A New Correction to the Rytov Approximation for Strongly Scattering Lossy Media

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Abstract—We propose a correction to the conventional Rytov approximation (RA) and investigate its performance for predicting wave scattering under strong scattering conditions. An important motivation for the correction and investigation is to help in the development of better models for inverse scattering. The correction is based on incorporating the high-frequency theory of inhomogeneous wave propagation for lossy media into RA formulation. We denote the technique as the extended RA for lossy media (xRA-LM). xRA-LM significantly improves upon the existing noniterative linear scattering approximations such as RA and Born approximation (BA) by providing a validity range for the permittivity of objects of up to 50 times greater than RA. We demonstrate the technique by providing results for predicting wave scattering from piecewise homogeneous scatterers in a 2-D region. Numerical investigation of the performance of xRA-LM for solving direct problems show that xRA-LM can accurately predict wave scattering by electrically large, low-loss scatterers with high complex relative permittivity ($\epsilon_r > 50 + 5j$). To the best of our knowledge, this is the first noniterative, linear approximate wave scattering model which has a large validity range in terms of both permittivity and electrical size.

Index Terms—Born approximation (BA), indoor imaging, inverse scattering, Rytov approximation (RA), wave scattering.

I. INTRODUCTION

MODELING of electromagnetic wave scattering has provided many technological breakthroughs related to the solution of direct and inverse problems [1]–[16]. Important applications have included indoor propagation prediction, microwave imaging (inverse scattering), and antenna design. The exact solution of electromagnetic wave scattering can be found from Maxwell’s equations using formulations such as the volume source integral (VSI) (Lipmann–Schwinger equation) or eigenfunction expansions [1], [2], [7]. These exact formulations can be solved numerically using techniques such as method of moments (MoM), and finite-difference time domain (FDTD) [1], [3]–[5], [17] and these have revolutionized the design of antennas and radio frequency (RF) circuits in recent decades by providing very accurate predictions of wave scattering and radiation.

However, for applications where the domain of interest (DoI) is electrically large or contains very high-permittivity materials, solving exact models as a direct problem becomes computationally infeasible. As an inverse problem, the exact models become highly nonlinear and ill-posed and cannot handle the measurement data suffering from inaccuracies due to noise and the real-world data acquisition process [1], [2], [6], [9]–[12]. In practice, such conditions involving an electrically large DoI with strong scatterers and imperfect measurements are encountered in many applications such as indoor imaging, nondestructive evaluation, and microwave imaging [1], [2], [6], [9]–[12], [18]–[20]. Under these conditions, the exact models (such as VSI) have found limited practical application. This opens up a large research field for finding more straightforward approximations to these exact models which can be solved with feasible computational requirements and practical measurement systems.

Common approximate models include Born approximation (BA), Rytov approximation (RA), geometrical optics (GO), and uniform theory of diffraction (UTD) [1], [2], [6]–[10], [16], [21]–[27]. Many of these approximate techniques have provided the basis for well-known approximate inverse scattering techniques and for extending wave scattering theory to practical applications [1], [2], [6]–[12], [18], [28]–[33]. Among the approximate techniques, the computationally least complicated include noniterative linear approximations such as RA and BA and these are commonly used in inverse scattering [1], [2], [9], [16], [21], [24]. While these are useful, they have a limited range of validity. For example, BA fails if the scatterer is electrically large or has permittivity deviating significantly from unity while RA fails when permittivity deviates significantly from unity.

In this work, we propose a noniterative linear approximation which can estimate scattering from strongly scattering objects with very high relative permittivity (up to $\epsilon_r > 50 + 5j$) and large electrical size (greater than the incident wavelength). Our technique is based on a correction to conventional RA and we denote the technique as the extended RA for lossy media (xRA-LM). An important motivation for correction and investigation is to help in the development of better models for phaseless inverse scattering [34], [35]. xRA-LM incorporates the high-frequency theory of inhomogeneous wave propagation for lossy media [23], [36]–[40] into the formulation of
RA, resulting in a remarkably higher validity range. Another key aspect of xRA-LM is its validity for lossy media and this is important for applications in the everyday environment where most materials have a loss component. For example, the complex-valued relative permittivity, \( \varepsilon_r = \varepsilon_r + j\varepsilon_i \), of scatterers in the everyday environment at 2.4 GHz have \( \varepsilon_R \) ranging from \( 2 < \varepsilon_R \leq 50 \) (where \( \varepsilon_R \geq 20 \) for the human body, for example) [14], [41]–[43]) and \( \varepsilon_i \) as characterized by loss tangent is in the range \( \delta = \varepsilon_i/\varepsilon_R \in [10^{-3}, 10^{-1}] \) [14], [41]–[43].

We numerically investigate the performance of xRA-LM for solving direct problems by providing results of wave scattering from a 2-D region. The results show that xRA-LM can accurately predict wave scattering from electrically large scatterers with large complex permittivity (\( \varepsilon_r > 50 + 5j \)). Comparisons with RA and BA also show that xRA-LM significantly outperforms RA and BA while maintaining similar computational complexity. To the best of our knowledge, this is the first noniterative, linear approximate wave scattering model which has such a high validity range (in terms of permittivity and size of scatterer). It can open up a new paradigm of noniterative linear models that provide practically feasible solutions to both direct and inverse scattering problems in strong scattering environments [1], [2], [6]–[12], [18], [28]–[33]. For example, a simplified phaseless version of xRA-LM has been used for inverse scattering to obtain impressive reconstruction results [34]. However, the analysis of the accuracy of the underlying direct problem can provide a more comprehensive approach to analyzing the model. Unlike the inverse problem, the direct problem is not ill-posed, and hence the accuracy obtained is directly related to the accuracy of the model rather than the regularization methods deployed to tackle ill-posedness. The goal of this article is to investigate xRA-LM from the direct problem perspective to analyze its performance and accuracy.

**Organization of This Article:** The problem formulation is described in Section II followed by derivation of the proposed xRA-LM approximation in Section III. Section IV provides numerical results followed by conclusions. In the remainder of this article, lower and upper case boldfaced letters are used to represent vectors and matrices, respectively. Italic letters are used to represent scalar quantities.

## II. Problem Formulation

Consider the 2-D scenario shown in Fig. 1 where a transmitter Tx or source of electromagnetic radiation illuminates an arbitrary shaped scatterer \( S \) placed inside DoI \( D \). The scatterer is characterized by its complex-valued relative permittivity \( \varepsilon_r(r) = \varepsilon_r(r) + j\varepsilon_i(r) \) (assuming permeability to be \( \mu_0 = 1 \)), and the scattering from the DoI is collected by an array of receivers placed around a measurement boundary \( B \). The electromagnetic radiation from Tx is assumed to be monochromatic, time-harmonic, and vertically polarized which is often referred to as transverse magnetic (TM) in wave scattering contexts and is commonly used in real-world applications.

