A WEAK COMPARISON PRINCIPLE IN TUBULAR NEIGHBOURHOODS OF EMBEDDED MANIFOLDS

FRANCESCO POLIZZI, PIETRO SABATINO, AND BERARDINO SCIUNZI

ABSTRACT. We study weak solutions to degenerate quasilinear elliptic equations, involving first order terms, in unbounded tubular domains. In particular we show that, under suitable hypotheses, the weak comparison principle holds if the domain is narrow enough.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We want to study the weak comparison principle for degenerate quasilinear elliptic equations with first order terms. More precisely, we consider $u,v \in C^1(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$, weak solutions to the problem

$$\begin{cases}
-\text{div}(a(z)\nabla u) + \Lambda|\nabla u|^q \leq f(z,u) & \text{in } \Omega \\
-\text{div}(a(z)\nabla v) + \Lambda|\nabla v|^q \geq f(z,v) & \text{in } \Omega,
\end{cases}$$

where $q \geq 1$, $\Omega \subseteq \mathbb{R}^n$ and denoting with $z$ a point in $\mathbb{R}^n$. Equivalently for smooth solutions, by using the divergence theorem, (1) can be rephrased as

$$\int_{\Omega} a(z)\nabla u \cdot \nabla \psi + \Lambda|\nabla u|^q \psi \, dz \leq \int_{\Omega} f(z,u) \psi \, dz,$$

$$\int_{\Omega} a(z)\nabla v \cdot \nabla \psi + \Lambda|\nabla v|^q \psi \, dz \geq \int_{\Omega} f(z,v) \psi \, dz,$$

for every test-function $\psi \in C^\infty_c(\Omega)$ with $\psi \geq 0$. In all the paper $a(z)$ is a nonnegative weight satisfying A1. In particular, since $a(z)$ may vanish, the operator could be degenerate at some points, causing that solutions could be not of class $C^2$. In such a case (2) and (3) represent the

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right way of understanding our problem and in fact all our proofs are carried out exploiting
only the weak formulation.

As customary in the literature, we say that the weak comparison principle for problem (1) holds in \( \Omega \) if, imposing that \( u \leq v \) on \( \partial \Omega \), then it follows that \( u \leq v \) in the whole of \( \Omega \). A leading example is provided by the weak maximum principle, namely by the case when \( v = 0 \). It is well-known that the maximum principle does not hold in general domains, but it holds if the domain is sufficiently small or after imposing suitable a-priori estimates on the solutions. For weak solutions this can be seen e.g. as a consequence of the Poincaré inequality; on the other hand, if one considers smooth solutions then direct methods are also available, see [GT83].

In the present work we deal with the case of unbounded domains. Here the situation is quite well understood for domains with flat boundary and in particular for narrow strips. In fact, maximum and comparison principles in strips have many applications to the study of symmetry and monotonicity properties of the solutions, see for instance [BCN1, BCN2, BCN3, BCN4] and especially [BNV]. The case of domains with a possible different geometry has been considered in [Cab95], where deep results have been obtained by means of Alexandroff-Bakelmann-Pucci-type estimates.

In our analysis, we suppose that the (possibly unbounded and) smooth domain \( \Omega \subset \mathbb{R}^n \) is contained in a normal tube \( \mathcal{B}(M, \varepsilon) \), where \( M \subset \mathbb{R}^n \) is a smoothly embedded submanifold. Our crucial idea is to manipulate the inequalities (2) and (3) by using the so-called Fermi coordinates associated with the normal tube \( \mathcal{B}(M, \varepsilon) \), see [G04, Chapter 2], [PPS14, Section 3]. The weak comparison principle then follows by showing that \( \mathcal{L}_R = 0 \) for all \( R > 0 \), where

\[
\mathcal{L}_R := \int_{\Omega_{2R}} a(z)(u - v)^*|\nabla(u - v)|^2\,d\text{vol}_g,
\]

see Section 3 for notation and details. The desired vanishing is established by proving that \( \mathcal{L}_R \) can be estimated in terms of \( \mathcal{L}_{2R} \), see Equation (35), allowing us to exploit the simple but useful Lemma (2.1).

This kind of technique seems to be very flexible. In particular, having in mind the fact that the technique is manly based on the weighted Sobolev inequality, we believe that it could be also used when dealing with \( p \)-Laplace inequalities as well as with fully nonlinear problems.

Let us now introduce some notation and terminology that will be used in the sequel. Let \( M \) be a smooth submanifold of \( \mathbb{R}^n \) of dimension \( n - k \), and let \( i : M \hookrightarrow \mathbb{R}^n \) be the embedding map. By a normal tube of constant radius \( \varepsilon > 0 \) centred at \( M \) we mean a tubular neighbourhood of \( M \) given by a disjoint union

\[
\mathcal{B}(M, \varepsilon) = \bigcup_{p \in M} B(p, \varepsilon),
\]

where \( B(p, \varepsilon) \) is a \( k \)-dimensional ball of radius \( \varepsilon \) centred at \( p \in M \) and contained in the normal subspace \( N_p M \subset \mathbb{R}^n \). Moreover, given any subset \( A \subseteq M \) we define

\[
\mathcal{B}(A, \varepsilon) = \bigcup_{p \in A} B(p, \varepsilon) \subseteq \mathcal{B}(M, \varepsilon).
\]

We are now ready to make our basic hypothesis on the domain \( \Omega \).

**Main Assumption.** The submanifold \( M \) admits a normal tube \( \mathcal{B}(M, \varepsilon) \) for some \( \varepsilon > 0 \) and we have \( \Omega \subset \mathcal{B}(M, \varepsilon) \).
Remark 1.1. Let us consider the normal exponential map
\[ \exp^\perp: NM \longrightarrow \mathbb{R}^n, \]
sending the pair \((p,v), \) with \(p \in M\) and \(v \in T_pM, \) to \(p + v\) (here we are identifying \(T_p\mathbb{R}^n\) with \(\mathbb{R}^n\) in the standard way). We can define the normal injectivity radius of \(M\) as the strictly positive function \(\rho: M \rightarrow \mathbb{R}\) that associates to \(p \in M\) the supremum \(\rho(p)\) of all \(\delta \leq 1\) such that the restriction of \(\exp^\perp\) to the neighbourhood of the zero section
\[ V_0(p) := \{(p', v') \in NM : |p - p'| < \delta, |v'| < \delta\} \]
is a diffeomorphism onto its image in \(\mathbb{R}^n.\) Therefore \(M\) admits a normal tube \(\mathcal{B}(M, \epsilon)\) of radius \(\epsilon\) if and only if \(\rho(p) \geq \epsilon\) for all \(p \in M.\) In fact, in this case we can identify \(\mathcal{B}(M, \epsilon)\) with the open neighbourhood of \(M\) in \(\mathbb{R}^n\) defined by
\[ \{x \in \mathbb{R}^n : d(x, M) < \epsilon\}, \]
see [O'Neill83, Proposition 7.26] and [Lee03, Theorem 10.89] for more details.

In particular, if \(M\) is compact and \(\epsilon\) is sufficiently small then \(\mathcal{B}(M, \epsilon)\) always exists, cf. [Con01, Proposition 3.7.18] and [PPS14, Section 3]. On the other hand, there are also plenty of non-compact submanifolds of \(\mathbb{R}^n\) admitting normal tubes of constant radius, for instance all the linear subspaces have this property.

