Flavor Symmetry $L_e - L_\mu - L_\tau$, Atmospheric Neutrino Mixing and CP Violation in the Lepton Sector

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Abstract

The PMNS neutrino mixing matrix is given, in general, by the product of two unitary matrices associated with the diagonalization of the charged lepton and neutrino mass matrices. Assuming that the active flavor neutrinos possess a Majorana mass matrix which is diagonalized by a bimaximal mixing matrix, we give the allowed forms of the charged lepton mixing matrix and the corresponding implied forms of the charged lepton mass matrix. We then assume that the origin of bimaximal mixing is a weakly broken flavor symmetry corresponding to the conservation of the non–standard lepton charge $L' = L_e - L_\mu - L_\tau$. The latter does not predict, in general, the atmospheric neutrino mixing to be maximal. We study the impact of this fact on the allowed forms of the charged lepton mixing matrix and on the neutrino mixing observables, analyzing the case of CP–violation in detail. When compared with the case of exact bimaximal mixing, the deviations from zero $U_{e3}$ and from maximal atmospheric neutrino mixing are typically more sizable if one assumes just $L'$ conservation. In fact, $|U_{e3}|^2$ can be as small as 0.007 and atmospheric neutrino mixing can take any value inside its currently allowed range. We discuss under which conditions the atmospheric neutrino mixing angle is larger or smaller than $\pi/4$. We present also a simple see–saw realization of the implied light neutrino Majorana mass matrix and consider leptogenesis in this scenario.

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1 Introduction

There has been a remarkable progress in the studies of neutrino oscillations in the last several years. The experiments with solar, atmospheric and reactor neutrinos [1, 2, 3, 4, 5, 6, 7, 8] have provided compelling evidences for the existence of neutrino oscillations driven by non–zero neutrino masses and neutrino mixing. Evidences for oscillations of neutrinos were obtained also in the first long baseline accelerator neutrino experiment K2K [9].

The interpretation of the solar and atmospheric neutrino, and of K2K and KamLAND data in terms of neutrino oscillations requires the existence of 3–neutrino mixing in the weak charged lepton current:

\[ \nu_{\ell L} = \sum_{j=1}^{3} U_{\ell j} \nu_{jL} . \]  

Here \( \nu_{\ell L} \), \( \ell = e, \mu, \tau \), are the three left–handed flavor neutrino fields, \( \nu_{jL} \) is the left–handed field of the neutrino \( \nu_j \) having a mass \( m_j \) and \( U \) is the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) neutrino mixing matrix [10]. Actually, all existing neutrino oscillation data, except the data of the LSND experiment [11], can be described if we assume the existence of 3–neutrino mixing in vacuum, Eq. (1), and we will consider this possibility in what follows.

In the standardly used parametrization, the PMNS mixing matrix has the form:

\[
U_{\text{PMNS}} = \begin{pmatrix}
                     c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}e^{i\delta}
\end{pmatrix} \text{diag}(1, e^{i\alpha}, e^{i\beta}),
\]  

where we have used the usual notations \( c_{ij} = \cos \theta_{ij} \), \( s_{ij} = \sin \theta_{ij} \), \( \delta \) is the Dirac CP violation phase, \( \alpha \) and \( \beta \) are two possible Majorana CP violation phases [13, 14].

The ranges of values of the three neutrino mixing angles, which are allowed at 3 s.d. by the most recent solar, atmospheric (and long–baseline accelerator) neutrino data and by the data from the reactor antineutrino experiments CHOOZ [15] and KamLAND, read [6, 8, 16]:

\[
0.27 \leq \tan^2 \theta_\odot \equiv \tan^2 \theta_{12} \leq 0.58 ,
\]

\[
|U_{e3}|^2 = \sin^2 \theta_{13} < 0.048 ,
\]

\[
\sin^2 2\theta_{\text{atm}} \equiv \sin^2 2\theta_{23} \geq 0.85 .
\]

The values of \( \tan^2 \theta_{12} \), \( \sin^2 \theta_{13} \) and \( \sin^2 2\theta_{23} \) suggested by the data, are relatively close to those obtained assuming that \( U_{\text{PMNS}} \) has bimaximal mixing form: the bimaximal mixing “scenario” [17] would correspond to \( \theta_{12} = \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \). Whereas data favors maximal atmospheric neutrino mixing, and allows for \( \theta_{13} = 0 \), maximal solar neutrino mixing is ruled out at close to 6\( \sigma \) [8, 16]. Though definitely large, the atmospheric neutrino mixing angle can deviate significantly from being maximal, a fact seen clearly in re–writing the 99.73% (90%) C.L. allowed range of \( \sin^2 2\theta_{\text{atm}} \geq 0.85 \) (0.92) as 0.44 (0.55) \( \leq \tan^2 \theta_{\text{atm}} \leq 2.26 \) (1.79).  

\(^1\)In the LSND experiment indications for oscillations \( \bar{\nu}_e \to \bar{\nu}_\mu \) with \( (\Delta m^2)_{\text{LSND}} \simeq 1 \text{ eV}^2 \) were obtained. The LSND results are being tested in the MiniBooNE experiment at Fermilab [12].
In what regards the neutrino mass squared differences driving the solar and atmospheric neutrino oscillations, $\Delta m^2_\odot$ and $\Delta m^2_A$, the best–fits of the current data are obtained for $(\Delta m^2_\odot)_{BF} = 8.0 \cdot 10^{-5}$ eV$^2$ [16] and $(\Delta m^2_A)_{BF} = 2.1 \cdot 10^{-3}$ eV$^2$ [6]. A phenomenologically very interesting quantity is the ratio $R$ of $\Delta m^2_\odot$ and $\Delta m^2_A$, whose “best–fit value” is

\[
R_{BF} \equiv \frac{(\Delta m^2_\odot)_{BF}}{(\Delta m^2_A)_{BF}} = \frac{8.0 \cdot 10^{-5}}{2.1 \cdot 10^{-3}} \simeq 0.040 .
\]

Using for $\Delta m^2_A$ and $\Delta m^2_\odot$ the $3\sigma$ allowed ranges from [6] and [16], $\Delta m^2_A = (1.3 - 4.2) \cdot 10^{-3}$ eV$^2$ and $\Delta m^2_\odot = (7.2 - 9.5) \cdot 10^{-5}$ eV$^2$, we find that $R$ lies approximately between 0.017 and 0.073.

The understanding of the origin of the patterns of neutrino mixing and of neutrino mass squared differences suggested by the data, is one of the central problems in today’s neutrino physics (see, e.g., [18]). Consider the Lagrangian which is compatible with the low energy squared differences suggested by the data, is one of the central problems in today’s neutrino physics (see, e.g., [18]). Consider the Lagrangian which is compatible with the low energy

\[
\mathcal{L} = -\frac{1}{2} (\nu'\ell)_L m_\nu (\nu'\ell)_R - (\ell'\nu)_L m_\ell (\ell'\nu)_R + \frac{g}{\sqrt{2}} W_\mu (\ell')_L \gamma^\mu (\nu')_L + h.c. ,
\]

where $(\ell'_{L(R)})^T \equiv (\ell'_{L(R)}, \nu'_{L(R)}, e^\ast_{L(R)})$, $(\nu')^T \equiv (\nu'_{\nu L}, \nu'_{\mu L}, \nu'_{\tau L})$, and $\nu'_{L(R)}$ being the weak eigenstate charged lepton and neutrino fields, $\nu'_{L(R)} \equiv C(\nu')_{L(R)}^T$, and $C$ is the charge conjugation matrix. The neutrino mass matrix $m_\nu$ and the charged lepton mass matrix $m_\ell$ are diagonalized through the well–known congruent and bi–unitary transformations, respectively:

\[
m_{\nu} = U_{\nu} m^\text{diag}_{\nu} U_{\nu}^T , \quad m_{\ell} = U_{\ell} m^\text{diag}_{\ell} U_{\ell}^T .
\]

Therefore, the matrix $m_{\ell} m_{\ell}^T$ is diagonalized by $U_{\ell}$. Written in terms of the new fields $\nu_L = U_{\nu}^\dagger (\nu')_L$, $\ell_L = U_{\ell}^\dagger (\ell')_L$ and $\ell_R = U_{\ell}^\dagger (\ell')_R$, the mass terms are diagonal and the weak charged lepton current takes the form: $\bar{\nu}_L \gamma^\mu U_{\ell}^\dagger U_{\nu} \nu_L$. We therefore identify the PMNS matrix as

\[
U_{\text{PMNS}} = U_{\ell}^\dagger U_{\nu} .
\]

In the present article we extend the analysis performed in [19] on a possible origin of the patterns of neutrino mixing and of neutrino mass squared differences emerging from the data. Following [19], we assume that $U_{\nu}$ has bimaximal mixing form and that the deviations from this form are induced by $U_{\ell}^\dagger$. Such a possibility has been considered phenomenologically a long time ago (in a different context) in [20]. More recently it has been investigated in some detail also in Refs. [21, 22, 23, 24]. Certain aspects of it have been discussed in the context of GUT theories, e.g., in Ref. [23]. The alternative hypothesis, namely, that $U_{\ell}$ has bimaximal mixing form and the observed deviations from it are due to $U_{\nu}$, was considered recently in Refs. [26, 27]. The present study concentrates on the possibility that the non–standard lepton charge $L' = L_e - L_\mu - L_\tau$ is responsible for the structure of $U_{\nu}$. We note that this flavor symmetry does not predict the atmospheric neutrino mixing to be necessarily maximal, but rather to be determined by ratio of two numbers of order one. This has some impact on the neutrino mixing observables obtained after including the corrections from the charged lepton sector.
In Section 2 we review the basics of the approach to understand the observed neutrino mixing as deviation from bimaximal mixing. We then concentrate on the case when the charged lepton sector is responsible for the deviation and give a “catalog” (although not complete) of the implied possible structures of the corresponding charged lepton and neutrino mass and mixing matrices. Considering the approximate conservation of the non–standard lepton charge $L' = L_e - L_\mu - L_\tau$ by $m_\nu$ as a possible origin of maximal and/or zero neutrino mixing we generalize the preceding analysis to the case of a priori non–maximal atmospheric neutrino mixing in Section 3. A very simple see–saw realization of the neutrino mass matrix having the indicated approximate symmetry is given in Section 4 where we discuss also its implications for leptogenesis. Section 5 summarizes the results of this work.

