THE RIEMANNIAN HEMISPHERE IS ALMOST CALIBRATED
IN THE INJECTIVE HULL OF ITS BOUNDARY

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ABSTRACT. An exact differential two-form in the injective hull of the Riemannian circle is constructed and its norm is shown to be stationary at points of the open hemisphere spanned by the circle. The norm employed is the co-mass with respect to the inscribed Riemannian definition of area on normed planes. This implies that in any metric space, the induced Finsler mass of a two-dimensional Ambrosio-Kirchheim rectifiable metric current with a Riemannian circle of length $2\pi$ as a boundary can be estimated from below by $2\pi$ plus a term of second order in the Hausdorff distance to an isometric copy of the hemisphere with the same boundary. These competitors contain the class of oriented Lipschitz surfaces of arbitrary topological type and as such the result provides positive evidence for Gromov’s filling area conjecture.

CONTENTS

1. Introduction 2
2. Setting 5
2.1. Metric currents and Finsler mass 5
2.2. Injective hulls of Riemannian spheres 10
2.3. Representation of the hemisphere 12
3. Definition of omega 15
3.1. Geometric interpretation 15
3.2. Coefficient estimates 17
3.3. Action on currents and paths 20
3.4. Coefficients of product type 24
4. Variation calculation 26
4.1. Structure of maximizing paths 26
4.2. Variation of paths and coefficients 30
4.3. Proof of the main theorems 41
5. Open questions 45
5.1. Other definitions of area 45
5.2. Lower bounds on the filling area 46
References 47
1. Introduction

Let $S^1$ be the Riemannian circle of length $2\pi$ equipped with the intrinsic geodesic distance function $d$. A filling of $S^1$ is a compact, oriented Riemannian surface $M$ with intrinsic distance $d_M$ such that the restriction $(\partial M, d_M|_{\partial M \times \partial M})$ is isometric to $S^1$. It is an open question of Gromov [12, §2.2] whether $\text{Area}(M) \geq 2\pi$ for such fillings of $S^1$. Equality does hold for the Riemannian hemisphere $S^2_+ \cap \{\text{constant curvature } 1\}$. There are some partial answers available in the literature. As a consequence of Pu's systolic inequality [22], it is true that $\text{Area}(M) \geq 2\pi$ whenever $M$ is a Riemannian disk that fills $S^1$. There is also a generalization due to Ivanov [16] for Finsler disks $M$ in case $\text{Area}(M)$ is interpreted as the Holmes-Thompson or Busemann-Hausdorff definition of area. In [4] the authors obtain the same inequality. The question is widely open. In this paper, we propose a more general approach to this problem using metric currents due to Ambrosio and Kirchheim [2].

Two-dimensional real rectifiable currents $\mathcal{R}_2(X)$ in a metric space $X$ are generalizations of compact, oriented Lipschitz surfaces and thus can model all the competitors that are of interest. In this generalization, the question strongly depends on the particular notion of area, respectively mass for currents, that is used. As we will show in subsection 2.1, a Finsler definition of area induces a corresponding Finsler mass on real rectifiable currents. More generally, this applies to rectifiable sets and to rectifiable chains in metric spaces with coefficients in a normed abelian group as introduced by De Pauw and Hardt [6]. The inner Riemannian definition of area $\mu^{\text{int}}$ introduced by Ivanov [15] is the largest possible choice, see Lemma 2.5. It assigns a Haar measure $\mu_V$ to every normed plane $V$ with normalization $\mu_V(\mathcal{E}_V) = \pi$ for the Löwner-John ellipse $\mathcal{E}_V$ of maximal area inside the unit ball of $V$. The corresponding mass on real rectifiable currents is denoted by $M_{ir}$. A possible generalization of Gromov’s question is the following.

**Question 1.** Let $X$ be a metric space and $T \in \mathcal{R}_2(X)$ be a real rectifiable current with boundary $\partial T$ isometric to $[S^1]$. Is $M_{ir}(T) \geq 2\pi$ with equality if and only if $T$ is isometric to $[S^2_+]$?

Note that in case $[M]$ is induced by a compact, oriented Riemannian surface $M$, then $M_{ir}([M])$ agrees with the usual area of $M$. Instead of working in an arbitrary metric space $X$, it is sufficient to work in the injective hull $E(S^1)$ of $S^1$. Roughly speaking, this is the smallest injective metric space that contains $S^1$ isometrically. Injective hulls were introduced independently by Isbell [14] and Dress [8]. An injective metric space $Y$ has the defining property that whenever $\varphi : A \to Y$ is a 1-Lipschitz map defined on a subset $A \subset X$ of a metric space $X$, there exists a 1-Lipschitz extension $\bar{\varphi} : X \to Y$. So whenever $T \in \mathcal{R}_2(X)$ with $\partial T$ isometric to $[S^1]$ in a metric space $X$, there exists a 1-Lipschitz map $\bar{\varphi} : X \to E(S^1)$ such that $\bar{\varphi}|_{\text{spit}(\partial T)}$ is an isometry. The push-forward $\bar{\varphi}_#T \in \mathcal{R}_2(E(S^1))$ is also a filling of $[S^1]$ with $M_{ir}(\bar{\varphi}_#T) \leq M_{ir}(T)$. The injective hull of the sphere $S^n$ can be characterized explicitly as those 1-Lipschitz functions $f : S^n \to \mathbb{R}$ with $f(x) + f(-x) = \pi$, see Proposition 2.7. By fixing a base point and an orientation of $S^1$ we can identify $E(S^1)$ with the space of 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f_{\alpha + \pi} + f_{\alpha} = \pi$ for all $\alpha \in \mathbb{R}$. This space is contained isometrically as a compact and convex subset of $L^\infty([0, \pi])$. Although $E(S^1)$ spans an infinite dimensional subspace, it may be possible to find a calibration for the isometric copy of $S^2_+ \cap \{\text{constant curvature } 1\}$ that sits in $E(S^1)$ by
employing a notion of differential form in an infinite dimensional setting. Such a calibration would answer Gromov’s question in the positive. In this direction, we study in detail the differential two-form defined by

\[ \hat{\omega}_f := \frac{1}{\pi} \int_{0}^{\pi} \int_{0}^{\pi} p_{\alpha,\beta}(f) \, df_\alpha \wedge df_\beta \, d\alpha d\beta \]

with coefficients

\[ p_{\alpha,\beta}(f) = \frac{1 - \cos(\beta - \alpha)^2 - \cos(f_\alpha)^2 - \cos(f_\beta)^2 + 2 \cos(\beta - \alpha) \cos(f_\alpha) \cos(f_\beta)}{\sin(\beta - \alpha)^2 \sin(f_\alpha)^2 \sin(f_\beta)^2} . \]

This definition is partly motivated by the differential form in [5] used to show that planes contained in a normed space are calibrated with respect to the Hausdorff measure. If \( E(S^1) \) is realized as subset of \( L^\infty([0, \pi]) \), we adopt \( df_\alpha : L^\infty([0, \pi]) \to \mathbb{R} \) to denote the coordinate projections \( df_\alpha(g) = g_\alpha \) for almost every \( \alpha \). This is justified in analogy to finite dimensions, where the notation \( dx_i(v) = v_i \) is standard in presence of linear coordinates \( x_1, \ldots, x_n \). Although ill-defined as proper linear functionals for fixed \( \alpha \), in contrast to say \( df_\alpha : C_0([0, \pi]) \to \mathbb{R} \), the definition of \( \hat{\omega}_f \) as an integral is meaningful.

The coefficients \( p_{\alpha,\beta}(f) \) are nonnegative, as we will see, and well-defined because any \( f \in E(S^1) \setminus S^1 \) takes values in \( (0, \pi) \) only. If \( T \in \mathcal{M}_2(\epsilon S(S^1)) \) is a metric current with finite mass and support away from \( S^1 \), the action \( T(\hat{\omega}) \) is defined by integrating \( T(p_{\alpha,\beta}(f) \, df_\alpha \wedge df_\beta) \) with respect to \( \alpha \) and \( \beta \), see subsection 3.3. This integral makes sense because the coefficients are uniformly bounded on \( \text{spt}(T) \) due to an interpretation of \( p_{\alpha,\beta}(f) \) in spherical geometry, Lemma 3.1. Further, \( \hat{\omega} \) has a properly defined comass \( \|\hat{\omega}_f\|_{\text{ir}} \) as the infimum over \( M \geq 0 \) such that

\[ |\hat{\omega}_f(v \wedge w)| \leq M \mu^R(v \wedge w) \]

for all \( v, w \in L^\infty([0, \pi]) \). Here \( \mu^R(v \wedge w) \) is the inscribed Riemannian area of the parallelogram spanned by \( v \) and \( w \). Calibrations as defined by Harvey and Lawson [13] are special differential forms in Riemannian manifolds. They are exact and have comass 1. \( \hat{\omega} \) is exact since \( p_{\alpha,\beta}(f) \) depends only on the evaluations \( f_\alpha \) and \( f_\beta \) of \( f \). Our main result is that \( \|\hat{\omega}_f\|_{\text{ir}} \) is close to 1 up to second order in the distance of \( f \) to \( S^2_\epsilon \).

**Theorem 1.1.** For every \( r > 0 \) and \( \xi \in (1, 2) \) there exists \( C > 0 \) such that for all \( f \in E(S^1) \) with \( \text{dist}(f, S^1) \geq r \) it holds

\[ \|\hat{\omega}_f\|_{\text{ir}} - 1 \leq C \text{dist}(f, S^2_\epsilon) \xi . \]

In particular, \( \|\hat{\omega}_f\|_{\text{ir}} = 1 \) whenever \( f \in S^2_\epsilon \setminus S^1 \). In this sense, even though \( \hat{\omega} \) is not a calibration, it almost calibrates the hemisphere. This allows to estimate the filling area of \( S^1 \) among surfaces that are close to \( S^2_\epsilon \) with respect to the Hausdorff distance in an arbitrary metric space.

**Theorem 1.2.** For every \( r \in (0, \frac{\pi}{2}) \) and \( \xi \in (1, 2) \), there exists \( C > 0 \) such that the following holds. If \( X \) is a metric space and \( S, T \in \mathcal{R}_2(X) \) are real rectifiable currents (with compact support) that satisfy

1. \( S \) is isometric to \( [S^2_\epsilon] \),
2. \( \partial T = \partial S \) (which is isometric to \( [S^1] \) by (1)),
3. \( T \perp N_r = S \perp N_r \) for the \( r \)-neighbourhood \( N_r \) of \( \text{spt}(\partial S) \) inside \( \text{spt}(S) \),

then
then
\[ 2\pi \leq M_{ir}(T) \left( 1 + C d_{Haus}(\text{spt}(S), \text{spt}(T)) \right) . \]

This theorem essentially tells that whenever a compact patch \( K \) of \( S^2 \setminus S^1 \) is replaced by a new surface close to \( K \) and with the same boundary as \( K \), then the new filling of \( S^1 \) can’t have much smaller area. The crucial observations are \( \xi > 1 \) and that competing surfaces are allowed to have arbitrary topological type. As a consequence, the hemisphere \( S^2 \) is stationary among variations with surfaces of arbitrary topological type that fix a collar of the boundary.

**Corollary 1.3.** Assume that \( X \) is a metric space and \( r \in (0, \frac{\pi}{2}) \) is fix. If \( T_n, S \in \mathcal{B}_2(X) \) for \( n \in \mathbb{N} \) satisfy

1. \( S \) is isometric to \([S^2_1]\).
2. \( \partial T_n = \partial S \).
3. \( T_n \cap N_r = S \cap N_r \) for the \( r \)-neighbourhood \( N_r \) of \( \text{spt}(\partial S) \) inside \( \text{spt}(S) \),
4. \( \lim_{n \to \infty} d_{Haus}(spt(T_n), spt(S)) = 0 \),

then
\[ \limsup_{n \to \infty} \frac{M_{ir}(S) - M_{ir}(T_n)}{d_{Haus}(\text{spt}(S), \text{spt}(T))} \leq 0 . \]
Specifically, the limit exists and is equal 0 in case \( \lim_{n \to \infty} M_{ir}(T_n) = M_{ir}(S) = 2\pi \).

Next we present a short overview of the proof. The hemisphere
\[ S^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \]
with its intrinsic Riemannian metric is represented isometrically in \( E(S^1) \) as those functions \( h : \mathbb{R} \to \mathbb{R} \) with \( h_\alpha = \arccos (\cos(d) \cos(\alpha - \tau)) \) for parameters \( \tau \in (-\pi, \pi] \) and \( d \in [0, \frac{\pi}{2}] \), Lemma 2.9. As its boundary, \( S^1 \subset E(S^1) \) corresponds to those functions with \( d = 0 \). The coefficients of \( \tilde{\omega} \) at a point \( h \in S^2_+ \setminus S^1 \) with respect to parameters \( \tau \) and \( d \) have the product structure
\[ p_{\alpha, \beta}(h) = \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \tau)^2} \frac{\sin(d)}{1 - \cos(d)^2 \cos(\beta - \tau)^2} = p_{\alpha}(h)p_{\beta}(h) , \]
with
\[ \int_0^{2\pi} p_{\alpha}(h) \, d\alpha = 2\pi , \]
see Lemma 2.10 and Lemma 3.1. As an element of \( \Lambda_2 L^\infty([0, \pi]), \) the inner Riemannian comass of \( \tilde{\omega} \) at \( f \in E(S^1) \setminus S^1 \) has the beneficial characterization
\[ ||\tilde{\omega}_f||_{ir} = \sup \{ \tilde{\omega}_f(v \wedge w) : (v, w) \in L^\infty([0, \pi)^2), ||v^2 + w^2||_\infty \leq 1 \} , \]
Proposition 3.8. Due to the product structure of \( p_{\alpha, \beta}(h) \), the maximization problem for \( ||\tilde{\omega}_h||_{ir} \) boils down to the plane isoperimetric inequality, Lemma 3.9, and has a unique maximizer \( (v, w) : [0, \pi) \to \mathbb{R}^2 \) up to rotations of \( \mathbb{R}^2 \). In particular, \( ||\tilde{\omega}_h||_{ir} = 1 \) for \( h \in S^2_+ \setminus S^1 \) as a consequence of (1.1).

For general \( f \in E(S^1) \setminus S^1 \), the existence of a maximizing path \((v, w)\) follows from the theorem of Banach–Alaoglu due to weak-star compactness in \( L^\infty([0, \pi]) = L^1([0, \pi])^* \), Lemma 4.2. Such a maximizer further satisfies \( v_\alpha^2 + w_\alpha^2 = 1 \) for almost all \( \alpha \), at least with mild assumptions on \( f \), Lemma 4.3. For a fixed \( h \in S^2_+ \setminus S^1 \) there exists a unique bi-Lipschitz function \( \nu_h : [0, \pi) \to [0, \pi) \) such that \( e^{i\nu_h} \) is a maximizing path for \( ||\tilde{\omega}_h||_{ir} \). A maximizing path \((v, w)\) at an arbitrary \( f \in E(S^1) \setminus S^1 \)
$S^1$ can be written as $e^{it(\nu_0 + \nu)}$ for some square integrable function $\eta : [0, \pi) \to \mathbb{R}$. In order to break the rotation invariance we assume that $\int_0^\pi \eta = 0$ and call the resulting space $L_0^2(\mathbb{R}/\mathbb{Z})$. With this preparation, the maximization problem is transferred to the function $\Psi_h : E(S^1) \setminus S^1 \times L_0^2(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ defined by

$$\Psi_h(f, \eta) := \frac{1}{\pi} \int_0^\pi \int_0^\pi \rho_{a,\beta}(f) \sin(\nu_h(\beta) - \nu_h(\alpha) + \eta(\beta) - \eta(\alpha)) \, d\beta \, d\alpha .$$

A direct variation calculation reveals that $\Psi_h$ is indeed stationary at $(h, 0)$. The main technical part in the proof of Theorem 1.1 is the justification of this method. So the question is, how does a maximizer $\eta$, which we already know exists, depend on $f$ if $\|f - h\|_\infty$ is small? There are two main ingredients. First, due to the stability of the plane isoperimetric inequality, we can restrict to the set $B_\varepsilon := \{ \eta \in L_0^2(\mathbb{R}/\mathbb{Z}) : \|\eta\|_\infty \leq \varepsilon \}$ in case $\|f - h\|_\infty$ is small enough, Lemma 4.10. The particular stability result for parametrized plane paths we use is due to Fuglede [11]. Second, assuming $\|f - h\|_\infty$ is small, $\eta \mapsto \Psi_h(f, \eta)$ is strictly concave when restricted to $B_\varepsilon$. More precisely,

$$\frac{d^2}{dt^2} \Psi_h(f, \eta_t) \leq -c\|\eta_t - \eta_0\|^2_2$$

holds for some $c > 0$ and variations $\eta_t = (1 - t)\eta_0 + t\eta_1$ whenever $\eta_0, \eta_1 \in B_\varepsilon$, Proposition 4.11. Thus there exists a unique maximizer $\eta_f$ of $\Psi_h(f, \cdot)$ in $B_\varepsilon$. We further need to resolve the interplay between the $L^\infty$ and the $L^2$ norm to conclude that locally around $h$, the map $f \mapsto \eta_f$ is Hölder continuous in the sense that $\|\eta_f\|_\infty \leq C\|f - h\|_\infty$ for any $\xi \in (0, 1)$. This then allows to apply a variation argument to $f \mapsto \Psi_h(f, \eta_f)$ to conclude the proof of Theorem 1.1. The second main Theorem 1.2 is a direct consequence of the first one and the results about rectifiable currents and Finsler mass in subsection 2.1.

2. Setting

2.1. Metric currents and Finsler mass. Metric currents, as developed in the original paper by Ambrosio and Kirchheim [2], are functionals on tuples of Lipschitz functions and extend the classical Euclidean currents introduced by Federer and Fleming [9, 10] and de Rham [7]. Since we only need currents with compact support, an equivalent definition is due to Lang [20]. See [24, Definition 2.2] for the same set of axioms used below.

**Definition 2.1 (Metric currents with compact support).** Let $X$ be a metric space and $n \geq 0$. A multilinear map $T : \text{Lip}(X)^{n+1} \to \mathbb{R}$ is a current in $\mathcal{D}_n(X)$ if the following axioms hold:

1. $T(f, g_1, \ldots, g_n) = 0$ if some $g_i$ is constant in a neighbourhood of $\text{spt}(f)$.
2. $\lim_{k \to \infty} T(f_k, g_1, \ldots, g_n, k) = T(f, g_1, \ldots, g_n)$ if $f_k \to f$, $g_i \to g_i$ uniformly for all $i$ and $\sup_{i,k} \{ \text{Lip}(f_k) \cap \text{Lip}(g_{i,k}) \} < \infty$.
3. There exists a compact set $K \subset X$ such that $T(f, g_1, \ldots, g_n) = 0$ whenever $\text{spt}(f) \cap K = \emptyset$.

The support $\text{spt}(T)$ of $T$ is the intersection of all closed sets $A \subset X$ with the property that $T(f, g_1, \ldots, g_n) = 0$ whenever $\text{spt}(f) \cap A = \emptyset$. See [24, Lemma 2.3] for more details on the support related to the axioms above. Assuming $n \geq 1$, the boundary $\partial T \in \mathcal{D}_{n-1}(X)$ of $T \in \mathcal{D}_n(X)$ is defined by

$$\partial T(f, g_1, \ldots, g_{n-1}) := T(1, f, g_1, \ldots, g_{n-1}) .$$
If \( \varphi : X \to Y \) is a Lipschitz map between compact spaces, then the push forward \( \varphi_\# : \mathcal{P}_n(X) \to \mathcal{P}_n(Y) \) is defined by
\[
(\varphi_\# T)(f, g_1, \ldots, g_n) \coloneqq T(f \circ \varphi, g_1 \circ \varphi, \ldots, g_n \circ \varphi).
\]
The mass of a current \( T \in \mathcal{P}_n(X) \) is defined by
\[
M(T) \coloneqq \sup_{\lambda \in \Lambda} \sum_{i \in \Lambda} T(f_{\lambda,i}, g_{1,\lambda}, \ldots, g_{n,\lambda}),
\]
where the supremum is taken over all finite collections \( \Lambda \) such that \( (f_{\lambda,i}, g_{1,\lambda}, \ldots, g_{n,\lambda}) \) is in \( \text{Lip}(X)^{n+1} \), each \( g_{i,\lambda} \) is 1-Lipschitz and \( \sum_{\lambda \in \Lambda} |f_{\lambda}| \leq 1 \).

By combining [2, Theorem 4.5] with [19, Lemma 4], rectifiable currents can be characterized as follows.

**Definition 2.2.** \( T \in \mathcal{P}_n(X) \) is an \( n \)-dimensional real rectifiable current in \( \mathcal{R}_n(X) \) if it has finite mass and for any \( \lambda > 1 \) there exists a sequence \( K_i \) of compact sets in \( \mathbb{R}^n \), functions \( \varphi_i \in L^1(K_i) \), norms \( \| \cdot \|_i \) on \( \mathbb{R}^n \) and maps \( \varphi_i : K_i \to K \) into some compact set \( K \subset X \) such that the sets \( \varphi_i(K_i) \) are pairwise disjoint,
\[
\lambda^{-1} \| x - y \|_i \leq d(\varphi_i(x), \varphi_i(y)) \leq \lambda \| x - y \|_i,
\]
\[
T = \sum_{i=0}^{\infty} \varphi_i[\theta_i] \quad \text{and} \quad M(T) = \sum_{i=0}^{\infty} M(\varphi_i[\theta_i]).
\]

Next we want to define a notion of Finsler mass on rectifiable currents that depends on a specific definition of volume. As we will see, the Ambrosio-Kirchheim mass is induced by the Gromov-mass star (or Benson) definition of volume.

**Definition 2.3.** Given \( n \in \mathbb{N} \), a definition of volume assigns to every \( n \)-dimensional normed space \( V \) a Haar measure \( \mu_v \) with the properties:

1. If \( V \) and \( W \) are \( n \)-dimensional normed spaces and \( A : V \to W \) is a linear map with \( \| A \| \leq 1 \), then \( A \) is volume decreasing, i.e. \( \mu_w(A(B)) \leq \mu_v(B) \) for all Borel sets \( B \subset V \).
2. If \( V \) is Euclidean, then \( \mu_v \) is the standard Lebesgue measure.

This is equivalent to the definition given in [1, §3], where instead of a Haar measure a norm, also denoted by \( \mu_v \), is assigned to the 1-dimensional space \( \Lambda_n V \).

The equivalence is induced by the identity
\[
\mu_v(P(v_1, \ldots, v_n)) = \mu_v(v_1 \wedge \cdots \wedge v_n),
\]
where \( P(v_1, \ldots, v_n) \) is the parallelepiped spanned by the vectors \( v_1, \ldots, v_n \in V \).

If \( s \) is a seminorm on \( \mathbb{R}^n \) with canonical basis \( e_1, \ldots, e_n \), the Jacobian of \( s \) is
\[
J_s(s) \coloneqq \begin{cases} \mu_s(e_1 \wedge \cdots \wedge e_n) & \text{if } s \text{ is a norm}, \\ 0 & \text{otherwise}. \end{cases}
\]
Or equivalently, in case \( s \) is a norm,
\[
J_s(s) = \frac{\mu_s(B)}{\mathcal{L}^n(B)}
\]
for every Borel set \( B \subset \mathbb{R}^n \) of positive and finite measure, where \( \mathcal{L}^n \) is the Lebesgue measure of \( \mathbb{R}^n \).

Let \( V \) be a normed space of dimension \( n \) with unit ball \( B_V \) and dual space \( V^* \). \( \alpha(n) \) denotes the Lebesgue measure of the Euclidean unit ball and \( \mathcal{E}_V \subset B_V \) is the inscribed Löwner-John ellipsoid. This is the unique ellipsoid of largest volume.
THE HEMISPHERE IS ALMOST CALIBRATED

contained in $B_V$. Below is a list of defining properties for those definitions of volume we need, see e.g. [1, §3]:

- (Gromov-mass star/Benson)
  $$\mu^\text{gm}(V) = \operatorname{sup}\{|\xi_1 \wedge \cdots \wedge \xi_n| : \|\xi_i\| \leq 1\},$$

- (Busemann-Hausdorff)
  $$\mu^\text{bh}(B_V) = \alpha(n),$$

- (Inner Riemannian)
  $$\mu^\text{ir}(E_V) = \alpha(n).$$

The inner Riemannian definition is due to Ivanov [15] and the main volume definition in the present work. Normalized properly, the $n$-dimensional Hausdorff measure $H^n$ agrees with $\mu^\text{bh}$, i.e. $H^n(B) = \mu^\text{bh}(B)$ for all Borel sets $B$ in an $n$-dimensional normed space $V$, see e.g. [19, Lemma 6] and the references therein.

