On discrete Gibbs measure approximation to runs

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ABSTRACT
A Stein operator for the runs is derived as a perturbation of an operator for discrete Gibbs measure. Due to this fact, using perturbation technique, the approximation results for runs arising from identical and non-identical Bernoulli trials are derived via Stein’s method. The bounds obtained are new and their importance is demonstrated through an interesting application.

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1. Introduction

Runs and patterns is an important topic in the areas related to probability and statistics, such as reliability theory, meteorology and agriculture, statistical testing and quality control among many others (see Balakrishnan and Koutras (2002), Upadhye and Kumar (2018) and Dafnis, Antzoulakos, and Philippou (2010) for more details). The research on this topic started with runs related to success/failure (see Philippou, Georgiou, and Philippou 1983; Philippou and Makri 1986) and series of articles later followed in this area, see Aki (1997), Aki, Kuboki, and Hirano (1984), Antzoulakos, Bersimis, and Koutras (2003), Antzoulakos and Chadjiconstantinidis (2001), Balakrishnan and Koutras (2002), Makri, Philippou, and Psillakis (2007) and references therein. Huang and Tsai (1991) and Dafnis, Antzoulakos, and Philippou (2010) extended the study by considering failures and successes together which is known as \((k_1, k_2)\)-runs or modified distribution of order \(k\). In this paper, we consider the approximation problem related to \((k_1, k_2)\)-runs.

Let \(\eta_1, \eta_2, \ldots, \eta_n\) be a finite sequence of independent Bernoulli trials with \(\mathbb{P}(\eta_i = 1) = p_i = 1 - q_i = \mathbb{P}(\eta_i = 0)\). Define, \(\eta_0 = 1\).
\[ \mathbb{I}_l := \eta_{l-1} \left[ \prod_{i=1}^{k_1} (1 - \eta_{l+i-1}) \right] \left( \prod_{j=1}^{k_2} \eta_{l+k_1+j-1} \right) (1 - \eta_{l+k_1+k_2}) , \]

(1)

Then, \( M_{k_1,k_2}^n \) denotes the number of occurrences of exactly \( k_1 \) consecutive failures followed by exactly \( k_2 \) consecutive successes in \( n \) trials with \( k_1, k_2 \geq 1 \). We denote \( M_{k_1,k_2}^n = S_{k_1,k_2}^n \) if \( p_i = p \) (i.e., identical Bernoulli trials). The distributional properties, such as probability generating function (PGF), probability mass function (PMF) and moments, are studied by Dafnis, Antzoulakos, and Philippou (2010) for \( S_{k_1,k_2}^n \) and are intractable for \( M_{k_1,k_2}^n \), in general.

Probabilistic approximations play a crucial role in understanding and comparing the behavior of the distributions. Therefore, it is suggested to approximate the distribution of \( M_{k_1,k_2}^n \) to the set of some well-known distributions so that the distribution of \( M_{k_1,k_2}^n \) can be characterized using these distributions at the cost of error in approximation. Recently, approximation problems, such as Poisson approximation (Vellaisamy 2004), binomial convoluted Poisson approximation (Upadhye, Čekanavičius, and Vellaisamy 2017), pseudo-binomial approximation and negative binomial approximation (Kumar and Upadhye 2017; Upadhye and Kumar 2018), related to runs are studied in the literature.

In this paper, we study three approximation problems related to the distribution of \( M_{k_1,k_2}^n \), namely, discrete Gibbs measure (DGM) approximation to \( S_{k_1,k_2}^n \), \( S_{k_1,k_2}^n \) approximation to \( M_{k_1,k_2}^n \) and DGM approximation to \( M_{k_1,k_2}^n \), using Stein’s method. It is important to mention that Stein’s method is one of the most powerful tool to study probabilistic approximations, due to its applicability for dependent setups. However, the application of Stein’s method is heavily dependent on finding a suitable Stein operator. There have been several approaches for identifying a suitable operator (see Stein 1972; Barbour 1990; Götze 1991; Diaconis and Zabell 1991). Recently, an interesting approach is presented by Upadhye, Čekanavičius, and Vellaisamy (2017), that uses PGF to obtain a Stein operator. This approach is most suitable for the distributions that arise out of dependent setups. We use PGF approach to derive Stein operator for DGM and the distribution of \( S_{k_1,k_2}^n \). Note that DGM is a general class of distributions which include Poisson, binomial, geometric, negative binomial and logarithmic series distribution among many other distributions. To the best of our knowledge, these approximation results are new and not studied in the literature. So it is worth exploring this problem, as it may benefit the readers to understand the properties of the distribution of \( M_{k_1,k_2}^n \).

This paper is organized as follows. In Section 2, we discuss some known results related to Stein’s method along with some notations. In Section 3, we present new error bounds for total variation distance between \( S_{k_1,k_2}^n, M_{k_1,k_2}^n \) and DGM, and also obtain Poisson, pseudo-binomial and negative binomial approximation results as corollary to main results. In Section 4, we show the importance of approximation results to two-stage startup demonstration testing. In Section 5, we derive some useful results related to the distributional properties of \( S_{k_1,k_2}^n \). Finally, in Section 6, we derive the proofs for the results presented in Section 3.
2. Preliminaries and notations

In this section, we discuss some known results related to Stein’s method and also define notations to simplify presentation of the paper.

Let $X$ and $Y$ be any two random variables, concentrated on $\mathbb{Z}$ (the set of integers), defined on some probability space. Suppose that $X$ has well-known distribution and the distribution of $Y$ is intractable, then Stein’s method can be used to study $X$-approximation to $Y$. The following are the important steps involved in this approximation.

1. We obtain a Stein operator (denoted by $A_X$ for a random variable $X$) which acts on a large class of functions $G_X$ such that

$$\mathbb{E}[A_X g(X)] = 0, \quad \text{for } g \in G_X$$

where $G_X = \{g : g \in G \text{ such that } g(0) = 0 \text{ and } g(x) = 0, \text{ for } x \notin \text{Supp}(X)\}$, $G$ be the set of all bounded functions and $\text{Supp}(X)$ denotes the support of a random variable $X$. We obtain the solution to Stein equation

$$A_X g(m) = f(m) - \mathbb{E}f(X), \quad m \in \mathbb{Z} \text{ and } f \in G$$

(2)

3. We replace $m$ with a random variable $Y$ in Stein equation and take expectations and supremum to find

$$d_{TV}(X, Y) := \sup_{f \in J} \left| \mathbb{E}f(X) - \mathbb{E}f(Y) \right| = \sup_{f \in J} \left| \mathbb{E}[A_X g(Y)] \right|$$

where $J = \{1(A) | A \subseteq \mathbb{Z}\}$ and $1(A)$ is the indicator function of the set $A$.

For more details, see Barbour (1990), Barbour and Chen (2014), Barbour, Čekanavičius, and Xia (2007), Barbour, Chen, et al. (1992), Barbour, Holst, et al. (1992), Čekanavičius (2016), Chen, Goldstein, and Shao (2011), Eichelsbacher and Reinert (2008), Nourdin and Peccati (2012), Ley, Reinert, and Swan (2017), Reinert (2005) and references therein.

Throughout this paper, let $X_1$, $X_2$ and $X_3$ have Poisson (with parameter $\lambda$), pseudo-binomial (with parameter $\tilde{\alpha}$ and $\hat{p}$) and negative binomial (with parameter $\tilde{\alpha}$ and $\hat{p}$) distribution, respectively, with PMFs

$$\mathbb{P}(X_1 = m) = \frac{e^{-\lambda} \lambda^m}{m!}, \quad m = 0, 1, 2...$$

(3)

$$\mathbb{P}(X_2 = m) = \frac{1}{R} \binom{\tilde{\alpha}}{m} \hat{p}^m \hat{q}^{\tilde{\alpha}-m}, \quad m = 0, 1, 2, ..., \lfloor \tilde{\alpha} \rfloor$$

(4)

$$\mathbb{P}(X_3 = m) = \binom{\tilde{\alpha} + m - 1}{m} \hat{p} \tilde{\alpha} \hat{q}^m, \quad m = 0, 1,...$$

(5)

where $\lambda, \tilde{\alpha}, \hat{\alpha} > 0$ and $0 < \hat{p}, \hat{\alpha} < 1$ with $\hat{q} = 1 - \hat{p}, \tilde{\alpha} = 1 - \hat{\alpha}, \lfloor \tilde{\alpha} \rfloor$ is the greatest integer part of $\tilde{\alpha}$ and $R = \sum_{m=0}^{\lfloor \tilde{\alpha} \rfloor} \binom{\tilde{\alpha}}{m} \hat{p}^m \hat{q}^{\tilde{\alpha}-m}$. From (34) of Upadhye, Čekanavičius, and Vellaisamy (2017), the bounds for the solution to the Stein equation for Poisson, pseudo-binomial and negative binomial distributions, respectively, are given by
\[
\|\Delta g\| \leq \frac{2\|f\|}{\max(1, \lambda)}, \quad \|\Delta g\| \leq \frac{2\|f\|}{\lambda\hat{p}\hat{q}} \quad \text{and} \quad \|\Delta g\| \leq \frac{2\|f\|}{\lambda q} \quad (6)
\]

where \(\|\Delta g\| := \sup_{j \in \mathbb{Z}_+} |\Delta g(j)|\) and \(\Delta g(j) = g(j + 1) - g(j)\).

