \theta = d\theta/\sigma_{n-1} \) for the normalized surface element on \( S^{n-1} \).

The letter \( c \) denotes an inessential positive constant which may vary at each occurrence. Dealing with integrals, we say that an integral exists in the Lebesgue sense if it is finite when the expression under the sign of integration is replaced by its absolute value.

2.2. Gegenbauer and Chebyshev polynomials. The Gegenbauer polynomials \( C^\lambda_m(t) \) form an orthogonal system in the weighted space \( L^2([-1, 1]; w_\lambda), w_\lambda(t) = (1-t^2)^{\lambda-1/2}, \lambda > -1/2 \). In the case \( \lambda = 0 \), they are usually substituted by the Chebyshev polynomials \( T_m(t) \). Below we review some properties of the polynomials \( C^\lambda_m(t) \) and \( T_m(t) \).

For \( |t| \leq 1 \) and \( \lambda > -1/2 \),

\[
|C^\lambda_m(t)| \leq c \left\{ \begin{array}{ll}
1, & \text{if } m \text{ is even,} \\
|t|, & \text{if } m \text{ is odd,}
\end{array} \right.
\quad c \equiv c(\lambda, m) = \text{const.} \quad (2.1)
\]
The same inequality holds for $T_m(t)$; cf. 10.9(18) and 10.11(22) in [20].

The following equalities for the Mellin transforms are simple consequences of 47(1) and 48(4) from [35, Sec. 10 (10)]. Let $\eta = 0$ if $m$ is even and $\eta = 1$ if $m$ is odd,

$$c_{\lambda,m} = \frac{\Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda)}, \quad \lambda > -1/2, \quad \lambda \neq 0. \quad (2.2)$$

Then\(^1\)

$$\alpha_m(z) \equiv \int_0^1 u^{z-1}(1-u^2)^{\lambda-1/2} C_m^\lambda(u) \, du \quad (2.3)$$

$$= \frac{c_{\lambda,m} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\lambda + \frac{z+1+m}{2}\right) \Gamma\left(\frac{z+1-m}{2}\right)}, \quad Re z > -\eta; \quad (2.4)$$

$$\beta_m(z) \equiv \int_0^1 u^{z-1}(1-u^2)^{\lambda-1/2} C_m^\lambda(1/u) \, du \quad (2.5)$$

$$= \frac{c_{\lambda,m} \Gamma\left(\frac{z-m}{2}\right) \Gamma\left(\lambda + \frac{z+m}{2}\right)}{\Gamma\left(\lambda + \frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)}, \quad Re z > m. \quad (2.6)$$

These formulas can be equivalently written in a different form; see 2.21.2(5) and 2.21.2(25) in [41]. Similarly, by 18(1) and 19(4) from [35, Sec. 10 (10)], we have

$$\int_0^1 u^{z-1}(1-u^2)^{-1/2} T_m(u) \, du = \frac{\pi^{1/2} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)}{2 \Gamma\left(\frac{z+1+m}{2}\right) \Gamma\left(\frac{z+1-m}{2}\right)}, \quad (2.6)$$

$$\int_0^1 u^{z-1}(1-u^2)^{-1/2} T_m(1/u) \, du = \frac{\pi^{1/2} \Gamma\left(\frac{z-m}{2}\right) \Gamma\left(\frac{z+m}{2}\right)}{2 \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)}, \quad (2.7)$$

where $Re z > -\eta$ and $Re z > m$, respectively.

\(^1\)If $m$ is odd, then $\eta = 1$ and (2.4) is understood for $-1 < Re z \leq 0$ by continuity (the same for (2.6)).
2.3. Riemann-Liouville and Erdélyi-Kober Fractional Integrals.

We briefly review some facts from [49, 51]. For a function \( f \) on \( \mathbb{R}_+ \), the fractional integrals of the Riemann-Liouville type are defined by

\[
(I^\alpha_+ f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s) \, ds}{(t-s)^{1-\alpha}}, \quad (I^\alpha_- f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s) \, ds}{(s-t)^{1-\alpha}},
\]

where \( t > 0 \) and \( \alpha > 0 \). Both integrals are one-sided. Hence, the behavior of \( f(s) \) is irrelevant for \( s \to \infty \) in \( I^\alpha_+ f \) and \( s \to 0 \) in \( I^\alpha_- f \). One integral can be obviously expressed through another if the argument of \( f \) is replaced by its reciprocal.

**Lemma 2.1.** Let \( a > 0 \). If

\[
\int_a^\infty |f(s)| \, s^{\alpha-1} \, ds < \infty,
\]

then \( (I^\alpha_- f)(t) \) is finite for almost all \( t > a \). If \( f \) is non-negative, locally integrable on \([a, \infty)\), and (2.8) fails, then \( (I^\alpha_- f)(t) = \infty \) for every \( t \geq a \).

**Proof.** The first statement is a consequence of the inequality

\[
\int_a^b (I^\alpha_- |f|)(t) \, dt < \infty \quad \forall \ a < b < \infty.
\]

The latter can be checked by changing the order of integration. To prove the second statement, we assume the contrary, that is, \( (I^\alpha_- f)(t) \) is finite, but (2.8) fails. Let first \( \alpha \leq 1 \). Then for any \( N > t \),

\[
\int_t^N \frac{f(s) \, ds}{(s-t)^{1-\alpha}} > \int_t^N \frac{f(s) \, ds}{s^{1-\alpha}} = \left( \int_a^N - \int_a^t \right) \frac{f(s) \, ds}{s^{1-\alpha}}.
\]

If \( N \to \infty \), then, by the assumption, the left-hand side remains bounded whereas the right-hand side tends to infinity. If \( \alpha > 1 \), we proceed as follows. Fix any \( b > t \). Then, for any \( N > 0 \),

\[
\int_t^{2b+N} \frac{f(s) \, ds}{(s-t)^{1-\alpha}} > \int_t^{2b+N} \frac{f(s) \, ds}{(s-t)^{1-\alpha}} > 2^{1-\alpha} \int_{2b}^{2b+N} \frac{f(s) \, ds}{s^{1-\alpha}}
\]

(note that \( s-t > s-b > s/2 \)). The rest of the proof is as before. \( \square \)

The corresponding operators \( D^\alpha_{\pm} \) of fractional differentiation are defined as left inverses of \( I^\alpha_{\pm} \), so that \( D^\alpha_{\pm} I^\alpha_{\pm} f = f \). The operators \( D^\alpha_{\pm} \) may
have different analytic forms. For example, if $\alpha = m + \alpha_0$, $m = \lfloor \alpha \rfloor$ (the integer part of $\alpha$), $0 \leq \alpha_0 < 1$, then

$$D_{\pm}^{\alpha} \varphi = (\pm d/dt)^{m+1} I_{\pm}^{1-\alpha_0} \varphi. \quad (2.9)$$

The equality $D_{\pm}^{\alpha} I_{\pm}^{\alpha} f = f$ must be justified at each occurrence.

The Erdélyi-Kober type fractional integrals are defined by

$$\left( I_{+,-}^{\alpha_{+,-},2} f \right)(t) = \frac{2}{\Gamma(\alpha)} \int_0^t \frac{f(s) s \, ds}{(t^2-s^2)^{1-\alpha}}, \quad (I_{+,-}^{\alpha_{+,-},2} f)(t) = \frac{2}{\Gamma(\alpha)} \int_0^\infty \frac{f(s) s \, ds}{(s^2-t^2)^{1-\alpha}}, \quad (2.10)$$

so that $I_{+,-}^{\alpha_{+,-},2} f = A^{-1} I_{\pm}^{\alpha_{+,-},2} f$ where $(Af)(t) = f(\sqrt{t})$. The following statement follows from Lemma 2.1.

**Lemma 2.2.** Let $a > 0$. If

$$\int_a^\infty |f(s)| \, s^{2\alpha-1} \, ds < \infty, \quad (2.11)$$

then $(I_{+,-}^{\alpha_{+,-},2} f)(t)$ is finite for almost all $t > a$. If $f$ is non-negative, locally integrable on $[a, \infty)$, and (2.11) fails, then $(I_{+,-}^{\alpha_{+,-},2} f)(t) = \infty$ for every $t \geq a$.

Fractional derivatives $D_{\pm}^{\alpha_{+,-}}$ of the Erdélyi-Kober type are defined as the left inverses of $(I_{\pm}^{\alpha_{+,-},2})^{-1}$.

**Theorem 2.3.** [49] Let $\varphi = I_{+,-}^{\alpha_{+,-},2} f$, where $f$ satisfies (2.11) for every $a > 0$. Then $f(t) = (D_{\pm}^{\alpha_{+,-},2} \varphi)(t)$ for almost all $t \in \mathbb{R}_{+}$, where $D_{\pm}^{\alpha_{+,-}} \varphi$ has one of the following forms.

(i) If $\alpha = m$ is an integer, then

$$D_{\pm}^{\alpha_{+,-}} \varphi = (-D)^{m} \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}. \quad (2.12)$$

(ii) If $\alpha = m + \alpha_0$, $m = \lfloor \alpha \rfloor$, $0 \leq \alpha_0 < 1$, then

$$D_{\pm}^{\alpha_{+,-}} \varphi = t^{2(1-\alpha+m)} (-D)^{m+1} t^{2m} \psi, \quad \psi = I_{+,-}^{1-\alpha+m} t^{-2m-2} \varphi. \quad (2.13)$$

Alternatively,

$$D_{\pm}^{\alpha_{+,-}} \varphi = 2^{-2\alpha} D_{\pm}^{2\alpha} t I_{+,-}^{\alpha_{+,-},1} t^{-2\alpha-1} \varphi, \quad (2.14)$$

where $D_{\pm}^{2\alpha}$ denotes the Riemann-Liouville derivative of order $2\alpha$, which can be computed according to (2.9).

(iii) If, moreover, $\int_1^\infty |f(t)| \, t^{2m+1} \, dt < \infty$, then

$$D_{\pm}^{\alpha_{+,-}} \varphi = (-D)^{m+1} I_{+,-}^{1-\alpha+m} \varphi. \quad (2.15)$$
The powers of \( t \) in this theorem denote the corresponding multiplication operators. An advantage of the inversion formula (2.14) in comparison with (2.12), (2.13), and (2.15), is that it employs the derivative \( d/dt \) rather than \( D = (2t)^{-1}d/dt = d/dt^2 \).

2.4. A Simple Lemma. The following lemma, which connects the integration over \( S^{n-1} \subset \mathbb{R}^n \) with the integration over the coordinate hyperplane \( \mathbb{R}^{n-1} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-1} \), is useful in different occurrences.

**Lemma 2.4.**

(i) If \( f \in L^1(S^{n-1}) \), then

\[
\int_{S^{n-1}} f(\theta) \, d\theta = \int_{\mathbb{R}^{n-1}} \left[ f\left( \frac{x + e_n}{|x + e_n|} \right) + f\left( \frac{x - e_n}{|x - e_n|} \right) \right] \frac{dx}{|x + e_n|^n}. \tag{2.16}
\]

In particular, if \( f \) is even, then

\[
\int_{S^{n-1}} f(\theta) \, d\theta = 2 \int_{\mathbb{R}^{n-1}} f\left( \frac{x + e_n}{|x + e_n|} \right) \frac{dx}{|x + e_n|^n}. \tag{2.17}
\]

(ii) Conversely, if \( f \in L^1(\mathbb{R}^{n-1}) \), \( S^+_{n-1} = \{ \theta \in S^{n-1} : \theta_n > 0 \} \), then

\[
\int_{\mathbb{R}^{n-1}} f(x) \, dx = \int_{S^+_{n-1}} f\left( \frac{\theta'}{\theta_n} \right) \frac{d\theta}{\theta_n^n} \quad \theta' = (\theta_1, \ldots, \theta_{n-1}). \tag{2.18}
\]

**Proof.** The slice integration yields

\[
\int_{S^{n-1}} f(\theta) \, d\theta = \int_0^\pi \sin^{n-2} \varphi \, d\varphi \int_{S^{n-2}} f(\omega \sin \varphi + e_n \cos \varphi) \, d\omega.
\]

Set \( s = \tan \varphi \) on the right-hand side to obtain

\[
\int_0^\infty \frac{s^{n-2}}{(1 + s^2)^{n/2}} ds \int_{S^{n-2}} \left[ f\left( \frac{s\omega + e_n}{\sqrt{1 + s^2}} \right) + f\left( \frac{s\omega - e_n}{\sqrt{1 + s^2}} \right) \right] \, d\omega.
\]

This coincides with (2.16). Similarly,

\[
\int_{\mathbb{R}^{n-1}} f(x) \, dx = \int_0^\infty s^{n-2} ds \int_{S^{n-2}} f(s\omega) \, d\omega
\]

\[
= \int_0^{\pi/2} \frac{\sin^{n-2} \varphi}{\cos^n \varphi} d\varphi \int_{S^{n-2}} f\left( \frac{\omega \sin \varphi}{\cos \varphi} \right) \, d\omega = \int_{S^+_{n-1}} f\left( \frac{\theta'}{\theta_n} \right) \frac{d\theta}{\theta_n^n}.
\]

\( \square \)
The Radon Transforms. We recall some facts which are needed for our treatment. More information can be found, e.g., in [21, 22, 27, 37, 48, 49]. Let \( \Pi_n \) be the set of all unoriented hyperplanes in \( \mathbb{R}^n \). The Radon transform of a function \( f \) on \( \mathbb{R}^n \) is defined by the formula

\[
(Rf)(\xi) = \int_{\xi} f(x) \, dx, \quad \xi \in \Pi_n,
\]

provided that this integral exists. Here \( dx \) denotes the Euclidean volume element in \( \xi \). Every hyperplane \( \xi \in \Pi_n \) has the form \( \xi = \{ x : x \cdot \theta = t \} \) where \( \theta \in S^{n-1} \), \( t \in \mathbb{R} \). Thus, we can write (2.19) as

\[
(Rf)(\theta, t) = \int_{\theta^\perp} f(t\theta + u) \, du,
\]

where \( \theta^\perp = \{ x : x \cdot \theta = 0 \} \) is the hyperplane orthogonal to \( \theta \) and passing through the origin, \( du \) is the Euclidean volume element in \( \theta^\perp \).

We denote \( Z_n = S^{n-1} \times \mathbb{R} \). Clearly, \( (Rf)(\theta, t) = (Rf)(-\theta, -t) \) for every \( (\theta, t) \in Z_n \). Using (2.18) and assuming \( t \neq 0 \), one can also write (2.20) as an integral over the hemisphere:

\[
(Rf)(\theta, t) = |t|^{n-1} \int_{v\in S^{n-1} : v \cdot \theta > 0} f\left( \frac{tv}{v \cdot \theta} \right) \frac{dv}{(v \cdot \theta)^n},
\]

see also [37, p. 26]. If \( f \) is a radial function, that is, \( f(x) \equiv f_0(|x|) \), then \( (Rf)(\theta, t) \equiv F_0(t) \) where

\[
F_0(t) = \sigma_{n-2} \int_{|r|}^\infty f_0(r)(r^2 - t^2)^{(n-3)/2}rdr.
\]

The next theorem shows for which functions \( f \) the Radon transform \( Rf \) does exist (cf. [49, Theorem 3.2]).

**Theorem 2.5.** If

\[
\int_{|x|>a} \frac{|f(x)|}{|x|} \, dx < \infty \quad \forall a > 0,
\]

then \( (Rf)(\xi) \) is finite for almost all \( \xi \in \Pi_n \). If \( f \) is nonnegative, radial, and (2.23) fails for some \( a > 0 \), then \( (Rf)(\xi) \equiv \infty \).

The following equality is a particular case of [48, formula (2.19)]:

\[
\int_{Z_n} \frac{(Rf)(\theta, t)}{(1 + t^2)^{n/2}} \, d\theta dt = \int_{\mathbb{R}^n} \frac{f(x)}{(1 + |x|^2)^{1/2}} \, dx
\]
provided that the right-hand side exists in the Lebesgue sense.

The dual Radon transform takes a function \( \varphi(\theta, t) \) on \( Z_n \) to a function \( (R^* \varphi)(x) \) on \( \mathbb{R}^n \) by the formula

\[
(R^* \varphi)(x) = \int_{S^{n-1}} \varphi(\theta, x \cdot \theta) \, d_s \theta. \tag{2.25}
\]

The operators \( R \) and \( R^* \) can be expressed one through another.

