Z-COMPLEMENTARY PAIRS WITH FLEXIBLE LENGTHS AND LARGE ZERO ODD-PERIODIC CORRELATION ZONES

LIQUN YAO
School of Mathematics, Southwest Jiaotong University
Chengdu, 610031, China
State Key Laboratory of Cryptology, P. O. Box 5159, Beijing 100878, China

WENLI REN
College of Mathematical sciences, Dezhou University
Dezhou 253023, China

YONG WANG*
School of Physics, Southwest Jiaotong University
Chengdu 610031, China

CHUNMING TANG
School of Mathematics and Information, China West Normal University
Nanchong 637002, China

(Communicated by Sihem Mesnager)

Abstract. Z-complementary pairs (ZCPs) have been widely used in different communication systems. In this paper, we first investigate the odd-periodic correlation property of ZCPs, and propose a new class of ZCPs, called ZOC-ZCPs with zero correlation zone (ZCZ) width $Z$ and zero odd-period correlation zone (ZOCZ) width $Z_{\text{odd}} = Z$ by horizontal concatenation of a certain combination of some known ZCPs. Particularly, based on any known Golay pair, we can generate a class of GCPs of more flexible length whose ZOCZ width is larger than a quarter of the sequence length.

1. Introduction

During the design of infrared multislit spectrometry, a Golay complementary pair (GCP) was first introduced by Golay in 1951, for which the sum of out-of-phase aperiodic autocorrelation values equals to zero. And Golay further studied their mathematical properties in 1961 [9]. Because of the desired correlation property, GCPs have been found extensive engineering applications in intersymbol interference channel, radar waveform designs, asynchronous multi-carrier code division multiple access, peak-to-mean envelope power ratio control, and so on [29, 25, 35, 32, 5, 30, 26, 24, 36]. However, the lengths of the exiting known binary GCPs are limited of the form $2^a10^b26^c$, where $a$, $b$ and $c$ are nonnegative integers [15]. For this reason, motivated by the scarcity of GCPs, Fan et al. [8] proposed the concept of binary Z-complementary pairs (ZCPs) in which their aperiodic autocorrelation sums are zero for certain time-shifts within a region around the origin,

2020 Mathematics Subject Classification: Primary: 94A05; Secondary: 60G35.
Key words and phrases: Aperiodic correlation, odd-periodic correlation, Golay complementary pair, Z-complementary pair, zero correlation zone, zero odd-periodic correlation zone.
* Corresponding author: Yong Wang.
called zero correlation zone (ZCZ) [7]. The ZCPs exist for various lengths with flexible ZCZ widths, and have been utilized in quasi-synchronous code-division multiple access (QS-CDMA) to eliminate multiple access interference and multipath interference. As a result, the design of ZCPs received more and more attention. Please refer to [38, 6, 14] for more constructions of ZCPs.

In particular, Gong et al. [11] first studied Golay sequences from the perspective of the periodic autocorrelation behavior and proposed three different constructions of Golay sequences whose zero autocorrelation zone (ZACZ) is about a half, a quarter or one eighth of the period. Then, Chen et al. [4] further studied the periodic cross-correlation property of Golay sequences and proposed GCPs with a large ZCZ width. After the construction of binary ZCPs proposed in [18] based on linear combination of some existing ZCPs, Liu et al. [20] not only gave a construction of even-length binary Z-complementary pairs (EB-ZCPs) length of $N = 2^{m+1} + 2^m$ with large ZCZ width $Z = 2^{m+1}$, but they also gave a system structure of an optimal odd-length binary ZCPs (OB-ZCPs) with the maximum ZCZ width $Z = (N + 1)/2$ by insertion and deletion of certain binary GCPs, where $N$ is odd [21]. Chen et al. presented a novel construction of ZCPs based on generalised Boolean functions, where the sequence length, the ZCZ width, and the constellation size are very flexible [3]. Very recently, by concatenating several different complementary pairs of length $2^m$, Xie et al. [37] proposed a construction of even-length binary ZCP with length $N = 2^{m+3} + 2^{m+2} + 2^{m+1}$ and ZCZ width $Z = 2^{m+3}$. Inserting method is a useful method for constructing ZCPs proposed by Adhikary et al. in [1]. Using this method, Adhikary et al. inserted two elements in GCPs, and then constructed ZCPs with length $4N + 2$ and ZCZ width $3N + 1$. Inspired by a recent work of [1], Gu et al. [12] gave a construction of EB-ZCPs with lengths $8N + 4$ and ZCZ widths $5N + 2$ (where $N = 2^\alpha 10^{\beta 26^\gamma}$, and $\alpha, \beta, \gamma$ are nonnegative integers). Avik et al. [2] further obtained flexible CZCPs by using the inserting method.

Yet it is worth pointing out that all of the above researches on ZCPs and GCPs are on their aperiod or period correlation properties and have got a great achievement so far. In practice an odd-periodic correlation function is of same importance with a periodic correlation function [27]. On certain conditions, periodic binary complementary sequences sets and odd-periodic binary sequences sets can transform into each other. More importantly, sequences with large odd periodic autocorrelations centering around the origin could be used to reduce the multipath interference at the receiver end and thus improve the performance of the communication system. In contrast to periodic sequences, the sign of odd periodic sequence has to alternate in each period and then also the polarity of the correlation peaks alternates in the same manner. Therefore, their technical implementation is no more complicated than in the case of periodic sequences [31, 22].

