A tensor form of the Dirac equation

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Abstract

We prove the following theorem: the Dirac equation for an electron (invented by P.A.M. Dirac in 1928) can be written as a linear tensor equation. An equation is called a tensor equation if all values in it are tensors and all operations in it take tensors to tensors.

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Introduction

The Dirac equation for an electron [9] can be written in several different but equivalent forms. In this paper we consider two forms of the Dirac equation. Namely Hestenes’ form of the Dirac Equation (HDE) and the Tensor form of the Dirac Equation (TDE). The aim of this paper is to prove the following theorem.

**Theorem 1.** *The Dirac equation for an electron* (2) *(see [9]) can be written in a form of linear tensor equation* (58).

An equation is said to be written in a form of tensor equation if all values in it are tensors and all operations in it take tensors to tensors.

A column of four complex valued functions (a bispinor) represents the wave function of the electron in the Dirac equation. This wave function has unusual (compared to tensors) transformation properties under Lorentz changes of coordinates. These properties are investigated in the theory of spinors (see [1],[2],[3],[20],[23]).
There were attempts to find tensor equations equivalent to the Dirac equation \([11],[12],[13]\). The resulting equations were nonlinear. This fact leads to difficulties with the superposition principle, etc. The tensor equation under consideration in our paper is linear.

In the first part of paper we consider the Clifford algebra \(\mathcal{O}(1,3)\) and in the second part we consider the exterior algebra of Minkowski space \(\Lambda(\mathcal{E})\). Elements of \(\Lambda(\mathcal{E})\) are covariant antisymmetric tensors and elements of \(\mathcal{O}(1,3)\) are not tensors.

The tensor form of the Dirac equation (58) under consideration has three sources. The first source is the Ivanenko-Landau-Kähler equation (65) (see [21],[14]). This is a tensor equation and a wave function of the electron is represented in it by a nonhomogeneous covariant antisymmetric tensor field with 16 complex valued components (four times more than in the Dirac equation).

The second source is Hestenes’ form of the Dirac equation (46) (see [6],[7]). The wave function of the electron is represented in it by a real even element of the Clifford algebra \(\mathcal{O}^{\text{even}}(1,3)\) and has 8 real valued components. This equation is equivalent to the Dirac equation (see the proof in [8]). HDE contains nontensor values namely elements of the Clifford algebra \(\mathcal{O}(1,3)\). Hence it is not a tensor equation.

The third source is the constructions developed in [15], [16],[17], [18].

Also we want to mention an interesting approach to the construction of the 2D Dirac tensor equation suggested by D.Vassiliev [19].

1 Part I.

1.1 The Dirac equation for an electron.

Let \(\mathcal{E}\) be Minkowski space with the metric tensor

\[
g = \|g_{\mu\nu}\| = \|g^{\mu\nu}\| = \text{diag}(1, -1, -1, -1),
\]

\(x^\mu\) coordinates, and \(\partial_\mu = \partial/\partial x^\mu\). Greek indices run over (0, 1, 2, 3). Summation convention over repeating indices is assumed. Our system of units is such that speed of light \(c\), Plank’s constant \(\hbar\), and the positron charge have the value 1.
The Dirac equation for an electron [9] has the form
\[ \gamma^\mu (\partial_\mu \psi + ia_\mu \psi) + im \psi = 0, \] (2)
where \( \psi = (\psi_1 \, \psi_2 \, \psi_3 \, \psi_4)^T \) is a column of four complex valued functions of \( x = (x^0, x^1, x^2, x^3) \) (\( \psi \) is the wave function of an electron), \( a_\mu = a_\mu(x) \) is a real valued covector of electromagnetic potential, \( m > 0 \) is a real constant (the electron mass), and \( \gamma^\mu \) are the Dirac \( \gamma \)-matrices
\[ \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \] (3)
\[ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \]

The equation (2) is invariant under the gauge transformation
\[ \psi \to \psi \exp(i\lambda), \quad a_\mu \to a_\mu - \partial_\mu \lambda, \] (4)
where \( \lambda = \lambda(x) \) is a smooth real valued function.

The matrices (3) satisfy the relations
\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbf{1}, \] (5)
where \( \mathbf{1} \) is the 4×4-identity matrix and
\[ \|\eta^{\mu\nu}\| = \text{diag}(1, -1, -1, -1). \]

Note that if 4×4-matrix \( T \) is invertible, then the matrices \( \gamma^\mu = T^{-1} \gamma^\mu T \) also satisfy (5).

Let us denote
\[ \gamma^{\mu_1 \ldots \mu_k} = \gamma^{\mu_1} \ldots \gamma^{\mu_k}, \quad \text{for} \quad 0 \leq \mu_1 < \cdots < \mu_k \leq 3. \] (6)

Then the 16 matrices
\[ 1, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^{01}, \gamma^{02}, \gamma^{03}, \gamma^{12}, \gamma^{13}, \gamma^{23}, \gamma^{012}, \gamma^{013}, \gamma^{023}, \gamma^{123}, \gamma^{0123} \] (7)
form a basis of the matrix algebra \( \mathcal{M}(4, \mathbb{C}) \) (\( \mathbb{C} \) is the field of complex numbers and \( \mathcal{R} \) is the field of real numbers).
Remark. We assume that the matrix $\|\eta^{\mu\nu}\|$ is independent of $g$ from (1). And more than this, the values $g^{\mu\nu}$ and $\eta^{\mu\nu}$ have different mathematical meaning. Namely metric tensor $g^{\mu\nu}$ is a geometrical object which is attribute of Minkowski space. But the matrix $\|\eta^{\mu\nu}\|$ is an algebraic object which contains structure constants of an underlying algebra (as we see in a moment this algebra is the Clifford algebra). Accidentally, the matrices $\|\eta^{\mu\nu}\|$ and $\|g^{\mu\nu}\|$ have the same diagonal form $\text{diag}(1,-1,-1,-1)$.

1.2 The Clifford algebra and the spinor group.

Let $\mathcal{L}$ be a real 16-dimensional vector space with basis elements enumerated by ordered multi-indices

$$\ell, \ell^0, \ell^1, \ell^2, \ell^3, \ell^{01}, \ell^{02}, \ell^{03}, \ell^{12}, \ell^{13}, \ell^{23}, \ell^{012}, \ell^{013}, \ell^{023}, \ell^{123}, \ell^{0123}. \tag{8}$$

Suppose the multiplication of elements of $\mathcal{L}$ is defined by the following rules:

(i) $\mathcal{L}$ is an associative algebra (with the unity element $\ell$) w.r.t. this multiplication;

(ii) $\ell^\mu \ell^\nu + \ell^\nu \ell^\mu = 2\eta^{\mu\nu} \ell$;

(iii) $\ell^{\mu_1} \ldots \ell^{\mu_k} = \ell^{\mu_1 \ldots \mu_k}$, for $0 \leq \mu_1 < \cdots < \mu_k \leq 3$.

Then this algebra $\mathcal{L}$ is called the (real) Clifford algebra$^1$ and is denoted by $\mathcal{Cl}(1,3)$, where the numbers 1 and 3 determine the signature of the matrix $\|\eta^{\mu\nu}\|$. The complex Clifford algebra is denoted by $\mathcal{Cl}_C(1,3)$.

