Adjacency energy of pebbling graphs

Sakunthala Srinivasan, Vimala Shanmugavel

Department of Mathematics, Mother Teresa Women’s University, Attuvampatty, Kodaikanal, Tamil Nadu.

E-mail: sakusivakumar@gmail.com

Abstract: This research paper presents the idea of energy for pebbling graphs. Pebbling is transportation of two pebbles from one vertex (place) to another vertex (place). During transformation, one pebble is lost as toll and only one pebble reaches the end. The lower bounds and upper bounds of energy levels were investigated for various types of simple and undirected graphs through pebbling concept.

Keywords: pebbling, energy, lower bound, upper bound, adjacency matrix, vertex, huckel molecular orbital theory, HMO

AMS 2010 Classification: 05C50, 05C38, 05C05.

1. Introduction

Graph pebbling is an addictive game, which attracted the research interest of many researchers all over the world. The term Pebbling of graphs was coined by Chung in 1989 [8]. Ever since the introduction of graph pebbling in early nineties, there has been extensive development in this area. In a graph consisting of two vertices, a pebbling move can be demonstrated by evicting two pebbles from the first vertex and accommodating one pebble on neighbouring vertex. Conventionally, the solution to a pebbling problem can be constituted by arriving at the graph’s well-defined vertex by a series of definite pebbling moves.

The footprints in energy of graph were started timely in 1940’s. In 1978, Gutman [10] instigated the idea of energy graph mathematically. The idea of energy stems from chemistry to estimate the total π-electron energy of a molecule. The energy of π electrons of the molecule was something like the energy of its molecular graph was established by Balakrishnan [5]. In an energy graph, every vertex is taken as a carbon atom and every edge is considered as carbon-carbon bond. The eigenvalues of the molecular graph entitle the energy level of the electron in a molecule. One of the amazing chemical applications of spectral graph theory is to build the close correlation in the middle of the graph eigenvalues and the molecular orbital energy levels of π-electrons in conjugated hydrocarbons [13]. In 2017, PG.Bhat and S D’Souza introduced a new concept of color signless energy and also computed color signless laplacian energy of families of graph with the minimum number of colors. The color signless laplacian energy for the complement of some colored graphs was also obtained [7]. The relation between the energy of wheel graph and fan graph was obtained and also generated MATLAB code for the adjacency matrix of wheel graph and fan graph by Sophia Shalini G.S and Mayamma Joseph [19].

In [21] Vaidya S.K and Kalpesh M.Popat discussed about the adjacency matrix of splitting graph and shadow graph and proved the $E(S(G)) = \sqrt{5E(G)}$ and $E(D_{n}(G)) = 2E(G)$. In 2018, the lower bounds for $\varepsilon(G)$ in terms number of vertices, edges, Radic index, minimum degree, diameter, walk and determinant of the adjacency matrix was done by Akbar Jahanbani [4]. The degree based energies of graphs was briefly described by Kinkar Ch.Das, Ivan Gutman and et.al [16]. In 2019, T.K. Mathew Varkey and John K.Rajan described about the nonhypoenergetic and hypoenergetic graphs among singular graphs [17]. In the same year, S.Nayak, et.al introduced the new concept of energy of partial complements of graph and partial complement energy was computed for few classes of graphs. Some bounds were obtained for partial complement energy of a graph G [18]. In 2020, Acharya B.D, Rao S.B, Sumathi.P and Swamianathan.V introduced the notion of robust domination energy and shear domination energy of G as the maximum energy of a minimal dominating set in G and they raised several open problems [1]. Misbalance degree matrix and related energy of graphs were discussed by Amitav Doley, Budheswar Deka and Bharali.A [3]. In particular in chemistry, energy has its applications in the theory of Hückel Molecular Orbital (HMO) theory. HMO theory is an approximate
method to treat planar conjugated hydrocarbons. Properties of the conjugated molecules are primarily determined by π-electrons. The idea connected to the spectrum of a graph is the energy. Nowadays approaches are deeply studied in physics, chemistry, computer science and in other limb of mathematics. In particular in chemistry, energy has its applications in the theory of Huckel Molecular Orbital Theory and is popularly called as HMO. The applications of energy was characterised by Cvetkovic and I.Gutman in [6].

