FINITE ENERGY GLOBAL WELL-POSEDNESS OF THE YANG-MILLS EQUATIONS ON $\mathbb{R}^{1+3}$: AN APPROACH USING THE YANG-MILLS HEAT FLOW.

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Abstract. In this work, along with the companion work [21], we propose a novel approach to the problem of gauge choice for the Yang-Mills equations on the Minkowski space $\mathbb{R}^{1+3}$. A crucial ingredient is the associated Yang-Mills heat flow. As this approach does not possess the drawbacks of the previous approaches (as in [13], [26]), it is expected to be more robust and easily adaptable to other settings.

Building on the results proved in the companion article [21], we prove, as one of the first applications of our approach, finite energy global well-posedness of the Yang-Mills equations on $\mathbb{R}^{1+3}$. This is a classical result first proved by S. Klainerman and M. Machedon [13] using local Coulomb gauges. As opposed to their method, the present approach avoids the use of Uhlenbeck’s lemma [29], and hence does not involve localization in space-time.

1. Introduction

In this work, along with the companion work [21], we propose a novel approach to the problem of gauge choice for the Yang-Mills equations

$$D^\mu F_{\nu\mu} = 0$$

on the Minkowski space $\mathbb{R}^{1+3}$ with a non-abelian structural group $\mathfrak{g}$. (For the notations, we refer the reader to §1.1.) An essential ingredient of our approach is the celebrated Yang-Mills heat flow

$$\partial_s A_i = D^\ell F_{i\ell},$$

which, first proposed by Donaldson [7], is a well-studied equation in the field of geometric analysis. (See [22], [3] and etc.) The idea of using the associated heat flow to deal with the problem of gauge choice had been first put forth by Tao [27], [28] in the context of energy critical wave maps on $\mathbb{R}^{1+2}$, and has been also adapted to the related energy critical Schrödinger maps by [1], [24], [25].

The novel approach using the Yang-Mills heat flow does not possess the drawbacks of the previous choices of gauge. As such, it is expected to be more robust and easily applicable to other problems. Building on the results proved in [21] (where, in particular, a new proof of $H^1_1$ local well-posedness of the Yang-Mills equations has been given), we will provide in this paper an alternative proof of finite energy global well-posedness of the Yang-Mills equations on $\mathbb{R}^{1+3}$, a classical result of S. Klainerman and M. Machedon [13], as one of the first demonstrations of the power of the new approach.

1.1. Background: The Yang-Mills equations on $\mathbb{R}^{1+3}$. We will work on the Minkowski space $\mathbb{R}^{1+3}$, equipped with the Minkowski metric of signature $(-+++)$. All tensorial indices will be raised and lowered by using the Minkowski metric. Moreover, we will adopt the Einstein summation convention of summing up repeated upper and lower indices. Greek indices, such as $\mu, \nu, \lambda$, will run over $x^0, x^1, x^2, x^3$, whereas latin indices, such as $i, j, k, \ell$, will run only over the spatial indices $x^1, x^2, x^3$. We will often use $t$ for $x^0$.

Let $\mathfrak{g}$ be a Lie group with the Lie algebra $\mathfrak{g}$, which is equipped with a bi-invariant inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to [0, \infty)$. The bi-invariant inner product will be used to define the absolute value of elements in $\mathfrak{g}$, and moreover will be used in turn to define the $L^2_\mathfrak{g}$-norm of $\mathfrak{g}$-valued functions.

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1 A bi-invariant inner product is an inner product on $\mathfrak{g}$ invariant under the adjoint map. A sufficient condition for the existence of such an inner product is that $\mathfrak{g}$ is a product of an abelian and a semi-simple Lie groups.
For simplicity, we will assume that $\mathfrak{g}$ is a matrix group. An explicit example which is useful to keep in mind is the group of special unitary matrices $\mathfrak{g} = \text{SU}(n)$, in which case $\mathfrak{g} = \text{su}(n)$ is the set of complex traceless anti-hermitian matrices and the bi-invariant metric is given by $(A, B) := \text{tr}(AB^*)$.

Consider a $\mathfrak{g}$-valued 1-form $A_\mu$ on $\mathbb{R}^{1+3}$, which we will call a connection 1-form, or connection coefficient. For any $\mathfrak{g}$-valued tensor field $B$ on $\mathbb{R}^{1+3}$, we define the associated covariant derivative $D = (A)D$ by

$$D_\mu B := \partial_\mu B + [A_\mu, B], \quad \mu = 0, 1, 2, 3$$

where $\partial_\mu$ is the ordinary directional derivative on $\mathbb{R}^{1+3}$.

The commutator of two covariant derivatives gives rise to a $\mathfrak{g}$-valued 2-form $F_{\mu\nu}$, called the curvature 2-form associated to $A_\mu$, in the following fashion.

$$D_\mu D_\nu B - D_\nu D_\mu B = [F_{\mu\nu}, B].$$

Using the definition, it is not difficult to verify that $F_{\mu\nu}$ is expressed directly in terms of $A_\mu$ by the formula

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

From the way $F_{\mu\nu}$ arises from $A_\mu$, it follows that the following Bianchi identity holds.

(Bianchi) $$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0.$$ A connection 1-form $A_\mu$ is said to be a solution to the Yang-Mills equations (YM) on $\mathbb{R}^{1+3}$ if the following equation holds for $\nu = 0, 1, 2, 3$.

(YM) $$D^\mu F_{\mu\nu} = 0.$$ Note the similarity of (Bianchi) and (YM) with the Maxwell equations.

An essential feature of (YM) is the gauge structure, which we explain now. Let $U$ be a smooth $\mathfrak{g}$-valued function. This $U$ may act on $A, D, F$ as a gauge transform according to the following rules:

$$\tilde{A}_\mu = UA_\mu U^{-1} - \partial_\mu UU^{-1}, \quad \tilde{D}_\mu = UD_\mu U^{-1}, \quad \tilde{F}_{\mu\nu} = UF_{\mu\nu}U^{-1}.$$ If a $\mathfrak{g}$-valued tensor transforms in the fashion $\tilde{B} = UBU^{-1}$, then we say that it is gauge covariant, or covariant under gauge transforms. Note that the curvature 2-form is gauge covariant. Given a gauge covariant $B$, its covariant derivative $D_\mu B$ is also gauge covariant, as the following formula shows:

$$D_\mu B = UD_\mu BU^{-1}.$$ Due to bi-invariance, we furthermore have $(\tilde{B}, \tilde{B}) = (B, B)$.

Note that (YM) evidently covariant under gauge transforms. It has the implication that a solution to (YM) makes sense only as a class of gauge equivalent connection 1-forms. Accordingly, we make the following definition.

**Definition 1.1.** A classical solution to (YM) is a class of gauge equivalent smooth connection 1-forms $A$ satisfying (YM). A generalized solution to (YM) is defined to be a class of gauge equivalent connection 1-forms $A$ for which there exists a sufficiently smooth representative $A$ (say $\partial_t, \partial_x A \in C^1_t L^2_x, A \in C^1_t L^3_x$) which satisfies (YM) in the sense of distributions.

A choice of a particular representative will be referred to as a gauge choice. A gauge is usually chosen by imposing a condition, called a gauge condition, on the representative. Some classical examples of gauge conditions are the temporal gauge $A_0 = 0$, or the Coulomb gauge $\partial^2 A_\ell = 0$, where $\ell$, being a latin index, is summed only over the spatial indices 1, 2, 3.

In this work, as well as the companion paper [21], we study the Cauchy problem associated to (YM). As in the case of Maxwell equations, the initial data set consists of $(\tilde{A}_i, \tilde{E}_i)$ for $i = 1, 2, 3$, where $\tilde{A}_i = A_i(t = 0)$ (magnetic potential) and $\tilde{E}_i = F_{0i}(t = 0)$ (electric field). Note that one
component of \((YM)\), namely \(\nu = 0\), imposes the following constraint equation on the possible initial data set \((A_i, F_i)\):

\[
\partial^\nu \nabla_\ell + [A_\ell', \nabla_\ell] = 0.
\]

The system \((YM)\) possesses a positive definite conserved quantity \(E[F_{\mu\nu}](t)\), called the conserved energy of \(F_{\mu\nu}\) at time \(t\), defined by

\[
E[F_{\mu\nu}](t) := \frac{1}{2} \int_{\mathbb{R}^3} \sum_{\ell=1,2,3} (F_{\mu\nu}(t,x), F_{\mu\nu}(t, x)) + \sum_{k,\ell=1,2,3} (F_{k\ell}(t,x), F_{k\ell}(t, x)) \, dx
\]

Note that \((YM)\) remain invariant under the scaling

\[
x^\alpha \to \lambda x^\alpha, \quad A \to \lambda^{-1} A, \quad F \to \lambda^{-2} F.
\]

We remark that the conserved energy \(E(t)\) and also the norms \(\|\partial_\alpha A_\mu\|_{L^2}, \|F_{\mu\nu}\|_{L^2}\) decrease as \(\lambda\) increases according to the above scaling. This reflects the sub-criticality of these quantities compared to the scaling property of \((YM)\).

1.2. The problem of gauge choice and previous approaches. We will begin with a discussion on the importance and difficulty of the problem of choosing an appropriate gauge in the study of the Yang-Mills equations. Our discussion will revolve around the following concrete example, which is a classical result of Klainerman-Machedon \([13]\), stated in a simplified form.

**Theorem 1.2** (Klainerman-Machedon \([13]\)). Let \((A_i, F_i)\) be a smooth initial data set satisfying the constraint equation (1.1). Consider the Cauchy problem for these data.

1. (\(H^1\) local well-posedness) There exists a classical solution \(A_\mu\) to the Cauchy problem for \((YM)\) on a time interval \((-T^*, T^*)\), where \(T^* > 0\) depends only on \(\|A_i\|_H^1, \|F_i\|_{L^2}\). The solution is unique in an appropriate gauge, e.g. in the temporal gauge \(A_0 = 0\).

2. (Finite energy global well-posedness) Furthermore, if the initial data set possesses finite conserved energy \(E(0) < \infty\), then the solution \(A_\mu\) extends globally.

After explaining the importance of gauge choice for proving Theorem 1.2, we will briefly summarize the previous approaches to the problem of gauge choice, namely the (local) Coulomb gauge \([13]\) and the temporal gauge \([23], [8], [26]\). It will be seen that each has its own set of drawbacks, which in fact makes Theorem 1.2 the best result so far in terms of the regularity condition on the initial data, concerning local and global well-posedness of \((YM)\) for possibly large initial data. This will motivate us to propose a novel approach to the problem of gauge choice in [1.3].

Importance of gauge choice. There are at least three reasons why a judicious choice of gauge is needed in order to prove Theorem 1.2:

A. To reveal the hyperbolicity\(^3\) of \((YM)\);

B. To exhibit the ‘special structure’ (namely, the null structure) of \((YM)\);

C. To utilize the conserved energy \(E(t)\) to control \(\|\partial_\alpha A_\mu(t)\|_{L^2}\).

In the future, we will refer to these as Issues A, B and C. Let us discuss each of the further.

Concerning Issue A, observe that the top order terms of \((YM)\) at the level of \(A_\mu\) have the form

\[
\Box A_\mu - \partial^\nu \partial_\nu A_\mu = (\text{lower order terms}).
\]

In an arbitrary gauge, due to the presence of the undesirable second order term \(-\partial^\nu \partial_\nu A_\mu\), it is even unclear whether the equation for \(A_\mu\) is hyperbolic (i.e. a wave equation). Therefore, in order to study \((YM)\) as a hyperbolic system of equations, the gauge should be chosen, at the very least,
in a way to reveal the hyperbolicity of \( \text{YM} \). We remark that this is analogous to the issue that the Yang-Mills heat flow is only \textit{weakly-parabolic}, to be discussed in [1.4].

Resolution of Issue A suffices to prove local well-posedness of \( \text{YM} \) for sufficiently regular initial data (see [23, 8]). However, it is still insufficient for Theorem 1.2 because of Issue B. After an appropriate choice of gauge, which does not have to be precise for the purpose of this heuristic discussion, the wave equation for the connection 1-form \( A \) satisfying \( \text{YM} \) becomes of the form

\[
\Box A = \mathcal{O}(A, \partial A) + (\text{cubic and higher})
\]

where \( \mathcal{O}(A, \partial A) \) refers to a linear combination of bilinear terms in \( A \) and \( \partial_x A \).

At this point, we encounter an important difficulty of proving Theorem 1.2: Strichartz estimates (barely, but in an essential way) fail short of proving \( H^1_x \) local well-posedness of (1.3), due to the well-known failure of the endpoint \( L^2_x L^{\infty}_t \) estimate on \( \mathbb{R}^{1+3} \). In fact, a counterexample, given by Lindblad [20], demonstrates that even local existence may fail at this regularity for a general equation of the form (1.3). Such considerations indicate that a proof of Theorem 1.2 necessarily has to exploit the ‘special structure’ of \( \text{YM} \), which distinguishes \( \text{YM} \) from a general system of semi-linear equations of the similar form. As we will see in sequel, this ‘special structure’ will go under the name \textit{null form}. Since the precise form of the wave equation for the connection 1-form \( A \) is highly dependent on the gauge, it is crucial to make a suitable choice of gauge so as to reveal the structure needed to establish Theorem 1.2.

Once Issues A and B are addressed, low regularity local well-posedness of \( \text{YM} \) (in particular, Part (1) of Theorem 1.2) can, in principle, be established. However, yet another difficulty remains in proving Part (2) of Theorem 1.2 namely Issue C. Had the conserved energy \( E(t) \) directly controlled \( \| \partial_x A_i(t) \|_{L^2_x} \), finite energy global well-posedness would have followed immediately from \( H^1_x \) local well-posedness. However, recalling the expression for the conserved energy

\[
E(t) = \frac{1}{2} \sum_{\mu, \nu} \| \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \|^2_{L^2_x},
\]

we see that in an arbitrary gauge, \( E(t) \) can only control a part of the full gradient of \( A_i \); Namely, the curl of \( A_i \), or \( \| \partial_x A_j - \partial_j A_i \|_{L^2_x} \). Therefore, in order to prove Part (2) of Theorem 1.2 as well, the chosen gauge must have a structure which allows for utilizing \( E(t) \) to control the \( L^2_x \) norm of the full gradient \( \partial_x A_i(t) \).

**Approach using the (local) Coulomb gauge: Proof of Klainerman-Machedon [13].** We will now discuss the approach of Klainerman-Machedon [13] using the local Coulomb gauge. As we will see, this approach addresses all of the issues A–C, but possesses the drawback of requiring localization in space-time, causing technical difficulties on the boundaries.

A key observation of Klainerman-Machedon [13] (which in fact goes back to the previous work [12] of Klainerman-Machedon on the related Maxwell-Klein-Gordon equations) was that under the (global) Coulomb gauge \( \partial^j A_j = 0 \) imposed everywhere on \( \mathbb{R}^{1+3} \), Issues A and B are simultaneously resolved. That is:

- After solving elliptic equations for \( A_0 \) and \( \partial_0 A_0 \), \( \text{YM} \) reduce to a system of wave equations for \( A_i \), and
- The most dangerous quadratic nonlinearities of the wave equations can be shown to be composed of \textit{null forms}.

More precisely, the wave equation for \( A_i \) takes the form

\[
\Box A_i = \mathcal{Q}(|\partial_x|^{-1} A_i, A_i) + |\partial_x|^{-1} \mathcal{Q}(A_i, A_i) + \text{(Less dangerous terms)},
\]

where each \( \mathcal{Q} \) is a linear combination of bilinear forms

\[
Q_{ijk}(\phi_1, \phi_2) = \partial_j \phi_1 \partial_k \phi_2 - \partial_k \phi_1 \partial_j \phi_2, \quad 1 \leq j < k \leq 3,
\]

which are particular examples of a \textit{null form}, introduced by Klainerman [10] and Christodoulou [3] in the context of small data global existence problem for nonlinear wave equations, and first used by Klainerman-Machedon [11] in the context of low regularity well-posedness. Improved estimates
are available for such class of bilinear interactions (see \cite{11,14} and etc.), and therefore the desired local well-posedness can be proved.

The Coulomb gauge has an additional benefit that \(\|\partial_x A_i(t)\|_{L^2_x}^2\) may be estimated by \(E(t)\) (provided that \(A_i\) is sufficiently regular to start with), as the the Coulomb gauge condition \(\partial^2 A_i = 0\) sets the part of \(\partial_x A_i\) which is not controlled by \(E(t)\) (namely the divergence of \(A_i\), or \(\partial^2 A_i\), according to Hodge decomposition) to be exactly zero. In other words, the Coulomb gauge settles Issue C as well.

Unfortunately, when the structural group \(G\) is non-abelian, there is a fundamental difficulty in imposing the Coulomb gauge globally in space (i.e. on \(\mathbb{R}^3\) for each fixed \(t\)). Roughly speaking, it is because when \(G\) is non-abelian, a gauge transform into the Coulomb gauge is given as a solution to a nonlinear elliptic system of PDEs, for which no good regularity theory is available in the large. A closely related phenomenon is the Gribov ambiguity \cite{9}, which asserts non-uniqueness of representative satisfying the Coulomb gauge equation \(\partial^2 A_i = 0\) in some equivalence class of connection \(1\)-forms on \(\mathbb{R}^3\) when \(G\) is non-abelian.

At a more technical level, this difficulty manifests in the fact that Uhlenbeck's lemma \cite{29}, which is a standard result asserting the existence of a gauge transform (possessing sufficient regularity) into the Coulomb gauge, requires the the curvature \(F\) to be small in \(L^3_t L^{3/2}_x\). Note that this norm is invariant under the scaling \(L^3_t L^{3/2}_x\), and therefore cannot be assumed to be small by scaling, unlike the energy \(E(F)\). To get around this problem, the authors of \cite{13} work in what they call local Coulomb gauges in small domains of dependence (in which the required norm of \(F\) can be assumed small), and glue the local solutions together by exploiting the finite speed of propagation. The execution of this strategy is quite involved due to the presence of the constraint equations \(\square A_i = 0\). In particular, it requires a delicate boundary condition for \(\square A_i\) in order to mesh the analyses of the elliptic and hyperbolic equations arising from \((YM)\) in the local Coulomb gauge.

**Approach using the temporal gauge.** A different route to the problem of gauge choice in the context of low regularity well-posedness has been suggested by Tao in his paper \cite{26}, where he proved \(H^s_x\) local well-posedness for \(s > 3/4\) (thus going even below the energy regularity) by working in the temporal gauge \(A_0 = 0\), under the restriction that the \(H^s_x \times H^{-1}_x\) norm of \((A_i, E_i)\) is small. This gauge has the advantage of being easy to impose globally (as gauge transforms into the temporal gauge can be found by solving an ODE), and thus does not have the problem that the Coulomb gauge possesses. Indeed, it had been used by other authors, including Segal \cite{23} and Eardley-Moncrief \cite{8}, to prove local and global well-posedness of \((YM)\) for (large) initial data with higher regularity (namely, \(s \geq 2\)). To reiterate this discussion in our framework, the temporal gauge resolves Issues A and \(C\) raised above.

However, this gauge possesses its own drawback that it fails to cope with initial data sets with a large \(H^s_x\) norm, when \(3/4 < s \leq 1\). Moreover, another drawback is that it is unclear how to deal with Issue C, namely how \(\|\partial_x A_i(t)\|_{L^2_x}\) may be controlled for every \(t\) using the conserved energy \(E\).

**1.3. Main idea of the novel approach.** The purpose of this paper, along with the companion paper \cite{21}, is to present a novel approach to the problem of gauge choice, which does not possess the drawbacks of the previous methods. As such, this approach does not involve localization in space-time and works well for large initial data. Nevertheless, it is (at the very least) as effective as the previous choices of gauge, as we will see that it addresses all of the issues A–C discussed above. As a demonstration of the power of the novel approach, we provide a new proof of finite

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\(^5\)In fact, it is possible to show, by a variational argument, that any \(A_i \in L^2_x\) may be gauge transformed to a weak solution \(\tilde{A} \in L^2_x\) to the Coulomb gauge equation \(\partial^2 \tilde{A} = 0\); see \cite{23}. The problem is that no further regularity of the gauge transform and \(\tilde{A}\) may be inferred, due to the lack of an appropriate regularity theory.

\(^6\)However, Issue B is not addressed fully in the sense that smallness of the initial data is needed.

\(^7\)One reason is that it still relies on a Uhlenbeck-type lemma to set \(\partial^2 A_i = 0\) at \(t = 0\), which requires some sort of smallness of the initial data. There is also a technical difficulty in the Picard iteration argument which does not allow one to use the smallness of the length of the time-interval; ultimately, this originates from the presence of a time derivative on the right-hand side of the equation \(\partial_t (\partial^2 A_i) = -[A', \partial_t A_i]\) (which is equivalent to the equation \(D^i F_{t0} = 0\)). See \cite{26} for more details.
energy global well-posedness of \((YM)\) in this work. This, combined with the new proof of \(H^1_x\) local well-posedness of \((YM)\) given in [21], will constitute an alternative proof of Theorem 1.2.

