Static Output Feedback Revisited

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Abstract—The synthesis problem of static output feedback controllers is revisited. Specifically an anisotropic norm setup for synthesis is considered. A tractable synthesis approach involving iterations over a convex optimization problem is suggested. The resulting optimization scheme can be applied using existing iterative methods for designing fixed structure controllers for various applications such as flight control.

Index Terms—Static Output Feedback, $H_{\infty}$, Anisotropic Norm.

I. INTRODUCTION

The problems of optimal control and filtering received much attention over the years. Solutions for these problems were presented by Kwakernaak and Sivan [11]. Modelling errors were considered in [17]. When the external input signals are of white noise type, $H_2$-norm minimization is applied, leading to the Kalman filter [7] and Linear Quadratic Gaussian (LQG) control. An alternative modelling of the exogenous inputs is based on deterministic bounded energy signals associated with the $H_\infty$-norm based framework ([21]) applicable both to filtering ([6], [16]) and control ([22]). Since many practical applications require an intermediate solution between $H_2$ and $H_\infty$. Since $H_2$ is not entirely suitable when signals are strongly coloured (e.g. white noise), mixed $H_2/H_\infty$ norm minimization becomes useful (see, e.g. [2], [13]). A promising alternative to accomplish such compromise is to use the so-called a-anisotropic norm ([8], [20], [10]) offering an intermediate topology between the $H_2$ and $H_\infty$ norms. More precisely, consider the $m$ dimensional coloured signal $w(t), t=0,1,...$ generated by the discrete-time stable filter $G$

$$
x(t+1) = A_f x(t) + B_f v(t)$$
$$w(t) = C_f x(t) + D_f v(t), t=0,1,...$$  \(1\)

where $A_f \in \mathbb{R}^{n_f \times n_f}$, $B_f \in \mathbb{R}^{n_f \times m}$, $C_f \in \mathbb{R}^{m \times n_f}$, $D_f \in \mathbb{R}^{m \times m}$ and where $v \in \mathcal{N}$ are independent Gaussian white noises with $E[v(t)] = 0$ and $E[v(t)v^T(t)] = I_m$. Then, the $a$-anisotropic norm $|||F|||_a$ of a discrete-time stable system $F$ with the state-space realisation

$$
x(t+1) = A x(t) + B u(t)$$
$$y(t) = C x(t) + D u(t), t=0,1,...$$  \(2\)

is defined as

$$|||F|||_a = \sup_{G \in \mathcal{G}_a} \frac{|||FG|||_2}{||G||_2},$$  \(3\)

$\mathcal{G}_a$ denoting the set of all stochastic systems of form \(1\) with the mean anisotropy $\tilde{A}(G) \leq a$. The mean anisotropy of stationary Gaussian sequences was introduced in [8] and it represents an entropy theoretic measure of the deviation of a probability distribution from Gaussian distributions with zero mean and scalar covariance matrices. In [7], it is proved based on the Szegö-Kolmogorov theorem ([14]) that the mean anisotropy of a signal generated by an $m$-dimensional Gaussian white noise $v(t)$ with zero mean and identity covariance applied to a stable linear system $G$ with $m$ outputs has the form

$$\tilde{A}(G) = \frac{1}{2} \ln \det \left( m E[\tilde{w}(0)\tilde{w}(0)^T] \right)$$  \(4\)

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|w(k), k < 0]$. In the case when the output $w$ of the filter $G$ is a zero mean Gaussian white noise (i.e. its optimal estimate is just zero), $w(0)$ cannot be estimated from its past values and $\tilde{w}(0) = w(0)$ which leads to $\tilde{A}(G) = 0$.

