An Infeasibility Result for the Multiterminal Source-Coding Problem

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Abstract

We prove a new outer bound on the rate-distortion region for the multiterminal source-coding problem. This bound subsumes the best outer bound in the literature and improves upon it strictly in some cases. The improved bound enables us to obtain a new, conclusive result for the binary erasure version of the “CEO problem.” The bound recovers many of the converse results that have been established for special cases of the problem, including the recent one for the Gaussian version of the CEO problem.

1 Introduction

In their lauded paper [1], David Slepian and Jack K. Wolf characterize the information rates needed to losslessly communicate two correlated, memoryless information sources when these sources are encoded separately. Their well-known result states that two discrete sources \(Y_1\) and \(Y_2\) can be losslessly reproduced if

\[
R_1 > H(Y_1|Y_2) \\
R_2 > H(Y_2|Y_1) \\
R_1 + R_2 > H(Y_1, Y_2),
\]

where \(R_1\) is the rate of the encoder observing \(Y_1\) and \(R_2\) is the rate of the encoder observing \(Y_2\). Conversely, lossless reproduction is not possible if \((R_1, R_2)\) lies outside the closure of this region. See Cover and Thomas [2, Section 14.4] or Csiszár and Körner [3, Section 3.1] for precise statements of the result and modern proofs. This result is naturally viewed as a multi-source generalization of...
the classical result of Shannon [4], which says that, loosely speaking, a discrete memoryless source with known law can be losslessly reproduced if and only if the data rate exceeds the entropy of the source. Shannon too studied a generalization of this result, albeit in a different direction. He studied the problem of reproducing a source imperfectly, subject to a minimum fidelity constraint, and showed that the required rate is given by the well-known rate-distortion formula [4, 5]. One of the central problems of Shannon theory is to understand the limits of source coding for models that combine the two generalizations. That is, we seek to determine the rates required to reproduce two correlated sources, each subject to a fidelity constraint, when the sources are encoded separately (see Fig. 1). Determining the set of achievable rates and distortions for this setup is often called the multiterminal source-coding problem, even though this name suggests a more elaborate network topology. This problem has been unsolved for some time.

The model we consider in this paper is slightly more general and is depicted in Fig. 2. Beyond considering an arbitrary number of encoders, \( L \), we also allow for a hidden source, \( Y_0 \), which is not directly observed by any encoder or the decoder, and a “side information” source, \( Y_{L+1} \), which is observed by the decoder but not by any encoder. We also permit arbitrary functions of the sources to be reproduced, in addition to, or in place of, the sources themselves. We will therefore use \( Z_1, Z_2, \) etc., to denote the instantaneous estimates instead of \( \hat{Y}_1, \hat{Y}_2, \) etc., as before. In this paper, we will refer to this more general problem as the multiterminal source-coding problem.
One might doubt the wisdom of embellishing the model when even the basic form shown in Fig. 1 is unsolved. But one of the contributions of this paper is to show that far from obscuring the problem, the added generality actually illuminates it. Of course, the more general problem is also unsolved.

Many special cases have been solved, however. For these, the reader is referred to the classical papers of Slepian and Wolf [1], mentioned earlier; Wyner [2]; Ahlswede and Körner [3]; Wyner and Ziv [4]; Körner and Marton [5]; and Gel’fand and Pinsker [6]; and to the more recent papers of Berger and Yeung [7]; Gastpar [8]; Oohama [9]; and Prabhakaran, Tse, and Ramchandran [10]. While all of these papers contain conclusive results, these results are established using coding theorems that are tailored to the special cases under consideration.

The solutions to these solved special cases suggest a coding technique for the general model [11, 12]. The idea is this. Each encoder first quantizes its observation as in single-user rate-distortion theory. The quantized processes are then losslessly communicated to the decoder using the binning scheme of Cover [12]. The decoder uses the quantized processes to produce the desired estimates. The set of rate-distortion vectors that can be achieved using this scheme is described in Section 3. This inner bound to the rate-distortion region is tight in all of the special cases listed above except that of Körner and Marton [5]. Indeed, the Körner-Marton problem seems to require a custom coding technique that relies on the problem’s unique structure. This suggests that the multiterminal source-coding problem may not have a classical single-letter solution.

We attack this problem, therefore, by proving single-letter inner and outer bounds on the rate-distortion region. The best inner bound in the literature has just been described. The best outer bound, which is due to Berger [13] and Tung [16], is described in Section 3. In light of the result of Körner and Marton, it is clear that the two bounds must not coincide in all cases. This gap cannot be entirely attributed to the inner bound, however, as there are instances of the problem that can be solved from first principles for which the Berger-Tung outer bound is strictly bigger than the true rate-distortion region (see Section 3.1 of this paper).

Our aim is to provide an improved outer bound for the problem. We prove such a bound in the next section, following a precise formulation of the problem. We show that our bound is contained in (i.e., subsumes) the Berger-Tung outer bound in Section 3. In that section, we also provide several examples for which the containment is strict.

One example is the binary erasure version of the “CEO problem,” the general version of which was introduced by Berger, Zhang, and Viswanathan [17]. The CEO problem is a special case of the multiterminal source-coding problem in which the observed processes $Y_1, \ldots, Y_L$ are conditionally independent given the hidden process $Y_0$ and in which the decoder (the CEO) is only interested in estimating the hidden process. This definition is not as restrictive as it might seem. Indeed, any instance of the multiterminal source-coding problem with a single distortion constraint can be transformed into...
the tradeoff between sum rate and Hamming distortion in the high-rate and many-encoder limit. Gel'fand and Pinsker [10] had earlier found the rate region in the lossless reproduction case. We consider the problem in which $Y_0$ is binary and uniform, and the encoders observe $Y_0$ through independent binary erasure channels. The decoder reproduces $Y_0$ subject to a constraint on the “erasure distortion” (see Section 3.2 or Cover and Thomas [2, p. 370]). For this problem, we show that our outer bound is tight in the sum rate for any number of users. In contrast, the Berger-Tung outer bound contains points whose sum rate is strictly smaller than the optimum.

In our view, this result is of interest in its own right. The binary erasure CEO problem arises naturally in sensor networks in which the sensors occasionally “sleep” to conserve energy. This application is described in Section 3.2. The result also provides an example for which the binning-based coding scheme mentioned earlier is optimum. Finally, this is one of relatively few conclusive results for the multiterminal source-coding problem in general, and the CEO problem in particular. These problems are considered sufficiently difficult that it is worth reporting solutions to special cases.

One of the few other conclusive results available is for the Gaussian version of the CEO problem, which was first studied by Viswanathan and Berger [19]. Here the encoders observe a hidden Gaussian source through independent Gaussian additive-noise channels. The distortion measure is expected squared error. The rate-distortion region for this problem was recently found by Oohama [20, 13] and independently by Prabhakaran, Tse, and Ramchandran [14]. We show that the converse result of these four authors can be recovered from our single-letter outer bound, while the Berger-Tung outer bound contains points that lie outside the true rate-distortion region.

The converse results used to solve all of the other special cases mentioned so far are also consequences of our bound. This is discussed in Section 4. Our outer bound therefore serves to unify most of what is known about the nonexistence of multiuser source codes. This unification is noteworthy in the case of Oohama [20] and Prabhakaran, Tse, and Ramchandran [14] because the connection between their remarkable converse result and the classical discrete results in this area is not immediately apparent. As we will see, subject to some technical caveats, most of the key results in multiterminal source coding can be recovered by combining the general inner bound described earlier with the outer bound described next.

### 2 Formulation and Main Result

We work exclusively in discrete time. We use uppercase letters to denote random variables and vectors, lowercase letters to denote their realizations, and script letters to denote their ranges. Let \( \{Y_0^n(t), Y_1^n(t), \ldots, Y_L^n(t), Y_{L+1}^n(t)\}_{t=1}^n \)

an instance of the CEO problem without changing the rate-distortion region by lumping $Y_1, \ldots, Y_L$ into $Y_0$ and redefining the distortion measure as needed. Nonetheless, it defines a useful special case.
Fig. 3: Notation for the encoding and decoding rules.

\[ Y_n^0 \]

\[ Y_n^1 \]

\[ Y_n^2 \]

\[ Y_n^L \]

Encoder 1 \[ f_1^{(n)}(Y_n^1) \]

Decoder \[ Z_1 = \phi_1^{(n)} \left( Y_{L+1}^n, \left( f_1^{(n)}(Y_n^1) \right)_t \right) \]

Encoder 2 \[ f_2^{(n)}(Y_n^2) \]

Decoder \[ Z_2 = \phi_2^{(n)} \left( Y_{L+1}^n, \left( f_2^{(n)}(Y_n^2) \right)_t \right) \]

: \[ : \]

Encoder L \[ f_L^{(n)}(Y_n^L) \]

Decoder \[ Z_K = \phi_K^{(n)} \left( Y_{L+1}^n, \left( f_L^{(n)}(Y_n^L) \right)_t \right) \]

Figure 3: Notation for the encoding and decoding rules.

be a vector-valued, finite-alphabet memoryless source. For \( A \subset \{1, \ldots, L\} \), we denote \( (Y_n^\ell(t))_{\ell \in A} \) by \( Y_n^A(t) \). If \( A = \{1, \ldots, L\} \), we write this simply as \( Y_n(t) \). In this context, the set \( A^c \) should be interpreted as \( \{1, \ldots, L\} \setminus A \) rather than \( \{0, \ldots, L + 1\} \setminus A \). When \( A = \{\ell\} \), we shall write \( Y_n^\ell(t) \) and \( Y_n^{\ell^c}(t) \) in place of \( Y_{\ell^c}(t) \) and \( Y_{\ell^c}(t) \), respectively. Also, we use \( Y_n^\ell(t_1 : t_2) \) to denote \( \{Y_n^\ell(t)\}_{t_1 \leq t \leq t_2} \), \( Y_n^\ell \) to denote \( Y_n^\ell(1 : n) \), and \( Y_n^{\ell^c}(t^e) \) to denote

\[ (Y_n^\ell(1), \ldots, Y_n^\ell(t - 1), Y_n^\ell(t + 1), \ldots, Y_n^\ell(n)). \]

Similar notation will be used for other vectors that appear later.

The notation for the encoding and decoding rules is shown in Fig. 3. For each \( \ell \in \{1, \ldots, L\} \), encoder \( \ell \) observes \( Y_n^\ell \), then employs a mapping

\[ f_\ell^{(n)} : Y_n^\ell \rightarrow \{1, \ldots, M_\ell^{(n)}\} \]

to convey information about it to the decoder. The decoder observes \( Y_{L+1}^\ell \) and uses it and the received messages to estimate \( K \) functions of the vector-valued source according to the mappings

\[ \varphi_k^{(n)} : Y_{L+1}^\ell \times \prod_{\ell=1}^L \{1, \ldots, M_\ell^{(n)}\} \rightarrow Z_k^n \text{ for } k = 1, \ldots, K. \]

We assume that \( K \) distortion measures \( d_k : \prod_{\ell=0}^{L+1} Y_\ell \times Z_k \rightarrow \mathbb{R}_+ \) are given.

We mention at this point that while the generality of this setup will be useful later when studying examples, it is not needed to appreciate the bounding technique itself. The reader is welcome to focus on the basic model shown in Fig. 4 for that purpose.

**Definition 1** The rate-distortion vector

\[ (R, D) = (R_1, R_2, \ldots, R_L, D_1, D_2, \ldots, D_K) \]

is achievable if there exists a block length \( n \), encoders \( f_\ell^{(n)} \), and a decoder

\[ (\varphi_1^{(n)}, \ldots, \varphi_K^{(n)}) \]
such that
\[ R_\ell \geq \frac{1}{n} \log M_\ell^{(n)} \] for all $\ell$, and
\[ D_k \geq E \left[ \frac{1}{n} \sum_{t=1}^{n} d_k(Y^n_0(t), Y^n(t), Y^n_{L+1}(t), Z^n_k(t)) \right] \] for all $k$.

Let $\mathcal{RD}_*$ be the set of achievable rate-distortion vectors. Its closure, $\overline{\mathcal{RD}_*}$, is called the rate-distortion region.

We will sometimes be concerned with projections of the rate-distortion region. We denote these by, for example, $\mathcal{RD}_* \cap \{R_1 = 0\}$, meaning
\[ \{ (R_2, \ldots, R_L, D_1, \ldots, D_K) : (0, R_2, \ldots, R_L, D_1, \ldots, D_K) \in \mathcal{RD}_* \}. \]

In this paper, we view lossless compression as a limit of lossy compression with the distortion tending to zero. More precisely, if we wish to reproduce $Y_1$ losslessly, we will set, say, $Z_1 = Y_1$ with $d_1$ equal to Hamming distance, and then examine $\mathcal{RD}_* \cap \{D_1 = 0\}$. This convention and Definition 1 together yield a notion of lossless compression that is weaker than the one traditionally used.

It is common instead to require that for all sufficiently large block lengths, there exists a code for which the probability of correctly reproducing the entire vector $Y^n$ is arbitrarily close to 1. But a weaker notion is desirable here since we are trying to prove an outer bound or "converse" result.

To state our result, let $Y_0, \ldots, Y_{L+1}$ be generic random variables with the distribution of the source at a single time. Let $\Gamma_o$ denote the set of finite-alphabet random variables $\gamma = (U_1, \ldots, U_L, Z_1, \ldots, Z_K, W, T)$ satisfying

(i) $(W, T)$ is independent of $(Y_0, Y, Y_{L+1})$,
(ii) $U_\ell \leftrightarrow (Y_\ell, W, T) \leftrightarrow (Y_0, Y_\ell', Y_{L+1}, U_\ell')$, shorthand for \"$U_\ell$, $(Y_\ell, W, T)$ and $(Y_0, Y_\ell', Y_{L+1}, U_\ell')$ form a Markov chain in this order\", for all $\ell$, and
(iii) $(Y_0, Y, W) \leftrightarrow (U, Y_{L+1}, T) \leftrightarrow Z$.

