CROSSED PRODUCTS OF THE CANTOR SET BY FREE MINIMAL ACTIONS OF \( \mathbb{Z}^d \)

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Abstract. Let \( d \) be a positive integer, let \( X \) be the Cantor set, and let \( \mathbb{Z}^d \) act freely and minimally on \( X \). We prove that the crossed product \( C^*(\mathbb{Z}^d, X) \) has stable rank one, real rank zero, and cancellation of projections, and that the order on \( K_0(C^*(\mathbb{Z}^d, X)) \) is determined by traces. We obtain the same conclusion for the C*-algebras of various kinds of aperiodic tilings.

In [35], Putnam considered the C*-algebra \( A \) associated with a substitution tiling system satisfying certain additional conditions, and proved that the order on \( K_0(A) \) is determined by the unique tracial state \( \tau \) on \( A \). That is, if \( \eta \in K_0(A) \) satisfies \( \tau(\eta) > 0 \), then there is a projection \( p \in M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A) \) such that \( \eta = [p] \).

In this paper, we strengthen Putnam’s theorem, obtaining Blackadar’s Second Fundamental Comparability Question ([7], 1.3.1) for \( A \), namely that if \( p, q \in M_\infty(A) \) are projections such that \( \tau(p) < \tau(q) \) for every tracial state \( \tau \) on \( A \), then \( p \preceq q \), that is, that \( p \) is Murray-von Neumann equivalent to a subprojection of \( q \). We further prove that the C*-algebra \( A \) has real rank zero [10] and stable rank one [37]. We also extend the theorem: the same conclusions hold for the C*-algebras of some other kinds of aperiodic tilings, and when \( A \) is the transformation group C*-algebra \( C^*(\mathbb{Z}^d, X) \) of an arbitrary free and minimal action of \( \mathbb{Z}^d \) on the Cantor set \( X \).

We should also mention the recent proof of the gap labelling conjecture for the Cantor set ([3], [5], [21]), which states that the image of \( K_0(C^*(\mathbb{Z}^d, X)) \) under the map to \( \mathbb{R} \) induced by a trace is the subgroup generated by the values of the corresponding invariant measure on compact open subsets of \( X \).

Our results are loosely related to the Bethe-Sommerfeld Conjecture for quasicrystals in the tight binding approximation. The tight binding Hamiltonian for a quasicrystal coming from an aperiodic tiling is a selfadjoint element of the C*-algebra of the tiling. When this C*-algebra has real rank zero, any selfadjoint element has arbitrarily small perturbations which have finite spectrum, and moreover selfadjoint elements with totally disconnected spectrum are generic (form a dense \( G_\delta \)-set) in the set of all selfadjoint elements.

Our proofs are based on the methods of Section 3 of [35]. These methods require the presence of a “large” AF subalgebra. For the substitution tilings of [35], a suitable subalgebra is constructed there. For transformation group C*-algebras of free minimal actions of \( \mathbb{Z}^d \) on the Cantor set, we obtain this subalgebra by reinterpreting the main result of Forrest’s paper [13] in terms of groupoids. We actually prove our main results for the reduced C*-algebras of what we call almost AF Cantor groupoids. These form a class of groupoids to which the methods of
Section 3 of [35] are applicable. Forrest in effect shows that the transformation group groupoid of a free minimal action of $\mathbb{Z}^d$ on the Cantor set is almost AF, and Putnam in effect shows in Section 2 of [35] that the groupoids of the substitution tiling systems considered there are almost AF.

There are three reasons for presenting our main results in terms of almost AF Cantor groupoids. First, the abstraction enables us to focus on just a few key properties. In particular, as will become clear, for actions of $\mathbb{Z}^d$ we do not need the full strength of the results obtained in [35]. Second, it seems plausible that groupoids arising in other contexts might turn out to be almost AF, so that our work would apply elsewhere. Third, we believe that the methods will work for actions of much more general discrete groups, and for actions that are merely essentially free. Proving this is, we hope, primarily a matter of generalizing Forrest’s construction of Kakutani-Rokhlin decompositions [15], and we want to separate the the details of the generalization from the methods used to obtain from it results for the crossed product C*-algebras.

This paper is organized as follows. The first section presents background material on principal r-discrete groupoids and their C*-algebras, especially in the case that the unit space is the Cantor set, in a form convenient for later use. In the second section, we define almost AF Cantor groupoids and present some basic results. The key technical result, Lemma 2.7, appears here. In Sections 3–5, we prove results for the reduced C*-algebra of an almost AF Cantor groupoid when the C*-algebra is simple, proving (most of) Blackadar’s Second Fundamental Comparability Question in Section 3, real rank zero in Section 4, and stable rank one in Section 5. The full statement of Blackadar’s Second Fundamental Comparability Question is obtained by combining the result of Section 3 with stable rank one. In Section 6, we use [15] to show that free minimal actions of $\mathbb{Z}^d$ on the Cantor set yield almost AF Cantor groupoids. The structural results above therefore hold for their C*-algebras. In Section 7, we do the same for the groupoids associated with several kinds of aperiodic tilings, and discuss the relation to the Bethe-Sommerfeld Conjecture. The last section contains some open problems and an example related to the nonsimple case.

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1. Cantor groupoids

In this section, we fix notation, recall some important definitions, and establish a few elementary facts. For groupoid notation and terminology, we will generally follow Renault’s book [36], with two exceptions. If $G$ is a groupoid with unit space $G(0)$, we will refer to its range and source maps $r, s : G \rightarrow G(0)$, given by

$$r(g) = gg^{-1} \quad \text{and} \quad s(g) = g^{-1}g$$

for $g \in G$. Also, transformation groups will normally act on the left; see Example [3] below for notation.
We will recall many of the relevant definitions from [36], since they are scattered through the book.

It is convenient to have a term to describe the basic assumptions we will be imposing on our groupoids.

**Definition 1.1.** A topological groupoid $G$ equipped with a Haar system (which will be suppressed in the notation) is called a *Cantor groupoid* if the following conditions are satisfied:

1. $G$ is Hausdorff, locally compact, and second countable.
2. The unit space $G^{(0)}$ is compact, totally disconnected, and has no isolated points (so is homeomorphic to the Cantor set).
3. $G$ is $r$-discrete in the sense of Definition 2.6 in Chapter 1 of [36], that is, $G^{(0)}$ is open in $G$.
4. The Haar system consists of counting measures.

Using Lemma 1.2 below, we can rephrase this in terminology that has recently become common (see for example [19] and [29]) by saying that a Cantor groupoid is a second countable locally compact Hausdorff etale groupoid whose unit space is the Cantor set, equipped with the Haar system of counting measures.

The following lemma describes some of the immediate properties of Cantor groupoids.

**Lemma 1.2.** Let $G$ be a Cantor groupoid. Then:

1. For any $x \in G^{(0)}$, the sets $r^{-1}(x), s^{-1}(x) \subset G$ are discrete.
2. The range and source maps $r, s : G \to G^{(0)}$ are local homeomorphisms.
3. $G$ is totally disconnected.

*Proof:* (1) This is Lemma 2.7(i) in Chapter 1 of [36].
(2) This is Lemma 2.7(iii) in Chapter 1 of [36].
(3) It follows from Part (2) that every point $g \in G$ has an open neighborhood which is homeomorphic to an open subset of $G^{(0)}$. Now use the fact that $G^{(0)}$ is assumed to be totally disconnected.

We now give the motivating example.

**Example 1.3.** Let $X$ be the Cantor set, and let $\Gamma$ be a countable discrete group which acts on $X$. Then the transformation group groupoid $\Gamma \times X$, equipped with the Haar system consisting of counting measures, is a Cantor groupoid.

For reference, and to establish conventions, here are the groupoid operations. The pairs $(\gamma_1, x_1)$ and $(\gamma_2, x_2)$ are composable exactly when $x_1 = \gamma_2 x_2$, and then $(\gamma_1, x_1)(\gamma_2, x_2) = (\gamma_1 \gamma_2, x_2)$. The range, source, and inverse are given by

$$r(\gamma, x) = (1, \gamma x), \quad s(\gamma, x) = (1, x), \quad \text{and} \quad (\gamma, x)^{-1} = (\gamma^{-1}, \gamma x).$$

**Example 1.4.** Let $G$ be a Cantor groupoid, and let $H$ be an open subgroupoid of $G$ which contains the entire unit space $G^{(0)}$ of $G$ (or, more generally, whose unit space is a compact open subset of $G^{(0)}$ with no isolated points). Equip $H$ with the Haar system of counting measures. Then $H$ is a Cantor groupoid.

To verify this, the only nonobvious condition from Definition 1.1 is that the counting measures form a Haar system, and for this the only issue is the second condition (continuity) in Definition 2.2 of Chapter 1 of [36]. For $x \in H^{(0)}$, let $\mu^x$ and $\nu^x$ be the counting measures on

$$\{g \in G : r(g) = x\} \quad \text{and} \quad \{g \in H : r(g) = x\}. $$
Let $f \in C_c(H)$. We have to show that $x \mapsto \int_H f \, d\nu^x$ is continuous. Since $H$ is open in $G$, the function $f$ extends to a function $\tilde{f} \in C_c(G)$ by setting $\tilde{f}(g) = 0$ for $g \in G \setminus H$. Further,

$$\int_H f \, d\nu^x = \int_G \tilde{f} \, d\mu^x,$$

which is known to be continuous in $x$. So $H$ is a Cantor groupoid.

As a specific example, we mention the subgroupoids of the transformation group groupoid of a minimal homeomorphism of the Cantor set implicit in [32]. For the groupoid interpretation, see Example 2.6 of [34].

For convenience, we will reformulate several standard definitions and constructions in our restricted context.

**Remark 1.5.** Let $G$ be a Cantor groupoid, or, more generally, a locally compact $r$-discrete groupoid with counting measures as the Haar system.

(1) (See the beginning of Section 1 in Chapter 2 of [36].) The product and adjoint in the convolution algebra $C_c(G)$ (the space of continuous functions on $G$ with compact support) are given by:

$$(f_1 f_2)(g) = \sum_{h \in G: r(h) = s(g)} f_1(gh) f_2(h^{-1}) \quad \text{and} \quad f^*(g) = \overline{f(g^{-1})}.$$  

(Note that we write $f_1 f_2$ rather than $f_1 * f_2$.) The C*-algebra $C^*(G)$ is the completion of $C_c(G)$ in a suitable C* norm; see Definition 1.12 in Chapter 2 of [36].

(2) (See Definitions 3.2, 3.4, and 3.12 in Chapter 1 of [36].) A Borel measure $\mu$ on $G^{(0)}$ is invariant if and only if for every $f \in C_c(G)$, the numbers

$$\int_{G^{(0)}} \left( \sum_{g \in G: r(g) = x} f(g) \right) \, d\mu(x) \quad \text{and} \quad \int_{G^{(0)}} \left( \sum_{g \in G: s(g) = x} f(g) \right) \, d\mu(x)$$

are equal. (The difference in the expressions is that one sum is over $r(g) = x$ and the other is over $s(g) = x$.)

(3) (See the discussion preceding Definition 2.8 in Chapter 2 of [36], but note that Renault seems to reserve the term “regular representation” for the case that the measure $\mu$ is quasiinvariant.) Let $\mu$ be a Borel measure on $G^{(0)}$. Then the regular representation $\pi$ of $C^*(G)$ associated with $\mu$ is constructed as follows. Define a measure $\nu$ on $G$ by

$$\int_G f \, d\nu = \int_{G^{(0)}} \left( \sum_{g \in G: s(g) = x} f(g) \right) \, d\mu(x)$$

for $f \in C_c(G)$. (This measure is called $\nu^{-1}$ in [31].) Then $\pi$ is the representation on $L^2(G, \nu)$ determined by the formula

$$\langle \pi(f)\xi, \eta \rangle = \int_{G^{(0)}} \left( \sum_{g, h \in G: s(g) = r(h) = x} f(gh) \xi(h^{-1}) \overline{\eta(g)} \right) \, d\mu(x).$$

(4) By comparing formulas, we see that the regular representation as defined here is the same as the representation called $\text{Ind}_\mu$ in [25] before Corollary 2.4. (See Section 1A of [27].)

Because of the way the relevant material is presented in the literature, and to call attention to the neat formulation of [25], we take some care with the definition of the reduced C*-algebra.
Lemma 1.7. Let $G$ be a locally compact r-discrete groupoid with counting measures as the Haar system. We define the reduced C*-algebra $C^*_r(G)$ to be the completion of $C_c(G)$ in the supremum of the seminorms coming from the representations $\lambda_x$, for $x \in G^{(0)}$, defined as follows: let $G_x = \{g \in G : s(g) = x\}$, and let $C_c(G)$ act on the Hilbert space $l^2(G_x)$ by

$$\lambda_x(f)(g) = \sum_{h \in G : s(h) = x} f(gh^{-1}) \xi(h).$$

As shown in Theorem 2.3 of [30], this norm comes from a single canonical regular representation of $C^*_r(G)$ on a Hilbert module over $C_0(G^{(0)})$. This might appropriately be used as the definition of $C^*_r(G)$.

**Lemma 1.7.** The reduced C*-algebra $C^*_r(G)$ as defined above is the same as the reduced C*-algebra as in Definition 2.8 in Chapter 2 of [36].

**Proof:** The representation $\lambda_x$ used in Definition 1.6 is easily checked to be the representation of Remark 1.6(3) coming from the point mass at $x$. Given this, and Remark 1.6(4), the result follows from Corollary 2.4(b) of [25].

The following two results are well known (in fact, in greater generality), but we have been unable to locate references. The version of the first for full groupoid C*-algebras and full crossed products is in [36] (after Definition 1.12 in Chapter 2), but we have been unable to find a statement of the reduced case.

**Proposition 1.8.** Let $X$ be a locally compact Hausdorff space, let the discrete groupoid $G$ act on $X$, and let $G = \Gamma \times H$ be the transformation group groupoid (as in Example 1.3). Then the reduced groupoid C*-algebra $C^*_r(G)$ is isomorphic to the reduced crossed product C*-algebra $C^*_r(\Gamma, X)$.

**Proof:** Recall that $C^*_r(\Gamma, X)$ is the completion of $C_c(\Gamma, C(X))$ in a suitable norm (see 7.6.5 and 7.7.4 of [31]), with multiplication and adjoint on $C_c(\Gamma, C(X))$ given by twisted versions of those in the ordinary group algebra (see 7.6.1 of [31]). Define $\varphi : C_c(G) \to C_c(\Gamma, C(X))$ by $\varphi(f)(\gamma)(x) = f(\gamma, \gamma^{-1}x)$. One checks that $\varphi$ is a bijective *-homomorphism. Moreover, when one identifies $G_{(1,x)}$ with $\Gamma$ in the obvious way, one finds that $\varphi$ transforms the representation $\lambda_{(1,x)}$ of Definition 1.6 into the regular representation $\pi_x$ of $C^*_r(\Gamma, C(X))$ determined as in 7.7.1 of [31] by the point evaluation $\text{ev}_x$, regarded as a one dimensional representation of $C(X)$. By Definition 1.6, the representation $\bigoplus_{x \in X} \lambda_{(1,x)}$ is injective on $C^*_r(G)$, and by Theorem 7.7.5 of [31] the representation $\bigoplus_{x \in X} \pi_x$ is injective on $C^*_r(\Gamma, X)$. Therefore $\varphi$ determines an isomorphism $C^*_r(G) \to C^*_r(\Gamma, X)$.  

**Proposition 1.9.** Let $G$ be a locally compact r-discrete groupoid with counting measures as the Haar system. Let $H$ be an open subgroupoid, and give $H$ also the Haar system consisting of counting measures. (See Example 1.4.) Then the inclusion $C_c(H) \subset C_c(G)$ defines an injective homomorphism $C^*_r(H) \to C^*_r(G)$.

**Proof:** Let $\lambda^G_x$ and $\lambda^H_y$ denote the representations of $C_c(G)$ and $C_c(H)$ used in Definition 1.6. It suffices to show that, for every $x \in G^{(0)}$, the representation $\lambda^G_x |_{C_c(H)}$ is a direct sum of representations $\lambda^H_y$ for various $y \in H^{(0)}$, and perhaps a copy of the zero representation.

Define a relation on $G_x$ by $g \sim h$ exactly when $gh^{-1} \in H$. Let $Y = \{g \in G_x : g \sim g\}$, which is equal to $\{g \in G_x : r(g) \in H\}$. Restricted to $Y$, the relation $\sim$ is an equivalence relation.
Let $E$ be an equivalence class. One easily checks that the subspace $l^2(E)$ is invariant for $\lambda^G_x|_{C_c(H)}$. Choose $g_0 \in E$ and let $y = r(g_0)$. Then the formula $h \mapsto h g_0$ defines a bijection $H_y \to E$. (To check surjectivity: if $g \sim g_0$, then $g g_0^{-1}$ is in $H$ and has source $y$, and $q = (g g_0^{-1}) g_0$.) Further, if we define a unitary $u : l^2(E) \to l^2(H_y)$ by the formula $(u \xi)(h) = \xi(h g_0)$, then we get $u (\lambda^G_x(f)|_H) u^* = \lambda^H_y(f)$ for all $f \in C_c(H)$.

Finally, one easily checks that $l^2(G_x \setminus Y)$ is an invariant subspace for $\lambda^G_x|_{C_c(L)}$, and that the restriction of the representation to this subspace is zero.}

**Definition 1.10.** (Definition 1.10 in Chapter 1 of 39.) Let $G$ be a groupoid. A $G$-set is a subset $S \subset G$ such that the restrictions of both the range and source maps to $S$ are injective.

**Lemma 1.11.** Let $G$ be a Cantor groupoid. Let $K \subset G$ be a compact set. Then $K$ is a finite disjoint union of compact $G$-sets, which are open if $K$ is open.

**Proof:** Since $r$ and $s$ are local homeomorphisms, for each $g \in K$ there is a compact open subset $E(g)$ such that the restrictions of both $r$ and $s$ to $E(g)$ are injective. Since $K$ is compact, there are $g_1, \ldots, g_n \in K$ such that $E(g_1), \ldots, E(g_n)$ cover $K$. Then set $K_1 = E(g_1) \cap K$ and, inductively,

$$K_l = (E(g_1) \cup \cdots \cup E(g_{l-1})) \cap K.$$  

The $K_l$ are compact, and open if $K$ is open, because the $E(g_i)$ are compact and open, they are $G$-sets because the $E(g_i)$ are $G$-sets, and clearly $K$ is the disjoint union of $K_1, \ldots, K_n$.

**Lemma 1.12.** Let $G$ be a Cantor groupoid, and let $S \subset G$ be a compact $G$-set. Then there exists a compact open $G$-set $T$ which contains $S$.