Let G be a finite, directionless simple graph. The Adjacency matrix A(G) is defined as

\[
A(G) = a_{ij} = \begin{cases} 
1 & \text{if } i \text{ and } j \text{ are adjacent} \\
0 & \text{otherwise}
\end{cases}
\]

In this paper, a new idea of energy in pebbling graph has been discussed and applied for finite simple graphs. The notepaper is partitioned into five sections. The first section is the introduction and the second section is the preliminaries, in which the basic definitions and bounds was given. The third section comprises of findings of the energy of pebbling graphs; fourth section envisages the energy levels of pebbling graphs and the fifth section concludes with the research findings carried out in this article. In the third section, the adjacency matrix is placed alongside the corresponding pebbling graph for the sake of convenience and clarity.

2. Preliminaries
2.1 Pebbling graph
In a simple graph, consisting of two vertices, a pebbling move can be demonstrated by evicting two pebbles from the first vertex and accommodating one pebble on the neighbouring vertex is called a pebbling graph.

2.2 Energy graph
Energy graph \( E(G) \) is the total of singular values of the Eigen values of adjacency matrix \( A(G) \) and it is expressed as \( E(G) = \sum_{i=1}^{n} |\lambda_i| \).

2.3 Bounds
- Applying the Cauchy-Schwartz inequality to \((1,1,...,1)\) and \((|\lambda_1|,|\lambda_2|,...,|\lambda_n|)\) we get the upper bound,

\[
E(G) \leq \sqrt{n} \sum_{i=1}^{n} \sqrt{\lambda_i^2} = \sqrt{n} \sqrt{\sum_{i=1}^{n} \lambda_i^2} = \sqrt{2mn}.
\]

- Lower bound: Using the arithmetic-geometric means inequality,

\[
\sqrt{2m + n(n-1)|\det(A)|^2}
\]

- The superior and bottommost bounds were given by McClelland’s bounds (1971).

3. Comparison between Graph Theoretical Terms and Chemical Terms
The graph theoretical terms and its analogous chemical terms are indicated in the Table 1 mentioned below.

| S.No. | Graph Theoretical Terms | Chemical Terms                           |
|-------|-------------------------|------------------------------------------|
| 1     | Chemical (molecular) graph | Structural formula                      |
| 2     | Vertex (Atom)           | Weighted vertex atom of a defined element(mostly other than carbon) |
| 3     | Edge (line) Chemical bond | Weighted edge chemical bond between the defined elements |
| 4     | Degree of a vertex      | Valency of an atom                       |
| 5     | Tree graph              | Acyclic structure                        |
| 6     | Chain                   | Linear alkane or polyene                 |
| 7     | Cycle                   | Cycloalkane or annulene                  |
| 8     | Adjacency matrix (A)    | Huckel (topological matrix)              |
Characteristic polynomial
Secular polynomial
Eigen value
Eigen value of Hückel matrix
Eigen factor
HMO (topological)
Zero eigen value
Level of Nonbonding energy
Positive eigen value
Energy level of Bonding
Negative eigen value
Level of energy antibonding

4. Energy of Pebbling Graphs

4.1 Bull Graph
The pebbling bull graph with five vertices (i.e., ‘a’ through ‘e’) is shown in figure 4.1. The assumed transit direction ‘b’ to ‘d’ is effected in two ways, viz., ‘bcd’ or ‘bd’ directly; however, the later transit is considered in this graph for convenience. Initially twelve pebbles are present at vertex ‘a’. The transit takes place from ‘a’ to ‘b’ and only six pebbles reaches ‘b’; while the remaining six pebbles are lost as fuel during transit. Vertex ‘b’ is connected to the vertices ‘c’ and ‘d’. Hence during transit from ‘b’ to ‘c’, one pebble is lost as fuel and one pebble reaches ‘c’. Similarly, during transit from ‘b’ to ‘d’, two pebbles are consumed as fuel and two pebbles reaches ‘d’. Finally, one pebble reaches the vertex ‘e’ from ‘d’, while one is lost during transit.

\[
A = \begin{bmatrix}
0 & 6 & 0 & 0 & 0 \\
6 & 0 & 1 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Figure 4.1 Pebbling bull graph

The Eigen values are 0, 6.4109, -6.4109, -0.9488, 0.9488. The Energy \( E(G) = 14.7194 \).

