Occupation measures arising in finite stochastic games

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Abstract

Shapley [5] introduced two-player zero-sum discounted stochastic games, henceforth stochastic games, a model where a state variable follows a two-controlled Markov chain, the players receive rewards at each stage which add up to 0, and each maximizes the normalized $\lambda$-discounted sum of stage rewards, for some fixed discount rate $\lambda \in (0,1]$. In this paper, we study asymptotic occupation measures arising in these games, as the discount rate goes to 0.

1 Introduction

Let $\Omega$ be a finite set of states and let $Q$ be a stochastic matrix over $\Omega$. A classical result is the existence of the weak ergodic limit $\Pi := \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} Q^m$. The sensitivity of the ergodic limit to small perturbations of $Q$ goes back to [4]. The simplest case is that of a linear perturbation of $Q$, $Q_\varepsilon := \frac{1}{1+\varepsilon} (Q + \varepsilon P)$ ($\varepsilon \geq 0$), where $P$ is another stochastic matrix over $\Omega$. A perturbation is said to be regular if the recurrence classes remain constant in a neighbourhood of 0. When the perturbation is not regular, the ergodic limit of $Q_\varepsilon$ may fail to converge, as $\varepsilon$ tends to 0, to that of $Q = Q_0$.

Example 1. Let $\Omega = \{1,2\}$, $Q = \text{Id}$, and $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so that $Q_\varepsilon = \frac{1}{1+\varepsilon} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $\varepsilon \geq 0$. Then $\Pi_\varepsilon = \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} Q_\varepsilon^m = \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$ for all $\varepsilon > 0$, whereas $\Pi_0 = \text{Id}$.

The study of regular and nonregular perturbations has been widely treated in the literature. The aim of this paper is to study a Markov chain perturbation problem arising in the asymptotic study of two-person zero-sum stochastic games. An important aspect in this model (see Section 1.1) is the discount rate $\lambda > 0$ which models the impatience of the players. As a consequence, we will consider from now on the Abel mean $\sum_{m\geq 0} \lambda(1-\lambda)^m Q^m$ instead of the Cesaro mean. Notice that, for any fixed stochastic matrix $Q$, Hardy-Littlewood’s Tauberian Theorem [2] gives the equality:

$$\Pi = \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} Q^m = \lim_{\lambda \to 0} \sum_{m\geq 0} \lambda(1-\lambda)^m Q^m.$$
Suppose now that \((Q_\lambda)_\lambda\) is a family of stochastic matrices, for \(\lambda \in [0,1]\). It is not hard to see that if \(Q_\lambda\) is a regular perturbation of \(Q_0\) in a neighborhood of 0, then again:

\[
\Pi = \lim_{\lambda \to 0} \sum_{m \geq 0} \lambda (1 - \lambda)^m Q_\lambda^m.
\]

It is enough to write:

\[
\sum_{m \geq 0} \lambda (1 - \lambda)^m Q_\lambda^m = \sum_{m \geq 0} \lambda^2 (1 - \lambda)^m (m + 1) \left( \frac{1}{m + 1} \sum_{k=0}^{m} Q_\lambda^k \right),
\]

and use the fact that \(\lim_{m \to \infty} \frac{1}{m + 1} \sum_{k=0}^{m} Q_\lambda^k = \Pi_\lambda\), which converges to \(\Pi\) as \(\lambda\) tends to 0. The case of non-regular perturbation is most interesting, as shows the following example.

**Example 2.** For any \(a \geq 0\), let \(Q_\lambda(a) := \begin{pmatrix} 1 - \lambda^a & \lambda^a \\ \lambda^a & 1 - \lambda^a \end{pmatrix}\). Note that \(Q_\lambda(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) is periodic and that, for any \(a > 0\), \(Q_\lambda(a)\) is a non-regular perturbation of \(Q_0(a) = \text{Id}\). Computation yields:

\[
\lim_{\lambda \to 0} \sum_{m \geq 0} \lambda (1 - \lambda)^m Q_\lambda(a)^m = \begin{cases} 1/2 & 1/2, \quad \text{if } 0 \leq a < 1; \\ 2/3 & 1/3, \quad \text{if } a = 1; \\ 1/3 & 2/3, \quad \text{if } a > 1. \\ \text{Id}, \quad \text{if } a = 1. 
\end{cases}
\]

The case \(a = 1\) appears as the critical value and can be explained by the fact that the perturbation and the discount rate are “of same order”.

The convergence, as \(\lambda\) tends to 0, of \(\sum_{m \geq 0} \lambda (1 - \lambda)^m Q_\lambda^m\) needs some regularity of the family \((Q_\lambda)_\lambda\). The following condition is a natural regularity requirement:

**Assumption 1:** There exist \(c_{\omega,\omega'}, c_{\omega,\omega'} > 0\) \((\omega, \omega' \in \Omega)\) such that:

\[
Q_\lambda(\omega, \omega') \sim_{\lambda \to 0} c_{\omega,\omega'} \lambda^{c_{\omega,\omega'}}.
\]

The constants \(c_{\omega,\omega'}\) and \(c_{\omega,\omega'}\) are referred as the coefficient and the exponent of the transition \(Q_\lambda(\omega, \omega')\). By convention, we set \(c_{\omega,\omega'} = \infty\) whenever \(c_{\omega,\omega'} = 0\).

Assumption 1 holds in the rest of the paper. Note that a perturbation satisfying this assumption can be regular or non-regular.

### 1.1 From stochastic games to occupation measures

Two-person zero-sum stochastic games were introduced by Shapley [5]. They are are described by a 5-tuple \((\Omega, \mathcal{I}, \mathcal{J}, q, g)\), where \(\Omega\) is a finite set of states, \(\mathcal{I}\) and \(\mathcal{J}\) are finite sets of actions, \(g : \Omega \times \mathcal{I} \times \mathcal{J} \to [0,1]\) is the payoff, \(q : \Omega \times \mathcal{I} \times \mathcal{J} \to \Delta(\Omega)\) the transition and, for any finite set \(X, \Delta(X)\) denotes the set of probability distributions over \(X\). The functions \(g\) and \(q\) are bilinearly extended to \(\Omega \times \Delta(\mathcal{I}) \times \Delta(\mathcal{J})\). The stochastic game with initial state \(\omega \in \Omega\) and discount rate \(\lambda \in (0,1]\) is denoted by \(\Gamma_\lambda(\omega)\) and is played as follows: at stage \(m \geq 1\), knowing the current state \(\omega_m\), the players choose actions \((i_m, j_m) \in \mathcal{I} \times \mathcal{J}\); their choice produces a stage payoff \(g(\omega_m, i_m, j_m)\) and influences the transition: a new state \(\omega_{m+1}\) is chosen according to the probability distribution \(q(\cdot | \omega_m, i_m, j_m)\). At the end of the game, player 1 receives \(\sum_{m \geq 1} \lambda(1 - \lambda)^m g(\omega_m, i_m, j_m)\) from player 2. The game \(\Gamma_\lambda(\omega)\) has a value \(v_\lambda(\omega)\), and the vector \(v_\lambda = (v_\lambda(\omega))_{\omega \in \Omega}\) is the unique fixed point of the so-called Shapley operator [5]: \(\Phi_\lambda : \mathbb{R}^\Omega \to \mathbb{R}^\Omega\),

\[
\Phi_\lambda(f)(\omega) = \text{val}(s,t) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{J}) \{ \lambda g(\omega, s, t) + (1 - \lambda) \mathbb{E}_q(\cdot | \omega, s, t) [f(\omega)] \}.
\]
From \(1.3\), one deduces the existence of optimal stationary strategies \(x : \Omega \to \Delta(\mathcal{I})\) and \(y : \Omega \to \Delta(\mathcal{J})\). The convergence of the discounted values as \(\lambda\) tends to 0 is due to Bewley and Kohlberg \([1]\). An alternative proof was recently obtained in \([3]\). Let \(v := \lim_{\lambda \to 0} v_\lambda \in \mathbb{R}^\Omega\) be the vector of limit values.

If both players play stationary strategies \(x\) and \(y\) in \(\Gamma_\lambda\), then every visit to \(\omega\) produces an expected payoff of \(g(\omega, x(\omega), y(\omega))\), and a transition \(Q(\omega, \cdot) := g(\omega, x(\omega), y(\omega))\). Thus, the expected payoff induced by \((x, y)\), denoted by \(\gamma_\lambda(\omega, x, y)\), satisfies:

\[
\gamma_\lambda(\omega, x, y) = \sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} Q_{\lambda}^{m-1}(\omega, \omega') \sum_{\omega' \in \Omega} g(\omega', x(\omega'), y(\omega')).
\]  

Consider a family of stationary strategies \((x_\lambda, y_\lambda)_\lambda\), and let \((g_\lambda)_\lambda\) and \((Q_\lambda)_\lambda\) be the corresponding families of state-payoffs and transition matrices. Provided that the limits exist, the boundedness of \(\sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} Q_{\lambda}^{m-1}\) yields:

\[
\lim_{\lambda \to 0} \gamma_\lambda(\cdot, x_\lambda, y_\lambda) = \left( \lim_{\lambda \to 0} \sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} Q_{\lambda}^{m-1} \right) \left( \lim_{\lambda \to 0} g_\lambda \right),
\]

where the existence of the limits clearly requires some “regularity” of \((x_\lambda)_\lambda\) and \((y_\lambda)_\lambda\) in a neighbourhood of 0.

**Definition 1.** A family \((x_\lambda)_\lambda\) of stationary strategies of player 1 is:

1. Regular if there exists coefficients \(c_{\omega,i}, c_{\omega,j} > 0\) and exponents \(e_{\omega,i}, e_{\omega,j} \geq 0\) such that \(x_\lambda(\omega) \sim_{\lambda \to 0} c_{\omega,i} \lambda^{-e_{\omega,i}}\), for all \(\omega \in \Omega\), \(i \in \mathcal{I}\) and \(j \in \mathcal{J}\).
2. Asymptotically optimal if, for any \(j : \Omega \to \mathcal{J}\) pure stationary strategy of player 2 (or equivalently, for any \(y : \Omega \to \Delta(\mathcal{J})\), or any \((y_\lambda)_\lambda)\):

\[
\liminf_{\lambda \to 0} \gamma_\lambda(\omega, x_\lambda, j) \geq v(\omega).
\]

Similar definitions hold for families of stationary strategies of player 2. Regular, asymptotically strategies exists \([1]\). Suppose that \((x_\lambda)_\lambda\) and \((y_\lambda)_\lambda\) are regular. A direct consequence is that \((Q_\lambda)_\lambda\) satisfies Assumption 1. On the other hand, the existence of \(\lim_{\lambda \to 0} g_\lambda\) is then straightforward. These observations motivate the study of \((Q_\lambda)_\lambda\) under Assumption 1. We are interested in describing the distribution over the state space at any fraction of the game \(t \in [0, 1]\), given a pair of regular stationary strategies.

