Additional fermionic fields onto parallelizable 7-spheres

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The geometric Fierz identities are here employed to generate new emergent fermionic fields on the parallelizable (curvatureless, torsionfull) 7-sphere ($S^7$). Employing recently found new classes of spinor fields on the $S^7$ spin bundle, new classes of fermionic fields are obtained from their bilinear covariants by a generalized reconstruction theorem, on the parallelizable $S^7$. Using a generalized non-associative product on the octonionic bundle on the parallelizable $S^7$, these new classes of algebraic spinor fields, lifted onto the parallelizable $S^7$, are shown to correctly transform under the Moufang loop generators on $S^7$.

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I. INTRODUCTION

(Classical) spinor fields are well known to be elements in the carrier space of the Spin group irreducible representations on any given spacetime that admits a spin structure, namely, if the second Stiefel–Whitney class vanishes. Spinor fields are, in particular, employed for constructing the so called bilinear covariants, consisting of tensorial quadratic forms involving the spinors. The bilinear covariants were shown to be the homogeneous part of a multivector Fierz aggregate [1]. Particularizing for 4D the Minkowski spacetime case, spinor fields were classified with respect to their bilinear covariants in the Dirac–Clifford algebra, by the so called Lounesto’s spinor field classification [2]. Further lattice generalizations in the context of quantum Clifford algebras were also studied [3]. The bilinear covariants are not independent, but constrained by the Fierz identities [2, 4]. Reciprocally, given the bilinear covariants, their associated spinor fields can be re-obtained up to a phase, by the reconstruction theorem [6, 7]. The Lounesto’s classification is based upon the U(1) gauge symmetry of the first-order equations of motion that rule spinor fields in each spinor class. However, a more general classification has been proposed in Ref. [8] encompassing spinor multiplets as realizations of (non-Abelian) gauge fields. In this more general classification, composed flagpoles, dipoles, and flag-dipoles naturally descend within fourteen disjoint classes of spinor fields, under the gauge symmetry SU(2) × U(1). In this setup, the spinor fields in the standard Lounesto’s classification were shown to be a limiting case, equivalent to Pauli singlets [8]. Further spinor representations were studied in Refs. [9, 10], with also other proposals to construct the bilinear covariants for flagpole spinors [11]. An analogous classification in the framework of second quantization and a quantum reconstruction algorithm was also proposed, being the Feynman propagator extended for regular and singular spinor fields, in Ref. [12].

In any fixed spacetime dimension, $n$, and signature, $(p, q)$, the very construction of the bilinear covariants depends on the existence of either real, or complex, or even quaternionic structures. Hence, the existence of non null bilinear covariants can be impeded by the geometric Fierz identities. Despite the natural obstructions due to the existence of algebraic and geometric structures on a given spacetime dimension/signature, the Lounesto’s spinor field classification on 4D Minkowski spacetime was successfully generalized to other spacetime dimensions and signatures, of relevance in their applications, as the emergence of fermionic fields in the respective spacetime compactifications. Spinor fields on the 7-sphere $S^7$, as an Einstein space composing the compactification $\text{AdS}_4 \times S^7$, were studied in Ref. [1], where new spinor classes were derived. On the other hand, new spinor field classes in the compactification $\text{AdS}_5 \times S^5$ were derived and investigated in Ref. [12].
representing new recently obtained fermionic solutions in string theory. More precisely, Ref. [1] proposed new classes of spinor fields on $S^7$, based on the geometric Fierz identities in Ref. [14]. The underlying structure of the geometric Fierz identities on $S^7$ was shown to sternly obstruct the amount of non null bilinear covariants were found on $S^7$. Nevertheless, further three new emergent classes of fermionic fields on $S^7$. From a more physical point of view, investigating these new classes of spinors $S^7$ may afford new fermionic solutions of first order equations of motion, that can play an important role on supergravity. In fact, one of the spontaneous compactification schemes on $n = 11$ supergravity can be implemented by the so called Freund–Rubin–Englert solution, obtained on a product manifold $AdS_4 \times S^7$ [15]. As important as the standard $S^7$, the so called parallelizable $S^7$, a curvatureless manifold that has torsion, emerges when the antisymmetric gauge field strength in the Englert’s solution exceeds the Freund–Rubin one, being identified with the Cartan–Schouten torsion on the 7-sphere.

Our main aim here is to construct new fermionic fields on the parallelizable $S^7$, that can be then obtained when new classes of $S^7$ spinor fields are lifted onto the parallelizable $S^7$. This paper is organised as follows: in Sect. II, after briefly reviewing how the geometric Fierz identities are used to derive additional spinor field classes on $S^7$, we propose a reconstruction procedure for obtaining the spinor fields, in these new classes, from the bilinear covariants and the geometric Fierz identities. Sect. III is then devoted to briefly review the parallelizable sphere, whose torsion is defined with respect to the non-associative $X$-product on the octonionic bundle. The geometric Fierz identities are used to derive the spinor field classes on $S^7$, that are going to be lifted onto the parallelizable $S^7$, whereon new fermionic fields can be then constructed through the introduction of a generalized octonionic law of transformation.

### II. GEOMETRIC FIERZ IDENTITIES AND BILINEAR COVARIANTS

Let $(M, g)$ be a manifold endowed with a metric tensor. The exterior bundle $\Omega(M) = \bigoplus_{i=0}^{\infty} \Omega_i(M)$ has endomorphisms that come from the tensor algebra quotient construction. Given a $k$-form field\(^1\) $a \in \sec \Omega^k(M)$, the grade involution, $\hat{a} = (-1)^k a$, is an automorphism; the reversion, $\tilde{a} = (-1)^{[(k/2)]} a$, for $[(k)]$ denoting the integer part of the degree $k$, is an antiautomorphism. These composition of these two morphisms define the conjugation, denoted by $\bar{a}$. The Clifford bundle can be obtained by equipping the exterior bundle with the universal Clifford product $u \diamond a = u \wedge a + u \lrcorner a$, for all 1-forms $u \in \sec \Omega^1(M)$, where $\lrcorner$ is the left contraction.

