Elastic scattering and path integral

Efimov G.V.

LTP JINR,

141980 Dubna, Russia

May 27, 2013

Abstract

Representation of the elastic scattering amplitude in the form of the path integral is obtained using the stationary Schroedinger equation. A few methods of evaluation of path integrals for large coupling constants are formulated. The methods are based on the uncertainty correlation. The scattering lengths and cross sections are calculated for the right angled and singular potentials and the Yukawa potential. The comparison with exact results is made.

1 Introduction

It is always pleasant to have a solution of a differential equation in a closed form which permits both to investigate general properties of the solution and perform numerical calculations. In quantum mechanics, such a closed representation is the solution of the nonstationary Schroedinger equation in the form of the Feynman path or functional integral. In this direction, enormous work is performed and many monographs and textbooks are published (see, for example, [1, 2, 3, 4] and so on). However, in quantum mechanics there are plenty important problems with fixed energy, which require solution of the stationary Schroedinger equation. In particular, all scattering processes are described by the stationary Schroedinger equation. At present, there are several approaches (see [5, 6, 7, 8, 9]), in which the representation of the solution of the nonstationary Schroedinger equation in the path integral form is used to obtain scattering amplitudes.
I would like to remind you what is the difference in the formulation of the scattering problem in the temporal and stationary quantum formalism in the case of rapidly decreasing potentials.

Stationary picture. A potential decreases very rapidly so that it acts in some bounded region only. Outside of this region particles can be considered as free ones. Any scattering looks like: at large distances outside of potential influence there is a current of free particles in a plane wave form (before scattering) and in a spherical wave form (after scattering). This physical picture is really realistic.

Nonstationary picture. It is supposed that in some infinitely past \( t \to -\infty \) and some infinitely future \( t \to \infty \) the potential does not exist and all particles are free and are described by plane waves in all space. The potential is "switched on" adiabatically in some past and then is "switched off" adiabatically in some future. A factor \( e^{-\epsilon |t|} \) is introduced to describe this process mathematically. This procedure is called the hypothesis of switching on interaction. This hypothesis is the basis of the scattering theory in quantum field theory. Physically, this picture is not correct because any interaction cannot be switched off. Besides, in quantum field theory space-time regions, where interaction is absent, do not exist. However, as it turned out this approach works well. Mathematically, this approach requires an accurate way to the limiting procedure \( \epsilon \to 0 \), some rules should be formulated and some counter terms should be introduced.

Therefore, we consider the nonstationary picture as absolutely unnecessary to describe scattering in quantum mechanics.

In works [5, 6, 7, 8, 9], where the subsequent references can be found, the path integral method is applied to potential scattering in nonrelativistic quantum mechanics to get the representation of the scattering amplitude in the path integral form. The argumentation is based on the nonstationary picture of scattering where the hypothesis of switching on interaction plays an essential role. The temporal Schrödinger equation is considered, the representation for the \( S \)-matrix as a transition amplitude from time \( t = -\infty \) to time \( t = \infty \) is obtained. The \( T \)-matrix is extracted by using the corresponding \( \delta \)-functions. Besides, all calculations on this way are quite cumbersome. From my point of view, this approach is not the best way to get scattering amplitudes. We will show that the desired representation can be obtained directly from the stationary Schrödinger equation.

After that the problem arises how to calculate the obtained path integral. The point is that we can calculate the Gaussian integrals only. But our path
integrals are not the Gaussian type. All known calculation methods are reduced in any case to appropriate Gaussian type integrals.

Thus, our problem consists of two points:

1. to get a representation for scattering amplitudes in the form of the path integral using the stationary Schroedinger equation;

2. to work out comparatively simple methods to evaluate the derived path integrals.

Our estimations of the path integrals are based on the uncertainty correlation, which is one of fundamental principles of quantum mechanics.

It is known that the uncertainty coordinate-time correlation permits us to evaluate qualitatively and semiquantitatively the spectrum of any hamiltonian. The main point is very simple. Let a hamiltonian $H = \frac{p^2}{2m} + V(x)$ have a discrete spectrum. Between the middle size of the region $\Delta x$, where a particle oscillates, and the middle of its momentum $\Delta p$ the connection $\Delta p \Delta x \sim \hbar n$, where $n$ is a number of excited states. The spectrum can be evaluated qualitatively and semiquantitatively by formula

$$E_n \sim \min_{\Delta x} \left[ \frac{\hbar^2 n^2}{2m(\Delta x)^2} + V(\Delta x) \right].$$

In the scattering case, when particles belong to the continuous spectrum, and are free in principle, the phenomenon of the expansion of the wave packet takes place, i.e. the uncertainty of the coordinate increases as $(\Delta x)^2 \sim t$, when time $t$ increases. We shall see how this property of the wave packet can be used.

So our problem is to get simple formulas for qualitative and semiquantitative estimation of the length of scattering and the cross section in the case of positive repulsing potentials with a large coupling constant using the path integral representation of the scattering amplitude. The Born approximation is not applicable in these cases. We shall use the results of [10], where the representation of the solution of the stationary Schroedinger equation in the path integral form is obtained.
Elastic scattering amplitude

The scattering problem is formulated in the following way. A potential is supposed to be short-range one. The steady-state Schrödinger equation

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} + V(x) - \frac{k^2}{2m} \right) \Psi(x) = 0
\]

should be solved for the continuous spectrum \( E = \frac{k^2}{2m} \) and the solution should satisfy the asymptotic boundary conditions for \( r \to \infty \)

\[
\Psi(x) \to e^{ikx} + f(k, \theta) \frac{e^{ikr}}{r},
\]

where \( f(k, \theta) \) is the desired scattering amplitude, which should be found. Let us look for the solution in the form

\[
\Psi(x) = e^{ikx} + \Phi(x).
\]

The function \( \Phi(x) \) satisfies the equation

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} + V(x) - \frac{k^2}{2m} \right) \Phi(x) = -V(x) e^{ikx},
\]

the solution of which can be written in the form

\[
\Phi(x) = -\frac{1}{-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) - \frac{k^2}{2m} - i0} V(x) e^{ikx}.
\]

Let us represent this solution in the path integral form using the standard calculations (see [12, 13, 10])

\[
\Phi(x) = -i \int_0^\infty dt \ e^{-it\left[-\frac{1}{2m} \frac{d^2}{dx^2} - \frac{k^2}{2m} + V(x) - i0\right]} V(x) e^{ikx}
\]

\[
= -i \int_0^\infty \frac{m^2}{(2\pi it)^2} e^{\frac{ik^2}{2m}} \int dy e^{i\frac{y^2m}{2}} \int \frac{D\xi}{C} e^{i \int_0^t d\tau \left[ \frac{m}{2} \xi^2(\tau) - V(x+(1-\frac{i}{\tau})y-\xi(\tau)) \right]}
\]

\[
\cdot V(x+y) e^{ik(x+y)} = -i \int dy V(y) e^{iky} I(y, x, k),
\]
where

\[
I(y, x, k) = \int_0^\infty \frac{m^2}{(2\pi it)^{\frac{3}{2}}} dt \ e^{i \left( \frac{\xi^2}{m^2} + \frac{(x-y)^2}{2m^2} \right)} \int \frac{D\xi}{C} \ e^{i \int_0^t dr \left[ \frac{m^2}{2} \dot{\xi}^2(\tau) - V\left( \frac{\xi}{\tau} + (1-\frac{r}{\tau})y - \xi(\tau) \right) \right]}
\]

with the boundary conditions \( \xi(0) = \xi(t) = 0 \). We look for the behavior of the function \( I(y, x, k) \) in the limit \( |x| = r \to \infty \). Let us introduce new variables \( t = rs \) and \( n = \frac{x}{r} \), then for large \( r \) one obtains

\[
I(y, x, k) = \int_0^\infty \frac{ds}{(2\pi is)^{\frac{3}{2}}} \ e^{i \int_0^t d\tau \left[ \frac{m^2}{2} \dot{\xi}^2(\tau) - V\left( \frac{n}{\tau} + (1-\frac{r}{\tau})y - \xi(\tau) \right) \right]}
\]

with the boundary conditions \( \xi(0) = \xi(R) = 0 \), where \( R = rs \to \infty \).

