Modulated pair condensate of $p$-orbital ultracold fermions

Zixu Zhang,1 Hsiang-Hsuan Hung,2 Chiu Man Ho,3,4,5 Erhai Zhao,1 and W. Vincent Liu1

1Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260
2Department of Physics, University of California, San Diego, CA 92093
3Department of Physics, University of California, Berkeley, CA 94720
4Theoretical Physics Group, Lawrence Berkeley National Laboratory, Berkeley, CA 94720
5Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235

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We show that an interesting of pairing occurs for spin-imbalanced Fermi gases under a specific experimental condition—the spin up and spin down Fermi levels lying within the $p_x$ and $s$ orbital bands of an optical lattice, respectively. The pairs condense at a finite momentum equal to the sum of the two Fermi momenta of spin up and spin down fermions and form a $p$-orbital pair condensate. This $2k_F$ momentum dependence has been seen before in the spin- and charge-density waves, but it differs from the usual $p$-wave superfluids such as $^3$He, where the orbital symmetry refers to the relative motion within each pair. Our conclusion is based on the density matrix renormalization group analysis for the one-dimensional (1D) system and mean-field theory for the quasi-1D system. The phase diagram of the quasi-1D system is calculated, showing that the $p$-orbital pair condensate occurs in a wide range of fillings. In the strongly attractive limit, the system realizes an unconventional BEC beyond Feynman’s no-node theorem. The possible experimental signatures of this phase in molecule projection experiment are discussed.

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I. INTRODUCTION

Pairing with mismatched Fermi surfaces has long fascinated researchers in the fields of heavy fermion and organic superconductors, color superconductivity in quark matter [1], and, most recently, ultracold Fermi gases with spin imbalance [2,3]. In a classic two-component model for superconductivity, the mismatch arises from the spin polarization of fermions in the same energy band. Its effect was predicted to produce intriguing, unconventional superfluids such as the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) [4,5] deformed Fermi surface [6,7], and breached pair phases [8,9]. The limiting case of large spin imbalance was also studied to explore the formation of Fermi polarons [10]. In parallel, the behavior of particles in the higher orbital bands of optical lattices, due to large filling factors, thermal excitations or strong interactions, is widely studied for novel orbital orderings of both bosons [11,12] and fermions [13,14] with repulsive interactions. Recently, interband pairing of unpolarized fermions was shown theoretically to give rise to Cooper pair waves [15].

In this article, we report a fermion pairing phase resulting from the interplay of Fermi surface mismatch and $p$-orbital band physics. In such a phase, the pair condensate wave function is spatially modulated and has a $p$-wave symmetry. This phase arises in an attractive two-component Fermi gas on anisotropic optical lattices under a previously unexplored condition of spin imbalance. Namely the majority (↑) spin and the minority (↓) spin occupy up to Fermi levels lying in the $p_x$ and $s$ bands, respectively. We show that pairings take place near the respective Fermi surfaces of the spin ↑ fermions in $p_x$ band and ↓ fermions in $s$ band. This induces a modulated $p$-orbital pair condensate that differs from the usual $p$-wave superfluids such as $^3$He. The state requires only an on-site isotropic contact interaction and the pair is a spin singlet, while the $^3$He $p$-wave superconductivity has to involve anisotropic interaction and spin triplet. The modulation wave vector of the order parameter is $Q ≈ k_{F↑} + k_{F↓}$, where $k_{F↑}$, $k_{F↓}$ are Fermi momenta for spin ↑ and ↓ species, respectively. This $2k_F$ momentum dependence is an unprecedented signature in superfluids other than the spin- and charge-density waves. In the strongly attractive limit, tightly bounded pairs condense at finite momentum $Q$, which realizes an unconventional Bose-Einstein condensate beyond Feynman’s no-node theorem [13,15,19,20].

II. MODEL

The system under consideration is at zero temperature and consists of two-component fermions in a three-dimensional (3D) cubic optical lattice with lattice constant $a$, described by the Hamiltonian

$$H = \sum_\sigma \int d^3x \psi_\sigma^\dagger (x) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \mu_\sigma \right] \psi_\sigma (x) + g \int d^3x \psi_{\uparrow}^\dagger (x) \psi_{\downarrow}^\dagger (x) \psi_{\downarrow} (x) \psi_{\uparrow} (x).$$  \hspace{1cm} (1)

Here $\psi_\sigma (x)$ is the fermionic field operator at $x$ with spin $\sigma = \uparrow, \downarrow$, $V(x)$ is the lattice potential, $\mu_\sigma$ is the chemical potential for spin $\sigma$ fermions, and $g < 0$ is the contact attraction which can be tuned by the Feshbach resonance. In particular, we consider the case where the lattice potential in the $x$ (parallel) direction is much weaker than
the other two (transverse) directions, so the system behaves quasi-one-dimensionally.

We expand \( \psi_r(x) = \sum_n c_n(x-r)c_{n;r}^\dagger \), where \( c_n(x-r) \) is the \( n \)th band Wannier function at lattice site \( r \) with \( c_{n;r} \) the annihilation operator in Wannier basis. As a result, we obtain the usual attractive Hubbard model with nearest-neighbor hopping between \( i \)th site with orbital band \( \alpha \) and \( j \)th site with orbital band \( \beta \)

\[
t_{\alpha\beta} = -\int d^3x \phi^\ast_{\alpha}(x-r_i) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V(x) \right] \phi_{\beta}(x-r_j)
\]

and on-site attraction between orbitals

\[
U_{\alpha\beta\gamma\eta} = g \int d^3x \phi^\ast_{\alpha}(x-r_i) \phi^\ast_{\beta}(x-r_i) \phi_{\gamma}(x-r_i) \phi_{\eta}(x-r_i).
\]

The lowest two energy bands are the \( s \) and \( p_x \) band (the \( p_y \) and \( p_z \) band are much higher in energy because of tighter confinement in the transverse directions). For brevity the \( p_x \) band is simply called \( p \) band in the following. By filling fermions with spin \( \uparrow \) to the \( p \) band and spin \( \downarrow \) to the \( s \) band, the Hamiltonian becomes

\[
H_{sp} = -\sum_{(r,r')} \left( t^\parallel s_{s,r}^\dagger S_{s,r'} - t^\perp p_{p,r}^\dagger P_{p,r'} + h.c. \right) - \mu_s \sum_r n^s_r
\]

\[ -\sum_{(r,r')} \left( t^\parallel s_{s,r}^\dagger S_{s,r'} + t^\perp p_{p,r}^\dagger P_{p,r'} + h.c. \right) - \mu_p \sum_r n^p_r
\]

\[ + \omega_h \sum_r n^p_r + U_{sp} \sum_r n^s_r n^p_r. \]

Here, \( \langle r, r' \rangle \) and \( \langle r, r'' \rangle \) denote the nearest neighboring lattice sites in parallel and transverse directions. \( t^\parallel s \) and \( t^\parallel p \) are the hopping amplitudes along the parallel direction for the \( s \) and \( p \)-band fermions respectively, while \( t^\perp s = t^\perp p \) are the hopping amplitudes in transverse directions. \( S_{s} \) (\( P_{p} \)) is the annihilation operator at lattice site \( r \) for \( s \)-band \( \downarrow \) (\( p \)-band \( \uparrow \)) fermions. \( n^s_r = S^\dagger_s S_r \), \( n^p_r = P^\dagger_p P_r \) are the number operators, and \( \mu_s, \mu_p \) are the corresponding chemical potentials. \( U_{sp} \) is the attractive on-site interaction between \( s \) and \( p \)-band fermions and can be tuned by changing the scattering length using Feshbach resonance. \( \omega_h \) is related to the band gap. In the tight binding region we assume \( \omega_h \gg |U_{sp}| \), and consequently the \( s \)−band fully filled spin \( \uparrow \) fermions are dynamically inert and not included in the \( H_{sp} \).

