On the integrability of Hamiltonian 1:2:2 resonance

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Abstract We study the integrability of the Hamiltonian normal form of 1:2:2 resonance. It is known that this normal form truncated to order three is integrable. The truncated to order four normal form contains many parameters. For a generic choice of parameters in the normal form up to order four, we carry on non-integrability analysis, based on the Morales–Ramis theory using only first variational equations along certain particular solutions. The non-integrability obtained by algebraic proofs produces dynamics illustrated by some numerical experiments. We also isolate a non-trivial case of integrability.

Keywords Hamiltonian 1:2:2 resonance · Liouville integrability · Differential Galois groups · Morales–Ramis theory

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1 Introduction

For an analytic Hamiltonian $H(q, p)$ with an equilibrium at the origin, we have the following expansion

$$H = H_2 + H_3 + H_4 + \cdots,$$

where $H_2 = \sum \omega_j (q_j^2 + p_j^2)$, $\omega_j > 0$, that is, $H_2$ is a positive-definite form and $H_j$ are homogeneous of degree $j$.

It is said that the frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ satisfies a resonant relation if there exists a vector $k = (k_1, \ldots, k_n)$, $k_j \in \mathbb{Z}$, such that $(\omega, k) = \sum k_j \omega_j = 0$, $|k| = \sum |k_j|$ being the order of the resonance.

There exists a procedure called normalization, which simplifies the Hamiltonian function in a neighborhood of the equilibrium and is achieved by means of canonical near-identity transformations [1,22,27]. When resonances appear, this simplified Hamiltonian is called Birkhoff–Gustavson normal form. To study the behavior of a given Hamiltonian system near the equilibrium, one usually considers the normal form truncated to some order

$$\overline{H} = H_2 + H_3 + \cdots + H_m.$$  \hspace{1cm} (2)

Note that by construction $\{\overline{H}_j, H_2\} = 0$ ($\{\}$ being the Poisson bracket). This means that the truncated resonant normal form has at least two integrals—$\overline{H}$ and $H_2$.

The first integrals for the resonant normal form $\tilde{H}$ are approximate integrals for the original system, see Verhulst [27] for the precise statements. If the truncated normal form happens to be integrable, then the original Hamiltonian system is called near-integrable. A recent review of some known results on integrability of the Hamiltonian normal forms can be found in [28].

In this paper, we study the integrability of the semi-simple Hamiltonian 1:2:2 resonance. The classical
water molecule model and concrete models of coupled rigid bodies serve as examples which are described by the Hamiltonian systems in 1:2:2 resonance, see Haller [7].

When studying normal forms, it is natural to introduce the complex coordinates

$$z_j = q_j + ip_j, \quad \zeta_j = q_j - ip_j.$$  

The generating functions of the algebra of the elements which are Poisson-commuting with

$$H_2 = (q_1^2 + p_1^2) + 2(q_2^2 + p_2^2) + 2(q_3^2 + p_3^2) = z_1\xi_1 + 2z_2\xi_2 + 2z_3\xi_3$$

can be found, for example, in [8, 22]:

$$z_j\xi_j, \quad j = 1, 2, 3, \quad z_2\xi_3, \quad z_2\xi_3, \quad z_2\xi_3, \quad z_2\xi_3.$$  

Then, the Hamiltonian normal form up to order four becomes

$$\overline{H} = H_2 + \overline{H}_3 + \overline{H}_4,$$  

where $H_2$ is as above and

$$\overline{H}_3 = az_1^2\xi_2 + az_2^2\xi_2 + az_3^2\xi_3 + b\xi_2^2\xi_3,$$

$$\overline{H}_4 = c_1(z_1\xi_1)^2 + c_2z_1\xi_2z_2 + c_3z_1\xi_3z_3 + c_4z_2\xi_2z_3 + c_5(z_2\xi_2)^2 + c_6(z_3\xi_3)^2 + d(z_2\xi_3)^2 + d(z_2\xi_3)^2 + (z_1\xi_1)(ez_2\xi_3 + e\xi_2z_3) + (z_2\xi_2)(z_2\xi_3 + \bar{e}\xi_2z_3) + (z_3\xi_3)(z_2\xi_3 + \bar{e}\xi_2z_3).$$  

For the normal form of the 1:2:2 resonance, normalized to degree three

$$\overline{H} = H_2 + \overline{H}_3$$  

symplectic coordinate changes allow to make the coefficients in $\overline{H}_3$ real and additionally to achieve $b = 0$ [11, 25]. This makes the cubic normal form integrable (see also [24] where results on the integrability for other first-order resonances are given). In particular, detailed geometric analysis based on 1:2 resonance is given in [6, 7, 11]. To see what happens in the normal form normalized to order four, van der Aa and Verhulst [25] consider a particular potential problem (we slightly change their notations and do certain rescaling)

$$H = H_2 + U(q_1, q_2, q_3)$$

It turns out that certain terms with the coefficient $\beta_3$ in $\overline{H}_4$ destroy the intrinsic symmetry of (6), which in turn causes the loss of the extra integral.

More geometric approach is taken by Haller and Wiggins [6, 7]. For a class of resonant Hamiltonian normal forms (1:2:2 among them) normalized to degree four, assuming that the coefficients in $\overline{H}_4$ are small enough, they prove the following results. First, most invariant 3-tori of the cubic normal form (6) survive on all but finite number of energy surfaces. Second, there exist whiskered 2-tori which intersect in a non-trivial way giving rise to multi-pulse homoclinic and heteroclinic connections. The existence of these whiskered 2-tori is an indication for nonexistence of an additional analytic integral and can be considered as a “geometric” source of non-integrability.

To study the integrability of the normal form (4), (5) we adopt more algebraic approach. To overcome the difficulty of dealing with too many parameters in the normal form, some assumptions on them are in order.

Firstly, we assume that at least one of $(a, b)$ is different from zero, say $a \neq 0, b = 0$ (see the above explanations). On the contrary, if $a = b = 0$, i.e., there is no cubic part, there exists an additional integral $I_1 := z_1\xi_1 = p_1^2 + q_1^2$ and the normal form truncated
to order 4 is integrable. In particular, the normal form

\[
\mathcal{H} = \frac{1}{2}(p_1^2 + q_1^2) + \frac{2}{2}(q_2^2 + p_2^2 + q_3^2 + p_3^2) \\
+ a_1(p_1^2 + q_1^2)^2 + a_2(p_2^2 + q_2^3 + p_3^2 + q_3^2)^2 \\
+ a_3(p_1^2 + q_1^2)(p_2^2 + q_2^3 + p_3^2 + q_3^2)^2 \\
+ a_4(p_3q_2 - q_3p_2)^2
\]

(9)
is integrable with the following quadratic first integrals

\[
I_1 := p_1^2 + q_1^2, \quad F_1 := p_2^2 + q_2^2 + p_3^2 + q_3^2, \\
F_2 := p_3q_2 - q_3p_2.
\]

(10)

For this case, the action-angle variables are introduced in a similar way as in [21] and the KAM theory conditions can be verified upon certain restrictions on the coefficients \(a_j\) of the normal form (9).

Secondly, the terms \(c_j\) in (4), (5) are called self-interaction terms [22]. They are not expected to be obstacles to integrability (see example (9)), so \(c_5 = c_6 = 0\) are taken for simplicity, but we keep \(c_1 \neq 0\) to verify this assumption.

