DENSITY OF CLOSED ORBITS IN THE NON-WANDERING SET FOR FLOWS ON SURFACES

TOMOO YOKOYAMA

Abstract. The density of closed orbits in the non-wandering set for a flow is one of the essential properties of Axiom A and chaos in the sense of Devaney. In this paper, we topologically characterize the density of closed orbits in the non-wandering set for a flow on a surface. More precisely, for a flow with finitely many connected components of the set of singular points on a compact surface, the density of closed orbits in the non-wandering set holds if and only if there are neither non-closed recurrent orbits, strict limit cycles, nor circuits with wandering holonomy. Moreover, we characterize the correspondence between the closed point set and the non-wandering point set for such a flow. In addition, the analogous results hold for flows on non-compact surfaces with finite genus and finite ends. On the other hand, we construct flows with countable singular points on tori satisfying such non-existence but not the density condition. Furthermore, using the characterization of the density of closed orbits in the non-wandering set, we show that the non-wandering set of a flow on a compact surface consists of finitely many orbits if and only if there are no non-closed recurrent orbits and are at most finitely many limit circuits and any limit circuit consists of at most finitely many orbits.

1. Introduction

In 1927, Birkhoff introduced the concepts of non-wandering points and recurrent points [7]. Using these concepts, we can describe and capture dynamical behaviors. The density of closed orbits in the non-wandering set for a flow is one of the essential properties of Axiom A and chaos in the sense of Devaney. It is also a necessary condition of topological stability for a differentiable flow on a closed manifold. Such density condition is called Axiom A(b). The sufficient conditions and genericity for Axiom A(b) are studied in hyperbolic dynamical systems [11, 12, 17, 21, 23, 26, 27, 30]. For instance, Smale asked whether hyperbolicity of the non-wandering set for a diffeomorphism on a compact manifold implies the density of closed orbits in the non-wandering set [30]. In other words, does Axiom A(a) imply Axiom A(b)? In the surface diffeomorphism case, Newhouse and Palis answered affirmatively [23]. On the other hand, the answer is no in general [11, 12, 17, 18]. In fact, Dankner constructed an example of a diffeomorphism on \( \mathbb{R}^3 \) which has infinite hyperbolic non-wandering sets but only finitely many periodic orbits [11, 12]. It is also shown that there is such a diffeomorphism for any isotopy class of the set of diffeomorphisms on a compact manifold whose dimension is more
than two. The analogous results for flows on manifolds of dimension at least four are also shown by suspension operations.

Moreover, topological stability for a homeomorphism on a closed manifold and the uniform periodic shadowing property for a homeomorphism on a compact Hausdorff space are sufficient conditions of Axiom A(b) [1,14].

The characterizations for Axiom A(b) are known in the low-dimensional dynamical system case. The following are equivalent for an orientation-preserving circle homeomorphism \( f : S^1 \rightarrow S^1 \): (1) \( \text{Per}(f) = \Omega(f) \); (2) There are periodic orbits; (3) The rotation number of \( f \) is rational (cf. [16, Proposition 11.1.4]). This equivalence means that the existence of non-periodic minimal sets is an obstruction of the density of closed orbits in the non-wandering set for a circle homeomorphism. Moreover, the non-existence of non-periodic recurrent points for a circle homeomorphism \( f \) is equivalent to the density condition \( \text{Per}(f) = \Omega(f) \). This equivalence holds for a regular curve homeomorphism, because the set of periodic points of a regular curve homeomorphism is either empty or dense in the non-wandering set [10]. Here a regular curve is a compact connected metric space such that for any point \( x \in X \) and any open neighborhood \( U \) of \( x \) there is a neighborhood \( V \subseteq U \) of \( x \) with finite boundary. Furthermore, the equivalence for the existence of non-periodic minimal sets and Axiom A(b) is studied for local dendrites [1,19,20].

In this paper, we consider necessary and sufficient conditions that the closure of the union of closed orbits becomes the non-wandering set. In other words, we pose the following characterization problem.

**Problem 1.** Find a topological characterization of Axiom A(b).

Under the finite existence of connected components of the singular point set, we construct the following answer for the previous problem for a flow on a compact surface, which is a generalization of the above equivalence for a circle homeomorphism.

**Theorem A.** The following conditions are equivalent for a flow \( v \) with finitely many connected components of the set of singular points on a compact surface:

1. \( \overline{\text{Cl}(v)} = \Omega(v) \).
2. There are neither non-closed recurrent points, strict limit circuits, nor circuits with wandering holonomy.

The finite existence of connected components of the singular point set in Theorem A is necessary (see Example 2 for details). In addition, we show the analogous results for flows on surfaces with finite genus (see Theorem 3.10 for details).

In [8], the following question is posed: If the set of closed orbits is closed, under what additional conditions on the dynamical system can we conclude that any non-wandering orbits are closed? The answer is negative in general. In fact, there is a spherical diffeomorphism with hyperbolic periodic points such that the non-wandering set is not the set of periodic points [8]. On the other hand, there are affirmative results for mappings on closed intervals and closed disks as follows. A continuous map of a closed interval to itself whose periodic point set is closed satisfies Axiom A(b), because the following are equivalent for a continuous map \( f \) of a closed interval to itself: (1) \( \text{Per}(f) \) is closed; (2) \( \text{Per}(f) = \Omega(f) \), where \( \text{Per}(f) \) is the set of periodic points of \( f \) and \( \Omega(f) \) is the set of non-wandering points [25,32]. The same statement holds for a \( C^1 \) self-map \( f(x, y) = (f_1(x), f_2(x, y)) \) on the unit square such that \( \text{Per}(f) \) is closed [5,13]. Applying Theorem A we
obtain the following affirmative statement, which generalizes the above result for a continuous map of a closed interval \([2\pi, 32]\).

**Corollary B.** The following conditions are equivalent for a flow \(v\) with finitely many connected components of the set of singular points on a compact surface:

1. \(\text{Cl}(v) = \Omega(v)\).
2. The closed point set \(\text{Cl}(v)\) is closed and there are neither non-closed recurrent points, non-periodic limit circuits, nor circuits with wandering holonomy.
3. The set \(P(v)\) of non-recurrent points are open and there are neither non-closed recurrent points, non-periodic limit circuits, nor circuits with wandering holonomy.

In any case, the flow \(v\) satisfies Axiom A(b).

The finite existence of connected components of the singular point set in Corollary \([4]\) is necessary (see Example \([2]\) for details). Using blow-up deformations of singular points, we can obtain a more general result for surfaces. By the end completion of surfaces of finite genus, we can obtain an analogous result for surfaces of finite genus (see Theorem \([3,17]\) for details).

In addition, the finiteness of the non-wandering set is characterized as follows.

**Theorem C.** The following conditions are equivalent for a flow \(v\) on a compact surface:

1. The non-wandering set \(\Omega(v)\) consists of finitely many orbits.
2. Each recurrent orbit is a closed orbit, any limit circuits consist of at most finitely many orbits, and there are at most finitely many closed orbits and limit circuits.
3. Each recurrent orbit is a closed orbit, any limit circuits consist of at most finitely many orbits, and there are at most finitely many closed orbits and strict limit circuits.

In any case, the non-wandering set \(\Omega(v)\) consists of closed orbits, strict limit circuits, and circuits with wandering holonomy.

The present paper consists of four sections. In the next section, as preliminaries, we introduce fundamental concepts. In §3, we topologically characterize the density of closed orbits in the non-wandering set for a flow with finitely many connected components of the singular point set on compact and non-compact surfaces with finite genus and finite ends. Moreover, the topological characterization of the finiteness of the non-wandering set is described. Furthermore, we topologically characterize the correspondence between the closed point set and the non-wandering point set for such flows. In the final section, some examples of flows are described to show the necessity of conditions in Theorem A and lemmas. Furthermore, we observe that the density of periodic orbits for non-wandering flows on compact surfaces corresponds to the non-existence of locally dense orbits, and that any Hamiltonian flows on compact surfaces and any gradient flows on manifolds satisfy the density of periodic orbits in the non-wandering set.

2. Preliminaries

2.1. Topological notion. Denote by \(\overline{A}\) the closure of a subset \(A\) of a topological space, by \(\text{int}A\) the interior of \(A\), and by \(\partial A := \overline{A} - \text{int}A\) the boundary of \(A\), where \(B - C\) is used instead of the set difference \(B \setminus C\) when \(B \subseteq C\). We define the border \(\partial^-A\) by \(A - \text{int}A\) of \(A\). A boundary component of a subset \(A\) is a connected component of the boundary of \(A\). A subset is locally dense if its closure has a
nonempty interior. Recall that a boundary component of a subset \( A \) is a connected component of the boundary of \( A \).

