Abstract

We discuss the thermal (or gravitational) responses in topological superconductors and in topological phases in general. Such thermal responses (as well as electromagnetic responses for conserved charge) provide a definition of topological insulators and superconductors beyond the single-particle picture. In two-dimensional topological phases, the Strădă formula for the electric Hall conductivity is generalized to the thermal Hall conductivity. Applying this formula to the Majorana surface states of three-dimensional topological superconductors predicts cross-correlated responses between the angular momentum and thermal polarization (entropy polarization). We also discuss a use of D-branes in string theory as a systematic tool to derive all such topological terms and topological responses. In particular, we relate the $\mathbb{Z}_2$ index of topological insulators introduced by Kane and Mele (and its generalization to other symmetry classes and to arbitrary dimensions) to the K-theory charge of non-BPS D-branes, and vice versa. We thus establish a link between the stability of non-BPS D-branes and the topological stability of topological insulators.

Résumé

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Key words: Keyword1 ; Keyword2 ; Keyword3
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and the thermal Hall effect [6]. These quantum transport phenomena of topological origin (or: topological currents) are regarded as a promising candidate for spintronics with low energy cost.

In metallic systems the usual transport currents with dissipation are dominant over these topological currents, and hence it is rather difficult to identify the latter. A topological current, however, can be non-vanishing even in insulating systems – such systems on general ground are called topological insulator. This is so since it is the topological properties of electronic wavefunctions, rather than a band structure of energy levels in solids, that are responsible for generation of a topological current.

The quantum Hall effect (QHE) is a canonical example of topological insulators characterized by a topological (i.e., quantized) transport law [7,8,9]. Recently topological insulators realized by strong spin-orbit interactions in two and three dimensions have been discovered [10,11,12,13,14,15,16,17]. In particular, the three-dimensional (3d) topological insulator is characterized by the topological magnetoelectric effect [18,19] (the axion electrodynamics [20]). (See below for more details.) As a consequence of non-trivial electrical wavefunctions in the bulk, these topological insulators support anomalous boundary (edge or surface) modes, whose gapless nature is topologically protected.

Analogously, for superconductors (SCs) and superfluids, one can consider topological properties associated with the wavefunctions of fermionic quasiparticles; Within the BCS mean-field theory, the BCS (Bogoliubov-de Gennes) Hamiltonian is a fermion bilinear in the Nambu spinor. The case where we have a quasi-particle gap everywhere in momentum space is an analogue of the band insulator. A topological SC is a SC with a full gap and topologically nontrivial quasiparticle wavefunctions. Canonical examples of topological SCs include, e.g., the 2d chiral $p$-wave SC [21,22].

In three dimensions, the B phase of superfluid $^3$He was recently identified as a new topological SC (superfluid) [23,24,25,26,27]. A recent surface transverse acoustic impedance measurement reported in Refs. [28,29] revealed a signature of the surface Majorana fermion mode on the surface of $^3$He-B. A copper-doped topological insulator (Cu$_x$Bi$_2$Se$_3$) has been discussed as a candidate of a 3d topological SC [30,31,32].

The purpose of this article is to describe the response theory of topological phases with a special focus on their thermal, rather than electrical, transport. A motivation for the thermal response is that it is well-defined even for phases in which the electrical charge is not conserved, such as topological SCs, or topological phases in spin systems. While topological SCs (as well as topological insulators) can be defined in terms of a topological invariant built out of, within the BCS mean-field theory, quasiparticle wavefunctions, the response theory gives a physically measurable definition of topological phases, which can be largely insensitive to microscopic details – a lesson we have learned from the physics of the QHE, or the topological magnetoelectric effect in the 3d topological insulator. In particular, formulating the thermal response as a response to external gravitational field, we will derive a thermal analogue of the topological magnetoelectric effect, which uncovers an interesting cross-correlation between thermal and mechanical responses, in terms of the temperature gradient, and an applied angular velocity, respectively (Sec. 3).

For the bulk of this article, we will focus on topological SCs in two and three dimensions, such as the 2d chiral $p$-wave SC or $^3$He B. However, we will also briefly describe how such response theory can be systematically constructed for a wider class of topological insulators and SCs in the “periodic table” [23,33,34] (Sec. 4). In fact, we will illustrate that responses in topological phases are closely related to quantum anomalies in field theories. Such characterization of topological phases in terms of anomalies is expected to incorporate arbitrary strong interactions as far as a bulk gap is not destroyed.

Interestingly, such considerations lead to a natural link to topological objects in string theory – D-branes. We will demonstrate, by considering a particular configuration of D-branes, we can realize a field theory model of topological insulators and superconductors with desired discrete symmetries (Sec. 4). The stability criterion of D-branes against Tachyon condensation is in one-to-one correspondence with the classification of topological insulators and superconductors (i.e., the periodic table [23,33,34]). From condensed matter point of view, the D-brane construction can be thought of as a convenient tool that bridges K-theory classification of fermionic Hamiltonians, and linear response theory.
2. electromagnetic response in the QHE and 3d topological insulator

2.1. the QHE in two dimensions

Let us start this review by first illustrating the usefulness of the effective field theory of linear response, by taking the QHE as an example. The electromagnetic response of the quantum Hall fluid is described by the effective Chern-Simons action

\[ I_{\text{eff}} = \frac{e^2 k}{4\pi \hbar} \int dt d^2 x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad k \in \mathbb{Z}, \]

where \( A_\mu \) is an external electromagnetic gauge field. The Chern-Simons action can be derived by considering a coupling of the (topological) insulator in question to the external (background) electromagnetic field \( A_\mu \), and then by integrating out fermions to derive the effective action for \( A_\mu \).

The Chern-Simons action encodes all types of topological responses in the QHE. (i) The current \( \delta j^k(t,x) \) induced by \( A_\mu \) is computed, for time-independent vector potential, as

\[ \delta j^k(t,x) = \frac{\delta I_{\text{eff}}}{\delta A_k(t,x)} = \frac{e^2 k}{2\pi \hbar} \epsilon^{kij} \partial_i A_j. \]

This is the QHE, \( J^x = \sigma H E^y \), with the Hall conductivity \( \sigma_H = \frac{e^2}{2\pi \hbar} \).

