Propagating wave in the flock of self-propelled particles

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We investigate the linearized hydrodynamic equations of interacting self-propelled particles. It found that the small perturbations of density and polarization fields satisfy the hyperbolic partial differential equations—that admit analytical propagating wave solutions. These solutions uncover the questionable traveling band formation in the flocking state of self-propelled particles. Below a critical noise strength, the unstable disordered state (random motion) undergoes the transient vortex and evolves to the ordered state (flocking motion) as unidirectional traveling waves. There appears two possible longitudinal wave patterns depending on the noise strength, including single band in stable state and multiple bands in unstable state.

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The onset of collective motion can be found in various systems of the self-propelled objects, ranging from macromolecules, to microorganism, animal, human, and swarming robots (see Ref. [1] and references therein). The physics aspect of this phenomenon has been a current active research topic.

The minimal paradigm that can be used to describe this dynamics successfully is acknowledged to the Vicsek model [2]. In this model, the point-like self-propelled particles (SPP) move at constant velocity in the direction of their orientation unit vector [2]. The heading direction of each particle is aligned by noisy mutual polar interaction. For sufficient small noise strength, below a critical value, the particles transit from random motion (disordered state) to flocking (ordered state), where they form the coherence clusters that the individual members trend to move together in the same direction. The Vicsek model can be viewed as the flying XY spin model where the phenomenological hydrodynamic equations have been proposed for description at continuum level by Toner and Tu [3, 4]. Recently, the hydrodynamic equations of SPP can be derived from specified individual-based dynamics by using several coarse-gaining frameworks, such as the Smoluchowski equation [5–7] and the Boltzmann equation [8–10].

Based on hydrodynamic theory, the propagating as sound wave of the long wavelength mode fluctuations of density and velocity fields in SPP has been predicted by the work of Tu and Toner [11]. Later, the moving bands of the ordered state in the disordered state background was obviously found in the large-scale simulations of SPP by Refs [3, 12–14]. Two distinguish robust pattern forms of the traveling waves in SPP have been explored, including solitary moving band [3, 12–14] and moving multi-stripes [15]. Apart from wave patterns, fluctuating flocking states [16] and stationary radially symmetric asters [17] in SPP have been also presented.

The analytical works have been carried out in order to gain deep insight into wave propagating dynamics in SPP. The standard method is the linear stability analysis of the hydrodynamic equations [9, 15, 17]. Several authors agree that the emergence of traveling waves in SPP is from instability of the homogeneous states [3, 12–17]. However, the linear stability analysis provides only the dispersion relation [9] that is inadequate to characterize the spatiotemporal wave patterns. In more rigorous study, the propagative Ansatz, in which the wave profile travels along a direction with constant speed, has been postulated to be a solution for hydrodynamic equations of SPP in one-dimensional space (1D) [9, 17–19]. This approach reduces the hydrodynamic equations from nonlinear partial differential equations (PDEs) to nonlinear ordinary differential equations (ODEs). The nonlinear ODEs can be recast further into equivalent Newton’s equation of motion for single particle moving in potential field in the presence of a friction by using a dynamical framework [18, 19]. This approach seems likely to classify the three different types of propagating patterns in SPP successfully, including solitary wave, multi-stripes wave and polar-liquid droplet. Nonetheless, the exact wave profiles are unable to solve explicitly by using this framework due to its nonlinearity.

As classified in the textbook of Whitham [20], there are two classes of wave solutions for the linear or nonlinear PDEs which consist of hyperbolic wave and dispersive wave solutions. The difference is that the hyperbolic wave propagates in two opposite directions along arbitrary axis, in which the speeds are unnecessary to be equal [21, 22]. Obviously, the previous analyses rely on the dispersive wave solution that propagates in a direction [3, 15, 17–19]. In contrast to, it is found in our research here that the linearized hydrodynamic equations of SPP can formulate the hyperbolic-type PDEs. Therefore, the previous propagating wave assumption, which belongs to the dispersive wave solution [3, 15, 17–19], is still incomplete wave feature of SPP.
In this work, we investigate the linearized hydrodynamic equations of SPP which can be combined into the linear wave PDEs \[21\,22\]. Instead of performing mode analysis as in the conventional works \[9\,15\,17\], we solve for the exact space and time dependent solutions of these equations by using the Riemann method \[21\,22\]. These linear analytical solutions are plausible to capture the dynamics of SPP in vicinity of early and final state of system. Especially, they can be used to classify the wave pattern formation in the flocking state of SPP clearly.