2 Deviations from Bimaximal Neutrino Mixing

2.1 General considerations

The primary goal of the approaches used to produce deviations from bimaximal mixing is to generate a deviation from maximal solar neutrino mixing \(^2\). The latter can be parameterized through a small parameter $\lambda \sim 0.2$ as \(^{22}\)

$$\sin \theta_\odot = \sqrt{\frac{1}{2}} (1 - \lambda). \quad (8)$$

Note that the parameter $\lambda$ is very similar in value to the Cabibbo angle \(^{22}\): $\lambda \sim \sin \theta_C \sim \theta_C$. The implied relation $\theta_\odot = \pi/4 - \theta_C$, if confirmed experimentally, might be linked to Grand Unified Theories \(^{29, 27, 30}\).

When one starts from bimaximal neutrino mixing and introduces some “perturbation” to generate non–maximal solar neutrino mixing, typically also $|U_{e3}|$ and the atmospheric neutrino mixing will deviate from their “bimaximal” values. These additional deviations will, in general, be proportional to some power of $\lambda$. Thus, a useful parametrization taking into account this fact is \(^{22}\)

$$U_{e3} = A_\nu \lambda^n e^{i\beta}, \quad U_{\mu3} = \sqrt{\frac{1}{2}} (1 - B_\nu \lambda^m) e^{i(\delta + \beta)}, \quad (9)$$

where $m, n$ are integer, and $A_\nu, B_\nu$ are real numbers, $A_\nu \geq 0$. Once $\sin \theta_\odot$, and correspondingly $\lambda$, is determined with relatively high precision, $m, n, A_\nu$ and $B_\nu$ can be fixed using the measured values of $|U_{e3}|$ and $|U_{\mu3}|$. For further details, we refer to \(^{22}\).

Deviations from bimaximal neutrino mixing can be generated, e.g., via radiative corrections \(^{31}\) or via small perturbations to a zeroth order neutrino mass matrix corresponding to bimaximal mixing. The latter can be achieved through “anarchical” perturbations \(^{32}\) or, e.g., by the presence of a small contribution from the conventional see–saw term in type II see–saw models \(^{33}\).

\(^2\)It is even possible to construct a model in which maximal solar neutrino mixing would be linked to a vanishing baryon asymmetry of the Universe \(^{28}\).
The ratio $R$ of $\Delta m^2_\odot$ and $\Delta m^2_A$ numerically turns out to be of order $\lambda^2$. Indeed, using the definition of $\lambda$ in Eq. (8) and the range of $\theta_{12}$ as given in Eq. (3), one finds that the $3\sigma$ allowed range of $\lambda$ is

$$\lambda \approx 0.16 - 0.37.$$  

The best-fit value of $\sin^2 \theta_{12} \approx 0.28$, obtained in the 3-neutrino oscillations analysis in [16], corresponds to $\lambda \approx 0.25$. Remembering that $R$ lies between 0.017 and 0.073, one sees that to a relatively good precision we have $R \approx \lambda^2$. This can be used to analyze the structure of the neutrino mass matrix in a simple manner [22].

When the deviations from bimaximal mixing stem from charged lepton mixing, and the mass matrices are not implemented in a specific model, the ratio $R$ is independent of the deviations and the relation between $R$ and $\lambda$ is purely accidental $^3$. If alternatively bimaximal mixing and zero $\Delta m^2_\odot$ are due to some symmetry, and the symmetry breaking involves just one “small” parameter, all deviations from bimaximal mixing and the ratio $R$ will be linked to $\lambda$. An example of such a scenario can be found in [33].

Assuming that the neutrino mass matrix generates bimaximal neutrino mixing $^4$, the allowed structures of the charged lepton mixing matrix have been analyzed in a model-independent way in [19]. In the next Section we will briefly summarize the obtained results. We will give also the implied structures of the charged lepton mass matrix, which were not presented in [19].

### 2.2 Deviations Due to Charged Lepton Mixing and Implied Structures of the Charged Lepton Mass Matrix

The goal is to generate successful neutrino mixing phenomenology by using the relation $U_{\text{PMNS}} = U^\dagger_L U_\nu$, where $U_L$ is the mixing matrix diagonalizing $m_\ell m_\ell^\dagger$ and $U_\nu$ diagonalizes the neutrino mass matrix. The matrix $U_\nu$ is assumed to have exact bimaximal mixing form, namely

$$U_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (11)$$

Three generic structures of $U_L$ can be identified in this case [19]. They can be classified by the magnitude of the sines of the three Euler angles $\theta_{ij}$ in $U_L$, $\sin \theta_{ij} \equiv \lambda_{ij}$, where $ij = 12, 13, 23$ in a parametrization of $U_L$ analogous to that in Eq. (2).

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$^3$As is well-known, if the approximate conservation of the lepton charge $L' = L_e - L_\mu - L_\tau$ [20] is at the origin of bimaximal neutrino mixing and zero $\Delta m^2_\odot$, it is possible to generate the observed values of neutrino mass and mixing parameters by adding small perturbations to the neutrino mass matrix only if the effect of $U_L \neq I$, arising from the charged lepton sector, is taken into account [24], see below.

$^4$Obviously, in the Standard Theory there would be no difference whether the bimaximal mixing originates from the neutrino or charged lepton mass term. The underlying theory whose low energy limit is the Standard Theory is assumed here to have “chosen” the neutrino mass term as the source of bimaximal mixing, i.e., neutrinos to be “special”.

5
• All $\lambda_{ij}$ are small: $\lambda_{ij} \lesssim 0.35$.

The matrix $U_L$ has a hierarchical “CKM–like” form, naturally expected given the hierarchical structure of the charged lepton masses. In this case relatively large values of $\theta_\odot$ and of $|U_{e3}|^2$ are typically predicted, $\tan^2 \theta_\odot \gtrsim 0.42$, $|U_{e3}|^2 \gtrsim 0.02$, while the atmospheric neutrino mixing angle $\theta_{\text{atm}}$ can deviate — depending on the hierarchy of the three $\lambda_{ij}$ — noticeably from $\pi/4$, $\sin^2 2\theta_{\text{atm}} \gtrsim 0.95$.

• $\lambda_{23} \simeq 1$, $\lambda_{12,13}$ are small: $\lambda_{12,13} \lesssim 0.35$.

The main consequence of this scenario is practically maximal atmospheric neutrino mixing $\sin^2 2\theta_{\text{atm}} \simeq 1$.

• All $\lambda_{ij}$ are large: $\lambda_{ij} \gtrsim 0.4$.

No interesting correlation of parameters or preferred values of the observables is found in this case.

We shall focus in what follows on the first two possibilities. To estimate the form of the relevant matrices, we can use the characteristic quantity $\lambda \simeq 0.20 – 0.25$, which typically appears when the magnitude of the angles in $U_L$ is constrained. Numerically, we also have

$$m_\mu \sim m_\tau \lambda^2 \text{ and } m_e \sim m_\tau \lambda^6,$$

(12)
a fact which makes it possible to express the elements of the charged lepton mass matrix in terms of powers of $\lambda$. Recall that $m_\ell$ is in general diagonalized by a bi–unitary transformation defined by $U_L$ and $U_R$. Since only $U_L$ appears in the PMNS matrix, we will analyzed two mass matrices associated with the charged leptons. The first one is $m_\ell m_\ell^T$, which is diagonalized by $U_L$. The second one is defined as $U_L m_\ell \text{diag}\ U_L^T$, which would correspond to the charged lepton mass matrix if it was symmetrical, a possibility which can occur in certain GUT models.

2.2.1 Small $\lambda_{ij}$

The characteristic value of $\lambda_{12}$ is approximately 0.23. For the other two $\lambda_{ij}$ there are basically two possibilities [19]. Both $\lambda_{23}$ and $\lambda_{13}$ can be much smaller than $\lambda_{12}$: $\lambda_{12} = \lambda$, $\lambda_{13} = A \lambda^3$ and $\lambda_{23} = B \lambda^2$. Here, the parameters $A, B$ are positive numbers of order one.