Compared with other correlation properties of ZCPs and GCPs, the studies of ZCPs or GCPs in the past took relatively less attention to their odd-period correlation, and some progress was made in relevant research, for more details see [23, 34, 16, 19, 17, 33, 28, 10, 13]. Sequences with good odd-period correlation also have very important significance in practical applications. Luke et al. [23] used almost perfect binary odd-periodic sequence sets to construct a class of binary odd-periodic complementary sequence sets. Wen et al. [34] pointed out the the relation of the sets of odd-periodic complementary binary sequences with the sets of periodic complementary binary sequences. The theory of families of odd-periodic perfect complementary sequence pairs (OPCSPF) was given by Jin et al.
Z-complementary pairs are introduced by Liu et al. [16]. Liu et al. [19] introduced the optimal odd-period binary Z-complementary pairs (OB-ZCPs). 8-QAM+ odd-periodic complementary sequences and sequence sets with zero correlation zone (ZCZ) were presented by Kai et al. [17]. These constructed perfect odd-periodic complementary sequence and sequence sets with ZCZ can be applied to provide more extensive signals for wireless communication with high bandwidth efficiency and low error rate. To be specific, Yang et al. [10] first studied the zero odd-period autocorrelation zone (ZOACZ) of the Golay sequences. Then Hu et al. [13] further investigated the odd-period cross-correlation properties of GCPs and proposed two classes of GCPs with larger zero odd-period correlation zone (ZOCZ) in which the length of GCPs is limited of the form $2^m$ for some integer $m$. Up to now, there is no study on ZCPs with odd-period zero correlation zone in the literature yet. Inspired by the work of [10] and [13], we will propose a system construction of ZCPs with large ZOCZ using a combination of a ZCP and its Z-complementary mate including GCPs with large ZOCZ as a special case.

The paper is organized as follows. In Section 2, we will introduce some basic definitions and properties. In Section 3, using combination of a known ZCP of length $N$ with ZCZ width $Z$ and one of Z-Complementary mates of the ZCP, we will propose a new class of ZCPs of length $4N$ with ZCZ width $Z$ and ZOCZ width $Z_{odd} = Z$, which includes GCPs of length $4N$ with ZOCZ width $N+1$ as a special case. Finally, we will summarize this paper in Section 4.

2. Notation and preliminaries

In this section, we recall some basic definitions and properties. Before that, fix some notations which will be used throughout this paper.

- $0_n$ denotes the vector of length $n$ where the elements are all 0.
- $\overline{a} = (a_{N-1}, a_{N-2}, \cdots, a_0)$ denotes the reverse of sequence $a = (a_0, a_1, \cdots, a_{N-1})$.
- $x^*$ denotes the complex conjugate of $x$.
- $a \parallel b$ denotes the horizontal concatenation of sequences $a$ and $b$.

Hereinafter, let $a = (a_0, a_1, \cdots, a_{N-1})$ and $b = (b_0, b_1, \cdots, b_{N-1})$ be two complex-valued sequences of length $N$, where $a_i$ and $b_i$ are the complex numbers with absolute value of 1.

**Definition 2.1.** The aperiodic cross-correlation function (ACCF) of $a$ and $b$ at shift $\tau$ is given as follows:

$$\rho_{a,b}(\tau) = \begin{cases} 
\sum_{k=0}^{N-1-\tau} a_k b^*_k + \tau, & 0 \leq \tau \leq N - 1; \\
\sum_{k=0}^{N-1+\tau} a_k b^*_k - (N - 1) \leq \tau \leq N - 1; \\
0, & |\tau| \geq N.
\end{cases}$$

When $a = b$, $\rho_{a,b}(\tau)$ is called aperiodic autocorrelation function (AACF) of $a$ and is denoted as $\rho_a(\tau)$.

**Definition 2.2.** The odd-periodic cross-correlation function (OCCF) of $a$ and $b$ at shift $\tau$ is given as follows:

$$\hat{R}_{a,b}(\tau) = \sum_{k=0}^{N-1-\tau} a_k b^*_k + \tau - \sum_{k=N-\tau}^{N-1} a_k b^*_k + \tau,$$
where the addition operation in the subscript is performed modulo $N$. When $a = b$, $	ilde{R}_{a,b}(\tau)$ is called odd-periodic autocorrelation function (OACF) of $a$ and denoted by $\tilde{\rho}_a(\tau)$ for short.

Obviously, the ACCF $\rho_{a,b}(\tau)$ and OCCF $\tilde{R}_{a,b}(\tau)$ have the following property.

**Property 1.** Let $a = (a_0, a_1, \cdots, a_{N-1})$ and $b = (b_0, b_1, \cdots, b_{N-1})$ be as above, then we have

\[
\begin{align*}
(i) \quad & \rho_{-a,b}(\tau) = \rho_{a,-b}(\tau) = -\rho_{a,b}(\tau); \\
(ii) \quad & \rho_{a,b}(\tau) = i\rho_{a,-b}(\tau); \\
(iii) \quad & \rho_{a,ib}(\tau) = -i\rho_{a,b}(\tau); \\
(iv) \quad & \rho_{ia}(\tau) = \rho_{-ia}(\tau) = \rho_{a}(\tau); \\
(v) \quad & \tilde{R}_{a,b}(\tau) = \rho_{a,b}(\tau) - \rho_{a,-a}(N-\tau).
\end{align*}
\]

**Definition 2.3.** A pair of binary sequences $a$ and $b$ of length $N$ is called a Z-complementary pair (ZCP) with a zero correlation zone (ZCZ) width $Z$, if

\[
\rho_a(\tau) + \rho_b(\tau) = 0.
\]

for all $1 \leq \tau \leq Z - 1$, and $\rho_a(Z) + \rho_b(Z) \neq 0$.

