A forest formula for the antipode in incidence Hopf algebras

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Abstract

We present a new formula for the antipode of incidence Hopf algebras. This formula is expressed as an alternating sum over forests. First, we prove the formula for incidence Hopf algebras of families of lattices by exhibiting a map from chains of a lattice to forests. Then, we extend the definition and present an analogous formula for the antipode of incidence Hopf algebras of families of posets. We characterize those families for which our formula is cancellation-free.

1 Introduction

Many combinatorial Hopf algebras can be realized as incidence Hopf algebras of families of posets. The antipode of incidence Hopf algebras, when expressed in terms of the canonical basis of indecomposable posets in the family, is generally highly non-trivial; i.e., the antipode is a sum with many terms. In [4], Haiman and Schmitt presented a closed formula for the antipode of the Faà di Bruno Hopf algebra. Their formula was expressed as a sum over trees (which we reinterpret as a sum over forests.) They proved that this antipode formula is equivalent to Lagrange inversion. In [3], Figueroa found a similar forest formula for the antipode of the incidence Hopf algebra of distributive lattices and presented applications in quantum field theory. In this paper, we introduce a forest formula for the antipode of an arbitrary incidence Hopf algebra. Both Figueroa’s and Haiman and Schmitt’s results are special cases of this new formula.

Haiman and Schmitt’s formula for the antipode of the Faà di Bruno Hopf algebra is cancellation-free, while Figueroa’s formula for the antipode of the incidence Hopf algebra of distributive lattices is generally not cancellation-free. We characterize those families of posets for which our forest formula for the antipode is cancellation-free. The formula is cancellation-free for all indecomposable posets in a hereditary family if and only if every upper interval of every indecomposable interval in the family is indecomposable. This condition is equivalent to the right-sided condition defined by Loday and Ronco in [5], which, in turn, is equivalent to the Lie algebra of primitive elements in fact being a pre-Lie algebra.
In Section 2, we recall the definitions of hereditary families of posets and incidence Hopf algebras, and we give the formula for the antipode as an alternating sum over chains. In Section 3, we present several lemmas concerning the center and decomposition of posets. In Section 4, we define forests of lattices. In Section 5, we prove our forest formula for the antipode of incidence Hopf algebras of lattices. In Section 6, we characterize those families for which the forest formula is cancellation-free. In Section 7, we generalize our results from families of lattices to analogous results for incidence Hopf algebras of families of posets.

2 Hereditary families

An interval is a partially ordered set $P$ with unique maximal and minimal elements, which we denote $\hat{1}_P$ and $\hat{0}_P$, respectively. We eliminate the subscript when there is no chance of ambiguity. In this paper, we will assume all intervals are finite.

We slightly modify the definition of hereditary families of posets from [4].

Definition 1. A hereditary family $(P, \sim, \cdot)$ is a family $P$ of finite intervals with a product operation $\cdot : P \times P \to P$, where we write $PQ$ for $P \cdot Q$; and an equivalence relation $\sim$ such that $P$ is closed under the formation of subintervals and, for all $P, Q, R \in P$,

1. If $P \sim Q$, then $PR \sim QR$.

2. $(PQ)R \sim P(QR)$, and if $I$ is any single-element interval, then $IP \sim PI \sim P$. Also, $PQ \sim QP$.

3. If $P \sim Q$, then there is a bijection $x \mapsto x'$ from $P$ to $Q$ such that $[\hat{0}_P, x] \sim [\hat{0}_Q, x']$ and $[x, 1_P] \sim [x', 1_Q]$ for all $x \in P$.

4. There is a poset isomorphism $\psi$ from the Cartesian product $P \times Q$ to $PQ$ such that $[\psi(\hat{0}_P, 0_Q), \psi(x, y)] \sim [\hat{0}_P, x][0_Q, y]$ and $[\psi(x, y), \psi(1_P, 1_Q)] \sim [x, 1_P][y, 1_Q]$ for all $(x, y) \in P \times Q$.

We use $\prod$ to denote the iterated $\cdot$ product.

Let $k$ be a commutative ring with 1, and let $P = (P, \sim, \cdot)$ be a hereditary family. Conditions (1) and (2) of Definition 1 imply that the quotient $P/\sim$ is a commutative monoid with product induced by the product in $P$. The incidence Hopf algebra of $P$, denoted $H(P)$, is the monoid algebra of $P/\sim$ over $k$, with coalgebra structure given by

$$\delta(P) = \sum_{x \in P} [\hat{0}, x] \otimes [x, 1],$$

and

$$\epsilon(P) = \begin{cases} 1 & \text{if } |P| = 1 \\ 0 & \text{otherwise}. \end{cases}$$
This coproduct is clearly coassociative. Condition (3) ensures that $\delta$ is well-defined on $\mathcal{P}/\sim$. Condition (4) guarantees that $\delta(PQ) \sim \delta(P)\delta(Q)$, and so $H(\mathcal{P})$ is a bialgebra.

We do not distinguish notationally between elements of $\mathcal{P}$ and $\mathcal{P}/\sim$.

**Definition 2.** Let $\mathcal{P}$ be a hereditary family of posets. We say that a poset is **decomposable** in $\mathcal{P}$ if it is the non-trivial product of non-singleton posets in $\mathcal{P}$, and we say that it is indecomposable otherwise. If $P \in \mathcal{P}$, then we write $D(P)$ for the set of all $x \in P \setminus \{\hat{0}, \hat{1}\}$ such that $[\hat{0}, x]$ is decomposable in $\mathcal{P}$, and we write $I(P)$ for the set of all $x \in P \setminus \{\hat{0}, \hat{1}\}$ such that $[\hat{0}, x]$ is indecomposable in $\mathcal{P}$.

Let $P_0$ be the set of all indecomposable posets in $\mathcal{P}$. Then $P_0/\sim$ is the free commutative monoid on $\mathcal{P}/\sim$. So, as an algebra, $H(\mathcal{P})$ is isomorphic to the polynomial algebra $k[P_0/\sim]$.