In the absence of any scatterers, the incident field at any point inside \( D \) is denoted by \( E_i(r) \). It satisfies the free-space wave equation

\[
(\nabla^2 + k_0^2)E_i(r) = 0, \quad r \in D
\]

where \( k_0 = 2\pi/\lambda_0 \) is the free-space wavenumber and \( \lambda_0 \) is the free-space wavelength. The total field at any point is the sum of the incident and scattered field (i.e., \( E(r) = E_i(r) + E_s(r), \quad r \in D \)) and satisfies the inhomogeneous Helmholtz wave equation

\[
(\nabla^2 + k_0^2v^2(r))E(r) = 0, \quad r \in D
\]

where \( v(r) \) is the refractive index and is related to relative permittivity through \( \varepsilon_r = v^2(r) \). Subtracting (1) and (2) provides a wave equation in terms of scattered field \( E_s = E - E_i \), and it is written as a Fredholm integral equation of the second kind

\[
E(r) = E_i(r) + k_0^2 \int_S g(r, r') (v^2(r') - 1) E_i(r') dr'^2,
\]

where \( r \in B, r' \in S \) and \( S \subseteq D \). Equation (3) is also known as the Lipmann–Schwinger equation or VSI and provides an exact description for wave scattering [1], [5].

Solving VSI as a direct problem implies estimation of \( E(r) \) given \( v(r) \) and \( E_i(r) \). This is computationally expensive when \( D \) is electrically large because it requires estimation of \( E(r) \) over \( D \). Solving VSI as an inverse problem implies solving it for \( v(r) \) given \( E(r) \) and \( E_i(r) \) on \( B \) which results in a nonlinear, ill-posed problem since both \( v(r) \) and \( E(r) \) need to be found inside \( D \). To overcome these challenges, noniterative linear approximations have been proposed to simplify VSI. The most extensively researched techniques are RA and BA. BA straightforwardly approximates the total field \( E(r) \) inside the integral (3) by the incident field \( E_i(r) \) to give

\[
E(r) = E_i(r) + k_0^2 \int_S g(r, r') (v^2(r') - 1) E_i(r') dr'^2,
\]

where \( r \in B \) and \( r' \in S \). Therefore, for the direct problem, unlike VSI, BA does not have any unknown field inside the integral, removing the need for any expensive matrix inversion step. It is also linear as an inverse problem (unlike VSI). However, BA has a poor range of validity as it fails for even a small permittivity contrast or if the size of the scatterer is comparable or larger than \( \lambda_0 \) [1], [2], [9], [16], [21], [24].

RA, on the other hand, uses the Rytov transformation to arrive at an approximate method that can handle electrically large objects but with a similar range of validity on permittivity as BA. The Rytov transformation normalizes the total field
$E(r)$ by the incident field $E_i(r)$ to express the scattering by a complex phase $\phi_s(r)$

$$\frac{E(r)}{E_i(r)} = e^{i\phi_s(r)}. \quad (5)$$

Intuitively, the complex phase $\phi_s(r)$ represents the phase and log amplitude deviations from the incident field (caused by scattering). Substituting (5) in (2) and using (1) gives a nonlinear differential equation (Riccati equation in $E_i\phi_s$) [24]

$$\left(\nabla^2 + k_0^2\right)(E_i(r)\phi_s(r)) = -k_0^2 E_i(r) \left[v'^2(r) - 1 + \frac{\nabla\phi_s(r) \cdot \nabla\phi_s(r)}{k_0^2}\right]. \quad (6)$$

Equation (6) can be written in integral form [which we call the Rytov integral (RI)] to obtain an expression for total field

$$E(r) = E_i(r) \exp\left(\frac{k_0^2}{E_i(r)} \int_D g(r, r') \chi_{RI}(r') E_i(r') d^2 r'\right) \quad (7a)$$

$$\chi_{RI}(r') = v'(r')^2 - 1 + \frac{\nabla\phi_s(r) \cdot \nabla\phi_s(r)}{k_0^2} \quad (7b)$$

where $\chi_{RI}$ is the contrast function of RI. The term $\nabla\phi_s \cdot \nabla\phi_s$ in RI is then neglected under a weak scattering assumption to arrive at RA as

$$E(r) = E_i(r) \exp\left(\frac{k_0^2}{E_i(r)} \int_D g(r, r') \chi_{RA}(r') E_i(r') d^2 r'\right) \quad (8a)$$

$$\chi_{RA}(r') = v'(r')^2 - 1. \quad (8b)$$

Neglecting $\nabla\phi_s \cdot \nabla\phi_s$ makes RA useful only for weak scattering with $\epsilon_R \approx 1$ (similar to BA). However, RA does not impose a restriction on the size of the scatterer unlike BA [24]. For high-permittivity variations, $\nabla\phi_s \cdot \nabla\phi_s$ cannot be neglected and estimation of $\nabla\phi_s \cdot \nabla\phi_s$ is difficult as it requires solving the intractable nonlinear equation (7) [10], [44]. To the best of our knowledge, this has not been done for strongly scattering lossy media. Therefore, approximating $\nabla\phi_s \cdot \nabla\phi_s$, instead of completely neglecting it, can provide improvement over conventional RA [45].

### III. CORRECTIONS TO THE RA

In this section, we derive the proposed xRA-LM technique by providing corrections to RA using the characterization of inhomogeneous waves in lossy media [23], [36]–[40].

#### A. High-frequency Approximations in Lossy Media

High-frequency approximations treat waves as straight rays and are used in approximations such as GO and GTD to describe scattering from objects that are larger in size compared with $\lambda_0$. However, ray formulations inside lossy media (with complex-valued refractive index) are intricate as the waves become inhomogeneous inside lossy media [23], [36]–[40]. Due to this, there are surprisingly limited ray formulations for lossy media [21], [23] even after decades of research. Inhomogeneous waves exhibit the property that the planes of constant phase are no longer parallel to the planes of constant amplitude [23], [36]–[40].

In this work, we deal with homogeneous plane waves (HPWs) that are incident on lossy media and become inhomogeneous plane waves (IPWs) inside the lossy media.

Fig. 2 illustrates a vacuum/air to lossy dielectric interface. The lossy medium is characterized by a constant complex refractive index $\nu$ and constant relative permittivity ($\epsilon_r$) [21] defined by

$$v = v_R + jv_I; \quad \epsilon_r = \epsilon_R + j\epsilon_I \quad (9)$$

and are related as $v^2 = \varepsilon_r$ [21]. Equating real and imaginary parts gives relations $v_R^2 - v_I^2 = \varepsilon_R$ and $v_R v_I = \varepsilon_I/2$. Using this, and defining loss tangent of the medium as $\delta = \varepsilon_I/\varepsilon_R$, we can express $v_R$ and $v_I$ as

$$v_R = \sqrt{\frac{\varepsilon_R(\sqrt{1 + \delta^2} + 1)}{2}}, \quad v_I = \sqrt{\frac{\varepsilon_R(\sqrt{1 + \delta^2} - 1)}{2}} \quad (10)$$

