Instanton analysis of hysteresis in the 3D Random-Field Ising Model

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We study the magnetic hysteresis in the random-field Ising model in 3D. We discuss the disorder dependence of the coercive field $H_c$, and obtain an analytical description of the smooth part of the hysteresis below and above $H_c$, by identifying the disorder configurations (instantons) that are the most probable to trigger local avalanches. We estimate the critical disorder strength at which the hysteresis curve becomes continuous. From an instanton analysis at zero field we obtain a description of local two-level systems in the ferromagnetic phase.

In disordered systems whose pure counterparts undergo a first order phase transition as an external parameter (e.g., field or pressure) is tuned, the presence of randomness leads to interesting hysteretic phenomena. Those arise from local modifications of the spinodal point due to disorder, as well as from the presence of many metastable states. The simplest system exhibiting such phenomena is the random field Ising model (RFIM), often studied as a representative model for hysteresis in the presence of random fields.

In the RFIM, the gradual transition between states of negative and positive magnetization proceeds via avalanches triggered by the increase of the external magnetic field. Previous studies have shown that in the presence of strong disorder all avalanches are bounded and the hysteresis loop is macroscopically smooth. In weak disorder, however, there is a finite coercive field beyond which the ferromagnetic coupling induces macroscopic avalanches that span large parts of the sample and lead to a sharp jump in the hysteresis curve. The two disorder regimes are separated by a critical point which is believed to control the power law distribution of Barkhausen noise. Recently, indications for such a disorder induced transitions have been reported in disordered magnets as well as in the context of capillary condensation in aerogels.

The hysteresis in the RFIM is intimately tied to the complexity of its free energy landscape, possessing an exponential number of metastable minima. However, the precise connection between the disorder induced energy landscape and the nature and extent of the avalanches has not been established. So far, studies of the hysteresis in the RFIM concentrated on lattice simulations at zero temperature, and most analytical insight was restricted to the Bethe lattice or the extreme mean field limit where all spins interact with each other. In this Letter, we make a first step towards an analytical theory for 3D systems. In particular, we analyze the features of the hysteresis loop as a function of disorder strength and sample size and characterize the typical avalanches occurring along the hysteresis loop. This provides a new perspective on the above-mentioned disorder induced transition. Further, by studying typical configurations that trigger avalanches at zero field we obtain insight into metastability in the ferromagnetic phase.

Before giving the details of our analysis, we summarize the qualitative picture we have obtained, concentrating on the raising branch of the saturation hysteresis loop (see Fig. 1). In a pure system, hysteresis arises due to the metastability of the negatively magnetized state in external fields below the spinodal field $H_{sp}$. This simple scenario is strongly modified by disorder: Due to rare regions with strong positive fields, bubbles of positive magnetization are created spontaneously at any small $H$. However, they are prevented from spreading by disorder induced pinning. The latter leads to a finite coercive field $H_c$ which grows with disorder strength. For fields below $H_c$, the magnetization curve is determined by small avalanches, triggered by the increase of $H$. At $H_c$ sample spanning avalanches occur, while beyond $H_c$, only small regions with strong negative random fields may resist the invasion of the positively magnetized phase, finally collapsing under further increase of the field. With increasing disorder strength, the local avalanches become so dense that the emerging bubbles already percolate before the threshold for unlimited bubble spread, $H_c$, is reached. In this situation, there is no room for avalanches to spread at $H_c$, and the hysteresis loop becomes continuous.

In order to study the nature of avalanches in the 3D RFIM, we have analyzed its continuum version, described by the standard Landau-Ginzburg free energy

$$\bar{F} = F_0 \int dx \left[ \frac{1}{2} (\nabla \phi)^2 + \bar{V}(\phi) - \bar{H} \phi - \bar{h}(x) \phi \right]. \quad (1)$$

Here $\phi(x)$ is the coarse-grained magnetization whose tendency towards ferromagnetism is described by the potential $\bar{V}(\phi) = m^2 \phi^2 + \frac{1}{2} \phi^4$ with the reduced temperature $m^2 \sim T \equiv -1/(1 - T/T_c)$. $\bar{H}$ is the uniform external magnetic field, and $\bar{h}(x)$ is a Gaussian random field with zero mean and variance $\langle \bar{h}(x) \bar{h}(x') \rangle = \Delta \delta(x - x')$. The prefactor $F_0$ depends on the microscopic details and increases with the range of ferromagnetic interactions.

