The Roles Leon Henkin Played in Mathematics Education

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Abstract

This paper is divided into two sections. In the first I give reasons for strongly recommending reading some of Henkin’s expository papers. In the second I describe Leon Henkin’s work as a social activist in the field of mathematics education, as he labored in much of his career to boost the number of women and underrepresented minorities in the upper echelons of mathematics.

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1 Some of Henkin’s expository papers

Henkin was an extraordinary insightful professor with a talent for exposition, and he devoted considerable effort to writing expository papers. I will mention three of them, while trying to convince you to read them with your students as a source of mutual inspiration.

1.1 Are Logic and Mathematics Identical?

This is the title of a wonderful expository paper [5], which Leon Henkin published in Science in 1962, and subtitled: An old thesis of Russell’s is reexamined in the light of subsequent developments in mathematical logic.

I recommend that you to give this paper to your students, not only because the historical view provided is comprehensive and synthetic but also because it shows the Henkin’s characteristic style; namely, the ability to strongly catch your attention from the start. How does he achieve it? you might wonder. In that particular paper, Henkin tells us that his interest in logic began at the age of 16, when ‘(I) came across a little volume of Bertrand Russell entitled Mysticism and Logic’. In the introduction Henkin cites Russell’s radical thesis, that ‘mathematics was nothing but logic’ together with the companion thesis ‘that logic is purely tautological’, and he describes the strong reaction against his thesis by the academic community: ‘Aux armes, citoyens du monde mathématique!’

Henkin then devotes the first section of the paper to explain the two main ideas that could help explain how Russell arrived at his conclusion. The first was the lengthy effort to achieve a ‘systematic reduction of all concepts of mathematics to a small number of them’, and the second was ‘the systematic study by mathematical means of the laws of logic which entered into mathematical proofs’. Henkin relates the work of Frege, Peirce, Boole and Schröder, during the second half of the nineteenth century, with the two efforts mentioned above, and identifies them as the primary raison d’être of Principia Mathematica.

In the following section, entitled From Russell to Gödel, Henkin explains the introduction of semantic notions by Tarski, as well as the formulation and proof of the completeness theorem for propositional logic by Post and for first-order logic by Gödel. ‘This result of Gödel’s is among the most basic and useful theorems we have in the whole subject of mathematical logic’. But, Henkin also explains how, in 1931, the hope of further extension of
this kind of completeness was ‘dashed by Gödel himself [...] (he) was able to demonstrate that the system of Principia Mathematica, taken as a whole was incomplete’. Immediately after, and anticipating what the reader might be thinking, Henkin dispels the hope of finding new axioms to repair the incompleteness phenomenon.

In the section entitled Consistency and the Decision Problem, Henkin analyzes these important notions and also explains how ‘Gödel was able to show that the questions of consistency and completeness were very closely linked to one another. [...] if a system such as the Principia were truly consistent, then in fact it would not be possible to produce a sound proof of this fact!’. In the following section, named Logic after 1936, Henkin describes how Alonzo Church proved that no decision procedure is available for first-order logic, and he devotes the rest of the paper to set theory, recursive functions, and algebraic logic. Henkin ends the paper with a section where he analyzes Russell’s Thesis in Perspective.

Henkin was awarded the Chauvenet Prize in 1964 for this paper. The prize is described as a Mathematical Association of America award to the author of an outstanding expository article on a mathematical topic by a member of the Association.

1.1.1 Bertrand Russell’s request

In April 1, 1963, Henkin received a very interesting letter from Bertrand Russell. In it, Russell thanked Henkin for ‘your letter of March 26 and for the very interesting paper which you enclosed.’ Right at the beginning Russell declared:

It is fifty years since I worked seriously at mathematical logic and almost the only work that I have read since that date is Gödel’s. I realized, of course, that Gödel’s work is of fundamental importance, but I was puzzled by it. It made me glad that I was no longer working at mathematical logic. If a given set of axioms leads to a contradiction, it is clear that at least one of the axioms is false. Does this apply to school-boys’ arithmetic, and if so, can we believe anything that we were taught in youth? Are we to think that 2+2 is not 4, but 4.001?

He then went on explaining his ‘state of mind’ while Whitehead and he were doing the Principia and added: ‘Both Whitehead and I were disappointed that the Principia was almost wholly considered in connection with the question whether mathematics is logic.’

Russell ended the letter with a request: ‘If you can spare the time, I should like to know, roughly, how, in your opinion, ordinary mathematics — or, indeed, any deductive system — is affected by Gödel’s work.’

According to Annellis: ‘Henkin replied to Russell at length with an explanation of Gödel’s incompleteness results, in a letter of July 1963, specifically explaining that Gödel’s showed, not the inconsistency, but the incompleteness of the [Principia] system.’