It is important to observe that it is not possible to give sufficient conditions for the existence of \(\mathcal{B}(M, \epsilon)\) in terms of curvature bounds for \(M.\) To see this, consider the smooth curve \(C \subset \mathbb{R}^3\) parametrized by \(x \mapsto (\sin x, \cos x, \arctan x),\) namely a spiral with the distance between the coils tending to 0. A straightforward computation shows that its curvature \(\kappa(x)\) satisfies \(\frac{1}{2} \leq \kappa(x) < 1,\) hence it is globally bounded, but \(M\) admits no tubular neighbourhood of constant radius.

In addition to our Main Assumption, will also need Assumptions A1, A2, A3, A4 stated below.

**Assumption A1:** For all \(p \in M,\) write
\[ \Omega_p := \Omega \cap B(p, \epsilon). \]
We require that the weight \(a\) in (1) is a non-negative function such that \(a \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)\) and we suppose that there exists a constant \(C_a\) such that
\[ \int_{\Omega_p} (a|_{\Omega_p})^{-t} \, d y^1 \ldots d y^k \leq C_a \quad \text{for some } t > k, \text{ uniformly with respect to } p. \]
Here \(a|_{\Omega_p}\) denotes the restriction of \(a\) to \(\Omega_p\) and \(y^1, \ldots, y^k\) are the Euclidean coordinates in the \(k\)-dimensional linear subspace \(N_pM\) of \(\mathbb{R}^n\) containing \(\Omega_p.\) We observe that this kind of condition is quite well-known in the context of weighted Sobolev spaces since of the works [MuSt68, Tr73].

**Assumption A2:** \(f(z, \cdot)\) is a continuous function, uniformly Lipschitz with respect to \(z.\) In other words, for every \(m > 0\) there is a positive constant \(L_f = L_f(m)\) such that for every \(z \in \Omega\) and every \(u, v \in [-m, m]\) we have
\[ \|f(z, u) - f(z, v)\| \leq L_f |u - v|. \]
As a leading example we may consider the case
\[ f(z, u) = h(z)g(u), \]
where \(h \in L^\infty(\Omega)\) and \(g\) is a locally Lipschitz, continuous function.
Assumption A3: We consider an oriented atlas of $M$

$$((U_a, \varphi_a)), \quad M = \bigcup_a U_a, \quad \varphi_a : U_a \xrightarrow{\cong} V_a \subset \mathbb{R}^{n-k}$$

such that the functions $i_a \circ \varphi_a^{-1} : V_a \to \mathbb{R}^n$ (where $i_a : U_a \to \mathbb{R}^n$ denotes the restriction of $i : M \to \mathbb{R}^n$ to $U_a$), together with their first and second derivatives, are uniformly bounded with respect to $a$.

Remark 1.2. An atlas as above exists for every smooth manifold $M$. In fact, let us start with an atlas $((U_a, \varphi_a))$ given by a countable basis of precompact smooth coordinate balls (it exists by [Lee03, Lemma 1.11]). On each chart $U_a$, the functions $i_a \circ (\varphi_a)^{-1}$ and their derivatives are bounded. We now set $\varphi_a := A_a \varphi_a'$, where $A_a$ is a positive constant; then, provided that we choose the $A_a$ large enough, we can make the bounds above uniform on $a$.

Assumption A4: There exist a point $\bar{p} \in M$ and constants $C_1, \gamma, R_0 > 0$ such that

$$\text{vol}(B_M(\bar{p}, R)) \leq C_1 R^\gamma$$

for all $R > R_0$. Here we define

$$B_M(\bar{p}, R) := \{ p \in M \mid d_M(\bar{p}, p) < R \},$$

where $d_M$ denotes the distance function induced on $M$ by the pull-back of the Euclidean metric of $\mathbb{R}^n$, and $\text{vol}(B_M(\bar{p}, R))$ stands for the measure of $B_M(\bar{p}, R)$, computed again with respect to the Euclidean metric on $M$.

Remark 1.3. It is known that estimate (6) holds (at every point $\bar{p} \in M$) if $M$ is a complete Riemannian manifold with non-negative Ricci curvature (since the volume is an intrinsic geometric invariant, the fact that $M$ is a submanifold of $\mathbb{R}^n$ is irrelevant here). Indeed, in this case, the Bishop-Gromov inequality gives

$$\text{vol}(B_M(\bar{p}, R)) \leq \Gamma_{\dim(M)},$$

where $\Gamma_{\dim(M)}$ stands for the volume of the Euclidean ball of $\mathbb{R}^{\dim(M)}$, see [Pet98, Lemma 1.5 p. 247].

On the other hand, without the non-negativity assumption on the Ricci curvature, (6) in general does not hold. For instance, the volume of a ball of radius $R$ in a symmetric space $M$ of non-compact type grows exponentially with $R$, cf. [HTT3, p. 209]. In order to have an embedded example of the last situation in any dimension $m$, recall that the hyperbolic space $\mathbb{H}^m$ admits a smooth, isometric embedding into $\mathbb{R}^n$ with $n = 6m - 5$, see [Bl55].

We are now ready to state our main result.

Theorem 1.4. Let $\Omega \subseteq \mathcal{B}(M, \varepsilon) \subseteq \mathbb{R}^n$ be a smooth domain and let $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ be weak solutions to (1), with

$$u \leq v \quad \text{on } \partial \Omega.$$

Assume moreover that Assumptions A1, A2, A3 and A4 are satisfied. Then there exists a positive constant

$$\varepsilon_0 := \varepsilon_0(\|u\|_{W^{1,\infty}}, \|v\|_{W^{1,\infty}}, t, a, q, n, k, f, \Lambda, \gamma)$$

such that, if $0 < \varepsilon < \varepsilon_0$, then

$$u \leq v \quad \text{in } \Omega.$$

Apart from this Introduction, the paper contains two more sections. More precisely, in Section 2 we collect some preliminary results that are needed in what follows, whereas Section 3 is devoted to the proof of Theorem 1.4.
2. Preliminaries

2.1. A useful lemma. We start by recalling a useful lemma already contained and exploited in \cite{FMS12}. For the reader’s convenience we also provide the proof.

**Lemma 2.1.** Let \( \theta > 0 \) and \( \gamma > 0 \) be such that \( \theta < 2^{-\gamma} \). Moreover, let \( R_0 > 0 \), \( C > 0 \) and

\[
\mathcal{L} : (R_0, +\infty) \to \mathbb{R}
\]

be a non-negative and non-decreasing function such that

\[
\begin{cases}
\mathcal{L}(R) \leq \theta \mathcal{L}(2R) + g(R) & \text{for all } R > R_0 \\
\mathcal{L}(R) \leq CR^\gamma & \text{for all } R > R_0,
\end{cases}
\]

where \( g : (R_0, +\infty) \to \mathbb{R}^+ \) is such that

\[
\lim_{R \to +\infty} g(R) = 0.
\]

Then

\[
\mathcal{L}(R) = 0 \quad \text{for all } R > R_0.
\]

**Proof.** Choose \( \theta_1 \) such that \( \theta < \theta_1 < 2^{-\gamma} \). Then there exists \( R_1 = R_1(\theta_1) \geq R_0 \) such that

\[
\theta \mathcal{L}(2R) + g(R) \leq \theta_1 \mathcal{L}(2R) \quad \text{for all } R \geq R_1,
\]

and so, using the first inequality in (7), we get

\[
\mathcal{L}(R) \leq \theta_1 \mathcal{L}(2R) \quad \text{for all } R \geq R_1.
\]