Elements of $\mathcal{Cl}(1,3)$ of the form

$$U = \sum_{\mu_1 < \cdots < \mu_k} u_{\mu_1 \ldots \mu_k} \ell^{\mu_1 \ldots \mu_k} \tag{9}$$

are said to be elements of rank $k$. For every $k = 0, 1, 2, 3, 4$ the set of elements of rank $k$ is a subspace $\mathcal{Cl}^k(1,3)$ of $\mathcal{Cl}(1,3)$ and

$$\mathcal{Cl}(1,3) = \mathcal{Cl}^0(1,3) \oplus \ldots \oplus \mathcal{Cl}^4(1,3) = \mathcal{Cl}^{\text{even}}(1,3) \oplus \mathcal{Cl}^{\text{odd}}(1,3),$$

$^1$The Clifford algebra was invented in 1878 by the English mathematician W.K.Clifford [10]
where
\[
\mathcal{A}^{\text{even}}(1,3) = \mathcal{A}^0(1,3) \oplus \mathcal{A}^2(1,3) \oplus \mathcal{A}^4(1,3),
\]
\[
\mathcal{A}^{\text{odd}}(1,3) = \mathcal{A}^1(1,3) \oplus \mathcal{A}^3(1,3).
\]

The dimensions of the spaces \(\mathcal{A}^k(1,3)\), \(k = 0, 1, 2, 3, 4\) are equal to 1, 4, 6, 4, 1 respectively and the dimensions of \(\mathcal{A}^{\text{even}}(1,3)\) and \(\mathcal{A}^{\text{odd}}(1,3)\) are equal to 8.

Four elements of \(\mathcal{A}(1,3)\) are called generators of the Clifford algebra if any element of \(\mathcal{A}(1,3)\) can be represented as a linear combination of products of these generators. Four generators \(L^\mu \in \mathcal{A}(1,3)\) are said to be primary generators if
\[
L^\mu L^\nu + L^\nu L^\mu = 2\eta^{\mu\nu}.\]

Hence, the basis elements \(\ell^\mu\) are primary generators of the Clifford algebra \(\mathcal{A}(1,3)\).

Let us define the trace of a Clifford algebra element as a linear operation
\[
\text{Tr} : \mathcal{A} \to \mathbb{R} \text{ or } \text{Tr} : \mathcal{A}_C \to \mathcal{C} \text{ such that } \text{Tr}(\ell) = 1 \text{ and } \text{Tr}(\ell^{\mu_1\cdots\mu_k}) = 0, \ k = 1, 2, 3, 4.
\]

The reader can easily prove that
\[
\text{Tr}(UV - VU) = 0, \quad \text{Tr}(V^{-1}UV) = \text{Tr}U, \quad U, V \in \mathcal{A}_C.
\]

For
\[
U = \sum_{\mu_1 < \cdots < \mu_k} u_{\mu_1\cdots\mu_k} \ell^{\mu_1\cdots\mu_k} \in \mathcal{A}_C^k(1,3)
\]
we may define an involution \(* : \mathcal{A}_C^k \to \mathcal{A}_C^k, \ k = 0, \ldots, 4\) by
\[
U^* = \sum_{\mu_1 < \cdots < \mu_k} \bar{u}_{\mu_1\cdots\mu_k} \ell^{\mu_k \cdots \mu_1}, \quad (10)
\]
where the bar means complex conjugation. For \(U \in \mathcal{A}^k(1,3)\) we have
\[
U^* = (-1)^{\frac{k(k-1)}{2}} U.
\]

It is readily seen that
\[
U^{**} = U, \quad (UV)^* = V^* U^*, \quad U, V \in \mathcal{A}(1,3). \quad (11)
\]

Let us define the group with respect to multiplication
\[
\text{Spin}(1,3) = \{ S \in \mathcal{A}^{\text{even}}(1,3) : S^* S = \ell \},
\]
which is called the spinor group. For any $F \in C^{\text{even}}(1,3)$ consider the linear operator $G_F : C(1,3) \to C(1,3)$ such that

$$G_F(U) = F^*UF. \quad (12)$$

In the sequel we use the following well known propositions (see proofs in [23]).

**Proposition 1.** If $T \in C^{\text{even}}(1,3)$ or $T \in C^{\text{odd}}(1,3)$, then

$$G_T : C^k(1,3) \to C^k(1,3), \quad k = 1, 2, 3;$$

$$G_T : C^0(1,3) \oplus C^4(1,3) \to C^0(1,3) \oplus C^4(1,3).$$

**Proposition 2.** If $S \in \text{Spin}(1,3)$, then

$$G_S : C^k(1,3) \to C^k(1,3), \quad k = 0, 1, 2, 3, 4.$$  

and

$$S^* \ell^\nu S = p_\mu^\nu \ell^\mu, \quad \text{for} \quad S \in \text{Spin}(1,3), \quad (13)$$

where the matrix $P = \|p_\mu^\nu\|$ satisfies the relations

$$P^T gP = g, \quad \det P = 1, \quad p_0^0 > 0. \quad (14)$$

Therefore if we transform the coordinate system

$$\tilde{x}^\nu = p_\mu^\nu x^\mu, \quad (15)$$

with the aid of this matrix $P = P(S)$, then we get the proper orthochronous Lorentz transformation from the group $\text{SO}^+(1,3)$. Conversely, if some matrix $P$ specifies a transformation (15) from the group $\text{SO}^+(1,3)$, then there exist two elements $\pm S \in \text{Spin}(1,3)$ such that the formula (13) is satisfied (in other words Spin(1,3) is a double covering of SO$^+(1,3)$). We say that the transformation of coordinates (15),(14) from the group $\text{SO}^+(1,3)$ is associated with the element $S \in \text{Spin}(1,3)$ if (13) holds.
1.3 Secondary generators of the Clifford algebra.

Denote $\ell^5 = \ell^{0123} = \ell^0\ell^1\ell^2\ell^3$. Then $(\ell^5)^2 = -\ell$ and $\ell^5$ commutes with all even elements and anticommutes with all odd elements of $\mathcal{O}(1,3)$.

**Definition 2.** If elements $H \in \mathcal{O}^1(1,3)$ and $I, K \in \mathcal{O}^2(1,3)$ satisfy the relations

$$H^2 = \ell, \quad I^2 = K^2 = -\ell, \quad [H, I] = [H, K] = 0, \quad \{I, K\} = IK + KI = 0,$$

then the elements $H, \ell^5, I, K$ are said to be secondary generators of the Clifford algebra $\mathcal{O}(1,3)$.

In particular, if we take

$$\dot{H} = \ell^0, \quad \dot{I} = -\ell^{12}, \quad \dot{K} = -\ell^{13},$$

then these elements satisfy (16) and hence, the elements $\dot{H}, \ell^5, \dot{I}, \dot{K}$ are secondary generators of $\mathcal{O}(1,3)$.

If $H, \ell^5, I, K$ are secondary generators of $\mathcal{O}(1,3)$, then the 16 elements

$$\ell, H, I, K, H1, HK, IK, HIK, \ell^5, \ell^5H, \ell^5I, \ell^5K, \ell^5HK, \ell^5IK, \ell^5HIK$$

are the basis elements of $\mathcal{O}(1,3)$ (linear independent) and the trace $\text{Tr}$ of every element of this basis, except $\ell$, is equal to zero.

Let $H, \ell^5, I, K$ be secondary generators of $\mathcal{O}(1,3)$. The first pair $H, \ell^5$ is such that

$$H^2 = \ell, \quad (\ell^5)^2 = -\ell, \quad \{H, \ell^5\} = 0.$$  

Thus the elements $H, \ell^5$ are generators of the Clifford algebra $\mathcal{O}(1,1)$.

The second pair $I, K$ is such that

$$I^2 = K^2 = -\ell, \quad \{I, K\} = 0.$$  

Therefore the elements $I, K$ are generators of the Clifford algebra $\mathcal{O}(0,2)$ (which is isomorphic to the algebra of quaternions). Furthermore, the elements $H, \ell^5$ commute with the elements $I, K$

$$[H, I] = [H, K] = [\ell^5, I] = [\ell^5, K] = 0.$$  

Consequently the Clifford algebra $\mathcal{O}(1,3)$ is isomorphic to the direct product

$$\mathcal{O}(1,3) \simeq \mathcal{O}(1,1) \otimes \mathcal{O}(0,2).$$
This relation leads to the well known fact that $\mathcal{O}(1,3)$ can be represented by the algebra $\mathcal{M}(2,\mathbb{H})$ of $2 \times 2$ matrices with quaternion elements.

