ABSTRACT. We generalize the concept of the pointwise supremum of real-valued functions to the pointfree setting. The concept itself admits a direct and intuitive formulation which makes no mention of points. But our aim here is to investigate pointwise suprema of subsets of \(R_L\), the family of continuous real valued functions on a locale, or pointfree space.

Thus our setting is the category \(W\) of archimedean lattice-ordered groups (\(\ell\)-groups) with designated weak order unit, with morphisms which preserve the group and lattice operations and take units to units. This is an appropriate context for this investigation because every \(W\)-object can be canonically represented as a subobject of some \(R_L\).

We show that the suprema which are pointwise in the Madden representation can be characterized purely algebraically. They are precisely the suprema which are context-free, in the sense of being preserved in every \(W\) homomorphism out of of \(G\). We show that closure under such suprema characterizes the \(W\)-kernels among the convex \(\ell\)-subgroups. And we prove that all existing (countable) joins in \(R_L\) are pointwise iff \(L\) is boolean (a \(P\)-frame).

This leads up to the appropriate analog of the Nakano-Stone Theorem: a (completely regular) locale \(L\) has the feature that \(R_L\) is conditionally pointwise complete (\(\sigma\)-complete), i.e., every bounded (countable) family from \(R_L\) has a pointwise supremum in \(R_L\), iff \(L\) is boolean (a \(P\)-locale).

It is perhaps surprising that pointwise suprema can be characterized purely algebraically, without reference to a representation. They are the context-free suprema, in the sense that the pointwise suprema are precisely those which are preserved by all morphisms out of \(G\). We adopt the latter attribute as the final, representation-free definition of pointwise suprema.

Thus emboldened, we adopt a maximally broad definition of unconditional pointwise completeness (\(\sigma\)-completeness): a divisible \(W\)-object \(G\) is pointwise complete (\(\sigma\)-complete) if it contains a pointwise supremum for every subset which has a supremum in any extension. We show that the pointwise complete (\(\sigma\)-complete) \(W\)-objects are those of the form \(R_L\) for \(L\) a boolean locale (\(P\)-locale). Finally, we show that a \(W\)-object \(G\) is pointwise \(\sigma\)-complete iff it is epicomplete.

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1. Introduction

When considering the suprema of real-valued functions, it is often relevant to know whether this supremum coincides with the function obtained by taking the supremum of the real values at each point. Here we propose a natural generalization of this notion to the pointfree setting. We first define pointwise suprema in $RL$, where the classical definition can be naturally articulated. We then show that this concept is actually independent of the representation of a particular $W$-object as a subobject of $RL$. For it is precisely the pointwise suprema in $RL$ which are preserved by every $W$-morphism out of $RL$. We take advantage of this unexpected information by adopting the latter attribute as the final, purely algebraic definition of pointwise supremum: an element $g \in G$ is the pointwise join of a subset $K \subseteq G^+$ iff $\theta(g) = \bigvee \theta[K]$ for all $W$-morphisms $\theta$ out of $G$.

The notion of pointfree pointwise suprema has several useful applications. For example, a convex $\ell$-subgroup $K$ of a $W$-object $G$ is a $W$-kernel iff it is pointwise closed, i.e., iff $K$ contains any pointwise join of a subset of $K$ which exists in $G$. And all existing (countable) suprema in $RL$ are pointwise iff $L$ is boolean (a $P$-frame).

This leads directly to Nakano-Stone type theorems. One of our main results is Theorem 5.2.1: $RL$ is conditionally pointwise complete ($\sigma$-complete) iff $L$ is boolean (a $P$-frame).

Unconditional pointwise completeness requires that certain unbounded subsets of a given $W$-object $G$ have pointwise suprema in $G$, but, of course, not all subsets can have suprema in $G$. The most permissive criterion for a subset to have a pointwise supremum in $G$ is that the subset have a supremum in some extension of $G$. We adopt this criterion as our definition of unconditional pointwise completeness in Section 7 and then show that the $W$-objects which enjoy this attribute are precisely those of the form $RL$ for $L$ a Boolean frame, and those which enjoy the corresponding unconditional $\sigma$-completeness are those of the form $RL$ for $L$ a $P$-frame. Finally, we show that the pointwise complete ($\sigma$-complete) objects form a full bireflective subcategory of $W$.

The paper is organized as follows. After a preliminary Section 2, we briefly outline Madden’s pointfree representation for $W$ in Section 3. We define pointwise suprema in Section 4 first in $RL$ in Subsection 4.1 and then in $W$ in Subsection 4.2. In Section 5 we give several different applications of the notion of pointwise suprema. We show that the $W$-kernels of a
particular $W$-object are precisely the pointwise closed convex $\ell$-subgroups. We show that all existing (countable) suprema in $RL$ are pointwise iff $L$ is boolean (a P-frame). And we show that any element of a $W$-object is the pointwise join of its truncates, we characterize the sequences that can be realized as the truncates of an element of an extension, and we show that every such sequence has a pointwise supremum in $RMG$. (Notation to be introduced subsequently.)

Section 6 is devoted to conditional pointwise completeness; the main result here is the pointfree Nakano-Stone Theorem for conditional pointwise completeness, Theorem 6.2.1. This result makes heavy use of the pointfree generalization of the classical theorem, a beautiful result of Banaschewski and Hong [11] which appears here in embellished form as Theorem 6.1.2.

Section 7 takes up unconditional pointwise completeness. A review of the well known facts concerning essential extensions constitutes Subsection 7.1, and a review of the less well known facts concerning cuts occupies Subsection 7.2. The section culminates in Subsection 7.3 in which we summarize our findings as they pertain to unconditional pointwise completeness in Theorem 7.3.2.

2. PRELIMINARIES

Our notation and terminology is conventional for the most part, save only for our notation for downsets. A subset $K$ of a poset $G$ is a downset if $g \leq k \in K$ implies $g \in K$. We write

$$\downarrow K \downarrow_G \equiv \{ g \in G : g \leq k \text{ for some } k \in K \}$$

for the downset in $G$ generated by a subset $K \subseteq G$. We drop the subscript whenever it is unambiguous to do so. Upsets are defined and denoted dually.

For a $W$-object $G$, we denote by $R^+(G)$ the set of those positive real numbers for which the corresponding constant function is present in $G$. Thus to say that $\bigwedge R^+(G) = 0$ is to say that $G$ contains arbitrarily small positive multiples of 1. This is a weakening of the condition of being divisible which plays a prominent role in our results.

Good general references are [2] and [14] for $\ell$-groups, [15] for $C(X)$, [17] for an introduction to $W$, [19] and [5] for the pointfree, or Madden representation for $W$, [18] and [20] for general frame theory, and the many papers of Bernhard Banaschewski, the tireless fount of knowledge of pointfree topology.

In spite of our use of the localic terminology in the abstract and introduction, we prefer the algebraic language of frames and frame morphisms. Henceforth, $RL$ stands for the $W$-object of frame maps $g : OR \to L$, where $OR$ is the frame of open subsets of the real numbers $\mathbb{R}$ and $L$ is a frame, assumed completely regular unless otherwise explicitly stipulated.

3. A BRIEF SYNOPSIS OF THE MADDEN REPRESENTATION

We mention here some of the technical results, familiarity with which will be assumed in the sequel. The reader may skip this section upon a first reading, returning to it as necessary.

3.1. Calculation in $RL$. The arithmetic operations on $\mathbb{R}$ beget corresponding operations on $RL$ as follows. We write $\vec{f}$ for $(f_1, f_2, \ldots, f_n) \in (RL)^n$, $\vec{U}$ for $(U_1, U_2, \ldots, U_n) \in (OR)^n$. For a continuous function $w : \mathbb{R}^n \to \mathbb{R}$ we write $w(\vec{U}) \subseteq U$ to mean $U_1 \times U_2 \times \cdots \times U_n \subseteq w^{-1}(U)$. 
3.1.1. Theorem ([7] 3.1.1). The canonical lifting \( w' : (\mathcal{RL})^n \to \mathcal{RL} \) of a continuous function \( w : \mathbb{R}^n \to \mathbb{R} \) is given by the formula
\[
\overline{w(f)}(U) = \bigvee_{w(U) \leq U} \bigwedge_{1 \leq i \leq n} f_i(U_i), \quad f_i \in \mathcal{RL}, \quad U \in \mathcal{OR}.
\]
The formula also applies to constant functions; the frame map lifted from the constant function \( x \mapsto r \) is given by
\[
r(U) = \begin{cases} 
\top & \text{if } r \in U \\
\bot & \text{if } r \notin U
\end{cases}.
\]

Theorem 3.1.1 provides a ready proof of a special case of Weinberg’s Theorem ([21]). A term is an expression built up from variables and constants using the operations \(+, -, \lor, \land\). An identity is an equation with terms on either side. Weinberg’s Theorem asserts that an equation holds in \( \mathbb{R} \) iff it holds in every abelian \( \ell \)-group.

3.1.2. Corollary. Any identity which holds in \( \mathbb{R} \) also holds in any \( \mathcal{RL} \).

Proof. The terms on either side of the identity determine two functions \( w_i : \mathbb{R}^n \to \mathbb{R} \), and these functions coincide because the identity holds in \( \mathbb{R} \). Therefore the liftings \( w'_i \) of these functions to \( \mathcal{RL} \) coincide by Theorem 3.1.1. \( \square \)

3.2. A few useful formulas. We record here a small number of formulas which will be especially useful in what follows. They may be derived using from Theorem 3.1.1 or even Corollary 3.1.2. Details can be found in the literature by following the references. Lemma 3.2.1 implies that a frame map \( f : \mathcal{OR} \to \mathcal{L} \) is completely determined by its values on the right rays. For \( f \) is clearly determined by its values on the base for \( \mathcal{OR} \) consisting of the open intervals \((r, s)\), \( r < s \), and \( f(r, s) = f(-\infty, s) \land f(r, \infty) \) and the left ray \( f(-\infty, s) \) can be expressed in terms of the right rays using the pseudocomplementation operator in the frame:
\[
a^+ = \bigvee_{a^\land b = \bot} b.
\]

3.2.1. Lemma ([5] 3.1.1). For any \( f, g \in \mathcal{RL} \) and \( r \in \mathbb{R} \),
\[
(1) \quad f(-\infty, r) = \bigvee_{s < r} f(s, \infty)^*,
(2) \quad f \leq g \iff f(r, \infty) \leq g(r, \infty) \text{ for all } r \in \mathbb{R} \iff f(-\infty, r) \geq g(-\infty, r) \text{ for all } r \in \mathbb{R}.
\]

Lemma 3.2.2 gives necessary and sufficient conditions for a function on right rays to be extended to a frame map.