**Proof:** Write $S = \bigcap_{n=1}^\infty V_n$ for compact open subsets $V_n \subset G$ with $V_1 \supset V_2 \supset \cdots \supset S$. We claim that some $V_n$ is a $G$-set. Suppose not. Then there are infinitely many $n$ such that $s|_{V_n}$ is not injective, or there are infinitely many $n$ such that $r|_{V_n}$ is not injective. We assume the first case. (The proof is the same for the second case.) Then for all $n$, the restriction $s|_{V_n}$ is not injective. Choose $g_n, h_n \in V_n$ such that $s(g_n) = s(h_n)$ and $g_n \neq h_n$. By compactness, we may pass to a subsequence and assume that $g_n \to g$ and $h_n \to h$. Then $g, h \in S$ and $s(g) = s(h)$. If $g \neq h$, we have contradicted the assumption that $S$ is a $G$-set. If $g = h$, then every neighborhood of $g$ contains two distinct elements, namely $g_n$ and $h_n$ for sufficiently large $n$, whose images under $s$ are equal; this contradicts the fact (Lemma 1.2(2)) that $s$ is a local homeomorphism. Thus in either case we obtain a contradiction, so some $V_n$ is a $G$-set.

**Lemma 1.13.** Let $G$ be a Cantor groupoid. Let $\mu$ be a Borel measure on $G^{(0)}$, and let $\nu$ be the measure on $G$ of Remark 1.3(3). Let $L \subset G$ be a compact $G$-set. Then:

1. $\nu(L) = \mu(s(L))$.
2. If $\mu$ is $G$-invariant, then $\nu(L) = \mu(r(L))$.

**Proof:** Use Lemma 1.12 to choose a compact open $G$-set $V$ which contains $L$. Then $s|_{V} : V \to s(V)$ and $r|_{V} : V \to r(V)$ are homeomorphisms. It suffices to
prove that if \( f : G \to [0, 1] \) is any continuous function with \( \text{supp}(f) \subset V \) and \( f = 1 \) on \( L \), then
\[
\int_G f \, d\nu = \int_{G^{(0)}} f \circ (s|_V)^{-1} \, d\mu
\]
and, when \( \mu \) is \( G \)-invariant,
\[
\int_G f \, d\nu = \int_{G^{(0)}} f \circ (r|_V)^{-1} \, d\mu.
\]
(We take \( f \circ (s|_V)^{-1} = 0 \) off \( s(V) \) and \( f \circ (r|_V)^{-1} = 0 \) off \( r(V) \).) Because \( V \) is a \( G \)-set, the first equation is just the definition of \( \nu \). For the second, we use invariance of \( \mu \) to rewrite
\[
\int_G f \, d\nu = \int_{G^{(0)}} \left( \sum_{g \in G : r(g) = x} f(g) \right) \, d\mu(x)
\]
(changing the condition \( s(g) = x \) in the original sum to the condition \( r(g) = x \)). Now the second equation follows in the same way as the first.

At this point, we recall some further definitions from [36].

**Definition 1.14.**
(1) (Definition 1.1 in Chapter 1 of [36]) Let \( G \) be a groupoid, and let \( x \in G^{(0)} \). The *isotropy subgroup* of \( x \) is the set \( \{ g \in G : r(g) = s(g) = x \} \).

(2) (Definition 1.1 in Chapter 1 of [36]) A groupoid \( G \) is *principal* if every isotropy subgroup is trivial (has only one element). Equivalently, whenever \( g_1, g_2 \in G \) satisfy \( r(g_1) = r(g_2) \) and \( s(g_1) = s(g_2) \), then \( g_1 = g_2 \).

(3) (See Page 35 of [36].) Let \( G \) be a groupoid. A subset \( E \subset G^{(0)} \) is *invariant* if whenever \( g \in G \) with \( s(g) \in E \), then also \( r(g) \in E \).

(4) (Definition 4.3 in Chapter 2 of [36]) A locally compact groupoid \( G \) is *essentially principal* if for every closed invariant subset \( E \subset G^{(0)} \), the set of \( x \in E \) with trivial isotropy subgroup is dense in \( E \).

The groupoids appearing in the following definition will play a crucial role in what follows.

**Definition 1.15.** A Cantor groupoid \( G \) is called *approximately finite* (AF for short), if it is the increasing union of a sequence of compact open principal Cantor subgroupoids, each of which contains the unit space \( G^{(0)} \).

In Definition 3.7 of [18], and with a weaker condition (designed to allow unit spaces which are only locally compact), such a groupoid is called an AF equivalence relation.

The next proposition is included primarily to make the connection with earlier work. The corollary will be essential, but it is easily proved directly.

**Proposition 1.16.** An AF Cantor groupoid is an AF groupoid in the sense of Definition 1.1 in Chapter 3 of [36]. An AF groupoid as defined there is an AF Cantor groupoid if and only if its unit space is compact and has no isolated points.

**Proof:** The first statement follows easily from Lemma 3.4 of [18].

The “only if” part of the second statement is clear.

To prove the rest, let \( G \) be an AF groupoid in the sense of [36], and assume its unit space is compact and has no isolated points. By definition, we can write \( G \) as the increasing union of a sequence of open subgroupoids \( H_n \), each of which has the same
unit space $G^{(0)}$, and each of which is a disjoint union of a sequence of elementary groupoids (Definition 1.1 in Chapter 3 of [36]) $H_{n,k}$ of types $N_{n,k} \in \{1, 2, \ldots, \infty\}$.

Since the unit spaces $H^{(0)}$ are compact, for each $n$ there are only finitely many $H_{n,k}$, that is, $H_n = \bigsqcup_{k=1}^{t(n)} H_{n,k}$ with $t(k) < \infty$. If all $N_{n,k}$ are finite, then each $H_n$ is compact, and we are done. Otherwise, write $H_{n,k} = X_{n,k} \times F_{n,k}^2$ with card $(F_{n,k}) = N_{n,k}$, where the groupoid structure is $(x, r, s)(x, s, t) = (x, r, t)$ and other pairs are not composable. Further write $F_{n,k}$ as an increasing union $F_{n,k} = \bigcup_{d=1}^{\infty} F_{n,k,d}$, with card $(F_{n,k,d}) = \min(d, \text{card}(F_{n,k}))$. Set $H_{n,k,d} = X_{n,k} \times F_{n,k,d}$, which is a subgroupoid of $H_{n,k}$, and set

$$G_{n,d} = \bigsqcup_{k=1}^{t(n)} H_{n,k,d},$$

which is a compact open subgroupoid of $G$ with the same unit space $G^{(0)}$. Moreover,

$$G = \bigcup_{n=1}^{\infty} \bigcup_{d=1}^{\infty} G_{n,d}.$$  

We construct inductively a sequence $n \mapsto d(n)$ such that

$$G_{1,d(1)} \subset G_{2,d(2)} \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} G_{n,d(n)} = G.$$

Take $d(1) = 1$. Given $d(1), d(2), \ldots, d(n)$, note that all $G_{k,r}$, for $1 \leq k \leq n + 1$ and $r \in \mathbb{N}$, are compact and contained in the increasing union of open sets $H_{n+1} = \bigcup_{r=1}^{\infty} G_{n+1,r}$. Therefore we can choose $d(n+1)$ so large that $G_{n+1,d(n+1)}$ contains all of

$$G_{1,n+1}, G_{2,n+1}, \ldots, G_{n+1,n+1}, G_{1,d(1)}, G_{2,d(2)}, \ldots, G_{n,d(n)}.$$

This gives the desired sequence.

Now set $G_n = G_{n,d(n)}$. \[\square\]

**Corollary 1.17.** Let $G$ be an AF Cantor groupoid. Then $C^*_r(G)$ is an AF algebra.

**Proof:** By Proposition 1.15 in Chapter 3 of [36], the full C*-algebra $C^*(G)$ is AF. The reduced C*-algebra $C^*_r(G)$ is a quotient (actually, in this case equal to the full C*-algebra). \[\square\]

### 2. Almost AF Groupoids

In this section we introduce almost AF Cantor groupoids, and prove some basic properties. An almost AF Cantor groupoid contains a “large” AF Cantor subgroupoid, and its reduced C*-algebra contains a corresponding “large” AF subalgebra. We establish one to one correspondences between the sets of normalized traces on the two C*-algebras, and between them and the sets of invariant Borel probability measures on the unit spaces of the two groupoids. In addition, if the reduced C*-algebra of the groupoid is simple, so is the AF subalgebra. Moreover, an almost AF Cantor groupoid is essentially principal, and has an invariant measure whose associated regular representation is injective on the reduced C*-algebra.

Although our main results involve only almost AF Cantor groupoids whose reduced C*-algebras are simple, and we don’t know how to generalize them, it seems worthwhile to attempt to give a definition which is also appropriate for the non-simple case. For more details, see the discussion after Definition 2.2.
Definition 2.1. Let $G$ be a Cantor groupoid, and let $K \subset G^{(0)}$ be a compact subset. We say that $K$ is thin if for every $n$, there exist compact $G$-sets $S_1, S_2, \ldots, S_n \subset G$ such that $s(S_k) = K$ and the sets $r(S_1), r(S_2), \ldots, r(S_n)$ are pairwise disjoint.

Definition 2.2. Let $G$ be a Cantor groupoid. We say that $G$ is almost $AF$ if the following conditions hold:

1. There is an open AF subgroupoid $G_0 \subset G$ which contains the unit space of $G$ and such that whenever $K \subset G \setminus G_0$ is a compact set, then $s(K) \subset G^{(0)}$ is thin in $G_0$ in the sense of Definition 2.1.
2. For every closed invariant subset $E \subset G^{(0)}$, and every nonempty relatively open subset $U \subset E$, there is a $G$-invariant Borel probability measure $\mu$ on $G^{(0)}$ such that $\mu(U) > 0$.

This definition is an abstraction of the key ideas in the argument of Section 3 of [35]. Note that $G_0$ is not uniquely determined by $G$.

We will see in Proposition 2.13 that condition (2) is redundant when $C^*_r(G)$ is simple (or when $C^*_r(G_0)$ is simple). In the nonsimple case, we would still like the definition to imply that $C^*_r(G)$ has stable rank one and real rank zero. We have three motivations for condition (2). First, it seems to be exactly what is needed to guarantee that the groupoid is essentially principal. Second, it allows products with a totally disconnected compact metric space, regarded as a groupoid in which every element is a unit. (For a transformation group groupoid, this corresponds to forming the product with the trivial action on such a space.) Third, there is a free nonminimal action of $\mathbb{Z}$ on the Cantor set whose transformation group groupoid $G$ satisfies condition (1) but has an open subset in its unit space which is null for all $G$-invariant Borel probability measures, and for which $C^*_r(G)$ does not have stable rank one. See Example 8.8.

Lemma 2.3. Let $G$ be a second countable locally compact Hausdorff r-discrete groupoid with Haar system consisting of counting measures. Suppose that for every nonempty open subset $U \subset G^{(0)}$, there is a $G$-invariant Borel probability measure $\mu$ on $G^{(0)}$ such that $\mu(U) > 0$. Then there exists a $G$-invariant Borel probability measure on $G^{(0)}$ such that the regular representation it determines (Remark 1.5(3)) is injective on $C^*_r(G)$.

Proof: Let $U_1, U_2, \ldots$ form a countable base for the topology of $G^{(0)}$ consisting of nonempty open sets. Choose a $G$-invariant Borel probability measure $\mu_n$ on $G^{(0)}$ such that $\mu_n(U_n) > 0$. Set $\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_n$, which is a $G$-invariant Borel probability measure on $G^{(0)}$ such that $\mu(U_n) > 0$ for all $n$. Then supp($\mu$) is a closed subset of $G^{(0)}$ such that supp($\mu$) $\cap U_n \neq \emptyset$ for all $n$. Therefore supp($\mu$) $= G^{(0)}$. Now apply Corollary 2.4 of [25] and Remark 1.5(4).

Corollary 2.4. Let $G$ be an almost AF Cantor groupoid. Then there exists a $G$-invariant Borel probability measure on $G^{(0)}$ such that the regular representation it determines, as in Remark 1.5(3), is injective on $C^*_r(G)$.

We now need a lemma on thin sets.

Lemma 2.5. Let $G$ be a Cantor groupoid, and let $K \subset G^{(0)}$ be a compact subset which is thin in the sense of Definition 2.1. Then:

1. For every $n$, there exist compact open sets $W_1, W_2, \ldots, W_n \subset G$, such that $s(W_k) = W$ for all $k$, and such
that the sets \( r(W_1), r(W_2), \ldots, r(W_n) \) are pairwise disjoint compact open subsets of \( G^{(0)} \).

(2) For every \( \varepsilon > 0 \), there is a compact open subset \( V \) of \( G^{(0)} \) such that \( K \subset V \) and \( \mu(V) < \varepsilon \) for every invariant Borel probability measure \( \mu \) on \( G^{(0)} \).

(3) For every invariant Borel probability measure \( \mu \) on \( G^{(0)} \), we have \( \mu(K) = 0 \).

**Proof:** (1) Using Definition 2.1(2), choose compact \( G \)-sets \( S_1, S_2, \ldots, S_n \subset G \) such that \( s(S_k) = K \) and such that the sets \( r(S_1), r(S_2), \ldots, r(S_n) \) are pairwise disjoint. Choose disjoint compact open sets \( U_1, U_2, \ldots, U_n \subset G^{(0)} \) such that \( r(S_k) \subset U_k \). Use Lemma 2.5 to choose compact open \( G \)-sets \( V_1, V_2, \ldots, V_n \subset G \) such that \( S_k \subset V_k \). Replacing \( V_k \) by \( V_k \cap r^{-1}(U_k) \), we may assume that \( r(V_1), r(V_2), \ldots, r(V_n) \) are pairwise disjoint. Since \( s \) is a principal (Definition 1.14(4)).

(2) Let \( \varepsilon > 0 \). Choose \( n \in \mathbb{N} \) with \( \frac{1}{n} < \varepsilon \). Let \( W \subset G^{(0)} \) and \( W_1, W_2, \ldots, W_n \subset G \) be as in Part (1). Let \( \mu \) be any invariant Borel probability measure on \( G^{(0)} \). Let \( \nu \) be the measure in Remark 1.13.

\[
\mu(r(W_k)) = \nu(W_k) = \mu(s(W_k)) = \mu(W)
\]

for all \( k \). Since the \( r(W_k) \) are disjoint and \( \mu(G^{(0)}) = 1 \), it follows that \( \mu(W) \leq \frac{1}{n} < \varepsilon \).

(3) This is immediate from Part (2).

**Lemma 2.6.** Let \( G \) be an almost AF Cantor groupoid. Then \( G \) is essentially principal (Definition 1.14(4)).

**Proof:** Let \( E \subset G^{(0)} \) be a closed \( G \)-invariant subset.

Let \( G_0 \) be as in Definition 2.2(1). Note that \( G_0 \) is principal. If \( x \in G^{(0)} \) has nontrivial isotropy, then there is \( g \in G \) with \( g \neq x \) such that \( r(g) = s(g) = x \). So \( g \notin G_0 \), whence \( x \in s(G \setminus G_0) \).

Now \( G \setminus G_0 \) is a closed subset of a locally compact second countable Hausdorff space, and therefore is a countable union of compact subsets: \( G \setminus G_0 = \bigcup_{n=1}^\infty K_n \). Each \( s(K_n) \) is thin relative to \( G^{(0)} \). Let \( U_n \) be the interior of \( s(K_n) \cap E \) relative to \( E \). Then Lemma 2.3(3) implies that \( \mu(U_n) = 0 \) for every \( G_0 \)-invariant Borel probability measure \( \mu \) on \( G^{(0)} \), and hence for every \( G \)-invariant Borel probability measure \( \mu \) on \( G^{(0)} \). So \( U_n = \emptyset \) by Definition 2.2(2). Thus \( s(K_n) \cap E \) is nowhere dense in \( E \). It follows that \( s(G \setminus G_0) \cap E \) is meager in \( E \), and in particular that its complement is dense in \( E \). So the points in \( E \) with trivial isotropy are dense in \( E \).

Now we start work toward the correspondences between the sets of invariant measures. The following lemma is the key technical result for taking advantage of the structure of an almost AF Cantor groupoid, not only here but in later sections as well. The main part is (3), in which the products are in the \( C^* \)-algebra of the AF subgroupoid. The other parts are given for easy reference.

**Lemma 2.7.** Let \( G \) be a Cantor groupoid, and let \( G_0 \) be an AF subgroupoid satisfying Part (1) of the definition of an almost AF Cantor groupoid (Definition 2.2).
Let $f \in C_c(G)$. Let $K$ and $L$ be compact open subsets of $G^{(0)}$ such that
\[ K \cap s(\text{supp}(f) \cap [G \setminus G_0]) = \emptyset \quad \text{and} \quad L \cap r(\text{supp}(f) \cap [G \setminus G_0]) = \emptyset. \]
Let $p = \chi_K$ and $q = \chi_L$. Then (with convolution products evaluated in $C_c(G)$), we have:

1. $p, q \in C_c(G)$ are projections.
2. 
\[
(fp)(g) = \begin{cases} f(g) & s(g) \in K \\ 0 & s(g) \notin K \end{cases} \quad \text{and} \quad (qf)(g) = \begin{cases} f(g) & r(g) \in L \\ 0 & r(g) \notin L \end{cases}.
\]
3. $fp, qf \in C_c(G_0)$.

**Proof:** Part (1) is obvious.

To prove Parts (2) and (3) for $fp$, we evaluate $(f\chi_K)(g)$ following Remark 1.3(1). We have $\chi_K(h) = 0$ for $h \notin G^{(0)}$, so the formula reduces to
\[
(f\chi_K)(g) = f(g)\chi_K(s(g))
\]
for $g \in G$. This is the formula for $fp$ in Part (2). Now suppose $g \in G \setminus G_0$. If $f(g) \neq 0$, then $s(g) \in s(\text{supp}(f) \cap [G \setminus G_0])$, so $\chi_K(s(g)) = 0$. Thus $g \in G \setminus G_0$ implies $(f\chi_K)(g) = 0$. Certainly $\text{supp}(f\chi_K)$ is compact, so $f\chi_K \in C_c(G_0)$, which is Part (3).

The proof of Parts (2) and (3) for $qf$ is similar, or can be obtained from the case already done by applying it to $f^*$ and taking adjoints.

**Lemma 2.8.** Let $G$ be a Cantor groupoid, and let $G_0$ be an AF subgroupoid satisfying Part (1) of the definition of an almost AF Cantor groupoid (Definition 2.2). Then every $G_0$-invariant Borel probability measure on $G^{(0)}$ is $G$-invariant.

**Proof:** Let $\mu$ be a $G_0$-invariant probability measure on $G^{(0)}$. By assumption, we have
\[
\int_{G^{(0)}} \left( \sum_{g \in G: r(g) = x} f(g) \right) d\mu(x) = \int_{G^{(0)}} \left( \sum_{g \in G: s(g) = x} f(g) \right) d\mu(x)
\]
for all $f \in C_c(G_0)$. We need to verify this equation for all $f \in C_c(G)$. It suffices to do this for nonnegative functions $f$.

