Characteristic Lie rings, finitely-generated modules and integrability conditions for (2 + 1)-dimensional lattices

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Abstract

Characteristic Lie rings for Toda and Volterra type (2 + 1)-dimensional lattices are defined. Some properties of these rings are studied. Infinite sequence of special kind subrings are introduced. It is proved that for known integrable lattices these subrings are of finite dimension. A classification algorithm based on this observation is suggested.

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1. Introduction

In the last decade the problem of classification of integrable multidimensional models has been intensively studied. An approach to the problem based on hydrodynamic type reductions was suggested several years ago in [1]. This approach allowed one to find new classes of integrable partial differential equations (PDEs) [1–3]. An alternative method of integrable classification suitable for multidimensional equations on quad graphs using consistency around a multidimensional cube was developed in [4].

In the present article we propose a new classification algorithm of integrable (2 + 1)-dimensional lattices of the form

\[
\frac{\partial^2}{\partial x \partial y} u_n = f_n(u_n, u_{n,x}, u_{n,y}, u_{n+x}, u_{n+y}, u_{n+x,y}, u_{n+x,y}, \ldots u_{n+x+y}, \ldots) ,
\]

(1.1)

where \(-\infty < n < \infty\) and \(u = u_n(x, y)\) is a function of three independent variables \(x, y, n\). Recall that the well-known two dimensional Toda lattice having important applications in Liouville and conformal field theories (see [5–8]) belongs to this class. In our opinion class (1.1) should contain new interesting integrable models like two dimensional generalizations of the so-called relativistic Toda lattice [9].

Our approach is based on the notion of characteristic vector fields introduced and applied to the classification problem of integrable (1 + 1)-dimensional hyperbolic type PDE by Goursat [10]. In the modern context of integrability the concept of characteristic Lie rings was revived in [11], where it was applied to Darboux integrable exponential type systems. Characteristic Lie rings of (1 + 1)-dimensional continuous and discrete soliton equations are studied in [12–14], (see also survey [15]). In a sense the present article is a prolongation of [13] where characteristic Lie rings for discrete (1 + 1)-dimensional models are investigated. However, here we consider characteristic vector fields in essentially different situation, since the lattice (1.1) actually is (2 + 1)-dimensional. Note that multidimensional models do not admit local integrals and therefore Darboux integrability loses its sense in this case. Nevertheless to any lattice of the form (1.1) one can assign two Lie rings \(L_x\) and \(L_y\) which provide an effective integrability criterion for the lattice.

The article is organized as follows. In section 2 we introduce Lie rings \(L_x\) and \(L_y\), describe their subrings and formulate our integrability test (conjecture). In section 3 we recall the definition of characteristic Lie rings for systems of hyperbolic type equations. This is used in section 5 but also it explains the motivation of introducing similar objects in the (2 + 1)-dimensional case. In section 4 we adopt our definitions to Volterra type lattices. In section 5 we prove theorem 2 and...
formulate theorem 3, approving the integrability conjecture given in definitions 1 and 2.

2. Preliminaries

Evidently operator \( D_\gamma = \frac{\partial}{\partial y} \) of the total derivative with respect to \( y \) acting on the subset of dynamical variables \( u_i, u_{i,x}, u_{i,xx}, \ldots \) defines a vector field given as a formal series

\[
Y = \sum_{i=-\infty}^{\infty} u_{i,y} \frac{\partial}{\partial u_i} + f_{i,1} \frac{\partial}{\partial u_{i,x}} + f_{i,2} \frac{\partial}{\partial u_{i,xx}} + f_{i,3} \frac{\partial}{\partial u_{i,xxx}} + \cdots.
\]

(1.2)

Recall the characteristic vector field in the \( y \)-direction for the lattice (1.1). Denote through \( L_\gamma \) the Lie ring generated by the operators \( Y, X_i = \frac{\partial}{\partial u_{i,y}} \), where \( i \) ranges in \((-\infty, \infty)\). Therefore the characteristic Lie ring of an infinite lattice has an infinite set of generators, thus it is rather complicated to work with\(^1\). However it has a sequence of relatively simple subrings, which can be used as indicators of integrability. For integrable lattices these subrings are of finite dimension.

