THE STABILITY PROBLEM FOR CHARACTERS ON
WEIGHTED SEMILATTICE ALGEBRAS

YEMON CHOI, MAHYA GHANDEHARI, HUNG LE PHAM

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Abstract

Let $S$ be a semilattice. We characterise all submultiplicative weights $\omega$ on $S$ for
which the weighted semilattice algebra $\ell^1(S,\omega)$ has stable characters, in the sense that
if a bounded linear functional on $\ell^1(S,\omega)$ is locally almost multiplicative, then it is
globally near a multiplicative linear functional. This result is then used to complete
the proof that $\ell^1(S,\omega)$ has stable characters for every submultiplicative weight $\omega$ if
and only if $S$ has finite breadth, answering a question from [Cho13] where the “if”
direction has been proved. The proof here is carried out through a detailed study of
the stability problem for filters in semilattices, relative to a given weight function.
Our method heavily relies on a new structure theory for infinite breadth semilattices
developed in [CGPpre].

Keywords: AMNM, breadth, semilattice, stable characters, stable filters, Ulam stabil-
ity.

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1 Introduction

A fundamental question in various branches of mathematics is to determine whether “lo-
cally approximate versions” of a given structure are small perturbations of that structure
in the global sense. Many variations of this question have been studied, often under the
name “Ulam stability”, although Hyers and Rassias also deserve mention in this context;
we shall not attempt a comprehensive history here.

In this article, we study a form of this question regarding multiplicative functionals
that arises in the theory of Banach algebras, and which seems to be less well-known than
the classical Ulam stability problem. A multiplicative functional on a Banach algebra $A$ is
a (bounded) linear functional $\psi : A \rightarrow \mathbb{C}$ satisfying $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in A$. The
set of multiplicative functionals on $A$ is denoted as $\text{Mult}(A, \mathbb{C})$ and the non-zero elements
of $\text{Mult}(A, \mathbb{C})$ are the characters of $A$. The following notion measures how much a bounded
linear functional on $A$ resembles being multiplicative when examined locally.

**Definition 1.1 (Multiplicative defect).** Let $A$ be a commutative Banach algebra and let
$\psi \in A^*$. The multiplicative defect of $\psi$ is the non-negative real number

\[ \text{def}(\psi) = \|\psi \circ \pi_A - \psi \otimes \psi\|_{(A \hat{\otimes} A)^*} = \sup\{|\psi(xy) - \psi(x)\psi(y)| : x, y \in A, \|x\| \leq 1, \|y\| \leq 1\}, \tag{1} \]

where $A \hat{\otimes} A$ is the projective tensor product of $A$ with itself, and $\pi_A : A \hat{\otimes} A \rightarrow A$ is the
multiplication map.
Note that our definition also makes sense for noncommutative Banach algebras. However, since character theory for Banach algebras is most useful and relevant when those algebras are commutative, we shall follow [Joh86] in restricting our discussion to the commutative setting.

The character stability problem for a commutative Banach algebra \( A \) then asks whether a bounded linear functional on \( A \) with a small multiplicative defect must necessarily be close to an actual multiplicative linear functional, i.e. whether an approximately multiplicative functional is near a multiplicative one. This inspires the notation “AMNM” in the following definition.

**Definition 1.2 (AMNM algebras, [Joh86])**. A commutative Banach algebra \( A \) is said to be **AMNM**, or have the **AMNM property**, or have **stable characters**, if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \phi \in A^* \) and \( \text{def}(\phi) < \delta \) then \( \inf\{\|\phi - \psi\| : \psi \in \text{Mult}(A, \mathbb{C})\} < \varepsilon \).

Many examples are studied in [Joh86], and we mention just a few results: it is shown there that abelian C\(^*\)-algebras, \( L^1 \)-convolution algebras of locally compact abelian groups, the Banach spaces \( \ell^p \) with pointwise product, and certain algebras of holomorphic functions (including the disc algebra and \( \ell^1(\mathbb{Z}_+) \) and \( L^1(\mathbb{R}_+) \)), are all AMNM. On the other hand, it is also shown in [Joh86] that the classical Volterra algebra \( L^1(0,1) \) is not AMNM. For further positive and negative results, the reader is referred to [Cho13, How03, Jar97, Joh86, Sid97]; the general impression is that the AMNM property often holds for “natural examples”, and it can be tricky to construct non-AMNM examples within a given class of Banach algebras.

The present paper is concerned with the AMNM property – or equivalently, the character stability problem – for the classe of weighted convolution algebras \( \ell^1(S, \omega) \), where \( S \) is a semilattice and \( \omega \) is a submultiplicative weight on \( S \). Recall that a semilattice is a commutative semigroup in which each element is idempotent. Classically, these objects play an important role in the structure theory of general semigroups (see e.g. [How76, Chapter IV]); they are also a fundamental concept for dataflow analysis in computer science [KU76, KSS09]. Recently semilattices have been proposed as models for distributed data structures [ASB, SPB+11], an active research topic in modern software engineering.

The AMNM property of weighted semilattice algebras \( \ell^1(S, \omega) \) has been studied extensively in [Cho13], where the following was proved.

**Theorem A** ([Cho13, Example 3.13 and Theorem 3.14]). Let \( S \) be a semilattice that has “finite breadth”. Then \( \ell^1(S, \omega) \) is AMNM for every submultiplicative weight \( \omega \).

It was then asked whether the converse of this theorem is true. We shall prove that this is indeed the case:

**Theorem B.** Let \( S \) be a semilattice that has “infinite breadth”. Then a submultiplicative weight \( \omega \) can be constructed such that \( \ell^1(S, \omega) \) is not AMNM.

We thus have a source of new examples of commutative (semisimple) Banach algebras that are not AMNM, which may be worth investigating further.

Our results will be interpreted as problems about the stability of filters on semilattices, and most of the proofs will be carried out in this setting. It should be noted that the task is difficult because \( \omega \) is required to be submultiplicative, which is a global condition on the function \( \omega : S \to [1,\infty) \).

The proofs and constructions here are far from routine, as semilattices with infinite breadth can display a diverse range of behaviour. More specifically, for a semilattice \( S \), the
condition of having infinite breadth does not really restrict its form, as this condition only
means that there is a sequence \((E_n)\) of finite subsets of \(S\) of increasing sizes, where the
product of elements in each \(E_n\) cannot be “compressed”. (The notions of compressibility
and of breadth in a semilattice will be defined later in Definition 4.1.) Not only do we not
have a priori control on the relative position of the subsets \(E_n\), but we have essentially
with no control on the remaining part \(S \setminus \bigcup E_n\); this makes it impossible to build a desired
weight on \(S\) from bottom-up (i.e. to piece together local configurations afforded by such a
sequence \((E_n)\) as in the example from [Cho13, Section 3.2]) while still ensuring the global
condition of submultiplicativity.

For this reason, we use a top-down approach, embedding a given semilattice \(S\) into one
of the form \((\mathcal{P}(\Omega), \cup)\) for some set \(\Omega\), and defining our weight function on all of \(\mathcal{P}(\Omega)\). The
construction of a suitable weight still turns out to be very delicate, for technical reasons
that are explained in §7. We shall proceed in two stages.

1. Our first main result, Theorem 3.6, gives an equivalent characterisation of the
AMNM property of \(\ell^1(S, \omega)\) for a weighted semilattice \((S, \omega)\) entirely in terms of
a growth condition on \(\omega\) called propagation (see Definition 3.3). Roughly speaking,
propagation means that whenever an element \(z\) is a divisor of a product \(x_1 \cdots x_n\) in
\(S\), then \(z\) can be obtained from \(x_1,\ldots, x_n\) by repeatedly taking divisors of products
of pairs while maintaining a control on the weight of any resulting element as a
function of the weights of \(z, x_1,\ldots, x_n\).

2. Our second main result, Theorem 5.1, states that an infinite breadth semilattice
\(S\) always possesses a weight \(\omega\) without propagation, thereby completing the proof
of Theorem B. To prove Theorem 5.1, we make use of the new structure theory
of infinite breadth semilattices, developed in [CGPpre] to overcome the apparent
arbitrariness that the condition of infinite breadth for \(S\) allows outside of a sequence
\((E_n)\) of finite subsets of \(S\) that witnesses its infinite breadth. The necessary details
from [CGPpre] are summarized in §4, and the actual construction of suitable \(\omega\),
using this structure theory, is carried out in Section 6.

