Entropy and Grand Lebesgue Spaces approach for Prokhorov-Skorokhod continuity of random processes, with tail estimates.

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We present in this paper a new sufficient condition for the so-called Prokhorov-Skorokhod continuity of random processes. Our conditions will be formulated in the terms of metric entropy generated by three-dimensional distribution of the considered random process (r.p.) in the parametric set, have a convenient and closed form, and generalize some previous results.

We study also the conditions for weak compactness of the sequence of random processes in this space and as a consequence the Central Limit Theorem.

Our consideration based on the theory of Prokhorov-Skorokhod spaces of random processes and on the theory of Banach spaces of random variables with exponential decreasing tails of distributions, namely, on the theory of Grand Lebesgue Spaces (GLS) of random variables.

Key words and phrases: Random variable and random vector (r.v.), random processes (r.p.), moment generating function, Cadlag functions, rearrangement invariant Banach spaces on random variables, ordinary and exponential moments, covering numbers and metric entropy, Grand Lebesgue Spaces (GLS), weak compactness and Central Limit Theorem (CLT), and module of continuity in the Prokhorov-Skorokhod space, gradual and exponential decreasing for tail of distribution.

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1 Introduction. Previous works.

Let \( f = f(t), \ t \in [0,1] \) be real (or complex) valued measurable function. Recall that the Prokhorov-Skorokhod module \( \kappa[f](\delta) \) for the function \( f(\cdot) \) at the point \( \delta, \ \delta \in [0,1] \) is defined as follows:
\[
\kappa[f](\delta) \overset{def}{=} \sup\{\min |f(t) - f(t_1)|, |f(t_2) - f(t)| : 0 \leq t_1 \leq t \leq t_2 \leq t_1 + \delta \leq 1\}.
\]

By definition, the function \( f : [0, 1] \to \mathbb{R} \) belongs to the Prokhorov-Skorokhod space \( D[0, 1] \) iff

\[
\lim_{\delta \to 0^+} \kappa[f](\delta) = 0
\]

and in addition

\[
\lim_{t \to 0^+} f(t) = f(0), \quad \lim_{t \to 1^-} f(t) = f(1) -
\]

unilateral continuity at both the boundary points \( t = 0 \) and \( t = 1 \).

Other name: "Cadlag" functions. These functions have in each point left and right limits; on the other words, without the points of discontinuity of a second kinds, as in the classical book of I.I.Gikhman and A.V.Skorokhod, see [20], chapters 4, 9.

We will take as ordinary that these functions are right continuous.

There are (as a minimum) two version of the relation (1.3) in the case when \( f(t) \) is a random process: \( f(t) = \xi(t), \ t \in [0, 1] \):

\[
\lim_{t \to 0^+} \mathbb{E} \arctan |\xi(t) - \xi(0)| = 0, \quad \lim_{t \to 1^-} \mathbb{E} \arctan |\xi(t) - \xi(1)| = 0,
\]

convergence in probability;

\[
\mathbb{P}(\lim_{t \to 0^+} \xi(t) = \xi(0)) = 1, \quad \mathbb{P}(\lim_{t \to 1^-} \xi(t) = \xi(1)) = 1,
\]

convergence almost everywhere.

It is known that the function \( f : [0, 1] \to \mathbb{R} \) belongs to the space \( D[0, 1] \) iff it is right continuous, has a left limits in each interior point, \( f(0+) = f(0), \ f(1^-) = f(1) \). For instance, the trajectories of separable square integrable martingales and empirical function of distribution are elements of this space with probability one.

The theory of this spaces is thoroughly outlined in the source articles and famous books [4], [5], [20], [25], [37], [40]; see also [2], [6], [35], [39] and so one.

In particular, this space is complete metrizable relative appropriate distance and is herewith separable.

Let a separable numerical valued random process \( \xi = \xi(t), \ t \in [0, 1] \) be given; for example, is given its consistent family of finite-dimensional distributions.
Our target in this article is finding of simple closed sufficient conditions in the entropy and Grand Lebesgue Spaces (GLS) terms for belonging of almost all paths of considered random process to the Prokhorov-Skorokhod space:

\[ P(\xi(\cdot) \in D[0,1]) = 1 \]  

and deriving the non-asymptotical estimates for tail distribution of the Prokhorov-Skorokhod module

\[ T_{\Delta[\xi]}(u) = P(\Delta[\xi] > u). \]  

We recall the needed definitions further, in the second and third sections.

The exact exponential non-asymptotical estimates for tail distribution of an uniform norm for the random field \( \xi(\cdot) \)

\[ T_{||\xi||}(u) := P(\sup_{t \in [0,1]} |\xi(t)| > u), \quad u \geq 1 \]  

are obtained, e.g. in the authors preprint [29].

Note that in contradistinction to the tail estimate in (1.5), the estimate in (1.5a) is important not only at \( u \to \infty \), but also at \( u \to 0 + \).

We will touch also briefly on the topic of weak compactness of the sequence \( \xi_n(\cdot) \) of random processes in the space \( D[0,1] \) with application to the Central Limit Theorem (CLT).

We will mention aside from the well-known applications of the considered here problem in the theory of martingales and in the non-parametrical statistics also very interest applications, indeed: in physics [7], [11], and in the Monte-Carlo method [21] by computation of integrals from discontinuous integrand functions.

2 Auxiliary notions and facts.

We present here for beginning some known facts from the theory of one-dimensional random variables with exponential decreasing tails of distributions, see [26], [27], chapters 1,2.

Especially we menton the authors preprints [28], [29]; we offer in comparison with existing there results a more fine approach.

Let \((\Omega, F, P)\) be a probability space, \( \Omega = \{\omega\} \).

Let also \( \phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \quad \lambda_0 = \text{const} \in (0, \infty) \) be certain even strong convex which takes positive values for positive arguments twice continuous differentiable function, briefly: Young - Orlicz function, such that

\( \phi(0) = \phi'(0) = 0, \quad \phi''(0) > 0, \quad \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty. \)  

(2.1)
For instance: $\phi(\lambda) = 0.5\lambda^2$, $\lambda_0 = \infty$.

We denote the set of all these Young-Orlicz function as $\Phi$: $\Phi = \{\phi(\cdot)\}$.

We say by definition that the centered random variable (r.v) $\xi = \xi(\omega)$ belongs to the space $B(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \max_{\pm} \mathbb{E} \exp(\pm \lambda \xi) \leq \exp[\phi(\lambda \tau)].$$

(2.2)

Obviously, this condition is quite equivalently to the well-known Kramer’s condition

$$\exists \mu = \text{const} > 0 \Rightarrow \max(\mathbb{P}(\xi > x), \mathbb{P}(\xi < -x)) \leq \exp(-\mu x), x > 0.$$ (2.2a)

The minimal non-negative value $\tau$ satisfying (2.2) for all the values $\lambda \in (-\lambda_0, \lambda_0)$, is named a $B(\phi)$ norm of the variable $\xi$, write

$$||\xi||_{B(\phi)} \overset{def}{=} \inf\{\tau, \tau > 0 : \forall \lambda : |\lambda| < \lambda_0 \Rightarrow \max_{\pm} \mathbb{E} \exp(\pm \lambda \xi) \leq \exp[\phi(\lambda \tau)]\}.$$ (2.3)

These spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $|| \cdot ||_{B(\phi)}$ and ordinary algebraic operations is a rearrangement invariant Banach space which is isomorphic to the subspace consisting on all the centered variables of Orlicz’s space $(\Omega, F, \mathbb{P}), N(\cdot)$ with $N$ – function

$$N(u) = \exp \phi^*(u) - 1, \quad \phi^*(u) \overset{def}{=} \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \rightarrow \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [26], [27], p. 19-40:

$$\xi \in B(\phi) \iff \mathbb{E} \xi = 0, \text{ and } \exists C = \text{const} > 0,$$

$$U(\xi, x) \leq \exp(-\phi^*(C x)), x \geq 0,$$ (2.4)

where $U(\xi, x)$ denotes in this section the one-dimensional tail of distribution of the r.v. $\xi$:
\[ U(\xi, x) = \max (P(\xi > x), P(\xi < -x)), \quad x \geq 0, \]

and this estimation is in general case asymptotically as \( x \to \infty \) exact.