2.1.1. **Curves and loops.** A curve is a continuous mapping \( C : I \to X \) where \( I \) is a non-degenerate connected subset of a circle \( S^1 \). A curve is simple if it is injective. We also denote by \( C \) the image of a curve \( C \). Denote by \( \partial C := C(\partial I) \) the boundary of a curve \( C \), where \( \partial I \) is the boundary of \( I \subset S^1 \). Put \( \text{int}C := C \setminus \partial C \). A simple curve is a simple closed curve if its domain is \( S^1 \) (i.e. \( I = S^1 \)). A simple closed curve is also called a loop. Two disjoint loops are parallel if there is an open annulus whose boundary is the union of the two loops. An arc is a simple curve whose domain is an interval. An orbit arc is an arc contained in an orbit.

2.1.2. **Surfaces and their subsets.** By a surface, we mean a two dimensional para-compact manifold, that does not need to be orientable. A subset of a surface is circular if it is homeomorphic to a circle. A subset of a surface is annular (resp. spherical, toral) if it is homeomorphic to an annulus (resp. sphere, torus).

2.2. **Notions of dynamical systems.** A flow is a continuous \( \mathbb{R} \)-action on a manifold. From now on, we suppose that flows are on surfaces unless otherwise stated. Let \( v : \mathbb{R} \times S \to S \) be a flow on a surface \( S \). For \( t \in \mathbb{R} \), define \( v_t : S \to S \) by \( v_t := v(t, \cdot) \). For a point \( x \) of \( S \), we denote by \( O(x) \) the orbit of \( x \), \( O^+(x) \) the positive orbit (i.e. \( O^+(x) := \{ v_t(x) \mid t > 0 \} \)), and \( O^-(x) \) the negative orbit (i.e. \( O^-(x) := \{ v_t(x) \mid t < 0 \} \)). Recall that a point \( x \) of \( S \) is singular if \( x = v_t(x) \) for any \( t \in \mathbb{R} \), is non-singular if \( x \) is not singular, and is periodic if there is a positive number \( T > 0 \) such that \( x = v_T(x) \) and \( x \neq v_t(x) \) for any \( t \in (0, T) \). An orbit is singular (resp. periodic) if it contains a singular (resp. periodic) point. An orbit is closed if it is singular or periodic. Denote by \( \text{Sing}(v) \) the set of singular points and by \( \text{Per}(v) \) (resp. \( \text{Cl}(v) \)) the union of periodic (resp. closed) orbits. Recall that the \( \omega \)-limit (resp. \( \alpha \)-limit) set of a point \( x \) is \( \omega(x) := \bigcap_{n \in \mathbb{R}} \{ v_t(x) \mid t > n \} \) (resp. \( \alpha(x) := \bigcap_{n \in \mathbb{R}} \{ v_t(x) \mid t < n \} \)), where the closure of a subset \( A \) is denoted by \( \overline{A} \).

For an orbit \( O \), define \( \omega(O) := \omega(x) \) and \( \alpha(O) := \alpha(x) \) for some point \( x \in O \). Note that an \( \omega \)-limit (resp. \( \alpha \)-limit) set of an orbit is independent of the choice of point in the orbit. A point \( x \) of \( S \) is recurrent if \( x \in \omega(x) \cup \alpha(x) \). A quasi-minimal set (or a Q-set for brevity) is defined to be the orbit closure of a non-closed recurrent point. Denote by \( R(v) \) the set of non-closed recurrent points.

A point is wandering if there are its neighborhood \( U \) and a positive number \( N \) such that \( v_t(U) \cap U = \emptyset \) for any \( t > N \). A point is non-wandering if it is not wandering (i.e. for any its neighborhood \( U \) and for any positive number \( N \), there is a number \( t \in \mathbb{R} \) with \( |t| > N \) such that \( v_t(U) \cap U = \emptyset \)). Denote by \( \Omega(v) \) the set of non-wandering points, called the non-wandering set. A flow is non-wandering if any points are non-wandering. Recall that a non-wandering flow on a compact surface has no exceptional orbits (i.e. \( E(v) = \emptyset \) because of [39, Lemma 2.3].

A subset \( A \) is positive (resp. negative) invariant if \( v_t(A) \subseteq A \) for any \( t \in \mathbb{R}_{>0} \) (resp. \( t \in \mathbb{R}_{<0} \)). A subset is invariant (or saturated) if it is a union of orbits. The saturation of a subset is the union of orbits intersecting it. Denote by \( \text{Sat}_v(A) = v(A) \) the saturation of a subset \( A \). Then \( \text{Sat}_v(A) = \bigcup_{a \in A} O(a) \). A nonempty closed invariant subset is minimal if it contains no proper nonempty closed invariant subsets.

A separatrix is a non-singular orbit whose \( \alpha \)-limit or \( \omega \)-limit set is a singular point.
2.2.1. **Topological properties of orbits.** An orbit is proper if it is embedded, locally dense if its closure has a nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if its orbit is proper (resp. locally dense, exceptional). Denote by $\text{LD}(v)$ (resp. $\text{E}(v)$, $\text{P}(v)$) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Note that an orbit on a paracompact manifold (e.g., a surface) is proper if and only if it has a neighborhood in which the orbit is closed \[34\]. This implies that a non-recurrent point is proper and so that a non-proper point is recurrent. In \[9,\text{Theorem VI}\], Cherry showed that the closure of a non-closed recurrent orbit $O$ of a flow on a manifold contains uncountably many non-closed recurrent orbits whose closures are $\overline{O}$. This means that each non-closed recurrent orbit of a flow on a manifold has no neighborhood in which the orbit is closed, and so is not proper. In particular, a non-closed proper orbit is non-recurrent. Therefore the union $\text{P}(v)$ of non-closed proper orbits is the set of non-recurrent points and that $\text{R}(v) = \text{LD}(v) \sqcup \text{E}(v)$, where $\sqcup$ denotes a disjoint union. Hence we have a decomposition $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \text{P}(v) \sqcup \text{R}(v)$.

2.2.2. **Flow box.** Let $I, J$ be intervals which are either $(0, 1)$, $(0, 1]$, $[0, 1)$, or $[0, 1]$ and $\mathbb{D} := I \times J \subset \mathbb{R}^2$ be a disk whose orbits are of form $I \times \{t\}$ for some $t \in J$. Then $D$ is called a trivial flow box. A disk $B$ is a flow box if there is an annular neighborhood $\mathcal{U}_B$ of the boundary $\partial B$ such that the intersection $B \cap \mathcal{U}_B$ is the intersection $D \cap \mathcal{U}_D$ of a trivial flow box $D$ and its neighborhood $\mathcal{U}_D$ of the boundary $\partial D$ up to topological equivalence as in Figure 1. In other words, a disk $B$ is a flow box if there is a homeomorphism $f : B \cap \mathcal{U}_B \rightarrow D \cap \mathcal{U}_D$ such that the images of orbit arcs are orbit arcs.

2.2.3. **Circuits.** A trivial circuit is a singular point. An open annular subset $\mathcal{A}$ of a surface is a collar of a singular point $x$ if the union $\mathcal{A} \cup \{x\}$ is a neighborhood of $x$. A trivial circuit $x$ is attracting (resp. repelling) if there is its collar which is contained in the stable (resp. unstable) manifold of $x$. In other words, an attracting trivial circuit is either a $\partial$-source or a source, and a repelling trivial circuit is either a $\partial$-sink or a sink. By a cycle or a periodic circuit, we mean a periodic orbit. By a non-trivial circuit, we mean either a cycle or an image of an oriented circle by a continuous orientation-preserving mapping which is a directed graph but not a singleton and which is the union of separatrices and finitely many singular points. A circuit is either a trivial or non-trivial circuit. Note that there are circuits with infinitely many edges and that any non-trivial non-periodic circuit contains non-closed proper orbits as in Figure 2. An open annular subset $\mathcal{A}$ of a surface is a collar of a non-trivial circuit $\gamma$ if $\gamma$ is a boundary component of $\mathcal{A}$ and there is a neighborhood $U$ of $\gamma$ such that $\mathcal{A}$ is a connected component of the complement.
Figure 2. A strict limit circuit that consists of a degenerate singular point and infinitely many connecting separatrices, and its neighborhood which consists of a singular point and non-recurrent orbits.

$U - \gamma$. A circuit $\gamma$ is a semi-attracting (resp. semi-repelling) circuit with respect to a small collar $\mathcal{A}$ if $\omega(x) = \gamma$ (resp. $\alpha(x) = \gamma$) and $O^+(x) \subset \mathcal{A}$ (resp. $O^-(x) \subset \mathcal{A}$) for any point $x \in \mathcal{A}$. Then $\mathcal{A}$ is called an attracting (resp. repelling) collar basin of $\gamma$. A non-trivial circuit $\gamma$ is a limit circuit if it is a semi-attracting or semi-repelling circuit. A limit cycle is a limit circuit in $\text{Per}(v)$.