We finish the paper with some examples in Section 7 that demonstrate how propagation
need not pass to quotients or sub-semilattices.

2 Weighted stability of filters in semilattices

Recall that a semilattice is a commutative semigroup \(S\) satisfying \(x^2 = x\) for all \(x \in S\).
Such an \(S\) has a standard and canonical partial order: if \(x, y \in S\) we write \(x \preceq y\) whenever
\(xy = x\). It is sometimes useful to read \(x \preceq y\) as: “\(x\) is a multiple of \(y\)” or “\(y\) is a factor of
\(x\)” or “\(y\) is a divisor of \(x\)”, in which case, we also write \(y \mid x\) to emphasise this algebraic
interpretation. In this language, \(xy\) is the “largest common multiple” of \(x\) and \(y\).

A filter in a semilattice \(S\) is defined to be a nonempty subset \(F \subseteq S\) which is closed
under taking binary products \((x, y \in S \implies xy \in S)\) and under taking divisors \((x \in S, y \in F\)
and \(x \mid y \implies x \in F)\). It is easily checked that this is equivalent to the condition
\[
\forall x, y \in S, \quad (xy \in F) \iff (x \in F \text{ and } y \in F).
\]
Filters in \(S\) can be naturally identified with its semicharacters, i.e. the nonzero multi-
licative functions \(S \rightarrow \mathbb{C}\). Thus, to analyze a semilattice \(S\), it is important to study its
filters.
For filters (or equivalently, for semicharacters) the stability question becomes: if a subset \( G \subseteq S \) satisfies the condition \((*)\) in an approximate sense, is it necessarily close (for some appropriate metric) to a genuine filter \( F \)?

There are different ways to interpret this question, but we restrict attention to the following one. Fix some \( p : S \to (0, \infty) \), which we regard as a penalty function. Given \( G \subseteq S \), we may use \( p \) to define two natural ways of quantifying the extent to which \( G \) fails to be a filter.

Let \( \mathcal{F}_S \) be the set of all filters in \( S \), and let \( 1_G \) denote the characteristic function of \( G \). We define

\[
\text{def}(G) := \sup_{x,y \in S} |1_G(xy) - 1_G(x)1_G(y)| p(x)p(y) \tag{2}
\]

\[
\text{dist}(G) := \inf_{F \in \mathcal{F}_S \cup \{\emptyset\}} \sup_{x \in S} |1_G(x) - 1_F(x)| p(x) \tag{3}
\]

In both cases, the quantity is 0 if and only if \( G \in \mathcal{F}_S \cup \{\emptyset\} \) (see Lemma 2.1 for a proof).

Informally, \( \text{def}(G) \) is obtained by measuring the worst outcome over all “local tests”, while \( \text{dist}(G) \) is given by the best outcome of all “global tests”. It is natural to restrict attention to those penalty functions which satisfy

\[
p(x)p(y) \leq p(xy) \quad \text{for all } x, y \in S, \tag{4}
\]

since this condition ensures that \( \text{dist}(G) \) being small (the “global condition”) implies that \( \text{def}(G) \) is also small (the “local condition”); see Lemma 2.1. The question we raised earlier now becomes:

does a sufficiently small value of \( \text{def}(G) \) force \( \text{dist}(G) \) to be small?

Informally: can we get global control from sufficiently good local control?

As standard in the literature on semigroups, instead of a fixed “penalty function” \( p : S \to (0, \infty) \), we shall work with \( \omega = 1/p \), which we view as a “weight function” on \( S \). So, given \( G \subseteq S \), instead of (2), we define the filter defect of \( G \) to be

\[
\text{def}_\omega(G) := \sup_{x,y \in S} \frac{|1_G(xy) - 1_G(x)1_G(y)|}{\omega(x)\omega(y)}. \tag{5}
\]

If \( F, G \subseteq S \), let

\[
\text{dist}_\omega(F,G) := \sup_{x \in S} \frac{|1_F(x) - 1_G(x)|}{\omega(x)},
\]

and then define instead of (3), \( \text{dist}_\omega(G) \) to be \( \inf_{F \in \mathcal{F}_S \cup \{\emptyset\}} \text{dist}_\omega(F,G) \).

Condition (4) for \( p \) is now equivalent to asking that the function \( \omega = 1/p : S \to (0, \infty) \) be submultiplicative – which implies, in particular, that \( 0 < \omega(x) \leq \omega(x)^2 \) for all \( x \), so that \( \omega \geq 1 \). We now show that submultiplicativity of \( \omega \) ensures that the local measure \( \text{def}_\omega \) is dominated by a constant multiple of the global measure \( \text{dist}_\omega \).

**Lemma 2.1.** Let \( S \) be a semilattice and let \( X \subseteq S \). Let \( \omega : S \to (0, \infty) \). Then for each \( s, t \in S \)

\[
\frac{|1_X(s)1_X(t) - 1_X(st)|}{\omega(s)\omega(t)} \leq \text{dist}_\omega(X) \left[ \frac{1}{\omega(t)} + \frac{1}{\omega(s)} + \frac{\omega(st)}{\omega(s)\omega(t)} \right]. \tag{5}
\]

Consequently:
(i) If $X \notin \mathfrak{F}_S \cup \emptyset$, then $\text{def}_\omega(X) > 0$ and $\text{dist}_\omega(X) > 0$.

(ii) If $\omega$ is submultiplicative, then $\text{def}_\omega(X) \leq 3 \text{dist}_\omega(X)$.

Proof. Let $F \in \mathfrak{F}_S \cup \emptyset$. Then by a 3e-style argument, followed by Hölder’s inequality,

$$|1_X(s)1_X(t) - 1_X(st)| \leq |1_X(s)(1_X(t) - 1_F(t))| + |(1_X(s) - 1_F(s))1_F(t)| + |1_F(st) - 1_X(st)|$$

$$\leq \sup_{x \in X} \frac{|1_X(x) - 1_F(x)|}{\omega(x)} (\omega(s) + \omega(t) + \omega(st)).$$

Taking the infimum over all $F$ and dividing by $\omega(s)\omega(t)$ yields the inequality (5).

If $X \notin \mathfrak{F}_S \cup \emptyset$ then there exist $s, t \in S$ for which the left-hand side of (5) is strictly positive. This proves (i). Finally, if $\omega$ is submultiplicative then $\omega \geq 1$, as previously remarked, so (ii) follows from (5).

Terminology. Henceforth, whenever we refer to a weight on a semilattice $S$, we always assume it is submultiplicative. This condition guarantees that $\ell^1(S, \omega)$ forms a Banach algebra in a natural way [Cho13].

For a weight $\omega$ on $S$, Lemma 2.1(ii) shows that a sequence $(G_n)$ with $\text{dist}_\omega(G_n) \to 0$ must satisfy $\text{def}_\omega(G_n) \to 0$.

Definition 2.2 (The stable filter property). Let $S$ be a semilattice and let $\omega$ be a weight. We say that $S$ has $\omega$-stable filters if every sequence $(G_n) \subseteq \mathcal{P}(S)$ with $\text{def}_\omega(G_n) \to 0$ satisfies $\text{dist}_\omega(G_n) \to 0$.

We note below an easy reformulation of this property.

Lemma 2.3. Let $S$ be a semilattice and let $\omega$ be a weight. Then the following are equivalent:

(i) $S$ has $\omega$-stable filters;

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $G \subseteq S$ satisfies $\text{def}_\omega(G) < \delta$, then there exists $F \subseteq S$ such that $F \in \mathfrak{F}_S \cup \emptyset$ and $\text{dist}_\omega(G, F) < \varepsilon$.

We now relate $\omega$-stability of filters on $S$ to the stability of characters on $\ell^1(S, \omega)$.

Lemma 2.4. To verify the AMNM property for $\ell^1(S, \omega)$, it suffices to check the stability condition for only those functionals $\phi \in \ell^1(S, \omega)^* = \ell^\infty(S, \omega^{-1})$ which are $\{0, 1\}$-valued.