It is straightforward to verify that $\Gamma_o$ is precisely the set of finite-alphabet random variables $(U_1, \ldots, U_L, Z_1, \ldots, Z_K, W, T)$ whose joint distribution with $(Y_0, Y, Y_{L+1})$ factors as
\[
p(y_0, y, y_{L+1}, u, z, w, t) = p(y_0, y, y_{L+1})p(w, t) \prod_{\ell=1}^{L} p(u_\ell|y_\ell, w, t)p(z|u, y_{L+1}, t).
\]

This description is helpful in that it suggests a parametrization of the space $\Gamma_o$.

Let $\chi$ denote the set of finite-alphabet random variables $X$ with the property that $Y_1, \ldots, Y_L$ are conditionally independent given $(X, Y_{L+1})$. Note that $\chi$ is nonempty since it contains, e.g., $X = (Y_1, \ldots, Y_{L-1})$.

\(2\) All logarithms and exponentiations in this paper have base $e$. 
There are many ways of coupling a given $X$ in $\chi$ and $\gamma$ in $\Gamma_o$. In this paper, we shall only consider the unique coupling for which $X \leftrightarrow (Y_0, Y, Y_{L+1}) \leftrightarrow \gamma$, which we call the Markov coupling. Whenever the joint distribution of $X$, $(Y_0, Y, Y_{L+1})$, and $\gamma$ arises, we assume that this coupling is in effect.

It is evident from the definition of $\chi$ that there is considerable latitude in choosing how $X$ depends on $Y_0$. This is because the sole constraint on the choice of $X$ only depends on the joint distribution of $X$ and $(Y, Y_{L+1})$. But as the following definition makes clear, this freedom is inconsequential since our outer bound only depends on the distributions of $(Y_0, Y, Y_{L+1}, \gamma)$ and $(X, Y, Y_{L+1}, \gamma)$ separately.

**Definition 2** Let

\[
\mathcal{R}D_o(X, \gamma) = \left\{(R, D) : \sum_{\ell \in A} R_\ell \geq I(X; U_A|U_{A^c}, Y_{L+1}, T) \right. \\
+ \sum_{\ell \in A} I(Y_\ell; U_\ell|X, Y_{L+1}, W, T) \text{ for all } A \subset \{1, \ldots, L\}, \left. \right. \\
and D_k \geq E[d_k(Y_0, Y, Y_{L+1}, Z_k)] \text{ for all } k \right\}.
\]

Then define

\[
\mathcal{R}D_o = \bigcap_{X \in \chi} \bigcup_{\gamma \in \Gamma_o} \mathcal{R}D_o(X, \gamma).
\]

The first theorem is our main result.

**Theorem 1** The rate-distortion region is contained in $\mathcal{R}D_o$. In fact, $\mathcal{R}D_o \subset \mathcal{R}D_o$.

**Proof.** It suffices to show the second statement. Suppose $(R, D)$ is achievable. Let $f_1^{(n)}$, $\ldots$, $f_L^{(n)}$ be encoders and $(\varphi_1^{(n)}, \ldots, \varphi_K^{(n)})$ a decoder satisfying [1]. Take any $X$ in $\chi$ and augment the sample space to include $X^n$ so that

\[
(X^n(t), Y_0^n(t), Y^n(t), Y_{L+1}^n(t))
\]

is independent over $t$. Next let $T$ be uniformly distributed over $\{1, \ldots, n\}$, independent of $X^n$, $Y_0^n$, $Y^n$, and $Y_{L+1}^n$. Then define

\[
X = X^n(T) \\
Y_\ell = Y_\ell^n(T) \text{ for each } \ell \in \{0, \ldots, L+1\} \\
U_\ell = \left(j_\ell^{(n)}(Y_\ell^n), X^n(1 : T - 1), Y_{L+1}^n(T^c) \right) \text{ for each } \ell \\
Z_k = Z_k^n(T) \text{ for each } k \\
W = (X^n(T^c), Y_{L+1}^n(T^c)).
\]
It can be verified that $\gamma = (U, Z, W, T)$ is in $\Gamma_\alpha$ and that, together with $Y_0$, $Y$, $Y_{L+1}$, and $X$, it satisfies the Markov coupling. It suffices to show that $(R, D)$ is in $\mathcal{RD}_\alpha(X, \gamma)$. First, note that (1) implies

$$D_k \geq E[d_k(Y_0^n(T), Y^n(T), Y_{L+1}^n(T), Z_k^n(T))]$$

for all $k$, i.e.,

$$D_k \geq E[d_k(Y_0, Y, Y_{L+1}, Z_k)]$$

for all $k$.

Second, let $A \subset \{1, \ldots, L\}$. Then by the cardinality bound on entropy,

$$n \sum_{\ell \in A} R_\ell \geq H\left(\sum_{\ell \in \Gamma} f^{(n)}(Y^n_\ell) \bigg| \sum_{\ell \in \Gamma} f^{(n)}(Y^n_\ell), Y^n_{L+1}\right)$$

Since conditioning reduces entropy, this implies

$$n \sum_{\ell \in A} R_\ell \geq H\left(\sum_{\ell \in \Gamma} f^{(n)}(Y^n_\ell) \bigg| \sum_{\ell \in \Gamma} f^{(n)}(Y^n_\ell), Y^n_{L+1}\right)$$

Applying the chain rule for mutual information,

$$\begin{align*}
I \left( X^n, Y^n_A; \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, Y^n_{L+1} \right) \\
= I \left( X^n; \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, Y^n_{L+1} \right) \\
+ I \left( Y^n_A; \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, X^n, Y^n_{L+1} \right). 
\end{align*}$$

By the chain rule for mutual information,

$$\begin{align*}
I \left( X^n; \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, Y^n_{L+1} \right) \\
= \sum_{t=1}^n I \left( X^n(t); \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, X^n(1 : t-1), Y^n_{L+1} \right). 
\end{align*}$$

Consider next the second term on the right-hand side of (3). Since $X \in \chi$,

$$\begin{align*}
I \left( Y^n_A; \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A} \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A^c}, X^n, Y^n_{L+1} \right) \\
= \sum_{\ell \in A} I \left( Y^n_A; \left( f^{(n)}_\ell(Y^n_\ell) \right) \bigg| X^n, Y^n_{L+1} \right).
\end{align*}$$
Applying the chain rule once more gives

\[ I \left( Y^n_\ell; f^{(n)}_\ell \right) | X^n, Y^n_{L+1} = \sum_{t=1}^n I \left( Y^n_\ell(t); f^{(n)}_\ell(Y^n_\ell) \right) | X^n, Y^n_\ell(1 : t-1), Y^n_{L+1} \].

But

\[ I \left( Y^n_\ell(t); f^{(n)}_\ell(Y^n_\ell) \right) | X^n, Y^n_\ell(1 : t-1), Y^n_{L+1} = I \left( Y^n_\ell(t); Y^n_\ell(1 : t-1) | X^n, Y^n_{L+1} \right) = I \left( Y^n_\ell(t); f^{(n)}_\ell(Y^n_\ell) \right) | X^n, Y^n_{L+1} \]

and the second term on the left-hand side is zero. Thus

\[ I \left( Y^n_\ell(t); f^{(n)}_\ell(Y^n_\ell) \right) | X^n, Y^n_\ell(1 : t-1), Y^n_{L+1} \geq I \left( Y^n_\ell(t); f^{(n)}_\ell(Y^n_\ell) \right) | X^n, Y^n_{L+1} \].

Substituting the results of these various calculations into (2) gives

\[ \sum_{\ell \in A} R_{\ell} \geq \frac{1}{n} \sum_{t=1}^n \left[ I \left( X^n(t); f^{(n)}_\ell(Y^n_\ell) \right) \bigg| \left( f^{(n)}_\ell(Y^n_\ell) \right) \bigg| X^n, Y^n_{L+1}(1 : t-1), Y^n_{L+1} \right] \]

If $A^e$ is nonempty, this can be rewritten as

\[ \sum_{\ell \in A} R_{\ell} \geq I(X^n(T); U_A | U_{A^c}, Y^n_{L+1}(T), T) + \sum_{\ell \in A} I(Y^n_\ell(T); U_\ell | X^n(T), X^n(T^c), Y^n_{L+1}(T), Y^n_{L+1}(T^c), T) = I(X; U_A | U_{A^c}, Y_{L+1}, T) + \sum_{\ell \in A} I(Y_\ell; U_\ell | X, Y_{L+1}, W, T) \].
The case \( A = \{1, \ldots, L\} \) is handled separately. In this case, observe that

\[
I \left( X^n(t); \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A}; X^n(1:t-1), Y^n_{L+1} \right)
= I \left( X^n(t); \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A}; X^n(1:t-1), Y^n_{L+1} \right)
+ I( X^n(t); X^n(1:t-1), Y^n_{L+1}(t^c) | Y^n_{L+1}(t) )
= I \left( X^n(t); \left( f^{(n)}_\ell(Y^n_\ell) \right)_{\ell \in A}; X^n(1:t-1), Y^n_{L+1}(t^c) | Y^n_{L+1}(t) \right).
\]

Substituting this into (4) and proceeding as in the \( A^c \neq \emptyset \) case completes the proof. □

It is worth noting that the proof uses classical techniques. Most of the manipulations in the latter part of the proof can be viewed as versions of the chain rule for mutual information. Since this chain rule holds in abstract spaces \[21, (3.6.6)\], the proof can be readily extended to more general alphabets.

The key step in the proof is the introduction of \( X^n \) in (2). Unlike the other auxiliary random variables, \( X^n \) does not represent a component of the code. Rather, it is used to aid the analysis by inducing conditional independence among the messages sent by the encoders. This technique of augmenting the source to induce conditional independence was pioneered by Ozarow \[22\], who used it to solve the Gaussian two-descriptions problem. Wang and Viswanath \[23\] used it to determine the sum rate of the Gaussian vector multiple-descriptions problem with individual and central decoders. It was also used by Wagner, Tavildar, and Viswanath \[24\] to solve the Gaussian two-terminal source-coding problem. A step that is similar to (2) appeared in Gel’fand and Pinsker \[10\] and in later papers on the Gaussian CEO problem \[13, 14\], although in these works \( X^n \) is part of the source, so no augmentation is involved.

The significance of conditional independence has long been known in the related field of distributed detection (e.g., \[25\]). Given the similarity between distributed detection and the multiterminal source-coding problem, one expects conditional independence to play a significant role here as well. Indeed, most conclusive results for the multiterminal source-coding problem require a conditional independence assumption \[10, 13, 12, 13, 14\]. The motivation for introducing \( X^n \) is that it allows one to apply the approach used in these works to problems that lack conditional independence.

We do not consider the problem of computing \( \mathcal{R}_D \) in this paper. Note that we have not specified the alphabet sizes of the auxiliary random variables \( U, W, \) and \( T \). As such, the outer bound provided by Theorem 1 is not computable \[3, p. 259\] in the present form. One might question the utility of an outer bound that cannot be computed. The remainder of the paper, however, will show that the bound is still useful as a theoretical tool. In addition, cardinality
bounds might be found later, although obtaining such bounds appears to be more difficult in this case than for related bounds.

It should be mentioned that the time-sharing variable $T$ is unnecessary; it can be absorbed into the other variables. We have included it to ease the comparison with existing inner and outer bounds, to which we turn next.

3 Relation to Existing Bounds

The coding scheme described in the introduction gives rise to the following inner bound on the rate-distortion region.

Definition 3 Let $\Gamma_i^{BT}$ denote the set of finite-alphabet random variables

$$\gamma = (U_1, \ldots, U_L, Z_1, \ldots, Z_K, T)$$

satisfying

(i) $T$ is independent of $(Y_0, Y, Y_{L+1})$, 

(ii) $U_\ell \leftrightarrow (Y_\ell, T) \leftrightarrow (Y_0, Y_\ell, Y_{L+1}, U_\ell)$ for all $\ell$, and 

(iii) $(Y_0, Y) \leftrightarrow (U, Y_{L+1}, T) \leftrightarrow Z$.

Then define

$$\mathcal{R}_i^{BT}(\gamma) = \left\{ (R, D) : \sum_{\ell \in A} R_\ell \geq I(Y_A; U_A|U_{A^-}, Y_{L+1}, T) \text{ for all } A, \right.$$ 

$$\text{and } D_k \geq E[d_k(Y_0, Y, Y_{L+1}, Z_k)] \text{ for all } k \right\}.$$ 

Finally, let

$$\mathcal{R}_i^{BT} = \bigcup_{\gamma \in \Gamma_i^{BT}} \mathcal{R}_i^{BT}(\gamma).$$

Proposition 1 ([15, 16]) $\mathcal{R}_i^{BT} \subset \mathcal{R}_*.$

In Appendix we show that $\mathcal{R}_i^{BT}$ is in fact closed. We call $\mathcal{R}_i^{BT}$ the Berger-Tung [15, 16] inner bound, since although these authors prove a bound that is less general than the one given here, their proof can be extended to prove Proposition 1. See Chen et al. [20] or Gastpar [12] for recent sketches of the proof that accommodate some of the generalizations included here.