\(^1\) One calls a $k$-form field a section of an homogeneous space of the exterior bundle.
The spinor bundle of the Minkowski spacetime $\mathbb{R}^{1,3}$ is composed by spinor fields, $\psi$, carrying the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representations of the Lorentz group. The bilinear covariants are sections of the exterior bundle $\Omega(M)$. With respect to a basis $\{e^\mu\}$, they read

\begin{align}
\sigma &= \bar{\psi}\psi \in \text{sec} \Omega^0(M), \\
J &= J_\mu e^\mu \in \text{sec} \Omega^1(M), \\
S &= S_{\mu\nu} e^\mu \wedge e^\nu \in \text{sec} \Omega^2(M), \\
K &= K_\mu e^\nu \in \text{sec} \Omega^3(M), \\
\omega &= \bar{\psi}\gamma^0\gamma^1\gamma^2\gamma^3\psi \in \text{sec} \Omega^4(M),
\end{align}

where $J_\mu = \bar{\psi}\gamma_\mu \psi$, $S_{\mu\nu} = \bar{\psi}\sigma_{\mu\nu} \psi$, $K_\mu = i\bar{\psi}\gamma^0\gamma^1\gamma^2\gamma^3\gamma_\mu \psi$, are the respective components in Eqs. (1b) – (1d); $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\bar{\psi} = \psi^\dagger \gamma_0$. Besides, $\sigma_{\mu\nu} := \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. Gamma matrices satisfy a Clifford algebra named $\mathcal{C}^{1,3}$, $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I$, where $g_{\mu\nu}$ denotes the Minkowski spacetime metric components. When not both $\sigma$ and $\omega$ vanish altogether, the bilinear covariants are governed by the Fierz identities

\begin{align}
K^2 + J^2 &= 0 = J \cdot K, \quad (\omega + \sigma\gamma^0\gamma^1\gamma^2\gamma^3)S = K \wedge J, \quad \omega^2 + \sigma^2 = J^2. \quad (2)
\end{align}

Lounesto derived, from the bilinear covariants, a classification of spinor fields [2], for $\mu \neq 0$. However, this condition that was firstly motivated by the Dirac electron theory, and can be circumvented in three additional classes that were recently derived in Minkowski spacetime [16], conjectured to consist of ghost spinors. Apart from these classes of ghost spinors, the original Lounesto’s classification splits the spinor fields on Minkowski spacetime into six disjoint classes. In Eqs. (3b) – (3f) below, we just denote the bilinear covariants that do not vanish:

\begin{align}
1) \quad &\omega \neq 0, \quad \sigma \neq 0, \quad K \neq 0, \quad S \neq 0 \quad (3a) \\
2) \quad &\sigma \neq 0, \quad K \neq 0, \quad S \neq 0 \quad (3b) \\
3) \quad &\omega \neq 0, \quad K \neq 0 \quad S \neq 0 \quad (3c) \\
4) \quad &K \neq 0, \quad S \neq 0 \quad (3d) \\
5) \quad &K = 0, \quad S \neq 0 \quad (3e) \\
6) \quad &K \neq 0, \quad S = 0 \quad (3f)
\end{align}

Classes 1, 2, and 3 consist of regular spinor fields, since not both the scalar and the pseudoscalar vanish. Classes 4, 5, and 6 realize singular spinor fields, where both $\sigma$ and $\omega$ are null. Refs. [3, 16–18] illustrate a vast range of applications of these classes in quantum field theory and gravity. Ref.
introduced more two exclusive classes into the Lounesto’s classification, through a generally-relativistic gauge classification, whereas the most general spinor field class in each spinor class was derived in Ref. [5] as a prominent computational tool for the reconstruction theorem.

The Fierz identities (2) are well known not to be valid for the case of singular spinors. In this case, based upon a Fierz aggregate, 

\[ Z = \frac{1}{2}(\omega_{0\gamma_1\gamma_2\gamma_3} + iK_{0\gamma_1\gamma_2\gamma_3} + iS + J + \sigma), \]  

the Fierz identities (2) can be replaced by

\begin{align*}
Z^2 &= \sigma Z, \\
Z_{\gamma\mu}Z &= J_\mu Z, \\
Z_{\sigma\mu\nu}Z &= S_{\mu\nu}Z, \\
Zi\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu Z &= K_\mu Z, \\
-Z\gamma_0\gamma_1\gamma_2\gamma_3Z &= \omega Z.
\end{align*}

Fierz aggregates that are self-adjoint multivectors, \( \gamma_0Z\gamma_0 = Z^\dagger \), are better known as boomerangs [2].

Given any spinor \( \upsilon \in \mathbb{C}^4 \) such that \( \bar{\upsilon}\gamma_0\psi \neq 0 \), the non trivial spinor \( \psi \) can be, then, reconstructed by the inversion theorem, as \( \psi = \frac{1}{2\sqrt{\bar{\upsilon}Z\upsilon}} e^{-i\alpha}Z\upsilon \), for an arbitrary phase \( \alpha \), such that \(-i\alpha = \ln (2\sqrt{\bar{\upsilon}\psi\upsilon Z\upsilon})\). In particular, any regular spinor can be reconstructed as [6, 7]

\[ \psi = \frac{1}{2} \sqrt{J_0 + \sigma - K_3 + S_{12}} Ze^{i\alpha} (1, 0, 0, 0)^T. \]

Heretofore spinor fields were approached without mentioning the spinor bundle. We denoted the Minkowski spacetime manifold by \( M \simeq \mathbb{R}^{1,3} \). Since it is an affine space, being isomorphic to its own tangent spaces, a lot of important structures were hidden throughout the text, for simplicity. Nevertheless, to approach spinor fields on higher dimensions, we should recall the spinor structures of Minkowski spacetime. Spinor fields are sections of the so called spinor bundle. For defining it, some underlying structures are introduced in the Appendix A.

