Multidimensional Manhattan Preferences

Jiehua Chen    Martin Nöllenburg    Sofia Simola    Anaïs Villedieu
Markus Wallinger
TU Wien, Austria
jiehua.chen@tuwien.ac.at,
{noellenburg,ssimola,avilledieu,mwallinger}@ac.tuwien.ac.at

Abstract

A preference profile with $m$ alternatives and $n$ voters is $d$-Manhattan (resp. $d$-Euclidean) if both the alternatives and the voters can be placed into the $d$-dimensional space such that between each pair of alternatives, every voter prefers the one which has a shorter Manhattan (resp. Euclidean) distance to the voter. Following Bogomolnaia and Laslier [Journal of Mathematical Economics, 2007] and Chen and Grottke [Social Choice and Welfare, 2021] who look at $d$-Euclidean preference profiles, we study which preference profiles are $d$-Manhattan depending on the values $m$ and $n$.

First, we show that each preference profile with $m$ alternatives and $n$ voters is $d$-Manhattan whenever $d \geq \min(n, m-1)$. Second, for $d = 2$, we show that the smallest non $d$-Manhattan preference profile has either three voters and six alternatives, or four voters and five alternatives, or five voters and four alternatives. This is more complex than the case with $d$-Euclidean preferences (see [Bogomolnaia and Laslier, 2007] and [Bulteau and Chen, 2020]).

1 Introduction

Modelling voters’ linear preferences as geometric distances is an approach popular in many research fields such as economics [15, 10, 12], political and social sciences [21, 19, 14, 2], and psychology [8, 3]. The idea is to consider the alternatives and voters as points in the $d$-dimensional space such that

for each two alternatives, each voter prefers the one which is closer to her.\(^1\) (CLOSE)

If the proximity is measured via the Euclidean distance, preference profiles obeying (CLOSE) are called $d$-Euclidean. While the $d$-Euclidean model seems to be canonical, in real life, however, the shortest path between two points may be Manhattan rather than Euclidean. For instance, in urban geography, the alternatives (e.g., a shop or a supermarket) and the voters (e.g., individuals) are often located on grid-like streets. That is, the distance between an alternative and a voter is more likely to be measured according to the Manhattan distance (aka. Taxicab distance), i.e., the sum of the the absolute differences of the coordinates of the facility and the individual. Similarly to the Euclidean preference notion, we call a preference profile $d$-Manhattan if there exists an embedding for the voters and the alternatives which satisfies the condition (CLOSE) under the Manhattan distance. Besides the previously mentioned literature,

\(^1\)Throughout the paper, we use ”she” to refer to a voter.
Euclidean and Manhattan preferences have been studied for a wide range of applications such as facility location [17], group decision making [20], and voting and committee elections [11]. Due to their practical relevance, Bogomolnaia and Laslier [2] studied how restrictive the assumption of Euclidean preferences is. They showed that an arbitrary preference profile with \( n \) voters and \( m \) alternatives is \( d \)-Euclidean if and only if \( d \geq \min(n, m - 1) \). For \( d = 1 \), their smallest counter-example of a non-1-Euclidean profile consists of either 3 voters and 3 alternatives or 2 voters and 4 alternatives, which is tight according to [6]. For \( d = 2 \), their smallest counter-example of a non-2-Euclidean profile consists of either 4 voters and 4 alternatives or 3 voters and 8 alternatives, which is also tight by Bulteau and Chen [5]. To the best of our knowledge, there is no such kind of characterization result on the \( d \)-Manhattan preferences.

In this paper, we aim to close this gap and study how restrictive the assumption that a preference profile is \( d \)-Manhattan is. First, we prove that, similarly to the Euclidean preferences case, a preference profile with \( m \) alternatives and \( n \) voters is \( d \)-Manhattan if \( d \geq \min(m - 1, n) \) (Theorem 1 and Theorem 2). Then, focusing on the two-dimensional case, we investigate how restricted two-dimensional \( d \)-Manhattan preferences are. More precisely, we seek to determine tight bounds on the smallest number of either alternatives or voters of a non-\( d \)-Manhattan profile. We find that the result is not comparable with the one for the \( d \)-Euclidean case. More precisely, we show that an arbitrary preference profile is 2-Manhattan if and only if it either has at most three alternatives (Theorem 2 and Theorem 5), or at most two voters (Theorem 1), or at most three voters and at most five alternatives (Theorem 3 and Proposition 1), or at most four voters and at most four alternatives (Theorem 4 and Proposition 1).

The paper is organized as follows. Section 2 introduces necessary definitions and notations. In Section 3, we examine positive findings. In Section 4, we examine the negative findings. In Section 5 we present our experimental results. We conclude with a few future research directions in Section 6.

## 2 Preliminaries

Given a non-negative integer \( t \), we use \([t]\) to denote the set \( \{1, 2, \ldots, t\} \). Let \( \mathbf{x} \) denote a vector of length \( d \) or a point in a \( d \)-dimensional space, and let \( i \) denote an index \( i \in [d] \). We use \( \mathbf{x}[i] \) to refer to the \( i \)th value in \( \mathbf{x} \). Given three values \( x, y, z \) we say that \( y \) is between \( x \) and \( z \) if either \( x \leq y \leq z \) or \( z \leq y \leq x \) holds.

Let \( A := \{1, \ldots, m\} \) be a set of alternatives. A preference order \( \succ \) of \( A \) is a linear order of \( A \); a linear order is a binary relation which is total, irreflexive, and transitive. For two distinct alternatives \( a \) and \( b \), the relation \( a \succ b \) means that \( a \) is preferred to \( b \). An alternative \( c \) is the most-preferred alternative in \( \succ \) if for any alternative \( b \in A \setminus \{c\} \) it holds that \( c \succ b \). Let \( \succ \) be a preference order over \( A \). We use \( \succeq \) to denote the binary relation which includes \( \succ \) and preserves the reflexivity, i.e., \( \succeq := \succ \cup \{(a, a) \mid a \in A\} \).

For a subset \( B \subseteq A \) of alternatives and an alternative \( c \) not in \( B \), we use \( B \succ c \) to denote that in the preference order \( \succ \) each \( b \in B \) is preferred to \( c \), i.e., \( b \succ c \).

A preference profile \( \mathcal{P} \) specifies the preference orders of a number of voters over a set of alternatives. Formally, \( \mathcal{P} := (A, V, \mathcal{R} := (\succ_1, \ldots, \succ_n)) \), where \( A \) denotes the set of \( m \) alternatives, \( V \) denotes the set of \( n \) voters, and \( \mathcal{R} \) is a collection of \( n \) preference orders such that each voter \( v_i \in V \) ranks the alternatives according to the preference order \( \succ_i \) on \( A \). Throughout the paper, if not explicitly stated otherwise, we assume \( \mathcal{P} \) is a preference profile of the form \( (A, V, \mathcal{R}) \).

For notational convenience, for each alternative \( a \in A \) and each voter \( v_i \in V \), let \( r_{k_i}(a) \) denote
the rank of alternative \( a \) in the preference order \( \succ_i \), which is the number of alternatives which are preferred to \( a \) by voter \( v_i \), i.e., \( r_i(a) = |\{ b \in A \mid b \succ_i a \}| \).

Given two points \( p, q \) in the \( d \)-dimensional space \( \mathbb{R}^d \), we write \( \| p - q \|_2 \) to denote the Euclidean distance of \( p \) and \( q \), i.e., \( \| p - q \|_2 = \sqrt{\sum_{i=1}^{d} (p[i] - q[i])^2} \), and we write \( \| p - q \|_1 \) to denote the Manhattan distance of \( p \) and \( q \), i.e., \( \| p - q \|_1 = \sum_{i=1}^{d} |p[i] - q[i]| \).

For \( d = 2 \), the Manhattan distance of two points is equal to the length of a path between them on a rectilinear grid. Hence, under Manhattan distances, a circle is a square rotated at a 45° angle from the coordinate axes. The intersection of two Manhattan-circles can range from two points to two segments as depicted in Figure 1.

### 2.1 Basic geometric notation

Throughout this paper, we use lower case letters in boldface to denote points in a space. Given two points \( p \) and \( q \), we introduce the following notions: Let \( \text{BB}(p, q) \) denote the set of points which are contained in the (smallest) rectilinear bounding box of points \( p \) and \( q \), i.e., \( \text{BB}(p, q) := \{ x \in \mathbb{R}^d \mid \min\{p[i], q[i]\} \leq x[i] \leq \max\{p[i], q[i]\} \text{ for all } i \in [d] \} \). In a \( d \)-dimensional space, a bisector of two points under Manhattan distances can itself be a \( d \)-dimensional object, while a bisector under Euclidean distances is always \( (d-1) \)-dimensional. See the right-most figure in Figure 2 for an illustration.