It is proved (see, for instance [20]) that the anisotropic norm has the property:

$$\frac{1}{\sqrt{m}} ||F||_2 = |||F|||_a \leq ||F|||_a \leq ||F|||_\infty = \lim_{a \to \infty} |||F|||_a$$  \(5\)

In [9], conditions for the anisotropic norm boundedness are given in terms of a non convex optimization problem while in [10] a convex form of the Bounded Real Lemma (BRL) type result with respect to the anisotropic norm was obtained. One of the leading motivations to use the anisotropic norm is the fact $|||F|||_a \leq ||F|||_\infty$ making it a relaxed version of the $H_\infty$-norm for many practical cases in which the driving noise signals can be characterised not just by their finite energy, but as outputs of a colouring linear systems in a certain class, where the colouring filters are of a finite anisotropy. In a case study presented in [19] it is shown that for a TU-154 type aircraft landing system, the $H_\infty$ controller is more efficient than the corresponding $H_2$ controller for a windshear profile (which is a coloured rather than a white noise process) but, as could be expected, is more conservative, in the sense of higher gains and subsequently larger control actions; moreover, the anisotropic-norm based controller (based on an appropriate anisotropic norm bound) is less conservative than the $H_\infty$ controller and requires significantly smaller control actions.
The aim of the present paper is to derive a tractable characterisation of the static output feedback synthesis problem. Static Output Feedback (SOF) synthesis is very useful, when the design of fixed structured controllers such as PID (Proportional, Integral, Derivative) is required. Such controllers are very common both in the process control and aerospace control applications. However, it is well known that SOF synthesis is an NP-hard problem. Nevertheless, under the $H_2$ and $H_\infty$ setups, the software package MATLAB provides efficient solutions such as hinfsyn procedure [1] and HIFOO Toolbox ([2]). Although the SOF problem within the anisotropic-norm setup has already been considered in [3], the solution there involves a couple of Linear Matrix Inequalities (LMI) with a non-convex coupling condition, limiting its application to some practical problems.

We, therefore, consider the following plant

$$x(t + 1) = Ax(t) + B_1 w(t) + B_2 u(t)$$

where we seek for a static control matrix $K$, so that $u(t) = Ky(t)$, where

$$y(t) = C_2 x(t)$$

will minimize

$$z(t) = C_1 x(t) + D_{12} u(t) + D_{11} w(t)$$

in the sense of bounded anisotropic norm, which is yet to be specified.

**Notation.** Throughout the paper the superscript ‘$T$’ stands for matrix transposition, $\mathcal{R}$ denotes the set of scalar real numbers whereas $\mathbb{Z}_+$ stands for the non-negative integers. Moreover, $\mathcal{R}^n$ denotes the $n$ dimensional Euclidean space, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$ ($P \succeq 0$), for $P \in \mathcal{R}^{n \times n}$ means that $P$ is symmetric and positive definite (positive semi-definite). The trace of a matrix $Z$ is denoted by $Tr(Z)$, and $||v||$ denotes the Euclidian norm of an $n$-dimensional vector $v$. Finally note that the terms Lyapunov and Riccati equations in this paper, refer to generalised versions of the standard equations appearing in the $H_2$ and $H_\infty$ control literature.

We next state in Section II the problem and provide some preliminaries. An approximate solution is suggested in Section III, whereas an exact but iterative solution is offered in Section IV. Finally, Section V includes some concluding remarks.

**II. Preliminaries and Motivation**

Consider the following discrete-time system $F$, described by

$$\begin{align*}
x(t + 1) &= Ax(t) + Bw(t) \\
y(t) &= Cx(t) + Dw(t), \ t = 0, 1, \ldots
\end{align*}$$

(8)

In the following, some known useful results are briefly reminded.

**Definition 1:** The $H_2$-type norm of the ES system (8) is defined as

$$\|F\|_2 = \left[ \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{\ell=0}^{\ell} E \left[ y^T(t)y(t) \right] \right]^{\frac{1}{2}},$$

where $\{y(t)\}_{t \in \mathbb{Z}_+}$ is the output of the system (1) with zero initial conditions generated by the sequence $\{w(t)\}_{t \in \mathbb{Z}_+}$ of independent random vectors with the property that $E[w(t)] = 0$ and $E[w(t)w^T(t)] = I_m$, $\{w(t)\}_{t \in \mathbb{Z}_+}$.

