TWISTING OF PARAMODULAR VECTORS

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Abstract. Let $F$ be a non-archimedean local field of characteristic zero, let $(\pi, V)$ be an irreducible, admissible representation of $\text{GSp}(4, F)$ with trivial central character, and let $\chi$ be a quadratic character of $F^\times$ with conductor $c(\chi) > 1$. We define a twisting operator $T_\chi$ from paramodular vectors for $\pi$ of level $n$ to paramodular vectors for $\chi \otimes \pi$ of level $\max(n + 2c(\chi), 4c(\chi))$, and prove that this operator has properties analogous to the well-known $\text{GL}(2)$ twisting operator.

1. Introduction

Let $k$ and $M$ be positive integers, and let $\chi$ be a quadratic Dirichlet character mod $C$. If $f \in S_k(\Gamma_0(M))$ is a cusp form of weight $k$ with respect to $\Gamma_0(M)$ with Fourier expansion

$$f(z) = \sum_{m=1}^{\infty} a(m) e^{2\pi i m z},$$

then the twist $f_\chi$ of $f$ by $\chi$ is the element of $S_k(\Gamma_0(MC^2))$ with Fourier expansion

$$f_\chi(z) = \sum_{m=1}^{\infty} \chi(m) a(m) e^{2\pi i m z}.$$ 

See, for example, Proposition 3.64 of [S]. In fact, twisting of cusp forms is a local operation when cusp forms are identified as automorphic forms on the adeles of $\text{GL}(2)$ over $\mathbb{Q}$.

Let $F$ be a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$, let $(\pi, V)$ be a smooth representation of $\text{GL}(2, F)$ for which the center of $\text{GL}(2, F)$ acts trivially, and let $\chi$ be a quadratic character of $F^\times$. For $n$ a non-negative integer, we let $V(n)$ and $V(n, \chi)$ be the spaces of $v \in V$ such that $\pi(k)v = v$ and $\pi(k)v = \chi(\det(k))v$, respectively, for $k \in \Gamma_0(p^n)$; here $\Gamma_0(p^n)$ is the subgroup of $\text{GL}(2, \mathfrak{o})$ of elements which are upper triangular mod $\mathfrak{p}$. For $v \in V$, define the $\chi$-twist $T_\chi(v)$ of $v$ as in (2). The main result about $\text{GL}(2)$ twisting is summarized by the following known theorem. See section 2 for further definitions and section 3 for a proof.

Theorem (GL(2) twisting). Let $(\pi, V)$ be a smooth representation of $\text{GL}(2, F)$ for which the center of $\text{GL}(2, F)$ acts trivially, and let $\chi$ be a quadratic character of $F^\times$ with conductor $c(\chi) > 0$. Let $n$ be a non-negative integer and define $N = \max(n, 2c(\chi))$. If $v \in V(n)$, then $T_\chi(v) \in V(N, \chi)$. Moreover, assume that $\pi$ is generic, irreducible and admissible with Whittaker model $W(\pi, \psi)$. Let $W \in V(n)$. The $\chi$-twisted zeta integral (3) of $T_\chi(W)$ is

$$Z(s, T_\chi(W), \chi) = (1 - q^{-1}) G(\chi, -c(\chi)) W(1).$$

For $n \geq N_\pi$, the image of $T_\chi : V(n) \rightarrow V(N, \chi)$ is spanned by the non-zero vector $T_\chi(\beta^n - N_\pi W_\pi)$, where $W_\pi$ is a newform for $\pi$. 

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The goal of this paper is to construct an analog of quadratic twisting for paramodular vectors in representations of GSp(4, F) with trivial central character. Let \((\pi, V)\) be a smooth representation of GSp(4, F) for which the center of GSp(4, F) acts trivially. Let \(V(n)\) and \(V(n, \chi)\) be the spaces of \(v \in V\) such that \(\pi(k)v = v\) and \(\pi(k)v = \chi(\lambda(k))v\), respectively, for \(k\) in the paramodular subgroup \(K(p^n)\) of GSp(4, F) of level \(p^n\). For \(v \in V\), we define the \(\chi\)-twist \(T_\chi(v)\) of \(v\) as in (9). Our main result is the following theorem. We refer to section 2 for more definitions and section 4 for the proof.

**Main Theorem.** Let \((\pi, V)\) be a smooth representation of GSp(4, F) for which the center of GSp(4, F) acts trivially, and let \(\chi\) be a quadratic character of \(F^\times\) with conductor \(c(\chi) > 0\). Let \(n\) be a non-negative integer and define \(N = \max(n + 2c(\chi), 4c(\chi))\). If \(v \in V(n)\), then \(T_\chi(v) \in V(N, \chi)\).

Moreover, assume that \(\pi\) is generic, irreducible and admissible with Whittaker model \(W(\pi, \psi_{c_1, c_2})\) where \(c_1, c_2 \in \sigma^\times\). If \(W \in V(n)\), then the \(\chi\)-twisted zeta integral (7) of \(T_\chi(W)\) is

\[
Z(s, T_\chi(W), \chi) = (q - 1)q^{c(\chi)2}G(\chi, \sigma(c(\chi)))W(1).
\]

For \(n \geq N\), the image of \(T_\chi : V(n) \to V(N, \chi)\) is spanned by the non-zero vector \(T_\chi(\theta^n_{N\pi}W_\pi)\), where \(W_\pi\) is a newform for \(\pi\).

In another work we will consider the application of the paramodular twisting operator \(T_\chi\) to Siegel modular forms and the resulting Fourier coefficients. One reason that Siegel paramodular forms of degree 2 are of interest is that their conjectural connection to abelian surfaces over \(Q\). This is discussed in [BK]; see also [PY].

We note that the integer \(N\) in the Main Theorem is optimal in the following sense. We may identify the space \(V(n, \chi)\) with the space \(V_{\chi\otimes\pi}(n)\) of \(K(p^n)\) fixed vectors in the twisted representation \(\chi\otimes\pi\). Then there exist generic, irreducible, and admissible representations \(\pi\) such that \(N = N_{\chi\otimes\pi}\).

For example, if \(n\) is a type I representation \(\chi_1 \times \chi_2 \times \sigma\) with \(\chi_1, \chi_2\) and \(\sigma\) unramified, then \(N_{\chi\otimes\pi} = 4c(\chi) = \max(0 + 2c(\chi), 4c(\chi))\). Further, suppose that \(\pi\) is a type X representation \(\pi_1 \times \sigma\) with \(\pi_1\) having trivial central character, \(\sigma\) unramified, and \(2c(\chi) < N_{\pi_1}\). Then \(N_{\chi\otimes\pi} = N_{\pi} + 2c(\chi) = \max(N_{\pi} + 2c(\chi), 4c(\chi))\). It is interesting to observe, as in this last example, that \(N_{\chi\otimes\pi} > N_{\pi}\) no matter how large \(N_{\pi}\) is.

