One possible application of the Chronometric Theory of I.E. Segal: a toy model of quarks and gluons

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Abstract. A fundamental role of linear-fractional transformations in different spaces of matrices is well-known. The current research mainly deals with one such space: $D$. As a space-time, $D$ can be viewed as a Lie group with causal structure determined by an invariant Lorentzian form on the Lie algebra $u(2)$. Recently, the author suggested a matrix multi-level model of quarks and gluons. As a basis for this model, the Segal’s compact cosmos $D$ has been employed as well as the sequence of canonical (that is, corresponding to principal minors of appropriate matrices) group embeddings: $U(2)$ into $U(3)$, $U(2)$ into $U(4)$, etc. These groups were called the levels (of matter): $U(2)$ – the 0th (that is, our mundane), $U(3)$ – the 1st, $U(4)$ – the 2nd, etc. Such a convention matches the standard quarks’ generations’ list. Seemingly, the multi-level model is the only known construct where such notions as flavour and colour are defined mathematically. According to the model, a quark can be interpreted as a ‘sank’ proton (during the beginning of the reaction process, proton ‘is pushed’ into a ‘deeper’ level). At each level, a gluon can be interpreted as a colored and flavored photon. Not each and every feature of the model coincides with the corresponding standard assumption about quarks and gluons. In particular, the total number of colors is level-dependent. The model predicts THREE new quarks of the 4th generation (whereas currently there is a search for TWO).

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth.

1. Introduction

In [1] a model of the quark-gluon media has been put forward. This model is based on the sequence of canonical (that is, corresponding to principal minors of appropriate matrices) group embeddings: $U(2)$ into $U(3)$, $U(2)$ into $U(4)$, etc. These groups are called the levels (of matter): $U(2)$ – the 0th (that is, our mundane), $U(3)$ – the 1st, $U(4)$ – the 2nd, etc. Such a convention matches the standard quarks generations’ list. As far as the author is informed, the multi-level model is the only known construct where such notions from physics as flavor and color are defined mathematically. However, it might take a while before other mathematical features of the model will be figured out and compared to their counterparts in modern physics. That is why it is probably safer (for now) to call it a toy model of the quark-gluon media.

According to Segal (Irving E. Segal, 1918-1998), the global fractional linear conformal SU(2,2)-action on $D = U(2)$ is a fundamental ingredient in order to determine the list of all spin $\frac{1}{2}$ elementary particles in such a space-time. There are four of those and they differ by an order in which they (or rather their representation spaces) enter the composition series: $p < p_0 < p_e < e$ (see [2, 3]). This is where the proton stability comes from: proton’s states belong to the SU(2,2) invariant subspace. According to the model, a quark can be interpreted as a ‘sank’ proton (during the beginning of the reaction process, proton ‘is pushed’ into a deeper level). The model seems to be compatible with detection of point-like constituents within the proton in highly inelastic electron-proton scattering, see [4] (and with elastic electron-quark scattering). To introduce gluons, deal with proton-antiproton pairs. At each level, a gluon can be interpreted as a flavored and colored photon. Not each and every feature of the model coincides with the corresponding standard assumption about quarks and gluons. In particular, the total number of colors is level-dependent. The model predicts THREE new quarks of the 4th generation (whereas currently there is a search for just TWO).
2. Transition from the Minkowski space-time M to Segal’s compact cosmos D

First, recall the following model of the Minkowski space-time M (see [5, c.81]). Points (events) in M, are Hermitian two by two matrices. The Lie algebra u(2) consists of all skew-Hermitian matrices h; they satisfy the condition \( h^* + h = 0 \), where \( h^* \) is the transpose and complex conjugate of \( h \). A generic element \((t, L_j)\) of the (eleven-dimensional) Poincare group \( P \) takes \( h \) into \( e^{tL_j}hL_j^* + j \), where \( t \) is a real number, \( L_j \) is a matrix from \( SL(2, C) \), \( j \) is a skew-Hermitian matrix. This \( P \)-action on \( u(2) \) is well-known. The Caley map \( C \) is defined as follows:

\[
h \to (1 + h/2)(1 - h/2)^{-1}.
\]

It is known to be defined everywhere on \( u(2) \), that is, the determinant of the matrix \( 1 - h/2 \) in (2.1) is never zero. Also, the image of the Caley map is open and dense in \( U(2) \).

