Thermodynamic stability criteria for a quantum memory based on stabilizer and subsystem codes

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Abstract. We discuss several thermodynamic criteria that have been introduced to characterize the thermal stability of a self-correcting quantum memory. We first examine the use of symmetry-breaking fields in analyzing the properties of self-correcting quantum memories in the thermodynamic limit; we show that the thermal expectation values of all logical operators vanish for any stabilizer and any subsystem code in any spatial dimension. On the positive side, we generalize the results of Alicki et al to obtain a general upper bound on the relaxation rate of a quantum memory at nonzero temperature, assuming that the quantum memory interacts via a Markovian master equation with a thermal bath. This upper bound is applicable to quantum memories based on either stabilizer or subsystem codes.
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### 1. Introduction

Thermal fluctuations pose a serious problem for reliable, passive, information storage since any open system eventually reaches a thermal equilibrium state in which all encoded information is lost. Fortunately, it has been shown that quantum information can be reliably stored for arbitrary long times in, say, a two-dimensional (2D) quantum memory \[1\] by means of active error correction and entropy removal. However, the implementation of active error correction implying extensive and fast classical input/output to the quantum memory poses a serious (but hopefully not insurmountable) experimental challenge.

The central idea behind self-correcting classical or quantum memories is to do without active error-correction and prevent thermalization and build-up of entropy by the presence of macroscopic ‘energy barriers’ separating encoded states.

The idea of such self-correcting quantum memory was first introduced in \[1\] and can be viewed as an extension of the ideas of topological protection developed by Kitaev \[2\]. In \[1\], it was argued that the 2D surface or toric code (2D Kitaev model) would not be a self-correcting memory, but a 4D surface code (4D Kitaev model) generalization was presented which would be thermally stable. Unfortunately, three spatial dimensions is all the room that the natural world seems to provide.

It is thus of interest to (1) either come up with models for self-correcting quantum memories in three or fewer dimensions, or (2) show that low-dimensional quantum physics does not allow for such passive stability. The latter possibility would provide evidence that genuine quantum phases of nature, such as topological phases, would be confined to the domain of finite systems and low temperatures: in the thermodynamic limit thermal fluctuations would destroy the quantum order at any nonzero temperature. Such a no-go possibility would also lend support to the thought that macroscopic quantum states suffer from intrinsic decoherence (see \[3\] for thoughts in this direction). In this sense, we believe that the question of thermal stability of a passive quantum memory is one of fundamental interest.
In fact, the thermal stability question can also be viewed as a question about the nature of the excitations of the quantum memory model. For 2D topological models, these excitations are point-like pairs of anyons. If we paraphrase the macroscopic energy barrier requirement of [4] in terms of the nature of excitations, it relates to a condition that the elementary excitations are extended objects; they are the boundary of a two- or higher-dimensional surface.

In [5] the subject of self-correcting quantum memories was brought to the fore. Bacon introduced two models, now called the 2D Bacon–Shor code or quantum compass model and the 3D Bacon–Shor code, which are examples of quantum subsystem codes. The 3D Bacon–Shor model may or may not be an example of a self-correcting quantum memory; it is an open question as to how to analyze its thermal stability.

The analysis of the thermal stability of a quantum stabilizer or quantum subsystem code model in a thermodynamic sense is the subject of this paper. Let us discuss some of the literature on this subject.

Necessary criteria for thermal stability of a quantum memory were formulated in [4] (see also [6]) in terms of a macroscopic distance of the underlying quantum code (i.e. zero temperature topological order) and the presence of macroscopic energy barriers. It was shown in that paper that all 1D and 2D local stabilizer codes fail to meet these criteria. The advantage of this approach is that it allows for very general no-go results. A disadvantage is that it does not make contact with any operational or thermodynamic expression of thermal stability. In particular, to prove positive results on particular quantum memory models, it is necessary to more thoroughly analyze the thermodynamics of an open quantum memory. The intuition that underlies the idea of a self-correcting quantum memory is that errors of increasing weight should map encoded states onto excited states with increasingly higher energy. If the quantum code has a macroscopic distance which scales with system size, then high-weight errors will have to happen in order to map one encoded state onto another. But such high-weight errors will, if the memory is self-correcting, correspond to high-energy states, hence there would be (macroscopic) energy barriers between different encoded states. In the second part of our paper, section 7 we will indeed see that the energy associated with high-weight errors corresponding to so-called bad syndromes, will play a crucial role in bounding the quantum memory relaxation rate.

Specific results ruling out the existence of finite temperature topological order for, e.g. 2D toric code, were obtained in [7]–[9], using in [8] an interesting finite-temperature extension of the topological entanglement entropy. Remarkably, these limitations can be overcome by including repulsive long-range interactions with bounded strength. Such extensions of the 2D toric code were proposed in [10] and are characterized by a diverging relaxation time in the thermodynamic limit. Since the requirement of a macroscopic energy barrier between logical states [4, 6] is violated in these models, the increase of the lifetime with the system size is only polynomial. However, the scaling power is very sensitive to the physical features of the thermal bath and becomes especially favorable for super-ohmic reservoirs. Such properties needed to be established in [10] by the explicit analysis of the non-equilibrium time evolution, instead of being addressed via a suitable equilibrium quantity as in the present work.

In [9], a thermodynamic criterion was presented for the existence of topological order at finite temperature. There, it was discussed whether the thermal expectation value of logical qubit operators could serve as a stability criterion for a quantum memory against thermal fluctuations. Specifically, following the reasoning used in the discussion of spontaneous symmetry breaking, a small perturbation (external field) is applied to the system, which breaks explicitly the symmetry of the Hamiltonian and the state of the system. Then, the thermodynamic limit is
taken before the external field is taken to zero. If the expectation values of the logical operators vanish in this order of limits, then, according to the argument given in [9], the information in the quantum memory will be lost after a finite, size-independent relaxation time at any finite temperature. This concept was demonstrated explicitly for the Kitaev model in 2D and for some generalizations of it to higher dimensions [9].

In this paper, we will discuss and analyze this criterion. We show that the line of reasoning in [9] leads to zero thermal averages for the logical operators for stabilizer codes as well as subsystem codes in any spatial dimension. We will discuss the root cause of these problems and discuss possible ways to extend the traditional analysis of spontaneous symmetry breaking to detecting a finite temperature quantum order.

The analysis of thermal stability of a quantum memory within the formalism of the thermodynamics of open quantum systems was seriously undertaken in a series of papers by Alicki et al [11]–[13]. In [12], it was demonstrated that for the 2D surface code model weakly coupled to a Markovian environment, the relaxation rate of any logical state is bounded from below by a constant independent of system size [12]. This result implies that increasing the system size does not increase the lifetime (stability) of the memory, but that the relaxation rate is an intrinsic feature of the model. In [13], the authors considered the 4D Kitaev model and explicitly proved that the relaxation times were exponentially increasing with system size, hence confirming the anticipated thermal stability in the thermodynamic limit.

In the second part of our paper (sections 6 and 7) we will present a formal analysis of the thermal stability of a quantum memory based on subsystem (stabilizer) codes [14]. The difficulty for Hamiltonian models based on subsystem codes (see discussions in [4]) is that the Hamiltonian is a sum of non-commuting terms, hence spectral information for such systems is not readily available analytically. Using some of the ideas developed in [9], [11]–[13] we will construct a simple observable whose expectation value on the thermal Gibbs state provides an upper bound on the relaxation rate thus determining how long quantum information can be stored in a given system. Our formalism will be general enough to cover both stabilizer as well as subsystem codes. In addition, we can use this formalism to provide a simple bound on the memory relaxation time of stabilizer of subsystem code which is not self-correcting, but is ‘protected by a gap’. In section 8, we prove that the memory relaxation time scales as $n^{-1} \exp(\beta \Delta)$ where $n$ is the system size, $\beta$ is the inverse temperature and $\Delta$ is the spectral gap of the memory Hamiltonian. For sufficiently small temperature, e.g. logarithmically scaling with the system size, such models may still be of practical interest.

At the end of the paper, we will show that the bound on the relaxation rate only depends on an induced temperature-dependent distribution associated with the Abelian stabilizer group of the subsystem gauge group.

2. Stabilizer and subsystem codes

We assume that the system chosen as the storage medium is represented by an $n$-qubit Hilbert space $\mathcal{H}$. Let $\mathcal{P}_n = \{iI, X_1, Z_1, \ldots, X_n, Z_n\}$ be the Pauli group on $n$ qubits generated by single-qubit Pauli operators and the phase factors $\pm 1, \pm i$. We envision that the quantum data is stored in the degenerate ground states of a quantum Hamiltonian acting on the $n$ physical qubits. The Hamiltonian will be associated with a quantum stabilizer or subsystem code.

