Positivity Criteria Generalizing the Leading Principal Minors Criterion

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Abstract. An $n \times n$ Hermitian matrix is positive definite if and only if all leading principal minors $\Delta_1, \ldots, \Delta_n$ are positive. We show that certain sums $\delta_l$ of $l \times l$ principal minors can be used instead of $\Delta_l$ in this criterion. We describe all suitable sums $\delta_l$ for $3 \times 3$ Hermitian matrices. For an $n \times n$ Hermitian matrix $A$ partitioned into blocks $A_{ij}$ with square diagonal blocks, we prove that $A$ is positive definite if and only if the following numbers $\sigma_l$ are positive: $\sigma_l$ is the sum of all $l \times l$ principal minors that contain the leading block submatrix $[A_{ij}]_{k,j=1}^{k-1}$ (if $k > 1$) and that are contained in $[A_{ij}]_{i,j=1}^k$, where $k$ is the index of the block $A_{kk}$ containing the $(l,l)$ diagonal entry of $A$. We also prove that $\sigma_l$ can be used instead of $\Delta_l$ in other inertia problems.

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1. Introduction

Let $A = [a_{ij}]$ be an $n \times n$ Hermitian complex matrix. By the leading principal minors criterion, $A$ is positive definite if and only if

$$\Delta_1 := a_{11} > 0, \quad \Delta_2 := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots, \quad \Delta_n := \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} > 0. \quad (1)$$

It is also known that $A$ is positive definite if and only if

$$\delta_1 > 0, \quad \delta_2 > 0, \quad \ldots, \quad \delta_n > 0, \quad (2)$$

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where \( \delta_i \) is the sum of all \( i \times i \) principal minors of \( A \) (a minor is called principal if its diagonal entries lie on the diagonal of \( A \)). Indeed, the characteristic polynomial of \( A \) is equal to

\[
(x - \lambda_1) \cdots (x - \lambda_n) = x^n - \delta_1 x^{n-1} + \delta_2 x^{n-2} - \cdots + (-1)^n \delta_n,
\]

and so the condition (2) implies the positivity of all eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( A \), which ensures the positive definiteness of \( A \) since \( A \) is unitarily similar to a real diagonal matrix. The conditions (2) are symmetric in the sense of [5]: permutations of rows and the same permutations of columns of \( A \) do not change \( \delta_1, \ldots, \delta_n \).

In this paper, we give other examples of criteria of positive definiteness of the form

\[
\Sigma_1 > 0, \quad \Sigma_2 > 0, \ldots, \quad \Sigma_n = \det A > 0,
\]

where each \( \Sigma_i \) is a sum of some \( i \times i \) principal minors.

In Section 2, for each partition of \( A \) into blocks with square diagonal blocks, we construct a criterion of positive definiteness of the form (4). In particular, the criteria (1) and (2) are obtained from the partitions, in which the number of diagonal blocks is \( n \) and, respectively, 1. We show that the obtained sums \( \Sigma_i \) can be used instead of the leading principal minors \( \Delta_i \) in other inertia problems.

It would be interesting to describe all principal minors criteria of positive definiteness of the form (4). In Section 3 we describe them for \( 3 \times 3 \) matrices. There are 6 such criteria; 4 criteria can be obtained from the partitions into blocks (as in Section 2) and the remaining 2 criteria are new.

This research was inspired by Stepanov’s paper [5], in which the criterion of positivity from Theorem 1(c) was proved for real symmetric block matrices whose diagonal blocks are at most 3-by-3.

2. Symmetric criteria of positive definiteness for block matrices

Every \( n \times n \) Hermitian complex matrix \( A = A^* \) defines the Hermitian form \( x^* Ax \) with \( x = [x_1, \ldots, x_n]^T \). Forms \( x^* Ax \) and \( x^* Bx \) are said to be equivalent if their matrices \( A \) and \( B \) are congruent, i.e., \( S^* AS = B \) for some nonsingular \( S \).