We omit proofs of the following three propositions.

**Proposition 3.** Suppose elements $H \in \mathcal{O}^{\text{odd}}(1,3)$ and $I \in \mathcal{O}^{\text{even}}(1,3)$ satisfy the relations

$$H^2 = \ell, \quad I^2 = -\ell, \quad [H, I] = 0. \quad (21)$$

Then there exists an invertible element $T \in \mathcal{O}^{\text{even}}(1,3)$ such that

$$T^{-1}HT = \ell^0, \quad T^{-1}IT = -\ell^{12}.$$

**Proposition 4.** Suppose elements $H \in \mathcal{O}^{\text{1}}(1,3)$ and $I \in \mathcal{O}^{\text{2}}(1,3)$ satisfy the relations (21). Then there exists an element $S \in \text{Spin}(1,3)$ such that

$$S^* HS = \ell^0, \quad S^* IS = -\ell^{12}.$$

**Proposition 5.** Suppose $H, \ell, I, K$ are secondary generators of $\mathcal{O}(1,3)$; then there exists a unique element $S \in \text{Spin}(1,3)$ such that

$$S^* HS = \ell^0, \quad S^* IS = -\ell^{12}, \quad S^* KS = -\ell^{13}.$$

### 1.4 Idempotents, left ideals, and matrix representations of $\mathcal{O}(1,3)$.

An element $\tau$ of an algebra $\mathcal{A}$ is said to be an idempotent if $\tau^2 = \tau$. A subspace $\mathcal{I} \subseteq \mathcal{A}$ is called a left ideal of the algebra $\mathcal{A}$ if $au \in \mathcal{I}$ for all $a \in \mathcal{A}, u \in \mathcal{I}$. Every idempotent $\tau \in \mathcal{A}$ generates the left ideal

$$\mathcal{I}(\tau) = \{a\tau : a \in \mathcal{A}\}.$$

Consider the idempotent

$$t = \frac{1}{4}(\ell + H)(\ell - iI) \quad (22)$$
and the left ideal $\mathcal{I}(t)$ of $\mathcal{A}_C(1,3)$. The complex dimension of $\mathcal{I}(t)$ is equal to 4. Let us denote

$$t_k = F_k t, \quad k = 1, 2, 3, 4,$$

where

$$F_1 = \ell, \quad F_2 = K, \quad F_3 = -I\ell^5, \quad F_4 = -KI\ell^5.$$  

(23)

(24)

Elements $t_k \in \mathcal{I}(t), k = 1, 2, 3, 4$ are linear independent and form a basis of $\mathcal{I}(t)$. It is easy to check that $t_k t_1 = t_k$ and $t_k t_n = 0$ for $n \neq 1$.

Let us define an operation of Hermitian conjugation

$$U^\dagger := HU^* H, \quad U \in \mathcal{A}_C(1,3)$$

(25)

such that $(UV)^\dagger = V^\dagger U^\dagger$, $U^{\dagger\dagger} = U$.

Now we may introduce a scalar product of elements of the left ideal $\mathcal{I}(t)$

$$(U,V) := 4 \text{ Tr}(UV^\dagger), \quad U,V \in \mathcal{I}(t).$$

(26)

This scalar product converts the left ideal $\mathcal{I}(t)$ into the four dimensional unitary space.

**Theorem**. The basis elements $t_k = t^k, k = 1, 2, 3, 4$ of $\mathcal{I}(t)$ are mutually orthogonal w.r.t. the scalar product (26)

$$(t_k, t^n) = \delta_{kn}, \quad k,n = 1, 2, 3, 4$$

where $\delta_{kk} = 1$ and $\delta_{kn} = 0$ for $k \neq n$.

**Proof** is by direct calculation.

In what follows we use the formulas

$$\Psi = (\Psi, t^k) t_k \quad \text{for} \quad \Psi \in \mathcal{I}(t)$$

(27)

and

$$(KU, V) = (U, K^t V) \quad \text{for} \quad U,V \in \mathcal{I}(t), K \in \mathcal{A}_C(1,3).$$

(28)

Now we introduce the following concept. We claim that the set of secondary generators $H, \ell^5, I, K$ uniquely defines a matrix representation for $\mathcal{A}(1,3)$. Indeed, for $U \in \mathcal{A}(1,3)$ the products $Ut_k$ belong to $\mathcal{I}(t)$ and can be represented as linear combinations of basis elements $t_n$ with certain coefficients $\gamma(U)_k^n$

$$Ut_k = \gamma(U)_k^n t_n, \quad k = 1, 2, 3, 4.$$
Thus, the matrix $\gamma(U)$ with the elements $\gamma(U)_n^k$ (an upper index enumerates lines and a lower index enumerate columns of a matrix) is associated with the element $U \in \mathcal{A}(1, 3)$. In particular, the matrices $\gamma^\mu = \gamma(\ell^\mu)$ are defined by the formulas
\[
\ell^\mu t_k = \gamma(\ell^\mu)_k^n t_n, \quad k = 1, \ldots, 4; \quad \mu = 0, \ldots, 3.
\] (29)
Considering scalar products of the left and right hand sides of (28),(29) by $t^n$ and using mutual orthogonality of $t_k$, we get
\[
\gamma(U)_n^k = (U t_k, t^n)
\] (30)
and, in particular
\[
\gamma^\mu_n = \gamma(\ell^\mu)_n^k = (\ell^\mu t_k, t^n).
\] (31)
It can be shown that
\[
\gamma(UV) = \gamma(U)\gamma(V), \quad \gamma(\ell) = 1, \quad \gamma(\alpha U) = \alpha\gamma(U), \quad \alpha \in \mathbb{C}.
\]
Therefore the map $\gamma : \mathcal{A} \rightarrow \mathcal{M}(4, \mathbb{C})$ defined by (28) is a matrix representation of the Clifford algebra $\mathcal{A}$.

If we take secondary generators of the form (17), then the matrices $\gamma^\mu$ from (31) are equal to the matrices from (3).

Let $S$ be an element of the group Spin(1, 3) and $\gamma(S)$, $\gamma(S^*)$ be the matrix representation of $S$, $S^*$ given by (28). From the formula (13) we have
\[
S^* \ell^\mu S = p^\mu_\nu \ell^\nu, \quad S \ell^\nu S^* = q^\nu_\mu \ell^\mu,
\] (32)
where
\[
p^\mu_\nu q^\nu_\lambda = \delta^\mu_\lambda, \quad q^\mu_\nu p^\nu_\lambda = \delta^\mu_\lambda.
\]
For the secondary generators $H, \ell^5, I, K$ consider the transformation
\[
(H, \ell^5, I, K) \rightarrow (S^* HS, \ell^5, S^* IS, S^* KS),
\]
which leads to the transformation of the left ideal $\mathcal{I}(t) \rightarrow \mathcal{I}(S^* t S)$ and the basis elements
\[
t_k \rightarrow t'_k = S^* t_k S.
\]
Now we may define a new matrix representation of the Clifford algebra $\gamma : \mathcal{A} \rightarrow \mathcal{M}(4, \mathbb{C})$ with the aid of the formula
\[
U t'_k = \gamma(U)_n^k t_n.
\] (33)
Theorem. For every $U \in \mathcal{A}$ the matrix $\gamma(U)$ defined with the aid of (30) connected with the matrix representation $\hat{\gamma}(U)$ by the formula

$$\hat{\gamma}(U) = \gamma(S)\gamma(U)\gamma(S^*)$$

Proof. It is sufficient to prove this theorem for the primary generators $\ell^\mu$. We have

$$\ell^\mu t_k = \gamma(\ell^\mu)^n_k t_n.$$

Multiplying both sides of this relation from the left by $S^*$ and from the right by $S$, we obtain

$$(S^*\ell^\mu S)(S^*t_kS) = \gamma(\ell^\mu)^n_k(S^*t_nS).$$

Substituting $S^*t_kS = \mathit{t}_k$ and $S^*\ell^\mu S = p^\mu_\nu \ell^\nu$ from (32), we get

$$p^\mu_\nu \ell^\nu \mathit{t}_k = \gamma(\ell^\mu)^n_k \mathit{t}_n.$$

Multiplying both sides by $q^\lambda_\mu$ from (32) and summing over $\mu$, we obtain

$$q^\lambda_\mu p^\mu_\nu \ell^\nu \mathit{t}_k = \ell^\lambda \mathit{t}_k = \gamma(q^\lambda_\mu)^n_k \mathit{t}_n = \gamma(S\ell^\lambda S^*)^n_k \mathit{t}_n = \gamma(\gamma(S)\gamma(S^*))^n_k \mathit{t}_n.$$

This completes the proof.

