Matrix Kendall’s tau in High-dimensions:

A Robust Statistic for Matrix Factor Model

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In this article, we first propose generalized row/column matrix Kendall’s tau for matrix-variate observations that are ubiquitous in areas such as finance and medical imaging. For a random matrix following a matrix-variate elliptically contoured distribution, we show that the eigenspaces of the proposed row/column matrix Kendall’s tau coincide with those of the row/column scatter matrix respectively, with the same descending order of the eigenvalues. We perform eigenvalue decomposition to the generalized row/column matrix Kendall’s tau for recovering the loading spaces of the matrix factor model. We also propose to estimate the pair of the factor numbers by exploiting the eigenvalue-ratios of the row/column matrix Kendall’s tau. Theoretically, we derive the convergence rates of the estimators for loading spaces, factor scores and common components, and prove the consistency of the estimators for the factor numbers without any moment constraints on the idiosyncratic errors. Thorough simulation studies are conducted to show the higher degree of robustness of the proposed estimators over the existing ones. Analysis of a financial dataset of asset returns and a medical imaging dataset associated with COVID-19 illustrate the empirical usefulness of the proposed method.

Keyword: Elliptical distribution; Matrix Factor Model; Kendall’s tau; Principle Component Analysis.

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1 Introduction

The spatial/multivariate Kendall’s tau, as a powerful nonparametric/robust tool for analyzing random vectors, is first introduced in Choi and Marden (1998) for testing independence, and henceforth has been used for estimating covariance matrices, principal components and factor models in both low and high-dimensions. However, it remains unclear how to extend such a tool to analyze matrix-variate data, which are now ubiquitous in areas such as finance, macroeconomics and biology. The naive vectorization technique ignores the intrinsic matrix structure, and thus is less efficient for analyzing matrix-variate data. In the current work, we introduce a new concept, named matrix Kendall’s tau, for analyzing random matrices, and use this tool to estimate the loading spaces of the matrix elliptical factor model introduced in Li et al. (2022). Although we focus on the application of this new tool to factor models, the proposed matrix Kendall’s tau has wide applications for analyzing matrix-variate data including but not limited to performing 2-Dimensional Principal Component Analysis (PCA) and estimating separable covariance matrices robustly (Yu et al., 2021; Ke et al., 2019).

1.1 Literature Review on Related Works

Multivariate rank-based statistics are widely discussed in robust statistics, see for example Tyler (1987); Oja (2010); Hallin and Paindaveine (2002); Hallin et al. (2010); Han and Liu (2014); Zhou et al. (2019), among many others. Among various multivariate rank-based statistics, the multivariate Kendall’ tau is attractive due to its robustness property in dealing with elliptically contoured random vectors as Marden (1999) and Croux et al. (2002) showed that the population multivariate Kendall’s tau shares the same eigenspace as the scatter/covariance matrix. The elliptical family provides more flexibility in modeling modern complex data than the Gaussian family in terms of capturing heavy-tail property and nontrivial tail dependence between variables (variables tend to go to extremes together), which are important in areas such as finance and macroeconomics. It is well known that financial data such as portfolio/asset returns are heavy-tailed with nontrivial tail dependence. With the aid of multivariate Kendall’s tau, Taskinen et al. (2012) characterized the robustness and efficiency properties of Elliptical Component Analysis (ECA) in low dimensions, and
Han and Liu (2018) proposed ECA procedures for both non-sparse and sparse settings in high dimensions. Moreover, Fan et al. (2018) proposed Principal Orthogonal complEment Thresholding (POET) for large-scale covariance matrix estimation based on the approximate elliptical factor model, and Yu et al. (2019) studied the estimation of factor number for large-dimensional elliptical factor model. More recently, He et al. (2022) proposed a Robust Two-Step (RTS) procedure for estimating loading and factor spaces.

Matrix-variate data arise when one observes a group of variables structured in a well defined matrix form, and have been frequently observed in various research areas such as finance, signal processing and medical imaging. The naive vectorization technique vectorizes the matrix observations into long vectors and thus ignores the intrinsic matrix structure. As a result, it may lead to inefficient statistical inference and also suffers from heavy computational burden. Modelling matrix-valued data by Matrix-elliptical family not only enjoys the flexibility in modeling heavy-tail property and tail dependencies of matrix-variate data, but also maintains the inherent matrix structures. A natural question arises: can we define a similar Kendall’s tau matrix for analyzing random elliptical matrices, parallel to the multivariate Kendall’s tau for elliptical vectors? In this work, we give an affirmative answer to this question by proposing a new type of Kendall’s tau, named “matrix Kendall’s tau”. In detail, assume \( \mathbf{X}_{p \times q} \) is a random matrix and \( \tilde{\mathbf{X}}_{p \times q} \) is an independent copy of \( \mathbf{X}_{p \times q} \). We define the row/column matrix Kendall’s tau \( K_r, K_c \) as

\[
K_r = \mathbb{E} \left( \frac{(\mathbf{X} - \tilde{\mathbf{X}})(\mathbf{X} - \tilde{\mathbf{X}})^\top}{\|\mathbf{X} - \tilde{\mathbf{X}}\|_F^2} \right), \quad K_c = \mathbb{E} \left( \frac{(\mathbf{X} - \tilde{\mathbf{X}})^\top(\mathbf{X} - \tilde{\mathbf{X}})}{\|\mathbf{X} - \tilde{\mathbf{X}}\|_F^2} \right).
\]

The “matrix Kendall’s tau” can be viewed as a generalization of multivariate Kendall’ tau to the random matrix setting. When \( q = 1 \), \( K_r \) is reduced to the conventional multivariate Kendall’ tau proposed by Choi and Marden (1998). The matrix Kendall’s tau is particularly suitable for analyzing matrix-elliptical distributions. Given a random matrix \( \mathbf{X} \sim E_{p,q}(\mathbf{M}, \Sigma \otimes \Omega, \psi) \) (see Definition 2.1), we show that \( K_r \) and \( \Sigma \) share the same eigenspace while \( K_c \) and \( \Omega \) share the same eigenspace, with the same descending order of the eigenvalues.

A closely related line of research is on large-dimensional factor model, which is a powerful tool for
summarizing and extracting information from large datasets and draws growing attention in the “big-data” era. In the context of large-dimensional approximate vector factor models, we refer to Bai and Ng (2002), Stock and Watson (2002), Onatski (2009), Ahn and Horenstein (2013), Fan et al. (2013), Trapani (2018), Barigozzi et al. (2018), Barigozzi and Hallin (2020), Aït-Sahalia and Xiu (2017), Yu et al. (2019), Aït-Sahalia et al. (2020), Chen et al. (2021) and He et al. (2022). Although (approximate) vector factor models have been extensively studied, they are inadequate to model matrix-variate observations that have drawn growing attention in recent years. Wang et al. (2019) first proposed the following matrix factor model for matrix series \( \{X_t, 1 \leq t \leq T\} \):

\[
X_t \overset{p_1 \times p_2}{\sim} P_{p_1 \times k_1} \times P_{k_1 \times k_2} \times C^{T}_{k_2 \times p_2} + E_t \overset{p_1 \times p_2}{\sim} E_t,
\]

(1.1)

where \( R \) is the row factor loading matrix exploiting the variations of \( X_t \) across the rows, \( C \) is the \( p_2 \times k_2 \) column factor loading matrix reflecting the differences across the columns of \( X_t \), \( F_t \) is the common factor matrix for all cells in \( X_t \), and \( E_t \) is the idiosyncratic component. This model along with its variant has been studied by Chen et al. (2020b), Chen and Chen (2020), Gao et al. (2021), Chen and Fan (2021), He et al. (2021a), He et al. (2021c), Yu et al. (2022). Li et al. (2022) proposed a Matrix Elliptical Factor Model (MEFM), assuming that \( \{(\text{Vec}(F_t)^T, \text{Vec}(E_t)^T)^T, t = 1, \ldots, T\} \) in (1.1) follow some vector elliptical distribution. They further proposed a Manifold Principal Component Analysis (MPCA) method to estimate the loading spaces. However, the MPCA method would result in loading space estimators with a slow \( \sqrt{T} \) convergence rate compared to the typical convergence rate of \( \sqrt{Tp_1} \) (or \( \sqrt{Tp_2} \)) for matrix factor model (Chen et al., 2020a; Yu et al., 2022). For matrix factor analysis, it is also critical to determine the pair of factor numbers \( (k_1, k_2) \). Chen et al. (2020a) proposed an \( \alpha \)-PCA based eigenvalue-ratio method; Yu et al. (2022) proposed a projection-based iterative eigenvalue-ratio method; He et al. (2021b) proposed a robust iterative eigenvalue-ratio method to estimate the numbers of factors using the Huber loss (Huber, 1964). All these works are based on the eigenvalue ratio idea from Ahn and Horenstein (2013) with an exception of He et al. (2021a) who proposed to determine the factor numbers from the perspective of sequential hypothesis testing.
1.2 Contributions and Paper Organization

The contributions of the current work lie in the following aspects. Firstly, we propose a new type of Kendall’s tau, named matrix Kendall’s tau, that generalizes the multivariate Kendall’ tau to the random matrix setting. We show that $K_r$ and $\Sigma$ share the same eigenspace while $K_c$ and $\Omega$ share the same eigenspace, with the same descending order of the eigenvalues. The sample version of matrix Kendall’s tau is a U-statistic with a bounded kernel under operator norm and enjoys the same distribution-free property as multivariate Kendall’s tau, which can also be directly extended to analyze high dimensional data. Secondly, we apply this new type of Kendall’s tau to the Matrix Elliptical Factor Model (MEFM), and propose a Matrix-type Robust Two Step (MRTS) method to estimate the loading and factor spaces under MEFM. The proposed estimator is “unbiased” and achieves faster convergence rates ($\sqrt{T_{p_1}}$ or $\sqrt{T_{p_2}}$) than Manifold Principal Component Analysis (MPCA) for estimating the loading spaces. As an illustration, we check the empirical performances of some state-of-the-art methods to the heavy-tailedness of the factor and idiosyncratic errors with a synthetic dataset. Figure 1 shows that the proposed MRTS method performs comparably with the $\alpha$-PCA method by Chen and Fan (2021), PE method by Yu et al. (2022) and RMFA by He et al. (2021b), and outperforms the MPCA by Li et al. (2022). As the tails become heavier, the performance of $\alpha$-PCA and PE deteriorates quickly, while MRTS, RMFA and MPCA exhibit robustness to different extents, and RMTS always performs the best in heavy-tailed cases. Thirdly, we provide consistent estimators of the pair of factor numbers without any moment constraints on the underlying distribution. At last, we point out that the matrix Kendall’s tau has wide applications to analyze matrix-variate data in addition to the MEFM discussed here, such as robust estimation of separable covariance matrix or robust 2-dimensional PCA.

The rest of the article proceeds as follows. In Section 2, we introduce a new type of Kendall’s tau and briefly discuss its properties along with its sample version. We then introduce a Matrix-type Robust Two Step (MRTS) procedure for the MEFM with the aid of matrix Kendall’s tau, parallel to the RTS procedure for vector elliptical factor model with the aid of multivariate Kendall’s tau (He et al., 2022). We further
propose robust estimators of the pair of factor numbers by exploiting the eigenvalue-ratios of the sample matrix Kendall’s tau. In Section 3, we investigate the theoretical properties of the proposed estimators for MEFM. In Section 4, thorough simulation studies are conducted to illustrate the advantages of the proposed estimators over the state-of-the-art methods. In Section 5, we analyze a financial dataset and a medical imaging dataset associated with COVID-19 to illustrate the empirical performance/usefulness of the proposed methods. We discuss possible future research directions and conclude the article in Section 6. The proofs of the main theorems and additional details are collected in the supplementary materials.

Notations adopted throughout the paper are as follows. For any vector $\mathbf{\mu} = (\mu_1, \ldots, \mu_p)^\top \in \mathbb{R}^p$, let $\|\mathbf{\mu}\|_2 = (\sum_{i=1}^{p} \mu_i^2)^{1/2}$, $\|\mathbf{\mu}\|_\infty = \max_i |\mu_i|$. For a real number $a$, denote $\lfloor a \rfloor$ as the largest integer smaller than or equal to $a$, let $\text{sgn}(a) = 1$ if $a \geq 0$ and $\text{sgn}(a) = -1$ if $a < 0$. Let $I(\cdot)$ be the indicator function. Let $\text{diag}(a_1, \ldots, a_p)$ be a $p \times p$ diagonal matrix, whose diagonal entries are $a_1, \ldots, a_p$. For a matrix $A$, let $A_{ij}$ (or $A_{i,j}$) be the $(i,j)$-th entry of $A$, $A_i$ be the $i$-th row and $A_{j,\cdot}$ be the $j$-th column, $A^\top$ the transpose of $A$, $\text{Tr}(A)$ the trace of $A$, $\text{rank}(A)$ the rank of $A$ and $\text{diag}(A)$ the diagonal matrix composed of the diagonal elements of $A$. $A \succcurlyeq 0$ means $A$ is a non-negative definite matrix. Denote $\lambda_j(A)$ as the $j$-th largest eigenvalue of a nonnegative definitive matrix $A$, and let $\|A\|$ be the spectral norm of matrix $A$ and $\|A\|_F$ be the Frobenius norm of $A$. For matrices $A$ and $B$, let $\text{diag}(A,B)$ be a block diagonal matrix with diagonal
matrices $A$ and $B$. For two series of random variables, $X_n$ and $Y_n$, $X_n \asymp Y_n$ means $X_n = O_p(Y_n)$ and $Y_n = O_p(X_n)$. For two random variables (vectors) $X$ and $Y$, $X \overset{d}{=} Y$ means the distributions of $X$ and $Y$ are the same. The constants $c, C_1, C_2$ in different lines can be nonidentical.

## 2 Methodology

In this section, we introduce the generalized Kendall’s tau, i.e., the Matrix Kendall’s tau and then propose a Matrix-type Robust Two Step (MRTS) procedure for estimating row/column loading and factor spaces for the Matrix Elliptical Factor Model (MEFM).

### 2.1 Matrix Kendall’s tau

Prior to introducing the Matrix Kendall’s tau, we first briefly review the definition of Matrix Elliptical Distribution (MED). We refer to Gupta and Nagar (2018) for further details.

**Definition 2.1.** A random matrix $Y$ of size $p \times q$ follows the matrix elliptical distribution if its characteristic function has the form $\varphi_Y(T) = \exp\{\text{Tr}(iT^\top M)\psi[\text{Tr}(T^\top \Sigma T \Omega)]\}$ with $T: p \times q$, $M: p \times q$, $\Sigma: p \times p$, $\Omega: q \times q$, $\Sigma \succeq 0, \Omega \succeq 0$ and $\psi: [0, \infty) \to \mathbb{R}$, briefly denoted as $Y \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi)$. Assume $\text{rank}(\Sigma) = m$, $\text{rank}(\Omega) = n$, the random matrix $Y \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi)$ if and only if

$$Y \overset{d}{=} rAUB^\top + M,$$

where $M$ is a deterministic location matrix of size $p \times q$, $U$ is a random matrix of dimension $m \times n$ and $\text{Vec}(U)$ is uniformly distributed on the unit sphere in $\mathbb{R}^{mn}$, $r$ is a nonnegative random variable independent of $U$, $\Sigma = AA^\top$ and $\Omega = BB^\top$ are rank factorizations of $\Sigma$ and $\Omega$.

Throughout the paper, we call $\Sigma$ as row scatter matrix and $\Omega$ as column scatter matrix. We first introduce the matrix Kendall’s tau for random matrices, which generalizes the multivariate Kendall’s tau for random vectors by Marden (1999).
Definition 2.2. Let $Y \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi)$ be a continuous random matrix, and $\tilde{Y}$ is an independent copy, the row/column matrix Kendall’s tau are defined as:

$$K_r = \mathbb{E} \left( \frac{(Y - \tilde{Y})(Y - \tilde{Y})^\top}{\|Y - \tilde{Y}\|_F^2} \right), \quad K_c = \mathbb{E} \left( \frac{(Y - \tilde{Y})^\top (Y - \tilde{Y})}{\|Y - \tilde{Y}\|_F^2} \right). \quad (2.1)$$

Both $K_r$ and $K_c$ are positive semidefinite (PSD) matrices of trace 1. The following proposition illustrates the connections between $K_r$ ($K_c$) and $\Sigma$ ($\Omega$).

Proposition 2.1. Let $Y \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi)$ be a continuous random matrix, $K_r$ be the row matrix multivariate Kendall’s tau. Then if rank($\Sigma$) = $m$, we have

$$\lambda_j(K_r) = \mathbb{E} \left( \frac{\lambda_j(\Sigma)U_j, \Omega^*U_j^\top}{\lambda_1(\Sigma)U_1, \Omega^*U_1^\top \cdots \lambda_m(\Sigma)U_m, \Omega^*U_m^\top} \right),$$

where $\Omega^* = B^\top B$. In addition, $K_r$ and $\Sigma$ share the same eigenspace with the same descending order of the eigenvalues. Furthermore, let $K_c$ be the column matrix Kendall’s tau, if rank($\Omega$) = $n$, we have

$$\lambda_j(K_c) = \mathbb{E} \left( \frac{\lambda_j(\Omega)U_{j,1}^\top \Sigma^*U_{j,1} + \cdots + \lambda_n(\Omega)U_{n,1}^\top \Sigma^*U_{n,1}}{\lambda_1(\Omega)U_{1,1}^\top + \cdots + \lambda_n(\Omega)U_{n,1}^\top} \right),$$

where $\Sigma^* = A^\top A$. In addition, $K_c$ and $\Omega$ share the same eigenspace with the same descending order of the eigenvalues. Without loss of generality, we assume that $\Omega^*$ and $\Sigma^*$ are both diagonal matrices. Otherwise, we can always find orthogonal matrices $P$ and $Q$ such that $B^* = BP$ and $A^* = AQ$ satisfy $B^*^\top B^*$ and $A^*^\top A^*$ are diagonal, and $Y \overset{d}{=} rA^*(Q^\top UP)B^*^\top + M \overset{d}{=} rA^*UB^*^\top + M$.

Proposition 2.1 shows that, to recover the eigenspace of the covariance matrix $\Sigma$ ($\Omega$), we can resort to recovering the eigenspace of $K_r$ ($K_c$), which, as will be discussed in the following, can be efficiently estimated by U-statistics.

Let $Y_1, \ldots, Y_T \in \mathbb{R}^{p \times q}$ be $T$ independent data points of a random matrix $Y \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi)$, and the definition of the matrix Kendall’s tau in (2.1) motivates one to consider the following sample matrix
Kendall’s tau estimator, which are second-order matrix U-statistics:

\[
\hat{K}_r = \frac{2}{T(T-1)} \sum_{t<t'} (Y_t - Y_{t'}) (Y_t - Y_{t'})^T, \quad \hat{K}_c = \frac{2}{T(T-1)} \sum_{t<t'} (Y_t - Y_{t'})^T (Y_t - Y_{t'}) / \|Y_t - Y_{t'}\|_F^2.
\]

It can be easily derived that \(E(\hat{K}_r) = K_r\) and \(E(\hat{K}_c) = K_c\) and both \(\hat{K}_r\) and \(\hat{K}_c\) are positive semidefinite (PSD) matrices of trace 1. The kernels of the U-statistics \(k_{MK}^r(\cdot) : \mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{p \times p}\) and \(k_{MK}^c(\cdot) : \mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{q \times q}\), defined as

\[
k_{MK}^r(Y_t, Y_{t'}) = \frac{(Y_t - Y_{t'}) (Y_t - Y_{t'})^T}{\|Y_t - Y_{t'}\|_F^2}, \quad k_{MK}^c(Y_t, Y_{t'}) = \frac{(Y_t - Y_{t'})^T (Y_t - Y_{t'})}{\|Y_t - Y_{t'}\|_F^2},
\]

are bounded under the spectral norm, i.e., \(\|k_{MK}^r(Y_t, Y_{t'})\| \leq 1, \|k_{MK}^c(Y_t, Y_{t'})\| \leq 1\). Intuitively, such a boundedness property makes the U-statistics \(\hat{K}_r\) and \(\hat{K}_c\) more robust to heavy-tailed distributions. We also prove that the kernels \(k_{MK}^r(Y_t, Y_{t'})\), \(k_{MK}^c(Y_t, Y_{t'})\) are distribution-free kernel, i.e., for any continuous \(Y \sim \mathcal{E}_{p,q}(M, \Sigma \otimes \Omega, \psi)\),

\[
k_{MK}^r(Y_t, Y_{t'}) \overset{d}{=} k_{MK}^r(Z_t, Z_{t'}), \quad k_{MK}^c(Y_t, Y_{t'}) \overset{d}{=} k_{MK}^c(Z_t, Z_{t'}),
\]

where \(Z_t, Z_{t'}\) follow matrix normal distribution \(Z \sim \mathcal{MN}(0, \Sigma, \Omega)\) (check Lemma C.2 for details). Therefore, \(\hat{K}_r\) and \(\hat{K}_c\) enjoy the same distribution-free property as Tyler’s M-estimator (Tyler, 1987) and multivariate Kendall’s tau (Marden, 1999). However, the matrix Kendall’s tau can be directly extended to analyze high dimensional data as multivariate Kendall’s tau, while Tyler’s M-estimator cannot (Han and Liu, 2018).

### 2.2 Matrix Elliptical Factor Model

For \(p_1 \times p_2\) matrix sequences \(\{X_t, t = 1, \ldots, T\}\), the matrix factor model is as follows:

\[
X_t = RF_t C^T + E_t, t = 1, \ldots, T,
\]
where \( R \) is the \( p_1 \times k_1 \) row factor loading matrix, \( C \) is the \( p_2 \times k_2 \) column factor loading matrix, \( F_t \) is the common factor matrix and \( E_t \) is the idiosyncratic components. In the definition of MEFM, we assume the factor matrix \( F_t \) and noise matrix \( E_t \) are from a joint matrix elliptical distribution with location parameter zero, that is,

\[
\begin{pmatrix}
\text{Vec}(F_t) \\
\text{Vec}(E_t)
\end{pmatrix} = r_t \begin{pmatrix}
I_{k_2} \otimes I_{k_1} & 0 \\
0 & \Omega^{1/2} \otimes \Sigma^{1/2}_e
\end{pmatrix} \frac{Z_t}{\|Z_t\|_2},
\]

where \( Z_t \) is a \((k_1k_2 + p_1p_2)\)-dimensional isotropic Gaussian vector, and \( r_t \) is a positive random variable independent of \( Z_t \). In fact, \( F_t \) and \( E_t \) are both matrix elliptically distributed, that is,

\[
F_t = \frac{r_t}{\|Z_t\|_2}Z_t^F \sim E_{k_1,k_2}(0, I_{k_1} \otimes I_{k_2}, \psi_F), \\
E_t = \frac{r_t}{\|Z_t\|_2}Z_t^E \Omega^{1/2}_e \sim E_{p_1,p_2}(0, \Sigma_e \otimes \Omega_e, \psi_E),
\]

where \( \Sigma_e \) of size \( p_1 \times p_1 \) and \( \Omega_e \) of size \( p_2 \times p_2 \) are positive-definite matrices , \( Z_t^F \) is a \( k_1 \times k_2 \) random matrix by stacking the first \( k_1 \times k_2 \) elements of \( Z_t \), and \( Z_t^E \) is of size \( p_1 \times p_2 \) by stacking all the elements left.

Under the above assumption, and by the property of the elliptical vectors, we have that the scatter matrix of \( \text{Vec}(X_t) \), denoted as \( \Sigma_{\text{Vec}(X_t)} \), has the form

\[
\Sigma_{\text{Vec}(X_t)} = (RR^T) \otimes (CC^T) + \Sigma_e \otimes \Omega_e,
\]

or equivalently, \( \Sigma_{\text{Vec}(X_t)} \) has a low-rank matrix plus sparse matrix structure. Neglecting the sparse term \((\Sigma_e \otimes \Omega_e)\), one would find that \( X \sim E_{p_1,p_2}(0, (RR^T) \otimes (CC^T), \psi_X) \), which motivates one to estimate \( R \) and \( C \) by the leading eigenvectors of the row/column matrix Kendall’s tau, as Proposition 2.1 shows that the row/column matrix Kendall’s tau matrix shares the same eigenspace with the row/column scatter matrix with the same descending order of the eigenvalues.
2.3 Matrix-type Robust Two step Procedure for MEFM

In this section, we introduce a Matrix-type Robust Two step (MRTS) procedure for fitting large-dimensional matrix elliptical factor model. In the first step, we propose to estimate $R$ and $C$ by the eigenvectors of the row/column matrix Kendall’s tau. Let $\hat{K}_r^X$ and $\hat{K}_c^X$ be the sample matrix Kendall’s tau based on the observations $\{X_t, t = 1, \ldots, T\}$, i.e.,

$$
\hat{K}_r^X = \frac{2}{T(T-1)} \sum_{t < t'} (X_t - X_{t'})^\top (X_t - X_{t'}) / \|X_t - X_{t'}\|_F^2,
\hat{K}_c^X = \frac{2}{T(T-1)} \sum_{t < t'} (X_t - X_{t'})^\top (X_t - X_{t'}) / \|X_t - X_{t'}\|_F^2.
$$

As the eigenvectors of the matrix Kendall’s tau matrix $K_r^X (K_c^X)$ are identical to the eigenvectors of the scatter matrix $\Sigma (\Omega)$, we estimate the factor loading matrix $R$ ($C$) by $\sqrt{p_1} (\sqrt{p_2})$ times the leading $k_1 (k_2)$ eigenvectors of $\hat{K}_r^X (\hat{K}_c^X)$. In detail, let $\{\hat{r}_1, \ldots, \hat{r}_{k_1}\}$ be the leading $k_1$ eigenvectors of $\hat{K}_r^X$, $\{\hat{c}_1, \ldots, \hat{c}_{k_2}\}$ be the leading $k_2$ eigenvectors of $\hat{K}_c^X$, We set $\hat{R} = \sqrt{p_1}(\hat{r}_1, \ldots, \hat{r}_{k_1})$ and $\hat{C} = \sqrt{p_2}(\hat{c}_1, \ldots, \hat{c}_{k_2})$ as the estimators of the factor loading matrices $R$ and $C$ respectively. The numbers of factors $(k_1, k_2)$ are relatively small compared with $p_1, p_2$ and $T$. We first assume that $(k_1, k_2)$ are known and fixed. In section 2.4, we provide consistent estimators for $(k_1, k_2)$.

In the second step, we estimate the factor matrices $\{F_1, \ldots, F_T\}$ by regressing $\text{Vec}(X_t)$ on $(\hat{C} \otimes \hat{R})$. In detail, $\text{Vec}(F_t)$ is estimated by the following least-squares-type estimator

$$
\text{Vec}(\hat{F}_t) = \arg \min_{\beta \in \mathbb{R}_{k_1,k_2}} \|\text{Vec}(X_t) - (\hat{C} \otimes \hat{R})\beta\|^2.
$$

In fact, it can be shown that by solving the above optimization problem would lead to $\hat{F}_t = \hat{R}^\top X_t \hat{C} / (p_1p_2)$.

An alternative approach is to extend the RTS procedure for vector elliptical factor model (He et al., 2022) by directly vectorizing matrix-variate observations. This, however, would ignore the intrinsic matrix structure, lead to inefficient statistical inference and bring in heavy computational burden. For estimating the loading spaces, applying the previous RTS procedure to vectorized data has a complexity of order $O(T^2p_1^2p_2^2)$, while the complexity of the proposed MRTS procedure is $O(T^2p_1p_2 \max(p_1, p_2))$. 

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2.4 Model Selection

Thus far we have assumed that the pair of factor numbers \((k_1, k_2)\) is known a priori. In practice, both both \(k_1\) and \(k_2\) are unknown and thus need be estimated before implementing the RMTS procedure. In this section, we introduce a robust method for determining the pair of factor numbers, which does not require any moment constraints and is of independent interest. Our “Matrix Kendall’s tau Eigenvalue-Ratio” (MKER) method is motivated by Ahn and Horenstein (2013) and Yu et al. (2019).

Given matrix observations \(\{X_1, \ldots, X_T\}\), let \(\hat{K}_r^X\) and \(\hat{K}_c^X\) be the sample matrix Kendall’s tau given in (2.3) with eigenvalues \(\lambda_j(\hat{K}_r^X), j = 1, \ldots, p_1\) and \(\lambda_j(\hat{K}_c^X), j = 1, \ldots, p_2\), respectively. For a prescribed maximum number of factors \(k_{\text{max}}\), we construct the Matrix Kendall’s tau Eigenvalue Ratio (“MKER”) estimators as

\[
\hat{k}_1 = \arg \max_{1 \leq j \leq k_{\text{max}}} \frac{\lambda_j(\hat{K}_r^X)}{\lambda_{j+1}(\hat{K}_r^X)}, \quad \hat{k}_2 = \arg \max_{1 \leq j \leq k_{\text{max}}} \frac{\lambda_j(\hat{K}_c^X)}{\lambda_{j+1}(\hat{K}_c^X)}.
\]  

(2.4)

To ensure that the denominators are away from zero, in practice one may add a positive but asymptotically negligible term to each \(\lambda_j(\hat{K}_r^X)\) and \(\lambda_j(\hat{K}_c^X)\). For instance, take \(\delta_1 = 1/\sqrt{\min(p_2, T^{1-\epsilon})}\), \(\delta_2 = 1/\sqrt{\min(p_1, T^{1-\epsilon})}\) for a small \(\epsilon > 0\) and let \(\hat{\lambda}_j(\hat{K}_r^X) = \lambda_j(\hat{K}_r^X) + c\delta_1, \hat{\lambda}_j(\hat{K}_c^X) = \lambda_j(\hat{K}_c^X) + c\delta_2\) with a small positive constant \(c\). We then replace \(\lambda_j(\hat{K}_r^X)\) and \(\lambda_j(\hat{K}_c^X)\) with \(\hat{\lambda}_j(\hat{K}_r^X)\) and \(\hat{\lambda}_j(\hat{K}_c^X)\) in (2.4), respectively.

3 Theoretical Analysis

In this section, we investigate the theoretical properties of the proposed estimators. We first formally introduce the matrix elliptical factor model along with some structural assumptions.

Assumption A (Joint Matrix Elliptical Model). We say the matrix-variate observations \(X_t \in \mathbb{R}^{p_1 \times p_2}\) follow a matrix elliptical factor model if

\[
X_t = RF_tC^T + E_t, \quad t = 1, \ldots, T,
\]

where \((\text{Vec}(F_t)^T, \text{Vec}(E_t)^T)^T\) satisfies (2.2), and \(Z_t\)’s therein are i.i.d. \((k_1k_2 + p_1p_2)\)-dimensional standard
multivariate Gaussian vectors, \( r_t \)'s are \( i.i.d. \) positive random scalars that are independent of \( Z_t \)'s. We consider the high-dimensional regime that \( p_1, p_2 \to \infty \) and \( \log(\max(p_1, p_2)) = o(T) \). In addition, we assume \( r_t = O_p(\sqrt{p_1 p_2}) \) as \( p_1, p_2 \to \infty \).

**Assumption B** (Strong Factor Conditions). We assume \( R^\top R/p_1 \to V_1 \) and \( C^\top C/p_2 \to V_2 \) for some positive definite matrices \( V_1 \) and \( V_2 \) as \( p_1, p_2 \to \infty \). There exist positive constants \( c_1, c_2 \) such that \( c_1 \leq \lambda_{k_1}(V_1) < \cdots < \lambda_1(V_1) \leq c_2 \), \( c_1 \leq \lambda_{k_2}(V_2) < \cdots < \lambda_1(V_2) \leq c_2 \).

**Assumption C** (Regular Noise Conditions). We assume there exist positive constants \( c_1 \) and \( C_1 \) such that \( c_1 \leq \lambda_{p_1}(\Sigma_e) \leq \lambda_1(\Sigma_e) \leq C_1 \), \( c_1 \leq \lambda_{p_2}(\Omega_e) \leq \lambda_1(\Omega_e) \leq C_1 \) as \( p_1, p_2 \to \infty \).

Assumption A is the joint matrix elliptical model assumption adopted in Li et al. (2022). The condition \( r_t = O_p(\sqrt{p_1 p_2}) \) essentially assumes that \( r_t = \sqrt{p_1 p_2} \times r^*_t \) for some random variable \( r^*_t \); see (2.2). For instance, assume \( (\text{Vec}(F_t)^\top, \text{Vec}(E_t)^\top)^\top \) follows a multivariate \( t \) distribution with degrees of freedom \( v \). Then we have \( r^*_t \overset{d}{=} \sqrt{F_{d,v}} \), where \( F_{d,v} \) stands for \( F \)-distribution with degrees of freedom \( d, v \) where \( d = (p_1 p_2 + k_1 k_2) \). It is worth noting that no moment constraint is exerted on \( r^*_t \) or \( r_t \), thus allowing for Cauchy-type distributions that do not even have finite means. This differentiates the current work from the literature on matrix factor models such as Chen and Fan (2021) and Yu et al. (2022). Assumptions B and C are standard in large-dimensional matrix factor analysis literature, see for example Chen and Fan (2021) and Yu et al. (2022). The eigenvalues of \( V_i, i = 1, 2 \) are assumed to be distinct such that the corresponding eigenvectors are identifiable. We assume strong factor conditions in Assumption B, indicating that the row and column factors are pervasive along both dimensions.

As our first main result, the following theorem provides the convergence rates of the factor loading matrix estimators \( \hat{R} \) and \( \hat{C} \) under both the Frobenious and spectral norms.

**Theorem 3.1.** Under Assumptions A-C with \( (k_1, k_2) \) fixed, and \( p_1, p_2, T \to \infty \), there exist matrices \( \hat{H}_R \) and \( \hat{H}_C \) such that \( \hat{H}_R^\top V_1 \hat{H}_R \to I_{k_1}, \hat{H}_C^\top V_2 \hat{H}_C \to I_{k_2} \) and

\[
\frac{1}{p_1} \| \hat{R} - R \hat{H}_R^\top \|_F^2 = O_p \left( \frac{1}{T p_2} + \frac{1}{p_1^2} \right), \quad \frac{1}{p_2} \| \hat{C} - C \hat{H}_C^\top \|_F^2 = O_p \left( \frac{1}{T p_1} + \frac{1}{p_2^2} \right).
\]
Moreover,

$$\frac{1}{p_1}\|\hat{R} - RH\|_F^2 = O_p\left(\frac{1}{T p_2^2} + \frac{1}{p_1^2}\right), \quad \frac{1}{p_2}\|\hat{C} - CH\|_F^2 = O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right).$$

Table 1: A comparison of convergence rates of loading matrices estimators under squared Frobenius norm. “IE” and “PE” denote the initial estimator and projection estimator by Yu et al. (2022), “α-PCA” denotes the estimator by Chen and Fan (2021), MPCA\_F denotes the estimator by Li et al. (2022).

| Working model | MRTS | IE | α-PCA | PE | MPCA\_F |
|---------------|------|----|-------|----|---------|
| MRTS          | Matrix Elliptical Factor Model | Matrix Factor Model | Matrix Factor Model | Matrix Factor Model | Matrix Elliptical Factor Model |
| $\frac{1}{p_1}\|\hat{R} - RH\|_F^2$ | $O_p\left(\frac{1}{T p_2^2} + \frac{1}{p_1^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ |
| $\frac{1}{p_2}\|\hat{C} - CH\|_F^2$ | $O_p\left(\frac{1}{T p_2^2} + \frac{1}{p_1^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ | $O_p\left(\frac{1}{T p_1^2} + \frac{1}{p_2^2}\right)$ |

Assume we vectorize the matrix observations by stacking the columns, model (2.2) degenerates to the vector elliptical factor model in He et al. (2022) with loading matrix $(C \otimes R)$. Following the theoretical analysis by He et al. (2022), the convergence for the loading matrix under squared Frobenius norm will be $O_p\left(T^{-1} + (p_1 p_2)^{-2}\right)$. Therefore, the matrix elliptical factor model will be more efficient as long as $T \ll \min\{p_1^2, p_2^2\}$, i.e., under “large dimension versus small sample size” regimes. Table 1 compares the convergence rates of loading matrices estimators under squared Frobenius norm. The derived rates for MRTS in Theorem 3.1 are much faster than those of the MPCA\_F method by Li et al. (2022). Indeed, the derived rates for MRTS match with those of the Initial Estimators (“IE” in Table 1) in Yu et al. (2022), which are slightly better than the convergence rates of α-PCA methods proposed by Chen and Fan (2021). The rates of the projected estimators (“PE” in Table 1) by Yu et al. (2022) are faster under light-tailed cases due to the projection technique. It is possible to further improve the convergence rates of MRTS using the projection technique proposed by Yu et al. (2022). This is beyond the scope of this work, and will be left for future research.
Theorems 3.2 and 3.3 below show consistency of the factor scores \( \hat{F}_t \) and the common components \( \hat{S}_t = \hat{R}_F \hat{C}^\top \), both under the Frobenius norm.

**Theorem 3.2.** Under Assumptions A-C with \((k_1, k_2)\) fixed, and \( \min\{T, p_1, p_2\} \to \infty \), we have

\[
\|\hat{F}_t - \hat{H}_R^{-1} \hat{F}_t \hat{H}_C^{-1\top}\|^2_F = O_p\left(\frac{1}{p_1 p_2}\right).
\]

**Theorem 3.3.** Under Assumptions A-C with \((k_1, k_2)\) fixed, and \( \min\{T, p_1, p_2\} \to \infty \), we have

\[
\frac{1}{p_1 p_2} \|\hat{S}_t - S_t\|^2_F = O_p\left(\frac{1}{p_1 p_2} + \frac{1}{T p_1} + \frac{1}{T p_2}\right),
\]

where \( S_t = R F_t C^\top \).

The convergence rates of factor scores and signal matrices are comparable to those in He et al. (2022), Yu et al. (2022) and Chen et al. (2020a). This is because that as long as the the loadings are estimated accurately, the estimation errors for \( F_t \) mainly depend on the idiosyncratic errors \( E_t \). Even if under the oracle case that the loading matrices are given, the estimation error is still of order \((p_1 p_2)^{-1}\) under the squared Frobenius norm. The estimation errors from \( \hat{F}_t \) will further affect the estimation accuracy for the signal matrices \( S_t \).

The following theorem shows the consistency of the MKER estimators introduced in Section 2.4.

**Theorem 3.4.** Under Assumptions A-C with \( k_1, k_2 \geq 1 \), we have \( P(\hat{k}_1 = k_1) \to 1 \) and \( P(\hat{k}_2 = k_2) \to 1 \) as \( \min\{T, p_1, p_2\} \to \infty \).

## 4 Simulation Study

### 4.1 Data Generation

We first describe the data generation mechanism of the synthetic dataset to assess the performance of the proposed Matrix-Robust-Two-Step (MRTS) method compared with some state-of-the-art methods.
We use similar data-generating models as in He et al. (2021b). We set \( k_1 = k_2 = 3 \), draw the entries of \( R \) and \( C \) independently from the uniform distribution \( U(-1,1) \), and let

\[
\begin{align*}
\text{Vec}(F_t) &= \phi \times \text{Vec}(F_{t-1}) + \sqrt{1-\phi^2} \times \text{Vec}(\epsilon_t), \\
\text{Vec}(E_t) &= \psi \times \text{Vec}(E_{t-1}) + \sqrt{1-\psi^2} \times \text{Vec}(U_t),
\end{align*}
\]

where \( \phi \) and \( \psi \) controls the temporal correlations, \( \text{Vec}(\epsilon_t) \) and \( \text{Vec}(U_t) \) are jointly generated from elliptical distributions. Before we give the data generating scenarios, we first review the matrix normal distribution and matrix \( t \)-distribution. In detail, when a random matrix \( U_t \) is from a matrix normal distribution \( MN(0, U_E, V_E) \), then \( \text{Vec}(U_t) \sim N(0, V_E \otimes U_E) \). If \( U_t \) is from a matrix \( t \)-distribution \( t_{p_1, p_2}(\nu, M, U_E, V_E) \), it has probability density function

\[
f(U_t; \nu, M, U_E, V_E) = K \times \left| I_{p_1} U^{-1}_E (U_t - M) V^{-1}_E (U_t - M)^\top \right|^{\frac{\nu + p_1 + p_2 - 1}{2}},
\]

where \( K \) is the regularization parameter. In our simulation study, we set \( M = 0 \) and let \( U_E \) and \( V_E \) be matrices with ones on the diagonal, and with \( 1/p_1 \) and \( 1/p_2 \) as off-diagonal entries, respectively.

### 4.2 Estimation error for loading spaces

In this section, we compare the MRTS method with the RMFA method by He et al. (2021b), the \( \alpha \)-PCA (\( \alpha = 0 \)) method by Chen and Fan (2021), the PE method by Yu et al. (2022) and the MPCA \( F \) by Li et al. (2022). We consider the following two scenarios:

**Scenario A**: \( \phi = 0 \) and \( \psi = 0 \), \( T \in \{20, 50, 100\}, p_1 = p_2 \in \{20, 50\}, (\text{Vec}(\epsilon_t)^\top, \text{Vec}(U_t)^\top)^\top \) are generated in the following ways: (i) \( (\text{Vec}(\epsilon_t)^\top, \text{Vec}(U_t)^\top)^\top \) are i.i.d. jointly Gaussian distributions \( \mathcal{N}(0, \text{diag}(I_{k_2} \otimes I_{k_1}, U_E \otimes V_E)) \); (ii) \( (\text{Vec}(\epsilon_t)^\top, \text{Vec}(U_t)^\top)^\top \) are i.i.d. jointly \( t \) distributions \( t_{\nu, \text{diag}(I_{k_2} \otimes I_{k_1}, U_E \otimes V_E)}(0, 1) \), \( \nu = 3, 2, 1 \).

**Scenario B**: \( \phi = 0.1 \) and \( \psi = 0.1 \), \( T \in \{20, 50\}, p_1 = p_2 \in \{20, 50\}, (\text{Vec}(\epsilon_t)^\top, \text{Vec}(U_t)^\top)^\top \) are generated in the same ways as in Scenario A.

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We adopt a metric between linear spaces which was also utilized in He et al. (2021b). For two column-wise orthogonal matrices \((Q_1)_{p \times q_1}\) and \((Q_2)_{p \times q_2}\), we define

\[
D(Q_1, Q_2) = \left( 1 - \frac{1}{\max(q_1, q_2)} \text{Tr}\left( Q_1 Q_1^\top Q_2 Q_2^\top \right) \right)^{1/2}.
\]

By the definition of \(D(Q_1, Q_2)\), we can easily deduce that its value lies in the interval \([0, 1]\), which measures the distance between the column spaces spanned by \(Q_1\) and \(Q_2\). The column spaces spanned by \(Q_1\) and \(Q_2\) are the same when \(D(Q_1, Q_2) = 0\), while are orthogonal when \(D(Q_1, Q_2) = 1\). The Gram-Schmidt orthonormalizing transformation can be applied when \(Q_1\) and \(Q_2\) are not column-orthogonal matrices.

Tables 2 and 3 show the averaged estimation errors of \(D(\hat{R}, R)\) with standard errors in parentheses for joint normal distribution and joint \(t\) distribution under Scenarios A and B, respectively. All methods benefit from large dimensions, and for the light-tailed normal distribution, the proposed MRTS method performs comparably with the \(\alpha\)-PCA method by Chen and Fan (2021), PE method by Yu et al. (2022) and RMFA by He et al. (2021b), while much better the MPCA by Li et al. (2022) especially in small \(T\) case due to faster convergence rate. What we want to emphasize is that the MRTS method shows great advantage over other methods under heavy-tailed \(t\) distributions. This shows that the MRTS performs robustly and much better when the data are heavy-tailed, and performs comparably with others when data are light-tailed. By comparing the results in Table 2 and Table 3, we can see that the proposed MRTS method is not sensitive to the weak temporal dependency of the factors and noises, although we need the independent condition for theoretical analysis.

### 4.3 Estimation error for common components

In this section, we compare the performances of the MRTS method with those of the RMFA method, the \(\alpha\)-PCA method, the PE method and the MPCA\(_F\) in terms of estimating the common component matrices.
Table 2: Averaged estimation errors and standard errors of $D(\hat{R}, R)$ for Scenario A under joint normal distribution and joint $t$ distribution over 500 replications.

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS | RMFA | $\alpha$-PCA | PE | MPCA$_F$ |
|------------|-----|-------|-------|------|------|--------------|----|----------|
| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1189(0.0363) | 0.0922(0.0162) | 0.1127(0.0352) | 0.0927(0.0166) | 0.1407(0.0223) |
| | 50  | 50    | 0.0622(0.0075) | 0.0569(0.0056) | 0.0596(0.0070) | 0.0568(0.0060) | 0.0985(0.0093) |
| $\mathcal{D}(\hat{R}, R)$ | 50  | 20    | 0.0811(0.0227) | 0.0574(0.0089) | 0.0774(0.0215) | 0.0577(0.0090) | 0.0868(0.0121) |
| | 50  | 50    | 0.0384(0.0041) | 0.0351(0.0032) | 0.0374(0.0039) | 0.0350(0.0032) | 0.0614(0.0050) |
| $\mathcal{D}(\hat{R}, R)$ | 100 | 20    | 0.0654(0.0208) | 0.0405(0.0065) | 0.0619(0.0202) | 0.0406(0.0067) | 0.0699(0.0088) |
| | 50  | 50    | 0.0279(0.0032) | 0.0247(0.0021) | 0.0271(0.0031) | 0.0246(0.0022) | 0.0430(0.0031) |

Normal Distribution

$t_1$ Distribution

| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1300(0.0357) | 0.0668(0.0080) | 0.2686(0.1428) | 0.4307(0.1770) | 0.4239(0.1893) | 0.1395(0.0213) |
| | 50  | 50    | 0.0689(0.0100) | 0.1336(0.1010) | 0.2870(0.1799) | 0.2817(0.1866) | 0.0992(0.0094) |
| $\mathcal{D}(\hat{R}, R)$ | 50  | 20    | 0.0880(0.0249) | 0.1881(0.1446) | 0.4173(0.1926) | 0.4041(0.2075) | 0.0861(0.0117) |
| | 50  | 50    | 0.0427(0.0045) | 0.1068(0.0814) | 0.2672(0.1749) | 0.2642(0.1828) | 0.0611(0.0047) |
| $\mathcal{D}(\hat{R}, R)$ | 100 | 20    | 0.0693(0.0239) | 0.1781(0.1475) | 0.4319(0.1912) | 0.4251(0.2056) | 0.0609(0.0088) |
| | 50  | 50    | 0.0309(0.0037) | 0.0946(0.0846) | 0.2656(0.1723) | 0.2627(0.1797) | 0.0430(0.0033) |

$t_2$ Distribution

| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1257(0.0349) | 0.1277(0.0517) | 0.2645(0.1507) | 0.2429(0.1592) | 0.1386(0.0208) |
| | 50  | 50    | 0.0669(0.0076) | 0.0783(0.0179) | 0.1472(0.1026) | 0.1408(0.1028) | 0.0992(0.0087) |
| $\mathcal{D}(\hat{R}, R)$ | 50  | 20    | 0.0852(0.0231) | 0.0851(0.0256) | 0.2125(0.1403) | 0.1942(0.1500) | 0.0860(0.0118) |
| | 50  | 50    | 0.0417(0.0041) | 0.0530(0.0188) | 0.1205(0.0987) | 0.1158(0.1009) | 0.0611(0.0047) |
| $\mathcal{D}(\hat{R}, R)$ | 100 | 20    | 0.0706(0.0249) | 0.0624(0.0193) | 0.1944(0.1321) | 0.1712(0.1380) | 0.0613(0.0095) |
| | 50  | 50    | 0.0302(0.0036) | 0.0402(0.0238) | 0.1011(0.0840) | 0.0966(0.0860) | 0.0429(0.0033) |

$t_3$ Distribution

| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1272(0.0391) | 0.1117(0.0229) | 0.1935(0.0978) | 0.1679(0.0946) | 0.1410(0.0226) |
| | 50  | 50    | 0.0657(0.0083) | 0.0676(0.0100) | 0.0974(0.0372) | 0.0926(0.0365) | 0.0988(0.0092) |
| $\mathcal{D}(\hat{R}, R)$ | 50  | 20    | 0.0840(0.0220) | 0.0711(0.0142) | 0.1415(0.0814) | 0.1186(0.0753) | 0.0868(0.0125) |
| | 50  | 50    | 0.0411(0.0046) | 0.0439(0.0063) | 0.0752(0.0427) | 0.0718(0.0478) | 0.0612(0.0049) |
| $\mathcal{D}(\hat{R}, R)$ | 100 | 20    | 0.0686(0.0232) | 0.0507(0.0111) | 0.1153(0.0691) | 0.0951(0.0678) | 0.0605(0.0086) |
| | 50  | 50    | 0.0296(0.0034) | 0.0312(0.0044) | 0.0601(0.0405) | 0.0571(0.0431) | 0.0429(0.0032) |

We evaluate the performance of different methods by the Mean Squared Error, i.e.,

$$\text{MSE} = \frac{1}{T_{p_1} p_2} \sum_{t=1}^{T} \| \hat{S}_t - S_t \|_F^2,$$

where the $\hat{S}_t$ refers to an arbitrary estimate and $S_t$ is the true common component matrix at time point $t$.

Table 4 shows that averaged MSEs with standard errors in parentheses under Scenario A for normal distribution and $t_3$ distribution. From Table 4, we can see that our MRTS method performs very well in
Table 3: Averaged estimation errors and standard errors of $\mathcal{D}(\hat{R}, R)$ for Scenario B under joint normal distribution and joint $t$ distribution over 500 replications.

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS | RMFA | $\alpha$-PCA | PE | MPCA$_F$ |
|------------|-----|-------|-------|------|------|--------------|----|----------|
|            |     |       |       |      |      |              |    |          |
| **Normal Distribution** |     |       |       |      |      |              |    |          |
| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1194(0.0358) | 0.0930(0.0164) | 0.1135(0.0348) | 0.0936(0.0168) | 0.1412(0.0219) |
|           | 50  | 50    | 0.0627(0.0076) | 0.0574(0.0060) | 0.0602(0.0071) | 0.0574(0.0060) | 0.0988(0.0095) |
|            | 50  | 20    | 0.0816(0.0230) | 0.0580(0.0090) | 0.0779(0.0217) | 0.0583(0.0091) | 0.0872(0.0122) |
|            | 50  | 50    | 0.0388(0.0041) | 0.0354(0.0032) | 0.0377(0.0040) | 0.0354(0.0032) | 0.0617(0.0049) |
|            | 100 | 20    | 0.0656(0.0207) | 0.0409(0.0065) | 0.0621(0.0202) | 0.0410(0.0067) | 0.0612(0.0087) |
|            | 50  | 50    | 0.0281(0.0033) | 0.0249(0.0022) | 0.0274(0.0031) | 0.0249(0.0022) | 0.0431(0.0031) |
| **$t_1$ Distribution** |     |       |       |      |      |              |    |          |
| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1445(0.0426) | 0.2365(0.1476) | 0.4310(0.1766) | 0.4242(0.1885) | 0.1520(0.0247) |
|           | 50  | 50    | 0.0769(0.0111) | 0.1391(0.1041) | 0.2867(0.1795) | 0.2813(0.1861) | 0.1061(0.0121) |
|            | 50  | 20    | 0.0968(0.0274) | 0.1959(0.1471) | 0.4178(0.1927) | 0.4049(0.2077) | 0.0924(0.0131) |
|            | 50  | 50    | 0.0484(0.0058) | 0.1123(0.0874) | 0.2676(0.1750) | 0.2647(0.1830) | 0.0651(0.0056) |
|            | 100 | 20    | 0.0759(0.0269) | 0.1857(0.1503) | 0.4320(0.1912) | 0.4252(0.2056) | 0.0658(0.0093) |
|            | 50  | 50    | 0.0347(0.0040) | 0.0995(0.0899) | 0.2656(0.1719) | 0.2628(0.1796) | 0.0457(0.0037) |
| **$t_2$ Distribution** |     |       |       |      |      |              |    |          |
| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1305(0.0371) | 0.1312(0.0568) | 0.2648(0.1503) | 0.2435(0.1597) | 0.1416(0.0217) |
|           | 50  | 50    | 0.0691(0.0083) | 0.0801(0.0197) | 0.1476(0.1027) | 0.1412(0.1028) | 0.1005(0.0094) |
|            | 50  | 20    | 0.0876(0.0236) | 0.0875(0.0275) | 0.2125(0.1398) | 0.1946(0.1502) | 0.0877(0.0122) |
|            | 50  | 50    | 0.0432(0.0044) | 0.0545(0.0206) | 0.1209(0.0992) | 0.1161(0.1013) | 0.0622(0.0050) |
|            | 100 | 20    | 0.0721(0.0255) | 0.0643(0.0206) | 0.1946(0.1322) | 0.1712(0.1378) | 0.0624(0.0096) |
|            | 50  | 50    | 0.0313(0.0037) | 0.0415(0.0255) | 0.1012(0.0840) | 0.0966(0.0860) | 0.0436(0.0034) |
| **$t_3$ Distribution** |     |       |       |      |      |              |    |          |
| $\mathcal{D}(\hat{R}, R)$ | 20  | 20    | 20    | 0.1295(0.0401) | 0.1135(0.0237) | 0.1941(0.0983) | 0.1685(0.0945) | 0.1436(0.0233) |
|           | 50  | 50    | 0.0669(0.0086) | 0.0686(0.0104) | 0.0975(0.0371) | 0.0930(0.0361) | 0.0999(0.0097) |
|            | 50  | 20    | 0.0853(0.0227) | 0.0723(0.0150) | 0.1420(0.0817) | 0.1189(0.0752) | 0.0877(0.0128) |
|            | 50  | 50    | 0.0419(0.0047) | 0.0447(0.0068) | 0.0755(0.0431) | 0.0723(0.0502) | 0.0617(0.0051) |
|            | 100 | 20    | 0.0694(0.0230) | 0.0516(0.0117) | 0.1155(0.0689) | 0.0952(0.0677) | 0.0615(0.0084) |
|            | 50  | 50    | 0.0302(0.0034) | 0.0318(0.0047) | 0.0602(0.0406) | 0.0572(0.0432) | 0.0434(0.0033) |

all settings. Firstly, the MRTS methods perform comparably with the PE method in the normal case, and MRTS, RMFA and MPCA$_F$ perform better than the other methods in the heavy-tailed case. Especially when $p_1$ and $p_2$ are large, the advantages of MRTS over the other methods are more obvious.

### 4.4 Estimating the numbers of factors

In this section, we compare the empirical performances of the proposed MKER method with the $\alpha$-PCA based ER method ($\alpha$-PCA-ER) by Chen and Fan (2021), the IterER method by Yu et al. (2022), the IC and
Table 4: Mean squared error and its standard under Scenario A over 500 replications.

| Distribution | $p_1$ | MRTS | RMFA | $\alpha$-PCA | PE | MPCA$_F$ |
|--------------|-------|------|------|---------------|----|----------|
| **$T = 20, p_2 = p_1$** |       |      |      |               |    |          |
| Normal       | 20    | 0.0453(0.0065) | 0.0369(0.0036) | 0.0429(0.0060) | 0.0371(0.0037) | 0.0557(0.0067) |
|              | 50    | 0.0105(0.0008) | 0.0099(0.0007) | 0.0099(0.0008) | 0.0094(0.0007) | 0.0211(0.0024) |
| $t_3$        | 20    | 0.1385(0.1263) | 0.1477(0.2537) | 0.3388(0.9222) | 0.3224(0.9629) | 0.1618(0.1434) |
|              | 50    | 0.0304(0.0207) | 0.0357(0.0370) | 0.0720(0.1233) | 0.0704(0.1347) | 0.0580(0.0399) |
| **$T = 50, p_2 = p_1$** |       |      |      |               |    |          |
| Normal       | 20    | 0.0337(0.0042) | 0.0283(0.0024) | 0.0327(0.0040) | 0.0283(0.0024) | 0.0355(0.0032) |
|              | 50    | 0.0064(0.0004) | 0.0059(0.0003) | 0.0062(0.0004) | 0.0059(0.0003) | 0.0106(0.0008) |
| $t_3$        | 20    | 0.1037(0.0803) | 0.1099(0.1106) | 0.2777(0.9210) | 0.2654(0.9586) | 0.1059(0.0839) |
|              | 50    | 0.0199(0.0147) | 0.0243(0.0415) | 0.0642(0.2261) | 0.0624(0.2367) | 0.0315(0.0228) |
| **$T = 100, p_2 = p_1$** |       |      |      |               |    |          |
| Normal       | 20    | 0.0300(0.0035) | 0.0253(0.0017) | 0.0292(0.0033) | 0.0254(0.0017) | 0.0289(0.0019) |
|              | 50    | 0.0051(0.0003) | 0.0047(0.0002) | 0.0050(0.0003) | 0.0047(0.0002) | 0.0071(0.0004) |
| $t_3$        | 20    | 0.0897(0.0503) | 0.0853(0.0906) | 0.1964(0.6719) | 0.1834(0.7119) | 0.0855(0.0509) |
|              | 50    | 0.0161(0.0119) | 0.0183(0.0246) | 0.0515(0.1750) | 0.0499(0.1815) | 0.0217(0.0153) |

ER method based on iTIPUP by Han et al. (2022) (denote as iTIP-IC and iTIP-EC), the TCorTh by Lam (2021) and the Rit-ER method by He et al. (2021b) in terms of estimating the pair of factor numbers. Table 5 presents the frequencies of exact estimation and underestimation over 500 replications under Scenario A by different methods. We set $k_{\text{max}} = 8$. Under the normal case, we see that the Rit-ER and IterER perform comparably and both perform better than the others. However, as $p_1$ and $p_2$ increase, it can be seen that all the methods’s performances get better. The proposed MKER method performs robustly and always performs the best for the heavy-tailed $t$ distribution cases. And it can also be seen that as the dimension of $p_1$ and $p_2$ increase, the proportion of exact estimation by MKER has the tendency to converge to 1, which is consistent with our theoretical analysis.

5 Real Data Examples

5.1 COVID-CT dataset

Coronavirus disease 2019 (COVID-19) is a contagious disease caused by a virus, the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). The first known case was identified in Wuhan, China, in December
Table 5: The frequencies of exact estimation and underestimation of the numbers of factors under Scenario A over 500 replications.

| Distribution | $p_1$ | MKER | Rit-ER | IterER | $\alpha$-PCA-ER | iTIP-IC | iTIP-EC | TCorTh |
|--------------|-------|------|--------|--------|----------------|--------|--------|--------|
| $T = 20, p_2 = p_1$ |
| Normal       | 20    | 0.676(0.046) | 0.990(0.000) | 0.990(0.000) | 0.594(0.070) | 0.000(1.000) | 0.110(0.386) | 0.680(0.048) |
|              | 50    | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.000(1.000) | 0.174(0.178) | 1.000(0.000) |        |
| $t_1$        | 20    | 0.620(0.066) | 0.330(0.256) | 0.226(0.470) | 0.060(0.774) | 0.090(0.370) | 0.565(0.538) | 0.272(0.380) |
|              | 50    | 0.998(0.000) | 0.666(0.094) | 0.532(0.304) | 0.302(0.548) | 0.146(0.478) | 0.118(0.406) | 0.700(0.162) |
| $t_2$        | 20    | 0.638(0.050) | 0.706(0.068) | 0.634(0.182) | 0.228(0.464) | 0.012(0.970) | 0.094(0.310) | 0.928(0.028) |
|              | 50    | 1.000(0.000) | 0.906(0.002) | 0.880(0.066) | 0.680(0.168) | 0.012(0.970) | 0.094(0.310) | 0.928(0.028) |
| $t_3$        | 20    | 0.646(0.050) | 0.870(0.016) | 0.806(0.276) | 0.000(1.000) | 0.090(0.430) | 0.482(0.148) |        |
|              | 50    | 1.000(0.000) | 0.906(0.002) | 0.880(0.066) | 0.680(0.168) | 0.012(0.970) | 0.094(0.310) | 0.928(0.028) |
| $T = 50, p_2 = p_1$ |
| Normal       | 20    | 0.772(0.014) | 1.000(0.000) | 1.000(0.000) | 0.734(0.032) | 0.090(0.276) | 0.954(0.000) |        |
|              | 50    | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.090(0.168) | 0.168(0.158) | 1.000(0.000) |        |
| $t_1$        | 20    | 0.742(0.012) | 0.420(0.186) | 0.292(0.466) | 0.106(0.740) | 0.088(0.194) | 0.062(0.552) | 0.556(0.054) |
|              | 50    | 1.000(0.000) | 0.688(0.060) | 0.560(0.294) | 0.356(0.494) | 0.176(0.266) | 0.128(0.370) | 0.698(0.014) |
| $t_2$        | 20    | 0.796(0.020) | 0.832(0.022) | 0.738(0.124) | 0.314(0.338) | 0.014(0.946) | 0.064(0.468) | 0.786(0.014) |
|              | 50    | 1.000(0.000) | 0.938(0.006) | 0.916(0.058) | 0.802(0.136) | 0.006(0.974) | 0.170(0.242) | 0.932(0.004) |
| $t_3$        | 20    | 0.760(0.018) | 0.956(0.000) | 0.914(0.032) | 0.530(0.164) | 0.000(1.000) | 0.090(0.402) | 0.876(0.012) |
|              | 50    | 1.000(0.000) | 0.990(0.000) | 0.988(0.006) | 0.954(0.030) | 0.000(1.000) | 0.156(0.204) | 0.992(0.000) |
| $T = 100, p_2 = p_1$ |
| Normal       | 20    | 0.824(0.010) | 1.000(0.000) | 1.000(0.000) | 0.786(0.016) | 0.000(1.000) | 0.076(0.334) | 0.996(0.000) |
|              | 50    | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.000(1.000) | 0.186(0.130) | 1.000(0.000) |        |
| $t_1$        | 20    | 0.810(0.010) | 0.402(0.202) | 0.278(0.438) | 0.092(0.724) | 0.078(0.046) | 0.052(0.560) | 0.458(0.014) |
|              | 50    | 1.000(0.000) | 0.702(0.044) | 0.568(0.304) | 0.348(0.514) | 0.196(0.122) | 0.134(0.366) | 0.468(0.000) |
| $t_2$        | 20    | 0.806(0.008) | 0.874(0.012) | 0.780(0.110) | 0.400(0.294) | 0.012(0.968) | 0.072(0.376) | 0.864(0.002) |
|              | 50    | 1.000(0.000) | 0.964(0.000) | 0.938(0.034) | 0.852(0.094) | 0.010(0.980) | 0.166(0.254) | 0.900(0.000) |
| $t_3$        | 20    | 0.818(0.008) | 0.966(0.000) | 0.950(0.014) | 0.640(0.080) | 0.000(1.000) | 0.082(0.376) | 0.960(0.000) |
|              | 50    | 1.000(0.000) | 0.992(0.000) | 0.992(0.006) | 0.968(0.016) | 0.000(1.000) | 0.196(0.182) | 0.980(0.000) |

2019. The disease then spreads worldwide rapidly, leading to the COVID-19 pandemic and causing millions of deaths until now. It is critical to identify patients infected with COVID-19 and isolate them. Chest computed tomography (CT) scans can be used to screen the positives from the population. We use the open-source chest CT dataset called COVID-CT which was ever analyzed in Zhang et al. (2022) and is available from https://github.com/UCSD-AI4H/COVID-CT. The dataset is consisted of 2D grayscale images mainly in the format of PNG and JPEG, including 349 COVID-19 positive CT scans and 397 negative CT scans in the early pandemic. Figure 2 illustrates the original CT scans of one COVID-19 positive case (the left panel) and one negative case (the right panel).
0 to 255, with 255 for white and 0 for black. The brighter the original region is, the larger pixel value it has. By contrast, the darker the original region is, the smaller pixel value it has. Each pixel value can be rescaled by dividing 255, therefore a grayscale image is converted into a matrix with all entries in the interval [0,1]. We read these grayscale images by R package EBImage. Then we compress images into uniform dimensions (height, width) = (150, 150) by upsampling and downsampling. As a result, we have the dataset \( \{Y_i, X_i\}_{1 \leq i \leq 746} \), where \( Y_i \) is the class label. Denote \( Y_i = 1 \) as a positive case and \( Y_i = 0 \) as a negative case, and \( X_i \in \mathbb{R}^{150 \times 150} \) represents CT scan image of subject \( i \). Then we construct the matrix factor model and estimate the loading matrices as well as factor score matrices by different methods (e.g., MRTS) by setting \( k_1 = k_2 = 3 \). Then we get the dataset \( \{Y_i, \text{Vec}(\hat{F}_i)\}_{1 \leq i \leq 746} \), where \( \hat{F}_i \) is the estimated factor score matrix for subject \( i \). We randomly select 70% of the samples as the training set and 30% of the samples as the testing set, and repeat this process 500 times. In each replication we put the estimated factor scores into Support Vector Machine (SVM) for classification.

Table 6: The means (standard errors) of classification metrics on real COVID-CT data with 500 replicates.

| Method        | MRTS         | \( \alpha \)-PCA | PE          | RMFA         | MPCA \( \alpha \) |
|---------------|--------------|------------------|-------------|--------------|------------------|
| **Original dataset** | **Precision** | 0.7335(0.0586) | 0.7243(0.0594) | 0.7218(0.0615) | 0.7224(0.0618) | 0.7289(0.0598) |
|               | **AUC**      | 0.7299(0.0283) | 0.7242(0.0295) | 0.7244(0.0297) | 0.7250(0.0296) | 0.7277(0.0289) |
|               | **Error Rate** | 0.3646(0.0330) | 0.3750(0.0334) | 0.3737(0.0331) | 0.3730(0.0335) | 0.3676(0.0340) |
| **Contaminated dataset** | **Precision** | 0.7244(0.0554) | 0.7120(0.0596) | 0.7085(0.0597) | 0.7084(0.0588) | 0.7131(0.0580) |
|               | **AUC**      | 0.7288(0.0287) | 0.7221(0.0293) | 0.7213(0.0290) | 0.7223(0.0290) | 0.7225(0.0293) |
|               | **Error Rate** | 0.3653(0.0322) | 0.3769(0.0335) | 0.3767(0.0339) | 0.3764(0.0340) | 0.3741(0.0343) |

Meanwhile, to measure the robustness of different methods, we contaminated the original dataset in the
following way. We replaced 30\% entries of the matrix $X_i$ with random numbers generated from uniform distribution $U(0.8, 1)$. Subfigure (a) and Subfigure (b) in Figure 3 are heatmaps of CT images of one positive case without/with contamination respectively. In order to give a direct illustration of the effect of our MRTS method in extracting CT image information, we display the heatmaps of reconstructed CT images using the estimated common components in subfigures (c) and (d) of Figure 3, from which we can see intuitively that the reconstructed image is very similar to the original image thereby indicating that the common components extracted by our method is a good approximation of the original image. The classification results are reported in Table 6, from which we see that MRTS performs the best in terms of the metrics Precision, AUC and Error rate. MRTS, RMFA and MPCA$_F$ are all robust methods, and outperform the other two non-robust
methods for both the original and contaminated datasets.

5.2 Portfolio returns dataset

In this section, we illustrate the empirical performance of the MRTS method by analyzing a heavy-tailed financial portfolio dataset, which was also studied in Wang et al. (2019), Yu et al. (2022), and He et al. (2021b). The dataset contains monthly returns of 100 portfolios ranging from January 1964 to December 2019, which are structured into a 10 × 10 matrix at each time point, with rows corresponding to 10 levels of market capital size (denoted as S1-S10) and columns corresponding to 10 levels of book-to-equity ratio (denoted as BE1-BE10). The dataset is available at the website http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. We refer to He et al. (2021b) for further details on this dataset and the preprocessing procedure (stationarity and missing values).

He et al. (2021b) showed the heavy-tailed property of the returns data and concluded that robust analysis methods are more appropriate. Moreover, He et al. (2021b) showed that it is reasonable to adopt a matrix factor model for this real dataset by using the test proposed in He et al. (2021a).

For the preprocessed monthly returns dataset, MKER method suggests that $(k_1, k_2) = (1, 1)$ while the Rit-ER suggests that $(k_1, k_2) = (1, 2)$. The difference between the estimates by different methods may be explained by the heavy-tailedness of the returns data. As overestimation is better than underestimation, and for better illustration, we take $(k_1, k_2) = (1, 2)$.

The estimated front and back loading matrices after varimax rotation and scaling are reported in Table 7, from which we observe that the PE, $\alpha$-PCA lead to very similar estimated loadings while the robust methods MRTS, RMFA, MPCA lead to very similar estimated loadings. To compare the matrix factor analysis methods, we also adopt a rolling-validation procedure as in Wang et al. (2019) and He et al. (2021b). For each year $t$ from 1996 to 2019, we recursively use the $n$ (bandwidth) half years observations before year $t$ to train the matrix factor model, then we estimate the row/column loading matrices. The loadings are then used to train regression model to derive the factor scores and the corresponding residuals of the 12 months in the current year. In detail, let $Y_i^t$ and $\hat{Y}_i^t$ be the observed and estimated price matrix of month $i$ in year
Table 7: Loading matrices for Fama–French data set, after varimax rotation and scaling by 30.

| Size | Method | Factor | S1 | S2 | S3 | S4 | S5 | S6 | S7 | S8 | S9 | S10 |
|------|--------|--------|----|----|----|----|----|----|----|----|----|-----|
|      | MRTS   | 1      | -41| -42| -41| -38| -32| -23| -14| -5 | 8  | 24  |
|      |        | 2      | 11 | 6  | -3 | -6 | -15| -26| -35| -45| -53| -44 |
|      | RMFA   | 1      | -16| -15| -13| -11| -8 | -6 | 10 | 12 | 14 | 15  |
|      |        | 2      | -6 | -2 | 2  | 5  | 8  | 10 | 12 | 13 | 15 | 10  |
|      | PE     | 1      | -16| -15| -12| -10| -8 | -5 | 12 | 12 | 13 | 15  |
|      |        | 2      | -6 | -1 | 3  | 5  | 8  | 11 | 12 | 13 | 15 | 10  |
|      | α-PCA  | 1      | -14| -14| -13| -11| -9 | -7 | 12 | 12 | 13 | 16  |
|      |        | 2      | -4 | -2 | 1  | 3  | 6  | 9  | 12 | 13 | 16 | 14  |
|      | MPCA_F | 1      | -47| -46| -40| -34| -29| -22| -12| -5 | 7  | 20  |
|      |        | 2      | 19 | 9  | -6 | -11| -20| -27| -36| -44| -51| -38 |

| Book-to-Equity | Method | Factor | BE1 | BE2 | BE3 | BE4 | BE5 | BE6 | BE7 | BE8 | BE9 | BE10 |
|----------------|--------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
|                | MRTS   | 1      | 17  | 8  | -7  | -18 | -27 | -35 | -39 | -41 | -40 | -37  |
|                |        | 2      | 53  | 55 | 41  | 30  | 19  | 10  | 0   | -3  | -3  | -4   |
|                | RMFA   | 1      | 6   | 1  | -3  | -6  | -9  | -11 | -12 | -13 | -12 | -11  |
|                |        | 2      | 19  | 17 | 12  | 9   | 5   | 3   | 0   | -1  | -1  | 0    |
|                | PE     | 1      | 6   | 1  | -4  | -7  | -10 | -11 | -12 | -12 | -12 | -10  |
|                |        | 2      | 20  | 17 | 11  | 8   | 4   | 2   | 0   | -1  | -1  | 0    |
|                | α-PCA  | 1      | 6   | 2  | -4  | -7  | -10 | -11 | -12 | -13 | -12 | -11  |
|                |        | 2      | 19  | 18 | 12  | 8   | 4   | 2   | 0   | -1  | -1  | -1   |
|                | MPCA_F | 1      | 19  | 7  | -9  | -18 | -28 | -35 | -38 | -41 | -41 | -37  |
|                |        | 2      | 58  | 54 | 38  | 29  | 18  | 11  | 1   | -2  | -3  | -4   |

$t$, and let $\bar{Y}_t$ be the mean price matrix. Further define

$$\text{MSE}_t = \frac{1}{12 \times 10 \times 10} \sum_{i=1}^{12} \| \hat{Y}_i^t - Y_i^t \|_F^2$$

and

$$\rho_t = \frac{\sum_{i=1}^{12} \| \hat{Y}_i^t - Y_i^t \|_F^2}{\sum_{i=1}^{12} \| Y_i^t - \bar{Y}_t \|_F^2}$$

as the mean squared pricing error and unexplained proportion of total variances, respectively. For year $t$ in the rolling-validation procedure, we measure the variation of loading space by $v_t := D(\hat{C}_t \otimes \hat{R}_t, \bar{C}_t \otimes \bar{R}_{t-1})$.

We report the results of the means of MSE, $\rho$ and $v$ by RMFA, PE, ACCE and α-PCA in Table 8. Firstly, we see that in all combinations considered, the robust methods MRTS, RMFA, MPCA$_F$ methods always have lower pricing errors than the non-robust methods PE and α-PCA. In terms of estimating factor loading spaces, MRTS, RMFA, MPCA$_F$ methods always perform more stably than the non-robust methods.

Since heavy tailedness is a well-known stylized fact of financial returns and stock-level predictor variables, the robust MRTS method is strongly recommended to cope with matrix-variate financial data.
Table 8: Rolling validation with $k_1 = 1$ and $k_2 = 2$ for the Fama–French portfolios, the sample size of training set is $6n$. $\bar{\text{MSE}}, \bar{\rho}, \bar{\upsilon}$ are the mean pricing error, mean unexplained proportion of total variances and mean variation of the estimated loading space.

| n  | MRTS | $\alpha$-PCA | PE | RMFA | MPCA$_{F}$ |
|----|------|--------------|----|------|-------------|
| 17 | 0.745 | 0.749        | 0.746 | 0.744 | **0.741**   |
| 19 | 0.747 | 0.751        | 0.748 | 0.746 | **0.743**   |
| 21 | 0.747 | 0.751        | 0.748 | 0.746 | **0.743**   |

| n  | MRTS | $\alpha$-PCA | PE | RMFA | MPCA$_{F}$ |
|----|------|--------------|----|------|-------------|
| 17 | 0.710 | 0.715        | 0.710 | 0.706 | **0.701**   |
| 19 | 0.711 | 0.717        | 0.711 | 0.707 | **0.701**   |
| 21 | 0.711 | 0.718        | 0.711 | 0.707 | **0.701**   |

| n  | MRTS | $\alpha$-PCA | PE | RMFA | MPCA$_{F}$ |
|----|------|--------------|----|------|-------------|
| 17 | 0.093 | 0.182        | 0.094 | 0.086 | **0.079**   |
| 19 | 0.079 | 0.177        | 0.082 | 0.073 | **0.067**   |
| 21 | 0.073 | 0.172        | 0.073 | 0.067 | **0.062**   |

6 Conclusion and Discussion

In this paper we propose a new type of Kendall’s tau for random matrices, named as matrix Kendall’s tau. We show that the row/column matrix Kendall’s tau share the same eigenspace with the row/column scatter matrix for matrix-elliptical distribution, with the same descending order of the eigenvalues. The sample version of the row/column matrix Kendall’s tau is a U-statistic with a bounded kernel (under operator norm) and enjoys the same distribution-free property as multivariate Kendall’s tau. Secondly, we propose a Matrix-type Robust Two Step (MRTS) method to estimate the loading and factor spaces for MEFM by matrix Kendall’s tau, which achieves faster convergence rates than the Manifold Principal Component Analysis (MPCA) for estimating the loading spaces. We also propose robust and consistent MKER estimators for the pair of factor numbers by exploiting the eigenvalue-ratios of the sample matrix Kendall’s tau. Numerical results show that the proposed MRTS/MKER methods outperform the existing methods particularly in heavy-tailed cases. The current work can be extended along several directions. Firstly, the matrix Kendall’s tau is a useful tool for matrix elliptical distribution, and provides a robust approach to perform 2-Dimensional Principal Component Analysis (2D-PCA) and to estimate separable covariance matrices in high dimensions. We leave these extensions as future work. Secondly, for matrix factor model, the projection technique is
attractive as it increases the signal-to-noise ratio and leads to more accurate estimators. The projection technique can be incorporated in the current framework. In detail, given observations \( \{X_t\} \), we first obtain a projection matrix \( \hat{C} \) that consists of the leading \( k_2 \) eigenvectors of \( \hat{K}^X_c \), multiplied by \( \sqrt{p_2} \). Then we project the original matrix observations to a lower dimensional space, i.e., \( Y_t = X_t \hat{C} / p_2 \). We further construct row matrix Kendall’s tau \( \hat{K}^Y_r \) based on \( Y_t \)'s, and obtain a projected estimator \( \tilde{R} \) by the leading \( k_1 \) eigenvectors of \( \hat{R}^Y_r \), multiplied by \( \sqrt{p_1} \). The projected estimator \( \tilde{C} \) can be obtained by a similar procedure applied to matrix observations \( \{X_t^\top\} \). The theoretical analysis is more challenging and will be left for future work.

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Supplementary Materials for “Matrix Kendall’s tau in High-dimensions: A Robust Statistic for Matrix Factor Model”

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This document provides detailed proofs and additional simulation results to the main paper. The notation X and ̃X are used repeatedly in the proof, but can represent different matrices in different lemmas.

A Proof of Proposition

Proof of Proposition 2.1

Proof. We only show the results for K_r and the results for K_c can be derived in a similar way.

By Lemma C.1, it is equivalent to consider $K_r = \mathbb{E}\left(\frac{(X-M)(X-M)^\top}{\|X-M\|_F^2}\right)$. Let $R = (u_1(\Sigma), \ldots, u_p(\Sigma))$ be the eigenvector matrix of $\Sigma$. Then, we have

$$R^\top X - M \|X-M\|_F = R^\top (X-M) \|X-M\|_F = Z,$$

where $Z = DUB^\top$ with $D = \left(\text{diag}(\sqrt{\lambda_1(\Sigma)}, \ldots, \sqrt{\lambda_m(\Sigma)}), 0\right)^\top \in \mathbb{R}^{p \times m}$. Therefore,

$$K_r = \mathbb{E}\left(\frac{(X-M)(X-M)^\top}{\|X-M\|_F^2}\right) = R\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)R^\top.$$

Next, we prove that $\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)$ is a diagonal matrix. For any matrix $P = \text{diag}(v)$, where $v = (v_1, \ldots, v_m)^\top$ satisfies that $v_j = 1$ or $-1$ for $j = 1, \ldots, m$, we have

$$P \frac{Z}{\|Z\|_F} = \frac{PZ}{\|PZ\|_F} \overset{d}{=} \frac{Z}{\|Z\|_F} \Rightarrow \mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right) = P\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)P.$$

This result holds if and only if $\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)$ is a diagonal matrix. In other words, $K_r$ shares the same eigenvector space as $\Sigma$.

The last step is to show that the diagonals of $\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)$ are decreasing. Recall that $Z = DUB^\top$, so

$$\mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right) = \mathbb{E}\left(\frac{DUB^\top BU^\top D^\top}{\|DUB\|^2_F}\right) = \mathbb{E}\left(\frac{DU\Omega^* U^\top D^\top}{\|DUB\|^2_F}\right),$$

$$\Rightarrow \mathbb{E}\left(\frac{ZZ^\top}{\|Z\|^2_F}\right)_{jj} = \mathbb{E}\left(\frac{\lambda_j(\Sigma)U_{j1}, \Omega^* U_{j1}^\top + \cdots + \lambda_m(\Sigma^*)U_{mj}, \Omega^* U_{mj}^\top}{\lambda_1(\Sigma)U_{11}, \Omega^* U_{11}^\top + \cdots + \lambda_m(\Sigma^*)U_{mm}, \Omega^* U_{mm}^\top}\right).$$

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Denote Lemma C.4 shows that

According to equation (B.1), we have

Define

Proof of Theorem 3.1:

This completes the proof. □

B Proof of the Main Theorems

Proof of Theorem 3.1: \( \hat{R} \) and \( \hat{C} \) converge in Frobenius norm

Proof. Define \( \hat{\Lambda} \) as the diagonal matrix composed of the leading \( k_1 \) eigenvalues of \( \hat{K}_r \). Lemma C.8 implies that \( \hat{\Lambda} \) is asymptotically invertible and \( \| \Lambda^{-1} \|_F = O_p(1) \). Because \( \hat{R} / \sqrt{p_1} \) is composed of the leading eigenvectors of \( \hat{K}_r \), we have

\[
\hat{K}_r \hat{R} = \hat{R} \hat{\Lambda} .
\]  (B.1)

Expand \( \hat{K}_r \) by its definition

\[
\hat{K}_r = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(X_t - X_s)(X_t - X_s)^\top}{\|X_t - X_s\|_F^2} = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(R(F_t - F_s)C^\top + (E_t - E_s))(R(F_t - F_s)C^\top + (E_t - E_s))^\top}{\|X_t - X_s\|_F^2} .
\]

Denote

\[
M_1 = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(F_t - F_s)C^\top C(F_t - F_s)^\top}{\|X_t - X_s\|_F^2} ,
\]

\[
M_2 = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(E_t - E_s)C(F_t - F_s)^\top}{\|X_t - X_s\|_F^2} ,
\]

\[
M_3 = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(F_t - F_s)C(E_t - E_s)^\top}{\|X_t - X_s\|_F^2} ,
\]

\[
M_4 = \frac{2}{T(T-1)} \sum_{1 \leq t < s \leq T} \frac{(E_t - E_s)(E_t - E_s)^\top}{\|X_t - X_s\|_F^2} .
\]

Then,

\[
\hat{K}_r = RM_1R^\top + M_2R^\top + RM_3 + M_4 .
\]

According to equation (B.1), we have

\[
\hat{R} = (RM_1R^\top + M_2R^\top + RM_3 + M_4)\hat{R} \hat{\Lambda}^{-1} .
\]

Let \( \hat{H}_R = M_1R^\top \hat{\Lambda}^{-1} \), so

\[
\hat{R} - \hat{R} \hat{H}_R = (M_2R^\top + RM_3 + M_4)\hat{R} \hat{\Lambda}^{-1} .
\]  (B.2)

Lemma C.4 shows that \( \| \hat{H}_R \|_F^2 \leq O_p(1) \). Lemmas C.3 and C.5 show that

\[
\|M_2\|_F^2 = O_p \left( \frac{1}{T p_1 p_2} + \frac{1}{p_1^2 p_2^2} \right) .
\]
while

$\frac{1}{p_1} \| M_3 \|_F^2 = O_p \left( \frac{1}{T_{p_1}p_2} + \frac{1}{p_1^2p_2^2} \right),$

\[
\frac{1}{p_1} \| M_3 \hat{R} \|_F^2 = O_p \left( \frac{1}{T_{p_1}p_2} + \frac{1}{p_1^2} \right) + o_p(1) \times \frac{1}{p_1} \| \hat{R} - R \hat{H}_R \|_F^2.
\]

Therefore, it is easy to prove that

\[
\frac{1}{p_1} \| \hat{R} - R \hat{H}_R \|_F^2 = O_p \left( \frac{1}{T_{p_1}p_2} + \frac{1}{p_1^2} \right).
\]

The proof of $\frac{1}{p_2} \| \hat{C} - CH_C \|_F^2 = O_p \left( \frac{1}{T_{p_1}p_2} + \frac{1}{p_1^2p_2^2} \right)$ is similar. It remains to show the properties of $\hat{H}_R$ and $\hat{H}_C$. This is easy because

\[
\hat{H}_R \hat{V}_1 \hat{H}_R = \hat{H}_R \left( V_1 - \frac{1}{p_1} R^T R \right) \hat{H}_R + \frac{1}{p_1} (R \hat{H}_R - R) \left( R \hat{H}_R - R \right) - \frac{1}{p_1} R^T \hat{R} + \frac{1}{p_1} \hat{R}^T \hat{R} + o_p(1) + \frac{1}{p_1} \hat{R}^T \hat{R} = a_p(1) + I_{k_1},
\]

where the $o_p(1)$ holds in terms of Frobenius norm. The proof of $\hat{H}_C$ is similar and omitted. \hfill \square

**Proof of Theorem 3.2: Convergence rate of $\hat{F}_t$**

*Proof.* By the least square optimization, we can get that

\[
\hat{F}_t = \frac{1}{p_1p_2} \hat{R}^T X_t \hat{C} = \frac{1}{p_1p_2} \hat{R}^T RF_t C^T \hat{C} + \frac{1}{p_1p_2} \hat{R}^T E_t \hat{C}.
\]

Writing $R = \hat{R} \hat{H}_R^{-1} - (\hat{R} \hat{H}_R^{-1} - R)$ and $C = \hat{C} \hat{H}_C^{-1} - (\hat{C} \hat{H}_C^{-1} - C)$, we obtain

\[
\hat{F}_t - \hat{H}_R^{-1} F_t \hat{H}_C^{-1T} = \frac{1}{p_1p_2} \hat{R}^T (\hat{R} - R) \hat{H}_R^{-T} F_t \hat{H}_C^{-1T} (\hat{C} - C) + \frac{1}{p_1p_2} \hat{R}^T E_t \hat{C} \quad \text{(B.3)}
\]

For $\hat{R}^T E_t \hat{C}$, further write $\hat{R} = (\hat{R} - R \hat{H}_R) + R \hat{H}_R$ and $\hat{C} = (\hat{C} - C \hat{H}_C) + C \hat{H}_C$. Then,

\[
\hat{F}_t - \hat{H}_R^{-1} F_t \hat{H}_C^{-1T} = \frac{1}{p_1p_2} \hat{R}^T (\hat{R} - R \hat{H}_R) \hat{H}_R^{-T} F_t \hat{H}_C^{-1T} (\hat{C} - C \hat{H}_C) + \frac{1}{p_1p_2} \hat{R}^T E_t \hat{C}
\]

\[
- \frac{1}{p_1} \hat{R}^T (\hat{R} - R \hat{H}_R) \hat{H}_R^{-T} F_t \hat{H}_C^{-1T} (\hat{C} - C \hat{H}_C) - \frac{1}{p_1} \hat{H}_R^{-T} F_t \hat{H}_C^{-1T} (\hat{C} - C \hat{H}_C)^T \hat{C} + \frac{1}{p_1p_2} (\hat{R} - R \hat{H}_R)^T E_t (\hat{C} - C \hat{H}_C) + \frac{1}{p_1p_2} (\hat{R} - R \hat{H}_R)^T E_t C \hat{H}_C
\]

\[
+ \frac{1}{p_1p_2} \hat{H}_R^T R^T E_t (\hat{C} - C \hat{H}_C) + \frac{1}{p_1p_2} \hat{H}_R^T R^T E_t C \hat{H}_C
\]

\[
= \sum_{i=1}^7 I_i.
\]
Since $\frac{1}{p_1}\|\hat{R} - R\hat{H}_R\|_F^2 = o_p(1)$ and $\frac{1}{p_2}\|\hat{C} - C\hat{H}_C\|_F^2 = o_p(1)$ by Theorem 3.1, term $I_1$ is dominated by $I_2$ and $I_3$. By Lemma C.7, we have

$$\|I_2\|_F^2 = O_p\left(\frac{1}{T_p^1p_2}\right), \quad \|I_3\|_F^2 = O_p\left(\frac{1}{T_p^1p_2}\right).$$

For $I_4$, we have

$$\|I_4\|_F^2 \leq \frac{1}{p_1^2p_2^2}\|\hat{R} - R\hat{H}_R\|_F^2 \|E_i\|_F^2 \|\hat{C} - C\hat{H}_C\|_F^2 = O_p\left(\frac{1}{T_p^2} + \frac{1}{p_1^1} + \frac{1}{p_2^2}\right) = o_p\left(\frac{1}{p_1p_2}\right).$$

For $I_5$, we have

$$\|I_5\|_F^2 = O_p\left(\frac{1}{T_p^2} + \frac{1}{p_1p_2}\right) \Rightarrow \|I_5\|_F^2 = o_p\left(\frac{1}{p_1p_2}\right).$$

Similarly,

$$\|I_6\|_F^2 = O_p\left(\frac{1}{T_p^2} + \frac{1}{p_1p_2}\right) \Rightarrow \|I_6\|_F^2 = o_p\left(\frac{1}{p_1p_2}\right).$$

On the other hand, Lemma C.6 indicates that $\|I_7\|_F^2 = O_p(1/(p_1p_2))$, which dominates in the errors. Then, we conclude that

$$\|\hat{F}_t - \hat{H}_R^{-1}\hat{F}_t\hat{H}_C^{-1}\|_F^2 = O_p\left(\frac{1}{p_1p_2}\right),$$

which concludes the theorem.

\[ \square \]

**Proof of Theorem 3.3:** Convergence rate of $\hat{S}_t$

By definition,

$$\hat{S}_t - S_t = \hat{R}\hat{F}_t\hat{C}^\top - RF_tC^\top = (\hat{R} - R\hat{H}_R + R\hat{H}_R)\hat{F}_t(\hat{C} - C\hat{H}_C + CH^_C)^\top - RF_tC^\top$$

$$= (\hat{R} - R\hat{H}_R)\hat{F}_t(\hat{C} - C\hat{H}_C)^\top + (R - R\hat{H}_R)\hat{F}_t\hat{H}_C^\top + R\hat{H}_R\hat{F}_t(\hat{C} - C\hat{H}_C)^\top$$

$$+ R(\hat{H}_R\hat{F}_t\hat{H}_C - F_tC)^\top.$$

By Theorems 3.1 and 3.2, we have

$$\frac{1}{p_1p_2}\|\hat{S}_t - S_t\|_F^2 = O_p\left(\frac{1}{p_1p_2} + \frac{1}{T_p^1} + \frac{1}{T_p^2}\right).$$

**Proof of Theorem 3.4:** Consistency of $\hat{k}_1$ and $\hat{k}_2$

According to Lemma C.8, we have $\hat{\lambda}_j(\hat{K}^X_j) \approx 1, j \leq k_1$ and $\hat{\lambda}_j(\hat{K}^X_j) = O_p(\delta_1), j > k_1$. It is easy to check that

for $j < k_1$ or $j > k_1$, $\hat{\lambda}_j(\hat{K}^X_j) / \hat{\lambda}_{j+1}(\hat{K}^X_j) = O(1)$, while for $j = k_1$ we have $\hat{\lambda}_j(\hat{K}^X_j) / \hat{\lambda}_{j+1}(\hat{K}^X_j) \to \infty$. Then $\hat{k}_1$ is consistent. The proof of the consistency of $\hat{k}_2$ can be similarly derived and is thus omitted.
C  Technical Lemmas

Lemma C.1. Let \( X \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi) \) be a continuous random matrix, \( \tilde{X} \) be an independent copy of \( X \). Then we have

\[
K_r = \mathbb{E} \left( \frac{(X - \tilde{X})(X - \tilde{X})^T}{\|X - \tilde{X}\|_F^2} \right) = \mathbb{E} \left( \frac{(X - M)(X - M)^T}{\|X - M\|_F^2} \right).
\]

\[
K_c = \mathbb{E} \left( \frac{(X - \tilde{X})^T(X - \tilde{X})}{\|X - \tilde{X}\|_F^2} \right) = \mathbb{E} \left( \frac{(X - M)^T(X - M)}{\|X - M\|_F^2} \right).
\]

Proof. By the definition of the elliptical distribution, \( X \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi) \), \( \tilde{X} \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi) \). We have \( \varphi_{X - \tilde{X}}(T) = \psi^2[\text{Tr}(T^T \Sigma T \Omega)] \), implying that \( X - \tilde{X} \sim E_{p,q}(0, \Sigma \otimes \Omega, \psi^2) \). Hence, there exists a nonnegative random variable \( r' \), such that \( X - \tilde{X} \overset{d}{=} r'AUB^\top \). Because \( X \) is continuous, we have \( \mathbb{P}(r' = 0) = 0 \). Therefore,

\[
K_r = \mathbb{E} \left( \frac{(X - \tilde{X})(X - \tilde{X})^T}{\|X - \tilde{X}\|_F^2} \right) = \mathbb{E} \left( \frac{(r'AUB^\top)(r'AUB^\top)^T}{\|r'AUB^\top\|_F^2} \right)
\]

\[
= \mathbb{E} \left( \frac{(r'AUB^\top)(AUB^\top)^T}{\|AUB^\top\|_F^2} \right) = \mathbb{E} \left( \frac{(rAUB^\top)(rAUB^\top)^T}{\|rAUB^\top\|_F^2} \right)
\]

\[
= \mathbb{E} \left( \frac{(X - M)(X - M)^T}{\|X - M\|_F^2} \right).
\]

The proof process for \( K_c \) is similar. \( \square \)

Lemma C.2. Let \( X \sim E_{p,q}(M, \Sigma \otimes \Omega, \psi) \) be a continuous random matrix, \( \Sigma = AA^\top \) and \( \Omega = BB^\top \). It takes another stochastic representation:

\[
X \overset{d}{=} M + r \frac{Z}{\|A^\dagger ZB\|_F^2},
\]

where \( Z \sim \mathcal{MN}(0, \Sigma, \Omega) \), \( A^\dagger \) and \( B^\dagger \) are the Moore-Penrose pseudoinverse of \( A \) and \( B \) respectively.

Proof. Let \( X = M + rAUB^\top \), \( \text{rank} (\Sigma) = \text{rank} (A) = m \), \( \text{rank} (\Omega) = \text{rank} (B) = n \). Let \( U = \frac{\mathbb{E}}{\|\mathbb{E}\|_F} \), so that \( \text{Vec}(U) = \frac{\text{Vec}(\varepsilon)}{\|\text{Vec}(\varepsilon)\|_2} \), where \( \text{Vec}(\varepsilon) \) is a standard normal vector in \( \mathbb{R}^{mn} \). Note that if \( A = V_1 \Sigma V_2^\top \) is the singular value decomposition of \( A \in \mathbb{R}^{p \times m} \) with \( V_1 \in \mathbb{R}^{p \times m} \) and \( D, V_2 \in \mathbb{R}^{m \times m} \), then \( A^\dagger = V_2 D^{-1} V_1^\top \). Since \( \text{rank} (A) = m \), we have \( A^\dagger A = I_m \). \( B \) is similar. Let \( Z = A\mathbb{E} B^\top \),

\[
\text{Vec}(Z) = (B \otimes A)\text{Vec}(\varepsilon) \sim \mathcal{N}(0, \Omega \otimes \Sigma) \iff Z \sim \mathcal{MN}(0, \Sigma, \Omega)
\]

then we have \( X = M + rA \frac{\mathbb{E}}{\|\mathbb{E}\|_F} B = M + r \frac{Z}{\|A^\dagger ZB\|_F^2} \). The proof is complete. \( \square \)
The following Lemma C.3 to C.5 establish the stochastic order of the following quantities:

\[
M_1 = \frac{2}{T(T - 1)} \sum_{1 \leq t < s \leq T} \frac{(F_t - F_s)C^T C(F_t - F_s)^T}{\|X_t - X_s\|_F^2},
\]

\[
M_2 = \frac{2}{T(T - 1)} \sum_{1 \leq t < s \leq T} \frac{(E_t - E_s)C(F_t - F_s)^T}{\|X_t - X_s\|_F^2},
\]

\[
M_3 = \frac{2}{T(T - 1)} \sum_{1 \leq t < s \leq T} \frac{(F_t - F_s)C(E_t - E_s)^T}{\|X_t - X_s\|_F^2},
\]

\[
M_4 = \frac{2}{T(T - 1)} \sum_{1 \leq t < s \leq T} \frac{(E_t - E_s)(E_t - E_s)^T}{\|X_t - X_s\|_F^2}.
\]

**Lemma C.3.** Under Assumption A-C, as \(\min\{T, p_1, p_2\} \to \infty\), we have

\[
\|M_2\|_F^2 = O_p\left(\frac{1}{T^{p_1p_2}} + \frac{1}{p_1^2 p_2^2}\right),
\]

\[
\|M_3\|_F^2 = O_p\left(\frac{1}{T^{p_1p_2}} + \frac{1}{p_1^2 p_2^2}\right).
\]

**Proof.** Note that \(M_3\) is \(M_2\)'s transpose, so we only show the result with \(M_2\). Similar to the proof in He et al. (2022), we assume \(T\) is even and \(T = T/2\), otherwise we can delete the last observation. Given a permutation \(\sigma\) of \(\{1, \ldots, T\}\), let \(F^\sigma_t, E^\sigma_t\) and \(X^\sigma_t\) be the rearranged factors, errors, observations, and

\[
M_2^\sigma = \frac{1}{T} \sum_{s=1}^T (E^\sigma_t - E^\sigma_s)C(F^\sigma_t - F^\sigma_s)^T \frac{1}{\|X^\sigma_t - X^\sigma_s\|_F^2}.
\]

Denote \(\mathcal{S}_T\) as the set containing all the permutations of \(\{1, \ldots, T\}\). After some elementary calculations,

\[
\sum_{\sigma \in \mathcal{S}_T} \frac{T}{2} M_2^\sigma = T \times (T - 2)! \times \frac{T(T - 1)}{2} M_2.
\]

That is,

\[
M_2 = \frac{1}{T!} \sum_{\sigma \in \mathcal{S}_T} M_2^\sigma \Rightarrow \mathbb{E}\|M_2\|_F \leq \frac{1}{T!} \sum_{\sigma \in \mathcal{S}_T} \mathbb{E}\|M_2^\sigma\|_F = \mathbb{E}\|M_2\|_F \leq \sqrt{\mathbb{E}\|M_2\|_F^2}.
\]

Now we assume \(\sigma\) is given. By the definition of joint matrix elliptical distribution, for any \(s = 1, \ldots, T\),

\[
\begin{pmatrix}
\text{Vec}(F_{2s-1} - F_{2s}) \\
\text{Vec}(E_{2s-1} - E_{2s})
\end{pmatrix}
\overset{d}{=} \begin{pmatrix}
I_{k_2} \otimes I_{k_1} \\
0 \\
0 \\
\Omega_{\epsilon}^{1/2} \otimes \Sigma_{\epsilon}^{1/2}
\end{pmatrix}
\begin{pmatrix}
Z \\
\|Z\|_2
\end{pmatrix},
\]

where \(Z = (\text{Vec}(Z_1)^T, \text{Vec}(Z_2)^T)^T \sim \mathcal{N}(0, I_{k_1 k_2 + p_1 p_2})\) and \(Z\) is independent of \(r\). \(Z_1\) and \(Z_2\) are \(k_1 \times k_2\) matrices, respectively. Hence,

\[
X_s \overset{d}{=} \frac{\Sigma_{\epsilon}^{1/2} Z_2 \Omega_{\epsilon}^{1/2} C Z_1^T}{\|R Z_1 C^T + \Sigma_{\epsilon}^{1/2} Z_2 \Omega_{\epsilon}^{1/2}\|_F^2}.
\]

Note that \(X_s\) and \(X_t\) are independently and identically distributed when \(s \neq t\), so

\[
\mathbb{E}\|M_2^\sigma\|_F^2 = \mathbb{E}\left\|\frac{1}{T} \sum_{s=1}^T X_s\right\|_F^2 = \frac{1}{T\mathbb{E}\|X_1\|_F^2} + \frac{T(T - 1)}{T^2} \mathbb{E}\|X_1\|_F^2.
\]

(C.1)
Hence, for the second term, according to Lemma S2 in He et al. (2022), we have
\[
\begin{bmatrix}
u_1 \\ u_2
\end{bmatrix} = \begin{bmatrix}
I_{k_2} \otimes I_{k_1} \\ 0
\end{bmatrix} \begin{bmatrix}
-(C \otimes R)^T (CC^T \otimes \Omega_e \otimes \Sigma_e)^{-1} \\ 0
\end{bmatrix} \begin{bmatrix}
I_{k_2} \otimes I_{k_1} \\ C \otimes R \quad \Omega_e \otimes \Sigma_e^{1/2}
\end{bmatrix} \begin{bmatrix}
\text{Vec}(Z_1) \\ \text{Vec}(Z_2)
\end{bmatrix}
\sim \mathcal{N} \left( 0, \begin{bmatrix}
\Sigma_{u_1} & 0 \\ 0 & \Sigma_x
\end{bmatrix} \right),
\]
where \(\Sigma_x = CC^T \otimes RR^T + \Omega_e \otimes \Sigma_e, \Sigma_{u_1} = I_{k_1} \otimes I_{k_2} - (C \otimes R)^T \Sigma_x^{-1} (C \otimes R). u_1, u_2\) are independent and
\[
\begin{bmatrix}
\text{Vec}(Z_1) \\ \text{Vec}(Z_2)
\end{bmatrix} = \begin{bmatrix}
I_{k_1} \\ -(\Omega^{1/2} \otimes \Sigma^{1/2})^{-1} (C \otimes R) \quad \Sigma_x^{-1}
\end{bmatrix} \begin{bmatrix}
\text{Vec}(Z_1) \\ \text{Vec}(Z_2)
\end{bmatrix} = \begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}.
\]
Since \(k_1, k_2\) are fixed numbers, without loss of generality, we let \(k_1 = k_2 = 1\) in the following to simplify notation. For general fixed \(k_1, k_2\), the error bounds are the same. As a result,
\[
X_1 \overset{d}{=} \frac{\Sigma_x^{1/2} Z_1 \Omega_x^{1/2} C Z_1^T}{\|RZ_1 C^T + \Sigma_x^{1/2} Z_2 \Omega_x^{1/2}\|_F^2}
\Rightarrow \text{Vec}(X_1) = \frac{(Z_1 C^T \Omega_e^{1/2}) \otimes \Sigma_x^{1/2} \text{Vec}(Z_2)}{\|\text{Vec}(RZ_1 C^T + \Sigma_x^{1/2} Z_2 \Omega_x^{1/2})\|_F^2}
= \frac{(u_1 + (C \otimes R)^T \Sigma_x^{-1} u_2) - (C \otimes I_{p_1}) (C \otimes R) u_1 + (C \otimes I_{p_1}) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} u_2}{\|u_2\|^2}.
\]
By Lemma S1 in He et al. (2022), \(\|\Sigma_{u_1}\|_F^2 = O\left( \frac{1}{p_1 p_2} \right)\), \(\|C \otimes R\|^T \Sigma_x^{-2} (C \otimes R)\|_F^2 = O\left( \frac{1}{p_1 p_2} \right)\), \(\Sigma_x\) and \(C \otimes R\) are equivalent to \(\Sigma_y\) and \(L\) respectively defined in He et al. (2022).
Because \(u_1\) and \(u_2\) are zero-mean independent Gaussian vectors, we have
\[
\mathbb{E} \frac{u_1 u_2}{\|u_2\|^2} = 0, \quad \mathbb{E} \|u_2\|^{-2} \leq \frac{1}{\lambda_{p_1 p_2}(\Sigma_x)} \mathbb{E} \frac{1}{\lambda_{p_1 p_2}^2} \approx \frac{1}{p_1 p_2}.
\]
Hence,
\[
\mathbb{E} \text{Vec}(X_1) = \text{Vec} \left( -\mathbb{E}(\|u_2\|^{-2}) ((C^T C) \otimes R) \Sigma_{u_1} + (C \otimes I_{p_1}) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} \mathbb{E} \frac{u_2 u_2^T}{\|u_2\|^2} \Sigma_x^{-1} (C \otimes R) \right).
\] (C.2)
For the first term,
\[
\left\| -\mathbb{E}(\|u_2\|^{-2}) ((C^T C) \otimes R) \Sigma_{u_1} \right\|_F^2 \leq \|\Sigma_{u_1}\|_F^2 \|C^T C\|_F^2 \|R\|_F^2 \left( \mathbb{E}(\|u_2\|^{-2}) \right)^2 = O\left( \frac{1}{p_1 p_2} \right). \] (C.3)
For the second term, according to Lemma S2 in He et al. (2022), we have
\[
\left\| \Sigma_x^{-1} \mathbb{E} \frac{u_2 u_2^T}{\|u_2\|^2} \Sigma_x^{-1} (C \otimes R) \right\|_F^2 = O\left( \frac{1}{p_1 p_2} \right).
\]
Therefore,
\[
\left\| (C \otimes I_{p_1}) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} \mathbb{E} \frac{u_2 u_2^T}{\|u_2\|^2} \Sigma_x^{-1} (C \otimes R) \right\|_F^2 \leq \|C \otimes I_{p_1}\|_F^2 \|\Omega_e \otimes \Sigma_e\| \left\| \Sigma_x^{-1} \mathbb{E} \frac{u_2 u_2^T}{\|u_2\|^2} \right\|_F \left\| \Sigma_x^{-1} (C \otimes R) \right\|_F^2 = O\left( \frac{1}{p_1 p_2} \right). \] (C.4)
Combing (C.2), (C.3) and (C.4), we have \( \| \text{Evec}(X_1) \|_{F}^{2} = O \left( \frac{1}{p_{1}p_{2}} \right) \).

Now we consider \( E \| X_1 \|_{F}^{2} \). According to Lemma S2 in He et al. (2022),

\[
E \| u_1 \|^{2} \leq \frac{1}{p_{1}p_{2}}, \quad E \| u_1 \|^{4} \leq \frac{1}{p_{1}p_{2}}, \quad E \| u_2 \|^{-4} \asymp \frac{1}{p_{1}p_{2}^{2}}, \quad E \| u_2 \|^{-8} \leq \frac{1}{\lambda_{p_{1}p_{2}}^{4}(\Sigma_{x})} E \chi_{p_{1}p_{2}}^{-4} \asymp (p_{1}p_{2})^{-4}.
\]

Then

\[
E \| X_1 \|_{F}^{2} = E \left\| \frac{\Sigma_{e}^{1/2} Z_{2} \Omega_{e}^{1/2} C Z_{1}^{T}}{\| R Z_{1} C^{T} + \Sigma_{e}^{1/2} Z_{2} \Omega_{e}^{1/2} \|_{F}^{2}} \right\|^{2} = E \left( \frac{E}{\| R Z_{1} C^{T} + \Sigma_{e}^{1/2} Z_{2} \Omega_{e}^{1/2} \|_{F}^{4}} \right) \| \Sigma_{e}^{1/2} Z_{2} \Omega_{e}^{1/2} C Z_{1}^{T} \|_{F}^{2}
\leq C \sqrt{E(\| Z_{2} \Omega_{e}^{1/2} C \|_{F}^{2}) E(\| u_2 \|^{-8})} = O \left( \frac{1}{p_{1}p_{2}} \right).
\]

As a result,

\[
E \| M_{2} \|_{F}^{2} \leq \sqrt{\frac{1}{T_{p_{1}p_{2}}} + \frac{1}{p_{1}^{2}p_{2}^{2}}} \Rightarrow \| M_{2} \|_{F}^{2} = O_{p} \left( \frac{1}{T_{p_{1}p_{2}}} + \frac{1}{p_{1}^{2}p_{2}^{2}} \right).
\]

\( \| M_{3} \|_{F}^{2} = O_{p} \left( \frac{1}{T_{p_{1}p_{2}}} + \frac{1}{p_{1}^{2}p_{2}^{2}} \right) \) can be derived similarly.

\( \text{Lemma C.4.} \) Under Assumptions A-C, we have \( \| M_{1} \|_{F}^{2} = O_{p} \left( \frac{1}{p_{1}} \right) \) and \( \| \hat{H}_{R} \|_{F}^{2} = O_{p}(1) \).

\( \text{Proof.} \) Similarly to equation (C.1) in Lemma C.3, it is easy to verify that

\[
E \| M_{1} \|_{F} \leq \sqrt{\frac{1}{T} E \| X \|_{F}^{2} + \frac{T(T - 1)}{T^{2}}} \| \text{Evec}(X) \|_{F}^{2},
\]

where \( X = \frac{Z C^{T} C Z^{T}}{\| R Z_{1} C^{T} + \Sigma_{e}^{1/2} Z_{2} \Omega_{e}^{1/2} \|_{F}^{2}} \), \( \text{Vec}(X) = \left( u_{1} + (C^{T} \Sigma_{x}^{-1} u_{2}) C^{T} C \left( u_{1} + (C^{T} R) \Sigma_{x}^{-1} u_{2} \right) \right) \) with \( Z_{1}, Z_{2}, u_{1}, u_{2} \) defined in Lemma C.3. Therefore,

\[
\| \text{Evec}(X) \|_{F}^{2} \leq \left( E \| u_{2} \|^{2} \right)^{2} \| C^{T} C \|_{F}^{2} \| \Sigma_{u_{1}} \|_{F}^{2} \| C \circ R \|_{F}^{2} \left( \Sigma_{x}^{-1} \left( \frac{u_{2} u_{2}^{T}}{\| u_{2} \|^{2}} \right) \Sigma_{x}^{-1} (C \circ R) \right) \| \| C^{T} C \|_{F}^{2} = O \left( \frac{1}{p_{1}} \right).
\]

On the other hand,

\[
E \| X \|_{F}^{2} \leq E(\| u_{1} \|^{4}) E(\| u_{2} \|^{-4}) \| C^{T} C \|_{F}^{2} \leq O \left( \frac{1}{p_{1}} \right).
\]

As a result, \( \| M_{1} \|_{F}^{2} = O_{p} \left( \frac{1}{p_{1}} \right) \) and by the definition of \( \hat{H}_{R} \) we have

\[
\| \hat{H}_{R} \|_{F}^{2} \leq \| M_{1} \|_{F}^{2} \| R \|_{F}^{2} \| \hat{H}_{R} \|_{F}^{2} \| \hat{A}^{-1} \|_{F}^{2} = O_{p}(1),
\]

which completes the proof.

\( \text{Lemma C.5.} \) Under Assumption A-C, we have

\[
\frac{1}{p_{1}} \| M_{4} \|_{F}^{2} = O_{p} \left( \frac{1}{T_{p_{2}}} + \frac{1}{p_{1}} \right) + o_{p}(1) \times \frac{1}{p_{1}} \| \hat{R} - R \hat{H}_{R} \|_{F}^{2}.
\]

\( \text{Proof.} \) By the decomposition \( \hat{R} = \hat{R} - R \hat{H}_{R} + R \hat{H}_{R} \), we have

\[
\frac{1}{p_{1}} \| M_{4} \|_{F}^{2} \leq \frac{1}{p_{1}} \| M_{4} \|_{F}^{2} \| \hat{H}_{R} \|_{F}^{2} + \| M_{4} \|_{F}^{2} \times \frac{1}{p_{1}} \| \hat{R} - R \hat{H}_{R} \|_{F}^{2}.
\]  (C.5)
We start with $\|M_4\|_F^2$. Similarly to the proof of equation C.1 in Lemma C.3,

$$
E\|M_4\|_F \leq \sqrt{\frac{1}{T}E\|X\|_F^2 + \frac{T(T - 1)}{T^2}E\|X\|_F^2},
$$

where $X \overset{d}{=} \frac{\Sigma_1^{1/2}Z_2\Omega_2\Sigma_2^{1/2}}{\|RZ_1C^\top + \Sigma_1^{1/2}Z_2\Omega_2\Sigma_2^{1/2}\|_F}$. Let $N = \Sigma_1^{1/2}Z_2\Omega_2^{1/2}$, so

$$
\text{Vec}(X) = \frac{\sum_{j=1}^{p_2} N_{:j} \otimes N_{:j}}{\|u_2\|^2}.
$$

Denote $E_j((\Omega_2^{1/2} \otimes \Sigma_2^{1/2})\text{Vec}(Z_2))$ to extract the $(j-1)p_1+1$-th to $jp_1$-th entries of $(\Omega_2^{1/2} \otimes \Sigma_2^{1/2})\text{Vec}(Z_2)$, where

$$
E_j = (0, \ldots, \underbrace{I_{p_1}}_{j\text{-th block}}, \ldots, 0)_{p_1 \times (p_1p_2)}.
$$

Then,

$$
\text{Vec}(X) = \frac{\sum_{j=1}^{p_2} \left( E_j((\Omega_2^{1/2} \otimes \Sigma_2^{1/2})\text{Vec}(Z_2)) \otimes (E_j((\Omega_2^{1/2} \otimes \Sigma_2^{1/2})\text{Vec}(Z_2)) \right)}{\|u_2\|^2} = \frac{\sum_{j=1}^{p_2} \left( E_j\left(- (C \otimes R)u_1 + (\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2\right) \otimes (E_j\left(- (C \otimes R)u_1 + (\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2\right) \right)}{\|u_2\|^2}.
$$

Therefore,

$$
E\text{Vec}(X) = \sum_{j=1}^{p_2} \left( E(\|u_2\|^2) - \left( E_j(C \otimes R) \otimes (E_j(C \otimes R) \right) \Sigma_{u_1} + E\left( \frac{(E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2) \otimes (E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2))}{\|u_2\|^2} \right) \right).
$$

For the first term,

$$
\left\| - E(\|u_2\|^2) \left( E_j(C \otimes R) \otimes (E_j(C \otimes R) \right) \Sigma_{u_1} \right\|_F \leq \|E_j(C \otimes R)\|_F \|\Sigma_{u_1}\|_F E(\|u_2\|^2) \leq O\left( \frac{1}{p_1p_2} \right).
$$

For the second term,

$$
E\left( \frac{(E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2) \otimes (E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2))}{\|u_2\|^2} \right) = \text{Vec}\left( (E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1})E\left( \frac{u_2u_2^\top}{\|u_2\|^2} \right)(E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1})^\top) \right).
$$

Therefore,

$$
\left\| E\left( \frac{(E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2) \otimes (E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}u_2))}{\|u_2\|^2} \right) \right\|_F^2 \leq \left\| E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}(\Omega_e \otimes \Sigma_e)^\top E_j \right\|_F^2 \left\| E\left( \frac{(\Sigma_x^{-1/2}u_2u_2^\top)\Sigma_x^{-1/2}}{\|u_2\|^2} \right) \right\|_F^2 \leq \frac{1}{p_1p_2} \left\| E_j((\Omega_e \otimes \Sigma_e)\Sigma_x^{-1}(\Omega_e \otimes \Sigma_e)^\top E_j \right\|_F^2 \leq \frac{1}{p_1p_2} \left\| E_j\Sigma_x \right\|_F^2 = O\left( \frac{1}{p_1p_2} \right),
$$

(C.6)
Hence, denote so that
\[
X = \sum_{i=1}^{\infty} \left\| R_{ij} \right\|^2 Z_{ij}^1 \top + \Omega \Lambda \frac{\left\| Z_{ij}^1 \right\|}{\parallel \gamma_i \parallel} \frac{\left\| \gamma_i \right\|}{\parallel \gamma_i \parallel} Z_{ij}^1 \top.
\]
where we use the fact that \[\left\| \mathbb{E} \left( \frac{\sum_{i=1}^{\infty} u_i u_i \Sigma_{ij}^{1/2}}{\left\| u_i \right\|^2} \right) \right\|^2 = O \left( \frac{1}{p_1 p_2} \right) \]. To prove this, denote the spectral decomposition of \( \Sigma_x \) as \( \Gamma_x \lambda_x \Gamma_x \top \) where \( \Gamma_x \) is an orthonormal matrix. Then
\[
\left\| \mathbb{E} \left( \frac{\sum_{i=1}^{\infty} u_i u_i \Sigma_{ij}^{1/2}}{\left\| u_i \right\|^2} \right) \right\|^2 = \left\| \lambda_x^{-1/2} \mathbb{E} \left( \frac{\Gamma_x \top u_i u_i \Gamma_x}{\left\| \Gamma_x \top u_i \right\|^2} \right) \lambda_x^{-1/2} \right\|^2.
\]
Denote
\[
\bar{\lambda} := \mathbb{E} \left( \frac{\Gamma_x \top u_i u_i \Gamma_x}{\left\| \Gamma_x \top u_i \right\|^2} \right),
\]
so that
\[
\bar{\lambda} \leq \lambda_x^{1/2} \mathbb{E} \left( \frac{\lambda_x^{-1/2} \Gamma_x \top u_i u_i \Gamma_x \lambda_x^{-1/2}}{\lambda_{p_1 p_2}(\Sigma_x) \parallel \lambda_x^{-1/2} \Gamma_x \top u_i \parallel^2} \right) \lambda_x^{1/2} = \frac{1}{p_1 p_2 \lambda_{p_1 p_2}(\Sigma_x) \lambda_x}.
\]
Consequently,
\[
\left\| \mathbb{E} \left( \frac{\sum_{i=1}^{\infty} u_i u_i \Sigma_{ij}^{1/2}}{\left\| u_i \right\|^2} \right) \right\|^2 = O \left( \frac{1}{p_1 p_2} \right).
\]
Hence,
\[
\parallel \text{vec}(X) \parallel_F^2 = p_2 \left( \frac{1}{p_1 p_2} \right) + \left( \frac{1}{p_1 p_2} \right) = O \left( \frac{1}{p_1} \right).
\]
On the other hand,
\[
\mathbb{E} \parallel X \parallel_F^2 = \mathbb{E} \left\| \Sigma_x^{1/2} Z \Omega \Sigma_x^{1/2} \right\|_F^2 = \mathbb{E} \left\| \Sigma_x^{1/2} Z \Omega \Sigma_x^{1/2} \right\|_F^2,
\]
\[
\leq \mathbb{E} \left( \left\| \Sigma_x^{1/2} Z \Omega \right\|_F^4 \right) \mathbb{E} \left( \parallel u_i \parallel^{-8} \right) = O \left( \frac{1}{p_2} \right).
\]
As a result,
\[
\parallel M_4 \parallel_F^2 = O \left( \frac{1}{T p_2} + \frac{1}{p_1} \right) = o_p(1).
\]
Next, we consider \( \parallel M_4 R \parallel_F^2 \). Indeed, the proof is very similar to that of \( \parallel M_4 \parallel_F^2 \). We have
\[
\mathbb{E} \parallel M_4 R \parallel_F^2 \leq \frac{1}{T} \mathbb{E} \parallel \tilde{X} \parallel_F^2 + \frac{T (T - 1)}{T^2} \mathbb{E} \parallel \tilde{X} \parallel_F^2,
\]
where \( \tilde{X} = \frac{\Sigma_x^{1/2} Z \Omega \Sigma_x^{1/2} \tilde{R}}{\parallel R Z_1 C + \Sigma_x^{1/2} Z \Omega \Sigma_x^{1/2} \parallel_F^2} \). Define
\[
\tilde{E}_j = (0, \ldots, I_{k_1}, \ldots, 0)_{k_1 \times (k_1 p_2)},
\]
\( j \)-th block.
Then,

\[
\text{Vec}(\tilde{X}) = \frac{\sum_{j=1}^{p_2} \left( E_j \left( (\Omega c^{1/2} \otimes \Sigma c^{1/2}) \text{Vec}(Z_2) \right) \otimes \left( \tilde{E}_j \left( (\Omega c^{1/2} \otimes R^T \Sigma c^{1/2}) \text{Vec}(Z_2) \right) \right) \right)}{\|u_2\|^2} \\
= \frac{\sum_{j=1}^{p_2} \left( \left( E_j ( - (C \otimes R) u_1 + (\Omega c \otimes \Sigma c) \Sigma_x^{-1} u_2) \right) \otimes \left( \tilde{E}_j ((C \otimes R^T R) u_1 + (\Omega c \otimes R^T \Sigma c) \Sigma_x^{-1} u_2) \right) \right)}{\|u_2\|^2},
\]

\[
E(\text{Vec}(\tilde{X})) = \sum_{j=1}^{p_2} E \left( \|u_2\|^{-2} \left( E_j (C \otimes R) \right) \otimes \left( \tilde{E}_j (C \otimes R^T R) \right) \right) \Sigma_{u_4} \\
+ E \left( \|u_2\|^2 \left( E_j (\Omega c \otimes \Sigma c) \Sigma_x^{-1} u_2) \otimes \left( \tilde{E}_j (\Omega c \otimes R^T \Sigma c) \Sigma_x^{-1} u_2) \right) \right) \right) \right),
\]

For the first term, we see that

\[
\left\| E \left( \|u_2\|^{-2} \left( E_j (C \otimes R) \right) \otimes \left( \tilde{E}_j (C \otimes R^T R) \right) \right) \Sigma_{u_4} \right\|_F^2 \leq E \|u_2\|^{-4} \|\Sigma_{u_4}\|_F^{-2} \|C\|_F^2 \|R\|_F^2 \|R^T R\|_F^2 = O \left( \frac{1}{p_1 p_2^2} \right).
\]

For the second term,

\[
E \left( \|u_2\|^2 \left( E_j (\Omega c \otimes \Sigma c) \Sigma_x^{-1} u_2) \otimes \left( \tilde{E}_j (\Omega c \otimes R^T \Sigma c) \Sigma_x^{-1} u_2) \right) \right) \right) \right),
\]

which implies that

\[
\left\| E \left( \|u_2\|^2 \left( E_j (\Omega c \otimes \Sigma c) \Sigma_x^{-1} u_2) \otimes \left( \tilde{E}_j (\Omega c \otimes R^T \Sigma c) \Sigma_x^{-1} u_2) \right) \right) \right) \right),
\]

\[
\leq \frac{1}{p_1 p_2} \|R\|_F \|\Sigma_{u_4}\|_F^{-2} \leq O \left( \frac{1}{p_1 p_2^2} \right).
\]

Therefore, we conclude that

\[
\|E \tilde{X}\|_F^2 \leq O \left( \frac{1}{p_1} \right).
\]

On the other hand,

\[
E \|\tilde{X}\|_F^2 \leq E \|X\|_F^2 \|R\|^2 \leq O \left( \frac{p_1}{p_2} \right).
\]

Consequently, we conclude that

\[
\frac{1}{p_1} \|M_2 R\|_F^2 \leq O_p \left( \frac{1}{p_1} + \frac{1}{p_1^2} \right).
\]

Hence,

\[
\frac{1}{p_1} \|M_2 \hat{R}\|_F^2 = O_p \left( \frac{1}{T p_2} + \frac{1}{p_1} \right) + o_p(1) \times \frac{1}{p_1} \|\hat{R} - R \hat{H}\|_F^2,
\]

which concludes the lemma.
Lemma C.6. Under Assumptions A-C, we have
\[
\left\| \frac{1}{p_1p_2} \hat{H}^\top_R R^\top E_t C \hat{H}_C \right\|^2_F = O_p\left(\frac{1}{p_1p_2}\right).
\]

Proof. According to equation 2.2, we have \( E_t = \frac{r_t}{\|Z_t\|_2} \Sigma_e^{1/2} Z_t^E \Omega_e^{1/2} \), then
\[
\left\| R^\top E_t C \right\|^2_F = \left\| \frac{r_t}{\|Z_t\|_2} R^\top \Sigma_e^{1/2} Z_t^E \Omega_e^{1/2} C \right\|^2_F = \left\| \frac{r_t}{\|Z_t\|_2} (C^\top \Omega_e^{1/2}) \otimes (R^\top \Sigma_e^{1/2}) \text{Vec}(Z_t^F) \right\|^2_F
\]
\[
= \left\| \frac{r_t}{\|Z_t\|_2} (0 \otimes (R^\top \Sigma_e^{1/2}) Z_t^F) \right\|^2_F = O_p(p_1p_2). \tag{C.7}
\]

Since \( \|\hat{H}_R\|^2_F = O_p(1) \), \( \|\hat{H}_C\|^2_F = O_p(1) \), we have
\[
\left\| \frac{1}{p_1p_2} \hat{H}_R^\top R^\top E_t C \hat{H}_C \right\|^2_F = O_p\left(\frac{1}{p_1p_2}\right).
\]

Lemma C.7. Under Assumptions A-C, we have
\[
\left\| \frac{1}{p_1} \hat{R}^\top (\hat{R} - R\hat{H}_R) \right\|^2_F = O_p\left(\frac{1}{T p_1 p_2}\right), \quad \left\| \frac{1}{p_2} (\hat{C} - C\hat{H}_C) \hat{C} \right\|^2_F = O_p\left(\frac{1}{T p_1 p_2}\right).
\]

Proof. Note that
\[
\frac{1}{p_1} \hat{R}^\top (\hat{R} - R\hat{H}_R) = \frac{1}{p_1} (\hat{R} - R\hat{H}_R)^\top (\hat{R} - R\hat{H}_R) + \frac{1}{p_1} \hat{H}_R R^\top (\hat{R} - R\hat{H}_R)
\]
while \( \frac{1}{p_1} \|\hat{R} - R\hat{H}_R\|^2_F = O_p\left(\frac{1}{T p_2} + \frac{1}{p_1}\right) \) and \( \hat{H}_R = O_p(1) \). Hence it suffices to bound \( R^\top (\hat{R} - R\hat{H}_R) \). By equation (B.2),
\[
R^\top (\hat{R} - R\hat{H}_R) = (R^\top M_2 R^\top + R^\top R M_3 + R^\top M_4) \hat{R} \hat{A}^{-1}.
\]

We will bound the three error terms separately.

Firstly, similar to the proof of equation C.1 in Lemma C.3, we have
\[\mathbb{E}\|R^\top M_2\|^2_F \lesssim \sqrt{\frac{1}{T} \mathbb{E}\|X\|^2_F + \|EX\|^2_F},\]
where \( X = \frac{R^\top \Sigma_x^{1/2} Z_x \Omega_x^{1/2} C \Sigma_x^{-1} Z_x}{\|RZ_x\|_2 + \Sigma_x^{1/2} Z_x \Omega_x^{1/2}}. \) We still assume \( k_1 = k_2 = 1 \). Then,
\[
\text{Vec}(X) = \frac{Z_1(C^\top \Omega_e^{1/2}) \otimes (R^\top \Sigma_e^{1/2}) \text{Vec}(Z_2)}{\|\text{Vec}(RZ_x C^\top + \Sigma_x^{1/2} Z_x \Omega_x^{1/2})\|^2} \times \left( u_1 + (C \otimes R)^\top \Sigma_x^{-1} u_2 \right) \left( C \Omega_e^{1/2} \otimes R \Sigma_e^{1/2} \right) \left( - (\Omega_e^{1/2} \otimes \Sigma_x^{-1}) (C \otimes R) u_1 + (\Omega_e^{1/2} \otimes \Sigma_e^{1/2}) \Sigma_x^{-1} u_2 \right)
\]
\[
= \frac{-u_1 (C^\top C \otimes R^\top R) u_1 + u_1 (C \otimes R)^\top (\Omega_x \otimes \Sigma_x) \Sigma_x^{-1} u_2}{\|u_2\|^2_F} - (C \otimes R)^\top \Sigma_x^{-1} u_2 (C^\top C \otimes R^\top R) u_1 + (C \otimes R)^\top \Sigma_x^{-1} u_2 (C \otimes R)^\top (\Omega_x \otimes \Sigma_x) \Sigma_x^{-1} u_2.
\]
Following similar technique as in the proof of Lemma C.3,

\[ \| \mathbb{E} \text{Vec}(X) \|_F^2 \lesssim \| C \otimes R \|_F^2 \| \Sigma_{u_1} \|_F^2 (\mathbb{E} \| u_2 \|^{-2})^2 + \| C \otimes R \|_F^2 \| \Omega_e \otimes \Sigma_e \|_2 \| \Sigma_x^{-1} \mathbb{E} (u_2 u_2^\top / \| u_2 \|^2) \Sigma_x^{-1} (C \otimes R) \|_F^2 = \mathcal{O} \left( \frac{1}{p_1^2 p_2^2} \right). \]

On the other hand,

\[ \mathbb{E} \| X \|_F^2 \lesssim \mathbb{E} (\| u_2 \|^{-4}) \| R^\top \Sigma_e^{1/2} Z_2 \Sigma_e^{1/2} C Z_1 \|_2^2 \leq \mathcal{O} \left( \frac{1}{p_1 p_2} \right). \]

As a result,

\[ \| R^\top M_2 \|_F^2 = O_p \left( \frac{1}{p_1^2 p_2^2} + \frac{1}{p_1^2 p_2^2} \right). \]

Similarly, \( \| M_3 R \|_F^2 = O_p \left( \frac{1}{p_1^2 p_2^2} + \frac{1}{p_1^2 p_2^2} \right) \) and

\[ \| M_3 \hat{R} \|_F^2 \lesssim \| M_3 \|_F^2 + \| M_3 \|_F^2 \| \hat{R} - R \hat{H} R \|_F^2 = O_p \left( \frac{1}{p_1^2 p_2^2} + \frac{1}{p_1^2 p_2^2} + \frac{1}{p_1^2 p_2^2} \right). \]

For the last term, we can verify that \( \mathbb{E} \| R^\top M_4 R \|_F \lesssim \sqrt{\frac{1}{p_1^2 p_2^2} \mathbb{E} \| X \|_F^2 + \| \mathbb{E} X \|_2^2} \), where \( X \triangleq \frac{R^\top \Sigma_e^{1/2} Z_2 \Sigma_e^{1/2} R}{\| R^\top \Sigma_e^{1/2} Z_2 \Sigma_e^{1/2} R \|_F} \) by slightly abusing the notation \( X \). Let \( N = R^\top \Sigma_e^{1/2} Z_2 \Sigma_e^{1/2} \), \( \text{Vec}(N) = (\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2})) \text{Vec}(Z_2) \). Then we have

\[ \text{Vec}(X) = \frac{\sum_{j=1}^{p_1} N_{j,j} \otimes N_{j,j}}{\| u_2 \|^2}. \]

Let \( E_j \left( (\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2})) \text{Vec}(Z_2) \right) \) extract the \((j-1)p_1 + 1\)-th to \(jp_1\)-th entries of \((\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2})) \text{Vec}(Z_2)\). Then,

\[ \text{Vec}(X) = \sum_{j=1}^{p_1} \left( E_j \left( (\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2})) \text{Vec}(Z_2) \right) \right) \otimes \frac{E_j \left( (\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2})) \text{Vec}(Z_2) \right)}{\| u_2 \|^2}, \]

where

\[ E_j (\Omega_e^{1/2} \otimes (R^\top \Sigma_e^{1/2}) \text{Vec}(Z_2)) = E_j \left( -(C \otimes R^\top) u_1 + (I_{p_2} \otimes R^\top) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} u_2 \right). \]

Therefore,

\[ \mathbb{E} \text{Vec}(X) = \sum_{j=1}^{p_1} \mathbb{E}(\| u_2 \|^{-2}) \left( E_j (C \otimes R^\top R) \right) \otimes \left( E_j (C \otimes R^\top R) \right) \Sigma_{u_1} + \sum_{j=1}^{p_1} \mathbb{E} \left( \left( E_j (I_{p_2} \otimes R^\top) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} u_2 \right) \otimes \left( E_j (I_{p_2} \otimes R^\top) (\Omega_e \otimes \Sigma_e) \Sigma_x^{-1} u_2 \right) \right) / \| u_2 \|^2. \]

It is easy to verify that

\[ \left\| \mathbb{E}(\| u_2 \|^{-2}) \left( E_j (C \otimes R^\top R) \right) \otimes \left( E_j (C \otimes R^\top R) \right) \Sigma_{u_1} \right\|_F^2 = \mathcal{O} \left( \frac{1}{p_2^2} \right). \]
and

\[ \left\| \mathbb{E}\left( \frac{(E_j ((IP_2 \otimes R^\top)(\Omega_c \otimes \Sigma_c) \Sigma^{-1}_x u_2)) \otimes (E_j ((IP_2 \otimes R^\top)(\Omega_c \otimes \Sigma_c) \Sigma^{-1}_x u_2))}{\|u_2\|^2} \right) \right\|_F^2 \]

\[ \leq \left\| \mathbb{E}\left( (E_j ((IP_2 \otimes R^\top)(\Omega_c \otimes \Sigma_c)) \Sigma^{-1}_x (E_j ((IP_2 \otimes R^\top)(\Omega_c \otimes \Sigma_c))^\top \|_F \mathbb{E}\left( \Sigma^{-1/2}_x u_2 u_2^\top \Sigma^{-1/2}_x \right) \right) \right\|_F^2 \]

\[ \leq \|E_j (IP_2 \otimes R^\top)\|_F^2 \|\Omega_c \otimes \Sigma_c\|^2 \|\Sigma^{-1}_x \|^2 \mathbb{E}\left( \left\| \frac{\Sigma^{-1/2}_x u_2 u_2^\top \Sigma^{-1/2}_x }{\|u_2\|^2} \right\|_F^2 \right) \]

\[ = O\left( \frac{1}{p_2^2} \right). \]

Then, \( \|\text{EVec}(X)\|_F^2 = p_2^2 O\left( \frac{1}{p_1} + \frac{1}{p_2} \right) = O(1). \) On the other hand,

\[ \mathbb{E}\|X\|_F^2 \leq C \sqrt{\mathbb{E}\left( \|R^\top \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e R\|_F^4 \right) \mathbb{E}\left( \|u_2\|^{-8} \right)} = O(1). \]

Hence, \( \|R^\top M_4 R\|_F^2 = O_p(1) \) and

\[ \left\| \frac{1}{p_1} R^\top M_4 R^\top \right\|_F^2 \leq \frac{1}{p_1^2} \left( \|R^\top M_4 R\|_F^2 \|\hat{H}_R\|_F^2 + \|R^\top M_4\|_F^2 \|\hat{R} - \hat{R}_R\|_F^2 \right) = O\left( \frac{1}{p_1^2} + \frac{1}{T^2 p_2^2} \right). \]

As a result,

\[ \left\| \frac{1}{p_1} \hat{R}^\top (\hat{R} - \hat{R}_R) \right\|_F^2 = O_p\left( \frac{1}{T^2 p_1 p_2} + \frac{1}{p_1^2} + \frac{1}{T^2 p_2^2} \right) = O_p\left( \frac{1}{T^2 p_1 p_2} \right). \]

Similarly, we can get

\[ \left\| \frac{1}{p_2} (\hat{C} - C \hat{H}_C^\top \hat{C}) \right\|_F^2 = O_p\left( \frac{1}{T^2 p_1 p_2} \right), \]

which concludes the lemma. \( \square \)

**Lemma C.8.** Under Assumptions A-C, for any constant \( \epsilon > 0 \) we have

\[ \begin{cases} 
\lambda_j(\hat{K}_r) \approx 1, & j \leq k_1, \\
\lambda_j(\hat{K}_r) \leq O_p(T^{-1/2+\epsilon} + p_2^{-1/2}), & j > k_1, \\
\lambda_j(\hat{K}_c) \approx 1, & j \leq k_2, \\
\lambda_j(\hat{K}_c) \leq O_p(T^{-1/2+\epsilon} + p_1^{-1/2}), & j > k_2.
\end{cases} \]

**Proof.** We start with the population Kendall’s tau matrix \( K_r \). By definition,

\[ K_r = \mathbb{E}\left( RZ_1 C^\top + \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e \right)^\top \mathbb{E}\left( RZ_1 C^\top + \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e \right)^\top \|RZ_1 C^\top + \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e\|_F^2. \] \hspace{1cm} (C.8)

Write

\[ K_r^4 := \mathbb{E}\left( \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e \right)^\top \mathbb{E}\left( \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e \right)^\top \|RZ_1 C^\top + \Sigma^{1/2}_e Z_2 \Omega Z_2^\top \Sigma^{1/2}_e\|_F^2. \]
and we aim to provide an upper bound for \(\|K_4^4\|\). Using the formula \(a^{-1} = b^{-1} - (ab)^{-1}(a - b)\),

\[
K_4^4 = 2\mathbb{E} \frac{\Sigma^1/2 Z_2 \Omega \Sigma^1/2}{\|RZ_1C^\top\|^2_F + \|\Sigma^1/2 Z_2 \Omega^1/2\|^2_F} - \mathbb{E} \frac{\Sigma^1/2 Z_2 \Omega \Sigma^1/2 \times 2\text{tr}(RZ_1C^\top \Omega^1/2 Z_2 \Sigma^1/2)}{\|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2\|^2_F (\|RZ_1C^\top\|^2_F + \|\Sigma^1/2 Z_2 \Omega^1/2\|^2_F)} \\
:= K_{41} + K_{42}.
\]

Note that the error term \(K_{42}\) satisfies

\[
\|K_{42}\| \leq \mathbb{E} \frac{2\text{tr}(RZ_1C^\top \Omega^1/2 Z_2 \Sigma^1/2)}{|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2|_F^2} \leq \sqrt{\mathbb{E} \|u_2\|^{-4} \times \mathbb{E} \text{tr}^2(RZ_1C^\top \Omega^1/2 Z_2 \Sigma^1/2)} \leq O(p_2^{-1/2}),
\]

where \(u_2\) are defined in Lemma C.3. Therefore, by Weyl’s theorem, we can write

\[
\|K_4^4\| \leq \|K_{41}\| + O(p_2^{-1/2}) \leq \mathbb{E} \frac{\Sigma^1/2 Z_2 \Omega \Sigma^1/2}{\|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2\|^2_F} + O(p_2^{-1/2}) \\
\leq \lambda_{\min}(\Sigma_e)^{-1} \lambda_{\min}(\Omega_e)^{-1} \lambda_{\max}(\Sigma_e) \mathbb{E} \frac{Z_2 \Omega Z_2}{\|Z_2\|^2_F} + O(p_2^{-1/2}) \leq O(p_2^{-1/2}).
\]

Next, consider the leading term in \(K_r\), denoted as

\[
K_1^r := \mathbb{E} \frac{RZ_1C^\top CZ_1^\top R^\top}{\|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2\|^2_F}.
\]

Similarly, we can write

\[
K_1^r = \mathbb{E} \frac{RZ_1C^\top CZ_1^\top R^\top}{\|RZ_1C^\top\|^2_F + \|\Sigma^1/2 Z_2 \Omega^1/2\|^2_F} - \mathbb{E} \frac{RZ_1C^\top CZ_1^\top R^\top \times 2\text{tr}(RZ_1C^\top \Omega^1/2 Z_2 \Sigma^1/2)}{\|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2\|^2_F (\|RZ_1C^\top\|^2_F + \|\Sigma^1/2 Z_2 \Omega^1/2\|^2_F)} \\
:= K_{11}^r + K_{12}^r,
\]

and claim that \(\|K_{12}^r\| \leq O(p_2^{-1/2})\) while \(\|K_{11}^r\| \leq C\) for some constant \(C > 0\). Then, by Cauchy-Schwartz inequality, the interaction terms in (C.8) satisfy

\[
\mathbb{E} \frac{\|RZ_1C^\top \Omega^1/2 Z_2 \Sigma^1/2\|^2}{\|RZ_1C^\top + \Sigma^1/2 Z_2 \Omega^1/2\|^2_F} \leq \sqrt{\mathbb{E} \|K_1^r\|^2 \|K_1^r\|^2} = o(1).
\]

Further note that the rank of \(K_1^r\) is at most \(k_1\). Therefore, it remains to prove that \(\lambda_{k_1}(K_{11}^r) \geq C^{-1}\) for some constant \(C > 0\).

Due to the orthogonal invariant property of Gaussian distributions, we assume \(C^\top C\) and \(R^\top R\) to be diagonal matrices. By Assumptions B and C, we can write

\[
\lambda_{k_1}(K_{11}^r) \geq \lambda_{k_1} \left( \mathbb{E} \frac{RZ_1C^\top CZ_1^\top R^\top}{C_2 p_1 p_2 \|Z_1\|^2_F + C_2^2 \|Z_2\|^2_F} \right) \geq \lambda_{k_1} \left( R^\top R \right) A_{k_1 k_1},
\]

where the second “\(\geq\)” comes from the facts that columns of \(R\) are orthogonal and \(A_{ii} = 0\) for \(i \neq j\). See also the proof of Proposition 2.1. Note that \(p_1 A_{k_1 k_1}\) is similar to \(M_{ij}\) in the proof of Lemma 3.1, Yu et al. (2019). According to their results, we have \(\lambda_{k_1}(K_{11}^r) \geq C^{-1}\) for some constant \(C > 0\).

