On the spectral sequence of the Swiss-cheese operad

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We prove that the homology of the Swiss-cheese operad is a Koszul operad. As a consequence, we obtain that the spectral sequence associated to the stratification of the compactification of points on the upper half plane collapses at the second stage, proving a conjecture by A Voronov in [17]. However, we prove that the operad obtained at the second stage differs from the homology of the Swiss-cheese operad.

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1 Introduction

An operad $\mathcal{O}$ is called differentiable when it is defined on the symmetric monoidal category of differentiable manifolds. If each $\mathcal{O}(n)$ is a manifold with corners whose connected boundary components are cartesian products of $\mathcal{O}(k)$ (with $k < n$) and the operad structure is given by the inclusion map of the boundary strata, then the operad is called stratified. To every stratified operad are associated dg-operads given by the spectral sequence induced by a natural filtration on its singular chain complex: the boundary strata filtration. Since that filtration is given by the codimension of the boundary components, it is finite and hence converges to the homology $H_*(\mathcal{O})$ at some finite stage. One would naturally wonder whether the spectral sequence degenerates and if the operad structure on the $E^\infty$–term is isomorphic to the operad structure on $H_*(\mathcal{O})$.

In the present paper we study the spectral sequence of a stratified operad, given by Kontsevich’s compactification [14], which is homotopically equivalent to the Swiss-cheese operad $\mathcal{SC}$. Among the related algebraic structures are Kajiura and Stasheff’s OCHA [10; 12], Leibniz pairs (see Flato, Gerstenhaber and Voronov [3]) and extensions of those considered by Dolgushev [2]. The relation between OCHAS, Leibniz pairs and the Swiss-cheese operad has been carefully studied by the authors in [9], where the 0th homology of the Swiss-cheese operad $\mathcal{SC}$ was related to the first row of the spectral sequence associated to the Kontsevich compactification. One of the purposes of [9] was to prove an $\mathcal{SC}$ analogue of the following fact concerning the little disks operad $\mathcal{D}_2$. 

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**Proposition**  The ⁰th homology of the operad \( \mathcal{D}_2 \) is Koszul dual to a suspension of the top homology.

In the case of little disks, the ⁰th homology is the operad \( \text{Com} \) and the top homology is a desuspension of the operad \( \text{Lie} \). In fact, the above Proposition is a consequence of a theorem, proved by Getzler and Jones, according to which the Gerstenhaber operad \( H_*(\mathcal{D}_2) \) is, up to suspension, a self-dual Koszul operad.

We proved in [9] that the ⁰th homology of \( \mathcal{S}C \) is a Koszul quadratic-linear operad, and that its Koszul dual \( H_0(\mathcal{S}C) \), which is a dg-operad, has for homology a suspension of the top homology. Note that in the context of \( \mathcal{S}C \), the top homology does not form an operad, so by top homology we mean the smallest operad containing the top degrees generators.

The little disks operad \( \mathcal{D}_2 \) is not stratified. However, by considering the real Fulton–MacPherson compactification of the moduli space of points in the complex plane, we get a homotopically equivalent stratified operad sometimes denoted \( \mathcal{F}_2 \) (see Salvatore [16]). The same compactification procedure can be applied to the Swiss-cheese operad \( \mathcal{S}C \) (the homotopy equivalent stratified operad obtained is sometimes denoted \( \mathcal{H}_2 \)). So, by passing to a homotopy equivalent operad, we can assume that both little disks and Swiss-cheese operads are stratified. Furthermore, two homotopy equivalent operads give isomorphic homology operads. Hence, to avoid cumbersome notation we will work with the stratified versions of little disks and Swiss-cheese, while keeping the notation: \( \mathcal{D}_2 \) and \( \mathcal{S}C \).

The main result of this paper is Theorem 4.2.2, where we prove the conjecture by A Voronov in [17] stating that the spectral sequence \( E(\mathcal{S}C) \) of the Swiss-cheese operad collapses at the second stage. This is done by proving that the homology of the Swiss-cheese operad is a quadratic-linear Koszul operad in the sense of Galvez-Carillo, Tonks and Vallette in [5]. The same result is true for the homology of \( \mathcal{S}C^{vor} \), a variant of \( \mathcal{S}C \). The relation (modulo (de)suspension) between \( E^1(\mathcal{S}C) \) and the cobar construction of the cohomology cooperad \( H^*(\mathcal{S}C) \) is well known [2], but in our setting it is slightly different, so a proof is given in Lemma 4.1.1. We compute in Proposition 4.2.1 the operad structure on \( E^2(\mathcal{S}C) = E^\infty(\mathcal{S}C) \). To sum up we get the following.

*Algebras over \( H_*(\mathcal{S}C) \) are triples \((G, A, f)\) where \( G \) is a Gerstenhaber algebra, \( A \) is an associative algebra and \( f: G \to A \) is a central map such that \( f(gg') = f(g) f(g') \), whereas algebras over \( E^\infty(\mathcal{S}C) \) are triples \((G, A, f)\), where \( G \) is a Gerstenhaber algebra, \( A \) is an associative algebra and \( f: G \to A \) is a central map such that \( f(gg') = 0 \).*

We finally prove in Proposition 4.3.1 that the two operads are not isomorphic as operads, though the \( S \)–module structures are the same.
Note that this case differs from the little disks case: Getzler and Jones [6] have proven that the spectral sequence associated to the stratification of $D_2$ collapses at the second stage and that $E^2(D_2) = \Lambda^{-1}(H_*(D_2)^1)$ is isomorphic to $e_2 = H_*(D_2)$.

The plan of the paper is the following. Section 2 is devoted to preliminaries and notation. Section 3 is devoted to the homology of the operads $\mathcal{SC}^\text{vor}$ and $\mathcal{SC}$. As in [9], in order to understand the structure of the operad $\mathcal{sc} = H_*(\mathcal{SC})$, it is necessary to first understand the operad $\mathcal{SC}^\text{vor}$, another version of the Swiss-cheese operad, whose homology is quadratic. The end of Section 3 is devoted to the structure of $H_*(\mathcal{sc})$. We use the techniques of distributive laws, as well as the results obtained in [9]. Section 4 concentrates on the spectral sequence of $\mathcal{SC}$.

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2 Preliminaries

2.1 On differential graded vector spaces

We work on a ground field $k$ of characteristic 0. The category $\text{dgvs}$ is the category of lower graded $k$–vector spaces together with a differential of degree $-1$. Objects in $\text{dgvs}$ are called for short dgvs. The degree of $x \in V$, where $V$ is a dgvs is denoted by $\deg x$. We say that a dgvs $V$ is finite dimensional if for each $n$, the vector space $V_n$ is finite dimensional.

The vector space $\text{hom}_k(V, W)$ denotes the $k$–linear morphisms between two vector spaces $V$ and $W$. When $V$ and $W$ are objects in $\text{dgvs}$, then we have that the differential graded vector space of maps from $V$ to $W$ is $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_i(V, W)$, where $\text{Hom}_i(V, W) = \prod_n \text{hom}_k(V_n, W_{n+i})$ together with the differential $(\partial f)(v) = d_W(f(v)) - (-1)^{|f|} f(d_V v)$.

The graded linear dual of $V$ in $\text{dgvs}$ is $V^* = \text{Hom}(V, k)$, where $k$ is concentrated in degree 0 with 0–differential. Consequently we have that $(V^*)_n = (V_{-n})^*$ and $(\partial f)(x) = (-1)^n f(d_V x)$ for any $f \in (V^*)_n$ and $x \in V_{-n+1}$. The suspension of a dgvs $V$ is denoted by $sV$ and defined as $(sV)_n = V_{n-1}$.