We only consider $T < T_c$ ($m^2 < 0$) which is neces-
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FIG. 1: Top: Hysteresis curves for increasing disorder. At

Hc, the magnetization jumps discontinuously. In

strong disorder (∆ > ∆c ≈ ∆s) the jump disappears since

bubbles are already dense at Hdp. Bottom: The shaded areas

indicate where local avalanches are observed in the H − ∆

plane (in finite volumes V1 < V2 < ∞ [lighter shaded], and for

V = ∞ [dark area, H < Hdp = α∆]). Macroscopic avalanches

occur at the coercive field Hc (thick blue line). The horizontal

lines correspond to the hysteresis curves in the top panel.

sary to ensure the presence of metastable states. It is

convenient to rescale all quantities according to natural

units, r = x/ℓ, Ψ(r) = ϕ(x)/ϕ0, h(r) = (ℓ2h(x))/(2ϕ0),

and H = (ℓ2H)/(2ϕ0), where ϕ0 = (m2)/g1/2 is the

equilibrium magnetization of the pure system and ℓ =

(2/m2)1/2 ∼ τ−1/2 is the mean field correlation length.

The reduced disorder strength is ∆ = (∆Δ)/(4ϕ0) ∼

(∆)τ−3/2. Further, we denote by Ψ±(H) the minimum

with positive/negative magnetization V(Ψ) − HΨ,

where V(Ψ) = −1

2

Ψ2 + 1

4

Ψ4.

In order to study the free energy landscape of this

model, we restrict ourselves to a mean field analysis and

concentrate on local extrema of Ψ, satisfying

\[ \frac{δF}{δΨ(r)} = -\frac{1}{2}∇^2Ψ(r) + V'(Ψ(r)) - H - h(r) = 0. \] (2)

In the spirit of the rate independent hysteresis approxi-
mation [12] we focus on spontaneously occurring events

and neglect thermal activation. This is justified suffi-
ciently far below Tc, where typical free energy barriers

between local minima are larger than T [12].

We may restrict the analysis of the hysteresis loop to

the raising branch where the magnetization is everywhere

close to Ψ−(H) at sufficiently negative H. As H is in-

creased, the magnetization either remains in its local free

energy minimum, responding paramagnetically to the

field change, or it becomes locally unstable, and a bub-

ble with higher magnetization is created spontaneously.

This requires a sufficiently strong local excursion h(r) of

the random field. For weak disorder such configurations

are rare, their density scaling as ρ ∝ \exp[−S], where

S = \int dr[h(r)]^2/2∆. Hence, we may think of them as
dilute islands each of which can be analyzed separately.

Further, we may assume that at some distance from a

rare excursion, the random fields are weak such that the

magnetization tends to Ψ−.

The fate of a spontaneously emerging bubble depends

on its size. We will see that typical bubbles are small and
do not spread much since they are held back by a surface

tension barrier. However, this does not apply to larger

(and hence exponentially rarer) bubbles. Instead, their

dynamics is governed by the competition between the ex-

ternal field exerting a driving force 2H on the surface of

the bubble, and the random field disorder, which tends

to pin it. Collective pinning theory predicts the critical

field for domain wall depinning to scale as Hdp = α\Delta[15].

From a high velocity expansion [16] we obtained the es-

imate α ≈ 0.055, which we confirmed by numerical sim-

ulations [17]. For H < Hdp, all avalanches are bounded,

and the hysteresis loop is dominated by typical, i.e., most

abundant, small avalanches. Beyond Hdp, the first un-

stable bubble which is not constrained by surface tension

induces a sample spanning avalanche and a macroscopic

jump in magnetization, however low the probability per

unit volume for such an event may be. In the thermody-

namic limit, the coercive field Hc thus coincides with the

depinning field Hdp. We will discuss mesoscopic fluctua-

tions of Hc in weakly disordered small samples further

below. The approach to complete saturation beyond Hc

is governed by small bubbles of negative magnetization

that spontaneously implode upon an increase of field.