(ii) Similarly, one can compute the charge \( \delta \rho(t,x) \) induced by \( A_\mu \) as:

\[ c \delta \rho(t,x) = \delta j^0(t,x) = \frac{\delta I_{\text{eff}}}{\delta A_0(t,x)} = \frac{e^2 k}{2\pi \hbar} \epsilon^{0ij} \partial_i A_j, \]

where \( c \) is the speed of light. This means that a (solitonic) flux tube binds \( k \) units of charge, which is, in cylinder geometry, nothing but charge-pumping in Laughlin’s thought experiment. For uniform magnetic field \( B^z \), the change in total electron charge is

\[ c \delta Q_e = \sigma_H \delta B^z \Rightarrow c \frac{\delta Q_e}{\delta B^z} = e \frac{\delta N_e}{\delta B^z} = \sigma_H, \]

where \( N_e \) is the total electron number. This is nothing but the Streda formula [35,36]. (iii) The Chern-Simons term can be written, for static configurations of \( A_\mu \), as

\[ I_{\text{eff}} = \frac{\sigma H}{c} \int dt d^3 x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \frac{\sigma H}{c} \int dt d^2 x \phi B^z, \]

where \( A_0 =: \phi/c \). If we define (the change in) magnetization by

\[ \delta M^z = \frac{\delta I_{\text{eff}}}{\delta B^z} = (\sigma_H/c) \phi \Rightarrow c \frac{\delta M^z}{\delta \mu} = \sigma_H, \]

where \( \mu = e\phi \) is the chemical potential \([\mu/c] e N_e = \phi Q_e\).

2.2. topological insulator in three dimensions

Similarly, for the 3d topological insulator, the electromagnetic response is encoded in the effective action [18,19,20]

\[ I_{\text{eff}} = \frac{\theta e^2}{32\pi^2 \hbar c} \int d^4 x \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} = \frac{\theta e^2}{4\pi^2 \hbar c} \int dt d^3 x E \cdot B. \]

This “axion” term can be derived, similarly to the Chern-Simons action, by integrating out fermions in the presence of the background electromagnetic field. The Dirac quantization condition and time-reversal symmetry (TRS) restrict \( \theta \) to be quantized, \( \theta = 0, \pi \mod 2\pi \); \( \theta = 0 \) for trivial insulators or vacuum whereas \( \theta = \pi \) inside topological insulators.

As inferred from the axion term, the topological insulator features topological magnetoelectric (ME) effect

\[ M = \frac{\delta I_{\text{eff}}}{\delta B^z} = \frac{\theta}{\pi} \frac{e^2}{2\hbar c} E = \frac{\theta}{\pi} \frac{e}{4\pi} E, \]

\[ P = \frac{\delta I_{\text{eff}}}{\delta E} = \frac{\theta}{\pi} \frac{e^2}{2\hbar c} B = \frac{\theta}{\pi} \frac{e}{4\pi} B, \]
or, quantized electromagnetic polarizability,
\[
\frac{\delta P_i}{\delta B_j} = \frac{\delta M_i}{\delta E_j} = \delta_{ij} \frac{\theta}{\pi} \frac{\alpha}{4\pi},
\]
where \(\alpha = e^2/(hc)\) is the fine structure constant. Equations (8) and (7) are the 3d analogue of Eqs. (4) and (5), respectively.

The magnetization \(M\) in Eq. (7), which follows the direction of the external electric field \(E\), is generated by the surface QHE (Fig. 1a); when a topological insulator is in contact with a topologically trivial insulator (or simply vacuum), the \(\theta\)-angle in the axion term jumps by \(\pi\) at the interface. As the axion term is the total derivative of the Chern-Simons term, such interface (where TRS is weakly broken) is accompanied by the half quantized surface Hall current \(\sigma_H = \pm e^2/(2h)\) which generates a bulk magnetization \(M\). Similarly, when an external magnetic field \(B\) is applied, according to Eq. (4), it induces an excess or a deficit of charge on the surfaces which are orthogonal to \(B\) – this is the source of a bulk electric polarization \(P\) in Eq. (8) (Fig. 1b).

3. gravitational response in topological superconductors

3.1. gravitoelectromagnetism

3.1.0.1. energy-gravity coupling  We now develop the thermal response theory of topological phases, with an eye, in particular, on applications to topological SCs. Our strategy here is, based on Luttinger’s idea [38], to follow as close as possible the linear response theory for the electric charge: When we study charge response, we consider a small external probe field for the charge density, \(H_I \sim \int d^d x \phi(x) \rho(x)\). The same strategy can be adopted since energy is conserved; as in the charge response, we can consider an external (fictitious) source term which couples to the energy (Hamiltonian) density \(\varepsilon(x)\), \(H_I \sim \int d^d x \phi_g(x) \varepsilon(x)/v^2\), where we have introduced \(v\) which has the dimension of velocity to assign a proper dimension to \(\phi_g\) (see below).

The external source \(\phi_g\) can be thought of as a fictitious gravitational field in the presence of the Lorentz invariance, as it allows us to identify the energy as the mass, \(\varepsilon(x) = m(x)v^2\), where \(m(x)\) is the mass density. For the gravitational theory of nature (not for Luttinger’s fictitious gravity that is a mere device to develop a linear response theory), \(v\) should be replaced by the speed of light \(c\), whereas in condensed matter systems with emergent Lorentz symmetry, \(v\) is the “Fermi” velocity (see below for more discussion). The coupling
of the system to the potential \( \phi_g \) can then be written as \( H_1 \sim \int d^4x \phi_g(x)m(x) \), where, from the analogy to the electromagnetism, mass can be thought of as a “charge” coupled to gravitational field.

With Lorentz invariance, in more covariant language (in the Lagrangian language) the gravitational coupling with the (thermal) energy current is introduced as follows: The energy density \( \varepsilon \) and the energy current \( j_E \) are components of the energy-momentum tensor \( T^{\mu\nu} \), \( \varepsilon = T^{00} \) and \( j_E^\mu = \varepsilon T^{0\mu} \). They are thus coupled to the variation of spacetime metric \( g_{\mu\nu} \) as \( -(1/2) \int dt d^3x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \) in the Lagrangian, where \( g = \det g_{\mu\nu} \) – in a way analogous to the way the charge current \( j^\mu \) couples with the external electromagnetic potential as \( -\int dt d^3x j^\mu A_\mu \).

### 3.1.0.2. gravitoelectric field

A spatial gradient in energy density inevitably causes a temperature gradient, as one can infer from the thermodynamic equality \( dU = TdS \) as follows. (Here, \( U \) is the internal energy, \( S \) the entropy, and \( T \) the temperature). Let us divide the total system into two subsystems (subsystems 1 and 2). Equilibrium between the two is achieved when the total entropy is maximized \( dS = dS_1 + dS_2 = 0 \). Since energy is conserved, \( dE_2 = -dE_1 \), and hence \( dS_1/dE_1 - dS_2/dE_2 = 0 \). I.e., \( T_1 = T_2 \). Let us now turn on a gradient in the gravitational potential, so that the gravitational potential felt by Subsystem 1 and 2 differs by \( \delta \phi_g \). In this case, we have \( dE_2 = -dE_1(1 + \delta \phi_g/v^2) \). This suggests the generation of a temperature difference \( T_2 = T_1(1 + \delta \phi_g/v^2) \). In other words, we can view the “electric” field \( E_g \) associated with the gradient of \( \phi_g \), which we call “gravitoelectric field”, as temperature gradient,

\[
E_g := -\nabla \phi_g = v^2 T^{-1} \nabla T. \tag{10}
\]