Next step, we consider the sample Kendall’s tau matrix and show that \(\|\hat{K}_r - K_r\| = o_p(p_2^{1/2})\). Assume
That is, $T$ is even and $\bar{T} = T/2$, otherwise we can delete the last observation. For any permutation $\sigma$ of $\{1, \ldots, T\}$, define $\omega^T_s = X_{2s-1} - X_{2s+1}$ for $s \in \{1, \ldots, \bar{T}\}$ and $\bar{K}^T_{\sigma} = \bar{T}^{-1} \sum_{s=1}^T \omega^T_s \omega^T_s \sigma$. Denote $S_T$ as the set containing all the permutations of $\{1, \ldots, T\}$, then we can get
\[
\sum_{\sigma \in S_T} \bar{K}^T_{\sigma} = T \times (T - 2)! \times \frac{T(T - 1)}{2} \bar{K}_r \Rightarrow \bar{K}_r = \frac{1}{T} \sum_{\sigma \in S_T} \bar{K}^T_{\sigma}.
\]
because
\[
\|\bar{K}_r - K_r\| \leq \frac{1}{T} \sum_{\sigma \in S_T} \|\bar{K}^T_{\sigma} - K_r\|
\]
and
\[
\|\bar{K}^T_{\sigma} - K_r\| = \left\| T^{-1} \sum_{s=1}^T \omega^T_s \omega^T_s \sigma - E\omega^T_s \omega^T_s \sigma \right\| = \left\| T^{-1} \sum_{s=1}^T (\omega^T_s \omega^T_s \sigma - E\omega^T_s \omega^T_s \sigma) \right\|.
\]
Define $\text{Vec}(Z_t) = (\text{Vec}(Z_{t1})^T, \text{Vec}(Z_{t2})^T)^T$ as i.i.d. $(k_1k_2 + p_1p_2)$ standard Gaussian random vectors, and $Z_{t1}$, $Z_{t2}$ are $k_1 \times k_2$, $p_1 \times p_2$ matrices, respectively. Let
\[
K_t = \frac{(RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2})(RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2})}{\|RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2}\|^2_F} = \frac{RZ_{t1}C^T C_1 Z_{t2}^T R^T}{\|RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2}\|^2_F} + \frac{RZ_{t1}C^T \Sigma_c^{1/2}Z_{t2}C_1^{1/2}}{\|RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2}\|^2_F} + \frac{\Sigma_c^{1/2}Z_{t2}C_1 \Sigma_c^{1/2} \Sigma_c^{1/2}}{\|RZ_{t1}C^T + \Sigma_c^{1/2}Z_{t2}C_1^{1/2}\|^2_F}
\]
\[
:= K^1_t + K^2_t + K^3_t + K^4_t.
\]
Then, the large sample properties of $\|\bar{K}_r - K_r\|$ is similar to those of $\|\bar{T}^{-1} \sum_{t=1}^T K_t - E K_1\|$. Similarly to $\|K_r\|$, one can verify that
\[
E\|K^1_t\| \leq C, \quad E\|K^j_t\|^2 \leq O(p_2^{-1}), j = 2, 3, 4.
\]
Therefore, according to matrix concentration inequality (Theorem 5.4.1 in Vershynin (2018)), we have
\[
\mathbb{P}\left(\left\|\sum_{t=1}^T (K^1_t - EK^1_t)\right\| \geq s\right) \leq k_1 \exp\left\{- \frac{s^2}{cT}\right\} \Leftrightarrow \mathbb{P}\left(\left\|\frac{1}{T} \sum_{t=1}^T (K^1_t - EK^1_t)\right\| \geq T^{-1/2 + \epsilon}\right) \leq k_1 \exp\left\{- \frac{T^{2\epsilon}}{c}\right\},
\]
for any $\epsilon > 0$. Hence, when we have $\|\bar{T}^{-1} \sum_{t=1}^T (K^1_t - EK^1_t)\| = O_p(T^{-1/2 + \epsilon})$ for any $\epsilon > 0$. On the other hand, for $j = 2, 3, 4$,
\[
\mathbb{P}\left(\left\|\sum_{t=1}^T (K^j_t - EK^j_t)\right\| \geq s\right) \leq p_1 \exp\left\{- \frac{p_2 s^2}{cT}\right\} \Leftrightarrow \mathbb{P}\left(\left\|\frac{1}{T} \sum_{t=1}^T (K^j_t - EK^j_t)\right\| \geq p_2^{-1/2}\right) \leq p_1 \exp\left\{- \frac{T^{2\epsilon}}{c}\right\}.
\]
That is, $\|\bar{T}^{-1} \sum_{t=1}^T (K^j_t - EK^j_t)\| = O(p_2^{-1/2})$ as long as $\log p_1 = o(T)$. As a result,
\[
\left\|\frac{1}{T} \sum_{t=1}^T (K_t - EK_t)\right\| = O_p(T^{-1/2 + \epsilon} + p_2^{-1/2}),
\]
for any $\epsilon > 0$, which concludes the lemma. □
## D Additional Simulation Results

Table 9: Averaged estimation errors and standard errors of $\mathcal{D}(\hat{C}, C)$ for Settings A under joint Normal distribution and joint $t$ distribution over 500 replications.

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS       | RMFA       | $\alpha$-PCA | PE          | MPCA$_F$ |
|------------|-----|-------|-------|------------|------------|--------------|-------------|-----------|
| $\mathcal{D}(\hat{C}, C)$ | 20  | 20    | 20    | 0.1192(0.0331) | 0.0914(0.0151) | 0.1128(0.0307) | 0.0921(0.0157) | 0.1400(0.0206) |
|              | 50  | 50    |       | 0.0623(0.0074) | 0.0569(0.0060) | 0.0598(0.0070) | 0.0568(0.0060) | 0.0986(0.0093) |
| $\mathcal{D}(\hat{C}, C)$ | 50  | 20    | 20    | 0.0807(0.0228) | 0.0573(0.0099) | 0.0771(0.0221) | 0.0575(0.0099) | 0.0866(0.0132) |
|              | 50  | 50    |       | 0.0389(0.0043) | 0.0352(0.0033) | 0.0378(0.0041) | 0.0351(0.0032) | 0.0610(0.0053) |
| $\mathcal{D}(\hat{C}, C)$ | 100 | 20    | 20    | 0.0661(0.0256) | 0.0405(0.0067) | 0.0627(0.0245) | 0.0406(0.0069) | 0.0607(0.0088) |
|              | 50  | 50    |       | 0.0279(0.0033) | 0.0246(0.0021) | 0.0271(0.0032) | 0.0246(0.0021) | 0.0429(0.0033) |

**Normal Distribution**

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS       | RMFA       | $\alpha$-PCA | PE          | MPCA$_F$ |
|------------|-----|-------|-------|------------|------------|--------------|-------------|-----------|
| $\mathcal{D}(\hat{C}, C)$ | 20  | 20    | 20    | 0.1294(0.0370) | 0.2298(0.1444) | 0.4353(0.1753) | 0.4246(0.1900) | 0.1400(0.0221) |
|              | 50  | 50    |       | 0.0689(0.0084) | 0.1327(0.0976) | 0.2845(0.1782) | 0.2829(0.1872) | 0.0989(0.0095) |
| $\mathcal{D}(\hat{C}, C)$ | 50  | 20    | 20    | 0.0884(0.0256) | 0.1884(0.1431) | 0.4187(0.1871) | 0.4069(0.2023) | 0.0865(0.0121) |
|              | 50  | 50    |       | 0.0428(0.0044) | 0.1070(0.0811) | 0.2694(0.1771) | 0.2641(0.1824) | 0.0610(0.0049) |
| $\mathcal{D}(\hat{C}, C)$ | 100 | 20    | 20    | 0.0685(0.0221) | 0.1767(0.1485) | 0.4304(0.1918) | 0.4232(0.2068) | 0.0606(0.0085) |
|              | 50  | 50    |       | 0.0308(0.0032) | 0.0939(0.0825) | 0.2678(0.1740) | 0.2636(0.1814) | 0.0431(0.0033) |

**$t_1$ Distribution**

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS       | RMFA       | $\alpha$-PCA | PE          | MPCA$_F$ |
|------------|-----|-------|-------|------------|------------|--------------|-------------|-----------|
| $\mathcal{D}(\hat{C}, C)$ | 20  | 20    | 20    | 0.1263(0.0368) | 0.1296(0.0549) | 0.2655(0.1521) | 0.2445(0.1603) | 0.1402(0.0210) |
|              | 50  | 50    |       | 0.0667(0.0086) | 0.0783(0.0173) | 0.1484(0.1011) | 0.1416(0.1015) | 0.0996(0.0099) |
| $\mathcal{D}(\hat{C}, C)$ | 50  | 20    | 20    | 0.0843(0.0236) | 0.0853(0.0255) | 0.2153(0.1402) | 0.1946(0.1464) | 0.0861(0.0121) |
|              | 50  | 50    |       | 0.0415(0.0045) | 0.0525(0.0171) | 0.1193(0.0983) | 0.1148(0.0992) | 0.0613(0.0047) |
| $\mathcal{D}(\hat{C}, C)$ | 100 | 20    | 20    | 0.0689(0.0270) | 0.0629(0.0209) | 0.1925(0.1338) | 0.1723(0.1399) | 0.0610(0.0091) |
|              | 50  | 50    |       | 0.0300(0.0034) | 0.0400(0.0240) | 0.1010(0.0862) | 0.0961(0.0856) | 0.0429(0.0033) |

**$t_2$ Distribution**

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS       | RMFA       | $\alpha$-PCA | PE          | MPCA$_F$ |
|------------|-----|-------|-------|------------|------------|--------------|-------------|-----------|
| $\mathcal{D}(\hat{C}, C)$ | 20  | 20    | 20    | 0.1248(0.0329) | 0.1111(0.0251) | 0.1947(0.1000) | 0.1669(0.0938) | 0.1409(0.0220) |
|              | 50  | 50    |       | 0.0657(0.0079) | 0.0674(0.0100) | 0.0974(0.0400) | 0.0924(0.0376) | 0.0988(0.0096) |
| $\mathcal{D}(\hat{C}, C)$ | 50  | 20    | 20    | 0.0844(0.0233) | 0.0703(0.0154) | 0.1406(0.0788) | 0.1192(0.0789) | 0.0869(0.0125) |
|              | 50  | 50    |       | 0.0411(0.0045) | 0.0438(0.0061) | 0.0756(0.0488) | 0.0715(0.0481) | 0.0616(0.0050) |
| $\mathcal{D}(\hat{C}, C)$ | 100 | 20    | 20    | 0.0677(0.0211) | 0.0508(0.0104) | 0.1149(0.0694) | 0.0944(0.0637) | 0.0607(0.0079) |
|              | 50  | 50    |       | 0.0297(0.0034) | 0.0311(0.0042) | 0.0604(0.0410) | 0.0571(0.0432) | 0.0429(0.0030) |
Table 10: Averaged estimation errors and standard errors of $D(\hat{C}, C)$ for Scenario B under joint Normal distribution and joint $t$ distribution over 500 replications.

| Evaluation | $T$ | $p_1$ | $p_2$ | MRTS | RMFA | $\alpha$-PCA | PE | MPCA_F |
|------------|-----|-------|-------|------|------|--------------|----|--------|
| Normal Distribution | | | | | | | | |
| $D(\hat{C}, C)$ | 20 | 20 | 20 | 0.1202(0.0331) | 0.0923(0.0151) | 0.1139(0.0305) | 0.0930(0.0157) | 0.1410(0.0209) |
| | 50 | 50 | 0.0627(0.0074) | 0.0575(0.0061) | 0.0604(0.0071) | 0.0574(0.0061) | 0.0993(0.0094) |
| $D(\hat{C}, C)$ | 50 | 20 | 20 | 0.0813(0.0231) | 0.0578(0.0100) | 0.0778(0.0224) | 0.0580(0.0100) | 0.0872(0.0133) |
| | 50 | 50 | 0.0392(0.0044) | 0.0356(0.0033) | 0.0382(0.0042) | 0.0355(0.0033) | 0.0613(0.0052) |
| $D(\hat{C}, C)$ | 100 | 20 | 20 | 0.0663(0.0255) | 0.0408(0.0067) | 0.0629(0.0245) | 0.0409(0.0069) | 0.0609(0.0088) |
| | 50 | 50 | 0.0282(0.0033) | 0.0249(0.0021) | 0.0274(0.0032) | 0.0248(0.0021) | 0.0430(0.0034) |
| $t_1$ Distribution | | | | | | | | |
| $D(\hat{C}, C)$ | 20 | 20 | 20 | 0.1442(0.0410) | 0.2400(0.1498) | 0.4359(0.1751) | 0.4249(0.1894) | 0.1516(0.0259) |
| | 50 | 50 | 0.0771(0.0114) | 0.1385(0.1010) | 0.2841(0.1776) | 0.2827(0.1866) | 0.1059(0.0121) |
| $D(\hat{C}, C)$ | 50 | 20 | 20 | 0.0976(0.0273) | 0.1960(0.1450) | 0.4193(0.1871) | 0.4075(0.2025) | 0.0938(0.0136) |
| | 50 | 50 | 0.0484(0.0061) | 0.1125(0.0870) | 0.2696(0.1770) | 0.2646(0.1826) | 0.0651(0.0058) |
| $D(\hat{C}, C)$ | 100 | 20 | 20 | 0.0751(0.0236) | 0.1845(0.1517) | 0.4303(0.1917) | 0.4234(0.2067) | 0.0655(0.0094) |
| | 50 | 50 | 0.0345(0.0036) | 0.0989(0.0888) | 0.2678(0.1740) | 0.2639(0.1817) | 0.0459(0.0036) |
| $t_2$ Distribution | | | | | | | | |
| $D(\hat{C}, C)$ | 20 | 20 | 20 | 0.1303(0.0378) | 0.1333(0.0610) | 0.2662(0.1520) | 0.2449(0.1606) | 0.1430(0.0220) |
| | 50 | 50 | 0.0690(0.0092) | 0.0801(0.0189) | 0.1490(0.1023) | 0.1418(0.1013) | 0.1009(0.0102) |
| $D(\hat{C}, C)$ | 50 | 20 | 20 | 0.0864(0.0235) | 0.0877(0.0275) | 0.2157(0.1404) | 0.1947(0.1461) | 0.0881(0.0123) |
| | 50 | 50 | 0.0430(0.0047) | 0.0540(0.0188) | 0.1196(0.0986) | 0.1151(0.0998) | 0.0623(0.0049) |
| $D(\hat{C}, C)$ | 100 | 20 | 20 | 0.0704(0.0271) | 0.0649(0.0223) | 0.1926(0.1336) | 0.1722(0.1397) | 0.0623(0.0091) |
| | 50 | 50 | 0.0311(0.0035) | 0.0413(0.0257) | 0.1012(0.0863) | 0.0962(0.0856) | 0.0437(0.0033) |
| $t_3$ Distribution | | | | | | | | |
| $D(\hat{C}, C)$ | 20 | 20 | 20 | 0.1272(0.0351) | 0.1129(0.0262) | 0.1958(0.1008) | 0.1679(0.0943) | 0.1430(0.0224) |
| | 50 | 50 | 0.0668(0.0082) | 0.0683(0.0105) | 0.0978(0.0400) | 0.0927(0.0374) | 0.0998(0.0099) |
| $D(\hat{C}, C)$ | 50 | 20 | 20 | 0.0857(0.0235) | 0.0715(0.0160) | 0.1409(0.0791) | 0.1193(0.0788) | 0.0879(0.0126) |
| | 50 | 50 | 0.0419(0.0046) | 0.0446(0.0065) | 0.0760(0.0502) | 0.0718(0.0487) | 0.0623(0.0049) |
| $D(\hat{C}, C)$ | 100 | 20 | 20 | 0.0685(0.0213) | 0.0518(0.0109) | 0.1152(0.0693) | 0.0945(0.0636) | 0.0615(0.0079) |
| | 50 | 50 | 0.0303(0.0034) | 0.0317(0.0044) | 0.0605(0.0411) | 0.0572(0.0432) | 0.0433(0.0030) |
Table 11: Mean squared error and its standard under Scenario B over 500 replications.

| Distribution | $p_1$ | $T = 20, p_2 = p_1$ | $T = 50, p_2 = p_1$ | $T = 100, p_2 = p_1$ |
|--------------|-------|---------------------|---------------------|---------------------|
| Normal       |       |                     |                     |                     |
| 20           | 0.0461(0.0066) | 0.0376(0.0037) | 0.0438(0.0061) | 0.0378(0.0038) | 0.0567(0.0068) |
| 50           | 0.0107(0.0009) | 0.0096(0.0007) | 0.0101(0.0008) | 0.0096(0.0007) | 0.0216(0.0026) |
| $t_3$        | 20    | 0.1446(0.1434) | 0.1542(0.2754) | 0.3455(0.9260) | 0.3291(0.9691) | 0.1675(0.1603) |
| 50           | 0.0319(0.0229) | 0.0370(0.0392) | 0.0730(0.1243) | 0.0716(0.1375) | 0.0595(0.0405) |
| Normal       |       |                     |                     |                     |
| 20           | 0.0340(0.0043) | 0.0285(0.0024) | 0.0330(0.0040) | 0.0286(0.0025) | 0.0358(0.0033) |
| 50           | 0.0065(0.0004) | 0.0060(0.0003) | 0.0063(0.0004) | 0.0060(0.0003) | 0.0107(0.0008) |
| $t_3$        | 20    | 0.1056(0.0829) | 0.1033(0.1173) | 0.2794(0.9220) | 0.2676(0.9615) | 0.1073(0.0838) |
| 50           | 0.0205(0.0155) | 0.0251(0.0451) | 0.0648(0.2269) | 0.0628(0.2375) | 0.0319(0.0226) |
| Normal       |       |                     |                     |                     |
| 20           | 0.0301(0.0035) | 0.0254(0.0017) | 0.0293(0.0033) | 0.0254(0.0017) | 0.0290(0.0019) |
| 50           | 0.0051(0.0003) | 0.0048(0.0002) | 0.0050(0.0003) | 0.0048(0.0002) | 0.0071(0.0004) |
| $t_3$        | 20    | 0.0905(0.0512) | 0.0868(0.0998) | 0.1968(0.6717) | 0.1837(0.7117) | 0.0862(0.0502) |
| 50           | 0.0163(0.0123) | 0.0188(0.0263) | 0.0516(0.1748) | 0.0500(0.1813) | 0.0219(0.0153) |
Table 12: The frequencies of exact estimation and underestimation of the numbers of factors under Scenario B over 500 replications.

| Distribution | $p_1$ | MKER | Rit-ER | IterER | $\alpha$-PCA-ER | iTIP-IC | iTIP-EC | TCorTh |
|--------------|-------|------|--------|--------|-----------------|--------|--------|--------|
| $T = 20, p_2 = p_1$ | | | | | | | | |
| Normal | 20 | 0.664(0.046) | 0.990(0.000) | 0.990(0.000) | 0.594(0.076) | 0.000(1.000) | 0.668(0.050) |
| | 50 | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.000(1.000) | 0.000(1.000) | 0.182(0.198) |
| $t_1$ | 20 | 0.568(0.09) | 0.312(0.264) | 0.220(0.484) | 0.064(0.776) | 0.082(0.288) | 0.278(0.378) |
| | 50 | 0.992(0.000) | 0.664(0.098) | 0.534(0.306) | 0.296(0.544) | 0.114(0.388) | 0.700(0.162) |
| $t_2$ | 20 | 0.614(0.048) | 0.702(0.072) | 0.630(0.174) | 0.224(0.470) | 0.024(0.926) | 0.398(0.224) |
| | 50 | 0.998(0.000) | 0.908(0.002) | 0.880(0.066) | 0.682(0.166) | 0.018(0.952) | 0.920(0.028) |
| $t_3$ | 20 | 0.636(0.050) | 0.866(0.018) | 0.806(0.066) | 0.364(0.286) | 0.004(0.996) | 0.486(0.152) |
| | 50 | 0.998(0.000) | 0.990(0.000) | 0.978(0.014) | 0.882(0.048) | 0.002(0.996) | 0.990(0.000) |
| $T = 50, p_2 = p_1$ | | | | | | | | |
| Normal | 20 | 0.772(0.010) | 1.000(0.000) | 1.000(0.000) | 0.732(0.028) | 0.000(1.000) | 0.954(0.000) |
| | 50 | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.000(1.000) | 0.234(0.130) | 1.000(0.000) |
| $t_1$ | 20 | 0.708(0.022) | 0.424(0.194) | 0.292(0.468) | 0.110(0.738) | 0.072(0.102) | 0.558(0.058) |
| | 50 | 1.000(0.000) | 0.686(0.062) | 0.558(0.294) | 0.354(0.492) | 0.164(0.128) | 0.696(0.014) |
| $t_2$ | 20 | 0.802(0.02) | 0.834(0.026) | 0.736(0.126) | 0.316(0.338) | 0.034(0.878) | 0.790(0.014) |
| | 50 | 1.000(0.000) | 0.936(0.004) | 0.912(0.056) | 0.804(0.136) | 0.018(0.944) | 0.930(0.004) |
| $t_3$ | 20 | 0.758(0.022) | 0.952(0.000) | 0.908(0.034) | 0.522(0.168) | 0.002(0.994) | 0.878(0.012) |
| | 50 | 1.000(0.000) | 0.990(0.000) | 0.988(0.006) | 0.956(0.030) | 0.000(1.000) | 0.992(0.000) |
| $T = 100, p_2 = p_1$ | | | | | | | | |
| Normal | 20 | 0.824(0.010) | 1.000(0.000) | 1.000(0.000) | 0.786(0.016) | 0.000(1.000) | 0.994(0.000) |
| | 50 | 1.000(0.000) | 1.000(0.000) | 1.000(0.000) | 0.000(1.000) | 0.374(0.084) | 1.000(0.000) |
| $t_1$ | 20 | 0.790(0.012) | 0.304(0.218) | 0.280(0.440) | 0.100(0.722) | 0.030(0.004) | 0.458(0.016) |
| | 50 | 1.000(0.000) | 0.702(0.052) | 0.568(0.308) | 0.350(0.512) | 0.134(0.020) | 0.466(0.002) |
| $t_2$ | 20 | 0.802(0.012) | 0.870(0.016) | 0.780(0.110) | 0.406(0.300) | 0.038(0.860) | 0.870(0.002) |
| | 50 | 1.000(0.000) | 0.962(0.002) | 0.938(0.032) | 0.854(0.096) | 0.038(0.896) | 0.424(0.178) |
| $t_3$ | 20 | 0.820(0.006) | 0.966(0.000) | 0.952(0.014) | 0.636(0.080) | 0.000(0.996) | 0.960(0.000) |
| | 50 | 1.000(0.000) | 0.992(0.000) | 0.992(0.006) | 0.968(0.016) | 0.000(1.000) | 0.980(0.000) |