ON THE HILBERT GEOMETRY OF PRODUCTS

CONSTANTIN VERNICOS

Abstract. We prove that the Hilbert geometry of a product of convex sets is bi-lipschitz equivalent the direct product of their respective Hilbert geometries. We also prove that the volume entropy is additive with respect to product and that amenability of a product is equivalent to the amenability of each terms.

INTRODUCTION AND STATEMENT OF RESULTS

Hilbert geometries are simple metric geometries defined in the interior of a convex set thanks to cross-ratios. They are generalisations of the projective model of the Hyperbolic Geometry. Because of their definition they are invariant by the action of projective transformations. Among all these geometries, those admitting a discrete subgroup of their isometries acting co-compactly, commonly known as divisible Hilbert Geometries or divisible convex sets, play an important part. For instance we can find examples of such geometries, which are Hyperbolic in the sense of Gromov with a quotient which does not admit any Riemannian Hyperbolic metric [Ben06].

The present paper focuses on product of Hilbert Geometries, and takes its roots in the following question: Does the product of two divisible convex sets give a divisible convex set ? The answer to that question is no and is given by a very simple example, the Hilbert geometry of the square. Indeed, the Hilbert geometry of the segment $[-1,1]$, which is isometric to the real line, is divisible. However the product of two such segments, which is a square in $\mathbb{R}^2$, endowed with its Hilbert geometry is not a divisible convex set, which is related to the fact that one can’t immerse $PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$ into $PGL(3,\mathbb{R})$.

However following B. Colbois, C. Vernicos, P. Verovic [CV11] the Hilbert Geometry of a polygon is bi-lipschitz equivalent to $\mathbb{R}^2$. In the light of that example we asked ourselves what is the relation between the Hilbert Geometry of a product and the product of Hilbert Geometries, our main theorem gives a complete answer to that question:

Theorem 1 (Main Theorem). The Hilbert Geometry of a product of open convex sets is bi-lipschitz equivalent to the direct metric product of the Hilbert Geometries of those convex sets.

The proof of that theorem is surprisingly simple but it allows us to get an impressive range of corollaries. Noticeably with respect to the
volume entropy (see also G. Berck, A. Bernig and C. Vernicos [BBV10], and M. Crampon [Cra]) and amenability (see C. Vernicos [Ver09]) we obtain the following consequences.

**Proposition 2** (Main consequences). Consider the two bounded open convex sets $A$ and $B$, then

1. The volume entropy is additive:
   
   $\text{Ent}(A \times B) = \text{Ent}(A) + \text{Ent}(B)$;

2. The product Hilbert geometry $(A \times B, d_{A \times B})$ is amenable if and only if both Hilbert geometries $(A, d_A)$ and $(B, d_B)$ are amenable.

Although the product of divisible convex sets need not be divisible itself thanks to the Main consequences one can apply M. Crampon [Cra] theorem on volume entropy to get new rigidity results (see corollaries 8 and 10 in section 4).

Let us conclude by an ”opening” remark. Our Main theorem shows that a second family of Hilbert Geometry seems to play a similar role to the divisible one, those we could call the lip-divisible ones, i.e., whose group of bi-Lipschitz bijections admits a discrete subgroup acting co-compactly. Noticeably, following our theorem, this family is closed under product.

1. **Definitions and notations**

A proper open set in $\mathbb{R}^n$ is a set not containing a whole line.

A Hilbert geometry $(C, d_C)$ is a non empty proper open convex set $C$ on $\mathbb{R}^n$ (that we shall call convex domain) with the Hilbert distance $d_C$ defined as follows: for any distinct points $p$ and $q$ in $C$, the line passing through $p$ and $q$ meets the boundary $\partial C$ of $C$ at two points $a$ and $b$, such that one walking on the line goes consecutively by $a$, $p$, $q$, $b$. Then we define

$$ d_C(p, q) = \frac{1}{2} \ln[a, p, q, b], $$

where $[a, p, q, b]$ is the cross ratio of $(a, p, q, b)$, i.e.,

$$ [a, p, q, b] = \frac{\|q - a\|}{\|p - a\|} \times \frac{\|p - b\|}{\|q - b\|} > 1, $$

with $\| \cdot \|$ the canonical euclidean norm in $\mathbb{R}^n$. If either $a$ or $b$ is at infinity the corresponding ratio will be taken equal to 1.

