Strings, matrix models, and meanders*†

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I briefly review the present status of bosonic strings and discretized random surfaces in $D > 1$ which seem to be in a polymer rather than stringy phase. As an explicit example of what happens, I consider the Kazakov–Migdal model with a logarithmic potential which is exactly solvable for any $D$ (at large $D$ for an arbitrary potential). I discuss also the meander problem and report some new results on its representation via matrix models and the relation to the Kazakov–Migdal model. A supersymmetric matrix model is especially useful for describing the principal meanders.

1. Introduction

I begin this talk with a brief review of the present status of the bosonic Polyakov string and discretized random surfaces in $D > 1$ dimensional embedding space. As an explicit example of what happens, I consider then the Kazakov–Migdal model with a logarithmic potential which is exactly solvable for any $D$ (at large $D$ for an arbitrary potential). I discuss at the end the challenging meander problem which is more complicated than the ones solved before by means of the matrix-model technique but maybe is simpler than the large-$N$ QCD.

2. Bosonic string in $D > 1$

2.1. The $D = 1$ barrier

The $D = 1$ barrier is associated with the KPZ (Knizhnik–Polyakov–Zamolodchikov) formula \[ \gamma_{\text{str}} = \frac{D - 1}{12} \] (which was in fact known [3,4] before KPZ)

for the critical index of string susceptibility of the bosonic Polyakov string in a $D$-dimensional embedding space. Alternatively, it describes two-dimensional quantum gravity interacting with conformal matter of the central charge $c = D$.

The right-hand-side (RHS) of Eq. (2.1) is well-defined for $D \leq 1$, where it is associated with topological theories of gravity which can be described also by (multi-) matrix models. The RHS becomes complex for $1 < D < 25$ which is physically unacceptable. There are two alternatives of how to interpret this fact:

- KPZ-approach is not applicable for $D > 1$ (say other structures of conformal theory may become relevant).
- KPZ-approach is correct but the interpretation of the result is not straightforward (say the theory no longer describes a string).

I discuss below that the second alternative realizes: the theory is in a branched polymer rather than stringy phase.

2.2. The DDSW-mechanism

The mechanism which governs the Polyakov string (= discretized random surfaces) in $D > 1$ was discovered by Das, Dhar, Sengupta and Wadia [5]. They considered the curvature matrix models, where the propagator is modified as

\[ \langle \Phi_{ij} \Phi_{kl} \rangle_{\text{Gauss}} \Rightarrow A_{ij}^{-1} A_{kl}^{-1} \] (2.2)

with the Hermitean $N \times N$ matrix $A$ describing an external field.

The modification (2.2) of the propagator results in the following modification of the

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quadratic part of the potential
\[ \text{tr} \Phi^2 \Rightarrow \text{tr} A \Phi A \Phi. \tag{2.3} \]

These models are solved as \( N \to \infty \) for a logarithmic (Penner) potential in [3] and for an arbitrary potential in [4].

The partition function in the external field \( A \) acquires the extra multiplier
\[ Z_A \propto \prod_{a=\text{vertex}} \frac{\text{tr}}{N} \left( A^{-1} \right)^{\Delta_a}, \tag{2.4} \]
where \( \Delta_a \) is the coordination number of a given vertex \( a \) (which describes intrinsic curvature). Representing the matrix \( \Phi \) in the form \( \Phi = \Omega^\dagger \Omega \) with diagonal \( \Lambda \) and integrating over the unitary matrix \( \Omega \), one gets the action which involves interaction terms in the form of the products \( \text{tr} \Phi^k \cdot \cdot \cdot \text{tr} \Phi^2 \text{tr} \Phi^2 \).

The simplest such a modification of the action reads
\[ S = \frac{\Lambda}{2} \text{tr} \Phi^2 + \frac{N}{2} \text{tr} \Phi^4 + g' \text{tr} \Phi^2 \text{tr} \Phi^2. \tag{2.5} \]
The coupling constant \( g' \) describes a new kind of interaction due to touching of surfaces. These touching diagrams do not vanish as \( N \to \infty \) because of the Weingarten arguments [7]: the smallness \( (N^{-2}) \) of connected correlators is compensated by the fact that the action \( \sim N^2 \). More than one touching of the same surfaces is suppressed as \( N^{-2} \).

As was shown by DDSW, the critical index \( \gamma_{\text{str}} \) can be expressed at some value of \( g' \) via that \( (\gamma_{c<1}) \) without the touching interaction by the formula
\[ \gamma_{\text{str}} = \frac{\gamma_{c<1}}{\gamma_{c<1} - 1}. \tag{2.6} \]
While only \( \gamma_{c<1} = -1/2 \) is possible for the quartic self-interaction \( (2.5) \) with \( g' = 0 \), this formula can be extended to arbitrary \( \gamma_{c<1} \) which is associated with the \( c < 1 \) KPZ-formula \( (2.1) \).

2.3. Trees of baby universes

Equation \( (2.6) \) can alternatively be obtained by making a resummation in the sum over surfaces with touching included. Typical surfaces which lead to the critical behavior \( (2.6) \) in \( D > 1 \) are trees constructed from closed two-dimensional surfaces (baby universes) as is depicted in Fig. 1.

At each of 2d surfaces, the continuum limit with \( \gamma_{c<1} < 0 \) is achieved by tuning the cosmological constant while Eq. \( (2.6) \) with \( \gamma_{\text{str}} > 0 \) in the \( D \)-dimensional embedding space can be reached by tuning the coupling of the touching interaction.

The case of \( \gamma_{c<1} = -1 \), which is associated with no critical behavior at the 2d surface at all, results due to Eq. \( (2.6) \) in \( \gamma_{\text{str}} = 1/2 \) — the typical mean-field value for pure branched polymers.

