EXPONENTIAL CONDITION NUMBER OF SOLUTIONS OF THE DISCRETE LYAPUNOV EQUATION

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Abstract—The condition number of the $n \times n$ matrix $P$ is examined, where $P$ solves $P - APA^* = BB^*$, and $B$ is a $n \times d$ matrix. Lower bounds on the condition number, $\kappa$, of $P$ are given when $A$ is normal, a single Jordan block or in Frobenius form. The bounds show that the ill-conditioning of $P$ grows as $\exp(n/d) >> 1$. These bounds are related to the condition number of the transformation that takes $A$ to input normal form. A simulation shows that $P$ is typically ill-conditioned in the case of $n >> 1$ and $d = 1$. When $A_{ij}$ has an independent Gaussian distribution (subject to restrictions), we observe that $\kappa(P)^{1/n} \sim 3.3$. The effect of autocorrelated forcing on the conditioning on state space systems is examined.

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Key words. Condition number, discrete Lyapunov equation, input normal, orthonormal filters, balanced systems, system identification.

I. INTRODUCTION

In system identification, one needs to solve linear algebraic systems: $PC = f$, where $c$ and $f$ are $n$-vectors and $P$ is the controllability Grammian, i.e. $P$ is the $n \times n$ positive definite matrix that solves

$$P - APA^* = BB^*.$$  

Equation (I.1) is known as the discrete Lyapunov equation and is more properly called Stein’s equation. In (I.1), the $n \times n$ matrix $A$ and the $n \times d$ matrix $B$ are given. The matrix $A$ is known as the state advance matrix and the matrix $B$ is known as the input matrix. Together, $(A, B)$ is known as an input pair. We assume that $A$ is stable and that $(A, B)$ is controllable. In this case, there is an unique selfadjoint solution of (I.1) and it is positive definite [18]. We denote the solution of (I.1) as a function of $A$ and $B$ by $P(A, B)$.

We study the condition number of $P(A, B)$, $\kappa(P) \equiv \kappa(P(A, B)) \equiv \sigma_1(P(A, B))/\sigma_n(P(A, B))$, where $\sigma_1(P)$ is the largest singular value of $P$ and $\sigma_n(P)$ is the smallest. We consider cases where the system input dimension, $d$, is smaller than the state space dimension, $n$. In this case, we claim that the condition number of $P$, $\kappa(P)$ can be exponentially large in

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Since the case \( n >> d \) is common in signal processing and system identification, our results put strong limitations on the applicability of high order arbitrary state space realizations.

A number of bounds on either \( \sigma_1(P(A, B)) \) or \( \sigma_n(P(A, B)) \) exist in the literature [20], [15], [16], [10], [19], [33]. Many of these bounds require that \( \det(BB^*) > 0 \) to be nontrivial. Theorem 2.2 of [33] can be used to bound the ratio of \( \sigma_1(P)/\sigma_n(P) \). (See also [32].) The existing bounds on \( \sigma_i(P(A, B)) \) generally make no assumptions on \( (A, B) \) and therefore tend to be weak or hard to evaluate. If \( A \) is real, symmetric, and stable, Penzl [30] gives a bound which we describe in Section V. For the continuous time case, interesting results on the condition number may be found in [2].

Our lower bounds on \( \kappa(P(A, B)) \) are for specific, commonly considered classes of input pairs, \( (A, B) \), such as companion matrices and normal matrices and when \( A \) is a single Jordan block.

Our results are based on transforming the input pair, \( (A, B) \), into input normal (IN) form. Input normal form implies that the identity matrix solves the discrete Lyapunov equation. Input normal pairs have special representations that allow for fast matrix-vector operations when \( A \) is a Hessenberg matrix [22], [23], [31].

In [27], a numerical simulation shows that input normal filters perform well in the presence of autocorrelated noise. We examine the condition number of the controllability Grammian when forcing term is autocorrelated. We derive a bound that explains the good performance of IN filters [27]. Other advantages of IN filters are described in [31], [28], [26].

The condition number of \( P \) is related to two other well-known problems: a) the distance of an input pair to the set of uncontrollable pairs [8] and b) the sensitivity of the \( P \) to perturbations in \( (A, B) \) [9], [10]. It is well known that \( 1/\kappa(P) = \min\{\|E\|_2/\|P\|_2 : (P + E) \text{ is singular}\} \) [11]. Thus we can lower bound the distance to uncontrollability by the \( 1/\kappa(P) \) times the sensitivity of the discrete Lyapunov equation. Our results indicate that \( 1/\kappa(P) \) is typically exponentially small in \( n/d \).

We present numerical simulations which compute the distribution of \( \kappa(P(A, B)) \) for several classes of input pairs, \( (A, B) \). When the elements of \( \sqrt{n + 2A, B} \) are independently distributed as Gaussians with unit variance, our simulation shows that the ensemble average of \( \kappa(P)^{1/n} \) tends to a constant for \( d = 1 \). We observe that \( \log(\log(\kappa(P))) \) is approximately Gaussian. These numerical results indicate that the ill-conditioning problems of \( \kappa(P) \) are probably generic when \( n/d << 1 \). To accurately solve (I.1), we use a novel QR iteration to precondition (I.1) and then apply a square root version of the doubling method [3].

Section II presents our computation of \( \kappa(P(A, B)) \) for an ensemble of stable controllable input pairs. Section III defines IN form and present new results on the properties of IN pairs. Section IV gives lower bounds on the condition number based on the transformation to an IN matrix and applies the bound to the case when \( A \) is normal. Section V gives abstract bounds based on the ADI iteration. Section VI gives lower bounds when \( A \) is in companion form. Sections VII and IX give additional bounds for normal \( A \). Section VIII gives...
bounds when $A$ is a single Jordan block. Section X examines condition numbers of the state covariance when the system is forced with colored noise.

**Notation:** Here $A$ is a $n \times n$ matrix with eigenvalues $\{\lambda_i\}$ ordered as $1 \geq |\lambda_1| \geq |\lambda_2| \ldots \geq |\lambda_n|$ and singular values, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. Depending on context, $\Lambda$ is the $n$-vector of $\lambda_i$ or the corresponding diagonal matrix. The matrix $B$ has dimension $n \times d$. When $A$ is stable and $(A, B)$ is controllable, we say that the input pair $(A, B)$ is CS. If $A$ is also invertible, we say $(A, B)$ is CIS. For us, ‘stable” means $|\lambda_1| < 1$, sometimes known as strict exponential stability. We let $D(A, B)$ denote the set of stable, controllable $(A, B)$ input pairs of dimension $n \times n$ and $n \times d$. The $n \times n$ identity matrix is denoted $I_n$. The transformation matrices, denoted by $T$ and $U$, have dimension $r \times n$ and rank $r$, where $r \leq n$. The Moore-Penrose inverse of $T$ is denoted by $T^+$. Here $\| \cdot \|_2$ is the Frobenius norm while $\| \cdot \|$ is any unitarily invariant matrix norm.