Next, let \(Z_1, Z_2, Z_3\) be three random variables defined on common probability space and we are interested in studying \(Z_2\) approximation to \(Z_3\). The distribution and Stein equation for \(Z_3\) is well-known and the bounds are available in the literature. Suppose a Stein operator for \(Z_2\) can be derived and observed as a perturbation of operator for \(Z_1\). Then, the following result suggests the method to obtain error bounds between \(Z_2\) and \(Z_3\).

**Lemma 2.1.** [Lemma 3.1, Upadhye, Čekanavičius, and Vellaisamy 2014] Let \(Z_1\) be a random variable with support \(S\), Stein operator \(\mathcal{A}_{Z_1}\) and \(g_0\) be the solution to Stein equation (2) satisfying

\[
\|\Delta g_0\| \leq w_1\|f\|[\min(1, \gamma^{-1})]
\]

where \(w_1, \gamma > 0\). Also, let \(Z_2\) be a random variable with Stein operator \(\mathcal{A}_{Z_2} = \mathcal{A}_{Z_1} + U_1\) and \(Z_3\) be another random variable such that, for \(g \in \mathcal{G}_{Z_1} \cap \mathcal{G}_{Z_2}\),

\[
\|U_1g\| \leq w_2\|\Delta g\|, \quad \|\mathbb{E}\mathcal{A}_{Z_2}g(Z_3)\| \leq \varepsilon\|\Delta g\|
\]

where \(w_1w_2 < \gamma\). Then

\[
d_{TV}(Z_2, Z_3) \leq \frac{\gamma}{2(\gamma - w_1w_2)} (\varepsilon w_1\min(1, \gamma^{-1}) + 2\mathbb{P}(Z_2 \in S^c) + 2\mathbb{P}(Z_3 \in S^c))
\]

where \(S^c\) denote the complement of set \(S\).

For more details about these results, we refer the reader to Barbour, Čekanavičius, and Xia (2007), Čekanavičius and Roos (2004), Upadhye, Čekanavičius, and Vellaisamy (2017) and Vellaisamy, Upadhye, and Čekanavičius (2013) and references therein.

Next, we say that the distribution of the random variable \(Z\) belongs to DGM, a family of discrete distributions, if it has the PMF which can be represented as

\[
\mathbb{P}(Z = m) := \Lambda(m) = \frac{e^{U(m)}w^m}{m!}, \quad m \in S
\]

where \(S \equiv \text{Supp}(Z)\), \(w > 0\) is fixed, \(U : S \rightarrow \mathbb{R}\) be a function and \(\beta = \sum_{m \in S} \frac{e^{U(m)}w^m}{m!}\).

Next, we give some examples of well-known distributions that belong to DGM family.

**O1** If \(\beta = e^{U(b)}/b!\), \(w = 1\) and \(S = \{b\}\), where \(b\) is a constant then \(Z\) follows degenerate distribution.

**O2** If \(\beta = 1\), \(U(m) = -\lambda\), \(w = \lambda\) and \(S = \{0, 1, 2, \ldots\}\) then \(Z\) follows Poisson distribution with parameter \(\lambda\).

**O3** If \(\beta = e^\lambda - 1\), \(U(m) = 0\), \(w = \lambda\) and \(S = \{1, 2, 3, \ldots\}\) then \(Z\) follows zero-truncated Poisson distribution with parameter \(\lambda\).

**O4** If \(\beta = 1\), \(U(m) = \ln((pm)!), w = (1 - p)\) and \(S = \{0, 1, 2, \ldots\}\) then \(Z\) follows geometric distribution with parameter \(p\).
(O5) If \( \beta = 1 \), \( U(m) = \ln[n(n - 1) \cdots (n - m + 1)] + n \ln q, w = p/q \) and \( S = \{0, 1, 2, \ldots, n\} \) then \( Z \) follows binomial distribution with parameters \( n \in \mathbb{N} = \{1, 2, \ldots\} \) and \( p \).

(O6) If \( \beta = R, U(m) = \ln[n(n - 1) \cdots (n - m + 1)] + n \ln q, w = p/q \) and \( S = \{0, 1, 2, \ldots, |n|\} \) then \( Z \) follows pseudo-binomial distribution with parameters \( n \in (0, \infty) \) and \( p \) with \( R \) as a normalizing constant.

(O7) If \( \beta = 1, U(m) = \ln[n(n + 1) \cdots (n + m - 1)] + n \ln p, w = 1 - p \) and \( S = \{0, 1, 2, \ldots\} \) then \( Z \) follows negative binomial distribution with parameters \( \alpha \) and \( p \).

(O8) If \( \beta = -\ln(1 - p), U(m) = \ln(m - 1)! \), \( w = p \) and \( S = \{1, 2, 3, \ldots\} \) then \( Z \) follows logarithmic series distribution with parameter \( p \).

Note also that the representation is not unique, for example, other representation for Poisson distribution is \( w = \lambda, \ U(m) = 0, \ \beta = e^{\lambda} \).

Eichelsbacher and Reinert (2008) derived the following Stein operator for DGM using generator approach (Barbour 1990; Götze 1991).

\[
A_Z g(m) = w e^{U(m+1) - U(m)} g(m+1) - mg(m), \quad m \in S
\]  

(8)

where \( S = \{0, 1, 2, \ldots, N\} \) and \( N \) can take the value infinity. This operator can also be derived using PGF approach as follows:

The PGF of \( Z \), whenever exists, is given by

\[
G(t) = \sum_{m=0}^{N} \Lambda(m) t^m
\]  

(9)

Therefore,

\[
G'(t) = \sum_{m=0}^{N} m \Lambda(m) t^{m-1} = \sum_{m=0}^{N} (m+1) \Lambda(m+1) t^m = \sum_{m=0}^{N} w e^{U(m+1) - U(m)} \Lambda(m) t^m
\]  

(10)

Comparing the coefficients of \( t^m \), we get

\[
(m + 1) \Lambda(m + 1) = w e^{U(m+1) - U(m)} \Lambda(m).
\]  

(11)

Let \( g \in G_Z \), then

\[
\sum_{m=0}^{N} g(m+1)(m+1) \Lambda(m+1) = \sum_{m=0}^{N} g(m+1) \Lambda(m)
\]

(12)

This implies \( \mathbb{E}[A_Z g(Z)] = \sum_{m=0}^{N} [w e^{U(m+1) - U(m)} g(m+1) - mg(m)] \Lambda(m) = 0 \)

Hence, (8) follows.

**Remark 2.1.** Note that the relation (11) can be computed from (7) and therefore, PGF approach is not more relevant as the Stein operator can be computed directly.

Next, we introduce some notations to improve the readability of the paper. Define \( k := k_1 + k_2 \),

\[
a_1 := 1, \ a_2 := -1, \ a_3 := q p, \ d_1 = d_3 := n - k - 2, \ d_2 := n - k - 1
\]

\[
b_1(n) := n + 1, \ b_3(n) := n \quad \text{and} \quad b_2(n) := \begin{cases} -q(k + 2) & n = k + 1 \\ n + 1 - q & n \geq k + 2 \end{cases}
\]
Also, define

\[
\begin{aligned}
\c_i^{(n,k)} := \begin{cases}
(n - 2k - 2)(k + 1) + \frac{(k + 1)^{n-k+1}}{(k + 2)^{n-k-1}}, & i = 1 \\
n - 3k + k^{(k+1)}_{k+2} n^{-2k}, & i = 2 \\
n - 5k + k(nk + 6k + 4 - k^2) \frac{(k + 1)^{n-3k-1}}{(k + 2)^{n-3k+1}}, & i = 3 \\
n(3k + 1) - (11k^2 + 9k + 2) + (2nk + k^2 + 7k + 2) \frac{(k+1)^{n-2k}}{(k + 2)^{n-k-1}}, & i = 4 \\
n(2k + 3) - 6k^2 - 16k - 10 + (2n + 2k + 5) \frac{(k + 1)^{n-k}}{(k + 2)^{n-k-1}}, & i = 5
\end{cases}
\]

\[
\delta = 2 + \delta^*(k + 2), \quad \delta_1 = \delta^* \left(5k + 6 + \frac{3k^2 + 11k + 10}{2} \delta^* \right), \quad \delta_2 = \delta^* \left(3 + \frac{4k + 7}{2} \delta^* \right)
\]

with \(\delta^* = 1 + q + q^p, h_1(n,k,p) := (n - k)(4 + \delta_1) + \delta_2 c_{n,k}^{(1)} + \delta_2 c_{m,k}^{(2)} + \delta^* c_{n,k}^{(3)} + \delta^* c_{n,k}^{(4)}
\]

\[
+ \frac{\delta^*}{2} c_{n,k}^{(5)} \text{ and } h_2(n,k,p) := (n - k)\delta + \delta^* c_{n,k}^{(1)} + c_{n,k}^{(2)}
\]

(12)

3. Main results

In this section, we study approximation problem related to \(M_{k_1,k_2}^n, S_{k_1,k_2}^n\) and DGM using Stein’s method. We also show that the approximation results for Poisson, pseudo-binomial and negative binomial distributions follow as special cases. The proofs of the results are given in Section 6.

