An Efficient Concurrent Graph Data-Structure which maintains Acyclicity

Sathya Peri, Muktikanta Sa, Nandini Singhal
Department of Computer Science & Engineering
Indian Institute of Technology Hyderabad, India
{sathya_p, cs15resch11012, cs15mtech01004}@iith.ac.in

Abstract

In this paper, we propose an algorithm for maintaining a concurrent directed graph (for shared memory architecture) that is concurrently being updated by threads adding/deleting vertices and edges. The update methods of the algorithm are deadlock-free while the contains methods are wait-free. To the best of our knowledge, this is the first work to propose a concurrent data structure for an adjacency list representation of the graphs. We extend the lazy list implementation of concurrent set for achieving this.

We believe that there are many applications that can benefit from this concurrent graph structure. An important application that inspired us is SGT in databases and Transactional Memory. Motivated by this application, on this concurrent graph data-structure, we pose the constraint that the graph should always be acyclic. We ensure this by checking for graph acyclicity whenever we add an edge. To detect the cycle efficiently we have proposed a \textit{Wait-free} reachability algorithm. We have compared the performance of the proposed concurrent data structure with coarse-grained locking implementation which has been traditionally used in implementing SGT. We show that our algorithm achieves on an average 8x improvement in throughput as compared to coarse-grained and sequential implementations.

Keywords: concurrent data structure; acyclicity; lazy list; directed graph; wait-free; locks;

1 Introduction

Graph is a common data-structure that can model many real world objects & relationships. A graph represents pairwise relationships between objects along with their properties. Due to their usefulness, graphs are being used in various fields like genomics various kinds of networks such as social, semantic etc. Generally, these graphs are very large and dynamic in nature. Dynamic graphs are the one’s which are subjected to a sequence of changes like insertion, deletion of vertices and/or edges [1]. Online social networks (facebook, linkedin, google+, twitter, quora, etc.), are dynamic in nature with the users and the relationships among them changing over time. There are several important problems that can become challenging in such a dynamic setting: finding cycles, graph coloring, minimum spanning tree, shortest path between a pair of vertices, strongly connected components, etc.

We have been specifically motivated by the problem of \textit{Serialization Graph Testing (SGT)} scheduler [15] from Databases and Transactional Memory [13]. A database scheduler (as the name suggests) handles the concurrency control over a set of transactions running concurrently. A transaction is a block of code invoked by a thread/process to access multiple shared memory variables atomically. The scheduler commits a transaction if it does not violate correctness (which is serializability/opacity); otherwise the transaction is aborted.

\textit{Conflict-Serializability (CSR)} [15] is the standard correctness-criterion for deciding the correctness of single-version databases schedulers. Almost all single-version databases schedulers including SGT implement CSR. Although these schedulers are correct, they differ w.r.t progress in terms of number of aborts. Some schedulers cause more transactions to abort (unnecessarily) while others cause lesser aborts for the same input history. It can be seen that the higher the number of unnecessary aborts the lesser is the efficacy of the database. Among the various schedulers that implement CSR, it was shown that SGT causes the least number of transaction aborts [13]. In fact, due to the effectiveness of SGT, several \textit{Software Transactional Memory Systems (STMs)} have been developed based on this scheduler.
The working of SGT scheduler is as follows: it maintains a graph called as conflict-graph (CG) over all transactions. The conflict-graph characterizes every transaction as a vertex and all conflicts between transactions as directed edges \[^{[15]}\]. The conflict graph gets modified dynamically with time by the arrival, deletion of transactions causing deletion of vertices respectively and by the addition of conflicts between the transactions causing addition of edges between the vertices. It must be noted that although the set of threads that invoke the transactions are finite, the number of transactions and the corresponding operations can grow to be very large over time.

Figure 1: An example of a directed acyclic graph in the shared memory which is being accessed by multiple threads. Thread \(T_1\) is trying to add a vertex 10 to the graph. Thread \(T_2\) is concurrently invoking a remove vertex 3. Thread \(T_3\) is also concurrently performing an addition of directed edge from vertex 9 to vertex 8 and will later create a cycle.

The SGT scheduler maintains the property that conflict graph always remains acyclic to ensure correctness. When a transaction requests an operation, the SGT scheduler checks whether the requested operation can cause a cycle in the dynamic conflict graph. If it does not then the transaction is allowed to perform the operation; otherwise it is aborted.

Apart from SGT, other well-known applications that require maintaining dynamic graphs while ensuring acyclicity are deadlock avoidance and detection algorithms for concurrently executing processes \[^{[12]}\]. But unlike SGT, there is a theoretical upper limit on the number of vertices in the graph for these applications which is the total number of threads/processes in the system.

The traditional solution employed by SGT in Databasese & STMs to maintain dynamic graphs is to use a single coarse lock to protect the entire graph. Clearly, this implementation can be made more efficient by providing finer granularity of synchronization. Each thread which has invoked a transaction should independently be able to add/delete from vertices/edges to independent parts of the graph. It can be seen that this problem gets complicated if the graph is huge and multiple threads are concurrently accessing the graph and performing some operations. Further, on the top of such a dynamically changing graph it is not clear how to check for cycles without locking the entire graph using coarse locks.

In this paper, we propose a solution for this problem. We develop a concurrent directed graph data structure which allows threads to concurrently add/delete/contains on vertices/edges while ensuring linearizability. The update methods add/delete on vertices/edges are deadlock-free while the contains methods on vertices/edges are wait-free. We then present a wait-free algorithm for preserving acyclicity of this concurrent graph which is based on reachability. To show their correctness we discuss the linearization points of all these methods. We also show experimental analysis of the presented concurrent data structure under varying workload distributions which demonstrate the concurrency obtained against the coarse locking strategy.

The main contributions of this paper are as follows:

- A concurrent directed graph data structure represented by adjacency list that has been implemented using the idea of concurrent set based on linked list \[^{[2]}\].
- An algorithm: wait-free reachability for detecting cycle in the fully dynamic concurrent graph.
- Experimental analysis of the presented concurrent data structure under varying workload distributions which demonstrate on an average 8x higher throughput obtained by the proposed algorithm over the coarse locking strategy.
• We show that all the presented methods are linearizable by using linearization points. We then give proof-sketch to show that the methods of the graph data-structure which maintains acyclicity are linearizable. We also give a proof-sketch of the progress conditions of all the methods.

Related Work. There are many efficient well-known algorithms for solving these problems in the sequential world. Also, several parallel tools have been developed for operating on graphs and several algorithms based on these tools [7,11]. However, there are a very few work in the area of concurrent graphs shared in memory setting that work with linearizability. There has been a recent interesting and relevant work on the area of concurrent graphs by Kallimanis and Kanellou [6]. They consider dynamic graphs with edge weights changing dynamically. They also support addition/deletion of edges and dynamic traversal with all of the methods being wait-free. But in their system, the total number of vertices in the graph is fixed. Hence, they represent the dynamic graph in the form of adjacency matrix which assumes a upper-limit on the total number of vertices. Such a system cannot be used for representing a SGT scheduler in which the set of vertices changes dynamically over time. Moreover, it is not clear how to ensure graph acyclicity with their algorithm.

Roadmap. The Section 2 describes the system model and underlying assumptions. In the Section 3 we define the problem along with the underlying assumptions. The Section 4 describes the construction of a fully dynamic concurrent graph based adjacency list data structure along with the working of each of its methods. Section 5 presents the solution approach for preserving acyclicity in this dynamically changing graph. Results and experimental analysis are given in the Section 7 and finally we conclude and present the future direction in the Section 8.

2 System Model & Preliminaries

In this paper, we assume that our system consists of finite set of \( p \) processors, accessed by a finite set of \( n \) threads that run in a completely asynchronous manner and communicate using shared objects. The threads communicate with each other by invoking higher-level methods on the shared objects and getting corresponding responses. Consequently, we make no assumption about the relative speeds of the threads. We also assume that none of these processors and threads fail.

Events. We assume that the threads execute atomic events. Similar to Lev-Ari et. al.’s work, [8,9] we assume that these events by different threads are (1) read, write on shared/local memory objects; (2) method invocations or inv event & responses or resp event on higher level shared-memory objects.

Global States, Execution and Histories. We define the global state or state of the system as the collection of local and shared variables across all the threads in the system. The system starts with an initial global state. We assume that all the events executed by different threads are totally ordered. Each event changes possibly the global state of the system leading to a new global state.

The events read, write on shared/local memory objects changes the global state. The events inv event & resp event on higher level shared-memory objects do not change the contents of the global state. Although we denote the resulting state with a new label.

We denote an execution of a concurrent threads as a finite sequence of totally ordered atomic events. We formally denote an execution \( E \) as the tuple \( \langle \text{evts}, <_E \rangle \), where \( E.\text{evts} \) denotes the set of all events of \( E \) and \( <_E \) is the total order among these events. A history corresponding to an execution consists only of method inv and resp events (in other words, a history views the methods as black boxes without going inside the internals). Similar to an execution, a history \( H \) can be formally denoted as \( \langle \text{evts}, <_H \rangle \) where \( \text{evts} \) are of type inv & resp and \( <_H \) defines a total order among these events.

With this definition, it can be seen that a history uniquely characterizes an execution and vice-versa. Thus we use these terms interchangeably in our discussion. For a history \( H \), we denote the corresponding execution as \( E^H \).

Next, we relate executions (histories) with global states. An execution takes the system through a series of global states with each event of the execution stating from the initial state takes the global state from one to the next. We associate the state of an execution (or history) to be global state after the last event of the execution. We denote this final global state \( S \) of an execution \( E \) as \( S = E.\text{state} \) (or \( H.\text{state} \)).

We refer to the set of all the global states that a system goes through in the course of an execution as \( E.\text{allStates} \) (or \( H.\text{allStates} \)). It can be seen that for \( E, E.\text{state} \in E.\text{allStates} \). Figure 2 show an execution and the corresponding history. In the figure the curve line is an event, the vertical line is a state. The open[] & close([]) square bracket are the separator and it has no meaning in the figure.
Given an event \( e \) of an execution \( E \), we denote global state just before the \( e \) as the pre-state of \( e \) and denote it as \( e.pre \). Similarly, we denote the state immediately after \( e \) as the the post-state of \( e \) or \( e.post \). Thus if an event \( e \) is in \( E.evts \) then both \( e.pre \) and \( e.post \) are in \( E.allStates \).

The notion of pre & post states can be extended to methods as well. We denote the pre-state of a method \( m \) or \( m.pre \) as the global state just before the invocation event of \( m \) whereas the post-state of \( m \) or \( m.post \) as the global state just after the return event of \( m \).

**Notations on Histories.** We now define a few notations on histories which can be extended to the corresponding executions. We say two histories \( H_1 \) and \( H_2 \) are equivalent if the set of events in \( H_1 \) are the same as \( H_2 \), i.e., \( H_1.evts = H_2.evts \) and denote it as \( H_1 \equiv H_2 \). We say history \( H_1 \) is a sub-history of \( H_2 \) if all the events of \( H_1 \) are also in \( H_2 \) in the same order, i.e., \( \langle (H_1.evts \subseteq H_2.evts) \wedge (<_{H_1} <_{H_2}) \rangle \).

Let a thread \( T_i \) invoke some methods on a few shared memory objects in a history \( H \) and \( o \) be a shared memory object whose methods have been invoked by threads in \( H \). Using the notation of [5], we denote \( H|T_i \) to be the sub-history all the events of \( T_i \) in \( H \). Similarly, we denote \( H|o \) to be the sub-history all the events involving \( o \).

We say that a resp event matches an inv event in an execution (as the name suggests) if the both resp and inv events are on the same method of the object, have been invoked by the same thread and the resp event follows inv event in the execution.

We assume that a history \( H \) as well-formed if a thread \( T_i \) does not invoke a method on a shared object until it obtains the matching response for the previous invocation. We assume that all the executions & histories considered in this paper are well-formed. We denote an inv event as pending if it does not have a matching resp event in the execution. Note that since an execution is well-formed, there can be at most only one pending invocation for each thread.

We say a history \( H \) is complete if for every method inv event there is a matching resp event. The history \( H \) is said to be sequential if every inv event, except possibly the last, is immediately followed by the matching resp event. Note that a complete history is not sequential and the vice-versa. It can be seen that in a well-formed history \( H \), for every thread \( T_i \), we have that \( H|T_i \) is sequential. Figure 3 shows a sequential history.

**Sequential Specification.** We next discuss about sequential-specification [5] of shared memory objects. The sequential-specification of a shared memory object \( o \) is defined as the set of (all possible) sequential histories involving the methods of \( o \). Since all the histories in the sequential-specification of \( o \) are sequential, this set captures the behavior of \( o \) under sequential execution which is believed to be correct. A sequential history \( S \) is said to be legal if for every shared memory object \( o \) whose method is invoked in \( S \), \( S|o \) is in the sequential-specification of \( o \).

**Safety: A safety property is defined over histories (and the corresponding executions) of shared objects and generally states which executions of the shared objects are acceptable to any application. The safety property that we consider is linearizability [5]. A history \( H \) is said to be linearizable if (1) there exists a completion \( \overline{H} \) of \( H \) in which some pending inv events are completed with a matching response and some other pending inv events are discarded; (2) there exists a sequential history \( S \) such that \( S \) is equivalent
3 CONSTRUCTION OF CONCURRENT GRAPH DATA-STRUCTURE

Another way to say that history $H$ is linearizable if it is possible to assign an atomic event as a linearization point (LP) inside the execution interval of each method such that the result of each of these methods is the same as it would be in a sequential history in which the methods are ordered by their LPs \[3\].

A concurrent object is linearizable if all the histories generated by it are linearizable. We prove the linearizability of a concurrent history by defining a LPs for each method. The LP of a method implies that the method appears to take effect instantly at its LP.

**Progress:** The progress properties specifies when a thread invoking methods on shared objects completes in presence of other concurrent threads. Some progress conditions used in this paper are mentioned here which are based on the definitions in Herlihy & Shavit \[4\]. The progress condition of a method in concurrent object is defined as:

1. **Blocking:** In this, an unexpected delay by any thread (say, one holding a lock) can prevent other threads from making progress.

2. **Deadlock-Free:** This is a blocking condition which ensures that some thread (among other threads in the system) waiting to get a response to a method invocation will eventually receive it.

3. **Starvation-Free:** This is a blocking condition which ensures that every thread trying to get a response to a method, eventually receives it.

4. **Non-Blocking:** In this, a failure or suspension of a thread cannot prevent some operation of another thread from making progress.

5. **Lock-Free:** This is a non-blocking condition which ensures that some thread waiting to get a response to a method (among multiple other threads), eventually receives it.

6. **Wait-Free:** This is a non-blocking condition which ensures that every thread trying to get a response to a method, eventually receives it.

It can be seen that wait-free methods are desirable since they can complete regardless of the execution of other concurrent methods. On the other hand, deadlock-free methods are system (or underlying scheduler) dependent progress condition since they involve blocking. It ensures that among multiple threads in the system, at least one of them will make progress.

The update methods exported by the graph data-structure discussed in this paper are deadlock-free and the contains methods are wait-free.

3 Construction of Concurrent Graph Data-Structure

3.1 Overview

In this section we describe the graph data structure. It is based on the adjacency list representation. Hence, it is implemented as a collection (list) of vertices wherein each vertex in turn holds a list of vertices to which it has outgoing edges. The implementation is a linked list of `vnode` and `enode` as shown.

![Figure 3: An illustration of a Sequential execution $E^S$.](image-url)
in the Table 1. The implementation of each of these lists are based on the lazy-list implementation of
the a concurrent-set [2]. The enode class has four fields. The val field is the key value of the edge(u, v)
(edge from u to v), stores the key value of v. The edge nodes are sorted in order of the val field. This
helps efficiently detect when a enode is absent in the edge list. The marked field is of type boolean which
indicates whether that enode is in the edge list or not. The enext field is a reference to the next enode
in the edge list. The lock field is for ensuring access to a shared enode happens in a mutually exclusion
manner. We say a thread acquires a lock and releases the lock when it executes a lock.acquire() and
lock.release() method call respectively.

Similarly, the vnode class has six fields. The val field is the key value of the vertex u. The vertex
d nodes are sorted in the order of the val field which helps detect presence/absence of a vnode in the vertex
list (like the the sorted enode list). The marked field is a boolean marked field indicating whether that
vnode is in the vertex list or not. The vnext field is a reference to the next vnode in the vertex list. The
EdgeHead field is a sentinel enode at the start of the each edge list for each vnode has the smallest possible
key value (−∞). The EdgeTail field is sentinel enode at the end of the each edge list has the largest
possible key value (+∞). The lock field to add the mutual exclusion to a shared vnode implementation.
We say a thread acquires a lock and releases the lock when it executes a lock.acquire() and lock.release()
method call respectively. We assume the enext and marked fields of the enode structure are atomic.
Similarly, the vnext and marked fields of a vnode are atomic.

| Edge Node | Vertex Node |
|-----------|-------------|
| class enode{
    int val;
    enode enext;
    boolean marked;
    Lock lock;
    enode(int key){
        val = key;
        marked = false;
        enext = null;
        lock = new Lock();
    }
}; |
| class vnode{
    int val;
    vnode vnext;
    enode EdgeHead;
    enode EdgeTail;
    boolean marked;
    Lock lock;
    vnode(int key){
        val = key;
        marked = false;
        vnext = null;
        EdgeHead = new enode(-infinity);
        EdgeTail = new enode(+infinity);
        EdgeHead.enext = EdgeTail;
        lock = new Lock();
    }
}; |

Table 1: Structure of Vertex and Edge Node.