In this paper, we consider a particular variant Vicsek model that has been studied by Farrell et al. \[24\]. Adapted from Ref. \[24\], the hydrodynamic equations, that describe evolution of particle number density field \(\rho(r, t)\) and polarization field \(\mathbf{W}(r, t)\) in two-dimensional space (2D), are given by

\[
\rho_t = -v_0 \nabla \cdot \mathbf{W},
\]

\[
\mathbf{W}_t = -\frac{v_0}{\gamma} \nabla \rho + \left(\frac{\gamma^2}{8\epsilon} - \epsilon \nabla W^2\right) \mathbf{W} - \frac{3\gamma}{16\epsilon} \mathbf{W} \cdot \nabla \mathbf{W} - \frac{5\gamma}{16\epsilon} v_0 \nabla \mathbf{W} \cdot \nabla \mathbf{W} + \frac{5\gamma}{32\epsilon} \nabla (|\mathbf{W}|^2) + \frac{\epsilon_v^2}{16\epsilon} \nabla^2 \mathbf{W},
\]

where \(v_0\) is particle moving speed, \(\epsilon\) describes noise strength and \(\gamma\) describes the strength of alignment. The polarization field is associated with the particle velocity field \(\mathbf{V}(r, t)\) in such a way that \(v_0 \mathbf{W}(r, t) = \rho(r, t) \mathbf{V}(r, t)\). These equations are coarse-gained dynamics of a \(N\) point-like SPP system that particles move at constant speed \(v_0\) in the direction of their orientation unit vector and interact to each other with noisy alignment rule \[2\]. The position \(r_i(t)\) and the orientation angle \(\theta_i(t)\) of \(i\)th particle at time \(t\) evolve with the following equations of motion: \(\dot{r}_i = v_0 \hat{\mathbf{p}}_i\) and \(\dot{\theta}_i = \sum_{j \neq i} F(\theta_i - \theta_j, r_i - r_j) + \sqrt{2\eta_i(t)}\), where the unit vector \(\hat{\mathbf{p}}_i(\theta_i) = \cos \theta_i \hat{x} + \sin \theta_i \hat{y}\) and \(\eta_i(t)\) is a white noise with zero mean and unit variance. The local pairwise alignment interaction is given by \(F(\theta, r) = \gamma \sin(\theta)/(\pi l^2)\), if \(|r| < l\) (otherwise \(F = 0\)), where \(l\) is interaction range \[24\]. The advantage of this variant model is that it can map the microscopic physical parameters into the hydrodynamic equations through a coarse-grained process explicitly. Although differences in physical parameters, Eq. [1] and Eq. [2] have identical form of the phenomenological model proposed by Toner and Tu \[3\,4\] and the coarse-grained equations obtained by using the Boltzmann theory \[4\].

The homogeneous states of Eq. [1] and Eq. [2] admit arbitrary constant density \(\rho_0\) with two possible values of polarization \(W_0\), given by

\[
|W_0| = W_0 = \begin{cases} 0, & \epsilon \geq \epsilon_0 \\ \sqrt{8\epsilon (\epsilon_0 - \epsilon)}/\gamma, & \epsilon < \epsilon_0 \end{cases}
\]

where \(\epsilon_0 = \gamma\rho_0/2\) which is defined as the critical noise strength value. Above the critical point \((\epsilon > \epsilon_0)\), the system is in disordered state with zero polarization where the SPP move in random direction. Whereas below the critical point \((\epsilon < \epsilon_0)\), the system undergoes ordered state where the SPP tend to move together in the same direction with nonzero polarization, called the flocking.

Now we study the dynamics of SPP in vicinity of the homogeneous states. We suppose that the homogeneous polarization aligns in \(x\)-direction. Thus we define the solutions as follows: \(\rho(r, t) = \rho_0 + n(r, t)\) and \(\mathbf{W}(r, t) = W_0 \hat{x} + \mathbf{u}(r, t)\), where \(n(r, t)\) and \(\mathbf{u}(r, t)\) respectively are small perturbations in density and polarization fields, called the perturbations for short. Substituting these solutions into Eq. (1) and Eq. (2) by retaining the first-order terms, then we obtain the linearized hydrodynamic equations of SPP,

\[
n_t = -v_0 \nabla \cdot \mathbf{u},
\]

\[
\mathbf{u}_t = -\frac{v_0}{2} \nabla n + \alpha_0 \mathbf{u} + \frac{\gamma}{2} W_0 \mathbf{h} + \frac{\epsilon_v^2}{16\epsilon} \nabla^2 \mathbf{u},
\]