One essential remark should be made about the boundary condition for \( R \to \infty \). The quadratic form in the path integral measure can written in the form

\[
\int_0^R d\tau \dot{\xi}^2(\tau) = \int \int d\tau d\tau' \xi(\tau) D^{-1}(\tau, \tau') \xi(\tau')
\]

where

\[
D^{-1}(\tau, \tau') = -\frac{d^2}{d\tau^2} \delta(\tau - \tau').
\]

The Green function of the operator \( D^{-1} \), which satisfies zero boundary conditions, takes the form

\[
D(\tau, \tau') = -\frac{1}{2} |\tau - \tau'| + \frac{1}{2}(\tau + \tau') - \frac{\tau \tau'}{R}
\]

This Green function is connected with the solution of the equation

\[
-\ddot{u}(\tau) = J(\tau), \quad u(0) = u(R) = 0.
\]

The solution for the finite \( R \) is

\[
u(\tau) = \int_0^R d\tau' D(\tau, \tau') J(\tau') = \left( 1 - \frac{\tau}{R} \right) \int_0^\tau d\tau' \tau' J(\tau') + \tau \int_\tau^R d\tau' \left( 1 - \frac{\tau'}{R} \right) J(\tau')
\]

5
and satisfies zero boundary conditions. However for $R \to \infty$ it equals

$$u(\tau) \to \int_{0}^{\tau} d\tau' \tau' J(\tau') + \tau \int_{\tau}^{\infty} d\tau' J(\tau') \to u(\infty) = \int_{0}^{\infty} d\tau' \tau' J(\tau') \neq 0,$$

and the corresponding Green function is

$$D(\tau, \tau') = -\frac{1}{2} |\tau - \tau'| + \frac{1}{2}(\tau + \tau').$$

Therefore, in the limit $R \to \infty$ we get $u(0) = 0$, but $u(\infty) \neq 0$, and, in principle, $u(\infty)$ can be an arbitrary number.

For large $r$ the integral in (5) over $s$ can be calculated by the saddle-point method, the result reads

$$I(y, x, k) \to \frac{m}{2\pi i r} e^{ikr} \int C e^{i \int_{0}^{\infty} d\tau \left[ m \dot{\xi}^2(\tau) - V (\frac{m k}{r} \tau + y - \xi(\tau)) \right]}$$

where we have only one condition $\xi(0) = 0$.

Thus, the scattering amplitude can be represented as

$$f(k, \theta) = -\frac{m}{2\pi} \int dr \ V(r) e^{iqr + \Phi(r, k)}$$

with

$$e^{\Phi(r, k)} = \int C e^{i \int_{0}^{\infty} d\tau \left[ m \dot{\xi}^2(\tau) - V (\frac{m k}{r} \tau + r - \xi(\tau)) \right]}$$

$$= \int C e^{\frac{1}{2} \int_{0}^{\infty} d\tau \dot{\xi}^2(\tau) - im \int_{0}^{\infty} d\tau V(\frac{m k}{r} \tau + r - \xi(\tau))}, \quad \xi(0) = 0.$$ 

The argument in the potential $\frac{k}{m} \tau + r - \xi(\tau)$ can be interpreted as a linear motion $\frac{k}{m} \tau + r$ plus all possible quantum fluctuations, which are described by the functional variable $\xi(\tau)$. These quantum fluctuations grow in time: it is the phenomenon of the expansion of the wave packet. This circumstance explains the disappearance of the boundary condition for $\tau = \infty$. 

6
2.1 Scattering amplitude and "imaginary" time

Let us come back to the representation (3). In the case of positive potentials ($V(x) > 0$) the operator

$$-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) \geq 0$$  \hspace{1cm} (9)

is positive. Let us introduce the complex variable $z = \kappa + ik$. The operator

$$-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) + \frac{z^2}{2m}$$  \hspace{1cm} (10)

on the real part of the positive real axis $z = \kappa > 0$ is positive. The analytical continuation $z \to -i(k + i0)$ gives the initial operator in equation (3). For $z = \kappa > 0$ the inverse operator can be represented as

$$G(x, y, \kappa)$$  \hspace{1cm} (11)

$$= \frac{1}{-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) + \frac{z^2}{2m}} \delta(x - y) = \int_0^\infty ds e^{-s\left[-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) + \frac{z^2}{2m}\right] \delta(x - y)}$$

$$= \int_0^\infty \frac{m^2 s}{(2\pi s)^{\frac{3}{2}}} e^{-s\left(\frac{m^2}{2m} + \frac{(x - y)^2}{2m}\right)} \frac{D\xi}{C} e^{-\int_0^s dv \left[\frac{m^2}{2m} \xi^2(\nu) + V(n\nu + (1 - \nu)y - \xi(\nu))\right]}$$

with the boundary condition $\xi(0) = \xi(s) = 0$.

In this representation the variable $s$ has the meaning of the imaginary time.

In order to find the behavior of the function $I(y, x, \kappa)$ in the limit $|x| = r \to \infty$, we introduce new variables $s = rv$ and $n = \frac{x}{r}$. Then for large $r$ one obtains

$$G(x, y, \kappa)$$  \hspace{1cm} (12)

$$= \frac{m^2}{\sqrt{r}} \int_0^\infty \frac{dv}{(2\pi v)^{\frac{3}{2}}} e^{-\frac{m^2}{2m} + \frac{(n - \frac{x}{r})^2}{2m}} \frac{D\xi}{C} e^{-\int_0^{rv} dv \left[\frac{m^2}{2m} \xi^2(\nu) + V(n\nu + (1 - \nu)y - \xi(\nu))\right]}$$

$$\to \frac{e^{-\kappa r}}{r} \cdot \frac{m}{2\pi} e^{\kappa ny} \int_0^R \frac{D\xi}{C} e^{-\int_0^R dv \left[\frac{m^2}{2m} \xi^2(\nu) + V(n\nu + y - \xi(\nu))\right]}$$

with the boundary conditions $\xi(0) = \xi(R) = 0$, where $R = rv \to \infty$. 7
Thus, the scattering amplitude in the Euclidean region can be represented as

\[ F(k, \kappa n) = -\frac{m}{2\pi} \int dy \: V(y) \: e^{iky + \kappa ny + \Phi(y, \kappa n)} \]  \hspace{1cm} (13)

for

\[ e^{\Phi(y, \kappa n)} = \int \frac{D\xi}{C} e^{-\int_0^\infty ds \left[ \frac{1}{2} \dot{\xi}^2(s) + mV(\kappa n \nu + y - \xi(s)) \right]} \right] , \quad \xi(0) = 0. \] \hspace{1cm} (14)

Here the substitution \( \nu \to m\nu \) is made.