Proof. This is essentially [Cho13, Corollary 3.7]. We take the opportunity to clarify a point of possible confusion in the earlier paper. It follows from the proof of [Cho13, Lemma 3.6] that any $\psi \in C^S$ satisfies $|\psi(x)| \leq (1 + \text{def}_\omega(\psi)^{1/2})\omega(x)$ for all $x \in S$. Hence in the reasoning which follows that lemma, one does not need the a priori assumption that $\psi : S \to \mathbb{C}$ is “$\omega$-bounded” — it comes for free. This observation is needed to fully justify [Cho13, Corollary 3.7] in its stated generality.

Proposition 2.5. Let $S$ be a semilattice, and let $\omega$ be a weight on $S$. Then the algebra $\ell^1(S, \omega)$ is AMNM if and only if $S$ has $\omega$-stable filters.

Proof. To show the “only if” assertion, assume that $S$ does not have $\omega$-stable filters. Examining the definitions of $\text{def}_\omega(G)$ and $\text{dist}_\omega(G)$, for $G \subseteq S$, one sees they correspond to the local and global quantities in Definition 1.2, for $A = \ell^1(S, \omega)$. Thus, a sequence $(G_n)$ which witnesses the failure of $\omega$-stability for filters, gives a sequence $(1_{G_n})$ in $\ell^1(S, \omega)^*$ that witnesses the failure of the AMNM property.

The “if” assertion follows similarly, using Lemma 2.4.
3 Weighted stability and propagation on a given semilattice

The function $\text{dist}_\omega : \mathcal{P}(S) \to [0, \infty)$ is defined in terms of the set $\mathcal{F}_S$ of filters of $S$. In this subsection, to make further progress on the filter stability problem, we introduce a criterion which is equivalent to the property of having stable filters, but which does not require prior knowledge of $\mathcal{F}_S$. This in turn gives us an equivalent criterion for $\ell^1(S, \omega)$ to be AMNM without referring to its set of characters or its bounded linear functionals. The precise statement is in Theorem 3.6, but first we need some preparation.

It is convenient to work with log-weights, by which we mean functions $\lambda : S \to [0, \infty)$ that satisfy $\lambda(xy) \leq \lambda(x) + \lambda(y)$ for all $x, y \in S$.

**Example 3.1.** Let $\Omega$ be a non-empty set and consider $\mathcal{P}^{\text{fin}}(\Omega)$, the set of all finite subsets of $\Omega$. This becomes a semilattice when equipped with binary union as our semigroup operation. Let $a \in (1, \infty)$: then the function $x \mapsto |x|$ is a log-weight on $\mathcal{P}^{\text{fin}}(\Omega)$, and the function $x \mapsto a^{|x|}$ is a weight on $\mathcal{P}^{\text{fin}}(\Omega)$.

**Notation.** Let $(S, \omega)$ be a semilattice, and let $\lambda = \log \omega$ be the associated log-weight. For a nonempty subset $E$ of $S$, we write $\text{fac}(E)$ for the set of factors of elements of $E$, or more formally,

$$\text{fac}(E) := \bigcup_{y \in E} \{ x \in S : x | y \}. \quad (6)$$

If $E$ is a non-empty subset of $S$ and $n \in \mathbb{N}$, let

$$(E)_n := \{ x_1 \cdots x_n : x_1, \ldots, x_n \in E \}. \quad (7)$$

We write $\text{Filt}(E)$ for the filter generated by $E$, i.e. the smallest filter in $S$ which contains $E$. It is easy to verify that

$$\text{Filt}(E) = \text{fac} \left( \bigcup_{n \geq 1} (E)_n \right) = \bigcup_{n \geq 1} \text{fac}( (E)_n ). \quad (8)$$

For the log-weight $\lambda : S \to [0, \infty)$ on the semilattice $S$, define $W_L(S, \lambda) = \{ x \in S : \lambda(x) \leq L \}$; when there is no danger of confusion we abbreviate this to $W_L$.

Let $C \geq 0$. For $E \subseteq S$ we define $\text{FBP}_C(E) := \text{fac}( (E)_2 ) \cap W_C$; in other words

$$\text{FBP}_C(E) := \{ z \in W_C : \text{there exist } x, y \in E \text{ such that } z | xy \}. \quad (9)$$

(FBP stands for “factors of binary products”. ) For sake of book-keeping, we note that $\text{FBP}_C(\emptyset) = \emptyset$. Finally, we say that $E$ is $\text{FBP}_C$-stable if $\text{FBP}_C(E \cap W_C) = E \cap W_C$.

**Lemma 3.2.** Let $S$ be a semilattice.

(i) Let $C \geq 0$ and suppose $X \subseteq S$ is $\text{FBP}_C$-stable. Then $\text{def}_\omega(X) \leq e^{-C}$.

(ii) Let $X \subseteq S$ and $C \geq 0$. If $\text{def}_\omega(X) < e^{-3C}$, then $X$ is $\text{FBP}_C$-stable.

**Proof of (i).** Assume $X$ is $\text{FBP}_C$-stable. Let $x, y \in S$: we must show that $|1_X(xy) - 1_X(x)1_X(y)| \leq e^{-C}\omega(x)\omega(y)$.

If $\lambda(x) + \lambda(y) \geq C$, then the inequality follows since

$$|1_X(xy) - 1_X(x)1_X(y)| \leq 1 \quad \text{and} \quad 1 \leq e^{-C} e^{\lambda(x)+\lambda(y)} = e^{-C}\omega(x)\omega(y).$$

On the other hand, suppose that $\lambda(x) + \lambda(y) \leq C$. We will show that this forces $1_X(xy) = 1_X(x)1_X(y)$. First, note that $x, y$ and $xy$ all belong to $W_C$. Now observe:
exists a constant $\lambda$ on $z$ is non-empty and $C$ – if $x \in W$ – if $xy$ lies in $X$, then both $x$ and $y$ belong to $\text{FBP}_C(\{xy\}) \subseteq \text{FBP}_C(X) = X$.

Thus $xy \in X$ if and only if $x$ and $y$ belong to $X$. \hfill \Box$

**Proof of (ii).** If $X \cap W_C = \emptyset$ then we are done. So suppose $X \cap W_C \neq \emptyset$. Let $x, y \in X \cap W_C$. Suppose $z \in W_C$ and $z \mid xy$. Then

\[
|1_X(xy) - 1_X(x)1_X(y)| \leq \text{def}_\omega(X)\omega(x)\omega(y) \leq \text{def}_\omega(X)e^{2C} < 1,
\]

\[
|1_X(xy) - 1_X(xy)1_X(z)| \leq \text{def}_\omega(X)\omega(xy)\omega(z) \leq \text{def}_\omega(X)e^{3C} < 1.
\]

Since $1_X(x) = 1 = 1_X(y)$, the first formula implies $1_X(xy) = 1$; feeding this into the second formula we deduce that $z \in X$. This shows that $\text{FBP}_C(X \cap W_C) \subseteq X \cap W_C$, and the converse inclusion is trivial. \hfill \Box

An obvious way to obtain $\text{FBP}_C$-stable sets is by iteration. For $k \geq 2$ we recursively define $\text{FBP}_C^k(E) = \text{FBP}_C(\text{FBP}_C^{k-1}(E))$, noting that

\[
E \cap W_C \subseteq \text{FBP}_C(E) \subseteq \text{FBP}_C^2(E) \subseteq \text{FBP}_C^3(E) \subseteq \ldots,
\]

and define $\text{FBP}_C^\infty(E)$ to be the inductive limit $\bigcup_{k \geq 1} \text{FBP}_C(E)$. By induction,

\[
\text{FBP}_C^\infty(E) \subseteq \text{Filt}(E) \cap W_C \quad \text{for all } E \subseteq S \text{ such that } E \cap W_C \neq \emptyset. \quad (10)
\]

For a given $E$ and $C$, the inclusion in (10) can be proper, since when constructing $\text{FBP}_C^\infty(E)$ we are only allowed to take binary products at each stage, and only allowed to consider factors which have log-weight at most $C$. On the other hand, if $E$ is non-empty and $z \in \text{Filt}(E)$, there always exists some $C \geq 0$, possibly depending on $z$, such that $z \in \text{FBP}_C^\infty(E)$. (For instance, if $x_1, \ldots, x_k \in E$ and $z \mid x_1 \cdots x_k$ then $C = \max\{\sum_{i=1}^k \lambda(x_i), \lambda(z)\}$ suffices.) These considerations lead naturally to the following definition.