A non-trivial circuit $\gamma$ is one-sided if for any small neighborhood $U$ of $\gamma$ there is a collar $V \subset U$ of $\gamma$ such that the union $V \cup \gamma$ is a neighborhood of some point in $P(v) \cap \gamma$. A non-trivial circuit $\gamma$ is two-sided if it is not one-sided (i.e. there is a small neighborhood $U$ of $\gamma$ such that the union $V \cup \gamma$ for any collar $V \subset U$ of $\gamma$ is not a neighborhood of any point in $P(v) \cap \gamma$). For a circuit $\mu$ which is a simple closed curve, notice that the circuit $\mu$ is one-sided if and only if it is either a boundary component of a surface or has a small neighborhood which is a Möbius band, and that the circuit $\mu$ is two-sided if and only if it has an open small annular neighborhood $\mathcal{A}$ such that the complement $\mathcal{A} - \mu$ consists of two open annuli.

A non-trivial non-periodic limit circuit $\gamma$ is a strict limit circuit if either $\gamma$ is one-sided, or there are a separatrix $\mu \subseteq \gamma$ and a transverse open arc $T$ intersecting $\mu$ such that $f_v|T_1$ is either attracting or repelling, and that $f_v|T_2$ has no fixed points where $f_v : T \to T$ is the first return map, $T_1$ and $T_2$ are the two connected components of $T \setminus \mu$. A non-trivial non-periodic circuit $\gamma$ is a boundary circuit if there are a non-singular point $x \in \gamma$ and an open transverse arc $I$ one of whose boundary components is $x$ such that a boundary component of the saturation $\text{Sat}_v(I)$ contains $\gamma$ (i.e. $\gamma \subseteq \partial(\text{Sat}_v(I))$). If the saturation $\text{Sat}_v(I)$ is annular, then we call it an associated collar of $\gamma$. A non-trivial circuit is a circuit $\gamma$ with wandering holonomy map if there are a non-singular point $x \in \gamma$ and arbitrarily small open transverse arc $I$ containing $x$ such that the first return map on $I$ is orientation reversing, the domain is nonempty, and the intersection of the domain and the codomain is empty. For instance, there are flows with circuits with wandering holonomy as in Figure 3 and Figure 4. We have the following observation.

**Lemma 2.1.** Let $v$ be a flow on a surface $S$. Then each circuit with wandering holonomy is not periodic and contains points in $\Omega(v) - \overline{\text{Cl}(v)}$.

**Proof.** Let $\gamma$ be a circuit with wandering holonomy. We claim that $\gamma$ is non-periodic. Indeed, assume that $\gamma$ is periodic. Fix a small open transverse arc $I$ intersecting $\gamma$. By the flow box theorem (cf. [3, Theorem 1.1, p.45]) to $\gamma$, the intersection of the domain and the codomain for the first return map to $I$ contains a nonempty open
interval, which contradicts that the intersection of the domain and the codomain is empty.

Fix a non-singular point $x \in \gamma$. The flow box theorem to $x$ implies that there is a neighborhood of $x$ containing no singular points. Since the intersection of the domain and the codomain for the first return map to arbitrarily small open transverse arc $I$ containing $x$ contains a nonempty open interval, the point $x$ is non-wandering. The empty intersection of the domain and the codomain of the first return map implies that there is a neighborhood of $x$ containing no periodic points. This means that $x \in \Omega(v) - \overline{v}$.

3. Topological characterization of density of periodic orbits in the non-wandering set

In this section, we topologically characterize Axiom A(b) for a flow with finitely many connected components of the singular point set on surfaces with finite genus and finite ends.

3.1. Necessary conditions. We show that Axiom A(b) implies the non-existence of Q-sets.

**Lemma 3.1.** Let $v$ be a flow on a compact surface $S$. If $\overline{\Omega(v)} = \Omega(v)$, then $R(v) = \emptyset$ (i.e. $S = \overline{v} \sqcup P(v)$).

**Proof.** Suppose that $\overline{\Omega(v)} = \Omega(v)$. Since recurrent orbits are non-wandering, we have $R(v) \subseteq \Omega(v) = \overline{v}$. Lemma 2.3 implies that $\text{Per}(v) \cap R(v) = \emptyset$. The closedness of the singular point set $\text{Sing}(v)$ implies that $\overline{\Omega(v)} \cap R(v) = \emptyset$ and so that $R(v) = \emptyset$. Then $S = \overline{v} \sqcup P(v)$. □

The converse of Lemma 3.1 is not true in general (see Example 1 in §4.1 for details). We show that Axiom A(b) implies the non-existence of circuits with wandering holonomy and strict limit circuits.

**Lemma 3.2.** Let $v$ be a flow on a compact surface $S$. If $\overline{\Omega(v)} = \Omega(v)$, then there are neither strict limit circuits nor circuits with wandering holonomy.

**Proof.** Lemma 2.1 implies that there are no circuits with wandering holonomy. By Lemma 3.1 there are no non-closed recurrent orbits and so $S = \overline{v} \sqcup P(v)$. Then $\Omega(v) \cap \text{int}(P(v)) = \Omega(v) - \overline{v}$. We claim that there are no strict limit circuits. Indeed, assume that there is a strict limit circuit $\gamma$. By definition, the circuit $\gamma$...
contains a point in $\Omega(v) \cap \text{intP}(v) = \Omega(v) - \overline{\text{Cl}(v)}$ and so $\overline{\text{Cl}(v)} \neq \Omega(v)$, which contradicts $\overline{\text{Cl}(v)} = \Omega(v)$. □

The converse of Lemma 3.2 is not true in general (see Example 2 in §4.1 for details). We have the following necessary condition for Axiom A(b) with respect to a flow on a surface.

**Proposition 3.3.** Let $v$ be a flow on a compact surface. If $\overline{\text{Cl}(v)} = \Omega(v)$, then there are neither non-closed recurrent points, strict limit circuits, nor circuits with wandering holonomy.

3.2. **Sufficient conditions under the finite existence of singular points.** We show the following existence of a closed transversal which is parallel to a limit circuit.

**Lemma 3.4.** For any limit circuit, there is a closed transversal that is not contractible in its associated collar.

*Proof.* Let $\gamma_0$ be a limit circuit for a flow $v$ on a surface. By time reversion if necessary, we may assume that $\gamma_0$ is semi-attracting. Let $x_0 \notin \gamma_0$ be a point with $\gamma_0 = \omega(x_0)$, $A$ its small associated collar, $y \in \gamma_0$ a non-recurrent point, $I \subset A$ an oriented open transverse arc such that $y$ is a boundary component of $I$, and $f_v : I \to I$ the first return map on $I$ induced by $v$, $x_1 := (f_v)^i(x_0)$ the $i$-th return of $x_0$, $C_{a,b} \subset O^+(x)$ the orbit arc from $a$ to $b$, and $I_{a,b} \subset I$ the subinterval between $a$ and $b$ of $I$. We may assume that $x_0 < x_1$. Suppose that the restriction of $f_v$ to a neighborhood of $x_i$ for some $i \in \mathbb{Z}_{\geq 0}$ is orientation-preserving. Then put $C := C_{x_i,x_{i+1}}$ and $J := I_{x_i,x_{i+1}}$. By the waterfall construction to the loop $\mu := C \cup J$ (see Figure 4), there is a closed transversal $\gamma$ intersecting $O(x)$ near $\mu$ which is parallel to $\partial A$.

We claim that there is a natural number $i$ such that $x_{i+1} < x_i$. Indeed, otherwise $x_i < x_{i+1}$ for any $i \in \mathbb{Z}_{\geq 0}$. Then each pair of loops $\gamma_i := C_{x_{2i},x_{2i+1}} \cup I_{x_{2i+1},x_{2i}}$ has disjoint neighborhoods each of which is a Möbius band. This means that $S$ has infinitely many non-orientable genus, which contradicts the compactness of $S$.