### 1.2 Main results

Let \((Q_\lambda)_\lambda\) be a fixed family of stochastic matrices over \(\Omega\) satisfying Assumption 1. Let \((X^\lambda_m\)\) be a Markov chain with transition \(Q_\lambda\). Extend the notation \(X^\lambda_m\) to make sense for integer times, to any real positive time by setting \(X^\lambda_{\lfloor t \rfloor} := X^\lambda_{\lfloor t \rfloor}\), where \(\lfloor t \rfloor = \max\{k \in \mathbb{N} \mid t \geq k\}\). The process \((X^\lambda_t)_{t \geq 0}\) is a continuous time inhomogeneous Markov chain which jumps at integer times.

For any \(\omega \in \Omega\), and \(\lambda \in (0, 1]\): \(\lambda(1 - \lambda)^{m-1} Q_{\lambda}^{m-1}(\omega, \omega')\) is the expected time spent in state \(\omega'\), starting from \(\omega\), if the weight given to stage \(m\) is \(\lambda(1 - \lambda)^{m-1}\). Thus, for any \(\lambda\) and \(n \in \mathbb{N}\), the weight given to the first \(n\) stages for a discount rate \(\lambda\) is

\[
\varphi(\lambda, n) := \sum_{k=1}^{n} \lambda(1 - \lambda)^{m-1}.
\]

In particular, note that \(\lim_{\lambda \to 0} \varphi(\lambda, \lfloor t/\lambda \rfloor) = 1 - e^{-t}\) so that, asymptotically, the first \(\lfloor t/\lambda \rfloor\) stages represent a fraction \(1 - e^{-t}\) of the play (see Figure 1). We denote this fraction of the game by “time \(t\)” and the limit, as \(t\) tends to 0, by “time 0”.


\[
\begin{array}{c|c|c|c}
\text{Stage} & 1 & |t/\lambda| & \infty \\
\text{as } \lambda \to 0 & & & \\
\text{Fraction of the game} & 0 & 1 - \epsilon^{-t} & 1 \\
\end{array}
\]

Figure 1: Relation between the number of stages, the fraction of the game and the time.

In Sections 2 and 3 we study \(P_t := Q^{(t/\lambda)} \in \Delta(\Omega)\), for any \(t > 0\), interpreted as the (distribution of the) instantaneous position at time \(t\). The existence of the limit is obtained, in some “extended sense” (see Section 1.4) under Assumption 1. Section 2 explores two particular cases of the family \((Q_\lambda)_λ\): absorbing and critical, respectively. In these cases, an explicit computation of \(P_t\) is obtained. Furthermore, we prove the convergence in distribution of the Markov chains with transition \(Q_\lambda\), to a Markov process in continuous time.

The general case is studied in Section 3. For some \(L \leq |\Omega|\) and some set \(\mathcal{R} = \{R_1, \ldots, R_L\}\) of subsets of \(\Omega\), we prove (see Theorem 3.1) that the instantaneous position admits the following expression:

\[
P_t = \mu e^{At} M,
\]

where \(\mu : \Omega \to \Delta(\mathcal{R})\), \(A : \mathcal{R} \times \mathcal{R} \to \mathbb{R}\) and \(M : \mathcal{R} \to \Delta(\Omega)\). The elements of \(\mathcal{R}\) are subsets of states such that, once they are reached, the probability of staying a strictly positive fraction of the play in them is strictly positive. They are the recurrent classes of a Markov chain defined in Section 3.5, which converges to a continuous time Markov process. Its infinitesimal generator is \(A\), while \(\mu\) represents the entrance laws to each of these subsets, and \(M\) gives the frequency of visits to each state in the subsets of \(\mathcal{R}\). From (1.6), it follows (see Corollary 3.2) that for any \(t > 0\),

\[
\lim_{\lambda \to 0} \sum_{m=1}^{\lfloor t/\lambda \rfloor} \lambda(1 - \lambda)^{m-1}Q_\lambda^{m-1} = \mu \left( \int_0^t e^{-s} e^{As} ds \right) M.
\]

In Section 3.7 we illustrate the computation of \(\mu\), \(A\) and \(M\) in an example. Finally, using the fact that \(A - \text{Id}\) is invertible (by Gershgorin’s Circle Theorem, for instance), we also obtain the following expression for the asymptotic payoff:

\[
\lim_{\lambda \to 0} \gamma_\lambda(\cdot, x_\lambda, y_\lambda) = \mu(A - \text{Id})^{-1} Mg,
\]

where \(g := \lim_{\lambda \to 0} g_\lambda \in \mathbb{R}^\mathcal{R}\).

### 1.3 Notation

For any \(\nu \in \Delta(\Omega)\) and \(B \subset \Omega\) let \(\nu(B) := \sum_{k \in B} \nu(k)\). Let \(P\) be some stochastic matrix over \(\Omega\), and let \((X_m)_{m \geq 0}\) be a Markov chain with transition \(P\). Let \(B^c := \Omega \setminus B\) and, for any \(k \in \Omega\), \(k^c := \{k\}^c\). In particular, \(P(k, B) = \sum_{k' \in B} P(k, k')\). Denote by \(\tilde{P}\) the stochastic matrix obtained by \(P\) as follows. For any \(k, k' \in \Omega\), \(k \neq k'\):

\[
\tilde{P}(k, k') = \begin{cases} 
P(k, k')/P(k, k^c) & \text{if } P(k, k^c) > 0; \\
0 & \text{if } P(k, k^c) = 0, 
\end{cases}
\]

and \(\tilde{P}(k, k) = 1 - \sum_{k' \neq k} \tilde{P}(k, k')\). Note that \(\tilde{P}(k, k) = 0\) whenever \(P(k, k^c) > 0\), and that \(P\) and \(\tilde{P}\) have the same recurrence classes \(R \in \mathcal{R}\). Denote by \(T\) the set of transient states. Let \(\mu : \Omega \to \mathcal{R}\) be, for any \(k \in \Omega\) and \(R \in \mathcal{R}\), be the entrance probability from \(k\) to the recurrence class \(R\):

\[
\mu(k, R) := \lim_{n \to \infty} P^n(k, R) = \lim_{n \to \infty} \tilde{P}^n(k, R).
\]
Let $\pi^R$ be the invariant measure of the restriction of $P$ to $R$, seen as a probability measure over $\Omega$. If the restriction $(X_m)_m$ to $R$ is $d$-periodic ($d \geq 2$), let $\pi_k^R$ ($k = 1, \ldots, d$) be the invariant measure of $(X_{md+k})_m$. Note that, in this case, $\pi^R = \frac{1}{d} \sum_{k=1}^d \pi_k^R$.

For any $\omega \in \Omega$, the probability of quitting $\omega$ satisfies $Q_{\lambda}(\omega, \omega') \sim_{\lambda \to 0} c_{\omega} \lambda^{\omega'}$, where:

$$e_\omega := \min\{e_{\omega, \omega'} \mid \omega' \neq \omega, c_{\omega, \omega'} > 0\} \quad \text{and} \quad c_{\omega} := \sum_{\omega' \neq \omega} c_{\omega, \omega'} 1_{\{e_{\omega, \omega'} = e_\omega\}}.$$  

(1.9)

For any $\omega \in \Omega$, let $P^\lambda_\omega$ be the unique probability distribution over $\Omega^N$ induced by $Q_{\lambda}$ and the initial state $\omega$, i.e. $P^\lambda_\omega(X^\lambda_1 = \omega) = 1$ and, for all $m \geq 1$ and $\omega', \omega'' \in \Omega$:

$$P^\lambda_\omega(X^\lambda_{m+1} = \omega'' | X^\lambda_m = \omega') = Q_{\lambda}(\omega', \omega'').$$

For any $t > 0$ and $\omega \in \Omega$, let $P^t_\omega$ be a shortcut for $P^\lambda_{\omega_0}(\cdot \mid X^\lambda_{t/\lambda} = \omega)$, for some fixed initial state $\omega_0$. By the Markov property, the choice of the initial state is irrelevant. Finally, let $t^\lambda_B := \inf\{m \geq 1 \mid X^\lambda_m \in B\}$ be the first arrival to $B$ and let us end this section with a useful Lemma.

**Lemma 1.** Let $P$ be an irreducible stochastic matrix over $\{1, \ldots, n\}$ with invariant measure $\pi$, and let $S$ be a diagonal matrix with diagonal coefficients in $(0, 1]$ such that $P = \text{Id} - S + SP$. Then $P$ is irreducible with invariant measure $\pi$ and:

$$\pi(k) = \frac{\hat{\pi}(k)/S(k, k)}{\sum_{k'=1}^n \hat{\pi}(k')/S(k', k')}, \quad \text{for all} \ 1 \leq k \leq n.$$  

**Proof.** It is enough to check that the right-hand side of the equality is invariant by $P$, which is equivalent to $\pi S^{-1} P = \pi S^{-1}$. We easily compute:

$$\pi S^{-1} P = \pi S^{-1} (\text{Id} - S + SP) = \pi S^{-1} - \hat{\pi} + \hat{\pi} P = \pi S^{-1},$$

which completes the proof. $\square$

### 1.4 The instantaneous position at time $t$

For any $t > 0$, let $Q^{t/\lambda}_\lambda := Q^{t/\lambda}_{\lambda}$ when there is no risk of confusion.

**Definition 2.** If the limit exists, let $P_t : \Omega \to \Delta(\Omega)$ such that for all $\omega, \omega' \in \Omega$:

$$P_t(\omega, \omega') = \lim_{\lambda \to 0} P^{\lambda}_{\omega_0}(X^\lambda_{t/\lambda} = \omega') = \lim_{\lambda \to 0} Q^{t/\lambda}_{\lambda}(\omega, \omega').$$

$P_t$ is the vector of (distributions of the) positions at time $t > 0$. Let $P_0 := \lim_{t \to 0} P_t$ be the position at time 0.

Assumption 1 does not ensure the existence of $P_t$: consider a constant, periodic family $(Q_{\lambda})_{\lambda} \equiv P := (1/2)$. Then for any initial state:

$$P^{\lambda}_{\omega_0}(X^\lambda_{t/\lambda} = \omega_0) = \begin{cases} 0 & \text{if } |t/\lambda| \equiv 0 \text{ (mod 2)} \\ 1 & \text{if } |t/\lambda| \equiv 1 \text{ (mod 2)} \end{cases}$$  \hspace{1cm} (1.10)

so that the limit does not exist. On the other hand, however, as $\lambda$ tends to 0, the frequency of visits to both states before stage $|t/\lambda|$ converges to $1/2$ for any $t > 0$, and

$$\lim_{\lambda \to 0} \delta_{\omega_0} \frac{1}{2} (Q^{t/\lambda}_{\lambda} + Q^{(t/\lambda)+1}_{\lambda}) = (1/2, 1/2).$$

Moreover, both $|t/\lambda|$ and $|t/\lambda| + 1$ represent the same fraction of the game. These observations motivate the following definitions. Let $Q_{\lambda}$ be such that $Q_{\lambda}(\omega, \omega') \sim_{\lambda \to 0} Q_{\lambda}(\omega, \omega') 1_{\{e_{\omega, \omega'} < 1\}}$, for all $\omega, \omega' \in \Omega$, and let $N$ be the product of the periods of its recurrence classes.
Definition 3. The extended position $\mathcal{P}_t$ at time $t \geq 0$ is obtained by averaging over $N$, i.e.,
$$
\mathcal{P}_t(\omega, \omega') = \lim_{\lambda \to 0} \frac{1}{N} \sum_{m=0}^{N-1} Q_\lambda^{\lfloor t/\lambda \rfloor + m}(\omega, \omega'),
$$
and $\mathcal{P}_0 := \lim_{t \to 0} \mathcal{P}_t$.