By Sylvester’s Inertia Law, every Hermitian form \( x^* Ax \) is equivalent to the form

\[
\bar{x}_1 x_1 + \cdots + \bar{x}_p x_p - \bar{x}_{p+1} x_{p+1} - \cdots - \bar{x}_{p+q} x_{p+q},
\]

where \( p \) and \( q \) do not depend on the method of reduction. The numbers \( p \) and \( q \) are equal to the numbers of positive and negative eigenvalues of \( A \) since \( A \) is unitarily congruent to a real diagonal matrix \( D \) (i.e., \( U^* AU = D \) for some unitary \( U \)), see [3, Theorem 4.1.5]. If \( r \) is the rank of \( A \) and the leading principal minors \( \Delta_1, \ldots, \Delta_r \) are all nonzero, then the numbers \( p \) and \( q \) can be calculated using the Jacobi formula [2, Chapter X, §9, Formula (133)]: \( x^* Ax \) is equivalent to

\[
\frac{\Delta_1}{\Delta_r} \bar{x}_1 x_1 + \frac{\Delta_2}{\Delta_{r-1}} \bar{x}_2 x_2 + \cdots + \frac{\Delta_r}{\Delta_{r-1}} \bar{x}_r x_r.
\]
In this section, we consider an $n \times n$ Hermitian matrix $A$ partitioned into $t \times t$ blocks with square diagonal blocks:

$$A = \begin{bmatrix}
A_{11} & \cdots & A_{1t} \\
\vdots & \ddots & \vdots \\
A_{t1} & \cdots & A_{tt}
\end{bmatrix}, \quad A_{ii} \text{ is } k_i \text{-by-} k_i. \quad (6)$$

We say that $A$ is block-unitarily congruent to $B$ if $U^*AU = B$, where $U$ is a direct sum of $t$ unitary matrices of sizes $k_1 \times k_1$, \ldots, $k_t \times k_t$.

Let us denote by $A_i$ the leading principal block submatrix of (6) formed by the first $i \times i$ blocks, i.e.,

$$A_1 = A_{11}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ldots, A_t = A. \quad (7)$$

Each $(l, l)$ diagonal entry of $A$ belongs to some diagonal block $A_{kk}$. Denote by $\sigma_l$ the sum of all $l \times l$ principal minors that contain $A_{k-1}$ (if $k > 1$) and that are contained in $A_k$. For example, if

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 6 \\
3 & 4 & 5 & 6 & 7 \\
4 & 5 & 6 & 7 & 8 \\
5 & 6 & 7 & 8 & 9
\end{bmatrix},$$

then

$$\sigma_1 = 1 + 3, \quad \sigma_2 = \Delta_2, \quad \sigma_3 = \Delta_3,$$

$$\sigma_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 9
\end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 6 \\ 3 & 4 & 5 & 7 \\ 5 & 6 & 7 & 8
\end{vmatrix}, \quad \sigma_5 = \Delta_5 = \det A.$$

**Theorem 1.** Let $A$ be a Hermitian matrix (6) partitioned into blocks such that the leading principal block submatrices $A_1, \ldots, A_{t-1}$ (see (7)) are nonsingular. Then

(a) The number $p$ of positive eigenvalues of $A$ is equal to the number of sign changes (ignoring zeros) in the sequence

$$1, -\sigma_1, \sigma_2, -\sigma_3, \ldots, (-1)^n \sigma_n. \quad (8)$$

(b) The number $q$ of negative eigenvalues of $A$ is equal to the number of sign changes (ignoring zeros) in the sequence $1, \sigma_1, \sigma_2, \ldots, \sigma_n$.

(c) The form $x^*Ax$ is positive definite if and only if

$$\sigma_1 > 0, \quad \sigma_2 > 0, \ldots, \sigma_n > 0. \quad (9)$$

(d) The form $x^*Ax$ is positive semidefinite if and only if all $\sigma_i \geq 0$. If all $\sigma_i \geq 0$, then either all $\sigma_i > 0$, or all $\sigma_i = 0$, or

$$\sigma_1 > 0, \ldots, \sigma_{l-1} > 0, \quad \sigma_l = \cdots = \sigma_n = 0 \quad (10)$$

for some $l > 1$. 


The number \( r := \max \{ i \mid \sigma_i \neq 0 \} \) is equal to the rank of \( A \). If \( \sigma_1, \ldots, \sigma_r \) are nonzero then \( x^* A x \) is equivalent to
\[
\sigma_1 \bar{x}_1 x_1 + \frac{\sigma_2}{\sigma_1} \bar{x}_2 x_2 + \cdots + \frac{\sigma_r}{\sigma_{r-1}} \bar{x}_r x_r.
\]
\[ (11) \]

(f) The numbers \( \sigma_1, \ldots, \sigma_n \) are invariant with respect to transformations of block-unitary \*congruence with \( A \) (in particular, with respect to any permutation of rows of \( A \) within horizontal strips and the same permutation of its columns).

