Fairness in Risk Assessment Instruments: Post-Processing to Achieve Counterfactual Equalized Odds

Alan Mishler* Edward H. Kennedy†

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Abstract

Algorithmic fairness is a topic of increasing concern both within research communities and among the general public. Conventional fairness criteria place restrictions on the joint distribution of a sensitive feature $A$, an outcome $Y$, and a predictor $S$. For example, the criterion of equalized odds requires $S \perp \perp A \mid Y$, or equivalently, when all three variables are binary, that the false positive and false negative rates of the predictor be the same for two levels of $A$ [Hardt et al., 2016].

However, fairness criteria based around observable $Y$ are misleading when applied to Risk Assessment Instruments (RAIs), such as predictors designed to estimate the risk of recidivism or child neglect. It has been argued instead that RAIs ought to be trained and evaluated with respect to potential outcomes $Y^0$ [Coston et al., 2020]. Here, $Y^0$ represents the outcome that would be observed under no intervention—for example, whether recidivism would occur if a defendant were to be released pretrial.

In this paper, we develop a method to post-process an existing binary predictor to satisfy approximate counterfactual equalized odds, which requires $S$ to be nearly conditionally independent of $A$ given $Y^0$, within a tolerance specified by the user. Our predictor converges to an optimal fair predictor at $\sqrt{n}$ rates under appropriate assumptions. We propose doubly robust estimators of the risk and fairness properties of a fixed post-processed predictor, and we show that they are $\sqrt{n}$-consistent and asymptotically normal under appropriate assumptions.

*Department of Statistics & Data Science, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. Email: amishler@stat.cmu.edu
†Assistant Professor, Department of Statistics & Data Science, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. Email: edward@stat.cmu.edu
1 Introduction

Machine learning is increasingly involved in high stakes decisions in domains such as healthcare, criminal justice, and consumer finance. In these settings, ML models often take the form of Risk Assessment Instruments (RAIs): given covariates such as demographic information and an individual’s medical/criminal/financial history, the model predicts the likelihood of an adverse outcome, such as a dangerous medical event, recidivism, or default on a loan. Rather than rendering an automatic decision, the model produces a “risk score,” which a decision maker may take into account when deciding whether to initiate a medical treatment, release a defendant on bail, or issue a personal loan.

The proliferation of machine learning has raised concerns that learned models may be discriminatory with respect to protected features like race, sex, age, and socioeconomic status. For example, there has been vigorous debate about whether a widely used recidivism prediction tool called COMPAS is biased against black defendants [Angwin et al., 2016; Angwin and Larson, 2016; Dieterich et al., 2016; Larson and Angwin, 2016; Lowenkamp et al., 2016]. Concerns have also been raised about risk assessments used to identify high risk medical patients [Obermeyer et al., 2019] and about common credit scoring algorithms such as FICO [Rice and Swesnik, 2012], among many others. Collectively, these types of algorithms arguably impact the lives of the vast majority of Americans.

These concerns have led to an explosion of methods in recent years for developing fair models and auditing the fairness of existing models. These efforts are complicated by the fact that there is no consensus on how to quantify (un)fairness. Researchers have proposed a wide range of fairness criteria, many of which turn out to be mutually unsatisfiable under real-world conditions [Chouldechova, 2017; Kleinberg et al., 2017].

The most widely discussed fairness criteria impose constraints on the joint distribution of a sensitive feature \( A \), an outcome \( Y \), and a predictor \( S \). These criteria are inappropriate for RAIs, however. RAIs are not concerned with the observable outcomes \( Y \) in the training data (“Did patients of this type historically experience serious complications?”), which are themselves a product of historical treatment decisions. Rather, they are concerned with the potential outcomes associated with available treatment decisions (“Would patients of this type experience complications if not treated?”). Because treatments are not assigned at random—doctors naturally treat the patients they think are at high risk—these are distinct questions.

Coston et al. [2020] showed how RAIs that are optimized to predict observable rather than potential outcomes systematically underestimate risk for units that have historically been receptive to treatment, leading to suboptimal treatment decisions. They further

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\(^1\) See [Imai and Jiang, 2020] for a set of sufficient conditions under which these unsatisfiability results disappear.
showed how evaluations of the performance and fairness properties of RAIs with respect to observable outcomes are misleading. However, they left open the question of how to develop predictors that achieve fairness with respect to potential outcomes.

The notion of fairness we target is *counterfactual equalized odds*. Conventional equalized odds requires that risk predictions be independent of the sensitive feature conditional on observed outcomes [Hardt et al., 2016]. In the setting that we consider, in which the sensitive feature, predictor, and outcome are all binary, this is equivalent to insisting that the false positive and false negative rates be equal for both groups. The counterfactual version of equalized odds simply substitutes a particular potential outcome for the observed outcome [Coston et al., 2020]. We argue that this fairness criterion is intuitively appealing in the RAI setting, and that it is preferable to counterfactual versions of other candidate fairness criteria.

Hardt et al. [2016] showed how an existing binary predictor can be post-processed to yield randomized classifiers that satisfy equalized odds. We extend their method to the counterfactual setting. Like their approach, our method yields a randomized classifier that requires access at runtime only to the sensitive feature and the output of the previously trained predictor. Our randomized classifier is likewise parameterized as the solution to a simple linear program. We further provide estimators for the loss and fairness properties of our derived predictor. We show that our estimators are $\sqrt{n}$-consistent and asymptotically normal under appropriate assumptions.

### 1.1 Outline and summary of contributions

In Section 2, we propose a relaxation of counterfactual equalized odds, *approximate counterfactual equalized odds*, that allows RAI designers to control the tradeoff between fairness and predictive performance. We also discuss related work and explain our choice to focus on equalized odds vs. other available fairness criteria. In Section 3, we motivate the use of counterfactuals through an example that illustrates how observable equalized odds fails to reduce disparities between groups and can cause harm to an already disadvantaged group. In Section 4, we define and provide identifying expressions for our main estimand, a loss-optimal derived predictor that satisfies approximate counterfactual equalized odds. In Section 5, we provide an estimator of this predictor and show that it converges to the optimum at $\sqrt{n}$ rates in a particular sense. Once a specific derived predictor has been chosen, it is of interest to estimate the loss and fairness properties of that predictor. We provide efficient doubly robust estimators for these properties and show that they yield asymptotically valid confidence intervals and hypothesis tests. In Section 6, we illustrate our theoretical results via simulations. We conclude with discussion in Section 7.
2 Background and Related Work

2.1 Basic notation and fairness definitions

A table listing all notational choices can be found in Appendix D.

Let $A, D, Y$ denote a sensitive feature, decision, and outcome, respectively. We consider the setting in which all three are binary, though most of the definitions below extend readily to continuous settings. Denote by $Y^0, Y^1$ the potential outcomes $Y^{D=0}, Y^{D=1}$. $Y^d_i$ is the outcome that would be observed for unit $i$ if, possibly contrary to fact, the decision were set to $D = d$ [Neyman, 1923; Holland, 1986; Rubin, 2005]. We refer to the two levels of the sensitive feature $A$ as the two “groups,” and we use “treatment” and “intervention” synonymously with “decision.” Let $S$ be any random variable that maps to $\{0, 1\}$.

Under typical assumptions, the potential outcome $Y^D$ associated with the actual decision that is made is rendered observable, while the other outcomes remain counterfactual. However, we follow convention in using the term “counterfactual” as a synonym for “potential,” covering both observable and unobservable cases.

In most RAI settings, one of the decision options is a natural baseline corresponding to “no intervention” ($D = 0$). Examples include the risk of recidivism if a defendant is released pretrial, or the risk of neglect or abuse if a child welfare call is not screened in for further investigation. Many or most RAI s do not generate a separate risk score for the outcome associated with intervention. (In the case of child welfare, for example, call screeners have to make a binary decision about whether to screen a call in or out. It’s not clear that a prediction that a child is at high risk with or without intervention would be more useful with respect to that decision than a prediction that a child is at high risk with no intervention.) We therefore restrict attention to this baseline potential outcome $Y^0$, though extensions to contrasts between potential outcomes are also possible.

Denote the observational and counterfactual false positive rates of $S$ for group $a$ by $\text{FPR}(S, a) = P(S = 1 \mid Y = 0, A = a)$ and $\text{cFPR}(S, a) = P(S = 1 \mid Y^0 = 0, A = a)$. For example, $\text{cFPR}(S, 0)$ could represent the chance of being falsely labeled high-risk if released pre-trial, among those black defendants who would not actually go on to recidivate, while $\text{cFPR}(S, 1)$ could represent the corresponding chance for white defendants who would not recidivate. Let $\text{FNR}$, $\text{cFNR}$, $\text{TPR}$, $\text{cTPR}$, $\text{TNR}$, $\text{cTNR}$ denote the corresponding false negative, true positive, and true negative rates, respectively.

**Definition 2.1.** A predictor $S$ satisfies observational equalized odds (oEO) with respect to $A$ and $Y$ if $S \perp A \mid Y$. It satisfies counterfactual equalized odds (cEO) if $S \perp A \mid Y^0$.

When $A, Y,$ and $S$ are all binary, equalized odds is equivalent to requiring that the corre-
sponding false positive and false negative rates be equal for the two levels of $A$. Our derived predictor will be designed to satisfy a relaxation of this criterion, defined below.

**Definition 2.2.** The *counterfactual error rate differences* for a predictor $S$ are the differences $\Delta^+$ and $\Delta^-$ in the cFPR and cFNR for the two groups $A = 0, A = 1$, defined as follows:

$$\Delta^+(S) = \text{cFPR}(S, 0) - \text{cFPR}(S, 1)$$
$$\Delta^-(S) = \text{cFNR}(S, 0) - \text{cFNR}(S, 1)$$

**Definition 2.3.** When $A, Y$, and $S$ are all binary, $S$ satisfies *approximate counterfactual equalized odds* with *unfairness tolerances* $\epsilon^+, \epsilon^- \in [0, 1]$ if

$$|\Delta^+(S)| \leq \epsilon^+$$
$$|\Delta^-(S)| \leq \epsilon^-$$

In general, a fairness-constrained predictor would not outperform an optimal unconstrained predictor, and in some cases, satisfying cEO exactly might degrade performance to the point that the RAI is no longer useful. This relaxation of cEO allows RAI designers to negotiate this tradeoff.

### 2.2 Ways of achieving fairness

There are three broad approaches to developing fair models: (1) preprocessing the input data to remove bias [Kamiran and Calders, 2012; Calmon et al., 2017], (2) constraining the learning process [Zafar et al., 2017; Donini et al., 2018; Narasimhan, 2018], and (3) post-processing a model to satisfy fairness constraints [Hardt et al., 2016; Kim et al., 2019]. Our approach can be viewed as belonging to class (3). Each approach has advantages and disadvantages, and in some cases the distinctions between the approaches may not be clear. Many widely used RAIs are proprietary tools developed by for-profit companies, so they are not amenable to internal tinkering. Developing new, fair(er) RAIs would be costly and perhaps infeasible from a policy perspective. The advantage of post-processing in this setting is that it can be applied to models that are already in use. Our method requires access at runtime only to the sensitive feature and the output of the existing predictor, so in principle, it could easily be incorporated into existing risk assessment pipelines. We refer to the predictor that our method returns equivalently as a “post-processed” or “derived” predictor [Hardt et al., 2016].
2.3 Why equalized odds?