4.2 Banner Graph
The pebbling banner graph with five vertices (i.e., ‘a’ through ‘e’) is presented in figure 4.2. The transit direction from ‘a’ to ‘c’ is effected in two ways, viz., ‘abde’ or ‘ae’; however, the transit ‘abde’ is considered for convenience. Initially twelve pebbles are present at vertex ‘a’. The transit takes place from ‘a’ to ‘b’ and only six pebbles reaches ‘b’; while six pebbles are lost as fuel during transit. Vertex ‘b’ is connected to the vertex ‘c’ and vertex ‘d’.

\[
A = \begin{bmatrix}
0 & 6 & 0 & 0 & 0 \\
6 & 0 & 1 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Figure 4.2 Pebbling banner graph

Hence during transit from ‘b’ to ‘c’, one pebble is lost as fuel and one pebble reaches ‘c’. Similarly, during transit from ‘b’ to ‘d’, two pebbles are consumed as fuel and two pebbles reaches ‘d’. Finally, one pebble reaches the vertex ‘e’ from ‘d’, while one is lost during transit. The Eigen values for the above mentioned graphs are 0, 6.4109, -6.4109, -0.9488, 0.9488. The Energy \( E(G) = 14.7194 \).

4.3 Caterpillar Graph
The pebbling caterpillar graph has nine vertices (i.e., ‘a’ through ‘i’) is depicted in figure 4.3. The transit is from ‘a’ to all other vertices. Vertex ‘b’ is connected to a non-end vertex ‘c’ and end vertices ‘e’ and ‘d’.

Likewise the end vertices ‘i’ and ‘h’ are connected to ‘g’, while the end vertex ‘f’ is connected to ‘e’. In the present graph, 48 pebbles are present at vertex ‘a’. The transit takes place from ‘a’ to ‘b’ and only twenty 24 reaches ‘b’ while the remaining 24 pebbles are lost as fuel. Vertex ‘b’ is connected to the vertex ‘c’, ‘d’ and ‘e’. Hence during transit from ‘b’ to ‘c’; 10 pebbles are lost as fuel and 10 pebbles reaches ‘e’. Similarly, during transit from ‘b’ to ‘d’ and ‘b’ to ‘c’, one pebble is lost as fuel while one pebble reaches ‘d’ and ‘c’ respectively. The transit from ‘e’ to ‘f’ and ‘e’ to ‘g’, one pebble is lost as fuel, one pebble reaches ‘f’; and four pebbles are lost as fuel and four pebbles reach ‘g’ respectively. Finally, the transit from ‘g’ to ‘h’ and ‘g’ to ‘i’, one pebble is lost as fuel while one pebble reaches ‘i’ and ‘h’ respectively. The Eigen values of the matrix A are 0, 0, -26.0877, 26.0877, -4.0406, 4.0406, -0.3225, 0.3225. The Energy of A is E(G)=60.9016.

4.4 Firecracker Graph

The pebbling firecracker graph with four vertices (i.e., ‘a’ through ‘d’) is displayed in figure 4.4. The transit direction is taken from ‘abc’. Initially six pebbles are present at vertex ‘a’.

The transit takes place from ‘a’ to ‘b’ and only four pebbles reaches ‘b’; while four pebbles are lost as fuel during transit. During transit from ‘b’ to ‘c’, two pebbles are lost as fuel and two pebbles reaches ‘c’. Finally, one pebble reaches the vertex ‘d’ while one is lost during transit. The Eigen Values are 4.4954, -4.4954, -0.8898, 0.8898. The Energy, E(G) = 10.7704.

4.5 Sunlet Graph

The pebbling sunlet graph with six vertices (i.e., ‘a’ through ‘f’) is laid out in figure 4.5. The transit is originates at ‘a’ and should terminate at ‘d’ and ‘f’. At ‘b’, the graph diverges in to two paths each to terminate at ‘d’ and ‘f’. Despite there is connected between ‘c’ and ‘e’, yet there is no transit between, them. Initially sixteen pebbles are present at vertex ‘a’.
During the first transit from eight pebbles are lost while the remaining 8 pebbles reach ‘b’. Between ‘bc’ and ‘bc’ two pebbles are lost during each transit and only two reaches ‘c’ and ‘e’. Further, one pebble is lost during each transit from ‘c’ and ‘e’ and finally one pebble reaches both ‘d’ and ‘f’. The Eigen values of pebbling sunlet graph is $-8.4919, 8.4919, -1, 1, -0.9421, 0.9421$. The Energy, $E(G) = 20.868$.