Note that, it is not possible to obtain a Stein operator for \(S_{k_1,k_2}^n,\) as a perturbation of an operator for DGM, in general, as Stein operator for DGM contains the term \(U(m)\) which is not known in general. However, if the following condition is satisfied

\[
\begin{array}{c}
\epsilon^{U(m+1)-U(m)} = a + bm
\end{array}
\] (13)

then a Stein operator can be derived. Also, observe that (13) is similar to a well-known Panjer’s recursion (see Panjer and Wang (1995) for more details) and a large class of distributions, for example, the cases \((O1)-(O8)\) among many others, satisfy (7) and (13).

Next, we first compute the mean and variance for the distribution of DGM, \(S_{k_1,k_2}^n\) and \(M_{k_1,k_2}^n,\) which can be used for the choice of parameters. From (9, 10) and (13), we get \(G'(t) = waG(t)/(1 - wbt).\) Therefore, the mean and variance of DGM \((Z\text{ as defined in (7)})\) is given by

\[
\begin{array}{c}
\mathbb{E}(Z) = G'(1) = \frac{wa}{1 - wb} \quad \text{and} \quad \text{Var}(Z) = G'(1) + G''(1) - (G'(1))^2 = \frac{wa}{(1 - wb)^2}
\end{array}
\]
Also, from (1), it is clear that

$$E(M_{k_1,k_2}^n) = \sum_{i=1}^{n-k} E(I_i) \quad \text{and} \quad \text{Var}(M_{k_1,k_2}^n) = \sum_{i=1}^{n-k} E(I_i) + \sum_{i=1}^{n-k} \sum_{j \neq i}^{n-k} \text{Var}(I_i I_j)$$

and, for $p_i = p, \ i = 1, 2, ..., n$, let $a(p) = q^i p^{k_2}$, then

$$E(S_{k_1,k_2}^n) = q [1 + (n - k - 1)p] a(p) \quad \text{and} \quad \text{Var}(S_{k_1,k_2}^n) = q [1 + (n - k - 1)p] a(p) - s_{n,k}$$

where $s_{n,k} = [(n(2k+3) - (3k+5)(k+1))q^2p^2 - 2(k+1)q^3 + (2n - 2k + 1)q^2 - 2(n - 2k) q]a(p)^2$.

Next, we present the approximation results between DGM and $S_{k_1,k_2}^n$. 

**Theorem 3.1.** Let $n \geq 5k, E(Z) = E(S_{k_1,k_2}^n)$ and $\phi = \text{Var}(Z) - \text{Var}(S_{k_1,k_2}^n)$. Then

$$d_{TV}(Z, S_{k_1,k_2}^n) \leq \| \Delta g \| \left\{ 2(2 + q\phi)a(p)^2 \{ |1 - wb|h_1(n,k,p)a(p) + |wb|h_2(n,k,p) \} ight\}$$

(14)

where $Z$ as defined in (7), and $h_1(n,k,p)$ and $h_2(n,k,p)$ as defined in (12).

**Remarks 3.1.**

i. Note that bounds obtained in (14) are of constant order and new to the best of our knowledge. Though the bound is of constant order, due to the presence of the term $(\frac{k+1}{k+2})^n$, the bounds decrease with increase in the value of $n$ (see Table 1).

ii. Observe that, if $\text{Var}(Z) = \text{Var}(S_{k_1,k_2}^n)$ (i.e., $\phi = 0$) then the bound become sharper, as expected.

iii. Note that, if $\text{Var}(Z) = \text{Var}(S_{k_1,k_2}^n)$ (i.e., $\phi = 0$) then the validity of the bound depends on admissibility of parameters. For example, $s_{n,k} > 0$ (mean larger than variance) and $s_{n,k} < 0$ (mean smaller than variance) pseudo-binomial and negative binomial approximations are valid, respectively. However, negative binomial (for $s_{n,k} > 0$) and pseudo-binomial (for $s_{n,k} < 0$) approximations are not valid as these conditions yield inadmissible parameters.

Next, we present the results for Poisson, pseudo-binomial and negative binomial distributions and the proofs follow directly from (O2), (O6), and (O7), and (6).
Corollary 3.1. Assume the conditions of Theorem 3.1 hold. Then, we have the following results

i. \[d_{TV}(X_1, S^n_{k_1, k_2}) \leq \frac{1}{\max(1, \gamma)} \left\{ 2(2 + qp)h_1(n, k, p)a(p)^3 + \varphi \right\}, \text{ for } X_1 \text{ defined in (3).} \]

ii. \[d_{TV}(X_2, S^n_{k_1, k_2}) \leq \frac{1}{\gamma q} \left\{ 2(2 + qp)a(p)^3 \left( h_1(n, k, p)a(p) + \hat{p}h_2(n, k, p) \right) + \varphi \right\}, \text{ for } X_2 \text{ defined in (4).} \]

iii. \[d_{TV}(X_3, S^n_{k_1, k_2}) \leq \frac{1}{\gamma q} \left\{ 2(2 + qp)a(p)^3 \left( \hat{p}h_1(n, k, p)a(p) + \hat{q}h_2(n, k, p) + \hat{p}\varphi \right) \right\}, \text{ for } X_3 \text{ defined in (5).} \]

Remark 3.1. In a similar spirit, the approximation results for various distributions satisfying (7) and (13) can be derived.

Next, we present the approximation results for the distribution of \(M^n_{k_1, k_2}, S^n_{k_1, k_2}\), and \(Z\).

Theorem 3.2. Let \(\bar{g}_0\) be the solution to the Stein equation for DGM (\(Z\) as defined in (7)) with

\[\|\Delta \bar{g}_0\| \leq \bar{w}_1 \|f\| \min(1, \gamma^{-1})\]

where \(\bar{w}_1, \gamma > 0\). Assume that \(n \geq 5k, \bar{w}_1 \bar{w}_2 < \gamma, E(S^n_{k_1, k_2}) = E(M^n_{k_1, k_2})\) and \(\tau = \text{Var}(M^n_{k_1, k_2}) - \text{Var}(S^n_{k_1, k_2})\). Then

\[d_{TV}(Z, M^n_{k_1, k_2}) \leq \frac{\gamma}{2(\gamma - \bar{w}_1 \bar{w}_2)} \left( \bar{w}_1 \bar{e}^* \min(1, \gamma^{-1}) + 2\mathbb{P}(S^n_{k_1, k_2} > N) + 2\mathbb{P}(M^n_{k_1, k_2} > N) \right) + d_{TV}(Z, S^n_{k_1, k_2})\]

where \(d_{TV}(Z, S^n_{k_1, k_2})\) is given in (14), and

\[\bar{w}_2 = (2 + qp)a(p)\{(n - k)(1 - wb)a(p)\delta + |wb|) + |1 - wb|a(p)(\delta^* c^{(1)}_{n, k} + c^{(2)}_{n, k})\} \text{ and}
\]
\[\bar{e}^* = 2\left\{ 1 - wb \left[ \sum_{i=1}^{n-k} \sum_{j=1}^{i} \left( \sum_{u=i-2k-2}^{i+2k+2} \sum_{u=i+k+2}^{i} \mathbb{E}(\mathbb{I}_i)\mathbb{E}(\mathbb{I}_j) + \mathbb{E}(\mathbb{I}_i)\mathbb{E}(\mathbb{I}_j) \right) \right] + \mathbb{E}(\mathbb{I}_i)\mathbb{E}(\mathbb{I}_j) \sum_{|u-i| \leq 2k+2} \mathbb{E}(\mathbb{I}_u) \right\}\]

\[+ \frac{\tau}{2} + (2 + qp)h_1(n, k, p)a(p)^3 + |wb| \left( \sum_{i=1}^{n-k} \sum_{j=1}^{i} \mathbb{E}(\mathbb{I}_i) + \mathbb{E}(\mathbb{I}_i) \right) \sum_{|j-i| \leq k+1} \mathbb{E}(\mathbb{I}_j) \right) + (2 + qp)h_2(n, k, p)a(p)^2 \right\}.\]

Remark 3.2.

i. The results for Poisson, pseudo-binomial and negative binomial approximations follow from Theorem 3.2, Corollaries 3.1, (O2), (O6) and (O7), and (6). Also, note that similar bounds can be obtained for other distributions satisfying (7) and (13).
ii. Observe that $P(S_{k_1,k_2}^n > N)$ and $P(M_{k_1,k_2}^n > N)$ are zero for Poisson and negative binomial distributions. Also, if we take $\bar{x} > \lfloor n/k \rfloor$ then these probabilities are zero for pseudo-binomial distribution.

iii. Observe that, if $\text{Var}(S_{k_1,k_2}^n) = \text{Var}(M_{k_1,k_2}^n)$ (i.e., $\tau = 0$) then the bound become sharper, as expected.

iv. Using the condition $E(S_{k_1,k_2}^n) = E(M_{k_1,k_2}^n)$ (i.e., $q(1 + (n - k - 1) \ p a(p) = \sum_{i=1}^{n-k} E(\ell))$, for any values of $n$, $k$ and $p$, we can estimate the value of $p$ or $q$ (see Table 2). Hence, with the estimated value of $p$ or $q$, the bounds can be easily computed.