### 2. Notation and preliminaries

In this paper \(F\) is a nonarchimedean local field of characteristic zero, with ring of integers \(\mathfrak{o}\) and generator \(\varpi\) of the maximal ideal \(p\) of \(\mathfrak{o}\). We fix a non-trivial continuous character \(\psi\) of \((F, +)\) such that \(\psi(\mathfrak{o}) = 1\) but \(\psi(p^{-1}) \neq 1\). We let \(q\) be the number of elements of \(\mathfrak{o}/p\) and we use the absolute value on \(F\) such that \(|\varpi| = q^{-1}\). We use the Haar measure on the additive group \(F\) that assigns \(\varpi\) measure 1 and the Haar measure on the multiplicative group \(F^\times\) that assigns \(\varpi^x\) measure \(1 - q^{-1}\). Throughout the paper \(\varpi\) is a quadratic character of \(F^\times\) with conductor \(c(\chi)\), i.e., \(c(\chi)\) is the smallest non-negative integer \(n\) such that \(\chi(1 + p^n) = 1\), where we take \(1 + p^0 = \varpi^x\).

If \(n\) is a non-negative integer, then we let \(\Gamma_0(p^n)\) be the subgroup of \(GL(2, \mathfrak{o})\) of elements which are upper triangular mod \(p^n\); we will also write \(\Gamma_0(p^n)\) for the analogous subgroup of \(SL(2, \mathfrak{o})\) when there is no risk of confusion. Let \((\pi, V)\) be a smooth representation of \(GL(2, F)\) for which the center of \(GL(2, F)\) acts trivially, and let \(n\) be a non-negative integer. The subspace \(V(n)\) consists of the vectors in \(V\) fixed by \(\Gamma_0(p^n)\) and \(V(n, \chi)\) is the subspace of vectors \(v \in V\) such that \(\pi(k)v = \chi(\det(k))v\) for \(k \in \Gamma_0(p^n)\). We define the level raising operators \(\beta, \beta' : V(n) \to V(n + 1)\) and \(\beta, \beta' : V(n, \chi) \to V(n + 1, \chi)\) by \(\beta(v) = \pi(1\varpi)v\) and \(\beta'v = v\). If \(\pi\) is generic, irreducible and admissible, then \(V(n)\) is non-zero for some \(n\); we let \(N_{\pi}\) be the smallest such \(n\). The space \(V(N_{\pi})\) is one-dimensional; if \(W_\pi\) is a non-zero element of \(V(N_{\pi})\) so that \(V(N_{\pi}) = \mathbb{C} \cdot W_\pi\), then
we refer to $W_\pi$ as a newform. The space $V(n)$ for $n \geq N_\pi$ is spanned by the vectors $\beta^i \beta^j W_\pi$ where $i$ and $j$ are non-negative integers with $i + j = n - N_\pi$. If $W_\pi$ is viewed as an element of the Whittaker model $W(\pi, \psi)$ of $\pi$, then $W_\pi(1) \neq 0$. As usual, the elements $W$ of $W(\pi, \psi)$ satisfy $W([1 \quad \xi]) g = \psi(x)W(g)$ for $x \in F$ and $g \in \text{GL}(2, F)$. See [C] and [D].

The theory of paramodular newforms is developed in [RS], and we will use the notation of [RS] concerning GSp$(4, F)$. We recall some necessary definitions and results. In particular, GSp$(4, F)$ is the subgroup of $g \in \text{GL}(4, F)$ such that

$$\begin{align*}
g \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \lambda(g) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{for some } \lambda(g) \in F^\times.
\end{align*}$$

If $n$ is a non-negative integer, we let $\text{Kl}(p^n)$ (respectively $\text{K}(p^n)$) be the subgroup of $k \in \text{GSp}(4, F)$ such that $\lambda(k) \in \pi^\times$ and

$$k \in \begin{bmatrix} 0 & 0 & 0 & 0 \\ p^n & 0 & 0 & 0 \\ p^n & 0 & 0 & 0 \\ p^n & p^n & p^n & 0 \end{bmatrix} \quad \text{resp. } k \in \begin{bmatrix} 0 & 0 & 0 & p^{-n} \\ p^n & 0 & 0 & 0 \\ p^n & 0 & 0 & 0 \\ p^n & p^n & p^n & 0 \end{bmatrix}.$$ 

The group $\text{Kl}(p^n)$ is called the Klingen congruence subgroup of level $p^n$ and $\text{K}(p^n)$ is called the paramodular subgroup of level $p^n$. For $a, b, c, d \in F^\times$, we set

$$\text{diag}(a, b, c, d) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$ 

This element is in GSp$(4, F)$ if and only if $ad = bc$. Let $(\pi, V)$ be a smooth representation of GSp$(4, F)$ such that the center of GSp$(4, F)$ acts trivially. If $n$ is a non-negative integer, then $V_{\text{Kl}}(n)$ and $V(n)$ are the subspaces of vectors fixed by the Klingen congruence subgroup $\text{Kl}(p^n)$, and paramodular subgroup $\text{K}(p^n)$, respectively; additionally, we let $V_{\text{Kl}}(n, \chi)$ and $V(n, \chi)$ be the subspaces of vectors $v$ in $V$ such that $\pi(k)v = \chi(\lambda(k))v$ for $k \in \text{Kl}(p^n)$ and $k \in \text{K}(p^n)$, respectively. Also, we define

$$\eta = \begin{bmatrix} \omega^{-1} & 1 & 1 \\ 1 & \omega & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & \omega^{-1} & 1 \\ \omega & 1 & \omega^{-1} \end{bmatrix}, \quad t_n = \begin{bmatrix} 1 & -\omega^{-n} \\ \omega^n & 1 \end{bmatrix}. \quad (1)$$

Sometimes we will write $\eta$ and $\tau$ for $\pi(\eta)$ and $\pi(\tau)$, respectively. We define the level raising operators $\eta : V(n) \to V(n+2)$ and $\theta, \theta' : V(n) \to V(n+1)$ as in [RS]. Let $(\pi, V)$ be an irreducible, admissible representation of GSp$(4, F)$ with trivial central character. If $V(n)$ is non-zero for some non-negative integer $n$ then we say that $\pi$ is paramodular and let $N_\pi$ be the smallest such integer. It is known that if $\pi$ is paramodular, then $V(N_\pi)$ is one-dimensional; if $W_\pi$ is a non-zero element of $V(N_\pi)$ so that $V(N_\pi) = \mathbb{C} \cdot W_\pi$, then we refer to $W_\pi$ as a newform. The space $V(n)$ for $n \geq N_\pi$ is spanned by the vectors $\theta^i \theta^j \eta^k W_\pi$ where $i, j$ and $k$ are non-negative integers with $i + j + 2k = n - N_\pi$. It is known that if $\pi$ is generic, then $\pi$ is paramodular; in general, all paramodular, irreducible, admissible representations of GSp$(4, F)$ with trivial central character have been classified. If $\pi$ is a generic, irreducible, admissible representation of GSp$(4, F)$ with trivial central character then we let $W(\pi, \psi_{c_1, c_2})$ be the Whittaker model of $\pi$ with respect to the character $\psi_{c_1, c_2}$ of the unipotent
The lemma now follows from the fact that the function $o$ and similarly $\int$

**Proof.** Let $n$ be a positive integer such that $f(x + p^n) = f(x)$ for $x \in o^\times$. We have

$$\int_{o^\times} f(u(1 + bu^{-1}w^t)) \, du = q^{-n} \sum_{u \in o^\times/(1 + p^n)} f(u(1 + bu^{-1}w^t)).$$

and similarly

$$\int_{o^\times} f(u) \, du = q^{-n} \sum_{u \in o^\times/(1 + p^n)} f(u).$$

The lemma now follows from the fact that the function $o^\times/(1 + p^n) \to o^\times/(1 + p^n)$ defined by $u \mapsto u(1 + bu^{-1}w^t)$ is a well-defined bijection. \qed

The following lemma about Gauss sums is well-known.

