Rationalization, Quantal Response Equilibrium, and Robust Outcomes in Large Populations

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Abstract. This paper provides a robust epistemic foundation for predicting and implementing collective actions when only the proportions that take specific actions in the population matter. We apply $\Delta$-rationalizability to analyze strategic sophistication entailed in (structural) quantal response equilibrium (QRE); the former is called $\Delta(p)$-rationalization to emphasize the only requirement on first-order beliefs is that they should be consistent with the transparent knowledge of the distributions of errors in the population. We show that each QRE is a $\Delta(p)$-rationalizable outcome. We also give conditions under which the converse also holds, and prove that the condition is almost never satisfied in generic games. It implies that QRE may be too demanding as a predictor in general, and $\Delta(p)$-rationalizable outcomes can be a robust benchmark to start from.

1 Introduction

Policy-making needs prediction and implementation of collective actions. Sometimes, they concern only the proportion in a population instead of choices in the individual level; the circumstance may be uncommon so that no extant data are directly applicable, for example, the voluntary vaccination rate in the outbreak of a new pandemic. A model thrivingly used in the empirical literature is proposed by McKelvey and Palfrey [19] (referred as MP in the following). There, the population is decomposed into groups, each having representative payoffs. An individual has her idiosyncrasy, or payoff type, which influences her payoffs and is known to her only; the distributions of the idiosyncrasies are publicly known. MP introduced a solution concept called quantal response equilibrium (QRE), which is a probabilistic summary based on the commonly known idiosyncrasies distributions of pure-strategic optimal action under each type given the distribution of actions among other groups.

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1McKelvey and Palfrey [20] gives a conceptually different definition of QRE, which is latter renamed as regular QRE in Goeree et al. [11]. MP’s is then renamed as structural QRE. We focus on structural QRE; for simplicity, we omit the qualifier “structural” when no confusion is caused. Section 1.1 will discuss the difference between the two concepts.
However, QRE may not necessarily fit the problem here. In general, achieving an equilibrium requires players’ common correct beliefs about each other (Tan and Werlang [29], Aumann and Brandenburger [2], Polak [25], Battigalli and Siniscalchi [5]), yet correct beliefs are hardly guaranteed, especially in an unprecedented circumstance. An alternative is rationalizability (Bernheim [7], Pearce [24]): when a player is ignorant about others’ behaviors, she can only rely on her individual rationality, i.e., “making a choice which is justifiable by an internally consistent system of beliefs” (Bernheim [7], p.1007). Battigalli and Siniscalchi [5] generalize this idea into games with incomplete information. Their Δ-rationalization is a framework to study behavioral consequences under some explicit restrictions on the commonly known content of first-order beliefs without constraining the possible epistemic types à la Harsanyi [13]; in other words, it characterizes robustness in the sense of Bergemann and Morris [6].

In this paper, we apply Δ-rationalization into MP’s model. Instead of focusing on the classic rationalizable actions/strategies, we turn to rationalizable outcomes, i.e., distributions over actions resulting from distributions over errors and the rationalization procedure. By doing this, we make explicitly the epistemic structure behind the individual reasoning procedure that QRE entails. Since the model assumes that the distributions of the idiosyncrasies are publicly known, the only restriction on an individual’s initial belief about her opponents is that the marginal distribution on their types should coincide with it; there is no conditions on people’ beliefs on others’ behavior or the correlation between types and choices. Yet if the type spaces are large enough, that restriction leads to determinate choices under some types. Given the distribution over types, the consequence is an infimum of the proportion that some action is used. This updates the restriction on beliefs, and leads to a new infimum, etc. Finally, the iterative procedure of Δ-rationalization results in a limit of the infimums of the proportion that each action is adopted, which can be a benchmark for the estimation aforementioned.

As an illustration, suppose that a policymaker is considering whether the size of population who will voluntarily get vaccinated in a community is above the threshold of herd immunity. The situation is described as the game in Table 1: the population is separated into two groups, the row is more vulnerable than the column (for example, they correspond to the senior and younger citizens or the health-care workers and people outside the medical systems, respectively).

|               | Not vaccinated | Vaccinated |
|---------------|---------------|------------|
| Not vaccinated| 0, 1          | 7, 2       |
| Vaccinated    | 1, 16         | 3, 4       |

Table 1: A game of spontaneous vaccination

Due to the cost (e.g., risk of side effects and the time spent on administrative processes), a representative player would prefer others to get vaccinated and reduce the chance of virus spreading. Yet because of the difference in vulnerability, the benefits of “free-riding” are asymmetric. Each individual has some idiosyncrasies (e.g., having some underlying health condition which makes her eager to get a vaccinated, or feeling distrustful of vaccination due to some personal trauma) described as a real-valued random variable for each
Suppose that idiosyncrasies are independent and each has an extreme value distribution with parameter $\lambda = 0.5$, i.e., the cdf $F_i(\theta_{ik}) = \exp(-\exp(-0.5\theta_{ik}))$ for each $i \in \{\text{row, column}\}$ and $k \in \{\text{Not Vaccinated, Vaccinated}\}$. The infimum-updating process is illustrated in Figure 1. Numerically, $q_{\text{row, Not}}^n \to 0.396, q_{\text{row, Vac}}^n \to 0.604, q_{\text{col, Not}}^n \to 0.968,$ and $q_{\text{col, Vac}}^n \to 0.032$ as $n \to \infty$, which coincides with the unique QRE in this game. In other words, given that the distributions of types and people’s rationality are commonly known, intrapersonal reasoning alone leads to the QRE outcome. This provides a robust epistemic foundation for using QRE as a predictor in this case.

This paper discusses the general relationship between QRE and the reasoning structure; the latter is called $\Delta(p)$-rationalization procedure to emphasize $p$, the distribution of idiosyncrasies, which is the only restriction on people’s beliefs that does not concern the classic rationality, in the framework of $\Delta$-rationalization. Theorem 1 shows that every QRE is a $\Delta(p)$-rationalizable outcome, that is, given a QRE $\pi$, among the type-action pairs surviving the $\Delta(p)$-rationalization procedure, each type can be associated with an action optimal at it under a belief consistent with the transparent knowledge of $p$, the distribution of types, and rationality, such that based on $p$, the distribution on actions coincide with $q$.

However, not every $\Delta(p)$-rationalization procedure converges to a QRE as it does in the above example. Theorem 2 gives a sufficient condition on payoff structures for the convergence; further, they are necessary when there are multiple QREs. In $2 \times 2$ cases, they characterize a subset of games that are intensively studied in the literature. However, they are almost never satisfied in general cases. Therefore, QRE may be too demanding as a predictor in general, and the limit infimum generated by the $\Delta(p)$-rationalization procedure can be a robust benchmark to start from.

Our results may be seen as providing an epistemic foundation for applying QRE in empirical research. Also, it provides a method to test the hierarchical belief assumption in epistemic game theory. By generating a distributions of a real-valued error for each action in addition to a game (with representative payoff values), at each level in the belief hierarchy, a point in the error space of a player is associated to a subset of “consistent” action.
actions. Graphs depicting the change of the association is shown in Figure 3. Comparison between the theoretical prediction and the actual behavior in the laboratory may help to determine the depths of reasoning in a population and their relationship with the numerical values of the error, i.e., the payoff types. The QRE-inspired setting provides a baseline (i.e., common knowledge of the distributions of errors) for reasoning strategic uncertainty, which is different from Kosenkova [16]'s nonparametric inference about to quantify strategic sophistication (k-rationalizability) in first-price auctions.

1.1 Literature

As mentioned in Footnote 1, there are two versions of QRE in the literature. The original one is given in MP, which is renamed as structural QRE (sQRE) in Goeree et al [11]. MP generates McFadden [18]'s qualitative choice behavior model into quantal (i.e., discrete) choices in a game-theoretic framework for estimation using field and experimental data. MP adopts a large-population (or Nash’s mass action) scenario with private information. Each player is interpreted as a large population; the payoffs reflect “representative” or “typical” preferences of each population, while an individual may have some “idiosyncrasies” for each action and those idiosyncrasies follow a fixed (joint) distribution in the population. A sQRE is a profile of probability measures over actions generated from some Bayesian-Nash equilibrium with common prior on the idiosyncrasies.

Another version of QRE is introduced in McKelvey and Palfrey [20]. They use an axiomatic method to define quantal response functions, which describe players’ disturbed reactions to others’ (mixed) strategies, and the equilibrium is a fixed-point in the system. Later, it is renamed as regular QRE (rQRE) in Goeree et al [11].

McKelvey and Palfrey [20] claimed that the two definitions are equivalent and structural QRE is the foundation of regular QRE. Since then, researches applying QRE without distinguishing the two versions to interpreting observed behavior in various fields flourishes (see Goeree et al [12] for a survey). However, Haile et al. [15] questioned the empirical content of (structural) QRE by showing that sQRE is not falsifiable in any static game. This forces researchers to differentiate the two QREs explicitly. One solution is provided by Goeree et al [11], which redefined rQRE by putting additional restrictions on the quantal response functions and showed that some rQRE cannot be modeled as a sQRE and rQRE have empirical content.

From the decision-theoretical viewpoint, the two QREs correspond to the two models interpreting the phenomenon that in a population, the subjects’ responses over the set $A$ of alternatives to the same choice situation is governed by a probability mechanism $\pi$ (see Section 5 in Luce and Suppes [17] for a survey). Structural QRE corresponds to the random utility model, where the utility function is selected according to some probability mechanism $p$, i.e., $\pi(a_k) = p[U_k \geq U_t$ for each $t \neq k]$. Regular QRE corresponds to the constant utility model, where the utility function is fixed and the response probability is a function of it; formally, there is a fixed utility profile $u = (u(a))_{a \in A} \in \mathbb{R}^A$ and a function

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2 This differentiates MP’s model from Harsanyi [14]’s games with randomly disturbed payoffs. The latter associates a random error to each profile of actions.

3 A working paper version of Haile et al. [15] appeared in 2004.
Each response equilibrium (QRE) under a mixed-strategy profile.

Definition 1. That is, \( p \) and \( \alpha \) reflect individuals' idiosyncrasies toward each action. Formally, let preferences of the population; in addition, there is a random variable reflecting individuals' idiosyncrasies (referred to as MP in the following).

We start from a definition more general than the one given in McKelvey and Palfrey \[19\]. For simplicity, we omit "structural" and call it QRE when no confusion is caused.

In this paper, we focus on structural quantal response equilibrium (sQRE). In the following, our research can be seen as a exploration of rationalization based on sQRE.