A subsystem code is determined by its gauge group $\mathcal{G}$ which can be an arbitrary subgroup of $\mathcal{G} \subseteq \mathcal{P}_n$. The set of Pauli operators $P \in \mathcal{P}_n$ that commute with all elements of $\mathcal{G}$ is called the

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centralizer of $G$ and is denoted as $C(G)$. The Abelian group $S = G \cap C(G)$ is called the stabilizer group of $G$. If $G$ is Abelian, then obviously $G = S$ up to phase factors and we call $S$ a stabilizer code. To preclude $S$ from containing non-trivial phase factors one usually adds a requirement $-I \notin S$ in the case of stabilizer codes. If $G$ is non-Abelian, we refer to $G$ as a subsystem code.

Logical operators of a stabilizer code $S$ are elements of $C(S)$ which are not in $S$. One can always choose a set of logical Pauli operators $\tilde{X}_1, \tilde{Z}_1, \ldots, \tilde{X}_k, \tilde{Z}_k \in C(S) \setminus S$ obeying the usual Pauli commutation relations: $\tilde{X}_i^2 = \tilde{Z}_i^2 = 1$ and $\tilde{X}_i \tilde{Z}_j = (-1)^{\delta_{ij}} \tilde{Z}_j \tilde{X}_i$. Note that $C(S) = \langle S, \tilde{X}_1, \tilde{Z}_1, \ldots, \tilde{X}_k, \tilde{Z}_k \rangle$.

The code space of a stabilizer code is defined as the common $2^k$ dimensional +1 eigenspace of $S$. It can also be viewed as the ground space of a Hamiltonian acting on $n$ qubits:

$$H = \sum_{i=1}^{m} r_i S_i.$$  

Here the $r_i$ are some real negative coupling constants and the operators $S_i$ form an (over)complete set of generators of $S$. Note that the definition of a logical operator is not unique, since we can multiply any logical operator by an element in $S$ which acts trivially on any state in the code space/ground space. Note that the logical operators are symmetry operations of $H$ since they commute with all elements $S_i \in S$ thus each energy level of $H$ has a degeneracy $2^k$. Therefore, the choice of the ground space as coding space, instead of any of the higher energy levels, is somehow arbitrary and other forms of encoding might be more useful. An interesting example is the thermal state encoding which will be described in section 6.1.

Bare logical operators of a subsystem code $G$ are elements of the centralizer $C(G)$ which are not in $G$. One can always choose a set of bare logical Pauli operators $\bar{X}_1, \bar{Z}_1, \ldots, \bar{X}_k, \bar{Z}_k \in C(G) \setminus G$ obeying the usual Pauli commutation relations. Note that $C(G) = \langle S, \bar{X}_1, \bar{Z}_1, \ldots, \bar{X}_k, \bar{Z}_k \rangle$. We can multiply such bare logical operators by elements in $G$ to get so-called dressed logical operators, which act on the gauge qubits, in addition to the logical qubits. With the group $G$ and its local generators $G_i$ we can associate a Hamiltonian

$$H = \sum_{i=1}^{m} r_i G_i,$$  

where $r_i$ are some real coefficients. Since any $G_i$ commutes with all the bare logical operators $(\bar{X}_i, \bar{Z}_i)$, it follows that $H$ commutes with $(\bar{X}_i, \bar{Z}_i)$. In addition, $H$ commutes with all elements in the Abelian stabilizer group $S = G \cap C(G)$ of $G$, hence $H$ is block diagonal in sectors labeled by the quantum numbers (syndromes) of this stabilizer group $S$. Typically, ground states of $H$ are confined to a single syndrome sector.

For simplicity we will assume in the remainder of this paper that a single qubit is encoded in the quantum memory, i.e. $k = 1$.

3. Thermal fragility?

To get started, let us consider the thermal fragility criterion introduced in [9] and apply this to general stabilizer code Hamiltonians, equation (1). As in [9], we introduce $H_h = H - h \cdot S$ where the additional perturbation is a symmetry-breaking field, designed to produce a finite expectation value of the logical operators for the encoded qubit. Here $S = (\bar{X}, \bar{Y}, \bar{Z})$.

For simplicity, we will consider a perturbation $h$ along the $z$-direction (this can always be assumed by a suitable choice of the logical operators), i.e. $h = h \bar{n}_z$ and $h \cdot S = h \bar{Z}$.
We can write the degenerate eigenvectors of $H$ with energy $\epsilon_s$ as $|s, \alpha\rangle$ where $\alpha = \pm 1$ is the eigenvalue of $\bar{Z}$ (the $H$ and the $\bar{Z}$ operator can be diagonalized simultaneously).

Clearly, $H$ only acts on the $s$ quantum number (the error syndrome, see section 4) of the eigenfunctions $|s, \alpha\rangle$, while the logical operators, and in particular the perturbation $\bar{h}\bar{Z}$, only acts on the $\alpha$ quantum numbers. As a consequence, the canonical partition function $Z_h = \text{Tr}(e^{-\beta H})$ at temperature $1/\beta$ factorizes

$$Z_h = \sum_{s,\alpha} e^{-\beta \epsilon_s} \langle s, \alpha | e^{\bar{h}\bar{Z}} | s, \alpha \rangle = \left( \text{Tr} e^{-\beta H} \right) \cosh(\beta \bar{h}),$$

where we used $\sum_{\alpha} \langle s, \alpha | e^{\bar{h}\bar{Z}} | s, \alpha \rangle = 2 \cosh(\beta \bar{h})$, independent of $s$.

As was shown in [9], we immediately see that the average value $\langle \bar{Z} \rangle_h$ is independent of the unperturbed Hamiltonian $H$, and only reflects the finite degeneracy of the energy levels

$$\langle \bar{Z} \rangle_h = \frac{\sum_{s,\alpha} e^{-\beta \epsilon_s} \langle s, \alpha | \bar{Z} e^{\bar{h}\bar{Z}} | s, \alpha \rangle}{\sum_{s,\alpha} e^{-\beta \epsilon_s} \langle s, \alpha | e^{\bar{h}\bar{Z}} | s, \alpha \rangle} = \tanh(\beta \bar{h}).$$

The expectation value in equation (4) evidently goes to zero at small $\bar{h}$. It is clear that equation (4), being independent of the form of $H$, holds also if the unperturbed Hamiltonian refers to a macroscopic system. Therefore, this procedure yields vanishing averages also after the thermodynamic limit is taken. If the Hamiltonian $H$ involves $n$ physical qubits, we obtain

$$\lim_{\bar{h} \to 0} \lim_{n \to \infty} \langle \bar{Z} \rangle_h = 0.$$ (5)

Given that this argument is independent of dimensionality, and thus also holds for the 4D Kitaev model, which is believed to be thermally stable, the result suggests that the symmetry-breaking field used in $H_h$ is not strong enough to bias the thermal state $\exp(-\beta H_h)$ towards having a nonzero logical operator expectation value.

**Proof based on the Bogoliubov inequality.** We consider next an alternative approach based on the Bogoliubov inequality [15, 16] and show that we reach the same conclusion as before. This method can then be applied to subsystem codes (see section 3.1). We start from the well-known Bogoliubov inequality [15, 16]

$$\frac{\beta}{2} \langle [A, A^{+}] \rangle \langle [[C, H], C^{+}] \rangle \geq |\langle [C, A] \rangle|^{2},$$

(6)

where $A$ and $C$ are two arbitrary operators and $H$ is the system Hamiltonian, with the assumption that all expectation values exist. Here, we use the convention $\{A, B\} \equiv AB + BA$. We then set $A = \bar{X}$ and $C = \bar{Y}$. Clearly, $\{\bar{X}, \bar{X}\} = 2$ and the right-hand side of the Bogoliubov inequality (6) gives $4\langle \bar{Z} \rangle_h^2$. Therefore we obtain

$$\beta \langle [[\bar{Y}, H - h\bar{Z}], \bar{Y}] \rangle_h \geq 4\langle \bar{Z} \rangle_h^2,$$

(7)

where the $\bar{Y}$ operator commutes with $H$. Using the commutation relations for the logical operators, we obtain $4\beta \langle \bar{Z} \rangle_h \geq 4\langle \bar{Z} \rangle_h^2 \geq 0$. For $\langle \bar{Z} \rangle_h$ strictly positive (otherwise we are done) we can divide by $\langle \bar{Z} \rangle_h$, and then, by taking the thermodynamic limit on both sides of the resulting inequality, we eventually obtain

$$\lim_{n \to \infty} \langle \bar{Z} \rangle_h \leq \beta \bar{h}.$$ (8)

At any finite temperature, we thus obtain that the thermal expectation value of the logical operator vanishes when $\bar{h} \to 0$. 