For simplicity, a ZCP of length $N$ with ZCZ width $Z$ is denoted as the $(N, Z)$-ZCP throughout this paper without any ambiguity.

**Definition 2.4.** A sequence pair $(a, b)$ of length $N$ is called a Golay complementary pair (GCP) if

\[
\rho_a(\tau) + \rho_b(\tau) = 0,
\]

for all $1 \leq \tau \leq N - 1$.

**Definition 2.5.** (Z-Complementary Mate): Let $(a, b)$ and $(c, d)$ be two ZCPs of with ZCZ width $Z$. Then $(c, d)$ is called a Z-complementary mate of $(a, b)$ if

\[
\rho_{a,c}(\tau) + \rho_{b,d}(\tau) = 0,
\]

for all $0 \leq \tau \leq Z - 1$.

When $Z = N$, a Z-complementary mate is reduced to a Golay mate.

For any GCP $(a, b)$, there clearly exists the GCP $(c, d) = (\overline{b^*}, -\overline{a^*})$ as one of Golay mates of $(a, b)$ which will be used throughout this paper.

**Definition 2.6.** Let $(a, b)$ be a ZCP with length $N$ and ZCZ width $Z$. Then $(a, b)$ is a ZOC-ZCP with ZCZ width $Z$ and ZOCZ width $Z_{\text{odd}}$ if it satisfies the following conditions:

\[
\begin{align*}
(i) \quad & \rho_a(\tau) + \rho_b(\tau) = 0, \text{ for } 0 < \tau < Z; \\
(ii) \quad & \tilde{R}_{a,b}(\tau) = 0, \text{ for } 0 < \tau < Z_{\text{odd}}; \\
(iii) \quad & \tilde{R}_{a,b}(\tau) = 0, \text{ for } 0 \leq \tau < Z_{\text{odd}}.
\end{align*}
\]

When $Z = N$, a ZOC-ZCP with ZCZ width $Z$ and ZOCZ width $Z_{\text{odd}}$ is reduced to a ZOC-GCP with ZOCZ width $Z_{\text{odd}}$.

For convenience’s sake, an $(N, Z, Z_{\text{odd}})$-ZCP denotes a ZOC-ZCP of length $N$ with ZCZ width $Z$ and ZOCZ width $Z_{\text{odd}}$, and an $(N, Z_{\text{odd}})$-GCP denotes a ZOC-GCP of length $N$ with ZOCZ width $Z_{\text{odd}}$. 
3. NEW CONSTRUCTION OF Z-COMPLEMENTARY PAIRS WITH ZCZ AND ZOCZ

In this section, for a Z-Complementary Pair (ZCP), by investigating their odd-periodic correlation properties, we propose a new class of ZOC-ZCP with ZCZ width \( Z \) and ZOCZ width \( Z_{\text{odd}} \). In the following, \( \hat{R}_p(\tau), \hat{R}_q(\tau), \hat{R}_{p,q}(\tau) \) are represented by \( \text{odd}R_p(\tau), \text{odd}R_q(\tau), \text{odd}R_{p,q}(\tau) \) in Figures for short.

**Construction 1:** Let \((a, b)\) of length \( N \) be a ZCP with ZCZ width \( Z \), and \((c, d) = (\hat{b}^*, -\hat{a}^*)\) be a Z-complementary mate of \((a, b)\). Then a pair of sequences \( p \) and \( q \) of length \( 4N \) can be constructed as follows

\[
\begin{align*}
    p &= x_1 \cdot a \parallel x_2 \cdot b \parallel x_3 \cdot i \cdot a \parallel x_4 \cdot i \cdot b, \\
    q &= x_1 \cdot c \parallel x_2 \cdot d \parallel x_3 \cdot i \cdot c \parallel x_4 \cdot i \cdot d,
\end{align*}
\]

where \( x_i \in \{1, -1\} \) for \( 1 \leq i \leq 4 \), and \( x_1 x_2 + x_3 x_4 = 0 \) holds.

It is clear that the following facts hold.

**Fact 1:** If \((a, b)\) is a GCP and \((c, d) = (\hat{b}^*, -\hat{a}^*)\). Then we have

\[
\begin{align*}
    (i) & \quad \rho_a(\tau) + \rho_c(\tau) = 0, \quad \rho_b(\tau) + \rho_d(\tau) = 0 \text{ for all } 0 < \tau \leq N - 1; \\
    (ii) & \quad \rho_{a,b}(\tau) + \rho_{c,d}(\tau) = 0 \text{ for all } 0 \leq \tau \leq N - 1; \\
    (iii) & \quad x_1 x_2 + x_3 x_4 = 0 \iff x_1 x_3 + x_2 x_4 = 0 \text{ for all } x_i \in \{1, -1\}, 1 \leq i \leq 4.
\end{align*}
\]

