Maximal Violations of the Free CHSH-3 Inequality

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Abstract—In this paper we deal with a special version of the Clauser-Horne-Shimony-Holt inequalities, namely the free CHSH-3. Our method uses semidefinite programming relaxations to compute maximal violations of such inequality in a quantum setting. In a standard setting the CHSH-3 inequality involves two separated qudits and compatible measurements, that is, commuting with each other, yielding the known quantum bound of $1 + \sqrt{11}/3 \approx 2.9149$ [13]. In our framework, $d$-dimensional quantum systems (qudits) where $d$ is not fixed a priori, and measurement operators possibly not compatible, are allowed. This loss of constraints yields a higher value of $2 + 2\sqrt{11}/3 \approx 3.1547$ for the maximum expectation of CHSH-3, which is attained for qudits of dimension $d = 5$.

Keywords—Bell inequalities, CHSH, semidefinite programming, quantum non-commutative observables

I. INTRODUCTION

Bell inequalities are important relations involving the expected values (or probabilities) of outcomes of measurements. These inequalities hold in classical mechanics but that can be violated in a quantum setting [6]. This is the case for the CHSH-2 inequality [10], for which the quantum bound (2\sqrt{2}) is higher than the classical one, which is 2, see for instance [9], [24]. One of the motivations of our work and, in general, for studying Bell inequalities and their violations, is that these can be used in order to develop protocols for random numbers generators.

For CHSH-2, the violation is related to the non-locality of quantum physics. Indeed, the complete description of a quantum system is not only related to its local environment, but can be correlated to a very far system, due to entanglement. Non-locality, in addition to the default random character of quantum physics, is the basis of the random number generator in [21], where a protocol is developed that relies on a two-parties configuration whose security is yielded by the violation of CHSH-2. Moreover, thanks to the relation between the violation and the output entropy, the protocol is proven to be device-independent. This means that in a quantum setting, the user can have a guarantee on the quality of the randomness without knowledge on the precise states and measurements that have been performed.

The generation of random numbers can also be obtained with a one-party system, see for instance [12] for the case of a unique qutrit. This protocol is based on quantum contextuality, that is, on the property that the measurement result of a quantum observable depends on the set of compatible observers. In [12] non-contextuality is verified by the KCBS inequality [10]. The security of the protocol relies on the fact that a violation of the KCBS inequality yields a strictly positive entropy. Such an entropy reaches the maximum for the maximum value of the violation of the Bell inequality (see [10] Fig. 1).

In this work, we study violations of the free CHSH-3 inequality. The term free indicates the fact that the dimension of the Hilbert space the quantum operators act on is not fixed a priori and that the operators do not commute. As a result of our relaxations, we prove that working with states and measures in dimension $d = 5$ allows us to get the expected value of $2 + 2\sqrt{11}/3 \approx 3.1547$ for the CHSH-3 inequality. This value is greater than the quantum bound which is available in the literature [13], which is explained by the fact that we do not impose constraints on the dimension and on the commutativity. This viewpoint is motivated by the result in [3] where the it is shown how to implement the product of some non-commuting observables.

The paper is organized as follows: Section II contains a reformulation of the CHSH-3 inequality in a free setting, that is, without commutativity and dimensional constraints involving the observables. In Section III we describe an approach based on convex semidefinite relaxations to compute bounds on the violation of CHSH-3.

II. CHSH-3 INEQUALITY AND ITS FREE VARIANT

A. Original setting

Clauser-Horne-Shimony-Holt inequalities in the standard context involves 2 parties, 2 measurements per party and 2 outcomes per measurement (compactly named CHSH-2). Further, many authors have worked on generalizations with many outcomes per measurement (compactly named CHSH-2). Moreover, thanks to the relation between the violation and the output entropy, the protocol is proven to be device-independent. This means that in a quantum setting, the user can have a guarantee on the quality of the randomness without knowledge on the precise states and measurements that have been performed.

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The paper is organized as follows: Section II contains a reformulation of the CHSH-3 inequality in a free setting, that is, without commutativity and dimensional constraints involving the observables. In Section III we describe an approach based on convex semidefinite relaxations to compute bounds on the violation of CHSH-3.