The case of \( \gamma_{c<1} = -1/m \), which is associated with the standard critical behavior of 2d gravity (with matter), results in \( \gamma_{\text{str}} = 1/(m + 1) \) due to polymerization. It differs from the mean-field value \( \gamma_{\text{str}} = 1/2 \) due to effects of 2d gravity. This reminds of how the critical index \( \gamma_{c<1} = -1/3 \) appears for the Ising model on a random lattice.

The formula \( (2.6) \) for the critical behavior of random surfaces (= Polyakov string) in the \( D \)-dimensional embedding space

1) describes numerical data for \( c > 1 \) theories (see [3] and references therein),
2) can be derived \[9\] from the Liouville gravity (by changing the gravitational dressing from \(\exp(\alpha + \phi)\) to \(\exp(\alpha - \phi)\)),

3) is rigorously proven \[10\] assuming locality,

4) can be understood at large \(D\) (see Subsect. 3.6 below).

Thus the bosonic Polyakov string (= discretized random surfaces) are in a branched polymer rather than stringy phase for \(D > 1\). The typical surfaces are crumpled rather than smooth. Such a ground state is stable and has no tachionic excitations. This picture is expected for any matrix model describing discretized random surfaces in \(D > 1\). I consider in the next Section an explicit example of the Kazakov–Migdal model \[11\].

3. The Kazakov-Migdal-Penner model \[12\]

3.1. Three equivalent models

A natural multi-dimensional extension of the matrix chain is the Kazakov–Migdal (KM) model \[11\] which is defined by the partition function

\[
Z_{KM} = \int \prod_{\{x,y\}} dU_{xy} \prod_{x} d\phi_{x} \times e^{N_{\text{tr}} \left( -\sum_{x} V(\phi_{x}) + c \sum_{(x,y)} \phi_{x} U_{xy}^{\dagger} \phi_{y} U_{xy} \right)}. \tag{3.7}
\]

Here \(\phi_{x}\) and \(U_{xy}\) are \(N \times N\) Hermitean and unitary matrices, respectively, with \(x\) labeling lattice sites and \(\{xy\}\) labeling the link from the site \(x\) to a neighbor site \(y\).

The lattice can be one of the following:

i) an infinite \(D\)-dimensional hypercubic lattice \[11\],

ii) a Bethe tree \[13\],

iii) a \(q\)-simplex \[12\] (see Fig. 2).

These three models are equivalent as \(N \to \infty\) at strong coupling when the coordination numbers \(\Delta = 2D = q - 1\) coincide. In particular, the model on a triangle \((q = 3)\) is equivalent to that on a one-dimensional chain \((D = 1)\). The one-matrix model is associated with \(q = 1\) or \(D = 0\), while the two-matrix model is described by \(q = 2\) or \(D = 1/2\). The gauge field \(U_{12}\) can be absorbed in the latter case by a unitary transformation of \(\phi\) so that the standard Hermitean two-matrix model is recovered. The proof of equivalence is presented in \[12\].

The integration over the gauge field \(U_{xy}\) is over the Haar measure on \(SU(N)\) at each link of the lattice. The model (3.7) obviously recovers the standard open matrix chain if the lattice is just a one-dimensional sequence of points for which the gauge field can be absorbed by a unitary transformation of \(\phi_{x}\).

3.2. Loop equations

A convenient way of solving the KM model is via the loop equations which are written for the one-link correlator

\[
G_{\nu\lambda} = \left\langle \frac{1}{N} \left( \frac{1}{\nu - \phi_{x}} U_{xy}^{\dagger} \frac{1}{\lambda - \phi_{y}} U_{xy} \right) \right\rangle. \tag{3.9}
\]

One gets

\[
G_{\nu\lambda} \overset{\nu \to \infty}{\rightarrow} \frac{E_{\lambda}}{\nu} + \ldots, \quad E_{\lambda} \equiv \left\langle \frac{1}{N} \left( \frac{1}{\lambda - \phi_{x}} \right) \right\rangle \tag{3.10}
\]

at asymptotically large \(\nu\).

The loop equation has the same form \[14\] as the one for the two-matrix model

\[
\int_{C_{1}} \frac{d\omega}{2\pi i (\nu - \omega)} G_{\omega \lambda} \frac{\mathcal{V}'(\omega)}{2} = E_{\nu} G_{\nu\lambda} + c (\lambda G_{\nu\lambda} - E_{\nu}) \tag{3.11}
\]

with the potential

\[
\mathcal{V}'(\omega) \equiv \mathcal{V}'(\omega) - (\Delta - 1) F(\omega), \tag{3.12}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Lattice in the form of a \(q\)-simplex (depicted for \(q = 5\)).}
\end{figure}
where $F(\omega)$ is determined by the pair correlator of the gauge fields

$$F(\phi_{ij}) = \frac{\int dU e^{c N A \phi U^+ U (U^+ \psi U)} \omega_{ij}}{\int dU e^{c N A \phi U^+ U}}.$$  \hspace{1cm} (3.13)

The contour $C_1$ in Eq. (3.11) encircles counterclockwise singularities of the function $G_{\omega \lambda}$ so that the integration over $\omega$ plays the role of a projector picking up negative powers of $\nu$.

The $1/\lambda$ term of Eq. (3.11) reads

$$\int_{C_1} \frac{d\omega}{2\pi i} \frac{\tilde{V}'(\omega)}{\lambda - \omega} E_\omega = E_\lambda^2,$$  \hspace{1cm} (3.14)

which coincides with the loop equation for the Hermitian one-matrix model with the potential $V'(\omega) = V' - D F(\omega) - \Delta$.  \hspace{1cm} (3.15)

This does not mean that the KM model reduces in general to the one-matrix model since $V'(\omega)$ may have singularities outside of the support of eigenvalues of the master field $\phi_3$ whose spectral density $\rho(\omega)$ is parametrized by $\tilde{V}(\omega)$ as

$$\tilde{V}'(\omega) = 2 \int dx \frac{\rho(x)}{\omega - x}.$$  \hspace{1cm} (3.16)

The solution to Eq. (3.11) consists of the following steps.

