Frame independent cosmological perturbations

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Abstract. We compute the third order gauge invariant action for scalar-graviton interactions in the Jordan frame. We demonstrate that the gauge invariant action for scalar and tensor perturbations on one physical hypersurface only differs from that on another physical hypersurface via terms proportional to the equation of motion and boundary terms, such that the evolution of non-Gaussianity may be called unique. Moreover, we demonstrate that the gauge invariant curvature perturbation and graviton on uniform field hypersurfaces in the Jordan frame are equal to their counterparts in the Einstein frame. These frame independent perturbations are therefore particularly useful in relating results in different frames at the perturbative level. On the other hand, the field perturbation and graviton on uniform curvature hypersurfaces in the Jordan and Einstein frame are non-linearly related, as are their corresponding actions and n-point functions.
1 Introduction

The theory of General Relativity exhibits general covariance – physical laws and observables are invariant under coordinate reparametrizations. In a field-theoretic framework these physical laws are derived from an action, written in a manifestly covariant form in terms of tensors, covariant derivatives, 4-vectors and scalars. Even though the action is manifestly covariant, this is not at all clear when the action is perturbed around a fixed background. Here is where one runs into the gauge problem of General Relativity – the perturbations themselves are dependent on the choice of coordinate system. As a consequence it becomes complicated to extract physical observables from the perturbed action.

Fortunately it is possible to specify unambiguous quantities through the use of gauge invariant cosmological perturbations [1]. This has been applied to the action for a single scalar field in an expanding universe, by deriving the manifestly gauge invariant quadratic action for linearly gauge invariant scalar perturbations [2, 3]. In an inflationary context the primordial power spectrum for these fluctuations is predicted to be nearly scale invariant, which has recently been established at the 5 σ level by the Planck mission [4]. Gauge invariant perturbations can also
be constructed at second order in perturbation theory by fixing the gauge transformation order by order [5–8]. Moreover, in Ref. [9] the third order action for standard single field inflation was derived in different gauges, while Refs. [10, 11] derived the action for more general single field inflation in the uniform field gauge. Finally the fourth order action was computed in the uniform curvature gauge in [12, 13]. Only recently the third order action for scalar perturbations was treated in a completely gauge invariant way [14, 15], and it was demonstrated how to find the gauge invariant vertices and second order gauge invariant variables from the action.

Closely related to the coordinate invariance of General Relativity and the associated difficulties at the perturbative level, is the case of frame transformations and physical equivalence. A single scalar field may be non-minimally coupled to the Ricci scalar in the action, which is then said to be in the Jordan frame. It has been long known that such a theory may be rewritten into the standard, minimally coupled Einstein frame form by redefining the metric and scalar field. Since these are just field redefinitions, no physical content is lost in the frame transformation, and the Jordan and Einstein frame are in that sense physically equivalent. This proves very convenient in finding Jordan frame results via the simpler Einstein frame. For example, the concept of Higgs inflation [16, 17] (see also [18, 19]), where the Higgs field is coupled to the Ricci scalar by a large non-minimal coupling, is most easily illustrated in the Einstein frame, where the potential becomes exponentially flat for large field values.

Although the equivalence between the Jordan and Einstein frame is straightforwardly established for the complete action, this is not obvious for the perturbed action. The reason is that the perturbations in one frame do not coincide with those in the other frame, because the frame transformations are field dependent (note the similarity to the gauge dependent case). As such it becomes troublesome to relate certain quantity in one frame, for example an $n$-point function in the Jordan frame, to the corresponding quantity in the other frame.

Luckily, also here it is possible to define unambiguous perturbations which take the same form in either frame. We call these perturbations frame independent cosmological perturbations. In Refs. [20, 21] the frame independent scalar perturbation was derived, and it was shown that it coincides with the linear gauge invariant curvature perturbation. In terms of such a frame independent perturbation the equivalence between Jordan and Einstein frame is almost trivially established for the second order action, and it was used to derive the power spectrum in the Jordan frame from the well-known Einstein frame result. Also, the second order action was derived directly in the Jordan frame, and it was shown to be related to the Einstein frame action via a frame transformation [22]. Furthermore, the frame independence of the gauge invariant curvature perturbation has been demonstrated to second order, and was used to compute $f_{NL}$ in the Jordan frame via the Einstein frame [23]. The action for third order scalar perturbations in the Jordan frame was computed in Ref. [24] in the uniform field gauge, and it was shown in Ref. [25] that it can be derived from the Einstein frame action by frame transformations of the background quantities. In recent work [15] we took a gauge invariant approach and showed how to derive the cubic Jordan frame action for gauge invariant scalar perturbations from the Einstein frame.

In most of the works above the only perturbation considered was the scalar degree of freedom in the action. In this work we extend our previous work to include the graviton and scalar-graviton interactions in the action. We perform all computations directly in the Jordan frame with a general non-minimal coupling. In section 2 we introduce the model and perturbations. In section 3 we compute the gauge invariant cubic vertices for the graviton and for scalar-graviton interactions on different hypersurfaces. Finally, in section 4 we show how the gauge invariant cubic action in the Jordan frame is related to that in the Einstein frame.
The model

We consider a single scalar field in the Jordan frame, i.e. coupled non-minimally to the Ricci scalar,

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ RF(\Phi) - g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2V(\Phi) \right\}. \]  

(2.1)

The function \( F(\Phi) \) is a general function of the scalar field \( \Phi \). For a minimally coupled theory \( F(\Phi) = M_P^2 \equiv 1 \), where \( M_P \) is the reduced Planck mass. In the case of Higgs inflation the coupling is \( F(\Phi) = M_P^2 - \xi \Phi^2 \).

We are interested in finding the action (2.1) up to third order in perturbations. Since the theory of general relativity is covariant, some of the degrees of freedom in the action are not physical. In order to eliminate unphysical degrees of freedom from the action, and to separate dynamical from constraint degrees of freedom, it is most convenient to slice up spacetime into spatial hypersurfaces. This can be done using the Arnowitt-Deser-Misner (ADM) metric [26], which is defined through the line element

\[ ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt). \]  

(2.2)

Here \( g_{ij} \) is the spatial metric and \( N \) and \( N^i \) are the lapse and shift functions, respectively. In terms of the ADM metric the action (2.1) can be written (up to boundary terms) as [22]

\[ S = \frac{1}{2} \int d^3x dt \sqrt{g} \left\{ NRF(\Phi) + \frac{1}{N} (E^{ij} E_{ij} - E^2) F(\Phi) - \frac{2}{N} E F'(\Phi) (\partial_t \Phi - N^i \partial_i \Phi) \right. \]

\[ + 2g^{ij} \nabla_i N \nabla_j F(\Phi) + \frac{1}{N} (\partial_t \Phi - N^i \partial_i \Phi)^2 - N g^{ij} \partial_i \Phi \partial_j \Phi - 2NV(\Phi) \right\}, \]

(2.3)

where \( F'(\Phi) = dF(\Phi)/d\Phi \) and the measure \( \sqrt{g} \), the Ricci \( R \) and covariant derivatives \( \nabla_i \) are composed of the spatial part of the metric \( g_{ij} \) alone. The quantities \( E_{ij} \) and \( E \) are related to the extrinsic curvature \( K_{ij} \) as

\[ E_{ij} = \frac{1}{2} (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) \]

(2.4)

\[ E = g^{ij} E_{ij}. \]  

(2.5)

As one observes from Eq. (2.3), there are no kinetic-like terms for \( N \) and \( N^i \). These fields therefore act as constraint or auxiliary fields. The remaining dynamical variables are \( g_{ij} \) and \( \Phi \). Although they contain in principle seven dynamical degrees of freedom, only three of them are physical due to the additional gauge freedom in the action. Diffeomorphism invariance (or covariance) allows one to reparametrize coordinates, which means that four degrees of freedom are unphysical. They can be eliminated from the action by (a): fixing the gauge freedom, or (b): constructing gauge invariant perturbations. Gauge fixing has the advantage that a clever gauge choice can lead to results much faster. A problem is that there can be gauge artifacts due to incomplete gauge fixing. Also, it may prove difficult to relate results in one gauge to those in another, as is the case in gravity where gauge transformations are non-linear. The gauge invariant approach is on the one hand more tedious, as one should take all degrees of freedom into account and rewrite e.g. the action or equations of motion in terms of gauge invariant
variables. On the other hand, it is more thorough and one is guaranteed to obtain results for the physical variables.

In this work we take a mixed approach. We fix the gauge freedom with regards to spatial coordinate transformations, but treat time reparametrizations in a gauge invariant way. As is well known from work on first order scalar perturbation theory (and the second order action describing these perturbations) this describes dynamical scalar and tensor perturbations that are gauge invariant, and constraint perturbations that are partially gauge fixed. Thus, this approach treats the most relevant perturbations, that is the dynamical perturbations, in a gauge invariant manner.

Our goal is to construct the third order action for gauge invariant scalar and tensor perturbations, starting from the unperturbed Jordan frame action (2.3). The perturbations are considered on top of the homogeneous FLRW background.

\[ g_{ij} = a^2 e^{2\zeta} (e^\alpha)_{ij} \]
\[ \Phi = \phi + \varphi \]
\[ N = \tilde{N} (1 + n) \]
\[ N^i = a^{-1} \tilde{N}(t)(a^{-1} \partial_i s + n_T^i), \] (2.6)

where \( \partial_i n_T^i = 0 \) and the background quantities are \( a = a(t), \phi = \phi(t) \) and \( \tilde{N} = \tilde{N}(t) \). This background can be inserted in the action (2.3). The background Friedmann equations and field equations are obtained by a variation of this action with respect to \( \tilde{N}, a \) and \( \phi \). This gives, respectively,

\[ H^2 = \frac{1}{6F} \left[ \dot{\phi}^2 + 2V - 6H \dot{F} \right] \]
\[ \dot{H} = \frac{1}{2F} \left( -\dot{\phi}^2 + H \dot{F} - \ddot{F} \right) \]
\[ 0 = \ddot{\phi} + 3H \dot{\phi} + V' - 6 \left( 2H^2 + \dot{H} \right) \frac{1}{2} F'. \] (2.7)

Here we have defined the dotted derivative as \( \dot{a} \equiv da/(\tilde{N} dt) \), and the Hubble parameter is \( H = \dot{a}/a \).

Next we consider perturbations. As we are interested in finding the third order action, all perturbations are of second order. Any third order perturbation vanishes due to the background equations of motion. The perturbation \( \alpha_{ij} \) is part of the scalar-vector-tensor decomposition of the metric (2.6) and reads

\[ \alpha_{ij} = \frac{\partial_i \partial_j \hat{h}}{a^2} + \frac{\partial_i (h_T^j)}{a} + \gamma_{ij}, \] (2.8)

with

\[ \partial_i h_T^i = 0, \quad \partial_i \gamma_{ij} = 0, \quad \gamma_{ii} = \delta_{ij} \gamma_{ij} = 0. \] (2.9)

The perturbation \( \gamma_{ij} \) is thus the transverse traceless tensorial perturbation, and is often referred to as the graviton. Even though the complete action (2.1) is generally covariant, the

\[ ^1 \text{Some notes on conventions, such as indices placement and factors of } \tilde{N} \text{ and } a, \text{ as well as some intermediate expansions of quantities in (2.3) can be found in appendix A.} \]
perturbations themselves are not. An infinitesimal diffeomorphism generated by a vector field \( \xi^\mu = (\xi^0, \xi^i) \), \( \xi^i = a^{-1} \partial_i \tilde{\xi} + \xi^T \), induces a gauge transformation of the perturbed metric and scalar field, which for the particular (dynamical) perturbations in Eqs. (2.6) and (2.8) means that to first order

\[
\varphi \rightarrow \varphi + \frac{\varphi \xi^0}{N} \\
\zeta \rightarrow \zeta + H \frac{\xi^0}{N} \\
\frac{\partial_i \partial_j \tilde{h}}{a^2} \rightarrow \frac{\partial_i \partial_j \tilde{h}}{a^2} - \frac{2 \partial_i \partial_j \tilde{\xi}}{a^2} \\
\frac{\partial_i h^T_{j}}{a} \rightarrow \frac{\partial_i h^T_{j}}{a} - \frac{2 \partial_i \tilde{\xi}^T_{j}}{a} \\
\gamma_{ij} \rightarrow \gamma_{ij}.
\]

Here is where the partial gauge fixing comes in. The spatial gauge freedom can be fixed by setting

\[
\tilde{h} = 0, \quad h^T_i = 0,
\]

such that the perturbed metric can be written as \( g_{ij} = a^2 e^{2 \zeta} (e^\gamma)_{ij} \). The only remaining gauge freedom is the temporal gauge freedom, characterized by the gauge parameter \( \xi^0 \). At first order in perturbations the graviton \( \gamma_{ij} \) is gauge invariant, but the scalar perturbations \( \zeta \) and \( \varphi \) are not. However, it is possible to construct a scalar combination which is gauge invariant to first order. At second order both the graviton and scalar perturbations are gauge non-invariant, but also here it is possible to define non-linear variables which are gauge invariant to second order. In principle there are infinitely many gauge invariant variables at second order, but they are all related by non-linear transformations. In a previous work \[15\] we focused on the gauge invariant scalar perturbations and showed how to construct the gauge invariant action at third order. In this work we shall focus mostly on the graviton and scalar-graviton interactions.

