Mean Field Linear-Quadratic-Gaussian (LQG) Games for Stochastic Integral Systems

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Abstract

In this paper we discuss a class of mean field linear-quadratic-Gaussian (LQG) games for large population system which has never been addressed by existing literature. The features of our works are sketched as follows. First of all, our state is modeled by stochastic Volterra-type equation which leads to some new study on stochastic “integral” system. This feature makes our setup significantly different from the previous mean field games where the states always follow some stochastic “differential” equations. Actually, our stochastic integral system is rather general and can be viewed as natural generalization of stochastic differential equations. In addition, it also includes some types of stochastic delayed systems as its special cases. Second, some new techniques are explored to tackle our mean-field LQG games due to the special structure of integral system. For example, unlike the Riccati equation in linear controlled differential system, some Fredholm-type equations are introduced to characterize the consistency condition of our integral system via the resolvent kernels. Third, based on the state aggregation technique, the Nash certainty equivalence (NCE) equation is derived and the set of associated decentralized controls are verified to satisfy the $\epsilon$-Nash equilibrium property. To this end, some new estimates of stochastic Volterra equations are developed which also have their own interests.

Keywords: $\epsilon$-Nash Equilibrium, Fredholm Equation, Mean Field LQG Games, Stochastic Integral System, Stochastic Volterra Equation.

1 Introduction

The large-population systems have important applications in various fields including, but not limited to, social science \[12\], financial management, economics \[15\], [26], industry engineering (e.g., multi-agent systems \[17\], [24], coupled oscillators \[27\], wireless communication \[9\]), etc. The most significant feature of large population system lies in the existence of considerable negligible agents or players involved in their individual states or (and) cost functionals through the state average across the whole population. Due to such high complexity in dynamics modeling, it is intractable to study the “centralized” control strategy since it strongly requires the congregation of global information throughout all individual agents. In many cases, such information congregation procedure turns out to be impossible yet unnecessary due to the complex...
interaction of all agents. Alternatively, it will be more reasonable to study the “decentralized”
decision strategies of our controlled large population systems for sake of complexity deduction
and computation efficiency.

Recently, there has been increasing research interest in studying this type of stochastic
control problems as well as addressing their applications. In particular, the research in this
area has been well developed for the decentralized control of large population system during
the last few decades. Among them, one efficient methodology is to consider the related mean-
field game by transforming the large-population or multi-agent optimization into one single-
agent optimization. Concerning to this methodology, for each given agent, only decentralized
information of its own state is required to design the control policy while the effects of remaining
agents are summarized by some mean-field term to be determined by the consistency condition.
As a consequence, each agent aims to solve some Hamilton-Jacobi-Bellman (HJB) equation
which is coupled with the Fokker-Planck (FP) equation of controlled large population system.
By the coupling of FP equation, the population or global effects are combined thus each agent
only needs to derive the feedback strategy involving its own state. In case of the linear controlled
system, it leads to some LQG mean-field games where the Riccati equations are coupled with
the FP equation of our system. Some relevant literature of LQG mean-field game include [2], [8],
[10], [11], [16], [20] and the reference therein. In rough sense, its general setup can be presented
as follows. There exist a collection of large number negligible players where individual state
follows linear stochastic differential equations

\[ dx_i(t) = \left[ A_i x_i(t) + B_i u_i(t) + F_i x^{(N)}(t) \right] dt + D_i dW_i(t), \quad (1) \]

where \( x^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_i \) denotes the state average or mean field term. It characterizes the global
effects of all agents in our population. The cost functional to be minimized takes the following form:

\[ J_i(u_i(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[ Q(x_i(t) - x^{(N)}(t))^2 + Ru_i^2(t) \right] dt + H x_i^2(T) \right\}. \]

The scheme of mean-field LQG game is formulated into some limiting LQG problem which can be
solved using the own individual state without the complicated interactions of all agents.
Based on it, by reproducing the best response of the state, the decentralized control policy can be
derived and some Nash equilibrium property in near-optimal sense should be verified. Last,
we would like to mention some other works on mean-field (McKean-Vlasov) SDEs (MFSDEs):
for example, see [1] [3] for stochastic mean-field control and related maximum principles, [4]
for the large deviation principle (LDP) of weakly coupled MFSDEs, and [19] for the stochastic
control of MFSDEs using the Malliavin calculus and its applications to mean-variance problem.

Based on above works, herein we turn to study the large population system with stochastic
“integral” instead of “differential” dynamics. Accordingly, new methodologies are required to be
developed, which are significantly different from those in the standard stochastic control theory
in the literature, to cope with the mathematical difficulties involved in this mean-field LQG.
Our motivation to mean-field LQG integral games are mainly as follows. First, it is remarkable there
exists many stochastic or deterministic models arising from physical, economics and finance
which cannot be represented by differential systems, see for example, stochastic input-output
model [5], capital replacement model [13], models in nanoscale biophysics [14]. Therefore, in
terms of stochastic integral system, it is more appropriate to study the mean-field LQG integral
game emerged naturally in the mass behavior of large population. Second, our integral system
includes some cases of stochastic delay in state or control variable which play important roles in biology, social science and engineering. One example is advertising model for which there are some delay effects [6], [7]. When we consider the interactions effects of all small advertising firms through the whole industry, it is necessary to formulate some mean-field integral games. Another example is the dynamic optimization of large population wireless interaction knots where there exist some delay effects in signal transmission thus some large population stochastic delayed games should be formulated.

Motivated by above concerns, in this paper, we consider the following controlled stochastic Volterra integral equations

\[ x_i(t) = \varphi(t) + \int_0^t b(t,s)x_i(s)ds + \int_0^t f(t,s)x^N(s)ds + \int_0^t c(t,s)u_i(s)ds + \int_0^t \sigma(t,s)dW_i(s). \]

(2)

where \( x^N = \frac{1}{N} \sum_{j=1}^N x_j \) is the mean field term which characterizes average interaction and mass effects of our population in spatial variable, \( \varphi, b, c, \sigma \) are deterministic functions, called the kernels, which characterize path dependence of out state in its temporal variable. Compared with mean field LQG games in [2], [8]-[12], [20], there are several new features or obstacles arising in such general framework. The first one is concerned on the expression of optimal control where the skills of Riccati equations are no long useful. By means of series skills, resolvent kernel, and backward stochastic Volterra integral equations (BSVIEs for short), we obtain optimal control \( \hat{u}_i(t) \) where the path dependence on state \( x_i \) in \( [t, T] \) is shown explicitly. As to BSVIEs, we refer the readers to [23], [25], [28], [29]. Secondly, the equation satisfied by optimal state is no long a linear SDEs but a new kind of equation with double integrals. We call it stochastic Volterra-Fredholm equation, the solvability of which is discussed under certain conditions. Thirdly, in the procedure of deriving NCE equation, certain Fredholm integral equations work well while Riccati equation becomes unapplicable again. Such result is consistent with the arguments in [22] where Riccati equation is replaced by Fredholm integral equation to express result more clearly. Fourthly, in order to discuss asymptotic equilibrium analysis of decentralized control, some nontrivial extension of classical results are needed, and new tricks in doing these are developed. In addition, the optimal control, optimal state and NCE equation are derived in a new manner.

The rest of this paper is organized as follows. In Section 2, the mean field LQG game for stochastic integral system is formulated. Section 3 is devoted to the discussion of the NCE equation and the consistency condition. The \( \epsilon \)-Nash equilibrium property of decentralized strategy is also discussed therein. In Section 4, some special cases are discussed. Section 5 concludes our work.

2 Problem formulation

In this paper, we set the state equation to be one-dimensional for sake of notation simplicity. There has no essential difficulty to extend the results to the multi-dimensional case. Suppose \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) is a complete filtered probability space on which \( W(t) = (W_i(t))_{0 \leq t \leq T}, 1 \leq i \leq N \) are independent scalar-valued Brownian motions. It is also assumed \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the natural filtration generated by \( W(t) \) and \( \mathcal{F} = \mathcal{F}_T \). We also need to introduce the following
spaces.

\[ L^2(0, T; \mathbb{R}) = \{ x : [0, T] \to \mathbb{R} \mid x(\cdot) \text{ is deterministic such that } \int_0^T |x(s)|^2 ds < \infty \}, \]

\[ L^2_f(0, T; \mathbb{R}) = \{ x : [0, T] \times \Omega \to \mathbb{R} \mid x(\cdot) \text{ is } \mathcal{F}_T\text{-adapted process such that } \mathbb{E} \int_0^T |x(s)|^2 ds < \infty \}, \]

\[ C(0, T; \mathbb{R}) = \{ x : [0, T] \to \mathbb{R} \mid x(\cdot) \text{ is deterministic and continuous on } [0, T] \}. \]

Now consider the stochastic large population integral system which consists of \( N \) individual negligible players, denote respectively by \( A_i, i = 1, 2, \cdots, N \). The dynamics of \( A_i \) is given by the following stochastic Volterra equation

\[
\begin{align*}
  x_i(t) &= \varphi(t) + \int_0^t b(t, s)x_i(s)ds + \int_0^t f(t, s)x^N(s)ds \quad + \int_0^t c(t, s)u_i(s)ds + \int_0^t \sigma(t, s)dW_i(s). \quad (3)
\end{align*}
\]

All individual players are coupled in terms of their individual cost functional as follows:

\[
\mathcal{J}_i(u_i, u_{-i}) = \mathbb{E} \int_0^T \left[ (x_i(t) - \gamma x^N(t) - \eta)^2 + Ru_i^2(t) \right] dt, \quad (4)
\]

where \( u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N) \), and \( R > 0, \gamma, \eta \) are constants. Some special cases of our general state equations are as follows.