**Lemma 2.2.** Let $\chi$ be a character of $o^\times$ with conductor $c(\chi)$, and let $k$ be an integer. Define

$$G(\chi, k) = \int_{o^\times} \chi(u)\psi(uw^k) \, du.$$  

If $\chi$ is ramified, then $G(\chi, k)$ is non-zero if and only if $k = -c(\chi)$.

### 3. Twist in genus 1

Let $(\pi, V)$ be a smooth representation of $GL(2, F)$ for which the center of $GL(2, F)$ acts trivially, let $\chi$ be a quadratic character of $o^\times$ with conductor $c(\chi)$, and let $n$ be a non-negative integer. For $v \in V(n)$ we define

$$T_\chi(v) = \int_{o^\times} \chi(b)\pi\left[ \begin{array}{c} 1 \\ b w^{-c(\chi)} \end{array} \right] v \, db. \quad (2)$$

If $\chi$ is unramified, then $T_\chi(v) = (1 - q^{-1})v$ for $v \in V(n)$. Thus, we will usually assume that $\chi$ is ramified. Assume further that $\pi$ is generic, irreducible and admissible with Whittaker model $W(\pi, \psi)$. For $W \in W(\pi, \psi)$ we define the $\chi$-twisted zeta integral of $W$ as

$$Z(s, W, \chi) = \int_{F^\times} W\left( \begin{array}{c} t \\ 1 \end{array} \right) |t|^{s-1/2} \chi(t) \, dt. \quad (3)$$

**Theorem 3.1.** Let $(\pi, V)$ be a smooth representation of $GL(2, F)$ for which the center of $GL(2, F)$ acts trivially, let $\chi$ be a quadratic character of $o^\times$ with conductor $c(\chi) > 0$, and let $n$ be a non-negative integer. Let $N = \max(n, 2c(\chi))$. If $v \in V(n)$, then $\pi(k) T_\chi(v) = \chi(\det(k)) T_\chi(v)$ for $k \in \Gamma_0(p^N)$. Moreover, assume that $\pi$ is generic, irreducible and admissible with Whittaker model $W(\pi, \psi)$. For $W \in V(n)$. The $\chi$-twisted zeta integral of $T_\chi(W)$ is

$$Z(s, T_\chi(W), \chi) = (1 - q^{-1}) G(\chi, -c(\chi)) W(1).$$
For $n \geq N_\pi$, the image of $T_\chi : V(n) \to V(N, \chi)$ is spanned by the non-zero vector $T_\chi(\beta^n - N_\pi W_\pi)$.

Proof. The group $\Gamma_0(p^N)$ is generated by the elements contained in the sets
\[
\begin{bmatrix}
    o^x & o^x \\
    1 & 1 \\
    1 & p^N \\
\end{bmatrix},
\]
It is easy to verify that $\pi(k)T_\chi(v) = (\chi(\det(k))T_\chi(v)$ for generators $k$ of the first two types. Let $y \in p^N$. Noting that $N - c(\chi) \geq c(\chi) > 0$ and $N \geq n$, we have
\[
\pi\left(\begin{bmatrix} 1 \\ y \\ 1 \end{bmatrix}\right) \int_{o^x} \chi(b)\pi\left(\begin{bmatrix} 1 \\ b\omega^{-c(\chi)} \\ 1 \end{bmatrix}\right)v \, db
\]
\[=
\int_{o^x} \chi(b)\pi\left(\begin{bmatrix} 1 \\ 1 + \omega^{-c(\chi)}y \\ 1 \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 \\ b\omega^{-c(\chi)}y \\ 1 \end{bmatrix}\right)v \, db
\]
\[=
\int_{o^x} \chi(b)\pi\left(\begin{bmatrix} 1 \\ 1 + \omega^{-c(\chi)}y \\ 1 \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 \\ b\omega^{-c(\chi)}y \\ 1 \end{bmatrix}\right)v \, db
\]
\[=
\int_{o^x} \chi((1 + \omega^{-c(\chi)}y)b^{-1})\pi\left(\begin{bmatrix} 1 \\ 1 + \omega^{-c(\chi)}y \\ 1 \end{bmatrix}\right)v \, db
\]
\[=
\int_{o^x} \chi(b)\pi\left(\begin{bmatrix} 1 \\ b\omega^{-c(\chi)} \\ 1 \end{bmatrix}\right)v \, db
\]
\[= T_\chi(v).
\]
For the penultimate equality we applied Lemma 2.1. Assume now that $\pi$ is generic, irreducible and admissible as in the statement of the theorem. Then:
\[
Z(s, T_\chi(W), \chi) = \int_{F^x} T_\chi(W)(\begin{bmatrix} t \\ 1 \end{bmatrix})|t|^{s-1/2}\chi(t) \, d^\times t
\]
\[=
\int_{F^x} \int_{o^x} \chi(b)W(\begin{bmatrix} t \\ 1 \\ 1 + b\omega^{-c(\chi)} \\ 1 \end{bmatrix})|t|^{s-1/2}\chi(t) \, db \, d^\times t
\]
\[=
\int_{F^x} (\int_{o^x} \chi(b)\psi(tb\omega^{-c(\chi)}) \, db)W(\begin{bmatrix} t \\ 1 \end{bmatrix})|t|^{s-1/2}\chi(t) \, d^\times t
\]
\[=
\int_{o^x} (\int_{o^x} \chi(b)\psi(tb\omega^{-c(\chi)}) \, db)W(\begin{bmatrix} t \\ 1 \end{bmatrix})\chi(t) \, d^\times t
\]
\[=
\int_{o^x} \chi(t)G(\chi, -c(\chi))W(\begin{bmatrix} t \\ 1 \end{bmatrix})\chi(t) \, d^\times t
\]
\[=
(1 - q^{-1})G(\chi, -c(\chi))W(1).
\]
Here, we have used Lemma 2.2. \qed
4. Twist in genus 2

Let \((\pi, V)\) be a smooth representation of GSp\((4, F)\) for which the center of GSp\((4, F)\) acts trivially. Let \(\chi\) be a quadratic character. For \(v \in V\) we define

\[
v^\chi = \int \int \int \chi(ab)\pi\left(\begin{bmatrix} 1 & -aw^{-c(\chi)} & b w^{-2c(\chi)} & z \\ -aw^{-c(\chi)} & 1 & b w^{-2c(\chi)} & a w^{-c(\chi)} \\ b w^{-2c(\chi)} & 1 & a w^{-c(\chi)} & z \\ a w^{-c(\chi)} & b w^{-2c(\chi)} & z & 1 \end{bmatrix}\right) d^c(\chi) v \, dz \, da \, db.
\] (4)

Evidently, if \(\chi\) is unramified, then \(v^\chi = (1 - q^{-1})^2 v\).