In a similar way one can define the Lie ring \( L_\Lambda \) generated by the operators \( \hat{X}_i = \frac{\partial}{\partial u_{i,y}} \), where

\[
\hat{Y} = \sum_{i=-\infty}^{\infty} u_{i,y} \frac{\partial}{\partial u_i} + f_{i,1} \frac{\partial}{\partial u_{i,x}} + f_{i,2} \frac{\partial}{\partial u_{i,xx}} + f_{i,3} \frac{\partial}{\partial u_{i,xxx}} + \cdots.
\]

(1.3)

Construct a sequence of the operators

\[
X_1, X_{1,j}, X_{2,j}, X_{3,j}, \ldots
\]

where \( j \in (-\infty, \infty) \) and \( Y_{1,j} = [X_i, Y], \ Y_{2,j} = [Y_{1,j}, Y], \ Y_{3,j} = [Y_{2,j}, Y], \ldots \)

Let \( F \) be a ring of locally analytic functions depending on a finite number of the dynamical variables \( u_i, u_{i,x}, u_{i,y}, u_{i,xx}, u_{i,yy}, \ldots \). Fix a finite subset \( A_y \) of the operators in (1.4). Stress that \( A_y \) may contain the operators \( X_i, X_{1,j} \) with different values of their indices, for example, one can choose \( A_y = \{X_1, Y_{2,j}, X_{3,j}\} \). Denote through \( L_{0,y} \) the set containing all of the operators in this subset, all possible commutators of these operators and linear combinations of the operators in \( A_y \) and their commutators with the coefficients from \( F \). Actually \( L_{0,y} \) has a structure of the left \( F \)-module (see [17]). Recall that the left \( F \)-module \( L_{0,y} \) is called finitely generated if there exist \( Z_1, Z_2, \ldots, Z_m \) in \( L_{0,y} \) such that for all \( Z \) in \( L_{0,y} \), there exist \( r_1, r_2, \ldots, r_m \) in \( F \) with \( Z = r_1 Z_1 + r_2 Z_2 + \cdots + r_m Z_m \). The set \( Z_1, Z_2, \ldots, Z_m \) is called a finite generating set of the module \( L_{0,y} \).

Remark 1. The finite generating set of \( F \)-module allows one to define the dimension of the module. Suppose that equation

\[
r_1 Z_1 + r_2 Z_2 + \cdots + r_m Z_m = 0
\]

holds for the functions \( r_1, r_2, \ldots, r_m \) analytic in a domain of many variable complex space. By reducing the domain if necessary one can get only two possible choices: either \( r_1 \not= 0 \) everywhere in the domain or \( r_1 \equiv 0 \). In the former case we see that \( Z_1 \) is linearly expressed in terms of \( Z_2, \ldots, Z_m \). Continuing this way we find a subset of linearly independent operators. Thus the module has locally a structure of a finite dimensional linear space.

In a similar way one defines the left \( F \)-module \( L_{0,x} \) in the \( x \)-direction, by choosing a finite subset \( A_x \) of the operator sequence, defined as follows

\[
\hat{X}_i, \ Y_{1,i} = [\hat{X}_i, \ Y], \ Y_{2,i} = [Y_{1,i}, \ Y], \ Y_{3,i} = [Y_{2,i}, \ Y], \ldots
\]

The following definition seems to be reasonable.

Definition 1. A lattice of the form (1.1) is called ‘integrable’ if for any choice of the subsets \( A_y, A_x \), the corresponding left \( F \)-modules \( L_{0,y} \) and \( L_{0,x} \) are finitely-generated.

Theorems 2 and 3 below show that a large class of lattices known to be integrable are also ‘integrable’ in the sense of definition 1. Therefore definition 1 is regarded as an adequate formalization of the integrability property for the lattice (1.1). Actually it can be used as a classification tool for integrable lattices.

Let us formulate some useful commutativity relations between generators of the rings \( L_x, L_y \). For simplicity we concentrate on \( L_y \), because all the properties of \( L_x \) are exactly symmetrical.

Lemma 1. Operators \( X_i, Y \) acting on the variables \( u_i, u_i, u_{i,x}, u_{i,xx}, u_{i,xxx}, \ldots \) satisfy the relations

\[
[D_\gamma, X_i] = -X_i(f_j)X_i, \quad [D_\gamma, Y] = -\sum_i Y(f_j)X_i.
\]

(1.5)

Lemma can be proved by applying both sides of the relations to the dynamical variables listed.

