Quantization of Harer-Zagier formulas

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Abstract

We derive the analogues of the Harer-Zagier formulas for single- and double-trace correlators in the \(q\)-deformed Hermitian Gaussian matrix model. This fully describes single-trace correlators and opens a road to \(q\)-deformations of important matrix models properties, such as genus expansion and Wick theorem.

1 Introduction

Matrix models \cite{1,5} are now ubiquitous in mathematical and theoretical physics \cite{4}. Reduction (or reformulation) of a problem in matrix model terms often leads to significant progress, be it in the domain of SUSY gauge theories \cite{9–22}, enumerative geometry \cite{23–38}, the theory of symmetric functions \cite{39–45} or even the quantum computation \cite{46,47}.

An interesting built-in feature of matrix models is their genus expansion – the natural splitting of any correlator into the contributions that can be thought of as associated with Riemann surfaces of particular genera, endowed (colored) with certain extra data. From the QFT point of view this expansion is nothing but WKB (perturbative) expansion, and in the simplest case of Hermitian Gaussian matrix model (HGMM, see Section 2 for a definition) it literally takes the form of the summation over fat (ribbon) graphs, each living on a particular Riemann surface.

For example, the average of \(\langle \text{tr} (X^4) \rangle\) in HGMM is equal to

\[
\langle \text{tr} (X^4) \rangle = 2N^3 + N
\] (1-1)

and the three summands are related to three fat graphs, two of which live on the sphere, and one on the torus:

There are many ways to derive this formula in case of HGMM, for instance, using the Wick theorem, which in this case takes the form of gluing ribbons (MM propagator) to the discs with marked boundary points (for the details see, for instance, \cite{26,48}).

This simplicity, however, is lost in the case of \(q\)-deformed Hermitian Gaussian matrix model (qHGMM, see Section 2), where the corresponding average is equal to

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\footnote{For a review of various interconnections, applications and notation, relevant for the present paper, see \cite{6–8}.}
\[ \langle \text{tr} (X^4) \rangle_q = \frac{1}{[2][4]} \left( q^4 \frac{[N+3]!}{[N-1]!} + \frac{[N+2]!}{[N-2]!} - \frac{[N+1]!}{[N-3]!} + \frac{1}{q^4} \frac{[N]!}{[N-4]!} \right) = \\
= - \frac{(q^6-q^8)}{(q^4-q^6)^2(q^2-q^4)^2(q^2-1)^2} q^{4N} + \\
+ \frac{(q^6-q^8)(q^2-q^4)(q^2-1)^2}{2} q^{4N} - \frac{(q^6-q^8)(q^2-1)^2}{(q^4-q^2)^2} q^{2N} + \\
+ \frac{1}{q^4} \left( q^2-1 \right)^2 q^{-4N} \]

Here we introduce \( q \)-numbers, \([n] = \frac{q^n-1}{q-1} \) and \( q \)-factorials, \([n]! = [1][2] \ldots [n] \). Not only does this formula feature strikingly new dependence on \( N \) (in the form of \( q^N \)), the leading power of \( q^N \) is no longer equal to the Euler characteristic of neither sphere nor torus. The complexity persists for more complicated correlators and this makes fainting the hope to generalize Wick’s theorem, genus expansion and other traditional matrix model structures beyond \( q = 1 \) case.

However, the hope is revived by the main observation the present paper – explicit \( q \)-deformation of Harer-Zagier \([49]\) formula.

This formula tells that Laplace transform in the variable \( N \) of the 1-point (single-trace) correlators can be explicitly presented as a fully-factorized rational function (see, also \([50]\) for a 2-point analog)

\[ \sum_{N=0}^{\infty} \lambda^N \langle \text{tr} (X^4) \rangle = \frac{3\lambda(1+\lambda)^2}{(1-\lambda)^4} \]

We find that its \( q \)-analogue is not any more complicated

\[ \sum_{N=0}^{\infty} \lambda^N \langle \text{tr} (X^4) \rangle_q = \frac{[3]_q \lambda q^2(\lambda^2-\lambda)(\lambda^4+\lambda)}{(q^3-\lambda)(1-\lambda)(1-q^2\lambda)(1-q^4\lambda)} \]

where \([3]_q = q^2+1+q^{-2}\) is a \( q \)-number. That is, \( q \)-deformation of Laplace transforms is just a judicious insertion of \( q \)-monomials into some of the brackets of \( q = 1 \) answer. This is an example of \( q \)-generalization of Harer-Zagier formula, see Section \([3]\) for the general case.

The simplest way to derive the Harer-Zagier generating function (see Section \([4]\), is by extensive use of the superintegrability of the model – an important property which takes form of exact solvability of correlators of Schur polynomials \( \chi \chi \),

\[ \langle \chi \chi \{X\} \rangle_q \sim \chi \chi \left( p_k^* \right) \]

where \( p_k^* \) is a distinguished point in the space of arguments (power sums, or time-variables) of Schur polynomials. It is important to emphasize that superintegrability does persist in models where genus expansion and Wick theorem are not readily available. Study of Harer-Zagier functions can therefore shed light on potential generalizations and interplay of these structures.

With help of this observation, it is straightforward to split the \( q \)-average of \( \text{tr} X^4 \) into the contributions of different powers of \( q^N \). Indeed, because of a general property of Fourier transform, expansion in powers of \( q^N \) is essentially the expansion of the Harer-Zagier function in partial fractions:

\[ \frac{[3]_q \lambda q^2(\lambda^2-\lambda)(\lambda^4+\lambda)}{(q^3-\lambda)(1-\lambda)(1-q^2\lambda)(1-q^4\lambda)} = \frac{1}{1-q^4\lambda} \text{res}_{\lambda=q^{-4}} \]

\[ + \frac{1}{1-q^4\lambda} \text{res}_{\lambda=q^{-2}} + \frac{1}{1-\lambda} \text{res}_{\lambda=1} + \frac{q^2}{q^4-\lambda} \text{res}_{\lambda=q^4} \]

where in residues we recognize the familiar coefficients from [1-2]

\[ \text{res}_{\lambda=q^{-4}} = - \frac{(q^8-q^6)(q^6-q^4)}{(q^4-q^2)^2(q^2-q^4)^2(q^2-1)^2} \]

\[ \text{res}_{\lambda=q^{-2}} = \frac{(q^6-q^8)(q^4-q^2)}{(q^3-q^4)(q^2-q^4)(q^2-1)^3} \]

\[ \text{res}_{\lambda=1} = - \frac{(q^4-q^4)(q^3-q^2)}{(q^2-q^4)(q^2-1)^3} \]

\[ \text{res}_{\lambda=q^4} = \frac{2}{(q^4-q^4)(q^2-1)^2} \]
We interpret this as follows: the q-generalized Harer-Zagier provides a simple way to compute (1-point) correlators in qHGMM, by means of a partial fractions expansion. This is the main practical result of the present paper, see Section [3] Equations (3-7), (3-8). This result allows for efficient and rapid calculation of arbitrary 1-point correlators and thus, in particular, opens a way to make educated guesses about the shape of quantum spectral curve \[51–62\].