1.2 On Mathematical Induction

In a personal communication Henkin affirmed that On mathematical induction, published in 1960, was the favorite among his articles because it had a somewhat panoramic nature and was not directed exclusively to specialists. He wrote: ‘[...] but my little paper on induction models from 1960, which has always been my favorite among my expository papers’. In it, the relationship between the induction axiom and recursive definitions is studied in depth.

why do I so strongly recommend that you ask your students to read this paper? From my point of view, it is the best paper on logic to offer students as a first reading of a “real-life”
article. The paper is especially interesting because Henkin describes something that would never appear in a formal article: his motivation. It seems that Henkin was trying to convince a mathematical colleague about why a given argument about the existence of recursive operations was completely wrong, even though at first sight it might seem convincing.

Before going into the details of the wrong argument that motivated the whole paper, Henkin explains Peano arithmetic.

Peano axiomatized the theory of natural numbers. To do so, he started out from indefinable primitive terms— in particular, those of natural number, zero, and the successor function— and by means of three axioms he synthesized the main facts. Among those axioms is that of induction, which states that any subset of natural numbers, closed by the successor operation and to which zero belongs, is precisely the set of all natural numbers. Although the axioms for the theory of natural numbers are very important, the most interesting theorems of the theory did not stem from them alone because in most of the theorems, operations of addition, multiplication, etc. are used.

Peano thought that after axiomatizing a theory it did not suffice to organize the facts by means of axiomatic laws; it was also necessary to organize the concepts, using definition laws. In particular, we can define addition recursively by means of:

1. \( x + 0 = x \)
2. \( x + Sy = S(x + y) \), for all \( x, y \) of \( \mathbb{N} \)

However, this definition must be justified by a theorem in which the existence of a unique operation that will satisfy the previous equations must be established.

A poor argument to prove this is as follows. Let us choose any \( x \in \mathbb{N} \). We define a subset \( G \) of \( \mathbb{N} \), by placing all \( y \in \mathbb{N} \) elements in \( G \) for which \( x + y \) is defined by the previous equations. It is not difficult to see that \( 0 \in G \) and that \( \forall (y \in G \rightarrow Sy \in G) \). Now using the induction axiom, we conclude that \( G = \mathbb{N} \) and thus that \( x + y \) is defined for all \( y \in \mathbb{N} \).

Why is this proof wrong? This was the question that Henkin's colleague posed him. Henkin tried to convince him that because the argument was designed to establish the existence of a function \( f \) (in the example), it is incorrect to assume in the course of the argument that we have such a function. Henkin made him see that in the proof only the third axiom was used and that, if correct, the same reasoning could be used not only for models that satisfy all Peano axioms but also for those that satisfy only the induction one. Henkin called these “Induction Models” and proved that in them not all recursive operations are definable. For example, exponentiation fails.

Induction models turn out to have a fairly simple mathematical structure: there are standard ones— that is, isomorphic to natural numbers— but also non-standard ones. The latter also have a simple structure: either they are cycles, in particular \( \mathbb{Z} \) modulo \( n \), or they are what Henkin calls “spoons” because they have a handle followed by a cycle. The reason is that the induction axiom is never fulfilled alone, since it requires Peano's first or second axiom. This does not mean that Peano's axioms are redundant, as it is well known that they are formally independent; i.e., each one is independent of the other two.

1.3 Completeness

If you take a look at the list of documents Leon Henkin left us, the first published paper, *The completeness of first order logic* [2], corresponds to his well known result, while the last, *The discovery of my completeness proof* [7], is a extremely interesting as autobiography, thus ending his career with a sort of fascinating loop.
I claim that reading the last paper is a must. Why? As you know, Leon Henkin left us an important collection of papers, some of them so exciting as his proof of the completeness theorem both for the theory of types and for first-order logic. He did so by means of an innovative and highly versatile method, which was later to be used in many other logics, even in those known as non-classical. In his 1996 paper, we learn about the process of discovery, which observed facts he was trying to explain, and why he ended up discovering things that were not originally the target of his enquiries. Thus, in this case we do not have to engage in risky hypotheses or explain his ideas on the mere basis of the later, cold elaboration in scientific articles. It is well known that the *logic of discovery* differs from what is adopted on organizing the final exposition of our research through the different propositions, lemmas, theorems and corollaries.

We also learn that the publication order of his completeness results ([2] and [3]) is the reverse of his discovery of the proofs. The completeness for first-order logic was accomplished when he realized he could modify the proof obtained for type theory in an appropriate way. We consider this to be of great significance, because the effort of abstraction needed for the first proof (that of type theory) provided a broad perspective that allowed him to see beyond some prejudices and to make the decisive changes needed to reach his second proof. In [7] you can find a detailed commentary of Henkin’s contribution to the resolution and understanding of the completeness phenomena.