4.6 Gear Graph

The pebbling gear graph has nine vertices (i.e, ‘a’ through ‘i’) is shown in figure 4.6. The assumed transit direction is taken from ‘b’ to ‘d’ is effected in two ways, viz., ‘bcd’ or from ‘b’ to ‘d’. However, the transit from ‘b’ to ‘c’ is considered in this graph for convenience. Initially one hundred and thirty six pebbles are present at vertex ‘a’.

The Eigen Values are $0, -76.3799, 76.3799, -16.4607, 16.4607, -4.0452, 4.0452, -0.8901, 0.8901$. The Energy, $E(G) = 195.5518$.

4.7 Paw Graph

The pebbling paw graph with four vertices (i.e, ‘a’ through ‘d’) is depicted in figure 4.7. The assumed transit direction is taken from ‘b’ to ‘d’ is effected in two ways, viz., ‘bcd’ or from ‘b’ to ‘d’. However, the transit from ‘b’ to ‘c’ is considered in this graph for convenience. Initially eight pebbles are present at vertex ‘a’.
Figure 4.7 Pebbling paw graph

The transit takes place from 'a' to 'b' and only four pebbles reaches 'b'; while four pebbles are lost as fuel during transit. Vertex 'b' is connected to the vertex 'c' and vertex 'd'. Hence during transit from 'b' to 'c', two pebbles are lost as fuel and two pebble reaches 'c'. Finally, one pebble reaches the vertex 'd' from 'c', while one is lost during transit. The Eigen Values are 4.4954, -4.4954, -0.8898, 0.8898. The Energy, $E(G) = 10.7704$.

4.8 Pan Graph

The pebbling pan graph with five vertices (i.e., 'a' through 'e') is presented in figure 4.8. The assumed transit direction is taken from 'b' to 'e' is effected in two ways, viz., 'bcde' or from 'b' to 'e'. However, the transit from 'bcde' is considered in this graph for convenience. Initially sixteen pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and only eight pebbles reaches 'b'; while eight pebbles are lost as fuel during transit. Vertex 'b' is connected to the vertex 'c' and vertex 'e'. Hence during transit from 'b' to 'c', four pebbles are lost as fuel and four pebbles reaches 'c'. Similarly, during transit from 'c' to 'd', two pebbles are consumed as fuel and two pebbles reaches 'd'. Finally, one pebble reaches the vertex 'e' from 'd', while one is lost during transit. The Eigen Values are 0, 8.9913, -8.9913, -2.0387, 2.0387. The Energy, $E(G) = 22.06$.

4.9 Comb Graph

The pebbling comb graph with ten vertices (i.e., 'a' through 'j') is indicated in figure 4.9. The vertex 'a' is connected to 'c' and 'b'; vertex 'c' is connected to 'd' and 'e'; the vertex 'e' is connected 'f' and 'g' and the vertex 'g' is connected to 'h' and 'i'. A total of sixty two pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and only one pebble reaches 'b'; while one pebble is lost as fuel during transit. Hence during transit from 'a' to 'c', thirty pebbles are lost as fuel and thirty pebbles reaches 'c'. Similarly, during transit from 'c' to 'd', one pebble is consumed as fuel and one pebble reaches 'd'. The transit from 'c' to 'e', fourteen pebbles are lost and fourteen pebbles reaches 'e'. Vertex 'e' is connected
to ‘g’ and ‘f’. During the transit from ‘e’ to ‘f’, one pebble is lost and only one pebble reaches ‘f’. The transit from ‘e’ to ‘g’, six pebbles are lost and only six pebbles reaches ‘g’. Vertex ‘g’ is connected to ‘h’ and ‘i’. The transit from ‘g’ to ‘h’, one pebble is lost and one pebble reaches ‘h’. During transit from ‘g’ to ‘i’, two pebbles are lost and two pebbles reaches ‘i’. Finally, one pebble reaches the vertex ‘j’ from ‘i’, while one is lost during transit. The Eigen values are -33.2361, 33.2361, -5.9440, 5.9440, -1, 1, -0.1682, 0.1682, -0.0301, 0.0301. The Energy, E(G) = 80.7568.