Equalized odds is one of several popular fairness criteria that impose constraints on the joint distribution of \((A,Y,S)\). Equalized odds is known more generally as separation, a term which covers settings in which these variables are not necessarily binary. The other two popular criteria in this class are independence \((S \indep A)\) and sufficiency \((Y \indep A \mid S)\); sufficiency is equivalent to calibration or predictive parity when all three variables are binary. Variants of all three criteria may be defined for example by conditioning on additional variables. The counterfactual versions of these criteria defined in [Coston et al., 2020] simply replace \(Y\) with \(Y^0\).

Except in highly constrained, unrealistic conditions, these three criteria are pairwise unsatisfiable, regardless of whether they are defined with respect to \(Y\) or \(Y^0\) [Kleinberg et al., 2017; Chouldechova, 2017; Barocas et al., 2018]. We must therefore choose which criterion we wish to target.

When evaluating a predictive system, it seems natural to focus on its real-world impact rather than its outputs per se. One desirable property of a decision process might be \(D \indep A \mid Y^0\). In the context of recidivism prediction, for example, this would mean no correlation between sentence length \((D)\) and race \((A)\) among defendants who would recidivate if released \((Y^0 = 1)\) or among those who wouldn’t recidivate if released \((Y^0 = 0)\). By way of shorthand, we will say that if \(D \not\indep A \mid Y^0\), then the system exhibits discriminatory disparate impact, meaning an unjustifiable difference in the distribution of benefits or burdens across groups.

In the context of RAIs, decision makers typically have wide latitude in how they interpret and act on the risk scores, so constraining the RAI does not enforce fairness with respect to their decisions. However, if decision makers, after the introduction of the RAI, make their decisions only on the basis of the RAI scores and other variables \(U\) which are independent of the RAI and \(A\) given \(Y^0\), then equalized odds will imply \(D \indep A \mid Y^0\). That is, let \(D = f(S,U)\) represent the function \(f\) describing the decision process after the RAI \(S\) is introduced. If cEO is satisfied and \(U \indep (S,A) \mid Y^0\), then it follows that \(D \indep A \mid Y^0\). Even if \(U \not\indep (S,A) \mid Y^0\), it is easy to see that if the conditional independence statement nearly holds, or if \(f\) depends primarily on \(S\) rather than \(U\), then discriminatory disparate impact can be small.

By contrast, for a predictor satisfying either independence or sufficiency, there is no mapping from predictions to decisions that satisfies this property. Chouldechova [2017] in particular showed how predictors which satisfy sufficiency (predictive parity) are likely to yield decisions such that \(D \not\indep A \mid Y\); these arguments are unchanged when we substitute \(Y^0\) for \(Y\). Though there is no consensus about how to quantify fairness, this is at least one consideration in favor of equalized odds over predictive parity or independence.
2.4 Other causal fairness criteria

There is another set of fairness criteria that is motivated by causal considerations. These criteria consider counterfactuals of the sensitive feature or proxy, and they characterize a decision or prediction as fair if the sensitive feature or proxy does not “cause” the decision or prediction, either directly or along a prohibited pathway [Kilbertus et al., 2017; Kusner et al., 2017; Nabi and Shpitser, 2018; Zhang and Bareinboim, 2018; Nabi et al., 2019; Wang et al., 2019]. There is some controversy over whether it is meaningful to discuss a counterfactual of a feature like race or gender [VanderWeele and Robinson, 2014; Glymour and Glymour, 2014; Hu and Kohler-Hausmann, 2020]. Furthermore, it is not clear that these metrics are appropriate in the context of risk assessment, where emphasis is on assessing risk regardless of the causes of that risk. Finally, satisfying these metrics typically precludes use of most of the features that go into risk assessment, like prior history, which is not tenable in practice [Coston et al., 2020].

For all these reasons, we focus on (counterfactual) equalized odds; however our approach can be adapted to other fairness criteria, and we do not advocate for the use of equalized odds in all settings.

3 Motivating Example

We now motivate the use of counterfactual rather than observational equalized odds. The following example illustrates how predictors which satisfy oEO (observational equalized odds) rather than cEO (counterfactual equalized odds) will in general not eliminate discriminatory disparate impact, and how they can unintentionally reduce rates of appropriate intervention.

Consider a school district that assigns tutors to students who are believed to be at risk of academic failure. The school district wishes to develop a RAI $S$ to better identify students who need tutors while ensuring that this resource is allocated fairly across two levels of the sensitive feature $A$. Let $D \in \{0, 1\}$ represent the decision to assign (1) or not assign (0) a tutor, and let $Y \in \{0, 1\}$ represent academic success (0) or failure (1).

Let $W$ represent the set of covariates available as input to the RAI, with $A$ possibly in $W$. The quantity of interest is $S = \hat{P}(Y^0 = 1 \mid W)$, i.e., the estimated probability of failure if no tutor is assigned. A cEO predictor satisfies $\hat{P}(S \mid Y^0, A) = \hat{P}(S \mid Y^0)$, while an oEO predictor satisfies $\hat{P}(S \mid Y, A) = \hat{P}(S \mid Y)$. Divergence in these predictors is driven by the extent to which $Y \neq Y^0$ in the training data. In order to parameterize this divergence, we introduce the following definitions.

**Definition 3.1.** The *need rate* for group $a$ is $\mathbb{P}(Y^0 = 1 \mid A = a)$, the probability that a
student from group $a$ would fail without a tutor.

**Definition 3.2.** The *opportunity rate* for group $a$ is $\mathbb{P}(D = 1 \mid Y^0 = 1, A = a)$, the probability that a student in group $a$ who needs a tutor receives one.

**Definition 3.3.** The *intervention strength* for group $a$ is $\mathbb{P}(Y^1 = 0 \mid Y^0 = 1, A = a)$, the probability that a student in group $a$ who would fail without a tutor would succeed with a tutor.

We simulate a simple data generating process in which we allow the intervention strength to vary, while constraining it to be equal for the two groups. We fix all other parts of the distribution. In particular, we set $\mathbb{P}(A = 1) = 0.7$, set the need rates to 0.4 and 0.2 for groups 0 and 1, and set the opportunity rates to 0.6 and 0.4. We set the probabilities that a tutor is assigned when it is not needed to $\mathbb{P}(D = 1 \mid Y^0 = 0, A = 0) = 0.3$ and $\mathbb{P}(D = 1 \mid Y^0, A = 1) = 0.2$. This represents a scenario in which the minority group has greater need, perhaps due to socioeconomic factors or prior educational opportunities, and also is likelier to receive resources. Finally, we set $\mathbb{P}(Y^1 = 0 \mid Y^0 = 0) = 1$, meaning that tutoring never *increases* the risk of failure.

We consider a hypothetical oEO predictor $S$ with fixed false positive rate $\mathbb{P}(S = 1 \mid Y = 0, A) = \mathbb{P}(S = 1 \mid Y = 0) = 0.1$ and false negative rate $\mathbb{P}(S = 0 \mid Y = 1, A) = \mathbb{P}(S = 0 \mid Y) = 0.2$. We assume $S \perp \perp Y^0 \mid A, Y$, as would be the case for example when $S$ is a high quality predictor of $Y$. Figure 1 shows the cTPRs for this predictor as a function of intervention strength, relative to the baseline opportunity rates for the two groups. When the intervention has no effect (strength 0), the cTPRs are equal because $Y \equiv Y^0$, so the cTPR and TPR are identical. (Of course, a strength of 0 means the tutoring is worthless.) For all strength values $> 0$, the cTPR of the minority groups is lower than for the majority group. The difference in error rates increases as intervention strength increases. A cEO predictor avoids this problem by design: the cTPRs for the two groups are constrained to be equal.

The actual effect of the RAI on opportunity rates depends on how human decision makers interpret and respond to the RAI, but this example makes it clear that oEO predictors in general will not prevent discriminatory disparate impact, whereas, as discussed in section 2.3, counterfactual EO predictors have at least the potential to mitigate or avoid it.

This example also illustrates how observational EO predictors can reduce rates of appropriate intervention. For example, suppose that decision makers, after the introduction of the RAI, set $D = S$, i.e. they assign tutors precisely to students whom the RAI labels as high risk. Then, for any intervention strength $> 0.5$, the opportunity rate for the minority group decreases below baseline: the RAI *harms the minority group.*

As described in section 2.2, there are many ways to arrive at a predictor which satisfies a
Figure 1: Counterfactual true positive rates for a predictor satisfying observable equalized odds, as a function of the intervention strength $\mathbb{P}(Y^1 = 0 \mid Y^0 = 1)$. Dashed lines indicate opportunity rates $\mathbb{P}(D = 1 \mid Y^0 = 0)$ prior to the development of the RAI.

given fairness criterion. We now define our estimand, which accomplishes this by taking in a previously trained predictor and post-processing it to satisfy cEO.

4 Estimand: an optimal fair derived predictor

We expand our notation in order to fully describe our problem setting. Consider a random vector $Z = (A, X, D, S, Y) \sim \mathbb{P}$, where in addition to the binary sensitive feature $A$, decision $D$, and outcome $Y$, we have covariates $X \in \mathbb{R}^p$ and a previously trained predictor $S$. Initially, we will consider binary predictors $S \in \{0, 1\}$. We require only that $S$ is observable; we do not require access to its inputs or internal structure. $S$ in practice could represent a RAI that is already in use, such as a recidivism prediction tool. The covariates $X$ may or may not overlap with the inputs to $S$. Their role in the analysis is to render counterfactual quantities identifiable.

Our target is a derived predictor that satisfies approximate cEO. As in the case of observable equalized odds considered by [Hardt et al., 2016], we achieve this by randomly flipping $S$ with probabilities that depend only on $S$ and $A$. Consider a vector $\theta =$
\((\theta_{0,0}, \theta_{0,1}, \theta_{1,0}, \theta_{1,1})^T \in [0,1]^4\). We define an associated derived predictor \(S_\theta\):

\[
S_\theta \sim \text{Bern}(\theta_{A,S})
\]

where \(\theta_{A,S} = \sum_{a,s \in \{0, 1\}} 1\{A = a, S = s\} \theta_{a,s}\)

In other words, the \(\theta_{a,0}\) parameters represent conditional probabilities that \(S\) flips, while the \(\theta_{a,1}\) parameters represent conditional probabilities that \(S\) doesn’t flip. Notice that for \(\tilde{\theta} = (0, 1, 0, 1)\), we have \(S_{\tilde{\theta}} = S\): the derived predictor is equal to the input predictor.\(^2\)

Our target is a loss-optimal fair predictor \(S_{\theta^*}\), where the fairness criterion is approximate cEO. The loss function we consider is MSE, which in the binary setting is equivalent to prediction error.\(^3\) For fixed \(\theta\), denote the loss by \(L(S_\theta) = \mathbb{E}[(Y^0 - S_\theta)^2]\). The estimand is

\[
\theta^* \in \arg \min_{\theta} L(S_\theta)
\]

subject to \(\theta \in [0,1]^4\)

\[
|\Delta^+(S_\theta)| \leq \epsilon^+
\]

\[
|\Delta^-(S_\theta)| \leq \epsilon^-
\]

where the unfairness tolerances \(\epsilon^+, \epsilon^- \in [0,1]\) are chosen by the user. Setting both these constraint parameters to 0 requires cEO to be satisfied exactly, while setting them to 1 allows \(S_{\theta^*}\) to be arbitrarily unfair.

For any fixed \(\theta \in [0,1]^4\)—say, for an estimate \(\hat{\theta}\) of the target parameter \(\theta^*\)—it is of interest to estimate the loss and fairness properties of the associated derived predictor.

Since our estimands involve counterfactual quantities, distributional assumptions are required in order to equate them to observable quantities.