### 1.3.1 Henkin’s expository papers on completeness

In 1967 Henkin published two very relevant expository papers on for the subject we are considering here, *Truth and Provability* and *Completeness*, which were published in *Philosophy of Science Today* [10].

#### 1.3.1.1 Truth and Provability

In less than 10 pages, Henkin gives a very intuitive introduction to the concept of truth and its counterpart, that of provability, in the same spirit of Tarski’s expository paper *Truth and Proof* [11]. The latter was published in *Scientific American* two years after Henkin’s contribution. This not so surprising as Henkin had by then been in Berkeley working with Tarski for about 15 years and the theory of truth was Tarski’s contribution.

The main topics Henkin introduces (or at least touches upon) are very relevant. They include the *use/mention* distinction, the desire for languages with infinite sentences and the need for a recursive definition of truth, the language/metalanguage distinction, the need to avoid reflexive paradoxes, the concept of denotation for terms, and the interpretation of quantified formulas. He also explains what an axiomatic theory is and how it works in harmony with a deductive calculus. Properties such as decidability and completeness/incompleteness of a theory are mentioned at the end. I admire the way these concepts are introduced, with such élan, and the chain Henkin establishes, which shows how each concept is needed to support the next.

#### 1.3.1.2 Completeness

In this short expository paper Henkin explores the complex landscape of the notions of completeness. He introduces the notion of logical completeness —both weak and strong—as an extension of the notion already introduced of “completeness of an axiomatic theory”. This presentation differs notably from the standard way these notions are introduced today where, usually, the completeness of the logic precedes the notion of completeness of a theory.
and, often, to avoid misunderstandings, both concepts are separated as much as possible, as if relating them were some sort of terrible mistake or even anathema. Gödel's incompleteness theorem is presented, as well as its negative impact on the search for a complete calculus for higher-order logic. The paper ends by introducing his own completeness result for higher-order logic with general semantics. The utilitarian way Henkin uses to justify his general models as a way of sorting the provable sentences from the unprovable ones in the class of valid sentences (in standard models) is very peculiar.

2 The Roles of Action and Thought in Mathematics Education

Henkin was often described as a social activist, he labored much of his career to boost the number of woman and underrepresented minorities in the upper echelons of mathematics. He was also very aware that we are beings immersed in the crucible of history from which we find it hard to escape, an awareness he brought to the very beginning of his interesting article about the teaching of mathematics [6]:

Waves of history wash over our nation, stirring up our society and our institutions. Soon we see changes in the way that all of us do things, including our mathematics and our teaching. These changes form themselves into rivulets and streams that merge at various angles with those arising in parts of our society quite different from education, mathematics, or science. Rivers are formed, contributing powerful currents that will produce future waves of history.

The Great Depression and World War II formed the background of my years of study; the Cold War and the Civil Rights Movement were the backdrop against which I began my career as a research mathematicians, and later began to involve myself with mathematics education.

(In [6] page 3)

In this paper he gave both a short outline of the variety of educational programs he created and/or participated in, and interesting details about some of them. In particular he discussed the following six:

1. 1957-59, NSF Summer Institutes. The National Science Foundation is an independent federal agency created by Congress in 1950. As you can read in their web page, http://www.nsf.gov/about/, its aim was ‘to promote the progress of science; to advance the national health, prosperity, and welfare; to secure the national defense…’. Nowadays, NSF is the ‘only federal agency whose mission includes support for all fields of fundamental science and engineering, except for medical sciences.’ NSF’s Strategic Plan includes Investing in Science, Engineering, and Education for the Nation’s Future. In [6] Henkin related this initiative to historical facts: ‘The launching of Sputnik demonstrated superiority in space travel, and our country responded in a variety of ways to improve capacity for scientific and technical developments’ In 1957, Henkin involved himself in several NSF’s programs, he served as a lecturer of several courses. These programs were designed to improve high school and college mathematics instruction and were directed to mathematics teachers. Henkin explained that the variety of attitudes toward mathematics of the teachers attending the courses was amazing, and that the experience gave him a view of the nature of instruction around the country. The subject of his courses was the axiomatic foundation of number systems. One of his aims was to get students to understand “the idea of a proof” because he believed that it could help students in the
effort of finding proofs of their own, in a much better way than the mere understanding of the steps that constitutes a proof.