### III. DMRG CALCULATION FOR 1D CASE

First we consider the pairing problem in the simplest case of 1D (\( t^\perp = 0 \)), which is schematically shown in Fig. 1(a). The two relevant Fermi momenta are \( k_{F\uparrow} \) (for \( s \)-band \( \downarrow \) fermions) and \( k_{F\uparrow} \) (for \( p \)-band \( \uparrow \) fermions). From a weak coupling point of view, to pair fermions of opposite spin near their respective Fermi surfaces, the Cooper pairs have to carry finite center-of-mass momentum (CMM) due to Fermi surface mismatch. Furthermore, in order for all Cooper pairs to have roughly the same CMM, the only choice is to pair fermions of opposite chirality. Note that the dispersion of \( p \) band is inverted with respect to the \( s \) band, so pairing occurs between fermions with momenta of the same sign but opposite group velocities. These elementary considerations show that the CMM of the pair should be approximately the sum of two Fermi momenta,

\[
Q \approx k_{F\uparrow} + k_{F\downarrow}.
\]

This result differs from that of the usual one-dimensional spin imbalanced fermions within the same band, where the FFLO pair momentum is the difference \( Q \approx |k_{F\uparrow} - k_{F\downarrow}| \), as found in a two-leg-ladder system [21].

Mean-field theory and weak coupling consideration can provide only a qualitative picture for 1D problems. To unambiguously identify the nature of the ground state, we use density matrix renormalization group (DMRG) to compute the pair correlation function. In the numerical calculations, we used parameters \( t^\parallel s = 1 \) as the unit of energy, \( t^\perp p = 8 \), \( \mu_s = 1.7 \), \( \mu_p - \omega_h = -11 \), in which the ratio between \( t_s \) and \( t_p \) is chosen according to typical tight-binding bandwidth ratio. \( U_{sp} \) is tunable with Feshbach resonance and in the following calculation we will focus on \( U_{sp} = -9 \) [22]. The truncation error is controlled in the order of \( 10^{-7} \) or less. Equation (5) predicts \( Q \approx k_{F\uparrow} + k_{F\downarrow} = 0.435\pi/a \). Figure 1(b) shows the pairing correlation function in real space \( C_{ij} = \langle S_i^\dagger P_j P_j S_j \rangle \) as a function of \( x = |i-j| \) for a chain of \( N = 60 \) sites with open boundary condition, where the indices \( i \) and \( j \) are real space positions. Since the system only has algebraic order, \( C(x) \) decays with \( x \) according to a power law. On
top of this, however, there is also an obvious oscillation. A curve fit with formula \( C(x) = a \cos(qx + b)/x^n + c \), shown in Fig. 1(b), yields a period of \( q = 0.438\pi/a \), which is in good agreement with the wave number given by Eq. 4 before. The Fourier transform of the pair correlation function

\[
C_q = \frac{1}{N} \sum_{i,j} e^{iq(t-j)} C_{ij}
\]

is peaked at \( q = 0.426\pi/a \) (to be plotted in Sec. VII). These features of the pair correlation function are the signature of the existence of the 2kF CMM pairing in our system. 2426

IV. MEANFIELD ANALYSIS FOR QUASI 1D CASE

Now we move on to the quasi-1D system where a weak transverse hopping \( t^\perp \ll t^\parallel \) is added. We carry out a mean-field analysis of Hamiltonian \( H_{sp} \) by introducing the s-p pairing order parameter

\[
\Delta_T = U_{sp}(S_T P_T),
\]

where \( \langle \cdots \rangle \) means the ground-state expectation value. Two different trial ground states are investigated, the exponential wave \( \Delta_T = e^{iQr} \), which is analogous to the Fulde-Ferrell phase and the cosine wave \( \Delta_T = \cos Q \cdot r \), which is analogous to the Larkin-Ovchinnikov phase. \( Q \) and \( \Delta \) are determined self-consistently by minimization of ground-state free energy \( \langle H_{sp} \rangle \). Transverse hopping introduces a small Fermi surface curvature and spoils the perfect nesting condition as in the pure 1D problem above. However, the curvature is small for weak \( t^\perp \). Thus, we expect \( Q \) pointing almost along the parallel direction, \( Q = Q(1,0,0) \), in order to maximize the phase space of pairing.

The mean-field Hamiltonian for the exponential wave can be diagonalized in momentum space by standard procedure. We get the ground state energy

\[
\langle H_{sp} \rangle = \sum_{k,\gamma = \pm} \Theta(-\lambda_k^{(\gamma)})\lambda_k^{(\gamma)} + \sum_k \xi_k^p - \frac{N^3\Delta^2}{U_{sp}}
\]

and the gap equation

\[
1 = \frac{U_{sp}}{N^3} \sum_k \frac{\Theta(-\xi_k^p + \xi_{Q-k}^p)}{\sqrt{4\Delta^2 + (\xi_k^p + \xi_{Q-k}^p)^2}}
\]

Here, \( k \) is lattice momentum, \( N^3 \) is the total number of sites, \( \Theta \) is a step function, and \( \lambda_k^{(\pm)} = \frac{1}{2}(\xi_k^p - \xi_{Q-k}^p) \pm \sqrt{4\Delta^2 + (\xi_k^p + \xi_{Q-k}^p)^2} \) is the eigenenergy of the Bogoliubov quasiparticles. As evident from these formulas, the pairing occurs between an s-band fermion of momentum \( k \) and a p-band fermion of momentum \( Q-k \) with dispersion \( \xi_k^p = 2t_p^\parallel \cos kt_x - 2t^\perp \cos kt_y - 2t^\perp \cos kt_z - \mu_p + \omega_k \)

and \( \xi_{Q-k}^p = 2t_p^\parallel \cos kt_x - 2t^\perp \cos kt_y - 2t^\perp \cos kt_z - \mu_p + \omega_k \), respectively.

The cosine wave is spatially inhomogeneous. A full mean-field analysis requires solving the Bogoliubov-de Gennes equation to determine the gap profile self-consistently. Here we are interested only in computing the free energy for the ansatz \( \Delta_T = \cos Q \cdot r \) to compare with the exponential wave case. Thus, it is sufficient to numerically diagonalize the full Hamiltonian Eq. 4 for a finite size lattice. We introduce a vector of dimension \( 2N \)

\[
\alpha_{k,k'} = \langle S_{k,l}^{(m_1)} P_{k',l}^{(m_2)} \rangle, \quad \beta_{k,k'} = \langle \bar{S}_{k,l}^{(m_1)} \bar{P}_{k',l}^{(m_2)} \rangle,
\]

where \( k'' = 2\pi n/Na \) is the discrete momentum in the \( x \) direction. The components of \( \alpha \) obey anticommutation relation \( \{\alpha_k^{(m_1)}, \alpha_k^{(m_2)}\} = \delta_{m_1,m_2} \), where \( m_1, m_2 \) labels the corresponding operator component of \( \alpha \). The Hamiltonian takes the compact form

\[
H_{sp} = \sum_{k,k'} \alpha_k^{(m_1)} \bar{H}_{k,k'} \alpha_k^{(m_2)} + \sum_k \xi_k^p - (1 + \delta_{Q,-Q})N^3\Delta^2/2U_{sp}.
\]

Since \( \bar{H}_{k,k'} \) is real and symmetric, it can be diagonalized by an orthogonal transformation \( \alpha_k^{(m_1)} = D_{k,k''} \beta_k^{(m_2)} \) to yield \( 2N \) eigenvalues \( E_{k,k''} \). The new operators \( \beta_k^{(m_2)} \) automatically obey the fermionic anticommutation relationship \( \{\beta_k^{(m_2)}, \beta_k^{(m_2)\dagger}\} = \delta_{m_1,m_2} \). We get the ground state energy,

\[
\langle H_{sp} \rangle = \sum_{k,k'} \sum_{l=1}^{2N} \Theta(-E^l_{k,k''}) + \sum_k \xi_k^p - \frac{N^3\Delta^2}{2U_{sp}}(1 + \delta_{Q,-Q}),
\]

and the gap equation,

\[
\Delta = \frac{2U_{sp}}{N^3(1 + \delta_{Q,-Q})} \sum_k \sum_l D_k^{m_1,l} D_{k,k''}^{m_1,l} \Theta(E_{l,k,k''}).
\]

Here, \( l \) labels the eigenenergy, and \( m_1, m_1' \) labels the matrix elements corresponding to the original \( S, P \) operators in the gap equation.

