Massive Unsourced Random Access Based on Uncoupled Compressive Sensing: Another Blessing of Massive MIMO

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Abstract—We put forward a new algorithmic solution to the massive unsourced random access (URA) problem, by leveraging the rich spatial dimensionality offered by large-scale antenna arrays. This paper makes an observation that spatial signature is key to URA in massive connectivity setups. The proposed scheme relies on a slotted transmission framework but eliminates completely the need for concatenated coding that was introduced in the context of the coupled compressive sensing (CCS) paradigm. Indeed, all existing works on CCS-based URA rely on an inner/outer tree-based encoder/decoder to stitch the slot-wise recovered sequences. This paper takes a different path by harnessing the nature-provided correlations between the slot-wise reconstructed channels of each user in order to put together its decoded sequences. The required slot-wise channel estimates and decoded sequences are first obtained through the powerful hybrid generalized approximate message passing (HyGAMP) algorithm which systematically accommodates the multiantenna-induced group sparsity. Then, a channel correlation-aware clustering framework based on the expectation-maximization (EM) concept is used together with the Hungarian algorithm to find the slot-wise optimal assignment matrices by enforcing two clustering constraints that are very specific to the problem at hand. Stitching is then readily accomplished by associating the decoded sequences to their respective users according to the ensuing assignment matrices. Exhaustive computer simulations reveal that the proposed scheme outperforms by far the only known work from the open literature, which investigates the use of large-scale antenna arrays in the context of massive URA. More precisely, it will be seen that our scheme can accommodate a very large number of active users with much higher total spectral efficiency while using a remarkably smaller number of receive antennas and achieving a low decoding error probability.

Index Terms—Unsourced random access, massive connectivity, massive MIMO, channel estimation, clustering, hybrid approximate message passing.

I. INTRODUCTION

A. Background and Motivation

Massive random access in which a base station (BS) equipped with a large number of antennas is serving a large number of contending users has recently attracted considerable attention. This surge of interest is fuelled by the need to satisfy the soaring demand in wireless connectivity for many envisioned IoT applications such as massive machine-type communication (mMTC). MTC have two distinct features [1] that make them drastically different from human-type communications (HTC) around which previous cellular systems have mainly evolved: i) machine-type devices (MTDs) require sporadic access to the network and ii) MTDs usually transmit small data payloads using short-packet signaling. The sporadic access leads to the overall mMTC traffic being generated by an unknown and random subset of active MTDs (at any given transmission instant or frame). This calls for the development of scalable random access protocols that are able to accommodate a massive number of MTDs. Short-packet transmissions, however, make the traditional grant-based access (with the associated scheduling overhead) fall short in terms of spectrum efficiency and latency, which are two key performance metrics in next-generation wireless networks. Hence, a number of grant-free random access schemes have been recently investigated within the specific context of massive connectivity (see [2] and references therein). In sourced, random access, grant-free transmissions often require two phases: i) pilot sequences are first used to detect the active users and estimate their channels, then ii) the identified active users are scheduled to transmit their messages. In this context, it was shown that the joint device activity detection and channel estimation task can be cast as a compressed sensing (CS) problem: more precisely a useful CS variant called the multiple-measurement vector (MMV) in presence of multiple receive antennas.

Among a plethora of CS recovery techniques, the approximate message passing (AMP) algorithm [3] has attracted considerable attention within the framework of massive random access mainly due to the existence of simple scalar equations that track its dynamics, as rigorously analyzed in [4]. Besides CS-based schemes, there is another line of work which has investigated the use of random access strategies based on conventional ALOHA [5] and coded slotted ALOHA [6]. In many applications, however, the BS is interested in the

1As opposed to the unsourced case, sourced multiple access refers to the case where the BS is interested in both the messages and the identities of the users that generated them.
transmitted messages only and not the IDs of the users, thereby leading to the so-called unsourced random access (URA). The information-theoretic work in [7] introduced a random coding existence bound for URA using a random Gaussian codebook with maximum likelihood-decoding at the BS. Moreover, popular multiple access schemes, e.g., ALOHA, coded-slotted ALOHA, and treating interference as noise (TIN) were compared against the established fundamental limit, showing that none of them achieves the optimal predicted performance. The difficulty in achieving the underlying bound stems from the exponential (in blocklength) size of the codebooks analyzed in [7].

Recent works on URA have focused more on the algorithmic aspect of the problem by relying on the CS-based encoding/decoding paradigm in conjunction with a slotted transmission framework. The use of slotted transmissions is driven by the need to alleviate the inherent prohibitive computational burden of the underlying index coding problem. More specifically, in the coded/coupled compressive sensing (CCS) scheme [8], the binary message of each user is partitioned into multiple information bit sequences (or chunks). Then binary linear block coding is used to couple the different sequences before using a random Gaussian codebook for actual transmissions over multiple slots. At the receiver side, inner CS-based decoding is first performed to recover the slot-wise transmitted sequences up to an unknown permutation. An outer tree-based decoder is then used to stitch the decoded binary sequences across different slots. A computationally tractable URA scheme has been recently proposed in [9] — based on the CCS framework — wherein the authors exploit the concept of sparse regression codes (SPARCs) [10] to reduce the size of the required codebook matrix. The main idea of SPARCs is to encode information in structured linear combinations of the columns of a fixed codebook matrix so as to design a polynomial-time complexity encoder over the field of real numbers. In [9], the AMP algorithm was used as inner CS decoder and the state-evolution framework was utilized to analyze the performance of the resulting URA scheme. Further extensions of the CCS framework for URA were also made in [11] where a low-complexity algorithm based on chirps for CS decoding was introduced. A number of other algorithmic solutions to the unsourced random access problem were also reported in [12]–[14]. However, all the aforementioned works assume a single receive antenna at the BS and it was only recently that the use of large-scale antenna arrays in the context of URA has been investigated in [15]. There, the authors use a low-complexity covariance-based CS (CB-CS) recovery algorithm [16] for activity detection, which iteratively finds the large-scale fading coefficients of all the users. Within the specific context of massive connectivity, CB-CS has the best known scaling law in terms of the required number of observations versus the number of active users. As there is no free lunch, however, the cost is to use a very large number of receive antennas as compared to what traditional CS methods would require. In plain English, while traditional CS algorithms require more pilot symbols than active users, CB-CS requires more antennas than active users. In this regard, CB-CS is better suited to the sourced random access scenario wherein the coherence time of the channel is split into a training period (for user identification) and a data period (for packets transmission). In this situation, existing CS-based algorithms which reconstruct the entire channel matrix require large-size pilot sequences (i.e., a prohibitively large overhead) in presence of a large number of active users. To sidestep this problem, CB-SC relies rather on the use of more number of receive antennas to identify the more active users by estimating their large-scale coefficients only. In the URA scenario, however, no user identification is required and the entire coherence block is dedicated to data communication thereby eliminating the need for using more antenna elements than the number of active users.