3.2.2. Lemma ([5] 3.1.2). A function \( f : \{(r, \infty) : r \in \mathbb{R}\} \to \mathcal{L} \) can be extended to an element of \( \mathcal{RL} \) iff it satisfies the following conditions for all \( r, s \in \mathbb{R} \). The extension is unique when it exists.
\[
(1) \quad f(s, \infty) < f(r, \infty) \text{ whenever } r < s.
(2) \quad f(r, \infty) = \bigvee_{s > r} f(s, \infty).
(3) \quad f(r, \infty) = f(r, \infty)^* = \top.
\]

We provide a proof of Corollary 3.2.3 in order to illustrate the use of Theorem 3.1.11 in calculations. This sort of reasoning will get heavy use in what follows. For \( f \in \mathcal{RL} \), the cozero element of \( f \) is
\[
\text{coz } f \equiv f(\mathbb{R} \setminus \{0\}) = |f|(0, \infty).
\]
3.2.3. Corollary. For \( f, g \in \mathcal{R} \) and \( c, r \in \mathbb{R} \),

1. \( (f - c)(r, \infty) = f(c + r, \infty) \).
2. \( \text{coz } f^+ = (f \setminus 0)(\mathbb{R} \setminus \{0\}) = f(0, \infty) \).
3. \( \text{coz } (f - c)^+ = (f - c)(0, \infty) = f(c, \infty) \).
4. \( (f \wedge g)(r, \infty) = f(r, \infty) \wedge g(r, \infty) \).
5. For \( f, g \geq 0 \), \( \bigvee_{n} \text{coz } (nf - g)^+ = \text{coz } f \).

Proof. To prove (1), consider \( \mathcal{U} \in \mathcal{O} \). Then by Theorem 3.1.1 we have

\[
(f - c)(r, \infty) = \bigvee_{U_1 - \mathcal{U} \subseteq (r, \infty)} (f(U_1) \wedge c(U_2)).
\]

But if \( \mathcal{U}_1 - \mathcal{U}_2 \subseteq (r, \infty) \) then \( \mathcal{U}_1 \) is bounded below and \( \mathcal{U}_2 \) is bounded above, say \( \mathcal{U}_1 \subseteq (t, \infty) \) and \( \mathcal{U}_2 \subseteq (-\infty, t - r) \) for some \( t \in \mathbb{R} \). And if, in addition, and \( c(U_2) > \bot \) then \( c \in \mathcal{U}_2 \), hence \( t > c + r \). That is to say that

\[
(f - c)(r, \infty) = \bigvee_{t > c + r} f(t, \infty) = f(c + r, \infty).
\]

The proof of (2) is similar to the proof of (1), and (3) follows from (1) and (2). To prove (4), again consider \( \mathcal{U} \in \mathcal{O} \).

\[
(f \wedge g)(r, \infty) = \bigvee_{U_1 \wedge \mathcal{U}_2 \subseteq (r, \infty)} (f(U_1) \wedge g(U_2)).
\]

But \( \mathcal{U}_1 \wedge \mathcal{U}_2 \subseteq (r, \infty) \) iff \( \mathcal{U}_1 \subseteq (r, \infty) \) and \( \mathcal{U}_2 \subseteq (r, \infty) \). Hence \( (f \wedge g)(r, \infty) = f(r, \infty) \wedge g(r, \infty) \).

To verify (5), note that

\[
\text{coz } (nf - g)^+ = (nf - g)(0, \infty) = \bigvee_{n \mathcal{U}_1 - \mathcal{U}_2 \subseteq (0, \infty)} (f(U_1) \wedge g(U_2)).
\]

But if \( n \mathcal{U}_1 - \mathcal{U}_2 \subseteq (0, \infty) \) then \( \mathcal{U}_1 \subseteq (r, \infty) \) and \( \mathcal{U}_2 \subseteq (-\infty, nr) \) for some \( r \in \mathbb{R} \), so that

\[
\bigvee_{n} \text{coz } (nf - g)^+ = \bigvee_{n} f(r, \infty) \wedge g(-\infty, nr) = \bigvee_{n} f(r, \infty) \wedge g(-\infty, nr) = f(r, \infty) = \text{coz } f.
\]

3.3. The frame of \( W \)-kernels of \( A \). Most of the calculation takes place in the frame of \( W \)-kernels of \( G \). The basic facts concerning this frame are well known; we briefly review them here to fix notation.

3.3.1. Lemma. Let \( K \) be a convex \( \ell \)-subgroup of \( G \).

1. \( G/K \) is archimedean iff

\[
(\forall n \in \mathbb{N} \ (\text{coz } (nf - g)^+ \in K) \implies f \in K), \quad f, g \in G^+.
\]

2. \( K \) is a \( W \)-kernel if, in addition,

\[
g \wedge 1 \in K \implies g \in K, \quad g \in G^+.
\]

Proof. (1) We have

\[
\text{coz } (nf - g)^+ \in K \iff K + (nf - g)^+ = K \iff K + nf \setminus b = K + g \iff K + nf \subseteq K + g.
\]

This makes it clear that the condition displayed in (1) is equivalent to the archimedean property of the quotient \( G/K \).
(2) This is evidently a reformulation of the requirement that \( K + u \) should function as a weak unit of the quotient, i.e., that \((K + g) \land (K + u) = 0 \) imply \( K + g = 0 \). □

3.3.2. Corollary. Suppose \( G \) is bounded. Then a convex \( \ell \)-subgroup \( K \) is a proper \( W \)-kernel iff

\[
\begin{align*}
(1) & \forall \, n \in \mathbb{N} \, ((n \ell - 1)^+ \in K) \implies f \in K, \ f \in G^+, \text{ and} \\
(2) & 1 \notin K.
\end{align*}
\]

In particular, \( [g] = \{ h : \forall \, n \exists \, m \, (n|h| - 1)^+ \leq mg \}, \, g \in G^+ \).

Proof. It is straightforward to show that in condition (1) of Lemma 3.3.1 the element \( g \) may be chosen to be 1 if \( G \) is bounded. What we must also demonstrate is that condition (2) above implies condition (2) of Lemma 3.3.1. Given \( g \in G^+ \), find a positive integer \( n \) such that \( g \leq n \). Then \( g \land 1 \in K \) implies \( n g \land n \in K \) because \( K \) is a group, hence \( g \land n \in K \) because \( K \) is convex, with the result that \( g \in K \). □

Since \( W \) is closed under products, the intersection of an arbitrary family of \( W \)-kernels is itself a \( W \)-kernel. We denote the \( W \)-kernel generated by a subset \( S \subseteq G \) by

\[
[S] \equiv \bigcap \{ K : K \text{ is a } W\text{-kernel and } S \subseteq K \}.
\]

3.3.3. Definition. The frame of \( W \)-kernels of \( G \) is called the Madden frame of \( G \); we denote it by \( MG \).

3.3.4. Lemma. \( MG \) forms a regular Lindelöf frame under the inclusion order. Its operations are

\[
K_1 \land K_2 = K_1 \cap K_2 \text{ and } \bigvee_i K_i = [K_i : i \in I] = \bigcup_i K_i.
\]

Proof. [5, 3.2.2, 3.25]. □

3.4. The Madden representation for \( W \). Let \( G \) be a \( W \)-object with \( L \equiv MG \) its frame of \( W \)-kernels. For each \( g \in G \) and \( r \in \mathbb{R} \), define

\[
\hat{g}(r, \infty) \equiv ([g - r]^+).
\]

Thus defined, \( \hat{g} \) satisfies the requirements of Lemma 3.2.2 and thus extends to a unique frame map \( \hat{0} \mathbb{R} \to L \), which we also denote \( \hat{g} \). We write \( \hat{G} \) for \( \{ \hat{g} : g \in G \} \), and \( \mu_G : G \to \hat{G} \) for the mapping \( g \mapsto \hat{g} \).

We say that a \( W \)-morphism \( \vartheta : H \to \mathbb{R}M \) is cozero dense if

\[
a = \bigvee_{h \in H} \text{coz } \vartheta(h), \quad a \in M.
\]

Note that it is enough for this condition to hold for each \( a \in \text{coz } M \) because \( M \) is assumed to be completely regular.

3.4.1. Theorem ([19]). Let \( G, L, \hat{G}, \) and \( \mu_G \) have the meaning above.

1. Then \( \mu_G \) is a cozero dense \( W \)-injection, and its range restriction \( G \to \hat{G} \) is a \( W \)-isomorphism.
2. \( L, \hat{G}, \) and \( \mu_G \) are unique up to isomorphism with respect to their properties in (1).
3. For any frame \( M \) and \( W \)-morphism \( \vartheta \) there is a unique frame map \( k \) making the diagram commute.
4. \( k \) is surjective iff \( \vartheta \) is cozero dense, and \( k \) is one-one iff for all \( K \subseteq G^+ \), \( \bigvee_K \text{coz } \vartheta(g) = \top \) in \( M \) implies \( \bigvee_K \text{coz } \hat{g} = \top \) in \( L \).
Proof. A detailed proof may be found in [5]; we comment only on part (4). For any \( W \)-kernel \( K \subseteq G \),
\[
k(K) = k \left( \bigvee_{k^+} [g] \right) = k \left( \bigvee_{k^+} \coz \hat{g} \right) = k \left( \bigvee_{k^+} \hat{g} (0, \infty) \right) = \bigvee_{k^+} k \circ \hat{g} (0, \infty) = \bigvee_{k^+} \theta(g)(0, \infty)
\]
\[
= \bigvee_{k^+} \coz \theta(g).
\]
This makes the surjectivity condition clear; the injectivity condition follows from the fact that a frame morphism between regular frames is one-one iff it is codense, i.e., iff the only element taken to the top of the codomain is the top element of the domain. \( \square \)

4. Pointwise suprema defined

In dealing with continuous real-valued functions on a Tychonoff space \( X \), it is often important to know whether a given function \( f \) is the supremum of a given subset \( K \subseteq \mathcal{C}X \), and, if so, whether this supremum is pointwise, i.e., whether \( \bigvee_K k(x) = f(x) \) for all \( x \in X \). In terms of the frame \( \mathcal{O}X \) of open sets of \( X \), \( f \) is the pointwise supremum of \( K \) iff \( \bigcup k^{-1}(r, \infty) = f(r, \infty) \) for all \( r \in \mathbb{R} \). It is the latter formulation which generalizes directly to the pointfree setting.