1) Given $V'(\omega)$ find $G_{\omega \lambda}$ and $E_\lambda$.

2) Calculate $\tilde{V}'(\omega) = 2 \Re E_\lambda$ at the cut.

3) Then $V' = \Delta V' - (1 - \Delta) \tilde{V}'$ and $F = V' - \tilde{V}'$ are completely determined.

It is evident that $V = \tilde{V}$ for the one-matrix model ($\Delta = 0$) and $V = V$ for the two-matrix model ($\Delta = 1$) when the solutions are known.

3.3. Two explicit solutions

The exact solutions of the KM model are known for

- the Gaussian (quadratic) potential \[15\],
- the Penner (logarithmic) potential \[16\].

The quadratic potential can be conveniently parametrized as

$$V(\phi) = \frac{m^2}{2} \phi^2 = \frac{1}{2} \left[ a \frac{b}{b} + (2D - 1)c^2 \frac{b^2}{a} \right] \phi^2,$$  \hspace{1cm} (3.17)

where the parameter $a/b$ is $D$-independent. The solution is then described by the one-matrix model with the potential

$$\tilde{V}(\phi) = \frac{1}{2} \left[ \frac{a}{b} - c^2 \frac{b^2}{a} \right] \phi^2,$$  \hspace{1cm} (3.18)

which can be obtained substituting $D = 0$ in Eq. (3.17). The function (3.13) reads

$$F(\phi) = c^2 \frac{b}{a} \phi.$$  \hspace{1cm} (3.19)

The exactly solvable logarithmic potential reads in the same notations

$$V(\phi) = -a \ln (b - \phi) - (2D - 1)(\alpha + 1) \times \ln (a + c\phi) + [(2D - 1)c b - a] \phi$$  \hspace{1cm} (3.20)

and

$$F(\phi) = c \left[ b - \frac{\alpha + 1}{a + c \phi} \right].$$  \hspace{1cm} (3.21)

The Gaussian solution is recovered when $\alpha = ab$, $a \sim b \to \infty$.  \hspace{1cm} (3.22)

As is shown in [16], the quartic potential is reproduced in the naive continuum limit for $D < 4$.

Introducing new variables

$$\phi \to \phi + \frac{c b - a}{2 c} 1, \quad \beta = \frac{a + c b}{2},$$  \hspace{1cm} (3.23)

one gets

$$\tilde{V}(\phi) = -a \ln (\beta - \phi) + (\alpha + 1) \ln (\beta + \phi) - 2 \beta \phi.$$  \hspace{1cm} (3.24)

The one-cut solution of the one-matrix model with the potential (3.24) reads \[14 \& 17\]

$$E_\lambda = \tilde{V}'(\lambda) \frac{2}{\sqrt{(\lambda - x_-)(\lambda - x_+)}},$$  \hspace{1cm} (3.25)

where $z$ is determined by the cubic equation

$$z^3 - z \left( \beta^2 - \alpha - \frac{1}{2} \right) + \frac{\beta}{2} = 0$$  \hspace{1cm} (3.26)

and

$$x_{\pm} = \frac{1}{2 \beta} \pm \sqrt{\frac{(\beta^2 - z^2)(4 \beta z - 1)}{2 \beta z}}$$  \hspace{1cm} (3.27)

are the ends of the cut. The behavior of the eigenvalue support and branched cuts of the logarithms are depicted in Fig. 3 for various values of $\alpha$.\]
3.4. Critical behavior

While the only continuum limit of the KM model with the quadratic potential is possible in $D = 1$, the logarithmic potential (3.20) reveals a rich phase structure. The critical behavior emerges when:

i) The spectral density ceases to be positive at the interval $[x_-, x_+]$ as for the one-matrix models with a polynomial potential.

ii) $x_-$ approaches $-\beta$.

iii) $x_-$ approaches $x_+$. The critical behavior the type i) occurs along the line

$$\alpha_c = \beta^2 - 3 \left( \frac{\beta^2}{4} \right)^{\frac{1}{2}} - \frac{1}{2},$$

(3.28)

where $x_+ = -z$. At this line the discriminant of the cubic equation (3.26) vanishes and the one-cut solution is not applicable for $\alpha > \alpha_c$. The critical behavior of the types ii) and iii) occurs for $\alpha = 0$ and $\alpha = -1$ respectively. These critical lines are depicted in Fig. 4. The critical behavior agrees with that for the one-matrix model with the cubic and Penner potentials which can be obtained from of the potential (3.24) in the limits

$$\alpha \sim \beta^2 \rightarrow \infty, \quad (\beta^2 - \alpha) \sim \beta^{2/3} \quad \text{(cubic limit)} \quad (3.29)$$

and

$$\beta \rightarrow \infty, \quad \alpha \sim 1 \quad \text{(Penner limit),} \quad (3.30)$$

respectively.

Figure 3. Eigenvalue support of the spectral density (the bold line) and the branch cuts of the logarithms (the thin lines): a) for $\alpha > 0$, b) for $\alpha \rightarrow +0$, c) for $-1 < \alpha < 0$, d) for $\alpha \rightarrow -1$ and e) for $\alpha < -1$.