To understand the difference between $\mathcal{R}_i^{BT}$ and $\mathcal{R}_o$, suppose that

$$(U, Z, W, T)$$
is in \( \Gamma_o \) and \( W \) is deterministic. Then \((U, Z, T)\) is in \( \Gamma_i^{BT} \), and for all \( A \subset \{1, \ldots, L\} \) and all \( X \in \chi \),

\[
I(X; U_A|U_{A^c}, Y_{L+1}, T) + \sum_{\ell \in A} I(Y_\ell; U_\ell|X, Y_{L+1}, W, T)
\]

\[
= I(X; U_A|U_{A^c}, Y_{L+1}, T) + I(Y_A; U_A|U_{A^c}, X, Y_{L+1}, W, T)
\]

\[
= I(X; Y_A; U_A|U_{A^c}, Y_{L+1}, T) + I(Y_A; U_A|U_{A^c}, X, Y_{L+1}, T)
\]

\[
= I(Y_A; U_A|U_{A^c}, Y_{L+1}, T).
\]

Thus

\[
\mathcal{RD}_o(X, U, Z, W, T) = \mathcal{RD}_i^{BT}(U, Z, T).
\] (5)

Conversely, if \((U, Z, T)\) is in \( \Gamma_i^{BT} \), then for any deterministic \( W \), \((U, Z, W, T)\) is in \( \Gamma_o \) and (5) holds for any \( X \). It follows that \( \mathcal{RD}_i^{BT} \) is equal to \( \mathcal{RD}_o \) with \( W \) restricted to be deterministic in the definition of \( \Gamma_o \).

In particular, to obtain coincident inner and outer bounds, it suffices to show that restricting \( W \) to be deterministic in the definition of \( \Gamma_o \) does not reduce \( \mathcal{RD}_o \). We will see later how this can be accomplished in several examples. Of course, it is not possible for the problem solved by Körner and Marton [9], since they show that the inner bound is not tight in that case.

The best outer bound in the literature is the following.

**Definition 4** Let \( \Gamma_i^{BT} \) denote the set of finite-alphabet random variables \( \gamma = (U, Z, T) \) satisfying

(i) \( T \) is independent of \((Y_0, Y, Y_{L+1})\),

(ii) \( U_\ell \leftrightarrow (Y_\ell, T) \leftrightarrow (Y_0, Y_\ell, Y_{L+1}) \) for all \( \ell \), and

(iii) \((Y_0, Y) \leftrightarrow (U, Y_{L+1}, T) \leftrightarrow Z\).

Then let

\[
\mathcal{RD}_o^{BT}(\gamma) = \left\{ (R, D) : \sum_{\ell \in A} R_\ell \geq I(Y; U_A|U_{A^c}, Y_{L+1}, T) \text{ for all } A,
\right.

and \( D_k \geq E[d_k(Y_0, Y, Y_{L+1}, Z_k)] \) for all \( k \) \}

Finally, let

\[
\mathcal{RD}_o^{BT} = \bigcup_{\gamma \in \Gamma_i^{BT}} \mathcal{RD}_o^{BT}(\gamma).
\]

**Proposition 2** ([15], [16]) \( \mathcal{RD}_o \subset \mathcal{RD}_o^{BT} \).
As with the inner bound, Berger [15] and Tung [16] prove the result for a model that is more restrictive than the one considered here, but their proof can be extended to this setup (c.f. [26, 12]). The difference between \( RD_{BT} \) and \( RD_o \) is that condition (ii) has been weakened in the latter. We next show that the Berger-Tung outer bound is subsumed by the one in the previous section.

**Proposition 3** \( RD_o \subset RD_{BT} \).

**Proof.** First observe that for any \((U, Z, W, T)\) in \( \Gamma_o \),

\[
U_\ell \leftrightarrow (Y_\ell, W, T) \leftrightarrow (Y_0, Y_\ell, Y_{\ell+1}) \text{ for each } \ell.
\]

Since \((Y_0, Y_\ell, Y_{\ell+1}) \leftrightarrow (Y_\ell, T) \leftrightarrow W\), it holds

\[
U_\ell \leftrightarrow (Y_\ell, T) \leftrightarrow (Y_0, Y_\ell, Y_{\ell+1}) \text{ for each } \ell.
\]

Thus \((U, Z, T)\) is in \( \Gamma_{BT}^o \) and in particular,

\[
RD_o(U, Z, W, T) = RD_{BT}^o(U, Z, T).
\]

It follows that

\[
RD_o \subset \bigcup_{\gamma \in \Gamma_o} RD_o(Y, \gamma) \subset \bigcup_{\gamma \in \Gamma_{BT}^o} RD_{BT}^o(\gamma) = RD_{BT}^o.
\]

The proof reveals that \( RD_o \) improves upon \( RD_{BT}^o \) in two ways. The first is that \( RD_o \) allows for optimization over \( X \) while \( RD_{BT}^o \) effectively requires the choice \( X = Y \). The second is that \( \Gamma_o \) is “smaller” than \( \Gamma_{BT}^o \) in the sense that if \((U, Z, W, T)\) is in \( \Gamma_o \) then \((U, Z, T)\) is in \( \Gamma_{BT}^o \). The balance of this section is devoted to showing that these improvements make the containment in Proposition 3 strict in some cases. As the reader will see, the former difference is entirely responsible for the gap that we expose between the two bounds in our examples. We hasten to add, however, that the latter improvement is not an empty one in that Anantharam and Borkar [27] have shown that there can exist a \((U, Z, T)\) in \( \Gamma_{BT}^o \) with the property that there does not exist a \( W \) such that \((U, Z, W, T)\) is in \( \Gamma_o \). It is interesting to note that the Anantharam-Borkar example arose independently of this work in the context of distributed stochastic control.

We will exhibit three examples for which \( RD_{BT}^o \) strictly contains \( RD_o \). The first is rather contrived and can be solved from first principles. It is included to illustrate the difference between the two bounds.

### 3.1 Toy Example

Let \( Y_{11}, Y_{12}, Y_{21}, \) and \( Y_{22} \) be independent and identically distributed (i.i.d.) random variables, uniformly distributed over \( \{0, 1\} \). Consider two encoders \((L = 2)\) with \( Y_1 = (Y_{11}, Y_{12}) \) and \( Y_2 = (Y_{21}, Y_{22}) \) (there is no hidden source.
or side information in this example). We have a single distortion constraint \((K = 1)\) with \(Z_1 = \{0, 1\}^2\) and

\[
d_1(Y, Z_1) = \begin{cases} 0 & \text{if } Z_1 = (Y_{11}, Y_{21}) \text{ or } Z_1 = (Y_{12}, Y_{22}) \\ 1 & \text{otherwise.} \end{cases}
\]

In words, the decoder attempts to guess either the first or the second coordinate of both encoders’ observations. It incurs a distortion of zero if it guesses correctly the same coordinate of the two sources and one otherwise. Note that the decoder need not declare which coordinate it is attempting to guess.

**Proposition 4** For this problem,

\[\mathcal{RD}_o \cap \{D_1 = 0\} \subset \{(R_1, R_2) : R_1 \geq \log 2, R_2 \geq \log 2\}.\]

**Proof.** Suppose \((R_1, R_2, \epsilon)\) is in \(\mathcal{RD}_o\), and \(\epsilon \leq 1/2\). Observe that since \(Y_1\) and \(Y_2\) are independent, deterministic random variables are in \(\chi\). Thus there exists \(\gamma\) in \(\Gamma_o\) such that

\[
\epsilon \geq E[d_1(Y, Z_1)]
\]

\[
R_1 \geq I(Y_1; U_1|W, T)
\]

\[
R_2 \geq I(Y_2; U_2|W, T).
\]

By condition \((ii)\) defining \(\Gamma_o\),

\[U_1 \leftrightarrow (Y_1, W, T) \leftrightarrow (Y_2, U_2).\]  

(6)

Since \(Y_2\) is independent of \((Y_1, W, T)\) in this example, \(Y_2\) must be independent of \((Y_1, U_1, W, T)\). Thus

\[I(Y_1; U_1|W, T) = I(Y_1; U_1|W, T, Y_{21} \neq Y_{22}).\]  

(7)

Likewise, \(Y_1\) is independent of \((Y_2, U_2, W, T)\) and hence given \((W, T)\), \(Y_1\) is independent of \((Y_2, U_2)\). This observation combined with \((6)\) implies \((Y_1, U_1) \leftrightarrow (W, T) \leftrightarrow (Y_2, U_2)\). In particular, \(Y_1 \leftrightarrow (U_1, W, T) \leftrightarrow (Y_2, U_2)\). By condition \((iii)\) defining \(\Gamma_o\), \(Y_1 \leftrightarrow (Y_2, U_1, U_2, W, T) \leftrightarrow (Y_2, Z_1)\). These last two chains imply that

\[Y_1 \leftrightarrow (U_1, W, T) \leftrightarrow (Y_2, Z_1).\]

Thus conditioned on \((W, T)\) and the event \(\{Y_{21} \neq Y_{22}\}\), we have \(Y_1 \leftrightarrow U_1 \leftrightarrow (Y_2, Z_1)\). It follows that

\[
I(Y_1; U_1|W, T, Y_{21} \neq Y_{22})
\]

\[
\geq I(Y_1; Y_2, Z_1|W, T, Y_{21} \neq Y_{22})
\]

\[
= I(Y_1; Y_2, Z_1, d_1(Y, Z_1)|W, T, Y_{21} \neq Y_{22})
\]

\[= I(Y_1; d_1(Y, Z_1)|W, T, Y_{21} \neq Y_{22})
\]

\[
\geq H(Y_1|W, T, Y_{21} \neq Y_{22}) - H(Y_1|Y_2, Z_1, d_1(Y, Z_1), W, T, Y_{21} \neq Y_{22})
\]

\[= H(d_1(Y, Z_1)|W, T, Y_{21} \neq Y_{22}),
\]
since $d_1(Y, Z_1)$ is a function of $Y$ and $Z_1$. Next, observe that on the events \(\{Y_{21} \neq Y_{22}\}\) and \(\{d_1(Y, Z_1) = 0\}\), $Y_2$ and $Z_1$ together must reveal one of the two bits of $Y_1$. Thus

\[
H(Y_1 | Y_2, Z_1, d_1(Y, Z_1) = 0, W, T, Y_{21} \neq Y_{22}) \leq \log 2.
\]

Continuing our chain of inequalities,

\[
I(Y_1; U_1 | W, T, Y_{21} \neq Y_{22}) \\
\geq 2 \log 2 - \log 2 \cdot \Pr(d_1(Y, Z_1) = 0 | Y_{21} \neq Y_{22}) \\
- (2 \log 2) \cdot \Pr(d_1(Y, Z_1) = 1 | Y_{21} \neq Y_{22}) \\
- H(d_1(Y, Z_1) | Y_{21} \neq Y_{22}) \\
\geq 2 \log 2 - 2 \log 2 \cdot \Pr(d_1(Y, Z_1) = 1 | Y_{21} \neq Y_{22}) \\
- H(d_1(Y, Z_1) | Y_{21} \neq Y_{22}).
\]

Now

\[
\frac{1}{2} H(d_1(Y, Z_1) | Y_{21} \neq Y_{22}) + \frac{1}{2} H(d_1(Y, Z_1) | Y_{21} = Y_{22}) \\
= H(d_1(Y, Z_1) | 1(Y_{21} = Y_{22}) \leq H(d_1(Y, Z_1)) \leq h(\epsilon),
\]

where, here and throughout, $h(\cdot)$ is the binary entropy function with natural logarithms. We conclude that

\[
H(d_1(Y, Z_1) | Y_{21} \neq Y_{22}) \leq 2h(\epsilon).
\]

Similarly,

\[
\Pr(d_1(Y, Z_1) = 1 | Y_{21} \neq Y_{22}) \leq 2\epsilon.
\]

Substituting these two observations into (8) and recalling (9) yields

\[
I(Y_1; U_1 | W, T) \geq \log 2 - 4\epsilon \log 2 - 2h(\epsilon).
\]

By symmetry, $I(Y_2; U_2 | W, T)$ must satisfy the same inequality. This implies the desired conclusion. \(\square\)

It is easy to see that the point $(\log 2, \log 2, 0)$ is achievable. Using rate log $2$, each encoder can send, say, the first coordinate of its observation. The decoder can then realize zero distortion by repeating the two bits it receives. This fact and the above proposition together imply

\[
\overline{RD}_o \cap \{D_1 = 0\} = \overline{RD}_o \cap \{D_1 = 0\} = \{(R_1, R_2): R_1 \geq \log 2, R_2 \geq \log 2\}.
\]

In particular, $\overline{RD}_o$ is tight in the zero-distortion limit. In contrast, we show next that the Berger-Tung outer bound is not.