**Lemma 4.1.** Let \((\pi, V)\) be a smooth representation of GSp\((4, F)\) for which the center of GSp\((4, F)\) acts trivially. Let \(\chi\) be a quadratic character with \(c(\chi) > 0\). Let \(n\) be a non-negative integer. Let \(v \in V_{Ki}(n)\). We have \(\pi(k)v^\chi = \chi(\lambda(k))v^\chi\) for the subgroup of \(k \in GSp(4, F)\) such that \(\lambda(k) \in \o^\times\) and

\[
k \in \begin{bmatrix} 0 & o & o & p^{-2c(\chi)} \\ 0 & o & o & 0 \\ p^{2c(\chi)} & 0 & o & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

**Proof.** The subgroup in the statement of the lemma is generated by the elements of the form \(\text{diag}(w_1 w_2 w, w_1 w, w_2 w, w)\) for \(w, w_1, w_2 \in \o^\times\), and elements of the subgroups

\[
\begin{bmatrix} 1 & p^{-2c(\chi)} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},
\]

Using \(v \in V_{Ki}(n)\), the definition of \(v^\chi\) and Lemma 2.1 one can verify that \(\pi(k)v^\chi = \chi(\lambda(k))v^\chi\) for each type of generator \(k\). As an illustration, let \(x \in \o\). Then

\[
\pi\left(\begin{bmatrix} 1 & -x & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & x & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right)v^\chi
\]

\[
= \int \int \int \chi(ab)\pi\left(\begin{bmatrix} 1 & -a w^{-c(\chi)} - x & b w^{-2c(\chi)} & z - x b w^{-2c(\chi)} \\ -a w^{-c(\chi)} & 1 & b w^{-2c(\chi)} & a w^{-c(\chi)} + x \\ b w^{-2c(\chi)} & 1 & a w^{-c(\chi)} & z \\ a w^{-c(\chi)} & b w^{-2c(\chi)} & z & 1 \end{bmatrix}\right) d^c(\chi) v \, dz \, da \, db
\]

\[
= \pi v^\chi.
\]

This completes the proof. \(\square\)

**Lemma 4.2.** Let \((\pi, V)\) be a smooth representation of GSp\((4, F)\) for which the center of GSp\((4, F)\) acts trivially. Let \(\chi\) be a quadratic character and let \(n\) be a non-negative integer. Let \(v \in V_{Ki}(n)\)
and define $v^x$ as in (4). Then $v^x$ is invariant under the subgroup

$$GSp(4, F) \cap \begin{pmatrix}
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{p} & 0 \\
0 & 0 & 0 & 1
\end{array}
\end{pmatrix}.$$

Proof. This is clear if $\chi$ is unramified; assume that $c(\chi) > 0$. Let $a, b \in o^x$ and $c \in o$, and set

$$g = \begin{bmatrix}
1 & -aw^{-c(\chi)} & bw^{-2c(\chi)} & cw^{-2c(\chi)} \\
0 & 1 & \frac{b}{aw^{-c(\chi)}} & \frac{c}{aw^{-c(\chi)}} \\
0 & 0 & 1 & \frac{a}{aw^{-c(\chi)}} \\
0 & 0 & 0 & 1
\end{bmatrix}^{c(\chi)}.
$$

Let $L$ be an integer and $y \in o$. We have the following identities:

$$\begin{bmatrix}
1 \\
yw^L \\
1 \\
yw^L
\end{bmatrix} g = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & acyw^{-4c(\chi)} & bcyw^{-3c(\chi)} & cyw^{-4c(\chi)} \\
0 & 1 & -bcyw^{-3c(\chi)} & -cyw^{-4c(\chi)} \\
0 & 0 & 1 & cyw^{-4c(\chi)} \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

These identities prove that $v^x$ is invariant under the group

$$GSp(4, F) \cap \begin{pmatrix}
\begin{array}{cccc}
p \max(n+c(\chi),3c(\chi)) & 1 & 0 & 0 \\
0 & p \max(n+c(\chi),3c(\chi)) & 0 & 0 \\
0 & 0 & p \max(n+c(\chi),3c(\chi)) & 1 \\
0 & 0 & 0 & p \max(n+c(\chi),3c(\chi))
\end{array}
\end{pmatrix}.$$

To prove the remaining invariance, set $L = \max(n + c(\chi), 3c(\chi))$. A calculation shows that

$$\begin{bmatrix}
1 \\
yw^L \\
yw^L
\end{bmatrix} g = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & abyw^{-1} & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.$$
for some $k \in \text{Kl}(p^n)$ with $u = 1 + by\omega L^{-2c(\chi)}$. Therefore,

$$
\pi(\begin{bmatrix}
1 & 1 \\
y\omega^L & y\omega^L
\end{bmatrix}) v^\chi = q^{2c(\chi)} \int \int \int \chi(ab) \pi(\begin{bmatrix}
1 & -a\omega^{-c(\chi)} & b\omega^{-2c(\chi)} & y\omega L^{-4c(\chi)} \\
1 & c\omega^{-2c(\chi)} & b\omega^{-2c(\chi)} & 1
\end{bmatrix}) \tau^c(\chi)
$$

Taking $v = (-2byc + ab^3y^2\omega L^{-3c(\chi)}u^{-2}\omega L^{-4c(\chi)})$ and $\chi$ be a quadratic character, we obtain $v^\chi$.

Let $(\pi, V)$ be a smooth representation of $\text{GSp}(4, F)$ for which the center of $\text{GSp}(4, F)$ acts trivially, let $\chi$ be a quadratic character, and let $n$ be a non-negative integer. For $v \in V_{\text{Kl}}(n)$ we define

$$
T^\text{Kl}_{\chi}(v) = \int \pi(\begin{bmatrix}
1 & 1 \\
x & 1
\end{bmatrix}) v^\chi \, dx + \int \pi(\begin{bmatrix}
1 & -1 \\
y & 1
\end{bmatrix}) v^\chi \, dy.
$$

(5)

Here, $v^\chi$ as in (4). If $\chi$ is unramified, then $T^\text{Kl}_{\chi}(v) = (1 + q^{-1})(1 - q^{-1})^2 v$.