2.2 On operads, 2–colored operads, cooperads

2.2.1 On the symmetric group The symmetric group acting on $n$ elements is denoted by $S_n$. An element $\sigma \in S_n$ will be denoted by its image notation $(\sigma(1) \cdots \sigma(n))$. The trivial representation of $S_n$ is denoted by $k$, the signature representation by $\text{sgn} \, n$ and the regular representation by $k[S_n]$. 

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2.2.2 Collections and $S$–modules In this article, we work with 2–colored (co)–operads, either in the category of spaces, or in dgvs. The colors we consider are denoted by $c$ (for closed) and $o$ (for open). A 2–collection $\mathcal{P}$ is a family of dgvs, given by $(\mathcal{P}(c; d) = \mathcal{P}(c_1, \ldots, c_n; d))_{\{c_i, d \in \{c, o\}\}}$. Let $c = (c_1, \ldots, c_n)$ be an $n$–tuple of colors. The symmetric group $S_n$ acts on the set of $n$–tuples of colors by $\zeta \cdot \sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. An $S$–module is a 2–collection $\mathcal{P}$ endowed with an action of the symmetric groups, sending $(x \in \mathcal{P}(c_1, \ldots, c_n; d), \sigma \in S_n)$ to $x \cdot \sigma \in \mathcal{P}(\zeta \cdot \sigma; d)$.

Note that it is an extension of the usual definition of an $S$–module, which is a family of dgvs $(Q(n))_{n \geq 1}$ such that for each $n$, $Q(n)$ is a right $S_n$–module. We can consider this collection as an $S$–module, where $Q(c; d)$ is $Q(n)$ if for all $i, c_i = c$ and $d = c$ and is 0 otherwise.

Given an $S$–module $M$, we may want to consider some truncation $N$ of it, invariant under the action of the symmetric groups. By definition a sub-$S$–module of $N$ is a sub-dgvs invariant under the induced action of the symmetric groups.

2.2.3 Operads The 2–collection $I$ defined by $I(c; c) = k, I(o, o) = k$ and $I(c; d) = 0$ elsewhere, plays a special role. Indeed, a 2–colored operad is an $S$–module together with a unit map $\eta: I \to \mathcal{P}$ and composition maps

$$\gamma: \mathcal{P}(c_1, \ldots, c_n; d) \otimes \mathcal{P}(b^1; c_1) \otimes \cdots \otimes \mathcal{P}(b^n; c_n) \to \mathcal{P}(b^1, \ldots, b^n; d),$$

which are associative, unital and respects the action of the symmetric groups.

We write $f(g_1, \ldots, g_n)$ for the image of $f \otimes g_1 \otimes \cdots \otimes g_n$ or $f(\text{id} \otimes i \otimes g \otimes \text{id} \otimes \cdots \otimes n^{-1} \otimes i)$ whenever every $g$ except one is the identity. We often use the same notation for $f$ in $\mathcal{P}$ or for $f$ seen as an operation on variables. In that context, we use the Koszul sign convention

$$(f \otimes g)(a \otimes b) = (-1)^{|a||g|} f(a) \otimes g(b).$$

Because of the action of the symmetric groups, one may only consider the spaces

$$\mathcal{P}(n, m; d) = \mathcal{P}(e, \ldots, e, o, \ldots, o; d).$$

In this paper, we only consider 2–colored operads such that $\mathcal{P}(0, 0; x) = 0$ and $\mathcal{P}(1, 0; c) = k = \mathcal{P}(0, 1; o)$. They are naturally augmented, that is, there is a morphism of operads $\mathcal{P} \to I$ and $\overline{\mathcal{P}}$ denotes the kernel of this map.

Any operad $\mathcal{P}$ can be considered as a 2–colored operad with $\mathcal{P}(c; d) = \mathcal{P}(n)$ if for all $i, c_i = c$ and $d = c$, $\mathcal{P}(o; o) = k$ and $\mathcal{P}(c; d) = 0$ otherwise.

In the sequel, we often use the generic terminology of operads for either operads or 2–colored operads, or operads seen as 2–colored operads.
2.2.4 Suspension of $S$–modules and operads The suspension of the $S$–module $P$ is

$$\Lambda P(n, m; x) = s^{1-n-m}P(n, m; x) \otimes \text{sgn}_{n+m}.$$  

If $P$ is an operad, then the structure of $P$–algebra on the pair $(V_c, V_o)$ is equivalent to the structure of $\Lambda P$–algebra on the pair $(sV_c, sV_o)$.

The suspension of the $2$–collection $P$ with respect to the color $c$ is

$$\Lambda_c P(n, m; x) = s^{\delta_{c,c}-n}P(n, m; x) \otimes \text{sgn}_{n},$$

where $\delta$ denotes the Kronecker symbol. If $P$ is an operad, then the structure of $P$–algebra on the pair $(V_c, V_o)$ is equivalent to the structure of $\Lambda_c P$–algebra on the pair $(sV_c, V_o)$.

2.2.5 Operads defined by generators and relations The free operad generated by an $S$–module $E$ is denoted by $\mathcal{F}(E)$. It is weight graded by the number $n$ of vertices of the underlying trees and $\mathcal{F}(n)(E)$ denotes the component of weight $n$.

A quadratic operad $\mathcal{F}(E, R)$ is an operad of the form $\mathcal{F}(E)/(R)$, where $E$ is an $S$–module, $R$ is a sub-$S$–module of $\mathcal{F}(2)(E)$ and $(R)$ is the ideal generated by $R$. There are analogous notions of cooperads, free cooperads $\mathcal{F}^c(E)$ cogenerated by $E$, and of cooperads cogenerated by an $S$–module $V$ with correlation $R$ denoted by $C(V, R)$.

Describing an operad is equivalent to describing algebras over it. In the text, we say that an operad $P$ is generated by $E$ with relations $R$ written as

\[(*) \quad r_1 = r_2.\]

This notation means that any $P$–algebra satisfies the relation $(*)$. At the level of operads, this is understood as $R$ contains the element $r_1 - r_2$.

2.2.6 Koszul dual Any quadratic operad $P = \mathcal{F}(E, R)$ admits a Koszul dual coop- erad given by $P^! = C(sE, s^2 R)$.

The Koszul dual operad $P^!$ of a finite dimensional quadratic operad $P$ is

\[(1) \quad P^! := (\Lambda P^\dagger)^* \quad \text{or equivalently,} \quad P^\dagger = (\Lambda P^!)^*.\]

When $P$ is a binary quadratic operad, we can use the original definition of Ginzburg and Kapranov [7] (see also Loday and Vallette [15, chapter 7]) to compute its Koszul dual operad. Namely, if $P = \mathcal{F}(E, R)$, then $P^! = \mathcal{F}(E^\vee, R^\perp)$, where $E^\vee = E^* \otimes \text{sgn}_2$ and $R^\perp$ denotes the orthogonal of $R$ under the pairing $\mathcal{F}(2)(E) \otimes \mathcal{F}(2)(E^\vee) \rightarrow k$. 

2.2.7 Bar and cobar constructions

As it is the case for algebras and coalgebras, there is a pair of adjoint functors between operads and cooperads given by the bar and cobar construction. We refer to [6] for a detailed account on this topic. Let us only recall that the cobar construction of a cooperad $C$ is denoted by $\Omega C$ and as a (nondifferential) operad, is the free operad $\mathcal{F}(s^{-1}\tilde{C})$ where $\tilde{C}$ is the coaugmentation ideal of the cooperad. The differential is computed from the differential of $C$ and the cooperad structure. The bar construction is denoted $B\mathcal{P}$ and it is defined similarly. When $C$ is finite dimensional one has

$$(\Omega C)^* = B(C^*).$$

2.3 Two versions of the Swiss-cheese operad

Here we recall the two definitions for the Swiss-cheese operad we have introduced in [9]. We denote by $\mathcal{D}_2$ the little disks operad.