**Definition 3.3 (Propagation).** For $z \in \text{Filt} E$, let

\[
V_E(z) = \inf\{C \geq 0 : z \in \text{FBP}_C^\infty(E)\}.
\]

Given $L \geq 0$, we say that $(S, \lambda)$ propagates at level $L$, or has $L$-propagation, if

\[
\sup_{\emptyset \neq E \subseteq W_L} \sup_{z \in \text{Filt}(E) \cap W_L} V_E(z) < \infty.
\]

It is convenient to set $V_E(z) := +\infty$ whenever $z \notin \text{Filt}(E)$. Note for future reference that $V_E(z) \geq \lambda(z)$.

**Remark 3.4.** Note that if $(S, \lambda)$ propagates at a level $L$, then it also does it at every lower level. Moreover, in the formula defining $L$-propagation, we could restrict $E$ to the finite subsets of $W_L$ without altering the value of the double supremum.

**Remark 3.5.** By definition, a log-weighted semilattice $(S, \lambda)$ propagates at level $L$ if there exists a constant $C$ such that if an element $z$ is a divisor of a product $x_1 \cdots x_n$, where all $z$, $x_1, \ldots, x_n$ have $\lambda$-weights at most $L$, then one can get to $z$ from $x_1, \ldots, x_n$ by repeatedly taking divisors of products of pairs while never using any resulting element unless its $\lambda$-weight is at most $C$. 7
If $V_E(z)$ is large, then to obtain $z$ from $E$ by repeatedly taking divisors of binary products, we must allow elements with large log-weight along the way, even if $z$ itself has small log-weight. Thus, understanding the behaviour of $V_E$ is the key to showing instability of filters for certain log-weights. This is demonstrated in the next theorem, which provides the reformulation of the stable filter and the stable character properties that we promised earlier.

**Theorem 3.6.** Let $\omega$ be a weight on a semilattice $S$. The following are equivalent.

(i) The algebra $\ell^1(S, \omega)$ is AMNM.

(ii) $S$ has $\omega$-stable filters.

(iii) Let $\lambda = \log \omega$. Then $(S, \lambda)$ has $L$-propagation for all $L \geq 0$.

**Proof.** The equivalence of (i) and (ii) was shown in Proposition 2.5.

We now prove (iii)$\Rightarrow$(ii). Suppose $(S, \lambda)$ has $L$-propagation for every $L \geq 0$. Let $\varepsilon > 0$ be given, and consider a fixed positive number $L > \ln(\varepsilon^{-1})$. From the definition of $L$-propagation, choose $C > L$ such that

$$C \geq \sup_{\emptyset \neq \omega \subseteq W_L} \sup_{z \in \text{Filt}(E) \cap W_L} V_E(z).$$

This is possible as the double supremum is finite by assumption. Finally, let $\delta := e^{-6C}$.

Suppose $G$ is a subset of $S$ for which $\text{def}_\omega(G) < \delta$. Then by Lemma 3.2, $G$ is FBP$_{2C}$-stable, i.e. FBP$_{2C}(G \cap W_{2C}) = G \cap W_{2C}$. If $G \cap W_L = \emptyset$ then $\text{dist}_\omega(G, \emptyset) < 1/e^L < \varepsilon$, and we are done. So suppose that $G \cap W_L \neq \emptyset$, and let $F = \text{Filt}(G \cap W_L)$. By the definition of $C$,

$$F \cap W_L \subseteq \text{FBP}_{2C}(G \cap W_L).$$

On the other hand,

$$\text{FBP}_{2C}(G \cap W_L) \cap W_L \subseteq \text{FBP}_{2C}(G \cap W_{2C}) \cap W_L = G \cap W_{2C} \cap W_L = G \cap W_L.$$

Hence, $G \cap W_L \subseteq F \cap W_L \subseteq G \cap W_L$, and so $G \cap W_L = F \cap W_L$ and $\text{dist}_\omega(F, G) < 1/e^L < \varepsilon$. Thus $(S, \omega)$ satisfies the equivalent condition, given in Lemma 2.3, of (ii).

Finally, we prove (ii)$\Rightarrow$(iii). Let $\lambda = \log \omega$ be the log-weight, and let $L > 0$ be given. Set $\varepsilon := e^{-L}$, and let $\delta < 2e^{-L}$ be chosen so that it satisfies the condition in Lemma 2.3. Put $C = -\log \frac{\delta}{2}$, and note that $C > L$. Let $E$ be a non-empty subset of $W_L$, and put $X = \text{FBP}_{2C}(E)$.

It suffices to show that $\text{Filt}(E) \cap W_L \subseteq X$ (since this implies the supremum in Definition 3.3 is $\leq C$ and hence finite). Note that $X$ is non-empty (since $X \cap W_L$ contains $E$) and FBP$_C$-stable. Hence, by Lemma 3.2, $\text{def}_\omega(X) \leq e^{-C} = \frac{\delta}{2} < \delta$. Since $(S, \omega)$ satisfies (ii), there exists $G \in \mathcal{F} \cup \{\emptyset\}$ such that $\text{dist}_\omega(X, G) < \varepsilon = 1/e^L$. This means that $X \cap W_L = G \cap W_L$. But since $E \subseteq X \cap W_L$, $G$ contains $E$ and hence contains $\text{Filt}(E)$. Therefore

$$\text{Filt}(E) \cap W_L \subseteq G \cap W_L = X \cap W_L \subseteq X,$$

as required. \qed
4 A structural theorem for semilattices with infinite breadth

In this section, we review the results of [CGPpre] that will be needed to prove our second main theorem. First, we recall some notational conventions.

Let $\Omega$ be a non-empty set. We write $\mathcal{P}(\Omega)$ for its power set, and $\mathcal{P}^{\text{fin}}(\Omega)$ for the set of all finite subsets of $\Omega$. Elements of $\Omega$ will usually be denoted by lower-case Greek letters.

In this section, we review the results of [CGPpre] that will be needed to prove our second main theorem. First, we recall some notational conventions.

Set systems on $\Omega$ (i.e. subsets of $\mathcal{P}(\Omega)$) will usually be denoted by letters such as $\mathcal{B}$, $\mathcal{S}$, etc. If $\mathcal{B}$ is such a set system, we refer to members of $\mathcal{B}$ rather than elements. Members of a set system $\mathcal{S}$ will be denoted by letters such as $a$, $b$, $p$, etc. Given $\mathcal{F} \subseteq \mathcal{P}(\Omega)$, the join of $\mathcal{F}$ is the set $\text{join}(\mathcal{F}) := \bigcup_{x \in \mathcal{F}} x$.

The set system $\mathcal{P}(\Omega)$ is considered as a semilattice where set-union serves as the binary operation. More generally, a union-closed set system or concrete semilattice on $\Omega$ is defined to be a subsemilattice of $\mathcal{P}(\Omega)$.

Conversely, every semilattice $\mathcal{S}$ can be viewed as a concrete semilattice, using the following construction. For $x \in \mathcal{S}$ let $E_x := \mathcal{S} \setminus \{ y \in \mathcal{S} : x \not\subseteq y \}$. It is easily checked that $E_x \cup E_y = E_{xy}$ for all $x, y \in \mathcal{S}$. Therefore, the function $E_\bullet : \mathcal{S} \to \mathcal{P}(\mathcal{S})$, $x \mapsto E_x$, defines an injective semilattice homomorphism from $\mathcal{S}$ into $(\mathcal{P}(\mathcal{S}), \cup)$. This is sometimes known as the Cayley embedding of a semilattice.

**Definition 4.1.** Let $\mathcal{S}$ be a semilattice. A finite, non-empty subset $E \subseteq \mathcal{S}$ is compressible if there exists a proper subset $E' \subset E$ such that $\prod_{x \in E} x = \prod_{x \in E'} x$; otherwise, we say $E$ is incompressible. The breadth of $\mathcal{S}$ is defined to be

$$b(\mathcal{S}) = \inf\{n \in \mathbb{N} : \text{every subset } E \subseteq \mathcal{S} \text{ with } n + 1 \text{ elements is compressible} \} = \sup\{n \in \mathbb{N} : \mathcal{S} \text{ has an incompressible subset with } n \text{ elements} \}$$

Incompressibility is referred to in [LLM77, Mis86] as “meet irredundant”. However, “incompressible” seemed to be better terminology for union-closed set systems, so we use the same terminology here to be consistent.

Note that diverse behaviour occurs even among semilattices of small breadth. For instance, every infinite $k$-ary rooted tree ($k \geq 2$) is a semilattice with breadth 2 that contains infinite chains and infinite antichains (see [CGPpre, Example 2.5]).

**Definition 4.2.** A spread is a sequence $\mathcal{E} = (E_n)_{n \geq 1}$ of finite non-empty subsets of $\Omega$ which are pairwise disjoint and satisfy $|E_n| \to \infty$. Given a spread $\mathcal{E} = (E_n)_{n \geq 1}$, for each $n \in \mathbb{N}$, let $E_{\leq n} := E_1 \cup \cdots \cup E_{n-1}$ (with the convention that $E_{\leq 1} = \emptyset$) and let $E_{>n} := \bigcup_{j \geq n+1} E_j$. Now define the following set systems on $\text{join}(\mathcal{E})$:

$$\mathcal{T}_{\text{max}}(\mathcal{E}) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{ E_{<n} \cup a \},$$

$$\mathcal{T}_{\text{min}}(\mathcal{E}) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{ a \cup E_{>n} \},$$

$$\mathcal{T}_{\text{ort}}(\mathcal{E}) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{ E_{<n} \cup a \cup E_{>n} \}.$$
Recall that the use of $\cup$ is to signify that the union is a disjoint one. Typical members of these semilattices are illustrated in Figure 1.

![Figure 1: Typical members of $T_{\text{max}}$, $T_{\text{min}}$, and $T_{\text{ort}}$ at level $n$](image)

The set systems $T_{\text{max}}$, $T_{\text{min}}$ and $T_{\text{ort}}$ are concrete realizations of standard examples, in the lattice theoretic literature, of semilattices with infinite breadth and “locally finite breadth”. For instance, $T_{\text{min}}$ corresponds to [Mis86, Example 1]. None of these three examples contain copies of $\{0, 1\}^N \cong \mathcal{P}(N)$, a property which is relevant to harmonic analysis on these semilattices: see [Mis86] for some further details and references.

Another concept introduced in [CGPpre] that plays an important role in the global structure of semilattices is the following.

**Definition 4.3 (Decisive colourings).** Given a spread $\mathcal{E}$, a partition of $\Omega$ into finitely many subsets $\Omega = C_1 \cup \cdots \cup C_d$ is said to colour $\mathcal{E}$ if $\lim_n |C_j \cap E_n| = \infty$ for each $j$. We call $\mathcal{C} = \{C_1, \ldots, C_d\}$ a colouring of $\mathcal{E}$.

Given a set system $S \subseteq \mathcal{P}(\Omega)$, we say that the colouring $\mathcal{C}$ decides $S$ with respect to $\mathcal{E}$ if there exists a colour class $C_0 \in \mathcal{C}$ with the following property: every $x \in S$ satisfies

$$\sup_{n \geq 1} \min \{|x \cap C_0 \cap E_n| : |x \cap \mathcal{E} \cap C \cap E_n|, C \in \mathcal{C}\} < \infty.$$  

(11)

In this context we say $C_0$ is a decisive colour class.

Informally speaking, when we have a decisive colour class $C_0$, we know that a set $x \in S$ must either have small intersection with $C_0 \cap E_n$, or else have large intersection with $C \cap E_n$ for some $C \in \mathcal{C}$, once $n$ is sufficiently large.

To state the required result from [CGPpre], we also need the following notation: Given $S \subseteq \mathcal{P}(\Omega)$, $a \in \mathcal{P}(\Omega)$, we define

$$S \boxdot a := \{x \cap a : x \in S\} \subseteq \mathcal{P}(\Omega).$$  

(12)

If $S$ is a concrete semilattice on $\Omega$, then $S \boxdot a$ is a concrete semilattice on $a$, and the obvious map $S \to S \boxdot a$ is a semilattice epimorphism.

The following is a consequence of [CGPpre, Corollary 5.8]. This is the bare minimum that we need from [CGPpre] for our construction (its proof would still require the full power of [CGPpre] though).

**Theorem 4.4.** Let $S$ be a semilattice with infinite breadth. Then there is a concrete representation $S$ of $\mathcal{S}$ on some set $\Omega$, and a spread $\mathcal{E}$ in $\Omega$, such that at least one of the following statements holds.

(i) $S \boxdot \text{join}(\mathcal{E}) \supseteq T_{\text{max}}(\mathcal{E})$.

(ii) $S \boxdot \text{join}(\mathcal{E}) \supseteq T_{\text{min}}(\mathcal{E})$.

(iii) $S \boxdot \text{join}(\mathcal{E}) \supseteq T_{\text{ort}}(\mathcal{E})$ and there is an $S$-decisive colouring of $\mathcal{E}$.
5 Infinite breadth semilattices and instability

Suppose $S$ is a semilattice with finite breadth. It is not hard, given Theorem 3.6, to see that $(S, \omega)$ has stable filters for every choice of submultiplicative weight $\omega : S \to [1, \infty)$. In other words, we obtain another proof of Theorem A from the Introduction (recall that this result was originally shown in [Cho13]). We can now state the second main theorem of this paper, which proves Theorem B.

**Theorem 5.1.** Let $S$ be a semilattice with infinite breadth. Then there is a log-weight $\lambda$ on $S$ such that $(S, \lambda)$ does not have 1-propagation, and so there is a weight $\omega$ on $S$ such that $\ell^1(S, \omega)$ is not AMNM.

The proof of Theorem 5.1 will be given in the next section. Before doing so, we give a proof for $S = \mathcal{P}^\text{fin}(\Omega)$ on any infinite set $\Omega$ (Proposition 5.3). This is for two reasons: firstly, we can prove the result for $\mathcal{P}^\text{fin}(\Omega)$ without using Theorem 4.4; secondly, the argument serves as a prototype for what we shall do in the proof of the general case.

Let $\Omega_{00}$ be a finite set and consider the union-closed set system $\mathcal{P}_*(\Omega_{00}) := \mathcal{P}(\Omega_{00}) \setminus \{\emptyset\}$. Define

$$\lambda(z) := |z| \text{ if } z \subseteq \Omega_{00}, \quad \lambda(\Omega_{00}) := 0. \quad (z \in \mathcal{P}_*(\Omega_{00})). \quad (13)$$

It is easily checked that $\lambda$ is a log-weight. Let $\mathcal{E} = \{\omega : \omega \in \Omega_{00}\}$, and note that $\mathcal{E} \subseteq W_1(\lambda)$ and $\{\Omega_{00}\} = W_0(\lambda) \subseteq W_1(\lambda)$. Intuitively, for this log-weight and sufficiently small $C$, the constraint in the FBP$_C$ operation that we can only multiply elements of log-weight $\leq C$ creates a barrier separating us from join($\mathcal{E}$); since join($\mathcal{E}$) $\in W_1(\lambda)$, this prevents propagation. More precisely, we have the following lemma.

**Lemma 5.2.** $V_{\mathcal{E}}(\Omega_{00}) \geq \frac{1}{2}|\Omega_{00}|$.

**Proof.** Let $C \geq 1$ be such that $\Omega_{00} \in \text{FBP}^C_{\infty}(\mathcal{E})$. Note that $\Omega_{00} \notin \mathcal{E} = \text{FBP}^0_C(\mathcal{E})$. Let $m \in \mathbb{N}$ be minimal such that $\Omega_{00} \in \text{FBP}^m_C(\mathcal{E})$. Then there exist $a_1, a_2 \in \text{FBP}^{m-1}_C(\mathcal{E})$ such that $\Omega_{00} \subseteq a_1 \cup a_2$. By minimality of $m$, both $a_1$ and $a_2$ are proper subsets of $\Omega_{00}$, and so

$$|\Omega_{00}| \leq |a_1| + |a_2| = \lambda(a_1) + \lambda(a_2) \leq 2C.$$

Hence $2V_{\mathcal{E}}(\Omega_{00}) \geq |\Omega_{00}|$, as required. \hfill $\square$

**Proposition 5.3.** Let $\Omega$ be an infinite set, and let $\mathcal{S} = \mathcal{P}^\text{fin}(\Omega)$. Then there is a log-weight $\lambda : \mathcal{S} \to [0, \infty)$ such that $(\mathcal{S}, \lambda)$ does not have 1-propagation.

