Evolution of parton distributions with truncated Mellin moments

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Abstract
Evolution equations for parton distributions can be approximately diagonalized and solved in moment space without assuming any knowledge of the parton distribution in the region of small \( x \). The evolution algorithm for truncated moments is simple and rapidly converging. Examples of applications are outlined.

1. Introduction

Altarelli–Parisi evolution equations for parton distributions can be exactly diagonalized and analytically solved by taking a Mellin transform, which turns the \( x \)-space integro-differential equation into a set of decoupled ordinary first order differential equations. This procedure, although elegant, suffers from a practical drawback: Mellin moments are integrals over the entire allowed kinematical range for the parton momentum fraction, \( 0 < x < 1 \), whereas in practice parton distributions are measured only in a finite subinterval of that range, say \( x_0 < x < x_1 \). Furthermore, solving the equation through a Mellin transform somewhat obscures the directionality in \( x \)-space of parton evolution, namely the fact that the evolution in \( Q^2 \) of the parton distribution at a given value of the momentum fraction \( z \) depends only on the values of the distribution for \( x > z \). In this note I present a method to evolve parton distributions in Mellin space without using any assumptions on the small–\( x \) behavior of parton distributions. The method is based on the notion of “truncated” moments, defined as integrals over a region \( x_0 < x < 1 \). It turns out that (non–singlet) truncated moments are coupled by evolution via a triangular matrix whose effective size can be kept small without loss of accuracy, and that can be diagonalized analytically with little effort; numerical implementation is thus relatively simple and efficient. The method addresses a specific source of uncertainty in the determination of parton distributions (the lack of small–\( x \) data, which are abundant only for \( F_2(x, Q^2) \)). This uncertainty is apparent in Mellin space, but is present also in \( x \)-space evolution codes, which operate by choosing a fixed parametrization for the parton distribution and subsequently fitting the parameters to the data. Any parametrization has a degree of rigidity and theoretical prejudice, so that tacit assumption made about the \( x \to 0 \) region affect also the region where the data are fitted. This kind of uncertainty, in principle very difficult to quantify, is replaced in the present method by a small, quantifiable and systematically reducible uncertainty due to the finite size chosen for the triangular matrix of couplings between truncated moments.

2. Truncated moments

For the sake of simplicity, in the following I will consider the nonsinglet quark distribution, \( q(x, Q^2) \). The extension of the method to singlet and gluon distributions requires some work, which has recently been completed. Consider the evolution equation

\[
\frac{d}{dt} q(x) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} P \left( \frac{x}{y} \right) q(y) ,
\]

where the evolution kernel \( P(z) \) is a perturbative series in \( \alpha_s(Q^2) \), and all dependence on \( Q^2 \) has been suppressed. Define the truncated moment of \( q(x) \) as

\[
q_N(x_0) \equiv \int_{x_0}^1 dx x^{N-1} q(x) .
\]

Then the evolution equation for truncated moments reads

\[
\frac{d}{dt} q_N(x_0) = \frac{\alpha_s}{2\pi} \int_{x_0}^1 dy y^{N-1} q(y) G_N \left( \frac{x_0}{y} \right) ,
\]

where \( G_N(z) \) is the evolution kernel for truncated moments.
where

\[ G_N(x) = \int_x^1 dzz^{N-1} P(z) \]  \hspace{1cm} (4)

It is easy to show that the \( N \)-th truncated moments is coupled by Eq. (3) to all moments with index \( N + K \), \((K > 0)\). Furthermore, the strength of this coupling decreases rapidly with \( K \). To prove it, one expands \( G_N(x_0/y) \) in Taylor series around \( y = 1 \). This expansion has radius of convergence \( 1 - x_0 \), and can be truncated, say at order \( M \). Reorganizing the polynomial in \( y - 1 \) thus obtained to collect the different powers of \( y \), and performing the \( y \) integration, one ends up with an equation coupling the \( N \)-th moment to the subsequent \( M \) moments,