Note that the invariance of the cross ratio by a projective map implies the invariance of $d_C$ by such a map.

These geometries are naturally endowed with a $C^0$ Finsler metric $F_C$ as follows: if $p \in C$ and $v \in T_p C = \mathbb{R}^n$ with $v \neq 0$, the straight line passing by $p$ and directed by $v$ meets $\partial C$ at two points $p_C^+$ and $p_C^-$. Then let $t^+$ and $t^-$ be two positive numbers such that $p + t^+ v = p_C^+$ and $p - t^- v = p_C^-$. 


and \( p - t^\cdot v = p^+_C \), in other words these numbers corresponds to the time necessary to reach the boundary starting at \( p \) with the speed \( v \) and \(-v\). Then we define

\[
F_C(p, v) = \frac{1}{2} \left( \frac{1}{t^+} + \frac{1}{t^-} \right) \quad \text{and} \quad F_C(p, 0) = 0.
\]

Should \( p^+_C \) or \( p^-_C \) be at infinity, then corresponding ratio will be taken equal to 0.

The Hilbert distance \( d_C \) is the length distance associated to \( F_C \).

Thanks to that Finsler metric, we can built a Borel measure \( \mu_C \) on \( C \) (which is actually the Hausdorff measure of the metric space \((C, d_C)\), see [BBI01], exemple 5.5.13) as follows.

To any \( p \in C \), let \( B_C(p) = \{ v \in \mathbb{R}^n \mid F_C(p, v) < 1 \} \) be the open unit ball in \( T_pC = \mathbb{R}^n \) of the norm \( F_C(p, \cdot) \) and \( \omega_n \) the euclidean volume of the open unit ball of the standard euclidean space \( \mathbb{R}^n \). Consider the (density) function \( h_C: C \to \mathbb{R} \) given by

\[
h_C(p) = \frac{\omega_n}{\text{Leb}(B_C(p))},
\]

where \( \text{Leb} \) is the canonical Lebesgue measure of \( \mathbb{R}^n \) equal to 1 on the unit "hypercube". We define \( \mu_C \), which we shall call the Hilbert Measure on \( C \), by

\[
\mu_C(A) = \int_A h_C(p) \text{dLeb}(p)
\]

for any Borel set \( A \) of \( C \).

The bottom of the spectrum of \( C \), denoted by \( \lambda_1(C) \), and the Sobolev constant \( S_\infty(C) \) are defined as in a Riemannian manifold of infinite volume, thanks to the Raleigh quotients as follows

\[
\lambda_1(C) = \inf_C \frac{\int_C ||df_p||^2 \; d\mu_C(p)}{\int_C f^2(p) d\mu_C(p)}, \quad S_\infty(C) = \inf_C \frac{\int_C ||df_p|| \; d\mu_C(p)}{\int_C |f| (p) d\mu_C(p)},
\]

where the infimum is taken over all non zero lipschitz functions with compact support in \( C \).

Finally the Cheeger constant of \( C \) is defined by

\[
I_\infty(C) = \inf_U \frac{\mu_C(\partial U)}{\mu_C(U)},
\]

where \( U \) is an open set in \( C \) whose closure is compact and whose boundary is a \( n-1 \) dimensional submanifold, and \( \mu_C \) is the Hausdorff measure associated to the restriction of the Finsler norm \( F_C \) to hypersurfaces. Thanks to [CV06] we know that there is a constant \( c \) such that

\[
\frac{1}{c} \cdot S_\infty(C) \leq I_\infty(C) \leq c \cdot S_\infty(C).
\]
2. The decomposition lemma

