On Sample Functions Behavior of Stable Processes

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Abstract

We investigate the asymptotic behavior of sample functions of stable processes when $t \to \infty$. We compare our results with the iterated logarithm law, results for the first hitting time and most visited sites problems.

1 Main Notions

1. Let $X_1, X_2, ...$ be mutually independent random variables with the same law of distribution $F(x)$. The distribution $F(x)$ is called strictly stable if the random variable

$$X = (X_1 + X_2 + ... + X_n)/n^{1/\alpha}$$

is also distributed according to the law $F(x)$. The number $\alpha$ $(0 < \alpha \leq 2)$ is called a characteristic exponent of the distribution. When $\alpha \neq 1$ the characteristic function of $X$ has the form (see [2])

$$E[\exp (i\xi X)] = \exp \{-\lambda |\xi|^\alpha [1 - i\beta (\text{sign} \xi)(\tan \frac{\pi \alpha}{2})]\}$$

where $-1 \leq \beta \leq 1$, $\lambda > 0$. When $\alpha = 1$ we have

$$E[\exp (i\xi X)] = \exp \{-\lambda |\xi|[1 + \frac{2i\beta}{\pi} (\text{sign} \xi)(\log |\xi|)]\}$$

where $-1 \leq \beta \leq 1$, $\lambda > 0$. The homogeneous process $X(\tau)(X(0) = 0)$ with independent increments is called a stable process if

$$E[\exp (i\xi X(\tau))] = \exp \{-\tau |\xi|^\alpha [1 - i\beta (\text{sign} \xi)(\tan \frac{\pi \alpha}{2})]\}$$

where $0 < \alpha < 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$, $\tau > 0$. When $\alpha = 1$ we have

$$E[\exp (i\xi X(\tau))] = \exp \{-\tau |\xi|[1 + \frac{2i\beta}{\pi} (\text{sign} \xi)(\log |\xi|)]\}$$
where $-1 \leq \beta \leq 1, \ \tau > 0$. The stable processes are a natural generalization of the Wiener processes. In many theoretical and applied problems it is important to estimate the value

$$p_\alpha(t, a) = P(\sup_{0 \leq \tau \leq t} |X(\tau)| < a), \ \ 0 \leq \tau \leq t. \quad (1.6)$$

It is proved (see [6],[7],[9]) that the value of $p_\alpha(t, a)$ decreases very quickly by the exponential law when $t \to \infty$. This fact prompted the idea to consider the case when the value of $a$ in (1.6) depends on $t$ and $a(t) \to \infty, \ t \to \infty$. In this paper we deduce the conditions under which one of the following three cases is realized:

1) $\lim p_\alpha(t, a(t)) = 1, \ t \to \infty.$
2) $\lim p_\alpha(t, a(t)) = 0, \ t \to \infty.$
3) $\lim p_\alpha(t, a(t)) = p_\infty, \ 0 < p_\infty \leq 1, \ t \to \infty.$

The value of $p_\alpha(t, a(t))$ is the probability of the trajectory of the process $X(\tau)$ that remains inside the corridor $|X(\tau)| < a(t)$ when $0 \leq \tau \leq t$.

We investigate the situation when $t \to 0$ too. We consider separately the classical case when $\alpha = 2$. We think that even in this case our results are new. In the last part of the paper we compare our results with the classical iterated logarithm law (see [10]), the first hitting time problem (see [2],[5],[10]) and the most visited sites results (see [1]).

**Remark 1.1.** In the famous work by M.Kac [3] the connection of the theory of stable processes and the theory of integral equations was shown. M.Kac considered in detail only the case $\alpha = 1, \ \beta = 0$. The case $0 < \alpha < 2, \ \beta = 0$ was later studied by H.Widom [11]. As to the general case $0 < \alpha < 2, \ -1 \leq \beta \leq 1$ it was investigated in our works [6],[7],[9]. In all the mentioned works the parameter $a$ in (1.6) was fixed. The present paper is dedicated to the important case when $a$ depends on $t$ and $a(t) \to \infty, \ t \to \infty$.

