Fundamental groups and group presentations with bounded relator lengths.
Sergio Zamora
zamora@mpim-bonn.mpg.de

Abstract

We study the geometry of compact geodesic spaces with trivial first Betti number admitting large finite groups of isometries. We show that if a finite group $G$ acts by isometries on a compact geodesic space $X$ whose first Betti number vanishes, then $\frac{\text{diam}(X)}{\text{diam}(X/G)} \leq 4\sqrt{|G|}$.

For a group $G$ and a finite symmetric generating set $S$, $P_k(\Gamma(G, S))$ denotes the 2-dimensional CW-complex whose 1-skeleton is the Cayley graph $\Gamma$ of $G$ with respect to $S$ and whose 2-cells are $m$-gons for $0 \leq m \leq k$, defined by the simple graph loops of length $m$ in $\Gamma$, up to cyclic permutations.

Let $G$ be a finite abelian group with $|G| \geq 3$ and $S$ a symmetric set of generators for which $P_k(\Gamma(G, S))$ has trivial first Betti number. We show that the first nontrivial eigenvalue $-\lambda_1$ of the Laplacian on the Cayley graph satisfies $\lambda_1 \geq 2 - 2 \cos(2\pi/k)$.

We also give an explicit upper bound on the diameter of the Cayley graph of $G$ with respect to $S$ of the form $O(k^2|S|\log|G|)$. Related explicit bounds for the Cheeger constant and Kazhdan constant of the pair $(G, S)$ are also obtained.

1 Introduction

1.1 Diameter

Compact geodesic spaces equipped with large discrete groups of isometries have been objects of great interest for a long time and several problems can be formulated in this setting [5, 9, 21, 22, 37]. One natural source of such spaces are finite-sheeted Galois covers of compact Riemannian manifolds. In 2009, Petrunin asked if one can control in an interesting way the diameter of a compact universal cover $\tilde{M}$.

Problem 1 (Petrunin). Let $M$ be a compact Riemannian manifold and assume it admits a compact universal cover $\tilde{M}$. What is the smallest upper bound of $\frac{\text{diam}(\tilde{M})}{\text{diam}(M)}$ in terms of $|\pi_1(M)|$?

It is not hard to show that $\frac{\text{diam}(\tilde{M})}{\text{diam}(M)} \leq |\pi_1(M)|$ [31], but getting a better bound is non-trivial matter. The goal of this paper is to study this question and the global shape of compact universal covers in general. One of our main results is the following.

Theorem 2. Let $X$ be a compact geodesic space and $G \leq \text{Iso}(X)$ a finite group of isometries. If the first Betti number $b_1(X)$ vanishes, then

$$\frac{\text{diam}(X)}{\text{diam}(X/G)} \leq 4\sqrt{|G|}.$$
Asymptotically as $|G| \to \infty$, there is a stronger yet non-effective bound [5].

**Theorem 3** (Benjamini–Finucane–Tessera). Let $X_n$ be a sequence of compact geodesic spaces and $G_n \leq \text{Iso}(X_n)$ a sequence of finite groups with $|G_n| \to \infty$ as $n \to \infty$. If the first Betti numbers $b_1(X_n)$ vanish, then for each $\varepsilon > 0$ one has

$$\frac{\text{diam}(X_n)}{\text{diam}(X_n/G_n)} = O(|G_n|^\varepsilon).$$

Problem 1 is better handled when reformulated in terms of Cayley graphs. For a group $G$ and a finite symmetric generating set $S$ we denote by $\Gamma(G,S)$ the Cayley graph of $G$ with respect to $S$.

For a graph $\Gamma$ and an integer $k \in \mathbb{N}$, as in [18] we denote by $P_k(\Gamma)$ the 2-dimensional CW-complex whose 1-skeleton is $\Gamma$ and whose 2-cells are $m$-gons for $0 \leq m \leq k$, defined by the simple graph loops of length $m$ in $\Gamma$, up to cyclic permutations.

**Proposition 4** (Švarc–Milnor Lemma). Let $X$ be a proper geodesic space, $p \in X$, $G \leq \text{Iso}(X)$ a discrete group, $\delta \geq 0$, and $r \geq 2 \cdot \text{diam}(X/G) + \delta$. Then $S := \{g \in G \mid d(gp,p) \leq r\}$ generates $G$. Moreover, if we equip $G$ with the metric induced from $\Gamma := \Gamma(G,S)$, for all $g, h \in G$ one has

$$\delta \cdot [d_{\Gamma}(g, h) - 1] \leq d_X(gp, hp) \leq r \cdot d_{\Gamma}(g, h).$$

**Proposition 5.** Let $X$ and $\Gamma$ be as in Proposition 4. Then $\pi_1(P_3(\Gamma))$ is a quotient of $\pi_1(X)$.

A proof of the Švarc–Milnor Lemma can be found in [17], and Proposition 5 will be proven in Section 3.3. Using these well known results, Theorem 2 becomes a corollary of its Cayley graph counterpart.

**Theorem 6.** Let $k \geq 3$, $G$ be a finite group, and $S \subset G$ a finite symmetric set of generators for which $P_k(\Gamma(G,S))$ has trivial first Betti number. Then

$$\text{diam}(\Gamma(G,S)) \leq \left(\sqrt{4|G| + 1} - 2\right) \left\lceil \frac{k+2}{3} \right\rceil.$$  \hspace{1cm} (1)

**Remark 7.** It is well known that for $k \geq 3$, a group $G$ and a finite symmetric set of generators $S$, the complex $P_k(\Gamma(G,S))$ is simply connected if and only if $G$ admits a presentation $\langle S \mid R \rangle$ with $R$ consisting of words of length $\leq k$ [18, Section 2][1]. Moreover, if one considers the abstract group $\tilde{G} = \langle S \mid R_k \rangle$, where $R_k$ consists of the words of length $\leq k$ representing the identity in $G$, then $P_k(\Gamma(G,S))$ is the universal cover of $P_k(\Gamma(G,S))$ and the fundamental group of $P_k(\Gamma(G,S))$ is precisely the kernel of the natural map $\tilde{G} \to G$.

By Remark 7, Theorem 6 has the following implication.

**Corollary 8.** Let $k \geq 3$, $G$ be a finite group, and $S \subset G$ a finite symmetric set of generators for which $G$ admits a presentation $\langle S \mid R \rangle$ with $R$ consisting of words of length $\leq k$. Then (1) holds.

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[1] see also the primer by Yann Ollivier [http://www.yann-ollivier.org/maths/primer.php](http://www.yann-ollivier.org/maths/primer.php)
1.2 Kazhdan constant, Cheeger constant, and spectral gap

The Švarc–Milnor Lemma implies that the medium-scale geometric features of $X$ and $\Gamma$ are closely related to each other. We now focus on such properties. Recall that for a finite group $G$ and a finite symmetric set of generators $S$, the Kazhdan constant $K(G, S)$, Cheeger constant $h$, and spectral gap $\lambda_1$ are related by the following inequalities

$$\frac{h^2}{|S|^2} \leq \frac{2\lambda_1}{|S|} \leq K(G, S)^2 \leq 2\lambda_1 \leq 4h.$$  \hfill (2)

We refer the reader to Section 3.5 for the definition of such quantities and further comments on (2). For now we just mention that the three non-negative quantities $K(G, S)$, $h$, and $\lambda_1$ measure the connectivity of $\Gamma(G, S)$ in different ways. The other main result of this paper concerns finite abelian groups.

**Theorem 9.** Let $k \geq 3$, $G$ a finite abelian group with $|G| \geq 3$, and $S \subset G$ a symmetric set of generators for which $P_k(\Gamma(G, S))$ has trivial first Betti number. Then the Kazhdan constant satisfies

$$K(G, S) \geq 2 \cdot \sin(\pi/k).$$ \hfill (3)

Consequently, the Cheeger constant, spectral gap, and diameter satisfy

$$2h \geq \lambda_1 \geq 2 - 2\cos(2\pi/k),$$ \hfill (4)

$$\text{diam}(\Gamma(G, S)) \leq \frac{|S| + 1 - \cos(2\pi/k)}{2(1 - \cos(2\pi/k))} \log |G| + 1.$$ \hfill (5)

A consequence of Theorem 9 is an upper bound on the mixing time of the random walk in the corresponding Cayley graph (see Remark 33). We refer the reader to Section 3.6 for the definitions of random walk and mixing time. For now we just mention that $\tau_\Gamma(c)$ is an estimate of how long does one have to wait for heat to propagate evenly (how evenly? quantified by $c$) along the network $\Gamma$.

**Corollary 10.** Let $k \geq 4$, $G$, and $S$ be as in Theorem 9. If $\tau_\Gamma : [0, 2] \rightarrow \mathbb{N}$ denotes the mixing time of the Cayley graph $\Gamma(G, S)$, then

$$\tau_\Gamma(c) \leq \frac{k^2|S|}{32} \left[ \log |G| - 2\log(c) \right] + 1.$$  

**Corollary 11.** Let $M$ be a closed $n$-dimensional Riemannian manifold with $\text{diam}(M) = D$, Ricci curvature $\geq \kappa(n-1)$ for some $\kappa \in \mathbb{R}$, and having a point whose injectivity radius is $\geq 2r_0 > 0$. If its fundamental group $\pi_1(M)$ is finite and abelian, then the universal cover $\tilde{M}$ satisfies

$$\frac{\text{diam}(\tilde{M})}{\text{diam}(M)} \leq 4 + \left[ \frac{2v_n^\kappa(2D + r_0)}{3v_n^\kappa(r_0)} + \frac{1}{3} \right] \log |\pi_1(M)|,$$

where $v_n^\kappa(r)$ denotes the volume of a ball of radius $r$ in the $n$-dimensional simply connected space of constant sectional curvature $\kappa$.  