3 The gauge invariant action in the Jordan frame

We now wish to find the gauge invariant action at third order for the dynamical scalar and tensor. Of course, the action (2.3) is gauge invariant by definition, since it originates from the manifestly covariant action (2.1). However, the perturbations (2.6) are not gauge invariant (2.10). Thus, we set out to find the manifestly gauge invariant action at the perturbative level.

In order to do so, we first have to deal with the lapse function \( N \) and shift vector \( N^i \). As was mentioned in the previous section, these fields act as auxiliary fields. Thus, they can be solved for and their solution can be inserted back into the action\[9\]. For the second order action, it is only necessary to find the solution for \( N \) and \( N^i \) to first order in perturbations. Any second order term for \( N \) multiplies the background equation for \( H^2 \) (2.7), as this equation is precisely derived by varying the action with respect to \( N \). A second order perturbation of \( N^i \) appears as a total spatial derivative, and such a term vanishes likewise.

\[\text{The gauge fixing essentially means that we choose } \tilde{\xi} = \frac{1}{2} \tilde{h} + O(\xi^2) \text{ and } \xi^T = \frac{1}{2} h^T + O(\xi^2). \text{ Therefore, in any other place where } \xi^T \text{ and } \tilde{\xi} \text{ appear, such as the second order gauge transformations of } \zeta, \varphi \text{ and } \gamma_{ij}, \text{ or first order transformation of the constraint fields, the perturbations } \tilde{h} \text{ and } h^T_i \text{ reappear. However, in this work we actually set } \tilde{h} \text{ and } h^T_i \text{ to zero everywhere. In a sense, we thus only consider temporal gauge transformations.}\]
For the third order action, it is also sufficient to know the solutions for $N$ and $N^i$ to first order in perturbations. The third order terms vanish due to the background equations of motion, and the second order perturbations of $N$ and $N^i$ multiply the first order solutions of the constraint equations.

The first order solutions for $N$ and $N^i$ are found from the equations of motion for these variables. Starting from the action (2.3) we find

$$0 = \frac{1}{2} \sqrt{g} \left\{ [R - N^{-2} (E^{ij} E_{ij} - E^2)] F(\Phi) + 2 N^{-2} E F'(\Phi) (\partial_i \Phi - N^i \partial_i \Phi) - 2 g^{ij} \nabla_i \nabla_j F(\Phi) ight\}$$

$$0 = \nabla_j \left[ N^{-1} (E^{jk} g_{ik} - E \delta_{ij}) F(\Phi) - \delta_{ij} N^{-1} F'(\Phi) (\partial_i \Phi - N^i \partial_i \Phi) \right]$$

$$- N^{-1} (-EF'(\Phi) + \partial_i \Phi - N^i \partial_i \Phi) \partial_j \Phi. \tag{3.1}$$

The second equation gives, to first order in perturbations

$$0 = \partial_i \left[ 2 (2 HF + F') n - 4 F \dot{\zeta} - 2 \dot{\Phi} \varphi - 2 F' \dot{\varphi} + 2 HF' \varphi - 2 F'' \dot{\varphi} \right] - \frac{1}{2} \frac{\nabla^2}{a} n^T, \tag{3.2}$$

which present the solutions

$$n = \frac{1}{2 HF + F'} \left[ 2 F \dot{\zeta} + \dot{\Phi} \varphi + F' \dot{\varphi} - HF' \varphi + F'' \dot{\varphi} \right]$$

$$n^T_i = 0. \tag{3.3}$$

The other constraint in Eq. (3.1) can also be expanded to first order in perturbations, and this presents the solution for the scalar shift perturbation $\nabla^2 s$. By using the equations of motion (2.7) and the first order solution for $n$ we find

$$\frac{\nabla^2 s}{a^2} = \frac{-1}{H + \frac{1}{2} F'} \frac{\nabla^2}{a^2} \left( \zeta + \frac{1}{2} \frac{F'}{F} \varphi \right) + \frac{1}{2} \frac{\dot{\Phi}^2 + \frac{3}{2} \frac{F^2}{F'} (H + \frac{1}{2} F')}{H + \frac{1}{2} F'} \left( \zeta - \frac{H}{\dot{\varphi}} \right).$$

$$= \frac{\nabla^2}{a^2} \left[ \frac{-1}{H + \frac{1}{2} F'} \left( \zeta + \frac{1}{2} \frac{F'}{F} \varphi \right) + \chi \right]. \tag{3.4}$$

Here we have implicitly defined the scalar quantity $\chi$. The solutions (3.3) and (3.4) may be inserted into the action (2.3) perturbed to second and third order. The remaining terms in the action only depend on $\varphi$, $\zeta$ and $\gamma_{ij}$. \footnote{It is possible to keep the other scalar and vectorial part of the metric perturbation, $\tilde{h}$ and $h_i^T$, see Eq. (2.8) In that case the first order solutions of the transverse part of the shift is $n^T_i/a = \frac{1}{2} (h_i^T/a^2)$, and the longitudinal part gets an extra contribution $\nabla^2 s/a^2 = \frac{1}{2} (\nabla^2 \tilde{h}/a^2) + \ldots$. When these solutions are inserted in the second order perturbed action, the quadratic parts in $\tilde{h}$ and $h_i^T$ disappear completely. This shows that these variables are not propagating degrees of freedom.}
extra terms appear in the second order action for the remaining dynamical variables, which, not surprisingly, are precisely those that one would get after replacing \( n \) and \( s \) by their first order solutions. The decoupled perturbations are gauge invariant and their equation of motion is the first order solution of the constraint equation. At third order the perturbations for \( N \) and \( N^i \) can also be decoupled, with the decoupled fields giving the second order solutions for the constraint equations, and extra terms in the third order action corresponding to the substitution of the first order solutions of the constraint fields in the action. The procedure is explained in appendix B.

3.1 Second order gauge invariant action

Here we briefly review the quadratic computation, which was done before in the Jordan frame [22]. The scalar field action in the Jordan frame (2.3) can be expanded to second order by inserting the perturbations (2.6). Details of expansions of individual terms can be found in appendix A. Next, the perturbations of \( n \) and \( s \) can be replaced by its first order solutions (3.3) and (3.4). This immediately yields the gauge invariant action for the graviton,

\[
S^{(2)}[\gamma^2] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 F \left[ \tilde{\gamma}_{ij} \tilde{\gamma}_{ij} - \left( \frac{\partial \gamma_{ij}}{a} \right)^2 \right].
\] (3.5)

Any cross terms \( \varphi \gamma \) and \( \zeta \gamma \) vanish as total derivatives. The remaining terms in the action are quadratic \( \zeta^2 \) and \( \varphi^2 \), or mixed terms \( \zeta \varphi \). They may be combined into the action for a single scalar perturbation,

\[
S^{(2)}[w_\zeta^2] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left( \frac{\dot{\varphi}^2}{H + 1} + \frac{3 \dot{F}^2}{2 F} \right) \left[ w_\zeta^2 - \left( \frac{\partial w_\zeta}{a} \right)^2 \right],
\] (3.6)

where \( w_\zeta \) is defined as

\[
w_\zeta = \zeta - \frac{H}{\dot{\varphi}} \varphi.
\] (3.7)

This variable is gauge invariant to first order in perturbations, as can be checked straightforwardly from Eq. (2.10). \( w_\zeta \) is commonly called the curvature perturbation on uniform field hypersurfaces, since it reduces to \( \zeta \) in the gauge \( \varphi = 0 \). Alternatively, we can define another gauge invariant variable, the field perturbation on uniform curvature hypersurfaces, by

\[
w_\varphi = \varphi - \frac{\dot{\varphi}}{H} \zeta.
\] (3.8)

Also this variable is gauge invariant, and in terms of it the action can be written as

\[
S^{(2)}[w_\varphi^2] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \frac{\dot{\varphi}^2}{H + 1} + \frac{3 \dot{F}^2}{2 F} \left( \frac{H w_\varphi}{\dot{\varphi}} \right)^2 - \left( \frac{\partial H w_\varphi}{a \dot{\varphi}} \right)^2 \nabla^2 a^2 w_\zeta.
\] (3.9)

The first order equations for motion for the graviton and the gauge invariant curvature perturbation are

\[
\frac{\delta S^{(2)}}{\delta \gamma_{ij}} = 0 = a^3 F \left( -\frac{1}{a^3 F} \left( a^3 F \tilde{\gamma}_{ij} \right) \cdot + \nabla^2 \frac{\gamma_{ij}}{a^2} \right)
\]

\[
\frac{\delta S^{(2)}}{\delta w_\zeta} = 0 = -a^3 \left( \frac{\dot{\varphi}^2}{H + 1} + \frac{3 \dot{F}^2}{2 F} w_\zeta \right) + a^3 \left( \frac{\dot{\varphi}^2}{H + 1} + \frac{3 \dot{F}^2}{2 F} \right) \nabla^2 a^2 w_\zeta.
\] (3.10)
The first order equation of motion for the gauge invariant field perturbation is related to that of the curvature perturbation as

$$\frac{\delta S^{(2)}}{\delta w_\varphi} = 0 = -H \frac{\delta S^{(2)}}{\delta w_\zeta} \bigg|_{w_\zeta = -\frac{H w_\varphi}{\dot{a}}}.$$  \hspace{1cm} (3.11)

This is of course true since at first order the gauge invariant curvature perturbation and field perturbation are related via a rescaling by background quantities

$$w_\zeta = -\frac{H}{\dot{a}} w_\varphi.$$  \hspace{1cm} (3.12)

The fact that the linear gauge invariant variables on different hypersurfaces are related by time dependent rescalings, plus the fact that the graviton decouples and is gauge invariant by itself, makes it relatively easy to find scalar and graviton 2-point functions on different hypersurfaces. At higher order this is no longer the case, and we shall discuss this next.

### 3.2 Third order gauge invariant action

We continue by finding the gauge invariant vertices for scalar and graviton interactions in the Jordan frame. Due to the non-linear nature of general relativity the second order perturbations (2.6) transform non-linearly under gauge transformations. As such, the gauge invariant scalar and tensor perturbations receive quadratic contributions. This implies that some of the vertices for scalar and graviton interactions that naively appear in the third order action after inserting the perturbations (2.6) in (2.3), are actually non-physical. When the naive third order action is expressed in terms of second order gauge invariant variables, some of the cubic vertices are absorbed into the quadratic action for these variables. This greatly complicates the construction of the third order action for scalar and graviton interactions. However, there is a systematic way to isolate the physical cubic vertices and at the same time find the second order gauge invariant scalar and tensor perturbation. The procedure was outlined in Ref. [15] where the cubic vertices for scalar interactions were found. Here we apply the procedure to the cubic action for scalar-graviton interactions.

#### 3.2.1 Scalar-scalar-graviton vertices on uniform field hypersurfaces

The naive scalar-scalar-graviton vertices are found by collecting those third order terms from the action (2.3) expanded up to third order in perturbations (2.6). This gives

$$S[ssg] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ 4 \left[ F\zeta + F'\varphi + Fn \right] \gamma_{ij} \frac{\partial_i \partial_j \zeta}{a^2} + 2F\gamma_{ij} \frac{\partial_i \zeta}{a} \frac{\partial_j \zeta}{a} + \left[ -3F\zeta - F'\varphi + nF \right] \gamma_{ij} \frac{\partial_i \partial_j s}{a^2} + F\frac{\partial_i \partial_j s \partial_k \gamma_{ij}}{a^2} \frac{\partial_k s}{a} \\
- 2\gamma_{ij} \frac{\partial_i n}{a} F' \frac{\partial_j \varphi}{a} + \gamma_{ij} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} \right\} \hspace{1cm} (3.13)$$

Note that $s$ and $n$ are linear in scalar perturbations according to (3.3) and (3.4). Gauge invariance is not manifest in the action (3.13), because the scalar perturbations $\zeta$ and $\varphi$ are not linearly gauge invariant. The curvature perturbation on uniform field hypersurfaces, $w_\zeta$, on the
other hand is gauge invariant to first order. So let us find the gauge invariant cubic vertices for scalar-scalar-graviton interactions by inserting into the action above

$$\zeta = w_\zeta + \frac{H}{\phi} \varphi.$$  \hfill (3.14)

This substitution separates the action (3.13) into a manifestly gauge invariant part with $w^2\gamma$ vertices, and a gauge dependent part with $\varphi^2\gamma$ and $w_\zeta \varphi \gamma$ vertices. Schematically

$$S[ssg] = S_{GI}[w^2\gamma] + S_{GD}[\varphi^2\gamma + w_\zeta \varphi \gamma].$$  \hfill (3.15)