Lemma 2. Suppose that the vector field

\[
Z = \sum_j a_j(1) \frac{\partial}{\partial u_{j,x}} + a_j(2) \frac{\partial}{\partial u_{j,xx}} + a_j(3) \frac{\partial}{\partial u_{j,xxx}} + \cdots
\]

satisfies the equation \([D_\gamma, Z] = 0\), then \( Z = 0\).

Proof. Apply both sides of the equation \([D_\gamma, Z] = 0\) to the variables \( u_j, u_{j,x}, u_{j,xx}, \ldots \) and get an infinite system of equations \( a_j(1) = 0 \) and \( a_j(k+1) = D_\gamma a_j(k) \) for \( k \geq 1 \) which immediately implies that all of the coefficients \( a_j(k) \) vanish. The lemma is proved.

3. Characteristic Lie rings for systems of hyperbolic type PDEs

Consider a system of hyperbolic type PDEs of the following form

\[
\frac{\partial^2}{\partial x^2} u_i = g_i(u, u_x, u_y), \quad i = 1, 2, \ldots, N.
\]

(2.1)
where \( \mathbf{u} = (u_1, u_2, \ldots, u_N) \) and \( \mathbf{v}_x = \frac{\partial}{\partial u} \mathbf{u}, \mathbf{v}_y = \frac{\partial}{\partial v} \mathbf{u} \). Define a set of standard dynamical variables \( S = S_1 \cup S_y \), where
\[
S_1 = \{ [u, u_x, u_{xx}, \ldots] \}, \quad S_y = \{ [u, u_y, u_{yy}, \ldots] \}.
\]
(2.2)
As usual, we consider dynamical variables as independent ones.

Recall that a function \( I \) depending on a finite number of the variables in \( S_i \) is called \( y \)-integral if \( D_y I = 0 \) by means of the equation system (2.1). A set of \( y \)-integrals \( I_{(1)}, I_{(2)}, \ldots, I_{(N)} \) constitutes a complete set of integrals if none of these integrals are expressed in terms of the other ones and their total derivatives with respect to \( y \). In a similar way the complete set of \( x \)-integrals for (2.1) is defined.

System (2.1) is called Darboux integrable if it admits complete sets of integrals in both the \( x \)- and \( y \)-directions.

According to the definition any \( y \)-integral solves the following equation \( Y_i I := D_y I (u, u_x, u_{xx}, \ldots) = 0 \), where, due to the chain rule, the characteristic vector field \( Y_h \) is defined as follows
\[
Y_h = \sum_{i=1}^{N} u_{i,y} \frac{\partial}{\partial u_i} + g_i \frac{\partial}{\partial u_{i,x}} + g_{i,x} \frac{\partial}{\partial u_{i,xx}} + \cdots. \tag{2.3}
\]
Since the \( y \)-integral does not depend on the variables \( u_{i,y} \) while the coefficients of \( Y_h \) depend on them, we have to write in addition to the equation \( Y_i I = 0 \) also equations \( X_i I = 0 \) for \( i = 1, 2, \ldots, N \), where \( X_j = \frac{\partial}{\partial u_{j,y}} \).

Let \( I_{(1)} \) be the set containing all of the operators \( X_1, X_2, \ldots, X_N, Y_h \) all possible commutators of these operators and linear combinations of the operators and their commutators with the variable coefficients in \( F \). Then \( I_{(1)} \) has a structure of the left \( F \)-module. Thus any \( y \)-integral is annihilated by the elements of \( I_{(1)} \). In a similar way the module \( I_{(2)} \) is defined.

The following theorem establishes a correspondence between integrals and the characteristic Lie rings (see, for instance, survey [15]).

**Theorem 1.** System (2.1) admits a complete set of \( y \)-integrals (or \( x \)-integrals) iff its characteristic \( F \)-module \( I_{(1)} \) (respectively, characteristic \( F \)-module \( I_{(2)} \)) is finitely generated.