We set $P_t := \mathcal{P}_t$ when the latter exists.

Averaging over $N$, one avoids irrelevant pathologies related to periodicity. In the previous example, for instance, $N = 2$ settled the problem. It clearly extends the previous definition since the existence of $P_t$ implies the existence of $\mathcal{P}_t$ and their equality. We prove in Theorem 5.1 that the latter always exists under Assumption 1. The following example shows why $N$ depends on $(Q_\lambda)_\lambda$, rather than on $(Q_\lambda)_\lambda$.

Example 3. Fix $t > 0$. Let $Q_\lambda(a) := \left( \frac{\lambda^a}{1 - \lambda} \frac{1 - \lambda^a}{\lambda} \right)$, for $\lambda \in [0, 1]$ and some $a \geq 0$.

Clearly, $Q_\lambda(a)$ is aperiodic for all $\lambda > 0$. However, $P_t$ exists only for all $0 \leq a < 1$. Indeed, consider the transition $Q_\lambda(a)$ for some $a > 1$. The probability that $X_{m+1}^\lambda = X_m^\lambda$ for some $m \leq \lfloor t/\lambda \rfloor$ is bounded by $(1 - \lambda^a)^{\lfloor t/\lambda \rfloor}$ which converges to 0 with $\lambda$. Thus, asymptotically, the chain behaves like the 2-periodic matrix $(0 1; 1 0)$ before time $t$, for any $t \geq 0$.

2 Characterization of two special cases

In this section we study two special families of stochastic matrices.

Definition 4. A stochastic matrix over $\Omega$ is absorbing if $Q(\omega, \omega) = 1$ for all $\omega \in \Omega \setminus \{\omega_0\}$. An absorbing matrix $Q$ will be identified with the vector $Q(\omega_0, \cdot) \in \Delta(\Omega)$.

Definition 5. $(Q_\lambda)_\lambda$ is absorbing if $Q_\lambda(\omega, \omega) = 1$ for all $\lambda > 0$, for all $\omega \neq \omega_0$.

Definition 6. $(Q_\lambda)_\lambda$ is critical if $e_{\omega, \omega'} \geq 1$ for all $\omega, \omega' \in \Omega$, $\omega \neq \omega'$.

Definition 7. The infinitesimal generator $A : \Omega \times \Omega \to \mathbb{R}$ corresponding to a critical family $(Q_\lambda)_\lambda$ is defined as follows:
$$
A(\omega, \omega') := \lim_{\lambda \to 0} \frac{Q_\lambda(\omega, \omega')}{\lambda} \quad (\omega' \neq \omega) \quad \text{and} \quad A(\omega, \omega) := - \sum_{\omega' \neq \omega} A(\omega, \omega').
$$

Note that $A = 0$ if and only if $e_{\omega, \omega} > 1$ for all $\omega \neq \omega'$. Absorbing families are treated in Section 2.1, critical families in Section 2.2. In both cases, $P_t$ exists and its computation can be carried explicitly.

2.1 Absorbing case

Let $(Q_\lambda)_{\lambda \in (0, 1]}$ be absorbing and let $\omega_0$ be non-absorbing state. To simplify the notation, let $Q_\lambda$ stand for $Q(\omega_0, \cdot)$. For any $\omega \neq \omega_0$, let $c_\omega := c_{\omega_0, \omega}$ and $e_\omega := e_{\omega_0, \omega}$. Let also $e := \min\{e_{\omega_0, \omega} \mid \omega \neq \omega_0\}$ and $c := \sum_{\omega' \neq \omega_0} e_{\omega_0, \omega'} I_{\{e_{\omega, \omega} = e\}}$. Finally, let $P_t := P_t(\omega_0, \cdot) \in \Delta(\Omega)$.

Proposition 2.1. $P_t$ exists for any $t > 0$. Moreover, one has:
$$
P_t(\omega_0) = \begin{cases}
1, & \text{if } e > 1; \\
0, & \text{if } 0 \leq e < 1; \\
e^{-ct}, & \text{if } e = 1.
\end{cases}
$$

For any $\omega \neq \omega_0$:
$$
P_t(\omega) = \begin{cases}
0, & \text{if } e > 1; \\
\frac{e}{c} I_{\{e_{\omega, \omega} = e\}}, & \text{if } 0 \leq e < 1; \\
(1 - e^{-ct}) \frac{e}{c} I_{\{e_{\omega, \omega} = e\}}, & \text{if } e = 1.
\end{cases}
$$
Proof. The equalities in (2.2) are immediate since, by the definition of \( e \):

\[
\mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda = \omega_0) = (1 - e^{\lambda t} + o(\lambda t))^{\frac{1}{\lambda}} \sim_{\lambda \to 0} e^{-\lambda t} \lambda^{-1}.
\]

It follows that, for \( e > 1 \), \( \sum_{\omega \neq \omega_0} P_t(\omega) = \lim_{\lambda \to 0} \mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda \neq \omega_0) = 0 \), so that \( P_t(\omega) = 0 \) for all \( \omega \neq \omega_0 \), in this case. Similarly if \( e \leq 1 \) then for any \( \omega \neq \omega_0 \):

\[
\mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda = \omega) = \mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda \neq \omega_0)\mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda = \omega | X_{t/\lambda}^\lambda \neq \omega_0),
\]

(2.4)

\[
\mathbb{P}_{\omega_0}^\lambda(X_{t/\lambda}^\lambda \neq \omega_0)\sum_{\omega \neq \omega_0} Q_{\lambda}(\omega). 
\]

(2.5)

Taking the limit, as \( \lambda \) tends to 0 gives (2.3).

\[\square\]

We can clearly distinguish three cases, depending on \( e \), as in Example 2.

(a) \textbf{Stable} (\( e > 1 \)). \( P_t(\omega_0) = 1 \) for all \( t > 0 \), so that \( \omega_0 \) is “never” left.

(b) \textbf{Unstable} (\( 0 \leq e < 1 \)). \( P_t(\omega_0) = 0 \) for all \( t > 0 \), so that \( \omega_0 \) is left “immediately”.

(c) \textbf{Critical} (\( e = 1 \)). \( P_t(\omega_0) \in (0,1) \) for all \( t > 0 \). From (2.2), one deduces that \( \omega_0 \) is left at time \( t \) with probability (density) \( ce^{-ct} dt \).

\[\text{2.2 Critical case}\]

Let \((Q_{\lambda})_{\lambda \in (0,1)}\) be critical. The next result explains why the matrix \( A \), defined in (2.1), is denoted the infinitesimal generator.

\textbf{Proposition 2.2.} \textit{For any} \( t \geq 0 \) \textit{and} \( h > 0 \) \textit{and} \( \omega' \neq \omega \) \textit{we have, as} \( h \to 0 \):

\begin{enumerate}[label=(\roman*)]
\item \( \lim_{\lambda \to 0} \mathbb{P}_{\omega}^\lambda(X_{(t+h)/\lambda}^\lambda = \omega) = 1 + A(\omega, \omega)h + o(h) \),
\item \( \lim_{\lambda \to 0} \mathbb{P}_{\omega}^\lambda(X_{(t+h)/\lambda}^\lambda = \omega') = A(\omega, \omega')h + o(h) \).
\end{enumerate}

\textbf{Proof.} \textit{Notation.} Define two deterministic times \( T_0^\lambda := t/\lambda \) and \( T_h^\lambda := (t + h)/\lambda \). For any \( k \in \mathbb{N} \), let \( F_{0,h}^\lambda(k) \) be the event that \( (X_{m/\lambda}^\lambda)_{m \geq 1} \) changes \( k \) times of state in the interval \([T_0^\lambda, T_h^\lambda]\), and let \( F_{0,h}^\lambda(k) = \cup_{f \leq k} F_{0,h}^\lambda(f) \).

Notice that, conditional to \( \{X_{t/\lambda}^\lambda = \omega\} \), the following disjoint union holds:

\[
\{X_{(t+h)/\lambda}^\lambda = \omega\} = F_{0,h}^\lambda(0) \cup \left( \{X_{(t+h)/\lambda}^\lambda = \omega\} \cap F_{0,h}^\lambda(2+) \right).
\]

The following computation is straightforward:

\[
\lim_{\lambda \to 0} \mathbb{P}_{\omega}^\lambda(F_{0,h}^\lambda(0)) = \lim_{\lambda \to 0} \prod_{m = [t/\lambda]}^{|(t+h)/\lambda|} \mathbb{P}_{\omega}^\lambda(X_{m+1}^\lambda = X_m^\lambda),
\]

(2.6)

\[
= \lim_{\lambda \to 0} \left( 1 - \sum_{\omega' \neq \omega} Q_{\lambda}(\omega, \omega') \right)^{h/\lambda},
\]

(2.7)

\[
= \exp\left( - \sum_{\omega' \neq \omega} A(\omega, \omega')h \right),
\]

(2.8)

\[
= 1 + A(\omega, \omega)h + o(h), \quad \text{as} \ h \to 0.
\]

(2.9)

On the other hand:

\[
\mathbb{P}_{\omega}^\lambda(F_{0,h}^\lambda(2+)) \leq \max_{\omega' \in \Omega} \mathbb{P}_{\omega}^\lambda(F_{0,h}^\lambda(1+))^2 = \max_{\omega' \in \Omega} \left( 1 - \mathbb{P}_{\omega}^\lambda(F_{0,h}^\lambda(0)) \right)^2.
\]

(2.10)
Therefore, \( \lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (2+)) = o(h) \) as \( h \) tends to 0 which, together with (2.9), proves (i). Similarly, conditional on \( \{ X^\lambda_{t/\lambda} = \omega \} \):

\[
\{ X^\lambda_{(t+h)/\lambda} = \omega' \} = \{ X^\lambda_{(t+h)/\lambda} = \omega' \} \cap (F_{0,h}^\lambda (1) \cup F_{0,h}^\lambda (2+)),
\]

so that by (2.10):

\[
\lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega (X^\lambda_{(t+h)/\lambda} = \omega') = \lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (1), X^\lambda_{(t+h)/\lambda} = \omega') + o(h).
\]

On the other hand, for any \( \lambda \) and \( m \):

\[
\mathbb{P}^\lambda (X^\lambda_{m+1} = \omega' | X^\lambda_m = \omega, X^\lambda_{m+1} \neq \omega) = \frac{Q\lambda (\omega, \omega')}{Q\lambda (\omega, \omega')}.
\]

Finally, note that by (2.10), \( \lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (1)) = \lim_{\lambda \to 0} 1 - \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (0)) + o(h) \), and that:

\[
\lim_{\lambda \to 0} \frac{Q\lambda (\omega, \omega')}{Q\lambda (\omega, \omega')} A(\omega, \omega') = A(\omega, \omega') \quad \text{for all } \omega \neq \omega'.
\]

Consequently, as \( h \) tends to 0:

\[
\lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (1), X^\lambda_{(t+h)/\lambda} = \omega') = \lim_{\lambda \to 0} \frac{Q\lambda (\omega, \omega')} {Q\lambda (\omega, \omega') (1 - \mathbb{P}^\lambda_\omega (F_{0,h}^\lambda (0)) + o(h))} \cdot \left(1 - e^{A(\omega, \omega')h}\right)
\]

\[
= A(\omega, \omega') h + o(h).
\]

\( \square \)

**Corollary 2.1.** The processes \( (X^\lambda_{t/\lambda})_{t \geq 0} \) converge, as \( \lambda \) tends to 0, to a Markov process \( (Y_t)_{t \geq 0} \) with generator \( A \).