**Proposition 5** The point $((3/4) \log 2, (3/4) \log 2, 0)$ is contained in $\overline{RD}_o^{BT}$. 


Proof. Let the random variable \( W \) be uniformly distributed over \( \{1, 2\} \), and let \( U_1 = Y_{1W} \) and \( U_2 = Y_{2W} \). Let \( Z_1 = (U_1, U_2) \). It is straightforward to verify that \( (U_1, U_2, Z_1) \) is in \( \Gamma_{oBT} \) (the time-sharing random variable \( T \) is unneeded and can be taken to be constant). Next note that \( E[d_1(Y, Z_1)] = 0 \). Finally, one can compute

\[
I(Y; U_1, U_2) = \frac{5}{4} \log 2
\]

and

\[
I(Y; U_1) = \frac{1}{2} \log 2.
\]

This implies that

\[
I(Y; U_1|U_2) = I(Y; U_2|U_1) = \frac{3}{4} \log 2 > \frac{5}{8} \log 2.
\]

The conclusion follows. \( \Box \)

3.2 Binary Erasure CEO Problem

Here \( Y_0 \) is uniformly distributed over \( \{-1, 1\} \), and \( Y_\ell = N_\ell Y_0 \) for \( \ell \) in \( \{1, \ldots, L\} \), where \( N_1, \ldots, N_L \) are i.i.d. with \( 0 < \Pr(N_1 = 0) = p < 1 \) and \( \Pr(N_1 = 1) = 1 - p \). Let \( Z_1 = \{-1, 0, 1\} \). We will assume that there is no side information and that the decoder is only interested in reproducing the hidden process \( Y_0 \). We measure the fidelity of its reproduction using a family of distortion measures, \( \{d_1^\lambda\}_{\lambda > 0} \), where

\[
d_1^\lambda(Y_0, Y, Z_1) = \begin{cases} 
0 & \text{if } Y_0 = Z_1 \\
1 & \text{if } Z_1 = 0 \\
\lambda & \text{otherwise.}
\end{cases}
\]

We are particularly interested in the large-\( \lambda \) limit. In this regime, \( d_1^\lambda \) approximates the “erasure distortion measure” [2] p. 370],

\[
d_1^{\infty}(Y_0, Y, Z_1) = \begin{cases} 
0 & \text{if } Y_0 = Z_1 \\
1 & \text{if } Z_1 = 0 \\
\infty & \text{otherwise.}
\end{cases}
\]

We use a finite approximation because an infinite distortion measure causes difficulties in the proof of the Berger-Tung inner bound.

This example is motivated by the following problem arising in energy-limited sensor networks. We seek to monitor a remote source, \( Y_0 \). To this end, we deploy an array of sensors, each of which is capable of observing the source with negligible probability of error. To lengthen the lifetime of the network, each sensor spends a fraction \( p \) of the time in a low-power “sleep” state. We assume that the sensors cycle between the awake and sleep states independently of each
other and on a faster time scale than the sampling; at each discrete time, each sensor sleeps with probability \( p \), independently of the other sensors and the past. Sensors do not make any observations while they are asleep, resulting in erasures. We permit the coding process to introduce additional erasures, but not errors, yielding the erasure distortion measure. What sum rate is required in order for the decoder to reproduce a fraction \( 1 - D \) of the \( Y_0^n \) variables while almost never making an error? Of course, \( D \) must satisfy \( D \geq pL \).

Define
\[
R_\star(D, \lambda) = \inf \left\{ \sum_{\ell=1}^{L} R_\ell : (R_1, \ldots, R_L, D) \in \mathcal{RD}_\star(\lambda) \right\},
\]
where \( \mathcal{RD}_\star(\lambda) \) is the rate-distortion region when the distortion measure is \( d_1^\lambda \). We define \( R_o(D, \lambda) \) and \( R_{BT}(D, \lambda) \) analogously.

In Appendix A we show that if \( pL \leq D \leq 1 \), then
\[
\lim_{\lambda \to \infty} R_{BT}(D, \lambda) \leq (1 - D) \log 2 + L \left[ h\left(\frac{D^{1/L}}{1-p}\right) - (1-p) h\left(\frac{D^{1/L} - p}{1-p}\right)\right]. \tag{9}
\]
In Appendix B we show that the quantity on the right-hand side is also a lower bound to \( \lim_{\lambda \to \infty} R_o(D, \lambda) \). Hence it must equal \( \lim_{\lambda \to \infty} R_\star(D, \lambda) \). That is, the improved outer bound and the Berger-Tung inner bound together yield a conclusive result for the sum rate of the binary erasure CEO problem. Evidently this problem was previously unsolved. In Appendix C we show that \( \mathcal{RD}_o^{BT} \) contains points with a strictly smaller sum rate in general. Fig. 4 shows the correct sum rate for \( p = 0.5 \) and several values of \( L \).

### 3.3 Gaussian CEO Problem \[19, 28, 13, 14\]

We turn to a continuous example. Here \( Y_0, \ldots, Y_L \) are jointly Gaussian and \( Y_1, \ldots, Y_L \) are conditionally independent given \( Y_0 \). For \( \ell \geq 1 \), let us write \( Y_\ell = Y_0 + N_\ell \), where \( Y_0, N_1, \ldots, N_L \) are mutually independent and
\[
E[N_\ell^2] = \sigma_\ell^2 > 0 \text{ for all } \ell.
\]
We will denote the variance of \( Y_0 \) by \( \sigma^2 \). Again there is no side information, and the decoder is only interested in reproducing the hidden process \( Y_0 \),
\[
d_1(Y_0, Y, Z_1) = (Y_0 - Z_1)^2.
\]
The rate-distortion region for this problem was recently found by Oohama \[20\] \[13\] and Prabhakaran, Tse, and Ramchandran \[14\]. The two proofs are nearly the same, and build on earlier work of Oohama \[28\]. The primary contribution is the converse result, which makes heavy use of the entropy power inequality \[2, \text{ Theorem 16.6.3}\]. The Berger-Tung inner bound is used for achievability.

It is straightforward to extend Theorem 1 to this continuous setting. A statement of the continuous version is given in Appendix D where we also use the techniques of Oohama \[13\] and Prabhakaran, Tse, and Ramchandran \[14\] to prove the following.
Proposition 6  
For the Gaussian CEO problem,

\[
\mathcal{RD}_o \subset \{(R_1, \ldots, R_L, D) \in \mathbb{R}^{L+1}_+ : \text{there exists } (r_1, \ldots, r_L) \in \mathbb{R}_+^L : \text{for all } A,
\sum_{\ell \in A} R_\ell \geq \frac{1}{2} \log^+ \left[ \frac{1}{D} \left( \frac{1}{\sigma^2} + \sum_{\ell \in A^c} \frac{1 - \exp(-2r_\ell)}{\sigma^2_\ell} \right)^{-1} \right] + \sum_{\ell \in A} r_\ell \right), \tag{10}
\]

where \( \log^+ x = \max(\log x, 0) \). Since this expression equals \( \overline{RD} \), we conclude that \( \mathcal{RD}_o \) is tight in this example. It also follows that the converse result of Oohama and Prabhakaran, Tse, and Ramchandran is a consequence of the outer bound provided in this paper. This does not imply, however, that the task of proving the converse result is made any easier by our bound. In fact, comparing Appendix D to the original works shows that proving Proposition 6 is as formidable a task as proving the converse result unaided. But this is still an improvement over the Berger-Tung outer bound, the closure of which we show in Appendix D contains points outside the rate-distortion region.

We end this section by mentioning that Oohama’s converse is actually more general than the result described here, in that Oohama permits one of the encoders to make noise-free observations (i.e., \( \sigma^2_1 = 0 \)). Comparing Oohama’s proof to Appendix D shows that the outer bound supplied in this paper also recovers this more general result.
4 Recovery of Discrete Converse Results

Having seen that the new outer bound recovers the converse of Oohama [13] and Prabhakaran, Tse, and Ramchandran [14] for the Gaussian CEO problem, we show in this final section that it also recovers the converse results for the discrete problems of Slepian and Wolf [1], Wyner [6], Ahlswede and Körner [7], Wyner and Ziv [8], Gel’fand and Pinsker [10], Berger and Yeung [11], and Gastpar [12].

The outer bound also recovers the converse result for the problem studied by Körner and Marton [9], although the proof of this fact is not as interesting. We shall therefore focus on the others. To recover these converse results, we shall use the following conclusive result for a special case of the problem.

Suppose that there exists a function \( g : Y_0 \mapsto Y_0 \) such that \( Y_1, \ldots, Y_L \) are conditionally independent given \( (g(Y_0), Y_{L+1}) \). Also let \( Z_1 = Y_0 \) and

\[
d_1(Y_0, Y, Y_{L+1}, Z_1) = \begin{cases} 
0 & \text{if } Z_1 = Y_0 \\
1 & \text{otherwise.}
\end{cases}
\]

We make no other assumptions about the problem. We would like to characterize the set \( \mathcal{R}D_o \cap \{ D_1 = 0 \} \). In words, conditioned on the side information and some function of the hidden variable, the observations are independent, and the hidden variable must be reproduced losslessly. Note that \( \mathcal{R}D_o \cap \{ D_1 = 0 \} \) will be empty unless \( H(Y_0|Y, Y_{L+1}) = 0 \). Gel’fand and Pinsker [10] refer to this condition as “completeness of observations.”

Proposition 7 For this problem,

\[
\mathcal{R}D_o \cap \{ D_1 = 0 \} = \mathcal{R}D_{BT}^o \cap \{ D_1 = 0 \} = \mathcal{R}D_{\gamma} \cap \{ D_1 = 0 \} \quad (11)
\]

\[
= \bigcup_{\gamma \in \Gamma_{\gamma}^{BT}} \mathcal{R}D_{\gamma}^{BT} \cap \{ D_1 = 0 \}. 
\quad (12)
\]

Proof: To show (11), it suffices to show that \( \mathcal{R}D_o \cap \{ D_1 = 0 \} \) is contained in \( \mathcal{R}D_{BT}^o \cap \{ D_1 = 0 \} \). Suppose \( (R, \epsilon, D_2, \ldots, D_K) \) is a point in \( \mathcal{R}D_o \) and \( \epsilon \leq 1/2 \). By choosing \( X = g(Y_0) \) in Definition [2] we see that there exists \( (U, Z, W, \tilde{T}) \) in \( \Gamma_o \) such that

\[
\Pr(Z_1 \neq Y_0) \leq \epsilon
\]

\[
E[d_k(Y_0, Y, Y_{L+1}, Z_k)] \leq D_k \text{ for all } k \geq 2,
\]

and for all \( A \),

\[
\sum_{\ell \in A} R_{\ell} \geq I(g(Y_0); U_A|U_{A^c}, Y_{L+1}, \tilde{T}) + \sum_{\ell \in A} I(Y_\ell; U_\ell|g(Y_0), Y_{L+1}, W, \tilde{T}).
\]

Now

\[
I(g(Y_0); U_A|U_{A^c}, Y_{L+1}, \tilde{T}) = H(g(Y_0)|U_{A^c}, Y_{L+1}, \tilde{T})
\]

\[
- H(g(Y_0)|U, Y_{L+1}, \tilde{T}) \\
\geq H(g(Y_0)|U_{A^c}, Y_{L+1}, W, \tilde{T}) - H(g(Y_0)|g(Z_1)),
\]

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where we have used the fact that
\[ g(Y_0) \leftrightarrow Y_0 \leftrightarrow (U, Y_{L+1}, \tilde{T}) \leftrightarrow Z_1 \leftrightarrow g(Z_1). \]

By Fano’s inequality [3, Lemma 1.3.8],
\[ H(g(Y_0)|g(Z_1)) \leq h(\epsilon) + \epsilon \log(|Y_0|). \]

Thus
\[ I(g(Y_0); U_A|U_{A'}, Y_{L+1}, \tilde{T}) \geq H(g(Y_0)|U_{A'}, Y_{L+1}, W, \tilde{T}) - h(\epsilon) - \epsilon \log(|Y_0|) \]
\[ \geq I(g(Y_0); U_A|U_{A'}, Y_{L+1}, W, \tilde{T}) - h(\epsilon) - \epsilon \log(|Y_0|). \]

It follows that
\[ \sum_{\ell \in A} [R_{\ell} + h(\epsilon) + \epsilon \log(|Y_0|)] \geq I(g(Y_0); U_A|U_{A'}, Y_{L+1}, W, \tilde{T}) \]
\[ = I(g(Y_0); U_A|U_{A'}, Y_{L+1}, W, \tilde{T}) + I(Y_A, Y; U_A|U_{A'}, g(Y_0), Y_{L+1}, W, \tilde{T}) \]
\[ = I(Y_A, Y; U_A|U_{A'}, Y_{L+1}, W, \tilde{T}). \]

If we now define \( T = (W, \tilde{T}), \) it is evident that \((U, Z, T)\) is in \( \Gamma_{BT}^i \) and the point
\( (R_1 + h(\epsilon) + \epsilon \log(|Y_0|), \ldots, R_L + h(\epsilon) + \epsilon \log(|Y_0|), \epsilon, D_2, \ldots, D_K) \)

is in \( RD_i^{BT} \). This implies that
\[ RD_o \cap \{ D_1 = 0 \} \subset RD_i^{BT} \cap \{ D_1 = 0 \}, \]
which proves (11). To prove (12), it suffices to show that \( RD_i^{BT} \) is closed. This is shown in Appendix F.

The differences between this result and that of Gel’fand and Pinsker [10] are numerous but minor. The most visible differences are that Gel’fand and Pinsker’s model does not allow for side information at the decoder or distortion constraints beyond the one on \( Y_0 \). Indeed, the region given here reduces to theirs when these extensions are ignored. Thus this result seems to be a generalization of theirs, albeit a trivial one since their proof can be modified to handle these extensions. A closer comparison, however, reveals that they define the rate region more stringently than we do here. Thus, our result does not recover theirs, strictly speaking, although it does recover the converse component of their result since our definitions are weaker.
The reason for including side information and additional distortion constraints in the model is that they enable us to also recover the converse results for the other problems mentioned earlier. For instance, Gastpar\cite{gastpar} considers the problem of reproducing the observations individually, subject to separate distortion constraints, under the assumption that the decoder is provided with side information that makes the observations conditionally independent. His converse result can be recovered by setting $Y_{0} = Y_{L+1}$. It is easily verified that, under this condition, our region coincides with his. The classical Wyner-Ziv problem\cite{wynerziv} can be viewed as Gastpar’s problem with a single encoder ($L = 1$). So that converse result is recovered too.