Further, a time-dependent canonical transformation as in [3,5] is performed to eliminate the quadratic part of (4). Let us recall it briefly. With the so-called action-angle variables

\[
q_j = \sqrt{2I_j} \sin \varphi_j, \quad p_j = \sqrt{2I_j} \cos \varphi_j, \quad j = 1, 2, 3,
\]

the truncated normal form (4) reads

\[
\mathcal{H} = I_1 + 2I_2 + 2I_3 + \mathcal{H}_3 + \mathcal{H}_4.
\]

Now, we introduce the following time-dependent canonical transformation

\[
I_j \rightarrow I'_j, \quad \phi_j \rightarrow \phi'_j + t, \\
\mathcal{H} \rightarrow \mathcal{H}', \quad \varphi_j \rightarrow \varphi'_j + 2t, \\
\varphi_j \rightarrow \varphi'_j + 2t.
\]

Then, \(\mathcal{H}'\) is merely \(\mathcal{H}_3 + \mathcal{H}_4\). We drop the primes and return to Cartesian coordinates.

Finally, influenced by the particular example (8), we consider the following generic case of the normal form (4), (5) which in Cartesian coordinates reads

\[
\mathcal{H} = a(q_2(q_1^2 - p_1^2) + 2q_1p_1p_2) + \frac{1}{2}(c_1(q_1^2 + p_1^2))^2 \\
+ c_2(q_1^2 + p_1^2)(q_2^2 + p_2^2) + c_3(q_1^2 + p_1^2)(q_3^2 + p_3^2) \\
+ c_4(q_1^2 + p_1^2)(q_3^2 + p_3^2) \\
+ \frac{\alpha}{2}(q_2q_3 + p_2p_3)^2 - (q_2q_3 - q_2p_3)^2 \\
+ \frac{\beta}{2}(q_2q_3 + p_2p_3)(q_1^2 + p_1^2).
\]

(11)

Our primary goal is to find some cases of integrability, that is, to find an additional integral \(G\) that is independent of \(\mathcal{H}, H_2\). Notice that if such integral \(G\) exists, it should be a combination of the generators (3) of the normal form since \(\{H_2, G\} = 0\).

Meanwhile, we want rigorously to establish domains of the parameters for which there is no such first integral. For all these reasons, the following observation is important.

Remark 1 It is clear that for this normal form, there exists an extra integral \(I_3 = p_3^2 + q_3^2\) when \(\alpha = \beta = 0\). Hence, the coefficients \(c_j, j = 1, \ldots, 4\) would be an obstacle for integrability only in certain combination with \(\alpha \neq 0, \beta \neq 0\).

Further, we will use the Hamilton’s equations corresponding to (11) repeatedly. That is why we write them down.

\[
\dot{q}_1 = 2a(q_1p_2 - q_2p_1) + p_1\left[2c_1(q_1^2 + p_1^2) \\
+ c_2(q_1^2 + p_1^2)(q_2^2 + p_2^2) + c_3(q_1^2 + p_1^2)(q_3^2 + p_3^2) \\
+ c_4(q_1^2 + p_1^2)(q_3^2 + p_3^2)\right],
\]

\[
\dot{p}_1 = -2a(q_1q_2 + p_1p_2) - q_1\left[2c_1(q_1^2 + p_1^2) \\
+ c_2(q_1^2 + p_1^2)(q_2^2 + p_2^2) + c_3(q_1^2 + p_1^2)(q_3^2 + p_3^2) + c_4(q_1^2 + p_1^2)(q_3^2 + p_3^2)\right],
\]

\[
\dot{q}_2 = 2aq_1p_1 + q_2\left[c_1(q_1^2 + p_1^2) + c_2(q_2^2 + p_2^2) \\
+ c_3(q_2^2 + p_2^2)(q_3^2 + p_3^2) + c_4(q_2^2 + p_2^2)(q_3^2 + p_3^2)\right] \\
+ \alpha[p_3(p_3q_3 + p_2p_3) - q_3(p_3q_3 - p_2p_3)] \\
- \frac{\beta}{2}p_3(q_1^2 + p_1^2),
\]

\[
\dot{p}_2 = -a(q_1^2 - p_1^2) - q_2\left[c_2(q_1^2 + p_1^2) + c_3(q_2^2 + p_2^2) + c_4(q_2^2 + p_2^2)\right] \\
- \alpha[p_3(q_3q_3 + p_2p_3) + p_3(p_3q_3 - q_2p_3)] \\
- \frac{\beta}{2}q_3(q_1^2 + p_1^2),
\]

\[
\dot{q}_3 = p_3\left[c_2(q_1^2 + p_1^2) + c_3(q_2^2 + p_2^2) + c_4(q_2^2 + p_2^2)\right] \\
+ \alpha[p_2(p_2q_3 + p_2p_3) + q_2(p_2q_3 - q_2p_3)] \\
+ \frac{\beta}{2}p_2(q_1^2 + p_1^2),
\]

\[
\dot{p}_3 = -q_3\left[c_2(q_1^2 + p_1^2) + c_3(q_2^2 + p_2^2) + c_4(q_2^2 + p_2^2)\right] \\
- \alpha[p_2(q_2q_3 + p_2p_3) - p_2(p_2q_3 - q_2p_3)]
\]
Our first result is the following

**Theorem 1** Assume \( \alpha \neq 0, \alpha \neq \pm c_4 \) and \( i[(c_3 - c_2)(A^2 + B^2) + 2iAB\beta] \notin \mathbb{Z} \), where

\[
B^2 = \frac{\sqrt{k_*}}{c_4(k_* + 1) + \alpha(3k_* - 1)}, \quad A^2 = k_*B^2
\]

and \( k_* \) is any solution of the equation

\[
k^2(c_4 - \alpha) + 2k(c_4 + 3\alpha) + (c_4 - \alpha) = 0.
\]

Then, the system (12) is non-integrable.

Then, we study the cases \( i[(c_3 - c_2)(A^2 + B^2) + 2iAB\beta] \in \mathbb{Z} \) numerically and \( \alpha = \pm c_4 \) analytically and numerically. Again the Hamiltonian system (12) turns out to be non-integrable. This suggests that the condition \( \alpha \neq 0 \) is the true reason for the non-integrability. Indeed, we have

**Theorem 2** Assume \( \alpha \neq 0 \) and \( c_1 = c_2 = c_3 = c_4 = \beta = 0 \). Then, the system (12) is non-integrable.

It remains to deal with the case \( \alpha = 0 \) which somehow is more definitive.

**Theorem 3** Suppose \( \alpha = 0, \beta \neq 0 \). Then, the Hamiltonian system (12) is integrable if and only if

\[
c_2 = 4c_1, \quad c_3 = c_4 = 0
\]

with an additional integral

\[
G = 4a^2(q_3^2 + p_3^2) - a\beta[q_3(q_1^2 - p_1^2) + 2q_1p_1p_3]
+ \frac{\beta}{16}(q_1^2 + p_1^2)^2 + \beta^2(q_2q_3 + p_2p_3)^2
+ 4c_1\beta(q_2q_3 + p_2p_3)[(q_1^2 + p_1^2) + 2(q_2^2 + p_2^2)]
+ 4c_1^2[(q_1^2 + p_1^2) + 2(q_2^2 + p_2^2)]^2.
\]

Obviously, this integral descends to \( I_3 \) as \( \beta = 0 \).