4.1. Pointwise suprema in \( \mathcal{RL} \).

4.1.1. Definition. Let \( L \) be a frame, and let \( K \) be a subset and \( f \) an element of \( \mathcal{RL} \). We say that \( f \) is the pointwise supremum (infimum) of \( K \), and write \( f = \bigvee^* K \) (\( f = \bigwedge^* K \)), provided that \( f(r, \infty) = \bigvee_K k(r, \infty) \) (\( f(-\infty, r) = \bigvee_K k(-\infty, r) \)) holds in \( L \) for all \( r \in \mathbb{R} \).

4.1.2. Remarks. A few remarks about this definition are in order.
(1) Observe that the frame definition coincides with the spatial definition in case \( L \) is the topology of a Tychonoff space.
(2) Recall that by Lemma 3.2.1 a frame map is completely determined by its values on the right or left rays alone. This makes the appearance of only the rays in this definition less mysterious.
(3) Recall that by Lemma 3.2.1 an element \( g \in \mathcal{RL} \) lies above (below) each \( k \in K \) iff \( k(r, \infty) \leq g(r, \infty) \) (\( g(-\infty, r) \leq k(-\infty, r) \)) for all \( r \in \mathbb{R} \) and all \( k \in K \).
(4) It follows from the preceding remarks that \( f = \bigvee K \) whenever \( f = \bigvee^* K \), and dually.
(5) It follows from the preceding remarks that if \( f = \bigvee^* K = g \) then \( f = g \).

We list some of the nice properties of pointwise suprema and infima.

4.1.3. Proposition. Let \( F \) and \( K \) be subsets and let \( f_0 \) and \( k_0 \) be elements of \( \mathcal{RL} \).
(1) \( f_0 = \bigvee^* F \) iff \( f_0 = \bigwedge^* (-F) \equiv \bigwedge^* (-f) \), and dually.
(2) \( f_0 = \bigvee^* (f_0) = \bigwedge^* (f_0) \).
(3) If \( f_0 = \bigvee^* F \) and \( k_0 = \bigvee^* K \) then \( f_0 \ominus k_0 = \bigvee^* (f \ominus k) \), where \( \ominus \) stands for one of the t-group operations \( +, \vee, \) or \( \wedge \).
(4) If \( f_0 = \bigvee^* F \) and \( 0 \leq r \in \mathbb{R} \) then \( rf_0 = \bigvee^* rf \).
4.2.1. \( R \)

\( 4.2. \)

using Theorem 3.1.1. \( \square \)

(1) follows from the fact that \( (-f) (-\infty, r) = f (-r, \infty) \) for any \( f \in \mathcal{R} \) and \( r \in \mathbb{R} \), as can be readily checked with the aid of Theorem 3.1.1. (2) is trivial. To prove part (3) for the \( + \) operation, first observe that for \( r \in \mathbb{R} \),

\[
\bigvee_{F,K} (f + k) (r, \infty) = \bigvee_{F,K U_1+U_2 \subseteq (r, \infty)} (f(U_1) \land k(U_2)).
\]

But if \( U_1 + U_2 \subseteq (r, \infty) \) then both \( U_1 \)'s are bounded below, say \( U_1 \subseteq (s, \infty) \) and \( U_2 \subseteq (r-s, \infty) \) for some \( s \in \mathbb{R} \). Therefore this join works out to

\[
\bigvee_{s} (f(s, \infty) \land k(r-s, \infty)) = \bigvee_{s} (f(s, \infty) \land k_\infty(r-s, \infty)) = (f_\infty(s, \infty) \land k_\infty(r-s, \infty)) = (f_\infty + k_\infty) (r, \infty).
\]

The proofs of (3) for the join and meet operations are similar. Finally, to verify (4) simply note that if \( r > 0 \) then \( (r f) (s, \infty) = f(s/r, \infty) \) for \( f \in \mathcal{R} \) and \( s \in \mathbb{R} \), as may be easily seen using Theorem 3.1.1. \( \square \)

4.2. Pointwise suprema in \( \mathcal{W} \). Having formulated the notion of pointwise supremum in \( \mathcal{R} \), let us now generalize it to abstract \( \mathcal{W} \)-objects.

4.2.1. Definition (First definition of pointwise supremum in \( \mathcal{W} \)). For \( F \subseteq G \in \mathcal{W} \) and \( f_0 \in G \), we shall say that \( f_0 \) is the pointwise supremum (infimum) of \( F \), and write \( f_0 = \bigvee^* F \) (\( f_0 = \bigwedge^* F \)), if the corresponding statement holds in \( \mathcal{G} \), i.e., if \( f_0 = \bigvee^* \hat{f} (\hat{f}_0 = \bigwedge^* \hat{f}) \).

Pointwise suprema can be characterized concretely by use of the details of the Madden representation (see Subsection 3.4).

4.2.2. Proposition. Let \( F \) be a subset and \( f_0 \) an element of a \( \mathcal{W} \)-object \( G \). Then

\[
\begin{align*}
\hat{f}_0 &= f_0 \iff \forall r \in \mathbb{R} \quad \left[ (f-r) + : f \in F \right] = \left[ (f_0-r) + \right], \\
\hat{f} &= (f-r) + \iff \forall r \in \mathbb{R} \quad \left[ (r-f) + : f \in F \right] = \left[ (r-f_0) + \right].
\end{align*}
\]

\( \hat{f} \) is the pointwise supremum and \( \hat{f}(-\infty, r) = \left[ (r-f) + \right] \).

\( \square \)

\( \mathcal{W} \)-morphisms preserve pointwise suprema.

4.2.3. Proposition. If \( \theta : G \rightarrow H \) is a \( \mathcal{W} \)-morphism and if \( f_0 = \bigvee^* F \) in \( G \) then \( \theta (f_0) = \bigvee^* \theta (f) \) in \( H \).

\( \square \)

It is a surprising fact that the converse of Proposition 4.2.3 holds as well. In general, the supremum of a subset \( F \) of a \( \mathcal{W} \)-object \( G \) depends on the context. If, for instance, \( G \) is a subobject of \( H \), it may well happen that \( f_0 = \bigvee F \) for some \( f_0 \in G \) but \( f_0 \neq \bigvee F \) in \( H \). The point of Proposition 4.2.4 is that it is precisely the pointwise suprema which are context free.
4.2.4. **Proposition.** Let $F$ be a subset and $f_0$ an element in some $W$-object $G$. Then $f_0 = \bigvee^* F$ iff $\theta(f_0) = \bigvee_F \theta(f)$ for every $W$-morphism $\theta$ out of $G$, and dually.

**Proof.** Proposition 4.2.3 is the forward implication of this equivalence. So suppose $f_0$ is not the pointwise supremum of $F$, let $L$ be the Madden frame of $G$, and identify $G$ with its Madden representation $\hat{G} \leq RL$. We may assume that $f_0 = 0$, since otherwise we may replace $F$ by $F - f_0 = \{ f - f_0 : f \in F \}$ by Proposition 4.1.3. We must find a $W$-morphism $\theta : G \to H$ such that $\bigvee_F \theta(f) \neq 0$.

Let $i : L \to M$ be a frame embedding of $L$ into a boolean frame $M$. (Such an embedding exists; see [18, II, 2.6].) Since $\bigvee^* F \neq 0$ there exists some $r \in \mathbb{R}$ such that

$$a \equiv \bigvee_F f(r, \infty) < 0(r, \infty) = \begin{cases} \top & \text{if } r \geq 0 \\ \perp & \text{if } r < 0 \end{cases}.$$

Note that $0(r, \infty)$ must be $\top$, hence $r < 0$. Now $i(a)$ has complement $b$ in $M$; note that $b > \perp$ because $i(a) < \perp$ since $a < \perp$ and $i$ is one-one.

Let $k : M \to \downarrow b$ designate the open quotient frame map $c \mapsto c \wedge b$, $c \in M$, let $H \equiv RL(\downarrow b)$, and let $\psi \equiv RL(k \circ i) : RL \to H$. We claim that the desired map $\theta$ is the restriction of $\psi$ to $\hat{G} \cong G$. For if $f \in F$ then

$$\psi(f)(r, \infty) = k \circ i \circ f(r, \infty) = i \circ f(r, \infty) \wedge b = \perp$$

since $i \circ f(r, \infty) \leq i(a)$ and $i(a) \wedge b = \perp$. It follows that for $s \in \mathbb{R}$,

$$\psi(f)(s, \infty) \leq (r/2)(s, \infty) = \begin{cases} \perp & \text{if } s \geq r/2 \\ \top & \text{if } s < r/2 \end{cases},$$

which implies by Lemma 3.2.1(2) that $\psi(f) \leq r/2 < 0$ for all $f \in F$, meaning that $\bigvee_F \psi(f) \neq 0$. This completes the proof.

Some caution is required when dealing with pointwise suprema. If $F \subseteq G$ and $f_0 \in G$ are such that $f_0 = \bigvee^* F$ in some $W$-extension $H \geq G$ then $f_0 = \bigvee F$ in $G$, of course, but the join may not be pointwise in $G$.

4.2.5. **Example.** Let $X$ be $\omega + 1$, the one-point compactification of the discrete space of finite ordinals. Let $G$ be $\mathbb{C}X$ and let

$$H \equiv \{ g + rh_0 : g \in G, \ r \in \mathbb{R} \},$$

where $h_0 \equiv (n \mapsto n)$ and $h_0(\omega) = \infty$. $H$ is a $W$-object in $DX$. Let $F$ be the family of functions

$$f_n(k) \equiv \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}.$$

Then it is not hard to check that $\bigvee^* F = 1$ in $H$ and $\bigvee F = 1$ in $G$ but the latter join is not pointwise.