B. Contributions

Motivated by all the aforementioned facts, we devise in this paper for the first time an algorithmic solution to the URA problem that can accommodate much more active users than the number of receive antenna elements at BS. The proposed scheme exploits the spatial channel statistics to stitch the decoded binary sequences among different slots thereby eliminating completely the need for concatenated coding as was done in all existing works on CCS-based URA [8], [11], [15]. In fact, the strong correlation between the slot-wise reconstructed channel vectors pertaining to each active device already provides sufficient information for stitching its decoded sequences across the different slots. It is the task of the inner CS-based decoder to recover the support of the unknown sparse vector and to estimate the users’ channels in each slot. Each recovered support is used to decode the associated information bit sequences that were transmitted by all the active users. Then, by clustering together the slot-wise reconstructed channels of each user, it will be possible to cluster/stitch its decoded sequences in order to recover its entire packet. Our CS-based decoder is based on a recent CS technique called the HyGAMP algorithm, which is able to account for the group sparsity in the underlying MMV model by incorporating latent Bernoulli random variables. HyGAMP runs loopy belief propagation coupled with Gaussian and quadratic approximations, for the propagated messages, which become increasingly accurate in the large system limits (i.e., large codebook sizes). At convergence, HyGAMP provides MMSE and MAP estimates of the users’ channels and their activity-indicator Bernoulli random variables. We further resort to the Gaussian-mixture expectation-maximization principle for channel clustering in combination with an integer optimization framework to embed two clustering constraints that are very specific to our problem. It will be seen that the newly proposed algorithm outperforms the state-of-the-art CB-CS URA scheme that has been recently investigated in [15]. In particular, our algorithm makes it possible to accommodate a much larger number of active users with reasonable antenna array sizes while bringing in remarkable performance advantages in terms of the decoding error probability. This is mainly due to the good-quality channel estimates that HyGAMP is able to return owing to the postulated Laplacian prior that can better capture the statistics of the channel coefficients in presence of large-scale fading.
C. Organization of the Paper and Notations

We structure the rest of this paper as follows. In Section II, we introduce the system model. In Section III, we describe the clustering-based stitching procedure of the decoded sequences. In Section IV, we assess the performance of the proposed URA scheme using exhaustive computer simulations. Finally, we draw out some concluding remarks in Section V.

We also mention the common notations used in this paper. Lower- and upper-case bold fonts, \( x \) and \( X \), are used to denote vectors and matrices, respectively. Upper-case calligraphic font, \( \mathcal{X} \) and \( \mathcal{X}^c \), is used to denote single and multivariate random variables, respectively, as well as for sets notation (depending on the context). The \((m,n)\)th entry of \( X \) is denoted as \( X_{mn} \), and the \( n \)th element of \( x \) is denoted as \( x_n \). The identity matrix is denoted as \( I \). The operator \( \text{vec}(X) \) stacks the columns of a matrix \( X \) one below the other. The shorthand notation \( \mathcal{X} \sim \mathcal{CN}(x; m, R) \) means that the random vector \( \mathcal{X} \) follows a complex circular Gaussian distribution with mean \( m \) and auto-covariance matrix \( R \). Likewise, \( \mathcal{X} \sim \mathcal{CN}(x,m,\mu) \) means that the random vector \( \mathcal{X} \) follows a Gaussian distribution with mean \( m \) and variance \( \mu \). Moreover, \{ \}^T \text{ and } \{ \}^H \text{ stand for the transpose and Hermitian (transpose conjugate) operators, respectively. In addition, }|\cdot| \text{ and } ||\cdot|| \text{ stand for the modulus and Euclidean norm, respectively. Given any complex number, } \Re\{\cdot\}, \Im\{\cdot\}, \text{ and } \{\cdot\}^* \text{ return its real part, imaginary part, and complex conjugate, respectively. The Kronecker function and product are denoted as } \delta_{m,n} \text{ and } \otimes, \text{ respectively. We also denote the probability distribution function (pdf) of single and multivariate random variables (RVs) by } p_X(x) \text{ and } p_{X}(x), \text{ respectively. The statistical expectation is denoted as } \mathbb{E}\{\cdot\}, \text{ } j \text{ is the imaginary unit (i.e., } j^2 = -1), \text{ and the notation } \triangleq \text{ is used for definitions.}

II. SYSTEM MODEL AND ASSUMPTIONS

Consider a single-cell network consisting of \( K \) single-antenna devices which are being served by a base station located at the center of a cell of radius \( R \). Devices are assumed to be uniformly scattered inside the cell, and we denote by \( r_k \) (measured in meters) the distance from the \( k \)-th device to the base station. This paper assumes sporadic device activity thereby resulting in a small number, \( K_a \ll K \), of devices being active over each coherence block. The devices communicate to the base station through the uplink uncoordinated scheme, in which every active device wishes to communicate \( B \) bits of information over the channel in a single communication round. The codewords transmitted by active devices are drawn uniformly from a common Gaussian codebook \( \mathcal{C} = \{ \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{2^B} \} \subset \mathbb{C}^n \). More precisely, \( \tilde{c}_b \sim \mathcal{CN}(0, P) \) where \( n \) is the blocklength and \( P \) is the transmit power. We model the device activity and codeword selection by a set of \( 2^B \) Bernoulli random variables \( \delta_{b,k} \) for \( k = 1, \ldots, K \) and \( b = 1, \ldots, 2^B \).

\[
\delta_{b,k} = \begin{cases} 
1 & \text{if user}\ k \text{ is active and transmits codeword } \tilde{c}_b, \\
0 & \text{otherwise.}
\end{cases}
\]

We consider a Gaussian multiple access channel (MAC) with a block fading model and a large-scale antenna array consisting of \( M_r \) receive antenna elements at the BS. The uplink received signal at the \( m \)th antenna element can be expressed as follows:

\[
\tilde{y}^{(m)} = \sum_{k=1}^{K} \sum_{b=1}^{2^B} \sqrt{g_{k,m}} h_{k,m} \delta_{b,k} \tilde{c}_b + \tilde{w}^{(m)}, \quad m = 1, \ldots, M_r.
\]

The random noise vector, \( \tilde{w}^{(m)} \), is modeled by a complex circular Gaussian random vector with independent and identically distributed (i.i.d.) components, i.e., \( \tilde{w}^{(m)} \sim \mathcal{CN}(0, \sigma_w^2 I) \). In addition, \( h_{k,m} \) stands for the small-scale fading coefficient between the \( k \)-th user and the \( m \)th antenna. We assume Rayleigh block fading, i.e., the small-scale fading channel coefficients, \( h_{k,m} \sim \mathcal{CN}(0,1) \), remain constant over the entire observation window which is smaller than the coherence time. Besides, \( g_{k} \) is the large-scale fading coefficient of user \( k \) given by (in dB scale):

\[
g_{k} [\text{dB}] = -\alpha - 10\beta \log_{10} (r_k),
\]

where \( \alpha \) is the fading coefficient measured at distance \( d = 1 \) meter and \( \beta \) is the pathloss exponent. For convenience, we also define the effective channel coefficient by lumping the large- and small-scale fading coefficients together in one quantity, denoted as \( h_{k,m} \triangleq \sqrt{g_{k} h_{k,m}} \), thereby yielding the following equivalent model:

\[
\tilde{y}^{(m)} = \sum_{k=1}^{K} \sum_{b=1}^{2^B} h_{k,m} \delta_{b,k} \tilde{c}_b + \tilde{w}^{(m)}, \quad m = 1, \ldots, M_r.
\]