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3.1. Subsystem codes

Let us use the Bogoliubov inequality to argue about the thermal fragility criterion for subsystem codes. The bare logical operators of the encoded qubit $S = (\bar{X}, \bar{Y}, \bar{Z})$ commute with all $G_i \in \mathcal{G}$, hence with $H$ in equation (2). We can choose the symmetry-breaking Hamiltonian as
\begin{equation}
H_h = H - h\bar{Z}G,
\end{equation}
for some choice of $G$ which dresses the bare logical operator $\bar{Z}$. Let us thus consider the thermal expectation value of $\langle \bar{Z}G' \rangle_h$ where $G'$ does not need to be the same as $G$.

We use the Bogoliubov inequality, with $A = \bar{X}G'$ and $C = \bar{Y}$. Since $G'$ commutes with $\bar{X}$ and $G'G'^\dagger = 1$ (valid for every Pauli operator), one obtains $[\bar{X}G', G'^\dagger\bar{X}] = 2$. This gives
\begin{equation}
\beta\langle [\bar{Y}, H - h\bar{Z}G], \bar{Y} \rangle_h \geq 4|\langle \bar{Z}G' \rangle_h|^2.
\end{equation}
and since $\bar{Y}$ commutes with $H$ and $G$, we can easily compute the left-hand side, to obtain
\begin{equation}
\beta h|\bar{Z}G \rangle_h \geq |\langle \bar{Z}G' \rangle_h|^2.
\end{equation}
We now note that $\bar{Z}G$ is a Pauli operator and thus has eigenvalues $\pm 1$. Hence, the thermal expectation value on the left side is always less then 1 independently of any details of $H$, which gives
\begin{equation}
\lim_{n \to \infty} |\langle \bar{Z}G' \rangle_h|^2 \leq \beta h,
\end{equation}
for any choice of $G$ and $G'$. Therefore, the same considerations valid for the stabilizer codes can be repeated in this case and we again conclude that the thermal expectation value of any logical operator vanishes at any finite temperature for vanishing field $h$.

4. Error correction

Let us pause for a moment and discuss our somewhat naive-looking approach. It seems that there are at least two issues at stake here. Let us assume that by choosing the right symmetry-breaking field, we are able to concentrate the weight of $\exp(-\beta H_h)$ around a logical $|\bar{0}\rangle$. Consider this $+1$ eigenstate $|\bar{0}\rangle$ of the $\bar{Z}$ logical operator and a state with a single-qubit error, $E|\bar{0}\rangle$, such that $E$ anti-commutes with $\bar{Z}$. Obviously, if at equilibrium the system is in a statistical mixture of $|\bar{0}\rangle$ and $E|\bar{0}\rangle$ with equal probability, one has $\langle \bar{Z} \rangle = 0$. However, the information in the memory is still preserved as long as we correct for errors such as $E$ when we determine what logical state has been stored. For a generic stabilizer code, the probability of the $E|\bar{0}\rangle$ states is small at low temperature (below the gap), but the statistical weight of all correctable errors might be very large in the thermodynamic limit. Therefore, $\langle \bar{Z} \rangle$ does not represent a meaningful stable order parameter for this problem; the value of $\bar{Z}$ has to be modified depending on the error syndrome. For this reason, the authors of [13] consider so-called error-corrected logical operators$^3$. Let us properly define these for stabilizer and subsystem codes.

For stabilizer codes, error correction consists of measuring the $\pm 1$ eigenvalues of the stabilizer generators; these sets of eigenvalues form the error syndrome. The error syndrome is used as input to a classical decoding algorithm which determines which errors have most likely taken place. For subsystem codes, error correction may proceed by measuring the eigenvalues

$^3$ In [13] these are called dressed logical operators, but we prefer to reserve the notion of ‘dressing’ for the multiplication of bare logical operators with elements of the gauge group $\mathcal{G}$. 

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of the local generators $G_i$. Since the operators $G_i$ do not commute, these eigenvalues cannot be simultaneously measured, nonetheless these (random) values of the generators of $\mathcal{G}$ will fix the eigenvalues of the stabilizer group $\mathcal{S}$. These eigenvalues of the stabilizer group $\mathcal{S}$ again form the error syndrome.

More precisely, any error $E \in \mathcal{P}_n$ determines a syndrome $s_E$: $\mathcal{S} \rightarrow \mathbb{Z}_2$ such that

$$E Q = (-1)^{s_E(Q)} Q E,$$

for all $Q \in \mathcal{S}$.

We can assume that there is some deterministic decoding algorithm, which assigns a correcting Pauli operator $C(s) \in \mathcal{P}_n$ to every syndrome $s$. An error $E \in \mathcal{P}_n$ is correctable iff $C(s_E)$ coincides with $E$ up to a gauge operator, that is, $E C(s_E) \in \mathcal{G}$.

We can define a subspace projector $P_s$ associated with every syndrome (quantum number) $s$. Let $P_0$ be the projector onto the $\mathcal{S}$-invariant code space in which $S_i |\psi\rangle = |\psi\rangle$ for all $i = 1, \ldots, p$. (By abuse of notations let us assume from now on that $-I \notin \mathcal{S}$.) For any syndrome $s$ we can define $P_s = E P_0 E^\dagger$ where $E \in \mathcal{P}_n$ is any error with syndrome $s$ (note that the projector $P_s$ does not depend on the choice of such $E$). Clearly $\sum_s P_s = I$. We define an error-correcting transformation for observables on $\mathcal{H}$ as

$$\Phi_{\text{ec}}(O) = \sum_s P_s C(s)^\dagger O C(s) P_s.$$  \hfill (13)

Note that $\Phi_{\text{ec}}(I) = I$, so the adjoint transformation $\Phi_{\text{ec}}^*$ acting on states is a trace-preserving completely positive (TPCP) map. Following [13] we can define the error-corrected logical operators as

$$\tilde{Z}_{\text{ec}} = \Phi_{\text{ec}}(\tilde{Z}), \quad \tilde{X}_{\text{ec}} = \Phi_{\text{ec}}(\tilde{X}).$$  \hfill (14)

for a pair of bare anti-commuting logical operators $(\tilde{X}, \tilde{Z})$. Note that $\tilde{Z}_{\text{ec}}, \tilde{X}_{\text{ec}}$ are not necessarily Pauli operators. However, it is not hard to show that the error-corrected logical operators obey the relations $\tilde{Z}_{\text{ec}}^2 = \tilde{X}_{\text{ec}}^2 = I$ and $\tilde{Z}_{\text{ec}} \tilde{X}_{\text{ec}} = -\tilde{X}_{\text{ec}} \tilde{Z}_{\text{ec}}$. We can understand this by defining coefficients $\lambda_z(s), \lambda_x(s) \in \{+1, -1\}$ such that

$$C(s)^\dagger \tilde{X} C(s) = \lambda_x(s) \tilde{X}, \quad C(s)^\dagger \tilde{Z} C(s) = \lambda_z(s) \tilde{Z}.$$  \hfill (15)

Any syndrome projector $P_s$ belongs to the algebra generated by $\mathcal{S}$ and thus commutes with $\tilde{Z}, \tilde{X}$. It follows that

$$\tilde{Z}_{\text{ec}} = \tilde{Z} D_z, \quad \text{where} \quad D_z = \sum_s \lambda_z(s) P_s,$$

$$\tilde{X}_{\text{ec}} = \tilde{X} D_x, \quad \text{where} \quad D_x = \sum_s \lambda_x(s) P_s.$$  \hfill (16)

The commutation relations for $\tilde{Z}_{\text{ec}}, \tilde{X}_{\text{ec}}$ follow directly from equation (16). Note also that the error-corrected logical operators commute with all elements in $\mathcal{G}$.

We can immediately check whether the use of error-corrected logical operators would change the analysis of the thermal expectation values. As observable, we choose, say, $\tilde{Z}_{\text{ec}} G'$ for some $G'$ whereas for the symmetry-breaking field we choose some $\tilde{Z}_{\text{ec}} G$. Using the properties of $\tilde{Z}_{\text{ec}}$ stated above, we can repeat the proof of the previous subsection to obtain again a vanishing expectation value

$$\lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \langle \tilde{Z}_{\text{ec}} \rangle_h = 0.$$  \hfill (17)
5. Analogy with the 2D Ising model: choice of symmetry-breaking field

We emphasize that the conclusions above are valid for arbitrary dimensions of any stabilizer or subsystem code. Although (or since) the argument is so universal it also appears to be exceedingly oversimplified. In the previous section, we have discussed the necessity to choose a stable logical observable, which includes the process of error correction. Let us now more closely examine the choice for the symmetry-breaking field.