Now, to define and construct analogous classifications on spacetimes of any dimension and signature, when it is possible, the geometric Fierz identities can be analyzed when a spin structure endows an \( M \) manifold. For it, the so called Kähler-Atiyah bundle introduced, which consists of the exterior bundle endowed with the Clifford product, denoted in this section by \( \diamond \). Considering our case of interest, consisting of the 7-sphere \( S^7 \), its \( S \) spin bundle is, thus, equipped with the induced \( \circ : S \rightarrow S \) product, accordingly [14]. This composition just indicates the product between spinor
fields, usually denoted by juxtaposition, when Minkowski spinor fields are regarded. Denoting by $\text{End}(S)$ all the linear mappings from $S$ to $S$ and by “sec” any section of a bundle, a bilinear pairing $B : \text{sec} S \times \text{sec} S \to \mathbb{R}$ can define a bilinear mapping \cite{1, 14}. Indeed, given sections $\psi, \Psi$ on the spin bundle, a bilinear mapping $B_0 : \text{sec} S \times \text{sec} S \to \mathbb{R}$, on the $S^7$ spinor bundle, reads \cite{14}

$$B_0(\psi, \Psi) = B(\Re \psi, \Re \Psi) - B(\Im \psi, \Im \Psi) + i[B(\Re \psi, \Im \Psi) + B(\Im \psi, \Re \Psi)],$$

(7)

for the real, $\Re \psi$, and the imaginary, $\Im \psi$, components of the spinor field $\psi$ \cite{14}. This bilinear mapping is the one that shall generalize the bilinear covariants $(3b - 3f)$ scalar components, that were constructed on $\mathbb{R}^{1,3}$ to the 7-sphere. This can be implemented by the bilinear mapping on the $S^7$ spin bundle:

$$B_k(\psi, \Psi) = B(\psi, \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi) = \bar{\psi} \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi.$$

(8)

To define new spinor classes on $S^7$, when $k$ is odd, the bilinear mapping $B(\psi, \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi)$ is not equal to zero \cite{14}. Defining

$$A_{\psi|\Psi}(\psi) := B(\psi, \Psi)\psi,$$

(9)

given $\bar{\psi}, \bar{\psi}, \bar{\Psi} \in \text{sec} S$, then the (geometric) Fierz identities then read \cite{14}

$$A_{\psi|\Psi} \circ A_{\bar{\psi}|\bar{\Psi}} = B(\bar{\psi}, \bar{\Psi})A_{\psi|\Psi}.$$

(10)

Given the structure $D$ that defines the complex conjugate on $S$ by $D(\psi) = \Im \psi$, the elements $A_{\psi|\Psi}$ are differential forms that can be always split into $A_{\psi|\Psi} = D \circ A_D^{1|\psi} + A_D^0|\psi$ \cite{14}, where

$$A_{\psi|\Psi}^{1|\psi} = \sum_{k=0}^{7} \frac{1}{k!} (-1)^k B(\psi, \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi) e^{\tau_1} \wedge \cdots \wedge e^{\tau_k},$$

(11a)

$$A_{\psi|\Psi}^0|\psi = \sum_{k=0}^{7} \frac{(-1)^k}{k!} B(\psi, D \circ \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi) e^{\tau_1} \wedge \cdots \wedge e^{\tau_k}.$$  

(11b)

The geometric Fierz identities then follow for $S^7$ \cite{14}:

$$A_{\psi|\Psi} \circ A_{\bar{\psi}|\bar{\Psi}} + A_{\psi|\Psi}^{1|\psi} \circ A_{\bar{\psi}|\bar{\Psi}} = B(\bar{\psi}, \bar{\Psi})A_{\psi|\Psi}^{1|\psi},$$

(12a)

$$A_{\psi|\Psi} \circ A_{\bar{\psi}|\bar{\Psi}} + (-1)^k A_{\psi|\Psi}^{k|\psi} \circ A_{\bar{\psi}|\bar{\Psi}} = B(\bar{\psi}, \bar{\Psi})A_{\psi|\Psi}^{k|\psi}.$$  

(12b)

These equations are the equivalent of Eqs. (2), for $S^7$.

Moreover, the bilinear covariants on $S^7$ emulate the ones of Minkowski spacetime $(3b - 3f)$, by

$$\Phi_k = \frac{1}{k!} B(\psi, \gamma_{\tau_1} \cdots \gamma_{\tau_k} \psi) e^{\tau_1} \wedge \cdots \wedge e^{\tau_k}$$

(13)
It is worth to emphasize that the bilinear covariants construction on $\mathbb{R}^{1,3}$ are not obstructed by a dimensional accident. However, on $S^7$ (and also on other specific dimensions), the geometric Fierz identities (12a, 12b) severely obstruct the very existence of homogeneous bilinear covariants [14]. In fact, spinors on $S^7$ have the bilinear covariants $\phi_k$ equal to zero, with the exceptions when $k = 0$ or $k = 4$ [1, 14], namely,

$$\phi_0 = B(\psi, \psi),$$

$$\phi_4 = \frac{1}{4!}B(\psi, \gamma_1 \gamma_2 \gamma_3 \gamma_4 \psi) e^{\tau_1} \wedge e^{\tau_2} \wedge e^{\tau_3} \wedge e^{\tau_4}.$$  

(15)