To define the random access code for this channel, let \( W_k \in \{2^B\} \triangleq \{ 1, 2, \ldots, 2^B \} \) denote the message of user \( k \), such that for some encoding function \( f : \{2^B\} \rightarrow \mathbb{C}^n \), we have \( f(W_k) = \tilde{c}_{b_k} \). By recalling that \( K_a \) stands for the number of active users, the decoding map \( g : \mathbb{C}^n \times K_a \rightarrow \{ \{2^B\} \} \) outputs a list of \( K_a \) decoded messages with the probability of error being defined as:

\[
P_e = \frac{1}{K_a} \sum_{k=1}^{K_a} \Pr(E_k),
\]

and \( E_k \triangleq \{ W_k \notin g(\tilde{y}^{(1)}, \tilde{y}^{(2)}, \ldots, \tilde{y}^{(M_r)}) \} \). Notice here that \( P_e \) depends solely on the number of active users, \( K_a \), instead of the total number of users \( K \). With this formulation in mind, we rewrite (3) in a more succinct matrix-vector form as follows:

\[
\tilde{y}^{(m)} = \tilde{C} \Delta h^{(m)} + \tilde{w}^{(m)}, \quad m = 1, \ldots, M_r,
\]

in which \( \tilde{C} = [\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{2^B}] \in \mathbb{C}^n \times 2^B \) is the codebook matrix, which is common to all the users and \( h^{(m)} \in \{h_{1,m}, h_{2,m}, \ldots, h_{K,m}\}^T \in \mathbb{C}^K \) is the multi-user channel vector at the \( m \)th antenna which incorporates the small- and large-scale fading coefficients. The matrix \( \Delta \in \{0,1\}^{2^B \times K} \) contains only \( K_a \) non-zero columns each of which having a single non-zero entry. Observe here that both \( \Delta \) and \( h^{(m)} \) are

\[\text{The notation } \{2^B\} \text{ stands for choosing } K_a \text{ different elements from the set } [2^B].\]
unknown to the receiver. Hence, by defining $\bar{x}^{(m)} = \Delta h^{(m)}$, it follows that:

$$\bar{y}^{(m)} = \bar{C} \bar{x}^{(m)} + \bar{w}^{(m)}, \quad m = 1, \ldots, M_r. \quad (6)$$

Note here that each active user contributes a single non-zero coefficient in $\bar{x}^{(m)}$ thereby resulting in $K_a$-sparse $2^B$-dimensional vector. Since $K_a$ is much smaller than the total number of codewords $2^B$, $\bar{x}^{(m)}$ has a very small sparsity ratio $\lambda = \frac{K_a}{2^B}$. Observe also that the formulation in (6) belongs to the MMV class in compressed sensing terminology, which can be equivalently rewritten in a more succinct matrix/matrix form as follows:

$$\bar{Y} = \bar{C} \bar{X} + \bar{W}, \quad (7)$$

in which $\bar{Y} = [\bar{y}^{(1)}, \bar{y}^{(2)}, \ldots, \bar{y}^{(M_r)}]$ is the entire measurement matrix and $\bar{X} = [\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots, \bar{x}^{(M_r)}]$. With this formulation, the unknown matrix $\bar{X}$ is row-sparse and we aim to exploit this structure by casting our task into the problem of estimating a group-sparse vector from a set of linear measurements. The theory of sparse reconstruction from noisy observations has been a hot research topic in statistics and we refer the theoretically inclined reader to chap. 7-9 in [17], for elaborate discussions on the theoretical guarantees for sparse reconstruction.

III. PROPOSED UNSOURCED RANDOM ACCESS SCHEME BASED ON COMPRESSIVE SENSING

A. Slotted Transmission Model and Code Selection

By revisiting (1), we see that the number of codewords grows exponentially with the blocklength $n$. Indeed, for a fixed rate $R = \frac{B}{n}$, we have $2^B = 2^{nR}$ — which becomes extremely large even at moderate values of $n$ — thereby making any attempt to directly use standard sparse recovery algorithms computationally prohibitive. Practical approaches have been introduced to alleviate this computational burden, including the slotted transmissions framework also adopted in this paper. Indeed, similar to [3], each active user partitions its $B$-bit message into $L$ equal-size information bit sequences (or chunks). As opposed to [3], however, our approach does not require concatenated coding to couple the sequences across different slots (i.e., no outer binary encoder). Therefore, we simply share the bits uniformly between the $L$ slots and there is no need to optimize the sizes of the $L$ sequences. In this way, there is a total number of $J = \frac{B}{n}$ bits in each sequence (i.e., associated to each slot).

Let the matrix $\bar{A} \in \mathbb{C}^{2^J \times 2^J}$ denote the common codebook for all the users (over all slots). That is, the columns of $\bar{A} = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{2^J}]$ form a set of codewords that each $\{k^{th}\}_{k=1}^{K_a}$ active user chooses from in order to encode its $\{l^{th}\}_{l=1}^{J}$ sequence before transmitting it over the $\{l^{th}\}_{l=1}^{J}$ slot. Notice here that, in such a slotted transmission framework, the size of the codebook is $2^J$. This is much smaller the original codebook of size $2^B$ which was actually used to prove the random coding achievability bound in [7]. Sloting is, however, a necessary step towards alleviating the computational burden as mentioned previously.

After partitioning each packet/message into $L$ $J$-bit information sequences, the latter are encoded separately using the codebook, $\bar{A} \in \mathbb{C}^{2^J \times 2^J}$, which will serve as the sensing matrix for sparse recovery. Conceptually, we operate on a per-slot basis by associating to every possible $J$-bit information sequence a different column in the codebook matrix $\bar{A}$. Thus, we can view this matrix as a set of potentially transmitted messages over the duration of a slot. The multiuser CS encoder can be visualized as an abstract multiplication of $\bar{A}$ by an index vector $\mathbf{v}$. The positions of non-zero coefficients in $\mathbf{v}$ are nothing but the decimal representations of the information bit sequences/chunks being transmitted by the active users over a given slot. Thus, the slotted transmission of the $B$-bit packets of all the active users gives rise to $J$ small-size compressed sensing instances (one per each slot). Now, after encoding its $J$-bit sequence, user $k$ modulates the corresponding codeword and transmits it over the channel where it is being multiplied by a complex coefficient $h_{k,m}$ before reaching the $m$th antenna. Hence, the overall baseband model over each slot reduces to the MAC model discussed in Section II. Hence, by recalling (7), the received signal over the $l$th slot is given by:

$$\bar{Y}_l = \bar{A} \bar{X}_l + \bar{W}_l, \quad l = 1, \ldots, L. \quad (8)$$
Vectorizing (3) yields:
\[
\text{vec}(\tilde{\mathbf{Y}}_l^T) = (\bar{\mathbf{A}}^T \otimes \mathbf{I}) \text{vec}(\tilde{\mathbf{X}}_l^T) + \text{vec}(\tilde{\mathbf{W}}_l^T),
\]
in which \(\otimes\) denotes the Kronecker product of two matrices. Then, by defining \(\bar{\mathbf{A}} \triangleq \mathbf{A}^T \otimes \mathbf{I} \in \mathbb{C}^{2M \times 2^M M^r}\),
\(\tilde{\mathbf{y}}_l \triangleq \text{vec}(\tilde{\mathbf{Y}}_l^T) \in \mathbb{C}^{2^M M^r}\), \(\tilde{\mathbf{x}}_l \triangleq \text{vec}(\tilde{\mathbf{X}}_l^T) \in \mathbb{C}^{2^M M^r}\), and
\(\tilde{\mathbf{w}}_l \triangleq \text{vec}(\tilde{\mathbf{W}}_l^T)\), we recover the problem of estimating a sparse vector, \(\tilde{\mathbf{h}}_l\), from its noisy linear observations:
\[
\tilde{\mathbf{y}}_l = \bar{\mathbf{A}} \tilde{\mathbf{x}}_l + \tilde{\mathbf{w}}_l, \quad l = 1, \ldots, L. \tag{10}
\]

Fig. 1 schematically depicts the proposed URA scheme and contrasts it to the existing coded/coupled compressed sensing-based scheme. As seen there, the proposed scheme completely eliminates the need for concatenated coding, i.e., the outer tree encoder which leads to tremendous gains in spectral efficiency. Instead, instead of coupling the slot-wise sequences through additional parity-check bits in order to stitch them at the receiver, the proposed scheme leverages the inherent coupling provided by nature in the form of channel correlations across slots. In other words, if one is able to find the assignment matrix that clusters the slot-wise reconstructed channels for each user together, then the decoded sequences can also be clustered (i.e., stitched) in the same way.