## 2 Auxillary results

1. In this section we formulate some results from our paper [7] (see also [9], Ch.7). An important role in our approach is played by formula [3]

$$\int_0^\infty e^{-su}p_\alpha(u, a)du = \int_{-a}^a \psi_\alpha(x, s, a)dx. \quad (2.1)$$

Here $\psi_\alpha(x, s, a)$ is defined by relation

$$\psi_\alpha(x, s, a) = (I + sB^{-1}_\alpha)^{-1}\Phi_\alpha(0, x, a), \quad (2.2)$$
(The operator $B_{\alpha}$ and its kernel $\Phi_{\alpha}(x, y, a)$ will be introduced later. Further we consider the three cases.

Case 1. $0 < \alpha < 2$, $\alpha \neq 1$, $-1 < \beta < 1$.

Case 2. $1 < \alpha < 2$, $\beta = \pm 1$.

Case 3. $\alpha = 1$, $\beta = 0$.

Now we introduce the operators

$$B_{\alpha}f = \int_{-a}^{a} \Phi_{\alpha}(x, y, a)f(y)dy$$

(2.3)

acting in the space $L^2(-a, a)$.

In case 1 the kernel $\Phi_{\alpha}(x, y, a)$ has the following form (see [7], [9])

$$\Phi_{\alpha}(x, y, a) = C_{\alpha}(2a)^{\mu-1} \int_{a|x-y|}^{a^2-xy} \left[z^2 - a^2(x-y)^2\right]^{-\mu} \left[z - a(x-y)\right]^{2\rho-\mu} dz, \quad (2.4)$$

where the constants $\mu, \rho$, and $C_{\alpha}$ are defined by the relations $\mu = 2 - \alpha$,

$$\sin \pi \rho = \frac{1 - \beta}{1 + \beta} \sin \pi (\mu - \rho), \quad 0 < \mu - \rho < 1, \quad (2.5)$$

$$C_{\alpha} = \frac{\sin \pi \rho}{(\sin \pi \alpha/2)(1 - \beta)\Gamma(1 - \rho)\Gamma(1 + \rho - \mu)}. \quad (2.6)$$

Here $\Gamma(z)$ is Euler’s gamma function. We remark that the constants $\mu, \rho$, and $C_{\alpha}$ do not depend on parameter $a$.

In case 2 when $\beta = 1$ we have [7], [9]

$$\Phi_{\alpha}(x, y, a) = \frac{(\cos \pi \alpha/2)}{(2a)^{\alpha-1}\Gamma(\alpha)} \left\{ [a|x-y| + y-x]^{\alpha-1} - (a-x)^{\alpha-1}(a+y)^{\alpha-1} \right\} \quad (2.7)$$

In case 2 when $\beta = -1$ we have [7], [9]

$$\Phi_{\alpha}(x, y, a) = \frac{(\cos \pi \alpha/2)}{(2a)^{\alpha-1}\Gamma(\alpha)} \left\{ [a|x-y| + x-y]^{\alpha-1} - (a+x)^{\alpha-1}(a-y)^{\alpha-1} \right\} \quad (2.8)$$

Finally, in case 3 according to M.Kac [3] we have

$$\Phi_{1}(x, y, a) = \frac{1}{4} \log \left[ \frac{a^2 - xy + \sqrt{(a^2 - x^2)(a^2 - y^2)}}{a^2 - xy - \sqrt{(a^2 - x^2)(a^2 - y^2)}} \right] \quad (2.9)$$
3 The asymptotic behavior of $p_\alpha(t, a)$

1. In this section we use the assertion, which immediately follows from relations (2.4),(2.7)-(2.9).

**Proposition 3.1.** In all the three cases the relation

$$\Phi_\alpha(ax, ay, a) = a^{\alpha-1}\Phi_\alpha(x, y, 1) \quad (3.1)$$

is valid.

