ON THE FUNDAMENTAL REPRESENTATION OF BORCHERDS ALGEBRAS WITH ONE IMAGINARY SIMPLE ROOT

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Borcherds algebras represent a new class of Lie algebras which have almost all the properties that ordinary Kac-Moody algebras have, and the only major difference is that these generalized Kac-Moody algebras are allowed to have imaginary simple roots. The simplest nontrivial examples one can think of are those where one adds “by hand” one imaginary simple root to an ordinary Kac-Moody algebra. We study the fundamental representation of this class of examples and prove that an irreducible module is given by the full tensor algebra over some integrable highest weight module of the underlying Kac-Moody algebra. We also comment on possible realizations of these Lie algebras in physics as symmetry algebras in quantum field theory.

1 Introduction

Studying the Lie algebra of physical states for the 26-dimensional bosonic string compactified on a torus, Borcherds discovered his celebrated fake Monster Lie algebra as the first generic example of a generalized Kac-Moody algebra (cf. [1], [2] or the review [3] for physicists). Up to this point it was only known that the tachyonic ground states give rise to an infinite rank Lie algebra $L_\infty$ with a set of simple roots isometric to the Leech lattice and with certain bounds on the dimension of the root spaces coming from the “no-ghost” theorem (cf. [1], [6], [7], [8]). It was Borcherds’ great achievement to observe that this upper bound can be satisfied by adding

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a certain set of photonic physical states as additional generators to the set of generators for $L_\infty$ \cite{8}. Mathematically speaking, he adjoined a set of \textit{imaginary simple roots} (where ‘imaginary’ means ‘negative norm’) to the set of real simple roots for the Kac-Moody algebra $L_\infty$. In the sequel Borcherds was able to axiomatize his ideas and he developed a theory of generalized Kac-Moody algebras in terms of generators and relations (cf. \cite{9}, \cite{11}).

To get a grasp of these new Lie algebras Slansky \cite{12} investigated the Borcherds extensions of the Lie algebras $\mathfrak{su}(2)$, $\mathfrak{su}(3)$, and affine $\mathfrak{su}(2)$ by a single lightlike (≡ norm zero) simple root. Computer calculations of the first few weight multiplicities of the basic representations suggested that the latter might be written as the tensor algebra over some module for the underlying nonextended Kac-Moody algebras. In the following we shall prove that this is true for any Kac-Moody algebra extended by an arbitrary imaginary simple root.

2 Definitions

Let us begin with a review of the definition of Borcherds algebras. As already mentioned, the original references on the subject are \cite{2}, \cite{9} and \cite{11}.

\textbf{Definition 1}

Let $\hat{A} = (a_{ij})$ be a real symmetric $n \times n$ matrix satisfying the following properties:

(i) either $a_{ii} = 2$ or $a_{ii} \leq 0$,

(ii) $a_{ij} \leq 0$ if $i \neq j$,

(iii) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.

Then the \textbf{Borcherds algebra} (\textbf{generalized Kac-Moody algebra}) associated to $\hat{A}$ is defined to be the Lie algebra $\hat{\mathfrak{g}}(\hat{A})$ given by the following generators and relations:

\textbf{Generators:} Elements $e_i$, $f_i$, $h_i$ for every $i$;

\textbf{Relations:}

(0) $[h_i, h_j] = 0$,

(1) $[e_i, f_j] = \delta_{ij} h_i$,

(2) $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$,

(3) $e_{ij} := (\text{ad } e_i)^{1-a_{ij}} e_j = 0$, $f_{ij} := (\text{ad } f_i)^{1-a_{ij}} f_j = 0$ if $a_{ii} = 2$ and $i \neq j$,

(4) $e_{ij} := [e_i, e_j] = 0$, $f_{ij} := [f_i, f_j] = 0$ if $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$.

The elements $h_i$ form a basis for an abelian subalgebra of $\hat{\mathfrak{g}}(\hat{A})$, called its \textbf{Cartan subalgebra} $\hat{\mathfrak{h}}(\hat{A})$. $\hat{\mathfrak{g}}(\hat{A})$ has a triangular decomposition

$$\hat{\mathfrak{g}}(\hat{A}) = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+,$$

where $\hat{\mathfrak{n}}_-$ (resp. $\hat{\mathfrak{n}}_+$) is the algebra obtained by dividing the free algebra $\hat{\mathfrak{n}}_- \oplus (\hat{\mathfrak{n}}_+)$ generated by the $f_i$ ($e_i$) by the ideal $\mathfrak{r}_-$ ($\mathfrak{r}_+$) generated by the $f_{ij}$ ($e_{ij}$).