For emphasis, we recast the definition of pointwise supremum in an arbitrary $W$-object.

4.2.6. **Definition** (Second definition of pointwise supremum in $W$). For $F \subseteq G \in W$ and $f_0 \in G$, we shall say that $f_0$ is the pointwise supremum (infimum) of $F$, and write $f_0 = \bigvee^* F$ ($f_0 = \bigwedge^* F$, if $\bigwedge \theta(F) = \theta(f_0)$ ($\bigwedge \theta(F) = \theta(f_0)$) for all $W$-homomorphisms $\theta : G \to H$. 

5. **Pointwise suprema applied**

In this section we aim to show that pointwise suprema are useful for characterizing important attributes of a \(W\)-object and its Madden frame. We begin by using them to characterize those \(W\)-objects in which every (countable) supremum is pointwise. Throughout this section \(G\) will represent a \(W\)-object with Madden frame \(L\).

5.1. **When all existing (countable) suprema are pointwise.** Pointwise suprema are useful for detecting whether the Madden frame of a given \(W\)-object is boolean or a \(P\)-frame. We shall require this information in Section 7.

5.1.1. **Theorem.** Suppose that \(\bigwedge R^+(G) = 0\). Then all existing (countable) suprema in \(G\) are pointwise iff \(L\) is boolean (a \(P\)-frame).

**Proof.** We prove this theorem in the boolean case; the same proof, mutatis mutandis, works in the \(P\)-frame case. Assume \(L\) is boolean, suppose \(f\) is an element and \(K\) is a subset of \(G\) such that \(f = \bigvee K\) in \(G\), and assume for the sake of argument that \(f(r, \infty) > \bigvee_{K} k(r, \infty) = b\) for some real number \(r\). Since \(\bigvee_{s \geq r} f(s, \infty) = f(r, \infty)\), there is some \(s > r\) for which \(f(s, \infty) \not< b\). Because \(L\) is boolean \(a = f(s, \infty) \wedge b^* > \bot\); define the ‘characteristic function’

\[
\chi(t, \infty) = \begin{cases} 
\top & \text{if } t < 0, \\
a & \text{if } 0 \leq t < s - r, \\
\bot & \text{if } t \geq 1,
\end{cases} \quad t \in \mathbb{R},
\]

and check that \(\chi\) satisfies the hypotheses of Lemma 3.2.2 and so extends to a unique member of \(\mathcal{R}\), and that, moreover, \(\chi > 0\).

We claim that \(f - k \geq \chi\) for all \(k \in K\). To verify the claim first note that by Theorem 3.1.1 \((f - k)(t, \infty) = \bigvee_{U_1} (f(U_1) \wedge k(U_2))\), where the join ranges over open subsets \(U_1 \subseteq \mathbb{R}\) such that \(U_1 - U_2 \subseteq (0, \infty)\). This condition implies that \(U_1\) is bounded below and \(U_2\) is bounded above, say \(U_1 \subseteq (u, \infty)\) and \(U_2 \subseteq (-\infty, u - t)\). Therefore

\[
(f - k)(t, \infty) = \bigvee_{u}(f(u, \infty) \wedge k(-\infty, u - t)),
\]

If \(t < 0\) then, since \(f \geq k\) implies \((\neg\infty, u - t) \leq (\neg\infty, u - t)\), we get for any choice of \(u \in \mathbb{R}\) that

\[
(f - k)(t, \infty) \geq f(u, \infty) \wedge k(-\infty, u - t) \geq f(u, \infty) \wedge f(\neg\infty, u - t) = \top.
\]

If \(0 \leq t < s - r\) then \(s - t > r\), hence \(k(-\infty, s - t) \geq k(r, \infty)^*\) since

\[
k(-\infty, s - t) \vee k(r, \infty) = k((-\infty, s - t) \cup (r, \infty)) = \top.
\]

This is relevant because \(b^* = (\bigvee_{K} k(r, \infty))^* = \bigwedge_{K} k(r, \infty)^*\) as a result of the fact that \(L\) is a complete boolean algebra, so that

\[
(f - k)(t, \infty) \geq f(s, \infty) \wedge k(-\infty, s - t) \geq f(s, \infty) \wedge k(r, \infty)^* \geq f(s, \infty) \wedge \bigwedge_{K} k(r, \infty)^* = f(s, \infty) \wedge b^* = a = \chi(t, \infty).
\]

This proves the claim, which implies that \(f > f - \chi \geq K\). Since \(\mu_G : G \to \mathcal{R}\) is cozero dense, there exists \(0 < g \in G\) such that \(\text{coz } g \leq a\), and, by meeting \(g\) with the appropriate constant function \(r \in R^+(G)\) we may assume that \(g \leq \chi\). In sum, we have \(f > f - g \geq K\), a violation of the assumption that \(f = \bigvee K\) in \(G\). Our only recourse is to conclude that \(f(r, \infty) = \bigvee_{K} k(r, \infty)\) for all \(r \in \mathbb{R}\), i.e., \(f = \bigvee^* K\).
Now suppose that all existing suprema in $G$ are pointwise; we aim to show that an arbitrary element $a \in L \equiv RL$ is complemented. For that purpose define subsets

$$U \equiv \{ g \in G : \text{coz } g \leq a \text{ and } 0 \leq g \leq 1 \}, \quad \text{and} \quad V \equiv \{ g \in G : \text{coz } g \leq a^* \text{ and } 0 \leq g \leq 1 \}.$$

By suitably augmenting $U$ we may assume that $0 \leq k \leq g \in U$ implies $k \in U$, and that $g \in U$ implies $mg \land 1 \in U$ for all $n$, and similarly for $V$.

We claim that $\bigvee (U \cup V) = 1$. If not then there exists some $k \in G$ such that $g \leq k < 1$ for all $g \in U \cup V$. This means that $1 - k > 0$, hence $b = \text{coz}(1 - k) > \top$. Since $\bigvee_{U \cup V} \text{coz } g$ is a dense element of $L$, $b$ meets either $\bigvee_U \text{coz } g$ or $\bigvee_V \text{coz } g$ nontrivially, say $0 < g \in U$ is such that $\text{coz } g \leq b$. Since $\text{coz } g = g(0, \infty) = \bigvee_m g\left(\frac{1}{m}, \infty\right)$, there exists some positive integer $m$ for which $\perp < g\left(\frac{1}{m}, \infty\right) = \text{coz}(g - \frac{1}{m}) = \text{coz}(mg - 1)$. But then

$$\text{coz}(mg \land 1 - k)^+ = \text{coz}(mg - k)^+ = \text{coz}(mg - k)^+ \land \text{coz}(1 - k) = \text{coz}(mg - k)^+ \land b = \text{coz}(mg - 1)^+ > \perp.$$ 

This is a contradiction, since $g \in U$ implies $mg \land 1 \in U$, hence $mg \land 1 \leq k$. A similar argument covers the case in which there exists some $0 < g \in V$ such that $\text{coz } g \leq a^*$, and the two cases together prove the claim.

Let $\bigvee_U \text{coz } h \equiv u$ and $\bigvee_V \text{coz } h \equiv v$. Then because $1 = \bigvee^*(U \cup V)$ we have

$$\top = 1(0, \infty) = \bigvee_{U \cup V} g(0, \infty) = \bigvee_{U \cup V} \text{coz } g = \bigvee_U \text{coz } g \cup \bigvee_V \text{coz } g = u \lor v.$$

Since $u \leq a$ and $v \leq a^*$ by construction, we see that $u$ and $v$ are complementary, and that $u = a$. \qed

5.2. **Truncate sequences.** The following fact plays an important role in our analysis of unconditional pointwise completeness in Section [7].

5.2.1. **Proposition.** For any $f \in G$, $\bigvee_n^*(f \land n) \equiv f$.

**Proof.** Identify $G$ with $\hat{G} \equiv RL$. For any $r \in R$ we have by Corollary [5.2.3](4) that

$$(f \land n)(r, \infty) = f(r, \infty) \land n(r, \infty) = \left\{ \begin{array}{ll} \perp & \text{if } r \geq n \\ f(r, \infty) & \text{if } r < n \end{array} \right..$$

Hence $\bigvee_n (f \land n)(r, \infty) = f(r, \infty)$ for all $r \in R$. \qed

Proposition 5.2.1 raises an important question: which sequences in $G$ are sequences of truncates of a member of $RL$, so called truncate sequences?

5.2.2. **Proposition.** Let $\{g_n\} \subseteq G^+$ be the sequence of truncates of $h \in RL^+$, i.e., $\hat{g}_n = h \land n$ for all $n$. Then

1. $g_{n+1} \land n = g_n$ in $G$, and
2. $\bigvee_n \hat{g}_n(-\infty, n) = \top$ in $L$.

Conversely, any sequence in $G$ having these two properties is the sequence of truncates of some $h \in RL$.

**Proof.** If $g_n = h \land n$ for all $n$ then (1) obviously holds, and

$$\bigvee_n g_n(-\infty, n) = \bigvee_n (h \land n)(-\infty, n) = \bigvee_n h(-\infty, n) = h \left(\bigvee_n (-\infty, n)\right) = h(-\infty, \infty) = \top.$$
Suppose now that \( \{g_n\} \) is a sequence in \( RL^+ \) satisfying (1) and (2). Put \( h(-\infty, r) = g_n(-\infty, r) \) for any \( n > r \). This definition is independent of the choice of \( n \) by (1). We must show that \( h \) satisfies the properties in the (up-down dual of) Lemma 3.2.2. It is clear that \( h \) satisfies the first of these properties, namely that \( h(-\infty, s) \prec h(-\infty, r) \) whenever \( r < s \), because it reduces to the same property of \( g_n \) for sufficiently large \( n \), and \( h \) satisfies the second property for similar reasons. Since \( \bigvee_r h(-\infty, r) = \bigvee_n h(-\infty, n) = \bigvee_n g_n(-\infty, n) \), \( h \) also satisfies half of the third property. But \( h \) also satisfies the other half because \( \bigvee_r h(-\infty, r)^* = \bigvee_r g_1(-\infty, r)^* = \top. \)

**5.2.3. Definition** (Truncate sequence). We shall refer to a sequence \( \{g_n\} \subseteq G \) satisfying Proposition 5.2.2 as a truncate sequence.