The gauge invariant vertices are

$$S_{GI}[w^2\gamma] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 F \left\{ 4 \left[ w_\zeta + n_{w_\zeta} \right] \gamma_{ij} \frac{\partial_i \partial_j w_\zeta}{a^2} + 2 \gamma_{ij} \frac{\partial_i w_\zeta}{a} \frac{\partial_j w_\zeta}{a} \right. \right.$$ \left.
$$\left. - \left[ 3 w_\zeta - n_{w_\zeta} \right] \frac{\partial_i \partial_j w_\zeta}{a^2} + \frac{\partial_i \partial_j w_\zeta}{a^2} \right. \right.$$ \left.
$$\left. + \frac{\partial_i \partial_j w_\zeta}{a^2} \right. \right.$$ \left.
$$\left. \frac{\partial_k \gamma_{ij}}{a} \frac{\partial_k w_\zeta}{a} \right\},$$  \hfill (3.16)

where for convenience we have separated the first order solutions of the constraint equations in a gauge invariant and gauge dependent part,

$$n = \frac{1}{H + \frac{1}{2} \dot{F}} \dot{w}_\zeta + \left( \frac{\varphi}{\phi} \right) \equiv n_{w_\zeta} + \left( \frac{\varphi}{\phi} \right).$$

$$\nabla^2 s = \frac{\nabla^2}{a^2} \left( \frac{-1}{H + \frac{1}{2} \dot{F}} w_\zeta + \chi \right) - \frac{\nabla^2}{a^2} \frac{\varphi}{\phi} \equiv \frac{\nabla^2}{a^2} s_{w_\zeta} - \frac{\nabla^2}{a^2} \frac{\varphi}{\phi}.$$  \hfill (3.17)

Note from Eq. (3.4) that the $\chi$ part of the first order solution is gauge invariant by itself. Although we did not write the gauge dependent part of the third order action $S_{GD}[\varphi^2\gamma + w_\zeta \varphi \gamma]$ explicitly, it is easy to see that it is nonzero. This seems to pose a major problem, since it suggests that gauge invariance is broken at the perturbative level. However, we should remind ourselves that in the third order action we deal with second order perturbations. Likewise, these perturbations transform to second order under gauge transformations. As a consequence, the first order gauge invariant curvature perturbation $w_\zeta$ and graviton $\gamma_{ij}$ are no longer gauge invariant. Schematically we can write these second order gauge transformations as (see Refs. \cite{14} \cite{15})

$$w_\zeta \rightarrow w_\zeta + \Delta^2 w_\zeta$$

$$\gamma_{ij} \rightarrow \gamma_{ij} + (\Delta^2 \gamma)_{ij}.$$  \hfill (3.18)

Under such gauge transformations the second order actions for $w_\zeta$ and $\gamma_{ij}$ transform as

$$S^{(2)}[\gamma^2] \rightarrow S^{(2)}[\gamma^2] + \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta \gamma_{ij}} (\Delta^2 \gamma)_{ij} \right\},$$

$$S^{(2)}[w_\zeta^2] \rightarrow S^{(2)}[w_\zeta^2] + \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta w_\zeta} \Delta^2 w_\zeta \right\},$$  \hfill (3.19)
up to total derivatives. This second order gauge transformation of the quadratic action can only be balanced by gauge dependent terms in the third order action which are proportional to the equation of motion. So in fact, in order for the third order action to be gauge invariant under second order transformations, we expect to have a gauge dependent part in the third order action after the substitution (3.14) into (2.3). This gauge dependent part must be proportional to the linear equations of motion. Indeed, after many partial integrations we find that

\[
S_{GD}[\varphi^2 \gamma + w_\zeta \varphi \gamma] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta \gamma_{ij}} \left[ \frac{\partial_i \varphi}{a \phi} \frac{\partial_j \varphi}{a \phi} - \left( \frac{\partial_i \varphi}{a \phi} \frac{\partial_j s_{w_\zeta}}{a} + \frac{\partial_j \varphi}{a \phi} \frac{\partial_i s_{w_\zeta}}{a} \right) \right] \right. \\
+ 2 \frac{\delta S^{(2)}}{a^3} \frac{\delta \varphi}{\nabla^2} \left( \frac{1}{4} \frac{\varphi}{\phi} \gamma_{ij} \right) \right\}.
\]  

The complete action to third order in perturbations can now be written in a manifestly gauge invariant way by defining second order gauge invariant variables,

\[
\tilde{\gamma}_{\zeta,ij} = \gamma_{ij} + \frac{\partial_i \varphi}{a \phi} \frac{\partial_j \varphi}{a \phi} - \left( \frac{\partial_i \varphi}{a \phi} \frac{\partial_j s_{w_\zeta}}{a} + \frac{\partial_j \varphi}{a \phi} \frac{\partial_i s_{w_\zeta}}{a} \right) + O(\varphi \gamma)
\]

\[
W_\zeta = w_\zeta + \frac{\partial_i \varphi}{\nabla^2} \left( \frac{1}{4} \frac{\varphi}{\phi} \gamma_{ij} \right) + O(\varphi^2).
\]

The first variable is the second order gauge invariant tensor perturbation, and may be called the graviton on uniform field hypersurfaces, since it reduces to \(\gamma_{ij}\) in the gauge \(\varphi = 0\). Similarly, \(W_\zeta\) is the second order curvature perturbation on uniform field hypersurfaces. The manifestly gauge invariant quadratic actions for these perturbations have precisely the same form as Eqs. (3.5) and (3.6), i.e.

\[
S^{(2)}[\tilde{\gamma}_\zeta] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \frac{F}{4} \left[ \tilde{\gamma}_{\zeta,ij} \tilde{\gamma}_{\zeta,ij} - \left( \frac{\partial \tilde{\gamma}_{\zeta,ij}}{a} \right)^2 \right]
\]

\[
S^{(2)}[W_\zeta^2] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \frac{\dot{H}^2 + \frac{3}{2} \dot{F}^2}{(H + \frac{1}{2} \dot{F})^2} \left[ \dot{W}_\zeta^2 - \left( \frac{\partial W_\zeta}{a} \right)^2 \right].
\]

The vertices for scalar-scalar-graviton interactions can be found immediately from Eq. (3.16) by replacing the first order gauge invariant variables by their second order generalizations (which does not change anything for the cubic vertices). Thus we get

\[
S[W_\zeta^2 \tilde{\gamma}_\zeta] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 F \left\{ 4 \left[ W_\zeta + n W_\zeta \right] \tilde{\gamma}_{\zeta,ij} \frac{\partial_i \partial_j W_\zeta}{a^2} + 2 \tilde{\gamma}_{\zeta,ij} \frac{\partial_i W_\zeta}{a} \frac{\partial_j W_\zeta}{a} \\
- \left[ 3 W_\zeta - n W_\zeta \right] \tilde{\gamma}_{\zeta,ij} \frac{\partial_i \partial_j s_{w_\zeta}}{a^2} + \frac{\partial_i \partial_j s_{w_\zeta}}{a^2} \frac{\partial_k \tilde{\gamma}_{\zeta,ij}}{a} \frac{\partial_k s_{w_\zeta}}{a} \right\}.
\]

Note that the gauge invariant variables can still contain other terms at second order, as can be seen from Eqs. (3.22). The \(O(\gamma \varphi)\) terms in the definition for \(\tilde{\gamma}_\zeta\) can only originate from the scalar-graviton-graviton action at third order, which is precisely what we shall discuss in the next section. The \(O(\varphi^2)\) terms in the definition for \(W_\zeta\) originate from the cubic scalar interactions. They were discussed (in the long wavelength limit) in Refs. [14] and [15].
In conclusion, we have presented and applied a systematic method to find the manifestly gauge invariant action at third order. The trick is to simply insert the first order gauge invariant variable (3.14) into the naive cubic vertices. Not only does this method give the physical cubic vertices, but it also gives the correct form of the second order gauge invariant variables. Usually the second order gauge invariant variables are found by fixing the second order gauge transformation of the perturbations (see e.g. [7, 8]). Here no explicit second order gauge transformations were necessary. The only ingredient that we used is the manifest general covariance of the complete action (2.1), and the notion that a manifestly gauge invariant can be written down order by order in perturbation theory.

The method can also be generalized to higher order perturbation theory. Take the manifestly gauge invariant action for second order perturbations, Eqs. (2.23) and (2.24). The action (2.3) expanded to fourth order in perturbation theory contains again \( \zeta, \varphi \) and \( \gamma_{ij} \). Now replace \( \zeta = W_\zeta + H\varphi/\phi \) and \( \gamma_{ij} = \tilde{\gamma}_{\zeta,ij} \). The fourth order action separates into a gauge invariant part with second order gauge invariant variables \( W_\zeta \) and \( \tilde{\gamma}_{\zeta,ij} \), and a gauge dependent part. The gauge dependent part must be proportional to the linear equations of motion, and to \( \delta S^3/\delta w_\zeta \) and \( \delta S^3/\delta \gamma_{ij} \). The first part must be there to balance the third order gauge transformation of the quadratic action, and can be absorbed into the definition of third order gauge invariant variables. The second part must be there to balance the second order gauge transformation of the cubic action, and can be absorbed into the definition of second order gauge invariant variables in the cubic action. Note that we replaced \( w_\zeta \to W_\zeta \) in the cubic vertices, which is only allowed when we consider the action up to third order in perturbations. One could go to even higher order in perturbation theory, where in general the gauge dependent part of the \( n \)th order action can be absorbed into lower order actions by defining gauge invariant variables at \( n - 1 \)th order.

### 3.2.2 Scalar-graviton-graviton vertices on uniform field hypersurfaces

So far we have only discussed the scalar-scalar-graviton vertices. The gauge invariant scalar-graviton-graviton and pure graviton vertices can be computed using the same method as described in the previous section. We start with the naive scalar-graviton-graviton vertices from the expansion of (2.3)

\[
S[\text{sgg}] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ -2F_{\gamma_{ik}} \gamma_{kj} \frac{\partial \partial_{\gamma_{ik}}} {a^2} - 2F_{\gamma_{ik}} \gamma_{ij} \frac{\partial \partial_{\gamma_{ik}}} {a} \gamma_{ij} \right. \nonumber \\
+ \left. \left( 3F_\zeta + F'_\varphi \right) \frac{1}{4} \gamma_{kl} \gamma_{kl} - \left( F_\zeta + F'_\varphi \right) \frac{1}{4} \frac{\partial \partial_{\gamma_{kl}}} {a} \gamma_{kl} \right. \nonumber \\
- \left. nF \left( \frac{1}{4} \gamma_{kl} \gamma_{kl} + \frac{1}{4} \frac{\partial \partial_{\gamma_{kl}}} {a} \gamma_{kl} \right) - \frac{1}{2} \frac{\partial \gamma_{kl}} {a} \frac{\partial \partial_{s_{\gamma_{kl}}}} {a} \right\}. \tag{3.25}
\]

Again we separate the cubic terms in manifestly gauge invariant vertices composed of \( w_\zeta \) and \( \gamma_{ij} \), plus gauge non-invariant terms, by using (3.14). The gauge invariant vertices are

\[
S[G]_{\text{GI}}[w_\zeta, \gamma^2] = \frac{1}{2} \int d^3x dt \tilde{N} a^3 F \left\{ -2\gamma_{ik} \gamma_{kj} \frac{\partial \partial_{\gamma_{ik}}} {a^2} - 2\gamma_{ik} \frac{\partial \partial_{\gamma_{ik}}} {a} \gamma_{ij} \right. \nonumber \\
+ \left. \left( 3w_\zeta - n_{w_\zeta} \right) \frac{1}{4} \gamma_{kl} \gamma_{kl} - \left( w_\zeta + n_{w_\zeta} \right) \frac{1}{4} \frac{\partial \partial_{\gamma_{kl}}} {a} \gamma_{kl} \right\}. \tag{3.26}
\]

\(^4\)Of course, one could check that the second order gauge invariant variables which follow from the action procedure agree with those constructed via the gauge transformation method as in [8].
and the gauge dependent part of the action (3.25) (with $\varphi \gamma^2$ terms) becomes after several partial integrations

$$S_{GD}[\varphi \gamma^2] = \frac{1}{2} \int d^3 x dt \bar{N} a^3 \left\{ -\frac{\varphi}{\dot{\phi}} \frac{2}{a^3} \delta S^{(2)}_{\gamma} + \frac{2}{a^3} \Delta S^{(2)} \right\}, \quad (3.27)$$

up to boundary terms. Hence, this gauge dependent part of the scalar-graviton-graviton action may be absorbed by a redefinition of $\gamma_{ij}$. Considering the previously defined second order graviton on uniform field hypersurfaces $\tilde{\gamma}_{c,ij}$, this precisely describes the $O(\varphi \gamma)$ contribution. Thus, the complete gauge invariant graviton to second order is

$$\tilde{\gamma}_{c,ij} = \gamma_{ij} + \frac{\partial_i \varphi}{a \dot{\phi}} \frac{\partial_j \varphi}{a \dot{\phi}} - \left( \frac{\partial_i \varphi}{a \dot{\phi}} \frac{s_{w_{c}}}{a} + \frac{\partial_j \varphi}{a \dot{\phi}} \frac{s_{w_{c}}}{a} \right) \Delta \gamma_{ij}. \quad (3.28)$$