4. An application to two-stage startup demonstration testing

Let us consider a scenario in which a customer is interested in buying a certain equipment. In such a case, Balakrishnan and Chan (1999) recommend two-stage startup demonstration test (see also Balakrishnan and Koutras (2002), Balakrishnan, Koutras, and Milienos (2014), and references therein). During the second stage of this test, we need to count specific pattern of consecutive failures (say $k_1$) and successes (say $k_2$). The customer accepts the equipment under testing, whenever a recommended count of this pattern is observed. Using this mechanism, the customer can check the performance of the equipment, and can decide to reject bad equipment at early (first) stage or put strict norms for acceptance of the equipment in second stage. In particular, the decision criteria in two-stage startup test, for the customer, proposed by Balakrishnan and Chan (1999) (see also Balakrishnan and Koutras 2002, 281) is as follows:

i. Accept the equipment (in the first stage) if a run of $k_1^*$ successes occurs before $\ell_1^*$ failures.

ii. Accept the equipment if a run of $k_1^*$ successes does not occur before $\ell_1^*$ failures, but a run of $k_1$ successes occur before $k_2$ failures; and

iii. Reject the item if a run of $k_1^*$ successes does not occur before $\ell_1^*$ failures, and also a run of $k_1$ successes does not occur before $k_2$ failures.

Assume that there is no run of $k_1^*$ successes followed by $\ell_1^*$ failures in the first stage. In second stage, the problem studied in this paper becomes relevant, whenever $k_1$ consecutive failures followed by $k_2$ consecutive successes are considered. Then, by interchanging the role of success and failure, the setup can be adapted to our setting and the approximation results can be applied rather than computing complicated distributions. Now, we compare approximation results for particular values of $n,k_1,k_2$ and $p$, and

| $(k_1, k_2)$ | $n$  | $q$             | $d_{IV}(X_1, S_{k_1,k_2}^n)$ | $d_{IV}(S_{k_1,k_2}^n, M_{k_1,k_2}^n)$ | $d_{IV}(X_1, M_{k_1,k_2}^n)$ |
|-------------|------|----------------|-------------------------------|--------------------------------------|-------------------------------|
| (3, 4)      | 50   | 0.0652339752   | $1.05 \times 10^{-3}$         | $1.44 \times 10^{-6}$                | $1.44105 \times 10^{-6}$     |
| (3, 5)      | 150  | 0.0289666557   | $2.07 \times 10^{-3}$         | $6.79 \times 10^{-9}$                | $6.79000 \times 10^{-9}$     |
| (4, 5)      | 250  | 0.026121138    | $1.20 \times 10^{-12}$        | $3.10 \times 10^{-9}$                | $3.10120 \times 10^{-9}$     |
Let Lemma 5.1.

where $cx$ of (1994), Koutras (1997), Balakrishnan and Koutras (2002), and Dafnis, Antzoulakos, and Sn derive exact distribution of

with setup and similarly, we can obtain bounds for other random variables which satisfy (13) PMF of $Sn$

In this section, we derive some results related to the distribution of $5$. Auxiliary results

| $i$  | $p_i$ | $i$  | $p_i$ | $i$  | $p_i$ | $i$  | $p_i$ |
|------|-------|------|-------|------|-------|------|-------|
| 1-30 | 0.15  | 61-90| 0.17  | 121-150| 0.19  | 181-210| 0.21  |
| 31-60| 0.16  | 91-120| 0.18 | 151-180| 0.20  | 211-240| 0.22  |

Now, we demonstrate the results for Poisson approximation under non-identical setup and similarly, we can obtain bounds for other random variables which satisfy (13) with $w_1w_2 < \gamma$. To compute $d_{TV}(S_{k_1,k_2}^n,M_{k_1,k_2}^n)$ and $d_{TV}(X_1,S_{k_1,k_2}^n)$, we estimate the value of $q$ from the relation $q(1+(n-k-1)p)a(p) = \sum_{i=1}^{n-k} E(I_i)$ and then obtain the bounds by adding these two bounds.

Note that the bounds for non-identical trials are larger than the bounds for identical trials, as expected.

5. Auxiliary results

In this section, we derive some results related to the distribution of $S_{k_1,k_2}^n$. We first derive exact distribution of $S_{k_1,k_2}^n$ (using Markov chain approach, see Fu and Koutras (1994), Koutras (1997), Balakrishnan and Koutras (2002), and Dafnis, Antzoulakos, and Philippou (2010)) in the following lemma.

Lemma 5.1. Let $\phi_n(\cdot), \Phi(\cdot, \cdot)$ and $p_n = \mathbb{P}(S_{k_1,k_2}^n = \cdot)$ denote single PGF, double PGF and PMF of $S_{k_1,k_2}^n$. Then,

i. $\Phi(t,z) = \sum_{n=0}^{\infty} \phi_n(t) z^n = \frac{1+aq(p)z^{q-1}(1-qz)(1-t)}{1-z^a(p)z^{q-1}(1-t)(1-qz)(1-pz)}$.

ii. $p_{m,n} = \begin{cases} 0 & n \leq k, \ m > 0 \\ 1 & n \leq k, \ m = 0 \\ qa(p) & n = k+1, \ m = 1 \\ 1 - qa(p) & n = k+1, \ m = 0 \\ p_{m,n-1} - a(p)[(p_{m,n-k} - p_{m-1,n-k}) \\ -(p_{m,n-k-1} - p_{m-1,n-k-1}) + q(p_{m,n-k-2} - p_{m-1,n-k-2})] & n \geq k+2, \ m \geq 0 \end{cases}$

Proof. Let $\ell_n = \sup\{x : \mathbb{P}(S_{k_1,k_2}^n = x) > 0\} = \lfloor n/k \rfloor$ and $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k_1+k_2+1}\}$, where $c_{x,i} = (x,i), \ 0 \leq i \leq k_1 + k_2 + 1$. Also, define a Markov chain $\{Y_t : t \geq 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ as $Y_t = (x,j)$ if $(k_1, k_2)$-event has occurred $x$ times in the first $t$ outcomes and
Also, we say the pattern is complete if exactly \( k_1 \) failures before \( \eta_t \),

\( j = i, 1 \leq i \leq k_1, \) if \( \eta_t = \eta_{t-1} = \cdots = \eta_{t-i+1} = 0 \) and the \( (t-i) \)-th outcome is a success (if exists).

\( j = k_1^1, \) if there are more than \( k_1 \) consecutive failures. i.e., there exists a positive integers \( l \geq k_1 + 1 \) such that \( \eta_t = \eta_{t-1} = \cdots = \eta_{t-l+1} = 0 \).

\( j = k_1 + 1, 1 \leq i \leq k_2, \) if \( \eta_t = \eta_{t-1} = \cdots = \eta_{t-i+1} = 1, \eta_{t-i} = \eta_{t-i-1} = \cdots = \eta_{t-i-(k_1-1)} = 0 \) and the \( (t-i-k_1) \)-th outcome is a success (if exists).

Also, we say the pattern is complete if exactly \( k_1 \) consecutive failures followed by exactly \( k_2 \) consecutive successes and a failure occurs at \( (k_1 + k_2 + 1) \)-th position. Now, \( S_{k_1,k_2}^n \) becomes Markov chain embeddable variable of binomial type (MVB, see Dafnis, Antzoulakos, and Philippou (2010) for more details) with this setup and \( \pi_0 = (1, 0, \ldots, 0)_{1 \times (k_1+k_2+2)} \):

\[
A = \begin{pmatrix}
(\cdot, 0) & (\cdot, 1) & (\cdot, k_1 - 1) & (\cdot, k_1) & (\cdot, k_1^1) & (\cdot, k_1 + 1) & (\cdot, k_1 + 2) & (\cdot, k_1 + k_2 - 1) & (\cdot, k_1 + k_2) \\
p & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\
p & 0 & q & 0 & 0 & 0 & 0 & p & 0 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( A \) is \((k_1 + k_2 + 2) \times (k_1 + k_2 + 2)\) matrix and \( B \) is the matrix with zero entries except at \((k_1 + k_2 + 2, 2)\) where it is equal to \( q \). Therefore, from Theorem 2.1 of Dafnis, Antzoulakos, and Philippou (2010), with some calculations, we get the required result. Also, \((ii)\) directly follows from \((i)\).

Next, using Lemma 5.1 \((i)\), we can derive the following relation between \( \phi_n \) and \( \phi'_n \).