For the proof of these results, we use mainly the Morales–Ramis result for the integrability of Hamiltonian systems, based on studying of the differential Galois group of the first variational equations along certain particular solution.

The paper is organized as follows. In Sect. 2, we recall some fact related to integrability of Hamiltonian systems in the complex domain, and in particular the methods of Ziglin [29,30], Morales–Ramis [16] and Lyapounov [12]. Theorems 1, 2 and 3 are proved in the next sections. We finish with some remarks about the possible application of this approach to the study of other Hamiltonian resonances.

### 2 Theoretical background

In this section, we summarize some notions and results related to Ziglin–Morales–Ramis theory which deals with integrability in complex domains.

We are given a Hamiltonian system

\[
\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in M
\]

corresponding to an analytic Hamiltonian \( H \), defined on the complex \( 2n \)-dimensional manifold \( M \). We call such Hamiltonian system integrable in sense of Liouville if there exist \( n \) independent (almost everywhere) first integrals in involution.

In most of the integrable Hamiltonian systems, the known first integrals when are considered in the complex domain are holomorphic or meromorphic functions, that is, single-valued functions. It was established that branching of solutions of Hamiltonian systems in complex time plane is an obstruction to the existence of new first integrals (see, e.g., Ziglin [29,30] and Kozlov [10]). We will return to that shortly after recalling some notions.

Suppose the system (15) has a non-equilibrium solution \( \Phi(t) \). Denote by \( \Gamma \) its phase curve. Along this solution, we can write the variational equations (VE)

\[
\dot{\xi} = DX_H(\Phi(t))\xi, \quad \xi \in T\Gamma M.
\]

The first integral \( H \) gives rise to a linear integral \( dH \) of the variational equations. Using the integral \( dH \), we can reduce the variational equations. Consider the normal bundle of \( \Gamma, F := T\Gamma M/T M \), and let \( \pi : T\Gamma M \rightarrow F \) be the natural projection. The system of equations (16) define a system of equations on \( F \)

\[
\eta = \pi_*(DX_H(\Phi(t))(\pi^{-1}\eta)), \quad \eta \in F.
\]

which is called the normal variational equations (NVEs). Each meromorphic first integral of the Hamiltonian system (15) in the neighborhood of the curve \( \Gamma \) corresponds to a meromorphic first integral of (NVE) [29,30]. Hence, the problem of complete integrability of the Hamiltonian system (15) reduces to the study of integrability of the linear system (17) (or 16).
Consider such a linear non-autonomous system
\[ \dot{\xi} = A(t)\xi, \quad \xi \in \mathbb{C}^n, \] (18)
with \( t \) defined on some Riemann surface \( \Gamma \). From the existence theorem, there is a fundamental matrix solution \( \mathcal{S}(t) \), analytic in a vicinity of any non-singular point \( t_0 \). The continuation of \( \mathcal{S}(t) \) along non-trivial loops on \( \Gamma \) defines a linear automorphism of the vector space of all solutions analytic in the vicinity of \( t_0 \), called the monodromy transformation. This linear automorphism \( \Delta_\gamma \) associated with a loop \( \gamma \in \pi_1(\Gamma, t_0) \) corresponds to multiplication of \( \mathcal{S}(t) \) from the right by a constant matrix \( M_\gamma \), called monodromy matrix
\[ \Delta_\gamma \mathcal{S}(t) = \mathcal{S}(t)M_\gamma. \]
The set of these matrices forms the monodromy group [26,31].

Next, we recall briefly the necessary notions and results from the differential Galois theory in order to understand the applications to the integrability of Hamiltonian systems. The detailed statements and proofs can be found in [13,15,23,26].

Denote the coefficient field in (18) by \( K \). A differential field \( K \) is a field with a derivation \( \partial = \partial_t \), i.e., an additive mapping satisfying Leibnitz rule. A differential automorphism of \( K \) is an automorphism commuting with the derivation.

Let \( \xi_{ij} \) be the elements of the fundamental matrix \( \mathcal{S}(t) \). Let \( F(\xi_{ij}) \) be the extension of \( K \) generated by \( K \) and \( \xi_{ij} \)—a differential field. This extension is called Picard–Vessiot extension. The Galois group \( G := Gal(\mathbb{F}/K) \) is defined to be the group of all differential automorphisms of \( \mathbb{F} \) leaving the elements of \( K \) fixed. The Galois group is an algebraic group. It has a unique connected component \( G^0 \) which contains the identity and which is a normal subgroup of finite index. The Galois group \( G \) can be represented as an algebraic linear subgroup of \( GL(n, \mathbb{C}) \) by
\[ \sigma \mathcal{S}(t) = \mathcal{S}(t)R_\sigma, \]
where \( \sigma \in G \) and \( R_\sigma \in GL(n, \mathbb{C}) \).

One should note that by its definition, the monodromy group is contained in the differential Galois group of the corresponding system.

In 1982, Ziglin [29,30] obtained necessary conditions for integrability of complex-analytical Hamiltonian systems by investigating the monodromy group of normal variational equations: The monodromy group manifests the branching of solutions of (NVE). Notice that Ziglin’s result does not assume that the existing \( n \) independent first integrals are in involution.

A more general approach based on the differential Galois theory was taken a decade later. The solutions of (16) define an extension \( \mathbb{F}_1 \) of the coefficient field \( K \) of (VE). This naturally defines a differential Galois group \( G = Gal(\mathbb{F}_1/K) \). Then, a fundamental result of the Morales–Ramis theory is

**Theorem 4** (Morales-Ruiz–Ramis [16]) Suppose that a Hamiltonian system has \( n \) meromorphic first integrals in involution. Then, the identity component \( G^0 \) of the Galois group \( G = Gal(\mathbb{F}_1/K) \) is abelian.

Now, we recall a method that preceded the Ziglin’s theory. In 1894, Lyapounov proposed a method for studying integrability in complex domains based on the following observation.

If the linear system (16) has a multi-valued solution, then the same holds for the nonlinear system (15) (see [10,12]) for the proof.

Hence, if the general solution of (VE) is not single-valued, then no additional analytic first integral for the nonlinear system exists.

Using this argument Lyapounov proved that the only cases of the system describing the motion of a rigid body around a fixed point with a general solution single-valued over entire complex time plane are the four integrable cases: isotropic, Euler, Lagrange and Kovalevskaia (see the extended discussion in [12,17]). Notice that for the above example considered by Lyapounov, the Galois group (and hence, the monodromy group) of the corresponding variational equations is abelian. Further, we find ourselves in a similar situation.