4.10 Centipede Graph
The pebbling centipede graph with six vertices (i.e., ‘a’ through ‘f’) is portrayed in figure 4.10. Initially twenty four pebbles are present at vertex ‘a’. The transit takes place from ‘a’ to ‘b’ and twelve pebbles reaches ‘b’; while twelve pebbles are lost as fuel during transit. During transit from ‘b’ to ‘c’, six pebbles are lost as fuel and six pebbles reaches ‘c’. The vertex ‘c’ is connected to vertex ‘d’ and ‘e’. The transit from ‘c’ to ‘d’, one pebble is lost and one pebble reaches ‘d’. The transit from ‘c’ to ‘e’, two pebbles are lost and two pebbles reaches ‘e’. Finally, one pebble reaches the vertex ‘f’ from ‘e’, while one is lost during transit. The Eigen values are -13.4546, 13.4546, -2.1926, 2.1926, -0.4068, 0.4068. The Energy, E(G) = 32.108.

4.11 Ladder Graph
The pebbling ladder graph with eight vertices (i.e., ‘a’ through ‘h’) is pictured in figure 4.11. Vertex ‘a’ is connected to ‘b’ and ‘c’. Initially twenty four pebbles are present at vertex ‘a’. The transit takes place from ‘a’ to ‘b’ and one pebble reaches ‘b’; while one pebble are lost as fuel during transit. During transit from ‘a’ to ‘c’, fourteen pebbles are lost as fuel and fourteen pebbles reaches ‘c’. The vertex ‘c’ is connected to vertex ‘d’ and ‘e’. The transit from ‘c’ to ‘d’, one pebble is lost and one pebble reaches ‘d’. During the transit from ‘c’ to ‘e’, six pebbles are lost and six pebbles reaches ‘e’. The vertex ‘c’ is connected to vertex ‘f’ and ‘g’. The transit from ‘c’ to ‘f’, one pebble is lost and one pebble reaches ‘f’. During the transit from ‘c’ to ‘g’, two pebbles are lost and two pebbles reaches ‘g’. Finally, one pebble reaches the vertex ‘h’ from ‘g’, while one is lost during transit. The Eigen values are -15.3175, 15.3175, -2.2753, 2.2753, -0.4395, 0.4395, -0.0653, 0.0653. The Energy, E(G) = 36.1952.

4.12 Banana tree Graph
The pebbling banana tree graph with nine vertices (i.e., 'a' through 'i') is shown in figure 4.12. Vertex 'a' is connected to 'b' and 'f'. Initially thirty two pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and eight pebbles reaches 'b'; while eight pebbles are lost as fuel during transit.

During transit from 'a' to 'f', eight pebbles are lost as fuel while eight pebbles reaches 'f'. The vertex 'b' is connected to 'c', four pebbles reaches 'c' and four pebbles are lost during transit. The vertex 'c' is connected to vertex 'd' and 'e'. One pebble is lost in transit from 'c' to 'd', and one pebble reaches 'd'. During the transit from 'c' to 'e', one pebble is lost and one pebble reaches 'e'. The vertex 'f' is connected to 'g', four pebbles reaches 'g' and four pebbles are lost during transit. Finally, the vertex 'g' is connected to vertex 'h' and 'i'. The transit from 'g' to 'h', one pebble is lost and one pebble reaches 'h'. During the transit from 'g' to 'i', one pebble is lost and one pebble reaches 'i'. The Eigen Values of the Pebbling Banana tree graph is 0, 0, 0, -12.0094, 12.0094, -4.2426, 4.2426, -1.3323, 1.3323. The Energy, $E(G) = 35.1686$.

4.13 Net Graph
The pebbling sunlet graph with six vertices (i.e., 'a' through 'f') is depicted in figure 4.13. The assumed transit direction is taken from 'b' is effected in two ways, viz., 'b' to 'c', 'b' to 'e'. Initially sixteen pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and only eight pebbles reaches 'b'; while eight pebbles are lost as fuel during transit. Vertex 'b' is connected to the vertex 'c' and vertex 'e'. Hence during transit from 'b' to 'e', two pebbles are lost as fuel and two pebbles reaches 'c'. Similarly, during transit from 'b' to 'e', two pebbles are consumed as fuel and two pebbles reaches 'e'. Finally, one pebble reaches the vertex 'd' from 'c' and one pebble reaches the vertex 'f' from 'e', while one is lost during each transit.