For low-loss media ($\delta \ll 1$), (10) can be simplified (for practical use) using the binomial expansion as

$$v_R \approx \sqrt{\varepsilon_R + \frac{1}{4} \delta^2 \varepsilon_R} \approx \sqrt{\varepsilon_R} \quad (11a)$$

$$v_I \approx \frac{\varepsilon_I}{2 \sqrt{\varepsilon_R}} = \frac{1}{2} \delta \sqrt{\varepsilon_R} \quad (11b)$$

When the HPW field is incident on the air–lossy media interface (in Fig. 2), it becomes partially reflected (as HPW) and transmitted (as IPW) at the interface. Snell’s law $\sin \theta_i = (v_R + jv_I)$ shows that the angle of refraction becomes a complex quantity in this case. Such complex angles are not geometrically intuitive in conventional GO, and hence, a new concept of effective refractive index has been introduced [36], [37], [40] which can be used to remove complex angles and is also used in our work. The concept of effective refractive index allows us to decompose the mathematical form of IPW as a linear combination of vectors normal to the planes of constant phase and constant amplitude. Then, real-valued Snell’s law angle can be applied separately for the refraction and attenuation components of IPW (see [36]–[40]). Using this concept, we can express the wave vector of transmitted IPW field inside the lossy medium as

$$k_l = k_0 \left( V_R \hat{k}_t + j V_I \hat{k}_a \right) \quad (12)$$
where, as shown in Fig. 2, the unit vectors \( \hat{k}_i \) and \( \hat{k}_a \) are the normal vectors, respectively, to the planes of constant phase and constant amplitude. \( V_R \) and \( V_I \) are the scalar constants of these unit vectors and are called effective real and imaginary parts of the refractive index, respectively [36]–[40].

In the free-space half of the interface, there will be two HPW fields, namely, incident and reflected fields which can be written as

\[
E_i(r) = A_0 \exp \left(j k_0 \hat{k}_i \cdot r \right) \quad \text{(13a)} \\
E_r(r) = A_r \exp \left(j k_0 \hat{k}_r \cdot r \right) \quad \text{(13b)}
\]

where \( \hat{k}_i \) and \( \hat{k}_r \) are the wave vectors of the incident and reflected fields, respectively. Furthermore, using (12), we can express the transmitted IPW field inside the lossy medium as

\[
E_t(r) = A_r \exp \left(j k_0 \left(V_R \hat{k}_i \cdot r + j V_I \hat{k}_a \cdot r \right) \right) \quad \text{(14)}
\]

The phase of the incident, reflected, and transmitted fields should match tangentially at the media interface, which gives

\[
\hat{p} \cdot \hat{k}_i = \hat{p} \cdot \hat{k}_r = \hat{p} \cdot \left(V_R \hat{k}_i + j V_I \hat{k}_a \right) \quad \text{(15)}
\]

Equating real parts of (15) provides real-valued Snell’s law angles for the interface

\[
\cos \theta_i = \sin \theta_r = V_R \sin \theta_t \quad \text{(16)}
\]

and equating imaginary parts of (15) shows that the vector \( \hat{k}_a \) is normal to the interface, that is,

\[
\hat{p} \cdot (V_I \hat{k}_a) = 0 \implies \hat{p} \perp \hat{k}_a. \quad \text{(17)}
\]

This is an interesting result as it implies that whenever an HPW field enters a lossy medium and becomes IPW field, the planes of constant amplitude become parallel to the interface. This result is key to our derivation of xRA-LM as we shall show later.

In the above results, it can be seen that there are no complex angles due to the use of effective refractive index expressions (14). The effective refractive index \( (V_R, V_I) \) has to be related to the actual refractive index \( (\nu_R, \nu_I) \) so that it can be used for scattering estimation in xRA-LM. This can be performed by substituting (14) into (2) which gives

\[
\frac{V_R^2 - V_I^2}{V_R V_I \cos \theta_t} = \nu_R \nu_I \quad \text{(18a)}
\]

where, from Fig. 2, \( \hat{k}_i \cdot \hat{k}_a = \cos \theta_i \). \( V_R \) can be estimated by eliminating \( V_I \) from (18) which gives

\[
V_R = \left\{ \frac{1}{2} \left[ \sqrt{\left( \frac{V_R^2 - V_I^2}{\cos \theta_t} \right)^2 + 4 \left( \frac{V_R V_I}{\cos \theta_i} \right)^2} \right] \right\}^{1/2} \quad \text{(19)}
\]

where \( \cos \theta_t \) can be expressed in terms of \( \sin \theta_t \) using (16). Similarly, \( V_I \) can be obtained by eliminating \( V_R \) from (18).

Under low-loss assumption \( (\epsilon_R \gg \epsilon_I) \), (19) can be simplified (using binomial expansion \([39]\)) to approximate \( V_R \) as

\[
V_R \approx v_R \left( 1 + \frac{\sin^2 \theta_i}{2(v_R^2 - \sin^2 \theta_i)} \delta^2 \right) \approx v_R. \quad \text{(20)}
\]

Similarly, \( V_I \) can be approximated as

\[
V_I \approx \frac{v_R V_I}{\sqrt{v_R^2 - \sin^2 \theta_i}} \left( 1 - \frac{v^2 I}{2(v_R^2 - \sin^2 \theta_I)} \delta^2 \right) \approx \frac{v_R V_I}{\sqrt{v_R^2 - \sin^2 \theta_I}}. \quad \text{(21)}
\]

Using these results, we can rewrite the ray formulation (14) inside an extended lossy scatterer with a piecewise homogeneous distribution of complex-valued refractive index. This can be performed by rewriting (14) using the path integral along the ray direction \( (dr = dr \hat{k}_i) \) as

\[
E_i = A_i \exp \left(-k_0 \int_{\hat{k}_i} V_R \hat{k}_i \cdot (\hat{k}_a dr) \right)
\times \exp \left(j k_0 \int_{\hat{k}_i} V_R \hat{k}_i \cdot (\hat{k}_r dr) \right) \quad \text{(22)}
\]

where \( E_i, V_R, V_I \) are the functions of \( r \), and for brevity, this is implicitly assumed in the rest of this article. In Section III-B, we use ray formulation in (22) for IPW field inside the extended scatterer for deriving corrections to RA.

Note that (14) is an approximation to the IPW field inside the scatterer as it only includes the first-order ray inside the homogeneous scatterer. Due to multiple scattering inside the scatterer’s boundaries, there will be higher order rays inside the scatterer, which are ignored in (14). Fortunately, for a large, lossy scatterer (considered in this work), it is known that the first-order ray is a good approximation [36] as higher order rays will contain low energy.