In Sections [5], [6] and [7] we outline possible further directions and applications, concentrating on problems and partial successes, but the actual development of these topics is postponed till future.

2 Background

2.1 Hermitian Gaussian matrix model (HGMM)

HGMM can be defined as the following integral over the space of eigenvalues

\[
Z\{p_k\} = \int_{-\nu}^{\nu} e^{-x_i^2/2} \prod_{i \neq j}(x_i - x_j) \exp\left(\sum_k \frac{p_k}{k}(x_1^k + \ldots + x_N^k)\right) \, dx_1 \ldots dx_N
\]

(2-1)

and admits a well-known exact solution in terms of Schur polynomials \(\chi_R(p_k)\) (here \(R\) are Young diagrams and \(p_k\)-the time-variables), which can be also taken for the definition of the matrix model \[63–65\]

\[
Z\{p_k\} = \langle \exp\left(\sum_k \frac{p_k}{k}(x_1^k + \ldots + x_N^k)\right) \rangle = \sum_R \chi_R(p_k) \chi_R(p_k)
\]

(2-2)

where we applied the Cauchy formula. The special points in time-variables space are \(p_k = \delta_{k,n} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}\)

(2-3)

while

\[p_k^* = N\]

is another distinguished point (the \(q = 1\) limit of topological locus, see below). This exact solution can be conveniently memorized as a character preservation property

The average of a character is again a character

\[\langle \chi_R[X]\rangle = \langle \chi_R(x_1^k + \ldots + x_N^k)\rangle = \frac{\chi_R(\delta_{k,2})}{\chi_R(\delta_{k,1})} \chi_R(p_k^*) \chi_R(p_k)\]

(2-4)

and by itself has deep and far going consequences \[64, 66–71\].

This apparent simplicity is obscured when looking at correlators of other symmetric quantities, different from Schur polynomials. Consider, for instance, the first few 1-point correlators, i.e. averages of power sums

\[\langle \text{tr} \{X^2\}\rangle = \langle x_1^2 + \ldots + x_N^2\rangle = N^2\]

(2-5)

\[\langle \text{tr} \{X^4\}\rangle = \langle x_1^4 + \ldots + x_N^4\rangle = 2N^3 + N\]

\[\langle \text{tr} \{X^6\}\rangle = \langle x_1^6 + \ldots + x_N^6\rangle = 5N^4 + 10N^2\]

and so on. There is no closed expression for these correlators, in contrast with the Schur averages presented above. However, various generating functions exist – and these are exactly the Harer-Zagier formulas.

2.2 q-deformation of HGMM

The Hermitian Gaussian matrix model admits a q-generalization

\[
Z_q\{p_k\} = \int_{-\nu}^{\nu} \prod_{i=1}^{N} \rho(x_i) \prod_{i \neq j}(x_i - x_j) \exp\left(\sum_k \frac{p_k}{k}(x_1^k + \ldots + x_N^k)\right) \, dx_1 \ldots dx_N
\]

(2-6)
where \( \nu = (1 - q)^{-1/2} \) and \( \rho(x) \) is a \( q \)-deformation of the Gaussian distribution and the integral \( \int_{-\nu}^{\nu} dx \) is, in fact, the \( q \)-deformed (so-called, Jackson) integral (for details see [17][69]). Similarly to the original model there is a nice solution in terms of Schur functions:

\[
Z_q \{ p_k \} = \left\langle \exp \left( \sum_k \frac{p_k}{k!} (x^k_1 + \ldots + x^k_N) \right) \right\rangle_q = \sum_R \frac{\chi_R(\delta_{k|2})}{\chi_R(\delta_{k|1})} \chi_R(p^*_k) \chi_R(p_k) \tag{2-7}
\]

where points \( \delta_{k|n} = n(q^{-1} - q^n)^{1/n} \) if \( k \) is divisible by \( n \) and 0 otherwise, and

\[
p^*_k = \frac{q^{-Nk} - q^{Nk}}{q^{-k} - q^k} \tag{2-8}
\]
is an important distinguished point (the so-called topological locus that plays an important role in knot theory [72][75]).

In the language of correlators, a correlator of a Schur function is nicely expressed in terms of Schur functions again:

\[
\langle \chi_R \{ X \} \rangle_q = \left\langle \chi_R(x^k_1 + \ldots + x^k_N) \right\rangle_q = \frac{\chi_R(\delta_{k|2})}{\chi_R(\delta_{k|1})} \chi_R(p^*_k) \tag{2-9}
\]

### 2.3 The triad of definitions

In this subsection we do not pay so much attention to the difference between usual \( (1-q^\nu)/(1-q) \) and symmetric \( (q^{\nu - n}/(q^{- \nu - 1}) \) definitions of the \( q \)-numbers.