To better reconstruct the channels, this paper postulates a Bernoulli-Laplacian distribution as a heavy-tailed prior on the entries of the covariance matrix that clusters the slot-wise reconstructed channels for each active user, that is, the nonzero blocks in the slot-wise decoded sequences of each user, so as to recover all the original transmitted \(B\)-bit messages (cf. Fig. 1 for more details).

For each \(l\)th slot, the task is then to first reconstruct \(\tilde{\mathbf{x}}_l\) from \(\tilde{\mathbf{y}}_l = \bar{\mathbf{A}} \tilde{\mathbf{x}}_l + \tilde{\mathbf{w}}_l\) given \(\tilde{\mathbf{y}}_l\) and \(\bar{\mathbf{A}}\). To solve the joint activity detection and channel estimation problem, we resort to the powerful HyGAMP CS algorithm \(\cite{13}\) and we will also rely on the EM-concept \(\cite{19}\) to learn the unknown hyperparameters of the model. In particular, we embed the EM algorithm inside HyGAMP to learn the variances of the additive noise and the postulated prior, which are both required to execute HyGAMP itself. HyGAMP makes use of large-system Gaussian and quadratic approximations for the messages of loopy belief propagation on the factor graph. As opposed to GAMP \(\cite{20}\), HyGAMP is able to accommodate the group sparsity structure in \(\tilde{\mathbf{x}}_l\) by using a dedicated latent Bernoulli random variable, \(\varepsilon_{j,l}\), for each \(j=1, \ldots, L\)

\[
\tilde{\mathbf{h}}_{k,l} = \tilde{\mathbf{h}}_k + \tilde{\mathbf{w}}_{k,l}, \quad k = 1, \ldots, K_a, \quad l = 1, \ldots, L. \tag{16}
\]
in which \(\tilde{\mathbf{h}}_k \triangleq [\mathbb{R}\{\mathbf{h}_k\}, \mathbb{I}\{\mathbf{h}_k\}]^T\) with \(\mathbf{h}_k \triangleq [h_{k,1}, h_{k,2}, \ldots, h_{k,M^r}]^T\) is the true complex channel vector for user \(k\). The outer clustering-based decoder takes the \(LK_a\) reconstructed channels in (16) — which are slot-wise permuted — and returns one cluster per active user, that contains its \(L\) noisy channel estimates.

To cluster the reconstructed channels into \(K_a\) different groups, we resort to the Gaussian mixture expectation maximization procedure which consists in fitting a Gaussian mixture distribution to the data points in (16) under the assumption of Gaussian residual noise. Note here the fact that the reconstruction noise is Gaussian is common to all of the
approximate message passing algorithms including HyGAMP. Moreover, we will devise an appropriate constrained clustering procedure that enforces the following two constraints that are very specific to our problem: i) each cluster must have exactly \( L \) data points, and ii) channels reconstructed over the same slot must not be assigned to the same cluster.

### D. Hybrid Approximate Message Passing

In this section, we describe the HyGAMP CS algorithm by which we estimate the channels and decode the data in each slot. As a matter of fact, decoding the transmitted messages in slot \( l \) comes as a byproduct of reconstructing the entire group-sparse vector \( \mathbf{x}_l \). This is because there is a one-to-one mapping between the positions of the non-zero blocks in \( \mathbf{x}_l \) and the transmitted codewords as being drawn from the common codebook \( \mathbf{A} \). As mentioned earlier, HyGAMP finds asymptotic MMSE estimates for the entries of the group-sparse vector, \( \mathbf{x}_l \), in each slot \( l \). To capture the underlying group sparsity structure, HyGAMP uses the following set of Bernoulli latent random variables:

\[
\epsilon_j = \begin{cases} 
1 & \text{if group } j \text{ is active,} \\
0 & \text{if group } j \text{ is inactive,}
\end{cases}
\]

which are i.i.d with the common prior \( \lambda \triangleq \Pr(\epsilon_j = 1) = \frac{\sigma_x}{\lambda^2} \). The marginal posterior probabilities, \( \Pr(\epsilon_j = 1 \mid y_t) \), for \( j = 1,2,\ldots,2^J \) are given by:

\[
\Pr(\epsilon_j = 1 \mid y_t) = \frac{\exp(\text{LLR}_{q,j})}{1 + \exp(\text{LLR}_{q,j})},
\]

where \( \text{LLR}_{q,j} \) is updated in line 19 of Algorithm 1 (cf. next page) while trying to reconstruct the unknown channels. The posterior probabilities, \( \{\Pr(\epsilon_j = 1 \mid y_t)\}_{j=1}^{2^J} \), are used by the receiver to infer which of the codewords were transmitted by the active users over slot \( l \). This is done by simply returning the columns in \( \mathbf{A} \) that correspond to the \( K_a \) largest values among the posterior probabilities in (18).

Note here that for the sake of simplicity, we assume the number of active users, \( K_a \), to be known to the receiver as is the case in all existing works on unsourced random access. Yet, we emphasize the fact that it is straightforward to generalize our approach to also detect the number of active users by learning the hyperparameter \( \lambda = \frac{K_a}{L} \) using the EM procedure as done in [21]. Motivated by our recent results in [22], we also postulate a Bernoulli-Laplacian prior to model the channel coefficients. The main rationale behind choosing this prior is the need for using a heavy-tailed distribution to capture the number of active users, \( K_a \), to be known to the receiver as is the case in all existing works on unsourced random access. Yet, we will explain how to also learn these two parameters from the data using the EM algorithm. For ease of exposition, the vector \( \mathbf{x}_l \) to be reconstructed, in slot \( l \), will be generically denoted as \( \mathbf{x} \) since HyGAMP will be executed in each slot separately. The underlying block-sparse vector, \( \mathbf{x} \), consists of \( 2^J \) blocks each of which consisting of \( 2M_r \) components, i.e.,

\[
\mathbf{x} \triangleq [x_1, x_2, \ldots, x_{2^J}]^T,
\]

with

\[
x_j \triangleq [x_{1,j}, x_{2,j}, \ldots, x_{2M_r-j}]^T,
\]

Similarly, the known sensing matrix, \( \mathbf{A} \), in (15) is partitioned into the corresponding \( 2^J \) blocks as follows:

\[
\mathbf{A} = [\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(2^J)}] \text{ with } \mathbf{A}^{(j)} \in \mathbb{R}^{M \times 2M_r} \forall j.
\]

Recall also that \( M = \frac{2nM_r}{J} \) and \( N = 2^J (2M_r) \) denote the number of rows and columns in \( \mathbf{A} \), respectively.