For $m, n \geq 0$ such that $m + n > 0$, let us define $SC(n, m; o)$ as the space of those configurations $d \in (2n + m)$ such that its image in the disk $D^2$ is invariant under complex conjugation and exactly $m$ little disks are left fixed by conjugation. A little disk that is fixed by conjugation must be centered at the real line, in this case it is called open. Otherwise, it is called closed. The little disks in $SC(n, m; o)$ are labelled according to the following rules.

(i) Open disks have labels in $\{1, \ldots, m\}$ and closed disks have labels in $\{1, \ldots, 2n\}$.

(ii) Closed disks in the upper half plane have labels in $\{1, \ldots, n\}$. If conjugation interchanges the images of two closed disks, their labels must be congruent modulo $n$.

There is an action of $S_n \times S_m$ on $SC(n, m; o)$ extending the action of $S_n \times \{e\}$ on pairs of closed disks having modulo $n$ congruent labels and the action of $\{e\} \times S_m$ on open disks. Figure 1 illustrates a point in the space $SC(n, m; o)$.

The 2–collection $SC$ is defined as follows. For $m, n \geq 0$ with $m + n > 0$, $SC(n, m; o)$ is the configuration space defined above and $SC(0, 0; o) = \emptyset$. For $n \geq 0$, $SC(n, 0; e)$ is defined as $\mathcal{D}_2(n)$ and $SC(n, m; e) = \emptyset$ for $m \geq 1$. The 2–colored operad structure in $SC$ is given, as usual, by insertion of disks.

There is a suboperad $SC^{vor}$ of $SC$ defined by $SC^{vor}(n, m; x) = SC(n, m; x)$, if $x = c$ or $m \geq 1$ and by $SC^{vor}(n, m; x) = \emptyset$, otherwise. The above definition says that $SC^{vor}$ coincides with $SC$ except for $m = 0$ and $x = o$, where $SC^{vor}(n, 0, o) = \emptyset$ for any $n \geq 0$. The operad $SC^{vor}$ is equivalent to the one defined by Voronov in [17], while $SC$ coincides with the one defined by Kontsevich in [13].
The homology of $SC$ is denoted by $sc$ while that of $SC^{vor}$ is denoted by $sc^{vor}$.

### 2.4 Conventions and notation

#### 2.4.1 Generators

In the paper we will have specific generators in the different operads considered, mainly two families of elements. The first is \{\(f_2, g_2, e_{0,2}, e_{1,1}, e_{1,0}\)\} and the second family is \{\(l_2, c_2, n_{0,2}, n_{1,1}, n_{1,0}\)\}.

The following array sum up the properties of the elements. The array must be read as follows: $f_2 \in M(c, c; e)$ means that it is an operation on two closed variables giving a closed variable; the representation is $k$, that is, \(f_2\) is a symmetric operation. The degree is 0.

| Element | $f_2$ | $g_2$ | $e_{0,2}$ | $e_{1,1}$ | $e_{1,0}$ |
|---------|-------|-------|-----------|-----------|-----------|
| Color   | $M(c, c; e)$ | $M(e, c; c)$ | $M(o, o; o)$ | $M(c, o; o)$ | $M(c; o)$ |
| Degree  | $k$  | $k$  | $k[2]$  | $k[2]$ in $M(c, o; o) \oplus M(o, c; o)$ | $k$  |

| Element | $l_2$ | $c_2$ | $n_{0,2}$ | $n_{1,1}$ | $n_{1,0}$ |
|---------|-------|-------|-----------|-----------|-----------|
| Color   | $M(c, c; c)$ | $M(e, c; c)$ | $M(o, o; o)$ | $M(c, o; o)$ | $M(c; o)$ |
| Degree  | $sgn_2$ | $sgn_2$ | $k[2]$  | $k[2]$ in $M(c, o; o) \oplus M(o, c; o)$ | $k$  |

Given elements $\{x_1, \ldots, x_n\}$ with specific colors, representation and degrees, the $S$–module $\langle x_1, \ldots, x_n \rangle$ is the $S$–module generated by these elements, with the action of the symmetric group indicated by the representation of the elements. For example $\langle e_{1,1} \rangle$ is the $S$–module $M$ where $M(c, o; o) = ke_{1,1}$, $M(o, c; o) = ke_{1,1} \cdot (21)$ and is zero elsewhere.
2.4.2 Notation for operads  The operad Ger, whose algebras are Gerstenhaber algebras is the operad \( \mathcal{F}(E_{Ger}, R_{Ger}) \) with \( E_{Ger} = \langle f_2, g_2 \rangle \) and \( R_{Ger} \) is the space of relations given by
\[
    f_2(id \otimes f_2) = f_2(f_2 \otimes id),
\]
\[
    g_2(g_2 \otimes id) \cdot ((123) + (231) + (312)) = 0,
\]
\[
    g_2(id \otimes f_2) = f_2(g_2 \otimes id) + f_2(id \otimes g_2) \cdot (213).
\]
The suboperad generated by \( f_2 \) is the operad Com = \( \mathcal{F}((f_2), R_{Com}) \) where \( R_{Com} \) is the first relation. The suboperad generated by \( g_2 \) is the operad \( \Lambda^{-1}\Lie \).

The Koszul dual of the operad Ger is \( \text{Ger}^! = \Lambda\text{Ger} \) (see eg [6]). It is described as \( \mathcal{F}(E_{\Lambda\text{Ger}}, R_{\Lambda\text{Ger}}) \) with \( E_{\Lambda\text{Ger}} = \langle l_2, c_2 \rangle \) and \( R_{\Lambda\text{Ger}} \) is the space of relations given by
\[
    c_2(id \otimes c_2) = -c_2(c_2 \otimes id),
\]
\[
    l_2(l_2 \otimes id) \cdot ((123) + (231) + (312)) = 0,
\]
\[
    l_2(id \otimes c_2) = c_2(l_2 \otimes id) + c_2(id \otimes l_2) \cdot (213).
\]
The suboperad generated by \( l_2 \) is the operad \( \text{Lie} = \mathcal{F}((l_2), R_{\text{Lie}}) \), where \( R_{\text{Lie}} \) is the second relation. The suboperad generated by \( c_2 \) is the operad \( \Lambda\text{Com} \). The operad \( \text{Ass} \) is described as \( \mathcal{F}((e_{0,2}), R_{\text{Ass}}) \) where \( R_{\text{Ass}} \) is the relation \( e_{0,2}(id \otimes e_{0,2}) = e_{0,2}(e_{0,2} \otimes id) \).

Note that we also use this notation replacing \( e_{0,2} \) by \( n_{0,2} \).

3  The homology operads sc\textsuperscript{vor} and sc

We prove in this section that the homology operad sc\textsuperscript{vor} is a quadratic Koszul operad and that the homology operad sc is a quadratic-linear Koszul operad, extending the results obtained for the 0\textsuperscript{th} homology of \( SC\textsuperscript{vor} \) and SC in [9].

3.1  The operad sc\textsuperscript{vor} is Koszul

Recall the following theorem.

**Theorem 3.1.1**  (A Voronov [17])  An algebra over sc\textsuperscript{vor} is a pair \((G, A)\), where \( G \) is a Gerstenhaber algebra and \( A \) is an associative algebra over the commutative ring \( G \).