Proof. We begin with a general result on \( \sigma_i \) which will be used in the proof of (a)–(f). Let \( t > 1 \). Represent \( A \) in the form
\[
A = \begin{bmatrix} A_{t-1} & B \\ B^* & A_t \end{bmatrix}, \quad B^* = [A_{t1} | \ldots | A_{tt-1}]. \]
\[ (12) \]
The size of \( A_{t-1} \) is \( k \times k \), where
\[
k := k_1 + k_2 + \cdots + k_{t-1}
\]
(see (6)). By the assumption of the theorem, \( A_{t-1} \) is nonsingular. Adding linear combinations of columns of \( A_{t-1} = A^*_t \) to columns of \( B \) and performing the \*congruent transformations of rows, we reduce \( A \) to the block-diagonal matrix
\[
\begin{bmatrix} A_{t-1} & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} I \\ - (A_{t-1} B)^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} A_{t-1} & B \\ B^* & A_t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\]
\[ (14) \]
which is \*congruent to \( A \).

If \( \Delta \) is a principal minor of \( A \) and \( \Delta \) contains \( A_{t-1} \), then \( \Delta \) is not changed by transformations (14). So \( \Delta = \Delta_k \Delta' \), where \( \Delta_k = \det A_{t-1} \) (\( k \) is defined in (13)) and \( \Delta' \) is a principal minor of \( A' \). We have
\[
\sigma_k = \Delta_k, \quad \sigma_{k+1} = \Delta_k \sigma_1', \quad \ldots, \quad \sigma_n = \sigma_{k+k} = \Delta_k \sigma_k',
\]
where \( \sigma_j' \) is the sum of all \( j \)-by-\( j \) principal minors of the matrix \( A' \).

(a) We prove the statement (a) using induction on \( t \). Let first \( t = 1 \) and let
\[
\chi_A(x) = x^n + c_1 x^{n-1} + \cdots + c_n
\]
be the characteristic polynomial of \( A \). Then
\[
c_1 = -\sigma_1, \quad c_2 = \sigma_2, \quad c_3 = -\sigma_3, \quad \ldots, \quad c_n = (-1)^n \sigma_n,
\]
and the sequence (8) takes the form 1, \( c_1, \ldots, c_n \). So the statement (a) follows from Descartes’ Sign Rule ([11, § 55] or [4, Chapter 6, § 4]): if all of the roots of a polynomial
\[
f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{R}[x]
\]
are real, then the number of its positive roots is equal to the number of sign changes in the sequence of coefficients 1, \( a_1, \ldots, a_n \).
Let now $t > 1$. Reduce $A$ to the form (13). By induction hypothesis, the statement (a) holds for $A_{t-1}$ and $A'$. Hence, the number $p_{t-1}$ of positive eigenvalues of $A_{t-1}$ is equal to the number of sign changes in the sequence

$$1, -\sigma_1, \sigma_2, -\sigma_3, \ldots, (-1)^k\sigma_k,$$

and the number $p'$ of positive eigenvalues of $A'$ is equal to the number of sign changes in the sequence

$$1, -\sigma'_1, \sigma'_2, -\sigma'_3, \ldots, (-1)^{k'}\sigma'_{k'},$$

In view of (15), the multiplication of the last sequence by $(-1)^k\Delta_k$ gives the sequence

$$(-1)^{k}\sigma_k, (-1)^{k+k+1}\sigma_{k+1}, \ldots, (-1)^{n}\sigma_n.$$ 

Therefore, the number $p_{t-1} + p'$ of positive eigenvalues of (14) is equal to the number of sign changes in the sequence (8). This proves (a) since by Sylvester’s Inertia Law the matrices (14) and $A$ have the same number of positive eigenvalues.

(b) Property (b) is evident from property (a) with changing $A$ by $-A$.

(c) The form $x^*Ax$ is positive definite if and only if all the eigenvalues of $A$ are positive. So (c) follows from (a).

(d) The form $x^*Ax$ is positive semidefinite if and only if all the eigenvalues of $A$ are nonnegative. So the first statement in (d) follows from (b).