### 3.1.0.3. gravitomagnetic field

The analogy with electromagnetism can be further put forward – such formalism is called gravitoelectromagnetism [39]. For our purposes to develop a linear response theory for thermal transport, we put an external gravity field which is infinitely small. We can thus write the metric as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) where \( \eta_{\mu\nu} \) is the metric of flat space-time. In the presence of matter, in the post-Newtonian limit, we keep only \( h_{00}, h_{ii}, \) and \( h_{0i} \) [which are of order \( \mathcal{O}(v^{-2}) \)], whereas \( h_{ij} = \mathcal{O}(v^{-4}) \) \((i \neq j)\) are neglected [39]:

\[
ds^2 = v^2(-1 + 2\phi_g/v^2)dt^2 - \frac{4}{v} A_g \cdot dx dt + (1 + 2\phi_g/v^2)dx \cdot dx, \tag{11}
\]

where we have introduced the gravitomagnetic potential, \( A_{g,i} := -(v^2/2)h_{0i} \). For small \( h_{\mu\nu} \), the Einstein equation can be linearized up to these non-zero components of \( h_{\mu\nu} \) kept in Eq. \( \text{(11)} \), and the resulting equation looks structurally identical to the Maxwell equation. (However, note that we are considering an external gravity field as a source which does not have its own dynamics). As we have seen, the gravitoelectric field \( E_g \) corresponds to temperature gradient. What does the gravitational analogue of magnetic field, the gravitomagnetic field, \( B_g \), correspond to? It turns out that \( B_g \) can be understood as the angular velocity vector of rotating systems,

\[
B_g := \nabla \times A_g = v\Omega. \tag{12}
\]

For a system rotating with the frequency \( \Omega = \Omega^z z \), this can be understood by making a coordinate transformation from the rest frame to the rotating frame in which the metric in the polar coordinates \((t, r, \varphi, z)\) takes the form \( ds^2 \simeq -v^2 dt^2 + 2\Omega^z r^2 d\varphi dt + r^2 d\varphi^2 + dr^2 + dz^2 \). One can then read off, from the definition of the gravitoelectromagnetic field, the non-zero gravito gauge potential \( A^z_g = (v/2)\Omega^z r \). In Cartesian coordinates, \( A_g = (v/2)\Omega^z \hat{z} \times \hat{x} \), and \( B_g = \nabla \times A_g = v\Omega^z \hat{z} \).

### 3.2. thermal Strˇ eda formula for 2d topological SCs

We now use the formalism described above to study thermal transport of topological SCs in two and three dimensions. The goal here is to establish a gravitational analogue of the Strˇ eda formula \((4-5)\) and the topological ME effect \((7-8)\) [40].

Let us start with 2d topological fluid which does not have time-reversal symmetry. To draw a parallelism with the charge response, we introduce the moment of the thermal current \( M_T \) as \( M_T = M_E - (\mu/e) M \),
of free energy is given by $dF$. According to the relation, we thus obtain
\[ \partial_p \text{at} \text{the} \text{surface} \text{and} \text{hence} \text{the} \text{surface} \text{is} \text{gapped}. \text{Such} \text{thought} \text{experiments} \text{lead to,} \text{for} \text{the} \text{(induced) thermal gravitomagnetic fields,} \text{this} \text{can be written as} \]
\[ \frac{\partial}{\partial \Omega} \text{geometry} \text{(as} \text{in} \text{Fig.} \text{1)}, \text{apply} \text{the} \text{temperature} \text{gradient} \text{essentially} \text{be} \text{discussed by} \text{looking} \text{at} \text{the} \text{surface} \text{transport.} \text{For} \text{simplicity,} \text{one} \text{can} \text{consider} \text{a} \text{cylindrical topological SCs} \text{to} \text{the} \text{temperature} \text{gradient} \text{and} \text{the} \text{rotation;} \text{Given} \text{that} \text{the} \text{fermionic quasiparticles} \text{are}
\]
\[ \text{related to the orbital angular momentum as} \text{M}_E = v \text{L}. \text{Thus,} \kappa_H = (\frac{v^2}{2}) \frac{(\partial L^z/\partial T)}{\partial \Omega}. \text{This} \text{is an analogue of the electromagnetic Str\'eda formula (5)).}
\]
\[ \text{To derive “the other half” of the Str\'eda formula [a thermal analogue of Eq. (4)]}, \text{we note that} \text{the variation} \text{of} \text{free energy} \text{is given by} \text{dF} = -SdT - L \cdot d\Omega \text{where} \text{S is} \text{the} \text{entropy.} \text{In} \text{terms} \text{of} \text{the} \text{gravitoelectric and gravitomagnetic fields, this} \text{can be written} \text{as} \text{dF} = -(T S/v^2)(v^2 T^{-1} dT) - (L^z/v) d(\phi g) - (\text{M}_E/v^2) \cdot dB_g, \text{where} \text{Q}_T = T S \text{is the thermal energy density,} \text{couples to} \phi g/v^2. \text{From} \text{the} \text{Maxwell relation,} \text{we} \text{thus} \text{obtain}
\]
\[ \kappa_H = \frac{v^3}{2T} \left( \frac{\partial M^z_{\phi g}}{\partial \phi g} \right) B^z_g \frac{\partial Q_T}{\partial B^z_g} \phi_g. \]
\[ \text{This} \text{is the thermal analogue of the Str\'eda formula for} \text{the} \text{charge} \text{Hall conductivity, in} \text{that} \text{Q}_T \text{is the} \text{zeroth component} \text{of} \text{the} \text{energy} \text{current} \text{as} \epsilon N_c \text{is in the} \text{charge} \text{current.} \text{With} \text{the} \text{pseudo-relativistic invariance, this} \text{can be written as}
\]
\[ \kappa_H = \frac{v^2}{2} \left( \frac{\partial L^z}{\partial T} \right) \frac{\partial S}{\partial \Omega}, \text{with} \text{the} \text{pseudo-relativistic invariance, this} \text{can} \text{be written as}
\]
\[ \kappa_H = \frac{v^2}{2} \left( \frac{\partial L^z}{\partial T} \right) \frac{\partial S}{\partial \Omega}. \]

3.3. cross-correlated response of 3d topological SC

The thermal Str\'eda formula derived above for 2d topological fluid can be used to study the response of 3d topological SCs to the temperature gradient and the rotation; Given that the fermionic quasiparticles are fully gapped in the bulk, all (topological) transport phenomena, in the presence of a boundary (surface), can essentially be discussed by looking at the surface transport. For simplicity, one can consider a cylindrical geometry (as in Fig. 1), apply the temperature gradient $\partial_z T$ or external rotation $\Omega^z$, and discuss the responses, which are mediated by the surface. (As in the electromagnetic responses, we weakly break TRS at the surface and hence the surface is gapped). Such thought experiments lead to, for the (induced) thermal polarization $P_E$ defined by $\delta Q_T = -\nabla \cdot P_E$, and for the moment of the energy current $M_E$.