Here are some other important constructs. Let \( C_i \) in \( M \) stand for the set \( y + C_0 \), where \( C_0 \) is an ‘upper’ half of the light cone at the origin \( O \) of the Minkowski space-time. In other words, \( C_i \) is the image of \( C_0 \) under parallel translation (defined by a vector \( y \)). Let \( K_y \) stand for the corresponding convex closed cone. The totality of all cones \( K_y \) in \( M \) is called the causal structure in \( M \). At each point \( y \), the differential of the Caley map takes a cone \( K_y \) onto a cone in the corresponding tangent space of \( U(2) \). Hence, the (local) causal structure on \( U(2) \) is defined and we denote \( U(2) \) as \( D \). In \( M \) there was no difference between left and right translations. However, one has to distinguish between left and right shifts in \( D \). It turns out that the (above introduced) causal structure in \( D \) is bi-invariant. Also, these cones (in tangent spaces of \( D \)) are determined by a certain bi-invariant Lorentzian metric on \( U(2) \): see [2] for details.

Denote by \( G \) the matrix group \( SU(2,2) \). Recall the famous linear-fractional \( G \)-action on \( U(2) \):

\[
g(z) = (Az + B)(Cz + D)^{-1}.
\]

Here a four by four matrix \( g \) in \( G \) is determined by its two by two blocks \( A, B, C, D \). In terms of the compact Segal’s cosmos \( D \) (which means that \( U(2) \) is equipped with the above mentioned Lorentzian metric), the transformations (2.2) are conformal (and \( G \) is sometimes referred to as the conformal group). To put it very briefly, the following is Segal’s main claim (with the goal to improve the state of things in theoretical physics): to adequately model particles and interactions, the Minkowski space-time \( M \) has to be replaced by \( D \). Such a replacement resulted in his Chronometric Theory. The relation of his chronometric spin \( \frac{1}{2} \) particles (they were listed above, see Section 1) to a standard list of such particles was traced on the basis of the fundamental

**Theorem 1** ([5, p.83]). A stationary subgroup (of any event \( z \) in \( D \)) is isomorphic to \( P \). The \( P \)-action (2.2), the (above introduced) \( P \)-action on \( M \), and the Caley map (2.1) form a commutative diagram.

Let us now continue with a brief summary of the Chronometry. The universal cover of \( D \) is \( D^+ \) having the \( R^+ \times S^3 \)-topology. As a conformal Lorentzian manifold it coincides with the Einstein static universe. However, the metric \( dt^2 - ds^2 \) (which defines this conformal structure) is not fixed, rather it is related to the choice of an ‘observer’. Here \( t \) is (a ‘global’) time, \( ds \) is the metric element on \( S^3 \) (the latter represents the 3-dimensional physical space). The radius \( R \) of \( S^3 \) is a conformal invariant (hence, it is observer-independent). The \( R \) goes to infinity limit is known as the relativistic limit of Chronometry. That is, one returns to the standard theory based on \( M \). Let \( K \) stand for the 7-dimensional isometry group of the Einstein static universe. It is generated by time \( t \) translations and by rotations in \( S^3 \). The chronometric (or Einstein’s) energy is related to the generator of the temporal evolution in \( D^+ \). A reference observer chosen, the Minkowski space-time \( M \) is imbedded into \( D^+ \) (for the first step to proceed with such an imbedding, refer to the above Theorem 1). The corresponding relativistic energy is related to the generator of the time evolution in \( M \): here one has in mind a certain Lorentzian reference frame, the one which maximally touches (at the reference event) the global reference frame which is defined when the
splitting $R^1 \times S^1$ is chosen. In any unitary (and $\mathbf{M}$-energy positive) representation of $G$, the Einstein’s energy of a particle exceeds its $\mathbf{M}$-energy. This energy difference is locally negligible but it increases as the support of the wave function in question is getting larger. The inertial mass of a particle can be introduced as the energy of its interaction with the entire cosmos content. It is strictly $K$-invariant, which implies its approximate invariance w.r.t. the (10-dimensional) Poincare group.