**Proof.** The limit is identified by Proposition 2.2. The tightness is a consequence of the bound in Proposition 2.2 (ii), which implies that for any \( T > 0 \), uniformly in \( \lambda > 0 \):

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \exists t_1, t_2 \in [0, T] \mid t_1 < t_2 < t_1 + \varepsilon, X^\lambda_{t_i/\lambda} \neq X^\lambda_{t_j/\lambda}, i \in \{1, 2\} \right) = 0,
\]

which is precisely the tightness criterion for càdlàg process with discrete values. \( \square \)

The following result is both a direct consequence of Proposition 2.2 or Corollary 2.1:

**Corollary 2.2.** \( P_t \) exists for any \( t > 0 \) and satisfies \( P_t = e^{At} \).

## 3 The general case

In this section we drop the assumption of \((Q\lambda)\lambda\) being critical or absorbing. Let us start by noticing that in Proposition 2.2 the time \( t/\lambda \) may be replaced by \( t/\lambda \pm 1/\lambda^\delta \), for any \( 0 < \delta < 1 \). That is:

\[
P_t(\omega, \omega') = \lim_{\lambda \to 0} \mathbb{P}^\lambda_\omega \left( X^\lambda_{t/\lambda \pm 1/\lambda^\delta} = \omega' \right) = e^{At}(\omega, \omega').
\]

(3.1)

Note that, as \( \lambda \) tends to 0, \( t/\lambda \pm 1/\lambda^\delta \) also corresponds to time \( t \). This remark gives an idea of the flexibility to the terminology “position at time \( t' \)”. Our main result is the following:

**Theorem 3.1.** There exists \( L \leq |\Omega| \), subsets \( \mathcal{R} = \{ R_1, \ldots, R_L \} \) of \( \Omega \), \( \mu : \Omega \to \Delta(\mathcal{R}) \), \( A : \mathcal{R} \times \mathcal{R} \to \mathbb{R} \) and \( M : \mathcal{R} \times \Omega \to \Delta(\Omega) \) such that \( P_t = \mu e^{At} M \), for all \( t \geq 0 \).
The proof of this result is constructive, and is left to Section 3.3 together with an algorithm for the computation of $L$, $R$, $\mu$, $A$ and $M$. An illustration of the algorithm is provided in Section 3.7 by means of an example. Note that if $Q_\lambda$ were critical, then the results in Section 2.2 yield Theorem 3.1 with $L = |\Omega|$, $R = \Omega$, $\mu = \text{Id} = M$ and $A$ is defined in 2.1.

The following two results are direct consequences of Theorem 3.1.

**Corollary 3.2.** For any $t > 0$:

$$
\lim_{\lambda \to 0} \sum_{m=1}^{\lfloor t/\lambda \rfloor} \lambda(1 - \lambda)^{m-1}Q_\lambda^{m-1} = \mu \left( \int_0^t e^{-s}e^{\lambda s}ds \right) M.
$$

In particular, $\lim_{\lambda \to 0} \sum_{m\geq 1} \lambda(1 - \lambda)^{m-1}Q_\lambda^{m-1} = \mu(\text{Id} - A)^{-1}M$.

For any $t \in [0,1)$, let $p_t := P_\lambda^{-\infty t} \ln(1 - t)$ be the position at the fraction $t$ of the game.

**Corollary 3.3.** Let $v_i$ be the eigenvalues of $A$ and let $m_i$ be the size of the Jordan box corresponding to $v_i$, in the canonical form of $A$. Then for any $t \in [0,1)$ and $\omega \in \Omega$, $p_t(\omega)$ is linear in $(1 - t)^{-m_i} \ln(1 - t)^k$, $0 \leq k \leq m_i - 1$.

The analogue of Corollary 3.3 holds here, yet with some slight modifications. Unlike in Section 2.2 it is not the processes $X^\lambda$ which converge, but rather their restriction to the set $R$ obtained in Theorem 3.1. The proof is then, word for word, as in Corollary 3.3.

**Definition 8.** The restriction of $X^\lambda$ to $R$ is:

$$
\hat{X}^\lambda_m := \Phi(X^\lambda_{\ell_m}), \quad m \geq 1,
$$

where $V^\lambda_m$ is the time of the $m$-th visit of $X^\lambda$ to $R$ and $\Phi$ is a mapping which associates, to any state $\omega \in \bigcup_{\ell=1}^{\infty} R_\ell \subset \Omega$, the subset $R_\ell$ which contains it. Let $\hat{X}_t := \hat{X}^\lambda_{\lfloor t \rfloor}$, $t \geq 0$.

**Corollary 3.4.** Let $R$, $\mu$ and $A$ be given in Theorem 3.1. The processes $\hat{X}^\lambda$ converge, as $\lambda$ tends to $0$, to a Markov process with initial distribution $\mu$ and generator $A$.

### 3.1 The order of a transition

A natural way to rank the transitions of the Markov chains $(X^\lambda_m)_m$ is in terms of their (asymptotic) order of magnitude. For that purpose, it is useful to define the following notion.

**Definition 9.** The order of the transition from $\omega$ to $\omega'$ is defined as follows:

$$
r_{\omega,\omega'} := \inf \left\{ \alpha \geq 0 \left| \liminf_{\lambda \to 0} \mathbb{P}_\omega^\lambda \left( \tau_\lambda \leq \frac{1}{\lambda^\alpha} \right) > 0 \right. \right\}.
$$

Let us present an example to illustrate this definition.

**Example 4.** Let $0 \leq a < b$, $\Omega = \{1, \ldots, n\}$ and suppose that $Q(1,2) = \lambda^a$, $Q(1,3) = \lambda^b$ and $Q(1,k) = 0$ for all $k = 3, \ldots, n$. On the one hand, for any $\delta < a$, $\mathbb{P}^\lambda_1(\tau_2 > 1/\lambda^\delta) \geq (1 - \lambda^a - \lambda^b)^{1/\lambda^\delta}$. Taking the limit yields:

$$
\lim_{\lambda \to 0} \mathbb{P}^\lambda_1(\tau_2 \leq 1/\lambda^\delta) \leq 1 - \lim_{\lambda \to 0} (1 - \lambda^a - \lambda^b)^{1/\lambda^\delta} = 0,
$$

which implies that $r_{1,2} \geq a$. On the other hand, starting from state 1, for any $m$:

$$
\{\tau_{1c} \leq m\} \cap \{X^\lambda_{\ell_1} = 2\} \subset \{\tau_2 \leq m\}.
$$

Taking the limit yields $r_{1,2} \leq a$ since:

$$
\lim_{\lambda \to 0} \mathbb{P}^\lambda_1(\tau_1 \leq 1/\lambda^\delta) \geq \lim_{\lambda \to 0} \left( 1 - (1 - \lambda^a - \lambda^b)^{1/\lambda^\delta} \right) \frac{\lambda^a}{\lambda^a + \lambda^b} = 1 - 1/e > 0.
$$
The previous example exhibits an explicit computation for the order of a transition. Note, however, that \( r_{1,3} \) cannot be computed with the data we provided, for it depends on other entries of \( Q_\lambda \). This is due to the fact that, conditional to leaving state 1, the probability of going to state 2 converges to 1, so that the future behaviour of the chain depends on the vector \( Q_\lambda(2,\cdot) \). Let us give an example where the computation of the order of a transition is a bit more involved.

**Example 5.** Let \( 0 \leq a < b, c \geq 0, \) and \( \Omega = \{1, 2, 3\} \). For any \( \lambda \in [0, 1], \) let:

\[
Q_\lambda = \begin{pmatrix}
1 - (\lambda^a + \lambda^b) & \lambda^a & \lambda^b \\
\lambda^c & 1 - \lambda^c & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The computation of \( r_{1,2} = a \) and of \( r_{2,1} = c \) is the same as in the previous example. Also, \( r_{3,2} = r_{3,1} = \infty \) because 3 is absorbing. Let us compute \( r_{1,3} \) and \( r_{2,3} \) heuristically. In average, state 1 is left after \( 1/(\lambda^a + \lambda^b) \) stages, then state 2 is left after \( 1/\lambda^c \) stages, and we are back in state 1 again. Hence, in average, an exit from state 1 occurs every \( \frac{1}{\lambda^a + \lambda^b} + \frac{1}{\lambda^c} \) stages. Consequently, state 1 is left \( 1/\lambda^b \) times, after:

\[
\frac{1}{\lambda^a + \lambda^b} + \frac{1}{\lambda^c}
\]

stages, and thus the probability of reaching state 3 is strictly positive. The relation \( \frac{1}{\lambda^a + \lambda^b} + \frac{1}{\lambda^c} \sim_{\lambda \to 0} \frac{1}{\max(\lambda^a, \lambda^c)}, \) which holds because \( a < b, \) yields \( r_{1,3} = r_{2,3} = \max\{a, c\} - b. \)

### 3.2 Fastest and secondary transitions

**Definition 10.** Let \( \alpha_1 := \min\{\epsilon_{\omega,\omega'} \mid \omega \neq \omega' \in \Omega\} \) be the order of the fastest transitions. A transition from \( \omega \) to \( \omega' \) is primary if \( \epsilon_{\omega,\omega'} = \alpha_1. \)

Note that if \( \alpha_1 \geq 1, \) \( Q_\lambda \) is critical. The results in Section 2.2 apply and yield (see Corollary 2.2) Theorem 3.1 with \( L = \Omega, \) \( R = \Omega, \mu = M = \text{Id} \) and \( A \) defined in (2.1).

