Large deviation estimates for a Non-Markovian Lévy generator of big order

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Abstract. We give large deviation estimates for a non-markovian convolution semi-group with a non-local generator of Lévy type of big order and with the standard normalisation of semi-classical analysis. No stochastic process is associated to this semi-group.

1. Introduction
There are much more semi-groups than semi-groups which are represented by stochastic processes. On the other hand, there are a lot of formulas in stochastic analysis which are natural. The theory of pseudodifferential operators [1–3] allow to understand a lot of partial differential equations, including parabolic equations. On the other hand we have imported in the theory of non-markovian semi-groups a lot of tools of stochastic analysis [4–17]. Stochastic analysis formulas are valid for the whole process. Their interpretation for non-markovian semi-groups work only for the semi-group.

In [16] and [17], we have done with the classical normalization of semi-classical analysis [18] Wentzel-Freidlin estimates [19] for four order differential operators. Here we extend the method of [16] to the case of an integro-differential operator of big order which generates a non-markovian convolution semi-group. Normalisation are of Maslov type [18].

2. Statement of the theorems
Let $C_0^\infty(R)$ the set of smooth functions on $R$ with bounded derivatives at each order endowed with its natural topology. $C_0(R)$ is the space of bounded continuous functions endowed with the uniform norm. $L^2$ is the space of square integrable norms for the Lebesgue measure. This is an Hilbert space endowed with its natural scalar product $\langle,\rangle$.

Let $h$ be a smooth positive function on $R$ with compact support such that $h(y) = h(-y)$ and such that $h(y) = 1$ is equal to 1 on $[-\beta, \beta]$.

Let be $\alpha \in [0, 1]$. We introduce the Levy generator acting on $C_0^\infty(R)$:

$$L f(x) = (-1)^{d+1} \int_R (f(x + y) - f(x) - \sum_{i=1}^d \frac{y^{2i}}{2i!} f^{(2i)}(x)) \frac{h(y)}{|y|^{2d+\alpha}} dy$$

(1)

$h(y)/|y|^{2d+\alpha}$ dy is called the Lévy measure.
**Theorem 1:** $L$ is symmetric positive on $L^2$. It has therefore a natural essentially-self adjoint extension which generates a semi-group of contraction $P_t$ on $L^2$.

We consider the Hamiltonian

$$H(\xi) = \int_R \left( \exp[\xi y] - 1 - \sum_{i=1}^{d} \frac{(\xi y)^{2i}}{2i!} \right) \frac{h(y)}{|y|^{2d+a}} dy$$

(2)

**Theorem 2:** $H(\xi)$ is a smooth convex function equals to 1 in 0.

Associate to it, we consider its Legendre transform:

$$L(p) = \sup_{\xi \in R} (\xi p - H(\xi))$$

(3)

If $\phi$ is a finite energy function in $R$, we consider the action functional

$$S(\phi) = \int_0^1 L(\frac{d\phi}{dt}) dt$$

(4)

Let us recall some basis of the pseudodifferential calculus. $\hat{f}$ is the Fourier transform of $f$. Let $L_1$ be an operator acting on $C^\infty_b(R^d)$ by

$$L_1 f(x) = \int_R a(x, \xi) \hat{f}(\xi) \exp[2\pi i \xi x] d\xi$$

(5)

We say that $a(\cdot, \cdot)$ is its symbol. If

$$\left| \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial \xi^m} a(x, \xi) \right| \leq C|\xi|^{r-m}$$

(6)

and if for $|\xi| > C_0$

$$|a(x, \xi)| \geq C|\xi|^r$$

(7)

we say that $L_1$ is an elliptic operator of order $r$. Let us recall that our thesis underline the relationship between pseudodifferential operators and Poisson processes [20].

**Theorem 3:** $L$ is an elliptic pseudodifferential operator.

By elliptic theory, it generates a semi-group on $C^\infty_b(R^d)$.