During transit from 'a' to 'f', eight pebbles are lost as fuel while eight pebbles reaches 'f'. The vertex 'b' is connected to 'c', four pebbles reaches 'c' and four pebbles are lost during transit. The vertex 'c' is connected to vertex 'd' and 'e'. One pebble is lost in transit from 'c' to 'd', and one pebble reaches 'd'. During the transit from 'c' to 'e', one pebble is lost and one pebble reaches 'e'. The vertex 'f' is connected to 'g', four pebbles reaches 'g' and four pebbles are lost during transit. Finally, the vertex 'g' is connected to vertex 'h' and 'i'. The transit from 'g' to 'h', one pebble is lost and one pebble reaches 'h'. During the transit from 'g' to 'i', one pebble is lost and one pebble reaches 'i'. The Eigen Values of the Pebbling Banana tree graph is 0, 0, 0, -12.0094, 12.0094, -4.2426, 4.2426, -1.3323, 1.3323. The Energy, $E(G) = 35.1686$.

4.14 Butterfly Graph
The pebbling butterfly graph with four vertices (i.e, 'a' through 'd') is presented in figure 4.14. The assumed transit direction is taken from 'a' are 'a' to 'c' or 'b' to 'c'. However, the transit from 'b' to
'c' is considered in this graph for convenience. Initially sixteen pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and only eight pebbles reaches 'b'; while eight pebbles are lost as fuel during transit.

![Pebbling butterfly graph](image)

During transit from 'b' to 'c', four pebbles are lost as fuel and four pebbles reaches 'c'. Vertex 'c' is connected 'd' and 'e'. The transit from 'c' to 'd', only one pebble is lost and one pebble reaches 'd'. Finally, one pebble reaches the vertex 'e' from 'c', while one is lost during transit. The Eigen values of pebbling butterfly graph are 0, 8.9671, -8.9671, -1.2617, and 1.2617. The Energy, \( E(G) = 20.4576 \).

### 4.15 Sun Graph

The pebbling sun graph with six vertices (i.e., 'a' through 'f') is pictured in figure 4.15. Vertex 'a' is connected to 'b' and 'c'. Initially sixteen pebbles are present at vertex 'a'. The transit takes place from 'a' to 'b' and only eight pebbles reaches 'b'; while eight pebbles are lost as fuel during transit. Vertex 'b' is connected to the vertex 'c', vertex 'd' and vertex 'e'. Hence during transit from 'b' to 'c', one pebble is lost as fuel and one pebble reaches 'c'. The transit from 'b' to 'd', only one pebble is lost and one pebble reaches 'd'. Similarly, during transit from 'b' to 'e', two pebbles are consumed as fuel and two pebbles reaches 'e'. Finally, one pebble reaches the vertex 'f' from 'e', while one is lost during each transit.

![Pebbling sun graph](image)

The Eigen values of the Pebbling Sun Graph is 0, 0, -8.3701, 8.3701, -0.9706, 0.9706. The Energy, \( E(G) = 18.6814 \).

### 5. Energy levels of graphs considered in this article

The Table 2 illustrates the lower bound, energy and upper bound for the various types of graphs considered in this article.

| Figure | Graph          | Lower Bound | Energy     | Upper Bound |
|--------|----------------|-------------|------------|-------------|
| 3.1    | Bull Graph     | 9.16515139  | 14.7194    | 20.4939015  |
| 3.2    | Banner Graph   | 9.16515139  | 14.7194    | 20.4939015  |
| 3.3    | Caterpillar Graph | 37.3363094 | 60.9016    | 112.008928  |
| 3.4    | Firecracker Graph | 9.48683298 | 10.7704    | 12.9614814  |
| 3.5    | Sunlet Graph   | 5.29150262  | 20.8680    | 29.7993288  |
| 3.6    | Gear Graph     | 110.652610  | 195.5518   | 366.993188  |
| 3.7    | Paw Graph      | 9.48683298  | 10.7704    | 12.9614814  |
| 3.8    | Pan Graph      | 13.0384048  | 22.0600    | 29.1547595  |
The energy of pebbling graphs and their bounds is discussed in the paper. From the above graphs, the authors obtain the relation that 

\[ \sqrt{2 \sum_{i=1}^{m} n_i^2 + n(n-1)|\Delta|^2} \leq E(G) \leq \sqrt{2 \sum_{i=1}^{m} n_i^2} n. \]