Then, the geometric Fierz identities yield a single class Majorana spinors on $S^7$, given by $\phi_0 \neq 0$ and $\phi_4 \neq 0$, being all other bilinears $\phi_k = 0$, for $\{k\} \neq \{0, 4\}$. A higher order generalization of Eq. (8) is then necessary, to encompass new classes of fermionic fields on $S^7$:

$$\beta_k(\psi, \gamma_1 \ldots \gamma_k \psi) = B(\psi, \gamma_1 \ldots \gamma_k \psi) \gamma_k - B(\gamma_k \psi, \gamma_1 \ldots \gamma_k \psi) + i \left[ B(\psi, \gamma_1 \ldots \gamma_k \psi) - B(\gamma_k \psi, \gamma_1 \ldots \gamma_k \psi) \right].$$

(16)

Hence, the complex bilinear covariants can be defined [1],

$$\Phi_k = \frac{1}{k!} \beta_k(\psi, \gamma_1 \ldots \gamma_k \psi)e^{\tau_1} \wedge \ldots \wedge e^{\tau_k},$$

(17)

yielding three (non-trivial) classes of spinor fields on the $S^7$ spin bundle [1],

1) $\Phi_0 = 0$, $\Phi_4 \neq 0$  

2) $\Phi_0 \neq 0$, $\Phi_4 = 0$  

3) $\Phi_0 \neq 0$, $\Phi_4 \neq 0$.

(18a)

(18b)

(18c)

The Fierz aggregate (4) in the $\mathbb{R}^{1,3}$ Minkowski spacetime can be now emulated for the 7-sphere. In fact, the reconstruction theorem can be then employed for constructing the original spinor field as a section of the spin bundle, from the corresponding Fierz aggregate

$$\hat{Z} = \Phi_0 + \Phi_4,$$

(19)

that is simpler than its 4D Minkowski counterpart Fierz aggregate, defined in Eq. (4). Hence, when an arbitrary spinor $\xi \in S^7$ satisfies $\xi^\dagger (\sigma_2 \otimes \gamma_0) \psi \neq 0$, where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, the original $S^7$ spinor $\psi$ can be obtained from its Fierz aggregate (19),

$$\psi = \frac{1}{2} \left( (\xi^\dagger (\sigma_2 \otimes \gamma_0) \hat{Z} \xi)_0 \right)^{-1/2} e^{-i\theta} \hat{Z} \xi,$$

(20)

where $e^{-i\theta} = 2(\xi^\dagger (\sigma_2 \otimes \gamma_0) \hat{Z} \xi)_0^{-1/2}$. 


III. LIFTING NEW SPINOR FIELDS ON THE PARALLELIZABLE $S^7$

Heretofore, new classes of $S^7$ spinors were derived into the classes $(18a - 18c)$, whose representative spinor fields can be reconstructed by Eq. (20). These representative spinor fields are now aimed to be lifted onto the so called parallelizable $S^7$, that can be regarded as the manifold of unit octonions. Among the parallelizable spheres, $S^7$ is the sole one that does not carry a Lie group structure, however a Moufang loop structure, instead. The (sub)bundle of octonionic sections on $S^7$ presents a Moufang loop underlying structure, which is fiberwise. From a geometric point of view, the $S^7$ algebra is a natural stage to generalize the concept of a Lie algebra, wherein the structures constants are substituted by the parallelizable torsion.

The octonionic algebra $\mathbb{O}$ is constituted by a 8-dimensional vector space, with basis $\{e_0, \ldots, e_7\} \subset \mathbb{R}^8$, with $e_0^2 = 1$, $e_a^2 = -1$, for $a = 1, \ldots, 7$, endowed with the octonionic multiplication, denoted by “$\circ$”, which is ruled by $e_a \circ e_b = f_{ab}^c e_c - \delta_{ab}$, where $f_{ab}^c = 1$ for the cyclic permutations $\{(abc)\} = \{(126), (237), (341), (452), (563), (674), (715)\}$. Every octonion $X \in \mathbb{O}$ can be, thus, written as $X = X_0 e_0 + \sum_{a=1}^7 X^a e_a$. Instead of the vector space $\mathbb{R}^8$, one can take the paravector space $V_7 := \mathbb{R} \oplus \mathbb{R}^{0,7}$ endowed with the octonionic standard product $\circ: V_7 \times V_7 \rightarrow V_7$. In fact, the scalar part, $X^0$ does correspond to the real part of an octonion, whereas the vector component, $\sum_{a=1}^7 X^a e_a$, regards the imaginary part. In this case, the identity $e_0 = 1$ and an orthonormal basis $\{e_a\}_{a=1}^7$ of $V_7 \hookrightarrow \mathcal{C}l_{0,7}$ generate the octonion algebra $\mathbb{O}$. The octonionic product can be emulated at the Clifford algebra $\mathcal{C}l_{0,7}$ as

$$A \circ B = (AB(1-\mathbb{U}))_{0\oplus 1}, \quad A, B \in V_7,$$

where $\mathbb{U} = e_7 e_1 e_5 + e_6 e_7 e_4 + e_5 e_6 e_3 + e_4 e_5 e_2 + e_3 e_4 e_1 + e_2 e_3 e_7 + e_1 e_2 e_6$ is a 3-form, and the juxtaposition denotes the Clifford product. The symbol $\langle \chi \rangle_{0\oplus 1}$ denotes the projection of a multivector $\chi \in \mathcal{C}l_{0,7}$ onto its paravector components. For the underlying Lie algebra $\mathfrak{g}_7$, the Lie bracket satisfies

$$[[e_i, e_j], e_k] + [[e_k, e_i], e_j] + [[e_j, e_k], e_i] = (\delta_{i[k]} \delta_{j]} e_p + \epsilon_{mij} e_{mkp})e_p,$$

where $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$, for any tensor $A_{ab}$, and the Einstein’s summation convention is used hereon. The (Clifford) conjugation of $X = X^0 + X^b e_b \in \mathbb{O}$ reads $\bar{X} = X^0 - X^b e_b$, for $X^0, X^a$ real coefficients. Given $X \in S^7$, the $X$-product is defined by [22]

$$A \circ_X B := (A \circ X) \circ (\bar{X} \circ B).$$
The expressions below are shown, e.g., [22]

\[(A \circ X) \circ (\bar{X} \circ B) = X \circ ((\bar{X} \circ A) \circ B) = (A \circ (B \circ X)) \circ \bar{X}.\] (24)