An algebra over the commutative ring \( G \) corresponds to a degree 0 map \( \lambda: G \otimes A \to A \) satisfying
\[
    \lambda(cc', a) = \lambda(c, \lambda(c', a)) = (-1)^{|c||c'|}\lambda(c', \lambda(c, a)),
\]
\[
    \lambda(c, aa') = \lambda(c, a)a' = (-1)^{|a||c|}\lambda(c, a').
\]
As a consequence we have the following.
Corollary 3.1.2  The operad $\text{sc}^{\lor}$ has a quadratic presentation $\mathcal{F}(E_v, R_v)$ where

$$E_v = \langle f_2, g_2, e_{0,2}, e_{1,1} \rangle$$

and $R_v$ is the sub-$S$–module of $\mathcal{F}(2)(E_v)$ generated by the relations

- $R_{Ger}$, for the Gerstenhaber structure defined by $f_2$ and $g_2$ and $R_{Ass}$ for the associativity of $e_{0,2}$;
- $e_{1,1}$ is an action, with

\[
\begin{align*}
e_{1,1}(id \otimes e_{1,1}) &= e_{1,1}(f_2 \otimes id), \\
e_{1,1}(id \otimes e_{0,2}) &= e_{0,2}(e_{1,1} \otimes id) = e_{0,2}(id \otimes e_{1,1}) \cdot (213). \end{align*}
\]

Lemma 3.1.3  Algebras over the Koszul dual operad $(\text{sc}^{\lor})!$ of $\text{sc}^{\lor}$ are of the form $(H, A, \rho)$ where $(H, [\cdot, \cdot], \times)$ is a $\Lambda Ger$–algebra, $A$ is an associative algebra, and $\rho: H \otimes A \to A$ is a map of degree 0 that satisfies the relations

\[
\begin{align*}
\rho([h, h'], a) &= \rho(h, \rho(h', a)) - (-1)^{|h||h'|} \rho(h', \rho(h, a)), \\
\rho(h, a \cdot a') &= \rho(h, a) \cdot a' + (-1)^{|a||h|} a \cdot \rho(h, a'), \\
\rho(h \times h', a) &= 0.
\end{align*}
\]

Note that the first two equations indicate that the map induced by $\rho$ from $H$ to $\text{End}(A)$ has values in $\text{Der}(A)$ and is a morphism of Lie algebras.

Proof  Because $\text{sc}^{\lor}$ has a binary quadratic presentation, we can use the direct computation of its Koszul dual operad presented in Section 2.2.6. Let us denote by $(l_2, c_2, n_{0,2}, n_{1,1})$ the dual basis of $(f_2, g_2, e_{0,2}, e_{1,1})$ in $E_v^\lor$. The degree of $c_2$ is $-1$ and all the other elements have degree 0.

The Koszul dual operad of $\text{sc}^{\lor}$ is $(\text{sc}^{\lor})! = \mathcal{F}(E_v^\lor)/(R_v^\perp)$. The pairing between $E_v$ and $E_v^\lor$ induces a pairing between $\mathcal{F}(2)(E_v)$ and $\mathcal{F}(2)(E_v^\lor)$. One gets $R_v^\perp(c, c, c; c)$ is the ideal defining $Ger^!$, that is, $R_{\Lambda Ger}$. Similarly, $R_v^\perp(o, o, o; o)$ is the orthogonal of the associativity relation for $e_{0,2}$, that is, the associativity relation for $n_{0,2}$.

The space $\mathcal{F}(E_v)(c, c, o; o)_0$ has dimension 3 and $R_v(c, c, o; o)_0$ has dimension 1. As a consequence, the dimension of $R_v^\perp(c, c, o; o)_0$ is 2 and corresponds to the first relation.

The space $\mathcal{F}(E_v)(c, o, o; o)$ has dimension 6 and $R_v(c, o, o; o)$ has dimension 2. Hence the dimension of $R_v^\perp(c, o, o; o)$ is 4 and corresponds to the second relation.

The space $\mathcal{F}(E_v)(c, c, o; o)_1$ has dimension 1 and $R_v(c, c, o; o)_1$ has dimension 0. As a consequence, the dimension of $R_v^\perp(c, c, o; o)_{-1}$ is 1 and corresponds to the third relation.  

\[\square\]
In terms of generators and relations, it expresses as the following.

**Corollary 3.1.4** The operad \((\text{sc}^\text{vor})^!\) has a binary quadratic presentation \(\mathcal{F}(E_{\text{ vor}}, R_{\text{ vor}})\), where

\[ E_{\text{ vor}} = \langle l_2, c_2, n_{0,2}, n_{1,1} \rangle \]

and \(R_{\text{ vor}}\) is the sub-\(S\)–module of \(\mathcal{F}^{(2)}(E_{\text{ vor}})\) generated by the relations

- \(R_{\Lambda \text{Ger}}\), for the \(\Lambda \text{Ger}\)–structure defined by \(l_2\) and \(c_2\) and \(R_{\text{Ass}}\) for the associativity of \(n_{0,2}\);
- relations for \(n_{1,1}\) are

\[
\begin{align*}
n_{1,1}(l_2 \otimes \text{id}) &= n_{1,1}(\text{id} \otimes n_{1,1}) \cdot (\text{id} - (213)), \\
n_{1,1}(\text{id} \otimes n_{0,2}) &= n_{0,2}(n_{1,1} \otimes \text{id}) + n_{0,2}(\text{id} \otimes n_{1,1}) \cdot (213), \\
n_{1,1}(c_2 \otimes \text{id}) &= 0.
\end{align*}
\]

**Theorem 3.1.5** The operad \(\text{sc}^\text{vor}\) is Koszul.

**Proof** In order to prove that \(\text{sc}^\text{vor}\) is Koszul, we prove that \((\text{sc}^\text{vor})^!\) is Koszul, using the rewriting method explained in [15], and using a part of the computation made by Alm in [1, AppendixA]. Recall that an algebra over \((\text{sc}^\text{vor})^!\) is given by the following data.

- A \(\Lambda \text{Ger}\)–algebra \(H\). We denote by \([x_1, x_2]\) the degree 0 bracket and by \(x_1 \times x_2\) the degree \(-1\) product.
- An associative algebra \(A\). We denote by \(a_1 \cdot a_2\) the degree 0 product.
- A map \(\rho: H \otimes A \to A\). We denote by \(x \bullet a\) the element \(\rho(x, a)\).

The rewriting rules are

\[
\begin{align*}
(a_1 \cdot a_2) \cdot a_3 &\leftrightarrow a_1 \cdot (a_2 \cdot a_3), \\
(x_1 \times x_2) \times x_3 &\leftrightarrow -x_1 \times (x_2 \times x_3), \\
[[x_1, x_2], x_3] &\leftrightarrow -[[x_2, x_3], x_1] - [[x_3, x_1], x_2], \\
[x_1, x_2 \times x_3] &\leftrightarrow [x_1, x_2] \times x_3 + x_2 \times [x_1, x_3], \\
x_1 \bullet (a_1 \cdot a_2) &\leftrightarrow (x_1 \bullet a_1) \cdot a_2 + a_1 \cdot (x_1 \bullet a_2), \\
(x_1 \times x_2) \bullet a &\leftrightarrow 0, \\
[x_1, x_2] \bullet a_1 &\leftrightarrow x_1 \bullet (x_2 \bullet a_1) - x_2 \bullet (x_1 \bullet a_1).
\end{align*}
\]

In order to study the confluence of critical monomials, it is enough to study the one involving both \(x\)'s and \(a\)'s because the one involving only \(a\)'s corresponds to the
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computation for the operad $\text{Ass}$, and the one involving only $x$’s corresponds to the computation for the operad $\Lambda \text{Ger}$. We know that a way to prove the Koszulity of these 2 operads is precisely to use the confluence of the critical monomials.