**Proof.** Fix a spread $\mathcal{E} = (E_n)_{n \geq 1}$ in $\Omega$. Let $\Omega_0 = \text{join}(\mathcal{E}) \subseteq \Omega$. For $x \in \mathcal{P}^\text{fin}(\Omega)$, let $C(x) := \{n \in \mathbb{N} : E_n \subseteq x\}$. Now define $\lambda : \mathcal{P}^\text{fin}(\Omega) \to [0, \infty)$ by

$$\lambda(x) := |x \setminus \bigcup_{n \in C(x)} E_n|.$$

That is, we throw away all the $E_n$ which are contained in $x$, and then we count the number of remaining elements. Given $x, y \in \mathcal{P}^\text{fin}(\Omega)$ we have $C(x \cup y) \supseteq C(x) \cup C(y)$, and so

$$(x \cup y) \setminus \left( \bigcup_{n \in C(x \cup y)} E_n \right) \subseteq \left( x \setminus \bigcup_{n \in C(x)} E_n \right) \cup \left( y \setminus \bigcup_{n \in C(y)} E_n \right).$$

Hence $\lambda(x \cup y) \leq \lambda(x) + \lambda(y)$, and so we have a valid log-weight.
To show failure of 1-propagation, we construct a sequence \((\mathcal{E}_n)\) in \(W_1\) such that

\[
\sup_{z \in \text{Filt}(\mathcal{E}_n) \cap W_1} V_{\mathcal{E}_n}(z) \to \infty.
\]

For \(n \in \mathbb{N}\), let \(\mathcal{E}_n = \{\{\delta\} : \delta \in E_n\} \subseteq \mathcal{S}\) and note that \(\text{join}(\mathcal{E}_n) = E_n\). Note also that if \(a \in \text{Filt}(\mathcal{E}_n)\) then \(a \subseteq E_n\), and if \(a\) is a proper subset of \(E_n\) then \(\lambda(a) = |a|\). We have \(\mathcal{E}_n \subseteq W_1\) and \(E_n \in \text{Filt}(\mathcal{E}_n) \cap W_0 \subseteq \text{Filt}(\mathcal{E}_n) \cap W_1\). Now, let \(C \geq 0\) be such that \(E_n \in \text{FBP}_C^\infty(\mathcal{E}_n)\). By the same “barrier argument” as in the proof of Lemma 5.2, we deduce that \(|E_n| \leq 2^C\). Thus \(V_{\mathcal{E}_n}(E_n) \geq \frac{1}{2^C}|E_n|\), which by assumption tends to infinity as \(n \to \infty\).

### 6 The proof of Theorem 5.1

Let \(S\) be a semilattice with infinite breadth. The last assertion of Theorem 5.1, which is the conclusion of Theorem B, follows from the first and Theorem 3.6, so it remains for us to prove the first assertion. For that purpose, by the Cayley embedding, we can always assume that \(S = S\) is a concrete subsemilattice of \((\mathcal{P}(\Omega), \cup)\) for some set \(\Omega\). Moreover, by Theorem 4.4, there are three cases to consider; in each case, we define a suitable log-weight on \(S\) using the subquotient structure (MAX, MIN or ORT) present in it.

Note that it is relatively easy to construct log-weights on these subquotients for which propagation fails. However, although log-weights on quotients of \(S\) pull back to give log-weights on \(S\), not every log-weight on a substructure of \(S\) can be extended to a log-weight on \(S\); this is one of the major sources of difficulty in proving Theorem 5.1. (Another subtlety is that having \(L\)-propagation need not pass to substructures or quotient structures, as will be seen in Section 7.)

Before starting the case-by-case analysis, we record a trivial observation for later reference.

**Remark 6.1.** Let \(\mathcal{F} \subseteq \mathcal{S}\). Then any factor of any product of members of \(\mathcal{F}\) is a subset of \(\text{join}(\mathcal{F})\). That is,

\[
\text{Filt}_S(\mathcal{F}) \subseteq \text{Filt}_{\mathcal{P}(\Omega)}(\mathcal{F}) \subseteq \mathcal{P}(\text{join}(\mathcal{F})).
\]

### 6.1 Proof of Theorem 5.1: Case I

Suppose that there is a spread \(\mathcal{E} = (E_n)_{n \geq 1}\) in \(\Omega\) such that \(\mathcal{S} \boxtimes \text{join}(\mathcal{E}) \supseteq \mathcal{T}_{\text{max}}(\mathcal{E})\). For each \(x \in \mathcal{P}(\Omega)\), define

\[
\lambda(x) := \begin{cases} 
0 & \text{if there are no or infinitely many } n \text{ such that } E_n \cap x \neq \emptyset \\
0 & \text{if } E_n \subseteq x \\
|\cdot \cap E_n| & \text{otherwise};
\end{cases}
\]  

(14)

where in the last two cases \(n\) is the largest natural number such that \(E_n \cap x \neq \emptyset\).

**Lemma 6.2.** Let \(x, y \in \mathcal{P}(\Omega)\). Then \(\lambda(x \cup y) \leq \lambda(x) + \lambda(y)\).

**Proof.** Fix \(x, y \in \mathcal{P}(\Omega)\). In the following, any case where some condition on \(x\) is satisfied has a corresponding case for \(y\), which is simply omitted.

The case when \(x \cap \text{join}(\mathcal{E}) = \emptyset\) (or \(y \cap \text{join}(\mathcal{E}) = \emptyset\)) is obvious.

If there are infinitely many \(n\) such that \(E_n \cap x \neq \emptyset\), then the same is true for \(x \cup y\), and so the inequality follows, since \(\lambda(y) \geq 0\).
Otherwise, let \( m \) be the largest natural number such that \( E_m \cap x \neq \emptyset \), and let \( n \) be the corresponding number for \( y \). If \( m \neq n \), then without loss of generality we suppose \( m > n \). Then \( m \) is also the largest number such that \( E_m \cap (x \cup y) \neq \emptyset \). Moreover \( E_m \cap (x \cup y) = E_m \cap x \), so the inequality follows, since \( \lambda(y) \geq 0 \).

If not, \( m = n \) is the largest number such that \( E_m \cap (x \cup y) \neq \emptyset \). If furthermore \( E_m \subseteq x \), then \( E_m \subseteq x \cup y \), and the inequality is again obvious. Otherwise, we see that

\[
\lambda(x \cup y) \leq |(x \cup y) \cap E_m| \leq |x \cap E_m| + |y \cap E_m| = \lambda(x) + \lambda(y).
\]

This completes the proof. \(\square\)

We will now show that \( (S, \lambda) \) does not have 1-propagation.

**Proof.** Let \( n \in \mathbb{N} \). Since \( S \sqsupseteq \text{join}(\mathcal{E}) \supseteq T_{\text{max}}(\mathcal{E}) \), there is a size-\( n \) subset \( \mathcal{F}_n \subseteq S \) such that:

- for each \( a \in \mathcal{F}_n \), \( E_j \subseteq a \) for \( j < n \), \( a \cap E_n \) is a singleton, and \( a \cap E_j = \emptyset \) for \( j > n \);
- \( a \cap E_n \) gives different singletons for different \( a \in \mathcal{F}_n \).

From our construction, \( \lambda(a) = 1 \) for every \( a \in \mathcal{F}_n \).