The manifestly gauge invariant cubic vertices for scalar-graviton-graviton interactions are then

$$S[W_{c}^2 \tilde{\gamma}_{c}] = \frac{1}{2} \int d^3 x dt \bar{N} a^3 F \left\{ -2 \tilde{\gamma}_{c,ik} \tilde{\gamma}_{c,jk} - 2 \tilde{\gamma}_{c,ik} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} - \frac{1}{2} \tilde{\gamma}_{c,ik} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} \right\} + (3W_{c} - nW_{c}) \frac{1}{4} \tilde{\gamma}_{c,kl} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} \right\}. \quad (3.29)$$

### 3.2.3 Pure graviton vertices on uniform field hypersurfaces

Since the second order tensor perturbation $\gamma_{ij}$ is gauge invariant to first order, the pure graviton vertices are automatically gauge invariant. Thus the $\gamma^3$ found after an expansion of (2.3) are the physical vertices. In the cubic vertices we can replace $\gamma_{ij}$ by its second order gauge invariant generalization, $\tilde{\gamma}_{c,ij}$, which gives the manifestly gauge invariant vertices

$$S[\gamma^3_{c}] = \frac{1}{2} \int d^3 x dt \bar{N} a^3 F \left\{ \frac{1}{4} \tilde{\gamma}_{c,ij} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} + \frac{1}{4} \tilde{\gamma}_{c,kl} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} - \frac{1}{4} \tilde{\gamma}_{c,ik} \frac{\partial_i \varphi}{a} \frac{\partial_j \varphi}{a} \right\}. \quad (3.30)$$

### 3.2.4 Scalar-graviton interactions on uniform curvature hypersurfaces

In the previous sections we we constructed in a systematic the cubic gauge invariant action for the second order graviton and curvature perturbation on uniform field hypersurfaces. Similarly, we can find the gauge invariant action for scalar field and tensor perturbations on uniform curvature hypersurfaces. The starting point is again the action (2.3), which expanded to third order for scalar-graviton interactions gives Eqs. (3.13) and (3.25). Instead of replacing $\zeta$ by its linearly gauge invariant variable $w_{c}$, we now make use of Eq. (3.8) to replace

$$\varphi = w_{c} + \frac{\dot{\varphi}}{H} \zeta. \quad (3.31)$$

This separates the cubic vertices into a manifestly gauge invariant part involving linearly gauge invariant scalar perturbation $w_{c}$ and tensor $\gamma_{ij}$, plus gauge dependent vertices that involve the gauge dependent variable $\zeta$. Analogous to the previous sections, the gauge dependent part of the cubic action can be brought to a form which is proportional to the linear equations of motion.
for \( w_\varphi \) and \( \gamma_{ij} \). These terms may be absorbed into the quadratic action by defining second order variables

\[
\tilde{\gamma}_{\varphi,ij} = \gamma_{ij} - \frac{\zeta}{H} \tilde{\gamma}_{ij} + \frac{\partial \zeta}{aH} \tilde{\gamma}_{ij} + \frac{\partial \zeta}{aH} \left( \frac{\partial w_\varphi}{aH} \right) - \frac{\partial \zeta}{aH} \tilde{\gamma}_{ij} \left( \frac{\partial s_{w_\varphi}}{aH} + \frac{\partial s_{w_\varphi}}{aH} \right)
\]

\[
W_\varphi = w_\varphi - \frac{\partial \zeta}{\sqrt{a^2}} \left( \frac{1}{4H^2} \tilde{\gamma}_{ij} + \frac{1}{4H^2} \tilde{\gamma}_{ij} \right) + O(\zeta^2),
\]

where we have separated the first order constraint solutions (3.3) and (3.4) into a gauge invariant and gauge dependent part

\[
\begin{align*}
\tilde{\gamma}_{\varphi,ij} & = \gamma_{ij} - \frac{\zeta}{H} \tilde{\gamma}_{ij} + \frac{\partial \zeta}{aH} \tilde{\gamma}_{ij} + \frac{\partial \zeta}{aH} \left( \frac{\partial w_\varphi}{aH} \right) - \frac{\partial \zeta}{aH} \tilde{\gamma}_{ij} \left( \frac{\partial s_{w_\varphi}}{aH} + \frac{\partial s_{w_\varphi}}{aH} \right) \\
W_\varphi & = w_\varphi - \frac{\partial \zeta}{\sqrt{a^2}} \left( \frac{1}{4H^2} \tilde{\gamma}_{ij} + \frac{1}{4H^2} \tilde{\gamma}_{ij} \right) + O(\zeta^2),
\end{align*}
\]

In the gauge \( \zeta = 0 \) the perturbations \( \tilde{\gamma}_{\varphi,ij} \) and \( W_\varphi \) reduce to the graviton \( \gamma_{ij} \) and field perturbation \( \varphi \). Therefore, they are called the second order gauge invariant graviton on uniform curvature hypersurfaces and field perturbation on uniform curvature hypersurfaces, respectively. The \( O(\zeta^2) \) terms in the second order perturbation \( W_\varphi \) follow from the pure scalar interactions, which have been discussed (in part) in [14] and [15]. The quadratic action for the perturbations (3.32) is

\[
S^{(2)}[\tilde{\gamma}_{\varphi,ij}] = \frac{1}{2} \int d^3x dt \tilde{\gamma}_{\varphi,ij} \tilde{\gamma}_{\varphi,ij} + \left( \frac{\partial \zeta}{a^2} \right)^2
\]

\[
S^{(2)}[W_\varphi^2] = \frac{1}{2} \int d^3x dt \tilde{\gamma}_{\varphi,ij} \left( \frac{\partial \zeta}{a^2} \right)^2 - \frac{\partial \zeta}{a^2} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2.
\]

The interaction vertices are found by replacing \( w_\varphi \) and \( \gamma_{ij} \) in the gauge invariant part of the third order action by their second order generalizations. This gives the scalar-graviton-graviton vertices

\[
S[W_\varphi^2 \tilde{\gamma}_{\varphi,ij}] = \frac{1}{2} \int d^3x dt \tilde{\gamma}_{\varphi,ij} \left( n_{W_\varphi} F - \frac{\partial \zeta}{a^2} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 - 2 \tilde{\gamma}_{\varphi,ij} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 \right)
\]

\[
S[W_\varphi^2 \tilde{\gamma}_{\varphi,ij}] = \frac{1}{2} \int d^3x dt \tilde{\gamma}_{\varphi,ij} \left( n_{W_\varphi} F - \frac{\partial \zeta}{a^2} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 - 2 \tilde{\gamma}_{\varphi,ij} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 \right)
\]

and the scalar-graviton-graviton vertices

\[
S[W_\varphi^2 \tilde{\gamma}_{\varphi,ij}] = \frac{1}{2} \int d^3x dt \tilde{\gamma}_{\varphi,ij} \left( \frac{\partial \zeta}{a^2} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 - 2 \tilde{\gamma}_{\varphi,ij} \left( \frac{\partial s_{w_\varphi}}{a^2} \right)^2 \right)
\]
and finally the pure graviton vertices

$$S[\tilde{\gamma}^3_{\varphi}] = \frac{1}{2} \int d^3 x d t N a^3 F \left\{ \frac{1}{4} \tilde{\gamma}_{\varphi,ij} \frac{\partial \hat{\gamma}_{\varphi,kl}}{a} \frac{\partial \hat{\gamma}_{\varphi,kl}}{a} + \frac{1}{4} \tilde{\gamma}_{\varphi,kl} \frac{\partial \hat{\gamma}_{\varphi,kl}}{a} \right\} .$$

(3.37)

In these interaction actions the $n_{W_{\varphi}}$ and $s_{W_{\varphi}}$ are obtained by replacing $w_{\varphi} \to W_{\varphi}$ in Eqs. (3.33).

### 3.3 Uniqueness of the scalar-graviton action

We have now computed two gauge invariant actions for cubic scalar and graviton interactions. Eqs. (3.24), (3.29) and (3.30) describe the interactions for the curvature perturbation and graviton on uniform field hypersurfaces, $W_{\zeta}$ and $\tilde{\gamma}_{\zeta,ij}$. Eqs. (3.35), (3.36) and (3.37) describe the interactions for the field perturbation and graviton on uniform curvature hypersurfaces, $W_{\varphi}$ and $\hat{\gamma}_{\varphi,ij}$. If we compare the gauge invariant vertices on different hypersurfaces, we see that they are not the same. They are however related in a relatively simple way. We can show this by writing the cubic actions on different hypersurfaces in the same form. For example, the scalar-scalar-graviton interactions Eqs. (3.24) and (3.35) may be partially integrated in order to find, up to boundary terms,

$$S[W_{\zeta}^2 \tilde{\gamma}_{\zeta}] = \frac{1}{2} \int d^3 x d t N a^3 \left\{ -\frac{1}{2} \left( \frac{\phi^2 + 3 \frac{E^2}{2 \mathcal{F}}} {H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \right) \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} + \frac{1}{4} \left( \frac{\phi^2 + 3 \frac{E^2}{2 \mathcal{F}}} {H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \right) \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \right\} + \frac{1}{H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \left( \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \right) \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} + \frac{1}{H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} 

(3.38)

and

$$S[W_{\varphi}^2 \hat{\gamma}_{\varphi}] = \frac{1}{2} \int d^3 x d t N a^3 \left\{ \frac{1}{2} \left( \frac{\phi^2 + 3 \frac{E^2}{2 \mathcal{F}}} {H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \right) \frac{W_{\varphi} \tilde{\gamma}_{\varphi,ij}}{a} \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} + \frac{1}{4} \left( \frac{\phi^2 + 3 \frac{E^2}{2 \mathcal{F}}} {H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \right) \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} \right\} + \frac{1}{H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \left( \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} \right) \frac{\partial \hat{\gamma}_{\varphi,ij}}{a} + \frac{1}{H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} 

(3.39)

Similarly, the scalar-graviton-graviton actions (3.29) and (3.36) become after some partial integrations,

$$S[W_{\zeta} \tilde{\gamma}_{\zeta}^2] = \frac{1}{2} \int d^3 x d t N a^3 \left\{ \frac{\phi^2 + 3 \frac{E^2}{2 \mathcal{F}}} {H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \frac{W_{\zeta}}{2} \left( \frac{1}{4} \tilde{\gamma}_{\zeta,ij} \tilde{\gamma}_{\zeta,ij} + \frac{1}{4} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \right) \right\} - \frac{F}{2} \tilde{\gamma}_{\zeta,ij} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} \frac{\partial \hat{\gamma}_{\zeta,ij}}{a} - \frac{2}{H + \frac{1}{2} \frac{E^2}{\mathcal{F}}} \delta S^{(2)} \hat{\gamma}_{\zeta,ij} ,$$

(3.40)
and

\[ S[W_\phi, \tilde{\gamma}_{\phi,ij}^2] = \frac{1}{2} \int d^3x dt N a^3 \left\{ \frac{\dot{\phi}^2 + \frac{3 F^2}{2 H}}{H + \frac{3 F}{2 H}} \frac{1}{2} \left( -\frac{H W_\phi}{\dot{\phi}} \right) \left( \frac{1}{4} \tilde{\gamma}_{\phi,ij} \tilde{\gamma}_{\phi,ij} + \frac{1}{4} \frac{\partial \tilde{\gamma}_{\phi,ij}}{a} \frac{\partial \tilde{\gamma}_{\phi,ij}}{a} \right) \right. \]

\[ - \frac{F}{2} \frac{\partial_k \tilde{\gamma}_{\phi,ij}}{a} \frac{\partial_k \chi}{a} - \frac{\tilde{\gamma}_{\phi,ij}}{H + \frac{3 F}{2 H}} \frac{1}{H} \left( -\frac{H W_\phi}{\dot{\phi}} \right) \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta \tilde{\gamma}_{\phi,ij}} \right\}. \]

(3.41)

We see that the actions for \( W_\zeta \) and \( \tilde{\gamma}_{\zeta,ij} \) on the one hand, are almost of the same form of those for \( W_\phi \) and \( \tilde{\gamma}_{\phi,ij} \) on the other hand. They only differ by terms proportional to the equation of motion and boundary terms (which can be found in appendix C). In fact, we can see that the actions (3.38) and (3.39) become of the same form once we identify

\[ \tilde{\gamma}_{\phi,ij} = \tilde{\gamma}_{\zeta,ij} - \frac{\partial_i W_\zeta}{a H} \frac{\partial_j W_\zeta}{a H} - \left( \frac{\partial_i W_\zeta}{a H} \frac{\partial_j s W_\zeta}{a} + \frac{\partial_j W_\zeta}{a H} \frac{\partial_i s W_\zeta}{a} \right) \]

\[ - \frac{H W_\zeta}{\dot{\phi}} = W_\zeta + \frac{\partial_i \partial_j \frac{1}{\nabla^2} \tilde{\gamma}_{\zeta,ij} W_\zeta}{H} + O(W_\zeta^2), \] (3.42)

The action for the curvature and graviton perturbation on uniform field hypersurfaces is thus related to the action for the field and graviton perturbation on uniform curvature hypersurfaces via non-linear field redefinitions. Of course, Eqs. (3.42) are nothing more than non-linear relations between gauge invariant variables on different hypersurfaces. It can be checked straightforwardly that the gauge invariant variables (3.22) and (3.32) satisfy the relation (3.42). The exercise in this section shows that the non-linear relation between different hypersurfaces also follows from the action. Note that we did not consider the boundary terms here. They are discussed in the next section.