Note that if $a_{ii} = 2$ for all $i$ then $\hat{\mathfrak{g}}(\hat{A})$ is the same as the ordinary Kac-Moody algebra with symmetrized Cartan matrix $\hat{A}$. In general, $\hat{\mathfrak{g}}(\hat{A})$ has almost all the properties that ordinary Kac-Moody algebras have, and the only major difference is that generalized Kac-Moody algebras
are allowed to have imaginary simple roots. In what follows we will exclusively deal with the case of a Borcherds algebra with one imaginary simple root.

It is clear that if we delete in \( \hat{A} \) the row and the column corresponding to the imaginary root then the resulting submatrix \( A \) is a generalized Cartan matrix in the sense of Kac [13] with associated Kac-Moody algebra \( \mathfrak{g}(A) \). Recall the triangular decomposition

\[
\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. 
\]

and the induced decomposition of the universal enveloping algebra:

\[
\mathcal{U}(\mathfrak{g}(A)) = \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+). 
\]

An irreducible \( \mathfrak{g}(A) \)-module \( \mathcal{F}_\lambda \) is called integrable highest-weight module if there exists a dominant integral weight \( \lambda \in \mathfrak{h}^* \) and a nonzero vector \( \omega \in \mathcal{F}_\lambda \) such that

\[
h(\omega) = \lambda(h)\omega \quad \text{for } h \in \mathfrak{h},
\]

\[
n_+(\omega) = 0,
\]

\[
\mathcal{U}(\mathfrak{n}_-)(\omega) = \mathcal{F}_\lambda.
\]

We denote by \( \mathfrak{T}(\mathcal{F}_\lambda) \) the tensor algebra over \( \mathcal{F}_\lambda \),

\[
\mathfrak{T}(\mathcal{F}_\lambda) := \bigoplus_{n=0}^{\infty} \mathcal{F}_\lambda^n \equiv \mathbb{C} \cdot 1 \oplus \mathcal{F}_\lambda \oplus (\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda) \oplus (\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda \otimes \mathcal{F}_\lambda) \oplus \ldots
\]

3 The theorem

Theorem 1

Let \( \hat{A} = (a_{ij}) \), \( 0 \leq i, j \leq n \), be a symmetric integer matrix satisfying the following properties:

(i) \( a_{00} \leq 0 \), \( a_{ii} = 2 \) for \( 1 \leq i \leq n \),

(ii) \( a_{ij} \leq 0 \) if \( i \neq j \).

Let \( \mathcal{F}_\lambda \) be the integrable highest weight module over the Kac-Moody algebra \( \mathfrak{g}(A) \) associated to the Cartan matrix \( A = (a_{ij}) \), \( 1 \leq i, j \leq n \), with highest weight \( \lambda \) defined by \( \lambda(h_i) := -a_{0i} \), \( 1 \leq i \leq n \), and highest weight vector \( \omega \). Then the tensor algebra \( \mathfrak{T}(\mathcal{F}_\lambda) \) over \( \mathcal{F}_\lambda \) is \( \hat{\mathfrak{g}}(\hat{A}) \)-module isomorphic to the highest weight module \( \mathcal{L}(\Lambda) \), \( \Lambda(h_i) = \delta_{i0} \), \( 0 \leq i \leq n \) of \( \hat{\mathfrak{g}}(\hat{A}) \).

Proof:

We define an action of the generators of \( \hat{\mathfrak{g}}(\hat{A}) \) on the tensor algebra \( \mathfrak{T}(\mathcal{F}_\lambda) \) as follows. Our convention for indices will be that \( i, j, k \) run from \( 1 \) to \( n \) unless otherwise stated!

The Kac-Moody generators \( e_i, h_i, f_i \) act trivially on the “vacuum” vector \( 1 \) and as highest weight representation on \( \mathcal{F}_\lambda \). We extend this action to the tensor algebra \( \mathfrak{T}(\mathcal{F}_\lambda) \) by Leibnitz’ rule.