### 3 Proof of Theorem 1

In this section, we prove Theorem 1. The following assertion is immediate

**Proposition 1** Suppose \( \alpha \neq 0 \) and \( \alpha \neq \pm c_4 \). Then, the system (12) has a particular solution of the form
\[ \Gamma_1 : q_1 = p_1 = 0, q_3 = A\varphi(t), p_3 = B\varphi(t), \]
\[ q_2 = iB\varphi(t), p_2 = iA\varphi(t), \] (19)
where \( \varphi \) satisfies
\[
\dot{\varphi} = -\varphi^3, \quad \varphi(t) = \frac{1}{2t^2}.
\]
(20)
k* is a solution of the equation
\[
k^2(c^4 - \alpha) + 2k(c_4 + 3\alpha) + (c_4 - \alpha) = 0
\]
and
\[
B^2 = \frac{\sqrt{k*}}{c_4(k_1 + 1) + 3k_1 - 1}, \quad A^2 = k_3 B^2.
\]
(22)

Denote \( dq_j = \xi_j, dp_j = \eta_j, j = 1, 2, 3 \). Then, normal variational equation (NVE) along the solution (19) is written in \( \xi_1, \eta_1 \) variables
\[
\dot{\xi}_1 = \frac{\sqrt{3iA}}{\sqrt{\tau}} \xi_1 + \left[ \frac{(c_3 - c_2)(A^2 + B^2) + 2iAB^2}{2\tau} - \frac{\sqrt{3iA}}{\sqrt{\tau}} \right] \eta_1
\]
\[
\dot{\eta}_1 = -\left[ \frac{(c_3 - c_2)(A^2 + B^2) + 2iAB^2}{2\tau} + \frac{\sqrt{3iA}}{\sqrt{\tau}} \right] \xi_1 - \frac{\sqrt{3iA}}{\sqrt{\tau}} \eta_1
\]
(23)

Further, we put \( t = \tau^2 \). (In fact, this transformation is a two-branched covering mapping, which in general changes the differential Galois group, but preserves the identity component.) Denote \( Q := 2\sqrt{2}iaA, P := (c_3 - c_2)(A^2 + B^2) + 2iAB^2 \) and \( R := 2\sqrt{2}iaB \) \((Q, R \neq 0 \text{ since } a, A, B \neq 0 \)). Then, the system (23) becomes (\( \frac{d}{dt}p \))
\[
\begin{bmatrix}
  \dot{\xi}_1 \\
  \dot{\eta}_1
\end{bmatrix} = \begin{bmatrix}
  Q & -R \\
  -R & -Q
\end{bmatrix} + \frac{1}{\tau} \begin{bmatrix}
  0 & P \\
  -P & 0
\end{bmatrix} \begin{bmatrix}
  \xi_1 \\
  \eta_1
\end{bmatrix}.
\]
(24)

We perform a linear change \( \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} \) with a constant matrix \( T \) which transforms the leading matrix in (24) into diagonal form. Additionally, we scale the independent variable
\[
\tau \rightarrow \tau 2\sqrt{2a} \sqrt{A^2 + B^2}.
\]
Then, we obtain
\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  -i & 0 \\
  0 & i
\end{bmatrix} + \frac{1}{\tau B} \begin{bmatrix}
  P & 0 \\
  0 & -P
\end{bmatrix} \begin{bmatrix}
  0 & A - \sqrt{A^2 + B^2} \\
  A - \sqrt{A^2 + B^2} & 0
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}.
\]
(25)

For the system (25), \( \tau = 0 \) is a regular singular point and \( \tau = \infty \) is an irregular singular point.

Now, we study the local Galois group \( G_\infty \). By a Theorem of Ramis [14, 15], this group is topologically generated by the formal monodromy, the exponential torus and the Stokes matrices. One can find the formal solutions near \( \tau = \infty \), and then, the exponential torus \( T \) turns out to be isomorphic to \( \mathbb{C}^* \), i.e., \( T = \{ \text{diag}(c, c^{-1}), c \neq 0 \} \) and the formal monodromy is trivial.

In fact, for a general system of that kind Balser et al. [2] have obtained the actual fundamental matrix solution in terms of exponentials and Kummer’s functions and as a result, they have got the Stokes matrices. The detailed calculations can be found in [2] or [14].

In our particular case (25), the Stokes matrices are
\[
S_{11} = \begin{bmatrix} 1 & 0 \\ s_1 & 1 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix},
\]
(26)
where
\[
s_1 = 2\pi i \frac{P}{B} \frac{(A + \sqrt{A^2 + B^2})}{\Gamma(1 - Pi)\Gamma(1 + Pi)}, \quad s_2 = 2\pi i \frac{P}{B} \frac{(A - \sqrt{A^2 + B^2})}{\Gamma(1 + Pi)\Gamma(1 - Pi)}.
\]
(27)
\( \Gamma(z) \) being the Euler’s Gamma function. The monodromy around the regular singular point \( \tau = 0 \) is \( M_0 \simeq \text{diag}(e^{2\pi i(Pi)} e^{-2\pi i(Pi)}) \). Since the local differential Galois group \( G_0 \) is topologically generated by \( M_0, G_0 \subset G_\infty \) or \( G = G_\infty \), but this is known result for the considered linear systems.

It is clear from (26) and (27) that the identity component \( G^0 \) of the Galois group of the system (25) is abelian if and only if \( s_1 = s_2 = 0 \), that is, \( iP \in \mathbb{Z} \) or \( i[(c_3 - c_2)(A^2 + B^2) + 2iAB\beta] \in \mathbb{Z} \).

Therefore, if
\[
\alpha \neq 0, \alpha = \pm c_4, i[(c_3 - c_2)(A^2 + B^2) + 2iAB\beta] \notin \mathbb{Z},
\]
(28)
the Galois group of (25) is \( G = G^0 = SL(2, \mathbb{C}) \) and the non-integrability of the Hamiltonian system (12) follows from the Morales–Ramis theorem.

Remark 2 Alternatively, one can reduce the system (25) to a particular Whittaker equation and study its Galois group with the same end result (see [16]).

An example. Let us return to the normal form (8). As it was explained above, the coefficient \( \beta_2 \) can be made
zero. Recall that $\beta_1, \beta_3$ are real and nonzero by assumption. Then, for this normal form

$$c_2 = \left(\beta_1^2 + \frac{8}{15} \beta_3^2\right), \quad c_3 = \frac{8}{15} \beta_3^2,$$

$$c_4 = \frac{28}{15} \beta_3^2, \quad \alpha = 2 \beta_3^2, \quad \beta = 0$$

and $k_3$ is any of the real roots of the equation $k^2 - 118k + 1 = 0$. Then, the condition (28) reduces to

$$iP = -\frac{15 \beta_3^2}{2 \beta_3^2} \frac{\sqrt{k_3}(k_3 + 1)}{9k_3 - 1} \in \mathbb{Z}.$$  

But this is possible only when $P = 0$, or equivalently $\beta_1 = 0$, which contradicts with the assumptions. Therefore, the normal form (8) is not integrable; that is, there is no additional first integral, meromorphic with respect to the phase variables.

It is natural to ask what happens when $i [(c_3 - c_2) (A^2 + B^2) + 2iAB] \in \mathbb{Z}$. A meticulous analysis would probably require the use of higher variations [19,20]. However, it is highly unlikely to get an occurrence of integrability. Numerical experiments in the simplest case $iP = 0 (c_2 = c_3, \beta = 0)$ indicate chaotic behavior which contradicts integrability (see Fig. 1).