Given a global state S, we define a few structures and notations as follows:

1. We denote vertex node, say v, as a vnode class object. Similarly, we denote edge node, say e, as a
   enode class object.

2. S.vnodes as a set of vnode class objects that have been created so far in S. Its structure is defined
   in the Table 1. Each vnode object v in S.vnodes is initialized with key, vnext to null, marked
   field initially set to false. Two sentinel edge nodes are created: EdgeHead and EdgeTail assigned
   with val −∞ and +∞ respectively; EdgeTail.enext to null and EdgeHead.enext to EdgeTail.

3. S.enodes is a set of enode class objects and its structure is defined in the Table 1. Each enode
   object e in S.enodes is initialized with key, enext to null, with marked field initially set to false.

4. S.VertexHead is a vnode class object (called sentinel head vertex node), which is initialized with
   a val −∞. This sentinel node is never removed from the vertex list. Similarly, S.VertexTail is
   a vnode class object (called sentinel tail vertex node), which is initialized with a val +∞. This
   sentinel node is never removed from the vertex list.
5. The contents & the status of a vnode $v$ keeps changing with different global states. For a global state $S$, we denote $S.v$ as its current state and the individual fields of $v$ in $S$ as $S.v.val, S.v.vnext, ...$ etc.

6. Similar to the contents of a vnode, it can be seen that the contents & the status of a enode $e$ keeps changing with different global states. Similarly, for a global state $S$, we denote $S.e$ as its current state and the individual fields of $e$ in $S$ as $S.e.val, S.e.vnext, ...$ etc.

Having defined a few notions on $S$, we now define the notion of an abstract graph, $AbG$ for a global state $S$ which we will use for guiding us in correctness of our methods. $AbG$ for a global state $S$ is the collection of $S.AbG(V)$ and $S.AbG(E)$ as defined below:

**Definition 1** $S.AbG(V) \equiv \{v|(v \in S.vnodes) \land (S.VertexHead \rightarrow^* S.v) \land \neg S.v.marked\}$.

This definition of AbG(V) captures the set of all vertices of AbG for the global state $S$. It consists of all the vnodes that are reachable from S.VertexHead and are not marked for deletion.

**Definition 2** $S.AbG(E) \equiv \{e|(e \in S.enodes) \land ((u,v) \subseteq S.AbG(V)) \land (S.u.EdgeHead \rightarrow^* S.e) \land \neg S.e.marked) \land (S.e.val = S.e.val)\}$.

This definition of AbG(E) captures the set of all edges of AbG for the global state $S$. Informally it consists of all the enodes that connects two vnodes $u, v$ with the edge going form $u$ to $v$.

The problems addressed in this paper are defined as here:

1. A concurrent directed graph $G = (V, E)$, which is dynamically being modified by a fixed set of concurrent threads. In this setting, threads may perform insertion / deletion of vertices or edges to the graph. We develop this data-structure in Section 3.

2. We also maintain an invariant that the concurrent graph $G$ updated by concurrent threads should be acyclic. This means that the graph should preserve acyclicity at the end of every operation in the generated equivalent sequential history. We describe the modified data-structure in Section 4.

We assume that all the vertices have unique identification key. Serialization Graph Testing Algorithm which is our motivating example, assumes that all the transactions have unique ids. Once a transaction is deleted it does not come back again into the system. As a result, we assume that all the vertices are assigned unique keys and duplicate vertices are not allowed. We assume that once a vertex id has been added, it will not be added again to the concurrent graph $G$.

### 3.2 Methods Exported & Sequential Specification

In this section, we describe the methods exported by the concurrent directed graph data structure along with their sequential specification. This specification as the name suggests shows the behaviour of the graph when all the methods are invoked sequentially.

1. The *AddVertex*(u) method adds a vertex $u$ to the graph, returning *true*. This follows directly from our assumption that all the vertices are assigned distinct keys. Once added, the method will never invoke addition on this key again.

2. The *RemoveVertex*(u) method deletes vertex $u$ from the graph, if it is present in the graph and returns true. By deleting this vertex $u$, this method ensures that all the incoming and outgoing vertices of $u$ are deleted as well. If the vertex is not in the graph, it returns *false*.

3. The *AddEdge*(u, v) method adds a directed edge $(u, v)$ to the concurrent graph if the edge $(u, v)$ is not already present in the graph and returns *true*. If the edge is already in the graph it simply returns *true*. But if either the vertices $u$ or $v$ is not present, it returns *false*. Section 3 presents the construction of concurrent data structure based upon this specification.

To maintain the graph acyclicity invariant which is described in section 4, we change the specification of the *AddEdge* method as follows: if either the vertices $u$ or $v$ is not present, it returns *false*, like in the earlier case. Similarly, if the edge is already present in the graph, it returns *true*. If both the vertices $u$ & $v$ are present and the edge is not in the graph already, this method tests to see if this edge $(u, v)$ will form a cycle in the graph by invoking *CycleDetect* method. If it does not form a cycle, the edge is added and it returns *true*. Otherwise, it returns *false*.
4. The `RemoveEdge(u,v)` method deletes the directed edge \((u,v)\) from the graph structure if it is present and returns `true`. If the edge \((u,v)\) is not present in the graph but the vertices \(u\) \& \(v\) are in the graph it still returns `true`. But, if either of the vertices \(u\) \or \(v\) is not present in the graph it returns `false`.

5. The `ContainsEdge(u,v)` returns `true`, if the graph contains the edge \((u,v)\); otherwise returns `false`.

6. The `ContainsVertex(u)` returns `true`, if the graph contains the vertex \(u\); otherwise returns `false`.

Table 2 describes the sequential specification of each method formally in any given global state \(S\) before the execution of the method and future state \(S'\) after executing it sequentially. The `Pre-state` is the shared state before `inv` event and the `Post-state` is also the shared state just after the `resp` event of a method, which is depicted in the Figure 2. For the vnodes \(u\) \and \(v\), we represent a particular `enode e` as the edge \((u,v)\), from \(u\) to \(v\), such that \(e.val = v.val\). We assume that a typical application invokes significantly more contains methods (`ContainsEdge` and `ContainsVertex`) than the update methods (`AddVertex`, `RemoveVertex`, `AddEdge`, `RemoveEdge`).

## 4 Working of Concurrent Graph Methods

In this section, we describe the implementation of the concurrent graph structure and the working of the various methods. We represent the graph using adjacency list representation, which is a list of linked lists as illustrated in the Figure 3. The underlying adjacency list implementation is an adaptation of the lazy-list based concurrent set \(\mathcal{P}\). All the fields in the structure are declared atomic. This ensures that operations on these variables happen atomically. In the context of a particular application, the node structure can be easily modified to carry useful data (like weights etc.).
The algorithm uses two nodes structures: \textit{vnode} & \textit{enode} which are described in Table 1. As can be seen from Figure 4, we have a list of all \textit{vnodes} (or vertices) denoted as \textit{vertex list}. All the vertices in this list are sorted by their keys. We maintain two sentinel nodes, called \textit{VertexHead} and \textit{VertexTail}, at the start and end of vertex list having the smallest and largest possible key values respectively. Each vertex \(v\) in the vertex list has an associated list of \textit{enodes} called as edge list to keep track of all the outgoing edges from this vertex. Similar to vertex list, the edge list is also sorted by their keys. We again maintain two sentinel nodes in each edge list, \textit{EdgeHead} and \textit{EdgeTail} at the start and end of the list having the smallest and largest possible key values respectively. None of the sentinel nodes are deleted. All the fields in the structure are declared atomic. This ensures that operations on these variables happen atomically.

In the context of a particular application, the node structure can be easily modified to carry useful data (like weights etc).

Till now, no fully dynamic concurrent adjacency list data structure has been proposed. Hence, when multiple threads simultaneously update the graph data structure, it is done by using coarse locks on the method calls. The problem with this is, when multiple threads try to access the concurrent data structure at the same time, the data structure becomes a sequential hold-up, forcing threads to wait in line for access to the lock. This graph data structure can be used as a fundamental building unit for different granularities of synchronization. In this paper, we consider lazy synchronization of concurrent set implemented using linked list [2] to implement dynamically changing adjacency list data structure.

Instead of maintaining a coarse lock on the adjacency list as a whole, we maintain locks for individual vertex and edge nodes in the adjacency list to increase the concurrency. Lazy synchronization further increases concurrency by allowing traversals to occur without acquiring locks. Once the correct nodes have been found and locks have been acquired, then the thread validates if the locked nodes are indeed correct. If before acquiring locks, some other thread has updated the data structure and wrong nodes were locked, then the locks are released and traversals start over.

**Notations used in PseudoCode:**

\(\downarrow, \uparrow\) denote input and output arguments to each method respectively. The shared memory is accessed only by invoking explicit \textit{read()} and \textit{write()} methods. The \textit{flag} is a local variable which returns the status of each operation. We use \(e_1, e_2\) to represent the \textit{enode} reference nodes and \(v_1, v_2, v\) to represent \textit{vnode} references.
Algorithm 1 ValidateVertex Method: Takes two vertices, $v_1, v_2$, each of type $vnode$ as input and validates for presence in vertex list and returns true or false.

1: procedure ValidateVertex($v_1 \downarrow, v_2 \downarrow, flag \uparrow$)
2: \hspace{1em} if ($\text{read}(v_1, \text{marked}) = \text{false}) \land (\text{read}(v_2, \text{marked}) = \text{false}) \land (\text{read}(v_1, v_{\text{next}}) = v_2)$ then
3: \hspace{2em} $flag \leftarrow \text{true};$ // validation successful
4: \hspace{2em} else
5: \hspace{3em} $flag \leftarrow \text{false};$ // validation fails
6: \hspace{1em} end if
7: \hspace{1em} return; // return flag
8: end procedure

Algorithm 2 LocateVertex Method: Takes key as input and returns the corresponding pair of neighboring $vnode$ ($v_1, v_2$). Initially $v_1$ and $v_2$ are set to null.

9: procedure LocateVertex (key $\downarrow, v_1 \uparrow, v_2 \uparrow$)
10: \hspace{1em} while (true) do
11: \hspace{2em} $v_1 \leftarrow \text{read}(\text{VertexHead});$
12: \hspace{2em} $v_2 \leftarrow \text{read}(v_1, v_{\text{next}});$ 
13: \hspace{2em} while ($\text{read}(v_2, \text{val}) < \text{key}$) do // search without acquiring lock
14: \hspace{3em} $v_1 \leftarrow v_2;$
15: \hspace{3em} $v_2 \leftarrow \text{read}(v_2, v_{\text{next}});$ 
16: \hspace{2em} end while
17: \hspace{2em} lock.acquire($v_1$);
18: \hspace{2em} lock.acquire($v_2$);
19: \hspace{2em} /* If ValidateVertex returns true, then it returns. */
20: \hspace{2em} if (ValidateVertex($v_1 \downarrow, v_2 \downarrow, flag \uparrow$)) then
21: \hspace{3em} return;
22: \hspace{2em} else
23: \hspace{3em} lock.release($v_1$);
24: \hspace{3em} lock.release($v_2$); // validation failed, try again
25: \hspace{2em} end if
26: \hspace{1em} end while
27: end procedure

4.1 Working of the Update Vertex methods - AddVertex & RemoveVertex

The $\text{AddVertex}(u)$ method is similar to the add method of concurrent set implemented using lazy linked list $^2$. When a thread wants to add a vertex to the concurrent graph, it traverses the vertex list without acquiring any locks until it finds a vertex with its key greater than or equal to $u$, say $u_{\text{curr}}$ and it’s predecessor, say $u_{\text{pred}}$. It acquires locks on the vertices $u_{\text{pred}}$ and $u_{\text{curr}}$ itself. It validates to check if $u_{\text{curr}}$ is reachable from $u_{\text{pred}}$, and if both the vnodes have not been deleted (marked). The algorithm maintains an invariant that all the unmarked vertex nodes are reachable. If the validation succeeds, the thread adds the vertex $u$ between $u_{\text{pred}}$ and $u_{\text{curr}}$ in the vertex list and returns true after unlocking the vertices. If it fails, the thread starts the traversal over after unlocking the locked vertices. The $\text{AddVertex}$ method is described in the Algorithm $^3$.

From the structure of $vnode$ class, we know that each $vnode$ has a boolean marked field in Table $^1$. The removal of a $vnode$, say $u$ happens in two steps like in the lazy linked list $^2$: (1) The $vnode$ $u$’s marked field is first set to true and it referred to as logical removal. This ensures that if any edge is being added or removed concurrently corresponding to that vertex, it would fail in their validation after encountering it to be marked at Line $^2$ in the Algorithm $^1$. (2) Then the pointers are changed so that $u$ is removed from the vertex list (Line $^67$). This is referred to as physical deletion, then it changes the pointer (Line $^68$) of the predecessor of the marked node to its successor so that the deleted node is no longer reachable in the vertex list. To achieve this, $\text{RemoveVertex}(u)$ method proceeds similar to the $\text{AddVertex}(u)$. The thread iterates through vertex list until it identifies the vertex $u$ to be deleted. Then after $u$ and its predecessor are locked, then logical removal occurs by setting the marked field to true. The $\text{RemoveVertex}$ method is described in the Algorithm $^5$.

After the physical deletion of the vertex $u$ in the vertex list, its incoming edges must also be deleted. This
is described by the RemoveIncomingEdge method (Algorithm 4). This is done by performing a traversal of the entire vertex list, to check if any of the existing vertices contain an edge node corresponding to the deleted vertex in their edge list. If such an edge node is present, locks are obtained on the enodes $e_1$ and $e_2$ in the Line 49 and 50 respectively of the vertex $v$ and then this edge node $e_2$ is deleted by setting the marked to true (logical removal) and then physically deleted. The physical deletion does not remove the vnode from the memory as we are not doing any garbage collection. It is to be noted that performing the deletion of incoming edges of deleted vertices is an optional step as this does not affect the correctness of the algorithm. In other words, even if edges nodes corresponding to the deleted vertices are still unmarked and reachable, no other method’s correctness is affected by their presence. In later section, we present results of variants with this method optional.

Algorithm 3 AddVertex Method: Successfully adds VNode(key) to the vertex list, if it is not present earlier.

28: procedure AddVertex (key ↓)
29:  LocateVertex(key ↓, v₁ ↑, v₂ ↑);
30:  if (read(v₂.val) ≠ key) then // not present
31:      write(v₃, new vnode(key)); // new vnode created
32:      write(v₃.vnext, v₂);
33:      write(v₁.vnext, v₃); // added in the vertex list
34:  end if
35:  lock.release(v₁);
36:  lock.release(v₂);
37:  return;
38: end procedure

If the validation succeeds, the thread adds the vertex $u$ between upred and ucurr in the vertex list and returns true after unlocking the vertices. If it fails, the thread starts the traversal over after unlocking the locked vertices. This is described in Algorithm 4.

Algorithm 4 RemoveIncomingEdge Method: This method helps remove all the incoming edges of a deleted vertex vnode(key) from the graph.

39: procedure RemoveIncomingEdge (key ↓)
40:  temp ← read(VertexHead); // Starting from VertexHead
41:  while (read(temp.vnext) ≠ NULL) do
42:      while (true) do
43:          e₁ ← read(temp.EdgeHead);
44:          e₂ ← read(e₁.enext);
45:          while (read(e₂.val) < key) do
46:              e₁ ← e₂;
47:              e₂ ← read(e₂.enext);
48:          end while
49:          lock.acquire(e₁);
50:          lock.acquire(e₂);
51:          if (ValidateEdge(e₁, e₂)) then
52:              if (read(e₂.val) = key) then // key is present
53:                  write(e₂.marked, true); // logically removed
54:                  write(e₁.enext, e₂.enext); // physically removed
55:              end if
56:          end if
57:          lock.release(e₁);
58:          lock.release(e₂);
59:      end while
60:  temp ← temp.vnext
61:  end while
62:  return;
63: end procedure
4 WORKING OF CONCURRENT GRAPH METHODS

Algorithm 5 RemoveVertex Method: \texttt{vnode}(key) gets removed from the vertex list if it is already present. Initially \texttt{flag} is set to \texttt{true}.