The next result provides a method to compute the $H_2$ norm of the system of (8) (see e.g. (3)).

**Lemma 1:** The $H_2$ type norm of the ES system (8) is given by $\|F\|_2 = (Tr(B^T XB + D^T D))^{\frac{1}{2}}$ where $X \geq 0$ is the solution of the generalised Lyapunov equation $X = A^T X A + C^T C$.

**Definition 2:** The $H_\infty$ norm of the stable discrete-time system of form (8) is defined as

$$\|F\|_\infty = \sup_{\theta \in [0, 2\pi]} \lambda_{\max} \left( F^T (e^{-j\theta}) F (e^{j\theta}) \right),$$

where $\lambda_{\max}$ denotes the maximal eigenvalue and $F(\cdot)$ is the transfer function of the system.

The $H_\infty$ norm is characterised by the following result, well-known as the Bounded Real Lemma (BRL).

**Lemma 2:** The stable system (8) has the norm $\|F\|_\infty < \gamma$ for a certain $\gamma > 0$ if and only if the Riccati equation

$$P = A^T PA + (A^T PB + C^T D) \times (\gamma^2 I - B^T PB - D^T D)^{-1} (A^T PB + C^T D)^T + C^T C$$

has a stabilising solution $P \geq 0$ such that $\Psi_{1/\gamma^2} := \gamma^2 I - B^T PB - D^T D > 0$.

It is recalled ([5]) that a symmetric solution $P$ of the above Riccati equation is called a *stabilising solution* if the system

$$x(t + 1) = (A + BK) x(t)$$

is stable, where by definition

$$K := \Psi_{1/\gamma^2}^{-1} (A^T PB + C^T D)^T.$$

To conclude this section, we state the BRL-like result to characterize the anisotropic norm ([9]). Note that for $a$ tending to infinity, the result of Lemma 2 is recovered.

**Theorem 3:** The system of (11) satisfies $||F||_a \leq \gamma$ for a given $\gamma > 0$ if and only if there exists $q \in (0, \min (\gamma^{-2}, \|F\|_\infty^2))$ such that the Riccati equation

$$X = A^T X A + (A^T XB + C^T D) \times \left( \frac{1}{q} I - B^T XB - D^T D \right)^{-1} \times (A^T XB + C^T D)^T + C^T C$$

has a stabilising solution $X \geq 0$ satisfying the following conditions

$$\Psi_q := \frac{1}{q} I - B^T XB - D^T D > 0$$

(10)

and

$$\det \left( \frac{1}{q} - \gamma^2 \right) \Psi_q^{-1} \leq e^{-2q}.$$

(11)

We conclude this section by providing motivation to use the anisotropic norm. To this end, we denote $\eta = \sqrt{1/q}$ and restate the result of Theorem 3 above, as $\|F\|_\infty < \eta$...
so that \( \det(\eta^2 - \gamma^2)\Psi_q^{-1} \leq e^{-2a} \). Namely, \( \eta^2 - \gamma^2 \leq (\det\Psi_q)^{1/m}e^{-2a/m} \). Using the general inequality (see [3]) \( (\det\Psi_q)^{1/m} \leq \frac{1}{m}e^{-\gamma q} \) valid for any \( \Psi_q \geq 0 \), and noting that \( \Psi_q = \eta^2 I - B^TXB - D^TD \), the following motivating result of [24] was obtained:

**Lemma 4:** Consider the system \( F \) of (1). Let \( \eta \) and \( \sigma \) respectively satisfy

\[ \|F\|_{\infty} < \eta \] and \( \|F\|_2 < \sigma \)

The anisotropic norm of the system (1) is then upper bounded by the following linear interpolation between its \( H_{\infty} \) and \( H_2 \) norms. Namely,

\[ \gamma^2 \geq \eta^2(1 - e^{-2a/m}) + \frac{\sigma^2}{m}e^{-2a/m} \]

We, therefore, see that in view of (3) one may interpret the anisotropic norm, the following approximate relation

\[ \|F\|_2^2 \approx \|F\|_{\infty}^2(1 - e^{-2a/m}) + \|F\|_2^2e^{-2a/m} \]

providing a useful insight to the anisotropic norm, which approximation can be interpreted also as just mixed \( H_{\infty}/H_2 \) optimization, however, in the exact proportions dictated by the Lemma, in terms of \( e^{-2a/m} \).