Let us note that $\hat{\gamma}(S) = \gamma(S)$ and $\hat{\gamma}(S^*) = \gamma(S^*)$

1.5 A one-to-one correspondence between $\mathcal{I}(t)$ and $\mathcal{C}_{\mathcal{E}}(1, 3)$.

The dimension of the linear space $\mathcal{C}_{\mathcal{E}}(1, 3)$ is equal to 8. The left ideal $\mathcal{I}(t)$ has the complex dimension 4 and, thus, the real dimension 8. We may consider the map $\mathcal{C}_{\mathcal{E}}(1, 3) \to \mathcal{I}(t)$ given by the formula

$$\Psi \to \Psi t = \phi^k t_k,$$

where $\Psi \in \mathcal{C}_{\mathcal{E}}(1, 3)$ and $\phi^k = (\Psi t, t^k)$. Let us prove that this map gives the one-to-one correspondence between the even Clifford algebra $\mathcal{C}_{\mathcal{E}}(1, 3)$ and the left ideal $\mathcal{I}(t)$. 

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Theorem 3. Suppose $\Phi = \phi^k t_k \in \mathcal{I}(t)$. Then the equation for $\Omega \in \mathcal{C}^\text{even}(1,3)$

$$\Omega t = \Phi$$

has a unique solution

$$\Omega = F_k(\alpha^k \ell + \beta^k I),$$

where $\phi^k = \alpha^k + i\beta^k$ and $F_k$ are defined in (24).

Proof. Multiplying both sides of (35) from the right by $t$ and using the relation $It = it$, we see that the formula (35) really gives the solution of (34). Now we must prove that the homogeneous equation $\Omega t = 0$ has only trivial solution $\Omega = 0$. Firstly we give the proof for the secondary generators from (17). Suppose $\Omega$ is of the form

$$\Omega = \omega \ell + \sum_{\mu < \nu} \omega_{\mu \nu} \ell_{\mu \nu} + \omega_5 \ell_5.$$ 

Then the element $\Omega t \in \mathcal{I}(t)$ can be expanded in the basis $t_k$

$$\Omega t = (\omega - i\omega_{12}) t_1 + (-\omega_{13} - i\omega_{23}) t_2 + (-\omega_{03} + i\omega_5) t_3 + (-\omega_{01} - i\omega_{02}) t_4.$$ 

Therefore from $\Omega t = 0$ we get $\Omega = 0$. For the case of general secondary generators we must represent $\Omega$ as the linear combination of products of the generators $\ell_5, I, K$ and arguing as above. So the solution (35) of (34) is unique. This completes the proof.

Theorem. If $\Psi \in \mathcal{C}^\text{even}(1,3)$ and $S \in \text{Spin}(1,3)$, then the relation

$$\Psi t = \psi^k t_k,$$

where $\psi^k = (\Psi t, t^k) \in \mathcal{C}$, is invariant under the following transformation:

$$
\begin{align*}
t_k & \rightarrow t'_k = S^* t_k S; \\
\Psi & \rightarrow \Psi' = \Psi S; \\
\psi^k & \rightarrow \psi' = \gamma(S)^k \psi^l = (St_l, t^k) \psi^l.
\end{align*}
$$

Proof. We have

$$
(\Psi t - \psi^k t_k) S^* = \Psi t - (St_l, t^k) \psi^l S^* t_k = \\
= \Psi t - \psi^l (St_l, t^k) (S^* t_k, t^n) t_n = \\
= \Psi t - \psi^l t_n.
$$
Here we use the formulas

\[ S^* t_k = (S^* t_k, t^n) t_n \]

and

\[ (St_1, t^k)(S^* t_k, t^n) = \gamma(S)^k_1 \gamma(S^*)_k^n = \delta^n_1. \]

This completes the proof.

Consider the correspondence \( \ell^\mu \to \gamma^\mu \) between primary generators of the Clifford algebra and Dirac’s \( \gamma \)-matrices. Let us extend this correspondence to the map \( \gamma : \mathcal{O}(1, 3) \to \mathcal{M}(4, \mathbb{C}) \) or \( \gamma : \mathcal{O}_{\mathbb{C}}(1, 3) \to \mathcal{M}(4, \mathbb{C}) \) in such a way that any element of the Clifford algebra

\[ U = u \ell + u_\mu \gamma^\mu + \sum_{\mu_1 < \mu_2} u_{\mu_1 \mu_2} \ell^\mu_1 \ell^\mu_2 + \sum_{\mu_1 < \mu_2 < \mu_3} u_{\mu_1 \mu_2 \mu_3} \ell^{\mu_1 \mu_2 \mu_3} + u_{0123} \ell^{0123} \quad (38) \]

corresponds to the matrix \( U \in \mathcal{M}(4, \mathbb{C}) \)

\[ U = u 1 + u_\mu \gamma^\mu + \sum_{\mu_1 < \mu_2} u_{\mu_1 \mu_2} \gamma^\mu_1 \gamma^\mu_2 + \sum_{\mu_1 < \mu_2 < \mu_3} u_{\mu_1 \mu_2 \mu_3} \gamma^{\mu_1 \mu_2 \mu_3} + u_{0123} \gamma^{0123} \quad (39) \]

And let \( \ell : \mathcal{M}(4, \mathbb{C}) \to \mathcal{O}_{\mathbb{C}}(1, 3) \) or \( \ell : \mathcal{M}(4, \mathbb{C}) \to \mathcal{O}(1, 3) \) be the inverse map such that any matrix (39) corresponds to the element (38) of the (complex) Clifford algebra.

### 1.6 The Covariance of the Dirac Equation.

Let us consider the transformation of the Dirac equation (2) under a change of coordinates \( (x) \to (\tilde{x}) \) from the group \( \text{SO}^+(1, 3) \). This change of coordinates (15),(14) is associated with an element \( S \in \text{Spin}(1, 3) \) in accordance with (13). By \( S = \gamma(S) \) denote the matrix representation of the element \( S \). Then the matrix \( S \) satisfies

\[ S^* \gamma^\nu S = p^\nu_{\mu} \gamma^\mu. \quad (40) \]

The covectors \( \partial_\mu \) and \( a_\mu \) are transformed under the change of coordinates (15) as

\[ \partial_\mu = p^\nu_\mu \tilde{\partial}_\nu, \quad a_\mu = p^\nu_\mu \tilde{a}_\nu, \quad (41) \]
where \( \tilde{\partial}_\nu = \partial/\partial \tilde{x}^\nu \) and \( \tilde{a}_\nu \) are components of the covector \( a_\mu \) in coordinates \( (\tilde{x}) \). By substituting (41),(40) into (2), we obtain

\[
\gamma^\mu (\partial_\mu \psi + ia_\mu \psi) + im\psi = \gamma^\mu (\tilde{\partial}_\nu \psi + i\tilde{a}_\nu \psi) + im\psi = S^* \gamma^\nu (\tilde{\partial}_\nu (S\psi) + i\tilde{a}_\nu (S\psi)) + im(S\psi)
\]

Hence, if a column of four complex valued functions \( \psi = \psi(x) \) in the coordinates \( (x) \) satisfies the equation (2), then the column \( \tilde{\psi} = S\psi(x(\tilde{x})) \) in coordinates \( (\tilde{x}) \) satisfies the equation

\[
\gamma^\nu (\tilde{\partial}_\nu \tilde{\psi} + i\tilde{a}_\nu \tilde{\psi}) + im\tilde{\psi} = 0,
\]

which has the same form as (2).