**Theorem 3.1.** Let the symbols be given in Construction 1. Then the pair \((p, q)\) is a ZCP with ZCZ width \( Z \).

**Proof.** For \( 0 < \tau \leq N - 1 \), we first compute the aperiodic autocorrelation function of \( p \).

\[
\begin{align*}
    \rho_p(\tau) &= x_1^2 \rho_a(\tau) + x_1 x_2 \rho_{a,b}^*(N - \tau) + x_2^2 \rho_b(\tau) + x_2 x_3 \rho_{a,b}^*(N - \tau) + x_3^2 \rho_{b,a}(\tau) \\
    &\quad + x_3 x_4 \rho_{b,a}^*(N - \tau) + x_4^2 \rho_{a,b}(\tau) \\
    &= \rho_a(\tau) + x_1 x_2 \rho_{a,b}^*(N - \tau) + x_2 x_3 \rho_{a,b}^*(N - \tau) + \rho_b(\tau) \\
    &\quad + x_3 x_4 \rho_{b,a}^*(N - \tau) + \rho_{b,a}(\tau) \\
    &= 2(\rho_a(\tau) + \rho_{b,a}(\tau)) + x_1 x_2 x_3 x_4 \rho_{a,b}^*(N - \tau) + x_2 x_3 i \rho_{a,b}^*(N - \tau) \\
    &= 2(\rho_a(\tau) + \rho_{b,a}(\tau)) + x_2 x_3 i \rho_{a,b}^*(N - \tau).
\end{align*}
\]

Similarly, one gets

\[
\rho_q(\tau) = 2(\rho_c(\tau) + \rho_{d,a}(\tau)) + x_2 x_3 i \rho_{c,d}^*(N - \tau).
\]

By Fact 1, combining (9) and (10), we have

\[
\rho_p(\tau) + \rho_q(\tau) = \begin{cases} 
    0, & 0 < \tau \leq Z - 1; \\
    4(\rho_a(\tau) + \rho_{b,a}(\tau)), & Z \leq \tau \leq N - 1.
\end{cases}
\]

The proof of this theorem is done.

Furthermore, setting the ZCZ width \( Z = N + 1 \), similar to the proof of Theorem 3.1, we have the following result.

**Corollary 1.** Let a pair \((a, b)\) be a GCP of length \( N \) and \((c, d) = (\hat{b}^*, -\hat{a}^*)\). Then the pair \((p, q)\) is a GCP of length \( 4N \).
Proof. According to Theorem 3.1, \( \rho_p(\tau) + \rho_q(\tau) = 0 \) holds, for \( 0 < \tau \leq N - 1 \). We divide the remainder of the proof into three cases as follows:

Case 1: \( N \leq \tau \leq 2N - 1 \). Then we have

\[
\rho_p(\tau) = x_1x_2\rho_{a,b}(\tau - N) + x_1x_3\rho_{a,a}^*(2N - \tau) + x_2x_3\rho_{b,a}(\tau - N) + x_2x_4\rho_{i,b,ib}(\tau - N) + x_3x_4\rho_{a,ia}(\tau - N) = x_1x_2\rho_{a,b}(\tau - N) + x_1x_3\rho_{a,a}^*(2N - \tau) - x_2x_3\rho_{b,a}(\tau - N) + x_2x_4\rho_{i,b,ib}(\tau - N) + x_3x_4\rho_{a,ia}(\tau - N).
\]

Case 2: \( 2N \leq \tau \leq 3N - 1 \). Then we have

\[
\rho_p(\tau) = x_1x_3\rho_{a,a}^*(\tau - 2N) + x_1x_4\rho_{b,a}^*(3N - \tau) + x_2x_4\rho_{b,b}(\tau - 2N) = -x_1x_3\rho_{a,a}(\tau - 2N) + x_1x_4\rho_{b,a}^*(3N - \tau) - x_2x_4\rho_{b,b}(\tau - 2N) = -x_1x_3\rho_{a,a}(\tau - 2N) + x_1x_4\rho_{b,a}^*(3N - \tau) - x_2x_4\rho_{b,b}(\tau - 2N).
\]

Case 3: \( 3N \leq \tau \leq 4N - 1 \). Then we have

\[
\rho_p(\tau) = x_1x_4\rho_{a,b}(\tau - 3N) = -x_1x_4\rho_{a,b}(\tau - 3N).
\]

Combining all cases, we have

\[
(11) \quad \rho_p(\tau) = \begin{cases} 
  x_1x_3\rho_{a,a}^*(2N - \tau) + x_2x_4\rho_{i,b,ib}^*(2N - \tau) - x_2x_3\rho_{b,a}(\tau - N), & N \leq \tau < 2N; \\
  -x_1x_3\rho_{a,a}(\tau - 2N) + x_1x_4\rho_{b,a}^*(3N - \tau) - x_2x_4\rho_{b,b}(\tau - 2N), & 2N \leq \tau < 3N; \\
  -x_1x_4\rho_{a,b}(\tau - 3N), & 3N \leq \tau < 4N. 
\end{cases}
\]

Similarly, we get

\[
(12) \quad \rho_q(\tau) = \begin{cases} 
  x_1x_3\rho_{c,c}^*(2N - \tau) + x_2x_4\rho_{d,d}^*(2N - \tau) - x_2x_3\rho_{a,a}(\tau - N), & N \leq \tau < 2N; \\
  -x_1x_3\rho_{c,c}(\tau - 2N) + x_1x_4\rho_{d,d}^*(3N - \tau) - x_2x_4\rho_{d,d}(\tau - 2N), & 2N \leq \tau < 3N; \\
  -x_1x_4\rho_{c,c}(\tau - 3N), & 3N \leq \tau < 4N. 
\end{cases}
\]

