HADRONIC STRING FROM CONFINEMENT

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Abstract

The existence of hadronic string is derived as a consequence of confinement in QCD. A state of “stretched glue”, created by a Polyakov loop operator, is shown to have translational zero-modes which are stringy degrees of freedom. These modes are described by an effective string theory, valid for worldsheets which are locally flat on length scales of order $\Lambda_{QCD}^{-1}$, the dominant behavior being given by the Nambu-Goto action. In a subsequent paper, these effective strings will be shown to emerge in mesons stretched by their orbital angular momentum, thereby deriving some aspects of Regge phenomenology directly from QCD and confinement.

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1 Introduction

Historically, the string model of hadrons emerged as a beautiful realization of dual models and Regge phenomenology. Prior to the ascendance of QCD, it was considered a prime candidate for the theory of the strong interactions. However, as is well known, when the Nambu-Goto string is taken as a fundamental principle, it leads to twenty-six dimensions, gravitons, tachyons, and other disturbing consequences [1]. Furthermore, it fails to reproduce the partonic behavior found in deep-inelastic scattering experiments.

Of course, the modern viewpoint is that QCD is the fundamental theory of the strong interactions. Strings may then provide an effective theory of some sector of QCD. Just like the chiral lagrangian description of pions, the effective theory will have a restricted domain of validity, beyond which it breaks down. A common picture is that QCD strings are vibrations of color-electric flux tubes which connect and confine quarks inside hadrons. Such flux tubes do indeed appear in the strong-coupling limit of lattice QCD and are responsible for the area law of confinement [2]. In the continuum, we expect that the flux-tubes will look stringy when they are much longer than their characteristic thickness, $\Lambda_{QCD}^{-1}$. This suggests that the effective theory is valid for strings much longer than $\Lambda_{QCD}^{-1}$. See ref. [3] for a discussion of effective string theory from this perspective. This interpretation bypasses the various embarrassments of fundamental hadronic strings. As a simple example, the massless graviton of fundamental string theory is a short closed string, and therefore is not predicted within the effective theory.

In this paper, I show from first principles (Yang-Mills theory) that effective string degrees of freedom must exist as a necessary consequence of confinement, and that the string description is dominated by the Nambu-Goto action. The derivation clarifies and justifies the string picture within QCD. To illustrate the relationship between strings and confinement, consider the analogous relationship between Coulomb’s law and photons in quantum electrodynamics. Suppose experimentally, we had only observed the static force between electric charges, given by Coulomb’s law. However let us say that one firmly believes in the principles of special relativity, charge conservation, and quantum mechanics. To describe electrostatics, one could invent the concept of an electric field satisfying Gauss’ law,

$$\partial_i E_i(x) = \rho(x).$$

\[\text{Comment:} \text{The strong-coupling limit is a rather drastic approximation, but is usually assumed to give a reasonable qualitative guide to the physics of confinement.}\]
Now, since charge is conserved, charge-density, $\rho(x)$, must transform as the time-component of a 4-vector, $J_\mu$. Therefore to accommodate Gauss’ law in a relativistic context we must take $\vec{E}$ to be part of an antisymmetric tensor satisfying Maxwell’s equation,

$$\partial_\mu F^{\mu\nu} = J^\nu.$$  \hspace{1cm} (2)

In particular, magnetic fields are necessary. Thus the static Coulomb law leads to a new dynamical object: the electromagnetic wave. In the quantum theory the associated particle is the photon. Similarly, reconciling Newton’s law of gravitation with relativity led Einstein to General Relativity as the unique low-energy theory of gravity. The resulting gravitational waves appear in the (effective) quantum theory as gravitons. To summarize, a static long-range force between charges always has associated waves in a relativistic theory. In QCD the static force is confining. That is, it costs an infinite energy to pull apart the quarks of a single hadron. In the real world there is the complication that a color-singlet hadron can break into two color-singlets given only a finite amount of energy by virtue of quark pair production. We can postpone this problem by either considering a world where there are no light dynamical quarks, or working in the large-$N_c$ limit, where pair production is suppressed. In this paper we will do the former. As will be seen in section 2, general principles and confinement imply a linear potential at long distances. This restriction is analogous to electrodynamics, where a $1/r^3$ potential, say, would be inconsistent with general principles. What are the waves that correspond to the confining force in QCD? I will show that they are string vibrations.