From equalities (2.1), (2.2), and (3.1) we deduce the assertion.

**Corollary 3.1.** In all the three cases the following relations

$$\psi_\alpha(ax, s, a) = a^{\alpha-1}\psi_\alpha(x, sa^\alpha, 1), \quad (3.2)$$

$$p_\alpha(t, a) = p_\alpha(t/a^\alpha, 1) \quad (3.3)$$

hold.

2. The kernel $\Phi_\alpha(x, y, a)$ of the operator $B_\alpha$ is non-negative [7],[9]. So the operator $B_\alpha$ has the non-negative eigenfunction $g_\alpha(x, a)$. The corresponding eigenvalue $\lambda_\alpha(a)$ is positive and larger than the modules of any other eigenvalues of the operator $B_\alpha$. The eigenvalue of the adjoint operator $B_\alpha^*$ with the largest modulus is also $\lambda_\alpha(a)$. We denote the corresponding non-negative eigenfunction by $h_\alpha(x, a)$. We normalize the functions $g_\alpha(x, a)$ and $h_\alpha(x, a)$ by the condition $(g_\alpha, h_\alpha) = 1$. Now we can formulate the following result.

**Theorem 3.1.** Let one of the following conditions be fulfilled:

I. $0 < \alpha < 2, \quad \alpha \neq 1, \quad -1 < \beta < 1$.

II. $1 < \alpha < 2, \quad \beta = \pm 1$.

III. $\alpha = 1, \quad \beta = 0$.

Then the asymptotic equality holds

$$p_\alpha(t, a) = e^{-t/\lambda_\alpha(a)}g_\alpha(0, a) \int_{-a}^{a} h_\alpha(x, a)dx[1 + o(1)], \quad t \to \infty. \quad (3.4)$$

We introduce the notations

$$\lambda_\alpha(1) = \lambda_\alpha, \quad p_\alpha(t, 1) = p_\alpha(t), \quad g_\alpha(x, 1) = g_\alpha(x), \quad h_\alpha(x, 1) = h_\alpha(x). \quad (3.5)$$

Using relation (3.3) and notations (3.5) we can rewrite formula (3.4) in the following way

$$p_\alpha(t, a) = e^{-t/\alpha^\alpha\lambda_\alpha}g_\alpha(0) \int_{-1}^{1} h_\alpha(x)dx[1 + o(1)], \quad t \to \infty. \quad (3.6)$$
**Remark 3.1.** The operator $B$ is self-adjoint when $\beta = 0$. In this case $h_\alpha = g_\alpha$.

**Remark 3.2.** The value $\lambda_\alpha$ characterizes how fast $p_\alpha(t, a)$ converges to zero when $t \to \infty$. The two-sided estimation for $\lambda_\alpha$ when $\beta = 0$ is given in [4] (see also [9], p.150).

3. Now we consider the case when the parameter $a$ depends on $t$. From Theorem 3.1. we deduce the assertion.

**Corollary 3.2.** Let one of conditions I-III of Theorem 3.1 be fulfilled and

$$
\frac{t}{a^\alpha(t)} \to \infty, \quad t \to \infty.
$$

Then the following equalities are true:

1) $p_\alpha(t, a(t)) = e^{t/[a^\alpha(t)\lambda_\alpha]}g_\alpha(0) \int_{-1}^{1} h_\alpha(x)dx[1 + o(1)], \quad t \to \infty.$

2) $\lim p_\alpha(t, a) = 0, \quad t \to \infty.$

3) $\lim P[\sup |X(\tau)| > a(t)] = 1, \quad 0 \leq \tau \leq t, \quad t \to \infty.$

**Remark 3.3.** Condition (3.7) is equivalent to the condition

$$
\frac{a(t)}{t^{1/\alpha}} \to 0, \quad t \to \infty.
$$

**Corollary 3.3.** Let one of conditions I-III of Theorem 3.1 be fulfilled and

$$
\frac{t}{|a(t)|^\alpha} \to 0, \quad t \to \infty.
$$

Then the following equalities are true:

1) $\lim p_\alpha(t, a(t)) = 1, \quad t \to \infty.$

2) $\lim P[\sup |X(\tau)| > a(t)] = 0, \quad 0 \leq \tau \leq t, \quad t \to \infty.$

Corollary 3.3 follows from (3.3) and the relation

$$
\lim p_\alpha(t) = 1, \quad t \to 0
$$
Corollary 3.4. Let one of conditions I-III of Theorem 3.1 be fulfilled and
\[
\frac{t}{|a(t)|^{\alpha}} \to T, \quad 0 < T < \infty, \quad t \to \infty.
\] (3.16)
Then the following equality is true:
\[
\lim_{t \to \infty} p_{\alpha}(t, a(t)) = p_{\alpha}(T), \quad t \to \infty.
\] (3.17)
Corollary 3.4 follows from (3.3).

4 Wiener Process

We consider separately the important special case when \(\alpha = 2\) (Wiener process). In this case the kernel \(\Phi_2(x, t, a)\) of the operator \(B_{\alpha}\) coincides with Green’s function (see [3]) of the equation
\[
-\frac{d^2y}{dx^2} = f(x), \quad -a \leq x \leq a
\] (4.1)
with the boundary conditions
\[
y(-a) = y(a) = 0.
\] (4.2)
It is easy to see that
\[
\Phi_2(x, t, a) = \frac{1}{2a} \left\{ \frac{(t + a)(a - x)}{(a - t)(a + x)} \right\} x \leq t \leq a
\] (4.3)
According to formula (4.3) equality (3.1) is true when \(\alpha = 2\) too. Hence the formula
\[
p_2(t, a) = p_2(t/a^2, 1)
\] (4.4)
is true as well.
Let \(a = 1\). The eigenvalues of problem (4.1),(4.2) have the form
\[
\mu_n = (n\pi/2)^2, \quad n = 1, 2, 3...
\] (4.5)
The corresponding normalized eigenfunctions are defined by the equality
\[
g_n(x) = \sin [(n\pi/2)(x - 1)].
\] (4.6)
The following formula

\[ p_2(t) = \sum_{j=1}^{\infty} g_j(0) \int_{-1}^{1} g_j(x) dx e^{-t \mu_j} \]  

(4.7)

holds. Formula (4.7) can be deduced in the same way as in the case \((0 < \alpha < 2, \beta = 0)\) (see [3], [6]). Using (4.5) and (4.6) we rewrite (4.7) in the form

\[ p_2(t) = \sum_{m=0}^{\infty} (-1)^m \frac{2}{\pi(m + 1/2)} e^{-t \pi(m + 1/2)^2}. \]  

(4.8)

The series (4.8) satisfies the conditions of Leibniz theorem. It means that \(p_2(t,a)\) can be calculated with a given precision when the parameters \(t\) and \(a\) are fixed. From (4.4) and (4.8) we deduce that

\[ p_2(t,a) = \frac{4}{\pi} e^{-t \pi^2/[2a(t)]^2} (1 + o(1)), \]  

(4.9)

where \(t/[a(t)]^2 \rightarrow \infty\).

**Proposition 4.1.** Theorem 3.1 and Corollaries 3.1-3.4 are true in case when \(\alpha = 2\) too.

**Remark 4.1** From the probabilistic point of view it is easy to see that the function \(p_2(t), (t > 0)\) is monotonic decreasing and

\[ 0 < p_2(t) \leq 1; \lim p_2(t) = 1, \quad t \to 0. \]  

(4.10)

5 \hspace{1em} Iterated logarithm law, most visited sites and first hitting time.

It is interesting to compare our results (Theorem 3.1, Corollaries 3.1-3.4 and Proposition 4.1) with the well-known results mentioned in the title of the section.