The parameters used in the mean-field calculations are the same as in the 1D case with small \( t^\perp \)'s added, and we still expect that the order parameter has the momentum around 0.435\pi/a as before. By self-consistently solving for \( Q \) and \( \Delta \), in the case \( t^\perp = 0.05 \), the ground state is the cosine wave phase with \( Q = 0.433\pi/a \) and \( \Delta = 0.822 \). The ground state energy per site is \(-2.5927\), lower than the noninteracting value \(-2.5896\). When \( t^\perp = 0.1 \), the ground state is also the cosine wave phase with \( Q = 0.433\pi/a \) and \( \Delta = 0.542 \). The ground state energy per site is \(-2.5955\), lower than the noninteracting value \(-2.5949\). These results confirm that (i) the cosine wave state has lower energy than the exponential wave state, (ii) the order parameter has the momentum close to the prediction of Eq. 4, and (iii) larger transverse hopping tends to destroy the \( p \)-orbital pair condensate since
the energy gain for larger transverse hopping is much smaller than for smaller transverse hopping.

An interesting feature of the \( p \)-orbital pair condensate in quasi-1D is the possible existence of Fermi surfaces with gapless energy spectrum. We monitor the fermion occupation number, i.e. \( \langle S^\dagger_k S_k \rangle \) and \( \langle P^\dagger_k P_k \rangle \) for increasing transverse hopping. The results are shown in Fig. 2. For small \( t^\perp \), they take the usual BCS form and vary smoothly from 1 (red) to 0 (blue) across the bare Fermi surface (with interaction turned off), as shown in Figs. 2(a) and 2(c) for \( t^\perp = 0.05 \). For larger transverse hopping, sharp Fermi surfaces characterized by a sudden jump in \( \langle S^\dagger_k S_k \rangle \) and \( \langle P^\dagger_k P_k \rangle \) appear. This is clearly shown in Figs. 2(b) and 2(d) for \( t^\perp = 0.1 \) as the occupation number changes discontinuously from 1 (red) to 0 (blue). It can be understood qualitatively as follows. As \( t^\perp \) increases, the original Fermi surfaces acquire a larger curvature in the transverse directions and the pairing condition in Eq. (4) cannot be satisfied everywhere anymore. Therefore in some regions fermions are not paired and Fermi surfaces survive. One should also note that the calculation is based on the assumption that \( t^\perp \ll t^\parallel \), which predicts that \( Q \) is in the parallel direction. This prediction should fail as \( t^\perp \) increases beyond certain critical values.

V. PHASE DIAGRAM

Now, we systematically explore the phases of our system for general band filling and spin imbalance. Since we have \( s \) - and \( p \) - bands with different bandwidths, we introduce two dimensionless quantities for the chemical potentials \( \mu_s \) and \( \mu_p \)

\[
\tilde{\mu}_s = \frac{\mu_s}{2t_s} = \frac{\mu_s}{2}, \\
\tilde{\mu}_p = \frac{\mu_p - \omega_b}{2t_p} = \frac{\mu_p - \omega_b}{16}.
\]

Thus, for a non-interacting system, \(-1 < \tilde{\mu}_s, \tilde{\mu}_p < 1\) control the filling for the \( s \) and \( p \)-band fermions respectively. We then define the quantities

\[
\mu = \frac{\tilde{\mu}_s + \tilde{\mu}_p}{2}, \\
h = \frac{\tilde{\mu}_s - \tilde{\mu}_p}{2},
\]

as the parameters controlling the average filling and polarization in the phase diagram. The phase at \(-\mu, -h\) is the same as the state at \( \mu, h \), since the transformation \( \mu, h \rightarrow -\mu, -h \) gives \( \mu_s, \mu_p \rightarrow -\mu_s, -\mu_p \), and the mean-field Hamiltonian with \( \mu_s, \mu_p \) is identical to Hamiltonian with \(-\mu_s, -\mu_p \) via a particle-hole transformation up to a constant.