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Considering the situation when $\text{diam}(\Gamma) \to \infty$, there are bounds similar to the ones in Theorem 9 for groups that are not necessarily abelian [9].

**Theorem 12** (Breuillard–Tointon). Let $G_n$ be a sequence of finite groups, $S_n \subset G_n$ a sequence of finite symmetric sets of generators, and $\Gamma_n := \Gamma(G_n, S_n)$ the corresponding Cayley graphs. Assume there is a sequence $k_n = o(\text{diam}(\Gamma_n))$ such that the first Betti numbers $b_1(P_{k_n}(\Gamma_n))$ vanish. Then for each $\varepsilon > 0$, the quantities

$$K(G_n, S_n), \lambda_1(G_n, S_n), h(G_n, S_n),$$

cannot go to zero faster than $(|S_n|/|G_n|)^\varepsilon$ as $n \to \infty$.

### 1.3 Outline

In Section 2 we present some computations and examples, and discuss related open problems and potential lines of research.

In Section 3 we introduce our notation and the standard theory we will need.

In Section 4 we give the proofs of Theorems 2, and 6. Theorem 2 follows from Theorem 6 which in turn depends on an elementary combinatorial argument. We also present a proof of Theorem 3 since it is currently stated in the literature only in the setting of vertex-transitive graphs [5, Theorem 1].

In Section 5 we give the proofs of Theorem 9 and Corollaries 10 and 11. An elementary geometric observation yields estimate (3), from which all other results follow.

### 2 Examples and further problems

#### 2.1 Diameter

Theorem 3 implies that the explicit bound in Theorem 2 is far from being sharp as $|G| \to \infty$. By a fundamental domain argument, even without the first Betti number assumption, one always has

$$\frac{\text{diam}(X)}{\text{diam}(\widehat{X}/G)} \leq 2 \cdot |G|,$$

so Theorem 2 says nothing new for $|G| \leq 4$. However, for a larger number, say, 120, Theorem 2 gives a meaningful bound (again, likely far from sharp). The following example was pointed out by Kuperberg [32].

**Example 13.** Let $\widehat{X} = S^3$ equipped with its usual metric, and consider $X = (\widehat{X} / \sim)$ the Poincaré sphere [36, Example 1.4.4 and Problem 4.4.17]. Then $\widehat{X}$ is the 120-sheeted universal cover of $X$, and

$$\frac{\text{diam}(\widehat{X})}{\text{diam}(X)} = \frac{\pi}{\arccos(\varphi^2/\sqrt{8})} \approx 8.09,$$

where $\varphi$ is the golden ratio. On the other hand, the bound provided by Theorem 2 is $4\sqrt{120} \approx 43.81$. 

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Proof sketch: A Voronoi domain of the quotient \( \tilde{X} \to X \) is a regular dodecahedron \( K \subset S^3 \) with dihedral angles equal to \( \pi/3 \) [30, Section 3.2.4]. Let \( O \in K \) be the center of the dodecahedron, \( F_1, F_2 \) be the centers of two adjacent faces of \( K \), and \( V_1, V_2 \) be the vertices shared by such faces. Also let \( P \) be the midpoint between \( V_1 \) and \( V_2 \). The diameter of \( X \) is attained by \( d_{S^3}(O, V_1) \) (see Figure 1).

![Diagram](image)

Figure 1: We can use the knowledge of the angles in the triangle \( OF_1 P \) to deduce the length of the segment \( OP \). We then proceed to compute the length of the segment \( OV_1 \) using the length of the segment \( OP \) and the known angles of the triangle \( OPV_1 \).

By the symmetry of the dodecahedron,

\[
\angle OF_1 P = \angle OF_2 P = \angle OPV_1 = \angle OPV_2 = \frac{\pi}{2}.
\]

Using elementary geometry one can also compute the angles

\[
\angle F_1 OF_2 = \arccos \frac{1}{3}, \quad \angle V_1 OV_2 = \arccos \frac{\sqrt{5}}{3}.
\]

By the spherical laws of sines and cosines [35], this is enough information to recover the length \( d_{S^3}(O, V_1) = \arccos(\varphi^2/\sqrt{8}) \).

It would also be interesting to investigate how sharp is Theorem 3. The known example in which \( \text{diam}(X_n)/\text{diam}(X_n/G_n) \) grows the fastest with respect to \( |G_n| \) is the following (again pointed out by Kuperberg [32]), which naturally leads to Conjecture 15 below.

**Example 14.** Let \( G_n \) be the symmetric group (the set of bijections of the set \( \{1, \ldots, n\} \)), and \( S_n \) the set of transpositions of consecutive elements of \( \{1, \ldots, n\} \) (we consider \( n \) and 1 not to be consecutive). Setting \( \Gamma_n := \Gamma(G_n, S_n) \), we have:

1. \( P_6(\Gamma_n) \) is simply connected for all \( n \).
2. \( \text{diam}(\Gamma_n) = o \left( (\log |G_n|)^2 \right) \).
3. \((\log |G_n|)^{2-\varepsilon} = o(\text{diam}(\Gamma_n))\) for every \(\varepsilon > 0\).

Proof sketch: 1: The group \(G_n\) can be presented as \(\langle S_n|R_n\rangle\), with \(S_n = \{\sigma_1, \ldots, \sigma_{n-1}\}\), and \(R_n\) consisting of the words \(\sigma_i^2\) for all \(i\), \((\sigma_i\sigma_{i+1})^3\) for all \(i = 1, \ldots, n-2\), and \((\sigma_i\sigma_j)^2\) with \(|i-j| \geq 2\). Since each word in \(R_n\) has length \(\leq 6\), the complex \(P_6(\Gamma_n)\) is simply connected.

2 and 3: Every permutation in \(G_n\) can be written as a composition of at most \(n(n-1)/2\) elements in \(S_n\), where the maximum is achieved by the permutation

\[i \rightarrow (n + 1 - i), \ i \in \{1, \ldots, n\}\]

that “reverses” the order. This means that

\[\text{diam}(\Gamma_n) = n(n-1)/2,\]

while

\[\log |G_n| = \log(n!),\]

which is of the order of \(n \log n\). \(\square\)

**Conjecture 15** (Petrunin). There is \(C > 0\) such that if \(G\) is a finite group and \(S\) a set of generators for which \(P_3(\Gamma(G, S))\) is simply connected, then

\[\text{diam}(\Gamma(G, S)) = O\left( (\log |G|)^C \right).\]

This question draws resemblance to another well known problem [2].

**Conjecture 16** (Babai). There is \(C > 0\) such that if \(G\) is a finite non-abelian simple group and \(S \subset G\) is any set of generators, then

\[\text{diam}(\Gamma(G, S)) = O\left( (\log |G|)^C \right).\]

Note however, that Babai’s Conjecture concerns any set of generators, while Petrunin’s Conjecture is about geometrically chosen sets of generators. It would be interesting to investigate how intertwined these two problems are. For instance, does the hypothesis in Conjecture 15 of \(P_3(\Gamma(G, S))\) being simply connected imply that the graph \(\Gamma(G, S)\) looks like the Cayley graph of a non-abelian simple group? We refer the reader to [20, 24] for recent updates on the state of Conjecture 16.

We would like to also point out that Conjecture 15 is still very interesting when the group \(G\) is abelian, in which case \(C\) could even be 1. For abelian groups, the known example in which \(\text{diam}(\Gamma(G, S))\) grows the fastest with respect to \(|G|\) is the following (pointed out by Petrunin [32]).

**Example 17.** Let \(G_n = \mathbb{Z}/(2^n)\mathbb{Z}\), and

\[S_n = \{\pm 1, \pm 2, \pm 2^2, \ldots, \pm 2^{n-1}\}.

Setting \(\Gamma_n := \Gamma(G_n, S_n)\), we have:
1. $P_3(Γ_n)$ is simply connected for all $n$.
2. $\text{diam}(Γ_n) = O(\log |G_n|)$.
3. $\log |G_n| = O(\text{diam}(Γ_n))$.

**Proof sketch:** 

1: The group $G_n$ can be presented as $\langle S_n | R_n \rangle$, where $R_n$ consists of the expressions $2^j - 2^{j-1} - 2^{j-1} = 0$ for $j \in \{1, \ldots, n-1\}$, and $2^{n-1} + 2^{n-1} = 0$. Since each word in $R_n$ has length $\leq 3$, the complex $P_3(Γ_n)$ is simply connected.