The first one is the differential case, i.e., the kernels \( \varphi, b, c, \sigma \) are independent of \( t \). In this case, equation (3) becomes the following stochastic integral equation,

\[
\begin{align*}
  x_i(t) &= x(0) + \int_0^t b(s)x_i(s)ds + \int_0^t c(s)u_i(s)ds + \int_0^t \sigma(s)dW_i(s). \quad (5)
\end{align*}
\]

Note that all the previous mentioned literature of large population LQG games are based on equation (5). The second case is the stochastic differential equation with delay in state which can be described as

\[
\begin{align*}
  dx_i(t) &= \left[ A(t)x_i(t-h) + \int_{t-h}^t B(t, s)x_i(s)ds \right] dt + C(t)u_i(t)dt + D(t)dW_i(t), \quad (6)
\end{align*}
\]

where \( h > 0 \) is the delay or lag parameter, \( x_i(t) = k(t), t \in [-h, 0] \). The third one includes stochastic differential equations with delay in control variable which evolves as

\[
\begin{align*}
  dx_i(t) &= A(t)x_i(t)dt + C(t)u_i(t-h)dt + D(t)dW_i(t), \quad (7)
\end{align*}
\]

where \( h > 0, x_i(t) = k(t), t \in [-h, 0] \). Under certain conditions, we can transform equation (4) and (7) into the following form of stochastic Volterra integral equation,

\[
\begin{align*}
  x_i(t) &= \psi(t) + \int_0^t K(t, s)u_i(s)ds + \int_0^t \Phi(t, s)D(s)dW_i(s). \quad (8)
\end{align*}
\]

We refer to Section 4 for more details on these procedures.
3 NCE equation system and asymptotic equilibrium analysis

Given the state dynamics (3) and cost functional (4), this section aims to derive the associated Nash certainty equivalence (NCE) equation. To this end, some decentralized asymptotic equilibrium analysis should be given.

3.1 NCE equation

First, let us formulate one auxiliary limiting control problem via the approximation of state average \( x^N \) by some deterministic function \( a(t) \). Now we introduce the following state equation

\[
x_i(t) = \varphi(t) + \int_0^t b(t,s)x_i(s)ds + \int_0^t f(t,s)a(s)ds + \int_0^t c(t,s)u_i(s)ds + \int_0^t \sigma(t,s)dW_i(s), \quad t \in [0,T],
\]

and the cost functional for player \( \mathcal{A}_i \) \( (i = 1, 2, \cdots, N) \),

\[
J_i(u_i) = E \int_0^T \left[ (x_i(t) - \gamma a(t) - \eta)^2 + R|u_i(t)|^2 \right] dt.
\]

In this case, it follows from [23] or [29] that the optimal control \( u_i \) can be represented by

\[
u_i(t) = -\frac{1}{2R} E \int_t^T c(s,t)\overline{y}_i(s)ds
\]

where \( \overline{y}_i \) satisfies the following linear Backward Stochastic Volterra Integral Equation (BSVIE)

\[
\overline{y}_i(t) = 2\overline{\pi}_i(t) - 2\gamma a(t) - 2\eta + \int_t^T b(s,t)\overline{y}_i(s)ds - \int_t^T \overline{z}_i(t,s)dW_i(s),
\]

and \( \overline{\pi}_i \) is the solution of (8) with respect to \( \overline{u}_i \). Due to the special form of (11), by using Lemma 1.1 in [22] we can formulate \( \overline{y}_i \) as

\[
\overline{y}_i(t) = 2\overline{\pi}_i(t) - 2\gamma a(t) - 2\eta + \int_t^T 2P(s,t)[\overline{\pi}_i(s) - \gamma a(s) - \eta]ds,
\]

where for any \( 0 \leq t \leq s \leq T \),

\[
P(s,t) = \sum_{k=1}^{\infty} \Lambda^k(s,t), \quad \Lambda^1(s,t) = b(s,t), \quad \Lambda^{k+1}(s,t) = \int_t^s b(s,r)\Lambda^k(r,t)dr.
\]

Remark 3.1 Note that (12) can be viewed as a constant variation formula in our setup and \( P \) is called “resolvent kernel”. In the special multi-dimensional differential case, \( P \) becomes the solution of fundamental matrix. The method of infinite series will be frequently adopted in our following analysis. On the other hand, it is straightforward to verify

\[
|\Lambda^k(s,t)| < \frac{M^k|t-s|^{k-1}}{(k-1)!}, \quad t, s \in [0,T], \quad k \geq 1,
\]

where \( M \) is the upper bound of \( b \), thus \( P \) is bounded. Moreover, if \( b \) is continuous in \( t \), then one can see that \( P \) is also continuous in \( t \).
Substituting (12) into (10), we can rewrite the optimal control in (10) as
\[
\mathbf{u}_i(t) = \frac{1}{R} \int_t^T \mathbb{E}^F \bar{c}(s,t) \mathbf{u}_i(s) - \gamma a(s) - \eta ds,
\]
where
\[
\bar{c}(s,t) = c(s,t) + \int_t^s P(s,r)c(r,t)dr, \quad s \geq t.
\] (15)

Note that if \(b, c\) are continuous in \(t\), by using Remark 3.1 one can see that \(\bar{c}(t,s)\) is also continuous in \(t\). On the other hand, by using Lemma 1.1. of [22], we can rewrite (8) as
\[
x_i(t) = \hat{\varphi}(t) + \hat{\sigma}_i(t) + \int_0^t \hat{f}(t,s)a(s)ds + \int_0^t \hat{c}(t,s)u_i(s)ds,
\]
where \(P\) is defined in (13) and
\[
\hat{\varphi}(t) = \varphi(t) + \int_0^t P(t,s)\varphi(s)ds,
\]
\[
\hat{\sigma}_i(t) = \int_0^t \left[ \sigma(t,s) + \int_s^t P(t,r)\sigma(r,s)dr \right] dW_i(s),
\]
\[
\hat{f}(t,s) = f(t,s) + \int_s^t P(t,r)f(r,s)dr.
\] (17)

Consequently, by combining (14) and (16), as well as stochastic Fubini theorem, the optimal state equation of this limit control problem can be described by
\[
\mathbf{x}_i(t) = \hat{\varphi}(t) + \hat{\sigma}_i(t) - \frac{1}{R} \int_0^t \int_0^{s \wedge T} \hat{c}(t,r)\hat{c}(s,r)\mathbb{E}^{\mathbb{F}_r} \mathbf{x}_i(s)dr ds
\]
\[
+ \int_0^t \hat{f}(t,s)a(s)ds + \frac{1}{R} \int_0^T \int_0^{s \wedge T} \hat{c}(t,r)\hat{c}(s,r)dr \cdot [\gamma a(s) + \eta] ds.
\] (18)

In particular, by taking expectation,
\[
\mathbb{E}\mathbf{x}_i(t) = \hat{\varphi}(t) + \int_0^t \hat{f}(t,s)a(s)ds + \frac{1}{R} \int_0^T \int_0^{s \wedge T} \hat{c}(t,r)\hat{c}(s,r)dr \cdot [\gamma a(s) + \eta] ds,
\]
where
\[
M(t,s) = \int_0^{s \wedge T} \hat{c}(t,r)\hat{c}(s,r)dr, \quad t, s \in [0,T].
\] (20)

Given \(a(s)\), (19) can be regarded as a Fredholm integral equation with solution being \(\mathbb{E}x_i(\cdot)\). Before analyzing, we present one result on the solvability of Fredholm integral equation which is easy to check.