Heuristically speaking, the key idea of the novel approach is to ‘smooth out’ the problem at hand in a ‘geometric fashion’. The expectation is that the problem of gauge choice for the ‘smoothed out problem’ would be much easier thanks to the additional regularity. All the difficulties, then, are shifted to the problem of controlling the error generated by the smoothing procedure. That this is possible for a certain choice of smoothing procedure, based on a geometric (weakly-)parabolic PDE called the Yang-Mills heat flow, is the main thesis of this work and [21].

In the following three subsections (§1.4–1.6), we will discuss how the novel approach deals with Issues A–C listed above. After a discussion on the Yang-Mills heat flow in §1.4, we will give a summary of the companion paper [21] in §1.5 in which we explain how Issues A and B are resolved. Then an overview of the main ideas of the present paper in §1.6 will follow, addressing Issue C.

Remark 1.3. The present work advances a relatively new idea in the field of hyperbolic PDEs, which is to use a geometric parabolic equation to better understand a hyperbolic equation. To the author’s knowledge, the first instance of this idea occurred in the work of Klainerman-Rodnianski [16], in which the linear heat equation on a compact 2-manifold was used to develop an invariant form of Littlewood-Paley theory. This was applied in [15] and [17] to study the causal geometry of solutions to the Einstein’s equations under very weak hypotheses.

Such idea was carried further by Tao [27], who proposed to use a nonlinear geometric heat flow to deal with the problem of gauge choice in the context of the energy critical wave map problem. This was put into use in the series of preprints [28] to develop a large energy theory of wave maps into a hyperbolic space \(\mathbb{H}^n\). In this setting, one begins by solving the associated heat flow, in this case the harmonic map flow, starting from a wave map restricted to a fixed \(t\)-slice. Then the key idea is that the harmonic map flow converges (under appropriate conditions) to a single point, same for every \(t\), in the target as the heat parameter goes to \(\infty\). For this trivial map at infinity, the canonical choice of gauge is clear, this choice is then parallel-transported back along the harmonic map flow. The resulting gauge is dubbed the caloric gauge. This gauge proved to be quite useful, and the use of such gauge has also been successfully extended to the related problem of energy critical Schrödinger maps as well, through the works [1], [24] and [25].

1.4. The Yang-Mills heat flow. Before delving into a more detailed exposition of our approach, let us first introduce the Yang-Mills heat flow (or \((YMHF)\) in short), which will play an important role in this series of works. Consider a spatial connection 1-form \(A_i(s,t)\) \((i = 1, 2, 3)\) on \(\mathbb{R}^3\) parametrized by \(s \in [0, s_0]\) \((s_0 > 0)\). We say that \(A_i(s)\) is a Yang-Mills heat flow if it satisfies the equation

\[
\partial_s A_i = D^j F_{ji}, \quad i = 1, 2, 3.
\]

First introduced by Donaldson [7], the Yang-Mills heat flow is the gradient flow for the Yang-Mills energy on \(\mathbb{R}^3\) (also referred to as the magnetic energy)

\[
B[A_i] := \frac{1}{2} \sum_{1 \leq i < j \leq 3} \|F_{ij}\|_{L^2}^2,
\]

and plays an important role in differential geometry. It has been a subject of an extensive research by itself; see, for example, [7], [22], [24] and etc.

Our intention is to use \((YMHF)\) as a geometric smoothing device for \((YM)\). One must be careful, however, since \((YMHF)\) is not strictly parabolic as it stands at the level of \(A_i\). Indeed, expanding \((YMHF)\) in terms of \(A_i\), the top order terms look like

\[
\partial_s A_i = \Delta A_i - \partial_i \partial^j A_j + \text{(lower order terms)},
\]

where \(\Delta A_i - \partial_i \partial^j A_j\) possesses non-trivial kernel (any \(A_i = \partial_i \phi\), for \(\phi\) a \(g\)-valued function). Due to this fact, the Yang-Mills heat flow is said to be only weakly-parabolic.

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8We remark, however, that the both of this work will rely on results proved in [21] other than \(H^1\) local well-posedness as well. On the other hand, [21] may be read independently of the present paper.

9Namely, one chooses the same orthonormal frame at each point on the domain.
The culprit of the non-parabolicity of (YM) turns out to be the gauge covariance of the term $D\ell F_{\ell i}$, which suggests that it can be remedied by studying the gauge structure of the Yang-Mills heat flow in detail. Upon inspection, we see that the gauge structure of the equations (YM) is somewhat restrained, as it is covariant only under gauge transforms that are independent of $s$. To deal with the problem of non-parabolicity, we will begin by fixing this issue, i.e. reformulating the Yang-Mills heat flow in a way that is covariant under gauge transforms which may as well depend on the $s$-variable.

Along with $A_i$, let us also add a component $A_s$ and consider $A_a$ ($a = x^1, x^2, x^3, s$), which is a connection 1-form on the product manifold $\mathbb{R}^3 \times [0, s_0]$. Corresponding to $A_s$, we also introduce the covariant derivative along the $\partial_s$-direction

$$D_s := \partial_s + [A_s, \cdot].$$

A covariant Yang-Mills heat flow is a solution $A_a$ to the following system of equations.

\begin{equation}
(cYM) \quad F_{si} = D\ell F_{\ell i}, \quad i = 1, 2, 3,
\end{equation}

where $F_{si}$ is the commutator between $D_s$ and $D_i$, given by the formula

\begin{equation}
F_{si} = \partial_s A_i - \partial_i A_s + [A_s, A_i].
\end{equation}

The system $(cYM)$ is underdetermined for $A_a$, and therefore requires an additional gauge condition (typically on $A_s$) in order to be solved. Note that the original Yang-Mills heat flow (YM) is a special case of $(cYM)$, namely when $A_s = 0$. On the other hand, choosing $A_s = \ell' A_\ell$, the top order terms of $(cYM)$ becomes

$$\partial_s A_i - \partial_i \ell' A_\ell = \Delta A_i - \partial_i \partial_s A_\ell + \text{(lower order terms)}.$$

The term $\partial_s \partial_i A_i$ on each side are cancelled, and we are consequently left with a strictly parabolic system of equations for $A_i$. In other words, the weakly-parabolic system (YM) equivalent to a strictly parabolic system of equations, connected via gauge transforms for $(cYM)$ back and forth $A_s = 0$ and $A_s = \ell' A_\ell$.

Henceforth, the gauge condition $A_s = 0$ will be referred to as the caloric gauge, in deference to the term introduced by Tao in his work [27]. The condition $A_s = \ell' A_\ell$ will be dubbed the DeTurck gauge, as the procedure outlined above may be viewed as a geometric reformulation of the standard DeTurck’s trick, introduced first by DeTurck [6] in the context of the Ricci flow and adapted to the Yang-Mills heat flow by Donaldson [7].

1.5. Overview of [21]: Proof of local-wellposedness. Acquainted with the covariant formulation of the Yang-Mills heat flow, we are ready to return to the task of describing our approach in more detail. We will begin by providing a short overview of [21], in which local well-posedness is proved for initial data sets with $H^1_x$ regularity; for a more precise statement, see Theorem 1.6. In particular, we will explain how issues A, B raised in $[1.2]$ are resolved in the novel approach.

To avoid too much technical details, we will treat here the simpler problem of proving an a priori bound of a solution to (YM) in the temporal gauge. That is, for some interval $I := (-T_0, T_0) \subset \mathbb{R}$, we will presuppose the existence of a solution $A^i_\mu$ to (YM) in the temporal gauge on $I \times \mathbb{R}^3$ and aim to establish an estimate of the form

$$\|\partial_{t,x} A^i_\mu\|_{C_t(L_2)} \leq C \sum_{i=1,2,3} \|\partial_i E_i\|_{L_2} \quad \text{for} \quad s_1 \leq s_2,$$

where $A^i_\mu(t = 0) = A_i, \partial_t A^i_\mu(t = 0) = E_i$.

Geometric smoothing of $A^i_\mu$ by the (dynamic) Yang-Mills heat flow. The first step of the proof is to smooth out the solution $A^i_\mu$, essentially using the covariant Yang-Mills heat flow. Let us introduce a new variable $s \in [0, s_0]$, and extend $A^i_\mu = A^i_\mu(t, x)$ to $A_a = A_a(t, x, s)$ (where $a = x^0, x^1, x^2, x^3, s$) on $I \times \mathbb{R}^3 \times (0, s_0]$ by solving the equations

\begin{equation}
(dYM) \quad F_{s\mu} = D\ell F_{\ell \mu}, \quad \mu = 0, 1, 2, 3
\end{equation}

with the initial data $F_{s\mu}(t = 0) = A^\mu_{\ell\mu}(t, x), F_{s\mu}(t = 0, x, s) = 0$ for $\mu = 0, 1, 2, 3$.
with an appropriate choice of $A_s$, starting with $A_\mu(s = 0) = A_{\mu}^0$. Note that that this system is $\text{dYMHF}$ appended with the equation $F_{s0} = \text{D}^i F_{i0}$ for $A_0$; it will be referred to as the *dynamic Yang-Mills heat flow* or, in short, $\text{dYMHF}$. Using Picard iteration, these equations can be solved provided that $s_0 > 0$ is small enough.

The hyperbolic-parabolic-Yang-Mills system. As a result, we arrive at a connection 1-form $A_\mu$ (where $a = x^0, x^1, x^2, x^3, s$) on $I \times \mathbb{R}^3 \times [0, s_0]$ which solves the following system of equations.

$\text{(HPYM)}$

\[
\begin{cases}
F_{\mu\nu} = \text{D}^i F_{i\mu} & \text{on } I \times \mathbb{R}^3 \times [0, s_0], \\
\text{D}^\mu F_{\mu\nu} = 0 & \text{along } I \times \mathbb{R}^3 \times \{0\}.
\end{cases}
\]

We will refer to this as the *Hyperbolic-Parabolic-Yang-Mills system* or, in short, *HPYM*. This will be the system of equations that we will mainly work with in place of *YM*. Accordingly, instead of $A_{\mu}^\dagger$, we will estimate $A_\ell := A_{\mu}(s = s_0)$, which may be viewed as a smoothed-out version of $A_{\mu}^\dagger$, and the error $\partial_s A_\mu(s)$ (for $s \in (0, s_0)$) in between.

**Gauge choices for HPYM:** DeTurck and caloric-temporal gauges. The next step consists of estimating $\partial_s A_\mu$ and $A_{\mu}^\dagger$ by using the equations arising from *HPYM*. Basically, the strategy is to first use the parabolic (in the $s$-direction) equations to estimate the new variables $\partial_s A_\mu, A_{\mu}^\dagger$ at $t = 0$, and then to use the hyperbolic (in the $t$-directions) equations to estimate their evolution in $t$. As *HPYM* is manifestly gauge covariant (under gauge transforms fully dependent on all the variables $x^0, x^1, x^2, x^3, s$), we need to fix a gauge in order to carry out such analyses.

As it turns out, a different gauge choice is needed to achieve each goal. For the purpose of deriving estimates at $t = 0$, which should be compatible with the analysis of the $t$-evolution to follow, it is essential to exploit the smoothing property of $\text{dYMHF}$. As such, the gauge of choice here is the *DeTurck gauge* $A_s = \partial^i A_i$. On the other hand, completely different considerations are required for estimating the $t$-evolution, and here the gauge condition we impose is

\[
\begin{cases}
A_s = 0 & \text{on } I \times \mathbb{R}^3 \times (0, s_0), \\
A_0 = 0 & \text{on } I \times \mathbb{R}^3 \times \{s_0\}.
\end{cases}
\]

which will be referred to as the *caloric-temporal gauge*. In practice, the DeTurck gauge will be first used to obtain estimates at $t = 0$, and then we will perform a gauge transformation\(^\text{10}\) into the caloric-temporal gauge to carry out the analysis of the evolution in $t$. We remark that finding such a gauge transform is always possible, as it amounts to simply solving a hierarchy of ODEs.

A brief discussion on the motivation behind our choice of the *caloric-temporal gauge* is in order. For $\partial_s A_\mu$ on $I \times \mathbb{R}^3 \times (0, s_0)$, let us begin by considering the following rearrangement of the formula (1.4) for $F_{si}$:

\[
\partial_s A_i = F_{si} + \text{D}_i A_s.
\]

A simple computation\(^\text{11}\) shows that $F_{si}$ is *covariant-divergence-free*, i.e. $\text{D}^i F_{s\ell} = 0$. This suggests that (1.5) may be viewed (heuristically) as a *covariant Hodge decomposition* of $\partial_s A_i$, where $F_{si}$ is the covariant-divergence-free part and $\text{D}_i A_s$, being a pure covariant-gradient term, may be regarded as the ‘covariant-curl-free part’ (although, strictly speaking, the covariant-curl does not vanish but is only of lower order for this term). Recall that the Coulomb gauge condition, which had a plenty of good properties as discussed earlier, is equivalent to having zero curl-free part. Therefore, to

\(^{10}\) A technical remark: Performing a gauge transformation $U = U(t, x, s)$ from the DeTurck gauge to the caloric gauge, with the additional condition that $U(t = 0, s = 0) = \text{Id}$, corresponds exactly to carrying out the standard DeTurck trick\(^\text{8}\). However, this is inappropriate for our purposes, as it turns out that this gauge transform is not bounded on $H^m$ for $m > 1$. As such, it cannot retain the smoothing estimates proved in the DeTurck gauge. Instead, we will use the gauge transform for which $U(t = 0, s = 1) = \text{Id}$. Under such gauge transform, $A_\ell(t = 0)$ remains the same, and thus smooth, at the cost of introducing a non-trivial gauge transform for the initial data at $t = 0, s = 0$. In some sense, this procedure is an analogue of the Uhlenbeck’s lemma in our approach.

\(^{11}\) The identity $\text{D}^i F_{s\ell} = 0$ follows from *YMHF* and $\text{D}^i \text{D}^k F_{i\ell} = 0$, which is proved simply by anti-symmetrizing the indices $\ell, k$. 

---
imitate the Coulomb gauge as closely as possible, we are motivated to set \( A_s = 0 \) on \( I \times \mathbb{R}^3 \times (0, s_0) \); incidentally, this turns out to be the caloric gauge condition discussed earlier.

On the other hand, at \( s = s_0 \), the idea is that \( A_s \) possesses smooth initial data \((A_s, F_{sp})(t = 0)\), thanks to the smoothing property of \((YM)\). Therefore, we expect that the problem of gauge choice for \( A_s \) is not as delicate as the original problem; as such, we choose the temporal gauge condition \( A_0 = 0 \), which is easy to impose yet sufficient for the analogous problem with smoother initial data, as the previous works \([23], [8]\) had shown.

**Resolution of Issues A and B.** With the caloric-temporal gauge, we are finally ready to describe how Issues A and B are resolved in the novel approach. Let us begin by introducing the Yang-Mills tension field \( w_\nu(s) := D^\mu F_{\nu\mu}(s) \), which measures the extent of failure of \( A_\mu(s) \) to satisfy \((YM)\).

Then we may derive the following system of equations \([21], Appendix A\):

\[
\begin{align*}
\mathbf{D}_s w_\nu &= D^\mu D_\nu w_\nu + 2[F_{\nu\mu}, w_\ell] + 2[F^\mu\ell, D_\mu F_{\nu\ell} + D_\ell F_{\nu\mu}], \\
\mathbf{D}_s F_{si} &= 2[F_{s\mu}, F_{i\mu}] - 2[F^\mu, D_\mu F_{s\ell} + D_\ell F_{s\mu}] - D^\ell D_\ell w_i + D_i D_\ell w_\ell - 2[F_{i\ell}, w_\ell], \\
\mathbf{D}_s F_{\nu\mu} &= w_\nu.
\end{align*}
\]

The underlines of \((1.3)\) signify that each variable is restricted to \( s = s_0 \). Furthermore, \( w_\nu = 0 \) at \( s = 0 \), for all \( \nu = 1, 2, 3 \).

The parabolic equation \((1.6)\) can be used to derive estimates for the Yang-Mills tension field \( w_\nu \). It is important to note that its data at \( s = 0 \) is zero, thanks to the fact that \( A_\mu(s = 0) \) satisfies \((YM)\). Moreover, note that \( w_0 = -F_0 \), which is equal to \(-\partial_s A_0\) thanks to the caloric gauge condition \( A_s = 0 \). In conclusion, after solving the parabolic equation \((1.6)\), the dynamics of \((HYYM)\) is reduced to that of the variables \( F_{si} = \partial_s A_i \) (again due to \( A_s = 0 \)) and \( A_s \). These are, in turn, estimated by \((1.7)\), which is a wave equation for \( F_{si} \), and \((1.8)\), which is the Yang-Mills equation with a source \( w_\nu \), for \( A_s \), under the temporal gauge \( A_0 = 0 \). This shows the hyperbolicity of \((YM)\), which takes care of Issue A.

Next, let us address the issue of exhibiting null forms (i.e. Issue B). Let us begin by observing that for \((1.3)\), no null form is needed to close the estimates; this is because \((A_s, F_{0s})(t = 0)\) has been smoothed out by \((HYYM)\) as mentioned earlier. For \((1.7)\), on the other hand, there turns out to be a single term which cannot be dealt with simply by Strichartz estimates, namely

\[ 2[A_{s} - A_{s}, \partial_s F_{si}] \]

If \( A_{s} - A_{s} \) were divergence-free, i.e. \( \partial_s (A_{s} - A_{s}) = 0 \), then an argument of Klainerman-Machedon \([12], [13]\) would show that this nonlinearity may be rewritten in as a linear combination of null forms \( Q_{jk}(\partial_x^{-1}(A - A), F_{si}) \). Although this is not strictly true, we have

\[ A_{s} - A_{s} = -\int_{0}^{s_0} F_{s\ell}(s) \, ds \]

thanks to the condition \( A_s = 0 \), where \( F_{s\ell} \) is covariant-divergence-free, i.e. \( D^\ell F_{s\ell} = 0 \). This suffices for a variant of the argument of Klainerman-Machedon to work, settling Issue B.

Provided that \( s_0, |I| \) are sufficiently small\(^12\), an analysis of \((HYYM)\) using the gauge conditions indicated above leads to estimates for \( \partial_s A_i \), in the caloric-temporal gauge, such as

\[
\begin{align*}
\sup_{0 < s < s_0} s^{-(m+1)/2} ||D_x^{(m+1)} \partial_t \partial_s (\partial_s A_i)(s)||_{C^1(I, L^2_x)} &\leq C_m \sum_{j=1,2,3} \left( \int_{0}^{s_0} \frac{s^{-(m+1)} ||D_x^{(m+1)} \partial_t \partial_s (\partial_s A_i)(s)||_{C^1(I, L^2_x)}^2 \, ds}{s} \right)^{1/2} \leq C_m \sum_{j=1,2,3} ||(\mathcal{A}_j, \mathcal{E}_j)||_{H^2_t \times L^2_x} \\
\int_{0}^{s_0} s^{-(k+1)/2} ||D_x^{(k+1)} \partial_t \partial_s A_i||_{C^1(I, L^2_x)} &\leq C_k \sum_{j=1,2,3} ||(\mathcal{A}_j, \mathcal{E}_j)||_{H^2_t \times L^2_x} \\
n_0^{-(k-1)/2} ||D_x^{(k-1)} \partial_t \partial_s A_i||_{C^1(I, L^2_x)} &\leq C_k \sum_{j=1,2,3} ||(\mathcal{A}_j, \mathcal{E}_j)||_{H^2_t \times L^2_x}
\end{align*}
\]

\(^12\)In the Coulomb gauge, the equation for \( A_0 \) is elliptic and therefore smallness of the time interval \( I \) cannot be utilized to solve for \( A_0 \) using perturbation; in \([13]\), the authors exploits the spatial localization to overcome this issue. For \((HYYM)\) in the caloric-temporal gauge, \( A_0 \) estimated by integrating \( F_{0s} = \partial_s A_0 \), where the latter variable satisfies a parabolic equation. For this, smallness of \( s_0 \) can be used, and thus the estimates are still global on \( \mathbb{R}^3 \).
up to some integers \( m_0, k_0 > 1 \), i.e. \( 1 \leq m \leq m_0, 1 \leq k \leq k_0 \). We remark that the weights of \( s \) are dictated by scaling.

Returning to \( A_\mu^{0} \). The only remaining step is to translate (1.9) to the desired estimate for \( \| \partial_t x A_\mu^{0} \|_{C_I(\ell^2, L^2)} \).