B. Corrections to Conventional RA

To approximate \( \nabla \phi_i \cdot \nabla \phi_j \), we start by equating the total field inside the scatterer (5) to the ray equation (22)

\[
E_i(r) e^{i \phi_i(r)} = A_i \exp \left(-k_0 \int V_I(r) \cos \theta_i \ dr \right) \exp \left(j k_0 \int V_R(r) \ dr \right) \quad \text{(23)}
\]

where \( \hat{k}_a \cdot \hat{k}_i = \cos \theta_i \) from Fig. 2. Substituting the incident field from (14a) as \( E_i(r) = A_0(0) e^{i k_0 \hat{k}_i \cdot r} \) gives

\[
\phi_i(r) = \ln \left[ \frac{A_i}{A_0} \right] + k_0 \left[ j \int V_R(r) dr - j \hat{k}_i \cdot r - \int V_I(r) \cos \theta_i dr \right]. \quad \text{(24)}
\]

Note that the quantities \( V_R, V_I, E_i \) are the functions of \( r \) in (24), and for brevity we do not show this dependence in
the remainder of this article. Taking the gradient of (24) gives (recall from (22), \(d\mathbf{r} = d\mathbf{r} \hat{k}_i\))

\[
\nabla \phi_s(\mathbf{r}) = \left[ \nabla \ln \left( \frac{A_i}{A_0} \right) \right] + k_0 \left[ j \left( V_R \hat{k}_i - \hat{k}_i \right) - V_I \hat{k}_a \right]
\]

(25)

so that

\[
\nabla \phi_s(\mathbf{r}) \cdot \nabla \phi_s(\mathbf{r}) = \left[ V_i^2 - V_R^2 - 1 + 2V_R \cos \theta_i + \frac{\nabla \hat{\mathbf{A}} \cdot \nabla \hat{\mathbf{A}}}{k_0^2} \right] + 2 j \left[ (V_I \cos \theta_i - V_R V_I \cos \theta_i) + \frac{1}{k_0} \nabla \hat{\mathbf{A}} \left( V_R \hat{k}_i - \hat{k}_i \right) \right].
\]

(26)

where \(\hat{\mathbf{A}} = \ln(A_i/A_0)\). We note from Fig. 2 that \(\hat{k}_i \cdot \hat{k}_a = \cos \theta_i\) and \(\hat{k}_i \cdot \hat{k}_a = \cos \theta_i\), where \(\theta_i\) is the scattering angle. Using this and separating out real and imaginary terms, we can now write (26) as

\[
\nabla \phi_s(\mathbf{r}) \cdot \nabla \phi_s(\mathbf{r}) = \left[ V_i^2 - V_R^2 - 1 + 2V_R \cos \theta_i + \frac{\nabla \hat{\mathbf{A}} \cdot \nabla \hat{\mathbf{A}}}{k_0^2} \right] + 2 j \left[ (V_I \cos \theta_i - V_R V_I \cos \theta_i) + \frac{1}{k_0} \nabla \hat{\mathbf{A}} \left( V_R \hat{k}_i - \hat{k}_i \right) \right].
\]

(27)

Equation (27) provides an expression for \(\nabla \phi_s \cdot \nabla \phi_s\) which is required in RI (7b) [neglected in RA (8b)]. Expanding the contrast function (7b) of RI using (18) gives

\[
\chi_{RI}(r) = (V_R + jV_I)^2 - 1 + \frac{\nabla \phi_s(\mathbf{r}) \cdot \nabla \phi_s(\mathbf{r})}{k_0^2}
\]

\[
= V_i^2 - V_R^2 + 2 V_I V_R \cos \theta_i + \frac{\nabla \phi_s(\mathbf{r}) \cdot \nabla \phi_s(\mathbf{r})}{k_0^2}.
\]

(28)

Substituting \((\nabla \phi_s \cdot \nabla \phi_s)/k_0^2\) from (27) to (28) leads to cancellation of several terms and gives

\[
\chi_{RI}(r) = \left[ 2V_R \cos \theta_i - 2 + \frac{1}{k_0^2} (\nabla \hat{\mathbf{A}} \cdot \nabla \hat{\mathbf{A}}) - \frac{2}{k_0} (\nabla \hat{\mathbf{A}}) V_R \hat{k}_a \right]
\]

\[
+ j \left[ 2 V_I \cos \theta_i + \frac{2}{k_0} (\nabla \hat{\mathbf{A}}) \left( V_R \hat{k}_i - \hat{k}_i \right) \right].
\]

(29)

Equation (29) can be further modified using (20) and (21) to replace \(V_R\) and \(V_I\) in terms of \(v_R\) and \(v_I\) under low-loss conditions as

\[
\chi_{RI}(r) = \left[ 2(v_R \cos \theta_i - 1) + \frac{1}{k_0^2} (\nabla \hat{\mathbf{A}} \cdot \nabla \hat{\mathbf{A}}) - \frac{2}{k_0} (\nabla \hat{\mathbf{A}}) v_R \hat{k}_a \right]
\]

\[
+ j \left[ 2 - \frac{v_R v_I}{\sqrt{v_R^2 - \sin^2 \theta_i}} \cos \theta_i + \frac{2}{k_0} (\nabla \hat{\mathbf{A}}) \left( v_R \hat{k}_i - \hat{k}_i \right) \right].
\]

(30)

Complex refractive index can also be expressed in terms of complex permittivity using (11). Using this, the final expression for the contrast function in RI under low-loss, high-frequency conditions is given by

\[
\chi_{RI}(r) = \left( \sqrt{\epsilon_R \cos \theta_i - 1} \right) + \frac{1}{k_0} \left( \frac{(\nabla \hat{\mathbf{A}} \cdot \nabla \hat{\mathbf{A}}) - \frac{2}{k_0} (\nabla \hat{\mathbf{A}}) V_R \hat{k}_a}{\sqrt{\epsilon_R - \sin^2 \theta_i}} \right)
\]

\[
+ j \left( \frac{\epsilon_I}{\sqrt{\epsilon_R - \sin^2 \theta_i}} \right) \left( v_R \hat{k}_i - \hat{k}_i \right).
\]

(31)

Unlike contrast function \(\chi_{RA}\) of conventional RA, the derived, corrected contrast \(\chi_{RI}\) is a nonlinear function of permittivity. Furthermore, the imaginary part \(\text{Im}(\chi_{RI})\) depends on both the real and imaginary parts of the permittivity and this describes the “crosstalk” where even when the permittivity is real, there will be a component in the imaginary part of the contrast function. Similarly, the real part of the contrast function also depends on both the real and imaginary parts of the permittivity.