The charged lepton mass matrices $m_\ell m_\ell^T$ and $m_\ell = m_\ell^T$ have the form:

$$m_\ell m_\ell^T \sim m_\tau^2 \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad U_L m_\ell \text{diag}\ U_L^T \sim m_\tau \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}.$$

(13)

The expressions for the neutrino mixing parameters read [19]

$$\tan^2 \theta_\odot \simeq 1 – 2\sqrt{2} \lambda + 4\lambda^2 – 2\sqrt{2}(2 – A + B)\lambda^3,$$

$$|U_{e3}| \simeq \left| \frac{\lambda}{\sqrt{2}} + \frac{A – B}{\sqrt{2}} \lambda^3 \right|,$$

$$\sin^2 2\theta_{\text{atm}} \simeq 1 – \frac{(1 + 4B)^2}{4} \lambda^4.$$
Thus, to leading order, the following correlations hold:

\[ \tan^2 \theta_\odot \simeq 1 - 4 \left| U_{e3} \right| (1 - 2|U_{e3}|) - 16 \left| U_{e3} \right|^3, \quad \sin^2 2\theta_{\text{atm}} \simeq 1 - (1 + 4B)^2 \left| U_{e3} \right|^4. \quad (15) \]

The first relation implies that future more precise measurements on the magnitude of \( |U_{e3}| \) can confirm or disfavor, or even rule out, this scenario (see also [19, 21]).

The second possibility corresponds to \( \lambda_{12} \) and \( \lambda_{23} \) having similar magnitudes, say, \( \lambda_{12} = \lambda \) and \( \lambda_{23} = B \lambda/2 \), but \( \lambda_{13} = A \lambda^3 \). One finds in this case

\[
\begin{pmatrix}
\lambda^6 & \lambda^4 & \lambda^3 \\
\lambda^4 & \lambda^2 & \lambda \\
\lambda^3 & \lambda & 1
\end{pmatrix}
= U_L m^\dagger_\ell U_L^T \sim m_\tau
\]

and

\[
\tan^2 \theta_\odot \simeq 1 - 2\sqrt{2} \lambda + (4 - \sqrt{2}B)\lambda^2 - \left(4\sqrt{2} - 2\sqrt{2}A - 4B - \frac{B^2}{2\sqrt{2}}\right) \lambda^3,
\]

\[
\left| U_{e3} \right| \simeq \left| \frac{\lambda}{\sqrt{2}} - \frac{B}{2\sqrt{2}} \lambda^2 \right|,
\]

\[
\sin^2 2\theta_{\text{atm}} \simeq 1 - B^2 \lambda^2.
\]

A similar correlation between \( \tan^2 \theta_\odot \) and \( |U_{e3}| \) as in the previous case holds (see Eq. (15)), but now the deviations from maximal atmospheric neutrino mixing can be larger: \( \sin^2 2\theta_{\text{atm}} \simeq 1 - 2B^2 \left| U_{e3} \right|^2 \).

### 2.2.2 \( \lambda_{23} \approx 1 \) and \( \lambda_{12,13} \) small

The case of \( \lambda_{23} \approx 1 \) and \( \lambda_{12,13} \) being small is counterintuitive, but nevertheless allowed by the data [19]. There are three different possibilities for the hierarchy between \( \lambda_{12} \) and \( \lambda_{13} \): i) \( \lambda_{12} \equiv \lambda, \lambda_{13} = A \lambda^2 \), ii) \( \lambda_{12} \equiv \lambda^2, \lambda_{13} = A \lambda \) and iii) \( \lambda_{12} \equiv \lambda/2, \lambda_{13} = A \lambda/2 \), where \( A \) is a real parameter of order one. All lead to expressions for, and relations between, the neutrino mixing parameters similar to those found in the “Wolfenstein–case” and given in Eqs. (14) and (15). The only particularity can occur in the case \( \lambda_{12} \equiv \lambda/2, \lambda_{13} = A \lambda/2 \): the two terms in the expression for \( |U_{e3}| \) can cancel and produce \( |U_{e3}| \simeq 0 \).

We give below the form of the corresponding charged lepton mass matrices \( m_\ell m^\dagger_\ell \) and \( m_\ell = m^T_\ell \) in each of these three cases. For \( \lambda_{12} \equiv \lambda \) and \( \lambda_{13} = A \lambda^2 \), we find

\[
\begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^5 \\
\lambda^2 & 1 - \lambda^4 & \lambda^7 \\
\lambda^5 & \lambda^7 & \lambda^4
\end{pmatrix},
U_L m^\dagger_\ell U_L^T \sim m_\tau
\]

\[
\begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^3 \\
\lambda^2 & 1 - \lambda^4 & \lambda^5 \\
\lambda^3 & \lambda^5 & \lambda^2
\end{pmatrix}.
\]

(18)
In the case of $\lambda_{12} \equiv \lambda^2$ and $\lambda_{13} = A \lambda$, we get:

$$m_{\ell}m_{\ell}^\dagger \sim m_\tau^2 \left( \begin{array}{ccc} \lambda^2 & \lambda & \lambda^6 \\ \lambda & 1 - \lambda^2 & \lambda^7 \\ \lambda^6 & \lambda^7 & \lambda^4 \end{array} \right), \quad U_L m_{\ell}^{\text{diag}} U_L^T \sim m_\tau \left( \begin{array}{ccc} \lambda^2 & \lambda & \lambda^4 \\ \lambda & 1 - \lambda^2 & \lambda^5 \\ \lambda^4 & \lambda^5 & \lambda^2 \end{array} \right). \quad (19)$$

Finally, if $\lambda_{12} \equiv \lambda/2$ and $\lambda_{13} = A \lambda/2$, one finds

$$m_{\ell}m_{\ell}^\dagger \sim m_\tau^2 \left( \begin{array}{ccc} \lambda^2 & \lambda & \lambda^5 \\ \lambda & 1 - \lambda^2 & \lambda^6 \\ \lambda^5 & \lambda^6 & \lambda^4 \end{array} \right), \quad U_L m_{\ell}^{\text{diag}} U_L^T \sim m_\tau \left( \begin{array}{ccc} \lambda^2 & \lambda & \lambda^3 \\ \lambda & 1 - \lambda^2 & \lambda^4 \\ \lambda^3 & \lambda^4 & \lambda^2 \end{array} \right). \quad (20)$$

Note that the largest entry is always the 22 element.

### 2.3 The $L_e - L_\mu - L_\tau$ Flavor Symmetry

We shall consider next the possibility that maximal and/or zero mixing in $U_{\nu}$ are a consequence of the flavor symmetry associated with the conservation of non-standard lepton charge \[20, 25\]

$$L' = L_e - L_\mu - L_\tau.$$  

The most general neutrino Majorana mass matrix obeying this symmetry reads \[20\] (see also \[35\])

$$m_\nu = m_0 \left( \begin{array}{ccc} 0 & -\cos \theta & \sin \theta \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{array} \right), \quad (21)$$

where $m_0$ denotes the typical neutrino mass scale in this scheme, $m_0 = \sqrt{2\Delta m^2_{\odot}}$. The fact that atmospheric neutrino mixing is observed to be relatively large implies that $|\cos \theta| \sim |\sin \theta|$. The neutrino mass matrix Eq. (21) predicts \[20\] a neutrino mass spectrum with inverted hierarchy, $m_3 = 0$ (since its rank is 2), maximal solar neutrino mixing, $|U_{e3}| = 0$, $R = 0$ and atmospheric neutrino mixing corresponding to $\theta^\nu_{23} = \theta$. Without applying further symmetries, e.g., a $\mu - \tau$ symmetry in the neutrino sector \[36\], the atmospheric mixing is not maximal, and furthermore, \textit{a priori} is unconstrained. Replacing a zero entry with a small entry proportional to $\epsilon^2$ will generate a non–zero $\Delta m^2_{\odot}$, according to $R \propto \epsilon^2$. We are considering only the case of adding one such non–zero entry (similar results occur for two or more), since this is the simplest possibility. In this case one also finds interesting relations for the $CP$–violating observables, see \[19\] and below. Table \[I\] summarizes the possibilities for adding one entry in $m_\nu$, which breaks the $L_e - L_\mu - L_\tau$ symmetry. As can be seen from Table \[I\] for all cases the corrections to $|U_{e3}| = 0$ and to $\tan^2 \theta_\odot = 1$ are at most proportional to $R$ and therefore especially for $\tan^2 \theta_\odot$ they are negligible.
Table 1: Possible ways of breaking of the flavor symmetry implying the conservation of $L_e - L_\mu - L_\tau$, by adding in $m_\nu$ one small (but non-zero) entry $m_0 \epsilon^2$ which generates non–vanishing $R \equiv \Delta m^2_3 / \Delta m^2_0 \sim 0.04$. Shown are the expressions for the ratio $R$, the smallest neutrino mass $m_3$ (in units of $m_0^2 = 2|\Delta m^2_0|$), and the mixing parameters $|U_{e3}|$ and $\tan^2 \theta_{\odot}$. The “large” $e\mu$ and $e\tau$ entries of $m_\nu / m_0$ are denoted by $c = \cos \theta$ and $s = \sin \theta$. In all cases except the first, one has $\tan^2 \theta_{\nu23} = \tan^2 \theta + O(\epsilon^4)$; in the first case the relation $\tan^2 \theta_{\nu23} = \tan^2 \theta$ holds exactly.