Waves on a string naturally have an interpretation as Nambu-Goldstone (NG) modes of the spontaneous breaking of translational symmetry transverse to the classical string ground state.\footnote{Quantum-mechanically this is a little trickier, because strictly speaking, fluctuations in this effectively $(1+1)$-dimensional system wipe out the symmetry breaking. This subtlety will be properly treated in the later sections.} This strongly suggests that if we wish to derive strings as a general consequence of confinement, we should demonstrate that the state of a stretched hadron spontaneously breaks spacetime symmetries in the pattern for which strings are the requisite NG modes. This is the approach taken in this paper. The hadron will be a pure glue state created by a large Wilson loop (a “Polyakov loop” to be precise, as explained later). The symmetry-breaking will be derived as a consequence of confinement. An important tool for describing the resulting NG modes
will be the chiral lagrangian approach. The chiral lagrangian construction for internal symmetry breaking by Callan, Coleman, Wess and Zumino [6] has been generalized by Volkov [7] to incorporate spontaneous breaking of spacetime symmetries. I will use a version of this approach, streamlined for the application to strings. As in the case of internal symmetries, the chiral lagrangian is expected to (model-independently) enforce the constraints of unitarity and (spontaneously broken) symmetry on the properties of the NG modes. In the present case the Poincare symmetries themselves are realized non-linearly in the chiral lagrangian.

In the interests of simplicity, the arguments presented in this paper are made for pure Yang-Mills dynamics, without light dynamical quarks. In a second paper [8], I will make contact with phenomenology, showing that strings emerge when a meson is stretched by orbitally exciting it. In certain kinematic regimes the resulting Regge trajectories are linear. Though the derivation of Regge trajectories from the string model is hardly new, what is new is their derivation directly from QCD and confinement, without model assumptions, and with an understanding of the domain of validity. The large-$N_c$ approximation is used in the second paper to focus on confinement and to treat the breaking of strings as a finite-$N_c$ correction.

Finally, though the subject of this paper is the effective string, I wish to briefly mention the ongoing endeavor to demonstrate an exact duality between QCD and some fundamental string theory (differing from Nambu-Goto strings). The basis for this duality is the Gauss law constraint on physical states of a gauge theory (which is also implicit in the present derivation of the effective string). The constraint is satisfied by eigenstates of the color-electric field where the field lines are closed or end on quarks. Gauge theory written in terms of the evolution of these flux-line states has the form of a string field theory. A simple way to see this concretely is provided by the hamiltonian lattice formulation of QCD. See for example refs. [9] [10]. Unfortunately, the explicit form of the fundamental dual string theory in the continuum is still unknown. Ref. [11] provides a brief and useful review of the progress and problems in this area, as well as references. More recently, it has been shown that 2-dimensional large-$N_c$ QCD can be exactly reformulated as a string theory [12].

This paper is organized as follows. Section 2 is a derivation of the area law (the euclidean spacetime version of a linear potential) from the assumption of confinement of static charges. In section 3, I consider glue “wrapped around the universe” and show that there is “almost” a spontaneous symmetry-breaking of transverse translations as a direct consequence
of the area law. This is sufficient to derive an analog of Goldstone’s theorem, demonstrating the existence of stringy NG modes. This is done in section 4. In section 5, I construct the chiral lagrangian description of the NG modes, dominated by the Nambu-Goto term. Section 6 provides discussion as well as connections to the second paper and future work.

2 The Area Law

For the remainder of the paper we consider the euclidean spacetime formulation of Yang-Mills theory for $SU(N_c)$ gauge group, with $N_c$ finite (say 3). In this section the area law is derived as the only possible static law of confinement. It is the euclidean reflection of a linear potential. Confinement is approached in the standard way, by considering the response to test charges. Consider a meson made from a static (infinitely massive) quark and anti-quark pair, both at rest in some frame. The dynamics is described by a heavy quark effective lagrangian,

$$
\mathcal{L} = \bar{Q}_+ D^+_t Q_+ + \bar{Q}_- D^-_t Q_- + \frac{1}{4g^2} \text{tr} G_{\mu\nu} G^{\mu\nu},
$$

where $Q_+/-$ are the quark and anti-quark fields. In this static limit the quarks can have fixed positions, with separation $L$. We are using the static quarks to stretch a meson and examine confinement, so consider $L \gg \Lambda_{QCD}^{-1}$. The dynamical object is the state of glue connecting the quarks. The energy eigenvalue of the lowest such state, $E_0(L)$, is then naturally identified as the potential energy needed to achieve the quark separation, $L$. Confinement corresponds to the statement,