Here and further \( C, C_j, C(i) \) will denote the non-essentially positive finite “constructive” constants; \( f^{-1}(\cdot) \) denotes the inverse function to the function \( f \) on the left-side half-line \((C, \infty)\).

Let \( F = \{\xi(t)\}, \ t \in T, \ T \) is an arbitrary set, be the family of somehow dependent mean zero random variables. The function \( \phi(\cdot) \) may be constructive introduced by the formula

\[ \phi(\lambda) = \phi_F(\lambda) \overset{\text{def}}{=} \max \log \sup_{t \in T} \mathbb{E} \exp(\pm \lambda \xi(t)), \quad (2.5) \]

if obviously the family \( F \) of the centered r.v. \( \{\xi(t), \ t \in T\} \) satisfies the so-called uniform Kramer’s condition:

\[ \exists \mu \in (0, \infty), \ \sup_{t \in T} U(\xi(t), x) \leq \exp(-\mu x), \quad x \geq 0. \]

In this case, i.e. in the case the choice the function \( \phi(\cdot) \) by the formula (2.5), we will call the function \( \phi(\lambda) = \phi_0(\lambda) \) a natural function, and correspondingly the function

\[ \lambda \to \mathbb{E} e^{\lambda \xi} \]

is named often as a moment generating function for the r.v. \( \xi \), if of course there exists in some non-trivial neighborhood of origin.

Further, define the function \( \psi(p) = p / \phi^{-1}(p), \ p \geq 2 \).

Let us introduce a new norm, the so-called moment norm, or equally Grand Lebesgue Space (GLS) norm, on the set of r.v. defined in our probability space by the following way: the space \( G(\psi) \) consist, by definition, on all the centered (mean zero) r.v. with finite norm

\[ \| \xi \|_{G(\psi)} \overset{\text{def}}{=} \sup_{p \geq 1} [\| \xi \|_p / \psi(p)], \quad (2.6) \]

here and in what follows as ordinary

\[ \| \xi \|_p := \mathbb{E}^{1/p} |\xi|^p = \left[ \int_{\Omega} |\xi(\omega)|^p \, \mathbb{P}(d\omega) \right]^{1/p}. \]

It is proved that the spaces \( B(\phi) \) and \( G(\psi) \) coincides: \( B(\phi) = G(\psi) \) (set equality) and both the norms \( \| \cdot \|_{B(\phi)} \) and \( \| \cdot \| \) are linear equivalent: \( \exists C_1 = C_1(\phi), C_2 = C_2(\phi) = \text{const} \in (0, \infty), \ \forall \xi \in B(\phi) \Rightarrow \)

\[ \| \xi \|_{G(\psi)} \leq C_1 \| \xi \|_{B(\phi)} \leq C_2 \| \xi \|_{G(\psi)}. \quad (2.7) \]

In particular, let \( \eta \) be a numerical mean zero r.v. and let \( m = \text{const} > 1 \). The following assertions are equivalent:
\[ A. \exists C_1 \in (0, \infty) \Rightarrow U(\eta, x) \leq \exp(-C_1 x^m), \ x \geq 0. \]  
\[ (2.8a) \]

\[ B. \sup_{p \geq 1} \left[ \frac{|\eta|^p}{p^{1/m}} \right] < \infty. \]  
\[ (2.8b) \]

\[ C. \exists C_2 \in (0, \infty) \Rightarrow \mathbf{E} \exp(\lambda \eta) \leq \exp \left( C_2 |\lambda|^{m/(m-1)} \right), \ |\lambda| \geq 1. \]  
\[ (2.8c) \]

The definition (2.6) may be extended as follows. Recently, see [17], [18], [23], [26], [27], chapters 1,2; [29], [34] appears the so-called Grand Lebesgue Spaces (GLS) \( G(\psi) = G(\psi; b) \) spaces consisting on all the measurable functions (random variables) \( \xi : \Omega \to R \) with finite norms

\[ ||\xi||_{G(\psi) \overset{def}{=} \sup_{p \in (1,b)} [||\xi|| p / \psi(p)]}. \]

Here \( \psi(\cdot) \) is some continuous positive on the open interval \( (1, b) \), \( b = \text{const} \in (1, \infty) \) function such that

\[ \inf_{p \in (1,b)} \psi(p) > 0. \]

It is evident that \( G(\psi; b) \) is a rearrangement invariant space.

Let now \( l \) be arbitrary number from the interval \( [1, \infty) \). We define a so-called degenerate GLS as follows. Put

\[ \psi_l(p) = \infty, \ p \neq l; \ \psi_l(l) = 1 \]

and define formally const \( l / \infty = 0 \). Then the \( G\psi_l \) norm of arbitrary r.v. \( \xi \) coincides with the classical \( L_l \) its norm:

\[ ||\xi||_{G\psi_l} = \sup_{p \geq 1} \left[ \frac{|\xi|^p}{\psi_l(p)} \right] = |\xi|_l, \]

if of course there exists. Thus, the classical Lebesgue - Riesz spaces \( L_p \) are particular, more precisely, extremal case of Grand Lebesgue Spaces.

These spaces are used, for example, in the theory of probability, theory of PDE, functional analysis, theory of Fourier series, theory of martingales etc.

Let us consider more detail example. The inequality of a form

\[ |\xi|^p \leq C_1 p^{1/m} \log^s p, \ p \geq 2, \ C_1 = \text{const} \in (0, \infty) \]  
\[ (2.9) \]

where \( m = \text{const} > 0, \ s = \text{const} \in R \) is completely equivalent to the tail estimate

\[ U(\xi, x) \leq \exp \left( -C_2(m, C_1) x^m (\ln x)^{-ms} \right), \ x \geq e. \]  
\[ (2.9a) \]

See, for example, [33]; [27], chapter 1.8, theorem 1.8.1.
The theory of multidimensional $B(\phi) = B(\vec{\phi})$ spaces and correspondingly of multidimensional $G(\psi) = G(\vec{\psi})$ ones is represented in the recent articles [28], [29]; it is quite analogous to the explained one.

3 Main results.

Let us return now to the introduced random processes, say $\xi(t)$, $t \in [0, 1]$. Define for this r.p. and for the fixed (non-random) triplet of numbers $(r, s, t)$ for which $0 \leq r \leq s \leq t \leq 1$

$$\delta[\xi](r, s, t) \overset{\text{def}}{=} \min(|\xi(s) - \xi(r)|, |\xi(t) - \xi(s)|), \quad (3.1)$$

We will denote the set of all such a triplets by $R$:

$$R = \{(r, s, t) : 0 \leq r \leq s \leq t \leq 1\}$$

and define for the fixed value $s \in (0, 1)$ the appropriate set

$$R(s) = \{r, t : 0 \leq r \leq s, \ s \leq t \leq 1 \}.$$

Define also

$$\Delta[\xi] \overset{\text{def}}{=} \sup_{s \in (0, 1)} \sup_{(r, t) \in R(s)} \delta[\xi](r, s, t). \quad (3.2)$$

**Proposition 3.0.** If (in our notations)

$$\sup_{s \in (0, 1)} \mathbb{P}(\delta[\xi](r, s, t) > u) \leq [G(t) - G(r)]^\alpha u^{-2\beta}, \quad (3.3)$$

where $\alpha = \text{const} > 1$, $\beta = \text{const} > 0$, $u = \text{const} > 0$, and $G : [0, 1] \to R$ is continuous increasing deterministic function, then

$$\mathbb{P}(\Delta[\xi] > u) \leq K(\alpha, \beta) u^{-2\beta} [G(1) - G(0)]^\alpha, \ K(\alpha, \beta) < \infty \quad (3.4)$$

and correspondingly

$$\mathbb{P}(\kappa[\xi](h) > u) \leq 2 K(\alpha, \beta) u^{-2\beta} [G(1) - G(0)]^\alpha (\omega[G](2h))^{\alpha-1}, \quad (3.5)$$

where $\omega[G](h)$ is ordinary module of continuity of the function $G(\cdot)$:

$$\omega[G](h) \overset{\text{def}}{=} \sup\{|G(t) - G(r)| : r, t \in [0, 1], |r - t| \leq h\}, \ h \in [0, 1].$$

If in addition the r.p. $\xi(t)$ satisfies the "boundary" conditions (1.3a), then almost all the trajectories of $\xi(t)$ belongs to the space $D[0, 1]$.