Berger and Yeung\cite{bergeryeung} solve the two-encoder problem in which the observations are to be reproduced individually, with at least one of the two being reproduced losslessly. In our notation, this corresponds to setting $L = 2$ and $Y_{1} = Y_{0}$. Note that our conditional independence assumption necessarily holds in this case.

To see that under these assumptions, our region reduces to theirs, suppose $(R_{1}, R_{2}, D_{2}) \in \mathcal{RD}_{BT}^{PT}(\gamma) \cap \{D_{1} = 0\}$ for some $\gamma \in \Gamma_{i}^{BT}$. Then

$$R_{1} \geq I(Y_{1}; U_{1} | U_{2}, T) = H(Y_{1} | U_{2}, T).$$  \hspace{1cm} (13)

Also,

$$R_{2} \geq I(Y_{2}; U_{2} | U_{1}, T) \geq I(Y_{2}; U_{2} | U_{1}, Y_{1}, T) = I(Y_{2}; U_{2} | Y_{1}, T),$$  \hspace{1cm} (14)

where we have used the fact that

\[ U_{2} \leftrightarrow (Y_{2}, U_{1}, T) \leftrightarrow Y_{1} \]

(see Cover and Thomas\cite{coverthomas} p. 33). Finally,

$$R_{1} + R_{2} \geq I(Y_{1}, Y_{2}; U_{1}, U_{2} | T) \geq I(Y_{1}; U_{1}, U_{2} | T) + I(Y_{2}; U_{1}, U_{2} | Y_{1}, T) \geq H(Y_{1}) + I(Y_{2}; U_{2} | Y_{1}, T).$$ \hspace{1cm} (15)

It is now evident that the two regions are identical (c.f.\cite{bergeryeung} p. 230). Thus the converse result of Berger and Yeung is a consequence of the outer bound provided here.

The classical problem of source coding with side information\cite{bergeryeung, ahlswede} can be viewed as a special case of the Berger-Yeung problem in which $D_{2}$ exceeds the maximum value of $d_{2}$, the distortion measure for $Y_{2}$. Berger and Yeung demonstrate how, under this assumption, the region described above reduces to the one given by Wyner\cite{wyner} and Ahlswede and Körner\cite{ahlswede}. \textit{Ipso facto}, the converse result for this problem is also recovered.

This paper ends the way it began, with the result of Slepian and Wolf\cite{slepianwolf}. Here the aim is to losslessly reproduce all of the observations. For two encoders ($L = 2$), this can be viewed as a special case of the problem of Berger and
Yeung. These authors show how the region described in Eqs. (13)–(15) reduces to the one given at the beginning of the paper. The result for more than two encoders can be viewed as a special case of Proposition 7 in which $Y_0 = Y$. In this case, if $R \in \mathbb{R}^{D_0} \cap \{D_1 = 0\}$, then for any $A$,

$$\sum_{\ell \in A} R_\ell \geq I(Y_A; U_A | Y_{A^c}^-, T)$$

$$= H(Y_A | U_{A^c}^-, T)$$

$$\geq H(Y_A | Y_{A^c}^-, T),$$

since $Y_A \leftrightarrow (Y_{A^c}^-, T) \leftrightarrow (U_{A^c}^-, T)$. Now $Y$ is independent of $T$, so

$$\sum_{\ell \in A} R_\ell \geq H(Y_A | Y_{A^c}^-),$$

which is the well-known rate region for this problem. Thus the converse of Slepian and Wolf is also recovered. For this result, as with the others, our outer bound dispenses with the need to prove a custom converse coding theorem. In fact, Proposition 7 can be viewed as unifying all of the results in this discussion, assuming one is willing to ignore the discrepancies in the definition of the rate-distortion region mentioned earlier.

A Sum-Rate Achievability for the Binary Erasure CEO Problem

Showing that a particular rate-distortion vector is achievable using the Berger-Tung inner bound is mostly a matter of finding the proper “test channels” $Y_\ell \rightarrow U_\ell$ for the encoders. To prove (9), we use binary erasure test channels that are identically distributed across the encoders. In this appendix and the next two, the notation is drawn from Section 3.2.

Lemma 1 For any $p^L \leq D \leq 1$,

$$\lim_{\lambda \to \infty} R_i^{BT}(D, \lambda) \leq (1 - D) \log 2 + L \left[ h \left( D^{1/L} \right) - (1 - p) h \left( \frac{D^{1/L} - p}{1 - p} \right) \right].$$

Proof. Fix $D$ and let $\tilde{N}_1, \ldots, \tilde{N}_L$ be i.i.d., independent of $Y_0, \ldots, Y_L$, with

$$\Pr \left( \tilde{N}_1 = 0 \right) = \frac{D^{1/L} - p}{1 - p}$$

$$\Pr \left( \tilde{N}_1 = 1 \right) = 1 - \frac{D^{1/L} - p}{1 - p}.$$

For $\ell$ in $\{1, \ldots, L\}$, let $U_\ell = Y_\ell \cdot \tilde{N}_\ell$. Then let

$$Z_1 = \text{sgn} \left( \sum_{\ell=1}^L U_\ell \right) = \begin{cases} -1 & \text{if } \sum_{\ell=1}^L U_\ell < 0 \\ 0 & \text{if } \sum_{\ell=1}^L U_\ell = 0 \\ 1 & \text{otherwise.} \end{cases}$$
Then for all $\lambda$,

$$E \left[ d_1^*(Y_0, Y, Z_1) \right] = \Pr(U_\ell = 0 \text{ for all } \ell) = [\Pr(U_1 = 0)]^L = D.$$  

Thus $(R, D)$ is contained in $RD_i^{BT}$ for all $\lambda$ if for all $A \subset \{1, \ldots, L\}$,

$$\sum_{\ell \in A} R_\ell \geq I(Y_A; U_A|U_{A^c}).$$  

(16)

The rate vectors satisfying this collection of inequalities are known to form a contrapolymatroid [29, 26]. As such, there exist rate vectors $R$ satisfying (16) such that

$$\sum_{\ell=1}^L R_\ell = I(Y; U).$$

In particular, this holds for any vertex of [16] [29, 26]. Now

$$I(Y; U) = I(Y_0, Y; U)$$

$$= I(Y_0; U) + I(Y; U|Y_0)$$

$$= I(Y_0; U) + L I(Y_1; U_1|Y_0).$$

But $I(Y_0; U) = (1 - D) \log 2$ and

$$I(Y_1; U_1|Y_0) = H(U_1|Y_0) - H(U_1|Y_1)$$

$$= h(D^{1/L}) - (1 - p) h \left( \frac{D^{1/L} - p}{1-p} \right).$$

Then for any $\lambda$, there exist vectors $(R, D)$ in $RD_i^{BT}(\lambda)$ such that

$$\sum_{\ell=1}^L R_\ell = (1 - D) \log 2 + L \left[ h(D^{1/L}) - (1 - p) h \left( \frac{D^{1/L} - p}{1-p} \right) \right].$$

The conclusion follows. \hfill \Box

### B Sum-Rate Converse for the Binary Erasure CEO Problem

We evaluate the outer bound’s sum-rate constraint for the binary erasure CEO problem via a sequence of lemmas. Throughout this appendix, $g(\cdot)$ will denote the function on $[p, \infty)$ defined by

$$g(x) = \begin{cases} h(x) - (1 - p) h \left( \frac{x - p}{1-p} \right) & p \leq x \leq 1 \\ 0 & x > 1. \end{cases}$$

We begin by proving several facts about $g(\cdot)$. For this, the following calculations are useful.
Lemma 2  For all $x$ in $(\log p, 0]$, 
\[ e^x \log(e^x - p) - xe^x \leq -p \]  \hspace{1cm} (17) 
and 
\[ e^x \log(e^x - p) - e^x(x + 1) + \frac{e^{2x}}{e^x - p} \geq 0. \]  \hspace{1cm} (18) 

Proof. It is well known that 
\[ \log \left( \frac{1}{1 - z} \right) \geq z \text{ for all } z < 1. \]
Replacing $z$ with $pe^{-x}$ and rearranging yields (17). To see (18), note that (17) implies that the first derivative of 
\[ (e^x - p) \log(e^x - p) - (e^x - p)(x + 1) + e^x \]  \hspace{1cm} (19) 
is nonpositive on $(\log p, 0]$. Since the function in (19) is nonnegative at $x = 0$, it follows that 
\[ (e^x - p) \log(e^x - p) - (e^x - p)(x + 1) + e^x \geq 0 \]  \hspace{1cm} (20) 
for all $x$ in $(\log p, 0]$. One can now obtain (18) by multiplying both sides by $e^x$ and dividing both sides by $(e^x - p)$. □

Lemma 3  The function $g(e^x)$ is nonincreasing and convex as a function of $x$ on $[\log p, \infty)$. 

Proof. The first derivative of $g(e^x)$ on $(\log p, 0)$ is 
\[ e^x \log(e^x - p) - xe^x. \]
This observation, the first conclusion of Lemma 2 and the continuity of $g(\cdot)$ together imply that $g(e^x)$ is nonincreasing on $[\log p, 0)$. Since $g(e^x)$ is constant on $[0, \infty)$, it follows that $g(e^x)$ is nonincreasing on $[\log p, \infty)$. The second derivative of $g(e^x)$ on $(\log p, 0)$ is 
\[ e^x \log(e^x - p) - e^x(x + 1) + \frac{e^{2x}}{e^x - p}. \]
This observation, the second conclusion of Lemma 2 and the continuity of $g(\cdot)$ together imply that $g(e^x)$ is convex on $[\log p, 0]$. Since $g(e^x)$ is nonincreasing on $[\log p, \infty)$ and constant on $[0, \infty)$, it follows that $g(e^x)$ is convex on $[\log p, \infty)$. □

Corollary 1  The function $g(y^{1/L})$ is nonincreasing and convex in $y$ on $[p^L, \infty)$. 

Proof. $g(y^{1/L}) = g(e^x)$ with $x = (1/L) \log y$, and $g(e^x)$ is convex and nonincreasing while $(1/L) \log(\cdot)$ is concave and nondecreasing. □

The next lemma is central to our evaluation of the outer bound’s sum rate. Note that condition $(i)$ in the hypothesis implies that $\Pr(Y_0 \cdot Z_1 < 0) = 0$. That is, the reproduction $Z_1$ is never in error (although it may be an erasure).
Lemma 4 Suppose $p^L \leq D$ and $(U, Z_1)$ is such that

(i) $E[d_\lambda^L(Y_0, Z_1)] \leq D$ for all $\lambda$,
(ii) $U_\ell \leftrightarrow Y_\ell \leftrightarrow (Y_0, Y_{\ell'}, U_{\ell'})$ for all $\ell$, and
(iii) $(Y_0, Y) \leftrightarrow U \leftrightarrow Z_1$.

Then

$$\frac{1}{L} \sum_{\ell=1}^L I(Y_\ell; U_\ell | Y_0) \geq g(D^{1/L}).$$

Proof. For each encoder $\ell$, let

$$A_{\ell,+} = \{ u \in U_\ell : \Pr(U_\ell = u | Y_0 = 1) > 0 \}$$
$$A_{\ell,-} = \{ u \in U_\ell : \Pr(U_\ell = u | Y_0 = -1) > 0 \}.$$

Then define

$$\tilde{U}_\ell = \begin{cases} 1 & \text{if } U_\ell \in A_{\ell,+} \setminus A_{\ell,-} \\ -1 & \text{if } U_\ell \in A_{\ell,-} \setminus A_{\ell,+} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let

$$\delta_{\ell,+} = \Pr(\tilde{U}_\ell = 0 | Y_\ell = 1)$$
$$\delta_{\ell,-} = \Pr(\tilde{U}_\ell = 0 | Y_\ell = -1).$$

Then

$$\frac{1}{L} \sum_{\ell=1}^L I(Y_\ell; U_\ell | Y_0) \geq \frac{1}{L} \sum_{\ell=1}^L I(Y_\ell; \tilde{U}_\ell | Y_0)$$

$$= \frac{1}{L} \sum_{\ell=1}^L \left[ H(\tilde{U}_\ell | Y_0) - H(\tilde{U}_\ell | Y_\ell) \right]$$

$$= \frac{1}{L} \sum_{\ell=1}^L \left[ \frac{1}{2} h(p + (1-p)\delta_{\ell,+}) + \frac{1}{2} h(p + (1-p)\delta_{\ell,-}) - \frac{1}{2} (1-p) h(\delta_{\ell,+}) - \frac{1}{2} (1-p) h(\delta_{\ell,-}) \right].$$

Since $Y_0 \cdot Z_1 \geq 0$ a.s., on the event $Z_1 = 1$ we must have $Y_0 = 1$ and hence $U_\ell \in A_{\ell,+}$ for all $\ell$. In addition, the condition $Y_0 \leftrightarrow U \leftrightarrow Z_1$ dictates that when $Z_1 = 1$ we must have $U_\ell \in A_{\ell,+} \setminus A_{\ell,-}$ for some $\ell$, for otherwise we would have $\Pr(Y_0 \cdot Z_1 = -1) > 0$. All of this implies that $\text{sgn}(\sum_{\ell=1}^L \tilde{U}_\ell) = 1$ on the event that $Z_1 = 1$. Similarly, $\text{sgn}(\sum_{\ell=1}^L \tilde{U}_\ell) = -1$ on the event $Z_1 = -1$. Thus $\text{sgn}(\sum_{\ell=1}^L \tilde{U}_\ell) = 0$ implies that $Z_1 = 0$, so

$$\Pr \left( \text{sgn} \left( \sum_{\ell=1}^L \tilde{U}_\ell \right) = 0 \right) \leq \Pr(Z_1 = 0) = D.$$
This implies that
\[
\frac{1}{2} \prod_{\ell=1}^{L} (p + (1 - p)\delta_{\ell,+}) + \frac{1}{2} \prod_{\ell=1}^{L} (p + (1 - p)\delta_{\ell,-}) \leq D.
\]