Further simplification of (31) is possible. For the high-frequency regime where \(k_0 \) is large, we can approximate (31) by ignoring the cross terms (\(R_2, R_3, \) and \(I_2\)). This approximation will be valid as long as the spatial variation in the term \(\hat{\mathbf{A}} = \ln(A_i/A_0)\) is small (so that \(\nabla \hat{\mathbf{A}}\) is small). Even for moderately high frequencies, the cross terms will be smaller due to division by \(k_0 \) terms. Furthermore, for large homogeneous scatterers, the gradient of \(\hat{\mathbf{A}}\) will be minimal inside and outside the objects. On the boundaries there will be a discontinuity, and hence our approximations will generally be accurate everywhere except at the boundaries of the objects where we can expect some errors. Based on these approximations, we can ignore the cross terms and rewrite (31) as

\[
\chi_{RI}(r) = 2 \left( \sqrt{\epsilon_R \cos \theta_i - 1} \right) + j \frac{\epsilon_I}{\sqrt{\epsilon_R - \sin^2 \theta_i}} \cos \theta_i.
\]

(32)

Using Fig. 2, (16), and (20), we can write the scattering angle as

\[
\cos \theta_i = \cos (\theta_i - \theta_t)
\]

\[
= \cos \theta_t \cos \theta_i + \sin \theta_t \sin \theta_i
\]

\[
= \cos \theta_t \sqrt{\frac{v_R - \sin^2 \theta_i}{v_R}} + \frac{\sin^2 \theta_i}{v_R}.
\]

(33)

Substituting (33) back in (32) gives the final expression for \(\chi_{RI}\) as

\[
\chi_{RI} = 2 \cos \theta_t \left( \sqrt{\epsilon_R - \sin^2 \theta_i} - \cos \theta_i \right) + j \frac{\epsilon_I \cos \theta_t}{\sqrt{\epsilon_R - \sin^2 \theta_i}}.
\]

(34)
This is our proposed contrast function which is also valid under strong scattering as the term \(\nabla \phi_s \cdot \nabla \phi_s\) is not neglected unlike in RA.

In the derivation of \(\chi_{RI}\), we do not impose any restriction on its permittivity value. We only impose a low-loss condition \((\epsilon_R \gg \epsilon_l)\) so that \(|\epsilon_l|\) can be arbitrarily large. As a result, unlike RA, the contrast function \(\chi_{RI}\) in (34) becomes a nonlinear function of permittivity and this provides a fundamentally new extension to RA which is valid for even strongly scattering objects that have a small loss tangent.

We can also look at (34) from the perspective of Fermat’s principle to gain more insight. Under strong scattering \((\epsilon_R \gg 1)\), and for the special case of normal incidence \((\theta_i = 0)\), our result (34) reduces to refractive index \(\chi_{RI} \approx 2(\nu_R - 1) + 2j\nu_I = 2(\nu - 1)\). This agrees with Fermat’s principle where the incremental phase change in a ray is directly related to the product of the path length along the ray and refractive index contrast \((\nu(r) - 1)\). In other words, the incremental phase change in a ray per wavelength should be proportional to \(k_0(\nu^2 - 1)\) [45]. For conventional RA, it is known (using asymptotic techniques) that the incremental phase change per wavelength is \((1/2)k_0(\nu^2 - 1)\) which does not match the expected phase change as per Fermat’s principal (34). Therefore, xRA-LM also appears to better satisfy the underlying physics of the problem.

C. Simplification

To further simplify the contrast function, \(\chi_{RI}\) in (34), we remove its dependence on \(\theta_i\) without significantly compromising accuracy. Estimation of \(\theta_i\) is plausible in direct problems as the information about the scatterer’s shape is known. But it will require intricate geometric calculations which defeats one of the purposes of using xRA, i.e., a computationally straightforward, noniterative linear alternative to VSI.

In a typical scattering setup, the incident rays enter the scattering object from a wide range of incidence directions in the range \(\theta_i \in [−\pi/2, \pi/2]\). Therefore, we can remove the dependence on \(\theta_i\) by averaging \(\chi_{RI}\) uniformly over this range of \(\theta_i\) to obtain

\[
\tilde{\chi}_{RI} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \chi_{RI} \, d\theta_i
\]

\[
= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left( 2 \cos \theta_i \left( \sqrt{\nu_R - \sin^2 \theta_i} - \cos \theta_i \right) + j \frac{\epsilon_l \cos \theta_i}{\sqrt{\nu_R - \sin^2 \theta_i}} \right) \, d\theta_i. \tag{35}
\]

This integral can be solved analytically (using substitution \(u = \sin \theta_i\)) to obtain

\[
\tilde{\chi}_{RI} = \frac{2}{\pi} \left( \sqrt{\nu_R - 1} + \sin^{-1} \left( \frac{1}{\sqrt{\nu_R}} \right) - \frac{\pi}{2} \right) + j \frac{2}{\pi} \epsilon_l \sin^{-1} \left( \frac{1}{\sqrt{\nu_R}} \right). \tag{36}
\]

It should be noted that averaging over the range of incident angles will lead to errors and can be considered as localizing the effect of the incident angles [1], [26], [27]. However, we show in the simulation that this error is acceptable (< 15%) even under strong scattering conditions. Also, even though \(\tilde{\chi}_{RI}\) is derived using a ray approximation inside the lossy media, it is correct for background vacuum/air where \(\tilde{\chi}_{RI} = \Im + j \Re\) since \(\nu_R = 1\) and \(\epsilon_l = 0\). Therefore, \(\tilde{\chi}_{RI}\) can be used as a physical parameter to characterize the permittivity distribution inside the DoI.

Substituting \(\tilde{\chi}_{RI}\) as \(\chi_{RI}\) in (7) gives the proposed xRA-LM approximation which can be written as

\[
E(r) = E_i(r) \exp \left( \int_D \mathcal{H}(r, r') \tilde{\chi}_{RI}(r') \, dr'^2 \right) \tag{37}
\]

where \(\mathcal{H}(r, r')\) is denoted here as the sensitivity kernel

\[
\mathcal{H}(r, r') = \frac{k_0^2}{E_i(r)} g(r, r') E_i(r'), \quad r' \in D, r \in B. \tag{38}
\]

The total field in (37) can also be decomposed in terms of attenuation and phase change components as

\[
E(r) = E_i(r) \cdot \exp \left( \int_D [\mathcal{H}^R\tilde{\chi}_{RI} - \mathcal{H}^I\tilde{\chi}_{RI}] \, dr'^2 \right) \cdot \exp \left( j \int_D [\mathcal{H}^R\tilde{\chi}_{RI} + \mathcal{H}^I\tilde{\chi}_{RI}] \, dr'^2 \right) \tag{39}
\]

where \(\mathcal{H}^R(r, r')\) and \(\mathcal{H}^I(r, r')\) are the real and imaginary parts of the sensitivity kernel \(\mathcal{H}(r, r')\), respectively, whereas \(\tilde{\chi}_{RI}(r')\) and \(\tilde{\chi}_{RI}(r')\) are the real and imaginary part of contrast \(\tilde{\chi}_{RI}(r')\), respectively.

D. Modifications for Extremely Strong Scattering

As we show later, our xRA-LM formulation in (39) is accurate even under strong scattering (large scatterers with \(2 \leq \nu_R < 5\)) and provides error less than 15% in predicting wave scattering. Hence, xRA-LM already surpasses the existing noniterative linear scattering models. However, for extremely strong scattering (large scatterers with \(5 \leq \nu_R \leq 50\)), the errors increase (20%–30%), but it is still significantly better than any other existing noniterative linear methods [1], [19]. In this section, we suggest a minor modification to (39) to increase its accuracy for extremely strong scattering conditions.