### 4. Volterra type lattices

In the case of the Volterra type lattices
\[
u_{n,y} = p(u_n, v_{n+1}, v_n), \quad \nu_{n,x} = q(v_n, u_n, u_{n-1}) \tag{3.1}
\]
our definitions are slightly changed. The characteristic vector fields in the \( y \)-direction are defined as follows
\[
Y_h = \sum_{i=-\infty}^{\infty} p_i \frac{\partial}{\partial u_i} + p_{i,x} \frac{\partial}{\partial u_{i,x}} + p_{i,xx} \frac{\partial}{\partial u_{i,xx}} + \cdots, \tag{3.2}
\]
and \( X_{e,i} = \frac{\partial}{\partial u_i} \). Now define the sequence
\[
X_{e,i}, X_{e,1,i} = [X_{e,i}, Y_e], X_{e,2,i} = [Y_e, X_{e,1,i}], X_{e,3,i} = [X_{e,2,i}, Y_e], \ldots, X_{e,k,i} = [Y_{e,k-1,i}, Y_e], \ldots, \tag{3.3}
\]
where integer \( i \) ranges in \((\infty, \infty)\). Having the operator sequence we choose its finite subset \( A_{e,i} \) and construct the characteristic module \( L_{(1),0,y} \).

Similarly we define the characteristic vector fields in the \( x \)-direction
\[
\tilde{X}_{e,i} = \sum_{i=-\infty}^{\infty} q_i \frac{\partial}{\partial u_i} + q_{i,y} \frac{\partial}{\partial u_{i,y}} + q_{i,yy} \frac{\partial}{\partial u_{i,yy}} + \cdots. \tag{3.4}
\]
\[
\tilde{X}_{e,i} = \frac{\partial}{\partial u_i}, \tag{3.5}
\]
Afterwards we choose a finite set \( A_{e,i} \) and construct the corresponding characteristic module \( L_{(2),0,x} \).

**Definition 2.** A lattice of the form (3.1) is called ‘integrable’ if for any choice of the subsets \( A_{e,i} \) the corresponding modules \( L_{(1),0,y} \) and \( L_{(2),0,x} \) are finitely-generated.

**5. A remarkable property of integrable Toda and Volterra type lattices**

Sequences of characteristic modules introduced in the previous sections are approved to provide an effective tool to justify integrability of a given lattice (see definitions 1 and 2 above). It is observed that for a large class of integrable lattices of the forms (1.1) and (3.1) the characteristic modules in both directions are finitely generated for any choice of the operator sets. Let us illustrate the statement with the example of the Toda lattice [5]:
\[
\frac{\partial^2}{\partial x \partial y} u_n = \exp(2u_n - u_{n-1} - u_{n+1}), \quad -\infty < n < \infty. \tag{4.1}
\]

**Theorem 2.** For any choice of the sets \( A_{e,i} \) for the Toda lattice (4.1) the corresponding modules \( L_{(1),0,y} \) and \( L_{(2),0,x} \) are finitely-generated.

**Proof.** Reduce the infinite lattice (4.1) to a finite system of hyperbolic type PDEs by imposing the following cutting off boundary conditions \( u_{-N-1} = 0 \) and \( u_{N+1} = 0 \) that are known to preserve integrability of lattice (4.1). As a result we find a finite system of hyperbolic type PDEs:
\[
\frac{\partial^2}{\partial x \partial y} u_{-N} = \exp(2u_{-N} - u_{-N+1}), \tag{4.2}
\]
\[
\frac{\partial^2}{\partial x \partial y} u_n = \exp(2u_n - u_{n-1} - u_{n+1}), \quad -N < n < N, \tag{4.2}
\]
\[
\frac{\partial^2}{\partial x \partial y} u_{N} = \exp(2u_{-N} - u_{-N-1}). \tag{4.2}
\]
Consider the Lie ring generated by the operators \( X_{-N}, X_{-N+1}, \ldots, X_{N}, Y_e \) where \( Y_e = \frac{\partial}{\partial u_{-N}} \), and
\[
Y_e = \sum_{i=-N}^{N} u_{i,y} \frac{\partial}{\partial u_i} + g_i \frac{\partial}{\partial u_{i,x}} + g_{i,x} \frac{\partial}{\partial u_{i,xx}} + \cdots, \tag{4.3}
\]
where
\[ g_{i-N} = \exp[2u_{i-N} - u_{i-N+1}], \]
\[ g_i = \exp[2u_i - u_{i-1} - u_{i+1}], \quad -N < i < N, \]
\[ g_N = \exp[2u_N - u_{N-1}]. \]

System (4.2) is known to be the Darboux integrable (see [11]), i.e. it admits complete sets of integrals in both characteristic directions. Hence due to theorem 1, characteristic Lie rings \( L_{i,y} \) and \( L_{i,x} \) have a structure of finitely generated \( \mathbb{F} \)-modules. Moreover in the case of the Toda lattice these modules are finite dimensional linear spaces, they admit finite basis (see remark 1).