Set \( b_n := \text{join}(\mathcal{F}_n) \). Then \( \lambda(b_n) = 0 \), and so \( b_n \in \text{Filt}(\mathcal{F}_n) \land W_1 \). To finish the proof it suffices to show that \( V_{\mathcal{F}_n}(b_n) \to \infty \) as \( n \to \infty \), which we do using our barrier argument (cf. the proof of Lemma 5.2). Let \( C \geq 1 \) be such that \( b_n \in \text{FBP}^\infty_C(\mathcal{F}_n) \). Let \( m \geq 1 \) be minimal with respect to the following property:

there exists \( a \in \text{FBP}^m_C(\mathcal{F}_n) \) which contains \( E_n \).

(Such an \( m \) exists by our assumption, since \( E_n \subseteq b_n \).) By minimality there are \( a_1 \) and \( a_2 \) in \( \text{FBP}^{m-1}_C(\mathcal{F}_n) \) such that \( E_n \) is contained in \( a_1 \cup a_2 \), yet \( E_n \not\subseteq a_1 \) and \( E_n \not\subseteq a_2 \). (When \( m = 1 \), our convention here is that \( \text{FBP}^0_C(\mathcal{F}_n) = \mathcal{F}_n \).

Let \( i \in \{1,2\} \). By the previous remarks, \( a_i \cap E_n \) is a proper, nonempty subset of \( E_n \).

By Remark 6.1, \( a_i \subseteq b_n \). Thus \( n \) is the largest natural number \( k \) such that \( a_i \cap E_k \neq \emptyset \), and so \( \lambda(a_i) = |a_i \cap E_n| \). Hence \( |E_n| \leq \lambda(a_1) + \lambda(a_2) \leq 2C \), with the last inequality following because \( a_1 \cup a_2 \in W_C \). Therefore, \( V_{\mathcal{F}_n}(b_n) \geq \frac{1}{2}|E_n| \to \infty \), completing the proof. \(\square\)

### 6.2 Proof of Theorem 5.1: Case II

This case is similar to (and somewhat easier than) the first case. Suppose that there is a spread \( \mathcal{E} = (E_n)_{n \geq 1} \) in \( \Omega \) such that \( S \sqsupseteq \text{join}(\mathcal{E}) \supseteq T_{\text{min}}(\mathcal{E}) \). Then, for each \( x \in \mathcal{P}(\Omega) \), define

\[
\lambda(x) := \begin{cases} 
0 & \text{if } x \cap \text{join}(\mathcal{E}) = \emptyset \\
0 & \text{if } E_n \subseteq x \\
|x \cap E_n| & \text{otherwise};
\end{cases}
\]

where in the last two cases \( n \) is the smallest natural number such that \( E_n \cap x \neq \emptyset \).

**Lemma 6.3.** In this case, \( \lambda \) is a log-weight on \( S \), and \( (S, \lambda) \) does not have 1-propagation.

**Proof.** This is similar to Case I. \(\square\)
6.3 Proof of Theorem 5.1: Case III

In this final case, by Theorem 4.4, we suppose that there exists a spread \( \mathcal{E} \) with a colouring of \( \mathcal{E} \) which is \( S \)-decisive such that \( S \sqcup \text{join}(\mathcal{E}) \supseteq \mathcal{T}_{\text{ort}}(\mathcal{E}) \).

**Lemma 6.4** (Creating a log-weight from a decisive colouring). Let \( S \subseteq \mathcal{P}(\Omega) \) be union-closed, and let \( \mathcal{E} = (E_n)_{n \geq 1} \) be a spread in \( \Omega \). Suppose there is an \( S \)-decisive colouring of \( \mathcal{E} \), call it \( C \), with a decisive colour class \( C_0 \).

For each \( x \in S \) define

\[
T(x) = \{ n \in \mathbb{N} : |x \cap C \cap E_n| \leq \frac{1}{2} |C \cap E_n| \text{ for all } C \in \mathcal{C} \}
\]

and define \( \lambda(x) = \sup_{n \in T(x)} |x \cap C_0 \cap E_n| \). Then \( \lambda(x) < \infty \), and \( \lambda : S \to [0, \infty) \) is a log-weight.

Note that if \( T(x) = \emptyset \), then \( \lambda(x) = 0 \), i.e. we take the usual convention when considering the least upper bounds of subsets of \([0, \infty)\).

**Proof.** The first step is to show \( \lambda(x) < \infty \). If \( T(x) \) is finite there is nothing to prove; so assume \( T(x) \) is infinite. Since \( C \) is a colouring of the spread \( \mathcal{E} \), we have \( \min_{C \in \mathcal{C}} |C \cap E_n| \to \infty \). Therefore, since \( T(x) \) is infinite, \( |x^c \cap C \cap E_n| \to \infty \) along \( n \in T(x) \). Now it follows from the condition (11) and the definition of \( \lambda \) that \( \lambda(x) < \infty \).

Finally, given \( x \) and \( y \) in \( S \), observe that \( T(x \cup y) \subseteq T(x) \cap T(y) \). Hence

\[
\lambda(x \cup y) \leq \sup_{n \in T(x \cup y)} (|x \cap C_0 \cap E_n| + |y \cap C_0 \cap E_n|) \leq \lambda(x) + \lambda(y),
\]

as required. \( \square \)

**Proposition 6.5.** Let \( S, \mathcal{E}, C \) and \( \lambda \) be as in the previous lemma with \( S \sqcup \text{join}(\mathcal{E}) \supseteq \mathcal{T}_{\text{ort}}(\mathcal{E}) \). Then \( (S, \lambda) \) does not have 1-propagation.

**Proof.** Fix \( n \in \mathbb{N} \). Let \( C_0 \) be the decisive colour class used to define \( \lambda \), and enumerate the elements of \( E_n \cap C_0 \) as \( \gamma_1, \ldots, \gamma_M \). Since \( S \sqcup \text{join}(\mathcal{E}) \supseteq \mathcal{T}_{\text{ort}}(\mathcal{E}) \), there exist \( x_1, \ldots, x_M \in S \) such that

\[
x_j \cap \text{join}(\mathcal{E}) = E_{<n} \cup \{ \gamma_j \} \cup E_{\geq n} \quad (1 \leq j \leq M).
\]

In particular \( x_i \cap E_n = x_i \cap C_0 \cap E_n = \{ \gamma_i \} \), while \( x_i \cap E_j = E_j \) for all \( j \neq i \). It follows that \( \lambda(x_i) = 1 \) for \( i = 1, \ldots, M \).

Let \( F_n = \{ x_1, \ldots, x_M \} \subseteq W_1 \), and let \( b_n = \text{join}(F_n) \). Since \( b_n \cap E_n = C_0 \cap E_n \), and since \( b_n \cap E_m = E_m \) for all \( m \neq n \), we have \( \lambda(b_n) = 0 \). Hence \( b_n \in \text{Filt}(F_n) \cap W_1 \).

Let \( K \geq 1 \) and suppose \( b_n \in \text{FBP}_K(F_n) \). Then, in particular, there exists \( m \geq 1 \) with the following property:

there exists \( y \in \text{FBP}_K^n(F_n) \) such that \( |y \cap C_0 \cap E_n| \geq \frac{1}{2} |C_0 \cap E_n| \).

Let \( m \) be minimal with respect to this property, and let \( y \) be the corresponding member of \( \text{FBP}_K^n(F_n) \). Then there exist \( y_1 \) and \( y_2 \) in \( \text{FBP}_K^{m-1}(F_n) \) such that \( y \subseteq y_1 \cup y_2 \); again, when \( m = 1 \), our convention here is that \( \text{FBP}_K(F_n) = F_n \). By the minimality of \( m \),

\[
|y_1 \cap C_0 \cap E_n| < \frac{1}{2} |C_0 \cap E_n| \quad \text{and} \quad |y_2 \cap C_0 \cap E_n| < \frac{1}{2} |C_0 \cap E_n|.
\]

14
By Remark 6.1, \( \text{Filt}(F_n) \subseteq \mathcal{P}(\text{join}(F_n)) = \mathcal{P}(b_n) \). In particular, for \( i \in \{1, 2\} \), we have \( y_i \subseteq b_n \) and hence \( y_i \cap E_n \subseteq b_n \cap E_n = C_0 \cap E_n \).