The generator \( h_0 \) acts diagonal:

\[
h_0(1) := 1, \tag{1}
\]

\[
h_0(\omega) := (1 - a_{00})\omega, \tag{2}
\]

\[
h_0(f_k\varphi) := -a_{0k}f_k\varphi + f_kh_0(\varphi) \quad \text{for } \varphi \in \mathcal{F}_\lambda, \tag{3}
\]

\[
h_0(\Phi \otimes \Psi) := h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi \quad \text{for } \Phi, \Psi \in \mathfrak{T}(\mathcal{F}_\lambda). \tag{4}
\]
The “imaginary” generator $f_0$ adjoins one tensor factor of the highest weight vector $\omega$, i.e.,

$$f_0(\Psi) := \omega \otimes \Psi \quad \text{for} \quad \Psi \in \mathfrak{T}(F_{\lambda}). \quad (5)$$

For $e_0$ we put

$$e_0(1) := 0, \quad (6)$$

while for the definition on $F_{\lambda}^n$, $n \geq 1$, we observe that $F_{\lambda}^n = \Omega(n_-)(\omega \otimes F_{\lambda}^{n-1})$, so that it is sufficient to require, inductively,

$$e_0(f_i(\Psi)) := f_i(e_0(\Psi)), \quad (7)$$

$$e_0(\omega \otimes \Psi) := h_0(\Psi) + \omega \otimes e_0(\Psi), \quad (8)$$

for $\Psi \in F_{\lambda}^n$, $n \geq 0$.

Having defined the action of the generators on the tensor algebra we will now check that $\mathfrak{T}(F_{\lambda})$ carries the claimed $\hat{g}(\hat{A})$-module structure. First we note that $h_0$ and the $h_i$’s are defined to act diagonal on the tensor algebra. Hence the $h$’s commute with each other. Secondly, all commutation relations involving only Kac-Moody generators $e_i$, $h_i$, $f_i$ are valid by assumption. Next, we have a look at those commutation relations which are more or less trivial since they can be checked immediately on the whole tensor algebra.

$$(e_0 f_0 - f_0 e_0)(\Psi) = e_0(\omega \otimes \Psi) - \omega \otimes e_0(\Psi) = h_0(\Psi),$$

$$(e_0 f_i - f_i e_0)(\Psi) = 0,$$

$$(e_i f_0 - f_0 e_i)(\Psi) = e_i(\omega \otimes \Psi) - \omega \otimes e_i(\Psi) = 0,$$

$$(h_0 f_0 - f_0 h_0)(\Psi) = h_0(\omega \otimes \Psi) - \omega \otimes h_0(\Psi) = (h_0 - 1)(\omega \otimes \Psi) = -a_{00} f_0(\Psi),$$

$$(h_i f_0 - f_0 h_i)(\Psi) = h_i(\omega \otimes \Psi) - \omega \otimes h_i(\Psi) = h_i(\omega \otimes \Psi) = -a_{0i} f_0(\Psi).$$

Finally we check the remaining four types of commutators:

$$(h_0 f_i - f_i h_0)(1) = -f_i(1) = 0 = -a_{0i} f_i(1),$$

$$(h_0 f_i - f_i h_0)(\varphi) = -a_{0i} f_i \varphi,$$

$$(h_0 f_i - f_i h_0)(\Phi \otimes \Psi) = h_0(f_i(\Phi) \otimes \Psi + \Phi \otimes f_i(\Psi)) - f_i(h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi)$$

$$= (h_0 f_i - f_i h_0)(\Phi) \otimes \Psi + \Phi \otimes (h_0 f_i - f_i h_0)(\Psi)$$

$$= -a_{0i} f_i(\Phi \otimes \Psi) \quad \text{by induction},$$

$$(h_0 e_i - e_i h_0)(1) = 0 = a_{0i} e_i(1),$$

$$(h_0 e_i - e_i h_0)(\omega) = 0 = a_{0i} e_i(\omega),$$

$$(h_0 e_i - e_i h_0)(f_k \varphi) = h_0(\delta_{ik} h_i(\varphi) + f_k e_i(\varphi)) - e_i(-a_{0k} f_k \varphi + f_k h_0(\varphi))$$