The analytical continuation \( \kappa \to -i(k + i0) \) gives the desired amplitude for physical momenta. Formally, this representation can be obtained if in the path integral (8) one goes to integration over imaginary "time" \( \tau \to -im\nu \):

\[ e^{\Phi(y, k)} = \int \frac{D\xi}{C} e^{-\int_0^\infty ds \left[ \frac{1}{2} \dot{\xi}^2(s) + mV(-iks + y - \xi(s)) \right]} , \quad \xi(0) = 0. \] \hspace{1cm} (15)

The point is that the physical amplitude is an analytical continuation of (14) for \( \kappa \to -i(k + i0) \). But the question arises, what properties of the potential should provide the representation (15) to be such continuation.

The representation (14) can be applied directly to scattering length

\[ a = F(0, 0, \theta) = -\frac{m}{2\pi} \int dr \: V(r) \int \frac{D\xi}{C} e^{-\int_0^\infty dv \left[ \frac{1}{2} \dot{\xi}^2(v) + mV(r - \xi(v)) \right]} \] \hspace{1cm} (16)

for \( \xi(0) = 0 \).

2.2 The coordinate system

Let us choose the following coordinate system:

\[ \mathbf{q} = \mathbf{k}_{in} - \mathbf{k}_{out}, \quad \mathbf{v}_{out} = -\frac{\mathbf{k}_{out}}{m} = \frac{k}{m} \mathbf{n}, \quad \mathbf{q}^2 = 4k^2 \sin^2 \frac{\theta}{2}, \]

\[ k = \mathbf{k}_{out} = (0, 0, k), \quad \mathbf{q} = \left( 0, k \sin \theta, 2k \sin^2 \frac{\theta}{2} \right), \]

\[ \mathbf{r} = (\rho, z) = (\rho \sin \phi, \rho \cos \phi, z). \]

The function \( \Phi \) in these coordinates can be written as

\[ \Phi(\mathbf{r}, \mathbf{k}) = \Phi(\rho, z, k). \] \hspace{1cm} (17)
The elastic scattering amplitude has the form
\[ f(k, \theta) = -m \int_0^\infty d\rho \rho J_0(k\rho \sin \theta) \int_{-\infty}^\infty dz V\left(\sqrt{\rho^2 + z^2}\right) e^{2ikz \sin \frac{\theta}{2} + \Phi(\rho, z, k)}, \]

(18)

where in the representation (8) we get \( k\tau + r = (\rho, k\tau + z) \).

The representation (7) permits us to get an exact inequality for the scattering length for any coupling constants. The scattering length is defined by the integral
\[ a = f(0, \theta) = -2m \int_0^\infty dr r^2 V(r) e^{\Phi(r)} \]

(19)

where
\[ e^{\Phi(r)} = \int \frac{D\xi}{C} e^{\int_0^\infty d\tau \left[ \Phi(\xi(\tau)) + mV(r-\xi(\tau)) \right]} = \int \frac{D\xi}{C} e^{\int_0^\infty d\tau \left[ \Phi(\xi(\tau)) + mV(r-\xi(\tau)) \right]} \]

Using the Yensen inequality one can get
\[ e^{\Phi(r)} \geq e^{\Phi_1(r)}, \]
\[ \Phi(r) \geq \Phi_1(r) = -\int \frac{D\xi}{C} e^{-\int_0^\infty d\tau \frac{\Phi(\xi(\tau))}{2m}} m \int_0^\infty d\tau V(r - \xi(\tau)) \]
\[ = -2m \int \frac{dp}{(2\pi)^3} \frac{\tilde{V}(p)}{p^2} e^{ipr} = -2m \left[ \frac{1}{r} \int_0^r dy y^2 V(y) + \int_0^\infty dy yV(y) \right] \]

and finally
\[ |a| \geq 2m \int_0^\infty dr r^2 V(r) e^{\Phi_1(r)}. \]

(20)

Thus, the problem of search for the scattering amplitude is reduced to calculation of the path integral (8) or (15). This integral is complicated enough.

One of the effective methods of the path integral evaluation is the variational method. However, the variational method based on the Yensen inequality cannot be applied in our case because the integral (8) is complex.
The generalization of the variational method is the method of the Gaussian equivalent representation (see [12]). The application of this method to the integral (8) leads to very cumbersome equations. The solution of these equations requires many efforts which should be made for solution of a particular task. Therefore, this method will not be considered in this paper. We refer a reader to [12, 13].

In this paper we formulate method based on the uncertainty correlation. So let us consider the integral (8).

3 Perturbation theory

Let us shortly remind perturbation method. The coupling constant should be small in this case. It means that in the representation

\[ e^{\Phi(g)} = \int d\sigma e^{gW} = \sum_{n=0}^{\infty} \frac{g^n}{n!} \langle W^n \rangle = e^{\sum_{n=0}^{\infty} g^n \Phi_n} , \]

\[ g\Phi_1 = g\langle W \rangle , \quad g^2\Phi_2 = \frac{g^2}{2} \left[ \langle W^2 \rangle - \langle W \rangle^2 \right] , ... \]

some few lowest terms should be taken into consideration only.

To perform the calculations of the perturbation terms in (8) we will use the standard Gaussian integral which in our case has the form

\[ \int D\xi \frac{1}{C} e^{ \frac{i}{2} \int_0^\infty d\tau \left( \xi^2 (\tau) - \int_0^\infty d\tau' (J(\tau)\xi(\tau)) \right) - \frac{i}{2} \int_0^\infty d\tau d\tau' (J(\tau)D(\tau,\tau')J(\tau')) } = e^{-\frac{i}{2} \int_0^\infty d\tau \int_0^\infty d\tau' (J(\tau)D(\tau,\tau')J(\tau'))} , \]

\[ D(\tau,\tau') = \frac{1}{2} [\tau + \tau' - |\tau - \tau'|] , \quad D(\tau,\tau) = \tau. \]

The potential in (8) can be represented as

\[ V (|k\tau + r - \xi(\tau)|) = \int \frac{dp}{(2\pi)^3} \tilde{V}(p) e^{ip(k\tau + r - \xi(\tau))} \] (21)