The two-dimensional case. In the two-dimensional space, the vertical line and the horizontal line crossing any point divide the space into four non-disjoint quadrants: the north-east, south-east, north-west, and south-west quadrants. Hence, given a point \( p \), we use \( \text{NE}(p) \), \( \text{SE}(p) \), \( \text{NW}(p) \), and \( \text{SW}(p) \) to denote these four quadrants. Formally, \( \text{NE}(p) := \{ x \in \mathbb{R}^2 \mid x[1] \geq p[1] \land x[2] \geq p[2] \} \), \( \text{SE}(p) := \{ x \in \mathbb{R}^2 \mid x[1] \geq p[1] \land x[2] \leq p[2] \} \), \( \text{NW}(p) := \{ x \in \mathbb{R}^2 \mid x[1] \leq p[1] \land x[2] \geq p[2] \} \), and \( \text{SW}(p) := \{ x \in \mathbb{R}^2 \mid x[1] \leq p[1] \land x[2] \leq p[2] \} \). As a convention, we use \( x \) and \( y \) to refer to the first dimension and the second dimension in the two-dimensional space, respectively.

### 2.2 Embeddings

Generally speaking, the Euclidean (resp. Manhattan) representation models the preferences of voters over the alternatives using the Euclidean (resp. Manhattan) distance between an alternative and a voter. A shorter distance indicates a stronger preference.

**Definition 1** (\( d \)-Euclidean and \( d \)-Manhattan embeddings). Let \( \mathcal{P} := (A, V := \{ v_1, \ldots, v_n \}, \mathcal{R} := (\succ_1, \ldots, \succ_n) \) be a preference profile. Let \( E : A \cup V \to \mathbb{R}^d \) be an embedding of the alternatives.
Figure 2: The bisector (in green) between two points in Manhattan distances. The green lines and areas extend to infinity.

![Figure 2](image)

Figure 3: Two possible embeddings illustrating the properties in Definition 2 (the numbering will be used in some proofs). (I) means “inside of” while (O) “outside of”.

![Figure 3](image)

and the voters into the $d$-dimensional space. A voter $v_i \in V$ is $d$-Euclidean with respect to $E$ if for each two distinct alternatives $a, b \in A$ voter $v_i$ strictly prefers the one closer to her, that is,

$$a \succ_i b\text{ if and only if } \|E(a) - E(v_i)\|_2 < \|E(b) - E(v_i)\|_2.$$  

Similarly, $v_i$ is $d$-Manhattan with respect to $E$ if for each two distinct alternatives $a, b \in A$ voter $v_i$ strictly prefers the one closer to her, that is,

$$a \succ_i b\text{ if and only if } \|E(a) - E(v_i)\|_1 < \|E(b) - E(v_i)\|_1.$$  

An embedding $E$ of the alternatives and voters is a $d$-Euclidean (resp. $d$-Manhattan) embedding of profile $\mathcal{P}$ if each voter in $V$ is $d$-Euclidean (resp. $d$-Manhattan) with respect to $E$. A preference profile is $d$-Euclidean (resp. $d$-Manhattan) if it admits a $d$-Euclidean (resp. $d$-Manhattan) embedding.

To characterize necessary conditions for 2-Manhattan profiles, we need to define several notions which describe the relative orders of the points in the $x$- and $y$-coordinates.

**Definition 2 (BE- and EX-properties).** Let $\mathcal{P}$ be a preference profile containing at least three voters called $u, v, w$ and let $E$ be an embedding for $\mathcal{P}$.

- We say that $E$ satisfies the $(v, u, w)$-BE-property if $E(v) \in BB(E(u), E(w))$ (see Figure 3(I)). For brevity’s sake, by symmetry, we sometimes omit voters $u$ and $w$ and just speak of the $v$-BE-property if $u, v, w$ are the only voters contained in $\mathcal{P}$.
We say that $E$ satisfies the $(v, u, w)$-EX-property if either “$E(u)[1]$ is between $E(v)[1]$ and $E(w)[1]$” and $E(u)[2]$ is between $E(v)[2]$ and $E(w)[2]$” or “$E(v)[1]$ is between $E(u)[1]$ and $E(w)[1]$” and $E(u)[2]$ is between $E(v)[2]$ and $E(w)[2]$” (see Figure 3(O)). Once again, we sometimes omit voters $u$ and $w$ and just call it the $v$-EX-property if $u, v, w$ are the only voters contained in $P$.

Note that there are four possible embeddings which satisfy the $(v, u, w)$-BE-property while there are eight possible embeddings which satisfy the $(v, u, w)$-EX-property. However, each of these embeddings satisfying the $(v, u, w)$-BE-property (resp. the $(v, u, w)$-EX-property) forbids certain types of preference profiles, specified below. The following two configurations describe preferences whose existence precludes an embedding from satisfying one of the properties defined in Definition 2, as we will show in Lemmas 3 and 4.

**Definition 3 (BE-configurations).** A preference profile $P$ with three voters $u, v, w$ and three alternatives $a, b, x$ is a $(v, u, w)$-BE-configuration if the following holds:

$$u: b \succ_u x \succ_u a, \quad v: a \succ_v x \succ_v b, \quad w: b \succ_w x \succ_w a.$$ 

**Definition 4 (EX-configurations).** A preference profile $P$ with three voters $u, v, w$ and six alternatives $x, a, b, c, d, e$ ($c, d, e$ not necessarily distinct) is a $(v, u, w)$-EX-configuration if the following holds:

$$u: a \succ_u x \succ_u b, \quad c \succ_u x, \quad d \succ_u x$$
$$v: \{a, b\} \succ_v x, \quad x \succ_v \{d, e\},$$
$$w: b \succ_w x \succ_w a, \quad c \succ_w x, \quad e \succ_w x.$$ 

**Example 1.** Consider the following two preference profiles $Q_1$ and $Q_2$:

$Q_1$:

$v_1: 1 \succ_1 2 \succ_1 3$,  
$v_2: 3 \succ_2 2 \succ_2 1$,  
$v_3: 3 \succ_3 2 \succ_3 1$,

$Q_2$:

$v_1: \{1, 2\} \succ_1 3 \succ_1 4$,  
$v_2: \{1, 4\} \succ_2 3 \succ_2 2$,  
$v_3: \{2, 4\} \succ_3 3 \succ_3 1$.

One can verify that $Q_1$ is a $(v_1, v_2, v_3)$-BE-configuration. Further, $Q_2$ is a $(v_1, v_2, v_3)^-$, $(v_2, v_1, v_3)^-$, and $(v_3, v_1, v_2)^-$EX-configuration. One can verify this by setting $(a, b, x, c, d, e) = (1, 2, 3, 4, 4, 4)$, $(a, b, x, c, d, e) = (1, 4, 3, 2, 2, 2)$, $(a, b, x, c, d, e) = (2, 4, 3, 1, 1, 1)$, respectively.

3 Manhattan Preferences: Positive Results

In this section, we show that for sufficiently high dimension $d$, i.e., $d \geq \min(n, m-1)$, a preference profile with $n$ voters and $m$ alternatives is always $d$-Manhattan.

**Theorem 1.** Every preference profile with $n$ voters is $n$-Manhattan.

**Proof.** Let $P = (A, V, (\succ_i)_{i \in [p]})$ be a preference profile with $m$ alternatives and $n$ voters $V$ such that $A = \{1, 2, \ldots, m\}$. The idea is to first embed the voters from $V$ onto $n$ carefully selected vertices of an $n$-dimensional hypercube, and then embed the alternatives such that each coordinate of an alternative reflects the preferences of a specific voter. More precisely, define an
embedding $E: \mathcal{A} \cup \mathcal{V} \to \mathbb{R}_0$ such that for each voter $v_i \in \mathcal{V}$ and each coordinate $z \in [n]$, we have $E(v_i)[z] := -m$ if $z = i$, and $E(v_i)[z] := 0$ otherwise.