$$= a_{0k} (e_i f_k - f_k e_i)(\varphi) + f_k (h_0 e_i - e_i h_0)(\varphi)$$

$$= a_{0k} \delta_{ik} h_i(\varphi) + a_{0i} f_k e_i(\varphi) \quad \text{by induction}.$$
\[
\begin{align*}
(h_0e_i - e_ih_0)(\Phi \otimes \Psi) &= h_0(e_i(\Phi) \otimes \Psi + \Phi \otimes e_i(\Psi)) \\
&\quad - e_i(h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi)) - \Phi \otimes \Psi) \\
&= (h_0e_i - e_ih_0)(\Phi) \otimes \Psi + \Phi \otimes (h_0e_i - e_ih_0)(\Psi) \\
&= a_0e_i(\Phi \otimes \Psi) \text{ by induction,}
\end{align*}
\]

\[
\begin{align*}
(h_i e_0 - e_0h_i)(\mathbf{1}) &= 0 = a_{i0}e_0(\mathbf{1}), \\
(h_i e_0 - e_0h_i)(\omega) &= h_i(\mathbf{1}) + a_{i0}e_0(\omega) = a_{i0} = a_{i0}e_0(\omega), \\
(h_i e_0 - e_0h_i)(f_k\varphi) &= 0 = a_{i0}e_0(f_k\varphi), \\
(h_i e_0 - e_0h_i)(\omega \otimes \Psi) &= h_i(h_0(\Psi) + \omega \otimes e_0(\Psi)) - e_0(h_i(\omega) \otimes \Psi + \omega \otimes h_i(\Psi)) \\
&= a_{i0}h_0(\Psi) + \omega \otimes (h_0e_0 - e_0h_i)(\Psi) + (h_i e_0 - e_0h_i)(\Psi) \\
&= a_{i0}e_0(\omega \otimes \Psi), \\
(h_i e_0 - e_0h_i)(f_k\varphi \otimes \Psi) &= h_i(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \\
&\quad - e_0(h_i(f_k\varphi) \otimes \Psi + f_k\varphi \otimes h_i(\Psi)) \\
&= -h_i(e_0(\varphi \otimes f_k(\Psi))) + h_i(f_k(e_0(\varphi \otimes \Psi))) \\
&\quad - a_{ik}e_0(\varphi \otimes f_k(\Psi)) + a_{ik}f_k(e_0(\varphi \otimes \Psi)) \\
&\quad + e_0(h_i(\varphi) \otimes f_k(\Psi)) - f_k(e_0(h_i(\varphi) \otimes \Psi)) \\
&\quad + e_0(\varphi \otimes f_k(h_i(\Psi))) - f_k(e_0(\varphi \otimes h_i(\Psi))) \\
&= (e_0h_i - e_0h_i)(\varphi \otimes f_k(\Psi)) \\
&\quad + f_k((h_0e_0 - e_0h_i)(\varphi \otimes \Psi)) \\
&= a_{i0}(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \text{ by induction} \\
&= a_{i0}e_0(f_k\varphi \otimes \Psi),
\end{align*}
\]

\[
\begin{align*}
(h_0e_0 - e_0h_0)(\mathbf{1}) &= 0 = a_{00}e_0(\mathbf{1}), \\
(h_0e_0 - e_0h_0)(\omega) &= h_0(\mathbf{1}) - (1 - a_{00})e_0(\omega) = a_{00} = a_{00}e_0(\omega), \\
(h_0e_0 - e_0h_0)(f_k\varphi) &= 0 = a_{00}e_0(f_k\varphi), \\
(h_0e_0 - e_0h_0)(\omega \otimes \Psi) &= h_0(h_0(\Psi) + \omega \otimes e_0(\Psi)) - e_0(-a_{00}\omega \otimes \Psi + \omega \otimes h_0(\Psi)) \\
&= a_{00}h_0(\Psi) + \omega \otimes (h_0e_0 - e_0h_0)(\Psi) \\
&= a_{00}e_0(\omega \otimes \Psi) \text{ by induction,} \\
(h_0e_0 - e_0h_0)(f_k\varphi \otimes \Psi) &= h_0(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \\
&\quad - e_0(h_0(f_k\varphi) \otimes \Psi + f_k\varphi \otimes h_0(\Psi) - f_k\varphi \otimes \Psi) \\
&= -h_0(e_0(\varphi \otimes f_k(\Psi))) + h_0(f_k(e_0(\varphi \otimes \Psi))) \\
&\quad - a_{0k}e_0(\varphi \otimes f_k(\Psi)) + a_{0k}f_k(e_0(\varphi \otimes \Psi)) \\
&\quad + e_0(h_0(\varphi) \otimes f_k(\Psi)) - f_k(e_0(h_0(\varphi) \otimes \Psi)) \\
&\quad + e_0(\varphi \otimes f_k(h_0(\Psi))) - f_k(e_0(\varphi \otimes h_0(\Psi))) \\
&\quad + e_0(\varphi \otimes f_k(\Psi)) - f_k(e_0(\varphi \otimes \Psi)) \\
&= (e_0h_0 - e_0e_0)(\varphi \otimes f_k(\Psi)) \\
&\quad + f_k((h_0e_0 - e_0h_0)(\varphi \otimes \Psi))
\end{align*}
\]
for all $\varphi \in F_\lambda$ and $\Phi, \Psi \in \mathfrak{T}(F_\lambda)$.