### 4.1 Identification

In this subsection we show that counterfactual error rates and loss can be identified under standard assumptions. All the quantities to be identified can be written in terms of the loss and the counterfactual error rates of \(S_\theta\). For ease of notation, we first define two nuisance parameters that appear in the estimand and associated estimators, namely the outcome

\(^2\)In the previous section, we used \(S\) to refer to an arbitrary RAI. From here forward, we use the notation \(S_\theta\) to indicate the dependence of the derived predictor on the input predictor \(S\) and the parameter \(\theta\).

\(^3\)We refer to MSE as “loss” instead of the conventional “risk” in order to avoid confusion between risk assessment and the error rate of a predictor.
regression and propensity score functions. We generally drop the arguments from these functions in subsequent usage for the sake of conciseness.

\[
\mu_0(A, X, S) = \mathbb{E}[Y \mid A, X, S, D = 0]
\]
\[
\pi(A, X, S) = \mathbb{P}(D = 1 \mid A, X, S)
\]

We make the following standard “no unmeasured confounding”-type causal inference assumptions:

Assumption 1 (Consistency).
\[Y = D\mu_1 + (1 - D)\mu_0\]

Assumption 2 (Positivity).
\[
\exists \delta \in (0, 1) \text{ s.t. } \mathbb{P}(\pi(A, X, S) \leq 1 - \delta) = 1
\]

Assumption 3 (Ignorability).
\[Y_0 \perp \perp D \mid A, X, S\]

Satisfying ignorability assumptions typically requires collecting a rich enough set of deconfounding covariates. In the present case, even if \(X\) is low dimensional, the ignorability assumption is plausible if the input predictor \(S\) substantially drives decision making, or if it happens to be an accurate (if not fair) predictor of \(Y_0\).

Before giving the identifying expressions for the loss \(\mathcal{L}(S_\theta)\) and the fairness constraints \(\Delta^+, \Delta^-\), we give identifying expressions for the error rates of the input predictor \(S\), which themselves appear in the expressions for \(\Delta^+, \Delta^-\).

**Proposition 1.** Under assumptions 1-3, the counterfactual error rates of the input predictor \(S\) are identified as follows:

\[
cFPR(S, a) = \frac{\mathbb{E}[S(1 - \mu_0)1\{A = a\}]}{\mathbb{E}[(1 - \mu_0)1\{A = a\}]}
\]
\[
cFNR(S, a) = \frac{\mathbb{E}[(1 - S)\mu_01\{A = a\}]}{\mathbb{E}[\mu_01\{A = a\}]}
\]

All proofs are given in the appendix. We now define several quantities that appear in the identifying expressions for the linear program:

\[
\beta_{a,s} = \mathbb{E}[1\{A = a, S = s\}(1 - 2\mu_0)], \text{ for } a, s \in \{0, 1\}
\]
\[
\beta = (\beta_{0,0}, \beta_{0,1}, \beta_{1,0}, \beta_{1,1})
\]
\[
\beta^+ = (1 - cFPR(S, 0), cFPR(S, 0), cFPR(S, 1) - 1, -cFPR(S, 1))
\]
\[
\beta^- = (-cFNR(S, 0), cFNR(S, 0) - 1, cFNR(S, 1), 1 - cFNR(S, 1))
\]
Proposition 2. Under assumptions 1-3, the loss and error rates of the derived predictor \(S_\theta\) are identified as:

\[
\mathcal{L}(S_\theta) = \theta^T \beta + \mathbb{E}[\mu_0]
\]
\[
\Delta^+(S_\theta) = \theta^T \beta^+
\]
\[
\Delta^-(S_\theta) = \theta^T \beta^-
\]

Since the term \(\mathbb{E}[\mu_0]\) in the loss is fixed, we can drop it without changing the minimizer of the loss. We can therefore rewrite the estimand as

\[
\theta^* \in \arg \min_{\theta} \theta^T \beta
\]
subject to \(\theta \in [0, 1]^4\)

\[
|\theta^T \beta^+| \leq \epsilon^+
\]
\[
|\theta^T \beta^-| \leq \epsilon^-
\]

In other words, the optimal fair derived predictor is the solution to a linear program (LP). We refer to this as the “true LP” since it defines the estimand. We now define an estimator \(\hat{\theta}\) as the solution to an “estimated LP.”

5 Estimation

5.1 Sample splitting

There are two broad estimation tasks. The first is to construct a derived predictor \(S_{\hat{\theta}}\) that approximates \(S_{\theta^*}\). The second is to estimate properties of \(S_{\hat{\theta}}\), conditional on \(\hat{\theta}\). We use sample splitting to generate two datasets: \(D_{\text{train}}\) is for the first task, and \(D_{\text{test}}\) is for the second.

Within each dataset, further splitting is used to separate the estimation of nuisance and target parameters, avoiding the need for potentially restrictive empirical process assumptions. The nuisance parameters are \(\mu_0\) and \(\pi\). The target parameters can in each case be written as functions of the nuisance parameters. In our analysis, each dataset is split once: one fold is used to estimate nuisance parameters, and the other fold is used to estimate the target parameter(s). We refer to these folds as \(D_{\text{train}}^{\text{nuis}}, D_{\text{target}}^{\text{target}}, D_{\text{train}}^{\text{nuis}}, D_{\text{test}}^{\text{target}}, D_{\text{target}}^{\text{target}}\). To regain full sample size efficiency, one can swap the folds, repeat the procedure, and average the results, an approach that is popularly called cross-fitting [Bickel and Ritov, 1988; Robins et al., 2008; Zheng and van der Laan, Mark, 2010; Chernozhukov et al., 2018]. A \(k\)-fold version of cross-fitting with \(k > 2\) is also possible. To simplify notation, we analyze
a single split procedure, with the understanding that extending to the $k$-fold setting is straightforward.

The sample splitting procedure is illustrated in Figure 2. For convenience, we assume that each of the four samples is of size $n$, though our results require only that each sample is $O(n)$. Note that our results obtain even without sample splitting if the nuisance parameters belong to Donsker classes, but sample splitting allows us to avoid this assumption.

![Figure 2: Sample splitting diagram.](image)

$D_{\text{train}}$ is used to compute $\hat{\theta}$, the estimate of the optimal predictor parameter $\theta^*$. Once $\hat{\theta}$ has been computed, $D_{\text{test}}$ is used to estimate the loss and fairness properties of the derived predictor $S_{\hat{\theta}}$. Within each dataset, sample splitting is used to separate estimation of the nuisance parameters $\mu_0, \pi$ from estimation of the target parameters.

**Remark 1. (Notation)** To avoid excessive notation, we use hats (as in $\hat{\pi}$) for quantities estimated on both $D_{\text{train}}$ and $D_{\text{test}}$. Likewise, for both $D = D_{\text{train}}$ and $D = D_{\text{test}}$, we use $\mathbb{P}_n$ to denote the empirical measure over $D_{\text{target}}$, so that for any function $\hat{f}(Z)$, $\mathbb{P}_n(\hat{f}(Z)) = \int \hat{f}(Z) d\mathbb{P}_n(Z) = n^{-1} \sum_{i=1}^n \hat{f}(Z_i)$ denotes the average of $\hat{f}(Z)$ taken over $D_{\text{target}}$. The norm of a fixed function $f$ is the $L_2$ norm with respect to $\mathbb{P}$, i.e., $\|f\|^2 = \int f(z)^2 d\mathbb{P}(z)$.

Quantities that refer to $D_{\text{train}}$ occur only in the context of estimating $\hat{\theta}$, while quantities that refer to $D_{\text{test}}$ occur only in the context of estimating properties of $S_{\hat{\theta}}$ (or of $S_\theta$ for any fixed parameter $\theta$). The intended usage should therefore be clear from context, but we make it explicit in cases where it might otherwise be unclear.

Both $\mu_0$ and $\pi$ can be estimated with arbitrary nonparametric learners. We will require only that they are estimated consistently at certain rates. Our theoretical results utilize the following additional assumptions:

**Assumption 4 (Bounded propensity estimator).** $\exists \delta \in [0, 1]$ s.t. $\hat{\pi} \leq 1 - \delta$

**Assumption 5 (Nuisance estimator rates).**

- $\|\hat{\mu}_0 - \mu_0\| = o_p(1)$,
- $\|\hat{\pi} - \pi\| = o_p(1)$,
- $\|\hat{\mu}_0 - \mu_0\|\|\hat{\pi} - \pi\| = o_p(1/\sqrt{n})$.
Assumption 4 can be trivially satisfied by truncating $\hat{\pi}$ at $1 - \delta$. Under assumption 2, this will not prevent $\hat{\pi}$ from being consistent for $\pi$. Assumption 5 states that $\hat{\mu}_0$ and $\hat{\pi}$ are consistent for $\mu_0$ and $\pi$, and that the product of their errors is smaller than $1/\sqrt{n}$. Assumption 5 can be satisfied under relatively weak and nonparametric smoothness or sparsity assumptions [Györfi et al., 2002; Raskutti et al., 2011]. For example, let $d = p + 2$ be the dimension of $(A, X, S)$. If $\mu_0$ and $\pi$ are in Hölder classes with smoothness index $s > d/2$, then there exist nonparametric estimators $\hat{\mu}_0$ and $\hat{\pi}$ such that $\|\hat{\mu}_0 - \mu\| = o_P(n^{-1/4})$ and $\|\hat{\pi} - \pi\| = o_P(n^{-1/4})$, in which case Assumption 5 would be satisfied. The two parameters naturally do not have to be estimated at the same rate, as long as they are both estimated consistently and the product of the rates is $o_P(1/\sqrt{n})$.

5.2 Estimating the optimal derived predictor

An estimator for $\theta^*$ is derived by computing estimates $\hat{\beta}, \hat{\beta}^+, \hat{\beta}^-$ of the true LP coefficients and then solving the resulting estimated LP:

$$\hat{\theta} = \arg \min_{\theta} \quad \theta^T \hat{\beta}$$

subject to

$$\theta \in [0, 1]^4$$
$$|\theta^T \hat{\beta}^+| \leq \epsilon^+$$
$$|\theta^T \hat{\beta}^-| \leq \epsilon^-$$

There are many possible estimators of the LP coefficients. We propose doubly robust estimators, which allow for fast rates of convergence under relatively mild assumptions [van der Vaart, 2002; Tsiatis, 2006]. These rates propagate to the loss and fairness properties of $S_{\hat{\theta}}$. For ease of notation, let

$$\phi = \frac{1 - D}{1 - \pi} (Y - \mu_0) + \mu_0$$
$$\hat{\phi} = \frac{1 - D}{1 - \hat{\pi}} (Y - \hat{\mu}_0) + \hat{\mu}_0$$

denote the uncentered efficient influence function for $E(Y^0)$ and its estimate, respectively [Bickel et al., 1993; Hahn, 1998; van der Laan and Robins, 2003; Kennedy, 2016]. The
The vectors $\widehat{\beta}, \hat{\beta}^+, \hat{\beta}^-$ are formed by plugging in $\widehat{\beta}_{a,s}$, $\widehat{\text{cFPR}}(S,a)$, $\widehat{\text{cFNR}}(S,a)$ into expressions (1), (3), and (4). We now give theoretical results for the derived predictor $S_{\widehat{\theta}}$. We show that $S_{\widehat{\theta}}$ approaches optimal behavior at fast rates\(^4\). We define two quantities of interest: the loss gap and the excess unfairness, and give accompanying theorems.

**Definition 5.1.** The loss gap is $\mathcal{L}(S_{\widehat{\theta}}) - \mathcal{L}(S_{\theta^*})$, i.e., the difference in loss between the derived predictor and the optimal derived predictor.