**Lemma 3.1** Consider the following Fredholm integral equation
\[
x(t) = \psi(t) + \frac{1}{R} \int_0^T A(t,s)x(s)ds, \quad t \in [0,T],
\]
where \(\psi, A\) are bounded and continuous functions in \(t\). If
\[
\sup_{t \in [0,T]} \int_0^T |A(t,s)|^2 ds < R, \quad \sup_{t \in [0,T]} |\psi(t)|^2 < \infty,
\]
then there exists a unique continuous solution \(x(\cdot)\) of (21).
From Lemma 3.1, we can get the following result directly.

**Theorem 3.1** Suppose \( \varphi, b, f, c \) are bounded and continuous in \( t \) such that

\[
\sup_{t \in [0,T]} \int_0^T |M(t,s)|^2 ds < R.
\]

Then, given \( a \in C[0,T] \), there exists a unique continuous function \( \mathbb{E} \gamma(t) \) satisfying equation \( 19 \).

Given \( a(s) \), using Theorem 3.1, we can define \( \Gamma a \) by

\[
[\Gamma a](t) = \hat{\varphi}(t) + \int_0^t \hat{f}(t,s)a(s)ds + \frac{1}{R} \int_0^T M(t,s)\{\gamma a(s) + \eta - [\Gamma a](s)\}ds.
\]

The Nash certainty equivalence (NCE) equation in our setting is thus

\[
\hat{a}(t) = \hat{\varphi}(t) + \int_0^t \hat{f}(t,s)\hat{a}(s)ds + \frac{1}{R} \int_0^T M(t,s)[(\gamma - 1)\hat{a}(s) + \eta]ds,
\]

where \( \hat{a} \) is the corresponding solution. Note that \( 23 \) is a linear deterministic Volterra-Fredholm integral equation. Before going further, similar to \( 16 \) above, we transform \( 23 \) into a Fredholm integral equation. More precisely, by using Lemma 1.1. in \( 22 \), we can rewrite \( 23 \) as

\[
\hat{a}(t) = \hat{\varphi}(t) + \frac{1}{R} \int_0^T \tilde{M}(t,s)\{[(\gamma - 1)\hat{a}(s) + \eta]ds,
\]

where

\[
\hat{\varphi}(t) = \hat{\varphi}(t) + \int_0^t \tilde{P}(t,s)\hat{\varphi}(s)ds,
\]

\[
\tilde{M}(t,s) = M(t,s) + \int_0^t \tilde{P}(t,r)M(r,s)dr,
\]

and

\[
\tilde{P}(t,s) = \sum_{k=1}^\infty \tilde{\Lambda}_k(t,s), \quad \tilde{\Lambda}_1(t,s) = \hat{f}(t,s), \quad \tilde{\Lambda}_k^{k+1}(t,s) = \int_s^t \tilde{f}(t,r)\tilde{\Lambda}_k(r,s)dr.
\]

Note that the technique applied here is similar to that in \( 13 \). To summarize, we have the following theorem.

**Theorem 3.2** Suppose \( \varphi, b, c, f \) are bounded and continuous in \( t \). If

\[
\sup_{t \in [0,T]} \int_0^T \left| (\gamma - 1)\tilde{M}(t,s) \right|^2 ds < R,
\]

then \( 25 \) admits a unique solution \( a \in C[0,T] \).
Before concluding this section, let us make some remarks on (14), (18) and (23). Firstly, note that \( \bar{u}_i(t) \) depends on the path of \( \bar{v}_i \) in \([t, T]\). Moreover, \( \bar{c} \) is a general non-separable function of \((t, s)\). In particular, such a general feature can naturally degenerate and transform into the form with Riccati equation involved, see Section 4.1 below. Secondly, equation (18) is a new type of stochastic equation with double integrals, which is essentially different from the corresponding equation in stochastic differential equations (SDEs) case. Since it has both the characters of Volterra and Fredholm equation, we then call it a stochastic Volterra-Fredholm equation. We will study its solvability under certain conditions in next section by noting our main concern here is NCE equation. Thirdly, we express the NCE equation in our setting by means of deterministic Volterra-Fredholm equation. This idea is also different to other literature on SDEs.

### 3.2 Asymptotic equilibrium analysis

Given the NCE equation, we need to discuss the asymptotic equilibrium property of the associated decentralized control strategies. To start, let us first introduce the definition of \( \epsilon \)-Nash equilibrium:

**Definition 3.1** Given a set of controls \( \hat{u}_i \in L^2_T((0, T; \mathbb{R}) \) with \( i = 1, 2, \ldots, N \), if for any \( i \), \( 1 \leq i \leq N \), \( J_i(\hat{u}_i, \hat{v}_i) \leq J_i(u_i, \hat{v}_i) + \epsilon \), where \( \epsilon > 0 \), \( \hat{v}_i = (\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots, \hat{u}_N) \), \( u_i \in L^2_T(0, T; \mathbb{R}) \), then we call this set of controls \( \hat{u}_i \), \( 1 \leq i \leq N \) an \( \epsilon \)-Nash equilibrium with respect to costs \( J_i \) with \( 1 \leq i \leq N \).

Suppose \( \hat{a} \) is the solution of NCE equation (23), then we can define the decentralized control \( \hat{u}_i \) for agent \( A_i \) as follows:

\[
\hat{u}_i(t) = -\frac{1}{R} E^F_i \int_t^T \hat{c}(s, t)[x_i(s) - \gamma \hat{a}(s) - \eta] ds \tag{27}
\]

where \( x_i \) is the solution of (3) corresponding to \( \hat{u}_i \). Note that \( \hat{u}_i \) is different from the above \( u_i \) in (14). Substituting (27) into equation (3), together with the transformation in (16), we get

\[
x_i(t) = \hat{\varphi}(t) + \hat{\sigma}_i(t) + \int_0^t \hat{f}(t, s)x^N(s) ds - \int_0^t \hat{c}(t, s) \frac{1}{R} E^F_i \int_s^T \hat{c}(r, s)[x_i(r) - \gamma \hat{a}(r) - \eta] dr ds, \tag{28}
\]

where \( \hat{\varphi}, \hat{\sigma}_i \) and \( \hat{f} \) are defined in (17). On the other hand, by replacing \( a \) by \( \hat{a} \) in (18), we get

\[
\bar{x}_i(t) = \hat{\varphi}(t) + \hat{\sigma}_i(t) - \frac{1}{R} \int_0^T \int_0^{s \wedge t} \hat{c}(t, r) \hat{c}(s, r) E^F_i \bar{x}_i(s) dr ds + \int_0^t \hat{f}(t, s) \hat{a}(s) ds + \frac{1}{R} \int_0^T \int_0^{s \wedge t} \hat{c}(t, r) \hat{c}(s, r) dr \cdot [\gamma \hat{a}(s) + \eta] ds. \tag{29}
\]

To derive the asymptotic equilibrium, we need first to prove several lemmas. In the following, we denote by \( C \) a generic positive constant independent of \( N \) that may change from line to line.

**Lemma 3.2** Suppose

\[
x(t) = \varphi(t) + \int_0^t A(t, s)x(s) ds - \frac{1}{R} \int_0^T \int_0^{s \wedge t} B(t, r) B(s, r) E^F x(s) dr ds, \tag{30}
\]
where $\varphi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$, $A, B$ are deterministic and bounded functions such that

$$T \int_0^T \int_0^T \int_0^{s \wedge t} |\tilde{B}(t, r)B(s, r)|^2 dr ds dt < \frac{R^2}{3}. \quad (31)$$

Here we denote by

$$\tilde{B}(t, r) = B(t, r) + \int_r^t P_0(t, v)B(v, r)dv, \quad t, v \in [0, T],$$

$$P_0(t, v) = \sum_{k=1}^{\infty} \Lambda^k(t, v), \quad \Lambda^1(t, t) = A(t, v), \quad \Lambda^{k+1}(t, v) = \int_v^t A(t, r)\Lambda^k(r, v)dr. \quad (32)$$

Then (30) admits a unique solution $x(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ satisfying

$$E \int_0^T |x(s)|^2 ds \leq 6E \int_0^T |\hat{\varphi}(s)|^2 ds, \quad (33)$$

where

$$\hat{\varphi}(t) = \varphi(t) + \int_0^t P_0(t, s)\varphi(s)ds, \quad t \in [0, T].$$

**Proof.** Given $y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$, consider the following Volterra equation

$$x(t) = \varphi(t) + \int_0^t A(t, s)x(s)ds - \frac{1}{R} \int_0^T \int_0^{s \wedge t} B(t, r)B(s, r)E_{\mathcal{F}_r} y(s)dr ds, \quad (34)$$

which obviously admits a unique solution $x(\cdot)$ under above assumptions. By using Lemma 1.1 in [22], we can rewrite (34) as

$$x(t) = \hat{\varphi}(t) - \frac{1}{R} \int_0^T \int_0^{s \wedge t} \tilde{B}(t, r)B(s, r)E_{\mathcal{F}_r} y(s)dr ds.$$

Then using the idea of contraction mapping and Lemma 3.1 together with above requirements, we can obtain the existence of $x(\cdot)$ in $L^2_{\mathcal{F}}(0, T; \mathbb{R})$ satisfying (33). \qed

**Remark 3.2** Some comment to (30). To be shown later, it plays some key role in our remaining analysis. Since our first concern here is the solvability of (30) under certain conditions. In such sense, we do not pursue the most general assumptions here and we hope to weaken these conditions in our future work.