**Lemma 5.2.** The PGF of \( S_{k_1,k_2}^n, \phi_n(\cdot) \), satisfies the following recursive relation

\[
\phi'_n(t) = a(p) \sum_{l=1}^{3} a_\ell \sum_{j=0}^{d(l)} b_j(n - s) C_s(t) \phi_{n-k-j-i+1}(t)
\]

where \( C_s(t) = \sum_{l=0}^{[s/k]} \sum_{m=0}^{[l+1]} \binom{s-l(k-1)-mk}{s-lk-m(k+1),l,m} \frac{(k+1)^{l} (k+1)^{m} (s-lk-m(k+1))}{(k+2)^{l+m} (s-lk-m(k+1))} \). \(-1\)\(^{m}2[a(p)(t-1)]^{l+m}

**Proof.** From Lemma 5.1 \((i)\), the double PGF of \( S_{k_1,k_2}^n \) can be written as

\[
[1 - z - a(p)(t-1)z^l(1-qz)(1-pz)] \sum_{n=0}^{\infty} \phi_n(t)z^n = 1 - a(p)(t-1)z^l(1-qz) \] (15)

Differentiating (15) w.r.t. $t$ and $z$, we have

$$[1 - z - a(p)(t - 1)z^{k}(1 - z + qpz^2)] \sum_{n=0}^{\infty} \phi_n'(t)z^n - a(p)z^k[1 - z + qpz^2] \sum_{n=0}^{\infty} \phi_n(t)z^n$$

$$= -a(p)z^k(1 - qz)$$ \hspace{1cm} (16)

$$[1 - z - a(p)(t - 1)z^{k}(1 - z + qpz^2)] \sum_{n=0}^{\infty} n\phi_n(t)z^n - [z + a(p)(t - 1)]$$

$$z^k(k - (k + 1)z + qp(k + 2)z^2)] \sum_{n=0}^{\infty} \phi_n(t)z^n = -a(p)(t - 1)z^{k}(k - q(k + 1)z)$$ \hspace{1cm} (17)

Multiplying by $(z + a(p)(t - 1)z^{k}(k - (k + 1)z + qp(k + 2)z^2))$ in (16), $a(p)z^{k}(1 - z + qpz^2)$ in (17) and subtracting, we get

$$\sum_{n=0}^{\infty} n\phi_n(t)z^n = \frac{a(p)z^{k+1}(1 - qz)(pa(p)(t - 1)z^{k}(1 - qz) - 1)}{1 - z - a(p)(t - 1)z^{k}(1 - qz)(1 - pz)}.$$ \hspace{1cm} (18)

Multiplying by $(k + 2)$ in (16) and adding with (18), we have

$$[(k + 2) - (k + 1)z - a(p)(t - 1)z^{k}(2 - z)] \sum_{n=0}^{\infty} \phi_n'(t)z^n = qa(p)(k + 2)z^{k+1}$$

$$+a(p) \left[ \sum_{n=k+2}^{\infty} (n + 1)\phi_{n-k}(t)z^n - \sum_{n=k+2}^{\infty} (n + 1 - q)\phi_{n-k-1}(t)z^n + qp \sum_{n=k+2}^{\infty} n\phi_{n-k-2}(t)z^n \right]$$ \hspace{1cm} (19)

With some algebraic calculations, it can be easily seen that

$$1 \over [(k + 2) - (k + 1)z - a(p)(t - 1)z^{k}(2 - z)] = \sum_{n=0}^{\infty} C_n(t)z^n$$

Substituting in (19) and comparing the coefficients of $z^n$, we get

$$\phi_n'(t) = a(p) \left[ \sum_{s=0}^{n-k-2} (n - s + 1)C_s(t)\phi_{n-k-s}(t) - \sum_{s=0}^{n-k-1} b_2(n - s)C_s(t)\phi_{n-k-s-1}(t)$$

$$+qp \sum_{s=0}^{n-k-2} (n - s)C_s(t)\phi_{n-k-s-2}(t) \right] = a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_{i}(n - s)C_s(t)\phi_{n-k-s-i+1}(t)$$

This proves the result. \hspace{1cm} $\square$
Remark 5.1. Observe that
\[
C_s(t) = \sum_{m=0}^{[s/k]} B_s(m) t^m
\]
where
\[
B_s(m) = \sum_{l=m(k+1)-s}^{[s/k]} \sum_{r=m-l}^{[s/k]} \left( \begin{array}{c} s - l(k - 1) - rk \\ s - lk - r(k + 1) \end{array} \right) \left( \begin{array}{c} s \\ r\end{array} \right) \frac{(k+1)^{s-lk-r(k+1)}}{(k+2)^{s-lk-r(k+1)}} (-1)^{l-m} 2a(p)^{l-r} \tag{20}
\]

Further, the expression can be expressed as
\[
C_n(t) = c_0 + c_1 (t - 1) + \cdots + c_{[n/k]} (t - 1)^{[n/k]} = \sum_{m=0}^{[n/k]} B_n(m) t^m
\]
Though the form looks complicated, we only need
\[
C_n(1) = \sum_{m=0}^{[n/k]} B_n(m) = c_0,
\]
\[
C'_n(1) = \sum_{m=0}^{[n/k]} mB_n(m) = c_1 \quad \text{and} \quad C''_n(1) = \sum_{m=0}^{[n/k]} m( m - 1) B_n(m) = 2c_2
\]
to derive the approximation results and they are easy to compute.

6. Proofs

Proof of Theorem 3.1. We know that
\[
\phi_n(t) = \sum_{m=0}^{[n/k]} P_{m,n} t^m, \quad \phi'_n(t) = \sum_{m=0}^{[n/k]-1} (m+1)P_{m+1,n} t^m \quad \text{and} \quad C_s(t) = \sum_{m=0}^{[s/k]} B_s(m) t^m
\]
where \(C_s(t)\) and \(B_s(\cdot)\) as defined in Lemma 5.2 and (20), respectively. Substituting (21) in the recursive relation derived in Lemma 5.2, we have
\[
\phi'_n(t) = a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \left( \sum_{m=0}^{\infty} B_s(m) \left( m \leq \left\lfloor \frac{s}{k} \right\rfloor \right) t^m \right) \left( \sum_{m=0}^{\infty} P_{m,n-k-s-i+1} \left( m \leq \left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor \right) t^m \right)
\]
\[
= a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{m=0}^{\infty} \sum_{l=0}^{m} p_{l,n-k-s-i+1} B_s(m-l) \left( l \leq \left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor \right) \left( m-l \leq \left\lfloor \frac{s}{k} \right\rfloor \right) t^m
\]
Multiplying by \((1 - wbt)\) and collecting the coefficients of \(t^m\), we get

\[
(m + 1) p_{m+1, n} 1(m \leq \lfloor n/k \rfloor - 1) - wbmp_{m, n} 1(m \leq \lfloor n/k \rfloor)
\]

\[
= a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{m} p_{l, n-k-s-i+1} B_s(m-l)
\]

\[
1 \left( l \leq \left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor \right) 1 \left( m-l \leq \left\lfloor \frac{s}{k} \right\rfloor \right)
\]

\[
-wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{m-1} p_{l, n-k-s-i+1} B_s(m-l-1)
\]

\[
1 \left( l \leq \left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor \right) 1 \left( m-l-1 \leq \left\lfloor \frac{s}{k} \right\rfloor \right)
\]

Let \(g \in G_{s_{1}, s_{2}}\), then

\[
\sum_{m=0}^{\infty} g(m+1) \left[ (m+1) p_{m+1, n} 1(m \leq \lfloor n/k \rfloor - 1) - wbmp_{m, n} 1(m \leq \lfloor n/k \rfloor) \right]
\]

\[
= \sum_{m=0}^{\infty} g(m+1) \left[ a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{m} p_{l, n-k-s-i+1} B_s(m-l)
\]

\[
1 \left( l \leq \left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor \right) 1 \left( m-l \leq \left\lfloor \frac{s}{k} \right\rfloor \right)
\]

\[-wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{m-1} p_{l, n-k-s-i+1} B_s(m-l-1)
\]

\[
1 \left( m-l-1 \leq \left\lfloor \frac{s}{k} \right\rfloor \right)
\]

This implies

\[
\sum_{m=0}^{\lfloor n/k \rfloor} [wbmg(m+1) - mg(m)] p_{m, n} + a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor}
\]

\[
\sum_{m=0}^{\lfloor s/k \rfloor} g(m) p_{l, n-k-s-i+1} B_s(m)
\]

\[
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\left\lfloor \frac{n-k-s-i+1}{k} \right\rfloor}
\]

\[
\sum_{m=0}^{\lfloor s/k \rfloor} g(m+1) p_{l, n-k-s-i+1} B_s(m)
\]

\[
\sum_{m=0}^{\lfloor s/k \rfloor} g(m+2) p_{l, n-k-s-i+1} B_s(m) = 0
\]
Interchanging \( m \) and \( l \) for second and third terms, we get

\[
\sum_{m=0}^{\lfloor n/k \rfloor} \left[ w^\mathcal{U}(m+1) - w^\mathcal{U}(m) \right] g(m+1) - mg(m) \right] p_{m,n} - aw \sum_{m=0}^{\lfloor n/k \rfloor} g(m+1)p_{m,n} \\
\begin{align*}
&+ a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{k} g(m+l+1)p_{m,n-k-s-i+1} \\
&- wba(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{k} g(m+l+2)p_{m,n-k-s-i+1} = 0. \\
\end{align*}
\tag{22}
\]