Although the thermal fragility criterion is patterned along the lines of standard symmetry-breaking arguments, it is only so on a formal level. It is instructive to compare the argument of Nussinov and Ortiz [9] with the standard example of spontaneous symmetry-breaking in the 2D Ising model [19] (see e.g. [20]):

\[ H_b = -J \sum_{\langle i,j \rangle} Z_i Z_j - b \sum_{i \in \Lambda} Z_i, \]  

(18)

where \( i, j \) label the 2D sites of the full lattice \( \Lambda \), the first sum is over pairs \( \langle i, j \rangle \) of nearest-neighbor sites, and an external magnetic field \( b \) is included. For the 2D Ising model one obtains at the low temperature

\[ \lim_{b \to 0} \lim_{n \to \infty} \langle Z_j \rangle_b \neq 0, \]  

(19)

at every lattice site \( j \), where the expectation value above is taken with respect to the Hamiltonian (18). This appearance of a symmetry-breaking order should be contrasted with the lack of such order in the 1D Ising model which has \( T_c = 0 \).

Note that, although the 2D Ising model does not display topological order, it does define a proper stabilizer code with logical operators \( \bar{X} = \prod_{i \in \Lambda} X_i \) and \( \bar{Z} = Z_j \), where \( j \) is a fixed (arbitrary) site in the lattice.\(^4\)

The arguments discussed in the previous sections consider a perturbation \( h \bar{Z} = h Z_j \) which leads to \( \langle \bar{Z} \rangle_h = \langle Z_j \rangle_h \to 0 \) and does not show that the value of the \( z \)-polarization is robust. In fact, the field \( h \) only acts on a single site, whereas in the standard case the symmetry-breaking field \( b \) acts on all sites of the lattice \( \Lambda \) simultaneously, see equation (18). The reason for the failure of the stability criterion appears thus to be that the chosen symmetry-breaking perturbation is not extensive. Although for topological memories the support of a logical operator \( \bar{Z} \) (i.e. the number of physical spins on which the operator acts nontrivially) becomes larger with the size of the system, the perturbation \( h \bar{Z} \) is bounded in norm by \( h \) and becomes irrelevant in the thermodynamic limit.

The analogy with the 2D Ising model suggests that the symmetry-breaking field should be chosen as a sum over different incarnations of a logical operator, i.e. we can multiply a logical \( \bar{Z} \) by elements of the stabilizer code \( S \) and obtain an extensive operator. It may be possible to salvage this symmetry-breaking route to getting a quantum order parameter, but of course any construction should ultimately be motivated operationally. This is the reason that we now switch to explicitly deriving a memory relaxation rate.

\(^4\) Of course, the other two expectation values \( \langle \bar{X} \rangle \) and \( \langle \bar{Y} \rangle \) are vanishing in the appropriate thermodynamic limit. This stabilizer code does not provide a good quantum memory since the distance of the code is 1 independent of lattice size.
6. Relaxation rate for general quantum memory Hamiltonians

The goal of this section is provide a criterion for thermal stability for a large class of quantum systems that can be described by subsystem codes [14]. This is a generalization of the work in [13] in which the thermal stability of the 4D Kitaev model was analyzed by considering the dynamics of the quantum memory in contact with a thermal bath.

Let \( \mathcal{H} \) be the Hilbert space describing the system chosen as a storage media and \( \mathcal{A} \) be the algebra of operators acting on \( \mathcal{H} \). The following definition will play an important role in this section.

**Definition 1.** Let \( O \in \mathcal{A} \) be an observable and let \( P \) be a projector onto some subspace of \( \mathcal{H} \) which is invariant under \( O \), that is, \( PO = OP \). We shall say that the observable \( O \) is protected from a set of errors \( E \subset \mathcal{A} \) on a subspace \( P \) iff

\[
[E, O] P = 0, \quad \text{for all } E \in \mathcal{E}.
\]  
(20)
(Here and below we use the notation \( P \) both for a subspace and the corresponding projector.)

Consider as example the case when \( E \) includes all single-qubit Pauli operators. Suppose \( O|\psi\rangle = \lambda|\psi\rangle \) for some \( |\psi\rangle \in P \). Then equation (20) implies that \( OE|\psi\rangle = \lambda E|\psi\rangle \) for all \( E \in \mathcal{E} \), that is, a single-qubit error cannot change the eigenvalue of \( O \) for any eigenvector that belongs to \( P \).

Quantum error-correcting codes provide a systematic way of constructing observables protected from low-weight errors on a code subspace, see below.

Suppose for simplicity that our goal is to encode a single qubit. We shall need a pair of observables \( \tilde{X}, \tilde{Z} \in \mathcal{A} \) obeying the canonical commutation rules of the Pauli operators,

\[
\tilde{X}^2 = I, \quad \tilde{Z}^2 = I, \quad \tilde{X}\tilde{Z} = -\tilde{Z}\tilde{X}.
\]  
(21)

In the following, we shall refer to \( \tilde{X} \) and \( \tilde{Z} \) obeying equation (21) as Pauli-like observables. (Note that Pauli-like observables need not to be single-qubit Pauli operators or tensor products of Pauli operators.)

Assume that the system evolves according to a Markovian master equation

\[
\dot{\rho} = -i[H, \rho] + \mathcal{L}(\rho),
\]  
(22)
where \( \mathcal{L} : \mathcal{A} \rightarrow \mathcal{A} \) is the Lindblad operator defined by

\[
\mathcal{L} = \sum_a \mathcal{L}_a, \quad \mathcal{L}_a(\rho) = S_a \rho S_a^\dagger - \frac{1}{2} \{\rho, S_a^\dagger S_a\}.
\]  
(23)

The operators \( S_a \) will be referred to as quantum jump operators. For any Lindblad operator \( \mathcal{L} \), let \( \mathcal{E}_\mathcal{L} \subset \mathcal{A} \) be the set of all quantum jump operators involved in \( \mathcal{L} \). Integrating equation (22) one arrives at

\[
\rho(t) = \Phi_t(\rho(0)), \quad \Phi_t = \exp(-it[H, \cdot] + t\mathcal{L}).
\]  
(24)

We shall measure the strength of \( \mathcal{L} \) using the norm

\[
\|\mathcal{L}\|_1 = \max_{F \in \mathcal{A}} \|\mathcal{L}(F)\|_1, \quad \text{subject to } \|F\|_1 \leq 1.
\]  
(25)

Here the maximization is over all self-adjoint operators \( F = F^\dagger \) acting on the system Hilbert space and \( \|F\|_1 \) is the trace norm of \( F \), i.e. \( \|F\|_1 = \text{Tr} \sqrt{FF^\dagger} \). Note that \( \|F\|_1 \) is distinct from the spectral norm \( \|F\| \).

The following theorem is the main result of this section.
Theorem 1. Let $\mathcal{L}$ be an arbitrary Lindblad operator with a set of quantum jump operators $\mathcal{E}_L$ such that the Gibbs state $\rho_\beta \sim \exp(-\beta H)$ is the fixed point of $\mathcal{L}$, $\mathcal{L}(\rho_\beta) = 0$. Suppose one can choose Pauli-like observables $\tilde{X}, \tilde{Z} \in \mathcal{A}$ that are protected from the set of errors $\mathcal{E}_L$ on some subspace $P$. Suppose also that $\tilde{X}, \tilde{Z}$ and $P$ commute with the system Hamiltonian $H$. Then there exist TPCP encoding and decoding maps $\Phi_{\text{in}} : \mathbb{L}(\mathbb{C}^2) \to \mathcal{A}$ and $\Phi_{\text{out}} : \mathcal{A} \to \mathbb{L}(\mathbb{C}^2)$ such that

$$\| \Phi_t \circ \Phi_{\text{in}}(\eta) - \Phi_{\text{in}}(\eta) \|_1 \leq 8t \| \mathcal{L} \|_1 \text{Tr}(I - P)\rho_\beta$$

and

$$\Phi_{\text{out}} \circ \Phi_{\text{in}}(\eta) = \eta$$

for all one-qubit states $\eta$ and for all $t \geq 0$.