3. The first level quarks’ flavors and colors

Let us first describe embeddings under (each of) which a matrix $Z$ from $\mathbf{D}=\mathbf{U}(2)$ is mapped onto a principal minor of the corresponding 3 by 3 matrix from $\mathbf{U}(3)$. Namely, let us denote by $\mathbf{D}_{12}$ the image of an embedding $\mathbf{A}_{12}$ of the original $\mathbf{D}$, such that:

(1) Each matrix $Z$ from $\mathbf{D}$ is now an upper 2 by 2 principal minor of a 3 by 3 matrix $\mathbf{A}_{12}(Z)$ in $\mathbf{U}(3)$,

(2) The third diagonal entry of $\mathbf{A}_{12}(Z)$ is 1,

(3) In the $\mathbf{A}_{12}(Z)$, all other entries vanish.

The two remaining embeddings, $\mathbf{A}_{13}$ and $\mathbf{A}_{23}$ are defined quite similarly. Clearly, $\mathbf{D}_{12}$, $\mathbf{D}_{13}$, and $\mathbf{D}_{23}$ are $\mathbf{U}(2)$–subgroups in $\mathbf{U}(3)$. Recall that the group $\mathbf{U}(2)$ is closed w.r.t. the complex conjugation, and w.r.t. the matrix transposition. The transposed matrix, $Z^T$ can be viewed as a symmetric one to $Z$ w.r.t. the principal diagonal. From where it follows that each of the $\mathbf{D}_{12}$, $\mathbf{D}_{13}$, $\mathbf{D}_{23}$ is invariant w.r.t. any of the two mentioned operations in $\mathbf{U}(3)$. Also, to enumerate all $\mathbf{D}_i$, it is enough to consider cases $i < j$, only.

In the totality of all $m$ by $m$ matrices, introduce $\mathbf{P}_m$, the symmetry w.r.t. the second diagonal. Clearly, when $Z$ is from $\mathbf{U}(2)$, then $\mathbf{P}_3(Z)$ is also in $\mathbf{U}(2)$. From here it follows that the subgroup $\mathbf{D}_{13}$ is $\mathbf{P}_3$-invariant in $\mathbf{U}(3)$ whereas $\mathbf{P}_3(\mathbf{D}_{12})=\mathbf{D}_{23}$, $\mathbf{P}_3(\mathbf{D}_{23})=\mathbf{D}_{12}$. That is, embeddings $\mathbf{A}_{12}$ and $\mathbf{A}_{23}$ are equivalent (one becomes the other when composed with $\mathbf{P}_3$). This relates to the presence of two $\mathbf{u}$-quarks ‘in’ a proton, whereas the $\mathbf{A}_{13}$ relates to the presence of a $\mathbf{d}$-quark in that proton. A notion of quark’s flavor has thus been defined for quarks of level (generation) one.