Hence, Stein operator of \( S_{k_1, k_2}^n \) is given by

\[
\begin{align*}
\mathcal{A}_{S_{k_1, k_2}^n}(g(m)) &= w^\mathcal{U}(m+1) - w^\mathcal{U}(m) g(m+1) - mg(m) - awg(m+1) \\
&+ a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \mathbb{E} \left\{ g(S_{k_1, k_2}^{n-k-s-i+1} + l + 1) \mid S_{k_1, k_2}^n = m \right\} \\
&- wba(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \mathbb{E} \left\{ g(S_{k_1, k_2}^{n-k-s-i+1} + l + 2) \mid S_{k_1, k_2}^n = m \right\} \\
&= \mathcal{A}_Z g(m) + \mathcal{U} g(m) \\
\tag{23}
\end{align*}
\]

where \( \mathcal{A}_Z \) is a Stein operator for DGM and \( \mathcal{U} \) is a perturbed operator. Taking expectation of the perturbed operator \( \mathcal{U} \) w.r.t. \( S_{k_1, k_2}^n \) defined in (23), we have

\[
\mathbb{E}[\mathcal{U} g(S_{k_1, k_2}^n)] = -wa \sum_{m=0}^{\lfloor n/k \rfloor} g(m+1)p_{m,n} + a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \\
\sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{k} g(m+l+1)p_{m,n-k-s-i+1} \\
- wba(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{k} g(m+l+2)p_{m,n-k-s-i+1}
\]

Observe that \( \lfloor n-k-s-i+1 \rfloor \leq \lfloor n/k \rfloor \) for all \( s \) and \( i \), hence, we replace \( \lfloor n-k-s-i+1 \rfloor \) by \( \lfloor n/k \rfloor \) as \( p_{m,n-k-s-i+1} \) become zero outside of its range. Hence,
\[
\mathbb{E} [\mathcal{U}_g(S^n_{k_1, k_2})] = -wa \sum_{m=0}^{\lfloor n/k \rfloor} g(m + 1)p_{m,n} + a(p) \sum_{i=1}^{d_i} a_i \sum_{s=0}^{d_i} b_i(n - s)
\]
\[
\sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} g(m + l + 1)p_{m,n-k-s-l+1}
\]
\[
-wba(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor m+1/k \rfloor} g(m + l + 2)p_{m,n-k-s-l+1}
\]
\[
= -wa \sum_{m=0}^{\lfloor n/k \rfloor} g(m + 1)p_{m,n} - wba(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l)
\]
\[
\sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + l + 1)p_{m,n-k-s-l+1}
\]
\[
+(1 - wb)a(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} g(m + l + 1)p_{m,n-k-s-l+1}
\]

It is known that
\[
g(m + l + 1) = \sum_{j=1}^{l} \Delta g(m + j) + g(m + 1)
\]

Substituting (25) in (24), we have
\[
\mathbb{E} [\mathcal{U}_g(S^n_{k_1, k_2})] = -wa \sum_{m=0}^{\lfloor n/k \rfloor} g(m + 1)p_{m,n} - wba(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l)
\]
\[
\sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + l + 1)p_{m,n-k-s-l+1}
\]
\[
+(1 - wb)a(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} g(m + l + 1)p_{m,n-k-s-l+1}
\]
\[
+(1 - wb)a(p) \sum_{i=1}^{\lfloor s/k \rfloor} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor n/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \sum_{j=1}^{l} \Delta g(m + j)p_{m,n-k-s-l+1}
\]

Now, from Lemma 5.1 (ii), it can be verified that
\[
p_{m,n} = p_{m,n-l} - a(p)(p_{m,n,l} - p_{m-1,n,l}^*), \quad \text{for } l \geq 1, \ n \geq k + 2
\]

where \( p_{m,n,l}^* = p_{m,n-k} - p_{m,n-k-l} + \sum_{u=0}^{l-1} [qpp_{m,n-k-u-2} - p_{m,n-k-u} 1(u = n - k) + qpp_{m,n-k-u-1} 1(u = n - k - 1)] \). Using (26) for the third term, this expression leads to
We know that
\[
\mathbb{E}[Ug(S_{k_1, k_2}^n)] = (1 - wb) \left( -\frac{wa}{1 - wb} + a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} g(m + 1) p_{m,n} \right) 
+ (1 - wb)a(p)^2 \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} g(m + 1) (p_{m,n}^{*} - p_{m,n,k+s+i-1}^{*} - p_{m,n,k+s+i-1}) 
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + l + 1) p_{m,n-k-s-i+1} 
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + j) p_{m,n-k-s-i+1} 
\] 

Using the fact that \( \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) = C_s(1) = \frac{(k+1)^{s}}{(k+2)^{s+1}} \), with \( \mathbb{E}(Z) = \mathbb{E}(S_{k_1, k_2}^n) \), it can be verified that
\[
-\frac{wa}{1 - wb} + a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) = 0 
\] 

The expression (27) now becomes
\[
\mathbb{E}[Ug(S_{k_1, k_2}^n)] = -(1 - wb)a(p)^2 \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + 1) p_{m,n,k+s+i-1}^{*} 
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + l + 1) p_{m,n-k-s-i+1} 
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + j) p_{m,n-k-s-i+1} 
\] 

We know that
\[
\Delta g(m + j) = \sum_{v=1}^{j-1} \Delta^2 g(m + v) + \Delta g(m + 1) 
\] 

Substituting (25) and (29) in (28), we get
\[
\mathbb{E}[Ug(S_{k_1, k_2}^n)] = -(1 - wb)a(p)^2 \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta g(m + 1) p_{m,n,k+s+i-1}^{*} 
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta^2 g(m + j) p_{m,n-k-s-i+1} 
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{\lfloor s/k \rfloor} B_s(l) \sum_{m=0}^{\lfloor n/k \rfloor} \Delta^2 g(m + v) p_{m,n-k-s-i+1} 
\]
Using (26), we have

$$
\mathbb{E}[Ug(S^n_{k_1,k_2})] = \left\{ -(1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l)[1 - (s \leq n - 2k - i + 1) \\
+ \sum_{u=0}^{k+s+i-2} (qp1(u \leq n - k - 2) - 1(u = n - k) + q1(u = n - k - 1))] \\
- wba(p) \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \\
+ (1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1) p_{m,n} \\
- (1 - wb)a(p)^3 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1) (p^{**}_{m,n,k+s+i-1} - p^{**}_{m-1,n,k+s+i-1}) \\
- wba(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1) (p^{*}_{m,n,k+s+i-1} - p^{*}_{m-1,n,k+s+i-1}) \\
+ (1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} l B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1) (p^{*}_{m,n,k+s+i-1} - p^{*}_{m-1,n,k+s+i-1}) \\
- wba(p) \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \sum_{j=1}^{l-1} \Delta^2 g(m+j) p_{m,n-k-s-i+1} \\
+ (1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \sum_{j=1}^{l-1} \sum_{v=1}^{j-1} \Delta^2 g(m+v) p_{m,n-k-s-i+1}
\right\} (30)
$$

where

$$
p^{**}_{m,n,l} = p^{*}_{m,n,k} - p^{*}_{m,n,k+l} + \sum_{u=0}^{l-1} (qp1_{m,n,k+u+2} - p^{*}_{m,n,k+u+1}) \mathbb{1}(u = n - k) + q1_{m,n,k+u+1} \mathbb{1}(u = n - k - 1)].
$$

Using \( \phi = \text{Var}(Z) - \text{Var}(S^n_{k_1,k_2}) \) with \( \mathbb{E}(Z) = \mathbb{E}(S^n_{k_1,k_2}) \), it can be easily verified that

$$
-(1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l)[1 - (s \leq n - 2k - i + 1) \\
+ \sum_{u=0}^{k+s+i-2} (qp1(u \leq n - k - 2) - 1(u = n - k) + q1(u = n - k - 1))] \\
- wba(p) \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} B_s(l) + (1 - wb)a(p)^2 \sum_{i=1}^{k} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{s/k} l B_s(l) = -\phi(1 - wb)
$$
Therefore,

\[
\mathbb{E}[Ug(S_{k_1,k_2}^n)] = -(1 - wb)a(p)^2 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1)
\]

\[
(\rho_{m,n,k+s+i-1} - \rho_{m-n,1,n,k+s+i-1})
\]

\[
- wba(p)^2 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1)(\rho_{m,n,k+s+i-1} - \rho_{m-n,1,n,k+s+i-1})
\]

\[
+ (1 - wb)a(p)^2 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1)(\rho_{m,n,k+s+i-1} - \rho_{m-n,1,n,k+s+i-1})
\]

\[
- wba(p)^3 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \sum_{j=1}^{l} \Delta^2 g(m + j) p_{m,n-k-s-i+1}
\]

\[- \phi(1 - wb) \sum_{n=0}^{\infty} \Delta g(m + 1) p_{m,n}
\]

\[
+ (1 - wb)a(p)^2 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \sum_{j=1}^{l} \sum_{v=1}^{j-1} \Delta^2 g(m + v) p_{m,n-k-s-i+1}
\]