Let $f \in C_c(G)$ be nonnegative, and let $\varepsilon > 0$. By Lemma 1.11, we can write $\text{supp}(f)$ as the disjoint union of finitely many compact $G$-sets, say $N$ of them. It follows that for any $x \in G^{(0)}$, we have
\[
\text{card}(\{g \in \text{supp}(f): r(g) = x\}) \leq N \quad \text{and} \quad \text{card}(\{g \in \text{supp}(f): s(g) = x\}) \leq N.
\]
Set
\[
K_1 = r(\text{supp}(f) \cap [G \setminus G_0]) \quad \text{and} \quad K_2 = s(\text{supp}(f) \cap [G \setminus G_0]).
\]
Then $K_1$ and $K_2$ are thin subsets of $G^{(0)}$, so Lemma 2.5(2) provides compact open subsets $V_1, V_2 \subset G^{(0)}$ such that
\[
K_j \subset V_j \quad \text{and} \quad \mu(V_j) < \frac{\varepsilon}{N\|f\|_{\infty} + 1}
\]
for $j = 1, 2$. Define $f_1 = \chi_{G^{(0)} \setminus V_1} f$. By Lemma 2.7(2),
\[
f_1(g) = \begin{cases} 0 & r(g) \in V_1 \\ f(g) & \text{otherwise} \end{cases}.
\]
Therefore
\[
\left| \int_{G^{(0)}} \left( \sum_{g \in G: \ r(g)=x} f(g) \right) \, d\mu(x) - \int_{G^{(0)}} \left( \sum_{g \in G: \ r(g)=x} f_1(g) \right) \, d\mu(x) \right| \\
= \int_{V_1} \left( \sum_{g \in G: \ r(g)=x} f(g) \right) \, d\mu(x) \\
\leq \mu(V_1) \|f\|_{\infty} \sup_{x \in G^{(0)}} \card \{g \in G: \ r(g)=x\} \leq \mu(V_1) \|f\|_{\infty} N < \varepsilon.
\]

In the following calculation, for the first step we use the previous estimate, for the second we use \( f_1 \in C_c(G_0) \) (which holds by Lemma 2.9(3)) and the \( G_0 \)-invariance of \( \mu \), and for the third we use \( f_1 \leq f \) (which holds because \( f \) is nonnegative):
\[
\int_{G^{(0)}} \left( \sum_{g \in G: \ r(g)=x} f(g) \right) \, d\mu(x) < \varepsilon + \int_{G^{(0)}} \left( \sum_{g \in G: \ s(g)=x} f_1(g) \right) \, d\mu(x) \\
= \varepsilon + \int_{G^{(0)}} \left( \sum_{g \in G: \ s(g)=x} f_1(g) \right) \, d\mu(x) \\
\leq \varepsilon + \int_{G^{(0)}} \left( \sum_{g \in G: \ s(g)=x} f(g) \right) \, d\mu(x).
\]

A similar argument, using \( f_2 = f \chi_{G^{(0)} \setminus V_2} \) in place of \( f_1 \), gives the same inequality with the range and source maps exchanged. Since \( \varepsilon > 0 \) is arbitrary, we get
\[
\int_{G^{(0)}} \left( \sum_{g \in G: \ s(g)=x} f(g) \right) \, d\mu(x) = \int_{G^{(0)}} \left( \sum_{g \in G: \ r(g)=x} f(g) \right) \, d\mu(x),
\]
as desired. \( \blacksquare \)

Lemma 2.9. Let \( G \) be a locally compact \( r \)-discrete groupoid with counting measures as the Haar system.

(1) Let \( \mu \) be an invariant Borel probability measure on \( G^{(0)} \). Then the formula
\[
\tau(f) = \int_{G^{(0)}} (f|_{G^{(0)}}) \, d\mu,
\]
for \( f \in C_c(G) \), defines a normalized trace on the \( C^* \)-algebra \( C^*_r(G) \). Moreover, the assignment \( \mu \mapsto \tau \) is injective.

(2) Suppose in addition that \( G \) is principal. Then every normalized trace on \( C^*_r(G) \) is obtained from an invariant Borel probability measure \( \mu \) on \( G^{(0)} \) as in (1).

Proof: This is a special case of Proposition 5.4 in Chapter 2 of [36]. \( \blacksquare \)

Lemma 2.10. Let \( G \) be a Cantor groupoid, and let \( G_0 \) be an AF subgroupoid satisfying Part (1) of the definition of an almost AF Cantor groupoid (Definition 2.2). Let \( \tau_1 \) and \( \tau_2 \) be two normalized traces on \( C^*_r(G) \) whose restrictions to \( C^*_r(G_0) \) are equal. Then \( \tau_1 = \tau_2 \).

Proof: Let \( f \in C_c(G) \), and let \( \varepsilon > 0 \). We prove that \( |\tau_1(f^* f) - \tau_2(f^* f)| < \varepsilon \). Such elements are dense in the positive elements of \( C^*_r(G) \), so their linear span is dense in \( C^*_r(G) \), and the result will follow.

Without loss of generality \( \|f\| \leq 1 \). Let
\[
K = r(\text{supp}(f^* f) \cap [G \setminus G_0]).
\]
Since \( f^* f \in C_c(G) \), the set \( K \) is thin in \( G_0 \) (Definition 2.1). Also, since \( f^* f \) is selfadjoint, we have \( s(\text{supp}(f^* f) \cap [G \setminus G_0]) = K \). Choose \( n \in \mathbb{N} \) with \( n >
2ε^{-1}. Use Lemma 2.3(1) to choose a compact open set \( W \) containing \( K \), and compact open \( G \)-sets \( W_1, W_2, \ldots, W_n \subset G \) such that \( s(W_k) = W \) and the sets \( r(W_1), r(W_2), \ldots, r(W_n) \) are pairwise disjoint compact open subsets of \( G^{(0)} \). Let \( p = \chi_W \), which is a projection in \( C_c(G) \). The function \( v_k = \chi_{W_k} \) defines an element of \( C_c(G) \) such that \( v_k^* v_k = p \) and \( v_k v_k^* = \chi_{r(W_k)} \). Since the projections \( \chi_{r(W_1)}, \chi_{r(W_2)}, \ldots, \chi_{r(W_n)} \) are pairwise orthogonal, it follows that

\[
\tau_1(p), \tau_2(p) \leq \frac{1}{n} < \frac{1}{2}\varepsilon.
\]

By Lemma 2.7(3), the products \((1-p)f^*f\) and \(f^*f(1-p)\) are in \( C_c(G_0) \). Since \( p \in C_c(G_0) \), it follows that

\[
f^*f - p f^* f p = (1 - p) f^* f + p f^* f (1 - p) \in C_c(G_0).
\]

Therefore

\[
\tau_1(f^*f - p f^* f p) = \tau_2(f^*f - p f^* f p).
\]

On the other hand,

\[
p f^* f p \leq \|f\|^2 p \leq p,
\]

so

\[
0 \leq \tau_1(p f^* f p) \leq \tau_1(p) < \frac{1}{2}\varepsilon \quad \text{and} \quad 0 \leq \tau_2(p f^* f p) \leq \tau_2(p) < \frac{1}{2}\varepsilon.
\]

It follows that

\[
|\tau_1(f^*f) - \tau_2(f^*f)| = |\tau_1(p f^* f p) - \tau_2(p f^* f p)| < \varepsilon,
\]

as desired. \( \blacksquare \)

**Proposition 2.11.** Let \( G \) be a Cantor groupoid, and let \( G_0 \) be an AF subgroupoid satisfying Part (1) of the definition of an almost AF Cantor groupoid (Definition 2.2). Then the following sets can all be canonically identified:

- The space \( M \) of \( G \)-invariant Borel probability measures on \( G^{(0)} \).
- The space \( M_0 \) of \( G_0 \)-invariant Borel probability measures on \( G^{(0)} \).
- \( T(C^*_r(G)) \), the space of normalized traces on \( C^*_r(G) \).
- \( T(C^*_r(G_0)) \), the space of normalized traces on \( C^*_r(G_0) \).

The map from \( M \) to \( M_0 \) is the identity. (Both are sets of measures on \( G^{(0)} \)) The map from \( T(C^*_r(G)) \) to \( T(C^*_r(G_0)) \) is restriction of traces (using Lemma 1.3). The maps from \( M \) to \( T(C^*_r(G)) \) and from \( M_0 \) to \( T(C^*_r(G_0)) \) are as in Lemma 2.9.

**Proof:** The map from \( M_0 \) to \( T(C^*_r(G_0)) \) is bijective by Lemma 2.9, because \( G_0 \) is principal. The map from \( M \) to \( M_0 \) is well defined because \( G \)-invariant measures are obviously \( G_0 \)-invariant, and is then trivially injective. It is surjective by Lemma 2.8. The map from \( T(C^*_r(G)) \) to \( T(C^*_r(G_0)) \) is injective by Lemma 2.10, and the map from \( M \) to \( T(C^*_r(G)) \) is injective by Lemma 2.9. The composite \( M \to T(C^*_r(G)) \) is bijective by what we have already done, so both these maps must be bijective. \( \blacksquare \)

It is apparently not known whether Lemma 2.9(2) can be generalized to essentially principal groupoids, but this proposition shows that its conclusion is valid for almost AF Cantor groupoids.

We next show how to simplify the verification that a groupoid is almost AF when its reduced C*-algebra is simple. We need the following well known lemma. We have been unable to find a suitable reference in the literature, so we include a proof for completeness.
Lemma 2.12. Every unital AF algebra $B$ has a normalized trace.

Proof: Write $B = \lim_{n \to \infty} B_n$ for finite dimensional $C^*$-algebras $B_n$ and injective unital homomorphisms $B_n \to B_{n+1}$. Let $\tau_n$ be a normalized trace on $B_n$, and use the Hahn-Banach Theorem to extend $\tau_n$ to a state $\omega_n$ on $B$. Use Alaoglu’s Theorem to find a weak* limit point $\tau$ of the sequence $(\omega_n)$. It is easily checked that $\tau$ is a trace on $B$.

Proposition 2.13. Let $G$ be a Cantor groupoid, and let $G_0$ be an AF subgroupoid satisfying Part (1) of the definition of an almost AF Cantor groupoid (Definition 2.2). Assume that $C^*_r(G)$ is simple, or that $C^*_r(G_0)$ is simple. Then $G$ is an almost AF Cantor groupoid.

Proof: We must verify Part (2) of Definition 2.2. First, if $C^*_r(G)$ is simple, Proposition 4.5(i) in Chapter 2 of [36] implies that there are no nontrivial closed $G$-invariant subsets in $G^{(0)}$. If $C^*_r(G_0)$ is simple, then for the same reason there are no nontrivial closed $G_0$-invariant subsets in $G^{(0)}$, and so certainly no nontrivial closed $G$-invariant subsets in $G^{(0)}$. Therefore, in either case, it suffices to find a $G$-invariant Borel probability measure $\mu$ on $G^{(0)}$ such that $\mu(U) > 0$ for every nonempty open subset $U \subset G^{(0)}$.

The $C^*$-algebra $C^*_r(G_0)$ is AF by Corollary 1.17. It is unital, so by Lemma 2.12 it has a normalized trace $\tau$. Proposition 2.13 therefore implies the existence of a $G$-invariant Borel probability measure $\mu$ on $G^{(0)}$.

Let $U \subset G^{(0)}$ be open and nonempty, and suppose that $\mu(U) = 0$. Let $V = r(s^{-1}(U))$, using the range and source maps of $G$. Then $V$ is a $G$-invariant subset of $G^{(0)}$ (Definition 1.14(2)) which contains $U$, and it is open because $r$ is a local homeomorphism (Lemma 1.2(2)). Therefore $V = G^{(0)}$. Since every element of a Cantor groupoid $G$ is contained in a compact open $G$-set, and since $G$ is second countable, there exists a countable base for the topology of $G$ consisting of compact open $G$-sets. In particular, there is a countable collection of compact open $G$-sets, say $W_1, W_2, \ldots$, such that $s(W_n) \subset U$ for all $n$ and $V = \bigcup_{n=1}^{\infty} r(W_n)$. Using Lemma 1.13, we get

$$
\mu(r(W_n)) = \mu(s(W_n)) \leq \mu(U) = 0,
$$

whence

$$
\mu(G^{(0)}) = \mu(V) \leq \sum_{n=1}^{\infty} \mu(r(W_n)) = 0.
$$

This contradicts the assumption that $\mu$ is a probability measure.

We now show that simplicity of $C^*_r(G)$ implies that of $C^*_r(G_0)$. We need a lemma.

Lemma 2.14. Let $A$ be a unital AF algebra. Suppose $\tau(p) > 0$ for every normalized trace $\tau$ on $A$ and every nonzero projection $p \in A$. Then $A$ is simple.

Proof: Suppose $A$ is not simple. Let $I$ be a nontrivial ideal in $A$. Then $A/I$ is a unital AF algebra. By Lemma 2.12, there is a normalized trace $\tau$ on $A/I$. Let $\pi : A \to A/I$ be the quotient map. Since $I$ is AF, there is a nonzero projection $p \in I$. Then $\tau \circ \pi$ is a normalized trace on $A$ and $p$ is a nonzero projection in $A$ such that $\tau \circ \pi(p) = 0$. 

Proposition 2.15. Let $G$ be an almost AF Cantor groupoid, with AF subgroupoid $G_0$ as in Definition 2.2(1). Suppose $C^*_r(G)$ is simple. Then $C^*_r(G_0)$ is a simple AF algebra.

Proof: The algebra $C^*_r(G_0)$ is AF because $G_0$ is an AF groupoid. (See Proposition 1.15 in Chapter 3 of [36].) It is unital because the unit space of $G_0$ is compact. By Proposition 2.11, every trace on $C^*_r(G_0)$ is the restriction of a trace on $C^*_r(G)$. Since this algebra is simple, every normalized trace on it is strictly positive on every nonzero projection in $C^*_r(G_0)$, and in particular on every nonzero projection in $C^*_r(G_0)$. So $C^*_r(G_0)$ is simple by Lemma 2.14.

We close this section with one significant unanswered question.

Question 2.16. Is an almost AF Cantor groupoid necessarily amenable?

Remark 2.17. If the almost AF Cantor groupoid $G$ is a transformation groupoid $\Gamma \times X$ (as in Example 1.3), then the answer is yes; in fact, the group $\Gamma$ is necessarily amenable. See Example 2.7(3) of [1].

3. Traces and order on K-theory

In this section, we prove that if $G$ is an almost AF Cantor groupoid such that $C^*_r(G)$ is simple, then the traces on $C^*_r(G)$ determine the order on $K_0(C^*_r(G))$, that is, if $\eta \in K_0(C^*_r(G))$ and $\tau_*(\eta) > 0$ for all normalized traces, then $\eta > 0$. When $G$ is the groupoid of a substitution tiling system as in [35], this is the main result of that paper. Theorem 7.1 below implies that such a groupoid is in fact an almost AF Cantor groupoid.

The result of this section will be strengthened in Section 5 below.

Although the proofs are a bit different (and, we hope, conceptually simpler), the basic idea of this section is entirely contained in Section 3 of [35].

Lemma 3.1. Let $G$ be an almost AF Cantor groupoid, with open AF subgroupoid $G_0 \subset G$ as in Definition 2.2(1). Let $F \subset C_c(G)$ be a finite set, and let $\varepsilon > 0$. Then for every $\varepsilon > 0$ there exists a compact open subset $V$ of $G^{(0)}$ such that, with

$$p = \chi V \in C(G^{(0)}) \subset C^*_r(G_0),$$

we have:

1. $r(supp(f) \cap [G \setminus G_0]) \cup s(supp(f) \cap [G \setminus G_0]) \subset V$ for all $f \in F$.
2. $\| (1 - p)f(1 - p) \& \| > \| f \& \| - \varepsilon$ for all $f \in F$.
3. $\tau(p) < \varepsilon$ for every normalized trace $\tau$ on $C^*_r(G)$.

Proof: We start by choosing $V$ so that (1) and (2) are satisfied.

Let $F = \{f_1, f_2, \ldots, f_n\}$. By Corollary 2.4, there is a $G$-invariant probability measure $\mu$ on $G^{(0)}$ whose associated regular representation $\pi$ (see Remark 1.5(3)) is faithful on $C^*_r(G)$. Let $\nu$ be as in Remark 1.5(3). Choose

$$\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n \in C_c(G) \subset L^2(G, \nu)$$

such that

$$\| \xi_k \& = \| \eta_k \& = 1 \quad \text{and} \quad |\langle \pi(f_k) \xi_k, \eta_k \rangle| > \| f_k \& \| - \frac{1}{2} \varepsilon$$

for $1 \leq k \leq n$. 

Let
\[ K = \bigcup_{k=1}^{n} \left( \text{supp}(f_k) \cup \text{supp}(\xi_k) \cup \text{supp}(\eta_k) \right). \]

Then \( K \cap (G \setminus G_0) \) is a compact subset of \( G \setminus G_0 \), so \( s(K \cap (G \setminus G_0)) \) is a thin set in \( G^{(0)} \) by Definition 2.2(1) and has measure zero by Lemma 2.5(3). Considering \( \{g^{-1} : g \in K\} \) in place of \( K \), we also get \( \mu(r(K \cap (G \setminus G_0))) = 0 \). Therefore
\[ L = s(K \cap (G \setminus G_0)) \cup r(K \cap (G \setminus G_0)) \]
is a compact subset of \( G^{(0)} \) with \( \mu(L) = 0 \). Choose a decreasing sequence of compact open sets \( V_i \subset G^{(0)} \) such that \( \bigcap_{i=1}^{\infty} V_i = L \). Set \( p_l = \chi_{G^{(0)} \setminus V_i} \in C(G^{(0)}) \). Then one checks, for example by using the formula for \( \langle \pi(p_l)\xi, \eta \rangle \) in Remark 1.5(3), that if \( \xi, \eta \in C_c(G) \subset L^2(G, \nu) \) then
\[ \langle \pi(p_l)\xi(g), \eta(g) \rangle = \left\{ \begin{array}{ll} 0 & \text{if } g \in V_i \\ f(g) & \text{otherwise} \end{array} \right. \]

Use Lemma 1.11 to write \( K \) as the union of compact \( G \)-sets \( K_1, K_2, \ldots, K_N \). From Lemma 1.13, and because \( \mu \) is \( G \)-invariant, we get
\[ \nu(r^{-1}(V_i) \cap K) = \sum_{j=1}^{N} \nu(r^{-1}(V_i) \cap K_j) = N \mu(V_i). \]

Set \( W_i = r^{-1}(V_i) \cap K \). Then \( W_1 \supset W_2 \supset \cdots \), we have \( \nu(W_i) \leq N \mu(V_i) \to 0 \) (because \( \mu(L) = 0 \)), and for each \( k \) we have \( \pi(p_l)\xi_k = \chi_{G \setminus V_i} \xi_k \) and \( \pi(p_l)\eta_k = \chi_{G \setminus V_i} \eta_k \) (pointwise product on the right). So \( \pi(p_l)\xi_k \to \xi_k \) and \( \pi(p_l)\eta_k \to \eta_k \) almost everywhere \([\nu] \) as \( l \to \infty \). Applying the Dominated Convergence Theorem, we get
\[ \lim_{l \to \infty} \|\pi(p_l)\xi_k - \xi_k\| = 0 \quad \text{and} \quad \lim_{l \to \infty} \|\pi(p_l)\eta_k - \eta_k\| = 0 \]
for \( 1 \leq k \leq n \). Therefore there is \( l \) such that
\[ |\langle \pi(f_k)\pi(p_l)\xi_k, \pi(p_l)\eta_k \rangle| > \|f_k\| - \varepsilon \]
for \( 1 \leq k \leq n \). So, using \( \|\xi_k\| = \|\eta_k\| = 1 \), we get
\[ \|p_l f_k p_l\| \geq |\langle \pi(p_l f_k p_l)\xi_k, \eta_k \rangle| = |\langle \pi(f_k)\pi(p_l)\xi_k, \pi(p_l)\eta_k \rangle| > \|f_k\| - \varepsilon. \]

Take \( V = V_i \). With this choice, parts (1) and (2) of the conclusion hold.