Now we shall prove that $\mathfrak{T}(F_\lambda)$ is indeed isomorphic to $L(\Lambda)$ as a $\mathfrak{g}(\hat{A})$-module. Denote the highest weight vector of $L(\Lambda)$ by $v_\Lambda$. Define a map $\nu : \mathfrak{U}(\tilde{n}_-) v_\Lambda \to \mathfrak{T}(F_\lambda)$ by

$$\nu(f_{i_1} \ldots f_{i_n} v_\Lambda) := f_{i_1} \ldots f_{i_n}(1)$$

where $i_1 \ldots i_n \in \{0, \ldots, n\}$, and linearity. To prove that $\nu$ reduces to a well defined $\mathfrak{g}(\hat{A})$-module homomorphism $\nu' : \mathfrak{U}(\tilde{n}_-) v_\Lambda \to \mathfrak{T}(F_\lambda)$, one has to check that the action of elements of $\mathfrak{r}_-$ on $\mathfrak{T}(F_\lambda)$ vanishes, i.e. that the Serre-relations are valid. For $f_{ij}, i,j = 1 \ldots n$, this is part of the definition. To check the remaining ones, observe that

$$(\text{ad} f_i)^m f_0)(\Psi) = f_i^m \omega \otimes \Psi,$$

so that for $i = 1 \ldots n$

$$f_{i0}(\Psi) = f_i^{1+\Lambda(h_i)} \omega \otimes \Psi = 0$$

because of lemma 10.1 of [13]. According to [9] (see also [10]) the irreducible module $L(\Lambda)$ is obtained from the Verma-module $M(\Lambda)$ by dividing out the subspace generated by the primitive vectors $f_i^{1+\Lambda(h_i)} v_\Lambda, i = 1 \ldots n$. Because of $f_i^{1+\Lambda(h_i)}(1) = f_i(1) = 0$, $\nu'$ reduces further to a map $\nu'' : L(\Lambda) \to \mathfrak{T}(F_\lambda)$. $\nu''$ is injective because the kernel of $\nu''$ would be a proper submodule of $L(\Lambda)$, and surjective because $\mathfrak{T}(F_\lambda)$ is spanned by vectors of the form

$$u_1 \omega \otimes \cdots \otimes u_n \omega = \nu(u_1(f_0) \ldots u_n(f_0) v_\Lambda)$$

where

$$u_i = F_{n_1(i)} \ldots F_{n_k(i)(i)}, \quad F_{n_1(i)} \in \mathfrak{g}(A), \quad u_i(f_0) = [F_{n_1(i)}, \ldots [F_{n_k(i)(i)}, f_0] \ldots].$$

We observe that the theorem is not altered if we replace $a_{00}$ by any nonpositive real number or $\Lambda(h_0)$ by any positive real number.

## 4 Outlook

According to a conjecture of Ginsparg [12], the special class of Borcherds algebras considered in the theorem might play a role in second quantization of a single particle theory. In this interpretation we regard the module $F_\lambda$ from above as one-particle Fock space so that $\mathfrak{T}(F_\lambda)$ comprises all multiparticle states. In other words, within a single irreducible representation of the Borcherds algebra we encounter all possible multiparticle excitations. Thus the “imaginary” generators $f_0$ and $e_0$ act as particle creation and particle annihilation operators, respectively, whereas the vector $1$ indeed deserves the name “true vacuum” in contrast to the “ground state” $\omega \in F_\lambda$.