II. GENERIC CONDITION NUMBER

We begin by examining the probability distribution of condition number of $P(A, B)$ as $A$ and $B$ are varied over a probability distribution, $\nu(A, B)$, on stable, controllable input pairs. We limit ourselves to single input pairs ($d = 1$).

A common class of random matrices is $\{A| A_{ij} \sim N(0, 1)/\sqrt{n}\}$, with the probability measure $\mu(A)$. The Girko law [6] states that the eigenvalues of such random $A$ are uniformly distributed on the complex disk $|\lambda_i| < 1$ as $n \rightarrow \infty$. For finite $n$, the distribution of eigenvalues is given in Theorem 6.2 of [6]. We exclude unstable $A$ and uncontrollable $(A, B)$ in our studies. We normalize $A$ by $1/\sqrt{n+2}$ instead of $1/\sqrt{n}$ to improve the odds of obtaining stable $A$. Specifically, we define the distribution:

**Definition 2.1:** Let $\nu(A, B)$ be the probability measure induced on $D(A, B)$ by letting the matrix elements $A_{ij} \sqrt{(n + 2)}$ and $B_{ij}$ have independent Gaussian distributions, $N(0, 1)$, subject to the CS restriction.

Each probability distribution on $(A, B)$ induces a distribution of $P(A, B)$ and $\kappa(P(A, B))$. We simulate the induced distribution by solving the discrete Lyapunov equation for 2,500 $(A, B)$ pairs chosen from the distribution $\nu$.

Inaccurate numerics will tend to underestimate $\kappa(P)$. Even for $n \approx 10$, these systems can be so ill-conditioned that existing numerical methods inaccurately determine the condition number. Therefore, we developed new numerical algorithms for the solution of (I.1) [25]. To solve the discrete Lyapunov equation, we use a novel square root version of the doubling method. For ill-conditioned problems, we find that preconditioning the discrete Lyapunov equation is important to accurately evaluate the condition number of $P$ [25].

Table 1 gives the quantile distribution of $\log(\kappa(P))$ as a function of $n$ for our numerical simulation with $(A, B)$ distributed in $\nu(A, B)$. The median condition number scales as $\log(\kappa(P)) \approx 1.2n$. The interquartile distance is approximately independent of $n$ with a value of $\approx 4.4$. (The interquartile distance is the distance between the 75th percentile and the 25th percentile and is a measure of the width of the distribution.) If the distribution were normal, the interquartile distance would be...
roughly 1.35 standard deviations. We plotted the quantiles of log(log(κ(P))) and of log(κ(P)) versus the quantiles of the Gaussian distribution. These quantile-quantile plots show that log(log(κ(P))) has an approximately Gaussian distribution and that log(κ(P)) has wide tails. Naturally, the tails of the empirical distribution are more poorly determined than the median and the quartiles.

Table 1: Quantile distribution of log(κ(P)) for (A, B) distributed in ν(A, B) as a function of n.

| n  | 1%  | 10% | 25% | 50% | 75% | 90% | 99% |
|----|-----|-----|-----|-----|-----|-----|-----|
| 8  | 6.23| 8.22| 9.62| 11.3| 13.4| 15.6| 20.2|
| 16 | 14.2| 16.5| 18.2| 20.3| 22.5| 24.6| 28.7|
| 24 | 22.3| 25  | 26.7| 28.7| 31.1| 33.6| 38  |

In [7], it is shown that a random matrix, A, has E[log(κ(A))] \sim \log(n), where E denotes the expected value. Thus P(A, B) is typically much more poorly conditioned than A is. In [34], it is shown that a random lower triangular matrix, L, has κ(L)^{1/n} \approx 2 with probability tending to 1 as n \to \infty. For the median value of κ(P) in our computation, the Cholesky factor of P, L scales as κ(L)^{1/n} \approx 1.8, which is nearly as badly conditioned as those in [34].

Table 1 displays results for d = 1. Empirically, we observe that the condition number grows at least as fast as n/d. In Section IV, we derive lower bound for the condition number when A is normal. We apply this bound to each matrix in our simulation. Table 3 shows that the actual condition number is worse than the normal bound by a factor of roughly 100 on average.

III. INPUT NORMAL PAIRS

In examining the condition number of solutions of the discrete Lyapunov equation, it is natural to begin with input pairs that admit solutions with condition number one.

Definition 3.1: A input pair, (\tilde{A}, \tilde{B}), is input normal (IN) of grade d if and only if \tilde{A} is stable, rank(\tilde{B}) = \text{column dim}(\tilde{B}) = d, and

\[ \tilde{A}\tilde{A}^* = I - \tilde{B}\tilde{B}^* . \]

A matrix, \tilde{A}, is a IN matrix of grade d if and only if there exists a n \times d-matrix \tilde{B} such that (\tilde{A}, \tilde{B}), is an IN pair. If \tilde{A} is lower (upper) triangular as well, (\tilde{A}, \tilde{B}) is a triangular input normal pair. If \tilde{A} is Hessenberg as well, (\tilde{A}, \tilde{B}) is a Hessenberg input normal pair.

In [31], ‘input normal pairs” are called orthogonal filters. In [21], ‘input normal” has a more restrictive definition of (III.1) and the additional requirement that the observability Grammian is diagonal. In our definition of ‘input normal”, we do not impose any such condition on the observability Grammian. We choose this language so that ‘normal” denotes restrictions on only one Grammian while “balanced” denotes simultaneous restrictions on both Grammians [17], [21]. This usage is consistent with the definitions of [5]. Input normal A are generally not normal matrices.

By Theorem 2.1 of [1], if the controllability Grammian is positive definite, then the input pair is stable. In [29], Ober shows that stability plus a positive definite solution to the discrete Lyapunov equation, (I.1), implies that the input pair is controllable. Thus for IN pairs, stability is equivalent to controllability. We now show that any CS input pair may be transformed to an IN pair.
Theorem 3.2: [31] Every stable, controllable input pair \((A, B)\), is similar to a input normal pair \((A \equiv TAT^{-1}, B \equiv TB)\) with \(\|B\|^2 \leq 1\).