This statement is proved, for example, in the preprint [39], lemma 7.15; see also [4], chapters 2,3.
Remark 3.1. We will repeatedly apply the following obvious extension of proposition 3.0. Suppose the inequality (3.3) is true for some set of the values $(\alpha, \beta)$ for which

$$D \subset \{(\alpha, \beta) : \alpha > 1, \beta > 0\},$$

with at the same function $G(\cdot)$. Then

$$P(\Delta[\xi] > u) \leq \inf_{(\alpha,\beta) \in D} \left\{ K(\alpha, \beta) \ u^{-2\beta} [G(1) - G(0)]^\alpha \right\}, \quad (3.4a)$$

$$P(\kappa[\xi](h) > u) \leq 2 \inf_{(\alpha,\beta) \in D} \left\{ K(\alpha, \beta) \ u^{-2\beta} [G(1) - G(0)]^\alpha (\omega[G](2h))^{\alpha-1} \right\}. \quad (3.5a)$$

Of course, one can that the function $G(\cdot)$ also dependent on the parameters $(\alpha, \beta)$ from this set.

Lemma 3.1. The "constant" $K(\alpha, \beta)$ in (3.4), (3.5) allows the following estimate

$$K(\alpha, \beta) \leq \frac{1 - 2^{(1-\alpha)/(4\beta)}}{2^{(\alpha-1)/2} - 1} = \overline{K}(\alpha, \beta). \quad (3.6)$$

Proof. Serik Sagitov in [39] proved that

$$\forall \theta \in \left(2^{(1-\alpha)/(2\beta)}, 1\right) \Rightarrow K(\alpha, \beta) \leq \frac{2^{(1-\alpha)/(2\beta)} \theta^{-2\beta} (1 - \theta)^{-2\beta}}{1 - 2^{1-\alpha}\theta^{-2\beta}}, \quad (3.7)$$

therefore

$$K(\alpha, \beta) \leq \inf_{\theta \in \left(2^{(1-\alpha)/(2\beta)}, 1\right)} \left[ \frac{2^{(1-\alpha)/(2\beta)} \theta^{-2\beta} (1 - \theta)^{-2\beta}}{1 - 2^{1-\alpha}\theta^{-2\beta}} \right]. \quad (3.7a)$$

The required estimate (3.6) can be obtained after substituting

$$\theta_0 = 2^{(1-\alpha)/(4\beta)}$$

into the right-hand side of inequality (3.7). Note that this value $\theta = \theta_0$ is asymptotical as $\alpha \to 1 + 0$ optimal.

Note in addition that as $\alpha \to 1 + 0 \Rightarrow$

$$\overline{K}(\alpha, \beta) \sim 2^{4\beta+1} \beta^{2\beta} (\ln 2)^{-2\beta-1} (\alpha - 1)^{-2\beta-1}.$$  

The claim of this section is obtaining the tails estimated for $\Delta[\xi]$ and $\kappa[\xi](h)$ in the terms of Grand Lebesgue Space norm for $\delta(r, s, t)$ and consequently via generated by its metric entropy.

To be more precise, we introduce the following semi-distance on the set $[0, 1]$ by means of some $\psi -$ function:
\[ \rho_\psi(r, t) \overset{def}{=} \sup_{s \in (0,1)} ||\delta(r, s, t)||G_\psi \] (3.8)

and as a particular case

\[ \rho_{\psi(p)}(r, t) \overset{def}{=} \sup_{s \in (0,1)} |\delta(r, s, t)|_p. \] (3.9)

Herewith \( r < t \) and \( s \in [r, t] \); we define in the case \( t < r \) \( d_p(t, r) := d_p(r, t) \) and \( \rho_\psi(t, r) := \rho_\psi(r, t) \), so that \( d_p(t, r) = d_p(r, t) \) and \( \rho_\psi(t, r) = \rho_\psi(r, t) \). For reasons of continuity \( d_p(t, t) := 0 \) and \( \rho_\psi(t, t) := 0 \).

We intend to deduce in this section the sufficient conditions for the Prokhorov-Skorokhod continuity of the r.p. \( \xi(t) \) in the terms of metric entropy of the set \([0, 1]\) generated by the semi-distance functions \( \rho_\psi(t, r) \) and \( d_p(t, r) \), likewise the problem of natural continuity of the r.p., see e.g. [9], [12], [13], [14], [26], [27], chapter 3, [36].

Recall for reader convenience that the so-called covering numbers \( N(X, q, \epsilon) \) of a (compact) metric space \((X, q)\) relative the semi-distance function \( q = q(x_1, x_2), \ x_1, x_2 \in X \) are defined as a minimal amount of closed balls

\[ B(x_i, q, \epsilon) = B(x_i, \epsilon) \overset{def}{=} \{ y, \ y \in X, q(x_i, y) \leq \epsilon \}, \] (3.10)

which cover all the set \( X \):

\[ N(X, q, \epsilon) = \min\{ N : \exists x_j, \ j = 1, 2, \ldots, N : \cup_{j=1}^{N} B(x_j, q, \epsilon) = X \}. \] (3.11)

The metric entropy \( H(X, q, \epsilon) \) of a (compact) metric space \((X, q)\) is by definition the natural logarithm of the covering number:

\[ H(X, q, \epsilon) := \ln N(X, q, \epsilon). \]

**Remark 3.2.** At the same definition may be used still in the case when the function \( q = q(x_1, x_2) \) is symmetrical, non negative, but does not satisfy in general case the triangle inequality, for example, when

\[ q(t, r) = |t - r|^{\alpha}, \ t, r \in [0, 1], \ \alpha = \text{const} > 1. \]

Denote also \( \Theta = \{ \bar{\theta} \}, \ \epsilon = \{ \bar{\epsilon} \}, \)

\[ \bar{\theta} = \theta = \{ \theta(1), \theta(2), \ldots, \theta(k), \ldots \} : \ \theta(j) > 0, \ \sum_{k=1}^{\infty} \theta(k) = 1; \]

\[ \bar{\epsilon} = \epsilon = \{ \epsilon(1), \epsilon(2), \ldots, \epsilon(k), \ldots \} : \ \epsilon(1) = 1, \ k \rightarrow \infty \Rightarrow \epsilon(k) \downarrow 0. \]

The next fact is a slight and closed generalization of a statement 7.15 in the article [39].
Proposition 3.1. Suppose the considered separable random process $\xi = \xi(t)$, $t \in [0, 1]$ is such that for some symmetrical and non negative numerical function $q = q(t, r)$, $t, r \in (0, 1)$ and for strictly increasing positive numerical function $\lambda = \lambda(u)$, $u > 0$ for which

$$\lim_{u \to \infty} \lambda(u) = \infty$$

there holds

$$\sup_{s \in (0, 1)} \sup_{(r, t) \in R(s)} \mathbb{P} (\delta[\xi](r, s, t) > u) \leq \frac{q(r, t)}{\lambda(u)}. \quad (3.12)$$

Then

$$\mathbb{P}(\Delta[\xi] > 2u) \leq Q(q(\cdot), \lambda(\cdot); u), \quad (3.13a)$$

where

$$Q(q(\cdot), \lambda(\cdot); u) \stackrel{def}{=} \inf_{\{\epsilon(k)\} \in \{\theta(k)\} \in \Theta} \inf_{k=1}^\infty N([0, 1], q, \epsilon(k + 1)) \cdot \frac{\epsilon(k)}{\lambda(u \cdot \theta(k))}. \quad (3.13b)$$

if of course the right-hand side of estimate (3.13b) tends to zero as $u \to \infty$

Denote also

$$\sigma[q](h) = h^{-1} \sup_{(r, t) : |r-t| \leq 2h} q(r, t).$$

It is assumed that

$$\lim_{h \to 0^+} \sigma[q](h) = 0.$$

We assert also under these conditions

$$\mathbb{P}(\kappa[\xi](h) > u) \leq Q(q(\cdot), \lambda(\cdot); u) \cdot \sigma[q](h).$$

If in addition the r.p. $\xi(t)$ is continuous at the extremal points $t = 0$, $t = 1$ in the sense (1.3a), then the random process $\xi(\cdot)$ belongs to the Prokhorov-Skorokhod space $D[0, 1]$ with probability one.