A chain $C$ in an interval $P$ is a set $\hat{0} = c_0 < c_1 < \cdots < c_n = \hat{1}$ of elements of $P$, and $\ell(C) = n$ is the length of $C$. The length of an interval $P$ is the length of its longest chain. Let $H(\mathcal{P})_n$ be the submodule of $H(\mathcal{P})$ spanned by intervals of length less than or equal to $n$. These define a bialgebra filtration $H(\mathcal{P})_0 \subseteq H(\mathcal{P})_1 \subseteq H(\mathcal{P})_2 \subseteq \ldots$. Condition (2) of Definition 1 ensures that $H(\mathcal{P})_0 = k \cdot 1$, and so $H(\mathcal{P})$ is a connected bialgebra, and thus it is a Hopf algebra.

**Definition 3.** The convolution algebra of $H(\mathcal{P})$ is $\text{Hom}_k(H(\mathcal{P}), H(\mathcal{P}))$ with operation convolution defined by

$$(f * g)(P) = \sum_{x \in P} f([\hat{0}, x])g([x, \hat{1}]).$$

Since $H(\mathcal{P})$ is a Hopf algebra, the identity map $id : H(\mathcal{P}) \rightarrow H(\mathcal{P})$ in the convolution algebra has a two-sided convolution inverse $\chi$, known as the antipode.

For $P \in \mathcal{P}$, let $\mathcal{C}(P)$ be the set of all chains of $P$. For each $P \in \mathcal{P}$, define $\Omega : \mathcal{C}(P) \rightarrow P$ by

$$\Omega(C) = \prod_{i=1}^{\ell(C)} [c_{i-1}, c_i].$$

**Proposition 4.** The antipode of $H(\mathcal{P})$ is given by

$$\chi(P) = \sum_{C \in \mathcal{C}(P)} (-1)^{\ell(C)} \Omega(C).$$

### 3 The center of a poset

In this section, we consider $\mathcal{P}$ to be a hereditary family of posets.
Definition 5. Let \( P \in \mathcal{P} \). The center of \( P \), denoted \( Z(P) \), is the set of all \( a \in P \) such that there is some \( a' \in P \) such that \( [0, a][0, a'] \sim [0, 1] \). The prime center of \( P \), denoted \( \hat{Z}(P) \), is the set of all minimal non-zero elements of \( Z(P) \).

In [1], Birkhoff described several properties of the centers of posets. In his work, two posets were considered to be equivalent if they were isomorphic, and the product considered was Cartesian product. We show that these properties hold in the more general case of hereditary families of posets.

For \( P \in \mathcal{P} \), let \( I_\approx(P) \) be the set of all \( x \in P \) such that \([0, x]\) is indecomposable as a Cartesian product. Let \( D_\approx(P) \) be the set of all \( x \in P \) such that \([0, x]\) is decomposable as a Cartesian product, and let \( Z_\approx(P) \) be the set of all \( a \in P \) such that there is some \( a' \in P \) such that \([0, a][0, a'] \cong [0, 1] \), where the product is Cartesian product.

It is clear from Condition (4) of the hereditary family definition that \( D(P) \subseteq D_\approx(P) \) and \( I_\approx(P) \subseteq I(P) \) for any \( P \in \mathcal{P} \). It then follows that \( Z(P) \subseteq Z_\approx(P) \). However, it is not necessarily true that \( Z'(P) \subseteq Z_\approx(P) \), since the minimal elements of \( Z(P) \) may not be minimal in the larger set \( Z_\approx(P) \).

Lemma 6. If \( a \in Z(P) \), where \( P \in \mathcal{P} \), then \( a \lor z \) and \( a \land z \) exist for any \( z \in P \). Additionally, if \( a \in Z(P) \), then \( P \sim [0, a][a, 1] \) by the map \( z \rightarrow (z \land a, z \lor a) \), where \( z \in P \).

Proof. Birkhoff shows the first assertion for any \( a \in Z_\approx(P) \), and so it is true for any \( a \in Z(P) \).

The second assertion is a modification of a lemma from Birkhoff. Since \( a \in Z(P) \), we know that \( P \sim XY \) where \( X \sim [0, a] \) and \( Y \) is some poset. So, in the \( \psi \) map described in Condition (4) of Definition [1], we have \( a = \psi(\hat{X}, \hat{Y}) \).

Let \( z \in P \). Then \( z = \psi(x, y) \) for some \( x \in X \) and \( y \in Y \). Then, by the given map,

\[
z = \psi(x, y) \rightarrow (\psi(x, y) \land \psi(\hat{X}, \hat{0}Y), \psi(x, y) \lor \psi(\hat{X}, \hat{0}Y)) = (\psi(x, \hat{0}Y), \psi(\hat{1}X, y)).
\]

The elements of the form \( \psi(\hat{1}X, y) \) in the factorization \( P \sim XY = [0, a]Y \) are exactly the elements of \([a, 1]\) in \( P \). So then we must have \( Y \sim [a, 1] \), and so the given map sends \( P \) to \([0, a][a, 1]\), as needed. \( \square \)

Lemma 7. If \( P \in \mathcal{P} \) and \( b \in Z(P) \), then \( b \) has a unique complement \( b' \in Z(P) \), and \([0, b][0, b'] \sim P \).

Proof. Birkhoff shows the existence of the unique complement in \( Z_\approx(P) \), and the existence of the unique complement in \( Z(P) \) follows from the same reasoning.

The previous lemma shows that \( P \sim [0, b][b, 1] \) by the map \( z \rightarrow (z \land b, z \lor b) \). Then \( b' \rightarrow (0, 1) \), and so we must have \([0, b'][0, b'] \sim [b, 1] \), and thus \( P \sim [0, b][0, b]' \). \( \square \)

Birkhoff uses the analogues of the previous two lemmas to prove the following three lemmas when the equivalence relation is isomorphism and the product is Cartesian product. The general hereditary family case follows from the same reasoning.
Lemma 8. $Z(P)$ is a Boolean lattice and a sublattice of $P$.

Lemma 9. If $a \in Z(P)$, then $a \in Z'(P)$ if and only if $a \in I(P)$.

Lemma 10. $P \sim \prod_{a \in Z'(P)}[0, a]$.

Definition 11. An element $a$ of a lattice $P$ is said to be distributive if the identities
\begin{align*}
    a \land (x \lor y) &= (a \land x) \lor (a \land y) \\
    x \land (a \lor y) &= (x \land a) \lor (x \land y)
\end{align*}
and their duals hold for all $x, y \in P$. An element $a$ is complemented in $P$ if there exists an element $a' \in P$ such that $a \lor a' = \hat{1}$ and $a \land a' = \hat{0}$.