This implies that \( n \in T(y_i) \), so that
\[
\lambda(y_i) = \sup_{j \in T(y_i)} |y_i \cap C_0 \cap E_j| \geq |y_i \cap C_0 \cap E_n|.
\]

Putting this all together, and remembering that \( y_1 \) and \( y_2 \) belong to \( W_K \),
\[
\frac{1}{2} |C_0 \cap E_n| \leq |(y_1 \cup y_2) \cap C_0 \cap E_n| \leq |y_1 \cap C_0 \cap E_2| + |y_2 \cap C_0 \cap E_2| \leq \lambda(y_1) + \lambda(y_2) \leq 2K.
\]

Hence \( K \geq \frac{1}{4} |C_0 \cap E_n| \). It follows that \( V_{\mathcal{F}_n}(b_n) \geq \frac{1}{4} |C_0 \cap E_n| \to \infty \), as required. \( \square \)

7 Some further examples

The proof of Theorem 5.1 could have been streamlined if the following claims were true:

(i) if \( q : S \to T \) is a surjective homomorphism of semilattices, and \( (T, \lambda^*) \) fails \( L \)-propagation, then \( (S, \lambda^* \circ q) \) fails \( L \)-propagation;

(ii) if \( (S, \lambda) \) has \( L \)-propagation then so does \( (R, \lambda) \) for every subsemilattice \( R \subset S \).

We present examples here to show that both claims are false.

Let \( \Omega = \{0, 1, 2\} \times \mathbb{N} \). Consider two sequences \( (x_j)_{j=1}^{\infty} \) and \( (a_n)_{n=1}^{\infty} \) defined as follows:
\[
x_j := \{(1,j), (2,j)\} ; \quad a_n := \{(0,k), (1,k) : n^2 \leq k < (n+1)^2\}.
\]

It may be helpful to picture these sets as certain rectangular “tiles”, as in Figure 2.

![Figure 2: Sets x_j and a_n](image)

Let \( \mathcal{X} = \{x_j : j \in \mathbb{N}\} \). We define \( \mathcal{S} \) to be the semilattice generated inside \( \mathcal{P}(\Omega) \) by \( \mathcal{A} \) and \( \mathcal{X} \). It is straightforward to show that every member of \( \mathcal{S} \) has a unique decomposition, up to ordering, as a union of members of \( \mathcal{A} \) and members of \( \mathcal{X} \).

We now take \( \lambda^*(z) := |\{j \in \mathbb{N} : (2,j) \in z\}| \) which is clearly subadditive, and so gives us a log-weight on \( \mathcal{P}(\Omega^*) \). We now take
\[
\lambda^*(z) := |\{j \in \mathbb{N} : (2,j) \in z\}|
\]
which is clearly subadditive, and so gives us a log-weight on \( \mathcal{P}(\Omega^*) \). Note that \( \lambda \) is given by the same formula as \( \lambda^* \). Moreover, if \( z \in \mathcal{S} \), then \( \lambda(z) \) counts how many factors from \( \mathcal{X} \) occur in the “prime factorization” of \( z \).
Example 7.1. Define $\mathcal{T} = q(\mathcal{S}) \subset \mathcal{P}(\Omega^*)$. We will show that $(\mathcal{T}, \lambda^*)$ does not have 1-propagation.

Let $b_n := q(a_n) = \{(1,k): n^2 \leq k < (n+1)^2\}$ and let $\mathcal{B} = \{b_n: n \in \mathbb{N}\}$. Then $\mathcal{T}$ is the subsemilattice of $\mathcal{P}(\Omega^*)$ generated by $\mathcal{X}$ and $\mathcal{B}$. For each $n$, let $\mathcal{E}_n = \{x_k: n^2 \leq k < (n+1)^2\}$. Then $\mathcal{E}_n \subseteq W_1(\mathcal{T})$ and $b_n \in \text{Filt}(\mathcal{E}_n) \cap W_1(\mathcal{T})$. Therefore, it suffices to prove that $\mathcal{V}_{\mathcal{E}_n}(b_n) \to \infty$ as $n \to \infty$. In fact, we will show that $\mathcal{V}_{\mathcal{E}_n}(b_n) \geq n$, by proving the following result.

Claim. Let $1 \leq C \leq n$. Then $\text{FBP}^m_C(\mathcal{E}_n) \subseteq \langle \mathcal{E}_n \rangle = \mathcal{P}(\mathcal{E}_n)$.

This claim implies the desired inequality, since $b_n \notin \langle \mathcal{E}_n \rangle$. We prove the claim by induction on $m$, using the familiar barrier argument. The case $m = 0$ is trivial. If the claim holds for $m = k - 1$ where $k \in \mathbb{N}$, then let $y, z \in \text{FBP}^{k-1}_C(\mathcal{E}_n)$ and let $a \in W_C(\mathcal{T})$ satisfy $a \subseteq y \cup z$. By the inductive hypothesis, $y$ and $z$ are subsets of $\mathcal{E}_n$. Since $\lambda^*(y) \leq C \leq n$ and $\lambda^*(z) \leq C \leq n$, while $|\mathcal{E}_n| = (n+1)^2 - n^2 = 2n + 1$, $y \cup z$ must be a proper subset of $\mathcal{E}_n$. Hence, no member of $\mathcal{B}$ is contained in $y \cup z$, and clearly the only members of $\mathcal{X}$ contained in $y \cup z$ are the members of $\mathcal{E}_n$, so $a \in \mathcal{P}(\mathcal{E}_n)$. Thus, the claim holds for $m = k$, completing the inductive step.

Example 7.2. $(\mathcal{S}, \lambda)$ has $L$-propagation for all $L \geq 0$.

Proof. Fix $L \geq 0$ and let $\mathcal{E}$ be a non-empty subset of $W_L$. Given $z \in \text{Filt}(\mathcal{E}) \cap W_L$, we will show that $\mathcal{V}_z(z) \leq L$.

First note that each “prime factor” of $z$ must be a factor of some element in $\mathcal{E}$ (by unique factorization). Hence, $z = a' \cup x'$ where $a'$ is a product of members of $\mathcal{A} \cap \text{fac}(\mathcal{E})$ and $x'$ is a product of members of $\mathcal{X} \cap \text{fac}(\mathcal{E})$.

Write $x' = x_{n(1)} \cup x_{n(2)} \cup \cdots x_{n(k)}$ where $n(1) < n(2) < \cdots < n(k)$. Then $k = \lambda(z) \leq L$. If $L < 1$ then $z = a'$; by induction on the number $m$ of the “prime factors” of $a'$, we obtain $a' \in \text{FBP}^m_0(\mathcal{E}) \subseteq \text{FBP}^m_L(\mathcal{E})$, and so we’re done. If $L \geq 1$, then by inductively considering $y_0 := a'$, $y_1 := y_0 \cup x_{n(1)}$, $y_2 := y_1 \cup x_{n(2)}$, etc., we obtain $y_j \in \text{FBP}^m_L(\mathcal{E})$ for all $j = 0, 1, \ldots, k$. Since $z = y_k \in \text{FBP}^m_L(\mathcal{E})$, this completes the proof.

Now consider the sets $g_j := \{(j, k): k \in \mathbb{N}\}$. We let $\mathcal{G}$ be the set of all $g_j$ and define $\mathcal{R}$ to be the semilattice generated by $\mathcal{X}$ and $\mathcal{G}$. Since each $b_n$ is the union of finitely many members of $\mathcal{G}$, $\mathcal{T}$ is a subsemilattice of $\mathcal{R}$.

Example 7.3. We claim that $(\mathcal{R}, \lambda^*)$ has $L$-propagation for all $L \geq 0$. The proof is very similar to the proof for $(\mathcal{S}, \lambda)$. Unlike $\mathcal{S}$, the semilattice $\mathcal{R}$ does not have “unique factorization”; but each $z \in \mathcal{R}$ has a largest factor belonging to $\langle \mathcal{X} \rangle$, and this factor has a unique decomposition as a union of members of $\mathcal{X}$. Using this, one can carry out the same kind of argument that was used to show $(\mathcal{S}, \lambda)$ has $L$-propagation. We leave the details to the reader.

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Yemon Choi, Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom.

y.choi1@lancaster.ac.uk

Mahya Ghandehari, Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, United States of America.

mahya@udel.edu

Hung Le Pham, School of Mathematics and Statistics, Victoria University of Wellington, Wellington 6140, New Zealand.

hung.pham@vuw.ac.nz