Let us stress again that in the general case of $L_e - L_\mu - L_\tau$ conservation, the atmospheric neutrino mixing is unconstrained, i.e., $U_\nu$ has the form

$$U_\nu = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{\cos \theta_{\nu23}}{\sqrt{2}} & \frac{\cos \theta_{\nu23}}{\sqrt{2}} & \sin \theta_{\nu23} \\
\frac{\sin \theta_{\nu23}}{\sqrt{2}} & -\frac{\sin \theta_{\nu23}}{\sqrt{2}} & \cos \theta_{\nu23}
\end{pmatrix}.$$  \hspace{1cm} (22)

It holds $\sin \theta_{\nu23} = (m_\nu)_{e\tau} / (m_\nu)_{e\mu}$ and with no additional contribution from the charged lepton sector this angle would correspond to the atmospheric neutrino mixing angle. In order to find the allowed forms of the charged lepton mixing matrix in this case, we should therefore redo the analysis performed in [19] with $\theta_{\nu23}$ not necessarily equal to $\pi/4$. This will be done in the next Section.

### 3 Generalization to $\theta_{\nu23} \neq \pi/4$

For illustrative reasons we will consider first the case of $CP$ invariance and analyze after that the more interesting and realistic case of $CP$ nonconservation.
3.1 The Case of CP Invariance

We repeated the analysis performed in [19], the only difference being that \( \tan \theta_{23}^\nu \) was allowed to vary in a certain range. Since \( \tan \theta_{23}^\nu \) is the ratio of the two “large” entries in \( m_\nu \), expected to be \( \sim m_0 \), \( \tan \theta_{23}^\nu \) was allowed to take values between 1/3 and 3. As in the case of \( \theta_{23}^\nu = \pi/4 \) analyzed in [19], the parameter regions favored by the data were those corresponding to all \( \lambda_{ij} \) being “small”, all \( \lambda_{ij} \) being “large” and \( \lambda_{23} \simeq 1 \) with the other two \( \lambda_{ij} \) “small”. The same parameter regions are present also if CP is not conserved, as is discussed below. Only the case of all \( \lambda_{ij} \) being small exhibits interesting correlations of neutrino mixing observables and we display some of them in Fig. 1.

We constrained \( \lambda_{ij} \) to vary in the interval \( 0 \leq \lambda_{ij} \leq 1/\sqrt{10} \), and required that the neutrino mixing observables are inside their 3\( \sigma \) allowed ranges given in Eq. 5. Again, \( \lambda_{12} \equiv \lambda \) is centered around 0.24. For \( \lambda_{23} \) there are the possibilities of \( \lambda_{23} = O(\lambda) \), in which case typically \( \theta_{23}^\nu \gtrsim \pi/4 \), and of \( \lambda_{23} = O(\lambda^2) \), leading typically to \( \theta_{23}^\nu \lesssim \pi/4 \).

For \( \theta_{23} \) we encountered in Ref. 19 the typical correlation \( \sin^2 2\theta_{\text{atm}} \sim 1 - 2 |U_{e3}|^2 \) or \( \sin^2 2\theta_{\text{atm}} \sim 1 - |U_{e3}|^4 \), which implies that at most \( \sin^2 2\theta_{\text{atm}} \gtrsim 0.9 \) [19]. As seen in Fig. 1, atmospheric neutrino mixing can now be anything between its lowest allowed value of 0.85 and 1. The parameter \( |U_{e3}|^2 \) is bounded to be larger than approximately 0.01, a limit weaker by a factor of two with respect to the case of \( \theta_{23}^\nu = \pi/4 \).

As a typical example, let us choose \( \lambda_{12} \equiv \lambda \), \( \lambda_{23} = B \lambda^2 \) and small \( \lambda_{13} = A \lambda^3 \). We find

\[
\begin{align*}
\tan^2 \theta_{12} &\simeq 1 - 4 \cos \theta_{23}^\nu \lambda + 8 \cos^2 \theta_{23}^\nu \lambda^2 ,
|U_{e3}| &\simeq |\sin \theta_{23}^\nu \lambda + (A - B) \cos \theta_{23}^\nu \lambda^3| ,
\tan^2 \theta_{\text{atm}} &\simeq \tan^2 \theta_{23} \frac{\sin \theta_{23}^\nu}{\cos^2 \theta_{23}^\nu} (4B + \sin 2\theta_{23}^\nu) \lambda^2 , \quad \text{or}
\sin^2 2\theta_{\text{atm}} &\simeq \sin^2 2\theta_{23} \frac{1}{2} (4B + \sin 2\theta_{23}^\nu) \sin 4\theta_{23}^\nu \lambda^2 \\
&\text{plus higher order terms. Setting } \theta_{12} \text{ to } \pi/4 \text{ reproduces Eq. 14. To lowest order in } \lambda \text{ we have } \\
\theta_{\text{atm}} = \theta_{23}^\nu, \text{ i.e., the atmospheric neutrino mixing angle corresponds to the } \lambda_{23} \text{ mixing angle in } U_{\nu}. \text{ We explicitly gave also the expression for } \tan^2 \theta_{\text{atm}} \text{ to address the question of whether } \theta_{\text{atm}} > \pi/4 \text{ or } \theta_{\text{atm}} < \pi/4. \text{ This ambiguity is part of the “eightfold degeneracy” from which the interpretation of the results of future long baseline neutrino experiments may suffer 37].}
\end{align*}
\]

There would be physically interesting consequences for the oscillations of atmospheric neutrinos, if \( \theta_{\text{atm}} \) differs significantly from \( \pi/4 \). A value of \( \theta_{\text{atm}} \gtrsim \pi/4 \) would imply larger (than in the case \( \theta_{\text{atm}} < \pi/4 \)) probabilities of the subdominant \( \nu_e \rightarrow \nu_\mu \) (\( \nu_e \rightarrow \bar{\nu}_\mu \)) and \( \nu_\mu \rightarrow \nu_e \) (\( \nu_\mu \rightarrow \bar{\nu}_e \)) oscillations of the multi–GeV atmospheric neutrinos \( \nu_e \) (\( \nu_\mu \)) and \( \nu_\mu \) (\( \nu_\mu \)) [38]. For sufficiently large values of \( \theta_{13} \), \( \sin^2 2\theta_{13} \gtrsim 0.05 \), these oscillations can be strongly enhanced by the Earth matter and their effects on Nadir angle distributions of the multi–GeV samples of atmospheric neutrino \( e^- \)-like and \( \mu^- \)-like (or \( \mu^- \) and \( \mu^+ \)) samples of events could be observable in the operating and future atmospheric neutrino detectors 39. For non–maximal atmospheric neutrino mixing \( \theta_{\text{atm}} < \pi/4 \), the subdominant \( \nu_e \rightarrow \nu_\mu \) and \( \nu_\mu \rightarrow \nu_e \) oscillations driven by \( \Delta m_{12}^2 \) can lead to relatively large effects in the sub–GeV sample of atmospheric \( e^- \)-like events.
measured in the SuperKamiokande experiment [10].

It follows from Eq. (23) that for $\theta^\nu_{23} = \pi/4$, the multiplication of $U_\nu$ with $U^\dagger_{lep}$ drags the atmospheric neutrino mixing towards $\theta_{atm} < \pi/4$. The presence of $CP$–violating phases can change this behavior, as will be shown below.

The quadratic correction to $\sin^2 2\theta_{atm} = \sin^2 2\theta^\nu_{23}$ vanishes for $\theta^\nu_{23} = \pi/4$, the next term being quartic in $\lambda$. Since to lowest order in $\lambda$ we have $\sin \theta^\nu_{23} \simeq \sin \theta_{atm}$, we can write using Eq. (23)

$$\tan^2 \theta_\odot \simeq 1 - 4 \left| U_{e3} \right| \cot \theta_{atm}.$$ (24)

In [23] it was stated correctly that the lowest order relation $\tan^2 \theta_\odot \simeq 1 - 4 \left| U_{e3} \right|$ can make this scenario problematic when improved limits on $\left| U_{e3} \right|$ will be available. Allowing $\theta^\nu_{23}$ to differ from $\pi/4$ makes it easier to satisfy the conflicting requirements in the scheme discussed of small $\left| U_{e3} \right|$ and $\tan^2 \theta_\odot \simeq 0.40$. Note, though, that the constraints on the atmospheric neutrino mixing still have to be satisfied.