Some final words on uniqueness of the scalar-graviton interactions. We have seen that the actions on different hypersurfaces differ by terms proportional to the linear equations of motion and boundary terms. This means that the complete non-linear equations of motion for \( W_\zeta \) and \( \tilde{\gamma}_{\zeta,ij} \) actually equal to those for \( -\frac{H W_\phi}{\dot{\phi}} \) and \( \tilde{\gamma}_{\phi,ij} \), since the boundary terms do not contribute and the terms proportional to the equation of motion vanish at linear order. Thus, the evolution of the fields, and of \( n \)-point functions, is the same whether one works on one hypersurface or another. In that sense, there is a unique bulk action that describes the evolution of non-Gaussianity [15]. Still, \( n \)-point functions in the interaction can most certainly receive contributions from boundary terms in the action, which we shall discuss next.

### 3.3.1 3-point functions and boundary terms

Due to the non-linear relation between different variables on different hypersurfaces, the 3-point functions on different hypersurfaces are related via disconnected pieces. For example, in the case of pure scalar interactions, the disconnected pieces are proportional to the square of the scalar power spectrum, see for example [9]. In the case of 3-point functions for scalar-graviton interactions, also the 3-point function on one hypersurface differs from that on another hypersurface by a product of the scalar and gravitational power spectrum. For example, if we consider only the local contributions in the non-linear relations (3.42), we find that the
scalar-graviton-graviton 3-point function on different hypersurfaces are related as

\[
\langle - \frac{H W_\varphi(x_1)}{\dot{\varphi}} \hat{\gamma}_\varphi(x_2) \hat{\gamma}_\varphi(x_3) \rangle = \langle W_\zeta(x_1) \hat{\gamma}_\zeta(x_2) \hat{\gamma}_\zeta(x_3) \rangle 
- \frac{1}{H} \left( \langle W_\zeta(x_1) W_\zeta(x_2) \rangle \langle \hat{\gamma}_\zeta(x_2) \hat{\gamma}_\zeta(x_3) \rangle + \langle W_\zeta(x_1) W_\zeta(x_3) \rangle \langle \hat{\gamma}_\zeta(x_2) \dot{\hat{\gamma}}_\zeta(x_3) \rangle \right) + \ldots .
\]

(3.43)

Here we used that \( \hat{\gamma}_\varphi,ij = \hat{\gamma}_\zeta,ij - \dot{\hat{\gamma}}_\zeta,ij W_\zeta/H + \ldots \), and the dots denote non-local contributions. Furthermore, we have made use of the Wick contraction and the fact that \( \langle W_\zeta \hat{\gamma}_\zeta \rangle = 0 \). The non-local terms are difficult to deal with in coordinate space, but can be treated consistently in momentum space. Considering the relation (3.43), one can show that in the so-called squeezed limit – where one of the momenta is much smaller than the other two – the scalar-graviton-graviton reduces to a product of the scalar and graviton 2-point correlators multiplied by the spectral index for tensor fluctuations, \( n_T \), see Ref. [9]. For the case of the scalar-scalar-graviton correlator, it can be seen from the non-linear relations (3.42) that the correlator on different hypersurfaces differs by non-local terms.

Eq. (3.42) is just an example of two gauge invariant variables that are non-linearly related, but in principle it can be used for any non-linearly related variables. This is very useful when computing 3-point functions for a certain variable from an action, since one can choose to work with a non-linearly related variable for which the action takes a form which is most suitable in specific situations. For example, during the slow-roll inflationary expansion it is useful to use the action for \( W_\varphi \) and \( \hat{\gamma}_\varphi,ij \), since the orders in slow-roll are separated and it is easy to see what terms are dominant. This is what was used in Ref. [9] to compute the bispectrum for a minimally coupled scalar field.

Now some words on the boundary terms. We have derived Eq. (3.43) based purely on the non-linear relations(3.42). Eq. (3.43) should also follow from the action, but this is not obvious when we consider the rewritten actions on the uniform field hypersurface (3.38) and (3.40) versus the actions on uniform curvature hypersurfaces (3.39) and (3.41). The bulk actions are of the same form, and therefore lead to similar 3-point functions. Terms proportional to the equations of motion do not contribute to bispectrum, as they vanish when evaluated on the solutions of the first order equation of motion. This suggests that the disconnected contributions should follow from the boundary terms that are obtained after partial integrations. Indeed, in the so-called in-in formalism temporal boundary terms containing time derivatives of fields can contribute to the bispectrum. These boundary terms were discussed for the pure scalar case in Refs. [28][29][14][15]. There, it was found that the temporal boundary terms precisely provide the dominant contributions to the disconnected pieces of the bispectrum. Thus, in the slow-roll regime the 3-point function for \( W_\zeta \) can be computed via the 3-point function for \( W_\varphi \).

So, in the pure scalar case the non-linear field redefinition (3.42) takes into account the dominant contributions coming from the boundary terms in the action for \( W_\zeta \). However, in previous work [15] we have argued that the field redefinition should completely take care of all boundary terms (by which the action on uniform field hypersurfaces differs from that on uniform curvature hypersurfaces). The argument is straightforward: the gauge invariant action for \( W_\zeta \) and \( \hat{\gamma}_\zeta,ij \), and the action for \( W_\varphi \) and \( \hat{\gamma}_\varphi,ij \), are both manifestly gauge invariant actions at third order that follow from the same original covariant action (2.1). The variables are related via a specific non-linear relation (3.42). Thus, also the manifestly gauge invariant actions should be related by this non-linear relation. To be more precise, if we write the non-linear transforma-
tions (3.42) schematically as \( \tilde{\gamma}_{\varphi,ij} = \tilde{\gamma}_{\zeta,ij} + Q_{ij}(W_\zeta, \tilde{\gamma}_\zeta) \) and \(-HW_\varphi/\dot{\phi} = W_\zeta + Q_{ij}(W_\zeta, \tilde{\gamma}_\zeta)\) the quadratic actions for \( \tilde{\gamma}_{\varphi,ij} \) and \( W_\varphi \) (3.34) change under the non-linear transformation as

\[
S^{(2)}[\tilde{\gamma}_{\varphi,ij}] = S^{(2)}[\tilde{\gamma}_{\zeta,ij}] + \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ Q_{ij}(W_\zeta, \tilde{\gamma}_\zeta) \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta \tilde{\gamma}_{\zeta,ij}} + \frac{1}{a^3} \left[ a^3 F \frac{\dot{\gamma}_{\zeta,ij} Q_{ij}(W_\zeta, \tilde{\gamma}_\zeta)}{\dot{\phi}} \right] - \frac{\partial_i}{a} \left[ F \frac{\partial_j \tilde{\gamma}_{\zeta,ij}}{\dot{\phi}} Q_{ij}(W_\zeta, \tilde{\gamma}_\zeta) \right] \right\},
\]

(3.44)

and

\[
S^{(2)}[W_\varphi^2] = S^{(2)}[W_\zeta^2] + \frac{1}{2} \int d^3x dt \tilde{N} a^3 \left\{ Q(W_\zeta, \tilde{\gamma}_\zeta) \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta W_\zeta} + \frac{1}{a^3} \left[ 2a^3 \frac{\dot{\phi}^2 + \frac{3}{2} \dot{F}^2}{(H + \frac{1}{2} \dot{F})^2} W_\zeta Q(W_\zeta, \tilde{\gamma}_\zeta) \right] - \frac{\partial_i}{a} \left[ 2a^3 \frac{\dot{\phi}^2 + \frac{3}{2} \dot{F}^2}{(H + \frac{1}{2} \dot{F})^2} \frac{\partial_j W_\zeta}{a} Q(W_\zeta, \tilde{\gamma}_\zeta) \right] \right\},
\]

(3.45)

We have seen that the third order action for \( W_\zeta \) and \( \tilde{\gamma}_{\zeta,ij} \) can be written as the third order action for \(-HW_\varphi/\dot{\phi} \) and \( \tilde{\gamma}_{\varphi,ij} \), plus terms proportional to the equation of motion and boundary terms, see Eqs. (3.38) and (3.40). The terms by which the actions differ are precisely the equation of motion terms as expected by the non-linear transformations (3.44)–(3.45). The boundary terms are however not precisely the expected ones. In appendix C we give explicit expressions for the boundary terms and show that the non-linear the redefinitions (3.42) do not completely take care of the boundary terms. Based on gauge invariance of the action, all boundary terms must be taken into account by the non-linear field redefinition. We suspect that the discrepancy is due to two factors that we did not consider. First, boundary terms are already present in the second order action (3.9). Under a non-linear redefinition (3.42) these second order boundary terms generate third order boundary terms. Second, boundary terms are also present in the ADM formulation of the action, i.e. by going from Eq. (2.1) to Eq. (2.3). In total, these boundary terms could make sure that the action on one hypersurface is related to that on another, at the bulk as well as at the boundary level.

4 Frame independent cosmological perturbations

4.1 The conformal transformation

So far we have worked exclusively in the Jordan frame and derived the physical vertices for the manifestly gauge invariant variables. It is however well known that the Jordan frame action with non-minimal coupling (2.1) can be brought to the Einstein frame action with minimal coupling

\[
S = \frac{1}{2} \int d^4x \sqrt{-g_E} \left\{ R_E - g_{\mu\nu}^{\prime} \partial_\mu \Phi_E \partial_\nu \Phi_E - 2V_E(\Phi_E) \right\}.
\]

(4.1)

by a combined redefinition of the metric and scalar field

\[
g_{\mu\nu,E} = \Omega^2 g_{\mu\nu},
\]

\[
\left( \frac{d\Phi_E}{d\Phi} \right)^2 = \frac{1}{\Omega^2} \left( 1 + 6\Omega^2 \right)
\]

\[
V_E(\Phi_E) = \frac{1}{\Omega^3} V(\Phi),
\]

(4.2)
with
\[ \Omega^2 = \Omega^2(\Phi) = F(\Phi). \] (4.3)

No physical content is lost by field redefinitions, and thus, in that sense, the Jordan and Einstein frame are physically equivalent. Nonetheless, there is a defining frame for e.g. the Hubble parameter that describes the rate of expansion of space. Take the example of Higgs inflation \cite{16, 17}. Here the defining frame is the Jordan frame, where \( V(\Phi) \) is the Standard Model Higgs potential and \( \Phi \) is the Higgs field that is non-minimally coupled to the Ricci scalar via \( F = 1 + \xi \Phi^2 \). The metric \( g_{\mu\nu} \) describes the curvature of spacetime on which the Higgs field lives.

Generally speaking, it is complicated to do computations directly in the Jordan frame due to the non-minimal coupling. Fortunately we can exploit the physical equivalence of Jordan and Einstein frame. It is possible to first transform to the Einstein frame, then perform the relevant computations in that simpler frame, and finally transform the results back to the Jordan frame. For instance, for the background alone one can straightforwardly show that
\[
\dot{N}_E = F^{\frac{1}{2}} \ddot{N}
\]
\[
a_E = F^{\frac{1}{2}} a
\]
\[
H_E = \frac{1}{F^{\frac{1}{2}}} \left( H + \frac{1}{2} \dot{F} \right)
\]
\[
\dot{\phi}_E = \frac{1}{F^{\frac{1}{2}}} \frac{d\phi_E}{d\phi} \dot{\phi} = \frac{1}{F^{\frac{1}{2}}} \sqrt{\frac{1}{F} + \frac{3 F^2}{2 F^2} \dot{\phi}},
\] (4.4)

where the dotted derivative in Einstein frame is defined as \( \dot{\phi}_E = d\phi_E/(\ddot{N}_E dt) \), and \( H_E = \ddot{a}_E/a_E \).