Note that the right-hand side of equation (26) provides an upper bound on the precision up to which the decoded state $\Phi_{\text{out}} \circ \Phi_t \circ \Phi_{\text{in}}(\eta)$ approximates the initial state $\eta$. Thus assuming that the system consists of $n$ qubits and that the norm of the Lindblad operator grows at most as $\text{poly}(n)$ we can store a qubit reliably for a time of order

$$\tau_{\text{qmem}} \sim (\text{poly}(n)\epsilon_{\text{qmem}})^{-1},$$

where

$$\epsilon_{\text{qmem}} = \text{Tr}(I - P)\rho_\beta.$$  

We shall refer to $\tau_{\text{qmem}}$ as the storage time and to the quantity $\epsilon_{\text{qmem}}$ as the relaxation rate. One can envision two scenarios when the bound equation (28) on the storage time can be useful: (i) the relaxation rate $\epsilon_{\text{qmem}}$ is exponentially small as a function of $n$, that is, $\epsilon_{\text{qmem}} \leq \exp(-n^\gamma)$ for some $\gamma > 0$; (ii) the relaxation rate $\epsilon_{\text{qmem}}$ is only polynomially small but the degree is sufficiently large, such that $\tau_{\text{qmem}}$ grows fast with $n$. The first scenario can be realized for systems featuring a macroscopic (growing as $n^\gamma$) energy barrier surrounding the states orthogonal to the protected subspace $P$. The 4D toric code model analyzed in [13] provides an example of such a system. The second scenario could be realized if the energy barrier grows only logarithmically as a function of $n$ as in [17, 18]. In this case, the exponent is controlled by the temperature, that is, $\epsilon_{\text{qmem}} \leq n^{-\rho\beta} = e^{-\beta\gamma\log(n)}$ for some $\gamma > 0$. If the temperature is smaller than a critical value, the relaxation rate $\epsilon_{\text{qmem}}$ decays sufficiently fast to yield a storage time $\tau_{\text{qmem}}$ increasing with $n$. A polynomial increase of the storage time is also obtained in [10] at any temperature, from the logarithmic divergence of a self-consistent gap. It is tempting to conjecture that such system may exist in lower spatial dimensions.

The proof of theorem 1 involves two ingredients: (i) constructing the encoding and decoding maps (see section 6.1), and (ii) proving that the encoded states are approximate fixed points of the Lindblad operator (see section 6.2). Our construction of encoding and decoding maps is identical to the one used by Alicki et al [11, 13]. It is described in section 6.1 which can be regarded as an overview of section IA in [13]. The second part of the proof is presented in section 6.2. Our approach here is quite different from the one taken in [13]. It yields a much simpler proof and requires less assumptions about the Lindblad operator compared with [13] (for instance, we do not need the detailed balance condition).

Following [11]–[13], we can specialize theorem 1 to the Markovian master equation due to Davies [21], which describes the dynamics induced by a weak coupling between the system...
and a thermal bath. It involves a coupling Hamiltonian

$$H_{\text{int}} = \sum_{k=1}^{K} A_k \otimes B_k,$$

(30)

where $A_k$ are some local few-qubit operators acting on the system and the operators $B_k$ act on the bath.

It was shown by Davies [21] that in the weak-coupling limit the system evolves according to the Markovian master equation, equation (22), where the Lindblad operator is defined as

$$\mathcal{L}(\rho) = \sum_{k} \sum_{\omega} h(k, \omega) \left( A_{k,\omega} \rho A_{k,\omega}^\dagger - \frac{1}{2} \{ \rho, A_{k,\omega}^\dagger A_{k,\omega} \} \right).$$

(31)

Here $A_{k,\omega}$ are the Fourier components of $A_k(t) \equiv e^{iHt} A_k e^{-iHt}$, that is,

$$A_k(t) = \sum_{\omega} A_{k,\omega} e^{-i\omega t}.$$

One can think about $A_{k,\omega}$ as the part of $A_k$ transferring energy $\omega$ from the system to the bath. The bath temperature enters into the equation only through the function $h(k, \omega)$, which has to obey the detailed balance condition,

$$h(k, -\omega) = e^{-\beta \omega} h(k, \omega).$$

(32)

The coefficient $h(k, \omega)$ is defined as the Fourier transform of the autocorrelation function of $B_k$ with respect to the bath state. One can regard $h(k, \omega)$ as a probability (per unit of time) of quantum jumps induced by the coupling operator $A_k$, which transfer energy $\omega$ from the system to the thermal bath. The detailed balance condition guarantees that the Gibbs state $\rho_\beta$ is a fixed point of $\mathcal{L}$.

It is important to discuss how the quantum jump operators $A_{k,\omega}$ depend on the original coupling operators $A_k$.

For stabilizer code Hamiltonians as in equation (1) the time-dependent operator $A_k(t) = \exp(iHt) A_k \exp(-iHt)$ acts only on a few qubits since all the terms in $H$ pairwise commute and thus $A_k(t) = \exp(iH't) A_k \exp(-iH't)$ where $H'$ includes only those terms of $H$ that act on the same qubits as $A_k$. Note that $H'$ has only a few Bohr frequencies since it acts only on a few qubits. It means that any quantum jump operator $A_{k,\omega}$ in the Davies master equation acts only on a few qubits and the total number of the quantum jump operators is roughly the same as the number of the coupling operators $A_k$.

This issue is more subtle for subsystem codes, since $A_k(t)$ may be a highly non-local operator for long times $t$ and the number of Bohr frequencies may be exponentially large. However, it is also clear that the non-locality of $A_k(t)$ is only due to multiplying it with non-local elements in the gauge group $\mathcal{G}$. Hence $A_k(t)$ remains local modulo gauge group transformations.

Let us specialize the theorem 1 to the Davies master equation, see equations (22) and (31). The condition that the observables $\tilde{X}$ and $\tilde{Z}$ are protected from all quantum jump operators in $\mathcal{E}_\mathcal{L}$ might seem too demanding since the operators $A_{k,\omega}$ may be highly non-local, see the remark above. Fortunately, it is sufficient to require that $\tilde{X}$ and $\tilde{Z}$ are protected from a set of errors $\mathcal{E}_{\text{int}} = \{ A_k \}$ including all coupling operators $A_k$. Indeed, since, by assumption, $H$ commutes with $P$ and $\tilde{X}$, $\tilde{Z}$, the condition $[\tilde{X}, A_k] P = 0$ implies $[\tilde{X}, A_{k,\omega}] P = 0$ for any frequency $\omega$. (The same remark applies to $\tilde{Z}$.)
Next, we need an upper bound on the norm of the Davies generator $\mathcal{L}$, see equation (31).

**Proposition 1.** Assuming that $\|A_k\| \leq 1$ for all $k$, $\omega$ one has

$$\|\mathcal{L}\|_1 \leq 2K h_{\text{max}},$$

where $h_{\text{max}} = \max_{k,\omega}|h(k, \omega)|$ and $K$ is the total number of terms in the interaction Hamiltonian equation (30).

**Proof.** Indeed, let $F = F^\dagger$ be an operator such that $\|F\|_1 \leq 1$ and $\|\mathcal{L}\|_1 = \|\mathcal{L}(F)\|_1$, see equation (25). Fix some $k$ and let $A = A_k$, $A_{\omega} = A_{k,\omega}$, and $h(\omega) = h(k, \omega)$. Let us bound the trace norm of a single term

$$\mathcal{L}_k(F) \equiv \sum_{\omega} h(\omega) A_{\omega} F A_{\omega}^\dagger - \frac{h(\omega)}{2} \{A_{\omega}^\dagger A_{\omega}, F\}.$$  

(34)

Note that

$$\sum_{\omega} A_{\omega}^\dagger A_{\omega} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, A(t)^\dagger A(t)$$

(35)

and

$$\sum_{\omega} A_{\omega} F A_{\omega}^\dagger = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, A(t) F A(t)^\dagger.$$  

(36)

Using the bound $\|AB\|_1 \leq \|A\| \cdot \|B\|_1$ valid for any operators $A$, $B$ we obtain

$$\|\sum_{\omega} (1/2) h(\omega) [A_{\omega}^\dagger A_{\omega}, F]\|_1 \leq \|\sum_{\omega} h(\omega) A_{\omega}^\dagger A_{\omega}\| \cdot \|F\|_1$$

$$\leq h_{\text{max}} \|\sum_{\omega} A_{\omega}^\dagger A_{\omega}\| \cdot \|F\|_1 \leq h_{\text{max}} \|F\|_1.$$  

(37)

Here the second line used equation (35), convexity of the norm, and the fact that $\|A(t)\| = \|A\| \leq 1$.