**Definition 2.4.** Any definition of volume $\mu$ on $n$-dimensional normed spaces gives rise to a definition of Finsler mass $M_\mu$ for rectifiable currents $T \in R_n(X)$ as follows. Given bi-Lipschitz parametrizations $\varphi_i : K_i \to X$ and densities $\theta_i$ for $T \in R_n(X)$ as in Definition 2.2,

$$M_\mu(T) := \sum_i \int_{K_i} |\theta_i(x)J_\mu(\operatorname{md}(\varphi_i)_x) dL^n(x),$$

where $\operatorname{md}(\varphi_i)_x$ is the (approximate) metric derivative of $\varphi_i$ at $x$ as defined in [19].

The note that in case $X$ is a Banach space, the maps $\varphi_i : K_i \to X$ can be extended to Lipschitz maps $\bar{\varphi}_i : \mathbb{R}^n \to X$ due to [18, Theorem 2]. Since $X$ has an isometric embedding into $\ell_\infty(X)$ via the Kuratowski embedding we can always assume that the maps $\varphi_i$ are defined on all of $\mathbb{R}^n$ for which the metric derivative exists almost everywhere.

We leave the it to the reader to show that this definition does not depend on the particular parametrization. With a decomposition argument, it boils down to an application of the area formula [10, Theorem 3.2.3] and the chain rule

$$\operatorname{md}(\varphi \circ \psi)_x(v) = \operatorname{md}(\psi(x))(D\psi(x))$$

for almost all $x \in K_1$ and all $v \in \mathbb{R}^n$, whenever $\psi : K_1 \to K_2$ is bi-Lipschitz, $K_1, K_2 \subset \mathbb{R}^n$ are compact and $\varphi : K_2 \to K$ is Lipschitz. The chain rule follows quite directly from the definition of the metric derivative in [19]. The definition of the Jacobian (2.1) then implies

$$J_\mu(\operatorname{md}(\varphi \circ \psi)_x) = J_\mu(\operatorname{md} \varphi(x))(D\psi_x)$$

for almost all $x \in K_1$. Note that the same definition of Finsler mass/volume also applies to rectifiable sets in metric spaces and more generally to rectifiable chains with coefficients in a normed abelian group due to De Pauw and Hardt [6].

**Lemma 2.5.** Assume that $\mu$ is a definition of volume on $n$-dimensional normed spaces, $V$ is an oriented $n$-dimensional normed space and $X$ is a metric space. The following properties hold for currents in $R_n(V)$ and $R_n(X)$ respectively.

1. If $\theta \in L^1(V)$ has compact support, then

$$M_\mu(\theta) = \int_V |\theta(x)| d\mu_V(x).$$
\( M = M_{m^{*}} \) \\
(3) \( C_n^2 M \leq M_{\mu} \leq C_n M \) for some \( C > 1 \) that depends only on \( n \). \\
(4) \( M_{\mu} \geq M_{\mu} \).

(5) \( M_{\mu}(\psi # T) \leq \text{Lip}(\psi)^n M_{\mu}(T) \) if \( \psi \in \text{Lip}(X, Y) \) and \( T \in \mathcal{R}_n(X) \).

**Proof.** (1): We first assume that \( \theta = \chi_B \) for some bounded Borel set \( B \subset V \) of positive measure. If a coordinate system on \( V \) is fixed via an isomorphism \( I : \mathbb{R}^n \to V \), then \( \text{md} \, L_1 \) is the pull-back norm on \( \mathbb{R}^n \) denoted by \( s \). Let \( B' := I^{-1}(B) \) and rewrite the definition of \( M_{\mu}([B]) \) using (2.2) as

\[
M_{\mu}([B]) = \int_{B'} \frac{\mu_s(B')}{\mathcal{L}^n(B')} \, d\mathcal{L}^n(x) = \mu_s(B') = \mu_V(B).
\]

For the last equality it is used that \( I : (\mathbb{R}^n, s) \to V \) is a linear isometry. The result for a weight function \( \theta \) as in the statement follows by approximation with step functions.

(2): Fix a basis \( \xi_1, \cdots, \xi_n \) of the dual space \( V^* \) with \( ||\xi_i||^* = 1 \) and the property that \( ||\xi_1 \wedge \cdots \wedge \xi_n, \nu|| = \mu_{V^n}^{\nu}(\nu) \) for one (and hence all) \( \nu \in \Lambda_n V \setminus \{0\} \). Let \( v_1, \ldots, v_n \in V \) be the predual basis and \( P := P(v_1, \ldots, v_n) \) be the a parallelepiped spanned by it. The set

\[
\{ v \in V : |\xi_i(v)| \leq 1 \text{ for all } i \} = \left\{ \sum_i x_i v_i : |x_i| \leq 1 \text{ for all } i \right\}
\]

contains the unit ball \( B_V \) and is a homothetic copy of \( P \). By (1) and the definitions,

\[
M_{m^{*}}([P]) = \mu_{V^n}^{\nu^*}(P) = \mu_{V^n}^{\nu^*}(v_1 \wedge \cdots \wedge v_n) = 1.
\]

\( P \) is parametrized by \( [0, 1]^n \ni (x_1, \ldots, x_n) \mapsto x_1 v_1 + \cdots + x_n v_n \). With a standard linearization argument, the Ambrosio-Kirchheim mass of \([P]\) can be expressed as

\[
M([P]) = \sup_g |P|_{(1, g)} = \sup_g \int_{[0, 1]^n} |g_1 \wedge \cdots \wedge g_n, v_1 \wedge \cdots \wedge v_n| \, d\mathcal{L}^n = 1,
\]

where the supremum is taken over all linear \( g = (g_1, \ldots, g_n) : V \to \mathbb{R}^n \) with \( ||g_i||^* \leq 1 \) for all \( i \). This implies that \( M_{m^{*}}([P]) = M([P]) \) and thus \( M([\theta]) = \int_X |\theta(x)| \, d\mu_{V^n}(x) \) for all \( \theta \) by approximation with step functions. For the general statement, let \( T \in \mathcal{R}_n(X) \) and for \( \lambda > 1 \) choose a parametrization \( (\phi_i, K_i, \theta_i) \) of \( T \) as in Definition 2.2. By the definition of the metric derivative in [19], it holds

\[
\lambda^{-1} ||v||_i \leq \text{md}(\phi_i)_x(v) \leq \lambda ||v||_i
\]

for all \( i, v \in \mathbb{R}^n \) and almost every \( x \in K_i \). As a consequence of the first property in Definition 2.3 and the scaling property of a Haar measure, the estimate above implies \( \lambda^{-n} \mu_{m^{*}}^{\mu_{V^n}}(\phi_i)_x \leq \mu_{m^{*}}^{\mu_{V^n}}(\phi_i)_x \leq \lambda^n \mu_{m^{*}}^{\mu_{V^n}}(\phi_i)_x \) and with (2.2) also \( \lambda^{-n} \mu_{m^{*}}^{\mu_{V^n}}(|| \cdot ||_i) \leq \mu_{m^{*}}^{\mu_{V^n}}(\text{md}(\phi_i)_x) \leq \lambda^n \mu_{m^{*}}^{\mu_{V^n}}(|| \cdot ||_i). \) Since

\[
\int_{K_i} |\theta_i(x)| \, \mu_{m^{*}}^{\mu_{V^n}}(|| \cdot ||_i) \, d\mathcal{L}^n(x) = \int_{K_i} |\theta_i(x)| \, \mu_{m^{*}}^{\mu_{V^n}}(\text{md}(\phi_i)_x) \, d\mathcal{L}^n(x) = M([\theta_i])
\]

we conclude

\[
M(T) \leq \lambda^n \sum_i M([\theta_i]) \leq \lambda^{2n} \sum_i \int_{K_i} |\theta_i(x)| \, \mu_{m^{*}}^{\mu_{V^n}}(|| \cdot ||_i) \, d\mathcal{L}^n(x) = \lambda^{2n} M_{m^{*}}(T)
\]

and the lower bound \( \lambda^{-2n} M_{m^{*}}(T) \) is obtained analogously. Since this holds for all \( \lambda > 1 \), we conclude (2).
(3): By a result of John [17], it holds
\[ \mathcal{E}_V \subset B_V \subset n^{\frac{n}{2}} \mathcal{E}_V. \]
Let \( e \) be the Euclidean norm on \( V \) such that its unit ball is given by \( B_e = \mathcal{E}_V \). The inclusions above imply
\[ e(v) \geq \| v \| \geq n^{-\frac{1}{2}} e(v) \]
for all \( v \in V \). The two properties of volumes in Definition 2.3 justify
\[ \alpha(n) = \mu_e(\mathcal{E}_V) \leq n^{\frac{n}{2}} \mu_V(\mathcal{E}_V) \leq n^{\frac{n}{2}} \mu_e(\mathcal{E}_V) = n^{\frac{n}{2}} \alpha(n) . \]
Hence \( \mu_1 \leq n^{\frac{n}{2}} \mu_2 \) for any two definitions of volume, and by (2.2), the statement follows for \( C_n = n^{\frac{n}{2}} \).

(4): If \( e \geq \| \cdot \| \) is the Euclidean norm on \( V \) as above, then
\[ \mu_V(\mathcal{E}_V) = \mu_e(\mathcal{E}_V) \geq \mu_V(\mathcal{E}_V) . \]
Consequently, \( \mu^{ir} \geq \mu \) and also \( M_{ir} \geq M_u \).

(5): This is a consequence of Definition 2.3(1). More precisely, with a decomposition of a parametrization into smaller compact sets it boils down to the following chain rule argument. Assume that \( K_X \subset V_X \) and \( K_Y \subset V_Y \) are compact subsets of \( n \)-dimensional normed spaces, \( \varphi_X : K_X \to X \) and \( \varphi_Y : K_Y \to Y \) are bi-Lipschitz embeddings with bi-Lipschitz constants bounded by \( \lambda > 1 \) and assume that \( \psi : \varphi_X(K_X) \to \varphi_Y(K_Y) \) is bi-Lipschitz too. Then \( \varphi := \varphi_Y^{-1} \circ \psi \circ \varphi_X : K_X \to K_Y \) satisfies \( \text{Lip}(\varphi) \leq \lambda^2 \text{Lip}(\psi) \). Assume that \( x \in K_X \) is a point of approximate differentiability of \( \varphi \), then for any Borel set \( B \subset V_X \) of positive and finite measure
\[ \frac{\mu_V(D\varphi_x(B))}{\mu_V(B)} \leq \| D\varphi_x \|^n \leq \text{Lip}(\varphi)^n \leq \lambda^2 \text{Lip}(\psi)^n . \]
Since for parametrizations as in Definition 2.2 we can choose \( \lambda > 1 \) arbitrary close to 1, the result follows from (1). The details for this decomposition argument into bi-Lipschitz pieces \( \psi : \varphi_X(K_X) \to \varphi_Y(K_Y) \) is given in [6, §3.5] in a more general setting and builds on [6, Lemma 3.1.1] applied to \( \psi \circ \varphi_i \), where \( \varphi_i \) is part of a parametrization for \( T \) as in Definition 2.2. \( \square \)

In order to understand the action of differential forms, we need tangent spaces in suitable infinite dimensional Banach spaces. For this, further results of [2] and [3] are needed. These build on the fact that any Lipschitz map \( f : \mathbb{R}^n \to Y \), where \( Y = X^* \) for \( X \) separable, is weak-star differentiable at almost every point \( x \in \mathbb{R}^n \) in the sense that there exists a linear map \( wd_f : \mathbb{R}^n \to Y \) such that
\[ w^*-\lim_{y \to x} \frac{f(y) - f(x) - wd_f(y-x)}{|y-x|} = 0 \]
and \( \| wd_f(v) \| = md f(x) \) for all \( v \in \mathbb{R}^n \) by [3, Theorem 3.5]. Due to [2, Theorem 9.1] any \( T \in \mathcal{H}_n(Y) \) can be represented as \( [S, \theta, \tau] \), where \( S \subset Y \) is a countably \( \mathcal{H}^n \)-rectifiable set, \( \theta : S \to (0, \infty) \) is a Borel function with \( \int_S \theta \ d\mathcal{H}^n < \infty \) and \( \tau : S \to \Lambda_n Y \) is an orientation such that
\[ T(f, g_1, \ldots, g_n) = \int_S \theta(x) f(x) \langle \Lambda_n d^S g, \tau(x) \rangle \ d\mathcal{H}^n(x) . \]
Here are some details. By the weak-star differentiability of Lipschitz maps, the set \( S \) has an \( n \)-dimensional approximate tangent space \( \text{Tan}(S, x) \) for \( \mathcal{H}^n \)-almost all \( x \in S \). An orientation is given by Borel maps \( \tau_1, \ldots, \tau_n : S \to Y \) such that for \( \mathcal{H}^n \)-almost all \( x \in S \):
The injective hull of the Riemannian circle $S^2$. Injective hulls of Riemannian spheres. For any Borel set $B$, the linear maps $\mu$ spaces used to transform from the Haar measure $\lambda$ where $\mu_B = \frac{\lambda_B}{\lambda}$. The factor on normed spaces used to transform from the Haar measure $\mu_{\text{bh}}$ to the Haar measure $\mu_{\text{ms}}$. The reason to base this on the Busemann-Hausdorff definition of volume is because it is induced by the $n$-dimensional Hausdorff measure of the ambient space. A similar factor can be calculated for all definitions of volume and together with Lemma 2.5 we obtain the following characterization of Finsler mass.

Corollary 2.6. Assume $Y$ is a Banach space with $Y = X^*$ for $X$ separable and $T \in \mathcal{R}_n(Y)$ is represented by $[S, \theta, \tau]$. Then
\[
\mathbf{M}_\mu(T) = \int_S \theta(x)\lambda_{\text{Tan}(S,x)} \, d\mathcal{H}^n(x),
\]
for every definition of volume $\mu$, where $\lambda_{\mu}$ is defined by
\[
\lambda_{\mu} := \frac{\mu_{\text{bh}}(B)}{\mu_{\text{ms}}(B)}
\]
for any Borel set $B \subset V$ of finite and positive measure.

2.2. Injective hulls of Riemannian spheres. Although we will only work with the injective hull of the Riemannian circle $S^1$, the main result of this section holds more generally for the Riemannian sphere $S^n$ of arbitrary dimension. $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is the standard Euclidean sphere of radius 1 endowed with the intrinsic geodesic distance $d$. As a specific subset of the Banach space $\ell_\infty(S^n)$ of bounded functions $S^n \to \mathbb{R}$, the injective hull $E(S^n)$ can be identified with the collection of 1-Lipschitz functions $f : S^n \to \mathbb{R}$ (denoted by $\text{Lip}_1(S^n)$) with the properties:

1. $d(x, y) \leq f(x) + f(y)$ for all $x, y \in S^n$.
2. For all $x \in S^n$ there exists $y \in S^n$ with $f(x) + f(y) = d(x, y)$.

Injective hulls were introduced independently by Isbell [14] and Dress [8]. See for example [21, §3] for more details and consequences of the definition. It is also shown there that $E(S^n)$, as defined above, is indeed an injective metric space. This means that for all 1-Lipschitz maps $\varphi : A \to E(S^n)$ defined on a subset $A$ of a metric space $X$, there is a 1-Lipschitz extension $\bar{\varphi} : X \to E(S^n)$. The map
\( \iota : S^n \to E(S^n) \) defined by \( \iota(x) := d_x \), where \( d_x(y) := d(x, y) \) for all \( y \in S^n \), is an isometry. This is known as the Kuratowski embedding. We identify \( S^n \) with the image of \( \iota \). As in the proof of Lemma 2.9 below, it can be shown that \( \iota(S^n) \) is the only isometric copy of \( S^n \) in \( E(S^n) \). With this identification, properties (1) and (2) directly imply

\[
\|d_x - f\|_\infty = f(x),
\]

for every \( x \in S^n \). This means that the distance of \( f \) to points in \( S^n \) is given by the evaluations of \( f \).

The crucial observation for the characterization of \( E(S^n) \) below is that for all pairs \( x, y \in S^n \), the point \( y \) is on a geodesic from \( x \) to \(-x\).

**Proposition 2.7.** The following properties hold:

1. For \( f \in \text{Lip}_1(S^n) \) it holds that \( f \in E(S^n) \) if and only if \( f(x) + f(-x) = \pi \) for all \( x \in S^n \).
2. \( E(S^n) \) is a compact, convex subset of \( \ell_\infty(S^n) \).
3. \( f \) takes values in \([0, \pi]\) for all \( f \in E(S^n) \).
4. For \( f \in E(S^n) \) it holds that \( f \in E(S^n) \setminus S^n \) if and only if \( f \) takes values in \((0, \pi)\).

**Proof.** (1): First let \( f \in E(S^n) \). For any \( x \in S^n \) there exists \( y \in X \) with \( f(x) + f(y) = d(x, y) \) by the definition of \( E(S^n) \). Hence

\[
d(x, -x) \leq f(x) + f(-x) = f(x) + f(y) + f(-x) - f(y) \\
\leq d(x, y) + d(-x, y) = d(x, -x),
\]

and therefore \( f(x) + f(-x) = d(x, -x) = \pi \).

On the other hand, let \( f \in \text{Lip}_1(S^n) \) with \( f(x) + f(-x) = \pi \) for some \( x \in S^n \) and assume by contradiction that \( y \in S^n \) is such that \( d(x, y) > f(x) + f(y) \). Because \( f \) is 1-Lipschitz

\[
d(x, -x) - d(-x, y) = d(x, y) > f(x) + f(y) \\
= f(x) + f(-x) + f(y) - f(-x) \\
\geq d(x, -x) - d(y, -x).
\]

This is not possible. Hence \( f(x) + f(y) \geq d(x, y) \) for all \( y \in S^n \). If \( f(x) + f(-x) = \pi \) holds for all \( x \in S^n \), then \( f(x) + f(y) \geq d(x, y) \) for all \( x, y \in S^n \) with equality for \( y = -x \). This shows that \( f \in E(S^n) \) and establishes (1).

(2): Because all the functions in \( E(S^n) \) are 1-Lipschitz and \( S^n \) is compact, \( E(S^n) \) is itself compact as a consequence of the Arzelà-Ascoli theorem. If \( f, g \in E(X) \) and \( t \in [0, 1] \), then \( tf + (1 - t)g \) is 1-Lipschitz and moreover

\[
(tf(x) + (1 - t)g(x)) + (tf(-x) + (1 - t)g(-x)) = t\pi + (1 - t)\pi = \pi,
\]

for all \( x \in X \). Hence \( tf + (1 - t)g \in E(X) \) by (1). This shows (2).

(3): Observe that \( f(x) + f(-x) = \pi \) by (1) and since \( 0 \leq \frac{1}{2}d(x, x) \leq f(x) \) by the definition of \( E(S^n) \), it follows that \( f(x) \in [0, \pi] \) for all \( x \in S^n \).

(4): If \( f \in S^n \subset E(S^n) \), then \( f = d_x \) for some \( x \in S^n \) and hence \( f(x) = d(x, x) = 0 \). On the other hand, if \( f(x) = 0 \) for \( f \in E(S^n) \) and \( x \in S^n \), then \( \|f - d_x\|_\infty = f(x) = 0 \) by (2.4). Hence \( f = d_x \in S^n \). Similarly if \( f(x) = \pi \), then \( f(-x) = 0 \) by (1) and therefore \( f = d_{-x} \) by the same argument. \( \square \)
For our main applications we have to choose an orientation on $S^1$. We do this by fixing a base point $p_0$ in $S^1$ and an arc length preserving $2\pi$-periodic parametrization $\gamma : \mathbb{R} \to S^1$ with $\gamma(0) = p_0$.

Identifying $t$ with $\gamma(t)$, it is a consequence of Proposition 2.7 that $E(S^1)$ can be identified isometrically with the space of functions $f : \mathbb{R} \to \mathbb{R}$ with the properties

(1) $f$ is 1-Lipschitz,
(2) $f_{\alpha+\pi} + f_{\alpha} = \pi$ for all $\alpha \in \mathbb{R}$.

Any such function is $2\pi$-periodic as a consequence of (2). This identification will be fixed once and for all. A priori, the differential form $\tilde{\omega}$ defined in the introduction seems to depend on the choice of the base point, but it does not as we will see in (3.10) below. In this notation, points in $S^1$ are identified with functions of the type $\alpha \mapsto \arccos(\cos(\alpha - \tau))$ for some parameter $\tau \in \mathbb{R}$.

Since the coefficients of $\tilde{\omega}$ are not bounded in a neighbourhood of $S^1$ we also will use the truncated injective hulls

\[
E_\varepsilon(S^1) := \{ f \in E(S^1) : f_\alpha \in [\varepsilon, \pi - \varepsilon] \text{ for all } \alpha \}\text{ for } \varepsilon \in (0, \tfrac{\pi}{2}).
\]

The following observations are easy to check and left to the reader.

**Lemma 2.8.**

1. $E_\varepsilon(S^1) = \{ f \in E(S^1) : \text{dist}(f, S^1) \geq \varepsilon\}$.
2. $E(S^1) \setminus S^1 = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}}(S^1)$.
3. $E_\varepsilon(S^1)$ is a compact and convex subset of $L^\infty(\mathbb{R})$.
4. $E_\varepsilon(S^1)$ is a 1-Lipschitz retract of $E(S^1)$.

### 2.3. Representation of the hemisphere.

The hemisphere $S^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$ is equipped with the induced intrinsic metric denoted by $d$. For $p \in S^2_+$ let $x_p \in S^1$ be a point with intrinsic distance $d(x_p, p) = \text{dist}(S^1, p)$. This point is unique unless $p$ is the north pole $N := (0, 0, 1)$. For any $x \in S^1$, the spherical Pythagorean theorem reads

\[
\cos(d(p, x)) = \cos(d(p, x_p)) \cos(d(x_p, x)) .
\]

So $p$ can be identified with the function $f_p : S^1 \to \mathbb{R}$ given by

\[
f_p(x) := d(p, x) = \arccos(\cos(d(p, x_p)) \cos(d(x_p, x))) .
\]

If $p = N$, it obviously holds

\[
f_N(x) = \frac{\pi}{2} = \arccos(0) = \arccos(\cos(\frac{\pi}{2}) \cos(d(x_p, x)))
\]

for all $x \in S^1$.

**Lemma 2.9.** The map $\iota : p \mapsto f_p$ is an isometric embedding of $S^2_+$ into $E(S^1)$. Moreover, $\iota(S^2_+)$ is the only isometric copy of $S^2_+$ in $E(S^1)$.

**Proof.** As a distance function it is clear that $f_p$ is 1-Lipschitz since

\[
|f_p(x) - f_p(y)| = |d(p, x) - d(p, y)| \leq d(x, y)
\]

for all $x, y \in S^1$. Further

\[
f_p(x) + f_p(-x) = d(p, x) + d(p, -x) = \pi
\]

for all $x \in S^1$ since $p \in S^2_+$ lies on a minimizing geodesic connecting $x$ with $-x$. Thus $f_p$ is in $E(S^1)$ by Proposition 2.7.
It remains to show that the intrinsic distance $d(p, q)$ is given by

\begin{equation}
(2.6) \quad d(p, q) = \|f_p - f_q\|_\infty
\end{equation}

for all pairs of different points $p, q \in S^2_R \setminus S^1$. With the preparation above,

$$d(p, q) \geq |d(p, x) - d(x, q)| = |f_p(x) - f_q(x)|$$

for all $x \in S^1$. On the other hand, a geodesic from $p$ to $q$ can be extended until it reaches $S^1$, say in $x$. This extended geodesic is minimizing and therefore

$$f_p(x) - f_q(x) = d(p, x) - d(q, x) = d(p, q).$$

This shows (2.6).