It remains to specify the embedding of the alternatives. To ease notation, for each alternative $j \in \mathcal{A}$, let $mk_j$ denote the maximum rank of the voters towards $j$, i.e., $mk_j := \max_{v_i \in \mathcal{V}} rk_i(j)$. Further, let $n_j$ denote the index of the voter who has maximum rank over $j$; if there are two or more such voters, then we fix an arbitrary one. That is, $v_{n_j} := \arg\max_{v_i \in \mathcal{V}} rk_i(j)$. Then, the embedding of each alternative $j \in \mathcal{A}$ is defined as follows:

$$
E(j)[z] := \begin{cases} \mk_j - mk_j, & \text{if } z \neq n_j, \\
M + 2\mk_j + \sum_{k \in [n]} (rk_k(j) - mk_j), & \text{otherwise.}
\end{cases}
$$

Herein, $M$ is a large but fixed value such that the second term in the above definition is non-negative. For instance, we can set $M := n \cdot m$. Notice that by definition, the following holds for each alternative $j \in \mathcal{A}$.

$$
\begin{align*}
-m & \leq \mk_j - mk_j \leq 0, & (1) \\
M + 2\mk_j + \sum_{k \in [n]} (rk_k(j) - mk_j) & \geq M - n \cdot m \geq 0. & (2)
\end{align*}
$$

In other words, it holds that

$$
\begin{align*}
|E(j)[i] - E(v_i)[i]| & = m + E(j)[i], & (3) \\
\|E(j)\|_1 & = \sum_{z \in [n]} |E(j)[z]| & \overset{(1),(2)}{=} M + 2\mk_j + \sum_{k \in [n]} (rk_k[j] - mk_j) + \sum_{z \in [n] \setminus \{n_j\}} -(rk_z[j] - mk_j) \\
& \overset{mk_j = rk_{n_j}(j)}{=} M + 2\mk_{n_j}(j). & (4)
\end{align*}
$$

Now, in order to prove that this embedding is $n$-Manhattan we show that the Manhattan distance between an arbitrary voter $v_i$ and an arbitrary alternative $j$ is linear in the rank value $rk_i(j)$. By definition, this distance is:

$$
\|E(v_i) - E(j)\|_1 = \sum_{k \in [n]} |E(j)[k] - E(v_i)[k]| = |E(j)[i] - E(v_i)[i]| + \sum_{k \in [n] \setminus \{i\}} |E(j)[k]| \\
\overset{(3)}{=} m + E(j)[i] + \|E(j)\|_1 - |E(j)[i]|. & (5)
$$

We distinguish between two cases.

**Case 1:** $i \neq n_j$. Then, by definition, it follows that

$$
\|E(v_i) - E(j)\|_1 \overset{(5)}{=} m + E(j)[i] + \|E(j)\|_1 - |E(j)[i]| \\
\overset{(1),(4)}{=} m + 2(rk_i(j) - mk_j) + M + 2\mk_{n_j}(j) \\
\overset{rk_{n_j}(j) = mk_j}{=} m + M + 2\mk_i(j).
$$

**Case 2:** $i = n_j$. Then, by definition, it follows that

$$
\|E(v_i) - E(j)\|_1 \overset{(5)}{=} m + E(j)[i] + \|E(j)\|_1 - |E(j)[i]| \\
\overset{(1),(4)}{=} m + 2(rk_i(j) - mk_j) + M + 2\mk_i(j).
$$
In both cases, we obtain that \( \|E(v_i) - E(j)\|_1 = m + M + 2r_k \hat{n}(j) \), which is linear in the ranks, as desired.

By Theorem 1, we obtain that any profile with two voters is 2-Manhattan. The following example provides an illustration.

**Example 2.** Consider a profile \( P_1 \) with two voters and five alternatives.

\[
\begin{align*}
v_1 &: 1 \succ 2 \succ 3 \succ 4 \succ 5, \\
v_2 &: 5 \succ 4 \succ 3 \succ 1 \succ 2.
\end{align*}
\]

By the proof of Theorem 1, the maximum ranks and the voters with maximum rank are defined as follows:

\[
\begin{array}{c|ccccc}
j & 1 & 2 & 3 & 4 & 5 \\
\hline
m_k_j & 3 & 4 & 2 & 3 & 4 \\
\hat{n}_j & 2 & 2 & 1 & 1 & 1
\end{array}
\]

The embedding of the voters and alternatives (according to the proof of Theorem 1) is as follows, where we let \( M := n \cdot m = 10 \).

\[
\begin{array}{c|ccccc|ccccc}
x \in V \cup A & v_1 & v_2 & 1 & 2 & 3 & 4 & 5 \\
\hline
E(x)[1] & -5 & 0 & -3 & -3 & 14 & 14 & 14 \\
E(x)[2] & 0 & -5 & 13 & 15 & 0 & -2 & -4
\end{array}
\]

Figure 4 depicts the corresponding embedding.

**Theorem 2.** Every preference profile with \( m + 1 \) alternatives is \( m \text{-Manhattan} \).
Proof. Let $\mathcal{P} = (A, V, (\triangleright_i)_{i \in [n]})$ be a preference profile with $m + 1$ alternatives and $n$ voters $V$ such that $A = \{1, 2, \cdots , m + 1\}$. The idea is to first embed the alternatives from $A$ onto $m + 1$ carefully selected vertices of an $m$-dimensional hypercube, and then embed the voters such that the $m$-Manhattan distances from each voter to the alternatives increase as the preferences decrease. More precisely, define an embedding $E: A \cup V \rightarrow \mathbb{R}^m$ such that alternative $m + 1$ is embedded in the origin coordinate, i.e., $E(m + 1) = (0)_{z \in [m]}$. For each alternative $j \in [m]$ and each coordinate $z \in [m]$, we have $E(j)[z] := 2m$ if $z = j$, and $E(j)[z] := 0$ otherwise.

Then, the embedding of each voter $v_i \in V$ is defined as follows:

$$
\forall j \in [m]: E(v_i)[j] := \begin{cases} 
2m - rk_i(j), & \text{if } rk_i(j) < rk_i(m + 1), \\
m - rk_i(j), & \text{if } rk_i(j) > rk_i(m + 1).
\end{cases}
$$

Observe that $0 \leq E(v_i)[j] \leq 2m$. Before we show that $E$ is a Manhattan embedding for $\mathcal{P}$, let us establish a simple formula for the Manhattan distance between a voter and an alternative.

Claim 1. For each voter $v_i \in V$ and each alternative $j \in A$, we have

$$
\|E(v_i) - E(j)\|_1 = \begin{cases} 
\|E(v_i)\|_1 + 2(m - E(v_i)[j]), & \text{if } j \neq m + 1, \\
\|E(v_i)\|_1, & \text{otherwise}.
\end{cases}
$$

Proof. The case with $j = m + 1$ is straightforward since alternative $m + 1$ is embedded at the origin. The proof for $j \neq m + 1$ is also straightforward by a direct application of the definition:

$$
\|E(v_i) - E(j)\|_1 = \sum_{z \in [m]} |E(v_i)[z] - E(j)[z]| = \left( \sum_{z \in [m] \setminus \{j\}} |E(v_i)[z]| \right) + |E(v_i)[j] - E(j)[j]|
$$

$$
= \left( \sum_{z \in [m] \setminus \{j\}} |E(v_i)[z]| \right) + (2m - E(v_i)[j])
$$

$$
= \|E(v_i)\|_1 + 2(m - E(v_i)[j]).
$$

(\text{of Claim 1}) \diamond

Now, we proceed with the proof. Consider an arbitrary voter $v_i \in V$ and let $j, k \in [m + 1]$ be two consecutive alternatives in the preference order $\triangleright_i$ such that $rk_i(j) = rk_i(k) - 1$. We aim to show that $\|E(v_i) - E(j)\|_1 < \|E(v_i) - E(k)\|_1$, and we distinguish between three cases.

Case 1: $rk_i(k) < rk_i(m + 1)$ or $rk_i(j) > rk_i(m + 1)$. Then, by Claim 1 and by definition, it follows that

$$
\|E(v_i) - E(j)\|_1 - \|E(v_i) - E(k)\|_1 = 2(E(v_i)[k] - E(v_i)[j]) = 2(rk_i(j) - rk_i(k)) < 0,
$$
as desired.

Case 2: $rk_i(k) = rk_i(m + 1)$, i.e., $k = m + 1$ and $E(v_i)[j] = 2m - rk_i(j)$. Then, by Claim 1 and by definition, it follows that

$$
\|E(v_i) - E(j)\|_1 - \|E(v_i) - E(k)\|_1 = 2(m - E(v_i)[j]) = 2rk_i(j) - 2m < 0.
$$

Note that the last inequality holds since $rk_i(j) = rk_i(k) - 1 < m$. 

8
1. Case 3: \( \text{rk}_i(j) = \text{rk}_i(m + 1) \), i.e., \( j = m + 1 \) and \( E(v_i)[k] = m - \text{rk}_i(k) \). Then, by Claim 1 and by definition, it follows that

\[
\|E(v_i) - E(j)\|_1 - \|E(v_i) - E(k)\|_1 = -2(m - E(v_i)[k]) = -2\text{rk}_i(k) < 0.
\]

Note that the last inequality holds since \( \text{rk}_i(k) = \text{rk}_i(j) + 1 > 0 \).

Since in all cases, we show that \( \|E(v_i) - E(j)\|_1 - \|E(v_i) - E(k)\|_1 < 0 \), \( E \) is indeed a Manhattan embedding for \( P \).

By Theorem 2, we obtain that any profile for 3 alternatives is 2-Manhattan. The following example illustrates how the voters and alternatives are embedded according to the proof of Theorem 2.