Deeper, it is easier to make mistakes and harder to accurately predict others' behavior. In this vein, our research can be seen as a exploration of rationalization based on sQRE.

The rest of the paper is organized as follows. Section 2 provides preliminaries about sQRE. Section 3 introduces \( \Delta(p) \)-rationalization procedure; we use some example to show how it works. Section 4 contains the main results. Section 5 concludes the paper.

2 Quantal response equilibrium

In this paper, we focus on structural quantal response equilibrium (sQRE). In the following, for simplicity, we omit "structural" and call it QRE when no confusion is caused. We start from a definition more general than the one given in McKelvey and Palfrey \[19\] (referred to as MP in the following).

Let \( G = (I, (A_i, u_i))_{i \in I} \) be a static game, where \( I \) is the finite set of players and for each \( i \in I, A_i = \{a_{i1}, ..., a_{ik_i}\} \) with \( K_i \geq 2 \) is the set of actions and \( u_i : A := \prod_{j \in I} A_j \to \mathbb{R} \) is the von Neumann-Morgenstern payoff function of player \( i \). We adopt a large-population scenario, where each player \( i \in I \) is interpreted as a population and \( u_i \) the "representative" preferences of the population; in addition, there is a random variable reflecting individuals' idiosyncrasies toward each action. Formally, let \( \Theta_i = \mathbb{R}^{A_i} \) for each \( i \in I, \Theta = \prod_{j \in I} \Theta_j \), and \( p \) a probability distribution on \( \Theta \). For each \( \pi = (\pi_j)_{j \in I} \in \prod_{j \in I} \Delta(A_j) \), each \( i \in I \), and each \( k \in \{1, ..., K_i\} \), we define

\[ E_{ik}(\pi) = \{ (\theta_{il})_{l=1}^{K_i} \in \Theta_i : u_i(a_{ik}, \pi_{-i}) + \theta_{ik} \geq u_i(a_{il}, \pi_{-i}) + \theta_{il} \text{ for all } l = 1, ..., K_i \}. \]

That is, \( E_{ik}(\pi) \) is the set of realization of \( \theta_i \) under which \( a_{ik} \) is a best response.

**Definition 1.** A mixed-strategy profile \( \pi^* = (\pi^*_j)_{j \in I} \in \prod_{j \in I} \Delta(A_j) \) is called a quantal response equilibrium (QRE) under \( (G, (\Theta_j)_{j \in I}, p) \) iff for each \( i \in I \) and each \( k \in \{1, ..., K_i\} \), \( \pi^*_i(a_{ik}) = p_i(E_{ik}(\pi^*)) \).
Since \( \pi^*_i \) is a probability distribution, the following observation holds.

**Observation 1.** If \( \pi^* \) is a QRE, then for each \( i \in 1 \) and \( k, \ell \in \{1, \ldots, K_i\} \) with \( k \neq \ell \),
\[
p_i(E_{ik}(\pi^*) \cap E_{i\ell}(\pi^*)) = 0.
\]

Observation 1 suggests that, under our definition, given a static game and \( p \), QRE may not exist. The following example gives an illustration.

**Example 1.** Consider the two-person game in Table 2. Let \( p_i \) be the Dirac measure on \( \{(0,0)\} \), \( i = 1, 2 \). This game has no QRE. Indeed, for each \( \pi_2 \in \Delta(A_2) \), \( p_1(E_{1H}(\pi) \cap E_{1T}(\pi)) = 1 > 0 \), which violates Observation 1. ▲

|     | H | T |
|-----|---|---|
| H   | 1,-1| 1,1 |
| T   | 1,1 | 1,-1 |

Table 2: Example 1

To guarantee the existence of a QRE, some restrictions are needed. MP requires \( p \) to be admissible, i.e., for each \( i \), \( p_i \) has a density function \( f_i \) such that the marginal distribution of \( f_i \) exists for each \( \theta_{ik} \), \( k = 1, \ldots, K_i \) and \( E_{f_i}(\theta_i) = 0 \). In our paper, some results (e.g., Theorem 2) requires that \( f_i \) to be continuous and have the full support (i.e., \( \text{supp}f_i = \Theta_i \)) while dispenses with restrictions on \( E_{f_i}(\theta_i) \). By Brouwer’s fixed-point theorem, QRE exists under both conditions.

**Example 2.** When \( p \) has an extreme value distribution with parameter \( \lambda \geq 0 \), for each \( \pi = (\pi_i)_{i \in I} \in \prod_{i \in I} \Delta(A_i) \), each \( i \in I \), and each \( k \in \{1, \ldots, K_i\} \),
\[
p_i(E_{ik}(\pi)) = \frac{e^{\lambda u_i(a_{ik}, \pi_{-i})}}{\sum_{t=1}^{K_i} e^{\lambda u_i(a_{it}, \pi_{-i})}}
\]

This is a logistic quantal response function, one of the most popular model in the literature. MP shows that when \( \lambda \to \infty \), the corresponding QREs converge to a Nash equilibrium of \( G \). ▲

3 **\( \Delta \)-Rationalizability and QRE

3.1 Definitions

We can take \( \Theta \) as the set of payoff types. By integrating it into the “representative” model, we obtain a game with incomplete information. Formally, given a static game \( G = \langle I, (A_j, u_j)_{j \in I} \rangle \) and \( \Theta = \prod_{j \in I} \Theta_i = \prod_{j \in I} \mathbb{R}^{A_j} \), the game of incomplete information based on \( G \) is a tuple
\[
\Gamma(G) = \langle I, (A_j, \Theta_j, U_j)_{j \in I} \rangle,
\]
where for each \( \theta = (\theta_j)_{j \in I} \in \Theta \), \( a = (a_{jk})_{j \in I} \in A \), and \( i \in I \), \( U_i(\theta, a) = u_i(a) + \theta_{ik} \). Since player \( i \)'s payoff is only influence by her own payoff type \( \theta_i \), this is a game with private value.

Let \( p \) be a probability measure on \( \Theta \). We now have to incorporate \( p \) into this framework. At first glance, it looks like that no one needs to know \( p \). Indeed, for each \( i \in I \), an individual in population \( i \) even does not need to know \( p_i \); if she knows \( \pi^*_i \), she just chooses a best response in \( A_i \) based on the realization of her payoff types. However, achieving an equilibrium requires individual’s common correct beliefs about others’ behavior (Tan and Werlang [29], Aumann and Brandenburger [2], Polak [25]), and justification of \( \pi^*_i \) needs the knowledge of the distribution of \( p \) on \( \Theta_{-i} \) conditional on each \( \theta_i \in \Theta_i \). This is a transparent restriction on each individual’s first-order beliefs. Formally, for each \( i \in I \) and \( \theta_i \in \Theta_i \), the set of beliefs at \( \theta_i \) consistent with \( p \) is

\[
\Delta^\theta_i(p) = \{ \mu^i \in \Delta(\Theta_{-i} \times A_i) : \text{marg}_{\Theta_{-i}} \mu^i = p(\cdot | \theta_i) \}. \tag{1}
\]

Here, \( \mu^i \) is a belief about the types and actions of \( i \)'s opponents under payoff type \( \theta_i \). There is no restriction on the joint distribution; the only requirement is that \( \mu^i \)'s marginal distribution on \( \Theta_{-i} \) should coincides with the distribution of \( p \) on \( \Theta_{-i} \) conditional on \( \theta_i \).

When \( p_i \)'s are independent, \( \mu^i \in \Delta^\theta_i(p) \) if and only if \( \text{marg}_{\Theta_{-i}} \mu^i = p_{-i} = \prod_{j \neq i} p_j \). Definition 1 suggests that \( p_{-i} \) does not vary along \( \theta_i \), otherwise the opponents’ distributions may not be stable as \( \pi^*_i \). Hence we focus on cases with independent \((p_j)_{j \in I}\).

**Definition 2.** Consider the following procedure, called \( \Delta(p) \)-rationalization procedure:

**Step 0.** For each \( i \in I \), \( \Sigma^0_{i, \Delta(p)} = \Theta_i \times A_i \).

**Step \( n + 1 \).** For each \( i = 1, 2 \) and each \( (\theta_i, a_i) \in \Theta_i \times A_i \) with \( (\theta_i, a_i) \in \Sigma^n_{i, \Delta(p)} \), \( (\theta_i, a_i) \in \Sigma^{n+1}_{i, \Delta(p)} \) iff there is some \( \mu^i \in \Delta^\theta_i(p) \) such that

1. \( a_i \) is a best response to \( \mu^i \) under \( \theta_i \), and
2. \( \mu^i (\Sigma^n_{i, \Delta(p)}) = 1 \), where \( \Sigma^n_{i, \Delta(p)} := \prod_{j \neq i} \Sigma^n_{j, \Delta(p)} \).

Finally, let \( \Sigma^\infty_{i, \Delta(p)} = \cap_{n \geq 0} \Sigma^n_{i, \Delta(p)} \) for \( i = 1, 2 \).

\( \Delta(p) \)-rationalization procedure is a special case of Battigalli and Siniscalchi [5]'s \( \Delta \)-rationalization procedure; \( p \) emphasizes that the restriction on the first-order belief is given by \( p \) as defined in \( \{1\} \). This procedure iteratively removes \( (\theta_i, a_i) \) where \( a_i \) cannot be rationalized under \( \theta_i \) by any belief in \( \Delta^\theta_i(p) \) supported by the outcomes in the previous stage. Finally, every \( (\theta_i, a_i) \in \Sigma^\infty_{i, \Delta(p)} \) is rationalizable based on common knowledge of rationality and the transparent restriction in \( \{1\} \).

Figure 2 illustrates how this procedure works on a game with \( A_i = \{T, U\} \) for some player \( i \in I \). Suppose that \( p_j \)'s are independent and the square (a subset of \( \mathbb{R}^{A_i} \)) is the support of \( p_j \). Since \( \Sigma^0_{i, \Delta(p)} = \Theta_i \times A_i \), the support of others’ belief about \( i \) can be arbitrary; on the left-hand side of Figure 2 we list four candidates. However, under some type \( \theta_i \in \Theta_i \), whatever \( i \)'s belief \( \mu^i \) is, the best response can only be \( T \) (or \( U \)). Definition 2 implies that \( \Sigma^1_{i, \Delta(p)} \) may look like something on the right-hand side of Figure 2, which
impose restrictions on other players’ second order belief about $i$ (the white area are still “free”, that is, under each payoff type the best response can be both $T$ and $U$, depending on the belief). Since the marginal distribution of those beliefs on $\Theta_i$ should coincide with $p_i$, it implies that the first step of the $\Delta(p)$-rationalization procedure generates infimums for the probabilities that $T$ and $U$ are used. Repeating this argument, we can see that the $\Delta(p)$-rationalization procedure generates sequences for the infimums of $T$ and $U$, respectively. The next subsection gives several numerical examples to show how those sequences behave.