**Lemma 2.6.** Let \( x \neq 0, \; t \neq 0, \)

\[
(A \varphi)(x) = \frac{1}{|x|^n} \varphi \left( \frac{x}{|x|}, \frac{1}{|x|} \right), \quad (B f)(\theta, t) = \frac{1}{|t|^n} f \left( \frac{\theta}{t} \right). \tag{2.26}
\]

The following equalities hold provided that the expressions on either side exist in the Lebesgue sense:

\[
(R^* \varphi)(x) = \frac{2}{|x| \sigma_{n-1}} (RA \varphi) \left( \frac{x}{|x|}, \frac{1}{|x|} \right), \tag{2.27}
\]

\[
(R f)(\theta, t) = \frac{\sigma_{n-1}}{2 |t|} (R^* B f) \left( \frac{\theta}{t} \right). \tag{2.28}
\]

**Proof.** The proof relies on Lemma 2.4. By (2.20),

\[
(RA \varphi)(\theta, t) = \int_{\theta^\perp} \varphi \left( \frac{t \theta + u}{|t \theta + u|}, \frac{1}{|t \theta + u|} \right) \frac{d_u u}{|t \theta + u|^n}
\]

\[
= \int_{\mathbb{R}^{n-1}} \varphi \left( \frac{\gamma(e_n + y)}{|e_n + y|}, \frac{1}{t|e_n + y|} \right) \frac{dy}{t|e_n + y|^n},
\]

where \( \theta = \gamma e_n, \; \gamma \in O(n) \). Setting \( \theta = x/|x|, \; t = 1/|x| \), we note that

\[
\frac{1}{t|e_n + y|} = |x| \left( e_n \cdot \frac{e_n + y}{|e_n + y|} \right) = |x| \left( \gamma e_n \cdot \frac{\gamma(e_n + y)}{|e_n + y|} \right) = x \cdot \frac{\gamma(e_n + y)}{|e_n + y|}.
\]

Hence,

\[
(RA \varphi) \left( \frac{x}{|x|}, \frac{1}{|x|} \right) = \int_{\mathbb{R}^{n-1}} \varphi \left( \frac{\gamma(e_n + y)}{|e_n + y|}, \frac{\gamma(e_n + y)}{|e_n + y|} \right) \frac{|x| \, dy}{|e_n + y|^n}.
\]

Now (2.17) yields

\[
(RA \varphi) \left( \frac{x}{|x|}, \frac{1}{|x|} \right) = \frac{|x| \sigma_{n-1}}{2} \int_{S^{n-1}} \varphi(\gamma \theta, x \cdot \gamma \theta) \, d_s \theta = \frac{|x| \sigma_{n-1}}{2} (R^* \varphi)(x),
\]

which gives (2.27). The second equality can be obtained from the first one if we change the notation and assume \( \varphi \) in (2.27) to be even. Here the cases \( t > 0 \) and \( t < 0 \) should be considered separately. \( \square \)
Theorem 2.5 combined with (2.27) gives the following

**Corollary 2.7.** If \( \varphi(\theta, t) \) is locally integrable on \( \mathbb{Z}_n \), then the dual Radon transform \((R^* \varphi)(x)\) is finite for almost all \( x \in \mathbb{R}^n \). If \( \varphi(\theta, t) \) is nonnegative, independent of \( \theta \), i.e., \( \varphi(\theta, t) \equiv \varphi_0(t) \), and such that
\[
\int_0^a \varphi_0(t) \, dt = \infty,
\]
for some \( a > 0 \), then \((R^* \varphi)(x) \equiv \infty\).

The following function spaces are important in the theory of Radon transforms. Let \( S(\mathbb{R}^n) \) be the Schwartz space of \( C^\infty \)-functions which together with their derivatives of all orders are rapidly decreasing. We supply \( S(\mathbb{R}^n) \) with the standard topology and denote by \( S'(\mathbb{R}^n) \) the corresponding space of tempered distributions. The following spaces were introduced by Semyanistyi [53] and extensively studied by Lizorkin [30]-[33]; see also Helgason [27] and Samko [50].

Let
\[
\Psi(\mathbb{R}^n) = \{ \psi(\theta, t) \in S(\mathbb{R}^n) : (\partial_j \psi)(0) = 0 \text{ for all } j \in \mathbb{Z}_n^+ \}.
\]
We denote by \( \Phi(\mathbb{R}^n) \) the Fourier image of \( \Psi(\mathbb{R}^n) \) and supply \( \Psi(\mathbb{R}^n) \) and \( \Phi(\mathbb{R}^n) \) with the topology of \( S(\mathbb{R}^n) \). The corresponding spaces of distributions are denoted by \( \Psi'(\mathbb{R}^n) \) and \( \Phi'(\mathbb{R}^n) \).

**Proposition 2.8.** Two \( S' \)-distributions that coincide in the \( \Phi' \)-sense differ from each other by a polynomial.

The analogues of the Semyanistyi-Lizorkin spaces for \( \mathbb{Z}_n = S^{n-1} \times \mathbb{R} \) are defined as follows. The derivatives of a function \( g \) on \( S^{n-1} \) will be defined as the restrictions onto \( S^{n-1} \) of the corresponding derivatives of \( \tilde{g}(x) = g(x/|x|) \), namely,
\[
(\partial^\alpha g)(\theta) = (\partial^\alpha \tilde{g})(x)|_{x=\theta}, \quad \alpha \in \mathbb{Z}_n^+, \quad \theta \in S^{n-1}.
\]  
(2.29)

We denote by \( S(Z_n) \) the space of all functions \( \varphi(\theta, t) \) on \( Z_n = S^{n-1} \times \mathbb{R} \), which are infinitely differentiable in \( \theta \) and \( t \) and rapidly decreasing as \( t \to \pm \infty \) together with all derivatives. The topology in \( S(Z_n) \) is defined by the sequence of norms
\[
||\varphi||_m = \sup_{|\alpha| + j \leq m} \sup_{\theta, t} (1 + |t|)^m |(\partial^\alpha_\theta \partial_j^t \varphi)(\theta, t)|, \quad m \in \mathbb{Z}_+.
\]  
(2.30)

The corresponding space of distributions is denoted by \( S'(Z_n) \). We set
\[
\Psi(Z_n) = \{ \psi(\theta, t) \in S(Z_n) : (\partial_j^t \psi)(\theta, 0) = 0, \quad \text{for all } \alpha \in \mathbb{Z}_n^+, \quad j \in \mathbb{Z}_+, \quad \theta \in S^{n-1} \},
\]
\[
\Phi(Z_n) = F_1 \Psi(Z_n) = \{ \varphi(\theta, t) \in S(Z_n) : \}
\]  
(2.31)
\[
\int_{-\infty}^{\infty} t^j (\partial_\theta^\alpha \partial_\phi^k) (\theta, t) \, dt = 0, \text{ for all } j \in \mathbb{Z}_+, \ \alpha \in \mathbb{Z}_n^+, \ k \in \mathbb{Z}_+, \ \theta \in S^{n-1} \}.
\]

Here \( F_1 \) denotes the one-dimensional Fourier transform in the \( t \)-variable. We supply \( \Psi(\mathbb{Z}_n) \) and \( \Phi(\mathbb{Z}_n) \) with the topology of the ambient space \( S(\mathbb{Z}_n) \). The corresponding spaces of distributions are denoted by \( \Psi'(\mathbb{Z}_n) \) and \( \Phi'(\mathbb{Z}_n) \). The notation \( S_e(\mathbb{Z}_n), \Psi_e(\mathbb{Z}_n), \text{ and } \Phi_e(\mathbb{Z}_n) \) is used for the corresponding spaces of even functions.

**Theorem 2.9.** [53, 27] The operator \( R \) is an isomorphism from \( \Phi(\mathbb{R}^n) \) onto \( \Phi_e(\mathbb{Z}_n) \). The operator \( R^* \) is an isomorphism from \( \Phi_e(\mathbb{Z}_n) \) onto \( \Phi(\mathbb{R}^n) \).

### 3. GEGENBAUER-CHEBYSHEV FRACTIONAL INTEGRALS

#### 3.1. The Right-sided Integrals.

In this section we consider the following integral operators on \( \mathbb{R}_+ \) indexed by \( \lambda > -1/2 \) and a nonnegative integer \( m \). Let first \( \lambda \neq 0 \). We set

\[
(G^\lambda_m f)(t) = \frac{1}{c_{\lambda,m}} \int_t^{\infty} (t^2 - r^2)^{\lambda - 1/2} C_m^\lambda \left( \frac{r}{t} \right) f(r) \, dr, \tag{3.1}
\]

\[
(G^\lambda_0 f)(t) = \frac{t}{c_{\lambda,0}} \int_t^{\infty} (t^2 - r^2)^{\lambda - 1/2} C_m^\lambda \left( \frac{r}{t} \right) f(r) \, \frac{dr}{r^{2\lambda+1}}, \tag{3.2}
\]

\[
c_{\lambda,m} = \frac{\Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda)}.
\]

In the cases \( m = 0 \) and \( m = 1 \), when \( C_0^\lambda(t) = 1 \) and \( C_1^\lambda(t) = 2\lambda t \), these operators are expressed through the Erdélyi-Kober type integrals (2.10) by the formulas

\[
G^\lambda_0 f = I_{-2}^{\lambda+1/2} f, \quad G^\lambda_1 f = t I_{-2}^{\lambda+1/2} t^{-1} f, \tag{3.4}
\]

\[
G^\lambda_0 f = t I_{-2}^{\lambda+1/2} t^{-2\lambda-2} f, \quad G^\lambda_1 f = t I_{-2}^{\lambda+1/2} t^{-2\lambda-1} f. \tag{3.5}
\]

Here, as usual, the powers of \( t \) denote the corresponding multiplication operators.

In the case \( \lambda = 0 \), when the Gegenbauer polynomials are substituted by the Chebyshev ones, we set

\[
(T_m^m f)(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} (r^2 - t^2)^{-1/2} T_m \left( \frac{t}{r} \right) f(r) \, dr, \tag{3.6}
\]
\[ (\mathcal{T}_m^*f)(t) = \frac{2t}{\sqrt{\pi}} \int_t^\infty (r^2 - t^2)^{-1/2} T_m \left( \frac{r}{t} \right) f(r) \frac{dr}{r}. \]  

(3.7)

As in (3.4) and (3.5).

\[ \mathcal{T}_0 f = I_{1/2}^{1/2} f, \quad \mathcal{T}_1 f = t I_{1/2}^{1/2} t^{-1} f, \]  

(3.8)

\[ \mathcal{T}_0^* f = t I_{1/2}^{1/2} t^{-2} f, \quad \mathcal{T}_1^* f = I_{1/2}^{1/2} t^{-1} f. \]  

(3.9)

We call (3.1)-(3.2) and (3.6)-(3.7) the right-sided Gegenbauer and Chebyshev fractional integrals, respectively. The next proposition contains information about the existence of these integrals.

**Proposition 3.1.** Let \( \eta = 0 \) if \( m \) is even and \( \eta = 1 \) if \( m \) is odd. Suppose that \( a > 0, \lambda > -1/2 \).

(i) If

\[ \int_a^\infty |f(t)| t^{2\lambda - \eta} dt < \infty, \]  

(3.10)

then \( (\mathcal{G}_{-m}^\lambda f)(t) \) is finite for almost all \( t > a \).

(ii) If

\[ \int_a^\infty |f(t)| t^{m-2} dt < \infty, \]  

(3.11)

then \( (\mathcal{G}_{m}^\lambda f)(t) \) is finite for almost all \( t > a \).

The case \( \lambda = 0 \) gives the similar statement for \( \mathcal{T}_{-m} f \) and \( \mathcal{T}_{m}^* f \).

**Proof.** (i) follows immediately from Lemma 2.2 and (2.1). To prove (ii), changing the order of integration, for any \( b \in (a, \infty) \) we have

\[
\int_a^b (|G_{-m}^\lambda f)(t)| dt \leq c \int_a^b \frac{dt}{t^2} \int_t^\infty (r^2 - t^2)^{-1/2} \left( \frac{r}{t} \right)^m |f(r)| \frac{dr}{r^{2\lambda + 1}}
\]

\[
\leq c \int_a^b \frac{|f(r)| dr}{r^{2\lambda + 1 - m}} \int_a^b (r^2 - t^2)^{-1/2} \frac{dt}{t^{m+2}}
\]

\[
\leq c \int_a^b \frac{|f(r)| dr}{r^{3}} \int_a^1 (1 - s^2)^{-1/2} \frac{ds}{s^{m+2}}
\]

\[
= c \int_a^b f(r) r^{m-2} \eta(r) dr, \quad \eta(r) = r^{-m-1} \int_a^1 (1 - s^2)^{-1/2} \frac{ds}{s^{m+2}}.
\]
Since the function \( \eta(r) \) is bounded, the result follows.

Remark 3.2. The conditions (3.10) and (3.11) are sharp. Suppose, for example, that \( m \) is even and let \( f_\varepsilon(t) = t^{-2\lambda-1+\varepsilon} \). Then (3.10) fails if \( f = f_\varepsilon \) with \( \varepsilon = 0 \). The Gegenbauer integral \( (G_{\lambda,m} f_\varepsilon)(t) \), which can be explicitly evaluated by (2.3) if \( \varepsilon < 0 \), does not exist for \( \varepsilon = 0 \) too. Other cases in Proposition 3.1 can be considered similarly.

Our main concern is the operators \( G_{\lambda,m} \) and \( T_m \), which play an important role in the study of Radon transforms. Below we discuss the injectivity of these operators and inversion formulas.

Lemma 3.3. Let \( \lambda > -1/2 \). If \( m = 0, 1 \), then \( G_{\lambda,m} \) is injective in the class of functions satisfying (3.10) for every \( a > 0 \). If \( m \geq 2 \), then \( G_{\lambda,m} \) is non-injective in this class of functions. Specifically, let \( f_k(t) = t^{-2\lambda-k-2} \), where \( k \) is a nonnegative integer such that \( m - k = 2, 4, \ldots \). Then \( (G_{\lambda,m} f_k)(t) = 0 \) for all \( t > 0 \). The case \( \lambda = 0 \) gives the similar statement for \( T_m f \).

Proof. The first statement is obvious from (3.4) and (3.8) thanks to the injectivity of the Erdélyi-Kober operators. In the case \( m \geq 2 \), changing variables, we get

\[
(G_{\lambda,m} f_k)(t) = t^{-k-1} \frac{1}{c_{\lambda,m}} \int_0^1 u^k (1-u^2)^{\lambda-1/2} C^\lambda_m(u) \, du
\]

\[
= t^{-k-1} \frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{k+2}{2} \right)}{\Gamma \left( \lambda + 1 + \frac{k+m}{2} \right) \Gamma \left( \frac{k-m+2}{2} \right)};
\]

cf. (2.3). Since the gamma function \( \Gamma((k-m+2)/2) \) has a pole when \( k - m + 2 = 0, -2, -4, \ldots \), the result follows. If \( \lambda = 0 \) the reasoning is similar and relies on (2.6).

Regarding inversion formulas, if \( m = 0 \) and 1, then \( G_{\lambda,m} \) and \( T_m \) are expressed through the Erdélyi-Kober integrals (see (3.4)) and can be explicitly inverted using Theorem 2.3 on the class of functions satisfying (3.10). We observe that this condition is necessary for the existence of these integrals.

In the case \( m \geq 2 \) some preparation is needed.
Lemma 3.4. Let $\lambda > -1/2$, $m \geq 2$. Suppose that
\[
\int_a^\infty |f(t)| t^{2\lambda + m - 1} \, dt < \infty \quad \forall a > 0.
\]  
(3.12)

(i) If $\lambda \neq 0$, then
\[
G_{\lambda,m}^* G_{\lambda,m} f = 2^{2\lambda+1} I_{2\lambda+1} f.
\]  
(3.13)

(ii) In the case $\lambda = 0$ we have
\[
\mathcal{T}_m^* \mathcal{T}_m f = 2 I_1^* f.
\]  
(3.14)

Proof. We change the order of integration on the left-hand side of (3.13). To justify application of Fubini’s theorem, let us replace all functions on the left-hand side of (3.13) by their absolute values and make use of Proposition 3.1 (ii) together with (2.1). For any $a > 0$ and $m$ even, we obtain
\[
I \equiv \int_a^\infty \int_t^\infty (r^2 - t^2)^{\lambda - 1/2} |C_m^\lambda \left( \frac{t}{r} \right) | |f(r)| r \, dr \, dt
\]  
(3.15)

\[
\leq c \int_a^\infty |f(r)| r \, dr \int_a^\infty (r^2 - t^2)^{\lambda - 1/2} t^{m-2} \, dt
\]

\[
= c \int_a^\infty |f(r)| r^{2\lambda + m - 1} \varphi_1(r) \, dr, \quad \varphi_1(r) = \int_{a/r}^1 s^{m-2}(1-s^2)^{\lambda - 1/2} \, ds.
\]

Since $\varphi_1(r)$ is bounded, then $I < \infty$. If $m$ is odd, then $|C_m^\lambda (t/r)| \leq c t/r$ in (3.15) and we proceed as above with
\[
\varphi_2(r) = \int_{a/r}^1 s^{m-1}(1-s^2)^{\lambda - 1/2} \, ds.
\]

The latter is bounded. For (3.14) the argument is similar.