The first issue arising in this step is that the naive approach of integrating the estimates (1.9) in \( s \) fails to bound \( \| \partial_t x A_\mu^{0} \|_{C_I(\ell^2, L^2)} \), albeit only by a logarithm. To resolve this issue, we take the weakly-parabolic equations

\[
\partial_s A_i = \Delta A_i - \partial^\mu \partial_i A_\mu + (\text{lower order terms}).
\]

differentiate by \( \partial_{t,x} \), multiply by \( \partial_{t,x} A_i \) and then integrate the highest order terms by parts over \( \mathbb{R}^3 \times [0, s_0] \). This procedure, combined with the \( L^2_\ell \)-type estimates of (1.9), overcome the logarithmic divergence.

Another issue is that (1.9), being in the caloric-temporal gauge, is in a different gauge from the temporal gauge along \( s = 0 \). Therefore, we are required to control the gauge transform back to the temporal gauge along \( s = 0 \), for which appropriate estimates for \( A_0(s = 0) \) in the caloric-temporal gauge are needed. These are obtained ultimately as a consequence of the analysis of the hyperbolic equations of (HPYM) (Strichartz estimates, in particular, are used).

1.6. Overview of the present paper: Finite energy global well-posedness. In the work of Klainerman-Machedon [8], as pointed out earlier, finite energy global well-posedness was a rather easy corollary of the \( H^2 \) local well-posedness proof thanks to the fact that in the (local) Coulomb gauge, the conserved energy \( E(t) \) essentially controls \( \| (A_i, F_0)(t) \|_{H^1_\ell \times L^2} \). However, in the temporal gauge, making use of the conserved energy \( E(t) \) is not as straightforward since \( E(t) \) only controls certain components (namely, the curl) of the full gradient of \( A_i(t) \). We remind the reader that this was referred to as Issue C in [1,2].

Nevertheless, it is another remarkable property of the novel approach that Issue C can also be resolved, and therefore finite energy global well-posedness of (YM) can be proved. Our proof proceeds roughly in three steps, each of which uses the conserved energy \( E(t) \) in a crucial way.

Step 1. Let us start with a solution \( A_\mu^{0} \) to (YM) in the temporal gauge on \((-T_0, T_0) \times \mathbb{R}^3 \). As in the proof of local well-posedness, the first step is to solve (YMHPF) to extend \( A_\mu^{0} \) to a solution \( A_\mu \) to (HPYM). A priori, however, it is not clear whether this is possible when \( T_0 \) is large.

To illustrate, suppose that \( A_\mu^{0} \) does not extend past the time \( T_0 \). Then from the local well-posedness statement, it is necessary that

\[
\| \partial_t x A_\mu^{0}(t) \|_{L^2} \to \infty \quad \text{as } t \to T_0.
\]

Because of this, the size of the \( s \)-interval on which (YMHPF) can be solved by perturbative methods shrinks as \( t \to T_0 \). As a consequence, there might not exist a non-trivial interval \([0, s_0]\) on which (YMHPF) can be solved for every \( t \in (-T_0, T_0) \).

However, such a scenario is ruled out, thanks to the conserved energy \( E(t) \), and (YMHPF) can be solved in a uniform manner globally in time. More precisely, it is possible to show that there exists \( s_0 > 0 \) depending only on \( E(t) \) such that (YMHPF) on a fixed \( t \)-slice can be solved on an interval \([0, s_0]\). As \( E(t) \) is conserved, this shows that \( A_\mu^{0} \) can be extended to a solution \( A_\mu \) to (HPYM) on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1] \).

Step 2. With a solution \( A_\mu \) of (HPYM) in hand, let us impose the caloric-temporal gauge condition via an appropriate gauge transform. We wish to demonstrate that the conserved energy \( E(t) \) controls the appropriate fixed-time norms of the dynamic variables, which in this case are \( A_\mu \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

The key observation is that \( \| D^{(k)}_{\mu} F_{\mu\nu}(t, s) \|_{L^2} \) is estimated (with an appropriate weight of \( s \)) by \( E(t) \), thanks to covariant parabolic estimates. In particular, \( \| D^{(k)}_{\mu} F_0(t) \|_{L^2} \) is under control, where \( F_{\mu\nu} \) is the connection 2-form restricted to \( \{ s = s_0 \} \). As the temporal gauge condition \( A_\mu^{0} = 0 \) is

\[\text{[13]}\]

For a solution \( A_\mu \) to (HPYM), \( E(t) \) is defined to be the conserved energy of \( A_\mu \) at \((t, s = 0)\). We remark that this is a gauge-invariant quantity.
enforced, we have \( E_{m} = \partial_{t} A_{\mu} \); therefore, the preceding norm may be integrated in \( t \) to control the size of \( \| \partial_{t}^{(b)} A_{\mu}(t) \|_{L_{\mu}^{2}} \) for \( t \in (-T_{0}, T_{0}) \). On the other hand, as \( F_{\mu \nu} = D^{i} F_{\mu i} \) is already of the form \( D_{\mu} F_{\mu \nu} \), we can use the conserved energy \( E(t) \) to control the appropriate (fixed-time) norms of \( F_{\mu i}(t) \) as well, for each \( t \in (-T_{0}, T_{0}) \).

**Step 3.** Finally, we must unwind all the gauge transformations which have been done and return to \( A_{\mu} \). As in the last step of the proof of local well-posedness, this requires estimating \( A_{\mu} \) along \( s = 0 \) in the calorric-temporal gauge, where an important ingredient for the latter is the estimates obtained from the hyperbolic equations of (HPYM). Iterating the techniques developed in [21] for proving local well-posedness on a short time interval, coupled with some new estimates arising from the conserved energy \( E \), leads to the desired estimates.

For a more rigorous overview of the whole argument of the present paper, we refer the reader to Section 3. There, the Main Theorem is reduced to Theorems A–C, which essentially correspond to Steps 1–3 in the respective order.

### 1.7. Statement of the Main Theorem

We will now give the precise statement of our main result. Let us begin by defining the class of initial data sets of interest.

**Definition 1.4 (Admissible \( H^{1} \) initial data set).** We say that a pair \( (\mathcal{A}_{i}, E_{i}) \) of 1-forms on \( \mathbb{R}^{3} \) is an admissible \( H^{1} \) initial data set for the Yang-Mills equations if the following conditions hold:

1. \( \mathcal{A}_{i} \in H^{1}_{x} \cap L^{3}_{x} \) and \( E_{i} \in L^{2} \),
2. The constraint equation
   \[
   \partial^{\nu} E_{\nu} + [\mathcal{A}^{i}, E_{i}] = 0, \]

holds in the distributional sense.

Let us also define the notion of admissible solutions.

**Definition 1.5 (Admissible solutions).** Let \( I \subset \mathbb{R} \). We say that a generalized solution \( A_{\mu} \) to the Yang-Mills equations (YM) in the temporal gauge \( A_{0} = 0 \) defined on \( I \times \mathbb{R}^{3} \) is admissible if

\[
A_{\mu} \in C_{t}(I, H^{1}_{x} \cap L^{3}_{x}), \quad \partial_{t} A_{\mu} \in C_{t}(I, L^{2}_{x})
\]

and \( A_{\mu} \) can be approximated by classical solutions in the temporal gauge in the above topology.

We begin with a \( H^{1} \) local well-posedness theorem, whose proof using the Yang-Mills heat flow has been given in the companion paper [21].

**Theorem 1.6 (\( H^{1} \) local well-posedness [21 Main Theorem]).** Let \( (\mathcal{A}_{i}, E_{i}) \) be an admissible \( H^{1} \) initial data set, and define \( \mathcal{I} := \| \mathcal{A} \|_{H^{1}_{x}} + \| E \|_{L^{2}_{x}} \). Consider the initial value problem (IVP) for (YM) with \( (\mathcal{A}_{i}, E_{i}) \) as the initial data. Then the following statements hold.

1. There exists \( T^{\ast} = T^{\ast}(\mathcal{I}) > 0 \), which is non-increasing in \( \mathcal{I} \), such that a unique admissible solution \( A_{\mu} = A_{\mu}(t, x) \) to the IVP in the temporal gauge \( A_{0} = 0 \) exists on the \( t \)-interval \( I := (-T^{\ast}, T^{\ast}) \). Furthermore, the following estimates hold.
   \[
   \sup_{i} \| \partial_{t_{x}} A_{i} \|_{C_{t}(I, L^{2}_{x})} \leq C \mathcal{I}, \quad \sup_{i} \| A_{i} \|_{C_{t}(I, L^{2}_{x})} \leq C_{\mathcal{I}} \sup_{i} \| \mathcal{A}_{i} \|_{L^{2}_{x}}.
   \]

2. Let \( (\mathcal{A}_{i}', E_{i}') \) be another admissible \( H^{1} \) initial data set such that \( \| \mathcal{A} \|_{H^{1}_{x}} + \| E \|_{L^{2}_{x}} \leq \mathcal{I} \), and let \( A_{\mu}' \) be the corresponding solution given by (1). Then the following estimates for the difference hold.
   \[
   \sup_{i} \| \partial_{t_{x}} A_{i} - \partial_{t_{x}} A_{i}' \|_{C_{t}(I, L^{2}_{x})} \leq C_{\mathcal{I}} (\sup_{i} \| \mathcal{A}_{i} - \mathcal{A}_{i}' \|_{H^{1}_{x}} + \sup_{i} \| E_{i} - E_{i}' \|_{L^{2}_{x}}),
   \]
   \[
   \sup_{i} \| A_{i} - A_{i}' \|_{C_{t}(I, L^{2}_{x})} \leq C_{\mathcal{I}} (\sup_{i} \| \mathcal{A}_{i} - \mathcal{A}_{i}' \|_{H^{1}_{x} \cap L^{3}_{x}} + \sup_{i} \| E_{i} - E_{i}' \|_{L^{2}_{x}}).
   \]
(3) Finally, the following version of persistence of regularity holds: if \( \partial_x A_i, E_i \in H^m_x \) for an integer \( m \geq 0 \), then the corresponding solution given by (1) satisfies
\[
\partial_t \partial_x A_i \in C^{k_1}_i((-T^*, T^*), H^{k_2}_x)
\]
for every pair \( (k_1, k_2) \) of nonnegative integers such that \( k_1 + k_2 \leq m \).

The Main Theorem of this paper is a global well-posedness statement, which (in essence) says that the solution given by Theorem 1.6 can be extended globally in time. It uses crucially the fact that an admissible initial data set always possesses finite conserved energy, which whose precise definition is as follows. Given a space-time 2-form \( F = F_{\mu \nu} \), we define its conserved energy to be
\[
E[F] := \frac{1}{2} \sum_{\mu < \nu} \| F_{\mu \nu} \|^2_{L^2}.
\]

We are ready to state our Main Theorem.

**Main Theorem** (Finite energy global well-posedness). Let \( (A_i, E_i) \) be an admissible \( H^1 \) initial data set, and consider the initial value problem (IVP) for (YM) with \( (A_i, E_i) \) as the initial data. Note that by admissibility, \( (A_i, E_i) \) always possesses finite conserved energy, i.e. \( E[F] < \infty \). Then the following statements hold.

1. The admissible solution given by Theorem 1.6 extends globally in time, uniquely as an admissible solution in the temporal gauge \( A_0 = 0 \).

2. Moreover, if \( A_i, E_i \) are smooth and \( \partial_x A_i, E_i \in H^m_x \) for an integer \( m \geq 0 \), then the corresponding solution given by (1) is also smooth and satisfies
\[
\partial_t \partial_x A_i \in C^{k_1}_i(\mathbb{R}, H^{k_2}_x)
\]
for every pair \( (k_1, k_2) \) of nonnegative integers such that \( k_1 + k_2 \leq m \).

We remark that quantitative estimates as in Parts (1), (2) of Theorem 1.6 can be obtained by applying Theorem 1.6 repeatedly. We have omitted these statements for the sake of brevity.

**Remark 1.7.** The temporal gauge in the statements of Theorem 1.6 and the Main Theorem does not play an essential role. We have used this mainly because it is a well-known gauge condition that is easy to impose. In fact, most of the analysis in this paper takes place under the caloric-temporal gauge condition which has been introduced above.

### 1.8. Outline of the paper.
After setting up the notations and conventions in Section 2, we begin the proof of the Main Theorem in Section 3 by reducing it to establishing Theorems A, B and C, all of which concern the system (HPYM). We remark that Theorems A, B and C will correspond to Steps 1, 2 and 3 which have been discussed in 1.6, in the respective order.

The rest of the paper is devoted to proofs of Theorems A, B and C. In Section 4, we gather some preliminary definitions and results needed in the remainder of the paper. In particular, we present an array of techniques for dealing with covariant parabolic equations in 4.2. These techniques are put into use in the following section (Section 5), where we study the covariant parabolic equations satisfied by the curvature 2-form \( F \) of a solution to (cYMHF) or (dYMHF). As a result, we derive covariant parabolic estimates, on which the whole paper is based. Then in Section 6, we study the systems (cYMHF) and (dYMHF) themselves under the caloric gauge condition \( A_0 = 0 \). Then in the final three sections of this paper, we finally give proofs of Theorems A, B and C in order.

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14 As a byproduct of our analysis, we obtain an independent proof of global existence of solutions to the original Yang-Mills heat flow \( \partial_s A_i = D^s F_{l_i} \) with finite Yang-Mills energy, which is a result originally due to Råde [22]. See Corollary 6.7.
2. Notations and Conventions

In this paper, we will use bold kernel letters to refer to all space-time components; more precisely, \( \mathbf{F} \) denotes any of the 6 components of \( F_{\mu\nu} \), and \( \mathbf{A}, \mathbf{F} \), denote any of the 4 components of \( A_\nu, F_{\kappa\nu} \), respectively. On the other hand, plain kernel letters will refer to only spatial components, i.e. \( F = F_{ij}, A = A_1, F_3 = F_{03} \) for \( i, j = 1, 2, 3 \). A norm of such an expression, such as \( \|A\| \) or \( \|F\| \), is to be understood as the maximum over the respective range of indices, i.e. \( \|A\| = \sup_{\mu = 0, 1, 2, 3} \|A_\mu\|, \|F\| = \sup_{\mu = 0, 1, 2, 3} \|F_{\kappa\mu}\| \) and etc.

We will use the notation \( \mathcal{O}(\phi_1, \ldots, \phi_k) \) to denote a \( k \)-linear expression in the values of \( \phi_1, \ldots, \phi_k \). For example, when \( \phi_1 \) and the expression itself is scalar-valued, then \( \mathcal{O}(\phi_1, \ldots, \phi_k) = C\phi_1\phi_2 \cdots \phi_k \) for some constant \( C \). In many cases, however, each \( \phi_i \) and the expression \( \mathcal{O}(\phi_1, \ldots, \phi_k) \) will actually be matrix-valued. In such case, \( \mathcal{O}(\phi_1, \ldots, \phi_k) \) will be a matrix, whose each entry is a \( k \)-linear functional of the matrices \( \phi_i \).

In stating various estimates, we will adopt the standard convention of denoting finite positive constants which are different, possibly line to line, by the same letter \( C \). Dependence of \( C \) on other parameters will be made explicit by subscripts. Furthermore, we will adopt the convention that \( C \) always depends in a non-decreasing manner with respect to each of its parameters, in its respective range, unless otherwise specified. For example, \( C_{E}(\omega, T) \), where \( E(\omega, T) \) range over positive real numbers, is a positive, non-decreasing function of both \( E \) and \( (\omega, T) \).

Finally, in addition to plain greek and latin indices, we will utilize bold latin indices, such as \( \mathbf{a}, \mathbf{b} \), which will refer to all possible indices \( x^0, x^1, x^2, x^3, s \).

3. Reduction of the Main Theorem

3.1. Preliminaries. Before we begin, let us borrow a few definitions and lemmas from \([21]\), which will be useful in the proof of the Main Theorem. We start with the notion of regular functions and initial data sets.

**Definition 3.1** (Regular functions). Let \( I \subset \mathbb{R}, J \subset [0, \infty) \) be intervals.

1. A function \( \phi = \phi(x) \) defined on \( \mathbb{R}^3 \) is regular if \( \phi \in H_x^\infty := \bigcap_{m=0}^{\infty} H_x^m \).
2. A function \( \phi = \phi(t, x) \) defined on \( I \times \mathbb{R}^3 \) is regular if \( \phi \in C_t^\infty(I, H_x^\infty) := \bigcap_{m=0}^{\infty} C_t^m(I, H_x^m) \).
3. A function \( \psi = \psi(t, x, s) \) defined on \( I \times \mathbb{R}^3 \times J \) is regular if \( \psi \in C_{t, x, s}^\infty(I \times J, H_x^\infty) := \bigcap_{k,m=0}^{\infty} C_{t, x, s}^k(I \times J, H_x^m) \).

**Definition 3.2** (Regular initial data sets). We say that an initial data set \( (\overline{A}_i, \overline{E}_i) \) to \([YM]\) is regular if, in addition to satisfying the constraint equation \( (1.1) \), \( \overline{A}_i, \overline{E}_i \) are smooth and \( \partial_x \overline{A}_i, \overline{E}_i \) are regular.

The first lemma tells us that an admissible initial data set may be approximated by a sequence of regular initial data sets.

**Lemma 3.3** (Approximation lemma \([21]\) Lemma 4.5)). Any admissible \( H^1 \) initial data set \( (\overline{A}_i, \overline{E}_i) \in (H_x^1 \cap L_x^5) \times L_x^2 \) can be approximated by a sequence of regular initial data sets \( (\overline{A}_{i(n)}, \overline{E}_{i(n)}) \) satisfying the constraint equation \( (1.1) \). More precisely, the initial data sets \( (\overline{A}_{i(n)}, \overline{E}_{i(n)}) \) may be taken to satisfy the following properties.

1. \( \overline{A}_{i(n)} \) is smooth, compactly supported, and
2. \( \overline{E}_{i(n)} \) is regular, i.e. \( \overline{E}_{i(n)} \in H_x^k \) for every integer \( k \geq 0 \).

The idea is that a regular initial data set leads to, by persistence of regularity, a regular solution to \([YM]\) and \([HPYM]\), which is defined as follows.

**Definition 3.4** (Regular solutions). We say that a representative \( A_\mu \) of a classical solution to \([YM]\) is regular if \( A_\mu \) is smooth and \( \partial_{t, x} A_\mu \) is regular. Furthermore, we say that a smooth solution \( A_\alpha \) to \([HPYM]\) is regular if \( A_\alpha \) is smooth and \( \partial_{t, x} A_\mu, A_\alpha \) are regular.

Related to the notion of a regular solution, we also introduce the definition of a regular gauge transform.
Definition 3.5 (Regular gauge transform). We say that a gauge transform $U$ on $I \times \mathbb{R}^3 \times J$ is a regular gauge transform if $U, U^{-1}$ are smooth and furthermore

$$U, U^{-1} \in C_c^\infty(I \times J, L^\infty_x), \quad \partial_{t,x} U, \partial_{t,x} U^{-1} \in C_c^\infty(I \times J, L^3_x), \quad \partial_{t,x}^{(2)} U, \partial_{t,x}^{(2)} U^{-1} \in C_c^\infty(I \times J, H^\infty_x).$$

A gauge transform $U$ defined on $I \times \mathbb{R}^3$ is a regular gauge transform if it is a regular gauge transform viewed as an $s$-independent gauge transform on $I \times \mathbb{R}^3 \times J$ for some $J \subset [0, \infty)$.

Note that a regular solution (either to $(\text{YM})$ or to $(\text{HPYM})$) remains regular under a regular gauge transform.

Let us also define a norm $A_0[A_0](I)$ (for $I \subset \mathbb{R}$ an interval) for a $g$-valued function $A_0$ on $I \times \mathbb{R}^3$ as follows:

$$A_0[A_0](I) := \|A_0\|_{L^\infty_t L^2_x(I)} + \|\partial_x A_0\|_{L^\infty_t L^2_x(I)} + \|A_0\|_{L^1_t L^2_x(I)} + \|\partial_x A_0\|_{L^1_t L^2_x(I)} + \|\partial_x^{(2)} A_0\|_{L^1_t L^2_x(I)}.$$ 

The next lemma shows that this norm is exactly what one needs in order to estimate gauge transforms into the temporal gauge.

Lemma 3.6 (Estimates for gauge transform to temporal gauge [21, Lemma 4.6]). Let $T_0 > 0$, and consider the following ODE on $(-T_0, T_0) \times \mathbb{R}^3$.

$$
\begin{cases}
\partial_t V = VA_0 \\
V(t = 0) = \nabla,
\end{cases}
$$

where we assume that $A_0$ is smooth and $A_0(-T_0, T_0) < \infty$.

1. Suppose that $\nabla = \nabla(x)$ is a smooth $\mathcal{G}$-valued function on $\{t = 0\} \times \mathbb{R}^3$ such that

$$\nabla, \nabla^{-1} \in L^\infty_x, \quad \partial_x \nabla, \partial_x \nabla^{-1} \in L^3_x, \quad \partial_x^{(2)} \nabla, \partial_x^{(2)} \nabla^{-1} \in H^m_x$$

for all integers $m \geq 0$.

