EXISTENCE, LOCAL UNIQUENESS AND PERIODICITY OF BUBBLING SOLUTIONS FOR A CRITICAL NONLINEAR ELLIPTIC EQUATION

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Abstract. We revisit the following nonlinear critical elliptic equation

$$-\Delta u + Q(y)u = u^{N+2\over N-2}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 5$. There seems to be no results about the periodicity of bubbling solutions. Here we try to investigate some related problems. Assuming that $Q(y)$ is periodic in $y_1$ with period 1 and has a local minimum at 0 satisfying $Q(0) = 0$, we prove the existence and local uniqueness of infinitely many bubbling solutions of the problem above. This local uniqueness result implies that some bubbling solutions preserve the symmetry of the potential function $Q(y)$, i.e. the bubbling solution whose blow-up set is $\{ (jL, 0, ..., 0) : j = 0, \pm 1, \pm 2, ..., \pm m \}$ must be periodic in $y_1$ provided that $L$ is large enough, where $m$ is the number of the bubbles which is large enough but independent of $L$. Moreover, we also show a non-existence of this bubbling solutions for the problem above if the local minimum of $Q(y)$ does not equal to zero.

Key words: Multi-bubbling solutions, Critical exponents, Prescribed scalar curvature, Local uniqueness, Periodicity.

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1. Introduction and the main results

In this paper, we consider the following nonlinear elliptic equations with critical exponent:

$$-\Delta u + Q(y)u = u^{N+2\over N-2}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N,$$  \hspace{1cm} (1.1)

where $N \geq 5$, and $Q(y)$ is a bounded nonnegative smooth function.

This problem corresponds to the following well-known Brezis-Nireberg problem in $S^N$,

$$-\Delta_{S^N} u = u^{N+2\over N-2} + \mu u, \quad u > 0,$$ \hspace{1cm} (1.2)

after using the stereographic project, problem (1.2) can be reduced to (1.1) with $Q(y) = -4\mu - N(N-2)\over (1 + |y|^2)^2$ and $Q(y) > 0$ if $\mu < -N(N-2)\over 4$. Problem (1.2) has been studied widely, one can refer to [5, 6, 13, 15].

When there is no linear term $Q(y)u$ in equation (1.1), i.e the following well-known prescribed scalar curvature problem

$$-\Delta u = K(y)u^{N+2\over N-2}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N,$$ \hspace{1cm} (1.3)

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where \( N \geq 5 \), and \( K(y) \) is a bounded nonnegative smooth function. Compared with problem (1.1), there are many results about problem (1.3). When \( K(y) \) is positive and periodic, Li proved that (1.3) has infinitely many multi-bump solutions for \( N \geq 3 \) in \([20, 21, 22]\) by gluing approximate solutions into genuine solutions with masses concentrating near isolated sets of maximum points of \( K(y) \). Similar results were obtained by Yan in \([29]\) provided that \( K(y) \) has a sequence of strict local maximum points tending to infinity. When \( K(y) \) is positive and periodic, in \([28]\) Xu constructed multi-bump solutions with mass concentrating near critical points of \( K(y) \) including saddle points. When \( K(y) \) is a positive radial function with a strict local maximum at \(|y| = r_0 > 0\) and satisfies

\[
K(r) = K(r_0) - c_0|r - r_0|^\alpha + O(|r - r_0|^{\alpha + \epsilon})
\]

for some constants \( c_0 > 0, \epsilon > 0 \) and \( \alpha \in [2, N - 2) \) near \(|x| = r_0\), Wei and Yan obtained in \([27]\) solutions with a large number of bumps concentrating near the sphere \(|x| = r_0\) and \( N \geq 5 \). Assuming that \( K(y) > 0 \) is periodic in \((x_1, ..., x_k)\) with \( 1 \leq k < \frac{N-2}{2} \), under some natural conditions on \( K(y) \) near a critical point, Li, Wei and Xu in \([23]\) proved the existence of multi-bump solutions where the centers of bumps can be placed in some lattices in \( \mathbb{R}^k \), including infinite lattices. They also showed that for \( k \geq \frac{N-2}{2} \), no such solutions exist. In \([11]\), Deng, Lin and Yan obtained a local uniqueness and periodicity result of bubbling solutions for (1.3) under the basic assumptions of \([23]\). For poly-harmonic prescribed scalar curvature problem, under some similar assumptions of the prescribed scalar curvature as in \([11]\), in \([18]\) Guo, Peng and Yan studied the existence, local uniqueness and periodicity of a bubbling solutions where the bubble set are a \( k \)-dimensional lattice. About more other related results, one can refer to \([1, 2, 3, 10, 14]\).

Since when \( