A central limit like theorem for Fourier sums

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July 21, 2017

Abstract

We consider the probability distributions of values in the complex plane attained by Fourier sums of the form

\[ \hat{a}_n(\nu) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} a_j e^{-2\pi i \nu j} \]

when the frequency \( \nu \) is drawn uniformly at random from an interval of length 1. If the coefficients \( a_j \) are i.i.d. drawn with finite third moment, the distance of these distributions to an isotropic two-dimensional Gaussian on \( \mathbb{C} \) converges in probability to zero for any pseudometric on the set of distributions for which the distance between empirical distributions and the underlying distribution converges to zero in probability.

The classical version of the central limit theorem states that for a series of real-valued independent identically distributed (iid) random variables \( X_1, X_2, \ldots \) with \( \mathbb{E}[X_j] = 0 \) and finite variance \( \sigma^2 \) the sequence

\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \]

converges in distribution to a Gaussian random variable with zero mean and variance \( \sigma^2 \) [1, p.357]. Formulations for random vectors \( X_j \) state convergence to multi-variate Gaussians [2]. Other well-known generalizations drop the assumption ‘identically distributed’ and replace it, for instance, with the Lyapunov condition

\[ \lim_{n \to \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}[X_j^2+\delta] = 0, \]

for some \( \delta > 0 \), where \( s_n \) denotes the sum of all variances of \( X_1, X_2, \ldots, X_n \) [1, p.362]. Then

\[ \frac{1}{\sqrt{s_n}} \sum_{j=1}^{n} X_j \]

converges in distribution to a standard Gaussian. It is also known that the independence assumption can be replaced with appropriate notions of weak dependence, e.g. [1, Theorem 27.4]. However, significantly more general scenarios yield Gaussians as limiting distributions. Here we consider sequences of Fourier sums of the form

\[ \hat{a}_n(\nu) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} a_j e^{-2\pi i \nu j}, \]

and show that sampling from random frequencies yields asymptotically a Gaussian – in a sense to be specified below – if the coefficients \( a_j \) are i.i.d. drawn. More precisely, let \( A_1, A_2, \ldots \) be a sequence of

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real-valued i.i.d. variables on a probability space \((\Omega, \Sigma, P_1)\). Then we first define for each frequency \(\nu\) the sequence \((\hat{A}_\nu^n)_{n \in \mathbb{N}}\) of random variables via

\[
\hat{A}_\nu^n \equiv \frac{1}{\sqrt{n}} \sum_{j=1}^{n} A_j e^{-2\pi i \nu j}.
\]

Using known vector-valued central limit theorems one can easily show that, under some technical condition of Lyapunov type detailed below, \(\hat{A}_\nu^n\) converges to a Gaussian on the complex plane for each \(\nu\) since it is obtained by a sum of the independent (but not identically distributed) complex-valued random variables \(X_j \equiv A_j e^{-2\pi i \nu j}\).

Here, however, we define for each \(\omega \in \Omega\) the sequence \((\hat{A}_\nu^n(\omega))_{n \in \mathbb{N}}\) of random variables on the probability space \([-1/2, 1/2], B, \lambda\), with \(B\) denoting the Borel sigma algebra and \(\lambda\) the Lebesque measure (to formalize the random choice of a frequency), via

\[
\hat{A}_\nu^n(\omega) \equiv \nu \mapsto \hat{A}_\nu^n(\omega) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} A_j(\omega) e^{-2\pi i \nu j}.
\]

The problem is motivated by Ref. \([3]\) which considers linear time invariant filters whose coefficients are randomly chosen. The question arising there was how the filter’s frequency response \((2)\) behaves in the limit \(n \to \infty\) for ‘typical’ choices of filter coefficients \(a_j\) when the latter are randomly drawn.

Note that the problem would become simple if we were to consider \(\hat{a}^n(\nu)\) at the discrete frequencies \(\nu_l := j/n\) for \(l = 1, \ldots, n/2\) (in signal processing \(|\hat{a}^n(\nu)|^2\) is also known as the periodogram of a signal \([4]\) and \(a_j\) were assumed to be drawn from independent Gaussians. For Gaussian \(A_j\), the random variables \(\hat{A}_\nu^n\) defined on the probability space \((\Omega, \Sigma, P_1)\) are also independent Gaussians for these different discrete frequencies, which can easily checked by computing the covariances. The question changes drastically when we consider the full continuum of frequencies, because \(\hat{A}_\nu^n\) and \(\hat{A}_{\nu'}^n\) are not in general independent for \(\nu \neq \nu'\). We will show, however, that they become asymptotically independent, which then results in an appropriate limit theorem for \(\hat{A}_\nu^n(\omega)\).

Crucial to phrase our limit theorem is the following type of distance measures on probability distributions:

**Definition 1 (well-behaved pseudometric)** For an arbitrary sequence \(Z_1, Z_2, \ldots\) of i.i.d. random variables on the probability space \((\Omega', \Sigma', P_{1'})\) let \(P_Z\) denote the distribution of each \(Z_j\) and \(P_{Z_1(\omega')}, \ldots, Z_k(\omega')\) denote the empirical distribution after the first \(k\) samples.

Let \(M_1\) denote the set of probability measures on the Borel-measurable subsets of \(\mathbb{R}^l\). Then a pseudometric \(d : M_1 \times M_1 \to \mathbb{R}_+^0\) is called ‘well-behaved’ if the distance between \(P_Z\) and \(P_{Z_1(\omega')}, \ldots, Z_k(\omega')\) converges in probability to zero uniformly over all i.i.d. sequences. More precisely, for every \(\epsilon, \delta > 0\) there is a \(k_0\) such that for all \(k \geq k_0\)

\[
P_{1'} \left\{ d(P_Z, P_{Z_1(\omega'), \ldots, Z_k(\omega')}) \geq \epsilon \right\} \leq \delta,
\]

holds for all sequences \(Z_1, Z_2, \ldots\) and probability spaces \((\Omega', \Sigma', P_{1'})\).

For distributions \(Q, R\) on \(\mathbb{R}\) with cumulative distribution functions \(F_Q\) and \(F_R\), respectively, \(d(Q, R) := \|F_Q - F_R\|_\infty\) provides a simple example of a well-behaved distance since

\[
\|F_{Q_\delta} - F_Q\|_\infty \geq \epsilon,
\]

occurs with probability at most \(2e^{-2k\epsilon^2}\) due to Massart’s formulation \([5]\) of the Dvoretzky-Kiefer-Wolfowitz (DKW) inequality. Another example is given by \(d(Q, R) := \sup_B |Q(B) - R(B)|\) where \(B\) runs over some set of sets whose indicator functions have finite VC-dimension. This follows from Vapnik and Chervonenkis’ uniform bound on the deviation of empirical frequencies of events from the corresponding probabilities \([6]\). Using Reproducing Kernel Hilbert Spaces (RKHS) one can construct a further example: the so-called kernel mean embedding \([7]\) represents distributions as vectors in a Hilbert space. Then the
Hilbert space distance is a well-behaved metric. This follows easily from the uniform consistency result in [8, Theorem 4] for the empirical estimator of this distance.

The purpose of this article is to show the following result:

**Theorem 1** Let \( P_{\hat{A}^{\nu}(\omega)} \) denote the distribution of \( \hat{A}^{\nu}(\omega) \) and \( G \) the distribution on \( \mathbb{C} \) for which real and imaginary parts are independent Gaussians with mean zero and variance \( 1/2 \). Then the distance between \( P_{\hat{A}^{\nu}(\omega)} \) and \( G \) converges to zero in probability for every well-behaved pseudometric \( d \). More, precisely, the random variable

\[
\omega \mapsto d(P_{\hat{A}^{\nu}(\omega)}, G)
\]

converges to zero in probability.

Obviously, the interval \([-1/2, 1/2]\) can be replaced by any interval of length 1, as stated in the abstract.