Let $F = F_+ - F_-$ be the decomposition of $F$ into positive and negative parts, that is, $F_\pm \geq 0$ and $\|F\|_1 = \text{Tr} F_+ - \text{Tr} F_- = \|F_+\|_1 + \|F_-\|_1$. Then

$$\|\sum_{\omega} h(\omega) A_{\omega} F A_{\omega}^\dagger\|_1 \leq \|\sum_{\omega} h(\omega) A_{\omega} F_+ A_{\omega}^\dagger\|_1 + \|\sum_{\omega} h(\omega) A_{\omega} F_- A_{\omega}^\dagger\|_1$$

$$\leq h_{\text{max}} \|\sum_{\omega} A_{\omega} F_- A_{\omega}^\dagger\|_1 + h_{\text{max}} \|\sum_{\omega} A_{\omega} F_+ A_{\omega}^\dagger\|_1$$

$$\leq h_{\text{max}} \|F_-\|_1 + h_{\text{max}} \|F_+\|_1 = h_{\text{max}} \|F\|_1.$$  

(38)

Here the last line used equation (36), convexity of the norm, and inequality $\|A(t) F_\pm A(t)^\dagger\|_1 \leq \|F_\pm\|_1$. Combining equations (37) and (38), we arrive to $\|\mathcal{L}_k(F)\|_1 \leq 2h_{\text{max}} \|F\|_1$ which leads to equation (33). \hfill \Box

To conclude, theorem 1 can be specialized to the Davies master equation as follows. Suppose the system interacts with a thermal bath at the inverse temperature $\beta$ via a Hamiltonian $H_{\text{int}} = \sum_{k=1}^{K} A_k \otimes B_k$, where $\|A_k\| \leq 1$ and $B_k$ are normalized via the condition $h(k, \omega) \leq h_{\text{max}}$. Suppose one can choose Pauli-like observables $\tilde{X}$ and $\tilde{Z}$ that are protected from any coupling

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operator $A_k$ on some subspace $P$. Suppose that $\tilde{X}, \tilde{Z}$ and $P$ commute with the system Hamiltonian $H$. Then theorem 1 implies that a qubit can be stored in the system reliably for a time $\tau_{\text{qmem}} \sim (K h_{\text{max}} \epsilon_{\text{qmem}})^{-1}$, where

$$\epsilon_{\text{qmem}} \equiv \text{Tr}(I - P)\rho_{\beta}.$$  

(39)

Note that $K$ will be $O(n)$ for local couplings $A_k$. We will discuss how to evaluate the relaxation rate $\epsilon_{\text{qmem}}$ in more detail in section 7.

6.1. Proof of theorem 1: part I

Let us start from defining the encoding and decoding maps $\Phi_{\text{in}}$ and $\Phi_{\text{out}}$. Let $A_Q \subseteq A$ be the algebra generated by $I$, $\tilde{X}$, $\tilde{Z}$ and $\tilde{Y} \equiv i\tilde{X}\tilde{Z}$. For any algebra $A$ let us define the center of $A$ as

$$Z(A) = \{ A \in A : AB = BA \text{ for all } B \in A \}.$$  

Clearly $Z(A_Q) = \mathbb{C} \cdot I$, that is, $A_Q$ has trivial center.

For any finite-dimensional Hilbert space let $\mathbb{L}(H)$ be the algebra of linear operators acting on $H$. We shall use the following fact (see for instance theorem 5 in [22], or a book [23]):

**Fact 1.** Let $A_Q \subseteq \mathbb{L}(H)$ be any algebra such that (i) $A_Q$ contains the identity operator; (ii) $A_Q$ is closed under hermitian conjugation; (iii) $A_Q$ has a trivial center. Then there exists a (virtual) tensor product structure $H = H_Q \otimes H_A$ such that

$$A_Q = \mathbb{L}(H_Q) \otimes I_A.$$  

(40)

It implies that there is a decomposition $H = H_Q \otimes H_A$ such that $H_Q$ describes a qubit $Q$ and the operators $\tilde{X}, \tilde{Y}, \tilde{Z}$ are the Pauli operators acting on $H_Q$, that is,

$$\tilde{X} = X_Q \otimes I_A, \quad \tilde{Y} = Y_Q \otimes I_A, \quad \tilde{Z} = Z_Q \otimes I_A.$$  

(41)

By assumption, the system’s Hamiltonian $H$ commutes with $\tilde{X}, \tilde{Y}, \tilde{Z}$. Therefore, $H$ acts trivially on $H_Q$ and thus there exists $H_A \in \mathbb{L}(H_A)$ such that

$$H = I_Q \otimes H_A.$$  

(42)

Note that $\text{Tr} \exp(-\beta H) = 2 \text{Tr} \exp(-\beta H_A)$. Thus the Gibbs state $\rho_{\beta}$ can be written as

$$\rho_{\beta} = \frac{1}{2} I_Q \otimes \eta_A, \quad \eta_A = \frac{\exp(-\beta H_A)}{\text{Tr} \exp(-\beta H_A)}.$$  

(43)

Define the encoding map $\Phi_{\text{in}} : \mathbb{L}(C^2) \to A$ as

$$\Phi_{\text{in}}(\eta) = \eta \otimes \eta_A.$$  

(44)

Using equations (41)–(43) one obtains

$$\Phi_{\text{in}}(I) = 2 \rho_{\beta}, \quad \Phi_{\text{in}}(Q) = 2 \tilde{Q} \rho_{\beta} = 2 \rho_{\beta} \tilde{Q} \quad \text{for any } Q \in \{X, Y, Z\}.$$  

(45)

Define the decoding map $\Phi_{\text{out}} : A \to \mathbb{L}(C^2)$ formally as the partial trace over the subsystem $H_A$,

$$\Phi_{\text{out}}(\rho) = \text{Tr}_{A}\rho.$$  

(46)

Clearly, $\Phi_{\text{out}} \circ \Phi_{\text{in}}$ is the identity map.
To demonstrate this formalism, let us explain how the encoding map \( \Phi_{in} \) is constructed for the special case of stabilizer (subsystem) codes. Imagine that one needs to store a single-qubit state \( \eta = \frac{1}{2} (I + \gamma \cdot S) \) with \( S = (X, Y, Z) \). We encode into the thermal state \( \Phi_{in}(\eta) = 2\rho_\beta \eta_{ec} \) where \( \eta_{ec} = \frac{1}{2} (I + \gamma \cdot S_{ec}) \) with the error-corrected logical operators \( S_{ec} = (\bar{X}_{ec}, \bar{Y}_{ec}, \bar{Z}_{ec}) \). Note that \( \eta_{ec} \) commutes with \( \rho_\beta \).

The central idea underlying the encoding into the thermal state is that \( \Phi_{in}(\eta) \) is the same as the stationary state \( \rho_\beta \) satisfying \( \Phi_t(\rho_\beta) = \rho_\beta \) on \( \mathcal{H}_A \). Thus, we can expect that if thermal fluctuations do not build up to logical errors, the state \( \Phi_t \circ \Phi_{in}(\eta) \) would remain close to the initial state \( \Phi_{in}(\eta) \).

Note that \( \Phi_{in} \) is quite different from the standard encoding into the ground state subspace, for which the requirement of a Hamiltonian with finite excitation gap appears most natural. Instead, the stability criterion of theorem 1 using the encoding in a thermal state does not explicitly involve the spectral gap. This is an interesting point, since it has become clear now that the presence of a gap does not imply robustness of topological protection. On the other hand, it might be possible to obtain a self-correcting quantum memory for a Hamiltonian with vanishing gap at large \( n \).

### 6.2. Proof of theorem 1: part II

Let \( \rho = \Phi_{in}(\eta) \) be any encoded state. Using equation (45) one check that \( \rho \) can be represented as

\[
\rho = O \rho_\beta = \rho_\beta O, \quad O \in \mathcal{A}_Q, \quad \|O\| \leq 2. \tag{47}
\]

Consider a family of states

\[
\rho(t) = \Phi_t(\rho), \quad \Phi_t = \exp(-it[H, \cdot] + \mathcal{L}t), \quad t \geq 0. \tag{48}
\]

Taking into account that \( H \) commutes with \( \rho \) we can represent the derivative \( \dot{\rho} \) as

\[
\dot{\rho}(s) = \Phi_s(\mathcal{L}(\rho)), \quad s \geq 0. \tag{49}
\]

Using the fact that \( \|\Phi(A)\|_1 \leq \|A\|_1 \) for any TPCP map \( \Phi \) and any operator \( A \) we obtain

\[
\|\dot{\rho}(s)\|_1 \leq \|\mathcal{L}(\rho)\|_1, \quad s \geq 0. \tag{50}
\]

Therefore

\[
\|\rho(t) - \rho(0)\|_1 = \int_0^t \|s\dot{\rho}(s)\|_1 \leq t \|\mathcal{L}(\rho)\|_1. \tag{51}
\]

Thus we have to prove an upper bound on the norm of \( \mathcal{L}(\rho) = \mathcal{L}(O\rho_\beta) \). Inserting twice the decomposition \( P + P^\perp = I \) we obtain

\[
\mathcal{L}(O\rho_\beta) = L_1 + L_2 + L_3 + L_4, \quad \text{where} \quad L_1 = P\mathcal{L}(O\rho_\beta), \quad L_2 = P^\perp \mathcal{L}(O\rho_\beta), \tag{52}
\]

and

\[
L_3 = PL\mathcal{L}(O P^\perp \rho_\beta), \quad L_4 = P^\perp \mathcal{L}(O P^\perp \rho_\beta). \tag{53}
\]