The formula takes place

\[ \int \frac{D\xi}{C} e^{ \frac{i}{2} \int_0^\infty d\tau \left( \xi^2 (\tau) - \sum_{j=1}^{N} (p_j \xi(\tau_j)) \right) - \frac{i}{2} \sum_{i,j=1}^{N} (p_i p_j)D(\tau_i,\tau_j) } = e^{-\frac{i}{2} \sum_{i,j=1}^{N} (p_i p_j)D(\tau_i,\tau_j)} . \]
For $\Phi_1$ and $\Phi_2$ one can get

$$g\Phi_1 = -im\int_0^{\infty} d\tau \int \frac{dp}{(2\pi)^3} \tilde{V}(p)e^{ip(k\tau+r) - i\frac{p^2}{2}\tau} = -2m \int \frac{dp}{(2\pi)^3} \tilde{V}(p)e^{ip(k\tau+r) - i0}$$

$$g^2\Phi_2 = -4m^2 \int \frac{dp_1dp_2}{(2\pi)^6} \frac{\tilde{V}(p_1)\tilde{V}(p_2)e^{i(p_1+p_2)r}}{(p_1^2 - 2(kp_2) - i0)(p_2^2 - 2(kp_2) - i0)} \frac{(p_1p_2)}{((p_1+p_2)^2 - 2((p_1+p_2)k) - i0)}$$

These functions for $k = 0$ define the scattering length and they are

$$g\Phi_1(r) = -2m \left[ \frac{1}{r} \int_0^{r} dy \, y^2V(y) + \int_r^{\infty} dy \, V(y) \right]$$

and

$$g^2\Phi_2(r) = \frac{1}{r} \int_0^{r} dy \, y^2 \left( \frac{d}{dy}g\Phi_1(y) \right)^2 + \int_r^{\infty} dy \, y \left( \frac{d}{dy}g\Phi_1(y) \right)^2.$$}

All calculations are very simple and we will not consider any examples.

### 4 Linear way approximation and eiconal

As said above, the argument of the potential in the representation (8)

$$k\frac{\tau}{m} + r - \xi(\tau) = v\tau + r - \xi(\tau)$$

can be interpreted as a linear motion of a particle $v\tau + r$ from the point $r$ to infinity plus quantum fluctuations $\xi(\tau)$ around the straight way. These fluctuations can be evaluated using the uncertainty correlation for a free motion. In this case, the uncertainty correlations lead to the following correlations

$$\Delta p \Delta r \sim h, \quad \Delta E \Delta t \sim h, \quad E = \frac{p^2}{2m}, \quad \frac{(\Delta p)^2}{2m} \Delta t \sim \frac{h^2}{m(\Delta r)^2} \Delta t \sim h, \quad (\Delta r)^2 \sim \frac{h}{m} \Delta t.$$
The last correlation \( (\Delta r)^2 \sim \frac{\hbar}{m} \Delta t \) is known as extension of the wave packet.

Let us come back to the path integral
\[
e^{\Phi(\rho, z, k)} = \int \frac{D\xi}{C} e^{\frac{i}{\hbar} \int_0^\infty d\tau \left[ \frac{m}{2} \dot{\xi}^2(\tau) - V(k \frac{\tau}{m} + \mathbf{r} - \xi(\tau)) \right]}.
\]

The main contribution to this integral comes from \( |\xi(\tau)| \sim \sqrt{\hbar \frac{\tau}{m}} \). Therefore, in order to neglect the function \( \xi(\tau) \) in the potential, the next condition should be satisfied
\[
\frac{|k|}{m} \tau \gg |\xi(\tau)| \sim \sqrt{\hbar \frac{\tau}{m}}.
\]

Let \( r_0 \) be of an order of the size of potential action. Then into integral over \( \tau \) the main contribution comes from the regions
\[
k \frac{\tau}{m} \sim r_0, \quad |\xi(\tau)| \sim \sqrt{\hbar \frac{\tau}{m}} \sim r_0.
\]

These correlations show that for large momenta \( k \) when \( k \gg \frac{\hbar}{r_0} \) one can neglect quantum fluctuations in the potential, i.e.
\[
V \left( k \frac{\tau}{m} + \mathbf{r} - \xi(\tau) \right) \approx V \left( k \frac{\tau}{m} + \mathbf{r} \right).
\]

As a result, we get the so called linear way or eiconal approximation:
\[
e^{\Phi(\rho, z, k)} \approx e^{-\frac{m}{\hbar} \int_0^\infty d\tau V \left( \sqrt{\rho^2 + z^2} \right)} = e^{-\frac{m}{\hbar} \int_0^\infty ds \sqrt{s^2 + \rho^2}}. \tag{23}
\]

It should be noted that the function \( \Phi(\rho, z, k) \) is diverged for \( k = 0 \), i.e. the eiconal approximation is valid for large momenta only.

The elastic scattering amplitude takes the form
\[
f(k, \theta) \approx -m \int_0^\infty d\rho \rho J_0(k \rho \sin \theta) \int_{-\infty}^\infty dz V \left( \sqrt{\rho^2 + z^2} \right) e^{2ikz \sin^2 \theta - \frac{m}{\hbar} \int_{-\infty}^\infty ds \sqrt{s^2 + \rho^2}}. \tag{24}
\]

For small scattering angles \( \theta \) the representation \([24]\) turns into the well-known quasiclassical approximation. Really, for large momenta and small angles one can estimate
\[
k \theta \sim 1, \quad \theta \sim \frac{1}{k} \ll 1,
\]

12
so that in the representation (24) one can put
\[ e^{2ikz\sin^2(\theta/2)} \approx 1. \]

As a result, the well known eiconal approximatin arises
\[
f(k, \theta) \approx -m \int_{0}^{\infty} d\rho \rho J_0(k\rho\theta) \int_{-\infty}^{\infty} dz V(\sqrt{\rho^2 + z^2}) e^{-imk} \int_{-\infty}^{\infty} ds V(\sqrt{s^2 + \rho^2})
\]
\[
= ik \int_{0}^{\infty} d\rho \rho J_0(\rho k\theta) \left[ 1 - e^{-imk} \int_{-\infty}^{\infty} ds V(\sqrt{s^2 + \rho^2}) \right].
\] (25)

This amplitude leads to the cross section
\[
\sigma(k) = 2\pi \int_{0}^{\pi} d\theta \sin(\theta)|f(k, \theta)|^2 = 2\pi \int_{0}^{\infty} d\rho \rho \left[ 1 - e^{-imk} \int_{-\infty}^{\infty} ds V(\sqrt{s^2 + \rho^2}) \right]^2
\]
\[
= 8\pi \int_{0}^{\infty} d\rho \rho \sin^2 \left( \frac{m}{2k} \int_{-\infty}^{\infty} d\tau V(\sqrt{\tau^2 + \rho^2}) \right).
\] (26)

It should be noted that formula (24) can be considered as generalization of the standard quasiclassical approximation for all angles.

For large momenta one gets
\[
\text{Im} f(k, 0) = k \int_{0}^{\infty} d\rho \rho \left[ 1 - \cos \left( \frac{m}{k} Y(\rho) \right) \right] \approx \frac{m^2}{2k} \int_{0}^{\infty} d\rho \rho Y^2(\rho), \quad (27)
\]
where
\[
Y(\rho) = \int_{-\infty}^{\infty} ds V(\sqrt{s^2 + \rho^2}).
\]

The cross section for large momenta equals
\[
\sigma(k) = 2\pi \frac{m^2}{k^2} \int_{0}^{\infty} d\rho \rho Y^2(\rho) = \frac{4\pi}{k} \text{Im} f(k, 0)
\] (28)
in complete correspondence with the unitary requirement.