**Lemma 4.3.** Let $(\pi, V)$ be a smooth representation of $\text{GSp}(4, F)$ for which the center of $\text{GSp}(4, F)$ acts trivially, let $\chi$ be a quadratic character, and let $n$ be a non-negative integer. Let $v \in V_{\text{Kl}}(n)$. Then

$$
\pi(k) T^\text{Kl}_{\chi}(v) = \chi(\lambda(k)) T^\text{Kl}_{\chi}(v)
$$

(6)

for $k \in \text{Kl}(p^N)$ where $N = \max(n + 2c(\chi), 4c(\chi))$. Moreover, $\pi(k) T^\text{Kl}_{\chi}(v) = T^\text{Kl}_{\chi}(v)$ for $k \in \text{GSp}(4, F)$ such that

$$
k \in \begin{bmatrix}
p^{N-2c(\chi)} & 1 \\
p^{N-2c(\chi)} & p^{N-2c(\chi)}
p^{N-2c(\chi)} & p^{N-2c(\chi)}
p^{N-2c(\chi)} & 1
\end{bmatrix}.
$$
Proof. The group $Kl(p^N)$ is generated by its elements contained in the sets
\[
\begin{bmatrix}
1 & 1 & 1 \\
p^N & p^N & p^N \\
p^N & p^N & p^N
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
\sigma^x & 0 & 0 \\
0 & \sigma & 0 \\
0 & 0 & \sigma^x
\end{bmatrix},
\]
and there is a disjoint decomposition
\[
SL(2, \mathfrak{o}) = \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}^{2c(x)}} \Gamma_0(p^{2c(x)}) \sqcup \bigsqcup_{y \in \mathfrak{p}/\mathfrak{p}^{2c(x)}} \left[ -1 \begin{bmatrix} 1 & 1 \\ y & 1 \end{bmatrix} \Gamma_0(p^{2c(x)}) \right].
\]
The lemma follows from these two facts, Lemma 4.1, and Lemma 4.2. □

Let $(\pi, V)$ be a generic, irreducible, admissible representation of $GSp(4, F)$ with trivial central character with Whittaker model $W(\pi, \psi_{c_1, c_2})$; we take $c_1, c_2 \in \mathfrak{o}^\times$. Let $\chi$ be a quadratic character of $F^\times$. If $W \in W(\pi, \psi_{c_1, c_2})$ we define the zeta integral of $W$ twisted by $\chi$ to be
\[
Z(s, W, \chi) = \int_{F^\times} \int_{F^\times} W\left( \begin{bmatrix}
t & 1 \\
z & 1
\end{bmatrix} \right) |t|^{s-3/2} \chi(t) dt dt.
\]
This is the same as the zeta integral of $W$ in the twist $\chi \otimes \pi$ of $\pi$ by $\chi$. See [RS].

Lemma 4.4. Let $(\pi, V)$ be a generic, irreducible, admissible representation of $GSp(4, F)$ with trivial central character with Whittaker model $W(\pi, \psi_{c_1, c_2})$; we take $c_1, c_2 \in \mathfrak{o}^\times$. Let $\chi$ be a quadratic character of $F^\times$ such that $c(\chi) > 0$. Let $n$ be a non-negative integer. Let $W \in V_{Kl}(n)$, and define $T_{\chi}^{Kl}(W)$ as in (5). We have
\[
Z(s, T_{\chi}^{Kl}(W), \chi) = (1 - q^{-1}) q^{c(\chi)} \chi(c_2) G(\chi, -c(\chi))^3 W(1).
\]
In particular, if $W$ is the newform of $\pi$, then $T_{\chi}^{Kl}(W) \neq 0$.

Proof. To begin, we note that by Lemma 4.1.1 of [RS] we have
\[
Z(s, T_{\chi}^{Kl}(W), \chi) = \int_{F^\times} T_{\chi}^{Kl}(W)\left( \begin{bmatrix}
t & 1 \\
p & 1
\end{bmatrix} \right) |t|^{s-3/2} \chi(t) dt dt.
\]
Therefore, the first part of $Z(s, T_{\chi}^{Kl}(W), \chi)$ is:
\[
\int_{F^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{p}^{-2c(\chi)}} W\left( \begin{bmatrix}
t & 1 \\
z & 1
\end{bmatrix} \begin{bmatrix}
1 & x & 1 \\
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 & \sigma & 0 \\
\sigma & \sigma^x & 0 \\
\sigma & 0 & \sigma^x
\end{bmatrix} \begin{bmatrix}
1 & \sigma & 0 \\
\sigma & \sigma^x & 0 \\
\sigma & 0 & \sigma^x
\end{bmatrix} \right) |t|^{s-3/2} \chi(t) \chi(ab) dt da db dx dt.
\]
By Lemma 2.2 the integral in the $b$ variable is zero unless $v(x) = c(\chi)$. Continuing,

$$= q^{2c(\chi)} \int_{F^\times} \int_{\phi^\times} \int_{\phi^\times} W(t) \begin{bmatrix} t & 1 & 1 \\ t & 1 & x \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ w^{-c(\chi)} \\ \chi \end{bmatrix} \int |t|^{s-3/2} \chi(t) \chi(ab) da \, db \, dx \, d^xt$$

Continuing,

$$= q^{c(\chi)} \int_{\phi^\times} \chi(a) \psi(-c_1 a w^{-c(\chi)}) da \int_{\phi^\times} \chi(b) \psi(-c_1 b x w^{-c(\chi)}) db \int_{F^\times} W(t) \begin{bmatrix} t & 1 & 1 \\ t & 1 & x \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ w^{-c(\chi)} \\ \chi \end{bmatrix} \int |t|^{s-3/2} \chi(t) d^xt dx$$

$$= q^{c(\chi)} G(\chi, -c(\chi))^2 \int_{\phi^\times} \chi(x) \int_{F^\times} W(t) \begin{bmatrix} t & 1 & 1 \\ t & 1 & x^{-1} w^{-c(\chi)} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -x^{-1} w^{-c(\chi)} \\ -x w^{-c(\chi)} \end{bmatrix} \int |t|^{s-3/2} \chi(t) d^xt dx$$

$$= q^{c(\chi)} G(\chi, -c(\chi))^2 \int_{\phi^\times} \chi(x) \int_{F^\times} W(t) \begin{bmatrix} t & 1 & 1 \\ t & 1 & x^{-1} w^{-c(\chi)} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -x^{-1} w^{-c(\chi)} \\ -x w^{-c(\chi)} \end{bmatrix} \int |t|^{s-3/2} \chi(t) d^xt dx$$
Finally, we prove that the second part of $Z(x, c(x))$ is zero unless $v(t) = 0$. Thus, our quantity is:

$$
= q^{c(x)}G(x, -c(x))^2 \int_{\mathbb{A}} \chi(x) \int_{\mathbb{F}^\times} \psi(c_2 t x^{-1} \varpi^{-c(x)}) W\left( \begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \, d^x t \, dx
$$