Suppose, on the contrary, that \( \alpha_1 < 1. \) Define a stochastic matrix \( P^{[1]}_\lambda \) which is the restriction of \( Q_\lambda \) to its fastest transitions. For any \( \omega \neq \omega' \in \Omega \) set:

\[
P^{[1]}_\lambda(\omega, \omega') := \begin{cases}
\epsilon_{\omega,\omega'} \lambda^\epsilon_{\omega,\omega'}, & \text{if } \epsilon_{\omega,\omega'} = \alpha_1; \\
0, & \text{otherwise};
\end{cases}
\]

and let \( P^{[1]}_\lambda(\omega, \omega) := 1 - \sum_{\omega' \neq \omega} P^{[1]}_\lambda(\omega, \omega'). \) Let \( R^{[1]} \) and \( T^{[1]} \) be, respectively, the set of its recurrence classes and transient states. Note that these sets are independent of \( \lambda > 0. \) Let \( \pi^{[1], R} \) be the invariant measures of the restriction of \( P^{[1]}_\lambda \) to the recurrence class \( R \in R^{[1]} \).

We can now define secondary transitions.

**Definition 11.** The order of secondary transitions is:

\[
\alpha_2 := \min\{\epsilon_{\omega,\omega'} \mid \omega \in R, \omega' \notin R, : R \in R^{[1]}\}.
\]

A transition from \( \omega \) to \( \omega' \) is secondary if \( \epsilon_{\omega,\omega'} \geq \alpha_2, \) and \( \omega \in R, \omega' \notin R, \) for some \( R \in R^{[1]} \).

The definition of \( P^{[1]}_\lambda \) implies that \( P^{[1]}_\lambda \) is independent of \( \lambda \) and has the same recurrence classes as \( P^{[1]}_1. \) Denote this matrix by \( \overline{P}^{[1]} \) and let \( \overline{\pi}^{[1], R} \) be the invariant measures of the restriction of \( \overline{P}^{[1]} \) to \( R. \) The restriction of \( P^{[1]}_\lambda \) to \( R \) is irreducible and, consequently, we may apply Lemma 4 with the diagonal matrix \( S^{[1], R}_\lambda, \) defined for each \( \omega \in R \) as follows:

\[
S^{[1], R}_\lambda(\omega, \omega) := P^{[1]}_\lambda(\omega, \omega').
\]

Note that either \( R = \{\omega\} \) is a singleton and \( S^{[1], R}_\lambda(\omega, \omega) = 1 \) or there are at least two states in \( R \) and \( S^{[1], R}_\lambda(\omega, \omega) := \epsilon_{\omega} \lambda^{\alpha_1} \) for each \( \omega \in R. \) The following result is thus a direct consequence of Lemma 4.
Corollary 3.5. Let $R \in \mathcal{R}$. Then there exist $c^{[1],R}(\omega) > 0 (\omega \in R)$ such that:

$$
\pi^{[1],R}_{\lambda}(\omega) = \frac{\pi^{[1],R}(\omega)/S^{[1],R}_{\lambda}(\omega,\omega)}{\sum_{\omega' \in R} \pi^{[1],R}(\omega')/S^{[1],R}_{\lambda}(\omega',\omega')} = c^{[1],R}(\omega).
$$

Since $\pi^{[1],R}_{\lambda}$ is independent of $\lambda$, we will denote it from now on simply by $\pi^{[1],R}_{\lambda}$. Conditional on having no transitions of order higher than $\alpha_1$ and on being in $R$, the frequency of visits to $\omega \in R$ converges (exponentially fast) to $\pi^{[1],R}(\omega)$. Consequently, the probability of a transition of higher order going out from $R$ converges to $\sum_{\omega \in R} \pi^{[1],R}(\omega)\hat{Q}_{\lambda}(\omega,\cdot')$. Aggregation is thus natural, in order to study phenomena of order strictly bigger than $\alpha_1$.

3.3 Aggregating the recurrence classes

Aggregating the recurrence classes stand to considering the state space $\Omega^{[1]} := \mathcal{T}^{[1]} \cup \mathcal{R}^{[1]}$, i.e. an element $\omega^{[1]} \in \Omega^{[1]}$ is either a transient state $\omega^{[1]} \in \mathcal{T}^{[1]}$ (in this case $\omega^{[1]} = \omega \in \Omega$) or a recurrence class $\omega^{[1]} = R \in \mathcal{R}^{[1]}$ (in this case $\omega^{[1]} \subset \Omega$). In particular, the states of $\Omega^{[1]}$ can be seen as a partition of the states of $\Omega^{[0]} := \Omega$ (see Figure 3 for an illustration). To avoid cumbersome notation, let $\omega, \omega'$ stand for states in $\Omega^{[1]}$ when there is no confusion. One can then define an “aggregated” stochastic matrix $Q^{[1]}_{\lambda}$ over $\Omega^{[1]}$ as follows.

$$
Q^{[1]}_{\lambda}(\omega,\omega') := \begin{cases} Q_{\lambda}(\omega,\omega') & \text{if } \omega, \omega' \in \mathcal{T}^{[1]}; \\
\sum z' \in \omega' Q_{\lambda}(\omega, z') & \text{if } \omega \in \mathcal{T}^{[1]}, \text{ and } \omega' \in \mathcal{R}^{[1]}; \\
\sum z \in \omega \pi^{[1],\omega}(z)Q_{\lambda}(z, \omega') & \text{if } \omega \in \mathcal{R}^{[1]}, \text{ and } \omega' \in \mathcal{T}^{[1]}; \\
\sum z \in \omega, z' \in \omega' \pi^{[1],\omega}(z)Q_{\lambda}(z, z') & \text{if } \omega, \omega' \in \mathcal{R}^{[1]}. 
\end{cases}
$$

Clearly, Assumption 1 ensures the existence of $c^{[1],\omega,\omega'}_{\omega,\omega'}$ and $c^{[1],\omega,\omega'}_{\omega,\omega'}$ such that

$$
Q^{[1]}_{\lambda}(\omega,\omega') \sim_{\lambda \to 0} c^{[1],\omega,\omega'}_{\omega,\omega'}, \quad \forall \omega, \omega' \in \Omega^{[1]}.
$$

An explicit computation of $c^{[1],\omega,\omega'}_{\omega,\omega'}$ and $c^{[1],\omega,\omega'}_{\omega,\omega'}$ can be easily deduced from (3.4), in terms of the coefficients and exponents of $(Q_{\lambda})_{\lambda}$ and of the invariant measures of $P^{[1]}_{\lambda}$. The matrix $Q^{[1]}_{\lambda}$ arises by aggregating the state in the recurrence classes of $P^{[1]}_{\lambda}$. Define the entrance laws $\mu^{[1]} : \Omega \to \Delta(\mathcal{R}^{[1]})$ as follows:

$$
\mu^{[1]}(\omega, R) := \lim_{n \to \infty} (P^{[1]}_{\lambda})^{n}(\omega, R).
$$

If $\alpha_2 \geq 1$, define $A^{[1]} : \mathcal{R}^{[1]} \times \mathcal{R}^{[1]} \to [0, \infty)$ the infinitesimal generator corresponding to $Q^{[1]}_{\lambda}$, as follows. For any $R, R' \in \mathcal{R}^{[1]}$:

$$
A^{[1]}(R, R') := \lim_{\lambda \to 0} \frac{1}{\lambda} \left( \sum_{\omega \in \Omega^{[1]} \setminus R} Q^{[1]}_{\lambda}(R, \omega)\mu^{[1]}(\omega, R') \right),
$$

and $A^{[1]}(R, R) = -\sum_{R' \neq R} A^{[1]}(R, R')$. Note that $A^{[1]}$ admits the following useful equivalent expression:

$$
A^{[1]}(R, R') = \lim_{\lambda \to 0} \frac{1}{\lambda} \left( \sum_{\omega \in R, \omega' \notin R} \pi^{[1],R}_{\lambda}(\omega)Q_{\lambda}(\omega, \omega')\mu^{[1]}(\omega', R') \right).
$$

Finally, let $M^{[1]} : \mathcal{R}^{[1]} \to \Delta(\Omega)$ be such that, for any $\omega \in R \in \mathcal{R}$:

$$
M^{[1]}(R, \omega) := \pi^{[1],R}(\omega).
$$
For any notation: deterministic times, and take into account periodic issues. Introduce some notation.

Proof of Proposition 3.1

Let \( \alpha \) be the event that \( \lambda \) is smaller than \( C_\lambda \). For any \( \lambda \), one needs however, to consider two more transitions of the Markov chain \( \lambda \). Moreover, the following disjoint union holds:

\[
T^\lambda_0 \quad T^\lambda_\delta \quad T^\lambda_{h-\delta} \quad T^\lambda_h
\]

Figure 2: The four deterministic times, for fixed \( t, h \geq 0 \).

### 3.4 One-step dynamics

The following intermediary step will be very useful in proving Theorem 3.1 Assume in this section that \( \alpha_2 \geq 1 \).

**Remark 3.6.** Notice that \( \alpha_2 > 1 \) if and only if \( A^{[1]} = 0 \).

**Proposition 3.1.** If \( \alpha_2 \geq 1 \), then \( P_\lambda = \mu^{[1]} e^{A^{[1]} t} M^{[1]} \), \( \forall t \geq 0 \).

Before getting into the proof, let us notice the following flexibility of the notion "the position at time \( t \)."

**Remark 3.7.** As in [3.6], we will actually prove a slightly stronger statement: for any \( \lambda \in \mathbb{R} \), \( t \geq 0 \), \( \omega \in \Omega \) and \( \delta' \) satisfying \( 0 \leq \alpha_1 < \delta' < 1 \leq \alpha_2 \):

\[
P_\lambda(\omega, R) = \lim_{\lambda \to t} \mathbb{P}_\omega^\lambda \left( X^\lambda_{T^\lambda_t \pm 1/\lambda \delta'} \in R \right) = \mu^{[1]} e^{A^{[1]} t}(\omega, R).
\]

**Proof of Proposition 3.1**

Let \( \delta \in (\alpha_1, 1), t, h > 0 \) and \( R, R' \in \mathbb{R}^{[1]} \) be fixed. The idea of the proof is similar to that of Proposition 2.2. One needs, however, to consider two more deterministic times, and take into account periodicity issues. Introduce some notation.