Example 3.1. Suppose $\lambda(u) = u^{2\beta}$, $\beta = \text{const} > 0$. The conditions of proposition 3.1. are satisfied if $N([0, 1], q, \epsilon) \leq C \epsilon^{-\gamma}$, $\epsilon \in (0, 1)$, where $\gamma = \text{const} < 1$.

Namely, it is sufficient to choose in (3.13) the values

$$\epsilon(k) = s^{k-1}, \quad \theta(k) = (1-\theta) \theta^k,$$

where

$$0 < s, \theta < 1, \quad s^{1-\gamma} < \theta^{2\beta},$$

wherein
\[
P(\Delta[\xi] > 2u) \leq \frac{L(\beta, \gamma, \theta, C, s)}{u^{2\beta}}, \quad u > 0.
\]

**Example 3.2.** Suppose alike the example 3.1 \(\lambda(u) = u^{2\beta}, \quad \beta = \text{const} > 0\). We retain also the condition (3.12). But we suppose now

\[
N([0, 1], q, \epsilon) \leq C \epsilon^{-1} |\ln \epsilon|^{-\gamma_1}, \quad \epsilon \in (0, 1/e),
\]

where \(\gamma_1 = \text{const} > 1\). If we choose in the estimate (3.13b)

\[
\theta(k) = C(\nu) k^{-\nu}, \quad \nu = \text{const} > \max(1, (\gamma_1 - 1)/(2\beta))
\]

and \(\epsilon(k) = \exp(-k + 1)\), then there holds as before

\[
P(\Delta[\xi] > 2u) \leq \frac{L_1(\beta, \gamma_1, C, \nu)}{u^{2\beta}}, \quad u > 0.
\]

The last example does not be obtained from the proposition 7.15 in [39].

4 **Constructive building of distance function. Grand Lebesgue Spaces approach.**

We discuss in this section the possibility of constructive building of the functions \(q(\cdot, \cdot), \lambda(\cdot)\) for the condition (3.12) in the terms of the source random process \(\xi = \xi(t)\) and following through the random process (field) \(\delta = \delta(r, s, t)\).

We recall for beginning the analogous natural approach for the problem of continuity relative appropriate distance function and finding of the tail estimation for the maximum distribution for the random process (field) \(\eta(t), \quad t \in T\), where \(T\) is arbitrary set, see [14], [26], [27], chapters 3,4; [36] etc.

Introduce the natural function for the r.p. \(\eta(t)\)

\[
\gamma(p) := \sup_{t \in T} |\eta(t)|_p,
\]

and suppose its finiteness at last for one value \(p\) greatest than one. Then \(\gamma(\cdot)\) is some \(\psi -\) function and one can to construct the semi-distance function \(d_\gamma(r, t)\) also by natural way

\[
d_\gamma(r, t) := ||\eta(r) - \eta(t)||_{G\gamma},
\]

or equivalently

\[
\tilde{d}(r, t) := ||\eta(r) - \eta(t)||_{B\phi}
\]

with appropriate Young-Orlicz exponential type function \(\phi = \phi(\lambda)\).
For instance, in $\eta(t)$ is (separable) Gaussian random field, then one can choose $\phi(\lambda) = 0.5 \lambda^2$ and

$$\bar{d}(r, t) = \sqrt{\text{Var}(\eta(r) - \eta(t))}$$

is the so-called Dudley’s distance.

The majority of results in described above problem were obtained in the terms of metric entropy $H(S, d_\gamma, \epsilon)$ of arbitrary subsets $S \subset T$ generated by the distance $d_\gamma(r, t)$.

This approach was introduced by R.M.Dudley, see e.g. [12], and X.Fernique [13] - [16]; it was applied in particular to the problem of Central Limit Theorem in the space of continuous functions.

An another approach is closely related with the too modern notion ”majorizing measures”, see [13] - [16], [34], [41] - [44].

We return now to the formulated above problem for (discontinuous, in general case) random process $\xi = \xi(t)$. Let us introduce the natural function for the r.p.

$$\nu(p) := \sup_{s \in (0, 1)} \sup_{(r, t) \in R(s)} \| \delta(r, s, t) \|_p,$$

and suppose its finiteness for some value $p_0 > 2$. It is not excluded herewith that $\nu(b) < \infty$, where as before $b = \sup\{p, \nu(p) < \infty\}$, if of course $2 < b < \infty$.

The function $\nu = \nu(p)$ belongs to the set $G\Psi = G\Psi(b)$ and herewith

$$\sup_{s \in (0, 1)} \sup_{(r, t) \in R(s)} \| \delta(r, s, t) \|_{G\nu} = 1.$$  \quad (4.2)

The pseudo - distance $w = w(r, t)$ may be defined by the formula

$$w(r, t) := \sup_{s \in (0, 1)} \| \delta(r, s, t) \|_{G\nu}, \; 0 \leq r < t \leq 1,$$

and we put by definition

$$w(r, t) := w(t, r), \; r > t; \; \; w(r, r) = 0; \quad (4.3a)$$

so that

$$\sup_{s \in (0, 1)} \| \delta(r, s, t) \|_p \leq w(r, t) \nu(p)$$

and by virtue of Tchebychev’s inequality

$$\sup_{s \in (0, 1)} \mathbf{P}(\delta(r, s, t) > u) \leq \frac{\nu^p(p) \cdot w^p(r, t)}{u^p}, \; u > 0, \; p \in [2, b].$$

We intend now to use the statement of proposition (3.0) taking into account Remark 3.1. Namely, we suppose that the (deterministic!) function $w(r, t)$ is continuous and hence it allows [30] an estimation of a form.
\[ w(r, t) \leq | G(t) - G(r) |, \quad (4.6) \]

where \( G : [0, 1] \to \mathbb{R} \) is some continuous increasing deterministic function. We can and will suppose without loss of generality \( G(0) = G(0^+) = 0 \).

We substitute into inequality (3.3) the values \( \alpha = p, \beta = p/2 \) and recall that here \( p \geq 2 \):

\[
\sup_{s \in (0,1)} P(\delta[\xi](r, s, t) > u) \leq \nu^p(p) [G(t) - G(r)]^p u^{-p}, \quad (4.7)
\]

then in accordance with proposition (3.0)

\[
P(\Delta[\xi] > u) \leq K(p, p/2) \nu^p(p) [G(1)]^p u^{-p}, \quad u > 0. \quad (4.8)
\]

It is easy to estimate

\[
K(p, p/2) \leq 3p, \quad p \geq 2.
\]

So, we obtained in fact the following statement.