Let us now analyze in more detail the properties of typ-

ical avalanches as a function of disorder and field, both

for H < Hc and H > Hc. An avalanche is triggered sponta-

neously when a local free energy minimum (with mag-

netization Ψ(r)) becomes unstable upon increase of H.

This occurs when Ψ(r) merges with an adjacent saddle

point, Ψsp(r), the magnetization difference δΨ = Ψsp − Ψ

turning into a soft mode of the Hessian,

\[ (−1/2∇^2 + V''(Ψ)) \deltaΨ = 0. \] (3)

Let us denote by M the set of all states Ψ, such that

there is a soft mode δΨ satisfying Eq. (3). These states

are “marginal” in the sense that they are unstable with

respect to the smallest perturbations.

Typical avalanches are triggered by the most proba-

ble configurations (“instantons”) of random fields, h =

hinst(r|H'), such that the current magnetization state,

Ψ(r), becomes marginal (Ψ ∈ M) when the external
field reaches \( H' \). These instantons thus correspond to non-trivial local minima of the action

\[
S = \frac{1}{2\Delta} \int dr h^2(r) = \frac{1}{2\Delta} \int dr \left[ \frac{1}{2} \nabla^2 \Psi - V'(\Psi) + H \right]^2
\]

within the set of marginal states \( \Psi \in \mathcal{M} \). Here, we used Eq. (2) to express the random field in terms of \( \Psi \). Assuming spherically symmetric instantons, the Euler-Lagrange equation takes the form

\[
\frac{-\Delta (d)}{2} + V''(\Psi) = \frac{\Delta (d+2)}{2} + V''(\Psi) + \Psi C_n \Psi
\]

where \( \Delta (d) = \partial_r^2 + \frac{d-1}{r} \partial_r \) denotes the radial part of the Laplacian in \( d \) dimensions, \( C \) is a Lagrange multiplier, and \( n_{\Psi} \) is the local normal to the manifold \( \mathcal{M} \).

Performing an extensive search for local minima of \( S \) (analytically by solving \( \Psi \)) and numerically by variational minimization of \( \Psi \), we found two types of optimal random field configurations. For \( H < H_1 = 0.0763 \), the optimal random field is such that the state \( \Psi \) corresponds to a threefold degenerate free energy minimum. However, slight deviations \( \delta h(r) \) from the instanton configuration \( h_{\text{inst}} \) lead to a local free energy landscape where the state with lowest magnetization is marginally unstable with respect to a nearby minimum with locally higher magnetization \( \delta M \propto (\int \delta h^2)^{1/4} \). An essentially identical analysis for \( H_c < H < +\infty \), but with inverse boundary conditions \( \Psi_- \rightarrow \Psi_+ \), yields similar instantons corresponding to the dominant bubbles which implode upon increase of the external field beyond \( H_c \).

A second type of instants describes the typical avalanches in the interval \( H_1 < H < H_c \). These instants can be found analytically by solving the “dimensionally reduced” equations of motion, \(-1/2 \Delta (d-2) \Psi + V'(\Psi) - H = 0\), which indeed yield a solution of Eq. (5) with \( C = 0 \). The corresponding random field is \( h_{\text{inst}}(r)H = -(\partial_r \Psi)/r \). In \( d = 3 \) dimensions, this equation is readily integrated with boundary conditions \( \Psi(r = 0) = 0 \) and \( \Psi(r \rightarrow \infty) \rightarrow \Psi_{\infty} \). The action Eq. (4) associated to these field configurations decreases monotonically from \( S(H_1) = 5.7/\Delta \) to \( S(H_{\text{sp}}) = 0 \).

An analysis of the local free energy landscape associated to this type of instants shows that for \( H_1 < H < H_2 = 0.173 \) the typical bubbles are constrained by a surface tension barrier, the change of magnetization remaining local (see Fig. 2(a)). This is illustrated in the lower panel of Fig. 2 where we plot the instanton \( h_{\text{inst}}(r)H \) together with all solutions of Eq. (2) for \( H = 0.13 \). Initially, the system is in the marginal state (I) with the lowest magnetization. Upon slight increase of \( H \), it becomes unstable and the system runs to the nearby lower lying minimum (II) which represents a localized bubble of positive magnetization. Further spread of this bubble is prevented by the surface tension of the domain wall that separates the positively and negatively magnetized regions. Indeed, a further saddle point (III) separates the local minimum from the global minimum (IV). Note that as long as \( H < H_{\text{sp}} \), domain wall pinning by typical random fields will split the state (IV) into a set of metastable states which differ on larger scales which are not shown in Fig. 2.