According the theory of semi-classical analysis [19], we consider the symbol $L_1^\epsilon$ associated to the symbol $\epsilon^{-1}a(x, \epsilon \xi)$. This leads to the operator

$$L^\epsilon f(x) = (-1)^{d+1} \epsilon \int_R (f(x + \epsilon y) - f(x) - \sum_{i=1}^{d} \frac{(\epsilon y)^{2i}}{2i!} f^{(2i)}(x)) \frac{h(y)}{|y|^{2d+a}} dy$$

(8)

By elliptic theory $L^\epsilon$ generates a semi-group on $L^2$ and even on $C^\infty_b(R^d)$. We consider its absolute value $|P_t^\epsilon|$. We have

**Theorem 4 (Wentzel-Freidlin estimates):** Let $O$ be the complement in $R$ of the interval $[x - \delta, x + \delta]$. We have when $\epsilon \to 0$

$$\lim_{\epsilon \to 0} \log |P_t^\epsilon|_{|O|} \leq - \inf_{\phi(x) \in O} S(\phi)$$

(9)

if $d+1$ is even.

The proof is very similar to the proof of [16], the only difference being in the algebraic treatment of Davies method [21].
3. Proofs of Theorem 1, Theorem 2 and Theorem 3

Proof of theorem 1: Let us show that $L$ is symmetric. Let $f$ and $g$ be smooth with compact supports:

$$ (-1)^{d+1} < Lf, g > = \int_{R^2} g(x)((f(x + y) - f(x)) - \sum_{i=1}^{d} \frac{y^{2i} f^{(2i)}(x)}{2i!} \frac{h(y)}{|y|^{2d+\alpha}}) dxdy $$

(10)

The symmetry holds by integrating by parts and since $h(-y) = h(y)$.

Let us show that $L$ is positive. We have if $y > 0$

$$ f(x+y) - f(x) = \sum_{i=1}^{2d-1} \frac{y^{2i} f^{(2i)}(x)}{2i!} + \int_{0<s_1<...<s_{2d}<y} f^{(2d)}(x+s_1)ds_1...ds_{2d} $$

(11)

Due to the parity of $h$, we have only to look at

$$ \int_{R^d} f(x) \int_{0<s_1<...<s_{2d}<y} f^{(2d)}(x+s_1)ds_1...ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy $$

(12)

By integrating by parts, it is equal to:

$$ (-1)^d \int_{R^d} f^{(d)}(x) \int_{0<s_1<...<s_{2d}<y} f^{(d)}(x+s_1)ds_1...ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy $$

(13)

By Cauchy-Schwarz inequality,

$$ \int_{R} f^{(d)}(x) f^{(d)}(x+s_1)dx \leq \int_{R} (f^{(d)}(x))^2 dx $$

(14)

Therefore

$$ (-1)^{d+1} \int_{R^d} f(x) \int_{0<s_1<...<s_{2d}<y} (f^{(2d)}(x+s_1) - f^{(2d)}(x))ds_1...ds_{2d} \frac{h(y)}{|y|^{2d+\alpha}} dxdy \geq 0 $$

(15)

This shows the result. The fact that the operator has a natural self-adjoint extension which is essentially self-adjoint holds by standard results.

Proof of theorem 2: $H(\xi)$ is smooth. We have clearly

$$ H^{(1)}(\xi) = \int_{R} y(y^{2\xi - 1}) \frac{h(y)}{|y|^{2d+\alpha}} dy $$

(16)

$$ H^{(2)}(\xi) = \int_{R} y^2(y^{2\xi - 1} - 1) \frac{h(y)}{|y|^{2d+\alpha}} dy $$

(17)

Due to the fact that $h(y) = h(-y)$, the result holds from the fact by induction on $l$ that

$$ \exp[y \xi] + \exp[-y \xi] - \sum_{i=0}^{l} \frac{\xi^{2i} y^{2i}}{2i!} $$

(18)
is positive convex in $\xi$. □

**Proof of theorem 3:** Let us compute the symbol of $L$.

$$f(x) = C \int_R \hat{f}(\xi) \exp[\sqrt{-1}x\xi]d\xi$$  (19)

Therefore

$$Lf(x) + \int_R \frac{h(y)}{|y|^{2d+\alpha}} dy \int_R (\exp[\sqrt{-1}(x+y)\xi] - \sum_{i=0}^{2d} \frac{(-1)^i y^{2i}}{2i!} \exp[\sqrt{-1}x\xi]) \hat{f}(\xi)d\xi = \int_R \hat{f}(\xi) \exp[\sqrt{-1}x\xi]d\xi \int_R (\exp[\sqrt{-1}y\xi] - 1 - \sum_{i=1}^{d} \frac{(-1)^i y^{2i}}{2i!} \frac{h(y)}{|y|^{2d+\alpha}} dy)$$  (20)