**III. Static Output Feedback**

We next provide a solution to the problem of synthesis of SOF control synthesis under the anisotropic norm, using iterative solution for LMIs. To this end, we define the cost function to be

\[ J(K) = |||F_{cl}(K)|||_a \]  

where \( F_{cl}(K) \) denotes the closed loop system obtained from (6) and (7) with the static output feedback \( u_2(t) = K(y(t) \]

Using Theorem 3, if follows that the above closed loop system \( F_{cl}(K) \) is stable and it has the anisotropic norm less than a given \( \gamma > 0 \) if and only if there exist a \( q \in (0, \min(\gamma^{-2}, \|F_{cl}\|_2^2)) \) and a symmetric matrix \( X > 0 \) such that

\[ \begin{bmatrix} \mathcal{E}_1(X, K) & \mathcal{E}_2(X, K) \\ (1, 2)^T & -\frac{1}{q}I + B_1^TXB_1 + D_{11}^1D_{11}^1 \end{bmatrix} < 0 \]  

where

\[ \mathcal{E}_1(X, K) := -X + (A + B_2KC_2)^TX(A + B_2KC_2) + (C_1 + D_{12}KC_2)^T(C_1 + D_{12}KC_2) \]

\[ \mathcal{E}_2(X, K) := (A + B_2KC_2)^TXB_1 + (C_1 + D_{12}KC_2)^TD_{11} \]

and

\[ \frac{1}{q} - \gamma^2 < e^{-2a} \left(\det\left(\frac{1}{q}I - B_1^TXB_1 - D_{11}^T D_{11}\right)\right)^{1/m} \]  

Based on Schur complements arguments, one can see that the inequality (13) is equivalent with

\[ \begin{bmatrix} -X & 0 & (3, 1)^T & (4, 1)^T \\ 0 & -\frac{1}{q}I & B_1^TX & D_{11}^T \\ A + B_2KC_2 & B_1 & -X^{-1} & 0 \\ C_1 + D_{12}KC_2 & D_{11} & 0 & -I \end{bmatrix} < 0 \]  

Multiplying the above inequality to the left and to the right by \( diag(I, I, X, I) \) one obtains that is is equivalent with

\[ \begin{bmatrix} -X & 0 & (3, 1)^T & (4, 1)^T \\ 0 & -\frac{1}{q}I & B_1^TX & D_{11}^T \\ X(A + B_2KC_2) & XB_1 & -X & 0 \\ C_1 + D_{12}KC_2 & D_{11} & 0 & -I \end{bmatrix} < 0 \]

which may be re-written as

\[ Z + \mathcal{P}^TKQ + Q^TK^T\mathcal{P} < 0, \]

where we denoted

\[ Z := \begin{bmatrix} -X & 0 & A^TX & C_1^T \\ 0 & -\frac{1}{q}I & B_1^TX & D_{11}^T \\ XA & XB_1 & -X & 0 \\ C_1 & D_{11} & 0 & -I \end{bmatrix}, \]

\[ \mathcal{P}^T := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

According with the so-called Projection lemma (see e.g. [15]), the inequality (17) is feasible with respect to \( K \) if and only if the following conditions are accomplished

\[ W^T\mathcal{P}ZW < 0 \]

and

\[ W^TQZ\mathcal{Q} < 0, \]