We use the term loss gap rather than excess loss to acknowledge that the loss of $S_{\widehat{\theta}}$ can be less than the loss of $S_{\theta^*}$, if $\widehat{\theta}$ falls outside the true constraints. Of course, this can only occur if $S_{\widehat{\theta}}$ violates the true fairness constraints, which can happen because the constraints are estimated.

**Theorem 1.** (Loss gap.) Under Assumptions 1-5:

$$\mathcal{L}(S_{\widehat{\theta}}) - \mathcal{L}(S_{\theta^*}) = O_p(1/\sqrt{n})$$

**Definition 5.2.** The excess unfairness of $S_{\theta}$ in the cFPR is

$$\text{UF}^+(S_{\theta}) := \max\{|\text{cFPR}(S_{\theta},0) - \text{cFPR}(S_{\theta},1)| - \epsilon^+, 0\},$$

and the excess unfairness of $S_{\theta}$ in the cFNR is

$$\text{UF}^-(S_{\theta}) := \max\{|\text{cFNR}(S_{\theta},0) - \text{cFNR}(S_{\theta},1)| - \epsilon^-, 0\}.$$ 

Intuitively, there ought to be roughly a 50% chance that $\text{UF}^+(S_{\widehat{\theta}}) = 0$, because the random constraint set should fluctuate around true constraint set. Hence this definition is nontrivial.

**Theorem 2.** (Excess unfairness.) Under assumptions 1-5:

$$\max\{\text{UF}^+(S_{\widehat{\theta}}), \text{UF}^-(S_{\widehat{\theta}})\} = O_p(1/\sqrt{n})$$

\(^4\)We ignore optimization error, since this is a function of the number of optimization iterations and can be made arbitrarily small [Boyd and Vandenberghe, 2004].
Remark 2. (\(\hat{\theta} \) vs. the behavior of \(S_{\hat{\theta}}\)). Without assumptions about how the loss and fairness of \(S_{\hat{\theta}}\) depend on \(\hat{\theta}\), there is no guarantee about the rate at which \(\hat{\theta}\) will approach \(\theta^*\). This is not a concern, since the object of interest is not \(\theta^*\) per se but a predictor that behaves like \(S_{\theta^*}\).

We now turn toward estimation of properties of a fixed derived predictor. These results are valid for any derived predictor \(S_{\theta}\). In practice, the most obvious use would be to estimate properties of \(S_{\hat{\theta}}\) once \(\hat{\theta}\) has been computed.

5.3 Estimating properties of a fixed derived predictor

Given fixed \(\theta\), we are interested in estimating the loss \(L(S_{\theta})\) and the error rate differences \(\Delta^+(S_{\theta}), \Delta^-(S_{\theta})\). We define one additional quantity of interest related to \(S_{\theta}\).

Definition 5.3. The change in loss for a derived predictor \(S_{\theta}\) relative to an input predictor \(S\) is \(\Gamma(S_{\theta}) = L(S_{\theta}) - L(S)\).

We refer to a change in loss rather than an increase in loss because it is possible for \(S_{\theta}\) to have smaller loss than \(S\). This is not a typical expectation: in fair prediction problems, the set of fair classifiers is necessarily smaller than the set of fair and unfair classifiers. Hence, there is a fairness-accuracy tradeoff: satisfying fairness comes at the cost of a reduction in predictive performance. In the RAI setting, however, since predictors are typically trained to predict observable outcomes, their performance may be arbitrarily bad with respect to the potential outcome \(Y^0\). It is therefore not implausible than a derived fair predictor could have higher accuracy than the input predictor it is derived from.

The estimators used here are essentially identical to the estimators of the LP coefficients used in the previous section. Here, however, our aim is to demonstrate properties of these estimators, rather than properties of our derived predictor \(S_{\hat{\theta}}\). In particular, we are interested in deriving confidence intervals, in addition to guaranteeing rates of convergence.
The estimators are

\[ \hat{L}(S_\theta) = \theta^T \hat{\beta} + \mathbb{P}_n(\hat{\phi}) \]  \hspace{1cm} \text{(loss)}

\[ \hat{\Gamma}(S_\theta) = (\theta - \hat{\theta})^T \hat{\beta} \]  \hspace{1cm} \text{(loss change)}

\[ \hat{cFPR}(S_\theta, a) = \theta_{a,0} \left(1 - \hat{cFPR}(S, a)\right) + \theta_{a,1} \hat{cFPR}(S, a) \]  \hspace{1cm} \text{(cFPR)}

\[ \hat{cFNR}(S_\theta, a) = (1 - \theta_{a,0}) \left(1 - \hat{cFNR}(S, a)\right) + \theta_{a,1} (1 - \hat{cFNR}(S, a)) \]  \hspace{1cm} \text{(cFNR)}

\[ \hat{\Delta}^+(S_\theta) = \theta^T \hat{\beta}^+ \]  \hspace{1cm} \text{(error rate difference in cFPR)}

\[ \hat{\Delta}^-(S_\theta) = \theta^T \hat{\beta}^- \]  \hspace{1cm} \text{(error rates difference in cFNR)}

where recall \( \hat{\theta} = (0, 1, 0, 1)^T \), so that \( S_{\hat{\theta}} = S \). Note that the loss estimator adds back in the portion of the loss that doesn’t depend on \( \theta \) and that we consequently removed from the LP in (5).

We now give theoretical results for these estimators. The estimators are built around influence functions for the parameters and consequently are efficient when the assumptions above are satisfied: no other estimator has smaller asymptotic variance uniformly around \( \mathbb{P} \) [van der Vaart, 2002].

**Theorem 3.** (Loss and loss change) Fix \( \theta \in [0, 1]^4 \). Under assumptions 1-5:

\[ \sqrt{n} \left( \hat{L}(S_{\theta}) - \mathcal{L}(S_{\theta}) \right) \rightsquigarrow N(0, \text{var}(f_\theta)) \]

\[ \sqrt{n} \left( \hat{\Gamma} - \Gamma \right) \rightsquigarrow N(0, \text{var}(f_\theta - f_{\hat{\theta}})) \]

where

\[ f_\theta = (1 - 2\mu_0)\theta_{A,S} + \phi \]

\[ \theta_{A,S} = \sum_{a,s \in 0,1} \theta_{a,s,1}\{A = a, S = s\} \]

and the estimators \( \hat{L}, \hat{\Gamma} \) attain the nonparametric efficiency bound.

**Corollary 3.1.** Given a consistent estimator for \( \text{var}(f_\theta) \), an asymptotically valid 95\% confidence interval for \( \mathcal{L}(S_{\theta}) \) is given by \( \hat{\mathcal{L}}(S_{\theta}) \pm 1.96 \cdot \sqrt{\text{var}(f_\theta)}/\sqrt{n} \). An asymptotically valid test of the hypothesis \( \mathcal{L}(S_{\theta}) = C \) for any \( C \) consists of evaluating whether \( C \) is in the confidence interval. An analogous result holds for \( \Gamma \).
Theorem 4. (Fairness estimators) Fix $\theta \in [0, 1]^4$. Under assumptions 1-5:

$$\sqrt{n} \left( \widehat{cFPR}(S_\theta, a) - cFPR(S_\theta, a) \right) \rightsquigarrow N(0, \text{var}(g_a))$$

$$\sqrt{n} \left( \widehat{cFNR}(S_\theta, a) - cFNR(S_\theta, a) \right) \rightsquigarrow N(0, \text{var}(h_a))$$

$$\sqrt{n} \left( \widehat{\Delta}^+ - \Delta^+ \right) \rightsquigarrow N(0, \text{var}(g_0 - g_1))$$

$$\sqrt{n} \left( \widehat{\Delta}^- - \Delta^- \right) \rightsquigarrow N(0, \text{var}(h_0 - h_1))$$

where

$$g_a = (\theta_{a,1} - \theta_{a,0}) \mathbb{E}[1\{A = a\}(1 - \phi)]^{-1} 1\{A = a\}(1 - \phi)(R - \widehat{cFPR}(S_\theta, a))$$

$$h_a = (\theta_{a,1} - \theta_{a,0}) \mathbb{E}[1\{A = a\}\phi]^{-1} 1\{A = a\}\phi(R - \widehat{cFNR}(S_\theta, 0))$$

and the estimators $\widehat{cFPR}$, $\widehat{cFNR}$, $\widehat{\Delta}^+$, $\widehat{\Delta}^-$ attain the nonparametric efficiency bound.

Asymptotically valid confidence intervals and hypothesis tests can be obtained in the manner described in Corollary 3.1. In order to construct confidence intervals for the loss and fairness quantities, we require estimators of the asymptotic variances. Our estimators are the sample variances of $\widehat{g}_a, \widehat{h}_a, \widehat{g}_0 - \widehat{g}_1, \widehat{h}_0 - \widehat{h}_1$, where these quantities are defined by the following:

$$\widehat{g}_a = (\theta_{a,1} - \theta_{a,0}) \mathbb{P}_n[1\{A = a\}(1 - \widehat{\phi})]^{-1} 1\{A = a\}(1 - \widehat{\phi})(R - \widehat{cFPR}(S_\theta, a))$$

$$\widehat{h}_a = (\theta_{a,1} - \theta_{a,0}) \mathbb{P}_n[1\{A = a\}\widehat{\phi}]^{-1} 1\{A = a\}\widehat{\phi}(R - \widehat{cFNR}(S_\theta, 0))$$

The quantities $f_\theta, g_a,$ and $h_a$ are the efficient influence functions (EIFs) for the loss and error rates.

In the subsequent section, we illustrate these theorems using simulated data.

6 Simulation Study

We use one set of simulations to illustrate theorems 1 and 2, which capture properties of $S_\theta$. We use another set of simulations to illustrate theorems 3 and 4, which capture properties of our loss and fairness estimators. Finally, we simulate the fairness-accuracy tradeoff for different values of the unfairness tolerances $\epsilon^+, \epsilon^-$. 

Data generating process First, we define a pre-RAI data generating process. Using this data, we train a predictor $S$ to predict observable outcomes $Y$, mirroring how RAIs
are typically constructed in practice. The predictor is a random forest trained on \((A,X)\). We then define a *post-RAI* data generating process. The only difference relative to the pre-RAI process is that the predictor \(S\) now affects the decisions \(D\). This emulates the way RAI s are intended to work in practice; for example, a criminal defendant labeled high-risk \((S = 1)\) might be less likely to be released pre-trial \((D = 0)\) than they would have prior to the introduction of the RAI. The data generating process is designed to meet assumptions 1-3, with \(\pi(D \mid A,X,S)\) upper bounded at 0.975 (Figure 3).

\[
\begin{align*}
\mathbb{P}(A = 1) &= 0.3 \\
X \mid A &\sim N(A \ast (1, -0.8, 4, 2)^T, I_4) \\
\mathbb{P}_{\text{pre}}(D = 1 \mid A, X) &= \min\{0.975, \expit((A, X)^T(0.2, -1, 1, -1, 1))\} \\
\mathbb{P}_{\text{post}}(D = 1 \mid A, X, S) &= \min\{0.975, \expit((A, X, S)^T(0.2, -1, 1, -1, 1))\} \\
\mathbb{P}(Y^0 = 1 \mid A, X) &= \expit((A, X)^T(-5, 2, -3, 4, -5)) \\
\mathbb{P}(Y^1 = 1 \mid A, X) &= \expit((A, X)^T(1, -2, 3, -4, 5)) \\
Y &= (1 - A)Y^0 + AY^1
\end{align*}
\]

Figure 3: Data generating process for simulations. \(I_4\) denotes the \(4 \times 4\) identity matrix. \(\mathbb{P}_{\text{pre}}\) and \(\mathbb{P}_{\text{post}}\) refer to the decision process before and after development of the RAI \(S\), which itself serves as input to our estimand \(S_{\theta^*}\). No other parts of the data generating process change in response to \(S\).