In fact, the definitions can be lifted one level up, to the \( (q, t) \)-deformation of the Hermitian Gaussian matrix model (qtHGMM)

The first one is with help of explicit Jackson integration ([17], Equation (2.6))

\[
Z_{(q, t)} \{ p_k \} = \int_{-\nu}^{\nu} \prod_{i=1}^N d_q x_i \prod_{j=1}^N (x^2_j q^2 \nu^2; q^2) \prod_{k \neq 1} x^2_j \left( x_k x_j; q^2 \right)_\infty \left( t x_k x_j; q^2 \right)_\infty \exp \left( \sum_k \frac{p_k}{k!} (x^k_1 + \ldots + x^k_N) \right), \tag{2-10}
\]

where, as usual, \( q = q^\beta, \nu = 1/\sqrt{1-q}, \left( x; q \right)_\infty = \prod_{m=0}^\infty (1 - xq^m) \) is the \( q \)-Pochhammer symbol and \( \int_{-\nu}^{\nu} d_q x \) is the \( q \)-Jackson integral

\[
\int_{-\nu}^{\nu} d_q x f(x) := (1-q) \sum_{n=0}^\infty \nu q^n \left[ g(\nu q^n) + g(-\nu q^n) \right] \tag{2-11}
\]

While this definition may seem ad hoc, in fact the Jackson integral can be reinterpreted as a sum over poles of some meromorphic function, after which \( Z_{(q, t)} \{ p_k \} \) becomes nothing but the \( N = 2 \) SUSY Yang-Mills-Chern-Simons partition function on \( D_2 \times S^1 \).

This definition can be used to directly evaluate any trace product correlator only at integer values of \( \beta \), nevertheless, it is useful for cross-checks.

A non-trivial corollary of (2-10) is that \( Z \{ p_k \} \) satisfies the system of \( (q, t) \)-deformed Virasoro constraints [17], Equation (4.20). Moreover, by clever choice of the equations from the system one can obtain a decomposition formula for any trace product correlator \([17], Equation (5.4)\), which is a bit lengthy to provide it here. Supplemented with initial conditions \( C_0 = 1 \) and \( C_1 = 0 \) this decomposition formula can be used as an alternative definition for (normalized) correlators of qtHGMM.

### 2.4 On \( q, t \)-deformed HGMM

The \( q \)-deformation can be further promoted to a \( q, t \)-analogue of the Gaussian matrix model (\( \beta \)-ensemble), where the measure \( \prod (x_i - x_j) \) is \( \beta \)-deformed with \( t = q^\beta \), and Schur functions deform to Macdonald functions. We consider this far-going generalization only briefly in this paper, see also Section 6.

As with other Gaussian models, the most practical definition of qtHGMM is by means of the \( (q, t) \)-character expansion property ("superintegrability") [64][67] – the approach taken in [69]

\[
\langle \text{Mac}_R \rangle_{q, t} = \text{Mac}_R \left( \delta_{k|2}^{(qt)} \right) \frac{\text{Mac}_R(\pi^*)}{\text{Mac}_R \left( \delta_{k|1}^{(qt)} \right)} \tag{2-12}
\]
where the $\pi^*$ is the $(q, t)$ topological locus

$$\pi^* : \ p_k = \frac{t^{-Nk} - t^{Nk}}{t^{-k} - t^k} \quad (2-13)$$

and the “special” loci $\delta_{k[1]}^{(q)}$ and $\delta_{k[2]}^{(q)}$ are defined by

$$\delta_{k[n]}^{(q)} = \frac{q^{-1} - q^{k/n}}{t^{-k} - t^k} \quad \left\{ \begin{array}{ll} 1 & \text{if } k \text{ is divisible by } n \\ 0 & \text{otherwise} \end{array} \right. \quad (2-14)$$

This choice is somewhat ad hoc – it is partly justified in [69], but can be easily changed to fit other interesting cases, up to $q$thGMM, inspired by arbitrary knot superpolynomials in the spirit of [76]. In the present paper we limit ourselves to [2-14].

The relation between the three definitions (integral, Virasoro and Macdonald) is as follows. Ref. [17] provides strict derivation of the Virasoro definition from the eigenvalue-integral definition (2-10). However, there is no way to restore the partition function at $p_k = 0$ from the Virasoro definition. The coincidence of Macdonald definition with the first two is a long-known experimental fact, but currently proving it directly from the definitions remains an open problem. Note, however, that the analogous equivalence is proved for the Selberg integral in the Macdonald book [77], Chapter VI, Section 9, Example 3.

### 3 Harer-Zagier formulas

The initial Harer-Zagier statement [49] is, that the generating function for HGMM one-point correlators, normalized by a peculiar double-factorial factors, is an elementary function

$$\sum_{k=0}^{\infty} \langle \text{tr } X^{2k} \rangle \frac{x^{2k}}{(2k-1)!!} = \frac{1}{2x^2} \left( \frac{1 + x^2}{1 - x^2} \right)^N - 1 \quad (3-1)$$

Subjected to additional Laplace transform in $N$, this function becomes even simpler

$$\sum_{N=0}^{\infty} \lambda^N \sum_{k=0}^{\infty} \langle \text{tr } X^{2k} \rangle \frac{x^{2k}}{(2k-1)!!} = \frac{\lambda}{1 - \lambda^2} \frac{1}{1 - \lambda - (1 + \lambda)x^2} \quad (3-2)$$

so that, in fact, its expansion coefficients in $x$ – the Laplace transforms of individual one-point correlators – are also very simple expressions. Explicitly, the first few correlators, summed over $N$ with weight $\lambda^N$ give

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^2 \rangle = \sum_{N=0}^{\infty} N^2 \lambda^N = \frac{\lambda(1 + \lambda)}{(1 - \lambda)^3} \quad (3-3)$$
$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^4 \rangle = \sum_{N=0}^{\infty} (2N^3 + N) \lambda^N = 3 \frac{(1 + \lambda)^2}{(1 - \lambda)^4}$$
$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^6 \rangle = \sum_{N=0}^{\infty} (5N^4 + 10N^2) \lambda^N = 15 \frac{(1 + \lambda)^3}{(1 - \lambda)^5}$$

$$\cdots$$

It is straightforward to imply (and then prove, for instance, with help of Toda equation) the general formula for Laplace transform of 1-point correlator:

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^{2m} \rangle = (2m - 1)!! \frac{\lambda(1 + \lambda)^m}{(1 - \lambda)^{m+2}} \quad (3-4)$$

These kind of formulas we also call Harer-Zagier formulas.