2: Any number in $\{1, 2, 3, \ldots, 2^n - 1\}$ can be written (using binary base) as a sum of at most $n$ summands of the form $2^j$, $j \in \{1, \ldots, n-1\}$. Hence

$$\text{diam}(Γ_n) = O(n) = O(\log |G_n|).$$

3: Given a sequence of length $n$ of 0’s and 1’s, one could count the number of “jumps” from one digit to another. E.g., 00011111 has 1 jump, 00110110 has 3 jumps, 1010110 has 5 jumps, etc. By writing an element $x \in G_n$ in binary base, we obtain a sequence of length $n$ of 0’s and 1’s. One can check that the effect of adding or subtracting a power of 2 to $x$ increases the number of such jumps by at most 2.

Expressing $x = 1 + 2^2 + \ldots + 2^{\lfloor \frac{n}{2} \rfloor}$ in binary we find $n - 1$ jumps. Hence at least $\lfloor \frac{n}{2} \rfloor$ elements of $S_n$ are required to write down $x$. This implies,

$$\log |G_n| = O \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = O(\text{diam}(Γ_n)).$$

We conclude this section by pointing out that examples of spaces $X$ with $b_1(X) = 0$ and finite groups $G \leq \text{Iso}(X)$ with $\text{diam}(X)/\text{diam}(X/G)$ of order $\log |G|$ arise in number theory.

**Example 18** (Calegari–Dunfield, Boston–Ellenberg). There is a closed hyperbolic 3-manifold $M_0$ admitting a tower of regular $m_n$-sheeted covers $M_n \to M_0$ with $b_1(M_n) = 0$ for all $n$, and

$$c \cdot \log(m_n) \leq \frac{\text{diam}(M_n)}{\text{diam}(M_0)} \leq C \cdot \log(m_n) \tag{6}$$

for some $C > c > 0$.

**Proof sketch:** In [15, Theorem 1.6], a sequence of regular finite-sheeted covers $M_n \to M_0$ is constructed, with $M_0$ a quotient of the hyperbolic space $\mathbb{H}^3$ by an arithmetic lattice $Γ \leq \text{PGL}_2(\mathbb{C})$, which possesses the Selberg property (see [27, Section 2]).

Since these covers correspond to congruence subgroups of $Γ$, by the Selberg property their (analytic) spectral gaps satisfy $\lambda_1(M_n) \geq \varepsilon$ for some $\varepsilon > 0$. Then by the work of Brooks [12, Theorem 1], the estimates (6) follow.

In [7] it was then proven that $b_1(M_n) = 0$ for all $n$ (this fact was proven initially in [15] assuming the Generalized Riemann Hypothesis and Langlands-type conjectures). \qed
2.2 Spectral gap

The bounds in Theorem \ref{thm:main} and Corollary \ref{cor:main} are rather general, so we don’t expect them to be fully sharp.

**Example 19.** Let $G_n = (\mathbb{Z}/2\mathbb{Z})^n$, $S_n = \{e_i \in G_n | i \in \{1, \ldots, n\}\}$, where $e_i$ is the $i$-th basis vector, and $\Gamma_n = \Gamma(G_n, S_n)$. Then $P_4(\Gamma_n)$ is simply connected for each $n$ so by Corollary \ref{cor:main}

$$\tau_{\Gamma_n}(c) \leq \frac{n}{2} [n \log 2 - 2 \log(c)] + 1.$$ 

For fixed $c > 0$, this bound grows quadratically in $n$. However, a careful computation \cite{19} shows that

$$\tau_{\Gamma_n}(c) = O(n \log n).$$

**Example 20.** Let $G_n = \mathbb{Z}/n\mathbb{Z}$, $S_n = \{-1, 1\}$, and $\Gamma_n = \Gamma(G_n, S_n)$. For fixed $\varepsilon > 0$, consider the sequence

$$t_n := \left\lfloor n^{2-\varepsilon} \right\rfloor.$$ 

By the Central Limit Theorem \cite[Section 7.3]{1}, the random walk in $\Gamma_n$ after $t_n$ steps will be concentrated in the interval $[-n/\sqrt{t_n}, n/\sqrt{t_n}]$. That is,

$$W^{t_n} \left( \left[ -\frac{n}{\sqrt{t_n}}, \frac{n}{\sqrt{t_n}} \right] \right) \to 1 \text{ as } n \to \infty.$$ 

This implies that for fixed $c < 2$, and large enough $n$,

$$\tau_{\Gamma_n}(c) \geq t_n \geq n^{2-\varepsilon} - 1.$$ 

Notice $P_n(\Gamma_n)$ is simply connected for each $n$, so the bound given by Corollary \ref{cor:main} is the following, not far from being sharp:

$$\tau_{\Gamma_n}(c) \leq \frac{n^2}{16} [\log n - 2 \log(c)] + 1.$$ 

It would be desirable to find explicit bounds similar to the ones of Theorem \ref{thm:main} in the non-abelian setting. However, at the moment the topological condition of $P_k(\Gamma(G, S))$ having trivial first Betti number (or even being simply connected) seems very hard to use when studying isometric actions $G \to \text{Iso}(S^n)$ with $n \geq 2$. In this direction, there is a universal control on the diameters of quotients of spheres by group actions \cite{22}.

**Theorem 21** (Gorodski–Lange–Lytechak–Mendes). There is $\delta > 0$ such that for any $n \geq 2$ and any compact group $G \leq \text{Iso}(S^n)$ not acting transitively, one has

$$\text{diam}(S^n/G) \geq \delta.$$ 

With techniques similar to the ones in the work of Mantuano \cite{29}, it seems possible to recover, using Theorem \ref{thm:main}, effective estimates on spectral gaps and medium-scale isoperimetric inequalities for compact Riemannian manifolds with trivial first Betti number and actions by finite abelian groups with small quotient. Successful results in similar programs have been obtained by Brooks \cite{10, 11}, Buser \cite{14}, Burger \cite{13}, Magee \cite{28}, and several others, mostly for surfaces.
2.3 (Lack of) Gromov–Hausdorff precompactness

An interesting problem in the theory of finite groups was to understand the possible limits of finite homogeneous spaces. For instance, can one find a sequence of compact geodesic spaces $X_n$ and finite groups $G_n \leq \text{Iso}(X_n)$ with $\text{diam}(X_n/G_n) \to 0$ such that $X_n$ converges to $S^2$ in the Gromov–Hausdorff sense? This question was answered negatively by Turing [37], and building upon his work, Gelander [21] proved the following.

**Theorem 22** (Gelander). Let $X_n$ be a sequence of compact geodesic spaces and $G_n \leq \text{Iso}(X_n)$ a sequence of finite groups with $\text{diam}(X_n/G_n) \to 0$. If $X_n$ converges in the Gromov–Hausdorff sense to a compact space $X$, then $X$ is a (possibly infinite-dimensional) torus.

A consequence of Theorem 22 is that a sequence of normalized universal covers cannot have a “limit shape”.

**Corollary 23.** Let $X_n$ be a sequence of compact geodesic spaces and $G_n \leq \text{Iso}(X_n)$ a sequence of finite groups with $\text{diam}(X_n/G_n)/\text{diam}(X_n) \to 0$. If $b_1(X_n) = 0$ for all $n$, then the sequence $X_n/\text{diam}(X_n)$ diverges in the Gromov–Hausdorff sense.

**Proof.** Assuming the contrary, $X_n/\text{diam}(X_n)$ converges to a space $X$ of diameter 1. By Theorem 22, $X$ is a torus so it admits a regular covering with Galois group $\mathbb{Z}$. Then by the work of Sormani–Wei [34, Theorem 3.4], there are surjective morphisms $\pi_1(X_n) \to \mathbb{Z}$ for $n$ large enough contradicting the assumption $b_1(X_n) = 0$.

**Remark 24.** Theorems 3 and 12 are proven in a similar fashion. In [5, 9], building upon the structure of approximate groups by Breuillard–Green–Tao [8], it is proven that if one had contradicting subsequences, then the normalized spaces would converge to a finite-dimensional torus, contradicting the lower-semi-continuity of the first Betti number [34].

It would be interesting to further understand what causes the behavior of the sequences $X_n/\text{diam}(X_n)$ in Corollary 23. Recall that some known families of Gromov–Hausdorff divergent sequences such as $X_n = S^n$ or $X_n = (\mathbb{Z}/2\mathbb{Z})^n$ present a concentration of measure property [25].

**Definition 25.** We say that a sequence $(X_n,d_n,\mu_n)$ of metric probability spaces of diameter 1 is a *Levy family* if for any sequence of 1-Lipschitz maps $f_n : X_n \to \mathbb{R}$, the sequence of variances $\text{Var}((f_n)_*\mu_n)$ goes to 0 as $n \to \infty$.

**Conjecture 26** (Petrunin). Let $(X_n,d_n,\mu_n)$ be a sequence of compact simply connected geodesic probability spaces and $G_n \leq \text{ Iso}(X_n)$ a sequence of finite groups of measure preserving isometries with $\text{diam}(X_n/G_n) \to 0$. Then $X_n$ is a Levy family.