The first step of the proof will be to investigate the asymptotics of the variances and covariances of real and imaginary part of \( \hat{A}^{\nu} \) and the covariances between real and imaginary parts \( \hat{A}^{\nu} \) and \( \hat{A}^{\nu'} \) for different frequencies \( \nu, \nu' \). To this end, we represent complex numbers as vectors in \( \mathbb{R}^2 \) and obtain the following result:

**Lemma 1 (asymptotic covariances)** Let \( \nu_1, \ldots, \nu_l \) be some arbitrary non-zero frequencies in \((-1/2, 1/2)\) with \( |\nu_i| \neq |\nu_j| \) for \( i \neq j \). Let \( C^{(n)} \) denote the covariance matrix of the random vector \( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} S_j \) with

\[
S_j := A_j \begin{pmatrix} \cos(2\pi \nu_1 j) & \sin(2\pi \nu_1 j) \\ \cos(2\pi \nu_2 j) & \sin(2\pi \nu_2 j) \\ \vdots & \vdots \\ \cos(2\pi \nu_k j) & \sin(2\pi \nu_k j) \end{pmatrix}^T.
\]

Then

\[
\lim_{n \to \infty} C^{(n)} = \frac{1}{2} \mathbf{1},
\]

where \( \mathbf{1} \) denotes the identity in \( 2k \) dimensions.

Proof: We first introduce the vector \( \mathbf{1} := (1, 0)^T \) and the rotation matrix

\[
D_\nu := \begin{pmatrix} \cos(2\pi \nu) & -\sin(2\pi \nu) \\ \sin(2\pi \nu) & \cos(2\pi \nu) \end{pmatrix}.
\]

Using powers of these rotations, we can write the random vector \( S_j \) as the direct sum

\[
S_j := A_j \left[ D_{\nu_1}^j \mathbf{1} \oplus D_{\nu_2}^j \mathbf{1} \oplus \cdots \oplus D_{\nu_k}^j \mathbf{1} \right].
\]

Its covariance matrix reads

\[
C_j := \begin{pmatrix} D_{\nu_1}^j & D_{\nu_2}^j & \cdots & D_{\nu_k}^j \\ D_{\nu_2}^j & D_{\nu_1}^j & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ D_{\nu_k}^j & \cdots & \cdots & D_{\nu_k}^j \end{pmatrix}. 
\]

Since the random vectors \( S_1, \ldots, S_n \) are uncorrelated (because the variables \( A_j \) are independent and thus uncorrelated), the weighted sum

\[
S^{(n)} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} S_j
\]

has the covariance matrix

\[
C^{(n)} := \frac{1}{n} \sum_{j=1}^{n} D_j^j C_0 D_j^{-j}.
\]

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1 If both samples in [8, Theorem 4] are drawn from the same distribution and one of the sample sizes tends to infinity the bound describes the distance between empirical and true distribution.
Block $ll'$ within the $k \times k$ block matrices of format $2 \times 2$ reads

$$C_{ll'}^{(n)} = \frac{1}{n} \sum_{j=1}^{n} F_{ll'}^{j}(cc^T),$$

where $F_{ll'}^{j}$ denotes the $j$th power of the map $F_{ll'}$ on the space $\mathcal{M}_2(\mathbb{C})$ of complex-valued $2 \times 2$-matrices defined via

$$F_{ll'}(M) := D_{\nu_l} M D_{\nu_l}^{-1}.$$

Note that $F_{ll'}$ is a unitary map on $\mathcal{M}_2(\mathbb{C})$ with respect to the inner product $\langle A, B \rangle := \text{tr}(B^T A)$, where $\dagger$ denotes the Hermitian conjugate. Therefore, von Neumann’s mean ergodic theorem \cite{9} implies

$$\frac{1}{n} \sum_{j=1}^{n} F_{ll'}^{j}(cc^T) = Q_{ll'}(cc^T),$$

where $Q_{ll'}$ denotes the orthogonal projection\footnote{We could have also applied the mean ergodic theorem to unitary map $C_0 \mapsto D C_0 D^{-1}$ instead of applying it to each block separately, but finally this would not have simpliﬁed the analysis.} onto the $F_{ll'}$-invariant subspace of $\mathcal{M}_2(\mathbb{C})$. If $r_1 := (1, i)^T$ and $r_2 := (1, -i)^T$ denote the joint eigenvectors of all $D_{\nu_l}$ with eigenvalues $e^{\pm 2\pi \nu_l}$ then $F_{ll'}$ has the 4 eigenvectors $r_j, r'_j$ with $j, l = 1, 2$ and eigenvalues $e^{2\pi (\pm \nu_l \pm \nu_l)}$, $e^{\pm 2\pi (\nu_l + \nu_l)}$. For $l \neq l'$ the $F_{ll'}$-invariant subspace is 0 because all eigenvalues differ from 1 due to $0 \neq |\nu_l| \neq |\nu_{l'}| \neq 0$. Hence, the non-diagonal blocks of $C^{(n)}$ vanish in the limit. To consider the diagonal blocks, note that $F_{ll'}$ is then just the adjoint map of $D_{\nu_l}$, which can be restricted to the space of real-valued symmetric matrices $\mathcal{M}_2^{sym}(\mathbb{R})$. We then conclude

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F_{ll'}^{j}(M) = Q_{ll'}(M) = \frac{1}{2} \text{tr}(M) 1 \quad \forall M \in \mathcal{M}_2^{sym}(\mathbb{R}).$$

This is since multiples of the identity are the only real symmetric matrices that commute with $D_{\nu}$ for $\nu \in (-1/2, 1/2) \setminus \{0\}$ and because $F_{ll'}$ preserves the trace. $\square$

We now state the following central limit theorem for random vectors \cite{10}:

**Lemma 2 (CLT for random vectors with explicit bound)** Let $X_1, \ldots, X_n$ be independent random vectors in $\mathbb{R}^d$ such that $\mathbb{E}[X_j] = 0$ for all $j$. Write $S := \sum_{j=1}^{n} X_j$ and assume that the covariance matrix $C_S$ of $S$ is invertible. Let $Z$ be a centered Gaussian random vector with covariance matrix $C_S$. Let $C$ denote the set of convex subsets of $\mathbb{R}^d$. Then,

$$\sup_{B \in C} \left| P\{S \in B\} - P\{Z \in B\} \right| \leq \eta d^{1/4} \sum_{j=1}^{n} \beta_j, \quad (3)$$

with

$$\beta_j := \mathbb{E} \left[ \left\| \sqrt{C_S^{-1}} X_j \right\|^3 \right],$$

for some $\eta > 0$.

Since we will not use the explicit bound we derive a simpler asymptotic statement as implication:

**Lemma 3 (simplified CLT for random vectors)** Let $Y_j$ with $j \in \mathbb{N}^*$ independent random vectors with covariance matrices $C_j$ such that $C^{(n)} := \frac{1}{n} \sum_{j=1}^{n} C_j$ converges to some invertible matrix $C$ with respect to any matrix norm. Assume, moreover, that there exists a constant $b < \infty$ such that $\mathbb{E}[\|Y_j\|^3] \leq b$ for all $j$. Then

$$S^{(n)} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j$$

converges in distribution to a multivariate Gaussian with covariance matrix $C$.\footnote{We could have also applied the mean ergodic theorem to unitary map $C_0 \mapsto D C_0 D^{-1}$ instead of applying it to each block separately, but finally this would not have simpliﬁed the analysis.}
where \( \| \cdot \| \) denotes the operator norm. Here we have assumed that \( C(n) \) is invertible, which is certainly true for sufficiently large \( n \) since \( C \) is invertible. Since \( \| \sqrt{C(n)}^{-1} \| \) converges to the constant \( \gamma := \| \sqrt{C}^{-1} \| \), we can bound \( \sum_{n=1}^{N} | \beta_j | \) for all \( n \geq n_0 \) for sufficiently large \( n_0 \) by \((\gamma + \epsilon) b/\sqrt{n} \) with some fixed \( \epsilon \). Let \( Z_n \) be a Gaussian with covariance matrix \( C(n) \) and \( Z \) be a Gaussian with covariance matrix \( C \). Since the right hand side of (3) converges to zero, we have

\[
\sup_{B \in \mathbb{C}} | P\{ S(n) \in B \} - P\{ Z_n \in B \} | \rightarrow 0. \tag{4}
\]

Since \( P\{ Z_n \in B \} \) converges to \( P\{ Z \in B \} \) uniformly in \( B \) (this is because the mapping of the covariance matrix to its Gaussian density is continuous at \( C \) for the uniform norm topology on the mapping’s codomain) (4) remains true when \( Z_n \) is replaced with \( Z \). Hence, \( S(n) \) converges in distribution to \( Z \). \( \square \)

We now combine Lemma 3 and Lemma 1 and obtain:

**Lemma 4 (independence and Gaussianity of frequencies)** Given \( k \) frequencies \( \nu_1, \nu_2, \ldots, \nu_k \in [-1/2, 1/2] \setminus \{0\} \) with \( |\nu_j| \neq |\nu_j'| \) for \( j \neq j' \) and assume \( E[|A_j|^3] \) to be finite. Then the sequence of random vectors

\[
(\tilde{A}_{n\nu_1}, \tilde{A}_{n\nu_2}, \ldots, \tilde{A}_{n\nu_k}) \in \mathbb{C}^k
\]

covers in distribution to \((W_1, \ldots, W_k)\) where \( W_i \) are i.i.d. random variables with distribution \( G \) as in Theorem 1.

The proof is immediate after representing real and imaginary parts of each \( \tilde{A}_{n\nu} \) by an \( \mathbb{R}^2 \)-valued random variable as in Lemma 1 and applying Lemma 3 to the random vector in \( \mathbb{R}^{2k} \). A uniform bound for \( E[|S_j|^3] \) follows easily from finiteness of \( E[|A_j|^3] \).

The fact that different \( \tilde{A}_{n\nu} \) are asymptotically independent and identically distributed for different \( \nu \) has a very intuitive consequence: computing the Fourier sum \( \hat{a}_n(\nu) \) for different frequencies and one fixed instance \( a = a_1, a_2, \ldots \) resembles the distribution of \( \tilde{A}_{n\nu} \) for fixed \( \nu \). This suggests that the distribution of \( \tilde{A}_{n\nu}(\omega) \) with \( \nu \) uniformly chosen from \([-1/2, 1/2]\) and fixed \( \omega \) yields asymptotically also a Gaussian. To formally phrase this idea we first need the following result:

**Lemma 5 (distribution over a path)** Let \( (\tau, \Sigma_{\tau}, P_{\tau}) \) denote a probability space and for each \( t \in \tau \), let \( (X^n_t) \) be a sequence of random vectors on the probability space \((\Omega, \Sigma, P_\Omega)\). Further, assume that the map

\[
X^n(\omega) : t \mapsto X^n_t(\omega)
\]

is \( \Sigma_{\tau} \)-measurable for all \( n \in \mathbb{N} \) and \( \omega \in \Omega \) and thus defines a random variable on \((\tau, \Sigma_{\tau}, P_{\tau})\) whose distribution we denote by \( P_{X^n(\omega)} \).

For every \( k \in \mathbb{N} \) and \( P_{\Omega}^k \)-almost all \( k \)-tuples \((t_1, \ldots, t_k)\) let the sequence of random vectors \((X^n_{t_1}, \ldots, X^n_{t_k})\) converge in distribution to \((Z_1, \ldots, Z_k)\) for \( n \rightarrow \infty \) where \( Z_1, \ldots, Z_k \) are i.i.d. random variables on \((\Omega', \Sigma', P_{\Omega'})\) with distribution \( P_Z \).

Then the distance \( d(X^n(\omega), P_Z) \) converges in probability to zero for any well-behaved pseudometric \( d \). More precisely, the random variable

\[
\omega \mapsto d(X^n(\omega), P_Z)
\]

on \((\Omega, \Sigma, P_\Omega)\) converges to zero in probability.

Proof: We have to show that for every \( \epsilon, \delta > 0 \) there is an \( n_0 \) such that

\[
P_\Omega \{ \omega \mid d(P_{X^n(\omega)}, P_Z) \geq \epsilon \} \leq \delta,
\]
for all \( n \geq n_0 \). For any \( k \in \mathbb{N} \), let \( t_1, \ldots, t_k \) be i.i.d. drawn from \( P_\tau \). Since \( d \) is well-behaved, the distance between \( \hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega) \) and \( P_X^n(\omega) \) converges to zero in probability uniformly in \( n \). Thus, we can choose \( k \) such that for all \( n \)

\[
d(P_X^n(\omega), \hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega)) \leq \epsilon/2
\]

holds with probability at least \( 1 - \delta/3 \), and that, at the same time,

\[
d(\hat{P}_{Z_1, \ldots, Z_k}, P_Z) \leq \epsilon/2
\]

also holds with probability at least \( 1 - \delta/3 \). Using the triangle inequality for \( d \) we obtain

\[
d(P_X^n(\omega), P_Z) \leq d(P_X^n(\omega), \hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega)) + d(\hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega), P_Z).
\]

The sequence of random vectors \( (X_1^n, \ldots, X_k^n) \) converge in distribution to \( (Z_1, \ldots, Z_k) \) for each fixed \( k \)-tuple \((t_1, \ldots, t_k)\). In other words, for any measurable set \( B \) in \( \mathbb{R}^k \) we have

\[
\lim_{n \to \infty} P_{\Omega} \{ \omega \mid (X_1^n(\omega), \ldots, X_k^n(\omega)) \in B \} = P_{\Omega'} \{ \omega' \mid (Z_1(\omega'), \ldots, Z_k(\omega')) \in B \}.
\]

Setting

\[
B := \{ z_1, \ldots, z_k \mid d(\hat{P}_{z_1, \ldots, z_k}, P_Z) \leq \epsilon/2 \},
\]

we conclude from (8) that we can find an \( n_0 \) such that

\[
P_{\Omega} \{ \omega \mid d(\hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega), P_Z) \leq \epsilon/2 \} \geq P_{\Omega'} \{ \omega' \mid d(\hat{P}_{Z_1, \ldots, Z_k}(\omega'), P_Z) \leq \epsilon/2 \} - \delta/3,
\]

for all \( n \geq n_0 \). Using (9) and (6) we can thus choose \( n_0 \) such that for all \( n \geq n_0 \)

\[
d(\hat{P}_{X_1^n}(\omega), \ldots, X_k^n(\omega), P_Z) \leq \epsilon/2
\]

with probability at most \( 1 - 2\delta/3 \). Combining (10) with (5) we have thus ensured that the right hand side of (7) is smaller than \( \epsilon \) with probability at least \( 1 - \delta \). \( \square \)

We are now able to prove Theorem 1 via Lemma 5. To this end, let \((\tau, \Sigma_\tau, P_\tau) = ([-1/2, 1/2], B, \lambda)\) and set

\[
X^n(\omega) := (\text{Re}\hat{A}^n(\omega), \text{Im}\hat{A}^n(\omega)),
\]

where \( \text{Re} \) and \( \text{Im} \) denote real and imaginary part, respectively. If we then draw \( d \) frequencies \( \nu_1, \ldots, \nu_d \) for arbitrarily large \( d \), we satisfy \( P^k \) - almost surely the condition \( 0 \neq |\nu_j| \neq |\nu_j'| \neq 0 \) required by Lemma 4. Thus, the sequence of random vectors \( (\hat{A}^n_1, \ldots, \hat{A}^n_d) \) converges in distribution to \((W_1, \ldots, W_k)\) where \( W_j \) are distributed according to \( G \). Therefore, the random variable

\[
\omega \mapsto d(P_{\hat{A}^n(\omega)}, G)
\]

converges to zero in probability due to Lemma 5.

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