Approximately Optimal Scheduling of an M/G/1 Queue with Heavy Tails

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Distributions with a heavy tail are difficult to estimate. If the design of an optimal scheduling policy is sensitive to the details of heavy tail distributions of the service times, an approximately optimal solution is difficult to obtain. This paper shows that the optimal scheduling of an M/G/1 queue with heavy tailed service times does not present this difficulty and that an approximately optimal strategy can be derived by truncating the distributions.

1. Introduction. The past two decades have seen considerable research activity around the study of the effects of long range dependent traffic on the performance measures of queueing systems. The interest was sparked by the observations made from real measurements of internet traffic from a variety of sources, which revealed characteristics of self similarity, long range dependence and heavy tailed response times, see [6, 14]. The prime objective has been to study the performance of systems under different scheduling policies in the presence of such traffic and to identify optimal policies for various notions of optimality.

In the classical probabilistic setting, a natural objective is to find a policy which minimizes the mean sojourn time of a job in the system. For this objective, when the service duration of the jobs are known at the time of the arrival, the policy of serving the job with the shortest remaining processing time (SRPT) is optimal for any distribution of the service time (see [19]). But when this information is not known, an optimal policy depends on the distribution and is generally difficult to obtain. The critical pitfall that needs to be avoided by a good policy is the starving of small jobs due to the time spent on serving longer jobs. Hence such a policy needs to be sensitive to the distributions of the residual service times of the jobs in the system at any given time. This problem is especially grave when the distribution is heavy tailed. Indeed, it was shown in an example in [3] that when the service times have an infinite variance, the mean sojourn time of a job is infinite under any

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non-preemptive scheduling policy, although there exist preemptive policies which are able to keep it finite.

Another objective that has received considerable attention in literature is that of optimizing the tail of the sojourn time of a job, where a scheduling policy needs to asymptotically minimize the probability of facing a long sojourn time. In that case, even when the information about the job service times is available on arrival, the results are very interesting. In contrast to the objective of mean optimality, the performance of different policies crucially depend on the tail characteristics of the service time distributions. For this objective, the first come first serve policy (FCFS) is asymptotically optimal for jobs with light tailed service times (see [20]), whereas it performs arbitrarily bad for heavy tailed jobs (see [3]). On the other hand SRPT performs very well for heavy tailed jobs but performs very poorly if the jobs are light tailed (see [16] and [17]). A survey of these results can be found in [7] and [10]. Indeed, it was recently proved in [23] that there can be no scheduling policy in a broadly defined class, which uniformly optimizes the tail of the sojourn time under every distribution of the service time of jobs. The design of a tail optimal scheduling policy thus critically depends on the details of the distribution of the service times. This sensitivity is not desirable since it is practically difficult to estimate the tail characteristics of heavy tailed distributions.

Does this sensitivity also hold for the objective of mean optimality when the service times are not known on arrival? If so, how bad is the effect of errors in our estimates? Can the loss be unbounded (especially in the view of observations in [3])? These are the central questions that we try to answer in this paper. Our result is positive: for the specific case of an M/G/1 queue, we show that the dependence of the optimal policy on the service time distribution is only through the first and second order moments of the tail, and that an approximately optimal policy can be found to the desired magnitude of diminishing additive error by appropriately truncating the distribution.

1.1. Overview of results. We consider the scheduling of jobs in an M/G/1 queue with a service time distribution $F$. At any given time, the server remembers how long he has worked on each job in the queue and chooses which job to work on, possibly in a preemptive way. The objective of the server is to find a scheduling policy that minimizes the mean sojourn time of the jobs in the system.

Let $X$ be a service time of a typical job and let the rate of arrival be $\lambda$. If the distribution of the service time has an infinite variance, we can derive
an approximately optimal policy from the optimal policy for a related queue with truncated jobs. This truncation is defined as follows.

**Definition 1.1 (Truncation of Type A).** Let $X$ be the random variable denoting the service time of a typical job with a c.d.f. $F$. The for any $s \in \mathbb{R}^+$, we define a random variable $X^s$ with a c.d.f. $F^s$ defined as

$$F^s(a) = P(X^s \leq a) = \begin{cases} F(a) & \text{if } a \in [-\infty, s) \\ 1 & \text{if } a \geq s \end{cases}$$

which implies that

$$P(X^s = s) = F^s(s) - \lim_{a \to s^-} F^s(a) = P(X \geq s)$$

We show that the following policy is approximately optimal. Fix a truncation duration $s$, and compute the optimal policy for scheduling jobs with these truncated service times. Use this policy for scheduling all jobs which have been served for a duration less than $s$. For any job longer than $s$, after working on it for a duration $s$, place it in a secondary low priority queue. Finally use Last-come-first-serve (LCFS) policy to serve this secondary queue as long as there are no jobs left in the primary queue. For this policy, termed as priority optimal-LCFS (PO-LCFS), we show that as the truncation $s$ goes to infinity, the expected contribution to the delay cost from the secondary queue goes to zero under the following technical condition.

$$\lim_{s \to \infty} P(X > s)E(X^21_{\{X<s\}}) = 0$$

If the job sizes have a finite variance i.e. $E(X^2) < \infty$, then we obtain a stronger result. We show that there exist constants $K_1 > 0$ and $K_2 > 0$ dependent only on $E(X)$, $E(X^2)$ and $\lambda$ such that if $s > 0$ is such that

$$K_1E[X1_{\{X>s\}}] + K_2E[X^21_{\{X>s\}}] \leq \epsilon$$

for $\epsilon > 0$, then one can again derive an $\epsilon$-optimal scheduling policy by truncating the distribution of $X$ to $s$. In this case the truncation is different from the truncation of type A and is defined as follows.

**Definition 1.2 (Truncation of Type B).** Let $X$ be the random variable denoting the service time of a typical job with a c.d.f. $F$. Then for any $s \in \mathbb{R}^+$, we define a random variable $\tilde{X}^s$ with a c.d.f. $\tilde{F}^s$ defined as

$$\tilde{F}^s(a) = P(\tilde{X}^s \leq a) = \frac{F(a \wedge s)}{F(s)} \text{ for } a \in [-\infty, \infty]$$
Specifically, this $\epsilon$-optimal policy agrees with the optimal policy for the distribution $F$ truncated (type B) at $s$, except that it switches to any arbitrary work-conserving policy when it discovers a job whose service time exceeds $s$.

In both these cases, finding the value of $s$ that corresponds to a desired approximation is simpler than estimating the details of the tail of the distribution of $X$, so this result has practical significance.

2. The infinite variance case. Consider the optimal scheduling problem of an M/G/1 queue where the service durations of the jobs have a c.d.f. $F$ and the rate of arrival of these jobs is $\lambda$. Since the process describing the total workload in the system is a renewal process that does not depend on the work-conserving policy, we consider a single busy period and our aim is to find a work conserving scheduling strategy $u$ which minimizes the expected value of the sum of sojourn times of all jobs in the busy period. Formally, for a busy period labeled $B$, we want to minimize $E(C(u))$ where

\[(2.1)\quad C(u) = \sum_{i=1}^{N_B} S_i(u)\]

where $N_B$ is a random variable denoting the number of jobs that arrive in the busy period $B$, and $S_i(u)$ is a random variable denoting the sojourn time of job $i$ when policy $u$ is used to schedule the jobs. Let $u^*$ be the optimal policy with service times distributed according to $F$, and $V^*$ be the corresponding optimal expected cost. We want to define an approximately optimal policy for this scheduling problem, which is derived from the optimal policy for a related problem where the distribution of the job sizes are truncated according to truncation of type A defined in (1.1).

Consider those jobs arriving in a busy period $B$ whose service time is larger than $s$. Let $N_B^s$ be the number of these jobs. We can consider each such job to be composed of two smaller jobs, one of length $s$ which arrives when the original job arrives and a second job of the residual length which arrives when the first job of length $s$ is finished. For any fixed scheduling policy, the sojourn time of the original job is exactly the sum of sojourn times of the two smaller jobs. Preserving the label of each original large job for the first small job of service time $s$ that it is composed of and defining new labels for the residual small jobs, we can express the objective in (2.1) as minimizing $E(C^s(u))$ where

\[(2.2)\quad C^s(u) = \sum_{i=1}^{N_B} S_i(u) + \sum_{j=1}^{N_B^s} S_j^s(u).\]
The first term of this cost is the contribution by all the jobs whose service time is smaller than or equal to $s$. The distribution of the service time of each of these jobs is $F^s$. This term can be minimized by using the optimal scheduling policy for the M/G/1 queue where the service times of the jobs are distributed as $F^s$. In the implementation, the server uses this optimal policy to schedule the jobs and whenever the service time of any job reaches $s$, he stops serving that job and places it in a low priority queue. This event is equivalent to the arrival of the smaller residual job. The low priority queue is thus composed of all the residual jobs and it is served only when there is no job left which has been served for a duration less than $s$. Thus the residual jobs which contribute to the second term in the cost do not interfere with the first term. Define the following policy for the original scheduling problem.

**Definition 2.1.** Priority optimal + LCFS (PO-LCFS) A ‘priority optimal + LCFS’ policy for a truncation parameter $s \in \mathcal{R}^+$ gives a high priority to jobs which have been served for a duration less than or equal to $s$, while using the optimal scheduling policy for the M/G/1 queue where the service times of the jobs are distributed as $F^s$ for these jobs, and uses the LCFS policy on jobs which have been served for a duration greater than $s$ only when no high priority jobs remain.

We will show that under the PO-LCFS policy, denoted by $w^*$, the expected contribution to the total cost in (2.2) from the residual jobs diminishes to 0 as the truncation parameter $s$ grows to infinity. This implies that an approximately optimal policy can be found for the original problem to the desired degree of approximation by choosing an appropriate truncation parameter. Let $V^s$ be the corresponding expected average of sojourn times of the jobs in a busy period under this policy.

**Theorem 2.1.** Let $X$ be the service time of a typical job. Suppose that

1. $E(X) < \infty$ and $\lambda E(X) < 1$,
2. $\lim_{s \to \infty} E(X^2 1_{\{X<s\}}) P(X > s) = 0$.

Then under the PO-LCFS policy,

$$(2.3) \quad \limsup_{s \to \infty} V^s - V^* \leq \limsup_{s \to \infty} E(\sum_{j=1}^{N^a_{\mu}} S'_j(u^s)) = 0$$
PROOF. First observe that

\[ V^s - V^* \leq \sum_{i=1}^{N_B} S_i(u^s) - \sum_{i=1}^{N_B} S_i(u^*) + \sum_{j=1}^{N^*_B} S'_j(u^s) - \sum_{j=1}^{N^*_B} S'_j(u^*) \]

(2.4)

\[ \leq \sum_{j=1}^{N^*_B} S'_j(u^s) \]

where the inequality follows since the PO-LCFS policy is optimal for jobs with service times smaller than or equal to \( s \) and moreover we can ignore the negative term at the end. Now let the sum of sojourn times of all the residual files under the PO-LCFS policy be denoted by \( R \) i.e. \( R = \sum_{j=1}^{N^*_B} S'_j(u^s) \). Then \( R \) is bounded by the first step decomposition

(2.5)

\[ R \leq ML + MQ + M \sum_{i=1}^{K(Q)} W_i + \sum_{i=1}^{K(Q)} R_i, \]

where

- \( M \) is the number of residual jobs in the low priority queue when it is served for the first time in the busy period,
- \( L \) is the time spent on the high priority jobs before the low priority queue is served for the first time,
- \( Q \) is the total workload of the \( M \) jobs in the low priority queue at the first time that it is served,
- \( K(Q) \) is the number of digressions that occur of complete sub-busy periods that interrupt the service of the \( M \) jobs due to the LCFS policy in the low priority queue (see for example fig. 1),
- \( W_i \) is the length of the \( i \)th sub-busy period which interrupts the service of the \( M \) jobs in the low priority queue,
- and \( R_i \) is the sum of the sojourn times of all the residual files under the PO-LCFS policy in a sub-busy period \( i \).

We thus have

(2.6)

\[ E(R) \leq E(ML) + E(MQ) + E(M \sum_{i=1}^{K(Q)} W_i) + E(\sum_{i=1}^{K(Q)} R_i) \]

We find lower bounds for each of these terms separately. First, let us consider \( E(ML) \). Observe that the random variables \( M \) and \( L \) do not depend on the particular policy used to schedule the high priority (small) jobs as long as this policy is work conserving. Thus to estimate these random variables, we
can assume that LCFS policy is used for the high priority jobs. Let $X^s$ be the random variable denoting the duration of the first job that arrives in the busy period $B$. Then $M$ can be expressed as

$$M = \mathbf{1}_{\{X^s = s\}} + \sum_{i=1}^{K(X^s)} M_i,$$

where $K(X^s)$ is the number of high priority sub-busy periods that interrupt the service of the first job due to the LCFS policy and $M_i$ is the number of residual jobs brought in by the $i$th sub-busy period. Thus

$$E(M) = P(X^s = s) + E(E[\sum_{i=1}^{K(X^s)} M_i | X^s])$$

$$= P(X > s) + E(\lambda X^s E(M)).$$

Here the second equality holds because $M_i$ and $X$ are mutually independent and conditional on $X$, $K(X^s)$ is a poisson random variable with mean $\lambda X^s$. Further each $M_i$ has the same distribution as $M$. And hence,

$$E(M) = \frac{P(X > s)}{1 - \lambda E(X^s)}.$$

Similarly, $L$ can be expressed as

$$L = X^s + \sum_{i=1}^{K(X^s)} L_i,$$

where $L_i$ is the duration of the $i$th high-priority sub-busy period. We can again derive

$$E(L) = \frac{E(X^s)}{1 - \lambda E(X^s)}.$$

which is finite by our assumptions. Further, we have

$$E(ML) = E[(\mathbf{1}_{\{X^s = s\}} + \sum_{i=1}^{K(X^s)} M_i)(X^s + \sum_{i=1}^{K(X^s)} L_i)]$$

$$= sP(X^s = s) + E[\sum_{i=1}^{K(X^s)} L_i | X^s = s]P(X^s = s)$$

$$+ E(X^s \sum_{i=1}^{K(X^s)} M_i) + E((\sum_{i=1}^{K(X^s)} M_i)(\sum_{i=1}^{K(X^s)} L_i)).$$
\[ sP(X > s) + s\lambda E(L)P(X > s) + \lambda E((X^s)^2)E(M) \]
\[ + E(\sum_{i=1}^{K(X^s)} M_i L_i) + E(\sum_{i=1}^{K(X^s)} \sum_{j=1; j \neq i}^{K(X^s)} L_j) \]
\[ \leq sP(X > s) + s\lambda E(L)P(X > s) + \lambda E((X^s)^2)E(M) \]
\[ + \lambda E(X^s)E(ML) + \lambda^2 E((X^s)^2)E(M^2)E(L). \]

We thus have
\[ (2.11) \quad E(ML) \leq \frac{(sP(X > s) + \lambda E((X^s)^2)E(M)) (1 + \lambda E(L))}{1 - \lambda E(X^s)} \]

Let us now consider \( E(MQ) \). Notice that we can express \( Q \) as,
\[ (2.12) \quad Q = \sum_{i=1}^{M} X_i. \]

where \( X_i \) is the (residual) service time of the \( i \)th job in the low priority queue. Thus we have
\[ E(MQ) = E(M \sum_{i=1}^{M} X_i) \]
\[ = E(E[M \sum_{i=1}^{M} X_i | M]) \]
\[ (2.13) \quad = E(M^2)E(X) \]

Now from (2.7), we can compute \( E(M^2) \) as
\[ E(M^2) = E[(1_{\{X^s=s\}} + \sum_{i=1}^{K(X^s)} M_i)^2] \]
\[ = P(X > s) + 2sP(X > s)\lambda E(M) \]
\[ + E(\sum_{i=1}^{K(X^s)} M_i^2) + 2E(\sum_{1 \leq i < j \leq K(X^s)} M_i M_j) \]
\[ = P(X > s) + 2sP(X > s)\lambda E(M) \]
\[ + \lambda E(X^s)E(M^2) + E(K(X^s)^2)E(M^2) \]
\[ = P(X > s) + 2sP(X > s)\lambda E(M) \]
\[ + \lambda E(X^s)E(M^2) + \lambda^2 E((X^s)^2)E(M^2) + \lambda E(X^s)E(M^2). \]
And thus we have
\[
E(MQ) = E(M^2)E(X)
\]
\[
\leq \frac{E(X)P(X > s)(1 + 2s\lambda E(M))}{1 - \lambda E(X^*)}
\]
\[
+ \frac{\lambda E(X)E(M)[E((X^*)^2)\lambda E(M) + E(X^*)E(M)]}{1 - \lambda E(X^*)}
\]
(2.14)

Now we turn to find a bound for the term \(E(M \sum_{i=1}^{K(Q)} W_i)\). We have
\[
E(M \sum_{i=1}^{K(Q)} W_i) = E(E[M \sum_{i=1}^{K(Q)} W_i | M, Q])
\]
\[
= E(\lambda MQE(W))
\]
\[
= \lambda E(MQ) \frac{E(X)}{1 - \lambda E(X)}
\]
(2.15)

again because \(W_i\) are i.i.d. and conditional on \(Q\), \(K(Q)\) is a poisson random variable with rate \(\lambda\). The second equality follows since \(E(W)\), which is the expected duration of the busy period \(B\), does not depend on the scheduling policy and it is well known, again using first step arguments for the LCFS policy to be \(E(W) = \frac{E(X)}{1 - \lambda E(X)}\), which is finite by our assumptions. Similarly for the last term we have
\[
E(\sum_{i=1}^{K(Q)} R_i) \leq E(E[\sum_{i=1}^{K(Q)} R_i | Q])
\]
\[
= E(\lambda QE(R))
\]
\[
= \lambda E(M)E(X)E(R).
\]
(2.16)

Here the second equality follows from (2.12).

Thus from (2.15) and (2.16) and the expression for \(E(R)\) in (2.6), we finally have
\[
E(R) \leq \frac{E(ML) + E(MQ)(1 + \frac{\lambda E(X)}{1 - \lambda E(X)})}{1 - \lambda E(M)E(X)}
\]
(2.17)

where \(E(ML)\) and \(E(MQ)\) is bounded by the expressions in (2.11) and (2.14) respectively. The convergence of this upper bound for increasing threshold parameter \(s\) is governed by the following terms.

1. \(E(M)E(X) = O(P(X > s)E[X - s | X > s]) = O(P(X > s)E[X | X > s])\) from (2.8).
2. $E(M)E(X^*) = O(P(X > s)(E[X \mid X < s] + sP(X > s))) = O(sP(X > s))$.

3. $E(M)E((X^*)^2) = O(P(X > s)(E(X^2 I_{X < s}) + s^2P(X > s))) = O(P(X > s)E(X^2 I_{X < s}) + s^2P(X > s)^2)$.

Now since $E(X) < \infty$ by our assumption, $\lim_{s \to \infty} P(X > s)E(X \mid X > s) = 0$. This also implies that $\lim_{s \to \infty} sP(X > s) = 0$. Further, by our second assumption, $\lim_{s \to \infty} P(X > s)E(X^2 I_{X < s}) = 0$.

### 2.1. An example.

Consider a shifted Pareto distribution with parameter $\alpha > 1$ where $F(x) = \frac{1}{(x+1)^\alpha}$ for $x \geq 0$ and 1 for $x \leq 0$. The density is given by $f(x) = \frac{\alpha}{(x+1)^{\alpha+1}}I_{x \geq 0}$. Note that since $\alpha > 1$, $E(X) < \infty$. Then

$$P(X > s)E(X^2 I_{X < s}) = \frac{1}{(s+1)^\alpha} \int_0^s \frac{\alpha x^2}{(x+1)^{\alpha+1}}dx \leq \frac{\alpha s}{(s+1)^\alpha}.$$ 

Here the inequality follows because the integrand is always bounded by $\alpha$, since $\alpha > 1$. Now

$$\lim_{s \to \infty} \frac{\alpha s}{(s+1)^\alpha} = 0$$

again because $\alpha > 1$. Thus the condition of the theorem is satisfied.

### 3. The finite variance case.

We again consider the optimal scheduling problem of an M/G/1 queue, but now we constrain the variance of the service durations of the jobs to be finite. We look at a single busy period and our aim is to find a work conserving scheduling strategy $u$ which minimizes the expected value of the sum of sojourn times of all jobs in the busy period. For a busy period labeled $B$, we want to minimize $E(C(u))$ where $C(u)$ is given by equation (2.1).

Let $u^*$ be the optimal policy with service times distributed according to $F$, and $V^*$ be the corresponding optimal expected cost. We again want to define an approximately optimal policy for this scheduling problem, which is derived from the optimal policy for a related problem where the distribution of the job sizes are truncated. But this truncation is of type B, defined in (1.2), which is different from the type A truncation described for the PO-LCFS policy.

Fix a threshold $s > 0$ and let $u^s$ be the optimal policy when all service times are distributed according to the truncated distribution $F^s$. Also, let $\tilde{u}^s$ be the policy that uses the policy $u^s$ till it discovers a job whose service time exceeds $s$, after which it switches to an arbitrary but work conserving
policy \( \tilde{u} \). Let \( \tilde{V}^s \) be the corresponding expected average of sojourn times of the jobs in a busy period under this policy.

**Theorem 3.1.** Let \( X \) be the service time of a typical job such that \( E(X) < \infty \) and \( \text{var}(X) < \infty \). Then, there exist constants \( K_1 \) and \( K_2 \) depending only on the mean and variance of \( F \), such that

\[
\tilde{V}^s - V^* \leq K_1 E[X1_{\{X>s\}}] + K_2 E[X^2 1_{\{X>s\}}].
\]

Hence

\[
\limsup_{s \to \infty} \tilde{V}^s - V^* = 0.
\]

To prove the theorem, we will first need the following lemma:

**Lemma 3.1.** Let \( X \) be the service time of a typical job. Consider a busy period \( B \). Let \( W \) be the duration of the busy period and \( N_B \) be the number of jobs that arrive during the busy period. For a threshold \( s \), let \( A \) be the event that the service time of every job in the busy period is less than the threshold. Then

1. \( E[W \mid A^c] \leq \frac{E[X \mid X>s]}{D} \).
2. \( E[N_B \mid A^c] \leq \frac{1}{D} + \frac{\lambda E[X \mid X>s]}{D^2} \).

where \( D = 1 - \lambda E(X) \).

**Proof.** We first compute an upper bound for \( P(A^c) \). This is the probability that at least one job in the busy period has a service time that exceeds the threshold. Suppose that the policy is LCFS. Let \( X \) denote the service time of the first job. Then we have the following expression:

\[
P(A^c) = P(X > s) + P(X \leq s)E[1 - \exp(-\lambda P(A^c)X) \mid X \leq s].
\]

This holds because while working on the first job with service time \( X \), new busy periods with at least one job with service time that exceeds the threshold \( s \) and busy periods with no job with service time exceeding \( s \) arrive as independent Poisson processes with rate \( \lambda P(A^c) \) and \( \lambda P(A) \) respectively. These “sub-busy periods” interrupt the service of the first job until it finishes. But since \( 1 - \exp(-x) \leq x \), we have

\[
P(A^c) \leq P(X > s) + P(X \leq s)E[\lambda P(A^c)X \mid X \leq s].
\]

Thus we have

\[
P(A^c) \leq \frac{P(X > s)}{1 - \lambda P(X \leq s)E[X \mid X \leq s]}.
\]
But since $P(X \leq s)E[X \mid X \leq s] \leq E(X)$, we have

\[(3.1) \quad P(A^c) \leq \frac{P(X > s)}{1 - \lambda E(X)}.\]

Now to prove part (1), we find an upper bound for $E[W \mid A^c]$. We again assume that the policy is LCFS. We use a first step argument that decomposes $W$ into $X$ plus the duration of the busy periods that arrive while the server processes the first job. Assume that $K(X)$ busy periods arrive during the processing of the first job. Among these $K(X)$ sub-busy periods, $N(X)$ have at least one job with service time that exceeds $s$ and $M(X) = K(X) - N(X)$ have all jobs with service times less than $s$. Note that as mentioned before, conditional on $X$, the random variables $N(X)$ and $M(X)$ are Poisson with mean $\lambda X P(A^c)$ and $\lambda X P(A)$, respectively. Define $\alpha := E[W \mid A^c]$, $\delta = E[W \mid A]$ and $\gamma = P[X > s \mid A^c] = \frac{P(X > s)}{P(A^c)}$. Then

$$\alpha = E[W \mid X > s, A^c] \gamma + E[W \mid X \leq s, A^c](1 - \gamma).$$

Now,

$$E[W \mid X > s, A^c] = E[W \mid X > s] = E[X \mid X > s] + \lambda E[X \mid X > s]E(W).$$

Also,

$$E[W \mid X \leq s, A^c] = E[X + \alpha E[N(X) \mid N(X) > 0, X] \mid X \leq s, N(X) > 0] + E[\delta E[M(X) \mid N(X) > 0, X] \mid X \leq s, N(X) > 0].$$

Furthermore,

$$E[N(X) \mid N(X) > 0, X] = \frac{\lambda P(A^c)X}{1 - \exp(-\lambda P(A^c)X)}.$$ 

Since $\frac{x}{1 + \exp(-x)} \leq 1 + x$, we have

\[(3.2) \quad E[N(X) \mid N(X) > 0, X] \leq 1 + \lambda P(A^c)X.\]

Also $M(X)$ and $N(X)$ are independent conditional on $X$. Hence,

$$\alpha \leq \gamma (E[X \mid X > s] + \lambda E[X \mid X > s]E(W))$$
$$+(1 - \gamma) (E[(1 + \lambda P(A^c)X)\alpha + \lambda P(A)X \delta \mid X \leq s, N(X) > 0])$$
$$= \gamma (1 + \lambda E(W))E[X \mid X > s]$$
$$+(1 - \gamma) (\alpha + (1 + \lambda E(W))E[X \mid X \leq s, N(X) > 0]).$$
Now denote \( E[X \mid X \leq s, N(X) > 0] = \phi(s) \). Then we have,

\[
\alpha \leq \gamma^{-1}(1 + \lambda E(W)) \left( \gamma E[X \mid X > s] + (1 - \gamma)\phi(s) \right). 
\]

Now from the the definition of \( \gamma \) we have

\[
E[W \mid A^c] \leq P(A^c) \left( \frac{P(X > s)}{P(A^c)} E[X \mid X > s] \right. 
\]

\[
+ \left. (1 - \frac{P(X > s)}{P(A^c)}) \phi(s) \right) .
\]

Now, (a) from the upper bound on \( P(A^c) \) in (3.1), (b) since \( E(W) = E(X) \frac{E(X | X > s)}{1 - \lambda E(X)} \) and (c) since \( \phi(s) = E[X \mid X \leq s, N(X) > 0] \leq E[X \mid X > s] \), we finally have

\[
E[W \mid A^c] \leq C
\]

where

\[
C = \frac{1}{D^2} E[X \mid X > s] \quad \text{and} \quad D = 1 - \lambda E(X).
\]

We thus have the required result.

Now to prove part (2), we find an upper bound for \( E[N_B \mid A^c] \). Again define \( \beta := E[N_B \mid A^c], \rho = E[N_B \mid A] \) and \( \gamma = P[X > s \mid A^c] = \frac{P(X > s)}{P(A^c)} \).

Then,

\[
\beta = E[N_B \mid X > s, A^c] \gamma + E[N_B \mid X \leq s, A^c](1 - \gamma).
\]

Now,

\[
E[N_B \mid X > s, A^c] = E[N_B \mid X > s] = 1 + \lambda E[X \mid X > s]E(N_B).
\]

Also,

\[
E[N_B \mid X \leq s, A^c] 
\]

\[
= E[1 + \beta E[N(X) \mid N(X) > 0, X] \mid X \leq s, N(X) > 0] 
\]

\[
+ E[\rho E[M(X) \mid N(X) > 0, X] \mid X \leq s, N(X) > 0].
\]

Hence by same arguments as in the case before,

\[
\beta \leq \gamma (1 + \lambda E[X \mid X > s]E(N_B)) \\
+ (1 - \gamma) \left( E[(1 + \lambda P(A^c) X) \beta + 1 + \lambda P(A) X \rho \mid X \leq s, N(X) > 0] \right) \\
= \gamma (1 + \lambda E[X \mid X > s]E(N_B)) + (1 - \gamma) (\beta + 1 + \lambda E(N_B) \phi(s)).
\]
Therefore,

\[ \beta \leq \gamma^{-1} \left[ 1 + \lambda E(N_B) (\gamma E[X \mid X > s] + (1 - \gamma) \phi(s)) \right]. \tag{3.5} \]

Now from the definition of \( \gamma \) we have

\[
E[N_B \mid A^c] \leq P \left( \frac{P(A^c)}{P(X > s)} \left[ 1 + \lambda E(N_B) \left( \frac{P(X > s)}{P(A^c)} E[X \mid X > s] \right. \right.ight.

\[ \left. \left. + (1 - \frac{P(X > s)}{P(A^c)}) \phi(s) \right] \right] \right] \text{.}
\]

Now again (a) from the upper bound on \( P(A^c) \) in (3.1), (b) since \( E(N_B) = \frac{1}{1 - \lambda E(X)} \) and (c) since \( \phi(s) = E[X \mid X \leq s, N(X) > 0] \leq E[X \mid X > s] \), we finally have

\[ E[N_B \mid A^c] \leq P \tag{3.6} \]

where

\[
P = \frac{1}{D} + \frac{\lambda}{D^2} E[X \mid X > s] \text{ and } D = 1 - \lambda E(X) \]

which is the required result. \( \square \)

We now prove the main theorem.

**Proof.** The system starts empty and we consider the first busy period. Let \( X \) be the service time of the first job that arrives. Let \( A \) be the event that every service time in the busy period is less than the threshold \( s \). Then we have

\[ V^* = E[C(u^*)] = E[C(u^*) \mid A]P(A) + E[C(u^*) \mid A^c]P(A^c) \]

\[ \geq E[C(u^*) \mid A]P(A) \geq E[C(u^*) \mid A^c]P(A^c). \]

The first inequality holds because we ignore the second term in the first expression while the second inequality holds because, on the event \( A \), policy \( u^* \) gives a lower cost than policy \( u^* \). By definition of \( u^* \) and \( \tilde{u}^* \), one has

\[ \tilde{V}^* = E[C(u^*) \mid A]P(A) + E[C(\tilde{u}^*) \mid A^c]P(A^c). \]

We thus have

\[ \tilde{V}^* - V^* \leq E[C(\tilde{u}^*) \mid A^c]P(A^c). \]
Under any work conserving policy, the sum of sojourn times of the jobs in a busy period, is upper bounded by the total duration of the busy period times the number of files that arrived in the busy period, and this random variable is independent of the scheduling policy. Let $W$ be the random variable denoting the total duration of the busy period and $N_B$ is the number of files that arrived in the busy period. Then we have

\begin{equation}
\hat{V}_s - V^* \leq E[W N_B \mid A^c] P(A^c).
\end{equation}

The rest of the proof is providing an upper bound on this quantity. We already have an upper bound for $P(A^c)$ from (3.1). We move on to find an upper bound for $E[W N_B \mid A^c]$. Again we use a technique similar to the one used in proving the lemma. Say that the policy is LCFS. We use a first step argument that decomposes $W N_B$ into the contribution to this quantity by this first job in the busy period plus the contribution from all the sub-busy periods that arrive and interrupt while the server processes the first job. As in the proof of the lemma, assume that $K(X)$ sub-busy periods arrive during the processing of the first job. Among these $K(X)$ busy periods, $N(X)$ have at least one job with service time that exceeds $s$ and $M(X) = K(X) - N(X)$ have all jobs with service times less than $s$. Here as before, given $X$, the random variables $N(X)$ and $M(X)$ are independent poisson with mean $\lambda X P(A^c)$ and $\lambda X P(A)$ respectively. We define two sets of notation for the durations of these busy periods. The duration of all the busy periods are denoted by $\{W_k; k = 1, \cdots, K(X)\}$ and the number of jobs in each of these busy periods is denoted by $\{N_{B_k}; k = 1, \cdots, K(X)\}$. We will find it convenient to also denote the busy periods with at least one job with service time exceeding threshold $s$ by $\{\hat{W}_i; i = 1, \cdots, N(X)\}$ and the number of jobs in these busy periods by $\{\hat{N}_{B_i}; i = 1, \cdots, N(X)\}$. Similarly, the busy periods with no job with service time exceeding threshold $s$ are denoted by $\{\tilde{W}_j; j = 1, \cdots, M(X)\}$ and the number of jobs in these busy periods are denoted by $\{\tilde{N}_{B_j}; j = 1, \cdots, M(X)\}$. Also define $\gamma = P[X > s \mid A^c] = \frac{P(X > s)}{P(A^c)}$. Then,

\begin{equation}
E[W N_B \mid A^c] = E[W N_B \mid X > s, A^c] \gamma + E[W N_B \mid X \leq s, A^c](1 - \gamma).
\end{equation}

We now have the following decomposition:

$$WN_B = (X + \sum_{j=1}^{K(X)} W_i)(1 + \sum_{j=1}^{K(X)} N_{B_i})$$
\[ = \sum_{i=1}^{K(X)} W_i + \sum_{j=1}^{K(X)} W_i N_{B_i} + \sum_{k=1}^{K(X)} N_{B_k} (X + \sum_{j \neq k,j=1}^{K(X)} W_j) \]
\[ = \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{j=1}^{M(X)} \tilde{W}_j + \sum_{i=1}^{N(X)} \tilde{W}_i N_{\tilde{B}_i} + \sum_{j=1}^{M(X)} \tilde{W}_j N_{\tilde{B}_j} \]
\[ + \sum_{i=1}^{M(X)} N_{\tilde{B}_i} (X + \sum_{l \neq i}^{N(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) \]
\[ + \sum_{j=1}^{M(X)} N_{\tilde{B}_k} (X + \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{m \neq j,m=1}^{M(X)} \tilde{W}_m) \]

Now,
\[ E[W N_B \mid X > s, A^c] = E[W N_B \mid X > s] \]
\[ = E[X + \sum_{i=1}^{K(X)} W_i + \sum_{j=1}^{K(X)} W_i N_{B_i} \mid X > s] \]
\[ + E[\sum_{k=1}^{K(X)} N_{B_k} (X + \sum_{j \neq k,j=1}^{K(X)} W_j) \mid X > s]. \]

Now for each sub-busy period \( i \), \( N_{\tilde{B}_i} \) is independent of \( X \) and \( \sum_{j \neq k,j=1}^{K(X)} W_j \).

We thus have
\[ E[\sum_{k=1}^{K(X)} N_{B_k} (X + \sum_{j \neq k,j=1}^{K(X)} W_j) \mid X] \]
\[ = X E[\sum_{k=1}^{K(X)} N_{B_k} \mid X] + E[\sum_{k=1}^{K(X)} N_{B_k} (\sum_{j \neq k,j=1}^{K(X)} W_j) \mid X] \]
\[ = \lambda X^2 E(N_B) + E[K(X) (K(X) - 1) E(W) E(N_B) \mid X] \]
\[ = \lambda X^2 E(N_B) + E(W) E(N_B) \lambda^2 X^2 \]

Where again the last equality follows since \( K(X) \) is a poisson random variable conditional on \( X \). Thus we have
\[ E[W N_B \mid X > s, A^c] = E[X \mid X > s] (1 + \lambda E(W) + \lambda E(W N_B)) \]
\[ + E[X^2 \mid X > s] \lambda E(N_B)(1 + \lambda E(W)). \]

Further,
\[ E[W N_B \mid X \leq s, A^c] = E[W N_B \mid X \leq s, N(X) > 0] \]
\[(3.10) \quad E[X + \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{j=1}^{M(X)} \tilde{W}_j \mid X \leq s, N(X) > 0]
\]
\[+ E[\sum_{i=1}^{N(X)} \tilde{W}_i N_{\tilde{B}_i} + \sum_{j=1}^{M(X)} \tilde{W}_j N_{\tilde{B}_j} \mid X \leq s, N(X) > 0]
\]
\[+ E[\sum_{i=1}^{N(X)} N_{\tilde{B}_i} (X + \sum_{l \neq i,l=1}^{M(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) \mid X \leq s, N(X) > 0]
\]
\[+ E[\sum_{j=1}^{M(X)} N_{\tilde{B}_j} (X + \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{m \neq j,m=1}^{M(X)} \tilde{W}_m) \mid X \leq s, N(X) > 0].\]

Now from (3.2) we know that

\[E[N(X) \mid N(X) > 0, X] \leq 1 + \lambda P(A^c)X.\]

Hence,

\[
\begin{align*}
& E[X + \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{j=1}^{M(X)} \tilde{W}_j \mid X \leq s, N(X) > 0] \\
& + E[\sum_{i=1}^{N(X)} \tilde{W}_i N_{\tilde{B}_i} + \sum_{j=1}^{M(X)} \tilde{W}_j N_{\tilde{B}_j} \mid X \leq s, N(X) > 0] \\
& \leq E[X + (1 + \lambda P(A^c)X)E[W \mid A^c] \mid X \leq s, N(X) > 0] \\
& + E[\lambda P(A)XE[W \mid A] \mid X \leq s, N(X) > 0] \\
& + E[(1 + \lambda P(A^c)X)E[W N_B \mid A^c] \mid X \leq s, N(X) > 0] \\
& + E[\lambda P(A)XE[W N_B \mid A] \mid X \leq s, N(X) > 0] \\
& (3.11) = \phi(s) (1 + \lambda E[W] + \lambda E(W N_B)) + E[W \mid A^c] + E[W N_B \mid A^c]
\end{align*}
\]

where \(\phi(s) = E[X \mid X \leq s, N(X) > 0].\) Now

\[
\begin{align*}
& E[\sum_{i=1}^{N(X)} N_{\tilde{B}_i} (X + \sum_{l \neq i,l=1}^{M(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) \mid X, N(X) > 0] \\
& = X (1 + \lambda P(A^c)X)E[N_B \mid A^c] \\
& + E[\sum_{i=1}^{N(X)} N_{\tilde{B}_i} (\sum_{l \neq i,l=1}^{M(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) \mid X, N(X) > 0].
\end{align*}
\]

Now since for each busy period \(i, N_{\tilde{B}_i} \) is independent of \(X, N(X), \sum_{l \neq i,l=1}^{N(X)} \tilde{W}_l \)
and $\sum_{j=1}^{M(X)} \tilde{W}_j$, we have

$$
E[\sum_{i=1}^{N(X)} N_{B_i}(\sum_{l\neq i,l=1}^{N(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) | X, N(X) > 0]
\leq E[N(X)(N(X) - 1)E[N_B | A^c]E[W | A^c] | X, N(X) > 0]
+ E[N(X)M(X)E[N_B | A^c]E[W | A] | X, N(X) > 0]
= \lambda P(A^c)X(1 + \lambda P(A^c)X)E[N_B | A^c]E[W | A^c]
+ \lambda P(A)X(1 + \lambda P(A^c)X)E[N_B | A^c]E[W | A]
= \lambda X(1 + \lambda P(A^c)X)E[N_B | A^c]E(W)
$$

where the first equality follows because $N(X)$ and $M(X)$ are independent conditional on $X$ and

$$
E[N(X)^2 - N(X) | N(X) > 0, X] = \frac{E[N(X)^2 - N(X) | X]}{1 - \exp(-\lambda P(A^c)X)}
= \frac{\lambda^2 P(A^c)^2X^2}{1 - \exp(-\lambda P(A^c)X)}
\leq \lambda P(A^c)X(1 + \lambda P(A^c)X).
$$

(3.12)

And hence,

$$
E[\sum_{i=1}^{N(X)} N_{B_i}(X + \sum_{l\neq i,l=1}^{N(X)} \tilde{W}_l + \sum_{j=1}^{M(X)} \tilde{W}_j) | X \leq s, N(X) > 0]
\leq E[X(1 + \lambda P(A^c)X) | X \leq s, N(X) > 0]E[N_B | A^c](1 + \lambda E(W))
\leq \phi'(s)\lambda P(A^c)E[N_B | A^c](1 + \lambda E(W))
\leq \phi(s)E[N_B | A^c](1 + \lambda E(W))
$$

(3.13)

where $\phi'(s) = E[X^2 | X \leq s, N(X) > 0]$. Similarly we have

$$
E[\sum_{j=1}^{M(X)} N_{B_k}(X + \sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{m\neq j,m=1}^{M(X)} \tilde{W}_m) | X, N(X) > 0]
\leq \lambda P(A)X^2E[N_B | A]
+ E[\sum_{j=1}^{M(X)} N_{B_k}(\sum_{i=1}^{N(X)} \tilde{W}_i + \sum_{m\neq j,m=1}^{M(X)} \tilde{W}_m) | X, N(X) > 0]
= \lambda P(A)X^2E[N_B | A] + \lambda P(A)X(1 + \lambda P(A^c)X)E[N_B | A]E[W | A^c]
+ \lambda^2 P(A)^2X^2E[N_B | A]E[W | A]
= \lambda P(A)X^2E[N_B | A](1 + \lambda E(W)) + \lambda P(A)XE[N_B | A]E[W | A^c]
$$
And hence,
\[
E\left[ \sum_{j=1}^{M(X)} N_{B_k}(X) + \sum_{i=1}^{N(X)} \bar{W}_i + \sum_{m \neq j; m=1}^{M(X)} \bar{W}_m \right] | X \leq s, N(X) > 0] \\
\leq \phi'(s)\lambda P(A)E[N_B | A](1 + \lambda E(W)) \\
+ \phi(s)\lambda P(A)E[N_B | A]E[W | A^c] \\
(3.14)
\]

Now combining equations (3.10), (3.11), (3.13) and (3.14), we have
\[
E[W N_B | X \leq s, A^c] \leq \phi(s)\left(1 + \lambda E[W] + \lambda E(W N_B)\right) \\
+ E[N_B | A^c](1 + \lambda E(W)) \\
+ \lambda P(A)E[N_B | A]E[W | A^c]) \\
+ \phi'(s)(\lambda E(N_B)(1 + \lambda E(W))) \\
(3.15)
\]
Finally, combining equations (3.9) and (3.15), we have
\[
E[W N_B | A^c] \leq \gamma\left( E[X | X > s](1 + \lambda E(W) + \lambda E(W N_B)) \\
+ E[X^2 | X > s]\lambda(1 + \lambda E(W))E(N_B)\right) \\
+ (1 - \gamma)\left( E[W N_B | A^c] + E[W | A^c]\right) \\
+ \phi(s)(1 + \lambda E[W] + \lambda E(W N_B) \\
+ E[N_B | A^c](1 + \lambda E(W)) \\
+ \lambda P(A)E[N_B | A]E[W | A^c]) \\
+ \phi'(s)(\lambda E(N_B)(1 + \lambda E(W))) \right) \\
(3.16)
\]
And thus we get the upper bound :
\[
E[W N_B | A^c] \leq \gamma^{-1}\left[ \gamma\left( E[X | X > s](1 + \lambda E(W) + \lambda E(W N_B)) \\
+ E[X^2 | X > s]\lambda(1 + \lambda E(W))E(N_B)\right) \\
+ (1 - \gamma)\left( E[W | A^c]\right) \\
+ \phi(s)((E[N_B | A^c] + 1)(1 + \lambda E(W)) + \lambda E(W N_B) \\
+ \lambda E[W | A^c]E(N_B)) + \phi'(s)\lambda E(N_B)(1 + \lambda E(W)) \right]. \\
(3.16)
\]
Here we used the fact that $E(N_B) \geq E[N_B | A]P(A)$. Now from the definition of $\gamma$, we have

$$
E[W N_B | A^c]P(A^c)
\leq \frac{P(A^c)}{P(X > s)} \left[ P(X > s) \left( E[X | X > s](1 + \lambda E(W)
+ \lambda E(W N_B)) + E[X^2 | X > s]\lambda(1 + \lambda E(W))E(N_B) \right)
+ (P(A^c) - P(X > s)) \left( E[W | A^c]
+ \phi(s)((E[N_B | A^c] + 1)(1 + \lambda E[W])
+ \lambda E(W N_B) + \lambda E[W | A^c]E(N_B))
\right. \\
\left. + \phi'(s)\lambda E(N_B)(1 + \lambda E(W)) \right].
$$