To obtain part (3), apply Lemma 2.5(2) with \( \varepsilon \) as given, and taking for \( K \) the set
\[ [G \setminus G_0] \cap \bigcup_{k=1}^{n} \left( \text{supp}(f_k) \cup \text{supp}(f_k^*) \right) \]
(which is thin in \( G_0 \)). Call the resulting set \( W \). Then replace \( V \) by \( V \cap W \). This clearly does not affect the validity of parts (1) and (2), and we get part (3) by Proposition 2.11.

**Lemma 3.2.** Let \( A \) be a \( C^* \)-algebra with real rank zero. Let \( a, b \in A \) be positive elements with
\[ \|a\|, \|b\| \leq 1 \quad \text{and} \quad ab = b. \]
Let \( \varepsilon > 0 \). Then there is a projection \( p \in \overline{bAb} \) such that
\[
ap = p \quad \text{and} \quad \|pb - b\| < \varepsilon.
\]

Proof: Let \( B = \overline{bAb} \). Then \( ax = x \) for all \( x \in B \). Since \( A \) has real rank zero, the hereditary subalgebra \( B \) has an approximate identity consisting of projections. Since \( b \in B \), there is \( p \in B \) with \( \|pb - b\| < \varepsilon \).

Lemma 3.3. Let \( G \) be an almost AF Cantor groupoid, with open AF subgroupoid \( G_0 \subset G \) as in Definition (2.2.1). Let \( e \in C^*_r(G) \) be a projection, and let \( \varepsilon > 0 \). Then there is a projection \( q \in C^*_r(G_0) \) which is Murray-von Neumann equivalent to a subprojection of \( e \) and such that \( \tau(e) - \tau(q) < \varepsilon \) for every normalized trace \( \tau \) on \( C^*_r(G) \).

Proof: Without loss of generality \( \varepsilon < 6 \).

Choose \( \delta_0 > 0 \) such that whenever \( A \) is a \( C^* \)-algebra and \( p_1, p_2 \in A \) are projections such that \( \|p_1 - p_2\| < \delta_0 \), then \( p_2 \) is Murray-von Neumann equivalent to a subprojection of \( p_1 \). Define a continuous function \( f : [0, \infty) \to [0, 1] \) by
\[
f(t) = \begin{cases} 6\varepsilon^{-1}t & 0 \leq t \leq \frac{t}{6}\varepsilon \\ 1 & \frac{t}{6}\varepsilon \leq t \end{cases}.
\]

Choose \( \delta > 0 \) such that whenever \( A \) is a \( C^* \)-algebra and \( a_1, a_2 \in A \) are positive elements with \( \|a_1\|, \|a_2\| \leq 1 \) and \( \|a_1 - a_2\| < \delta \), then \( \|f(a_1) - f(a_2)\| < \frac{1}{2}\delta_0 \).

Since \( C^*_r(G) \) is a dense \( * \)-subalgebra of \( C^*_r(G) \), there is a selfadjoint element \( d \in C^*_r(G) \) with
\[
\|d - d\| < \min\left(\frac{1}{2}\delta, \frac{1}{2}\varepsilon\right) \quad \text{and} \quad \|d\| \leq 1.
\]

Apply Lemma 2.7 with \( F = \{d\} \), obtaining a projection
\[
p = \chi V \in C(G^{(0)}) \subset C^*_r(G_0)
\]
for a suitable compact open set \( V \subset G^{(0)} \), such that
\[
r(supp(d) \cap [G \setminus G_0]) \cup s(supp(d) \cap [G \setminus G_0]) \subset V
\]
and \( \tau(p) < \varepsilon \) for every \( \tau \in T(C^*_r(G)) \), the space of normalized traces on \( C^*_r(G) \).

Lemma 2.7(3) gives
\[
(1 - p)d, d(1 - p) \in C^*_r(G_0).
\]

For every \( \tau \in T(C^*_r(G)) \), we have
\[
\tau(pe(1 - p)) = \tau((1 - p)ep) = 0,
\]
so
\[
\tau((1 - p)e(1 - p)) = \tau(e) - \tau(pep) \geq \tau(e) - \tau(p) > \tau(e) - \frac{1}{2}\varepsilon,
\]
and (using \( \|d^2 - e\| < \frac{1}{2}\varepsilon \))
\[
\tau(d(1 - p)d) = \tau((1 - p)d^2(1 - p)) > \tau(e) - \frac{1}{2}\varepsilon.
\]

Also, \( d(1 - p)d \) is a positive element in \( C^*_r(G_0) \).

Let \( f : [0, \infty) \to [0, 1] \) be as above, and define continuous functions \( g, h : [0, \infty) \to [0, 1] \) by
\[
g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{t}{6}\varepsilon \\ 6\varepsilon^{-1}t - 1 & \frac{t}{6}\varepsilon \leq t \leq \frac{1}{3}\varepsilon \\ 1 & \frac{1}{3}\varepsilon \leq t \end{cases}
\]
and
\[
h(t) = \begin{cases} t & 0 \leq t \leq \frac{1}{6}\varepsilon \\ \frac{1}{6}\varepsilon & \frac{1}{6}\varepsilon \leq t \end{cases}.
\]
Define
\[ a = f(d(1-p)d), \quad b = g(d(1-p)d), \quad \text{and} \quad c = h(d(1-p)d). \]

Then \( a, b, c \in C^*_r(G_0) \) are positive, and
\[ ab = b, \quad b + c \geq d(1-p)d, \quad \|a\| \leq 1, \quad \|b\| \leq 1, \quad \text{and} \quad \|c\| \leq \frac{1}{b} \varepsilon. \]

In particular, every \( \tau \in T(C^*_r(G)) \) satisfies \( \tau(c) \leq \frac{1}{b} \varepsilon \), so
\[ \tau(b) = \tau(b + c) - \tau(c) \geq \tau(d(1-p)d) - \frac{1}{b} \varepsilon > \tau(e) - \frac{3}{b} \varepsilon. \]

Since \( C^*_r(G_0) \) is an AF algebra, we can apply Lemma 3.2 to find a projection \( q \in bC^*_r(G_0)b \) such that \( aq = q \) and \( \|qb - b\| < \frac{1}{b} \varepsilon \). Then \( \|qbq - b\| < \frac{1}{b} \varepsilon \), so that for every \( \tau \in T(C^*_r(G)) \),
\[ \tau(q) \geq \tau(qbq) > \tau(b) - \frac{1}{b} \varepsilon > \tau(e) - \varepsilon. \]

This is one half of the desired conclusion.

From \( \|d - e\| < \frac{1}{b} \delta \) and \( \|d\| \leq 1 \) we get \( \|d(1-p)d - e(1-p)e\| < \delta \), so the choice of \( \delta \) at the beginning of the proof gives \( \|a - f(e(1-p)e)\| < \frac{1}{b} \delta_0 \). Since
\[ e(f(e(1-p)e) = f(e(1-p)e), \]
we get \( \|ea - a\| < \delta_0 \). Also \( aq = q \), so \( \|aq - q\| < \delta_0 \). The choice of \( \delta_0 \) at the beginning of the proof implies that \( q \) is Murray-von Neumann equivalent to a subprojection of \( e \). This the other half of the desired conclusion.

The following result is a K-theoretic version of Blackadar’s Second Fundamental Comparability Question ([1], 1.3.1). We will get the full result in Section 3 after we have proved stable rank one.

**Theorem 3.4.** Let \( G \) be an almost AF Cantor groupoid. Suppose that \( C^*_r(G) \) is simple. If \( \eta \in K_0(C^*_r(G)) \) satisfies \( \tau_\eta(\eta) > 0 \) for all normalized traces \( \tau \) on \( C^*_r(G) \), then there is a projection
\[ e \in M_\infty(C^*_r(G)) = \bigcup_{n=1}^\infty M_n(C^*_r(G)) \]
such that \( \eta = [e] \).

**Proof:** Write \( \eta = [q] - [p] \) for projections \( p, q \in M_\infty(C^*_r(G)) \). Choose \( n \) large enough that both \( p \) and \( q \) are in \( M_n(C^*_r(G)) \). Then \( G \times \{1, 2, \ldots, n\}^2 \), with the groupoid structure being given by \( (g, j, k)(h, k, l) = (gh, j, l) \) when \( q \) and \( h \) are composable in \( G \), and all other pairs not composable, is an almost AF Cantor groupoid whose reduced C*-algebra is the simple C*-algebra \( M_n(C^*_r(G)) \). Replacing \( G \) by \( G \times \{1, 2, \ldots, n\}^2 \), we may therefore assume \( p, q \in C^*_r(G) \).

Since \( C^*_r(G) \) is simple, all traces are faithful. Because \( T(C^*_r(G)) \) is weak* compact, there is \( \varepsilon > 0 \) such that \( \tau(q) - \tau(p) > \varepsilon \) for all \( \tau \in T(C^*_r(G)) \). Apply Lemma 3.3 twice, once to find a projection \( q_0 \in C^*_r(G_0) \) which is Murray-von Neumann equivalent to a subprojection \( q_1 \) of \( q \) and such that \( \tau(q) - \tau(q_0) < \frac{1}{b} \varepsilon \) for all \( \tau \in T(C^*_r(G)) \), and once to find a projection \( f_0 \in C^*_r(G_0) \) which is Murray-von Neumann equivalent to a subprojection \( f_1 \) of \( 1 - p \) and such that \( \tau(1-p) - \tau(f_0) < \frac{1}{b} \varepsilon \) for all \( \tau \in T(C^*_r(G)) \). Then (using Proposition 2.11) we get \( 0 < \tau(q_0) - \tau(1-f_0) < \frac{1}{b} \varepsilon \) for all \( \tau \in T(C^*_r(G)) \). Since \( C^*_r(G_0) \) is a simple AF algebra (by Proposition 2.13),
we can apply the last part of Corollary 6.9.2 of [4] to find a projection $p_0 \leq q_0$ which is Murray-von Neumann equivalent to $1 - f_0$. Now we write, in $K_0(C^*_r(G))$,

$$
\eta = [q] - [p] = ([q] - [q_0]) + ([q_0] - [p_0]) + ([p_0] - [p])
$$

$$
\quad = [q - q_0] + [q_0 - p_0] + [1 - p - f_1] > 0,
$$
as desired.  

4. **Real rank of the C*-algebra of an almost AF groupoid**

In this section, we prove that if $G$ is an almost AF Cantor groupoid such that $C^*_r(G)$ is simple, then $C^*_r(G)$ has real rank zero.

**Lemma 4.1.** Let $G$ be an almost AF Cantor groupoid, with open AF subgroupoid $G_0 \subset G$ as in Definition 2.2(1). Let $F \subset C_c(G)$ be a finite set, and let $n \in \mathbb{N}$. Then there exists a compact open subset $V$ of $G^{(0)}$ such that the projection $p = \chi_V \in C(G^{(0)})$ satisfies:

1. $(1 - p)f, f(1 - p) \in C_c(G_0)$ for all $f \in F$.
2. There are $n$ mutually orthogonal projections in $C(G^{(0)})$, each of which is Murray-von Neumann equivalent to $p$ in $C^*_r(G_0)$.

**Proof:** Let $K = \bigcup_{f \in F} [\text{supp}(f) \cup \text{supp}(f^*_k)]$.

Then $K \cap (G \setminus G_0)$ is a compact subset of $G \setminus G_0$, so by Definition 2.2, the set $s(K \cap (G \setminus G_0))$ is thin in $G^{(0)}$. By Lemma 2.5(1), there exist a compact open set $W$ containing $K$ and compact open $G_0$-sets $W_1, W_2, \ldots, W_n \subset G_0$, such that $s(W_k) = W$ and the sets $r(W_1), r(W_2), \ldots, r(W_n)$ are pairwise disjoint compact open subsets of $G^{(0)}$. Then the functions $\chi_{W_k} \in C_c(G_0)$ are partial isometries which implement Murray-von Neumann equivalences between $p$ and the $n$ mutually orthogonal projections $\chi_{r(W_1)}, \chi_{r(W_2)}, \ldots, \chi_{r(W_n)}$.

We clearly have $s(\text{supp}(f) \cap (G \setminus G_0)) \subset K$ for all $f \in F$. Also,

$$
\text{supp}(f^*) \cap [G \setminus G_0] = \{g^{-1}: g \in \text{supp}(f) \cap [G \setminus G_0]\},
$$

so

$$
r(\text{supp}(f) \cap [G \setminus G_0]) = s(\text{supp}(f^*) \cap [G \setminus G_0]) \subset K
$$

for all $f \in F$. Therefore $(1 - p)f, f(1 - p) \in C_c(G_0)$ by Lemma 2.5(3).  

**Lemma 4.2.** Let $A$ be a finite dimensional C*-algebra. Let $a \in A_{sa}$, let $p \in A$ be a projection, and let $n \in \mathbb{N}$. Then there is a projection $q \in A$ such that $p \leq q$, $[q] \leq 2n[p] \in K_0(A)$, and $\|qa - aq\| \leq \frac{1}{n}\|a\|$.

**Proof:** The result is trivial if $a = 0$, so, by scaling, we may assume $\|a\| = 1$. Without loss of generality $A = M_m$, which we think of as operators on $C^m$, and $a$ is diagonal. Making suitable replacements of the diagonal entries of $a$, we find $b \in (M_m)_{sa}$ such that

$$
\|a - b\| \leq \frac{1}{2n} \quad \text{and} \quad \text{sp}(b) \subset \left\{ -\frac{2n - 1}{2n}, -\frac{2n - 3}{2n}, \ldots, \frac{2n - 3}{2n}, \frac{2n - 1}{2n} \right\}.
$$
Define $u = \exp(\pi i b)$. Then $u$ is a unitary in $M_n$ with $u^{2n} = 1$. Moreover, with log being the continuous branch with values such that $\text{Im}(\log(\zeta)) \in (-\pi, \pi)$, we have $b = \frac{1}{2\pi} \log(u)$.

Let $H \subset \mathbb{C}^m$ be the linear span of the spaces $u^k p \mathbb{C}^m$ for $0 \leq k \leq 2n - 1$. Then $u H = H$ since $u^{2n} = 1$, and $\dim(H) \leq 2n \cdot \text{rank}(p)$. Let $q \in M_n$ be the orthogonal projection onto $H$. Then $p \leq q$ and $[q] \leq 2n[p] \in K_0(M_n)$. Since $uq = qu$, functional calculus gives $b q = q b$. Since $\|a - b\| \leq \frac{1}{2n}$, this gives $\|qa - aq\| \leq \frac{1}{n}$.

**Lemma 4.3.** Let $A$ be a unital AF algebra. Let $a \in A_{sa}$ be nonzero, and let $p \in A$ be a projection. Let $\varepsilon > 0$, and let $n \in \mathbb{N}$ satisfy $n > \frac{1}{\varepsilon}$. Then there is a projection $q \in A$ such that $p \leq q$, $[q] \leq 2n[p] \in K_0(A)$, and $\|qa - aq\| < \varepsilon\|a\|$.

**Proof:** There are finite dimensional subalgebras of $A$ which contain projections arbitrarily close to $p$ and selfadjoint elements arbitrarily close to $a$ and with the same norm. Conjugating by suitable unitaries, we find that there are finite dimensional subalgebras of $A$ which exactly contain $p$ and which contain selfadjoint elements arbitrarily close to $a$ and with the same norm. Choose such a subalgebra $B$ which contains a selfadjoint element $b$ with

$$\|b\| = \|a\| \quad \text{and} \quad \|a - b\| < \frac{1}{2}(\varepsilon - \frac{1}{n})\|a\|.$$  

Apply Lemma 1.2 to $B$, $b$, $p$, and $n$. The resulting projection $q \in B \subset A$ has the required properties.

**Lemma 4.4.** Let $r > 0$, let $f_1, f_2, \ldots, f_n : [-r, r] \to [0, 1]$ be continuous functions, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that, whenever $A$ is a unital C*-algebra, and whenever a projection $p \in A$ and a selfadjoint element $a \in A$ satisfy $\|a\| \leq r$ and $\|pa - ap\| < \delta$, and

$$\tau(f_k(a)) - \tau(1 - p) > \varepsilon$$

for $1 \leq k \leq n$ and all normalized traces $\tau$ on $A$, then (using functional calculus in $p A p$)

$$\tau(f_k(pap)) > \tau(f_k(a)) - \tau(1 - p) - \varepsilon$$

for $1 \leq k \leq n$ and all normalized traces $\tau$ on $A$.

**Proof:** Approximating the functions $f_k$ uniformly on $[-r, r]$ by polynomials, we can choose $\delta > 0$ so small that whenever $A$ is a unital C*-algebra, and whenever a projection $p \in A$ and a selfadjoint element $a \in A$ satisfy $\|a\| \leq r$ and $\|pa - ap\| < \delta$, then

$$\|f_k(pap) - pf_k(a)p\| < \varepsilon$$

for $1 \leq k \leq n$. (The expression $f_k(pap)$ is evaluated using functional calculus in $p A p$.)