HyGAMP passes messages on a probabilistic graphical model wherein the components of each block \( \mathbf{x}_j \) are connected to the same latent variable \( \epsilon_j \). The latter sends its belief (updated in line 20 of Algorithm 1) about each component of the block, being zero or non-zero. This updated belief is based on the information harvested from the other components of the same block (line 19 of Algorithm 1). The other updates in lines 2-18 are standard GAMP updates. The estimates of the unknown components in the group sparse vector are updated through the MMSE denoising step in line 17 and the associated variances are updated in line 18. As a starting point, we initialize all the posterior means, \( \tilde{x}_{q,j}(t) \), and variances,
Algorithm 1 Sum-Product GAMP for MMSE estimation

Require: $A \in \mathbb{R}^{M \times N}$; $y \in \mathbb{R}^M$; $\sigma_x$, $\lambda$, $\sigma_w^2$, precision tolerance ($\epsilon$), maximum number of iterations ($T_{\text{MAX}}$)

Ensure: MMSE estimates, $\{\hat{x}_{q,j}\}_{q=1}^{2M}$, of $\{x_{q,j}\}_{q=1}^{2M}$ and MAP estimates, $\{\varepsilon_j\}_{j=1}^{2J}$, of $\{\varepsilon_j\}_{j=1}^{2J}$

1: Initialization
2: $t \leftarrow 1$
3: $\forall q, j : \hat{x}_{q,j}(t) = 0$
4: $\forall q, j : \mu_{q,j}^{2}(t) = 1$
5: $\forall j : \hat{s}_j(t-1) = 0$
6: $\forall q, j : \text{LLR}_{q \rightarrow j}(t-1) \leftarrow \log(\lambda/(1-\lambda))$
7: $\forall q, j : \hat{\rho}_{q,j}(t) \leftarrow 1/(1 + \exp(\text{LLR}_{q \rightarrow j}(t)))$
8: repeat
9: $\forall q, j : \mu_{q,j}^{2}(t) = \sum_{q,j} |A_{iq}^{(j)}|^2 \mu_{q,j}^2(t)$
10: $\forall i : \hat{p}_i(t) = \sum_{q,j} A_{iq}^{(j)} \hat{x}_{q,j}(t) - \mu_{q,j}^{2}(t) \hat{s}_i(t-1)$
11: $\forall i : \mu_{i}^{2}(t) = \text{var} \left\{ z_i | y ; \hat{p}_i(t), \mu_{q,j}^{2}(t), \sigma_w^2 \right\}$
12: $\forall i : \hat{z}_i^{0}(t) = \mathbb{E}_{z_i | y} \left\{ z_i | y ; \hat{p}_i(t), \mu_{q,j}^{2}(t), \sigma_w^2 \right\}$
13: $\forall i : \mu_{i}^{2}(t) = \frac{1}{\mu_{i}^{2}(t)} \left\{ 1 - \frac{\mu_i^2(t)}{\mu_i^2(t)} \right\}$
14: $\forall i : \hat{s}_i(t) = \frac{1}{\mu_i^2(t)} \left\{ \hat{z}_i(t) - \hat{p}_i(t) \right\}$
15: $\forall q, j : \mu_{q,j}^{2}(t) = \left( \sum_{i=1}^{M} |A_{iq}^{(j)}|^2 \mu_{i}^{2}(t) \right)^{-1}$
16: $\forall q, j : \hat{\tau}_{q,j}(t) = \hat{x}_{q,j}(t) + \mu_{q,j}^{2}(t) \sum_{i=1}^{M} A_{iq}^{(j)} \hat{s}_i(t)$
17: $\forall q, j : \hat{x}_{q,j}(t+1) = \mathbb{E}_{x_{q,j} | y} \left\{ x_{q,j} | y ; \hat{\tau}_{q,j}(t), \mu_{q,j}^{2}(t), \hat{\rho}_{q,j}(t) \right\}$
18: $\forall q, j : \mu_{q,j}^{2}(t+1) = \text{var}_{x_{q,j} | y} \left\{ x_{q,j} | y ; \hat{\tau}_{q,j}(t), \mu_{q,j}^{2}(t), \hat{\rho}_{q,j}(t) \right\}$
19: $\forall q, j : \text{Compute LLR}_{q \rightarrow j}(t)$ using (27)
20: $\forall q, j : \text{LLR}_{q \rightarrow j}(t) \leftarrow \log(\lambda/(1-\lambda)) + \sum_{q' \neq q} \text{LLR}_{q' \rightarrow j}(t)$
21: $\forall q, j : \hat{\rho}_{q,j}(t+1) \leftarrow 1/(1 + \exp(\text{LLR}_{q \rightarrow j}(t)))$
22: $t \leftarrow t + 1$
23: until $\|\hat{x}(t) - \hat{x}(t-1)\|^2 \leq \epsilon \|\hat{x}(t)\|^2$ or $t > T_{\text{MAX}}$

The updates in lines 17 and 18, however, depend on the particular choice of the prior and as such need to be expressed as function of the other outputs of HyGAMP. In this paper, we only provide the final expressions of the required updates under the Bernoulli-Laplacian prior given in (23). We omit the derivation details for sake of brevity since they are based on some equivalent algebraic manipulations as recently done in [22] in the absence of group sparsity. For notational convenience, we also introduce the following intermediate quantities that are needed to express the required updates, for $q = 1, \ldots, 2M$, and $j = 1, \ldots, 2J$ (some variables are defined in Algorithm 1):

$$\theta_{q,j} \triangleq \frac{2\sigma_x (1 - \hat{\rho}_{q,j})}{\hat{\rho}_{q,j} \sqrt{2\pi \mu_{q,j}^2}}$$

$$\alpha_{q,j}^- \triangleq \frac{\hat{\tau}_{q,j} - \mu_{q,j}^2}{\sigma_x - \frac{\mu_{q,j}^2}{2\sigma_x^2}}$$

$$\alpha_{q,j}^+ \triangleq \frac{\hat{\tau}_{q,j} - \mu_{q,j}^2}{\sigma_x - \frac{\mu_{q,j}^2}{2\sigma_x^2}}$$

The updates in lines 17 and 18, however, depend on the particular choice of the prior and as such need to be expressed as function of the other outputs of HyGAMP. In this paper, we only provide the final expressions of the required updates under the Bernoulli-Laplacian prior given in (23). We omit the derivation details for sake of brevity since they are based on some equivalent algebraic manipulations as recently done in [22] in the absence of group sparsity. For notational convenience, we also introduce the following intermediate quantities that are needed to express the required updates, for $q = 1, \ldots, 2M$, and $j = 1, \ldots, 2J$ (some variables are defined in Algorithm 1):
\[ \gamma_{q,j}^- \triangleq \hat{\gamma}_{q,j} + \frac{\mu_{q,j}}{\sigma_x}, \]
\[ \gamma_{q,j}^+ \triangleq \hat{\gamma}_{q,j} - \frac{\mu_{q,j}}{\sigma_x}. \]

It is worth mentioning here that those quantities depend on the unknown parameter \( \sigma_x \) of the Laplacian distribution. Therefore, on top of being updated by HyGAMP, these \( \sigma_x \)-dependent quantities must also be updated locally by the nested EM algorithm that learns the unknown parameter \( \sigma_x \) itself. We also define the following two intermediate \( \sigma_x \)-independent quantities:

\[ \nu_{q,j}^+ \triangleq Q \left( -\frac{\gamma_{q,j}^+}{\sqrt{\mu_{q,j}^+}} \right) e^{\frac{\gamma_{q,j}^+}{\mu_{q,j}^+}}, \]
\[ \nu_{q,j}^- \triangleq Q \left( -\frac{\gamma_{q,j}^-}{\sqrt{\mu_{q,j}^-}} \right) e^{\frac{\gamma_{q,j}^-}{\mu_{q,j}^-}}, \]

in which \( Q(.) \) is the standard Q-function, i.e., the tail of the normal distribution:

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}} dt. \]