Birkhoff proves the following lemma.

Lemma 12. If $a \in P$, then $a \in Z_{\sim}(P)$ if and only if $a$ is both distributive and complemented in $P$.

4 Forests of lattices

For the next several sections, we consider $P$ to be a family of lattices. The more general poset case is considered in Section 7.

Definition 13. A forest of a lattice $P \in P$ is a set $F \subseteq I(P)$, with $\bigvee F \neq \hat{1}$ and $0 \notin F$, such that:

1. if $a_1, a_2 \in F$, then either $a_1 \leq a_2$, $a_2 \leq a_1$, or $a_1 \land a_2 = \hat{0}$. (This condition is referred to as “non-overlapping.”)

2. if $\{b_i\}_i$ is an antichain in $F$, then $\prod_{i}[\hat{0}, b_i] \sim [\hat{0}, \bigvee_i b_i]$.

Example 1. In [3], Figueroa defined forests of distributive lattices. Figueroa’s definition relied on the fundamental theorem of distributive lattices: If $L$ is a finite distributive lattice, then $L$ is isomorphic to the poset of order ideals $J_P$ of some finite poset $P$. He then defined a forest $F$ of $L$ as a collection of connected order ideals of $P$ where $\emptyset \notin F$ and $\bigcup F \neq F$, such that if $I_1, I_2 \in F$, then either $I_1 \cap I_2 = \emptyset$, or $I_1 \subseteq I_2$, or $I_2 \subseteq I_1$. In the map sending order ideals of $P$ to $J_P$, connected order ideals are sent to $I(J_P)$, and so this non-overlapping condition on order ideals is equivalent to Condition 1 of Definition [3]. The second condition of Definition [3] holds for any antichain in a distributive lattice.

Example 2. In the lattice shown in Figure 1, the forests of the interval are the empty forest; the single-element forests $\{a\}$ and $\{b\}$; and the two-element forest $\{a, b\}$. Note that $c$ cannot be in any forest since the interval $[\hat{0}, c]$ is decomposable.

Example 3. In the lattice shown in Figure 2, the forests of the interval are the empty forest; the single-element forests $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$; and the three two-element forests $\{a, d\}$, $\{b, d\}$, and $\{c, d\}$. The set $\{a, b\}$, for example, is not a forest, because even though it satisfies the first condition of the forest definition, it does not satisfy the second condition, since $a \lor b = d$ and $[\hat{0}, a][\hat{0}, b] \sim [\hat{0}, d]$. 

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5 Forest formula for antipode in incidence Hopf algebras of hereditary families of lattices

If $F$ is a forest of a lattice $P$ and $b \in F \cup \{1\}$, then we say $a$ is a predecessor of $b$ in $F$ if $a \in F$, $a < b$, and there is no $a' \in F$ such that $a < a' < b$. If $b$ has no predecessors in $F$, then we consider $0$ to be its predecessor.

**Definition 14.** If $P \in \mathcal{P}$, then let $\mathcal{F}(P)$ be the set of all forests of $P$. If $F \in \mathcal{F}(P)$, then let

$$\Theta(F) = \prod_{b \in F \cup Z'(P)} [\tilde{b}, b]$$

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**Figure 1: Example 2**

**Figure 2: Example 3**
where \( \hat{b} \) is the join of all the predecessors of \( b \) in \( F \). Note that

\[
\Theta(F) = \prod_{b \in F \cup \{1\}} [\hat{b}, b]
\]

is an equivalent definition.

**Proposition 15.** Let \( P \in \mathcal{P} \) be a finite lattice. There is a surjection \( \phi : C(P) \to \mathcal{F}(P) \), with \( C \mapsto F_C \), such that \( \Omega(C) \sim \Theta(F_C) \) for all \( C \in \mathcal{C}(P) \).

**Proof.** If \( C : \hat{0} = c_0 < c_1 < \cdots < c_{\ell(C)} = \hat{1} \) is a chain of \( P \), then let

\[
F_C := \bigcup_{i=1}^{\ell(C)-1} Z'(\hat{0}, c_i) - 1).
\]

We want to show that \( F_C \) satisfies Definition 13. We know that \( [0, a] \) is indecomposable for any \( a \in F_C \), since \( a \) is in the prime center of some \( [\hat{0}, c_i] \).

Also, we know that \( \bigvee F_C = c_{\ell(C)-1} < \hat{1}_P \), and so \( F_C \) satisfies the preliminary conditions of Definition 13.

Now, we need to show that \( F_C \) satisfies the first condition of Definition 13. Let \( a, b \in F_C \). We want to show that either \( a \leq b \) or \( a \wedge b = 0 \).

Let \( i \) and \( j \) be the smallest indices such that \( a \) and \( b \) are in the prime centers of \( [\hat{0}, c_i] \) and \( [\hat{0}, c_j] \), respectively. Without loss of generality, assume \( i \leq j \). We know that \( a \leq c_i \leq c_j \).

If \( a \leq b \), then we are done. If not, then we must have \( b < c_j \). Since \( b \in Z'(\hat{0}, c_j) \), there is a unique \( b' < c_j \) such that \( b \wedge b' = \hat{0} \) and \( \hat{0}, c_j \sim [\hat{0}, b][\hat{0}, b'] \). Since \( [0, a] \) is indecomposable and \( a < c_j \), we know that either \( a \leq b \) or \( a \leq b' \).

By assumption, the first possibility is false, and so we must have \( a \leq b' \), and thus \( a \wedge b = 0 \).

Next, we want to show that, if \( \{b_i\}_i \) is an antichain in \( F_C \), then each \( b_i \) is in the center of \( [\hat{0}, \bigvee b_i] \). We induct on the size of the antichain.

First, suppose we have an antichain \( \{b_1, b_2\} \) in \( F_C \). As before, let \( i \) and \( j \) be the smallest indices such that \( b_1 \) and \( b_2 \) are in the prime centers of \( [\hat{0}, c_i] \) and \( [\hat{0}, c_j] \), respectively. Assume \( i \leq j \).