For the second statement assume that $X$ is an isometric copy of $S^2_R$ in $E(S^1)$. Its (surface) boundary $\partial X$ is isometric to $S^1$ and we claim that $\partial X$ agrees with the natural isometric copy $S := \{d_x : x \in S^1\}$ of $S^1$ in $E(S^1)$. Indeed, any point $f \in \partial X$ has a corresponding point $g \in \partial X$ with $\|f - g\|_\infty = \pi$. But, as a consequence of Proposition 2.7, the only pair of points in $E(S^1)$ that have distance $\pi$ are antipodal pairs $d_x, d_{-x}$ of $S$. Thus $\partial X$ is contained in $S$ and because both $\partial X$ and $S$ are topological circles, equality $S = \partial X$ holds. Any point $f \in E(S^1)$ is uniquely determined by the distance functions $f(x) = \|f - d_x\|_\infty$ for $x \in S^1$ due to (2.4) and any $f \in X$ as a point of an isometric copy of $S^2_R$ is uniquely determined by the distance functions $\|f - g\|_\infty$ to points $g$ of the boundary $\partial X$. Thus $X \subset \iota(S^2_R)$ because $S = \partial X$. Now $X$ can’t be isometric to a proper subset of $\iota(S^2_R)$ and hence $X = \iota(S^2_R)$.

This lemma shows that there is a unique subset of $E(S^1)$ which is isometric to $S^2_R$. The proof above also shows that its boundary $\{d_x : x \in S^1\}$ is the only isometric copy of $S^1$ in $E(S^1)$. Both metric spaces $S^1$ and $S^2_R$ will therefore be identified with the corresponding subsets of $E(S^1)$.

As functions on $\mathbb{R}$ any $p \in S^2_R$ is represented by a function $f : \mathbb{R} \to \mathbb{R}$ with

$$f_\alpha = \arccos(\cos(d) \cos(\alpha - \tau))$$

for some parameters $\tau \in \mathbb{R}$ and $d \in [0, \frac{\pi}{2}]$. Actually, the assignment $(\frac{\pi}{2} - d, \tau) \mapsto f$ is the polar normal coordinates with the north pole $N$ as center.

For later use we want to understand the variations in $\tau$ and $d$. Consider

$$\Gamma_\alpha(\tau, d) := \arccos(\cos(d) \cos(\alpha - \tau))$$

for $\alpha, \tau \in \mathbb{R}$ and $d \in [0, \frac{\pi}{2}]$. Since $\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$, we get

$$\frac{\partial}{\partial \tau} \Gamma_\alpha(\tau, d) = -\frac{\cos(d) \sin(\alpha - \tau)}{(1 - \cos(d)^2 \cos(\alpha - \tau)^2)^{\frac{3}{2}}},$$

$$\frac{\partial}{\partial d} \Gamma_\alpha(\tau, d) = \frac{\sin(d) \cos(\alpha - \tau)}{(1 - \cos(d)^2 \cos(\alpha - \tau)^2)^{\frac{3}{2}}}.$$
Lemma 2.10. For fixed $d \in (0, \frac{\pi}{2})$ and $\tau \in \mathbb{R}$, the tangent plane of $S^2_d$ at $f = \arccos(\cos(d) \cos(-\tau))$ is represented by $\gamma = \gamma_{\tau,d} : \mathbb{R} \to \mathbb{R}^2$. Further, $\gamma$ is a counterclockwise parametrization of the unit circle,

$$\gamma(\alpha) \times \gamma(\beta) = \frac{\sin(d) \sin(\beta - \alpha)}{(1 - \cos(d)^2 \cos(\alpha - \tau)^2)^{\frac{1}{2}} \cos(\alpha - \tau)^2 (1 - \cos(d)^2 \cos(\beta - \tau)^2)^{\frac{1}{2}}}$$

where $v \times w := v_1 w_2 - v_2 w_1$ and

$$|\gamma'(\alpha)| = \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \tau)^2} = \frac{\sin(\text{dist}(f, S^1))}{\sin(f_\alpha)^2}$$

with integral

$$2\pi = \int_0^{2\pi} \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \tau)^2} \, d\alpha.$$  

Proof. For $\alpha, \beta \in \mathbb{R}$,

$$\gamma(\alpha) \times \gamma(\beta) = \frac{\sin(d) \cos(\alpha - \tau) \sin(\beta - \tau) - \sin(d) \cos(\beta - \tau) \sin(\alpha - \tau)}{(1 - \cos(d)^2 \cos(\alpha - \tau)^2)^{\frac{1}{2}} (1 - \cos(d)^2 \cos(\beta - \tau)^2)^{\frac{1}{2}}}$$

$$= \frac{\sin(d) \sin(\beta - \alpha)}{(1 - \cos(d)^2 \cos(\alpha - \tau)^2)^{\frac{1}{2}} (1 - \cos(d)^2 \cos(\beta - \tau)^2)^{\frac{1}{2}}}.$$  

Here $v \times w = v_1 w_2 - v_2 w_1$ is the signed area spanned by the parallelogram of two vectors $v, w \in \mathbb{R}^2$.

Thus $\gamma(\alpha) \times \gamma(\beta) > 0$ if $\beta \in (\alpha, \alpha + \pi)$. Together with $|\gamma'(\alpha)| = 1$ for all $\alpha$, it follows that $\gamma$ is a smooth counterclockwise parametrization of $S^1$. Its speed is...
calculated by
\[ |\gamma'(\alpha)| = \gamma(\alpha) \times \gamma'(\alpha) = \lim_{\varepsilon \to 0} \frac{\gamma(\alpha) \times \gamma(\alpha + \varepsilon)}{\varepsilon} = \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \tau)^2} \]
and the length of \( \gamma_{[0,2\pi]} \) is given by
\[ 2\pi = \int_0^{2\pi} |\gamma'(\alpha)| \, d\alpha = \int_0^{2\pi} \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \tau)^2} \, d\alpha \]
as claimed. \( \square \)

The integral identity above is the primary motivation for the definition of the differential form \( \tilde{\omega} \).

3. Definition of Omega

For \( f \in E(S^1) \setminus S^1 \) and \( \alpha, \beta \in \mathbb{R} \) with \( \alpha \neq \beta \mod \pi \mathbb{Z} \), coefficients are defined by
\[ p_{\alpha,\beta}(f) : = \frac{1 - \cos(\beta - \alpha)^2 - \cos(f_\alpha)^2 - \cos(f_\beta)^2 + 2 \cos(\beta - \alpha) \cos(f_\alpha) \cos(f_\beta)}{\sin(\beta - \alpha)^2 \sin(f_\alpha)^2 \sin(f_\beta)^2} . \]
First note that because \( f \notin S^1 \), \( f \) takes values in \((0, \pi)\) by Proposition 2.7 and thus \( p_{\alpha,\beta}(f) \) is well-defined. We decompose \( p_{\alpha,\beta}(f) \) as
\[ p_{\alpha,\beta}(f) = \sin(f_\alpha)^{-2} q_{\alpha,\beta}(f) = \sin(f_\alpha)^{-2} \sin(f_\beta)^{-2} e_{\alpha,\beta}(f) , \]
where
\[ e_{\alpha,\beta}(f) : = 1 - \frac{1}{\sin(\beta - \alpha)^2} (\cos(f_\alpha)^2 + \cos(f_\beta)^2 - 2 \cos(\beta - \alpha) \cos(f_\alpha) \cos(f_\beta)) \]
and
\[ q_{\alpha,\beta}(f) : = \frac{e_{\alpha,\beta}(f)}{\sin(f_\beta)^2} . \]
The differential two-form \( \omega \in \Omega^2(E(S^1) \setminus S^1) \) is defined by
\[ \omega_f : = \int_0^\pi \int_\alpha^\beta p_{\alpha,\beta}(f) \, df_{\alpha} \wedge df_{\beta} \, d\alpha . \]
The precise meaning of \( \omega_f \) will become clear later in subsection 3.3 when the action of \( \omega \) on currents will be defined. Note that \( \omega \) is off by a factor of \( \pi \) from \( \tilde{\omega} \) in the introduction. We work with \( \omega \) for simplicity of notation until the proofs of the main theorems.

3.1. Geometric interpretation. The coefficients \( p_{\alpha,\beta}(f) \) for \( f \in E(S^1) \) have geometric meaning. Fix \( f \in E(S^1) \) and \( \alpha, \beta \in \mathbb{R} \) that represent points in \( S^1 \) also denoted by \( \alpha \) and \( \beta \) with distance \( d(\alpha, \beta) = |\beta - \alpha| \mod \pi \) not contained in \( \{0, \pi\} \). The three numbers \( f_\alpha, f_\beta \) and \( d(\alpha, \beta) \) are in \( [0, \pi] \) and satisfy the triangle inequality by the definition of \( E(S^1) \). Thus there exists a unique point \( p \in S^2_+ \subset \mathbb{R}^3 \) with spherical distances \( d(p, \alpha) = f_\alpha \) and \( d(p, \beta) = f_\beta \). Let \( C \) be the angle at \( p \) of the spherical triangle with vertices \( \alpha, \beta \) and \( p \) and the height of \( p \) above the horizontal plane \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \) is denoted by \( h_{\alpha,\beta}(f) \geq 0 \). The spherical law of sines states
\[ \frac{\sin(C)^2}{\sin(\beta - \alpha)^2} = \frac{V^2}{\sin(\beta - \alpha)^2 \sin(f_\alpha)^2 \sin(f_\beta)^2} . \]
where \( V \geq 0 \) is the volume of the parallelepiped in \( \mathbb{R}^3 \) spanned by the position vectors of \( \alpha, \beta \) and \( p \). Therefore
\[
V^2 = h_{\alpha, \beta}(f)^2 \sin(\beta - \alpha)^2
\]
and
\[
(3.3) \quad \frac{\sin(C)^2}{\sin(\beta - \alpha)^2} = \frac{h_{\alpha, \beta}(f)^2}{\sin(f_\alpha)^2 \sin(f_\beta)^2}.
\]

Next we derive a formula for the height function \( h_{\alpha, \beta}(f) \). The point \( p \in S^2_+ \) is represented by \( g \in E(S^1) \) with \( g_\alpha = f_\alpha \), \( g_\beta = f_\beta \) and satisfies \( \cos(g_\alpha) = \cos(d) \cos(x - \tau) \) for some \( \tau \in \mathbb{R} \) and \( d \in [0, \frac{\pi}{2}] \). Here \( d \) is the intrinsic distance in \( S^2_+ \) of \( p \) to \( S^1 \) and hence \( \sin(d) \) is the height above \( \mathbb{R}^2 \times \{0\} \). Because \( g_\alpha = f_\alpha \) and \( g_\beta = f_\beta \) it follows that
\[
\cos(d) \cos(\alpha - \tau) = \cos(f_\alpha) \quad \text{and} \quad \cos(d) \cos(\beta - \tau) = \cos(f_\beta).
\]
We can rewrite \( \cos(g_\alpha) = a \cos(x) + b \sin(x) \) for \( a, b \in \mathbb{R} \) that satisfy \( \cos(d)^2 = a^2 + b^2 \). The two equations
\[
a \cos(\alpha) + b \sin(\alpha) = \cos(f_\alpha) \quad \text{and} \quad a \cos(\beta) + b \sin(\beta) = \cos(f_\beta)
\]
imply
\[
\cos(d)^2 = a^2 + b^2 = \frac{\cos(f_\alpha)^2 + \cos(f_\beta)^2 - 2 \cos(\beta - \alpha) \cos(f_\beta) \cos(f_\alpha)}{\sin(\beta - \alpha)^2}.
\]
Therefore
\[
h_{\alpha, \beta}(f)^2 = \sin(d)^2 = 1 - \cos(d)^2
\]
\[
(3.4) \quad = 1 - \frac{\cos(f_\alpha)^2 + \cos(f_\beta)^2 - 2 \cos(\beta - \alpha) \cos(f_\beta) \cos(f_\alpha)}{\sin(\beta - \alpha)^2}.
\]
Together with (3.3) we obtain the following interpretation of the coefficients \( p_{\alpha, \beta}(f) \) in terms of spherical geometry.

**Lemma 3.1.** For all \( f \in E(S^1) \) and \( \alpha, \beta \in S^1 \) with \( d(\alpha, \beta) \notin \{0, \pi\} \),
\[
p_{\alpha, \beta}(f) = \frac{\sin(\angle_p(\alpha, \beta))^2}{\sin(\beta - \alpha)^2} = \frac{h_{\alpha, \beta}(f)^2}{\sin(f_\alpha)^2 \sin(f_\beta)^2},
\]
where \( p \in S^2_+ \subset \mathbb{R}^3 \) satisfies \( d(p, \alpha) = f_\alpha \), \( d(p, \beta) = f_\beta \), \( h_{\alpha, \beta}(f) = e_{\alpha, \beta}(f)^{\frac{1}{2}} \) is the height of the point \( p \) above \( \mathbb{R}^2 \times \{0\} \) and \( \angle_p(\alpha, \beta) \) is the angle at \( p \) of the spherical triangle induced by \( \alpha \), \( \beta \) and \( p \).

With this geometrical interpretation at hand, we can extract some information about the coefficients.

**Lemma 3.2.** For \( f \in E(S^1) \setminus S^1 \) and \( \alpha, \beta \in S^1 \) with \( d(\alpha, \beta) \notin \{0, \pi\} \) it holds:
\begin{enumerate}
\item \( p_{\alpha, \beta}(f) = p_{\beta, \alpha}(f) \).
\item \( p_{\alpha, \beta}(f) \) is \( \pi \)-periodic in \( \alpha \) and \( \beta \).
\item \( p_{\alpha, \beta} \geq 0 \) with equality if and only if one the numbers \( f_\alpha \), \( f_\beta \) and \( d(\alpha, \beta) \) is the sum of the other two.
\item \( \sup_{\alpha \neq \beta \mod \pi} p_{\alpha, \beta}(f) \leq \sin(\text{dist}(f, S^1))^2 \).
\end{enumerate}
Proof. (1) is obvious by definition and (2) is a consequence of the antipodal symmetry of $E(S^1)$. Namely, $f_\alpha + f_{\alpha+\pi} = \pi$ implies
\[
\cos(f_{\alpha+\pi}) = \cos(\pi - f_\alpha) = -\cos(-f_\alpha) = -\cos(f_\alpha),
\]
and $\cos(\beta - \alpha - \pi) = -\cos(\beta - \alpha)$ is clear. So the sign changes that occur when transforming $p_{\alpha,\beta}(f)$ to $p_{\alpha+\pi,\beta}(f)$ cancel.

(3) and (4) follow from the geometric interpretation of $p_{\alpha,\beta}(f)$ in Lemma 3.1. □

For technical reasons we distinguish the subset $E^+(S^1)$ of functions $f \in E(S^1)$ that satisfy $p_{\alpha,\beta}(f) > 0$ for all $\alpha \neq \beta \bmod \pi$.

**Lemma 3.3.** If $f \in E(S^1)$ satisfies $\text{Lip}(f) < 1$, then $f \in E^+(S^1)$. In particular, $E^+(S^1)$ is dense in $E(S^1)$.

**Proof.** Assume that $f \notin E^+(S^1)$. According to Lemma 3.2 there are $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta \bmod \pi$ such that the numbers $f_\alpha$, $f_\beta$ and $d(\alpha, \beta) := |\beta - \alpha| \bmod \pi$ form a degenerate triangle in $\mathbb{R}^2$. Note that they form a triangle by the definition of $E(S^1)$ since $d(\alpha, \beta) \leq f_\alpha + f_\beta$ and $|f_\alpha - f_\beta| \leq d(\alpha, \beta)$. In case $f_\alpha = f_\beta + d(\alpha, \beta)$ or $f_\beta = f_\alpha + d(\alpha, \beta)$, then $|f_\alpha - f_\beta| = d(\alpha, \beta)$ and this implies that $\text{Lip}(f) = 1$.

Otherwise, $d(\alpha, \beta) = f_\alpha + f_\beta$. For $\alpha' := \alpha + \pi$ we obtain
\[
d(\alpha', \beta) = \pi - d(\alpha, \beta) = \pi - f_\alpha - f_\beta = f_{\alpha'} - f_\beta
\]
which again implies that $\text{Lip}(f) = 1$. This proves the first statement.

The second statement is obvious since for any $\lambda \in (0, 1)$ and $f \in E(S^1)$, the function $f_\lambda := (1 - \lambda)\frac{x}{2} + \lambda f$ is in $E(S^1)$ by the convexity of $E(S^1)$ and satisfies $\text{Lip}(f_\lambda) \leq \lambda$ as well as
\[
\|f - f_\lambda\|_\infty = (1 - \lambda)\|f - \frac{x}{2}\|_\infty \leq (1 - \lambda)\frac{x}{2}.
\]

### 3.2. Coefficient estimates.

Fix $\varepsilon \in (0, \frac{\pi}{2})$. For two functions $f^0, f^1 \in E_\varepsilon(S^1)$, the convex combination $[0, 1] \ni t \mapsto f^t := (1 - t)f^0 + tf^1$ is also in $E_\varepsilon(S^1)$ by Lemma 3.3. Since we want to interchange integrals over $p_{\alpha,\beta}(f^t)$ with derivatives in $t$ we are interested in uniform bounds of $p_{\alpha,\beta}(f^t)$ and its derivatives. $p_{\alpha,\beta}(f^t)$ is clearly bounded by Lemma 3.2(4) with a constant depending only on $\varepsilon$.

The function
\[
p(a, x, y) := \frac{1 - \cos(a)^2 - \cos(x)^2 - \cos(y)^2}{\sin(a)^2 \sin(x)^2 \sin(y)^2} + 2 \cos(a) \cos(x) \cos(y)
\]
is defined for $a, x, y \in \mathbb{R} \setminus \pi \mathbb{Z}$. It is clear that $p$ is symmetric and smooth. The partial derivatives are stated below.

**Lemma 3.4.** The first and second derivatives of $p$ in $(a, x, y)$ are given by
\[
px = 2\frac{\cos(a) \cos(x) - \cos(y))}{\sin(a)^2 \sin(x)^2 \sin(y)^2} \cos(a) \cos(x) \cos(y)
\]
\[
pxx = 2\frac{(\cos(a) \cos(x) \cos(y)(5 + \cos(x)^2 - (1 + 2 \cos(x)^2)(\cos(a)^2 + \cos(y)^2))}{\sin(a)^2 \sin(x)^4 \sin(y)^2}
\]
\[
pxy = 2\frac{\cos(a)(1 + \cos(x)^2)(1 + \cos(y)^2) - 2 \cos(x) \cos(y)(1 + \cos(a)^2)}{\sin(a)^2 \sin(x)^3 \sin(y)^3}
\]
The proof is left to the reader. Next is a lemma in which we establish uniform bounds of the first two derivatives of the variations $t \mapsto p_{\alpha,\beta}(f^t)$.
Lemma 3.5. There is a constant $C > 0$ with the following property. If $\varepsilon \in (0, \frac{\pi}{2})$ and $f : (1 - t)f + g$ for $f, g \in E_c(S^1)$ and $t \in [0, 1]$, then

$$\sup_{t \in [0,1], \alpha \neq \beta \text{ mod } \pi} \left| \frac{d}{dt} p_{\alpha, \beta}(f^t) \right| \leq C \sin(\varepsilon)^{-6},$$

$$\sup_{t \in [0,1], \alpha \neq \beta \text{ mod } \pi} \left| \frac{d^2}{dt^2} p_{\alpha, \beta}(f^t) \right| \leq C \sin(\varepsilon)^{-8},$$

$$\sup_{t \in [0,1], \alpha \neq \beta \text{ mod } \pi} \left| \sin(\beta - \alpha) \frac{d}{dt} p_{\alpha, \beta}(f^t) \right| \leq C \sin(\varepsilon)^{-5} \| f - g \|_{\infty},$$

$$\sup_{t \in [0,1], \alpha \neq \beta \text{ mod } \pi} \left| \sin(\beta - \alpha)^2 \frac{d^2}{dt^2} p_{\alpha, \beta}(f^t) \right| \leq C \sin(\varepsilon)^{-6} \| f - g \|_{\infty}^2.$$

Proof. Because $p_{\alpha, \beta}(f^t) = p_{\alpha, \beta + k\pi}(f^t)$ for $k \in \mathbb{Z}$ by Lemma 3.2, it is enough to consider the case $0 < |\delta| \leq \frac{\pi}{2}$ for $\delta := \beta - \alpha$. This assumption implies $|\sin(\delta)| \leq |\delta| \leq \frac{\pi}{2} |\sin(\delta)|$ and $1 - \cos(\delta) \leq \sin(\delta)^2$.

For simplicity of notation, we only estimate the derivatives at $t = 0$ and abbreviate $\Delta := f^1 - f^0$, $c_x := \cos(f_x)$ and $s_x := \sin(f_x)$ for $x \in \{\alpha, \beta\}$, $c_\delta := \cos(\beta - \alpha)$, $s_\delta := \sin(\beta - \alpha)$, $q_x := \Delta_x \sin(f(x))^{-1}$.

For the first estimate we need to bound

$$S_1 := p_x(\beta - \alpha, f, f_\beta) \Delta_{\alpha} + p_y(\beta - \alpha, f, f_\beta) \Delta_{\beta}$$

by Lemma 3.4. We note that $\alpha \mapsto q_{\alpha}$ satisfies the Lipschitz condition

$$|q_{\alpha} - q_{\beta}| \leq \frac{1}{s_\alpha s_\beta} |s_\alpha \Delta_\beta - s_\beta \Delta_\alpha| = \frac{1}{s_\alpha s_\beta} |s_\alpha \Delta_\beta - s_\alpha \Delta_\alpha + s_\alpha \Delta_\alpha - s_\beta \Delta_\alpha|$$

$$\leq \sin(\varepsilon)^{-2} (\text{Lip}(\Delta) + \text{Lip}(f) \| \Delta \|_{\infty}) |\delta|$$

$$\leq \sin(\varepsilon)^{-2} (2 + 2 \cdot 2 \pi) |\delta|$$

$$\leq 3\pi \sin(\varepsilon)^{-2} |\delta|.$$

First with $\delta = 0$ in the numerator of $S_1$,

$$\frac{1}{s_\delta^2} |(c_\alpha - c_\beta) q_{\alpha} + (c_\beta - c_\alpha) q_{\beta}| \leq \frac{1}{s_\delta} |(c_\beta - c_\alpha) (q_{\alpha} - q_{\beta})|$$

$$\leq 4(1 \cdot 3 \pi) \sin(\varepsilon)^{-2} \leq 50 \sin(\varepsilon)^{-2}$$

and

$$\frac{1}{s_\delta^2} |(1 - c_\delta) c_\alpha q_{\alpha} + (1 - c_\delta) c_\beta q_{\beta}| \leq |c_\alpha q_{\alpha} + c_\beta q_{\beta}| \sin(\varepsilon)^{-1} \leq 2 \| \Delta \|_{\infty} \sin(\varepsilon)^{-1}$$

$$\leq 4\pi \sin(\varepsilon)^{-1} \leq 20 \sin(\varepsilon)^{-1}.$$

With these estimates,

$$|S_1| \leq \frac{2}{s_\delta^2} (50 + 20) \sin(\varepsilon)^{-2} |c_\delta - c_\alpha c_\beta| \leq 280 \sin(\varepsilon)^{-6}.$$

For the second estimate in the lemma we need to bound the absolute value of

$$S_2 := p_{xx}(\beta - \alpha, f, f_\beta) \Delta_\alpha^2 + 2 p_{xy}(\beta - \alpha, f, f_\beta) \Delta_\alpha \Delta_\beta + p_{yy}(\beta - \alpha, f, f_\beta) \Delta_\beta^2.$$
If we plug in \( \delta = 0 \) in the numerator of \( p_{xx}, p_{xy} \) and \( p_{yy} \) and multiply with \( \frac{1}{2} s_\alpha^2 s_\beta^2 s_\delta^2 \), the resulting term to be estimated is

\[
(c_\alpha c_\beta (5 + c_\alpha^2) - (1 + 2c_\alpha^2)(1 + c_\beta^2))q_\alpha^2 + (c_\alpha c_\beta (5 + c_\beta^2) - (1 + c_\alpha^2)(1 + 2c_\alpha^2))q_\beta^2
+ 2((1 + c_\alpha^2)(1 + c_\beta^2) - 4c_\alpha c_\beta)q_\alpha q_\beta
=: X + Y + Z
\]

by Lemma 3.4 with three terms given by

\[
X = ((1 + c_\alpha^2 - 2c_\alpha c_\beta)(c_\alpha c_\beta - 1) - (c_\alpha - c_\beta)^2)q_\alpha^2,
Y = ((1 + c_\beta^2 - 2c_\alpha c_\beta)(c_\alpha c_\beta - 1) - (c_\alpha - c_\beta)^2)q_\beta^2,
Z = 2(c_\alpha - c_\beta)^2 + (c_\alpha c_\beta - 1)^2)q_\alpha q_\beta.
\]

Collecting matching expressions, we get

\[
|X + Y + Z| \leq 2((1 + c_\alpha^2 - 2c_\alpha c_\beta)(q_\alpha - q_\beta)^2 + (c_\alpha - c_\beta)((c_\beta q_\beta^2 - c_\alpha^2 q_\alpha^2)
+ (c_\alpha - c_\beta)^2(q_\alpha - q_\beta)^2
\leq 2(6 \text{Lip}(q)^2 + \text{Lip}(\cos(f)q^2))\delta^2
\leq C_1 \sin(\varepsilon)^{-1} s_\delta^2.
\]

for some \( C_1 > 0 \). In the last line we used \( \text{Lip}(q)^2 \leq C_2 \sin(\varepsilon)^{-4} \) and

\[
\text{Lip}(\cos(f)q^2) \leq \text{Lip}(f)||q||_\infty + 2||\cos(f)||_\infty ||q||_\infty \text{Lip}(q) \leq C_2 \sin(\varepsilon)^{-3}
\]

for some \( C_2 > 0 \). Since we plugged \( \delta = 0 \) in the numerators of \( S_2 \), the absolute value of the difference

\[
X + Y + Z - \frac{1}{2} s_\alpha^2 s_\beta^2 s_\delta S_2 = ((1 - c_\delta)c_\alpha c_\beta (5 + c_\alpha^2) - (1 + 2c_\alpha^2)(1 + c_\beta^2))q_\alpha^2
+ ((1 - c_\delta)c_\alpha c_\beta (5 + c_\beta^2) - (1 + 2c_\alpha^2)(1 + c_\beta^2))q_\beta^2
+ 2((1 - c_\delta)(1 + c_\alpha^2)(1 + c_\beta^2) - 2c_\alpha c_\beta(1 - c_\delta^2))q_\alpha q_\beta
\]

has a factor of \( 1 - c_\delta \) and thus an estimate of the form \( C_3 \sin(\varepsilon)^{-2} s_\delta^2 \) for some \( C_3 > 0 \). Therefore

\[
|S_2| \leq C_3 \sin(\varepsilon)^{-8},
\]

for some \( C_3 > 0 \) as claimed.