Figure 4. Phase diagram of the KM model with the logarithmic potential (3.20). The bold line which starts at $\beta = 0$, $\alpha = -1/2$ corresponds to Eq. (3.28). The one-cut solution realizes for $\alpha < \alpha_c$. The critical lines $\alpha = \alpha_c$ and $\alpha = -1$ correspond to $\gamma_{str} = -1/2$ and $\gamma_{str} = 0$, respectively, while the tricritical point $\beta = 1/2, \alpha = -1$ is associated with a phase transition of the Kosterlitz–Thouless type.
The character of the critical behavior can be find out by calculating the susceptibility
\[
\chi \equiv -\frac{1}{N^2 \text{Vol.} \frac{d^2}{da^2}} \ln Z_{KM} \tag{3.31}
\]
where Vol. stands for the volume of the system and the number of sites of the lattice. The result in genus zero reads explicitly
\[
\chi_0 = (D-1) \ln \left\{ \frac{1}{4} \left( \sqrt{\frac{\beta + x_+ (-\beta - x_+)}{\beta + x_- (-\beta - x_+)}} \right) \pm \sqrt{\frac{(\beta + x_+)(-\beta - x_-)}{(\beta + x_-)(-\beta - x_+)}} \right\}
+ \ln \left\{ \left( \sqrt{\frac{\beta + x_-}{\beta + x_+}} + \sqrt{\frac{\beta + x_+}{\beta + x_-}} \right)^2 \right\}
- \ln \left\{ \left( \sqrt{\frac{\beta + x_+}{\beta - x_-}} \pm \sqrt{\frac{\beta - x_-}{\beta + x_+}} \right)^2 \right\} \tag{3.32}
\]
where the positive sign in \(\pm\) corresponds to \(\alpha > 0\) while the minus sign should be substituted for \(\alpha < 0\).

Having the explicit formula (3.32) for \(\chi_0\), we can find out which \(\gamma_{str}\) is associated with each type of the critical behavior. Near the line (3.28) where \(\chi_0\) is not singular and equals some value \(\chi_0^c\), one gets
\[
\chi_0 - \chi_0^c \sim (x_+ - x_-^c) \quad \text{for} \quad \alpha \approx \alpha_c. \tag{3.33}
\]

Since \((x_+ - x_-^c) \sim (\alpha_c - \alpha)^{1/2}\), one obtains \(\gamma_{str} = -1/2\) near the critical line (3.28).

On the contrary, \(\chi_0\) given by Eq. (3.32) has logarithmic singularities for \(\alpha = 0\) and \(\alpha = 1\):
\[
\chi_0 \approx \frac{1}{2} \ln (\beta - x_+) \quad \text{for} \quad \alpha \approx 0 \tag{3.34}
\]
and
\[
\chi_0 \approx -\ln (x_+ - x_-) \quad \text{for} \quad \alpha \approx 0. \tag{3.35}
\]

Near the critical lines \(\alpha = 0\) and \(\alpha = 1\), one gets \(\gamma_{str} = 0\). While (3.33) is positive, (3.34) is negative. For this reason we exclude the critical line \(\alpha = 0\) from the consideration.

### 3.5. Continuum limits

#### 3.5.1. \(\gamma_{str} = -1/2\)

The continuum theory near \(\alpha = \alpha_c\) given by Eq. (3.28) reminds 2d gravity and can be obtained expanding near the edge singularity:
\[
z = \left( \frac{\beta}{4} \right)^{1/2} + \varepsilon \sqrt{\Lambda}, \quad \lambda = \left( \frac{\beta}{4} \right)^{1/2} + \varepsilon \xi, \tag{3.36}
\]
where \(\varepsilon \to 0\), \(\Lambda\) is the cosmological constant, and \(\xi\) is the continuum momentum variable.

The continuum spectral density is determined by Eq. (3.25) to be
\[
\rho_c(\xi) \propto \frac{1}{\pi} \left( \xi + \sqrt{\Lambda} \right) \sqrt{\xi - 2 \sqrt{\Lambda}} \tag{3.37}
\]
and describes all continuum correlators of the trace of powers of the (renormalized) field \(\phi_x\) at some point \(x\), which is the standard set of observables of 2d gravity.

The critical behavior of matter is described by observables associated with extended objects — the open-loop averages
\[
G_{\nu\lambda}(C_{xy}) = \left\langle \frac{1}{N} \left( \frac{1}{\nu - \phi_x} U_{xy} \frac{1}{\lambda - \phi_y} U_{xy}^\dagger \right) \right\rangle, \tag{3.38}
\]
where \(C_{xy}\) goes from \(x\) to \(y\) along some path and the average is w.r.t. the same measure as in (3.7). They depend at large \(N\) only on the algebraic length \(L(C_{xy})\) of the contour \(C_{xy}\).

The double discontinuity
\[
\begin{align*}
C(\nu, \lambda; L) &\equiv \frac{1}{\pi^2 \rho(\nu) \rho(\lambda)} \text{Disc}_\nu \text{Disc}_\lambda G_{\nu\lambda}(C_{xy}). \tag{3.39}
\end{align*}
\]
across the cut determines \(C(\nu, \lambda; 1)\) — the one-link Itzykson–Zuber correlator of the gauge fields which reads
\[
C(\nu, \lambda; 1) \propto \frac{1}{\varepsilon^2 (\xi_\nu - \xi_\lambda)^2} \tag{3.40}
\]
as \(\varepsilon \to 0\). It is quite similar to the Gaussian one in the naive continuum limit at \(D = 1\).

The RHS of Eq. (3.40) describes \(C(\nu, \lambda; L)\) for \(L \ll 1/\sqrt{\varepsilon}\) while a nontrivial continuum limit of the matter correlator (3.38) sets in for \(L \sim 1/\sqrt{\varepsilon}\)
to be
\[ C_c(x, y; \sqrt{u}) = \frac{2\sqrt{u}}{(x - y)^2 + 2u(x + y)xy + u^2x^2y^2} \] (3.41)
with \( u \propto L^2 \varepsilon \). The expression (3.41) obeys the following convolution property
\[ \frac{1}{\pi} \int_0^\infty dt \, t^2 C_c(x, t; \sqrt{u}) C_c(t, y; \sqrt{u}) = C_c(x, y; \sqrt{u} + \sqrt{v}) \] (3.42)
which is analogous to that \([18]\) for the Gaussian case.