The surprising observation of the present paper is that $q$-analogues of Harer-Zagier formulas are equally simple,
Using the explicit expression for the correlator of a Schur polynomial (2-9), we find

\[ \sum_{N=0}^{\infty} \lambda^N \langle \text{tr} \{ X^{2m} \} \rangle_q = [2m - 1]!! \frac{q^m \lambda \prod_{n=1}^{m} (q^{2n} + \lambda)}{(q^{2m} - \lambda) \prod_{n=0}^{m} (1 - q^{2n}\lambda)} \]  (3-6)

where \([2m - 1]!! = [1][3][5]...[2m - 1]\) is the q-double factorial. Using this result, it is straightforward to derive a simple formula for the 1-point correlators:

\[ \langle \text{tr} \{ X^{2m} \} \rangle_q = q^{-2Nm} \text{res}_{\lambda=q^{2m}} \sum_{a=0}^{m} q^{2Na} \text{res}_{\lambda=q^{-2a}} \]  (3-7)

by means of evaluating the contributions of all the residues

\[ \text{res}_{\lambda=q^{-2a}} = [2m - 1]!! \frac{q^{m-2a} \prod_{n=1}^{m} (q^{2n} + q^{-2n})}{(q^{2m} - q^{-2a}) \prod_{n\neq a}^{m} (1 - q^{2n-2a})} \]  (3-8)

\[ \text{res}_{\lambda=q^{2m}} = [2m - 1]!! \frac{q^{m} \prod_{n=1}^{m} (q^{2n} + q^{2n})}{\prod_{n=0}^{m} (1 - q^{2n+2m})} \]

Note that such a formula was impossible on the \(q = 1\) level. There, in (3-4), all the simple poles collided at \(\lambda = 1\) and there was no way to isolate their individual contributions and get the simple factorized expressions. Moreover, each of the residues (3-8) is singular at \(q = 1\), while \(N\)-dependence of each separate summand in (3-7) vanishes at \(q = 1\). It is the competition between these two effects that restores the well-known polynomial dependence on \(N\) at \(q = 1\).

4 Derivation of Harer-Zagier formulas

To derive the q-generalized Harer-Zagier formula, we use the following well-known expansion of power sums in Schur polynomials,

\[ \text{tr} \ X^{2m} = \sum_{\ell=0}^{2m-1} (-1)^\ell \chi_{[2m-\ell,1^\ell]}[X] \]  (4-1)

where \(1^\ell = 1, \ldots, 1\). Therefore,

\[ \langle \text{tr} \ X^{2m} \rangle_q = \sum_{\ell=0}^{2m-1} (-1)^\ell \langle \chi_{[2m-\ell,1^\ell]}[X] \rangle_q \]  (4-2)

Using the explicit expression for the correlator of a Schur polynomial (2-9), we find

\[ \langle \chi_{[2m-\ell,1^\ell]}[X] \rangle_q = (-1)^{\ell+\ell_{even}/2} q^{-\ell_{even}/4+(2m-\ell_{even})^2/4} [2m - \ell_{even} - 1]!! \ [\ell_{odd}]!! \times \]  (4-3)

\[ \prod_{i=2}^{\ell+1} [N + 1 - i] [1 - i] \prod_{j=2}^{2m-\ell} [N + j - 1] [j - 1] \]

where \(\ell_{even} = 2\lceil \frac{\ell}{2} \rceil\) and \(\ell_{odd} = \ell_{even} - 1\). This can be rewritten as

\[ \langle \chi_{[2m-\ell,1^\ell]}[X] \rangle_q = (-1)^{\ell_{even}/2} q^{-\ell_{even}/4+(2m-\ell_{even})^2/4} [2m - \ell_{even} - 1]!! \ [\ell_{odd}]!! \times \]  (4-4)

\[ \prod_{i=2}^{\ell+1} \frac{[N + 2m - \ell - 1]}{[2m - \ell - 1]!! [N - \ell - 1]!!} \]
Summing over $N$, we find

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr} X^{2m} \rangle_q = \sum_{\ell=0}^{2m-1} (-1)^{\ell+\ell_{\text{even}}/2} q^{-\ell_{\text{even}}/4+(2m-\ell_{\text{even}})^2/4} [2m-\ell_{\text{even}}-1]!! \ell_{\text{odd}}!! \times
$$

$$\times \frac{1}{[2m]} \frac{1}{[2m-\ell-1]!![\ell]!!} \sum_{N=0}^{\infty} \lambda^N \frac{[N+2m-\ell-1]!}{[N-\ell-1]!} = (4-5)$$

$$= \sum_{\ell=0}^{2m-1} (-1)^{\ell+\ell_{\text{even}}/2} q^{-\ell_{\text{even}}/4+(2m-\ell_{\text{even}})^2/4} [2m-\ell_{\text{even}}-1]!! \ell_{\text{odd}}!! \times
$$

$$\times [2m-\ell]!! \frac{\lambda^{\ell+1}}{[2m-\ell-1]!![\ell]!!} \prod_{k=-m}^{m} 1 - q^{2k} \lambda =$$

$$= \prod_{k=-m}^{m} \frac{1}{1 - q^{k} \lambda} \sum_{\ell=0}^{2m-1} (-1)^{\ell+\ell_{\text{even}}/2} q^{-\ell_{\text{even}}/4+(2m-\ell_{\text{even}})^2/4} \frac{[2m-\ell_{\text{even}}-1]!! \ell_{\text{odd}}!! [2m-1]!! \ell_{\text{even}}!! (q^{2m}-\lambda^{\ell+1})}{[2m-\ell-1]!![\ell]!!}. (4-6)$$

which is a relative of $q$-Newton’s binomial formula (see, for instance, [78]).

Putting everything together, we obtain our main result

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr} X^{2m} \rangle_q = (2m-1)!! q^{m^2} \lambda \prod_{n=1}^{m} (q^{2n} - \lambda^2). (4-7)$$

which is the $q$-generalization of Harer-Zagier formula.