Hence the critical monomials left are $(x_1 \cdot ((a_1 \cdot a_2) \cdot a_3))$, $([x_1, x_2] \cdot (a_1 \cdot a_2))$, $((x_1, x_2, x_3)\cdot a_1)$, $((x_1 \cdot x_2)\cdot (a_1 \cdot a_2))$, $(((x_1 \cdot x_2) \cdot x_3)\cdot a)$ and $([x_1, x_2 \cdot x_3] \cdot a)$. The first three have been proven to be confluent by J Alm. The fourth critical monomial can be rewritten either as

$$(x_1 \times x_2) \cdot (a_1 \cdot a_2) \mapsto ((x_1 \times x_2) \cdot a_1) \cdot a_2 + a_1 \cdot ((x_1 \times x_2) \cdot a_2)$$

$\Rightarrow 0$

or $(x_1 \times x_2) \cdot (a_1 \cdot a_2) \mapsto 0$. The same is true for the fifth critical monomial.

The critical monomial $[x_1, x_2 \cdot x_3] \cdot a$ can be rewritten either as

$$[x_1, x_2 \cdot x_3] \cdot a \mapsto x_1 \cdot ((x_2 \cdot x_3) \cdot a) - (x_2 \cdot x_3) \cdot (x_1 \cdot a)$$

$\Rightarrow 0$

or

$$[x_1, x_2 \cdot x_3] \cdot a \mapsto ([x_1, x_2] \cdot x_3) \cdot a + (x_2 \cdot [x_1, x_3]) \cdot a$$

$\Rightarrow 0$.

Hence, all the critical monomials are confluent and $(\text{sc}^\text{vor})^!$ is Koszul. As a consequence $\text{sc}^\text{vor}$ is a Koszul operad.


3.2 The operad $\text{sc}$ is Koszul

In this section we follow closely the article by Imma Galvez-Carrillo, Andy Tonks and Bruno Vallette [5] and our paper [9] in order to prove that the homology operad $\text{sc}$ is Koszul. Recall from the computation of F Cohen and A Voronov and from [9] the following.

Proposition 3.2.1 An $\text{sc}$–algebra $(G,A,f)$ is a Gerstenhaber algebra $G$ and an associative algebra $A$ together with a central morphism of associative algebras $f : G \rightarrow A$.

Corollary 3.2.2 The operad $\text{sc}$ has a presentation of the form $\mathcal{F}(E', R')$ where

$$E' = \langle f_2, g_2, e_{0,2}, e_{1,0} \rangle$$

and the space of relations $R'$ is the sub-$\mathcal{S}$–module of $\mathcal{F}^{(2)}(E) \oplus \mathcal{F}^{(3)}(E)$ defined by the relations.
• $R_{\text{Ger}}$ for the Gerstenhaber structure induced by $f_2$ and $g_2$ and $R_{\text{Ass}}$ for the associativity of $e_{0,2}$;
• centrality of $e_{1,0} : e_{0,2}(e_{1,0} \otimes \text{id}) = e_{0,2}(\text{id} \otimes e_{1,0})$ · (21);
• a quadratic-cubical relation: $e_{1,0}(f_2) = e_{0,2}(e_{1,0} \otimes e_{1,0})$.

This corollary shows clearly that this presentation is quadratic and cubic. In order to apply the theory of [5], one needs a presentation which is quadratic and linear. However, we will see in Proposition 4.2.1 that the quadratic operad $\mathcal{F}(E')/(qR')$ obtained by killing the cubical elements in the relations of $R'$ plays also an important role for the study of $\mathfrak{sc}$.

The idea to obtain a presentation with quadratic-linear relations of $\mathfrak{sc}$ is to add a new generator, in order to replace the quadratic-cubical relation by quadratic-linear relations. This new generator $e_{1,1}$, will correspond at the level of algebras to the operation $\lambda(c, a) := f(c)a$. Consequently, we introduce new relations in the operad corresponding to the relations $f(c)a = af(c) = \lambda(c, a)$ and $\lambda(c, f'(c')) = f(cc') = f(c)f(c')$, that are present in the algebra setting.

Recall the theory explained in [5] for quadratic-linear operads. A quadratic-linear operad is of the form $\mathcal{F}(E)/(R)$ with $R \subset \mathcal{F}(1)(E) \oplus \mathcal{F}(2)(E)$. Such an $R$ is called quadratic-linear. We also ask the presentation to satisfy

(ql1) \( R \cap E = \{0\} \),
(ql2) \( (R \otimes E + E \otimes R) \cap \mathcal{F}(2)(E) \subset R \cap \mathcal{F}(2)(E) \).

**Proposition 3.2.3** The operad $\mathfrak{sc}$ has a presentation $\mathcal{F}(E, R)$, where

$$E = \langle f_2, g_2, e_{0,2}, e_{1,1}, e_{1,0} \rangle$$

and the space of relations $R$ is the sub-$S$–module of $\mathcal{F}(1)(E) \oplus \mathcal{F}(2)(E)$ defined by $R = R_v \oplus R(e_{1,0})$, where $R_v$ is the space of quadratic relations of $\mathfrak{sc}^{\text{vor}}$ and $R(e_{1,0})$ is the sub-$S$–module of $\mathcal{F}(E)$ generated by the following relations:

• two quadratic-linear relations: $e_{1,1} = e_{0,2}(e_{1,0} \otimes \text{id})$ and $e_{1,1} = e_{0,2}(\text{id} \otimes e_{1,0})$ · (21),
• a new quadratic relation: $e_{1,1}(\text{id} \otimes e_{1,0}) = e_{1,0}(f_2)$.

Moreover this presentation satisfies (ql1) and (ql2).

Here we recall the definition of a Koszul quadratic-linear operad given in [5].
**Definition 3.2.4** Let \( q \) denote the projection \( \mathcal{F}(E) \to \mathcal{F}^{(2)}(E) \) and let \( qR \) be the image of \( R \) under this projection. A quadratic-linear operad \( \mathcal{P} = \mathcal{F}(E)/(R) \) satisfying (ql1) and (ql2) is said to be Koszul if \( q\mathcal{P} := \mathcal{F}(E)/(qR) \) is a quadratic Koszul operad. Its Koszul dual cooperad is \( (\mathcal{P})^! = ((q\mathcal{P})^!, \partial_{\varphi}) \) where the differential \( \partial_{\varphi} \) depends on the quadratic-linear relations.

In the case of \( \text{sc} \) presented as in **Proposition 3.2.3**, the projection of \( R = R_v \oplus R(e_{1,0}) \) onto \( \mathcal{F}^{(2)}(E) \) is \( qR = R_v \oplus qR(e_{1,0}) \), where \( qR(e_{1,0}) \) is the sub-\( S \)-module of \( \mathcal{F}^{(2)}(E) \) generated by the relations \( 0 = e_{0,2}(e_{1,0} \otimes \text{id}), 0 = e_{0,2}(e_{1,0} \otimes \text{id})(12) \) and \( e_{1,1}(\text{id} \otimes e_{1,0}) = e_{1,0}(f_2) \).

Consequently a \( q\text{sc} \)-algebra is an \( \text{sc}^{\text{vor}} \)-algebra \( (G, A, \lambda) \) endowed with a degree 0 linear map \( f: G \to A \) satisfying \( f(c)a = af(c) = 0 \) and \( \lambda(c, f(c')) = f(cc') \) for all \( c, c' \in G, a \in A \). As in [9], the operad \( q\text{sc} \) is obtained as the result of a distributive law between the operad \( \text{sc}^{\text{vor}} \) and \( \mathcal{F}(e_{1,0}) \). The distributive law is given by

\[
\text{sc}^{\text{vor}} \circ \mathcal{F}(e_{1,0}) \to \mathcal{F}(e_{1,0}) \circ \text{sc}^{\text{vor}},
\]

\[
(3) \quad e_{0,2}(e_{1,0} \otimes \text{id}), \quad e_{0,2}(e_{1,0} \otimes \text{id})(12) \mapsto 0,
\]

\[
e_{1,1}(\text{id} \otimes e_{1,0}) \mapsto e_{1,0}(f_2).
\]

**Proposition 3.2.5** The operad \( q\text{sc} \) is identical to the operad \( \mathcal{F}(e_{1,0}) \circ \text{sc}^{\text{vor}} \), with composition given by the distributive law (3).