### 3 Preliminaries

#### 3.1 Notation

For a finite-dimensional $\mathbb{C}$-Hilbert space $V$, we denote by $\text{End}(V)$ the space of linear maps $V \to V$ and by $U(V) \subset \text{End}(V)$ the set of unitary automorphisms. If $V = \mathbb{C}^n$, then we
denote \( U(V) \) also by \( U(n) \). For \( A \in \text{End}(V) \), we denote its spectrum by \( \sigma(A) \subset \mathbb{C} \) and its adjoint by \( A^* \). The trace operator is denoted by \( \text{Tr} : \text{End}(V) \to \mathbb{C} \). When \( V \) is 1-dimensional, we will identify \( \text{End}(V) \) with \( \mathbb{C} \) via \( \text{Tr} : \text{End}(V) \to \mathbb{C} \).

For a path connected topological space \( X \), we denote its first Betti number by \( b_1(X) \). Recall that it equals the supremum of the \( m \) for which there is a surjective morphism \( \pi_1(X) \to \mathbb{Z}^m \).

For a metric space \( X, p \in X, \) and \( r > 0 \), we denote by \( B(p, r) \) the open ball of radius \( r \) around \( p \). For two metric spaces \( X \) and \( Y \), we denote their Gromov–Hausdorff distance by \( d_{GH}(X, Y) \).

### 3.2 Graphs and CW-complexes

For the purposes of this paper, a graph always means a locally finite undirected graph without loops or multiple edges. For vertices \( x, y \) in a graph, we write \( x \sim y \) if there is an edge connecting \( x \) to \( y \). For an edge \([x, y]\), we denote by \((x, y)\) its interior.

For a sequence of vertices \( v_0, v_1, \ldots, v_m \) in a graph \( \Gamma \) such that \( v_{i-1} \sim v_i \) for each \( i \in \{1, \ldots, m\} \), we denote by \([v_0, \ldots, v_m]\) the curve \( \gamma : [0, m] \to \Gamma \) with \( \gamma(i) = v_i \) for every \( i \in \{0, \ldots, m\} \), so that \( \gamma_{|[i-1,i]} \) travels along the edge \( v_{i-1}v_i \). A curve (loop) of this form is called a graph curve (loop) of length \( m \).

For vertices \( x, y \) in a connected graph \( \Gamma \), the graph distance \( d_{\Gamma}(x, y) \) between \( x \) and \( y \) is the minimum \( m \) for which there is a graph curve of length \( m \) connecting them.

For a graph \( \Gamma \) and an integer \( k \in \mathbb{N} \), we denote by \( P_k(\Gamma) \) the 2-dimensional CW-complex whose 1-skeleton is \( \Gamma \) and whose 2-cells are \( m \)-gons for \( 0 \leq m \leq k \), defined by the simple graph loops of length \( m \) in \( \Gamma \), up to cyclic permutations.

**Remark 27.** It is not hard to equip \( P_k(\Gamma) \) with a geodesic metric that restricted to \( \Gamma \) coincides with its original metric, and such that \( d_{GH}(P_k(\Gamma), \Gamma) \leq k \). For instance; for \( k = 3 \) one can make each 2-cell a Reuleaux triangle.

Let \( G \) be a group and \( S \subset G \) a symmetric generating subset. The Cayley graph \( \Gamma(G, S) \) of \( G \) with respect to \( S \) is defined to be the one with \( G \) as its vertex set and such that two distinct elements \( g, h \in G \) are adjacent if and only if \( g = hs \) for some \( s \in S \).

We now state a trivial observation. We include its proof since this same counting argument will be used later (see Claim 2 in the proof of Theorem 6).

**Lemma 28.** Let \( G \) be a finite group with \( |G| \geq 3 \), \( S \subset G \) a symmetric set of generators, \( \Gamma := \Gamma(G, S) \) the Cayley graph, and \( t \in S \setminus \{e\} \). Then \( \Gamma \setminus (e, t) \) is connected.

**Proof.** If \( t \) has order \( m \geq 3 \), then the path \([t, t^2, \ldots, t^m]\) connects the endpoints of the removed edge, so we can assume \( t = t^{-1} \). Let \( C_1 \) and \( C_2 \) denote the connected components of \( \Gamma \setminus (e, t) \) containing \( e \) and \( t \), respectively. Since multiplication by \( t \) exchanges \( e \) and \( t \), it sends \( C_1 \) to \( C_2 \) and vice-versa, so \( |C_1| = |C_2| \).

Since \( |G| \geq 3 \), \( S \setminus \{e\} \) contains an element \( s \neq t \). Multiplication by \( s \) sends \((e, t)\) to \((s, st)\), so \( sC_1 \) is the connected component of \( \Gamma \setminus (s, st) \) containing \( s \). Since \( s \neq t = t^{-1} \), the three segments \((s, e), (s, st), \) and \((e, t)\) are distinct. Hence,
• the path \([s, e, t]\) lies entirely in \(sC_1 \subset \Gamma \setminus (s, st)\).

• the path \([e, s, st]\) lies entirely in \(C_1 \subset \Gamma \setminus (e, t)\).

If \(C_1 \neq C_2\), then \(C_1 \cap C_2 = \emptyset\) and the above implies that

• the connected set \([s, e, t] \cup C_2\) lies entirely in \(sC_1 \subset \Gamma \setminus (s, st)\).

• \(\{s, e\} \cap C_2 = \emptyset\).

Therefore

\[|C_1| = |sC_1| \geq |[s, e, t] \cup C_2| = 2 + |C_2| = 2 + |C_1|\]

This contradiction finishes the proof of Lemma 28.

\[\square\]

### 3.3 Constructing covering spaces

In this section we prove Proposition 5. In order to do so, we present a general construction (cf. [23, Section 5D]).

**Proposition 29.** Let \(X\) be a proper geodesic space, \(p \in X\), \(G \leq \text{Iso}(X)\) a discrete group of isometries, and \(r \geq 2 \cdot \text{diam}(X/G)\). Then set \(S := \{g \in G|d(gp,p) \leq r\}\), and let \(\tilde{G}\) be the abstract group generated by \(S\), with relations

\[s = s_1s_2 \text{ in } \tilde{G}, \text{ whenever } s, s_1, s_2 \in S \text{ and } s = s_1s_2 \text{ in } G.\]

Denote the canonical embedding \(S \hookrightarrow \tilde{G}\) as \((s \mapsto s^\sharp)\), and by \(\Phi : \tilde{G} \rightarrow G\) the unique morphism with \(\Phi(s^\sharp) = s\) for all \(s \in S\). Then there is a regular covering \(\tilde{X} \rightarrow X\) with Galois group \(\text{Ker}(\Phi)\).

**Proof.** In order to construct the space \(\tilde{X}\), notice that by discreteness of \(G\), there is \(\eta > 0\) with \(S = \{g \in G|d(gp,p) < r + 2\eta\}\). Set \(B := B(p, r/2 + \eta)\). Then \(S = \{g \in G|B \cap gB \neq \emptyset\}\). Equip \(\tilde{G}\) with the discrete topology, and consider the topological space

\[\tilde{X} := (\tilde{G} \times B) / \sim,\]

where \(\sim\) is the minimal equivalence relation such that

\[(gs^\sharp, x) \sim (g, sx) \text{ whenever } s \in S, x, sx \in B.\]  

(7)

We then obtain a continuous map \(\Psi : \tilde{X} \rightarrow X\) given by

\[\Psi(g, x) := \Phi(g)(x).\]

Fix \(g_0 \in \tilde{G}\) and set \(U := \Phi(g_0)(B)\). The proof of [38, Theorem 2.32] carries over (with \(V = B\) and \(\Gamma = G\)) to show that \(U\) is evenly covered. As \(g_0\) ranges over \(\tilde{G}\), the sets \(\Phi(g_0)(B)\) cover \(X\), so \(\Psi\) is a covering map. The proof of [38, Theorem 2.32] again carries over to show that \(\Psi\) is regular with Galois group \(\text{Ker}(\Phi)\).

\[\square\]
We now prove Proposition 5. Let $X, G, S, \Gamma$ be as in the statement of the proposition. Let $\tilde{G}$ be the group with presentation $(S \mid R)$, where $R$ consists of the words of length $\leq 3$ that represent the trivial element of $G$. Then $P_3(\Gamma(\tilde{G}, S))$ is the universal cover of $P_3(\Gamma)$, and $\pi_1(P_3(\Gamma))$ is isomorphic to the kernel of the natural map $\tilde{G} \to G$ (see Remark 7). By Proposition 29, there is a regular covering map $\tilde{X} \to X$ with Galois group $\pi_1(P_3(\Gamma))$, so there is a surjective map $\pi_1(X) \to \pi_1(P_3(\Gamma))$.

3.4 Representation theory of finite groups

In this section we recall the results from representation theory we will need. We refer the reader to [33, Chapters 1-2] for proofs and further discussion. Throughout this section, let $G$ be a finite group.

For our purposes, a (unitary) representation is a morphism $\rho : G \to U(V)$ for some finite-dimensional $\mathbb{C}$-Hilbert space $V$. The dimension of $V$ is called the dimension of the representation and will be denoted by $d_\rho$. We say that such representation is irreducible if whenever there is a subspace $W \leq V$ invariant under the $G$-action, either $W = \{0\}$ or $W = V$. The representation $\rho$ is said to be trivial if $\rho(g) = \text{Id}_V$ for all $g \in G$.