**Remark 3** Here and hereafter, the figures are drown with the help of MAPLE procedure “poincare”, together with the procedure “generate-ic” which gives on the prescribed energy level up to 75 initial conditions. The corresponding trajectories are naturally on different levels of the integral $H_2$. Sometimes it is difficult to find a recurrent behavior in the particular dynamics; that is why different scenes for the Poincaré maps are used. Recall that we have eliminated the quadratic part in the normal form $H$ that is why the typical behavior around the elliptic equilibrium is not observed.

This observation leads us to the assumption that the obstacle to the integrability is probably due to the other two conditions: $\alpha \neq 0$ and $\alpha \neq \pm c_4$.

Now, we will consider the cases $\alpha = \pm c_4 \neq 0$.

First, we deal with $\alpha = c_4$.

**Proposition 2** Suppose $\alpha = c_4$. Then, the system (12) has the following particular solution

$$\begin{align*}
\Gamma_2 : q_1 &= p_1 = 0, \quad q_2 = ip_3, \quad p_2 = iq_3, \\
q_3 &= e^{-4\alpha t}, \quad p_3 = e^{4\alpha t}.
\end{align*}$$  

(29)

The proof is straightforward. $\square$

In the above notations, the normal variational equation (NVE) along the solution (29) is written in $\xi_1, \eta_1$ variables

$$\begin{align*}
\dot{\xi}_1 &= 2iaq_3 \xi_1 + (c_3 - c_2)(q_3^2 + p_3^2) + 2i \beta - 2iap_3 \eta_1, \\
\dot{\eta}_1 &= -((c_3 - c_2)(q_3^2 + p_3^2) + 2i \beta + 2iap_3) \xi_1 - 2iap_3 \eta_1.
\end{align*}$$  

(30)

Next, in order to get an algebraic form of this system we put $z := q_3(t)$. Denoting

$$\begin{align*}
A := \frac{a}{2\alpha} \neq 0, \quad P := \frac{c_3 - c_2}{4\alpha}, \quad Q := \frac{\beta}{2\alpha},
\end{align*}$$  

(31)

we obtain ($' = d/dz$)

$$\begin{align*}
\xi_1 &= -iA \xi_1 - \left[ P \left( \frac{1}{z^3} + \frac{1}{z^2} \right) + iQ \frac{1}{z^2} - iA \frac{1}{z^2} \right] \eta_1, \\
\eta_1 &= \left[ P \left( \frac{1}{z^3} + \frac{1}{z^2} \right) + iQ \frac{1}{z^2} + iA \frac{1}{z^2} \right] \xi_1 + iA \eta_1.
\end{align*}$$  

(32)

**Remark 4** Notice that for this system $z = 0$ and $z = \infty$ are irregular singular points. It is more complicated than (25), and it is unlikely to find the actual fundamental solution. Moreover, the presence of many parameters makes it difficult to apply the approach for finding

![Fig. 1](image-url)  

Fig. 1 Poincaré cross section for $\beta = 0$ and $c_2 = c_3$
the Stokes matrices, based on the Laplace transformations, see, e.g., [14, 23]. Even if we succeed in finding these Stokes matrices, the non-integrability condition would seem like the last part in (28). Therefore, we take another road.

To simplify the things, we consider the case $P = Q = 0$, that is, $c_2 = c_3, \beta = 0$ where the analysis is more conclusive. Then, the system (32) becomes

$$
\begin{align*}
\xi'_1 &= -iA\xi_1 + iA\frac{1}{z^2}\eta_1, \\
\eta'_1 &= iA\frac{1}{z^2}\xi_1 + iA\eta_1.
\end{align*}
$$

(33)

After the scaling $z \rightarrow iAz$, we reduce this system to a single second-order equation

$$
\xi'' + p(z)\xi' + q(z)\xi = 0,
$$

(34)

where $p(z) = \frac{2}{z}$ and $q(z) = -\frac{1}{z^2} + \frac{2}{z} - 1$.

The term involving $\xi'_1$ can be eliminated by performing the change $y = \exp\left(\frac{1}{2} \int p(z)dz\right)\xi_1$. Then, one gets the standard form

$$
y'' + \left(-\frac{1}{z^2} + \frac{2}{z} - 1\right)y = 0,
$$

(35)

which is a particular case of the double confluent Heun equation. For the Heun family, Duval and Loday-Richaud [4] have studied the existence of closed-form solutions by means of the Kovacic algorithm. In our case, no such solutions can be found. This implies that the Galois group of (35) is $\text{SL}(2, \mathbb{C})$, but one can verify that directly by following the steps of the Kovacic algorithm. We do not repeat this famous algorithm here since it is described in many works, see, e.g., [4, 9, 16].

Therefore, if

$$
\alpha = c_4 \neq 0, \quad c_2 = c_3, \quad \beta = 0,
$$

the identity component of the Galois group of (35) is not abelian and the Hamiltonian system (12) is non-integrable by Morales–Ramis theorem.

For the cases $c_2 \neq c_3$ and $\beta \neq 0$, numerical experiments reveal chaotic behavior which excludes integrability (see Fig. 2, where we have taken $\alpha = c_4 = 1, a = 1, c_1 = 0$).

Next, we study the case $\alpha = -c_4$ in the same lines.

**Proposition 3** Suppose $\alpha = -c_4$. Then, the system (12) admits the following particular solution

$$
\Gamma_3 : q_1 = p_1 = 0, \quad q_2 = ip_3, \quad p_2 = iq_3,
$$

The (NVE) along the above solution looks exactly like (30), but with different $q_3$ and $p_3$ in this case. Using the expressions

$$
q_3 = \sinh(2\alpha t), \quad p_3 = \cosh(2\alpha t).
$$

(36)

one gets
\[\dot{\xi}_1 = ia(e^{2\alpha t} - e^{-2\alpha t})\xi_1 + \left[\frac{c_3 - c_2 + i\beta}{2} e^{4\alpha t} + \frac{c_3 - c_2 - i\beta}{2} e^{-4\alpha t}\right]\eta_1,\]
\[\dot{\eta}_1 = -\left[\frac{c_3 - c_2 + i\beta}{2} e^{4\alpha t} + \frac{c_3 - c_2 - i\beta}{2} e^{-4\alpha t}\right]\xi_1 - ia(e^{2\alpha t} - e^{-2\alpha t})\eta_1.\]

As previously, we change the independent variable \(z := e^{2\alpha t}\) and use the same notations \(A, P, Q\) as in (31) to obtain
\[\dot{\xi}'_1 = iA\left(1 - \frac{1}{z^2}\right)\xi_1 - \left[(P + iQ)z + (P - iQ)\frac{1}{z}\right]\eta_1,\]
\[\dot{\eta}'_1 = -\left[(P + iQ)z + (P - iQ)\frac{1}{z}\right]\xi_1 - iA\left(1 - \frac{1}{z^2}\right)\eta_1.\]  
(37)

Due to the same arguments as in Remark 3, we consider the case \(P = Q = 0\). After stretching the independent variable \(z \rightarrow iz\) and reducing the corresponding system to a second-order equation in standard form, we get
\[y'' = r(z) y,\]  
(38)

where
\[r(z) = 2 + \frac{4}{z} + \frac{2}{z^3} - \left(2 - \frac{3i}{4}\right)\frac{1}{z - i} - \left(2 + \frac{3i}{4}\right)\frac{1}{z + i} + \frac{3}{4}\left(\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2}\right).\]

Now, simple calculations by hand along the steps of the Kovacic algorithm give that the Galois group of (38) is again \(\text{SL}(2, \mathbb{C})\). Therefore, the Morales–Ramis theorem applies. Alternatively, one can use the MAPLE package \texttt{kovacicSols} with the same end result.