The notion of quark’s color was introduced in [1] as follows. Recall that each matrix $g_n$ (in $G_n=\mathbf{SU}(n,n)$) has block structure (with $n$ by $n$ blocks $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$) and that $G_n$ acts linear-fractionally on $\mathbf{U}(n)$; see [5, Section 2.1] for all related definitions and conventions. Let us start with the embedding $\mathbf{A}_{12}$. That is, rows one and two (as well as columns one and two) are selected. This choice determines an $\mathbf{SU}(2,2)$-subgroup $\mathbf{G}_{12}$ in $\mathbf{G}_2$. Namely, each $g_2$ in $\mathbf{G}_{12}$ is built (with the help of blocks $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{C}_2$, $\mathbf{D}_2$, of the original $g_2$ from $\mathbf{G}_2 = \mathbf{SU}(2,2)$) as follows: $\mathbf{A}_2$ is the upper left principal minor of the 6 by 6 matrix $g_2$; $\mathbf{D}_2$ is the principal minor which occupies the intersection of rows and columns with numbers four and five; the (non-principal) minor $\mathbf{B}_2$ occupies the intersection of rows one and two and columns four and five; the minor, $\mathbf{C}_2$ occupies the intersection of rows four and five and columns one and two. Each of the remaining entries of $g_2$ is 1(if on the principal diagonal) or 0 (if off-diagonal). The subgroups $\mathbf{G}_{13}$ and $\mathbf{G}_{23}$ are defined quite similarly. A linear fractional action of each of these three subgroups on any of the $\mathbf{U}(2)$-subgroups $\mathbf{D}_{12}$, $\mathbf{D}_{13}$ and $\mathbf{D}_{23}$ is (naturally) defined. Color (one of the three possible at level 1) of a quark is determined by the choice of a single subgroup: $\mathbf{G}_{12}$, $\mathbf{G}_{13}$, or $\mathbf{G}_{23}$.

4. Description of deeper levels and the number of quarks

To deal with deeper levels, consider embeddings of $\mathbf{D}=\mathbf{U}(2)$ into $\mathbf{U}(4)$, first. Here they are: $\mathbf{A}_{12}$, $\mathbf{A}_{13}$, $\mathbf{A}_{14}$, $\mathbf{A}_{23}$, $\mathbf{A}_{24}$, $\mathbf{A}_{34}$; notation mimics the one which has been already used in the $\mathbf{U}(3)$-case. To determine equivalencies, consider the (earlier defined) operator $\mathbf{P}_4$. Clearly, $\mathbf{A}_{12}$ is equivalent to $\mathbf{A}_{14}$, and $\mathbf{A}_{13}$ is equivalent to $\mathbf{A}_{34}$. Each of the subgroups $\mathbf{D}_{14}$ and $\mathbf{D}_{23}$ is $\mathbf{P}_4$-invariant. Relate $\mathbf{A}_{14}$ to an $\mathbf{s}$-quark, and $\mathbf{A}_{23}$ – to a $\mathbf{c}$-quark. At this (=second) level, $\mathbf{A}_{12}$ (which is equivalent to $\mathbf{A}_{34}$) is associated with a $\mathbf{u}$-quark whereas $\mathbf{A}_{13}$ (equivalent to $\mathbf{A}_{24}$) – with a $\mathbf{d}$-quark. Hence, quarks of both generations (one and two) ‘live’ on level two. Colors can be likewise introduced and the total number of (level two) colors is six.
To make the definition of color (at an arbitrary U(n)-level) mathematically rigorous, let us understand by $G_{ij}$ the following SU(2,2)-subgroup in $G_n=SU(n,n)$. It is assumed that an embedding $A_{ij}$ from $D=U(2)$ into $U(n)$ is chosen. Each element $g_{ij}$ of $G_{ij}$ is determined as the image of a unique $g_2$ from $G_2$. Each $g_2$ is specified by its 2 by 2 blocks $A_2, B_2, C_2, D_2$. To define each (of the four: $A_n, B_n, C_n, D_n$) n by n block of $g_n$, proceed as follows. The choice of $A_{ij}$ specifies a certain 2 by 2 principal minor (in any n by n matrix). The block $A_n$ is defined by the demand that $A_2$ "occupies" exactly that minor whereas the remaining entries are ones (if on the principal diagonal) or zero (if off-diagonal) – compare to how $G_{12}$ of Section 2 has been introduced. The block $D_n$ is defined similarly: with the help of $D_2$. The remaining two blocks, $B_n$ and $C_n$, are defined (with the help of $B_2$ and $C_2$) slightly differently; namely, each entry outside of the preferred 2 by 2 minor should vanish.