Theorem 3. Consider the family of convex sets $A_i \in \mathbb{R}^{n_i}$, for $i = 1, \ldots, k$ and $n_i \in \mathbb{N}^*$, then for any point $p = (p_1, \ldots, p_k)$ of the convex set $A_1 \times \cdots \times A_k$ and any vector $v = (v_1, \ldots, v_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ one has

$$\max_{1 \leq i \leq k} F_{A_i}(p_i, v_i) \leq F_{A_1 \times \cdots \times A_k}(p, v) \leq \sum_{i=1}^k F_{A_i}(p_i, v_i),$$

therefore the identity restricted to $A_1 \times \cdots \times A_k$ is a bi-lipschitz map between $(A_1 \times \cdots \times A_k, d_{A_1 \times \cdots \times A_k})$ and the direct product of the metric spaces $(A_i, d_{A_i})$ for $i = 1, \ldots, k$.

Proof. Consider a point $p = (p_1, \ldots, p_k)$ of the convex set $A_1 \times \cdots \times A_k$ and a vector $v = (v_1, \ldots, v_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$. If the two positive numbers $t^+$ and $t^-$ are such that

$$p + t^+ v \in \partial (A_1 \times \cdots \times A_k) \quad \text{and} \quad p - t^- v \in \partial (A_1 \times \cdots \times A_k)$$

then $F_{A_1 \times \cdots \times A_k}(p, v) = \frac{1}{2} \left( \frac{1}{t^+} + \frac{1}{t^-} \right)$. This implies that for given $i, j \in \{1, \ldots, k\}$, $p_i + t^+ v_i \in \partial A_i$ and $p_j - t^- v_j \in \partial A_j$.

Hence, should we define for each integer $i \in \{1, \ldots, k\}$ the positive numbers $t^+_i$ and $t^-_i$ by asking that

$$p_i + t^+_i v_i \in \partial A_i \quad \text{and} \quad p_i - t^-_i v_i \in \partial A_i,$$

we would then obtain $t^+ = \min\{t^+_1, \ldots, t^+_k\}$ and $t^- = \min\{t^-_1, \ldots, t^-_k\}$, which would imply that

$$F_{A_1 \times \cdots \times A_k}(p, v) = \frac{1}{2} \left( \max\{1/t^+_1, \ldots, 1/t^+_k\} + \max\{1/t^-_1, \ldots, 1/t^-_k\} \right)$$

and therefore using the classical comparison between the $l^1$ and $l^\infty$ norm in $\mathbb{R}^k$ we get

$$\frac{1}{2} \max_{1 \leq i \leq k} \left( \frac{1}{t^+_i} + \frac{1}{t^-_i} \right) \leq F_{A_1 \times \cdots \times A_k}(p, v) \leq \frac{1}{2} \left( \sum_{i=1}^k \frac{1}{t^+_i} + \sum_{i=1}^k \frac{1}{t^-_i} \right)$$

which we can rewrite by associativity of the addition in the following form:

$$\max_{1 \leq i \leq k} \frac{1}{2} \left( \frac{1}{t^+_i} + \frac{1}{t^-_i} \right) \leq F_{A_1 \times \cdots \times A_k}(p, v) \leq \sum_{i=1}^k \frac{1}{2} \left( \frac{1}{t^+_i} + \frac{1}{t^-_i} \right).$$

□

Corollary 4. Consider the family of convex sets $A_i \in \mathbb{R}^{n_i}$, for $i = 1, \ldots, k$ and $n_i \in \mathbb{N}^*$. Then at any point $p = (p_1, \ldots, p_k) \in A_1 \times \cdots \times A_n$ we have the following inequality:

$$\prod_{i=1}^k d\mu_{A_i}(p_i) \leq d\mu_{A_1 \times \cdots \times A_n}(p) \leq k^{n_1 + \cdots + n_k} \prod_{i=1}^k d\mu_{A_i}(p_i).$$