By renumbering, we may assume that $x_2 < x_1$. From $x_0 < x_1$, the first return map for $I_{x_2,x_0}$ along $C_{x_0,x_2}$ is orientation-preserving such that a pair of $C := C_{x_0,x_2}$ and $J := I_{x_2,x_0}$. As above, the waterfall construction to the loop $\mu := C \cup J$ there is a closed transversal $\gamma$ intersecting $O(x)$ near $\mu$ which is parallel to $\partial A$. □
Lemma 3.5. Let \( v \) be a flow with finitely many singular points on a compact surface \( S \), \( x \) non-recurrent point, \( I \) an open transverse arc containing \( x \), and \( f_I \) the first return map on \( I \). Suppose that \( f_I \) is orientation-reversing and \( f_I(I_-) = I_+ \), where \( I_- \), \( I_+ \) are the connected component of \( I - \{x\} \). Denote by \( D \) the union of orbit arcs from points in \( I_- \) to the image of \( f_I \). The connected component of \( \partial D - (I_- \cup I_+) \) containing \( x \) is a circuit.

Proof. By definition, the union \( D \) is an open flow fox as in the left on Figure 5. Let \( \gamma \) be the connected component of \( \partial D - (I_- \cup I_+) \) containing \( x \). Replace a small trivial open flow fox containing a point in \( O^-(x) \) by a one-punctured Möbius band as in Figure 6, the resulting surface is a compact surface and the resulting flow \( w \) is a flow with finitely many singular points such that the first return map on \( I_- \) by \( w \) is repelling. Then \( \gamma \) is the \( \alpha \)-limit set of a point in \( I_- \) near \( x \) with respect to \( w \) and is a boundary component of an open annulus contained in \( w(I_-) \). By the generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points, the \( \alpha \)-limit set \( \gamma \) with respect to \( w \) is a semi-repelling limit circuit with respect to \( w \). Since the replacement preserve \( \gamma \), the subset \( \gamma \) is also a circuit with respect to \( v \).

□
We show that any non-singular points in the boundary of the periodic point set are contained in a circuit under the finite existence of singular points.

**Lemma 3.6.** Let $v$ be a flow with finitely many singular points on a compact surface $S$. Each non-singular point in $\text{Per}(v)$ is contained in a circuit.

**Proof.** Fix a non-singular point $x_0 \in \overline{\text{Per}(v)}$. Then there are a small open transverse arc $I$ whose boundary contains $x_0$, and a sequence $(x_i)_{i \in \mathbb{Z}_{>0}}$ of fixed points of the first return map on $I$ which converges to $x_0$ such that one of the boundary components of the saturation $v(I)$ is the periodic orbit $O(x_1)$. Let $I_{a,b}$ be the open subarc in $I$ from a point $a \in I$ to a point $b \in I$. Denote by $A_i$ the connected component of $S - \bigcup_{j \in \mathbb{Z}_{>0}} O(x_j)$ containing $I_{x_i, x_{i+1}}$. By the finite existence of genus and singular points, taking a subsequence of $(x_i)_{i \in \mathbb{Z}_{>0}}$, we may assume that any $A_i$ is an open invariant annulus intersecting no singular points. Then the union $A := \bigcup_{i \in \mathbb{Z}_{>0}} A_i \cup O(x_{i+1})$ is an open invariant annulus intersecting no singular points. Let $\partial$ be the boundary component of $A$ containing $x$. Then $\partial A = O(x_1) \cap \partial$. By [23] Lemma 2.3 implies that $\overline{\text{Per}(v)} \cap \text{R}(v) = \emptyset$ and so that $\text{R}(v) \cap \partial = \emptyset$. Since $v$ is smooth, using a bump function on a flow box $U$ with $I \subset U \subset A$ and $x_0 \in \partial U$, we can modify the vector field generating the flow $v$ into the resulting vector field. We obtain $v|_{S - \partial} = v|_{S - A}$. Moreover, we may assume that $A \cap \text{Sing}(w) = \emptyset$. From $\text{R}(v) \cap \partial = \emptyset$ and $\partial \subset S - A$, we have $\text{R}(w) \cap \partial = \emptyset$. Therefore $\partial$ is the $\omega$-limit set of the flow $w$ with finitely many singular points. Since a non-singular point $x_0 \notin \text{Sing}(w)$ is contained in $\partial$, by $\text{R}(w) \cap \partial = \emptyset$, the generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points (cf. [24] Theorem 2.6.1)) implies that the $\omega$-limit set $\partial$ is a limit circuit. By construction, the non-singular point $x_0$ is contained in the circuit $\partial$ with respect to $v$. \hfill \Box

We have the following classification of a non-recurrent point under the non-existence of non-closed recurrent orbits.

**Lemma 3.7.** Let $v$ be a flow with finitely many singular points on a compact surface $S$. Suppose that there are no non-closed recurrent points. Then one of the following statements holds exclusively for a non-recurrent point $x$:

1. The point $x$ is contained in a strict limit circuit and $x \in \Omega(v)$.
2. The point $x$ is contained in a circuit with wandering holonomy and $x \in \Omega(v)$.
3. There is a circuit $\gamma \in \overline{\text{Per}(v)} \subseteq \Omega(v)$ containing $x$.
4. The point $x$ is wandering (i.e. $x \notin \Omega(v)$).

**Proof.** By Gutierrez’s smoothing theorem [15], we may assume that $v$ is smooth. Since $\text{Per}(v) \subseteq \Omega(v)$, the closedness of $\Omega(v)$ implies that $\overline{\text{Per}(v)} \subseteq \Omega(v)$. We may assume that there is no circuit $\gamma \in \overline{\text{Per}(v)}$ containing $x$. The closedness of $\text{Sing}(w)$ implies that there is no circuit $\gamma \in \overline{\text{Cl}(v)}$ containing $x$. Lemma [33] implies $x \notin \overline{\text{Per}(v)}$. If $x$ is contained in a circuit with wandering holonomy, then $x \in \Omega(v)$. Thus we may assume that $x$ is not contained in a circuit with wandering holonomy.

We claim that a limit circuit containing $x$ is a strict limit circuit. Indeed, suppose that $x$ is contained in a limit circuit $\gamma$. Then $x \in \Omega(v)$ and $\gamma \setminus \overline{\text{Cl}(v)} \neq \emptyset$. If $\gamma$ is one-sided, then it is a strict limit circuit. Thus we may assume that $\gamma$ is two-sided. Let $A_+$ be a small associated collar of the limit circuit $\gamma$ such that $\gamma$ is either
semi-attracting or semi-repelling with respect to $A_+$. Since $\gamma \setminus \text{Cl}(v) \neq \emptyset$, there is a small open transverse arc $I$ containing a non-recurrent point $x_0 \in \gamma \setminus \text{Cl}(v)$. Denote by $I_-$ the connected component of $I - \{x_0\}$ which does not intersect $A_+$. By $\gamma \setminus \text{Fer}(v) \neq \emptyset$, the first return map on $I_-$ has no fixed points, and so the circuit $\gamma$ is a strict limit circuit.

Thus we may assume that $x$ is contained in neither a limit circuit nor a circuit with wandering holonomy.

Assume that $x$ is non-wandering. Let $d$ be a Riemannian distance on $S$ and $T$ a small open transverse arc intersecting $x$ whose closure contains no singular points. Denote by $T_-$ and $T_+$ the connected components of $T - \{x\}$. Since $x \notin \text{Per}(v)$ is non-wandering, there is a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ of non-periodic points in $T$ with $O^+(a_n) \cap T \neq \emptyset$ and $O(a_n) \neq O(a_m)$ for any $n \neq m \in \mathbb{Z}_{>0}$ such that the sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ converges to $x$, $d(x, a_n) < 1/n$ and $d(x, b_n) < 1/n$ for any $n \in \mathbb{Z}_{>0}$, where $b_n$ is the first return of $a_n$ for the first return map $f_T$ on the open transverse arc $T$. Renaming $T_+$ and $T_-$ and taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that $a_n \in T_+$ for any $n \in \mathbb{Z}_{>0}$ and that the sequences $(d(x, a_n))_{n \in \mathbb{Z}_{>0}}$ and $(d(x, b_n))_{n \in \mathbb{Z}_{>0}}$ of distances monotonically decrease.