**Doubly robust vs. plugin estimators**  We contrast the performance of our approach using doubly robust estimators with an approach that uses plugin estimators everywhere instead, both in the linear program used to compute \(\hat{\theta}\) and in the estimators of the loss and fairness properties for the fixed derived predictor \(S_{\hat{\theta}}\). The plugin estimators simply substitute \(\hat{\mu}_0\) for \(\hat{\phi}\); they are otherwise identical to the doubly robust estimators. None of theorems 1-5 will typically hold with plugin estimators: unlike with the doubly robust estimators, assumptions 4-5 are not sufficient to guarantee \(\sqrt{n}\)-rates in any of the quantities of interest. In general, one would only expect these results to hold for plugin estimators if \(\|\hat{\mu}_0 - \mu_0\| = o_P(n^{-1/2})\). Importantly, in a nonparametric model, this condition generally cannot be satisfied, unlike assumption 5 [Györfi et al., 2002].

**Simulating \(\hat{\mu}_0\) and \(\hat{\pi}\)**  To simulate estimators that are consistent for \(\mu_0\) and \(\pi\) at different rates, we add random noise \(\epsilon\) of different magnitudes to \(\mu_0\) and \(\pi\), as described in Figure 4. The noise is added on the logit scale to ensure that \(\hat{\mu}_0, \hat{\pi}\) remain in \([0, 1]\), and \(\hat{\pi}\) is again truncated to 0.975.

**Sample sizes and Monte Carlo iterations**  Each estimation procedure was run 500
\[ \epsilon \sim N(0, 30n^{-0.26}) \]
\[ \logit(\hat{\pi}) = \min\{\logit(0.975), \logit(\pi) + \epsilon\} \]
\[ \logit(\hat{\mu}_0) = \logit(\mu_0) + \epsilon \]

Figure 4: Procedure for simulating \( \sqrt{n} \)-consistency of the nuisance parameter estimators \( \hat{\mu}_0 \) and \( \hat{\pi} \)

times for each of \( n \in \{100, 200, 500, 1000, 5000, 20000\} \). The “true” loss and fairness values were computed on a separate validation set of size 500,000, using the plugin estimators with the true \( \mu_0 \). These approximately true values showed negligible variation over many repetitions.

6.1 Simulation Results

Simulations 1

Figure 5 shows the true loss \( L(S_\hat{\theta}) \) and the excess unfairness values \( UF^+(S_\hat{\theta}), UF^-(S_\hat{\theta}) \) for a predictor \( S_\hat{\theta} \) with \( \epsilon^+ = 0.10, \epsilon^- = 0.20 \). Following Theorems 1 and 2, the loss and excess unfairness values converge to the loss \( L(S_{\theta^*}) \) of the optimal derived predictor and 0, respectively. Only the doubly robust estimators yield guaranteed \( \sqrt{n} \) convergence.

Simulations 2

To illustrate that Theorems 3 and 4 apply to arbitrary derived predictors, the second set of simulations involves a fixed derived predictor \( S_{\theta} \), with \( \theta \) randomly set to \( (0.74, 1.0, 0, 0.8) \). (Note that solutions to a linear program with a compact feasible set must occur at an extreme point of the set, so the presence of 0 and/or 1 in \( \hat{\theta} \) and \( \theta^* \) is virtually guaranteed.) Figure 6 illustrates estimators \( \hat{L}(S_\theta), \hat{\Gamma}(S_\theta), \hat{\Delta}^+(S_\theta), \hat{\Delta}^-(S_\theta) \) of the loss, loss change, and error rate differences for \( S_\theta \). Following Theorems 3 and 4, the estimators converge to their target values. Only the doubly robust estimators yield guaranteed \( \sqrt{n} \) convergence.

Table 1 contains coverage results of 95% confidence intervals for the error rates, error rate differences, loss, and loss change for the same arbitrary \( S_\theta \). The CIs were constructed using sample variances. To ensure that they did not exceed the bounds of the possible parameter values (i.e. \([0, 1]\) for the loss and error rates, \([-1, 1]\) for the error rate differences and loss change), the CIs were constructed using the Delta method, via the transformations \( \hat{\psi} \mapsto \logit(\hat{\psi}) \) (for \( \hat{\psi} \in \{cFPR, cFNR, \hat{L}\} \)) or \( \hat{\psi} \mapsto \logit((\hat{\psi} + 1)/2) \) (for \( \hat{\psi} \in \{\hat{\Delta}^+, \hat{\Delta}^-, \hat{\Gamma}\} \)). Nominal coverage is achieved for various quantities at various sample sizes, but since the coverage guarantees are asymptotic, it is not surprising that it is not achieved everywhere. Interestingly, the median coverage rate in the table is 0.95.
A separate set of CIs was computed without using the Delta method; those results did not differ substantially and are therefore omitted here.

**Simulations 3** Finally, the third set of simulations illuminates the fairness-accuracy tradeoff. Figure 7 shows the loss change $\Gamma(S_{\theta^*}) = \mathcal{L}(S_{\theta^*}) - \mathcal{L}(S)$ for each point in a grid of unfairness tolerances $\epsilon^+, \epsilon^-$. Here, $S$ is the Bayes-optimal predictor of $Y^0$ in our data generating scenario, meaning $S(A, X) = \mathbb{E}[Y^0 | A, X]$. Since any derived predictor necessarily has greater loss than the Bayes-optimal predictor, we refer to the loss change here as the **performance cost**.

In the setup described above, the Bayes-optimal predictor has a loss of only 0.07 and absolute error rate differences of only 0.05 ($\Delta^+$) and 0.04 ($\Delta^-$), which leaves little room to illustrate the potential cost of fairness. For these simulations, therefore, we set $\mathbb{P}(Y^0 = 1 | A, X) = \expit((A, X)^T(-4, 0.4, 0.6, 0.8, -1))$, leaving the rest of the data generating process as is. These yields a Bayes-optimal predictor with loss 0.24 and absolute error rate differences of 0.23 ($\Delta^+$) and 0.40 ($\Delta^-$), which are plausible values for a real predictor.

As expected, when $\epsilon^+ \geq \Delta^+(S)$ or $\epsilon^- \geq \hat{\Delta}^-(S)$, the performance cost is 0: the input predictor already falls within the fairness constraints. As the tolerances tighten towards 0, the performance declines, though never substantially. For $\epsilon^+ = \epsilon^- = 0$, when the derived predictor is constrained to exactly satisfy cEO, the loss increases by 0.10, to 0.34. The different values for $\Delta^+(S)$ and $\Delta^-(S)$ are reflected in the differing costs of satisfying fairness along the two axes: the cost of controlling $\Delta^+(S_{\theta})$ are lower than the costs of controlling $\Delta^+(S_{\theta})$.

[Woodworth et al., 2017] showed that post-processing can result in predictors with poor performance, but it is unclear how likely this is to be a problem in practice. While the fairness-accuracy tradeoff naturally depends on the data generating process, our example illustrates that fairness can in some cases be achieved without substantial performance costs.
|                | 100  | 200  | 500  | 1000 | 5000 | 20000 |
|----------------|------|------|------|------|------|-------|
| Loss           | \(\hat{L}(S_\theta)\) | 0.98  | 0.92  | 0.87  | 0.84  | 0.84  | 0.85  |
| Loss Change    | \(\hat{\Gamma}(S_\theta)\) | 1.00  | 0.99  | 0.93  | 0.94  | 0.71  | 0.78  |
| Error rates    | cFPR\((S_\theta,0)\) | 0.99  | 0.98  | 0.98  | 0.96  | 0.94  | 0.95  |
|                | cFPR\((S_\theta,1)\) | 0.90  | 0.89  | 0.93  | 0.95  | 0.96  | 0.57  |
|                | cFNR\((S_\theta,0)\) | 0.99  | 0.99  | 0.98  | 0.99  | 0.92  | 0.93  |
|                | cFNR\((S_\theta,1)\) | 0.99  | 0.99  | 0.99  | 1.00  | 0.98  | 0.71  |
| Error rate diffs | \(\hat{\Delta}^+(S_\theta)\) | 0.98  | 0.98  | 0.97  | 0.99  | 0.97  | 0.92  |
|                | \(\hat{\Delta}^-(S_\theta)\) | 0.99  | 1.00  | 0.99  | 0.99  | 0.94  | 0.94  |

Table 1: 95% CI coverage at sample sizes ranging from 100 to 20,000 for the loss, loss change, error rates, and error rate differences, for an arbitrary derived predictor \(S_\theta\) with parameter \(\theta = (0.74, 1.0, 0, 0.8)\). Coverage varies by estimator and sample size, though the median coverage is 95%.
Figure 5: (Illustration of Theorems 1 and 2) Loss and excess unfairness for the derived predictor $\hat{S}_\theta$ for samples of size 100 to 20,000, using doubly robust (DR) vs. plugin (PI) estimators for the parameters of the linear program that defines $\hat{\theta}$. Each vertical line represents a mean ±1 sd over 500 simulations. Black horizontal lines represent the loss of the optimal derived predictor $S_{\theta^*}$ (first column) and the target excess unfairness value of 0 (other columns). The values in the bottom two rows represent the values in the top two rows transformed by $\psi(S_{\hat{\theta}}) \mapsto \sqrt{n}(\psi(S_{\hat{\theta}}) - \psi(S_{\theta^*}))$, where $\psi$ is $\mathcal{L}$ or $\text{UF}^+$ or $\text{UF}^-$, as appropriate. The top two rows show that the loss and excess unfairness converge to their target values for both DR and SR estimators. The bottom two rows illustrate that $\sqrt{n}$-convergence is only guaranteed for $\hat{\theta}_{DR}$: the scaled values for $\hat{\theta}_{DR}$ do not grow in $n$, while the scaled values for $\hat{\theta}_{SR}$ begin to diverge.
Figure 6: (Illustration of Theorems 3 and 4) Doubly robust (DR) vs. plugin (PI) estimates of the loss, loss change, and error rate differences for an arbitrary derived predictor $S_{\theta}$, with $\theta = (0.74, 1.0, 0, 0.8)$, for samples of size 100 to 20,000. Each vertical line represents a mean ±1 sd over 500 simulations. Black horizontal lines represent the true values that the estimates should converge to. The values in the bottom two rows represent the values in the top two rows transformed by $\hat{\psi}(S_{\theta}) \mapsto \sqrt{n}(\hat{\psi}(S_{\theta}) - \psi(S_{\theta}))$, where $\hat{\psi}$ is $\hat{L}$ or $\hat{\Delta}^+$ or $\hat{\Delta}^-$, as appropriate. The top two rows show that the DR and SR estimators converge. The bottom two rows illustrate that $\sqrt{n}$-convergence is only guaranteed for the DR estimators: the scaled values for the DR estimators do not appear to grow in $n$, while the scaled values for $\hat{L}_{DR}(S_{\theta})$ and $\hat{\Gamma}_{DR}(S_{\theta})$ begin to diverge.
Figure 7: Loss change $\Gamma(\theta^*) = \mathcal{L}(S_\theta) - \mathcal{L}(S)$ for the Bayes-optimal input predictor $S(A, X) = \mathbb{E}[Y^0 \mid A, X]$ and $\theta^*$ corresponding to different unfairness tolerances $\epsilon^+, \epsilon^-$. The black area represents fairness constraints that are looser than the error rate differences of the input predictor ($\Delta^+(S) = 0.24$, $\Delta^-(S) = 0.40$), which incur no performance cost. The highest performance cost (0.10) occurs when the error rates differences are both constrained to be 0, meaning the derived predictor $S_\theta$ satisfies cEO exactly.