Then one has

\[
(13) \quad \rho_p(\tau) + \rho_q(\tau) = \begin{cases} 
  x_1x_3\rho_{a,a}^*(2N - \tau) + \rho_{c,c}^*(2N - \tau) + x_2x_4\rho_{i,b,ib}^*(2N - \tau) + x_2x_3\rho_{b,a}(\tau - N) + \rho_{d,d}^*(\tau - N), & N \leq \tau < 2N; \\
  -x_1x_3\rho_{a,a}(\tau - 2N) + \rho_{c,c}(\tau - 2N) + x_1x_4\rho_{b,a}^*(3N - \tau) + x_2x_4\rho_{b,b}(\tau - 2N) + \rho_{d,d}(\tau - 2N), & 2N \leq \tau < 3N; \\
  -x_1x_3\rho_{c,c}(\tau - 3N) + \rho_{c,c}(\tau - 3N) + x_1x_4\rho_{a,a}(\tau - 3N) + \rho_{d,d}(\tau - 3N), & 3N \leq \tau < 4N. 
\end{cases}
\]

It follows that \( \rho_p(\tau) + \rho_q(\tau) = 0 \) holds for all \( N \leq \tau < 4N \) from (13).

The proof is finished. \( \square \)

In what follows, we further investigate the property of odd-periodic correlation of the pair \((p, q)\) in Construction 1, and obtain the following result.
Theorem 3.2. Let the symbols be given in Construction 1. Then the pair \((p, q)\) is a ZOC-ZCP with ZCZ width \(Z\) and ZOCZ width \(Z\). That is, the pair \((p, q)\) is a \((4N, Z, Z)\)-ZCP.

Proof. According to Theorem 3.1, it is shown that \((p, q)\) is a ZCP of length \(4N\) with ZCZ width \(Z\). Then we divide the calculation of the odd-periodic correlation of the pair \((p, q)\) in the following three cases.

Case 1. In this case, we calculate the odd-periodic autocorrelation function of \(p\) and \(q\), respectively.

For \(0 < \tau \leq N - 1\), we have

\[
\tilde{R}_p(\tau) = x_1^2 \rho_a(\tau) + x_1 x_2 \rho_{b,a}(N - \tau) + x_2^2 \rho_b(\tau) + x_2 x_3 \rho_{c,b}(N - \tau) + x_3^2 \rho_a(\tau) \\
+ x_3 x_4 \rho_{d,a}(N - \tau) + x_4^2 \rho_{d,c}(\tau - N) - x_1 x_4 \rho_{a,b}(N - \tau) \\
= \rho_a(\tau) + x_1 x_2 \rho_{b,a}(N - \tau) + \rho_b(\tau) + x_2 x_3 \rho_{c,b}(N - \tau) + \rho_a(\tau) \\
+ x_3 x_4 \rho_{d,a}(N - \tau) + \rho_{c,b}(\tau) + x_1 x_4 \rho_{a,b}(N - \tau) \\
= 2(\rho_a(\tau) + \rho_b(\tau)) + (x_1 x_2 + x_3 x_4) \rho_{b,a}(N - \tau) + (x_2 x_3 + x_1 x_4) \rho_{a,b}(N - \tau) \\
= 2(\rho_a(\tau) + \rho_b(\tau)).
\]

Thus we have

\[
\tilde{R}_p(\tau) = \begin{cases} 
0, & 0 < \tau \leq Z - 1; \\
2(\rho_a(Z) + \rho_b(Z)), & \tau = Z.
\end{cases}
\]

Similarly, one has

\[
\tilde{R}_q(\tau) = \begin{cases} 
0, & 0 \leq \tau \leq Z - 1; \\
2(\rho_c(Z) + \rho_d(Z)), & \tau = Z.
\end{cases}
\]

Case 2. In this case, we will show \(\tilde{R}_{p,q}(\tau) = 0\) for \(0 \leq \tau \leq N\) and \(\tilde{R}_{p,q}(\tau) \neq 0\) for \(N + 1 \leq \tau \leq 2N - 1\).

For \(0 \leq \tau \leq N - 1\), it follows that

\[
\tilde{R}_{p,q}(\tau) = x_1^2 \rho_{a,c}(\tau) + x_1 x_2 x_3^2 \rho_{d,a}(N - \tau) + x_2^2 \rho_{d,b}(\tau) + x_2 x_3 \rho_{c,b}(N - \tau) \\
+ x_3^2 \rho_{a,c}(\tau) + x_3 x_4 \rho_{d,c}(N - \tau) + x_4^2 \rho_{d,b}(\tau) - x_1 x_4 \rho_{a,b}(N - \tau) \\
= 2(\rho_{a,c}(\tau) + \rho_{b,d}(\tau)) + (x_1 x_2 + x_3 x_4) \rho_{d,a}(N - \tau) \\
+ (x_2 x_3 + x_1 x_4) \rho_{a,b}(N - \tau) \\
= 2(\rho_{a,c}(\tau) + \rho_{b,d}(\tau))
= 0,
\]

where the last equality holds due to Fact 1.