where \(\alpha_0 = (\epsilon_0 - \epsilon - \epsilon_0^2/2\delta)^2\) and \(\mathbf{h} = n \hat{x} - \frac{\delta}{\sqrt{\gamma}} W_0 (\hat{x} \cdot \mathbf{u}) \hat{x} - \frac{\sqrt{\gamma}}{2} (\hat{x} \cdot \nabla) \mathbf{u} - 5\sqrt{\gamma} (\nabla \cdot \mathbf{u}) + 5\sqrt{\gamma} (\hat{x} \cdot \mathbf{u})\). The vector field \(\mathbf{h}\) tends to drive the polarization field to the mean direction and has the effect only in the flocking state \((W_0 \neq 0)\). Operating Eq. [1] with \(\mathbf{h}\) and using Eq. [6] (similarly, operating Eq. [5] with \(\partial_t\) and using Eq. [4]), we evaluate that

\[
n_{tt} - \alpha_0 n_{tt} = c^2 \nabla^2 n - \frac{\gamma}{2} W_0 v_0 \nabla \cdot \mathbf{h} + O(\kappa \nabla^2 n),
\]

\[
\mathbf{u}_{tt} - \alpha_0 \mathbf{u}_t = c^2 \nabla^2 \mathbf{u} + c^2 \nabla \times (\nabla \times \mathbf{u}) + \frac{\gamma}{2} W_0 \mathbf{h}_t + O(\kappa \nabla^2 \mathbf{u}_t),
\]

where \(c = \frac{\sqrt{\gamma}}{\sqrt{2\epsilon}}\) and \(\kappa = \frac{\epsilon_v^2}{16\epsilon}\). Noting that the third-order derivative terms in Eq. [6] and Eq. [7] can be ignored since we are interested in evolution of large flocking cluster or long wavelength (\(\lambda\)) mode, that \(\lambda \gg \frac{\pi v_0}{2\sqrt{\epsilon \alpha_0}}\). Obviously, Eq. [6] and Eq. [7] belong to the wave equations or hyperbolic PDEs \[21\,22\].

From Eq. [6] and Eq. [7], the perturbations of the disordered state \((W_0 = 0)\) satisfy the telegraph equations \[21\,22\]. Specially, Eq. [7] generates the vortex, \(\omega = \nabla \times \mathbf{u}\), that can be observed in the transient stage of simulations of the Vicsek-type model \[22\]. By taking the curl operator \(\nabla \times\) to Eq. [4] with \(W_0 = 0\), we obtain the governing equation for the perturbed vorticity

\[
\omega_t = (\epsilon_0 - \epsilon) \omega + \kappa \nabla^2 \omega.
\]

By using the following solutions \(n = e^{(\epsilon_0 - \epsilon)/\epsilon} \hat{n}\) and \(\omega = e^{(\epsilon_0 - \epsilon)/\epsilon} \hat{\omega}\), we found that \(\hat{n}\) satisfies the Klein-Gordon equation \(\hat{n}_{tt} = c^2 \nabla^2 \hat{n} + \frac{1}{2} (\epsilon_0 - \epsilon)^2 \hat{n}\) while \(\hat{\omega}\) satisfies the diffusion equation \(\hat{\omega}_t = \kappa \nabla^2 \hat{\omega}\) \[22\]. So that, \(c\) is interpreted as the speed of sound in disordered phase that has the magnitude about 0.707 of the individual particle...
velocity $v_0$. And, $\kappa$ is diffusion constant. It implies that the disordered state is unstable below the critical point ($\varepsilon < \varepsilon_0$) and it evolves to the ordered state to form the flocking.

As shown in Eq. (9) and Eq. (7), the perturbations around the flocking or the ordered state ($W_0 \neq 0$) tend to be biased to the mean direction by vector field $h$. It is observed, at least in simulations, that the moving bands are unidirectional waves. Such a symmetry-broken field, we rewrite \( \mathbf{u}(r, t) = w(r, t) \mathbf{x} + v(r, t) \mathbf{y} \), where $w$ and $v$ are $x$- and $y$-component of the small perturbed polarization field, respectively. Now we consider the longitudinal mode, where the wave profiles propagate in the same direction of mean polarization ($n_y = w_y = v_y = 0$). From Eq. (9) and Eq. (7), the wave equations for the ordered state in this case read