By comparing (23) and (29), we have

**Lemma 3.3** Suppose NCE equation (23) admits a unique solution $\hat{a}(\cdot)$, and (31), (32) hold with $A = 0$, $B = \hat{c}$, then

$$E \int_0^T |\hat{a}(s) - \overline{a}^N(s)|^2 ds = O\left(\frac{1}{N}\right), \quad (35)$$

where $\overline{a}^N(t) = \frac{1}{N} \sum_{i=1}^{N} \overline{a}_i(t)$ and $\overline{a}_i$ satisfies (24).
Proof. From (23) and (29), we know the difference between \( \pi_i \) and \( \tilde{a} \) satisfies the type of Equation (30) with \( A = 0, B = \tilde{c}, \varphi = \tilde{\sigma}_i \). Then by means of (33),

\[
E \int_0^T |\pi^N(t) - \tilde{a}(t)|^2 dt \leq C E \int_0^T \left\{ \frac{1}{N} \sum_{i=1}^{N} \tilde{\sigma}_i(t) \right\}^2 dt.
\]

It then follows from the boundedness of \( \sigma \) and \( P \) in (17), together with the independence of \( W_i (i = 1, 2, \cdots, N) \) that

\[
E \int_0^T |\pi^N(t) - \tilde{a}(t)|^2 dt = O \left( \frac{1}{N} \right),
\]

which implies the desired result. \( \square \)

Lemma 3.4 Suppose NCE equation (23) admits a solution \( \tilde{a} \). (31) and (32) hold for two cases of \( A = \hat{f}, B = \hat{c}, \) and \( A = 0, B = \hat{c}, \) then

\[
\text{E} \int_0^T |\tilde{a}(s) - x^N(s)|^2 ds + \sup_{1 \leq i \leq N} \text{E} \int_0^T |x_i(t) - \pi_i(t)|^2 dt = O \left( \frac{1}{N} \right)
\]

(36)

where \( x^N(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t), x_i \) satisfies (23).

Proof. Comparing (28) and (29), we can rewrite \( x_i(t) \) as

\[
x_i(t) = \pi_i(t) + \int_0^t \hat{f}(t, s) \left[ x^N(s) - \tilde{a}(s) \right] ds - \frac{1}{R} \int_0^T \int_0^{s \wedge t} \hat{c}(t, r) \hat{c}(s, r) E^F \left[ x_i(s) - \pi_i(s) \right] dr ds.
\]

Therefore,

\[
x^N(t) - \tilde{a}(t) = \pi^N(t) - \tilde{a}(t) + \int_0^t \hat{f}(t, s) \left[ x^N(s) - \tilde{a}(s) \right] ds - \frac{1}{R} \int_0^T \int_0^{s \wedge t} \hat{c}(t, r) \hat{c}(s, r) E^F \left[ x^N(s) - \tilde{a}(s) \right] dr ds + \frac{1}{R} \int_0^T \int_0^{s \wedge t} \hat{c}(t, r) \hat{c}(s, r) E^F \left[ \pi^N(s) - \tilde{a}(s) \right] dr ds.
\]

By using Lemma 3.2, Lemma 3.3 together with the boundedness of \( \hat{f} \) and \( \hat{c} \), we can obtain the desired result (36). \( \square \)

We also need the following result.

Theorem 3.3 Suppose \( \varphi, f, b \) and \( c \) are bounded and continuous in \( t \), NCE equation (23) admits a unique continuous solution \( \hat{a} \in C[0, T], \pi_i \) is the optimal control with state equation (3), cost functional (2) that is parameterized by \( \hat{a} \), (31) and (32) hold for two cases of \( A = \hat{f}, B = \hat{c}, \) and \( A = 0, B = \hat{c} \). Then for \( 1 \leq i \leq N \), we have

\[
| J_i(\hat{a}_i, \tilde{a}_{-i}) - J_i(\pi_i) | = O \left( \frac{1}{\sqrt{N}} \right) .
\]

(37)
**Proof.** First from the definition of $J_i$ in (41) and $\mathcal{J}_i$ in (49),

\[
J_i(\tilde{u}_i, \tilde{u}_{-i}) - \mathcal{J}_i(\bar{u}_i) \leq C \mathbb{E} \int_0^T |R[\tilde{u}_i(t)]^2 - R[\bar{u}_i(t)]^2| dt
\]

\[
+ C \mathbb{E} \int_0^T ||x_i(t) - \gamma x^N(t) - \eta||^2 - ||\bar{u}_i(t) - \gamma \hat{a}(t) - \eta||^2 | dt
\]

\[
\leq C \mathbb{E} \int_0^T R[||\tilde{u}_i(t) - \bar{u}_i(t)||^2 + 2|\bar{u}_i(t)| \cdot |	ilde{u}_i(t) - \bar{u}_i(t)|] dt
\]

\[
+ C \mathbb{E} \int_0^T |x_i(t) - \bar{u}_i(t) - \gamma (x^N(t) - \hat{a}(t))|^2 dt
\]

\[
+ C \mathbb{E} \int_0^T |\bar{u}_i(t) - \gamma \hat{a}(t) - \eta| \cdot |x_i(t) - \bar{u}_i(t) - \gamma (x^N(t) - \hat{a}(t))| dt,
\]

where $x_i$ and $\bar{u}_i$ satisfy (28) and (29) respectively. It follows from (10) and (27) that

\[
\mathbb{E} \int_0^T |\tilde{u}_i(t) - \bar{u}_i(t)|^2 dt \leq \frac{1}{2^2} \mathbb{E} \int_0^T \mathbb{E}^{\mathcal{F}_t} \left| \int_t^T \tilde{c}(s,t)[x_i(s) - \bar{x}_i(s)] ds \right|^2 dt
\]

\[
\leq C \mathbb{E} \int_0^T |x_i(s) - \bar{x}_i(s)|^2 ds.
\]

Note that equation (29) is one special case of (30) with $A = 0$, $B = \hat{c}$. Under above requirement, then from inequality (33) above we have the boundedness of $\bar{x}_i$ in $L^2_T(0,T; \mathbb{R})$. Recalling (14) with $a = \hat{a}$, we can also get similar boundedness result for $\bar{u}_i$. As a result, by Schwarz’s inequality,

\[
\mathbb{E} \int_0^T |\bar{u}_i(t)||\tilde{u}_i(t)| dt \leq \left[ \mathbb{E} \int_0^T |\bar{u}_i(t)|^2 dt \right]^{1/2} \left[ \mathbb{E} \int_0^T |\tilde{u}_i(t) - \bar{u}_i(t)|^2 dt \right]^{1/2}
\]

\[
\leq C \left[ \mathbb{E} \int_0^T |x_i(s) - \bar{x}_i(s)|^2 ds \right]^{1/2}.
\]

On the other hand,

\[
\mathbb{E} \int_0^T |x_i(t) - \bar{x}_i(t) - \gamma (x^N(t) - \hat{a}(t))|^2 dt \leq C \mathbb{E} \int_0^T |x_i(t) - \bar{x}_i(t)|^2 dt + C \mathbb{E} \int_0^T |x^N(t) - \hat{a}(t)|^2 dt.
\]

Similarly, we can use Schwarz’s inequality to estimate

\[
\mathbb{E} \int_0^T |\bar{x}_i(t) - \gamma \hat{a}(t) - \eta| \cdot |x_i(t) - \bar{x}_i(t) - \gamma (x^N(t) - \hat{a}(t))| dt
\]

\[
\leq C \left[ \mathbb{E} \int_0^T |x_i(t) - \bar{x}_i(t)|^2 dt \right]^{1/2} + C \left[ \mathbb{E} \int_0^T |x^N(t) - \hat{a}(t)|^2 dt \right]^{1/2}.
\]

Therefore, by using Lemma 3.4, we obtain the desired result. \(\square\)

In the following part, given two processes $\xi_a, \xi_b$ in $L^2_T(0,T; \mathbb{R})$, we denote by

\[
\epsilon_a = \left( \mathbb{E} \int_0^T |\xi_a(t)|^2 dt \right)^{1/2}, \quad \epsilon_b = \left( \mathbb{E} \int_0^T |\xi_b(t)|^2 dt \right)^{1/2} < \infty.
\]

(38)
First let us recall the optimal control problem of which optimal control \( \overline{u}_i \) and optimal state equation \( \overline{x}_i \) are given by (14) and (18) with \( a = \overline{a} \). For comparison, we also discuss one perturbed version with state equation

\[
x_i(t) = \varphi(t) + \int_0^t b(t, s)x_i(s)ds + \int_0^t f(t, s)\overline{a}(s)ds + \int_0^t c(t, s)u_i(s)ds + \int_0^t \sigma(t, s)dW_i(s), \tag{39}
\]

and the cost functional

\[
J_i^\xi(u_i) = \mathbb{E} \int_0^T [x_i(t) - \gamma \overline{a}(t) - \eta + \xi_b(t)]^2 + R|u_i(t)|^2 \] dt.