3.5.2. \( \gamma_{\text{str}} = 0 \)

The continuum theory near the critical line \( \alpha \approx -1 \) looks like 2d gravity + 1d matter. The continuum spectral density reads
\[ \rho_c(\xi) \propto \frac{1}{\pi} \frac{\sqrt{\xi^2 - 4\Lambda}}{\xi^2} \] (3.43)
The Itzykson–Zuber correlator
\[ C(\nu, \lambda; 1) = \frac{4\beta^2}{4\beta^2 - 1} + O(\varepsilon) \] (3.44)
is non-singular if \( \beta \neq 1/2 \). There is, hence, no unusual behavior of the matter correlators in this case.

3.5.3. Tricritical point

At the tricritical point \( \beta = 1/2 \) \( \alpha = -1 \), the continuum system undergoes a phase transition of the Kosterlitz–Thouless type between the phases with \( \gamma_{\text{str}} = -1/2 \) and \( \gamma_{\text{str}} = 0 \).

This domain is most interesting since in the vicinity of the tricritical point the singular part of the susceptibility
\[ \chi_0 = -\log \left( \frac{1 - \kappa}{1 + 3\kappa} \right) - 2D \log (\sqrt{2}\delta) \] (3.45)
is \( D \)-dependent. Here the deviation of the tricritical point is parametrized by \( \delta \alpha = (3\kappa + 1)(1 - \kappa)(\delta \beta)^2 \).
\[ \delta \alpha = (3\kappa + 1)(1 - \kappa)(\delta \beta)^2. \] (3.46)
This region is the only one where \( \gamma_{\text{str}} \) might depend on \( D \) but it was not investigated in [12].

3.6. Large-\( D \) limit

The Itzykson–Zuber integral
\[ I[\phi_x, \phi_y] \equiv \int dU \, e^{c\text{tr} \, \phi_x U^\dagger \phi_y U}, \] (3.47)
which enters the partition functions (3.7), can easily be calculated as \( c \to 0 \):
\[ \ln I[\phi_x, \phi_y] = \text{ctr} \, \phi_x \text{tr} \, \phi_y \]
\[ + \frac{c^2 N^2}{2} \left[ \frac{\text{tr} \, \phi_x^2}{N} - \left( \frac{\text{tr} \, \phi_x}{N} \right)^2 \right]\]
\[ \times \left[ \frac{\text{tr} \, \phi_y^2}{N} - \left( \frac{\text{tr} \, \phi_y}{N} \right)^2 \right] + O(c^3). \] (3.48)

If \( V(\phi_x) \sim 1 \) as \( D \to \infty \), then \( c \sim 1/D \) for the kinetic term to be of order one and only the first term is left on the RHS of Eq. (3.48). The partition function (3.7) can be written as
\[ Z = \int \prod_x d\phi_x \]
\[ \times e^{-\frac{c}{N} \sum_x \text{tr} \, V(\phi_x) + c \sum_{\langle x, y \rangle} \text{tr} \, \phi_x \text{tr} \, \phi_y}. \] (3.49)

Further simplification occurs in the large-\( N \) limit when we can replace one trace in the product of two traces in the exponent in (3.49) by the average value due to factorization. One arrives, hence, at the one-matrix model whose potential \( \tilde{V}(\phi) \) is determined self-consistently from the equation
\[ \tilde{V}(\phi) = V(\phi) - 2cD \left\langle \frac{\text{tr} \, \phi}{N} \right\rangle \tilde{\phi}, \] (3.50)
where \( \left\langle \right\rangle_{\tilde{\phi}} \) stands for the averaging in the one-matrix model with the potential \( \tilde{V} \).

For the one-cut solution of the Hermitean one-matrix model, one rewrites Eq. (3.50) as
\[ \tilde{V}^\prime(\lambda) = V^\prime(\lambda) - 2cD \int_{C_1} \frac{d\omega}{4\pi i} \tilde{V}^\prime(\omega) \]
\[ \times \sqrt{\omega - x_+}(\omega - x_-) + \frac{x_- + x_+}{2} \] (3.51)
which is an equation for \( \tilde{V} \).

Let us identify the cosmological constant with \( g_1 \) — the coupling in front of the linear term of
the potential. For the susceptibility in genus zero one gets
\[ \chi_0 = \int_{C_1} \frac{d\omega}{2\pi i} V(\omega) \dot{E}_\omega = \frac{1}{2} \hat{g}_1 \frac{(x_--x_+)^2}{16} \] (3.52)
where \( \dot{f} \equiv \partial f/\partial g_1 \), while Eq. (3.51) yields
\[ \frac{\dot{g}_1}{1 + cD(x_--x_+)^2} \] (3.53).

To obtain the critical behavior, we expand near \( x_- = x^c \), which gives for a \( k \)-th multicritical point of the one-matrix model:
\[ x_- - x^c \sim (\hat{g}_1 - \hat{g}_1)^{1/k}. \] (3.54)
Under normal circumstances when Eq. (3.33) holds, one gets from (3.54)
\[ \chi_0 - \chi^c \sim (\hat{g}_1 - \hat{g}_1)^{1/k} \] (3.55)
so that \( \gamma_{str} = -1/k \) since
\[ (g_1^c - g_1) \sim (\hat{g}_1 - \hat{g}_1). \] (3.56)

This is not the case, however, for
\[ c = -\frac{8}{D(x_--x_+)^2} \] (3.57)
when the denominator in Eq. (3.53) vanishes. At this point one has
\[ \dot{g}_1 \sim (x_- - x^c)^{-1} \] (3.58)
so that
\[ (g_1^c - g_1) \sim (\hat{g}_1 - \hat{g}_1)^{(k+1)/k} \sim (x_- - x^c)^{(k+1)}. \] (3.59)
One gets, therefore, for the susceptibility
\[ \chi_0 \sim (x_- - x^c)^{-1} \sim (g_1^c - g_1)^{-1/(k+1)} \] (3.60)
which is associated with \( \gamma_{str} = 1/(k+1) \).