With these background relations the Friedman and field equations in the Jordan frame (2.7) can be derived immediately from the well-known Einstein frame equations,
\[
H_E^2 = \frac{1}{6} \left[ \frac{1}{F} + \frac{3 F^2}{2 F^2} \dot{\phi} \right]
\]
\[
\dot{H}_E = -\frac{1}{2} \dot{\phi}_E^2
\]
\[
0 = \ddot{\phi}_E + 3 H_E \dot{\phi}_E + V_E.
\] (4.5)

This demonstrates the equivalence at the level of the background equations of motion. In Higgs inflation, or more general theories with non-minimal coupling, the equivalence is frequently used to study the inflationary expansion of the early universe. In order to solve the horizon and flatness problems inflation must have lasted sufficiently long, which translates to the condition that the number of e-folds \( N_e \equiv \int H \ddot{N} dt \) exceeds 60. This means that \( H \) must be roughly constant and very slowly changing in time, which is not easy to see in the Jordan frame. However, in the Einstein frame it is easy to see that the potential becomes almost flat for very large field values, which leads to a slow-rolling trajectory and an approximately constant Hubble parameter \( H_E \), which in turn is closely related to the Jordan frame Hubble parameter \( H \). \footnote{It can be shown that in Higgs inflation the number of e-folds in Jordan and Einstein frame are approximately the same, \( N_e \sim N_{e,E} \).}

### 4.2 Frame independent perturbations at first order

Even though the equivalence between Jordan and Einstein frame can be easily demonstrated for the background fields and action, it is far from obvious for the perturbed action. The reason is
that the perturbations in Einstein frame do not coincide with those in the Jordan frame. For example, if we expand the complete field redefinitions (4.2) up to first order in perturbations, we find that

\[
\varphi_E = \frac{d\phi_E}{d\phi} \varphi
\]

\[
\zeta_E = \zeta + \frac{1}{2} \frac{F'}{F} \varphi
\]

\[
\gamma_{ij,E} = \gamma_{ij}.
\] (4.6)

Thus, if we take the quadratic Einstein frame action for perturbations \(\varphi_E, \zeta_E\) and \(\gamma_{ij,E}\), with background quantities \(H_E\) and \(\dot{\phi}_E\), it is difficult to see that we obtain the quadratic Jordan frame action when we insert the relations (4.4) and (4.6). Of course, we should obtain the perturbed Jordan frame action after transforming the action in Einstein frame, since we know that the complete Jordan and Einstein frame action Eqs. (2.1) and (4.1) are related via (unperturbed) field redefinitions. Thus, the frame transformation should also work order by order in perturbation theory.

At this point we can draw a nice analogy with the discussion of gauge invariance in the previous section. For the original, unperturbed action (2.1) general covariance was manifest. Thus gauge invariance should also be present order by order in perturbation theory. For the background the gauge invariance is trivial, since the background is by definition the part of the metric that does not transform under a gauge transformation. The gauge invariance at the perturbative level is not so obvious, because the perturbations themselves are not gauge invariant. However, we were able to write the perturbed action also in a manifestly gauge invariant way, by expressing it in terms of variables which are inherently gauge invariant.

Based on this analogy we can ask ourselves: is it possible to express the quadratic Einstein and Jordan frame action in terms of certain variables in which it is particularly simple to see that they are related via a frame transformation? The answer is yes. First of all, the graviton in the Einstein frame is equal to that in the Jordan frame, see Eq. (4.6). Thus, if we take the quadratic Einstein frame action

\[
S^{(2)}[\gamma_E^2] = \frac{1}{2} \int d^3x dt \epsilon_E^a a_E^3 \frac{1}{4} \left[ \dot{\gamma}_{ij,E} \dot{\gamma}_{ij,E} - \left( \frac{\partial \gamma_{ij,E}}{a_E} \right)^2 \right],
\] (4.7)

we immediately obtain the Jordan frame action (3.5) when we transform the background quantities according to Eq. (4.4) and replace \(\gamma_{ij,E} = \gamma_{ij}\). Now, for the scalar perturbations we can combine \(\zeta_E\) and \(\varphi_E\) in such a way that

\[
w_{\zeta,E} = \zeta_E - \frac{H_E}{\dot{\phi}_E} \varphi = \zeta - \frac{H}{\dot{\phi}} \varphi \equiv w_\zeta.
\] (4.8)

Thus, the linearly gauge invariant curvature perturbation in the Einstein frame coincides with the curvature perturbation in the Jordan frame \([20, 21]\). Thus, in terms of this variable, the quadratic Jordan frame action can be found straightforwardly from the Einstein frame action by transforming the background quantities, as we can see when we compare the Jordan frame action (3.6) with the Einstein frame action

\[
S^{(2)}[w_{\zeta,E}^2] = \frac{1}{2} \int d^3x dt \epsilon_E^a a_E^3 \frac{\dot{\phi}_E^2}{H_E^2} \left[ \dot{w}_{\zeta,E}^2 - \left( \frac{\partial w_{\zeta,E}}{a_E} \right)^2 \right].
\] (4.9)
The gauge invariant graviton $\gamma_{ij}$ and curvature perturbation $w_{\zeta}$ take the same form after a frame transformation. Thus, they can be called *frame independent cosmological perturbations*, at least to first order. Analogous to the discussion of gauge invariance, where a manifestly gauge invariant form of the action is reached when using gauge invariant variables, here the action is “manifestly equivalent” in terms of frame independent perturbations. What we mean is that, when expressed in terms of $w_{\zeta}$ and $\gamma_{ij}$, the actions in Jordan and Einstein frame are directly related via (trivial) frame transformations of the background alone. Thus, if one computes certain results in the Einstein frame using these frame independent perturbations, such as the scalar or tensorial power spectrum, the corresponding Jordan frame result is immediately obtained by transforming the background quantities. This was used in, for example [16, 20, 21, 30]. Note that the power spectrum, for example, is expressed in terms of $H$ and $\dot{\phi}$ in the Jordan frame, but in terms of $\dot{H}_E$ and $\dot{\phi}_E$ in the Einstein frame. Of course, the actual value of the amplitude and spectral index is independent of the choice of frame, once the equations of motion for $H$ and $\phi$, or $H_E$ and $\dot{\phi}_E$ are solved and their values computed at horizon crossing. This shows that physical observables are independent of the choice of frame.

It is no surprise that the gauge invariant curvature perturbation on uniform field hypersurfaces is a frame independent perturbation as well. In the gauge $\varphi = 0$ the curvature perturbation $w_{\zeta}$ reduces to $\zeta$. Also, in this gauge the frame transformation, which is a function of $\Phi$, becomes a function of the background field alone. Thus the frame transformation does not affect the perturbations, and the perturbed action in Jordan frame is simply obtained from the Einstein frame by transforming the background fields alone.

### 4.3 Frame independent perturbations at second order

As we have seen in section refsec: Third order gauge invariant action, at higher order the gauge transformations become non-linear, such that the manifestly gauge invariant action at third order is expressed in terms of second order gauge invariant variables. In analogy to this, we expect that a manifestly equivalent action can be expressed in terms of second order frame independent perturbations. Based on the previous section, we are lead to believe that the second order curvature perturbation and graviton on uniform field hypersurfaces are also frame independent to second order.

Based on the previous section, we are lead to believe that the second order curvature perturbation and graviton on uniform field hypersurfaces are also frame independent. In the gauge $\varphi = 0$ they take the same form after a frame transformation. Thus, if one computes certain results in the Einstein frame using these frame independent perturbations, such as the scalar or tensorial power spectrum, the corresponding Jordan frame result is immediately obtained by transforming the background quantities. This was used in, for example [16, 20, 21, 30]. Note that the power spectrum, for example, is expressed in terms of $H$ and $\dot{\phi}$ in the Jordan frame, but in terms of $\dot{H}_E$ and $\dot{\phi}_E$ in the Einstein frame. Of course, the actual value of the amplitude and spectral index is independent of the choice of frame, once the equations of motion for $H$ and $\phi$, or $H_E$ and $\dot{\phi}_E$ are solved and their values computed at horizon crossing. This shows that physical observables are independent of the choice of frame.

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$$W_{\zeta,E} = w_{\zeta,E} + \frac{\partial_i \partial_j}{\nabla^2} \left( \frac{1}{4} \frac{\varphi}{\dot{\phi}_E} \gamma_{ij,E} \right) + O(\varphi^2) = w_{\zeta} + \frac{\partial_i \partial_j}{\nabla^2} \left( \frac{1}{4} \frac{\varphi}{\dot{\phi}} \gamma_{ij} \right) + O(\varphi^2) = W_{\zeta}. \quad (4.10)$$

Similarly we can show for the graviton on uniform field hypersurfaces

$$\tilde{\gamma}_{\zeta,ij,E} = \gamma_{ij,E} + \frac{\partial_i \varphi}{a \Phi E} \frac{\partial_j \varphi}{a \Phi E} - \left( \frac{\partial_i \varphi}{a E \phi_E} \frac{\partial_j s_{w_{\zeta,E}}}{a E \phi_E} + \frac{\partial_i \varphi}{a E \phi_E} \frac{\partial_j s_{w_{\zeta,E}}}{a E \phi_E} \right) - \frac{\varphi}{\phi_E} \tilde{\gamma}_{ij} = \tilde{\gamma}_{\zeta,ij,E}. \quad (4.11)$$

Here we used that $a^{-1} \dot{s}_{w_{\zeta,E}} = a^{-1} s_{w_{\zeta}}$, where $s_{w_{\zeta}}$ is defined in (3.17). Thus $W_{\zeta,E} = W_{\zeta}$ and $\tilde{\gamma}_{\zeta,ij,E} = \tilde{\gamma}_{\zeta,ij}$, such that the gauge invariant scalar and tensor perturbations on uniform field hypersurfaces are frame independent to second order. Of course, in the gauge $\varphi = 0$ it becomes

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6The frame independence of the gauge invariant curvature perturbation can be demonstrated to all orders [31, 32] by using the fully non-linear generalization of the curvature perturbation [33–35].
particularly clear that $W_\zeta$ and $\tilde{\gamma}_{\zeta,ij}$ do not transform under a frame transformation, which can be generalized to arbitrary order to show that the curvature perturbation is frame independent [36]. By similar reasoning, also the graviton on uniform field hypersurfaces is frame independent to all orders.

Now that we have proven that $W_\zeta$ and $\tilde{\gamma}_{\zeta,ij}$ are frame independent variables, it is trivial to show that the cubic Jordan and Einstein frame actions, expressed in those variables, are physically equivalent. Take the cubic vertices for scalar-graviton interactions in the Jordan frame, Eqs. (3.24) and (3.29). The same vertices are obtained by taking the corresponding Einstein frame action (setting $F = 1$) and transforming the background quantities according to Eq. (4.4). Thus one can use the Einstein frame action to compute, for example, the 3-point function for $W_\zeta$ in terms of $H_E$ and $\dot{\phi}$, and then re-express everything in terms of $H$ and $\phi$ to find the 3-point function in the Jordan frame. Again, once these background quantities are solved for through their equations of motion and their values inserted in the 3-point function, we would find exactly the same number. For example, $f_{\text{NL}}$ computed in the Einstein frame coincides with that computed in Jordan frame, and is thus invariant under (unphysical) field redefinitions [23].

Although the cubic action for $W_\zeta$ and $\tilde{\gamma}_{\zeta,ij}$ is "manifestly equivalent", the perturbative physical equivalence of Jordan and Einstein frame is not so obvious when the action is expressed in terms of the gauge invariant field and graviton perturbations on uniform curvature hypersurfaces, $W_\varphi$ and $\tilde{\gamma}_{\varphi,ij}$. The cubic interaction vertices for these perturbations in the Einstein frame are (set $F = 1$ in Eqs. (3.35) and (3.36))

$$S[W_{\varphi,E}^2 \tilde{\gamma}_{\varphi,E}] = \frac{1}{2} \int d^3x dt N_E a_E^3 \left\{ \frac{\phi_E}{2 H_E^2} \frac{1}{\phi_E} H_E W_\varphi, E \tilde{\gamma}_{\varphi,ij,E} \frac{\partial_i \partial_j \chi_E}{a_E^2} + \frac{\phi_E^2}{H_E^2} \tilde{\gamma}_{\varphi,ij,E} \frac{\partial_i H_E W_\varphi, E \partial_j H_E W_\varphi, E}{a_E \phi_E} a_E \phi_E + \frac{\partial_i \partial_j \chi_E}{a_E^2} \frac{\partial k \tilde{\gamma}_{\varphi,ij,E}}{a_E} \right\},$$  

(4.12)

and

$$S[W_\varphi, E \tilde{\gamma}_{\varphi,E}^2] = \frac{1}{2} \int d^3x dt N_E a_E^3 \left\{ \frac{\phi_E}{2 H_E^2} \frac{1}{\phi_E} \left( \frac{1}{4} \tilde{\gamma}_{\varphi,ij,E} \tilde{\gamma}_{\varphi,ij,E} + \frac{1}{4} \frac{\partial^2 \tilde{\gamma}_{\varphi,ij,E}}{a_E} \partial k \chi_E \right) - \frac{1}{2} \tilde{\gamma}_{\varphi,ij,E} \frac{\partial ^2 \chi_E}{a_E} \right\}.$$  

(4.13)

Here $\chi_E$ is defined by

$$\frac{\nabla^2 \chi_E}{a_E^2} = \frac{1}{2} \frac{\phi_E^2}{H_E^2} \left( -\frac{H_E W_\varphi, E}{\phi_E} \right).$$  

(4.14)

Now, if we compare with the Jordan frame actions (3.35) and (3.36), we see that the actions in the two frames are not simply related by re-expressing the background fields as (4.4). The reason is that the second order field perturbation on uniform curvature hypersurfaces, are not frame independent, $W_\varphi, E \neq W_\varphi$ and $\tilde{\gamma}_{\varphi,ij,E} \neq \tilde{\gamma}_{\varphi,ij}$. Even so, it is not clear how exactly the different frames are related at the perturbative level when working on the uniform curvature hypersurface. Fortunately, we can make this more explicit by several partial integrations of the Jordan frame action, which make it look more like the transformed Einstein frame action. In fact, we have already performed these partial integrations in section 3.3, and the resulting actions are Eqs. (3.39) and (3.41). These actions are almost of the same form as the Einstein
frame actions (4.12)–(4.13) after a frame transformation of the background, but they differ by terms proportional to the linear equation of motion and boundary terms. In analogy to the relation between the actions on different hypersurfaces in section 3.3, we can now identify

\[
\tilde{\gamma}_{\phi,ij,E} = \gamma_{\phi,ij} + \frac{1}{2F} \left[ \frac{W_{\phi}}{\phi} \tilde{\gamma}_{\phi,ij} + \frac{1}{2F} \frac{H + 1}{H} \partial_i W_{\phi} \partial_j W_{\phi} - \left( \frac{\partial_i W_{\phi}}{\phi} \partial_j W_{\phi} + \frac{\partial_j W_{\phi}}{\phi} \partial_i W_{\phi} \right) \right] + O(W_{\phi}^2),
\]

(4.15)

such that the Jordan and Einstein frame actions on uniform curvature hypersurfaces are related by combined transformations of the background (4.4) and non-linear transformations of the perturbations (4.15) \(^7\). This demonstrates the physical equivalence for the third order action expressed in gauge invariant variables on the uniform curvature hypersurface, although the action is not "manifestly equivalent".