To sum up, the fact that $\theta^\nu_{23}$ has to be close to $\pi/4$, results in very similar correlations between the neutrino mixing observables, that is $\tan^2 \theta_\odot \sim 1 - 4 \left| U_{e3} \right|$ and $\sin^2 2\theta_{atm} \sim 1 - \left| U_{e3} \right|^2 (\sin^2 2\theta_{atm} \sim 1 - \left| U_{e3} \right|^4)$ for $\lambda_{23} \sim \lambda$ ($\lambda_{23} \sim \lambda^2$). Nevertheless, better limits on $\left| U_{e3} \right|$ will not automatically lead to a conflict with $\tan^2 \theta_\odot \simeq 0.40$, unless the atmospheric neutrino mixing turns out to be very close to maximal. For instance, for the minimally allowed at $1\sigma$ value of $\sin^2 2\theta_{atm} = 0.95 \simeq \sin 2\theta^\nu_{23}$, we have $\tan^2 \theta_\odot \simeq 1 - 5 \left| U_{e3} \right|$, to be compared with the lowest order relation in the case of maximal $\theta^\nu_{23}$: $\tan^2 \theta_\odot \simeq 1 - 4 \left| U_{e3} \right|$. Thus, the presence of a non–maximal $\theta^\nu_{23}$ allows for smaller values of $\left| U_{e3} \right|$ while at the same time being in agreement with the observed non–maximal solar neutrino mixing. This is basically independent on the precise form of $U_L$, as long as the angles in $U_L$ are all small, and holds also in the case of $CP$ nonconservation.

3.2 Violation of $CP$

3.2.1 General Considerations

As was shown in [12], there are six $CP$–violating phases, in general, from the neutrino and charged lepton sector contributing to the PMNS matrix $U_{PMNS} = U^\dagger_L U_\nu$. Using the general description of unitary matrices given in [11], one can show that

$$U_{PMNS} = \tilde{U}_{lep} P_\nu \tilde{U}_\nu Q_\nu.$$ (25)

Here, $P_\nu \equiv \text{diag}(1, e^{i\phi}, e^{i\omega})$ and $Q_\nu \equiv \text{diag}(1, e^{i\rho}, e^{i\sigma})$ are diagonal phase matrices having two phases each, and $\tilde{U}_{\nu,lep}$ are unitary matrices, each containing one phase and three angles. Only $\tilde{U}_{lep}$ stems from the diagonalization of the charged lepton matrix, all the other three matrices have their origin in the neutrino sector. Note that $Q_\nu$ is “Majorana–like”, i.e., its two phases will not affect the flavor neutrino oscillations. They can enter into the expressions of observables associated with processes in which the total lepton charge is not conserved, such as neutrinoless double beta decay, see, e.g., [12, 13]. Finally, the presence of one zero angle in $\tilde{U}_\nu$ means that the Dirac–like phase in $\tilde{U}_\nu$ can be set to zero. We note in passing that if the bimaximal mixing
would stem entirely from the charged lepton sector, the Dirac–like phase in $\tilde{U}_{lep}$ would not be physical and all leptonic CP violation would stem from the neutrino sector.

In the case of 3–neutrino mixing under discussion there are, in principle, three CP violation rephasing invariants. The first is the standard Dirac one $J_{CP}$ \[^{[14]}\] , associated with the Dirac phase $\delta$ and measured in neutrino oscillation experiments \[^{[15]}\] :

$$J_{CP} = \text{Im} \left\{ U_{e1} U_{\mu2}^* U_{e2}^* U_{\mu1} \right\}. \quad (26)$$

There are two additional invariants, $S_1$ and $S_2$, whose existence is related to the two Majorana CP violation phases in $U_{PMNS}$, i.e., to the Majorana nature of massive neutrinos. As can be shown, the effective Majorana mass measured in $(\beta \beta)$–decay experiments depends, in general, on these two invariants and not on $J_{CP}$ \[^{[13]}\]. The invariants $S_1$ and $S_2$ can be chosen as \[^{[46, 47, 43]}\] :

$$S_1 = \text{Im} \left\{ U_{e1} U_{e3}^* \right\}, \quad S_2 = \text{Im} \left\{ U_{e2} U_{e3}^* \right\}. \quad (27)$$

If $U_{e3} = 0$, the Majorana phases $\alpha$ and $\beta$ in $U_{PMNS}$ can still induce CP–violating effects as long as $\text{Im} \left\{ U_{e1} U_{e2}^* \right\} \neq 0$ and $\text{Im} \left\{ U_{\mu2} U_{\mu3}^* \right\} \neq 0$ \[^{[17]}\].

Turning to the $L_e - L_\mu - L_\tau$ symmetry, we can now include the possibility of CP violation \[^{[19]}\]. Suppose, for example, that all non–zero entries in $m_\nu$ are complex and consider, e.g., the second case in Table I i.e., a small perturbation in the $\mu\mu$ entry. The neutrino mass matrix can then be written as

$$m_\nu = m_0 \begin{pmatrix} 0 & c e^{i\beta} & s e^{i\gamma} \\ . & e^{i\alpha} & 0 \\ . & . & 0 \end{pmatrix} = m_0 \text{diag}(e^{i(\beta-\alpha/2)}, e^{i\alpha/2}, e^{i(\alpha/2+\gamma-\beta)}), \quad (28)$$

and the real (inner) matrix can be diagonalized by a real orthogonal matrix $O$. Following the procedure outlined in Section 4.6 of Ref. \[^{[19]}\], we can identify the phases $\phi$ and $\omega$ present in $P_\nu$, Eq. \[^{[25]}\], with $(\alpha - \beta)$ and $(\alpha - 2\beta + \gamma)$. The phase matrix $Q_\nu$ is equal to the unit matrix \[^{[19]}\].

### 3.2.2 First example

Let us express now the neutrino mixing observables in terms of the parameters in $U_\nu$ and $U_L$. As in the case of CP conservation we choose $\lambda_{12} = \lambda$, $\lambda_{23} = B \lambda^2$ and $\lambda_{13} = A \lambda^3$. For the

\[^{5}\] We assume that the fields of massive Majorana neutrinos satisfy Majorana conditions which do not contain phase factors.
mixing parameters we get

\[ \tan^2 \theta_\odot \simeq 1 - 4 c_\phi \cos \theta^\nu_{23} \lambda + 8 c^2_\phi \cos^2 \theta^\nu_{23} \lambda^2 , \]

\[ |U_{e3}| \simeq |\sin \theta^\nu_{23} \lambda - (B c_{\omega-\phi} - A c_{\omega-\psi}) \lambda^3| , \]

\[ \tan^2 \theta_{\text{atm}} \simeq \tan^2 \theta^\nu_{23} - \frac{\sin \theta^\nu_{23}}{\cos^4 \theta^\nu_{23}} (4B c_{\omega-\phi} + \sin 2\theta^\nu_{23}) \lambda^2 , \]

\[ \sin^2 2\theta_{\text{atm}} \simeq \sin^2 2\theta^\nu_{23} - \frac{1}{2} (4B c_{\omega-\phi} + \sin 2\theta^\nu_{23}) \sin 4\theta^\nu_{23} \lambda^2 . \]

(29)

Here \( \psi \) is the Dirac–like phase in \( \tilde{U}_{\text{lep}} \) and we used the obvious notation \( c_\phi = \cos \phi \), etc. Comments similar to those in the \( CP \)–conserving case discussed above can be made. In particular, we have now a similar correlation between solar neutrino mixing and \( |U_{e3}| \):

\[ \tan^2 \theta_\odot \simeq 1 - 4 \cos \phi \cot \theta_{\text{atm}} |U_{e3}| . \]

(30)

If \( |U_{e3}| \) is relatively small, say \( |U_{e3}| \lesssim 0.10 \), the phase \( \phi \) should be close to 0 or 2\( \pi \) in order to be compatible with the data on \( \tan^2 \theta_\odot \). In the previous Section we mentioned the possibility that non–maximal \( \theta^\nu_{23} \) might resolve the possible conflict between a relatively small value of \( |U_{e3}| \) and significantly non–maximal solar neutrino mixing. Such a possibility holds when atmospheric neutrino mixing is not too close to maximal. This is illustrated in Fig. 2 where the correlation between \( \tan^2 \theta_\odot \) and \( |U_{e3}|^2 \) is shown in the case of \( CP \) nonconservation for two different limits on \( \sin^2 2\theta_{\text{atm}} \) and for the cases of \( \theta^\nu_{23} = \pi/4 \) and of “free” \( \theta^\nu_{23} \), \( 1/3 \leq \tan \theta^\nu_{23} \leq 3 \). We limited \( \lambda_{12} \equiv \lambda \leq 1/\sqrt{10} \) and chose \( \lambda_{23} = B \lambda^2 \) and \( \lambda_{13} = A \lambda^3 \), with \( A, B \) varying between \( 1/\sqrt{3} \) and \( \sqrt{3} \). It is seen that in the case of “free” \( \theta^\nu_{23} \) and the current limit of \( \sin^2 2\theta_{\text{atm}} > 0.85 \), smaller values of both \( \tan^2 \theta_\odot \) and \( |U_{e3}| \) are possible, whereas in the case of \( \sin^2 2\theta_{\text{atm}} > 0.995 \) no difference exists between the two cases. In fact, the lower limit on \( |U_{e3}|^2 \) is roughly a factor of two weaker when \( \theta^\nu_{23} \) is not \( \pi/4 \) (\( |U_{e3}|^2 \gtrsim 0.013 \) and \( \gtrsim 0.007 \), respectively). Note that the exact expression for the observables were used to produce the plots in Fig. 2 (and not the leading order relations, Eq. (30)).