The corresponding Bell expression is

\[ I_3 = P(1,1|A_1B_1) + P(\omega,\omega|A_1B_1) + P(\omega^2,\omega^2|A_1B_1) \]
\[ + P(\omega^2,1|A_2B_1) + P(1,\omega|A_2B_1) + P(\omega,\omega^2|A_2B_1) \]
\[ + P(1,1|A_2B_2) + P(\omega,\omega|A_2B_2) + P(\omega^2,\omega^2|A_2B_2) \]
\[ + P(1,1|A_1B_2) + P(\omega,\omega|A_1B_2) + P(\omega^2,\omega^2|A_1B_2) \]
\[ - P(1,\omega|A_1B_1) - P(\omega,\omega^2|A_1B_1) - P(\omega^2,1|A_1B_1) \]
\[ - P(1,1|A_2B_1) - P(\omega,\omega|A_2B_1) - P(\omega^2,\omega^2|A_2B_1) \]
\[ - P(1,\omega|A_2B_2) - P(\omega,\omega^2|A_2B_2) - P(\omega^2,1|A_2B_2) \]
\[ - P(\omega,1|A_1B_2) - P(\omega^2,\omega|A_1B_2) - P(1,\omega^2|A_1B_2) \]  
(1)
where \( P(\omega^k, \omega^\ell | A_i B_j) \) denotes the probability of getting \( \omega^k, \omega^\ell \) with measurements \( A_i, B_j \). The classical bound of 2 is satisfied in a local realistic setting and establishes what one generally calls the CHSH-3 inequality: \( I_3 \leq 2 \).

In a quantum setting, \( A_1, A_2, B_1, B_2 \) are observables acting on a three-dimensional Hilbert space \( \mathbb{H} \) with eigenvalues \( 1, \omega, \omega^2 \) defined as above. The corresponding eigenvectors are denoted by \( |a_{i1}, a_{i2}, a_{i3}| \) for \( A_i \), \( i = 1, 2, \) and similarly for \( B_1, B_2 \). This allows us to define the projectors

\[
A_{11} = |a_{11}| \langle a_{11} |, \quad A_{21} = |a_{21}| \langle a_{21} |, \\
A_{1\omega} = |a_{1\omega}| \langle a_{1\omega} |, \quad A_{2\omega} = |a_{2\omega}| \langle a_{2\omega} |, \\
A_{1\omega^2} = |a_{1\omega^2}| \langle a_{1\omega^2} |, \quad A_{2\omega^2} = |a_{2\omega^2}| \langle a_{2\omega^2} |,
\]

and the corresponding decomposition for \( A_1 \) (similarly for \( A_2, B_1, B_2 \)):

\[
A_1 = 1 \cdot |a_{11}| \langle a_{11} | + \omega \cdot |a_{1\omega}| \langle a_{1\omega} | + \omega^2 \cdot |a_{1\omega^2}| \langle a_{1\omega^2} |.
\]

Under the assumption that the observables \( A_i \)'s commute with the \( B_j \)'s, the following equality holds:

\[
\langle \phi | A_i B_j | \phi \rangle = P(\omega^k, \omega^\ell | A_i B_j)
\]

for a state \( |\phi\rangle \in \mathbb{H}, i, j \in \{1, 2\} \) and \( k, \ell \in \{0, 1, 2\} \). Thus one can rewrite the expression in eq. (1) as

\[
\langle \phi | A_{11} B_{j1} + A_{11} B_{1j} - A_{11} B_{11} - A_{11} B_{22} + A_{1\omega} B_{1\omega} + A_{1\omega} B_{12} - A_{1\omega} B_{11} - A_{1\omega} B_{21} + A_{1\omega^2} B_{1\omega^2} + A_{1\omega^2} B_{12} - A_{1\omega^2} B_{11} - A_{1\omega^2} B_{21} + A_{21} B_{11} - A_{21} B_{11} - A_{21} B_{21} + A_{2\omega} B_{1\omega} + A_{2\omega} B_{12} - A_{2\omega} B_{11} - A_{2\omega} B_{21} + A_{2\omega^2} B_{1\omega^2} + A_{2\omega^2} B_{12} - A_{2\omega^2} B_{11} - A_{2\omega^2} B_{21}| \phi \rangle
\]

In this case (commutative observables) we recall that the quantum bound for \( I_3 \) is \( 1 + \sqrt{11}/3 \approx 2.9149 \), see [13], yielding a violation of \( \sqrt{11}/3 - 1 \approx 0.9149 \) for the CHSH-3. In this paper, we use a semidefinite-programming-based strategy to compute upper bounds on the violation of a special version of CHSH-3, which is described below in Section II-B.

B. Free CHSH-3 inequality

Let us describe the precise setting we are working on. Our goal is to consider a non-commutative version of Equation (2), and where the dimension of the Hilbert space the observables are operating on, is not fixed \textit{a priori}. That is we are interested in a free CHSH-3 inequality.