5 Towards multi-point Harer-Zagier formulas

We begin with reminding some results from [50] at $q = 1$. The $n$-point Harer-Zagier generating functions are made from irreducible correlators $K_{i_1, \ldots, i_n}(N)$

$$\rho_{HZ}(s_1, \ldots, s_n) = \sum_{N=0}^{\infty} \lambda^N \sum_{i_1, \ldots, i_n=0}^{\infty} K_{i_1, \ldots, i_n}(N) \cdot \frac{s_1^{i_1} \cdots s_n^{i_n}}{\text{double factorials}}. (5-1)$$

and they always are elementary functions. For instance, the 2-point functions are given by

$$\rho_{HZ}^{(\text{odd})}(s_1, s_2) = \sum_{N=0}^{\infty} \lambda^N \sum_{i_1, i_2=1}^{\infty} K_{2i_1-1, 2i_2-1}(N) \cdot \frac{s_1^{2i_1-1} s_2^{2i_2-1}}{(2i_1-1)!!(2i_2-1)!!} = \lambda \frac{\lambda}{\sqrt{\lambda-1 + (\lambda+1)(s_1^2 + s_2^2)}} \arctan\left(\frac{s_1 s_2 \sqrt{-\lambda}}{\sqrt{\lambda-1 + (\lambda+1)(s_1^2 + s_2^2)}}\right). (5-2)$$

$$\rho_{HZ}^{(\text{even})}(s_1, s_2) = \sum_{N=0}^{\infty} \lambda^N \sum_{i_1, i_2=1}^{\infty} K_{2i_1, 2i_2}(N) \cdot \frac{s_1^{2i_1} s_2^{2i_2}}{(2i_1-1)!!(2i_2-1)!!} = \frac{s_1 s_2}{s_1^2 - s_2^2} \left(\frac{\partial}{\partial s_1} - \frac{\partial}{\partial s_2}\right) \rho_{HZ}^{(\text{odd})}(s_1, s_2)$$

Extension to $q \neq 1$ is rather straightforward, for example the simplest odd correlators are:

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr} X^{2m-1} \text{tr} X \rangle_q = [2m-1]!! q^{m^2} \prod_{n=0}^{m} (q^{2n} - \lambda^2)$$

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr} X^{2m-1} \text{tr} X^3 \rangle_q = [2m-1]!! q^{m+1} \prod_{n=0}^{m} (q^{2n} - \lambda^2) \left\{ [2m+1] \left(\lambda^2 + q^{2m-1} [2] \lambda + q^{4m-2}\right) + 2q^{2m-1} [2m-2] \lambda \right\}$$

...
i.e. for $m_1 \geq m_2$

$$\sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^{2m_1-1} \text{ tr } X^{2m_2-1} \rangle = [2m_1 - 1]!! \cdot \frac{\prod_{i=0}^{m_1-m_2-1} (q^2i + \lambda)}{(1-q^{2i})^{m_1+m_2-1}} \cdot \text{Pol}_{m_2-2}(m_1|\lambda)$$

$$= [2m_1 + 2m_2 - 1]!! \cdot \frac{\prod_{i=0}^{m_1+m_2-1} (q^2i + \lambda)}{(1-q^{2i})^{m_1+m_2-2}} + \ldots \quad (5-3)$$

while the simplest even ones are

$$\sum_{N=0}^{\infty} \lambda^N \langle \langle \text{tr } X^{2m} \text{ tr } X^2 \rangle \rangle_q := \sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr } X^{2m} \text{ tr } X^2 \rangle_q - \langle \text{tr } X^{2m} \rangle_q \langle \text{tr } X^2 \rangle_q \right) =$$

$$= [2m][2m - 1]!! \cdot \frac{q^m \lambda}{(q^{2m} + \lambda)} \cdot \prod_{i=1}^{m} (q^2i + \lambda) = \frac{[2m]}{q \cdot [2m + 1]} \cdot \frac{q^{2m+2} + \lambda}{q^{2m} + \lambda} \cdot \sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^{2m+1} \text{ tr } X \rangle_q \quad (5-4)$$

$$\sum_{N=0}^{\infty} \lambda^N \langle \langle \text{tr } X^{2m} \text{ tr } X^4 \rangle \rangle_q := \sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr } X^{2m} \text{ tr } X^4 \rangle_q - \langle \text{tr } X^{2m} \rangle_q \langle \text{tr } X^4 \rangle_q \right) =$$

$$= \frac{(q^{2m+4} + \lambda)}{(q^{2m-2} + \lambda)} \cdot \frac{[2m - 2]}{q^3 \cdot [2m + 1]} \cdot \sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^{2m+1} \text{ tr } X^3 \rangle_q +$$

$$+ \frac{[3][2m + 2]}{q \cdot [2m + 1][2m + 3]} \cdot \frac{(1 + q^{2m-2} \lambda)}{(q^{2m+2} + \lambda)} \cdot \sum_{N=0}^{\infty} \lambda^N \langle \text{tr } X^{2m+3} \text{ tr } X \rangle_q \quad (5-5)$$

They are natural $q$-deformations of (5-2).

We now sketch the steps, leading to $q$-deformed formulas of this type. As a generalization of (5-2) the full correlators

$$C_\Delta = C_{i_1 \ldots i_n} := \langle \text{tr } X^{i_1} \ldots \text{tr } X^{i_n} \rangle_q$$

for a Young diagram $\Delta = (i_1 \geq i_2 \geq \ldots \geq i_n)$ are equal to

$$C_\Delta = \sum_{R \subseteq \Delta} \text{Chi}_{R,\Delta} \cdot <\chi_R >_q = \sum_{R \subseteq \Delta} \text{Chi}_{R,\Delta} \cdot \frac{\chi_R(\delta_{k|2})}{\chi_R(\delta_{k|1})} \cdot \chi_R(p^\Delta) \quad (5-7)$$

where $\text{Chi}_{R,\Delta}$ are easily available symmetric-group characters. Irreducible correlators arise after subtractions, e.g. $K_{i_1 i_2} := C_{i_1 i_2} - C_{i_1} C_{i_2}$ or taking a logarithm at the level of partition functions. For the pair of odd traces there is no difference, $K_{2m_1-1,2m_2-1} = C_{2m_1-1,2m_2-1}$ and for $q = 1$ their Laplace transform in $N$ is given by the double $s$-expansion of arctan in (5-2). It is convenient to introduce additional parameter $\beta$ in denominator of this formula, so that

$$\sum_{m_1 \geq m_2 \geq 1} \frac{s_1^{2m_1-1} s_2^{2m_2-1}}{(2m_1 - 1)!! (2m_2 - 1)!!} \langle \text{tr } X^{2m_1-1} \text{ tr } X^{2m_2-1} \rangle_{q=1} =$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{s_1^k s_2^{2k+1}}{2k + 1} \left( \frac{(1 - \lambda)(\frac{1}{1 - \beta(1 + \lambda)\frac{s_1^2 + s_2^2}{s_1^2 + s_2^2})^{k+1}} \right)_{\beta=1} =$$

$$\sum_{k,j=0}^\infty \frac{(-1)^k (k + j)!}{2k + 1} \beta^j \lambda^{(1 + \lambda)^j} (s_1 s_2)^{2k+1} (s_1^2 + s_2^2)^j \left( \frac{1}{1 - \lambda(\frac{s_1^2 + s_2^2}{s_1^2 + s_2^2})^{2j}} \right)_{\beta=1} \quad (5-8)$$