Using the identity \( \|AB\|_1 \leq \|A\| \cdot \|B\|_1 \) valid for any operators \( A \) and \( B \), taking into account that \( \|O\| \leq 2 \) and using equation (25), one easily gets

\[
\|L_3\|_1, \|L_4\|_1 \leq 2 \|\mathcal{L}\|_1 \operatorname{Tr} P^\perp \rho_\beta. \tag{53}
\]
We shall bound the norm of $L_1$ and $L_2$ using the fact that $\mathcal{L}(\rho_\beta) = 0$. Indeed, using the assumption that $[S_a, O]P = 0$ and $P[S_a^\dagger, O] = 0$ for all $a$ one can rewrite $L_1$ as

$$L_1 = O P \mathcal{L}(P \rho_\beta) = -O P \mathcal{L}(P \rho_\beta).$$

(54)

It follows that

$$\|L_1\|_1 \leq 2 \|\mathcal{L}\|_1 \text{ Tr } P \rho_\beta.$$

(55)

Using $P[O, S_a^\dagger] = 0$, $PP = 0$ and $\mathcal{L}(\rho_\beta) = 0$ we can rewrite $L_2$ as

$$L_2 = P \mathcal{L}(P \rho_\beta) O = -P \mathcal{L}(P \rho_\beta) O$$

and thus

$$\|L_2\|_1 \leq 2 \|\mathcal{L}\|_1 \text{ Tr } P \rho_\beta.$$

(56)

Combining equations (53), (55) and (57) we arrive at

$$\|\mathcal{L}(O \rho_\beta)\|_1 \leq 8 \|\mathcal{L}\|_1 \text{ Tr } P \rho_\beta.$$

(58)

Plugging it into equation (51) we get $\|\rho(t) - \rho\|_1 \leq 8 \|\mathcal{L}\|_1 \text{ Tr } (I - P) \rho_\beta$.

7. Relaxation rate for subsystem code Hamiltonians

In this section, we explain how to construct the protected Pauli-like observables and the subspace $P$ involved in theorem 1 using the formalism of subsystem codes. Let $G \subseteq \mathcal{P}_n$ be the gauge group of some subsystem code encoding one qubit into $n$ qubits. Assume that the system’s Hamiltonian $H$ is defined as in equation (2), so that $H$ is a linear combination of gauge operators. Suppose we seek protection from some set of elementary errors $E$. Assume without loss of generality that all elements of $E$ are Pauli operators, that is, $E \subseteq \mathcal{P}_n$. For example, $E$ may include all Pauli operators that appear in the decomposition of the operators $A_k$ coupling the system and the bath, see equation (30). In this case, any elementary error acts only on a few qubits.

Let us start from defining a notion of goodness of syndromes relative to the set of elementary error $E$. We will say that a syndrome $s$, see the definitions in section 4, is good iff

$$C(s + s_E)EC(s) \in G, \quad \text{for all } E \in E.$$

(59)

Remember that $C(s) \in \mathcal{P}_n$ is the correcting Pauli operator for a given syndrome $s$ which is determined by some deterministic error correction algorithm. To highlight the intuition behind the definition of good syndromes, let us assume that the syndrome $s$ has been caused by some pre-existing error $E'$. If the error $E'$ is correctable then we have $C(s)E' \in G$ and thus $EC(s)$ coincides with $EE'$ up to a gauge operator in $G$. Note that $EE'$ has syndrome $s + s_E$. Equation (59) says that $C(s + s_E)EE' \in G$, that is, the error $EE'$ is also correctable for all elementary errors $E \in E$.

We would like to point out that in the theory of quantum fault-tolerance and error correction, very similar notions are used to determine the correctness of an encoded logical gate (called a rectangle), see e.g. in the discussion at the bottom of page 12 in [24]. The correctness of the encoded logical gate depends on its incoming pre-existing syndrome in combination with new errors which occur during the execution of the encoded gate. In our definition of goodness,
no gate happens, but we allow for any elementary error \( E \) and determine whether the pre-existing syndrome in combination with the new error leads to making a good inference about the total error.

Thus, one can keep adding more and more elementary errors as long as the observed syndromes are good. On the other hand, if the observed syndrome becomes bad (that is, not good), it means that one has already reached the limits of the error-correcting capabilities of the code and the next elementary error can potentially destroy the encoded information. In this case, the operator \( C(s + s_E)EC(s) \in \mathcal{C}(S) \setminus \mathcal{G} \) becomes a nontrivial logical operator.

Now, in order to apply theorem 1, we pick some bare logical Pauli operators \( \bar{X}, \bar{Z} \in \mathcal{C}(\mathcal{G}) \setminus \mathcal{G} \), see section 2, and choose the Pauli-like observables \( \tilde{X} \) and \( \tilde{Z} \) as the error-corrected logical operators \( \tilde{X}_{ec} \) and \( \tilde{Z}_{ec} \) defined in section 4. Below we shall prove that \( \tilde{X}_{ec} \) and \( \tilde{Z}_{ec} \) are protected from the set of elementary errors \( \mathcal{E} \) on the subspace \( P \) spanned by good syndromes, that is,

\[
P = \sum_{\text{good } s} P_s. \tag{60}
\]

Note that by construction \( \tilde{X}_{ec}, \tilde{Z}_{ec} \), and \( P \) commute with any Hamiltonian \( H \) made up from gauge operators, see section 4, while \( P \) is an invariant subspace of \( X_{ec} \) and \( Z_{ec} \), see section 4, so all the conditions of theorem 1 are met.

**Lemma 1.** The observable \( \tilde{Z}_{ec} \) is protected from all elementary errors on the subspace spanned by good syndromes.

**Proof.** Let \( s \) be any good syndrome and \( E \in \mathcal{E} \) be any elementary error. It suffices to prove that

\[
[E, \tilde{Z}_{ec}] P_s = 0. \tag{61}
\]

Let \( t = s + s_E \) and \( A = EC(s)C(t) \). Goodness of \( s \) implies that \( A \in \mathcal{G} \). Since we have chosen \( \tilde{Z} \in \mathcal{C}(\mathcal{G}) \), it implies \( A\tilde{Z} = \tilde{Z}A \). Taking into account equation (15) we obtain

\[
\tilde{Z} E = E\tilde{Z}\lambda_z(s)\lambda_z(t). \tag{62}
\]

Thus, we see that

\[
[E, \tilde{Z}_{ec}] P_s = P_1[E, \tilde{Z}_{ec}] P_s = P_1(\lambda_z(s)E\tilde{Z} - \lambda_z(t)\tilde{Z}E) P_s = 0,
\]

where we have used equations (15), (16) and (62).

Recall that the syndrome subspaces \( P_s \) are well defined only if one fixes the subspace \( P_0 \) associated with the trivial syndrome. A natural choice of \( P_0 \) is dictated by the ground state of the system’s Hamiltonian \( H = \sum_i r_i G_i \). Typically, the degeneracy of ground subspace of \( H \) is the minimal degeneracy consistent with the symmetry of \( H \). In our case the non-Abelian symmetries of \( H \) include the bare logical operators \( \tilde{X} \) and \( \tilde{Z} \), so we should expect the ground state to have degeneracy 2. In this case the ground state determines a particular set of quantum numbers (irreducible representation) of the stabilizer group \( S = \mathcal{G} \cap \mathcal{C}(\mathcal{G}) \). We can choose the trivial syndrome subspace \( P_0 \) as the subspace spanned by all states that have the same quantum numbers as the ground state. Equivalently, \( P_0 \) includes all states that can be obtained from the ground state by applying gauge operators and logical operators.
The question of whether a particular family of subsystem codes is suitable for building a good quantum memory can now be reduced to bounding the relaxation rate defined in equation (29):

$$\epsilon_{\text{qmem}} = 1 - \text{Tr} \, \rho^H \, P = \frac{\sum_{\text{bad } s} Z_s}{\sum_s Z_s}, \quad \text{where } Z_s = \text{Tr} \, P_s \exp(-\beta H), \quad (63)$$

as a function of $n$. Recall that all terms in $H$ commute with $\mathcal{S}$ so that all syndrome subspaces $P_s$ are invariant under $H$. It is clearly desirable to have a Gibbs state $\exp(-\beta H)$ with support mostly concentrated on the good syndromes. Note that low-weight correctable errors which in addition remain correctable if any single additional error occurs, will have syndromes which are good. Hence the Gibbs state should be concentrated on the ground space and excited states that can be created from the ground state by these correctable errors. This type of property is what has been shown for the 4D Kitaev model in [13]. Bounding the relaxation rate for subsystem codes is a hard task, since it depends on the full partition function of the model. The following simple observations might be helpful for obtaining upper bound on $\epsilon_{\text{qmem}}$.