For small and intermediate momenta the eiconal approximation does not work. As the next step in taking into account quantum fluctuations we formulate two approaches which will be named *the quantum mean approximation* and *the unitary approximation.*
5 Quantum mean approximation

The linear way approximation supposes that all quantum fluctuations can be neglected. This approximation is justified for large momenta. If the momenta are small, quantum fluctuations play an essential role and should be taken into account. The natural generalization of the eiconal formulas will be suggested the following approach.

Let us consider the path integral (16), describing the scattering length. In this case, momenta equal zero \( k = 0 \), and quantum correlations are important. This integral is real and the main contribution to it comes from the region of the order

\[
\langle \xi(\nu) \rangle^2 \sim \nu
\]

It is in correspondence with the uncertainty correlation (22), i.e. quantum fluctuations increase as

\[
\langle \xi(\nu) \rangle^2 \sim \nu.
\]

The quantum mean approximation is a way to take into account these fluctuations. We suppose that the main contribution to the path integral comes from the region (29), and mathematically it is realized by formula

\[
e^{\Phi(r)} = \int \frac{D\xi}{C} e^{-\int_0^\infty d\nu \frac{1}{2} \xi^2(\nu) - mV(\xi+\xi(\nu))}
\approx e^{-m \int_0^\infty d\nu \left( \sqrt{r^2 + (\xi(\nu))^2} \right)} = e^{-m \int_0^\infty d\nu \left( \sqrt{r^2 + b\nu} \right)}.
\]

(30)

The value of the parameter \( b \sim 1 \) is of an order of unity and the variation \( b = 1 \pm \delta \) defines the accuracy of this approximation. So that we get

\[
e^{\Phi(r)} \approx e^{-m \int_0^\infty d\nu \left( \sqrt{r^2 + b\nu} \right)} = e^{-2m b \int_0^\infty ds \ sV(s)}.
\]

(31)

The representation for the scattering length reads

\[
a = 2m \int_0^\infty dt \ r^2 V(r) e^{-2m b \int_0^\infty ds \ sV(s)}.
\]

(32)
The cross section \( \sigma(k) \) for zero momentum \( k = 0 \) equals

\[
\sigma(0) = 4\pi a^2 = 4\pi \left[ 2m \int_0^\infty dr \, r^2 V(r) \, e^{-2\mu a \int_r^\infty ds \, V(s)} \right]^2.
\]  

(33)

In order to get the cross section for all momenta, we proceed in the following way. We know the behavior of the cross section for large (26) and zero (33) momenta. Therefore, we need to connect these sections in a smooth way. To do this we suggest two approaches which will be named the *quantum mean approximation* and *unitary approximation*.

The quantum mean approximation for the amplitude consists in that the scattering amplitude is represented in the form

\[
f(k, \theta) = -m \int_0^\infty d\rho \, \rho J_0(k\rho \sin \theta) \int_{-\infty}^\infty dz V\left(\sqrt{\rho^2 + z^2}\right) e^{2ikz \sin^2 \theta - \frac{m}{k_c} \int_{-\infty}^\infty ds \, V\left(\sqrt{s^2 + \rho^2}\right)}
\]

(34)

where the parameter \( k_c \) is introduced. This representation does not change high momentum behavior. The value of the parameter \( k_c \) is defined by the condition that for zero momenta the amplitude \( f(0, \theta) \) coincides with the scattering length (32):

\[
2m \int_0^\infty dr \, r^2 V(r) \, e^{-\frac{2\mu a}{k} \int_r^\infty ds \, V(s)} = k_c \int_0^\infty d\rho \, \rho \left[ 1 - e^{-\frac{m}{k_c} \int_{-\infty}^\infty ds \, V\left(\sqrt{s^2 + \rho^2}\right)} \right]
\]

(35)

This equation defines the parameter \( k_c \). Then the cross section equals

\[
\sigma(k) = 2\pi \int_0^\pi d\theta \sin \theta |f(k, \theta)|^2.
\]

(36)

The quantum mean approximation for the cross section consists in modi-
ification of the eiconal formula (26):
\[ \sigma_a(k) \approx 8\pi \int_0^\infty d\rho \rho \sin^2 \left( \frac{m}{\sqrt{k_c^2 + k^2}} \right) \int_0^\infty dz V \left( \sqrt{\rho^2 + z^2} \right). \] (37)

The parameter \( k_c \) is defined by the condition that for the zeroth momenta the cross sections in two formulas (33) and (37) coincide, i.e.
\[ \sigma_0(b) = \sigma_a(0, k_c), \] (38)
\[ 4\pi \left[ 2m \int_0^\infty dr r^2 V(r) e^{-\frac{2m}{b} \int ds V(s)} \right]^2 = 8\pi \int_0^\infty d\rho \rho \sin^2 \left( \frac{m}{k_c} \right) \int_0^\infty dz V \left( \sqrt{\rho^2 + z^2} \right). \]

It is the equation to calculate the parameter \( k_c \).

The accuracy of its approximation is controlled by the parameter \( b \) which is changed in the vicinity of the point \( b \sim 1 \).

It should be noted that these approximations are rough enough, but the general character of the behavior is described correctly.

Another approach of the quantum mean approximation can be the following approximation for the path integral
\[ e^{\Phi(k,r)} = \int \frac{D\xi}{C} e^{i \int_0^\infty d\tau \left[ \frac{m}{2} \xi^2 - V(k_m + r - \xi(\tau)) \right]} \] (39)
\[ \approx \int \frac{d\xi}{(2\pi i)^{\frac{3}{2}}} e^{\frac{1}{2} \xi^2 - \frac{1}{2} \int_0^\infty d\nu V(k_0 + r - b \xi(\nu))}. \]

Further work should be done to evaluate the effectiveness of this approach.
6 Unitary approximation

The cross section for small momenta can be obtained in another way which is named the unitary approximation. The scattering amplitude is supposed to have the form

\[ f(k, \theta) = -m \int_0^\infty d\rho \rho J_0(k\rho \sin \theta) \int_{-\infty}^\infty dz V(\sqrt{\rho^2 + z^2}) e^{-\frac{m k}{k_c - ik} \int_s^\infty ds V(\sqrt{s^2 + \rho^2})}, \]  

(40)

However, the parameter \( k_c \) and the cross section will be defined by the unitary condition.