Whereas the standard setting consists of two parties (Alice and Bob) with four given observables, two for each party \( (A_1, A_2, B_1, B_2) \) as previously discussed in Section II-A, our model consists of one single party with four observables \( X_1, X_2, X_3, X_4 \), acting on states \( |\phi\rangle \) living in a Hilbert space of unconstrained dimension.

The observables \( X_i \) are possibly not commuting to each other, they are unknown and will be explicitly constructed by solving a single semidefinite program, the details are given in Section III. For each \( i \in \{1, 2, 3, 4\} \), and \( j \in \{0, 1, 2\} \), as in Section II-A we decompose each \( X_i \) as follows:

\[
X_i = 1 \cdot X_{i1} + \omega \cdot X_{i\omega} + \omega^2 \cdot X_{i\omega^2}, \quad \text{for} \ i \in \{1, 2, 3, 4\}
\]

introducing 12 variables \( X_{i\omega^k}, i \in \{1, 2, 3, 4\}, k \in \{0, 1, 2\} \) corresponding to the projector \( |x_{i\omega^k}\rangle \langle x_{i\omega^k}| \) on the eigenvector \( |x_{i\omega^k}\rangle \) of the \( X_i \)'s (see [20, Sec. 2.2]).

Therefore the CHSH-3 quadratic form can be formally restated as a function of \( X = (X_{11}, X_{1\omega}, \ldots, X_{4\omega^2}) \) and of the state \( |\phi\rangle \) as \( \langle \phi | f(X) | \phi \rangle \) with

\[
f(X) = \\
= X_{11}X_{31} + X_{11}X_{41} - X_{11}X_{31} - X_{11}X_{41} + X_{1\omega}X_{3\omega} + X_{1\omega}X_{4\omega} - X_{1\omega}X_{3\omega} - X_{1\omega}X_{4\omega} + X_{21}X_{31} + X_{21}X_{41} - X_{21}X_{31} - X_{21}X_{41} + X_{2\omega}X_{3\omega} + X_{2\omega}X_{4\omega} - X_{2\omega}X_{3\omega} - X_{2\omega}X_{4\omega} + X_{2\omega^2}X_{3\omega} + X_{2\omega^2}X_{4\omega} - X_{2\omega^2}X_{3\omega} - X_{2\omega^2}X_{4\omega} - X_{11}X_{44} + X_{1\omega}X_{4\omega} - X_{1\omega}X_{4\omega} + X_{21}X_{44} + X_{2\omega}X_{4\omega} - X_{2\omega}X_{4\omega} + X_{2\omega^2}X_{4\omega} - X_{2\omega^2}X_{4\omega} - X_{2\omega^2}X_{4\omega} - X_{2\omega^2}X_{4\omega}
\]
The SDP hierarchy has been extended to the non-commutative setting and successfully applied to quantum information, see [22] and [5, Ch. 21]. The hierarchy in [22] allows one to get bounds on the minimum or maximum of the action of a non-commutative polynomial function of observables, possibly subject to equalities and inequalities. The key idea of such a hierarchy is to linearize the quantity

$$\langle \phi, f(X) \rangle = \langle \phi, \sum_w f_w w(X) \rangle = \sum_w f_w \langle \phi, w(X) \rangle$$

where \( f(X) = \sum_w f_w w(X) \) is a non-commutative polynomial function of \( n \) measurement operators \( X = (X_1, \ldots, X_n) \) defined on a Hilbert space \( \mathbb{H} \), \( w(X) \) is a monomial on \( X \), and \( \phi \in \mathbb{H} \) is a pure state. The linearization consists of replacing the action \( \langle \phi, w(X) \rangle \) of the monomial \( w(X) \) on the state \( \phi \), with a new variable, or moment, \( y_w \). In other words, one replaces the original non-linear operator on \( X \) with the following linear function on the space of variables \( y \):

$$\langle \phi, f(X) \rangle = \sum_w f_w \langle \phi, w(X) \rangle = \sum_w f_w y_w.$$

The moments \( y_w \) up to some order \( D \) are then organized in a symmetric multi-hankel moment matrix \( M_D(y) = (y_{vw})_{v,w} \) (that is, the entry of \( M_D(y) \) indexed by \( (v, w) \) is \( y_{vw} \)). By construction of \( y_w \), one gets the necessary condition that \( M_D(y) \) must be positive semidefinite, from the fact that \( z^* M_D(z) z \geq 0 \) for any complex vector \( z = (z_w) \). Similarly, non-linear constraints can be linearized and lead to additional linear and semidefinite constraints on variables \( X \) in the relaxation.