As we dealt with bundles in the previous sections, the octonion bundle

\[O S^7 \simeq (\mathbb{R} \times S^7) \oplus T S^7,\] (25)

shall be employed, where \(T S^7\) denotes the tangent bundle on \(S^7\), with fibers \(\mathbb{R} \oplus T_X S^7\) [21]. Hence, given \(A, B, C \in O S^7\), and the associator \([A, B, C] = A \circ (B \circ C) - (A \circ B) \circ C\), one can write [21]

\[A \circ_X B = A \circ B + [A, B, \bar{X}] \circ X.\] (26)

Although \(A \circ_X B \neq A \circ B\) in general, choosing \(X\) as being an element of the following sets of vector fields \(\{\pm e_b\}\), \(\{(\pm e_a \pm e_b)/\sqrt{2}\}\), \(\{(\pm e_a \pm e_b \pm e_c \pm e_d)/2\mid e_a \circ (e_b \circ (e_c \circ e_d)) = \pm 1\}\), makes the equality \(A \circ_X B = A \circ B\) to hold for such particular values of \(X\) [24].

Eq. (24) shows that the octonionic field \(X \in \sec(O S^7)\) determines two endomorphisms of the octonionic algebra, \(f_1, f_2 \in \End(\sec(O S^7))\), defined by \(A \circ_X B = f_1(A \circ f_1^{-1}(B)) = f_2(f_2^{-1}(A) \circ B)\), for all \(A, B \in \sec(O S^7)\). The **quasi-alternativity** of the \(\circ_X\)-multiplication then follows as

\[A \circ_X (A \circ_X B) = (A \circ A) \circ_X B, \quad (A \circ_X B) \circ_X B = A \circ_X (B \circ B).\] (27)

The \(X\)-product can be, thus, seen as the original octonionic product. In fact, there exists an orthogonal mapping \(T \in \SO(\mathbb{R}^{0,7})\), such that the mapping \(\rho : (V_7, \circ) = O \rightarrow (V_7, \circ_X) = O_X\), given by \(a + v \mapsto a + T(v)\), is an isomorphism, for all \(a \in \mathbb{R}\) and \(v \in \mathbb{R}^{0,7}\) [29]. The reciprocal statement is up to now a conjecture. Besides, an orbit whose elements are isomorphic copies of \(O\) obtained out of any fixed copy of \(O\) is an orbifold \(S^7/\mathbb{Z}_2 = \mathbb{R} P^7\), being diffeomorphic to \(SO(7)/G_2\). In fact, identifying two antipode points on \(S^7\) yields \(A \circ_{-X} B = A \circ_X B\). One of the most natural ways of obtaining a parallelizable \(S^7\) is choosing two non-canonical connections on \(\Spin(7)/G_2\) [23].

Besides, the sphere \(S^7\) plays a prominent role on the (quaternionic) Hopf fibration \(S^3 \hookrightarrow S^7 \xrightarrow{p} S^4\), [25]. In this sense, \(S^7\) can be realized as being the set \(\{(q_1, q_2) \in \mathbb{H}^2 \mid \|q_1\|^2 + \|q_2\|^2 = 1\}\), where \(p : S^7 \rightarrow S^4\) maps the pair \((q_1, q_2)\) to \(q_1/q_2\), an element in the projective line \(\mathbb{H} P^1 \simeq S^4\). Thus, each fiber is represented by a torsor that is parametrized by quaternions of unit norm, defining \(S^3\). A construction of this Hopf algebra was also realized using regular spinors, being the most important realization with respect to the Lounesto’s spinor field classification, in Refs. [16, 25, 26].

\[2\] We denote hereon by \(\mathbb{H}\) the ring of quaternions.
More generally speaking, without considering just the $S^7$ manifold, a $n$-manifold $M$ is said to have the property of global parallelizability if there are $n$ linearly independent vector fields defined on $M$. Thereupon, for each $X \in M$, one can linearly combine these fields to obtain an orthonormal basis for $T_X M$. Given one of these bases, since vectors are linear combinations of such elements, their covariant derivative in different points can be taken in a natural way, which results in path independence for the parallel transport. In fact, it follows that

$$[\hat{D}_\mu, \hat{D}_\nu] = 0 = R_{\mu\nu},$$

where $\hat{D}$ denotes the covariant derivative defined with respect to this parallel transport, whereas $R_{\mu\nu}$ denotes the curvature tensor. As usual, $\hat{D} = \partial + \hat{\Gamma} = D - T$, where $T$ denotes the parallelizing torsion and $\hat{\Gamma}$ is the parallelizing connection. Let $e_\nu^a$ indicate the vielbein, related to a non-coordinate basis, wherein roman letters indicate the indexes of the tangent spaces, accordingly. As $D_{\mu} e_\nu^a = 0$, the covariant derivative of the vielbein yields $\hat{D}_{\mu} e_\nu^a = -T_{\mu\nu}^a$. Now, one can look at a manifold $M$, that for our case is $S^7$, and consider the infinitesimal translations determined by the covariant derivatives. As may be seen, it is straightforward that these translations configure a closed algebra [22]:

$$[D_a, D_b] = [e_a^\mu D_{\mu}, e_b^\nu D_{\nu}] = 2e_a^\mu [D_{\mu}, e_b^\nu] D_{\nu} = 2e_a^\mu T_{\mu\nu}^b D_{\nu} = 2T_{ab}^c D_c.$$ 

When the manifold is also a group manifold, it is evident that the parallelizing torsion does not depend on the point chosen and is, thus, only expressed by the structure constants. Nonetheless, $S^7$ must be carefully considered, for the torsion varies at each point on the manifold. This fact is intrinsically related to the non-associativity of $\mathcal{O}$, as it can be seen in Ref. [27].