\[ \frac{d}{dt} q_{N}(x_0) = \frac{\alpha_s}{2\pi} \sum_{K=0}^{M} c_{K,N}^{(M)}(x_0) q_{N+K}(x_0) \]  \hspace{1cm} (5)

with coefficients \( c_{K,N}^{(M)}(x_0) \) that are perturbatively calculable for arbitrary \( x_0 \), given the evolution kernel \( P(z) \). For moderately small \( x_0 \), say \( x_0 \leq 0.1 \), one gets a very good approximation to the exact evolution for reasonably small values of \( M \), say \( M \sim 3-5 \), and furthermore the value of \( M \) required to keep the desired accuracy decreases with \( N \). This can be understood by noting that, for \( x_0 = 0 \), \( G_N(x_0/y) \) is independent of \( y \), so that for finite but small \( x_0 \), and over most of the \( y \) integration range, one is Taylor expanding a function which is approximately a constant. Furthermore, the contribution to the evolution of higher moments is suppressed by the fact that they are dominated by high values of \( x \), where the parton distribution themselves are decreasing as powers of \((1 - x)\). Notice that the accuracy of these approximations for each truncated moment can always be controlled by comparing numerically to the exact evolution, given by Eq. (3).

3. NLO evolution

Given the approximations discussed above, the evolution of a given truncated moment (say the \( N_0 \)-th) can be calculated by solving a linear system of differential equations coupling it to moments \( N_0 + 1 \) through \( N_0 + M \). Schematically, at NLO,

\[ \frac{d\hat{q}_K}{dt} = \frac{\alpha_s}{2\pi} \sum_{L=N_0}^{N_0+M} \left[ \hat{C}_{KL}^{(0)} + \frac{\alpha_s}{2\pi} \hat{C}_{KL}^{(1)} \right] q_L \]  \hspace{1cm} (6)

a situation which is formally identical to the coupled evolution of the gluon and singlet quark distributions. Here analytical and numerical computations are greatly simplified by the fact that the “anomalous dimension” matrix is triangular, so that its eigenvalues are simply given by the diagonal entries, while the eigenvectors are given by a recursion relation, without the need to compute any determinants. Denoting by \( R \) the matrix diagonalizing the LO anomalous dimension \( C^{(0)} \), one can use it to rotate the set of moments \( q_K \) under consideration, as well as the NLO matrix \( C^{(1)} \), according to

\[ \hat{q}_K = \sum_{L=0}^{N_0+M} R_{KL} q_L \]  \hspace{1cm} (7)

\[ \hat{D}_{KL} = \sum_{P,Q=N_0}^{N_0+M} R_{KP} C^{(1)}_{PQ} R^{-1}_{QL} \]  \hspace{1cm} (8)

Then the NLO solution is given by

\[ \hat{q}_K = \left[ \frac{\alpha_s^0}{\alpha_s} \right]^{\gamma_K} \left[ 1 - \frac{\gamma_K b_1}{2\pi b_0} (\alpha_s^0 - \alpha_s) \right] q_K^0 \]

\[ - \sum_{L=N_0}^{N_0+M} \hat{D}_{KL} \frac{1}{\gamma_K - \gamma_L + 1} \left[ \left( \frac{\alpha_s^0}{\alpha_s} \right)^{\gamma_L} \right] q_L^0 \]  \hspace{1cm} (8)

where the apex 0 denotes quantities evaluated at the input scale \( Q_0 \), and \( \gamma_M = C_{MM}/b_0 \).

4. Perspectives

The method presented in this note has been extended to parton distributions with coupled evolution \[ [3] \], and can be straightforwardly extended to polarized partons as well. It can be applied to derive estimates for all physical quantities depending on evolution, unbiased by the extrapolation of data to small \( x \). An example already studied in \[ [1] \] is the running coupling itself. Other significant example include the first moment of the polarized structure function \( g_1 \), appearing in the Bjorken sum rule \[ [3] \], and the behavior of the unpolarized gluon distribution at moderate to large values of \( x \). Work on these applications is in progress.

References

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