Let us mention the case of the CHSH-2 inequality for two space-like separated parties, many measurements settings with two outcomes, for which the first level of the hierarchy is sufficient to compute Tsirelson’s bounds, as shown in [24]. In this work, we use semidefinite programming in the spirit of [24], [22] to compute explicit (non-commuting) observables yielding a violation of the CHSH-3 inequality higher than the known quantum bound of \( 1 + \sqrt{11}/3 \).

### B. First relaxation of the free CHSH-3

Let us denote by

$$X = (X_{1,1}, X_{1,ω}, X_{1,ω^2}, X_{2,1}, \ldots, X_{4,ω}, X_{4,ω^2})$$

the (unknown) projectors on the eigenstates of operators \( X_1, X_2, X_3, X_4 \) related to eigenvalues \( 1, ω, ω^2 \), as defined in Section 4.11 and let \( f(X) \) be the non-commutative quadratic polynomial defined in Equation (4). Since our goal is to compute the maximal violation of CHSH-3 with no dimensional constraints, we let \( k_1, k_2, \ldots, k_d \) be the additional eigenvalues up to dimension \( d \) (see for instance [20, §2.2.6]) and similarly we denote by \( X_{i,j} \) the eigenstate corresponding to \( k_j \), \( j \in \{4, \ldots, d\} \).

Let us introduce the following compact notation for the indices of \( X_{i,j} \). We define the set

$$T = \{ (i, μ) \mid i = 1, 2, 3, 4, μ = 1, ω, ω^2, k_4, \ldots, k_d \}.$$  

Hence the variables \( X_{i,μ} \) are exactly those of the form \( X_α \) with \( α = (i, μ) \in T \) for some \( i, μ \). Thus the original problem can be stated as follows:

\[
\begin{align*}
  f^* := \sup_{y_0 = 1} & \langle φ, f(X) \rangle \\
  \text{s.t.} & \quad X_α X_β = δ_{α μ} X_α, \quad α = (i, μ), β = (i, ν) \in T \\
                & \quad \sum_μ X_α = 1, \quad i \in \{1, 2, 3, 4\}, \quad α = (i, μ)
\end{align*}
\]

where \( δ_{μν} \) is the Kronecker delta for indices \( μ, ν \in \{1, ω, ω^2, k_4, \ldots, k_d\} \). The last two constraints are related to the equality \( X_{i,μ} = |x_{i,μ}| |x_{i,μ}| \) that we want to impose, as discussed above.

We denote by \( y_α = \langle φ | X_α | φ \rangle \) for \( α \in T \), the moment of order one associated to the variable \( X_α \) and to state \( |φ \rangle \) (omitted in the notation). Similarly we denote by \( y_{α β} = \langle φ | X_α X_β | φ \rangle \) the moments of order two. Note that whereas \( X_α X_β \neq X_β X_α \), one has that the expected values are the same assuming \( X_α \) are projectors (hence Hermitian): indeed, for \( X_α = |ψ⟩⟨ψ|, X_β = |θ⟩⟨θ| \) one has

$$y_{α β} = \langle φ | X_α X_β | φ \rangle = \langle φ | X_α X_β | φ \rangle \mathbb{E} =$$

$$= (I_{d_t} (X_α X_β))^T (|φ⟩⟨φ|) \mathbb{E} = (I_{d_t} X_α^T X_β^T |φ⟩⟨φ|) \mathbb{E} =$$

$$= (I_{d_t} X_β X_α |φ⟩⟨φ|) \mathbb{E} = (X_β X_α |φ⟩ |φ⟩) \mathbb{E} = y_{β α}$$

for all \( |φ⟩ \), where \( (\cdot, \cdot) \mathbb{E} \) is the standard inner product in \( \mathbb{H} \).

The first moment relaxation of Equation (4) is thus expressed in the following form

\[
\begin{align*}
  f^*_1 := \sup_{y_0 = 1} & \sum_α c_α y_α \\
  \text{s.t.} & \quad y_0 = 1 \\
                & \quad y_{α β} = δ_{α μ} y_α, \quad α = (i, μ), β = (i, ν) \in T \\
                & \quad \sum_α y_α = 1, \quad i \in \{1, 2, 3, 4\}, \quad α = (i, μ) \\
                & \quad M_1(y) \succeq 0
\end{align*}
\]

where \( c_α \in \{-1, 0, 1\} \) are such that \( f(X) = \sum_α c_α X_α \), and \( M_1(y) \) is the moment matrix of order 1, namely the matrix

$$M_1(y) = \langle φ | v_1 v_1^T | φ \rangle = \begin{pmatrix}
  y_0 & y_{α_1} & y_{α_2} & \cdots & y_{α_4d} \\
  y_{α_1} & y_{α_1 α_1} & y_{α_1 α_2} & \cdots & y_{α_1 α_4d} \\
  y_{α_2} & y_{α_2 α_1} & y_{α_2 α_2} & \cdots & y_{α_2 α_4d} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  y_{α_4d} & y_{α_4d α_1} & y_{α_4d α_2} & \cdots & y_{α_4d α_4d}
\end{pmatrix}$$