Since \( b_2 \) is in the prime center of \( [\hat{0}, c_j] \), we know from Lemma 12 that there is some \( b' < c_j \) such that \( [\hat{0}, b_2][\hat{0}, b'] \sim [\hat{0}, c_j] \). We know that \( b_1 \leq c_i \leq c_j \), and, since we showed that \( F_C \) satisfies the first forest condition, we know that \( b_1 \wedge b_2 = 0 \), and so we must have \( b_1 \leq b' \). Then by Condition (4) of Definition 11 we must have \( [\hat{0}, b_1][\hat{0}, b_2] \sim [\hat{0}, b_1 \vee b_2] \).

Now assume that \( \prod [0, b_i] \sim [\hat{0}, \bigvee b_i] \) for all antichains in \( F_C \) up to size \( n - 1 \).

Let \( \{b_1, \ldots, b_n\} \) be an antichain in \( F_C \). For each \( 1 \leq i \leq n \), let \( k_i \) be the smallest index such that \( b_i \in Z'(\hat{0}, c_{k_i}] \). Without loss of generality, assume \( k_n \geq k_i \) for all \( i \). Since \( b_n \) is in the center of \( [\hat{0}, c_{k_n}] \), we know from Lemma 12 that \( b_n \) is distributive in \( [\hat{0}, c_{k_n}] \), and so the distributive property shows

\[
b_n \wedge \left( \bigvee_{j=1}^{n-1} b_j \right) = \bigvee_{j=1}^{n-1} (b_n \wedge b_j) = \hat{0},
\]
since, by the previous part, \( b_n \land b_j = \hat{0} \) for any \( 1 \leq j \leq n - 1 \).

Since \( b_n \) is in the center of \( [0, c_k] \), Lemma 7 shows that it must have a unique complement \( b' \) in \( [0, c_k] \) such that \( [0, b_n][0, b'] \sim [0, c_k] \). We know that \( \bigvee_{j=1}^{n-1} b_j \leq c_k \) and we just showed that \( b_n \land (\bigvee_{j=1}^{n-1} b_j) = \hat{0} \), and so we must have \( \bigvee_{j=1}^{n-1} b_j \leq b' \). So then, by induction and Condition (4) of the hereditary family definition, we get

\[
\prod_{j=1}^{n} [\hat{0}, b_j] = [\hat{0}, b_n]\prod_{j=1}^{n-1} [\hat{0}, b_j] \sim [\hat{0}, b_n]\bigvee_{j=1}^{n-1} b_j \sim [\hat{0}, b_n] \bigvee_{j=1}^{n-1} b_j = [\hat{0}, \bigvee_{j=1}^{n} b_j],
\]

as needed.

Next, we want to show that this map is surjective. Let \( F \) be a forest of \( P \). Let \( S_1 \) be the collection of all maximal elements of the forest, \( S_2 \) the collection of all maximal elements of \( F \setminus S_1 \), and so forth. Define \( C_F \) as the chain \( \bigvee S_1 \setminus \bigvee S_2 \setminus \cdots > 0 \). We need to show that \( \bigvee S_i > \bigvee S_{i+1} \) for all \( i \).

Clearly, \( \bigvee S_i \setminus \bigvee S_{i+1} \). Thus, \( Z'(\bigvee S_i) = S_i \) and \( Z'(\bigvee S_{i+1}) = S_{i+1} \). Since \( S_i \) and \( S_{i+1} \) are disjoint, we know that \( \bigvee S_i \setminus \bigvee S_{i+1} \), and so \( \bigvee S_i > \bigvee S_{i+1} \). So each forest is, in fact, associated to at least one chain.

Last, we want to show that \( \Omega(C) \sim \Theta(F_C) \) for any chain \( C \) of \( P \). Let \( \ell(C) = n \). For each \( 1 \leq i \leq n \), let \( F_i = Z'(\bigvee [\hat{0}, c_i]) \). Then \( \bigcup F_i = F_C \cup Z'(P) \).

For each \( i \), we know from Lemma 10 that

\[
[\hat{0}, c_i] \sim \prod_{a \in Z'(\bigvee [\hat{0}, c_i])} [\hat{0}, a].
\]

Then, from Lemma 10 and Condition (4) of the hereditary family definition, we know that

\[
[\hat{0}, c_{i-1}] \sim \prod_{a \in Z'(\bigvee [\hat{0}, c_i])} [\hat{0}, a \land c_{i-1}].
\]

We then get

\[
[c_{i-1}, c_i] \sim \prod_{a \in Z'(\bigvee [\hat{0}, c_i])} [a \land c_{i-1}, a].
\]

For \( a \in Z'(\bigvee [\hat{0}, c_i]) \), we find

\[
a \land c_{i-1} = a \land \left( \bigvee_{b \in Z'(\bigvee [\hat{0}, c_i])} b \right) = \bigvee_{b \in Z'(\bigvee [\hat{0}, c_{i-1}])} a \land b = \bigvee_{b \in Z'(\bigvee [\hat{0}, c_{i-1}])} b = \tilde{a}.
\]

So then \( [c_{i-1}, c_i] \sim \prod_{a \in F_i} [\tilde{a}, a] \), and so we get \( \Omega(C) \sim \Theta(F_C) \).
The first main result of this paper is a Zimmerman-type formula for the antipode of $H(\mathcal{P})$ in terms of forests. To derive that formula, we require one more proposition.

**Proposition 16.** If $F$ is a forest of $P$, then $\sum_{C \in \phi^{-1}(F)} (-1)^{\ell(C)} = (-1)^{d(F)}$.

**Proof.** We follow the similar argument of [4]. Let a filtration $G$ of the forest $F$ be a chain $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_k = F$ of lower order ideals of $F$ such that, for all $1 \leq j \leq k$, the set $I_j \setminus I_{j-1}$ is an antichain. The length of the filtration is $\ell(G) = k$. We claim that, for each forest $F$ of a poset $P$, there is a bijection between $\phi^{-1}(F)$ and the set $G(F)$ of all filtrations of $F$, such that, if a chain $C$ is mapped to the filtration $G$, then $\ell(C) = \ell(G) + 1$.