The literature of rationalization used to focus on rationalizable (pure) actions (see Battigalli and Bonanno [3], Perea [23], Dekel and Siniscalchi [9], and Battigalli et al. [4] for surveys). In the literature of QRE, it is usually assumed that each $p_i$ has a full support. Under this assumption, every action is rationalizable, which makes rationalizable actions less attractive. Instead, here our focus is rationalizable outcomes, which means a profile of distributions on $A_i$’s such that each is supportable by $\Sigma_{i,\Delta(p)}^\infty$. The formal definition is as follows.

**Definition 3.** A probability measure $\pi_i \in \Delta(A_i)$ is called a $\Delta(p)$-rationalizable distribution for (the population of) player $i$ iff there is a measurable $s_i : \Theta_i \rightarrow A_i$ such that

1. $p_i[s_i^{-1}(a_i)] = \pi_i(a_i)$, and
2. $(\theta_i, s_i(\theta_i)) \in \Sigma_{i,\Delta(p)}^\infty$.

A profile $\pi^*_j \in \prod_{j \in I} \Delta(A_j)$ is called a $\Delta(p)$-rationalizable outcome iff each $\pi_i (i \in I)$ is $\Delta(p)$-rationalizable distribution.
Our examples in the next subsections will illustrate what $\Delta(p)$-rationalizable distributions and outcomes look like and how they are connected with the aforementioned sequences of infimums.

3.2 Examples

Example 3. Consider the game $G$ in Table 3 and the $p = p_1 \times p_2$, where each $p_i$ is the uniform distribution on $([-2, 2] \times [-2, 2])$. The unique QRE of this game is $\pi^* = (0.5H + 0.5T, 0.5H + 0.5T)$, which coincides with the Nash equilibrium. Further, we show that, for each $i = 1, 2$, the only $\Delta(p)$-rationalizable distribution is $\pi_i^*$. In other words, $\Delta(p)$-rationalization procedure converges to $\pi^*$.

|     | $H$ | $T$ |
|-----|-----|-----|
| $H$ | 1,0 | 0,1 |
| $T$ | 0,1 | 1,0 |

Table 3: Example 3

![Figure 3: The $\Delta(p)$-rationalization procedure converges to the QRE](image)

First, let $i \in I$ and $\theta_i = (\theta_{iH}, \theta_{iT}) \in \Theta_i$. Then $(\theta_i, H) \in \Sigma^1_{i\Delta(p)}$ if and only if for some $\pi H + (1 - \pi) T \in \Delta(A_{-i})$, $\pi(1 + \theta_{iH}) + (1 - \pi) \theta_{iT} \geq \pi \theta_{iT} + (1 - \pi) (1 + \theta_{iT})$, or, equivalently

$$\theta_{iH} - \theta_{iT} \geq 1 - 2\pi \quad (2)$$

Since $\pi \in [0,1]$, it follows that when $\theta_{iH} - \theta_{iT} \geq 1 - 2 \times 0 = 1$, definitely $(\theta_i, H) \in \Sigma^1_{i\Delta(p)}$. Therefore, the types in the green area in in Figure 3(1) can only be associated with $H$ in each player’s second-order belief. Similarly, when $\theta_{iH} - \theta_{iT} \leq 1 - 2 \times 1 = -1$, definitely $(\theta_i, T) \in \Sigma^1_{i\Delta(p)}$; this is depicted by the red area in Figure 3(1). Therefore, the types in the green area in in Figure 3(1) can only be associated with $H$. Every type in between them (i.e., in the white area) can be associated with both $H$ and $T$. Therefore, the measures of the green and red areas provide the infimums of the probabilities that each action is used in players’ second-order beliefs.
To be more specific, note that the probability measure of each area is $\frac{9}{32}$. It means that to satisfy $\mu^i \in \Delta^{q_i}(p)$ and $\mu^i(\Sigma_{-i\Delta(p)}) = 1$, both $\text{margin}_{A_{i\Delta(p)}}(H)$ and $\text{margin}_{A_{i\Delta(p)}}(T)$ are at least $\frac{9}{32}$; in other words, the value of $\pi$ in inequality (2) is between $\frac{9}{32}$ and $\frac{23}{32}$. Under this new restriction, inequality (2) implies that when $\theta_{ii} - \theta_{IT} \geq 1 - 2 \times \frac{9}{32} = \frac{7}{16}$, $(\theta_i,H) \in \Sigma_{i\Delta(p)}$. Similarly, when $\theta_{ii} - \theta_{IT} \leq 1 - 2 \times \frac{23}{32} = -\frac{7}{16}$, $(\theta_i,T) \in \Sigma_{i\Delta(p)}$. Those areas are the green and red ones in Figure 3 (2). Now $\pi$ gains new restrictions and its range becomes narrower. In the same vein, the range of $\pi$ derived from $\Sigma_{i\Delta(p)}$ is shown in Figure 3 (3).

In general, let $\pi^n$ and $\pi^\|n$ be the infimum and the supremum of the probability of using $H$ at step $n$ (i.e., the greatest lower bound and the least upper bound of the range of $\pi$ in inequality (2)). By mathematical induction, we can show that (a) $\pi^n + \pi^\|n = 1$, (b) $\pi^n \leq \frac{1}{2} \leq \pi^\|n$, and (c) $(\pi^n)_n$ is non-decreasing and $(\pi^\|n)_n$ is non-increasing. The measure of the area in $\Theta_i$ which is only associated to $H$ is $\frac{(3 + 2\pi^n)^2}{32}$ and the measure of the area for $T$ is $\frac{(5 - 2\pi^n)^2}{32}$. Since, by inductive hypothesis, $\pi^n + \pi^\|n = 1$, it follows that $\frac{(3 + 2\pi^n)^2}{32} = \frac{(5 - 2\pi^n)^2}{32}$, and

$$\pi^{n+1} = \frac{(3 + 2\pi^n)^2}{32}, \pi^\|n+1 = 1 - \frac{(5 - 2\pi^n)^2}{32}.$$ 

Hence (a) - (c) hold. Due to (b) and (c), both $(\pi^n)_n$ and $(\pi^\|n)_n$ converge and $\lim_{n \to \infty} \pi^n \leq \frac{1}{2} \leq \lim_{n \to \infty} \pi^\|n$. Further, (a) implies that $\lim_{n \to \infty} \pi^n = \lim_{n \to \infty} \pi^\|n = \frac{1}{2}$. \(\blacktriangleleft\)

Since the QRE in Example 3 coincides with the Nash equilibrium, one may conjecture that the $\Delta(p)$-rationalization procedure converges to the latter as well. The following example shows that it is not the case.

**Example 4.** Consider the game in Table 4. We assume that each $\theta_{ik}$ has the extreme value distribution with $\lambda = 10$. Note that for each $\alpha, \beta \in [0,1]$, $(\alpha U + (1 - \alpha)D, \beta L + (1 - \beta)R)$ is a Nash equilibrium. The only perfect equilibrium is $(D, R)$. In contrast, the unique QRE is approximately $(0.5U + 0.5D, 0.5L + 0.5R)$. We show that the $\Delta(p)$-rationalization procedure converges to the QRE.

|   | L | C | R |
|---|---|---|---|
| U | 1,1 | 0,0 | 1,1 |
| M | 0,0 | 0,0 | 0,5 |
| D | 1,1 | 5,0 | 1,1 |

Table 4: Example 4

Let player 1’s belief about player 2’s behavior $\pi_2 = p_2L + q_2C + (1 - p_2 - q_2)R$. The probabilities that player 1’s best response is $U$, $M$, or $D$ are

$$p_1(R_{1U}(\pi)) = \frac{e^{10(1-q_2)}}{e^{10(1-q_2)} + 1 + e^{10(1+4q_2)}} (3)$$

---

This is a special case of the game in Table 1 of MP, with $A = B = 5$. 

10
\[ p_1(R_{1M}(\pi)) = \frac{1}{e^{10(1-q_2)} + 1 + e^{10(1+4q_2)}} \]  \tag{4} \\
\[ p_1(R_{1D}(\pi)) = \frac{e^{10(1+4q_2)}}{e^{10(1-q_2)} + 1 + e^{10(1+4q_2)}} \]  \tag{5} 

All of the functions rely upon \( q \) only. Since the game is symmetric, given player 2’s belief about player 1 is \( \pi_1 = p_1U + q_1M + (1-p_1-q_1)D \), the probabilities that player 2’s best response is \( L, C, R \) are obtained from (3) - (5) by replacing \( q_2 \) with \( q_1 \), respectively. Therefore, \( q_1 = q_2 \) in the QRE, or, equivalently, \( q = q_1 = q_2 \) is a fixed point of (4). So, we only need to study the behavior of \( q_1 \) and \( q_2 \) in the \( \Delta(p) \)-rationalization procedure. We rephrase (4) as a one-variable function \( f : [0, 1] \rightarrow [0, 1] \) such that for each \( q \in [0, 1] \),

\[ f(q) = \frac{1}{e^{10(1-q)} + 1 + e^{10(1+4q)}}. \]

Consider the \( \Delta(p) \)-rationalization procedure. We use \( \underline{q}^n_i \) and \( \overline{q}^n_i \) to denote the infimum and the supremum of \( q_i \), \( i = 1, 2 \). In the first step, each \( q_i \) can be any number in \([0, 1]\), so \( \underline{q}^0_i = 0 \) and \( \overline{q}^0_i = 1 \) for \( i = 1, 2 \). Since \( f \) is decreasing on \([0, 1]\), it follows that \( \underline{q}^1_i = f(\overline{q}^0_i) = f(q^0_i) \) and \( \overline{q}^1_i = f(\underline{q}^0_i) = f(q^0_i) \). The same argument implies that for each \( n \in \mathbb{N} \),

\[ q^{n+1}_i = f(\overline{q}^n_i) \quad \text{and} \quad \overline{q}^{n+1}_i = f(q^n_i). \]  \tag{6}

Since \(|f'(x)| < 1\) on \([0, 1]\), \( f \) is a contraction mapping on a compact set in \( \mathbb{R} \). Hence the process in (6) converges to the unique fixed point of \( f \) on \([0, 1]\), i.e., the probability of using \( M \) and \( C \) in QRE. Therefore, the \( \Delta(p) \)-rationalization procedure converges to the QRE. ▲

The following example differs from the previous two since, as the parameter varies, there may be multiple QREs. We apply the \( \Delta(p) \)-rationalization procedure on the game with different parameter values and see what kind of distributions are \( \Delta(p) \)-rationalizable. It will provide some hint for the main results in the next section.