Example 3. The following preference profile \( P_2 \) with 6 voters and 3 alternatives is 2-Manhattan.

\[
\begin{align*}
\text{v}_1: & \text{ } 1 \triangleright 2 \triangleright 3, & \text{v}_2: & \text{ } 1 \triangleright 3 \triangleright 2, \\
\text{v}_3: & \text{ } 2 \triangleright 1 \triangleright 3, & \text{v}_4: & \text{ } 2 \triangleright 3 \triangleright 1, \\
\text{v}_5: & \text{ } 3 \triangleright 1 \triangleright 2, & \text{v}_6: & \text{ } 3 \triangleright 2 \triangleright 1.
\end{align*}
\]

One can check that the following embedding \( E \) with

\[
\begin{align*}
E(1) &= (4, 0), & E(2) &= (0, 4), & E(3) &= (0, 0), \\
E(v_1) &= (4, 3), & E(v_2) &= (4, 0), & E(v_3) &= (3, 4), & E(v_4) &= (0, 4), & E(v_5) &= (1, 0), & E(v_6) &= (0, 1)
\end{align*}
\]

is 2-Manhattan embedding for \( P_3 \). Figure 5 depicts the corresponding embedding.

4 Manhattan Preferences: Negative Results

In this section, we prove results regarding minimally non-2-Manhattan preferences. We show that for \( n \in \{3, 4, 5\} \) voters, the smallest non-2-Manhattan preference profile has \( 9 - n \) alternatives. The first negative result is for the case with three voters.

Theorem 3. There exists a non 2-Manhattan preference profile with three voters and six alternatives.
The second negative result deals with four voters.

**Theorem 4.** There exists a non-2-Manhattan preference profile with four voters and five alternatives.

Finally, the last negative result is about the case where there are five voters.

**Theorem 5.** There exists a non-2-Manhattan preference profile with five voters and four alternatives.

Before we show the aforementioned results, we first go through some technical but useful statements for 2-Manhattan preference profiles in Subsection 4.1. Then, we show the proofs of the main results in Subsections 4.2 to 4.4.

### 4.1 Technical results

**Lemma 1.** Let $\mathcal{P}$ be a 2-Manhattan preference profile and let $E$ be a 2-Manhattan embedding for $\mathcal{P}$. For any two voters $r, s$ and two alternatives $x, y$ the following holds:

1. If $r, s: y \succ x$, then $E(x) \notin \mathbb{BB}(E(r), E(s))$.
2. If $r: x \succ y$ and $s: y \succ x$, then $E(s) \notin \mathbb{BB}(E(r), E(x))$.

**Proof.** Let $\mathcal{P}$, $E$, $r, s$, and $x, y$ be as defined. Both statements follow from using simple calculations and the triangle inequality of Manhattan distances.

For Statement (i), suppose, towards a contradiction, that $r, s: y \succ x$ and $E(x) \in \mathbb{BB}(E(r), E(s))$. By the definition of Manhattan distances, this implies that $\|E(r) - E(x)\|_1 + \|E(x) - E(s)\|_1 = \|E(r) - E(s)\|_1$. By the preferences of voters $r$ and $s$ we infer the following:

$$\|E(s) - E(y)\|_1 + \|E(r) - E(y)\|_1 < \|E(s) - E(x)\|_1 + \|E(r) - E(x)\|_1 = \|E(r) - E(s)\|_1,$$

a contradiction to the triangle inequality of $\| \cdot \|_1$.

For Statement (ii), suppose, towards a contradiction, that $r: x \succ y$ and $s: y \succ x$ and $E(s) \in \mathbb{BB}(E(r), E(x))$. By the definition of Manhattan distances, this implies that $\|E(r) - E(x)\|_1 = \|E(r) - E(s)\|_1 + \|E(s) - E(x)\|_1$. By the preferences of voters $r$ and $s$ we infer the following:

$$\|E(r) - E(s)\|_1 + \|E(s) - E(y)\|_1 < \|E(r) - E(s)\|_1 + \|E(s) - E(x)\|_1$$

$$= \|E(r) - E(x)\|_1$$

$$< \|E(r) - E(y)\|_1,$$

a contradiction to the triangle inequality of $\| \cdot \|_1$. \qed

The following is a summary of coordinate differences with regard to the preferences.

**Observation 1.** Let $\mathcal{P}$ be a 2-Manhattan preference profile and let $E$ be a 2-Manhattan embedding for $\mathcal{P}$. For each voter $s$ and any two alternatives $x, y$ with $s: x \succ y$, the following holds:

1. If $E(y) \in \text{NE}(E(s))$, then $E(y)[1] + E(y)[2] > E(x)[1] + E(x)[2]$.
2. If $E(y) \in \text{NW}(E(s))$, then $-E(y)[1] + E(y)[2] > -E(x)[1] + E(x)[2]$. 

10
(iii) If \( E(y) \in \text{SE}(E(s)) \), then \( E(y)[1] - E(y)[2] > E(x)[1] - E(x)[2] \).

(iv) If \( E(y) \in \text{SW}(E(s)) \), then \( -E(y)[1] - E(y)[2] > -E(x)[1] - E(x)[2] \).

Proof. All proofs are straightforward by evoking the definition of Manhattan embedding. We only show the first statement. Let \( \mathcal{P}, E, s, x, y \) be as defined. For brevity’s sake, let \( s, x, y \) denote the points \( E(s), E(x), \) and \( E(y) \), respectively.

Assume that \( y \in \text{NE}(s) \). Then, by the Manhattan property and the fact that \( s \succ x \), it follows that

\[
(y[1] - s[1]) + (y[2] - s[2]) = \|y - s\|_1 > \|x - s\|_1 = |x[1] - s[1]| + |x[2] - s[2]| \\
\geq (x[1] - s[1]) + (x[2] - s[2]) \\
\Rightarrow \quad y[1] + y[2] > x[1] + x[2],
\]
as desired. \[\square\]

The next technical lemma excludes two alternatives to be put in the same quadrant region of some voters.

**Lemma 2.** Let \( \mathcal{P} \) be a 2-Manhattan profile and let \( E \) be a 2-Manhattan embedding for \( \mathcal{P} \). Let \( r, s, t \) and \( x, y \) be three voters and two alternatives in \( \mathcal{P} \), respectively. The following holds.

(i) If \( r : x \succ y \) and \( s : y \succ x \), then for each \( \Pi \in \{\text{NE}, \text{NW}, \text{SE}, \text{SW}\} \) it holds that if \( E(x) \in \Pi(E(s)) \), then \( E(y) \notin \Pi(E(r)) \).

(ii) If \( r, t: x \succ y, s : y \succ x, E(r)[1] \leq E(s)[1] \leq E(t)[1] \), and \( E(r)[2] \leq E(s)[2] \leq E(t)[2] \), then the following holds.

- If \( E(x) \in \text{NW}(E(s)) \), then \( E(y) \notin \text{NW}(E(s)) \).
- If \( E(x) \in \text{SE}(E(s)) \), then \( E(y) \notin \text{SE}(E(s)) \).

Proof. Let \( \mathcal{P}, E, r, s, t, x, y \) be as defined. See Figure 3(I) (replacing \( u, v, w \) with \( r, s, \) and \( t \), respectively) for an illustration of the embedding of \( r, s, t \). For brevity’s sake, let \( r, s, t, x, y \) denote the points \( E(r), E(s), E(t), E(x), \) and \( E(y) \), respectively.

The first statement is straightforward to verify by using Observation 1. Hence, we only show the case for \( \Pi = \text{NW} \). Suppose, towards a contradiction, that \( x \in \text{NW}(s) \) and \( y \in \text{NW}(r) \). Then, since \( r: x \succ y \) and \( s : y \succ x \), by Observation 1(ii), it follows that

\[
y[2] - y[1] > x[2] - x[1] \quad \text{and} \quad x[2] - x[1] > y[2] - y[1],
\]
a contradiction.

For the second statement, the two parts are symmetric. Hence, we only show the first part. Suppose, towards a contradiction, that \( x, y \in \text{NW}(s) \).

Since \( r, t: x \succ y, s : y \succ x, x \in \text{NW}(s) \), by the first statement, it follows that \( y \notin \text{NW}(r) \cup \text{NW}(t) \). However, since \( y \in \text{NW}(s) \), it follows that \( y \in \text{BB}(r, t) \), a contradiction to Lemma 1(i). \[\square\]

**Lemma 3.** If a preference profile contains a \((v, u, w)-\text{BE}\)-configuration, then no 2-Manhattan embedding satisfies the \((v, u, w)-\text{BE}\)-property.
Proof. Suppose, towards a contradiction, that a preference profile, called \( \mathcal{P} \), contains a \((v, u, w)\)-BE-configuration, and there exists a 2-Manhattan embedding \( E \) for profile \( \mathcal{P} \) which satisfies the \((v, u, w)\)-BE-property, for three voters \( v, u, w \).

Let \( a, b, x \) be the three alternatives defined in the \((v, u, w)\)-BE-configuration (see Definition 3). For brevity’s sake, let \( u, v, w, a, b, x \) denote \( E(u), E(v), E(w), E(a), E(b), E(x) \), respectively. By symmetry and by the preferences of \( u \) and \( w \), the embedding \( E \) corresponds to Figure 3(I). Since there are three voters, we can divide the two-dimensional space into 16 subspaces by drawing a vertical and horizontal line through each voter’s embedded point. We enumerate these regions as in Figure 3(I) and use \( R_i \) to refer to region \( i, i \in [16] \).

First, using Lemma 1(i) (setting \( (r, s, y) := (u, w, b) \)), we infer that alternative \( x \) cannot be embedded in \( R_6, R_7, R_{10}, \) or \( R_{11} \). Moreover, using Lemma 1(ii) (setting \( (r, s, y) := (u, v, a) \)), we infer that alternative \( x \) cannot be embedded in \( R_3, R_4, R_7, \) or \( R_8 \). Similarly, using Lemma 1(ii) (setting \( (r, s, y) := (w, v, b) \)), we infer that alternative \( x \) cannot be embedded in \( R_9, R_{10}, R_{13}, \) or \( R_{14} \). This implies that \( x \) is in one of the regions \( R_1, R_2, R_5, R_{12}, R_{15} \) or \( R_{16} \). By exchanging the two coordinates and the roles of \( u \) and \( w \) and the roles of \( a \) and \( b \), we know that if \( E \) embeds alternative \( x \) in \( R_5 \) (resp. \( R_1 \) or \( R_2 \)), then there exists another Manhattan embedding which embeds \( x \) in \( R_{15} \) (resp. \( R_{16} \) or \( R_{12} \)), and vice versa. Hence, without loss of generality, assume that \( E \) embeds \( x \) in \( R_1, R_2, \) or \( R_5 \). Note that this implies that \( x \in NW(v) \).

Similarly, using Lemma 1(ii) (wrt. voters \( u \) and \( v \), and voters \( w \) and \( v \), respectively) we infer that \( b \notin NE(v) \cup SW(v) \). Since \( x \in NW(v) \), by Lemma 2(ii) (wrt. alternatives \( x \) and \( b \)), it follows that \( b \notin NW(v) \). This implies that \( b \in SE(v) \).

Let us consider alternative \( a \). On the one hand, since \( u, w : x \succ a \) and \( v : a \succ x \), by Lemma 1, it follows that \( a \notin BB(u, w) \cup NE(w) \cup SW(u) \). Altogether, it follows that \( a \in SE(w) \cup NW(w) \cup SE(u) \cup NW(u) \).

On the other hand, since \( v : a \succ b, u, w : b \succ a \), and \( b \in SE(v) \), by Lemma 2(i), it follows that \( a \notin SE(u) \cup SE(w) \). Analogously, since \( v : a \succ x, u, w : x \succ a \), and \( x \in NW(v) \), by Lemma 2(i), it follows that \( a \notin NW(u) \cup NW(w) \).

This results in having no place to embed alternative \( a \), a contradiction. \( \square \)

Lemma 4. If a preference profile contains a \((v, u, w)\)-EX-configuration, then no 2-Manhattan embedding satisfies the \((v, u, w)\)-EX-property.

Proof. Suppose, for the sake of contradiction, that a preference profile, called \( \mathcal{P} \), contains a \((v, u, w)\)-EX-configuration, and there exists a 2-Manhattan embedding, called \( E \), for profile \( \mathcal{P} \) which satisfies the \((v, u, w)\)-EX-property, for three voters \( v, u, w \).

Let \( x, a, b, c, d, e \) be the six alternatives defined in the \( v \)-EX-configuration (see Definition 4). For brevity’s sake, let \( u, v, w, a, b, x, c, d, e \) denote \( E(u), E(v), E(w), E(a), E(b), E(x), E(c), E(d), E(e) \), respectively. Observe that the preferences of \( u \) and \( w \) are symmetric in the sense that if we exchange the roles of \( a \) and \( b \), and also the roles of \( d \) and \( e \), then we arrive at a new \((v, u, w)\)-EX-configuration for \( \mathcal{P} \). Hence, up to rotation and mirroring, embedding \( E \) corresponds to Figure 3(O). Since there are three voters, we can divide the two-dimensional space into 16 subspaces by drawing a vertical and horizontal line through each voter’s embedded point. We enumerate these regions as in Figure 3(O) and use \( R_i \) to refer to region \( i, i \in [16] \).

First, using Lemma 1(i) (setting \( (r, s, y) := (u, w, c) \)), we infer that alternative \( x \) cannot be embedded in \( R_6 \). Analogously, repeatedly using Lemma 1(i) (setting \( (r, s, y) := (u, v, a) \) and \( (r, s, y) := (v, w, b) \), respectively), we infer that \( x \) cannot be embedded in regions \( R_7, R_{10} \) or \( R_{11} \). Further, using Lemma 1(ii) (setting \( (r, s, y) := (v, u, d) \)), we infer that alternative \( x \) cannot
be in regions $R_1$ and $R_5$. Again, using Lemma 1(ii) repeatedly (setting $(r, s, y) := (v, w, e)$, $(r, s, y) := (u, w, b)$, and $(r, s, y) := (w, u, a)$, and $(r, s, y) := (u, v, b)$, respectively), we further infer that alternative $x$ cannot be in regions $R_1-R_4$, $R_9$, $R_{13}$, and $R_{16}$.

This implies that $x$ is in one of the regions $R_8$, $R_{12}$, $R_{14}$ or $R_{15}$. We aim to show that none of the regions is possible for $x$. To this end, since $v \in SE(u) \cap SE(w)$, by Lemma 1(ii), we observe that

$$a[2] \leq w[2] \text{ and } b[1] \geq u[1].$$

If $E$ embeds $x$ in regions $R_{14}-R_{15}$, then

$$x \in SW(v) \cap SE(u).$$

Since $v: a \succ x$, $w: x \succ a$, by Lemma 2(i), it follows that $a \notin SW(w)$. By (6), it follows that $a \in SE(w)$. Since $w: x \succ a$ and $u: a \succ x$, by Lemma 2(i), it follows that $x \notin SE(u)$, a contradiction to (7).

Analogously, we also obtain a contradiction if $x$ is embedded in region $R_8$ or $R_{12}$ by focusing on voter $u$ and alternative $b$. Assume that

$$x \in NE(v) \cap SE(w).$$

Since $v: b \succ x$ and $u: x \succ b$, by Lemma 2(i), it follows that $b \notin NE(u)$. By (6), it follows that $b \in SE(u)$. Since $u: x \succ b$ and $w: b \succ x$, by Lemma 2(i), it follows that $x \notin SE(w)$, a contradiction to (8).

4.2 Proof of Theorem 3

Using Lemmas 3 and 4, we can prove Theorem 3 with the help of Example 4 below.

**Example 4.** The following preference profile $P_3$ with three voters and six alternatives is not 2-Manhattan.

- $v_1: 1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$,
- $v_2: 1 \succ 4 \succ 6 \succ 3 \succ 5 \succ 2$,
- $v_3: 6 \succ 5 \succ 2 \succ 3 \succ 1 \succ 4$.

**Proof of Theorem 3.** To show this, we consider profile $P_3$ given in Example 4. Suppose, towards a contradiction, that $E$ is a 2-Manhattan embedding for $P_3$. Since each embedding for three voters must satisfy one of the two properties in Definition 2, we distinguish between two cases: either there exists a voter whose embedding is inside the bounding box of the other two, or there is no such voter.

**Case 1:** There exists a voter $v_i$, $i \in [3]$, such that $E$ satisfies the $v_i$-BE-property. Since $P$ contains a $(v_1, v_2, v_3)$-BE-configuration with $a = 2$, $b = 6$, $x = 5$, by Lemma 3 it follows that $E$ does not satisfy the $v_1$-BE-property. Analogously, since $P$ contains a $(v_2, v_1, v_3)$-BE-configuration regarding $a = 2$, $b = 4$, $x = 3$, and $(v_3, v_1, v_2)$-BE-configuration with $a = 1$, $b = 5$, $x = 3$, neither does $E$ satisfy the $v_2$-BE-property or the $v_3$-BE-property.
Case 2: There exists a voter $v_i$, $i \in [3]$, such that $E$ satisfies the $v_i$-EX-property. Now, consider the subprofile $\mathcal{P}'$ restricted to the alternatives $1, 2, 3, 6$. We claim that this subprofile contains an EX-configuration, which by Lemma 4 precludes the existence of such a voter $v_i$ with the $v_i$-EX-property: First, since $\mathcal{P}'$ contains a $(v_3, v_1, v_2)$-EX-configuration (setting $(u, v, w) := (v_1, v_3, v_2)$ and $(x, a, b, c, d, e) = (3, 2, 6, 1, 1, 1)$), by Lemma 4, it follows that $E$ does not satisfy the $v_3$-EX-property. In fact, $\mathcal{P}'$ also contains a $v_2$-EX-configuration (setting $(u, v, w) := (v_1, v_2, v_3)$ and $(x, a, b, c, d, e) = (3, 1, 6, 2, 2, 2)$) and a $v_1$-EX-configuration (setting $(u, v, w) := (v_2, v_1, v_3)$ and $(x, a, b, c, d, e) = (3, 1, 2, 6, 6, 6)$). By Lemma 4, it follows that $E$ does not satisfy the $v_2$-EX-property nor the $v_1$-EX-property.