First, suppose $C \in \phi^{-1}(F)$ is $\hat{0} = c_0 < c_1 < \cdots < c_n = \hat{1}$. Define the filtration $G$ by setting $I_0 = \emptyset$ and $I_k = Z'(\{\hat{0}, c_k\}) \cup I_{k-1}$ for $1 \leq k \leq n-1$. Each $I_k$ must be a lower order ideal of $F$, and the $I_k$ must be strictly increasing. Since $I_k \setminus I_{k-1}$ is a subset of $Z'(\{\hat{0}, c_k\})$, it must be an antichain. Thus, $G$ is a filtration.

Conversely, given a filtration $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_n = F$ of $F$, define the chain $C$ by letting $c_0 = \hat{0}$ and $c_k = \bigvee I_k$ for $1 \leq k \leq n$, and let $c_{n+1} = \hat{1}$. Then, by our definition of a forest, $Z'(\{\hat{0}, c_k\})$ must be the set of maximal elements of $I_k$. Each element of $I_k \setminus I_{k-1}$ must be either greater than some maximal element of $I_{k-1}$ or not comparable to any element of $I_{k-1}$, and so we have $c_k > c_{k-1}$.

These constructions are clearly inverse to one another, and so they form a bijection.

Lemma 4 of [4] states that, if $Q$ is a finite poset and $G(Q)$ is the set of all filtrations of $Q$, then

$$\sum_{G \in G(Q)} (-1)^{\ell(G)} = (-1)^{|Q|}.$$  

By using this lemma and the given bijection, we get

$$\sum_{C \in \phi^{-1}(F)} (-1)^{\ell(C)} = \sum_{G \in G(F)} (-1)^{\ell(G)+1} = (-1)^{d(F)}.$$  

We now come to our first main result.

**Theorem 17.** If $\mathcal{P}$ is a hereditary family of lattices, then the antipode of $H(\mathcal{P})$ is given by

$$\chi(\mathcal{P}) = \sum_{F \in \mathcal{F}(\mathcal{P})} (-1)^{d(F)} \Theta(F)$$  

for all $P \in \mathcal{P}$.

**Proof.** We know from Proposition 4 that

$$\chi(\mathcal{P}) = \sum_{C \in \mathcal{G}(\mathcal{P})} (-1)^{\ell(C)} \Omega(C).$$  

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Using Proposition 15, we get
\[ \chi(P) = \sum_{F \in F(P)} \Theta(F) \sum_{C \in \phi^{-1}(F)} (-1)^{\ell(C)}. \]

Then Proposition 16 gives us
\[ \chi(P) = \sum_{F \in F(P)} (-1)^{d(F)} \Theta(F). \]

6 Conditions for non-cancellation in computation of antipode

Formula (11) is very similar to the formula of Zimmerman, explored in [3], for the antipode of the Hopf algebra of Feynman graphs. Zimmerman’s antipode formula has the useful property of being cancellation-free.

For the forest computation of \( \chi(P) \) given by (11) to be cancellation-free, it must be the case that \((-1)^{d(F)}\) and \((-1)^{d(F')}\) have the same sign whenever \( F \) and \( F' \) are forests of \( P \) such that \( \Theta(F) \sim \Theta(F') \). In general, Formula (11) is not cancellation-free. A simple example of an indecomposable lattice for which the forest computation of \( \chi \) is not cancellation-free is shown in Figure 3. If \( P \) is the lattice shown, then the forests \( F = \{a\} \) and \( F' = \{a, b\} \) will cancel each other in the computation of \( \chi(P) \).

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \\
1 \quad 2 \\
2 \quad a \\
1 \quad b
\]

Figure 3: Lattice without cancellation-free forest computation

It is also clear that the forest computation of \( \chi(P) \) will have cancellations if \( P \) is decomposable: if \( a \in Z'(P) \), then \( \Theta(\{a\}) = [0, a][a, \hat{1}] \sim P = \Theta(\emptyset) \). Note that, since \( \chi \) is multiplicative, it is determined by its value on indecomposables. We now characterize those indecomposable lattices in hereditary families for which the forest computation of the antipode is cancellation-free. We
then characterize those hereditary families for which the forest computation is cancellation-free for all indecomposable lattices.

**Definition 18.** An indecomposable lattice \( P \in \mathcal{P} \) is called upper-indecomposable if, for every \( x < 1 \in P \), the interval \([x, 1]\) is indecomposable. An indecomposable lattice \( P \in \mathcal{P} \) is called super-upper-indecomposable (s.u.i.) if every indecomposable interval of \( P \) is upper-indecomposable.

**Proposition 19.** A lattice \( P \in \mathcal{P} \) is s.u.i if and only if every indecomposable lower interval of \( P \) is upper-indecomposable in \( \mathcal{P} \).

**Proof.** Clearly, if \( P \) is s.u.i., then every indecomposable lower interval of \( P \) is upper-indecomposable.

We prove the converse by contradiction. We want to show that if every indecomposable lower interval of \( P \) is upper-indecomposable, and there are \( a, x, y \in P \) such that \( 0 \leq a < x < y \leq 1 \) and \([x, y]\) is decomposable, then \([a, y]\) is decomposable. So suppose \( a, x, y \) are as given, and \([x, y]\) is decomposable.

Then, since every indecomposable lower interval of \( P \) is upper-indecomposable, the interval \([0, y]\) must be decomposable. Let \( Z'([0, y]) = \{y_i\}_{i=1}^n \).

Let \( x_i = x \land y_i \) for all \( i \). Then Condition (4) of the hereditary family definition implies \([0, x] \sim \prod_{i=1}^n [0, x_i]\) and \([x, y] \sim \prod_{i=1}^n [x_i, y_i]\). Since \([0, y_i]\) is indecomposable for each \( i \), and every indecomposable lower interval of \( P \) is upper-indecomposable, we know that \([x_i, y_i]\) is indecomposable for each \( i \).

Since \([x, y]\) is decomposable, we must have \( x_i \leq y_i \) for at least two values of \( i \).

Let \( a_i = a \land y_i \) for all \( i \). Since \( a < x \), we must also have \( a_i \leq x_i \) for all \( i \). Since \( x_i \leq y_i \) for at least two values of \( i \), we must have \( a_i \leq y_i \) for at least two values of \( i \), and so, since \([a, y] \sim \prod_{i=1}^n [a_i, y_i]\), we conclude that \([a, y]\) is decomposable, as needed.

\[ \square \]

**Proposition 20.** Let \( P \in \mathcal{P} \) be an indecomposable lattice. The forest computation of \( \chi(P) \) given by Theorem 17 is cancellation-free if and only if \( P \) is s.u.i.