Thus
\[
\frac{1}{L} \sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}| Y_{0}) \geq \inf \left\{ \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{2} \left[ h(p + (1 - p)\delta_{\ell,+}) - (1 - p)h(\delta_{\ell,+}) 
+ h(p + (1 - p)\delta_{\ell,-}) - (1 - p)h(\delta_{\ell,-}) \right] : \right. \\
\delta_{\ell,+}, \delta_{\ell,-} \in [0, 1] \text{ for all } \ell \text{ and } \\
\frac{1}{2} \prod_{\ell=1}^{L} (p + (1 - p)\delta_{\ell,+}) + \frac{1}{2} \prod_{\ell=1}^{L} (p + (1 - p)\delta_{\ell,-}) \leq D \right\}.
\]

This optimization problem is not convex, but if we change variables to
\[
\Delta_{\ell,+} = \log(p + (1 - p)\delta_{\ell,+}) \\
\Delta_{\ell,-} = \log(p + (1 - p)\delta_{\ell,-}),
\]
then it can be rewritten as
\[
\inf \left\{ \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{2} \left[ h (e^{\Delta_{\ell,+}}) - (1 - p)h \left( \frac{e^{\Delta_{\ell,+}} - p}{1 - p} \right) 
+ h (e^{\Delta_{\ell,-}}) - (1 - p)h \left( \frac{e^{\Delta_{\ell,-}} - p}{1 - p} \right) \right] : \right. \\
\Delta_{\ell,+}, \Delta_{\ell,-} \in [\log p, 0] \text{ for all } \ell \text{ and } \\
\frac{1}{2} \exp \left( \sum_{\ell=1}^{L} \Delta_{\ell,+} \right) + \frac{1}{2} \exp \left( \sum_{\ell=1}^{L} \Delta_{\ell,-} \right) \leq D \right\}
\]
\[
= \inf \left\{ \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{2} \left[ g (e^{\Delta_{\ell,+}}) + g (e^{\Delta_{\ell,-}}) \right] : \right. \\
\Delta_{\ell,+}, \Delta_{\ell,-} \in [\log p, 0] \text{ for all } \ell \text{ and } \\
\frac{1}{2} \exp \left( \sum_{\ell=1}^{L} \Delta_{\ell,+} \right) + \frac{1}{2} \exp \left( \sum_{\ell=1}^{L} \Delta_{\ell,-} \right) \leq D \right\},
\]
which is convex by Lemma 3. Thus we may assume without loss of optimality that
\[
\Delta_{1,+} = \Delta_{2,+} = \cdots = \Delta_{L,+} =: \Delta_{+}
\]
and
\[ \Delta_{1,-} = \Delta_{2,-} = \cdots = \Delta_{L,-} =: \Delta_-. \]
This gives
\[
\frac{1}{L} \sum_{\ell=1}^{L} I(Y_\ell; U_\ell|Y_0) \geq \inf \left\{ \frac{1}{2} g \left( e^{\Delta_+} \right) + \frac{1}{2} g \left( e^{\Delta_-} \right) : \Delta_+, \Delta_- \in [\log p, 0] : \frac{1}{2} e^{L\Delta_+} + \frac{1}{2} e^{L\Delta_-} \leq D \right\} \\
\geq \inf \left\{ g \left( e^{\Delta} \right) : \Delta \in [\log p, 0] : e^{L\Delta} \leq D \right\} \\
\geq g(D^{1/L}),
\]
by Lemma 3. □

The quantity \( I(Y_\ell; U_\ell|Y_0) \) can be interpreted as the amount of information that the \( \ell \)th encoder sends about its observation noise. Lemma 4 then says that if a fraction \( D \) of the output symbols is allowed to be erased and no errors are allowed, then the amount of information that the average encoder must send about its observation noise is at least \( g(D^{1/L}) \). We would like to extend this last assertion to allow “few” decoding errors instead of none. To this end, we will employ the following cardinality bound on the alphabet size of the auxiliary random variables \( U_1, \ldots, U_L \).

**Lemma 5** Let \((U, Z_1)\) be such that
\[(i)\quad U_\ell \leftrightarrow Y_\ell \leftrightarrow (Y_0, Y_\ell^c, U_\ell^c) \text{ for all } \ell, \text{ and} \]
\[(ii)\quad Y_0, Y \leftrightarrow U \leftrightarrow Z_1. \]

Then for any \( \lambda \), there exist alternate random variables \( \tilde{U} \) and \( \tilde{Z}_1 \) also satisfying (i) and (ii) such that
\[ E[d_\lambda^1(Y_0, \tilde{Z}_1)] \leq E[d_\lambda^1(Y_0, Z_1)], \]
\[ I(Y_\ell; \tilde{U}_\ell|Y_0) = I(Y_\ell; U_\ell|Y_0) \text{ for all } \ell, \]
and
\[ |\tilde{U}| \leq |Y_\ell| + 1 \text{ for all } \ell. \]

See Wyner and Ziv [8, Theorem A2] or Csiszár and Körner [3, Theorem 3.4.6] for proofs of similar results. The next lemma is the desired extension of Lemma 4.

**Lemma 6** Suppose \( p^L \leq D \) and \((U, Z_1)\) is such that
\[(i)\quad E[d_\lambda^1(Y_0, Z_1)] \leq D, \]
\[(ii)\quad U_\ell \leftrightarrow Y_\ell \leftrightarrow (Y_0, Y_\ell^c, U_\ell^c) \text{ for all } \ell, \text{ and} \]

---

3This terminology is due to Prabhakaran, Tse, and Ramachandran.
(iii) $(Y_0, Y) \leftrightarrow U \leftrightarrow Z_1$.

If
\[
\frac{32L}{p(1-p)} \left( \frac{2D}{X} \right)^{1/L} \leq \delta \leq \frac{1}{2},
\]
then
\[
\frac{1}{L} \sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}|Y_0) \geq g \left( (D + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5}.
\]

Proof. By Lemma 5, we may assume that $U_{\ell} = \{1, \ldots, 4\}$ for each $\ell$. We may also assume that $Z_1$ is a deterministic function of $U$: $Z_1 = \phi(U)$. Define
\[
A_{\ell,+} = \left\{ u_{\ell} \in U_{\ell} : \exists u_{\ell} : \phi(u) = 1 \text{ and } \Pr(U_{\ell} = u_{\ell}|Y_0 = -1) = \min_{j \in \{1, \ldots, L\}} \Pr(U_{\ell} = u_j|Y_0 = -1) \right\}
\]
\[
A_{\ell,-} = \left\{ u_{\ell} \in U_{\ell} : \exists u_{\ell} : \phi(u) = -1 \text{ and } \Pr(U_{\ell} = u_{\ell}|Y_0 = 1) = \min_{j \in \{1, \ldots, L\}} \Pr(U_{\ell} = u_j|Y_0 = 1) \right\}.
\]

We now define random variables $(\tilde{U}, \tilde{Z})$ to replace $(U, Z)$. The replacements will be close to the originals in distribution but will have the property that $\Pr(Y_1 \cdot \tilde{Z}_1 < 0) = 0$. That is, $\tilde{Z}_1$ will never be in error. Set $U_{\ell} = \{1, \ldots, 5\}$ for each $\ell$, and let
\[
\Pr(\tilde{U}_{\ell} = i|Y_0 = 1) = \begin{cases} 0 & \text{if } i \in A_{\ell,+} \\ \Pr(U_{\ell} \in A_{\ell,+}|Y_0 = 1) & \text{if } i = 5 \\ \Pr(U_{\ell} = i|Y_0 = 1) & \text{otherwise} \end{cases}
\]
\[
\Pr(\tilde{U}_{\ell} = i|Y_0 = -1) = \begin{cases} 0 & \text{if } i \in A_{\ell,+} \\ \Pr(U_{\ell} \in A_{\ell,+}|Y_0 = -1) & \text{if } i = 5 \\ \Pr(U_{\ell} = i|Y_0 = -1) & \text{otherwise} \end{cases}
\]
\[
\Pr(\tilde{U}_{\ell} = i|Y_0 = 0) = \begin{cases} 0 & \text{if } i \in A_{\ell,+} \cup A_{\ell,-} \\ \Pr(U_{\ell} \in A_{\ell,+} \cup A_{\ell,-}|Y_0 = 0) & \text{if } i = 5 \\ \Pr(U_{\ell} = i|Y_0 = 0) & \text{otherwise} \end{cases}
\]

Then define
\[
\tilde{Z}_1 = \tilde{\phi}(\tilde{U}) := \begin{cases} \phi(\tilde{u}) & \text{if } \tilde{u}_\ell \leq 4 \text{ for all } \ell \\ 0 & \text{otherwise} \end{cases}
\]
There is a natural way of coupling $\tilde{U}$ to $U$ such that if $\tilde{U}_\ell$ is in $\{1, \ldots, 4\}$ then $\tilde{U}_\ell = U_\ell$. With this coupling in mind, it is evident that

$$E[d_1(\tilde{Y}_0, \tilde{Z}_1)] = E[d_1(\tilde{Y}_0, \tilde{Z}_1) \mathbb{1}(\max(\tilde{U}_1, \ldots, \tilde{U}_L) \leq 4)]$$

$$+ E[d_1(\tilde{Y}_0, \tilde{Z}_1) \mathbb{1}(\max(\tilde{U}_1, \ldots, \tilde{U}_L) = 5)]$$

$$\leq E[d_1(Y_0, Z_1) \mathbb{1}(\max(U_1, \ldots, U_L) \leq 4)]$$

$$+ \Pr(\max(\tilde{U}_1, \ldots, \tilde{U}_L) = 5)$$

$$\leq D + \Pr(\max(U_1, \ldots, U_L) = 5).$$

Now for any $\ell$ in $\{1, \ldots, L\}$,

$$\Pr(\tilde{U}_\ell = 5) = \frac{1-p}{2} \Pr(U_\ell \in A_{\ell,-}|Y_\ell = 1) + \frac{1-p}{2} \Pr(U_\ell \in A_{\ell,+}|Y_\ell = -1)$$

$$+ p \Pr(U_\ell \in A_{\ell,+} \cup A_{\ell,-}|Y_\ell = 0).$$

By the union bound, this is upper bounded by

$$\left[\frac{1-p}{2} \Pr(U_\ell \in A_{\ell,-}|Y_\ell = 1) + p \Pr(U_\ell \in A_{\ell,-}|Y_\ell = 0)\right]$$

$$+ \left[\frac{1-p}{2} \Pr(U_\ell \in A_{\ell,+}|Y_\ell = -1) + p \Pr(U_\ell \in A_{\ell,+}|Y_\ell = 0)\right].$$

Since $U_\ell \leftrightarrow Y_\ell \leftrightarrow Y_0$,

$$\Pr(\tilde{U}_\ell = 5) \leq \left[\frac{1-p}{2} \Pr(U_\ell \in A_{\ell,-}|Y_\ell = 1, Y_0 = 1)\right]$$

$$+ p \Pr(U_\ell \in A_{\ell,-}|Y_\ell = 0, Y_0 = 1)$$

$$+ \left[\frac{1-p}{2} \Pr(U_\ell \in A_{\ell,+}|Y_\ell = -1, Y_0 = -1)\right]$$

$$+ p \Pr(U_\ell \in A_{\ell,+}|Y_\ell = 0, Y_0 = -1)$$

$$\leq \Pr(U_\ell \in A_{\ell,-}|Y_0 = 1) + \Pr(U_\ell \in A_{\ell,+}|Y_0 = -1).$$

But

$$\sum_{u: \phi(u) = 1} \frac{1}{2} \prod_{j=1}^L \Pr(U_j = u_j|Y_0 = -1) \lambda \leq D,$$

which implies

$$\sum_{u: \phi(u) = 1, u_\ell \in A_{\ell,+}} \prod_{j=1}^L \Pr(U_j = u_j|Y_0 = -1) \leq \frac{2D}{\lambda}. \quad (21)$$
By the definition of $A_{\ell,+}$, for each $u_{\ell} \in A_{\ell,+}$, there exists at least one $u_{c}$ such that $\phi(u) = 1$ and

$$
\prod_{j=1}^{L} Pr(U_j = u_j|Y_0 = -1) \geq Pr(U_{\ell} = u_{\ell}|Y_0 = -1)^L.
$$

Together with (21), this implies

$$
\sum_{u_{\ell} \in A_{\ell,+}} Pr(U_{\ell} = u_{\ell}|Y_0 = -1)^L \leq \frac{2D}{\lambda}.
$$

Applying Hölder’s inequality [30, p. 121] gives

$$
Pr(U_{\ell} \in A_{\ell,+}|Y_0 = -1) = \sum_{u_{\ell} \in A_{\ell,+}} Pr(U_{\ell} = u_{\ell}|Y_0 = -1)
\leq \left[ \sum_{u_{\ell} \in A_{\ell,+}} Pr(U_{\ell} = u_{\ell}|Y_0 = -1)^L \right]^{1/L} \cdot |A_{\ell,+}|^{(L-1)/L}
\leq 4 \left( \frac{2D}{\lambda} \right)^{1/L}.
$$

Likewise,

$$
Pr(U_{\ell} \in A_{\ell,-}|Y_0 = 1) \leq 4 \left( \frac{2D}{\lambda} \right)^{1/L}.
$$