Suppose all $\sigma_i \geq 0$, there exist $\sigma_i > 0$, and there exist $\sigma_i = 0$. Write $l := \min\{i \mid \sigma_i = 0\}$. Let us prove (10) using induction on $l$.

If $t = 1$, then we reduce $A$ by transformations of unitary *congruence to a real diagonal matrix

$$D = \text{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0), \quad \lambda_1 > 0, \ldots, \lambda_s > 0.$$ 

These transformations do not change $\chi_A(x)$. By (17), they do not change all $\sigma_i$, and so we can calculate $\sigma_i$ using minors of $D$ instead of minors of $A$:

$$\sigma_1 = \sum_i \lambda_i, \quad \sigma_2 = \sum_{i<j} \lambda_i\lambda_j, \quad \sigma_3 = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k, \ldots$$

(19)

Since $\lambda_1, \ldots, \lambda_s$ are positive, we have (10) with $l = s$.

If $t > 1$, then we reduce $A$ to the form (14). By induction hypothesis, the statement (d) holds for $A_{t-1}$ and $A'$. Since $A_{t-1}$ is nonsingular, $\sigma_k = \Delta_k > 0$, hence all $\sigma_1, \ldots, \sigma_k$ are positive, and so $l > k$.

If $l = k + 1$, then $\sigma'_1 = 0$, and therefore all $\sigma'_i$ are zero. If $l = k + 1$, then

$$\sigma'_1 > 0, \ldots, \sigma'_{l-k-1} > 0, \quad \sigma'_{l-k} = \cdots = \sigma'_{k-1} = 0.$$ 

In view of (15), this proves (10).

(e) Let $r := \max\{i \mid \sigma_i \neq 0\}$. Since $A_{t-1}$ is nonsingular, $\sigma_k = \det A_{t-1} \neq 0$, thus $r \geq k$. Reduce $A$ to the form (14) and obtain (15). Then reduce $A'$ by transformations of unitary *congruence to a real diagonal matrix (18) and obtain (19) with $\sigma_i$ replaced by $\sigma'_i$. By (15),

$$r = k + \max\{i \mid \sigma'_i \neq 0\} = k + \text{rank} D = \text{rank} A_{t-1} + \text{rank} A' = \text{rank} A.$$
If all \( \sigma_1, \ldots, \sigma_r \) are nonzero, then the forms \( x^*Ax \) and (11) are equivalent. Indeed, their matrices have the same number of positive eigenvalues and the same number of negative eigenvalues due to (a), (b), and the equalities \( \sigma_{r+1} = \cdots = \sigma_n = 0. \)

(f) We use induction on \( t \). For \( t = 1 \), property (f) holds by (17) since the coefficients of \( \chi_A(x) \) are invariant with respect to similarity transformations with \( A \). For \( t > 1 \), consider \( \tilde{A} := U^*AU \), where \( U = U_1 \oplus \cdots \oplus U_t \) and each \( U_i \) is a \( k_i \times k_i \) unitary matrix. The sums \( \sigma_i \) were defined for \( A \); denote by \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n \) the corresponding sums for \( \tilde{A} \). Partition \( A \) into blocks as in (12) and partition \( \tilde{A} \) analogously:

\[
\tilde{A} = \begin{bmatrix} \tilde{A}_{t-1} & \tilde{B} \\ \tilde{B}^* & \tilde{A}_{tt} \end{bmatrix}, \quad \tilde{B}^* := [\tilde{A}_{t1}| \ldots |\tilde{A}_{tt-1}].
\]

Let \( V := U_1 \oplus \cdots \oplus U_{t-1} \). Then \( \tilde{A}_{t-1} = V^*A_{t-1}V \). By induction hypothesis, property (f) holds for \( A_{t-1} \), that is, \( \sigma_1 = \tilde{\sigma}_1, \ldots, \sigma_k = \tilde{\sigma}_k \), where \( k \) was defined in (13). It remains to prove that

\[
\sigma_{k+1} = \tilde{\sigma}_{k+1}, \ldots, \sigma_n = \tilde{\sigma}_n. \tag{20}
\]

Since \( U = (V \oplus I_{k_1})(I_{k_1} \oplus U_t) \), the transformation \( A \mapsto U^*AU \) is the composition of two transformations: \( A \mapsto (V \oplus I_{k_1})^*A(V \oplus I_{k_1}) \) and \( A \mapsto (I_{k_1} \oplus U_t)^*A(I_{k_1} \oplus U_t) \). The first transformation does not change \( \sigma_{k+1}, \ldots, \sigma_n \) since it does not change every minor of \( A \) containing \( A_{t-1} \). It remains to prove (20) for the second transformation.