**Theorem 3.2.6** The operad \( q\text{sc} \) is a quadratic Koszul operad. As a consequence, there exists a quadratic-linear presentation of the operad \( \text{sc} \) so that \( \text{sc} \) is a quadratic-linear Koszul operad.

**Proof** From [15, Chapter8], one has that \( q\text{sc} = \mathcal{F}(e_{1,0}) \circ \text{sc}^{\text{vor}} \) is Koszul since \( \text{sc}^{\text{vor}} \) and \( \mathcal{F}(e_{1,0}) \) are Koszul colored operads. By definition, it means that \( \text{sc} \) is a quadratic-linear Koszul operad. \( \square \)

### 3.3 Description of the Koszul dual operad \( (\text{sc})^! \) of \( \text{sc} \).

In **Proposition 3.2.5**, we have described \( q\text{sc} \) as a distributive law between \( \text{sc}^{\text{vor}} \) and \( \mathcal{F}(e_{1,0}) \). As a consequence \( (q\text{sc})^! = (\text{sc}^{\text{vor}})^! \circ \mathcal{F}(e_{1,0})^! \), with the operad structure given by the signed dual of the distributive law (3). Recall from Corollary 3.1.4 that \( \{f_2, c_2, n_{0,2}, n_{1,1}\} \) is the dual basis of \( \{f_2, g_2, e_{2,0}, e_{1,1}\} \) that generates \( E_v^! \). From relation (1), one has \( \mathcal{F}(e_{1,0})^! = \mathcal{F}(n_{1,0}) \) where \( n_{1,0} \) has degree \(-1\). The dual of the
The distributive law is given by

\[ \mathcal{F}(n_{1,0}) \circ (\text{sc}^{\text{vor}})^! \rightarrow (\text{sc}^{\text{vor}})^! \circ \mathcal{F}(n_{1,0}), \]

\[ n_{1,0}(I_2) \mapsto n_{1,1}(\text{id} \otimes n_{1,0}) \cdot (\text{id} - (21)), \]

\[ n_{1,0}(e_2) \mapsto 0. \]

Consequently, a \((q\text{sc})^!\)-algebra is an \((\text{sc}^{\text{vor}})^!\)-algebra \((H, A, \rho)\) satisfying conditions of Lemma 3.1.3, together with a linear map \(\beta: H \rightarrow A\) of degree \(-1\) satisfying

\begin{equation}
\beta([h, h']) = (-1)^{|h|} \rho(h, \beta(h')) - (-1)^{|h||h'| + |h'|} \rho(h', \beta(h)),
\end{equation}

\begin{equation}
\beta(h \times h') = 0.
\end{equation}

In order to understand the structure of an \((\text{sc})^!\)-algebra it is then enough to understand the differential on the operad \((q\text{sc})^!\) that comes from the nonquadraticity of the operad \text{sc}.

Let \(\varphi: qR \rightarrow E\) be defined by

\[ \varphi(e_{0,2}(e_{1,0} \otimes \text{id})) = \varphi(e_{0,2}(\text{id} \otimes e_{1,0}) \cdot (21)) = e_{1,1}, \]

\[ \varphi(R_v) = 0, \]

\[ \varphi(e_{1,1}(\text{id} \otimes e_{1,0}) - e_{1,0}(f_2)) = 0. \]

The Koszul dual cooperad of \(q\text{sc}\) is \((q\text{sc})^i = C(sE, s^2qR)\), with the notation of Section 2.2.6. To \(\varphi\) is associated the composite map

\[ (q\text{sc})^i \rightarrow s^2qR \xrightarrow{s^{-1}\varphi} sE. \]

There exists a unique coderivation \(\tilde{\partial}_\varphi: (q\text{sc})^i \rightarrow \mathcal{F}^c(sE)\) which extends this map. Moreover, \(\tilde{\partial}_\varphi\) induces a square zero coderivation \(\partial_\varphi\) on the Koszul dual cooperad \((q\text{sc})^i\).

The Koszul dual cooperad of \text{sc} is by definition \(\text{sc}^i = (C(sE, s^2qR), \partial_\varphi)\).

Recall from (1) that \((q\text{sc})^i = (\Lambda(q\text{sc})^i)^*\). As a consequence, \(\text{sc}^i = ((q\text{sc})^i, d_\varphi)\), where \(d_\varphi\) is obtained as a combination of transpose and signed suspension of \(\partial_\varphi\). Namely, \(\text{sc}^i\) is a differential graded operad and we have the following Proposition.

**Proposition 3.3.1** An algebra over \(\text{sc}^i\) consists in a \(\Lambda\text{Ger}\)–algebra \((H, [\cdot, \cdot], \times, d_H)\), a \(\Lambda\text{Ger}\)–algebra \((A, d_A)\), an action \(\rho: H \otimes A \rightarrow A\) and a degree \(-1\) map \(\beta: H \rightarrow A\) such that, for all \(h \in H, a \in A\), we have \(d_A(\beta(h)) = -\beta(d_H h)\) and that the relations (2) and (4) are satisfied. Moreover, the following relation is satisfied:

\begin{equation}
d_A \rho(h, a) = \rho(d_H h, a) + (-1)^{|h|} \rho(h, d_A a) + \beta(h)a - (-1)^{|a|(|h|+1)}a\beta(h).
\end{equation}
Note that relation (5) says that the map $\beta: H \to A$ is central up to homotopy having the map $\rho: H \otimes A \to A$ as the homotopy operator. For a geometrical description of the above relations in terms of the Kontsevich compactification [14], we refer the reader to the first author [8], Kajiura and Stasheff [11] and the authors [9].

**Remark 3.3.2** There is a more compact way to understand what are $\mathfrak{sc}^!$–algebras. Let $(A, d_A)$ be a dg associative algebra. Let $\text{Der}(A)$ be the dg Lie algebra of derivations of $A$. For a given $a \in A$ we denote by $D_a$ the inner derivation which is defined by $D_a(x) = ax - (-1)^{|a||x|}xa$. The graded $k$–vector space $sA$ is a module over $\text{Der}(A)$ via the action $[d, sa] = (-1)^{|d||s|}sd(a)$. Consequently, there is a structure of graded Lie algebra on $\text{Der}_+(A) = \text{Der}(A) \oplus sA$. A short computation shows that the differential $\partial_+(d + sa) = \partial d + D_a - sd_A(a)$ endows $\text{Der}_+(A)$ with a structure of dg Lie algebra. Furthermore, any dg Lie algebra is a dg $\Lambda Ger$–algebra, setting the product to be 0.

As a consequence, one has the following.

An algebra over $\mathfrak{sc}^!$ consists in a dg $\Lambda Ger$–algebra $(H, [\cdot, \cdot], \times, d_H)$, a dg associative algebra $(A, d_A)$, and a morphism of dg $\Lambda Ger$–algebras $\gamma: H \to \text{Der}_+(A)$.

Translating the proposition in the language of operads, one gets the following corollary.