64: \textbf{procedure} \texttt{RemoveVertex} (key ↓, flag ↑)
65: \hspace{1em} \texttt{LocateVertex}(key ↓, v₁ ↑, v₂ ↑)
66: \hspace{1em} if \texttt{(read}(v₂.val) = key) \texttt{then} \hspace{1em} // present
67: \hspace{1em} \texttt{write}(v₂.marked, true); \hspace{1em} // logically removed
68: \hspace{1em} \texttt{write}(v₁.vnext, v₂.vnext); \hspace{1em} // physically removed
69: \hspace{1em} \texttt{lock.release}(v₁);
70: \hspace{1em} \texttt{lock.release}(v₂);
71: \hspace{1em} \texttt{RemoveIncomingEdges}(key ↓);
72: \hspace{1em} /* \texttt{RemoveIncomingEdges} removes all incoming edges to \texttt{vnode}(key). */
73: \hspace{1em} else
74: \hspace{2em} \texttt{lock.release}(v₁);
75: \hspace{2em} \texttt{lock.release}(v₂);
76: \hspace{2em} flag ← \texttt{false}; \hspace{1em} // not present
77: \hspace{1em} end if
78: return; \hspace{1em} //return flag
79: \textbf{end procedure}

Each \texttt{vnode} of vertex list has a boolean marked field as can be seen in Table 1. The removal of a \texttt{vnode} \texttt{u} happens in two steps like in lazy linked list \cite{2}; (1) The \texttt{vnode} \texttt{u}’s marked field is first set to \texttt{true}. This is referred to as logical removal. This ensures that if any edge is being added or removed concurrently corresponding to that vertex will fail in the validation process after checking the marked field. (2) Then, the pointers are changed so that \texttt{u} is removed from the vertex list. This is referred to as physical deletion which involves changing the pointer of the predecessor of the marked node to its successor so that the deleted node is no longer reachable in the vertex list. To achieve this, \texttt{RemoveVertex}(\texttt{u}) method proceeds similar to the \texttt{AddVertex}(\texttt{u}). The thread iterates through vertex list until it identifies the vertex \texttt{u} to be deleted. Then after \texttt{u} and its predecessor have been locked, logical removal occurs by setting the marked field to \texttt{true}. This is described in Algorithm 5.

After the physical deletion of the vertex \texttt{u}, the vertex should be removed from the edge-list of all the other nodes \texttt{v} from whom there is an incoming edge to \texttt{u}. This is described by the \texttt{RemoveIncomingEdge} method. This is done by performing a traversal of the entire vertex list, to check if the deleted vertex is contained as an edge node in their edge list. If such an edge node is found, locks are obtained on the \texttt{enodes} \texttt{e₁} and \texttt{e₂} in the Line 49 and 50 in Algorithm 4 respectively of the vertex \texttt{v} and then this edge node \texttt{e₂} to be deleted is marked (logical removal) and then physically deleted. The implementation is available at \cite{10}. The physical deletion does not remove the \texttt{vnode} from the memory as we are not doing any garbage collection. All deleted vnodes are not reachable from the \texttt{VertexHead} but they are present in the system.

4.2 Working of the Update Edge methods - AddEdge & RemoveEdge

Algorithm 6 ValidateEdge Method: Takes two \texttt{ENode} \texttt{e₁}, \texttt{e₂} and validates for presence in edge list.

80: \textbf{procedure} \texttt{ValidateEdge}(e₁ ↓, e₂ ↓, flag ↑)
81: \hspace{1em} if \texttt{(read}(e₁.marked) = false) \land \texttt{(read}(e₂.marked) = false) \land \texttt{(read}(e₁.enext) = e₂) \texttt{then}
82: \hspace{2em} flag ← \texttt{true}; \hspace{1em} // validation successful
83: \hspace{1em} else
84: \hspace{2em} flag ← \texttt{false}; \hspace{1em} //validation fails
85: \hspace{1em} end if
86: return; \hspace{1em} //return flag
87: \textbf{end procedure}
Algorithm 7 HelpSearchEdge Method: This method helps to optimise searching of edge vertices. It compare the key1 and key2, starts searching based on smaller key value. Takes two keys, key1 and key2, as input and returns the vnode \((v_1, v_2)\) corresponding to them. Initially \(v_1, v_2\) are set to null and flag is set to true.

```plaintext
88: procedure HelpSearchEdge (key1 ↓, key2 ↓, v1 ↑, v2 ↑, flag ↑)
89:    if (key1 < key2) then
90:       v1 ← read(VertexHead);  // starting from VertexHead
91:       while (read(v1.val) < key1) do
92:          v1 ← read(v1.vnext);
93:       end while
94:       if (read(v1.val) ≠ key1) ∨ (read(v1.marked)) then  // vnode(key1) not present or marked
95:          flag ← false;
96:          return;
97:       end if
98:    end if
99:    v2 ← read(v1.vnext);
100:   while (read(v2.val) < key2) do
101:      v2 ← read(v2.vnext);
102:     end while
103:    if (read(v2.val) ≠ key2) ∨ (read(v2.marked)) then  // vnode(key2) not present or marked
104:       flag ← false;
105:      return;
106:   end if
107: else
108:    v2 ← read(VertexHead);  // starting from VertexHead
109:    while (read(v2.val) < key2) do
110:      v2 ← read(v2.vnext);
111:    end while
112:    if (read(v2.val) ≠ key2) ∨ (read(v2.marked)) then  // vnode(key2) not present or marked
113:      flag ← false;
114:     return;
115: end if
116:    v1 ← read(v1.vnext);
117:    while (read(v1.val) < key1) do
118:      v1 ← read(v1.vnext);
119:    end while
120:    if (read(v1.val) ≠ key1) ∨ (read(v1.marked)) then  // vnode(key1) not present or marked
121:      flag ← false;
122:     return;
123: end if
124: end procedure
```

The AddEdge\((u, v)\) method starts by checking for the presence of vertices \(u\) and \(v\) in the vertex list of the graph by invoking the HelpSearchEdge method (Algorithm 7). After this, once again \(u\) & \(v\) are validated to be unmarked in the Line 146. The reason for this is explained by an example in Figure 5. Once the vertices \(u\) and \(v\) have been validated to be reachable and unmarked in the vertex list, the thread traverses the edge list of vertex \(u\) until an edge node with key greater than \(v\) has been encountered, say ecurr and it’s predecessor say epred, locks are obtained on the epred and ecurr. The thread does all this without acquiring locks on vertex \(u\) and \(v\). After this, validation is performed to check if the respective edge nodes are unmarked and reachable. If the validation is successful, the new edge node is added in between epred and ecurr in the edge list of \(u\). The AddEdge method is described in the Algorithm 8. In RemoveEdge\((u, v)\), the enode \(v\) is removed from \(u\)’s edge list. This method works similar to RemoveVertex method. It proceeds in two phases: first logical removal of the enode \(v\) by setting the mark field. Then, \(v\) is removed physically from \(u\)’s edge list by changing the pointers. Unlike RemoveVertex, the physical removal is simpler as there is no other extra work to be done. This is because the edges in the concurrent graph are directed and only one edge needs to be removed from one list. To achieve this, the RemoveEdge\((u, v)\) method proceeds similar to the AddEdge\((u, v)\) by traversing the vertex list in the graph, without acquiring any locks and then verifying that the vertices \(u\) and
v are indeed present in the graph. The thread then traverses the edge list of u (without acquiring locks) to identify the epred & ecurr edge nodes. Once these nodes have been locked and validated (for reachability), logical removal occurs and then physical deletion of edge nodes place immediately after logical removal. The RemoveEdge method is described in the Algorithm 10

Figure 5: This figure depicts why we need an additional check to locate vertices in LocateEdge (Algorithm 9) in Line 146. A thread T1 trying to perform AddEdge (u, v, true), first invokes HelpLocateEdge. Just after T1 has verified vertex u (Line 102), thread T2 deletes vertex u. Also vertex v gets added by thread T3 just before T1 verifies it by executing Line 102. So, now thread T1 has successfully tested for the presence of vertices u and v in the vertex list, and then it proceeds to add edge (u, v), returning true. However, as is evident, no possible sequentially generated history of the given concurrent execution is correct. Hence an additional check must be performed before proceeding to actually add the edge as in Line 146.

Algorithm 8 AddEdge Method: enode(key2) gets added to the edge list of vnode(key1), if it is not present. Initially, flag is set to true.

```
125: procedure AddEdge (key1 ↓, key2 ↓, flag ↑)
126:  LocateEdge(key1 ↓, key2 ↓, v1 ↑, v2 ↑, e1 ↑, e2 ↑, flag ↑);
127:  if (flag = false) then // vnode(key1) or vnode(key2) not found
128:      return;
129:  end if
130:  /* After this line, the method does not return false. It does not matter if the node is there or not.*/
131:  if (read(e2.val) ≠ key2) then // enode(key2) not present
132:     write(e3, new enode(key2));
133:     write(e3.enext, e2);
134:     write(e1.enext, e3);
135:  end if
136:  lock.release(e1);
137:  lock.release(e2);
138:  return; // returns flag to be true
139: end procedure
```
Algorithm 9 LocateEdge Method: Takes two keys, $key_1$ and $key_2$, as input and returns the pair of adjacent enode $\langle e_1, e_2 \rangle$. If enode $v_1$ or $v_2$ or enode $e_2$ is not present, it returns false. Initially enodes $e_1, e_2$ are set to null and flag is set to true.

```plaintext
140: procedure LocateEdge (key_1↓, key_2↓, v_1↑, v_2↑, e_1↑, e_2↑, flag↑)
141: HelpSearchEdge(key_1↓, key_2↓, v_1↑, v_2↑, flag↑);
142: if (flag = false) then
143: return; // $v_1$ or $v_2$ not found, returns flag
144: end if
145: /*This lines ensures both the vertices $v_1$ & $v_2$ have been added to the system by this time and are not marked*/
146: if (read($v_1$.marked) ∨ read($v_2$.marked)) then
147: flag ← false;
148: return;
149: end if
150: /* Helping for search edge, is a supporting method for locate edge. It locates the vertices $v_1$ & $v_2$*/
151: while (true) do
152: $e_1$ ← read($v_1$.enext);
153: $e_2$ ← read($e_1$.enext);
154: /* Search enode ($key_2$) without acquiring any locks*/
155: while (read($e_2$.val) < key_2) do
156: $e_1$ ← $e_2$;
157: $e_2$ ← read($e_2$.enext);
158: end while
159: lock.acquire($e_1$);
160: lock.acquire($e_2$);
161: if (ValidateEdge($e_1$↓, $e_2$↓, flag↑)) then
162: return; // returns true if validation succeeds.
163: else
164: lock.release($e_1$);
165: lock.release($e_2$); // validation failed, try again
166: end if
167: end while
168: end procedure
```

In RemoveEdge ($u, v$), the enode $v$ is removed from $u$’s edge list. This method works similar to RemoveVertex method. It proceeds in two phases: first logical removal of the enode $v$ by setting the mark field. Then, $v$ is removed physically from $u$’s edge list by changing the pointers. Unlike RemoveVertex, the physical removal is simpler as there is no other extra work to be done. This is because the edges in the concurrent graph are directed and only one edge needs to be removed from one list.

To achieve this, the RemoveEdge ($u, v$) method proceeds similar to the AddEdge($u, v$) by traversing the vertex list in the graph, without acquiring any locks and then verifying that the vertices $u$ and $v$ are indeed present in the graph. The thread then traverses the edge list of $u$ (without acquiring locks) to identify the $epred$ & $ecurr$ edge nodes. Once these nodes have been locked and validated (for reachability), logical removal occurs. Physical deletion of edge nodes can take place immediately after logical removal. This is described in Algorithm 10.
Algorithm 10 RemoveEdge Method: enode(key2) gets removed from the edge list of vnode(key1), if it is present. Returns successful if the edge is not present earlier.

Procedure REMOVEEDGE (key1 ↓, key2 ↓, flag ↑)

/* vnode(key1) or vnode(key2) not found*/
if (flag = false) then
  return;
end if
if (read(e2.val) = key2) then  // enode(key2) present
  write(e2.marked, true);  // logically removed
  write(e1.enext, e2.enext);  // physically removed
end if  // not present
lock.release(e1);
lock.release(e2);
return;  // returns flag which is true
end procedure

4.3 Working of the Read-Only methods - ContainsVertex & ContainsEdge

Methods ContainsVertex(u) and ContainsEdge(u, v), as described in Algorithm 11 and 12 respectively, traverse the graph without acquiring any locks. These methods return true if the vertex/edge node it was searching for is present and unmarked in the graph and otherwise returns false.

Algorithm 11 ContainsVertex Method: Returns true if vnode(key) is present in vertex list and returns false otherwise.

Procedure CONTAINSVERT Ex (key ↓, flag ↑)

while (read(v.val) < key) do
  v ← read(v.vnext);
end while
if ((read(v.val) ≠ key) ∨ (read(v.marked))) then
  flag ← false;
else
  flag ← true;
end if
return;
end procedure
Algorithm 12 ContainsEdge Method: Returns true if enode(key2) is part of the edge list of vnode(key1) and returns false otherwise.

```
195: procedure CONTAINSEDGE (key1 ↓, key2 ↓, flag ↑)
196:   HelpSearchEdge(key1 ↓, key2 ↓, v1 ↑, v2 ↑, flag ↑);
197:   /* vnode(key1) or vnode(key2) not found */
198:   if (flag = false) then
199:     return;
200:   end if
201:   e ← read(v1.enext);
202:   while (read(e.val) < key2) do
203:     e ← read(e.enext);
204:   end while
205:   if (read(e.val) ≠ key) ∨ (read(e.marked))) then
206:     flag ← false;
207:   else
208:     flag ← true;
209:   end if
210:   return;
211: end procedure
```

4.4 Correctness: Linearization Points

In this subsection, we define the Linearization Points (LPs) of all methods of our concurrent graph data structure. The linearization point of AddVertex(u) method is write(v1.vnext, v3) event in Line 33. Line 33 implies that the key u is not already present and the effect of this method actually happens at this line, such that a new vertex node is now made reachable from VertexHead of the vertex list. It can be seen that AddVertex(u) never returns false which follows from the sequential-specification.

For a successful RemoveVertex(u) call, the linearization point occurs when the deletion of the key u succeeds i.e. write(v2.marked, true) in Line 67, this means the key u is already present in the vertex list. An unsuccessful call is linearized at read(v2.val) in Line 66 where the key u is found to be not present in the vertex list.

The successful return of ContainsVertex(u) method is linearized when the key u is found to be unmarked in the vertex list i.e. at read(u.marked) in Line 188. We linearize an unsuccessful ContainsVertex method call within its execution interval at the earlier of the following points: (1) if no successful concurrent add vertex (lying between the first read of a shared variable in the method until LP) on u, the LP is defined to be later of read(n.val) not equal to u or read(n.marked) equal to true in Line 188, depending upon the execution. (2) if there is a successful concurrent add vertex on u, the ContainsVertex(u) is linearized at the point immediately before the LP of the successful concurrent add vertex u. This LP is similar to LP of contains in lazy list implementation of the concurrent set. Figure 6 illustrates why a concurrent AddVertex must be considered while linearizing an unsuccessful ContainsVertex method.
We linearize a successful $AddEdge(u,v)$ method call within its execution interval at the earlier of the following points: (1) if there is no successful concurrent delete vertex on $u$ and $v$, the LP is defined as the last of $read(e_2.val)$ in Line 131 and $write(e_1.enext, e_3)$ in Line 134 depending upon the execution. If the last line to execute is Line 131 the edge $(u,v)$ is present in the edge list of the vertex $u$. Whereas, Line 134 implies that the edge $(u,v)$ was not present earlier and this line adds a new edge node $(u,v)$ such that it is now reachable from $VertexHead$ of the vertex list. (2) if there is a successful concurrent delete vertex on $u$ or $v$ or both, the LP is the point immediately before the first LP of successful concurrent delete on vertex $u$ or $v$. Figure 7 illustrates why a concurrent $RemoveVertex$ must be considered while linearizing a successful $AddEdge$ method. Suppose the point where the edge $(5,7)$ is added to the graph is considered as a LP while there is a concurrent $RemoveVertex(7)$. The example shown shows that this LP will be wrong, since the $AddEdge$ returns true after successful deletion of the corresponding vertex.
Figure 7: An execution of two concurrent operations $T_1.\text{RemoveVertex}(7, true)$ and $T_2.\text{AddEdge}(5, 7, true)$. Figure (a) shows that $T_2$ has validated vertices 5 and 7 for presence in the vertex list and proceeds to add an edge (5, 7). Meanwhile, when $T_1$ is traversing the edge list of vertex 5 and has marked the vertex 7. Figure (b) depicts that LP of $\text{RemoveVertex}(7, true)$ (logical removal) happens before the (wrong) LP of $\text{AddEdge}(5, 7, true)$. The sequential history: $T_1.\text{RemoveVertex}(7, true) \prec_H T_2.\text{AddEdge}(5, 7, true)$ (ordered by their execution) is not linearizable. Figure (c) depicts the correct LP order.

For an unsuccessful $\text{AddEdge}(u, v)$ call, the LP is defined to be within its execution interval at the earlier of the following points: (1) if there is no successful concurrent add vertex on $u$ and $v$, LP is the, (a) last of $\text{read}(u.\text{val})$ in Line 94/119 where $u$ is not found in vertex list, (b) $\text{read}(u.\text{marked})$ in Line 94/119 where $u$ is found to be marked, (c) $\text{read}(v.\text{val})$ in Line 102/111 where $v$ is not found in the vertex list, (d) $\text{read}(v_2.\text{marked})$ in Line 102/111 where $v$ is found to be marked, (e) $\text{read}(v_1.\text{marked})$ in Line 146 depending upon the execution. (2) if there is a successful concurrent add vertex on $u$ or $v$ or both, it is linearized at the point immediately before the LP of the first successful concurrent add vertex on $u$ or $v$. Figure 8 illustrates how to linearize an unsuccessful $\text{AddEdge}$ method in presence of a concurrent $\text{AddVertex}$. Also it shows the case of concurrent $\text{RemoveEdge}$ as well since its LP is very similar to $\text{AddEdge}$.