Then there exists a unique solution $V$ to the ODE, which obeys the following estimates.

$$
\begin{cases}
\|V\|_{L^\infty_t L^2_x(-T_0, T_0)} \leq C_{A_0(-T_0, T_0)} \cdot \|\nabla\|_{L^\infty_x}, \\
\|\partial_x V\|_{L^\infty_t L^2_x(-T_0, T_0)} \leq C_{A_0(-T_0, T_0)} \cdot \left(\|\partial_x \nabla\|_{L^2_x} + A_0(-T_0, T_0)\|\nabla\|_{L^\infty_x}\right), \\
\|\partial_x^{(2)} V\|_{L^\infty_t L^2_x(-T_0, T_0)} \leq C_{A_0(-T_0, T_0)} \cdot \left(\|\partial_x^{(2)} \nabla\|_{L^2_x} + A_0(-T_0, T_0)\|\nabla\|_{L^\infty_x}\right).
\end{cases}
$$

These estimates remain true with $V, \nabla$ replaced by $V^{-1}, \nabla^{-1}$, respectively.

3.2. Reduction of the Main Theorem to Theorems A, B, and C. The theorem we present below concerns two points: A) After scaling, it ensures that $(\text{HYME})$ can always be solved (from which we obtain a solution to $(\text{HPYM})$) on the unit $s$-interval provided that the conserved energy is small, and B) It gives a quantitative estimate for the data for $(\text{HPYM})$ at $t = 0$ (namely $\mathcal{I}(0)$) in terms of the size $\mathcal{F}$ of the initial data set $(\mathcal{A}, \mathcal{E})$.

Given a solution $A_0$ to $(\text{HPYM})$ on $I \times \mathbb{R}^3 \times [0,1]$, we assert the existence of a norm $\mathcal{I}(t)$ ($t \in I$) of $\partial_{t,x} F_{si}(t,s)$ ($0 < s < 1$) and $\partial_{t,x} A_0(t)$, for which the following theorem holds. The precise definition will be given in Section 7.

Theorem A (Transformation into the caloric-temporal gauge and estimates at $t = 0$). Consider a regular initial data set $(\mathcal{A}, \mathcal{E})$ to $(\text{YM})$ which satisfies[25]

$$
\|\mathcal{A}\|_{H^1} < \delta_0, \quad \mathcal{E}[\mathcal{F}] < \delta.
$$

where $\delta_0, \delta > 0$ are small absolute constants. Let $A_0$ be the corresponding regular solution to $(\text{YM})$ in the temporal gauge given by Theorem 7.4, which we assume to exist on $(-T_0, T_0) \times \mathbb{R}^3$ for some $T_0 > 0$. Then:

[25] We remind the reader the notation $\mathcal{F} = F_{\mu\nu}(t = 0, s = 0)$. 
(1) There exists a regular gauge transform \( V = V(t, x) \) on \((-T_0, T_0) \times \mathbb{R}^3\) and a regular solution \( A_a = A_a(t, x, s) \) to (HPYM) in the caloric-temporal gauge on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]\) such that \( F_{\mu\nu} \) is regular and

\[
A_{\mu}(s = 0) = V A_{\mu}^{-1} V^{-1} - \partial_{\mu} V V^{-1}.
\]

(2) With the notation \( \mathcal{I} := ||A||_{L^2_t} + ||\mathcal{E}||_{L^2_t} \), the following estimates hold.

\[
\mathcal{I}(0) \leq C \cdot T, \quad ||\nabla V||_{L^\infty_t} \leq C, \quad ||\partial_{\mu} V||_{L^2_t} + ||\partial_{\mu}^2 V||_{L^2_t} \leq C \cdot T \cdot \mathcal{I}.
\]

The identical estimates as the last two hold for \( V \) replaced by \( V^{-1} \) as well.

The non-trivial initial gauge transform \( V \) has been introduced to ensure that \( A_{\mu} \) is smoother than \( A_{\mu}^{t} \). It can be thought of as a substitute for the Uhlenbeck’s lemma in our setting; for more details, we refer the reader to [21, Proof of Theorem A]. The quantitative estimates \( (3.5) \), as well as the existence of \( V \), follows from \( [21, \text{Theorem A}] \).

The next theorem basically says that the conserved energy \( E(t) \) can be used to control \( \mathcal{I}(t) \) for every \( t \in (-T_0, T_0) \); we refer the reader to Step 2 in [1.6] for the basic idea behind the theorem.

**Theorem B** (Fixed time estimates by \( E \)). Let \( T_0 > 0 \), and consider a regular solution \( A_a \) to (HPYM) in the caloric-temporal gauge on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]\) satisfying \( \mathcal{I}(0) < \infty \) and \( E[\mathcal{F}] < \infty \).

Then for \( t \in (-T_0, T_0) \), \( \mathcal{I}(t) \) can be bounded in terms of the initial data and \( T_0 \), i.e.,

\[
\sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq C_{\mathcal{I}(0), E[\mathcal{F}], T_0} < \infty.
\]

From Theorems 1A and 1B we obtain a priori estimates on each fixed-time slice \( \{t\} \times \mathbb{R}^3 \times [0, 1] \). In order to estimate the gauge transform back to the temporal gauge, however, one needs to control \( A_0 \) (recall Lemma 3.6), and for this purpose it turns out that these fixed-time estimates are insufficient. In order to estimate \( A_0 \), we need to take advantage of the fact that the dynamic variables \( F, A \) satisfy wave equations, which is exactly what the next theorem achieves.

**Theorem C** (Short time estimates for (HPYM) in the caloric-temporal gauge). Let \( T_0 > 0 \), and consider a regular solution \( A_a \) to (HPYM) in the caloric-temporal gauge on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]\) such that \( F = F_{\mu\nu} \) is regular on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]\). Suppose furthermore that

\[
\sup_{t \in (-T_0, T_0)} E[\mathcal{F}(t, s = 0)] < \delta, \quad \sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq D,
\]

where \( D > 0 \) is an arbitrarily large finite number and \( \delta > 0 \) is an absolute small constant independent of \( D \).

Then there exists a number \( d = d(D, \delta) \), which depends on \( D, \delta \) in a non-increasing fashion, such that on every subinterval \( I_0 \subset I \) of length \( d \), the following estimate holds.

\[
\sup_{s \in [0, 1]} \|\partial_{i,a} A(s)\|_{C_t(I_0, L^2)} + \sup_{s \in [0, 1]} \mathcal{A}_0[A_0(s = 0)](I_0) \leq C_{D, \delta}.
\]

In essence, Theorem C is a result of a fairly standard local-in-time analysis of the wave equations of (HPYM). However, there is a little twist, which necessitates the extra hypotheses (3.7) and demands an explanation. Among the equations of (HPYM) is an equation for \( F_{\theta\theta} \) which, unlike the other components \( F_{s,s} \), is parabolic. As such, smallness of the time interval cannot be utilized to solve this equation in a perturbative manner[14]. What saves us is the fact that the parabolic equation for \( F_{\theta\theta} \) is covariant, and therefore can be analyzed using the covariant techniques presented in [4.2]. The first inequality of (3.7) provides the necessary smallness for this analysis, whereas the second one is needed to estimate the errors arising from switching covariant derivatives to usual derivatives. A rigorous proof will be given in Section 9.

We are now prepared to give a proof of the Main Theorem, under the assumption that Theorems 1A, 1B and 1C are true.

\[\text{[14]}\] In [21], this issue is bypassed by keeping the lengths of the \( s \)- and the time intervals fixed and requiring the size of the data to be small by scaling. If one unwinds the scaling, this amounts to taking the length of both the \( s \)- and the time intervals small.
Proof of the Main Theorem, assuming Theorems \( A \) and \( B \) and \( C \). To begin with, let us consider a regular initial data set with finite conserved energy, i.e., \( \mathbf{A}_0, \mathbf{E}_0 \) are smooth, \( \partial_x \mathbf{A}_0, \mathbf{E}_0 \) are regular and \( \mathbf{E}(\mathbf{F}) < \infty \). Applying Theorem \( 1.9 \) to \((\mathbf{A}_0, \mathbf{E}_0)\), there exists a unique regular solution to \((YM)\) in the temporal gauge on some time interval centered at 0, which we will denote by \( A_0^1 \). We will first show this solution exists globally in time.

For the purpose of contradiction, suppose that the solution \( A_0^1 \) cannot be extended globally as a unique regular solution to \((YM)\) in the temporal gauge. Then there exists a positive finite number \( 0 < T_0 < \infty \), which is the largest positive number for which the solution \( A_0^1 \) can be extended as a regular solution on \((-T_0, T_0)\). We claim that there exists a finite positive constant \( C = C_{\mathbf{T}, \mathbf{E}; \mathbf{F}, T_0} \), which depends only on the initial data and \( T_0 \), such that the following inequality holds.

\[
\sup_{t \in (-T_0, T_0)} \| A_0^1(t) \|_{\dot{H}^1_x} \leq C_{\mathbf{T}, \mathbf{E}; \mathbf{F}, T_0} < \infty.
\]

Let us complete the proof of the Main Theorem first, under the assumption that the claim is true. If the claim were true, then the solution may be extended as a unique regular solution to \((-T_0 - \epsilon, T_0 + \epsilon)\) for some \( \epsilon > 0 \) by Theorem \( 1.9 \) which is a contradiction. It follows that \( T_0 = \infty \), and thus \( A_0^1 \) can be extended globally in time as a unique regular solution to \((YM)\) in the temporal gauge. Observe that the estimate \((3.9)\) still holds for the global solution \( A_0^1 \) for every \( T_0 > 0 \).

Next, Lemma \( 3.8 \) implies that an admissible initial data can be approximated by a sequence of regular initial data sets \((\mathbf{A}_0(n), \mathbf{E}_0(n))\). Let us denote the corresponding unique global regular solutions by \( A_{0(n)}^1 \). Using Theorem \( 1.9 \) repeatedly (with the help of \( 3.8 \)), the following statement may be proved: For every \( T_0 > 0 \), the sequence of regular solutions \( A_{0(n)}^1 \) restricted to the time interval \((-T_0, T_0)\) is a Cauchy sequence in the topology \( C_t((-T_0, T_0) \times \mathbb{R}^3, L^2_x) \). Hence a limit \( A_0^1 \) exists on \((-T_0, T_0)\). Moreover, it is also possible to show that \( \partial_t A_{0(n)}^1 \rightarrow \partial_t A_0^1 \) in \( C_t((-T_0, T_0), L^2_x) \). Thus it follows that \( A_0^1 \) is admissible solution to \((YM)\) in the temporal gauge on \((-T_0, T_0)\). Uniqueness among the class of admissible solutions follows from the corresponding statement for regular solutions. As \( T_0 > 0 \) is arbitrary, \( A_0^1 \) is global, and the Main Theorem follows.

We are only left to establish the claim, which is a rather straightforward application of Theorems \( A \) and \( B \) and \( C \). First, by scaling, we may assume that \( \| A \|_{\dot{H}^1} < \delta \) and \( \mathbf{E}(\mathbf{F}) < \delta \), i.e., \( 3.8 \) holds. This allows us to apply Theorem \( A \) from which we obtain a regular gauge transform \( \mathbf{V} \) and a regular solution \( \mathbf{A}_0 \) to \((HPYM)\) in the caloric-temporal gauge on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1] \) such that \((3.4)\) holds. By Theorem \( B \) along with the estimate for \( \mathcal{I}(0) \) in \((3.5)\), we see that

\[
\sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq C_{\mathbf{T}, \mathbf{E}; \mathbf{F}, T_0} < \infty.
\]

To use Theorem \( C \) let us cover \((-T_0, T_0)\) by subintervals of length \( d \); the number of subintervals required can be bounded from above by, say, \( 10(T_0/d) \). Applying Theorem \( C \) on each subinterval, we are led to the estimate

\[
\sup_{s \in [0, 1]} \| \partial_{t,x} \mathbf{A}(s) \|_{C^1_1([t_0, t_1], L^2_x)} + \sup_{s \in [0, 1]} \mathbf{A}_0 [A_0(s = 0)] (I_0) \leq C_{\mathbf{T}, \mathbf{E}; \mathbf{F}, T_0} < \infty.
\]

The only remaining step is to transfer the above estimate to \( A_0^1 \); for this purpose, observe from \((3.4)\) that \( \mathbf{V} \) satisfies \( \partial_t \mathbf{V} = \mathbf{V} A_0(s = 0) \). Using Lemma \( 3.8 \) along with the previous estimates for \( \mathbf{A}_0 \) and \( \mathbf{V} \), we are led to the following estimate for the gauge transform \( \mathbf{V} \):

\[
\| \mathbf{V} \|_{L^\infty_t L^2_x} + \| \partial_{t,x} \mathbf{V} \|_{L^\infty_t L^2_x} + \| \partial_x^2 \mathbf{V} \|_{L^\infty_t L^2_x} \leq C_{\mathbf{T}, \mathbf{E}; \mathbf{F}, T_0} < \infty.
\]

Here, all norms have been taken over \((-T_0, T_0) \times \mathbb{R}^3 \). The preceding estimate, applied to the formula \((3.4)\), implies \((3.9)\) as desired. \( \square \)

4. Preliminaries

4.1. p-normalized norms and the Correspondence Principle. For the purpose of studying parabolic equations, it is quite convenient to utilize norms that are normalized according to the scaling properties of these equations. An estimate concerning homogeneous norms can be easily translated to the corresponding estimate in terms of the normalized norms, via a simple principle
we dub the *Correspondence Principle*. For a more detailed discussion, we refer the reader to [21] §3.3–3.5]; here, we will only give a brief summary which will suffice for the use in the present paper.

The basic idea is that $s^{1/2}$ scales like $x$, where $s$ is the time parameter for the parabolic equation. Therefore, whatever ‘dimension of $x$’ a norm has, we will normalize by compensating it with the appropriate factor of $s^{-1/2}$.

To be more precise, consider a norm $\| \cdot \|_X$ defined for functions on $\mathbb{R}^3$, which is homogeneous of degree $2\ell$ in the sense that $\|\phi(\lambda)\|_X = \lambda^{2\ell}\|\phi(\lambda)\|_X$. This indicates that the norm $\| \cdot \|_X$ has the ‘dimension of $x^{2\ell}$, and therefore we shall normalize it by multiplying by $s^{-\ell}$. Accordingly, we define the $p$-normalization of $X$ at $s$ (denoted by the calligraphic typeface $X(s)$) as

$$\| \cdot \|_{X(s)} := s^{-\ell}\| \cdot \|_X.$$  

Some examples of homogeneous norms which will be $p$-normalized are $L^p_x$, $H^m_x$, whose $p$-normalizations will be denoted by $L^p_x(s)$ and $H^m_x(s)$, respectively. These are, in fact, the only norms whose $p$-normalizations will be considered in this paper.

A derivative, such as $\partial_t$ or $D_i$, has the ‘dimension of $x^{-1}$’. Therefore, the $p$-normalizations of $\partial_t$ and $D_i$ at $s$ (denoted by $\nabla_i(s)$ and $D_i(s)$, respectively) are defined as

$$\nabla_i(s) := s^{1/2}\partial_i, \quad D_i(s) := s^{1/2}D_i.$$  

An estimate concerning homogeneous norms (e.g. Hölder, Sobolev Gagliardo-Nirenberg and etc.) naturally leads to a corresponding estimate in terms of the respective $p$-normalized norms; we will refer to this process as the *Correspondence Principle*. We will not make any effort to formulate a rigorous version of the principle, as it would be unpractical and overly complicated; instead, we will be satisfied with the following ‘cookbook-recipe’ type formulation, whose validity should be obvious every time the principle is invoked.

**Correspondence Principle.** Suppose that we are given an estimate in terms of norms $X_i$ of scalar- or $g$-valued functions $\sigma_i = \sigma_i(x)$, all of which are homogeneous. Suppose furthermore that the estimate is scale-invariant, in the sense that both sides transform the same under scaling.

Starting from such an estimate, make the following substitutions on both sides:

$$\sigma_i \rightarrow G_i(s), \quad \partial_x \rightarrow \nabla_x(s), \quad D_x \rightarrow D_x(s), \quad X_i \rightarrow X_i(s).$$

Then the resulting estimate still holds with the same constant for every $s \in J$.

The ‘proof’ of this principle is quite simple; it amounts to the observation that the weight of $s$ required to p-normalize each side is the same.

Next, let us define some norms with respect to the $s$-variable. Let $J \subset [0, \infty)$ be an interval, and consider a measurable function $f = f(s)$ defined on $J$. For $\ell \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the norm $L_s^{\ell,p}(J)$ by

$$\|f\|_{L_s^{\ell,p}(J)} := \left( \int_J \left( s^{\ell} f(s) \right)^p \frac{ds}{s} \right)^{1/p}.$$  

The following lemma, which is nothing but the Hölder inequality in the $s$-variable, is quite useful.

**Lemma 4.1** (Hölder for $L_s^{\ell,p}$). Let $\ell, \ell_1, \ell_2 \geq 0$, $1 \leq p, p_1, p_2 \leq \infty$ and $f, g$ functions on $J = (0, s_0]$ (or $J = (0, s_0]$) such that $\|f\|_{L_s^{\ell_1,p_1}}, \|g\|_{L_s^{\ell_2,p_2}} < \infty$. Then we have

$$\|fg\|_{L_s^{\ell,mp}(J)} \leq C s_0^{\ell_1 - \ell - \ell_2} \|f\|_{L_s^{\ell_1,p_1}(J)} \|g\|_{L_s^{\ell_2,p_2}(J)}$$

provided that either $\ell = \ell_1 + \ell_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, or $\ell > \ell_1 + \ell_2$ and $\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}$. In the former case, $C = 1$, while in the latter case, $C$ depends on $\ell - \ell_1 - \ell_2$ and $\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}$.

Finally, let us consider a mix of the preceding two types of norms. For a function $\psi = \psi(x,s)$ defined on $\mathbb{R}^3 \times J$ such that $s \rightarrow \|\psi(s)\|_{X(s)}$ is measurable on $J$, we define the norm $L_s^{\ell, p}(J)$ to be

$$\|\psi\|_{L_s^{\ell, p}(J)} := \|\psi(s)\|_{X(s)} \|\psi(s)\|_{L_s^{\ell, p}(J)}.$$
In order to derive estimates in terms of such norms, we will often use the Correspondence Principle and Lemma [1.1] in tandem. Let us demonstrate this with an example. Starting with a homogeneous estimate (which follows from Hölder and Corollary [1.3])

\[ \|O(\sigma_1, \sigma_2)\|_{L^2_x} \leq C\|\sigma_1\|_{L^2_x} \|\sigma_2\|_{L^2_x} \leq C\|\sigma_1\|^{1/2}_{L^2_x} \|D_x \sigma_1\|^{1/2}_{L^2_x} \|D_x \sigma_2\|_{L^2_x}, \]

applying the Correspondence Principle, taking the \(L^\ell_p(0, s_0)\) norm and using Lemma [1.1] we arrive at

\[ \|O(G_1, G_2)\|_{L^\ell_p L^2_x(0, s_0)} \leq C s_0^{\ell - \ell_1 - \ell_2} \|G_1\|^{1/2}_{L^{\ell_1, s_1} L^2_x(0, s_0)} \|D_x G_1\|^{1/2}_{L^{\ell_1, s_1} L^2_x(0, s_0)} \|D_x G_2\| \|L^{\ell_2, p_2} L^2_x(0, s_0)\|, \]

where \(C, \ell, \ell_1, \ell_2, p, p_1, p_2\) are as in Lemma [1.1].

4.2. Covariant techniques. Here, we collect some techniques which are applicable to the study of covariant parabolic equations. The use of such techniques, instead of those for handling the usual scalar heat equation, is the key analytic difference between this paper and [21].

Lemma 4.2 (Kato’s inequality). Let \(\sigma\) be a g-valued function. Then

\[ |\partial_x \sigma| \leq |D_x \sigma| \]

in the distributional sense.

Proof. Let \(\epsilon > 0\). We compute

\[ \partial_x \sqrt((\sigma, \sigma) + \epsilon) = \frac{(\sigma, D_x \sigma)}{\sqrt((\sigma, \sigma) + \epsilon)} \leq |\sqrt((\sigma, \sigma) + \epsilon)| \cdot |D_x \sigma| \leq |D_x \sigma|. \]

Testing against a positive test function and taking \(\epsilon \to 0\), we see that \(\partial_x \sigma \leq |D_x \sigma|\) in the distributional sense. Repeating the same argument to \(-\partial_x \sqrt((\sigma, \sigma) + \epsilon)\), we obtain (4.1). \(\square\)

The following Sobolev inequalities for covariant derivatives are easy consequences of Kato’s inequality.