**Proposition 4.1.** We deduce under formulated above in this section notations and conditions

\[
P(\Delta[\xi] > u) \leq 3p \nu^p(p) [G(1)]^p u^{-p}, \quad u > 0. \quad (4.9)
\]

As a slight consequence:

\[
P(\Delta[\xi] > u) \leq \inf_{p \in (1, b)} \left\{ 3p \nu^p(p) [G(1)]^p u^{-p} \right\}, \quad u > 0. \quad (4.9a)
\]

Evidently, the last estimations (4.9), (4.9a) are essentially non-improvable.

Let us estimate also in the introduced terms and conditions the Prokhorov-Skorokhod module \( \kappa[\xi](h), \ h \in (0, 1) \). We apply again the inequality (3.5), in which we substitute \( \alpha = p, \beta = p/2 \) and under at the same restriction \( p \geq 2 \):

\[
P(\kappa[\xi](h) > u) \leq 2 K(p, p/2) u^{-p} \nu^p(p) [G(1) - G(0)]^p (\omega[G](2h))^{p-1}. \quad (4.10)
\]

We proved in fact the following estimate.

**Proposition 4.2.** We deduce under formulated above in this section notations and conditions

\[
P(\kappa[\xi](h) > u) \leq 2 \inf_{p \in [2, b]} \left[ 3p u^{-p} \nu^p(p) [G(1) - G(0)]^p (\omega[G](2h))^{p-1} \right]. \quad (4.11)
\]

Evidently, under these conditions

\[
\forall \epsilon > 0 \Rightarrow \lim_{h \to 0^+} P(\kappa[\xi](h) > u) = 0, \quad (4.12)
\]
so that if as before in addition the r.p. \( \xi(t) \) is (unilateral) continuous at the extremal points \( t = 0, t = 1 \) in the sense (1.3a), then the random process \( \xi(\cdot) \) belongs to the Prokhorov-Skorokhod space \( D[0,1] \) with probability one.

We will use the last estimate further, in the seventh section.

**Example 4.1.** Suppose \( b = \infty \) and that

\[
\forall p \in [1, \infty) \Rightarrow \nu(p) \leq C_1 p^m, \quad C_1, m = \text{const} > 0;
\]

then we obtain the following exponential decreasing tail estimates

\[
P(\Delta[\xi] > u) \leq \exp \left( -C_2(m, C_1, G(\cdot)) u^{1/m} \right), \quad u \geq 1,
\]

and correspondingly for the values \( u \geq \left[ \omega[G](2h) \right] \ln \omega[G](2h)]^{-m} \)

\[
P(\kappa[\xi](h) > u) \leq 2 \left[ \omega[G](2h) \right]^{-1} \exp \left( -C_3(m, C_1, G(\cdot)) u^{1/m} \omega[G](2h) \right).
\]

### 5 Moment estimates for tail of minimum distribution.

In order to apply the results of the last section, we need to estimate the tail of distribution of minimum for the set random variables. The exponential ones were received in [28], [29].

Let us consider at first a two-dimensional case \( d = 2 \). Indeed, let \( (\xi, \eta) \) be a two-dimensional random vector. Introduce a so-called binary absolute moment

\[
\nu_{\xi,\eta}(p_1, p_2) = \nu(p_1, p_2) \overset{df}{=} \mathbb{E} |\xi|^{p_1} |\eta|^{p_2}, \quad p_1, p_2 = \text{const} > 0,
\]

and correspondent pseudo-norm

\[
| (\xi, \eta) |_{p_1, p_2} := [\mathbb{E} |\xi|^{p_1} |\eta|^{p_2}]^{1/(p_1 + p_2)} = [\nu_{\xi,\eta}(p_1, p_2)]^{1/(p_1 + p_2)}.
\]

Denote also

\[
D(\xi, \eta) = \{ (p_1, p_2) : \nu_{\xi,\eta}(p_1, p_2) < \infty \},
\]

\[
T_{\xi,\eta}(u, v) = \mathbb{P}(|\xi| > u, |\eta| > v) = \mathbb{P}((|\xi| > u) \cap (|\eta| > v)), \quad u, v > 0
\]

be a two-dimensional tail function for the random vector \( (\xi, \eta) \). We deduce alike the proof of Tchebychev’s inequality

\[
\nu_{\xi,\eta}(p_1, p_2) = \int_{\Omega} |\xi|^{p_1} |\eta|^{p_2} \mathbb{P}(d\omega) \geq
\]
\[
\int_{|\xi| > u, \ |\eta| > v} |\xi|^{p_1} |\eta|^{p_2} \, \mathbf{P}(d\omega) \geq \int_{|\xi| > u, \ |\eta| > v} u^{p_1} v^{p_2} \, \mathbf{P}(d\omega) = u^{p_1} v^{p_2} T_{\xi,\eta}(u, v).
\]

Therefore,
\[
T_{\xi,\eta}(u, v) \leq \frac{\nu_{\xi,\eta}(p_1, p_2)}{u^{p_1} v^{p_2}}; \quad u, v > 0.
\] (5.5)

As a slight consequence: introduce the following set
\[
D = D(\xi, \eta) := \{(p_1, p_2) : \nu_{\xi,\eta}(p_1, p_2) < \infty\};
\]
then
\[
T_{\xi,\eta}(u, v) \leq \inf_{(p_1, p_2) \in D(\xi, \eta)} \left[ \frac{\nu_{\xi,\eta}(p_1, p_2)}{u^{p_1} v^{p_2}} \right]; \quad u, v > 0. \quad (5.5a)
\]

We obtain as a consequence, choosing \(v = u > 0\):
\[
\mathbf{P}(\min(|\xi|, |\eta|) > u) \leq \frac{E |\xi|^{p_1} |\eta|^{p_2}}{u^{p_1+p_2}}. \quad (5.6)
\]

If we take in addition \(p_1 = p_2 = p > 0\), then
\[
\mathbf{P}(\min(|\xi|, |\eta|) > u) \leq \frac{E |\xi \eta|^p}{u^{2p}}. \quad (5.6a)
\]

We can transform the estimate (5.5a) as follows. Let us extend the function \(\nu_{\xi,\eta}(p_1, p_2)\) on the whole plane of the values \((p_1, p_2)\):
\[
\nu_{\xi,\eta}(p_1, p_2) := +\infty, \quad (p_1, p_2) \notin D(\xi, \eta),
\]
then we have for the values \(u > 1, \ v > 1\)
\[
T_{\xi,\eta}(u, v) \leq \inf_{(p_1, p_2) \in \mathbb{R}^2} \left[ \frac{\nu_{\xi,\eta}(p_1, p_2)}{u^{p_1} v^{p_2}} \right] = \\
\inf_{(p_1, p_2) \in \mathbb{R}^2} \exp \left( -p_1 \ln u - p_2 \ln v + \ln \nu_{\xi,\eta}(p_1, p_2) \right) = \\
\exp \left( -\sup_{(p_1, p_2) \in \mathbb{R}^2} \left( p_1 \ln u + p_2 \ln v - \ln \nu_{\xi,\eta}(p_1, p_2) \right) \right) = \\
\exp \left( -(\ln \nu_{\xi,\eta})^*(\ln u, \ln v) \right),
\]
where the notation \(f^*(\cdot, \cdot)\) stands for the two-dimensional Young-Fenchel transform
\[
f^*(\lambda_1, \lambda_2) \overset{\text{def}}{=} \sup_{x_1, x_2 \in \mathbb{R}^2} [x_1 \lambda_1 + x_2 \lambda_2 - f(x_1, x_2)].
\]
We list further some properties of the introduced pseudo-norm.

0. Note first of all that the expression for \(|(\xi, \eta)|_{p_1,p_2}\) does not represent in general case the really norm, still for the values \(p_1 = p_2 = 1\). For instance, the "unit sphere"

\[
S = \{(\xi, \eta) : |(\xi, \eta)|_{p_1,p_2} \leq 1\}
\]

is not convex set.