**Proof.** Suppose an indecomposable lattice \( P \) is s.u.i. Let \( F \) be a forest of \( P \). Then \( \Theta(F) = \prod_{b \in F \cup \{1\}} [b, b] \). Since \([0, b]\) is indecomposable for each \( b \) and \( P \) is s.u.i., we know that each \([0, b]\) is upper-indecomposable, and so each \([0, b]\) will be indecomposable. We also know that \( b \neq b \). So then \( d(F) \) is the number of indecomposable intervals in the unique factorization of \( \Theta(F) \). Thus, the computation of \( \chi(P) = \sum_{F \in F(P)} (-1)^{d(F)} \Theta(F) \) will be cancellation-free.

Conversely, suppose \( P \) is indecomposable but not s.u.i. Then there are some \( x, y \in P \) such that \( 0 < y < x \) and \([0, x]\) is indecomposable, but \([y, x]\) is decomposable. Say \([y, x] \sim [y, x_1] \cdots [y, x_n]\) is the factorization of \([y, x]\) into indecomposable intervals.

Assume \( x < 1 \). (The case where \( x = 1 \) is analogous.)

Case 1: Suppose \([0, x_i]\) is indecomposable for some \( i \). Let \( F = Z'([0, y]) \cup \{x\} \), and let \( F' = Z'([0, y]) \cup \{x_i, x\} \). Then

\[ \Theta(F) = [0, y][y, x][x, 1], \]
and
\[ \Theta(F') = [\hat{0}, y][y, x_1][x_1, x][x, \hat{1}] \sim [\hat{0}, y][y, x][x, \hat{1}] . \]

So \( \Theta(F) \sim \Theta(F') \), but \( d(F') = d(F) + 1 \), so these two forests cancel each other in the forest computation of \( \chi(P) \).

Case 2: Suppose \([\hat{0}, x_i]\) is decomposable for all \( i \). Let

\[ [\hat{0}, x_1] \sim [\hat{0}, x_{1,1}][\hat{0}, x_{1,2}] \cdots [\hat{0}, x_{1,k}] \]

be the unique factorization of \([\hat{0}, x_1]\) into indecomposables. We know that \([y, x_1]\) is indecomposable, and so, without loss of generality, we can say \( y = (y_{1,1}, x_{1,1}, x_{1,2}, \ldots, x_{1,k}) \) in this factorization of \([\hat{0}, x_1]\). We can see that \([y_{1,1}, x_{1,1}] \sim [y, x_1] \).

We can also see that \( Z'([\hat{0}, y]) = Z'([\hat{0}, x_{1,1}]) \cup \{x_{1,2}, \ldots, x_{1,k}\} \).

Let \( F = Z'([\hat{0}, y]) \cup \{x\} \) and let \( F' = Z'([\hat{0}, y]) \cup \{x_{1,1}, x\} \). These both satisfy Definition 13 and we have

\[ \Theta(F) = [\hat{0}, y][y, x][x, \hat{1}] , \]

and

\[ \Theta(F') = [\hat{0}, y][y_{1,1}, x_{1,1}][x_{1,1}, x][x, \hat{1}] \sim [\hat{0}, y][y, x_1][x_1, x][x, \hat{1}] \sim [\hat{0}, y][y, x][x, \hat{1}] . \]

So again, \( \Theta(F) \sim \Theta(F') \), but \( d(F') = d(F) + 1 \), and hence these forests cancel each other.

\[ \square \]

Definition 21. A hereditary family \( P \) is called upper-indecomposable if every indecomposable \( P \in P \) is upper-indecomposable.

Note that, since a hereditary family must be closed under the taking of intervals, the hereditary family \( P \) is upper-indecomposable if and only if every indecomposable \( P \in P \) is super-upper-indecomposable.

These propositions bring us to our next main result.

Theorem 22. Let \( P \) be a hereditary family. Then the forest computation of \( \chi(P) \) given by Theorem 17 will be cancellation-free for all indecomposable \( P \in P \) if and only if \( P \) is upper-indecomposable.

In [5], Loday and Ronco defined a cofree-coassociative combinatorial Hopf algebra as a cofree bialgebra \( H \) together with an isomorphism between \( H \) and the tensor coalgebra over the primitive elements of \( H \). Furthermore, such \( H \) satisfies the right sided condition if \( \delta(Q(H)) \subseteq H \otimes Q(H) \), where \( Q(H) \) denotes the subspace of irreducibles in \( H \). The upper-indecomposable hereditary families \( P \) are exactly those hereditary families for which \( H(P) \) satisfies the right-sided condition. Theorem 5.3 of [5] states that the right-sided cofree-coassociative combinatorial Hopf algebras are exactly those cofree-coassociative combinatorial Hopf algebras in which the primitive elements form a pre-Lie algebra, rather than merely a Lie algebra.
Example 4. The partition lattice $\Pi_n$ of the set $\{1, \ldots, n\}$ is the poset of all partitions of $\{1, \ldots, n\}$, ordered by refinement: if $x, y \in \Pi_n$, then $x \leq y$ if every block of $x$ is contained in a block of $y$. The Faà di Bruno Hopf algebra is the Hopf algebra $H(\mathcal{P})$, where $\mathcal{P}$ is the set of all finite products of finite partition lattices, with the equivalence relation $\sim$ given by isomorphism. In [4], Haiman and Schmitt define a surjection from the set of chains of $\Pi_n$ to the set of leaf-labelled trees with $n$ leaves and no vertices of degree 1. According to their definition, if $C$ is a chain in the partition lattice $\Pi_n$, then the tree $T(C)$ associated with $C$ is the poset of all subsets of $\{1, \ldots, n\}$ which appear as blocks of partitions in $C$, ordered by inclusion.