The formula (3.50), which describes the reduction of the KM model to a one-matrix model at large \( N \) in the large-\( D \) limit, explicitly holds for the potential (3.24) when the exact solution is known at any \( D \). As \( D \to \infty \) with \( c \sim 1/D \), the potential \( \dot{V} \) coincides with the Penner potential so that all the calculations can be explicitly done [22]. The only possible scaling behavior is with \( \gamma_{str} = 0 \) in perfect agreement with the results of Subsect. 3.4. This seems to be a \( k \to \infty \) limiting case of \( \gamma_{str} = 1/(k+1) \) which appears from the critical behavior with \( \gamma_{str} = -1/k \) of the one-matrix model with the potential \( \dot{V} \).

Nonvanishing results for continuum correlators can be obtained in the large-\( D \) limit only for those of operators living at the same lattice site while the Itzykson–Zuber correlator for a contour of the length \( L \) is suppressed as
\[ C(\nu, \lambda; L) \sim c^L \sim D^{-L}. \] (3.61)
Therefore extended correlators vanish in the large-\( D \) limit.

4. The meander problem [19]

The meander problem is known to people working on Quantum Field Theory since the Arnold’s question to V. Kazakov in the middle of the eighties. The problem is to calculate combinatorial numbers associated with the crossings of an infinite river (Meander) and a closed road by \( 2n \) bridges. Neither the river nor the road intersects with itself. These principle meander numbers, \( M_n \), obviously describe the number of different foldings of a closed strip of \( 2n \) stamps or of a closed polymer chain.

One can consider also a generalized problem of the multi-component meander numbers \( M_n^{(k)} \) which are associated with \( k \) closed loops of the road so that \( M_n = M_n^{(1)} \). The results of a computer enumeration of the meander numbers are presented in [24,20] up to \( n = 12 \).

4.1. Hermitian matrix model for meanders

Meanders can be described by the following Hermitian matrix model [22]
\[ \mathcal{F}_{N \times N}(c) = \frac{2}{N^2} \int \prod_{a=1}^{m} dW_a e^{-\frac{1}{2} \sum_{a=1}^{m} \text{tr} W_a^2} \cdot \ln \left( \int d\phi e^{-\frac{1}{2} \text{tr} \phi^2 + \sum_{a=1}^{m} \text{tr} \phi W_a \phi W_a} \right) \] (4.62)
where the integration goes over the \( N \times N \) Hermitian matrices \( W_a \) \( (a = 1, \ldots, m) \) and \( \phi \). The logarithm in Eq. (4.62) leaves only one closed loop of the field \( \phi \). The coupling constant \( c \) is associated with the (quartic) interaction between \( W_a \) and \( \phi \).

\[ \text{See [24] for an introduction to the subject.} \]
Expanding the generating function \( \langle 4.63 \rangle \) in \( c \) and identifying the diagrams with the ones for the meanders, one relates the large-\( N \) limit of \( \mathcal{F}_{N \times N}(c) \) with the following sum over the meander numbers

\[
\lim_{N \to \infty} \mathcal{F}_{N \times N}(c) = \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{k=1}^{n} M_n^{(k)} m^k. \quad (4.63)
\]

The \( N \to \infty \) limit is needed to keep only planar diagrams as in the meander problem.

The RHS of Eq. \( \langle 4.62 \rangle \) can be expressed entirely via the Gaussian averages of \( W \)'s. This leads to the following representation of the meander numbers:

\[
\sum_{k=1}^{n} M_n^{(k)} m^k = \sum_{a_1, a_2, \ldots, a_{2n-1}, a_{2n}=1}^{m} \times \left( \frac{1}{N} \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right)^2 \quad (4.64)
\]

where the average over \( W \)'s is calculated with the Gaussian weight — the same as in \( \langle 4.62 \rangle \). This formula can be proven by calculating the Gaussian integral over \( \phi \) in Eq. \( \langle 4.62 \rangle \), expanding the result in \( c \) and comparing with the RHS of Eq. \( \langle 4.63 \rangle \). The factorization at large \( N \) is also useful.

The principle meander numbers \( M_n \) are given by Eq. \( \langle 4.64 \rangle \) as the linear-in-\( m \)-terms, i.e. as linear terms of the expansion in \( m \). This looks like the replica trick which suppresses higher loops of the field \( W \).

The ordered but cyclic-symmetric sequence of indices \( a_1, a_2, \ldots, a_{2n-1}, a_{2n} \) is often called as a word constructed of \( m \) letters. The average on the RHS of Eq. \( \langle 4.64 \rangle \) is the meaning of a word. Thus, the meander problem in equivalent to summing over all the words with the Gaussian meaning.

Since for \( m = 1 \)

\[
\frac{1}{N} \text{tr} W^{2n} \quad \text{Gauss} = \frac{(2n)!}{(n+1)!n!} = C_n, \quad (4.65)
\]

which is known as the Catalan number of the order \( n \), one gets from Eq. \( \langle 4.64 \rangle \)

\[
\sum_{k=1}^{n} M_n^{(k)} = C_n^2. \quad (4.66)
\]

This is nothing but the first sum rule of \( \langle 20 \rangle \).