Proof: The unique solution of (I.1), \(P\), is strictly positive definite [18]. Let \(L\) be the unique Cholesky lower triangular factor of \(P\) with positive diagonal entries, \(P = LL^*\). We set \(T = L^{-1}, A = L^{-1}AL\), and \(B = L^{-1}B\). ■

Using the singular value decomposition, we have the following characterization of IN matrices:

Theorem 3.3: Let \(A\) be a stable \(n \times n\) matrix with \(\sigma_1(A) = 1\), and let \(d\) equal the number of singular values of \(A\) less than 1, \((d = \# \{k | \sigma_k(A) < 1\})\). There is an \(n \times d\) matrix \(B\) with \(\text{rank}(B) = d\) such that \(I - AA^* = BB^*\) and therefore \(A\) is an IN matrix. The smallest \(d\) singular values of \(A\) satisfy \(\prod_{j=n-d+1}^n \sigma_j^2(A) = \prod_{i=1}^d |\lambda_i|^2\), where \(\lambda_i\) are the eigenvalues of \(A\).

Proof: Let \(v_k\) be the \(k\)th singular vector of \(I - AA^*\) and define \(B = \frac{(\sqrt{1 - \sigma_n(AA^*)}v_n, \ldots, \sqrt{1 - \sigma_{n-d+1}(AA^*)}v_{n-d+1})}{|\text{det}(AA^*)|}. \)

This constructs \(B\). The singular value identity follows from \(\prod_{j=n-d+1}^n \sigma_j^2 = \prod_{i=1}^d |\lambda_i|^2\).

For input normal pairs, this yields the bound: \(\sigma_n^2(A) \leq \left( \prod_{j=n-d+1}^n \sigma_j^2(A) \right)^{1/d} = \left( \prod_{i=1}^n |\lambda_i|^2 \right)^{1/d}. \)

There are many similar input normal pairs since if \((A, B)\) is IN, then so is \((UAU^*, UB)\) for any orthogonal \(U\). This additional freedom may be used to simplify the input pair representation [31], [23], [24].

IV. CONDITION NUMBER BOUNDS AND THE TRANSFORMATION TO INPUT NORMAL PAIRS

In this section, we derive lower bounds on the condition number of \(P(A, B)\). Our bounds are based on transforming \((A, B)\) to an IN pair \((A', B')\). The following lemma describes the transformation of solutions under a linear change of coordinates.

Lemma 4.1: Let \(T\) be an \(r \times n\) matrix of rank-\(r\) with \(r \leq n\) and let the rows of \(T\) be a basis for a left-invariant subspace of \(A\). Define \(A'\) by \(TA = A'T\). Let \(\| \cdot \|\) be an unitarily invariant matrix norm and let \(\phi\) be an analytic function on the spectrum of \(A\) with \(\|\phi(A)\| > 0\). Then \(\kappa(T) \equiv \sigma_1(T)\sigma_1(T^+) \geq \|\phi(A')\|/\|\phi(A)\|\). When \(A\) is invertible \(\kappa(T) \geq \|A'^{-1}\|/\|A^{-1}\|\).

Also \(\|T\|\|T^+\| \geq \kappa(T)\|e_1e_1^*\|\), where \(e_1\) is the unit vector in the first coordinate.

Proof: Note \(\phi(A') = T\phi(A)T^+\) since \(TT^+ = I_r\). We apply the bound \(\|GH\| \leq \sigma_1(F)\sigma_1(H)\|G\|\) [13, p. 211] to \(\phi(A') = T\phi(A)T^+.\) When \(A\) is invertible, so is \(A'\) and \(A'^{-1} = TA^{-1}T^+.\) To bound \(\|T\|\|T^+\|\), we use the bound \(\|T\| > \sigma_1(T)\|e_1e_1^*\|\) [13, p. 206]. ■

A related result in [13, p. 162] is

\[\kappa(T) \geq \max\{\sigma_k(A)/\sigma_k(A'), \sigma_k(A')/\sigma_k(A)\}\],

(IV.1)

for invertible \(T\) and nonvanishing \(\sigma_k(A')\) and \(\sigma_k(A)\).

When \(r = n\), \(T\) is invertible and \(A'\) is similar to \(A: A' = TAT^{-1}\). In this case \((r = n)\), we can reverse the roles of \(A\) and \(A'\) in the bounds as well. The case \(r < n\) is of interest in model reduction problems, where one projects a system onto a left invariant subspace of \(A\).

In the remainder of this section, we use \(\phi(A) = A\) and \(\phi(A) = A^{-1}\). When
$A' = T AT^{-1}$ is input normal, we have the following bound for the condition number of the transformation of a stable matrix $A$ to input normal form.

**Theorem 4.2:** Let $A$ be stable and invertible and $A' \equiv TAT^{-1}$ be an input normal matrix of grade $d$, where $T$ is an invertible and $d < n$, then

$$\kappa(T) \geq \max \left\{ \frac{1}{\sigma_1(A)}, \frac{\sigma_n(A)}{\prod_{i=1}^{n} |\lambda_i(A)|} \right\} \tag{IV.2}$$

where $\{\lambda_i(A)\}$ are the eigenvalues of $A$ and $\sigma_1(A)$ and $\sigma_n(A)$ are the largest and smallest singular values of $A$. For $d = 1$, $\sigma_n(A') = \prod_{i=1}^{n} |\lambda_i(A)|$.

**Proof:** By Theorem 3.3, $\sigma_1(A') = 1$ and $\sigma_n(A') \leq \prod_{j=n-d+1}^{n} \sigma_j^{1/d}(A') = |\det(A'A^*)|^{1/2d} = |\det(A)|^{1/d} = \prod_{i=1}^{n} |\lambda_i(A)|^{1/d}$. □

Note that Theorem 4.2 does not depend on any specific input matrix $B$.

**Corollary 4.3:** Let $(A, B)$ be a CIS input pair, then the condition number of $P(A, B)$ satisfies the equality $\kappa(P(A, B)) = \kappa(T)^2$, where $\kappa(T)$ and $A'$ are defined in Theorem 4.2.

**Proof:** The unique solution of (I.1), $P$, is strictly positive definite [18]. Let $L$ be the Cholesky factor of $P(A, B)$; $LL^* = P(A, B)$, and set $T = L^{-1}$. Note $\kappa(P(A, B)) = \kappa(T^{-1}T^*) = \kappa(T)^2$. □

For normal advance matrices, $\sigma_n(A) = |\lambda_n(A)|$, the smallest eigenvalue of $A$. This simplifies Corollary 4.3.