**Notation:** For any \( h > 0 \), define four deterministic times (see Figure 2):

\[
T^\lambda_0 := \frac{t}{\lambda}, \quad T^\lambda_\delta := \frac{t}{\lambda} + \frac{1}{\lambda}, \quad T^\lambda_{h-\delta} := \frac{t + h}{\lambda} - \frac{1}{\lambda}, \quad T^\lambda_h := \frac{t + h}{\lambda}
\]

For any \( k \in \mathbb{N} \) and \( \alpha, \beta \in \{0, \delta, h-\delta, h\} \), denote by \( F_{\alpha, \beta}(k) \) the event that \( k \) secondary transitions of the Markov chain \( X^\lambda \) occur in the interval \( [T^\lambda_\alpha, T^\lambda_\beta] \). Let \( F_{\alpha, \beta}^k(k^+) := \bigcup_{\ell \geq k} F_{\alpha, \beta}(\ell) \) be the event corresponding to at least \( k \) secondary transitions. For any \( k_1, k_2, k_3 \in \mathbb{N} \), let \( F^\lambda_{0,h}(k_1, k_2, k_3) \) denote the event that \( k_1 \) \( \sim \) \( k_2 \) \( \sim \) \( k_3 \) secondary transitions. On the one hand, conditional to \( X^\lambda_{t/\lambda} \in R \), there is at least one secondary in order to leave the class \( R \):

\[
\{ X^\lambda_{(t+h)/\lambda} \in R' \} = \{ X^\lambda_{(t+h)/\lambda} \in R' \} \cap F^\lambda_{0,h}(1^+).
\]

Moreover, the following disjoint union holds:

\[
F^\lambda_{0,h}(1^+) = F^\lambda_{0,h}(2^+) \cup F^\lambda_{0,h}(1,0,0) \cup F^\lambda_{0,h}(0,1,0) \cup F^\lambda_{0,h}(0,0,1).
\]

**Claim 1.** For any \( \omega \in \Omega \) and \( R \in \mathbb{R}^{[1]} \) one has, as \( \lambda \) and \( h \) tend to 0:

(i) \( \lim_{\lambda \to t} \mathbb{P}_\omega^\lambda(F^\lambda_{0,h}(1^+)) = O(h) \) and \( \lim_{\lambda \to t} \mathbb{P}_\omega^\lambda(F^\lambda_{0,h}(2^+)) = o(h) \).

(ii) \( \mathbb{P}_\omega^\lambda(F^\lambda_{0,h}(1,0,0)) \) and \( \mathbb{P}_\omega^\lambda(F^\lambda_{0,h}(0,0,1)) \) are \( O(\lambda^{\alpha_2-\delta}) \).

(iii) \( \lim_{\lambda \to t} \mathbb{P}_\omega^\lambda(X^\lambda_{T^\lambda_3} \in R) = \mu^{[1]}(\omega, R) \).

**Proof of Claim 1**

(i) Let \( C \geq 0 \) be such that the probability of a secondary transition from any state is smaller than \( C \lambda^{\alpha_2} \). Then, the probability of having no secondary transition in \( [T^\lambda_0, T^\lambda_h] \) satisfies:

\[
\mathbb{P}_\omega^\lambda(F^\lambda_{0,h}(0)) \geq (1 - C \lambda^{\alpha_2})^{h/\lambda} \sim_{\lambda \to t} \exp(-Ch\lambda^{\alpha_2-1}).
\]
Taking the limit yields, due to \( \alpha_2 \geq 1 \), that \( \lim_{\lambda \to 0} \mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) = 1 - O(h) \), as \( h \) tends to 0. But then \( \lim_{\lambda \to 0} \mathbb{P}_\omega^t(F_{0,h}^\lambda(1^+)) = O(h) \) so that:

\[
\lim_{\lambda \to 0} \mathbb{P}_\omega^t(F_{0,h}^\lambda(2^+)) \leq \lim_{\lambda \to 0} \left( \max_{\omega \in \Omega} \mathbb{P}_\omega^t(F_{0,h}^\lambda(1^+)) \right)^2 = o(h), \quad \text{as } h \to 0.
\]

\( (ii) \) Clearly, \( \mathbb{P}_\omega^t(F_{0,h}^\lambda(1,0,0)) = \mathbb{P}_\omega^t(F_{0,h}^\lambda(1)) \leq 1 - \mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) = \mathbb{P}_\omega^t(F_{0,h}^\lambda(1^+)) \). As in (3.10), one has that:

\[
\mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) \geq (1 - C\lambda^{\alpha_2})^{1/\lambda^\delta} \sim_{\lambda \to 0} \exp(-C\lambda^{\alpha_2-\delta}) = 1 - O(\lambda^{\alpha_2-\delta}).
\]

Thus, \( \mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) \leq \mathbb{P}_\omega^t(F_{0,h}^\lambda(1^+)) = O(\lambda^{\alpha_2-\delta}). \) The proof for \( \mathbb{P}_\omega^t(F_{0,h}^\lambda(0,0,1)) \) is similar.

\( (iii) \) If \( \omega \in R \), then \( \mathbb{P}_\omega^t(X_{T_\delta}^\lambda \in R) \geq \mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) = 1 - O(\lambda^{\alpha_2-\delta}) \), where the last equality holds by (ii). Suppose that \( \omega \notin R \). By (ii), \( \lim_{\lambda \to 0} \mathbb{P}_\omega^t(F_{0,h}^\lambda(0)) = 1 \), so that:

\[
\lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{T_\delta}^\lambda \in R) = \lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{T_\delta}^\lambda \in R | F_{0,h}^\lambda(0)) = \mu^[[1]](\omega, R).
\]

The main consequences of Claim [1] are that, combined with [3.9], it yields, as \( h \) tends to 0:

\[
\begin{align*}
\lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R') &= \lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R' \cap F_{0,h}^\lambda(0,1,0)) + o(h), \\
(3.11) \\
\lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R) &= 1 - \sum_{R' \neq R} \lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R') + o(h).
\end{align*}
\]

We will need the following coupling result which implies that, up to an error which vanishes with \( \lambda \), the distribution at stage \( T_\delta^\lambda \) is \( \pi[1,R] \).

**Claim 2.** If \( R \) is aperiodic then, conditional to \( \{X_{t/\lambda}^\lambda \in \omega \in R\} \) and \( F_{0,h}^\lambda(0) \), the distance in total variation between the distribution of \( X_{T_\delta}^\lambda \) and \( \pi[1,R] \) is \( O(\lambda^\gamma) \) as \( \lambda \) tends to 0.

**Proof of Claim 2.** Let \( P_{[1]}^\lambda \) and \( \tilde{P}_{[1]}^\lambda \) be the restrictions of \( P_{[1]}^\lambda \) and \( \tilde{P}_{[1]}^\lambda \) to \( R \) respectively. Let \( S_{[1]}^\lambda \) be a diagonal matrix such that \( S_{[1]}^\lambda(\omega, \omega) = \frac{1}{\lambda^\alpha} P_{[1]}^\lambda(\omega, \omega)^\alpha \), for all \( \omega \in R \). It does not depend on \( \lambda \) and that, by Gershgorin Circle Theorem, all its eigenvalues have nonnegative real part. By construction \( \text{Id} - P_{[1]}^\lambda = \lambda^{\alpha_1} S_{[1]}(\text{Id} - \tilde{P}_{[1]}^\lambda) \). Thus, \( \rho \) is an eigenvalue of \( S_{[1]}(\text{Id} - \tilde{P}_{[1]}^\lambda) \) if and only if \( 1 - \rho \lambda^{\alpha_1} \) is an eigenvalue of \( P_{[1]}^\lambda \). By aperiodicity, \( 1 \) is a simple eigenvalue of \( P_{[1]}^\lambda \), so that the second largest eigenvalue is \( 1 - \rho \lambda^{\alpha_1} \) for some eigenvalue of \( S_{[1]}(\text{Id} - \tilde{P}_{[1]}^\lambda) \), \( \rho \neq 0 \). By Perron-Frobenius Theorem, the distance in total variation between the two distributions is thus of order \( |1 - \rho \lambda^{\alpha_1}|^{\lambda^{-\delta}} \sim_{\lambda \to 0} \exp(-\eta \lambda^{\alpha_1-\delta}) \), which is \( O(\lambda^\gamma) \) for any \( \varepsilon > 0 \) by the choice of \( \delta \).

\( \square \)

**Claim 3.** For any \( t \geq 0 \), \( h > 0 \) and \( \omega \in R \) we have, as \( h \) tends to 0:

\( (i) \) \( \lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R') = A[1](R, R')h + o(h); \)

\( (ii) \) \( \lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R) = 1 + A[1](R, R)h + o(h). \)

**Proof of Claim 3.** Assume first that \( R \) is aperiodic. Thanks Claim [1]-(ii) and Claim [2] we can define some auxiliary random variable \( \bar{X}_\delta^\lambda \) distributed as \( \pi_{[1,R]} \) and such that \( \mathbb{P}(X_{T_\delta}^\lambda \neq X_{T_\delta}^\lambda) = O(\lambda^{\alpha_2-\delta}) \) as \( \lambda \) tends to 0. Thus, up to an error which vanishes with \( \lambda \), the distribution at stage \( T_\delta^\lambda \) is \( \pi_{[1,R]} \). Combining this coupling result with (3.11) yields, as \( h \) tends to 0:

\[
\lim_{\lambda \to 0} \mathbb{P}_\omega^t(X_{(t+h)/\lambda}^\lambda \in R') = \lim_{\lambda \to 0} \mathbb{P}_{\pi_{[1,R]}}^t(X_{(t+h)/\lambda}^\lambda \in R', F_{0,h}(0,1,0)) + o(h). \quad (3.13)
\]
To compute the right-hand-side of (3.13), consider the following disjoint union:

$$F^λ_{0,h}(0, 1, 0) = \bigcup_{m = T^λ_{h-δ}}^{T^λ_{h-δ}} F^λ_{0,h}(m, \omega'),$$

where $F^λ_{0,h}(m, \omega')$ is the event of a secondary transition occurring at stage $m$, and not before nor after, to a state $\omega' \notin R$. Notice that, by the choice of $\pi^{[1],R}$ and Claim 1 for any $m \in [T^λ_{h-δ}, T^λ_{h}]$:

$$\mathbb{P}_ω(F^λ_{0,h}(m, \omega')) = (1 - O(h))^2 \sum_{\omega \in R} \pi^{[1],R}(\omega, \omega') = \sum_{\omega \in R} \pi^{[1],R}_\lambda(\omega, \omega') + o(h).$$

On the other hand, by Claim 1-(iii), for any $m \leq T^λ_{h-δ}$:

$$\lim_{λ \to 0} \mathbb{P}_ω\left(X^λ_{(t+h)/\lambda} \in R' \mid F^λ_{0,h}(m, \omega')\right) = \mu^{[1]}(\omega', R'). \quad (3.14)$$