**Proposition 1.** $G_{ij}$ is a subgroup of $G_n$ and it is isomorphic to SU(2,2).

**Proof.** It is an easy exercise to verify that each matrix $g_n$ from $G_{ij}$ satisfies conditions which define $SU(n,n)$: see Lemma 2.1.4 of [5], parts (ii), (iii). Also, by the very way of how each $G_{ij}$ is defined, the totality of all matrices in $G_{ij}$ is a group. Finally, the (above introduced) mapping from $G_2$ into $G_n$ is an isomorphism between $G_2$ and $G_{ij}$.

For each level $U(n)$, $n\geq 2$, a quark (having a certain flavor and a certain color) is now defined as an ordered triple $(D_{pq}, G_{ij}, f)$. Here $f$ is 1 or negative 1 (depending on whether we deal with a particle or with an antiparticle). The subgroup $D_{pq}$ of $U(n)$ defines flavor whereas $G_{ij}$ defines color. An ‘implicit’ part of such a definition is a specific representation space ($p$-space) with the group $G_{ij}$ acting in it accordingly.

It is an easy exercise (left to the reader) to verify the following

**Proposition 2.** The total number of colors at U(n)-level is $n(n-1)/2$.

Here is the list of all $D=U(2)$ embeddings into $U(5)$: $A_{12}, A_{13}, A_{14}, A_{15}, A_{23}, A_{24}, A_{25}, A_{34}, A_{35}, A_{45}$. Clearly, $P_5(D_{12}) = D_{45}$, $u$-quark; $P_5(D_{13}) = D_{35}$, $d$-quark; $P_5(D_{14}) = D_{25}$, $s$-quark; $P_5(D_{23}) = D_{34}$, $c$-quark. Each of the following two subgroups, $D_{15}$ ($t$-quark) and $D_{24}$ ($b$-quark), is $P_5$-invariant. The total number of (generation three) colors is 10.

Let us introduce the notation $t(n; i, j)$ for the corresponding quark of an n-th generation; one can always assume that $i=j$. Such a convention results in the following identifications:

$t(1; 1, 2) = t(1; 2, 3) = u, t(1; 1, 3) = d; t(2; 1, 2) = t(2; 3, 4) = u, t(2; 1, 3) = t(2; 2, 4) = d, t(2; 2, 3) = c, t(2; 1, 4) = s, t(3; 1, 2) = t(3; 4, 5) = u, t(3; 1, 3) = t(3; 3, 5) = d, t(3; 2, 3) = t(3; 3, 4) = c, t(3; 1, 4) = t(3; 2, 5) = s, t(3; 2, 4) = b, t(3; 1, 5) = t, “top quark”$.

It has been proven in [1] that there are three new flavors (the fourth generation quarks) at the level of $U(6)$. More generally, the following statement has been proven in [1]:

**Theorem 2.** On the level $U(n)$, let an U(2)-subgroup $D_{ij}$ be not $P_n$-invariant. Then $D_{ij}$ corresponds to a quark from a lower level. The recurrent (1) and explicit (2) formulas (for the total number of quarks at $U(n)$-level) hold:

$$m_2 = 1, m_n = m_{n-1} + \lfloor n/2 \rfloor. \quad (1)$$

$$m_n = \{n(n-1)/2 + \lfloor n/2 \rfloor\}/2. \quad (2)$$

By $[x]$, the value (at a real number $x$) of the greatest integer function is understood.
5. On Antiquarks and Gluons

Segal models photon on the basis of a tensor product of proton and anti-proton spaces (see [6], p. 37 and p. 56). For each level U(n), n>2, a quark (having a certain flavor and a certain color) has been defined as an ordered triple (Dpq, Gij, f). Each antiquark is formally a triple (Dpq, Gij, -f). Here the action of Gij from (Dpq, Gij, -f) is the complex-conjugate to the action of Gij from (Dpq, Gij, f) - according to the way how one gets an antiproton when the original proton is specified. Hence, each antiquark has an anti-flavor and an anti-color. The model allows the interpretation of gluons as flavored and colored photons. Namely, the pair color-anticolor (it characterizes a gluon) is introduced formally as ((Gij,f), (Gsk,-f)). There are eight gluons on U(3) level which is in compliance with standard chromo-dynamics.