The amplitude for the forward scattering \( \theta = 0 \) equals

\[ f(k, 0) = -m \int_0^\infty d\rho \int_{-\infty}^\infty dz V(\sqrt{\rho^2 + z^2}) e^{-\frac{m k}{k_c - ik} \int_s^\infty ds V(\sqrt{s^2 + \rho^2})} \]

\[ = (-k_c + ik) \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{m k}{k_c} Y(\rho)} \right], \]  

(41)

\[ Y(\rho) = \int_{-\infty}^\infty ds V(\sqrt{s^2 + \rho^2}). \]

The imaginary part of the amplitude reads

\[ \frac{\text{Im} f(k, 0)}{k} = \int_0^\infty d\rho \rho \left\{ 1 - e^{-\frac{m k}{k_c^2 + k^2} Y(\rho)} \left[ \cos \left( \frac{m k}{k_c^2 + k^2} Y(\rho) \right) + \frac{k_c}{k} \sin \left( \frac{m k}{k_c^2 + k^2} Y(\rho) \right) \right] \right\}, \]  

(42)

The unitary condition, or the optical theorem requires

\[ \sigma(k) = \frac{4\pi}{k} \text{Im}(k, 0) = 2\pi \int_0^{\pi} d\theta \sin \theta |f(k, \theta)|^2. \]  

(43)

The unitary approximation consists in that the parameter \( k_c \) is a function of momentum \( k_c = k_c(k) \), and the unitary condition (43) is the equation on this function.
Now we proceed in a simpler way. For the zeroth momentum ($k = 0$) the imaginary part of the scattering amplitude reads

$$\frac{\text{Im} f(k, 0)}{k} \bigg|_{k=0} = \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{m}{k_c} Y(\rho)} \left( 1 + \frac{m}{k_c} Y(\rho) \right) \right].$$

and, according to the unitary condition, the cross section equals

$$\sigma(0) = 4\pi \frac{\text{Im} f(k, 0)}{k} \bigg|_{k=0} = 4\pi A^2(k_c), \quad (44)$$

where $A(k_c)$ is the scattering length:

$$A(k_c) = f(0, 0) = -k_c \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{m}{k_c} Y(\rho)} \right].$$

The correlation $(44)$ leads to the equation for the parameter $k_c$:

$$4\pi \left( k_c \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{m}{k_c} Y(\rho)} \right] \right)^2 = 4\pi \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{m}{k_c} Y(\rho)} \left( 1 + \frac{m}{k_c} Y(\rho) \right) \right]. \quad (45)$$

Finally, the cross section looks like

$$\sigma(k) = \frac{4\pi}{k} \text{Im}(k, 0) \quad (46)$$

$$= 4\pi \int_0^\infty d\rho \rho \left\{ 1 - e^{-\frac{m}{k_c^2 + k^2} Y(\rho)} \left[ \cos \left( \frac{k m Y(\rho)}{k_c^2 + k^2} \right) + \frac{k_c}{k} \sin \left( \frac{k m Y(\rho)}{k_c^2 + k^2} Y \right) \right] \right\}.$$

Below we will test the effectiveness of these approximations.

**7 Examples**

Let us consider some examples to demonstrate the approaches formulated above. We calculate the scattering lengths and cross sections for the right angled and singular potentials and the Yukawa potential and compare these approximations with exact results.
7.1 Right angled potential

Let us consider the right angled repulsing potential

\[ V(r) = V \theta(R - r) = \begin{cases} 
V, & r < R; \\
0, & R < r. 
\end{cases} \]

The exact solution can be obtained in the standard way. The wave function is represented as a sum over partial waves:

\[ \Psi(r) = \sum_{\ell=0}^{\infty} \frac{\chi_{\ell}(r)}{\sqrt{r}} Y_{\ell m}(n). \]

The solution of the Schroedinger equation looks like

\[ \chi_{\ell}(r) = \begin{cases} 
C_{\ell} I_{\ell+\frac{1}{2}}(r \kappa), & r < R; \\
A_{\ell}(k) J_{\ell+\frac{1}{2}}(kr) + B_{\ell}(k) N_{\ell+\frac{1}{2}}(kr), & R < r. 
\end{cases} \]

Here the notions \( \kappa = \sqrt{G - k^2} \), \( G = 2mV \) are introduced. For large \( r \) we have

\[ \chi_{\ell}(r) \to \sqrt{\frac{2}{\pi r}} \sqrt{A_{\ell}^2(k) + B_{\ell}^2(k)} \sin \left( kr - \frac{\pi}{2} \ell + \delta_{\ell}(k) \right), \quad r \to \infty. \]

The boundary conditions for \( r = R \)

\[ C_{\ell} I_{\ell+\frac{1}{2}}(\kappa R) = A_{\ell}(k) J_{\ell+\frac{1}{2}}(kR) + B_{\ell}(k) N_{\ell+\frac{1}{2}}(kR) \]

\[ \kappa C_{\ell} I'_{\ell+\frac{1}{2}}(\kappa R) = k \left[ A_{\ell}(k) J'_{\ell+\frac{1}{2}}(kR) + B_{\ell}(k) N'_{\ell+\frac{1}{2}}(kR) \right] \]

define the phases

\[ \tan \delta_{\ell}(k) = \frac{B_{\ell}(k)}{A_{\ell}(k)} = - \frac{k I_{\ell+\frac{1}{2}}(kR) J'_{\ell+\frac{1}{2}}(kR) - \kappa I'_{\ell+\frac{1}{2}}(kR) J_{\ell+\frac{1}{2}}(kR)}{k I'_{\ell+\frac{1}{2}}(kR) N'_{\ell+\frac{1}{2}}(kR) - \kappa I_{\ell+\frac{1}{2}}(kR) N_{\ell+\frac{1}{2}}(kR)}. \]

The scattering amplitude has the standard form

\[ f(k, \theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1) \left( e^{2i\delta_{\ell}(k)} - 1 \right) P_{\ell}(\cos \theta) \]
The scattering length equals
\[ a(G) = f(0, 0) = \frac{\delta_t(k)}{k} \bigg|_{k=0} = R \left[ 1 - \frac{\tanh(\sqrt{G})}{\sqrt{G}} \right] \]

The cross section is
\[ \sigma(k) = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2(\delta_t(k)). \]

### 7.1.1 Quantum mean approximation

In the case of the right angled repulsing potential we get
\[ 2mV(\sqrt{\rho^2 + s^2}) = G\theta \left( R - \sqrt{\rho^2 + s^2} \right) = G\theta \left( \sqrt{R^2 - \rho^2} - s \right), \]
\[ 2m \int_{r}^{\infty} ds \ sV(s) = 2mV \int_{r}^{R} ds \ s = G(\frac{R^2}{2} - r^2), \]
\[ mY(\rho) = 2m \int_{0}^{\infty} ds \ V(\sqrt{s^2 + \rho^2}) = G\sqrt{R^2 - \rho^2}. \]

In the quantum mean approximation the scattering length equals
\[ a = G \int_{0}^{R} dr \ r^2 e^{-\frac{G}{2}(R^2 - r^2)} = bR^2 \left[ 1 - \frac{1}{2} \sqrt{\frac{\pi}{c}} e^{-c} \text{Erfi}(\sqrt{c}) \right], \quad c = \frac{GR^2}{b} \]

The scattering amplitude in this approximation takes the form
\[ f(k, \theta, ka) = \frac{G}{2i} \int_{0}^{R} d\rho \ \rho J_0(\rho \sin \theta) \left[ e^{2ik \sin^2 \frac{\theta}{2} \sqrt{R^2 - \rho^2}} - e^{-2i(\frac{k \sin^2 \frac{\theta}{2}}{2 + 2(ik + k)}) \sqrt{R^2 - \rho^2}} \right] \]
\[ \frac{G}{2(ik + k)} + 2k \sin^2 \frac{\theta}{2} \]

and the scattering length is
\[ f(0, 0, ka) = ka \int_{0}^{R} d\rho \ \rho \left[ 1 - e^{-\frac{G}{2k \sin^2 \frac{\theta}{2} \sqrt{R^2 - \rho^2}}} \right] \]
\[ = \frac{kaR^2}{2} \left[ 1 - \frac{2}{A^2} \left( 1 - (1 + A)e^{-A} \right) \right], \quad A = \frac{GR}{ka} \]
The cross section looks as

\[
\sigma(k, k_\alpha) = 8\pi \int_0^R dr \ r \sin^2 \left( \frac{G}{\sqrt{k^2 + k^2}} \sqrt{R^2 - r^2} \right)
\]

\[
= \frac{\pi R^2}{B^2} \left[ 1 + 2B^2 - \cos(2B) - 2B \sin(2B) \right], \quad B = \frac{GR}{\sqrt{k^2 + k^2}}
\]