Thus

$$
Pr(\tilde{U}_{\ell} = 5) \leq 8 \left( \frac{2D}{\lambda} \right)^{1/L}.
$$

By the union bound, it follows that

$$
Pr(\max(\tilde{U}_1, \ldots, \tilde{U}_L) = 5) \leq 8L \left( \frac{2D}{\lambda} \right)^{1/L}
$$

and therefore

$$
E[d_1^2(Y_0, \tilde{Z}_1)] \leq D + 8L \left( \frac{2D}{\lambda} \right)^{1/L} \leq D + \delta.
$$

Note that $\tilde{Z}_1 = 1$ only if $\tilde{U}_{\ell}$ is in $A_{\ell,+}$ for some $\ell$, and

$$
Pr(\tilde{U}_{\ell} \in A_{\ell,+}|Y_0 = -1) = 0 \text{ for all } \ell.
$$

Thus $Pr(Y_0 = -1, \tilde{Z}_1 = 1) = 0$ and similarly, $Pr(Y_0 = 1, \tilde{Z}_1 = -1) = 0$. It follows from Lemma 3 that

$$
\frac{1}{L} \sum_{\ell=1}^{L} I(Y_\ell; \tilde{U}_{\ell}|Y_0) \geq g \left( (D + \delta)^{1/L} \right).
$$

(22)
The remainder of the proof is devoted to showing that $I(Y_{\ell}; \tilde{U}_{\ell}|Y_{0})$ is close to $I(Y_{\ell}; U_{\ell}|Y_{0})$. For this we use the decomposition

$$I(Y_{\ell}; \tilde{U}_{\ell}|Y_{0}) = H(\tilde{U}_{\ell}|Y_{0}) - H(\tilde{U}_{\ell}|Y_{\ell}).$$

Observe that

$$\Pr(\tilde{U}_{\ell} = 5|Y_{\ell} = 0)p \leq \Pr(\tilde{U}_{\ell} = 5) \leq 8 \left(\frac{2D}{\lambda}\right)^{1/L}.$$ 

Thus

$$\Pr(\tilde{U}_{\ell} = 5|Y_{\ell} = 0) \leq \frac{\delta}{2}.$$ 

Similarly,

$$\Pr(\tilde{U}_{\ell} = 5|Y_{\ell} = 1) \leq \frac{\delta}{2}$$

and

$$\Pr(\tilde{U}_{\ell} = 5|Y_{\ell} = -1) \leq \frac{\delta}{2}.$$ 

Therefore if we view $U_{\ell}$ as a random variable on $\{1, \ldots, 5\}$, for any $i$ in $\{-1, 0, 1\}$,

$$\sum_{j=1}^{5} |\Pr(\tilde{U}_{\ell} = j|Y_{\ell} = i) - \Pr(U_{\ell} = j|Y_{\ell} = i)| = 2 \Pr(\tilde{U}_{\ell} = 5|Y_{\ell} = i) \leq \delta.$$ 

A standard result on the continuity of entropy [3, Lemma 1.2.7] now implies that (recall $\delta \leq 1/2$)

$$|H(\tilde{U}_{\ell}|Y_{\ell} = i) - H(U_{\ell}|Y_{\ell} = i)| \leq -\delta \log \frac{\delta}{5}$$

so

$$|H(\tilde{U}_{\ell}|Y_{\ell}) - H(U_{\ell}|Y_{\ell})| \leq -\delta \log \frac{\delta}{5}.$$ 

Likewise, for any $i$ in $\{-1, 1\}$,

$$\frac{1}{2} \Pr(\tilde{U}_{\ell} = 5|Y_{0} = i) \leq 8 \left(\frac{2D}{\lambda}\right)^{1/L}.$$ 

Thus

$$\Pr(\tilde{U}_{\ell} = 5|Y_{0} = i) \leq \frac{\delta}{2},$$

so

$$|H(\tilde{U}_{\ell}|Y_{0}) - H(U_{\ell}|Y_{0})| \leq -\delta \log \frac{\delta}{5}$$

as before. It follows that

$$|I(Y_{\ell}; \tilde{U}_{\ell}|Y_{0}) - I(Y_{\ell}; U_{\ell}|Y_{0})| \leq -2\delta \log \frac{\delta}{5}.$$ 

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Combining this with (22) yields
\[
\frac{1}{T} \sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}| Y_0) \geq g \left( (D + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5}.
\]

We are now in a position to prove the main result of this Appendix.

**Lemma 7** For any \( p_L \leq D \),
\[
\lim_{\lambda \to \infty} R_o(D, \lambda) \geq (1 - D) \log 2 + Lg \left( D^{1/L} \right).
\]

**Proof.** Fix \( p_L \leq D \leq 1 \) and \( \delta \in (0, 1/2] \), and suppose \( \lambda \) satisfies \( \lambda \geq \max \left[ 4 \left( \frac{32L}{\delta p(1-p)} \right)^{2L}, \left( \frac{D}{\delta} \right)^2 \right] \).

By (23)
\[
\sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}| Y_0) \geq g \left( (D + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5}.
\]

For each possible realization \((w, t)\) of \((W, T)\), let
\[
D_{w,t} = E[d_1(Y_0, Z_1)|W = w, T = t].
\]

Let \( S = \{(w, t) : D_{w,t} \leq \sqrt{A}\} \). Then by Markov’s inequality,
\[
Pr((W, T) \notin S) \leq \frac{D}{\sqrt{A}} \leq \delta.
\]

In particular, \( Pr((W, T) \in S) > 0 \). Also, for any \((w, t) \in S\),
\[
\frac{32L}{p(1-p)} \left( \frac{2D_{w,t}}{\lambda} \right)^{1/L} \leq \delta
\]
by (23). Thus, by Lemma 6 if \((w, t) \in S\),
\[
\frac{1}{L} \sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}| Y_0, W = w, T = t) \geq g \left( (D_{w,t} + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5}.
\]

By averaging over \((w, t) \in S\) and invoking Corollary 1, we obtain
\[
\sum_{(w, t) \in S} \frac{1}{L} \sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}| Y_0, W = w, T = t) \cdot \frac{Pr(W = w, T = t)}{Pr((W, T) \in S)} \geq g((D + \delta)^{1/L}) + 2\delta \log \frac{\delta}{5}.
\]
From (25), it follows that
\[
\sum_{\ell=1}^{L} I(Y_{\ell}; U_{\ell}|Y_{0}, W, T) \geq L(1 - \delta) \left[ g \left( (D + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5} \right].
\] (26)

Now by the data processing inequality,
\[
I(Y_{0}; U|T) = I(Y_{0}; U, T) \geq I(Y_{0}; Z_{1}).
\]

Let \( \varepsilon = 1(Y_{0} \cdot Z_{1} = -1) \). Continuing,
\[
I(Y_{0}; U|T) \geq H(Y_{0}) - H(Y_{0}|Z_{1})
\]
\[
= \log 2 - H(Y_{0}, \varepsilon|Z_{1})
\]
\[
= \log 2 - H(\varepsilon|Z_{1}) - H(Y_{0}|\varepsilon, Z_{1})
\]
\[
\geq \log 2 - h(D/\lambda) - \Pr(Z_{1} = 0) \log 2
\]
\[
\geq (1 - D) \log 2 - h(\delta).
\]

Substituting this and (26) into (24) yields
\[
R_{o}(D, \lambda) \geq (1 - D) \log 2 - h(\delta) + L(1 - \delta) \left[ g \left( (D + \delta)^{1/L} \right) + 2\delta \log \frac{\delta}{5} \right] - \delta.
\]

The proof is terminated by letting \( \lambda \to \infty \) and then \( \delta \to 0 \). \( \square \)

**C  The Berger-Tung Outer Bound is Loose for the Binary Erasure CEO Problem**

We will show numerically that for one instance of the binary erasure CEO problem, \( R_{o}^{BT} \) contains points with a strictly superoptimal sum rate. Let \( L = 2 \) and \( p = 1/2 \). Let \( W_{1} \) and \( W_{2} \) be \( \{0, 1\} \)-valued random variables with the joint distribution
\[
\begin{bmatrix}
1/5 & 2/5 \\
2/5 & 0
\end{bmatrix},
\]
i.e.,
\[
\Pr(W_{1} = 0, W_{2} = 0) = \frac{1}{5}
\]
\[
\Pr(W_{1} = 1, W_{2} = 0) = \Pr(W_{1} = 0, W_{2} = 1) = \frac{2}{5}.
\]

We assume that \((W_{1}, W_{2})\) is independent of \((Y_{0}, Y_{1}, Y_{2})\). Let \( U_{\ell} = Y_{\ell} \cdot W_{\ell} \) for \( \ell \) in \( \{1, 2\} \), and let \( Z_{1} = \text{sgn}(U_{1} + U_{2}) \). Since \( Y_{\ell} \) can be written as \( Y_{\ell} = Y_{0} \cdot N_{\ell} \) where \( N_{1} \) and \( N_{2} \) are i.i.d. with \( \Pr(N_{1} = 0) = \Pr(N_{1} = 1) = 1/2 \) (recall the
notation of Section 3.2), we have \( U_\ell = Y_0 \cdot N_\ell \cdot W_\ell \). Note that \( N_1 \cdot W_1 \) and \( N_2 \cdot W_2 \) have the joint distribution
\[
\begin{bmatrix}
3/5 & 1/5 \\
1/5 & 0
\end{bmatrix}.
\]
Thus for any \( \lambda \), \( E[d_1^\lambda (Y_0, Z_1)] = \Pr(N_1 \cdot W_1 = N_2 \cdot W_2 = 0) = 3/5 \). Now we can compute
\[
I(Y_1, Y_2; U_1, U_2) \leq 0.6273 \text{ nats}
\]
and
\[
I(Y_1, Y_2; U_1 | U_2) = I(Y_1, Y_2; U_1, U_2) - I(Y_1, Y_2; U_2) \\
\leq 0.3248 \text{ nats}.
\]
It follows that \( (0.3248, 0.3248, 3/5) \) is in \( \mathcal{RD}^\text{BT}_\lambda \) for any \( \lambda \). Thus
\[
\lim_{\lambda \to \infty} \inf \left\{ R_1 + R_2 : \left( R_1, R_2, \frac{3}{5} \right) \in \overline{\mathcal{RD}^\text{BT}_\lambda} \right\} \leq 0.6496 \text{ nats}.
\]
From the previous two appendices, the correct sum rate is
\[
\left( 1 - \frac{3}{5} \right) \log 2 + 2g \left( \sqrt{\frac{3}{5}} \right) \geq 0.6562 \text{ nats}.
\]

**D Evaluation of the Outer Bound for the Gaussian CEO Problem**

Two lemmas are needed for our proof of Proposition 6. The first is a simple extension of Theorem 1 to the Gaussian CEO problem setting of Section 3.3. For this appendix, let us redefine \( \chi \) to be the set of real-valued random variables \( X \) such that \( Y_1, \ldots, Y_L \) are conditionally independent given \( X \) (the side information \( Y_{L+1} \) is unneeded in this context and shall be ignored). Let us also redefine \( \Gamma_o \) to be the set of random variables \( (U_1, \ldots, U_L, Z_1, W, T) \) such that each takes values in a finite-dimensional Euclidean space, and collectively they satisfy the Markov conditions defining the original \( \Gamma_o \),

(i) \((W, T)\) is independent of \((Y_0, Y)\),

(ii) \(U_\ell \leftrightarrow (Y_\ell, W, T) \leftrightarrow (Y_0, Y_\ell, U_\ell)\) for all \( \ell \), and

(iii) \((Y_0, Y, W) \leftrightarrow (U, T) \leftrightarrow Z_1\),

and one new technical condition,

(iv) the conditional distribution of \( U_\ell \) given \( W \) and \( T \) is discrete for each \( \ell \).
Note that any conditional distribution involving these random variables is well-defined. As such, so is any conditional mutual information. 

Lemma 8 For the Gaussian CEO problem, $\mathcal{RD}_* \subset \mathcal{RD}_o$ if $\mathcal{RD}_o$ is defined using the $\chi$ and $\Gamma_o$ just described.

The proof follows the original and is omitted. The second ingredient is a consequence of an intriguing result of Oohama and Prabhakaran, Tse, and Ramchandran that relates information the encoders send about the hidden source to information they send about their observation “noise.”