We claim that there are at most finitely many $\omega$-limit and $\alpha$-limit sets such that each of $\omega(a_n)$ and $\alpha(a_n)$ is one of them. Indeed, assume that there are pairwise distinct infinitely many $\omega$-limit sets of any points $a_n$. By the generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points, the $\omega$-limit and $\alpha$-limit sets of any points $a_n$ are singular points or limit circuits. By the finite existence of $\text{Sing}(v)$, there are pairwise disjoint infinitely many limit circuits $C_n$. Taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$ and by time reversion if necessary, we may assume that $C_n = \omega(a_n)$. Since any collars for semi-attracting limit circuits $C_n$ are pairwise disjoint, the small associated collars are pairwise disjoint. By Lemma 3.4 there are infinitely many closed transversals $\gamma_n$ in the pairwise disjoint associated collars which are not contractible in their associated collars respectively. By the finite existence of $\text{Sing}(v)$, taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$, any closed transversals $\gamma_n$ are not contractible in $S$, because any contractible closed transversal bounds a disk containing a singular point by the Poincaré-Hopf theorem. Then any connected components of the complement $S - \bigcup_{i} \gamma_i$ have at most finitely many boundary components. Denote by $U_x$ the connected component of the complement $S - \bigcup_{i} \gamma_i$ containing $x$. Then $U_x$ is invariant. Let $\mu_1, \ldots, \mu_k$ be the boundary components of $U_x$ which are contained in $\bigcup_{i} \gamma_i$. Replacing $T$ with a small open subarc, we may assume that $T \cap \bigcup_{j=1}^{k} \mu_j = \emptyset$. Taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that $\gamma_i = \mu_i$ for any $j \in \{1, 2, \ldots, k\}$. Then $U_x \cap \bigcup_{n > k} \gamma_n = \emptyset$ and $\overline{U_x} = U_x \cup \bigcup_{j=1}^{k} \gamma_j$. Fix any $n > k$. Since $\gamma_n \cap O^+(a_n) \neq \emptyset$, we obtain that $O^+(a_n) \setminus U_x \neq \emptyset$. The invariance of $U_x$ implies that $a_n \notin U_x$, which contradicts $a_n \in T \subset U_x$. By symmetry of the time reversion, the claim is completed.

Taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that there are closed invariant subsets $\alpha$ and $\omega$ in $S - \overline{T}$ such that $\alpha = \alpha(a_n)$ and $\omega = \omega(a_n)$ for any $n \in \mathbb{Z}_{>0}$. Cutting limit circuits and collapsing the new boundary components into singletons if necessary, we may assume that each of $\alpha$ and $\omega$ is a singular point. Taking a subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that either $\{b_n \mid n \in \mathbb{Z}_{>0}\} \subset T_+$ or $\{b_n \mid n \in \mathbb{Z}_{>0}\} \subset T_-$. 

DENSITY OF CLOSED ORBITS IN THE NON-WANDERING SET 11
We claim that any \( O(a_n) \) intersects \( T \) at most finitely many times. Indeed, assume that \( O(a_n) \) intersects \( T \) infinitely many times. Let \( a_{n,k} \) be a sequence of pairwise distinct points in \( O(a_n) \cap T \) converging to \( a_\infty \in \mathcal{T} \). Since \( \mathcal{T} \cap \text{Sing}(v) = \emptyset \), the point \( a_\infty \) is non-singular. From \( \overline{O(a_n)} = O(a_n) \cup \{ \alpha, \omega \} \) and \( \alpha, \omega \in \text{Sing}(v) \), we obtain \( a_\infty \in \overline{O(a_n)} \setminus \text{Sing}(v) = O(a_n) \). Since \( a_\infty \in O(a_n) \) is an accumulation point of \( O(a_n) \cap T \), the non-closed point \( a_\infty \) is recurrent, which contradicts the non-existence of non-closed recurrent points.

Taking a subsequence of \( (a_n)_{n \in \mathbb{Z}_{>0}} \), we may assume \( O^\pm(a_n) \cap T_+ = \emptyset \) for any \( n \in \mathbb{Z}_{>0} \). Let \( f_T \) the first return map on \( T \) induced by \( v \) and \( T_{a,b} \subset T \) the subinterval between \( a \) and \( b \) of \( T \). Put \( \gamma_i := \{ \alpha \} \cup O^-(a_i) \cup O^-(a_{i+1}) \cup T_{a_i,a_{i+1}} \) for any \( i \in \mathbb{Z}_{>0} \). Then any \( \gamma_i \) is a loop. Denote by \( D_i \) the connected component of \( S - \bigcup_{i \in \mathbb{Z}_{>0}} \gamma_i \) such that \( O^+(a) \subset D_i \) for any \( a \in T_{a_i,a_{i+1}} \setminus \{ a_i, a_{i+1} \} \). By the finite existence of genus and singular points, taking a subsequence of \( (a_n)_{n \in \mathbb{Z}_{>0}} \), we may assume that any \( D_i \) is an invariant open disk intersecting no singular points. Then any \( D_i \) intersects no periodic points. Since \( O^-(a_n) \cap T_+ = \emptyset \) for any \( n \in \mathbb{Z}_{>0} \), we have \( T \cap \bigcup_{n \in \mathbb{Z}_{>0}} D_n = \emptyset \).

We claim that the \( \alpha \)-limit sets of any points in the negative invariant open disk \( D_i \) are \( \alpha \). Indeed, since \( D_i \) and the closure \( \overline{D_i} - \{ \alpha \} \) consist of non-recurrent points, the generalization of the Poincaré-Bendixon theorem for a flow with finitely many singular points implies that \( \alpha \)-limit sets of any points in \( D_i \) are singular points or limit circuits containing \( \alpha \). Assume that there is an \( \alpha \)-limit set \( \gamma_i \) of a point \( d_i \in D_i \) which is a limit circuit. Then \( \alpha \in \gamma_i \) and there is the associated collar \( \mathcal{A}_i \subset D_i \) of \( \gamma_i \) in \( D_i \). Lemma 3.4 implies that there is a closed transverse \( \mu_i \subset \mathcal{A}_i \subset D_i \). Since \( D_i \) is an open disk, the closed transverse \( \mu_i \subset \mathcal{A}_i \) is null homotopic in \( D_i \) and so bounds a negative invariant open disk \( B_i \subset D_i \). The Poincaré-Hopf theorem implies that \( B_i \) contains singular points, which contradicts the non-existence of singular points on \( D_i \).

By construction, the union \( D := \bigcup_{i \in \mathbb{Z}_{>0}} D_i \cup O^-(a_{i+1}) \) is a negative invariant open disk with \( T \cap \overline{D} = \emptyset \) such that \( O^-(x) \subset \partial D \) and \( \alpha = \alpha(a) \) for any \( a \in \overline{D} \). Then the saturation \( \overline{D} := v(D) \) is an invariant open box with \( O(x) \subset \partial \overline{D} \) and \( \alpha = \alpha(a) \) for any \( a \in \overline{D} \). Replacing \( T \) with the open subarc \( T_{b_n,b_0} \), we may assume that \( \text{dom}(f_T) = T_+ \subset \partial \overline{D} \). Since \( D \cap T = \emptyset \) and \( \{ \alpha \} = \bigcup_{a \in D} \alpha(a) \), we obtain \( \text{dom}(f_T) \cap \text{dom}(f_T) = T_+ \cap \text{codom}(f_T) = \emptyset \). Then \( \{ b_n \mid n \in \mathbb{Z}_{>0} \} \subset T_- \) and so the sequence \( (b_n)_{n \in \mathbb{Z}_{>0}} \) converges to \( x \). Since \( b_n \) is the first return of \( a_n \) by the first return map \( f_T \) on \( T \), the map \( f_T \) is orientation-reversing. By construction, Lemma 3.5 implies that there is a circuit \( \nu \) in \( \partial \overline{D} \) with wandering holonomy such that \( x \in \nu \), which contradicts that \( x \) is not contained in a circuit with wandering holonomy. Thus \( x \) is wandering.

We have the following sufficient condition for Axiom \( A(b) \) with respect to a flow with finitely many singular points on a surface.

**Lemma 3.8.** Let \( v \) be a flow with finitely many singular points on a compact surface \( S \) without non-closed recurrent points, strict limit circuits, nor circuits with wandering holonomy. Then \( \overline{\text{Cl}(v)} = \Omega(v) \).

**Proof.** The non-existence of non-closed recurrent points implies that \( S = \overline{\text{Cl}(v)} \cup \overline{\text{P}(v)} \). Fix a point \( x \in \overline{\text{P}(v)} \). From Lemma 3.4 either \( x \notin \Omega(v) \) or there is a circuit \( \gamma \in \overline{\text{Per}(v)} \subset \Omega(v) \) containing \( x \). Thus either \( x \notin \Omega(v) \) or \( x \in \overline{\text{Per}(v)} \subset \Omega(v) \).
Therefore $P(v) = (P(v) \setminus \Omega(v)) \cup (P(v) \cap \overline{\text{Per}(v)})$ and so $P(v) \cap \Omega(v) = P(v) \cap \overline{\text{Per}(v)}$. Since $S - P(v) = \text{Cl}(v) \subseteq \Omega(v)$, we obtain $\Omega(v) = S \cap \Omega(v) = (\text{Cl}(v) \cup P(v)) \cap \Omega(v) = (\text{Cl}(v) \cap \Omega(v)) \cup (P(v) \cap \Omega(v)) = \text{Cl}(v) \cup (P(v) \cap \text{Per}(v)) \subseteq \text{Cl}(v) \subseteq \Omega(v)$. This means that $\text{Cl}(v) = \Omega(v)$.