(i) Let us prove (3.13). The above estimates enable us to change the order of integration and we get
\[
\text{l.h.s.} = \frac{1}{c_{\lambda,m}^2} \int_t^\infty f(s) I(s,t) \, ds,
\]
\[ I(s, t) = \int_t^s (s^2 - r^2)^{\lambda-1/2} (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left( \frac{r}{s} \right) C_m^\lambda \left( \frac{t}{r} \right) \frac{dr}{r^{2\lambda+1}}. \]  

(3.16)

Let us show that

\[ I(s, t) = \frac{2^{2\lambda+1} c_{\lambda,m}^2}{\Gamma(2\lambda + 1)} (s - t)^{2\lambda} \]  

(3.17)

where \( c_{\lambda,m} \) is defined by (3.3). Once (3.17) is proved, the result follows.

Setting \( \xi = t/s \), we easily get

\[ I(s, t) = t^{2\lambda} I_0(\xi), \]  

(3.18)

\[ I_0(\xi) = \xi^{1-2\lambda} \int_1 (1 - u^2)^{\lambda-1/2} \left( 1 - \frac{\xi^2}{u^2} \right)^{\lambda-1/2} C_m^\lambda(u) C_m^\lambda \left( \frac{u}{\xi} \right) \frac{du}{u^2} \]

Here \( f_1 \circ f_2 \) denotes the Mellin convolution of functions

\[ f_1(u) = u^{-1} (1 - u^2)^{\lambda-1/2} C_m^\lambda(u), \quad f_2(u) = (1 - u^2)^{\lambda-1/2} C_m^\lambda(1/u). \]

Thus, we have to show that

\[ I_0(\xi) = \frac{2^{2\lambda+1} c_{\lambda,m}^2}{\Gamma(2\lambda + 1)} \left( \frac{1}{\xi} - 1 \right)^{2\lambda}. \]  

(3.19)

It suffices to establish the coincidence of the Mellin transform \( \tilde{I}_0(z) \) with the Mellin transform of the right-hand side of (3.19) for sufficiently large \( Re z \). The formulas (2.3) and (2.5) enable us to compute the Mellin transform of \( I_0(\xi) \). We have

\[ \tilde{I}_0(z) = \int_0^\infty \xi^{z-1} I_0(\xi) d\xi = \tilde{f}_1(z + 1 - 2\lambda) \tilde{f}_2(z + 1 - 2\lambda) \]

\[ = c_{\lambda,m}^2 \frac{\Gamma \left( \frac{z}{2} - \lambda \right) \Gamma \left( \frac{z + 1}{2} - \lambda \right)}{\Gamma \left( \frac{z + 1}{2} \right) \Gamma \left( \frac{z}{2} + 1 \right)} = 2^{2\lambda+1} c_{\lambda,m}^2 \frac{\Gamma(z - 2\lambda)}{\Gamma(z + 1)} \]

\[ = \frac{2^{2\lambda+1} c_{\lambda,m}^2}{\Gamma(2\lambda + 1)} \int_0^\infty \xi^{z-1} \left( \frac{1}{\xi} - 1 \right)^{2\lambda} d\xi. \]

Thus, the Mellin transforms of the both sides of (3.19) coincide and we are done.
(ii) Let us prove (3.14). As above,

\[ l.h.s. = \frac{4}{\pi} \int_{t}^{\infty} f(s) I(s, t) \, ds, \quad I(s, t) \equiv I_0(\xi) = \xi (f_1 \circ f_2)(\xi), \quad \xi = \frac{t}{s}, \]

where

\[ f_1(u) = u^{-1}(1 - u^2)^{-1/2} T_m(u), \quad f_2(u) = (1 - u^2)^{-1/2} T_m(1/u). \]

By (2.6) and (2.7),

\[ \tilde{I}_0(z) = \tilde{f}_1(z + 1) \tilde{f}_2(z + 1) = \frac{\pi}{2z}, \]

and therefore,

\[ I_0(\xi) = \frac{\pi}{2} H(1 - \xi) = \frac{\pi}{2} \begin{cases} 1 & \text{if } \xi < 1, \\ 0 & \text{if } \xi > 1. \end{cases} \]

This gives \( I(s, t) = (\pi/2) H(1 - t/s), \) and (3.14) follows. \( \square \)

The following inversion formulas for the Gegenbauer-Chebyshev integrals are immediate consequences of Lemma 3.4.

**Corollary 3.5.** Let \( m \geq 2, \lambda > -1/2, \) and suppose that \( f \) satisfies the conditions of Lemma 3.4. Then \( f(t) \) can be uniquely reconstructed for almost all \( t > 0 \) from the Gegenbauer-Chebyshev integrals \( G_{\lambda,m} f = g \) and \( T_m f = g \) by the formulas

\[ f(t) = 2^{-2\lambda-1} (\mathcal{D}^{2\lambda+1} \mathcal{G}_{\lambda,m}^{-} g)(t), \quad (3.20) \]

\[ f(t) = -\frac{1}{2} \frac{d}{dt} (\mathcal{T}^{-m} t^{-2} g)(t), \quad (3.21) \]

where \( \mathcal{D}^{2\lambda+1} \) stands for the corresponding Riemann-Liouville fractional derivative; see Section 2.3.

**Remark 3.6.** The assumption \( \int_{a}^{\infty} |f(t)| t^{2\lambda+m-1} \, dt < \infty \) in Corollary 3.5 is essentially stronger than (3.10) in Proposition 3.1 (i) which guarantees the existence of \( G_{\lambda,m} f \). The inversion problem for \( G_{\lambda,m} f \) under the less restrictive assumption (3.10) does not have a unique solution; cf. Lemma 3.3. We recall that for \( m = 0 \) and 1, unlike \( m \geq 2 \), the inversion formulas provided by Theorem 2.3 hold under the same assumptions which are necessary for the existence of the corresponding Gegenbauer-Chebyshev integrals.
The Left-sided Integrals. Let $\lambda > -1/2$, $m \in \mathbb{Z}_+$. The left-sided Gegenbauer and Chebyshev fractional integrals are defined as follows. For $\lambda \neq 0$, we set
\[
(G^\lambda_m f)(r) = \frac{r^{-2\lambda}}{c_{\lambda,m}} \int_0^r (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left( \frac{t}{r} \right) f(t) \, dt,
\]
(3.22)
\[
(G^\lambda_m f)(r) = \frac{1}{c_{\lambda,m}} \int_0^r (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left( \frac{r}{t} \right) f(t) \, t \, dt,
\]
(3.23)
c_{\lambda,m} being defined by (3.3). In the case $\lambda = 0$ we denote
\[
(T^m f)(r) = \frac{2}{\sqrt{\pi}} \int_0^r (r^2 - t^2)^{-1/2} T_m \left( \frac{t}{r} \right) f(t) \, dt,
\]
(3.24)
\[
(T^m f)(r) = \frac{2}{\sqrt{\pi}} \int_0^r (r^2 - t^2)^{-1/2} T_m \left( \frac{r}{t} \right) f(t) \, t \, dt.
\]
(3.25)
The left-sided integrals are expressed through the right-sided ones by the formulas
\[
(G^\lambda_m f)(r) = \frac{1}{r} (G^\lambda_m f_1) \left( \frac{1}{r} \right), \quad f_1(t) = \frac{1}{t^{2\lambda+2}} f \left( \frac{1}{t} \right); \quad (3.26)
\]
\[
(G^\lambda_m f)(r) = r^{2\lambda} (G^\lambda_m f_2) \left( \frac{1}{r} \right), \quad f_2(t) = \frac{1}{t} f \left( \frac{1}{t} \right). \quad (3.27)
\]
These formulas combined with Proposition 3.1 show that the integrals (3.22) and (3.24) are absolutely convergent for almost all $r > 0$ whenever
\[
t^{-\eta} f \in L^1_{\text{loc}}([0, \infty)).
\]
(3.28)
(we recall that $\eta = 0$ if $m$ is even and $\eta = 1$ if $m$ is odd). Similarly, (3.23) and (3.25) are well-defined a.e. on $\mathbb{R}_+$ provided that
\[
t^{1-m} f \in L^1_{\text{loc}}([0, \infty)).
\]
(3.29)
The conditions (3.28) and (3.29) are sharp; see Remark 3.2.
The following statement is a consequence of Lemma 3.3.

**Lemma 3.7.** If $m = 0, 1$, then $G^\lambda_m$ is injective in the class of functions satisfying (3.28). If $m \geq 2$, then $G^\lambda_m$ is non-injective in this class of functions. Specifically, let $f_k(t) = t^k$, where $k$ is a nonnegative integer such that $m - k = 2, 4, \ldots$. Then $(G^\lambda_m f_k)(t) = 0$ for all $t > 0$.

Similarly, Lemma 3.4 yields the following.
Lemma 3.8. Let $m \geq 2$, $\lambda > -1/2$. Suppose that $f$ satisfies (3.29). If $\lambda \neq 0$, then
\[
\mathcal{G}_+^{\lambda, m} f = 2^{2\lambda+1} I_+^{2\lambda+1} f.
\] (3.30)
In the case $\lambda = 0$ we have
\[
\mathcal{T}_+^m f = 2 I_+^1 f.
\] (3.31)

Corollary 3.9. Suppose that $\lambda > -1/2$ and let $f$ satisfy (3.29). Then $f(t)$ can be uniquely reconstructed for almost all $t > 0$ from the Gegenbauer-Chebyshev integrals by the formulas
\[
f(t) = 2^{-2\lambda-1} (\mathcal{D}_+^{2\lambda+1} \mathcal{G}_+^{\lambda, m} g)(t), \quad g = \mathcal{G}_+^{\lambda, m} f,
\] (3.32)
\[
f(t) = \frac{1}{2} d dt (\mathcal{T}_+^m g)(t), \quad g = \mathcal{T}_+^m f,
\] (3.33)
where $\mathcal{D}_+^{2\lambda+1}$ stands for the corresponding Riemann-Liouville fractional derivative.

It is natural to ask are there any other functions in the kernel of the operator $\mathcal{G}_+^{\lambda, m}$, rather than those mentioned in Lemma 3.7. Below we prove that the answer is “No”, at least, in the class of $m$ times continuously differentiable functions on $[0, \infty)$.

Lemma 3.10. Let $m \geq 2$, $0 < a < \infty$, $f \in C^m([0, a))$. If $(\mathcal{G}_+^{\lambda, m} f)(r) = 0$ for all $0 < r < a$, then
\[
f(t) = \sum_{k=0}^{m-2} c_k t^k,
\] (3.34)
where “$\,^\prime\,$” means that the sum contains only those terms for which $m - k$ is even.

Proof. By Taylor’s theorem,
\[
f(t) = \sum_{k=0}^{m-2} \frac{f^{(k)}(0)}{k!} t^k + q(t), \quad q(t) = \int_0^t \frac{f^{(m)}(s)}{(m-1)!} (t-s)^{m-1} ds.
\]
Hence, by Lemma 3.7, $(\mathcal{G}_+^{\lambda, m} f)(r) = (\mathcal{G}_+^{\lambda, m} q)(r)$. We apply the operator $\mathcal{G}_+^{\lambda, m}$ to both sides of this equality and make use of Lemma 3.8. The
conditions of this lemma are satisfied. Indeed,
\[
\int_0^a t^{1-m} |q(t)| \, dt \leq \int_0^a t^{1-m} \, dt \int_0^t \frac{|f^{(m)}(s)|}{(m-1)!} (t-s)^{m-1} \, ds \\
\leq \max_s \frac{|f^{(m)}(s)|}{m!} \int_0^a t \, dt < \infty.
\]
Hence,
\[
0 = \mathcal{G}_+^{\lambda_m} \mathcal{G}_+^{\lambda_m} f = \mathcal{G}_+^{\lambda_m} \mathcal{G}_+^{\lambda_m} q = 2^{2\lambda+1} f_+^{2\lambda+1} q = 2^{2\lambda+1} f_+^{2\lambda+1+m} f^{(m)},
\]
which implies \( f^{(m)}(t) = 0 \) for all \( 0 < t < a \). It follows that \( f \) is a polynomial of degree less than \( m \), that is,
\[
f(t) = \sum_{k=0}^{m-1} c_k t^k.
\]
(3.35)
The coefficients \( c_k \) corresponding \( m - k \) odd are zero. Indeed, applying \( \mathcal{G}_+^{\lambda_m} \) to both sides of (3.35), owing to Lemma 3.7, we get
\[
0 = (\mathcal{G}_+^{\lambda_m} f)(t) = \sum_{k=0}^{m-1} c_k \mathcal{G}_+^{\lambda_m}[t^k],
\]
where "\(^\prime\)" means that the sum includes only those terms which correspond to \( m - k \) odd. Note that \( (\mathcal{G}_+^{\lambda_m}[t^k])(r) = \alpha_{k,m} r^{k-2\lambda} \) where
\[
\alpha_{k,m} = \frac{1}{c_{\lambda,m}} \int_0^1 (1-s^2)^{\lambda-1/2} C_m^\lambda(s) s^k \, ds
\]
(3.36)
\[
= \frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{k+2}{2} \right)}{\Gamma \left( \lambda+1+\frac{k+m}{2} \right) \Gamma \left( \frac{k-m+2}{2} \right)};
\]
cf. (2.3). Since \( \alpha_{k,m} \neq 0 \) in our case, then
\[
\sum_{k=0}^{m-1} c_k \alpha_{k,m} r^k = 0 \quad \forall r > 0.
\]
It follows that all \( c_k \) in this sum are zero and \( f \) contains only those terms for which \( m - k \) is even. \( \square \)
4. Radon Transforms and Spherical Harmonics

We fix a real-valued orthonormal basis \( \{Y_{m,\mu}\} \) of spherical harmonics in \( L^2(S^{n-1}) \); see, e.g., [36]. Here \( m \in \mathbb{Z}_+ \) and \( \mu = 1, 2, \ldots, d_n(m) \) where

\[
d_n(m) = (n + 2m - 2) \frac{(n + m - 3)!}{m!(n - 2)!}
\]  

(4.1)

is the dimension of the subspace of spherical harmonics of degree \( m \).

The following Funk-Hecke Theorem is well-known in analysis on the sphere; see, e.g., [26, p. 18], [52, p. 117].

**Theorem 4.1.** Let \( h(s)(1 - s^2)^{(n-3)/2} \in L^1(-1,1) \). Then for every spherical harmonic \( Y_m \) of degree \( m \) and every \( \theta \in S^{n-1} \),

\[
\int_{S^{n-1}} h(\theta \cdot \xi) Y_m(\xi) \, d\xi = \lambda_m Y_m(\theta) \quad \text{(the Funk-Hecke formula)},
\]

(4.2)

where

\[
\lambda_m = \sigma_{n-2} \int_{-1}^{1} h(s) P_m(s) \left(1 - s^2\right)^{(n-3)/2} \, ds,
\]

(4.3)

\[
P_m(s) = \begin{cases} 
T_m(s) & \text{if } n = 2, \\
\frac{m!(n-3)!}{(m+n-3)!} C_{m/2-1}^n(s) & \text{if } n \geq 3.
\end{cases}
\]

(4.4)

Now let us consider the Radon transform and its dual; see Section 2.5. Since these operators commute with rotations, they can be diagonalized (at least formally) in terms of spherical harmonic expansions. Specifically, if

\[
f(x) \sim \sum_{m,\mu} f_{m,\mu}(r) Y_{m,\mu}(\theta), \quad r = |x| \neq 0, \quad \theta = x/r,
\]

(4.5)

then for \( \varphi(\theta,t) = (Rf)(\theta,t) \) we have

\[
\varphi(\theta,t) \sim \sum_{m,\mu} \varphi_{m,\mu}(t) Y_{m,\mu}(\theta).
\]

(4.6)

Similarly, if \( \varphi \) is a function on \( Z_n = S^{n-1} \times \mathbb{R} \) and \( f = R^* \varphi \), then (4.6) implies (4.5).

We will be using the same notation \( \mathcal{E}_m \) for the spaces of functions of the form \( f(x) = u(|x|) Y_m(x/|x|) \) and \( \varphi(\theta,t) = v(t) Y_m(\theta) \), where \( Y_m \) is a spherical harmonic of degree \( m \).
4.1. The Action of the Radon Transform and its Dual on the Spaces $\mathcal{E}_m$.