Corollary 4.3 (Sobolev and Gagliardo-Nirenberg inequalities for covariant derivatives). For a regular g-valued function \(\sigma\), the following estimates hold.

\begin{align*}
(4.2) & \quad \|\sigma\|_{L^3_x} \leq C \|\sigma\|_{L^2_x}^{1/2} \|D_x \sigma\|_{L^2_x}^{1/2}, \\
(4.3) & \quad \|\sigma\|_{L^6_x} \leq C \|D_x \sigma\|_{L^2_x}, \\
(4.4) & \quad \|\sigma\|_{L^\infty_x} \leq C \|D_x \sigma\|_{L^2_x}^{1/2} \|D_x^{(2)} \sigma\|_{L^2_x}^{1/2}. \\
\end{align*}

Next, consider a inhomogeneous covariant heat equation

\[ (D_s - D^t D_t) \sigma = N. \tag{4.5} \]

Adapting the usual proof of the energy integral inequality (integration by parts) for the ordinary heat equation to (4.5), we obtain the following gauge-invariant version of the energy integral inequality.

Lemma 4.4 (Energy integral inequality). Let \(\ell \in \mathbb{R}, (s_1, s_2) \subset (0, \infty)\) and suppose that \(\sigma\) and \(A_i\) are “sufficiently nice”. Then the following estimate holds.

\[ \|\sigma\|_{L^{\ell, \infty} L^2_x(0, s_2)} \|D_x \sigma\|_{L^{\ell, 2} L^2_x(0, s_2)} \leq C s_0^{\ell} \|\sigma(1)\|_{L^2_x(s_1)} + C(\ell - 3/4)\|\sigma\|_{L^{\ell, 2} L^2_x(0, s_2)} + C\|N\|_{L^{\ell+1, 1} L^2_x(0, s_2)}. \tag{4.6} \]

\(\text{A sufficient condition for (4.6) to hold, which will be verifiable in applications below, is that \(\sigma\) is smooth and the left-hand side of (4.6) is finite.}\)
Proof. We will carry out a formal computation, discarding all boundary terms at the spatial infinity which arise; it is easy to verify that for ‘sufficiently nice’ $\sigma$ and $A_1$, this can be made into a rigorous proof.

Let $s \in (s_1, s_2]$. Taking the bi-invariant inner product of the equation $(D_s - D^s \ell)\sigma = \mathcal{N}$ with $s^{2\ell-3/2}\sigma$ and integrating by parts over $(s_1, s)$, we arrive at

$$\frac{1}{2} s^{2\ell-3/2} \int (\sigma, \sigma)(s) \, ds + \int_{s_1}^s s^{2\ell-1/2}(D^s \ell, D^s \sigma)(s) \, ds = \frac{1}{2} s^{2\ell-3/2} \int (\sigma, \sigma)(s_1) \, ds + \int_{s_1}^s s^{2\ell-3/2}(\sigma, \sigma)(s) \, ds \, ds.$$  

Taking the supremum over $s_1 < s \leq s_2$ and rewriting in terms of p-normalized norms, we obtain

$$\frac{1}{2} \|\sigma\|^2_{L^p_{-\infty}L^2_{s}(s_1, s_2)} + \|\sigma\|^2_{L^p_{s}D^2_{s}(s_1, s_2)} \leq \frac{1}{2} s^{2\ell} \|\sigma(s_1)\|^2_{L^p_{s}L^2_{s}(s_1, s_2)} + (\ell - \ell_0) \|\sigma\|^2_{L^p_{s}D^2_{s}(s_1, s_2)} + \|\mathcal{N}, \sigma\|^2_{L^{p+1, 1}_{s}L^2_{s}(s_1, s_2)}.$$  

By Hölder, Lemma 4.4 and Cauchy-Schwarz, the last term can be estimated by $\|\mathcal{N}\|^2_{L^{p+1, 1}_{s}L^2_{s}(s_1, s_2)}$. Taking the square root of both sides, we obtain (4.7).

Proceeding as in the proof of Kato’s inequality, we can derive the following parabolic inequality for $|\sigma|$.  

**Lemma 4.5 (Bochner-Weitzenböck-type inequality).** The following inequality holds in the distributional sense.

$$\Delta \sigma + \sigma \leq |\mathcal{N}|.$$  

**Proof.** This lemma was essentially proved in [28]; we shall give a proof nevertheless for completeness. Let $\epsilon > 0$. We compute

$$\partial_s \sqrt{(\sigma, \sigma)} + \epsilon = \frac{1}{\sqrt{(\sigma, \sigma)} + \epsilon}(\sigma, D_s \sigma),$$

$$\Delta \sigma + \epsilon \sqrt{(\sigma, \sigma)} + \epsilon = \frac{1}{\sqrt{(\sigma, \sigma)} + \epsilon}((\sigma, D^s \ell \sigma) + (D^s \sigma, D^s \ell \sigma) - \frac{(\sigma, D^s \ell \sigma)(\sigma, D^s \ell \sigma)}{(\sigma, \sigma)} + \epsilon).$$

Therefore,

$$(\partial_s - \Delta) \sqrt{(\sigma, \sigma)} + \epsilon = \frac{1}{\sqrt{(\sigma, \sigma)} + \epsilon}((\sigma, \mathcal{N}) - (D^s \sigma, D^s \ell \sigma) + \frac{(\sigma, D^s \ell \sigma)(\sigma, D^s \ell \sigma)}{(\sigma, \sigma)} + \epsilon),$$

$$\leq \frac{1}{\sqrt{(\sigma, \sigma)} + \epsilon}(\sigma, \mathcal{N}) \leq |\mathcal{N}|.$$  

Testing against a positive test function and taking $\epsilon \to 0$, we obtain (4.7).  

The virtue of (4.7) is that it allows us to use estimates arising from the (standard) heat kernel. Before we continue, let us briefly recap the definition and basic properties of the heat kernel.

Let $e^{s\Delta}$ denote the solution operator for the free heat equation. It is an integral operator, defined by

$$e^{s\Delta} \psi_0(x) = \frac{1}{\sqrt{4\pi s}} \int e^{-|x-y|^2/4s} \psi_0(y) \, dy.$$  

The kernel on the right hand side is called the heat kernel on $\mathbb{R}^3$. Using Young’s inequality, it is easy to derive the following basic inequality for the heat kernel:

$$\|e^{s\Delta} \psi_0\|_{L^p_{s}} \leq C_{p, r} s^{-3/(2p) + 3/(2r)}\|\psi_0\|_{L^r_{s}},$$  

where $C_{p, r}$ is a constant depending on the dimension and the growth of the space.
where $1 \leq p \leq r$.

Now consider the initial value problem for the inhomogeneous heat equation $(\partial_s - \triangle)\psi = N$. Duhamel’s principle tells us that this problem can be equivalently formulated in an integral form as follows:

$$\psi(s) = e^{s\triangle} \psi(s = 0) + \int_0^s e^{(s-\tau)\triangle} N(\tau) \, d\tau.$$ 

With these prerequisites, we are ready to derive a simple comparison principle for $|\sigma|$, along with a simple weak maximum principle; both statements are easily proved using basic properties of the heat kernel.

**Corollary 4.6.** Let $\sigma := \sigma(s = 0)$. Then the following point-wise inequality holds.

$$|\sigma|(x, s) \leq e^{s\triangle} |\sigma|(x) + \int_0^s e^{(s-\tau)\triangle} |N(\tau)|(x) \, d\tau,$$

As a consequence, the following weak maximum principle holds.

$$\sup_{0 \leq \tau \leq s} \|\sigma(\tau)\|_{L^p_x} \leq \|\sigma(s = 0)\|_{L^p_x} + \int_0^s \|N(\tau)\|_{L^p_x} \, d\tau.$$ 

**Proof.** The first inequality is an immediate consequence of (4.7), Duhamel’s principle, and the fact that the heat kernel $K(x, y) = \frac{1}{(4\pi s)^{d/2}} e^{-|x-y|^2/4s}$ is everywhere positive. The second one follows by taking the $L^p_x$ norm of (4.9) and using (4.10). □

For later use, we need the following lemma for the Duhamel integral, whose proof utilizes the basic inequality (4.8) for the heat kernel.

**Lemma 4.7.** The following estimate holds.

$$\left\| \int_0^s e^{(s-\tau)\triangle} N(\tau) \, d\tau \right\|_{L^{1/2} L^{2(1/2)}_{t,x}(0,s)} \leq C \|N\|_{L^{1+1/2} L^{2}_{t,x}(0,s)}.$$ 

**Proof.** Unwinding the definitions of p-normalized norms, (4.11) is equivalent to

$$\left( \int_0^{s_0} s^{1/2} \left( \int_0^{s_0} e^{(s-\tau)\triangle} N(\tau) \, d\tau \right)^2 \frac{ds}{s} \right)^{1/2} \right) \leq C \left( \int_0^{s_0} s \|N(s)\|_{L^1_x}^2 \frac{ds}{s} \right)^{1/2}.$$

Let us put $f(s) = s^{1/2} \|N(s)\|_{L^1_x}$; then it suffices to estimate the left-hand side of (4.12) by $C \|f\|_{L^2_t L^2_{x}(0,s_0)}$. By Minkowski and (4.8), we have

$$\left\| \int_0^s e^{(s-\tau)\triangle} N(\tau) \, d\tau \right\|_{L^{1/2}_x} \leq C \int_0^s (s-\tau)^{-3/4} \left( \int_0^s f(\tau) \, d\tau \right)^{1/2} \frac{d\tau}{\tau}.$$ 

Therefore the left-hand side of (4.12) is bounded from above by

$$C \left( \int_0^{s_0} \left( \int_0^s s^{1/4} (s-\tau)^{-3/4} f(\tau) \frac{d\tau}{\tau} \right)^{1/2} \frac{ds}{s} \right)^{1/2}.$$ 

Observe that

$$\sup_{s \in (0,s_0)} \int_0^s s^{1/4} (s-\tau)^{-3/4} \frac{d\tau}{s} \leq C,$$ 

Therefore, by Schur’s test, (4.11) is estimated by $\|f(s)\|_{L^2_t L^2_x(0,s_0)}$ as desired. □

Finally, we end this section with a simple lemma which is useful for substituting covariant derivatives by usual derivatives and vice versa.

**Lemma 4.8.** For $k \geq 1$, and $\alpha$ be a multi-index of order $k$. Then the following schematic algebraic identities hold.

$$D^{(\alpha)}_x \sigma = \delta^{(\alpha)}_x \sigma + \sum_{\ell = 1}^k O_\alpha (\partial^{(\ell)}_x A, \partial^{(\ell)}_x A, \cdots, \partial^{(\ell)}_x A, \partial^{(\ell)}_x \sigma),$$

$$\delta^{(\alpha)}_x \sigma = D^{(\alpha)}_x \sigma + \sum_{\ell = 1}^k O_\alpha (\partial^{(\ell)}_x A, \partial^{(\ell)}_x A, \cdots, \partial^{(\ell)}_x A, D^{(\ell)}_x \sigma).$$
In both cases, the summation is over all $1 \leq j \leq k$ and $0 \leq \ell_1, \ldots, \ell_j, \ell \leq k - 1$ such that $j + \ell_1 + \cdots \ell_j + \ell = k$.

Proof. In the case $k = 1$, both (4.14) and (4.15) follow from the simple identity

$$D_i \sigma = \partial_i \sigma + [A_i, \sigma].$$

The cases of higher $k$ follow from a simple induction argument, using Leibniz's rule. We leave the easy detail to the reader. \qed

5. Analysis of covariant parabolic equations

5.1. Covariant parabolic equations of (dYMHF). Let $I \subset \mathbb{R}$ be an interval, and consider a smooth solution $A_a$ to the dynamic Yang-Mills heat flow

$$(dYMHF) \quad F_{\mu \nu} = D^\ell F_{\ell \mu}, \quad \mu = 0, 1, 2, 3,$$

on $I \times \mathbb{R}^3 \times [0, 1]$. Note that these equations are a part of $\text{(HPYM)}^{18}$

Let us derive the following parabolic equation satisfied by $F_{\mu \nu}$.

(5.1)

$$D_s F_{\mu \nu} - D^\ell D_{\ell \mu} F_{\nu} = -2[F_\mu^{\ell}, F_{\nu \ell}].$$

We start with the Bianchi identity

(5.2)

$$D_a F_{bc} + D_b F_{ca} + D_c F_{ab} = 0,$$

which easily follows from the formula $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$. Taking the case $a = s$, $b = \mu$ and $c = \nu$, we arrive at the identity

$$D_s F_{\mu \nu} = D_\mu F_{s \nu} - D_\nu F_{s \mu}.$$

Since we are considering a solution to (dYMHF), the right-hand side is equal to $D_\mu D^\ell F_{\ell \nu} - D_\nu D^\ell F_{\ell \mu}$. Commuting the covariant derivatives and applying

$$D_\mu F_{\ell \nu} - D_\nu F_{\ell \mu} = D_\ell F_{\mu \nu},$$

(which is again a consequence of the Bianchi identity) we arrive at (5.1).

Next, let us derive covariant parabolic equations satisfied by higher covariant derivatives of $F$.

Given a $g$-valued tensor $B$, we compute

$$D_i D_s B - D_i D^\ell D_{\ell s} B = D_s D_i B - D^\ell D_{\ell s} D_i B = 2[F_\mu^{\ell}, D_{\ell s} B].$$

Concisely, $[D_i, D_s - D^\ell D_{\ell s}] B = O(F_i D_s B)$. Using this, it is not difficult to prove the following proposition.

Proposition 5.1 (Covariant parabolic equations of (dYMHF)). Let $A_a$ be a solution to (dYMHF). Then the curvature 2-form $F_{\mu \nu}$ satisfies the following parabolic equation.

(5.3)

$$D_s F_{\mu \nu} - D^\ell D_{\ell \mu} F_{\nu} = -2[F_\mu^{\ell}, F_{\nu \ell}].$$

The covariant derivatives of $F_{\mu \nu}$ satisfy the following schematic equation.

(5.4)

$$(D_s - D^\ell D_{\ell s})(D_s^{(k)} F) = \sum_{j=0}^{k} \mathcal{O}(D_s^{(j)} F, D_s^{(k-j)} F).$$

Proceeding in the same manner for a solution $A_a$ $(a = x^1, x^2, x^3, s)$ to (dYMHE), we may derive the following equation for $D_s^{(k)} F_{ij}$:

(5.5)

$$(D_s - D^\ell D_{\ell s})(D_s^{(k)} F) = \sum_{j=0}^{k} \mathcal{O}(D_s^{(j)} F, D_s^{(k-j)} F).$$

\footnote{One could say that these are parabolic equations of the Hyperbolic-Parabolic-Yang-Mills system.}
5.2. Estimates for the covariant parabolic equations. Let us fix a time \( t \in I \). Let us denote the Yang-Mills energy of \( F(t) \) at \( s = 0 \) by \( E(t) \), i.e.

\[
E(t) := E[F(t, s = 0)] = \sum_{\mu < \nu} \frac{1}{2} \| F_{\mu \nu}(t, s = 0) \|_{L^2}^2.
\]

Recall that \( D_t := s^{1/2} D_t \). The following proposition, which is proved by applying covariant techniques to (5.3), is the analytic heart of this paper.

**Proposition 5.2** (Covariant parabolic estimates for \( F \)). Let \( I \subset \mathbb{R} \) be an interval, and \( t \in I \). Suppose that \( A_n \) is a smooth solution to (1.3) on \( I \times \mathbb{R}^3 \times [0, 1] \) such that \( F = F_{\mu \nu} \) is regular on \( I \times \mathbb{R}^3 \times [0, 1] \). There exists \( \delta > 0 \) such that if \( E(t) < \delta \), then the following estimate holds for each integer \( k \geq 1 \).

\[
\| D_x^{(k-1)} F(t) \|_{L_t^{3/4, \infty} L_x^2(0,1)} + \| D_x^{(k)} F(t) \|_{L_t^{3/4,2} L_x^2(0,1)} \leq C_{k, E(t) \cdot \sqrt{E(t)}}.
\]

(5.6)

**Proof.** Let us start with the cases \( k = 1, 2 \). Let \( s \in (0, 1] \). Applying the energy integral estimate (4.1) with \( \ell = 3/4 \) to (5.5) and (5.4) for \( D_x F \), we have

\[
\| F \|_{L_t^{3/4, \infty} L_x^2(0,1)} + \| D_x F \|_{L_t^{3/4,2} L_x^2(0,1)} \leq C \sqrt{E} + C \| \mathcal{O}(F, F) \|_{L_t^{3/4+1,1} L_x^2(0,1)}.
\]

Using Hölder and Corollary (4,3), we see that

\[
\| \mathcal{O}(\sigma_1, \sigma_2) \|_{L_x^2} \leq C \| \sigma_1 \|_{L_x^2}^{1/2} \| D_x \sigma_1 \|_{L_x^2}^{1/2} \| D_x \sigma_2 \|_{L_x^2}.
\]

By the Correspondence Principle, Lemma (4.1) and the fact that \( s \leq 1 \), we have

\[
\| \mathcal{O}(F, F) \|_{L_t^{3/4+1,1} L_x^2(0,1)} \leq C \| F \|_{L_t^{3/4, \infty} L_x^2(0,1)}^{1/2} \| D_x F \|_{L_t^{3/4+2} L_x^2(0,1)} \leq C B_2(s)^2.
\]

Similarly, we also have

\[
\| \mathcal{O}(D_x F, F) \|_{L_t^{3/4+1,1} L_x^2(0,1)} \leq C B_2(s)^2.
\]

Therefore, we obtain a bound of the form \( B_2(s) \leq C \sqrt{E} + C B_2(s)^2 \), for every \( s \in (0, 1] \). Then by a simple bootstrap argument, the bound \( B_2(1) \leq C \sqrt{E} \) follows, which implies the desired estimate.

Let us turn to the case \( k \geq 3 \), which is proved by induction. Fix \( k \geq 3 \), and suppose, for the purpose of induction, that (5.6) holds for up to \( k - 1 \). That is, defining

\[
B_{k-1} := \sum_{j=1}^{k-1} \left[ \| D_x^{(j-1)} F \|_{L_t^{3/4, \infty} L_x^2(0,1)} + \| D_x^{(j)} F \|_{L_t^{3/4,2} L_x^2(0,1)} \right],
\]

we will assume that \( B_{k-1} \leq C_{k, E \cdot \sqrt{E}} \).

Applying the energy integral estimate (4.1) with \( \ell = \frac{3}{4} + \frac{k-1}{2} \) to (5.4) for \( D_x^{(k-1)} F \), we obtain

\[
\| D_x^{(k-1)} F \|_{L_t^{3/4, \infty} L_x^2} + \| D_x^{(k)} F \|_{L_t^{3/4,2} L_x^2} \leq C \| D_x^{(k-1)} F \|_{L_t^{3/4,2} L_x^2} + C \sum_{j=0}^{k-1} \| \mathcal{O}(D_x^{(j)} F, D_x^{(k-1-j)} F) \|_{L_t^{3/4+1,1} L_x^2}.
\]

where we used the fact that \( \liminf_{s \to 0} s^{3/4} \| D_x^{(k-1)} F(s) \|_{L_x^2(s)} = 0 \).
The first term is bounded by $B_{k-1}$; therefore, (5.6) for $k$ will follow once we establish
\begin{equation}
(5.7) \quad \sum_{j=0}^{k-1} \|O(D_x^{(j)}F, D_x^{(k-1-j)}F)\|_{L^{3/4+1.1}_x L^{2}_t} \leq CB_{k-1}^2.
\end{equation}

By Leibniz’s rule, we see that (5.7) follows once we establish the estimates
\begin{equation}
(5.8) \quad \begin{cases}
\|O(D_x G_1, D_x G_2)\|_{L^{3/4+1.1}_x L^{2}_t} \leq CB_{k}^2 \\
\|O(G_1, D_x^{(2)} G_2)\|_{L^{3/4+1.1}_x L^{2}_t} + \|O(D_x^{(2)} G_1, G_2)\|_{L^{3/4+1.1}_x L^{2}_t} \leq CB_{k}^2.
\end{cases}
\end{equation}
for any $g$-valued 2-forms $G_i = G_i(x,s)$. Note that these roughly correspond to the case $k = 3$ of (5.7).

Using the Correspondence Principle, Lemma 4.1, and recalling the definition of $B_{k-1}$, it suffices to prove the estimates
\begin{align*}
\|O(D_x \sigma_1, D_x \sigma_2)\|_{L^2} &\leq C\|D_x \sigma_1\|_{L^{1/2}_x} \|D_x^{(2)} \sigma_1\|_{L^{2}_x} \|D_x^{(2)} \sigma_2\|_{L^2} , \\
\|O(\sigma_1, D_x^{(2)} \sigma_2)\|_{L^2} &\leq C\|D_x \sigma_1\|_{L^{1/2}_x} \|D_x^{(2)} \sigma_1\|_{L^{2}_x} \|D_x^{(2)} \sigma_2\|_{L^2}.
\end{align*}
The former is an easy consequence of Hölder, (4.2) and (4.3), whereas the latter is proved similarly by applying Hölder, (4.3) and (4.4).