1. Below, we will consider the part of the thermal current carried by $j_E$ (and hence $M_E$) specializing to the case of $\mu = 0$. This is valid in discussing topological SCs.

2. Our convention for $\phi_g$, $A_g$, and $M_E$ in this review differs from the one employed in Ref. [40]; there is a factor of two difference in defining $\phi_g$ and $A_g$ [see Eq. (11)]. Accordingly, there is a similar factor of 1/2 difference in defining $M_E$. These factors are due to the spin 2 nature of gravitons, and are convenient since the resulting linearized Einstein equation does not have such factors of two, and the similarity with the Maxwell equation is clearer. Also, in the definition of $\phi_g$, we have put an extra factor of $v^2$.

3. While in the bulk, in the absence of interactions, the Kubo formula gives us one of the most direct ways to get the Str\'eda formula for $\sigma_H$ and $\kappa_H$, the applicability of the Str\'eda formulas is not limited to non-interacting systems, as they can be also derived assuming the presence of chiral edge states. See Ref. [40] for details.
\[ M_E = (T\kappa_H/v^3)E_g, \]  
\[ P_E = (T\kappa_H/v^3)B_g, \]  
\[ \frac{\delta M_{E,i}}{\delta E_{g,j}} = \frac{\delta P_{E,i}}{\delta B_{g,j}} = \delta_{ij} \frac{\theta T\kappa_H}{\pi v^3}. \]  

The parallel between the electromagnetic and gravitational cases are obvious. Observe, however, that the gravitational response is not quantized as in the charge response, because of the presence of the velocity \( v \).

The situation is somewhat similar to the detection of the conformal anomaly (central charge) from specific heat in 1d quantum systems at criticality; while the central charge \( C \) for a given quantum critical system in 1d is a dimensionless universal parameter, it shows up in specific heat \( C_V \) together with the velocity \( v \) as \( C_V = C\pi T/(3v) \).

3.3.0.4. possible experiments The thermal (gravitational) analogue of the Strˇedá formula [Eq. (15)] and topological ME effect [Eq. (18)] can be tested experimentally. An external angular velocity \( \Omega^z \) results in the change in temperature (in 2d topological SCs) and thermal polarization (in 3d topological SCs). If the heat capacity of the system is sufficiently small, these may not be difficult to measure.

In three dimensions, the “dual” response, i.e., the response to the applied temperature gradient [Eq. (17)], can be detected by making use of the Einstein-de Hass effect. Let us assume a cylindrical 3d topological SC is suspended by a thin string and apply thermal gradient (as in Fig. 1a). This induces surface energy current with angular momentum \( L^z \), according to Eq. (17). By the conservation law of total angular momentum, it must be compensated by a mechanical angular momentum of the material, which can be directly measured in principle.

Note that in both responses to \( \Omega^z \) and to temperature gradients, the part of the responses of our interest are contributions from the fermionic quasiparticles. They should be distinguished from the contributions from bosonic excitations such as vortices. If \( \Omega^z \) is larger than the critical angular velocity \( \Omega_{c,1} \) above which vortices are introduced in the bulk of the sample, an extra contribution to thermal polarization would be generated.

4. anomaly ladder and D-branes

4.1. integrating out fermions and chiral anomaly

We have constructed the response theory of 3d topological SCs starting from the thermal Strˇedá formula of 2d topological SCs. We now discuss the response of topological SCs (and insulators) to external gravitational field from a field theoretical point of view [41,42]. Our goal here is to derive a gravitational analogue of the axion term in the electromagnetic response, and show that it is related to quantum anomaly (chiral anomaly). In this section, we will use natural units \( e = c = \hbar = 1 \), and set the Fermi velocity \( v \) to be unity for simplicity.

Let us work with an example: A canonical example of the 3d topological SC is the B phase of \(^3\)He, which is described, in momentum space, by the following BdG Hamiltonian: \( \hat{H} = (1/2) \int d^3k \Psi^\dagger(k)\hat{H}(k)\Psi(k) \), where \( \Psi^\dagger(k) = (c^\dagger_{s,k}, c^\dagger_{s,-k}, c^\dagger_{t,-k}, c^\dagger_{t,k}) \) is the Nambu spinor composed of fermionic creation/annihilation operators \( (c^\dagger_{s,k}, c_{s,k}) \) of a \(^3\)He atom with spin \( s \) and momentum \( k \), and the kernel \( \hat{H}(k) \) takes the following form:

\[
\hat{H}(k) = \begin{pmatrix}
\xi(k) & \Delta(k) \\
\Delta^\dagger(-k) & -\xi(-k)
\end{pmatrix}, \quad \xi(k) = \frac{k^2}{2M} - \mu, \quad \Delta(k) = |\Delta| k \cdot \hat{s}(\hat{i}s_y),
\]

where \( M \) is the mass of a \(^3\)He atom, \( \mu \) is the chemical potential, and \( |\Delta| \) is the amplitude of the pair potential. With the \( \hat{d} \)-vector pointing parallel to momentum, \( \hat{d}(k) = |\Delta| k \), there is an isotropic gap everywhere on the 3d fermi surface. The critical point at \( \mu = 0 \) separates topologically trivial (\( \mu < 0 \)) and non-trivial (\( \mu > 0 \)) phases, which are characterized by an integral topological invariant of symmetry class DIII (the winding number) \( \nu = 0 \) and \( \nu = 1 \), respectively [23]. We will henceforth set \( |\Delta| = 1 \) and drop the \( \mathcal{O}(k^2) \) term in \( \xi(k) \), \( \xi(k) \rightarrow -\mu \equiv m \). With a suitable unitary transformation, the BdG Hamiltonian is written in terms of the
The chiral transformation which rotates 
\[ \psi \rightarrow \psi' = e^{i\phi} \psi, \]
\[ \bar{\psi} \rightarrow \bar{\psi}' = e^{i\phi^{\gamma}} \bar{\psi}' , \]
under which the mass is rotated as
\[ m'(\phi) = m e^{i\phi^{\gamma}} = m \left( \cos \phi + i \gamma_5 \sin \phi \right), \]
so that \( m'(\phi = 0) = m \) and \( m'(\phi = \pi) = -m \). Since \( m \) can continuously be rotated into \(-m\), one would think, naively, \( I_{\text{eff}}[m] = I_{\text{eff}}[-m] \). This naive expectation is, however, not true because of chiral anomaly. The chiral transformation which rotates \( m \) continuously costs the Jacobian \( J \) of the path integral measure,
\[ J \left[ \bar{\psi}, \psi \right] = J \left[ \bar{\psi}', \psi' \right]. \]
This chiral anomaly (the chiral Jacobian \( J \)) is responsible for the gravitational analogue of the axion term (the \( \theta \)-term). The Jacobian \( J \) can be computed explicitly by the Fujikawa method [44] as
\[ I_{\text{eff}}^\theta = -\ln J = \frac{1}{2} \frac{\theta}{384\pi^2} \int d^4x \sqrt{-g} g^{cde} R^a_{bce} R^b_{aef}, \]
where \( R = d\omega + \omega \wedge \omega \) (\( R^a_{b\mu\nu} = \partial_{\mu} \omega^a_{\nu\rho} - \partial_{\nu} \omega^a_{\rho\mu} + [\omega_{\mu}, \omega_{\nu}]^a_{\rho} \)) is the Riemann curvature tensor; as in the electromagnetic response, the \( \theta \)-angle is fixed to either \( \theta = 0 \) or \( \theta = \pi \) by time-reversal symmetry. The former corresponds to a topologically trivial state, and \( \theta = \pi \) to a topologically non-trivial state. The theta term in the gravitational effective action (23) ("the gravitational instanton term") is an analogue of the axion term \( \propto 4E \cdot B \) in the electromagnetic effective action; there is an obvious structural parallelism between the electromagnetic and gravitational cases [45].