For the third estimate in the lemma, as for \( S_1 \) above,

\[
s_\delta \left. \frac{d}{dt} \right|_{t=0} p_{\alpha, \beta}(f^t) = \frac{2(c_\delta - c_\alpha c_\beta)}{s_\alpha^2 s_\beta^2} \left( \frac{c_\delta c_\alpha - c_\beta}{s_\alpha} \Delta_\alpha + \frac{c_\delta c_\beta - c_\alpha}{s_\beta} \Delta_\beta \right).
\]

Further

\[
|c_\delta c_\alpha - c_\beta| \leq |(1 - c_\delta)c_\alpha| + |c_\alpha - c_\beta|
\leq s_\delta^2 + |\delta| \leq s_\delta (1 + \frac{\theta}{2})
\leq 3|s_\delta|
\]

and similarly for \( |c_\delta c_\beta - c_\alpha| \). Consequently,

\[
|s_\delta \left. \frac{d}{dt} \right|_{t=0} p_{\alpha, \beta}(f^t)| \leq \frac{24}{\sin(\varepsilon)^8} ||f - g||_\infty.
\]
For the last estimate, the trivial bound
\[ \max\{\Delta_{\alpha}^2, |\Delta_{\alpha}\Delta_{\beta}|, \Delta_{\beta}^2\} \leq \|f - g\|_\infty^2 \]
is applied to \( S_2 \) as defined above.

3.3. **Action on currents and paths.** For \( \varepsilon \in (0, \frac{\pi}{2}) \) assume that \( T \in \mathcal{M}_2(E_\varepsilon(S^1)) \) is a metric current of finite mass in the sense of Ambrosio-Kirchheim [2] or [20]. The action of \( \omega \) is defined by
\[
T(\omega) := \int_0^\pi \int_0^\alpha T(p_{\alpha,\beta}(f) \, df_\alpha \wedge df_\beta) \, d\beta \, d\alpha .
\]

With the metric current notation, the integrand can be written as \( T(p_{\alpha,\beta}, \pi_\alpha, \pi_\beta) \), where \( \pi_x : E(S^1) \to \mathbb{R} \) is the evaluation map \( \pi_x(f) := f_x \) for \( x \in \mathbb{R} \).

**Lemma 3.6.** \( T(\omega) \) is well-defined and depends only on \( \partial T \). Moreover
\[
|T(\omega)| \leq C \, M(T),
\]
for some \( C(\varepsilon) > 0 \).

**Proof.** For fixed \( 0 < \alpha < \beta < \pi \), the function \( f \mapsto p_{\alpha,\beta}(f) \) depends only on \( f_\alpha \) and \( f_\beta \). Thus it is possible to write \( p_{\alpha,\beta}(f) \, df_\alpha \wedge df_\beta \) as \( dP(f_\alpha, f_\beta) \wedge df_\beta \) for some smooth \( P : (0, \pi)^2 \to \mathbb{R} \). For another argument, it holds
\[
T(p_{\alpha,\beta}(f) \, df_\alpha \wedge df_\beta) = (\pi_\alpha, \pi_\beta)_p T(g_{\alpha,\beta}(x, y) \, dx \wedge dy)
\]
for the smooth function \( g_{\alpha,\beta} : (0, \pi)^2 \to \mathbb{R} \) with \( g_{\alpha,\beta}(x, y) = p(\beta - \alpha, x, y) \) as in (3.5). \( g_{\alpha,\beta}(x, y) \, dx \wedge dy \) is an exact form in \((0, \pi)^2\) by the lemma of Poincaré. This shows the first statement.

The mass bound is clear since \( p_{\alpha,\beta}(f) \) is uniformly bounded for \( f \) in \( E_\varepsilon(S^1) \) by Lemma 3.2(4) and the evaluation maps \( \pi_x : E(S^1) \to \mathbb{R} \) are 1-Lipschitz. \( \square \)

\( E(S^1) \) can be identified isometrically with a subset of \( L^\infty([0, 2\pi]) \). Since this is the dual of the separable space \( L^1([0, 2\pi]) \), the results at the end of subsection 2.1 apply and we can represent \( T \in \mathcal{R}_2(E(S^1)) \) as a current \([S, \theta, \tau]\) in \( \mathcal{R}_2(L^\infty([0, 2\pi])) \).

**Lemma 3.7.** Let \( T \in \mathcal{R}_2(E_\varepsilon(S^1)) \) for some \( \varepsilon \in (0, \frac{\pi}{2}) \) with representation \([S, \theta, \tau]\) in \( \mathcal{R}_2(L^\infty([0, 2\pi])) \). Then
\[
T(\omega) = \int_S \theta(f) \int_0^\pi \int_0^\alpha p_{\alpha,\beta}(f)(\tau_1,\alpha(f)\tau_2,\beta(f) - \tau_1,\beta(f)\tau_2,\alpha(f)) \, d\beta \, d\alpha \, d\omega^2(f) .
\]

**Proof.** The smoothing operator \( A_\delta : L^\infty([0, 2\pi]) \to L^\infty([0, 2\pi]) \) for \( \delta \in (0, \frac{\pi}{2}) \) is defined by
\[
f_\alpha^\delta := A_\delta(f) := \frac{1}{2\delta} \int_{\alpha - \delta}^{\alpha + \delta} f_t \, dt .
\]
Here we assume that \( f \) is extended to a \( 2\pi \)-periodic function. The \( \delta \)-approximation of \( \omega \) is defined by
\[
\omega^\delta := \int_0^\pi \int_0^\alpha p_{\alpha,\beta}(f) \, df_\alpha^\delta \wedge df_\beta^\delta \, d\beta \, d\alpha ,
\]
with the obvious action on \( T \) given by
\[
T(\omega^\delta) := \int_0^\pi \int_0^\alpha T(p_{\alpha,\beta}, \pi_\alpha^\delta, \pi_\beta^\delta) \, d\beta \, d\alpha ,
\]
where \( \pi^2_\delta : L^\infty([0, 2\pi]) \to L^\infty([0, 2\pi]) \) is defined by \( \pi^2_\delta = \pi_x \circ A_\delta \) for all \( x \). It is easy to check that \( A_\delta \) maps \( E_\varepsilon(S^1) \) into \( E_\varepsilon(S^1) \), \( \|A_\delta\| \leq 1 \) and \( A_\delta \to \text{id} \) uniformly on \( E_\varepsilon(S^1) \). This allows to apply the continuity property of currents and the dominated convergence theorem to conclude that \( T(\omega^\delta) \to T(\omega) \) for \( \delta \to 0 \). The main point here is that \( \pi^2_\delta \) is defined and has finite operator norm for all \( x \). Indeed,

\[
|\pi^2_\delta(f)| \leq \frac{1}{2\delta} \int_{0}^{\alpha+\delta} |f(t)| \, dt \leq \|f\|_\infty .
\]

With (2.3) we can express

\[
T(\omega^\delta) = \int_{0}^{\pi} \int_{0}^{\pi} \int_{S} \theta(f) p_{\alpha, \beta}(f) \left( \Lambda_2 \delta d^2 \left( \pi^\delta_\alpha, \pi^\delta_\beta \right), \tau(f) \right) \, d\mathcal{H}^2(f) \, d\beta \, d\alpha .
\]

Note here that for fixed \( \alpha, \beta \) and \( \delta \), the evaluation \( (\pi^\delta_\alpha, \pi^\delta_\beta) : L^\infty([0, 2\pi]) \to \mathbb{R}^2 \) is well-defined, linear with operator bounded by \( \sqrt{2} \). Thus the map

\[
\Lambda_2 \delta d^2 \left( \pi^\delta_\alpha, \pi^\delta_\beta \right) : \Lambda_2 \text{Tan}(S, f) \to \Lambda_2 \mathbb{R}^2
\]

is given by

\[
\left\langle \Lambda_2 \delta d^2 \left( \pi^\delta_\alpha, \pi^\delta_\beta \right), v \wedge w \right\rangle = v^\delta \alpha w^\delta \beta - v^\delta \beta w^\delta \alpha
\]

for \( \mathcal{H}^2 \)-almost all \( f \in S, \) all \( \alpha, \beta \) and all vectors \( v, w \in \text{Tan}(S, f) \subset L^\infty([0, 2\pi]) \).

For simplicity we will write \( \tau(f) = v(f) \wedge w(f) \). The theorem of Fubini leads to

\[
T(\omega^\delta) = \int_{S} \int_{\Delta} I_\delta(q, f) \, d\mathcal{L}^2(q) \, d\mathcal{H}^2(f) ,
\]

where \( \Delta := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < \beta < \pi \} \) and

\[
I_\delta((\alpha, \beta), f) := \theta(f) p_{\alpha, \beta}(f) \left( v^\delta \alpha(f) w^\delta \beta(f) - v^\delta \beta(f) w^\delta \alpha(f) \right) .
\]

Here are the details for the prerequisites using the theorem of Fubini. The integrand \( \Delta \times S \ni (q, f) \mapsto I_\delta(q, f) \) is \( \mathcal{L}^2 \otimes \mathcal{H}^2 \)-measurable as a combination of the following two facts:

- For fixed \( q \), the function \( f \mapsto I_\delta(q, f) \) is Borel measurable for a fixed \( q \in \Delta \) by the measurability of \( v, w \) and \( \theta \), and the continuity of \( A_\delta \) and \( f \mapsto p_q(f) \) due to Lemma 4.5.
- For a fixed \( f \), the function \( q \mapsto I_\delta(q, f) \) is continuous because \( q \mapsto p_q(f) \) is continuous and \( \alpha \mapsto v^\delta \alpha(f) \) (and also \( \alpha \mapsto w^\delta \alpha(f) \)) is Lipschitz continuous since

\[
|v^\delta \alpha(f) - v^\delta \alpha(f)| \leq \frac{|f|_\infty}{\delta} |\beta - \alpha|
\]

as in the proof of \([23, \text{Theorem} \ 4.7]\).

Additionally, \( I_\delta(q, f) \leq \theta(f) M(\varepsilon) \) for all \( \delta \) because \( p_q(f) \) is uniformly bounded on \( \Delta \times E_\varepsilon(S^1) \) due to Lemma 3.2(4) and also

\[
|\left\langle \Lambda_2 \left( \pi^\delta_\alpha, \pi^\delta_\beta, \tau(f) \right) \right\rangle| \leq \|\left( \pi^\delta_\alpha, \pi^\delta_\beta \right)\| \mu^\text{bh}(\tau(f)) \leq 2
\]

for \( \mathcal{H}^2 \)-almost all \( f \in S \). This bound, in turn, is due to \( \mu^\text{bh}(\tau(f)) = 1, \|\left( \pi^\delta_\alpha, \pi^\delta_\beta \right)\| \leq \sqrt{2} \) and the properties of a definition of area.

For \( \delta \to 0 \), the dominated convergence theorem implies the limit identity

\[
T(\omega) = \int_{S} \int_{0}^{\pi} \int_{0}^{\pi} \theta(f) p_{\alpha, \beta}(f) (v^\alpha(f) w^\beta(f) - v^\beta(f) w^\alpha(f)) \, d\beta \, d\alpha \, d\mathcal{H}^2(f) .
\]
Here are the details. For the left hand side, \( T(\omega^\delta) \to T(\omega) \) by the continuity of \( T \) as observed before. For the right hand side, we know that

\[
S \ni f \mapsto \int_\Delta I_s(p, f) \, d\mathcal{L}^2(q)
\]

is \( \mathcal{H}^2 \)-measurable for all \( \delta \) by Fubini’s theorem, with pointwise \( \mathcal{H}^2 \)-almost everywhere, and thus \( \mathcal{H}^2 \)-measurable, limit

\[
S \ni f \mapsto \int_0^\pi \int_\alpha^\pi \theta(f)p_{\alpha,\beta}(f)(v_\alpha(f)w_\beta(f) - v_\beta(f)w_\alpha(f)) \, d\beta \, d\alpha ,
\]

for \( \delta \to 0 \). The existence of this limit is due to the bound \( I_\delta(q, f) \leq \theta(f)M(\varepsilon) \) for \( f \in S \subset E_\varepsilon(S^1) \) and

\[
\lim_{\delta \to 0} v^\delta_{\alpha}(f)w^\delta_{\beta}(f) - v^\delta_{\beta}(f)w^\delta_{\alpha}(f) = v_\alpha(f)w_\beta(f) - v_\beta(f)w_\alpha(f)
\]

if both \( \alpha \) and \( \beta \) are density points of \( v(f) \) and \( w(f) \).

\( E(S^1) \) has an isometric embedding into \( L^\infty([0, \pi]) \) because \( f_{\alpha+\pi} + f_\alpha = \pi \) for \( f \in E(S^1) \) and \( \alpha \in \mathbb{R} \). Further since differences \( g - f \) for \( f, g \in E(S^1) \) have the antipodal property \( (g - f)_{\alpha+\pi} = -(g - f)_\alpha \) for all \( \alpha \in \mathbb{R} \), the orienting vector fields \( \tau_1, \tau_2 : S \to L^\infty([0,2\pi]) \) in the lemma above can be assumed to have the same property since they arise as weak-star derivatives of a parametrization of \( S \). Accordingly, we can, and will, work in \( L^\infty([0, \pi]) \) instead of \( L^\infty([0,2\pi]) \) and assume that \( \tau_1, \tau_2 \) are maps into \( L^\infty([0, \pi]) \) or into the isometrically isomorphic Banach space

\[
L^\infty_{ap}(\mathbb{R}) := \{ f \in L^\infty(\mathbb{R}) : f_{\alpha+\pi} = -f_\alpha \text{ for all } \alpha \} .
\]

With this lemma, the pointwise definition

\[
\omega_f(v \wedge w) := \int_0^\pi \int_\alpha^\pi p_{\alpha,\beta}(f)(v_\alpha(f)w_\beta(f) - v_\beta(f)w_\alpha(f)) \, d\beta \, d\alpha
\]

for \( f \in E(S^1) \setminus S^1 \) and \( v, w \in L^\infty([0, \pi]) \) will turn out to be useful. The pointwise comass \( \|\omega_f\|_{ir} \) of \( \omega \) at \( f \in E(S^1) \setminus S^1 \) is defined as the infimum over all constants \( M \geq 0 \) such that

\[
|\omega_f(v \wedge w)| \leq M\mu^{ir}(v \wedge w)
\]

for all \( v, w \in L^\infty([0, \pi]) \). It is clear from this definition that \( \|\omega_f\|_{ir} \) depends only on the plane spanned by \( v \) and \( w \). Building on Lemma 3.7 and the definitions above we obtain the following mass and comass characterizations.

For \( T \in \mathcal{H}_2(E_\varepsilon(S^1)) \) with representation \( T = [S, \theta, \tau] \in \mathcal{H}_2(L^\infty([0, \pi])) \) it follows from Corollary 2.6 that

\[
M_{ir}(T) = \int_S \theta(f)\lambda^{ir}_{\tan(S,f)} \, d\mathcal{H}^2(f) = \int_S \theta(f)\mu^{ir}(\tau_1(f) \wedge \tau_2(f)) \, d\mathcal{H}^2(f),
\]

where

\[
\lambda^{ir}_{\tan(S,f)} = \frac{\mu^{ir}(\tau_1(f) \wedge \tau_2(f))}{\mu^{bh}(\tau_1(f) \wedge \tau_2(f))}
\]

for \( \mathcal{H}^2 \)-almost all \( f \in S \).
Proposition 3.8. Let $T \in \mathcal{R}_2(\mathcal{E}(S^1))$ for some $\varepsilon \in (0, \frac{\pi}{2})$ and representation $[S, \theta, r]$ in $\mathcal{R}_2(\mathcal{L}\infty([0, \pi]))$. Then
\[
|T(\omega)| \leq M_{ir}(T) \sup_{\omega \in \text{spt}(T)} \|\omega f\|_{ir} .
\]

Moreover,
\[(3.7) \quad \|\omega f\|_{ir} = \sup \{\omega f(v \wedge w) : v, w \in L^\infty([0, \pi]), \|v^2 + w^2\|_{\infty} \leq 1\}
\]
for all $f \in E(S^1) \setminus S^1$.

Proof. Due to Corollary 2.6,
\[
|T(\omega)| = \int_S \theta(f) \omega f(\tau_1(f) \wedge \tau_2(f)) d\mathcal{H}^2(f)
\]
\[
\leq \sup_{\omega \in \text{spt}(T)} \|\omega f\|_{ir} \int_S \theta(f) \mu^ir(\tau_1(f) \wedge \tau_2(f)) d\mathcal{H}^2(f)
\]
\[
= \sup_{\omega \in \text{spt}(T)} \|\omega f\|_{ir} \int_S \theta(f) \mu^ir_{\tan}(S, f) d\mathcal{H}^2(f)
\]
\[
= \sup_{\omega \in \text{spt}(T)} \|\omega f\|_{ir} M_{ir}(T) .
\]

For the second part we denote the right hand side of (3.7) by $A$ and fix a two-dimensional subspace $V \subset L^\infty([0, \pi])$. Let $\mathcal{E}_V \subset B_V$ be the inscribed Löwner-John ellipse contained in the closed unit ball of $V$ and $e$ be the Euclidean norm on $V$ with respect to which $\mathcal{E}_V$ is the unit ball. This implies that $e(v) \geq \|v\|_{\infty}$ for all $v \in V$. Assume that $v, w \in V$ form an orthonormal basis with respect to $e$. For all $t \in \mathbb{R}$ it holds
\[
1 = e(\cos(t)v + \sin(t)w)^2 \geq \|\cos(t)v + \sin(t)w\|_{\infty}^2 ,
\]
and this implies that $\|v^2 + w^2\|_{\infty} \leq 1$. As an orthonormal basis with respect to $e$, the parallelogram spanned by $v$ and $w$ has area
\[
\mu^v_1(v \wedge w) = e(v \wedge w) = 1 .
\]
Taking the supremum over all $V$ implies $\|\omega f\|_{ir} \leq A$ by the definition of $\|\omega f\|_{ir}$ in (3.6).

To establish $\|\omega f\|_{ir} \geq A$ assume that $v, w \in L^\infty([0, \pi])$ are linearly independent and satisfy $\|v^2 + w^2\|_{\infty} \leq 1$. Again, let $e$ be the Euclidean norm on $V$ with respect to which $\mathcal{E}_V$ is the unit ball. $\|v^2 + w^2\|_{\infty} \leq 1$ is equivalent to $\|\cos(t)v + \sin(t)w\|_{\infty}^2 \leq 1$ for all $t$ and thus $t \mapsto \cos(t)v + \sin(t)w$ traces the boundary of an ellipse $E$ contained in $B_V$. The area of $E$ (or of any origin symmetric ellipse for that matter) within the Euclidean space $(V, e)$ is given by
\[
\pi \max_{x, y \in E} e(x \wedge y) = \pi \max_{x, y \in E} \mu^v_1(x \wedge y) .
\]
Since $\mathcal{E}_V \subset B_V$ has maximal area it follows that
\[
\mu^v_1(v \wedge w) \leq \frac{1}{\pi} \mu^v_1(E) \leq \frac{1}{\pi} \mu^v_1(\mathcal{E}_V) = 1 .
\]
This shows $\|\omega f\|_{ir} \geq A$. \hfill \Box
We can view $\omega_f$ for $f \in E(S^1) \setminus S^1$ as an operator on plane paths. More precisely, we consider the path spaces

\[ L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2) := \{ (\gamma_1, \gamma_2) : \gamma_1, \gamma_2 \in L^\infty_{ap}(\mathbb{R}) \}, \]
\[ B^\infty_{ap}(\mathbb{R}, \mathbb{R}^2) := \{ \gamma \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2) : \|\gamma\|_\infty := \|\gamma_1^2 + \gamma_2^2\|_\infty \leq 1 \}. \]

The action of $\omega_f$ on $\gamma \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$ is defined by

\[
(3.8) \quad \omega_f(\gamma) := \int_0^\pi \int_0^\pi p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha,
\]
where $v \times w = v_1 w_2 - v_2 w_1$ is the signed area of the parallelogram spanned by $v, w \in \mathbb{R}^2$. With this notation it is understood that the coordinate functions of $\gamma$ represent vectors in $L^\infty([0, \pi])$ and $\|\omega_f\|_\infty$ can be written as

\[
(3.9) \quad \|\omega_f\|_\infty = \sup_{\gamma \in B^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)} \omega_f(\gamma)
\]
for $f \in E(S^1) \setminus S^1$ by the proposition above.

We finish this subsection by justifying that the action of $\omega_f$ on $\gamma \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$ does not depend on the choice of the base point in $S^1$ used to define $\omega_f$. By symmetry,

\[
0 = \int_0^\pi \int_0^\pi p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]
since $\gamma(\alpha) \times \gamma(\beta) = -\gamma(\beta) \times \gamma(\alpha)$ and $p_{\alpha, \beta}(f) = p_{\beta, \alpha}(f)$ by Lemma 3.2. Thus

\[
\omega_f(\gamma) = -\int_0^\pi \int_0^\alpha p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]
\[= \int_0^\pi \int_0^{\alpha+\pi} p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]
because $p_{\alpha, \beta+\pi}(f) = p_{\alpha, \beta}(f)$ and $\gamma(\alpha) \times \gamma(\beta + \pi) = -\gamma(\alpha) \times \gamma(\beta)$. Hence

\[
\omega_f(\gamma) = \frac{1}{2} \int_0^\pi \int_0^{\alpha+\pi} p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]
\[= \frac{1}{4} \int_0^{2\pi} \int_0^{\alpha+\pi} p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]
because $p_{\alpha+\pi, \beta+\pi}(f) = p_{\alpha, \beta}(f)$ and $\gamma(\alpha + \pi) \times \gamma(\beta + \pi) = \gamma(\alpha) \times \gamma(\beta)$. The last integral is obviously independent of the fixed base point in $S^1$.