**Corollary 3.3.3** The differential graded operad $(\mathfrak{sc})^!$ has a presentation $\mathcal{F}(E_1, R_1)$, where

$$E_1 = \langle l_2, c_2, n_{0,2}, n_{1,1}, n_{1,0} \rangle$$

and the vector space $R_1$ is the sub-$\mathcal{S}$–module of $\mathcal{F}^{(2)}(E_1)$ generated by the relations:

- $R_{\Lambda Ger}$, for the $\Lambda Ger$–structure defined by $l_2$ and $c_2$ and $R_{\text{Ass}}$ for the associativity of $n_{0,2}$;
- relations for $n_{1,1}$ are
  
  $$n_{1,1}(l_2 \otimes \text{id}) = n_{1,1}(\text{id} \otimes n_{1,1}) \cdot (\text{id} - (213)),$$
  $$n_{1,1}(\text{id} \otimes n_{0,2}) = n_{0,2}(n_{1,1} \otimes \text{id}) + n_{0,2}(\text{id} \otimes n_{1,1}) \cdot (213),$$
  $$n_{1,1}(c_2 \otimes \text{id}) = 0;$$
- relations for $n_{1,0}$ are
  
  $$n_{1,0}(l_2) = n_{1,1}(\text{id} \otimes n_{1,0}) \cdot ((12) - (21)),$$
  $$n_{1,0}(c_2) = 0.$$

The differential is given by $dn_{1,1} = n_{0,2}(n_{1,0} \otimes \text{id}) - n_{0,2}(\text{id} \otimes n_{1,0}) \cdot (21)$ and vanishes elsewhere.
3.4 On the homology of \( \mathsf{sc}^! \)

In [9], we have considered the 0\(^{th}\) homology operad of \( \mathsf{SC} \). In particular, the description of \( H_0(\mathsf{SC})^! \) ([9, Proposition 6.3.2]) is the following.

**Proposition 3.4.1**  The differential graded operad \( H_0(\mathsf{SC})^! \) has a presentation given by \( \mathcal{F}(E_0, R_0) \), where

\[
E_0 = \langle l_2, n_{0,2}, n_{1,1}, n_{1,0} \rangle
\]

and the vector space \( R_0 \) is the sub-\( \mathbb{S} \)–module of \( \mathcal{F}^{(2)}(E_0) \) generated by the relations:

- \( R_{\text{Lie}} \), for the \( \text{Lie} \)–structure defined by \( l_2 \) and \( R_{\text{Ass}} \) for the associativity of \( n_{0,2} \);
- relations for \( n_{1,1} \) are
  \[
  n_{1,1}(l_2 \otimes \text{id}) = n_{1,1}(\text{id} \otimes n_{1,1}) \cdot (\text{id} - (213)),
  \quad n_{1,1}(\text{id} \otimes n_{0,2}) = n_{0,2}(n_{1,1} \otimes \text{id}) + n_{0,2}(\text{id} \otimes n_{1,1}) \cdot (213);
  \]
- relations for \( n_{1,0} \) are
  \[
  n_{1,0}(l_2) = n_{1,1}(\text{id} \otimes n_{1,0}) \cdot ((12) - (21)).
  \]

The differential is given by \( d n_{1,1} = n_{0,2}(n_{1,0} \otimes \text{id}) - n_{0,2}(\text{id} \otimes n_{1,0}) \cdot (21) \) and is zero on all the other generators.

From this, it is easy to prove the following corollary.

**Corollary 3.4.2**  The dg operad \( \mathsf{sc}^! \) is the operad composite \( \Lambda \text{Com} \circ H_0(\mathsf{SC})^! \) together with the distributive law given by

\[
H_0(\mathsf{SC})^! \circ \Lambda \text{Com} \to \Lambda \text{Com} \circ H_0(\mathsf{SC})^!,
\]

\[
l_2(\text{id} \otimes c_2) \mapsto c_2(l_2 \otimes \text{id}) + c_2(\text{id} \otimes l_2) \cdot (213),
\]

\[
n_{1,1}(c_2 \otimes \text{id}) \mapsto 0,
\]

\[
n_{1,0}(c_2) \mapsto 0.
\]

As a consequence we get the following.

**Theorem 3.4.3**  Algebras over the homology of the operad \( \mathsf{sc}^! \) are triples \( (H, A, \beta) \) where \( H \) is a \( \Lambda \text{Ger} \)–algebra, \( A \) is an associative algebra and \( \beta : H \to A \) is a central map of degree \(-1\) satisfying \( \beta(x \times y) = 0 \).
Proof  Recall from [9, Theorem 7.2.5] that algebras over the homology of the operad $H_0(SC)$ are triples $(L, A, f)$ where $L$ is a Lie algebra, $A$ is an associative algebra and $f: L \to A$ is a central map of degree $-1$. Using the Künneth formula for the plethysm product $\circ$ of $S$–modules, as in Fresse [4, Lemma 2.1.3], we obtain that $H_*(SC^1) = \Lambda \text{Com} \circ H_*(H_0(SC))$ with the distributive law given by

\[
H_*(H_0(SC)) \circ \Lambda \text{Com} \to \Lambda \text{Com} \circ H_*(H_0(SC)),
\]

\[
[l_2](\text{id} \otimes c_2) \leftrightarrow c_2([l_2] \otimes \text{id}) + c_2(\text{id} \otimes [l_2]) \cdot (213),
\]

\[
[n_1,0](c_2) \leftrightarrow 0,
\]

where $[x]$ denotes the image of a cycle $x$ in $H_*(H_0(SC))$.

4 On the spectral sequence

In this section we will show that the spectral sequence $E(SC)$ associated to the stratification of the compactification of points in the upper half plane collapses at the second stage. We prove that, as an $S$–module, $E^2(SC)$ corresponds to the $S$–module defined by $SC$, but prove that the operad structures are not isomorphic.

4.1 On the first sheet of the spectral sequence

For this section, we refer to [17; 8; 2].

For the compactification of points we are considering two different spaces: the space $C(n)$ of configurations of $n \geq 2$ points in the disk modded out by the action of the group of dilatations and translations, of dimension $2n - 3$; the space $C(n,m)$, with $2n + m \geq 2$, of configurations of $n$ points in the upper half plane, $m$ points on the line, modded out by the action of the group of dilatations and translations along the line, of dimension $2n + m - 2$.

The operad $SC$ is homotopy equivalent to the Fulton–MacPherson compactification of $C(n)$ for $SC(n,0)$ and of $C(n,m)$ for $SC(n,m; o)$. Since $C(1)$ and $C(0,1)$ are not well defined, we introduce both $SC(1,0)$ and $SC(0,1; o)$ as the one point spaces containing the identity element of the closed and open colors, respectively. It has been proven by Getzler and Jones in [6], that the filtration associated to the stratification of the compactification of $C(n)$ induces a spectral sequence $E(D_2)$, whose first sheet coincides with the cobar construction of the cooperad $(Ger)^i$. Furthermore, the spectral sequence collapses at the second stage, and $E^2(D_2)$ coincides, as an operad, with $Ger$. We will then focus on the open part of the Swiss-cheese operad. From [8, Theorem 5.2],
there is a stratification of $C(n, m)$ indexed by partially planar trees, which induces a topological filtration

$$F^p := F^p(\overline{C}) = \{\text{closure of the union of strata of dimension } p\}.$$ 

It yields a spectral sequence, whose first sheet is given by

$$E^1(SC)_{p,q} = H_{p+q}(F^p, F^{p-1}) = H^{-q}(F^p \setminus F^{p-1}).$$ 

Let $\mathbb{T}(n, m)_p$ be the set of partially planar trees with $n$ closed inputs, $m$ open inputs, the output being open and $v := 2n + m - p - 1$ vertices. To any vertex $v_i$ of a tree $T \in \mathbb{T}(n, m)_p$ is associated the triple $(n_i, m_i, x_i)$ corresponding respectively, to its closed, open inputs, and its output. One has the relation

$$\sum_{i=1}^{v} n_i = n + \sum_{i=1}^{v} \delta_{x_i, c},$$

$$\sum_{i=1}^{v} m_i = m + \sum_{i=1}^{v} \delta_{x_i, o} - 1.$$ 

Since any tree $T \in \mathbb{T}(n, m)_p$ is responsible for a strata $C_T := \prod_{i=1}^{v} SC(n_i, m_i; x_i)$ one gets that

$$E^1(SC)(n, m)_{p,q} = \bigoplus_{T \in \mathbb{T}(n,m)_p, \quad i=1}^{v} H^{u_i}(SC(n_i, m_i; x_i)).$$

Because we are not exactly using the notation of [2], we need the following Lemma.