Let $\lambda = (n - 2)/2, f(x) = u(|x|) Y_m(x/|x|)$. For $n \geq 3$ we define

$$v(t) = \frac{\pi^{\lambda+1/2}}{c_{\lambda,m}} \int_0^\infty (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left( \frac{t}{r} \right) u(r) r \, dr,$$

(4.7)

$$c_{\lambda,m} = \frac{\Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda)} = \frac{(n + m - 3)! \Gamma((n - 1)/2)}{2m! (n - 3)!}.$$  

(4.8)

Similarly, for $n = 2$ we set

$$v(t) = 2 \int_0^\infty (r^2 - t^2)^{-1/2} T_m \left( \frac{t}{r} \right) u(r) r \, dr.$$  

(4.9)

Lemma 4.2. Let $f(x) = u(|x|) Y_m(x/|x|)$ where

$$\int_a^\infty |u(r)| r^{2\lambda} \, dr < \infty \quad \forall a > 0.$$  

(4.10)

Then $(Rf)(\theta, t)$ is finite for all $\theta \in S^{n-1}$ and almost all $t \in \mathbb{R}$. Furthermore,

$$(Rf)(\theta, t) = v(t) Y_m(\theta).$$  

(4.11)

The function $v(t)$ has the following properties:

(a) $v(-t) = (-1)^m v(t)$.

(b) If $t > 0$, then $v$ is represented by the Gegenbauer-Chebyshev integrals (3.1) and (3.6). Specifically,

$$v(t) = \pi^{\lambda+1/2} (G_m^\lambda u)(t) \quad \text{and} \quad v(t) = \pi^{1/2} (T_m u)(t)$$  

(4.12)

for $n \geq 3$ and $n = 2$, respectively.

(c) For any nonnegative integer $j < m$,

$$\int_{-\infty}^\infty t^j v(t) \, dt = 0 \quad \text{provided that} \quad \int_0^\infty |u(r)| r^{j+2\lambda+1} \, dr < \infty.$$  

(4.13)

Proof. Let first $t > 0$. By (2.21),

$$(Rf)(\theta, t) = t^{n-1} \int_{\omega \cdot \theta > 0} f \left( \frac{t \omega}{\omega \cdot \theta} \right) \frac{d\omega}{(\omega \cdot \theta)^n}, \quad f \left( \frac{t \omega}{\omega \cdot \theta} \right) = u \left( \frac{t}{\omega \cdot \theta} \right) Y_m(\omega).$$
Now (4.11) holds by the Funk-Hecke formula (4.2) (set $h(s) = s^{-n}u(t/s)$ if $s > 0$ and $h(s) \equiv 0$, otherwise) and

$$v(t) = \sigma_{n-2} \int_0^1 (1 - s^2)^{(n-3)/2} P_m(s) \frac{u(t/s) ds}{s^{n-1}}. \tag{4.14}$$

By (4.10) and Lemma 2.2, the condition $h(s)(1 - s^2)^{(n-3)/2} \in L^1(-1,1)$ in Theorem 4.1 is satisfied for almost all $t > 0$, so that (4.11) is valid for all $\theta \in S^{n-1}$ and almost all $t > 0$. The equality (4.14) implies (4.7) and (4.9). The formulas in (4.12) follow from (4.14) owing to (3.1) and (3.6). The equality $v(-t) = (-1)^m v(t)$ is a consequence of the formulas $(Rf)(\theta,t) = (Rf)(-\theta,-t)$ and $Y_m(-\theta) = (-1)^m Y_m(\theta)$. To prove (c), we first change the order of integration. This operation is possible thanks to the inequality in (4.13). Then the result follows by the orthogonality of Gegenbauer (or Chebyshev) polynomials.

For the dual Radon transform we have the following.

**Lemma 4.3.** Let $\lambda = (n - 2)/2$, $\varphi(\theta,t) = v(t)Y_m(\theta)$, where $Y_m$ is a spherical harmonic of degree $m$ and $v(t)$ is a locally integrable function on $\mathbb{R}$ satisfying $v(-t) = (-1)^m v(t)$. Then $(R^* \varphi)(x) \equiv (R^* \varphi)(r\theta)$ is finite for all $\theta \in S^{n-1}$ and almost all $r > 0$. Furthermore,

$$(R^* \varphi)(r\theta) = u(r) Y_m(\theta). \tag{4.15}$$

The function $u(r)$ is represented by the Gegenbauer integral (3.1) (or the Chebyshev integral (3.6)) as follows.

For $n \geq 3$:

$$u(r) = \frac{r^{-2\lambda}}{\tilde{c}_{\lambda,m}} \int_0^r (r^2 - t^2)^{-\lambda - 1/2} C_m^\lambda \left( \frac{t}{r} \right) v(t) dt = \pi^{\lambda+1/2} (G_{+}^{\lambda,m} u)(t), \tag{4.16}$$

$$\tilde{c}_{\lambda,m} = \frac{\pi^{1/2} \Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda) \Gamma(\lambda + 1)}.$$

For $n = 2$:

$$u(r) = \frac{2}{\pi} \int_0^r (r^2 - t^2)^{-1/2} T_m \left( \frac{t}{r} \right) v(t) dt = \pi^{1/2} (T^m_+ u)(t). \tag{4.17}$$

**Proof.** We first note that $\varphi$ is locally integrable on $Z_\theta$ and therefore, $(R^* \varphi)(x)$ is finite for almost all $x$. Then, by the Funk-Hecke formula,
we get (4.15) with 
\[
u(r) = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^{1} (1 - s^2)^{(n-3)/2} P_m(s) v(rs) ds.
\]
Since \(v(-s) = (-1)^m v(s)\) and \(P_m(-s) = (-1)^m P_m(s)\), the last formula gives the result. \(\square\)

**Theorem 4.4.** Suppose that
\[
\int |f(x)| \left\{ \begin{array}{ll}
|y|^{-1}, & \text{if } m = 0, 1, \\
|y|^{-m/2}, & \text{if } m \geq 2
\end{array} \right\} \, dx < \infty \quad \forall a > 0.
\] (4.18)

Then the Fourier-Laplace coefficients \(f_{m,\mu}(t)\) of \(f\) can be uniquely reconstructed for almost all \(t > 0\) from the corresponding coefficients \(\varphi_{m,\mu}\) of \(\varphi = Rf\) by the following formulas.

For \(n \geq 3:\)
\[
f_{m,\mu}(t) = c \left( -\frac{d}{dt} \right)^{n-1} t \int_{t}^{\infty} (r^2 - t^2)^{(n-3)/2} C_m^{n/2-1} \left( \frac{r}{t} \right) \varphi_{m,\mu}(r) \, r^{1-n} \, dr,
\] (4.19)
\[
c = \frac{\Gamma(n/2 - 1) \, m!}{2\pi^{(n-1)/2}(n - 3 + m)!}.
\]

For \(n = 2:\)
\[
f_{m,\mu}(t) = -\frac{1}{\pi} \frac{d}{dt} t \int_{t}^{\infty} (r^2 - t^2)^{-1/2} T_m \left( \frac{r}{t} \right) \varphi_{m,\mu}(r) \, dr.
\] (4.20)

**Proof.** By Lemma 4.2, \(\varphi_{m,\mu}(t) = \pi^{(n-1)/2}(G_m^{n/2-1,m} f_{m,\mu})(t)\) if \(n \geq 3\), and \(\varphi_{m,\mu}(t) = \pi^{1/2}(T_m f_{m,\mu})(t)\) if \(n = 2\). Hence, the result follows by Corollary 3.5, the conditions of which are satisfied, owing to (4.18). \(\square\)

4.2. **The Kernel and Support Theorems.**

4.2.1. **The Kernel of \(R^*\).** The next theorem gives the description of the kernel of \(R^*\) in terms of the Fourier-Laplace coefficients
\[
\varphi_{m,\mu}(t) = \int_{S^{n-1}} \varphi(\theta, t) \, Y_{m,\mu}(\theta) \, d\theta.
\] (4.21)

**Theorem 4.5.** Let \(\varphi\) be an even locally integrable function belonging to \(S'(Z_n)\), \(Z_n = S^{n-1} \times \mathbb{R}\). Then \((R^* \varphi)(x) = 0\) for almost all \(x \in \mathbb{R}^n\) if and only if the Fourier-Laplace coefficients (4.21) satisfy the following conditions:
(i) If $m = 0, 1$, then $\varphi_{m, \mu}(t) = 0$ for almost all $t \in \mathbb{R}$.

(ii) If $m \geq 2$, then $\varphi_{m, \mu}(t)$ is a linear combination of the power functions $t^k$ with $0 \leq k \leq m - 2$ and $m - k$ even.

The proof of Theorem 4.5 needs some preparation.

**Lemma 4.6.** If $\varphi \in L^1_{\text{loc}}(Z_n)$ is even, then for almost all $r > 0$,

$$
(R^* \varphi)_{m, \mu}(r) = \int_{S^{n-1}} (R^* \varphi)(r \theta) Y_{m, \mu}(\theta) d\theta = \pi^{\lambda + 1/2} (\mathcal{G}_+^{\lambda, m} \varphi_{m, \mu})(r) \quad (4.22)
$$

where $\lambda = (n - 2)/2$ and right-hand side represents the Gegenbauer integral (3.1) (or the Chebyshev integral (3.6)).

**Proof.** Since the integral in (4.22) exists in the Lebesgue sense for almost all $r > 0$, we can change the order of integration. Using the Funk-Hecke formula (4.2), we obtain

$$
(R^* \varphi)_{m, \mu}(r) \equiv \int_{S^{n-1}} d_\star \eta \int_{S^{n-1}} \varphi(\eta, r \theta \cdot \eta) Y_{m, \mu}(\theta) d\theta
$$

$$
= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^{1} (1 - s^2)^{(n-3)/2} P_m(s) \int_{S^{n-1}} \varphi(\eta, rs) Y_{m, \mu}(\eta) d\eta
$$

$$
= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^{1} (1 - s^2)^{(n-3)/2} P_m(s) \varphi_{m, \mu}(s) ds.
$$

Since $\varphi$ is even, then $\varphi_{m, \mu}(-t) = (-1)^m \varphi_{m, \mu}(t)$. Moreover, $P_m(-s) = (-1)^m P_m(s)$. Hence, the last integral equals $\pi^{\lambda + 1/2} (\mathcal{G}_+^{\lambda, m} \varphi_{m, \mu})(r)$; cf. the proof of Lemma 4.3. \hfill \Box

Lemma 4.6 implies the “if” part of Theorem 4.5. We formulate it as a separate statement.

**Proposition 4.7.** Let $\varphi$ be an even function in $L^1_{\text{loc}}(Z_n)$ with the Fourier-Laplace coefficients $\varphi_{m, \mu}(t)$ satisfying (i) and (ii) of Theorem 4.5. Then $(R^* \varphi)(x) = 0$ for almost all $x \in \mathbb{R}^n$.

**Proof.** By Lemma 3.7, the operator $\mathcal{G}_+^{\lambda, m}$ annihilates monomials $t^k$ provided that $0 \leq k \leq m - 2$ with $m - k$ even. Hence, by (4.22),

$$(R^* \varphi)_{m, \mu}(r) = 0 \text{ for almost all } r > 0.$$  

We recall that $R^* \varphi$ is locally integrable in $\mathbb{R}^n$. Hence, by Fubini’s theorem, the function $f_r(\theta) \equiv (R^* \varphi)(r \theta)$ belongs to $L^1(S^{n-1})$ for almost all $r > 0$. Let us consider
the Poisson integral
\[(\Pi_{\rho} f_r)(\theta) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \frac{1 - \rho^2}{|\theta - \rho\eta|^n} f_r(\eta) \, d\eta;\]

see, e.g., Stein and Weiss [55]. Since \((R^* \varphi)_{m,\mu}(r) = 0 \text{ a.e. for all } m, \mu,\)

\[(\Pi_{\rho} f_r)(\theta) = \sum_{m,\mu} \rho^m (f_r)_{m,\mu} Y_{m,\mu}(\theta) = \sum_{m,\mu} \rho^m (R^* \varphi)_{m,\mu}(r) Y_{m,\mu}(\theta) = 0\]

for almost all \(r > 0, \text{ all } \rho \in [0,1), \text{ and all } \theta \in S^{n-1}.\) Furthermore, since

\[f_r(\theta) = \lim_{\rho \to 1} (\Pi_{\rho} f_r)(\theta)\]

in the \(L^1\)-norm, then \(f_r(\theta) = (R^* \varphi)(r\theta) = 0 \text{ for almost all } \theta \in S^{n-1}\) and almost all \(r > 0.\) This gives the result. \(\square\)

In Proposition 4.7 we did not assume that \(\varphi \in S'(Z_n).\) This assumption is used in the proof of the “only if” part. The next lemma employs the distribution spaces \(S'(Z_n)\) and \(\Phi'(Z_n)\) from Section 2.5.

**Lemma 4.8.** Let \(\varphi\) be a locally integrable function in \(S'(Z_n).\) If \(\varphi = 0\) in the \(\Phi'(Z_n)\)-sense, then all the Fourier-Laplace coefficients \(\varphi_{m,\mu}(t)\) are polynomials.

**Proof.** Given \(\omega \in S(\mathbb{R}),\) let \(\psi(\theta, t) = \omega(t) Y_{m,\mu}(\theta).\) Keeping in mind that \(\varphi \in L^1_{loc}(Z_n) \cap S'(Z_n),\) we conclude that the expression

\[(\varphi_{m,\mu}, \omega) = \int_{Z_n} \varphi_{m,\mu}(t) \overline{\psi}(\theta, t) \, d\theta dt = (\varphi, \psi)\]

is meaningful. If, moreover, \(\omega \in \Phi(\mathbb{R}),\) then \(\psi \in \Phi(Z_n)\) and, by the assumption, \((\varphi_{m,\mu}, \omega) = (\varphi, \psi) = 0.\) Hence, by Proposition 2.8, \(\varphi_{m,\mu}(t)\) is a polynomial. \(\square\)

The next proposition completes the proof of Theorem 4.5.

**Proposition 4.9.** Let \(\varphi\) be a locally integrable even function in \(S'(Z_n)\) such that \(R^* \varphi)(x) = 0\) for almost all \(x \in \mathbb{R}^n.\) Then, for every \(m\) and \(\mu,\) the Fourier-Laplace coefficients \(\varphi_{m,\mu}(t)\) satisfy (i) and (ii) of Theorem 4.5.

**Proof.** Since \((R^* \varphi)(x) = 0,\) then \((R^* \varphi)_{m,\mu}(r) = 0\) and, by (4.22),

\[(G^\lambda_{m,\mu} \varphi_{m,\mu})(r) = 0\]  \hspace{1cm} (4.23)

for almost all \(r > 0\) and all \(m, \mu.\) Furthermore, if \(g \in \Phi(Z_n)\) and \(g_e \in \Phi_e(Z_n)\) is the even part of \(g,\) then \((\varphi, g) = (\varphi, g_e),\) because \(\varphi\)
is even. Since by Theorem 2.9, \( g_e = Rf \) for some \( f \in \Phi(\mathbb{R}^n) \), then \( R^* \varphi = 0 \) yields

\[
(\varphi, g) = (\varphi, g_e) = (\varphi, Rf) = (R^* \varphi, f) = 0.
\]

By Lemma 4.8 it follows that \( \varphi_{m,\mu}(t) \) is a polynomial. We write \( \varphi_{m,\mu}(t) = p(t) \). The structure of this polynomial is determined by the equality

\[
G^\lambda_m p = 0,
\]

which follows from (4.23). Specifically, by Lemma 3.7, if \( m = 0, 1 \), then \( p(t) = 0 \) for almost all \( t \in \mathbb{R}_+ \). If \( m \geq 2 \), then \( G^\lambda_m p \) is a finite sum of the form \( \sum_k c_k G^{\lambda_m}[k] \), where the terms corresponding to \( k \leq m - 2 \) with \( m - k \) even are zero. For all other \( k \) in this sum (we denote this set by \( K \)), we have

\[
(G^{\lambda_m}[k])(r) = \alpha_{k,m} r^{k-2\lambda}, \quad \alpha_{k,m} = \frac{1}{c_{\lambda,m}} \int_0^1 (1-s^2)^{\lambda-1/2} C^\lambda_m(s) s^k \, ds,
\]

where \( \lambda = (n-2)/2 \); cf. (3.36). By (2.3), \( \alpha_{k,m} \neq 0 \). Thus, (4.24) yields

\[
\sum_{k \in K} c_k \alpha_{k,m} r^k = 0 \quad \forall r > 0.
\]

It follows that all \( c_k \) with \( k \in K \) are zero and \( p(t) \) contains only terms corresponding to \( m - k \geq 2 \) even. This completes the proof. \( \square \)

4.2.2. The Kernel of \( R \). Theorem 4.5 combined with the formula

\[
(Rf)(\theta, t) = \frac{\sigma_{n-1}}{2|t|} (R^* Bf) \left( \frac{\theta}{t} \right), \quad (Bf)(\theta, t) = \frac{1}{|t|^n} f \left( \frac{\theta}{t} \right)
\]

(see Lemma 2.6), enables us to describe the kernel of the Radon transform \( R \). We first prove the following lemma.

