What is wrong with the Lax-Richtmyer
Fundamental Theorem of Linear
Numerical Analysis?

Elemer E Rosinger
Department of Mathematics
and Applied Mathematics
University of Pretoria, Pretoria
0002 South Africa
eerosinger@hotmail.com

Abstract

We show that the celebrated 1956 Lax-Richtmyer linear theorem in
Numerical Analysis - often called the Fundamental Theorem of Nu-
merical Analysis - is in fact wrong. Here ”wrong” does not mean
that its statement is false mathematically, but that it has a limited
practical relevance as it misrepresents what actually goes on in the
numerical analysis of partial differential equations. Namely, the as-
sumptions used in that theorem are excessive to the extent of being
unrealistic from practical point of view. The two facts which the men-
tioned theorem gets wrong from practical point of view are:
- the relationship between the convergence and stability of numerical
methods for linear PDEs,
- the effect of the propagation of round-off errors in such numerical
methods.
The mentioned theorem leads to a result for PDEs which is unrealis-
tically better than the well known best possible similar result in the
numerical analysis of ODEs. Strangely enough, this fact seems not to
be known well enough in the literature. Once one becomes aware of
the above, new avenues of both practical and theoretical interest can
open up in the numerical analysis of PDEs.
1. Towards a correct relationship between stability and convergence

It has been shown that in practically relevant situations the converse implication "convergent $\implies$ stable" in the Lax-Richtmyer theorem may fail to hold, see Rosinger [1-8], Rosinger & van Niekerk [1,2], Oberguggenberger & Wang. Thus there need not always be an equivalence between the convergence and stability of a numerical scheme. It may therefore happen that convergence is a weaker property than stability, which means that we may have convergent numerical schemes which nevertheless fail to be stable as well.

In this way, what has become a kind of "UNIVERSALLY RECITED MANTRA" in the numerical analysis of linear partial differential equations, namely that

( ? ) "stability and convergence are equivalent"

for linear numerical methods approximating such equations, does in fact lack a valid enough practical reality, and can be replaced with the far more convenient fact, according to which

( ! ) "convergence need not always imply stability".

Needless to say, the practical interest in such a possibility is significant, as it can enlarge the class of convergent numerical schemes beyond those which are stable. Examples in this regard are mentioned in Rosinger [1-8], Rosinger & van Niekerk [1,2], Oberguggenberger & Wang.

By the way, the well known necessary condition for stability, given by von Neumann prior to the Lax-Richtmyer theorem, and which does not require a Banach space setup, can be seen as a further indication of the rather involved relationship between stability and convergence.

A yet more important point about the above mantra is the following. Even if it were true in the linear case - which in fact is the only case addressed by the Lax-Richtmyer theorem - it would still lack rel-
evance in most of the cases when exact solutions of nonlinear PDEs are approximated by respective nonlinear numerical methods. Indeed, as is well known, Kreiss, Stetter, a local linearized stability analysis of nonlinear PDEs and of their nonlinear numerical methods need not in general lead either to necessary, or sufficient convergence conditions.

2. Questions about the implication ”convergent $\implies$ stable”

Briefly, the Lax-Richtmyer theorem, see below, states the equivalence between the convergence and stability of a linear numerical scheme which is consistent with a well posed linear PDE, see Lax & Richtmyer, Richtmyer, Richtmyer & Morton, as well as a fully detailed analysis and presentation in Rosinger [2, pp. 1-14].

The important fact to note is the following.

The proof of the implication ”stable $\implies$ convergent” is trivial, and certainly, it does in no way require the completeness of space in which it happens. Therefore, the crux of the Lax-Richtmyer theorem is solely in the proof of the converse implication, namely, ”convergent $\implies$ stable”.

That converse implication ”convergent $\implies$ stable”, however, is proved based on the celebrated Principle of Uniform Boundedness of Linear Operators in Banach spaces. And as is well known, see Appendix, that property of uniform boundedness does not necessarily hold in normed spaces which are not complete, thus fail to be Banach spaces.

It is precisely here, with the assumptions which are made in order to secure a Banach space framework, that the Lax-Richtmyer theorem goes twice wrong from practical point of view. Namely, it goes wrong both with respect to the relationship between stability and convergence, as well as regarding the treatment of the essentially nonlinear phenomenon of the propagation of round-off errors.

Furthermore, the proof of the implication ”convergent $\implies$ stable” is essentially linear, as it makes use of the mentioned linear principle, as well as of a linear concept of stability. This makes the extension of
that implication to the fully nonlinear case extremely difficult.