Again, by Lemma 2.2 the integral in the $x$ variable is zero unless $v(t) = 0$. Thus, our quantity is:

$$
= q^{c(x)}G(x, -c(x))^2 \int_{\mathbb{A}} \chi(x) \psi(c_2 t x \varpi^{-c(x)}) \, dx \int_{\mathbb{F}^\times} \chi(t) \chi(x) \psi(c_2 x \varpi^{-c(x)}) \, dx \, W(1) \chi(t) \, d^x t
$$

$$
= q^{c(x)} \chi(c_2)G(x, -c(x))^3 \int_{\mathbb{A}} W(1) \, d^x t
$$

$$
= (1 - q^{-1})q^{c(x)} \chi(c_2)G(x, -c(x))^3 W(1).
$$

Finally, we prove that the second part of $Z(s, T^{KL}(W), \chi)$ is zero:

$$
= q^{2c(x)} \int_{\mathbb{A}} \psi(-c_2 ty) W\left( \begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \chi(ab) \, dz \, da \, db \, dy \, d^x t
$$
\[
q^{2c(\chi)} \int_{F^\times} \int_p \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \psi(-c_2 t y) \psi(c_1 b \omega^{-2c(\chi)}) \left( \begin{array}{cc}
t & t \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{\chi}(v) \left( \begin{array}{c}
1 \\
\omega^{-c(\chi)} \\
\omega^{-c(\chi)} \\
1 \end{array} \right) |t|^{s-3/2} \chi(t) \chi(ab) \, da \, db \, dy \, dz
\]

\[
= 0.
\]

The last equality holds because \( \chi \) is ramified by assumption.

Let \((\pi, V)\) be a smooth representation of \( \text{GSp}(4, F) \) for which the center of \( \text{GSp}(4, F) \) acts trivially, let \( \chi \) be a quadratic character, and let \( n \) be a non-negative integer. Define \( N = \max(n + 2c(\chi), 4c(\chi)) \). For \( v \in V_{\text{Kl}}(n) \) we define

\[
T_{\chi}(v) = q \int_v \pi(\begin{array}{c}
1 \\
1 \\
1 \\
1 \end{array}) \left( \begin{array}{cccc}
1 & z \omega^{-N} & & \\
1 & 1 & & \\
1 & 1 & & \\
1 & 1 & & \\
\end{array} \right) T_{\chi}^{\text{Kl}}(v) \, dz + \pi(t_N) \int_v \pi(\begin{array}{c}
1 \\
1 \\
1 \\
1 \end{array}) \left( \begin{array}{cccc}
1 & z \omega^{-N+1} & & \\
1 & 1 & & \\
1 & 1 & & \\
1 & 1 & & \\
\end{array} \right) T_{\chi}^{\text{Kl}}(v) \, dz. \quad (9)
\]

Here, \( T_{\chi}^{\text{Kl}}(v) \) is defined as in (5). Explicitly,

\[
q^{-2c(\chi)} T_{\chi}(v)
\]

\[
= q \int_v \int_p \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(\begin{array}{c}
1 \\
x & 1 \\
1 & 1 \end{array}) \left( \begin{array}{cccc}
1 & -a \omega^{-c(\chi)} & b \omega^{-2c(\chi)} & z \omega^{-N} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \end{array} \right) \tau^{c(\chi)} v \, da \, db \, dx \, dz
\]

\[
+ q \int_v \int_p \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(\begin{array}{c}
1 \\
-1 & 1 \\
1 & 1 \end{array}) \left( \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & y & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \end{array} \right) \tau^{c(\chi)} v \, da \, db \, dy \, dz
\]

\[
+ \int_v \int_p \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(t_N) \pi(\begin{array}{c}
1 \\
x & 1 \\
1 & 1 \end{array}) \left( \begin{array}{cccc}
1 & -a \omega^{-c(\chi)} & b \omega^{-2c(\chi)} & z \omega^{-N} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \\
1 & 1 & 1 & a \omega^{-c(\chi)} \end{array} \right) \tau^{c(\chi)} v \, da \, db \, dx \, dz
\]

\[
(12)
\]
\[
+ \int_{\mathfrak{o} \times \mathfrak{o} \times \mathfrak{o}} \int \pi(t_N) \begin{bmatrix}
1 & 1 & 1 & 1
-1 & y & 0 & 0
0 & 0 & 1 & 1
-a_\mathfrak{m}^{-c(\chi)} & b_\mathfrak{m}^{-2c(\chi)} & z_\mathfrak{m}^{-N+1} & 1
b_\mathfrak{m}^{-2c(\chi)} & a_\mathfrak{m}^{-c(\chi)} & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1
0 & 0 & 1 & 1
0 & 0 & 1 & 1
0 & 0 & 1 & 1
\end{bmatrix}
\] (13)

Lemma 4.5. Let \((\pi, V)\) be a smooth representation of \(\text{GSp}(4, F)\) for which the center of \(\text{GSp}(4, F)\) acts trivially and let \(\chi\) be a quadratic character. Let \(n\) be a non-negative integer and define \(N = \max(n + 2c(\chi), 4c(\chi))\). Let \(v \in V_{\text{Kl}}(n)\).

i) We have \(\pi(k)T_\chi(v) = \chi(\lambda(k))T_\chi(v)\) for \(k \in \text{K}(\mathfrak{p}^N)\).
ii) Assume that \(c(\chi) > 0\). If \(T_\chi(v)\) is invariant under under the elements

\[
\begin{bmatrix}
1 & r_1_\mathfrak{m}^{-1} & r_2_\mathfrak{m}^{-1} & 1
1 & 1 & -r_1_\mathfrak{m}^{-1} & 1
1 & r_2_\mathfrak{m}^{-1} & 1 & 1
1 & 1 & 1 & 1
\end{bmatrix}
\] (14)

for \(r_1, r_2 \in \mathfrak{o}\), then \(T_\chi(v) = 0\).

Proof. i) Fix a Haar measure for the group \(\text{GSp}(4, F)\). By Lemma 3.3.1 of [RS] there is a disjoint decomposition

\[
\text{K}(\mathfrak{p}^N) = \bigsqcup_{z \in \mathfrak{o}/\mathfrak{p}^N} \begin{bmatrix}
1 & z_\mathfrak{m}^{-N}
1 & 1
1 & 1
1 & 1
\end{bmatrix}
\text{Kl}(\mathfrak{p}^N) = \bigsqcup_{z \in \mathfrak{o}/\mathfrak{p}^{N-1}} \text{Kl}(\mathfrak{p}^N) = \bigsqcup_{z \in \mathfrak{o}/\mathfrak{p}^{N-1}} t_N \begin{bmatrix}
1 & z_\mathfrak{m}^{-N+1}
1 & 1
1 & 1
1 & 1
\end{bmatrix}
\]