Beyond \( H_2 \), the confining surface tension barrier vanishes, and the avalanches spread further out (cf. Fig. 2(b,c)), leading to a marked jump in the slope of the hysteresis curve. However, this effect is only observable in a small window of disorder (see Fig. 1). Indeed, for \( \Delta < H_2/\alpha \), macroscopic avalanches set in before \( H_2 \), while for \( \Delta > H_2/\alpha \) the density of bubbles becomes rapidly high, preventing large avalanches as discussed below.

At a critical disorder \( \Delta_c \), the typical bubbles start to percolate at the depinning field \( H_{\text{dp}} \) (see Fig. 1). This marks the crossover towards a regime with a continuous hysteresis loop. Indeed, for \( \Delta > \Delta_c \), bubbles are dense already below the depinning threshold \( H_{\text{dp}} \), such that avalanches at \( H_{\text{dp}} \) run into already positively magnetized regions and thus die out. We can estimate \( \Delta_c \).
from the requirement that the density of typical bubbles is \( \rho(H_{\text{dp}}[\Delta_c]) = O(1) \), or \( S(H_{\text{dp}}[\Delta_c]) \approx 1 \). A more careful calculation including Gaussian prefactors yields \( \Delta_c \approx 3.2 \), which translates into \( \Delta_c \approx 9 \left( \langle m^2 \rangle \right)^{3/2}/g \approx \text{const} \times \tau^{3/2}/g \) for the transition line in the \( T - \Delta \) plane. It is interesting to note that for \( \Delta \approx \Delta_c \), the depinning field \( H_{\text{dp}} = \alpha \Delta \) is very close to the threshold field \( H_2 \) for typical avalanches to overcome surface tension barriers.

Above we used the identity of the coercive field and the depinning threshold. This applies to the infinite volume limit, as a result of which one finds \( H_c[\Delta] \to 0 \to 0 \) (see, e.g., [19]), instead of \( H_c[\Delta] \to 0 \to H_{\text{sp}} \), which holds for finite size systems (see bottom of Fig. 1). \( H_{\text{sp}} = 2/3\sqrt{3} \approx 0.385 \) being the spinodal field of the pure system. For a finite system of volume \( V \), the only type of bubbles occurring in the hysteresis loop are those with sufficiently large density, \( V \rho(H, \Delta) \gtrsim 1 \). This relation allows us to estimate the field \( H^* \) at which the first (typical) bubbles occur. If the disorder is so weak that \( H^* > H_2 \), there will be a single sample spanning avalanche around the coercive field \( H^* \approx H_{\text{sp}} - 0.5\Delta \log[V/\xi^3] \). For stronger disorder (\( H^* < H_2 \)) local bubbles occur at \( H^* \) while only at higher field a macroscopic avalanche is triggered by the first bubble that is not constrained by surface tension.

The analysis of instantons also provides information on the emergence of metastability conjectured by Villain in the context of pinned domain walls [12]. Over a range of order \( \Delta T \sim \delta^{3/2} \) (where \( \delta \approx (\langle \delta h^2 \rangle)^{1/2} \)) the two states coexist, being separated by a free energy barrier of order \( \Delta F \approx \delta^{2/3} \). The spatial density of such two-level systems with a given free energy barrier scales as \( \rho(\Delta F) \propto \exp[-S_0 - O(\Delta F^{3/2})] \) with \( S_0 = S[h_{\text{inst}}(r|H = 0)] \approx 8\pi/\Delta \). These two-level systems are very similar to the metastable bubbles observed numerically above the ferromagnetic transition [20]. We expect a deeper analysis of the bifurcation of states along the lines of this Letter to yield new insight into the nature of the phase transition in the RFIM.

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