Example 5. Consider the vaccination game in Section 1, whose game matrix is reposted in Table 5. From the viewpoint of QRE (and \( \Delta(p) \)-rationalization procedure) this game is equivalent to the asymmetric chicken game studied in Goeree et al. [12] (pp. 25-26). Here, as in Section 1, we follow them and assume that each \( \Theta_i = \mathbb{R}^2 \) and \( \theta_{ik} \)'s are independent and has an extreme value distribution with parameter \( \lambda \). The relationship between QREs and the value of \( \lambda \) is summarized in Figure 2, which is a copy of Figure 2.4 in Goeree et al. [12], p.25. When \( \lambda \) is small, e.g., \( \lambda = 1/2 \) (i.e., \( \lambda/(1+\lambda) = 1/3 \)), there is a unique QRE; while for big values of \( \lambda \), e.g., \( \lambda = 4 \) (i.e., \( \lambda/(1+\lambda) = 0.8 \)), there are multiple QREs. We choose these two values of \( \lambda \) and see what outcomes the \( \Delta(p) \)-rationalization procedure generates.

We use \( q^n_{ik} \) to denote the infimum of the probability that player \( i \) uses action \( k \) at round \( n \). As in the previous examples, \( q^0_{ik} = 0 \) for each \( i \) and \( k \). For \( n \geq 0 \), the recursive relations are:

\[ q^{n+1}_{1T} = \frac{1}{1 + e^{\lambda(1-5q^{n}_{2S})}}, \quad q^{n+1}_{1S} = \frac{1}{1 + e^{\lambda(4-5q^{n}_{2T})}}. \]
When $\lambda = 1/2$, they converge to the unique QRE. The convergence process has been shown in Figure 1. The speed is relatively fast: in less than 15 steps, all $q_{nk}^n$'s are quite close to the limit. When $\lambda = 4$, $q_{1T}^n \to 0.018, q_{1S}^n \to 1.613 \times 10^{-7}, q_{2T}^n \to 3.63 \times 10^{-21}$, and $q_{2S}^n \to 0.018$ as $n \to \infty$; by looking at Figure 4 one may notice that when $\lambda = 4$, the game has multiple QREs, and each $q_{nk}^n$ converges to the smallest probability that action $k$ of player $i$ is used in all QREs. ▲

One may imply from the above examples that $\Delta(p)$-rationalization procedure "converges" to QRE. To be specific, the infimum of the probability that an action is used in all $\Delta(p)$-rationalizable outcomes seems coincide with the infimum of the probability the action is used in all QRE; especially, when the QRE is unique, the $\Delta(p)$-rationalization procedure seems to converge to the QRE. This conjecture is rejected by the following example.

**Example 6.** Consider the asymmetric Matching-Pennies style game in Table 6. Here, $p = \bigotimes_{i=1,2,k=H,T} p_{ik}$, where each $p_{ik}$ is the extreme value distribution with $\lambda = 5$.

This game has only one QRE (see Goeree et al. [12], Chapter 2.2). The recursive func-
Table 6: Example 6

|   | \( H \) | \( T \) |
|---|---|---|
| \( H \) | 9, 0 | 0, 1 |
| \( T \) | 0, 1 | 1, 0 |

Equations are as follows:

\[
q_{1H}^{n+1} = \frac{1}{1 + e^{5-50q_{1T}^n}} \quad q_{1T}^{n+1} = 1 - \frac{1}{1 + e^{50q_{1T}^n - 45}} \tag{7}
\]

\[
q_{2H}^{n+1} = \frac{1}{1 + e^{5-10q_{1T}^n}} \quad q_{2T}^{n+1} = 1 - \frac{1}{1 + e^{10q_{1T}^n - 5}}
\]

Through some calculation, it can be seen that

\[
q_{1H}^n \to 0.00932805, \quad q_{1T}^n \to 1.2467 \times 10^{-24}, \quad q_{2H}^n \to 0.006692805, \quad q_{2T}^n \to 0.0073424,
\]

which is not a QRE since the sum of each pair is strictly less than 1. Actually, this is a fixed point of the system of equations in (7) near 0.\[5\]

\[\Delta(p)\]-rationalizability will be intensively investigated in the next section.

4 Relations between QRE and \( \Delta(p) \)-rationalizability

4.1 QRE \( \rightarrow \) \( \Delta(p) \)-rationalizability

Our first result is that every QRE is a \( \Delta(p) \)-rationalizable outcome, or, equivalently, each QRE mixed strategy is a \( \Delta(p) \)-rationalizable distribution. This can be seen as the parallel of the classic result that every Nash equilibrium action is rationalizable (see Bernheim [7]).

Theorem 1. Consider a static game \( G = (I, (A_j, u_j)_{j \in I}) \) and \( (\Theta_j)_{j \in I}, p \) with \( p_j \)'s independent. Each QRE is a \( \Delta(p) \)-rationalizable outcome.

Proof. Let \( \pi = (\pi_j)_{j \in I} \) be a QRE under \( (G, (\Theta_j)_{j \in I}, p) \). We will construct a a profile of random variables \( (s_j : \Theta_i \to A_j)_{j \in I} \) such that for each \( i \in I \), \( p_i(s_i^{-1}(a_i)) = \pi_i(a_i) \) for each \( a_i \in A_i \) and \( (\theta_i, s_i(\theta_i)) \in \sum_{i}^{\Delta(p)} \) for each \( \theta_i \in \Theta_i \).

Recall that for each \( i \in I \) and \( k \in \{1,\ldots,K_i\} \), \( E_{ik}(\pi) \) is the set of \( \theta_i \)'s under which \( a_{ik} \) is a best response to \( \pi_{-i} \). For each \( i \in I \) and \( \theta_i \in \Theta_i \), we define \( A_i^\pi(\theta_i) = \{a_{ik} \in A_i : \theta_i \in E_{ik}(\pi)\} \), i.e., the set of best responses of player \( i \) again \( \pi_{-i} \) under \( \theta_i \). We consider a mapping \( s_i : \Theta_i \to A_i \) such that \( s_i(\theta_i) \in A_i^\pi(\theta_i) \). Note that for a “boundary” \( \theta_i \), i.e., \( |A_i^\pi(\theta_i)| > 1 \), \( s_i(\theta_i) \) can be any action in \( A_i^\pi(\theta_i) \); this does not cause any problem since, due to Observation 1, those “boundary” states form a null set with respect to \( p_i \).

\[5\]The numerical calculation in Examples 5 and 6 are carried out by Mathematica. Codes are available under request.
and the probability the action is used in all QRE. This subsection is devoted to exploring that an action is used in all \( \Delta \).

Examples in Section 3.2 suggest that the converse of Theorem 1 may hold under some condition. As remarked after Example 5, it seems that the infimum of the probability that an action is used in all \( \Delta(p) \)-rationalizable distributions coincide with the infimum of the probability the action is used in all QRE. This subsection is devoted to exploring this conjecture.

We first have to formulate the conjecture. Consider a static game \( G = \langle I, (A_i, u_i)_{i \in I} \rangle \) and \( \langle (\Theta_i)_{i \in I}, p \rangle \). In this subsection, we assume that \( p_i \)'s are independent and for each \( i \in I \), \( p_i \) has a continuous density function \( f_i \) which has the full support (i.e., \( \text{supp} f_i = \mathbb{R}^{A_i} \)). We use \( Q(G, p) \) to denote the set of QREs. Brouwer’s fixed point theorem guarantees that \( Q(G, p) \neq \emptyset \). For each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), we define

\[
q_{ik} = \inf \{ q_i(a_{ik}) : q_i \in \text{Proj}_{\Delta(A_i)} Q(G, p) \}.
\]

For each \( i \in I \), we use \( S_i \) to denote the set of all random variables \( s_i : \Theta_i \rightarrow A_i \). For each \( n \geq 0 \), let \( S_i^n = \{ s_i \in S_i : (\theta_i, s_i(\theta_i)) \in \sum_{i, \Delta(p)}^n \text{ for each } \theta_i \in \Theta_i \} \), i.e., the “restriction” of \( S_i \) on the \( n \)-th order \( \Delta(p) \)-rationality. For each \( k \in \{1, \ldots, K_i\} \), we define

\[
q_{ik}^n = \inf \{ p_i(s_i^{-1}(a_{ik})) : s_i \in S_i^n \}.
\]

As in Section 3.2, \( q_{ik}^n \) is interpreted as the infimum of the probability that \( i \) uses action \( a_{ik} \) in step \( n \) of the \( \Delta(p) \)-rationalization process. It is clear that \( q_{ik}^0 = 0 \) since \( \sum_{i, \Delta(p)}^0 = \Theta_i \times A_i \) and the random variable \( s_i \) which assigns \( a_{it} \) (\( t \neq k \)) to each \( \theta_i \) is in \( S_i^0 \).

The investigation is decomposed into the following three steps:

(a) We show that for each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), the sequence \( (q_{ik}^n)_n \) is bounded and non-decreasing. Hence, there is some \( q_{ik}^* \) such that \( \lim_{n \rightarrow \infty} q_{ik}^n = q_{ik}^* \). Also, for each \( i \in I \) and \( n \in \mathbb{N} \), \( \sum_{k=1}^{K_i} q_{ik}^n \leq 1 \), and consequently \( \sum_{k=1}^{K_i} q_{ik}^* \leq 1 \).

(b) We show that for each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), \( q_{ik}^* \leq q_{ik}^* \).

(c) We gives a sufficient condition, under which for each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), \( q_{ik}^* \) is the probability that action \( a_{ik} \) is used in some QRE. Note that for \( t \neq s \), \( q_{ik}^* \) and \( q_{is}^* \)
may correspond to different QREs. In addition, we show that when \(|Q(G, p)| > 1\), the condition is also necessary.

Section 4.2.1 shows (a) (Proposition 1) and (b) (Proposition 2). Section 4.2.2 gives the condition in (c) (Theorem 2). Section 4.2.3 discusses how strict the condition is.

4.2.1 Properties of the infimum sequences

**Proposition 1.** For each \(i \in I \) and \( k \in \{1, \ldots, K_i\} \), the sequence \((q^n_{ik})_n\) is bounded and non-decreasing. Hence, there is some \(q^*_n\) such that \(\lim_{n \to \infty} q^n_{ik} = q^*_n\). Also, for each \(i \in I \) and \(n \in \mathbb{N}\), \(\sum_{k=1}^{K_i} q^n_{ik} \leq 1\).