**Definition 1.** A column of four complex valued functions \( \psi \) is called a **bispinor** if \( \psi \) transforms under the change of coordinates (15),(14),(13) as \( \psi \rightarrow \tilde{\psi} = S\psi(x(\tilde{x})) \), where \( S = \gamma(S) \).

### 1.7 Algebraic bispinors and the Dirac equation.

Consider the equation [4]

\[
\ell^\mu (\partial_\mu \rho + ia_\mu \rho) + im\rho = 0,
\]

where \( \rho = \rho(x) \in \mathcal{A}_c(1, 3) \). The element \( \rho \) has 16 complex components, i.e., four times more than the bispinor. Multiplying the equation (43) from the right by \( t \), we obtain that the element \( \theta = \rho t \in \mathcal{I}(t) \) satisfying the same equation

\[
\ell^\mu (\partial_\mu \theta + ia_\mu \theta) + im\theta = 0.
\]

**Theorem 2.** An element \( \theta = \psi_k t^k \in \mathcal{I}(t) \) satisfies the equation (44) iff the column \( \psi = (\psi_1 \psi_2 \psi_3 \psi_4)^T \) satisfies the Dirac equation (2).

**Proof.** Necessity. Substituting \( \theta = \psi_k t^k \) into (44) and using (28), we get

\[
\ell^\mu t^k (\partial_\mu \psi_k + ia_\mu \psi_k) + im\psi_l t^l = (\gamma^\mu_l (\partial_\mu \psi_k + ia_\mu \psi_k) + im\psi_l) t^l = 0.
\]

Taking into account the linear independence of \( t^k \), we obtain the equations

\[
\gamma^\mu_l (\partial_\mu \psi_k + ia_\mu \psi_k) + im\psi_l = 0, \quad l = 1, 2, 3, 4,
\]
which is evidently equivalent to the Dirac equation (2).

Arguing as above but in inverse order, we prove sufficiency. This completes the proof.

We say that for \( t \) from (22) the left ideal \( \mathcal{I}(t) \subset \mathcal{O}_C(1, 3) \) is the spinor space. Elements of the spinor space are called algebraic bispinors. The formula \( \theta = \psi_k t^k \) gives the relation between the algebraic bispinor \( \theta \in \mathcal{I}(t) \) and the bispinor \( \psi = (\psi_1 \psi_2 \psi_3 \psi_4)^T \).

1.8 Hestenes’ form of the Dirac equation.

Let \( H, \ell^5, I, K \) be secondary generators of \( \mathcal{O}(1, 3) \) and be independent of \( x \). Consider the equation for \( \Psi = \Psi(x) \in \mathcal{O}^{\text{even}}(1, 3) \)

\[
\ell^\mu (\partial_\mu \Psi + a_\mu \Psi I) + m\Psi HI = 0, \tag{46}
\]

which was invented by D.Hestenes [6] in 1966. This equation is called Hestenes’ form of the Dirac equation (HDE). Let us show that the equation (46) is equivalent to the equation (44) and consequently to the Dirac equation (2). To prove this fact we need the following theorem.

**Theorem 4.** An element \( \Psi = \Psi(x) \in \mathcal{O}^{\text{even}}(1, 3) \) satisfies the equation (46) iff the element \( \theta = \Psi t \in \mathcal{I}(t) \) satisfies the equation (44).

**Proof.** Necessity. Suppose \( \Psi \in \mathcal{O}^{\text{even}}(1, 3) \) satisfies the equation (46). Let us multiply (46) from the right by the idempotent \( t \) and use the relations

\[
Ht = t, \quad It = it.
\]

Then we get the equation (44) for \( \theta = \Psi t \in \mathcal{I}(t) \)

\[
\ell^\mu (\partial_\mu \Psi t + a_\mu \Psi It) + m\Psi HI t = \ell^\mu (\partial_\mu \Psi t + ia_\mu \Psi t) + im\Psi t = 0.
\quad \tag{47}
\]

Sufficiency. Suppose the element \( \theta \in \mathcal{I}(t) \) satisfies the equation (44). Let us multiply this equation from the right by \( t \) and use the relations

\[
t = tH, \quad it = tI, \quad \theta t = \theta.
\]

Then we obtain

\[
\ell^\mu (\partial_\mu \theta + a_\mu \theta I) + m\theta HI = 0. \quad \tag{47}
\]
Now using Theorem 3 we may take $\Psi \in C^\text{even}(1,3)$ as the solution of the equation $\Psi t = \theta$. We claim that this $\Psi$ satisfies the equation (46). In fact, substituting $\theta = \Psi t$ into the equation (47) and multiplying from the left by $H$, we get

$$0 = H(\ell^\mu(\partial_\mu\Psi + a_\mu \Psi I) + m\Psi HI)t \equiv \Omega t,$$

where $\Omega \in C^\text{even}(1,3)$. By Theorem 3 the equation $\Omega t = 0$ has only the trivial solution $\Omega = 0$. Hence $\Psi$ satisfies (46). The theorem is proved.

Thus, we have proved that HDE (46) for $\Psi \in C^\text{even}(1,3)$ is equivalent to the Dirac equation (2) for $\psi = (\psi_1 \psi_2 \psi_3 \psi_4)^T$, $\psi_k = \alpha_k + i\beta_k$ and the relation between these solutions is

$$\Psi = F^k(\alpha_k \ell + \beta_k I), \quad (48)$$

where the summation over $k = 1, 2, 3, 4$ is assumed.

Consider a change of coordinates (15),(14),(13) from the group $\text{SO}^+(1,3)$ associated with the element $S \in \text{Spin}(1,3)$. Arguing as in section 3, we see that HDE transforms under this change of coordinates as follows

$$\ell^\nu(\partial_\nu \Psi + a_\nu \Psi I) + m\Psi HI = S^*(\ell^\nu(\tilde{\partial}_\nu(S\Psi) + \tilde{a}_\nu(S\Psi) I) + m(S\Psi)HI).$$

Hence, if $\Psi = \Psi(x) \in C^\text{even}(1,3)$ in coordinates $(x)$ satisfies HDE (46), then the element $\tilde{\Psi} = S\Psi(x(\tilde{x}))$ in coordinates $(\tilde{x})$ satisfies the equation

$$\ell^\nu(\tilde{\partial}_\nu \tilde{\Psi} + \tilde{a}_\nu \tilde{\Psi} I) + m\tilde{\Psi} HI = 0,$$

which has the same form as (46). As was shown, the relation in coordinates $(x)$ between a solution $\psi$ of the Dirac equation and the solution $\Psi$ of HDE is given by the formula

$$\Psi t = \psi_k t^k, \quad \psi = (\psi_1 \psi_2 \psi_3 \psi_4)^T.$$ 

And the relation in coordinates $(\tilde{x})$ between the solution $\tilde{\psi} = S\psi$ of the Dirac equation and the solution $\tilde{\Psi} = S\Psi$ of HDE is given by the formula

$$\tilde{\Psi} t = \tilde{\psi} t^l, \quad \tilde{\psi} = (\tilde{\psi}_1 \tilde{\psi}_2 \tilde{\psi}_3 \tilde{\psi}_4)^T.$$ 

Indeed, using the formula $St = \gamma(S)t^k_i$, where $\gamma(S)t^k_i$ are elements of the matrix $S$, we obtain

$$\tilde{\Psi} t = S\Psi t = \psi_k St^k = \psi_k \gamma(S)t^k_i = \tilde{\psi} t^l.$$
1.9 The Grassmann-Clifford bialgebra.