$$
E_0(L) \rightarrow_{L \rightarrow \infty} \infty.
$$

A gauge-invariant interpolating operator for the meson (which is necessarily non-local due to the fixed quark separation) is given by

$$
\mathcal{O}(t) \equiv Q_+(0, t) P e^{\int_0^L dx A_x(x, t)} Q_-(L, t),
$$

where $y$ and $z$ are fixed on the right-hand side, so I have suppressed writing them. Remembering the rule for going from Minkowski to euclidean time-evolution, $e^{iHt} \rightarrow e^{-Ht}$, the spectral decomposition of the $\mathcal{O}$ correlator for $t > 0$ reads,

$$
< \mathcal{O}(t) \mathcal{O}^\dagger(0) > = \int_{E_0(L)}^\infty dE \rho(E, L) e^{-Et},
$$
where \( \rho \) is the spectral density,

\[
\rho(E, L) \equiv \sum_r |< 0|O(0)|r, L >|^2 \delta(E - E_r),
\]

and the \( |r, L> \) constitute the complete energy basis for the glue in the meson. For very large euclidean times the correlator is dominated by the lowest-lying intermediate state of glue.

The position-space static quark propagator in the gauge field background is given by the identity,

\[
D_t [\theta(t)\delta^3(\vec{x})Pe^i \int_0^t dt' A_t(\vec{x}_0, t')] = \delta^4(x).
\]

Using it, we can integrate out the quarks (static quarks having unit functional determinant) in the \( O \) correlator, giving

\[
< O(t) O^\dagger(0) > = < \text{tr}Pe^i \int_C dx \mu A_\mu >,
\]

where \( C \) is a rectangular loop with sides of length \( t \) and \( L \). In deriving this equation a trivial factor of \( (\delta^3(0))^2 \) has been dropped by a renormalization of \( O \), making \( O \) dimensionless. Combining this result with the spectral decomposition, we have,

\[
< \text{tr}Pe^i \int_C dx \mu A_\mu > \sim_{t \to \infty} e^{-E_0(L)t}.
\]

Now the left-hand side has an obvious symmetry to exploit to get information about \( E_0(L) \), namely it is invariant under a 90-degree rotation of the rectangle, \( C \), combined with exchanging \( t \) and \( L \). This is just a consequence of 4-dimensional euclidean rotational symmetry. One complication in using this is that, since there are two length scales, \( L \gg \Lambda_{QCD}^{-1} \), it is not immediately obvious whether the above asymptotic behavior in \( t \) sets in for \( t \gg \Lambda_{QCD}^{-1} \), or much later, say for \( t \gg L \). I will show that it is the former case, if I make the extremely plausible technical assumption that \( O \) is a “good” interpolating operator, as defined below.

First note that in the \( t \to 0 \) limit of eqs. (6) and (9) we find,

\[
\int_{E_0(L)}^\infty dE \rho(E, L) = < \text{tr}1 > = N_c.
\]
The precise technical assumption I need to make is that there is some non-zero fraction $k$, such that for sufficiently large $L$,

$$\int_{E_0(L)}^{E_0(L)+\Lambda_{QCD}} dE \rho(E, L) > kN_c. \quad (12)$$

In words, $O^1|0>$ has a non-vanishing overlap with low-lying states of glue for large $L$. Given this assumption, two simple bounds follow,

$$\int_{E_0(L)}^{E_0(L)+2\Lambda_{QCD}} dE \rho(E, L) \, e^{-Et} \geq \int_{E_0(L)}^{E_0(L)+\Lambda_{QCD}} dE \rho(E, L) \, e^{-Et} \geq kN_c \, e^{[E_0(L)+\Lambda_{QCD}]t}, \quad (13)$$

$$\int_{E_0(L)+2\Lambda_{QCD}}^{\infty} dE \rho(E, L) \, e^{-Et} < (1-k)N_c \, e^{[E_0(L)+2\Lambda_{QCD}]t}. \quad (13)$$

For $t \gg \Lambda_{QCD}^{-1}$, we see that the states between $E_0$ and $E_0+2\Lambda_{QCD}$ dominate. Therefore,

$$\langle \text{tr} P e^{i \int_C dx_{\mu} A_\mu} \rangle \sim t \gg \Lambda_{QCD}^{-1} e^{-E_0(L)t}. \quad (14)$$

Since confinement implies $E_0 \gg 2\Lambda_{QCD}$ for large $L$, we finally arrive at

$$\langle \text{tr} P e^{i \int_C dx_{\mu} A_\mu} \rangle \sim E_{0}(L) \gg \Lambda_{QCD}^{-1} e^{-E_0(t)L}. \quad (15)$$