Here \( g \) is bounded deterministic function. Then we have the following result.

**Lemma 3.5** Suppose \( \overline{u}_i \) is an optimal control given in (14) and

\[
\mathbb{E} \int_0^T |x_i(t)|^2 dt + \mathbb{E} \int_0^T |u_i(t)|^2 dt \leq C_0, \tag{40}
\]

where \( x_i, u_i \) are defined in (39), \( C_0 \) is a fixed constant. Moreover, \( \overline{a} \) is continuous and bounded, (27) and (32) hold for \( A = 0, B = \overline{c} \), then we have

\[
J_i^\xi(u_i) \geq \overline{J}_i(\overline{u}_i) - C(1 + \epsilon_b)(\epsilon_a + \epsilon_b).
\]

**Proof.** For convenience, we can rewrite (39) as

\[
x_i(t) = \varphi(t) + \int_0^t b(t, s)x_i(s)ds + \int_0^t f(t, s)\overline{a}(s)ds + \int_0^t c(t, s)u'_i(s)ds + \int_0^t g(t, s)\xi_a(s)ds + \int_0^t c(t, s)\overline{u}_i(s)ds + \int_0^t \sigma(t, s)dW_i(s).
\]

Here \( u'_i(s) = u_i(s) - \overline{u}_i(s) \),

\[
\overline{u}_i(s) = -\frac{1}{R} \mathbb{E} \int_t^T \overline{c}(r, t)[\gamma \overline{a}(r) - \eta]dr,
\]

where \( x_i \) satisfies (39), \( \overline{c} \) is given in (14). It then follows from inequality (10) that \( \mathbb{E} \int_0^T |u'_i(t)|^2 dt \leq C \). Next we construct one auxiliary optimal control problem with state equation and cost functional as

\[
\overline{x}_i(t) = \varphi(t) + \int_0^t b(t, s)\overline{x}_i(s)ds + \int_0^t f(t, s)\overline{a}(s)ds + \int_0^t c(t, s)u''_i(s)ds + \int_0^t c(t, s)\overline{u}_i(s, \overline{x}_i)ds + \int_0^t \sigma(t, s)dW_i(s), \tag{41}
\]

and

\[
\overline{J}_i(u''_i) = \mathbb{E} \int_0^T [\overline{x}_i(t) - \gamma \overline{a}(t) - \eta]^2 + R|u''_i(t) + \overline{u}_i(t, \overline{x}_i)|^2 \] dt.
where $u''_i$ is the control variable. For $t \in [0, T]$, $\pi_i(t, \tilde{x}_i)$ in (41) is defined as
\[
\pi_i(t, \tilde{x}_i) = -\frac{1}{R} E^{F_t} \int_t^T \tilde{c}(r, t)[\tilde{x}_i(r) - \gamma \tilde{a}(r) - \eta] dr.
\]

Obviously, $\tilde{J}_i(u''_i)$ attains its minimum when $u''_i = 0$, and the corresponding cost functional equals to $\tilde{J}_i(\pi_i)$. On the other hand, by Lemma 3.2
\[
M'_i(t) = \int_0^t c(t, s)u'_i(s)ds + \int_0^t b(t, s)M'_i(s)ds - \frac{1}{R} \int_0^t c(t, s)E^{F_t} \int_t^T \tilde{c}(r, t)M'_i(r)drds
\]
admits a unique solution $M'_i$ such that
\[
E \int_0^T |M'_i(t)|^2 dt \leq 6E \int_0^T \left| \int_0^t c(t, s)u'_i(s)ds \right|^2 dt \leq C.
\]

If we take $u''_i = u'_i(s) = u_i(s) - \pi_i(s)$, and $\tilde{x}'_i$ is the corresponding solution of (41), then $\tilde{x}'_i = \pi_i(t) + M'_i(t)$, where $\pi_i(t)$ is the optimal state in (18) with $a = \tilde{a}$. Moreover, recalling the boundedness of $\pi_i$ in $L^2_t(0, T; \mathbb{R})$, we also have $E \int_0^T |\tilde{x}'_i(t)|^2 dt \leq C$. Since $\tilde{x}_i$ and $\tilde{x}'_i$ are given in (39) and (41), thus their difference $d_i = \tilde{x}_i - \tilde{x}'_i$ satisfy
\[
d_i(t) = \int_0^t g(t, s)\xi_a(s)ds + \int_0^t b(t, s)d_i(s)ds - \frac{1}{R} \int_0^t c(t, s)E^{F_t} \int_t^T \tilde{c}(r, t)d_i(r)drds.
\]

Under above requirements, from Lemma 3.2 we have
\[
E \int_0^T |x_i(t) - \tilde{x}'_i(t)|^2 dt \leq 6E \int_0^T \left| \int_0^t g(t, s)\xi_a(s)ds \right|^2 dt \leq C\epsilon_a^2.
\]

Following the similar arguments in (20), we should have
\[
|\tilde{J}_i(u'_i) - \tilde{J}_i(u''_i)| \leq C(1 + \epsilon_b)(\epsilon_a + \epsilon_b),
\]
where $C$ does not depend on $\xi_a$ and $\xi_b$. After combining the estimate of $\tilde{J}_i(u'_i) \geq \tilde{J}_i(\pi_i)$, we can get the desired result. \hfill \Box

The next lemma is concerned about the boundedness of $\tilde{J}_i(\tilde{u}_i, \tilde{u}_{-i})$.

**Lemma 3.6** Suppose $\tilde{a}$ is solution of NCE equation (23), (31) and (32) hold for two cases of $A = \hat{f}$, $B = \hat{c}$, and $A = 0$, $B = \hat{c}$. Then $\tilde{J}_i(\tilde{u}_i, \tilde{u}_{-i}) \leq C$, where $C$ does not depend on $N$.

**Proof.** If we can prove
\[
E \int_0^T |\tilde{u}_i(s)|^2 ds \leq C, \quad E \int_0^T |x_i(s)|^2 ds \leq C,
\]
where $x_i$ satisfies (28), $C$ does not depend on $N$, then the result holds directly. From (28), we have
\[
x^N(t) = \hat{\varphi}(t) + \int_0^t \hat{f}(t, s)x^N(s)ds + \frac{1}{N} \sum_{i=1}^N \hat{\sigma}_i(t) - \int_0^t \hat{c}(t, s)\frac{1}{R} E^{F_s} \int_s^T \hat{c}(r, s)[x^N(r) - \gamma \hat{a}(r) - \eta] dr ds,
\]

(42)
where $x^N(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t)$. Note that (42) is one special case of equation (30), (31) and (32) hold for $A = \hat{f}, B = \hat{c}$, it then follows from Lemma 3.2 that

$$
\mathbb{E} \int_0^T |x^N(s)|^2 ds \leq C \mathbb{E} \int_0^T |\hat{\varphi}(s)|^2 ds + C \mathbb{E} \int_0^T \left| \frac{1}{N} \sum_{i=1}^N \hat{\sigma}_i(t) \right|^2 dt
$$

$$
+ C \mathbb{E} \int_0^T |\hat{a}(s)|^2 ds + C.
$$

(43)

As to $x_i$ in equation (28), using (43) and conditions (31), (32) with $A = 0, B = \hat{c}$, we should have $\mathbb{E} \int_0^T |x_i(s)|^2 ds \leq C$ where $C$ is independent of $N$. At last, we know that

$$
\mathbb{E} \int_0^T |\hat{u}_i(t)|^2 dt = \frac{1}{R^2} \mathbb{E} \int_0^T \int_t^T \hat{c}(r,t)|x_i(r) - \gamma \hat{a}(r) - \eta| dr dt
$$

$$
\leq \frac{1}{R^2} \int_0^T \int_t^T |\hat{c}(r,t)|^2 dr dt \cdot \mathbb{E} \int_0^T |x_i(r) - \gamma \hat{a}(r) - \eta|^2 dr.
$$

Hence $\mathbb{E} \int_0^T |\hat{u}_i(t)|^2 dt$ is bounded, too. Then the desired result holds.\hfill \square

If we apply control $\hat{u}_j$ in (28) to all the player except $A_i$, that is, for $j \neq i$,

$$
x_j(t) = \varphi(t) + \int_0^t b(t,s)x_j(s)ds + \int_0^t f(t,s)[x^N(s) - \hat{a}(s)]ds
$$

$$
+ \int_0^t f(t,s)\hat{a}(s)ds + \int_0^t c(t,s)\hat{u}_j(s)ds + \int_0^t \sigma(t,s)dW_j(s),
$$

(44)

while for player $i$,

$$
x_i(t) = \varphi(t) + \int_0^t b(t,s)x_i(s)ds + \int_0^t c(t,s)u_i(s)ds
$$

$$
+ \int_0^t \sigma(t,s)dW_i(s) + \int_0^t f(t,s)x^N(s)ds,
$$

(45)

then similar as Lemma 3.4 above, we also have the following result.