If $x$ is a partition in $\Pi_n$ with non-singleton blocks $B_1, \ldots, B_k$, then $Z'(\{0, x\})$ is the set of all partitions $\{a_1, \ldots, a_k\}$ of $\Pi_n$, where $a_i$ is the partition with block $B_i$ and all other elements as singleton blocks. For a chain $C$, the forest $\phi(C)$ given by Proposition 15 is the set of all partitions in $\Pi_n$ with one non-singleton block, such that the non-singleton block is a block of a partition in $C$. For each chain $C$ of $\Pi_n$, then, there is a clear bijection between the forest $\phi(C)$ and the tree $T(C)$, since the non-singleton blocks represented by the internal vertices of $T(C)$ are exactly the non-singleton blocks in the elements of the forest $\phi(C)$. The refinement order on $\Pi_n$ inherited by $\phi(C)$ is the same as the ordering of the blocks in $T(C)$ by inclusion.

The indecomposable $P \in \mathcal{P}$ are the members of the equivalence classes of the partition lattices $\Pi_n$ for all $n$. Each upper interval $[\rho, \hat{1}]$ in a partition lattice is equivalent to $\Pi_{|\rho|}$, and so $\mathcal{P}$ is an upper-indecomposable family, and thus the forest computation of $\chi(P)$ is cancellation-free for all $\Pi_n$. Haiman and Schmitt proved that the Lagrange inversion formula is equivalent to this antipode formula.

7 Forest formula for the antipode for hereditary families of posets

The antipode formula in Theorem 17 for hereditary families of lattices can be extended to a formula for the antipode of the Hopf algebra of any hereditary family of posets. Clearly, the second condition of Definition 13 cannot be applied to general posets, since general posets lack a join operation. A poset $P$ with non-overlapping indecomposable lower intervals $[0, a]$ and $[0, b]$ might have several elements $c > a, b$ such that $[0, a][0, b] \sim [0, c]$. Note that the converse, however, is not true. Any decomposable lower interval of a poset has a unique factorization as a product of indecomposable lower intervals.

We introduce a new definition of forest.

Definition 23. A forest of a poset $P \in \mathcal{P}$ is an ordered pair $(F, J_F)$ where $F \subseteq I(P) \setminus \{0, \hat{1}\}$ and $J_F : 2^F \to P \setminus \{\hat{1}\}$ such that:

1. If $a_1, a_2 \in F$, then either $a_1 \leq a_2$; $a_2 \leq a_1$; or $[0, a_1] \cap [0, a_2] = \{0\}$. 

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2. If \( G \subseteq F \) and \( \{a_i\} \) is the set of maximal elements of \( G \), then \([\emptyset, J_F(G)] \sim \prod [\emptyset, a_i]\).

3. If \( G' \subseteq G \subseteq F \), then \( J_F(G') \leq J_F(G) \).

Note that a single set \( F \) can have several different \( J_F \) functions which each satisfy this definition. Each of these \((F, J_F)\) pairs is regarded as a unique forest. If \( P \) is a lattice, however, then Conditions (2) and (3) of Definition 23 guarantee that \( J_F \) must be the join operation, and so Definition 13 can be seen as a special case of Definition 23.

Note that Conditions (1) and (2) of Definition 23 are essentially the same as the conditions of Definition 13. The join operation of a lattice always satisfies Condition (3) of Definition 23. It is possible, however, for a non-lattice finite \( G \)-colored partition poset to have a subset \( F \) and a function \( J : 2^F \rightarrow P \) such that \( F \) and \( J \) satisfy the first two conditions of Definition 23 but not the third, and so the third condition is not superfluous.

### 7.1 Motivating example: the \( N \)-colored Faà di Bruno Hopf algebra

We generalize the Faà di Bruno Hopf algebra described in Example 4. Following the example of [1], let \( N \in \mathbb{N} \). An \( N \)-colored set is a finite set \( X \) with a map \( \theta = \theta_X : X \rightarrow \{1, \ldots, N\} \), with \( \theta(x) \) called the color of \( x \). Let \( X_r = \{ x \in X | \theta(x) = r \} \), and if \( X \) is an \( N \)-colored set, let \( |X| \) be the vector \((|X_1|, \ldots, |X_N|)\).

An \( N \)-colored partition of an \( N \)-colored set \( X \) is a partition \( \pi \) of \( X \) such that each block of \( \pi \) is assigned a color and, if \( \{x\} \) is a singleton block of \( \pi \), then \( \theta_{\pi}(\{x\}) = \theta_X(x) \).

The poset \( \Pi_n \) of \( N \)-colored partitions of an \( N \)-colored set \( X \) with \( |X| = n = (n_1, \ldots, n_N) \) is formed by letting \( \pi \leq \rho \) if \( \pi \leq \rho \) in the refinement order of the partition lattice and, if \( B \) is a block of both \( \pi \) and \( \rho \), then \( \theta_{\pi}(B) = \theta_{\rho}(B) \). This poset has a \( \emptyset \) but it does not have a \( 1 \); the partition with a single block can occur with any of \( N \) colors. Let \( \Pi_n^\circ \) be the poset of \( N \)-colored partitions with all maximal elements except the one colored \( r \) deleted.

Let \( \mathfrak{S}^N \) be the family of all \( \Pi_n^\circ \) of \( N \)-colored partitions posets of \( N \)-colored sets. If \( \pi \leq \rho \) in some \( N \)-colored partition poset, the let \( \rho | \pi \) be the \( N \)-colored partition of the set \( \pi \) induced by the colors of the unions of the blocks of \( \pi \) in \( \rho \). Define the relation \( \sim \) as color-isomorphism: \([\pi, \rho] \sim [\pi', \rho']\) if \([\pi, \rho] \cong [\pi', \rho']\) and there is a bijection \( B \rightarrow B' \) from the non-singleton blocks of \( \rho | \pi \) to the non-singleton blocks of \( \rho' | \pi' \) such that

1. If \( B \) is the union of \( i \) blocks of \( \pi \), then \( B' \) is the union of \( i \) blocks of \( \pi' \);
2. \( \theta_{\rho | \pi}(B) = \theta_{\rho' | \pi'}(B') \);
3. For each non-singleton block \( B \) of \( \rho | \pi \), there is a bijection \( \varphi \) from the blocks of \( \pi \) in \( B \) to the blocks of \( \pi' \) in \( B' \) such that, if \( \beta \) is a block of \( \pi \) in \( B \), then \( \theta_{\pi}(\beta) = \theta_{\pi'}(\varphi(\beta)) \).
For example, if we let subscripts denote the color of an element or block, then
\[
\begin{align*}
[1, 2] / [3, 4] / [5, 1], \quad (13) / 2 / (12) \quad \sim \quad [1, 2] / [3, 4] / [5, 1], \quad (13) / 2 / (12) \\
\end{align*}
\]

Let the product on the equivalence classes be defined so that
\[
\pi, \rho \sim \prod_{B \in \rho} \Pi_{\Pi(B)}^{\#\Pi(B)}.
\]

For example,
\[
\begin{align*}
[1, 2] / [3, 4] / [5, 1], \quad (12) / 2 / (13) \\
\sim \quad [1, 2] / [3, 4] / [5, 1], \quad (12) / 2 / (13) \\
\end{align*}
\]

It is straightforward to verify that the family of all finite products of the elements of \( N \) is a hereditary family. The incidence algebra of \( N \) is known as the \( N \)-colored Faà di Bruno Hopf algebra.