2 These characters satisfy orthogonality conditions

$$\sum_{\Delta \uparrow [R]} \text{Chi}_{R,\Delta} \text{Chi}_{R',\Delta} = \delta_{R,R'} \iff \sum_{\Delta \uparrow [D_{\Delta\Delta}]} \text{Chi}_{R,\Delta} \text{Chi}_{R',\Delta} = z_\Delta = \delta_{\Delta,\Delta'}$$

and appear in expansions of Schur polynomials

$$\chi_R(p) = \sum_{\Delta \uparrow [R]} \text{Chi}_{R,\Delta} p^\Delta$$

They are related to characters $\varphi(R, \Delta)$ in [79] by rescaling $\text{Chi}_{R,\Delta} = z_\Delta \cdot \chi_R(\delta_{k,1}) \cdot \varphi(R, \Delta)$.
Picking up appropriate powers of $s_1$ and $s_2$ we get for $m_1 \geq m_2$

$$
\sum_{N=0}^{\infty} \lambda^N \sum_{m_1 \geq m_2 \geq 1} \frac{\langle \text{tr} X^{2m_1-1} \text{tr} X^{2m_2-1} \rangle_{q=1}}{(2m_1-1)!(2m_2-1)!} = 
\sum_{k=0}^{m_2-1} \frac{(-)^k \beta^{m_1+m_2-2k-2}}{(2k+1) \cdot k!} \cdot \frac{(m_1 + m_2 - 2)!}{(m_1 - k - 1)!(m_2 - k - 1)! \cdot k!} \cdot \frac{\lambda(1 + \lambda)^{m_1+m_2-2k-2}}{(1 - \lambda)^{m_1+m_2-2k}}
$$

Thus for $q = 1$ the polynomial in (5-9) is

$$
\text{Pol}_{2m_2-2}(m_1 | \lambda)_{q=1} = \sum_{j=0}^{m_2-1} \frac{(-)^j}{2j + 1} \cdot \frac{(2m_2 - 1)!!}{(m_2 - j - 1)!!} \cdot (1 + \lambda)^{(2m_2-1-j)} \cdot (1 - \lambda)^j \cdot \prod_{i=1}^{m_2-1} (m_1 + i - j - 1)
$$

$q$-deformation is relatively straightforward – all factorials are changed for $q$-factorials and powers of $(1 \pm \lambda)$ to appropriate Pochhammer symbols. Note only that $m_1$ and $m_2$ are everywhere converted to $2m_1$ and $2m_2$, and accordingly all factorials are double-factorials.

$$
P_{2m_2-2}(m_1 | \lambda) = \sum_{k=0}^{m_2-1} \frac{(-)^k}{[2k+1][2k]!!} \cdot \frac{[2m_1 - 1]!!}{[2m_1 - 2 - 2k]!!} \cdot \frac{[2m_1 + 2m_2 - 4 - 2k]!!}{[2m_1 - 2 - 2k]!![2m_1 - 2 - 2k]!!} \cdot \prod_{i=1}^{2m_2-2-2k} (q^{2m_1-2m_2+2i} + \lambda) \cdot \prod_{i=2m_2-2-2k}^{2m_2-2} \frac{2q_{m_1-2m_2+2i} - \lambda}{q_{m_1-2m_2+2i} - \lambda}
$$

The first two of these polynomials coincide with the examples in (5-3), but the structure is now clarified and inspired by (5-10). Odd double-trace correlators are now provided by the last expression in (5-3).

Irreducible even double-trace correlators are described by equally explicit, even if a bit lengthier, formula for $m_1 \geq m_2$

$$
\sum_{N=0}^{\infty} \lambda^N \langle \langle \text{tr} X^{2m_1} \text{tr} X^{2m_2} \rangle \rangle_q = \sum_{N=0}^{\infty} \lambda^N \langle \langle \text{tr} X^{2m_1} \text{tr} X^{2m_2} \rangle \rangle_q - \langle \langle \text{tr} X^{2m_1} \rangle \rangle_q \langle \langle \text{tr} X^{2m_2} \rangle \rangle_q = 
\sum_{k=0}^{m_2-1} \frac{(-)^k}{[2k+1][2k]!!} \cdot \frac{q^{m_1+m_2-1}(q^{1+m_2+2m_2-4-2k})!!}{[2m_1-2-2k][2m_1-2-2k]!!} \cdot \prod_{i=1}^{m_1+m_2-2-2k} (q^{2i} + \lambda) \cdot \prod_{i=m_1+m_2-2-2k}^{m_1+m_2-2} (q^{2i} - \lambda)
$$

with

$$
\text{Pol}_{2n}^{\text{even}}(\lambda | m_1) = \sum_{j=0}^{n-1} \frac{[2(2n - 2j)]!!}{[2j + 1][2j]!![2n - 2j][2n - 2j]!!} \cdot \frac{[2m_1 + 2n - 2j]!!}{[2m_1 - 2 - 2j]!!} \cdot \frac{(-1)^j}{2} P_{2n,2j}(\lambda | m_1)
$$

$$
+ \sum_{j=0}^{n-2} \frac{\{q\}}{[2j + 1][2j]!![2n - 2j][2n - 2j]!!} \cdot \frac{[2m_1 + 2n - 2j]!!}{[2m_1 - 2 - 2j]!!} \cdot \frac{(-1)^j}{2} P_{2n+2j}(\lambda | m_1)
$$

$$
+ \frac{1}{[2n + 1][2n]!!} \cdot \frac{[2m_1]!!}{[2m_1 - 2 - 2n]!!} \cdot (-1)^n P_{2n,2n}(\lambda | m_1),
$$

where $\{q\} = q - q^{-1}$ and the basis polynomials $P_{2n,k}$ are similar to the structures in the odd double-trace correlator:

$$
P_{2n,k}(\lambda | m_1) = \prod_{i=0}^{k-1} (\lambda - q^{2m_1+2n-2i}) \prod_{i=k}^{2n-1} (\lambda + q^{2m_1+2n-2i})
$$