1. The functional \((\xi, \eta) \mapsto |(\xi, \eta)|_{p_1,p_2}\) is positive homogeneous of a degree 1:

\[|(\lambda \xi, \lambda \eta)|_{p_1,p_2} = |\lambda| \cdot |(\xi, \eta)|_{p_1,p_2}, \quad \lambda = \text{const}.
\]

2. Non-negativity:

\[|(\xi, \eta)|_{p_1,p_2} \geq 0; \quad |(\xi, \eta)|_{p_1,p_2} = 0 \Leftrightarrow \xi \cdot \eta \overset{\text{a.e.}}{=} 0.
\]

The relation \(\xi \cdot \eta \overset{\text{a.e.}}{=} 0\) is named often as a *disjointness* of the random variables \(\xi\) and \(\eta\), especially for indicator functions.

3. A simple estimate.

\[
u_{\xi,\eta}(p_1, p_2) = \mathbb{E}|\xi|^{p_1} |\eta|^{p_2} \leq \left[\mathbb{E}|\xi|^{\alpha p_1}\right]^{1/\alpha} \cdot \left[\mathbb{E}|\eta|^{\beta p_2}\right]^{1/\beta} = |\xi|_{\alpha p_1}^{p_1} \cdot |\eta|_{\beta p_2}^{p_2}.
\]

We used the Hölder’s inequality; here \(\alpha, \beta, p_1, p_2 = \text{const} \geq 1, 1/\alpha + 1/\beta = 1\). Following

\[|(\xi, \eta)|_{p_1,p_2} \leq \inf \left\{ |\xi|_{\alpha p_1}^{p_1/(p_1+p_2)} \cdot |\eta|_{\beta p_2}^{p_2/(p_1+p_2)} : \alpha, \beta \geq 1, 1/\alpha + 1/\beta = 1 \right\}.
\]

For example,

\[|(\xi, \eta)|_{p_1,p_2} \leq |\xi|_{2 p_1}^{p_1/(p_1+p_2)} \cdot |\eta|_{2 p_2}^{p_2/(p_1+p_2)}.
\]

If in addition the r.v. \(\xi, \eta\) are independent, then \(\mathbb{E}|\xi|^{p_1} |\eta|^{p_2} = \mathbb{E}|\xi|^{p_1} \cdot \mathbb{E}|\eta|^{p_2}\) and hence

\[|(\xi, \eta)|_{p_1,p_2} \leq |\xi|_{p_1}^{p_1/(p_1+p_2)} \cdot |\eta|_{p_2}^{p_2/(p_1+p_2)}.
\]

4. Estimation of pseudo-norm for sum of random vectors.

\[p_1, p_2 \geq 1 \Rightarrow |\xi_1 + \xi_2, \eta_1 + \eta_2|_{p_1,p_2} \leq 2^{1-2/(p_1+p_2)} \times
\]

\[
\sup_{\alpha, \beta > 0, 1/\alpha + 1/\beta = 1} \left[ \left( |\xi_1|_{\alpha p_1}^{p_1} + |\xi_2|_{\alpha p_1}^{p_1} \right) \cdot \left( |\eta_1|_{\beta p_2}^{p_2} + |\eta_2|_{\beta p_2}^{p_2} \right) \right]^{1/(p_1+p_2)}.
\]
Proof. Let \( \alpha, \beta > 0, 1/\alpha + 1/\beta = 1 \). We observe applying the last estimate

\[
\nu_{\xi_1+\xi_2, \eta_1+\eta_2}(p_1, p_2) \leq |\xi_1 + \xi_2|_{p_1}^{p_1} \cdot |\eta_1 + \eta_2|_{p_2}^{p_2}.
\]

The triangle inequality for the classical Lebesgue-Riesz spaces \( L(p) \) together with an elementary inequality

\[
(a + b)^{p} \leq 2^{p-1} (a^{p} + b^{p}), \quad a, b > 0, \quad p \geq 1
\]
gives us the required estimate.

For example,

\[
|\xi_1 + \xi_2, \eta_1 + \eta_2|_{p_1, p_2} \leq 2^{1-2/(p_1+p_2)} \times \left[ \left( |\xi_1|_{p_1}^{p_1} + |\xi_2|_{p_1}^{p_1} \right) \cdot \left( |\eta_1|_{p_2}^{p_2} + |\eta_2|_{p_2}^{p_2} \right) \right]^{1/(p_1+p_2)}.
\]

It is not hard to generalize these propositions into the \( d \) - dimensional case. Namely, let \( \vec{\xi} = \{\xi(1), \xi(2), \ldots, \xi(d)\} \) be \( d \) - dimensional random vector and \( \vec{p} = \{p(1), p(2), \ldots, p(d)\}, \vec{u} = \{u(1), u(2), \ldots, u(d)\} \) be \( d \) - dimensional numerical vectors with positive entries. Then the tail function for \( \vec{\xi} \) may be easily estimated as follows

\[
T_{\vec{\xi}}(\vec{u}) \overset{\text{def}}{=} P \left( \bigcap_{j=1}^{d} \{|\xi(j)| > u(j)\} \right) \leq \frac{E \prod_{j=1}^{d} |\xi(j)|^{p(j)}}{\prod_{j=1}^{d} u(j)^{p(j)}}.
\]

In particular,

\[
P(\min_{j} |\xi(j)| > u) \leq \frac{E \prod_{j=1}^{d} |\xi(j)|^{p(j)}}{u \sum_{j} p(j)},
\]

and

\[
P(\min_{j} |\xi(j)| > u) \leq \frac{E \tau^{p}}{u^{d-p}},
\]

where

\[
\tau = \prod_{j=1}^{d} |\xi(j)|.
\]

We note in continuation of this theme. Assume that the r.v. \( \tau \) belongs to some Grand Lebesgue Space \( G\psi_b, b = \text{const} > 1 \); for instance, one can choose \( \psi(\cdot) \) as a natural function for the r.v. \( \tau : \psi(p) := |\tau|^p \). We derive for the values \( p \in [1, b] \)

\[
E \tau^{p} \leq \psi^{p}(p)
\]

and hence for the values \( u \geq 1 \)
\[ P(\min \mid \xi(j) \mid > u) \leq \frac{\psi(p)}{u^d} = \exp \left\{ -dp \ln u + p \ln \psi(p) \right\}, \]

\[ P(\min \mid \xi(j) \mid > u) \leq \inf_{p \in (1, \infty)} \exp \left\{ -dp \ln u + p \ln \psi(p) \right\} = \exp \left[ -\sup_p \{pd \ln u - \psi_1(p)\} \right], \]

where \( \psi_1(p) = p \ln \psi(p), \ p \in [1, b) \) and \( \psi_1(p) = +\infty \) when \( p \not\in [1, b) \).

The last term may be expressed in turn through the Young-Fenchel transform of the function \( \psi_1(p) \), as well:

\[ P(\min \mid \xi(j) \mid > u) \leq \exp \left[ -\psi_1^*(\ln(u^d)) \right], \ u > 1. \]

**Example 5.1: an application.** Let again \( \xi(t), t \in [0, 1] \) be a separable r.p. We deduce

\[ P(\delta(r, s, t) > u) \leq \frac{\mathbb{E}|\xi(r) - \xi(s)|^{p_1} |\xi(s) - \xi(t)|^{p_2}}{u^{p_1+p_2}}, \ p_1, p_2, u > 0, \]

and as a particular case

\[ P(\delta(r, s, t) > u) \leq \frac{\mathbb{E}|\xi(r) - \xi(s)|^p |\xi(s) - \xi(t)|^p}{u^{2p}}, \ p, u > 0. \]

and

\[ P(\delta(r, s, t) > u) \leq \inf_{p_1, p_2 > 0} \left[ \frac{\mathbb{E}|\xi(r) - \xi(s)|^{p_1} |\xi(s) - \xi(t)|^{p_2}}{u^{p_1+p_2}} \right], \ u > 0, \]

\[ P(\delta(r, s, t) > u) \leq \inf_{p > 0} \left[ \frac{\mathbb{E}|\xi(r) - \xi(s)|^p |\xi(s) - \xi(t)|^p}{u^{2p}} \right], \ u > 0. \]