3.4. Coefficients of product type. In this subsection it is assumed that the coefficient function $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of product type. More precisely, we assume that

\[
(1) \quad p_{\alpha, \beta} = p_\alpha p_\beta,
(2) \quad p : \mathbb{R} \to \mathbb{R} \text{ is locally integrable and } \pi \text{-periodic},
(3) \quad p_\alpha > 0 \text{ for almost all } \alpha.
\]

Note that the coefficients $p_{\alpha, \beta}(f)$ are of this type in case $f \in S^2_+ \setminus S^1$ by Lemma 3.1 and Lemma 3.2. These coefficients act on paths $\gamma \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$ by

\[
(3.11) \quad \omega_p(\gamma) := \int_0^\pi \int_0^\pi p_{\alpha, \beta}(f) \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha.
\]
Due to the plane isoperimetric inequality there exists, up to rotations of \( \mathbb{R}^2 \), a unique maximizer \( \gamma \in B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2) \) of \( \omega_p \).

**Lemma 3.9.** Assume that \( p_{\alpha, \beta} = p_\alpha p_\beta \) is as above and \( \nu : \mathbb{R} \to \mathbb{R} \) is the unique bi-Lipschitz function with \( \nu(0) = 0 \) and \( \nu'(\alpha) = p_\alpha (\frac{1}{2\pi} \int_0^{2\pi} p_\beta \, d\beta)^{-1} \) for almost all \( \alpha \). Then the path \( \gamma(\alpha) := e^{i\nu(\alpha)} \) is in \( B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2) \) and satisfies

\[
\omega_p(\gamma) = \sup_{\delta \in B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2)} \omega_p(\delta) = \text{Area}(\sigma|_{[0, 2\pi]}) = \frac{1}{4\pi} \left( \int_0^{2\pi} p_\alpha \, d\alpha \right)^2,
\]

where

\[
\sigma(\alpha) := \frac{1}{2} \int_0^{\alpha + \pi} p_\beta \gamma(\beta) \, d\beta.
\]

Moreover, any maximizer of \( \omega_p \) is of the form \( \alpha \mapsto e^{\nu(\alpha)+ic} \) for some \( c \in \mathbb{R} \) and satisfies

\[
|\gamma'(\alpha)| = \nu'(\alpha) = p_\alpha \left( \frac{1}{2\pi} \int_0^{2\pi} p_\alpha \, d\alpha \right)^{-1}
\]

for almost all \( \alpha \).

**Proof.** For an antipodal plane path \( \gamma \in B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2) \) define \( \sigma \) as in the statement. Note that \( \sigma \in L_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2) \) by the symmetries of \( p \) and \( \gamma \). Further, \( \sigma'(\alpha) = -p_\alpha \gamma'(\alpha) \) and thus \( |\sigma'(\alpha)| \leq p_\alpha \) for almost all \( \alpha \). By the properties of \( p \) and \( \gamma \),

\[
\omega_p(\gamma) = \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha
\]

\[
= \frac{1}{2} \int_0^\pi \int_\alpha^{\alpha + \pi} p_\alpha \gamma(\alpha) \times p_\beta \gamma(\beta) \, d\beta \, d\alpha
\]

\[
= \int_0^\pi p_\alpha \gamma(\alpha) \times \sigma(\alpha) \, d\alpha
\]

\[
= \int_0^\pi \sigma(\alpha) \times \sigma'(\alpha) \, d\alpha
\]

\[
= \frac{1}{2} \int_0^{2\pi} \sigma(\alpha) \times \sigma'(\alpha) \, d\alpha.
\]

This is the signed area \( \text{Area}(\sigma|_{[0, 2\pi]}) \) spanned by \( \sigma|_{[0, 2\pi]} \). With the isoperimetric inequality for plane paths

\[
\text{(3.12)} \quad \text{Area}(\sigma|_{[0, 2\pi]}) \leq \frac{1}{4\pi} \text{Length}(\sigma|_{[0, 2\pi]})^2 \leq \frac{1}{4\pi} \left( \int_0^{2\pi} p_\alpha \, d\alpha \right)^2.
\]

If \( |\gamma(\alpha)| < 1 \) on a set of positive measure, then inequality (3.12) is strict because \( L(\sigma|_{[0, 2\pi]}) < \int_0^{2\pi} p_\alpha \, d\alpha \) due to \( p_\alpha > 0 \) almost everywhere. So equality holds if and only if \( |\gamma(\alpha)| = 1 \) almost everywhere and \( \sigma \) is a counterclockwise parametrization of the circle with the origin as the center and radius \( r := \frac{1}{2\pi} \int_0^{2\pi} p_\alpha \, d\alpha \). Because \( |\sigma'(\alpha)| = p_\alpha = r\nu'(\alpha) \) for almost all \( \alpha \), it must hold \( \sigma(\alpha) = re^{i\nu(\alpha)+ic} \) for some \( c \in \mathbb{R} \). Then

\[-p_\alpha \gamma(\alpha) = \sigma'(\alpha) = r\nu'(\alpha)e^{i\nu(\alpha)+ic}.
\]
and in turn \( \gamma(\alpha) = -ie^{i\nu(\alpha)+ic} = e^{i\nu(\alpha)+i(c-\frac{\pi}{2})} \) for almost all \( \alpha \) with
\[
|\gamma'(\alpha)| = \nu'(\alpha) = p_\alpha r^{-1} = p_\alpha \left( \frac{1}{2\pi} \int_0^{2\pi} \rho_\beta d\beta \right)^{-1}.
\]

Thus \( \gamma \) is a counterclockwise parametrization of the unit circle with speed given by \( \nu'(\alpha) \). It is now easy to check that any path of the form \( \alpha \mapsto e^{i\nu(\alpha)+ic} \) achieves equality in (3.12).

\[
\square
\]

Such a product structure is in place for the coefficients induced by \( h \in S_+^2 \setminus S^1 \). Indeed,
\[
p_{\alpha,\beta}(h) = \frac{\sin(d(h))^2}{\sin(h_\alpha)^2 \sin(h_\beta)^2} = \frac{\sin(d(h)) \sin(d(h))}{\sin(h_\alpha)^2 \sin(h_\beta)^2} =: p_{\alpha}(h)p_{\beta}(h).
\]

By Lemma 2.10 it follows that \( 1 = \frac{1}{2\pi} \int_0^{2\pi} p_\alpha(h) \, da \). So a maximal \( \gamma \) for \( \omega_h \), which is unique up to rotations of \( \mathbb{R}^2 \), parametrizes the unit circle with \( |\gamma'(\alpha)| = p_\alpha \) and (3.13)
\[
\omega_h(\gamma) = \pi.
\]

The corresponding \( \sigma \) also parametrizes a unit circle.

4. Variation calculation

4.1. Structure of maximizing paths. For this subsection we assume that the measurable coefficient function \( p : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfies
\begin{enumerate}
\item \( p_{\alpha,\beta} > 0 \) almost everywhere,
\item \( p_{\alpha,\beta} = p_{\beta,\alpha} \),
\item \( p_{\alpha,\beta} \) is \( \pi \)-periodic in both arguments,
\item \( p_{\alpha,\beta} \) is (essentially) uniformly bounded.
\end{enumerate}

Properties (2),(3) and (4) are satisfied for \( p_{\alpha,\beta}(f) \) in case \( f \in E(S^1) \setminus S^1 \) by Lemma 3.2. (1) is satisfied too in case \( f \in E^+(S^1) \setminus S^1 \). As in (3.11) these coefficients define an action \( \omega_p \) on \( L_\text{ap}^\infty(\mathbb{R},\mathbb{R}^2) \) or \( L^\infty([0,\pi],\mathbb{R}^2) \) respectively. It is well-defined because the integrand is measurable and uniformly bounded. In order to show the existence of maximizing paths for \( \omega_p \) we first show that the action is weak-star continuous using the identity \( L^\infty([0,\pi]) = L^1([0,\pi])^* \).

Lemma 4.1. With \( p \) and \( \omega_p \) as above let \( (\gamma_n) \) be a sequence in \( L^\infty([0,\pi],\mathbb{R}^2) \) that converge with respect to the weak-star topology to \( \gamma \) (i.e. the coordinate functions of \( \gamma_n \) converge), then
\[
\lim_{n \to \infty} \omega_p(\gamma_n) = \omega_p(\gamma).
\]

Proof. We define the bilinear action
\[
B(\gamma, \delta) := \int_0^\pi \int_\alpha^\pi p_{\alpha,\beta} \gamma(\alpha) \times \delta(\beta) \, d\beta \, da
\]
on \( L^\infty([0,\pi],\mathbb{R}^2) \times L^\infty([0,\pi],\mathbb{R}^2) \) and show that \( B \) is weak-star continuous. So let \( (\gamma_n) \) and \( (\delta_n) \) be two sequences that converge to \( \gamma \) and \( \delta \) respectively with respect to the weak-star topology. In particular, both sequences are bounded in \( L^\infty([0,\pi],\mathbb{R}^2) \). With the bilinearity of \( B \),
\[
B(\gamma_n, \delta_n) - B(\gamma, \delta) = B(\gamma_n - \gamma, \delta) + B(\gamma_n, \delta_n - \delta),
\]

In order to show the existence of maximizing paths for \( \omega_p \) we first show that the action is weak-star continuous using the identity \( L^\infty([0,\pi]) = L^1([0,\pi])^* \).
and thus it is enough to consider the cases $\gamma = 0$ and $\delta = 0$. So let us assume first that $\delta_n \to 0$. Note that
\[
|B(\gamma_n, \delta_n)| = \left| \int_0^\pi \gamma_n(\alpha) \times \mu_n(\alpha) \, d\alpha \right| \leq \int_0^\pi |\gamma_n(\alpha)| \|\mu_n(\alpha)\| \, d\alpha ,
\]
where
\[
\mu_n(\alpha) := \int_0^\pi \chi_{[\alpha, \pi]}(\beta) p_{\alpha, \beta} \delta_n(\beta) \, d\beta .
\]
Now $(\mu_n)$ is a bounded sequence because $p_{\alpha, \beta}$ is bounded and $\mu_n(\alpha) \to 0$ for any $\alpha$ because $\delta_n \to 0$. Since $(\gamma_n)$ and $(\mu_n)$ are bounded sequences it follows that $B(\gamma_n, \delta_n) \to 0$ by the bounded convergence theorem.

In case $\gamma_n \to 0$, Fubini’s theorem implies
\[
B(\gamma_n, \delta_n) = \int_0^\pi \int_0^\beta p_{\alpha, \beta} \gamma_n(\alpha) \times \delta_n(\beta) \, d\alpha \, d\beta = -\int_0^\pi \delta_n(\beta) \times \int_0^\beta p_{\alpha, \beta} \gamma_n(\alpha) \, d\alpha \, d\beta .
\]
The argument now goes through as in the first case.

The direct method in the calculus of variations implies the existence of a maximizer for $\omega_p$.

**Lemma 4.2.** The functional $\gamma \mapsto \omega_p(\gamma)$ has a maximum in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$.

**Proof.** Note that $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ can be identified with those elements $\gamma = (\gamma_x, \gamma_y) \in L^\infty((0, \pi), \mathbb{R}^2)$ that satisfy $\|\gamma_x^2 + \gamma_y^2\|_\infty \leq 1$. It is clear that $\omega_p$ is bounded on $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ with say $S$ as the supremum. Choose a sequence $(\gamma_n)$ in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ with $\lim_{n \to \infty} \omega_p(\gamma_n) = S$. Because $L^1((0, \pi))^2 = L^\infty((0, \pi))$, the Banach–Alaoglu theorem applied to the coordinate functions of $\gamma_n$ guarantees a subsequence of $(\gamma_n)$ that converges to $\gamma \in L^\infty((0, \pi), \mathbb{R}^2)$ in the weak-star topology. Since $\omega_p$ is weak-star continuous by Lemma 4.1 it follows that $\omega_p(\gamma) = S$.

It remains to check that $\gamma$ is actually in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$. To this end, we note that for $v, w \in L^\infty((0, \pi))$,
\[
\|v^2 + w^2\|^{\frac{1}{4}} = \sup_{g,a,b} \int_0^\pi g(t)(a(t)v(t) + b(t)w(t)) \, dt ,
\]
where the supremum is taken over all $g \in L^1([0, \pi])$ and $a, b \in L^\infty([0, \pi])$ with $\|g\|_1 \leq 1$ and $\|a^2 + b^2\|_\infty \leq 1$ respectively. This identity is implied by the Cauchy-Schwarz inequality and the norm representation $\|f\|_\infty = \sup_{\|g\|_1 \leq 1} \int_0^\pi g(t)f(t) \, dt$ for $f \in L^\infty([0, \pi])$. If $(v_n, w_n) \to (v, w)$ in $L^\infty((0, \pi))^2$ and $g, a, b$ are as above, then $ga$ and $gb$ are in $L^1([0, \pi])$ and in turn
\[
\int_0^\pi g(t)(a(t)v(t) + b(t)w(t)) \, dt = \lim_{n \to \infty} \int_0^\pi g(t)(a(t)v_n(t) + b(t)w_n(t)) \, dt \leq \limsup_{n \to \infty} \|v_n^2 + w_n^2\|^{\frac{1}{4}} .
\]
Taking the supremum over all such $g, a, b$ we obtain
\[
\|v^2 + w^2\|^{\frac{1}{4}} \leq \limsup_{n \to \infty} \|v_n^2 + w_n^2\|^{\frac{1}{4}} .
\]
This shows that the maximizer $\gamma$ obtained before is in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ because the approximating sequence $(\gamma_n)$ is.

With a variation argument we show that any maximizer $\gamma$ of $\omega_p$ is contained in the unit circle. Here we use the strict positivity of the coefficients $p_{\alpha, \beta}$.

**Lemma 4.3.** Let $\gamma \in B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ be a maximizer of $\omega_p(\gamma)$. Then $|\gamma(\alpha)| = 1$ for almost all $\alpha$.

**Proof.** We want to show that the set $A = \{\alpha \in S^1 : |\gamma(\alpha)| < 1 \}$ has measure zero. Let $\delta \in L_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ be such that $|\delta(\alpha)| + |\gamma(\alpha)| \leq 1$ for almost all $\alpha$. Then $\gamma + t\delta$ is in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ for $t \in [-1, 1]$ and thus

$$0 \geq \frac{d^2}{dt^2} \bigg|_{t=0} \omega_f(\gamma + t\delta) = \frac{d^2}{dt^2} \bigg|_{t=0} \int_0^\pi \int_0^\pi p_{\alpha, \beta} (\gamma + t\delta)(\alpha) \times (\gamma + t\delta)(\beta) \, d\beta \, d\alpha = \int_0^\pi \int_0^\pi p_{\alpha, \beta} \delta(\alpha) \times \delta(\beta) \, d\beta \, d\alpha .$$

Because $p_{\alpha, \beta} > 0$ almost everywhere, we see by varying $\delta$ on $A$ that this set has measure zero. For example one can take $\delta(\alpha) = \frac{1}{\pi} e^{i\alpha}$ on the set $A_\alpha := \{\alpha \in [0, \pi) : |\gamma(\alpha)| \leq 1 - \frac{1}{\pi}\}$ and $\delta(\alpha) = 0$ on $[0, \pi) \setminus A_\alpha$, to conclude that $\mathcal{L}^2(A_\alpha) = 0$. □

We can extract more information from variations of $\gamma$.

**Lemma 4.4.** Let $\gamma \in B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ be a maximizer of $\omega_p$ and set

$$\mu(\alpha) := \int_\alpha^{\alpha+\pi} p_{\alpha, \beta} \gamma(\beta) \, d\beta .$$

Then $\mu \in L_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$, $\mu(\alpha)$ is orthogonal to $\gamma(\alpha)$ and $\gamma(\alpha) \times \mu(\alpha) \geq 0$ for almost every $\alpha$.

**Proof.** Because $p$ is bounded and since $p_{\alpha+\pi, \beta+\pi} \gamma(\beta + \pi) = -p_{\alpha, \beta} \gamma(\beta)$ it is clear that $\mu \in L_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$.

By Lemma 4.3 we can assume that $|\gamma(\alpha)| = 1$ almost everywhere and thus $\gamma(\alpha) = e^{i\eta(\alpha)}$ for some measurable function $\eta : \mathbb{R} \to \mathbb{R}$ with $\eta(\alpha + \pi) = \eta(\alpha) + \pi$ for all $\alpha$. We can further assume that $\eta(\alpha) \in (-\pi, \pi]$ if $\alpha \in [0, \pi)$. Let $\delta : \mathbb{R} \to \mathbb{R}$ be a $\pi$-periodic measurable function and consider the variation $\gamma_t := e^{i\eta t}$ in $B_{ap}^\infty(\mathbb{R}, \mathbb{R}^2)$ given by $\eta_t := \eta + t\delta$. Set

$$v(\alpha) := \left. \frac{d}{dt} \right|_{t=0} \gamma_t(\alpha) = i\delta(\alpha) e^{i\eta(\alpha)} ,$$

$$w(\alpha) := \left. \frac{d^2}{dt^2} \right|_{t=0} \gamma_t(\alpha) = -\delta(\alpha)^2 e^{i\eta(\alpha)} .$$
By the maximality of $\gamma$,
\[
0 = \frac{d}{dt} \bigg|_{t=0} \omega_p(\gamma_t) = \int_0^\pi \int_0^\pi p_{\alpha,\beta} [v(\alpha) \times \gamma(\beta) + \gamma(\alpha) \times v(\beta)] \, d\beta \, d\alpha \\
= \int_0^\pi v(\alpha) \times \int_0^\pi p_{\alpha,\beta} \gamma(\beta) \, d\beta \, d\alpha - \int_0^\pi v(\beta) \times \int_0^\pi p_{\alpha,\beta} \gamma(\alpha) \, d\beta \, d\alpha \\
= \int_0^\pi v(\alpha) \times \int_0^\pi p_{\alpha,\beta} \gamma(\beta) \, d\beta \, d\alpha - \int_0^\pi v(\alpha) \times \int_0^\pi p_{\alpha,\beta} \gamma(\beta) \, d\beta \, d\alpha \\
= \int_0^\pi v(\alpha) \times \mu(\alpha) \, d\alpha \\
= \int_0^\pi \delta(\alpha) (i\gamma(\alpha)) \times \mu(\alpha) \, d\alpha .
\]

Since this holds for all $\delta$ it follows that $\gamma$ and $\mu$ are orthogonal almost everywhere. This establishes the first statement.

The second variation satisfies
\[
0 \geq \frac{d^2}{dt^2} \bigg|_{t=0} \omega_p(\gamma_t) \\
= \int_0^\pi \int_0^\pi p_{\alpha,\beta} [\gamma(\alpha) \times w(\beta) + w(\alpha) \times \gamma(\beta) + 2v(\alpha) \times v(\beta)] \, d\beta \, d\alpha \\
= \int_0^\pi \int_0^\pi p_{\alpha,\beta} \left[ (-\delta(\alpha)^2 - \delta(\beta)^2) \gamma(\alpha) \times \gamma(\beta) + 2\delta(\alpha)\delta(\beta) \gamma(\alpha) \times \gamma(\beta) \right] \, d\beta \, d\alpha \\
= -\int_0^\pi \int_0^\pi p_{\alpha,\beta} (\delta(\alpha) - \delta(\beta))^2 \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha .
\]

Fix $0 < a < b < \pi$ and let $\delta$ be the $\pi$-periodic extension of $\chi_{[a,b]}$, then
\[
0 \leq \int_a^b \int_a^b p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha + \int_a^b \int_b^\pi p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha \\
= \int_a^b \int_a^b p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha + \int_a^b \int_b^\pi p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha \\
= \int_a^b \int_a^\pi p_{\alpha,\beta} \gamma(\beta) \times \gamma(\alpha) \, d\beta \, d\alpha + \int_a^b \int_b^\pi p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha \\
= -\int_a^b \int_a^\pi p_{\alpha,\beta} \gamma(\beta) \times \gamma(\alpha) \, d\beta \, d\alpha + \int_a^b \int_b^\pi p_{\alpha,\beta} \gamma(\alpha) \times \gamma(\beta) \, d\beta \, d\alpha \\
= \int_a^b \gamma(\alpha) \times \int_b^\pi p_{\alpha,\beta} \gamma(\beta) \, d\beta \, d\alpha .
\]
The inner integral in the last line we denote by $\mu_{a,b}(\alpha)$ for $\alpha \in [a,b]$. It follows
\[
|\mu_{a,b}(\alpha) - \mu(\alpha)| \leq \int_a^b p_{\alpha,\beta} \gamma(\beta) \, d\beta + \int_a^{a+\pi} p_{\alpha,\beta} \gamma(\beta) \, d\beta \leq C|a - b| ,
\]
where $p_{\alpha,\beta} \leq C$ is a uniform bound. Thus
\[
\int_a^b \gamma(\alpha) \times \mu(\alpha) \, d\alpha \geq \int_a^b \gamma(\alpha) \times \mu_{a,b}(\alpha) \, d\alpha - \int_a^b C|a - b| \, d\alpha \geq -C|a - b|^2 .
\]
If \( \alpha \in (0, \pi) \) is a density point of \( \gamma \times \mu \), \( a = \alpha - \varepsilon \), \( b = \alpha + \varepsilon \) and both sides are divided by \( 2\varepsilon \), then \( \gamma(\alpha) \times \mu(\alpha) \geq 0 \) is established in the limit for \( \varepsilon \to 0 \) as an application of the Lebesgue differentiation theorem. \( \square \)

By Lemma 4.2 there is a maximum \( \gamma \) of \( \omega_p \). By Lemma 4.3 and Lemma 4.4 this maximizer satisfies \( |\gamma(\alpha)| = 1 \) and \( \mu(\alpha) = |\mu(\alpha)|\tau(\alpha) \) for almost all \( \alpha \). With additional assumptions we obtain that this maximizer \( \gamma \) has a continuous representation.

**Lemma 4.5.** Assume that \( (\alpha, \beta) \mapsto p_{\alpha, \beta} \) is additionally continuous on \( \{(\alpha, \beta) \in (0, \pi)^2 : \alpha < \beta\} \). Then \( \mu \) is continuous and a maximizer \( \gamma \in B^p_{\text{per}}(\mathbb{R}, \mathbb{R}^2) \) of \( \omega_p \) has a continuous representation on \( F = \{ \alpha \in \mathbb{R} : \mu(\alpha) \neq 0 \} \).

**Proof.** The path \( \mu \) is continuous as a consequence of the continuity of \( p \) and the Lebesgue dominated convergence theorem. Hence \( F \) is open. For almost all \( \alpha \in F \) it then holds that

\[
-i \frac{\mu(\alpha)}{|\mu(\alpha)|} = \gamma(\alpha)
\]

by Lemma 4.4. The left hand side is continuous on \( F \), so \( \gamma \) has a continuous representation on \( F \) as well. \( \square \)

### 4.2. Variation of paths and coefficients.

Fix some \( h \in S^2 \setminus S^1 \). Then \( h_\alpha = \arccos(\cos(d)\cos(\alpha - \tau)) \) for \( \alpha, \tau \in \mathbb{R} \) and \( d \in (0, \frac{\pi}{2}] \), where \( d = \text{dist}(h, S^1) \geq r \) for some \( r \in (0, \frac{\pi}{2}) \). We know from Lemma 3.1 that \( p_{\alpha, \beta}(h) = p_\alpha(h)p_\beta(h) \) with

\[
p_\alpha(h) = \frac{\sin(d)}{\sin(h_\alpha)} = \frac{\sin(d)}{1 - \cos(d)\cos(\alpha - \tau)} \geq \sin(d)^{-1} \geq m
\]

for \( m = \sin(r)^{-1} > 0 \) and all \( \alpha \). These coefficients satisfy \( \int_0^{2\pi} p_\alpha(h) \, d\alpha = 2\pi \) by Lemma 2.10. As in Lemma 3.9, we assume that \( \nu_h : \mathbb{R} \to \mathbb{R} \) is the unique Lipschitz function that satisfies \( \nu_\alpha(0) = 0 \) and \( \nu'_\alpha(\alpha) = p_\alpha(h) \). Since \( p_\alpha(h) \) is \( \pi \)-periodic, it follows that \( \nu_\alpha(\alpha + \pi) = \nu_\alpha(\alpha) + \pi \). The path \( \gamma(\alpha) = e^{i\nu_\alpha(\alpha)} \) is a maximizer of \( \omega_h \) and

\[
\sin(\nu_\beta(\beta) - \nu_\alpha(\alpha)) = \gamma(\alpha) \times \gamma(\beta) = \frac{\sin(d)\sin(\beta - \alpha)}{\sin(h_\alpha)\sin(h_\beta)}.
\]

due to Lemma 2.10 and Lemma 3.9.