For a field $X \in \sec(O S^7)$, one can construct a parametrization of $S^7$ with respect to unitary octonionic fields $\frac{X}{|X|} \in \sec(O S^7)$. The tangent space $T_X S^7$ is spanned by the usual octonionic basis as $\{X \circ e_i\}_{i=1}^7$. Now, as introduced in Ref. [22], let us consider the infinitesimal operator $\delta_A$, where $A \in \sec(O S^7)$ is now a pure imaginary octonionic field, acting on $X$ as $\delta_A X = X \circ A$. This transformation defines the parallel transport on the basis spanned by the choice of $X$. An explicit derivation can be realized [22] to find the commutator of the defined transformations:

$$[\delta_A, \delta_B] X \equiv \delta_A (\delta_B X) - \delta_B (\delta_A X) = X \circ (\bar{X} \circ ((X \circ B) \circ A)) - \bar{X} \circ ((X \circ A) \circ B)).$$

(30)

It can be shown that the parameter $\bar{X} \circ ((X \circ B) \circ A) - \bar{X} \circ ((X \circ A) \circ B) = 2\{\bar{X} \circ ((X \circ B) \circ A))$ is twice the negative of the parallelizing torsion [27]. Componentwise,

$$T_{abc}(X) = [(e_a \circ \bar{X}) \circ (X \circ e_b) \circ e_c] \quad \text{and} \quad [\delta_A, \delta_B] = 2T_{abc}(X)\delta_c,$$

(31)
presenting, thus, a Moufang loop (or Moufang quasigroup) structure in the second equation in (31). Therefore, one can see that the operator \( \delta \) and the parallelizing covariant derivative are, in fact, in a 1-1 correspondence. Now, taking another field \( \zeta \in \mathcal{O} \), with \( \zeta|\mathcal{O}| = Y = \sec(\mathcal{O} \mathcal{S}^7) \), over the same orientation given by the choice of \( X \), and transforming it such that the relations in Eq. (31) are preserved, such properties preclude the straightforward ansatz \( \delta_A Y = Y \circ A \) [22]. The two regarded fields on \( \mathcal{S}^7 \) must, thus, transform according to another rule, that may seem at a first glance, not the simplest choice. Ref. [22] derived the appropriate transformation rule for fermionic fields on \( \mathcal{S}^7 \), taking into account its underlying parallelizable torsion, as \( \delta_A Y = Y \circ_X A \).

Now, the new classes of spinors on \( \mathcal{S}^7 \) can be lifted onto the parallelizable \( \mathcal{S}^7 \). In fact, for it we need to remember the equivalence between the classical and the algebraic spinor fields. Going back to the 4D Minkowski spacetime, the standard Dirac spinor \( \psi \) was identified, e. g., in Ref. [2] as an element of the minimal left ideal \( (\mathbb{C} \otimes \mathcal{C} \ell_{1,3}) f \) associated to the Dirac–Clifford algebra \( (\mathbb{C} \otimes \mathcal{C} \ell_{1,3}) \), generated by the primitive idempotent \( f = \frac{1}{4} (1 + \gamma_0) (1 + i\gamma_1 \gamma_2) \) yielding \( \psi \in (\mathbb{C} \otimes \mathcal{C} \ell_{1,3}) f \) is an algebraic spinor [2]. Hence, using the Dirac representation of the gamma matrices, the algebraic spinor

\[
\psi = \begin{pmatrix}
\psi_1 & 0 & 0 & 0 \\
\psi_2 & 0 & 0 & 0 \\
\psi_3 & 0 & 0 & 0 \\
\psi_4 & 0 & 0 & 0 \\
\end{pmatrix} \in (\mathbb{C} \otimes \mathcal{C} \ell_{1,3}) f \simeq \mathcal{M}(4, \mathbb{C}) f, \tag{32}
\]

is equivalent to the classical spinor \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \in \mathbb{C}^4 \).

Now, this concept can be extended for the parallelizable \( \mathcal{S}^7 \), emulating the transformation \( \delta_A \psi = \psi \circ_X A \) that can encompass algebraic spinor fields. For it, let us consider the Clifford algebra \( \mathcal{C} \ell_{0,7} \) on a tangent space \( T_X \mathcal{S}^7 \), at a point \( X \in \mathcal{S}^7 \). According to the Radon–Hurwitz theorem in the Appendix B, for \( k = 7 - r_7 = 4 \), one aims the set \( \{e_{I_1}, e_{I_2}, e_{I_3}, e_{I_4}\} \subset \mathcal{C} \ell_{0,7} \) that commute and squares the identity [28]. Identifying, for example [29] \( e_{I_1} = e_1 e_2 e_3, e_{I_2} = e_1 e_4 e_5, e_{I_3} = e_1 e_6 e_7, \) and \( e_{I_4} = e_3 e_4 e_7 \), yields that the idempotent

\[
f = \frac{1}{16} (1 + e_1 e_2 e_3)(1 + e_1 e_4 e_5)(1 + e_1 e_6 e_7)(1 + e_3 e_4 e_7) \in \mathcal{C} \ell_{0,7}
\]

is a primitive one. Hence, a spinor \( \hat{\psi} \in \mathcal{S}^7 \) has its algebraic version as the element \( \hat{\psi} f \) of the left ideal \( \mathcal{C} \ell_{0,7} f \), for some multivector \( \hat{\psi} \in \mathcal{C} \ell_{0,7} \). This is accomplished just for introducing the \( \mathcal{S}^7 \) spinor into the Clifford bundle itself, on \( \mathcal{S}^7 \).