**5.2.4. Corollary.** Every truncate sequence in \( G \) has a pointwise join in \( RL \).

In Section 10 of [16], the second author conducted an analysis of a construct which is closely related to truncate sequences, but stronger. His ‘expanding sequences’ have the first property of truncate sequences but satisfy \( \bigcap_n (u_{n+1} - u_n)_{\perp \perp} = 0 \) instead of the second property. The possession of a supremum for every such sequence turns out to be equivalent to the property of being *-maximum, or *-max for short. A \( W \)-object is *-max if it contains a copy of every other \( W \)-object with the same bounded part. This interesting attribute is not the same as requiring the truncate sequences to have joins, for it implies, inter alia, that the classical Yosida space of \( G \) be a quasi-F space. As is evident from Corollary 5.2.4, no such restriction applies to the \( W \)-objects in which the truncate sequences have joins.

We confess ignorance of the many questions that arise naturally here, postponing an investigation for the time being. But surely the first question is unavoidable, as it is motivated by the characterization of the divisible *-max \( W \)-objects as being precisely those in which every expanding sequence has a join ([16 Section 10]). See Theorem 7.2.7 below for further discussion of these topics.

**5.2.5. Question.** Which \( W \)-objects \( G \) have the feature that every truncate sequence in \( G \) has a pointwise join in \( G \)?

**5.3. Pointwise closure and \( W \)-kernels.** \( W \)-kernels are characterized by the property of being closed under pointwise joins. A convex \( \ell \)-subgroup \( K \leq G \) is said to be pointwise closed if \( K_0 \subseteq K^+ \) and \( \bigvee^* K_0 = g \) imply \( g \in K \).

**5.3.1. Proposition.** A convex \( \ell \)-subgroup \( K \) of a \( W \)-object \( G \) is a \( W \)-kernel iff it is pointwise closed.

**Proof.** Suppose \( K \) is a \( W \)-kernel with subset \( K_0 \) such that \( \bigvee^* K_0 = g \). According to Proposition 4.2.2, we are supposing that \( [(k-r)^+: k \in K_0] = [(g-r)^+] \) for all \( r \in R \). In particular, for \( r = 0 \) this says that \( [K_0] = [g] \), i.e., any \( W \)-kernel containing \( K_0 \) must also contain \( g \). But one such \( W \)-kernel is \( K \), hence \( g \in K \) and \( K \) is pointwise closed.

Now suppose that \( K \) is a pointwise closed convex \( \ell \)-subgroup of \( G \); we must show that \( K \) has properties (1) and (2) of Lemma 3.3.1. To check (2), suppose that \( g \wedge 1 \in K \) for some \( g \in G \). Then for each positive integer \( n \) we would have \( ng \wedge n \in K \) because \( K \) is a group, hence \( g \wedge n \in K \) because \( K \) is convex, with the result that \( g \in K \) by Proposition 5.2.1.

To check property (1) consider \( a, b \in G^+ \) such that \( (na - b)^+ \in K \) for all \( n \). We claim that \( \bigvee_n ((na - b)^+ \wedge a) = a. \) What we will actually prove is that

\[
\bigvee_n (((n-1)a - b) \lor (-a)) \wedge 0) = 0,
\]
the result of subtracting  a from the equation claimed. Since 0[(r, ∞) = T for r < 0 and otherwise, this amounts to showing that \( \bigvee_n ((n - 1)a - b) ∨ (-a) \cdot [r, ∞) = T \) for r < 0. According to Theorem 3.1.1, it is sufficient to demonstrate that \( \bigvee_n \bigvee_{u,v} (a(U) ∧ b(V)) = T \), where the inner join ranges over open subsets U, V ⊆ R for which

\( ((n - 1)U - V) ∨ (-U) ⊆ (r, ∞). \)

But if U and V are to satisfy this containment then U must be bounded both above and below, say U ⊆ (u, w), and V must be bounded above, say V ⊆ (−∞, v), where \( (n - 1)u > r + v \) or \( -w > r \). In sum, we must show that, for r < 0,

\[
T = \bigvee_n \left( \bigvee_{u < w < -r} \bigvee_{v} (a(u, w) ∧ b(−∞, v)) ∨ \bigvee_{w > u > -r} (a(u, w) ∧ b(−∞, v)) \right)
\]

\[
= \bigvee_n \left( \bigvee_{u < w < -r} \left( a(u, w) ∧ \bigvee_{v} b(−∞, v) \right) ∨ \bigvee_{w > u > -r} \left( a(u, w) ∧ b(−∞, v) \right) \right)
\]

\[
= \bigvee_n \left( \bigvee_{u < w < -r} a(u, w) ∨ \bigvee_{v} \left( a\left(\frac{r + v}{n - 1}, ∞\right) ∧ b(−∞, v) \right) \right)
\]

\[
= a(−∞, −r) ∨ \bigvee_{v} \left( a\left(\frac{r + v}{n - 1}, ∞\right) ∧ b(−∞, v) \right)
\]

But for v > −r we have \( \bigvee_n \left( a\left(\frac{r + v}{n - 1}, ∞\right) ∧ b(−∞, v) \right) = a(0, ∞) ∧ b(−∞, v) \), so that the last join displayed reduces to \( a(−∞, −r) ∨ a(0, ∞) = a(−∞, ∞) = T \), thereby proving the claim and the proposition.

\[\square\]

6. Conditional pointwise completeness

The classical Nakano-Stone Theorem asserts that every bounded (countable) subset of \( C(X) \) has a supremum in \( C(X) \) iff X is extremally disconnected (basically disconnected). In this section we prove the corresponding result for pointwise suprema, Theorem 6.2.1. Our analysis will be closely intertwined with the pointfree version of the classical theorem, a result of Banaschewski and Hong [11].

6.1. The Banaschewski-Hong Theorem. We begin with an observation.

6.1.1. Proposition. A conditionally pointwise complete (σ-complete) W-object is conditionally complete (σ-complete).

Proof. This follows from Remark 4.1.2(4).

The converse of Proposition 6.1.1 does not hold, even for W objects of the form \( C(X) \), X a Tychonoff space. In this case \( C(X) \) is conditionally σ-complete iff X is basically disconnected; this is the classical Nakano-Stone Theorem. On the other hand, if X is compact and basically disconnected then G ≡ \( C(X) \) is conditionally σ-complete. But L ≡ MG = \( C(X) \) is a P-frame iff X is a P-space, and a compact P-space is finite. The point is that, by Theorem 6.2.1, G is not conditionally pointwise σ-complete unless X is finite. See also [15, 4N].

Theorem 6.1.1 is a modestly embellished version of the pointfree Nakano-Stone Theorem, a beautiful result of Banaschewski and Hong [11]. We prove Theorem 6.1.1 in some detail, not just because we need the result but also because the proofs provide the basis for the corresponding result for pointwise completeness in Subsection 6.2.
Let us review the basic definitions: a frame is said to be extremally disconnected (basically disconnected) provided that $a^* \lor a^{**} = \top$ for all $a \in L$ ($a \in \text{coz } L$). And $\mathbb{R}^+ (G)$ stands for the set of positive real numbers such that the corresponding constant function lies in G.

6.1.2. **Theorem.** Let $G$ be a W-object with Madden frame $L$. Then conditions (1) and (2) together are equivalent to conditions (3) and (4).

1. $\bigwedge \mathbb{R}^+ (G) = 0$, i.e., $G$ contains arbitrarily small positive multiples of 1.
2. $G^*$ is conditionally complete ($\sigma$-complete).
3. $L$ is extremally disconnected (basically disconnected).
4. The Madden representation carries $G^*$ onto $\mathbb{R}^+ L$.

**Proof.** Since it is most relevant to our purposes, we prove the version of this theorem having to do with the conditional $\sigma$-completeness of $G^*$ versus the basic disconnectivity of $L$. The implication from (3) and (4) to (1) and (2) is provided by the result of Banaschewski and Hong ([11, Prop. 2]), since they prove that if $L$ is basically disconnected then $\mathbb{R}L$, and hence $\mathbb{R}^* L$, is conditionally $\sigma$-complete. The opposite implication is provided by Propositions 6.1.3 and 6.1.7. The running assumptions throughout are that $L$ is the Madden frame of $G$, and that $G$ has been identified with its Madden representation in $\mathbb{R}L$, i.e., $G$ is a W subobject of $\mathbb{R}L$. □

6.1.3. **Proposition.** If $\bigwedge \mathbb{R}^+ (G) = 0$ and $G^*$ is conditionally $\sigma$-complete then $L$ is basically disconnected.

**Proof.** Consider a cozero element $a \in L$, say $a = f (0, \infty)$ for $f \in C^+ L$. By replacing $f$ by $f \land 1$, we may assume that $f \in \mathbb{R}^* L = G^*$. Define the sequence $(g_n)$ in $G^*$ by setting

$$g_n = nf \land 1, \quad n \in \mathbb{N},$$

and let $g \in G^*$ be such that $g = \bigvee_N g_n$.

We aim to show that $g$ is a component of 1, i.e., that $(1 - g) \land g = 0$, by means of several claims. We first claim that $(1 - g) \land (nf - 1)^+ = 0$ for all $n$. For

$$g = \bigvee_N g_n \implies 1 - g = \bigwedge_N (1 - g_n) = \bigwedge_N (1 - nf)^+, \quad (1 - g) \land (nf - 1)^+ = 0,$$

and, since $(1 - nf)^+ \land (nf - 1)^+ = 0$,

$$(1 - g) \land (nf - 1)^+ \leq (1 - nf)^+ \land (nf - 1)^+ = 0.$$

We next claim that $(1 - g) \land f = 0$. For if not, then $x = (1 - g) \land f \land 1 > 0$. Since $G$ is archimedean, there exists $k \in \mathbb{N}$ such that $kx \not\leq 1$; let $k$ be the least such integer. Then

$$0 < (kx - 1)^+ \leq (kf - 1)^+ \implies (kx - 1)^+ \land (1 - g) = 0.$$

But

$$(k - 1) x \leq 1 \implies (kx - 1)^+ \leq x \leq (1 - g),$$

a contradiction. It now follows that $(1 - g) \land nf = 0$ for all $n$, hence $(1 - g) \land g_n = 0$ for all $n$. Upon recalling a basic fact about $\ell$-groups, namely that if $g = \bigvee_N g_n$ for $(g_n) \subseteq G^+$ and if $a \land g_n = 0$ for all $n$ then $a \land g = 0$, we reach the desired conclusion: $g \land (1 - g) = 0$.