Figure 8: An execution of three concurrent operations $T_1 : \text{AddVertex}(7, true)$, $T_2 : \text{AddEdge}(5, 7, false)$ and $T_3 : \text{RemoveEdge}(5, 7, false)$. Figure (a) shows that $T_1$ is traversing the vertex list looking for 7. Meanwhile, $T_2$ and $T_3$ searching for the presence of vertex 7 in vertex list. Suppose $T_2$ has reached Line 102 and before it could $\text{read}(v_2.\text{val})$ i.e. vertex 9, $T_1$ adds 7, which is shown in the Figure (b), so $T_2$ and $T_3$ return $false$ and $T_1$ successfully add the vertex 7. This situation precisely shows why the LP must be re-ordered. Figure (c) depicts the correct LP order.

We linearize a successful $\text{RemoveEdge}(u, v)$ method call within its execution interval at the earlier of the following points: (1) if there is no successful concurrent delete vertex, the LP is later of $\text{read}(e_2.\text{val})$ in Line 173 and $\text{write}(e_2.\text{marked}, true)$ in Line 176 based on the execution. If the last line to execute
is the edge \((u, v)\) is already present in the edge list of the vertex \(u\), the \(LP\) is the logically marked as deleted. If the last line to execute is the \(u, v\) is not present in the edge list of the vertex \(u\), the \(LP\) is the \(read(e_2.val)\). (2) if there is a successful concurrent delete vertex on \(u\) or \(v\) or both, it is linearized just before the LP of the first successful concurrent delete vertex on \(u\) or \(v\).

Figure 9 illustrates why a concurrent \(RemoveVertex\) must be considered while linearizing a successful \(RemoveEdge\) method. Suppose the point where the edge \((5, 9)\) is removed from the graph is considered as a LP while there is a concurrent \(RemoveVertex(9)\). The example shown shows that this LP will be wrong, since the \(RemoveEdge\) returns true after successful deletion of the corresponding vertex. For an unsuccessful \(RemoveEdge(u, v)\) method call, the LPs remain same as the LPs of the \(AddEdge(u, v)\) returning unsuccessfully, as described in Figure 8.

![Diagram](image_url)

Figure 9: An execution of two concurrent operations \(T_1.RemoveVertex(9, true)\) and \(T_2.RemoveEdge(5, 9, true)\). Figure (a) shows that \(T_2\) has validated vertices 5 and 9 for presence in the vertex list and proceeds to remove the edge \((5, 9)\). Meanwhile, when \(T_1\) is traversing the vertex list and it finds vertex 9 and has marked it (logically removal). Then \(T_1\) removes the enode 9 from 5’s list. Figure (b) shows that \(T_2\) has now removed the edge \((5, 9)\). The actual sequence: \(T_1.RemoveVertex(9, true) <_H T_2.RemoveEdge(5, 9, true)\) (ordered by their execution) is not linearizable. Figure (c) depicts the correct LP order.

For an unsuccessful \(RemoveEdge(u, v)\) method call, the LPs remain same as the LPs of the \(AddEdge(u, v)\) returning unsuccessfully, as described in Figure 8.

We linearize a successful \(ContainsEdge(u, v)\) method call within its execution interval at the earlier of the following points: (1) if there is no successful concurrent delete vertex on \(u\) and \(v\), the \(LP\) is \(read(e.marked)\) in Line 205 where the edge node with key \(v\) is found to be unmarked in the edge list of vertex \(u\). (2) if there is a successful concurrent delete vertex on \(u\) or \(v\), it is linearized immediately before the LP of the first successful concurrent delete on corresponding vertex. Figure 10 illustrates why a concurrent \(RemoveVertex\) must be considered while linearizing a successful \(ContainsEdge\) method. If the \(ContainsEdge(u, v)\) method returns unsuccessfully, it is linearized within its execution interval at the earlier of the following points: (1) if there is no successful concurrent add edge \((u, v)\), the \(LP\) is, (a) last of \(read(u.val)\) in Line 94\[119\] where \(u\) is not found in the vertex list, (b) \(read(u.marked)\) in Line 94\[119\] where \(u\) is found to be marked, (c) \(read(v.val)\) in Line 102\[111\] where \(v\) is not found in the vertex list, (d) \(read(v.marked)\) in Line 102\[111\] where \(v\) is found to be marked, depending upon the execution. (2) if there is a successful concurrent add edge on \((u, v)\), it is linearized immediately before the \(LP\) of that successful concurrent add edge. Figure 11 illustrates why a concurrent \(AddEdge\) must be considered while linearizing an unsuccessful \(ContainsEdge\) method. Suppose the point where the edge \((5, 9)\) is added to the graph is considered as a LP while there is a concurrent \(ContainsEdge(5, 9)\). The example shown shows that this LP will be wrong, since the \(ContainsEdge\) returns false after the successful addition of the corresponding edge.
As is evident, the LP of the methods have been defined in a dependent manner. To make the LPs precise, we define a total order on overlapping methods. Given a history, we first linearize all the vertex methods as follows: `AddVertex` $\rightarrow$ `RemoveVertex` $\rightarrow$ `ContainsVertex`. Since the LP of `ContainsVertex` is dependent on `RemoveVertex`, this ordering does not cause any ambiguity. After ordering the LPs of vertex method, we order the LPs of the edge methods, following a similar order: `AddEdge` $\rightarrow$ `RemoveEdge` $\rightarrow$ `ContainsEdge`. Thus with this order, by the time `ContainsEdge`'s LP needs to be fixed, all the other methods on whose LPs it could possibly depend on have already been fixed. An example of this is illustrated in Figure 12.
We now show how to construct a sequential history denoted as $H$ to show that every history generated by $g$ is linearizable. To prove the linearizability of a concurrent graph data structure, we assume following sequential execution for our concurrent graph data structure:

1. In any sequential execution, any method of the concurrent graph can be invoked in any global state and get a response.
2. Every sequential history $S$ generated by the concurrent graph is legal.
3. If a method $m_{i}(\text{inv-params} \uparrow, \text{rsp-params} \downarrow)$ of the concurrent graph data structure in a concurrent execution $E^{H}$. Then $m_{i}$ has a unique LP which is an atomic event within the inv and resp events of $m_{i}$ in $E^{H}$. The LP event can be identified based on the $\text{inv-params} \uparrow$, $\text{rsp-params} \downarrow$ and the execution $E^{H}$.
4. For any execution $E^{H}$ of a concurrent graph data structure, only the LP events of the methods can change the contents $AbG$ of the given concurrent graph data structure.

5.1 Constructing Sequential History

To prove the linearity of a concurrent graph $g$ which satisfies the above assumptions, we have to show that every history generated by $g$ is linearizable. To show this, we consider an arbitrary history $H$ generated by $g$. First we complete $H$, to form $\overline{H}$ if $H$ is incomplete. We then construct a sequential history denoted as $S$. $H$ is linealizable if (1) $S$ is equivalent to a completion of $H$; (2) $S$ respects the real-time order of $H$ and (3) $S$ is legal.

We now show how to construct $\overline{H}$, $S$. We then analyze some properties of $S$.

Completion of $H$. Suppose $H$ is not complete. This implies $H$ contains some incomplete methods. Note that since these methods are incomplete, they could have executed multiple possible LP events. Based on these LP events, we must complete them by adding appropriate resp event or ignore them. We construct the completion $\overline{H}$ and $E^{\overline{H}}$ as follows:

1. Among all the incomplete methods of $E^{H}$ we ignore those methods that did not execute a single LP event in $E^{H}$.
2. The remaining incomplete methods must have executed at least one possible LP event in $E^{H}$. For these methods, we consider the latest LP event executed in $E^{H}$ as the LP of the method. We build an ordered set consisting of all these incomplete methods which is denoted as partial-set. The methods in partial-set are ordered by their LPs.

Figure 12: Figure (a) shows an example of an execution with four concurrent operations and $T_{1} : \text{RemoveVertex}(7,\text{true})$ linearize first. Figure (b) $T_{2} : \text{AddEdge}(5,7,\text{true})$ linearize before $T_{1} : \text{RemoveVertex}(7,\text{true})$. Figure (c) $T_{3} : \text{RemoveEdge}(5,7,\text{true})$ linearize before $T_{1} : \text{RemoveVertex}(7,\text{true})$ and after $T_{2} : \text{AddEdge}(5,7,\text{true})$. Figure (d) $T_{4} : \text{ContainsEdge}(5,7,\text{true})$ linearize before $T_{1} : \text{RemoveEdge}(5,7,\text{true})$ and after $T_{2} : \text{AddEdge}(5,7,\text{true})$. The finally LP order, we obtain $T_{2} <_{H} T_{4} <_{H} T_{3} <_{H} T_{1}$.

5 Correctness of Concurrent Graph Object

In this section, we try to formalise the proof of our concurrent graph data structure based on LP events of the methods. Most of the notations used here are derived from [14].

We assume following sequential execution for our concurrent graph data structure:

1. In any sequential execution, any method of the concurrent graph can be invoked in any global state and get a response.
2. Every sequential history $S$ generated by the concurrent graph is legal.
3. If a method $m_{i}(\text{inv-params} \uparrow, \text{rsp-params} \downarrow)$ of the concurrent graph data structure in a concurrent execution $E^{H}$. Then $m_{i}$ has a unique LP which is an atomic event within the inv and resp events of $m_{i}$ in $E^{H}$. The LP event can be identified based on the $\text{inv-params} \uparrow$, $\text{rsp-params} \downarrow$ and the execution $E^{H}$.
4. For any execution $E^{H}$ of a concurrent graph data structure, only the LP events of the methods can change the contents $AbG$ of the given concurrent graph data structure.

5.1 Constructing Sequential History

To prove the linearity of a concurrent graph $g$ which satisfies the above said assumptions, we have to show that every history generated by $g$ is linearizable. To show this, we consider an arbitrary history $H$ generated by $g$. First we complete $H$, to form $\overline{H}$ if $H$ is incomplete. We then construct a sequential history denoted as $S$. $H$ is linealizable if (1) $S$ is equivalent to a completion of $H$; (2) $S$ respects the real-time order of $H$ and (3) $S$ is legal.

We now show how to construct $\overline{H}$, $S$. We then analyze some properties of $S$.

Completion of $H$. Suppose $H$ is not complete. This implies $H$ contains some incomplete methods. Note that since these methods are incomplete, they could have executed multiple possible LP events. Based on these LP events, we must complete them by adding appropriate resp event or ignore them. We construct the completion $\overline{H}$ and $E^{\overline{H}}$ as follows:

1. Among all the incomplete methods of $E^{H}$ we ignore those methods that did not execute a single LP event in $E^{H}$.
2. The remaining incomplete methods must have executed at least one possible LP event in $E^{H}$. For these methods, we consider the latest LP event executed in $E^{H}$ as the LP of the method. We build an ordered set consisting of all these incomplete methods which is denoted as partial-set. The methods in partial-set are ordered by their LPs.
3. To build $H$, for each incomplete method $m_i$ in partial-set considered in order, we append the appropriate resp event to $H$ based on the LP event of $m_i$ executed. Since the methods in partial-set are ordered by their final LP events, the appended resp events are also ordered by their LP events. Here, we have assumed that once the LP event is finalized its resp event can be determined.

4. To construct $E[H]$, for each incomplete method $m_i$ in partial-set considered in order, we sequentially append all the remaining events of $m_i$ (after its LP) to $E[H]$. All the appended events are ordered by the LPs of their respective methods.

From this construction, we can conclude that if $H$ is linearizable then $H$ is also linearizable. Formally, $(H$ is linearizable $) \implies (H$ is linearizable $)$.

For simplicity of presentation, we assume that all the concurrent histories & executions that we consider in the rest of this document are complete unless stated otherwise. Given any history that is incomplete, we can complete it by the transformation mentioned here. Next, we show how to construct a $S$ for a complete history $H$.

**Construction of $S$.** Given a complete history $H$ consisting of method inv & rsp events of a concurrent graph data structure $g$, we construct $S$ as follows: We have a single (hypothetical) thread invoking each method of $H$ (with the same parameters) on $g$ in the order of their inv events. Only after getting the response for the currently invoked method, the thread invokes the next method. From assumption we have that the methods are total, we get that for every method invocation $g$ will issue a response. Thus we can see that the output of these method invocations is the sequential history $S$. From the assumption we have that $S$ is legal. The histories $H$ and $S$ have the same inv events for all the methods. But, the resp events could possibly be different. Hence, they may not be equivalent to each other unless we prove otherwise.

In the sequential history $S$ all the methods are totally ordered. So we can enumerate all its methods as: $m_1$(inv-params, rsp-params) $m_2$(inv-params, rsp-params) ... $m_n$(inv-params, rsp-params). On the other hand, the methods in a concurrent history $H$ are not ordered. From our model, we have that all the events of the execution $E[H]$ are ordered. From the assumption, we have that each complete method has a unique LP event which is atomic. All the methods of $H$ and $E[H]$ are complete. Hence, we can order the LPs of all the methods in $E[H]$. Based on LP ordering, we can enumerate the corresponding methods of the concurrent history $H$ as $m_1$(inv-params, rsp-params), $m_2$(inv-params, rsp-params), ... $m_n$(inv-params, rsp-params). Note that this enumeration has nothing to do with the ordering of the inv and resp events of the methods in $H$.

Thus from the construction of $S$, we get that for any method $m_i$, $H.inv(m_i$(inv-params)) = $S.inv(m_i$(inv-params)) but the same need not be true for the resp events.

For showing $H$ to be linearizable, we further need to show $S$ is equivalent to $H$ and respects the real-time order $H$. Now, suppose $S$ is equivalent to $H$. Then from the construction of $S$, it can be seen that $S$ satisfies the real-time order of $H$.

![Figure 13](image-url)
5.2 Safety: Proof Skeleton

In this subsection, we describe the lemmas to prove the correctness of our concurrent graph data structure.

Observation 1 Only the LP events in the execution can change the AbG. □

Lemma 2 The global state of the graph after the resp event of a method is the same as the state before the inv event of the consecutive method in the sequential execution. Formally, \( \langle \text{PostM}[E^S.m_i].\text{AbG} = \text{PreM}[E^S.m_{i+1}].\text{AbG} \rangle \)

Proof Sketch: From the definition of Sequential Execution as depicted in Figure 13(b). □

Lemma 3 In every concurrent execution ordered by its events, the global state of the graph immediately after a LP event is the same as the state before the next LP event. Formally, \( \langle \text{PostE}[E^H.m_i.LP].\text{AbG} = \text{PreE}[E^H.m_{i+1}.LP].\text{AbG} \rangle \)

Proof Sketch: From Observation 1 as illustrated by Figure 13(a). □

Lemma 4 If some AddVertex(key) method returns true in \( E^H \), then

1. The vertex node corresponding to that key will not be present in the pre-state of the LP event of the method. Formally, \( \langle \text{AddVertex(key, true)} \implies (\text{vnode(key)} \notin \text{PreE}[E^H.\text{AddVertex(key, true)}.LP].\text{AbG}) \rangle \).

2. The vertex node corresponding to that key will be present in the post-state of the LP event of the method. Formally, \( \langle \text{AddVertex(key, true)} \implies (\text{vnode(key)} \in \text{PostE}[E^H.\text{AddVertex(key, true)}.LP].\text{AbG}) \rangle \).

Proof Sketch:

- 1 By observing the code, we know from Line 29 that when LocateVertex(key) returns, vertex nodes \( v_1 \) and \( v_2 \) are locked, \((v_1, v_2) \in S.vnodes\) and \( v_1.val < key \leq v_2.val \). We know that LP of this method is Line 33, which is reachable by evaluating condition on Line 30 as true. This means \( v_2.val \neq key \) and \( v_2 \in S.vnodes\), so, as \( v_1.val < key < v_2.val \) on the Line 30 the vnode(key) not belongs to \( S.AbG(V) \), where \( S.AbG(V) \) is the pre-state of the LP event of the method. Hence, \( \text{vnode(key)} \notin \text{PreE}[E^H.\text{AddVertex(key, true)}.LP].\text{AbG} \).

- 2 By observing the code, we know from Line 29 that when LocateVertex(key) returns, vertex nodes \( v_1 \) and \( v_2 \) are locked, \((v_1, v_2) \in S.vnodes\) and \( v_1.val < key \leq v_2.val \). We know that LP of this method is Line 33, which is reachable only by evaluating condition on Line 30 as true. After Line 33 the vnode(key) belongs to \( S.AbG(V) \), where \( S.AbG(V) \) is the post-state of the LP event of the method and the key value of the vnode(key) does not change after initialization. Hence, \( \text{vnode(key)} \in \text{PreE}[E^H.\text{AddVertex(key, true)}.LP].\text{AbG} \).

Lemma 5 If some RemoveVertex(key) method returns true in \( E^H \), then

1. The vertex node corresponding to that key must be present in the pre-state of LP event of the method. Formally, \( \langle \text{RemoveVertex(key, true)} \implies (\text{vnode(key)} \in \text{PreE}[E^H.\text{RemoveVertex(key, true)}.LP].\text{AbG}) \rangle \).