Consider the partition function $Z_s$ associated with some syndrome-sector $P_s$. Let $E \in \mathcal{P}_n$ be some error with the syndrome $s$. Note that $P_s = E P_0 E$. It implies

$$Z_s = \text{Tr} \, E P_0 E \exp(-\beta H) = \text{Tr} \, P_0 \exp(-\beta E H E)$$

$$= \text{Tr} \, P_0 \exp(-\beta H + \beta H_E) \leq \text{Tr} \, P_0 \exp(-\beta H) \exp(\beta H_E), \quad (64)$$

where

$$H_E = 2 \sum_{i : G_i E = -E G_i} r_i G_i \quad (65)$$

is a sum of all terms in $H$ that anticommute with the error $E$. The second line in equation (64) follows from the Golden–Thompson inequality. It implies

$$\epsilon_{\text{qmem}} \leq \sum_{\text{bad } s} \langle P_0 \exp(\beta H_{E(s)}) \rangle_\beta, \quad (66)$$

where $\langle \cdot \rangle_\beta$ is the average over the thermal Gibbs state and $E(s)$ is some fixed error causing syndrome $s$.

Let us now express our intuitive understanding of under what circumstances the bound in equation (66) could give rise to self-correction. For a self-correcting quantum memory, we expect that bad syndromes correspond to errors which anti-commute with a macroscopic number of terms in $H$, hence $H_E$ is a sum over a macroscopic number of terms, say $l(n)$. This type of requirement has been expressed in [4]. If this requirement is fulfilled, one can imagine that in ‘sufficiently high dimensions’, it is possible to use a mean-field approximation and approximate $\langle P_0 \exp(\beta H_{E(s)}) \rangle_\beta$ by $\exp(\beta \langle P_0 H_{E(s)} \rangle_\beta)$. It is of course important that fluctuations around such mean-field approximation die off sufficiently fast. Now, if one can upper bound each individual term in $\langle P_0 H_{E(s)} \rangle_\beta$ by some constant $-c$, then, because there are a macroscopic number of terms in $H_{E(s)}$, the factor $\langle P_0 \exp(\beta H_{E(s)}) \rangle_\beta$ would scale like $\exp(-\beta O(l(n)))$. The sum of the bad syndromes will multiply this exponential decay by some factor upper bounded by $2^{\text{rank} \, (S)}$ where $\text{rank} \, (S)$ is the minimal generating set of $S$. If $l(n)$ grows at least as fast with $n$ as $\text{rank} \, (S)$, then $\epsilon_{\text{qmem}}$ would be exponentially decaying. It is clear that many things have to ‘go right’ in order for self-correction to be feasible, in particular it is not clear whether three-dimensions would be sufficient for mean-field-type approximations with sufficiently small corrections.
Another simple observation shows that one may be able to make headway in computing the partition functions $Z_s$ by making use of the underlying symmetry. Recall that $P_s$ is the projector associated with some irreducible representation of the stabilizer group $S = G \cap C(G)$. Therefore we can write $P_s$ as a linear combination of elements of $S$,

$$P_s = \sum_{Q \in S} \sigma_s(Q) Q, \quad \sigma_s(Q) \in \{+1, -1\}. \quad (67)$$

Expanding the exponent $\exp(-\beta H)$ in powers of $\beta^k$ we will only get nonzero contributions to $Z_s$ from terms in $H^k$ that are elements of the stabilizer group $S$. We will leave a detailed analysis of the memory relaxation rate for a particular subsystem code to a future paper. □

8. Relaxation rate for gapped Hamiltonians

It is worth emphasizing that the lower bound on the storage time obtained in sections 6, 7 applies to both gapped and gapless memory Hamiltonians. It is natural to ask whether a stronger bound can be obtained if the memory Hamiltonian $H$ has a constant spectral gap $\Delta > 0$ separating the ground state from excited states while the bath temperature $T$ is small compared to $\Delta$. In this low-temperature regime, the rate of all processes exciting the system from the ground state is suppressed by the Boltzmann factor $e^{-\beta \Delta}$ and thus one should expect that the storage timescales as

$$\frac{1}{\tau_{qmem}} \leq O(n) e^{-\beta \Delta}. \quad (68)$$

Below we prove that this is indeed the case assuming that the interaction with the thermal bath can be described by the Davies equation, see equations (22) and (31). In contrast to our main result (see theorem 1), the proof of equation (68) will rely on the detailed balance condition equation (32). Our analysis will use the encoding and decoding maps $\Phi_{in}, \Phi_{out}$ defined in section 6.1, where we set $\beta = \infty$. In other words, we encode information into the ground state of $H$ rather than the thermal state. Accordingly, any encoded state $\rho = \Phi_{in}(\eta)$ can be represented as

$$\rho = O \rho_{\infty} = \rho_{\infty} O, \quad O \in A_Q, \quad \|O\| \leq 2. \quad (69)$$

The observable $O$ must be protected from all coupling operators $A_k$, see equation (30), on the ground-state subspace $P_0$, that is, one must have $[O, A_k] P_0 = 0$ for all $k$. As was mentioned in section 6, this is equivalent to the condition

$$[O, A_{k,\omega}] P_0 = 0, \quad \text{for all } k, \quad \text{for all } \omega. \quad (70)$$

Using equation (51) it suffices to prove that

$$\|L(\rho)\|_1 \leq O(n) e^{-\beta \Delta}, \quad (71)$$

where $L$ is the Davies generator defined in equation (31). Let $A_{k,\omega}$ be any quantum jump operator from equation (31). Since $A_{k,\omega}$ transfers energy $\omega$ from the system to the bath, we have

$$A_{k,\omega} \rho_{\infty} = 0, \quad \text{for } \omega > 0. \quad (72)$$
It follows that $\mathcal{L}(\rho)$ contains only the terms with non-positive Bohr frequencies. We claim that the zero-frequency term also does not contribute to $\mathcal{L}(\rho)$. Indeed, since $A_{k,0}$ commutes with $H$, we obtain

$$A_{k,0}P_0 = P_0 A_{k,0} = P_0 A_{k,0} P_0.$$  \hfill (73)

In addition, since $A_k$ is hermitian, we have

$$A_k^\dagger = A_{k,0}.$$  \hfill (74)

Combining equations (70), (73) and (74) one can easily check that

$$A_{k,0} \rho A_{k,0}^\dagger - \frac{1}{2} \left\{ \rho, A_{k,0} A_{k,0}^\dagger \right\} = 0.$$  \hfill (75)

Since any negative Bohr frequency is separated from 0 by the gap $\Delta_1$, we obtain

$$\mathcal{L}(\rho) = \sum_k \sum_{\omega \leq -\Delta} h(k, \omega) \left( A_{k,\omega} \rho A_{k,\omega}^\dagger - \frac{1}{2} \left\{ \rho, A_{k,\omega} A_{k,\omega}^\dagger \right\} \right).$$  \hfill (76)

The detailed balance condition equation (32) implies that for any $\omega \leq -\Delta$ one has

$$|h(k, \omega)| \leq e^{-\beta \Delta} |h(k, -\omega)| \leq e^{-\beta \Delta} h_{\max}.$$  \hfill (77)

Let $\mathcal{L}'$ be a superoperator obtained from $\mathcal{L}$ by setting $h(k, \omega) = 0$ for $\omega > -\Delta$. Using proposition 1 and equation (77) one infers that $\|\mathcal{L}'\|_1 \leq 2K h_{\max} e^{-\beta \Delta}$ for some $K = O(n)$. It means that

$$\|\mathcal{L}(\rho)\|_1 = \|\mathcal{L}'(\rho)\|_1 \leq \|\mathcal{L}'\|_1 \leq 2K h_{\max} e^{-\beta \Delta} = O(n) e^{-\beta \Delta}.$$  \hfill (78)

It proves equation (71) completing the proof of equation (68).

It is worth pointing out that the condition $[O, A_k] P_0 = 0$ used in the above analysis is satisfied whenever the coupling operators $\{A_k\}$ are linear combinations of correctable errors with respect to the code $P_0$. Since each coupling operator $A_k$ acts only on $O(1)$ qubits, the distance of the code $P_0$ need to be larger than some constant value depending on the locality of the coupling Hamiltonian $H_{int}$. Thus formally the bound in equation (68) applies even to microscopic systems that consist only of a few qubits. Note however that the degeneracy of the ground space for microscopic systems is not stable under small perturbations of the memory Hamiltonian $H$. Making the ground-space degeneracy insensitive to perturbations requires codes with a macroscopic distance, which can be achieved only for macroscopic systems.

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