7 Discussion

In this paper we considered fairness in risk assessment instruments (RAIs), which are naturally concerned with potential outcomes rather than strictly observable outcomes. We introduced the fairness criterion approximate counterfactual equalized odds (approximate cEO), which allows users to negotiate the tradeoff between fairness and performance. We argued that this fairness criterion is likelier than other candidate criteria to reduce discriminatory disparate impact, which we defined as $D \not\perp \perp A \mid Y^0$.

Our method extends the work of [Hardt et al., 2016] to the potential outcome setting: it post-processes an arbitrary binary classifier, yielding a randomized classifier that approximately satisfies approximate cEO. The post-processed classifier is the solution to a four-dimensional linear program with a compact feasible set, so it is easy to compute. We showed that our classifier converges in a certain sense to the optimal fair classifier at $\sqrt{n}$.
rates. We presented doubly robust estimators of the loss and fairness properties of our post-processed classifier and showed that these estimators are $\sqrt{n}$-consistent. We illustrated our results via simulations, and also illustrated the possibility of achieving fairness at a relatively small cost in predictive performance.

Unlike in [Hardt et al., 2016], we did not assume that the joint distribution of the sensitive feature, input predictor, and outcome was known. Our rate results can be readily translated to the setting of [Hardt et al., 2016], in which the outcome of interest is the observable $Y$ and the fairness criterion is (approximate) observational equalized odds.

As with any counterfactual quantity, our estimand requires for identification a set of covariates that are sufficient to deconfound the treatment and the potential outcome of interest. This raises the possibility that those same covariates could be used to train a fair predictor from scratch rather than to post-process an existing predictor. While this approach merits investigation, our approach has an advantage in that these covariates are required only to construct the post-processed predictor; at runtime, our predictor requires access only to the sensitive feature and the prediction from the input predictor. This makes it more feasible that our method could be implemented on top of existing RAIs. A predictor trained from scratch would be constrained by the set of covariates available in deployment, whereas our method would allow researchers to devise a set of suitable deconfounding covariates and then collect an appropriate dataset on a one-time basis.

In closing, we note that from our perspective, notions of fairness in predictive systems ought to be subordinate to notions of fairness grounded in the actual decisions or events that those systems inform, and the impact that those decisions have on people’s lives. It is perfectly possible for a RAI that is fair according to some formal criterion to result in greater unfairness in decision making, or vice versa. Though little is currently known about how decision makers respond to RAIs, there is some evidence that judges do not have much faith in recidivism predictions and that RAIs can have little impact on the decisions that are made [Johnson, 2018; Stevenson, 2018]. As RAIs and the general public’s understanding of how they function co-evolve, it is likely that the ways in which decision makers respond to them will evolve as well.

Nevertheless, it seems plausible that some fairness criteria for RAIs are likelier than others to lead to increased (un)fairness with respect to decisions and outcomes. While this is ultimately an empirical question, we believe that this kind of consideration ought to ground discussions of fairness in RAIs and predictive systems generally. As long as there are domains involving high stakes decisions that we do not wish to fully automate, RAIs will remain relevant, and so will the task of ensuring that they lead to a society that is more fair, not less.
7.1 Extensions to other settings

In ongoing work, we generalize this method within a single framework that can handle several variations: (1) input predictors that take values in $[0, 1]$ rather than $\{0, 1\}$ [Hardt et al., 2016], (2) approximate counterfactual sufficiency ($Y^0 \perp \perp A \mid S$) rather than equalized odds, and (3) either $Y^0$ or observable $Y$. Though we argued in section 2.3 that equalized odds has an appealing property that sufficiency lacks, we do not wish to claim definitively that one criterion is preferable to the other, and we are interested in methods that can accommodate either one. Variation (3) is designed to accommodate diagnostic settings rather than risk assessment settings, in which decisions do not change the estimated outcomes. We find preliminarily that we can construct predictors that reflect any combination of these variants as solutions to convex programs, and that many of the properties of the estimators defined above translate to this generalized setting.

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Appendix A: Proofs of propositions

For convenience, we restate the five assumptions from the main text that underlie our results:

Assumption 1 (Consistency). \( Y = D Y^1 + (1 - D) Y^0 \)

Assumption 2 (Positivity). \( \exists \delta \in (0, 1) : \mathbb{P}(\pi(A, X, S) \leq 1 - \delta) = 1 \)

Assumption 3 (Ignorability). \( Y^0 \perp D \mid A, X, S \)

Assumption 4 (Bounded propensity estimator). \( \exists \delta \in [0, 1] \text{ s.t. } \hat{\pi} \leq 1 - \delta \)

Assumption 5 (Nuisance estimator rates). \( \hat{\mu}_0 - \mu_0 = o_{\mathbb{P}}(1), \; \hat{\pi} - \pi = o_{\mathbb{P}}(1), \; \|\hat{\mu}_0 - \mu_0\| \|\hat{\pi} - \pi\| = o_{\mathbb{P}}(1/\sqrt{n}) \)

Proof of Proposition 1 (Identification of error rates for input predictor \( S \))

\[
cFPR(S, a) = \mathbb{P}(S = 1 \mid Y^0 = 0, A = a) \\
= \frac{\mathbb{P}(S = 1, Y^0 = 0, A = a)}{\mathbb{P}(Y^0 = 0, A = a)} \\
= \frac{\mathbb{E}[R(1 - Y^0)\mathbb{1}\{A = a\}]}{\mathbb{E}[(1 - Y^0)\mathbb{1}\{A = a\}]} \\
= \frac{\mathbb{E}[S(1 - \mathbb{E}[Y^0 \mid A, X, S, D = 0])\mathbb{1}\{A = a\}]}{\mathbb{E}[(1 - \mathbb{E}[Y^0 \mid A, X, S, D = 0])\mathbb{1}\{A = a\}]} \\
= \frac{\mathbb{E}(S(1 - \mu_0)\mathbb{1}\{A = a\})}{\mathbb{E}(1 - \mu_0)\mathbb{1}\{A = a\}}
\]

\[
cFNR(S, a) = \mathbb{P}(S = 0 \mid Y^0 = 1, A = a) \\
= \frac{\mathbb{P}(S = 0, Y^0 = 1, A = a)}{\mathbb{P}(Y^0 = 1, A = a)} \\
= \frac{\mathbb{E}[(1 - S)Y^0\mathbb{1}\{A = a\}]}{\mathbb{E}[Y^0\mathbb{1}\{A = a\}]} \\
= \frac{\mathbb{E}[(1 - S)\mathbb{E}[Y^0 \mid A, X, S, D = 0]\mathbb{1}\{A = a\}]}{\mathbb{E}[\mathbb{E}[Y^0 \mid A, X, S, D = 0]\mathbb{1}\{A = a\}]} \\
= \frac{\mathbb{E}[(1 - S)\mu_0\mathbb{1}\{A = a\}]}{\mathbb{E}[\mu_0\mathbb{1}\{A = a\}]} 
\]
The fourth equality in both derivations uses Assumptions 2 (positivity) and 3 (ignorability), and the fifth equality uses Assumption 1 (consistency).

Proof of Proposition 2 (Identification of the loss and fairness constraints)

Proof. Beginning with the loss, we have:

\[ L(S_\theta) := \mathbb{E}[(S_\theta - Y^0)^2] \]
\[ = \mathbb{E}[(1 - 2S_\theta)Y^0 + S_\theta] \]
\[ = \mathbb{E}[(1 - 2\mathbb{E}[S_\theta | A, S])\mathbb{E}[Y^0 | X, A, S, D = 0] + \mathbb{E}[S_\theta | A, S]] \]
\[ = \mathbb{E} \{ (1 - 2\mathbb{E}[S_\theta | A, S])\mu_0 \} + \mathbb{E}\{\mathbb{E}[S_\theta | A, S] \} \]
\[ = \mathbb{E}\{\mu_0 \} + \mathbb{E}\\{ \mathbb{E}[S_\theta | A, S](1 - 2\mu_0) \} \]
\[ = \mathbb{E}\{\mu_0 \} + \mathbb{E} \left\{ \sum_{a,s \in \{0,1\}} \theta_{a,s} \mathbb{I}\{A = a, S = s\}(1 - 2\mu_0) \right\} \]
\[ = \mathbb{E}\{\mu_0 \} + \theta^T \beta \]

where \( \beta \) is defined in (1). The second equality follows from the fact that \( S_\theta, Y^0 \in \{0,1\} \), the third uses ignorability and the fact that \( S_\theta \) depends only on \((A, S)\), the fourth uses consistency, and the fifth follows from the definition of \( \theta_{a,s} \).

We turn now to the fairness constraints. The error rates of the derived predictor \( S_\theta \) depend on the error rates on the input predictor \( S \) as follows. Beginning with \( c\text{FPR}(S_\theta, a) \), we have:

\[ \mathbb{P}(S_\theta = 1 | Y^0 = 0, A = a) = \sum_{r \in \{0,1\}} \mathbb{P}(S_\theta = 1 | Y^0 = 0, A = a, S = r) \mathbb{P}(S = r | Y^0 = 0, A = a) \]
\[ = \sum_{r \in \{0,1\}} \mathbb{P}(S_\theta = 1 | A = a, R = r) \mathbb{P}(S = r | Y^0 = 0, A = a) \]
\[ = \theta_{a,0}(1 - \text{cFPR}(S, a)) + \theta_{a,1} \text{cFPR}(S, a) \]

where the first equality simply involves conditioning on \( S \), and the second equality uses that \( S_\theta \perp Y^0 | A, S \). In other words, the false positive rate of \( S_\theta \) depends only on \( \theta \) and the false positive rate of the input predictor \( S \). For the cFNR, by similar reasoning, we
have:
\[
P(S_0 = 0 \mid Y^0 = 1, A = a) = \sum_{r \in \{0, 1\}} P(S_0 = 0 \mid Y^0 = 1, A = a, S = r)P(S = r \mid Y^0 = 1, A = a)
\]
\[
= \sum_{r \in \{0, 1\}} P(S_0 = 0 \mid A = a, R = r)P(S = r \mid Y^0 = 1, A = a)
\]
\[
= (1 - \theta_{a,0})cFNR(S, a) + (1 - \theta_{a,1})(1 - cFNR(S, a))
\]
\[
= -\theta_{a,0}(cFNR(S, a)) + \theta_{a,1}(cFNR(S, a) - 1) + 1
\]

The identification statements in the proposition follow by simply substituting in the expressions for \(cFPR(S, a), cFNR(S, a)\) from Proposition 1 and rearranging.

\[\square\]

9 Appendix B: Technical Lemmas

We introduce several lemmas used in the proofs of the theorems. Here and throughout, let \(P(\hat{f}) = E[\hat{f} \mid \mathcal{D}_{nuis}]\) denote the expected value of any random variable \(\hat{f}\) conditional on the data used to estimate the nuisance parameters (either \(\mathcal{D}_{nuis}^{\text{train}}\) or \(\mathcal{D}_{nuis}^{\text{test}}\), depending on context). Note that this usage is consistent with the usage of \(P\) throughout the paper.