Fix a trace $\tau$. We estimate:

$$\tau((1 - p)f_k(a)(1 - p)) \leq \tau(1 - p)\|f_k(a)\| \leq \tau(1 - p)$$

and

$$|\tau(f_k(pap)) - \tau(pf_k(a)p)| \leq \|f_k(pap) - pf_k(a)p\| < \varepsilon$$

for $1 \leq k \leq n$. Also,

$$\tau((1 - p)f_k(a)p) = \tau(pf_k(a)(1 - p)) = 0,$$
because \( \tau \) is a trace. So
\[
\tau(f_k(pa)) > \tau(pf_k(a)p) - \varepsilon \\
= \tau(f_k(a)) - \varepsilon \\
- \left[ \tau((1-p)f_k(a)(1-p)) + \tau((1-p)f_k(a)p) + \tau(pf_k(a)(1-p)) \right] \\
\geq \tau(f_k(a)) - \tau(1-p) - \varepsilon,
\]
as desired. \( \blacksquare \)

**Lemma 4.5.** Let \( X \subseteq \mathbb{R} \) be a compact set and let \( \varepsilon > 0 \). Let \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \) be real numbers such that for every \( \lambda \in X \) there is \( k \) with \( |\lambda - \lambda_k| < \varepsilon \). Define \( a \in C(X, M_{n+1}) \) by
\[
a(\lambda) = \text{diag}(\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n)
\]
for \( \lambda \in X \). Then there exists a selfadjoint diagonal element \( b \in C(X, M_{n+1}) \) and a unitary \( u \in C(X, M_{n+1}) \) such that
\[
\text{sp}(b) \subseteq \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \quad \text{and} \quad \|au - b\| < 2\varepsilon.
\]

**Proof:** We first prove this in the case that \( X \) is an interval \([\alpha, \beta]\) and \( \lambda_{k+1} - \lambda_k < 2\varepsilon \) for \( 1 \leq k \leq n - 1 \). Let
\[
\rho = \sup_{\lambda \in [\alpha, \beta]} \text{dist}(\lambda, \{ \lambda_1, \lambda_2, \ldots, \lambda_n \}).
\]
Then \( \rho < \varepsilon \) by assumption. Choose numbers \( \varepsilon_k \) with \( 0 < \varepsilon_k < \varepsilon - \rho \) such that
\[
\lambda_1 + \varepsilon_1 < \lambda_2 - \varepsilon_2 < \lambda_2 + \varepsilon_2 < \cdots < \lambda_{n-1} + \varepsilon_{n-1} < \lambda_n - \varepsilon_n.
\]
For any \( k \), let \( I_k \) denote the \( k \times k \) identity matrix. For \( 0 \leq k \leq n \), define a unitary \( v_k \in M_{n+1} \) by
\[
v_k = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \oplus I_{n-k},
\]
where the first block is a cyclic shift of size \((k + 1) \times (k + 1)\). Note that
\[
v_k \text{diag}(\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n)v_k^* = \text{diag}(\lambda_1, \ldots, \lambda_k, \lambda, \lambda_{k+1}, \ldots, \lambda_n).
\]
Choose a continuous path \( \lambda \mapsto z_k(\lambda) \) in the unitary group of \( M_2 \), for \( \lambda_k - \varepsilon_k \leq \lambda \leq \lambda_k + \varepsilon_k \), such that
\[
z_k(\lambda_k - \varepsilon_k) = 1 \quad \text{and} \quad z_k(\lambda_k + \varepsilon_k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Define
\[
w_k(\lambda) = I_{k-1} \oplus z_k(\lambda) \oplus I_{n-k} \in M_{n+1}
\]
for \( \lambda_k - \varepsilon_k \leq \lambda \leq \lambda_k + \varepsilon_k \). Define

\[
\begin{align*}
u(\lambda) &= \begin{cases} v_1 & \lambda \leq \lambda_1 + \varepsilon_1 \\ v_k & \lambda_k + \varepsilon_k \leq \lambda \leq \lambda_{k+1} - \varepsilon_{k+1} \text{ with } 1 \leq k \leq n-1 \\ w_k(\lambda)v_{k-1} & \lambda_k - \varepsilon_k \leq \lambda \leq \lambda_k + \varepsilon_k \text{ with } 2 \leq k \leq n \\ v_n & \lambda_n + \varepsilon_n \leq \lambda \end{cases}
\end{align*}
\]

Set

\[b_0 = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_n)\].

We claim that

\[\|u(\lambda)a(\lambda)u(\lambda)^*-b_0\| < 2\varepsilon\]

for all \( \lambda \in [\alpha, \beta] \). This will prove the special case, with \( b \) being the constant function \( b(\lambda) = b_0 \) for all \( \lambda \in [\alpha, \beta] \). For \( 1 \leq k \leq n-1 \) and \( \lambda_k + \varepsilon_k \leq \lambda \leq \lambda_{k+1} - \varepsilon_{k+1} \), this difference is

\[\|\text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k, \lambda, \lambda_{k+1}, \ldots, \lambda_n) - \text{diag}(\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n)\|,
\]

which is at most

\[\max_{1 \leq j \leq k-1} (\lambda_{j+1} - \lambda_j) < 2\varepsilon.
\]

Similar calculations show that \(\|u(\lambda)a(\lambda)u(\lambda)^*-b_0\| < 2\varepsilon\) for \( \lambda \leq \lambda_1 + \varepsilon_1 \) and for \( \lambda_n + \varepsilon_n \leq \lambda \). Using the same method for \( 2 \leq k \leq n \) and \( \lambda_k - \varepsilon_k \leq \lambda \leq \lambda_k + \varepsilon_k \), we get

\[\|u(\lambda)a(\lambda)u(\lambda)^*-b_0\| \leq \max \left( \max_{1 \leq j \leq k-2} (\lambda_{j+1} - \lambda_j), \left\| z_k(\lambda) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_k \end{pmatrix} z_k(\lambda)^* - \begin{pmatrix} \lambda_{k-1} & 0 \\ 0 & \lambda_k \end{pmatrix} \right\| \right).
\]

We have already seen that the first term is less than \( 2\varepsilon \). To estimate the other term, first observe that

\[\left\| z_k(\lambda) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_k \end{pmatrix} z_k(\lambda)^* - \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} \right\| < \varepsilon - \rho
\]

because \( z_k(\lambda) \) commutes with \( \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix} \). Therefore this term is less than

\[(\varepsilon - \rho) + (\lambda_k - \lambda_{k-1}) \leq \varepsilon - \rho + 2\rho < 2\varepsilon.
\]

So \(\|u(\lambda)a(\lambda)u(\lambda)^*-b_0\| < 2\varepsilon\) for all \( \lambda \in [\alpha, \beta] \). The special case is now proved.

In the general case, set

\[\rho = \sup_{\lambda \in X} \text{dist}(\lambda, \{\lambda_1, \lambda_2, \ldots, \lambda_n\}) < \varepsilon
\]

and

\[Y = \{\lambda \in \mathbb{R}: \text{dist}(\lambda, \{\lambda_1, \lambda_2, \ldots, \lambda_n\}) \leq \rho\}.
\]

Then \( X \subset Y \), and it suffices to prove the result for \( Y \) in place of \( X \). Now \( Y \) is the finite disjoint union of compact intervals. For each such interval \( J \), there are \( k \leq l \) such that

1. \( \{\lambda_k, \ldots, \lambda_l\} = J \cap \{\lambda_1, \ldots, \lambda_n\} \).
2. \( \lambda_{j+1} - \lambda_j < 2\varepsilon \) for \( k \leq j \leq l - 1 \).
(3) For every \( \lambda \in J \) there is \( j \) such that \( k \leq j \leq l \) and \( |\lambda - \lambda_k| < \varepsilon \).

Apply the result of the special case, obtaining \( a_J, b_J, \) and \( u_J \). Then on \( J \) define

\[
a_0(\lambda) = \text{diag}(\lambda_1, \ldots, \lambda_{k-1}) \oplus a_J(\lambda) \oplus \text{diag}(\lambda_{l+1}, \ldots, \lambda_n),
\]

\[
b_0(\lambda) = \text{diag}(\lambda_1, \ldots, \lambda_{k-1}) \oplus b_J(\lambda) \oplus \text{diag}(\lambda_{l+1}, \ldots, \lambda_n),
\]

and

\[
u_0(\lambda) = I_{k-1} \oplus u_J(\lambda) \oplus I_{n-l}.
\]

A suitable permutation matrix \( w \) conjugates \( a_0(\lambda) \) to \( a(\lambda) \) as defined in the statement, and we set \( u = wu_0w^* \) and \( b = wb_0w^* \).

The following lemma gives a suitable version of a now standard technique, which goes back at least to the proof of Lemma 1.7 of [12].

**Lemma 4.6.** Let \( A \) be a C*-algebra, let \( a \in A \) be normal, let \( \lambda_0 \in \mathbb{C} \), let \( \varepsilon > 0 \), and let \( f : \mathbb{C} \to [0, 1] \) be a continuous function such that \( \text{supp}(f) \) is contained in the open \( \varepsilon \)-ball with center \( \lambda_0 \). Let \( p \) be a projection in the hereditary subalgebra generated by \( f(a) \). Then

\[
\|pa - ap\| < 2\varepsilon \quad \text{and} \quad \|pap - \lambda_0 p\| < \varepsilon.
\]

**Proof:** Choose a continuous function \( g : \mathbb{C} \to \mathbb{C} \) such that \( |g(\lambda) - \lambda| < \varepsilon \) for all \( \lambda \in \mathbb{C} \) and \( g(\lambda) = \lambda_0 \) for all \( \lambda \in \text{supp}(f) \). Then

\[
\|g(a) - a\| < \varepsilon, \quad pg(a) = g(a)p, \quad \text{and} \quad pg(a)p = \lambda_0 p.
\]

**Theorem 4.7.** Let \( G \) be an almost AF Cantor groupoid. Suppose that \( C^*_r(G) \) is simple. Then \( C^*_r(G) \) has real rank zero in the sense of [3].

**Proof:** Let \( a \in C^*_r(G) \) be selfadjoint with \( \|a\| \leq 1 \). We approximate \( a \) by a selfadjoint element with finite spectrum. Since \( C_c(G) \) is a dense *-subalgebra of \( C^*_r(G) \), we can approximate \( a \) arbitrarily well by an element \( a_0 \in C_c(G) \); replacing \( a_0 \) by \( \frac{1}{n}(a_0 + a_0^*) \) and suitably scaling, we can approximate \( a \) arbitrarily well by a selfadjoint element \( a_0 \in C_c(G) \) with \( \|a_0\| \leq 1 \). Thus, without loss of generality \( a \in C_c(G) \).

Let \( \varepsilon > 0 \). Choose real numbers \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \) in \( \text{sp}(a) \) such that for every \( \lambda \in \text{sp}(a) \) there is \( k \) such that \( |\lambda - \lambda_k| < \frac{1}{4}\varepsilon \). Set

\[
\varepsilon_0 = \frac{\varepsilon}{2(n^2 + 2n + 3)},
\]

and choose continuous functions \( f_k, g_k : [-1, 1] \to [0, 1] \) such that the \( f_k \) have disjoint supports and

\[
g_k(\lambda_k) = 1, \quad f_k g_k = g_k, \quad \text{and} \quad \text{supp}(f_k) \subset \{ \lambda \in \mathbb{R} : |\lambda - \lambda_k| < \varepsilon_0 \}
\]

for \( 1 \leq k \leq n \).

Let \( T(C^*_r(G)) \) be the set of normalized traces on \( C^*_r(G) \), and define

\[
\alpha = \inf_{1 \leq k \leq n} \left( \inf_{\tau \in T(C^*_r(G))} \tau(g_k(a)) \right).
\]

Each \( g_k(a) \) is a nonzero positive element, all traces are faithful because \( C^*_r(G) \) is simple, and \( T(C^*_r(G)) \) is weak* compact. Therefore \( \alpha > 0 \).
Choose $\delta > 0$ as in Lemma 4.3, with $\frac{1}{2}\alpha$ in place of $\varepsilon$, with $r = 2$, and with $g_1, g_2, \ldots, g_n$ in place of $f_1, f_2, \ldots, f_n$. We also require $\delta < \varepsilon_0$. Choose $m \in \mathbb{N}$ with $\frac{2}{m} < \delta$, and use Lemma 4.3 to find a projection $p_0 \in C(G^{(0)})$ which is Murray-von Neumann equivalent in $C^*_r(G_0)$ to more than $8m\alpha^{-1}$ mutually orthogonal projections in $C(G^{(0)})$, and such that $(1 - p_0)a, a(1 - p_0) \in C^*_c(G_0)$. In particular,

$$\tau(p_0) < \frac{\alpha}{8m}$$

for all $\tau \in T(C^*_r(G_0))$.

Define $b = a - p_0ap_0$, which is a selfadjoint element of $C^*_c(G_0)$ with $\|b\| \leq 2$. Because $\frac{2}{m} < \delta$, we can apply Lemma 4.3 with $\frac{1}{2}\delta$ in place of $\varepsilon$ to obtain a projection $p \in C^*_r(G_0)$ such that $\|pb - bp\| < \delta$, $p_0 \leq p$, and $[p] \leq 2m[p_0]$ in $K_0(C^*_r(G))$. Now $p$ commutes with $b - a = p_0ap_0$, so also $\|pa - ap\| < \delta$. Furthermore, because $p \in C^*_r(G_0)$ and $p \geq p_0$, we get $(1 - p)a, a(1 - p) \in C^*_r(G_0)$.

Define $a_0 = (1 - p)a(1 - p)$. For every $\tau \in T(C^*_r(G_0))$, we have

$$\tau(p) \leq 2m\tau(p_0) < 2m \cdot \frac{\alpha}{8m} = \frac{1}{4}\alpha.$$

By the choice of $\delta$ and using Lemma 4.3 (with $1 - p$ in place of $p$ and $\frac{1}{2}\alpha$ in place of $\varepsilon$), we get

$$\tau(g_k(a_0)) > \tau(g_k(a)) - \frac{1}{2}\alpha \geq \alpha - \frac{1}{2}\alpha - \frac{1}{4}\alpha = \frac{1}{4}\alpha$$

for $1 \leq k \leq n$ and all $\tau \in T(C^*_r(G_0))$. Also $f_k(a_0)g_k(a_0) = g_k(a_0)$ for all $k$, and $C^*_r(G_0)$ is an AF algebra, so Lemma 3.2 provides projections $q_k \in C^*_r(G_0)$ such that

$$q_k \in g_k(a_0)A g_k(a_0), \quad f_k(a_0)q_k = q_k, \quad \text{and} \quad \|q_k g_k(a_0) - g_k(a_0)\| < \frac{1}{8}\alpha.$$

Therefore

$$\|q_k g_k(a_0)q_k - g_k(a_0)\| < \frac{1}{8}\alpha.$$

For all $\tau \in T(C^*_r(G_0))$, we have

$$\tau(q_k g_k(a_0)q_k) \leq \tau(q_k)$$

because $\|g_k(a_0)\| \leq 1$. Combining this with the previous estimates, it follows that

$$\tau(q_k) > \tau(g_k(a_0)) - \frac{1}{2}\alpha > \frac{1}{4}\alpha - \frac{1}{4}\alpha = 0.$$

Since $C^*_r(G_0)$ is simple (by Proposition 2.13), since traces determine order on the $K_0$ group of a simple AF algebra (see the last part of Corollary 6.9.2 of [3]), and since $\tau(p) < \frac{1}{4}\alpha$ for all $\tau \in T(C^*_r(G_0))$, it follows that there are projections $p_k \leq q_k$ with $p_k$ Murray-von Neumann equivalent to $p$. In particular, $p_k$ is in the hereditary subalgebra of $C^*_r(G_0)$ generated by $g_k(a_0)$. Since the $g_k$ have disjoint supports, the $p_k$ are orthogonal; since $a_0$ is orthogonal to $p$, so are the $p_k$. Moreover, Lemma 4.3 implies that

$$\|p_k a_0 - a_0 p_k\| < 2\varepsilon_0 \quad \text{and} \quad \|p_k a_0 p_k - \lambda_k p_k\| < \varepsilon_0.$$

Define

$$e = 1 - p - \sum_{k=1}^n p_k.$$
We estimate
\[ \left\| a - \left( eae + pap + \sum_{k=1}^{n} \lambda_k p_k \right) \right\|. \]

First,
\[ a - a_0 - pap = pa(1 - p) + (1 - p)ap. \]

Since \( \|pa - ap\| < \delta \leq \varepsilon_0 \), we get
\[ \|a - a_0 - pap\| < 2\varepsilon_0. \]

Next,
\[ a_0 - \left( eae + \sum_{k=1}^{n} p_k ap_k \right) = \sum_{k=1}^{n} p_k a_0 e + \sum_{k=1}^{n} e a_0 p_k + \sum_{j \neq k} p_j a_0 p_k. \]

Since \( \|p_k a_0 - a_0 p_k\| < 2\varepsilon_0 \), each of the terms has norm at most \( 2\varepsilon_0 \). There are
\[ 2n + (n^2 - n) = n^2 + n \text{ terms, so} \]
\[ \left\| a - \left( eae + \sum_{k=1}^{n} p_k ap_k \right) \right\| < [2(n^2 + n) + 2]\varepsilon_0. \]

Moreover, \( \|p_k a_0 p_k - \lambda_k p_k\| < \varepsilon_0 \) and the terms \( p_k a_0 p_k - \lambda_k p_k \) are orthogonal, so it follows that
\[ \left\| a - \left( eae + \sum_{k=1}^{n} \lambda_k p_k \right) \right\| < [2(n^2 + n) + 3]\varepsilon_0 = \frac{1}{2}\varepsilon. \]

There is a \( C^* \) subalgebra \( B \) of \( (1-e)C^*_r(G)(1-e) \) isomorphic to \( C(\text{sp}(pap), M_{n+1}) \) with identity
\[ 1 - e = p - \sum_{k=1}^{n} p_k \]
and containing
\[ pap + \sum_{k=1}^{n} \lambda_k p_k. \]

Using Lemma 4.3 and the choice of the numbers \( \lambda_k \), we can construct a selfadjoint element \( b \in (1-e)C^*_r(G)(1-e) \) with finite spectrum such that
\[ \left\| b - \left( pap + \sum_{k=1}^{n} \lambda_k p_k \right) \right\| < \frac{1}{2}\varepsilon. \]

Also, \( eae \in eC^*_r(G_0)e \), which is an AF algebra, so there is a selfadjoint element \( c \in eC^*_r(G_0)e \) with finite spectrum such that
\[ \|c - eae\| < \frac{1}{2}\varepsilon. \]

It follows that \( b + c \) is a selfadjoint element in \( C^*_r(G) \) with finite spectrum such that
\[ \|a - (b + c)\| < \varepsilon. \]

This completes the proof. \( \blacksquare \)
5. Stable rank of the C*-algebra of an almost AF groupoid

In this section, we prove that if \( G \) is an almost AF Cantor groupoid such that \( C^*_r(G) \) is simple, then \( C^*_r(G) \) has stable rank one. This implies that the projections in \( \mathcal{M}_\infty(C^*_r(G)) \) satisfy cancellation, and allows us to strengthen the conclusion of Theorem 3.4 to the full version of Blackadar’s Second Fundamental Comparability Question.