The non-linear relation (4.8) between the gauge invariant variables in different frames implies that the \(n\)-point functions in different frames are related via disconnected terms. For example, the 3-point function for \(W_{\phi,E}\) differs from the 3-point function for \(W_{\phi}\) by squares of the 2-point function [15]. Thus, care must be taken when computing Jordan frame quantities via the Einstein frame, if one works on the uniform curvature hypersurface. An example of a delicate situation is the computation of the naive cut-off in Higgs inflation [37–42], where the cut-off seems to depend on whether it is found directly in the Jordan frame, or via the Einstein frame. As we have discussed in previous work [15], and emphasize again here, the cut-off should not depend on the frame. This is most obvious when working with frame independent variables, but the frame equivalence is also non-linearly realized in the variables \(W_{\phi}\) and \(\tilde{\gamma}_{\phi,ij}\).

5 Summary and outlook

The aim of this work was to compute the gauge invariant action at third order for a single scalar field in the Jordan frame, with emphasis on the graviton and its interactions with the scalar perturbations. Even though the unperturbed action is manifestly covariant, it is not obviously so at the perturbative level. The reason is that the perturbations themselves are gauge dependent. We have demonstrated a method to find the manifestly gauge invariant level order by order in perturbation theory. The procedure relies on separating the higher order action in a gauge invariant plus a gauge dependent part and absorbing the latter into the definition of a new variable. By doing so one not only obtains the physical vertices, but at the same time finds the correct higher order gauge invariant variables. The method thus provides an alternative way to find gauge invariant variables directly from the action, without having to resort to non-linear gauge transformations.

In section 3 we computed the cubic gauge invariant action for the second order curvature perturbation and graviton on uniform field hypersurfaces, and for the second order field perturbation and graviton on uniform curvature hypersurfaces, both directly in the Jordan frame. We demonstrated that the different action are related via non-linear transformations of the gauge invariant variables, which are precisely the non-linear relations between the variables on different hypersurfaces. The actions on different hypersurfaces only differ by terms proportional to the

\(^7\)The non-linear relations (4.15) can also be derived from the definitions of the gauge invariant perturbations in the Einstein frame (3.32) and replacing (4.4) and (4.6).
equation of motion and boundary terms, such that the evolution of an \( n \)-point function is the same for one set of gauge invariant variables or another. In this sense the gauge invariant action is unique. Still, the \( n \)-point functions, for example the bispectrum, for one set of variables differs from that for another by disconnected pieces due to the non-linear relation.

The situation concerning gauge invariance is quite similar to that of the frame transformation. For the unperturbed action in the Jordan frame it is well known that it can be brought to the Einstein frame action via field redefinitions, and are in that sense physically equivalent. This is however not obvious at the perturbative level, since the perturbations themselves are not invariant under the frame transformation. However, the perturbed action can be written in a ”manifestly equivalent” form by expressing it in terms of frame independent perturbations. We have demonstrated that the gauge invariant curvature perturbation and graviton on uniform field hypersurfaces in the Jordan frame coincide with the corresponding perturbations in the Einstein frame, and are thus frame independent. In terms of these variables, the perturbed action in the Jordan frame can be obtained from the Einstein frame action by a frame transformation of the background fields alone. Moreover, we have shown that the field perturbation and graviton on uniform curvature hypersurfaces in the Einstein frame are non-linearly related to their counterparts in the Jordan frame. This can be derived from the perturbations themselves, but also follows from the action for these variables. As a consequence, \( n \)-point functions for perturbations on uniform curvature hypersurfaces differ by disconnected pieces between different frames.

In conclusion, we have shown that whether one takes the action for gauge invariant perturbations on different hypersurfaces in the same frame, or the action for the same gauge invariant perturbations in different frames, they are all related via non-linear transformations, which makes it very convenient to find \( n \)-point functions on a specific hypersurface/in a specific frame via the \( n \)-point functions on a different hypersurface/in a different frame.

The results in this work may be used to compute gauge invariant 3-point functions for scalar-graviton interactions, and can be extended to compute gauge invariant 1-loop quantum corrections to, for example, the scalar or tensorial power spectrum in a general non-minimally coupled theory. For such loop corrections one should derive the fourth order gauge invariant action, and the procedure outlined in this paper can be readily extended to do precisely that. Moreover, we can readdress quantum corrections and the naturalness problem in Higgs inflation in an unambiguous way by using physical, frame independent perturbations. We intend to address these questions in following work.

6 Acknowledgements

This research was supported by the Dutch Foundation for 'Fundamenteel Onderzoek der Materie' (FOM) under the program ”Theoretical particle physics in the era of the LHC”, program number FP 104.

A Conventions and expansions

In the main text we are using the action (2.3) and perturb it to third order using the second order perturbations defined in Eq. (2.6) with the spatial gauge fix (2.11). Thus the perturbed metric becomes

\[
g_{ij} = a^2 e^{2\xi} (\epsilon^\gamma)_{ij} = a^2 e^{2\xi} \left( \delta_{ij} + \gamma_{ij} + \frac{1}{2} \gamma_{ik} \gamma_{kj} \right),
\]  

(A.1)
where we have expanded to second order. The conventions we are using here is that, at the perturbative level, we define all quantities such as spatial derivatives, tensors, vectors and the Kronecker delta with lower indices. Thus, the Laplacian for perturbations is defined as \( \nabla^2 = \delta_{ij} \partial_i \partial_j \), and the inverse of the metric is

\[
g^{ij} = a^{-2} e^{-2\zeta} (e^{-\gamma})^{-1} j_{ij} = a^{-2} e^{-2\zeta} \left( \delta_{ij} - \gamma_{ij} + \frac{1}{2} \gamma_{ik} \gamma_{kj} \right). \tag{A.2} \]

It can be checked straightforwardly that \( g_{ik} g^{kj} = \delta_{ij} \). Moreover, we define the shift perturbation with an upper index according to (2.6). For the unperturbed quantities such as \( N_i \) and \( E_{ij} \) indices are raised and lowered by the unperturbed metric, i.e. \( N_i = g_{ij} N^j \) and \( E^{ij} = g^{ia} g^{jb} E_{ab} \).

With the definition of the perturbed metric (A.1) the determinant becomes

\[
\sqrt{g} = a^3 e^{3\zeta}. \tag{A.3} \]

Derivatives of the metric are also particularly simple, and at second order they are

\[
\dot{g}_{ij} = 2(H + \dot{\zeta}) g_{ij} + \frac{1}{2}(\dot{\gamma}_{ik} g_{kj} + g_{ik} \dot{\gamma}_{kj}) \]

\[
\partial_k g_{ij} = 2 \partial_k \zeta g_{ij} + \frac{1}{2}(g_{ij} \partial_k \gamma_{il} + g_{il} \partial_k \gamma_{ij}). \tag{A.4} \]

With this we can construct the Christoffel symbols

\[
\Gamma^k_{ij} = [\delta_{ik} \partial_j \zeta + \delta_{jk} \partial_i \zeta - \delta_{ij} \partial_k \zeta]
+ \frac{1}{4} \left[ \partial_j \gamma_{ik} + \partial_i \gamma_{jk} + g^{km} g_{id} (\partial_j \gamma_{lm} - \partial_m \gamma_{ij}) + g^{km} g_{jd} (\partial_i \gamma_{lm} - \partial_m \gamma_{ij}) \right] \]

\[
\Gamma^k_{ik} = 3 \partial_i \zeta, \tag{A.5} \]

which can be used to find for example

\[
\nabla_i N^i = \partial_i N^i + 3 \partial_i \zeta N^i, \tag{A.6} \]

and the Ricci scalar

\[
R = -4 g^{ij} \partial_i \zeta \partial_j \zeta - 2 g^{ij} \partial_i \zeta \partial_j \zeta + 2 g^{ij} \partial_i \gamma_{jk} \partial_j \zeta - \frac{1}{8} g^{ij} \partial_i \gamma_{kl} \partial_j \zeta - \frac{1}{8} g^{ij} \partial_k \gamma_{il} \partial_j \gamma_{jk} + \frac{1}{8} g^{ij} g^{kl} g_{mn} (-\partial_l \gamma_{km} \partial_j \gamma_{lm} + \partial_l \gamma_{mn} \partial_k \gamma_{jm}). \tag{A.7} \]

If we expand this to third order in perturbations we find

\[
\sqrt{g} R = a e^\zeta \left[ -4 \nabla^2 \zeta - 2(\partial \zeta)^2 + 4 \gamma_{ij} \partial_i \zeta \partial_j \zeta + 2 \gamma_{ij} \partial_i \zeta \partial_j \zeta - 2 \gamma_{ik} \gamma_{kj} \partial_i \partial_j \zeta - 2 \gamma_{ik} \partial_i \gamma_{kj} \partial_j \zeta
- \frac{1}{4} \partial_k \gamma_{kl} \partial_l \gamma_{kl} + \frac{1}{4} \gamma_{ij} \partial_l \gamma_{kl} \partial_j \gamma_{kl} + \frac{1}{4} \gamma_{kl} \partial_l \gamma_{kj} \partial_j \gamma_{kl} - \frac{1}{4} \gamma_{ik} \partial_l \gamma_{jl} \partial_j \gamma_{kl} \right]. \tag{A.8} \]

### B Decoupling the constraint fields in the action

We now present an alternative method to deal with the auxiliary fields in the action. Instead of solving for them to first order in perturbations, as was done in [9], we can decouple them from
the dynamical fields in the action. The method was explained in Refs. [22, 27]. After inserting
the perturbations (2.6) in the action (2.3) and expanding to second order, we can collect only
those terms that contain perturbations of the lapse and shift field. After performing some partial
integrations and making use of the background equations of motion the action takes the form

\[ S^{(2)}[n, s] = \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ -2Vn^2 + I_n n + 2 \left( 2HF + \dot{F} \right) n_{\text{sol}} \frac{\nabla^2 s}{a^2} \right\}, \tag{B.1} \]

where

\[ I_n = 6(2HF + \dot{F})\dot{\zeta} + (6H F' - \dot{\phi})\dot{\varphi} + (6H^2 F' + 6HF'' - 2V')\varphi \]

\[ - 2(2HF + \dot{F})\frac{\nabla^2 s}{a^2} - 2\frac{\nabla^2 s}{a^2}(2F'\zeta + F'\varphi), \tag{B.2} \]

and

\[ n_{\text{sol}} = \frac{1}{2HF + \dot{F}} \left[ 2F\dot{\zeta} + \phi\varphi + F'\dot{\varphi} - HF'\varphi + F''\phi\varphi \right]. \tag{B.3} \]

In terms of \( n_{\text{sol}} \) we may re-express \( I_n \) as

\[ I_n = 4Vn_{\text{sol}} - 2(2HF + \dot{F}) \left( \frac{\nabla^2 s}{a^2} - \frac{\nabla^2 s_{\text{sol}}}{a^2} \right), \tag{B.4} \]

where

\[ \frac{\nabla^2 s_{\text{sol}}}{a^2} = -\frac{1}{H + \frac{1}{2F}} \frac{\nabla^2 \left( \zeta + \frac{1}{2} F'\varphi \right)}{a^2} + \frac{1}{2F} \left( \frac{\dot{\varphi}^2 + \frac{3}{2} \frac{\dot{F}^2}{F}}{H + \frac{1}{2F}} \right) \left( \zeta - \frac{H}{\phi} \varphi \right). \tag{B.5} \]

Now we define new variables \( \tilde{s} \) and \( \tilde{n} \) as

\[ \frac{\nabla^2 \tilde{s}}{a^2} = \frac{\nabla^2 s}{a^2} - \frac{\nabla^2 s_{\text{sol}}}{a^2}, \]

\[ \tilde{n} \equiv n - \frac{1}{4V} I_n = n - n_{\text{sol}} - 2(2HF + \dot{F}) \frac{\nabla^2 \tilde{s}}{a^2}. \tag{B.6} \]

In terms of these variables, the action (B.1) can be rewritten as

\[ S^{(2)}[n, s] = \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ -2V\tilde{n}^2 + \frac{(2HF + \dot{F})^2}{V} \left( \frac{\nabla^2 \tilde{s}}{a^2} \right)^2 + 2Vn_{\text{sol}}^2 + 2 \left( 2HF + \dot{F} \right) n_{\text{sol}} \frac{\nabla^2 s_{\text{sol}}}{a^2} \right\}. \tag{B.7} \]

Now, the first two terms in the action are the completely decoupled quadratic actions for \( \tilde{n} \) and \( \tilde{s} \). The equations of motions for these variables are simply \( \tilde{n} = 0 \) and \( \nabla^2 \tilde{s}/a^2 = 0 \), which give the first order solutions of the constraint equations, \( n = n_{\text{sol}} \) and \( \nabla^2 s/a^2 = \nabla^2 s_{\text{sol}}/a^2 \). The additional two terms in the rewritten quadratic action are precisely the terms that we would have obtained if we would have solved the constraint equations to first order in perturbations and inserted the solution in the action. This is what we wanted to proof, although we have so far only shown it for the second order action.