That is, Lower bound \( \leq \) Energy \( \leq \) Upper bound. Our further work is to implement for different types of energy in pebbling graphs and also to find the relation between them. The principal application of the proposed method is to estimate the \( \pi \)-electron energy of a molecule. Several other application include model of surveillance or security system in civil engineering, circuit design and electrical switch boards. The application of bond energies are used for the determination of enthalpy of reaction, enthalpies of formation of compounds and resonance energy.

7. References

[1] Acharya B.D, Rao S.B, Sumathi P and Swaminathan.V 2020 Energy of a set of vertices in a graph, AKCE International Journal of Graphs and Combinatorics, published online on 10 March 2020, pp.145-152.

[2] Adiga C, Bayad A, Gutman I, Shrikanth Avant Srinivas 2012 The Minimum Covering Energy of a Graph Kragujevac J. Sci 34 p 39-56.

[3] Amitav Doley, Budheswar Deka and Bharati.A, 2020 Misbalance Degree matrix and related energy of graphs, J. Math. Comput. Sci. vol.10, issue.3, pp.436-447.

[4] Akbar Jahanbani 2018 Lower bounds for the energy of graphs, AKCE International Journal of Graphs and Combinatorics, vol.15, issue 1, pp.88-96.

[5] Balakrishnan R 2004 The energy of a graph, Linear Algebra Appl.387 p 287-295.

[6] Betten A, Kohnert A, Laue R, Wassermann A (Eds.), 2001 Algebraic Combinatorics and Applications, Springer Verlag, Berlin, pp.145-174.

[7] Bhat PG, S D’Souza 2017 Color Signless Laplacian energy of graphs, AKCE International Journal of Graphs and Combinatorics, vol.14, issue 2, pp.142-148.

[8] Chung F, 1989 Pebbling in hypercubes, SIAM J. Discrete Math. 2(4) pp. 461-472.

[9] Cvetkovic D, Dooband M, Sachs H, 1980 Spectra of graphs, Academic Press, New York.

[10] Cvetkovic D, Gutman I (Eds.), 2009 Applications of Graph Spectra, Math. Inst., Belgrade.

[11] Cvetkovi D, Gutman I (Eds.), 2011 Selected topics on applications of graph Spectra, Math. Inst., Belgrade.

[12] Glenn Hurlbert, 1999 A survey of graph pebbling. Proc. of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL vol 139) p 41-64.

[13] Glenn Hurlbert, 2005 Recent progress in graph pebbling. Graph Theory Notes N. Y., vol.49, p 25-37.

[14] Glenn Hurlbert, 2014 Graph pebbling. In Jonathan L. Gross, Jay Yellen, and Ping Zhang, editors, Handbook of Graph Theory, CRC Press, 2nd edition.

[15] Gutman I, 1978, The energy of a graph, Ber. Math. Stat. Sekt. Forschungsz. Graz. 103,p 1-22.

[16] Kinkar ch. Das, Ivan Gutman and et.al, 2018 Degree-based energies of graphs, Linear Algebra and its Applications, vol.554, pp.185-204.

[17] Mathew Varkey T.K and John K.Rajan, 2019 AKCE International Journal of Graphs and Combinatorics, vol.16, issue 3, pp.265-271.
[18] Nayak, S., D’Souza, Gortham HJ and Bhat PG 2019 Energy of partial complements of a graph, *Proceedings of the Jangjeon Mathematical Society*, volume 22, issue 3, pp.369-379.

[19] Sophia Shalini G.B and Mayamma Joseph, 2017 New results on energy of graphs of small order, *Global Journal of Pure and Applied Mathematics*, vol.13, no.7, pp.2837-2848.

[20] Stephan Wagner and Hua Wang 2019 *Introduction to Chemical graph Theory*, CRC Press, Taylor & Francis Group, A Chapman & Hall Book (2019).

[21] Vaidya S.K and Kalpesh M. Popat, 2017 Some new results on energy of graphs, *MATCH Commun. Math. Comput. Chem.* Vol.77, pp.589-594.