In this way, we have proved that if
\[\alpha = -c_4 \neq 0, \quad c_2 = c_3, \quad \beta = 0,\]
the Hamiltonian system (12) is not integrable.

For the cases \(c_2 \neq c_3\) and \(\beta \neq 0\), numerical experiments show chaotic behavior which contradicts integrability (see Fig. 3, where we have taken \(\alpha = -1, c_4 = 1, c_3 = \beta = 0\)).

4 Proof of Theorem 2

To this point, we have studied the cases \(i[(c_3 - c_2)(A^2 + B^2) + 2iAB\beta] \in \mathbb{Z}\) numerically and \(\alpha = \pm c_4 \neq 0\) analytically and numerically. The above considerations
suggest that $\alpha \neq 0$ is probably the true obstacle for integrability. To see this, we proceed with the simplest of the cases, namely $c_1 = c_2 = c_3 = c_4 = \beta = 0$.

To carry on the proof of Theorem 2, we need another particular solution.

**Proposition 4** Suppose $\alpha \neq 0$. Then, the system (12) admits the following solution

$$
\begin{align*}
\Gamma_4: \quad q_2 = \frac{2a}{\alpha}, & \quad q_1 = p_1 = \frac{a}{\alpha} \exp \left( -\frac{4a^2}{\alpha} t \right), \\
& \quad q_3 = p_3 = \frac{i}{\sqrt{2}} q_1, \quad p_2 = 0.
\end{align*}
$$

(39)

The proof is immediate. \qed

Denote again $\xi_j = dq_j, \eta_j = dp_j, j = 1, 2, 3$. Then, the variational equations (VE) along the solution (39) split nicely. Indeed, introducing the variables $v_1 = \xi_1 - \eta_1, v_3 = \xi_3 - \eta_3$ we have

$$
\begin{align*}
\dot{v}_1 &= 2aq_2 v_1 + 4aq_1 \eta_2, \\
\dot{v}_2 &= -2aq_1 v_1 - 2aq_2^2 \eta_2 - 2aq_2 q_3 v_3, \\
\dot{v}_3 &= aq_2^2 v_3 + 4aq_2 q_3 \eta_2.
\end{align*}
$$

(40)

The above system admits an integral

$$
dH = q_1 v_1 + q_3 v_3 := 0,
$$

(41)

which stems from the linearization of $H$ along the solution (39). With the help of (41), we remove $v_3$ from (40) and obtain the system

$$
\begin{align*}
\dot{v}_1 &= \frac{4a^2}{\alpha} \left( v_1 + e^{-\frac{4a^2}{\alpha} t} \eta_2 \right), \\
\dot{v}_2 &= \frac{2a^2}{\alpha} e^{-\frac{4a^2}{\alpha} t} v_1 + \frac{a^2}{\alpha} \left( e^{-\frac{4a^2}{\alpha} t} \right)^2 \eta_2.
\end{align*}
$$

(42)

Now, after introducing a new independent variable $z := e^{-\frac{4a^2}{\alpha} t}$, we get the algebraic form of the above system

$$
\begin{align*}
\dot{v}_1' &= \frac{1}{z} v_1 - \eta_2, \\
\dot{v}_2' &= -\frac{1}{2} v_1 - \frac{z}{4} \eta_2.
\end{align*}
$$

(43)

Fortunately, in this case we can find the fundamental system of solutions

$$
\begin{pmatrix}
\frac{v_1}{\eta_2}
\end{pmatrix} = \Phi \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},
$$

where $D_1, D_2$ are arbitrary constants and $\Phi$ is the fundamental matrix

$$
\Phi := (\Phi^{(1)}, \Phi^{(2)})
$$

$$
\begin{pmatrix}
-\frac{1}{2} - 2e^{\frac{4a^2}{\alpha} t} - \frac{z}{4} \int e^{\frac{4a^2}{\alpha} t} dz \\
1 + \int e^{\frac{4a^2}{\alpha} t} dz
\end{pmatrix},
$$

(44)

which in turn implies that its Galois group is solvable.

Let us see whether it is abelian. The coefficient field of the system (43) is $\mathbb{K} := \mathbb{C}(z)$ with the usual derivation. From the type of the solutions (44), we conclude that the corresponding Picard–Vessiot extension is

$$
\mathbb{L} := \mathbb{C}(z, e^{-\frac{2}{\alpha} \int z} \int e^{\frac{4a^2}{\alpha} t} dz).
$$

Let $G := Gal(\mathbb{L}/\mathbb{K})$ be the Galois group of (43) and $\sigma \in G$; that is, $\sigma$ is a differential automorphism of $\mathbb{L}$ fixing $\mathbb{K}$. Using that $\sigma \left( e^{-\frac{2}{\alpha} \int z} \right) = \delta e^{-\frac{2}{\alpha} \int z}$ and the well-known fact that $\int e^{-\frac{2}{\alpha} \int z}$ is not an elementary function, we have

$$
\sigma \Phi = \Phi R_\sigma, \quad R_\sigma = \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix}, \quad \delta, \gamma \in \mathbb{C}^*.
$$

Hence, the Galois group is represented by the matrix group $\{ \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix}, \gamma, \delta \neq 0 \}$, which is connected, solvable, but clearly non-commutative. As a matter of fact, $G$ is isomorphic to the semi-direct product of the additive group and the multiplicative group $G \cong \mathbb{A} \times \mathbb{G}_m$, see, e.g., Magid [13].

Therefore, when $\alpha \neq 0$ and $c_1 = c_2 = c_3 = c_4 = \beta = 0$, the Hamiltonian system (12) is non-integrable by the Morales–Ramis theory. This proves Theorem 2. \qed

**5 Proof of Theorem 3**

For the proof of Theorem 3, we use the approach from [18]. The idea is to bring the system of variational equations in upper triangular form by means of symplectic transformation, and then eventually to solve this transformed system. Here, the situation is easier, but there are parameters.

