ON A RECENTLY PROPOSED RELATION BETWEEN oHS AND ITO SYSTEMS

Atalay Karasu

Department of Physics, Faculty of Arts and Sciences
Middle East Technical University, 06531 Ankara-Turkey

Abstract

The bi-Hamiltonian structure of original Hirota-Satsuma system proposed by Roy based on a modification of the bi-Hamiltonian structure of Ito system is incorrect.
In a recent paper[1] a bi-Hamiltonian structure for the original Hirota-Satsuma(oHS) system where $a$ is arbitrary is proposed and a relation between oHS and Ito systems is introduced and a recursion operator is found for the oHS system. In this note we point out that the oHS system does not in fact admit a bi-Hamiltonian structure and the relation between the oHS and Ito systems claimed by the author of [1] is actually incorrect.

It is well known that the original Hirota-Satsuma system (oHS)[2]
\begin{align*}
  u_t &= a(u_{xxx} + 6uu_x) + 2bvv_x, \\
  v_t &= -v_{xxx} - 3uv_x 
\end{align*}
(1)
for all values of $a$ and $b$, possesses three conserved quantities.

\begin{align*}
  I_1 &= u, \quad I_2 = u^2 + \frac{2}{3}bv^2, \quad I_3 = (1 + a)(u^3 - \frac{1}{2}(u_x)^2) + b(uv^2 - (v_x)^2). \quad (2)
\end{align*}

Later Hirota-Satsuma [3] showed that oHS system has infinitely many conserved quantities for the choice of $a = \frac{1}{2}$ and conjectured that it is completely integrable. Dodd and Fordy [4] showed that the oHS system admits a Lax representation only for this particular value of $a = \frac{1}{2}$. Also Aiyer [5] proved that the oHS system possesses a recursion operator of degree four only for $a = \frac{1}{2}$. The same result has been recently reported by the author and Gürses [6] - [8] in the context of the integrable coupled KdV systems admitting recursion operators. Wilson [9] pointed out that the oHS system with $a = \frac{1}{2}$ belongs to the general construction of evolutionary equations possessing Lax-pair due to Drinfeld and Sokolov [10]-[11]. On the other hand the Ito system [12]
\begin{align*}
  u_t &= u_{xxx} + 6uu_x + 2vv_x,
\end{align*}
1
\[ v_t = 2(uv)_x. \] (3)

admits a bi-Hamiltonian structure

\[ B_{II} \delta \mathcal{H}_n = B_I \delta \mathcal{H}_{n+1} \] (4)

where

\[
B_{II} = \begin{pmatrix}
D^3 + 4uD + 2u_x & 2vD \\
2v_x + 2vD & 0
\end{pmatrix},
B_I = \begin{pmatrix}
D & 0 \\
0 & D
\end{pmatrix}.
\] (5)

with the Hamiltonian functionals

\[
\mathcal{H}_1[u,v] = \int \frac{1}{2}(u^2 + v^2)dx,
\]
\[
\mathcal{H}_2[u,v] = \int \frac{1}{2}(u^3 - \frac{1}{2}u_x^2 + uv^2)dx.
\] (6)

The recursion operator arising from a Hamiltonian pair

\[
R = B_{II}(B_I)^{-1} = \begin{pmatrix}
D^2 + 4u + 2u_x D^{-1} & 2v \\
2v + 2v_x D^{-1} & 0
\end{pmatrix}.
\] (7)

is a hereditary operator [7] which gives rise to infinitely many conserved quantities. The multi-Hamiltonian structure of this system was studied by Antonowicz and Fordy [13] and by Olver and Rosenau [14].

At this stage we have the following observations:

**observation 1:**

The author of [1] points out that there is a printing error in the conserved density \( I_3 \) (instead of 1 + \( a \) there is \( a \)) obtained in [2]. But this claim is incorrect. One can check this either using \( \frac{d}{dt} \int I_3 dx = 0 \) or finding its gradient \( \gamma_3 \) satisfies \( \gamma_3'[K] + (K')^\dagger[\gamma_3] = 0 \) and \( \gamma_3' = (\gamma_3')^\dagger[13] \). Here \( K \) is the right hand side of the oHS system.
observation 2:

The author of [1] claims that the oHS system can be written as

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = AB_1 \left( \begin{pmatrix}
  \frac{\delta H_1}{\delta u} \\
  \frac{\delta H_1}{\delta v}
\end{pmatrix}
\right)
\]

(8)

with

\[
A = \begin{pmatrix}
  a(D^2 + 4u + 2u_x D^{-1}) & 2v + v_x D^{-1} \\
  2v + v_x D^{-1} & -(D^2 + 4u + 2u_x D^{-1})
\end{pmatrix}
\]

(9)

and \(B_1\) given in (5). Here \(H_1\) is the Hamiltonian functional of the Ito system. First we have noticed that \(H_1\) is also a Hamiltonian functional of the oHS system \((b = 3/2)\). Second, the operator \(AB_1\) does not satisfy the Jacobi identity [16]. Therefore it is not a Hamiltonian operator for the oHS system although it is a skew-symmetric operator. Furthermore the author of [1] proposes a bi-Hamiltonian structure for the oHS system which is based on the bi-Hamiltonian form of Ito system (4) as

\[
AB_{II} \delta H_n = AB_I \delta H_{n+1}, \quad n = 0, 1, 2, ...
\]

(10)

First of all, the Hamiltonian functional \(H_2\) of the Ito system is not a Hamiltonian functional of the oHS system. Moreover neither \(AB_I\) nor \(AB_{II}\) are Hamiltonian operators. As a result the expression (10) does not constitute a bi-Hamiltonian form for the oHS system.

observation 3

The author of [1] claims that an infinite hierarchy of the oHS system is generated by the recursion operator

\[
\mathcal{R} = (AB_{II})(AB_I)^{-1} = ARA^{-1}
\]

(11)
where $\mathcal{R}$ is the recursion operator for Ito system. It is easy to see that the given recursion operator $\mathcal{R}$ has at most degree two. Therefore it does not generate the hierarchy of the oHS system. It has been recently reported that neither oHS nor HS with $a = \frac{1}{2}$ possess a recursion operator of degree two [6]-[8].

As a conclusion, contrary to the claims made in ref. [1], the oHS system is not a bi-Hamiltonian system. This system is integrable and admits a bi-Hamiltonian structure only when $a = \frac{1}{2}$ [7]-[8]. The hierarchy of this system was studied by Levi [19].

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