Now, to write the correct transformation of a fermionic field on the parallelizable \( \mathcal{S}^7 \), given an element of the vector space underlying \( \mathcal{C} \ell_{0,7} \), a non-associative product called the \( \xi \)-product was
introduced in [19] as a natural generalization for the $X$-product. For homogeneous multivectors
\[ \xi = u_1 \wedge \ldots \wedge u_k \in \sec \Lambda^k(\mathbb{R}^{0,7}) \hookrightarrow \sec \mathcal{C}_{0,7} \],
where \( \{u_p\}_{p=1}^k \subset \sec \mathcal{T}\mathbb{R}^{0,7} \) and \( A \in \sec(\mathbb{O}S^7) \), the
products \( \bullet_\xi \) and \( \bullet_\eta \) are defined (and extended by linearity) by [19, 29]
\[ \bullet_\xi : \sec(\mathbb{O}S^7) \times \sec \Lambda^k(\mathbb{R}^{0,7}) \rightarrow \sec(\mathbb{O}S^7) \]
\[ (A, \xi) \mapsto A \bullet_\xi \xi = (\cdots ((A \circ_X u_1) \circ u_2) \circ \cdots) \circ u_k, \quad (33) \]
\[ \bullet_\eta : \sec \Lambda^k(\mathbb{R}^{0,7}) \times \sec(\mathbb{O}S^7) \rightarrow \sec(\mathbb{O}S^7) \]
\[ (\xi, A) \mapsto \xi \bullet_\eta A = u_1 \circ (u_2 \circ (\cdots \circ (u_k \circ A)) \cdots). \quad (34) \]

Hence, within the above constructions, the transformation of the reconstructed spinor field on
\( S^7 \) from its bilinear covariants in Eq. (20), that is a representative of the new classes (18a – 18c)
of spinor fields on \( S^7 \), can be defined as
\[ \delta_A \psi = \psi \bullet_\eta A, \quad \forall A \in \sec(\mathbb{O}S^7). \quad (35) \]

In this way, the previous new classes of \( S^7 \) spinors are lifted onto the parallelizable \( S^7 \). This
transformation is compatible to the ones defined in Ref. [30].

**IV. CONCLUSIONS**

We have managed to establish the reconstruction theorem for the new classes of spinor fields on
\( S^7 \) using the generalized Fierz aggregate, for each recently found new class of spinor fields on the
\( S^7 \) spin bundle according to their bilinear covariants. Besides, this categorization has enabled
the construction of new fermionic fields on the parallelizable \( S^7 \), promoting the new classes of classical
spinor fields on \( S^7 \) to new classes of algebraic ones. Hence, the correct transformation of these
elements, generating a Moufang loop structure on the parallelizable \( S^7 \) was derived. Aiming to
this procedure, we briefly reviewed the parallelizability property on the parallelizable \( S^7 \), wherein
the parallel transport could be analyzed with respect to the torsion. Therein, the non-associativity
of the octonionic bundle on \( S^7 \) was related to the torsion tensor on the parallelizable \( S^7 \), as a
function dependent on each point on \( S^7 \), via the \( X \)-product. In this way, additional classes of
fermionic (spinor) fields on the parallelizable \( S^7 \) have been constructed, according to the classes
obtained heretofore, lifted from the \( S^7 \) spin bundle, with the right transformation under infinitesimal
transformations. Our results, thus, generalize the ones in Ref. [22], also proposing new classes
of fermionic fields that may play the role of the solutions in compactifications of supergravity.
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Appendix A: The spinor bundle

Here the spinor bundle of Minkowski spacetime is introduced. Such a structure can be emulated for any spacetime \((p, q)\) signature, when the regarded manifold \(M\) has a spin structure. In particular, the construction for the \((0, 7)\) and \((7, 0)\) signatures, regarding the \(S^7\) spinor bundle, are similarly constructed. As the Minkowski spacetime is the most illustrative and phenomenologically explored, we want to fix the notation and the intuitive setup for the spinor bundles.

In what follows, one denotes the connected component to the identity of the Spin group by \(\text{Spin}_{1,3}^e \simeq \text{SL}(2, \mathbb{C})\), being the universal covering group of the (restricted) Lorentz group \(\text{SO}_{1,3}^e\). Again we denote by \(M\) the Minkowski spacetime \(\mathbb{R}^{1,3}\). Given the Minkowski metric tensor and the principal bundle of frames on the manifold \(M\), the orthonormal frame [coframe] bundle shall be denoted by \(P_{\text{SO}_{1,3}}(M)\) [\(P_{\text{SO}_{1,3}^e}(M)\)]. Considering \(M\) a spin manifold, there exists the spin frame and the spin coframe bundles, respectively denoted by \(P_{\text{Spin}_{1,3}}(M)\) and \(P_{\text{Spin}_{1,3}^e}(M)\).

Sections of \(P_{\text{SO}_{1,3}}(M)\) and of \(P_{\text{Spin}_{1,3}^e}(M)\) are both orthonormal coframes. However, contrary to the \(P_{\text{SO}_{1,3}}(M)\) bundle, coframes on the bundle \(P_{\text{Spin}_{1,3}^e}(M)\) that differ by a \(2\pi\) rotation are considered to be distinct, whereas coframes that can be led into each other by a \(4\pi\) rotation are equivalent.

The fundamental mapping \(s : P_{\text{Spin}_{1,3}^e}(M) \to P_{\text{SO}_{1,3}^e}(M)\) completely defines the bundle \(P_{\text{Spin}_{1,3}^e}(M)\). A spin structure on \(M\) consists of a principal fiber bundle, endowed with a canonical projection \(\pi_s : P_{\text{Spin}_{1,3}^e}(M) \to M\), under the conditions:

(i) Given the projection mapping \(\pi : P_{\text{SO}_{1,3}^e}(M) \to M\), then \(\pi(s(p)) = \pi_s(p)\), for all element \(p \in P_{\text{Spin}_{1,3}^e}(M)\).