We can now straightforwardly use euclidean rotational symmetry to exchange “time” and “length” for the meson. That is, in euclidean space we can equally well choose the direction parallel to the $L$-length side of the rectangle to be “time” and interpret $t$ to be the inter-quark separation. Repeating all our steps we find,

$$\langle \text{tr} P e^{i \int_C dx_{\mu} A_\mu} \rangle \sim t, L \gg \Lambda_{QCD}^{-1} \Lambda_{QCD}^2 e^{-E_0(t)L}. \quad (16)$$

Comparing eqs. (15) and (16),

$$e^{-E_0(L)t} \sim t, L \gg \Lambda_{QCD}^{-1} \Lambda_{QCD}^2 e^{-E_0(t)L}. \quad (17)$$

This has a unique confining asymptotic solution,

$$E_0(L) \sim L \gg \Lambda_{QCD}^{-1} \Lambda_{QCD}^2 L. \quad (18)$$
which completes the proof that the static potential is linear at long distances.\(^6\) Eq. (15) then tells us that the expectation of a large rectangular Wilson loop is exponentially suppressed by its area. This is the standard statement of the area law of confinement.

### 3 “Symmetry-Breaking” by Stretched Glue

In the last section we obtained a stretched state of glue by pinning the ends down with infinitely massive quarks. These special endpoints are a nuisance, so in this section we will do away with them. (Of course phenomenologically dynamical quarks are attached at these ends, but this complication is deferred until the second paper.) Glue does not feel any other force, so to stretch it we will use spacetime itself. To this end, consider the spatial \(x\)-direction to be compact, with a large radius, \(R \gg \Lambda_{QCD}^{-1}\). Imagine taking the static quark ends of the meson of the last section to be further and further apart in the \(x\)-direction. As this is done, the energy of the glue in between grows linearly, as we saw. Finally, the static quarks come together after winding “around the universe”, since the \(x\)-direction is compact, with the lowest glue energy-eigenvalue being \(2\pi R\Lambda_{QCD}^2\), presumably. Let us now annihilate the quarks, leaving a state of pure glue wrapped around the \(x\)-axis. The glue cannot contract without breaking (and there are no light quarks so this is disallowed), so it remains conveniently stretched around the \(x\)-direction. Presumably its energy remains \(2\pi R\Lambda_{QCD}^2\). However now that it is free of the static quarks, it is completely free to move in the \(y\) and \(z\) directions. From the viewpoint of the effective \((2 + 1)\)-dimensional theory below the compactification scale, \(1/R\), the winding glue state looks like a point-like glueball, which is very heavy compared to \(\Lambda_{QCD}\).

To precisely state and prove the above intuitions, we proceed as follows. An interpolating operator for the winding glueball state is provided by a Polyakov loop, which is a winding Wilson loop operator parallel to the \(x\)-axis. It is given by

\[
P(y, z, t) \equiv \text{tr} Pe^{i \int_0^{2\pi R} dx A_x(x, y, z, t)}. \tag{19}
\]

The intermediate states in the spectral decomposition of the \(P\)-correlator

\(^6\)I have used “\(\Lambda_{QCD}\)” a few times to denote different quantities which are all parametrically set by the strong interaction scale. The different \(\mathcal{O}(1)\) prefactors are unimportant.
are the winding glue states of interest:

\[
\langle \mathcal{P}(y, z, t)\mathcal{P}^\dagger(0, 0, 0) \rangle = \int_{s_0(R)}^\infty ds \int \frac{dp_y dp_z}{2\sqrt{s + p_y^2 + p_z^2}} \rho(s, R) \times e^{i(p_y y + p_z z)} e^{-\sqrt{s + p_y^2 + p_z^2} t}.
\] (20)

The spectral density is a function of the (2+1)-dimensional mass-squared of the intermediate states, \(s\), since I have used translation symmetry in the \(y\) and \(z\) directions to sum the corresponding momentum eigenstates explicitly,

\[
\rho(s, R) \equiv \sum_a |\langle 0|\mathcal{P}^\dagger(0, 0, 0)|a, R \rangle|^2 \delta(s - m^2_a).
\] (21)

The fact that I have used (2+1)-dimensional notation is a matter of convenience. The formulas are exact and not low-energy approximations. The fact that the world is fundamentally (3+1)-dimensional should be reflected in the spectrum of glue states, which is as yet undetermined. If our earlier intuition is correct the correlator should obey the area law of confinement since all the intermediate states are stretched glueballs,

\[
\langle \mathcal{P}(y, z, t)\mathcal{P}^\dagger(0, 0, 0) \rangle \sim e^{-2\pi R \sqrt{t^2 + y^2 + z^2} \Lambda_{QCD}^{-1}}.
\] (22)