With \( L^p_0(\mathbb{R}) \) we denote the space of \( \pi \)-periodic functions \( \eta \in L^2_{\text{loc}}(\mathbb{R}) \) that satisfy \( \int_0^{\pi} \eta(t) \, dt = 0 \). For convenience sake, this space is equipped with the scalar product \( \langle \eta_1, \eta_2 \rangle := \pi \int_0^{\pi} \eta_1(t)\eta_2(t) \, dt \). The induced norm has the double integral expression

\[
\int_0^{\pi} \int_0^{\pi} (\eta(\beta) - \eta(\alpha))^2 \, d\beta \, d\alpha = \frac{1}{2} \int_0^{\pi} \int_0^{\pi} \eta(\beta)^2 + 2\eta(\alpha)\eta(\beta) + \eta(\alpha)^2 \, d\beta \, d\alpha
\]

\[
= \int_0^{\pi} \int_0^{\pi} \eta(\beta)^2 \, d\beta \, d\alpha
\]

\[
= \int_0^{\pi} \frac{1}{\pi} \|\eta\|^2_2 \, d\alpha = \|\eta\|^2_2
\]

(4.3)
for $\eta \in L^2_0(\mathbb{R})$. Because $\eta$ is $\pi$-periodic the following shifted identity holds for all $x \in \mathbb{R}$,
\[
\int_0^\pi \int_0^\pi (\eta(\beta + x) - \eta(\alpha))^2 \, d\beta \, d\alpha = \int_0^\pi \int_0^{\pi + x} (\eta(\beta) - \eta(\alpha))^2 \, d\beta \, d\alpha
\]
\[
= \int_0^\pi \int_0^\pi (\eta(\beta) - \eta(\alpha))^2 \, d\beta \, d\alpha
\]
\[
= 2\|\eta\|_2^2. \tag{4.4}
\]

Let $\Psi_h : E(S^1) \setminus S^1 \times L^2_0(\mathbb{R}) \to \mathbb{R}$ be defined by
\[
\Psi_h(f, \eta) := \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^v) \, d\beta \, d\alpha,
\]
where $\Delta_{\alpha, \beta}^h := \nu_h(\beta) - \nu_h(\alpha)$ and $\Delta_{\alpha, \beta}^v := \eta(\beta) - \eta(\alpha)$ for $\eta \in L^2_0(\mathbb{R})$.

**Lemma 4.6.** For fixed $h \in S^2_\mathbb{R} \setminus S^1$, $f \in E(S^1) \setminus S^1$ and $\eta, v \in L^2_0(\mathbb{R})$, the function $t \mapsto \Psi_h(f, \eta + tv)$ is in $C^2$ and has derivatives
\[
\frac{d}{dt}\Psi_h(f, \eta + tv) = \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \cos(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^{\eta + tv}) \Delta_{\alpha, \beta}^v \, d\beta \, d\alpha,
\]
\[
\frac{d^2}{dt^2}\Psi_h(f, \eta + tv) = -\int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^{\eta + tv})(\Delta_{\alpha, \beta}^v)^2 \, d\beta \, d\alpha,
\]
for $\eta, v \in L^2_0(\mathbb{R})$.

**Proof.** $p_{\alpha, \beta}(f)$ is uniformly bounded by Lemma 3.2(4), and $t \mapsto \Delta_{\alpha, \beta}^{\eta + tv}$ is smooth with
\[
\frac{1}{|s|} \left| \Delta_{\alpha, \beta}^{\eta + (t+s)v} - \Delta_{\alpha, \beta}^{\eta + tv} \right| \leq |v(\beta) - v(\alpha)|.
\]
Thus $(\alpha, \beta) \mapsto p_{\alpha, \beta}(f)|v(\beta) - v(\alpha)|$ is integrable by (4.3), and in turn $t \mapsto \Psi_h(f, \eta + tv)$ is differentiable with derivative
\[
\frac{d}{dt}\Psi_h(f, \eta + tv) = \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \frac{d}{dt}\sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^{\eta + tv}) \, d\beta \, d\alpha
\]
\[
= \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \cos(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^v) \Delta_{\alpha, \beta}^v \, d\beta \, d\alpha
\]
as a consequence of the dominated convergence theorem. The second derivative is calculated alike. Finally, $t \mapsto \frac{d^2}{dt^2}\Psi_h(f, \eta + tv)$ is continuous again by the dominated convergence theorem.

The following lemma is used to bound the second derivatives above.

**Lemma 4.7.** Let $h \in S^2_\mathbb{R} \cap E_r(S^1)$ for some $r \in (0, \frac{\pi}{2})$. Then there exist $\varepsilon(r), c(r) > 0$ such that if $\eta \in L^2_0(\mathbb{R})$ and $f \in E(S^1)$ satisfy $\max\{\|f - h\|_\infty, \|\eta\|_\infty\} \leq \varepsilon$, then
\[
\int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^v)(\Delta_{\alpha, \beta}^v)^2 \, d\beta \, d\alpha \geq c\|v\|_2^2
\]
for all $v \in L^2_0(\mathbb{R})$.

**Proof.** Assume $h_{\alpha} = \arccos(\cos(d) \cos(\alpha - \tau))$ for some $d \geq r$ and $\tau \in \mathbb{R}$. We define
\[
D := \{(\alpha, \beta) \in [0, \pi]^2 \setminus \{(0, \pi)\} : \alpha < \beta\},
\]
\[
A_\delta := \{ (\alpha, \beta) \in D : \inf_{k \in \mathbb{Z}} |\beta - \alpha + \pi k| < \delta \}, \tag{4.5}
\]
for \( \delta \in (0, \frac{\pi}{2}) \). Note that \( D \setminus A_\delta \) is compact and \( \mathcal{L}^2(A_\delta) = \pi \delta \). Before we begin we need an \( L^2 \) estimate of \( v \) over \( A_\delta \). With \((x + y)^2 \leq 2(x^2 + y^2)\) and (4.4) it follows

\[
\int_{A_\delta} (\Delta v_{\alpha, \beta})^2 \, d\beta \, d\alpha = \int_0^\pi \int_0^{\alpha+\delta} (v(\beta) - v(\alpha))^2 \, d\beta \, d\alpha \\
= \int_0^\pi \int_0^\delta (v(\alpha + x) - v(\alpha))^2 \, dx \, d\alpha \\
= \frac{1}{\pi} \int_0^\pi \int_0^\pi \int_0^\delta ((v(\alpha + x) - v(\zeta)) + (v(\zeta) - v(\alpha))^2 \, dx \, d\alpha \\
\leq \frac{2}{\pi} \int_0^\delta \int_0^\pi \int_0^\pi (v(\alpha + x) - v(\zeta))^2 + (v(\alpha) - v(\zeta))^2 \, d\alpha \, d\zeta \, dx \\
= \frac{8}{\pi} \int_0^\delta \|v\|_2^2 \, dx \\
\leq \frac{8}{\pi} \delta \|v\|_2^2.
\]

(4.6)

In the first line, the corner integral with \( \pi - \delta \leq \beta \leq \pi \) and \( 0 \leq \alpha \leq \delta + \beta - \pi \) is replaced by the integral over the corner with \( \pi \leq \beta \leq \alpha + \delta \) and \( \pi - \delta \leq \alpha \leq \pi \) using the isometry \((\alpha, \beta) \mapsto (\beta, \alpha + \pi)\) which preserves \((v(\beta) - v(\alpha))^2\) due to the periodicity of \( v \).

The following statements are true for \( m(r), M(r) > 0 \):

1. \( p_{\alpha, \beta}(h) \geq m^2 > 0 \) for all \((\alpha, \beta) \in D\) due to (4.1).
2. \((\alpha, \beta, f) \mapsto p_{\alpha, \beta}(f)\) is continuous on \( D \times B(h, \frac{\pi}{2})\) and bounded by \( M > 0 \) on \( B(h, \frac{\pi}{2})\) due to Lemma 3.2(4).
3. \( \nu_h : \mathbb{R} \to \mathbb{R} \) is an increasing and bi-Lipschitz with \( m|\beta - \alpha| \leq |\nu_h(\beta) - \nu_h(\alpha)|\) for all \( \alpha, \beta \in \mathbb{R} \) due to the lower bound \( \nu_h'(\alpha) = p_{\alpha}(h) \geq m \).

Accordingly, for any \( \delta \in (0, \frac{\pi}{2}) \) there exists \( \varepsilon(\delta, r) \in (0, \min(\frac{\pi}{2}, \delta)) \) such that for all \( f \in E(S^1) \) with \( \|f - h\|_\infty \leq \varepsilon \) it holds:

(a) \( p_{\alpha, \beta}(f) \geq p_{\alpha, \beta}(h) - \delta \) if \((\alpha, \beta) \in D \setminus A_\delta\),
(b) \( \max_{(\alpha, \beta) \in D} p_{\alpha, \beta}(f) \leq M \),
(c) \( \sin(\Delta_{\alpha, \beta}^h) \geq 2\varepsilon \) if \((\alpha, \beta) \in D \setminus A_\delta\).

As a consequence, if \( \|\eta\|_\infty \leq \varepsilon \), then

\[
\sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^\eta) \geq \sin(\Delta_{\alpha, \beta}^h) - 2\varepsilon \geq 0
\]
for all \((\alpha, \beta) \in D \setminus A_3\). For \(f, \eta\) with \(\max \{\|f - h\|_\infty, \|\eta\|_\infty\} \leq \varepsilon\) it follows from \(p_{\alpha, \beta}(f) \geq 0\) by Lemma 3.1, \(\sin(\Delta^h_{\alpha, \beta}) \geq 0\) by (3), (4.6), (a), (b) and (c) that

\[
\int_D p_{\alpha, \beta}(f) \sin(\Delta^h_{\alpha, \beta} + \Delta^\eta_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 \\
\geq \int_{D \setminus A_3} (p_{\alpha, \beta}(h) - \delta)(\sin(\Delta^h_{\alpha, \beta}) - 2\varepsilon)(\Delta^\nu_{\alpha, \beta})^2 - \int_{A_3} M(\Delta^\nu_{\alpha, \beta})^2 \\
\geq \int_{D \setminus A_3} p_{\alpha, \beta}(h) \sin(\Delta^h_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 - \delta(1 + 2M) \int_{D \setminus A_3} (\Delta^\nu_{\alpha, \beta})^2 - M \int_{A_3} (\Delta^\nu_{\alpha, \beta})^2 \\
\geq \int_D p_{\alpha, \beta}(h) \sin(\Delta^h_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 - \delta(1 + 2M) \int_D (\Delta^\nu_{\alpha, \beta})^2 - (M + 1) \int_{A_3} (\Delta^\nu_{\alpha, \beta})^2 \\
\geq \int_D p_{\alpha, \beta}(h) \sin(\Delta^h_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 - C\delta\|v\|_2^2
\]

for some \(C(r) > 0\). Because \(v_h\) is bi-Lipschitz by (3) with the given lower Lipschitz constant \(m\) and \(v_h(\alpha + \pi) = v_h(\alpha) + \pi\) for all \(\alpha\), it holds

\[
\sin(v_h(\beta) - v_h(\alpha)) \geq \frac{2}{\pi} am,
\]

whenever \((\alpha, \beta) \in D \setminus A_a\) for \(a \in (0, \frac{\pi}{r})\). Thus for \(a \in (0, \frac{\pi}{r})\) with (4.3) and (4.6),

\[
\int_D p_{\alpha, \beta}(h) \sin(\Delta^h_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 \geq m^2 \int_{D \setminus A_3} \sin(\Delta^h_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 \\
\geq \frac{2}{\pi} am^3 \int_{D \setminus A_3} (\Delta^\nu_{\alpha, \beta})^2 \\
\geq \frac{2}{\pi} am^3 (1 - \frac{8}{\pi} a)\|v\|_2^2.
\]

Choosing \(a = \frac{\pi}{32}\) and \(\delta(r) > 0\) small enough such that \(C\delta \leq \frac{1}{16m^3}\), it follows

\[
\int_D p_{\alpha, \beta}(f) \sin(\Delta^h_{\alpha, \beta} + \Delta^\eta_{\alpha, \beta}) (\Delta^\nu_{\alpha, \beta})^2 \geq \frac{1}{16m^3}\|v\|_2^2
\]

whenever \(\max \{\|f - h\|_\infty, \|\eta\|_\infty\} \leq \varepsilon(\delta, r)\) and \(v \in L^2_0(\mathbb{R})\) as claimed. \(
\square
\)

For \(f \in E(S^1) \setminus S^1\) and \(\gamma \in B^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)\), the path \(\mu_{f, \gamma} \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)\) is defined by

(4.7) \[
\mu_{f, \gamma}(\alpha) := \int_{-\pi}^{\alpha + \pi} q_{\alpha, \beta}(f)\gamma(\beta) \, d\beta,
\]

where \(q\) is defined in (3.1). Assume that \(h \in S^2_{2^3} \setminus S^1\) is represented by \(h_\alpha = \arccos(\cos(d) \cos(\alpha - \tau))\) for parameters \(\tau \in \mathbb{R}\) and \(d \in (0, \frac{\pi}{4})\). Recall that

\[
\nu_\alpha(h) = p_{\alpha}(h) = \frac{\sin(d)\sin(h_\alpha)}{\sin(h_\alpha)^2}.
\]

To each \(\pi\)-periodic and measurable function \(\eta : \mathbb{R} \rightarrow \mathbb{R}\) we associate the path \(\gamma_\eta \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)\) defined by \(\gamma_\eta := e^{i(\nu_\eta(\alpha) + \eta(\alpha))}\). For \(\gamma_0(\alpha) = e^{i\nu_\alpha(\alpha)}\) it holds

\[
\mu_{h, \gamma_0}(\alpha) = \int_{-\pi}^{\alpha + \pi} \frac{\sin(d)^2}{\sin(h_\beta)^2} \gamma_0(\beta) \, d\beta = -i \sin(d) \int_{-\pi}^{\alpha + \pi} i\nu_\beta(\alpha) e^{i\nu_\beta(\beta)} \, d\beta
\]

(4.8) \[
= -i \sin(d)(e^{i\nu_\beta(\alpha + \pi)} - e^{i\nu_\beta(\alpha)}) = 2i \sin(d) e^{i\nu_\beta(\alpha)}
\]
and $\omega_h(\gamma_0) = \pi$ by (3.13). Define

$$\sigma_\eta(\alpha) := \frac{1}{2} \int_0^{\alpha + \pi} p_\beta(h) \gamma_\eta(\beta) \, d\beta.$$  \hspace{1cm} (4.9)

All the paths $\mu_{f,\gamma}, \gamma_\eta$ and $\sigma_\eta$ are elements of $L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$. Moreover, $\sigma_\eta$ is Lipschitz and has length $\int_0^{2\pi} p_\alpha(h) \, d\alpha = 2\pi$, when restricted to $[0, 2\pi]$, by Lemma 2.10 since $|\sigma_\eta'(\alpha)| = |p_\alpha(h)\gamma_\eta(\alpha)| = p_\alpha(h)$ for almost all $\alpha$. Both paths $\gamma_\eta$ and $\sigma_\eta$ depend on $h \in S^2_\alpha \setminus S^1$ but the reference should clear from the context.

As a next step we show that the maps $f \mapsto \omega_f(\gamma)$ and $f \mapsto \mu_{f,\gamma}$ are Hölder continuous for fixed $\gamma \in B^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$. This also applies to $f \mapsto \Psi_h(f, \eta) = \omega_f(\gamma_\eta)$ for a given $\eta$.

**Lemma 4.8.** For any $\xi \in (0, 1)$ and $r \in (0, \frac{\pi}{2})$ there exists $H(\xi, r) > 0$ such that

$$\sup_{\alpha \in [0, \pi]} |\omega_f(\gamma) - \omega_g(\gamma)| \leq H \|f - g\|_{\infty}^\xi,$$

$$|\mu_{f,\gamma} - \mu_{g,\gamma}| \leq H \|f - g\|_{\infty}^\xi,$$

whenever $\gamma \in B^\infty_{ap}(\mathbb{R}, \mathbb{R}^2)$ and $f, g \in E_r(S^1)$. $\|\omega_f\|_{\infty}$ is characterized by (3.9).

**Proof.** Throughout the proof we fix $f, g \in E_r(S^1)$ and define $\phi^t := (1 - t) f + tg$ for $t \in [0, 1]$. The sets $D$ and $A_c$ for $c \in (0, \frac{\pi}{2})$ are defined as in (4.5).

From Lemma 3.2(4), Lemma 3.4 and Lemma 3.5 it follows that $p(\beta - \alpha, \phi^t_\alpha, \phi^t_\beta)$ is smooth in $t$ and both the function and its first derivative are bounded by a constant $M(r) > 0$. By Lemma 3.5 we can further assume that $M$ is large enough such that

$$|p_t p(\beta - \alpha, \phi^t_\alpha, \phi^t_\beta)| \leq \frac{M}{\sin(\beta - \alpha)} \|f - g\|_{\infty}$$

holds for all $t \in [0, 1]$ and all $(\alpha, \beta) \in D$.

Because all the terms we want to estimate are uniformly bounded for functions in $E_r(S^1)$ we can assume without loss of generality that $0 < \delta := \|f - g\|_{\infty}^{\xi} < \frac{\pi}{2}$. Since $\mathcal{L}^2(A_c) = \pi c$ and $\frac{\pi}{2} \sin(\beta - \alpha) \geq c$ for $(\alpha, \beta) \in D \setminus A_c$ the dominated convergence theorem implies

$$|\omega_f(\gamma) - \omega_g(\gamma)| \leq \int_D |p_{\alpha,\beta}(f) - p_{\alpha,\beta}(g)|$$

$$\leq \int_D \sup_{t \in [0, 1]} |p_t p(\beta - \alpha, \phi^t_\alpha, \phi^t_\beta)|$$

$$\leq \int_{A_c} M + \int_{D \setminus A_c} \frac{M}{\sin(\beta - \alpha)} \|f - g\|_{\infty}$$

$$\leq M \pi \delta + M \|f - g\|_{\infty}^\xi \int_{D \setminus A_c} \frac{1}{\sin(\beta - \alpha)} \|f - g\|_{\infty}^{\xi - \xi}$$

$$\leq \|f - g\|_{\infty}^\xi \left( M \pi + M \left( \frac{\pi}{2} \right)^{\xi - 1} \int_D \sin(\beta - \alpha)^{\xi - 2} \right).$$
In the last line it is used that
\[ \|f - g\|^1_\infty = \delta^{\frac{1}{2}} < \left( \frac{\pi}{2} \right)^{\frac{1}{2} - 1} \sin(\beta - \alpha)^{\frac{1}{2} - 1} \]
for \((\alpha, \beta) \in D \setminus A_3\). The statement follows if the integral above is bounded. For this we can assume that \(\xi > \frac{1}{2}\) because the case \(\xi \leq \frac{1}{2}\) is trivial. The boundedness follows from \(\frac{1}{2} - 2 > -1\) since
\[ \int_D \sin(\beta - \alpha)^{\frac{1}{2} - 2} = \frac{1}{2} \int_0^\pi \int_0^{\alpha + \pi} |\sin(\beta - \alpha)|^{\frac{1}{2} - 2} d\beta d\alpha 
\leq \int_0^\pi \int_0^{\frac{\pi}{2}} \sin(t)^{\frac{1}{2} - 2} dt d\alpha 
\leq \pi \int_0^{\frac{\pi}{2}} (\frac{2}{\alpha})^{\frac{1}{2} - 2} dt < \infty. \]
This proves the first two estimates in the statement for some \(H(\xi, r) > 0\).

For fixed \(f\) and \(g\) as above there exists \(\gamma \in B_\infty(\mathbb{R}, \mathbb{R}^2)\) such that \(\|\omega_f\|_r = \omega_f(\gamma)\) by Lemma 4.2. Thus
\[ \|\omega_f\|_r = \omega_f(\gamma) + H\|f - g\|_\infty \leq \|\omega_g\|_r + H\|f - g\|_\infty. \]
By changing the roles of \(f\) and \(g\), the third estimate follows.

Next we check that the paths \(\mu_{f, \gamma}\) as defined in (4.7) are continuous in the first argument. The coefficients in the definition of \(\mu_{f, \gamma}\) are given by
\[ q_{\alpha, \beta}(f) = \sin(f_\alpha)^2 p_{\alpha, \beta}(f) \]
according to (3.1). Inherited from \(p_{\alpha, \beta}(\phi^t)\) also \(q_{\alpha, \beta}(\phi^t)\) is smooth in \(t\) and there exists \(C(r) > 0\), such that
\[ \max\{|q_{\alpha, \beta}(\phi^t)|, |\partial_t q_{\alpha, \beta}(\phi^t)|\} \leq C \]
and
\[ |\partial_t q_{\alpha, \beta}(\phi^t)| \leq \frac{C}{|\sin(\beta - \alpha)|} \|g - f\|_\infty \]
if \(\alpha \neq \beta \mod \pi\). For \(\alpha \in \mathbb{R}\) and \(c \in (0, \frac{\pi}{2})\) set \(A_{\alpha, c} := (\alpha, \alpha + c) \cup (\alpha + \pi - c, \pi)\). Since \(L(A_{\alpha, c}) = 2c\) and \(\frac{\pi}{2} \sin(\beta - \alpha) \geq c\) for \(\beta \in (0, \pi) \setminus A_{\alpha, c}\), the dominated convergence theorem implies
\[ |\mu_{f, \gamma}(\alpha) - \mu_{g, \gamma}(\alpha)| \leq \int_\alpha^{\alpha + \pi} \sup_{t \in [0, 1]} |\partial_t q_{\alpha, \beta}(\phi^t)| 
\leq \int_{A_{\alpha, c}} C + \int_{(\alpha, \alpha + \pi) \setminus A_{\alpha, c}} \frac{C}{\sin(\beta - \alpha)} \|f - g\|_\infty 
\leq 2C\delta + C\|f - g\|_\infty \int_{(\alpha, \alpha + \pi) \setminus A_{\alpha, c}} \frac{1}{\sin(\beta - \alpha)} \|f - g\|_\infty^{1 - \xi} 
\leq \|f - g\|_\infty \left( 2C + C \left( \frac{\pi}{2} \right)^{\frac{1}{2} - 1} \int_\alpha^{\alpha + \pi} \frac{\|f - g\|_\infty^{1 - \xi}}{\sin(\beta - \alpha)^{\frac{1}{2} - 2}} \right). \]
In the last inequality the same estimate as for \(\omega\) before is used. Again we can assume that \(\xi > \frac{1}{2}\) and the boundedness of the integral follows by essentially the
same reason as before,
\[
\int_{\alpha}^{\alpha + \pi} \sin(\beta - \alpha)^{\frac{1}{2} - 2} d\beta = 2 \int_{0}^{\frac{\pi}{2}} \sin(t)^{\frac{1}{2} - 2} dt \\
\leq 2 \int_{0}^{\frac{\pi}{2}} (\frac{2t}{\pi})^{\frac{1}{2} - 2} dt < \infty.
\]

This concludes the proof. \(\square\)

For the following statement we use a stability result for the plane isoperimetric inequality due to Fuglede.