**Lemma 5.1.** Let \( G \) be an almost AF Cantor groupoid, with open AF subgroupoid \( G_0 \subseteq G \) as in Definition 2.2(1). For every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( a \in C_c(G) \) satisfies \( \|a\| \leq 1 \) and if \( e_0 \in C^*_r(G) \) is a nonzero projection such that \( \|ae_0\| < \delta \), then there exists a nonzero projection \( e \in C^*_r(G_0) \) and a compact open subset \( V \subset G^{(0)} \) such that, with \( p = \chi_V \in C^*_r(G_0) \), we have:

1. \( \text{r}(\text{supp}(a) \cap [G \setminus G_0]) \cup s(\text{supp}(a) \cap [G \setminus G_0]) \subset V \).
2. \( \|ae\| < \varepsilon \).
3. \( e \) and \( p \) are orthogonal.

**Proof:** Choose \( \delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right) > 0 \). Let \( a \in C_c(G) \) satisfy \( \|a\| \leq 1 \) and let \( e_0 \in C^*_r(G) \) be a nonzero projection such that \( \|ae_0\| < \delta \). Choose \( c \in C_c(G) \) with \( \|c\| = 1 \) and \( \|c - e_0\| < \delta^2 \). Apply Lemma 3.4 with \( F = \{a, c, c^*c\} \), and with \( \delta^2 \) in place of \( \varepsilon \). We obtain a compact open subset \( V \subset G^{(0)} \) such that, with \( p = \chi_V \), Condition (1) is satisfied, and also

\[
\|(1 - p)c^*c(1 - p)\| > 1 - \delta^2.
\]

Moreover, Lemma 2.7(3) implies that \( (1 - p)c^*, c(1 - p) \in C^*_r(G_0) \). It follows that \( (1 - p)c^*c(1 - p) \in C^*_r(G_0) \), which is an AF algebra.

Choose a continuous function \( f \) such that

\[
[1 - \delta^2, 1] \subset \text{supp}(f) \subset (1 - 2\delta^2, 1 + \delta^2).
\]

Choose a nonzero projection \( e \) in the hereditary subalgebra generated by

\[
f((1 - p)c^*c(1 - p))
\]

in the AF algebra \( C^*_r(G_0) \). Apply Lemma 4.6 in \( C^*_r(G_0) \) with this \( f \) and with \( \lambda_0 = 1 \), getting

\[
\|e - e(1 - p)c^*c(1 - p)e\| < 2\delta^2.
\]

Since \( 0 \notin \text{supp}(f) \), we have \( e \in (1 - p)C^*_r(G_0)(1 - p) \). So \( p \) and \( e \) are orthogonal, which is Part (3). We combine this with the estimate above and the estimate

\[
\|c^*c - e_0\| \leq \|c^*c - e\| + \|c^*c - e_0\| < 2\delta^2
\]

to obtain

\[
\|e - e_0e\|^2 = \|e(1 - e_0)e\| < \|e(1 - c^*c)e\| + 2\delta^2 = \|e - ec^*ce\| + 2\delta^2 < 4\delta^2.
\]

Therefore, since \( \|a\| \leq 1 \),

\[
\|ae\| \leq \|a\|\|e - e_0e\| + \|ae_0\|\|e\| < 2 \cdot \delta + \delta \leq \varepsilon.
\]

This is Part (2). ☐

**Theorem 5.2.** Let \( G \) be an almost AF Cantor groupoid. Suppose that \( C^*_r(G) \) is simple. Then \( C^*_r(G) \) has (topological) stable rank one in the sense of \[17\].
Proof: We are going to show that every two sided zero divisor in $C^*_r(G)$ is a limit of invertible elements. That is, if $a \in C^*_r(G)$ and there are nonzero $x, y \in C^*_r(G)$ such that $xa = ay = 0$, then we show that for every $\varepsilon > 0$ there is an invertible element $e \in C^*_r(G)$ such that $\|a - e\| < \varepsilon$. It will follow from Theorem 3.3(a) of \[6\] (see Definition 3.1 of \[6\]) that any $a \in C^*_r(G)$ which is not a limit of invertible elements is left or right invertible but not both. Since $C^*_r(G)$ is simple, the definition of an almost AF Cantor groupoid implies it has a faithful trace, and there are no such elements.

So let $a \in C^*_r(G)$, let $x, y \in C^*_r(G)$ be nonzero elements such that $xa = ay = 0$, and let $\varepsilon > 0$. Without loss of generality $\|a\| \leq 1$. Since $C^*_r(G)$ has real rank zero (Theorem 1.7), there are nonzero projections
\[ e_0 \in x^*C^*_r(G)x \quad \text{and} \quad f_0 \in yC^*_r(G)y, \]
and we have $e_0a = af_0 = 0$. Choose $\delta > 0$ in Lemma 5.1 for $\frac{1}{2}\delta$ in place of $\varepsilon$. Choose $b \in C_c(G)$ such that $\|b\| \leq 1$ and $\|a - b\| < \min(\delta, \frac{1}{4}\varepsilon)$. Then $\|e_0b\|, \|bf_0\| < \delta$.

Let
\[ K = r(\text{supp}(b) \cap [G \setminus G_0]) \cup s(\text{supp}(b) \cap [G \setminus G_0]). \]

Apply Lemma 5.1 to $b^*$ and $e_0$, obtaining a nonzero projection $e_1 \in C^*_r(G_0)$ and a compact open subset $V \subset G^{(0)}$ such that, with $g = \chi_V \in C^*_r(G)$, we have:
\[ K \subset V, \quad \|e_1b\| < \frac{1}{4}\varepsilon, \quad \text{and} \quad e_1g = 0. \]

Apply Lemma 5.1 to $b$ and $f_0$, obtaining a nonzero projection $f_1 \in C^*_r(G_0)$ and a compact open subset $W \subset G^{(0)}$ such that, with $h = \chi_W \in C^*_r(G)$, we have:
\[ K \subset W, \quad \|bf_1\| < \frac{1}{4}\varepsilon, \quad \text{and} \quad f_1h = 0. \]

Choose $\rho > 0$ with
\[ \rho < \min\left(\left.\inf_{\tau \in T(C^*_r(G))} \tau(e_1), \inf_{\tau \in T(C^*_r(G))} \tau(f_1)\right)\right). \]

As usual, $T(C^*_r(G))$ is the space of normalized traces. The set $K$ is thin in $G_0$, so Lemma 2.5(2) provides a compact open set $Z \subset G^{(0)}$ such that $K \subset Z$ and $\mu(Z) < \rho$ for every $G_0$-invariant Borel probability measure $\mu$ on $G^{(0)}$. Let $p = \chi_{V \cap W \cap \rho} \in C(G^{(0)})$. Then $p, e_1, f_1$ are all in $C^*_r(G_0)$, and $e_1p = f_1p = 0$. Proposition 2.11 implies that
\[ \tau(p) < \tau(e_1) \quad \text{and} \quad \tau(p) < \tau(f_1) \]
for all $\tau \in T(C^*_r(G_0))$. Since $C^*_r(G_0)$ is a simple AF algebra (by Proposition 2.13), it follows (see the last part of Corollary 6.9.2 of \[6\]) that there are projections $e_2, f_2 \in C^*_r(G_0)$ which are unitarily equivalent to $p$ and with $e_2 \leq e_1$ and $f_2 \leq f_1$. Furthermore, since $e_2$ and $f_2$ are both orthogonal to $p$, and $(1 - p)C^*_r(G_0)(1 - p)$ is a simple AF algebra, there is a unitary $w \in C^*_r(G_0)$ such that
\[ we_2w^* = f_2 \quad \text{and} \quad wpuw^* = p; \]
also, it is easy to find a unitary $u \in C^*_r(G_0)$ such that
\[ vf_2u^* = p \quad \text{and} \quad vp^* = f_2. \]

Set $u = vw$, which is a unitary in $C^*_r(G_0)$ such that
\[ uc_2u^* = p \quad \text{and} \quad upu^* = f_2. \]
Define
\[ b_0 = (1 - e_2)b(1 - f_2). \]

Since
\[ b_0 = b - e_2b(1 - f_2) - bf_2, \]
and since
\[ \|e_2b(1 - f_2)\| \leq \|e_1\| < \frac{1}{4} \varepsilon \quad \text{and} \quad \|bf_2\| \leq \|b_1\| < \frac{1}{4} \varepsilon, \]
we get \( \|b - b_0\| < \frac{1}{4} \varepsilon \), so \( \|a - b_0\| < \frac{3}{4} \varepsilon \). We have \( (1 - p)b(1 - p) \in C_1^*(G_0) \) by Lemma 2.7(3). With respect to the decomposition of the identity
\[ 1 = p + [1 - p - f_2] + f_2, \]
we claim that \( ub_0 \) has the block matrix form
\[
\begin{pmatrix}
0 & 0 & 0 \\
x & d_0 & 0 \\
y & z & 0
\end{pmatrix}
\]
with
\[ d_0 \in (1 - p - f_2)C_1^*(G_0)(1 - p - f_2). \]

To see this, we observe that \( ub_0f_2 = 0 \) because \( b_0f_2 = 0 \), that
\[ pub_0 = u(u^*pu)b_0 = ue_2b_0 = 0 \]
because \( e_2b_0 = 0 \), and that
\[
\begin{align*}
d_0 &= (1 - p - f_2)ub_0(1 - p - f_2) = u(1 - e_2 - p)b_0(1 - p - f_2) \\
&= u[1 - e_2][(1 - p)b_0(1 - p)][1 - f_2],
\end{align*}
\]
which is in \( C_1^*(G_0) \) because \( u \) and each of the three terms in brackets are in \( C_1^*(G_0) \). Since \( C_1^*(G_0) \) is an AF algebra, there exists an invertible element \( d \in C_1^*(G_0) \) such that \( \|d - d_0\| < \frac{1}{4} \varepsilon \). Then
\[
\begin{pmatrix}
\frac{1}{4} \varepsilon & 0 & 0 \\
x & d_0 & 0 \\
y & z & \frac{1}{4} \varepsilon
\end{pmatrix}
\]
is an invertible element of \( C_1^*(G) \) such that \( \|b_0 - c\| \leq \frac{1}{4} \varepsilon \), so \( \|a - c\| < \varepsilon \).

Recall that \( M_\infty(A) = \bigcup_{n=1}^\infty M_n(A) \).

**Corollary 5.3.** Let \( G \) be an almost AF Cantor groupoid. Suppose that \( C_1^*(G) \) is simple. Then the projections in \( M_\infty(C_1^*(G)) \) satisfy cancellation: if \( e, f, p \in M_\infty(C_1^*(G)) \) are projections such that \( e \oplus p \) is Murray-von Neumann equivalent to \( f \oplus p \), then \( e \) is Murray-von Neumann equivalent to \( f \).

**Proof:** This follows from the fact that \( C_1^*(G) \) has stable rank one (Theorem 5.2) and real rank zero (Theorem 4.7), using Proposition 6.5.1 of [6].

**Corollary 5.4.** Let \( G \) be an almost AF Cantor groupoid. Suppose that \( C_1^*(G) \) is simple. Let \( p, q \in M_\infty(C_1^*(G)) \) be projections such that \( \tau(p) < \tau(q) \) for all normalized traces \( \tau \) on \( C_1^*(G) \). Then \( p \) is Murray-von Neumann equivalent to a subprojection of \( q \).

**Proof:** This follows from Theorem 5.3 and Corollary 5.3.
6. Kakutani-Rokhlin decompositions

In this section, we show that the transformation group groupoid coming from a free minimal action of $\mathbb{Z}^d$ on the Cantor set is an almost AF Cantor groupoid. The argument is essentially a reinterpretation of some of the results of Forrest’s paper [15] in terms of groupoids. As can be seen by combining the results of this section and Section 2, Forrest does not need all the conditions he imposes to get his main theorem. We use only the Følner condition, not the inradius conditions.

We presume that the construction of [15] can be generalized to cover actions of more general groups. Accordingly, we state the definitions in greater generality, in particular using the word length metric rather than the Euclidean distance on $\mathbb{Z}^d$. This change has no effect on the results.

It should also be possible to generalize to actions that are merely essentially free. This generalization would require more substantial modification of Forrest’s definitions, and we do not carry it out here. Doing so would gain no generality for the groups we actually handle, namely $\mathbb{Z}^d$. An essentially free minimal action of an abelian group is necessarily free, because the fixed points for any one group element form a closed invariant set.

We begin by establishing notation for this section.

**Convention 6.1.** Throughout this section:

1. $\Gamma$ is a countable discrete group with a fixed finite generating set $\Sigma$ which is symmetric in the sense that $\gamma \in \Sigma$ implies $\gamma^{-1} \in \Sigma$.
2. $X$ is the Cantor set, and $\Gamma$ acts freely and continuously on $X$ on the left. The action is denoted $(\gamma, x) \mapsto \gamma x$.
3. $G = \Gamma \times X$ is the transformation group groupoid, and is equipped with the Haar system consisting of counting measures. Thus $G$ is a Cantor groupoid. (See Definition 1.1 and Example 1.3.)

For the convenience of the reader, we reproduce here the relevant definitions from [15], but stated for the more general situation of Convention 6.1. (Forrest considers the case in which $\Gamma = \mathbb{Z}^d$ and $\Sigma$ consists of the standard basis vectors and their inverses.)

**Definition 6.2.** (Definition 2.1 of [15].) Let the notation be as in Convention 6.1.

1. A *tower* is a pair $(E, S)$, in which $E \subset X$ is a compact open subset and $S \subset \Gamma$ is a finite subset, such that $1 \in S$ and the sets $\gamma E$, for $\gamma \in S$, are pairwise disjoint.
2. The *levels* of a tower $(E, S)$ are the sets $\gamma E \subset X$ for $\gamma \in S$.
3. A *traverse* of a tower $(E, S)$ is a set of the form $\{\gamma x: \gamma \in S\}$ with $x \in E$.
4. A *Kakutani-Rokhlin decomposition* $Q$ is a finite collection of towers whose levels form a partition $P_Q$ of $X$ (called the *partition determined by* $Q$).

In the following definition, we use the word length metric (as opposed to the Euclidean norm on $\mathbb{Z}^d$ used in Definition 3.2 of [15]). However, in that case the two metrics are equivalent in the usual sense for norms on Banach spaces, so the difference is not significant.

**Definition 6.3.** Let $l(\gamma)$ (or $l_{\Sigma}(\gamma)$) denote the word length of $\gamma \in \Gamma$ relative to the generating set $\Sigma$. Let $S \subset \Gamma$ be a nonempty finite subset. The *Følner constant* $c(S)$ (or $c_{\Sigma}(S)$) is the least number $c > 0$ such that

$$\text{card}(S \triangle \gamma S) \leq c \cdot l(\gamma) \text{card}(S)$$
for all $\gamma \in \Gamma$. Here $S \Delta \gamma S$ is the symmetric difference
\[
S \Delta \gamma S = (S \setminus \gamma S) \cup (\gamma S \setminus S).
\]
The Følner constant $c(Q)$ of a Kakutani-Rokhlin decomposition $Q$ is
\[
c(Q) = \sup_{(E,S) \in Q} c(S).
\]

**Definition 6.4.** (Definition 2.2 of [15].) Let $Q_1$ and $Q_2$ be Kakutani-Rokhlin decompositions. We say that $Q_2$ refines $Q_1$ if:
1. The partition $P_{Q_2}$ (Definition 6.2(4)) refines the partition $P_{Q_1}$.
2. Every traverse (Definition 6.2(3)) of a tower in $Q_2$ is a union of traverses of towers in $Q_1$.

**Definition 6.5.** Let $(E, S)$ be a tower as in Definition 6.2(1), and let $G = \Gamma \times X$ as in Convention 6.1(3). We define the subset $G((E,S)) \subset G$ by
\[
G((E,S)) = \{ (\gamma_1 \gamma_2^{-1}, \gamma_2 x) : x \in E \text{ and } \gamma_1, \gamma_2 \in S \}.
\]
We show below that it is a subgroupoid of $G$. We equip it with the Haar system consisting of counting measures; we show below that this really is a Haar system.

Let $Q$ be a Kakutani-Rokhlin decomposition as in Definition 6.2(4). We define the subgroupoid associated with $Q$ to be
\[
G_Q = \bigcup_{(E,S) \in Q} G((E,S)) \subset G.
\]
Again, we show below that it is a subgroupoid of $G$, and that we may equip it with the Haar system consisting of counting measures.

**Lemma 6.6.** Assume the hypotheses of Convention 6.1. Then the subset $G((E,S)) \subset G$ of Definition 6.5 is a compact open Cantor subgroupoid of $G$. Further, the subset $G_Q \subset G$ of Definition 6.5 is a compact open Cantor subgroupoid of $G$ which contains the unit space $G^{(0)}$ of $G$, and is the disjoint union of the sets $G((E,S))$ as $(E, S)$ runs through the towers of $Q$.

**Proof:** The subset $G((E,S))$ is a finite union of subsets of $G$ of the form
\[
\{ (\gamma_1 \gamma_2^{-1}, \gamma_2 x) : x \in E \}
\]
for fixed $\gamma_1, \gamma_2 \in \Gamma$. Since $E$ is compact and open in $X$, these sets are compact and open in $G$. Therefore $G((E,S))$ is compact and open in $G$. So $G_Q$, being a finite union of sets of the form $G((E,S))$, is also compact and open in $G$. It is immediate to check that the subset $G((E,S))$ is closed under product and inverse in the groupoid $G$, and so it is a subgroupoid. It is a Cantor groupoid by Example 1.4.

The subset $G_Q$ is the disjoint union of finitely many sets of the form $G((E,S))$, and is therefore also a Cantor groupoid. Its unit space, thought of as a subset of $X$, is
\[
\bigcup_{(E,S) \in Q} G^{(0)}((E,S)) = \bigcup_{(E,S) \in Q} \left( \bigcup_{\gamma \in S} \gamma E \right),
\]
which is the union of all the levels of towers in $Q$, and is therefore equal to $X$ (Definition 6.2(4)). Thus it is equal to $G^{(0)}$. \qed
Lemma 6.7. Assume the hypotheses of Convention 6.1. Let $Q_1$ and $Q_2$ be Kakutani-Rokhlin decompositions such that $Q_2$ refines $Q_1$ (Definition 6.4). Then $G_{Q_1} \subset G_{Q_2}$.

Proof: Let $(E, S)$ be a tower of $Q_1$ and let $g \in G_{(E,S)}$. Then $g = (\gamma_1 \gamma_2^{-1}, \gamma_2 x)$ for some $x \in E$ and $\gamma_1, \gamma_2 \in S$. Since $P_{Q_2}$ is a partition of $X$, there exist a tower $(F, T) \in Q_2$ and elements $\eta_1 \in T$ and $y \in F$ such that $\gamma_2 x = \eta_2 y$.