Here, the second disjoint union is not present if \(N = 0\). Therefore, by (6),

\[
\int_{\text{K}(\mathfrak{p}^N)} \chi(\lambda(k))\pi(k)T_\chi^{\text{Kl}}(v) \, dk = \text{vol(\text{Kl}(\mathfrak{p}^N))} \sum_{z \in \mathfrak{o}/\mathfrak{p}^N} \pi(\begin{bmatrix}
1 & z_\mathfrak{m}^{-N}
1 & 1
1 & 1
1 & 1
\end{bmatrix})T_\chi^{\text{Kl}}(v)
\]

\[
+ \text{vol(\text{Kl}(\mathfrak{p}^N))} \sum_{z \in \mathfrak{o}/\mathfrak{p}^{N-1}} \pi(t_N \begin{bmatrix}
1 & z_\mathfrak{m}^{-N+1}
1 & 1
1 & 1
1 & 1
\end{bmatrix})T_\chi^{\text{Kl}}(v)
\]

\[
= \text{vol(\text{Kl}(\mathfrak{p}^N))} q^N \int_{\mathfrak{o}} \pi(\begin{bmatrix}
1 & z_\mathfrak{m}^{-N}
1 & 1
1 & 1
1 & 1
\end{bmatrix})T_\chi^{\text{Kl}}(v) \, dz
\]

\[
+ \text{vol(\text{Kl}(\mathfrak{p}^N))} q^{N-1} \int_{\mathfrak{o}} \pi(t_N \begin{bmatrix}
1 & z_\mathfrak{m}^{-N+1}
1 & 1
1 & 1
1 & 1
\end{bmatrix})T_\chi^{\text{Kl}}(v) \, dz.
\]

This is a positive multiple of \(T_\chi(v)\), and thus implies the desired transformation rule.
ii) Assume that $T_\chi(v)$ is invariant under the elements in (14). Then

$$T_\chi(v) = \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & r_1 \omega^{-1} & r_2 \omega^{-1} \\ 1 & r_2 \omega^{-1} & -r_1 \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) T_\chi(v) \, dr_1 \, dr_2$$

$$= q \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & r_1 \omega^{-1} & r_2 \omega^{-1} \\ 1 & r_2 \omega^{-1} & -r_1 \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) T^{KL}_\chi(v) \, dr_1 \, dr_2 \, dz$$

$$+ \pi(t_N) \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & 1 & r_2 \omega^{-N-1} \\ -r_1 \omega^{-N-1} & 1 & 1 \\ z \omega^{-N+1} & -r_1 \omega^{-N-1} & -r_2 \omega^{-N-1} \end{array} \right) T^{KL}_\chi(v) \, dz \, dr_1 \, dr_2.$$  \hfill (15)

We claim that the first summand of (15) is zero. Now

$$\int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & r_1 \omega^{-1} & r_2 \omega^{-1} \\ 1 & r_2 \omega^{-1} & -r_1 \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) T^{KL}_\chi(v) \, dr_1 \, dr_2 \, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & (r_1 + x r_2) \omega^{-1} & r_2 \omega^{-1} \\ 1 & 1 & -(r_1 + x r_2) \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) v^x \, dr_1 \, dr_2 \, dx \, dz$$

$$+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 1 & 1 \\ y & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & (r_1 y - r_2) \omega^{-1} & r_1 \omega^{-1} \\ 1 & 1 & -(r_1 y - r_2) \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) v^x \, dr_1 \, dr_2 \, dy \, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & r_1 \omega^{-1} & r_2 \omega^{-1} \\ 1 & r_2 \omega^{-1} & -r_1 \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) v^x \, dr_1 \, dr_2 \, dx \, dz$$

$$+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 1 & 1 \\ y & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & r_2 \omega^{-1} & r_1 \omega^{-1} \\ 1 & 1 & -(r_1 y - r_2) \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) v^x \, dr_1 \, dr_2 \, dy \, dz.$$
Moreover,
\[
\int_0 \int_0 \int_0 \pi(\begin{bmatrix} 1 & r_1 \omega^{-1} & z \omega^{-N} \\ \frac{1}{r_2 \omega^{-1}} & 1 & r_2 \omega^{-1} \\ \frac{1}{r_1 \omega^{-1}} & \frac{1}{-r_1 \omega^{-1}} & 1 \end{bmatrix}) \)v^\chi \ dr_1 \ dr_2 \ dz \\
= q^{2c(\chi)} \int_0 \int_0 \int_0 \int_0 \int_0 \chi(ab) \pi(\begin{bmatrix} 1 & -a u_1 \omega^{-c(\chi)} & bu_2 \omega^{-2c(\chi)} & z \omega^{-N} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & 1 & bu_2 \omega^{-2c(\chi)} & (a - r_1) \omega^{-c(\chi)} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & \frac{1}{-r_1 \omega^{-c(\chi)}} & 1 & 1 \end{bmatrix}) \)v \ dr_1 \ dr_2 \ dz \ da \ db
\]

with
\[
u_1 = 1 - r_1 a^{-1} \omega^{-c(\chi)^{-1}} \quad \text{and} \quad u_2 = 1 + b^{-1} r_2 \omega^{-2c(\chi)^{-1}}.
\]
Assume first \(c(\chi) = 1\). Then this integral is:
\[
\int_0 \int_0 \int_0 \int_0 \int_0 \chi(ab) \pi(\begin{bmatrix} 1 & - (a - r_1) \omega^{-c(\chi)} & bu_2 \omega^{-2c(\chi)} & z \omega^{-N} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & 1 & bu_2 \omega^{-2c(\chi)} & (a - r_1) \omega^{-c(\chi)} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & \frac{1}{-r_1 \omega^{-c(\chi)}} & 1 & 1 \end{bmatrix}) \)v \ dr_1 \ dr_2 \ dz \ da \ db
\]
\[
= 0.
\]
Assume that \(c(\chi) > 1\). Changing variables in \(r_1\) and then in \(a\), this integral is:
\[
\int_0 \int_0 \int_0 \int_0 \int_0 \chi(ab) \pi(\begin{bmatrix} 1 & -a(1 + r_1 \omega^{-c(\chi)^{-1}}) \omega^{-c(\chi)} & bu_2 \omega^{-2c(\chi)} & z \omega^{-N} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & 1 & bu_2 \omega^{-2c(\chi)} & a(1 + r_1 \omega^{-c(\chi)^{-1}}) \omega^{-c(\chi)} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & \frac{1}{-r_1 \omega^{-c(\chi)}} & 1 & 1 \end{bmatrix}) \)v \ dr_1 \ dr_2 \ dz \ da \ db
\]
\[
= \int_0 \int_0 \int_0 \int_0 \int_0 \chi(a(1 + r_1 \omega^{-c(\chi)^{-1}}) b) \pi(\begin{bmatrix} 1 & -a \omega^{-c(\chi)} & bu_2 \omega^{-2c(\chi)} & z \omega^{-N} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & 1 & bu_2 \omega^{-2c(\chi)} & a \omega^{-c(\chi)} \\ \frac{1}{bu_2 \omega^{-2c(\chi)}} & \frac{1}{-r_1 \omega^{-c(\chi)}} & 1 & 1 \end{bmatrix}) \)v \ dr_1 \ dr_2 \ dz \ da \ db
\]
\[
= 0.
\]
This proves that the first summand of (15) is zero, as claimed. We now have:
\[
T_{\chi}(v) = \pi(t_N) \int_0 \int_0 \int_0 \pi(\begin{bmatrix} 1 & r_2 \omega^{-N-1} & 1 \\ -r_1 \omega^{-N-1} & 1 & r_2 \omega^{-N-1} \\ -r_1 \omega^{-N-1} & -r_2 \omega^{-N-1} & 1 \end{bmatrix})
\]
Applying \( \pi(t_N)^{-1} \) to both sides and using the invariance of \( T_\chi(v) \) under \( t_N \in K(p^N) \) from i), we see that \( T_\chi(v) \) is:

\[
\int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & r_2 z & r_2 z \\ 0 & -r_2 z & -r_1 z \\ 0 & 0 & 1 \end{array} \right) T^{K\ell}_{\chi}(v) \, dz \, dr_1 \, dr_2 = \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & z \omega^{-N+1} \\ 0 & r_2 z & r_2 z \\ 0 & -r_2 z & -r_1 z \\ 0 & 0 & 1 \end{array} \right) T^{K\ell}_{\chi}(v) \, dz \, dr_1 \, dr_2
\]

where we have used the invariance properties of \( T^{K\ell}_{\chi}(v) \) from Lemma 4.3. By assumption, \( T_\chi(v) \) is invariant under the elements of the form (14); integrating again over these elements we have

\[
T_\chi(v) = \int_0^1 \int_0^1 \int_0^1 \pi \left( \begin{array}{ccc} 1 & r_1 -1 & r_2 -1 \\ 0 & 1 & z \omega^{-N+1} \\ 0 & 1 & 1 \end{array} \right) T^{K\ell}_{\chi}(v) \, dz \, dr_1 \, dr_2 \, dr.
\]

This integral is zero by an argument analogous to the one above proving that the first term of (15) is zero. The proof is complete. \( \square \)

**Theorem 4.6.** Let \((\pi, V)\) be a smooth representation of \( \text{GSp}(4, F) \) for which the center of \( \text{GSp}(4, F) \) acts trivially, and let \( \chi \) be a quadratic character of \( F^\times \) with center of \( \chi > 0 \). Let \( n \) be a non-negative integer and define \( N = \max(n + 2c(\pi), 4c(\pi)) \). If \( v \in V(n) \), then \( T_\chi(v) \in V(N, \chi) \). Moreover, assume that \( \pi \) is generic, irreducible and admissible with Whittaker model \( W(\pi, \psi_{c_1, c_2}) \) where \( c_1, c_2 \in \mathfrak{o}^\times \). If \( W \in V(n) \), then the \( \chi \)-twisted zeta integral (7) of \( T_\chi(W) \) is

\[
Z(s, T_\chi(W), \chi) = (q - 1)^{c(\pi)} \chi(c_2) G(\chi, -c(\chi))^3 W(1).
\]

For \( n \geq N_\pi \), the image of \( T_\chi : V(n) \to V(N, \chi) \) is spanned by the non-zero vector \( T_\chi(\theta^n W_{\pi}) \), where \( W_{\pi} \) is a newform for \( \pi \).

**Proof.** The first assertion was proven in i) of Lemma 4.5. Assume now that \( \pi \) is generic and irreducible. We work in the Whittaker model \( W(\pi, \psi_{c_1, c_2}) \) with \( c_1, c_2 \in \mathfrak{o}^\times \). By Lemma 4.1.1 of [RS] we have

\[
Z(s, T_\chi(v), \chi) = \int_{F^\times} T_\chi(v)( \begin{array}{c} t \\ 1 \end{array} ) |t|^{s-3/2} \chi(t) \, d^\times t.
\]

By the definition of \( T_\chi(v) \), this is
We assert that the second summand is zero; it will suffice to prove that the integrand is zero. Let \( t \in F^\times \) and \( z \in \mathfrak{o} \). Let \( x \in \mathfrak{o} \). Then

\[
q \int_{F^\times} \int_{\mathfrak{o}} T_{\chi}^{Kl}(v) \left( \begin{bmatrix} t \\ 1 \\ t \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \, dz \, d^\times t
\]

\[
= \int_{F^\times} \int_{\mathfrak{o}} T_{\chi}^{Kl}(v) \left( \begin{bmatrix} t \\ 1 \\ t \\ 1 \\ t_N \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \, dz \, d^\times t.
\]

where the last equality follows from the invariance properties of Lemma 4.3. Since \( \psi(p^{-1}) \neq 1 \), this implies that the integrand is zero. The first summand is

\[
q \int_{F^\times} \int_{\mathfrak{o}} T_{\chi}^{Kl}(v) \left( \begin{bmatrix} t \\ 1 \\ 1 \\ 1 \\ z^w/2 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \, dz \, d^\times t
\]

\[
= q \int_{F^\times} \int_{\mathfrak{o}} T_{\chi}^{Kl}(v) \left( \begin{bmatrix} t \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) |t|^{s-3/2} \chi(t) \, dz \, d^\times t.
\]
\[
= q \int_{\mathbb{F}^\times} T_{\chi}^{Kl}(v)(\begin{bmatrix} t & t \\ 1 & 1 \end{bmatrix}) |t|^{s-3/2} \chi(t) dt
= q Z(s, T_{\chi}^{Kl}(v), \chi).
\]

The formula (16) follows now from (8). To prove the final assertion, we note first by Theorem 7.5.7 of [RS] that the space \( V(n) \) is spanned by the vectors \( \theta^i \theta^j \eta^k W_\pi \) with \( i + j + 2k = n - N_\pi \). The formula (3.7) of [RS] implies that

\[
Z(s, T_{\chi}(\theta^{n-N_\pi} W_\pi), \chi) = (q - 1) q^{c(\chi)} \chi(c_2) G(\chi, -c(\chi))^{3}(\theta^{n-N_\pi} W_\pi)(1)
= (q - 1) q^{c(\chi) + n-N_\pi} \chi(c_2) G(\chi, -c(\chi))^{3}W_\pi(1),
\]

and this is non-zero. To complete the proof, it will suffice to prove that \( T_{\chi}(\theta^{i} \theta^j \eta^k W_\pi) = 0 \) if \( j > 0 \) or \( k > 0 \). Let \( W = \theta^i \theta^j \eta^k W_\pi \) with \( j > 0 \) or \( k > 0 \). The \( \chi \)-twisted zeta integral of \( W \) is a constant times \( (\theta^i \theta^j \eta^k W_\pi)(1) \); this quantity is zero by the definitions of \( \eta, \theta \), and Lemma 4.1.2 of [RS]. Since \( Z(s, T_{\chi}(W), \chi) = 0 \), by Theorem 4.3.7 of [RS] there exists \( W' \in V(N-2, \chi) \) such that \( T_{\chi}(W) = \eta W' \). This implies that \( T_{\chi}(W) \) is invariant under the elements in (14). Therefore, by ii) of Lemma 4.5, \( T_{\chi}(W) = 0 \).

\[\square\]

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