2. The vertex node corresponding to that key not present in the post-state of LP event of the method. Formally, \( \langle \text{RemoveVertex(key, true)} \implies (\text{vnode(key)} \notin \text{PostE}[E^H.\text{RemoveVertex(key, true)}.LP].\text{AbG}) \rangle \).

Proof Sketch:
• By observing the code, we know from Line that when LocateVertex(key) returns, vertex nodes \( v_1 \) and \( v_2 \) are locked, \((v_1, v_2) \in S.vnodes \) and \( v_1.val < key \leq v_2.val \). We know that the LP of this method is write\((v_2, marked, true)\) on the Line which is reachable only by evaluating the condition in Line to true. This means that \( v_2.val = key \) is evaluated to true such that \( v_1.vnext = v_2 \) and \( v_2 \in S.vnodes \). Thus we know \( vnode(key) \) belongs to \( S.AbG(V) \), where \( S \) is the pre-state of the LP event of the method. Hence \( vnode(key) \notin PreE[E^H. RemoveVertex(key, true).LP].AbG \).

• By observing the code, we know from Line that when LocateVertex(key) returns, vertex nodes \( v_1 \) and \( v_2 \) are locked, \((v_1, v_2) \in S.vnodes \) and \( v_1.val < key \leq v_2.val \). We know that the LP of this method is write\((v_2, marked, true)\) on the Line After Line \( v_1.vnext = v_3 \) and \( v_2 \) is not belongs to \( S.vnodes \), where \( S \) is the post-state of the LP event of the method. Hence \( vnode(key) \notin PreE[E^H. RemoveVertex(key, true).LP].AbG \).

Lemma 6 If some RemoveVertex(key) method returns false in \( E^H \), then

1. The vertex node corresponding to that key will not present in the pre-state of the LP event of the method. Formally, \( (\text{RemoveVertex}(key, false) \implies (vnode(key) \notin \text{PreE}[	ext{E}^H \cdot \text{i.LP}].\text{AbG}).) \)

2. The vertex node corresponding to that key will not present in the post-state of the LP event of the method. Formally, \( (\text{RemoveVertex}(key, false) \implies (vnode(key) \notin \text{PostE}[	ext{E}^H \cdot \text{RemoveVertex}(key, false).\text{LP}].\text{AbG}).) \)

Proof Sketch:

• By observing the code, we know that from Line that when LocateVertex(key) returns, vertex nodes \( v_1 \) and \( v_2 \) are locked, \((v_1, v_2) \in S.vnodes \) and \( v_1.val < key \leq v_2.val \). We know that the LP of this method is Line where the condition is evaluated to false. This means that \( v_2.val \neq key \) such that \( v_1.vnext = v_2 \). So, \( v_2 \) is not belongs to \( S.vnodes \), where \( S \) is the pre-state of the LP event of the method. Hence, \( vnode(key) \notin \text{PreE}[	ext{E}^H \cdot \text{RemoveVertex}(key, false).\text{LP}].\text{AbG} \).

• It is similar argument as Lemma the LP of this method is Line by evaluating the condition \( \text{read}(v_2.val) = key \) to false. This means that \( v_2.val \neq key \), so \( v_2 \) is not belongs to \( S.vnodes \), where \( S \) is the post-state of the LP event of the method. Hence, \( vnode(key) \notin \text{PostE}[	ext{E}^H \cdot \text{RemoveVertex}(key, false).\text{LP}].\text{AbG} \).

Lemma 7 If some ContainsVertex(key) method returns true in \( E^H \), then

1. The vertex node corresponding to that key must be present in the pre-state of LP event of the method. Formally, \( (\text{ContainsVertex}(key, true) \implies (vnode(key) \in \text{PreE}[	ext{E}^H \cdot \text{ContainsVertex}(key, true).\text{LP}].\text{AbG}).) \)

2. The vertex node corresponding to that key must be present in the post-state of LP event of the method. Formally, \( (\text{ContainsVertex}(key, true) \implies (vnode(key) \in \text{PostE}[	ext{E}^H \cdot \text{ContainsVertex}(key, true).\text{LP}].\text{AbG}).) \)

Proof Sketch:

• By observing the code, we realise that at the end of while loop in Line of ContainsVertex(key) method, \( v.val \geq key \). To return true, \( v.marked \) should be set to false in the pre-state of LP. It is clear that any unmarked public node should be reachable from VertexHead and thereby belong to \( S.AbG(V) \). We also know that a node’s key value does not change after initialization. So, the \( v \) is belongs to \( S.vnodes \), where \( S \) is the pre-state of the LP event of the method. Hence, \( vnode(key) \in \text{PreE}[	ext{E}^H \cdot \text{ContainsVertex}(key, true).\text{LP}].\text{AbG} \).

• Similar argument as Lemma 71
The vertex node corresponding to that key will not be present in the pre-state of LP event of the method. Formally, \(\text{ContainsVertex}(\text{key}, \text{false}) \implies (\text{vnode}(\text{key}) \notin \text{PreE}[E^H.\text{ContainsVertex}(\text{key}, \text{false}).\text{LP}].\text{AbG})\).

The vertex node corresponding to that key will not be present in the post-state of LP event of the method. Formally, \(\text{ContainsVertex}(\text{key}, \text{false}) \implies (\text{vnode}(\text{key}) \notin \text{PostE}[E^H.\text{ContainsVertex}(\text{key}, \text{false}).\text{LP}].\text{AbG})\).

Proof Sketch:

1. No successful concurrent \text{AddVertex} on \text{key}: By observing the code, we realise that at the end of while loop at Line 157 of \text{ContainsVertex}(\text{key}) method, \(v.val \geq \text{key}\). Also \(v\) being a reference, does not change. To return false, \(v\text{.marked}\) should be set to true in the pre-state of \text{LP} in Line 158. Since a node once marked, cannot be unmarked, therefore it does not belong to \(S.AbG(V)\). Hence we conclude that, \(\text{vnode}(\text{key}) \notin \text{PreE}[E^H.\text{ContainsVertex}(\text{key}, \text{false}).\text{LP}].\text{AbG}\).

2. Concurrent successful \text{AddVertex}(\text{key}) method: We know that the \text{LP} in this case is immediately before the \text{LP} of \text{AddVertex method}. Also from Lemma 8.1 it follows that if \text{AddVertex}(\text{key}) returns true, then the vertex node corresponding to that key is not present in the pre-state of the \text{LP}. Hence we conclude that, \(\text{vnode}(\text{key}) \notin \text{PostE}[E^H.\text{ContainsVertex}(\text{key}, \text{false}).\text{LP}].\text{AbG}\).

Similar argument as Lemma 8.1.

**Lemma 9** If some \text{AddEdge}(\text{key}_1, \text{key}_2) method returns true in \(E^H\), then

1. The corresponding vertex nodes, \(\text{vnode}(\text{key}_1)\) and \(\text{vnode}(\text{key}_2)\) will be present in the pre-state of LP event of the method. Formally, \(\langle \text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}) \implies (\text{vnode}(\text{key}_1) \land \text{vnode}(\text{key}_2)) \in \text{PreE}[E^H.\text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}).\text{LP}].\text{AbG}\rangle\).

2. The corresponding vertex nodes, \(\text{vnode}(\text{key}_1)\) and \(\text{vnode}(\text{key}_2)\) will be present in the post-state of LP event of the method. Formally, \(\langle \text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}) \implies (\text{vnode}(\text{key}_1) \land \text{vnode}(\text{key}_2)) \in \text{PostE}[E^H.\text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}).\text{LP}].\text{AbG}\rangle\).

Proof Sketch:

1. No successful concurrent \text{RemoveVertex} on \text{key}_1 \& \text{key}_2: By observing the code, we realise that \text{HelpSearchEdge} returns successfully only when \(\text{vnode}(\text{key}_1)\) \& \(\text{vnode}(\text{key}_2)\) were found to be present in the vertex list at least in one state of the execution. Now after this, the vertex nodes are again tested for removal. Thus, we can conclude that in the post-state of Line 146 the vertex nodes \(\text{vnode}(\text{key}_1)\) and \(\text{vnode}(\text{key}_2)\) must belong to \(S.AbG(V)\). Since there is no concurrent deletion of these vertices, we conclude that both the vertex nodes are present in the pre-state of \text{LP} of this method. Hence \(\text{vnode}(\text{key}_1) \land \text{vnode}(\text{key}_2) \in \text{PreE}[E^H.\text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}).\text{LP}].\text{AbG}\).

2. Concurrent successful \text{RemoveVertex} on \text{key}_1 or \text{key}_2 or both: We know that the \text{LP} in this case is immediately before the first \text{LP} of \text{RemoveVertex method}. Also from Lemma 5 it follows that if \text{RemoveVertex}(\text{key}) returns true, then the vertex node corresponding to that key is present in the pre-state of the \text{LP}. If any of the vertices were not present, the method returns \text{false}. Hence we conclude that, \(\text{vnode}(\text{key}_1) \land \text{vnode}(\text{key}_2) \in \text{PreE}[E^H.\text{AddEdge}(\text{key}_1, \text{key}_2, \text{true}).\text{LP}].\text{AbG}\).

Similar argument as Lemma 9.1.

**Lemma 10** If some \text{AddEdge}(\text{key}_1, \text{key}_2) method returns false in \(E^H\), then

1. At least one of the vertex nodes corresponding to those keys will not be present in the pre-state of LP event of the method. Formally, \(\langle \text{AddEdge}(\text{key}_1, \text{key}_2, \text{false}) \implies (\text{vnode}(\text{key}_1) \lor \text{vnode}(\text{key}_2)) \notin \text{PreE}[E^H.\text{AddEdge}(\text{key}_1, \text{key}_2, \text{false}).\text{LP}].\text{AbG}\rangle\).
5 CORRECTNESS OF CONCURRENT GRAPH OBJECT

Proof Sketch:

- **[Lemma 11]**

1. No successful concurrent AddVertex on key1 and key2: By observing the code, we realise that in the Line 94-119 vertex node vnode(key1) ∉ S.AbG(V) because it is not reachable from VertexHead or marked. Also, in the Line 102-111 vertex node vnode(key2) ∉ S.AbG(V) because it is not reachable from VertexHead or marked. Consequently, the Line 146 returns false, if read(v1.marked) or read(v2.marked) is set depending upon the execution. Hence, ((vnode(key1) ∨ (vnode(key2)) ∉ PreE[E^H.AddEdge(key1, key2, false).LP].AbG).

2. Concurrent successful AddVertex on key1 or key2 or both: We know that the LP in this case is immediately before the first LP of AddVertex method on u and v. Also from Lemma 4, it follows that if AddVertex(key) returns true, then the vertex node corresponding to that key is not present in the pre-state of the LP. Hence we conclude that, ((vnode(key1) ∨ (vnode(key2)) ∉ PreE[E^H.AddEdge(key1, key2, false).LP].AbG).

- **[Lemma 12]**

1. At least one of the vertex nodes, vnode(key1) and vnode(key2) must be present in the pre-state of LP event of the method. Formally, (RemoveEdge(key1, key2, true) → (vnode(key1) ∧ vnode(key2)) ∈ PreE[E^H.RemoveEdge(key1, key2, true).LP].AbG).

2. The corresponding vertex nodes, vnode(key1) and vnode(key2) must be present in the post-state of LP event of the method. Formally, (RemoveEdge(key1, key2, true) → ((vnode(key1) ∧ vnode(key2)) ∈ PostE[E^H.RemoveEdge(key1, key2, true).LP].AbG)∧(vnode(key2) ∉ PostE[E^H.RemoveEdge(key1, key2, true).LP].AbG).

- **[Lemma 13]**

1. Both the vertex nodes, vnode(key1) & vnode(key2) and the edge node corresponding to it, enode(key2) must be present in the pre-state of LP event of the method. Formally, (ContainsEdge(key1, key2, true) → (vnode(key1) ∧ vnode(key2) ∧ enode(key2)) ∈ PreE[E^H.ContainsEdge(key1, key2, true).LP].AbG).

2. Both the vertex nodes, vnode(key1) & vnode(key2) and the edge node corresponding to it, enode(key2) must be present in the post-state of LP event of the method. Formally, (ContainsEdge(key1, key2, true) → (vnode(key1) ∧ vnode(key2) ∧ enode(key2)) ∈ PostE[E^H.ContainsEdge(key1, key2, true).LP].AbG).
1. No successful concurrent RemoveVertex on key1 and key2 and no successful RemoveEdge(key1, key2): By observing the code, we realise that HelpSearchEdge returns successfully only when enode(key1) & enode(key2) were found to be present in the vertex list at least in one state of the execution. Thus, we can conclude that in the post-state of Line 201 vertex nodes enode(key1) and enode(key2) belong to S.AbG(V) (since there is no concurrent RemoveVertex). Now in the end of the Line 204 key2 ≤ e.val. Now in Line 205 e.val = key2 is evaluated to true. Also e being a reference, does not change. It is clear that any unmarked public node should be reachable from VertexHead and thereby belong to S.AbG(E). \( \langle \text{enode(key1) } \land \text{enode(key2)} \rangle \in \text{PreE}[^{E^H}.\text{ContainsEdge}^{(key1, key2, true)}.LP].AbG \).

2. Successful concurrent RemoveVertex on key1 or key2 or Successful RemoveEdge(key1, key2): We know that the LP in this case is immediately before the first LP of RemoveVertex method on enode(key1) and enode(key2) and RemoveEdge of enode(key1, key2). Also from Lemma 5, it follows that if RemoveVertex(key1) or RemoveVertex(key2) returns true, then the vertex node corresponding to that key is present in the pre-state of the LP. Also from Lemma 11 it follows that if RemoveEdge(key1, key2) returns true, then the edge node enode(key2) corresponding to that key is present in the pre-state of the LP. Hence we conclude that, \( \langle \text{enode(key1) } \land \text{enode(key2)} \rangle \in \text{PreE}[^{E^H}.\text{ContainsEdge}^{(key1, key2, true)}.LP].AbG \).

- \ref{13.2} Similar argument as Lemma \ref{13.1}

**Lemma 14** If some ContainsEdge(key1, key2) method returns false in E^H, then

1. The edge node corresponding to it, enode(key2) will not be present in the pre-state of LP event of the method. Formally, \( \langle \text{ContainsEdge}^{(key1, key2, false)} \Rightarrow \text{enode(key2)} \notin \text{PreE}[^{E^H}.\text{ContainsEdge}^{(key1, key2, false)}.LP].AbG \rangle \).

2. The edge node corresponding to it, enode(key2) will not be present in the post-state of LP event of the method. Formally, \( \langle \text{ContainsEdge}^{(key1, key2, false)} \Rightarrow \text{enode(key2)} \notin \text{PostE}[^{E^H}.\text{ContainsEdge}^{(key1, key2, false)}.LP].AbG \rangle \).

**Proof Sketch:**

- \ref{14.1}

1. No successful concurrent AddEdge on (key1, key2): By observing the code, we realise that in the Line 94 vertex node enode(key1) \( \notin \text{S.AbG(V)} \) because it is not reachable from VertexHead or marked. Also, in the Line 102 vertex node enode(key2) \( \notin \text{S.AbG(V)} \) because it is not reachable from VertexHead or marked, depending upon the execution. Therefore, \( \text{enode(key1)} \lor \text{enode(key2)} \notin \text{S.AbG(V)} \). This implies that from the case 1 of Lemma 10 Hence \( \langle \text{enode(key2)} \notin \text{PreE}[^{E^H}.\text{m1}.LP].AbG \rangle \).

2. Concurrent successful AddEdge(key1, key2): We know that the LP in this case is immediately before the first LP of AddEdge method on (key1, key2). Also from Lemma 9 it follows that if AddEdge(key1, key2) returns true, then the vertex node corresponding to that key is not present in the pre-state of the LP. Hence \( \langle \text{enode(key2)} \notin \text{PreE}[^{E^H}.\text{m1}.LP].AbG \rangle \).

- \ref{14.1} Similar argument as Lemma \ref{14.1}
Proof Sketch: Let us prove by contradiction. So we assume that,

\[
\langle (\text{PreE}[E^H.m_x.LP].AbG = \text{PreM}[E^S.m_y.AbG]) \land (E^H.m_x.inv = E^S.m_y.inv) \Rightarrow (E^H.m_x.resp \neq E^S.m_y.resp) \rangle
\]

We have the following cases where \( E^H.m_x.inv \) is invocation of either of these methods:

1. \( m_x.inv \) is \text{AddVertex} (key) Method:
   - \( m_x.resp = true \): Given that the method \( m_x.resp \) is \text{AddVertex} (key) returns true, we know that from the Lemma 4 \( m_x(key) \notin \text{PreE}[E^H.\text{AddVertex}(key, true).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of AddVertex method} \), then the \text{AddVertex}(key, true) method must return \text{true} \text{ in } E^S. This is a contradiction.