The proof follows by using the old trick of exchanging euclidean time with the \(x\)-direction. That is, we can interpret the compact direction as an imaginary “time” and take all the non-compact directions to be spatial. This puts us in the framework of finite-temperature field theory, with \(\beta \equiv 2\pi R\). The \(\mathcal{P}\)-correlator is now precisely the finite temperature partition functional for the mesonic glue states of the previous section, where the static quarks are separated by

\[
L \equiv \sqrt{t^2 + y^2 + z^2}.
\] (23)

By turning our heads by 90 degrees, the zero-temperature problem of winding glue has transformed into a finite-temperature problem for the static-quark mesons! The two Polyakov loops are now interpreted as the worldlines for the static quarks of the meson, which are of course color sources for glue. Thus,

\[
\langle \mathcal{P}(y, z, t)\mathcal{P}^\dagger(0, 0, 0) \rangle = \sum_r e^{-\beta E_r(L)}.
\] (24)
(I am distinguishing the states of glue stretched between static quarks from
the states of glue wrapped around a compact spatial direction, by denoting
them \(|r>|\) and \(|a>|\) respectively.) For temperatures much smaller than
\(\Lambda_{QCD}\), the system is dominated by the lowest-lying states. We found in the
last section that their energies scale like \(\Lambda_{QCD}^2 L\) for \(L \gg \Lambda_{QCD}^{-1}\). Therefore,
writing \(\beta\) and \(L\) in terms of the original variables, \(R, y, z, t\), we arrive at the
area law for Polyakov loops, eq. (22).

We can deduce the nature of the lowest-lying states \(|a, R>|\), interpolated
by \(P\), by Fourier transforming the area law with respect to \(y\) and \(z\) (but
not \(t\)). Since small and large \(y\) and \(z\) must be considered, we must take
\(t \gg \Lambda_{QCD}^{-1}\) in order to remain in the domain of validity of the area law. We
get,

\[
\langle \mathcal{P}(t) \mathcal{P}^\dagger(0) \rangle (p_y, p_z) \sim \int dydz \ e^{-m_0 \sqrt{t^2 + y^2 + z^2}}
\]

\[
\sim \int dydz \ e^{-m_0 (1 + \frac{y^2 + z^2}{2t^2})}
\]

\[
\sim e^{-(m_0 + \frac{p_y^2 + p_z^2}{2m_0})t},
\]

where,

\[m_0 \equiv 2\pi R \Lambda_{QCD}^2.\]  

Comparing this result with the spectral decomposition, we deduce that for
\(R \gg \Lambda_{QCD}^{-1}\) the lowest-lying states created by the Polyakov loop are the
\(|m_0; p_y, p_z>|\), with mass eigenvalue \(m_0\). The derivation has implicitly ex-

danded for \(p_y, p_z \ll m_0\), but by \((2 + 1)\)-dimensional relativistic invariance,
the exact dispersion relation is given by,

\[E_0 = \sqrt{m_0^2 + p_y^2 + p_z^2}.\]  

The existence of the \(|m_0; p_y, p_z>|\) states is the phenomenon of “almost”
spontaneous symmetry-breaking of transverse translations referred to in the
introduction, in the sense that states with fixed momenta \((p_y, p_z)\), become
degenerate with the winding ground state \(|m_0, p_y = p_z = 0>|\), in the \(R \rightarrow \infty\)
limit. Strictly there is no symmetry-breaking\(^4\) since \(|m_0, p_y = p_z = 0>|\) is
translation invariant, but the near degeneracy will suffice to demonstrate the
presence of gapless “Nambu-Goldstone” modes with non-zero \(x\)-momentum,
by means closely analogous to the proof of Goldstone’s theorem. This is done
in the next section.