**Lemma 4.10.** If there exists \( N > 0 \) such that

\[
\int_{|x|<1} |x|^{N-1} |f(x)| \, dx + \int_{|x|>1} \frac{|f(x)|}{|x|} \, dx < \infty,
\]

then \( (Bf)(\theta, t) \) is a tempered locally integrable function on \( Z_n \).
Proof. Changing variables, we have
\[
\int_{Z_n} \frac{|(Bf)(\theta, t)|}{(1 + |t|)^N} \, d\theta dt \leq 2 \int_0^\infty \frac{dt}{t^n(1 + t)^N} \int_{S^{n-1}} \left| f \left( \frac{\theta}{t} \right) \right| \, d\theta
\]
\[
= \frac{2}{\sigma_{n-1}} \int_{\mathbb{R}^n} \frac{|x|^{N-1}}{(1 + |x|)^N} |f(x)| \, dx,
\]
which gives the result.

We note that the finiteness of the first integral in (4.26) means that \( f(x) \) is allowed to have a tempered growth as \( x \to 0 \). The finiteness of the second integral is necessary for the existence of the Radon transform; cf. Theorem 2.5.

**Theorem 4.11.** Let \( f \) satisfy (4.26) for some \( N > 0 \). Then \((Rf)(\theta, t) = 0\) almost everywhere on \( Z_n \) if and only if every Fourier-Laplace coefficient \( f_{m,\mu}(r) \) of the function \( f_r(\theta) = f(r\theta) \) has the form
\[
f_{m,\mu}(r) = \begin{cases} 0 & \text{if } m = 0, 1, \\ \sum_{k=0}^{m-2} \frac{c_{m,k}}{r^{m+k}} & \text{if } m \geq 2, \end{cases}
\]
for some constant coefficients \( c_{m,k} \). Here, “\( ^* \)” means that the sum includes only those terms, for which \( m - k \) is even.

Proof. If \( Rf = 0 \) a.e. on \( Z_n \), then \( R^*Bf = 0 \) a.e. on \( \mathbb{R}^n \). Hence, by Theorem 4.5, for \( t > 0 \) we have
\[
(Bf)_{m,\mu}(t) = t^{-n} \int_{S^{n-1}} f \left( \frac{\theta}{t} \right) Y_{m,\mu}(\theta) \, d\theta = \begin{cases} 0 & \text{if } m = 0, 1, \\ \sum_{k=0}^{m-2} c_{m,k} t^k & \text{if } m \geq 2. \end{cases}
\]
Changing variable \( t = 1/r \), we obtain (4.27). Conversely, if (4.27) holds, then \( (Bf)_{m,\mu}(t) = 0 \) if \( m = 0, 1 \), and \( (Bf)_{m,\mu}(t) = \sum_{k=0}^{m-2} c_{m,k} t^k \) if \( m \geq 2 \). The last equality is obvious for \( t > 0 \). If \( t < 0 \), then
\[
(Bf)_{m,\mu}(t) = \int_{S^{n-1}} (Bf)(\theta, t) Y_{m,\mu}(\theta) \, d\theta
\]
\[
= (-1)^m \int_{S^{n-1}} (Bf)(\theta, |t|) Y_{m,\mu}(\theta) \, d\theta
\]
\[
= (-1)^m \sum_{k=0}^{m-2} c_k |t|^k = \sum_{k=0}^{m-2} c_{m,k} t^k
\]
because \( m - k \) is even. Hence, by Theorem 4.5, \( R^* B f = 0 \) a.e. on \( \mathbb{R}^n \) and therefore, by (4.25), \( R f = 0 \) a.e. on \( Z_n \).

**Example 4.12.** Consider the function \( f(x) = |x|^{-n} Y_2(x/|x|), \ x \neq 0 \), where \( Y_2 \) is a spherical harmonic of degree 2. This function has a non-integrable singularity at the origin and the integrals of \( f \) over hyperplanes through the origin are not absolutely convergent. However, \((R f)(\theta, t)\) is represented by an absolutely convergent integral for all \((\theta, t)\) with \( t \neq 0 \) and is continuous on the open half-cylinders \( C_\pm = \{ (\theta, t) \in Z_n : \pm t > 0 \} \). Since \( f \) obeys (4.26) with any \( N > 1 \), then, by Theorem 4.11, \((R f)(\theta, t) \equiv 0\) in \( C_\pm \). The latter means that, by continuity, we can also set \((R f)(\theta, t) \equiv 0\) at the points \((\theta,0)\).

### 4.2.3. Support Theorems.

Theorems 4.5 and 4.11 yield the following versions of Helgason’s support theorem; cf. [27, p. 10]. For \( a > 0 \), we denote

\[
B_a^+ = \{ x \in \mathbb{R}^n : |x| < a \}, \quad B_a^- = \{ x \in \mathbb{R}^n : |x| > a \},
\]

\[
C_a^+ = \{ (\theta, t) \in Z_n : |t| < a \}, \quad C_a^- = \{ (\theta, t) \in Z_n : |t| > a \}.
\]

**Theorem 4.13.** Let \( a > 0 \). If \( f(x) = 0 \) for almost all \( x \in B_a^- \), then \((R f)(\theta, t) = 0\) a.e. on \( C_a^- \). Conversely, if

\[
\int_{|x| > a} |f(x)| |x|^m \, dx < \infty \quad \forall m \in \mathbb{N}
\]

and \((R f)(\theta, t) = 0\) a.e. on \( C_a^- \), then \( f(x) = 0 \) for almost all \( x \in B_a^- \).

**Proof.** The first statement is obvious if \( f \) is continuous. In the general case we set \( f_a(x) = f(x) \) if \( |x| > a \) and 0 otherwise. By (2.24),

\[
\int_{Z_n} \frac{(R[f_a])(\theta, t)}{(1 + t^2)^{n/2}} \, d_s \theta dt = \int_{\mathbb{R}^n} \frac{|f_a(x)|}{(1 + |x|^2)^{1/2}} \, dx.
\]

Since the right-hand side is zero, then so is the left-hand side, and, therefore, \( R[f_a] = 0 \) a.e. on \( Z_n \). Hence, \( R f = 0 \) a.e. on \( C_a^- \).

Conversely, if \( f \) obeys (4.28), then, by (2.24), the integral

\[
\int_a^b dt \int_{S^{n-1}} |(R f)(\theta, t)| \, d\theta
\]

is finite for all \( a < b < \infty \). Hence,

\[
\int_a^b |(R f)_{m, \mu}(t)| \, dt \leq c \int_a^b \int_{S^{n-1}} |(R f)(\theta, t)| \, d\theta dt < \infty.
\]
If \((Rf)(\theta, t) = 0\) for almost all \((\theta, t) \in C_a^-\), then the right-hand side of (4.30) equals zero for all \(b > a\). Hence, the left-hand side is also zero and, therefore, \((Rf)_{m, \mu}(t) = 0\) for almost all \(t \notin (-a, a)\). By Theorem 4.4, it follows that \(f_{m, \mu}(r) = 0\) for almost all \(r > a\). Invoking the Poisson integral, as in the proof of Lemma 4.7, we conclude that \(f(x) = 0\) a.e. whenever \(|x| > a\). □

Theorem 4.14. Let \(\varphi\) be an even function on \(Z_n\), \(a > 0\). If \(\varphi(\theta, t) = 0\) a.e. on \(C_a^+\), then \((R^*\varphi)(x) = 0\) a.e. on \(B_a^+\). Conversely, if

\[
\int_{C_a^+} |\varphi(\theta, t)| \cdot |t|^{-m} \, dtd\theta < \infty \quad \forall m \in \mathbb{N} \quad (4.31)
\]

and \((R^*\varphi)(x) = 0\) a.e. on \(B_a^+\), then \(\varphi(\theta, t) = 0\) a.e. on \(C_a^+\).

Proof. The statement follows from the previous theorem by (2.27). □

Remark 4.15. The condition (4.28) gives an example of a class of functions for which the implication

\[
(Rf)(\theta, t) = 0 \text{ on } C_a^- \implies f(x) = 0 \text{ on } B_a^- \quad (4.32)
\]

is true. However, in general, this implication does not hold. For example, every function of the form

\[
f(x) = Y_m(x') \sum' \frac{c_k}{|x|^{n+k}}, \quad x' = x/|x|,
\]

where \(\sum'\) includes only those terms for which \(m - k\) is even, has the vanishing Radon transform on \(C_a^-\); cf. Theorem 4.11. On the other hand, a rapid decrease of \(f\) is not necessary for (4.32), as can be easily seen, by taking functions of the form \(f(x) = Y_m(x/|x|) g(|x|)\) with \(m = 0, 1\); cf. Theorem 4.4. A similar remark can be addressed to Theorem 4.14.

5. Spheres through the Origin

Below we consider the spherical mean Radon-like transform which assigns to a function \(f\) on \(\mathbb{R}^n\) the integrals of \(f\) over spheres passing to the origin. This transform is defined by the formula

\[
(Qf)(x) = \int_{S^{n-1}} f(x + |x| \theta) \, d_s\theta, \quad (5.1)
\]

where \(d_s\theta\) is the normalized surface element, so that \(\int_{S^{n-1}} d_s\theta = 1\). Thus, \(f\) is integrated in (5.1) over the sphere of radius \(|x|\) with center at \(x\). There is a remarkable connection between (5.1) and the Darboux
equation. Specifically, in the classical Cauchy problem for the Darboux equation we are looking for a function \( u(x,t) \) satisfying

\[
\Delta u - u_{tt} - \frac{n-1}{t} u_t = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.
\]

(5.2)

Here \( x \in \mathbb{R}^n, \ t > 0, \ f \) is a given function. Now consider the inverse problem: Given the trace \( u(x, |x|) \) of the solution of (5.2) on the cone \( t = |x| \), reconstruct the initial function \( f(x) \). It is known that the solution of the Cauchy problem (5.2) has the form

\[
u(x,t) = \int_{S^{n-1}} f(x + t\theta) \, d_s \theta;
\]

(5.3)

see, e.g., [18, p. 699]. Hence, the above inverse problem reduces to reconstruction of \( f \) from \( (Qf)(x) \).

The study of the operator (5.1) relies on the following connection between \( (Qf)(x) \), the dual Radon transform (2.25), and the Radon transform (2.20).

**Lemma 5.1.** Let \( n \geq 2 \). Then

\[
(Qf)(x) = |x|^{2-n}(R^*\varphi)(x), \quad \varphi(\theta,t) = (2|t|)^{n-2} f(2t\theta),
\]

(5.4)

and

\[
(Qf)(x) = |x|^{-n}(R\psi) \left( \frac{x}{|x|}, \frac{1}{|x|} \right), \quad \psi(x) = \frac{2^{n-1}}{\sigma_{n-1}} |x|^{2-2n} f \left( \frac{2x}{|x|^2} \right),
\]

(5.5)

provided that either side of the corresponding equality exists in the Lebesgue sense.

**Proof.** The formula (5.4) is due to Cormack and Quinto up to a minor change of notation; cf. [17, formula (11)]. To prove it, let \( x = r\eta, \ r > 0, \ \eta \in S^{n-1} \). Then (5.4) becomes

\[
(Qf)(x) = 2^{n-2} \int_{S^{n-1}} f(2r(\eta \cdot \theta) \theta) |\eta \cdot \theta|^{n-2} \, d_s \theta.
\]

(5.6)

Choose \( \gamma \in O(n) \) so that \( \eta = \gamma e_n \). Changing variable \( \theta = \gamma \xi \) and setting \( f_{r,\gamma}(x) = f(r\gamma x) \), we have

\[
(Qf)(x) = \int_{S^{n-1}} f(r\gamma e_n + r\gamma \xi) \, d_s \xi = \int_{S^{n-1}} f_{r,\gamma}(e_n + \xi) \, d_s \xi
\]

\[
= \frac{1}{\sigma_{n-1}} \int_{-1}^{1} (1 - t^2)^{(n-3)/2} \, dt \int_{S^{n-2}} f_{r,\gamma}(\sqrt{1 - t^2} \eta + (1 + t) e_n) \, d\eta.
\]
Put $t = 2s^2 - 1$. This gives

$$(Qf)(x) = \frac{2^{n-1}}{\sigma_{n-1}} \int_0^1 (1-s^2)^{(n-3)/2} s^{n-2} ds \int_{S^{n-2}} f_{r,\gamma}(2s(\sqrt{1-s^2} \eta + se_n)) \, d\eta$$

$$= 2^{n-2} \int_{S^{n-1}} f_{r,\gamma}(2\xi (\xi \cdot e_n)) |\xi \cdot e_n|^{n-2} d\xi.$$ 

The last expression coincides with the right-hand side of (5.6). The equality (5.5) follows from (5.4) and (2.27). □

The following existence result is a consequence of (5.4) and Corollary 2.7 (one can alternatively use (5.5) and Theorem 2.5).

**Theorem 5.2.** If

$$\int_{|x|<a} \frac{|f(x)|}{|x|} \, dx < \infty \quad \forall \, a > 0,$$

(5.7)

then $(Qf)(x)$ is finite for almost all $x$. If $f$ is nonnegative, radial, and (5.7) fails for some $a > 0$, then $(Qf)(x) \equiv \infty$.

Since every function in $L^p(\mathbb{R}^n)$, $1 \leq p < n/(n - 1)$, can be uniquely reconstructed from its Radon transform, the equality (5.5) implies the following statement.

**Theorem 5.3.** If

$$|x|^{2(n-1-n/p)} f(x) \in L^p(\mathbb{R}^n), \quad 1 \leq p < \frac{n}{n-1},$$

(5.8)

then $f$ can be uniquely reconstructed from $Qf$ by the formula

$$f(x) = 2^{n-1} \sigma_{n-1} |x|^{2-n} (R^{-1} g) \left( \frac{2x}{|x|^2} \right), \quad g(\theta,t) = t^{1-n}(Qf) \left( \frac{\theta}{t} \right),$$

(5.9)

where $R^{-1}$ is the inverse Radon transform.

For example, $Q$ is injective on the class of functions $f$ for which $|x|^{-2} f(x) \in L^1(\mathbb{R}^n)$. It is also injective on the class of all compactly supported continuous function on $\mathbb{R}^n$. Every such function satisfies (5.8) with $p$ sufficiently close to $n/(n - 1)$.

The operator $Q$ is not injective on the class of all functions $f$ satisfying (5.7). The kernel of $Q$ is described by the next statement which follows from Theorem 4.5. We recall that a function $f$ on $\mathbb{R}^n$ is called tempered if $\int_{\mathbb{R}^n} (1 + |x|)^{-N} f(x) \, dx < \infty$ for some $N > 0$. 

Theorem 5.4. Let $f$ be a tempered function on $\mathbb{R}^n$ satisfying (5.7). Then $(Qf)(x) = 0$ for almost all $x \in \mathbb{R}^n$ if and only if the Fourier-Laplace coefficients

$$f_{m,\mu}(r) = \int_{S_{n-1}} f(r\theta) Y_{m,\mu}(\theta) \, d\theta, \quad r > 0,$$

(5.10)
satisfy the following conditions:

(i) If $m = 0, 1$, then $f_{m,\mu}(r) = 0$ for almost all $r > 0$.

(ii) If $m \geq 2$, then $f_{m,\mu}(r)$ is a linear combination of the power functions $r^k$ with $0 \leq k \leq m - 2$ and $m - k$ even.

Another consequence of (5.4) is the support theorem that follows from Theorem 4.11. Given $a > 0$, we denote by $B_a$ and $B_{2a}$ the balls centered at the origin of radius $a$ and $2a$, respectively.

Theorem 5.5. If $f = 0$ a.e. on $B_{2a}$, then $Qf = 0$ a.e. on $B_a$. If

$$\int_{|x|<2a} \frac{|f(x)|}{|x|^{m+1}} \, dx < \infty \quad \forall m \in \mathbb{N}$$

(5.11)
and $Qf = 0$ a.e. on $B_a$, then $f = 0$ a.e. on $B_{2a}$.