**Lemma 4.1.1** The operad $E^1(SC)$ coincides with the cobar construction of the cooperad $(\Lambda_\zeta \Lambda sc)^*$. More precisely, one has

$$E^1(SC)(n, m)_{p,q} = \Omega((\Lambda_\zeta \Lambda sc)^*)(n, m; o)^{(2n+m-p-1)},$$

where the upper index corresponds to the weight grading by the number of vertices of the trees involved and the lower index corresponds to the total degree.

**Proof** Recall that for any cooperad $\mathcal{C}$, $\Omega(\mathcal{C})$ is the free operad $\mathcal{F}(s^{-1}\mathcal{C})$, where $s\mathcal{C}$ is the coaugmentation ideal of the cooperad. Since $E^1(SC)$ is also a free $2$–colored
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They have the same description in terms of trees. One has

\[ H^u(\text{SC}(n, m; x)) = \text{sc}^*(n, m; x)_{-u} \]

\[ = (\Lambda^{-1} \text{sc}^*)(n, m; x)_{-u+n+m-1} \]

\[ = (\Lambda^{-1}_c \Lambda^{-1} \text{sc}^*)(n, m; x)_{-u+2n+m-1-\delta_{x,c}} \]

\[ = (s^{-1} \Lambda^{-1}_c \Lambda^{-1} \text{sc}^*)(n, m; x)_{-u+2n+m-1-\delta_{x,c}-1}. \]

Using the description of \( E^1(\text{SC}) \) in (7) and the formulas in (6), one gets

\[ \sum_{i=1}^{v} (-u_i + 2n_i + m_i - 1 - \delta_{x_i,c} - 1) = q + 2n + m + \sum_{i=1}^{v} \delta_{x_i,c} + \delta_{x_i,0} - 2v - 1 \]

\[ = q + 2n + m - v - 1 = q + p, \]

which explains the grading obtained. From [8; 2], we know the differentials of the two operads coincide. As a consequence, the two differential graded operads coincide. □

4.2 On the second sheet of the spectral sequence

Theorem 3.2.6 asserts that \( \text{sc} \) is a Koszul operad, which expresses that

\[ \Omega(\text{sc}^i) \to \text{sc} \]

is a quasi-isomorphism of operads. Since all the graded vector spaces involved are finite dimensional, there is a quasi-isomorphism of cooperads

\[ \text{sc}^* \to (\Omega(\text{sc}^i))^* = B((\text{sc}^i)^*) \overset{(1)}{=} B(\Lambda(\text{sc}^i)). \]

Applying the bar-cobar adjunction we have a sequence of quasi-isomorphisms,

\[ \Omega(\text{sc}^*) \to \Omega B(\Lambda(\text{sc}^i)) \to \Lambda(\text{sc}^i). \]

Now applying the functor \( \Lambda^{-1}_c \Lambda^{-1} \) to the above morphism and using Lemma 4.1.1, we finally have the quasi-isomorphism

\[ E^1(\text{SC}) = \Omega((\Lambda_\text{sc}^*)^*) \to \Lambda^{-1}_c(\text{sc}^i). \]

Proposition 4.2.1 The operad \( E^2(\text{SC}) \) is the quadratic operad \( \mathcal{F}(E', qR') \), where

\[ E' = \langle f_2, g_2, e_{0,2}, e_{1,0} \rangle \]

and the space of relations \( qR' \) is the sub-\( S \)-module of \( \mathcal{F}^{(2)}(E') \) defined by the relations

- \( R_{\text{Ger}} \) for the Gerstenhaber structure induced by \( f_2 \) and \( g_2 \) and \( R_{\text{Ass}} \) for the associativity of \( e_{0,2}; \)
centrality of $e_{1,0}: e_{0,2}(e_{1,0} \otimes \text{id}) = e_{0,2}(\text{id} \otimes e_{1,0})$, (21);

the quadratic relation: $e_{1,0}(f_{2}) = 0$.

Equivalently, algebras over the operad $E^{2}(SC)$ are triples $(G, A, f)$ where $G$ is a Gerstenhaber algebra, $A$ is an associative algebra, $f: G \to A$ is a central degree 0 map satisfying $f(gg') = 0$, for all $g, g' \in G$.

**Proof** The operad $E^{2}(SC)$ is the homology of the dg operad $E^{1}(SC)$. Due to the quasi-isomorphism (8), it is the homology of the operad $\Lambda_{c}^{-1}(\text{sc}^{l})$. From the computation of the homology of $\text{sc}^{l}$ obtained in Theorem 3.4.3, we get the result. □

**Theorem 4.2.2** The spectral sequence $E(SC)$ collapses at the second stage.

**Proof** Proposition 4.2.1 implies we have that, as an $S$–module, $E^{2}(SC)(n, m; o) = \text{Ger}(n) \otimes \text{Ass}(m) = H_{*}(SC)(n, m; o)$. Because the spectral sequence converges to the homology of $SC$, and because the dimension of the second sheet is the dimension of the target, one gets that $E(SC)$ collapses at the second stage. □

**4.3 Conclusion**

We have shown the following.

Algebras over $H_{*}(SC)$ are triples $(G, A, f)$ where $G$ is a Gerstenhaber algebra, $A$ is an associative algebra and $f: G \to A$ is a central map such that $f(gg') = f(g)f(g')$, whereas algebras over $E^{\infty}(SC)$ are triples $(G, A, f)$, where $G$ is a Gerstenhaber algebra, $A$ is an associative algebra and $f: G \to A$ is a central map such that $f(gg') = 0$.

Note that the operad $E^{\infty}(SC) = E^{2}(SC)$ obtained is exactly the quadratic operad associated to the quadratic-cubical presentation of the operad $\text{sc}$ of Corollary 3.2.2. This is not a surprise because it is the graded operad associated to a filtration of $\text{sc}$. Note also that there is no hope of having a theorem similar to the one obtained by Getzler and Jones in [6] for the little disks operad, that is, an isomorphism between $E^{\infty}(SC)$ and $\text{sc}$. Indeed, one has the following.

**Proposition 4.3.1** $E^{2}(SC)$ and $\text{sc}$ are not isomorphic.

**Proof** If they were, there would be a bijective morphism of operads $\varphi: E^{2}(SC) \to \text{sc}$. Let $f_{2}, g_{2}, e_{0,2}, e_{1,0}$ denote the generators of $E^{2}(SC)$ and $f'_{2}, g'_{2}, e'_{0,2}, e'_{1,0}$ the generators of $\text{sc}$.
The generators we are concerned with are $f_2$ and $e_{1,0}$. Note that $E^2(SC)(c, c; c)_0$ is 1–dimensional so $f_2$ is a generator of this $k$–vector space. The same argument holds for the choice of $e_{1,0}, f'_2$ and $e'_{1,0}$. Hence, because of degree and arity reasons, there exist $\lambda, \mu \in k$ such that $\phi(f_2) = \lambda f'_2$ and $\phi(e_{1,0}) = \mu e'_{1,0}$. But we have that $\phi(e_{1,0}(f_2)) = \phi(0) = \lambda \mu e'_{1,0}(f'_2)$ and $e'_{1,0}(f'_2) \neq 0 \in sc(c, c; c)$. So $\lambda \mu = 0$, which contradicts the fact that $\phi$ is bijective.

\[ \square \]

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