3. Is completeness an appropriate requirement?

The numerical analysis of a given PDE does typically assume the a priori knowledge of the existence of certain exact solutions of that equation. After all, in case an exact solution does not exist, it is of course nonsensical to try to approximate it numerically. Thus, if and when the existence of an exact solution is known a priori, then the aim of numerical analysis is to construct numerical solutions approximating one or another of such exact solutions. In this way, we are given an exact solution \( U \) and construct a sequence, say, \( U_{\Delta t} \), with \( \Delta t > 0 \), of numerical solutions. Thus the problem is whether or not we have the hoped for convergence property

\[
( \ast ) \quad \lim_{\Delta t \to 0} U_{\Delta t} = U
\]

where the limit holds in some appropriate sense.

Suppose now, as usual, that both the exact solution \( U \) and the numerical solutions \( U_{\Delta t} \) belong to a certain normed space \( (X, || \cdot ||) \).

A crucial observation here is the following one. And it is missed by the Lax-Richtmyer theorem.

Clearly, in order to establish whether the above convergence property \(( \ast )\) does, or for that matter, does not hold, one does not at all need to assume that the respective normed space \( (X, || \cdot ||) \) is complete. Indeed, we have started by assuming that the exact solution \( U \) exists, thus the hoped for limit value in \(( \ast )\) exists. Furthermore, the terms \( U_{\Delta t} \) of the sequence in \(( \ast )\) also exist, being the constructed numerical solutions. Finally, the normed space \( X \) is supposed to be chosen in such a way that both the exact solution \( U \) and its numerical approximations \( U_{\Delta t} \) do belong to it. And then the only problem is whether the constructed numerical solution \( U_{\Delta t} \) does indeed happen to converge to the exact solution \( U \).

Furthermore, often, when for instance the exact solution \( U \) is a classi-
cal solution of the PDE considered, one can choose the normed space $(X, \| \| )$ as constituted from sufficiently smooth functions, since the numerical solutions $U_{\Delta t}$ are typically defined at discrete points, thus they can be extrapolated to functions of required smoothness.

In this way, there does not appear to be any practical reason whatsoever why the normed space $(X, \| \| )$ in which the convergence property (*) is to be established should be complete.

The alleged reason why nevertheless the completeness of the normed space $(X, \| \| )$ is requested appears to be the claim that it is needed in order to handle the effect of round-off errors as well. Indeed, as it stands, the Lax-Richtmyer theorem is only supposed to deal with the effects of the propagation of truncation errors, since it does not anywhere mention directly round-off errors.

However, as seen in Rosinger [5], see also Rosinger [2-4,7], this claim that the completeness of $X$ will give the opportunity to deal as well with the effect of the propagation of round-off errors is simply unrealistic from the point of view of the way round-off errors actually propagate in the computations involved. In particular, this claim leads to the paradox that one obtains a result regarding the effect of round-off errors in the numerical solution of PDEs which is strictly better than the well known best possible corresponding result in the case of the numerical solution of ODEs.

Obviously, in the many usual cases when one approximates classical solutions $U$, one can choose the normed space $(X, \| \| )$ made up of sufficiently smooth functions. But then, the completeness requirement in the Lax-Richtmyer theorem obliges one to consider its completion $(X^#, \| \| )$. And typically, $X^#$ will be a much larger space, containing a considerable amount of non-smooth functions.

Two important points should be noted here.

First, within this larger and completed space $X^#$, the original convergence problem (*) will remain precisely the same. Indeed, the constructed sequence of numerical solutions $U_{\Delta t} \in X$ converges to the existing exact solution $U \in X$ in the space $X$, if and only if it con-
verges to $U$ in the space $X^\#$. 

On the other hand, the *stability* property of the respective numerical method may turn out to lead to a more *stringent* condition in the larger space $X^\#$, than in the original smaller space $X$. 

This is indeed of one the issues related to the assumption of completeness, an assumption which is essential in the particular method of proof of the implication ”convergent $\implies$ stable” in the Lax-Richtmyer theorem. 

Otherwise, one simply notes that, in general, the completeness condition does *not* necessarily belong to the problem of establishing the convergence property (*). 

4. Compactness or Boundedness ? 

The above *convergence* relation (*), whenever it holds, clearly implies that the subset 

$$\{ U_{\Delta t} \ | \ \Delta t > 0 \} \cup \{ U \} \subset X$$

is *compact* in $X$, regardless of $X$ being complete or not. And let us recall that all the elements of this subset are supposed to exist. Indeed, the exact solution $U$ of the PDE under consideration exists, otherwise the problem of its numerical approximation would be vacuous. Further, the approximating numerical solutions $U_{\Delta t}$ are effectively constructed by the numerical method employed. 