In the quantum mean approximation for amplitudes the parameter \( k_\alpha \) is defined by the equation

\[
a = f(0, 0, k_\alpha)
\]

or

\[
k_\alpha \int_0^R d\rho \ \rho \left[ 1 - e^{-\frac{G}{2\pi k_\alpha} \sqrt{R^2 - \rho^2}} \right] = G \int_0^R dr \ r^2 e^{-\frac{G}{2\pi}(R^2 - r^2)}.
\]

The cross section is defined by the standard formula

\[
\sigma(k, k_\alpha) = 2\pi \int_0^\pi d\theta \ \sin \theta |f(k, k_\alpha)|^2.
\]

In the quantum mean approximation for cross section the parameter \( k_\alpha \) is defined by the equation

\[
4\pi a^2 = \sigma(0, k_\alpha)
\]

In the unitary approximation the forward scattering amplitude equals

\[
f(k, 0) = (-k_c + ik) \int_0^R d\rho \ \rho \left[ 1 - e^{-\frac{G}{\pi k_c} \sqrt{R^2 - \rho^2}} \right]
\]

\[
= \frac{R^2}{2} (-k_c + ik) \left[ 1 - \frac{2}{D^2} \left( 1 - (1 + D)e^{-D} \right) \right], \quad D = \frac{GR}{k_c - ik}.
\]

The scattering length is

\[
A(k_c) = f(0, 0) = \frac{R^2}{2} (-k_c) \left[ 1 - \frac{2}{D^2} \left( 1 - (1 + D)e^{-D} \right) \right], \quad D = \frac{GR}{k_c}.
\]
The imaginary part of the amplitude reads

\[ \mathcal{I}(k_c) = \left. \frac{\text{Im} f(k, 0)}{k} \right|_{k=0} = R^2 \left[ \frac{1}{2} - \frac{3k_c^2}{G^2R^2} + e^{-\frac{c}{k_c}} \left( 1 + \frac{3k_c}{GR} + \frac{3k_c^2}{G^2R^2} \right) \right] \]  

(48)

The parameter \( k_c \) is defined by the equation

\[ 4\pi A^2(k_c) = 4\pi \mathcal{I}(k_c) \]  

(49)

Finally the cross section according to the optic theorem is

\[ \sigma(k) = 4\pi \frac{\text{Im} f(k, 0)}{k} \]  

(50)

The comparison of the exact cross section with the approximations are shown on Figures 1 and 2.

### 7.2 Singular repulsing potential

Let us calculate the scattering length and the cross section in the case of scattering on the singular repulsing potential. In the dimensionless variable \( r \rightarrow \frac{r}{R} \) the hamiltonian reads

\[ H = \frac{p^2}{2m} + \frac{g}{R^2} \left( \frac{R^2}{r^2} \right)^N = \frac{1}{mR^2} \left[ \frac{p^2}{2} + \frac{G^2}{r^{2N}} \right], \quad G = gm. \]  

(51)

In the limit \( N \rightarrow \infty \) the singular potential becomes the potential of the rigid sphere

\[ V(r) = \frac{g}{R^2} \left( \frac{R^2}{r^2} \right)^N \xrightarrow{N \rightarrow \infty} \begin{cases} +\infty & r < R, \\ 0 & r > R. \end{cases} \]  

(52)

The perturbation approximation does not work in this case. The scattering length is in the quantum mean approximation, according to formula (32)

\[ a(G) = G \int_0^\infty \frac{dr}{r^{2N-2}} e^{\frac{-G}{r}} \frac{d}{dr} \frac{1}{r} = \left( \frac{G}{N-1} \right)^{\frac{1}{N-1}} f_a(N), \]  

(53)

\[ f_a(N) = b^{\frac{1}{N-1}} \Gamma \left( 1 - \frac{1}{2(N-1)} \right). \]
For a given singular potential the exact result for the scattering length is known (see [11]):
\[ a = \left( \frac{G}{N-1} \right)^{\frac{1}{2(N-1)}} f(N), \quad f(N) = 2^{-\frac{1}{2(N-1)}} \frac{\Gamma \left( 1 - \frac{1}{2(N-1)} \right)}{\Gamma \left( 1 + \frac{1}{2(N-1)} \right)}. \quad (54) \]

The behavior of the functions \( f_a(N) \) for \( b = 1 \) and \( f_a(N) \) is shown on Figure 3. One can see that in the case of large \( N \gg 1 \) both the results coincide.

In the eiconal approximation the cross section equals
\[
\sigma_e(k) = 8\pi \int_0^\infty d\rho \sin^2 \left[ \frac{G}{k} \int_0^\infty \frac{d\tau}{(\tau^2 + \rho^2)^N} \right] \quad (55)
\]
\[
= 2\pi \left( \frac{G}{k} \right)^{\frac{2}{N-1}} \left[ \sqrt{\pi} \frac{\Gamma \left( N - \frac{1}{2} \right)}{\Gamma(N)} \right]^{\frac{2}{N-1}} \frac{\Gamma \left( 2N - 3 \right)}{\Gamma(2N - 1)} \sin \left( \frac{\pi}{2} \cdot \frac{2N + 1}{2N - 1} \right).
\]

For \( N \to \infty \) one has according (54) and (55)
\[
\sigma(k) \to \begin{cases} 
4\pi & k \to 0 \\
2\pi & k \to \infty
\end{cases} \quad (56)
\]

In this limit \( N \to \infty \) the dependence on momenta disappears in the eiconal approximation.

### 7.3 The Yukawa potential

Let us calculate the scattering length and the cross section on the Yukawa potential.

In the dimensionless variable \( r \to \frac{r}{\mu} \) the hamiltonian in this case reads
\[
H = \frac{P^2}{2m} + \frac{g}{4\pi} \frac{e^{-\mu r}}{r} = \frac{\mu^2}{m} \left[ \frac{P^2}{2} + G e^{-r} \right], \quad G = \frac{gm}{4\pi\mu}
\]

#### 7.3.1 Quantum mean approximation

The scattering length looks like
\[
a(b) = 2 \int_0^\infty dr \, re^{-r-\frac{2G}{b}e^{-r}} \quad (57)
\]
The elastic cross section takes the form

\[ \sigma_a(k, k_c) \approx 8\pi \int_0^\infty d\rho \rho \sin^2 \left( \frac{G}{\sqrt{k_c^2 + k^2}} K_0(\rho) \right). \]  

(58)

The parameter \( k_c \) is defined by the condition that for the zeroth momenta two formulas (57) and (58) for the cross section give the same result

\[ 4\pi a^2(b) = \sigma_a(0, k_c). \]  

(59)

The results of calculations are shown on Figures 4 and 5.