All these theorems can be easily reformulated in terms of the aforementioned inverse problem for the Darboux equation. For example, the solution to this problem is unique in the class of compactly supported continuous functions on $\mathbb{R}^n$ and also in the wider class determined by Theorem 5.3. Theorem 5.5 shows that if the trace $u(x, |x|)$ is zero for almost all $x \in B_a$, then $f(x) = 0$ for almost all $x \in B_{2a}$ provided that (5.11) holds.

6. The Funk Transform

The Funk transform of a function $f$ on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ has the form

$$(Ff)(\theta) = \int_{\{\sigma \in S^n : \theta \cdot \sigma = 0\}} f(\sigma) \, d\theta \sigma$$

(6.1)
where $d\theta \sigma$ stands for the $O(n+1)$-invariant probability measure on the $(n-1)$-sphere $\{\sigma \in S^n : \theta \cdot \sigma = 0\}$; see, e.g., [21, 27]. One can readily show that $Ff$ is well defined for all $f \in L^1(S^n)$ and annihilates odd functions. Below we essentially enrich this statement using the results of Section 4 and the link between the Funk transform and the Radon transform.
Let \( e_1, \ldots, e_{n+1} \) be the coordinate unit vectors in \( \mathbb{R}^{n+1} \),
\[
\mathbb{R}^{n+1} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-1}, \quad \mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n,
\]
\[
S^n_+ = \{ \theta = (\theta_1, \ldots, \theta_{n+1}) \in S^n : 0 < \theta_{n+1} \leq 1 \}. \quad (6.2)
\]
Consider the projection map
\[
\mathbb{R}^n \ni x \xrightarrow{\mu} \theta \in S^n_+, \quad \theta = \mu(x) = \frac{x + e_{n+1}}{|x + e_{n+1}|}, \quad (6.3)
\]
which is a bijection from \( \mathbb{R}^n \) onto the set of all lines through the origin that do not lie in the plane \( x_{n+1} = 0 \). A simple geometric argument shows that \( |x| = (1 - \theta_{n+1}^2)^{1/2}/|\theta_{n+1}| \) and the inequalities \( |x| > a \) and \( |\theta_{n+1}| < (1 + a^2)^{-1/2} \) are equivalent for every \( a \geq 0 \). Moreover, if \( f \) is even and \( g(x) = (1 + |x|^2)^{-n/2} f(\mu(x)) \), then (6.3) and (2.17) yield
\[
\int_{|x|>a} g(x) \, dx = \frac{1}{2} \int_{|\theta_{n+1}|<\alpha} f(\theta) \, d\theta, \quad \alpha = (1 + a^2)^{-1/2}, \quad a \geq 0, \quad (6.4)
\]
provided that at least one of these integrals exists in the Lebesgue sense.

The map \( \mu \) extends to the bijection \( \tilde{\mu} \) from the set \( \Pi_n \) of all unoriented hyperplanes in \( \mathbb{R}^n \) onto the set
\[
\tilde{S}_n^+ = \{ \omega = (\omega_1, \ldots, \omega_{n+1}) \in S^n : 0 \leq \omega_{n+1} < 1 \}. \quad (6.5)
\]
cf. (6.2). Specifically, if \( \tau = \{ x \in \mathbb{R}^n : x \cdot \eta = t \} \in \Pi_n, \, \eta \in S^{n-1} \subset \mathbb{R}^n, \, t \geq 0 \), and \( \tilde{\tau} \) is the \( n \)-dimensional subspace containing the lifted plane \( \tau + e_{n+1} \), then \( \omega \) is defined to be a normal vector to \( \tilde{\tau} \). A simple geometric consideration shows that
\[
\omega = -\eta \cos \alpha + e_{n+1} \sin \alpha, \quad \tan \alpha = t. \quad (6.6)
\]
The above notation is used in the following theorem.

**Theorem 6.1.** Let \( g(x) = (1 + |x|^2)^{-n/2} f(\mu(x)), \, x \in \mathbb{R}^n \), where \( f \) is an even function on \( S^n \). The Funk transform \( F \) and the Radon transform \( R \) are related by the formula
\[
(Ff)(\omega) = \frac{2}{\sigma_{n-1} \sin d(\omega, e_{n+1})} (Rg)(\tilde{\mu}^{-1} \omega), \quad \omega \in \tilde{S}_n^+, \quad (6.7)
\]
where \( d(\omega, e_{n+1}) \) is the geodesic distance between \( \omega \) and \( e_{n+1} \).

**Proof.** Since the operators on both sides of this equality commute with rotations about the \( x_{n+1} \) axis, it suffices to prove the theorem when \( \omega \) is the image of the hyperplane \( \tau = \{ x \in \mathbb{R}^n : x \cdot e_n = t \} \), that is, \( \omega = -e_n \cos \alpha + e_{n+1} \sin \alpha \) where \( \tan \alpha = t, \, 0 \leq \alpha < \pi/2 \).
Let \( \tilde{\omega} = e_n \sin \alpha + e_{n+1} \cos \alpha \). We denote by \( r_\omega \) a rotation in the \( (x_n, x_{n+1}) \)-plane that takes \( e_{n+1} \) to \( \tilde{\omega} \). Changing variables and using (2.17), we obtain

\[
(Ff)(\omega) = \int_{S^n \cap \omega^\perp} f(\sigma) \, d\sigma = \int_{S^n \cap \omega^\perp} f(r_\omega \zeta) \, d\zeta
\]

\[
= \frac{2}{\sigma_{n-1}} \int_{\mathbb{R}^{n-1}} f(r_\omega e_y) \, dy \frac{1}{|y + e_{n+1}|^n}, \quad e_y = \frac{y + e_{n+1}}{|y + e_{n+1}|}.
\]

Note that

\[
r_\omega e_y = \frac{y + r_\omega e_{n+1}}{\sqrt{1 + |y|^2}} = \frac{y + e_n \sin \alpha + e_{n+1} \cos \alpha}{\sqrt{1 + |y|^2}}
\]

\[
= \frac{z + e_n \tan \alpha + e_{n+1}}{|z + e_n \tan \alpha + e_{n+1}|}, \quad z = \frac{y}{\cos \alpha}.
\]

Hence,

\[
(Ff)(\omega) = \frac{2}{\sigma_{n-1} \cos \alpha} \int_{\mathbb{R}^{n-1}} f\left(\frac{z + e_n \tan \alpha + e_{n+1}}{|z + e_n \tan \alpha + e_{n+1}|}\right) \frac{dz}{(t^2 + |z|^2 + 1)^{n/2}}
\]

\[
= \frac{2}{\sigma_{n-1} \cos \alpha} \int_{\mathbb{R}^{n-1}} f(\mu(z + te_n)) \frac{dz}{(t^2 + |z|^2 + 1)^{n/2}}
\]

\[
= \frac{2}{\sigma_{n-1} \sin d(\omega, e_{n+1})} (Rg)(e_n, t).
\]

This gives the result. \( \square \)

Theorem 6.1 enables us to essentially extend the classes of function \( f \) for which the Funk transform \( Ff \) is finite a.e. on \( S^n \) and is injective. For example, Theorem 2.5 and (6.7) imply the following statement.

**Theorem 6.2.** Suppose that an even function \( f \) is integrable on any domain \( \Omega \subset S^n \) that lies away from the poles \( \pm e_{n+1} \) and the equator \( \theta_{n+1} = 0 \). If

\[
\int_{|\theta_{n+1}| < 1/2} |f(\theta)| |\theta_{n+1}| \, d\theta < \infty,
\]

(6.8)

then \( (Ff)(\omega) \) is finite for almost all \( \omega \in S^n \). If \( f \) is nonnegative, zonal\(^2\), even, and (6.8) fails, then \( (Ff)(\omega) \equiv \infty \).

Another statement, which similarly follows from the injectivity of the Radon transform on \( L^p(\mathbb{R}^n) \), \( 1 \leq p < n/(n-1) \), is the following.

\(^2\)A function \( f \) on \( S^n \subset \mathbb{R}^{n+1} \) is called zonal if it is invariant under all orthogonal transformations preserving the \( x_{n+1} \)-axis.
Theorem 6.3. The Funk transform $F$ is injective on the space of all even functions $f$ satisfying
\[ \int_{S^n} |f(\theta)|^p |\theta_{n+1}|^{n(p-1)} d\theta < \infty, \]
for some $1 \leq p < n/(n-1)$. The restriction $p < n/(n-1)$ is sharp.

It is clear that the class of functions $f$ described in this theorem and allowing non-integrable (but tempered) singularities on the equator $\theta_{n+1} = 0$ is bigger than the known injectivity class $L^1_e(S^n)$ of even integrable functions on $S^n$.

Combining Theorem 6.1, (6.6) and (6.4) with Theorem 4.13, we arrive at the support theorem for the Funk transform.

Theorem 6.4. For $\alpha \in (0,1)$, let
\[ \mathcal{O}_\alpha = \{ \theta \in S^n : |\theta_{n+1}| < \alpha \}, \quad \tilde{\mathcal{O}}_\alpha = \{ \omega \in S^n : |\omega_{n+1}| > \sqrt{1-\alpha^2} \}. \]
If $f = 0$ a.e. in $\mathcal{O}_\alpha$, then $Ff = 0$ a.e. in $\tilde{\mathcal{O}}_\alpha$. Conversely, if
\[ \int_{\mathcal{O}_\alpha} |f(\theta)||\theta_{n+1}|^{-m} d\theta < \infty \quad \forall m \in \mathbb{N} \]
and $Ff = 0$ a.e. in $\tilde{\mathcal{O}}_\alpha$, then $f = 0$ a.e. in $\mathcal{O}_\alpha$.

In a similar way, Theorem 4.11 yields the corresponding result for the kernel $\ker F$ of the operator $F$. We know that $\ker F = \{0\}$ if the action of $F$ is considered on even integrable functions. The situation changes if the functions under consideration allow non-integrable singularities at the poles $\pm e_{n+1}$ and the equator $\theta_{n+1} = 0$, so that the Funk transform still exists in the a.e. sense.

If $f$ is even, it suffices to consider the points $\theta \in S^n$ which are represented in the spherical polar coordinates as
\[ \theta = \eta \sin \psi + e_{n+1} \cos \psi, \quad \eta \in S^{n-1}, \quad 0 < \psi < \pi/2. \]
The corresponding Fourier-Laplace coefficients have the form
\[ f_{m,\mu}(\psi) = \int_{S^{n-1}} f(\eta \sin \psi + e_{n+1} \cos \psi) Y_{m,\mu}(\eta) d\eta. \]

Theorem 6.5. Let $f$ be an even function on $S^n$ such that
\[ \int_{|\theta_{n+1}|>1/2} |f(\theta)|(1-\theta_{n+1}^2)^\gamma d\theta + \int_{|\theta_{n+1}|<1/2} |f(\theta)||\theta_{n+1}| d\theta < \infty \]
(6.11)
for some $\gamma > -1/2$. Then $(Ff)(\omega) = 0$ almost everywhere on $S^n$ if and only if every Fourier-Laplace coefficient (6.10) has the form

$$f_{m,\mu}(\psi) = \begin{cases} 0 & \text{if } m = 0,1, \\ \cos^{-n}\psi \sum_{k=0}^{m-2} c_{m,k} \cot^{n+k}\psi & \text{if } m \geq 2, \end{cases}$$

(6.12)

where $c_{m,k}$ are constant coefficients and "$'$" means that the sum includes only those terms for which $m - k$ is even.

**Proof.** By Theorem 6.1, it suffices to reformulate our statement in terms of the function $g(x) = (1 + |x|^2)^{-n/2} f(\mu(x))$ and then apply Theorem 4.11. One can readily check that (6.11) is equivalent to (4.26) (with $f$ replaced by $g$) and (6.12) mimics (4.27). \hfill \Box

**Example 6.6.** Let $\{Y_{m,\mu}\}$ be a fixed real-valued orthonormal basis of spherical harmonics in $L^2(S^{n-1})$. Consider any function of the form

$$f(\theta) = \frac{Y_{2,\mu}(\theta'/|\theta'|)}{(1-\theta^2_{n+1})^{n/2}}, \quad \theta' = (\theta_1, \ldots, \theta_n), \quad \mu = 1, 2, \ldots, \frac{(n+2)(n-1)}{2}.$$

This function is even and satisfies (6.11) for any $\gamma > n/2$. Moreover, if $\theta = \eta \sin \psi + e_{n+1} \cos \psi, \eta \in S^{n-1}, 0 < \psi < \pi/2$, then

$$f_{2,\mu}(\psi) = \sin^{-n}\psi \int_{S^{n-1}} [Y_{m,\mu}(\eta)]^2 d\eta = \sin^{-n}\psi = \cos^{-n}\psi \cot^n\psi.$$

Hence, by Theorem 6.5, $Ff = 0$ a.e. on $S^n$. In fact, $(Ff)(\omega) = 0$ for all $\omega$ away from the poles $\pm e_{n+1}$. To see that, it suffices to smoothen $f$ in arbitrarily small neighborhoods of the poles.

### 7. The Spherical Slice Transform

Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}, n \geq 2$. We denote by $\Gamma(S^n)$ the set of all $(n-1)$-dimensional geodesic spheres $\gamma \subset S^n$ passing through the north pole $e_{n+1}$. Every such $\gamma$ is a cross-section of $S^n$ by the relevant hyperplane. Below we consider an integral transform that assigns to a function $f$ on $S^n$ a function $\mathcal{S}f$ on $\Gamma(S^n)$ by the formula

$$(\mathcal{S}f)(\gamma) = \int_{\gamma} f(\eta) d_\gamma \eta$$

(7.1)

where $d_\gamma \eta$ denotes the usual surface element on $\gamma$. The map $f \to \mathcal{S}f$ is called the spherical slice transform of $f$.

Every geodesic sphere $\gamma \in \Gamma(S^n)$ can be indexed by its center, so that

$$\gamma(\xi) = \{ \eta \in S^n : \eta \cdot \xi = e_{n+1} \cdot \xi \}, \quad \gamma(\xi) = \gamma(-\xi).$$
The operator (7.1) has close connection with the following Cauchy problem for the Darboux equation on $S^n$:

$$\delta_\xi u - u_{\omega\omega} - (n-1) \cot \omega \ u_\omega = 0, \quad u(\xi, 0) = f(\xi), \quad u_\omega(\xi, 0) = 0. \quad (7.2)$$

Here $\xi \in S^n$ is the space variable, $\omega \in (0, \pi)$ is the time variable, $\delta_\xi$ is the Beltrami-Laplace operator which is applied to $u(\xi, \omega)$ in the $\xi$-variable. If $(M_\xi f)(t)$ is the spherical mean

$$(M_\xi f)(t) = \frac{(1 - t^2)^{(1-n)/2}}{\sigma_{n-1}} \int_{\eta \cdot t} f(\eta) \, d\eta, \quad t \in (-1, 1), \quad (7.3)$$

then the function $u(\xi, \omega) = (M_\xi f)(\cos \omega)$ is the solution to the problem (7.2); see, e.g., [38, 39].

The corresponding inverse problem is formulated as follows:

Let $d(\xi, e_{n+1})$ be the geodesic distance between the point $\xi$ and the north pole $e_{n+1}$. Given the trace $u(\xi, d(\xi, e_{n+1}))$ of the solution $u(\xi, \omega)$ of (7.2) on the conical set $\{(\xi, \omega) : \omega = d(\xi, e_{n+1})\}$, reconstruct the initial function $f(\xi)$.

One can easily see that $u(\xi, d(\xi, e_{n+1}))$ is exactly our slice transform (7.1) with $\gamma = \gamma(\xi)$.

Since $\gamma(\xi) = \gamma(-\xi)$, we restrict our consideration to the case when the center $\xi$ is located on the closed hemisphere

$$S^n_+ = \{\xi = (\xi_1, \ldots, \xi_{n+1}) \in S^n : 0 \leq \xi_{n+1} \leq 1\}.$$

When the closedness is not essential, we simply write $S^n_+$. If $\gamma$ is centered at $\xi \in S^n_+$ and

$$\xi = \theta \sin \psi + e_{n+1} \cos \psi, \quad \theta \in S^{n-1} \subset \mathbb{R}^n, \quad 0 \leq \psi \leq \pi/2,$$

we write $\gamma = \gamma(\xi) = \gamma(\theta, \psi)$ and $(\mathcal{G} f)(\gamma) = (\mathcal{G} f)(\xi) = (\mathcal{G} f)(\theta, \psi)$, respectively. Clearly,

$$\gamma(\xi) = \{\eta \in S^n : \eta \cdot \xi = \cos \psi\}.$$

If $\xi = e_{n+1}$, then $\gamma(\xi)$ degenerates into one point, the north pole.