**Lemma 4.9.** Assume that \(\xi \in (0, 1)\) and \(h \in S_{+}^{2} \cap E_{r}(S^{1})\) for some \(r \in (0, \frac{\pi}{2})\). Then there exists \(C(\xi, r) > 0\) with the property that for all \(f \in E_{\pi}(S^{1})\) and all measurable \(\pi\)-periodic \(\eta : \mathbb{R} \to \mathbb{R}\) there exists \(\lambda(f, \eta) \in S^{1} \subset \mathbb{C}\) such that
\[
\|\mu_{f, \gamma_{\eta}} - \lambda \mu_{h, \gamma_{\eta}}\|_{\infty} \leq C \left( \|f - h\|_{\infty} + |\Psi_{h}(h, 0) - \Psi_{h}(f, \eta)|^{\frac{1}{2}} \right).
\]

**Proof.** Assume that \(h_{\alpha} = \arccos(\cos(d) \cos(\alpha - \tau))\) for parameters \(d \in [r, \frac{\pi}{2}]\) and \(\tau \in \mathbb{R}\) and let \(f \in E_{\pi}(S^{1})\). In Lemma 3.9 we realized that the signed area spanned by \(\sigma_{\eta}|_{[0, 2\pi]}\), as defined in (4.9), is given by
\[
(4.10) \quad A_{\eta} := \Psi_{h}(h, \eta) = \frac{1}{2} \int_{0}^{2\pi} \sigma_{\eta}(\alpha) \times \sigma'_{\eta}(\alpha) d\alpha.
\]

Clearly, \(\sigma'_{\eta}(\alpha) = -p_{\alpha}(h) \gamma_{\eta}(\alpha)\) and therefore also \(|\sigma'_{\eta}(\alpha)| = p_{\alpha}(h)\) for almost all \(\alpha\). In particular, the length of \(\sigma_{\eta}|_{[0, 2\pi]}\) is \(\int_{0}^{2\pi} p_{\alpha}(h) d\alpha = 2\pi\) and thus \(|A_{\eta}| \leq \pi\) by the plane isoperimetric inequality. The inverse function \(g = \nu_{h}^{-1} : \mathbb{R} \to \mathbb{R}\) is strictly increasing and satisfies, as \(\nu_{h}\) does, \(g(t + \pi) = g(t) + \pi\) for all \(t\). \(\tilde{\eta}(t) := \sigma_{\eta}(g(t))\) is parametrized by arc length because
\[
1 = \nu_{h}(g(t)) g'(t) = p_{g(t)}(h) g'(t) = |\sigma'_{\eta}(g(t)) g'(t)| = |\tilde{\sigma}'_{\eta}(t)|
\]
for almost all \(t\), and in turn can be written as
\[
\tilde{\sigma}_{\eta}(t) = \frac{1}{2} \int_{g(t)}^{g(t + \pi)} p_{\beta}(h) \gamma_{\eta}(\beta) d\beta
\]
\[
= \frac{1}{2} \int_{t}^{t+\pi} p_{g(s)}(h) \gamma_{\eta}(g(s)) g'(s) ds
\]
\[
= \frac{1}{2} \int_{t}^{t+\pi} e^{i(s + \eta(g(s)))} ds
\]
with derivative
\[
\tilde{\sigma}'_{\eta}(t) = -e^{i(t + \eta(g(t)))}
\]
for almost all \(t\). The stability result of Fuglede is formulated with respect to the dissimilarity function
\[
w(t) := c_{0}(\eta) + c_{1}(\eta)e^{it} - \tilde{\sigma}_{\eta}(t),
\]
where
\[
c_{n}(\eta) := \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{\sigma}_{\eta}(t) e^{-int} dt
\]
is the $n$th Fourier coefficient of $\tilde{\sigma}_\eta$. The coefficients of interest satisfy $c_0(\eta) = 0$ by symmetry and $|c_1(\eta)| \leq 1$ because
\[
1 = \frac{1}{2\pi} \int_0^{2\pi} |\tilde{\sigma}'_\eta(t)|^2 \, dt = \sum_{n \in \mathbb{Z}} n^2 |c_n(\eta)|^2.
\]
It holds
\[
\tilde{\sigma}_0(t) = \frac{1}{2} \int_t^{t+\pi} e^{is} \, ds = \frac{1}{2}(-ie^{i(s+\pi)} - (-ie^{is})) = ie^{is},
\]
so that $c_1(0) = i$. Due to a stability result of Fuglede [11, §1],
\[
(4.11) \quad \|w\|_\infty \leq 5\pi(\pi - A_\eta),
\]
and in turn because $\sigma_0(t) = ic_0(t) = ie^{i\nu_0(t)}$,
\[
5\pi(\pi - A_\eta) \geq \sup_t |\tilde{\sigma}_\eta(t) - c_1(\eta)e^{it}| = \sup_\alpha |\tilde{\sigma}_\eta(\nu_\eta(\alpha)) - c_1(\eta)e^{i\nu_\eta(\alpha)}| = \|\sigma_\eta + ic_1(\eta)\sigma_0\|_\infty.
\]
Next we want to show that $|c_1(\eta)|$ is close to 1 if $A_\eta$ is close to $\pi$. Indeed, (4.11) is a consequence of a stronger Sobolev-norm estimate for $w$ established in [11, §1], namely
\[
\int_0^{2\pi} |w|^2 + |w'|^2 \leq 5(\pi - A_\eta) .
\]
Since
\[
|w'(t)| \geq |\tilde{\sigma}'_\eta(t)| - |c_1(\eta)ie^{it}| = 1 - |c_1(\eta)| \geq 0, 
\]
it follows
\[
0 \leq 1 - |c_1(\eta)| \leq (\pi - A_\eta)^\frac{1}{2}.
\]
Set $\lambda := -\frac{ic_1(\eta)}{|c_1(\eta)|}$ if $c_1(\eta) \neq 0$ and $\lambda = 1$ if $c_1(\eta) = 0$. In any case, with $|\sigma_0| \equiv 1$,
\[
\|\sigma_\eta - \lambda \sigma_0\|_\infty \leq \|\sigma_\eta + ic_1(\eta)\sigma_0\|_\infty + \|ic_1(\eta)\sigma_0 + \lambda \sigma_0\|_\infty \\
\leq 5\pi(\pi - A_\eta) + (1 - |c_1(\eta)|) \\
\leq 5\pi(\pi - A_\eta) + (\pi - A_\eta)^\frac{1}{2} \\
\leq C_1(\pi - A_\eta)^\frac{1}{2}
\]
(4.12)
for $C_1 := 5(2\pi)^\frac{1}{2} + 1 > 0$.

As a consequence of $\mu_{h,\gamma_\eta} = 2\sin(d)\sigma_\eta$ from (4.8), (4.10), (4.12) and Lemma 4.8, there exists $H(\frac{\pi}{2}) > 0$ such that
\[
\|\mu_{f,\gamma_\eta} - \lambda \mu_{h,\gamma_0}\|_\infty \leq \|\mu_{f,\gamma_\eta} - \mu_{h,\gamma_\eta}\|_\infty + \|\mu_{h,\gamma_\eta} - \lambda \mu_{h,\gamma_0}\|_\infty \\
\leq H\|f - h\|_\infty + 2\sin(d)\|\sigma_\eta - \lambda \sigma_0\|_\infty \\
\leq H\|f - h\|_\infty + 2\sin(d)C_1(\pi - \Psi_h(h, \eta))^{\frac{1}{2}}.
\]
Further, again by Lemma 4.8 using that $\omega_f(\gamma_\eta) = \Psi_h(f, \eta)$,
\[
|\pi - \Psi_h(h, \eta)|^{\frac{1}{2}} \leq |\pi - \Psi_h(f, \eta)|^{\frac{1}{2}} + |\Psi_h(f, \eta) - \Psi_h(h, \eta)|^{\frac{1}{2}} \\
\leq |\Psi_h(h, 0) - \Psi_h(f, \eta)|^{\frac{1}{2}} + H^{\frac{1}{2}}\|f - h\|_\infty^{\frac{1}{2}}.
\]
This proves the lemma. □
Due to this global estimate by employing the stability of the plane isoperimetric inequality, a maximizer \( \eta_f \) of \( \Psi_h(f, \cdot) \) can be found locally around 0 in case \( \|f - h\|_\infty \) is small.

**Lemma 4.10.** For \( r \in (0, \frac{\pi}{2}) \) and \( \xi \in (0, 1) \) there exists \( \varepsilon \in (0, \frac{\xi}{2}) \) such that for all \( h \in S^2_+ \cap E_r(S^1) \) and \( f \in B(h, \varepsilon) \cap E^+(S^1) \) there exists \( \gamma_f \in L^2_0(\mathbb{R}) \) with the following properties:

1. \( \Psi_h(f, \eta_f) = \sup_{\gamma \in L^2_0(\mathbb{R})} \Psi_h(f, \eta) = \sup_{\gamma \in B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2)} \omega_f(\gamma) \),
2. \( \eta_f \) is continuous,
3. \( \|\mu_f, \eta_f\|_\infty \geq \sin(r) \) and \( \gamma_f = -i\left|\frac{\mu_f, \gamma_f}{\|\mu_f, \gamma_f\|}\right|_c \),
4. \( \|\eta_f\|_\infty \leq C(\xi, r)\|f - h\|_\infty^\xi \).

**Proof.** Because \( f \in E^+(S^1) \), it holds that that \( p_{\alpha, \beta}^c(f) > 0 \) for almost every \( \alpha, \beta \) by definition. According to Lemma 4.2 and Lemma 4.3, a maximizer \( \gamma \in B_{ap}^{\infty}(\mathbb{R}, \mathbb{R}^2) \) of \( \omega_f \) exists and satisfies \( |\gamma| = 1 \) almost everywhere. So there exists a measurable \( \pi \)-periodic function \( \eta : \mathbb{R} \to \mathbb{R} \) such that \( \gamma_0 = e^{i\langle \eta, f \rangle} \) is a maximizer of \( \omega_f \). By replacing \( \eta(\alpha) \) with multiples of \( 2\pi \) we may assume that \( \eta \) takes values in \((-\pi, \pi]\), and in turn its restriction to \([0, \pi]\) is in \(L^2\). Since \( \omega_f(\gamma) = \omega_f(e^{i\alpha}) \) for any \( \alpha \in \mathbb{R} \), by the rotation invariance of this action, it can be assumed that \( f_{\pi}^0 \eta = 0 \) by replacing \( \eta \) with \( \eta - \frac{1}{\pi}\int_{0}^{\pi} \eta \) if necessary. Thus for all \( f \in E^+(S^1) \) there exists a maximizer \( \eta_f \in L^2_0(\mathbb{R}) \) of \( \Psi_h(f, \eta) \) such that

\[
\|\omega_f\|_{ir} = \Psi_h(f, \eta_f).
\]

Set \( \varepsilon_0 := \frac{\pi}{2} < \text{dist}(h, S^1) \). According to Lemma 4.8 and Lemma 4.9, there exist \( H(\xi, r), C_0(\xi, r) > 0 \) such that for all \( \eta \in L^2_0(\mathbb{R}) \) and all \( f \in B(h, \varepsilon_0) \cap E^+(S^2) \),

\[
|\Psi_h(f, \eta_f) - \Psi_h(h, 0)| = \|\omega_f\|_{ir} - \|\omega_h\|_{ir} \leq H\|f - h\|_\infty^\xi
\]

and

\[
\|\mu_f, \gamma_f - \lambda(f)\mu_{h, \gamma_0}\|_\infty \leq C_0\left(\|f - h\|_\infty^\xi + |\Psi_h(f, \eta_f) - \Psi_h(h, 0)|^\xi\right)
\]

for some \( \lambda(f) \in \mathbb{C} \) with \( |\lambda(f)| = 1 \). Set \( \varepsilon := \|f - h\|_\infty \) for some \( f \in E^+(S^1) \). If \( \varepsilon \leq \varepsilon_0 \), then

\[
\|\mu_f, \gamma_f - \lambda(f)\mu_{h, \gamma_0}\|_\infty \leq C_1(\varepsilon^\xi + H^2\varepsilon^\xi) = C_1\varepsilon^\xi
\]

for some \( C_1(\xi, r) > 0 \). The path \( \mu_f, \gamma_f \) is continuous by Lemma 4.5 since the path \( \mu \in L^\infty_{ap}(\mathbb{R}, \mathbb{R}^2) \) defined there is given by

\[
\mu(\alpha) = \int_{\alpha}^{\alpha + \pi} p_{\alpha, \beta}(f)\gamma_f(\beta) d\beta
\]

and thus satisfies \( \mu(\alpha) = \sin(f_\alpha)^2\mu_f, \gamma_f(\alpha) \). Because \( \mu_{h, \gamma_0} = 2i\sin(d) e^{i\nu_h} \) by (4.8) for some \( d \geq r \), we can choose \( \varepsilon_1 \in (0, \varepsilon_0) \) small enough such that \( |\mu_f, \gamma_f(\alpha)| \geq \sin(r) \) for all \( \alpha \) if \( \varepsilon \leq \varepsilon_1 \). In this situation, also \( \mu(\alpha) \neq 0 \) for all \( \alpha \) and \( \gamma_f \) as a maximizer of \( \omega_f \) is continuous and satisfies \( i\gamma_f|\mu| = \mu \) by Lemma 4.5. Equivalently, we can also write \( i\gamma_f|\mu_f, \gamma_f| = \mu_f, \gamma_f \),
Because the map $\rho : \mathbb{C} \setminus U(0, \sin(r)) \to S^1$ given by $\rho(z) := -i\frac{z}{|z|}$ is Lipschitz it follows from (4.14) that

$$\|e^{i\eta f} - \lambda(f)\|_{\infty} = \|e^{i(\eta f + \eta r)} - \lambda(f) e^{i\eta r}\|_{\infty} = \|\gamma_{\eta f} - \lambda(f) e^{i\eta r}\|_{\infty} \leq C_2 \varepsilon^\frac{1}{2}$$

for some $C_2(\xi, r) > 0$ whenever $\varepsilon \leq \varepsilon_1$.

Let $c \in \mathbb{R}$ be such that $\lambda(f) = e^{-i c}$ and $e^{i \tilde{\eta} f} \lambda(f) = e^{i \eta f}$ for $\tilde{\eta} := \eta f + c$. By adding multiples of $2\pi$, $\tilde{\eta} f$ can be assumed to be a measurable $\pi$-periodic function that takes values in $(-\pi, \pi]$. Assume that $\varepsilon_2 \in (0, \varepsilon_1]$ is small enough such that $C_2 \varepsilon_2^\xi/2 \leq \frac{1}{2}$. It follows $\|e^{i \tilde{\eta} f} - 1\|_{\infty} \leq C_2 \varepsilon_2^\xi/2 \leq \sqrt{\varepsilon}$ and hence

$$\|\tilde{\eta} f\|_{\infty} \leq \frac{\varepsilon}{2} \|\sin(\tilde{\eta} f)\|_{\infty} \leq \frac{\varepsilon}{2} \|e^{i \tilde{\eta} f} - 1\|_{\infty} \leq \frac{\varepsilon}{2} C_2 \varepsilon_2^\xi \,$$

whenever $\varepsilon \leq \varepsilon_2$. $\tilde{\eta} f$ can be replaced by $\tilde{\eta} f - \frac{1}{\pi} \int_{0}^{\pi} \tilde{\eta} f$ to obtain a function in $L_0^2(\mathbb{R})$ with upper bound $\pi C_2 \varepsilon_2^\xi$. Note that $e^{i \tilde{\eta} f}$ is equal to $e^{i \eta f}$ up to a rotation and hence $\tilde{\eta} f$ is also a maximizer of $\Psi_h(f, \cdot)$. Because $\gamma_{\eta f}$ is continuous it follows that $\tilde{\eta} f$ is continuous since $\|\tilde{\eta} f\|_{\infty} < \pi$ if $\varepsilon \leq \varepsilon_2$. This proves the lemma. \hfill \Box

The restriction to $E^+(S^1)$ is not a problem by the density of $E^+(S^1)$ in $E(S^1)$ due to Lemma 3.3 in combination with the continuity of $f \mapsto \|\omega f\|_{ir}$ by Lemma 4.8. We need to improve on the Hölder exponent in the estimate of $\|\eta f\|_{\infty}$.

**Proposition 4.11.** For every $\xi \in (0, 1)$ and $r \in (0, \frac{\pi}{2})$ there exist $\varepsilon_1 \in (0, \frac{\pi}{2})$, $\varepsilon_2 > 0$ and $c, C > 0$ such that for all $h \in S^1_\xi \cap E_r(S^1)$ and $f \in B(h, \varepsilon_1) \cap E^+(S^1)$ the following properties hold:

1. $\eta \mapsto \Psi_h(f, \eta)$ is strictly concave on $L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$ with a unique maximizer $\eta f$ that satisfies $\Psi_h(f, \eta f) = \sup_{\eta \in L_0^2(\mathbb{R})} \Psi_h(f, \eta)$ and $\|\omega f\|_{ir}$. Further,

$$\frac{d^2}{dt^2} \Psi_h(f, \eta^t) \leq -c\|\eta^t\|_2^2$$

holds for $\eta^t = (1 - t)\eta^0 + t\eta^1$ if $\eta^0, \eta^1 \in L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$.

2. $f \mapsto \eta f$ is continuous at $h$ in the sense that

$$\|\eta f\|_{\infty} \leq C\|f - h\|_\xi$$

**Proof.** We can choose $0 < \varepsilon_1 \leq \varepsilon_2 < \frac{\pi}{2}$ small enough such that Lemma 4.10 holds for $\varepsilon_1$ with $\|\eta f\|_{\infty} \leq \varepsilon_2$ for $f \in B(h, \varepsilon_1) \cap E^+(S^1)$ and Lemma 4.7 holds for $\varepsilon_2$. This is possible due to Lemma 4.10(4).

Fix $f \in B(h, \varepsilon_1) \cap E^+(S^1)$ and consider the variation $\eta^t = (1 - t)\eta^0 + t\eta^1 \in L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$ for $\eta^0, \eta^1 \in L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$. From Lemma 4.6 and Lemma 4.7 it follows that

$$\frac{d^2}{dt^2} \Psi_h(f, \eta^t) = - \int_0^\pi \int_0^\pi p_{\alpha, \beta}(f) \sin \left( \Delta^h_{\alpha, \beta} + \Delta^\eta^t_{\alpha, \beta} \right) \left( \Delta^\eta^t_{\alpha, \beta} \right)^2 d\beta d\alpha$$

(4.15)

for some $c(r) > 0$. This implies that $\Psi_h$ is strictly concave in the second argument when restricted to $L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$ and thus $\eta f$ is the unique maximizer of $\eta \mapsto \Psi_h(f, \eta)$ in $L_0^2(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$. 

THE HEMISPHERE IS ALMOST CALIBRATED 39
If $\eta \in L^2_0(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$, it follows from Lemma 4.8, $q_{\alpha, \beta}(h) \leq M(r)$ by Lemma 3.2(4), the definition $\|\eta\|_2^2 = \pi \int_0^{\alpha + \pi} \eta$ and $\|\eta\|_\infty \leq \pi$ that
\[
|\mu_{f, \gamma}(\alpha) - \mu_{h, \gamma}(\alpha)| \leq |\mu_{f, \gamma}(\alpha) - \mu_{h, \gamma}(\alpha)| + |\mu_{h, \gamma}(\alpha) - \mu_{h, \gamma}(\alpha)| \\
\leq H \|f - h\|_\infty^\xi + \int_\alpha^{\alpha + \pi} q_{\alpha, \beta}(h) |\gamma_{\eta}(\beta) - \gamma_0(\beta)| \ d\beta \\
\leq H \|f - h\|_\infty^\xi + M \int_\alpha^{\alpha + \pi} |e^{i\eta(\beta)} - 1| \ d\beta \\
\leq H \|f - h\|_\infty^\xi + M \int_\alpha^{\alpha + \pi} |\eta(\beta)| \ d\beta \\
\leq H \|f - h\|_\infty^\xi + M \left( \pi \int_\alpha^{\alpha + \pi} |\eta(\beta)|^2 \ d\beta \right)^{1/2} \\
= H \|f - h\|_\infty^\xi + M \|\eta\|_2,
\]
Together with Lemma 4.10(3) and $\|\eta\|_\infty \leq \frac{\pi}{2}$ this implies
\[
(4.16) \quad \|\eta\|_\infty \leq \frac{\pi}{2} \ |e^{i\eta t} - 1|_\infty = \frac{\pi}{2} \ |\gamma_{\eta} - \gamma_0|_\infty \leq a(\|f - h\|_\infty^\xi + \|\eta\|_2)
\]
for some $a(\xi, r) > 0$.

Set
\[
F(f, \eta)(v) := \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \cos(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^\eta) \ d\beta \ d\alpha
\]
for $\eta \in L^2_0(\mathbb{R}) \cap B_{L^\infty(\mathbb{R})}(0, \varepsilon_2)$ and $v \in L^2_0(\mathbb{R})$ and consider the variation $\psi(t) := F(f, t\eta)(\eta)$ for $t \in [0, 1]$. The first derivative is
\[
\psi'(t) = - \int_0^\pi \int_\alpha^\pi p_{\alpha, \beta}(f) \sin(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^{t\eta}) \ d\beta \ d\alpha
\]
by Lemma 4.6 with
\[
|\psi'(t)| \geq c \|\eta\|_2^2
\]
for $c(r) > 0$ by (4.15). Since $\psi$ is in $C^1$, there is some $m \in (0, 1)$ with
\[
|F(f, \eta)(\eta) - F(f, 0)(\eta)| = |\psi'(m)|.
\]
It holds $F(f, \eta)(\eta) = 0 = F(h, 0)(\eta)$ because $\Psi_h(f, \cdot)$ is stationary at $\eta$ and $\Psi_h(f, \cdot)$ is stationary at 0. Together with Lemma 4.8,
\[
c \|\eta\|_2^2 \leq |\psi'(m)| = |F(f, \eta)(\eta) - F(f, 0)(\eta)| = |F(f, 0)(\eta) - F(h, 0)(\eta)|
\]
\[
= \left| \int_0^\pi \int_\alpha^\pi (p_{\alpha, \beta}(f) - p_{\alpha, \beta}(h)) \cos(\Delta_{\alpha, \beta}^h + \Delta_{\alpha, \beta}^{t\eta}) \ d\beta \ d\alpha \right|
\]
\[
\leq 2 \|\eta\|_\infty \int_0^\pi \int_\alpha^\pi |p_{\alpha, \beta}(f) - p_{\alpha, \beta}(h)| \ d\beta \ d\alpha \\
\leq 2 H \|\eta\|_\infty \|f - h\|_\infty^\xi.
\]
Collecting this with the other main estimate (4.16) there exists $a(\xi, r), b(\xi, r) > 0$ such that
\[
\|\eta\|_\infty \leq a(\|f - h\|_\infty^\xi + \|\eta\|_2),
\]
\[
\|\eta\|_2 \leq b \|\eta\|_2^2 \|f - h\|_\infty^\xi.
\]
Proposition 4.11(2). Applying Taylor’s theorem to \( \psi \) and all \( t \) with representation

Indeed, if \( 4.3. \)

Proof of the main theorems. This shows (2).

Fix \( \xi \in (0, 1) \), \( r \in (0, \frac{\pi}{2}) \) and \( h \in S^2_\pi \setminus S^1 \) with representation \( h = \arccos(\cos(d) \cos(-\tau)) \) for \( \tau \in \mathbb{R} \) and \( d \in [r, \frac{\pi}{2}] \). Let \( \varepsilon_1(\xi, r), \varepsilon_2(\xi, r) > 0 \) be as in Proposition 4.11 such that a maximizer \( \eta_f \) of \( \Psi_h(f, \cdot) \) exists in \( L^2_0(\mathbb{R}) \cap B L^\infty(\mathbb{R})(0, \varepsilon_2) \) if \( f \in B(h, \varepsilon_1) \). \( \Psi_h \) is defined as in the last subsection by

\[
\Psi_h(f, \eta) = \int_0^\pi \int_0^\pi p_{\alpha, \beta}(f) \sin(\nu_h(\beta) - \nu_h(\alpha + \eta(\beta) - \eta(\alpha))) \, d\beta \, d\alpha ,
\]

where \( \nu_h : \mathbb{R} \to \mathbb{R} \) is the unique bi-Lipschitz function that satisfies \( \nu_h(0) = 0 \), \( \nu'_h(\alpha) = p_{\alpha}(h), \nu_h(\alpha + \pi) = \nu_h(\alpha) + \pi \) and

\[
\nu'_h(\alpha) = p_{\alpha}(h) = \frac{\sin(d)}{\sin(h_\alpha)^2} = \frac{\sin(d)}{1 - \cos(d)^2 \cos(\alpha - \pi)^2} \in [m_1(r), m_2(r)] .
\]

We want to estimate

\[
(4.17) \quad |\Psi_h(h, 0) - \Psi_h(f, \eta_f)| \leq |\Psi_h(h, 0) - \Psi_h(f, 0)| + |\Psi_h(f, 0) - \Psi_h(f, \eta_f)|
\]

if \( f \in B(h, \varepsilon_1) \cap E^+(S^1) \) and start with the second term on the right hand side.