To make the connection with the existence of topologically protected surface modes, we note, when there are boundaries (say) in the \( x^i \)-direction at \( x^i = L_+ \) and at \( x^i = L_- \), the gravitational instanton term \( I_{\text{eff}}^\theta \), at the non-trivial time-reversal invariant value \( \theta = \pi \) of the angle \( \theta \), can be written in terms of the gravitational Chern-Simons terms at the boundaries, \( I_{\text{eff}}^\theta = I_{\text{CS}}|_{x^i=L_+} - I_{\text{CS}}|_{x^i=L_-} \), where \((i,j,k = 0,1,2)\)
\[ I_{\text{CS}} = \frac{c'}{24} \int d^3x e^{ijk} \left( \omega_i \partial_j \omega_k + \frac{1}{3} \omega_i \omega_j \omega_k \right), \]
with \( c' = 1/2 \). This value of the coefficient of the gravitational Chern-Simons term is one-half of the canonical value \((1/4\pi) \times (c'/24)\) with \( c' = 1/2 \). As discussed by Volovik [47] and by Read and Green [21] in the context of the 2d chiral \( p \)-wave SC, the coefficient of the gravitational Chern-Simons term is directly related to the thermal Hall conductivity, which in our case is carried by the topologically protected surface modes.

4.2. Anomaly ladder and periodic table

All types of responses we have discussed so far (the QHE, the topological ME, and their gravitational (thermal) analogues) are related to quantum anomaly; the Chern-Simons terms in the QHE (both electromagnetic and gravitational) is a manifestation of parity anomaly; the axion term can be derived, as we have seen above, from chiral anomaly. In fact, a wider class of topological insulators and SCs in the periodic table (Table 1) can be related to quantum anomalies.
Table 1

| Symmetry Class | d | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---------------|---|---|---|---|---|---|---|---|---|-----|
| A             |   | Z | 0 | Z | 0 | Z | 0 | Z | 0 | ... |
| AIII          |   |   | Z | 0 | Z | 0 | Z | 0 | Z | ... |
| AI            | Z♦| 0 | 0 | 0 | Z | Z | 2 | Z | 2 | ... |
| BDI           | Z♦| 0 | 0 | 0 | Z | Z | 2 | Z | 2 | ... |
| D             | Z♦| 0 | 0 | 0 | Z | Z | 2 | Z | 2 | ... |
| DIII          |   | Z | Z | 0 | 0 | Z | Z | 2 | Z | ... |
| AII           | Z♠| 0 | 0 | 0 | Z | Z | 2 | Z | 2 | ... |
| CII           | Z♠| 0 | 0 | Z | Z | Z | Z | Z | 2 | ... |
| C             | Z♠| 0 | 0 | Z | Z | Z | Z | Z | 2 | ... |
| CI            |   | Z | Z | Z | Z | Z | Z | Z | 2 | ... |

Periodic table of topological insulators and superconductors for the 10 symmetry classes in various spatial dimensions. Topological insulators (superconductors) in the complex symmetry classes (A and AIII) are related to the chiral U(1) anomaly.

The primary series of the topological insulators (superconductors) with an integer (Z) classification in the eight real symmetry classes are located on the diagonal in the table. In even space-time dimensions (odd space dimensions) they are predicted from the chiral anomaly in the presence of background gravity (Z♦), and from the chiral anomaly in the presence of both background gravity and U(1) gauge field (Z♣). The topological response of topological phases in odd space-time dimensions (Z♦ and Z♠) follows from their higher-dimensional ancestor (Z♠ and Z♣), respectively.

Let us consider topological insulators and SCs with an integer topological invariant located on the diagonal of the periodic table (Table 1), which we call the “primary series”. (The Z♦ topological phases which “descend” from the primary series can be called 1st and 2nd descendants [34].) For the primary series, the effective action for electromagnetic and gravitational response can be derived essentially by repeating the procedure discussed above; taking a Dirac representative in D′ = D + 1 = even, coupling it to the background electromagnetic and gravitational fields, and then integrate out fermions; chiral anomaly in D′ = D + 1 = even dimensions allows us to calculate the imaginary part of the action (i.e., the part which encodes topological part of the response). The result is summarized as follows:

\[ \delta \ln Z = 2\pi i \int_{M_D} \text{ch}(E) \hat{A}(R) \bigg|_D. \] (25)

\( \delta \ln Z \) represents the change in the effective action under the infinitesimal chiral transformation; integrating it one gets the topological action for the linear response. \( \text{ch}(E) \) denotes the Chern character of the vector bundle \( E \), which is explicitly given by \( \text{ch}(E) = \text{tr} [e^{F/(2\pi)}] \) in terms of its field strength \( F \). \( \hat{A}(R) \) is the A-roof genus and takes the form

\[ \hat{A}(R) = 1 + \frac{1}{192\pi^2} \text{tr} [R^2] + \cdots, \] (26)

where \( R \) is the curvature two-form on the manifold \( M_D \). Since (25) measures the number of chiral fermion zero modes minus that of anti-chiral ones as follows from the chiral rotation, (25) is equivalent to the index theorem in mathematics.

Topological terms for \( d = \text{even} \) can be derived from the topological terms in \( d + 1 \) dimensions considered above: They are all Chern-Simons type and obtained as a boundary contribution from a \( (d + 1) \)-dimensional topological term.

One could check that the topological response derived from chiral anomaly is fully consistent with the periodic table: (i) the topological terms derived in this way preserve correct discrete symmetries for primary series; (ii) for symmetry classes which are realized as topological SC (i.e., no charge conservation), the anomaly polynomial predicts that there is no topological response for EM field; only gravitational (thermal) response exists.

One could think of this as an alternative “derivation” of the periodic table (for the primary series).
...D-branes is directly related to the $Z$ deepening our understanding of the stability of D-branes. As we will argue below, the stability of the so-called why they give us an insight on the periodic table. In fact, the opposite is also true; topological phases role in AdS/CFT or holography. For our discussions in this article, however, we will not use AdS/CFT.)