Suppose that for elements of $\mathcal{C}(1,3)$ the exterior multiplication (denoted by $\wedge$) is defined by the following rules:

(i) $\mathcal{C}(1,3)$ is an associative algebra (with the unity element $\ell$) with respect to exterior multiplication;

(ii) $\ell^\mu \wedge \ell^\nu = -\ell^\nu \wedge \ell^\mu$, $\mu, \nu = 0, 1, 2, 3$;

(iii) $\ell^{\mu_1} \wedge \ldots \wedge \ell^{\mu_k} = \ell^{\mu_1 \ldots \mu_k}$ for $0 \leq \mu_1 < \ldots < \mu_k \leq 3$.

The resulting algebra (equipped with the Clifford multiplication and with the exterior multiplication) is called the Grassmann-Clifford bialgebra and is denoted by $\Lambda(1,3)$. The complex valued Grassmann-Clifford bialgebra is denoted by $\Lambda_C(1,3)$. Any element $U \in \Lambda(1,3)$ can be expanded in the basis as in (38). The coefficients $u_{\mu_1 \ldots \mu_k}$ in (38) are enumerated by ordered multi-indices. Let us take the coefficients that are antisymmetric w.r.t. all indices

$$u_{\mu_1 \ldots \mu_k} = u_{[\mu_1 \ldots \mu_k]}$$

where square brackets denote the operation of alternation (with the division by $k!$). Then elements of the form

$$\sum_{\mu_1 < \ldots < \mu_k} u_{\mu_1 \ldots \mu_k} \ell^{\mu_1} \ldots \ell^{\mu_k} = \frac{1}{k!} u_{\nu_1 \ldots \nu_k} \ell^{\nu_1} \wedge \ldots \wedge \ell^{\nu_k}$$

(49)

are said to be elements of rank $k$ and belong to $\Lambda^k(1,3)$, where $\Lambda^k(1,3)$, $\Lambda^{\text{even}}(1,3)$, $\Lambda^{\text{odd}}(1,3)$ are the same as $\mathcal{C}^k(1,3)$, $\mathcal{C}^{\text{even}}(1,3)$, $\mathcal{C}^{\text{odd}}(1,3)$. If $U \in \Lambda^r(1,3)$, $V \in \Lambda^s(1,3)$ then

$$U \wedge V = (-1)^{rs} V \wedge U \in \Lambda^{r+s}(1,3).$$

(50)

2 Part II.

2.1 The exterior algebra of Minkowski space.

Let $\mathcal{E}$ be Minkowski space with the metric tensor (1), with coordinates $x^\mu$, with basis coordinate vectors $e_\mu$, and with basis covectors $e^\mu = g^{\mu\nu} e_\nu$. Consider a covariant antisymmetric tensor field of rank $0 \leq k \leq 3$ on $\mathcal{E}$ with
components
\[ u_{\mu_1 \ldots \mu_k} = u_{[\mu_1 \ldots \mu_k]}(x). \]

It is suitable to write this field with the aid of the expression
\[ \frac{1}{k!} u_{\nu_1 \ldots \nu_k} e^{\nu_1} \wedge \ldots \wedge e^{\nu_k}, \tag{51} \]
where the expression \( e^{\nu_1} \wedge \ldots \wedge e^{\nu_k} \) in the fixed coordinates \( (x) \) can be considered as an element of the Grassmann algebra. Under a linear change of coordinates
\[ x^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \tilde{x}^\nu = q^\mu_\nu \tilde{x}^\nu \tag{52} \]
the transformation rule for the expression \( e^{\nu_1} \wedge \ldots \wedge e^{\nu_k} \) corresponds to the transformation rule for basis covectors \( e^\nu \). That is,
\[ e^\nu = q^\nu_\mu \tilde{e}^\mu, \quad e^{\nu_1} \wedge \ldots \wedge e^{\nu_k} = q^\nu_{\nu_1} \ldots q^{\nu_k}_{\nu_k} \tilde{e}^{\nu_1} \wedge \ldots \wedge \tilde{e}^{\nu_k}. \tag{53} \]

Therefore under the change of coordinates (52) the expression (51) is invariant
\[ \frac{1}{k!} u_{\nu_1 \ldots \nu_k} e^{\nu_1} \wedge \ldots \wedge e^{\nu_k} = \frac{1}{k!} \tilde{u}_{\mu_1 \ldots \mu_k} \tilde{e}^{\mu_1} \wedge \ldots \wedge \tilde{e}^{\mu_k}, \]
where
\[ \tilde{u}_{\mu_1 \ldots \mu_k} = p^\lambda_{\nu_1} \ldots p^\lambda_{\nu_k} u_{\lambda_1 \ldots \lambda_k} \]
are components of the tensor field \( u_{\nu_1 \ldots \nu_k} \) in the coordinates \( (\tilde{x}) \) and
\[ p^\lambda_\nu = \frac{\partial \tilde{x}^\lambda}{\partial x^\nu}, \quad p^\lambda_\nu q^\nu_\mu = \delta^\lambda_\mu \quad (\delta^\lambda_\lambda = 1, \, \delta^\lambda_\mu = 0 \text{ for } \mu \neq \lambda). \]

The expressions (51) are called exterior forms of rank \( k \) or \( k \)-forms. The set of all \( k \)-forms is denoted by \( \Lambda^k(\mathcal{E}) \). The formal sum of \( k \)-forms
\[ \sum_{k=0}^{4} \frac{1}{k!} u_{\nu_1 \ldots \nu_k} e^{\nu_1} \wedge \ldots \wedge e^{\nu_k} \tag{54} \]
are said to be (nonhomogeneous) exterior form. The set of all exterior forms is denoted by \( \Lambda(\mathcal{E}) \) and
\[ \Lambda(\mathcal{E}) = \Lambda^0(\mathcal{E}) \oplus \ldots \oplus \Lambda^4(\mathcal{E}) = \Lambda^{\text{even}}(\mathcal{E}) \oplus \Lambda^{\text{odd}}(\mathcal{E}). \]
It is well known that the exterior product of exterior forms is an exterior form.

Consider the Hodge star operator $\star : \Lambda^k(\mathcal{E}) \to \Lambda^{4-k}(\mathcal{E})$. If $U \in \Lambda^k(\mathcal{E})$ has the form (51), then

$$\star U = \frac{1}{k!(4-k)!} \varepsilon_{\mu_1...\mu_k} u^{\mu_1...\mu_k} e^{\mu_{k+1}} \wedge ... \wedge e^{\mu_4},$$

where

$$u^{\mu_1...\mu_k} = g^{\mu_1\nu_1} ... g^{\mu_k\nu_k} u_{\nu_1...\nu_k}$$

and $\varepsilon_{\mu_1...\mu_4}$ is the sign of the permutation $(\mu_1...\mu_4)$. $\star U$ is an exterior form (a covariant antisymmetric tensor) w.r.t. any change of coordinates with a positive Jacobian.

**Remark.** In this paper we consider changes of coordinates only from the group $SO^+(1,3)$ (a Jacobian is equal to 1) and do not distinguish tensors and pseudotensors.