The extra terms in Eq. (B.7) are needed to construct the manifestly gauge invariant action for \( w_\zeta \). This implies that the variables \( \tilde{n} \) and \( \tilde{s} \) are gauge invariant by themselves, which can be
shown explicitly [22]. The gauge invariance can be exploited to decouple non-dynamical degrees of freedom from dynamical ones at higher order. The reasoning is very similar to that in section 3.2.1.

Let us consider the schematic form of the cubic action for the lapse perturbation alone

\[ S^{(3)}[n] = \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ 2V n^3 + n^2 Q^{(1)} + n Q^{(2)} \right\}. \]  

(B.8)

Here the \( Q^{(1)} \) and \( Q^{(2)} \) are linear and quadratic functions of the dynamical perturbations \( \zeta, \phi \) and \( \gamma_{ij} \) (and of \( s \), but for simplicity we neglect it). The explicit form of these functions can be derived when expanding Eq. (2.3) up to third order in perturbations. We want to decouple the non-dynamical degrees of freedom from the dynamical ones, but this naively does not seem possible. Here is where we exploit the gauge invariance of the action. Although \( \tilde{n} \) is gauge invariant to first order, it transforms under second order gauge transformations as

\[ \tilde{n} \rightarrow \tilde{n} + \Delta \tilde{n}. \]  

(B.9)

This induces a change in the quadratic action (B.7)

\[ S^{(2)}[\tilde{n}] \rightarrow S^{(2)}[\tilde{n}] + \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ \frac{2}{a^3} \frac{\delta S^{(2)}}{\delta \tilde{n}} \Delta \tilde{n} \right\} = S^{(2)}[\tilde{n}] + \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ -4V \tilde{n} \Delta \tilde{n} \right\}. \]  

(B.10)

Such a second order gauge transformation was also shown for the dynamical perturbations (3.19), and a similar gauge transformation can be derived for the action for \( \tilde{s} \). Now, the only way in which such a second order gauge transformation of the quadratic action can be balanced, is by terms proportional to the equations of motion for \( \tilde{n} \) in the third order action. Since the equation of motion for a non-dynamical field is simply proportional to the field itself, we should find all the terms in the third order action proportional to \( \tilde{n} \). We can do this systematically for the action (B.8) by replacing \( n \) by \( \tilde{n} \) using Eq. (B.6). The result is

\[ S^{(3)}[\tilde{n}] = \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ 2V \tilde{n}^3 - 4V \tilde{n}\left[ -\frac{3}{2} \tilde{n}_{\text{sol}} - \frac{3}{2} \tilde{n}_{\text{sol}}^2 - \frac{Q^{(1)} \tilde{n}}{4V} - \frac{Q^{(1)} n_{\text{sol}}}{2V} - \frac{Q^{(2)}}{4V} \right] \right. \]

\[ + 2V n_{\text{sol}}^3 + n_{\text{sol}}^2 Q^{(1)} + n_{\text{sol}} Q^{(2)} \right\}. \]  

(B.11)

The terms on the first line is a gauge invariant vertex for the non-dynamical \( \tilde{n} \), and terms proportional to the linear equation of motion. The latter can be absorbed into the quadratic action by defining a new second order gauge invariant variable

\[ \tilde{N} = \tilde{n} - \frac{3}{2} \tilde{n}_{\text{sol}} - \frac{3}{2} \tilde{n}_{\text{sol}}^2 - \frac{Q^{(1)} \tilde{n}}{4V} - \frac{Q^{(1)} n_{\text{sol}}}{2V} - \frac{Q^{(2)}}{4V}, \]  

(B.12)

such that the cubic action (B.13) becomes

\[ S^{(3)}[\tilde{N}] = \frac{1}{2} \int d^3x dt \bar{N} a^3 \left\{ 2V \tilde{N}^3 + 2V n_{\text{sol}}^3 + n_{\text{sol}}^2 Q^{(1)} + n_{\text{sol}} Q^{(2)} \right\}. \]  

(B.13)
Thus we have found a decoupled cubic vertex for $\tilde{N}$, plus remaining terms which are the same as those obtained by replacing $n$ in Eq. (B.8) by its first order solution. In Eq. (B.13) we have neglected the $\tilde{s}$ terms, but in a very similar way they can be absorbed into the definition of a second order gauge invariant shift perturbation $\tilde{S}$. So, finally we are left with a decoupled, manifestly gauge invariant cubic action for the non-dynamical fields $\tilde{N}$ and $\tilde{S}$, and extra terms that we would have obtained when the constraints would have been replaced by their first order solution. This is precisely what we wanted to prove. The method outlined here is a nice and systematic way to decouple the non-dynamical sector from the dynamical one, while at the same time finding the second order gauge invariant constraint fields (which are the second order solutions of the constraint equations).

The procedure here can be extended to higher order, for example fourth order. First, we found the second order gauge invariant action in terms of the linearly gauge invariant $\tilde{n}$ and $\tilde{s}$. Next, we insert these quantities in the third and fourth order action. From the third order action we now find the definition for second order perturbations $\tilde{N}$ and $\tilde{S}$. We can now replace in the third and fourth order action $\tilde{n} = \tilde{N} + n^{(2)}_{\text{sol}} + \ldots$ and $\tilde{s} = \tilde{S} + s^{(2)}_{\text{sol}} + \ldots$, where $n^{(2)}_{\text{sol}}$ and $s^{(2)}_{\text{sol}}$ are the second order parts of the solution of the constraint equation. This gives gauge invariant cubic vertices for $\tilde{N}$ and $\tilde{S}$. Finally, the terms proportional to $\tilde{N}$ and $\tilde{S}$ in the fourth order action can be absorbed in the quadratic action for $\tilde{N}$ and $\tilde{S}$ by defining a third order gauge invariant variable. The remaining terms in the action are those where $n$ and $s$ are replaced by their second order solution $n^{(2)}_{\text{sol}}$ and $s^{(2)}_{\text{sol}}$. This was also found in Ref. [11], and applied in Refs. [12, 13] to find the fourth order action for perturbations in the uniform curvature gauge.

C Boundary terms

Here we discuss some of the boundary terms that appear due to the partial integrations section 3.3. We only show the temporal boundary terms, as these are terms that can possibly contribute to the bispectrum.

We start with the boundary terms in the scalar-graviton-graviton action. By going from Eq. (3.29) to Eq. (3.40), we find the following temporal boundary terms

$$S_0[W_\zeta \zeta^2_\zeta] = \frac{1}{2} \int d^3x \left\{ \frac{a^3 F}{H + \frac{1}{2} F} W_\zeta \left( -\frac{1}{2} \dot{\zeta}_{,ij} \dot{\zeta}_{,ij} - \frac{1}{4} \frac{\partial \zeta_{,ij}}{a} \frac{\partial \dot{\zeta}_{,ij}}{a} \right) \right\}. \tag{C.1}$$

Likewise, going from Eq. (3.36) to Eq. (3.41) we obtain

$$S_0[W_\varphi \dot{\varphi}^2_\varphi] = \frac{1}{2} \int d^3x \left\{ \frac{a^3 F}{H + \frac{1}{2} F} \frac{-\dot{F}}{2F H} \left( -\frac{H W_\varphi}{\phi} \right) \left( -\frac{1}{4} \dot{\varphi}_{,ij} \dot{\varphi}_{,ij} - \frac{1}{4} \frac{\partial \varphi_{,ij}}{a} \frac{\partial \dot{\varphi}_{,ij}}{a} \right) \right\}. \tag{C.2}$$

In section 3.3.1 it was argued that by the non-linear relation (3.42) the action for $W_\varphi$ and $\dot{\varphi}_\varphi$ should transform into that for $W_\zeta$ and $\dot{\zeta}_\zeta$, both at the bulk and boundary level. If we consider the scalar-graviton-graviton action alone, and see what boundary terms for $W_\zeta$ and $\dot{\zeta}_\zeta$ we get under the transformation (3.42), we find

$$\frac{1}{2} \int d^3x \left\{ \frac{a^3 F}{H + \frac{1}{2} F} \frac{-\dot{F}}{2F H} \left( -\frac{1}{4} \dot{\zeta}_{,ij} \dot{\zeta}_{,ij} - \frac{1}{4} \frac{\partial \zeta_{,ij}}{a} \frac{\partial \dot{\zeta}_{,ij}}{a} \right) - \frac{a^3 F}{2} \frac{W_\zeta}{H} \frac{\dot{\zeta}_{,ij}}{\zeta_{,ij}} \right\} = S_0[W_\zeta \dot{\zeta}^2_\zeta] + \frac{1}{2} \int d^3x \left\{ \frac{a^3 F}{H} W_\zeta \left( -\frac{1}{4} \dot{\zeta}_{,ij} \dot{\zeta}_{,ij} + \frac{1}{4} \frac{\partial \dot{\zeta}_{,ij}}{a} \frac{\partial \zeta_{,ij}}{a} \right) \right\}. \tag{C.3}$$
The first line contains terms coming from the boundary terms in Eq. (C.2), plus terms that are generated from the quadratic action through the transformation, see Eq. (3.44). Eq. (C.3) suggests that the scalar-graviton-graviton boundary terms are not related via the non-linear transformation (3.42). In the main text we argue why the boundary terms must be related as well, and we discuss what may cause the discrepancy.

Similarly, we can compute the boundary terms for scalar-scalar-graviton interactions. By going from Eq. (3.24) to Eq. (3.38), we find the following temporal boundary terms

\[ S_0 W_2 \tilde{\gamma}_\zeta = \frac{1}{2} \int d^3 x \left\{ -a^3 \frac{F}{H + \frac{1}{2} F} \left[ \tilde{\gamma}_{\zeta,ij} \frac{1}{2} \left( \frac{\partial_i W_\zeta}{a} \frac{\partial_j \chi}{a} + \frac{\partial_j W_\zeta}{a} \frac{\partial_i \chi}{a} - \frac{1}{H + \frac{1}{2} F} \frac{\partial_i W_\zeta}{a} \frac{\partial_j W_\zeta}{a} \right) \right. \right. 
\left. \left. + 4 \tilde{\gamma}_{\zeta,ij} \frac{\partial_i W_\zeta}{a} \frac{\partial_j W_\zeta}{a} \right] \right\}. \]  

Likewise, after many partial integrations to get Eq. (3.39) from Eq. (3.35), we find the following boundary terms,

\[ S_0 W_2 \tilde{\gamma}_\varphi = \frac{1}{2} \int d^3 x \left\{ \frac{a^3 \frac{1}{2} F}{H + \frac{1}{2} F} \left[ \tilde{\gamma}_{\varphi,ij} \frac{1}{2} \left( \frac{\partial_i W_\varphi}{a \phi} \frac{\partial_j W_\varphi}{a \phi} \right. \right. \right. 
\left. \left. \left. - \frac{\partial_i W_\varphi}{a \phi} \frac{\partial_j \chi}{a \phi} - \frac{\partial_j W_\varphi}{a \phi} \frac{\partial_i \chi}{a \phi} \right) \right. \right. \right. 
\left. \left. \left. - \tilde{\gamma}_{\varphi,ij} \frac{\partial_i W_\varphi}{a \phi} \frac{\partial_j W_\varphi}{a \phi} \right] \right\}. \]  

Also here we find that, after the redefinition of \( W_\varphi \) and \( \tilde{\gamma}_\varphi \) using Eq. (3.42), the total boundary terms for scalar-scalar-graviton interactions do not agree with those in (C.4).

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