On the other hand, the condition of *stability* of the numerical methods used in the Lax-Richtmyer theorem, see (5.18) below, is given in terms of *boundedness*, and as is well known, boundedness does *not* imply compactness in infinite dimensional normed spaces. 

This *discrepancy* between the association of convergence with compactness, and on the other hand, of stability with boundedness was first pointed out and dealt with in Rosinger [1], where with an appropriate compactness based definition of stability, a general *nonlinear equivalence* result was given between convergence and stability.
It should be mentioned here that the above arguments related to stability, convergence, completeness, compactness and boundedness were, back in the early summer of 1979, personally communicated by the author to P D Lax, at a conference at the Tel Aviv University, in Israel.

5. Some details of the Lax-Richtmyer theorem

Let us now, for convenience, recall the Lax-Richtmyer theorem as given in its original formulation, see Lax & Richtmyer, Richtmyer, Richtmyer & Morton, Rosinger [2, pp. 1-14]. We consider a linear evolution type PDE

\[ \frac{d}{dt} U(t) = A(U(t)), \; t \in [0, T] \]

with the initial value

\[ U(0) = u \]

where \( A : D \subseteq X \longrightarrow X \) is a linear operator defined on the subspace \( D \) of the Banach space \( X \), \( u \in D \), while \( U : [0, T] \longrightarrow D \) is the sought after solution. Since we deal with an evolution PDE, the operator \( A \) is in fact a linear partial differential operator in some space variable \( x \in \mathbb{R}^n \).

Further, one can assume that, when given, linear homogenous boundary conditions have already been incorporated in the definition of \( D \).

Typically, one can also assume that \( D \) is dense in \( X \) and we have satisfied the following exact solution property

\[ \forall \; u \in D : \]

\[ \exists \; U : [0, T] \longrightarrow X : \]

\[ (5.3) \]

\[ \ast ) \lim_{\Delta t \to 0} \| (U(t + \Delta t) - U(t))/\Delta t - A(U(t)) \| = 0, \; t \in [0, T] \]

\[ \ast \ast ) \; U(0) = u \]

Given now time, respectively space increments \( \Delta t \in (0, \infty) \) and \( \Delta x \in (0, \infty)^n \), we construct a finite difference method
which we assume to be a continuous linear mapping.

The numerical analysis problem we face in the above terms is to characterize the relations

\[(5.5) \quad \Delta x = \alpha(\Delta t) \]

where the mapping \( \alpha : (0, \infty) \rightarrow (0, \infty)^n \) is such that \( \lim_{\Delta t \rightarrow 0} \alpha(\Delta t) = 0 \in \mathbb{R}^n \), and the convergence property holds

\[(5.6) \quad \lim_{\Delta t \rightarrow 0, n \rightarrow \infty, n\Delta t \rightarrow t} \| U(t) - C_{\Delta t, \alpha(\Delta t)}^n u \| = 0 \]

uniformly for \( t \in [0, T] \), for every \( u \in D \), where \( U \) corresponds to \( u \) according to (5.3).

As is well known, in general, this is not a trivial problem. In Courant et.al., it was shown for the first time that one cannot in general expect instead of (5.6) the stronger convergence property

\[(5.7) \quad \lim_{\Delta t \rightarrow 0, n \rightarrow \infty, n\Delta t \rightarrow t, \Delta x \rightarrow 0} \| U(t) - C_{\Delta t, \Delta x}^n u \| = 0 \]

to hold uniformly for \( t \in [0, T] \).

Property (5.7), in which \( \Delta t \) and \( \Delta x \) can simultaneously and independently tend to 0, is called unconditional stability. On the other hand, property (5.6), in which the relation (5.5) ties \( \Delta x \) to \( \Delta t \) when they both tend to 0, is called conditional stability.

Obviously, in the case of conditional stability, one is interested in numerical methods (5.4) in which \( \alpha \) tend to 0 as fast as possible, when \( \Delta t \) tends to 0. Indeed, in such a situation one can obtain a good space accuracy without increasing too much the computation time.

As a simple and immediate illustration, let us consider the initial value problem for the heat equation

\[(5.4) \quad C_{\Delta t, \Delta x} : X \rightarrow X \]
\[ U_t = U_{xx}, \quad t \in [0, \infty), \; x \in \mathbb{R} \]

\[ U(0, x) = u(x), \quad x \in \mathbb{R} \]

In this case we can take \((X, || ||) = L^\infty(\mathbb{R})\) and \(A = \partial^2 / \partial x^2\), while

\[ D = \{ u \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R}) \mid Au \in L^\infty(\mathbb{R}) \} \]

and as is well known, John, the exact solution property (5.3) holds.