**Theorem 4.4:** Let $A$ be a normal matrix and $(A, B)$ be a CIS input pair, then the condition number of $P(A, B)$ satisfies the bound

$$\kappa(P(A, B)) \geq \max \left\{ \frac{\lambda_n(A)^2}{\prod_{i=1}^{n} |\lambda_i(A)|^{2/d}}, \frac{\sigma_n(A)^2}{\lambda_1(A)^2} \right\} \tag{IV.3}$$

where $\lambda_n(A)$ is the eigenvalue of $A$ with the smallest magnitude and $A'$ is the IN matrix generated in the map defined in the proof of Corollary 4.3. For $d = 1$, the lower bound simplifies to $\kappa(P) \geq 1/\prod_{i=1}^{n-1} |\lambda_i(A)|^2$.

We compare this bound to the condition number of $P(A, B)$ for an ensemble of input pairs where $A$ is a normal matrix; i.e. $A$ has orthonormal eigenvectors. We need to select a distribution on the set of eigenvalue $n$-tuples. A natural choice is the distribution $\nu_A(A)$ induced by the random distribution of $A$ given in Definition 2.1.

**Definition 4.5:** $D_{\mathcal{N}}(A, B)$ is the set of CS input pairs, $(\Lambda, B)$, where $\Lambda$ is diagonal.

Let $\nu_A(\Lambda, B)$ be the probability measure induced from eigenvalue n-tuple distribution, $\nu_{\lambda, n}(A)$ of $\nu_n(A, B)$ of Definition 2.1 and let $B_{ij}$ have the Gaussian distribution $\mathcal{N}(0, 1)$ subject to the controllability restriction.

| n   | 1%   | 10%  | 25%  | 50%  | 75%  | 90%  | 99%  |
|-----|------|------|------|------|------|------|------|
| 8   | 7.27 | 9.25 | 10.8 | 12.4 | 14.6 | 16.7 | 21.7 |
| 16  | 15.0 | 17.7 | 19.4 | 21.3 | 23.6 | 26.1 | 31.6 |
| 24  | 23.2 | 25.9 | 27.7 | 29.8 | 32.1 | 34.5 | 39.3 |

Table 2: Quantiles of $\log(\kappa)$ as function of $n$ for $d = 1$. Note that $\log(\log(\kappa(P)))$ has an approximately Gaussian distribution

As seen in Table 2, our numerical computations show that the distribution of $\kappa(P)$ for the normal matrices $D_{\mathcal{N}}(A, B)$ is virtually identical to that of our general random matrices $D_{\mathcal{N}}(A, B)$. Again, $\kappa(P)^{1/n}$ is approximately constant with median condition number scaling as $\kappa(P)^{1/n} \approx 3.4$. The interquartile distance is again nearly independent of $n$ with a value of $\approx 4.4$. 

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Table 3: Quantiles of $\log(\kappa/\kappa_{bd})$ as function of $n$. Here $\kappa_{bd} = 1/\prod_{i=1}^{n-1} |\lambda_i|^2$ is the bound given in Theorem IV.3, evaluated for each input pair.

Table 3 compares $\log(\kappa)$ versus our theoretical bound. The discrepancy is growing only slightly in $n$, in contrast to $\log(\kappa)$ which is growing linearly in $n$. A regression indicates that the median value of $\kappa/\kappa_{bd}$ is growing as $n^\alpha$ with $1 \leq \alpha \leq 2$. Plotting the quantiles of $\log(\kappa/\kappa_{bd})$ as a function of $\log(\kappa_{bd})$ shows that the residual error is a weakly decreasing function of $\log(\kappa_{bd})$. We also observe that the spread of $\log(\kappa/\kappa_{bd})$ is almost independent of $\log(\kappa_{bd})$, perhaps indicating a heuristic model: $\log(\kappa) \sim \log(\kappa_{bd}) + f(n) + X_n$, where the random variable $X_n$ barely depends on $n$. To model the long tails of $\log(\kappa)$, an analogous model for $\log(\log(\kappa))$ is probably called for. We have also compared the normal bound with the log-condition number for the ensemble of random matrices in Section II. Surprisingly, the agreement with the bound is even better in this case. However, there are many cases where the condition number of a random input pair is smaller than the bound for normal matrices predicts.

The bound (IV.3) indicates that $P$ can be quite ill-conditioned. Theorems 4.2 -4.4 do not use any property of $B$ (except controllability) nor of the complex phases of the eigenvalues, $\lambda_i$. Including this information in the bounds can only sharpen the lower bound. We believe that a significant fraction of the ill-conditioning that is not explained by our bound arises from using a random input pair is smaller than the bound given in Theorem IV.3. However, there are many cases where the condition number of a random input pair is smaller than the bound for normal matrices predicts.

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Theorem 5.1: In this section we present condition number bounds based on the alternating direction implicit (ADI) iteration for the solution of the continuous time Lyapunov solution. These results were formulated by T. Penzl in [30]. We restate his results in a more general context.

The results for the discrete Lyapunov equation follow by applying the bilinear transform. We define $f(A,\tau) \equiv (A+\tau I_n)^{-1}(A-\tau I_n)$. The Cayley transform corresponds to $\tau = 1$: $\hat{A} = f(A,1)$ and $\hat{B} = \sqrt{2}(I_n + A)^{-1}B$. The solution $P(A,B)$ of the discrete Lyapunov equation (I.1) for $(A,B)$ satisfies the Lyapunov equation

$$\hat{A} P + P \hat{A}^* = -\hat{B} \hat{B}^*.$$  \hspace{1cm} (V.1)

Following [30], we define the shifted ADI iteration on (V.1). To approximately solve (V.1), we let $P^{(0)} = 0$ and define $P^{(k)}$ by

$$P^{(k)} = f(\hat{A},\tau_k)P^{(k-1)}f(\hat{A},\tau_k)^* - 2 Re(\tau_k)(\hat{A} + \tau_k I_n)\hat{B} \hat{B}^*.$$ \hspace{1cm} (V.2)

where the $\tau_k$ are the shift parameters. Using the methodology of [30], we have the following bound:

**Theorem 5.1:** In the ADI iteration of (V.2), let $kd < n$. The $P^{(k)}$ has rank $kd$ and satisfies the approximation bound:

$$\frac{\lambda_{kd+1}(P)}{\lambda_1(P)} \leq \frac{\|P - P^{(k)}\|_2}{\|P\|_2} \leq \|F(\hat{A}; \tau_1 \ldots \tau_k)\|_2,$$