Therefore the symbol is given by

$$a(\xi) = H(\sqrt{-1}\xi)$$  (21)

By putting $y\xi = z$ if $\xi > 0$ we get that

$$a(\xi) = \xi^{(2d-1+\alpha)} \int_R h(\frac{z}{\xi})(\cos[z] - \sum_{i=0}^{2d} \frac{(-1)^i z^{2i}}{(2i)!})dz$$  (22)

In (22), we consider a smooth $h_1$ function which is equal to zero near 0 and which is equal to 1 in a neighborhood of the infinity and which takes its values in $[0, 1]$ and we write

$$a(\xi) = a_1(\xi) + a_2(\xi)$$  (23)

$$a_1(\xi) = \xi^{(2d-1+\alpha)} \int_R h_1(\frac{z}{\xi})h(\frac{z}{\xi})(\cos[z] - \sum_{i=0}^{2d} \frac{(-1)^i z^{2i}}{(2i)!})dz$$  (24)

By integrating by parts successively, $|a_1(\xi)| \leq C\xi^{-n}$ for all $n$. On the other if the support of $1 - h_1$ is small enough, we have

$$|a_2(\xi)| \geq C\xi^{(2d-1+\alpha)}$$  (25)

for some positive $C$. The result arise by symmetry for $\xi < 0$. □

**4. Proof of the Wentzel-Freidlin estimates**

Let us begin by some elementary remarks. We remark that

$$\hat{L} \hat{f} = H(\sqrt{-1}) \hat{f}$$  (26)

such that

$$\hat{P}_t f = \exp[-tH(\sqrt{-1})] \hat{f}$$  (27)

These elementary remarks (which are true a lot of convolution semi-groups) will allow us to adapt the proof of [16].

**Lemma 5:** For all $\delta > 0$, all $C$ there exist $t_\delta$ such that if $t < t_\delta$

$$|P_t^\delta|_{[1_{|x-\delta, x+\delta\rangle}])(x) \leq \exp[-\frac{C}{\epsilon}]$$  (28)

**Proof:** We consider the semi-group

$$\exp[-\frac{xe}{\epsilon}] P_t^\delta[\exp[-\frac{xe}{\epsilon}]f(x')]\hat{f}(x)$$  (29)
The symbol of its generator is

$$F^\epsilon_\xi(\xi') = \frac{1}{\epsilon} H(\sqrt{-1}\xi' + \xi) \quad (30)$$

This is the symbol of an elliptic operator which is positive if $|\xi'|$ is big. It generates therefore a semi-group on $C_b(R)$ $Q^\epsilon_\xi$. We get the expansion

$$F^\epsilon_\xi(\xi') = \frac{H(\xi)}{\epsilon} + H^{(1)}(\xi)\sqrt{-1}\xi' + \epsilon \int_{0<s_1<s_2<1} (\xi')^2 H^{(2)}(\epsilon s_1\sqrt{-1}\xi' + \xi)ds_1ds_2 + \frac{H(\xi)}{\epsilon} + H^{(1)}(\sqrt{-1}\xi' + R^\epsilon_\xi(\xi')) \quad (31)$$

Therefore We get

$$Q^\epsilon_\xi f = \exp[-t H(\xi)/\epsilon] \exp[-t (H^{(1)}(\sqrt{-1}\xi' + R^\epsilon_\xi(\xi')))] f \quad (32)$$

The uniform norm of $\exp[-t (H^{(1)}(\sqrt{-1}\xi' + R^\epsilon_\xi(\xi')))$ is bounded and the uniform norm of its derivative is bounded by $\exp[C|\xi|]/\epsilon$. Therefore the norm on $C_b(R)$ of $Q^\epsilon_\xi$ is bounded by $\exp[-CtH(\xi)/\epsilon] \exp[C|\xi|]$. Therefore

$$|P^\epsilon_t[|1_{[x-\delta,x+\delta]}\dot{c}](x)| \leq \exp[-CtH(\xi)/\epsilon] \exp[\delta|\epsilon|] \exp[C|\xi|] \quad (33)$$

But $H(\xi) \geq C|\xi|$ if $|\xi| > K(C)$ for all $C$. \[ Remark: \] This inequality where the classical Davies gauge transform plays a fundamental role [21] replace the role of exponential martingales of [19].