It proves useful to derive an expression for the difference between \( S_1 \) and \( S_2 \) in the case of \( \rho = \sigma = 0 \). As shown in [19] (see also the preceding discussion), these two phases vanish if, e.g., the neutrino mass matrix has the form given in Eq. (28). When all \( \lambda_{ij} \) are small, \( S_1 - S_2 \) will take its maximal value for \( \lambda_{12} \equiv \lambda, \lambda_{23} = B \lambda \) and \( \lambda_{13} = A \lambda \), i.e., when all three \( \lambda_{ij} \) have similar magnitudes. Then one finds:

\[ (S_1 - S_2)|_{\rho=\sigma=0} \simeq \sqrt{2}A \ s_{\omega-\phi-\psi} \lambda^2 + \sqrt{2}B \ (A^2 - 1) \ s_{\omega-\phi} \lambda^3 . \]

(31)

A definite hierarchy between the different \( \lambda_{ij} \) will be reflected in this relation by, e.g., replacing \( B \) with \( B \lambda \) for \( \lambda_{23} = \mathcal{O}(\lambda^2) \) and so on. Note that for the case we are interested in, \( \rho = \sigma = 0 \), the two invariants \( S_1 \) and \( S_2 \) are equal at least to order \( \lambda \).

We will give next expressions for the \( CP \)–violating rephasing invariants. For \( J_{CP} \) we find

\[ J_{CP} \simeq \frac{1}{2} \cos \theta^\nu_{23} \sin^2 \theta^\nu_{23} s_\phi \lambda + \mathcal{O}(\lambda^3) . \]

(32)
We therefore identify (to leading order in $\lambda$) $\phi$ as the Dirac phase, measurable in neutrino oscillation experiments.

As an example for the correlations between $CP$–conserving and $CP$–violating observables, consider the following case: if $|U_{e3}|$ is close to the minimal value in its allowed range, $\cos \phi$ has to be close to one, and thus $\phi$ close to zero or $2\pi$, in order for the relation $\tan^2 \theta_\odot \simeq 1 - 4 \cos \phi |U_{e3}|$ to be compatible with measured value of $\tan^2 \theta_\odot$. This implies that $J_{CP} \propto \sin \phi$ will be additionally suppressed.

Recall that the charged lepton sector contributes with only one phase to the PMNS matrix. The hierarchical structure of $U_L$, which is under discussion here, will suppress the effects of this phase and the main dependence of the $CP$–violating observables can be expected to come from the neutrino sector. To check this consideration, suppose now that the neutrino mass matrix $m_\nu$ conserves $CP$. In our framework this would mean $\phi = 0$. Since to leading order in $\lambda$, $J_{CP} \propto \lambda \sin \phi$, the $CP$–violating effects in neutrino oscillations will depend on the phase $\psi$ in $\tilde{U}_{lep}$ and appear only at the next higher order, which turns out to be $\lambda^3$.

The rephasing invariants associated with the Majorana nature of massive neutrinos are given by:

$$S_1 \simeq \frac{1}{\sqrt{2}} \sin \theta_{23}^\nu s_{\phi + \sigma} \lambda + \frac{1}{\sqrt{2}} \sin \theta_{23}^\nu \cos \theta_{23}^\nu s_\sigma \lambda^2$$

and

$$S_2 \simeq \frac{1}{\sqrt{2}} \sin \theta_{23}^\nu s_{\phi - \rho + \sigma} \lambda + \frac{1}{\sqrt{2}} \sin \theta_{23}^\nu \cos \theta_{23}^\nu s_\rho - \sigma \lambda^2.$$  

In the case of $\rho = \sigma = 0$, which is realized if, e.g., the neutrino mass matrix has the form of Eq. (21) modified by one additional small term (as summarized in Table I), to a good precision (i.e., up to corrections $\sim \lambda^3$) we have

$$S_1 \simeq S_2 \simeq \frac{2\sqrt{2}}{\sin 2\theta_{23}^\nu} J_{CP} \simeq \frac{2\sqrt{2}}{\sin 2\theta_{atm}^\nu} J_{CP},$$

where we have used the fact that $\theta_{atm}^\nu \simeq \theta_{23}^\nu$.

We will discuss next whether in the scheme we are analyzing one has $\theta_{atm}^\nu > \pi/4$ or $\theta_{atm}^\nu < \pi/4$. It is instructive to consider the difference of $|U_{\mu3}|$ and $|U_{\tau3}|$, which reads

$$|U_{\mu3}| - |U_{\tau3}| \simeq \sin \theta_{23}^\nu - \cos \theta_{23}^\nu - \frac{\lambda^2}{2} (\sin \theta_{23}^\nu + 2B c_{\omega - \phi} (\cos \theta_{23}^\nu + \sin \theta_{23}^\nu))$$

$$\theta_{23}^\nu \rightarrow \pi/4 \rightarrow - \frac{\lambda^2}{2\sqrt{2}} (1 + 4B c_{\omega - \phi}).$$

Thus, for $\theta_{23}^\nu = \pi/4$ and in the presence of $CP$ violation, it is possible to have both cases, $\theta_{atm}^\nu > \pi/4$ and $\theta_{atm}^\nu < \pi/4$. For $\theta_{23}^\nu = \pi/4$ and $CP$ conservation we have $|U_{\mu3}| < |U_{\tau3}|$ and thus typically $\theta_{atm}^\nu < \pi/4$. We will elaborate further on this issue in the next Subsection.

We will comment next on the effective Majorana mass $|<m>|$ measurable in neutrinoless double beta decay. This process is sensitive to the absolute value of the $ee$ element of the
the effective Majorana mass can be written as \[<m>| \approx \sqrt{\Delta m^2_{\text{A}}} \left| U^2_{e1} - U^2_{e2} \right|, \] (37)

In the case under discussion one finds \(U^2_{e1} - U^2_{e2} \approx 2 \cos \theta_{23}^\nu e^{i \phi} \lambda + O(\lambda^3)\). Using Eqs. (20) and (32), one can therefore express \(|<m>|\) as

\[
|<m>| \approx \sqrt{\Delta m^2_{\text{A}}} \left| \cos 2\theta_\odot + 4i \frac{J_{CP}}{\sin^2 \theta_{\text{atm}}} \right|, \tag{38}
\]

which contains both, the Dirac phase, i.e., \(J_{CP}\), and the atmospheric neutrino mixing angle. In the case of \(\theta_{23}^\nu = \pi/4\) we recover the expression for \(|<m>|\) found in 19]. Note that this relation between the effective mass, \(J_{CP}\) and the atmospheric neutrino mixing does not depend on where the small perturbation in \(m_{\nu}\) is introduced.

### 3.2.3 Second example

If we choose now a weaker hierarchy between the three \(\lambda_{ij}\), i.e., \(\lambda_{12} \equiv \lambda, \lambda_{23} = B \lambda\) and \(\lambda_{13} = A \lambda^3\), we find:

\[
\tan^2 \theta_\odot \approx 1 - 4 c_\phi \cos \theta_{23}^\nu \lambda + 4 \left(2 c_\phi^2 \cos^2 \theta_{23}^\nu - B c_\omega \sin \theta_{23}^\nu \right) \lambda^2, \\
|U_{e3}| \approx \left| \sin \theta_{23}^\nu \lambda - B c_{\omega-\phi} \cos \theta_{23}^\nu \lambda^2 \right|, \\
\tan^2 \theta_{\text{atm}} \approx \tan^2 \theta_{23}^\nu - \frac{\sin \theta_{23}^\nu}{\cos^2 \theta_{23}^\nu} 2B c_{\omega-\phi} \lambda \\
\text{or } \sin^2 2\theta_{\text{atm}} \approx \sin^2 2\theta_{23}^\nu - 2B c_{\omega-\phi} \sin 4\theta_{23}^\nu \lambda, \tag{39}
\]

where we omitted the lengthy expression for the sizable quadratic terms in \(\tan^2 \theta_{\text{atm}}\) and \(\sin^2 2\theta_{\text{atm}}\). The presence of the linear term in \(\lambda\), which is proportional to \(\sin \theta_{23}^\nu/\cos^3 \theta_{23}^\nu\), explains why for the choice of \(\lambda_{23} \propto \lambda\) the angle \(\theta_{23}^\nu\) has a tendency to be smaller than \(\pi/4\). We mentioned this trend while discussing Fig. 11 and can understand this now by noting that \(\sin \theta_{23}^\nu/\cos^3 \theta_{23}^\nu\) decreases when \(\theta_{23}^\nu\) decreases starting from \(\pi/4\).

Thus, when \(\theta_{23}^\nu > \pi/4\), it will be possible to have \(\theta_{\text{atm}} < \pi/4\) for \(\lambda_{23} \propto \lambda\). If \(\lambda_{23} \propto \lambda^2, \theta_{\text{atm}}\) prefers to stay above \(\pi/4\).