Given two representations $\rho_1 : G \to U(V_1)$, $\rho_2 : G \to U(V_2)$, we say a linear map $\lambda : V_1 \to V_2$ is equivariant if

$$\lambda \rho_1(g) = \rho_2(g) \lambda \text{ for all } g \in G.$$ 

We say that $\rho_1$ and $\rho_2$ are isomorphic if there is an equivariant linear isomorphism $\lambda : V_1 \to V_2$. It turns out there are only finitely many isomorphism classes of irreducible representations and they satisfy

$$|G| = \sum_{\rho} d_\rho^2,$$

where the sum is taken among the isomorphism classes of irreducible unitary representations of $G$.

Let $\mathbb{C}[G]$ be the space of functions $G \to \mathbb{C}$. For $f \in \mathbb{C}[G]$, we denote by $f^* \in \mathbb{C}[G]$ the function given by $f^*(g) := \overline{f(g^{-1})}$. For $f, h \in \mathbb{C}[G]$, their product is defined as

$$(f * h)(g) := \sum_{u \in G} f(gu)h(u^{-1}).$$

The Hermitian product

$$\langle f, h \rangle := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}$$

makes $\mathbb{C}[G]$ a Hilbert space called the convolution algebra of $G$. It admits a natural action $\rho_0 : G \to U(\mathbb{C}[G])$ given by

$$\rho_0(g)(f)(h) := f(g^{-1}h) \text{ for } f \in \mathbb{C}[G], g, h \in G.$$
For $f \in \mathbb{C}[G]$ and a representation $\rho : G \to U(V)$, the **Fourier transform** $\hat{f}(\rho) \in \text{End}(V)$ is defined as

$$\hat{f}(\rho) := \sum_{g \in G} f(g) \rho(g).$$

The Fourier transform is compatible with products and adjoints. That is,

$$\hat{f} \ast \hat{h}(\rho) = \hat{f}(\rho) \hat{h}(\rho)$$

$$\hat{f}^*(\rho) = [\hat{f}(\rho)]^*$$

for any $f, h \in \mathbb{C}[G]$ and any representation $\rho$.

It can be shown that for any irreducible representation $\rho : G \to U(V)$, there is a (non-unique) equivariant isometric embedding $t_V : V \to \mathbb{C}[G]$. Moreover, such embeddings among all irreducible representations span $\mathbb{C}[G]$. This leads to the following important result.

**Theorem 30** (Plancherel formula). For $f, h \in \mathbb{C}[G]$ one has

$$\langle f, h^* \rangle = \frac{1}{|G|^2} \sum_{\rho} d_\rho \text{Tr}(\hat{f}(\rho) \hat{h}(\rho)),$$

where the sum is taken among the isomorphism classes of irreducible unitary representations of $G$.

The following result follows from the fact that any unitary representation is totally reducible.

**Lemma 31.** For any unitary representation $\rho : G \to U(V)$, there is $x \in V$ with $|x| = 1$ such that the span of the $G$-orbit of $x$ is irreducible.

The ensuing result follows from the fact that commuting unitary automorphisms are simultaneously diagonalizable.

**Proposition 32.** If $G$ is abelian, then any irreducible unitary representation is 1-dimensional.

### 3.5 Kazhdan constant, Cheeger constant, and spectral gap

We refer the reader to [4] for a detailed introduction to the theory of Kazhdan’s property (T) and related topics. Throughout this section, let $G$ be a finite group, $S \subset G$ a symmetric generating set, and $\Gamma := \Gamma(G, S)$ the corresponding Cayley graph.

The **Kazhdan constant** of $G$ with respect to $S$ is defined as

$$K(G, S) := \inf_{\rho} \inf_{|x| = 1} \sup_{s \in S} d(\rho(s)x, x),$$

where the first infimum is taken among non-trivial irreducible unitary representations $\rho : G \to U(V)$ and the second infimum is taken among unit vectors $x \in V$. 

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The Laplacian is the map $\Delta : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by

$$\Delta(f)(g) := \sum_{s \in \mathcal{S}} [f(gs) - f(g)].$$

This map is equivariant, self adjoint, and its spectrum is a finite set of non-positive real numbers

$$\sigma(\Delta) := \{0 = \lambda_0 > -\lambda_1 \geq \ldots \geq -\lambda_{|G|-1}\}.$$  

We denote the $\Delta$-eigenspace of an eigenvalue $-\lambda$ by $E_\lambda$. The quantity $\lambda_1$ is called the spectral gap of $G$ with respect to $S$.

The Cheeger (isoperimetric) constant of $G$ with respect to $S$ is defined by

$$h(G, S) := \inf \left\{ \left| \frac{\partial A}{|A|} \right| : A \subset G, |A| \leq \frac{1}{2}|G| \right\},$$

where $\partial A$ denotes the set of edges in $\Gamma$ connecting a vertex in $A$ with a vertex in $G \setminus A$.

**Remark 33.** The Kazhdan constant $K(G, S)$ quantifies how easy it is to tell apart isometric actions on spheres with fixed points from isometric actions on spheres without fixed points, the spectral gap $\lambda_1$ quantifies how fast heat flows through $\Gamma$, and $h$ quantifies how bad the bottlenecks of $\Gamma$ are. Each quantity measures in some way how robust is the network $\Gamma$.

Recall that these quantities satisfy the well known relations

$$\frac{h^2}{|S|^2} \leq 2\lambda_1 \leq K(G, S)^2 \leq 2\lambda_1 \leq 4h$$  

(see [9, Section 1.2] for a similar expression involving $\text{diam}(\Gamma)$). A proof of the first and last inequalities, known as the discrete Cheeger–Buser inequalities, along with historical background can be found in [26, Section 4.2]. To verify the second inequality, take an arbitrary irreducible unitary representation $\rho : G \to U(V)$. Since there exists an equivariant embedding $i_V : V \to \mathbb{C}[G]$, we can assume $V \leq \mathbb{C}[V]$. Moreover, since $\Delta$ is equivariant, we can further assume $V \leq E_\lambda$ for some $\lambda \neq 0$. Then assume a unit vector $x \in V$ satisfies

$$\sup_{s \in \mathcal{S}} d(\rho_0(s)x, x) \leq 2 \cdot \sin(\theta/2)$$

for some $\theta \in [0, \pi]$. Then the angle between $x$ and $\rho_0(s)x$ is at most $\theta$ for each $s \in \mathcal{S}$ (see Figure 2). This implies

$$\lambda = \langle -\Delta x, x \rangle = \sum_{s \in \mathcal{S}} \langle x - \rho_0(s)x, x \rangle \leq |S|[1 - \cos(\theta)].$$

From the identity $1 - \cos(\theta) = 2 \cdot \sin^2(\theta/2)$, we deduce $[2 \cdot \sin(\theta/2)]^2 \geq 2\lambda/|S|$. Since $x$ was arbitrary and $\lambda \geq \lambda_1$, the second inequality of (10) follows.
Basic trigonometry shows that if the angle between the unit vectors \( x \) and \( \rho(s)x \) is \( \theta \), then the distance between the endpoints is \( 2 \sin(\theta/2) \).

To verify the third inequality of (10), recall that by Lemma 31, there is a unit vector \( x \in E_{\lambda_1} \) for which its \( G \)-orbit spans an irreducible representation. By definition there is \( t \in S \) with \( d(\rho_0(t)x, x) \geq K(G, S) \). Set \( K(G, S) = 2 \cdot \sin(\theta/2) \) with \( \theta \in [0, \pi] \). Then since \( \left| \langle (\rho(s)x, x) \rangle \right| \leq 1 \) for all \( s \in S \setminus \{t\} \), one has

\[
\lambda_1 = \langle -\Delta x, x \rangle = \sum_{s \in S} \langle x - \rho_0(s)x, x \rangle \geq 1 - \cos(\theta) = \frac{K(G, S)^2}{2}.
\]

### 3.6 Random walks on Cayley graphs

We refer to [19, Chapter 3] for an introduction on the theory of random walks in finite groups. Throughout this section, let \( G \) be a finite group, \( S \subset G \) a symmetric generating set, and \( \Gamma := \Gamma(G, S) \) the corresponding Cayley graph.