(4)
Proof. Let us denote by \( \text{Leb} \) the Lebesgue measure on \( \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \) normalised by 1 on the unit cube and by \( \text{Leb}_i \) the corresponding one on \( \mathbb{R}^{n_i} \). Then, for any point \( p = (p_1, \ldots, p_k) \) of the convex set \( A_1 \times \cdots \times A_k \), if \( T_p B(1) \) corresponds to the unit tangent ball at \( p \) and for all \( i, T_{p_i} B(1) \) to the unit tangent ball at \( p_i \) for \( A_i \) then one has

\[
\frac{1}{k^{n_1 + \cdots + n_k}} \prod_{i=1}^{k} \text{Leb}_i(T_p B(1)) \leq \text{Leb}(T_p B(1)) \leq \prod_{i=1}^{k} \text{Leb}_i(T_p B(1))
\]

and the corollary follows by definition of the measure. \( \square \)

3. Illustrations

In order to illustrate in a simple way our main theorem we apply it to two geometries: the \( n \)-dimensional cube and the \( n \)-dimensional simplex.

The next two applications, are useful to obtain qualitative information on volumes in the given Hilbert geometries (see for instance proposition 6 in [CVV04] and its corollaries 6.1 and 6.2).

**Proposition 5.** Let \( C^n = [-1, 1]^n \) be the \( n \)-dimensional cube. We have the following

1. \( C^n \) is bi-lipschitz equivalent to \( \mathbb{R}^n \).
2. For all \( x = (x_1, \ldots, x_n) \in [-1, 1]^n \), let \( T_x B(1) \) be the tangent unit ball for \( F_{C^n} \), then we have

\[
(2/n)^n \prod_{i=1}^{n} (1 - x_i^2) \leq \text{Leb}(T_x B(1)) \leq 2^n \prod_{i=1}^{n} (1 - x_i^2).
\]

Notice that actually there is a better lower bound because one can replace \((2/n)^n\) by \(2^n/(n!)\), using theorem \( \text{[B]} \) instead of its corollary.

**Proposition 6.** Let \( S^n = [0, +\infty]^n \) be the \( n \)-dimensional positive cone, whose Hilbert geometry is isometric to the Hilbert geometry of the simplex of \( \mathbb{R}^n \). We have the following

1. \( S^n \) is bi-lipschitz equivalent to \( \mathbb{R}^n \).
2. \( x = (x_1, \ldots, x_n) \in [0, +\infty]^n \), let \( T_x B(1) \) be the tangent unit ball for \( F_{S^n} \), then we have

\[
(4/n)^n \prod_{i=1}^{n} x_i \leq \text{Leb}(T_x B(1)) \leq 4^n \prod_{i=1}^{n} x_i.
\]

For the same reason, one can also replace \((4/n)^n\) by \(4^n/(n!)\) in this lower bound. In that case one can actually make a precise computation and obtain, for instance,

\[
\text{Leb}(T_x B(1)) = 12x_1 \cdot x_2
\]
4. THE VOLUME ENTROPY OF PRODUCTS

The general behaviour of the volume entropy is not yet completely understood, and the main conjecture, to prove that it is always less than that of the Hyperbolic geometry, is still open in dimension bigger than 2. Therefore the next result and the generalisation it implies validate this conjecture a little bit more. They also simplify and generalise result obtained in [Ver08]

Proposition 7. The volume entropy is subadditive with respect to product of convex sets: take a family of bounded open convex sets $A_i \in \mathbb{R}^{n_i}$, for $i = 1, \ldots, k$ and $n_i \in \mathbb{N}^*$, then one has

$$\max_{1 \leq i \leq k} \{\text{Ent}(A_i)\} \leq \text{Ent}(A_1 \times \cdots \times A_k) \leq \sum_{i=1}^{k} \text{Ent}(A_i).$$

If the convex sets are also bounded then we actually have additivity:

$$\text{Ent}(A_1 \times \cdots \times A_k) = \sum_{i=1}^{k} \text{Ent}(A_i).$$

Proof. We will do the proof for $k = 2$, the general case trivially follows. Let $A$ and $C$ be two open convex sets, respectively in $\mathbb{R}^n$ and $\mathbb{R}^m$, and let $p = (p_A, p_C) \in A \times C$.