where \( W_P \) and \( W_Q \) are any bases of the null spaces of \( \mathcal{P} \) and \( \mathcal{Q} \), respectively. Since a base of the null space of \( \mathcal{P} \) is:

\[ W_P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1}W_1 & 0 \\ 0 & 0 & W_2 & 0 \end{bmatrix} \]

where \( W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \) is the orthogonal complement of \( \begin{bmatrix} B_2^T \\ D_{12}^T \end{bmatrix} \). Similarly, a base of the null space of \( \mathcal{Q} \) is:

\[ W_Q = \begin{bmatrix} W_3 & 0 & 0 & 0 \\ W_4 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \]

where \( V := \begin{bmatrix} W_3 \\ W_4 \end{bmatrix} \) is the orthogonal complement of \( \begin{bmatrix} C_2 & 0 \end{bmatrix} \). In order to simplify the inequality of (19) we next express

\[ W_P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & W_1^T & W_2^T \\ 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \]
Namely
\[ W_P = \begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & X^{-1} \\ 0 & 0 & 0 \end{bmatrix} \]
with the above definition of \( W \). Therefore, (19) is simply expressed as
\[ \begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} < 0 \] (23)
where \( M \) is given by
\[ M := \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \]
with
\[ M_{11} = \begin{bmatrix} -X & 0 \\ 0 & -\frac{1}{\eta}I \end{bmatrix}, \quad M_{12} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \quad M_{22} = \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \] (24)
Then, from (23), using the Schur complement of \( M_{11} \) it follows that or by using Schur complements
\[ W^T(M_{22} - M_{12}^T M_{11}^{-1} M_{12})W < 0 \] (25)
Substituting the definition for \( M_{ij} \), \( i, j = 1, 2 \) and recalling the definition \( \eta^2 = \frac{1}{\gamma} \) we obtain the following convenient form of (25) which we denote by \( \mathcal{L}_Y < 0 \) the inequality
\[ W^T \begin{bmatrix} -Y + AY^T + B_1 B_1^T & AY C_1^T + B_1 D_{11}^T + \Phi_Y \\ C_1 Y A^T + D_{11} B_1^T & -\Phi_Y \end{bmatrix} W < 0 \]
where
\[ \Phi_Y := \eta^2 I - C_1 Y C_1^T D_{11}^T \]
and where we have defined
\[ \eta^{-2}Y = X^{-1} \] (26)
We next repeat the same lines to simplify (20) as well. To this end, we partition
\[ W_Q = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \]
and readily obtain using Schur complements, that (20) is equivalent to
\[ V^T(N_{11} - N_{12} N_{22}^{-1} N_{12}^T) V < 0 \]
where
\[ N_{11} = \begin{bmatrix} -X & 0 \\ 0 & -\frac{1}{\eta}I \end{bmatrix}, N_{12} = \begin{bmatrix} X A & X B_1 \\ C_1 & D_{11} \end{bmatrix}^T, \quad N_{22} = \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \] (27)
We, therefore, obtain the following form of (20)
\[ V^T \begin{bmatrix} -X + A^T A + C_1^T C_1 & A^T X B_1 + C_1^T D_{11} \\ B_1^T X A + D_{11}^T C_1 & -\Phi_X \end{bmatrix} V < 0 \]
where
\[ \Phi_X := \eta^2 I - B_1^T X B_1 - D_{11}^T D_{11} \]
We denote the latter inequality by :
\[ \mathcal{L}_X < 0 \]
We summarize the above results in the following result.
**Theorem 5:** The closed loop system \( \mathcal{F}_{cf}(K) \) is stable and it has the \( a \)-anisotropic norm less than a given \( \gamma > 0 \) if there exist symmetric matrices \( X > 0 \) and \( Y > 0 \) and a scalar \( \eta^2 \) which satisfy the following dual LMIs
\[ \mathcal{L}_X < 0, \mathcal{L}_Y < 0 \]
so both the following convex condition
\[ \eta^2 - det(\Phi_X)^{1/m} e^{-2n/m} < \gamma^2 \] (28)
and the additional bilinear condition :
\[ XY = \eta^2 I \] (29)
are satisfied.