2. 1959-63, MAA Math Films. The Mathematical Association of America was established one century ago, in 1915. As you can read in their web page, http://www.maa.org/, ‘Over our first century, MAA has certainly grown, but continues to maintain our leadership in all aspects of the undergraduate program in mathematics’. Long before internet resources became available, the MAA made movies. As Henkin said: ‘Sensing a potential infusion of technology into mathematics instruction, MAA set up a committee to make a few experimental films. [...] the committee approached me in 1959-60 with a request to make a filmed lecture on mathematical induction which could be shown at the high-school-senior/college-freshman level. I readily agreed.’ The film was part of the Mathematics Today series, and was shown on public television in New York City and in high schools. In [6] Henkin explained the preparation of the film, both from a technical point of view and from a methodological and pedagogical perspective. He attributed the lack of understanding of the induction principle at the undergraduate level to the current formulation as a mathematical principle, and he proposed to use it as ‘a statement about sets of numbers satisfying two simple conditions; formulated in this way, it is a fine vehicle for giving students practice in forming and using sets of numbers to show that all natural numbers possess various properties’

3. 1961-64 CUPM. The Mathematical Association of America’s Committee on the Undergraduate Program in Mathematics (CUPM) is charged with making recommendations to guide mathematics departments in designing curricula for their undergraduate students. In the sixties, the CUPM proposed courses to be taken by elementary teachers. In [6] Henkin said ‘Some of my colleagues and I began, for the first time, to have classroom contact with prospective elementary teachers, and that led, in turn, to in-service programs for current teachers. I learned a great deal from teaching teachers-students; I hope they learned at least half as much as I!’

4. 1964-. Activities To Broaden Opportunity. “The sixties” is the term used to describe the counterculture and revolution movement that took place in several places in the U.S.A. and Europe. Berkeley students were taking energetic actions against segregation in southeastern U.S.A. as well as against military actions in Vietnam. In [6] Henkin said ‘In the midst of this turmoil I joined in forming two committees at Berkeley which enlarged the opportunity of minority ethnic groups for studying mathematics and related subjects. [...] We noted that while there was a substantial black population in Berkeley and the surrounding Bay Area, our own university student body was almost “lily white” and the plan to undertake action through the Senate was initiated’ In 1964, Leon Henkin and Jerzy Neyman, a world-famous Polish-American statistician from Berkeley University, started a program at Berkeley to increase the number of minority students entering college from Bay Area high schools. Henkin told us that the initiative came after Neyman participation in the MAA’s Visiting Mathematician Program in Fall 1963. He lectured in southern states where, by law, whites and blacks studied in separate colleges. Upon returning to Berkeley he told some of his friend that “first-rate students were being given a third-rate education” ’ Henkin and Neyman undertake actions through the Senate, and in 1964 the Senate established a committee with the desired effect. The committee recruited promising students and offered them summer programs to study mathematics and English. If they persisted in the program, they were offered special scholarships. In the same year, 1964, Henkin heard a talk by a Berkeley High School teacher, Bill Johntz. After that, Henkin was invited to see him in action, while he was teaching
mathematics to elementary students from low-income neighborhoods, and realized that Johntz was able to raise great enthusiasm in the class. Significantly, students enjoyed and actively engaged in the process of learning, and they became integrally involved in their own education. He was using a Socratic group-discovery method modeled after the filmed teaching of David Page, a University of Illinois mathematics professor. The method was working well, and they recruited university mathematics students as well as engineers as teachers, after some training. The program was called Project SEED — Special Elementary Education for the Disadvantaged. This program is still alive, as you can see in their web page, http://projectseed.org/.

5. 1960-68. Teaching Teachers, Teaching Kids. In this paper Henkin described several conferences on school mathematics as well as several projects and courses he was involved in. The following paragraph caught my eye: ‘After I began visiting elementary school classes in connection with CTFO, I came to believe that the emotional response of the teachers to mathematics was of more importance to the learning process of the students than the teacher’s ability to relate the algorithms of arithmetic to the axioms of ring theory’.

6. 1968-70. Open Sesame: The Lawrence Hall Of Science. The Lawrence Hall of Science, a science museum in Berkeley, was created in honor of the 1939 Nobel prize winner Ernest Orlando Lawrence. As you can read in their web page, http://www.lawrencehallofscience.org/about

‘We have been providing parents, kids, and educators with opportunities to engage with science since 1968.’

According to Henkin’s tale ‘In 1968, the newly appointed director, Professor of Physics Alan Portis, decided to transform the museum into a center of science and mathematics education, whose functions would be integrated with graduate research programs directed by interdisciplinary group of faculty.’ To help in his endeavor, he gathered a group of faculty from a variety of science departments interested in science education. ‘These faculty members proposed a new, interdisciplinary Ph.D program under the acronym SESAME — Special Excellence in Science and Mathematics Education. Entering students were required to have a masters degree in mathematics or in one of the sciences. Courses and seminars in theories of learning, cognitive science, and experimental design were either identified in various departments, or created’. Nitsa Movshovitz-Hadar, a student from the Technion in Israel, was admitted in the SESAME program, she wrote her thesis under the direction of Leon Henkin. Nitsa is one of the contributors of the book, The Life and Work of Leon Henkin: Essays on His Contributions.

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