2. \( m_x.inv \) is \text{RemoveVertex} (key) Method:
   - \( m_x.resp = true \): Given that the method \( m_x.resp \) is \text{RemoveVertex} (key) returns true, we know that from the Lemma 5 \( m_x(key) \notin \text{PreE}[E^H.\text{RemoveVertex}(key, true).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of RemoveVertex method} \), then the \text{RemoveVertex}(key, true) method must return \text{true} \text{ in } E^S. This is a contradiction.
   - \( m_x.resp = false \): Given that the method \( m_x.resp \) is \text{RemoveVertex} (key) returns false, we know that from the Lemma 6 \( m_x(key) \notin \text{PreE}[E^H.\text{RemoveVertex}(key, false).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of RemoveVertex method} \), then the \text{RemoveVertex}(key, false) method must return \text{false} \text{ in } E^S. This is a contradiction.

3. \( m_x.inv \) is \text{ContainsVertex} (key) Method:
   - \( m_x.resp = true \): Given that the method \( m_x.resp \) is \text{ContainsVertex} (key) returns true, we know that from the Lemma 7 \( m_x(key) \notin \text{PreE}[E^H.\text{ContainsVertex}(key, true).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of ContainsVertex method} \), then the \text{ContainsVertex}(key, true) method must return \text{true} \text{ in } E^S. This is a contradiction.
   - \( m_x.resp = false \): Given that the method \( m_x.resp \) is \text{ContainsVertex} (key) returns false, we know that from the Lemma 8 \( m_x(key) \notin \text{PreE}[E^H.\text{ContainsVertex}(key, false).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of ContainsVertex method} \), then the \text{ContainsVertex}(key, false) method must return \text{false} \text{ in } E^S. This is a contradiction.

4. \( m_x.inv \) is \text{AddEdge} (key1, key2) Method:
   - \( m_x.inv = true \): Given that the method \( m_x.resp \) which is \text{AddEdge} (key1, key2) returns true, we know that from the Lemma 9 \( m_x(key) \notin \text{PreE}[E^H.\text{AddEdge}(key1, key2, true).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key) \notin \text{pre-state of LP of AddEdge method} \), then the \text{AddEdge}(key1, key2, true) method must return \text{true} \text{ in } E^S. This is a contradiction.
   - \( m_x.inv = false \): Given that the method \( m_x.resp \) which is \text{AddEdge} (key1, key2) returns false, we know that from the Lemma 10 \( m_x(key1) \) and \( m_x(key2) \notin \text{PreE}[E^H.\text{AddEdge}(key1, key2, false).LP].AbG \). But since from assumption in equation (1) \( (E^H.m_x.resp \neq E^S.m_y.resp) \), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if \( m_x(key1) \) and \( m_x(key2) \notin \text{pre-state of LP of AddEdge method} \), then the \text{AddEdge}(key1, key2, false) method must return \text{false} \text{ in } E^S. This is a contradiction.
5. \( m_x.inv \) is **RemoveEdge** \((key_1, key_2)\) Method:

- **\( m_x.inv = true \):** Given that the method \( m_x.resp \) which is **RemoveEdge** \((key_1, key_2)\) returns \( true \), we know that from the Lemma11 **enode** \((key_2)\) \( \in PreE[E^H.RemoveEdge(key_1, key_2, true).LP].AbG \). But since from assumption in equation 1 \((E^H.m_x.resp \neq E^S.m_y.resp)\), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if **enode**\((key2)\) \( \in \) pre-state of LP of **RemoveEdge** method, then the **RemoveEdge**\((key_1, key_2, true)\) method must return \( true \) in \( E^S \). This is a contradiction.

- **\( m_x.inv = false \):** Given that the method \( m_x.resp \) which is **RemoveEdge** \((key_1, key_2)\) returns \( false \), we know that from the Lemma11 **enode** \((key_1)\) and **enode** \((key_2)\) \( \notin PreE[E^H.RemoveEdge(key_1, key_2, false).LP].AbG \). But since from assumption in equation 1 \((E^H.m_x.resp \neq E^S.m_y.resp)\), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if **enode**\((key2)\) \( \notin \) pre-state of LP of **RemoveEdge** method, then the **RemoveEdge**\((key_1, key_2, false)\) method must return \( false \) in \( E^S \). This is a contradiction.

6. \( m_x.inv \) is **ContainsEdge** \((key_1, key_2)\) Method:

- **\( m_x.inv = true \):** Given that the method \( m_x.resp \) which is **ContainsEdge** \((key_1, key_2)\) returns \( true \), we know that from the Lemma13 **enode** \((key_2)\) \( \in PreE[E^H.ContainsEdge(key_1, key_2, true).LP].AbG \). But since from assumption in equation 1 \((E^H.m_x.resp \neq E^S.m_y.resp)\), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if **enode**\((key2)\) \( \in \) pre-state of LP of **ContainsEdge** method, then the **ContainsEdge**\((key_1, key_2, true)\) method must return \( true \) in \( E^S \). This is a contradiction.

- **\( m_x.inv = false \):** Given that the method \( m_x.resp \) which is **ContainsEdge** \((key_1, key_2)\) returns \( false \), we know that from the Lemma13 **enode** \((key_1)\) and **enode** \((key_2)\) **enode** \((key_2)\) \( \notin PreE[E^H.ContainsEdge(key_1, key_2, false).LP].AbG \). But since from assumption in equation 1 \((E^H.m_x.resp \neq E^S.m_y.resp)\), \( E^S.m_y.resp \) is false. However, from the sequential execution we know that, if **enode**\((key2)\) \( \notin \) pre-state of LP of **ContainsEdge** method, then the **ContainsEdge**\((key_1, key_2, false)\) method must return \( false \) in \( E^S \). This is a contradiction.

\[ \square \]

**Lemma 16** In every execution, the pre-state of LP event must be same as the pre-state of the method in the sequential execution. Formally, \((PreE[E^H.m_k,LP].AbG = PreM[E^S.m_k].AbG)\).

Proof Sketch: We prove by Induction on events which are the linearization points of the methods,

**Base Step:** Before the 1\( ^{st} \) LP event, the initial \( AbG \) remains same because all the events in the concurrent execution before the 1\( ^{st} \) LP do not change \( AbG \).

**Induction Hypothesis:** Let us assume that the first \( k \) LP events, we know that,

\( PreE[E^H.m_k,LP].AbG = PreM[E^S.m_k].AbG \)

**Induction Step:** We have to prove that: \( PreE[E^H.m_{k+1},LP].AbG = PreM[E^S.m_{k+1}].AbG \) holds true.

We know from Induction Hypothesis that for \( k^{th} \) method,

\( PreE[E^H.m_k,LP].AbG = PreM[E^S.m_k].AbG \)

From the construction of \( S \), we get that \( H.m_x.inv = S.m_x.inv \). Combining this with Lemma13 we have,

\[(H.m_x.inv = S.m_x.inv) \land (PreE[E^H.m_k,LP].AbG = PreM[E^S.m_k].AbG)\]

From the Lemma2 we have,

\[ PostM[E^S.m_k].AbG \quad \text{From Lemma2} \quad PreM[E^S.m_{k+1}].AbG \]

\[ \text{(2)} \]

\[ \text{(3)} \]
From the equation 2, we have,

\[ \text{PostE}[E^H.m_k.LP].AbG = \text{PostM}[E^S.m_k].AbG \]  

(4)

By combining the equation 4 and 3, we have,

\[ \text{PostE}[E^H.m_k.LP].AbG = \text{PreM}[E^S.m_{k+1}].AbG \]  

(5)

And from Lemma 3, we have,

\[ \text{PostE}[E^H.m_k.LP].AbG = \text{PreE}[E^H.m_{k+1}.LP].AbG \]  

(6)

So, by combining equations 6 and 5, we get,

\[ \text{PreE}[E^H.m_{k+1}.LP].AbG = \text{PreM}[E^S.m_{k+1}].AbG \]  

(7)

Hence this holds for all \( m_i \) in \( E^H \). Hence Proved.

Lemma 17 The return values for all the methods in \( H \) & \( S \) are the same. Formally, \( \forall m_i \in E^H \cup E^S: E^H.m_i.retval() = E^S.m_i.retval() \).

Proof Sketch: From the construction of \( S \), we get that for any method \( m_i \) in \( H \), \( S \) the invocation parameters are the same. From Lemma 16, we get that the pre-states of all these methods are the same. Combining this result with Lemma 15, we get that the responses parameters for all these methods are also the same.

Lemma 18 Every history \( H \) generated by the interleaving of any of the methods of the concurrent graph data structure, is linearizable.

Proof Sketch: From Lemma 17, we get that for all the methods \( m_i \), the responses in \( H \) and \( S \) are the same. This implies that \( H \) and \( S \) are equivalent to each other. We know that \( S \) respects the real-time order of \( H \) and \( S \) is legal history. Hence \( H \) is linearizable.

5.3 Liveness: Proof Outline

In this subsection we prove the progress guarantee of the methods of our proposed graph data structure.

Lemma 19 The methods AddVertex, RemoveVertex, AddEdge and RemoveEdge are deadlock-free.

Proof Sketch: We prove all the AddVertex, RemoveVertex, AddEdge and RemoveEdge methods are deadlock-free by direct argument based of the acquiring lock on both the current and predecessor nodes.

1. AddVertex: the AddVertex(key) method is deadlock-free because a thread always acquires lock on the vnode with smaller keys first. Which means, if a thread say \( T_1 \) acquired a lock on a vnode(key), it never tries to acquire a lock on a vnode with key smaller than or equal to vnode(key). This is true because the AddVertex method acquires lock on the predecessor vnode from the LocateVertex method.

2. RemoveVertex: the RemoveVertex(key) method is also deadlock-free, similar argument as AddVertex.

3. AddEdge: the AddEdge(key1, key2) method is deadlock-free because a thread always acquires lock on the enode with smaller keys first. Which means, if a thread say \( T_1 \) acquired a lock on a enode(key2), it never tries to acquire a lock on a enode of the vertex vnode(key1) with key smaller than or equal to enode(key2). This is true because the AddEdge method acquires lock on the predecessor edge nodes of the vertex vnode(key1) from the LocateEdge method.

4. RemoveEdge: the RemoveEdge(key1, key2) method is also deadlock-free, similar argument as AddEdge.

Lemma 20 The methods ContainsVertex and ContainsEdge are wait-free.
Proof Sketch: The ContainsVertex(key) method scans the vertex list of the graph starting from the VertexHead, ignoring whether enode are marked or not. It returns a boolean flag either true or false depending on vnode(key) greater than or equal to the sought-after key. If the desired vnode is unmarked, it simply returns true and this is correct because the vertex list is sorted. On the other hand, it returns false if vnode(key) is not present or has been marked. This ContainsVertex method is wait-free, because there are only a finite number of vertex keys that are smaller than the one being searched for. By the observation of the code, a new enode with lower or equal keys is never added ahead of it, hence they are reachable from VertexHead even if vertex nodes are logically removed from the vertex list. Therefore, each time the ContainsVertex moves to a new vertex node, whose key value is larger key than the previous one. This can happen only finitely many times, which says the traversal of ContainsVertex method is wait-free.

Similarly, the ContainsEdge(key1, key2) method first scans the vertex list of the graph starting from the VertexHead, ignoring whether vertex nodes are marked or not. It returns a boolean flag either true or false depending on vnode(key2) greater than or equal to the sought-after key in the edge list of the vertex vnode(key1). If the desired enode is unmarked, it simply returns true and this is correct because the vertex list is sorted as well as the edge list of the vertex vnode(key1) is also sorted. On the other hand it returns false if either vnode(key1) or vnode(key2) is not present or has been marked in the vertex list or enode(key2) is not present or has been marked in the edge list of the vertex vnode(key1). This ContainsEdge method is wait-free, because there are only a finite number of vertex keys that are smaller than the one being searched for as well as a finite number of edge keys that are smaller than the one being searched for in edge list of any vertex. By observation of the code, a new enode with lower or equal keys is never added ahead of enode(key2) in the edge list of the vertex vnode(key1), hence they are reachable from VertexHead even if vertex nodes or edge nodes of vnode(key1) are logically removed from the vertex list. Therefore, each time the ContainsEdge moves to a new edge node, whose key value is larger key than the previous one. This can happen only finitely many times, which says the traversal of ContainsEdge method is wait-free.

\[\square\]

6 Maintaining Graph Acyclicity

In this section, we consider the problem of maintaining an invariant of acyclicity in this concurrent dynamic graph data structure. As described earlier, the objective is to maintain an acyclic conflict graph of transactions for SGT. For a concurrent graph to be acyclic, the graph should maintain the acyclic property at the end of each operation in the equivalent sequential history. To achieve this, we try to ensure that the graph stays acyclic in all the global states.

It is easy to see that a cycle can be created only on addition of edge to the graph. We modify the concurrent graph data structure presented in the earlier section to support this acyclic property. The sequential specification of AcyclicAddEdge is relaxed as follows: after a new directed edge has been added to the graph (in the shared memory), we verify if the resulting graph is acyclic. If it is, we leave the edge. Otherwise we delete the edge from the shared memory. Thus AcyclicAddEdge(u, v) may fail even if the edge (u, v) was not already part of the graph. The sequential specification of the graph data structure with acyclic property is same as that of sequential specification of graph data structure as shown in the Table 2, except AddEdge method. This modified specification is shown in the Table 4.

Table 3 describes the modified fields of the edge and vertex nodes of the Table 1. Each node has a status field which can be in one of the three states: transit, marked, added. When a new edge node is added, its status is initially set to transit. The overall idea is that it is allowed to be logically added to the graph i.e. its status set to added only after it is ensured that the edge does not form any cycle. Thus, we ensure that in any global state all the edges in ‘added’ state do not form a cycle. Although, it is possible that the edges with status added and transit form a cycle in some global state.

A thread adds an edge to the graph in transit state. If the new transit edge does not cause a cycle, then its status is changed to added. Otherwise, the new edge is deleted by setting its status to marked. An edge node in transit state can not get removed by any other concurrent thread. However, every thread performing a cycle detect on the concurrent graph can see all edges in transit as well as added state.

A side-effect to be observed here is that this may allow false positives. This means that the algorithm may detect a cycle even though the graph does not contain one. This can happen in the following scenario: two threads \(T_1\) and \(T_2\) are adding edges lying in the path of a single cycle. In this case, both threads...
detect that the newly added edge (in transit) has led to formation of a cycle and both may delete their respective edges. However, in a sequential execution, only one of the edges will be removed. We allow this as the resulting graph at the end of each operation is acyclic. From the perspective of application SGT which uses the graph acyclicity, a false positive implies unnecessary abort of a transaction. But this does not violate the correctness, i.e., serializability in Databases and Opacity in STMs. On the other hand, having a cycle in the application violates correctness and thus should not be allowed.

Thus to incorporate these changes to preserve acyclicity, the methods exported by the algorithm are modified as: AcyclicAddEdge, AcyclicRemoveEdge, AcyclicContainsEdge. Algorithms 13-19 illustrate these methods. The methods ValidateVertex, AddVertex, RemoveVertex and ContainsVertex remain same as described in Algorithms 1, 3, 5 and 11 respectively.

6.1 Working of Concurrent Graph Methods preserving Acyclicity

In this section, we describe the implementation of the concurrent graph structure with acyclicity properties and the working of the various methods. For a concurrent graph to be acyclic, the graph should maintain the acyclic property at the end of each operation in the equivalent sequential history. Like concurrent graph data structure the fields in the acyclic structure also declared as atomic. This ensures that operations on these variables happen atomically. In the context of a particular application, the node structure can be easily modified to carry useful data (like weights etc).