### 4.2. Complex matrix model for meanders

The meander numbers can alternatively be represented as the Gaussian average over the complex matrices. The corresponding generating function reads

\[
\mathcal{F}(c) = \frac{1}{N^2} \left\langle \ln \left( \int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}} \quad (4.67)
\]

with

\[
S = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \sum_{a=1}^{m} \text{tr} \left( \phi_1 W_a \phi_2 W_a^\dagger \right). \quad (4.68)
\]

Here, \( \phi_1 \) and \( \phi_2 \) are Hermitean while \( W_a \ (a = 1, \ldots, m) \) are general complex matrices.

It is convenient to introduce one more generating function

\[
M(c) = c \times \left\langle \int d\phi_1 d\phi_2 e^{-S} \frac{\text{tr} \phi_1 W_1 \phi_2 W_1}{\int d\phi_1 d\phi_2 e^{-S}} \right\rangle_{\text{Gauss}} \quad (4.69)
\]

where only one component of \( W_a \), say the first one, enters the averaging expression. Differentiating the generating function \( \langle 4.67 \rangle \) with respect to \( c \) and noting that all \( m \) components of \( W_a \) are on equal footing, we get the relation

\[
\frac{d\mathcal{F}(c)}{dc} = mM(c) \quad (4.70)
\]

between the two generating functions.

In order to show how the complex matrix model recovers the meander numbers, let us replace \( \phi_1^{ij} \) or \( \phi_2^{ij} \) in the numerator of Eq. \( \langle 4.66 \rangle \) by \( N^{-1} \partial/\partial \phi_1^{ij} \) or \( N^{-1} \partial/\partial \phi_2^{ij} \), respectively, and integrate by parts. Repeating this procedure iteratively, we get

\[
M(c) = \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^{n} M_n^{(k)} m^{k-1}, \quad (4.71)
\]

with

\[
\sum_{k=1}^{n} M_n^{(k)} m^{k-1} = \sum_{a_2, \ldots, a_{2n-1}, a_{2n}=1} \times \left( \frac{1}{N} \text{tr} W_1 W_2^\dagger \cdots W_{a_{2n-1}} W_{a_{2n}}^\dagger \right)^2. \quad (4.72)
\]
Equation (4.72) can alternatively be derived by calculating the Gaussian integrals over \(\phi_1\) and \(\phi_2\) in Eq. (4.69) by virtue of
\[
\int d\phi_1 d\phi_2 e^{-S} = \det^{-1/2} \left[ I \otimes I - e^2 \sum_{a,b=1}^m W_a W_b^\dagger \otimes (W_b^\dagger W_a^\dagger)^t \right].
\]
(4.73)

For \(m = 1\) the formula
\[
\left\langle \frac{1}{N} \text{tr} (WW^\dagger)^n \right\rangle_{\text{Gauss}} = C_n,
\]
which is analogous to Eq. (4.66), holds for the complex matrices. This results again in Eq. (4.67).

It is instructive to consider also the case when \(W\) is a fermionic Grassmann valued matrix \(\text{à la 2}\). We shall denote the fermionic matrix as \(F\) and its conjugate as \(\bar{F}\). Then we get
\[
\left\langle \frac{1}{N} \text{tr} (FF^\dagger)^n \right\rangle_{\text{Gauss}} = \left\{ \begin{array}{ll}
0 & n = 2p \text{ (even)} \\
C_p & n = 2p + 1 \text{ (odd)} \end{array} \right.
\]
(4.75)

Since each loop of the fermionic field is accompanied by a factor of \((-1)\), we arrive at the sum rule
\[
\sum_{k=1}^n (-1)^{k-1} M_n^{(k)} = \left\{ \begin{array}{ll}
0 & n = 2p \text{ (even)} \\
C_p & n = 2p + 1 \text{ (odd)} \end{array} \right.
\]
(4.76)

This is nothing but the second sum rule of [20].

Note that the trace of the square of a fermionic matrix vanishes because of the anticommutation relation imposed on the components. This is why we did not consider Hermitian fermionic matrices and used first a representation of meanders in terms of complex matrices to discuss fermionic representation of meanders. Fermionic matrix models are a natural representation of the notion of the signature of arch configurations of [20].

\[\text{See [24] for a review.}\]

4.3. Supersymmetric matrix model for principle meander

Having the representation (4.71), (4.72) of meanders via general complex matrices (either bosonic or fermionic), we can utilize the idea of supersymmetry to kill the loops of the \(W\)-field instead of the replica trick. Let us consider the two-component \(W_a\) whose first component is bosonic while the second one is a fermionic matrix:
\[
W_a = (B, F), \quad \bar{W}_a \equiv W_a^\dagger = (B^\dagger, \bar{F}) .
\]
(4.77)

Then all the multi-component meanders in Eqs. (4.71) or (4.72) vanish and we get the following representation for the principle meander
\[
M_n = \sum_{a_2, \ldots, a_{2n-1}, a_{2n}=1}^2 \left\langle \frac{1}{N} \text{tr} BW_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right\rangle_{\text{Gauss}}
\]
\[
\times \left\langle \frac{1}{N} \text{tr} \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} B \right\rangle_{\text{Gauss}}
\]
(4.78)

where we kept trace of the order of matrices of how it appears from Eq. (4.71). The signs are essential for fermionic components.

The generating function (4.67) equals zero for the supersymmetric model since all the loops of the \(B\) and \(F\) fields are mutually cancelled. One should use alternatively the generating function (4.68) which can be represented for the supersymmetric matrix model as
\[
M(c) = \left\langle \frac{1}{N} \text{tr} BB^\dagger \ln \left( \int d\phi_1 d\phi_2 e^{-S} \right) \right\rangle_{\text{Gauss}}
\]
(4.79)

where \(S\) is explicitly given by
\[
S = \frac{N}{2} \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \text{tr} \left( \phi_1 B^\dagger \phi_2 B \right) - cN \text{tr} \left( \phi_1 \bar{F} \phi_2 F \right)
\]
(4.80)

as is prescribed by Eq. (4.68) with \(W_a\) substituted according to Eq. (4.77). The equivalence of Eqs. (4.69) and (4.79) in the supersymmetric case can be proven replacing \(B\) in the integrand on the RHS of Eq. (4.73) by \(N^{-1} \partial/\partial B^\dagger\), integrating by parts, and recalling that \(F(c) = 0\).
Equation (1.78) is a nice representation of the principle meander which looks more natural than the one based on the replica trick. A hope is that it will be simpler to solve the $m = 2$ supersymmetric model than a pure bosonic one at arbitrary $m$.