It is easy to prove that for any $U \in \Lambda^k(\mathcal{E})$

$$\star (\star U) = (-1)^{k+1} U. \quad (55)$$

Further on we consider the bilinear operator $\text{Com} : \Lambda^2(\mathcal{E}) \times \Lambda^2(\mathcal{E}) \to \Lambda^2(\mathcal{E})$ such that for basis 2-forms

$$\text{Com}(e^{\mu_1} \wedge e^{\mu_2}, e^{\nu_1} \wedge e^{\nu_2}) = -2g^{\mu_1\nu_1} e^{\mu_2} \wedge e^{\nu_2} - 2g^{\mu_2\nu_2} e^{\mu_1} \wedge e^{\nu_1} + 2g^{\mu_1\nu_2} e^{\mu_2} \wedge e^{\nu_1} + 2g^{\mu_2\nu_1} e^{\mu_1} \wedge e^{\nu_2}$$

Evidently

$$\text{Com}(U, V) = -\text{Com}(V, U), \quad U, V \in \Lambda^2(\mathcal{E}).$$

Now we define the Clifford multiplication of exterior forms with the aid of the following formulas:

$$\begin{align*}
0^k U V &= V U = U \wedge V = V \wedge U, \\
1^k U V &= U \wedge V - \star (U \wedge \star V), \\
k^1 U V &= U \wedge 1 + \star (\star U \wedge 1), \\
k^2 U V &= U \wedge 2 + \star (\star U \wedge 2) + \frac{1}{2} \text{Com}(U, V),
\end{align*}$$

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\[
\begin{align*}
\mathcal{U}^3 \mathcal{V} &= \star \mathcal{U} \wedge \star \mathcal{V} - \star (\mathcal{U} \wedge \star \mathcal{V}), \\
\mathcal{U}^4 \mathcal{V} &= \star \mathcal{U} \wedge \star \mathcal{V}, \\
\mathcal{V}^2 \mathcal{U} &= -\star \mathcal{V} \wedge \mathcal{U} - \star (\mathcal{V} \wedge \mathcal{U}), \\
\mathcal{V}^3 \mathcal{U} &= \star \mathcal{V} \wedge \mathcal{U} + \star (\mathcal{V} \wedge \mathcal{U}), \\
\mathcal{V}^4 \mathcal{U} &= \star \mathcal{V} \wedge \mathcal{U}, \\
\mathcal{V}^2 \mathcal{U} &= -\star \mathcal{V} \wedge \mathcal{U}, \\
\mathcal{V}^4 \mathcal{U} &= -\star \mathcal{V} \wedge \mathcal{U},
\end{align*}
\]

where ranks of exterior forms are denoted as $U \in \Lambda^k(E)$ and $k = 0, 1, 2, 3, 4$.

From this definition we may get some properties of the Clifford multiplication of exterior forms.

1. If $U, V \in \Lambda(E)$, then $UV \in \Lambda(E)$.
2. The axioms of associativity and distributivity are satisfied for the Clifford multiplication.
3. $e^\mu e^\nu = e^\mu \wedge e^\nu + g^{\mu\nu}e$, $e^\mu e^\nu + e^\nu e^\mu = 2g^{\mu\nu}e$.
4. $e^{\mu_1} \ldots e^{\mu_k} = e^{\mu_1} \wedge \ldots \wedge e^{\mu_k} = e^{\mu_1 \ldots \mu_k}$ for $0 \leq \mu_1 < \cdots < \mu_k \leq 3$.
5. If $U, V \in \Lambda^2(E)$, then $\text{Com}(U, V) = UV - VU$.

Taking into account these properties of Clifford multiplication, we may conclude that Propositions 1 – 5 of Part I initially formulated for elements of $\mathcal{C}(1, 3)$ are also valid for elements of $\Lambda(E)$.

In the sequel, we use the group

\[
\text{Spin}(E) = \{ S \in \Lambda^\text{even}(E) : S^*S = e \}.
\]

\[\text{(56)}\]

### 2.2 Operators $d, \delta, \Upsilon, \Delta$.

First consider the operator

\[
dV = e^\mu \wedge \partial_\mu V, \quad V \in \Lambda(E)
\]

such that
1) $d : \Lambda^k(\mathcal{E}) \to \Lambda^{k+1}(\mathcal{E})$;
2) $d^2 = 0$;
3) $d(U \wedge V) = dU \wedge V + (-1)^k U \wedge dV$ for $U \in \Lambda^k(\mathcal{E}), V \in \Lambda(\mathcal{E})$.

Secondly, consider the operator $\delta$

$$\delta U = \ast d \ast U \quad \text{for} \quad U \in \Lambda(\mathcal{E})$$

such that
1) $\delta : \Lambda^k(\mathcal{E}) \to \Lambda^{k-1}(\mathcal{E})$;
2) $\delta^2 = 0$.

Thirdly, consider the operator (Upsilon)

$$\Upsilon = d - \delta$$

such that
1) $\Upsilon : \Lambda^k(\mathcal{E}) \to \Lambda^{k+1}(\mathcal{E}) \oplus \Lambda^{k-1}(\mathcal{E})$;
2) $\Upsilon U = e^\mu \partial_\mu U$.

The second property in this list follows from the definition of Clifford multiplication

$$\frac{1}{k} \frac{1}{k} U \wedge V = U \wedge V - \ast (U \wedge \ast V)$$

if we formally substitute $\frac{1}{U} = e^\mu \partial_\mu$.

Finally, consider the Beltrami-Laplace operator

$$\Delta = \Upsilon^2 = (d - \delta)^2 = -(d\delta + \delta d) = g^{\mu\nu} \partial_\mu \partial_\nu$$

such that
1) $\Delta : \Lambda^k(\mathcal{E}) \to \Lambda^k(\mathcal{E})$;
2) $\Delta$ commutes with the operators $d, \delta, \Upsilon, \ast$. 

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2.3 A tensor form of the Dirac equation.

Let \( H \in \Lambda^1(\mathcal{E}) \), \( I \in \Lambda^2(\mathcal{E}) \) be two independent of \( x \) exterior forms such that

\[
H^2 = e, \quad I^2 = -e, \quad [H, I] = 0.
\]

(57)

Now we consider the equation

\[
\Upsilon \Phi + A\Phi I + m\Phi HI = 0,
\]

(58)

where \( \Phi = \Phi(x) \in \Lambda^{\text{even}}(\mathcal{E}) \), \( A = a_\mu(x)e^\mu \in \Lambda^1(\mathcal{E}) \), and \( m \geq 0 \) is a real constant. All the values (\( \Phi, A, I, H \)) in (58) are exterior forms (covariant antisymmetric tensors). Two operations are used in (58). Namely the differential operator \( \Upsilon = d - \delta = e^\mu \partial_\mu \) and the Clifford multiplication of exterior forms. Both operations take exterior forms to exterior forms. In other words, (58) is a tensor equation. We say that the equation (58) is the Tensor form of the Dirac Equation (TDE). The TDE is invariant under the following global (independent of \( x \)) transformation

\[
\begin{align*}
\Phi & \rightarrow \Phi S, \\
H & \rightarrow S^* HS, \\
I & \rightarrow S^* IS,
\end{align*}
\]

(59)

where \( S \in \text{Spin}(\mathcal{E}) \), \( \partial_\mu S = 0 \).

In a fixed coordinate system \( (x) \) the TDE is equivalent to HDE (46) and the connection between (58) and (46) is given by the formula

\[
(\Phi)_{\mu^* \rightarrow \nu^*} = \Psi.
\]

Let us remind that HDE (46) is equivalent to the Dirac equation (2) and the connection between them is given by the formula (48).