Thanks to the left hand side inequality of theorem 3, we obtain for any point $q = (q_A, q_C) \in A \times C$ that

$$d_A(p_A, q_A) \leq d_{A \times C}(p, q) \quad \text{and} \quad d_C(p_C, q_C) \leq d_{A \times C}(p, q),$$

which imply the next inclusion

$$B_{A \times C}(p, R) \subset B_A(p_A, R) \times B_C(p_C, R).$$

The right hand side inequality of theorem 3 yields in turn that for any $\varepsilon > 0$, $B_A(p_A, \varepsilon R) \times B_C(p_C, (1 - \varepsilon)R)$ is a subset of $B_{A \times C}(p, R)$.

Hence, computing the volumes, using the inequalities of corollary 4 we obtain that

$$\mu_A(B_A(\varepsilon R)) \times \mu_C(B_C((1 - \varepsilon)p, R)) \leq \mu_{A \times C}(B_{A \times C}(p, R)) \leq 2^{n+m} \mu_A(B_A(p, R)) \times \mu_C(B_C(p, R)).$$

Taking the logarithm of both inequalities, dividing by $R$ and taking the limit as $R \to +\infty$, gives the following inequality, for any $\varepsilon > 0$:

$$\varepsilon \text{Ent}(A) + (1 - \varepsilon) \text{Ent}(C) \leq \text{Ent}(A \times C) \leq \text{Ent}(A) + \text{Ent}(C),$$

which implies 6.

In case both $A$ and $C$ are bounded, we can work as in [CV04] with the asymptotic balls $A_{A \times C}(p, R)$, that is the image of $A \times C$ by
the dilation of ratio \( \tanh(R) \) centred at \( p \). Those asymptotic balls are exactly the product of the asymptotic balls of \( A \) and \( C \) respectively centred at \( p_A \) and \( p_C \). Therefore

\[
\mu_A(\AsB_A(p_A, R)) \times \mu_C(\AsB_C(p_C, R)) \\
\leq \mu_{A \times C}(\AsB_{A \times C}(p, R)) \\
\leq 2^{n+m} \mu_A(\AsB_A(p_A, R)) \mu_C(\AsB_C(p_C, R)).
\]

This inequality allows us to conclude using the fact shown in [CV04] that there exists some constant \( K \) such that

\[
B_{A \times C}(R - 1) \subset \AsB_{A \times C}(p, R) \subset B_{A \times C}(R + K).
\]

The following corollary is a straightforward application of M. Crampon [Cra] rigidity result and the subadditivity of entropy:

**Corollary 8.** Consider the family of divisible convex set with \( C^1 \) boundary \( A_i \in \mathbb{R}^{n_i} \), for \( i = 1, \ldots, k \) and \( n_i \in \mathbb{N}^* \), then one has

- \( \text{Ent}(A_1 \times \cdots \times A_k) \leq n_i - k \),
- \( \text{Equality occurs if and only all } A_i \text{ are ellipsoids.} \)

**Corollary 9.** Let \( C \) be a convex set in \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \) and let \( p \) be a point outside \( \mathbb{R}^n \) in \( \mathbb{R}^{n+1} \). Then \( \text{Ent}(p + C) = \text{Ent}(C) \).

**Proof.** This comes from the fact that \( p + C \) is projectively equivalent to \( C \times [0, +\infty[ \), and

\[
\max\{\text{Ent}(C), \text{Ent}([0, +\infty[)\} \leq \text{Ent}(C \times [0, +\infty[) \leq \text{Ent}(C) + \text{Ent}([0, +\infty[)
\]

by Proposition [7]. As \([0, +\infty[\) endowed with its Hilbert geometry is isometric to the real line we easily conclude. \( \square \)

M. Crampon [Cra] rigidity result applied to that case therefore implies:

**Corollary 10.** Let \( C \) be a divisible convex set with \( C^1 \) boundary in \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \) and let \( p \) be a point outside \( \mathbb{R}^n \) in \( \mathbb{R}^{n+1} \). Then