A simple numerical method for this heat equation is given by

\[ (C_{\Delta t}, \Delta x) u(x) = u(x) + (u(x + \Delta x) - 2u(x) + u(x - \Delta x)) \Delta t / \Delta x^2 \]

for \(u \in X, \; x \in \mathbb{R}, \; \Delta t, \; \Delta x > 0\). Also as is well known, Richtmyer, this explicit numerical method will not have the convergence property given by the unconditional stability (5.7), while the weaker convergence property (5.6), called conditional stability, will hold, if and only if

\[ 2\Delta t \leq \Delta x^2 \]

which in terms of (5.5) can be written as

\[ \alpha(\Delta t) \geq \sqrt{2\Delta t} \]

This obviously means that \(\alpha \to 0\) rather slowly, when \(\Delta t \to 0\), which is inconvenient, since a small \(\Delta x\) will impose the use of a quadratically smaller \(\Delta t\), leading thus to an increased number of time iterations.

Returning to the general case, in view of the exact solution property (5.3), we can define the family of linear mappings

\[ E_0(t) : D \rightarrow X, \quad t \in [0, T] \]

by

\[ E_0(t)(u) = U(t), \quad t \in [0, T] \]
The initial value problem (5.1), (5.2) is called *properly posed*, if and only if the family of linear mappings (5.8) is uniformly bounded, that is, for a certain $K > 0$, we have

\[(5.9) \quad ||E_0(t)|| \leq K, \quad t \in [0, T]\]

As is well known, since $D$ is a dense subspace in $X$, one can extend by continuity the family of linear mappings (5.8) to a unique family of linear mappings

\[(5.10) \quad E(t) : X \longrightarrow X, \quad t \in [0, T]\]

with the same uniform bound, namely

\[(5.11) \quad ||E(t)|| \leq K, \quad t \in [0, T]\]

In addition, we shall have the *semigroup* property

\[E(t)E(s) = E(t+s), \quad t, s \in [0, T], \quad t + s \leq T\]

\[(5.12) \quad E(0) = \text{id}_X\]

\[\lim_{\Delta t \to 0} ||E(u) - u|| = 0, \quad u \in X\]

We can note that the semigroup property (5.12) leads to a further extension, this time of the linear mappings (5.10), namely

\[(5.13) \quad E(t) : X \longrightarrow X, \quad t \in [0, \infty)\]

where

\[(5.13.1) \quad E(t) = E(t - [t/T]T)E(T)^{[t/T]}, \quad t \in [T, \infty)\]

with $[t/T]$ denoting the largest integer which is smaller than, or equal to $t/T$. In this case, instead of the corresponding above relations, we shall have
\[ E(t)E(s) = E(t+s), \; t, s \in [0, \infty) \]
(5.14)
\[ \| E(t) \| \leq K^{1+[t/T]}, \; t \in [0, \infty) \]

Returning now to the numerical method (5.4), (5.5), we shall consider as our finite difference scheme the family of continuous linear mappings

(5.15) \[ C_{\Delta t} = C_{\Delta t, \alpha(\Delta t)} : X \rightarrow X \]

Here it is important to note that, typically, this family of continuous linear mappings \( C_{\Delta t} \), with \( \Delta t > 0 \), is not uniformly bounded for small \( \Delta t \).

Now in view of (5.5), (5.6), the finite difference scheme (5.15) is called convergent to the semigroup (5.10) - (5.12) on the time interval \([0, T]\), where \( T > 0 \) is given, if and only if

\[ \forall \; u \in X, \; \epsilon > 0 : \]
\[ \exists \; \delta > 0 : \]
(5.16)
\[ \forall \; t \in [0, T], \; \Delta t > 0, \; n \in \mathbb{N}, \; n\Delta t \leq T : \]
\[ \Delta t, \; | t - n\Delta t | \leq \delta \implies \| E(t) - C_{\Delta t}^n u \| \leq \epsilon \]

Further, the finite difference scheme (5.15) is called consistent with the initial value problem (5.1), (5.2) on the same time interval \([0, T]\), if and only if

\[ \forall \; u \in X, \; \epsilon > 0 : \]
\[ \exists \; \theta > 0 : \]
(5.17)
\[ \forall \; t \in [0, T], \; \Delta t > 0 : \]
\[ \Delta t \leq \theta \implies \| C_{\Delta t} E(t) - E(t + \Delta t)u \| \leq \epsilon \]
Finally, the finite difference scheme (5.15) is called stable on the time interval \([0, T]\), if and only if

\[ \exists \ L > 0 : \]

\[ \forall \ \Delta t > 0, \ n \in \mathbb{N}, \ n\Delta t \leq T : \]

\[ ||C_{\Delta t}^n|| \leq L \]

With the above, we have the so called *Fundamental Theorem of Linear Numerical Analysis*

**Theorem (Lax-Richtmyer, 1956)**

Given a properly posed semigroup (5.10) - (5.12) and a finite difference scheme (5.15) which is consistent with it, then the finite difference scheme is convergent to the semigroup, if and only if it is stable.