Above we have chosen an order for indices \( α \) in \( T = \{α_1, \ldots, α_{4d}\} \), and denoted the vector of moments up to degree 1 by \( v_1 = (1, X_{α_1}, X_{α_2}, \ldots, X_{α_{4d}}) \in \mathbb{C}^{4d+1} \). Problem 5 is a relaxation of Problem 4 which implies that \( f^* \leq f^*_1 \).

For two symmetric matrices \( C_1, C_2 \), we denote by \( C_1 \cdot C_2 = \text{Trace}(C_1 C_2) \) the usual Euclidean inner product. Let \( C, A_0, A_β, A_β \) be the \( (1 + 4d) \times (1 + 4d) \) symmetric matrices such that \( \sum_α c_α y_α = C \cdot M_1(y), y_0 = C_0 \cdot M_1(y), y_{α β} - δ_{μ ν} y_α = A_α β \cdot M_1(y) \) and \( \sum_μ y_α = A_1 \cdot M_1(y) \). Thus the problem in Equation 5 is equivalent to the semidefinite program
We remark that the same value appears as an intermediate bound through the semidefinite programming hierarchy in [2]. It has to be compared also to the known commutative quantum bound of $(1 + \sqrt{1/3}) \approx 2.9149$ in [13].

The $13 \times 13$ submatrix $M^*$ of the optimal moment matrix $M_1(y^*)$, corresponding to variables $X$ occurring in the CHSH-3 inequality, has the following form:

$$M^* = \frac{1}{3} \begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

for $s = \sin(\pi/12), \ c = \cos(\pi/12), \ \tilde{s} = s \sqrt{2/3} \approx 0.211$ and $\tilde{c} = c \sqrt{2/3} \approx 0.788$. The matrix $M$ has rank five and it is positive semidefinite, with eigenvalues $\frac{7}{3}, \frac{2}{3}$ and $0$ of multiplicity $1, 4$ and $8$, respectively.

In order to retrieve the optimal projectors, we thus compute a factorization of $M^*$ of the form $M^* = B^T B$ (certifying that $M^* \succeq 0$), with $B$ the following $5 \times 13$ matrix:

$$B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}$$

As in [5, Ch. 21], the first column of $B$ is interpreted as the optimal state $|\phi^+\rangle$, and for $i \in \{1, 2, 3, 4\}$, the normalization of columns $3i - 2, 3i - 1$ and $3i$ of $B$ as the eigenstates $|x_{i,1}\rangle$, $|x_{i,\omega}\rangle$ and $|x_{i,\omega^2}\rangle$ corresponding to projective measurements $X_i^\phi$ that can be recovered as in [20, §2.2.6], as follows:

$$X_i^\phi = 1|\langle x_{i,1}| + |\langle x_{i,\omega}| + |\langle x_{i,\omega^2}| + 0\cdot E_{i,0},$$

Above $E_{i,0}$ is the rank two projector satisfying $|\langle x_{i,1}| + |\langle x_{i,\omega}| + |\langle x_{i,\omega^2}| + E_{i,0} = \Id_3$. 

We prove the following result concerning the relaxation in Equation (6).

**Theorem 1.** The optimal value of Problem (4) is $2 + 2\sqrt{1/3}$ and it is attained for the configuration in Equation (9) and for $|\phi^+\rangle = (1, 0, 0, 0, 0)^T$. 

**Proof:** First, we remark that the operators constructed in Equation (9) satisfy the constraints in Problem (4), which yields

$$2 + 2\sqrt{1/3} = C \cdot M_1(y^*) = \langle \phi^+| f(X^*)|\phi^+\rangle \leq f^*.$$

Moreover Equation (5) is a relaxation of Equation (4), that is, $f^* \leq f_1^* = 2 + 2\sqrt{1/3}$, and we conclude.

IV. Conclusions

In this work, we have considered a variant of the standard Clauser-Horne-Shimony-Holt inequality, the free CHSH-3, where the dimension of the observables and the commutativity constraints are relaxed. We proved that semidefinite programming relaxations allow to compute the maximal violation of $2 + 2\sqrt{1/3}$, in a quantum non-commutative setting, for such inequality.

A natural perspective, which is postponed to future work, is to use the result presented in this paper in order to design a random number generator based on the computed violations of the CHSH-3 inequality, and possibly to be made device-independent.

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