When we have proved this lemma, the estimates follow closely the lines of [16] and [19].

We cut the time interval $[0, 1]$ is small intervals of length $[t_i, t_{i+1}]$. By the semi group property we use that

$$|P^\epsilon_{t_{i+1}}[|1_{[x-\delta,x+\delta]}\dot{c}](x)| \leq |P^\epsilon_{t_{i+1}-t_i}||P^\epsilon_{t_i}||1_{[x-\delta,x+\delta]}\dot{c}](x) \quad (34)$$

In $P^\epsilon_{t_{i+1}-t_i}$, we distinguish if $x_{t_i-1}$ and $x_i$ are far or not. If they are we use the previous lemma. If they are close, we deduce a positive measures $|W_\epsilon|$ on polygonal paths $\phi_t$ which joins $x_i$ to $x_{t_{i+1}}$. By the previous lemma, it remains to estimate $|W_\epsilon||1_{[x-\delta,x+\delta]}\dot{c}](\phi_1)|$. But $|W_\epsilon|$ is a positive measure, we have

$$|W_\epsilon||1_{[x-\delta,x+\delta]}\dot{c}](\phi_1)| \leq |W_\epsilon||\exp[S(\phi)/\epsilon]1_{[x-\delta,x+\delta]}\dot{c}](\phi_1)| \exp[-\sup_{\phi_1 \in [x-\delta,x+\delta]} S(\phi)/\epsilon] \quad (35)$$

Therefore we have only to estimate $|W_\epsilon||\exp[S(\phi)/\epsilon]1_{[x-\delta,x+\delta]}\dot{c}](\phi_1)|$. The sequel follows [19] p 152 ( [16]. We can choose some $p_i$ in finite numberr such that if we put

$$L'(p) = \sup_i (L(p_i) + \partial_p L(p_i)(p - p_i)) \quad (36)$$

we have for all polygonal paths considered for a small $\chi$

$$L(\partial_t \phi_1) - L'(\partial_t \phi_1) \leq \chi \quad (37)$$
Let us put

\[ S^t(\phi) = \int_0^1 L^t(\frac{d\phi}{dt}) dt \]  

(38)

Since \(|W_\epsilon|\) is a positive measure, we have only to estimate the quantity

\[ |W_\epsilon||\exp\left[\frac{S^t(\phi)}{\epsilon}\right]|1_{[x-\delta,x+\delta]}(\phi_1) \]  

(39)

We remark that

\[ \exp[\sup a_i] \leq \sum \exp[a_i] \]  

(40)

Moreover

\[ L'(p) = \sup(\xi, p - H(\xi)) \]  

(41)

where \(\xi_i = \frac{\partial}{\partial p} L(p_i)\). Therefore it is enough to show that

\[ \sup_{x,|\xi|<C} |P_\epsilon \left[\exp[\xi'(x' - x) - t\delta H(\xi)]\right](x) \]  

(42)

has a small blowing up when \(\epsilon \to 0\). We do as in the previous lemma. We consider the generator of the semi group

\[ f \to P_\epsilon \left[\exp[\xi'(x' - x) - tH(\xi)]f\right](x) \]  

(43)

Its symbol is

\[ \frac{1}{\epsilon} H(\epsilon \sqrt{-1} \xi' + \xi) - \frac{1}{\epsilon} H(\xi) \]  

(44)

Its asymptotic expansion in \(\epsilon\) is

\[ (H^{(1)} \sqrt{-1} \xi' + R_\epsilon(\xi')) \]  

(45)

The result follows as in the lemma. ♦

5. Conclusion

We have adapted the standard proof of large deviation estimates of jump processes of [19] (with the standard normalisation of semi-classical analysis [18]) to the case of a non-markov Lévy generator of big order. The main difference with [16] is that the classical gauge transform of Davies [21] induces a simple transformation on the symbol of the Lévy generator [20].

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