\hspace{1cm} (V.3)
where \( F(t; \tau_1, \ldots, \tau_k) \equiv \prod_{j=1}^{k} f(t, \tau_j) \). Let \( \hat{A} \) have a complete set of eigenvectors and the eigenvalue decomposition \( \hat{A} = T \Lambda T^{-1} \), then
\[
\|F(\hat{A}; \tau_1 \ldots \tau_k)\|_2^2 \leq \kappa(T)^2 \max_{\lambda \in \text{spec}(\hat{A})} |F(\lambda, \tau_1, \ldots, \tau_k)|^2 .
\] (V.4)

We define \( F_k \equiv \min_{\tau_1 \ldots \tau_k} \max_{\lambda \in \text{spec}(\hat{A})} |F(\lambda, \tau_1, \ldots, \tau_k)|^2 \). Thus \( F_k \) is the best bound of the type in (V.3) The difficulty in using Theorem 5.1 is finding good shifts that come close to approximating \( F_k \). There are algorithms for selecting shifts, but only rarely have explicit upper bounds on \( F_k \) been given. Penzl simplified this bound for the case of real, symmetric, stable \( A \):

**Theorem 5.2 ([30]):** Let \( \hat{A} \) be real symmetric and stable, and define \( \hat{\kappa} = \lambda_1(A)/\lambda_n(A) \). Then
\[
\frac{\lambda_{kd+1}(P)}{\lambda_1(P)} \leq \left( \prod_{j=0}^{k-1} \frac{\hat{\kappa}(2j+1)/2k - 1}{\hat{\kappa}(2j+1)/2k + 1} \right)^2 .
\] (V.5)

Penzl’s proof is based on using a geometric sequence of shifts on the interval containing the eigenvalues of \( \hat{A} \). It is difficult to determine when the bound (V.5) is stronger or weaker than the bounds in Sections IV and IX since (V.5) is independent of the precise distribution of eigenvalues while (IV.3) uses the exact eigenvalues.

The bound (V.4) shows that well-conditioned input pairs \((A, B)\) (such as input normal pairs) have \( A \) and \( \hat{A} \) that are far from normal in the sense that \( \kappa(T) \) is large when \( F_k(A) \) is small.

**VI. CONDITION BOUNDS FOR COMPANION MATRICES**

We now specialize Corollary 4.3 to the case where the advance matrix, \( A \), is a companion matrix. Other names for this case are Frobenius normal form and Luenberger controller canonical form. The second direct form [31] and autoregressive (AR) models are special case of this type and correspond to \( d = 1 \), with \( B \) being the unit vector in the first coordinate direction: \( B = e_1 \). For autoregressive models, \( C = e_1 \), while the second direct form uses \( C \) to specify transfer function. Let \( A \) be of the form
\[
A_c \equiv \begin{pmatrix} -c^* & -e_0^* \\ \Pi_{n-1} & 0 \end{pmatrix} ,
\] (VI.1)
where \( \Pi_{n-1} \) is a \((n-1) \times (n-1)\) projection matrix of the form \( \Pi_{n-1} \equiv I_{n-1} - \gamma e_p e_p^* \) where \( 1 < p \leq n-1 \) and \( \gamma = 0 \) or 1. Note that \( \gamma = 0 \) corresponds to companion normal form. Here \( e \) is an \((n-1)\) vector.

Autoregressive moving average (ARMA) models of degree \((p, q)\) satisfy the advance equation
\[
x_{t+1} = c_1 x_t + c_2 x_{t-1} + \ldots + c_p x_{t+1-p} = e_{t+1} - c_{p+1} e_t + \ldots - c_{p+q} e_{t+1-q},
\] where \( \{e_t\} \) is a sequence of independent random variables with \( E[e_t] = 0 \) and \( E[e_t^2] = 1 \). The ARMA \((p, q)\) model has a state space representation with the state vector \( z_t^T = (x_t, x_{t-1}, \ldots, x_{t-p+1}, e_t, \ldots, e_{t-q+1}) \), \( B = e_1 + \gamma e_{p+1} \) and \( A \) given in (VI.1) with \( \gamma = 1 \) and \( n = p + q \). When \( p = q \), this is a matrix representation of the first direct form.

**Lemma 6.1:** Let \( A_c \) be an \( n \times n \) matrix of the form given in (VI.1) with \( n > 2 \), then \( A_c \) has singular values, \( \sigma_1 \) and \( \sigma_m \), that are the square roots of \( \mu_\pm \), where \( \mu_\pm \) are the two roots of the equation
\[
\mu^2 - (1 + |c_0|^2 + ||c||^2) \mu + |c_0|^2 + \gamma |c_p|^2 = 0 ,
\] (VI.2)
where \( c_0 \) and \( c \) are given in (VI.1) and \( c_p \) is the \( p \)th component of the vector \( c \).
If $\gamma = 1$, then $m = n - 1$ and $A_c$ has a zero singular value. Otherwise, $m = n$. The remaining singular values of $A_c$ are one with multiplicity $n - 2 - \gamma$ and zero if $\gamma = 1$.

For $\gamma = 0$, this result is in [14]. For $\gamma = 1$, $e_{p+1}$ is a null vector of $A_c$.

Proof: Note $A_cA_c^* = \alpha \oplus \Pi_{n-1} - we_1^* - e_1w^*$, where $\alpha \equiv |c_0|^2 + \|c\|^2$, $w \equiv (0, \Pi_{n-1}c)$. To compute the eigenvalues of $A_cA_c^*$ we define an orthogonal transformation to reduce $A_cA_c^*$ to the direct sum of a $2 \times 2$ matrix with roots given by (VI.2) and a projection matrix. Let $H$ be the $(n-1) \times (n-1)$ Householder transformation such that $H\Pi_{n-1}c = \beta e_p$, where $\beta = \|\Pi_{n-1}c\|$.

We define $v \equiv He_p = \Pi_{n-1}c/\beta$. Since $H\Pi_{n-1}H^* = \Pi_{n-1} - \gamma Hv^*$ and $A_cA_c^* = \alpha \oplus (\Pi_{n-1} - \gamma Hv^*)$, we define an orthogonal transform $A_cA_c^* = (1 0 \ 0 H)^* (1 0 \ 0 H) = \alpha \oplus (\Pi_{n-1} - \gamma Hv^*)$.

If $\gamma = 1$, then $v$ is a zero $p$th coordinate. For both the $\gamma = 0$ and the $\gamma = 1$ cases, the eigenvalue equation decouples into the eigenvalues of the $2 \times 2$ matrix of the first and $(p+1)$st rows and columns and the eigenvalues of $\Pi_{n-1} - \gamma Hv^*$ projected onto the space orthogonal to $e_p$. The eigenvalue equation for the $2 \times 2$ matrix is given by $(\mu - \alpha)(\mu - 1) - \beta^2 = 0$. We define $\Gamma \equiv 1 + |c_0|^2 + \|c\|^2$ and $\omega = |c_0|^2 + |c|^2$. We denote the largest root of (VI.2) by $\mu_+$ and the smallest by $\mu_-$. Note $\mu_- = \omega \mu_+^{-1}$. The bound of Corollary 4.3 reduces to

Theorem 6.2: Let $A_c$ be companion matrix as specified by (VI.1) and $(A_c, B)$ be a stable, controllable input pair with $n > 2$.