If the conditions of Theorem 5 are satisfied then the static output gain may be obtained solving (17) with respect to \( K \).

However, the above requires a solution of a set of Bilinear Matrix Inequalities (BMI) due to the convex problem comprised of the inequalities (11), (30) and (14) is solved for a given \( \gamma \), \( k = 0 \). \( X_k = 0 \) and \( Y_k = 0 \) are set. Next step where \( k \) is set to \( k + 1 \) and \( X, Y \) are found so as to minimize
\[ f_k := Tr\{X_k Y + XY_k\} \]
subject to \( \mathcal{L}_X < 0, \mathcal{L}_Y < 0, (28) \) and (30). Then \( X_k = X \) and \( Y_k = Y \) are set. This step is repeated until \( f_k \) is small enough. A related algorithm requiring also line search but with improved convergence properties has been suggested in [12].

**IV. APPLICATION TO FLIGHT CONTROL**

We next consider the numerical example of [27] with the synthesis of pitch control loop for the F4E aircraft. Consider
\[ \begin{bmatrix} \frac{d}{dt} \Delta e \\ \delta_e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} \Delta e \\ q \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \begin{bmatrix} u + 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \omega + \begin{bmatrix} 0 & 30 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \omega \\ z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 & 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \end{bmatrix} 0.001 \\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \]
The state-vector consists of the load-factor $N_z$, the pitch-rate $q$ and elevon angle $\delta_e$, where the latter relates to the elevon command $u$ via a first-order servo model of a bandwidth of $30\text{rad/sec}$. We note that we have deliberately chosen a small weight on $u$ in $z$ to obtain small damping ratio of the closed-loop poles, when no pole-placement restrictions are imposed. The parameters $a_{i,j}, i = 1, 2; j = 1, 2, 3$ and $b_1$ are given in [27] at the four operating points listed in the following table:

| Operating point | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|
| Mach number     | .5 | .9 | .85 | 1.5 |
| Altitude (ft)   | 5000 | 3500 | 5000 | 35000 |
| $a_{11}$        | -9896 | -.6607 | -1.702 | -.5162 |
| $a_{12}$        | 17.41 | 18.11 | 50.72 | 29.96 |
| $a_{13}$        | 96.15 | 84.34 | 263.5 | 178.9 |
| $a_{21}$        | .2648 | .08201 | .2201 | -.6896 |
| $a_{22}$        | -.8512 | -.6587 | -.1418 | -.1225 |
| $a_{23}$        | -11.39 | -10.81 | -.3199 | -.3038 |
| $b_1$           | -97.78 | -.272.2 | -.85.09 | -.175.6 |

Table 7: The parameters of the four operating points.

The static output controller $u = Ky$ will be designed for each of the four operating points, applying Theorem 5, applying the iterative procedure by [12] to deal with the bilinear equality. We will first take mean anisotropy level of $a$ that tends to $\infty$ to obtain the $H_{\infty}$ controller. Next we will take a finite value of $a$. The final paper will include a figure depicting the singular values of both cases.

V. CONCLUSIONS

A synthesis scheme for static output feedback controllers has been derived, under the setup of $a$-anisotropic norm which is based on an intermediate topology between $H_2$ and $H_{\infty}$. Given a required norm-bound, set of Linear Matrix Inequalities, along with a geometric-mean convex inequality, and an additional bilinear condition characterize sub-optimal controllers. To cope with the bilinear condition, existing algorithms suggested in the past for static output feedback $H_{\infty}$ synthesis, can be used. The resulting control scheme is useful in the aerospace industry for flight control loops, where controllers with classical "cook-book" structures in the style of Proportional-Integral-Derivative controllers which can also cope with flexible modes [25]. The final form of the paper will include the results of an example using the iterative algorithm of [12].

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