Applying this idea to string theory one should think about the underlying Kac-Moody algebra $\mathfrak{g}(A)$ as spectrum generating algebra for the physical states of the bosonic string. Consequently, the tensor algebra $\mathfrak{T}(F_\lambda)$ would be intimately related to a string field theory. Note that in the special case of an underlying affine Lie algebra $\mathfrak{g}(A)$ we would end up with a string field theory on the group manifold associated to $\mathfrak{g}(A)$ (cf. [14]).
It is clear that the emergence of Borcherds algebras in quantum field theory is just a naïve speculation since up to now at least one important point in dealing with particles is missing. The tensor algebra $\mathcal{T}(F_\lambda)$ carries no symmetry or antisymmetry constraints at all, which means that the concept of statistics is absent. At present, work is in progress to clarify how symmetrization of $\mathcal{T}(F_\lambda)$ can be implemented into $\hat{g}(\hat{A})$ algebraically via additional relations.

In view of these possible realizations of Borcherds algebras in physics we shall finish with the useful construction of a “number operator” which counts the number of $f_0$’s (number of particles/strings) occurring in the expression for a homogeneous state vector $\Psi \in \mathcal{T}(F_\lambda)$. We are looking for an element $N$ in the Cartan subalgebra $\hat{h}(\hat{A})$ satisfying

$$N(\Psi) = n\Psi \quad \forall \Psi \in F_\lambda^n, n \geq 1,$$

or, equivalently,

$$[N, f_j] = \delta_{j0}f_j \quad \text{for } 0 \leq j \leq n.$$

The ansatz $N = \sum_{i=0}^{n} N_i h_i$ yields the following system of linear equations for the rational coefficients $N_i$:

$$\sum_{i=0}^{n} a_{ij} N_i = -\delta_{j0} \quad \text{for } 0 \leq j \leq n.$$

If $\hat{A}$ is invertible we obtain a unique solution for the number operator $N$. Note, however, that the eigenvalues of $N$ give us the number of $f_0$’s shifted by $N_0$ since we have $N(1) = N_01$ instead of $N(1) = 0$. This annoying constant may be removed by defining the “renormalized” number operator $\hat{N} := N - N_0$.

References

[1] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the monster. Proceedings of the National Academy Society USA, 83 :3068–3071, 1986.

[2] R. E. Borcherds. Monstrous Lie superalgebras. Inventiones Mathematicae, 109 :405–444, 1992.

[3] R. W. Gebert. Introduction to vertex algebras, Borcherds algebras, and the monster Lie algebra. International Journal of Modern Physics, A , 1993, to appear.

[4] J. H. Conway. The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. Journal of Algebra, 80 :159–163, 1983.

[5] R. E. Borcherds, J. H. Conway, L. Queen, and N. J. A. Sloane. A monster Lie algebra? Advances in Mathematics, 53 :75–79, 1984.

[6] I. B. Frenkel. Representations of Kac-Moody algebras and dual resonance models. In Applications of Group Theory in Theoretical Physics, pages 325–353, American Mathematical Society, Providence, 1985. Lect. Appl. Math., Vol. 21.

[7] P. Goddard and D. Olive. Algebras, lattices and strings. In J. Lepowsky, S. Mandelstam, and I. M. Singer, editors, Vertex Operators in Mathematics and Physics – Proceedings of a Conference November 10-17, 1983, pages 51–96, Springer, New York, 1985. Publications of the Mathematical Sciences Research Institute #3.
[8] R. E. Borcherds. The monster Lie algebra. *Advances in Mathematics*, **83**:30–47, 1990.

[9] R. E. Borcherds. Generalized Kac-Moody algebras. *Journal of Algebra*, **115**:501–512, 1988.

[10] K. Harada, M. Miyamoto, H. Yamada. A generalization of Kac-Moody algebras. preprint

[11] R. E. Borcherds. Central extensions of generalized Kac-Moody algebras. *Journal of Algebra*, **140**:330–335, 1991.

[12] R. Slansky. *An Algebraic Role for Energy and Number Operators for Multiparticle States*. preprint LA-UR-91-3562, Los Alamos National Laboratory, 1991.

[13] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[14] D. Gepner and E. Witten. String theory on group manifolds. *Nuclear Physics*, **B278**:493–549, 1986.