The basic disconnectivity of $L$ follows immediately from the fact that $g$ is a component of 1, for

$$g \land (1 - g) = 0 \implies g \lor (1 - g) = g + (1 - g) = 1,$$

hence $\text{coz } g \lor \text{coz } (1 - g) = \text{coz } 1 = T$, which is to say that $a^{**} \lor a^* = T$. □
The proof of Theorem 6.1.2 is completed by Proposition 6.1.7, which requires two simple lemmas, the first of which is folklore. We say that an \( \ell \)-subgroup \( H \leq G \) is order dense in \( G \) if for every \( 0 < g \in G \) there is some \( h \in H \) such that \( 0 < h \leq g \).

6.1.4. Lemma. Suppose \( G \) is an order dense \( \ell \)-subgroup of \( H \).

(1) Then suprema and infima in \( G \) and \( H \) agree, and

(2) \( h = \bigvee \downarrow h_{\leq G} \) for all \( h \in H^+ \).

Proof. (1) Suppose \( \bigvee A = g_0 \) for some \( A \subseteq G^+ \) and \( g_0 \in G^+ \), but that \( A \leq h < g_0 \) for some \( h \in H \). Find \( g_1 \in G \) such that \( 0 < g_1 \leq g_0 - h \). Since \( g_0 - g_1 < g_0 \) there is some \( g \in A \) for which \( g \not> g_0 - g_1 \). But this flies in the face of the fact that \( g + g_1 \leq h + g_1 \leq g_0 \). We conclude that \( \bigvee A = g_0 \) in \( H \).

(2) Given \( h_0 \in H^+ \), let \( A \equiv \downarrow h_0 \downarrow G \), and suppose for the sake of argument that \( A \leq h_1 < h_0 \) for some \( h_1 \in H^+ \). Then find \( g_0 \in G \) such that \( 0 < g_0 \leq h_0 - h_1 \). But for any \( g \in A \) we have \( g + g_0 \leq h_1 + g_0 \leq h_0 \), i.e., \( A + g_0 \subseteq A \). It follows that \( n g_0 \in A \) for all \( n \), which is to say that \( n g_0 \leq h_0 \) for all \( n \), a violation of the archimedean property of \( H \).

6.1.5. Lemma. For \( h, k \in R^+ \) and \( 0 \leq q < \infty \), if \( \text{coz} \, k \leq h(q, \infty) \) and \( k \leq q \) then \( k \leq h \).

Proof. By Lemma 3.2.1(2) it is sufficient to show that for any \( r \in R \),

\[
h(r, \infty) \geq k(r, \infty) = (k \wedge q)(r, \infty) = k(r, \infty) \wedge q(r, \infty) = \begin{cases} k(r, \infty) & \text{if } r < q \\ \bot & \text{if } r \geq q. \end{cases}
\]

But this is clear, for if \( 0 \leq r < q \) then \( k(r, \infty) \leq k(0, \infty) = \text{coz} \, k \leq h(q, \infty) \leq h(r, \infty) \).

We remind the reader that a cozero element \( a \) of a Lindelöf frame \( L \) is Lindelöf, i.e., \( a = \bigvee A \) implies \( a = \bigvee A_0 \) for some countable subset \( A_0 \subseteq A \).

6.1.6. Lemma. If \( \bigwedge R^+(G) = 0 \) then every element of \( RL \) is the join of a countable subset of \( G^* \).

Proof. Given \( 0 < h \in R^+ L \) and \( q \in R^+(G) \), \( h(q, \infty) = \text{coz}(h - q)^+ \) is a cozero element of \( L \) and is therefore Lindelöf. Since \( \mu_G : G \rightarrow RL \) is cozero dense, this element is the join of those of the form \( \text{coz} \, g \), \( g \in G^+ \). Let \( G_q \) be a countable subset of \( G^+ \) such that \( h(q, \infty) = \bigvee_{G_q} \text{coz} \, g \). By suitably restricting and augmenting \( G_q \), we may assume that \( g \leq q \) for all \( g \in G_q \), and that \( g \in G_q \) implies \( mg \wedge q \in G_q \) for all integers \( m \). Finally, let \( G_0 \equiv \bigcup_{0 < q \in G} G_q \) for some countable dense subset \( R \subseteq R^+(G) \).

We claim that \( h = \bigvee G_0 \). If not then \( G_0 \leq k < h \) for some \( k \in RL \), so that by Lemma 3.2.1(2) there is some \( q \in R \) such that \( k(q, \infty) < h(q, \infty) \). More is true; there must be some \( s > q \) in \( R \) for which \( a \equiv k(-\infty, s) \wedge h(s, \infty) \neq \bot \); for otherwise \( h(s, \infty) \leq k(-\infty, s)^+ \) for all \( s > q \) would imply

\[
h(s, \infty) = \bigvee_{q < s} h(s, \infty) \leq \bigvee_{q < s} k(-\infty, s)^+ = k(q, \infty),
\]

contrary to assumption. Now \( h(s, \infty) = \bigvee_{G_s} \text{coz} \, g \), so we may find \( g \in G_s \) such that \( \text{coz} \, g \wedge a \neq \bot \). Since \( \text{coz} \, g = \bigvee_{m} \text{coz}(mg - k)^+ \) by Corollary 3.2.3(5), there exists some integer \( m \) for which \( \text{coz}(mg - k)^+ \wedge a > \bot \). In sum, we have arranged that

\[
\text{coz}(mg \wedge s - k)^+ = \text{coz}((mg - k)^+ \wedge (s - k)^+) = \text{coz}(mg - k)^+ \wedge \text{coz}(s - k)^+ = \text{coz}(mg - k)^+ \wedge k(-\infty, s) > \bot.
\]
But $g \in G_s$ implies $mg \wedge s \in G_s$, hence $mg \wedge s \leq k$, contrary to the information displayed above. This completes the proof of the claim and the lemma.

\[ \square \]

6.1.7. Proposition. If $\bigwedge R^+(G) = 0$ and $G^*$ is conditionally $\sigma$-complete then $\hat{G}^* = R^+ L$.

Proof. Given $0 < h \in R^+ L$, we know from Lemma 6.1.6 that $h = \bigvee A$ for some countable subset $A \subseteq \downarrow h_{\uparrow G}$. Since $h$ is bounded by some multiple of 1, so is $A$. By virtue of the conditional $\sigma$-completeness of $G$, $A$ has a supremum $g_0$ in $G$. Finally, $\hat{g}_0 = h$ by Lemmas 6.1.4 and 6.1.6. \[ \square \]

6.1.8. Corollary. A conditionally $\sigma$-complete $W$-object which contains arbitrarily small positive multiples of 1 contains all real multiples of 1. That is, $\bigwedge R^+(G) = 0$ implies $R^+(G) = R^+$.

The proof of Theorem 6.1.2 is complete.

It is worthwhile to restate Theorem 6.1.2 in the language of regular $\sigma$-frames. This is always possible, since the fact that $L$ is Lindelöf means that $L$ is isomorphic to $H$ coz $L$.

6.1.9. Theorem. Let $G$ be a $W$-object with Madden frame $L$. Then conditions (1) and (2) together are equivalent to conditions (3) and (4).

1. $\bigwedge R^+(G) = 0$.
2. $G^*$ is conditionally complete ($\sigma$-complete).
3. $L$ is extremally disconnected (basically disconnected).
4. $L$ is conditionally complete ($\sigma$-complete).

$L$ is isomorphic to $H(A)$ for some regular $\sigma$-frame $A$ such that for all $C \subseteq A$ there exists a complemented element $b \in A$ with

\[ \forall d \in A \ (\forall c \in C \ (d \wedge c = \bot) \iff d \leq b) . \]

$L$ is isomorphic to $H(A)$ for some regular $\sigma$-frame $A$ such that for all $c \in A$ there exists a complemented element $b \in A$ with

\[ \forall d \in A \ (d \wedge c = \bot \iff d \leq b) . \]

4. The Madden representation carries $G^*$ onto $R^+ L$.

6.2. P-frames and boolean frames. Proposition 6.1.1 holds that conditional pointwise completeness is stronger than conditional completeness. In view of Theorem 6.1.2, then, the question naturally arises as to what condition on $MG$ is equivalent to the conditional pointwise completeness of a $W$-object $G$. The answer is that $MG$ must be boolean in order for $G$ to be conditionally pointwise complete, and $MG$ must be a $P$-frame in order for $G$ to be conditionally pointwise $\sigma$-complete.

6.2.1. Theorem. Let $G$ be a $W$-object with Madden frame $L$. Then conditions (1) and (2) together are equivalent to conditions (3) and (4).

1. $G$ contains arbitrarily small positive multiples of 1.
2. $G^*$ is conditionally pointwise complete (conditionally pointwise $\sigma$-complete).
3. $L$ is boolean (a $P$-frame).

4. The Madden representation carries $G^*$ onto $R^+ L$.

Proof. We first prove the countable version of this theorem. Suppose $L$ is a $P$-frame, identify $G$ with its Madden representation $\hat{G} \subseteq R L$, and suppose $G^* = R^+ L$. Then $G$ certainly satisfies (1); in order to verify that $G^*$ is pointwise $\sigma$-complete, consider a countable subset $F \subseteq G$ with upper bound $g \in G^*$. Define a function $f_0$ on the right rays by the rule

\[ f_0 (r, \infty) = \bigvee_f (r, \infty) , \ r \in R . \]
We claim that $f_0$ extends to a unique member of $\mathcal{RL}$, which must then lie in $G^*$ by virtue of its convexity, since clearly $f \preceq f_0 \preceq g$ by Lemma \ref{3.2.12}. To establish this claim we need only check the three hypotheses of Lemma \ref{3.2.2}. The first hypothesis clearly holds, since a complemented element of any frame is rather below itself. To verify the second, simply observe that
\[
\bigvee_{s>r} f_0 (s, \infty) = \bigvee_{s>r} f (s, \infty) = \bigvee_{s>r} f (s, \infty) = \bigvee_{s>r} f (r, \infty) = f_0 (r, \infty).
\]
To verify the third hypothesis, note that
\[
\bigvee_{r} f_0 (r, \infty) = \bigvee_{r} f (r, \infty) = \bigvee_{r} f (r, \infty) = \top.
\]
And, since $f_0 (r, \infty) = \bigvee_{r} f (r, \infty) \leq g (r, \infty)$ for all $r \in \mathbb{R}$,
\[
\bigvee_{r} f_0 (r, \infty) \leq \bigvee_{r} f (r, \infty) \leq \bigvee_{r} g (r, \infty) \leq \bigvee_{r} g (-\infty, r) = \top.
\]
The second inequality holds because $g (-\infty, r) \wedge g (r, \infty) = g (0) = \bot$.