We have four possible phases in such a system as shown in Fig. 3. As before, we ignored the inert fully filled \( s \) band of spin \( \uparrow \) fermions. We consider the \( p \) band of spin \( \uparrow \) fermions and \( s \) band of spin \( \downarrow \) fermions. When one of these two bands is empty and the other is filled, the pairing does not happen and we call it normal phase I (N1) as in Fig. 3(a). When one of these two bands is fully filled and the other is partially filled, the pairing also does not happen since the fully filled band is inert. We call it normal phase II (N2) as in Fig. 3(b). When both of them are partially filled, fermions near Fermi surfaces from the two bands will be paired and the system is in superfluid phases as shown in Figs. 3(c) and 3(d). In the superfluid regime, when \( h \) is small, the pairing momentum prefers \( Q = \pi/a \) and we call it commensurate \( p \)-orbital pair condensate (CpPC). It is a special case of the \( p \)-orbital pair
condensate, where the occupation numbers of $s$-band spin down fermions and $p$-band spin up fermions are the same. It is similar to the conventional unpolarized pairing (BCS), where the spin up fermions and spin down fermions have the same population. However, in BCS pairing the CMM of the pair has the property $Q = 0$, while here $Q = \pi/a$. To understand the momentum $\pi/a$ preference, note that in conventional BCS case, the two species of fermions have the same energy spectrum and the pairing is between two fermions with opposite momenta, which leads to the CMM of pair $Q = 0$. Here, the structure of energy spectrum of $p$ band is different from $s$ band. The equal occupation numbers mean $k_{F\uparrow} = \pi/a - k_{F\downarrow}$, which gives rise to $Q = k_{F\uparrow} + k_{F\downarrow} = \pi/a$, as shown in Fig. 3(c). At last, when $h$ is large, the pairing momentum stays at a general $Q \approx k_{F\uparrow} + k_{F\downarrow}$ and the occupation number for the two species of fermions differ. We call it incommensurate $p$-orbital pair condensate (IpPC) as shown in Fig. 3(d).

To determine the phases, we minimize the free energy as a function of the pairing amplitude $\Delta$ and pairing momentum $Q$ by mean-field analysis using the cosine wave function as outlined in the previous section. When the minimum is realized at $\Delta = 0$, it is normal phase. When $\Delta$ is finite, there are two possibilities. When $Q = \pi/a$, it is CpPC. When $Q \neq \pi/a$, it is IpPC. For the transition between superfluid and normal phase, and the transition between CpPC and IpPC, the behaviors of free energy show that the phase transitions are first order in a lattice system. Between the superfluid and normal phases, near the phase transition, $\Delta$ changes suddenly from 0 to finite, and the free energy shows two local minima at $\Delta = 0$ and $\Delta \neq 0$. Between CpPC and IpPC, the pairing momentum changes from $Q = \pi/a$ to $Q \neq \pi/a$ discontinuously, and the free energy as a function of $Q$ also has two local minima at $Q = \pi/a$ and $Q \neq \pi/a$. Thus, they are first-order phase transitions according to our mean field analysis. Therefore, we can determine the phase boundaries by normal phase and superfluid phase by monitoring $\Delta$ changing from zero to finite. We can also monitor $Q$ changing from $Q = \pi/a$ to $Q \neq \pi/a$ to determine the phase boundaries between CpPC and IpPC.