**Update Vertex Methods:** - AcyclicAddVertex & AcyclicRemoveVertex

The working of the AcyclicAddVertex and AcyclicRemoveVertex methods are similar to the AddVertex and RemoveVertex methods of the concurrent graph data structure described in the Section 6.1

**Update Edge Method:** - AcyclicAddEdge
Algorithm 13 AcyclicValidateEdge Method: Takes two enode \( e_1, e_2 \) and validates for presence in the edge list.

```plaintext
212: procedure AcyclicValidateEdge(\( e_1 \downarrow, e_2 \downarrow, \text{flag} \uparrow \))
213: if \((\text{read}(e_1, \text{status}) = \text{added}) \land (\text{read}(e_2, \text{status}) = \text{added}) \land (\text{read}(e_1, \text{enext}) = e_2)\) then
214: \( \text{flag} \leftarrow \text{true} \); //validation successful
215: else
216: \( \text{flag} \leftarrow \text{false} \); //validation failed
217: end if
218: return //return flag
219: end procedure
```

Algorithm 14 Modified ValidateEdge Method: Takes two ENode \( e_1, e_2 \) and validates for presence in edge list.

```plaintext
220: procedure ModifiedValidateEdge(e_1 ↓, e_2 ↓, flag ↑)
221: if \((\text{read}(e_1, \text{status}) \neq \text{marked}) \land (\text{read}(e_2, \text{status}) \neq \text{marked}) \land (\text{read}(e_1, \text{enext}) = e_2)\) then
222: \( \text{flag} \leftarrow \text{true} \); //validation successful
223: else
224: \( \text{flag} \leftarrow \text{false} \); //validation fails
225: end if
226: return //return flag
227: end procedure
```

Algorithm 15 AcyclicLocateEdge Method: Takes two keys, \( k_{y_1} \) and \( k_{y_2} \), as input and returns the pair of adjacent enode \((e_1, e_2)\). If enode \( e_1 \) or \( e_2 \) or enode \( e_2 \) is not present, it returns false. Initially enodes \( e_1, e_2 \) are set to null and \( \text{flag} \) is set to true.

```plaintext
228: procedure AcyclicLocateEdge(\( k_{y_1} ↓, k_{y_2} ↓, \text{flag} \uparrow \))
229: HelpSearchEdge\( (k_{y_1} ↓, k_{y_2} ↓, v_1 \uparrow, v_2 \uparrow, e_1 \uparrow, e_2 \uparrow, \text{flag} \uparrow) \)
230: if \((\text{flag} = \text{false})\) then
231: \( \text{flag} \leftarrow \text{false} \); //return flag
232: end if
233: /*This lines ensures both the vertices \( v_1 \) & \( v_2 \) have been added to the system by this time and are not marked*/
234: if \((\text{read}(v_1, \text{marked}) \lor \text{read}(v_2, \text{marked}))\) then
235: \( \text{flag} \leftarrow \text{false} \);
236: return;
237: end if
238: /*Helping for search edge, is a supporting method for locate edge. It locates the vertices \( v_1 \) & \( v_2 \)*/
239: while (true) do
240: \( e_1 \leftarrow \text{read}(v_1, \text{enext}); \)
241: \( e_2 \leftarrow \text{read}(e_1, \text{enext}); \)
242: /* Search enode\( (k_{y_2}) \) without acquiring any locks*/
243: while \((\text{read}(e_2, \text{val}) < k_{y_2})\) do
244: \( e_1 \leftarrow e_2; \)
245: \( e_2 \leftarrow \text{read}(e_2, \text{enext}); \)
246: end while
247: \( \text{lock.acquire}(e_1); \)
248: \( \text{lock.acquire}(e_2); \)
249: if \((\text{val})\) then
250: if \((\text{ModifiedValidateEdge}(e_1 ↓, e_2 ↓, \text{flag} ↑))\) then
251: return;
252: else
253: \( \text{lock.release}(e_1); \)
254: \( \text{lock.release}(e_2); \)
255: end if
256: else if \((\neg \text{val})\) then
257: if \((\text{AcyclicValidateEdge}(e_1 ↓, e_2 ↓, \text{flag} ↑))\) then
258: return;
259: else
260: \( \text{lock.release}(e_1); \)
261: \( \text{lock.release}(e_2); \)
262: end if
263: else
264: \( \text{lock.release}(e_1); \)
265: \( \text{lock.release}(e_2); \)
266: end if
267: end while
268: end procedure
```

When a thread wants to add an edge \((u, v)\) to the concurrent graph, it invokes the AcyclicLocateEdge method in the Line 228. This AcyclicLocateEdge internally invokes the HelpSearchEdge method in the Line 229. The HelpSearchEdge described in the Algorithm 7. This method checks for the presence of vertices \( u \) and \( v \) in the vertex list of the graph. Now we add an additional check to verify that the two vertices are reachable from the VertexHeadin Line 233. The reason for this is as explained in Figure 1. Once the vertices have been found, the thread traverses the edge list of vertex \( u \) until an edge node with key greater than \( v \) has been encountered, say \( ecurr \) and it’s predecessor say \( epred \). The thread does all this without acquiring locks. It then acquires locks on both \( epred \) and \( ecurr \) and performs validation by invoking ModifiedValidateEdge in the Line 250. The ModifiedValidateEdge is described in the Algorithm 15. This method checks that the locked nodes have not been deleted and are reachable. In other words, it successfully validates even though the nodes are in transit state and returns the locked nodes. In case
the validation fails, it releases the locks and retries. Once the $e_{pred}$ and $e_{curr}$ edge nodes have been found, a new edge node (with status as transit) is added between them, if the key does not already exists. After this the acquired locks are released. If the node was newly added, then it invokes the $PathExists$ method in the Line 306 to check if there exists a path from vertex $v$ to $u$. The $PathExists$ method is described in the Algorithm 20.

Now, if there exists a path from $v$ to $u$, it means a cycle has been detected, then the edge $(u, v)$ must be removed by setting its status to marked, else it must be changed to added. However, for deletion of the node in transit, we must be able to get the pointer to the predecessor node. Hence it invokes the $NewLocateEdge$ in the Line 308. It is to be noted that during this phase, the vertex $u$ could have been deleted. Hence $NewLocateEdge$ method starts scans directly the edge list of the $u$ until it finds an enode with a key value greater than or equal to $e_{3}.key$ and it’s predecessor say $e_{pred}$. After this validation is performed by invoking the $ModifiedValidateEdge$ method in the Line 280 to check if the locked edge nodes are reachable and unmarked. If the validation is successful, the new edge node $e_3$ is removed by making status to marked in the Line 309. If there is no path from $v$ to $u$, means cycle has not been detected, so the status of the $e_3$ is simply changed to added in the Lineline:addec14. The $AcyclicAddEdge$ method is described in Algorithm 17.

Algorithm 16 New LocateEdge Method: Takes two keys, $key_1$ and $key_2$, as input and returns the pair of adjacent enode $(e_1, e_2)$. If $enode\ v_1$ or $enode\ v_2$ is not present, it returns false. Initially enodes $e_1, e_2$ are set to null and flag is set to true.

\begin{algorithm}
\begin{algorithmic}
\Procedure{NewLocateEdge}{key_1, key_2, v_1, v_2, e_1, e_2, flag}
\State $e_1 \gets \text{read}(v_1, \text{enext})$;
\State $e_2 \gets \text{read}(v_1, \text{enext})$;
\State /* Search \text{enode}(key_2) without acquiring any locks*/
\While{($\text{read}(e_2, \text{val}) < \text{key}_2$)}
\State $e_3 \gets e_2$;
\State $e_2 \gets \text{read}(e_2, \text{enext})$;
\EndWhile
\State $\text{lock.release}(e_2)$;
\State $\text{if } (\text{ModifiedValidateEdge}(v_1, v_2, \text{flag})) \text{ then}$
\State $\text{return};$ // returns true if validation succeeds.
\State \Else
\State $\text{lock.release}(e_1)$;
\State $\text{lock.release}(e_2)$; // validation failed, try again
\State $\text{end if}$
\State $\text{end while}$
\State $\text{end procedure}$
\EndProcedure
\end{algorithmic}
\end{algorithm}

Algorithm 17 AcyclicAddEdge Method: $enode(key_2)$ gets added to the edge list of $enode(key_1)$, if it is not already part of it and does not form a cycle. Initially flag is set to true.

\begin{algorithm}
\begin{algorithmic}
\Procedure{AcyclicAddEdge}{key_1, key_2, flag}
\State $\text{AcyclicLocateEdge}(key_1, \text{true}, v_1, v_2, e_1, e_2, flag)$ // Algorithm 15\State $\text{if } (\text{flag} = \text{false}) \text{ then}$
\State $\text{return};$
\State $\text{end if}$
\State $\text{if } (\text{read}(e_2, \text{val}) \neq \text{key}_2) \text{ then }$ //enode(key2) not found\State $\text{write}(e_3, \text{new enode(key_2)})$;
\State $\text{write}(e_3, \text{enext}, e_2)$;
\State $\text{write}(e_1, \text{enext}, e_3)$;
\State $\text{lock.release}(e_1)$;
\State $\text{lock.release}(e_2)$;
\State $\text{end if}$
\State $\text{else}$
\State $\text{lock.release}(e_1)$; // enode(key2) already present
\State $\text{lock.release}(e_2)$;
\State $\text{return}$;
\State $\text{end if}$
\State $\text{if } (\text{cycle}\_\text{flag} = \text{true}) \text{ then }$ //cycle detected
\State $\text{write}(e_3, \text{status}, \text{marked})$; //logical removal
\State $\text{write}(e_3, \text{enext}, e_2)$;
\State $\text{flag} \gets \text{false}$;
\State $\text{lock.release}(e_1)$;
\State $\text{lock.release}(e_2)$;
\State $\text{else}$
\State $\text{write}(e_3, \text{status}, \text{added})$; //logical addition
\State $\text{return};$ // return flag
\State $\text{end if}$
\State $\text{end procedure}$
\EndProcedure
\end{algorithmic}
\end{algorithm}
6 MAINTAINING GRAPH ACYCLICITY

Update Edge Method- AcyclicRemoveEdge
In AcyclicRemoveEdge(u, v), the enode v is removed from u’s edge list. It starts by invoking AcyclicValidateEdge method in the Line 320. This is same as described in the AcyclicAddEdge above, with the only difference that it performs validation by invoking AcyclicValidateEdge in Line 257. The role of this method is to check if the locked edge nodes are added in the concurrent graph data structure. This is important since transit edge nodes cannot be deleted. If the edge node to be deleted is already present then deletion of enode v takes in two phases: first logical removal of the enode v by setting the status field in the Line 326 to marked. Then, enode v is removed physically from u’s edge list by changing the pointers in the Line 327. The AcyclicRemoveEdge method is described in Algorithm 18.

Read-only Methods: - AcyclicContainsVertex & AcyclicContainsEdge
The AcyclicContainsVertex method works similar to the wait-free ContainsVertex of method described in the Algorithm 11. It proceeds by scanning the vertex list, without acquiring any locks until it encounters a vnode of key greater than or equal to u. It returns true if the vnode it was searching for is present and unmarked in the graph and otherwise returns false. Similarly, the AcyclicContainsEdge(u, v) method first scans the vertex list like ContainsVertex for each vertex u and v and checks both the vertices are present and unmarked. It traverse the edge list without acquiring any locks on enode. It returns true if the enode it was searching for is present and it’s status is added in the edge list of vnode u and otherwise returns false. The AcyclicContainsEdge method is described in the Algorithm 19.

6.2 Wait-Free Reachability
The following subsection presents an algorithm based on reachability for ensuring acyclicity in the concurrent graph developed. Given a directed graph, \( G = (V, E) \) and two vertices \( u, v \in V \), a vertex \( v \) is reachable from another vertex \( u \), if there exists a path from the vertex \( u \) to \( v \) in \( G \). By using this idea of reachability, we define a method for cycle detection in concurrent directed graphs. Before invoking this method, we must remember that AcyclicAddEdge(x, y) has already returned true. So this is to check if there exists a path from vertex y to x in the concurrent graph. The reachability method creates a local ReachSet of all vertices reachable from y, with an explored boolean field corresponding to each vertex in
this set. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

However, in the concurrent setting, the set of vertices in the path keep varying dynamically. Since the key size is finite and all keys are unique, the adjacency list will be traversed in finite number of steps. The reachability method begins by traversing the adjacency list of $y$ to find $x$ in the concurrent graph, without acquiring locks. All these traversed vertices edge nodes are added to the local \texttt{ReachSet} and the vertex $y$ is marked to be \textit{explored}. Now, the method recursively visits (similar to breadth-first traversal, BFS) the outgoing edges (which are not marked) from the neighbours of $y$ to find $x$. Clearly, this is done until all the vertices in all the paths from $y$ to $x$ in $G$ have been \textit{explored} or a cycle has been detected in the graph. This method is described in Algorithm 20.

6.3 Correctness: Linearization Points

In this subsection, we define the \textit{Linearization Points} (LP) of methods of the concurrent graph data structure with acyclicity. Only the LP of $\texttt{AcyclicAddEdge}(u, v)$ changes due to the introduction of acyclic invariant.

We linearize a successful $\texttt{AcyclicAddEdge}(u, v)$ method call within its execution interval at the earlier of the following points: (1) if there is no successful concurrent $\texttt{RemoveVertex}$ on $u$ and $v$, the LP is defined as the last of $\texttt{read}(e_2.\text{val})$ in Line 294 and $\texttt{write}(e_3.\text{status}, \text{added})$ in Line 315 depending upon execution. If the last line to execute is 294, the edge $(u, v)$ is already present in the edge list of the vertex
Figure 14: An execution of wait-free reachability for concurrent cycle detect. Figure (a) is the initial graph when a thread \( T_1 \) is trying to concurrently \( \text{AcyclicAddEdge}(3, 7) \) to the graph. Figure (b) depicts the graph when thread \( T_1 \) has finished adding edge \( (3, 7) \) and is invoking \( \text{PathExists} \). Concurrently, thread \( T_2 \) is trying to \( \text{AcyclicAddEdge}(4, 7) \). Figure (c) shows the reachability path from vertex 7 to 3 as computed by thread \( T_1 \). A cycle has now been detected and hence the edge \( (3, 7) \) will now be removed. Similarly, the thread adding edge \( (4, 7) \) will also invoke \( \text{PathExists} \) afterwards and remove it.

6.4 Safety: Proof Outline

In this subsection, we try to formalise the proof of our concurrent graph data structure with acyclic invariant based on \( LP \) events of the methods. Here we show that the \( \text{AcyclicAddEdge} \) maintains acyclicity property. The proof of other properties for maintaining sequential consistency (mentioned below) can be shown similar to the proof in SubSection 5.2.

**Lemma 21** The algorithms employed for our acyclic concurrent graph data structure are linearizable.

**Lemma 22** The methods \( \text{AddVertex} \), \( \text{RemoveVertex} \), \( \text{AcyclicAddEdge} \) and \( \text{AcyclicRemoveEdge} \) are deadlock-free and \( \text{ContainsVertex} \), \( \text{AcyclicContainsEdge} \) and \( \text{PathExists} \) are wait-free.

Now, we show the proof of graph acyclicity. For simplicity in this proof sketch, we assume that there are no concurrent add vertex and remove edges/vertices as they do not affect acyclicity property.

**Lemma 23** Consider a state \( S \) in which there is a path from vertex \( u \) to \( v \). Then if a thread invokes \( \text{PathExists}(u, v) \) in \( S \) and there are no edge deleted in this path between this method invocation and response then \( \text{PathExists}(u, v) \) will return true.

Proof Sketch: The proof of this lemma comes directly from the working of the algorithm.

**Lemma 24** In any global state \( S \), all the edges with the status added are acyclic in \( S.AbG \).
Proof Sketch: We initially start with an acyclic graph. Let the cycle be formed in $ABG$ after the successful invocation & return of $AcyclicAddEdge(u,v)$ by a thread $T_i$. This implies that in the post-state of the LP of $AcyclicAddEdge(u,v)$, say $S'$ there is a cycle consisting of edges with status added. From our assumption, $S'$ the first state in which such a cycle exists. Let $S$ be the pre-state of $AcyclicAddEdge(u,v)$. Then in $S.ABG$ there is a path from the $v$ to $u$.

W.l.o.g, let the path in $S$ consist of the following vertices: $v \xrightarrow{a} n_1 \xrightarrow{a} n_2 \ldots \xrightarrow{a} n_k \xrightarrow{a} u$ where $v \xrightarrow{a} n_1$ implies that the edge is in status added. From our assumption, all these edges are in state added in $S$. From the working of the algorithm all these edges must have been added as transit earlier. Among these set of edges, let us suppose that the edge $ni \xrightarrow{a} nj$ be the last edge to be added in the state transit by a thread $T_k$ (which could be same as $T_i$). Note that this edge need not be the last edge to be converted to the status added in this set.

Let $ni \xrightarrow{t/a} nj$ and $u \xrightarrow{t/a} v$ be the events of the corresponding edges getting added to the $ABG$ with status transit. Now, we have two cases:

- **$ni \xrightarrow{t/a} nj <_E u \xrightarrow{t/a} v$:** Here $ni \xrightarrow{t/a} nj$ got added before $u \xrightarrow{t/a} v$. Thus from our assumption after $ni \xrightarrow{t/a} nj$ has got added, there is a path from $v$ to $u$ in a state $S1$ consisting of edges with status either as transit or added in the path. As per the algorithm, $T_i$ will invoke $PathExists(v,u)$ after $u \xrightarrow{t/a} v$. This from Lemma 23 this method would then return true since there is a path from $v$ to $u$. Hence, $T_i$ will convert not the status of the edge from transit to added and thus deleting this edge directly. Hence this case is not possible.

- **$u \xrightarrow{t/a} v <_E ni \xrightarrow{t/a} nj$:** Here, $u \xrightarrow{t/a} v$ has got added before $ni \xrightarrow{t/a} nj$. Now consider the state $S1$ just after adding of $ni \xrightarrow{t/a} nj$. In this state from our assumption, we have that $u \xrightarrow{t/a} v$. Let us consider the connectivity between $nj$ and $ni$. Originally, we had assumed that there is a path from $v$ to $u$ through $ni$ and $nj$ with $ni$ to $nj$ being the last edge added. Thus from our assumption in $S1$,

$$v \xrightarrow{t/a} ni - 1 \xrightarrow{t/a} \ldots n_i$$

(8)

This equation implies that there is a path from $v$ to $ni$ through a series of edges which are in states either added or transit. Similarly, we have,

$$nj \xrightarrow{t/a} nj + 1 \xrightarrow{t/a} \ldots u$$

(9)

From the condition of this case, we have that an edge from $u$ to $v$ was added in transit state before. From our assumption in $S1$ this edge would still be transit. Thus we have that

$$u \xrightarrow{t/a} v$$

(10)

Combining Equations 8, 9, 10 we get that there is a path from $nj$ to $ni$ in $S1$. Thus when $T_k$ invokes $PathExists(nj,ni)$ after $S1$, from Lemma 23 we get that this method will return true. This implies that $T_k$ will not change the status of the edge $ni$ to $nj$ as added. Hence when $T_i$ changes the status of edge $u$ to $v$ to added the path $v$ to $u$ will not exist. Hence a cycle is not possible.