**Proof.** We first show the statement for \(2 \times 2\) games and then generalize the idea arbitrary cases.

|       | \(L\) | \(R\) |
|-------|-------|-------|
| \(T\) | \(\alpha_1, \alpha_2\) | \(\beta_1, \beta_2\) |
| \(U\) | \(\gamma_1, \gamma_2\) | \(\delta_1, \delta_2\) |

Table 7: A \(2 \times 2\) game

For each \(i \in I \) and \(s, t \in \{1, \ldots, K_i\} \), we let \(\overline{H}^{is}_i = \max_{a_{-i} \in A_{-i}}(u_i(a_{is}, a_{-i}) - u_i(a_{it}, a_{-i}))\) and \(\underline{H}^{is}_i = \min_{a_{-i} \in A_{-i}}(u_i(a_{is}, a_{-i}) - u_i(a_{it}, a_{-i}))\). It is easy to see that \(\overline{H}^{is}_i = -\underline{H}^{is}_i\), or, equivalently, \(H^{is}_i = -\overline{H}^{is}_i \). \(\overline{H}^{is}_i\)'s are the pillars to determine the area in \(\Theta_i\) that “definitely” associated with \(t\) in each step of the \(\Delta(p)\)-rationalization procedure.

Consider the game in Table 5. Without loss of generality, assume that \(\overline{H}^{TU}_1 = \gamma_1 - \alpha_1 \geq \overline{H}^{TU}_2 = \delta_1 - \beta_1\) and \(\overline{H}^{LR}_2 = \beta_2 - \alpha_2 \geq \overline{H}^{LR}_2 = \delta_2 - \gamma_2\). We take player 1’s viewpoint. For player 1’s belief \(qL + (1 - q)R, q \in [0, 1]\), about player 2, at each \(\theta_1 = (\theta_{1T}, \theta_{1U})\) satisfying \(q\alpha_1 + (1 - q)\beta_1 + \theta_{1T} \geq q\gamma_1 + (1 - q)\delta_1 + \theta_{1U}\), i.e., \(\theta_{1T} - \theta_{1U} \geq q(\gamma_1 - \alpha_1) + (1 - q)(\delta_1 - \beta_1)\), player 1 would choose \(T\). Since \(q \in [0, 1]\), when \(\theta_{1T} - \theta_{1U} \geq \max_{q \in [0, 1]}[q(\gamma_1 - \alpha_1) + (1 - q)(\delta_1 - \beta_1)] = \gamma_1 - \alpha_1 = \overline{H}^{TU}_1\), i.e., when

\[
\theta_{1T} - \theta_{1U} \geq (1 - q^0_{2R})\overline{H}^{TU}_1 + q^0_{2R}\overline{H}^{TU}_1,
\]

player 1 should choose \(T\). We use \(E^1_{TU}\) to denote the set of \(\theta_1\)'s satisfying \(9\). Therefore, for every interior \(\theta_1 \in E^1_{TU}\) (i.e., the strict inequality holds for \(\theta_1\) in \(9\)) \((\theta_1, U) \notin \sum^1_{1, \Delta(p)}\). Note that since \(p_1\) is absolutely continuous, for those boundary \(\theta_1\)'s, though \((\theta_1, U)\) still in \(\sum^1_{1, \Delta(p)}\), they have no essential influence to the outcome. This argument implies that, for any \(s_1 : \Theta_1 \to A_1 \in S^1, p_1(s_1^{-1}(T)) > p_1(E^1_{TU})\). Also, it is easy to see that for all \(\theta_1 \notin E^1_{TU}\), \((\theta_1, U) \in \sum^1_{1, \Delta(p)}\) because they can be supported by some belief \(\mu^1\). Therefore,

\[
q^1_{1T} = p_1(E^1_{TU}) = p_1[\theta_{1T} - \theta_{1U} \geq (1 - q^0_{2R})\overline{H}^{TU}_1 + q^0_{2R}\overline{H}^{TU}_1],
\]
In the same vein, we can see that

\[ q^{1}_{1U} = p_1[\theta_{1T} - \theta_{1U} \geq q^0_2H_1^{TU} + (1 - q^0_2)H_1^{TU}], \]
\[ q^{1}_{2L} = p_2[\theta_{2L} - \theta_{2R} \geq (1 - q^0_{1U})H_2^{LR} + q^0_{1U}H_2^{LR}], \]
\[ q^{1}_{2R} = p_2[\theta_{2L} - \theta_{2R} \leq q^0_1H_2^{LR} + (1 - q^0_1)H_2^{LR}]. \]

At the second round, each player has to update her belief based on the outcome of the first round. Again, we take the viewpoint of player 1. Still, player 1 chooses \( q \) by the inductive hypothesis, i.e., \( q \) becomes smaller: for each \( \mu \in \Delta(\Theta_2 \times A_2) \) with \( \mu(\Sigma_2\Delta(p)) = 1 \) and \( \text{marg}_{\Theta_2}H^1 = p_2 \), the \( q \) generated by \( \mu \) is in the interval \([q^2_{2R}, 1 - q^2_{2L}]\) \(^6\) Hence \( E^2_{TU} = \{\theta_1 \in \Theta_1 : \theta_{1T} - \theta_{1U} \geq (1 - q^2_{2R})H_1^{TU} + q^2_{2R}H_1^{TU}\} \). In general, for \( n \geq 0 \),

\[ q^{n+1}_{1T} = p_1[\theta_{1T} - \theta_{1U} \geq (1 - q^n_{2R})H_1^{TU} + q^n_{2R}H_1^{TU}], \quad (10) \]
\[ q^{n+1}_{1U} = p_1[\theta_{1T} - \theta_{1U} \leq q^n_{2L}H_1^{TU} + (1 - q^n_{2L})H_1^{TU}], \quad (11) \]
\[ q^{n+1}_{2L} = p_2[\theta_{2L} - \theta_{2R} \geq (1 - q^n_{1U})H_2^{LR} + q^n_{1U}H_2^{LR}], \quad (12) \]
\[ q^{n+1}_{2R} = p_2[\theta_{2L} - \theta_{2R} \leq q^n_{1T}H_2^{LR} + (1 - q^n_{1T})H_2^{LR}], \quad (13) \]

with \( q^0_i = 0 \) for each \( i \) and \( k \).

Since each \( q^n_{ik} \in [0, 1] \), it is clear that they are bounded. We show the monotonicity part, i.e., \( q^{n+1}_{ik} \geq q^n_{ik} \) for each \( i, k \) and \( n \), inductively. Since \( f_1 \) is continuous and has the full support, it is clear that \( q^1_{1T} = p_1[\theta_{1T} - \theta_{1U} \geq H_1^{TU}] \geq 0 = q^0_{1T} \). Similarly, it can be seen that \( q^1_{ik} \geq q^0_{ik} \) for all \( i \) and \( k \). Suppose that for some \( n > 0 \), \( q^n_{ik} \geq q^{n-1}_{ik} \) for all \( i \) and \( k \). Now we show that it also holds for \( n + 1 \). Since

\[ q^n_{1T} = p_1[\theta_{1T} - \theta_{1U} \geq (1 - q^{n-1}_{2R})H_1^{TU} + q^{n-1}_{2R}H_1^{TU}], \]

and

\[ q^{n+1}_{1T} = p_1[\theta_{1T} - \theta_{1U} \geq (1 - q^n_{2R})H_1^{TU} + q^n_{2R}H_1^{TU}], \]

by the inductive hypothesis, i.e., \( q^n_{2R} \geq q^{n-1}_{2R} \), that \( (1 - q^{n-1}_{2R})H_1^{TU} + q^{n-1}_{2R}H_1^{TU} \geq (1 - q^n_{2R})H_1^{TU} + q^n_{2R}H_1^{TU} \). Hence \( E^n_{TU} \subseteq E^{n+1}_{TU} \) i.e., \( q^n_{1T} \leq q^{n+1}_{1T} \). Similarly, we can see that \( q^n_{ik} \leq q^{n+1}_{ik} \) for all \( i \) and \( k \). By induction, we have shown the first statement. Hence, there is some \( q^*_ik \) such that \( \lim_{n \to \infty} q^n_{ik} = q^*_ik \).

It is easy to see that each \( q^*_ik \) is the measure of the area under or above some line on the \( \theta_{1T} - \theta_{1U} \) \( (\theta_{2L} - \theta_{2R}) \) space, as shown in Figure 2. Geometrically, the first statement implies that the intercepts of those boundary lines converge to a point “in the middle”.

\(^6\) Note that since \( (1 - q^0_{1U})H_2^{LR} + q^0_{1U}H_2^{LR} = H_2^{LR} \geq H_1^{LR} = q^0_{1T}H_2^{LR} + (1 - q^0_{1T})H_2^{LR} \), it follows that \( q^1_{2L} \geq q^1_{2R} \leq 1 \), i.e., \( q^1_{2L} = 1 - q^1_{2R} \). This outcome will be generalized in the following.
Next, we show that \( q_{1U}^n + q_{1L}^n \leq 1 \) and \( q_{2L}^n + q_{2R}^n \leq 1 \) for each \( n \in \mathbb{N} \). The statement holds for \( n = 1 \) since \( H_1^{TU} \leq H_1^{LU} \) and \( H_2^{LR} \leq H_2^{TR} \). Suppose it holds for some \( n \in \mathbb{N} \). Since \( q_{2L}^n + q_{2R}^n \leq 1 \), \( q_{2L}^n \leq 1 - q_{2R}^n \), it follows from (10) and (11) that \( q_{1U}^{n+1} + q_{1L}^{n+1} \leq 1 \). Similarly, \( q_{2L}^{n+1} + q_{2R}^{n+1} \leq 1 \). Here we have shown the first statement for \( 2 \times 2 \) games.

Now we show how to generalize the method. Consider a general game \( G, (\Theta_i)_{i \in I}, p(\cdot) \). For each \( i \in I \) and \( k, t \in \{1, \ldots, K_i\} \), let \( E_{i,1}^{kt} = \{ \theta_i : \theta_{ik} - \theta_{it} \geq H_i^{kt} \} \) and \( E_{i,1}^{k} = \cap_{s=1}^{K_i} E_{i,1}^{ks} = \cap_{s \neq k} E_{i,1}^{ks} \). As in \( 2 \times 2 \) cases, it is easy to see that \( q_{1i}^{1} = p_i(E_{i,1}^{1}) \). Also, since \( E_{i,1}^{kt} = \{ \theta_i : \theta_{ik} - \theta_{it} \leq H_i^{kt} \} \) and \( H_i^{kt} \leq H_i^{kt}, p_i(E_{i,1}^{kt} \cap E_{i,1}^{ks}) = 0 \) due to the absolute continuity of \( p_i \). Since \( E_{i,1}^{kt} \cap E_{i,1}^{1} = (\cap_{s=1}^{K_i} E_{i,1}^{ks}) \cap (\cap_{s=1}^{K_i} E_{i,1}^{ts}) \subseteq E_{i,1}^{kt} \cap E_{i,1}^{1} \), it follows that \( \sum_{s=1}^{K_i} q_{1s}^{1} = \sum_{s=1}^{K_i} p_i(E_{i,1}^{s}) = p_i(\cup E_{i,1}^{s}) \leq 1 \).