Recalling $F_{a\nu} = D^a F_{\ell \nu}$, we obtain the following estimates for $F_{a\nu}$.

**Corollary 5.3.** Under the same hypotheses as Proposition 5.2, the following estimates hold for every integer $k \geq 0$.
\begin{align}
(5.9) \quad &\|D^{(k)} F_x\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} + \|D^{(k)} F_s\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} \leq C_{k, E} \cdot \sqrt{E} , \\
(5.10) \quad &\|D^{(k)} F_x\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} + \|D^{(k)} F_s\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} \leq C_{k, E} \cdot \sqrt{E}.
\end{align}

**Proof.** The $L^2$-type estimate (5.9) follows immediately from Proposition 5.2 by the relation $F_{a\nu} = D^a F_{\ell \nu}$. The $L^{\infty}$-type estimate (5.10) then follows from (5.9) by (4.4) of Corollary 1.3 (covariant Gagliardo-Nirenberg) and the Correspondence Principle.

The above discussion may be easily restricted to spatial connection 1-form $A_i$ satisfying (CYMHF). Given a spatial 2-form $F = F_{ij}$ $(i,j = 1,2,3)$, let us define the magnetic energy $B[F]$ by
\[ B[F] := \sum_{i<j} \frac{1}{2} \|F_{ij}(s = 0)\|_{L^2}^2. \]

Repeating the proof of Proposition 5.2 the following proposition easily follows.

**Proposition 5.4.** (Covariant parabolic estimates for $F$). Let $\delta > 0$ be as in Proposition 5.2 and consider a smooth solution $(A_i, A_x)$ to the covariant Yang-Mills heat flow $F_{a\nu} = D^a F_{\ell \nu}$ on $\mathbb{R}^3 \times [0,1]$ such that $F = F_{ij}$ is regular on $\mathbb{R}^3 \times [0,1]$. If $B := B[F(s = 0)] < \delta$, then the following estimate holds for every integer $k \geq 1$.
\begin{equation}
(5.11) \quad \|D^{(k-1)} F_x\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} + \|D^{(k)} F_x\|_{L^{3/4}_x L^{\infty}_t L^2(0,1)} \leq C_{k, B} \cdot \sqrt{B}.
\end{equation}

6. Analysis of Yang-Mills heat flows in the caloric gauge

6.1. **Analysis of the Yang-Mills heat flow.** In this subsection, we will consider the following IVP for (CYMHF) in the caloric gauge $A_x = 0$, for $s_0 > 0$.
\[ \begin{cases}
\partial_s A_i = D^i F_{\ell i}, & \text{on } \mathbb{R}^3 \times [0,s_0] \\
A_i(s = 0) = \overline{\mathcal{A}}_i.
\end{cases} \]

As this system is simply the original Yang-Mills heat flow (YMHF), we will refer to it simply as (YMHF). We will mainly be concerned with the class of regular initial data sets and regular solutions to (YMHF), which are defined as follows.
Definition 6.1. We say that a connection 1-form $\tilde{A}_i$ on $\mathbb{R}^3$ is a regular initial data set for (YMHE) if $\partial_x A_i$ is regular on $\mathbb{R}^3$. Furthermore, we say that a smooth solution $A_i$ to (YMHE) defined on $\mathbb{R}^3 \times [0, s_0]$ is a regular solution to (YMHE) if $\partial_x A_i$ is regular on $\mathbb{R}^3 \times [0, s_0]$.

Our immediate goal is to establish a local well-posedness theorem (Theorem 6.5), where the interval of existence depends only on the magnetic energy $B(F)$ of the initial data. The starting point of our analysis is Theorem C from [21], which is a $H^1_x$ local existence statement. We restate the theorem below for the convenience of the reader.

Theorem 6.2 ([21 Theorem C]). Consider the above IVP for (YMHE) with initial data $\tilde{A}_i \in H^1_x$ at $s = 0$.

1. There exists a number $s^* = s^*(\|\tilde{A}\|_{H^1_x}) > 0$, non-increasing in $\|\tilde{A}\|_{H^1_x}$, such that there exists a solution $A_i \in C_s([0, s^*], H^1_x)$ to the IVP satisfying

$$\sup_{s \in [0, s^*]} \|A(s)\|_{H^1_x} \leq C\|\tilde{A}\|_{H^1_x}.$$  

2. Let $\tilde{A}_i \in H^1_x$ be another initial data set such that $\|\tilde{A}\|_{H^1_x} \leq \|\tilde{A}\|_{H^1_x}$, and $A_i'$ the corresponding solution to the IVP on $[0, s^*]$ given in (1). Then the following estimate for the difference $\delta A := A - A'$ holds.

$$\sup_{s \in [0, s^*]} \|\partial_x (\delta A)(s)\|_{H^1_x} \leq C\|\delta A\|_{H^1_x}.$$  

3. If $\tilde{A}_i$ is smooth and $\partial x \tilde{A}_i, F_{ij} := \partial_i \tilde{A}_j - \partial_j \tilde{A}_i + [\tilde{A}_i, \tilde{A}_j]$ are regular, then the solution $A_i \in C_s([0, s^*], H^1_x)$, with $A_i(x, s)$ given in (1) is smooth and $\partial_x A_i, F_{ij}$ are regular on $[0, s^*]$. Furthermore, if $\tilde{A}_i(t) \in H^1_x$ is a one parameter family of initial data such that $\partial_t \tilde{A}_i, F_{ij}$ are regular on $I \times \mathbb{R}^3$, then $A_i = A_i(t, x)$ is smooth and $\partial_t A_i, F_{ij}$ are regular on $I \times \mathbb{R}^3 \times [0, s^*]$.

According to our definition, the first statement of Part (3) of Theorem 6.2 states that if $\tilde{A}_i$ is a regular initial data set for (YMHE), then there exists a solution $A_i$ to (YMHE) which is regular in the sense of Definition 6.1.

Remark 6.3 (Remark on the proof of Theorem 6.2). Let us return to the perspective of viewing YMHE as the system (YMHE) with the caloric gauge condition imposed. The problem with the caloric gauge is that the equations for $A_i$ are not strictly parabolic, but only weakly-parabolic. As discussed in the introduction, we can make the system (YMHE) strictly parabolic by choosing a different gauge, namely the DeTurck gauge $A_i = \partial^\alpha A_i$. Local well-posedness for $H^1_x$ data (on some $s$-interval $[0, s^*]$) then follows by a rather standard parabolic theory. To come back to the caloric gauge $A_i = 0$, however, we must perform a gauge transform. The desired gauge transform can be obtained by solving the following ODE:

$$\begin{cases}
\partial_x U = U A_x & \text{on } \mathbb{R}^3 \times [0, s^*], \\
U(s = 0) = \text{Id}.
\end{cases}$$

This ODE can be easily derived by the gauge transform formula $\tilde{A}_i = U A_i U^{-1} - \partial_x U U^{-1}$. The initial condition $U(s = 0) = \text{Id}$ has been chosen to leave the initial data the same. It is then possible to show that the gauge transform is bounded on $C_s([0, s^*], H^1_x)$, which proves Theorem 6.2.

We add that this procedure ends up being the standard DeTurck trick, as in [7], applied to YMHE.

As a first step, we complement Theorem 6.2 with the following uniqueness statement in the class of regular solutions.

Lemma 6.4 (Uniqueness of regular solution to (YMHE)). Let $A_i, A_i'$ be two regular solutions to YMHE on a common $s$-interval $J = [0, s_0]$. If their initial data coincide, i.e. $\tilde{A}_i = \tilde{A}_i'$, then so do the solutions, i.e. $A_i = A_i'$.
Proof. By taking the difference between the parabolic equations satisfied by \( F_{ij} \) and \( F'_{ij} \), we obtain the following equation for \( \delta F_{ij} = F_{ij} - F'_{ij} \):

\[
(6.3) \quad (\partial_s - D^t D_t)(\delta F_{ij}) = -2[F_{ij}^t, \delta F_{ij}] - 2[\delta F_{ij}^t, F'_{ij}] + (\delta D^t D_t)F_{ij},
\]

where \( \delta D^t D_t \) is defined by

\[
(\delta D^t D_t)B := 2[\delta A^t, \partial_t B] + [\partial^t(\delta A_t), B] + [A^t, [\delta A_t, B]] + [\delta A^t, [A'_t, B]].
\]

Next, recall that \( F_{si} = D^t F_{ti} \) satisfies the equation

\[
(\partial_s - D^t D_t)F_{si} = -2[F_{si}^t, F_{st}].
\]

Taking the difference between the equations for \( F_{si} \) and \( F'_{si} \), we obtain

\[
(6.4) \quad (\partial_s - D^t D_t)(\delta F_{si}) = -2[F_{si}^t, \delta F_{si}] - 2[\delta F_{si}^t, F'_{si}] + (\delta D^t D_t)F_{si},
\]

As \( F_{si} = \partial_s A_i \), we have

\[
(6.5) \quad \partial_s(\delta A_i) = \delta F_{si}.
\]

Finally, thanks to the special identity \( D^t F_{st} = 0 \), we have the following equation for \( \partial^t(\delta A_t) \):

\[
(6.6) \quad \partial_t \partial^t(\delta A_t) = -[A^t, \delta F_{st}] - [\delta A^t, F'_{st}].
\]

Now for each \( s \in J \), let us define \( \delta B(s) \) to be

\[
\delta B(s) := \max_{i,j} \left( \|\delta A_i(s)\|_{L^\infty_t} + \|\partial^t(\delta A_t)(s)\|_{L^\infty_t} + \|\delta F_{ij}(s)\|_{L^\infty_t} + \|\delta F_{st}(s)\|_{L^\infty_t} \right).
\]

Let \( \epsilon > 0 \). Applying the weak maximum principle \( (4.10) \) of Corollary \( (4.6) \) to \( (6.3), (6.4) \) and integrating \( (6.5), (6.6) \), for each \( s \in [0, s_0 - \epsilon] \) we arrive at

\[
\delta B(s) < \delta B(0) + (C + C^2) \int_0^s \delta B(s) \, ds.
\]

where

\[
C = \max_{ij} (\|A_i\|_{L^\infty_{t,x}}, \|F_{ij}\|_{L^\infty_{t,x}}, \|F_{st}\|_{L^\infty_{t,x}}),
\]

with all norms taken over \( I \times \mathbb{R}^3 \times [0, s_0 - \epsilon] \). Note that \( C < \infty \), since \( A_i, A'_i \) are regular. Then as \( \delta B(0) = 0 \), by Gronwall’s inequality, it follows that \( \delta B(s) = 0 \) for all \( [0, s_0 - \epsilon] \). Taking \( \epsilon \to 0 \), we see that \( A_i = A'_i \) on \( I \times \mathbb{R}^3 \times [0, s_0] \). \( \square \)

**Theorem 6.5 (Improved local well-posedness for \( \text{(YMHF)} \)).** Consider the above IVP for \( \text{(YMHF)} \) with a regular initial data set \( \overline{\mathbf{A}}_i \) (in the sense of Definition \( 6.7 \)). Suppose furthermore that \( \mathbf{F}_{ij} := \partial_i \overline{\mathbf{A}}_j - \partial_j \overline{\mathbf{A}}_i + [\overline{\mathbf{A}}_i, \overline{\mathbf{A}}_j] \) belongs to \( L^2_t \) and the norm is sufficiently small, i.e.

\[
\mathcal{B}[\mathbf{F}] = \frac{1}{2} \sum_{i < j} \|\mathbf{F}_{ij}\|_{L^2_t}^2 < \delta,
\]

where \( \delta > 0 \) is the positive number as in Proposition \( 5.2 \).

Then there exists a unique regular solution \( A_i \) to the IVP on \( \mathbb{R}^3 \times [0, 1] \). Furthermore, this solution has the additional property that \( F_{ij} \) is regular on \( \mathbb{R}^3 \times [0, 1] \).

**Remark 6.6.** Other constituents of a local well-posedness statement, such as continuous dependence on the data, can be proved by a minor modification of the proof below. Also, the statement can be extended to a rougher class of initial data and solutions by an approximation argument. We shall not provide proofs for these as they are not needed in the sequel; we welcome the interested reader to fill in the details.

**Proof.** By Theorem \( 6.2 \) and Lemma \( 6.4 \) there exists \( s^* > 0 \) such that a unique regular solution \( A_i \) to the IVP for \( \text{(YMHF)} \) exists on \( [0, s^*] \) and obeys

\[
(6.7) \quad \sup_{0 \leq s \leq s^*} \|A(s)\|_{L^2_t} \leq C \sup_{0 \leq s \leq s^*} \|A(s)\|_{H^1_t} \leq C \|\overline{\mathbf{A}}\|_{H^1_t}.
\]

We remark that the first inequality holds by Sobolev embedding.
Let us denote by $s_{\text{max}}$ the largest $s$-parameter for which $A_i$ extends as a unique regular solution on $[0, s_{\text{max}})$. We claim that under the hypothesis that $B[F] < \delta$, the following statement holds:

\begin{equation}
\text{If } s_{\text{max}} \leq 1 \text{ then } \sup_{s \in [0, s_{\text{max}})} \|A_i\|_{H^2} < \infty.
\end{equation}

If this claim were true, then we may apply Theorem 6.2 and Lemma 6.4 to extend $A_i$ past $s_{\text{max}}$ if $s_{\text{max}} \leq 1$. Therefore, it would follow that $s_{\text{max}} > 1$.

Let us establish (6.8). The first step is to show that $\|A(s)\|_{L^p_t} \leq C_B \cdot \sqrt{\|A\|_{H^2}}$. We claim that under the hypothesis that $B[F] < \delta$, it suffices to restrict our attention to $s > s^*$; therefore, $s \in (s^*, s_{\text{max}})$. Since $s_{\text{max}} \leq 1$, by Proposition 5.4 and Corollary 1.3 we see that

\[ \|\partial_s A_i(s)\|_{L^p_t} = \|D_t F_i(s)\|_{L^p_t} \leq s^{-1} C_B \cdot \sqrt{B}. \]

Integrating from $s = s^*$ and using (6.7), we arrive at

\begin{equation}
\sup_{s^* < s < s_{\text{max}}} \|A(s)\|_{L^p_t} \leq \|A\|_{H^2} + C_B \cdot \|\log s^*\| \sqrt{B} < \infty.
\end{equation}

Next, let us show that $\|A_i(s)\|_{H^2} \leq C_B \cdot \sqrt{\|A\|_{H^2}}$ does not blow up on $[0, s_{\text{max}})$. Again, it suffices to consider $s \in (s^*, s_{\text{max}})$. Recall that $D_s D^s F_i = \partial_s D^s F_i + [A, D^s F_i]$; thus by triangle and Hölder,

\[ \|\partial_s A_i(s)\|_{H^2} \leq \|D_s D^s F_i(s)\|_{L^2_t} + \|A(s)\|_{L^p_t} \|D^s F_i(s)\|_{L^2_t}. \]

Using Proposition 5.4 and Corollary 1.3 we obtain

\[ \|\partial_s A_i(s)\|_{H^2} \leq s^{-1} C_B \cdot \sqrt{B} + s^{-3/4} C_B \cdot \sqrt{B} \left( \sup_{s^* < s < s_{\text{max}}} \|A(s)\|_{L^p_t} \right). \]

Recalling (6.8) and integrating from $s^*$, we see that $\sup_{s^* < s < s_{\text{max}}} \|A_i\|_{H^2} < \infty$, as desired. \hfill $\Box$

For any regular initial data with finite magnetic energy, we can use scaling to make $B(s = 0) < \delta$; thus, Theorem 6.2 applies also to initial data with large magnetic energy. Furthermore, using the fact that the magnetic energy $B(s)$ is non-increasing in $s$ under the Yang-Mills heat flow (which is formally obvious, as the Yang-Mills heat flow is the gradient flow of $B$; see [22]), we can in fact iterate Theorem 6.2 to obtain a unique global solution to the IVP, leading to an independent proof of the following classical result of [22].

**Corollary 6.7** (Råde [22]). Consider the IVP for $(\text{YMHF})$ with a regular initial data set $\overline{A}_i$ which possesses finite magnetic energy, i.e. $B[F] := (1/2) \sum_{1 < j} F_{ij} \|F_{ij}\|_{L^2} < \infty$. Then there exists a unique global regular solution $A_i$ to the IVP on $\mathbb{R}^3 \times [0, \infty)$, with the additional property that $F_{ij}$ is regular on $\mathbb{R}^3 \times [0, \infty)$.  

### 6.2. Analysis of the dynamic Yang-Mills heat flow in the caloric gauge.

Hereafter, we shall study the dynamic Yang-Mills heat flow $(d\text{YMHF})$ in the caloric gauge $A_s = 0$. Writing out the left-hand side of $(d\text{YMHF})$, we obtain

\begin{equation}
\partial_s A_{\mu} = D^\mu F_{i\mu}, \quad \mu = 0, 1, 2, 3.
\end{equation}

Let $I \subset \mathbb{R}$ be an interval. We will study the IVP associated to (6.10) on $I \times \mathbb{R}^3 \times [0, 1]$, with the initial data given by

\begin{equation}
A_{\mu}(t, x, 0) = \overline{A}_{\mu}(t, x) \quad \text{on } I \times \mathbb{R}^3.
\end{equation}

As in (6.1) we will be focusing on regular initial data sets and solutions to (6.10), whose definitions we give below.

**Definition 6.8.** Let $I \subset \mathbb{R}$ be an interval. We say that a connection 1-form $\overline{A}_{\mu} = \overline{A}_{\mu}(t, x)$ defined on $I \times \mathbb{R}^3$ is a regular initial data set for $(d\text{YMHF})$ in the caloric gauge if $\partial_{t,x} A_{\mu}$ is regular on $I \times \mathbb{R}^3$. Furthermore, we say that a smooth solution $A_{\mu}$ to (6.10) defined on $I \times \mathbb{R}^3 \times [0, s_0]$ is a regular solution to $(d\text{YMHF})$ in the caloric gauge if $\partial_{t,x} A_{\mu}$ is regular on $I \times \mathbb{R}^3 \times [0, s_0]$.

We begin with a uniqueness lemma for a regular solution to $(d\text{YMHF})$ in the caloric gauge.

\footnote{Note that integrating from $s^*$ allows us to bypass the issue of logarithmic divergence at $s = 0$.}
Lemma 6.9 (Uniqueness of regular solution to (YMHF) in the caloric gauge). Let $A_\mu, A'_\mu$ be two regular solutions to (YMHF) in the caloric gauge on $I \times \mathbb{R}^3 \times [0, s_0]$ for some $I \subset \mathbb{R}$ and $s_0 > 0$. If their initial data coincide, i.e. $A_\mu(s = 0) = A'_\mu(s = 0)$ on $I \times \mathbb{R}^3$, then so do the solutions, i.e. $A_\mu = A'_\mu$ on $I \times \mathbb{R}^3 \times [0, s_0]$.

Proof. Note that the spatial components $A_i(t)$ satisfy (YMHF) for each fixed $t$; therefore, by Lemma 6.3 it follows that $A_i = A'_i$. As a consequence, we are only left to show $A_0 = A'_0$.

As we already know $A_1 = A'_1$, observe that $\delta F_{0i} = F'_{0i} - F_{0i}$ now obeys the simple equation

$$\partial_s (\delta F_{0i}) (\delta F_{0i}) = -2[\delta F_0^\ell, F_{0\ell}].$$

Note furthermore that $\delta F_{0i}(s = 0) = 0$. Applying the weak maximum principle (4.10) and using the Gronwall's inequality as before, we see that $F_{0i} = F'_{0i}$ on $I \times \mathbb{R}^3 \times [0, s_0]$. Then, by the dynamic Yang-Mills heat flow, $\partial_s A_0 = D^\ell F_{0\ell} = \partial_s A'_0$ everywhere. Since $A_0(s = 0) = A'_0(s = 0)$, it follows that $A_0 = A'_0$ on $I \times \mathbb{R}^3 \times [0, s_0]$, which concludes the proof. \hfill $\Box$

We end this section with a 'well-posedness statement' (by which we mean existence and uniqueness here) for the IVP for (YMHF) in the caloric gauge with regular initial data set possessing finite energy,$^{20}$ $E(0) < \infty$.

Theorem 6.10 (Local well-posedness for (YMHF) in the caloric gauge). Let $I \subset \mathbb{R}$ be an interval and consider the above IVP for (YMHF) in the caloric gauge with a regular initial data set $A_{\mu}$ (in the sense of Definition 6.2). Suppose furthermore that $A_{\mu} \in L^\infty_t L^2_x$, $F_{\mu\nu} := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \in L^\infty_t L^2_x$, and the energy is uniformly small on $I$, i.e.

$$\sup_{t \in I} E[F(t)] = \sup_{t \in I} \frac{1}{2} \sum_{\mu < \nu} \|F_{\mu\nu}(t)\|^2_{L^2_x} < \delta,$$

where $\delta > 0$ is the positive number as in Proposition 5.2.