**Lemma 3.7** Suppose $\hat{a}$ is the solution of NCE equation (28), (31) and (32) hold for two cases of $A = \hat{f}, B = \hat{c}$ and $A = 0, B = \hat{c}$. Moreover, $(1 + K_2)K_1 \frac{300}{N^2} L^2 < \frac{1}{2}$, where

$$
L = \int_0^T \int_0^t |\hat{c}(t,s)|^2 ds dt, \quad K_1 = \int_0^T \int_0^t |\hat{f}(t,s)|^2 ds dt, \quad K_2 = \int_0^T \int_0^t |P_0(t,s)|^2 ds dt,
$$

$P_0$ is defined in (28) with $A = \hat{f}$. Then we have

$$
\mathbb{E} \int_0^T |\hat{a}(s) - x^N(s)|^2 ds = O\left(\frac{1}{N}\right),
$$

where $x^N(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t), x_i$ satisfies (44) and (45).
Proof. For convenience, we also rewrite (45) as

\[ x_i(t) = \varphi(t) + \int_0^t b(t, s)x_i(s)ds + \int_0^t f(t, s)\bar{a}(s)ds + \int_0^t c(t, s)\bar{u}_i(s)ds + \int_0^t \sigma(t, s)dW_i(s) + \int_0^t f(t, s)[x^N(s) - \bar{a}(s)]ds + \int_0^t c(t, s)[u_i(s) - \bar{u}_i(s)]ds, \]

where \( \bar{u}_i \) is defined as

\[ \bar{u}_i(t) = -\frac{1}{R}\Ex_F^F \int_t^T \varphi(r, t)[x_i(r) - \gamma \bar{a}(r) - \eta]dr, \]

and \( x_i \) satisfies (45). After some direct calculations,

\[ x^N(t) = \bar{x}(t) + \int_0^t \bar{f}(t, s)\bar{a}(s)ds + \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_i(t) \]

\[ - \int_0^t \bar{c}(t, s)\frac{1}{R}\Ex_F^F \int_s^T \bar{c}(r, s)[x^N(r) - \gamma \bar{a}(r) - \eta]drds \]

\[ + \int_0^t \bar{f}(t, s)[x^N(s) - \bar{a}(s)]ds + \frac{1}{N} \int_0^t \bar{c}(t, s)[u_i(s) - \bar{u}_i(s)]ds, \]

where \( \bar{x}, \bar{f}, \bar{\sigma}_i \) are defined in (17). Using Fubini theorem, it also follows from (18) (with \( a = \bar{a} \)) that

\[ \bar{x}^N(t) = \bar{x}(t) + \int_0^t \bar{f}(t, s)\bar{a}(s)ds + \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_i(t) \]

\[ - \int_0^t \bar{c}(t, s)\frac{1}{R}\Ex_F^F \int_s^T \bar{c}(r, s)[\bar{x}^N(r) - \gamma \bar{a}(r) - \eta]drds, \]

therefore we have

\[ x^N(t) - \bar{a}(t) = \bar{x}^N(t) - \bar{a}(t) + \int_0^t \bar{f}(t, s)[x^N(s) - \bar{a}(s)]ds \]

\[ - \int_0^t \bar{c}(t, s)\frac{1}{R}\Ex_F^F \int_s^T \bar{c}(r, s)[x^N(r) - \bar{a}(r)]drds \]

\[ + \int_0^t \bar{c}(t, s)\frac{1}{R}\Ex_F^F \int_s^T \bar{c}(r, s)[\bar{x}^N(r) - \bar{a}(r)]drds \]

\[ + \frac{1}{N} \int_0^t \bar{c}(t, s)[u_i(s) - \bar{u}_i(s)]ds. \]

Note that if \( A = 0, B = \hat{c}, \) then \( (31) \) and \( (32) \) imply \( \bar{x}^N \) is bounded in \( L^2_T(0, T; \mathbb{R}) \). Similarly, if \( (31) \) and \( (32) \) hold with \( A = \hat{f}, B = \hat{c}, \) then using Lemma 3.2 to (48), we have

\[ \Ex \int_0^T |x^N(t) - \bar{a}(t)|^2 dt \]

\[ \leq (1 + K_2) \left[ (36 + 12R^2)\Ex \int_0^T |\bar{x}^N(t) - \bar{a}(t)|^2 dt + \frac{36}{N^2}L \cdot \Ex \int_0^T [u_i(s) - \bar{u}_i(s)]^2 ds \right], \]
where $L$ and $K_2$ are defined before. On the other hand, from Lemma 3.3 $\mathcal{J}_i(\hat{u}_i, \hat{w}_i)$ is bounded, it then suffices to consider $u_i$ in (45) such that $\mathcal{J}_i(u_i, \hat{w}_i) \leq \mathcal{J}_i(\hat{u}_i, \hat{w}_i)$, which implies that $\mathbb{E} \int_0^T |u_i(t)|^2 dt \leq C$. As to $\hat{u}_i$ in (47), we have

$$\mathbb{E} \int_0^T |\hat{u}_i(t)|^2 dt = \frac{1}{R^2} \mathbb{E} \int_0^T \left| \int_t^T \hat{c}(r,t)[x_i(r) - \gamma \hat{a}(r) - \eta] dr \right|^2 dt \leq \frac{1}{R^2} L \cdot \mathbb{E} \int_0^T |x_i(r) - \gamma \hat{a}(r) - \eta|^2 dr,$$

therefore,

$$\mathbb{E} \int_0^T |u_i(s) - \hat{u}_i(s)|^2 ds \leq 2\mathbb{E} \int_0^T |u_i(s)|^2 ds + 2\mathbb{E} \int_0^T |\hat{u}_i(s)|^2 ds \leq C + \frac{2}{R^2} L \cdot \mathbb{E} \int_0^T |x_i(r)|^2 dr,$$  

(50)

where $C$ does not depend on $N$. Substituting (50) into (49), we have

$$\mathbb{E} \int_0^T |x^N(t) - \hat{a}(t)|^2 dt \leq (1 + K_2)(36 + 12R^2)\mathbb{E} \int_0^T |\mathcal{F}^N(t) - \hat{a}(t)|^2 dt + \frac{C}{N^2} + (1 + K_2) \frac{72}{N^2 R^2} L^2 \cdot \mathbb{E} \int_0^T |u_i(s) - \hat{u}_i(s)|^2 ds.$$  

(51)

After some transformations, we can rewrite (45) as

$$x_i(t) = \hat{\varphi}(t) + \int_0^t \hat{f}(t,s)[x^N(s) - \hat{a}(s)] ds + \int_0^t \hat{c}(t,s)u_i(s) ds + \hat{\sigma}_i(t),$$

therefore,

$$\mathbb{E} \int_0^T |x_i(t)|^2 dt \leq 5\mathbb{E} \int_0^T |\hat{\varphi}(t)|^2 dt + 5\int_0^T \int_0^t |\hat{f}(t,s)|^2 ds dt \cdot \mathbb{E} \int_0^T |x^N(s) - \hat{a}(s)|^2 ds + 5\int_0^T \int_0^t |\hat{c}(t,s)|^2 ds dt \cdot \mathbb{E} \int_0^T |u_i(s)|^2 ds + \mathbb{E} \int_0^T |\hat{\sigma}_i(t)|^2 dt.$$  

(52)

Since $\hat{\varphi}$, $\hat{f}$, $\hat{a}$, $\sigma$ are bounded, substituting (51) into (52) yields

$$\mathbb{E} \int_0^T |x_i(t)|^2 dt \leq C + C \mathbb{E} \int_0^T |\mathcal{F}^N(t) - \hat{a}(t)|^2 dt + (1 + K_2)K_1 \frac{360}{N^2 R^2} L^2 \cdot \mathbb{E} \int_0^T |x_i(s)|^2 ds.$$  

So if $(1 + K_2)K_1 \frac{360}{N^2 R^2} L^2 < \frac{1}{2}$, then we have

$$\mathbb{E} \int_0^T |x_i(t)|^2 dt \leq C + C \mathbb{E} \int_0^T |\mathcal{F}^N(t) - \hat{a}(t)|^2 dt,$$

which implies the boundedness of $\mathbb{E} \int_0^T |x_i(t)|^2 dt$. By combining (51), (52) and Lemma 3.3 we obtained the desired result. □
Remark 3.3 Compared with Lemma 3.4 above, our result here seems to be more general because \( u_i \) in [38] is not necessary to be \( \hat{u}_i \) in [27]. Moreover, our result also generalizes the corresponding result in [20] in that our state equation can degenerate to the linear SDEs there. In addition, our approach here is also different from the one used in [20]. At last, although the conditions imposed on above coefficients \( K_1, K_2 \) and \( L \) seem to be technical, it has two interesting features. First, the above inequality holds true in general because \( N \) is always sufficiently large. Second, such condition is not required in the nontrivial case \( \hat{f} = 0 \).