D-branes has been proved to be a useful tool to understand non-perturbative phenomena in gauge field theories, such as monopoles, Seiberg-Witten theory [48], etc. (Such dual nature of D-branes also plays a key

D-branes is a $(p+1)$-dimensional solitonic solution to the 10d Einstein equation (such D-brane is called D$p$-brane). Besides such geometrical attribute in gravity theory, an important property of D-branes for our purposes is the fact that open string excitations on D-branes give rise to a gauge field theory. This dual nature of D-branes has been proved to be a useful tool to understand non-perturbative phenomena in gauge field theories, such as monopoles, Seiberg-Witten theory [48], etc. (Such dual nature of D-branes also plays a key role in AdS/CFT or holography. For our discussions in this article, however, we will not use AdS/CFT.)

We will describe below, what D-branes are, how we can construct topological phases from D-branes, and why they give us an insight on the periodic table. In fact, the opposite is also true; topological phases deepen our understanding of the stability of D-branes. As we will argue below, the stability of the so-called “non-BPS” D-branes is directly related to the $Z_2$ topological index of topological insulators.

4.3. topological phases and D-branes

4.3.0.5. introduction We would now like to point out an interesting connection between topological phases in condensed matter and D-branes. D-branes are topologically stable objects in string theory; At the level of classical (super)gravity, which is a low-energy effective field theory of string theory, D-brane can be visualized as a $(p+1)$-dimensional solitonic solution to the 10d Einstein equation (such D-brane is called D$p$-brane).

What is behind the stability of D-branes is the fact that there is a “charge” associated to them. These charges are quantized, and hence they cannot change for a smooth deformation of field configurations; D-branes are thus stable.

The charge of a stable D-brane can be either integer- or $Z_2$-valued, depending on the types of D-branes and string theory. This is summarized in Table 2. Observe that for type IIA and IIB string theory, “0” and “Z” appear closely follows the Bott periodicity. Compare entries “type IIB” and “type IIA” in Table 2 with “complex” symmetry classes (A and AIII) in Table 1, and entries “type I” in Table 2 with eight “real” symmetry classes in Table 1. In fact, it was argued that D-brane charges are classified by K-theory [49,50,51]; one then cannot help speculating on a possible connection between D-branes and the periodic table of topological phases.

For some cases where a D-brane has an integral charge, D-branes are an electric or a magnetic source of an Abelian $p$-form gauge field $C^{(p)} = (1/p!) C^{x_1 \cdots x_p}$, the so-called “Ramond-Ramond” (RR) gauge field. An integral of the RR-gauge flux generated by a magnetic D-brane, $\int_S dC^{(p)}$, measures the charge of the D-brane, where $S$ is a hypersurface which encloses the D-brane. For D-branes with a $Z_2$ charge, while we do see K-theory charge exist, it is not possible to write it down as a quantized integral of the RR flux.

4.3.0.6. K-theory charge of D-branes As mentioned above, D-branes are a topologically stable object in string theory; D$p$-brane is a $(p+1)$-d solitonic object in 10d space-time of classical (super)gravity theory. What is behind the stability of D-branes is the fact that there is a “charge” associated to them. These charges are quantized, and hence they cannot change for a smooth deformation of field configurations; D-branes are thus stable.

The charge of a stable D-brane can be either integer- or $Z_2$-valued, depending on the types of D-branes and string theory. This is summarized in Table 2. Observe that for type IIA and IIB string theory, “0” and “Z” appear in an alternating fashion. On the other hand, for type I string theory, the way K-theory charges “0”, “Z” and “Z_2” appear closely follows the Bott periodicity. Compare entries “type IIB” and “type IIA” in Table 2 with “complex” symmetry classes (A and AIII) in Table 1, and entries “type I” in Table 2 with eight “real” symmetry classes in Table 1. In fact, it was argued that D-brane charges are classified by K-theory [49,50,51]; one then cannot help speculating on a possible connection between D-branes and the periodic table of topological phases.

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4.3.0.7. D-brane configurations for topological phases For our purposes, we consider D-brane configurations which consist of two types of D-branes, a D$p$-brane and a D$q$-brane. They are located in parallel and do not intersect (Fig. 2) [52,53,54]. One can then ask what kind of (gauge) field theory is realized in such D-brane configuration. In string theory, quantum fields are realized as a vibration of a fundamental string.

In our configuration, an open string can have its end points on D-branes. Let us first consider a string which has one end on a D$p$-brane and the other on a D$q$-brane. Analyzing the vibrations of such string, one finds a massive fermion in the string spectrum. The mass of the fermion is proportional to the distance...
between the D-branes. For a relativistic fermion which is fully gapped by a mass term, following the periodic
table of topological phases, one can discuss (under a suitable set of discrete symmetries) its topological
stability against adiabatic deformation; one can “compute” its topological invariant or topological charge.
On the other hand, for the D-brane configuration at hand, if it is stable, one can assign a topological (K-
theory) charge to it, following the K-theory charge of D-branes. One can check these two types of topological
charges agree precisely. This is the first implication of the correspondence between the periodic table and
D-branes.

The Dirac fermion is not the only quantum field realized in the D-brane configuration. Let us now turn
our attention to a string which has its both ends on the D\textsubscript{p}-brane. Such string vibration gives rise to a
gauge field \(A_\mu\) living on the D-brane. (Its gauge group depends on the type of string theory and D-branes.) What is
the dynamics of such gauge field on the D-brane? This is answered by the effective action of D-branes. The
topological part (the so-called “Wess-Zumino term”) of the effective action for a D\textsubscript{p}-brane in a flat space
is

\[
S_{\text{WZ}} = \sum_q \int_X C_{(q+1)} \wedge \text{ch}(E) \hat{A}(TX)
\]

(27)

where integration (\(\int_X\)) is over the \((p+1)\)-d world-volume \(X\) of the D\textsubscript{p}-brane. In this action, \(C_{(q+1)}\) is the
background \((q+1)\)-form RR gauge field (which is, in our situation, sourced by the D\textsubscript{q}-brane). On the other
hand, \(\text{ch}(E)\) and \(\hat{A}(TX)\) depends on the field configuration on the D\textsubscript{p}-brane: The U(1) gauge bundle on
the brane is denoted by \(E\); \(TX\) and \(NX\) are the tangent bundle and normal bundle. Plugging the RR-field
generated by the D\textsubscript{q}-brane, whose integral \(\int dC_{(q+1)}\) is quantized as it measures the topological charge of the
D\textsubscript{q}-brane, the Wess-Zumino term of the D\textsubscript{p}-brane recovers precisely the response theory that we discusse
d in terms of anomaly.