\(^4\)This is true even in the \(R \rightarrow \infty\) limit \([5]\).
Before going to the formal derivation it is useful to build some intuition. We can summarize our result thus far by writing an effective theory for the winding glue states we have found,

\[ S = 2\pi RA_{QCD}^2 \int dt \sqrt{1 + (\partial_t y)^2 + (\partial_t z)^2}. \]  

(28)

For momenta below the compactification scale \(1/R\), this effective theory looks very reasonable. The world is effectively \((2 + 1)\)-dimensional and we see a glueball state described by the action for a relativistic point-particle of mass \(m_0 = 2\pi R A_{QCD}^2\). There is nothing strange about the presence of the ultraviolet scale \(R\) in the lagrangian. However, our result is also valid for \(p_y, p_z \gg 1/R\), and therefore must somehow be part of a fully \((3 + 1)\)-dimensional theory. From this perspective eq. (28) seems more disturbing. The theory appears not to be local in \(x\). Instead it depends sensitively on the infrared length scale \(R\). The only cure for this state of affairs is if there are other modes which must be included in the effective theory. We see that this is allowed, because the \(|m_0; p_y, p_z>\) are necessarily \(x\)-translation invariant since they are created from the vacuum by \(\mathcal{P}\), which is invariant by the cyclic invariance of traces. There may be other gapless (relative to \(m_0\)) states as \(R \to \infty\), which carry non-zero \(x\)-momentum, and therefore are not created by \(\mathcal{P}\), but are nevertheless part of the full theory. They would presumably be created by Polyakov loops with “wiggles”, to break \(x\)-translation invariance.

A naive guess is that the point-particle state of the effective \((2 + 1)\)-dimensional theory remains a point-particle state in \(3 + 1\) dimensions. However, this does not work; as \(R \to \infty\) relativistic invariance gives the effective theory for such a state the standard form,

\[ S = m \int dt \sqrt{1 + (\partial_t x)^2 + (\partial_t y)^2 + (\partial_t z)^2}, \]  

(29)

which cannot match with eq. (28) without reintroducing large sensitivity to \(R\). Contrast this with the string solution,

\[ S = \Lambda_{QCD}^2 \int dt \int_0^{2\pi R} dx \sqrt{1 + (\partial_t y)^2 + (\partial_x y)^2 + (\partial_t z)^2 + (\partial_x z)^2}. \]  

(30)

This description of a string, \(y(t, x), z(t, x)\), is insensitive to large \(R\), and satisfies \((3 + 1)\)-dimensional Poincare invariance (see section 5).
4 The Effective String Emerges

The main plausible physical assumption in this section and the next is that local probes of the $|m_0>$ states are insensitive to the radius of the universe, $R$. This is closely analogous to the fact that accelerator experiments located on earth (local probes of the vacuum state) are insensitive to the size of our universe. Our first goal is to demonstrate the presence of NG states with non-zero eigenvalues of $P_x$, nearly degenerate with the $|m_0; p_y, p_z>$. The symmetries we will use are the spatial translations, with the conserved momentum generators,

$$P_i(t) = \int_0^{2\pi R} dx \int dy \int dz \, T_{ti}(t, \vec{x}),$$

(31)

where $T_{\mu\nu}$ is the energy-momentum tensor for the Yang-Mills theory. Let us begin with a $|m_0; p_y, p_z>$ state, with $p_y \neq 0$ say,

$$P_y(t)|m_0; p_y, p_z> = p_y|m_0; p_y, p_z> \neq 0.$$

(32)

By continuity, for small enough $x$-momentum, $q_x$, we have,

$$\int_0^{2\pi R} dx \int dy \int dz \, e^{iq_x x} \, T_{ty}(t, \vec{x})|m_0; p_y, p_z> \neq 0.$$

(33)

The operator on the left-hand side has injected momentum $q_x$ into the $|m_0>$ state, so the result has non-zero $x$-momentum and therefore has no overlap with any of the $|m_0>$ states, which were shown earlier to be $x$-translation invariant. On the other hand for small $q_x$, by continuity again, eq. (33) must have a non-vanishing overlap with some energy-eigenstate whose energy is near $\sqrt{m_0^2 + p_y^2 + p_z^2}$. These gapless (as $R \to \infty$) states with non-zero $x$-momentum are the NG modes we are seeking.

The NG states carrying $x$-momentum cannot be obtained by Lorentz transformations (for example an $x$-boost) from the $|m_0; p_y, p_z>$ states. The reason is that these modes are local in $x$, and by the assumption at the beginning of this section they should be insensitive to large $R$, so that they are constrained by (3 + 1)-dimensional Poincare invariance. Therefore if these states and the $|m_0; p_y, p_z>$ form a single multiplet they must have a common mass, $m$, which is insensitive to large $R$. This contradicts the strong $R$-dependence in $m_0$. This argument just rephrases the case of eq. (29) disposed of in the previous section. We conclude that applying the operator $\int_0^{2\pi R} dx \int dy \int dz \, e^{iq_x x} \, T_{ty}(t, \vec{x})$ to the $|m_0; p_y p_z>$ creates new Poincare
multiplets of gapless modes. Similarly, we can replace “y” by “z” in the above manipulations to get NG modes related to z-translations.