As usual we start with finding a particular solution.
Proposition 5 Suppose $\alpha = 0$, $\beta \neq 0$. Then, the system (12) admits the following solution

$$I_5 : q_1 = i \sqrt{\frac{1}{\beta \cosh(2\alpha t)}} \cdot p_2 = \frac{a}{\beta} \tanh(2\alpha t),$$

$$p_1 = i q_1, \quad q_2 = -ip_2, \quad q_3 = -1, \quad p_3 = i.$$  \hspace{1cm} (45)

The proof follows easily. □

With already introduced notations, the variational equations along $I_5$ are

$$\dot{\xi}_1 = (2ap_2 + 4icq_1^2)(\xi_1 + i\eta_1) + (2c_2q_1p_2 - 2iaq_1)(\xi_2 + i\eta_2) - t\beta p_2\xi_1 = -i\beta q_1(\xi_2 - i\eta_2) + \beta q_1p_2(\xi_3 + i\eta_3).$$

$$\dot{\eta}_1 = (2iap_2 - 4icq_1^2)(\xi_1 - i\eta_1) + (2icq_1p_2 - 2aq_1)(\xi_2 - i\eta_2) + 2c_3q_1(\xi_3 - i\eta_3) - 2i\beta p_2\xi_1 - \beta q_1(\xi_2 - i\eta_2) + i\beta q_1p_2(\xi_3 + i\eta_3).$$

$$\dot{\xi}_2 = 2iaq_1(\xi_1 - i\eta_1) + q_1(2c_2p_2 + i\beta)(\xi_1 + i\eta_1) - 2c_4p_2(\xi_3 - i\eta_3).$$

$$\dot{\eta}_2 = -2aq_1(\xi_1 - i\eta_1) + q_1(2icq_1p_2 + \beta)(\xi_1 + i\eta_1) - 2c_4p_2(\xi_3 + i\eta_3).$$

$$\dot{\xi}_3 = q_1(2ic + \beta p_2)(\xi_1 + i\eta_1) + 2c_4p_2(\xi_2 + i\eta_2).$$

$$\dot{\eta}_3 = q_1(2ic + i\beta p_2)(\xi_1 + i\eta_1) - 2ic_4p_2(\xi_2 + i\eta_2).$$  \hspace{1cm} (46)

As before, it is convenient to set a new independent variable $z := 2\alpha t$. The variational equations (46) suggest introducing the variables

$$v_j = \xi_j + i\eta_j, \quad w_j = \xi_j - i\eta_j, \quad j = 1, 2, 3.$$  \hspace{1cm} (47)

In these variables (little rearranged), the variational equations (46) become

$$\zeta' = \begin{pmatrix} \frac{a}{\beta} p_2 - i \frac{a}{\beta} q_1 & 0 \\
\frac{1}{2i} & 0 \\
0 & 0 \\
\end{pmatrix} \zeta, \quad \zeta = (w_1, w_2, v_3, w_3, v_1, v_2)^T$$

where $C_1 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
- \frac{a}{\beta} q_1 & 0 & 0 \\
\end{pmatrix}$.

The proof follows easily. □

In this way, we have transformed the variational equations in upper triangular form. Furthermore, we have gathered the parameters $c_1, c_2, c_3, c_4$ in the upper right block.

Lemma 5 The system (48) is solvable. Moreover, its Galois group is abelian.

Proof: We are looking for the fundamental matrix of the system in the form

$$\Psi = \begin{pmatrix} \Phi_1 & \Phi_2 \\
0 & \Phi_2 \\
\end{pmatrix}.$$  \hspace{1cm} (51)

First, to find $\Phi_2$ we solve the system

$$w'_2 = i \sqrt{\frac{a}{\beta} \sinh z} \cdot \frac{1}{\sqrt{\beta \cosh^2 z}} \cdot v_1,$$

$$v'_1 = \tanh z v_1 + 2 \sqrt{\frac{a}{\beta} \cosh z} \cdot v_2,$$

$$v'_2 = - \sqrt{\frac{\beta}{a} \cosh z} \cdot v_1.$$  \hspace{1cm} (52)

It has the following general solution

$$w_3 = \frac{i}{\beta} w_3^0 - \frac{1}{2a} \frac{1}{\beta} \cosh^2 z \cdot v_1^0$$

$$+ \frac{i}{\beta} \frac{1}{\cosh z} \left( 2z + \tanh z - \frac{z}{\cosh^2 z} \right) v_2^0,$$

$$v_1 = \sqrt{\frac{a}{\beta} \cosh z} \cdot v_1^0 + \sqrt{\frac{a}{\beta} \cosh z} \cdot \left( \sinh z + \frac{z}{\cosh z} \right) v_2^0,$$

$$v_2 = - \tanh z v_1^0 + (1 - z \tanh z) v_2^0.$$  \hspace{1cm} (53)

Therefore, the fundamental matrix of (52) reads

$$\Psi_2 = \begin{pmatrix} \frac{i}{\beta} \frac{1}{\cosh z} v_1^0 - \frac{1}{\beta} \frac{1}{\cosh^2 z} \frac{i}{\beta} \left( 2z + \tanh z - \frac{z}{\cosh^2 z} \right) v_2^0 \\
0 \sqrt{\frac{a}{\beta} \cosh z} \left( \sinh z + \frac{z}{\cosh z} \right) \sqrt{\frac{a}{\beta} \cosh z} \\
0 - \tanh z \sqrt{\frac{a}{\beta} \cosh z} \end{pmatrix}.$$  \hspace{1cm} (54)

The coefficient field $K$ is the field of hyperbolic functions. It can be seen from (54) that the solution of (52)
contains an element not belonging to $K$, namely
\[ z = \ln e^z = \ln(\sinh z + \cosh z). \]

By adjoining the element $z$ to the field $K$, we get the following Picard–Vessiot extension $F_1 := K(z)$. Let $\sigma \in \text{Gal}(F_1/K)$. Then,
\[ \sigma(z) = z + \gamma, \quad \gamma \in \mathbb{C}^*. \]

Therefore,
\[ \sigma \Psi_2 = \Psi_2 M_2, \quad \text{where} \quad M_2 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \ \gamma \\ 0 & 0 & 1 \end{pmatrix}. \]

The matrix group
\[ \left\{ \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \ \gamma \\ 0 & 0 & 1 \end{pmatrix} \bigg| \gamma \in \mathbb{C}^* \right\} \]

can be though as a representation of $\text{Gal}(F_1/K)$. This group is clearly commutative.

Next, to find $\Psi_1$ we have to solve the system
\begin{align*}
\frac{dz}{z} &= - \tanh z \, w_1 + \sqrt{\frac{1}{\beta}} w_2 + i \sqrt{\frac{1}{\beta}} \sinh z \, v_3, \\
\frac{dw_1}{z} &= -2 \sqrt{\frac{1}{\beta}} \cosh \frac{1}{z} w_1, \\
v_3 &= 0.
\end{align*}

Direct computation gives the general solution of the above system
\begin{align*}
w_1 &= i \sqrt{\frac{1}{\beta}} \left[ \sinh \frac{z}{\cosh^2 \frac{z}{2}} w_0 + \left( \frac{\sinh z}{\cosh^2 \frac{z}{2}} - \frac{1}{\cosh \frac{z}{2}} \right) w_2 \right] \\
&\quad + \left( \frac{\sinh z}{\cosh^2 \frac{z}{2}} - \frac{1}{2 \cosh \frac{z}{2}} \right) v_3, \\
w_2 &= i \sqrt{\frac{1}{\beta}} \left[ \frac{1}{\cosh \frac{z}{2}} w_0 + \left( \tanh z + \frac{z}{\cosh^2 \frac{z}{2}} \right) w_2 \right] \\
&\quad + \frac{z}{\cosh^2 \frac{z}{2}} v_3, \\
v_3 &= 0.
\end{align*}

Thus, the fundamental matrix of (57) is
\[ \Psi_1 = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\
\psi_{21} & \psi_{22} & \psi_{23} \\
0 & 0 & 1 \end{pmatrix}. \]