As an example, let us consider the case with \(p = q = 5\);\(^4\)\(^6\)\(^7\) this D\textsubscript{p}-brane configuration realizes the 2d
QHE. The vibration of an open string stretching between the D-branes gives rise to a (2+1)d massive Dirac
fermion, \(\mathcal{H} = \psi^\dagger (-i \sum_{i=x,y} \sigma_i \partial_i + m \sigma_z) \psi\) where \(\psi\) is the two-component complex Dirac fermion field. On
the other hand, the WZ action of D\textsubscript{p}-brane is given by the Chern-Simons term:

\[
S_{\text{WZ}} = \frac{1}{2(2\pi)^2} \int_X C_{(2)} \wedge F \wedge F = \frac{1}{2(2\pi)^2} \int_{Y \times Z} C_{(2)} \wedge \text{d}(A \wedge F)
\]

\[
= \frac{-1}{8\pi^2} \int_Z dC_{(2)} \int_Y A \wedge F = \frac{\pm m}{8\pi |m|} \int_Y A \wedge \text{d}A.
\]

(28)

\(^4\) For the discussion below, we will focus on the situation where the topological charge of D-branes is an integer, for
presentational simplicity.

\(^5\) This is obtained from a more general expression (A.1) by noting that in our situation \(TX \otimes NX\) is a trivial bundle, and
hence \(A(TX \otimes NX) = \hat{A}(TX) \wedge \hat{A}(NX) = 1\).

\(^6\) Since we can apply the T-duality equivalence which shifts the value of \(p\) and \(q\) by one [57], we can fix the values of \(p\), say
\(p = 5\).

\(^7\) This configuration has 6 ND directions.
Here, the integral is over the 6d world volume of the Dp-brane, which we split into the common directions of the Dp- and Dq-branes (Y) and the compliment thereof (Z), and we noted that the Dq-brane couples magnetically to the RR two form $C_{(2)}$, and hence the integral $(2\pi)^{-1} \int dC_{(2)} = \pm 1/2$ measures the RR-charge of the Dq-brane. (We have done a partial integration.) The sign $\pm$ in front of the Chern-Simons term corresponds to $\text{sgn} \Delta x = \pm 1$ (see Fig. 2). Similarly, from the WZ coupling $S_{WZ} = \int_X C_{(2)} \wedge \hat{A}(R) = \frac{1}{12} \int_X C_{(2)} \wedge \frac{R}{i\pi} \wedge \frac{R}{i\pi}$, we obtain the gravitational Chern-Simons term $\sim \omega \wedge d\omega + \frac{2}{3} \omega^3$, as expected from the responses.

Observe the structural parallelism between the topological terms in the response theory of topological phases, and the Wess-Zumino action of D-branes. In the former, the effective action looks, typically, as $I_{\text{eff}} \propto (\text{topological invariant}) \times \int d^{d+1}x$ (topological term in gauge theory), where “(topological invariant)” is the topological invariant of the topological phase in question, and “(topological term in gauge theory)” is the term of topological origin in gauge theories such as the Chern-Simons term, or the axion term. For example, $I_{\text{eff}} \propto \text{Ch} \times \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ in the QHE, where Ch is the TKNN integer. In the D-brane construction of topological phases, the coefficient in front of the topological term is given by the integral of the RR-field, and measures the K-theory charge of the Dq-brane [55,56].

5. conclusion

We have described, in the order of increasing spatial dimensions, from two, three, and to arbitrary dimensions, the theory of thermal response in topological phases. Emphasized is a close analogy to electromagnetic responses of topological insulators by adopting the language of gravitomagnetism, which revealed a cross-correlation between thermal and mechanical (rotational) responses.

Let us close with discussion on effects of interactions. The fact that topological currents (either in electromagnetic or thermal response) are related to anomalies in field theories suggests topological phases with topological currents should be stable against interactions. This can be seen simply by observing that a term of topological origin in the effective action, once exists, has its coefficient which is quantized (in some case in the presence of some discrete symmetry). Thus, small interactions should not destroy topological properties of a given topological phase.  

We have computed the theta term (both electromagnetic and gravitational ones) by making use of chiral anomaly (which arises as a Fujikawa Jacobian). Let us imagine repeating the same calculations in the presence of interactions such as $\bar{\psi} \gamma^\mu \psi$ (a coupling to a bosonic scalar field $\varphi$), or electromagnetic interactions. The Adler-Bardeen theorem says that the anomaly (the Fujikawa Jacobian) will not be altered even in the presence of such interactions. (Historically, this nonrenormalization of the chiral anomaly was important to predict the number of quark colors even at the time when the details of the strong interaction were not known.)

Insensitivity of anomalies to interactions can further be illustrated by the “anomaly matching condition” proposed by ’t Hooft: an anomaly (i.e., topological current) can be computed in terms of either the infra-red (IR) or ultra-violet (UV) degrees of freedom, and the results should be the same. For example, the degrees of freedom in solids at UV are of “free-fermion” type, while deep in the IR region, such description can be replaced by quasi-particles (such as excitons) that arise due to interactions.

The discussion above naturally echoes in the D-brane construction of topological phases; the field theories realized by Dp-Dq systems come with, in addition to massive fermions we discussed, other fields and interactions among them. As we can understand more or less geometrically, the topological phases are nevertheless stable.

While we have discussed the stability of non-interacting topological phases against interactions, it remains largely an open problem what is the nature of strongly interacting topological phases which arise solely because of interactions, if they exist at all beyond the fractional quantum Hall effect. The existence of a

8. An alternative derivation of the electrical/thermal Středa formula mentioned in Footnote 3 also provides another link between topological currents and anomalies, suggesting the stability of topological currents against interactions.
topological current, however, should be a hallmark that we can use to characterize even for these putative “fractional topological insulators”.

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Appendix A. a short course for D-branes

A.1. what is D-brane?

In string theory [57], we consider a string as a fundamental object instead of an elementary particle. The vibration of a closed string produces, among others, the gravitational field at low energies. On the other hand, the vibration of an open string, which looks topologically like an interval with two ends, produces, among others, gauge fields in the low-energy limit.

A D-brane is defined as an object where open strings can end on it [57,59]; D-brane can be considered as an object giving a boundary condition to string vibrations – the name D-brane originates from “Dilchlet” brane.

In terms of the 10d gravity theory, which is the low-energy limit of string theory, D-branes are a stable solitonic object which is quite massive. A Dp-brane extends in p-spatial directions and in time-direction; i.e, its world volume is (p+1)-d. For example, a D0-brane and a D1-brane look like a particle (called D-particle) and a string (called D-string), respectively.

For our purposes, we will note the following properties (see Ref. [59]): (i) On a D-brane, an abelian gauge theory is realized. The fluctuations of the gauge field correspond to the fluctuations of the end points of open string. If we have \( N \) coincident Dp-branes, we obtain a non-abelian \( U(N) \) gauge theory. (ii) An intersection of two D-branes realizes massless fermions. (iii) D-branes are characterized by a K-theory charge. The last point will further be discussed below.