In Fig. 4 we present a phase diagram for $t^+ = 0.05$. The x’s in Fig. 4 show the data points for the phase boundary obtained from the numerical procedure, and by connecting them we get the phase boundaries. An illustrative physical understanding about this phase diagram is as follows. In Fig. 4 when chemical potential difference $h$ is small and the two bands are still partially filled to ensure the pairing, the system tends to stay in CpPC where $Q = \pi/a$. It is similar to the conventional BCS superfluid case. As $h$ becomes larger, as long as the average filling $\mu$ is not too large or small and the two bands are still both partially filled, the pairing persists despite the spin imbalance and the system is in IpPC. If $\mu$ gets more and more negative, the average filling becomes smaller and smaller, and at certain $\mu, h$, $p$ band of spin up fermions will be empty and the system will become N1 without pairing. Similarly, when $\mu$ is large and positive, the average filling is very high and at certain $\mu, h$, the $s$ band of spin down fermions will be fully occupied, and the system becomes N2 without pairing. The almost straight phase boundaries in Fig. 4 between IpPC and normal phases indicate that these phase transitions are due to the change of band occupation as empty $\leftrightarrow$ partially filled $\leftrightarrow$ fully filled. In Fig. 4 the phase boundary between IpPC and N1 corresponds to the critical condition that the $s$ band of spin down fermions is partially filled while the $p$ band of spin up fermion becomes empty, and the almost straight phase boundary corresponds to the condition that $\tilde{\mu} = \mu - h = -1$ (but, as before, this is only an approximate argument due to the presence of interaction). Similarly, the almost straight phase boundary between IpPC and N2 corresponds to the condition that the $s$ band of spin down fermions becomes fully filled, while the $p$ band of spin up fermion is partially filled, or $\tilde{\mu} = \mu + h = 1$. All the phase transition lines in Fig. 4 are mean field results, and these straight lines are expected to be corrected by quantum critical fluctuations. The phase diagram shows that the $p$-orbital pair condensate happens in large parameter regimes and is closely related to the band and orbital properties in the optical lattice systems.

VI. SIGNATURE OF THE $P$-ORBITAL PAIR CONDENSATE IN MOLECULE PROJECTION EXPERIMENT

The $p$-orbital pair condensate phase can inspire important experimental signatures for finite momentum condensation of bosonic molecules in higher orbital bands.
sites obtained by DMRG. The peak is located at 0 in both figures, which corresponds to the value $k_{F}$ of 60 sites obtained by DMRG. The peak is located at 0.433$\pi/a$ in both figures, which corresponds to the value $k_{F} = (N_{+} + N_{-} - N_{p})\pi/Na$ for $N_{+} = 49$ and $N_{p} = 15$.

By fast sweeping the magnetic field (and thus the interaction) from the BCS region to the deep BEC region across a Feshbach resonance, the BCS pairs are projected onto Feshbach molecules, which can be further probed for example by time-of-flight images. The bosons produced effectively reside in $p$ band and are stable, since by Pauli blocking the filled $s$-band fermions will prevent the the $p$-wave bosons from decaying. Here, we use a simple model to evaluate the momentum distribution of molecules after projection

$$n_{n} = \sum_{k,k'} f_{k}^{*} f_{k'}^{*} \langle \hat{c}_{k+q/2}^{\dagger} \hat{c}_{-k+q/2} \rangle P_{k+q/2} S_{k+q/2},$$

where $f_{k}$ is the molecular wave function, and the correlation function can be evaluated within mean field theory. For fast sweeps, the molecular size is small compared to lattice constant and its wave function can be approximated by a delta function in real space (a constant $\sqrt{1/N}$ in momentum space). By this assumption, $n_{q}$ is the same quantity as $C_{q}$ in Eq. (6). Figure (a) shows the $n_{q}$ of $p$-wave Feshbach molecules and a peak is located at 0.433$\pi/a$. Figure (b) shows $C_{q}$ from Eq. (6), based on the DMRG results shown in Fig. (b). The time-of-flight experiment is predicted to distribute peaks corresponding to that in Fig. (a). Note that for the 1D problem (Fig. (b)), the delta-function peak is replaced by a cusp characteristic of power law due to the lack of long range order.

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