The right-hand side of the proposition (5.11) may be estimated in turn by means of Cauchy’s inequality as follows. Denote by \( d_p(s, t) \) the Pizier’s (semi - ) distance

\[ d_p(s, t) \overset{\text{def}}{=} |\xi(s) - \xi(t)|_p = [\mathbb{E}|\xi(s) - \xi(t)|^p]^{1/p}, \]

then

\[ P(\delta[\xi](r, s, t) > u) \leq u^{-2p} \sqrt{\mathbb{E}|\xi(r) - \xi(s)|^{2p}} \mathbb{E}|\xi(s) - \xi(t)|^{2p} = \frac{d_p^2(r, s) d_p^2(s, t)}{u^{2p}}, \ u > 0. \]
Of course,
\[ P(\delta[\xi](r, s, t) > u) \leq \inf_{p > 0} \left\{ \frac{d_{2p}^p(r, s) \ d_{2p}^p(s, t)}{u^{2p}} \right\}, \ u > 0 \] (5.14)
and following
\[ \sup_{s \in (0, 1)} P(\delta[\xi](r, s, t) > u) \leq \sup_{s \in (0, 1)} \inf_{p > 0} \left\{ \frac{d_{2p}^p(r, s) \ d_{2p}^p(s, t)}{u^{2p}} \right\}, \ u > 0 \] (5.15)

Let us impose the following condition on the r.p. $\xi(\cdot)$:
\[ d_p(r, s) \ d_p(s, t) \leq Z(p) \cdot |V(t) - V(r)|^l, \ l = \text{const} > 1/p \] (5.16)
for some function $Z = Z(p)$ from the set $[2, b)$, $b = \text{const} > 2$ and for some continuous increasing bounded function $V : [0, 1] \to R$ for which $V(0) = 0$. We deduce choosing
\[ \alpha = lp > 1, \ \beta = p, \ G(t) = Z^{1/l}(2p) \ V(t) : \]
\[ P(\kappa[\xi](h) > u) \leq 2 \ K(l \ p, p) \ Z^{1/l}(2p) \ V^l(1) \ u^{-2p} \ \{\omega[Z](2h)\}^{l-1}, \ u > 0, \ h \in [0, 1], \ p \in [2, \min(b, 1/l)] \] (5.17)
and hence $P(\kappa[\xi](h) > u) \leq \inf_{p \in [2, \min(b, 1/l)]} \left[ 2 \ K(l \ p, p) \ Z^{1/l}(2p) \ V^l(1) \ u^{-2p} \ \{\omega[Z](2h)\}^{l-1} \right]$. (5.18)

Note that under formulated above assumptions
\[ \forall \epsilon > 0 \Rightarrow \lim_{h \to 0^+} P(\kappa[\xi](h) > \epsilon) = 0. \] (5.19)

6 About boundary restrictions.

Let’s turn our attention to the condition (1.3) for the random process $\xi(t), \ t \in [0, 1]$. Question: under what conditions (sufficient or necessary or sufficient and necessary conditions) on the distribution on the $\xi(\cdot)$
\[ P(\lim_{t \to 0^+} (\xi(t) - \xi(0)) = 0) = 1 \] (6.1)
or analogously
\[ P(\lim_{t \to 1^-} (\xi(t) - \xi(1)) = 0) = 1. \] (6.1a)

Denote
\[ Z_0(\beta) = \mathbb{E} \arctan \sup_{t \in [0,\beta]} |\xi(t) - \xi(0)|, \]
\[ Z_1(\beta) = \mathbb{E} \arctan \sup_{t \in [1-\beta,1]} |\xi(t) - \xi(1)|, \beta = \text{const} \in (0,1/2). \]

**Proposition 6.1.**

A. The condition
\[ \lim_{\beta \to 0^+} Z_0(\beta) = 0 \quad (6.2) \]
is necessary and sufficient for the equality (6.1).

B. The condition
\[ \lim_{\beta \to 0^+} Z_1(\beta) = 0 \quad (6.2a) \]
is necessary and sufficient for the equality (6.1a).

**The proof** is quite analogously to one for a main result of the author’s preprint [31] and may be omitted.

But it is worth to note that if
\[ \lim_{h \to 0^+} \kappa[\xi](h) = 0 \]
with probability one and
\[ \lim_{t \to 0^+} (\xi(t) - \xi(0)) = 0, \quad (6.3) \]
\[ \lim_{t \to 1^-} (\xi(t) - \xi(1)) = 0 \quad (6.3a) \]
in the sense of convergence in probability, or equally (here) in the sense of convergence in distribution, then \( P(\xi(\cdot) \in D[0,1]) = 1 \), see [39].

7 **Conditions for weak compactness. CLT in this space.**

Let \( X(t); X_n(t), n = 1,2,\ldots, t \in [0,1] \) be a sequence of separable random processes. We will study in this section the problem of finding sufficient conditions for weak (in distribution) convergence in the Prokhorov-Skorokhod space
\[ \text{Law}(X_n(\cdot)) \overset{D[0,1]}{\to} \text{Law}(X(\cdot)). \quad (7.1) \]

We will suppose in this section that all the finite-dimensional distributions of r.p. \( X_n(t) \) converges to ones for the r.p. \( X(t) \). Assume also that the limit process \( X(t) \) satisfies the boundary conditions (1.3a).
We will study in this section the problem of finding sufficient conditions for weak (in distribution) convergence in the Prokhorov-Skorokhod space 

\[ \text{Law}(X_n(\cdot)) \overset{D[0,1]}{\Rightarrow} \text{Law}(X(\cdot)). \] (7.1)

Of course, we are forced to admit that \( X_n(\cdot), X(\cdot) \) are elements of the space \( D[0,1] \) with probability one.

We need to introduce some new notations. Define the following uniform natural \( G\Psi \) function

\[ \zeta(p) = \zeta([X_n]) \overset{d}{=} \sup_n \sup_{s \in (0,1)} \sup_{(r,t) \in R(s)} |\delta[X_n(\cdot)](r,s,t)|_p, \] (7.2)

and suppose its finiteness for the at last one value \( p = p_0 = \text{const} > 2 \); denote as before \( b = \sup\{p \colon \zeta(p) < \infty\} \); then \( b = \text{const} \in (2, \infty) \).

There exists a continuous increasing function \( Q : [0,1] \to \mathbb{R} \) for which

\[ \sup_n \sup_{s \in (0,1)} |\delta[X_n(\cdot)](r,s,t)|_p \leq |Q(t) - Q(r)| \cdot \zeta(p). \] (7.3)

**Proposition 7.1.** We deduce on the basis of proposition 4.1 under formulated above in this section notations and conditions

\[ \sup_n \mathbb{P}(\Delta[X_n] > u) \leq \inf_{p \in [2,b]} \left\{ 3^p \zeta^p(p) [Q(1)]^p u^{-p} \right\}, \quad u > 0; \] (7.4)

\[ \sup_n \mathbb{P}(\kappa[X_n](h) > u) \leq 2 \inf_{p \in [2,b]} \left\{ 3^p u^{-p} \zeta^p(p) [Q(1)]^p (\omega[Q](2h))^{p-1} \right\}. \] (7.5)

Furthermore, if in addition all the considered r.p. \( X_n(t) \), including the limiting random process \( X(t) \), are continuous at the extremal points \( t = 0, t = 1 \) in the sense (1.3a), then the sequence of r.p. \( X_n(\cdot) \) converges at the r.p. \( X(\cdot) \) weakly in distribution in the space \( D[0,1] \).

**Proof.** The estimates (7.4), (7.5) follows immediately from the proposition (4.1). Both these inequalities together with the convergence of r.v. \( X_n(0) - X(0) \to 0, X_n(1) - X(1) \to 0 \) guarantee us the weak compactness of the distributions \( X_n(\cdot) \) in the Prokhorov-Skorokhod space \( D[0,1] \).