The traverse $\{\eta y : \eta \in T\}$ (Definition 6.2(3)) of $(F,T)$ is, by Definition 6.4(2), a union of traverses of towers in $Q_1$. That is, there exist towers $(E_1, S_1), \ldots, (E_n, S_n) \in Q_1$ (not necessarily distinct) and points $x_k \in E_k$ for $1 \leq k \leq n$, such that

$$\{\eta y : \eta \in T\} = \bigcup_{k=1}^n \{\gamma x_k : \gamma \in S_k\}.$$

Therefore $\eta_2 y = \gamma x_k$ for some $k$ and some $\gamma \in S_k$. Now $\gamma x_k$ is in the level $\gamma E_k \in P_{Q_1}$. Since $\gamma x_k = \gamma_2 x$, the action is free, and the levels of all the towers of $Q_1$ form a partition of $X$, it follows that

$$(E_k, S_k) = (E, S), \quad x_k = x, \quad \text{and} \quad \gamma = \gamma_2.$$

In particular,

$$\{\gamma x : \gamma \in S\} \subset \{\eta y : \eta \in T\},$$

so there is $\eta_1 \in T$ such that $\gamma_1 x = \eta_1 y$. Substituting $y = \eta_2^{-1} \gamma_2 x$, we get $\gamma_1 x = \eta_1 \eta_2^{-1} \gamma_2 x$. Because the action is free, it follows that $\eta_1 \eta_2^{-1} = \gamma_1 \gamma_2^{-1}$. We now have

$$g = (\gamma_1 \gamma_2^{-1}, \gamma_2 x) = (\eta_1 \eta_2^{-1}, \eta_2 y) \in G_{(F,T)} \subset G_{Q_2}.$$

Theorem 6.8. (Forrest [13,1]) In the situation of Convention 6.1, assume that $\Gamma = \mathbb{Z}^d$, that $\Sigma$ consists of the standard basis vectors and their inverses, and that the action of $\mathbb{Z}^d$ on $X$ is minimal (as well as being free). Then there exists a sequence $Q_1$, $Q_2$, $Q_3$, \ldots of Kakutani-Rokhlin decompositions such that $Q_{n+1}$ refines $Q_n$ for all $n$, and such that the Følner constants obey $\lim_{n \to \infty} c(Q_n) = 0$.

Proof: Combine Proposition 5.1 of [13] with Lemma 3.1 of [13]. This immediately gives everything except for the estimate on the Følner constants. From [13] we get an estimate on constants slightly different from those given in Definition 6.3. Specifically, define $c_0(S)$ and $c_0(Q)$ as in Definition 6.3 but considering $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$ and using $\|\gamma\|_2$ in place of $l(\gamma)$. That is,

$$\text{card}(S \triangle \gamma S) \leq c_0(S) \cdot \|\gamma\|_2 \text{card}(S),$$

etc. That $\lim_{n \to \infty} c_0(Q_n) = 0$ follows from the fact that the atoms of the partitions $T_n(x)$ in Proposition 5.1 of [13], for fixed $n$ and as $x$ runs through $X$, are exactly the translates in $\Gamma$ of the sets $S$ appearing in the towers $(E, S)$ which make up $Q_n$. Now for any $\gamma \in \mathbb{Z}^d$ we have $l(\gamma) = \|\gamma\|_\infty \geq d^{-1} \|\gamma\|_2$. Therefore $c(Q_n) \leq d \cdot c_0(Q_n)$ for all $n$, whence $\lim_{n \to \infty} c(Q_n) = 0$, as desired.\[\square\]
**Theorem 6.9.** Assume the hypotheses of Convention 6.1 and assume moreover that the action of $\Gamma$ on $X$ is minimal (as well as being free). Assume that there exists a sequence

$$Q_1, Q_2, Q_3, \ldots$$

of Kakutani-Rokhlin decompositions such that $Q_{n+1}$ refines $Q_n$ for all $n$, and such that the Følner constants obey $\lim_{n \to \infty} c(Q_n) = 0$. Then $G = \Gamma \times X$ is almost AF in the sense of Definition 2.2 and $C^*_r(G)$ is simple.

**Proof:** The algebra $C^*_r(G)$ is the same as the reduced transformation group C*-algebra $C^*_r(\Gamma, X)$ by Proposition 1.3. So it is simple by the corollary at the end of 1.2. (See the preceding discussion and Definition 1 there.) To prove that $G$ is almost AF, it is by Proposition 2.13 now enough to verify Definition 2.2(1).

With $G_{Q_n}$ as in Definition 6.3, define $G_0 = \bigcup_{n=1}^{\infty} G_{Q_n}$. Using Lemma 6.6 and Lemma 6.7 we easily verify the conditions of Definition 1.15. Thus $G_0$ is an AF Cantor groupoid, which is open in $G$ because each $G_{Q_n}$ is open in $G$.

It remains to verify the condition that $s(K)$ be thin in $G_0$ (Definition 2.1) for every compact set $K \subset G \setminus G_0$. For this argument, we will identify the unit space $G^{(0)} = \{1\} \times X$ with $X$ in the obvious way. So let $K \subset G \setminus G_0$ be compact and let $n \in \mathbb{N}$. Let

$$T = \{\gamma \in \Gamma : ((\gamma) \times X) \cap K \neq \emptyset\} \subset \Gamma.$$ 

Then $T$ is finite because $K$ is compact. Let $l$ be the word length metric on $\Gamma$, as used in Definition 6.3. Let $\rho = \sup_{\gamma \in T} l(\gamma)$. Choose $m$ so large that

$$c(Q_m) < \frac{1}{\rho m \cdot \text{card}(T)}.$$ 

We now claim that if $(E, S) \in Q_m$ is a tower and

$$x \in s(K) \cap \bigcup_{\eta \in S} \eta E,$$

then there exist $\gamma \in T$ and $\eta \in S \setminus \gamma^{-1}S$ such that $(\gamma, x) \in \{\gamma\} \times \eta E$. To prove this, recall that $s(\gamma, x) = x$ (really $(1, x)$), and find $\gamma \in T$ such that $(\gamma, x) \in K$. Also choose $\eta \in S$ such that $\eta^{-1}x \in E$. Then write $(\gamma, x) = (|\eta\eta^{-1}|, \eta|\eta^{-1}x|)$. Since $(\gamma, x) \notin G_{(E,S)}$, we have $\gamma \eta \notin S$. So $\eta \in S \setminus \gamma^{-1}S$. This proves the claim.

It follows that the sets $\eta E$, for $(E, S) \in Q_m$, $\gamma \in T$, and $\eta \in S \setminus \gamma^{-1}S$, form a cover of $s(K)$ by finitely many disjoint compact open subsets of $X$. For

$$(E, S) \in Q_m \text{ and } \eta \in \bigcup_{\gamma \in T}(S \setminus \gamma^{-1}S),$$

define $L_{(E,S),\eta} = s(K) \cap \eta E$. The sets $L_{(E,S),\eta}$ are a cover of $s(K)$ by finitely many disjoint compact subsets of $X$.

For fixed $(E, S) \in Q_m$, we now claim that there are compact $G_{(E,S)}$-sets

$$M_{(E,S),1}, M_{(E,S),2}, \ldots, M_{(E,S),n} \subset G_{(E,S)}$$

such that

$$s(M_{(E,S),k}) = s(K) \cap \bigcup_{\eta \in S} \eta E.$$
and the sets \( r(M_{(E,S),1}) \), \( r(M_{(E,S),2}) \), \ldots, \( r(M_{(E,S),n}) \) are pairwise disjoint. Since the subgroupoids \( G_{(E,S)} \) are disjoint (Lemma 6.6) and their union is contained in \( G_0 \), we can take then the required \( G_0 \)-sets to be

\[
M_k = \bigcup_{(E,S) \in Q_m} M_{(E,S),k}.
\]

To find the \( M_{(E,S),k} \), first note that for \( \gamma \in T \), we have \( l(\gamma^{-1}) = l(\gamma) \leq \rho \), so, using Definition 6.3,

\[
\text{card}(S \setminus \gamma^{-1}S) \leq \text{card}(S \triangle \gamma^{-1}S) \leq c(Q_m)\rho \cdot \text{card}(S) \leq \frac{\text{card}(S)}{n \cdot \text{card}(T)}.
\]

Therefore, with

\[
R_{(E,S)} = \bigcup_{\gamma \in T} (S \setminus \gamma^{-1}S),
\]

we have \( \text{card}(R_{(E,S)}) < \frac{1}{n} \text{card}(S) \). So there exist \( n \) injective functions

\[
\sigma_1, \sigma_2, \ldots, \sigma_n : R_{(E,S)} \to S
\]

with disjoint ranges. Define

\[
M_{(E,S),k} = \bigcup_{\eta \in R_{(E,S)}} \left( \{ \sigma_k(\eta)\eta^{-1} \} \times L_{(E,S),\eta} \right),
\]

Then \( M_{(E,S),k} \subset G_{(E,S)} \) because \( L_{(E,S),\eta} \subset \eta E \). Clearly \( M_{(E,S),k} \) is compact. The restriction of the source map to \( M_{(E,S),k} \) is injective because the \( L_{(E,S),\eta} \) are disjoint. Further, \( \sigma_k(\eta)\eta^{-1}L_{(E,S),\eta} \subset \sigma_k(\eta)E \), so the sets \( \sigma_k(\eta)\eta^{-1}L_{(E,S),\eta} \), for \( \eta \in R_{(E,S)} \) and \( 1 \leq k \leq n \), are pairwise disjoint. It follows both that the restriction of the range map to \( M_{(E,S),k} \) is injective and that the sets

\[
r(M_{(E,S),1}), r(M_{(E,S),2}), \ldots, r(M_{(E,S),n})
\]

are pairwise disjoint. Thus the \( M_{(E,S),k} \) are \( G_{(E,S)} \)-sets with the required properties. We have verified Definition 5.3 for \( s(K) \).

**Corollary 6.10.** Let \( d \) be a positive integer, let \( X \) be the Cantor set, and let \( \mathbb{Z}^d \) act freely and minimally on \( X \). Then the transformation group groupoid \( G = \mathbb{Z}^d \times X \) is almost AF in the sense of Definition 2.2 and \( C^*_\tau(G) \) is simple.

**Proof:** This is immediate from Theorems 5.8 and 6.3.

**Theorem 6.11.** Let \( d \) be a positive integer, let \( X \) be the Cantor set, and let \( \mathbb{Z}^d \) act freely and minimally on \( X \). Then:

1. \( C^*(\mathbb{Z}^d, X) \) has (topological) stable rank one in the sense of 37.
2. \( C^*(\mathbb{Z}^d, X) \) has real rank zero in the sense of 10.
3. Let \( p, q \in M_\infty(C^*(\mathbb{Z}^d, X)) \) be projections such that \( \tau(p) < \tau(q) \) for all normalized traces \( \tau \) on \( C^*(\mathbb{Z}^d, X) \). Then \( p \) is Murray-von Neumann equivalent to a subprojection of \( q \).

**Proof:** Using Proposition 4.3 and Corollary 6.10, we obtain Part (1) from Theorem 5.4. Part (2) from Theorem 4.7 and Part (3) from Corollary 5.4.
7. Tilings and Quasicrystals

In this section, we show that the $C^*$-algebras associated with three different broad classes of aperiodic tilings have real rank zero and stable rank one, and satisfy Blackadar’s Second Fundamental Comparability Question. In particular, we strengthen the conclusion of the main result of [35]. Then we discuss the relationship with the Bethe-Sommerfeld Conjecture for quasicrystals.

We begin by showing that the groupoids of [35] are almost AF. The proof consists of assembling, in the right order, various results proved in [35].

**Theorem 7.1.** Consider a substitution tiling system in $\mathbb{R}^d$ as in Section 1 of [35], under the conditions imposed there:

- The substitution is primitive.
- The finite pattern condition holds.
- The system is aperiodic.
- The substitution forces its border.
- The capacity of the boundary of every prototile is strictly less than $d$.

Then the groupoid $\mathcal{R}_{\text{punc}}$ defined in Section 1 of [35] is an almost AF Cantor groupoid.

**Proof:** That the groupoid $\mathcal{R}_{\text{punc}}$ is a Cantor groupoid in the sense of Definition 1.1 follows from the construction as described on pages 594–595 of [35] and on page 187 of [24]. Note that the base for the topology described in [35] is countable, so that $\mathcal{R}_{\text{punc}}$ really is second countable.

We take the open AF subgroupoid required in Definition 2.2 to be the open subgroupoid $\mathcal{R}_{\text{AF}}$ from page 596 of [35]. It is easily seen from the construction there to be an AF Cantor groupoid which contains the unit space of $\mathcal{R}_{\text{punc}}$. (Also see pages 198–200 of [24].)

We must show that if $K \subset \mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{\text{AF}}$ is compact, then $s(K)$ is thin in the sense of Definition 2.1. Referring to the definition of $\mathcal{R}_{\text{punc}}$ and its topology (pages 594–595 of [35]), we see that $\mathcal{R}_{\text{punc}}$ consists of certain pairs $(T, T + x)$ in which $T$ is a tiling of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, and that there is a base for its topology consisting of sets of the form $\{(T, T + x) : T \in U\}$ for suitable $x \in \mathbb{R}^d$ and suitable sets $U$ of tilings. It is immediate from this that for any $r > 0$ the set

$$\{(T, T - x) \in \mathcal{R}_{\text{punc}} : \|x\| < r\}$$

is open in $\mathcal{R}_{\text{punc}}$. As $r$ varies, these sets cover $\mathcal{R}_{\text{punc}}$, and $K^{-1}$ is also a compact subset of $\mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{\text{AF}}$, so there is $r > 0$ such that $K^{-1}$ is contained in the set

$$L = \{(T, T - x) \in \mathcal{R}_{\text{punc}} \setminus \mathcal{R}_{\text{AF}} : \|x\| \leq r\}.$$ 

Since $s(K) \subset r(L)$, it suffices to show that $r(L)$ is thin.

The proof of Theorem 2.1 of [35], at the end of Section 2.1 there, consists of showing that there are compact open sets $U_n \subset \Omega_{\text{punc}}$, the unit space of $\mathcal{R}_{\text{punc}}$, for $n \in \mathbb{N}$, each containing $r(L)$, and homeomorphisms $\gamma_n$ from $U_n$ to disjoint subsets of $\Omega_{\text{punc}}$, each having graph contained in $\mathcal{R}_{\text{AF}}$. But it is immediate from this that $r(L)$ is thin. This completes the verification of Part (1) of Definition 2.2.

From the description of $C^*(\mathcal{R}_{\text{AF}})$ and its K-theory at the end of Section 1 of [35], we see that this algebra is a direct limit of a system of finite dimensional $C^*$-algebras in which the matrix $B$ of partial embedding multiplicities is the same at each stage. Moreover, there is $n$ such that $B^n$ has no zero entries. Therefore the direct limit
algebra is simple, whence $C^*_r(R_{AF})$ is simple. It now follows from Proposition 2.13 that $R_{punct}$ is almost AF.

The following corollary contains Theorem 1.1 of [35]. Using more recent results (see Theorem 5.1 of [21]), it is now also possible to obtain this result from Theorem 5.11 in the same way the that next two theorems are proved.

**Corollary 7.2.** For a substitution tiling system in $\mathbb{R}^d$ as in Theorem 7.1, the $C^*$-algebras of the associated groupoid $R_{punct}$ as in [35] has stable rank one and real rank zero, and satisfies Blackadar’s Second Fundamental Comparability Question (§1, 1.3.1).

**Proof:** Using Theorem 7.1, we obtain stable rank one from Theorem 5.2, real rank zero from Theorem 4.7, and Blackadar’s Second Fundamental Comparability Question from Corollary 5.4.

We can also obtain the same result for several other kinds of aperiodic tilings. As discussed on page 198 of [24], we reduce to the case of crossed products by free minimal actions of $\mathbb{Z}^d$ on the Cantor set.

**Theorem 7.3.** Consider a projection method pattern $T$ as in [17], with data $(E, K, u)$ (see Definitions I.2.1 and I.4.4 of [17]) such that $E \cap \mathbb{Z}^d = \{0\}$. Let $\mathcal{G}T$ be the associated groupoid (Definition II.2.7 of [17]). Then $C^*(\mathcal{G}T)$ has stable rank one and real rank zero, and satisfies Blackadar’s Second Fundamental Comparability Question.

**Proof:** It follows from Theorem II.2.9 and Corollary I.10.10 of [17] that $C^*(\mathcal{G}T)$ is strongly Morita equivalent to the transformation group $C^*$-algebra of a minimal action of $\mathbb{Z}^d$ on a Cantor set $X_T$, for a suitable $d$. The action is free since the action of $\mathbb{R}^d$ in the pattern dynamical system in that corollary is free by construction.

The required properties hold for $C^*(\mathbb{Z}^d, X_T)$ by Theorem 6.11. Since $C^*(\mathcal{G}T)$ is strongly Morita equivalent to $C^*(\mathbb{Z}^d, X_T)$, and both algebras are separable, these two algebras are stably isomorphic by Theorem 1.2 of [17]. So $C^*(\mathcal{G}T)$ has stable rank one by Theorem 3.6 of [17] and has real rank zero by Theorem 3.8 of [10]. It is clear that Blackadar’s Second Fundamental Comparability Question is preserved by stable isomorphism.

Theorem II.2.9 of [17] shows that several other groupoids associated to $T$ give $C^*$-algebras strongly Morita equivalent to $C^*(\mathcal{G}T)$. These $C^*$-algebras therefore also have stable rank one and real rank zero, and satisfy Blackadar’s Second Fundamental Comparability Question.

**Theorem 7.4.** Consider a minimal aperiodic generalized dual (or grid) method tiling [41], satisfying the condition (D2) before Lemma 10 of [23]. Then the $C^*$-algebra associated with the tiling has stable rank one and real rank zero, and satisfies Blackadar’s Second Fundamental Comparability Question.

**Proof:** Theorem 1 and Lemma 11 of [23] show that the $C^*$-algebra associated with such a tiling is stably isomorphic to the crossed product $C^*$-algebra for an action of $\mathbb{Z}^d$ on the Cantor set. (See Definition 12 of [24] and the following discussion, the discussion following Theorem 1 of [23], and also Definition 13 of [23].) The action is minimal (see the discussion following Definition 12 of [23]) and free (see Definition 8 and Lemma 3 of [23]). Therefore the crossed product has stable
rank one and real rank zero, and satisfies Blackadar’s Second Fundamental Comparability Question, by Theorem 6.11. It follows as in the proof of Theorem 7.3 that the C*-algebra of the tiling has these properties as well.