A.2. K-theory classification of stable D-branes

Gravitons and gauge fields are not the only fields that arise in superstring theory. In addition to them, the vibration of a closed string generates Abelian \( p \)-form gauge fields, the Ramond-Ramond (RR) gauge fields. (Here, we are focusing for a moment on a particular type of superstring theory, type IIA and IIB superstring theory (with or without orientifolds). Supersymmetric (or BPS) Dp-branes are charged under these gauge fields; they have the RR charges, and hence stable. A Dp-brane directly (electrically) couples to the \((p+1)\)-form RR field \( C_{(p+1)} \). An anti Dp-brane is defined to be the one with a negative RR charge. In type IIA superstring, \( p \) takes only even integer values i.e., \( p = 0, 2, 4, 6, 8 \) and in type IIB, \( p \) takes only odd ones \( p = -1, 1, 3, 5, 7, 9 \).

Moreover, a Dp-brane couples to other RR \( q \)-forms with \( q < p \) in the presence of the gauge flux on the brane. This is clearly described by the following formula of the RR couplings of a Dp-brane [58]:

\[
S_{\text{RR}} = \sum_q \int_X C_{(q+1)} \wedge \text{ch}(E_D) \sqrt{\frac{\hat{A}(TX)}{\hat{A}(NX)}},
\]

(A.1)

where \( TX \) and \( NX \) are the tangent bundle (i.e., the vector bundle tangent to the D-brane world-volume \( X \)) and normal bundle (i.e., the vector bundle which is normal to \( TX \)). The U(1) gauge bundle on the
brane is denoted by $E_D$. This formula (A.1) means that if there is a non-trivial gauge bundle on the Dp-brane, it is possible that there exist charges which correspond to lower dimensional D-brane charges. Such a configuration can be interpreted as a bound state of a Dp-brane and the lower dimensional D-branes.

In fact, there are D-branes which do not have any RR charges and which are nevertheless stable. They are not supersymmetric and are called non-BPS D-branes [60]. Also a system of a Dp-brane and an anti Dp-brane sometimes forms a stable bound state. Such a system is called a brane anti-brane system [61]. They typically exist in the presence of the special projection called the orientifold projection. These brane configurations exhaust all possible D-branes in string theory. The orientifold projection means that we require the invariance of string theory under the action $\Omega_q = I_q \cdot \Omega$, defined by the product of the parity $I$ with respect to $q$ spatial coordinates and the orientation reverse $\Omega$ of the string world-sheet. The set of fixed points of $I_q$ is called the orientifold $(9-q)$ plane.

Being a stable object in string theory, one could imagine what is protecting them to “decay”. It turns out that one can assign a K-theory charge to D-branes, which is the reason of stability. Indeed, the K-theory provides a very systematic classification of D-branes in string theory [50,51]. In mathematics, (topological) K-theory classifies vector bundles. More precisely, we start from a pair of two bundles $(E_1, E_2)$ on a manifold $X$ and consider its difference. In other words, we introduce the identification

$$ (E_1 \oplus H, E_2 \oplus H) \simeq (E_1, E_2). \quad (A.2) $$

This defines the K-group $K(X)$.

In string theory, this identification is naturally interpreted as follows. We start with a brane anti-brane system. The gauge bundles on the D-brane and the anti D-brane are regarded as $E_1$ and $E_2$. Typically a brane anti-brane system becomes unstable because the total RR charge is vanishing and it can pair-annihilate. In other words, there appears so-called a tachyon field in the open string between the brane and the anti-brane. The tachyon field has unstable potential energy and condenses like the Higgs effect, which makes the system decay into a lower-dimensional D-brane. This procedure is mathematically described by (A.2), which means that the charge is conserved under the tachyon condensation. If the gauge field configurations are the same i.e., $E_1 = E_2$, then the brane and anti-brane are completely annihilated, and nothing remains after the tachyon condensation. If $E_1 \neq E_2$, then eventually the system decays into a D-brane which corresponds to the difference between $E_1$ and $E_2$. We presented the K-theory classification of D-branes in type IIA, type IIB and type I string theory, where we take $X = S^{9-p}$ for a Dp-brane via a compactification procedure. Notice that type I string theory is defined to be the projection of type IIB string theory by an (SO type) orientifold 9-plane. For type IIA and type I, we need to use a different K-theory called $K^{-1}(X)$, which just shifts the dimension by one, and $KO(X)$, which is the real valued version of $K(X)$.

In particular, if we ignore the torsion of K-group $K(X)$, then it is known that $K(X)$ is reduced to even-dimensional cohomology $\oplus_{i \geq 0} H^{2i}(X, \mathbb{Q})$. Indeed, this is explicitly given by the Chern character and this nicely matches with the RR coupling formula (A.1) [49]. On the other hand, the argument based on K-theory with torsion is quite general and includes the case where the D-branes do not have any RR charges as is so for the non-BPS D-brane.

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It is, however, not clear if we can reduce the gravitational instanton term, by taking a weak gravitational field limit, to
\[ \propto \theta E_g \cdot B_g \]
which we would expect from the discussion in Subsec. 3.3. This seems related to the fact that the temperature gradient alone does not produce a non-trivial curvature background which is a source for an anomalous current (topological current) to flow \[46\]. While the cross-correlation discussed in Subsec. 3.3 is obtained from the application of the Kubo formula to a microscopic model, it may be the case that it has a different origin than that of the anomalous current captured by the gravitational instanton term (23). For our purposes in Subsecs. 4.2 and 4.3, however, we regard the presence/absence of the gravitational instanton term (and its analogues in other dimensions and in other symmetry classes) as a hallmark of the topologically non-trivial/trivial ground state, without making much contact with thermal/mechanical responses.

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\[54\] Dp-Dq systems have common \((d + 1)\) dimensions; this is where a \(d\)-dimensional topological insulator exists. From the viewpoint of open string, the \((d + 1)\) dimensions has the Neumann boundary condition at the both ends (called the NN direction). In the same way, \((p + q - 2d)\) coordinates are along Neumann-Dirichlet (ND) directions, while the remaining \((9 - p - q + d)\) ones are in the DD direction.

\[55\] In this section, for illustration purposes, we have not discussed how discrete symmetries are implemented in the D-brane realization of topological phases. We found that particle-hole symmetry corresponds to the orientation reverse \(\Omega\) of strings, while chiral (or sublattice) symmetry to the parity transformation in one of the DD directions, respectively. Time-reversal is nothing but the orientifold projection.

With these implementations of discrete symmetries, our D-brane systems are characterized by the directions of Dp, Dq and the orientifold. Classifying these conditions (aided by an analysis on the stability of boundary modes of these systems), we found that the allowed D-brane systems are in one-to-one correspondence to the ten-fold classification of topological phases.

\[56\] Their edge states are obtained by replacing one of DD directions by a ND direction. The chiral fermions typical in edge states appear due to the topological mechanism. This construction leads to the correct fermion spectra for the topological insulators and their edge states.

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