\[
\begin{align*}
n_{tt} + \alpha n_t &= c^2 n_{xx} - 2\nu n_{tx} - \beta n_x + O(\kappa n_{txx}), \quad (9) \\
\nu_{tt} + \alpha \nu_t &= c^2 \nu_{xx} - 2\nu \nu_{tx} - 3\nu_x + O(\kappa \nu_{txx}), \quad (10)
\end{align*}
\]

where $\alpha = 2(\varepsilon_0 - \varepsilon)$, $\beta = \frac{3}{2} W_0 v_0$ and $\nu = \frac{3\varepsilon}{2} W_0 v_0$. Noting that $v$ is decoupled and trends to decay to eventually small value by a bias-diffusion process.

Next, we consider the transverse mode where the wave profiles propagate in perpendicular direction of mean polarization ($n_x = w_x = v_x = 0$). From Eq. (9) and Eq. (7), the wave equations for the ordered state in this case read

\[
\begin{align*}
n_{tt} &= c^2 n_{yy} + O(\kappa n_{txx}), \quad (11) \\
\nu_{tt} &= c^2 \nu_{yy} + O(\kappa \nu_{txx}). \quad (12)
\end{align*}
\]

Noting that $w$ is assumed to relax to zero in the ordered state. By neglecting the third-order derivative term, Eq. (11) and Eq. (12) are the plane wave equations that have the well-known d’Alembert solution—where the initial condition splits into two waves that propagate in opposite directions along the $y$-axis with the speed of sound $c$. Since the perturbations do not change shape from the initial conditions for this sort of wave, we ignore this mode in this study.

We are now looking for the analytical space and time dependent solution of longitudinal waves. By dropping the third-order derivative term, we rewrite Eq. (9) or Eq. (10)

\[
\phi_{tt} + 2\nu \phi_{tx} - c^2 \phi_{xx} + \alpha \phi_t + \beta \phi_x = 0, \quad (13)
\]

where $\phi$ can refer to either $n$ or $w$, since all equations are in identical form. The initial conditions for Eq. (13) are given by $\phi(x, 0) = f(x)$, $D_0 \phi(x, 0) = \phi_0(x, 0) + \nu \phi_x(x, 0) \equiv g(x)$. Eq. (13) is a second-order PDE whose characteristic equation is given by \((\frac{dx}{dt})^2 - 2\nu (\frac{dx}{dt}) - c^2 = 0\). From the characteristic equation, obviously, Eq. (13) is a hyperbolic-type PDE and it can be reduced to a canonical form by introducing the curvilinear coordinates:

\[
\xi = x + c^{-1} t, \quad \eta = x - c^+ t, \quad (14)
\]

where $c^\pm = \sqrt{\nu^2 + c^2} \pm \nu$. So that, $\nu$ is exactly the collective speed of SPP induced by the alignment interaction. Applying the transformations in Eq. (14), we rewrite Eq. (13) in $\eta\xi$-plane

\[
\phi_{\eta\xi} + k^- \phi_\eta + k^+ \phi_\xi = 0, \quad (15)
\]

where $k^- = -\frac{\alpha - \beta}{4\Lambda^2}$, $k^+ = \frac{\alpha + \beta}{4\Lambda^2}$ and $\Lambda = \sqrt{\nu^2 + c^2} = \frac{1}{2}(c^- + c^+)$. Now the solutions of Eq. (15) depends on the two wave variables, $\phi(x, t) = \phi(\eta, \xi)$. According to $\nu > 0$ in the ordered state, the wave speeds $c^\pm$ are always positive so that $\eta$ and $\xi$, respectively, are left- and right-propagating wave variables. In the presence of collective motion, the wave speeds in the flocking state are larger than the speed of sound in disordered phase. This supports the supersonic wave structure as pointed out by Ihle [16].

Finding the solution of Eq. (15) subjected to the initial data is called a Cauchy problem, which can be solved by using the Riemann method [21, 22]. This approach can solve the general form of linear hyperbolic PDE in 1D, but case study in the presence of $\nu$ term is rare in many textbooks [21, 22]. Therefore, we provide the procedure for solving Eq. (15) in the Supplemental Material [26]. From [26], the analytical wave solution of Eq. (15) in space and time variables is provided by

\[
\phi(x, t) = \frac{1}{2} \left[ e^{-\alpha t} f(x + c^- t) + e^{\alpha t} f(x - c^- t) \right] + \frac{1}{2} e^{-\mu x - \sigma t} \int_{x-c^+ t}^{x+c^- t} e^{\mu \xi} \left[ F(x - \xi, t) f(\xi) + G(x - \xi, t) g(\xi) \right] d\xi, \quad (16)
\]