Returning to lattice (4.1), which is a particular case of (1.1). The following statement applies.

**Lemma 3.** For any natural \( k \) and any integer \( i \) we have \( Y_{i,k} \in L_{i,N} \) and \( Y_{i,k} \in L_{i,N} \) for \( N \geq |i| + k \).

**Proof of lemma 3.** By definition we have \( X_i \in L_{i,N} \) for \( |i| \leq N \). For (4.1) and its reduction (4.2) we have \( Y_{i,1} = [X, Y] = \frac{\partial}{\partial u_{i,x}} \) and \( Y_{1,i} = [X_i, Y] = \frac{\partial}{\partial u_{i,y}} \) hence \( Y_{1,i} \in L_{i,y} \) for \( |i| \leq N \).

Comparison of the operators \( Y_{i,1} = [X_{i}, Y] \) and \( Y_{1,2} = Y_{2,1} \in L_{i,N} \) given as a formal series
\[ Y_{i,2} = \sum_{j=1}^{i+1} f_{i,j} u_j \frac{\partial}{\partial u_{i,x}} + (f_{i,j}) u_j \frac{\partial}{\partial u_{i,x}} + \cdots, \]
(4.4)
and
\[ Y_{1,2} = \sum_{j=1}^{i+1} g_{i,j} u_j \frac{\partial}{\partial u_{i,y}} + (g_{i,j}) u_j \frac{\partial}{\partial u_{i,y}} + \cdots \]
(4.5)
shows that \( Y_{1,j} = Y_{i,2,j} \) for \( |i| \leq N - 2 \) since \( f_{i,j} = g_{i,j} \) for \( |i| \leq N - 1 \). Therefore, \( Y_{i,1} \in L_{i,N} \) for \( |i| \leq N - 2 \). Continuing this way one can complete the proof of the lemma by induction proving that \( Y_{i,k} \in L_{i,N} \) for \( |i| \leq N - k \), where \( N > k \). It implies that \( Y_{i,k} \in L_{i,N} \) for \( N \geq |i| + k \).

End of proof of theorem 2. Choose a finite set \( A_y \) of operators in (1.4) corresponding to the Toda lattice. Due to lemma 3 there is a sufficiently large natural \( N \) such that \( A_y \subseteq L_{i,N} \). As was noted above, \( L_{i,y} \) is a finite dimensional linear space, therefore \( L_{0,y} \) generated by the set \( A_y \) is also a finite dimensional linear space. Thus it is a finitely generated \( \mathbb{F} \)-module. The theorem is proved.

In a similar way one can prove the following statement concerning the lattices listed in [18] as the integrable ones.

**Theorem 3.** For an arbitrary choice of the sets \( A_y, A_x \) for any of the lattices below the corresponding modules \( L_{0,y} \) and \( L_{0,x} \) are finitely generated
\[ u_{aux} = u_{ax} [\exp(u_{a+1} - u_a) - \exp(u_a - u_{a-1})], \]
(4.6)
\[ u_{uy} = u_{ay} (v_{a+1} - v_a), \quad v_{ux} = u_{uy} (v_b - v_{a-1}), \]
(4.7)
\[ u_{ux} = u_{ay} (v_{a+1} - v_a), \quad v_{nx} = v_{nx} (u_a - u_{a-1}). \]
(4.8)

For the lattice (4.7) the proof of the statement easily follows from the results in [16].

6. Conclusions

The concept of characteristic Lie rings provides an effective tool to investigate the Darboux integrable (1 + 1)-dimensional nonlinear equations. Darboux integrable equations, called also Liouville type equations, are nonlinear hyperbolic type equations or systems of such equations admitting complete sets of integrals. Due to this circumstance they admit explicit formulas for general solutions. These kinds of models have important applications in physics. For example models of Liouville field theory corresponding to simple Lie algebras are all Darboux integrable. To the best of our knowledge there is no analogue of the characteristic Lie rings for multidimensional equations. In this article an attempt is undertaken to fill up this gap, the concept of characteristic Lie rings is generalized to (2 + 1)-dimensional lattices. A sequence of special type subrings is considered. It is shown that for the known integrable lattices of the form (1.1) listed in [18] all of these subrings are of finite dimension. This fact can be used as a basis of a classification criterion for integrable lattices.

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