Iterating (8) and using the second inequality in (7) we obtain, for all \( m \in \mathbb{N}^+ \) and \( R \geq R_1 \),

\[
0 \leq \mathcal{L}(R) \leq \theta_1^m \mathcal{L}(2^m R) \leq C \theta_1^m (2^m R)^\gamma = C(2^\gamma \theta_1^m) R^\gamma.
\]

Since \( 0 < 2^\gamma \theta_1 < 1 \), by (9) we deduce

\[
0 \leq \mathcal{L}(R) \leq \lim_{m \to +\infty} C(2^\gamma \theta_1^m) R^\gamma = 0 \quad \text{for all } R \geq R_1,
\]

that implies \( \mathcal{L}(R) = 0 \) for all \( R \geq R_1 \). By Assumption \( \mathcal{L} \) is non-decreasing, so the claim follows. \( \square \)

2.2. Weighted Sobolev Spaces. The weighted Sobolev space with weight \( \rho \) is defined as the space \( W^{1,2}(\Omega, \rho) \) of those functions in \( L^2(\Omega) \) having distributional derivative in \( \Omega \) for which the norm

\[
\|v\|_{W^{1,2}(\Omega, \rho)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega, \rho)} = \left( \int_{\Omega} v^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} |
abla v|^2 \rho \right)^{\frac{1}{2}}
\]

is bounded. The case \( \rho = 1 \) corresponds to the classical Sobolev space \( W^{1,2}(\Omega) \). We also define the space \( H^{1,2}(\Omega, \rho) \) as the closure of \( C_0^\infty(\Omega) \cap W^{1,2}(\Omega, \rho) \) and the space \( H_0^{1,2}(\Omega, \rho) \) as the closure of \( C_0^\infty(\Omega) \), with respect to the norm (10). Let us now recall from \cite{MuSt68, Tr73} the following

**Theorem 2.2.** Let \( \mathcal{D} \subset \mathbb{R}^k \) be a smooth, bounded domain, \( \rho \in L^1(\mathcal{D}) \) and \( C_\rho \) a constant such that

\[
\int_{\mathcal{D}} \frac{1}{\rho^t} \leq C_\rho \quad \text{for some } t > \frac{k}{2}.
\]

Then, for \( 1 + \frac{1}{t} > \frac{2}{k} \), setting

\[
\frac{1}{2^t(t)} := \frac{1}{2} - \frac{1}{k} + \frac{1}{2t}
\]
it results $2 < 2^*(t) < +\infty$ and it follows that the space $H^{1,2}_0(\mathcal{D}, \rho)$ is continuously embedded in $L^q(\mathcal{D})$ for $1 \leq q \leq 2^*(t)$. More precisely, there exists a constant $C_{t,k}$ such that

$$
\|w\|_{L^{2^*(t)}(\mathcal{D})} \leq C_{t,k} C_\rho^{-t} \|\nabla w\|_{L^2(\mathcal{D}, \rho)} = C_S \int_\mathcal{D} |\nabla w|^2 \rho,
$$

for any $w \in H^{1,2}_0(\mathcal{D}, \rho)$, where we set $C_S = C_S(t, k, \rho) =: C_{t,k} C_\rho^{-t}$.

2.3. Fermi coordinates for $\Omega$. A key ingredient for the proof of Theorem 1.4 is the choice of suitable local coordinates in order to parametrize the normal tube $\mathcal{B}(M, \varepsilon)$, such coordinates are sometimes called Fermi coordinates, cf. [GO4, Chapter 2] and [PPS14, Section 3].

Up to refinements of the atlas (5), we can assume that on each coordinate neighbourhood $U_a \subset \mathbb{R}^n$ is defined a moving orthonormal frame $e^1_a, \ldots, e^k_a$ for the normal bundle $N_M|U_a$. This means that, for all $p \in U_a$, the $k$ vectors

$$
e^1_a(p), \ldots, e^k_a(p) \in \mathbb{R}^n$$

are an orthonormal basis for the normal space $N_p M$ and that, for each $i$, the assignment $p \mapsto e^i_a(p)$ defines a smooth vector field. We take $\mathcal{B}(U_a, \varepsilon)$ as in (4), and we set

$$
\varphi_a = \phi_a^{-1}: V_a \rightarrow U_a, \quad E^i_a = e^i_a \circ \varphi_a.
$$

Thus we have a diffeomorphism

$$
\Phi_a: V_a \times B_{\mathbb{R}^k}(0, \varepsilon) \rightarrow \mathcal{B}(U_a, \varepsilon)
$$

$$
\{(x^1_a, \ldots, x^{n-k}_a), (y^1, \ldots, y^k)\} \rightarrow \varphi_a(x^1_a, \ldots, x^{n-k}_a) + \sum_{i=1}^k y^i E^i_a(x^1_a, \ldots, x^{n-k}_a),
$$

where we denoted by $x^1_a, \ldots, x^{n-k}_a$ the Euclidean coordinates in the open set $V_a \subset \mathbb{R}^{n-k}$ and by $y^1, \ldots, y^k$ the Euclidean coordinates in the open ball $B_{\mathbb{R}^k}(0, \varepsilon)$. This shows that

$$
\{\{(\mathcal{B}(U_a, \varepsilon), \Phi_a^{-1})\}_a, \quad \Phi_a^{-1}: \mathcal{B}(U_a, \varepsilon) \rightarrow V_a \times B_{\mathbb{R}^k}(0, \varepsilon) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k
$$

is an oriented atlas for $\mathcal{B}(M, \varepsilon)$.

Denoting by $z^1, \ldots, z^n$ the Euclidean coordinates in $\mathbb{R}^n$, in the local chart $\mathcal{B}(U_a, \varepsilon)$ we have the Euclidean metric

$$
g = (dz^1)^2 + \cdots + (dz^n)^2.
$$

On the other hand, on $V_a$ we have the induced metric

$$
h'_a = (i_a \circ \varphi_a)^* g,
$$

where $i_a: U_a \hookrightarrow \mathbb{R}^n$ denotes the embedding map as in Assumption A3, whereas on $B_{\mathbb{R}^k}(0, \varepsilon)$ we have the metric $h''$ induced by the Euclidean metric on $\mathbb{R}^k$, namely

$$
h'' = (dy^1)^2 + \cdots + (dy^k)^2.
$$

The next result allows us to compare the two metrics $\Phi_a^* g$ and $h_a = h'_a + h''$ on $V_a \times B_{\mathbb{R}^k}(0, \varepsilon)$, cf. [PPS14, Lemma 3.1].

**Lemma 2.3.** With the notation introduced above, we have

$$
\Phi_a^* g = h_a + \sum_{i=1}^k y^i (r^i_a + 2t^i_a) + \sum_{i,j=1}^k y^i y^j s^{ij}_a,
$$

where $r^i_a$, $t^i_a$ and $s^{ij}_a$ are 2-tensors on $V_a \times B_{\mathbb{R}^k}(0, \varepsilon)$ whose coefficients are smooth functions on $V_a$. 
Proof. For simplicity of notation, let us temporarily drop the subscript \( \alpha \). Given any pair \( X, Y \) of vector fields in \( V \times B_{\mathbb{R}^3}(0, \varepsilon) \) we have
\[
(\Phi^* g)(X, Y) = g(d\Phi(X), d\Phi(Y)) = d\Phi(X) \cdot d\Phi(Y),
\]
where the dot stands for the Euclidean scalar product. Then, in order to determine the coefficients of \( \Phi^* g \) with respect to the frame
\[
\partial_{x^1}, \ldots, \partial_{x^n}, \partial_{y^1}, \ldots, \partial_{y^k}
\]
of \( TV \times TB_{\mathbb{R}^3}(0, \varepsilon) \), we must compute the quantities
\[
(\Phi^* g)(\partial_{y^i}, \partial_{y^j}) = d\Phi(\partial_{y^i}) \cdot d\Phi(\partial_{y^j}) = \partial_{y^i} \cdot \Phi \cdot \partial_{y^j} \Phi,
\]
\[
(\Phi^* g)(\partial_{x^i}, \partial_{y^j}) = d\Phi(\partial_{x^i}) \cdot d\Phi(\partial_{y^j}) = \partial_{x^i} \cdot \Phi \cdot \partial_{y^j} \Phi,
\]
\[
(\Phi^* g)(\partial_{x^i}, \partial_{x^j}) = d\Phi(\partial_{x^i}) \cdot d\Phi(\partial_{x^j}) = \partial_{x^i} \cdot \Phi \cdot \partial_{x^j} \Phi.
\]
Observing that \( \partial_{x^i} \varphi \) is a tangent vector field on \( V \), whereas \( E^i \) is a normal vector field, we obtain
\[
\partial_{x^i} \varphi \cdot E^j = 0.
\]
Moreover, the fact that \( e^1, \ldots, e^k \) is an orthonormal frame implies \( E^i \cdot E^j = \delta_{ij} \).

Therefore we get
\[
\partial_{y^i} \Phi \cdot \partial_{y^j} \Phi = E^i \cdot E^j = \delta_{ij} = h_{ij}'',
\]
\[
\partial_{x^i} \Phi \cdot \partial_{y^j} \Phi = \sum_{i=1}^k y^i \partial_{x^i} E^i \cdot E^j
\]
and
\[
\partial_{x^i} \Phi \cdot \partial_{x^j} \Phi = \partial_{x^i} \varphi \cdot \partial_{x^j} \varphi + \sum_{i=1}^k y^i \left( \partial_{x^i} \varphi \cdot \partial_{x^j} E^i + \partial_{x^i} \varphi \cdot \partial_{x^i} E^i \right) + \sum_{i=1}^k y^i \left( \partial_{x^i} E^i \cdot \partial_{y^i} E^j \right)
\]
\[
= h_{ab}'' + \sum_{i=1}^k y^i \left( \partial_{x^i} \varphi \cdot \partial_{x^j} E^i + \partial_{x^i} \varphi \cdot \partial_{x^i} E^i \right) + \sum_{i=1}^k y^i \left( \partial_{x^i} E^i \cdot \partial_{x^j} E^j \right).
\]

Now, setting
\[
r^i = \sum_{a,b=1}^{n-k} \left( \partial_{x^a} \varphi \cdot \partial_{x^b} E^i + \partial_{x^a} \varphi \cdot \partial_{x^i} E^i \right) dx^a dx^b,
\]
(15)
\[
t^i = \sum_{a=1}^{n-k} \sum_{j=1}^k \left( \partial_{x^a} E^i \cdot E^j \right) dx^a dy^j,
\]
\[
s^{ij} = \sum_{a,b=1}^{n-k} \left( \partial_{x^a} E^i \cdot \partial_{x^b} E^j \right) dx^a dx^b,
\]
we obtain (14).

\[\Box\]

Example 2.4. Let us explicitly compute the expression (14) in three simple cases, namely when \( M \) is either a plane curve, or a space curve or a surface in \( \mathbb{R}^3 \).

(i) Let \( M \subset \mathbb{R}^2 \) be a plane curve. Then \( n = 2, k = 1 \) and the function \( \Phi \) can be chosen of the form
\[
\Phi(x, y) = \varphi(x) + yN(x),
\]
where \( \varphi \) is the arc-length parametrization and \( N \) is the principal normal vector, see [dC76, Chapter 1]. Then
\[
\partial_{x} \Phi = \partial_{x} \varphi + y \partial_{x} E = (1 - \kappa y)T, \quad \partial_{y} \Phi = N,
\]
where \( T = T(x) \) is the unit tangent vector and \( \kappa = \kappa(x) \) is the curvature. From this we get
\[
\Phi^* g = (d\kappa(x))^2 + (d(y)^2 - 2ykd(x)^2 + y^2\kappa^2(dx)^2
= h - 2ykd(x)^2 + y^2\kappa^2(dx)^2.
\]

(ii) Let \( M \subset \mathbb{R}^3 \) be a space curve. Then \( n = 3, k = 2 \) and the function \( \Phi \) can be chosen of the form
\[
\Phi(x, y^1, y^2) = \varphi(x) + y^1E^1(x) + y^2E^2(x),
\]
where \( \varphi \) is the arc-length parametrization, \( E^1 \) is the principal normal vector and \( E^2 \) is the binormal vector. Thus \( \partial_x \varphi = T \), where \( T = T(x) \) is the unit tangent vector, and by the Frénet formulas \([dC76, p. 19]\) we obtain
\[
\partial_x E^1 = -\kappa T - \tau E^2, \quad \partial_x E^2 = \tau E^1,
\]
where \( \kappa = \kappa(x) \) is the curvature and \( \tau = \tau(x) \) is the torsion. Now, using (15), we easily find
\[
\begin{align*}
  r^1 &= -2\kappa(d\kappa(x))^2, \\
  t^1 &= -\tau dxdy^2, \\
  r^2 &= 0, \\
  t^2 &= \tau dxdy^1, \\
  s^{11} &= (k^2 + \tau^2)(d\kappa(x))^2, \\
  s^{12} &= 0, \\
  s^{22} &= \tau^2(d\kappa(x))^2,
\end{align*}
\]
so we get
\[
\Phi^* g = h - 2y^1(\kappa(d\kappa(x))^2 + \tau dxdy^2) + 2y^2(\tau dxdy^1) + (y^1)^2((k^2 + \tau^2)(d\kappa(x))^2) + (y^2)^2(\tau^2(d\kappa(x))^2).
\]

(iii) Let \( M \subset \mathbb{R}^3 \) be a surface. Then \( n = 3, k = 1 \) and the function \( \Phi \) can be chosen of the form
\[
\Phi(x^1, x^2, y) = \varphi(x^1, x^2) + yE(x^1, x^2),
\]
where we take \( E(x^1, x^2) = \frac{\partial, \varphi \partial, \varphi}{|\partial, \varphi \partial, \varphi|} \). Denoting as customary by
\[
E = \partial_x \varphi \cdot \partial_x \varphi, \quad F = \partial_x \varphi \cdot \partial_x \varphi, \quad G = \partial_x \varphi \cdot \partial_x \varphi
\]
the coefficients of the first fundamental form \( I_M \) of \( M \) and by
\[
e = -\partial_x E \cdot \partial_x \varphi, \quad f = -\partial_x E \cdot \partial_x \varphi, \quad g = -\partial_x E \cdot \partial_x \varphi
\]
the coefficients of the second fundamental form \( II_M \), from the formulas in \([dC76, p. 92 \text{ and p. 154}]\) we deduce
\[
\begin{align*}
  \partial_x \varphi \cdot \partial_x \varphi \cdot \varphi = E - 2ye + y^2\psi_{11}, \\
  \partial_x \varphi \cdot \partial_x \varphi \cdot F = F - 2ye + y^2\psi_{12}, \\
  \partial_x \varphi \cdot \partial_x \varphi \cdot G = G - 2ye + y^2\psi_{22},
\end{align*}
\]
where
\[
\psi_{11} = \frac{e^2 - 2ef + f^2E}{E - F^2}, \quad \psi_{12} = \frac{f^2 + efG - egF + g^2E}{E - F^2}, \quad \psi_{22} = \frac{f^2G - 2fgF + g^2E}{E - F^2}.
\]
Then, using the expressions of the mean curvature \( H_M \) and the Gaussian curvature \( K_M \) of \( M \) in terms of the coefficients of the first two fundamental forms (cf. \([dC76, p. 155-156]\)), we can rewrite (14) as
\[
\Phi^* g = I_M + (dye)^2 = h - 2yH + y^2K_M + y^2H_M + y^2H_M = h - 2yH + y^2K_M + y^2H_M.
\]

As a consequence of Lemma 2.3 we have the result below, comparing the volume forms of the two metrics \( \Phi^* g \) and \( h_M \) on \( V_a \times B_\delta(0, \epsilon) \).
\textbf{Corollary 2.5.} With the notation introduced above, we have

\begin{equation}
\text{dvol}\Phi^*g = \lambda_a \text{dvol}h_a
\end{equation}

for a smooth function \(\lambda\) on \(\partial(M, \varepsilon)\) such that \(\lambda \equiv 1\) on \(M\). If moreover \(\varepsilon < 1\), for every

\[(x_a, y) = \left((x_a^1, \ldots, x_a^{n-k}), (y^1, \ldots, y^k)\right) \in V_a \times B_{\varepsilon^2}(0, \varepsilon)\]

we have

\[|\lambda_a(x_a, y) - 1| \leq K_1\|y\|,\]

where \(K_1\) is a positive constant, independent on \(a\).

It follows that there exists \(\varepsilon_1 = \min\left\{1, \frac{1}{2K_1}\right\}\), independent on \(a\), such that if \(\varepsilon < \varepsilon_1\) then

\begin{equation}
\frac{1}{2} \leq \lambda_a(x_a, y) \leq \frac{3}{2} \quad \text{for all } (x_a, y) \in V_a \times B_{\varepsilon^2}(0, \varepsilon).
\end{equation}

\textbf{Proof.} In order to simplify the notation, we drop the subscript \(a\). Since \(\det h \neq 0\) everywhere, by Lemma 2.3 we can write the following finite expansion of \(\det \Phi^*g\) in terms of the normal coordinates \(y^i:\)

\begin{equation}
\det \Phi^*g = \det h + \sum_{i=1}^k y^i A^i + \sum_{i,j=1}^k y^i y^j A^{ij} + \sum_{i,j,l=1}^k y^i y^j y^l A^{i jl} + \ldots
\end{equation}

where all the coefficients are smooth functions depending on the first and second derivatives of \(\Phi\). Now \(\varepsilon < 1\) implies \(\|y\| < 1\) and so \(|y^i| < 1\) for all \(i\), thus from (18) we obtain

\begin{equation}
\det \Phi^*g = \det h + \sum_{i=1}^k \tilde{K}_i^1(x, y)y^i
\end{equation}

where \(|\tilde{K}_i^1(x, y)| \leq \tilde{K}_1^1\) and \(\tilde{K}_1^1\) is a positive constant, which is moreover independent on the chart because of Assumption \(A3\). Setting \(\lambda(x, y) := \det \Phi^*g \cdot \det h^{-1}\), from (19) we get\footnote{Here we use the inequality \(|\sqrt{1 + x} - 1| \leq |x|\) for \(x \geq -1\).}

\[|\lambda(x, y) - 1| = \left|\sqrt{\det \Phi^*g \cdot \det h^{-1}} - 1\right| = \left|\sqrt{1 + \sum_{i=1}^k \frac{\tilde{K}_i^1(x, y)}{\det h} y^i} - 1\right|
\leq \sum_{i=1}^k \frac{\tilde{K}_i^1(x, y)}{\det h} |y^i| \leq \sum_{i=1}^k \frac{\tilde{K}_1^1}{\det h} |y^i| \leq k \frac{\tilde{K}_1^1}{\det h} \|y\| = K_1 \|y\|,
\]

where we put \(K_1 := k\tilde{K}_1^1/\det h\). Again by Assumption \(A3\), the function \(\det h\) is bounded, uniformly with respect to \(a\), so the positive constant \(K_1\) is also independent on the chart. The proof of the last statement is now straightforward. \(\square\)

\textbf{Remark 2.6.} Using the same arguments as in the proof of Corollary 2.5, we can show that on \(V_a\) we have

\begin{equation}
\text{dvol}h_a = \mu_a(x_a) dx_a
\end{equation}

for a smooth positive function \(\mu_a(x_a)\), uniformly bounded with respect to \(a\).
3. The weak comparison principle: Proof of Theorem 1.4

3.1. Test functions. For the sake of brevity, we write $B_R$ instead of $B_M(\bar{p}, R)$, see Assumption A4. Let $\varphi_R \in C_c^\infty(M)$ be such that

\[
\begin{align*}
0 &\leq \varphi_R \leq 1 \quad \text{in } M, \\
\varphi_R &\equiv 1 \quad \text{in } B_R, \\
\varphi_R &\equiv 0 \quad \text{in } M \setminus B_{2R}, \\
|\nabla \varphi_R| &\leq \frac{2}{R} \quad \text{in } B_{2R} \setminus B_R.
\end{align*}
\]

For instance we could take $\varphi_R$ of the form

\[
\varphi_R(p) = \begin{cases} 
1 & \text{in } B_R, \\
\left(\frac{2R-d_M(\bar{p}, p)}{R}\right)^2 & \text{in } B_{2R} \setminus B_R, \\
0 & \text{in } M \setminus B_{2R},
\end{cases}
\]

in fact the distance function $d_M(\bar{p}, p)$ has distributional gradient whose $L^\infty$-norm is at most 1, see [Gr09, Theorem 11.3 p. 296]. Since $\mathcal{B}(M, \epsilon)$ is locally trivial, in view of diffeomorphisms in (12), we can extend $\varphi_R$ to the whole tubular neighbourhood and we continue to denote this extension by $\varphi_R$. In what follows we will consider the bounded domain

\[\Omega_R := \Omega \cap \mathcal{B}(B_R, \epsilon),\]

see Figure 1 below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The bounded domain $\Omega_R$.}
\end{figure}

Moreover, for a fixed real number $\beta \geq 1$, we will define

\[\psi_R := [(u-v)^+]^\beta \varphi_R^2.\]

With this notation, we can state the following crucial result, whose proof relies on our assumption that $u \leq v$ on $\partial \Omega$.

**Lemma 3.1.** The function $\psi_R$ belongs to $H_0^1(\Omega_{2R})$, and it can be plugged into (2) and (3).

**Proof.** We have $\partial \Omega_{2R} \subseteq \partial \Omega \cup \partial \mathcal{B}(B_{2R}, \epsilon)$; furthermore, by assumption, $(u-v)^+ = 0$ on $\partial \Omega$, whereas $\varphi_R = 0$ on $\partial \mathcal{B}(B_{2R}, \epsilon)$. This means that $\psi_R = 0$ on $\partial \Omega_{2R}$, so the first claim follows from [GT83, Chapter 7], cf. also [Bre11, Theorem 8.12]. The second claim is an immediate consequence of the density of $H_0^1(\Omega_{2R})$ in $C_c^\infty(\Omega_{2R})$. \qed
Plugging $\psi_R$ as a test function into (2) and (3) and subtracting we get
\[
\beta \int_{\Omega_{2R}} a(z) \left[ (u-v)^+ \right]^{\beta-1} |\nabla (u-v)|^2 \varphi_R^2 \, dvol_g
\]
This implies that the functions $\rho$ via the Lagrange theorem applied to the function $t^g$ and we exploited the reverse triangular inequality. Moreover the constant $L_f = L_f \left( \|u\|_\infty + \|v\|_\infty \right)$ is the one contained in Assumption A2.

In the sequel, we will estimate the terms $A_{2R}$, $B_{2R}$ and $C_{2R}$ in (22), making repeated use of the Young’s inequality
\[
ab a \leq \tau a^2 + \frac{1}{4\tau} b^2, \quad a, b \in \mathbb{R}, \quad \tau > 0.
\]

3.2. Estimate for $A_{2R}$. Dealing first with the term $A_{2R}$, we obtain
\[
A_{2R} = \int_{\Omega_{2R}} \left[ (u-v)^+ \right]^{\beta-1} |\nabla (u-v)|^2 \varphi_R^2 \, dvol_g
\]
\[
= \int_{\Omega_{2R}} a(z)^{\frac{1}{2}} \left[ (u-v)^+ \right]^{\frac{\beta-1}{2}} |\nabla (u-v)|_{\alpha} \varphi_R a(z)^{-\frac{1}{2}} \left[ (u-v)^+ \right]^{\frac{\beta-1}{2}} \varphi_R \, dvol_g
\]
\[
\leq \tau \int_{\Omega_{2R}} a(z) \left[ (u-v)^+ \right]^{\beta-1} |\nabla (u-v)|^2 \varphi_R^2 \, dvol_g + \frac{1}{4\tau} \int_{\Omega_{2R}} a(z)^{-1} \left[ (u-v)^+ \right]^{\frac{\beta-1}{2}} \varphi_R^2 \, dvol_g
\]
\[
= \tau L_{2R} + \frac{1}{4\tau} \int_{\Omega_{2R}} a(z)^{-1} \left[ (u-v)^+ \right]^{\frac{\beta-1}{2}} \varphi_R^2 \, dvol_g.
\]

In order to estimate the integral on the right-hand side of the above equality, we will consider the oriented atlas of $B(M, \varepsilon)$ given by
\[
\{ \{ B(U, x, \varepsilon), \Phi^{-1}_{a} \} \},
\]
cf. (13). We choose a partition of unity $\{ \rho_{a} \}_{a}$, subordinate to the open cover $(B(U, \varepsilon))_{a}$ of $B(M, \varepsilon)$ and obtained by pulling back a partition of unity subordinate to the open cover $(U_{a})_{a}$ of $M$ via the projection map
\[
B(M, \varepsilon) \to M, \quad x \in B(p, \varepsilon) \to p.
\]

This implies that the functions $\rho_{a} := \Phi_{a}^{-1} \left( \rho_{a} \right)$, defined on $V_{a} \times B_{R_{a}}(0, \varepsilon)$, only depend on the first factor, namely $\rho_{a} = \rho_{a}(x_{a})$. Therefore for $\varepsilon < \varepsilon_{1}$ we can write, by using (16), (17), (20) and the
Fubini-Tonelli theorem,
\[
\int_{\Omega_{2R}} a(z)^{-1} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \rho^2_R \, d\nu_{\mathcal{G}}
= \sum_x \int_{V_x \times B_{\rho^2}(0, \epsilon)} \rho_a(x) a(x, y)^{-1} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \rho^2_R(x) \lambda(x, y) \, d\nu_{\mathcal{H}_a}
\leq \frac{3}{2} \sum_x \int_{V_x} \rho_a(x) \Phi_a^2(x) \left( \int_{B_{\rho^2}(0, \epsilon)} a(x, y)^{-1} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \, dy \right) \mu(x) \, dx .
\]

**Remark 3.2.** In order to simplify the notation, in the above equation and in the following computations we drop the subscript \( a \) whenever it is not strictly necessary. Moreover, with a slight abuse of notation, we write \( a(x, y), \rho^2_R(x) \) and \((u-v)^+\) instead of \( \Phi_a^*(a(z)), \Phi_a^*(\rho^2_R(x)) \) and \( \Phi_a^*((u-v)^+) \), respectively, highlighting for clarity the dependence on the variables \( x \) and \( y \) if necessary.

Now we estimate the integral on \( B_{\rho^2}(0, \epsilon) \) by exploiting Hölder inequality, obtaining
\[
\int_{B_{\rho^2}(0, \epsilon)} a(x, y)^{-1} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \, dy
\leq \left( \int_{B_{\rho^2}(0, \epsilon)} (a(x, y)^{-1})^{\frac{\beta_1}{\tau}} \, dy \right)^{\frac{\beta_1}{\beta_1 \tau}} \left( \int_{B_{\rho^2}(0, \epsilon)} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \, dy \right)^{\frac{\beta_1 \tau}{\beta_1}}
\]
By exploiting again Hölder inequality and making use of Assumption A1, we have
\[
\int_{B_{\rho^2}(0, \epsilon)} (a(x, y)^{-1})^{\frac{\beta_1}{\tau}} \, dy \leq \left( \int_{B_{\rho^2}(0, \epsilon)} a(x, y)^{-\epsilon} \, dy \right)^{\frac{\beta_1}{\beta_1 - \epsilon \tau}} \left( \int_{B_{\rho^2}(0, \epsilon)} 1 \, dy \right)^{\frac{\beta_1 \epsilon \tau}{\beta_1 - \epsilon \tau}}
\leq C_a^{\frac{\beta_1}{\tau}} \left( \Gamma_k \epsilon \right)^{\frac{\beta_1 \epsilon \tau}{\beta_1 - \epsilon \tau}}
= C_a^{\frac{\beta_1}{\tau}} \Gamma_k \epsilon^{\frac{\beta_1 \epsilon \tau}{\beta_1 - \epsilon \tau}},
\]
were \( \Gamma_k \) is the volume of the unit ball in \( \mathbb{R}^k \). Moreover we have, by using (11),
\[
\left( \int_{B_{\rho^2}(0, \epsilon)} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \, dy \right)^{\frac{2}{\beta_1 \tau}} = \| (u-v)^{\frac{\beta_1}{\tau}} \|_{L^2(B_{\rho^2}(0, \epsilon))}^2
\leq C_S^2 \| \nabla_{\rho^2}((u-v)^+) \|_{L^2(B_{\rho^2}(0, \epsilon), a)}^2
= C_S^2 \int_{B_{\rho^2}(0, \epsilon)} |\nabla_{\rho^2}((u-v)^+) \|^{\beta_1} 2(a, y) \, dy.
\]

**Remark 3.3.** By Theorem 2.2, see also [FMS13, proof of Theorem 5.1], it follows that the constant \( C_S \) in (26) does not depend on \( a \) because of Assumption A1.