Thus we can suppose that \( V = I_{k_1} \). Then

\[
\tilde{A} = U^*AU = \begin{bmatrix} A_{t-1} & BU_t \\ U_t^*B^* & U_t^*A_{tt}U_t \end{bmatrix}.
\]

Reduce \( A \) to the form (14) and \( \tilde{A} \) to the form

\[
\begin{bmatrix} A_{t-1} & 0 \\ 0 & \tilde{A}' \end{bmatrix} := \begin{bmatrix} A_{t-1} & 0 \\ 0 & U_t^*A_{tt}U_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ -(A_{t-1}^{-1}BU_t)^* & I \end{bmatrix} \begin{bmatrix} A_{t-1} & BU_t \\ U_t^*B^* & U_t^*A_{tt}U_t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{21}
\]

where \( \tilde{A}' = A_{tt} - B^*A_{t-1}^{-1}B \) was defined in (14).

Each \( \sigma_i \) with \( i \geq k \) has the form (15). Analogously, each \( \tilde{\sigma}_i \) with \( i \geq k \) has the form

\[
\tilde{\sigma}_i = \Delta_k, \quad \tilde{\sigma}_{k+1} = \Delta_k\tilde{\sigma}'_1, \quad \cdots, \quad \tilde{\sigma}_{k+r-1} = \Delta_k\tilde{\sigma}'_{r-1},
\]

where \( \tilde{\sigma}'_j \) is the sum of all \( j \)-by-\( j \) principal minors of the matrix \( \tilde{A}' \). Since the matrices \( \tilde{A}' \) and \( \tilde{A}' = U_t^*A_{tt}U_t \) are similar, \( \sigma_j = \tilde{\sigma}_j \) for all \( j \). This proves (20). \( \square \)
3. Principal minors criteria for $3 \times 3$ matrices

For each $n \times n$ matrix, denote by

$$P_{i_1 i_2 \ldots i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

its $k \times k$ principal minor that lies on the intersection of rows $i_1, i_2, \ldots, i_k$ with columns $i_1, i_2, \ldots, i_k$. Let (4) be a system of inequalities, in which every $\Sigma_k$ is a sum of some $P_{i_1 i_2 \ldots i_k}$ with distinct $(i_1, i_2, \ldots, i_k)$. We say that (4) ensures positive definiteness if every $n \times n$ Hermitian matrix is positive definite if and only if it satisfies (4).

**Theorem 2.** (a) Each of the following systems of inequalities ensures positive definiteness of $3$-by-$3$ Hermitian matrices:

(i) $P_1 > 0$, $P_{12} > 0$, $P_{123} > 0$;
(ii) $P_1 > 0$, $P_{12} + P_{13} > 0$, $P_{123} > 0$;
(iii) $P_1 + P_2 > 0$, $P_{12} > 0$, $P_{123} > 0$;
(iv) $P_1 + P_2 > 0$, $P_{12} + P_{13} + P_{23} > 0$, $P_{123} > 0$;
(v) $P_1 + P_2 + P_3 > 0$, $P_{12} + P_{13} > 0$, $P_{123} > 0$;
(vi) $P_1 + P_2 + P_3 > 0$, $P_{12} + P_{13} + P_{23} > 0$, $P_{123} > 0$.

Systems (i), (ii), (iii), and (vi) have the form (9) with respect to the partitions

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (22)$$

(b) If a system (4) with $n = 3$ ensures positive definiteness, then it can be obtained from one of (i)–(vi) by a permutation of the indexing set $\{1, 2, 3\}$.

**Proof.** If a 3-by-3 Hermitian matrix is positive definite, then its principal minors are positive, and so it satisfies each of systems (i)–(vi).