For \( n \geq 1 \), for each \( i \in I \) and \( k, t \in \{1, \ldots, K_i\} \), let

\[
E_{i,n}^{kt} = \{ \theta_i : \theta_{ik} - \theta_{it} \geq \max_{q_j(a_j)} \sum_{a_i \in A_i, j \neq i} \prod_{j \neq i} q_j(a_j)[u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i})]\},
\]

where the maximum is taken over the intervals determined by \( q_{1i}^{n-1} \)'s. The set \( E_{i,n}^{kt} \) contains all \( \theta_i \)'s where player \( i \) should choose \( a_{ik} \) against \( a_{it} \) (again, the boundary does not matter). The maximum exists since the product of intervals is compact. Define \( E_{i,n}^{k} = \cap_{s=1}^{K_i} E_{i,n}^{ks} \). It can be seen that \( q_{1i}^{n} = p_i(E_{i,n}^{k}) \). By the inductive hypothesis, the feasible region for \( (q_j(a_j))_{a_j \in A_j} (j \in I) \) is non-increasing from \( n - 1 \) to \( n \), hence the maximum value is non-increasing. Therefore, \( E_{i,n}^{kt} \subseteq E_{i,n+1}^{kt} \), and consequently \( q_{1i}^{n} \leq q_{1i}^{n+1} \). Since \( p_i(E_{i,n}^{k} \cap E_{i,n}^{1}) = 0 \), \( \sum_{k=1}^{K_i} q_{1i}^{n+1} \leq 1 \) for each \( n \geq 0 \). Therefore, the statement holds for general cases.

\[\square\]

**Proposition 2.** \( q_{ik}^{*} \leq q_{ik} \) for each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \).

**Proof.** Suppose that \( q_{ik}^{*} > q_{ik} \) for some \( i \in I \) and \( k \in \{1, \ldots, K_i\} \). Then for each \( s_i : \Theta_i \rightarrow A_i \) such that \( (\theta_i, s_i(\theta_i)) \in \sum_{i}^{\infty} \Delta(p) \) for each \( \theta_i \in \Theta_i \), \( p_i(s_i^{-1}(a_{ik})) \geq q_{ik}^{*} > q_{ik} \), which is contradictory to Theorem 1.

\[\square\]

**4.2.2 Conditions for \( \Delta(p) \)-rationalizability \( \rightarrow \) QRE**

Propositions 1 and 2 imply that for each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), \( q_{ik}^{*} \), the infimum of the probability that action \( a_{ik} \) is used in the outcome of the \( \Delta(p) \)-rationalization process, is no larger than \( q_{ik}^{*} \), the smallest probability that the action is used in all QREs. Our purpose now is to find conditions under which \( q_{ik}^{*} = q_{ik}^{*} \), or equivalently, the following statement holds.

**Statement 1.** For each \( i \in I \) and \( k \in \{1, \ldots, K_i\} \), \( q_{ik}^{*} \) represents the probability that action \( a_{ik} \) is used in some QRE.

Example 6 in the previous section shows that Statement 1 does not hold unconditionally. We use the game in Table 7 and discuss two cases where the conjecture holds and
does not, respectively. The discussion will give an intuition about the conditions under which the statement holds.

**Case 1 (Statement 1 holds).** As in the proof of Proposition 1, we assume that \( H_{1 \rightarrow U} = \gamma_1 - \alpha_1 \geq H_{1 \rightarrow U} = \delta_1 - \beta_1 \) and \( H_{2 \rightarrow LR} = \beta_2 - \alpha_2 \geq H_{2 \rightarrow LR} = \delta_2 - \gamma_2 \).

Consider an arbitrary \((\theta_i, a_i) \in \sum_{i, \Delta(p)}\). Since \((\theta_i, a_i)\) has not been eliminated in the \(\Delta(p)\)-rationalization procedure, \(a_i\) is optimal under \(\theta_i\) against on a belief \((q_j)_{j \neq i} \in \prod_{j \neq i} \Delta(A_j)\) of the distributions of others’ choices such that for each \(j \neq i\) and \(k \in \{1, ..., K_j\}\), \(q_j(a_{kj}) \geq q^*_{jk}\). In other words, if we use \(B = (B_i)_{i \in I}\) to denote the operator on subsets \(E \subseteq \Theta \times A\) such that \(B_i(E)\) is the set of \((\theta_i, a_i) \in E\) which can be supported by some belief \(\mu^i\) in Definition 2, then \(B(\prod_{i \in I} \sum_{\Delta(p)}\) = \(\prod_{i \in I} \sum_{\Delta(p)}\). This is called the fixed-point property of the outcome of a rationalization procedure (see Pearce [24], Battigalli and Bonanno [3]).

Return to the game in Table 7. The fixed-point property implies that at the limit,

\[
q^*_{1T} = p_1[\theta_{1T} - \theta_{1U}] \geq (1 - q^*_{2L})H_{1 \rightarrow U} + q^*_{2R}H_{1 \rightarrow U}, \tag{15}
\]

\[
q^*_{1U} = p_1[\theta_{1T} - \theta_{1U}] \leq q^*_{2L}H_{1 \rightarrow U} + (1 - q^*_{2R})H_{1 \rightarrow U}, \tag{16}
\]

\[
q^*_{2L} = p_2[\theta_{2L} - \theta_{2R}] \geq (1 - q^*_{1L})H_{2 \rightarrow LR} + q^*_{1U}H_{2 \rightarrow LR}, \tag{17}
\]

\[
q^*_{2R} = p_2[\theta_{2L} - \theta_{2R}] \leq q^*_{1T}H_{2 \rightarrow LR} + (1 - q^*_{1T})H_{2 \rightarrow LR}. \tag{18}
\]

Is there any QRE where player 1 using \(T\) with probability \(q^*_{1T}\)? The answer is yes. Consider \((q_1, q_2) \in \Delta(A_1) \times \Delta(A_2)\) such that

\[
q_1(T) = q^*_{1T}, \quad q_1(U) = 1 - q^*_{1T}, \quad q_2(L) = 1 - q^*_{2R}, \quad q_2(R) = q^*_{2R}.
\]

This is a QRE. Indeed, note that when \(q_2(L) = 1 - q^*_{2R}\), at each \(\theta_1 \in E_1^U(1 - q^*_{2R}, q^*_{2R}) := [\theta_{1T} - \theta_{1U}] \leq (1 - q^*_{2R})H_{1 \rightarrow U} + q^*_{2R}H_{1 \rightarrow U}\) player 1 choose \(U\). By [15] \(p_1(E_1^U(1 - q^*_{2R}, q^*_{2R})) = 1 - q^*_{1T}\). Also, at each \(\theta_2 \in E_2^L(q^*_{1T}, 1 - q^*_{1T}) := [\theta_{2L} - \theta_{2R}] \geq q^*_{1T}H_{2 \rightarrow LR} + (1 - q^*_{1T})H_{2 \rightarrow LR}\), player 2 chooses \(L\), and it follows from equation [16] that \(p_2(E_2^L(q^*_{1T}, 1 - q^*_{1T})) = 1 - q^*_{2R}\). In a similar manner, we can show that each \(q^*_{jk}\) is the probability that player \(i\) uses the action \(k\) in some QRE.

**Case 2 (Statement 1 does not hold).** Assume that \(H_{1 \rightarrow U} = \delta_1 - \beta_1 \geq H_{1 \rightarrow U} = \gamma_1 - \alpha_1\) and, still, \(H_{2 \rightarrow LR} = \beta_2 - \alpha_2 \geq H_{2 \rightarrow LR} = \delta_2 - \gamma_2\) at the limit, it should be

\[
q^*_{1T} = p_1[\theta_{1T} - \theta_{1U}] \geq (1 - q^*_{2L})H_{1 \rightarrow U} + q^*_{2L}H_{1 \rightarrow U}, \tag{19}
\]

\[
q^*_{1U} = p_1[\theta_{1T} - \theta_{1U}] \leq q^*_{2L}H_{1 \rightarrow U} + (1 - q^*_{2R})H_{1 \rightarrow U}, \tag{20}
\]

\[
q^*_{2L} = p_2[\theta_{2L} - \theta_{2R}] \geq (1 - q^*_{1L})H_{2 \rightarrow LR} + q^*_{1U}H_{2 \rightarrow LR}, \tag{21}
\]

\[
q^*_{2R} = p_2[\theta_{2L} - \theta_{2R}] \leq q^*_{1T}H_{2 \rightarrow LR} + (1 - q^*_{1T})H_{2 \rightarrow LR}. \tag{22}
\]

Is there any QRE where player 1 using \(T\) with probability \(q^*_{1T}\)? Not necessarily so. To be specific, when \(q^*_{2L} + q^*_{2R} = 1\) (which implies that \(q^*_{1T} + q^*_{1U} = 1\)), the answer is yes.
Yet if $q_{2L}^* + q_{2R}^* < 1$ (which implies that $q_{1T}^* + q_{1U}^* < 1$), as illustrated in Figure 5, the answer is definitely no. To see this, without loss of generality, suppose that in some QRE $q = (q_1, q_2)$, player 1 uses $T$ with probability $q_1(T) = q_{1T}^*$. To fulfill this, by equation (19), $q_2(L) = q_{2L}^*$, which implies that $q_1(U) = q_{1U}^*$ and consequently it leads to $q_2(R) = q_{2R}^*$. However, by assumption, $p_2(L) + p_2(R) = q_{2L}^* + q_{2R}^* < 1$, not even a probability distribution.