1. We begin with the famous Khinchine theorem (see[10]) about the iterated logarithm law.

**Theorem 5.1.** Let \(X(t)\) be stable process \((0 < \alpha < 2)\). Then almost surely \((a.s.)\) that

\[ \lim \sup \frac{|X(t)|}{(t \log t)^{1/\alpha} \log \log t} = \begin{cases} 0 & \epsilon > 0 \text{ a.s.} \\ \infty & \epsilon = 0 \text{ a.s.} \end{cases} \]  

(5.1)
We introduce the random process

\[ U(t) = \sup_{0 \leq \tau \leq t} |X(\tau)|, \quad 0 \leq \tau \leq t \]  

(5.2)

From Corollaries 3.1-3.4 and Proposition 4.1 we deduce the assertion.

**Theorem 5.2.** Let one of conditions I-III of Theorem 3.1 be fulfilled or let \( \alpha = 2 \) and

\[ b(t) \to \infty, \quad t \to \infty. \]  

(5.3)

Then

\[ b(t)U(t)/t^{1/\alpha} \to \infty \quad \text{a.s.}, \quad U(t)/[b(t)t^{1/\alpha}] \to 0 \quad \text{a.s.} \]  

(5.4)

In particular we have:

\[ [(\log t)U(t)]/t^{1/\alpha} \to \infty \quad \text{a.s.}, \quad U(t)/[(\log t)t^{1/\alpha}] \to 0 \quad \text{a.s.} \]  

(5.5)

when \( \epsilon > 0 \). We see that our approach and the classical one have some similar points (estimation of \( |X(\tau)| \)), but these approaches are essentially different.

We consider the behavior of \( |X(\tau)| \) on the interval \((0, t)\), and in the classical case \( |X(\tau)| \) is considered on the interval \((t, \infty)\).

2. We denote by \( V(t) \) the most visited site of stable process \( X \) up to time \( t \) (see [1]). We formulate the following result (see [1] and references therein).

Let \( 1 < \alpha < 2, \quad \beta = 0, \quad \gamma > 9/(\alpha - 1) \). Then we have

\[ \lim_{t \to \infty} \frac{(\log t)^\gamma}{t^{1/\alpha}} |V(t)| = \infty, \quad t \to \infty \quad \text{a.s.} \]  

(5.6)

To this important result we add the following estimation.

**Theorem 5.3.** Let one of the conditions I-III of Theorem 3.1 be fulfilled or let \( \alpha = 2 \) and

\[ b(t) \to \infty, \quad t \to \infty. \]  

(5.7)

Then

\[ |V(t)(t)/[b(t)t^{1/\alpha}] \to 0 \quad \text{a.s.} \]  

(5.8)

In particular we have when \( \epsilon > 0 \):

\[ |V(t)|/[(\log t)t^{1/\alpha}] \to 0 \quad \text{a.s.} \]  

(5.9)
The formulated theorem follows directly from the inequality $U(t) \geq |V(t)|$.

3. The first hitting time $T_a$ is defined by the formula

$$T_a = \inf \{ t \geq 0, X(t) \geq a \}.$$  \hfill (5.10)

It is obvious that

$$P(T_a > t) = P[\sup X(\tau) < a], \quad 0 \leq \tau \leq t.$$  \hfill (5.11)

We have

$$P(T_a > t) \geq P[\sup |X(\tau)| < a] = p_\alpha(t, a), \quad 0 \leq \tau \leq t.$$  \hfill (5.12)

So our formulas for $p_\alpha(t, a)$ estimate $P(T_a > t)$ from below.

**Remark 5.1.** Our results can be interpreted in terms of the first hitting time $T_{\pm a}$ one of the barriers $x = \pm a$. Namely, we have

$$P(T_{\pm a} > t) = p_\alpha(t, a).$$  \hfill (5.13)

The distribution of the first hitting time for the stable processes is an open problem.

**Remark 5.2.** Rogozin B.A. in his interesting work [5] established the law of the overshoot distribution for the stable processes when the existing interval is fixed.

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