We now turn to the Bethe-Sommerfeld conjecture for quasicrystals. Consider the Schrödinger operator $H$ for an electron moving in a solid in $\mathbb{R}^d$, which at this point may be crystalline, amorphous, or quasicrystalline. (The physical case is, of course, $d = 3$.) Associated with this situation there is a C*-algebra which contains all bounded continuous functions of $H$ and all its translates by elements of $\mathbb{R}^d$. See [4]. For crystals, in which the locations of the atomic nuclei form a lattice in $\mathbb{R}^d$, the Bethe-Sommerfeld conjecture asserts that if $d \geq 2$ then there is an energy above which the spectrum of $H$ has no gaps. See [13], [40], Corollary 2.3 of [20], and [22] for some of the mathematical results on this conjecture. According to the introduction to [4], it is expected that this property also holds for quasicrystals. We point out that quasicrystals actually occur in nature; see [39] for a survey. For the Schrödinger operator in the so-called tight binding representation for a quasicrystal whose atomic sites are given by an aperiodic tiling, the relevant C*-algebra is the C*-algebra of the tiling. See Section 4 of [4].

We have just seen that the C*-algebras associated with three different broad classes of tilings have real rank zero, that is, every selfadjoint element, including $H$, can be approximated arbitrarily closely in norm by selfadjoint elements with finite spectrum. In Proposition 7.5 below, we further show that among the selfadjoint elements of such an algebra, those with totally disconnected spectrum are generic, in the sense that they form a dense $G_δ$-set. That is, the selfadjoint elements of the relevant C*-algebra whose spectrum is not totally disconnected form a meager set in the sense of Baire category.

These results say nothing about the spectrum of $H$ itself. It is also not clear whether arbitrary norm small selfadjoint perturbations are physically relevant. Nevertheless, gap labelling theory (see [4] for a recent survey) exists in arbitrary dimensions, and the introduction of [4] raises the question of its physical interpretation when there are no gaps. Our results do suggest the possibility that this theory has physical significance in the general situation.

**Proposition 7.5.** Let $A$ be a C*-algebra with real rank zero. Then there is a dense $G_δ$-set $S$ in the selfadjoint part $A_{sa}$ of $A$ such that every element of $S$ has totally disconnected spectrum.

**Proof:** For a finite subset $F \subset \mathbb{R}$, let

$$S_F = \{a \in A_{sa} : \text{sp}(a) \cap F = \emptyset\}.$$ 

Then $S_F$ is open $A_{sa}$, since if $a, b \in A_{sa}$ satisfy $\|a-b\| < \varepsilon$, then $\text{sp}(b)$ is contained in the $\varepsilon$-neighborhood of $\text{sp}(a)$. We show that $S_F$ is dense in $A_{sa}$. Given any selfadjoint element $a \in A$ and $\varepsilon > 0$, the real rank zero condition provides a selfadjoint element $b \in A$ with finite spectrum and $\|a-b\| < \frac{1}{2}\varepsilon$. By perturbing eigenvalues, it is easy to find a selfadjoint element $c \in A$ with $\|b-c\| < \frac{1}{2}\varepsilon$ and such that $\text{sp}(c) \cap F = \emptyset$. Then $c \in S_F$ and $\|a-c\| < \varepsilon$. So $S_F$ is dense.

The set of selfadjoint elements $a \in A$ with $\text{sp}(a) \cap \mathbb{Q} = \emptyset$ is a countable intersection of sets of the form $S_F$, and is therefore a dense $G_δ$-set. Clearly all its elements have totally disconnected spectrum. □
8. Examples and open problems

In this section, we discuss some open problems, beginning with the various possibilities for improving our results. One of the most obvious questions is the following:

**Question 8.1.** Let $G$ be an almost AF Cantor groupoid. Does it follow that $C^*_r(G)$ is tracially AF in the sense of Definition 2.1 of \[26\]?

Lemma 2.7 and the thinness condition in Definition 2.2 come close to giving the tracially AF property; the main part that is missing is the requirement that the projections one chooses approximately commute with the elements of the finite set.

By Theorem 3.1 of \[16\], the K-theory of the crossed product of the Cantor set by any action of $\mathbb{Z}^d$ is torsion free. Therefore, by recent work of H. Lin \[27\], a positive answer to this question would imply a positive solution to the following conjecture:

**Conjecture 8.2.** Let $d$ be a positive integer, let $X$ be the Cantor set, and let $\mathbb{Z}^d$ act freely and minimally on $X$. Then $C^*(\mathbb{Z}^d, X)$ is an AT algebra, that is, isomorphic to a direct limit $\lim_{\rightarrow} A_n$ of C*-algebras $A_n$ each of which is a finite direct sum of C*-algebras of the form $M_k$ or $C(S^1, M_k)$. (The matrix sizes $k$ may vary among the summands, even for the same value of $n$.)

This conjecture is also predicted by the Elliott classification conjecture. It is true for $d = 1$ (essentially \[33\]). We point out here that Matui has shown in \[28\] (see Proposition 4.1 and Theorem 4.8) that when $d = 2$, the algebra $C^*(\mathbb{Z}^d, X)$ is often AF embeddable.

We next address the possibility of generalizing the group. For not necessarily abelian groups, we should presumably require that the action be minimal and merely essentially free, which for minimal actions means that the set of $x \in X$ with trivial isotropy subgroup is dense in $X$. This is the condition needed to ensure simplicity of the transformation group C*-algebra. See the final corollary of \[2\], noting that all actions of amenable groups are regular (as described before this corollary) and that essential freeness of an action of a countable discrete group on a compact metric space $X$ is equivalent to topological freeness, Definition 1 of \[2\], of the action on $C(X)$ (as is easy to prove). Thus, let $\Gamma$ act minimally and essentially freely on the Cantor set $X$. If $\Gamma$ is close to $\mathbb{Z}^d$, and especially if the action is actually free, then there seems to be a good chance of adapting the methods of Forrest \[15\] to show that the transformation group groupoid is still almost AF.

This might work, for example, if $\Gamma$ has a finite index subgroup isomorphic to $\mathbb{Z}^d$. In this context, we mention that crossed products of free minimal actions of the free product $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ on the Cantor set, satisfying an additional technical condition, are known to be AT, in fact AF \[8\]. The group $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ has an index two subgroup isomorphic to $\mathbb{Z}$. However, we believe the correct generality is to allow $\Gamma$ to be an arbitrary countable amenable group.

**Question 8.3.** Let $X$ be the Cantor set, and let the countable amenable group $\Gamma$ act minimally and essentially freely on $X$. Is the transformation group groupoid $\Gamma \times X$ almost AF?

Even if not, do the conclusions of Theorem 6.11 still hold? That is:

**Question 8.4.** Let $X$ be the Cantor set, and let the countable amenable group $\Gamma$ act minimally and essentially freely on $X$. Does it follow that $C^*(\Gamma, X)$ has stable rank one and real rank zero? If $p, q \in M_\infty(C^*(\mathbb{Z}^d, X))$ are projections such
that \( \tau(p) < \tau(q) \) for all normalized traces \( \tau \) on \( C^*(\mathbb{Z}^d, X) \), does it follow that \( p \) is Murray-von Neumann equivalent to a subprojection of \( q \)?

The other obvious change is to relax the condition on the space. If \( X \) is not totally disconnected, then even a crossed product by a free minimal action of \( \mathbb{Z} \) need not have real rank zero. Examples 4 and 5 in Section 5 of [11] have no nontrivial projections. However, it seems reasonable to hope that the other parts of the conclusion of Theorem 6.11 still hold, at least under mild restrictions.

**Question 8.5.** Let \( X \) be a compact metric space with finite covering dimension [14], and let the countable amenable group \( \Gamma \) act minimally and essentially freely on \( X \). Does it follow that \( C^*(\Gamma, X) \) has stable rank one? If \( p, q \in M_\infty(\mathbb{Z}^d, X) \) are projections such that \( \tau(p) < \tau(q) \) for all normalized traces \( \tau \) on \( C^*(\mathbb{Z}^d, X) \), does it follow that \( p \) is Murray-von Neumann equivalent to a subprojection of \( q \)?

We think we know how to prove stable rank one when \( \Gamma = \mathbb{Z}^d \), but a number of technical details need to be worked out.

The definition given for an almost AF Cantor groupoid whose reduced \( C^* \)-algebra is not simple is merely a guess, and it is also not clear what properties the reduced \( C^* \)-algebra of such a groupoid should have.

**Question 8.6.** What is the “right” definition of an almost AF Cantor groupoid in the case that the reduced \( C^* \)-algebra is not simple?

**Question 8.7.** Let \( G \) be an almost AF Cantor groupoid such that \( C^*_r(G) \) is not simple. What structural consequences does this have for \( C^*_r(G) \)?

The definition should surely exclude the transformation group groupoid coming from the action of \( \mathbb{Z} \) on \( \mathbb{Z} \cup \{\pm \infty\} \) by translation, even though this action is free on a dense set. It is not essentially free, which is equivalent to the transformation group groupoid \( G \) not being essentially principal (Definition [13] 4).

Suppose, however, that \( h_1, h_2 : X \to X \) are two commuting homeomorphisms of the Cantor set \( X \), such that the map \( (n_1, n_2) \mapsto h_1^{n_1} \circ h_2^{n_2} \) determines a free minimal action of \( \mathbb{Z}^2 \). Then the transformation group groupoid for the action of \( \mathbb{Z} \) on \( X \) generated by \( h_1 \) should be almost AF. It is possible for this action to be already minimal, or for \( X \) to be the disjoint union of finitely many minimal sets, in which case the situation is clear. It is possible that neither of these happens—consider the product of two minimal actions. It is not clear what additional structure the homeomorphism \( h_1 \) must have (although it can’t be completely arbitrary—see below), and it is also not clear whether the crossed product must necessarily have real rank zero, stable rank one, or order on its \( K_0 \)-group determined by traces.

Let \( h \) be an arbitrary aperiodic homeomorphism of the Cantor set \( X \). (That is, the action of \( \mathbb{Z} \) it generates is free, but need not be minimal.) Theorem 3.1 of [33] shows that if \( h \) has no nontrivial invariant subsets which are both closed and open, and if \( h \) has more than one minimal set, then \( C^*(\mathbb{Z}, X, h) \) does not have stable rank one. If \( h_1 \) and \( h_2 \) are as above, and \( h = h_1 \), then the existence of a unique minimal set \( K \) for \( h \) implies that \( h \) is minimal. (The set \( K \) is necessarily invariant under \( h_2 \).) On the other hand, we know of no examples of \( h_1 \) and \( h_2 \) as above in which \( h_1 \) is not minimal yet has no nontrivial invariant subsets which are both closed and open.
It follows from [30] that if \( h \) is an arbitrary aperiodic homeomorphism of the Cantor set \( X \), then \( C^*(Z, X, h) \) has the ideal property, that is, every ideal is generated as an ideal in the algebra by its projections. This certainly suggests that the reduced C*-algebra of an almost AF Cantor groupoid should have the ideal property. This condition is, however, rather weak in this context, since every simple unital C*-algebra, regardless of its real or stable rank, has the ideal property.

We now give an example of an aperiodic homeomorphism of the Cantor set whose transformation group groupoid is not almost AF, but only because of the failure of condition (2) in Definition 2.2. Its C*-algebra does not have stable rank one.

**Example 8.8.** Let \( X_1 = Z \cup \{ \pm \infty \} \) be the two point compactification of \( Z \), and let \( h_1 : X_1 \to X_1 \) be translation to the right (fixing \( \pm \infty \)). Let \( X_2 \) be the Cantor set, and let \( h_2 : X_2 \to X_2 \) be any minimal homeomorphism. Let \( X = X_1 \times X_2 \), and let \( h = h_1 \times h_2 : X \to X \). Let \( G = Z \times X \) be the transformation group groupoid, as in Example 1.3, for the corresponding action of \( Z \). Then the \( G \) has the following properties:

- It is a Cantor groupoid.
- It satisfies condition (1) of Definition 2.2, the main part of the definition of an almost AF Cantor groupoid.
- There is a nonempty open subset \( U \subset G^{(0)} \) which is null for all \( G \)-invariant Borel probability measures.
- The reduced C*-algebra \( C^*_r(G) \) does not have stable rank one.

For the first statement, observe that \( X \) is a totally disconnected compact metric space with no isolated points, and hence homeomorphic to the Cantor set. So \( G \) is a Cantor groupoid, as in Example 1.3.

The third statement is also easy to prove, with \( U = Z \times X_2 \). The \( G \)-invariant Borel probability measures on \( G^{(0)} \) are exactly the \( h \)-invariant Borel probability measures on \( X \). (We have not found this statement in the literature. However, our case follows immediately from Remark 1.3(2) and the formulas in Example 1.3 by considering functions supported on sets of the form \( \{ \gamma \} \times X \) and letting \( \gamma \) run through the group.) So let \( \mu \) be any \( h \)-invariant Borel probability measure on \( X \). Then \( \mu(\{n\} \times X_2) \) is independent of \( n \), and

\[
\sum_{n \in Z} \mu(\{n\} \times X_2) = \mu(Z \times X_2) \leq \mu(X) = 1.
\]

It follows that \( \mu(\{n\} \times X_2) = 0 \) for all \( n \in Z \), whence \( \mu(Z \times X_2) = 0 \).

We verify condition (1) of Definition 2.2. Fix any \( z \in X_2 \). Set \( Y = X_1 \times \{z\} \subset X \). Let \( G_0 \) be the subgroupoid \( G' \) as in Example 2.6 of [34] corresponding to this situation. (Also see the statement of Theorem 2.4 of [34].) Note that, in our notation, \( G_0 = G \setminus (L \cup L^{-1}) \), with

\[
L = \{(k, h^l(y)) : y \in Y, k, l \in Z, l > 0, \text{ and } k + l < 0 \} \subset G.
\]

To see that \( G_0 \) is AF (compare with [32]), we choose a decreasing sequence of compact open subsets \( Z_n \subset X_2 \) with \( \bigcap_{n=1}^{\infty} Z_n = \{z\} \). Set \( Y_n = X_1 \times Z_n \), define \( L_n \subset G \) by using \( Y_n \) in place of \( Y \) in the definition of \( L \), and set \( H_n = G \setminus (L_n \cup L_n^{-1}) \).

It is easy to check that \( H_n \) is a closed and open subgroupoid of \( G \) which is contained in \( G_0 \), and that \( G_0 = \bigcup_{n=1}^{\infty} H_n \). Since \( h_2 \) is minimal, there is \( l(n) \) such that the sets \( h_2(Z_n), h_2^2(Z_n), \ldots, h_2^{l(n)}(Z_n) \) cover \( X_2 \), and it follows that

\[
H_n \subset \{-n(l), -n(l) + 1, \ldots, n(l)\} \times X.
\]
So $H_n$ is compact. We have now shown that $G_0$ is in fact an AF Cantor groupoid.

We now show that if $K \subset G \setminus G_0$ is compact, then $s(K)$ is thin in $G_0$. We will identify the unit space $G^{(0)} = \{1\} \times X$ with $X$ in the obvious way. Compact subsets of thin sets are easily seen to be thin, so it suffices to consider sets of the form $K = (G \setminus G_0) \cap T$ with

$$
T = \{-m, -m + 1, \ldots, m - 1, m\} \times X
$$

for some $m \in \mathbb{N}$. Now

$$(G \setminus G_0) \cap T = (L \cup L^{-1}) \cap T,$$

so, following Example 3.3,

$$s(K) = s(L \cap T) \cup r(L \cap T).$$

Next, we calculate

$$L \cap T = \{(k, h'(y)): y \in Y, k, l \in \mathbb{Z}, l > 0, k + l \leq 0, \text{and } -m \leq k \leq m\}$$

$$= \{(-k, h'(y)): y \in Y, k, l \in \mathbb{Z}, \text{and } 1 \leq l \leq k \leq m\}.$$

It follows that

$$s(L \cap T) = h(Y) \cup h^2(Y) \cup \cdots \cup h^m(Y)$$

and

$$r(L \cap T) = \bigcup_{1 \leq i \leq k \leq m} h^{-k+i}(Y) = Y \cup h^{-1}(Y) \cup \cdots \cup h^{-m+1}(Y).$$

Now let $n \in \mathbb{N}$. We find compact $G$-sets $S_1, S_2, \ldots, S_n \subset G$ such that $s(S_k) = s(K)$ and the sets $r(S_1), r(S_2), \ldots, r(S_n)$ are pairwise disjoint. For $1 \leq j \leq n$, define

$$S_j = \{m(j - 1)\} \times [h(Y) \cup \cdots \cup h^m(Y)] \cup \{-m(j - 1)\} \times [Y \cup \cdots \cup h^{-m+1}(Y)]$$

$$\subset \mathbb{Z} \times X.$$

Then the $S_j$ are compact $\mathbb{Z} \times X$-sets, $s(S_j) = s(K)$, and

$$r(S_j) = \left[ h^{m(j-1)+1}(Y) \cup h^{m(j-1)+2}(Y) \cup \cdots \cup h^{mj}(Y) \right]$$

$$\cup \left[ h^{-m(j-1)}(Y) \cup h^{-[m(j-1)+1]}(Y) \cup \cdots \cup h^{-(mj-1)}(Y) \right]$$

$$= X_1 \times \left[ \{h_2^{m(j-1)+1}(z), \ldots, h_2^{mj}(z)\} \cup \{h_2^{-m(j-1)}(z), \ldots, h_2^{-(mj-1)}(z)\} \right].$$

Since the homeomorphism $h_2$ has no periodic points, these sets are pairwise disjoint. This completes the proof that if $K \subset G \setminus G_0$ is compact, then $s(K)$ is thin in $G_0$, and hence also the proof of condition (1) of Definition 2.2.

The reduced $C^\ast$-algebra $C^\ast(X \times X)$ is isomorphic to the transformation group $C^\ast$-algebra $C^\ast(X, X, h) = C^\ast(X, X, h)$, by Proposition 1.3. We can use Theorem 3.1 of 3.3 to show that $C^\ast(X, X, h)$ does not have stable rank one, but it is just as easy to do this directly. Let $u \in C^\ast(X, X, h)$ be the canonical unitary, satisfying $ufu^* = f \circ h^{-1}$ for $f \in C(X)$. Set

$$R = \{0, 1, \ldots, \infty\} \times X_2 \subset X \quad \text{and} \quad p = \chi_R \in C(X) \subset C^\ast(X, X, h).$$

Then one checks that $s = up + (1 - p)$ satisfies

$$s^*s = 1 \quad \text{and} \quad ss^* = 1 - \chi_{\{0\} \times X_2} \neq 1.$$
So $s$ is a nonunitary isometry in $C^*\langle \mathbb{Z}, X, h \rangle$. It follows that $s$ is not a norm limit of invertible elements of $C^*\langle \mathbb{Z}, X, h \rangle$, so this algebra does not have stable rank one. (Compare with Examples 4.13 of [37].)
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