where $a^- = 2\alpha k^-$, $a^+ = -2\alpha k^+$, $\mu = k^+ + k^-$ and $\sigma = k^- c^- - k^+ c^+$. For $-c^- t < x < c^+ t$ and $4k^- k^+ = k^2 > 0$, the propagators $F$ and $G$ are given by

\[
F(x, t) = -\Gamma J_0(k s(x, t)) + \Lambda t \frac{J_1(k s(x, t))}{s(x, t)}, \quad (17)
\]

\[
G(x, t) = \frac{1}{\Lambda} J_0(k s(x, t)), \quad (18)
\]
where \( s(x,t) = \sqrt{c^2t^2 + 2\nu xt - x^2} \) and \( \Gamma = k^+ - k^- \). Eq. (16) is valid only in the interval \( [x-c^-t, x+c^+t] \), called domain of dependence \([22, 23]\), that supports the finite bands formation and discontinuous front as found in simulations \([8, 13, 16]\).

From Eq. (19), the analytical solution indicates that initial profiles of the small perturbed density and polarization fields lose their configuration and propagate in both positive and negative direction of x-axis with unequal speed. Due to \( c^+ > c^- \), the propagation in the positive direction is faster than in the negative one. Below the critical point, \( \varepsilon < \varepsilon_0 \), we found that \( k^+ > 0 \) when \( \varepsilon < \frac{7}{11} \varepsilon_0 \) and \( k^- < 0 \) when \( \varepsilon > \frac{7}{11} \varepsilon_0 \) while \( k^- \) is always negative \([24]\).

Therefore, the propagating waves in SPP trend move in the direction of mean polarization vector as found in simulations \([9, 12, 14, 18, 19]\).

To this point, there exists another transition noise strength at \( \frac{7}{11} \varepsilon_0 \) that separates the spatiotemporal pattern formation of the propagating wave in the flocking state of SPP into two regimes as mentioned by Chaté \textit{et al.} \([13]\). Let us consider the unstable regime where \( 4k^-k^+ = k^2 > 0 \). The asymptotic Bessel function is given by \( J_m(ks) = \sqrt{\frac{2}{\pi ks}} \cos(ks - \frac{\pi}{4} - \frac{m\pi}{2}) \) for \( s \gg 1 \). With this character, it shows that the perturbations of the unstable ordered state propagate as waves with spatial oscillatory pattern or multiple-bands in 2D, that has been observed in simulations \([13, 17, 19]\). The wave profiles grow fastest in the position of the leading front and grow slower for the tandem position \([20]\), at least in the early stage. Therefore, \( k \) is equivalent to wavenumber which relates to the wavelength \( \lambda_w \) as follow: \( \lambda_w = \frac{2\pi}{k} = \frac{\pi}{\sqrt{k^-k^+}} \). That is, we have

\[
\lambda_w = 2\pi \frac{\frac{9}{11} \left(1-\varepsilon'\right) + 1}{\sqrt{\frac{11}{2} \left(\varepsilon' - \frac{7}{11} \right) \left(1-\varepsilon'\right) \varepsilon_0}} v_0 \quad (19)
\]

where \( \varepsilon' = \frac{\varepsilon_0}{\varepsilon_0} \). The wavelength in Eq. (19) can be used to approximate the stripes width and it shows that the particle moving speed \( v_0 \) has a role on regulation of the bands width. In the opposite situation, for the stable regime that \( 4k^-k^+ = -k^2 < 0 \), the propagators change to

\[
F(x, t) = -\Gamma I_0(ks(x,t)) - \Lambda k t \frac{I_1(ks(x,t))}{s(x,t)}, \quad \text{(20)}
\]

\[
G(x, t) = \frac{1}{\Lambda} I_0(ks(x,t)). \quad \text{(21)}
\]

From the asymptotic form of the modified Bessel functions, given by \( I_m(k) \sim \frac{1}{\sqrt{2\pi k}} e^{ks} \) for \( s \gg 1 \), it shows the non-oscillatory wave patterns or a single band in 2D.