For \( \alpha \geq |S| \), the random walk on \( \Gamma \) is the \( G \)-valued Markov process \( \{W^t_\alpha\}_{t \in \mathbb{N}} \) such that \( W^0_\alpha \equiv e \), and at each time, if the walker is at \( g \in G \), then it stays at \( g \) with probability \( 1 - |S|/\alpha \) and jumps to a neighbor uniformly at random with probability \( |S|/\alpha \). This gives rise to the law

\[
P \left[ W^t_{\alpha+1} = g \right] = \frac{1}{\alpha} \left( (\alpha - |S|)P \left[ W^t_\alpha = g \right] + \sum_{s \in S} P \left[ W^t_\alpha = gs \right] \right).
\]

We will denote \( W^t_1 \) simply by \( W_\alpha \). We can identify \( \mathbb{C}[G] \) with the set of complex valued measures on \( G \) via the correspondence

\[
\mu \in \mathbb{C}[G] \leftrightarrow \mu(A) := \sum_{g \in A} \mu(g).
\]

Note that after this identification,

\[
W_\alpha = \delta_e + \frac{1}{\alpha} \Delta(\delta_e), \quad (\text{11})
\]

where \( \delta_e \) denotes the Dirac mass at \( e \). It is also straightforward to verify that

\[
W^t_\alpha * W^t_\alpha = W^{t+s+t}_\alpha, \quad (W^t_\alpha)^* = W^t_\alpha \quad (\text{12})
\]
for all $s,t \in \mathbb{N}$. If $\alpha > |S|$, the distribution $W^t_\alpha$ converges to the uniform distribution $U$ on $G$ as $t \to \infty$. One can quantify how fast this convergence occurs with the quantity

$$\varepsilon_\alpha(t) := \sum_{g \in G} |W^t_\alpha(g) - \frac{1}{|G|}|.$$  

The mixing time $\tau_\alpha^\alpha : [0, 2] \to \mathbb{N}$ of the process $\{W^t_\alpha\}_{t \in \mathbb{N}}$ is defined as

$$\tau_\alpha^\alpha(c) := \inf \{ t \in \mathbb{N} | \varepsilon_\alpha(t) \leq c \}.$$  

A direct computation shows that for all $t \in \mathbb{N}$,

$$\langle W^t_\alpha, U \rangle = \langle U, U \rangle = \frac{1}{|G|^2}. \quad (13)$$

If $1 : G \to U(1)$ denotes the trivial representation, then (9) and (12) imply

$$\hat{W}^t_\alpha(1) = \hat{W}^t_\alpha(1) = \text{Id}_C \quad (14)$$

for all $t \in \mathbb{N}$. Then one has

$$\langle W^t_\alpha - U, W^t_\alpha - U \rangle = \langle W^t_\alpha, W^t_\alpha \rangle - \frac{1}{|G|^2}$$

$$= \frac{1}{|G|^2} \sum_\rho d_\rho \text{Tr}(\hat{W}^{2t}_\alpha(\rho)) - \frac{1}{|G|^2}$$

$$= \frac{1}{|G|^2} \sum_{\rho \neq 1} d_\rho \text{Tr}(\hat{W}^{2t}_\alpha(\rho)), \quad (15)$$

where the last sum is taken over isomorphism classes of non-trivial irreducible unitary representations; the first equality uses (13), the second one follows from the Plancherel formula and (12), and the third one uses (14). Finally, by the Cauchy-Schwarz inequality, (15) implies

$$|\varepsilon_\alpha(t)|^2 \leq \sum_{\rho \neq 1} d_\rho \text{Tr}(\hat{W}^{2t}_\alpha(\rho)). \quad (16)$$

When $\alpha = 2|S|$, we denote $W^t_\alpha$ by $W^t$, $W_\alpha$ by $W$, $\varepsilon_\alpha(t)$ by $\varepsilon(t)$, and $\tau_\alpha^\alpha$ by $\tau_\Gamma$.

4 Diameter bounds

In this section we prove Theorems 2, 3, and 6. For a group $G$ and a symmetric generating set $S \subset G$, we set $S_k := S \cup S^2 \cup \ldots \cup S^{\lfloor \frac{k+2}{2} \rfloor}$. It is straightforward to check that

$$\text{diam}(\Gamma(G,S)) \leq \text{diam}(\Gamma(G,S_k)) \left[ \frac{k+2}{3} \right]. \quad (17)$$

The ensuing result [3] reduces the proof of Theorem 6 to the case $k = 3$. 

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Lemma 34 (Behr). Let $G$ a group, and $S \subset G$ a finite symmetric set of generators. Then for each $k \geq 3$, there is a surjective map

$$\pi_1(P_k(\Gamma(G,S))) \to \pi_1(P_3(\Gamma(G,S_k))).$$

Proof. First notice that if an edge $\omega$ of $\Gamma(G,S_k)$ corresponds to an element of $S^m$ with $m \leq \lfloor \frac{k+2}{3} \rfloor$ then there is an endpoint-preserving homotopy in $P_3(\Gamma(G,S_k))$ taking $\omega$ to a concatenation of $m$ edges in $\Gamma(G,S)$ (see Figure 3). Since the fundamental group of a CW-complex is generated by the loops in its 1-skeleton, the above observation implies that the inclusion

$$\Gamma(G,S) \to P_3(\Gamma(G,S_k))$$

induces a surjective map at the level of fundamental groups. It remains to check that (18) extends to a continuous map $P_k(\Gamma(G,S)) \to P_3(\Gamma(G,S_k))$. This boils down to the fact that any word of length $\leq k$ representing the identity in $G$ using the elements of $S$ as letters, can be written as a concatenation of words of length $\leq 3$ representing the identity in $G$ using the elements of $S_k$ as letters (see [16, Lemma 7.A.8] for further details).

![Figure 3:](image)

Figure 3: If $\omega = s_1 \cdots s_m$, then the decomposition $w = (s_1 \cdots s_{m-1})(s_m)$ yields an endpoint preserving homotopy in $P_3(\Gamma(G,S_k))$ from $w$ to the concatenation of an edge corresponding to an element in $S^{m-1}$ and one in $S$. Proceeding inductively yields the desired homotopy. In the picture $m = 6$.

The following elementary observation will be required at the end of the proof of Theorem 6.

Lemma 35. Let $\Gamma$ be a connected graph, $T \subset \Gamma$ a connected subgraph, and $C_1, \ldots, C_\ell \subset \Gamma$ the connected components of $\Gamma \setminus T$. Then for each $j \in \{1, \ldots, \ell\}$, the graph $\Gamma \setminus C_j$ is connected.

Proof. If the result is false, there are $w_0, w_1 \in \Gamma \setminus C_j$ such that any path connecting them passes through $C_j$. Since $\Gamma$ is connected, there is a path $[w_0 = v_0, v_1, \ldots, v_k = w_1]$, which by assumption passes through $C_j$. Let $i_1$ be the first index such that $v_{i_1+1}$ is in $C_j$ and let $i_2$ be the last index such that $v_{i_2-1}$ lies in $C_j$. Then both $v_{i_1}$ and $v_{i_2}$ lie in $T$, and by connectedness of $T$ there is a path $[v_{i_1} = a_0, a_1, \ldots, a_m = v_{i_2}]$ in $T$. Then it is easy to check that the path $[w_0 = v_0, \ldots, v_{i_1}, a_1, \ldots, a_{m-1}, v_{i_2}, \ldots, v_k = w_1]$
we have \( \text{diam}(\Gamma) \) and a minimizing path \( v \).

Figure 4: The portion \([v_1, v_{i1+1}, \ldots, v_{i2-1}, v_{i2}]\) can be replaced by \([a_0, \ldots, a_m]\). In the picture the shadowed region represents \( C_j \), the solid line represents \( T \), \( i_1 = 4 \), and \( i_2 = 9 \).

\[ \text{Proof of Theorem 6:} \text{ By Lemma 34 we have } b_1(P_3(\Gamma(G, S_k))) = 0. \text{ Combining this with (17) we can assume } k = 3 \text{ without loss of generality.} \]

Let \( e \in G \) be the neutral element and \( \Gamma = \Gamma(G, S) \). Take \( h \in G \) with \( d_\Gamma(h, e) = m = \text{diam}(\Gamma) \) and a minimizing path \([e = g_0, g_1, \ldots, g_m = h]\). For each \( i \), set \( \Sigma_i \subset \Gamma \) as the subgraph induced by the set of vertices \( \{g \in G \mid d_\Gamma(g, e) = i\} \) and let \( T_i \) be the connected component of \( \Sigma_i \) containing \( g_i \).

Claim 1: For each \( i_0 \in \{1, \ldots, m - 1\} \), the vertices \( e \) and \( h \) lie in distinct connected components of \( \Gamma \setminus T_{i_0} \).

Let \( Y_0 = Y_1 \subset \Gamma \) be the subgraph induced by \( \bigcup_{j=0}^{i_0-1} \Sigma_j \), \( Y_{1/4} \subset \Gamma \) the subgraph induced by \( T_{i_0} \), \( Y_{1/2} \subset \Gamma \) the subgraph induced by \( \bigcup_{j=i_0+1}^{m} \Sigma_j \), and \( Y_{3/4} \subset \Gamma \) the subgraph induced by \( \Sigma_{i_0} \setminus T_{i_0} \). Since \( d_\Gamma(, e) \) is 1-Lipschitz, there are no edges between \( Y_0 \) and \( Y_{1/2} \), and by the definition of \( T_{i_0} \) there are no edges between \( Y_{1/4} \) and \( Y_{3/4} \).

Then construct a map \( \psi : \Gamma \rightarrow \mathbb{R}/\mathbb{Z} \) that restricted to \( Y_s \) equals \( s \) for \( s \in \{0, 1/4, 1/2, 3/4\} \), and takes all edges joining \( Y_0 \) with \( Y_{1/4} \) to the interval \([0, 1/4]\), doing the same for the intervals \([1/4, 1/2]\), \([1/2, 3/4]\), and \([3/4, 1]\).

By construction, \( \psi \) sends each edge of \( \Gamma \) to either a point or an interval of length 1/4 in \( \mathbb{R}/\mathbb{Z} \). Also recall that each 2-cell \( \alpha \) of \( P_3(\Gamma) \) is attached to \( \Gamma \) via a simple loop \( \partial \alpha \) of length \( \leq 3 \). Therefore, \( \psi(\partial \alpha) \subset \mathbb{R}/\mathbb{Z} \) is a loop of length \( \leq 3/4 \) hence nullhomotopic for each \( \alpha \), and \( \psi \) extends to a map \( \Psi : P_3(\Gamma) \rightarrow \mathbb{R}/\mathbb{Z} \) (see Figure 5).