We shall show that the spherical slice transform can be expressed through the hyperplane Radon transform on $\mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$ by making use of the stereographic projection of $S^n$ from the north pole $e_{n+1}$ onto $\mathbb{R}^n$. We denote this projection by $\nu$. If $\eta \in S^n \setminus \{e_{n+1}\}$ and $x = \nu(\eta) \in \mathbb{R}^n$, then

$$\eta = \nu^{-1}(x) = \frac{2x + (|x|^2 - 1) e_{n+1}}{|x|^2 + 1}. \quad (7.4)$$
Lemma 7.1. If \( f \in L^1(S^n) \), then
\[
\int_{S^n} f(\eta) \, d\eta = 2^n \int_{\mathbb{R}^n} (f \circ \nu^{-1})(x) \frac{dx}{(|x|^2 + 1)^n}. \tag{7.5}
\]

If \( g \in L^1(\mathbb{R}^n) \), then
\[
\int_{\mathbb{R}^n} g(x) \, dx = \int_{S^n} (g \circ \nu)(\eta) \frac{d\eta}{(1 - \eta_{n+1})^n}. \tag{7.6}
\]

Proof. We have
\[
\int_{S^n} f(\eta) \, d\eta = \int_0^\pi \sin^{n-1} \varphi \, d\varphi \int_{S^{n-1}} f(\omega \sin \varphi + e_{n+1} \cos \varphi) \, d\omega.
\]

Let
\[
\sin \varphi = \frac{2 \tan \frac{\varphi}{2}}{1 + \tan^2 \frac{\varphi}{2}}, \quad \cos \varphi = \frac{1 - \tan^2 \frac{\varphi}{2}}{1 + \tan^2 \frac{\varphi}{2}}, \quad \tan \frac{\varphi}{2} = \frac{1}{s}. \tag{7.7}
\]

Changing variables, we obtain
\[
l.h.s. = 2^n \int_0^\infty \frac{s^{n-1} \, ds}{(s^2 + 1)^n} \int_{S^{n-1}} f \left( \frac{2s \omega + (s^2 - 1)e_{n+1}}{s^2 + 1} \right) \, d\omega.
\]

This coincides with (7.5). The second equality follows from the first one if we note that
\[
|x|^2 + 1 = s^2 + 1 = \frac{1}{\tan^2 \frac{\varphi}{2}} + 1 = \frac{1}{\sin^2 \frac{\varphi}{2}} + 1 = \frac{2}{1 - \cos \varphi} = \frac{2}{1 - \eta_{n+1}}. \tag{7.8}
\]

□

We also need a modification of Lemma 7.1 which will be applied to the stereographic projections of slices. Consider a sphere \( S^n_r = \{ y \in \mathbb{R}^n : |y| = r \} \) and let \( \tilde{\nu} \) be the stereographic projection of \( S^n_r \) from the north pole \( re_n \) onto the hyperplane \( y_n = -q, q \geq 0 \).

Lemma 7.2. If \( f \in L^1(S^n_{r-1}) \), \( \tau = \{ y \in \mathbb{R}^n : y_n = -q \} \), then
\[
\int_{S^n_{r-1}} f(\eta) \, d\eta = (2r(r+q))^{n-1} \int_{\tau} (f \circ \tilde{\nu}^{-1})(y) \frac{dy}{(|y+qe_n|^2 + (r+q)^2)^{n-1}}. \tag{7.9}
\]
Proof. If \( \sigma = r(\omega \sin \varphi + e_n \cos \varphi) \), \( \omega \in S^{n-2}, 0 < \varphi \leq \pi \), then, by simple geometric reasons,
\[
y = \tilde{\nu}(\eta) = \omega(r + q) \cot(\varphi/2) - qe_n.
\]
(7.10)

As in the proof of Lemma 7.1,
\[
I = \int_{S^{n-1}} f(\eta) d\eta = (2r)^{n-1} \int_0^\infty \frac{s^{n-2}}{(s^2 + 1)^{n-1}} \int_{S^{n-2}} f(\eta_{r,\omega}(s)) d\omega,
\]
where
\[
\eta_{r,\omega}(s) = \frac{2rs \omega + r(s^2 - 1) e_n}{s^2 + 1}, \quad y = \omega(r + q) s - qe_n.
\]
Setting \((r + q) s = t\), we get
\[
I = (2(r+q))^{n-1} \int_0^\infty \frac{t^{n-2}}{(t^2 + (r+q)^2)^{n-1}} \int_{S^{n-2}} f(\eta_{r,\omega}(\frac{t}{r+q})) d\omega
\]
\[
= (2(r+q))^{n-1} \int_{\mathbb{R}^{n-1}} f\left(\eta_{r,\omega/|\omega|}(\frac{|\omega|}{r+q})\right) \frac{d\zeta}{(|\zeta|^2 + (r+q)^2)^{n-1}}.
\]
Since \( y = t \omega - qe_n = \zeta - qe_n \), the result follows. \( \square \)

Now we can establish connection between the slice transform \( \mathcal{S} \) in (7.1) and the hyperplane Radon transform \( R \) in (2.20).

**Theorem 7.3.** Let \( \nu \) be the stereographic projection of \( S^n \) from the north pole \( e_{n+1} \) onto the coordinate plane \( \mathbb{R}^n = e_{n+1}^\perp \), as in (7.4). Then for \( \theta \in S^{n-1}, 0 < \psi \leq \pi/2 \),
\[
(\mathcal{S} f)(\theta, \psi) = (Rg)(\theta, t), \quad t = \cot \psi,
\]
(7.11)
\[
g(x) = \left(\frac{2}{|x|^2 + 1}\right)^{n-1} (f \circ \nu^{-1})(x),
\]
(7.12)
provided that either side of (7.11) is finite when \( f \) is replaced by \( |f| \).

**Proof.** We transform \((\mathcal{S} f)(\theta, \psi)\) by making use of the formula (7.9) in which the sphere \( S^{n-1} \) in \( \mathbb{R}^n \) is replaced by the geodesic sphere \( \gamma = \gamma(\theta, \psi) \) in the hyperplane through \( e_{n+1} \). If \( \gamma \) is projected onto the \((n-1)\)-plane \( \tau' \subset \mathbb{R}^n \) and \( t \) is the Euclidean distance from the origin to \( \tau' \), then \( t = \cot \psi \) and the parameters \( r \) and \( q \) in (7.9) become
\[
r = \sin \psi = \frac{1}{\sqrt{1 + t^2}}, \quad q = t \cos \psi = \frac{t^2}{\sqrt{1 + t^2}},
\]
so that \( r(r + q) = 1 \) and \((r + q)^2 = 1 + t^2\). Instead of the \((n-1)\)-plane \( y_n = -q \) in (7.9), we have the plane \( \{ x \in \mathbb{R}^n : x \cdot \theta = t \} \). Furthermore, \( |y + qe_n| = |y - (-qe_n)| \) must be replaced by the distance between \( x \in \tau' \)
and the point \( tθ ∈ τ' \), which is \( \sqrt{|x|^2 - t^2} \). Making these changes, we obtain

\[
(\mathcal{S}f)(θ, ψ) = 2^{n-1} \int_{τ'} \frac{(f ◦ ν^{-1})(x)}{(|x|^2 + 1)^{n-1}} d_τ x,
\]

where \( d_τ x \) is the volume element in \( τ' = ν(γ) \). This gives (7.11). □

Theorem 7.3 enables us to investigate the slice transform \( \mathcal{S} \) using the properties of the hyperplane Radon transform \( R \).

**Theorem 7.4.** The integral \((\mathcal{S}f)(ξ)\) is finite for almost all \( ξ ∈ S^n_+ \) provided that

\[
\int_{S^n} \frac{|f(η)|}{(1 - η_{n+1})^{1/2}} dη < ∞. \tag{7.13}
\]

Under this condition,

\[
\int_{S^n_+} \frac{(\mathcal{S}f)(ξ)}{(1 - ξ^2)^{1/2}} dξ = 2^{-3/2} σ_{n-1} \int_{S^n} \frac{f(η)}{(1 - η_{n+1})^{1/2}} dη. \tag{7.14}
\]

**Proof.** The first statement can be easily derived from Theorem 2.5 by making use of (7.8) and (7.6). It shows that for the existence of the slice integral \( \mathcal{S}f \), the function \( f \) must have a better behavior at the north pole than just integrability. To prove (7.14), owing to (2.24), we have

\[
\frac{1}{σ_{n-1}} \int_{S^{n-1}} \int_{S^n} (Rg)(θ, t) \frac{(1 + t^2)^{n/2}}{1 + |x|^2} dθ dt = \int_{\mathbb{R}^n} g(x) \frac{1 + |x|^2}{(1 + |x|^2)^{1/2}} dx. \tag{7.15}
\]

It remains to rewrite this equality in terms of \( f \). Specifically, setting \( t = cot ψ \), by Theorem 7.3 we have

\[
l.h.s. = 2 \frac{π^2}{σ_{n-1}} \int_{S^{n-1}} (\mathcal{S}f)(θ, ψ) \sin^{n-2} ψ dθ dψ = \frac{2}{σ_{n-1}} \int_{S^n_+} \frac{(\mathcal{S}f)(ξ)}{(1 - ξ^2_{n+1})^{1/2}} dξ.
\]

For the right-hand side, applying Theorem 7.3 and the equalities (7.6) and (7.8), we obtain

\[
r.h.s. = 2^{n-1} \int_{\mathbb{R}^n} \frac{(f ◦ ν^{-1})(x)}{(1 + |x|^2)^{n-1/2}} dx = 2^{-1/2} \int_{S^n} \frac{f(η)}{(1 - η_{n+1})^{1/2}} dη.
\]

This gives the result. □

**Remark 7.5.** Theorem 2.5 also implies that the condition (7.13) is sharp. It means that there is a function \( \tilde{f} \) for which (7.13) fails and \( (\mathcal{S}\tilde{f})(ξ) ≡ ∞ \).
The following statement, which is a spherical analogue of Theorem 5.3, can be easily obtained using (7.6) and (7.8).

**Theorem 7.6.** If

\[(1 - \eta_{n+1})^{n-1-n/p} f(\eta) \in L^p(S^n), \quad 1 \leq p < \frac{n}{n-1},\]  

then \(f\) can be uniquely reconstructed from \(\mathcal{G} f\) by the formula

\[f(\eta) = (1 - \eta_{n+1})^{1-n} (R^{-1} F \circ \nu)(\eta), \quad F(\theta, t) = (\mathcal{G} f)(\theta, \cot^{-1} t),\]  

where \(R^{-1}\) is the inverse Radon transform.

A simple calculation shows that the injectivity condition (7.16) is stronger than the existence condition (7.13).

**Corollary 7.7.** The operator \(\mathcal{G}\) is injective on the class of functions \(f\) for which \((1 - \eta_{n+1})^{-1} f(\eta) \in L^1(S^n)\). Moreover, it is injective on \(L^\infty(S^n)\).

The first statement is contained in Theorem 7.6 (set \(p = 1\)). The second one follows from the observation that every bounded function satisfies (7.16) with \(p\) sufficiently close to \(n/(n-1)\).

If \(f(\eta)\) is zonal, then \((\mathcal{G} f)(\xi)\) is zonal too, because \(\mathcal{G}\) is invariant under rotations around the \(x_{n+1}\)-axis. To derive an explicit formula for \(\mathcal{G} f\) in this case, we set

\[\eta = \omega \sin \varphi + e_{n+1} \cos \varphi, \quad \omega \in S^{n-1}, \quad 0 < \varphi \leq \pi.\]

Since \(f\) is zonal, then \(f(\eta)\) depends only on \(\varphi\). We denote \(f(\eta) = f_0(\cot \varphi/2)\). Similarly, if \(\xi = \theta \sin \psi + e_{n+1} \cos \psi, \quad \theta \in S^{n-1}, \quad 0 \leq \psi \leq \pi/2\), then \((\mathcal{G} f)(\xi)\) depends only on \(\psi\). Let \((\mathcal{G} f)(\xi) = F_0(\cot \psi)\). Using this notation, we state the following

**Theorem 7.8.** If \(f\) is a zonal function satisfying (7.13), then

\[F_0(t) = 2^{n-1} \sigma_{n-2} \int_t^\infty \frac{f_0(r)}{(1 + r^2)^{n-1}} (r^2 - t^2)^{(n-3)/2} r \, dr.\]  

*Proof.* Since \(f\) is zonal, then the function \(g\) in (7.11) is radial, that is, \(g(x) \equiv \tilde{g}(|x|)\) for some one-variable function \(\tilde{g}\). By (2.22) and Theorem 7.3,

\[F_0(t) = \sigma_{n-2} \int_t^\infty \tilde{g}(r)(r^2 - t^2)^{(n-3)/2} r \, dr.\]
It remains to express \( \tilde{g} \) through \( f_0 \). We have

\[
g(x) = \frac{2^{n-1} (f \circ \nu^{-1})(x)}{(1 + |x|^2)^{n-1}}, \quad |x| = |\nu(\eta)| = \cot \varphi/2
\]

(the last equality becomes obvious if we draw the picture). Hence,

\[
\tilde{g}(r) = \frac{2^{n-1} f_0(r)}{(1 + r^2)^{n-1}},
\]

and we are done. \( \square \)

Another application of the Radon transform theory is related to the spherical harmonic decomposition of \( f(\eta) = f(\omega \sin \varphi + e_{n+1} \cos \varphi) \) in the \( \omega \)-variable. Let

\[
f_{m,\mu}(\varphi) = \int_{S^{n-1}} f(\omega \sin \varphi + e_{n+1} \cos \varphi) Y_{m,\mu}(\omega) \, d\omega.
\]

Then Theorem 4.11 together with Theorem 7.3 imply the following statement which describes the kernel of the operator \( \mathcal{S} \).

**Theorem 7.9.** Suppose that

\[
\int_{S^{n}_{\pm}} |f(\eta)| (1 - \eta_{n+1})^{-1/2} \, d\eta + \int_{S^{n}_{\pm}} |f(\eta)|(1 + \eta_{n+1})^{(N-1)/2} \, d\eta < \infty
\]

for some \( N > 0 \), where \( S^{n}_{\pm} \) denote the upper and the lower hemisphere, respectively. Then \( (\mathcal{S} f)(\xi) = 0 \) a.e. on \( S^{n} \) if and only every Fourier-Laplace coefficient \( f_{m,\mu}(\varphi) \) has the form

\[
f_{m,\mu}(\varphi) = \begin{cases} 
0 & \text{if } m = 0, 1, \\
(1 - \cos \varphi)^{n-1} \sum_{k=0}^{m-2} c_{m,k} \left(\tan \frac{\varphi}{2}\right)^{n+k} & \text{if } m \geq 2,
\end{cases}
\]

where \( c_{m,k} \) are constant coefficients and "\( \cdot \)" means that the sum includes only those terms, for which \( m - k \) is even.

Theorem 7.3 combined with Theorem 4.11 gives the support theorem for the slice transform \( \mathcal{S} f \).

**Theorem 7.10.** Let \( a \in (0,1) \) and consider the spherical caps

\[
\Omega_a = \{ \eta \in S^n : \eta_{n+1} > a \}, \quad \tilde{\Omega}_a = \{ \xi \in S^n : \xi_{n+1} > \sqrt{(1+a)/2} \}.
\]

If \( f = 0 \) a.e. in \( \Omega_a \), then \( \mathcal{S} f = 0 \) a.e. in \( \tilde{\Omega}_a \). Conversely, if

\[
\int_{\Omega_a} (1 - \eta_{n+1})^{-1-m/2} |f(\eta)| \, d\eta < \infty \quad \forall \, m \in \mathbb{N}
\]

and \( \mathcal{S} f = 0 \) a.e. in \( \tilde{\Omega}_a \), then \( f = 0 \) a.e. in \( \Omega_a \).
8. The Totally Geodesic Radon Transform on the Hyperbolic Space

We will be dealing with the hyperboloid model of the $n$-dimensional real hyperbolic space which is described in [22]; see also [6]. Let $\mathbb{E}^{n,1} \sim \mathbb{R}^{n+1}, n \geq 2,$ be the $(n+1)$-dimensional pseudo-Euclidean real vector space with the inner product

$$[x, y] = -x_1 y_1 - \ldots - x_n y_n + x_{n+1} y_{n+1}. \quad (8.1)$$

The $n$-dimensional real hyperbolic space $\mathbb{H}^n$ is realized as the upper sheet of the two-sheeted hyperboloid in $\mathbb{E}^{n,1}$, that is,

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : \|x\|^2 = 1, x_{n+1} > 0\}.$$ 

The corresponding one-sheeted hyperboloid is defined by

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : \|x\|^2 = -1\}.$$ 

Both $\mathbb{H}^n$ and $\mathbb{H}^n$ are orbits of the identity component $G = SO_0(n, 1)$ of the special pseudo-orthogonal group $SO(n, 1)$ of linear transformations preserving the bi-linear form $[x, y]$ and having the determinant 1.