Using the above notations, we establish the closed-form expressions for \( \tilde{x}_{q,j} \) and \( \mu_{q,j}^2 \) (required, respectively, in lines 17 and 18 of Algorithm 1) as given in (35) and (36) displayed on the top of the next page. The closed-form expression for the LLR updates in line 19 of Algorithm 1 was also established as follows:

\[ \text{LLR}_{q\rightarrow j} = \log \left( \frac{\sqrt{2\pi} \mu_{q,j}^2}{2\sigma_x} \left[ \nu_{q,j}^+ + \nu_{q,j}^- \right] \right). \]

We also resort to the maximum likelihood (ML) concept in order to estimate the unknown hyperparameters \( \sigma_x \) and \( \sigma_x^2 \). More specifically, we show that the ML estimate of the noise variance is given by:

\[ \hat{\sigma}_x^2 = \frac{1}{M} \sum_{i=1}^{M} (y_i - \tilde{z}_i)^2 + \mu_z^2, \]

where \( \tilde{z}_i \triangleq (A \hat{x})_i \). Unfortunately, the ML estimate (MLE), \( \hat{\sigma}_x \), of \( \sigma_x \), cannot be found in closed form and we use the EM algorithm instead to find the required MLE iteratively. Indeed, starting from some initial guess, \( \hat{\sigma}_{x,0} \), we establish the \((d+1)\)th MLE update as follows:

\[ \psi_{q,j;d} = \frac{\tilde{\rho}_{q,j}}{2 \sigma_x} e^{-\frac{\gamma_{q,j}}{\sigma_x}} Q \left( \frac{\gamma_{q,j}}{\sqrt{\mu_{q,j}}} \right) + e^{-\frac{\gamma_{q,j}}{\sigma_x}} Q \left( \frac{\gamma_{q,j}}{\sqrt{\mu_{q,j}}} \right) + (1 - \tilde{\rho}_{q,j}) e^{-\frac{\gamma_{q,j}}{2 \mu_{q,j}}}, \]
\[ \kappa_{q,j;d} = \gamma_{q,j;d} e^{-\frac{\gamma_{q,j;d}}{\sigma_x}} Q \left( \frac{\gamma_{q,j;d}}{\sqrt{\mu_{q,j}}} \right) - \gamma_{q,j;d} e^{-\frac{\gamma_{q,j;d}}{\sigma_x}} Q \left( \frac{\gamma_{q,j;d}}{\sqrt{\mu_{q,j}}} \right) + \frac{2 \mu_{q,j}^2}{\sqrt{2\pi \mu_{q,j}}} e^{-\frac{\gamma_{q,j;d}}{2 \mu_{q,j}}}. \]

Note here that \( \gamma_{q,j;d}, \gamma_{q,j;d}, \sigma_{q,j}^2 \), and \( \sigma_{q,j;d} \) involved in (39)-(40) are also expressed as in (33)-(34), except the fact that \( \sigma_x \) is now replaced by \( \hat{\sigma}_{x,d} \).

**E. Constrained Clustering-Based Stitching Procedure**

In this section, we focus on the problem of clustering the reconstructed channels from all the slots to obtain one cluster per user. By doing so, it will be easy to cluster (i.e., stitch) the slot-wise decoded sequences of all users so as to recover their transmitted messages/packets. To that end, we first estimate the large-scale fading coefficients from the outputs of HyGAMP as follows:

\[ \hat{g}_k = \frac{1}{2M_r L} \sum_{l=1}^{L} ||\hat{h}_{k,l}||_2^2, \]

where \( \hat{h}_{k,l} \) is the reconstructed active channel \( k \) in slot \( l \). The estimates of the different large-scale fading coefficients are required to re-scale the reconstructed channels before clustering. This is in order to avoid, for instance, having the channels of the cell-edge users clustered together due to their strong pathloss attenuation. We can now visualize (16) as a set of \( K_a L \) data points in \( \mathbb{R}^{2M_r} \): \n
\[ \mathcal{H} = \{ \hat{h}_{k,l} | k = 1, \ldots, K_a, \ l = 1, \ldots, L \}. \]

In other words, \( \mathcal{H} \) gathers the noisy reconstructed channels pertaining to all \( K_a \) active users and all \( L \) slots. After normalization by the large-scale fading coefficients, we are left with the small scale-fading coefficients which are assumed to be Gaussian distributed. Therefore, we propose to fit a Gaussian mixture distribution to this data set and use the EM algorithm to estimate the parameters of the involved mixture densities along with the mixing coefficients.

The rationale behind the use of clustering is our prior knowledge about the nature of the data set \( \mathcal{H} \). Indeed, we know that there are \( K_a \) users whose channels remain constant over all the slots. Therefore, each user contributes exactly \( L \) data points in \( \mathcal{H} \) which are noisy estimates of its true channel vector. Our goal is hence to cluster the whole data...
Taking the logarithm of (46) yields the following log-
likelihood function factorizes as follows:

\[
\frac{1}{\mu_{q,j}^+} = \left( \left( \gamma_{q,j}^- \right)^2 + \mu_{q,j}^+ \right) - \frac{2(\mu_{q,j}^-)^2}{\sigma_x \sqrt{2\pi \mu_{q,j}^+ \nu_{q,j}^-}} + \theta_{q,j} \left( \left( \gamma_{q,j}^- \right)^2 + \mu_{q,j}^+ \right) \nu_{q,j}^- + \frac{2(\mu_{q,j}^-)^2}{\sigma_x \sqrt{2\pi \mu_{q,j}^+ \nu_{q,j}^-}} \right) - 1.
\]

(35)

\[
\hat{x}_{q,j} = \left( \frac{1}{\gamma_{q,j}^- + \gamma_{q,j}^+ \nu_{q,j}^-} + \frac{1}{\gamma_{q,j}^- + \gamma_{q,j}^+ \nu_{q,j}^+} + \frac{\theta_{q,j}}{\gamma_{q,j}^- + \gamma_{q,j}^+ \nu_{q,j}^-} \right)^{-1}.
\]

(43)

Here, \( \pi \triangleq [\pi_1, \ldots, \pi_{K_a}]^T \) are the mixing coefficients, \( \mu \triangleq [\mu_1, \ldots, \mu_{K_a}]^T \) are the clusters’ means, and \( \Sigma \triangleq [\Sigma_1, \ldots, \Sigma_{K_a}]^T \) are their covariance matrices.

The assumption we make here is justified by the fact that the residual reconstruction noise of AMP-like algorithms (including HyGAMP) is Gaussian-distributed. Notice that in (43) we considered a mixture of \( K_a \) components, which amounts to assigning a Gaussian distribution to each active user. We now turn our attention to finding the likelihood function of all the unknown parameters, involved in (43).