Finally, the convergence of all the finite-dimensional distributions of r.p. \( X_n(t) \) to the ones for \( X(t) \) gives us what is required, see e.g. [39].

Let us now turn as a capacity of the particular case to the study of the Central Limit Theorem in this space.

We recall here the classical definition of the CLT in Prokhorov-Skorokhod (or more generally in arbitrary linear separable topological) space. Let \( \xi(t) = \xi_1(t) \) be centered (mean zero) separable random process with values in this space having
finite variance in weak sense. Let $\xi_i = \xi_j(t)$, $j = 1, 2, \ldots$ be independent copies of $\xi(t)$. Denote

$$S_n(t) = n^{-1/2} \sum_{j=1}^{n} \xi_j(t),$$

and let $S_\infty(t) = S(t)$ be centered separable Gaussian process with at the same covariation as $\xi(t)$.

It will be presumed that all the random processes $\xi_j(t)$, $S_\infty(t)$ are defined at the same sufficiently rich probability space.

By definition, the r.p. $\xi(\cdot)$, or equally the sequence of r.p. $\{\xi_j(\cdot)\}$ satisfies the Central Limit Theorem (CLT) in the space $D[0, 1]$, if all the considered r.p. $\{\xi_j(\cdot)\}$, $S_\infty(\cdot)$ belong to this space almost surely and if the sequence of the distributions $S_n(\cdot)$ converges weakly as $n \to \infty$ to the distribution of the $S_\infty(\cdot)$: for arbitrary bounded continuous functional $F : D[0, 1] \to \mathbb{R}$

$$\lim_{n \to \infty} \mathbb{E}F(S_n) = \mathbb{E}F(S_\infty).$$

In particular,

$$\lim_{n \to \infty} \mathbb{P}(||S_n|| > u) = \mathbb{P}(||S_\infty|| > u), \quad u > 0.$$

The latter circumstance is the basis not only in the non-parametrical statistics, but also in the Physic [11], and in the Monte-Carlo method for computation of multiple integrals from the discontinuous functions, see [19], [21].

Evidently, the finite-dimensional distributions of the r.p. $S_n(\cdot)$ converges as $n \to \infty$ to ones for the r.p. $S_\infty(t)$; it remains only to ground the weak compactness of the correspondent distributions in the space $D[0, 1]$.

We retain the definitions and result of the example 5.1., especially the estimates (5.12) and (5.13); recall only that $\mathbb{E}\xi(t) = 0$ and $p \geq 2$.

We will use the famous Rosenthal's inequality, see [38], [24], [22] etc. in the following form. Let $\tau$, $\tau_i = 1, 2, \ldots$ be a sequence of i., i.d. mean zero random variables with finite $p^{th}$ absolute moment. The following estimate holds true:

$$\sup_{n \geq 1} \left| n^{-1/2} \sum_{i=1}^{n} \tau_i \right|_p \leq K_R(p) \cdot \left| \tau \right|_p, \quad p \geq 2, \quad (7.6)$$

where $K_R(p)$ is so-called Rosenthal’s "constant", more precisely, function on $p$. It is known, see [32], that

$$K_R(p) \leq C_A \cdot \frac{p}{\ln p}, \quad p \geq 2, \quad (7.6a)$$

where $C_A$ is an absolute constant, with the following its value $C_A \approx 0.65349368 < 0.6535$.

We will apply the inequality (7.6) to the sequence of differences $\xi_j(t) - \xi_j(r)$ with correspondent norms.
\[ d_p[\xi](t, r) := |\xi(t) - \xi(r)|_p, \]

and suppose as before its finiteness for the values \( p \in [2, b) \), where \( b = \text{const} \in (2, \infty] \).

We conclude

\[ \sup_n d_p[S_n](t, r) \leq K_R(p) \cdot d_p[\xi](t, r). \quad (7.7) \]

Define analogously the following natural \( G\Psi \) function

\[ y(p) = y[\xi](p) \overset{\text{def}}{=} \sup_{s \in (0, 1)} \sup_{(r, t) \in R(s)} |\delta[\xi(\cdot)](r, s, t)|_p, \quad p \in [2, b), \quad (7.8) \]

then

\[ Y(p) \overset{\text{def}}{=} \sup_n y[S_n](p) \leq K_R(p) \cdot y(p). \quad (7.9) \]

There exists a continuous increasing function \( B : [0, 1] \to R \) for which \( B(0) = B(0+) = 0 \) and

\[ \sup_{s \in (0, 1)} |\delta[\xi(\cdot)](r, s, t)|_p \leq |B(t) - B(r)| \cdot y(p), \quad (7.10) \]

following

\[ \sup_n \sup_{s \in (0, 1)} |\delta[S_n(\cdot)](r, s, t)|_p \leq |B(t) - B(r)| \cdot K_R(p) \cdot y(p). \quad (7.11) \]

It remains to use the proposition 7.1.

**Proposition 7.2.** We deduce on the basis of proposition 7.1 under formulated above in this section notations and conditions

\[ \sup_n \mathbf{P}(\Delta[S_n] > u) \leq \inf_{p \in [2, b)} \left\{ 3^p K_R^p(p) \cdot y^p(p) \cdot [B(1)]^p \cdot u^{-p} \right\}, \quad u > 0; \quad (7.12) \]

\[ \sup_n \mathbf{P}(\kappa[S_n](h) > u) \leq 2 \inf_{p \in [2, b)} \left\{ 3^p u^{-p} \cdot K_R^p(p) \cdot y^p(p) \cdot [B(1)]^p \cdot (\omega[B](2h))^{p-1} \right\}. \quad (7.13) \]

As a consequence, the r.p. \( \xi(\cdot) \), or equally the sequence of r.p. \( \{\xi_j(\cdot)\} \) satisfies the Central Limit Theorem in the space \( D[0, 1] \).

**Example 7.1.** Assume that

\[ y(p) \leq C_1 \ p^{1/m} \ln^s p, \quad p \geq 2, \quad m = \text{const} > 0, \quad s = \text{const}; \quad (7.14) \]

then we have the following non-asymptotical tail estimates:

\[ \sup_n \mathbf{P}(\Delta[S_n] > u) \leq \]
\[ \exp \left\{ -C_2(C_1, m, s) \frac{u^{m/(m+1)}}{\ln u^{(m(s-1)/(m+1))}} \right\}, \quad u \geq e; \quad (7.15) \]

\[ \sup_n \mathbb{P}(\kappa[S_n](h) > u) \leq 2 \left( \omega[B](2h) \right)^{-1} \times \]

\[ \exp \left\{ -C_3(C_1, m, s) \left[ \frac{u}{\omega[B](2h)} \right]^{m/(m+1)} \left[ \ln \left( \frac{u}{\omega[B](2h)} \right) \right]^{m(s-1)/(m+1)} \right\}, \]

when

\[ u > e \cdot \omega[B](2h) \cdot \ln \omega[B](2h) \left[ 1 + 1/m \right]. \quad (7.16) \]

In turn, the condition (7.14) may be expressed in the terms of tail behavior for the r.p. \( \delta(r, s, t) \), see (2.9) - (2.9a).

8 Concluding remarks.

A. It is interest by our opinion to generalize obtained results into the multidimensional case, i.e. into the space \( D[0, 1]^d \). The non-asymptotical estimated for tail of uniform norm distribution for discontinuous random processes are obtained in [29].

B. It is interest also a generalization on the case when the sequence of r.p. \( \{ \xi_j(t) \} \) forms on the index \( j \) a sequence of martingale differences relative appropriate filtration.

C. Perhaps, the applying of the more modern technic, indeed the so-called majorizing measures, see [13], [41] - [44] one can give more exact estimated.

D. More interest new examples of CLT in the Prokhorov-Skorokhod space with applications may be found in the articles [2], [6] - [8], [21], [35] etc.

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