Then the condition number of $P(A_c, B)$ satisfies the bound $\kappa(P(A_c, B)) \geq \mu_+$. If $A_c$ is invertible,

$$\kappa(P(A_c, B)) \geq \max \left\{ \mu_+, \frac{\sigma_n(A')^2}{\mu_-} \right\},$$

where $A'$ is the IN matrix generated in the map defined in the proof of Corollary 4.3. Here $\mu_+$ satisfies

$$1 + \|\Pi_{n-1}c\|^2 \leq \Gamma - \frac{\omega}{\mu} \leq \mu_+ \leq \Gamma - \frac{\omega}{\mu} \leq \Gamma.$$

Proof: The bound (VI.5) is proven by rewriting (VI.2) as a sequence of continued fractions

$$\mu = \Gamma - \frac{\omega}{\mu} = \Gamma - \frac{\omega}{\mu} \cdots \frac{\omega}{\mu} \cdots \frac{\omega}{\mu}.$$

Applying simple bounds to the continued fractions yields (VI.5).

The bound in Theorem 6.2 also applies when $A'$ is in companion form, corresponding to Luenberger observer canonical form. If the eigenvalues of $A_c$ are prescribed, the coefficients in $A_c, \{c_k\}$, are the coefficients of the characteristic polynomial of $A_c$: $p(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) = \lambda^n - \sum_{i=1}^{n}c_i\lambda^{n-i} - c_0$ for $\gamma = 0$. When the eigenvalues of $A_c$ are positive real, a weaker but explicit bound is

Theorem 6.3: Let $A_c$ be invertible with positive real eigenvalues $\lambda_i$, then $\kappa(P(A_c, B)) > \prod_{i=1}^{n}(1 + |\lambda_i|)^2/((n + 1) - \sum_{i=1}^{n}|\lambda_i|^2$.

Proof: We evaluate the characteristic polynomial at $-1$: $|p(-1)| = \prod_{i=1}^{n}(1 + |\lambda_i|) = \sum_{j=0}^{n}|c_j|$, where $c_n \equiv 1$. Note $(\sum_{j=0}^{n}|c_j|)^2/(n + 1) < \sum_{j=0}^{n}|c_j|^2$ and $|c_0|^2 = \prod_{i=1}^{n}|\lambda_i|^2$. The bound (VI.5) implies $\mu \geq \sum_{j=1}^{n}|c_j|^2 \geq (\sum_{j=0}^{n}|c_j|)^2/(n + 1) - |c_0|^2$. 

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For $n \gg 1$, $|c_0|^2 << \prod_{i=1}^n (1 + |\lambda_i|)^2/(n + 1)$. When the eigenvalues of $A_c$ have random or near random complex phases, the value of $\mu_+$ is typically much less than $\prod_{i=1}^n (1 + |\lambda_i|)^2/n$, since generally $\sum_i \lambda_i << \sum_i |\lambda_i|$, $\sum_{i \neq j} \lambda_i \lambda_j << \sum_{i \neq j} |\lambda_i \lambda_j|$, ... We now compare the bound in Theorem 6.2 with a random distribution of $A_c$.

**Definition 6.4:** $\mathcal{D}_C(A_c, B)$ is the set of stable, controllable $(A_c, B)$, where $A_c$ is given by (VI.1) and $B = e_1$. The distribution $\nu_C(A, B)$ is defined by the eigenvalues of $A_c$ having the distribution $\nu_{\lambda,n}(A)$ of $\nu_n(A, B)$ of Definition 2.1 subject to the CS restriction.

Table 4a gives the quantiles of $\log(\kappa(P))$ for $A_c$ with random $B$ and Table 4b gives the corresponding quantiles for $B = e_1$. The random $B$ case has a very broad distribution with the interquartile distance two to three times larger than that of generic random $(A, B)$ case. The top quartile of the random $B$ Frobenius case is as badly conditioned as the random case although the bottom quartile is much better conditioned.

| n   | 1%  | 10% | 25% | 50% | 75% | 90% | 99% |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | 2.30| 4.08| 5.66| 8.68| 12.2| 15.1| 19.1|
| 16  | 4.10| 6.71| 9.59| 14.6| 20.1| 24.2| 29.6|
| 24  | 5.61| 8.57| 12.4| 19.3| 27.5| 32.9| 38.7|

Table 4a: Quantiles of $\log(\kappa(P))$ distribution for $\mathcal{D}_C(A_c, B)$ for randomly distributed $B$. The bound VI.4 significantly underestimates $\kappa(P)$ in many cases.

We have also computed the condition numbers when the eigenvalues are all positive: $\lambda_i \rightarrow |\lambda_i|$. These models are appreciably more ill-conditioned than in cases where the eigenvalues are distributed randomly in the complex plane. *This ill-conditioning corresponds to the difficulty in estimating the coefficients of a sum of decaying exponentials.* For positive $\{\lambda_i\}$, the observed ill-conditioning is usually much larger than the formula, $\kappa > \prod_{i=1}^n (1 + |\lambda_i|)^2/(n + 1) - \prod_{i=1}^n |\lambda_i|^2$.

The $B = e_1$ case may be interpreted as a random autoregressive model. Our distribution, $\mathcal{D}_C(A_c, B)$, corresponds to a random distribution of poles of the autoregressive transfer function: For autoregressive models, the observability Grammian corresponds to solving (I.1) with the pair $(A^*, A[1,:]^t)$, i.e. using the characteristic polynomial coefficients as $B$. Thus we may examine the condition numbers of both the controllability and observability Grammians.

| n   | 1%  | 10% | 25% | 50% | 75% | 90% | 99% |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | .825| 1.69| 2.45| 3.59| 5.05| 6.62| 9.89|
| 16  | 1.66| 2.88| 3.79| 5.08| 6.74| 8.48| 11.8|
| 24  | 2.36| 3.66| 4.64| 5.99| 7.66| 9.36| 13.1|

Table 4b: Quantiles of $\log(\kappa(P))$ distribution for $\mathcal{D}_C(A_c, B)$ for $B = e_1$.