The function \( \psi(t) := \Psi_h(f, (1 - t)\eta_f) \) is in \( C^2([0, 1]) \) by Lemma 4.6 and satisfies \( \psi'(0) = 0 \) because \( \eta_f \) is a maximizer. \( p_{\alpha, \beta}(f) \leq M(r) \) for all \( 0 < \alpha < \beta < \pi \) and all \( t \in [0, 1] \) due to Lemma 3.2(4) and \( \|\eta_f\|_\infty \leq C(\xi, r)\|f - h\|_\infty \) due to Proposition 4.11(2). Applying Taylor’s theorem to \( \psi \), there exists \( m \in (0, 1) \) such
that
\[
|\Psi_h(f,0) - \Psi_h(f,\eta_f)| = |\psi(1) - \psi(0) - \psi'(0)| = \frac{1}{2} |\psi''(m)|
\]
\[
= \frac{1}{2} \int_0^{\pi} \int_\alpha^\pi \partial_{\alpha,\beta}^2 p_{\alpha,\beta}(f) \sin(\Delta^h_{\alpha,\beta} + \Delta^{(1-m)\eta_f}_{\alpha,\beta})^2 d\beta d\alpha
\]
\[
\leq \frac{1}{2} M \int_0^{\pi} \int_\alpha^\pi (\Delta^{\eta_f}_{\alpha,\beta})^2 d\beta d\alpha
\]
\[
\leq \pi^2 M \|\eta_f\|_\infty^2
\]
(4.18)

For a bound of the first term on the right hand side of (4.17), let
\[
f_t := (1-t)h + tf
\]
for \( t \in [0,1] \). Similar to the estimate above, consider the function
\[
\phi(t) := \Psi_h(f_t,0) = \int_0^{\pi} \int_\alpha^\pi p_{\alpha,\beta}(f_t) \sin(\nu_h(\beta) - \nu_h(\alpha)) d\beta d\alpha .
\]
It is in \( C^2((0,1]) \) due to Lemma 3.4 and with Lemma 3.5 with second derivative bounded by
\[
|\phi''(t)| \leq \int_0^{\pi} \int_\alpha^\pi \sup_{t \in [0,1]} |\partial^2_{\alpha,\beta} p_{\alpha,\beta}(f_t)| \sin(\nu_h(\beta) - \nu_h(\alpha)) d\beta d\alpha .
\]
Because \( \sin(\nu_h(\beta) - \nu_h(\alpha)) \leq 2m_2 \sin(\beta - \alpha) \) for \( 0 < \alpha < \beta < \pi \), the integrand has an upper bound of
\[
C_1 \sin(\beta - \alpha)^{-1} \| f - h \|_\infty^2,
\]
for \( C_1(r) > 0 \) by Lemma 3.5. Exactly as in the proof of Lemma 4.8, this leads to the estimate
(4.19) \[
|\phi''(t)| \leq C_2 \| f - h \|_\infty^2
\]
for \( t \in [0,1] \) and \( C_2(r) > 0 \). We claim that \( \phi'(0) = 0 \) and this is the main part of this proof. Abbr.\viationg \( \delta := f - h \),
\[
\phi'(0) = \int_0^{\pi} \int_\alpha^\pi (\partial_x p(\beta - \alpha, h\alpha, h\beta) \delta_x + \partial_y p(\beta - \alpha, h\alpha, h\beta) \delta_y) \sin(\Delta^h_{\alpha,\beta}) d\beta d\alpha .
\]
For simplicity of notation, we assume that \( \tau = 0 \) and \( h\alpha = \arccos(\cos(d) \cos(\alpha)) \).
Because
\[
\cos(\beta - \alpha) \cos(\alpha) - \cos(\beta) = \sin(\beta - \alpha) \sin(\alpha) ,
\]
the first partial derivative is
\[
\frac{1}{2} \partial_x p(\beta - \alpha, h\alpha, h\beta)
\]
\[
= \frac{(\cos(\beta - \alpha) \cos(h\alpha) - \cos(h\beta))(\cos(\beta - \alpha) - \cos(h\alpha) \cos(h\beta))}{\sin(\beta - \alpha)^2 \sin(h\alpha)^3 \sin(h\beta)^2}
\]
\[
= \cos(d) \frac{\sin(\delta) \sin(\alpha) (\cos(\beta - \alpha) - \cos(h\alpha) \cos(h\beta))}{\sin(\beta - \alpha)^2 \sin(h\alpha)^3 \sin(h\beta)^2}
\]
for almost all \( \alpha, \beta \in \mathbb{R} \) according to Lemma 3.4. By (4.2),
\[
\sin(\Delta^h_{\alpha,\beta}) = \sin(\nu_h(\beta) - \nu_h(\alpha)) = \frac{\sin(d) \sin(\beta - \alpha)}{\sin(h\alpha) \sin(h\beta)}
\]
and in turn
\[ \frac{1}{2} \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \sin(\Delta_{\alpha,\beta}^h) = \frac{\cos(d)^2 \sin(\alpha) \cos(\beta - \alpha) - \cos(h_\alpha) \cos(h_\beta)}{\sin(h_\beta)^4 \sin(h_\beta)^3}. \]

In particular, \( \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \delta_\alpha \sin(\Delta_{\alpha,\beta}^h) \) is integrable over \([0, \pi]^2\). By symmetry, also \( \partial_y p(\beta - \alpha, h_\alpha, h_\beta) \delta_\beta \sin(\Delta_{\alpha,\beta}^h) \) is integrable and the integral can be expressed as

\[
\int_0^\pi \int_0^\pi \partial_x p(\beta - \alpha, h_\beta, h_\alpha) \delta_\alpha \sin(\Delta_{\alpha,\beta}^h) \, d\beta \, d\alpha
\]

\[
= \int_0^\pi \int_0^\pi \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \delta_\beta \sin(\Delta_{\alpha,\beta}^h) \, d\alpha \, d\beta
\]

\[
= \int_0^\pi \int_0^\pi \partial_x p(\alpha - \beta, h_\alpha, h_\beta) \delta_\alpha \sin(\Delta_{\alpha,\beta}^h) \, d\beta \, d\alpha
\]

\[
= -\int_0^\pi \int_0^\pi \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \delta_\alpha \sin(\Delta_{\alpha,\beta}^h) \, d\beta \, d\alpha
\]

\[
= \int_0^\pi \int_0^{\alpha + \pi} \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \delta_\alpha \sin(\Delta_{\alpha,\beta}^h) \, d\beta \, d\alpha.
\]

In the last line it is used that \( \partial_x p(\pi + \beta - \alpha, h_\alpha, h_{\beta + \pi}) = \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \) and \( \sin(\Delta_{\alpha,\beta+\pi}^h) = -\sin(\Delta_{\alpha,\beta}^h) \). It follows

\[
\phi'(0) = \int_0^\pi \delta_\alpha \int_0^{\alpha + \pi} \partial_x p(\beta - \alpha, h_\alpha, h_\beta) \sin(\Delta_{\alpha,\beta}^h) \, d\beta \, d\alpha.
\]

So the claim is proved if the integral

\[
(4.20) \int_0^{\alpha + \pi} \frac{\cos(\beta - \alpha) - \cos(h_\alpha) \cos(h_\beta)}{\sin(h_\beta)^3} \, d\beta
\]

vanishes for all \( \alpha \). The integrand is equal to

\[
\frac{\cos(\beta - \alpha) - \cos(h_\alpha) \cos(h_\beta)}{\sin(h_\beta)^3} = \frac{\cos(\beta - \alpha) - \cos(d)^2 \cos(\alpha) \cos(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{3}{2}}} = \frac{\sin(\alpha) \sin(\beta) + \sin(d)^2 \cos(\alpha) \cos(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{3}{2}}}
\]

and since

\[
\frac{\partial}{\partial \beta} \frac{\sin(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{3}{2}}} = \frac{\sin(d)^2 \cos(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{5}{2}}},
\]

\[
\frac{\partial}{\partial \beta} \frac{\cos(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{3}{2}}} = \frac{-\sin(\beta)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{5}{2}}},
\]

it follows that the integral in (4.20) is equal to

\[
-\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \bigg|_0^{\alpha + \pi} = \frac{\sin(\beta - \alpha)}{(1 - \cos(d)^2 \cos(\beta)^2)^{\frac{3}{2}}} \bigg|_0^{\alpha + \pi} = 0.
\]

This proves that \( \phi'(0) = 0 \). Again applying Taylor’s theorem, there exists \( m \in (0, 1) \) such that with (4.19),

\[
|\Psi_h(f, 0) - \Psi_h(h, 0)| = |\phi(1) - \phi(0) - \phi'(0)| = |\frac{1}{2} \phi''(m)| \leq C_2 \|f - h\|_{\infty}^{2\xi}.
\]
Together with (4.18) and Proposition 4.11 we obtain

\[ \|\omega_h\|_{ir} - \|\omega_f\|_{ir} = |\Psi_h(h,0) - \Psi_h(f,\eta_f)| \leq C_3 \|f - h\|^{\xi}_\infty \]

for some $C_3(\xi, r) > 0$ in case $f \in B(h, \varepsilon_1) \cap E^+ (S^1)$. Now $f \mapsto \|\omega_f\|_{ir}$ is uniformly continuous on $B(h, \varepsilon_1)$ by Lemma 4.8. Thus (4.21) holds for all $f \in B(h, \varepsilon_1)$ by the density of $E^+ (S^1)$ in $E(S^1)$ due to Lemma 3.3.

Fix $r \in (0, \frac{\pi}{2})$ and $\xi \in (1, 2)$ as in the statement of Theorem 1.1, choose $\varepsilon_1(\frac{\pi}{2}, \frac{\pi}{2}) > 0$ as above and assume that $f \in E_r(S^1) \cap B(S^1_+, \varepsilon_1)$. Note that $\varepsilon_1 \leq \frac{\xi}{r}$ as assumed in Proposition 4.11. Let $h \in S^2_r$ be such that $\|f - h\|_\infty = \text{dist}(f, S^2_r)$. It is clear that $h \in E_r(S^1)$ because $\|f - g\|_\infty > \frac{\pi}{2} > \varepsilon_1$ for all $g \in E(S^1) \setminus E_r(S^1)$. So with (4.21) it follows

\[ \|\omega_f\|_{ir} - \pi | \leq C_3(\xi, \frac{\pi}{2}) \|f - h\|^{\xi}_\infty = C_3(\xi, \frac{\pi}{2}) \text{dist}(f, S^1)^\xi. \]

Here we used that $\|\omega_h\|_{ir} = \pi$ for $h \in S^2_r \setminus S^1$ because of (3.13). Since $\|\omega_f\|_{ir}$ is uniformly bounded on $E_r(S^1)$ by Lemma 4.8 for example, the estimate above holds for all $f \in E_r(S^1)$ with a possibly larger constant. The two-form defined in the introduction is equal to $\dot{\omega} = \frac{1}{\pi} \omega$. This proves Theorem 1.1.

Theorem 1.2 is a rather direct consequence of Theorem 1.1 and Proposition 3.8. For the formulation of the theorem chosen we need the following fact. For $r \in (0, \frac{\pi}{2})$ as in the statement of the theorem, the radial retraction $\rho_r : S^2_r \to S^2_+ \cap E_r(S^1)$ is 1-Lipschitz. This is a consequence of Gauss’s lemma in differential geometry and the fact that whenever $\gamma_1, \gamma_2 : [0, \frac{\pi}{2}] \to S^2_r$ are two unit speed geodesics emitting from the north pole, then $d(\gamma_1(s), \gamma_2(s)) \leq d(\gamma_1(t), \gamma_2(t))$ whenever $s \leq t$. The image of this retraction is denoted by $S_r := \{ p \in S^2_r : \text{dist}(p, S^1) \geq r \}$. Because $E(S^1)$ is injective, there exists a 1-Lipschitz extension $\tilde{\rho}_r : E(S^1) \to E(S^1)$. Due to Lemma 2.8(4), we can assume that the image of $\tilde{\rho}_r$ is contained in $E_r(S^1)$ because $S_r$ is. Now assume that $T, S \in H_2(X)$ are as in the statement of Theorem 1.1 with respect to some $r \in (0, \frac{\pi}{2})$ and $\xi \in (1, 2)$. By assumption, $S$ is isometric to $[S^2_r]$ and hence there exists an isometric embedding $\varphi : \text{spt}(S) \to E(S^1)$ mapping $\text{spt}(S)$ to $S^2_+$ as characterized in Lemma 2.9. Since $E(S^1)$ is injective, there exists a 1-Lipschitz extension $\bar{\varphi} : X \to E(S^1)$. We also need the composition $\psi := \tilde{\rho}_r \circ \bar{\varphi} : X \to E_r(S^1)$ and the restrictions $S_i := S\cap(X \setminus N_r)$ and $T_i := T\cap(X \setminus N_r)$ to the parts away from the collar $N_r := B(\text{spt}(\partial S), r) \cap \text{spt}(S)$ of the boundary in $\text{spt}(S)$. By assumption, $\partial S_i = \partial T_i$, $\bar{\varphi}_#S_i = \psi_#S = [S_r]$, and $T' := \psi_#T = \psi_#T$. The latter two identities are based on $\tilde{\rho}_r([S^2_r \setminus S_r]) = 0$, because the image $\tilde{\rho}_r(S^2_r \setminus S_r)$ is a circle, and

\[ \bar{\varphi}_#(T - T_i) = \bar{\varphi}_#(S - S_i) = [S^2_r \setminus S_r]. \]

Set $d := d_{\text{dist}}(\text{spt}(S), \text{spt}(T))$, $\text{spt}(T')$ is contained in $B(S_r, d) \cap E_r(S^1)$ because $\psi$ is 1-Lipschitz. With Theorem 1.1 and Proposition 3.8 it follows

\[ T' (\bar{\omega}) \leq M_{ir}(T') \sup_{\text{spt}(T')} \|\omega_f\| \leq M_{ir}(T')(1 + Cd^\delta) \]
for $C(\xi, r) > 0$. $\tilde{\omega}$ is exact by Lemma 3.6 and hence two currents with the same boundary have the same evaluation on $\tilde{\omega}$. Now Lemma 2.5(5) applied to the 1-Lipschitz maps $\psi$ and $\bar{\phi}$ implies

$$2\pi = \left[ S^2_+ \right](\tilde{\omega}) = \left[ T' \right] + \left[ S^2_+ \setminus S_r \right](\tilde{\omega}) = T'(\tilde{\omega}) + M_{ir}(\left[ S^2_+ \setminus S_r \right])$$

$$\leq (M_{ir}(T') + M_{ir}(\left[ S^2_+ \setminus S_r \right]))(1 + Cd^\delta)$$

$$= (M_{ir}(\psi # T_i) + M_{ir}(\bar{\phi}_#(T - T_i)))(1 + Cd^\delta)$$

$$\leq (M_{ir}(T_i) + M_{ir}(T - T_i))(1 + Cd^\delta)$$

$$= M_{ir}(T)(1 + Cd^\delta).$$

In the last line we used that $T = T_i + (T - T_i) = T_{L}(X \setminus N_r) + T_{L}N_r$ is a disjoint decomposition. This proves Theorem 1.2.

5. Open questions

5.1. Other definitions of area. Gromov’s filling area conjecture as formulated in the introduction is for the inner Riemannian Finsler mass with the obvious reason that this is the largest Finsler mass on rectifiable currents as shown in Proposition 2.5. Having different definitions of area at our disposal it begs the question:

**Question 2.** For which definitions of Finsler mass is Gromov’s filling conjecture true?

This question aims more towards excluding certain choices. The conjecture indeed doesn’t hold for every Finsler mass even when only fillings with disks are allowed. To see this, consider the cone $C := N \times [S^1]$ as a current in $\mathcal{D}_2(E(S^1))$, where $N \in E(S^1)$ is the central function with $N \equiv \frac{\pi}{2}$. $C$ is a properly oriented Lipschitz disk that contains all the functions $f \in E(S^1)$ of the form

$$f = \frac{\pi}{2}(1 - r) + rd_\alpha,$$

where $d_\alpha$ is the distance function to $\alpha \in S^1$ and $r \in [0, 1]$. $C$ can be written as $\varphi_\#[[0, 1] \times [0, 2\pi]]$ with the parametrization $\varphi(r, \alpha)$ given on the right hand side of (5.1). Thus for a particular definition of area $\mu$, it holds

$$M_\mu(C) = \int_{[0,1] \times [0,2\pi]} J_\mu(\text{md} \varphi_x) \, dx.$$

Let $X$ be $\mathbb{R}^2$ equipped with the $L^1$-norm $|| (x, y) ||_1 = |x| + |y|$ and $B_X$ be the unit disk in $X$. Since

$$d(\varphi(r, \alpha), \varphi(r + h, \alpha + k)) = \frac{\pi}{2}|h| + r|k| + o(|h| + |k|),$$

it follows

$$\text{md} \varphi_{(r, \alpha)}(h, k) = \frac{\pi}{2}|h| + r|k|$$

and

$$J_\mu(\text{md} \varphi_{(r, \alpha)}) = \frac{\pi}{2}rJ_\mu(|| \cdot ||_1) = \frac{\pi}{2}r\mu_X(e_1 \wedge e_2)$$
for almost all $r$ and $\alpha$. Thus
\[
M_\mu(C) = \int_0^1 \int_0^{2\pi} J_\mu(\text{nd } \varphi_{(r,\alpha)}) \, d\alpha \, dr
= 2\pi \int_0^1 \frac{\pi}{2} r \mu_X(e_1 \wedge e_2) \, dr
= \frac{\pi^2}{2} \mu_X(e_1 \wedge e_2) .
\]

With the properties of different areas as stated in [1, §3] and noting that $\mu_X(B) = 2\mu_X([0,1]^2) = 2\mu_X(e_1 \wedge e_2)$:

- (Gromov-mass) $\mu_X^m(e_1 \wedge e_2) = \inf \{ \|v\|_1 \|w\|_1 : v \wedge w = e_1 \wedge e_2 \} = 1$
- (Gromov-mass star) $\mu_X^{m*}(e_1 \wedge e_2) = \inf \{ \langle \xi \wedge \eta, e_1 \wedge e_2 \rangle : \|\xi\|_\infty, \|\eta\|_\infty \leq 1 \} = 2$
- (Busemann-Hausdorff) $\mu_X^{bh}(B) = \pi$
- (Holmes-Thompson) $\mu_X^{ht}(e_1 \wedge e_2) = \frac{\pi}{4}$

This implies $M_m(C) = \frac{\pi^2}{2} < 2\pi = M_{ht}(C) < M_{bh}(C) = \frac{\pi^3}{4} < \pi^2 = M_{m*}(C)$.

In particular $M_m(C) < 2\pi$, so Gromov’s filling conjecture doesn’t hold for the Gromov-mass definition of area. This also implies that it fails for the outer Riemannian mass which can be defined similarly to the inner Riemannian mass with respect to the circumscribed Löwner-John ellipsoid. At least the uniqueness part in the conjecture also fails for the Holmes-Thompson definition of area.

### 5.2. Lower bounds on the filling area

Let $T \in \mathcal{R}_2(E(S^1))$ with $\partial T = [S^1]$ and fix $\alpha \in \mathbb{R}$. Then
\[
T(d\pi_\alpha \wedge d\pi_{\alpha+\frac{\pi}{2}}) = [S^1](\pi_\alpha d\pi_{\alpha+\frac{\pi}{2}})
= (\pi_\alpha, \pi_{\alpha+\frac{\pi}{2}}) \# [S^1](x \, dy)
= \frac{\pi^2}{2} .
\]

In the last line we used that $(\pi_\alpha, \pi_{\alpha+\frac{\pi}{2}}) \# [S^1]$ is a counterclockwise parametrization of the square in $\mathbb{R}^2$ with corners $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(\pi, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$ that has filling area $\frac{\pi^2}{2}$. With Lemma 2.5(2) this implies that
\[
M_{m*}(T) = M(T) \geq \frac{\pi^2}{2} \approx 4.9348
\]
and hence the Gromov-mass star filling area (and then also the inner Riemannian filling area) of $[S^1]$ is bounded from below by this number.

Computer assisted computations suggest that $\tilde{\omega}$ is not a calibration but it seems not too far off. In this direction, it is useful to get upper bounds on the $L^1$ norm
\[
\|p(f)\|_1 := \int_0^\pi \int_0^\pi p_{\alpha,\beta}(f) \, d\beta \, d\alpha .
\]
It is unclear to the author if this has an upper bound independent of $f \in E(S^1) \setminus S^1$ even though computer assisted computations suggest that the following question has a positive answer.

**Question 3.** Is it true that

$$\|p(f)\|_1 \leq \frac{\pi^2}{2}$$

with equality if and only if $f \in S^2_2 \setminus S^1$?

If the above question has a positive answer, then

$$\|\tilde{\omega}\|_m \leq \frac{\pi}{2}$$

because with a very crude estimate

$$\tilde{\omega} f(v \wedge w) = \frac{1}{\pi} \int_0^\pi \int_0^\pi p_{\alpha,\beta}(f)(v_\alpha w_\beta - v_\beta w_\alpha) \, d\beta \, d\alpha \leq \frac{1}{\pi} \|p(f)\|_1 \|v\|_\infty \|w\|_\infty$$

for all $v, w \in L^\infty([0, \pi])$. With Proposition 3.8, $M_m(T)\|\tilde{\omega}\|_m \geq T(\tilde{\omega}) = \|S^2_2\|(\tilde{\omega}) = 2\pi$. A positive answer to Question 3 would imply $M_m(T)^\ast \geq M_m(T) \geq 4$ with [1, Proposition 3.14]. This lower bound for the Gromov-mass star mass is weaker than $\frac{\pi^2}{2}$ obtained above, but the bound on $\|\tilde{\omega}\|$ can certainly be improved upon.

**References**

[1] J. C. Alvarez Paiva, A. C. Thompson. Volumes on normed and Finsler spaces. In A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ. vol. 50, pp. 1–48, Cambridge Univ. Press, 2004.

[2] L. Ambrosio, B. Kirchheim. Currents in metric spaces. Acta Math. 185 (2000), 1–80.

[3] L. Ambrosio, B. Kirchheim. Rectifiable sets in metric and Banach spaces. Math. Ann. 318 (2000), 527–555.

[4] V. Bangert, C. Croke, S. Ivanov, M. Katz. Filling area conjecture and ovalless real hyperelliptic surfaces. Geom. Funct. Anal. 15 (2005), 577–597.

[5] D. Burago, S. Ivanov. Minimality of planes in normed spaces. Geom. Funct. Anal. 22 (2012), 627–638.

[6] T. De Pauw, R. Hardt. Rectifiable and flat $G$ chains in a metric space. Amer. J. Math. 134 (2012), 1–69.

[7] G. de Rham. Variétés différentiables, formes, courants, formes harmoniques. Actualités Sci. Ind. vol. 1222, Hermann, 1955.

[8] A. W. M. Dress. Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces. Adv. in Math. 53 (1984), 321–402.

[9] H. Federer, W. H. Fleming. Normal and integral currents. Ann. of Math. 72 (1960), 458–520.

[10] H. Federer. Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften vol. 153, Springer, 1969.

[11] B. Fuglede. Stability in the isoperimetric problem. Bull. London Math. Soc. 18 (1986), 599–605.

[12] M. Gromov. Filling Riemannian manifolds. J. Diff. Geom. 18 (1983), 1–147.

[13] R. Harvey, H. B. Lawson. Calibrated geometries. Acta Math. 148 (1982), 47–157.

[14] J. R. Isbell. Six theorems about injective metric spaces. Comment. Math. Helv. 39 (1964), 65–76.

[15] S. V. Ivanov. Volumes and areas of Lipschitz metrics. Algebra i Analiz 20 (2008), 74–111.

[16] S. V. Ivanov. Filling minimality of Finslerian 2-discs. Proc. Steklov Inst. Math. 273 (2011), 176–190.

[17] F. John. Extremum problems with inequalities as subsidiary conditions. In Studies and essays presented to R. Courant on his 60th birthday, pp. 187–204, Interscience, 1948.
[18] W. B. Johnson, J. Lindenstrauss, G. Schechtman. Extension of Lipschitz maps into Banach spaces. Israel J. Math. 54 (1986), 129–138.
[19] B. Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. Proc. Amer. Math. Soc. 121 (1994), 113–123.
[20] U. Lang. Local currents in metric spaces. J. Geom. Anal. 21 (2011), 683–742.
[21] U. Lang. Injective hulls of certain discrete metric spaces and groups. J. Topol. Anal. 5 (2013), 297–331.
[22] P. M. Pu. Some inequalities in certain nonorientable Riemannian manifolds. Pacific. J. Math. 2 (1952), 55–71.
[23] R. Züst. Integration of Hölder forms and currents in snowflake spaces. Calc. Var. and PDE 40 (2011), 99–124.
[24] R. Züst. Functions of bounded fractional variation and fractal currents. Geom. Funct. Anal. 29 (2019), 1235–1294.

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