(3.17)

Now since $\phi'(s) \leq E[X | X > s]$ and $\phi'(s) \leq E[X^2 | X > s]$, we have

$$
E[W N_B | A^c]P(A^c)
\leq \frac{P(A^c)}{P(X > s)} \left[ P(A^c) \left( E[X | X > s](1 + \lambda E(W)
+ \lambda E(W N_B)) + E[X^2 | X > s]\lambda(1 + \lambda E(W))E(N_B) \right)
+ (P(A^c) - P(X > s)) \left( E[W | A^c]
+ E[X | X > s](E[N_B | A^c](1 + \lambda E[W])
\right. \\
\left. + \lambda E[W | A^c]E(N_B)) \right].
$$

(3.18)

Finally, using the bound on $P(A^c)$ in (3.1), and ignoring the negative term we have

$$
E[W N_B | A^c]P(A^c) \leq M
$$

(3.19)

where

$$
M = \frac{1}{D^2}P(X > s) \left[ E[X | X > s] \left( (1 + E[N_B | A^c])(1 + \lambda E[W])
+ \lambda E(W N_B) + \lambda E[W | A^c]E(N_B) \right) + E[W | A^c]
\right. \\
\left. + E[X^2 | X > s]\lambda(1 + \lambda E(W))E(N_B) \right]
$$
and

\[ D = 1 - \lambda E(X). \]

Now we have bounds for all the quantities in \( M \). First, from the lemma, we have

\[
E[W | A^c] \leq \frac{E[X | X > s]}{D^2} \]
\[
E[N_B | A^c] \leq \frac{1}{D} + \frac{\lambda E[X | X > s]}{D^2}.
\]

Further, we can compute an upper bound on \( E(WN_B) \) using our decomposition:

\[
E(WN_B) = E[X + \sum_{i=1}^{K(X)} W_i + \sum_{j=1}^{K(X)} W_i N_{B_i} + \sum_{k=1}^{K(X)} N_{B_k} (X + \sum_{j \neq k, j=1}^{K(X)} W_j)]
\]
\[
= E(X) (1 + \lambda E(W) + \lambda E(WN_B))
\]
\[
+ E[E[\sum_{k=1}^{K(X)} N_{B_k} (X + \sum_{j \neq k, j=1}^{K(X)} W_j) | X]]
\]
\[
= E(X) (1 + \lambda E(W) + \lambda E(WN_B))
\]
\[
+ \lambda \lambda E(W) + \lambda E(WN_B)
\]
\[
= E(X) (1 + \lambda E(W) + \lambda E(WN_B))
\]
\[
+ \lambda (\lambda E(W) + 1) E(N_B) E(X^2).
\]

We thus have

\[
E(WN_B) \leq \frac{(E(X) + \lambda E(N_B) E(X^2)) (\lambda E(W) + 1)}{1 - \lambda E(X)}.
\]

But again since \( E(W) = \frac{E(X)}{1 - \lambda E(X)} \) and \( E(N_B) = \frac{1}{1 - \lambda E(X)} \), we have

\[
E(WN_B) \leq \frac{E(X) + \lambda E(X^2) - \lambda E(X)^2}{D^3}
\]

and thus ignoring the negative term, we have

\[
E(WN_B) \leq \frac{E(X) + \lambda E(X^2)}{D^3}. \quad (3.20)
\]
We thus finally have

\[
E[WN_B \mid A^c]P(A^c) \leq M \\
\leq \frac{P(X > s)}{D^2} \left[ E[X \mid X > s] \left( \frac{D^2 + D + \lambda E[X \mid X > s]}{D^3} \right) + \frac{\lambda E(X) + \lambda^2 E(X^2)}{D^3} \right] \\
+ \frac{\lambda E[X^2 \mid X > s]}{D^2} + E[X \mid X > s]
\]

(3.21)

where we have substituted all of the computed upper bounds in (3.19). Now by Jensen’s inequality we have 

\[E[X \mid X > s]^2 \leq E[X^2 \mid X > s].\]

Thus we can further simplify the bound as follows:

\[
E[WN_B \mid A^c]P(A^c) \\
\leq \frac{P(X > s)}{D^4} \left[ E[X \mid X > s] \left( 2 + D + \frac{\lambda E(X) + \lambda^2 E(X^2)}{D} \right) \right] \\
+ \frac{E[X^2 \mid X > s]}{D^2} \left( \frac{2\lambda}{D} + \lambda \right).
\]

Thus we have

(3.22) \( \tilde{V}^s - V^* \leq E[WN_B \mid A^c]P(A^c) \leq K_1 E[X^1_{X > s}] + K_2 [X^2 1_{X > s}] \)

where

(3.23) \( K_1 = \frac{1}{D^4} \left( 2 + D + \frac{\lambda E(X) + \lambda^2 E(X^2)}{D} \right) \)

and

(3.24) \( K_2 = \frac{1}{D^4} \left( \frac{2\lambda}{D} + \lambda \right). \)

Now \( \lim_{s \to \infty} E[X^1_{X > s}] = \lim_{s \to \infty} E[X^2 1_{X > s}] = 0 \) since \( E(X) < \infty \) and \( E(X^2) < \infty \). Thus

\[ \lim_{s \to \infty} \tilde{V}^s - V^* = 0 \]

hence proving the result. \( \square \)
3.1. An example. Consider $\lambda = 1$ and assume that the service times of the jobs have the pareto distribution with the tail probability function $1 - F(x) = P(X > x) = \frac{1}{(x+1)^{3}}1_{\{x \geq 0\}}$. The density function is $f_X(x) = \frac{3}{(x+1)^{4}}1_{\{x \geq 0\}}$. Then we have $E(X) = \frac{1}{2}$ and $E[X^2] = 1$. Thus $D = 1 - \lambda E(X) = \frac{1}{2}$. Then from equations (3.23) and (3.24) we find that $K_1 = 120$ and $K_2 = 144$. Further we can compute

$$E[X 1_{\{X > s\}}] = \frac{3s + 1}{2(s + 1)^3}$$
$$E[X^2 1_{\{X > s\}}] = \frac{3s^2 + 3s + 1}{(s + 1)^3}.$$ 

Thus we have

\begin{equation} V^* - V^* \leq g(s) = 120\left(\frac{3s + 1}{2(s + 1)^3}\right) + 144\left(\frac{3s^2 + 3s + 1}{(s + 1)^3}\right). \end{equation}

This function is plotted in fig. 2.

![Figure 1](image_url)
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