In the case of \(CP\) violation, the same expressions for \(S_{1,2}\) and \(J_{CP}\) as those given in the previous Subsection will hold, when one formally replaces \(B\) with \(B/\lambda\). As can be seen from Eqs. (32) - (34), to leading order the results are the same. In particular, the relation Eq. (35) between the three \(CP\)-violating rephasing invariants \(J_{CP}, S_1\) and \(S_2\) holds again. Moreover, the effective Majorana mass \(|<m>|\) is given by the same expression (38).
Let us for illustration fix \( \lambda_{12} \equiv \lambda = 0.24 \) and choose \( \lambda_{13} = A \lambda^3 \) and \( \lambda_{23} = B \lambda^2 \). The parameters \( A, B \) are let to vary between \( 1/\sqrt{3} \) and \( \sqrt{3} \). The result of the numerical analysis is displayed in Fig. 3. It is seen again from the plots that in the case of “free” \( \theta_{23}^\nu \) and \( \lambda_{23} = B \lambda^2 \), the oscillation parameters can have significantly lower values within their allowed ranges. The reason is that for \( \lambda_{23} = \lambda \) there are more terms of the same order of \( \lambda \) contributing to the observables. Furthermore, as can be seen from the lower right plot, when \( \theta_{23}^\nu < \pi/4 \) (\( \theta_{23}^\nu > \pi/4 \)) and \( \lambda_{23} = B \lambda^2 \), the atmospheric neutrino mixing “prefers” to take a value \( \theta_{\text{atm}} < \pi/4 \) (\( \theta_{23} > \pi/4 \)), as discussed above.

4 Realization of the Neutrino Mass Matrix within the See–Saw Mechanism

We will present in this Section a very simple see–saw realization of the neutrino mass matrix of Eq. (21), including the requisite small perturbation. The problem of getting a neutrino mass matrix corresponding to an inverted hierarchy within the see–saw mechanism [49] has been considered, e.g., in [50, 51]. The Lagrangian leading to the see–saw mechanism [49] reads

\[
\mathcal{L} = \frac{1}{2} \left( N_R^T \right) C^{-1} M_R N_R - N_R m_D L + h.c.,
\]

(40)

where \( N_R \) are the heavy Majorana neutrinos and \( L \) the left–handed \( SU(2)_L \) lepton doublet. The most simple way to generate the requisite light neutrino mass matrix \( m_\nu \) is to assume that the Dirac mass matrix \( m_D \) is diagonal and that the inverse of the heavy Majorana mass matrix \( M_R^{-1} \) has the structure of \( m_\nu \) we would like to get. In this spirit, we take

\[
m_D = \begin{pmatrix} m_{D1} & 0 & 0 \\ 0 & m_{D2} & 0 \\ 0 & 0 & m_{D3} \end{pmatrix} \quad \text{and} \quad M_R^{-1} = \begin{pmatrix} 0 & b & d \\ \cdot & a & 0 \\ \cdot & \cdot & 0 \end{pmatrix},
\]

(41)

where \( a, b, d \) are complex, i.e., \( a = |a| e^{i\alpha}, \; b = |b| e^{i\beta} \) and \( d = |d| e^{i\gamma} \). The resulting light neutrino mass matrix reads

\[
m_\nu = - \begin{pmatrix} 0 & b m_{D1} & m_{D1} m_{D2} & d m_{D1} m_{D3} \\ \cdot & a m_{D2}^2 & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot \end{pmatrix},
\]

(42)

which can be identified with the mass matrix (21) with a small perturbation in the \( \mu\mu \) entry. In order to match the required ratio of the entries, the following conditions must hold:

\[
m_{e\mu} \sim m_{e\tau} \Rightarrow \frac{|b|}{|d|} \sim \frac{m_{D3}}{m_{D2}},
\]

(43)

\[
m_{\mu\mu} \sim \epsilon^2 m_{e\mu} \Rightarrow \epsilon^2 \frac{m_{D2}}{m_{D1}} \sim \frac{|a|}{|b|}.
\]

In case of a hierarchy between the Dirac masses, \( m_{D3} \gg m_{D2} \gg m_{D1} \), we would find that \( |b| \gg |d| \) and \( |b| \gg |a| \) must hold. The ratio between \( |d| \) and \( |a| \) required to reproduce the
desired structure of $m_\nu$ would then depend on the ratio of the Dirac masses. Looking at Table 11 we can identify $\tan \theta_{23} \simeq |d|/|b| m_{D3}/m_{D2}$, $R \simeq 2|a b^2| m^2_{D2}/m_{D1}$ and $m_3 \simeq |a d| m^2_{D2} m_{D3}$. The phase $\phi$, on which $J_{CP}$ depends in the case of hierarchical charged lepton mixing, coincides with $\alpha - \beta$, see Section 3.2.1. Note that although the structure of $M^{-1}_R$ is identical to that of $m_\nu$, the 12 element of $M^{-1}_R$ is, for hierarchical Dirac masses, much larger than the other two non–vanishing elements.

In analogy to the procedure outlined above in Section 3.2.1 we can simplify the diagonalization of $M^{-1}_R$ by writing it as

$$M^{-1}_R = \text{diag}(e^{i(\beta - \alpha/2)}, e^{i\alpha/2}, e^{i(\alpha/2 + \gamma - \beta)}) \begin{pmatrix} 0 & |b| & |d| \\ |a| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{diag}(e^{i(\beta - \alpha/2)}, e^{i\alpha/2}, e^{i(\alpha/2 + \gamma - \beta)})$$

$$\equiv P \begin{pmatrix} 0 & |b| & |d| \\ |a| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P \equiv P |M^{-1}_R| P,$$

so that we do not have to bother about the phases in the procedure. Diagonalizing $|M^{-1}_R|$ via $V_R |M^{-1}_R| V_R^T = 1/|M^{-1}_R|$ brings the Majorana neutrinos in their physical mass basis, in which we have to replace $m_D$ with $\tilde{m}_D \equiv V_R P m_D$. Defining $\epsilon_1 = |d/b|$ and $\epsilon_2 = |a/b|$, $V_R$ is approximately given by

$$V_R \simeq \begin{pmatrix} \sqrt{\frac{1}{2} + \frac{\epsilon_2}{4 \sqrt{2}}} - \sqrt{\frac{1}{2} + \frac{\epsilon_2}{4 \sqrt{2}}} - \frac{\epsilon_1}{\sqrt{2}} \\ \sqrt{\frac{1}{2} - \frac{\epsilon_2}{4 \sqrt{2}}} - \sqrt{\frac{1}{2} + \frac{\epsilon_2}{4 \sqrt{2}}} + \frac{\epsilon_1}{\sqrt{2}} \\ \epsilon_2 \epsilon_1 - \epsilon_1 \end{pmatrix} + O(\epsilon^2),$$

(45)

with the corresponding eigenvalues $\pm |b|(1 \mp \epsilon_2/2) = |a|/2 \pm |b|$, and $|a d^2/b^2|$.

Consider now leptogenesis [52] in the scenario under study. Decays of heavy Majorana neutrinos in the early Universe create a lepton asymmetry which is subsequently converted into a baryon asymmetry (for reviews see, e.g., [53]). The requisite $CP$–violating asymmetry is caused by the interference of the tree level contribution and the one–loop corrections in the decay rate of the three heavy Majorana neutrinos, $N_i \to \Phi^- \ell^+$ and $N_i \to \Phi^+ \ell^-$ and it reads [53]:

$$\varepsilon_i = \frac{\Gamma(N_i \to \Phi^- \ell^+) - \Gamma(N_i \to \Phi^+ \ell^-)}{\Gamma(N_i \to \Phi^- \ell^+) + \Gamma(N_i \to \Phi^+ \ell^-)}$$

$$\simeq \frac{1}{8 \pi v^2} \frac{1}{(\tilde{m}_D \tilde{m}^\dagger_D)_{ii}} \sum_{j=2,3} \text{Im}(\tilde{m}_D \tilde{m}^\dagger_D)_{ij}^2 f(M_j^2/M_i^2),$$

(46)

where $\Phi$ is the Higgs field, $v$ its vacuum expectation value and $f(x) \simeq -3/(2\sqrt{x})$ for $x \gg 1$. Explicitly, the function reads

$$f(x) = \sqrt{x} \left( 1 - (1 + x) \log(1 + 1/x) + \frac{1}{1 - x} \right).$$

(47)
From the above expression for $M_R^{-1}$ the right–handed Majorana masses are found to be

$$M_{1,2} \simeq \pm \frac{1}{|b|} (1 \pm \epsilon_2/2) \text{ and } M_3 \simeq \left| \frac{b^2}{a \ d^2} \right|,$$

(48)
displaying an “inverted–hierarchy–like” spectrum, where now however the two lighter neutrinos are close in mass with a very small mass difference of $\Delta M \simeq |a/b|^2$. This can lead to a resonant amplification of the decay asymmetry [53]. The relevant condition under which the formula Eq. (46) is valid (otherwise a more complicated formula is required) can be written as $\Delta M \gg \Gamma_i$, where $\Delta M$ is the mass difference of the two neutrinos and $\Gamma_i$ the tree level decay width, which is $\Gamma_i = M_i (\bar{m}_D m_D^\dagger)_{ii}/(16\pi v^2)$. In our case we find that

$$\frac{\Delta M}{\Gamma_1} \simeq \frac{\Delta M}{\Gamma_2} \simeq \frac{32\pi v^2}{m_{D2}^2} \epsilon_2.$$

(49)