Proposition 3.8 and Lemma 3.8 imply the following equivalence.

**Lemma 3.9.** The following conditions are equivalent for a flow $v$ with finitely many singular points on a compact surface:

1. $\text{Cl}(v) = \Omega(v)$.
2. There are neither non-closed recurrent points, strict limit circuits, nor circuits with wandering holonomy.

### 3.3. Topological characterization of finiteness of the non-wandering set.

We state the following statement.

**Lemma 3.10.** Let $v$ be a flow with finitely many singular points on a compact surface. If each recurrent orbit is a closed orbit, then any circuits with wandering holonomy contain at most finitely many non-wandering orbits.

**Proof.** Let $v$ be a flow with finitely many singular points on a compact surface with $R(v) = \emptyset$. Since any point whose $\omega$-limit or $\alpha$-limit set is a limit circuit is wandering, the generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points implies that the $\omega$-limit and $\alpha$-limit sets of any non-recurrent non-wandering points are singular points. Assume that there is a circuit $\gamma$ with wandering holonomy contains infinitely many non-wandering orbits $O_n \ (n \in \mathbb{Z}_{>0})$. Then the $\omega$-limit and $\alpha$-limit sets of $O_n$ are singular points for any $n \in \mathbb{Z}_{>0}$. By the finite existence of genus singular points, there is a singular point $\alpha$ (resp. $\omega$) which is the $\omega$-limit (resp. $\alpha$-limit) set of $O_n$ for infinitely many $n \in \mathbb{Z}_{>0}$. Taking a subsequence of $(O_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that $\omega(O_n) = \omega$ and $\alpha(O_n) = \alpha$ for any $n \in \mathbb{Z}_{>0}$. Denote by $D_n$ the connected component of $S - \{\alpha, \omega\} \cup \bigcup_{n \in \mathbb{Z}_{>0}} O_n$ with $\partial D_n = \{\alpha, \omega\} \cup O_n \cup O_{n+1}$ for any $n \in \mathbb{Z}_{>0}$. By the finite existence of genus and singular points, taking a subsequence of $(O_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that any $D_n$ is an invariant open disk. Then any $D_i$ intersects no periodic points and so consists of non-recurrent points. This implies that any $O_{n+1}$ for any $n \in \mathbb{Z}_{>0}$ has no non-orientable holonomy and so is not contained in a circuit with wandering holonomy, which contradicts the definition of $O_{n+1}$. \hfill \Box

#### 3.3.1. Proof of Theorem 3.4

In any case, there are at most finitely many closed orbits. Obviously, the assertion (2) implies the assertion (3). By the compactness of $S$, since a circuit with wandering holonomy has a small neighborhood which is a Möbius band, there are at most finitely many circuits with wandering holonomy.

Suppose that $\Omega(v)$ consists of finitely many orbits. By Lemma 3.1, since $R(v) \subseteq \Omega(v)$, the finiteness of $\Omega(v)$ implies that $R(v) = \emptyset$. Lemma 3.7 implies that any non-recurrent point in $\Omega(v)$ is contained in either a strict limit circuit or a circuit with wandering holonomy. This means that $\Omega(v)$ consists closed orbits, a strict limit circuit, and a circuit with wandering holonomy and that any limit circuits are strict limit circuits. Therefore the assertion (2) holds.

Suppose that $R(v) = \emptyset$ and there are at most finitely many closed orbits and strict limit circuits. Lemma 3.7 implies that any non-recurrent point in $\Omega(v)$ is contained in either a limit circuit or a circuit with wandering holonomy. This implies that $\Omega(v)$
Figure 7. Canonical quotient mappings induced by the metric completion and the collapse.

consists closed orbits, a strict limit circuit, and a circuit with wandering holonomy, and that any limit circuits are strict limit circuits. The finite existence of closed orbits, strict limit circuits, and circuits with wandering holonomy implies that $\Omega(v)$ consists of finitely many orbits. This complete the proof of Theorem C.

3.4. Topological characterization for the case with finitely many connected components of the singular point set. To generalize Lemma 3.9 into a characterization of Axiom A(b) for a flow with finitely many connected components of the singular point set, we introduce some concepts.

3.4.1. Blow-downs for singular points. We define blow-downs for the singular point set of a surface and their flows as follows (see Figure 7): Let $v$ be a flow on a surface $S$ whose singular point set has at most finitely many connected components. Since the singular point set $\text{Sing}(v)$ is closed, the complement $S - \text{Sing}(v)$ is open and so is a surface. In particular, the set difference $S - (\text{Sing}(v) \cup \partial S)$ is an open surface without boundary. Fix a Riemannian metric on $S$ such that $\text{Sing}(v)$ is bounded. Denote by $S_{\text{mc}}$ the metric completion of the complement $S - \text{Sing}(v)$.

Identifying the union $\partial$ of new boundary components with the new singular points, define a flow $v_{\text{mc}}$ on $S_{\text{mc}}$ such that $O(v)(x) = O(v_{\text{mc}})(x)$ for any point $x \in S - \text{Sing}(v) = S_{\text{mc}} - \text{Sing}(v_{\text{mc}})$ up to topological equivalence. Then $\partial = \text{Sing}(v_{\text{mc}})$ and so $S - \text{Sing}(v) = S_{\text{mc}} - \text{Sing}(v_{\text{mc}})$. By [28, Theorem 3], each connected component of the open surface $S - (\text{Sing}(v) \cup \partial S)$ without boundary is homeomorphic to the resulting surface from a closed surface by removing a closed totally disconnected subset. Therefore, collapsing each connected component of $\text{Sing}(v_{\text{mc}})$ into a singular point, we obtain the resulting flow $v_{\text{col}}$ with totally disconnected singular points, called the blow-down flow of $v$, on the resulting surface $S_{\text{col}}$, called the blow-down surface, up to topological equivalence. Then $O(v)(x) = O_{\text{mc}}(x) = O_{\text{col}}(x)$ for any point $x \in S - \text{Sing}(v) = S_{\text{mc}} - \text{Sing}(v_{\text{mc}}) = S_{\text{col}} - \text{Sing}(v_{\text{col}})$. Notice that $S_{\text{mc}}$ and $S_{\text{col}}$ may have infinitely many connected components but that each connected component of $S_{\text{col}}$ is a compact surface.

3.4.2. Concepts related to blow-downs. We recall some concepts to state the characterization of the density of closed orbits in the non-wandering set. An invariant subset $\gamma$ is a blow-up of a circuit with wandering holonomy if the image $q(\pi_{\text{mc}}^{-1}(\gamma))$ is a circuit with wandering holonomy for $v_{\text{col}}$. A closed connected invariant subset is a non-trivial quasi-circuit if it is a boundary component of an open annulus, contains a non-recurrent point, and consists of non-recurrent points and singular points. A non-trivial quasi-circuit $\gamma$ is a quasi-semi-attracting (resp. quasi-semi-repelling) limit quasi-circuit (with respect to a small collar $A$) if there is a point $x \in A$ with $O^+(x) \subset A$ (resp. $O^-(x) \subset A$) such that $\omega(x) = \gamma$ (resp. $\alpha(x) = \gamma$). A non-trivial quasi-circuit is a limit quasi-circuit (with respect to a small collar $A$) if
it is a quasi-semi-attracting/quasi-semi-repelling limit quasi-circuit with respect to \( A \). By construction, we have the following observation.

**Lemma 3.11.** The following statements hold for a flow \( v \) with finitely many connected components of the singular point set on a compact surface \( S \):

1. The resulting flow \( v_{\text{col}} \) is a flow with weakly sectored singular points on \( S_{\text{col}} \).
2. \( \text{Cl}(v) = \Omega(v) \) if and only if \( \text{Cl}(v_{\text{col}}) = \Omega(v_{\text{col}}) \).
3. \( R(v) = \emptyset \) if and only if \( R(v_{\text{col}}) = \emptyset \).
4. The \( \omega \)-limit set \( \omega_v(x) \) of a non-singular point \( x \) on \( S \) is a limit quasi-circuit if and only if the \( \omega \)-limit set \( \omega_{v_{\text{col}}}(x) \) is a limit circuit.
5. There are no strict limit quasi-circuits for \( v \) if and only if there are no strict limit circuits for \( v_{\text{col}} \).
6. There are no blow-ups of circuits with wandering holonomy for \( v \) if and only if there are circuits with wandering holonomy for \( v_{\text{col}} \).