- \( \text{Ent}(p + C) \leq n - 1 \),
- \( \text{Equality occurs if and only if } C \text{ is an ellipsoid.} \)

5. **Amenability of products**

Following our former work [Ver09] we say that a Hilbert geometry is amenable if and only if the bottom of its spectrum is null, which is equivalent to the nullity of its Cheeger constant. In this section we show how this property behaves with respect to product.
Proposition 11. Consider the family of convex sets $A_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, k$. The following are equivalent

(i) The Hilbert geometry of $A_1 \times \cdots \times A_k$ is amenable;
(ii) For all $i$, $A_i$ is amenable;

More precisely, with respect to the bottom of the spectrum and the Sobolev constants we have the following inequalities:

\begin{align}
\lambda_1(A_1 \times \cdots \times A_k) &\geq k^{-n_1 - \cdots - n_k} \max_{1 \leq i \leq k} \lambda_1(A_i) \\
S_\infty(A_1 \times \cdots \times A_k) &\leq k^{n_1 + \cdots + n_k} \sum_{1 \leq i \leq k} S_\infty(A_i).
\end{align}

Proof. Let us denote $A_1 \times \cdots \times A_k$ by $\Pi$. Consider a lipschitz function with compact support $f : A_1 \times \cdots \times A_k \to \mathbb{R}$, then we have for almost every point in $A_1 \times \cdots \times A_k$ the function $f$ admits a differential $df$ and for any $i$ we have $\|df\|_{A_i} \leq \|df\|_{\Pi}$, therefore for any $i$ we have

\begin{align}
\int_{A_1 \times \cdots \times A_k} \|df\|^2_{A_i} d\mu_{\Pi} &\geq \int_{A_1 \times \cdots \times A_k} \|df\|^2_{A_i} d\mu_{\Pi} \\
\int_{\Pi} \|df\|^2_{A_i} d\mu_{\Pi} &\geq \int_{\Pi} \|df\|^2_{A_i} d\mu_{A_1} \cdots d\mu_{A_k} \\
\int_{\Pi} \|df\|^2_{A_i} d\mu_{\Pi} &\geq \lambda_1(A_i) \int_{\Pi} f^2 d\mu_{A_1} \cdots d\mu_{A_k},
\end{align}

and thanks to corollary [4] we have

\begin{align}
\int_{\Pi} \|df\|^2_{A_i} d\mu_{\Pi} &\geq \frac{\lambda(A_i)}{k^{n_1 + \cdots + n_k}} \int_{\Pi} f^2 d\mu_{\Pi}.
\end{align}

which implies the inequality [10], and the implication (i) $\Rightarrow$ (ii).

For the other implication we will use the Cheeger constant and for better clarity, restrict ourselves to the product of two convex sets. Now let us suppose that $I(A) = I(C) = 0$ and let us prove that $I(A \times C) = 0$. To do so we will prove the inequality [11]. Let us consider two real valued lipschitz functions $f$ and $g$ with compact support respectively in $A$ and $C$. We then define the function $h : A \times C \to \mathbb{R}$ as follows: for any $p = (p_A, p_C) \in A \times C$, $h(p) = f(p_A)g(p_C)$. We first use the textbook equality $\text{dh} = gdf + f dg$.

Applying the right hand side inequality of Theorem [3] we obtain

\begin{align}
\|dh\|_{A \times C} &\leq \|dh\|_{A} + \|dh\|_{C} \leq |g| \cdot \|df\|_{A} + |f| \cdot \|dg\|_{C}.
\end{align}
The next step consists in integrating over $A \times C$ taking into account the right hand side inequality of (4) to obtain
\[
\int_{A \times C} ||dh||_{A \times C} d\mu_{A \times C} \leq 2^{n_A + n_C} \left( \int_C |g| d\mu_C \cdot \int_A ||df||_A d\mu_A \\
+ \int_A |f| d\mu_A \cdot \int_C ||dg||_C d\mu_C \right).
\]
(17)