And for \(\tau = N\), we have

\[
\tilde{R}_{p,q}(\tau) = x_1 x_2 \rho_{a,d}(\tau - N) - x_2 x_3 i \rho_{b,c}(\tau - N) + x_3 x_4 \rho_{a,d}(\tau - N) \\
- x_1 x_4 i \rho_{a,b}(\tau - N) \\
= (x_1 x_2 + x_3 x_4) \rho_{a,d}(\tau - N) - (x_1 x_4 + x_2 x_3) i \rho_{b,c}(\tau - N)
= 0.
\]

On the other hand, for \(N < \tau \leq 2N - 1\), one gets
\[ \hat{R}_{p,q}(\tau) = x_1x_2\rho_{a,d}(\tau - N) + x_1x_3\rho_{c,a}(2N - \tau) - x_2x_3\rho_{b,c}(\tau - N) \\
+ x_2x_4\rho_{d,b}^*(2N - \tau) \\
+ x_3x_4\rho_{a,d}(\tau - N) + x_1x_3\rho_{c,a}^*(2N - \tau) - x_1x_4\rho_{b,c}(\tau - N) \\
+ x_2x_4\rho_{d,b}^*(2N - \tau) \\
= 2x_1x_3\rho_{c,a}(2N - \tau) + 2x_2x_4\rho_{d,b}^*(2N - \tau) \\
= 4x_1x_3\rho_{c,a}^*(2N - \tau). \]

Remark 1. As far as we know, most studies of the known ZCPs are only on the aperiodic correlation properties of ZCPs, and no study of the odd-periodic property of ZCP is reported in the literature yet. This motivates us to study this subject. Based on two ZCPs of length \( N \) and ZCP width \( Z \), in which one pair is a specific Z-complementary mate of the other pair, Theorem 3.2 shows that the value of ZOCZ width of the sequence pair constructed in (7) is identical to the ZCP width of used ZCPs. In Table 1, we list the parameters of the resultant ZCPs by Construction 1 from the ZCPs available in the literature [20, 21, 37].

Table 1. The Parameters of Some ZCPs by Construction 1

| Sequence Length | ZOCZ width | ZOCZ width |
|-----------------|------------|------------|
| \( 4(2^{m+1} + 2^m) \) | \( 2^{m+1} \) | \( 2^{m+1} \) |
| \( 4(2^m + 1) \) | \( 2^{m-1} + 1 \) | \( 2^{m-1} + 1 \) |
| \( 4(2^m - 1) \) | \( 2^{m-1} \) | \( 2^{m-1} \) |
| \( 4(2^{m+3} + 2^{m+2} + 2^m) \) | \( 2^{m+3} \) | \( 2^{m+3} \) |
| \( 56 \cdot 2^a 10^b 2^c 6^d \) | \( 12 \cdot 2^a 10^b 2^c 6^d \) | \( 12 \cdot 2^a 10^b 2^c 6^d \) |
| \( 48 \cdot 2^a 10^b 2^c 6^d \) | \( 10 \cdot 2^a 10^b 2^c 6^d \) | \( 10 \cdot 2^a 10^b 2^c 6^d \) |

In the following, we show some illustrative examples.

Example 1. Let \((a, b)\) be a binary ZCP of length 24 given by

\[ a = (1, 1, 1, -1, 1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1), \]
\[ b = (1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1). \]

And \((c, d) = (\overline{b}^*, -\overline{a}^*)\). Choose \( x_1 = x_2 = x_3 = 1, x_4 = -1 \). A quaternary sequence pair \((p, q)\) in (7) is given by

\[ p = a \| b \parallel ia \parallel -ib, \]
\[ q = c \parallel d \parallel ic \parallel -id. \]

Then \((p, q)\) is a ZCP of length 96 with ZOCZ width \( Z_{odd} = 20 \), because

\[ (\rho_p(\tau) + \rho_q(\tau)) \bigg|_{\tau = 0}^{95} = (192, 0_{19}, 32, 0_{75}); \]
\[ (\hat{R}_p(\tau)) \bigg|_{\tau = 0}^{95} = (96, 0_{19}, 16, 0, 0, 0, 0, 4i, 0, -4i, 0, -4i, 0, 4i, 0, -20i, 0, 4i, 0, -4i, 0, -4i, 0, -4i, 0, 4i, 0, -20i, 0, 4i, 0, -4i, 0, -4i, 0, 4i, 0, 0, 0, 0, -16, 0_{19}); \]
(17) \[
\left( \hat{R_q}(\tau) \right)_{\tau=0}^{95} = (96, 0_{19}, 16, 0, 0, 0, -4i, 0, 4i, 0, 4i, 0, -4i, 0, 20i, 0, -4i, 0, 4i, 0, 12i, 0, 20i, 0, -4i, 0, 12i, 0, -4i, 0, 20i, 0, -4i, 0, 4i, 0, 4i, 0, 4i, 0, 4i, 0, 4i, 0, 20i, 0, -4i, 0, 12i, 0, 20i, 0, -4i, 0, 12i, 0, 20i, 0, -4i, 0, -4i, 0, 20i, 0, -4i, 0, -4i, 0, -4i, 0, -16, 0_{19});
\]
(\hat{R_{p,q}}(\tau))_{\tau=0}^{95} = (0_{25}, 4i, 0, -12i, 0, -12i, 0, 20i, 0, 4i, 0, -28i, 0, 4i, 0, 4i, 0, 36i, 0, -28i, 0, 4i, 0, -28i, 0, -20i, 0, 28i, 0, 12i, 0, -4i, 0, 28i, 0, 12i, 0, -4i, 0, -4i, 0, -4i, 0, -4i, 0, -4i, 0, -20i, 0, -4i, 0, -4i, 0, -4i, 0, -4i, 0, -16, 0_{24}).

The odd-periodic correlation magnitudes of (96, 20)-ZCP \((p, q)\) are shown in FIGURE 1.