It is very important to understand why we cannot inject momenta $q_y, q_z$ in creating these NG modes. Consider the state

$$\int_0^{2\pi R} dx \int dy \int dz \ e^{iq_y y} T_{ty}(t, \vec{x}) |m_0; p_y, p_z \rangle \neq 0, \quad (34)$$

which also reduces to eq. (32), as $q_y \to 0$. There is no reason why this state should not be a superposition of $|m_0; p_y + q_y, p_z \rangle$ plus other energy eigenstates lying above some finite gap. That is, there is no Goldstone theorem for new gapless modes carrying $q_y, q_z$ to be produced.

The properties we have found are the qualitative hallmarks of the spectrum of a (single) string wrapped around the x-axis: energy eigenstates are labelled by the transverse momenta of the whole string, $p_y, p_z$, while further NG quanta carrying only x-momentum can be added, in correspondence to the y and z vibrational degrees of freedom along the string. The precise form of the string spectrum will be deduced in the next section.

The derivation presented here is similar to Goldstone’s theorem for internal symmetries, with the broken internal symmetry charge and current replaced by the momentum generators, $P_y, P_z$, and $T_{\mu \nu}$ respectively, and with the non-invariant degenerate vacuum states replaced by $|m_0, p_y, p_z \rangle$. The main difference is that in the case of internal symmetry breaking the ground state is invariant under all spacetime symmetries, so that there are gapless NG modes with all components of momentum, corresponding to massless point-particle excitations.

5 The String Chiral Lagrangian

It is more convenient in constructing the effective lagrangian to translate our findings entirely into position space. To this end, note that the $|m_0; p_y, p_z \rangle$ states can be written,

$$|m_0; p_y, p_z \rangle = \int \int dydz \ e^{ip_y y + ip_z z} |m_0; y, z \rangle, \quad (35)$$

where the $|m_0; y, z \rangle$ are the conjugate position eigenstates for y and z, but remain x-translation invariant. Note that they are not energy eigenstates, though by taking $R$, and hence $m_0$, to be large we can make arbitrarily
The derivation of the last section can therefore be repeated by replacing $|m_0; p_y, p_z>$ by such $y, z$-localized wavepackets. Since $T_{ty}$ is the charge density for local $y$-translations (and similarly for $z$), we see that acting on the wavepacket localized at $y_0$ (for all $x$) with $\int_0^{2\pi R} dx \int dy \int dz \ e^{iq_xx} T_{ty}(t, \vec{x})$ creates a state with $y$ localized in an $x$-dependent fashion. These are the NG states of the last section. We can therefore denote the gapless NG states by fields $y(t, x), z(t, x)$, to describe the $x$-dependent localization in $y$ and $z$, at any time $t$. This is quite analogous to the interpretation of NG fields for internal symmetry breaking.

Thus the NG fields, $y(t, x), z(t, x)$, define a “world-sheet” surface in spacetime. This simple physical interpretation allows us to straightforwardly determine their transformations under $(3+1)$-dimensional Poincare symmetry. Under the Poincare subgroup associated with the $x$ and $t$ directions, they transform as scalar fields. Under $y$ and $z$ translations they shift, just as one would expect of NG modes associated with the spontaneous breaking of these symmetries. Under general Lorentz transformations $\Lambda^\nu_\mu$, the vector $(t, x, y(t, x), z(t, x))$ transforms to $(t', x', y'(t', x'), z'(t', x'))$ by multiplication by the tensor $\Lambda$. These results can also be derived more laboriously from the construction of the NG modes in terms of the energy-momentum tensor $T_{\mu\nu}$ in the last section, and using the $T_{\mu\nu}$ current algebra. What is noteworthy is that, though the Poincare algebra implies that the non-linear realization of the $y, z$ translations necessarily results in a non-linear realization of some of the Lorentz transformations, we only have NG modes corresponding to $P_y, P_z$. The reason is that the Poincare algebra has fewer independent currents than it has generators: $T_{\mu\nu}$ is used to construct both the translation and Lorentz generators.

The spacetime symmetry gives the central constraint on the construction of the effective lagrangian. The constraint is that the action must be constructed out of geometric invariants of the embedding of the world-sheet surface in spacetime. The simplest and most important of these is the sur-

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1. This is just the approach to the “static limit” $m_0 \to \infty$ in $2 + 1$ dimensions, so-called because in this limit position eigenstates are stationary states.