Plugging (25) and (26) into (24), we infer
\[
\int_{B_{\rho^2}(0, \epsilon)} a(x, y)^{-1} \left[ (u-v)^{\frac{\beta_1}{\tau}} \right]^2 \, dy \leq C_a^{-t} C_S^2 \Gamma_k^{2t - \beta_1 \tau} \epsilon^{\frac{2t - \beta_1 \tau}{\beta_1 - \epsilon \tau}} \int_{B_{\rho^2}(0, \epsilon)} a(x, y) |\nabla_{\rho^2}((u-v)+) \|^{\beta_1} \, dy
\leq C_a^{-t} C_S^2 \Gamma_k^{2t - \beta_1 \tau} \epsilon^{\frac{2t - \beta_1 \tau}{\beta_1 - \epsilon \tau}} \left( \frac{\beta_1 + 1}{2} \right)^2 \int_{B_{\rho^2}(0, \epsilon)} a(x, y) \left( (u-v)^{\frac{\beta_1}{\tau}} |\nabla_{\rho^2}((u-v)^+) \right) \, dy
\leq C_a^{-t} C_S^2 \Gamma_k^{2t - \beta_1 \tau} \epsilon^{\frac{2t - \beta_1 \tau}{\beta_1 - \epsilon \tau}} (\beta + 1)^2 \int_{B_{\rho^2}(0, \epsilon)} a(x, y) (u-v)^{\beta - 1} |\nabla_{\rho^2}((u-v)^+) \|^{2} \lambda(x, y) \, dy,
\]
where the last inequality follows because we have \( 1 \leq 2 \lambda(x, y) \) for \( \epsilon < \epsilon_1 \), see (17).
**Remark 3.4.** By using Lemma 2.3, we can write

\[ |\nabla \varphi_{gs}(u-v)|^2 = |\nabla R(u-v)|^2 + |\nabla K'(u-v)|^2 + o(\varepsilon) \]

in \( V_a \times B_{\varepsilon}(0, \varepsilon) \), uniformly on \( a \). So, making \( \varepsilon_1 \) smaller if necessary, we can assume that the inequality

\[ |\nabla R(u-v)|^2 \leq |\nabla \varphi_{gs}(u-v)|^2 \]

holds in every coordinate chart \( V_a \times B_{\varepsilon}(0, \varepsilon) \), as soon as \( \varepsilon < \varepsilon_1 \).

Moreover, since the gradient \( \nabla \) is the vector field representing the differential map with respect to the metric (see [Pet98, p. 20]), it is straightforward to check that

\[ \Phi^*_a \nabla g = \nabla \varphi_{gs} \Phi^*_a. \]

Using Remark 3.4, and noticing that all the constants involved in the computations are independent on the local chart \( U_a \), we obtain

\[
\begin{align*}
&\int_{\Omega_{2R}} a(x)^{-1} ((u-v)^+) \frac{\varepsilon_1}{2} \varphi_R^2 \mathrm{d}vol_g \\
&\leq 3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2} \sum_a \int_{V_a} \rho_a(x) \\
&\int_{B_{\varepsilon}(0, \varepsilon)} a(x, y)((u-v)^+) [\nabla \varphi_{gs}(u-v)]^2 \varphi_R^2(x) \lambda(x, y) dy \mu(x) dx \\
&\leq 3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2} \sum_a \int_{V_a} \rho_a(x) \\
&\int_{B_{\varepsilon}(0, \varepsilon)} a(x, y)((u-v)^+) [\nabla \varphi_{gs}(u-v)]^2 \varphi_R^2(x) \lambda(x, y) dy \mu(x) dx \\
&= 3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2} \sum_a \int_{V_a \times B_{\varepsilon}(0, \varepsilon)} \rho_a(x) \left( a(x, y)((u-v)^+) \right) \lambda(x, y) dy \\
&\int_{\Omega_{2R}} |\nabla \varphi_{gs}(u-v)|^2 \varphi_R^2 \mathrm{d}vol_g \\
&= 3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2} \int_{\Omega_{2R}} a(z)((u-v)^+) \left( -\frac{2k-2k}{4}/(\beta + 1)^2 \right) \mathrm{d}vol_g \\
&= 3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2} \mathcal{L}_{2R}.
\end{align*}
\]

Summing up, if \( \varepsilon \leq \varepsilon_1 \), from (23) and (27) we finally get

\[
\mathcal{A}_{2R} \leq \left( \tau + \frac{3C_a^{-1} C_4^2 \Gamma_h^{\frac{2k-2k}{4}} e^{-\frac{2k-2k}{4}/(\beta + 1)^2}}{16} \right) \mathcal{L}_{2R}.
\]

3.3. **Estimate for** \( \mathcal{B}_{2R} \). Let us now estimate the term \( \mathcal{B}_{2R} \). As before, for \( \varepsilon \leq \varepsilon_1 \) we have

\[
\begin{align*}
\mathcal{B}_{2R} &= \int_{\Omega_{3R}} [(u-v)^+]^{\beta - 1} \varphi_R^2 \mathrm{d}vol_g \\
&= \sum_a \int_{V_a \times B_{\varepsilon}(0, \varepsilon)} \rho_a(x)((u-v)^+) \varphi_R^2(x) \lambda(x, y) dy \\
&\leq \frac{3}{2} \sum_a \int_{V_a} \rho_a(x) \varphi_R^2(x) \left( \int_{B_{\varepsilon}(0, \varepsilon)} [(u-v)^+]^{\beta - 1} dy \right) \mu(x) dx.
\end{align*}
\]
Now we estimate the integral on $B_{R^2}(0, \varepsilon)$ by using Hölder inequality and (26), obtaining

\begin{align}
\int_{B_{R^2}(0, \varepsilon)} [(u-v)^+]^\beta + 11 d y &\leq \left( \int_{B_{R^2}(0, \varepsilon)} (u-v)^+ \right)^{\frac{3k-\beta}{4k}} \cdot \left( \int_{B_{R^2}(0, \varepsilon)} [(u-v)^+]^\frac{\beta+1}{2} d y \right)^{\frac{3k}{2k}} \\
&\leq \left( \frac{\epsilon}{\varepsilon} \right)^{\frac{3k-\beta}{4k}} C_{\beta}^2 \int_{B_{R^2}(0, \varepsilon)} |\nabla u| ((u-v)^+) \frac{\beta+1}{2} a(x, y) d y \\
&= \frac{\epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} [(u-v)^+]^\beta |\nabla u| a(x, y) d y \\
&\leq \frac{\epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} [(u-v)^+]^\beta - 1 |\nabla u| a(x, y) d y \\
&\leq \frac{\epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} [(u-v)^+]^\beta - 1 |\nabla u| a(x, y) d y.
\end{align}

(29)

Arguing as in (27) and using Remark 3.4, we deduce

\begin{align}
\mathcal{B}_{2R} &\leq \frac{3 \epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} C_{\beta}^2 \sum a \int_{V_a} \rho_a(x) \left( \int_{B_{R^2}(0, \varepsilon)} a(x, y) [(u-v)^+]^\beta - 1 |\nabla u| a(x, y) d y \right) \mu(x) d x \\
&\leq \frac{3 \epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} C_{\beta}^2 \sum a \int_{V_a \times B_{R^2}(0, \varepsilon)} \rho_a(x) \left( a(x, y) [(u-v)^+]^\beta - 1 |\nabla u| a(x, y) d y \right) \mu(x) d x.
\end{align}

Summing up, for $\varepsilon < \varepsilon_1$ we get

(30)

\begin{align}
\mathcal{B}_{2R} &\leq \frac{3 C_{\beta}^2 \epsilon}{\varepsilon} \left( \beta + 1 \right)^2 \frac{\epsilon}{\varepsilon} \int_{B_{R^2}(0, \varepsilon)} \rho_a(x) |\nabla u| a(x, y) d y.
\end{align}