Let

$$\Sigma_1 > 0, \quad \Sigma_2 > 0, \quad P_{123} > 0 \quad (23)$$

be a system of the form (4) with $n = 3$. For each substitution $\sigma$ on the indexing set $\{1, 2, 3\}$, we define the system

$$\Sigma_1^\sigma > 0, \quad \Sigma_2^\sigma > 0, \quad P_{123} > 0 \quad (24)$$

obtained from (23) by replacement of all the summands $P_i$ and $P_{ij}$ of $\Sigma_1$ and $\Sigma_2$ with $P_{\sigma(i)}$ and $P_{\sigma(i)\sigma(j)}$. A $3 \times 3$ Hermitian matrix $A$ satisfies (23) if and only if the matrix $A^\sigma$ obtained by the corresponding permutations of rows and columns satisfies (24). Hence (23) ensures positive definiteness if and only if the same holds for (24).
Each system of the form (23) determined up to substitutions \( \sigma \) is presented by one of the rows of the following table:

| \( P_1 \) | \( P_{12} \) | \( P_{23} \) | \( P_{12} + P_{13} \) | \( P_{12} + P_{13} + P_{23} \) | \( (i) \) | \( \text{diag}(1, -1, -1) \) |
|-----------|-------------|-------------|-----------------|-----------------|----------------|----------------|
| \( P_1 + P_2 \) | \( P_{12} \) | \( P_{13} \) | \( P_{12} + P_{13} \) | \( P_{12} + P_{13} + P_{23} \) | \( (ii) \) | \( \text{diag}(1, -1, -2) \) |
| \( P_1 + P_2 + P_3 \) | \( P_{12} \) | \( P_{13} \) | \( P_{12} + P_{13} \) | \( P_{12} + P_{13} + P_{23} \) | \( (iii) \) | \( \text{diag}(1, -2, -1) \) |

The first two entries of the row are \( \Sigma_1 \) and \( \Sigma_2 \), and the last entry is either a matrix that is not positive definite but fulfills \( \Sigma_1 > 0, \Sigma_2 > 0, P_{123} > 0 \) (which means that the system does not ensure positive definiteness) or the number ((i)–(vi)) of the corresponding system in Theorem 2.

It remains to prove that each of the systems (i)–(vi) ensures positive definiteness. This is true for (i), (ii), (iii), and (vi) due to Theorem 1(c) applied to 3 \( \times \) 3 matrices partitioned as in (22).

Let a 3 \( \times \) 3 Hermitian matrix \( A \) satisfy condition (iv). By a suitable transformation \((U \oplus I_1)^{-1}A(U \oplus I_1)\) with unitary \( U \), we reduce \( A \) to the form

\[
\begin{bmatrix}
a & 0 & \bar{x} \\
0 & b & \bar{y} \\
x & y & c
\end{bmatrix}
\]

(25)

This similarity transformation does not change the left-hand sides of the inequalities (iv) since \( P_1 + P_2 \) is the trace of the leading principal 2 \( \times \) 2 submatrix (whose determinant is \( P_{12} \)), and \( P_{12} + P_{13} + P_{23} \) is a coefficient of the characteristic polynomial of \( A \) (see (3)). Therefore, the matrix (25) fulfills (iv):

\[
a + b > 0, \quad ab + (ac - |x|^2) + (bc - |y|^2) > 0, \quad abc - |x|^2b - |y|^2a > 0.
\]

(26)

If \( c < 0 \) then \( ab > 0 \) by the first and the second inequalities in (26); since \( a + b > 0 \), we have \( a > 0 \) and \( b > 0 \), which contradicts the third inequality in (26). Thus \( c \geq 0 \), \( a + b + c > 0 \), \( A \) satisfies (vi), and so it is positive definite.

Let a 3 \( \times \) 3 Hermitian matrix \( A \) satisfy condition (v). By a suitable transformation \((I_1 \oplus U)^{-1}A(I_1 \oplus U)\) with unitary \( U \), we reduce \( A \) to the form

\[
\begin{bmatrix}
a & \bar{x} & \bar{y} \\
x & b & 0 \\
y & 0 & c
\end{bmatrix}
\]

(27)
This similarity transformation does not change $P_{23}$ and $P_{12} + P_{13} + P_{23}$, hence it preserves $P_{12} + P_{13}$. Therefore, the matrix $P$ fulfils (v):

$$a + b + c > 0, \quad ab - |x|^2 + ac - |y|^2 > 0, \quad abc - |x|^2c - |y|^2b > 0.$$ 

Since $a(b + c) > |x|^2 + |y|^2$, $a \neq 0$. If $a < 0$ then $b + c < 0$, which contradicts $a + b + c > 0$. Thus $a > 0$, $A$ satisfies (ii), and so it is positive definite. 

\[\square\]

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