Now let us state the main result in this subsection.

Theorem 3.4 Suppose \( \varphi, f, b \) and \( c \) are bounded and continuous in \( t \), the NCE equation (23) admits a unique continuous solution \( \hat{u} \in C[0, T] \). (37) and (32) hold with \( A = \hat{f}, B = \hat{c} \) and \( A = 0, B = \hat{c} \). Moreover, \( (1 + K_2)K_1 \frac{\bar{a}}{N^2 R^2} L^2 < \frac{1}{2} \), where \( L, K_1, K_2 \) are defined in Lemma 3.7. Then the set of control \( \hat{u}_i \) in (27) with \( 1 \leq i \leq N \) for \( N \) players is an \( \epsilon \)-Nash equilibrium, i.e., for \( 1 \leq i \leq N \),

\[
\mathcal{J}_i(\hat{u}_i, \hat{u}_{-i}) - \epsilon \leq \inf_{u_i \in L^2[0, T]} \mathcal{J}_i(u_i, \hat{u}_{-i}) \leq \mathcal{J}_i(\hat{u}_i, \hat{u}_{-i}).
\]

Proof. If we denote by \( \xi_a(t) = x^N(t) - \hat{a}(t), \xi_b(t) = \gamma(\hat{a}(t) - x^N(t)) \), then the dynamics and cost functional of \( \mathcal{A}_i \) can be rewritten as

\[
x_i(t) = \varphi(t) + \int_0^t b(t, s)x_i(s)ds + \int_0^t f(t, s)\hat{a}(s)ds
+ \int_0^t f(t, s)\xi_a(s)ds + \int_0^t c(t, s)u_i(s)ds + \int_0^t \sigma(t, s)dW_s(s),
\]

and

\[
\mathcal{J}_i(u_i) = \mathbb{E} \int_0^T \left[ |x_i(t) - \gamma \hat{a}(t) - \eta + \xi_b(t)|^2 + R|u_i(t)|^2 \right] dt.
\]

From Lemma 3.7 we have \( \epsilon_a + \epsilon_b = O(\frac{1}{\sqrt{N}}) \), and \( \mathbb{E} \int_0^T |x_i(t)|^2 dt + \mathbb{E} \int_0^T |u_i(t)|^2 dt \leq C \), where \( C \) is independent of \( N \), \( \epsilon_a, \epsilon_b \) are defined in [38], it then follows from Lemma 3.7 Theorem 3.3 that

\[
\mathcal{J}_i(u_i, \hat{u}_{-i}) \geq \mathcal{J}_i(\pi_i) - O \left( \frac{1}{\sqrt{N}} \right) \geq \mathcal{J}_i(\hat{u}_i, \hat{u}_{-i}) - O \left( \frac{1}{\sqrt{N}} \right).
\]

The conclusion is thus proved. \( \square \)

4 Some special cases

In this section, we want to show that our general results can include some special cases that are important for real applications. For sake of simplicity, we suppose \( f = 0 \) in the following.

4.1 The case of stochastic differential equation

In this case, \( \varphi, b, c, \sigma \) are independent of \( t \), and Equation (30) becomes

\[
x_i(t) = x(0) + \int_0^t b(s)x_i(s)ds + \int_0^t c(s)u_i(s)ds + \int_0^t \sigma(s)dW_s(s),
\]

(54)
Remark 4.1  
One can express advantage of state equation and optimal control (note that \([8], [11], \) and \([20]\). Such procedure is different from ours here. More precise, it makes use the \(s\) satisfy Riccati equation and backward ordinary differential equation respectively, see for example, Theorem 4.1

Suppose that \(N\) while the cost functional (4) still keeps the same. Most related literatures on mean field LQG games are discussed by the above linear SDEs. For \(s, t \in [0, T]\), we have

\[
\Lambda^{1}(s, t) = b(t), \quad \Lambda^{k+1}(s, t) = b(t) \left( \int_{t}^{s} b(r) dr \right)^{k}, \quad P(s, t) = b(t) e^{\int_{s}^{t} b(u) du}.
\]

Therefore,

\[
\tilde{\varphi}(t) = x(0) e^{\int_{0}^{t} b(r) dr}, \quad \tilde{\sigma}_{i}(t) = \int_{0}^{t} \sigma(s) e^{\int_{s}^{t} b(r) dr} dW_{i}(s),
\]

\[
\tilde{c}(s, t) = c(t) e^{\int_{s}^{t} b(r) dr}, \quad M(t, s) = \int_{0}^{s \wedge t} |c(r)|^{2} e^{\int_{s}^{r} b(v) dv + \int_{s}^{r} b(v) dv} dr.
\]

The NCE equation in this setting takes the following rather simple form

\[
\tilde{a}(t) = \tilde{\varphi}(t) + \frac{1}{R} \int_{0}^{T} \int_{0}^{s \wedge t} |c(r)|^{2} e^{\int_{s}^{r} b(v) dv + \int_{s}^{r} b(v) dv} dr \cdot \{ \gamma \tilde{a}(s) + \eta - \tilde{a}(s) \} ds. \tag{55}
\]

**Theorem 4.1** Suppose that \(b, c\) are bounded functions such that

\[
\sup_{t \in [0, T]} \left[ (\gamma - 1) \int_{0}^{s \wedge t} |c(r)|^{2} e^{\int_{s}^{r} b(v) dv + \int_{s}^{r} b(v) dv} dr \right]^{2} ds < R.
\]

Then NCE equation \((55)\) admits a continuous solution.

We define by

\[
\tilde{a}_{i}(t) = - \frac{1}{R} c(t) E^{\mathbb{F}_{i}} \int_{t}^{T} e^{\int_{s}^{t} b(r) dr} \left[ \int_{0}^{r} e^{\int_{s}^{r} b(v) dv + \int_{s}^{r} b(v) dv} dr ds \right] x_{i}(s) - \gamma \tilde{a}(s) - \eta \right) ds, \quad 1 \leq i \leq N, \tag{56}
\]

where \(\tilde{a}\) is the solution of NCE equation and

\[
x_{i}(t) = x(0) e^{\int_{0}^{t} b(r) dr} + \int_{0}^{t} \sigma(s) e^{\int_{s}^{t} b(r) dr} dW_{i}(s) - \frac{1}{R} \int_{0}^{t} |c(s)|^{2} E^{\mathbb{F}_{s}} \int_{s}^{T} e^{\int_{s}^{r} b(v) dv + \int_{s}^{r} b(v) dv} dr ds x_{i}(r) - \gamma \tilde{a}(r) - \eta dr ds, \tag{57}
\]

then we get the following asymptotic equilibrium analysis in present setting.

**Theorem 4.2** Suppose \(b, c\) and \(\sigma\) are bounded and deterministic, the NCE equation admits a solution \(\tilde{a} \in C[0, T]\). Moreover,

\[
T \int_{0}^{T} \int_{0}^{s \wedge t} |c(r)|^{4} e^{2 \int_{s}^{r} b(v) dv + 2 \int_{s}^{r} b(v) dv} dE dt < \frac{R^{2}}{3}.
\]

Then the set of controls defined in \((56)\) for \(N\) players is an \(\epsilon\)-Nash equilibrium.