Assume the claim is false and take a minimizing path \([e = g_0, g_1', \ldots, g_{m'} = h]\) in \( \Gamma \setminus T_{i_0} \). Consider the map \( \Phi : \mathbb{R}/\mathbb{Z} \rightarrow P_3(\Gamma) \) that sends the interval \([0, 1/2]\) to the path \([g_0, g_1, \ldots, g_m]\) and \([1/2, 1]\) to the path \([g_{m'}, \ldots, g_1, g_0]\).

Since \( \Psi \circ \Phi(0) = 0 \), \( \Psi \circ \Phi(1/2) = 1/2 \), \( \Psi \circ \Phi|_{[0, 1/2]} \) misses 3/4, and \( \Psi \circ \Phi|_{[1/2, 1]} \) misses 1/4, the composition \( \Psi \circ \Phi \) is homotopic to the identity in \( \mathbb{R}/\mathbb{Z} \), meaning that the induced map
\[ Y_0 = Y_1 \]

\[ Y_1^2 \]

\[ Y_3^4 \]

\[ \Phi \]

\[ \Psi \]

\[ 0 = 1 \]

\[ 1/4 \]

\[ 1/2 \]

\[ 3/4 \]

Figure 5: Each edge of \( \Gamma \) is sent via \( \psi \) to either a point or an interval of length \( \leq 1/4 \), so \( \psi \) extends to a map \( \Psi : P_3(\Gamma) \to \mathbb{R}/\mathbb{Z} \).

\[ \Psi_* : \pi_1(P_3(\Gamma)) \to \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z} \] is surjective, contradicting the hypothesis \( b_1(P_3(\Gamma)) = 0 \). This finishes the proof of the claim.

Claim 2: For each \( i_0 \in \{1, \ldots, m-1\} \), either \( g_{i_0}^{-1}T_{i_0} \) or \( hg_{i_0}^{-1}T_{i_0} \) intersect \( T_{i_0} \).

Let \( C_1, \ldots, C_\ell \subset \Gamma \) denote the connected components of \( \Gamma \setminus T_{i_0} \), with \( e \in C_1, h \in C_2 \). Observe that \( g_{i_0}^{-1}C_1, \ldots, g_{i_0}^{-1}C_\ell \) are the connected components of \( \Gamma \setminus g_{i_0}^{-1}T_{i_0} \), and \( hg_{i_0}^{-1}C_1, \ldots, hg_{i_0}^{-1}C_\ell \) are the connected components of \( \Gamma \setminus hg_{i_0}^{-1}T_{i_0} \).

Assume \( T_{i_0} \cap g_{i_0}^{-1}T_{i_0} = \emptyset \). Since \( g_{i_0}^{-1}T_{i_0} \) is connected and contains \( e \), it is contained in \( C_1 \).

By Lemma 35, \( \Gamma \setminus C_1 = T_{i_0} \cup C_2 \cup \ldots \cup C_\ell \) is connected, and since it doesn’t intersect \( g_{i_0}^{-1}T_{i_0} \), it is contained in \( g_{i_0}^{-1}C_{j_1} \) for some \( j_1 \). Therefore

\[ |C_j| < |C_{j_1}| \text{ for all } j \neq 1. \]

This is only possible if \( j_1 = 1 \), and in particular we have

\[ |C_2| < |C_1|. \]  \tag{19}

Similarly, if \( T_{i_0} \cap hg_{i_0}^{-1}T_{i_0} = \emptyset \), then \( hg_{i_0}^{-1}T_{i_0} \) is contained in \( C_2 \). By Lemma 35, \( \Gamma \setminus C_2 = T_{i_0} \cup C_1 \cup C_3 \cup \ldots \cup C_\ell \) is connected, and since it doesn’t intersect \( hg_{i_0}^{-1}T_{i_0} \), it is contained in \( hg_{i_0}^{-1}C_{j_2} \) for some \( j_2 \), meaning that

\[ |C_j| < |C_{j_2}| \text{ for all } j \neq 2. \]

This implies that \( j_2 = 2 \) and

\[ |C_1| < |C_2|. \]  \tag{20}

Assuming the claim is false, both (19) and (20) would hold; a contradiction.
From the second claim, we deduce \( \text{diam}(T_i) \geq \min\{i, m-i\} \) for each \( i \in \{1, \ldots, m-1\} \), and since the \( T_i \)'s are disjoint, we conclude that

\[
|G| \geq \Sigma_{i=0}^{m} |T_i| \geq \Sigma_{i=0}^{m} \min\{i+1, m-i+1\}.
\]

This implies that \( m \leq \sqrt{4|G|}+1 - 2 \), which is the required inequality. \( \square \)

**Proof of Theorem 2:** Let \( p \in X \) and define \( S \) as in Proposition 4 with \( \delta = 0 \). By Proposition 5, the first Betti number of \( P_3(\Gamma(G, S)) \) vanishes. For \( x, y \in X \), take \( g_1, g_2 \in G \) with \( d_X(g_1p, x), d_X(g_2p, y) \leq \text{diam}(X/G) \).

\[
d_X(x, y) \leq 2 \cdot \text{diam}(X/G) + d_X(g_1p, g_2p)
\]

\[
\leq 2 \cdot \text{diam}(X/G)[1 + d_L(g_1, g_2)]
\]

\[
\leq 2 \cdot \text{diam}(X/G)[\sqrt{4|G|} + 1 - 1],
\]

where the second inequality follows from Švarc–Milnor Lemma and the third one from Theorem 6. The result follows since \( \sqrt{4|G|} + 1 \leq 2\sqrt{|G|} + 1 \). \( \square \)

**Proof of Theorem 3:** Pick \( p_n \in X_n \), set \( S_n := \{g \in G_n | d(gp_n, p_n) \leq 2 \cdot \text{diam}(X_n/G_n)\} \), and let \( \Gamma_n := \Gamma(G_n, S_n) \). If the result fails, there is \( \varepsilon > 0 \) such that after taking a subsequence one has

\[
|G_n|^\varepsilon = O\left(\frac{\text{diam}(X_n)}{\text{diam}(X_n/G_n)}\right).
\]

By the Švarc–Milnor Lemma, this would imply \( |G_n|^\varepsilon = O(\text{diam}(\Gamma_n)) \). Then by [5, Theorem 1], after further taking a subsequence, \( \Gamma_n/\text{diam}(\Gamma_n) \) converges to an \( m \)-dimensional torus \( X \). By Remark 27, the sequence \( P_3(\Gamma_n)/\text{diam}(P_3(\Gamma_n)) \) also converges to \( X \) and by [34, Theorem 2.1] there are surjective morphisms \( \pi_1(P_3(\Gamma_n)) \rightarrow \pi_1(X) = \mathbb{Z}^m \) for large enough \( n \), contradicting Proposition 5. \( \square \)

## 5 Fourier analysis in abelian groups

In this section we prove Theorem 9 and Corollaries 10 and 11. Let \( k, G, S \) be as in the statement of Theorem 9 and let \( \Gamma := \Gamma(G, S) \). Assuming the estimate (3) fails to hold, there is an irreducible non-trivial unitary representation \( \rho : G \rightarrow U(m) \) and \( x \in S^{2m-1} \) with

\[
\sup_{s \in S} d(\rho(s)x, x) < 2 \cdot \sin(\pi/k).
\] (21)

By Proposition 32 we have \( m = 1 \), and since the metric on \( S^1 \) is bi-invariant, we can assume \( x = 1 \). Let \( \psi : \Gamma \rightarrow S^1 \) be the map that restricted to \( G \) coincides with \( \rho \), and restricted to an edge \([g, h] \subset \Gamma \) is a minimizing geodesic from \( \rho(g) \) to \( \rho(h) \).

If \( g, h \in G \) are such that \( g = hs \) for some \( s \in S \), then (21) implies that the angle between \( \rho(g) \) and \( \rho(h) \) is less than \( 2\pi/k \). Hence, for any simple loop of length \( \leq k \) in \( \Gamma \), its
image under $\psi$ has length less than $2\pi$ and is contractible. Therefore, $\psi$ extends to a map $\Psi : P_k(\Gamma) \to S^1$.

Since $\rho$ is non-trivial, there is $s \in S$ with $\rho(s) \neq 1$. Then the image under $\Psi$ of the loop $[e, s, s^2, \ldots, s^{[G]} = e]$ is a loop in $S^1$ that winds around at least once; counterclockwise if $\text{Re}(\rho(s)) > 0$ and clockwise if $\text{Re}(\rho(s)) < 0$. This means the map $\Psi_* : \pi_1(P_k(\Gamma)) \to \pi_1(S^1) = \mathbb{Z}$ is non-trivial, contradicting the assumption $b_1(P_k(\Gamma)) = 0$. This finishes the proof of (3).