Then there exists a unique regular solution $A_{\mu}$ to the IVP on $I \times \mathbb{R}^3 \times [0, 1]$. Furthermore, this solution has the additional property that $A_{\mu} \in C^\infty_t(I \times [0, 1], L^2_x)$ and $F_{\mu\nu}$ is regular on $I \times \mathbb{R}^3 \times [0, 1]$.

Proof. As in the proof of Lemma 6.3 we begin with the observation that the spatial components $A_i(t)$ satisfy (YMHF) for each fixed $t$. Observe furthermore that $B[F(t)] \leq E[F(t)] < \delta$, where $\delta > 0$ is the common small constant occurring in Propositions 5.2 and 5.4. Repeating the proof of Theorem 6.3 (but this time making use of the second statement of Part (3) of Theorem 6.2), we obtain a smooth spatial 1-form $A_i$ on $I \times \mathbb{R}^3 \times [0, 1]$ solving (YMHF) for each fixed $t$, such that $\partial_{i,x} A_i, F_{ij}$ are regular on $I \times \mathbb{R}^3 \times [0, 1]$. At this point, we are left to solve for $A_0$, making use of the equation

$$\partial_s A_0 = D^\ell F_{0\ell}.$$  \hfill (6.12)

Here, we follow Step 1 of [21] Proof of Theorem A. Note that $F_{0i}$ satisfies the covariant parabolic equation

$$\partial_s F_{0i} = D^\ell D_{i\ell} F_{0i} = 2[F_{0i}^\ell, F_{0\ell}].$$  \hfill (6.13)

Since $A_i$ and $F_{ij}$ have already been solved, we may view this as a linear system of parabolic equation for $F_{0i}$ with smooth coefficients, which should be amenable to standard techniques. Indeed, applying [21] Proposition 5.6, there exists a unique regular solution $F_{0i}' = -F_{0i}^\ell$ to (6.13) on $I \times \mathbb{R}^3 \times [0, 1]$ with $F_{0i}'(s = 0) = \overline{F_{0i}}$. Then motivated by (6.12), we extend $\overline{A_0}$ to $A_0$ on $I \times \mathbb{R}^3 \times [0, 1]$ by solving

$$\begin{cases}
\partial_s A_0 = D^\ell F_{0\ell}' & \text{on } I \times \mathbb{R}^3 \times [0, 1], \\
A_0(s = 0) = \overline{A_0} & \text{on } I \times \mathbb{R}^3.
\end{cases}$$

By construction, it is easy to verify that $A_0$ is smooth, $\partial_{i,x} A_0$ is regular and $A_0 \in C^\infty_t(I \times [0, 1], L^2_x)$. With $A_0$ defined on $I \times \mathbb{R}^3 \times [0, 1]$, we may define the curvature components $F_{0i} = \partial_{i,x} A_0 - \partial_{i,x} A_0 + [A_0, A_i]$ and ask whether $F_0 = F'_0$. This is indeed the case, by [21] Lemma 6.1,\hfill $^{20}$

\hfill $^{20}$The theorem itself concerns the case of small energy, but by scaling, of course, the theorem extends to the large energy case as well.
Taking multiplying both sides by of calculus and Minkowski’s inequality, we have the relation furthermore that in the caloric gauge Let and we have the desired solution to (dYMHF) in the caloric gauge. □

6.3. Substitution of covariant derivatives by usual derivatives. At several points below, we will need to transfer estimates for covariant derivatives to the corresponding estimates for usual derivatives. The purpose of this part is to develop a general technique for carrying out such procedures. Our starting point is the following proposition, which concerns estimates for the \( L^2_x \) norm of \( A \).

To state the following proposition, we need the following definition.

\[
(\Delta)I(t) := \sum_{k=1}^{31} \|\partial_{x^k}A(t)\|_{H^{k-1}}.
\]

In fact, this is a part of a larger norm \( I(t) \), whose definition will be given in Section 7

Proposition 6.11. Let \( I \subset \mathbb{R} \) be an interval, \( t \in I \), and consider a regular solution \( A_\mu \) to (dYMHF) in the caloric gauge \( A_s = 0 \) on \( I \times \mathbb{R}^3 \times [0, 1] \). Suppose that \( F = F_{\mu\nu} \) is regular on \( I \times \mathbb{R}^3 \times [0, 1] \), and \( E(t) := E[F(t, s = 0)] < \delta \), where \( \delta > 0 \) is the small constant in Proposition 5.2. Then the following estimate holds for all \( 0 \leq k \leq 29 \).

\[
\|\nabla_x^{(k)}A(t)\|_{L^{1/4}_{x,t}} L^\infty_x(0,1) \leq (\Delta)I + C_{k, E(t), (\Delta)I(t)} \cdot \sqrt{E(t)}.
\]

Proof. Henceforth, we shall fix \( t \in I \) and omit writing \( t \). By the caloric gauge condition \( A_s = 0 \), we have the relation \( \partial_s A_\nu = F_{s\nu} \), where the latter can be controlled by Corollary 5.3. Observe furthermore that

\[
\|\partial_x^{(k)}A\|_{L^\infty_x} \leq C(\Delta)I
\]

for \( 0 \leq k \leq 29 \), by Sobolev (or Gagliardo-Nirenberg). Now, the idea is to use the fundamental theorem of calculus of control \( \partial_x^{(k)}A(s) \) for \( 0 < s \leq 1 \).

We will proceed by induction on \( k \). Let us start with the case \( k = 0 \). By the fundamental theorem of calculus and Minkowski’s inequality, we have

\[
s^{1/2} \|A(s)\|_{L^\infty_x} \leq s^{1/4} \|A\|_{L^\infty_x} + \int_s^1 (s/s')^{1/4}(s')^{5/4} \|F_s(s')\|_{L^\infty_x} \frac{ds'}{s'}.
\]

As remarked earlier, the first term on the right-hand side may be estimated by \( (\Delta)I \) uniformly in \( s \in (0, 1] \). For the second term, we apply (6.11) of Corollary 5.3 and estimate \( (s')^{5/4} \|F_s(s')\|_{L^\infty_x} \) by \( C_{k, E} \cdot \sqrt{E} \). The case \( k = 0 \) of (6.14) follows, since

\[
\sup_{0 < s \leq 1} \int_s^1 (s/s')^{1/4} \frac{ds'}{s'} \leq C < \infty.
\]

Next, for the purpose of induction, assume that (6.14) holds for \( 0, 1, \ldots, k - 1 \), where \( 1 \leq k \leq 29 \). Taking \( \partial_x^{(k)} \) of \( \partial_s A_\nu = F_{s\nu} \), using the fundamental theorem of calculus, Minkowski’s inequality and multiplying both sides by \( s^{1/4+k/2} \), we arrive at

\[
s^{1/4} \|\nabla_x^{(k)}A(s)\|_{L^\infty_x} \leq s^{1/4+k/2} \|\partial_x^{(k)}A\|_{L^\infty_x} + \int_s^1 (s/s')^{1/4+k/2}(s')^{5/4} \|\nabla_x^{(k)}F_s(s')\|_{L^\infty_x} \frac{ds'}{s'}.
\]

Once we establish

\[
\sup_{0 < s \leq 1} \|s^{5/4} \nabla_x^{(k)}F_s\|_{L^\infty_x} \leq C_{k, E, (\Delta)I} \cdot \sqrt{E},
\]

then proceeding as in the previous case, (6.14) for \( k \) will follow, which completes the induction.

Fix \( 0 < s \leq 1 \). Applying (1.15) of Lemma 3.3 and multiplying both sides by \( s^{5/4+k/2} \), we see that

\[
s^{5/4} \nabla_x^{(k)}F_s(s) = s^{5/4} D_x^{(k)}F_s(s) + \sum_x s^{1/4} O(s^{1/4} \nabla_x^{(k)} A_{\ell_1}, \ldots, s^{1/4} \nabla_x^{(k)} A_{\ell_m} F_s(s)),
\]
where the range of the summation is as specified in Lemma 4.8. Let us take the $L^\infty_x$-norm of both sides; by the triangle inequality and (5.10) of Corollary 5.3 it suffices to control
$$s^{1/4} \| O(s^{1/4} \nabla_x^{\ell_1} A, \ldots, s^{1/4} \nabla_x^{\ell_j} A, s^{1/4} D_x^{(t)} F_x) \|_{L^\infty_x}$$
for each summand of $\sum_s$.

Let us throw away the extra power $s^{1/4}$ (which is okay as $0 < s \leq 1$).

Observe that $0 \leq \ell_1, \ldots, \ell_j \leq k - 1$; therefore, by the induction hypothesis, we have
$$\| s^{1/4} \nabla_x^{\ell_1} A(s) \|_{L^\infty_x}, \ldots, \| s^{1/4} \nabla_x^{\ell_j} A(s) \|_{L^\infty_x} \leq C_{k,E} \cdot \sqrt{E}.$$ 

Note furthermore that $\| s^{1/4} D_x^{(t)} F_x(s) \|_{L^\infty_x} \leq C_{l,E} \cdot \sqrt{E}$ by Corollary 5.3. Hence, by Hölder, each summand may be estimated by $C_{k,E} \cdot \sqrt{E}$, and thus (6.15) follows.

As a consequence, we obtain the following corollary which allows us to easily switch estimates for covariant derivatives to those for usual derivatives.

**Corollary 6.12 (Substitution of covariant derivatives by usual derivatives).** Assume that the hypotheses of Proposition 6.11 hold. Let $\sigma$ be a $g$-valued function on $\{t\} \times \mathbb{R}^3 \times [0, 1]$, $m \geq 0$ an integer, $b \geq 0, 1 \leq p, r \leq \infty$. Suppose that there exists $D > 0$ such that the estimate
$$\| D_x^{(t)} [g^{(t)}] \sigma \|_{L^p_x \cap L^r_x} \leq D$$
holds for $0 \leq k \leq m$. Then we have
$$\| \nabla_x^{(b)} \sigma \|_{L^p_x \cap L^r_x} \leq C_{(b,I),E} \cdot D$$
for $0 \leq k \leq \min(m, 30)$.

**Proof.** We will again omit $t$ in this proof. The case $k = 0$ is obvious; we thus fix $1 \leq k \leq \min(m, 30)$. Using (4.10) of Lemma 4.8 to $\sigma$ and multiplying by $s^{b+k/2}$, we get
$$s^b \nabla_x^{(k)} \sigma(s) = s^b D_x^{(t)} \sigma(s) + \sum_s s^{1/4} O(s^{1/4} \nabla_x^{(t)} A(s), \ldots, s^{1/4} \nabla_x^{(t)} A(s), s^b D_x^{(t)} \sigma(s)),$$
where the range of summation $\sum_s$ is as specified in Lemma 4.8. Taking the $L^p_x \cap L^r_x$ norm of both sides, applying triangle and using (6.10) to estimate $\| s^b D_x^{(t)} \sigma(s) \|_{L^p_x \cap L^r_x} = \| D_x^{(t)} [g^{(t)}] \sigma \|_{L^p_x \cap L^r_x} \leq D$, we are left to establish
$$s^{1/4} \| O(s^{1/4} \nabla_x^{(t)} A(s), \ldots, s^{1/4} \nabla_x^{(t)} A(s), s^b D_x^{(t)} \sigma(s)) \|_{L^p_x \cap L^r_x} \leq C_{(b,I),E} \cdot D$$
for each summand in $\sum_s$. Note that we have an extra power of $s^{1/4}$, which we can just throw away (as $0 < s \leq 1$). Let us use Hölder to put each $s^{1/4} \nabla_x^{(t)} A(s)$ in $L^p_x \cap L^r_x$ and $s^b D_x^{(t)} \sigma(s)$ in $L^p_x \cap L^r_x$. Then using Proposition 6.11 (This is possible since $k \leq 30$) and (6.10) to control the respective norms, we obtain (6.18). □

7. Transformation to the caloric-temporal gauge and estimates at $t = 0$:

**Proof of Theorem A**

The purpose of this section is to prove Theorem A. Let us begin by giving the precise definition of the norm $\mathcal{I}(t)$, which was alluded in Section 3.

For a solution $A_{\alpha}$ to (HPYM) on $I \times \mathbb{R}^3 \times [0, 1]$, define $\mathcal{I}(t) := (\mathcal{F}) \mathcal{I}(t) + (\Delta) \mathcal{I}(t)$ ($t \in I$), where
$$(\mathcal{F}) \mathcal{I}(t) := \sum_{k=1}^{10} \left[ \| \nabla_{t,x} F_x(t) \|_{L^{p/4}_{t,x} H^{k-1}_{t,x}} + \| \nabla_{t,x} F_x(t) \|_{L^{p/4}_{t,x} H^{k-1}_{t,x}} \right],$$
$$(\Delta) \mathcal{I}(t) := \sum_{k=1}^{31} \| \partial_{t,x} A(t) \|_{H^k_{t,x}}.$$ 

A key ingredient for proving Theorem A is Theorem 6.11 which has been proved in the previous section. The remaining analytic drudgery, on the other hand, is mostly contained in 21. Theorem

21 We gain an extra power of $s^{1/4}$ for each factor of $A_{\alpha}$ replacing $\partial_{t,x}$ thanks to the subcriticality of the problem at hand.
A]. Let us give a simplified version of [21, Theorem A], which suffices for our purpose and can be easily read off from the original version.

**Theorem 7.1** ([21, Theorem A], simplified). Let \( 0 < T \leq 1 \), and \( A^\dagger_\mu \) a regular solution to (YM) in the temporal gauge \( A^\dagger_0 = 0 \) on \( (-T, T) \times \mathbb{R}^3 \) with the initial data \((A_i, E_i)\) at \( t = 0 \). Define \( \mathcal{I} := \|A\|_{H^1_x} + \|E\|_{L^2_x} \). If

\[
\sup_{t \in (-T, T)} \|A^\dagger(t)\|_{H^1} < \delta_P
\]

where \( \delta_P \) is a small absolute constant, then the following statements hold.

1. There exists a regular gauge transform \( V = V(t, x) \) on \( (-T, T) \times \mathbb{R}^3 \) (in the sense of Definition 3.5) and a regular solution \( A_a \) to (HPYM) such that

\[
A_a(s = 0) = V(A^\dagger_0)V^{-1} - \partial_x V V^{-1}.
\]

2. Furthermore, the solution \( A_a \) satisfies the caloric-temporal gauge condition, i.e. \( A_x = 0 \) everywhere and \( A^\dagger_0 = 0 \).

3. The following initial data estimate holds.

\[
\mathcal{I}(0) \leq C_T \mathcal{T}.
\]

4. For \( \nabla := V(t = 0) \), the following estimates hold.

\[
\|\nabla\|_{L^\infty_x} \leq C_T, \quad \|\partial_x \nabla\|_{L^2_x} + \|\partial_x^2 \nabla\|_{L^2_x} \leq C_T \mathcal{T}.
\]

The same estimates with \( \nabla \) replaced by \( V^{-1} \), respectively, also hold.

With Theorem 7.1 in hand, we are ready to give a proof of \( A \).

**Proof of Theorem 7.1**. We begin with a regular solution \( A^\dagger_\mu \) to the hyperbolic Yang-Mills equations in the temporal gauge, defined on \((-T_0, T_0) \times \mathbb{R}^3\). Thanks to the regularity assumption, note that \( \partial_{t,x} A^\dagger_\mu \) and \( F^\dagger_{\mu\nu} \) are regular on \((-T_0, T_0) \times \mathbb{R}^3\).

**Step 1. Construction of regular solution to (HPYM) in caloric-temporal gauge.** Recall the hypothesis (3.3). By smoothness in \( t \) and conservation of energy, respectively, it follows that

\[
\sup_{t \in (-\epsilon_0, \epsilon_0)} \|A^\dagger\|_{H^1} < \delta_P, \quad \sup_{t \in (-T_0, T_0)} |E[F^\dagger(t)]| < \delta,
\]

for some small \( \epsilon_0 > 0 \). The second smallness condition allows us to apply Theorem 6.10 from which we obtain a unique regular solution \( \tilde{A}_\mu \) to (7.10) on \((-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]\).

Supplied with \( A_x = 0, A_a = (A_\mu, A_x) \) is a solution to (HPYM). We will apply a gauge transform \( V = V(t, x, s) \) to \( \tilde{A}_\mu \) to enforce the caloric-temporal gauge condition. Let us denote the resulting connection coefficients \( A_a \), i.e.

\[
A_a := V \tilde{A}_\mu V^{-1} - \partial_x V V^{-1}.
\]

In order for \( A_a \) to be in the caloric-temporal gauge, we need a gauge transform \( V \) which is \( A \) independent of \( s \) (to keep \( A_x = 0 \) and \( B \) makes \( A^\dagger_0 = 0 \). These two requirements are in fact equivalent (once one assumes enough regularity of \( V \)) to \( V \) solving the ODE

\[
\begin{cases}
\partial_t V = V \tilde{A}_0, \\
V(t = 0) = \nabla,
\end{cases}
\]

where \( \tilde{A}_0 := \tilde{A}_\mu(s = 1) \) and \( \nabla \) is a gauge transform on \( \mathbb{R}^3 \), to be specified in Step 2 in accordance to Theorem 7.1.

**Step 2. Application of Theorem 7.1.**

The next step of the proof is to apply Theorem 7.1 to choose \( \nabla \) and furthermore obtain a quantitative estimate for \( \mathcal{I}(0) \). Thanks to the first inequality of (7.3), we may apply Theorem
on the time interval $(-\epsilon_0, \epsilon_0)$. Let us mark the objects obtained from Theorem 7.1 with a prime, i.e. $A'_\mu$, $V'$ and $\overline{V'}$. Consider $\widetilde{A}_\mu' = \overline{A}_\mu'(t, x, s)$ defined by
\[
\widetilde{A}_\mu' := (V')^{-1}A'_\mu V' - \partial_\mu(V')^{-1}V',
\]
where we remind the reader that $V'$, $\overline{V'}$, $(V')^{-1}$ and $(\overline{V'})^{-1}$ are independent of $s$.

Note that $\widetilde{A}_\mu'$ is a regular solution to (AYMHP) in the caloric gauge, as is $\overline{A}_\mu'$. Moreover, their initial data sets coincide (both being $A'_\mu$). By the uniqueness lemma (Lemma 6.9), we conclude that $\widetilde{A}_\mu = \overline{A}_\mu'$ on the set on which $\overline{A}_\mu'$ is defined, i.e.
\[
\widetilde{A}_\mu = (V')^{-1}A'_\mu V' - \partial_\mu(V')^{-1}V',
\]
on $(-\epsilon_0, \epsilon_0) \times \mathbb{R}^3 \times [0, 1]$. As $A'_0 = 0$, we also see that
\[
\begin{cases}
\partial_t V' = V' \overline{A}_0, \\
V'(t = 0) = \overline{V}.
\end{cases}
\]
on $(-\epsilon_0, \epsilon_0) \times \mathbb{R}^3$.

At this point, let us make the choice $\overline{V} = V'$. Then the previous ODE is exactly that satisfied by $V$. Therefore, by uniqueness for ODE with smooth coefficients, $V = V'$ on $(-\epsilon_0, \epsilon_0) \times \mathbb{R}^3$, and hence we conclude that
\[
A_\mu = A'_\mu
\]
on $(-\epsilon_0, \epsilon_0) \times \mathbb{R}^3 \times [0, 1]$. From Parts (3), (4) of Theorem 7.1, the quantitative estimates in (8.3) follow. Moreover, it is not difficult to show that $V$ is a regular gauge transform on $(-T_0, T_0) \times \mathbb{R}^3$. It also follows that $\partial_{t,x}A_\mu$ and $F_{\mu\nu}$ are regular on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$, since $\overline{A}_\mu$ and $\overline{F}_{\mu\nu}$ were regular. This completes the proof of Theorem A. \square

8. Fixed-time estimates by $\mathbf{E}$: Proof of Theorem 13

Our aim in this section is to prove Theorem 13. Splitting $I(t) = (\Delta)I(t) + (F)I(t)$, we will reduce the theorem to establishing two inequalities, namely (8.1) and (8.6) of Propositions 8.1 and 8.2 respectively.

Throughout this section, we will be concerned with a regular solution $A_\alpha$ to (HPYM) in the caloric-temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$ $(T_0 > 0)$, which satisfies $I(0) = (\Delta)I(0) + (F)I(0) < \infty$ and $\mathbf{E}[\mathbf{F}] < \infty$. Then by conservation of energy for the hyperbolic Yang-Mills equations at $s = 0$, we see that
\[
\mathbf{E}[\mathbf{F}(t, s = 0)] = \mathbf{E}[\mathbf{F}] \quad \forall t \in (-T_0, T_0).
\]

We will denote the common value of $\mathbf{E}[\mathbf{F}(t, s = 0)]$ by $\mathbf{E}$.