Observe that

\[
d_i \leq n - k - 1 \quad \text{and} \quad |b_i(n - s)| \leq n - s + 1, \quad \text{for all } s, i
\]

Hence, for \( g \in \mathcal{G}_Z \cap \mathcal{G}_{S_{k_1,k_2}} \) and using (31), we get

\[
|\mathbb{E}[Ug(S_{k_1,k_2}^n)]| \leq \|\Delta g\| \{2(2 + q)p a(p)^2 |1 - wb| h_1(n,k,p) a(p) + |wb| h_2(n,k,p)
\]

\[+ |\phi(1 - wb)| \}

This proves result. \( \square \)

**Proof of Theorem 3.2.** Combining (22) and (27), we get

\[
0 = \mathbb{E}[A_{S_{k_1,k_2}} g(S_{k_1,k_2})] = \sum_{m=0}^{n/k} [wb mg(m + 1) - mg(m)]
\]

\[
+(1 - wb)q(1 + (n - k - 1)p) a(p) g(m + 1) p_{m,n}
\]

\[
-(1 - wb)a(p)^2 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + 1) p_{m,n,k+s+i-1}
\]

\[
-wba(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \Delta g(m + l + 1) p_{m,n-k-s-i+1}
\]

\[
+ (1 - wb)a(p)^3 \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n - s) \sum_{l=0}^{s/k} B_s(l) \sum_{m=0}^{n/k} \sum_{j=1}^{l} \Delta^2 g(m + j) p_{m,n-k-s-i+1}
\] (32)
Therefore, the Stein operator now becomes

\[
A_{S_{k_1,k_2}^m} g(S_{k_1,k_2}^n) = w \left[ \frac{(1 - wb)}{w} q(1 + (n - k - 1)p) a(p) + bm \right] g(m + 1) - mg(m)
- (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \mathbb{E} \left\{ \Delta g(S_{k_1,k_2}^{n-k-s-i+1}) | S_{k_1,k_2}^n = m \right\}
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \mathbb{E} \left\{ \Delta g(S_{k_1,k_2}^{n-k-s-i+1} + l) | S_{k_1,k_2}^n = m \right\}
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{j=1}^{l} \mathbb{E} \left\{ \Delta g(S_{k_1,k_2}^{n-k-s-i+1} + j) | S_{k_1,k_2}^n = m \right\}
= A_{Z}g(m) + \bar{U}_1g(m),
\]

where \(A_{Z}\) is a Stein operator for DGM with \(a = ((1 - wb)/w)q(1 + (n - k - 1)p)a(p)\) and

\[
(\Delta g(S_{k_1,k_2}^{n-k} + 1) | S_{k_1,k_2}^n = m) = [(\Delta g(S_{k_1,k_2}^{n-k} + 1) - \Delta g(S_{k_1,k_2}^{n-k-1} + 1)
+ \sum_{u=0}^{l-1} qp \Delta g(S_{k_1,k_2}^{n-k-u} + 1) - \Delta g(S_{k_1,k_2}^{n-k-u} + 1) 1(u = n - k)
+ q \Delta g(S_{k_1,k_2}^{n-k-u} + 1) 1(u = n - k - 1)) | S_{k_1,k_2}^n = m]
\]

Now, from the definition of \(\bar{U}_1\) in (33) and (34), it can be easily seen that

\[
\| \bar{U}_1g \| \leq \bar{w}_2 \| \Delta g \|
\]

Following the similar steps from (28) to (30) for the last three terms of (32), we get

\[
\mathbb{E}[A_{S_{k_1,k_2}^m} g(S_{k_1,k_2}^n)] = \sum_{m=0}^{[n/k]} \left[ \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \left[ 1 - 1(s \leq n - 2k - i + 1) \right] + \sum_{u=0}^{l-1} qp \Delta g(S_{k_1,k_2}^{n-k-u} + 1) - \Delta g(S_{k_1,k_2}^{n-k-u} + 1) 1(u = n - k)
+ q \Delta g(S_{k_1,k_2}^{n-k-u} + 1) 1(u = n - k - 1)) \right] - wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \Delta g(m + 1) p_{m,n}
\]

\[
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \Delta^2 g(m + 1) p_{m,n,k+s+i-1}
\]

\[
+ wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \Delta^2 g(m + 1) p_{m,n,k+s+i-1}
\]

\[
- (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \Delta^2 g(m + 1) p_{m,n,k+s+i-1}
\]

\[
- wba(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \Delta^2 g(m + j) p_{m,n-k-s-j+1}
\]

\[
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \sum_{j=1}^{l-1} \sum_{v=1}^{j-1} \Delta^2 g(m + v) p_{m,n-k-s-i+1}
\]

\[
+ (1 - wb)a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{m=0}^{[n/k]} \sum_{j=1}^{l-1} \sum_{v=1}^{j-1} \Delta^2 g(m + v) p_{m,n-k-s-i+1}
\]

\[
= A_{Z}g(m) + \bar{U}_1g(m),
\]

\[
(33)
\]

\[
(34)
\]

\[
(35)
\]
Therefore, the Stein operator now becomes
\[
A_{k_1,k_2}^n g(m) = wb mg(m+1) - mg(m) + (1 - wb) q(1 + (n - k_1 - 1)p) a(p) g(m + 1) \\
+ \left\{ - (1 - wb) a(p)^2 \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left[ 1 - 1(s \leq n - 2k + i + 1) \\
+ \sum_{u=0}^{k+s+i-2} (q p(1 \leq n - k - 2) - 1(u = n - k) + q(1(u = n - k - 1)) \right] \
- wa(p) \sum_{l=0}^{k+s+i-2} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = m \} \\
- (1 - wb) a(p)^2 \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = m \} \\
- wa(p) \sum_{l=0}^{k+s+i-2} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = m \} \\
+ (1 - wb) a(p) \sum_{i=1}^{3} a_i \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \sum_{j=1}^{l} j \sum_{v=1}^{l} i \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1} + v) |S_{k_1,k_2}^n = m \} \right. \\
+ \frac{wb}{1 - wb} a(p)^3 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta g(M_{k_1,k_2}^n + 1) \right\} \\
+ \frac{wb}{1 - wb} a(p)^3 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \\
+ wba(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \\
- (1 - wb) a(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left\{ \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right\} |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \\
Now, taking expectation w.r.t. \( M_{k_1,k_2}^n \), we have
\[
\mathbb{E}[A_{k_1,k_2}^n g(M_{k_1,k_2}^n)] = (1 - wb) q(1 + (n - k_1 - 1)p) a(p) \mathbb{E}[g(M_{k_1,k_2}^n + 1)] \\
+ b \mathbb{E}[M_{k_1,k_2}^n g(M_{k_1,k_2}^n + 1)] - \mathbb{E}[M_{k_1,k_2}^n g(M_{k_1,k_2}^n)] \\
+ (1 - wb) \left\{ - a(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left( 1 - 1(s \leq n - 2k + i + 1) \\
+ \sum_{u=0}^{k+s+i-2} (q p(1 \leq n - k - 2) - 1(u = n - k) + q(1(u = n - k - 1)) \right) \
- wba(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left( \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right) |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \right. \\
+ wba(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left( \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right) |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \\
- (1 - wb) a(p)^2 \sum_{i=1}^{3} a_i b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) l \left( \Delta^2 g(S_{k_1,k_2}^{n-k-s-i+1}) + 1 \right) |S_{k_1,k_2}^n = M_{k_1,k_2}^n \} \\
\]
\[-wba(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{j=1}^{l} \mathbb{E} \{ \Delta^2 g(S_{k_1, k_2}^{n-k-s-i+1} + j) \mid S_{k_1, k_2}^{n} = M_{k_1, k_2}^{n} \} \]

\[+ (1 - wba) a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) \sum_{j=1}^{l} \sum_{v=1}^{j-1} \mathbb{E} \{ \Delta^2 g(S_{k_1, k_2}^{n-k-s-i+1} + v) \mid S_{k_1, k_2}^{n} = M_{k_1, k_2}^{n} \} \}

(36)

where \((\Delta^2 g(S_{k_1, k_2}^{n-k-s-i+1} + 1) \mid S_{k_1, k_2}^{n} = M_{k_1, k_2}^{n})\) is defined according to \(p_{m,n,k+s+i-1}^{**}\) and similar to \((\Delta g(S_{k_1, k_2}^{n-l} + 1) \mid S_{k_1, k_2}^{n} = m)\) defined in (34).