Consequently, as $h$ tends to 0:

$$\lim_{λ \to 0} \mathbb{P}_{\pi^{[1],R}}\left(X^λ_{T^λ_{h}} \in R', F^λ_{0,h}(0, 1, 0)\right) = \lim_{λ \to 0} \sum_{m = T^λ_{h-δ}}^{T^λ_{h-δ}} \mathbb{P}_{\pi^{[1],R}}\left(X^λ_{T^λ_{h}} \in R', F^λ_{0,h}(m, \omega')\right) \quad \text{for any } m \leq T^λ_{h-δ},$$

which gives (i). Finally, (ii) is now a consequence of the (3.12). The case where $R$ is periodic, needs minor changes. Note that periodicity can only happen if $α_1 = 0$ for otherwise with positive probability the chain does not change of state. Now, $R$ is then a disjoint union of $R^1 \cup \cdots \cup R^d$, and the restriction of $P^{[1]}_λ$ to these sets is aperiodic, with invariant measure $π^{[1],R}_λ (k = 1, \ldots, d)$. One needs to take into account the subclass at time $T^λ_{h}$ and define, in the aperiodic case, some auxiliary random variable $\tilde{X}^λ_{\delta,k}$ distributed as $π^{[1],R}_λ$ and such that $\mathbb{P}(\tilde{X}^λ_{\delta,k} \neq X^λ_{T^λ_{h}}) = O(λ^{1-δ})$ as $λ$ tends to 0. The results then follows from the fact that $π^{[1],R}_λ = \frac{1}{d} \sum_{k=1}^{d} π^{[1],R}_λ$ and that, under the initial probability $π^{[1],R}_λ$, the distribution at stage $m$ is $π^{[1],R}_λ$, which is a shortcut for $π^{[1],R}_λ (\text{mod } d)$. The computation is now, for some $k$:

$$\lim_{λ \to 0} \mathbb{P}_{\pi^{[1],R}}\left(X^λ_{T^λ_{h}} \in R', F^λ_{0,h}(0, 1, 0)\right) = \lim_{λ \to 0} \sum_{m = T^λ_{h-δ}}^{T^λ_{h-δ}} \sum_{\omega \notin R} \mathbb{P}_{\pi^{[1],R}}\left(X^λ_{T^λ_{h}} \in R', F^λ_{0,h}(m, \omega')\right) \quad \text{for any } m \leq T^λ_{h-δ},$$

$$= \lim_{λ \to 0} \frac{h}{λ} \frac{1}{d} \sum_{k=1}^{d} \left( \sum_{\omega \in R, \omega' \notin R} π^{[1],R}_λ(\omega, \omega') μ^{[1]}(\omega', R') + o(h) \right),$$

$$= h \lim_{λ \to 0} \left( \frac{1}{λ} \sum_{\omega \in R, \omega' \notin R} π^{[1],R}_λ(\omega, \omega') μ^{[1]}(\omega', R') + o(h) \right),$$

$$= h A^{[1]}(R, R') + o(h),$$

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which proves the Claim. \hfill \Box

Let us go back to the proof of Proposition 3.1. Let \( R = R^1 \cup \ldots \cup R^d \) be a recurrence class of period \( d \geq 1 \). Consider four deterministic times as in Figure 2, with \( h > 0 \) and \( t = 0 \), i.e.

\[
T_0^\lambda := 1, \quad T_1^\lambda := \frac{1}{\lambda}, \quad T_{h-\delta}^\lambda := \frac{h}{\lambda} - \frac{1}{\lambda^*}, \quad T_h^\lambda := \frac{h}{\lambda}.
\]

For any \( m \in \mathbb{N} \), let \( T_h^\lambda + m = \lfloor h/\lambda \rfloor + m \). On the one hand, from Claim 3 (see Remark 3.1) one deduces that for any \( R' \in R^{[1]} \) and \( \delta' \) satisfying \( 1 > \delta' > \alpha_1 \):

\[
\lim_{\lambda \to 0} \mathbb{P}_\omega^\lambda \left( X_{T_{h-\delta}^\lambda}^\lambda \in R \middle| X_{T_h^\lambda}^\lambda \in R' \right) = e^{A^{(1)} h}(R', R),
\]

which, together with Claim 1, \( (iii) \), yields:

\[
\lim_{\lambda \to 0} \mathbb{P}_\omega^\lambda \left( X_{T_{h-\delta}^\lambda}^\lambda \in R \right) = \sum_{R' \in R^{[1]}} \mu^{[1]}(\omega, R') e^{A^{(1)} h}(R', R). \tag{3.15}
\]

By periodicity, if \( X_{T_{h-\delta}^\lambda}^\lambda \in R' \subset R \), then \( X_{T_{h-\delta}^\lambda}^\lambda \in R'^{[1/\lambda']} \mod d \). Consequently, for any \( \lambda > 0 \), \( r = 1, \ldots, d \) and \( D \in \mathbb{N}^* \):

\[
\sum_{m=1}^{Dd} 1_{\{X_{T_{h-\delta}^\lambda}^\lambda + m \in R^{[k]} \}} = D. \tag{3.16}
\]

Thus, by Perron-Frobenius Theorem, for any \( \omega' \in R^{[k]} \subset R \), and \( k = 1, \ldots, d \):

\[
\lim_{\lambda \to 0} \frac{1}{Dd} \sum_{r=1}^{Dd} \mathbb{P}_\omega^\lambda \left( X_{T_{h-\delta}^\lambda}^\lambda = \omega' \middle| X_{T_{h}^\lambda}^\lambda \in R \right) = \frac{D}{Dd} \pi^{[1],R}(\omega') = \pi^{[1],R}(\omega'), \tag{3.17}
\]

using the fact that \( \pi^{[1],R}(\omega') = 0 \), for all \( r' \neq r \). Combining (3.15) and (3.17) one has, thanks to the definition of \( N \), that for any \( \omega \in R \in R^{[1]} \):

\[
\lim_{\lambda \to 0} \frac{1}{N} \sum_{r=1}^{N} \mathbb{P}_\omega^\lambda \left( X_{T_{h-\delta}^\lambda}^\lambda = \omega' \right) = \sum_{R' \in R^{[1]}} \mu^{[1]}(\omega, R') e^{A^{(1)} h}(R', R) \pi^{[1],R}(\omega'), \tag{3.18}
\]

\[
= \mu^{[1]} e^{A^{(1)} h} M^{[1]}(\omega, \omega'), \tag{3.19}
\]

which proves Proposition 3.1. \hfill \Box

### 3.5 Proof of Theorem 3.1 and Algorithm

Theorem 3.1 can be proved using the same ideas, inductively. The first step is precisely Proposition 3.1. If \( Q^{[1]} \) is not critical, proceed by steps. The aggregation of states is illustrated in Figure 3.

#### Initialisation (Step 0).
Let \( \Omega^{-1} := \emptyset \), \( R^{[0]} = \Omega^{[0]} = \Omega \) and \( T^{[0]} := \emptyset \). Let \( Q^{[0]} := Q^{[1]} \), \( \pi^{[0],\omega} = \delta^{[\omega]} \), for any \( \omega \in \Omega \). The coefficients and exponents \( c^{[0]} \), \( \mu^{[0]} \), \( c^{[0],\omega} \), \( c^{[0],\omega'} \), \( c^{[0],\omega,\omega'} \) \( (\omega, \omega' \in \Omega) \) are deduced from the definitions of \( Q^{[0]} \) and \( \pi^{[0],\omega} \).

#### Induction (Step k, \( k \geq 1 \)).
The following quantities have already been defined, or computed, for \( \ell = 0, \ldots, k-1 \): \( R^{[\ell]}, T^{[\ell]}, \Omega^{[\ell]}, P^{[\ell]}, Q^{[\ell]} \), \( \mu^{[\ell]}(u, R), (c^{[\ell],v}, c^{[\ell],v'}) \), \( c^{[\ell],R} \), \( c^{[\ell],R} \), \( R \in R^{[\ell]} \). Define \( \alpha_k \) as follows:

\[
\alpha_k := \min \left\{ e^{[k-2],v} + e^{[k-1],R} \middle| u \in \Omega^{[k-2]}, u \in R \in R^{[k-1]}, v \in \Omega^{[k-2]} \setminus R \right\}. \tag{3.20}
\]
Figure 3: Example of aggregation of states, for $k = 3$. Since we only aggregate recurrence classes, we may deduce from the diagram that $\omega_1^{[1]}, \omega_1^{[1]} \in \mathcal{R}^{[1]}, \omega_0^{[2]}, \omega_1^{[2]} \in \mathcal{R}^{[2]}$ and $\omega_1^{[3]} \in \mathcal{R}^{[3]}$. Recurrent states are indicated with a thicker border ($\mathcal{R}^{[0]} = \Omega^{[0]} = \Omega$ by definition). The diagram does not tell whether any of the other states are recurrent or transient, in their corresponding state spaces.

Note that this definition coincides with $\alpha_1$ and $\alpha_2$ (defined in Section 3.2) for $k = 1, 2$.

Define a stochastic matrix by setting $P^k_\lambda(\omega, \omega') := Q_\lambda(\omega, \omega') \mathbb{I}_{\{\epsilon_{\omega, \omega'} \leq \alpha_k\}}$ for all $\omega' \neq \omega \in \Omega$.

Compute its recurrence classes $\mathcal{R}^{[k]}$, its invariant measures $\pi^{[k], R}$ and its transients states $\mathcal{T}^{[k]}$, and define $\Omega^{[k]} := \mathcal{R}^{[k]} \cup \mathcal{T}^{[k]}$. As in Corollary 3.3, there exists $e^{[k], R} > 0$ and $e^{[k], R} \in \{\alpha_k - \alpha_i | i = 0, \ldots, k\}$, for all $\omega \in \Omega^{[k-1]}$, $\omega, \omega' \in \mathcal{R}^{[k]}$, such that:

$$
\pi^{[k], R}_\lambda(\omega) \sim_{\lambda \to 0} c^{[k], R}_\omega \lambda^{e^{[k], R} - 1}.
$$

Define the aggregated stochastic matrix $Q^{[k]}_\lambda$ over $\Omega^{[k]}$ by setting:

$$
Q^{[k]}_\lambda(\omega, \omega') :=
\begin{cases}
Q_\lambda(\omega, \omega') & \text{if } \omega, \omega' \in \mathcal{T}^{[k]}, \\
\sum_{z' \in \omega} Q_\lambda(\omega, z') & \text{if } \omega \in \mathcal{T}^{[k]}, \text{ and } \omega' \in \mathcal{R}^{[k]}, \\
\sum_{z \in \omega} \pi^{[k], R}_\lambda(z) Q_\lambda(z, \omega') & \text{if } \omega \in \mathcal{R}^{[k]}, \text{ and } \omega' \in \mathcal{T}^{[k]}, \\
\sum_{z \in \omega, z' \in \omega} \pi^{[k], R}_\lambda(z) Q_\lambda(z, z') & \text{if } \omega, \omega' \in \mathcal{R}^{[k]}.
\end{cases}
$$

(3.21)

Deduce $c^{[k], R}_\omega$ and $c^{[k], R}_{\omega, \omega'}$ from (3.21), for all $\omega, \omega' \in \Omega^{[k]}$. If, for instance, $\omega, \omega' \in \mathcal{R}^{[k]}$:

$$
c^{[k], R}_\omega = \min_{z \in \omega, z' \in \omega} c^{[k-1], R}_{z, z'} e^{[k-1], R},
$$

(3.22)

$$
c^{[k], R}_{\omega, \omega'} = \sum_{z \in \omega, z' \in \omega} c^{[k-1], R}_{z, z'} c^{[k-1], R}_{z'} \mathbb{I}_{\{\epsilon_{z, z'}^{[k-1], R} + \epsilon_{z', z}^{[k-1], R} = e^{[k], R}_{\omega, \omega'}\}}.
$$

(3.23)

If $\alpha_k < 1$, let $k := k + 1$ and go back to Step k.