Finally, to find $\Psi_3$ we make use of variation of constants formula
\[ \Psi_3 = \Psi_1 \int Q(z) \, dz, \]

where
\begin{align*}
\psi_{11} &= \sqrt{\frac{a}{\beta}} \sinh \frac{z}{\cosh \frac{z}{2}}, \\
\psi_{12} &= \sqrt{\frac{a}{\beta}} \left( \frac{\sinh z}{\cosh^2 \frac{z}{2}} - \frac{1}{\cosh \frac{z}{2}} \right), \\
\psi_{13} &= \sqrt{\frac{a}{\beta}} \left( \frac{\sinh z}{\cosh \frac{z}{2}} - \frac{1}{2 \cosh \frac{z}{2}} \right), \\
\psi_{21} &= \sqrt{\frac{a}{\beta}} \left( \frac{\sinh z}{\cosh \frac{z}{2}} - \frac{1}{\cosh \frac{z}{2}} \right), \\
\psi_{22} &= \sqrt{\frac{a}{\beta}} \left( \tanh z + \frac{z}{\cosh^2 \frac{z}{2}} \right), \\
\psi_{23} &= \frac{a}{\beta} \sinh \frac{z}{\cosh \frac{z}{2}}.
\end{align*}

Since we have not adjoined any new element to the field $F_1$, in view of (55), if $\sigma \in \text{Gal}(F_1/K)$, then
\[ \sigma \Psi_1 = \Psi_1 M_1, \quad \text{where} \quad M_1 = \begin{pmatrix} 1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}. \]

Remark 5 Notice that the both functions $\ln(\cosh z)$ and $J = \int \tanh z \, dz$ do not belong to the filed $F_1$. In fact,
\[ J = \frac{z^2}{2} + z \ln 2 + z \ln(\cosh z) - \int \ln(e^{2z} + 1) \, dz. \]

The last integral can be represented as
\[ \int \ln(e^{2z} + 1) \, dz = - \frac{1}{2} \int_0^{e^{2z} + 1} \frac{\ln t}{1 - t} \, dt, \]

which in turn can be reduced to the dilogarithm $\text{Li}_2$. Hence, $\ln(\cosh z)$ and $J$ are multi-valued.
So far, we have solved the system \((48)\) which means that its Galois group is solvable. Now, we will show that this Galois group is commutative.

Adjoining the elements of \(\Psi_3\) and more precisely \((61)\), we get another Picard–Vessiot extension

\[
\mathbb{K} \subset \mathbb{F}_1 \subset \mathbb{F}_2 := \mathbb{F}_1 \left(< Q_{12}, \int Q_{13}, \int Q_{23} >\right). \quad (62)
\]

Every member of the above chain is obtained via adjoining quadratures, so the extension \(\mathbb{F}_2\) is a Liouville extension. Therefore, \(G = Gal(\mathbb{F}_2/\mathbb{K})\) is a solvable group (but we know that) and so is \(G^0\)—the identity component of \(G\). If \(\sigma \in G\), we have

\[
\sigma \Psi = \Psi R_\sigma, \quad R_\sigma = \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix}
\]

and just as in \([3, 18]\) \(M_3 = M_1 M\), where \(M\) is obtained in the following way

\[
\sigma \int Q(z) dz = \int Q(z) dz + M, \quad M \in Mat(3, \mathbb{C}). \quad (63)
\]

Taking into account \((61)\), \(M\) amounts to

\[
M = \begin{pmatrix} 0 & v & \delta \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, \quad v, \delta \in \mathbb{C}
\]

with

\[
\sigma(\ln(\cosh z)) = \ln(\cosh z) + v, \quad \sigma(J) = J + \delta.
\]

Notice that the constants \(v, \delta\) could be zero, because the linear combinations of the parameters in the coefficients in the expressions \((61)\) may vanish.

Then, the matrix \(M_3 = M_1 M\) results in

\[
M_3 = \begin{pmatrix} 1 & \gamma & v \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & v & \delta \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v & \delta + v \delta \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}.
\]

As a result of that \(G = G^0\) is represented by the matrix group consisting of unipotent matrices \(R_\sigma\),

\[
G^0 = \left\{ \begin{pmatrix} 1 & \gamma & v \gamma + \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & v \gamma & \delta \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix} \mid \gamma \neq 0, v, \delta \in \mathbb{C} \right\}. \quad (64)
\]

After some linear algebra, one can see that this group is abelian. This finishes the proof of the lemma. \(\square\)

Up to now, we have established that the Galois group of \((48)\) is abelian. The same holds for the monodromy group. Hence, they cannot serve as obstacles to integrability.

However, we are in situation where we can apply the Lyapounov’s method. Recall that the functions from \(\mathbb{F}_1\) are single-valued, while \(\ln(\cosh z)\) and \(J := \int z \tanh zdz\) are not.

In view of the Lyapounov’s observation, as shown in Sect. 2, the additional integral could exist if the solutions of \((46)\) or equivalently \((48)\) are single-valued in \(z\) plane. But from \((61)\), this happens exactly when the coefficients of \(\ln(\cosh z)\) and \(J\) vanish; that is, the parameters have to satisfy the linear system

\[
8c_1 - 2c_2 - 8c_3 + 3c_4 = 0, \\
8c_1 - 2c_2 - 6c_3 + 5c_4 = 0, \\
8c_1 - 2c_2 + 5c_4 = 0. \quad (65)
\]

Its solutions are

\[
c_2 = 4c_1, \quad c_3 = c_4 = 0. \quad (66)
\]

For these values of the parameters, we are lucky to find the integral \((14)\).

On the other hand, if the condition \((66)\) is not satisfied, then the general solution of \((46)\) is multi-valued, and hence, no new first integral exists. This ends the proof of Theorem 3. \(\square\)

To get a confirmation that the Hamiltonian system \((12)\) is non-integrable for values of the parameters \(c_1, c_2, c_3, c_4\) far from \((66)\), we perform numerical experiments. With the set of parameters \(a = 1, \beta = 2, \alpha = 0, c_1 = 1, c_2 = 4\), Poincaré cross sections in Fig. 4 indicate chaotic behavior. Similar behavior can be seen in Fig. 5 for \(c_1 = c_2 = 0\).

### 6 Concluding remarks

In this paper, we study the integrability of the truncated to order four normal form of the 1:2:2 resonance. This normal form contains too many parameters which makes the complete analysis of the problem difficult. For a large family of parameters, we prove that the corresponding normal form does not possess an additional first integral. As in the study of other first-order resonances \([3]\), we use the Morales–Ramis
theory. The presented algebraic proofs are another confirmation of non-integrability suggested in [28]. It is worth mentioning again that due to the works [6,7], the non-integrability has a clear geometric and dynamical meaning.

Perhaps, obtaining a couple of integrable cases is the more interesting result here. The first one (9) is natural and easy; moreover, it is KAM non-degenerate upon certain conditions on the parameters. The second one with the extra integral (14) is non-trivial and cannot be explained by obvious symmetry.

Let us note that the algebraic approach adopted here and in [3] can be applied to study integrability of other resonance Hamiltonian normal forms in three degrees of freedom. cf. [8,28]. We can remove the quadratic part of the normal form $H_2$ one way or another. This allows us to deal with the resonances $1:k:l$ even in the case when $k$ (or $l$) is negative. A systematic way how to obtain the generators of the corresponding normal forms is given in Hansmann [8]. Note that the resonances $1:2:-2$ and $1:-1:2$ are generically non-semisimple.
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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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