Next, define

\[M_i = M_{k_1, k_2}^{n} - I_i \quad \text{and} \quad N_i = M_{k_1, k_2}^{n} - \sum_{|s-i| \leq k+1} \mathbb{I}_s \]

Then \(N_i\) and \(\mathbb{I}_i\) are independent. It is given that

\[\mathbb{E}(S_{k_1, k_2}^{n}) = \mathbb{E}(M_{k_1, k_2}^{n}) \Rightarrow q(1 + (n - k - 1)p)a(p) = \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \]

Consider the first three terms of (36), we have

\[(1 - wba) q(1 + (n - k - 1)p)a(p) \mathbb{E} [\Delta g(M_{k_1, k_2}^{n} + 1)] + wba \mathbb{E} [M_{k_1, k_2}^{n} g(M_{k_1, k_2}^{n} + 1)]

- \mathbb{E} [M_{k_1, k_2}^{n} g(M_{k_1, k_2}^{n})]

= (1 - wba) \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \mathbb{E} [\Delta g(M_{k_1, k_2}^{n} + 1)] - (1 - wba) \mathbb{E} [M_{k_1, k_2}^{n} g(M_{k_1, k_2}^{n})]

+ wba \mathbb{E} [M_{k_1, k_2}^{n} \Delta g(M_{k_1, k_2}^{n})]

= (1 - wba) \sum_{i=1}^{n-k} \{ \mathbb{E}(\mathbb{I}_i) \mathbb{E} [g(M_{k_1, k_2}^{n} + 1)] - \mathbb{E}(\mathbb{I}_i g(M_{k_1, k_2}^{n} + 1)) \} + wba \mathbb{E} [M_{k_1, k_2}^{n} \Delta g(M_{k_1, k_2}^{n})]

= (1 - wba) \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \mathbb{E} [\Delta g(N_i + 1)] + (1 - wba) \sum_{i=1}^{n-k} \mathbb{E}[\mathbb{I}_i (g(N_i + 1) - g(M_i + 1))]

+ wba \mathbb{E} [M_{k_1, k_2}^{n} \Delta g(M_{k_1, k_2}^{n})]

= (1 - wba) \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{|j-i| \leq k+1} \mathbb{E} \left( \mathbb{I}_j \Delta g \left( N_i + \sum_{s=i-k}^{j-1} \mathbb{I}_s + 1 \right) \right)

- (1 - wba) \sum_{i=1}^{n-k} \sum_{|j-i| \leq k+1} \mathbb{E} \left( \mathbb{I}_j \Delta g \left( N_i + \sum_{s=i-k}^{j-1} \mathbb{I}_s + 1 \right) \right) + wba \mathbb{E} [M_{k_1, k_2}^{n} \Delta g(M_{k_1, k_2}^{n})]

(37)

Observe the terms involving \(\mathbb{E}(\Delta g(M_{k_1, k_2}^{n} + 1))\), we have

\[a(p) \sum_{i=1}^{3} \sum_{s=0}^{d_i} b_i(n-s) \sum_{l=0}^{[s/k]} B_s(l) = q(1 + (n - k - 1)p)a(p) = \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \]

(38)

and, using (38) with \(\tau = \text{Var}(M_{k_1, k_2}^{n}) - \text{Var}(S_{k_1, k_2}^{n})\),
Now, substituting these values in (36) and using (37), we get

\[
-\sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\mathbb{I}_j) + \sum_{i=1}^{n-k} \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\mathbb{I}_i \mathbb{I}_j) - \tau
\]

Now, substituting these values in (36) and using (37), we get

\[
\mathbb{E}[\mathcal{A}_{n,k_1} g(M_{k_1,k_2})] = -(1 - wb) \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \left[ \mathbb{E}(\mathbb{I}_j) \mathbb{E}(\Delta g(M_{k_1,k_2} + 1)) - \mathbb{E}(\mathbb{I}_i \mathbb{I}_j) \mathbb{E}(\Delta g(M_{k_1,k_2} + 1) + 1) \right] - \mathbb{E}(\mathbb{I}_i \Delta g(N_i + \sum_{s=i-k-1}^{i-1} \mathbb{I}_s + 1))
\]

\[
= -(1 - wb) \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \mathbb{E}(\Delta g(M_{k_1,k_2} + 1)) - \mathbb{E}(\mathbb{I}_i \Delta g(M_i + 1)) - \tau (1 - wb) \mathbb{E}(\Delta g(M_{k_1,k_2} + 1))
\]

\[
+ (1 - wb) a(p)^3 \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\Delta g(S_{k_1,k_2}^{n-k-s-i+1} + 1) | S_{k_1,k_2} = M_{k_1,k_2})
\]

\[
= -(1 - wb) a(p)^3 \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\Delta g(S_{k_1,k_2}^{n-k-s-i+1} + 1) | S_{k_1,k_2} = M_{k_1,k_2})
\]

\[
+ (1 - wb) a(p)^3 \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\Delta g(S_{k_1,k_2}^{n-k-s-i+1} + 1) | S_{k_1,k_2} = M_{k_1,k_2})
\]

\[
+ (1 - wb) a(p)^3 \sum_{i=1}^{n-k} \mathbb{E}(\mathbb{I}_i) \sum_{j \neq i} \sum_{|j-i| \leq k+1} \mathbb{E}(\Delta g(S_{k_1,k_2}^{n-k-s-i+1} + v) | S_{k_1,k_2} = M_{k_1,k_2})
\]

(39)

Define

\[
S_i = M_{k_1,k_2} - \sum_{|j-i| \leq 2k+2} \mathbb{I}_j.
\]
Then, \( S_i \) and \( \{ I_j : |j - i| \leq k + 1 \} \) are independent. Now, consider

\[
\begin{align*}
&\bigg| \mathbb{E}(I_j) \mathbb{E}(\Delta g(M_{kn}^{*}, k_2 + 1)) - \mathbb{E}\left(I_j \Delta g \left( N_i + \sum_{s=i-k-1}^{j-1} I_s + 1 \right) \right) \bigg| \\
&= \bigg| \mathbb{E}(I_j) \mathbb{E}\{ \Delta g(M_{kn}^{*}, k_2 + 1) - \Delta g(S_i + 1) \} \\
&+ \mathbb{E}\left\{ I_j \left( \Delta g(S_i + 1) - \Delta g \left( N_i + \sum_{s=i-k-1}^{j-1} I_s + 1 \right) \right) \right\} \bigg| \\
&\leq \mathbb{E}(I_j) \mathbb{E}\{ \Delta g(M_{kn}^{*}, k_2 + 1) - \Delta g(S_i + 1) \} + \bigg| \\
&\mathbb{E}\left\{ I_j \left( \Delta g(S_i + 1) - \Delta g \left( N_i + \sum_{s=i-k-1}^{j-1} I_s + 1 \right) \right) \right\} \bigg| \\
&= \bigg| \sum_{u=i-2k-2}^{i-1} \mathbb{E}(I_j) \mathbb{E}\left( I_u I_j \Delta^2 g \left( S_i + \sum_{s=i-2k-2}^{u-1} I_s + \sum_{s=i+k+2}^{i+2k+2} I_s + 1 \right) \right) \bigg| \\
&+ \bigg| \sum_{u=i+k+2}^{i+2k+2} \mathbb{E}(I_j) \mathbb{E}\left( I_u \Delta^2 g \left( S_i + \sum_{s=i+k+2}^{u-1} I_s + 1 \right) \right) \bigg| \\
&+ \mathbb{E}(I_j) \bigg| \sum_{|u-i| \leq 2k+2} \mathbb{E}(I_u) \sum_{u=i-2k-2}^{i-1} \mathbb{E}(I_u) \bigg| \\
&\leq 2\| \Delta g \| \left\{ \left( \sum_{u=i-2k-2}^{i-1} \mathbb{E}(I_j) \mathbb{E}(I_u) + \mathbb{E}(I_j) \sum_{|u-i| \leq 2k+2} \mathbb{E}(I_u) \right) \right\} \tag{40}
\end{align*}
\]

Similarly,

\[
\begin{align*}
&\bigg| \mathbb{E}(I_j) \mathbb{E}(\Delta g(M_{kn}^{*}, k_2 + 1)) - \mathbb{E}(I_j \Delta g \left( N_i + \sum_{s=i-k-1}^{j-1} I_s + 1 \right) \right) \bigg| \\
&\leq 2\| \Delta g \| \left\{ \left( \sum_{u=i-2k-2}^{i-1} \mathbb{E}(I_j) \mathbb{E}(I_u) + \mathbb{E}(I_j) \sum_{|u-i| \leq 2k+2} \mathbb{E}(I_u) \right) \right\} \tag{41}
\end{align*}
\]

and

\[
\begin{align*}
&\bigg| \mathbb{E}(I_j) \mathbb{E}(\Delta g(M_{kn}^{*}, k_2 + 1)) - \mathbb{E}(I_j \Delta g(M_i + 1) \right) \bigg| \leq 2\| \Delta g \| \left\{ \left( \sum_{|u-i| \leq 2k+2} \mathbb{E}(I_j) \mathbb{E}(I_u) + \mathbb{E}(I_j) \sum_{|u-i| \leq 2k+2} \mathbb{E}(I_u) \right) \right\} \tag{42}
\end{align*}
\]
Now, using (40, 41) and (42) in (39), we get

\[ E \left[ A_{k_1, k_2}^n g(M^n_{k_1, k_2}) \right] \leq \varepsilon^* \Delta g \]  

(43)

Using (35) and (43) with Lemma 2.1, we get the required result. □

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