![Figure 1](image_url)

FIGURE 1. The odd-periodic correlation magnitudes of ZCP in Example 1.

Let \(Z = N + 1\), then the following result is straightforward from Corollary 1 and Theorem 3.2 as a special case.

**Corollary 2.** Let \((a, b)\) be a GCP of length \(N\) and \((c, d) = (\overrightarrow{b^*}, -\overrightarrow{a^*})\). Then the quaternary sequence pair \((p, q)\) is a GCP of length \(4N\) with ZOCZ width \(N + 1\).

**Remark 2.** In [13], Hu et al. gave a construction of \((2^k; 2^m; 2^{\pi_1(2)-1})\)-Golay-ZOCZ sequence sets with flexible ZOCZ width, where the length of the sequence is limited to a power-of-two. When \(\pi_1(2) = m - 1\), the width of ZOCZ is one-fourth of the sequence length. Whereas, Corollary 2 gives a construction of GCPs of length \(4N\)
with ZOCZ width $Z_{odd} = N + 1$, where $N$ is a positive integer, and the ZOCZ width is approximately a quarter of the sequence length. Thus, our construction of $(4N, N + 1)$--GCP can not be covered by those in [13].

**Remark 3.** Compared with the known GCPs, most of which depend on Boolean functions, the proposed GCP in Construction 1 is independent of Boolean functions.

**Example 2.** Let $a = (1, 1, -1)$, $b = (1, i, 1)$ be a quadruphase GCP of length 3. Set $x_1 = x_2 = x_3 = 1, x_4 = -1$. A quaternary sequence pair $(p, q)$ in (7) is given by

\[ p = a \parallel i a \parallel -ib = (1, 1, -1, 1, i, i, -i, -i, 1, i), \]
\[ q = c \parallel d \parallel ic \parallel -id = (1, -i, 1, 1, -1, 1, i, i, -i, i, i). \]

Then $(p, q)$ is a GCP of length 12 with $Z_{odd} = 4$, because

\[ \hat{R}_p(\tau)_{\tau=0}^{11} = (12, 0, 0, 0, 4i, 0, 0, 4i, 0, 0, 0), \]
\[ \hat{R}_q(\tau)_{\tau=0}^{11} = (12, 0, 0, 0, -4i, 0, 0, 0, -4i, 0, 0, 0), \]
\[ \hat{R}_{p,q}(\tau)_{\tau=0}^{11} = (0, 0, 0, 0, 4i, -4 - 4i, 4 - 4i, -4i, 0, 0, 0). \]

**Figure 2.** The odd-periodic correlation magnitudes of a GCP in Example 2.
4. Concluding remarks

In this paper, we made two contributions to Z-complementary pairs. Firstly, we proposed a new class of ZCPs constructed by horizontal concatenation of a certain linear combination of some known ZCPs, which have ZCZ width Z and ZOCZ width $Z_{\text{odd}} = Z$. Secondly, the proposed construction is independent of Boolean functions and can be also utilized to generate more GCPs of length $4N$ with flexible parameters which have ZOCZ width $Z_{\text{odd}} = N + 1$ and are reported in the literature for the first time. It would be interesting to find more ZCPs with new parameters.

ACKNOWLEDGMENTS

The authors are very grateful to the reviewers and the Editor for their valuable comments and suggestions that improved the presentation and quality of this paper. The first author is grateful to her supervisor, Prof. Zhongchun Zhou, for his patient guidance and helpful discussion. This work was supported in part by the National Science Foundation of China under Grant 62071397.

REFERENCES

[1] A. R. Adhikary, S. Majhi, Z. L. Liu and L. G. Yong, New sets of even-length binary Z-complementary pairs with asymptotic ZCZ ratio of 3/4, IEEE Signal Processing Letters, 25 (2018), 970–973.
[2] A. R. Adhikary, Z. Zhou, Y. Yang and P. Fan, Constructions of cross Z-complementary pairs with new lengths, IEEE Transactions on Signal Processing, 68 (2020), 4700–4712.
[3] C.-Y. Chen, A novel construction of Z-complementary pairs based on generalized boolean functions, IEEE Signal Processing Letters, 24 (2017), 987–990.
[4] C.-Y. Chen and S.-W. Wu, Golay complementary sequence sets with large zero correlation zones, IEEE Transactions on Communications, 66 (2018), 5097–5204.
[5] L. Chen, Z. Luo, X. Cheng and S. Li, Golay sequence based time-domain compensation of frequency-dependent I/Q imbalance, Communications China, 11 (2014), 1–11.
[6] J. D. Coker and A. H. Tewfik, Simplified ranging systems using discrete wavelet decomposition, IEEE Transactions on Signal Processing, 58 (2010), 575–582.
[7] P. Z. Fan, N. Suehiro, N. Kuroyanagi and X. M. Deng, Class of binary sequences with zero correlation zone, Electronics Letters, 35 (1999), 777–779.
[8] P. Fan, W. Yuan and Y. Tu, Z-complementary binary sequences, IEEE Signal Processing Letters, 14 (2007), 509–512.
[9] M. J. E. Golay, Complementary series, IRE Transactions on Information Theory, IT-7 (1961), 82–87.
[10] G. Gong, F. Huo and Y. Yang, Large zero correlation zone of Golay pairs and QAM Golay pairs, Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on Information Theory, (2013), 3135–3139.
[11] G. Gong, F. Huo and Y. Yang, Large zero autocorrelation zones of Golay sequences and their applications, IEEE Transactions on Communications, 61 (2013), 3967–3979.
[12] Z. Gu, Y. Yang and Z. Zhou, New sets of even-length binary Z-complementary pairs, 2019 Ninth International Workshop on Signal Design and its Applications in Communications, (2019), 1–5.
[13] T. Hu, Y. Yang and Z. Zhou, Golay complementary sets with large zero odd-periodic correlation zones, Advances in Mathematics of Communications, 15 (2019), 23–33.
[14] K. M. Z. Islam, T. Y. Al-Naffouri and N. Al-Dhahir, On optimum pilot design for comb-type OFDM transmission over doubly-selective channels, IEEE Transactions on Communications, 59 (2011), 930–935.
[15] J. Jedwab and M. G. Parker, Golay complementary array pairs, Designs Codes Cryptography, 44 (2007), 209–216.
[16] H. L. Jin, G. D. Liang, C. Q. Xu and J. B. Zhang, The necessary condition of families of odd-periodic perfect complementary sequence pairs, Second International Workshop on Education Technology Computer Science, 2 (2009), 303–307.
[17] L. Kai, L. Hao and S. Yan, Construction of 8-QAM+ odd-periodic complementary sequences and sequence sets with zero correlation zone, *Electronics Communications and Networks IV*, 2015.