Now suppose that $G$ satisfies (1) and (2), and again identify it with its Madden representation. Then $G^*$ is $\sigma$-complete by Remark \ref{4.12.5}, so that Theorem \ref{6.1.2} allows us to conclude that $G^* = \mathcal{RL}$. To verify (3), consider a cozero element $a \in \mathcal{L}$, say $a = \coz f$ for some $f \in \mathcal{RL}$. By replacing $f$ by $f \wedge 1$ if necessary, we may assume that $1 \geq f \in G^*$. Define the sequence $\{g_n\}$ in $G^*$ by setting
\[
g_n \equiv nf \wedge 1, \ n \in \mathbb{N},
\]
and let $g \in G^*$ be such that $g = \bigvee \mathbb{N} g_n$. Since the $g_n$’s here are defined exactly as in the proof of Theorem \ref{6.1.2} and since $g = \bigvee g_n$, the argument given there applies here, and shows that
\[
\coz g \vee \coz (1 - g) = \coz 1 = \top.
\]
But when we observe that $\coz g = g (0, \infty) = a$ and $\coz (1 - g) = a^*$, we come to the desired conclusion: $a \vee a^* = \top$.

It remains to prove the version of the theorem in which the pointwise joins are of unrestricted cardinality. For the most part the argument goes along the lines of the countable case. The only significant departure is the implication from (1) and (2) to (3). So suppose $G$ contains arbitrarily small positive multiples of 1 and that $G^*$ is pointwise complete, and consider an arbitrary element $a_0 \in \mathcal{L}$. Express $a_0$ in the form $\bigvee \mathbb{I} a_i$, where $\{a_i : i \in \mathbb{I}\}$ is the set of cozero elements below $a_0$. For each $i \in \mathbb{I}$ let $g_i$ be the characteristic function of $a_i$, i.e.,
\[
g_i (U) = \begin{cases} \top & \text{if} \ 0, 1 \notin U \\ a_i & \text{if} \ 0 \notin U \ni 1 \\ a_i^* & \text{if} \ 1 \notin U \ni 0 \\ \bot & \text{if} \ 0, 1 \in U \end{cases}.
\]
These functions lie between 0 and 1 and hence are in $G^*$. Let $g_0 \equiv \bigvee \mathbb{I} g_i$. By inspection one sees that
\[
g_0 (U) = \begin{cases} \top & \text{if} \ r < 0 \\ a_0 & \text{if} \ 0 \leq r < 1 \\ \bot & \text{if} \ 1 \leq r \end{cases},
\]
the characteristic function of $a_0$. But since $a_0 = g_0 (3/4, \infty) \prec g_0 (1/4, \infty) = a_0$, it follows that $a_0$ is complemented. The shows that $\mathcal{L}$ is boolean, and completes the proof. \qed
A quotient of a P-frame need not be a P-frame (8). However, a C-quotient of a P-frame is clearly a P-frame, for a C-quotient \( f: L \rightarrow M \) is coz-onto, meaning every cozero element of \( M \) is the image under \( f \) of a cozero element of \( L \). Since the cozero elements of \( L \) are complemented, so are their images. An alternative argument can be made using Theorem 6.2.1.

6.2.2. **Corollary.** A C-quotient of a P-frame is a P-frame.

**Proof.** A C-quotient map \( f: L \rightarrow M \) induces a W-surjection \( Rf: RL \rightarrow RM \). If \( L \) is a P-frame then conditions (3) and (4) of Theorem 6.2.1 hold, and therefore conditions (1) and (2) are true of \( RL \). But the latter two conditions are clearly inherited by any quotient of \( RL \), and therefore are true of \( RM \). A second application of the theorem gives the desired conclusion. \( \square \)

7. **Unconditional pointwise completeness**

In this section we define and analyze the ultimate, or unconditional form of pointwise completeness. This naturally raises the question of precisely what unconditional pointwise completeness ought to mean. Our definition comes in Subsection 7.3, but it requires a digression to review essential extensions in Subsection 7.1 and cuts in Subsection 7.2. The reader may wish to skip this material upon a first reading, returning to it as necessary.

7.1. **Essential extensions and complete embeddings.** In this subsection we recall the basic facts concerning essential extensions in \( W \). We do so not only because we will make use of these facts in the sequel, but also for the reader's convenience, for these extensions appear in the literature under various names and with various definitions. Because this material is well known (see, e.g., (12)), we offer here only hints of proofs.

Recall that the booleanization of a frame \( M \) is the frame map

\[ b_M : M \rightarrow M^{**} = (a \mapsto a^{**}) \]

In spatial terms, this is the map which sends an open set to its regularization, i.e., the smallest regular open subset containing it.

7.1.1. **Lemma.** The following are equivalent for an extension \( G \leq H \) in \( W \).

1. The embedding \( G \rightarrow H \) is an essential monomorphism, i.e., any morphism out of \( H \) whose restriction to \( G \) is one-one is also one-one on \( H \).
2. Every nontrivial W-kernel of \( H \) meets \( G \) nontrivially.
3. Every nontrivial polar of \( H \) meets \( G \) nontrivially.
4. \( G \) is large in \( H \), i.e., every nontrivial convex \( \ell \)-subgroup of \( H \) meets \( G \) nontrivially.
5. If \( H \) is divisible then these conditions are equivalent to \( G \) being order dense in \( H \), i.e., for every \( 0 < h \in H \) there exists some \( 0 < g \in G \) such that \( g \leq h \).
6. The frame map \( f: L \equiv MG \rightarrow MH \equiv M \) which realizes the extension \( G \leq H \) 'drops' to an isomorphism of the booleanizations.
Proof. The equivalence of (1) with (2) is evident, and the implication from (2) to (3) is a consequence of the fact that polars are W-kernels. To show that (3) implies (4), one shows that \( K^\perp \cap G = 0 \) for any convex \( \ell \)-subgroup \( K \leq H \) such that \( K \cap G = 0 \). And (4) clearly implies (2).

Part (3) can be interpreted as saying that the extension is essential iff the polars of \( G \) and \( H \) are in bijective correspondence by intersection. But the polars of any W-object are in bijective correspondence with the elements of its booleanization via

\[
a'' \longrightarrow \{ g \in G : \text{coz} |g| \leq a'' \}
\]

\[
\bigvee_p \text{coz} g \leftarrow P
\]

This observation can be readily converted into a proof that (6) is equivalent to the other conditions.

From Lemma 7.1.1(6) we see that for any frame \( L \) the dual \( Rb_L \) of the booleanization map provides an essential extension \( RL \rightarrow RL^{**} \). It is this embedding which is meant whenever we write \( RL \leq RL^{**} \).

We say that an object is essentially complete if it has no proper essential extensions. A maximal essential extension of \( G \) is an essential extension \( G \leq H \) such that \( H \) is essentially complete.

7.1.2. Proposition. (1) A W-object is essentially complete iff it is of the form \( RL \) for a boolean frame \( L \).

(2) Every W-object \( G \) has a maximal essential extension, namely \( G \leq RL \leq RL^{**} \).

(3) Any two maximal essential extensions of \( G \) are isomorphic over \( G \).

(4) Let \( G \leq H \) be a maximal essential extension. Then an arbitrary extension \( G \leq K \) is essential iff \( K \) is isomorphic over \( G \) to an \( \ell \)-subgroup of \( H \).

Proof. (1) is a consequence of the fact that \( G \leq RL \leq RL^{**} \) is an essential extension which is an isomorphism iff \( G = RL = RL^{**} \). (2) is Proposition 2.1 of [10]. The rest is due to Conrad from his seminal article [13].

[10] provides an analysis of maximal essential extensions in categories related to W. See also [9] for a closely related analysis in the context of completely regular frames.

Essential extensions take their importance here from the fact that any extension may be ‘reduced’ to an essential extension by passage to an appropriate quotient. This is Lemma 7.1.4 which involves an attribute weaker than essentiality. Recall that an injective homomorphism \( \tau : H \rightarrow K \) is said to be complete if it preserves all suprema and infima that exist in \( H \). The following is folklore; see, e.g., [14].

7.1.3. Lemma. An essential injection is complete.

Proof. Let \( G \leq H \) be an essential extension, and let \( \bigvee Z = g \) for some subset \( Z \subseteq G^+ \) and element \( g \in G^+ \). If \( G \) fails to be the supremum of \( Z \) in \( H \), it is only because there is some \( h \in H \) such that \( Z \leq h < g \). Now the convex \( \ell \)-subgroup of \( H \) generated by \( g - h \) is nontrivial, and by Lemma 7.1.1(4) contains some \( 0 < g' \in G \), say \( g' \leq n(g - h) \) for a positive integer \( n \). This rearranges to

\[
ng > ng - g' \geq nh \geq nz
\]

for all \( z \in Z \).

But \( \bigvee Z = g \) implies \( \bigvee Z nz = ng \in G \), and this contradicts the displayed condition.

7.1.4. Lemma. For any injective homomorphism \( \gamma : G \rightarrow H \) there is a surjective homomorphism \( \tau : H \rightarrow K \) such that \( \tau \circ \gamma \) is an essential injection. Moreover, \( \tau \) may be chosen to be complete.
Proof. We claim that the family \( \mathcal{Q} \) of polars \( Q \) such that \( Q \cap \gamma(G) = 0 \) contains maximal elements. For if \( e \) is a nonempty chain in \( \mathcal{Q} \) then, since \( \gamma(G) \cap \bigcup e = 0 \) and \( \bigcup e \) is convex, \( \gamma(G) \subseteq (\bigcup e)^\perp \), hence \( \gamma(G) \cap (\bigcup e)^\perp = 0 \) and so \( (\bigcup e)^\perp \subseteq Q \). If we take \( \tau \) to be the quotient map \( H \to H/R \) for some polar \( K \) maximal in \( Q \) then it is clear that part (3) of Lemma 7.1.1 is satisfied by \( \tau \circ \gamma \). And \( \tau \) is complete because \( Q \) is order closed. \( \square \)

7.1.5. Lemma. If a subset \( Z \subseteq G \) has a supremum in some extension then it has a supremum in some essential extension.