We observe here an interesting phenomenon – conversion of the naive ratio with non-trivial switch in the numerator:

$$
\sum_{i=0}^{m_1} (q^{2i} + \lambda) \sum_{i=0}^{m_2} (q^{2i} + \lambda) \sum_{i=0}^{\text{max}+1} (1 - q^{2i}\lambda) \sum_{i=0}^{\text{max}+1} (1 - q^{2i}\lambda)
$$

Here $j = m_1 + m_2 - 2 - 2k$ with $k \geq 0$ and $j_{\text{max}} = m_1 + m_2 - 2$. Modulo a common power of $q$, the switch is basically the change in one of the products $q^{2i} \rightarrow q^{2-2i}$, i.e. a kind of a twisted conjugation in the case of a unimodular $q = e^{\pi \alpha}$. 
In general, relation between even and odd cases is:

$$\sum_{N=0}^{\infty} \lambda^N \langle \langle \text{tr} X^{2m_1} \text{tr} X^{2m_2} \rangle \rangle_q := \sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr} X^{2m_1} \text{tr} X^{2m_2} \rangle_q - \langle \text{tr} X^{2m_1} \rangle_q \langle \text{tr} X^{2m_2} \rangle_q \right) =$$

$$= \frac{q^{2m_1+2m_2} + \lambda}{q^{2m_1+2-2m_2} + \lambda} \sum_{j=0}^{m_2-1} \frac{[2m_1 + 2 - 2m_2 + 4j]}{q^{2m_2-2j-1}} \cdot \frac{[2m_1 - 1]!!}{[2m_2 - 1]!!} \cdot \prod_{i=1}^{j} \frac{1 + q^{2m_1-2m_2+4i-2} + \lambda}{q^{2m_1-2m_2+4i+2} + \lambda} \cdot \sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr} X^{2m_1+2j+1} \rangle_q \langle \text{tr} X^{2m_2-2j-1} \rangle_q \right)$$  \hspace{1cm} (5-16)

Explicit interpretation of the r.h.s. as an action of a difference operator, which would look as literal generalization of \([5-2]\), will be given elsewhere, together with analysis of the multi-trace case.

6 Towards \((q,t)\)-Harer-Zagier formulas

In this section we list our partial progress in understanding the \((q,t)\) generalization of the Harer-Zagier formulas. For the purposes of this section we define the \((q,t)\)-deformation of Hermitian Gaussian matrix model (qtHGMM) by means of its Macdonald averages, as was done in [99]. This definition implies poles at \(q = 1\) (for \(t\) fixed) and some other unexpected properties – which seem technically difficult to avoid, but do not have any clear conceptual explanation.

With this definition the simplest 1-point correlator is

$$\langle \text{tr} (X^2) \rangle_{q,t} := \left[ \text{Mac}[2] - \frac{\langle \text{tr} (q + q^{-1}) \rangle_{\text{Mac}[1,1]} }{\langle q \rangle_{\text{Mac}[1]} } \right]_{q,t} =$$

$$= \frac{\{t^N\}}{\{q\}\{t\}} \cdot \left( (q^2 - 1)t^{N-1} + \frac{2\{t^{N-1}\}}{t^2+1} \right),$$  \hspace{1cm} (6-1)

where \(\{x\} = x - x^{-1}\). We use notation with the trace, though in Macdonald case the proper notion of single-trace operators is not so obvious – and this can affect the interpretation of eq. (6-3) below.

The Laplace transform of \((6-1)\) does have reasonable poles, but the numerator is not a simple factorized expression

$$\sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr} (X^2) \rangle_{q,t} \right) = \frac{q\lambda}{(\lambda - 1)(\lambda t^2 - 1)} \left( 1 - \frac{2\lambda(t^2 - 1)}{(\lambda - t^2)(q^2 - 1)} \right) =$$

$$= - \frac{q\lambda}{(\lambda - t^2)(\lambda - 1)(\lambda t^2 - 1)} \left( \lambda + t^2 + \frac{2\lambda(t^2 - q^2)}{q^2 - 1} \right)$$  \hspace{1cm} (6-2)

Factorized is the answer for (so far mysterious) linear combination

$$\sum_{N=0}^{\infty} \lambda^N \left( \frac{q^2 - 1}{t^2 - 1} \cdot \langle \text{tr} (X^2) \rangle - \frac{q^2 - t^2}{t^2 - 1} \cdot \langle \text{tr} X \rangle^2 \right)_{q,t} = - \frac{q\lambda(\lambda + t^2)}{(\lambda - t^2)(\lambda - 1)(\lambda t^2 - 1)}$$  \hspace{1cm} (6-3)

Likewise, for the second non-trivial 1-point correlator

$$\langle \text{tr} (X^4) \rangle_{q,t} =$$

$$= \frac{\{t^N\}}{\{t\}(t^2)} \left( \frac{2t^2 q^6 + 2q^4 t^2 + q^6 + 2q^4 t^2 + q^6 t^2 + q^4 t^4 + q^6 t^4 + q^4 t^6 + q^6 t^6 + q^4 t^8 + q^6 t^8 + q^4 t^{10} + q^6 t^{10} + q^4 t^{12} + q^6 t^{12} + q^4 t^{14} + q^6 t^{14} + q^4 t^{16} + q^6 t^{16} + q^4 t^{18} + q^6 t^{18} + q^4 t^{20} + q^6 t^{20} + \cdots }{t^N + 2t^{N+2} + 2t^{N+3}} \right)$$  \hspace{1cm} (6-4)

and its Laplace transform

$$\sum_{N=0}^{\infty} \lambda^N \left( \langle \text{tr} (X^4) \rangle_{q,t} \right) = \frac{q^2\lambda}{(\lambda + t^2)(\lambda - 1)(\lambda t^2 - 1)} \left( t^4(q^4 + q^2 + 1) + \frac{2\lambda(\lambda + t^2)}{t^2(q^2 - 1)} \right) =$$

$$= \frac{q^2\lambda}{(\lambda + t^2)(\lambda - 1)(\lambda t^2 - 1)} \left( t^4(q^4 + q^2 + 1) + \frac{2\lambda(\lambda + t^2)}{t^2(q^2 - 1)} \right)$$  \hspace{1cm} (6-5)

In the absence of full understanding, let us speculate a little bit. We know, that the basis (in the space of symmetric functions) of Macdonald polynomials has simple factorized averages, as functions of \(q, t\) and \(Q = t^N\). At the same time, when we take the Laplace transform of the Macdonal averages, this simple factorizability is lost, and instead some (yet unknown) Harer-Zagier symmetric functions have simple factorized averages.