2. Of course the finite $x$ radius explicitly breaks this symmetry, but this is not significant for the local couplings of the NG fields, for large $R$. That is, $1/R$ is a “soft” explicit breaking of Poincare symmetry. Again this is completely analogous to the fact that Lorentz symmetry holds good in terrestrial experiments despite perhaps being invalid for our universe globally.
face area,
\[ S = \Lambda_{QCD}^2 \int dt \int_0^{2\pi R} dx \sqrt{1 + (\partial_t y)^2 + (\partial_x y)^2 + (\partial_t z)^2 + (\partial_x z)^2}, \]  
(36)

which is just the Nambu-Goto string action. Other invariants involve more
derivatives and are consequently less important at low energies in quantizing
about the classical string ground state where \( y, z \) are constants. Now that
we have derived the existence of the effective string theory within QCD, the
considerations of ref. [3] apply to its quantization.

Expanding the Nambu-Goto term in derivatives and continuing to Minkowski
space gives,
\[ S = \Lambda_{QCD}^2 \int dt \int_0^{2\pi R} dx \{ -1 + \frac{1}{2} [(\partial_t y)^2 - (\partial_x y)^2 + (\partial_t z)^2 - (\partial_x z)^2] + \ldots \}, \]  
(37)

which makes quantization straightforward (free \((1+1)\)-dimensional scalar
field theory). In the sector with \( y \) and \( z \) independent of \( x \) we recover the
spectrum of the \(|m_0>\) states found in section 3. Thus the string action eq.
(36) summarizes all the information we have learned in previous sections
about the lowest-lying winding glue states, plus it enforces the constraints
of Poincare invariance.

In the formalism of Volkov for non-linear realizations of spacetime sym-
metries [7][13] applied to the breaking of \( y, z \)-translations, \( y(x, t) \) and \( z(x, t) \)
provide the minimal NG field content. In ref. [14], a similar construction to
that in this section was used to describe effective superstrings that emerge
in the supersymmetric abelian Higgs model.

6 Discussion

In this paper I have paralleled the well-known route to deriving the existence
and behavior of Nambu-Goldstone (NG) modes associated to internal sym-
metries: dynamics \(\rightarrow\) (almost) degenerate (winding) ground states which are
not invariant under the symmetry \(\rightarrow\) “Goldstone theorem” proving the exis-
tence of gapless NG modes \(\rightarrow\) chiral lagrangian derivative expansion for the
NG modes, constrained by the non-linear realization of the full dynamical
symmetry. The NG modes are strings, not particles, because the “broken”
symmetries are spacetime symmetries, not internal symmetries.

Since this derivation of effective strings relied on \( R \gg \Lambda_{QCD}^{-1} \), it is valid
as a theory of long strings, Furthermore, since the general terms in the string
effective lagrangian are geometrical invariants of the world-sheet embedding in spacetime, the Nambu-Goto term (surface area) only dominates for world-sheets which are locally flat on $\Lambda_{QCD}^{-1}$ length scales. Thus there would appear to be a clear domain of validity for the effective theory. There is however one possible worry. An effective theory based only on the fact of spontaneous symmetry breaking, includes the minimal NG fields, but does not inform us about the presence or absence of other massless modes. For the case of hadronic strings I have only included the minimal NG fields in the chiral lagrangian. The more general case when there are other gapless modes can be treated by the formalism of Volkov. I have not yet resolved whether these extra modes will always decouple from the string modes at low energies, analogously to the case of internal symmetry breaking.

The remaining problem is that in this paper I only treated pure Yang-Mills theory, describing a closed string winding around the compact $x$-direction, whereas phenomenologically we want to consider mesons and baryons. The important observation is that the NG string modes are local excitations of the stretched glue, and therefore should be insensitive to boundary conditions if they are far-removed. Thus if a meson is stretched by its angular momentum, the glue in between the quarks should still be described by effective strings. The region near the quark ends however requires special consideration. One further complication when dynamical quarks are introduced is that strictly there is no confinement but rather screening, due to quark pair production. This can be avoided by using the large-$N_c$ limit to suppress the pair production. Finite-$N_c$ corrections will then correspond to the splitting of the effective strings. In a second paper, I will treat mesons in this manner, taking care to remain in the domain of validity of the effective string theory. Baryons are necessarily trickier, though there are some indications that their string picture may simplify in the large-$N_c$ limit. Their treatment is left for the future.

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