To that end, we use \( \hat{h}_n \) to denote a generic data point in \( \mathcal{H} \), i.e.:

\[
\mathcal{H} = \left\{ \hat{h}_{k,l} \mid k = 1, \ldots, K_a, \ l = 1, \ldots, L \right\},
\]

(44)

\[
\hat{h}_n = \left\{ \hat{h}_n \mid n = 1, \ldots, N_{tot} \right\}.
\]

(45)

Owing to the i.i.d. assumption on the data, the associated likelihood function factorizes as follows:

\[
p_{\hat{h}_1, \ldots, \hat{h}_{N_{tot}}}(\hat{h}_1, \ldots, \hat{h}_{N_{tot}}; \pi, \mu, \Sigma) = \prod_{n=1}^{N_{tot}} p_{\hat{h}_n}(\hat{h}_n; \pi, \mu, \Sigma).
\]

(46)

Taking the logarithm of (46) yields the following log-likelihood function (LLF):

\[
\mathcal{L}(\pi, \mu, \Sigma) \triangleq \ln p_{\hat{h}_1, \ldots, \hat{h}_{N_{tot}}}(\hat{h}_1, \ldots, \hat{h}_{N_{tot}}; \pi, \mu, \Sigma),
\]

\[
= \sum_{n=1}^{N_{tot}} \ln \left[ \pi_k N(\hat{h}_n; \mu_k, \Sigma_k) \right].
\]

(47)

Our task is to then maximize the LLF with respect to the unknown parameters, i.e.:

\[
\arg \max_{\pi_k, \mu_k, \Sigma_k} \sum_{n=1}^{N_{tot}} \ln \left( \sum_{k=1}^{K_a} \pi_k N(\hat{h}_n; \mu_k, \Sigma_k) \right).
\]

(48)

Unfortunately, it is not possible to obtain a closed-form solution to the above optimization problem. Yet, the EM algorithm can again be used to iteratively update the ML estimates of the underlying parameters. In the sequel, we will provide the resulting updates, and we refer the reader to Chap. 9 of [23] for more details.

We initialize the means \( \{\mu_k\}_{k=1}^{K_a} \), covariances \( \{\Sigma_k\}_{k=1}^{K_a} \), and mixing coefficients \( \{\pi_k\}_{k=1}^{K_a} \) using the well-known k-means++ algorithm. Then, the three steps needed to learn the parameters of the above Gaussian mixture model are as follows:

- **Expectation step (E-STEP)**

\[
\Pr(\hat{h}_n \in \text{cluster } k) = \frac{\pi_k N(\hat{h}_n; \mu_k, \Sigma_k)}{\sum_{k=1}^{K_a} \pi_k N(\hat{h}_n; \mu_k, \Sigma_k)}.
\]

(49)

- **Maximization step (M-STEP)**

\[
\mu_k^{\text{new}} = \frac{1}{N_k} \sum_{n=1}^{N_{tot}} \Pr(\hat{h}_n \in \text{cluster } k) \hat{h}_n,
\]

(50)

\[
\Sigma_k^{\text{new}} = \frac{1}{N_k} \sum_{n=1}^{N_{tot}} \Pr(\hat{h}_n \in \text{cluster } k) \left( \hat{h}_n - \mu_k^{\text{new}} \right) \left( \hat{h}_n - \mu_k^{\text{new}} \right)^T,
\]

(51)

\[
N_k = \sum_{n=1}^{N_{tot}} \Pr(\hat{h}_n \in \text{cluster } k),
\]

(52)

\[
\pi_k^{\text{new}} = \frac{N_k}{N_{tot}}.
\]

(53)

- **Evaluation step (EVAL-STEP)**

\[
\mathcal{L}(\pi^{\text{new}}, \mu^{\text{new}}, \Sigma^{\text{new}}) = \sum_{n=1}^{N_{tot}} \ln \left( \sum_{k=1}^{K_a} \pi_k^{\text{new}} N(\hat{h}_n; \mu_k^{\text{new}}, \Sigma_k^{\text{new}}) \right).
\]

(54)
In the E-STEP, we compute the probability of having a particular data point belong to each of the $K_a$ users. In the M-STEP, we update the means, covariances, and the mixing coefficients for each of the clusters. We further need to evaluate the LLF at each iteration to check the convergence of the EM-based algorithm, hence the Eval-STEP. Recall, however, that we are actually dealing with a constrained clustering problem since it is mandatory to enforce the following two intuitive constraints:

- **Constraint 1**: Channels from the same slot cannot be assigned to the same user.
- **Constraint 2**: Users/clusters should have exactly $L$ channels/data points.

At convergence, the EM algorithm returns a matrix, $P$, of posterior membership probabilities, i.e., whose $(n,k)$th entry is $P_{nk} = Pr(I_{nk} \in \text{cluster } k)$. Since the EM solves an unconstrained clustering problem, relying directly on $P$ would result in having two channels reconstructed from the same slot being clustered together, thereby violating “constraint 1” and/or “constraint 2”. In what follows, we will still make use of $P$ in order to find the best possible assignment of the $N_{tot}$ reconstructed channels to the $K_a$ users (i.e., the one that minimizes the probability of error) while satisfying the two constraints mentioned above.

To enforce “constraint 2”, we begin by partitioning $P$ into $L$ equal-size and consecutive blocks, i.e., $K_a \times K_a$ matrices $\{P^{(l)}\}_{l=1}^L$, as follows:

$$
P = \begin{bmatrix}
P^{(1)} \\
\vdots \\
P^{(L)}
\end{bmatrix}. \quad (56)
$$

Then, since each $k$th row in $P^{(l)}$ sums to one, it can be regarded as a distribution of some categorical random variable, $V_{k,l}$, that can take on one of $K_a$ possible mutually exclusive states. For convenience, we represent these categorical random variables by 1-of-$K_a$ binary coding scheme. That is, each $V_{k,l}$ is represented by a $K_a$-dimensional vector $v_{k,l}$ which takes values in $\{e_1, e_2, \ldots, e_{K_a}\}$ where $e_i = [0, \ldots, 1, \ldots, 0]^T$ has a single 1 located at position $i$. We also denote the set of all $K_a \times K_a$ permutation matrices by $P$.

We enforce “constraint 1” by using the following posterior joint distribution on $\{V_{k,l}\}_{k=1}^{K_a}$ in each $l$th slot:

$$
p_{V_1,l,\ldots,V_{K_a,l}}(v_{1,l},\ldots,v_{K_a,l}) \propto \mathbb{I}(V_l \in P) \prod_{k=1}^{K_a} p_{V_{k,l}}(v_{k,l}), \quad (57)
$$

where $\mathbb{I}(\cdot)$ is the indicator function and $V_l \triangleq [v_{1,l},\ldots,v_{K_a,l}]^T$. Moreover, it is clear that any categorical distribution with $K_a$ atoms can be parametrized in the following way:

$$
p_{V_{k,l}}(v_{k,l}) = \exp \left\{ \sum_{k'=1}^{K_a} \alpha_{k,k'}^{(l)} \mathbb{I}(v_{k,l} = e_{k'}) \right\}, \quad (58)
$$

in which

$$
\alpha_{k,k'}^{(l)} = \log p_{V_{k,l}}(v_{k,l} = e_{k'}) = \log P_{k,k'}^{(l)}. \quad (59)
$$

Since our optimality criteria is the largest-probability assignment, we need to maximize the distribution in (57) which when combined with (58) yields:

$$
p_{V_l}(V_l) = p_{V_{1,l},\ldots,V_{K_a,l}}(v_{1,l},\ldots,v_{K_a,l}),
\begin{align*}
&\propto \mathbb{I}(V_l \in P) \prod_{k=1}^{K_a} \exp \left\{ \sum_{k'=1}^{K_a} \alpha_{k,k'}^{(l)} \mathbb{I}(v_{k,l} = e_{k'}) \right\}, \\
&= \mathbb{I}(V_l \in P) \exp \left\{ \sum_{k=1}^{K_a} \sum_{k'=1}^{K_a} \alpha_{k,k'}^{(l)} \mathbb{I}(v_{k,l} = e_{k'}) \right\}. \quad (60)
\end{align*}
$$

Now, finding the optimal assignment inside slot $l$, subject to constraint 1, amounts to finding the optimal assignment matrix, $\hat{V}_l$, that maximizes the constrained posterior joint distribution, $p_{V_l}(V_l)$, established in (60), i.e.:

$$
\hat{V}_l = \arg\max_{V_l} p_{V_l}(V_l). \quad (61)
$$

Owing to (60), it can be shown that finding $\hat{V}_l$ is equivalent to solving the following constrained optimization problem:

$$
\arg\max_{V_l} \sum_{k=1}^{K_a} \sum_{k'=1}^{K_a} \alpha_{k,k'}^{(l)} \mathbb{I}(v_{k,l} = e_{k'}) \quad (62)
$$

subject to

$$
\begin{align*}
\sum_{k'=1}^{K_a} \mathbb{I}(v_{k,l} = e_{k'}) &= 1 \quad \text{for all } k, \\
\sum_{k=1}^{K_a} \mathbb{I}(v_{k,l} = e_{k'}) &= 1 \quad \text{for all } k'.
\end{align*} \quad (63)
$$

Note that the constraints in (63) enforce the solution to be a permutation matrix. This follows from the factor, $\mathbb{I}(V_l \in P)$, in the posterior distribution established in (60) which assigns zero probability to non-permutation matrices.