**Proposition 8.1.** There exist $22$ $N > 0$ such that for any $t \in (-T_0, T_0)$, we have
\[
(\Delta)I(t) \leq C_{(\Delta)I(0), \mathbf{E}} (1 + |t|)^N.
\]

**Proof.** By symmetry, it suffices to consider $t > 0$. The main idea is to use the relation
\[
F_{\alpha t} = \partial_t A_\alpha,
\]
which holds thanks to the fact that we are in the temporal gauge $A_0 = 0$ along $s = 1$, and proceed as in the proof of Proposition 6.11.

Let us begin by estimating the $L^\infty_x$ norms. We claim that
\[
\|\partial_t^{(k)} A(t)\|_{L^\infty_x} \leq C_{k, (\Delta)I(0), \mathbf{E}} (1 + t)^{k+1},
\]
for $0 \leq k \leq 29$.

Let us begin with the case $k = 0$ and proceed by induction. Note the inequality
\[
\|A(t)\|_{L^\infty_x} \leq \|A(t = 0)\|_{L^\infty_x} + \int_0^t \|F_{\alpha t}(t')\|_{L^\infty_x} dt'.
\]

22In the course of the proof, it will be clear that $N$ may be chosen to depend only on the number of derivatives of $A$, controlled. In our case, in which we control up to 31 derivatives of $A$, we may choose $N = 32$. 

Taking \( \partial_x \) of both sides of (8.2) and using the fundamental theorem of calculus, we obtain
\[
\| \partial_x A_i(t) \|_{L^\infty_x} \leq \| \partial_x A_i(t = 0) \|_{L^\infty_x} + \int_0^t \| \partial_x F_{0i}(t') \|_{L^\infty_x} \, dt'.
\]

The first term is estimated by \((8.2)\) at \( t = 0 \), as \( 1 \leq k \leq 29 \). For the second term, we apply (4.15) of Lemma 4.8. Then it suffices to estimate
\[
\int_0^t \left( \| D_x^{(k)} E_{0i}(t') \|_{L^\infty_x} + \sum_{\ell} \| O(\partial_x^{(\ell_1)} A_i, \ldots, \partial_x^{(\ell_j)} A_i, D_x^{(j)} F_{0i}(t')) \|_{L^\infty_x} \right) \, dt'
\]
where the range of the summation \( \sum_{\ell} \) is as in Lemma 4.8. In particular, \( 1 \leq \ell_1, \ldots, \ell_j, \ell \leq k - 1 \). Let us use Hölder to estimate each factor in \( L^\infty_x \), and estimate the derivatives of \( A \) and \( F_{0i} \) by the induction hypothesis and Proposition 5.2 respectively. Then it is not difficult to see that the worst term (in terms of growth in \( t \)) is of the size
\[
C_{k, (\omega \tau(0)), E} \int_0^t (1 + t')^k \, dt' = C_{k, (\omega \tau(0)), E} (1 + t)^{k+1}.
\]

Therefore, (8.3) for \( k \) follows. By induction, this establishes the claim.

With (8.3) in hand, we now proceed to prove
\[
\| \partial_x A_i(t) \|_{L^\infty_x} \leq C_{k, (\omega \tau(0)), E} (1 + t)^{k+1},
\]
for \( 1 \leq k \leq 31 \).

Arguing as in the proof of (8.3), we arrive at the inequality
\[
\| \partial_x A_i(t) \|_{L^\infty_x} \leq \| \partial_x A_i(t = 0) \|_{L^\infty_x} + \int_0^t \left( \| D_x^{(k)} E_{0i}(t') \|_{L^\infty_x} + \sum_{\ell} \| O(\partial_x^{(\ell_1)} A_i, \ldots, \partial_x^{(\ell_j)} A_i, D_x^{(j)} F_{0i}(t')) \|_{L^\infty_x} \right) \, dt'.
\]

It suffices to estimate the \( t' \)-integral. For \( 1 \leq k \leq 30 \), let us use Hölder to estimate each \( \partial_x^{(\ell_i)} A \) in \( L^\infty_x \) and \( D_x^{(j)} F_{0i} \) in \( L^2_x \). Then we estimate these by (8.3) and Proposition 5.2 respectively, from which (8.4) follows immediately for \( 1 \leq k \leq 30 \).

Next, proceeding similarly in the case \( k = 31 \), all terms are easily seen to be okay except
\[
\| O(\partial_x^{(30)} A, F_{0i}(t')) \|_{L^2_x},
\]
for which we cannot use (8.3). In this case, however, we may put \( \partial_x^{(30)} A \) in \( L^2_x \) and \( F_{0i} \) in \( L^\infty_x \). Then the former can be estimated by using the case \( k = 30 \) of (8.4) that we have just established, whereas the estimate for the latter follows from Proposition 5.2. It follows that this term is of size \( C_{k, (\omega \tau(0)), E} (1 + t')^{31} \). Integrating over \([0, t]\) gives the growth \( C_{k, (\omega \tau(0)), E} (1 + t)^{32} \).

Finally, we are left to prove estimates for \( \partial_x A_i \). For this purpose, we claim
\[
\| \partial_x A_i(t) \|_{L^2_x} \leq C_{k, (\omega \tau(0)), E} (1 + t)^{k-1},
\]
for \( 1 \leq k \leq 30 \).

To prove (8.5), recall that \( \partial_t A_i = E_{0i} \); therefore, the case \( k = 1 \) follows immediately from Proposition 5.2. For \( k > 1 \), we take \( \partial_x^{(k-1)} \) and use Lemma 4.8 to substitute the usual derivatives by covariant derivatives. Then by (8.3), (8.4) and Proposition 5.2 (8.5) follows.

Combining (8.4) and (8.5), we obtain (8.1) with \( N = 32 \).

**Proposition 8.2.** For any \( t \in (-T_0, T_0) \), we have
\[
(F_s) I(t) \leq C_{(\omega \tau(t)), E} \cdot \sqrt E.
\]
Proof. Throughout the proof, the time \( t \in (-T_0, T_0) \) will be fixed and thus be omitted.

Recalling the definition of \((F_s)I\), establishing (8.8) reduces to proving

\begin{align}
\| \nabla_x F_s \|_{L_t^{5/4, p} H_x^{-1}} & \leq C_{k, (\omega), E} \cdot \sqrt{E}, \\
\| \nabla_0 F_s \|_{L_t^{5/4, p} H_x^{-1}} & \leq C_{k, (\omega), E} \cdot \sqrt{E}
\end{align}

for \( 1 \leq k \leq 10 \) and \( p = 2, \infty \).

The estimate (8.7) is an easy consequence of (5.9) of Corollary 5.3 and Corollary 6.12. On the other hand, to prove (8.8), we use the formula

\[ \nabla_0 F_s = s^{-1/2} D_t D_t F_{0i} - 2s^{1/2} [F_t^2, F_{it}] + D_t F_{s0} - s^{1/2} [A_0, F_{si}], \]

which is an easy consequence of the Bianchi identity (5.2) and the parabolic equation for \( D_x F_{0i} \).

Taking \( D_x^{(k-1)} \) of both sides and using Proposition 5.2 and Corollary 5.3 and Proposition 6.11, we obtain

\[ \| D_x^{(k-1)} \nabla_0 F_s \|_{L_t^{5/4, p} L_x^{s}} + \| D_x^{(k-1)} \nabla F_s \|_{L_t^{5/4, p} L_x^{s}} \leq C_{k, E} \cdot \sqrt{E} \]

for \( k \geq 1 \). At this point, applying Corollary 6.12 we obtain (8.8). \( \square \)

Combining Propositions 8.1 and 8.2, Theorem B follows.

9. Short time estimates for (HPYM) in the caloric-temporal gauge:

Proof of Theorem C

The goal of this section is to prove Theorem C. As discussed in Section 3, this theorem follows from a local-in-time analysis of the wave equations of (HPYM). As such, its proof will follow closely that of [21] Theorem B, which is essentially a ‘\( H^1 \) local well-posedness (in time)’ statement for (HPYM) in the caloric-temporal gauge.

To begin with, let us borrow the following definition from [21].

\[ \mathcal{E}(t) := \sum_{m=1}^{3} \left( \| \nabla x (m-1) F_{s0} (t) \|_{L_x^{s} L_t^{s}} + \| \nabla x (m) F_{s0} (t) \|_{L_x^{s} L_t^{s}} \right). \]

Given a time interval \( I \subset \mathbb{R} \), we define \( \mathcal{E}(I) \) to be \( \sup_{t \in I} \mathcal{E}(t) \).

Moreover, also borrowing from [21], we assert the existence of the norms \( \mathcal{F}(I), \mathcal{A}(I) \) of \( F_{si}, A_{si} \), respectively, such that the following lemma holds.

Lemma 9.1. Let \( I \subset \mathbb{R} \) be a finite open interval centered at \( t = 0 \), and \( A_{si} \) a regular solution to (HPYM) in the caloric-temporal gauge on \( I \times \mathbb{R}^3 \times [0, 1] \). Then:

(1) The following estimates hold.

\[ \mathcal{A}_0 [A_0 (s = 0)] (I) \leq C_{\mathcal{F}(I), \mathcal{A}(I)} \cdot \mathcal{E}(I) + C_{\mathcal{F}(I), \mathcal{A}(I)} \cdot \left( \mathcal{F}(I) + \mathcal{A}(I) \right)^2, \]

\[ \sup_{t \in I} \| A_0 (s) \|_{L_x^{s} L_t^{s}} \leq C_{\mathcal{F}(I), \mathcal{A}(I)} \cdot \left( \mathcal{F}(I) + \mathcal{A}(I) \right). \]

(2) The norms \( \mathcal{F}(-T, T) \) and \( \mathcal{A}(-T, T) \) are continuous as a function of \( T \) (where \( 0 < T < T_0 \)). Furthermore, we have

\[ \lim_{T \to T_0^-} \left( \mathcal{F}(-T, T) + \mathcal{A}(-T, T) \right) \leq C \mathcal{I}, \]

\[ \lim_{T \to T_0^+} \left( \delta \mathcal{F}(-T, T) + \delta \mathcal{A}(-T, T) \right) \leq C \delta \mathcal{I}. \]

(3) The following estimates hold for \( \mathcal{A} \):

\[ \mathcal{A}(I) \leq C \mathcal{I} (0) + |I| \left( C_{\mathcal{F}(I), \mathcal{A}(I)} \cdot \mathcal{E}(I) + C_{\mathcal{E}(I), \mathcal{F}(I), \mathcal{A}(I)} \cdot \left( \mathcal{E}(I) + \mathcal{F}(I) + \mathcal{A}(I) \right)^2 \right). \]

(4) The following estimates hold for \( \mathcal{F} \):

\[ \mathcal{F}(I) \leq C \mathcal{I} (0) + |I|^{1/2} C_{\mathcal{E}(I), \mathcal{F}(I), \mathcal{A}(I)} \cdot \left( \mathcal{E}(I) + \mathcal{F}(I) + \mathcal{A}(I) \right)^2 \]

Proof. This is essentially a summary of [21] Propositions 7.1, 7.2, 7.4 and [21] Theorems D, E. \( \square \)
The main reason why the analysis in [21] is insufficient to prove Theorem C is because of [21, Proposition 7.3], which gives an estimate for $\mathcal{E}(t)$ only under the hypothesis that either the size of the initial data or the $s$-interval is small. The following proposition is a replacement of [21, Proposition 7.3], which utilizes the smallness of the conserved energy $E(t)$ instead, based on the covariant parabolic estimates derived in Section 5.

**Proposition 9.2.** Let $I \subset \mathbb{R}$ be an open interval and $t \in I$. Consider a regular solution $A_n$ to (HPYM) in the caloric-temporal gauge on $I \times \mathbb{R}^3 \times [0,1]$ such that $F = F_{\mu \nu}$ is regular on $I \times \mathbb{R}^3 \times [0,1]$ and satisfies

$$
\mathcal{E}(t) := \mathcal{E}[F(t, s = 0)] \leq \delta, \quad \Delta T(t) \leq D,
$$

where $D > 0$ is an arbitrarily large number and $\delta > 0$ is the small constant in Proposition 5.2. Then the following estimate holds.

$$
\mathcal{E}(t) \leq C_{D, \delta}.
$$

In 9.1 we will give a proof of Proposition 9.2. Assuming Proposition 9.2, the proof of Theorem C is a straightforward adaptation of [21, Proof of Theorem B]. We shall present a sketch in 9.2.

### 9.1. Improvement of estimates for $F_{s_0}$: Proof of Proposition 9.2

The goal of this subsection is to prove Proposition 9.2. Consider a regular solution $A_n$ to (HPYM) on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ ($T_0 > 0$) such that $F$ is regular. From Proposition 5.1 and the fact that $F_{s_0} = D^k F_{\ell_0}$, it is not difficult to verify that $F_{s_0}$ satisfies the covariant parabolic equation

$$
(D_s - D^i D_i) F_{s_0} = 2 [F_0^\ell, F_{s_0}].
$$

Recall furthermore that $[D_i, (D_s - D^i D_i)] B = O(F, D_x B)$. This implies that $D_x^{(k)} F_{s_0}$ for $k \geq 1$ satisfies the following schematic parabolic equation.

$$
(D_s - D^i D_i)(D_x^{(k)} F_{s_0}) = \sum_{j=0}^{k} O(D_x^{(j)} F, D_x^{(k-j)} F_s).
$$

Now recall that the hyperbolic Yang-Mills equation holds along $s = 0$. In particular, the constraint equation $D^i F_{\ell_0}(s = 0) = 0$ holds, which is equivalent to $F_{s_0}(s = 0) = 0$. Taking this extra ingredient into account, it follows that $F_{s_0}$ obeys an improved bound compared to the one proved in Section 5 as we state below.

**Proposition 9.3.** (Improved estimate for $F_{s_0}$, with covariant derivatives). Let $T_0 > 0$, $t \in (-T_0, T_0)$, and consider a regular solution $A_n$ to (HPYM) on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ such that $F = F_{\mu \nu}$ is regular on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$. If $E(t) := E[F(t, s = 0)] < \delta$, where $\delta > 0$ is the small constant in Proposition 5.2, then the following estimate holds for each integer $k \geq 0$.

$$
\|D_x^{(k-1)} F_{s_0}(t)\|_{L^{1;\infty}_x L^2(0,1)} + \|D_x^{(k)} F_{s_0}(t)\|_{L^{1+1/2;2}_x L^2(0,1)} \leq C_{k, E(t)} \cdot E(t),
$$

When $k = 0$, we omit the first term on the left-hand side.

**Proof.** We shall fix $t \in (-T_0, T_0)$ and therefore omit writing $t$. Let us begin with the case $k = 0$. Applying Lemma 1.7 to the covariant parabolic equation for $F_{s_0}$, along with the fact that $F_{s_0} = 0$ at $s = 0$ thanks to (YM), it follows that

$$
\|F_{s_0}\|_{L^{1;2}_x L^2} \leq 2 \int_0^s e^{(s-\tau)\Delta} \|F_{0 \ell_0} F_{s_0}\|_{(\sigma)} d\sigma \|_{L^{1;2}_x L^2}.
$$

Using Lemma 1.7, Hölder, (5.6) (Proposition 5.2) and (5.9) (Corollary 5.3), we have

$$
\| \int_0^s e^{(s-\tau)\Delta} \|F_{0 \ell_0} F_{s_0}\|_{(\sigma)} d\sigma \|_{L^{1;2}_x L^2} \leq \|F_{0 \ell_0} F_{s_0}\|_{L^{1+1/2;2}_x L^2} \leq \|F_{0 \ell_0} F_{s_0}\|_{L^{3/4;2}_x L^2} \leq C_E \cdot E.
$$

Therefore, we have proved $\|F_{s_0}\|_{L^{1;2}_x L^2} \leq C_E \cdot E$.

The latter case is not explicitly treated in [21], but follows essentially by a scaling argument.
For $k \geq 1$, we proceed by induction. Suppose, for the purpose of induction, that the cases $0, \ldots, k-1$ has already been established. Using the energy integral estimate (4.6) with $\ell = 1 + \frac{k-1}{2}$ to (9.3) for $D_x^{(k-1)}F_s0$, we see that
\[
\|D_x^{(k-1)}F_s0\|_{L^\infty_xL^2} + \|D_x^{(k)}F_s0\|_{L^{1/2}_xL^2} \leq C\|D_x^{(k-1)}F_s0\|_{L^{1/2}_xL^2} + C\sum_{j=0}^{k-1}\|C(D_x^{(j)}F, D_x^{(k-j)}F_s)\|_{L^{1/2}_xL^2}.
\]

The first term on the right-hand side is acceptable by the induction hypothesis; we therefore focus on the second term. Let us use Hölder to estimate $D_x^{(j)}F$ in $L^{3/4}_xL^6$ and $D_x^{(k-j)}F_s$ in $L^{5/4}_xL^3$. Next, we apply Corollary 4.3 to each. Then using Proposition 5.2 and Corollary 5.3, the sum is estimated by
\[
\sum_{j=0}^{k-1}\|D_x^{(j+1)}F\|_{L^{3/4}_xL^2} + \|D_x^{(k-j)}F_s\|^{1/2}_{L^{5/4}_xL^2} \leq C_{k,E}E.
\]

Therefore, (9.4) holds for the case $k$, which completes the induction.

Suppose furthermore that $A_0$ is in the caloric-temporal gauge, so that $A_0 = 0$ in particular. Combining Proposition 9.3 and Corollary 6.12, the corresponding estimate for usual derivatives. This is the content of the following corollary, whose easy proof we omit.

**Corollary 9.4** (Improved estimate for $F_s0$, with usual derivatives). Assume that the hypotheses of Proposition 9.2 hold. Furthermore, assume that $A_0$ satisfies the caloric-temporal gauge condition. Then the following estimate holds for $0 \leq k \leq 29$.

\[
\|D_x^{(k)}F_s0\|_{L^\infty_xL^2} + \|D_x^{(k)}F_s0\|_{L^{1/2}_xL^2} \leq C_{k,E,E(t)}E(t).
\]

The estimate (9.5) is more than sufficient to prove Proposition 9.2.

9.2. Proof of Theorem C

With Proposition 9.2, we are ready to give a proof of Theorem C. We shall basically follow [21, Proof of Theorem B], replacing [21, Proposition 7.3] by Proposition 9.2.

**Proof of Theorem C** Let $A_0$ be a regular solution to the hyperbolic-parabolic Yang-Mills equation in the caloric-temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$ such that $F$ is regular and (4.7) is satisfied. For simplicity, we will consider the case in which $I_0$ is centered at $t = 0$, i.e. $I_0 = (-d/2, d/2)$ for $d > 0$ to be determined. As we shall see, the proof only utilizes the hypotheses (3.7) on $I_0$; therefore, the same proof applies to other $I_0 \subset (-T_0, T_0)$ as well.

We claim that
\[
\mathcal{F}(I_0) + \mathcal{A}(I_0) \leq BD,
\]
for a large enough absolute constant $B$, to be determined later, provided that $|I_0| = d$ is small enough. Note that Theorem C then follows immediately from the claim, thanks to Part (1) of Lemma 9.1.

We will use a bootstrap argument. The starting point is provided by Part (2) of Lemma 9.1, which implies
\[
\mathcal{F}(I'_0) + \mathcal{A}(I'_0) \leq 2I(0),
\]
for some subinterval $I'_0 \subset I$ containing 0 such that $|I'_0| > 0$ is sufficiently small (by upper semi-continuity of $\mathcal{F}, \mathcal{A}$ at 0). Note that the right-hand side is estimated by $BD$, provided we choose $B \geq 2$.

Next, let us assume the following bootstrap assumption:
\[
\mathcal{F}(I'_0) + \mathcal{A}(I'_0) \leq 2BD
\]
for $I'_0 := (-T', T') \subset I$. Applying Parts (3), (4) of Lemma 9.1 and using Proposition 9.2 to control $E(I'_0)$, we obtain

$$
F(I'_0) + A(I'_0) \leq C I + (T')^{1/2} C_{\delta'} F(I'_0) \Delta(I'_0) (C_D + F(I'_0) + A(I'_0))^{2} 
$$

$$
+ (T') (C_{\delta'} F(I'_0) \Delta(I'_0) (C_D + F(I'_0) + A(I'_0))^{2} + F(I'_0) + A(I'_0)^{2}).
$$

Here, we used the hypotheses (9.7) on $I'_0 \subset I_0$. Using the bootstrap assumption and choosing $d$ small enough depending on $D, \delta$ and $B$ (note that $T' \leq d$), we can make the second and third terms on the right-hand $\leq \frac{2}{D} D$. Then choosing $B > 2C$, we see that

$$
F(I'_0) + A(I'_0) \leq BD,
$$

which ‘beats’ the bootstrap assumption. By a standard continuity argument, 9.6 then follows. □

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