3.4. Estimate for $\mathcal{C}_{2R}$. Finally, let us estimate the term $\mathcal{C}_{2R}$. Exploiting Young's inequality and the last inequality in (21), we get

\begin{align}
\mathcal{C}_{2R} &= \int_{\Omega_{2R}} a(z) [(u-v)^+]^\beta |\nabla(u-v)| a(z) \mu_{\varphi_{2R}} d y \\
&= \int_{\Omega_{2R}} a(z) [(u-v)^+]^\frac{\beta+1}{2} |\nabla(u-v)| a(z) [(u-v)^+]^\frac{\beta+1}{2} |\nabla \varphi_{2R} | d y \\
&\leq \tau' \int_{\Omega_{2R}} a(z) [(u-v)^+]^\beta - 1 |\nabla u| a(z) [(u-v)^+]^\beta - 1 |\nabla \varphi_{2R} | d y \\
&\leq \tau' \mathcal{L}_{2R} + \frac{1}{\tau R^2} \int_{\Omega_{2R}} [(u-v)^+]^\beta + 1 |\nabla \varphi_{2R} | d y.
\end{align}
If $\varepsilon < \varepsilon_1$ the integral at the right-hand side can be estimated by (29), obtaining
\[
\int_{\Omega_2^R} [(u-v)^+]^{\beta+1} \, d\mu_g = \sum_a \int_{V_a \times B_{h_a}(0, \varepsilon)} \rho_a(x) [(u-v)^+]^{\beta+1} \, \lambda(x, y) \, d\mu_h_a
\lesssim \frac{3}{2} \sum_a \rho_a(x) \left( \int_{B_{h_a}(0, \varepsilon)} [(u-v)^+]^{\beta+1} \, dy \right) \mu(x) \, dx
\lesssim \frac{3 \Gamma_{1/2}^2 \varepsilon}{2 \Gamma_{1/2}^2} C_S^2 (\beta + 1)^2 \sum_a \int_{V_a \times B_{h_a}(0, \varepsilon)} \rho_a(x) \left( a(x, y) [(u-v)^+]^{\beta-1} |\nabla \Phi_g(u-v)|^2 \right) \, dx
\lesssim \frac{3 \Gamma_{1/2}^2 \varepsilon}{2 \Gamma_{1/2}^2} C_S^2 (\beta + 1)^2 \sum_a \int_{V_a \times B_{h_a}(0, \varepsilon)} \rho_a(x) \left( a(x, y) [(u-v)^+]^{\beta-1} \right) \, d\mu_g.
\]
Summing up, for $\varepsilon < \varepsilon_1$ we get
\[
\mathcal{E}_{2R} \lesssim \tau' \mathcal{L}_{2R} + \frac{3 C_S^2 \Gamma_{1/2}^2 \varepsilon}{2 \Gamma_{1/2}^2} (\beta + 1)^2 \|a\|_\infty \int_{\Omega_2^R} a(z) [(u-v)^+]^{\beta-1} |\nabla (u-v)|^2 \, d\mu_g.
\]

3.5. **End of the proof.** We can now specialize the computations above, using particular values for the parameters. Fixing, for instance,
\[
\beta = 2, \quad \tau' = \frac{1}{4},
\]
we can rewrite inequalities (28), (30), (31) as follows:
\[
\mathcal{A}_{2R} \leq \left( \tau + \frac{\Theta_{\delta \delta}}{\tau} \right) \mathcal{L}_{2R},
\]
\[
\mathcal{B}_{2R} \leq \Theta_{\beta \beta} \mathcal{L}_{2R},
\]
\[
\epsilon_{2R} \leq \frac{1}{4} \mathcal{L}_{2R} + \frac{2 \Theta_{\delta \delta}}{\tau} \int_{\Omega_2^R} a(z) [(u-v)^+] \|\nabla (u-v)|^2 \, d\mu_g,
\]
where
\[
\Theta_{\delta \delta} = \frac{27 C_a^{-2} C_S^2 \Gamma_{1/2}^2 \varepsilon^{4/5} e^{4/5}}{16},
\]
\[
\Theta_{\beta \beta} = \frac{27 C_S^2 \Gamma_{1/2}^2 \varepsilon^{4/5} e^{4/5}}{4},
\]
\[
\Theta_{\delta \delta} = 27 C_S^2 \Gamma_{1/2}^2 \varepsilon^{4/5} e^{4/5} \|a\|_\infty,
\]
so that (22) becomes
\[
2 \mathcal{L}_{2R} \leq |\Lambda| q \left( \|\nabla u\|_\infty + \|\nabla v\|_\infty \right)^{q-1} \left( \tau + \frac{\Theta_{\delta \delta}}{\tau} \right) \mathcal{L}_{2R} + \frac{2 \Theta_{\delta \delta}}{\tau} \int_{\Omega_2^R} a(z) [(u-v)^+] \|\nabla (u-v)|^2 \, d\mu_g.
\]
Finally, we fix 
\[ \tau = \frac{1}{2|\Lambda|^q (\|\nabla u\|_\infty + \|\nabla v\|_\infty)^{q-1}} \]
and we resume the previous computations as

\[ L_2 \leq \Theta_1 L_2 + \frac{\Theta_2}{R^2} \int_{\Omega_{2R}} a(z)(u - v)^+ |\nabla(u - v)|^2 \, dvol_g, \]

where

\[ \Theta_1 := \left( 2|\Lambda|^2 q^2 (\|\nabla u\|_\infty + \|\nabla v\|_\infty)^{2q-2} \Theta_{sf} + L_f \Theta_{ef} \right), \quad \Theta_2 := 2\Theta_\phi. \]

Let us now set

\[ \mathcal{L}_R := \int_{\Omega_R} a(z)(u - v)^+ |\nabla(u - v)|^2 \, dvol_g, \]

so that, from the properties of \( \varphi_R \) stated in \( (21) \), it follows

\[ \mathcal{L}_R \leq L_2 \leq \mathcal{L}_{2R}. \]

Therefore by \( (33) \) we get

\[ \mathcal{L}_R \leq \theta \mathcal{L}_{2R}, \]

where

\[ \theta := \left( \Theta_1 + \frac{\Theta_2}{R^2} \right). \]

Looking at \( (32) \) and \( (34) \) we see that, taking \( \varepsilon < \varepsilon_1 \) sufficiently small and \( R \) sufficiently large, we have \( \theta < 2^{-\gamma} \), where \( \gamma \) is as in Assumption A4. Moreover, we are supposing \( a, u, v, \nabla u, \nabla v \in L_\infty(\Omega) \) so, using Assumption A4, we obtain

\[ \mathcal{L}_R \leq CR^\gamma, \]

where we set \( C := C_1 \|a(z)(u - v)^+ |\nabla(u - v)|^2\|_{L_\infty(\Omega)}. \)

Summing up, there exists \( 0 < \varepsilon_0 < \varepsilon_1 \) such that, if \( \varepsilon < \varepsilon_0 \), we can apply Lemma 2.1 in order to deduce \( \mathcal{L}_R = 0 \) for all \( R > 0 \). This in turn implies \( (u - v)^+ |\nabla(u - v)|^2 = 0 \) in the whole of \( \Omega \). Since \( (u - v)^+ = 0 \) on \( \partial\Omega \), we get \( u \leq v \) everywhere, that is \( u \leq v \) on the whole of \( \Omega \) and the weak comparison principle is proved.

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DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÀ DELLA CALABRIA
PONTE PIETRO Bucci 31B, I-87036 ARCAVACATA DI RENDE, COSENZA, ITALY
E-mail address: polizzi@mat.unical.it

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI ROMA TOR VERGATA
VIA DELLA RICERCA SCIENTIFICA - 00133 ROMA
E-mail address: pietrsabat@gmail.com

DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÀ DELLA CALABRIA
PONTE PIETRO Bucci 31B, I-87036 ARCAVACATA DI RENDE, COSENZA, ITALY
E-mail address: sciunzi@mat.unical.it