In (32), the real part of the contrast \(\tilde{\chi}_{RI}^R\) depends on the scattering angle \(\theta_i\) (which further depends on the incident angle \(\theta_i\) and the scatterer’s permittivity distribution), and we approximate it in (33) as its exact value is difficult to estimate. On the other hand, the imaginary part of contrast \(\tilde{\chi}_{RI}^I\) only depends on \(\theta_i\), and therefore the real component will likely be significantly less accurate than the imaginary part of the contrast function. This difference in accuracy will be severe under extremely strong scattering conditions where our approximation to \(\theta_i\) can be less accurate and the error in the real part of contrast \(\tilde{\chi}_{RI}^R\) corresponding large.

By considering the line of sight (LOS) path or region between the transmitter and the receiver, we can attempt to reduce the possible error in the real part of the contrast function \(\tilde{\chi}_{RI}^R\) by realizing the LOS region will be dominated by attenuation rather than more intricate scattering effects [36]. We can therefore take a straightforward step and neglect \(\tilde{\chi}_{RI}^R\).
within the LOS region and only rely on $\chi_{RI}^L$ for estimating the total field at the receiver. Intuitively, this means that for points within the LOS region, we rely on the absorption caused by the scatterer rather than estimating the effect of higher order scattering (inside scatterer). Even with small loss tangent $\epsilon_R \gg \epsilon_I$, since $\epsilon_R$ is large, the value of $\epsilon_I$ can be large enough to cause substantial absorption of electromagnetic energy (especially for scatterers large in size). This is also supported by numerical studies [36] which show that for scatterers with substantial loss, higher order scattered rays inside the scatterer are weak. Hence, the total field within LOS regions will be dominated by the attenuating incident field. For points outside the LOS region, the incident field will not dominate and we cannot rely on absorption alone and hence cannot ignore $\chi_{RI}^L$.

To define the LOS region in the scatterer, we use Fresnel zones. We use $E$ to denote the set of points lying inside the first Fresnel zone for a source and receiver pair. Using our approach, we substitute $\chi_{RI}^R = 0$ in (39) for points that lie inside the first Fresnel zone $E$ so that (39) becomes

$$E(r) = E_i(r) \exp\left(\int_D [\mathcal{H}^R (\beta \chi_{RI}^R) - \mathcal{H}^I (\chi_{RI}^I)] d\mathbf{r}'^2\right)$$

$$\times \exp\left(\int_D [\mathcal{H}^R (\beta \chi_{RI}^R) + \mathcal{H}^I (\chi_{RI}^I)] d\mathbf{r}'^2\right)$$

where

$$\beta(r, r') = \begin{cases} 0, & \text{if } r' \in E \text{ and } \epsilon_R \gg 1 \\ 1, & \text{otherwise} \end{cases}$$

Using (41), the real part of the contrast is set to zero for points of the scatterer in the first Fresnel zone when calculating the field. For all other points, (40) reduces exactly to (39).

While the correction (40) is somewhat heuristic in justification, we show in the next section that it performs extremely well for extremely strong scattering conditions (for large scatterers with $\epsilon_r > 5$), when compared with the results without correction (39). It is also important to note that even without LOS corrections in (40), our derived xRA-LM method performs accurately for $\epsilon_r \leq 5$ (as shown later) and for all scattering angles, which already significantly surpasses any other noniterative linear methods. Future work can focus further on improved approximations to $\theta_s$ to enhance the technique further.

The computational efficiency of using xRA-LM for direct problems is high because it does not require matrix inversion at any step. The key computation involved in solving xRA-LM as a direct problem is matrix-vector multiplication when the equation for the proposed model in (40) is applied to the discretized DoI and for multiple receivers. Let there be $N$ discretized grids in the DoI and in total $M$ receivers at the measurement boundary. Hence, the xPR-LM model matrix of size $M \times N$ needs to be multiplied with the contrast vector of size $N \times 1$ to generate the total field vector of size $M \times 1$. The complexity of this matrix-vector multiplication is $O(MN)$. RA and BA also have the same complexity. This is much less than for solving Lipmann–Schwinger equation (3) using MoM [1], [5]. Estimating the total field using (3) involves two key steps: 1) matrix inversion to find total field inside the

...
standard test object in wave scattering evaluation [5], [25], [46]–[48]. The circular cylinder is illuminated by a source (Tx) of monochromatic, time-harmonic, and vertically polarized electromagnetic radiation at Tx which is (−3, 0) m. An array of N receivers is placed along the semicircular boundary B with radius \( l = 24 \times \lambda_0 = 3 \) m around the scanner. The location of the \( n \)th receiver is \( (3 \cos \theta_n, 3 \sin \theta_n) \) m.

The second scatterer profile has two cylinders as shown in Fig. 4 and is exactly the same as Fig. 3 in terms of location of source, receivers, and size of DoI. Scatterer \( S_1 \) is a circular cylinder centered at \((0, 0.44)\) m, and \( S_2 \) is a square cylinder centered at \((0, -0.44)\) m (both with infinite height along the z-axis). The diameter of the circular cylinder and the side of the square cylinder are both \( 5 \times \lambda_0 = 0.625 \) m.

The third scatterer profile in Fig. 5 is the well-known Austria profile which is often used as a benchmark profile in inverse scattering literature [1], [19], [49]. The details of the Austria profile are provided in the caption of Fig. 5. (Note that the size and location of the two disks in the Austria profile are expressed in terms of the incident wavelength.)

For generating results, we vary the real part of the relative permittivity of the scatterers between \( 1 \leq \epsilon_R \leq 50 \) and \( \delta = \epsilon_I/\epsilon_R \in [10^{-4}, 10^{-1}] \). [The complex valued relative permittivity can be written as \( \epsilon_r = \epsilon_0(1 + \delta j) \)] We select this range of complex-valued permittivity for numerical tests based on real-world objects [14], [41]–[43]. The values of \( \epsilon_R \) for objects in the environment around us vary from \( 2 < \epsilon_R \leq 50 \) where \( \epsilon_R \geq 20 \) is for water and human body at 2.4 GHz, at room temperature [14], [41]–[43]. Therefore, we used this range of \( \epsilon_R \) in our numerical tests. The loss tangent (\( \delta = \epsilon_I/\epsilon_R \in [10^{-4}, 10^{-1}] \)) considered also represents realistic values for objects around us at 2.4 GHz [14], [41]–[43].

For numerical simulation, we divide the DoI (in both Figs. 3 and 4) into \( M \) small grids, each of size \( \lambda/10 \) where \( \lambda = \lambda_0/|\epsilon_0| \). The grid size of \( \lambda/10 \) is an accepted convention to ensure sufficient accuracy [1].

B. Numerical Results and Analysis

We first focus on the first profile (shown in Fig. 3) and analyze the effect of changing permittivity and loss tangent of the scatterer on the performance of xRA-LM. Later, we also investigate the effect of frequency on the performance.