Summarizing, we obtain a contradiction for $E$. 

4.3 Proof of Theorem 4

The proof will be based on the following example.

Example 5. Any profile $\mathcal{P}_4$ satisfying the following is not 2-Manhattan.

\begin{align*}
v_1 & : \{1, 2\} \succ 3 \succ 4 \succ 5, \\
v_2 & : \{1, 2\} \succ 3 \succ 5 \succ 4, \\
v_3 & : 1 \succ 4 \succ 5 \succ 3 \succ 2, \\
v_4 & : 2 \succ 4 \succ 5 \succ 3 \succ 1.
\end{align*}

Proof of Theorem 4. To show the statement, we show that profile $\mathcal{P}_4$ given in Example 5 is not 2-Manhattan.

Suppose, towards a contradiction, that $\mathcal{P}_4$ is 2-Manhattan and $E$ is a 2-Manhattan embedding for $\mathcal{P}_4$. For brevity’s sake, we will use $v_i$ and $c_j$, $i \in [4]$, $j \in [5]$ to refer to the embedded point of voter $v_i$ and alternative $j$, respectively. Note that since $\mathcal{P}_4$ has four voters, there are $4! \cdot 4! = 576$ possible combinatorially different embeddings of the voters. We observe however that there are only two groups of voter embeddings which are relevant.

First, we claim that for all $v \in \{v_1, v_2\}$ and $\{u, w\} = \{v_3, v_4\}$, $E$ cannot satisfy the $(v, u, w)$-EX-property, or the $(u, v, w)$-EX-property, or the $(w, u, v)$-EX-property: If we restrict the preference profile to voters $v, u, w$ and alternatives $\{1, 2, 3, 4\}$, then we obtain a profile which is equivalent to profile $Q_2$ from Example 1. Since $Q_2$ is an EX-configuration and, by Lemma 4, violates the EX-property, it follows that

\begin{align*}
v_1 & \in \text{BB}(v_3, v_4) \lor v_3 \in \text{BB}(v_1, v_4) \lor v_4 \in \text{BB}(v_1, v_3) \quad \text{and} \quad (9) \\
v_2 & \in \text{BB}(v_3, v_4) \lor v_3 \in \text{BB}(v_2, v_4) \lor v_4 \in \text{BB}(v_2, v_3). \quad (10)
\end{align*}

Second, it is straightforward to verify that by Lemma 3, embedding $E$ violates the $(v_2, v_3, v_4)$-BE-property (wrt. alternatives $\{3, 5, 4\}$), the $(v_3, v_1, v_2)$-BE-property (wrt. alternatives $\{2, 3, 5\}$), and the $(v_4, v_1, v_2)$-BE-property (wrt. alternatives $\{1, 3, 5\}$). Together with (9)–(10), this implies:

\begin{align*}
v_3 & \notin \text{BB}(v_1, v_2) \quad \text{and} \quad (11) \\
v_3 & \in \text{BB}(v_2, v_4) \quad \text{or} \quad v_4 \in \text{BB}(v_2, v_3). \quad (12)
\end{align*}

Based on (12), we distinguish between two cases.
Case 1: $v_3 \in \text{BB}(v_2, v_4)$. By (9) and (11), we can further infer that $v_3 \in \text{BB}(v_1, v_4)$. Without loss of generality, assume that $v_2[1] \leq v_3[1] \leq v_4[1]$, $v_2[2] \leq v_3[2] \leq v_4[2]$, and $v_1[1] \leq v_3[1] \leq v_4[1]$, $v_1[2] \leq v_3[2] \leq v_4[2]$. See Figure 6 for an illustration; note that the relative positions of $v_1$ and $v_2$ need not be exactly as depicted, but they both are embedded to the southwest of $v_3$. As in previous proofs, we divide the two-dimensional space into 16 subspaces by drawing a vertical and horizontal line through the embedded point of each voter $v_z$, $z \in \{2, 3, 4\}$. We enumerate these regions as in Figure 6 and use $R_i$ to refer to region $i$, $i \in [16]$.

Let us consider alternative 2 and its embedded point $c_2$. By Lemma 1(ii) (wrt. $(v_2, v_3, 2, 3)$), we infer that alternative 2 is not embedded to the northeast of $v_3$, i.e., $c_2 \notin \text{NE}(v_3)$. Again, by Lemma 1(ii) (wrt. $(v_4, v_3, 2, 3)$), we infer that $c_2 \notin \text{SW}(v_3)$. That is, $c_2 \in \text{NW}(v_3) \cup \text{SE}(v_3)$. By symmetry, we can assume that $c_2 \in \text{SE}(v_3)$.

Using a similar reasoning and considering the preferences over $\{x, 2\}$ for $x \in \{3, 4, 5\}$, we obtain that $c_3, c_4, c_5 \notin \text{BB}(v_2, v_1) \cup \text{NE}(v_4) \cup \text{SW}(v_2)$. Further, since $c_2 \in \text{SE}(v_3)$, by Lemma 2(i), it follows that $c_3, c_4, c_5 \notin \text{SE}(v_4) \cup \text{SW}(v_2)$. This implies that $c_3, c_4, c_5 \in R_1 \cup R_2 \cup R_3 \cup R_5 \cup R_9$. Since $v_3: 1 \succ \{4, 5, 3\}$ and $v_4: \{4, 5, 3\} \succ 1$, by Lemma 1(ii), it follows that $c_3, c_4, c_5 \notin \text{SW}(v_3)$. That is,

$$c_3, c_4, c_5 \in R_1 \cup R_2 \cup R_3 \cup R_5.$$ (13)

Since $v_2: \{3, 5\} \succeq 4$, $v_3: 4 \succeq \{5, 3\}$, by Lemma 1(ii), it follows that $c_3, c_5 \notin \text{NE}(v_3)$. By (13), we have that

$$c_3, c_5 \in R_1 \cup R_2 \cup R_5 \subseteq \text{NW}(v_3).$$ (14)

Since $v_3: 4 \succeq 3$ and $v_2: 3 \succeq 4$, by (14) and Lemma 2(i), it follows that $c_4 \notin \text{NW}(v_2)$. By (13), this implies that

$$c_4 \in R_2 \cup R_3 \subseteq \text{NE}(v_2).$$ (15)

By Lemma 2(i) regarding $v_1: 4 \succ 5$ and $v_2: 5 \succ 4$, it follows that $c_5 \notin \text{NE}(v_1)$. Since $v_1 \in \text{SW}(v_3)$, this implies that $c_5 \in \text{NW}(v_1)$. However, this contradicts (14) and Lemma 2(i) (setting $(r, s, x, y) = (v_3, v_1, 5, 3)$).

Case 2: $v_4 \in \text{BB}(v_2, v_3)$. Note that this case is analogous to the first case by exchanging the roles of alternatives 1 and 2 and the roles of voters $v_3$ and $v_4$, respectively.

Altogether, we prove that $\mathcal{P}_4$ is not 2-Manhattan. \hfill \Box

4.4 Proof of Theorem 5

Example 6. Any profile $\mathcal{P}_5$ satisfying the following is not 2-Manhattan.

\begin{align*}
v_1: & 1 \succ 2 \succ 3 \succ 4, \\
v_2: & 1 \succ 4 \succ 3 \succ 2, \\
v_3: & \{2, 4\} \succ 3 \succ 1, \\
v_4: & 3 \succ 2 \succ 1 \succ 4, \\
v_5: & 3 \succ 4 \succ 1 \succ 2.
\end{align*}
Case 1: \( r \in \text{NW}(v) \). Since \( r: a \succ d \) and \( v: d \succ a \), by Lemma 1(ii), it follows that \( a \not\in \text{SE}(v) \), a contradiction to our assumption.

Case 2: \( r \in \text{SE}(v) \). This case is analogous to the first case. We consider \( c \) and \( d \) instead. Since \( r: c \succ d \) and \( v: d \succ c \), by Lemma 1(ii), it follows that \( c \not\in \text{NW}(v) \), a contradiction to our assumption as well.

Case 3: \( r \in \text{NE}(v) \). Let us consider alternative \( d \). By the preferences of \( u \) and \( r \), and by Lemma 1(i), we obtain that \( d \not\in \text{BB}(u, r) \). By Lemma 1(ii) (considering the preferences
of $u$ and $r$ regarding $c$ and $d$), we further infer that $d \not\in \text{NE}(r) \cup \text{SW}(u)$. Moreover, by Lemma 2(i)–(ii) (considering the preferences of $u, v, r$ regarding $c$ and $d$) and since $c \in \text{NW}(v)$, we infer that $d \not\in \text{NW}(v) \cup \text{NW}(u) \cup \text{NW}(r)$. Hence,

$$d \in \text{SE}(u) \cup \text{SE}(r).$$

However, this is a contradiction: Since $v$: $d \succ a$, $u$: $a \succ d$, and $a \in \text{SE}(v)$, by Lemma 2(i), it follows that $d \not\in \text{SE}(u) \cup \text{SE}(r)$.