**Remark**

The practical interest in the above type of result is in the following. The **consistency** of a finite difference scheme with a semigroup generated by an initial value problem (5.1), (5.2) is typically easy to establish with the use of a finite Taylor series argument, in case we deal with smooth enough, or classical solutions. Also, what is practically particularly important, the consistency property can be established *without* the effective knowledge of any specific exact solution of the initial value problem, and only based on the knowledge of the regularity of such solutions, that is, the existence of smooth enough, or classical exact solutions.

The **convergence** property of such a finite difference scheme is, of course, the main and nontrivial issue, and just like the consistency property, it is a **relational** property, since it involves the semigroup, or the initial value problems as well. Furthermore, here the fact that, typically, the exact solution is only known to exist, but it is not known effectively - this being the very reason for using numerical analysis - makes it so much more difficult to establish convergence.

On the other hand, the **stability** property of a finite difference scheme
is no longer a relational property, but an \textit{intrinsic} property which is \textit{solely} of the finite difference scheme itself, therefore, at least in principle, it can be established alone on the information contained in that finite difference scheme.

In this way, in the study of the convergence of finite difference schemes there is clearly a major interest in establishing a certain connection between the \textit{relational} property of \textit{convergence} which is the sought after aim, and on the other hand, the \textit{intrinsic} property of \textit{stability}.

The above Lax-Richtmyer theorem does establish such a connection, in fact, an equivalence, between convergence and stability. Unfortunately however, it assumes the \textit{completeness} of the normed space in which all of this happens, in order to be able to prove the implication ”convergent $\implies$ stable”.

\textbf{Appendix}

We present a simple \textit{counterexample} to the celebrated Principle of Uniform Boundedness of Linear Operators in a Banach Space, based on the fact that the respective normed space fails to be complete, that is, Banach. This shows that in this principle, the completeness of the normed space involved is indeed essential.

We take the normed space $(X, \|\|)$ defined as follows

\begin{equation}
(A.1) \quad X = \left\{ x = (x_0, x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N} \left| \begin{array}{c} \exists \ m \in \mathbb{N} : \\
\forall \ n \in \mathbb{N}, \ n \geq m : \\
x_n = 0 \end{array} \right. \right\}
\end{equation}

with the norm given by

\begin{equation}
(A.2) \quad \|x\| = \sup \left\{ |x_n| : n \in \mathbb{N} \right\}
\end{equation}

for $x = (x_0, x_1, x_2, \ldots) \in X$.

Now, for every $k \in \mathbb{N}$, we define the linear operator $T_k : X \to X$ by

\begin{equation}
(A.3) \quad T_k(x_0, x_1, x_2, \ldots) = (y_0, y_1, y_2, \ldots)
\end{equation}
where for $x = (x_0, x_1, x_2, \ldots) \in X$, we have

\begin{equation}
(A.4) \quad y_n = \begin{cases} 
kx_k & \text{if } n = k \\
0 & \text{if } n \neq k\end{cases}
\end{equation}

It follows easily that

\begin{equation}
(A.5) \quad \| T_k \| = k, \quad k \in \mathbb{N}
\end{equation}

therefore, the family of linear operators $(T_k \mid k \in \mathbb{N})$ is not uniformly bounded.

On the other hand, given any fixed $x = (x_0, x_1, x_2, \ldots) \in X$, there exists $m \in \mathbb{N}$, such that $x_n = 0$, for $n \in \mathbb{N}, n \geq m$. Hence $T_k(x) = 0$, for $k \in \mathbb{N}, k \geq m$. Consequently

\begin{equation}
(A.6) \quad \sup \{ \| T_k(x) \| \mid k \in \mathbb{N} \} < \infty
\end{equation}

In this way, the family of linear operators $(T_k \mid k \in \mathbb{N})$ is bounded at each point $x \in X$, and yet it is not uniformly bounded on $X$.

The reason for that is obviously in the fact that the normed space $(X, \| \|)$ is not complete. Indeed, $(X, \| \|)$ is a strict and dense subspace of $l^\infty$.

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