6. Concluding Remarks

It follows from the above that (when trying to apply the model) one has to use new reactions’ cross sections formulas. Clearly, they must be level-dependent (which seems to be in accordance with a well-known leading particles and effective energy approach).

It is known that interaction between quarks is both flavor- and color-independent. This is quite understandable within the multi-level model.

Let us now try to get closer to possible experimental verification of the multi-level model. It is known that the ratio of (full) cross sections between πp- and pp-scatterings is in compliance with the (standard) quarks’ model. In terms of the standard model this is explained as follows. Proton is composed of two u-quarks and of a d-quark (p = uud). The π-meson consists of two quarks, namely π⁺ = u↑τ, and π⁻ = d↓τ. When π-meson collides with a proton, each quark in the π-meson can interact with each quark in the proton. The proton-proton collision is similarly described. Hence (under the assumption that interaction between hadrons is independent of the types of their quarks’ constituents), the ratio of full cross sections between πp- and pp-scatterings should be 2/3. Experiments (within the appropriate energy range) agree on 0.633, that is, pretty close to 2/3.

According to the multi-level model, interactions (of the type above mentioned) take place on the level of U(3). Let us prove that the ratio of full cross sections between pion-proton and pp-scatterings should be 2/3.

The multi-level model (staying in agreement with the Standard Model in this issue) assumes that the interaction is between quarks. In case of an unstable particle, the multi-level model is able to describe interactions ‘as if’ the particle in question has the same quarks ingredients as specified by the Standard Model. In order to “cook” such a particle (π⁺, say), one anti-proton has to be captured by a d-cell (there is just one d-cell ‘available’) while a proton has to be captured by an u-cell (there are two u-cells ‘at hand’). For the proton to be involved into scattering on the U(3)-level, it has to be captured by any (of the three possible) cell on that level. Conclusion: there are six options to choose a couple of interacting quarks. We have thus proven that the ratio of full cross sections between π⁺p- and pp-scatterings should be 6/9 = 2/3.

It is worth indicating that the model predicts a (small) correction to the graph of the following curve (the knowledge about that curve is fundamental for the current high-energy physics). The curve deals with the e⁺e⁻ annihilation. In order to plot the curve, the measure along horizontal axis is in terms of the square s of the total energy in the center of inertia. The vertical axis is for a certain ratio R, which (for long time) remains one of the main objects of experimental research (when dealing with quarks). Namely, R(s) = (total cross section to get hadrons) over (total cross section to get muon and anti-muon pair). Outside of the resonances, the curve is piece-wise constant. The correction occurs in the range of those values of s, where b- and t-quarks are first detected (it is the case of U(5)-level in terms of the model’s terminology). Introduce the following two cases. Case one: b-quarks are involved, only; case two: b-quarks as well as t-quarks are involved. In each of the two cases it is an ‘up-correction’. In case one, it is a jump from 1.22 (the accepted value) to 1.25; in case two: from 1.67 (the accepted value) to 1.74. In terms of the multi-level model, these two jumps are theoretically justified by non-standard quarks’ charges in the
corresponding energy range: 7/10 (instead of 2/3) for u-, c-, and t-quarks; negative 3/10 (instead of negative 1/3) for d-, s-, and b-quarks. These values are obtained on the basis of the Han-Nambu scheme. Interestingly, on the levels U(3) and U(4) the model predicts standard quarks’ charges. For the three ‘new’ quarks on the U(6)-level: t(4; 1, 6), t(4; 3, 4), and t(4; 2, 5), their charges are, correspondingly, 2/3, 2/3, and negative 1/3. To further support the claims in the current paragraph, more details are to be provided elsewhere.

Proton decay was one of the key predictions of several theories proposed in the 1970s. That probably explains why Segal himself (in late 80s) has not identified the lowest entry in the corresponding composition series with the proton (see above, the second paragraph of the Introduction where the current author claims stability of the proton). To try to find out whether Segal was thinking of a proton in this regard, one might be willing to get to Segal’s Archive which is materially located at Boston University (see http://math.mit.edu/segal-archive/guide.php for more details).

7. References
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