**Remark 4.1** One can express \(\tilde{a}_{i}\) in \((56)\) as \(\tilde{a}_{i}(t) = \frac{1}{R} c(t) \left[ -P(t) x_{i}(t) + \delta(t) \right]\) where \(P\) and \(\delta\) satisfy Riccati equation and backward ordinary differential equation respectively, see for example, \([8], [11]\) and \([20]\). Such procedure is different from ours here. More precise, it makes use the advantage of state equation and optimal control (note that \(s\) and \(t\) are separable). However, such skills are hard to be applied in more general model such as the state equations with delay below. In such sense, our approach applied here is more flexible which can fill this technical gap.
4.2 SDEs with delay in the state process

In this case, suppose cost functional is \(\Phi\) and the state equation is described by

\[
dx_i(t) = \left[ A(t)x_i(t-h) + \int_{t-h}^t B(t,s)x_i(s)ds \right] dt + C(t)u_i(t)dt + D(t)dW_i(t),
\]

where \(h > 0, x_i(t) = k(t), t \in [-h,0]\), \(k(t)\) is bounded. Hence the delay term appears in the state process. By introducing function \(\Phi\) as

\[
\begin{align*}
\frac{\partial \Phi_1(t,s)}{\partial t} &= A(t)\Phi_1(t-h,s) + \int_{t-h}^t B(t,r)\Phi_1(r,s)dr, \quad t \geq 0, \\
\Phi_1(0,0) &= 1, \quad \Phi_1(t,s) = 0, \quad t < 0,
\end{align*}
\]

we transform (58) into (see [18], [21])

\[
x_i(t) = \psi(t) + \int_0^t K(t,s)u_i(s)ds + \int_0^t \Phi_1(t,s)D(s)dW_i(s),
\]

where \(K(t,s) = \Phi_1(t,s)C(s)\), and

\[
\psi(t) = \Phi_1(t,0)k(0) + \int_{-h}^0 \left[ \Phi_1(t,s+h)A(s+h) + \int_0^h \Phi_1(t,u)B(u,s)du \right] k(s)ds.
\]

It follows that the above \(\Phi_1\) is bounded and continuous in \(t\). On the other hand, (60) is one special case of Equation (53) by letting \(b = 0\), hence for \(t,s \in [0,T]\),

\[
P(t,s) = 0, \quad \tilde{c}(s,t) = \Phi_1(t,s)C(s),
\]

\[
\tilde{\gamma}(t) = \psi(t), \quad \tilde{\sigma}_i(t) = \int_0^t \Phi_1(t,s)D(s)dW_i(s),
\]

\[
M(t,s) = \int_0^{s\land t} \Phi_1(t,r)\Phi_1(s,r)|C(r)|^2dr.
\]

The NCE equation in this situation becomes

\[
\tilde{a}(t) = \psi(t) + \frac{1}{R} \int_0^T \int_0^{s\land t} \Phi_1(t,r)\Phi_1(s,r)|C(r)|^2dr \cdot \{\gamma \tilde{a}(s) + \eta - \tilde{a}(s)\}ds.
\]

Theorem 4.3 Suppose \(A, B, C\) are bounded deterministic functions such that

\[
\sup_{t \in [0,T]} \int_0^T \left[ (\gamma - 1) \int_0^{s\land t} \Phi_1(t,r)\Phi_1(s,r)|C(r)|^2dr \right]^2 ds < R,
\]

where \(\Phi_1\) is defined by (59), then NCE equation (61) admits a continuous solution.

Given \(\tilde{a}\) being a solution of NCE equation, if we define

\[
\tilde{u}_i(t) = -\frac{1}{R} C(t)E\int_t^T \Phi_1(s,t)\{x_i(s) - \gamma \tilde{a}(s) - \eta\}ds, \quad 1 \leq i \leq N,
\]

\[\boxed{(62)}\]
where

\[ x_i(t) = \psi(t) + \int_0^t \Phi_1(t, s)D(s)dW_i(s) \]

\[ -R^{-1}\int_0^T \int_0^{s\wedge t} \Phi_1(t, r)\Phi_1(s, r)|C(r)|^2E^F_t[x_i(s) - \gamma\tilde{a}(s) - \eta]drds, \]

then we get the following asymptotic equilibrium analysis in such setting.

**Theorem 4.4** Suppose that A, B, C and D are bounded deterministic functions, the NCE equation admits a consistent solution \( \hat{a} \in C[0, T] \). Moreover,

\[ T\int_0^T \int_0^{s\wedge t} |C(r)|^4\Phi_1(t, r)\Phi_1(s, r)drdsdt < \frac{R^2}{3}. \]

Then the set of controls defined in (62) for N players is an \( \epsilon \)-Nash equilibrium.

### 4.3 SDEs with delay in the control

In this case, suppose the cost functional is (4) and the state equation is described by

\[ dx_i(t) = A(t)x_i(t)dt + C(t)u_i(t - h)dt + D(t)dW_i(t), \]

where \( h > 0, \ t \in [-h, 0], \ C(t) = 0 \) with \( t < h \). The delayed term appears in the control variable. Similarly, by introducing function \( \Phi_2 \) as

\[ \frac{\partial \Phi_2(t, s)}{\partial t} = A(t)\Phi_2(t, s) \quad \Phi_2(2, s) = 1, \quad t, s \in [0, T], \]

we can transform equation (63) into

\[ x_i(t) = \psi(t) + \int_0^t K(t, s)u_i(s)ds + \int_0^t \Phi_2(t, s)D(s)dW_i(s), \]

where

\[ \psi(t) = \Phi_2(t, 0)k(0), \quad K(t, s) = \Phi_2(t, s + h)C(s + h), \quad t, s \in [0, T]. \]

Therefore, (65) is one special case of (4) with \( b = 0 \), and

\[ P(t, s) = 0, \quad \tilde{c}(s, t) = \Phi_2(s, t + h)C(t + h), \]

\[ \tilde{\varphi}(t) = \Phi_2(t, 0)k(0), \quad \tilde{\sigma}_i(t) = \int_0^t \Phi_2(t, s)D(s)dW_i(s), \]

\[ M(t, s) = \int_0^{s\wedge t} \Phi_2(t, r + h)\Phi_2(s, r + h)|C(r + h)|^2dr. \]

The NCE equation in this situation becomes

\[ \hat{a}(t) = \psi(t) + \frac{1}{R}\int_0^T \int_0^{s\wedge t} \Phi_2(t, r + h)\Phi_2(s, r + h)|C(r + h)|^2dr \cdot \{ \gamma\hat{a}(s) + \eta - \hat{a}(s) \} ds. \]
**Theorem 4.5** Suppose $A$, $C$ and $D$ are bounded functions such that

$$
\sup_{t \in [0,T]} \int_0^T \left[ (\gamma - 1) \int_0^{s \wedge t} \Phi_2(t, r + h) \Phi_2(s, r + h) |C(r + h)|^2 dr \right]^2 ds < R,
$$

then NCE equation (66) admits a unique solution.

Given $\hat{a}$ being the NCE solution, we define

$$
\hat{u}_i(t) = -\frac{1}{R} C(t + h) \mathbb{E}^{\mathcal{F}_t} \int_0^T \Phi_2(s, t + h) [x_i(s) - \gamma \hat{a}(s) - \eta] ds, \quad t \in [0, T],
$$

while $x_i$ satisfies

$$
x_i(t) = \Phi_2(t, 0) k(0) + \int_0^t \Phi_2(s, t) D(s) dW_i(s)
- \frac{1}{R} \int_0^T \int_0^{s \wedge t} \Phi_2(t, r + h) \Phi_2(s, r + h) |C(r + h)|^2 \mathbb{E}^{\mathcal{F}_r} [x_i(s) - \gamma \hat{a}(s) - \eta] dr ds,
$$

then we get the following asymptotic equilibrium analysis.

**Theorem 4.6** Suppose that $b$ and $c$ are bounded, the NCE equation (23) admits a unique continuous solution $\hat{a} \in C[0, T]$. Moreover,

$$
T \int_0^T \int_0^T \int_0^{s \wedge t} |C(r + h)|^4 \Phi_2(t, r + h) \Phi_2(s, r + h) dr ds dt < \frac{R^2}{3},
$$

where $\Phi_2$ is defined in (64). Then the set of controls defined in (67) for $N$ players is an $\epsilon$-Nash equilibrium.

## 5 Concluding remark

Herein, we investigate a class of mean-field LQG games where state equation is some stochastic Volterra integral system. The NCE consistency condition is derived based on some Fredholm equations and the $\epsilon$-Nash equilibrium property of decentralized controls is also established. Our work is the first attempt to the LQG games with stochastic integral system and there arise various research directions upon it. On one hand, it is possible to include the Volterra integral kernel into the cost functional to be minimized. This provides some potential way to study the LQG game with time inconsistency. On the other hand, this paper also considers the mean-field games with stochastic delayed system. Actually, more extensive research can be given in this research line and it is anticipated some new consistency conditions can be given which depend on the delay characters.

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