We finish by dividing by the integral of $|h|$ over $A \times C$ using the right hand side inequality of (4) to finally get
\[
\frac{\int_{A \times C} ||dh||_{A \times C} d\mu_{A \times C}}{\int_{A \times C} |h| d\mu_{A \times C}} \leq 2^{n_A + n_C} \left( \frac{\int_A ||df||_A d\mu_A}{\int_A |f| d\mu_A} + \frac{\int_C ||dg||_C d\mu_C}{\int_C |g| d\mu_C} \right).
\]
(18)

This last inequality implies inequality (11), and allows us to conclude thanks to the main theorem of our paper [Ver09].

This proposition shades some light on the example given by proposition 4.1 in [CV07] of a Hilbert Geometry which is not Hyperbolic in the sense of Gromov, but which has positive bottom of the spectrum, and therefore allows us to get more example of the same kind. Indeed, it is straightforward that a product of convex set is never strictly convex which implies that it is never Hyperbolic in the sense of Gromov.

REFERENCES

[Ben06] Y. Benoist. Convexes hyperboliques et quasiisométries. (Hyperbolic convexes and quasiisometries.). Geom. Dedicata, 122:109–134, 2006.

[BBI01] D. Burago, Y. Burago and S. Ivanov. A Course in Metric Geometry, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, 2001.

[BBV10] G. Berck, A. Bernig and C. Vernicos. Volume Entropy of Hilbert Geometries. Pacific J. of Math. 245(2):201–225, 2010.

[Ber09] A. Bernig. Hilbert Geometry of Polytopes. Archiv der Mathematik, 92:314-324, 2009.

[CV06] B. Colbois and C. Vernicos. Bas du spectre et delta-hyperbolicité en géométrie de Hilbert plane. Bulletin de la Société Mathématique de France 134(3):357–381, 2006.

[CV07] —Les géométries de Hilbert sont à géométrie locale bornée. Annales de l’Institut Fourier 57(4):1359-1375, 2007.

[CV11] B. Colbois, C. Vernicos and P. Verovic. Hilbert Geometry for convex polygonal domains. preprint 2008, to appear in Journal of Geometry. arXiv:0804.1620v1

[CVV04] —L’aire des triangles idéaux en géométrie de Hilbert. Enseign. Math., 50(3–4):203–237, 2004.
B. Colbois and P. Verovic. Hilbert geometry for strictly convex domain. Geom. Dedicata, 105:29–42, 2004.

Hilbert domains quasi-isometric to normed vector spaces. preprint 2008, arXiv:0804.1619v1 [math.MG].

M. Crampon. Entropies of compact strictly convex projective manifolds. preprint arXiv:0904.2489 2009.

P. de la Harpe. On Hilbert’s metric for simplices. in Geometric group theory, Vol. 1 (Sussex, 1991), pages 97–119. Cambridge Univ. Press, 1993.

D. Hilbert. Les fondements de la Géométrie, édition critique préparée par P. Rossier. Dunod, 1971.

É. Socié-Méthou. Comportement asymptotiques et rigidités en géométries de Hilbert, thèse de doctorat de l’université de Strasbourg, 2000. http://www-irma.u-strasbg.fr/irma/publications/2000/00044.ps.gz.

—Caractérisation des ellipsoïdes par leurs groupes d’automorphismes. Ann. Sci. de l’ÉNS, 35(4):537–548, 2002.

C. Vernicos. Spectral Radius and amenability in Hilbert Geometry. Houston Journal of Math., 35(4):1143-1169, 2009.

—Sur l’entropie volumique des géométries de Hilbert. Sém. Th. Spe. et Geo. de Grenoble, 26:155-176, 2008.

—Lipschitz Characterisation of Polytopal Hilbert Geometries. arXiv:0812.1632v1 [math.DG], 2008.

Institut de mathématique et de modélisation de Montpellier, Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, F–34395 Montpellier Cedex, France

E-mail address: Constantin.Vernicos@math.univ-montp2.fr