Thus in both the cases, a cycle is not possible.

\[\square\]

**Lemma 25** If some $AcyclicAddEdge(key_1, key_2)$ method returns true in $E^H$, then

1. The corresponding vertex nodes, $vnode(key_1)$ and $vnode(key_2)$ will be present in the pre-state of LP event of the method and the graph is acyclic. Formally, \((vnode(key_1) \land \neg vnode(key_2)) \implies (vnode(key_1) \land vnode(key_2) \in \text{Pre}_E[E^H.AcylicAddEdge(key_1, key_2, true).LP].AbG \land (AbG \text{ is Acyclic})].\)

2. The corresponding vertex nodes, $vnode(key_1)$ and $vnode(key_2)$ will be present in the post-state of LP event of the method and the graph is acyclic. Formally, \((vnode(key_1) \land \neg vnode(key_2)) \implies (vnode(key_1) \land vnode(key_2) \in \text{Post}_E[E^H.AcylicAddEdge(key_1, key_2, true).LP].AbG \land (AbG \text{ is Acyclic})].\)

Proof Sketch:
If some vertex nodes corresponding to those keys will not be present in the pre-state of LP.

1. No successful concurrent RemoveVertex on key1 & key2: By observing the code, we realise that HelpSearchEdge returns successfully only when vnode(key1) & vnode(key2) were found to be present in the vertex list at least in one state of the execution. Now after this, the vertex nodes are again tested for removal. Thus, we can conclude that in the post-state of the Graph, vertex nodes vnode(key1) and vnode(key2) must belong to S.AbG(V). Since there is no concurrent deletion of these vertices, we conclude that both the vertex nodes are present in the pre-state of LP of this method. And from the Lemma 28, the graph already maintain the acyclic in the pre-state of the AbG. Hence (vnode(key1) ∧ vnode(key2) ∈ PreE[E^H.AcyclicAddEdge(key1, key2, true), LP].AbG) ∧ (AbG is Acyclic).

2. Concurrent successful RemoveVertex on key1 or key2 or both: We know that the LP in this case is immediately before the first LP of RemoveVertex method. The Lemma 5 shows, if RemoveVertex(key) returns true, the vertex node corresponding to that key is present in the pre-state of the LP. If any of the vertices were not present, the method returns false. So we conclude that, (vnode(key1) ∧ vnode(key2)) ∈ PreE[E^H.AcyclicAddEdge(key1, key2, true), LP].AbG. And from the Lemma 28, the graph already maintain the acyclic in the pre-state of the AbG. Hence, (vnode(key1) ∧ vnode(key2)) ∈ PreE[E^H.AcyclicAddEdge(key1, key2, true), LP].AbG ∧ (AbG is Acyclic).

\[\square\]

**Lemma 26** If some AcyclicAddEdge(key1, key2) method returns false in \(E^H\), then

1. At least one of the vertex nodes corresponding to those keys will not be present in the pre-state of LP event of the method and the graph is acyclic. Formally, \((\text{AcyclicAddEdge}(key1, key2, \text{false}) \implies (vnode(key1) \lor vnode(key2)) \notin \text{PreE}[E^H.\text{AcyclicAddEdge}(key1, key2, \text{false}), \text{LP}].\text{AbG}) \land (\text{AbG is Acyclic})\).

2. At least one of the vertex nodes corresponding to those keys will not be present in the post-state of LP event of the method and the graph is acyclic. Formally, \((\text{AcyclicAddEdge}(key1, key2, \text{false}) \implies (vnode(key1) \lor vnode(key2)) \notin \text{PostE}[E^H.\text{AcyclicAddEdge}(key1, key2, \text{false}), \text{LP}].\text{AbG}) \land (\text{AbG is Acyclic})\).

Proof Sketch:

1. No successful concurrent AddVertex on key1 and key2: By observing the code, we realise that in the Line 146 vertex node vnode(key1) is not in the S.AbG(V) because it is not reachable from VertexHead or marked. Also, in the Line 102 vnode(key2) is not in the S.AbG(V) because it is not reachable from VertexHead or marked. Consequently, the Line 146 returns false, if read(v1.marked) or read(v2.marked) is set depending upon the execution. So \((\text{vnode}(key1) \lor \text{vnode}(key2)) \notin \text{PreE}[E^H.m, \text{LP}].\text{AbG}\). If \((\text{vnode}(key1) \lor \text{vnode}(key2)) \notin \text{PreE}[E^H.m, \text{LP}].\text{AbG}\) checks in the Line 306 that there is a path from the vnode(key2) to vnode(key1), to check presence of a cycle in the graph. If cycle exists in the Line 306, the edge is removed and the graph maintain acyclic invariant like previous operations by the property of the Lemma 28. Hence, \((\text{vnode}(key1) \lor \text{vnode}(key2)) \notin \text{PreE}[E^H.\text{AcyclicAddEdge}(key1, key2, \text{false}), \text{LP}].\text{AbG} \land (\text{AbG is Acyclic})\)

2. Concurrent successful AddVertex on key1 or key2 or both: We know that the LP in this case is immediately before the first LP of AddVertex method on u and v. Also from the Lemma 3 it follows that if AddVertex(key) returns true, then the vertex node corresponding to that key is not present in the pre-state of the LP. And from the Lemma 25, the graph already maintain the acyclic in the pre-state of the AbG. Hence, \((\text{vnode}(key1) \lor \text{vnode}(key2)) \notin \text{PreE}[E^H.\text{AcyclicAddEdge}(key1, key2, \text{false}), \text{LP}].\text{AbG} \land (\text{AbG is Acyclic})\)

\[\square\]
Lemma 27 For some AcyclicAddEdge(key\(_1\), key\(_2\)) method the graph is acyclic in the pre-state of LP event of the method then in the post-state of LP event of the method of the graph is also acyclic. Formally, \( \langle \{ \text{Pre}E[H].\text{AcyclicAddEdge}(key\(_1\), key\(_2\), true).LP].\text{AbG \text{is acyclic} \rangle \rightarrow \text{Post}E[H].\text{AcyclicAddEdge}(key\(_1\), key\(_2\), true).LP].\text{AbG \text{is acyclic} \rangle \).

Proof Sketch: From the Lemma \( \ref{lemma:acyclic} \) after each methods operation the graph is acyclic invariant. And we know that only AcyclicAddEdge method can form the cycle in the graph. Before AddEdge in the pre-state of the LP event the graph is acyclic. After successful AcyclicAddEdge(key\(_1\), key\(_2\)) method from the Lemma \( \ref{lemma:acyclic} \) the graph is also acyclic in the post-state of the LP event. Hence, if the PreE[H].AcyclicAddEdge(key\(_1\), key\(_2\), true).LP].AbG is acyclic, after successful AcyclicAddEdge PreE[H].AcyclicAddEdge(key\(_1\), key\(_2\), true).LP].AbG is acyclic.

Lemma 28 In any execution of the reachability algorithm, all the edges of the graph with the state added do not form a cycle.

Proof Sketch: From the Lemma \( \ref{lemma:acyclic} \) after each methods operation the graph is acyclic invariant. And we know that only AcyclicAddEdge method can form the cycle in the graph. Before AddEdge in the pre-state of the LP event the graph is acyclic. After successful AcyclicAddEdge(key\(_1\), key\(_2\)) method from the Lemma \( \ref{lemma:acyclic} \) the graph is also acyclic in the post-state of the LP event. Hence, if the PreE[H].AcyclicAddEdge(key\(_1\), key\(_2\), true).LP].AbG is acyclic, after successful AcyclicAddEdge PreE[H].AcyclicAddEdge(key\(_1\), key\(_2\), true).LP].AbG is acyclic.

6.5 Liveness: Proof Outline

In this subsection we prove the progress guarantee of the methods of our proposed graph data structure with acyclic property.

Lemma 29 The methods AddVertex, RemoveVertex, AcyclicAddEdge and AcyclicRemoveEdge are deadlock-free.

Proof Sketch: The proof of the AddVertex, RemoveVertex, AcyclicAddEdge and AcyclicRemoveEdge methods are deadlock-free by direct argument based of the acquiring lock on both the current and predecessor nodes. The AddVertex and RemoveVertex proof are similar argument as Lemma \( \ref{lemma:deadlock} \) and \( \ref{lemma:deadlock} \) respectively. And the proof of AcyclicAddEdge and AcyclicRemoveEdge is given bellow.

1. AcyclicAddEdge: the AcyclicAddEdge(key\(_1\), key\(_2\)) method is deadlock-free because a thread always acquires lock on the enode with smaller keys first. Which means, if a thread say T\(_1\) acquired a lock on a enode(key\(_2\)), it never tries to acquire a lock on a enode of the vertex vnode(key\(_1\)) with key smaller than or equal to enode(key\(_2\)). This is true because the AcyclicAddEdge method acquires lock on the predecessor edge nodes of the vertex vnode(key\(_1\)) from the AcyclicLocateEdge method and returns after releasing the locks.

2. AcyclicRemoveEdge: the AcyclicRemoveEdge(key\(_1\), key\(_2\)) method is also deadlock-free, similar argument as AcyclicAddEdge.

Lemma 30 The methods ContainsVertex, AcyclicContainsEdge and PathExists are wait-free.

Proof Sketch: The proof of the ContainsVertex and AcyclicContainsEdge method is similar arguments as \( \ref{lemma:wait} \) and \( \ref{lemma:wait} \) respectively.

7 Simulation Results & Analysis

We performed our tests on 2 sockets & 14 cores per socket Intel Xeon (R) CPU E5-2660 v4 running at 2.00 GHz core frequency. Each core supports 2 hardware threads. Every core’s L1, L2 cache are private to that core; L3 cache (35MB) is shared across all cores of a processor. The tests were performed in a controlled environment, where we were the sole users of the system. The complete source code is
available at [10] and the implementation has been done in C++ (without any garbage collection) and threading is achieved by using Posix threads.

In the experiments conducted, we start with an initial complete acyclic graph. When the program starts, it creates threads and each thread randomly performs a set of operations chosen by a particular workload distribution. Here, the evaluation metric used is the time taken to complete all the operations. We measure throughput obtained on running the experiment for 20 seconds and present the results for the following workload distributions: (1) Update-dominated: 25% AddV, 25% AddE, 10% RemoveV, 10% RemoveE, 15% ContainsV and 15% ContainsE; (2) Contains-dominated: 40% ContainsV, 40% ContainsE, 7% AddV, 7% AddE, 3% RemoveV and 3% RemoveE; (3) Edge-updates: 40% AddE, 60% RemoveE and rest are 0%. Figure 15 depicts the results for the data structure methods. Figure 16, on the other hand, depicts the performance results for the acyclic methods. Each data point is obtained after averaging for 5 iterations. We assume that duplicate vertices are not inserted.

We tested different variants of the data structure for different number of threads - ConcGraph-NoDIE: the concurrent data structure presented without Deletion of Incoming Edges (DIE) for deleted vertices, ConcGraph-DIE: supporting Deletion of Incoming Edges of deleted vertices (Algorithm 4), CoarseLock: which supports concurrent operations by acquiring a global lock and the sequential implementation. The acyclicity variant has been implemented via reachability method as described in the Section 6. These are compared against a coarse lock & sequential implementation of the same method. The figures depict that the presented algorithm certainly outperforms the coarse lock and sequential counterpart. Also update methods of ConcGraph and the reachability based cycle detection (without deletion of incoming edges, DIE) give a significant increase in the throughput and scale well with increasing the number of threads. We noted on an average 8x increased throughput.

It is interesting to observe that ConcGraph-NoDIE achieves higher throughput than the one with DIE. This can be attributed to the observation that it is cost inefficient to traverse all the vertices to search for the incoming edges of the deleted vertices. However, in Figure 15c and 16c, ConcGraph-DIE performs similar to NoDIE. This is because in these experiments there are no deletion of vertices and hence no reason to perform DIE for deleted vertices. Hence the performance is very similar. We performed initial experiments to also test the number of failed AddEdge operations in our acyclic variant. Our preliminary results show that the percentage of false positives are very low.

8 Conclusion & Future Direction

In this paper, we have shown how to construct a fully dynamic concurrent graph data structure, which allows threads to concurrently add/delete vertices/edges. The update methods of the algorithm are
deadlock-free while the contains methods are wait-free. To the the best of our knowledge, this is the first work to propose a concurrent data structure for an adjacency list representation of the graphs. The other known work on concurrent graphs by Kallimanis & Kanellou \cite{6} works on adjacency matrix and assume an upper-bound on the number of vertices while we make no such assumption.

We believe that there are many applications that can benefit from this concurrent graph structure. An important application that inspired us is SGT in Databases and Transactional Memory. Motivated by this application, on this concurrent graph data-structure, we pose the constraint that the graph should be acyclic. We ensure this by checking for graph acyclicity whenever we add an edge. To detect the cycle efficiently, we have proposed a wait-free reachability algorithm. We have compared the performance of the concurrent data structure with the coarse-grained locking and sequential implementation and we achieve on an average 8x increased throughput. For proving linearizability, we have used linearization points.

As observed earlier, the current implementation of AddEdge using wait-free reachability might falsely not be able add an edge although adding the edge will not cause a cycle. Our preliminary results show that the number of total failed AddEdge operations are infact comparable to the sequential counterpart. We plan to explore this issue more deeply to see how adversely can it affect the progress of the algorithm and how can they be reduced.

Currently the update methods of our algorithm are blocking and deadlock free. We proposed a blocking approach to this complicated problem of concurrent graph acyclicity as a first cut. In the future, we plan to explore wait-free variant of all the methods of the graph data-structure. We also plan to propose solutions for the efficient deletion of incoming edges of deleted vertices.

References

[1] Camil Demetrescu, Irene Finocchi, and Giuseppe F. Italiano. Dynamic graphs. In *Handbook of Data Structures and Applications*. 2004.

[2] Steve Heller, Maurice Herlihy, Victor Luchangco, Mark Moir, William N. Scherer III, and Nir Shavit. A Lazy Concurrent List-Based Set Algorithm. *Parallel Processing Letters*, 17(4):411–424, 2007.

[3] Maurice Herlihy and Nir Shavit. *The Art of Multiprocessor Programming*. Morgan Kaufmann, 2008.

[4] Maurice Herlihy and Nir Shavit. On the Nature of Progress. In *Principles of Distributed Systems - 15th International Conference, OPODIS 2011, Toulouse, France, December 13-16, 2011. Proceedings*, pages 313–328, 2011.

[5] Maurice P. Herlihy and Jeannette M. Wing. Linearizability: a correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.

[6] Nikolaos D. Kallimanis and Eleni Kanellou. Wait-Free Concurrent Graph Objects with Dynamic Traversals. In *19th International Conference on Principles of Distributed Systems, OPODIS 2015, December 14-17, 2015, Rennes, France*, pages 27:1–27:17, 2015.

[7] Aapo Kyrola, Guy Blelloch, and Carlos Guestrin. GraphChi: Large-scale Graph Computation on Just a PC. In *Proceedings of the 10th USENIX Conference on Operating Systems Design and Implementation, OSDI’12*, pages 31–46, Berkeley, CA, USA, 2012. USENIX Association.

[8] Kfir Lev-Ari, Gregory V. Chockler, and Idit Keidar. On correctness of data structures under reads-write concurrency. In *Distributed Computing - 28th International Symposium, DISC 2014, Austin, TX, USA, October 12-15, 2014. Proceedings*, pages 273–287, 2014.

[9] Kfir Lev-Ari, Gregory V. Chockler, and Idit Keidar. A constructive approach for proving data structures linearizability. In *Distributed Computing - 29th International Symposium, DISC 2015, Tokyo, Japan, October 7-9, 2015, Proceedings*, pages 356–370, 2015.

[10] Sathya Peri, Muktikanta Sa, and Nandini Singhal. ConcurrentGraphDS. [https://github.com/PDCRL/ConcurrentGraphDS](https://github.com/PDCRL/ConcurrentGraphDS) 2017.

[11] Dimitrios Prountzos, Roman Manevich, and Keshav Pingali. Elixir: a system for synthesizing concurrent graph programs. In *Proceedings of the 27th Annual ACM SIGPLAN Conference on Object-Oriented Programming, Systems, Languages, and Applications, OOPSLA 2012*, pages 375–394, 2012.
REFERENCES

[12] Abraham Silberschatz, Peter Baer Galvin, and Greg Gagne. *Operating system concepts, 9th Edition*. Wiley, 2009.

[13] Arnab Sinha and Sharad Malik. Runtime checking of Serializability in Software Transactional Memory. In *24th IEEE International Symposium on Parallel and Distributed Processing, IPDPS 2010*, pages 1–12, 2010.

[14] Viktor Vafeiadis, Maurice Herlihy, Tony Hoare, and Marc Shapiro. Proving Correctness of Highly-Concurrent Linearisable Objects. In *Proceedings of the ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming, PPOPP 2006*, pages 129–136, 2006.

[15] Gerhard Weikum and Gottfried Vossen. *Transactional Information Systems: Theory, Algorithms, and the Practice of Concurrency Control and Recovery*. Morgan Kaufmann, 2002.