The autoregressive models ($B = e_1$) are much better conditioned than those with a random-righthand rank-one side. The scaling of the median of $\log(\kappa(P))$ versus $n$ is ambiguous. The interquartile distance of $\log(\kappa(P))$ is a weak function of $n$. Table 4c examines the observability Grammian of the autoregressive model. We find that the corresponding observability Grammians for our autoregressive models are very poorly conditioned. Thus these autoregressive models are nearly unobservable.
Table 4c: Quantiles of \( \log(\kappa(P)) \) for the observability Grammian of the autoregressive model.

The controllability is much better conditioned for the random autoregressive model than for the normal advance matrices with the same spectrum while the observability Grammian is grossly ill-conditioned in the random autoregressive model. These results may be influenced by the choice of \( D_C(A_c, B) \).

VII. POWER ESTIMATE

We now show that for certain classes of input pairs, the condition number, \( \kappa(P(A, B)) \), grows exponentially in \( n/d \). Specifically, let \( \{ (A_n, B_n) \} \) be CS with the uniform bound \( \sigma_1(A_n) \leq c < 1 \). The proof applies Lemma 4.1 with \( \phi(A) = A^k \). The theorem below shows that \( \kappa(P(A_n, B_n)) \geq c^{-2k} \), where \( k \) is the integral part of \( (n - 1)/d \).

Theorem 7.1: Let \( (A, B) \) be a stable, controllable input pair, then \( \kappa(P(A, B)) \geq |\sigma_1(A)|^{-2k} \), where \( kd < n \).

Proof: Let \( T \) be a \( \phi \)-INizing transformation, \( A' = TAT^{-1}, B' = TB \) with \( (A', B') \) being IN. Let \( \phi(A) = A^k \) for \( k \) such that \( kd < n \). By Lemma 4.1, \( \kappa(P(A, B)) \geq |\sigma_1(A^k)|^2/|\sigma_1(A^k)|^2 \). Note \( |\sigma_1(A^k)| < |\sigma_1(A)|^k \). The proof is completed when we prove the lemma below that \( |\sigma_1(A^k)| = 1 \).

Lemma 7.2: Let \( (A, B) \) be an \( \phi \)-IN pair, then \( \sigma_1(A^k) = 1 \) for \( kd < n \).
Lemma 8.3: [35] Let $H$ be a stable $n \times n$ matrix, $\sigma_1(H) \geq 1$, then
\[
\sup_{k \geq 0} \sigma_1(H^k) \geq \phi(H) \equiv \sup_{|z| > 1} \frac{|z| - 1}{\sigma_n(z\mathbb{I} - H)}.
\]
Solving yields the choice $|z_o| - \frac{1}{\sigma_n(z_o\mathbb{I} - J_o)} \geq |z_o| - \frac{1}{\sigma_n(z_o\mathbb{I} - \lambda_o)}$ for any $z_o \neq \lambda_o$ using the bound on $\sigma_n$ from Lemma 8.2. Maximizing the expression in $z_o$ yields the bound on $\sigma_n$ from Lemma 8.2. Choosing the phase of $z_o$ yields the bound in (VIII.1).

Table 5 gives the quantiles of $\log(\kappa(P))$ distribution over an ensemble of $B$ for fixed $\lambda$ and $n$. Here $\kappa_{bd}$ is the bound given in (VIII.1).

### IX. BILINEAR TRANSFORMATIONS BOUNDS

We now show that if $A$ is normal and all of the the eigenvalues of $A$ are approximately equal ($\lambda_k \approx x$) then $\kappa(P)$ is exponentially large in $n/d$. Our analysis is based on applying Theorem 4.1 to a fractional linear transformation of $A$.

We define the bilinear map of $A$: $A \rightarrow \hat{A} \equiv f(A, w), B \rightarrow \hat{B} \equiv g(B, w)$ defined by
\[
\hat{A} = f(A, w) = (I_n - w^*A)^{-1} (A - w\mathbb{I}_n), \quad \hat{B} = \sqrt{1 - |w|^2(I_n - \mathbb{I}_n)}.
\]
with $|w| < 1$. The bilinear map, $f(z, w) = (z - w)/(1 - w^*z)$, is a univalent function that maps the unit disk $|z| < 1$ onto itself and thus preserves stability. The bilinear map preserves solutions of the (1.1) $P(\hat{A}, \hat{B}) = P(A, B)$. Let $T$ be an INizing transformation of $(A, B)$ to $(\hat{A}', \hat{B}')$. Note $f(TA^{-1}, w) = TF(A, w)T^{-1}$ so $T$ is also an INizing transformation of $(\hat{A}, \hat{B})$ to the IN pair $(f(\hat{A}', w), g(\hat{B}', w))$.

We now bound the condition number of $P = T^{-1}T^*$ by applying Theorem 4.2 to $\hat{A} = f(A, w)$. Note $\hat{A}$ has eigenvalues, $\{f(\lambda_i(A), w)\}$ and is normal if $A$ is. Thus
\[
f(A, w)' \leq \prod_{i=1}^n |f(\lambda_i(A), w)|^{2/d}.
\]

Theorem 9.1: Let $A$ be a normal matrix and $(A, B)$ be a CS input pair with non-singular $f(A, w)$ and $|w| < 1$, then the condition number of $P(A, B)$ satisfies the bound
\[
\kappa(P(A, B)) \geq \min_k \{|f(\lambda_k, w)|^2\} \prod_{i=1}^n |f(\lambda_i, w)|^{2/d}.
\]
Theorem 9.1 allows us to optimize \( w \) in the bound (IX.2) for a given set of eigenvalues. Theorem 4.4 corresponds to \( w = 0 \). As \( w \) tends to zero, we see that Theorem 4.4 remains valid when \( A \) is singular.

We now assume that the eigenvalues are localized in a shifted disk: \( | \lambda_i - x | < \rho \) where \( | x | + \rho < 1 \). Here \( x \) is the center of the disk which contains all of the eigenvalues and \( \rho \) is the radius. We now return to the ansatz that all of the eigenvalues of \( A \) are contained in the disk of radius \( \rho \) centered about \( x \). Choosing \( w = x \), the bilinear transformation maps the circle, \( z(\theta) = x + \rho e^{i \theta} \), to the circle \( | \lambda | < \rho / (1 - | x | \rho - \rho^2) \).

Applying Theorem 7.1 yields

**Theorem 9.2:** Let \( A \) be a normal matrix and \( (A, B) \) be a stable, controllable input pair with the eigenvalues are localized in a shifted disk: \( | \lambda_i - x | < \rho \), where \( | x | + \rho < 1 \). Let \( k d < n \). The condition number of \( P(A, B) \) satisfies the bound \( \kappa(P(A, B)) \geq \left( \frac{\rho}{1 - | x | \rho - \rho^2} \right)^{-2k} \).