[18] X. Li, P. Fan, X. Tang and Y. Tu, Existence of binary Z-complementary pairs, *IEEE Signal Processing Letters*, 18 (2010), 63–66.

[19] Z. Liu, Y. Guan and U. Parampalli, On optimal binary Z-complementary pair of odd period, *IEEE International Symposium on Information Theory*, (2013), 3125–3129.

[20] Z. Liu, U. Parampalli and Y. L. Guan, On even-period binary Z-complementary pairs with large ZCZs, *IEEE Signal Processing Letters*, 21 (2014), 284–287.

[21] Z. Liu, U. Parampalli and Y. L. Guan, Optimal odd-length binary Z-complementary pairs, *IEEE Transactions on Information Theory*, 60 (2014), 5768–5781.

[22] H. D. Lüke and H. D. Schotten, Odd-perfect, almost binary correlation sequences, *IEEE Transactions on Aerospace and Electronic Systems*, 31 (1995), 495–498.

[23] H. D. Lüke, Binary odd-periodic complementary sequences, *IEEE Transactions on Information Theory*, 43 (1997), 365–367.

[24] V. Nee and D. J. R, OFDM codes for peak-to-average power reduction and error correction, *Global Telecommunications Conference*, (1996), 740–744.

[25] K. G. Paterson, Generalized Reed-Muller codes and power control in OFDM modulation, *IEEE Transactions on Information Theory*, 46 (2000), 104–120.

[26] A. Pezeshki, A. R. Calderbank, W. Moran and S. D. Howard, Doppler resilient Golay complementary waveforms, *IEEE Transactions on Information Theory*, 54 (2008), 4254–4266.

[27] M. Pursley, Performance evaluation for phase-coded spread-spectrum multiple-access communication—Part I: System analysis, *IEEE Transactions Communications*, 25 (1977), 795–799.

[28] M. B. Pursley and P. Hall, *Introduction to Digital Communications: United States Edition*, Pearson Schw Ag, 2004.

[29] P. Spasojevic and C. N. Georgiades, Complementary sequences for ISI channel estimation, *IEEE Transactions on Information Theory*, 47 (2011), 1145–1152.

[30] E. Spano and O. Ghebrebrhan, Complementary sequences with high sidelobe suppression factors for ST/MST radar applications, *IEEE Transactions on Geoscience Remote Sensing*, 34 (1996), 317–329.

[31] D. V. Sarwate and M. B. Pursley, Crosscorrelation properties of pseudorandom and related sequences, *Proceedings of the IEEE*, 68 (1980), 593–619.

[32] S. Wang and A. Abdi, MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences, *IEEE Transactions on Vehicular Technology*, 56 (2007), 3024–3039.

[33] J.-B. Wang, X.-X. Xie, Y. Jiao, X. Song, X. Zhao, M. Gu and M. Sheng, Optimal odd-periodic complementary sequences for diffuse wireless optical communications, *Optical Engineering*, 51 (2012), 095002.

[34] H. Wen, F. Hu and F. Jin, Design of odd-periodic complementary binary signal set, *Computers and Communications, 2004. Proceedings. ISCC 2004. Ninth International Symposium*, 2 (2004), 590–593.

[35] K. K. Wong and T. O’Farrell, Application of complementary sequences in indoor wireless infrared communications, *IEEE Proceedings Optoelectronics*, 150 (2003), 453–464.

[36] D. Wulich, Reduction of peak to mean ratio of multicarrier modulation using cyclic coding, *Electronics Letters*, 32 (1996), 432–432.

[37] C. Xie and Y. Sun, Constructions of even-period binary Z-complementary pairs with large ZCZs, *IEEE Signal Processing Letters*, 25 (2018), 1141–1145.

[38] J. D. Yang, X. Jin, K. Y. Song, J. S. No and D. J. Shin, Multicode MIMO systems with quaternary LCZ and ZCZ sequences, *IEEE Transactions on Vehicular Technology*, 57 (2008), 234–2341.

Received January 2021; revised May 2021; early access August 2021.

E-mail address: 970295503@qq.com
E-mail address: renwenli80@163.com
E-mail address: wangyonga@swjtu.edu.cn
E-mail address: tangchunmingmath@163.com