**Lemma 1.** Let \(W\) be a function of (at most) \(A, X, S\) such that \(\|W\| \leq \infty\). Then
\[
P(W(\widehat{\phi} - \phi)) = o_P(1/\sqrt{n})
\]
Proof.

\[ P(W(\hat{\phi} - \phi)) = P\left(W\left(1 - \frac{D}{1 - \pi}(Y - \mu_0) + \hat{\mu}_0 - \frac{1 - D}{1 - \pi}(Y - \mu_0) - \mu_0\right)\right) \]
\[ = P\left(W\left(1 - \frac{\pi}{1 - \pi}(\mu_0 - \hat{\mu}_0) + \hat{\mu}_0 - \frac{1 - D}{1 - \pi}(\mu_0 - \mu_0) - \mu_0\right)\right) \]
\[ = P\left(W\left(\frac{(\mu_0 - \hat{\mu}_0)(\hat{\pi} - \pi)}{1 - \pi}\right)\right) \]
\[ \leq \frac{1}{\delta} P(W(\mu_0 - \hat{\mu}_0)(\hat{\pi} - \pi)) \]
\[ \leq \frac{1}{\delta} \|W\|\|\mu_0 - \hat{\mu}_0\|\|\hat{\pi} - \pi\| \]
\[ = o_P(1/\sqrt{n}) \]

where the fourth line uses iterated expectation, conditioning on \((A,X,S)\); the fifth line uses iterated expectation again, conditioning again on \((A,X,S)\); the seventh line uses Assumption 2 (positivity); the eighth line uses the Cauchy-Schwarz inequality; and the last line uses Assumption 4 (nuisance parameter rates). \(\square\)

The next lemma gives sufficient conditions under which the optimal value of an estimated convex problem converges at \(\sqrt{n}\) rates to the optimal value of the target convex program. It is a simplification of Theorem 3.5 in [Shapiro, 1991]; we omit some details from that theorem that are not relevant to our problem context.

**Lemma 2.** (Shapiro, 1991) Let \(\Theta\) be a compact subset of \(\mathbb{R}^k\). Let \(C(\Theta)\) denote the set of continuous real-valued functions on \(\Theta\), with \(\mathcal{L} = C(\Theta) \times \ldots \times C(\Theta)\) the \(r\)-dimensional Cartesian product. Let \(\psi(\theta) = (\psi_0, \ldots, \psi_r) \in \mathcal{L}\) be a vector of convex functions. Consider the quantity \(\alpha^*\) defined as the solution to the following convex optimization program:

\[ \alpha^* = \min_{\theta \in \Theta} \psi_0(\theta) \]

subject to \(\psi_j(\theta) \leq 0, \ j = 1, \ldots, r\)

Assume that Slater’s condition holds, so that there is some \(\theta \in \Theta\) for which the inequalities are satisfied and non-affine inequalities are strictly satisfied, i.e. \(\psi_j(\theta) < 0\) if \(\psi_j\) is non-affine. Now consider a sequence of approximating programs, for \(n = 1, 2, \ldots:\)

\[ \hat{\alpha}_n = \min_{\theta \in \Theta} \hat{\psi}_0(\theta) \]

subject to \(\hat{\psi}_j(\theta) \leq 0, \ j = 1, \ldots, r\)
where $\hat{\psi}_n(\theta) = (\hat{\psi}_{0n}, \ldots, \hat{\psi}_{rn}) \in \mathcal{L}$. Assume that $\sqrt{n}(\hat{\psi}_n - \psi)$ converges in distribution to a random element $W \in \mathcal{L}$. Then:

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) \Rightarrow L$$

for a particular random variable $L$. It follows that $\hat{\alpha}_n - \alpha_0 = O_p(1/\sqrt{n})$.

**Lemma 3.** Let $\xi, W$ be constant vectors and $\hat{\xi}_n, \hat{W}_n$ be random variables, with $\xi - \hat{\xi}_n = O_p(1/\sqrt{n})$. If, for all $M > 0$, $P(\|W - \hat{W}_n\| > M) \leq P(||\xi - \hat{\xi}_n|| > CM)$ for some constant $C$, then $W - \hat{W}_n = O_p(1/\sqrt{n})$.

**Proof.** For any $\epsilon > 0$, there exists some $M_\epsilon > 0$ such that $P(\sqrt{n}||\xi - \hat{\xi}_n|| > M_\epsilon) < \epsilon$ for all $n$ large enough. Set $M = M_\epsilon/C$. Then $P(\sqrt{n}||W - \hat{W}_n|| > M) < \epsilon$ for all $n$ large enough. □

### 10 Appendix C: Proofs of Theorems

In each proof that follows, quantities with hats are derived from the same dataset $D$, with the nuisance parameters estimated on $D^{\text{nuis}}$ and the target parameters estimated on $D^{\text{target}}$ as usual. In the proofs of Theorems 1 and 2, $D$ is $D^{\text{train}}$; for Theorems 3 and 4, $D$ is $D^{\text{test}}$.

#### 10.1 Theorem 1 ($\sqrt{n}$-convergence of the loss gap to 0)

The proof of Theorem 1 relies on Lemma 2 and Theorems 3 and 4. We expand the loss by introducing the term $\hat{\beta}^T \hat{\theta}$, which is the quantity that is minimized in the course of computing $\hat{\theta}$. We proceed by splitting the loss into two terms and showing that each of those terms is $O_p(1/\sqrt{n})$.

**Proof.** First, note that, following the logic employed in the proofs of Theorems 3 and 4, we have

$$\hat{\beta} - \beta = O_p(1/\sqrt{n})$$

$$\hat{\Delta}^+(S_\theta) - \Delta^+(S_\theta) = O_p(1/\sqrt{n})$$

$$\hat{\Delta}^-(S_\theta) - \Delta^-(S_\theta) = O_p(1/\sqrt{n})$$

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The loss gap can be expanded as follows:

\[
\mathcal{L}(S_{\hat{\theta}}) - \mathcal{L}(S_{\theta^*}) = \beta^T \hat{\theta} - \beta^T \theta^* \\
= \left( \beta^T \hat{\theta} - \beta^T \hat{\theta} \right) + \left( \beta^T \hat{\theta} - \beta^T \theta^* \right)
\]

For term (1), we have

\[
\hat{\theta}^T (\beta - \hat{\beta}) \leq \|\hat{\theta}\| \|\beta - \hat{\beta}\| \\
\leq 2 \|\beta - \hat{\beta}\| \\
= O_P(1/\sqrt{n})
\]

where the first line uses Cauchy-Schwarz and the second line follows from the fact that \(\hat{\theta} \in [0, 1]^4\). For the second term in the loss gap, we rely on Lemma 2. Note that we can write

\[
\mathcal{L}(S_{\theta^*}) = \min_{\theta \in \Theta} \psi_0(\theta) \\
\text{subject to } \psi_j(\theta) \leq 0, \ j = 1, \ldots, r
\]

\[
\hat{\mathcal{L}}(S_{\hat{\theta}}) = \min_{\theta \in \Theta} \hat{\psi}_0(\theta) \\
\text{subject to } \hat{\psi}_j(\theta) \leq 0, \ j = 1, \ldots, r
\]

with \(\Theta = [0, 1]^4\), and

\[
\psi(\theta) = (\mathcal{L}(S_{\theta}), \Delta^+(S_{\theta}) - \epsilon^+, -\Delta^+(S_{\theta}) - \epsilon^+, \Delta^-(S_{\theta}) - \epsilon^-, -\Delta^-(S_{\theta}) - \epsilon^-) \\
\hat{\psi}(\theta) = (\hat{\mathcal{L}}(S_{\hat{\theta}}), \hat{\Delta}^+(S_{\hat{\theta}}) - \epsilon^+, -\hat{\Delta}^+(S_{\hat{\theta}}) - \epsilon^+, \hat{\Delta}^-(S_{\hat{\theta}}) - \epsilon^-, -\hat{\Delta}^-(S_{\hat{\theta}}) - \epsilon^-)
\]

Since these are linear programs, Slater’s condition is trivially satisfied. Again, following the logic in Theorems 3 and 4, we have

\[
\sqrt{n} \left( \hat{\psi}(\theta) - \psi(\theta) \right) = (\mathbb{P}_n - \mathbb{P})\psi(\theta) + \gamma(\theta) \\
\text{with } \gamma(\theta) = o_P(1/\sqrt{n})
\]

Since \(\psi_j(\theta)\) is parametric and Lipschitz in \(\theta\), with compact domain \(\Theta\), it follows that the set \(\{\psi_j(\theta) : \theta \in \Theta\}\) is a Donsker class. Therefore

\[
\sqrt{n}(\hat{\psi} - \psi) \rightsquigarrow W
\]
for a Gaussian process $W$ indexed by $\theta \in \Theta$, with mean 0 and finite covariance function given by $\text{cov}(W(\theta_1), W(\theta_2)) = \text{cov}(\psi(\theta_1), \psi(\theta_2))$. That is, the estimated linear program, considered as a vector of functions of $\theta$, converges at $\sqrt{n}$ rates to the true linear program. Per Lemma 2, it follows that $\hat{L}(S_{\hat{\theta}}) - L(S_{\theta^*}) = O_P(1/\sqrt{n})$.

The sum of the two terms in the loss gap is therefore also $O_P(1/\sqrt{n})$.

10.2 Theorem 2 ($\sqrt{n}$-convergence of the excess unfairness to 0)

The proof relies on Lemma 3 and the $\sqrt{n}$-convergence of the constraint quantities $\hat{\beta}^+, \hat{\beta}^-$. When $\hat{\beta}^+, \hat{\beta}^-$ are close to $\beta^+, \beta^-$, the excess unfairness must be small for any $\theta \in \Theta = [0,1]^4$, including of course $\hat{\theta}$. The $\sqrt{n}$-consistency of $\hat{\beta}^+, \hat{\beta}^-$ carries over into the excess unfairness.

Proof. Note that, following the logic employed in the proof of Theorem 4, we have

$$\hat{\beta}^+ - \beta^+ = O_P(1/\sqrt{n})$$
$$\hat{\beta}^- - \beta^- = O_P(1/\sqrt{n})$$

We have

$$P(\text{UF}^+(S_{\hat{\theta}}) > \delta \text{ or } \text{UF}^-(S_{\hat{\theta}}) > \delta)$$
$$\leq P\left(|\hat{\Delta}^+(S_{\theta})| - |\Delta^+(S_{\theta})| > \delta \text{ or } |\hat{\Delta}^-(S_{\theta})| - |\Delta^-(S_{\theta})| > \delta \text{ for some } \theta \in [0,1]^4\right)$$
$$\leq P\left(|\hat{\Delta}^+(S_{\theta}) - \Delta^+(S_{\theta})| > \delta \text{ or } |\hat{\Delta}^-(S_{\theta}) - \Delta^-(S_{\theta})| > \delta \text{ for some } \theta \in [0,1]^4\right)$$
$$= P\left(|\theta^T(\hat{\beta}^+ - \beta^+)| > \delta \text{ or } |\theta^T(\hat{\beta}^- - \beta^-)| > \delta \text{ for some } \theta \in [0,1]^4\right)$$
$$\leq P\left(\|\theta\| \cdot \|\hat{\beta}^+ - \beta^+\| > \delta \text{ or } \|\theta\| \cdot \|\hat{\beta}^- - \beta^-\| > \delta \text{ for some } \theta \in [0,1]^4\right)$$
$$\leq P\left(2\|\hat{\beta}^- - \beta^-\| > \delta \text{ or } 2\|\hat{\beta}^+ - \beta^+\| > \delta\right)$$
$$\leq P\left(2\|\hat{\beta}^- - \beta^-\| > \delta\right) + P\left(2\|\hat{\beta}^+ - \beta^+\| > \delta\right)$$
$$= P\left(\|\hat{\beta}^+ - \beta^+\| > \delta/2\right) + P(\|\hat{\beta}^- - \beta^-\| > \delta/2)$$

where the fifth line uses Cauchy-Schwartz, the sixth line uses that $\theta \in [0,1]^4 \implies \|\theta\| \leq 2$, and the seventh uses the union bound. (Note that the norm under consideration here is the Euclidean norm.) It follows from Lemma 3 that

$$\max \{\text{UF}^+(S_{\hat{\theta}}), \text{UF}^-(S_{\hat{\theta}})\} = O_P(1/\sqrt{n})$$

as claimed.
10.3 Theorem 3 (loss and loss change estimators for fixed $\theta$)