This bound illustrates the ill-conditioning that results when the eigenvalues of \( A \) are clustered in the complex plane.

**X. COLORED NOISE FORCING**

In [27], a computation is presented that shows that input normal filters perform well in the presence of autocorrelated noise. We now examine the condition number of the controllability Grammian when forcing term is autocorrelated. Our results help to explain the good performance of IN pairs observed in [27]. Let the state vector, \( z_t \), evolve as

\[
 z_{t+1} = Az_t + Bx_t = Bx_t + ABx_{t-1} + A^2 Bx_{t-2} + \cdots
\]

(X.1)

where \( x_t \) is a zero mean stationary sequence with the \( d \times d \) autocovariance, \( \phi_k = E[x_t x_{t-k}^*] \). The covariance of the state vector, \( W \equiv E[z_t z_t^*] \), satisfies

\[
 W = \begin{pmatrix}
 E[x_{t-1}x_{t-1}^*] & E[x_{t-2}x_{t-1}^*] \\
 E[x_{t-1}x_{t-2}^*] & E[x_{t-2}x_{t-2}^*]
\end{pmatrix}
\]

(X.2)

**Theorem 10.1:** Let \( (A, B) \) be a CS input pair and \( x_t \) a zero mean stationary sequence with autocovariance \( \Phi, \Phi_{jk} = E[x_{t-k}x_{t-j}^*] \). Let \( z_t \) be a sequence of state vectors satisfying (X.1), where \( x_t \) is a stationary sequence with a smooth spectral density. Let \( W = E[z_t z_t^*] \) be the state covariance. Then

\[
 \kappa(W) \leq \kappa(P) S_{\text{max}} / S_{\text{min}}, \quad (X.3)
\]

where \( S_{\text{min}} \) and \( S_{\text{max}} \) are the minimum and maximum modulus of the spectral density of \( x_t \) and \( P \) is the solution of \( P - APA^* = BB^* \).

Pessimists (and many realists) expect that \( \kappa(W) \sim \kappa(P) S_{\text{max}} / S_{\text{min}} \).

**Proof:** Let \( M_t \equiv (B\ AB\ A^2B\ \ldots \ A^{t-1}B) \) and let \( \Phi_t \) be the \( dt \times dt \) leading submatrix of the noise covariance matrix \( \Phi \). Define \( W_t \equiv M_t \Phi_t M_t^* \), by the lemma below we have

\[
 \kappa(W_t) \leq \kappa(M_t)^2 \kappa(\Phi_t) \quad (X.4)
\]

and by well-known result that \( \kappa(\Phi_t) \leq S_{\text{max}} / S_{\text{min}} \) [4, p. 137]

\[
 \kappa(W_t) \leq \kappa(M_t)^2 S_{\text{max}} / S_{\text{min}}. \quad (X.5)
\]

Let \( M \equiv (B\ AB\ A^2B\ \ldots) \); then \( MMM^* - A-MM^*A^* = BB^* \), and \( MM^* = P \). Since \( A \) is stable, \( M_t M_t^* = \sum_{k=0}^{t-1} A^k BB^* (A^*)^k \) converges to \( P \). Similarly \( W_t \) converges to \( W \) as \( t \) increases. Singular values are continuous functions of the matrices \( M_t \).
and $W_t$ so the result follows by taking the limit on both sides. ■

We now prove (X.4):

**Lemma 10.2:** Let $\Phi$ be Hermitian positive definite and $M$ be a $r \times n$ matrix of rank $r$, then $M\Phi M^*$ satisfies $\kappa(M\Phi M^*) \leq \kappa(\Phi) \kappa(M)^2$.

**Proof:** Clearly $\sigma_1(M\Phi M^*) \leq \sigma_1(M)^2 \sigma_1(\Phi)$. Note

$$\sigma_r(M\Phi M^*) = \min_v \|M\Phi M^* v\| = \min_v |v^* M\Phi M^* v| \geq \sigma_1(M)^2 \sigma_n(\Phi) \tag{X.6}$$

where $v$ has norm one. Dividing the first inequality by the second proves yields the lemma. ■

For input normal realizations, the bound is $\kappa(W) \leq \kappa(\Phi) \leq \frac{S_{\text{max}}}{S_{\text{min}}}$. Colored noise forcing arises when a signal is being under-modeled or modeled with uncertainty. The bound shows that input normal representations have state covariances that are well-conditioned even in the presence of colored noise. This property is independent of the system order and is important for practical applications. The lower bounds given in previous sections show that many common or random state space representations can be expected to fail for high order systems even for white noise.

**XI. SUMMARY**

We have examined the condition number, $\kappa(P)$, of solutions of the discrete Lyapunov equation. For random stable controllable input pairs, the $n$th root of the condition number, $\kappa^{1/n}(P)$, is approximately constant. When $A$ is normal with the same distribution of eigenvalues, $\kappa(P)$ has a very similar distribution. In both these cases, the median condition number grows exponentially while the interquartile distance of $\log(\kappa)$ has a weak dependence on $n$. Empirically, $\log(\log(\kappa(P)))$ has an approximately Gaussian distribution.

We have given analytic bounds for the conditioning of solutions of the discrete Lyapunov equation. For cases with $n >> d$, these bounds can be considerable. For both normal advance matrices and random advance matrices, the analytic bound for normal $A$ explains a large portion of the ill-conditioning. Nevertheless, the actual condition numbers are often several hundred times larger or more. The ill-conditioning, and the excess ill-conditioning, $\kappa(P)/\kappa_{\text{bd}}$, are larger when the eigenvalues cluster in the complex plane, (either as a single Jordan block or as multiple closely spaced eigenvalues).

For random autoregressive models, the controllability Grammian is usually well-conditioned and the observability Grammian is extremely ill-conditioned (for our ensemble of models).

Our analytic bounds do not use any property of $B$ except controllability. Thus our results are actually lower bounds on $\inf_B \kappa(P(A,B))$. Our bounds in Sections IV-VII do not utilize information on the complex phases of the eigenvalues, $\lambda_i$. Including additional information in the bounds can only sharpen the lower bound. Alternatively, we could compare our bounds versus the best possible $B$ matrix for a given $A$.

Finally, we have examined the covariance of the state vector in the presence of autocorrelated noise. Our bound depends on the ratio of the maximum to the minimum spectral density of the noise. When this ratio is not to large and an input normal representation is used, then the covariance
of the state vector is well-conditioned. This indicates that input normal representations are robust to undermodeling errors in filter design and system identification.

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