Fig. 6 provides the results for the total received field using xRA-LM, RA, and BA as well as the exact MoM result. The plots of total field in Fig. 6(a)–(h) are, respectively, for the real part of permittivity \( \epsilon_R = 1.1, 1.5, 2, 3, 4, 5, 10, 50 \) with loss tangent \( \delta = 0.1 \) or equivalently, relative permittivity \( \epsilon_r = \epsilon_R(1 + 0.1j) \). The RE between the estimated and exact field is shown in the legends of each plot.

Fig. 6(a) shows that for extremely weak scattering (\( \epsilon_R = 1.1 \)), both RA and xRA-LM provide comparable performance and are close to the exact field (RE \( < 5% \)). BA has large error (RE \( = 31\% \)) which is expected because BA has validity for scatterers smaller in size than \( \lambda_0 \), whereas the scatterer in this numerical test has diameter \( = 10\lambda_0 \).

Fig. 6(b) shows that as the permittivity is increased slightly to \( \epsilon_R = 1.5 \), there is a sudden increase in error (RE \( = 11\% \)) by RA. Whereas xRA-LM still provides accurate estimation with low error (RE \( = 3\% \)), BA again is worse as expected. Fig. 6(c) shows that as the permittivity increases to \( \epsilon_R = 2 \), the estimation error of RA increases rapidly (RE \( = 25\% \)), whereas xRA-LM still gives low error (RE \( = 5\% \)).

Fig. 6(d)–(f) shows the results for permittivity values of \( \epsilon_R = 3, 4, \) and 5, respectively. It can be seen that even for these large values of permittivity, our proposed xRA-LM method is able to predict the total field with low error (RE \( \leq 10\% \)) and outperforms RA and BA by a significantly large margin. Note that these values of relative permittivity are considered very large in the direct/inverse scattering community [1], [5], [19], and to the best of our knowledge, no other noniterative approximate linear model has been shown to work for this range of permittivity.

In Fig. 6(g) and (h), we further increase the permittivity to extremely large values of \( \epsilon_R = 10 \) and 50, respectively. Even for these extremely strong scattering conditions, xRA-LM provides significantly better performance than RA and BA and acceptable performance in terms of predicting the exact field. The RE is less than 20% even for extremely large permittivity at \( \epsilon_R = 10 \) which no other noniterative linear approximation (or even more intricate nonlinear models [1], [19]) has shown to provide.

Next, we investigate the effect of varying loss tangent on the accuracy of xRA-LM. Fig. 7 provides performance versus loss tangent in the range \( \delta \in [10^{-4}, 10^{-1}] \) for a fixed value of \( \epsilon_R \). We can see that there is increased error if we make the loss tangent extremely small. This happens because as the scatterer causes negligible absorption, higher order reflections inside the boundaries of the scatterer become stronger and create stronger multiple scattering inside the scatterer, and these higher order scattered rays are neglected in our derivation. Fortunately, there is a sufficient range of loss tangent 0.001 \( < \delta < 0.1 \) for which xRA-LM provides acceptable error (RE \( < 15\% \)) even for extremely strong scattering. This range of loss tangent covers the lossy behavior of most of the objects typically found around us (at 2.4 GHz or other microwave frequencies around it) [14], [41]–[43].

To provide results for xRA-LM using (40) when compared with (39), Fig. 8(a)–(c) provides total field estimation using: 1) xRA-LM without LOS correction (39) and 2) xRA-LM with LOS correction (40). Fig. 8(a) shows that for moderate...
Fig. 6. Results for the profile shown in Fig. 3. We vary the relative permittivity of the scatterer and plot the total field magnitude as a function of scattering angle. The plots (a)–(h) are, respectively, for real part of permittivity $\varepsilon_r = 1.1$, $\varepsilon_r = 1.5$, $\varepsilon_r = 2.0$, $\varepsilon_r = 3.0$, $\varepsilon_r = 4.0$, $\varepsilon_r = 5.0$, $\varepsilon_r = 10$, $\varepsilon_r = 50$ with loss tangent $\delta = 0.1$ so that relative permittivity $\varepsilon_r = \varepsilon_R(1.0 + 0.1j)$. (a) $\varepsilon_r = \varepsilon_R(1.0 + 0.11j)$. (b) $\varepsilon_r = \varepsilon_R(1.5 + 0.15j)$. (c) $\varepsilon_r = \varepsilon_R(2.0 + 0.2j)$. (d) $\varepsilon_r = \varepsilon_R(3.0 + 0.3j)$. (e) $\varepsilon_r = \varepsilon_R(4.0 + 0.4j)$. (f) $\varepsilon_r = \varepsilon_R(5.0 + 0.5j)$. (g) $\varepsilon_r = \varepsilon_R(10 + 1j)$. (h) $\varepsilon_r = \varepsilon_R(50 + 5j)$.

Fig. 7. Effect of variation in relative permittivity on estimation error of xRA-LM for different values of loss tangent $\delta$.

To summarize results from Figs. 6–8, the validity range of xRA-LM (40) is in the range $1 < \varepsilon_R < 50$ for all the scattering angles.

C. Effect of Frequency on Accuracy

In this section, we investigate the effect of frequency on the accuracy of xRA-LM. (Figs. 9–11 shows the results for profile shown in Figs. 3–5, respectively.) These results provide an analysis of the effect of scatterer’s electrical size on the performance of the proposed method.

permittivity value of $\varepsilon_r = 2 + 0.2j$, xRA-LM provides acceptable error even without LOS corrections. If we further increase the permittivity to $\varepsilon_r = 5 + 0.5j$ in Fig. 8(b), accuracy remains good without LOS correction. Finally, in Fig. 8(c), when permittivity is $\varepsilon_r = 10 + 1j$ to create extremely strong scattering, the LOS correction provides significant improvement in accuracy in the forward scattering angles. Whereas for other scattering angles there is no effect of LOS correction and the results are accurate even without it.

To summarize results from Figs. 6–8, the validity range of xRA-LM (40) is in the range $1 < \varepsilon_R < 50$ for all the scattering angles.
λ varies from frequency from 500 MHz to 8 GHz (corresponding wavelength range of frequencies rather than only at 2.4 GHz. We vary θ total field at 6 scattering angles.

itaration in our derivation of corrected contrast) are small and cross terms (which are ignored under high-frequency assumption in our derivation of corrected contrast) are small. However, now we generate results for a range of frequencies rather than only at 2.4 GHz. We vary frequency from 500 MHz to 8 GHz (corresponding wavelength varies from λ₁ = 0.6 m to 0.0375 m) and show the estimated total field at 6 scattering angles θₖ = 0°, 45°, 60°, 90°, 135°, and 150°. These scattering angles are selected such that we can see the accuracy of various methods for forward and backscattering regions. The results are shown in Fig. 9.

Note that for the largest wavelength of λ₁ max = 0.6 m, the scatterer diameter (= 1.25 m) is only twice the incident wavelength, and hence high-frequency assumption is not applicable. Whereas at the shortest wavelength of λ₁ min = 0.0375 m, the scatterer (1.25 m) is around 30 times larger, and hence high-frequency assumption is strongly justified. We expect xRA-LM to perform even when high-frequency assumption is not applicable because as we explained in Section III-B, the cross terms (which are ignored under high-frequency assumption in our derivation of corrected contrast) are small due to division by wavenumber k₀ and k₀² terms. Even for the lowest frequency (λ max = 0.6), k₀ = 10 and k₀² = 100 accuracy remains good and crosstalk terms do not appear to add significantly to error. Also, V₂A terms in the crosstalk terms are nonzero only at the boundaries which further minimizes the crosstalk terms.