Lemma 9 (c.f. Lemma 3) If $\gamma$ is in $\Gamma_o$, then for all $A \subset \{1, \ldots, L\}$,

$$\exp(2I(Y_0; U_A|W,T)) \leq 1 + \sum_{\ell \in A} \frac{1 - \exp(-2I(Y_0; U_\ell|Y_0, W,T))}{\sigma^2}.$$ 

Proof. For any realization of $(W,T)$, it follows from Lemma 3 in Oohama that

$$\exp(2I(Y_0; U_A|W=w, T=t)) \leq 1 + \sum_{\ell \in A} \frac{1 - \exp(-2I(Y_0; U_\ell|Y_0, W=w, T=t))}{\sigma^2}.$$ 

We now average over $(w, t)$ and invoke the convexity of $\exp(\cdot)$ twice, once on each side. □

Proof of Proposition 7 If $(R, D)$ is in $\mathcal{RD}_o$, then there exists $\gamma$ in $\Gamma_o$ such that $E[(Y_0 - Z_1)^2] \leq D$ and for all $A \subset \{1, \ldots, L\}$,

$$\sum_{\ell \in A} R_\ell \geq I(Y_0; U_A|U_{A^c}, T) + \sum_{\ell \in A} I(Y_0; U_\ell|Y_0, W, T). \tag{27}$$

Now

$$I(Y_0; U_A|U_{A^c}, T) + I(Y_0; U_{A^c}|T) = I(Y_0; U|T). \tag{28}$$

Since $Y_0 \leftrightarrow (U, T) \leftrightarrow Z_1$, the right-hand side can be lower bounded as follows

$$I(Y_0; U|T) = I(Y_0; U, T) \geq I(Y_0; Z_1) \geq \frac{1}{2} \log^{+} \frac{\sigma^2}{E[(Y_0 - Z_1)^2]},$$

where we have used the rate-distortion theorem for Gaussian sources. In particular,

$$I(Y_0; U|T) \geq \frac{1}{2} \log \frac{\sigma^2}{D}. \tag{29}$$

\footnote{Oohama’s result assumes that $U_\ell$ is a discrete, deterministic function of $Y_\ell$ for each $\ell$, but the proof shows that conditions (ii) and (iv) above are actually sufficient.}
Let us address the second term on the left-hand side of (28). Observe that
\[ I(Y_0; U_{A^c}|T) + I(Y_0; W|U_{A^c}, T) = I(Y_0; W|T) + I(Y_0; U_{A^c}|W, T). \]

Defining \( r_\ell = I(Y_\ell; U_\ell|Y_0, W, T) \) and applying Lemma 9 to the right-hand side gives
\[ I(Y_0; U_{A^c}|T) \leq \frac{1}{2} \log \left( 1 + \frac{1}{\sigma_W^2} \sum_{\ell \in A^c} \frac{1 - \exp(-2r_\ell)}{\sigma_\ell^2/\sigma_W^2} \right). \] (30)

Substituting (29) and (30) into (28) gives
\[ I(Y_0; U_{A^c}|U_{A^c}, T) \geq \frac{1}{2} \log \left( \frac{1}{D} + \frac{1}{\sigma_W^2} \sum_{\ell \in A^c} \frac{1 - \exp(-2r_\ell)}{\sigma_\ell^2} \right)^{-1}. \]

The conclusion follows upon substitution of this inequality and the definition of \( r_\ell \) into (27). \( \square \)

E The Berger-Tung Outer Bound is Loose for the Gaussian CEO Problem

We have just seen that the improved outer bound is capable of recovering the converse result of Oohama [13] and Prabhakaran, Tse, and Ramchandran [14] for the Gaussian CEO problem. Here we will show that the Berger-Tung outer bound does not recover this result. As with the binary erasure CEO problem, we will show that, in general, the Berger-Tung outer bound contains points with a strictly superoptimal sum rate.

Consider the case in which, in the notation of Section 3.3, \( L = 2 \) and \( \sigma^2 = \sigma_1^2 = \sigma_2^2 = 1 \). In words, two encoders each observe a unit variance, i.i.d. Gaussian process in additive Gaussian noise with a signal-to-noise ratio of unity. It follows from Proposition 6 that the minimum sum rate needed to achieve the distortion \( 1/2 \) is at least \( (3/2) \log 2 \) nats.

Let \( W, V_1, \) and \( V_2 \) be Gaussian random variables, independent of each other and \( Y_0, Y_1, \) and \( Y_2 \). Let \( V_1 \) and \( V_2 \) have unit variance; we denote the variance of \( W \) by \( \sigma_W^2 \). Let

\[ U_1 = Y_1 + V_1 + W, \]
\[ U_2 = Y_2 + V_2 - W. \]

Note that the sum of \( U_1 \) and \( U_2 \) is a sufficient statistic for \( Y_0 \) given \( U_1 \) and \( U_2 \). This observation makes it easy to verify that if \( Z_1 = E[Y_0|U_1, U_2] \), then \( E[(Y_0 - Z_1)^2] = 1/2 \). Note that this distortion is independent of \( \sigma_W^2 \).

It follows that for any value of \( \sigma_W^2 \), \( \mathcal{R}^{BT}_D \) contains points of the form \((R_1, R_2, 1/2)\) with

\[ R_1 + R_2 = \max(I(Y_1, Y_2; U_1, U_2), 2I(Y_1, Y_2; U_1|U_2)). \]
But [2, Theorem 9.4.1]

\[ I(Y_1, Y_2; U_1, U_2) = \frac{3}{2} \log 2 + \frac{1}{2} \log \frac{1 + \sigma_W^2}{1 + 2\sigma_W^2} \]

and

\[ I(Y_1, Y_2; U_1|U_2) = I(Y_1, Y_2; U_1, U_2) - I(Y_1, Y_2; U_2) \]
\[ = \frac{3}{2} \log 2 + \frac{1}{2} \log \frac{1 + \sigma_W^2}{1 + 2\sigma_W^2} - \frac{1}{2} \log \left(1 + \frac{2}{\sigma_W^2 + 1}\right). \]

Observe that, when viewed as functions of \( \sigma_W^2 \), \( I(Y_1, Y_2; U_1, U_2) \) is strictly decreasing and \( I(Y_1, Y_2; U_1|U_2) \) is continuous. Since \( \sigma_W^2 = 0 \) yields

\[ I(Y_1, Y_2; U_1, U_2) = \frac{3}{2} \log 2 \]
\[ 2I(Y_1, Y_2; U_1|U_2) < \frac{3}{2} \log 2, \]

it follows that there exists \( \sigma_W^2 > 0 \) such that

\[ \max(I(Y_1, Y_2; U_1, U_2), 2I(Y_1, Y_2; U_1|U_2)) < \frac{3}{2} \log 2. \]

\section{F \( \mathcal{RD}_i^{BT} \) is Closed}

The main step in proving that \( \mathcal{RD}_i^{BT} \) is closed is to show that one can limit the ranges of the auxiliary random variables without reducing the region.

\textbf{Definition 5} Let \( \hat{\Gamma}_i^{BT} \) denote the set of finite-alphabet random variables

\[ \gamma = (U_1, \ldots, U_L, Z_1, \ldots, Z_K, T) \]

in \( \Gamma_i^{BT} \) such that

\[ |U_\ell| = |Y_\ell| + 2^L + K - 1 \text{ for all } \ell \]

and

\[ |T| = 2^L + K. \]

Then let

\[ \hat{\mathcal{RD}}_i^{BT} = \bigcup_{\gamma \in \hat{\Gamma}_i^{BT}} \mathcal{RD}_i^{BT}(\gamma). \]

We shall show that \( \mathcal{RD}_i^{BT} = \hat{\mathcal{RD}}_i^{BT} \) in two steps, first handling the case in which \( T \) is deterministic, and then bootstrapping to the general case. Both steps involve now-standard uses of Carathéodory’s theorem [32, Theorem 17.1]. We give proofs of both steps, albeit condensed ones, due to the complexity of our setup.
Lemma 10 Suppose that \( \gamma = (U, Z, T) \) is in \( RD_i^{BT} \) and \( T \) is deterministic. Then there exists \( \hat{\gamma} = (\hat{U}, \hat{Z}, \hat{T}) \) in \( RD_i^{BT} \) such that \( \hat{T} \) is deterministic and \( RD_i^{BT}(\gamma) = RD_i^{BT}(\hat{\gamma}) \).

Proof. For any \( A \subset \{1, \ldots, L\} \) containing 1, we have

\[
I(Y_A; U_A|A^c) = H(Y_A|A^c) - H(Y_A|U) = H(Y_A|A^c) - \sum_{u_1} H(Y_A|U_1, U_1 = u_1) \Pr(U_1 = u_1),
\]

while for any nonempty \( A \) not containing 1, we have

\[
I(Y_A; U_A|A^c) = \sum_{u_1} I(Y_A; U_A|A^c \{1\}, U_1 = u_1) \Pr(U_1 = u_1).
\]

Carathéodory’s theorem guarantees that we can find a \( \hat{U}_1 \) with \( \hat{U}_1 \subset U_1 \) such that \(|\hat{U}_1| = |Y_1| + 2^L + K - 1\),

\[
\sum_{u_1 \in U_1} \Pr(Y_1 = y_1|U_1 = u_1) \Pr(\hat{U}_1 = u_1) = \Pr(Y_1 = y_1) \quad \text{for all } y_1 \text{ in } Y_1 \text{ but one},
\]

and similarly for \( I(Y_A; U_A|A^c \{1\}, U_1 = u_1) \) and \( E[d_k(Y_0, Y, Y_{L+1}, Z_k)|U_1 = u_1] \). Since \( U_1 \leftrightarrow Y_1 \leftrightarrow (Y_0, Y_{1c}, U_{1c}, Y_{L+1}) \), if we substitute \( \hat{U}_1 \) for \( U_1 \), the resulting \( \gamma \) is in \( \Gamma_i^{BT} \) and \( RD_i^{BT}(\gamma) \) is unchanged. Repeating this procedure for \( U_2, \ldots, U_L \) completes the proof. \( \square \)

Lemma 11 \( RD_i^{BT} = RD_i^{BT} \).

Proof. Let \( (U, Z, T) \) be in \( \Gamma_i^{BT} \). For each \( t \) in \( T \), let \((U, Z, t)\) denote the joint distribution of \((U, Z, T)\) conditioned on the event \( \{T = t\} \). By Lemma \( \text{[1]} \) for each \( t \), there exists \( (\hat{U}, \hat{Z}) \) such that \((\hat{U}, \hat{Z}, t)\) is in \( \Gamma_i^{BT} \) and \( RD_i^{BT}(U, Z, t) = RD_i^{BT}(\hat{U}, \hat{Z}, t) \). By replacing \((U, Z)\) with \((\hat{U}, \hat{Z})\) for each value of \( T \), we obtain \((\hat{U}, \hat{Z}, T)\) in \( \Gamma_i^{BT} \) such that \( |\hat{U}_t| = |Y_t| + 2^L + K - 1 \) for all \( t \) and \( RD_i^{BT}(U, Z, T) = RD_i^{BT}(\hat{U}, \hat{Z}, T) \). Now

\[
RD_i^{BT}(\hat{U}, \hat{Z}, T) = \left\{ (R, D) : \sum_{t \in A} R_t \geq \sum_{t \in T} I(Y_A; U_A|A^c, Y_{L+1}, T = t) \Pr(T = t) \quad \text{for all } A, \right. \quad \text{and} \quad \left. D_k \geq \sum_{t \in T} E[d_k(Y_0, Y, Y_{L+1}, Z_k)|T = t] \Pr(T = t) \quad \text{for all } k \right\}.
\]
Carathéodory’s theorem implies that we can find a $\hat{T}$ with $\hat{T} \subset T$ and $|\hat{T}| = 2^L + K$ such that

$$\sum_{t \in T} I(Y_A; U_{A^c}, Y_{L+1}, T = t) \Pr(T = t)$$

$$= \sum_{t \in \hat{T}} I(Y_A; U_{A^c}, Y_{L+1}, T = t) \Pr(\hat{T} = t)$$

for all $A$ and similarly for $E[d_k(Y_0, Y, Y_{L+1}, Z_k)|T = t]$. Then $(\hat{U}, \hat{Z}, \hat{T})$ is in $\hat{\Gamma}_i^{BT}$ and

$$\mathcal{R}\mathcal{D}_i^{BT}(U, Z, T) = \mathcal{R}\mathcal{D}_i^{BT}(\hat{U}, \hat{Z}, \hat{T}).$$

Since $(U, Z, T)$ in $\Gamma_i^{BT}$ was arbitrary, it holds $\mathcal{R}\mathcal{D}_i^{BT} \subset \mathcal{R}\mathcal{D}_i^{BT}$. This completes the proof since the reverse containment is obvious.

The cardinality bounds provided by the last two lemmas, while finite, are exponential in $L$ and hence impractical for moderate numbers of encoders. One can improve upon these bounds by exploiting the polymatroid structure \cite{29, 26} of $\mathcal{R}\mathcal{D}_i^{BT}$. While this would be useful if one wished to numerically evaluate the bound, our aim here is merely to show that it is closed.

**Lemma 12** $\hat{\mathcal{R}}\mathcal{D}_i^{BT}$ is closed.

**Proof.** The Markov conditions defining $\hat{\Gamma}_i^{BT}$ can be expressed as

$$I(T; Y_0, Y, Y_{L+1}) = 0$$

$$I(U_\ell; Y_0, Y, Y_{L+1}, U_{\ell^c}, T) = 0$$

for all $\ell$ and

$$I(Y_0, Y, Z|U, Y_{L+1}, T) = 0.$$

Since the conditional mutual information function is continuous, $\hat{\Gamma}_i^{BT}$ is compact when viewed as a subset of Euclidean space. Thus if $(\mathbf{R}^{(n)}, \mathbf{D}^{(n)})$ is a sequence in $\mathcal{R}\mathcal{D}_i^{BT}$ that converges to $(\mathbf{R}, \mathbf{D})$, by considering subsequences we may assume that $(\mathbf{R}^{(n)}, \mathbf{D}^{(n)})$ is in $\mathcal{R}\mathcal{D}_i^{BT}(\gamma^{(n)})$ for each $n$ and $\gamma^{(n)} \to \gamma = (U, Z, T) \in \hat{\Gamma}_i^{BT}$. By invoking the continuity of mutual information once again, we obtain

$$\sum_{\ell \in A} R_{\ell} \geq I(Y_A; U_{A^c}, T)$$

for each $A$. Likewise,

$$D_k \geq E[d_k(Y_0, Y, Y_{L+1}, Z_k)]$$

for all $k$.

It follows that $(\mathbf{R}, \mathbf{D})$ is in $\mathcal{R}\mathcal{D}_i^{BT}(\gamma)$ and therefore also in $\hat{\mathcal{R}}\mathcal{D}_i^{BT}$. □

**Corollary 2** $\mathcal{R}\mathcal{D}_i^{BT}$ is closed.
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