Summarizing, this implies that $r \in \text{SW}(v)$, and hence $v \in \text{BB}(r, w)$. \hfill \square

**Proof of Theorem 5.** To show the statement, we show that profile $P_5$ given in Example 6 is not 2-Manhattan.

Suppose, towards a contradiction, that $P_5$ is 2-Manhattan and $E$ is a 2-Manhattan embedding for $P_5$. For brevity’s sake, we use $v_i$ and $c_j$, $i \in [5]$, $j \in [4]$ to refer to the embedded points of voter $v_i$ and alternative $j$, respectively.

First, we observe that one of $v_1, v_2, v_3$ is embedded inside the bounding box defined by the other two since the subprofile of $P_5$ restricted to voters $v_1, v_2$ and $v_3$ is equivalent to profile $Q_2$ which, by Lemma 4, violates the EX-property (for each of $v_1, v_2$, and $v_3$, respectively). Hence, we distinguish between two cases.

**Case 1:** $v_2 \in \text{BB}(v_1, v_3)$ or $v_1 \in \text{BB}(v_2, v_3)$. Note that these two subcases are equivalent in the sense that if we exchange the roles of alternatives 2 and 4, i.e., $1 \mapsto 1$, $3 \mapsto 3$, $2 \mapsto 4$, and $4 \mapsto 2$, we obtain an equivalent (in terms of the Manhattan property) preference profile where the roles of voters $v_1$ and $v_2$ (resp. $v_4$ and $v_5$) are exchanged. Hence, it suffices to consider the case of $v_2 \in \text{BB}(v_1, v_3)$. Without loss of generality, assume that $v_2[1] \leq v_3[1] \leq v_1[1]$ and $v_2[2] \leq v_3[2] \leq v_1[2]$ (see Figure 7a). Then, by Lemma 5 (setting $u = v_1$, $v = v_2$, $w = v_3$, and $r = v_4$), we obtain that $v_2 \in \text{BB}(v_4, v_3)$. This implies that $v_4[1] \leq v_2[1]$ and $v_4[2] \leq v_2[2]$.

By the preferences of $v_4, v_2, v_3$ regarding alternatives 2 and 1, and by Lemma 1, it follows that $c_2 \in \text{NW}(v_2) \cup \text{SE}(v_2)$ and $c_1 \not\in \text{BB}(v_3, v_4) \cup \text{NE}(v_3) \cup \text{SW}(v_4)$. Similarly, regarding the preferences over 3 and 1, it follows that $c_3 \in \text{NW}(v_2) \cup \text{SE}(v_2)$. By Lemma 2(ii) (considering the preferences of $v_1, v_2$ and $v_3$ regarding alternatives 2 and 3), we further
infer that either \( c_2 \in NW(v_2) \) and \( c_3 \in SE(v_2) \) or \( c_2 \in SE(v_2) \) and \( c_3 \in NW(v_2) \). Without loss of generality, assume that \( c_2 \in NW(v_2) \) and \( c_3 \in SE(v_2) \).

By the preferences of \( v_3 \) and \( v_2 \) (resp. \( v_4 \) and \( v_2 \)) regarding 1 and 3 and by Lemma 2(i), it follows that \( c_1 \notin SE(v_3) \) (resp. \( c_1 \notin SE(v_4) \)). By prior reasoning, we have that \( c_1 \in NW(v_3) \cup NW(v_4) \). However, this is a contradiction due to the preferences of \( v_4 \) and \( v_2 \) (resp. \( v_3 \) and \( v_2 \)) regarding 1 and 2: By Lemma 2(ii), it follows that \( c_1 \notin NW(v_3) \cup NW(v_4) \).

**Case 2: \( v_3 \in BB(v_1, v_2) \).** Without loss of generality, assume that \( v_1[1] \leq v_3[1] \leq v_2[1] \) and \( v_1[2] \leq v_3[2] \leq v_2[2] \); see Figure 7b. Then, by Lemma 5 (setting \((u, v, w, r) = (v_1, v_3, v_2, v_4)\) and \((u, v, w, r) = (v_2, v_3, v_1, v_5)\), respectively), we obtain that \( v_3 \in BB(v_4, v_2) \) and \( v_3 \in BB(v_5, v_1) \). This implies that

\[
v_4[1] \leq v_3[1] \quad \text{and} \quad v_4[2] \leq v_3[2]. \tag{16}
\]

and

\[
v_5[1] \geq v_3[1] \quad \text{and} \quad v_5[2] \geq v_3[2]. \tag{17}
\]

By Lemma 1(ii) (setting \((r, s, x, y) = (v_1, v_3, 3, 2)\) and \((r, s, x, y) = (v_2, v_3, 3, 2)\)), we infer that \( c_3 \in NW(v_3) \cup SE(v_3) \). Similarly, using the preferences of \( v_1, v_3, v_2 \) regarding \( \{1, 4\} \), we infer that

\[
c_1 \in NW(v_3) \cup SE(v_3). \tag{18}
\]

By symmetry, assume that \( c_1 \in NW(v_3) \). Then, by Lemma 2(ii) (setting \((r, s, t, x, y) = (v_1, v_3, v_2, 1, 3)\)), we obtain that \( c_3 \notin NW(v_3) \). By (18), we infer that \( c_3 \in SE(v_3) \).

Next, we specify exactly the relative positions between alternative 2 (resp. 4) and voters \( v_1 \) and \( v_4 \).

Since \( v_3 : \{2, 4\} \succ 3 \) and \( v_4, v_5 : 3 \succ \{2, 4\} \), and \( c_3 \in SE(v_3) \), by Lemma 1 (resp. Lemma 2(i)–(ii)), it follows that \( c_2, c_4 \notin BB(v_4, v_5) \cup NE(v_5) \cup SW(v_4) \) (resp. \( c_2, c_4 \notin SE(v_4) \cup SE(v_5) \cup SE(v_3) \)). Analogously, since \( v_3 : 2 \succ 1, v_1, v_5 : 1 \succ 2 \), and \( c_1 \in NW(v_3) \), by Lemma 2(i)–(ii), it follows that \( c_2 \notin NW(v_1) \cup NW(v_5) \cup NW(v_3) \). Summarizing, it follows that

\[
c_2 \in SW(v_3) \cap NW(v_4). \tag{19}
\]

Consequently, since \( v_3 : 2 \succ 1, v_1, v_2 : 1 \succ 2 \) and \( c_1 \in NW(v_3) \) by Lemma 1 (resp. Lemma 2(i)), it follows that \( c_2 \notin BB(v_1, v_3) \cup SW(v_1) \) (resp. \( c_2 \notin NW(v_1) \)). Together with (19), we have that

\[
c_2 \in SE(v_1) \cap SW(v_3) \cap NW(v_4). \tag{20}
\]

Similarly, since \( v_3 : 4 \succ 1, v_1, v_2 : 1 \succ 4 \), and \( c_1 \in NW(v_3) \), by Lemma 2(i)–(ii), it follows that \( c_4 \notin NW(v_1) \cup NW(v_2) \cup NW(v_4) \cup NW(v_3) \). This implies that

\[
c_4 \in NE(v_3) \cap NW(v_5). \tag{21}
\]

Since \( v_3 : 4 \succ 1 \) and \( v_1, v_2 : 1 \succ 4 \), by Lemma 1 and Lemma 2(i), it follows that \( c_4 \notin BB(v_1, v_2) \cup NE(v_2) \cup NW(v_2) \). Together with (21), we infer that

\[
c_4 \in NE(v_3) \cap NW(v_5) \cap SE(v_2). \tag{22}
\]
See Figure 7c for an illustration.

In the remainder of the proof, we will derive several inequalities, which are mutually inconsistent. The main idea is that since \( v_2 \) and \( v_4 \) (which are on the opposite “diagonal” of \( v_3 \)) both have \( 1 \succ 4 \) and \( 3 \succ 2 \), it is necessary that the bisector between alternatives 1 and 4 and the one between alternatives 2 and 3 “cross” twice. Similarly, due to \( v_1 \) and \( v_5 \) the bisector between alternatives 1 and 2 and the one between alternatives 3 and 4 “cross” twice. This is, however, impossible.