![Figure 5: $\Delta(p)$ rationalization procedure does not converge to any QRE.](image)

One may notice that the difference is at the pattern of the influence relations between actions. Indeed, for each $i$ and $k$, the $\Delta(p)$-rationalization procedure is based on recursively determining the area in $\Theta_i$ where $\theta_{ik} - \theta_{it}(t \neq k)$ is bigger than the maximum value generated by the infimum of $q_{-i,k}$’s in the previous stage. To find the maximum value, one needs to make as small as possible the probabilities of the opponent’s actions which is associated with a value strictly less than $H_i^{kt}$. Those actions of the opponent are the “marginal” ones determining the value of $q_{ik}^n$ for each $n$. If we want one action $a_{ik}$ to be used by the probability $q_{ik}^n$, then every action $a_{it}$ directly or indirectly marginal for it should be used by $q_{it}^n$. Since all those are infimums, whether they are in some QRE depends on whether the remaining actions can decompose the residue and form an equilibrium.

We now formalize the idea in 2-person games. The condition will be formalized in Theorem 2. The generalization will be discussed in Section 4.2.3.

Consider a 2-person game $G = (A_1, A_2, u_1, u_2)$. For each $i = 1, 2$, we define a correspondence $\phi_i : A_i \times A_i \Rightarrow A_{-i}$ such that for each $(a_{ik}, a_{it}) \in A_i \times A_i$, $\phi_i(a_{ik}, a_{it}) = \{a_{-i} \in A_{-i} : u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i}) < H_i^{kt}\}$. We have some straightforward results for $\phi_i$’s.

**Lemma 1.** Consider $i \in \{1, 2\}$ and $a_{ik}, a_{it} \in A_i$ with $k \neq t$. Then either $\phi_i(a_{ik}, a_{it}) = \phi_i(a_{it}, a_{ik}) = \emptyset$, or $\phi_i(a_{ik}, a_{it}) \cup \phi_i(a_{it}, a_{ik}) = A_{-i}$.

**Proof.** If $\phi_i(a_{ik}, a_{it}) = \emptyset$, then $u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i}) = H_i^{kt}$ for each $a_{-i} \in A_{-i}$, which implies that $u_i(a_{ik}, a_{-i}) - u_i(a_{it}, a_{-i})$ is constant on $A_{-i}$, i.e., $\phi_i(a_{it}, a_{ik}) = \emptyset$.

Suppose that $\phi_i(a_{ik}, a_{it}) \neq \emptyset$. Note that $\phi_i(a_{ik}, a_{it})$ can never be $A_{-i}$, since there is some $a_{-i} \in A_{-i}$ such that $u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i}) = H_i^{kt}$. Hence $\phi_i(a_{ik}, a_{it}) \neq \emptyset$ implies that
Proof. If consequently every action in the game is eventually non-serial, for each \( i \in I \) and \( a_{ik} \in A_i \) for each \( a \), \( \Theta \) area in \( a \) for Lemma 2. If for some \( i \) non-serial is called \( \Phi \) reachable from \( a \) for the game in Example 5, and in Figure 6 (2) is that for Example 6. Also, we define an operator \( \Phi \) for each \( a \), \( A \) is optimal. Hence \( B \) encers are in \( \Phi \) (\( \Theta \)). We have the following result.

We call a game serial iff no action is non-serial. It can be seen that \( (\Phi_1, \Phi_2) \) defines a directed graph \( (\Rightarrow, A_1 \cup A_2) \) such that for each \( a_{ik}, a_{jt} \in A_1 \cup A_2, a_{ik} \Rightarrow a_{jt} \) iff \( a_{jt} \in \Phi_i(a_{ik}) \). For example, in Figure 6 (1) it is the directed graph for the game in Example 5, and in Figure 6 (2) is that for Example 6.

For each \( a_{ik}, a_{jt} \in A_1 \cup A_2, a_{jt} \) is reachable from \( a_{ik} \) iff there are \( a_{ik,k_0}, a_{ik,k_1}, ..., a_{ik,k_N} \in A_1 \cup A_2 (N \geq 0) \) such that \( a_{ik} = a_{ik,k_0} \Rightarrow a_{ik,k_1} \Rightarrow ... \Rightarrow a_{ik,k_N} = a_{jt} \). Note that \( a_{ik} \) is reachable from itself (i.e., when \( N = 0 \)). We use \( C(a_{ik}) \) to denote the set of all actions reachable from \( a_{ik} \).

Also, we define an operator \( \mathcal{L} \) on \( 2^{A_1 \cup A_2} \) as follows: for each \( B \subseteq A_1 \cup A_2, \mathcal{L}(B) := \{ a_j \in A_1 \cup A_2 : \Phi_j(a_j) \subseteq B \} \). Informally, \( \mathcal{L}(B) \) is the set of actions whose marginal influencers are in \( B \). We can repeatedly apply \( \mathcal{L} \) to a set, and we define \( \mathcal{L}^n(B) = \bigcup_{i=0}^{\infty} \mathcal{L}(B) \);

\[ \mathcal{H}_i^{kt} > H_i^{kt} \] Since \( u_i(a_{ik}, a_{-i}) - u_i(a_{it}, a_{-i}) = -(u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i})) \), it follows that \( (A_{-i} \setminus \Phi_i(a_{ik}, a_{it})) \subseteq \Phi_i(a_{it}, a_{ik}) \), and consequently \( \Phi_i(a_{ik}, a_{it}) \cup \Phi_i(a_{it}, a_{ik}) = A_{-i} \).

Based on \( \phi_i \), we define a correspondence \( \Phi_i : A_i \Rightarrow A_{-i} \) such that \( \Phi_i(a_{ik}) = \bigcup_{i=1}^{K_i} \Phi_i(a_{ik}, a_{it}) \) for each \( i \in I \) and \( a_{ik} \in A_i \). Informally, \( \Phi_i(a_{ik}) \) is the set of “marginal” actions of player \( -i \) for \( a_{ik} \) since the infimums of the probabilities that those actions are used determines the area in \( \Theta \), where \( a_{ik} \) is optimal. An action \( a_{ik} \in A_i \) is called non-serial iff \( \Phi_i(a_{ik}) = \emptyset \); it is called indirectly non-serial iff for some \( b_{-i} \in \Phi_i(a_{ik}) \), \( \Phi(b_{-i}) = \emptyset \). We have the following result.

**Lemma 2.** If for some \( i \in \{1, 2\} \), \( a_{ik} \in A_i \) is non-serial, then each action in \( A_i \) is non-serial, and consequently every action in the game is eventually non-serial.

Proof. If \( a_{ik} \) is non-serial, then for each \( a_{it} \in A_i \) and each \( a_{-i} \in A_{-i}, u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i}) = \mathcal{H}_i^{kt} \), which implies that for each \( a_{it}, a_{ik} \in A_i, u_i(a_{it}, a_{-i}) - u_i(a_{it}, a_{-i}) \) is constant on \( A_{-i} \). Hence \( \Phi_i(a_i) = \emptyset \) for each \( a_i \in A_i \), and consequently every action in the game is eventually non-serial.

We call a game serial iff no action is non-serial.

The names are borrowed from modal logic, a field devoted to the research of relational structures. See, for example, Chellas [8] and Battigalli and Bonanno [3].

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7 We call them marginal since their roles are similar to that of the marginal consumers/ producers in competitive market equilibrium models, who determines the market price.

8 The names are borrowed from modal logic, a field devoted to the research of relational structures. See, for example, Chellas [8] and Battigalli and Bonanno [3].
here, we stipulate that \( \mathcal{L}^0(B) = B \). \( \mathcal{L}^\infty(B) \) is the set of all actions directly and indirectly being influenced by and influencing actions in \( B \).

**Theorem 2.** Let \( a_{ik} \in A_i \). If one of the following condition is satisfied, then \( q^*_{ik} = q_{jk} \):

1. \( a_{ik} \) is eventually non-serial, or
2. \( \mathcal{L}^\infty(C(a_{ik})) \neq A_1 \cup A_2 \).

**Proof.** First, suppose that \( a_{ik} \) is non-serial. It follows that for each \( t \in \{1, ..., K_i\} \), \( u_i(a_{it}, a_{-i}) - u_i(a_{ikt}, a_{-i}) = \overline{H}_i^kt \) for each \( a_{-i} \in A_{-i} \). Therefore, for each \( n \geq 1 \), \( E_i^{kt} = E_1^{kt} := [\theta_{ik} - \theta_{it} \geq \overline{H}_i^{kt}] \) is fixed, and consequently \( q^*_{ik} = p_t(\cap_{s=1}^{K_i} E_i^{kt}) \). Also, it is clear that in every QRE, \( a_{ik} \) is used with probability \( p_t(\cap_{s=1}^{K_i} E_i^{kt}) \). Hence \( q^*_{ik} = q_{jk} \). Similarly, if \( a_{ik} \) is indirectly non-serial, since the probabilities of their marginal actions will be fixed from the first round, \( q^*_{ik} \) is fixed for each \( n \geq 2 \), and in every QRE \( a_{ik} \) is used by \( q^*_{jk} \). Hence we still have \( q^*_{ik} = q_{jk} \).

Now suppose that \( a_{ik} \) is not eventually non-serial and \( \mathcal{L}^\infty(C(a_{ik})) \neq A_1 \cup A_2 \). We define the following symbols: for each \( j \in \{1, 2\} \),

\[
A_j^0 := A_j \cap \mathcal{L}^\infty(C(a_{ik})), \quad A_j^\prime = A_j \setminus \mathcal{L}^\infty(C(a_{ik})).
\]

It is clear that for each \( j \in I \), \( A_j^0 \) and \( A_j^\prime \) form a partition of \( A_j \). We have the following observation.

**Observation 2.** When \( a_{ik} \) is not eventually non-serial, \( \mathcal{L}^\infty(C(a_{ik})) \neq A_1 \cup A_2 \) implies that \( A_j^\prime \neq \emptyset \) for each \( j = 1, 2 \).

To see this, suppose that \( A_j^\prime = \emptyset \) for some \( j \in \{1, 2\} \). Then \( A_j^\prime \neq \emptyset \) since \( A_j^\prime = (A_1 \cup A_2) \setminus \mathcal{L}^\infty(C(a_{ik})) \). Yet since no \( a_{-j} \in A_j^\prime \) is non-serial, there should be some \( b_j \in A_j \) marginal to some \( a_{-j} \in A_j^\prime \) which is not in \( \mathcal{L}^\infty(C(a_{ik})) \), otherwise by definition \( A_j^\prime = \emptyset \). Yet since \( A_j^\prime = \emptyset \), \( b_j \in \mathcal{L}^\infty(C(a_{ik})) \), and consequently \( a_{-j} \in \mathcal{L}^\infty(C(a_{ik})) \), which implies that \( A_j^\prime = \emptyset \), a contradiction. Therefore, \( A_j^\prime \neq \emptyset \) for both \( j = 1, 2 \).