**Lemma 3.9** and **Lemma 3.11** imply the following statement.

**Theorem 3.12.** Let \( v \) be a flow with finitely many connected components of the singular point set on a compact surface \( S \). The following conditions are equivalent:

1. \( \text{Cl}(v) = \Omega(v) \).
2. There are neither non-closed recurrent points, strict limit quasi-circuits, nor blow-ups of circuits with wandering holonomy.
3. Each orbit is proper, and there are neither strict limit quasi-circuits nor blow-ups of circuits with wandering holonomy.
4. The orbit space \( S/v \) is \( T_0 \), and there are neither strict limit quasi-circuits nor blow-ups of circuits with wandering holonomy.

The previous theorem implies Theorem A. Note the finiteness of connected components of the singular point set is necessary. In other words, \( \text{Cl}(v) \not\subseteq \Omega(v) \) for a flow \( v \) on a compact surface in general. Indeed, there is a flow \( v \) on a compact surface \( S \) with \( R(v) = \emptyset \) and \( \text{Cl}(v) \not\subseteq \Omega(v) \) but without strict limit circuits nor circuits with wandering holonomy (e.g. Example 4.1).

### 3.5. Topological characterization of correspondence between the periodic point set and the non-wandering set

**Proposition 3.13.** Let \( v \) be a flow on a compact surface. If \( \text{Cl}(v) = \Omega(v) \), then \( \text{Cl}(v) \) is closed, and there are neither non-closed recurrent points, non-periodic limit circuits, nor circuits with wandering holonomy.

**Proof.** Suppose that \( \text{Cl}(v) = \Omega(v) \). Since \( \Omega(v) \) is closed, the closed point set \( \text{Cl}(v) \) is closed with \( \text{Cl}(v) = \text{Cl}(v) = \Omega(v) \). **Proposition 3.3** implies that there are neither non-closed recurrent points, strict limit circuits, nor circuits with wandering holonomy. Since any points in \( \omega \)-limit and \( \alpha \)-limit sets of any points are non-wandering, there are no non-periodic limit circuits. \( \square \)

**Lemma 3.14.** Let \( v \) be a flow with finitely many connected components of the singular point set on a compact surface \( S \). Suppose that \( \text{Cl}(v) \) is closed and there are neither non-closed recurrent points, non-periodic limit circuits, nor circuits with wandering holonomy. Then \( \text{Cl}(v) = \Omega(v) \).

**Proof.** **Theorem 3.12** implies \( \text{Cl}(v) = \Omega(v) \). Since \( \text{Cl}(v) \) is closed, we obtain \( \text{Cl}(v) = \text{Cl}(v) = \Omega(v) \). \( \square \)
Since $S - \text{Cl}(v) = P(v)$ for any flow $v$ without non-closed recurrent orbits on a surface, Proposition 3.13 and Lemma 3.14 implies Corollary 13.

3.6. Density of closed orbits in the non-wandering set and correspondence for flows on non-compact surfaces. To describe flows on non-compact surfaces, we introduce end completions of surfaces with flows.

3.6.1. End completions of surfaces with finite genus. Recall the end completion as follows. Consider the direct system $\{K_\lambda\}$ of compact subsets of a topological space $X$ and inclusion maps such that the interiors of $K_\lambda$ cover $X$. There is a corresponding inverse system $\{\pi_0(X - K_\lambda)\}$, where $\pi_0(Y)$ denotes the set of connected components of a space $Y$. Then the set of ends of $X$ is defined to be the inverse limit of this inverse system. Notice that Lemma 3.15.

We have the following observation.

3.6.2. Virtually strict limit quasi-circuits and virtually quasi-circuits. An invariant subset is a virtually limit (resp. strict limit) quasi-circuit if it is the resulting subset from a limit (resp. strict limit) quasi-circuit on $S_{\text{end}}$ with respect to $v_{\text{end}}$ by removing all the ends. An invariant subset $\mu$ is a virtual blow-up of a circuit with wandering holonomy if there is a blow-up of a circuit $\gamma$ with wandering holonomy on $S_{\text{end}}$ with respect to $v_{\text{end}}$ such that the resulting subset from $\gamma$ by removing all the ends is $\mu$.

Theorem 3.12 and Lemma 3.15 imply the following characterization of Axiom A(b) for a flow with finitely many connected components of the singular point set on a surface $S$ with finite genus and finite ends.
Theorem 3.16. The following conditions are equivalent for a flow $v$ with finitely many connected components of the singular point set on a surface $S$ with finite genus and finite ends:

1. $\text{Cl}(v) = \Omega(v)$.
2. There are neither non-closed recurrent points, virtually strict limit quasi-circuits, nor virtual blow-up of circuits with wandering holonomy.
3. Every orbit is proper, and there are neither virtually strict limit quasi-circuits nor virtual blow-up of circuits with wandering holonomy.
4. The orbit space $S/v$ is $T_0$, and there are neither virtually strict limit quasi-circuits nor virtual blow-up of circuits with wandering holonomy.

Corollary 3.15 and Theorem 3.16 imply the following correspondence.

Theorem 3.17. The following conditions are equivalent for a flow $v$ with finitely many connected components of the singular point set on a surface $S$ with finite genus and finite ends:

1. $\text{Cl}(v) = \Omega(v)$.
2. The closed point set $\text{Cl}(v)$ is closed and there are neither non-closed recurrent points, non-periodic virtually limit quasi-circuits, nor virtual blow-up of circuits with wandering holonomy.
3. The closed point set $\text{Cl}(v)$ is closed, every orbit is proper, and there are neither non-periodic virtually limit quasi-circuits nor virtual blow-up of circuits with wandering holonomy.
4. The closed point set $\text{Cl}(v)$ is closed and the orbit space $S/v$ is $T_0$ and there are neither non-periodic virtually limit quasi-circuits nor virtual blow-up of circuits with wandering holonomy.

In any case, the set $P(v)$ of non-recurrent points are open.

4. Examples and observations on the density for gradient flows and non-wandering flows and examples

We describe concrete examples and (necessary and) sufficient conditions for Axiom A(b) with respect to some kind of classes of flows.

4.1. Examples. We describe some examples of flows to show necessity conditions in results. The following two examples imply the necessity of the finite existence of connected components of the singular point set.

Example 1. There is a toral flow $v$ with uncountable singular points satisfying the following three conditions:

1. $R(v) = \emptyset$ and there are no circuits.
2. $\text{Cl}(v) \neq \Omega(v)$.

Proof. Consider the suspension flow $v$ of a Denjoy homeomorphism $f: S^1 \to S^1$ with a minimal set $C \subset S^1$, which is a Cantor set, on the mapping torus $S := (S^1 \times [0, 1])/(x, 0) \sim (f(x), 1)$. Then the minimal set $M$ satisfies that $M \cap (S^1 \times 1/2) = C \times \{1/2\} \subset S$. Moreover, we obtain $\Omega(v) = M$. Using the bump function $\varphi$ with $\varphi^{-1}(0) = C \times \{1/2\}$, replace the orbits in $M$ with a union of singular points and multi-saddle separatrices of the resulting flow $v_\varphi$ (i.e. $M = \text{Sing}(v_\varphi) \cup \{\text{separatrix of } v_\varphi\}$) such that $O_{v_\varphi}(y) = O_v(y)$ for any point $y \in S - M$. Then $S = \text{Sing}(v_\varphi) \cup P(v_\varphi)$ and so $\overline{\text{Cl}(v)} = \text{Sing}(v) = C \times \{1/2\} \subset M = \Omega(v_\varphi)$. □
Example 2. There is a flow $v$ on a closed surface $S$ with countable singular points satisfying the following three conditions:

1) $R(v) = \emptyset$ and there are neither strict limit circuits nor circuits with wandering holonomy.

2) $\text{Cl}(v) = \text{Cl}(v) = \text{Sing}(v) \neq \Omega(v)$.

3) The flow $v$ has a non-periodic non-limit circuit in $\Omega(v)$.