First, let us consider the preferences over 3 and 4. Since \( v_5 : 3 \succ 4 \), by (22), we infer that

\[
(v_5[1] - c_4[1]) + (c_4[2] - v_5[2]) > |v_5[1] - c_3[1]| + |v_5[2] - c_3[2]|
\]

\[
\Rightarrow -c_4[1] + c_4[2] > -c_3[1] - c_3[2] + 2 \cdot v_5[2].
\] (23)

Since \( v_3 : 4 \succ 3 \), by the assumption that \( c_3 \in \text{SE}(v_3) \) and by (22), we infer that

\[
(c_3[1] - v_3[1]) + (v_3[2] - c_3[2]) > (c_4[1] - v_3[1]) + (c_4[2] - v_3[2])
\]

\[
\Rightarrow c_3[1] - c_3[2] > c_4[1] + c_4[2] - 2 \cdot v_3[2].
\] (24)

Now, let us consider the preferences for \( v_2 \) and \( v_4 \) over the pairs \{2, 3\} and \{1, 4\}, respectively. Since \( v_2, v_4 : 3 \succ 2 \) and \( v_2 \in \text{NE}(v_3) \), by (20), we infer that

\[
(v_2[1] - c_2[1]) + (v_2[2] - c_2[2]) > |v_2[1] - c_1[1]| + |v_2[2] - c_1[2]|
\]

\[
\Rightarrow 2 \cdot v_2[1] - c_2[1] - c_2[2] > c_1[1] - c_3[2],
\] (25)

\[
(v_4[1] - c_2[1]) + (c_2[2] - v_4[2]) > |v_4[1] - c_1[1]| + |c_3[2] - v_4[2]|
\]

\[
\Rightarrow 2 \cdot v_4[1] - c_2[1] + c_2[2] > c_1[1] + c_3[2].
\] (26)

Since \( v_2, v_4 : 1 \succ 4 \) and \( v_4 \in \text{SW}(v_3) \), by (22), we infer that

\[
(c_4[1] - v_2[1]) + (v_2[2] - c_4[2]) > |v_2[1] - c_1[1]| + |v_2[2] - c_1[2]|
\]

\[
\Rightarrow c_4[1] - c_2[1] > -c_1[1] - c_1[2] + 2 \cdot v_2[1],
\] (27)

\[
(c_4[1] - v_4[1]) + (c_4[2] - v_4[2]) > |v_4[1] - c_1[1]| + |c_1[2] - v_4[2]|
\]

\[
\Rightarrow c_4[1] + c_4[2] > -c_1[1] + c_1[2] + 2 \cdot v_4[1].
\] (28)

Adding inequalities (23)–(28) yields

\[
2 \cdot (c_1[1] - c_2[1]) > 2(v_5[2] - v_3[2]).
\] (29)

However, this contradicts the preferences of \( v_1 : 1 \succ 2 \) and \( v_3 : 2 \succ 1 \) as these, by (20) and the assumption \( c_1 \in \text{NW}(v_3) \), imply that

\[
(c_2[1] - v_1[1]) + (v_1[2] - c_2[2]) > |c_1[1] - v_1[1]| + |c_1[2] - v_1[2]|
\]

\[
\Rightarrow c_2[1] - c_2[2] > c_1[1] + c_1[2] - 2 \cdot v_1[2],
\] (30)

\[
(v_3[1] - c_1[1]) + (c_1[2] - v_3[2]) > (v_3[1] - c_2[1]) + (v_3[2] - c_2[2])
\]

\[
\Rightarrow -c_1[1] + c_1[2] > -c_2[1] - c_2[2] + 2 \cdot v_3[2].
\] (31)

Adding (29)–(31) yields \( v_1[2] > v_5[2] \), a contradiction to the assumption of \( v_1[2] \leq v_3[2] \) and inequality (17).

In summary, we show that it is not possible to find a Manhattan embedding for profile \( P_5 \).
5 Experimental Results

In this section, we discuss our experimental results.

**Proposition 1.** If \((n, m) = (3, 5)\) or \((n, m) = (4, 4)\), then each preference profile with at most \(n\) voters and at most \(m\) alternatives is 2-Manhattan.

**Proof.** Since the Manhattan property is monotone, to show the statement, we only need to look at preference profiles which have either three voters and five alternatives, or four voters and four alternatives. We achieve this by using a computer program employing the CPLEX solver that exhaustively searches for all possible profiles with either three voters and five alternatives, or four voters and four alternatives, and provide a 2-Manhattan embedding for each of them. Since the CPLEX solver accepts constraints on the absolute value of the difference between any two variables, our computer program is a simple one-to-one translation of the \(d\)-Manhattan constraints given in Definition 1.

Following a similar line as in the work of Chen and Grottke [6], we did some optimization to significantly shrink our search space on all preference profiles: We only consider preference profiles with distinct preference orders and assume that one of the preference orders is \(1 \succ 2 \succ \ldots \succ m\). Hence, the number of relevant preference profiles with \(n\) voters and \(m\) alternatives is \(\binom{m!}{n-1}\). For \((n, m) = (3, 5)\) and \((n, m) = (4, 4)\), we need to iterate through 7021 and 1771 preference profiles, respectively. We implemented a program which, for each of these produced profiles, uses the IBM ILOG CPLEX optimization software package to check and find a 2-Manhattan embedding. The verification is done by going through each voter’s preference order and checking the condition given in Definition 1. All generated profiles, together with their 2-Manhattan embeddings and the distances used for the verification, are available online at https://owncloud.tuwien.ac.at/index.php/s/s6t1vymD0x4EfU9.

\[\square\]

6 Conclusion

Motivated by the question of how restricted \(d\)-Manhattan preferences are, we initiated the study of the smallest dimension sufficient for a preference profile to be \(d\)-Manhattan.

This work opens up to several future research directions. One future research direction concerns the characterization of \(d\)-Manhattan preference profiles through forbidden subprofiles. Such work has been done for other restricted preference domain such as single-peakedness [1], single-crossingness [4] and 1-Euclideanness [7]. Another research direction would be to look into the computational complexity of determining whether a given preference profile is \(d\)-Manhattan. To this end, let us mention that 1-Euclidean preference profiles cannot be characterized via finitely many finite forbidden subprofiles [7], but they can be recognized in polynomial time [9, 16, 13]. As for \(d \geq 2\), recognizing \(d\)-Euclidean preference profile becomes notoriously hard (beyond NP) [18]. This is in stark contrast to recognizing \(d\)-Manhattan preferences, which is in NP. Finally, it would be interesting to see whether assuming \(d\)-Manhattan preferences can lower the complexity of some computationally hard social choice problems.

References

[1] M. Ballester and G. Haeringer. A characterization of the single-peaked domain. Social Choice and Welfare, 36(2):305–322, 2011. 20
[2] A. Bogomolnaia and J.-F. Laslier. Euclidean preferences. *Journal of Mathematical Economics*, 43(2):87–98, 2007. 1, 2

[3] I. Borg, P. J. Groenen, and P. Mair. *Applied Multidimensional Scaling and Unfolding*. Springer, 2018. 1

[4] R. Bredereck, J. Chen, and G. J. Woeginger. A characterization of the single-crossing domain. *Social Choice and Welfare*, 41(4):989–998, 2013. 20

[5] L. Bulteau and J. Chen. On the border between Euclidian and non-Euclidean preference profiles in 2-dimension, 2020. working paper. 2

[6] J. Chen and S. Grottke. Small one-dimensional euclidean preference profiles. *Social Choice and Welfare*, 57(1):117–144, 2021. 2, 20

[7] J. Chen, K. Pruhs, and G. J. Woeginger. The one-dimensional Euclidean domain: Finely many obstructions are not enough. *Social Choice and Welfare*, 48(2):409–432, 2017. 20

[8] C. H. Coombs. *A Theory of Data*. John Wiley and Sons, 1964. 1

[9] J. Doignon and J. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994. 20

[10] A. Downs. *An Economic Theory of Democracy*. Harper and Row, 1957. 1

[11] D. Eckert and C. Klamler. An equity-efficiency trade-off in a geometric approach to committee selection. *European Journal of Political Economy*, 26(3):386–391, 2010. 2

[12] J. X. Eguia. Foundations of spatial preferences. *Journal of Mathematical Economics*, 47(2):200–205, 2011. 1

[13] E. Elkind and P. Faliszewski. Recognizing 1-Euclidean preferences: An alternative approach. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT '14)*, volume 8768 of *Lecture Notes in Computer Science*, pages 146–157, 2014. 20

[14] J. M. Enelow and M. J. Hinich. *Advances in the Spatial Theory of Voting*. Cambridge University Press, 2008. 1

[15] H. Hotelling. Stability in competition. *Economic Journal*, 39(153):41–57, 1929. 1

[16] V. Knoblauch. Recognizing one-dimensional Euclidean preference profiles. *Journal of Mathematical Economics*, 46(1):1–5, 2010. 20

[17] R. C. Larson and G. Sadiq. Facility locations with the manhattan metric in the presence of barriers to travel. *Operation Research*, 31(4):652–669, 1983. 2

[18] D. Peters. Recognising multidimensional euclidean preferences. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI '17)*, pages 642–648, 2017. 20

[19] K. T. Poole. *Spatial Models of Parliamentary Voting*. Cambridge University Press, 1989. 1

[20] H.-S. Shiha, H.-J. Shyur, and S. Lee. An extension of TOPSIS for group decision making. *Mathematical and Computer Modelling*, 45(7–8):801–813, 2007. 2
[21] D. E. Stokes. Spatial models of party competition. *The American Political Science Review,* 57(2):368–377, 1963.