Unlike the boldfaced $x, y \in \mathbb{E}^{n,1},$ the usual letters $x, y$ will be used for points in $\mathbb{H}^n.$ The geodesic distance between the points $x$ and $y$ is defined by $d(x, y) = \cosh^{-1}[x, y].$ The $G$-invariant measure $dx$ on $\mathbb{H}^n$ is normalized so that for every $f \in L^1(\mathbb{H}^n),$

$$\int_{\mathbb{H}^n} f(x) \, dx = \int_0^\infty \sinh^{n-1} r \, dr \int_{S^{n-1}} f(\theta \sinh r + e_{n+1} \cosh r) \, d\theta. \quad (8.2)$$

The totally geodesic Radon transform of a function $f$ on $\mathbb{H}^n$ is defined by the formula

$$(\mathcal{R} f)(\xi) = \int_{\{x \in \mathbb{H}^n : [x, \xi] = 0\}} f(x) \, d_x x, \quad \xi \in \mathbb{H}^n, \quad (8.3)$$

and represents an even function on $\mathbb{H}^n.$ The corresponding dual transform of an even function $\varphi$ on $\mathbb{H}^n$ has the form

$$(\mathcal{R}^* \varphi)(x) = \int_{\{\xi \in \mathbb{H}^n : [x, \xi] = 0\}} \varphi(\xi) \, d_\xi \xi, \quad x \in \mathbb{H}^n. \quad (8.4)$$

The measures $d_x x$ and $d_\xi \xi$ are $G$-images of the corresponding measures on the sets

$$\mathbb{H}^{n-1} = \{y \in \mathbb{H}^n : y_n = 0\}, \quad S^{n-1} = \{\eta \in \mathbb{H}^n : \eta_{n+1} = 0\}.$$
Specifically, let $\omega_x$ and $\omega_\xi$ be hyperbolic rotations in $G$ satisfying
\[ \omega_x : e_{n+1} \to x, \quad \omega_\xi : e_n \to \xi. \tag{8.5} \]

If $f_\xi(y) = f(\omega_\xi y)$ and $\varphi_x(\eta) = \varphi(\omega_x \eta)$, then the precise meaning of the above integrals is the following:
\[ (\mathcal{R} f)(\xi) = \int_{\mathbb{H}^{n-1}} f_\xi(y) \, dy, \quad (\mathcal{R}^* \varphi)(x) = \int_{S^{n-1}} \varphi_x(\eta) \, d^* \eta. \tag{8.6} \]

Both $\mathcal{R}$ and $\mathcal{R}^*$ are $G$-invariant.

Let $S^{n-1}$ and $S^{n-2}$ be the unit spheres in the coordinate planes $\mathbb{R}^n = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_n$ and $\mathbb{R}^{n-1} = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{n-1}$, respectively. We use the notation
\[ a_r = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh r & \sinh r \\ 0 & \sinh r & \cosh r \end{bmatrix} \tag{8.7} \]
for a hyperbolic rotation in the plane $(x_n, x_{n+1})$. Given $x \in \mathbb{H}^n$ and $\xi \in \mathbb{H}^n$, we set
\[ x = \theta \sinh r + e_{n+1} \cosh r = \omega_\theta a_r e_{n+1}, \quad \theta \in S^{n-1}, \ r \in \mathbb{R}_+, \tag{8.8} \]
\[ \xi = \sigma \cosh \rho + e_{n+1} \sinh \rho = \omega_\sigma a_\rho e_n, \quad \sigma \in S^{n-1}, \ \rho \in \mathbb{R}. \tag{8.9} \]

Here $\omega_\theta$ and $\omega_\sigma \in SO(n)$ are arbitrary rotations satisfying $\omega_\theta e_n = \theta$, $\omega_\sigma e_n = \sigma$; $a_\rho$ has the same meaning as $a_r$ in (8.7).

**Lemma 8.1.** Let $f_\sigma(x) = f(\omega_\sigma x)$, $\varphi_\theta(\xi) = \varphi(\omega_\theta \xi)$. Then
\[ (\mathcal{R} f)(\xi) = \int_0^\infty \sinh^{n-2} s \, ds \tag{8.10} \]
\[ \times \int_{S^{n-2}} f_\sigma(\omega \sinh s + (e_n \sinh \rho + e_{n+1} \cosh \rho) \cosh s) \, d\omega, \]
\[ (\mathcal{R}^* \varphi)(x) = \int_{S^{n-1}} \varphi_\theta(\eta' + e_n \eta_n \cosh r + e_{n+1} \eta_n \sinh r) \, d^* \eta, \tag{8.11} \]

$\eta = (\eta', \eta_n)$, provided that the corresponding integrals exist in the Lebesgue sense.
Proof. Consider the totally geodesic Radon transform (8.3) and set 
\[ x = \omega_a(x) \] 
where \( \sigma \) and \( a_\rho \) are the same as in (8.9). We have 
\[ (Rf)(\xi) = \int_{H^n} f(\omega) d\omega. \]
This gives (8.10). Further, setting \( \xi = \omega a_t \eta \) in (8.4), we obtain 
\[ (R^* \varphi)(x) = \int_{S^{n-1}} \varphi(a_t \eta) d\eta. \]
\[ \square \]

The totally geodesic transform (8.3) and its dual (8.4) are intimately 
connected with the hyperplane Radon transform \( R \) and its dual \( R^* \). To 
establish this connection we fix the notation by setting 
\[ (Rg)(\sigma, t) = \int g(\sigma t + u) d\sigma u, \quad (R^* h)(y) = \int h(\sigma, y \cdot \sigma) d\sigma. \]
Here \( t \in \mathbb{R} \) and \( \sigma^\perp \) is the subspace of \( \mathbb{R}^n \) orthogonal to \( \sigma \in S^{n-1} \). Let 
\[ x = \theta \sinh r + e_{n+1} \cosh r, \quad \theta \in S^{n-1}, \quad r \in \mathbb{R}, \]
\[ \xi = \sigma \cosh \rho + e_{n+1} \sinh \rho, \quad \sigma \in S^{n-1}, \quad \rho \in \mathbb{R}. \]
We also write \( \bar{x} = (x_1, \ldots, x_n), \quad \bar{\xi} = (\xi_1, \ldots, \xi_n), \)
\[ x = (\bar{x}, x_{n+1}) = (\theta \sinh r, \cosh r) \in \mathbb{H}^n, \]
\[ \xi = (\bar{\xi}, \xi_{n+1}) = (\sigma \cosh \rho, \sinh \rho) \in \mathbb{H}^n, \]
\[ f(x) \equiv f(\theta \sinh r, \cosh r), \quad \varphi(\xi) \equiv \varphi(\sigma \cosh \rho, \sinh \rho). \]

To every \( x = (\bar{x}, x_{n+1}) \in \mathbb{H}^n \) we associate its image \( y \) in the tangent 
hyperplane to \( \mathbb{H}^n \) at the point \((0, \ldots, 0, 1) \in \mathbb{H}^n \) so that \( x \) and \( y \) lie on 
the same line through the origin \((0, \ldots, 0, 0) \) of \( E^{n-1} \). If this tangent 
hyperplane is identified with the Euclidean space \( \mathbb{R}^n \), then the map 
\[ x \rightarrow y \] 
is a bijection between \( \mathbb{H}^n \) and the unit ball \( B_n = \{ y \in \mathbb{R}^n : |y| < 1 \} \), so that 
\[ y = \frac{\bar{x}}{x_{n+1}}, \quad x = (\bar{x}, x_{n+1}) = \left( \frac{y}{\sqrt{1 - |y|^2}}, \frac{1}{\sqrt{1 - |y|^2}} \right). \]
Under this map, every totally geodesic submanifold of $\mathbb{H}^n$ is associated with a chord in $B_n$ of the same dimension. The corresponding functions on $\mathbb{R}^n$ and $Z_n = S^{n-1} \times \mathbb{R}$ are defined by

$$g(y) = (1 - |y|^2)^{-n/2}_+ f \left( \frac{y}{\sqrt{1 - |y|^2}}, \frac{1}{\sqrt{1 - |y|^2}} \right), \quad y \in \mathbb{R}^n, \quad (8.14)$$

$$h(\sigma, t) = (1 - t^2)^{-n/2}_+ \varphi \left( \frac{\sigma}{\sqrt{1 - t^2}}, \frac{t}{\sqrt{1 - t^2}} \right), \quad \sigma \in S^{n-1}, \ t \in \mathbb{R}, \quad (8.15)$$

so that

$$f(x) = \frac{g(\tilde{x}/x_{n+1})}{x_{n+1}} = \frac{g(\theta \tanh r)}{\cosh^n r}, \quad (8.16)$$

$$\varphi(\xi) = (1 + \xi_{n+1}^2)^{-n/2}_+ h \left( \frac{\tilde{\xi}}{\sqrt{1 + \xi_{n+1}^2}}, \frac{\xi_{n+1}}{\sqrt{1 + \xi_{n+1}^2}} \right) = \frac{h(\sigma, \tanh \rho)}{\cosh^n \rho}. \quad (8.17)$$

**Lemma 8.2.** For every $\delta \geq 0$,

$$\int_{d(x, e_{n+1}) > \delta} f(x) \frac{dx}{x_{n+1}} = \int_{\tanh \delta < |y| < 1} g(y) \, dy \quad (8.18)$$

provided that either integral exists in the Lebesgue sense.

**Proof.** We have

$$r.h.s. = \int_{\tanh \delta}^1 S^{n-1} \, ds \int_{S^{n-1}} g(\theta s) \, d\theta = \int_{\delta}^{\infty} \frac{\tanh^{n-1} r}{\cosh^n r} \, dr \int_{S^{n-1}} g(\theta \tanh r) \, d\theta.$$ 

By (8.2) and (8.16), this coincides with the left-hand side. \qed

**Lemma 8.3.** The following relations hold provided that the integral in either side of the corresponding equality exists in the Lebesgue sense:

$$(\mathcal{M} f)(\xi) = \frac{1}{\cosh \rho} (Rg)(\sigma, \tanh \rho), \quad (8.19)$$

$$(\mathcal{M}^* \varphi)(x) = \frac{1}{\cosh \rho} (R^* h)(\theta \tanh r). \quad (8.20)$$
Proof. Let $\omega_\theta, \omega_\sigma \in SO(n)$ be arbitrary rotations satisfying $\omega_\theta e_n = \theta, \omega_\sigma e_n = \sigma$. We write $g_\sigma(y) = g(\omega_\sigma y)$. By (8.10) and (8.16),

$$(\mathfrak{R}f)(\xi) = \int_0^\infty \sinh^{-2}s \, ds \int_{S^{n-2}} g_\sigma \left( \frac{\omega \tanh s + e_n \tanh \rho}{\cosh \rho} \right) \frac{d\omega}{(\cosh s \cosh \rho)^n}$$

(set $t = \tanh s / \cosh \rho$)

$$= \int_0^{t^{n-2}dt / \cosh \rho} \int_{S^{n-2}} g_\sigma(\omega t + e_n \tanh \rho) \, d\omega$$

$$= \frac{1}{\cosh \rho} \int_{\mathbb{R}^{n-1}} g_\sigma(v + e_n \tanh \rho) \, dv.$$ 

The last expression coincides with (8.19). Let us prove (8.20). Denoting $h_\theta(\sigma, t) = h(\omega_\theta \sigma, t)$ and using (8.11) and (8.17), we have

$$(\mathfrak{R}^* \varphi)(x) = \frac{1}{\sigma_{n-1}} \int_{-1}^1 (1 - \eta_n^2)^{(n-3)/2} d\eta_n \times \int_{S^{n-2}} \varphi_\theta \left( \omega \sqrt{1 - \eta_n^2} + e_n \eta_n \cosh r + e_{n+1} \eta_n \sinh r \right) \, d\omega$$

$$= \frac{1}{\sigma_{n-1}} \int_{-1}^1 (1 - \eta_n^2)^{(n-3)/2} d\eta_n \int_{S^{n-2}} h_\theta \left( \frac{\omega \sqrt{1 - \eta_n^2} + e_n \eta_n \cosh r}{\sqrt{1 + \eta_n^2 \sinh^2 r}}, \frac{\eta_n \sinh r}{\sqrt{1 + \eta_n^2 \sinh^2 r}} \right) \, d\omega$$

Setting $\eta_n = \zeta_n / \sqrt{\cosh^2 r - \zeta_n^2 \sinh^2 r}$, we continue

$$(\mathfrak{R}^* \varphi)(x) = \frac{1}{\sigma_{n-1} \cosh r} \int_{-1}^1 (1 - \zeta_n^2)^{(n-3)/2} d\zeta_n \times \int_{S^{n-2}} h_\theta(\omega \sqrt{1 - \zeta_n^2} + e_n \zeta_n, \zeta_n \tanh r) \, d\omega$$

$$= \frac{1}{\cosh r} \int_{S^{n-1}} h_\theta(\zeta, (\zeta \cdot e_n) \tanh r) \, d_s \zeta = \frac{1}{\cosh r} (R^* h)(\theta \tanh r).$$
Lemmas 8.2 and 8.3 combined with the support theorem for the Radon transform $R$ (see Theorem 4.13), give the corresponding result for $\mathfrak{R}$.

**Theorem 8.4.** Let $a > 0$ and let $\tau_\xi$ denote the totally geodesic sub-
manifold in $\mathbb{H}^n$ indexed by $\xi \in \mathbb{H}^n$. If $f(x) = 0$ for almost all $x \in \mathbb{H}^n$ satisfying $d(x, e_{n+1}) > a$, then $(\mathfrak{R} f)(\xi) = 0$ for almost all $\xi$ satisfying $d(\tau_\xi, e_{n+1}) > a$. Conversely, if

$$\frac{1}{a} \int_{d(x, e_{n+1}) > a} |f(x)| \frac{dx}{x_{n+1}} < \infty$$

and $(\mathfrak{R} f)(\xi) = 0$ for almost all $\xi$ with $d(\tau_\xi, e_{n+1}) > a$, then $f(x) = 0$ for almost all $x \in \mathbb{H}^n$ satisfying $d(x, e_{n+1}) > a$.

We observe an amazing fact that, unlike the Euclidean case in The-
orem 4.13, the above theorem does not require a rapid decay of $f$ at
infinity. This fact was first discovered by Kurusa [28] for continuous
functions $f$. The reason is that the function $g$ in (8.19) is supported
in the unit ball and therefore, the condition (4.28) holds automatic-
ally.

Note also that the condition $d(\tau_\xi, e_{n+1}) > a$ is equivalent to $|\xi_{n+1}| > \sinh a$ and $d(x, e_{n+1}) > a$ is equivalent to $x_{n+1} > \cosh a$.

In a similar way, Theorem 4.11 yields the corresponding result for
the kernel of the operator $\mathfrak{R}$. We write $x \in \mathbb{H}^n$ in the hyperbolic polar
coordinates as $x = \theta \sinh r + e_{n+1} \cosh r$, $\theta \in S^{n-1}$, $r > 0$, and compute
the Fourier-Laplace coefficients

$$f_{m,\mu}(r) = \int_{S^{n-1}} f(\theta \sinh r + e_{n+1} \cosh r) Y_{m,\mu}(\theta) d\theta. \quad (8.21)$$

**Theorem 8.5.** Suppose that

$$\int_{\theta_{n+1} < 2} |f(x)| (x_{n+1} - 1)^a dx + \int_{\theta_{n+1} > 2} |f(x)| \frac{dx}{x_{n+1}} < \infty$$

for some $a > -1/2$. Then $(\mathfrak{R} f)(\xi) = 0$ almost everywhere on $\mathbb{H}^n$ if
and only every Fourier-Laplace coefficient (8.21) has the form

$$f_{m,\mu}(r) = \begin{cases} 0 & \text{if } m = 0, 1, \\ \cosh^{-n} r \sum_{k=0}^{m-2} c_{m,k} \coth^{n+k} \psi & \text{if } m \geq 2, \end{cases} \quad (8.22)$$

where $c_{m,k}$ are constant coefficients and """ means that the sum in-
cludes only those terms for which $m - k$ is even.
Analogues of Theorems 8.4 and 8.5 for the dual transform $R^*$ can be similarly derived from Theorems 4.14 and 4.5, respectively, using the connection (8.20). We leave this exercise to the interested reader.

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