We now proceed to prove (4). Notice that if we simply apply (2) naively to (3) we would get a weaker result. Let $-\lambda \in \sigma(\Delta) \setminus \{0\}$, and $E_\lambda \leq \mathbb{C}[G]$ the corresponding eigenspace. By Lemma 31, there is $x \in E_\lambda$ with $|x| = 1$ such that the span of its orbit is an irreducible representation $\rho : G \to S^1$. Since $\lambda \neq 0$, it follows that $\rho$ is non-trivial.

Case 1: $\rho(s) \neq -1$ for all $s \in S$.

By (3), there is $t \in S$ with $d(\rho(t), 1) \geq 2 \cdot \sin(\pi/k)$. This implies $\text{Re}(\rho(t)) = \text{Re}(\rho(t^{-1})) \leq \cos(2\pi/k)$. Since $\rho(t) \neq -1$, one has $t \neq t^{-1}$ so

$$\lambda = \langle -\Delta x, x \rangle = \sum_{s \in S} [1 - \rho(s)]$$

$$= |S| - \sum_{s \in S} \text{Re}(\rho(s))$$

$$\geq 2 - 2\cos(2\pi/k),$$

where in the last line we used that $\text{Re}(\rho(s)) < 1$ for all $s \in S \setminus \{t, t^{-1}\}$.

Case 2: $k = 3$ and $\rho(t) = -1$ for some $t \in S$.

We claim there are $s_1, s_2 \in S \setminus \{e, t\}$ such that $s_1s_2 = t$. Assume otherwise; then the edge $[e, t] \subset \Gamma$ does not belong to a 2-cell of $P_3(\Gamma)$. Let $x$ be the midpoint of the edge $[e, t]$, $A := [e, t]$, and $B := P_3(\Gamma) \setminus \{x\}$. Then $A \cap B = [e, x] \cup (x, t]$ and by Lemma 28, $B$ is connected, so the portion of the Mayer–Vietoris sequence (with real coefficients)

$$H_1(P_3(\Gamma)) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(P_3(\Gamma)) \to 0$$

yields the exact sequence (using $b_1(P_3(\Gamma)) = 0$)

$$0 \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0.$$

This is impossible by dimension counting. Then there are $s_1, s_2 \in S \setminus \{e, t\}$ (not necessarily distinct) with $s_1s_2 = t$. Since $\rho(s_1)\rho(s_2) = -1$, without loss of generality we can assume $\text{Re}(\rho(s_1)) \leq 0$. Then

$$\lambda = |S| - \sum_{s \in S} \text{Re}(\rho(s)) \geq 2 - \text{Re}(\rho(t) + \rho(s_1))$$

$$\geq 3 = 2 - 2\cos(2\pi/3).$$
Case 3: $k \geq 4$ and $\rho(t) = -1$ for some $t \in S$.

Directly compute

$$\lambda = |S| - \sum_{s \in S} \text{Re}(\rho(s)) \geq 2 \geq 2 - 2 \cos(2\pi/k).$$

This, together with (2), finishes the proof of (4).

For notational convenience, we set

$$\xi_k := 1 - \cos(2\pi/k) \quad \text{and} \quad \alpha_k := |S| + \xi_k.$$

To prove (5), we look at the random walk $W'_{\alpha_k}$ in $\Gamma$.

**Lemma 36.** For any non-trivial irreducible unitary representation $\rho : G \to S^1$,

$$\hat{W}_{\alpha_k}(\rho) \in \left[-1 + \frac{2\xi_k}{\alpha_k}, 1 - \frac{2\xi_k}{\alpha_k}\right].$$

**Proof.** This is a direct computation using (11). On one hand we have

$$\hat{W}_{\alpha_k}(\rho) = 1 + \frac{1}{\alpha_k} \sum_{s \in S} [\rho(s) - 1] \leq 1 - \frac{2\xi_k}{\alpha_k},$$

where we first used the identity $\hat{\delta}_s(\rho) = \rho(s)$ and then the estimate $\lambda_1 \geq 2\xi_k$. For the other inequality, notice that $\alpha_k - 2|S| = 2\xi_k - \alpha_k$, then

$$\hat{W}_{\alpha_k}(\rho) = 1 + \frac{1}{\alpha_k} \sum_{s \in S} [\rho(s) - 1] \geq \frac{\alpha_k - 2|S|}{\alpha_k} = \frac{2\xi_k}{\alpha_k} - 1,$$

where we used $\text{Re}(\rho(s)) \geq -1$ for all $s \in S$ in the inequality. \qed

Then by (15), for $t \in \mathbb{N}$ we have

$$\langle W'_{\alpha_k}^t - U, W'_{\alpha_k}^t - U \rangle = \frac{1}{|G|^2} \sum_{\rho \neq 1} d_{\rho} \text{Tr} \left( \hat{W}_{\alpha_k}^{2t} \right) < \frac{1}{|G|} \left| 1 - \frac{2\xi_k}{\alpha_k} \right|^{2t},$$

where we used (8) for the inequality. Now assume $t \leq \text{diam}(\Gamma)$. Since $W'_{\alpha_k}^t$ is supported in the ball of radius $t$ around $e$, then the left hand side of the equation is at least $\frac{1}{|G|^2}$. Hence by taking logarithm and using the identity $\log(1 + u) \leq u$ we get

$$-2 \log |G| < 2t \log \left| 1 - \frac{2\xi_k}{\alpha_k} \right| \leq -\frac{4t\xi_k}{\alpha_k}.$$ 

Rearranging terms,

$$t < \frac{\alpha_k \log |G|}{2\xi_k}.$$ 

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This implies
\[ \text{diam}(\Gamma) \leq \frac{\alpha_k}{2\xi_k} \log |G| + 1. \]
This concludes the proof of Theorem 9. Notice that if \( k = 3 \), then (5) simplifies to
\[ \text{diam}(\Gamma) \leq \left[ \frac{|S|}{3} + \frac{1}{2} \right] \log |G| + 1. \]
(22)

In order to prove Corollary 10, we need to establish an analogue of Lemma 36.

**Lemma 37.** Under the hypothesis of Corollary 10, for any non-trivial irreducible unitary representation \( \rho : G \to S^1 \) one has
\[ \hat{W}(\rho) \in \left[ 0, 1 - \frac{\xi_k}{|S|} \right]. \]

**Proof.** This is again a direct computation. Using \( \lambda_1 \geq 2\xi_k \) we get
\[ \hat{W}(\rho) = 1 + \frac{1}{2|S|} \sum_{s \in S} [\rho(s) - 1] \leq 1 - \frac{\xi_k}{|S|}. \]
On the other hand, simply using \( \text{Re}(\rho(s)) \geq -1 \) for all \( s \in S \) we conclude
\[ \hat{W}(\rho) = 1 + \frac{1}{2|S|} \sum_{s \in S} [\rho(s) - 1] \geq 0. \]
\[ \square \]

**Proof of Corollary 10:** By (16), using Lemma 37 and (8), we have
\[ \varepsilon(t)^2 \leq \sum_{\rho \neq 1} d_\rho \text{Tr}(\hat{W}^{2t}) < |G| \left| 1 - \frac{\xi_k}{|S|} \right|^{2t} \]
for \( t \in \mathbb{N} \). If \( \varepsilon(t) \geq c \in [0, 2] \), then taking logarithms as above we get
\[ 2 \log(c) < \log |G| - \frac{2t\xi_k}{|S|}. \]
Rearranging terms we get
\[ t < \frac{|S|}{2\xi_k} \log |G| - 2 \log(c). \]
Since \( \xi_k \geq 16/k^2 \) the result follows. \[ \square \]
Proof of Corollary 11: Take a point \( p \in M \) with injectivity radius \( \geq 2r_0 \), \( \tilde{p} \in \tilde{M} \) in its preimage, and set
\[
S := \{ g \in \pi_1(M) \setminus \{ e \} \mid d(g\tilde{p}, \tilde{p}) \leq 2D \}.
\]
By the injectivity radius condition, for \( g, h \in \pi_1(M) \) distinct, the balls \( B(g\tilde{p}, r_0) \) and \( B(h\tilde{p}, r_0) \) are isometric and disjoint. Since \( g \in S \cup \{ e \} \) implies \( B(g\tilde{p}, r_0) \subset B(\tilde{p}, 2D + r_0) \), we have
\[
|S| + 1 \leq \frac{\text{Vol}(B(\tilde{p}, 2D + r_0))}{\text{Vol}(B(\tilde{p}, r_0))}.
\]
By the Bishop–Gromov inequality [6, Section 11.10], the right hand side of the equation is less or equal than \( v_n^\kappa(2D + r_0)/v_n^\kappa(r_0) \). By Proposition 5, \( P_3(\Gamma(\pi_1(M), S)) \) is simply connected, so (22) holds;
\[
\text{diam}(\Gamma(\pi_1(M), S)) \leq \left[ \frac{v_n^\kappa(2D + r_0)}{3v_n^\kappa(r_0)} + \frac{1}{6} \right] \log |\pi_1(M)| + 1. \tag{23}
\]
Arguing as in the proof of Theorem 2, we have
\[
\frac{\text{diam}(\tilde{M})}{\text{diam}(M)} \leq 2 + 2 \cdot \text{diam}(\Gamma(\pi_1(M), S)). \tag{24}
\]
Combining (23) and (24) the result follows.

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