To simplify the analysis, we can choose now $m_{D2} \simeq \lambda^n m_{D3}$ and $m_{D1} \simeq \lambda^m m_{D3}$, where the expansion parameter $\lambda \simeq 0.23$. To reproduce the mass matrix (28) for the entries in $M_R^{-1}$ it must hold: $|d| \simeq |b| \lambda^n$ and $|a| \simeq |b| \lambda^l$, with $l = 2 + m - n$, or $\epsilon_2 = \lambda^l$. For the lightest neutrino $M_1$ we have explicitly

$$\epsilon_1 \simeq \frac{1}{8\pi v^2} \frac{1}{(\bar{m}_D m_D^\dagger)_{11}} \left( \text{Im}(\bar{m}_D m_D^\dagger)_{12}^2 f(M_2^2/M_1^2) + \text{Im}(\bar{m}_D m_D^\dagger)_{13}^2 f(M_3^2/M_1^2) \right).$$

(50)

We can simplify the function $f$ in the two terms with $f(M_2^2/M_1^2) \simeq M_1^2/(M_1^2 - M_2^2) \simeq -1/2 \lambda^{-l}$ for the two neutrinos close in mass and $f(M_3^2/M_1^2) \simeq -3/2 M_1/M_3 \simeq -3/2 \lambda^{l+2n}$. After straightforward calculation, putting everything together yields:

$$\epsilon_1 \simeq -\frac{1}{32\pi} \frac{m_{D3}^2}{v^2} \frac{1}{\lambda^{2m} + 2\lambda^{2n}} \left\{ \frac{\lambda^{2m} - 2\lambda^{2n}}{\lambda^l} \sin 2(\alpha - \beta) + 6 \lambda^{l+2n} (\lambda^{3n} + \lambda^{l+2m+n})^2 \right\}.$$

(51)

Note that the first term is proportional to $\sin 2(\alpha - \beta) = \sin 2\phi$.

The important parameter governing the wash–out [53] of the generated lepton asymmetry is

$$\tilde{m}_1 \simeq \frac{\sqrt{\Delta m_3^2}}{2} (\lambda^{m-n} + 2\lambda^{n-m}) \ ,$$

(52)

where we have used that $m_{D3}^2 |b| \lambda^{m+n} \simeq \sqrt{\Delta m_3^2}$, which is obvious from Eqs. (21) and (43). A typical choice of parameters leading to reasonable heavy Majorana masses is $m_{D3} \simeq v$ and $m = n = 1$, which leads to $\Delta M/\Gamma_1 \simeq 96\pi$, $M_{1,2} \simeq 3 \cdot 10^{13}$ GeV and $M_3 \simeq 10^{16}$ GeV. The implied heavy Majorana mass matrix is

$$M_R \simeq \frac{1}{|b| \lambda^2} \left( \begin{array}{ccc} 0 & \lambda^3 & \lambda^3 \\ \lambda^2 & \lambda & \lambda \\ \lambda^2 & 1 & 1 \end{array} \right).$$

(53)

The decay asymmetry is dominated by the first term and given by

$$\epsilon_1 \simeq \frac{1}{96\pi \lambda^2} \sin 2\phi \simeq 6.3 \cdot 10^{-2} \sin 2\phi .$$

(54)
Hence, the decay asymmetry is (due to the two Majorana neutrinos close in mass) rather large and depends to leading order on two times the Dirac phase. Since typically negative and small $|\varepsilon_1| \gtrsim 10^{-5}$ is required for successful leptogenesis \[53\], one expects a phase closely below $\pi$ or $2\pi$ and therefore small $CP$–violating effects in neutrino oscillations. Since $\cos \phi$ has to be close to $+1$ in order to obey the constraints coming from the correlations between $|U_{e3}|$ and $\tan^2 \theta_\odot$, expressed in Eq. (30), $\phi$ consistent with all constraints should have a value close to, but somewhat smaller than, $2\pi$.

5 Conclusions

We analyzed the possibility that the observed pattern of neutrino mixing in the PMNS matrix arises as a result of an interplay between the two unitary matrices associated with the diagonalization of the charged lepton ($m_L$) and neutrino ($m_\nu$) mass matrices, $U_{PMNS} = U^\dagger_L U_\nu$. The matrix $U_\nu$ diagonalizing the neutrino Majorana mass matrix $m_\nu$, is first assumed to have an exact bimaximal mixing form. We have summarized the allowed structures of the charged lepton mixing matrix $U_L$ in this case and have given the implied forms of the matrices $m_\ell m^T_\ell$ and $m_\ell = m_\ell T$, diagonalized by $U_L$. We have assumed further that the origin of bimaximal mixing is a weakly broken flavor symmetry corresponding to the conservation of the non–standard lepton charge $L' = L_e - L_\mu - L_\tau$. The latter does not predict, in general, the mixing angle $\theta^e_{23}$ in $U_\nu$ to be maximal. We therefore generalized our analysis to the case of an a priori non–maximal mixing angle $\theta^e_{23}$. We performed a detailed study of the allowed structures of the charged lepton mixing and the implied correlations between the neutrino mixing and $CP$–violating observables. Considering the case when the $L_e - L_\mu - L_\tau$ flavor symmetry is broken by one small element in $m_\nu$, we found that the three $CP$–violating rephasing invariants, associated with the three $CP$–violating phases in the PMNS mixing matrix, are related. This result is valid independently of the location of the symmetry breaking element in $m_\nu$. We discussed the corrections from hierarchical charged lepton mixing and their impact on the precise value of $\theta_{\text{atm}}$. To leading order, $\theta^e_{23}$ corresponds to $\theta_{\text{atm}}$. The Dirac phase $\phi$, measurable in neutrino oscillation experiments, enters into the expression for $\tan^2 \theta_\odot$: $\tan^2 \theta_\odot \simeq 1 - 4 \cos \phi \cot \theta_{\text{atm}} |U_{e3}|$. Since $\tan^2 \theta_\odot$ is constrained from above by the data, $\tan^2 \theta_\odot \lesssim 0.58$, this result implies that the smaller $|U_{e3}|$, the closer to 0 or $2\pi$ the phase $\phi$ would be. Consequently, for $|U_{e3}| \lesssim 0.1$, $J_{CP}$ would be additionally suppressed by the fact that $\sin \phi$ has to be relatively small. Moreover, $J_{CP}$ is also suppressed when $m_\nu$ conserves $CP$. The possible “tension” between a relatively small value of $\theta_{13}$ and substantially non–maximal solar neutrino mixing is somewhat relaxed in the case of $\theta^e_{23} \neq \pi/4$. More specifically, the lower limit on $|U_{e3}|$ obtained for exact bimaximal mixing can be reduced by a factor of two, i.e., to $|U^2_{e3}| \gtrsim 0.007$. Moreover, in contrast to the case of $\theta^e_{23} = \pi/4$, $\sin^2 2\theta_{\text{atm}}$ can take any value inside its currently allowed range. However, if $\theta_{\text{atm}}$ is found to be very close to $\pi/4$, the limits corresponding to the case of exact bimaximal mixing are of course recovered. As a consequence of the obtained correlations between the neutrino mixing and $CP$–violating observables, the effective Majorana mass in neutrinoless double beta decay depends on $J_{CP}$ and $\sin^2 \theta_{\text{atm}}$. A very simple see–saw realization of a neutrino mass matrix conserving $L_e - L_\mu - L_\tau$ was written down, which contains one heavy Majorana neutrino.
with mass much larger than the other two, which in turn are close in mass. In this scheme, the Dirac phase responsible for \( \text{CP} \) violating effects in neutrino oscillations is also responsible for the baryon asymmetry of the Universe. It has to be closely below \( 2\pi \), which is consistent with the mentioned relation between this phase, \( \tan^2 \theta_\odot \) and \( |U_{\text{e}3}| \).

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Figure 1: Scatter plot of some of the $\lambda$ parameters, $\sin \theta_{23}$ and the neutrino mixing observables for the $3\sigma$ allowed ranges of values of the neutrino mixing parameters given in Eq. (3). All $\lambda_{ij}$ are small and conservation of $CP$ is assumed.
Figure 2: Scatter plot of $\tan^2 \theta \odot$ against $|U_{e3}|^2$ for two different limits on $\sin^2 2\theta_{\text{atm}}$ and for the cases $\theta_{23}^{\text{free}} = \pi/4$ and free $\theta_{23}^{\text{atm}}$. A “CKM–like” hierarchy of the $\lambda_{ij}$ is assumed and it is required that the neutrino mixing parameters $\tan^2 \theta \odot$ and $|U_{e3}|$ have values in their respective $3\sigma$ allowed intervals, given in Eq. (3).
Figure 3: Scatter plot of $\tan^2 \theta_{\odot}$ against $|U_{e3}|^2$, $|U_{e3}|^2$ against $\tan^2 \theta_{\text{atm}}$, $\tan^2 \theta_{\odot}$ against $J_{\text{CP}}$ and $\tan^2 \theta_{\text{atm}}$ against $\tan^2 \theta_{23}$. We choose $\lambda_{12} \equiv \lambda = 0.24$, $\lambda_{13} = \lambda^3$ and $\lambda_{23} = \lambda^2$ (except for the lower right plot, in which results for $\lambda_{23} = \lambda$ are also shown). The neutrino mixing parameters are required to lie within their respective $3\sigma$ allowed ranges of values, given in Eq. (3).