It can be seen in Fig. 9 that for all the frequencies and scattering angles considered, xRA-LM provides the best estimation of the exact field. Even for low frequencies of 500 MHz and 1 GHz, xRA-LM provides low RE (<13%) and significantly outperforms RA and BA. This shows that even when the scatterer size is smaller than or comparable to the incident wavelength (high-frequency assumption is not strongly imposed), xRA-LM provides good prediction of the exact field. This is an important result since it shows that even though xRA-LM is derived by introducing corrections to RA using a high-frequency approximation, xRA-LM can provide good results even when high-frequency assumption is not strongly present.

Next, we provide these results for the multiple scatterer profile (in Fig. 4) and also for the Austria profile (in Fig. 5). Both these profiles have scatterers of different shapes and different permittivity values and hence provide a good test for strong multiple scattering conditions. The results for the scatterer profile in Fig. 4 are shown in Fig. 10, whereas the results for the Austria profile (in Fig. 5) are shown in Fig. 11. These results show that xRA-LM provides highly accurate results and outperform RA and BA by a significant margin, even for these scatterers, for a wide range of frequencies. More broadly, we have found that for all the configurations that were considered for the single cylinder, the same conclusions for accuracy can be drawn for the two-cylinder and Austria profile configurations.

D. Results for Inhomogeneous Scatterer

As explained in Section III-A, the proposed method is derived under the assumption that scatterers are homogeneous or piecewise homogeneous, where the spatial variation in relative permittivity is larger than the incident wavelength inside the scatterer. All the results shown previously are for homogeneous and piecewise homogeneous scatterers. In this section, we also test how the proposed method performs if the scatterer is strongly inhomogeneous. Fig. 12 shows an inhomogeneous scatterer centered at origin with spatial permittivity distribution generated using a Gaussian function inside a 1.25 × 1.25 m² square region. The maximum relative permittivity (ε₂ = 20 + 2j) is at the origin, and the spatial standard deviation is 0.2 m. In this profile, the permittivity values change with each grid (each grid has a size which is 1/20th of the incident wavelength). The total field estimation results are shown in Fig. 13 for this Gaussian scatterer profile. It can be seen that the proposed method can predict the total field magnitude for the profile. The results are less accurate compared with homogeneous and piecewise homogeneous profiles, but still significantly better than other linear methods (RA and BA). Therefore, the proposed method provides a good estimate of total field for all three types of scatterers; homogeneous, piecewise homogeneous, and inhomogeneous profiles.

E. Phase Estimation Accuracy

The results shown so far in this work focus on the magnitude estimation of the total field. In this section, we provide an example to show the performance of the proposed method in estimating the phase of the total field under strong scattering.
Fig. 9. Effect of variation in frequency on RE for estimating total field using xRA-LM, RA, and BA. The simulation setup is the same as in Fig. 3. The scatterer size used for estimation is $\epsilon_r = 5 + j0.5$ to simulate strong scattering conditions. The scatterer size is 1.25 m. The RE versus frequency plots in (a)–(f) are for scattering angles $\theta_n = 0^\circ, 45^\circ, 60^\circ, 90^\circ, 135^\circ,$ and $150^\circ$.

Fig. 10. Effect of variation in frequency on the error for multiple scatterer profile in Fig. 4, where circular and square cylinders have permittivity $\epsilon_r = 4 + 0.4j$ and $\epsilon_r = 10 + 1j$, respectively. The RE versus frequency plots in (a)–(f) are for scattering angles $\theta_n = 0^\circ, 45^\circ, 60^\circ, 90^\circ, 135^\circ,$ and $150^\circ$. 
Fig. 11. Effect of variation in frequency on RE for the Austria profile shown in Fig. 5, where \( \varepsilon_r = 10 + 1j \). The RE versus frequency plots in (a)–(f) are for scattering angles \( \theta_n = 0^\circ, 45^\circ, 60^\circ, 90^\circ, 135^\circ, \) and \( 150^\circ \).

Fig. 12. Illustration of the wave scattering setup for an inhomogeneous scatterer with the measurement setup as in Fig. 3. The scatterer is centered at the origin with spatial permittivity distribution generated using a Gaussian function inside a \( 1.25 \times 1.25 \) m\(^2\) square region. The maximum relative permittivity (\( \varepsilon_r = 20 + 2j \)) is at the origin and the spatial standard deviation is 0.2 m. Note that the figure is scaled for better visualization and does not represent the actual distances between Tx, Rx, and the scatterer.

Fig. 13. Total field estimation for inhomogeneous scatterer profile shown in Fig. 12 (at 2.4 GHz).

Fig. 14. Magnitude and phase estimation when Austria profile with \( \varepsilon_r = 10 + 1j \) is used as the test scatterer. The incident field frequency is 5 GHz. (a) Magnitude Estimation. (b) Phase Estimation.

Austria profile (shown in Fig. 5) with relative permittivity \( \varepsilon_r = 10 + 1j \), which makes it an extremely strong scattering test object. It can be seen in Fig. 14(b) that the proposed method provides a very good estimate of phase and outperforms RA and BA by a significant margin. In fact, the estimated phase is more accurate than the estimated magnitude, which is interesting and can be explored in more detail in future work.

conditions. Fig. 14 provides the magnitude and the phase estimation results using the proposed method, RA, and BA. The test scatterer profile used to generate these results is the
V. Conclusion

In this article, we have presented fundamental corrections to conventional RA using a high-frequency approximation for lossy media. It combines the physical interpretation of inhomogeneous wave propagation as rays along with the diffractive modeling provided by RA. The simulation results demonstrate that for low-loss, piecewise homogeneous scatterers, the proposed xRA-LM method provides good accuracy with errors of 20% or less even under extremely strong scattering conditions \( \epsilon_R = 50 \) and outperforms RA and BA by a significant margin. To the best of our knowledge, xRA-LM is the first noniterative linear approximation for wave scattering that performs well even for extremely large values of permittivity. The technique can open up new paradigms in noniterative linear approximations that provide feasible solutions to both direct and inverse scattering problems in strong scattering environments with low computational load. It could be particularly important in inverse scattering contexts where the formulations are inherently nonlinear and ill-posed and the measurement data suffer from inaccuracies due to noise and real-world data acquisition.

We focused on TM polarization to demonstrate the feasibility and potential performance of the technique. Future work can focus on considering the accuracy of the proposed method under transverse electric (TE) polarization. It is also important to understand why there is a higher estimation error for the intermediate scattering angles \((40^\circ \leq \theta_s \leq 60^\circ)\) as can be seen in Figs. 6 and 8. This can also be considered as part of future work.

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