Now we return to the proof of Theorem 2. To show \( q^*_{ik} = q_{jk} \), we show that Statement 1 holds here, i.e., there is some QRE where \( a_{ik} \) is used by probability \( q^*_{ik} \). Combining it with Propositions 1 and 2 we obtain \( a^*_{ik} = q_{jk} \). Consider \( q = (q_1, q_2) \in \Delta(A_1) \times \Delta(A_2) \) defined as follows:

(a) For each \( a_{jt} \in A_j^0 \) for each \( j \in \{1, 2\} \), let \( q_j(a_{jt}) = q^*_{jt} \).

(b) Since we have shown above that \( A_j^\prime \neq \emptyset \) for both \( j = 1, 2 \), we can define

\[
B_j = \{ r_j \in [0, 1]^A_j^\prime : \sum_{a_{js} \in A_j^\prime} r_j(a_{js}) = 1 - \sum_{a_{jt} \in A_j^0} q^*_{jt} \text{ and } r_j(a_{js}) \geq q^*_{js} \text{ for each } s \text{ with } a_{js} \in A_j^0 \}.
\]

By Proposition 1, each \( B_j \) is well defined. It is clear that each \( B_j \) is compact and convex, so is \( B := B_1 \times B_2 \). Now consider \( g : B \to B \) such that for each \( j \in \{1, 2\} \), \( r \in B \), and \( s \) such that \( a_{js} \in A_j^\prime \),

\[
g_{js}(r) = p_j[E_j^s((q^*_{-j,t})_{t:a_{-js} \in A_j^\prime} (r_{-j,t})_{t:a_{-js} \in A_j^\prime})],
\]
where $E^s_j$ is a measurable function from $[0,1]^{A_{-j}}$ to $2^{\Theta_i}$ such that for each $\gamma \in [0,1]^{A_{-j}}$,

$$E^s_j(\gamma) = \bigcap_{i=1}^{K_i} \{ \theta_i \in \Theta_i : \theta_{is} - \theta_{it} \geq \sum_{a_{-i} \in A_{-i}} \gamma(a_{-j}) [u_i(a_{it}, a_{-i}) - u_i(a_{ik}, a_{-i})] \}$$

By our assumptions about $p_j$, $j = 1, 2$, $g$ is continuous. It follows from Brouwer’s fixed point theorem that $g$ has a fixed point $r^*$. Then for each $a_{js} \in A'_{js'}$, we let $q_j(a_{js}) = r^*$. It can be seen that $q$ is a QRE in which $a_{ik}$ is used by probability $q^*_{ik}$.

**Corollary 1.** If $|Q(G, p)| = 1$ and either (1) or (2) in Theorem 2 is satisfied, the QRE is the only $\Delta(p)$-rationalizable outcome.

The condition provided in Theorem 2 is sufficient. It is not necessary when the QRE is unique. To see this, consider games in Examples 3 and 6 in Section 3.2. For each game and each $a_{ik}$, $\mathcal{L}^\infty(C(a_{ik})) = A_1 \cup A_2$. However, the $\Delta(p)$-rationalization procedure converges to the QRE in Example 3 but fails to do so in Example 6. However, when there are multiple QREs, the conditions in Theorem 2 is also necessary. We have the following result.

**Proposition 3.** If $|Q(G, p)| > 1$ and both conditions in Theorem 2 are violated, $q^*_{ik} < q_{ik}$ for each $i \in I$ and $k \in \{1, ..., K_i\}$.

**Proof.** Since no action is eventually non-serial and $\mathcal{L}^\infty(C(a_{ik})) = A_1 \cup A_2$, it follows that for each distinct $\pi, \pi' \in Q(G, p)$, each $j \in I$ and $t \in \{1, ..., K_i\}$, $\pi_j(a_{jt}) \neq \pi'_j(a_{jt})$. Therefore, $\sum_{s \in \{1, ..., K_j\}} q^*_{js} < 1$ for each $j \in I$. Applying the fixed-point property as in Case 2 (Statement 1 does not hold) above, it follows that if $q^*_{jk} = q_{jk}$ for some $i \in I$ and $k \in \{1, ..., K_i\}$, it follows from $\mathcal{L}^\infty(C(a_{ik})) = A_1 \cup A_2$ that all $q^*_{ji} = q_{ji}$, which does not form a QRE. Hence $q^*_{ik} < q_{ik}$ for each $i \in I$ and $k \in \{1, ..., K_i\}$. \qed

### 4.2.3 The strictness of the conditions and a full convergence

Theorem 2 gives a sufficient condition for a $\Delta(p)$-rationalization process “locally” converging to some QRE (locally means that we only focus on an individual action $a_{ik}$); it is also necessary when the QRE is unique by Proposition 3. Now we face a problem: How “large” is the set of games satisfying condition (1) or (2) in Theorem 2? Or, how “special” such a game can be? This problem is also relevant to generalizing Theorem 2 to $n$-person games.

It is clear that condition (1) is quite strict and it does not hold for generic games. Indeed, as noted in Lemma 2, one action $a_{ik}$’s non-seriality implies that the payoff matrix of player $i$ has order 1. Condition (2) seems more general. In a $2 \times 2$ game, it implies a directed

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9Strictly speaking, Example 3 does not satisfy the assumption in this subsection since $f_j$’s there do not have full supports. Yet after reflecting on the proof of Theorem 2, one may notice that the full-support assumption can be relaxed; all results in this subsection hold as well if the support of each $f_j$ is large enough to rationalize each action of $i$. Therefore, Example 3 still serves as a good illustration here and does not need any essential change.
In this paper, we define a graph as in Figure 6 (1). Many games intensively studied in the literature satisfy the condition, for example, the asymmetric game of chicken (Goeree et al. [12], p.25-26), coordination games (Goeree et al. [12], p.29-30, Anderson et al. [1], Turocy [30]), and many (but not all) dominance-solvable games. However, it is impossible for a Matching-Pennies style game (MP, Ochs [22]) to satisfy condition (2). For a general 2-person game, we have the following result.

Lemma 3. Suppose that \( a_{ik} \in A_i \) is not eventually non-serial. Then if it satisfies condition (2) in Theorem 2, it satisfies the following two conditions:

(A) \( |\Phi_i(a_{ik})| = 1 \), and

(B) \( \Phi_{-i}(\Phi_i(a_{ik})) = \{a_{ik}\} \).

Proof. For (A), since \( a_{ik} \) is serial, \( |\Phi_i(a_{ik})| \geq 1 \). Suppose that \( |\Phi_i(a_{ik})| > 1 \) and let \( b_{-i}, b'_{-i} \in \Phi_i(a_{ik}) \) with \( b_{-i} \neq b'_{-i} \). Then Lemma 1 (1) implies that \( \Phi_{-i}(b_{-i}) \cup \Phi_{-i}(b'_{-i}) = A_i \), and consequently \( A_i \subseteq \mathcal{L}^\infty(C(a_{ik})) \). By Observation 2, it follows that \( \mathcal{L}^\infty(C(a_{ik})) = A_1 \cup A_2 \), a contradiction. Hence, \( |\Phi_i(a_{ik})| = 1 \).

For (B), let \( \Phi_i(a_{ik}) = \{b_{-i}\} \). Suppose that there is some \( a_{is} \in \Phi_{-i}(b_{-i}) \) with \( s \neq k \). By Lemma 1 (1), it follows that \( \Phi_i(a_{ik}) \cup \Phi_i(a_{is}) = A_{-i} \), and consequently \( A_{-i} \subseteq \mathcal{L}^\infty(C(a_{ik})) \). By Observation 2, it follows that \( \mathcal{L}^\infty(C(a_{ik})) = A_1 \cup A_2 \), a contradiction. Hence, \( \Phi_{-i}(\Phi_i(a_{ik})) = \{a_{ik}\} \).

Lemma 3 shows that condition (2) in Theorem 2 can be quite strict in general 2-person games. It actually implies that most games do not satisfy the condition. We have the following result.

Proposition 4. Consider a 2-person serial game with \( |A_i| \geq 2 \) for each \( i = 1, 2 \) and at least one player has more than two actions. Then no action satisfies condition (2) in Theorem 2.

Proof. Without loss of generality, suppose that player 1 has more than two actions, and \( a_1 \in A_1 \) satisfies condition (2). By Lemma 3, \( \Phi_i(a_1) = \{a_2\} \) for some \( a_2 \in A_2 \). So \( C(a_1) = \{a_1, a_2\} \). Let \( b_1, c_1 \in A_1 \setminus \{a_1\} \) be distinct. By Lemma 1, \( \Phi_1(b_1) \cup \Phi_1(c_1) = A_2 \). So \( a_2 \in \Phi_1(b_1) \) or \( a_2 \in \Phi_1(c_1) \). Hence \( \{b_1, c_1\} \cap \mathcal{L}(C(a_1)) \neq \emptyset \).

By Lemma 1, \( \{b_1, c_1\} \subseteq \Phi_2(b_2) \) for each \( b_2 \in A_2 \) with \( b_2 \neq a_2 \) (since \( |A_2| \geq 2 \), such \( b_2 \) exists), and consequently \( b_2 \in \mathcal{L}^2(C(a_1)) \). Hence \( A_2 \cap \mathcal{L}^\infty(C(a_1)) = A_2 \). By Observation 2, \( \mathcal{L}^\infty(C(a_1)) = A_1 \cup A_2 \), a contradiction.

Proposition 4 implies that condition (2) in Theorem 2 cannot be satisfied in a generic \( n \)-person game \((n > 2)\). Even if each player has only two actions, since \( \Phi_i(a_{ik}) \) now contains profiles of actions in \( A_{-i} = \prod_{j \neq i} A_j \) and \( |A_{-i}| > 2 \), through an argument similar to the proof of Proposition 4, it can be seen that no action satisfies condition (2) in Theorem 2.

5 Conclusion

In this paper, we define \( \Delta(p) \)-rationalization procedure, a special case of Battigalli and Siniscalchi [5]’s \( \Delta \)-rationalization procedure, to characterize robust outcomes in large
populations, and we investigate the relationship between $\Delta(p)$-rationalizable outcomes and MP’s QREs. Our results here have two implications. First, in a non-trivial class of $2 \times 2$ games, which is characterized in Theorem 2 and Proposition 3, QREs are informative for determining robust outcomes. Second, however, in general, when QRE is not unique, robust outcomes derived from the $\Delta(p)$-rationalization procedure can be larger than the QRE, which makes the former a better benchmark to estimate robust outcomes in large populations.\footnote{The condition for the uniqueness of QRE is not fully studied yet. See Melo \cite{Melo} for the latest progress in this direction. It is still hard to estimate the size of the set of game-distribution pairs with unique QRE.}

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