(ii) Denoting by \(\text{Aut}(\mathcal{C}_{1,3})\) the set of automorphisms (namely, isomorphisms from \(\mathcal{C}_{1,3}\) to itself), given the adjoint mapping

\[
\text{Ad} : \text{Spin}_{1,3}^e \to \text{Aut}(\mathcal{C}_{1,3})
\]

\[
\tau \mapsto \text{Ad}_\tau : \mathcal{C}_{1,3} \to \mathcal{C}_{1,3}
\]

\[
\xi \mapsto \tau\xi\tau^{-1},
\]

then \(s(p\tau) = s(p)\text{Ad}_\tau\), for all element \(p \in P_{\text{Spin}_{1,3}^e}(M)\).
The Clifford bundle of differential forms \( \mathcal{C} \ell(M, g) \) is a vector bundle associated with \( P_{\text{Spin}^c_{1,3}}(M) \), whose sections are sums of non-homogeneous differential forms. Hence \( \mathcal{C} \ell(M, g) \cong P_{\text{SO}^c_{1,3}}(M) \times_{\text{Ad}'} \mathcal{C} \ell_{1,3} \) is a bundle defined by:

1. Let \( \pi : \mathcal{C} \ell(M, g) \to M \) be the canonical projection and let \( \{ U_\alpha \} \) be an open covering of \( M \). There are trivialization mappings \( \psi_i : \pi^{-1}(U_i) \to U_i \times \mathcal{C} \ell_{1,3} \) of the form \( \psi_i(p) = (\pi(p), \psi_i(x)(p)) = (x, \psi_i(p)) \). If \( x \in U_i \cap U_j \) and \( p \in \pi^{-1}(x) \), then \( \psi_i(p) = h_{ij}(x) \psi_j(x)(p) \), for \( h_{ij}(x) \in \text{Aut}(\mathcal{C} \ell_{1,3}) \), where \( h_{ij} : U_i \cap U_j \to \text{Aut}(\mathcal{C} \ell_{1,3}) \) are the transition mappings of \( \mathcal{C} \ell(M, g) \). It is worth to emphasize that every automorphism of \( \mathcal{C} \ell_{1,3} \) can be written as \( h_{ij}(x) \psi_j(x)(p) = a_{ij}(x) \psi_i(x)(p) a_{ij}(x)^{-1} \) for some invertible element of \( a_{ij}(x) \in \mathcal{C} \ell_{1,3} \). In other words, \( \text{Ad}_{a_{ij}} = h_{ij} \) in all intersections \( U_i \cap U_j \).

2. Besides, when the adjoint mapping is restricted to the group \( \text{Spin}^c_{1,3} \), it defines the mapping \( \text{Ad}|_{\text{Spin}^c_{1,3}} : \text{Spin}^c_{1,3} \to \text{SO}^c_{1,3} \), with kernel \( \mathbb{Z}_2 \). Hence \( \text{Ad} : \text{Spin}^c_{1,3} \to \text{Aut}(\mathcal{C} \ell_{1,3}) \) descends to a representation \( \text{Ad}' : \text{SO}^c_{1,3} \to \text{Aut}(\mathcal{C} \ell_{1,3}) \) of \( \text{SO}^c_{1,3} \), yielding \( \text{Ad}'(\tau) = \tau \xi \tau^{-1} \).

3. The main group underlying the Clifford bundle \( \mathcal{C} \ell(M, g) \) is, thus, reducible to \( \text{SO}^c_{1,3} \). The transition mappings of \( P_{\text{SO}^c_{1,3}}(M) \) can be then regarded as transition mappings of the Clifford bundle, yielding [31, 32]

\[
\mathcal{C} \ell(M, g) = P_{\text{SO}^c_{1,3}}(M) \times_{\text{Ad}'} \mathcal{C} \ell_{1,3} = P_{\text{Spin}^c_{1,3}}(M) \times_{\text{Ad}} \mathcal{C} \ell_{1,3}.
\] (A2)

Hence, spinor fields are sections of vector bundles associated with the principal bundle of spinor coframes. The well known regular Minkowski spinor fields are sections of the bundle

\[
S = P_{\text{Spin}^c_{1,3}}(M) \times_\rho \mathbb{C}^4,
\] (A3)

with \( \rho \) being the \( D^{(1/2,0)} \oplus D^{(0,1/2)} \) representation of \( \text{Spin}^c_{1,3} \) onto the space of linear mappings on \( \mathbb{C}^4 \).

With what was exposed heretofore in the Appendix A, one straightforwardly introduces, *mutatis mutandis*, the analogous underlying spinor bundle on the 7-sphere.

**Appendix B: The Radon–Hurwitz theorem**

Let \( \mathcal{C} \ell_{p,q} \) be the Clifford algebra associated to \( \mathbb{R}^{p,q} \) and \( \{ e_i \} (i = 1, \ldots, n) \) an orthonormal basis of this quadratic space. A primitive idempotent of \( \mathcal{C} \ell_{p,q} \) is given by \( f = \frac{1}{2}(1 + e_{I_1}) \cdots \frac{1}{2}(1 + e_{I_k}) \), where \( \{ e_{I_1}, \ldots, e_{I_k} \} \) is a set of elements in \( \mathcal{C} \ell_{p,q} \) that commute and such that \( (e_{I_\alpha})^2 = 1 \) for \( \alpha = 1, \ldots, k \). It generates a group of order \( 2^k \), where \( k = q - r_{q-p} \), and \( r_j \) are the Radon–Hurwitz numbers defined by [28] with the recurrence relation \( r_{j+8} = r_j + 4 \).
| j  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|---|
| $r_j$ | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 |

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