**Proof.** Fix $\theta \in [0,1]^4$. It is straightforward to show that the identifying expressions in Proposition 2 hold if $\mu_0$ is replaced by $\phi$. We then have:

$$\hat{\mathcal{L}}(S_\theta) - \mathcal{L}(S_\theta) = (\theta^T \hat{\beta} + \mathbb{P}_n(\hat{\phi})) - (\theta^T \beta + \mathbb{P}(\phi)) \quad (16)$$

$$= \mathbb{P}_n(\hat{\theta}) - \mathbb{P}(\theta) \quad (17)$$

$$= (\mathbb{P}_n - \mathbb{P})f_\theta + (\mathbb{P}_n - \mathbb{P})(\hat{\theta} - \theta) + \mathbb{P}(\hat{\theta} - \theta) \quad (18)$$

Term (2) is $o_P(1/\sqrt{n})$ by Lemma 2 in [Kennedy et al., 2020], and term (3) is $o_P(1/\sqrt{n})$ by Lemma 1. We can therefore rewrite (18) as

$$\hat{\mathcal{L}}(S_\theta) - \mathcal{L}(S_\theta) = (\mathbb{P}_n - \mathbb{P})f_\theta + o_P(1/\sqrt{n})$$

$$\implies \sqrt{n}(\hat{\mathcal{L}}(S_\theta) - \mathcal{L}(S_\theta)) \rightsquigarrow N(0, \text{var}(f_\theta))$$

where the convergence follows from the central limit theorem and Slutsky’s theorem. By equivalent reasoning,

$$\hat{\Gamma}(\theta, S) - \Gamma(\theta, S) = (\mathbb{P}_n - \mathbb{P})(f_\theta - f_\theta) + o_P(1/\sqrt{n})$$

$$\implies \sqrt{n}(\hat{\Gamma}(\theta, S) - \Gamma(\theta, S)) \rightsquigarrow N(0, \text{var}(f_\theta - f_\theta))$$

per the second statement of the theorem. \qed

10.4 Theorem 4 (unfairness estimators for fixed $S_\theta$)

**Proof.** Fix $\theta \in [0,1]^4$. Once again, it is simple to show that the identifying expressions from Proposition 2 hold if $\mu_0$ is replaced with $\phi$. Per Proposition 1 and the definition of $\text{cFPR}$ given in (7), we have the following for the $\text{cFPR}$ of the input predictor $S$:

$$\text{cFPR}(S, a) - \text{cFPR}(S, a) = \frac{\mathbb{P}_n[S_\gamma a]}{\mathbb{P}_n[\gamma a]} - \mathbb{E}[S_\gamma a] \quad (19)$$

$$= \frac{\mathbb{P}_n[S_\gamma a]\mathbb{P}[\gamma a] - \mathbb{P}[S_\gamma a]\mathbb{P}_n[\gamma a]}{\mathbb{P}_n[\gamma a]\mathbb{P}[\gamma a]} \quad (20)$$

$$= \frac{\mathbb{P}[\gamma a]\left(\mathbb{P}_n[S_\gamma a] - \mathbb{P}[S_\gamma a]\right) - \mathbb{P}[S_\gamma a]\left(\mathbb{P}_n[\gamma a] - \mathbb{P}[\gamma a]\right)}{\mathbb{P}_n[\gamma a]\mathbb{P}[\gamma a]} \quad (21)$$

$$= \mathbb{P}_n[\gamma a]^{-1}\left\{\mathbb{P}_n[S_\gamma a] - \mathbb{P}[S_\gamma a] - \text{cFPR}(S, a)\left(\mathbb{P}_n[\gamma a] - \mathbb{P}[\gamma a]\right)\right\} \quad (22)$$
The two terms can be expanded as follows:

\[(1) = (P_n - P)S\gamma_a + (P_n - P)(S\tilde{\gamma}_a - S\gamma_a) + P(S\tilde{\gamma}_a - S\gamma_a) \quad (23)\]

\[(2) = (P_n - P)\gamma_a + (P_n - P)(\tilde{\gamma}_a - \gamma_a) + P(\tilde{\gamma}_a - \gamma_a) \quad (24)\]

The second term in both these expressions is \(o_P(1/\sqrt{n})\) per Lemma 2 in [Kennedy et al., 2020]. The third terms are \(o_P(1/\sqrt{n})\) per Lemma 1. Hence we can rewrite (22) as

\[(P_n - P)\left\{P_n[\tilde{\gamma}_a]^{-1}(S\gamma_a - c\text{FPR}(S,a))\gamma_a\right\} + o_P(1/\sqrt{n}) \quad (25)\]

Per the identifying expression for \(c\text{FPR}(S,a)\) in Proposition 1 and the definition of \(\hat{\text{cFPR}}\), we have

\[c\text{FPR}(S,\theta,a) - c\text{FPR}(S,\theta,a) = (\theta_{a,1} - \theta_{a,0})(c\hat{\text{cFPR}}(S,a) - c\text{FPR}(S,a))\]

Combining this with (25), we have

\[\sqrt{n}(c\text{FPR}(S,\theta,a) - c\text{FPR}(S,\theta,a)) = \sqrt{n}(P_n - P)g_a + o_P(1) \approx N(0, \text{var}(g_a))\]

where the convergence follows from the fact that \(\hat{\gamma}_a\) is consistent for \(\gamma_a\) (per Assumption 4), combined with the central limit theorem and Slutsky’s theorem. This establishes the first statement of the theorem. The second statement follows by identical reasoning, substituting \(h_a\) for \(g_a\) for the cFNR. The error rate differences can then be expressed as:

\[\sqrt{n}\left(\hat{\Delta}^+(\theta,S) - \Delta^+(\theta,S)\right) = \sqrt{n}(P_n - P)(g_0 - g_1) + o_P(1) \approx N(0, \text{var}(g_0 - g_1))\]

\[\sqrt{n}\left(\hat{\Delta}^-(\theta,S) - \Delta^-(\theta,S)\right) = \sqrt{n}(P_n - P)(h_0 - h_1) + o_P(1) \approx N(0, \text{var}(h_0 - h_1))\]

where the convergence once again follows from the central limit theorem and Slutsky’s theorem. \(\square\)

11 Appendix D: Notation
### Input data

\[ Z = (X, A, D, S, Y) \sim P \]

Data: covariates \( X \in \mathbb{R}^p \), sensitive feature \( A \), decision (treatment, intervention) \( D \), input predictor \( S \), outcome \( Y \)

| \( D_{\text{train}} \) | Sample used to construct \( S_{\tilde{\theta}} \) |
| \( D_{\text{test}} \) | Sample used to estimate properties of \( S_{\tilde{\theta}} \), given \( \tilde{\theta} \) |

### Derived predictor

\[
S_{\theta} = \sum_{a,s \in \{0,1\}} 1 \{ A = a, S = s \} B_{a,s} \sim \text{Bern}(\theta_{a,s})
\]

Predictor derived from \( S \)

\[
B_{a,s} \sim \text{Bern}(\theta_{a,s})
\]

Random variable that flips \( S \)

\[
\theta_{a,s} = \mathbb{P}(S_{\theta} = 1 \mid A = a, S = s)
\]

Conditional probability that defines \( S_{\theta} \)

\[
\theta_{A,S} = \sum_{a,s \in \{0,1\}} \theta_{a,s} 1 \{ A = a, S = s \}
\]

RV that takes value \( \theta_{a,s} \) with prob. \( \mathbb{P}(A = a, S = s) \)

\[
\theta = (\theta_{0,0}, \theta_{0,1}, \theta_{1,0}, \theta_{1,1})^T
\]

Optimization parameter

\[
\tilde{\theta} = (0, 1, 0, 1)
\]

The value such that \( S_{\tilde{\theta}} = S \)

### Nuisance parameters

\[
\pi = \pi(A, X, S) = \mathbb{P}(D = 1 \mid A, X, S)
\]

Propensity score for the decision

\[
\mu_0 = \mu_0(A, X, S, D) = \mathbb{E}[Y \mid A, X, S, D = 0]
\]

Outcome regression

\[
\phi = \frac{1-D}{1-\pi} (Y - \mu_0) + \mu_0
\]

Uncentered influence function for \( \mathbb{E}[Y^0] \)

### Loss parameters

\[
\beta_{a,s} = \mathbb{E}[1 \{ A = a, S = s \} (1 - 2\mu_0)]
\]

A coefficient in the loss

\[
\beta = (\beta_{0,0}, \beta_{0,1}, \beta_{1,0}, \beta_{1,1})^T
\]

Vector of loss coefficients

\[
f_{\theta} = (1 - 2\theta_{A,S}) \phi + \theta_{A,S}
\]

Uncentered IF for the loss of \( S_{\theta} \)

\[
\mathcal{L}(S_{\theta}) = \mathbb{E} \left[ (S_{\theta} - Y^0)^2 \right] = \theta^T \beta = \mathbb{E} \left[ f_{\theta} \right]
\]

Loss of \( S_{\theta} \), in several equivalent forms

\[
\Gamma(S_{\theta})
\]

Change in loss \( \mathcal{L}(S_{\theta}) - \mathcal{L}(S) \)

### Table 2: Notation.
### Fairness parameters

| Term | Definition |
|------|------------|
| cFPR($S_\theta, a$) | $P(S_\theta = 1 \mid Y^0 = 0, A = a)$ |
| cFNR($S_\theta, a$) | $P(S_\theta = 0 \mid Y^0 = 1, A = a)$ |
| $\beta^+$ | $(1 - cFPR(S, 0), cFPR(S, 0), cFPR(S, 1) - 1, -cFPR(S, 1))$ |
| $\beta^-$ | $(-cFNR(S, 0), cFNR(S, 0) - 1, cFNR(S, 1), 1 - cFNR(S, 1))$ |
| $\Delta^+(S_\theta)$ | $\theta^T \beta^+ = cFPR(S_\theta, 0) - cFPR(S_\theta, 1)$ |
| $\Delta^-(S_\theta)$ | $\theta^T \beta^- = cFNR(S_\theta, 0) - cFNR(S_\theta)$ |
| $\epsilon^+ , \epsilon^-$ | |
| $U^+(S_\theta)$ | $\max(|\Delta^+(S_\theta)| - \epsilon^+, 0)$ |
| $U^-(S_\theta)$ | $\max(|\Delta^-(S_\theta)| - \epsilon^-, 0)$ |

### Optimal fair derived predictor

| Term | Definition |
|------|------------|
| $\theta^*$ | $\arg \min_{\theta} \mathcal{L}(S_\theta)$ |
| s.t. | $|\Delta^+(S_\theta)| \leq \epsilon^+, |\Delta^-(S_\theta)| \leq \epsilon^-$ |

Table 3: Notation continued.