As in the Faà di Bruno Hopf algebra, if \( P = \Pi_n \in \mathfrak{F}^N \), then \( I(P) \) will be the set of all \( \pi \in P \) such that \( \pi \) has exactly one non-singleton block. Although the posets in \( \mathfrak{F}^N \) are not lattices, there is a unique choice of \( J_F \) for each possible forest set \( F \). If \( P = \Pi_n \), then, as in the Faà di Bruno Hopf algebra, a forest \((F, J_F)\) must be a set of partitions in \( P \) such that each partition has exactly one non-singleton block and, if \( \pi, \rho \in F \) such that \( B_\pi, B_\rho \) are their respective non-singleton blocks, then either \( B_\pi \subseteq B_\rho \), \( B_\rho \subseteq B_\pi \), or \( B_\pi \cap B_\rho = \emptyset \). The only possible map \( J_F \) which satisfies Condition (2) of Definition 23 is that in which \( J_F(G) \) is the \( N \)-colored partition whose non-singleton blocks are exactly the non-singleton blocks of the maximal elements of \( G \), with the same colors.

### 7.2 Forest antipode formula for hereditary families of posets

Using arguments similar to the proofs in Section 5, we find a forest formula for the antipode of incidence Hopf algebras of hereditary families of posets.

**Definition 24.** Let \( P \in \mathcal{P} \), let \((F, J_F)\) be a forest of \( P \), and let \( a \in F \). Then let \( \hat{a} = J_F(\{p(a)\}) \), where \( \{p(a)\} \) is the set of all predecessors of \( a \) in \( F \). As in Definition 14, we let \( \Theta(F, J_F) = \prod_{a \in F \cup Z'(P)} [\hat{a}, a] \).

**Theorem 25.** Let \( \mathcal{P} \) be a hereditary family of posets. A formula for the antipode in \( H(\mathcal{P}) \) is
\[
\chi(P) = \sum_{(F, J_F) \in \mathcal{F}(P)} (-1)^d(F) \Theta(F, J_F)
\]
for all \( P \in \mathcal{P} \).
Proof. First, as in Proposition 15 we find a surjection \( \phi : C(P) \to F(P) \). As in Proposition 15, if \( C \) is a chain of \( P \), then let
\[
F_C = \phi(C) := \bigcup_{i=1}^{\ell(C)-1} Z'(\hat{0}, c_i)
\]

Next, we define \( J_{F_C} \) by induction. First, \( J_{F_C}(\emptyset) = \hat{0} \), and, if \( a \in F_C \), then \( J_{F_C}(\{a\}) = a \).

Now, let \( \{b_i\}_{i=1}^n \) be an antichain in \( F_C \). For each \( 1 \leq i \leq n \), let \( c_k \) be the minimal chain element such that \( b_i \in Z'(\hat{0}, c_k) \). Assume \( k_n \geq k_i \) for all \( i \). Assume by induction that we have defined \( J_{F_C}(\{b_i\}_{i=1}^{n-1}) \), and that \( J_{F_C}(\{b_i\}_{i=1}^{n-1}) \leq c_{k_n-1} \leq c_{k_n} \). Since \( b_n \in Z'(\hat{0}, c_{k_n}) \), we know by Lemma 3 that \( b_n \lor J_{F_C}(\{b_i\}_{i=1}^{n-1}) \) is defined in \([\hat{0}, c_{k_n}]\), and so let \( J_{F_C}(\{b_i\}_{i=1}^n) = b_n \lor J_{F_C}(\{b_i\}_{i=1}^{n-1}) \) in \([\hat{0}, c_{k_n}]\). Since join is commutative and associative, \( J_{F_C}(\{b_i\}_{i=1}^n) \) is well-defined.

We can regard the elements of \( F_C \), together with the image of \( J_{F_C} \), as a subposet of \( P \). In fact, if \( G \) is a subset of \( F_C \), then \( J_{F_C}(G) \) is a minimal upper bound of \( G \). We can thus regard \( F_C \) together with the image of \( J_{F_C} \) as a “sublattice” of \( P \), in which joins in the sublattice correspond to \( J_{F_C} \) in \( P \). The proofs of Propositions 15 and 16 and Theorem 17 can thus be easily modified from the lattice case to the general poset case, completing the proof.

The conditions for non-cancellation in a hereditary family of lattices similarly generalize to families of posets.

**Theorem 26.** The antipode calculation in Theorem 25 is cancellation-free for all indecomposable \( P \) in a hereditary family of posets \( \mathcal{P} \) if and only if \( \mathcal{P} \) is upper-indecomposable.

**Example 5** In [2], Chapoton and Livernet define a hereditary family of posets from an set-operad \( P \). If \( \mathcal{P} \) is a set-operad, then \( \Pi_{\mathcal{P}} \) is the species \( \text{Comm} \circ \mathcal{P} \), where \( \text{Comm} \) is the species that maps a finite set \( I \) to the singleton \( \{I\} \). For each finite set \( I \), they introduce a partial order on \( \Pi_{\mathcal{P}}(I) \). Proposition 3.4 of [2] proves that, for each set-operad \( \mathcal{P} \), the family of all \( \Pi_{\mathcal{P}}(I) \) is a hereditary family. Proposition 3.3 shows that this family satisfies our upper-indecomposable condition. Therefore, the computation of \( \chi \) given by Theorem 25 is cancellation-free for the incidence Hopf algebra of this family.

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