Proof. Consider a toral flow $w$ which consists of one non-contractible limit cycle $C$ and non-closed proper orbits. Fix a point $z \in C$ and a non-closed proper orbit $O$. Write an open trivial flow box $D:= \mathbb{T}^2 - (C \cup O)$. Choose a closed transversal $T$ through $z$, a point $x \in O$, and a monotonic sequence $(t_n)_{n \in \mathbb{Z}}$ on $O$ with $T \cap O = \{x_n\}_{n \in \mathbb{Z}}$, $\lim_{n \to \infty} t_n = \infty$, $\lim_{n \to -\infty} t_n = -\infty$, and $\lim_{n \to \infty} x_n = \lim_{n \to -\infty} x_n = z$ such that any connected component of the set difference $D \setminus T$ are open trivial flow boxes, where $x_n := w_{t_n}(x)$. Write $D_n$ the open trivial flow box with four corners $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ such that $D \setminus T = \bigcup_{n \in \mathbb{Z}} D_n$. Write an open trivial flow box $D'_{2n} = D_{2n} \cup D_{2n+1} \cup ((T \setminus O) \cap \overline{D_{2n} \setminus D_{2n+1}}) \subset D$. Replacing the closure of each $D'_{2n}$ by a box with a flow as shown in Figure 8 we obtain the resulting flow $v$ on the torus $\mathbb{T}^2$ such that the singular point set $\text{Sing}(v) = \{z\} \cup \{x_n, y_n\}_{n \in \mathbb{Z}} \subset T$ is countable, $\mathbb{T}^2 = \text{Sing}(v) \cup P(v)$, and $\Omega(v) = C \cup \{x_n, y_n\}_{n \in \mathbb{Z}} \supset \text{Sing}(v) = \overline{\text{Cl}(v)}$. □

The following example implies the necessity of the non-existence of circuits with wandering holonomy in the results.

Example 3. There is a flow $w$ without non-degenerate singular points on a non-orientable closed surface $S$ with non-orientable genus four satisfying the following three conditions:

1) $\overline{\text{Cl}(w)} = \text{Cl}(w) = \text{Sing}(w) = \bigcup_{x \in S} \omega(x) \cup \alpha(x) \subset \Omega(w)$.

2) $\text{Per}(w) \cup R(w) = \emptyset$.

3) There are no strict limit quasi-circuits but a circuit with wandering holonomy.

Proof. Consider a flow $v_0$ on a non-orientable compact surface $S_0$ as in Figure 9 with $S_0 = \text{Sing}(w_0) \cup P(v)$ and $\bigcup_{x \in S_0} \omega(x) \cup \alpha(x) = \text{Sing}(w_0) \subset \text{Sing}(w_0) \cup O' = \Omega(w_0)$ such that $\Omega(w_0)$ does not contain limit circuits, where $O'$ is a proper orbit. Then the lift of the flow $w_0$ to the double $S$ of $S_0$ is desired. □
Figure 9. A flow on a compact surface with two non-orientable genus and with two boundary components consists of one sink, one source, six ∂-saddles, and non-closed proper orbits.

The non-existence of circuits with wandering holonomy in the results can not be weakened. In fact, the density condition does not inhibit the existence of non-orientable holonomy as follows.

Example 4. There is a flow \( v \) on a Klein bottle satisfying \( \overline{\text{Cl}(v)} = \Omega(v) \) with a circuit with fixed-point-free non-orientable holonomy.

Proof. Consider a flow on a Klein bottle consisting of periodic orbits. Replacing a periodic orbit with non-orientable holonomy with a 0-saddle with a homoclinic separatrices, the resulting flow \( v \) has a circuit with fixed-point-free non-orientable holonomy and consists of one non-closed proper orbits and closed orbits such that \( \mathbb{K} = \Omega(v) = \overline{\text{Cl}(v)} \).

4.2. Density of periodic orbits in the non-wandering set for non-wandering flows on compact surfaces and gradient flows on manifolds. Finally, we observe that Axiom A(b) for non-wandering flows on compact surfaces corresponds to the non-existence of locally dense orbits, and that any Hamiltonian flows on compact surfaces and any gradient flows on manifolds satisfy Axiom A(b).

4.2.1. Density for non-wandering flows on compact surfaces. We have the following characterization of non-wandering flows to characterize the density condition.

Proposition 4.1. The following are equivalent for a flow \( v \) on a compact surface \( S \):

1. The flow \( v \) is non-wandering.
2. \( \text{intP}(v) = \emptyset \).
3. \( \text{intP}(v) \cup \text{E}(v) = \emptyset \) (i.e. \( S = \text{Cl}(v) \cup \partial^{-}\text{P}(v) \cup \text{LD}(v) \)).
4. \( S = \text{Cl}(v) \cup \text{LD}(v) \).

Proof. The assertion (3) implies the assertion (2). Recall that \( S = \text{Cl}(v) \cup \text{P}(v) \cup \text{R}(v) \) and that the union \( \text{P}(v) \) is the set of non-recurrent points. Then the assertions (3) and (4) are equivalent. By [33, Lemma 2.3], the union \( \text{P}(v) \cup \text{E}(v) \) is a neighborhood of \( \text{E}(v) \). By the Maier theorem [22,21], the closure \( \overline{\text{E}(v)} \) is a finite union of closures of exceptional orbits and so is nowhere dense. This means that \( \text{E}(v) = \emptyset \) if \( \text{intP}(v) = \emptyset \). Therefore the assertion (2) implies the assertion (3).
Suppose that $v$ is non-wandering. By [6, Theorem III.2.12 and Theorem III.2.15], the set of recurrent points is dense in $S$. The density of recurrent points implies that $\text{intP}(v) = \emptyset$ and so $E(v) = \emptyset$. This implies that $S = \overline{\text{Cl}(v) \cup \text{LD}(v)}$.

Conversely, suppose that $\text{intP}(v) \cup E(v) = \emptyset$. Then $S = \overline{\text{Cl}(v) \cup \partial^- P(v) \cup \text{LD}(v)}$. Therefore the closure of the set of recurrent points is the whole surface (i.e. $S = \overline{\text{Cl}(v) \cup \text{LD}(v)}$) and so $v$ is non-wandering.

We have the following equivalence.

**Lemma 4.2.** The following are equivalent for a flow $v$ on a compact surface $S$:
1. The flow $v$ is non-wandering and $\overline{\text{Cl}(v)} = \Omega(v)$.
2. $\overline{\text{Cl}(v)} = S$.
3. $\text{intP}(v) = \emptyset$ and $\overline{\text{Cl}(v)} = \Omega(v)$.
4. $\text{LD}(v) \cup \text{intP}(v) = \emptyset$ (i.e. $S = \overline{\text{Cl}(v) \cup \partial^- P(v)}$).
5. The flow $v$ is non-wandering and $\text{LD}(v) = \emptyset$.

**Proof.** Recall that $S = \overline{\text{Cl}(v) \cup P(v) \cup R(v)}$. Obviously, the assertions (1) and (2) are equivalent. We show that the assertion (2) implies the assertion (3). Indeed, if $\overline{\text{Cl}(v)} = S$, then $\text{intP}(v) = \emptyset$ and $\Omega(v) \subseteq S = \overline{\text{Cl}(v)} \subseteq \Omega(v)$ because of definition of $\Omega(v)$.

We show that the assertion (3) implies the assertion (4). Indeed, if $\text{intP}(v) = \emptyset$ and $\overline{\text{Cl}(v)} = \Omega(v)$, then $\text{LD}(v) \subseteq \Omega(v) - \overline{\text{Cl}(v)} = \emptyset$.

We show that the assertion (4) is equivalent to the assertion (2). Indeed, suppose $\text{LD}(v) \cup \text{intP}(v) = \emptyset$. By [33, Lemma 2.3], we have $E(v) = \emptyset$ and so $S = \overline{\text{Cl}(v) \cup \partial^- P(v)} = \overline{\text{Cl}(v)}$.

Notice that the assertions (1)–(4) imply the assertion (5). Proposition 4.1 implies the assertion (5) implies the assertion (4).

In the non-wandering case, Axiom A(b) is characterized as follows.

**Proposition 4.3.** The following are equivalent for a non-wandering flow $v$ on a compact surface:
1. $\overline{\text{Cl}(v)} = \Omega(v)$.
2. $\text{LD}(v) = \emptyset$.

This implies the following observation.

**Corollary 4.4.** Each Hamiltonian flow on a compact surface satisfies Axiom A(b).

**Proof.** Let $v$ be a Hamiltonian flow on a compact surface. Since any Hamiltonian flow on a compact surface is an area-preserving and so has no wandering domain, the flow $v$ is non-wandering. The existence of the Hamiltonian of $v$ on the surface implies the non-existence of non-closed recurrent orbits. Proposition 4.3 implies the assertion.

4.2.2. Density for gradient flows on manifolds. We have the following observation.

**Lemma 4.5.** Each gradient flow on a manifold satisfies Axiom A(b).

**Proof.** Let $w$ be a gradient flow on a manifold $M$. The existence of the height function of $w$ implies that any non-singular points are non-recurrent and so wandering. The closedness of $\text{Sing}(w)$ implies $\overline{\text{Cl}(w)} = \text{Sing}(w) = \Omega(w)$.
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Applied Mathematics and Physics Division, Gifu University, Yanagido 1-1, Gifu, 501-1193, Japan.

Email address: tomoo@gifu-u.ac.jp