### 7.3.2 Unitary approximation

The scattering amplitude in this approximation equals

\[ f_a(k, \theta) = G \int_0^\infty d\rho \rho J_0(k \rho \sin \theta) \]

\[ \cdot \int_{-\infty}^{\infty} dz \ e^{2ikz \sin^2 \theta} \cdot e^{-\sqrt{z^2 + \rho^2}} \cdot e^{-\frac{ig}{\sqrt{z^2 + \rho^2}}} \]

\[ \cdot \int_{-\infty}^{\infty} dz \ e^{2ikz \sin^2 \theta} \cdot e^{-\sqrt{z^2 + \rho^2}} \cdot e^{-\frac{ig}{\sqrt{z^2 + \rho^2}}}. \]

(60)

For the scattering length one gets

\[ f_a(0, 0) = G \int_0^\infty d\rho \rho \int_{-\infty}^{\infty} dz \ e^{-\sqrt{z^2 + \rho^2}} \cdot e^{-\frac{ig}{\sqrt{z^2 + \rho^2}}}. \]

\[ = k_0 \int_0^\infty d\rho \rho \left[ 1 - e^{-\frac{ig}{k_0} K_0(\rho)} \right]. \]

(61)

because

\[ \int_0^\infty ds e^{-\sqrt{s^2 + \rho^2}} \sqrt{s^2 + \rho^2} = K_0(\rho). \]

where \( K_0(\rho) \) is the Bessel function of imaginary argument.
The imaginary part of the scattering amplitude equals

\[ \text{Im} f_a(k, 0) = -k \int_0^\infty d\rho \rho \left[ 1 - e^{-2Gk_0 K_0(\rho)} \cos \left( \frac{2Gk}{k_0^2 + k^2} K_0(\rho) \right) \right] \]

\[ + k_0 \int_0^\infty d\rho \rho e^{-2Gk_0 K_0(\rho)} \sin \left( \frac{2Gk}{k_0^2 + k^2} K_0(\rho) \right). \]  

For zero momenta \( k = 0 \) one has

\[ \left. \frac{\text{Im} f_a(k, 0)}{k} \right|_{k=0} = - \int_0^\infty d\rho \rho \left[ 1 - \left( 1 + \frac{2G}{k_0} K_0(\rho) \right) e^{-2Gk_0 K_0(\rho)} \right]. \]  

The unitary condition for zero momenta \( k = 0 \) gives the equation for the parameter \( k_0 \):

\[ \sigma(0) = 4\pi \left( k_c \int_0^\infty d\rho \rho \left[ 1 - e^{-2Gk_0 K_0(\rho)} \right] \right)^2 \]

\[ = 4\pi \int_0^\infty d\rho \rho \left[ 1 - e^{-2Gk_0 K_0(\rho)} \left( 1 + \frac{2G}{k_c} K_0(\rho) \right) \right]. \]

After calculation of the parameter \( k_c = k_c(G) \) as a function of the coupling constant one can compute the cross section

\[ \sigma(k) = 4\pi \int_0^\infty d\rho \rho \left\{ 1 - e^{-2Gk_0 K_0(\rho)} \left[ \cos \left( \frac{2Gk}{k_c^2 + k^2} K_0(\rho) \right) + k_c \sin \left( \frac{2Gk}{k_c^2 + k^2} K_0(\rho) \right) \right] \right\}. \]  

On Figures 6 and 7 the results of calculations are demonstrated.

8 Conclusion

The representation of the elastic scattering amplitude in the form of the path integral is obtained by using the stationary Schroedinger equation.

The methods of evaluation of the path integrals are based on the uncertainty correlation for a free motion.
Formulas for the scattering lengths and cross sections for any coupling constants are simple enough so that qualitative and semiquantitative estimations can be obtained without great efforts. The examined examples show the effectiveness of the proposed methods.

Generally speaking, other known and unknown methods of calculating of path integrals can be worked out and applied, for example, variation methods and method of the Gaussian equivalent representation (see, for example, [12 [13 8 9] ] and so on). All these problems require further investigations.

In conclusion I wish to thank V.S.Melezhik for many helpful discussions.

References

[1] R.P.Feynman and A.R.Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Company, N.Y. 1965.

[2] H.Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial marketing*, World Scientific, NY, 2006.

[3] J.Zinn-Justin, *Path integrals in quantum mechanics*, OXFORD University Press, 2005.

[4] A.Das, *Field Theory. A Path Integral Approach*, World Scientific, London, 2006.

[5] B.M.Barbashov and V.V.Nesterenko, *Eiconal approximation for high energy scattering of particles* (in russian), Moscow University, Moscow, 1977.

[6] Chiou-Lahanas C. and all, Phys.Rev **D52**, 5877-5882 (1995);

[7] Rembielinski J., [arXiv:hep-ph/9509219v2](http://arxiv.org/abs/hep-ph/9509219) (1995);

[8] Carron J., Rosenfelder R., arXiv: 1107.3034v2[nucl-th], 13 Oct 2011;

[9] Rosenfelder R., arXiv: 1302.3419[nucl-th], 2011;

[10] G.V.Efimov, Theoretical and Mathematical Physics, **171**:812-831, 2013.

[11] Fluegge, S., *Practical Quantum Mechanics. I.*, Springer-Verlag, Heidelberg, 1971;
[12] V.Dineykhan, G.V.Efimov, G.Ganbold and S.N.Nedelko, *Oscillator representation in quantum physics.*, Lecture Notes in Physics, m26, Springer-Verlag, Berlin, 1995;

[13] G.V.Efimov, *Method of functional integration*, (in russian), University ”Dubna”, Dubna, 2008.
Figure 1: The scattering length for the right angled potential as a function of the coupling constants $G$. Boldface lines - the quantum mean approximation. The upper line - $b = 1$, lower line - $b = 0.8$. Thin line - exact result.
Figure 2: The cross section $\sigma(k)$ for the right angled potential for the coupling constants $G = 5$, $G = 10$ and $G = 15$. Boldface lines - the quantum mean approximation. The upper line - $b = 1$, lower line - $b = 0.8$. Thin line - exact result.
Figure 3: The scattering length for the singular potential as a function of the coupling constant $G$. The upper line - quantum mean approximation, lower line - exact result.

Figure 4: Quantum mean approximation. The scattering length for the Yukawa potential as a function of the coupling constant $G$. Boldface lines - the approximation for $b = 1$, thinner lines - $b = 0.7$. Thin line - numerical result.
Figure 5: Quantum mean approximation. The cross section $\sigma(k)$ for the Yukawa potential for the coupling constants $G = 5$, $G = 10$ and $G = 15$. Boldface lines - the approximation, thin line - numerical result.
Figure 6: Unitary approximation. The scattering length for the Yukawa potential as a function of the coupling constant $G$. Boldface lines - the approximation for $b = 1$, thiner lines for $b = 0.7$. Thin line - numerical result.
Figure 7: Unitary approximation. The cross section $\sigma(k)$ for the Yukawa potential for the coupling constants $G = 5$, $G = 10$ and $G = 15$. Boldface lines - the approximation, thin line - numerical result.