If $\alpha_k \geq 1$, terminate.

Now, the matrix $Q^{[k]}_\lambda$ is critical, so that the result of Section 2.2 apply. By construction, $\Omega^{[k]}$ is a partition of $\Omega^{[k-1]}$ for any $\ell = 1, \ldots, k$ (see Figure 3). Thus, for any $\omega \in \Omega$, there exists a unique sequence $\omega = \omega^{[0]}, \omega^{[1]}, \ldots, \omega^{[k]}$ such that:

$$
\omega^{[\ell]} \in \Omega^{[\ell]}, \quad \text{and} \quad \omega^{[\ell]} \in \Omega^{[\ell - 1]}, \quad \text{for all } \ell = 0, \ldots, k.
$$

(3.24)

In particular, there exists a unique $\omega^{[k]} \in \Omega^{[k]}$ which contains the initial state $\omega$. Define the entrance distribution $\mu : \Omega \rightarrow \Delta(\mathcal{R}^{[k]})$ (see (3.5)) setting, for each $R \in \mathcal{R}^{[k]}$ and $\omega \in \Omega$:

$$
\mu(\omega, R) = \lim_{n \to \infty} (P^{[k]}_\lambda)^n(\omega, R).
$$
Thus, multiplying (3.27) and the latter over all \( \ell \), we obtain

\[
\mathcal{P}_t = \lim_{\lambda \to 0} \frac{1}{N} \sum_{r=1}^{N} \mathbb{P}_{\lambda}^{\lambda} \left( X_{t/\lambda + r} = \omega' \right) = \mu^{[k]} e^{A^{[k]} t} M^{[k]} (\omega, \omega').
\]

We finish this section by justifying \( \mathcal{P}_t = \mu e^{At} M \).

**Claim 4.** \( \mathcal{P}_t \) exists for all \( t \geq 0 \). For any \( \omega, \omega' \in \Omega \):

\[
\mathcal{P}_t (\omega, \omega') := \lim_{\lambda \to 0} \frac{1}{N} \sum_{r=1}^{N} \mathbb{P}_{\lambda}^{\lambda} \left( X_{t/\lambda + r} = \omega' \right) = \mu^{[k]} e^{A^{[k]} t} M^{[k]} (\omega, \omega').
\]

**Proof.** For any \( 0 < \delta < 1 \), let \( T(\lambda, \delta) := t/\lambda - 1/\lambda^{\delta} \). Let \( \omega' = \omega^{[0]}, \omega^{[1]}, \ldots, \omega^{[k]} \) be a sequence satisfying (3.24). Let \( \delta_\ell (\ell = 1, \ldots, k - 1) \) satisfy:

\[
0 \leq \alpha_1 < \delta_1 < \alpha_2 < \cdots < \alpha_{k-1} < \delta_{k-1} < 1 < \alpha_k.
\]

In particular, \( t/\lambda \gg T(\lambda, \delta_1) \gg \cdots \gg T(\lambda, \delta_{k-1}) \). On the one hand, by Proposition 2.2 (see Remark 3.7), for any \( t > 0 \) and \( R \in \mathcal{R}^{[k]} \):

\[
\lim_{\lambda \to 0} \mathbb{P}_{\lambda}^{\lambda} \left( X_{T(\lambda, \delta_{k-1})} \in R \right) = \mathcal{P}_t (\omega, R).
\]

Thus, multiplying (3.27) and the latter over all \( \ell = 2, \ldots, k \) yields:

\[
\lim_{\lambda \to 0} \mathbb{P}_{\lambda}^{\lambda} \left( X_{T(\lambda, \delta_{1})} = \omega^{[1]} \right) = \mu^{[k]} e^{A^{[k]} t} (\omega^{[k]}) \left( \lim_{\lambda \to 0} \prod_{\ell=2}^{k} \pi^{[\ell]}_\lambda (\omega^{[\ell-1]}) \right).
\]

To avoid the periodicity issues of the first order transitions, it is enough to consider the times \( t/\lambda + 1, \ldots, t/\lambda + N \). One obtains, exactly in the same way as in (3.16) and (3.17):

\[
\lim_{\lambda \to 0} \frac{1}{N} \sum_{r=1}^{N} \mathbb{P}_{\lambda}^{\lambda} \left( X_{t/\lambda + r} = \omega^{[0]} \mid X_{T(\lambda, \delta_1)} = \omega^{[1]} \right) = \pi^{[1]}_\lambda (\omega^{[0]}),
\]

for each \( \omega^{[0]} \in \Omega \) and \( \omega^{[1]} \in \mathcal{R}^{[1]} \), which concludes the proof. \( \square \)

### 3.6 Relaxed Assumption 1

Though quite natural, Assumption 1 can be relaxed by noticing that \( \mu, A \) and \( M \), and consequently, \( \mathcal{P}_t \), depend only on the relative speed of convergence of the mappings \( \lambda \mapsto Q_\lambda (\omega, \omega') \), for \( \omega \neq \omega' \) and \( \lambda \mapsto \lambda \), and of some products between them. Define:

\[
\mathcal{F}_Q := \{ \lambda \mapsto \lambda, (\lambda \mapsto Q_\lambda (\omega, \omega')), \omega \neq \omega \in \Omega \}.
\]
One can then replace Assumption 1 by:

**Assumption 1':** For any \(A, B \subset F_Q\), the \(\lim_{\lambda \to 0} \prod_{a \in A} a(\lambda)\) exists in \([0, \infty]\).

To perform the algorithm described in Section 3.5 it is enough to use [3, Proposition 2], which implies that if \((Q_\lambda)\) satisfies Assumption 1', there exists coefficients and exponents \((c_a, e_a)_{a \in F_Q}\) such that:

\[
\lim_{\lambda \to 0} \prod_{a \in A} a(\lambda) = \lim_{\lambda \to 0} \prod_{b \in B} b(\lambda), \quad \forall A, B \subset F_Q.
\] (3.30)

The coefficient and exponent corresponding to \(\lambda \mapsto \lambda\) are, of course, equal to 1.

### 3.7 Illustration of the algorithm

Suppose that \(Q_\lambda\) is a stochastic matrix over \(\Omega = \{1, \ldots, 8\}\) satisfying Assumption 1, such that, for some \(a, b, c, d, e, f, g, h > 0\):

\[
Q_\lambda \sim_{\lambda \to 0} \begin{pmatrix}
1 & a_1^\lambda/5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & b_2^\lambda/5 & f_1^\lambda/5 & 0 & 0 & 0 & 0 \\
c_3^\lambda/5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & g_4^\lambda & 0 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & d_5^\lambda/5 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

See Figure 4 for an illustration. For simplicity, let us fix an initial, say 5, and compute \(P_t(5, k) = \lim_{\lambda \to 0} Q^{t/\lambda}_\lambda(5, k)\), for any \(t > 0\) and \(k \in \Omega\). We use the definition (3.20) to compute \(0 \leq \alpha_1 \leq \cdots \leq \alpha_k\).

**Step 1.** \(\alpha_1 = 0\).

\(\mathcal{R}^{[1]} = \{1, 2, 3, 4, 6, u\}\) and \(\mathcal{T}^{[1]} = \{5\}\),

where \(u := \{7, 8\}\) is a 2-periodic recurrence class. Computes the (nontrivial) entrance law \(P^{[1]}(5, 1) = P^{[1]}(5, 6) = P^{[1]}(5, u) = 1/3\) and the invariant measure \(\pi^{[1]} = \frac{1}{3} \delta_7 + \frac{1}{3} \delta_8\). Since there are non-trivial recurrence classes, one defines the aggregated matrix \(Q^{[1]}_\lambda\).

**Step 2.** \(\alpha_2 = 1/5\). The transitions of order \(\alpha_2\) are \(1 \mapsto 2\) and \(6 \mapsto 4\).

\(\mathcal{R}^{[2]} = \{2, 3, 4, u\}\) and \(\mathcal{T}^{[2]} = \{1, 5, 6\}\).
Step 3. $\alpha_3 = 2/5$. The only transition of order $\alpha_3$ is $2 \rightarrow 3$. 

$$\mathcal{R}^[2] = \{3, 4, u\} \text{ and } \mathcal{T}^[2] = \{1, 2, 5, 6\}.$$ 

Step 4. $\alpha_4 = 3/5$. The only transition of order $\alpha_4$ is $3 \rightarrow 1$. The subset $v := \{1, 2, 3\}$ is now a recurrence class 

$$\mathcal{R}^[4] = \{v, 4, u\} \text{ and } \mathcal{T}^[4] = \{5, 6\}.$$ 

Compute the invariant measure $\pi^{[4],v}_\lambda$. Clearly, $\pi^{[4],v}_\lambda = \frac{1}{3} \delta_1 + \frac{1}{3} \delta_2 + \frac{1}{3} \delta_3$, for it is a cycle. By Corollary 3.5

$$\pi^{[4],v}_\lambda(1) \sim \lambda^{1/5} \frac{1/3}{a^{1/5}} \frac{1/3}{b^{1/5}} + \frac{1/3}{c^{1/5}} \sim \frac{c}{a} \lambda^{2/5},$$

and similarly state 2 and 3, so that $\pi^{[4],v}_\lambda \sim_{\lambda \to 0} \frac{c}{a} \lambda^{2/5} \delta_1 + \frac{c}{b} \lambda^{2/5} \delta_2 + \frac{c}{c} \lambda^{2/5} \delta_3$. Since there are non-trivial recurrence classes, one defines the aggregated matrix $Q^{[4]}_\lambda$ (see Figure 5).

Step 5. $\alpha_5 = 1$. Terminate. Compute the infinitesimal generator over $\mathcal{R}^[4]$: 

$$A = \begin{pmatrix} -\left(\frac{2}{a}e + \frac{2}{b}f\right) & \frac{2}{a}e + \frac{2}{b}f & 0 \\ \frac{1}{1/2} & -\frac{2}{b}g & \frac{2}{a}g \\ 0 & -\frac{2}{a}g & 0 \end{pmatrix}.$$ 

On the other hand, the entrance measure is $P = \frac{1}{3} \delta_v + \frac{1}{3} \delta_4 + \frac{1}{3} \delta_u$. Finally:

$$M(v, \cdot) = \lim_{\lambda \to 0} \pi^{[4],v}_\lambda = \delta_3, \ M(4, \cdot) = \delta_4 \text{ and } M(u, \cdot) = \lim_{\lambda \to 0} \pi^{[1],u}_\lambda = \frac{1}{2} \delta_7 + \frac{1}{2} \delta_8.$$ 

The periodicity yields $N = 2$, so that for any $k \in \Omega$:

$$P_t(5, k) = \lim_{\lambda \to 0} \frac{1}{2} \sum_{r=1}^2 \mathbb{P}^\lambda(X^\lambda_{t+r} = k) = \mu e^{At} M(5, k).$$

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