This optimization problem can be solved in polynomial time. The Hungarian algorithm is one of such algorithms which has an overall complexity in the order of $O(K_a^3)$. After finding the optimal assignment matrices, $\{\hat{V}_l\}_{l=1}^L$, the decoding task is completed since these very assignment matrices can be used to cluster the reconstructed sequences, thereby recovering the original transmitted messages.

IV. Simulation Results

A. Simulation Parameters

In this section, we assess the performance of the proposed URA scheme using exhaustive Monte-Carlo computer simulations. Our performance metric is the probability of error given in (4). We fix the number of information bits per user/packet to $B = 102$, which are communicated over $L = 6$ slots. This corresponds to $J = 17$ information bits per slot. We also fix the bandwidth to $W = 10$ MHz and the noise power to $P_w = 10^{-15.9} \times W$ [Watts]. The path-loss parameters in (2) are set to $\alpha = -15.3$ dB and $\beta = 3.76$. In the following, our baseline is the covariance-based scheme introduced recently in [15] which is simply referred to as CB-CS in this paper.
B. Results

Fig. 5 depicts the performance of both URA schemes as function of the number of active users \( K_a \). Here, we fix the transmit power to \( P = 15 \) dBm for all the users. For sake of fairness, we simulate both algorithms under the same spectral efficiency \( \mu \triangleq 0.015 \) bits/user/channel-use. That is, the 102 information bits of each active user are communicated over \( n = 6798 \) channel uses. The packets are communicated over \( L = 6 \) slots each of which spans over \( n_0 = 1133 \) channel uses. This yields a sensing matrix \( \mathbf{A} \) which has \( N_{\text{row}} = (2M_r) \times 1133 \) rows and \( N_{\text{col}} = (2M_r) \times 2^{17} \) columns. As a matter of fact, when \( M_r = 64 \) we have \( N_{\text{row}} \sim 10^5 \) and \( N_{\text{col}} \sim 10^7 \) which is a very large dimension for the multiple matrix-vector multiplications required inside HyGAMP. To alleviate this computational bottleneck, we use a circulant Gaussian codebook \( \tilde{\mathbf{A}} \) so as to perform these multiplications via the fast Fourier transform (FFT) algorithm. Huge computational savings also follow from taking advantage of the inherent Kronecker structure in \( \mathbf{A} \) as seen from [9].

![Graph](image)

Fig. 3: Performance of the proposed scheme as a function of the number of active users, \( K_a \), and receive antenna elements \( M_r \) with a fixed per-user spectral efficiency \( \mu = 0.015 \) bits/user/channel-use.

It is seen that the proposed scheme outperforms by far CB-CS especially when the number of active users becomes large as is the case in massive connectivity setups. The inefficiency of CB-CS stems from the fact that the number of active users exceeds the number of antenna branches at the BS as already mentioned in [15]. In fact, to accommodate \( K_a = 450 \) users, CB-CS would require as high as \( M_r = 450 \) receive antennas as opposed to only \( M_r = 64 \) or \( M_r = 32 \) with the proposed scheme. We achieve this by making use of the small-scale fading signatures of the different users, as well as, the CS-based decoding instead of relying solely on covariance matrix of the receive signal. In this respect, doubling the number of receive antennas consistently improves the performance of the proposed scheme as function of the number of active users as opposed to CB-CS. In fact, as the number of antennas increases, the users’ channels become almost orthogonal and users can be easily separated in the spatial domain due to the higher spatial resolution and the channel hardening effect, which is one of the blessings of massive MIMO. The quality of the CB-CS-based estimates of the large-scale fading coefficients also improves by increasing \( M_r \) thereby enhancing CB-CS performance but the higher spatial resolution of massive MIMO is still not fully exploited.

Note that in Fig. 3 the total spectral efficiency \( \mu_{\text{tot}} = K_a \mu \) increases with the number of active users and is as high as \( \mu_{\text{tot}} = K_a \times 0.015 = 6.75 \) bits/channel-use in presence of \( K_a = 450 \) active users. Now, we turn the tables and fix the total spectral efficiency to \( \mu_{\text{tot}} = 6.75 \) bits/channel-use while still varying the number of active users from \( K_a = 150 \) to \( K_a = 450 \). The performance of both URA schemes is shown in Fig. 4. There, the number of receive antennas is fixed to \( M_r = 128 \) and \( M_r = 32 \) for CB-CS and our scheme, respectively. At such high total spectral efficiency, the existing CB-CS-based URA scheme fails completely to decode the users’ messages even by using 4 times more antennas as compared to the proposed scheme. The latter, however, achieves a decoding error probability \( P_e \sim 10^{-2} \) by using 32 antennas only. The net spectral efficiency gains brought by our proposed scheme are of course a direct consequence of removing the concatenated coding that CB-CS relies on in order to stitch sequences.

Fig. 5 depicts the probability of error of the proposed scheme as a function of the transmit power. Here, we fix the number of active users to \( K_a = 150 \) and assess the performance for two antenna array sizes, namely \( M_r = 32 \) and \( M_r = 64 \). As expected, increasing the transmit power improves the performance appreciably. We also observe the 3-dB gain stemming from doubling the number of receive antenna branches. We believe that the performance can still be further improved for both by i) exploiting the distance-
based information in the large-scale fading coefficients and/or \textit{ii}) using high-rate concatenated codes combined with the proposed scheme.

V. CONCLUSION

We have introduced a new algorithmic solution to the unsourced random access problem that is also based on slotted transmissions. As opposed to all existing works, however, the proposed scheme relies purely on the rich spatial dimensionality offered by large-scale antenna arrays instead of coding-based coupling for sequence stitching purposes. HyGAMP CS recovery algorithm has been used to reconstruct the users channels and decode the sequences at a per-slot basis. Afterwards, the EM framework together with the Hungarian algorithm have been used to solve the underlying constrained clustering/stitching problem. The performance of the proposed approach has been compared to the only existing URA algorithm, in the open literature. The proposed scheme provides substantial performance enhancements especially in presence of a large number of active users. There are many possible avenues for future work. The two-step procedure of channel estimation and data decoding is overall sub-optimal. Therefore, it is desirable to devise a scheme capable of jointly estimating the random permutation and the support of the unknown block-sparse vector in each slot. In addition, it will be interesting to improve the proposed scheme by exploiting the fact that the same set of channels are being estimated across different slots. We also believe that making further use of the large-scale fading coefficients can be a fruitful direction for future research.

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