Proof. Let \( G \leq H \) be an extension such that \( \bigvee Z = h \) in \( H \), and let \( \tau : H \to K \) be the complete surjection of Lemma 7.1.4 such that \( \tau \) is one-one on \( Z \) and \( K \) is an essential extension of \( \tau[Z] \). Then \( \bigvee \tau[Z] = \tau(h) \) in \( K \) since \( \tau \) is complete. Identifying \( Z \) with its image under \( \tau \) provides the desired extension. \( \square \)

7.2. Mobile downsets and cuts. Many completion results are based on the technique of adjoining to \( G \) a supremum for each downset of a particular type. The downsets in play, usually called cuts, depend on the sort of completeness desired. The broadest notion of a cut was introduced in Section 4 of [9].

7.2.1. Definition. A downset \( Z \subseteq G \) is called a cut if it has a supremum in some extension of \( G \).

Observe that by Lemma 7.1.5 a downset \( Z \subseteq G^+ \) is a cut iff it has a supremum in some essential extension of \( G \).

Definition 7.2.1 is sufficiently opaque as to appear useless, but its utility is restored by Theorem 7.2.5 which gives a working criterion for a subset \( Z \subseteq G^+ \) to be a cut. That criterion involves the inability of the subset to remain stationary under addition by a positive element.

7.2.2. Definition. A downset \( Z \subseteq G \) is said to be mobile if \( Z + g \nsubseteq Z \) for all \( 0 < g \in G \).

7.2.3. Observations. Let \( Z \) be a downset in \( G \).

1. \( Z \) is mobile iff there is no \( 0 < g \in G \) for which \( Z + G(g) \subseteq Z \), where \( G(g) \) designates the convex \( \ell \)-subgroup of \( G \) generated by \( g \).

2. \( Z \) is mobile iff it is not the union of cosets of some nontrivial convex \( \ell \)-subgroup of \( G \).

Proof. If \( Z + g \subseteq Z \) then \( Z + ng \subseteq Z \) for all \( n \), hence \( Z + k \subseteq Z \) for all \( k \) such that \( |k| \leq ng \) for some \( n \).

The next proposition hints at why mobile downsets are relevant to our investigation, for it shows that two types of subsets which may have pointwise joins are mobile.

7.2.4. Proposition. (1) The downset \( \downarrow g_0 \wedge n : n \in \mathbb{N}, g_0 \in G^+ \) is mobile.

(2) A nonempty bounded downset is mobile.

Proof. (2) Suppose the downset \( \emptyset \neq Z \subseteq G \) is bounded above by \( g_0 \), and suppose for the sake of argument that \( Z + g \subseteq Z \) for some \( 0 < g \in G \). We may assume that \( 0 \in Z \), for we may always replace \( Z \) by \( Z - z_0 \) and \( g_0 \) by \( g_0 - z_0 \), where \( z_0 \) is any member of \( Z \). But then \( ng \in Z \) for all positive integers \( n \), with the result that \( ng \leq g_0 \) for all \( n \), a violation of the archimedean property of \( G \).

1. Let \( Z = \downarrow g_0 \wedge n : n \in \mathbb{N} \) be the set of lower bounds of the truncates of \( g_0 \in G^+ \), and suppose for the sake of argument that \( Z + g \subseteq Z \) for some \( 0 < g \in G \). Then by
the archimedean property there is a positive integer \( m \) such that \( mg \nleq g_0 \). It follows that \( mg \nleq n \land g_0 \) for any positive integer \( n \), which is to say that \( mg \notin \mathbb{Z} \), contrary to Observation 7.2.3(1).

\[ \Box \]

### 7.2.5. Theorem

The following are equivalent for a downset \( Z \subseteq G \).

1. \( Z \) is a cut in \( G \).
2. \( G \) has an essential extension \( H \) containing an element \( h \) such that \( \bigvee Z = h \) in \( H \).
3. \( G \) is completely embedded in an extension \( H \) containing an element \( h \) such that \( \bigvee Z = h \) in \( H \).
4. \( Z \) is not a union of cosets of a nontrivial convex \( \ell \)-subgroup of \( G \).
5. \( Z \) is mobile.

**Proof.** The equivalence of (1) and (2) is Lemma 7.1.5, the implication from (2) to (3) is Lemma 7.1.3, and the implication from (3) to (1) is trivial. The equivalence of (3) and (5) is Proposition 4.5 of [4], and the equivalence of (4) and (5) is Observation 7.2.3(2).

Our main Theorem 7.3.2 requires a technical lemma.

### 7.2.6. Lemma

Let \( G = \mathcal{R}L \) for a \( P \)-frame \( L \), and let \( G \leq H \) be its maximal essential extension (Proposition 7.1.2). If all of the truncates of an element \( h \in H^+ \) lie in \( G \) then \( h \) lies in \( G \).

**Proof.** Here \( G = \mathcal{R}L \leq \mathcal{R}L^{**} = H \), where the embedding \( \mathcal{R}L \to \mathcal{R}L^{**} \) is provided by \( \mathcal{R}b_L = (g \mapsto b_L \circ g) \). Suppose \( (g_n) \subseteq G^+ \). The condition that \( g_{n+1} \land n = g_n \) for all \( n \) clearly holds in \( H \) iff it holds in \( G \), and the condition that \( \bigvee_n g_n (\neg \infty, n) = \top \) implies that

\[
\top = b_L (\top) = b_L \left( \bigvee_n g (\neg \infty, n) \right) = \bigvee_n b_L \circ g (\neg \infty, n)
\]

because \( b_L \) is a frame morphism. We must demonstrate the converse, i.e., that the displayed condition implies that \( \bigvee_n g_n (\neg \infty, n) = \top \).

So assume that \( \bigvee_n b_L \circ g_n (\neg \infty, n) = \top \) holds in \( L^{**} \), i.e., that \( (\bigvee_n g_n (\neg \infty, n)^{**})^{**} = \top \) holds in \( L \). Note that \( g_n (\neg \infty, n) \), being a cozero element of a \( P \)-frame, is complemented, i.e., \( g_n (\neg \infty, n)^{**} = g_n (\neg \infty, n) \). Also note that, since the inclusion \( \text{coz } L \to L \) is a \( \sigma \)-frame homomorphism, the supremum \( \bigvee_n g_n (\neg \infty, n) \) in \( L \) agrees with its supremum in \( \text{coz } L \). But the latter is a cozero and is therefore complemented, so that we get \( (\bigvee_n g_n (\neg \infty, n))^{**} = \bigvee_n g_n (\neg \infty, n) \).

The hypotheses of the preceding lemma are more generous than necessary, so we digress briefly to tighten it up in the light of the analysis conducted by the second author in [16]. We refer the interested reader to that article for terminology and notation otherwise undefined here, and omit the details of proof.

### 7.2.7. Theorem

The following are equivalent for a \( W \)-object \( G \) with maximal essential extension \( G \leq H \).

1. Every element \( h \in H^+ \) which has all its truncates in \( G \) must lie in \( G \).
2. Every element \( h \in D^+ (\text{YG}) \) with all its truncates in \( G \) must lie in \( G \).
3. \( G \) is \( \ast \)-maximum, i.e., \( G \) contains a copy of every \( W \)-object with the same bounded part as \( G \).
7.3. Pointwise completeness

7.3.1. Definition. A W-object is pointwise complete (σ-complete) if every (countably generated) cut in G has a pointwise join in G.

7.3.2. Theorem. The following are equivalent for a W-object G.

1. G is pointwise complete (σ-complete).
2. Every (countably generated) mobile downset of G has a pointwise join in G.
3. G is of the form RL for a boolean frame (P-frame) L.

Proof. The equivalence of (1) with (2) follows from Theorem 7.2.5. If G satisfies (2) then by Lemma [7.2.4][2] every bounded (countable) subset of G has a pointwise join in G, hence by Theorem [6.2.1] G is a subobject of RL, L boolean (a P-frame), and G contains R*L. That G is actually all of RL follows from Proposition [5.2.1].

Let us show that (3) implies (1) when G is of the form RL for some P-frame L. Let Z be a countable subset of G+ such that ∨ Z = h0 in some extension G ≤ H. By Lemma [7.1.5] we may assume this extension to be essential and, in fact, it does no harm to assume that H is the maximal essential extension of G. That is because any essential extension of G is isomorphic to an ℓ-subgroup of H containing G, and suprema in all these extensions agree by Lemma [7.1.3].

Since G is conditionally pointwise σ-complete by Theorem [6.2.1] for each n the subset Z ∧ n has a pointwise supremum ḡn ∈ G. Of course, the same subset has supremum h0 ∧ n in H, and the two suprema coincide by Proposition [4.2.3]. Lemma [7.2.6] then implies that h0 ∈ G. Since h0 is the pointwise join of the ḡn’s by Proposition [5.2.1] and in light of the fact that each ḡn is the pointwise join of Z ∧ n, it follows that ∨ Z = h0 ∈ G.

It remains to show that (3) implies (1) when G is of the form RL for some P-frame L. Let Z ⊆ G+ be such that ∨ Z = h0 in some extension G ≤ H. By Lemma [7.1.5] again, we may assume this extension to be essential. Because G is essentially complete by Proposition [7.1.2] we know that G = H. And finally, the supremum ∨ Z = h0 ∈ G is pointwise by Proposition [5.2.2].

7.3.3. Corollary. G is pointwise σ-complete iff it is epicomplete in W.

Proof. Both conditions are equivalent to G being of the form RL for L a P-frame. One equivalence is provided by Theorem 3.4 of [8] and the other by Theorem [7.3.2].

7.3.4. Corollary. The full subcategory comprised of the pointwise σ-complete objects is reflective in W.

Proof. A complete description of the extension in (2) is G → RP L, where P L designates the P-frame reflection of the Madden frame L of G.

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