In the stable ordered state, the perturbations decay to eventually smaller values thus our linear approximation should be valid in long time scale. In long time scale \( t \to \infty \), we approximate \( s \approx ct + \frac{\varepsilon}{c^+}x - \frac{1}{2} \frac{\lambda^2}{c^+} x^2 \). Thus, the perturbations converge to the homogeneous ordered state as biased Gaussian waves for below noise threshold. However, the homogeneous ordered state does not observe in the simulation and this state is replaced by the fluctuating flocking state \([13, 17]\).

In conclusion, based on the linearized hydrodynamic theory of self-propelled particles, the small perturbed density and polarization fields are governed by the hyperbolic partial differential equations. Our analytical hyperbolic wave solutions reveal some different aspect of spatiotemporal pattern formations in self-propelled particles, as opposed to the previous analytical studies, that rely on the dispersive wave solution. Below critical noise strength, the homogeneous disordered state is unstable, growing to the ordered state, and generates the vortex flow of perturbation polarization field. The perturbations in the homogeneous ordered state evolve as two possible unidirectional longitudinal propagating waves, separated by a threshold noise strength. This includes single band in the stable state below the threshold value and multiple-bands in the unstable state above the threshold value. We believe that these special case solutions could provide the basic knowledge for study the dynamics of generic self-propelled particles in the future work.

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[26] Supplemental Material can be found at [URL will be inserted by publisher].
Supplemental Material “Propagating wave in the flock of self-propelled particles”

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RIEMANN METHOD

For convenience in further calculation, we rewrite equation (15) in the main text

\[ L[\phi] = \phi_{\eta \xi} + k^- \phi_{\eta} + k^+ \phi_{\xi} = 0, \]  

where \( L \) is linear operator. From equation (14) in the main text, we have that \( t = \frac{\eta \xi}{t^2 + t^2 + t^2} \).

When \( t = 0 \), we have \( \eta = \xi = x \), which is the straight line in the \( \eta \xi \)-plane. Therefore, the initial conditions in \( \eta \xi \)-plane (Cauchy data) are transformed to

\[ \phi|_{t=0} = f(\xi), \]  

\[ M[\phi]|_{t=0} = \frac{1}{\Lambda} (\phi_\eta(\xi) + \nu_\xi(\xi)) = \frac{1}{\Lambda} g(\xi), \]

where we define operator \( M[s] = \partial_s \ast \partial_\ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \a
which is first-order modified Bessel function [4]. And it shows that this Riemann function satisfies all required conditions.

As shown in the main text, we transform the solution in Eq. (7) back to \(xt\)-plane by using the Riemann function Eq. (8) subjected to the initial data Eq. (2) and Eq. (3). Since \(\xi\) becomes dummy variable now, we let \(\eta_0 = x+c^-t\) and \(\xi_0 = x-c^+t\).

**Stability analysis**

We now find stability of the obtained analytical solution by considering the exponential factor in Eq. (8) along \(\eta = \xi\) line that \(A(\xi, \xi) = \exp[k^+ (\xi - \eta_0) + k^- (\xi - \xi_0)]\). For \(\varepsilon < \varepsilon_0\), it found that \(k^- < 0\), due to \(c^\pm\), \(\alpha\) and \(\beta\) are always positive, while \(k^+\) can be either negative or positive. Solving the inequality, we found that \(k^+ < 0\) if \(\varepsilon > \frac{7}{11}\varepsilon_0\) and \(k^+ > 0\) if \(\varepsilon < \frac{7}{11}\varepsilon_0\). For \(\xi_0 \leq \xi \leq \eta_0\) or equivalent to \(-c^-t \leq x - \xi \leq c^+t\), therefore \(A(\xi, \xi)\) always decays when \(0 < \varepsilon < \frac{7}{11}\varepsilon_0\) (stable regime). In contrast, \(A(\xi, \xi)\) can grow when \(\frac{7}{11}\varepsilon_0 < \varepsilon < \varepsilon_0\) (unstable regime). The growth rate is highest at \(\xi = \xi_0\) and trends to decrease as \(\xi < \xi_0\). Consequently, \(k^2 > 0\) if \(\frac{7}{11}\varepsilon_0 < \varepsilon < \varepsilon_0\) (unstable regime) and \(k^2 < 0\) if \(0 < \varepsilon < \frac{7}{11}\varepsilon_0\) (stable regime). Using the relation \(J_m(s) = i^{-m}J_m(is)\) where \(J_m(s)\) is the Bessel function of order \(m\), the Riemann function for \(k^2 < 0\) changes to \(R(\eta, \xi) = A(\eta, \xi)J_0(k\sqrt{(\eta - \eta_0)(\xi - \xi_0)})\).

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