**Remark.** Taking into account the theorem 5, it is clear that we may use in the TDE \( H \in \Lambda^{\text{odd}}(\mathcal{E}), I \in \Lambda^{\text{even}}(\mathcal{E}) \) which satisfies (57) instead of \( H \in \Lambda^1(\mathcal{E}), I \in \Lambda^2(\mathcal{E}) \). In this case the equation (58) is invariant under the global transformation

\[
\begin{align*}
\Phi & \rightarrow \Phi T, \\
H & \rightarrow T^{-1} HT, \\
I & \rightarrow T^{-1} IT,
\end{align*}
\]

where \( T \in \Lambda^{\text{even}}(\mathcal{E}) \) is an invertible element.
Under a linear change of coordinates \((x) \rightarrow (\tilde{x})\) all exterior forms are invariants. Therefore in coordinates \((\tilde{x})\) the TDE has the form

\[
\tilde{\Upsilon} \tilde{\Phi} + \tilde{A} \tilde{\Phi} \tilde{I} + m \tilde{\Phi} \tilde{H} \tilde{I} = 0,
\]

where \(\tilde{\Upsilon} = \tilde{e}^\mu \partial/\partial \tilde{x}^\mu = \Upsilon\), \(\tilde{\Phi} = \Phi\), \(\tilde{A} = A\), \(\tilde{H} = H\), \(\tilde{I} = I\) and \(\tilde{\Phi}, \tilde{A}, \tilde{H}, \tilde{I}\) are the exterior forms written in coordinates \((\tilde{x})\).

Let us define the trace of an exterior form as a linear operation \(\text{Tr} : \Lambda(E) \rightarrow \mathbb{R}\) or \(\text{Tr} : \Lambda_\mathbb{C}(E) \rightarrow \mathbb{C}\) such that \(\text{Tr}(e) = 1\) and \(\text{Tr}(e_{\mu_1 \cdots \mu_k}) = 0, k = 1, 2, 3, 4\). The reader can easily prove that

\[
\text{Tr}(UV - VU) = 0, \quad \text{Tr}(V^{-1}UV) = \text{Tr}U, \quad U, V \in \Lambda(E).
\]

Now we can find the conservative current for the TDE. For this let us denote

\[
C = \Phi^*(\Upsilon \Phi + A \Phi I + m \Phi HI), \quad \Phi = H \Phi^*.
\]

Then

\[
HC = \tilde{\Phi}(e^\mu \partial_\mu \Phi + A \Phi I + m \Phi HI), \quad HC^* = (\partial_\mu \tilde{\Phi} e^\mu - I \Phi A - m I H \Phi) \Phi.
\]

Using the formula \(\text{Tr}(UV - VU) = 0\), we get

\[
\text{Tr}(\Phi A \Phi - \tilde{\Phi} A \Phi I) = 0, \quad \text{Tr}(H I \Phi \Phi - \tilde{\Phi} \Phi HI) = 0. \quad (60)
\]

Suppose \(\Phi \in \Lambda^\text{even}(\mathcal{E})\) is a solution of the TDE, then, with the aid of (60), we obtain

\[
0 = \text{Tr}(H(C + C^*)) = \text{Tr}(\Phi e^\mu \partial_\mu \Phi + \partial_\mu \tilde{\Phi} e^\mu \Phi) = \text{Tr}(\partial_\mu (\tilde{\Phi} e^\mu \Phi)) = \partial_\mu j^\mu,
\]

where

\[
j^\mu = \text{Tr}(\tilde{\Phi} e^\mu \Phi).
\]

Therefore the vector \(j^\mu\) is the conservative current. If we take the 1-form

\[
J = g_{\mu \nu} j^\nu e^\mu = j_\mu e^\mu = \Phi \Phi^*,
\]

then the divergence \(\partial_\mu j^\mu = 0\) can be rewritten in the form

\[
\delta J = 0.
\]
Finally let us define the Lagrangian from which the TDE can be derived

\[ \text{Lagr}_1 = \text{Tr}(HC_{1}). \]

Adding the term that describes the free field \( A \) to \( \text{Lagr}_1 \), we obtain

\[ \text{Lagr} = \text{Lagr}_1 + \text{Tr}(F^2), \quad (61) \]

where \( F = dA \) is a 2-form, \( \text{Tr}(F^2) = -\frac{1}{2}f^{\mu\nu}f_{\mu\nu} \), \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). Hence the Lagrangian \( \text{Lagr} \) depends on the following exterior forms: \( \Phi \in \Lambda^{\text{even}}(E) \), \( \Phi^* \in \Lambda^{\text{odd}}(E) \), \( A \in \Lambda^1(E) \). Using the variational principle [22] we suppose that the exterior forms \( \Phi \) and \( \Phi^* \) are independent and as the variational variables we take the 8 functions which are the coefficients of the exterior form \( \Phi \) and the 4 functions which are the coefficients of the exterior form \( A \). The Lagrange-Euler equations with respect to these variables give us the system of equations, which can be written in the form

\[ (d - \delta)\Phi + A \Phi I + m\Phi HI = 0, \]
\[ dA = F, \]
\[ \delta F = J, \quad (62) \]

where \( J = \Phi \Phi^* = \Phi H \Phi^* \). This system of equation can also be written in the form

\[ e^\mu (\partial_\mu \Phi + a_\mu \Phi I) + m\Phi HI = 0, \]
\[ \partial_\mu a_\nu - \partial_\nu a_\mu = f_{\mu\nu}, \quad (63) \]
\[ \partial_\mu f^{\mu\nu} = j^\nu, \]

where \( j^\nu = \text{Tr}(\Phi e^\nu \Phi) \), \( f^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}f_{\lambda\sigma} \).

The Lagrangian (61) and the systems of equations (62),(63) are invariant under the gauge transformation with the symmetry group \( U(1) \)

\[ \Phi \rightarrow \Phi \exp(\lambda I), \quad A \rightarrow A - d\lambda, \quad (64) \]

where \( \lambda = \lambda(x) \in \Lambda^0(E) \) and \( \exp(\lambda I) = \cos \lambda + I \sin \lambda \). And they are also invariant under the global (independent of \( x \)) transformation (59) with the symmetry group \( \text{Spin}(E) \). The gauge invariance (64) expresses the interaction of the electron (fermion) with an electromagnetic field. The global invariance
(59) leads to unusual (compared to tensors) transformation properties of bispinors under Lorentz changes of coordinates. The Dirac equation can be generalized to the (pseudo) Riemannian space $\mathcal{V}$ using a special technique known as the tetrad formalism. We suppose that the TDE gives another possibility to describe the electron in the presence of gravity. Here we must take into account the fact that in Riemannian space there is no invariance of equations under global transformations of the form (59). Consequently we must use a gauge (local) transformation instead of the global transformation (59). This leads to a new gauge field with the $\text{Spin}(\mathcal{V})$ symmetry group, which we interpret as the gravitational field. For details of such an approach see [18].

2.4 Other tensor equations.

Consider the Ivanenko-Landau-Kähler equation [21],[14]

$$\Upsilon \rho + i A \rho + im \rho = 0,$$  

(65)

where $\rho = \rho(x) \in \Lambda_C(\mathcal{E})$ and $A = A(x) \in \Lambda^1(\mathcal{E})$. We arrive at the TDE (58) by multiplying (65) from the right by the idempotent

$$t = \frac{1}{4}(e + H)(e - iI)$$

and using the relations

$$t = tH, \quad it = tI.$$  

Similarly, multiplying (65) by

$$t = \frac{1}{2}(e + H)$$

and using the relation $t = tH$, we arrive at the following equation:

$$\Upsilon \eta + i A \eta + im \eta H = 0,$$  

(66)

where $\eta \in \Lambda^\text{even}_C(\mathcal{E})$. In the same way, multiplying (65) by

$$t = \frac{1}{2}(e - ie^5), \quad (e^5 = e^{0123} = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = e^0 e^1 e^2 e^3),$$

and using the relation $it = te^5$ we arrive at the equation

$$\Upsilon \omega + A \omega e^5 + m \omega e^5 = 0,$$  

(67)

where $\omega \in \Lambda(\mathcal{E})$. Evidently, (65),(66),(67) are tensor equations.
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