The total number of nonvanishing terms on the RHS of Eq. (4.72) for the pure bosonic case, which we shall denote as $\#_n$, is given by the following generating function

$$
\sum_{n=0}^{\infty} \#_n c^{2n} = \frac{m^2}{2} \sqrt{1 - 4(m - 1)c^2} - \frac{m^2}{2} + 1.
$$

(4.81)

This formula can be derived [19] using noncommutative free random variables.

### 4.4. Relation to the KM model

Equation (4.81) is known from the solution [15] of the KM model with the Gaussian potential. There is the following reason for that. Suppose that the matrices $W_n$ are unitary instead of the general complex ones. Then one has

$$
\langle \frac{1}{N} \text{tr} U_{a_1} U_{a_2} \cdots U_{a_{2n-1}} U_{a_{2n}}^\dagger \rangle_{\text{Haar}}
$$

$$
= \begin{cases} 
1 & \text{for closed loops} \\
0 & \text{for open loops}
\end{cases}
$$

(4.82)

where the average over the unitary matrices $U$’s is over the Haar measure.

Here the loops represent the sequences of indices $\{a_1, a_2, \ldots, a_{2n-1}, a_{2n}\}$. The nonvanishing result is only when the loop is closed and encloses a surface of the vanishing minimal area, i.e. each link of the loop is passed at least twice. This is a reflection of the so-called local confinement in the KM model.

The generating function (4.81) coincides with the following correlator in the KM model with the Gaussian potential on an infinite $D$-dimensional lattice

$$
\sum_{n=0}^{\infty} \#_n c^{2n} = \langle \frac{1}{N} \text{tr} \phi^2(0) \rangle,
$$

(4.83)

where the average is defined with the same weight as in Eq. (3.7), provided that $m = 2D$.

The solution of the KM model with the Gaussian potential can be completely reformulated as a combinatorial problem of summing over all such closed loops of a given length with all possible backtracks (or foldings) included. Its solution [23] is given by Eq. (4.81).

By virtue of the Eguchi–Kawai reduction [26], the correlator (4.83) in the KM model on the infinite lattice is equivalent to that in the reduced model given by

$$
\sum_{n=0}^{\infty} \#_n c^{2n} = \frac{\int d\phi_1 d\phi_2 \prod_{a=1}^{m} dU_a e^{-S[\phi,U]}}{\int d\phi_1 d\phi_2 \prod_{a=1}^{m} dU_a e^{-S[\phi,U]}} e^{N \text{tr} \phi_1^2}
$$

with $m = 2D$ and the reduced action being

$$
S[\phi,U] = N \text{tr} \phi_1^2 + \frac{N}{2} \text{tr} \phi_2^2 - cN \sum_{a=1}^{m} \text{tr} (\phi_1 U_a^\dagger \phi_2 U_a).
$$

(4.85)

The representation (4.84), (4.85) can now be rewritten as

$$
\sum_{n=1}^{\infty} \#_n c^{2n} = e \sum_{a=1}^{m} \langle \frac{\int d\phi_1 d\phi_2 e^{-S[\phi,U]}}{\int d\phi_1 d\phi_2 e^{-S[\phi,U]}} \text{tr} \phi_1 U_a^\dagger \phi_2 U_a \rangle_{\text{Haar}}
$$

(4.86)

since the determinant (4.73) is equal to a constant when $W_n$ are unitary. The representation (4.84) looks very similar to the generating function (4.69) of the meander numbers. The difference is that the average is over the unitary matrices in Eq. (4.86) and over the Gaussian complex matrices in Eq. (4.69).

We can interpolate between the two cases by modifying the weight for averaging over $W$’s along the line of [28]. Let us introduce

$$
\langle F [W,W^\dagger] \rangle_\alpha \equiv \prod_{a=1}^{m} \left( dW_a^\dagger dW_a \right. \\
\left. \times \ e^{-\frac{N}{2} \text{tr} \left( W_a^\dagger W_a - 1 + \frac{1}{\alpha} \right)^2 + \frac{N}{2\alpha}} \right) F [W,W^\dagger].
$$

(4.87)

Then the averaging over the Gaussian complex matrices is reproduced as $\alpha \to 0$ while the average

See [27] for a review.
over the unitary matrices is recovered as $\alpha \to \infty$ since the matrix $W_\alpha$ is forced to be unitary as $\alpha \to \infty$.

We see, thus, that the words are the same both for the meander problem and for the KM model. The only difference resides in the meaning of non-vanishing words — it is equal to one for the unitary matrices.

5. Conclusions

- The Kazakov–Migdal–Penner model is an explicit example of crumpled strings (= discretized random surfaces) in the $D > 1$ embedding space. It possesses the only continuum limits associated with lower dimensional theories of $c = 0$ or $c = 1$.
- The meander problem results in a more complicated matrix model than those solved before. It belongs to the same generic class of problems of words as the large-$N$ QCD in $D = 4$ but is presumably simpler.
- The relation of the matrix models describing meanders with the KM model might be a hint on how to solve the former ones.
- The supersymmetric matrix models of the type discussed in connection with the meander problem could be useful for discretization of super-Riemann surfaces and super-strings.

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