In \(q = 1\), and even in \(q \neq 1, t = q\) case the one-point (symmetric) Harer-Zagier functions were just simple power sums. However, the generic \((q,t)\) case seems to tell us, that in general these functions are a little bit more complicated.

Perhaps, one can, indeed, look for a basis in the space of symmetric functions that have nicely factorized Laplace-transformed averages and (together with some form of orthogonality) this will be enough to fix them. The work in continuing in these directions.
7 Towards q-Wick theorem and q-genus expansion

At the $q = 1$ any trace product correlator (due to the Wick theorem) is a sum over the fat (ribbon) graphs, where the valencies of the graph’s vertices are equal to the exponents of the matrix under the trace. The weight of each particular fat graph is (up to an overall normalization factor)

$N_Euler$ characteristic

For instance the average of $\langle \text{tr} (X^4) \rangle$ is equal to the following sum

$$\langle \text{tr} (X^4) \rangle = N \cdot \left( N^2 + N^2 + N^0 \right)$$

(7-1)

In other words, this sum is a sum over gluings of Riemann surfaces with marked points from polygons, each polygon corresponding to a vertex of a fat graph (and to a trace of the matrix) and thus having as many edges as the corresponding exponent. Continuing the example, there are the following gluings

(7-2)

The factor of $N$ then comes from the additional sum over the labellings of the gluings’ distinct vertices with numbers $1 \ldots N$. In the example, the first and the second gluings have three distinct vertices, resulting in the factor $N \cdot N^2$, while the third gluing has one distinct vertex and has weight $N \cdot N^0$.

After going to $q \neq 1$, for each concrete integer $N$ any trace product correlator becomes a Laurent polynomial in $q$ with non-negative coefficients. For instance, for $\text{tr} (X^4)$ at first few $N$

$$\langle \text{tr} (X^4) \rangle_q \bigg|_{N = 1} = q^2 + q^4 + q^6$$

(7-3)

$$\langle \text{tr} (X^4) \rangle_q \bigg|_{N = 2} = 2q^{-2} + 3q^0 + 4q^2 + 3q^4 + 3q^6 + 2q^8 + q^{10}$$

$$\langle \text{tr} (X^4) \rangle_q \bigg|_{N = 3} = 2q^{-6} + 4q^{-4} + 6q^{-2} + 7q^0 + 9q^2 + 9q^4 + 8q^6 + 5q^8 + 4q^{10} + 2q^{12} + q^{14}$$

One can easily check with help of computer experiments (both for one-point and multi-point correlators) that

$$\left\{ \begin{array}{c} \# \text{ of } q\text{-monomials} \\ \text{in a polynomial,} \\ \text{with multiplicities} \end{array} \right\} = \left\{ \begin{array}{c} \# \text{ of labeled} \\ \text{gluings} \end{array} \right\}$$

(7-4)

It is, therefore, tempting to conjecture that for $q \neq 1$ some formula for trace product correlators in terms of labeled polygon gluings exists, namely, that there is some recipe to assign a $q$-monomial to each labeled gluing.

Such a formula would simultaneously provide a $q$-analogue of the Wick theorem, because it would assign a $q$-weight to (labelled) pairwise splitting of the sides of the polygons, and the $q$-analogue of genus expansion, because every Riemann surface, resulting from a gluing, naturally has a genus.

Obtaining such formula requires careful study (because straightforward attempts do not succeed) and is postponed for future research. Here are the first two examples, which illustrate some of the occurring complications. Ignoring the labels, each of these correlators should have exactly one gluing

$$\langle (\text{tr} X)^2 \rangle = N \rightarrow \langle (\text{tr} X)^2 \rangle_q = q^N\frac{(q^{-N} - q^N)}{(q^{-1} - q)}$$

(7-5)

$$\langle (\text{tr} X^2) \rangle = N^2 \rightarrow \langle (\text{tr} X^2) \rangle_q = \frac{(q^{-N} - q^N)(q^2Nq^2 + q^2Nq^{-2} - 2)}{(q^{-1} - q)q^N(q + q^{-1})}$$

One sees that two kinds of phenomena occur. First, spurious $q^N$ framing factors (vanishing at $q \rightarrow 1$) appear. Second, the deformation is clearly more complicated than the naive quantization $N \rightarrow \lfloor N \rfloor$. While our $q$-Harer-Zagier formulas (7-7), (7-8), together with the observation (7-4) do provide evidence that the correct quantization prescription can be found, this is neither immediate, nor obvious and, once available, will undoubtedly lead to better understanding of the relevant enumerative geometric structures.
8 Conclusion

To conclude, we derived an explicit q-deformation of Harer-Zagier formula for the single-trace operator averages in Gaussian Hermitian model, which looks somewhat involved for particular matrix sizes \( N \), but actually becomes as simple as it was at \( q=1 \) after Laplace transform in \( N \): see Eq. (3-6). We also conjectured, based on intuition from \( q=1 \) case, equally explicit formulas for q-deformation of connected double-trace correlators, see Eqs. (5-12)-(5-16).

The derivation can be based both on Jackson-integral version of the eigenvalue integral and on the super-integrability property of matrix models \([64,67]\), which is directly generalizable in many directions — from \( q,t \)-deformations to tensor models. However, four immediate generalizations of the remarkable results (3-6) and (5-12)-(5-16) — to multi-trace correlators, to \( t \) deformation, to \( q \)-genus expansion and to \( q \)-Wick theorem — appear somewhat problematic or at least less straightforward.

Resolution of emerging problems should shed additional light on the structure of matrix-model deformations and their relation to representation theory of double loop algebras from the DIM family. This, in turn, is important for understanding the relation between super-integrability (roughly, the existence of superficially large number of integrals of motion) and the more familiar KP/Toda/Hirota integrability in situations where the former one is obviously preserved, but the latter one needs serious and still unknown modification.

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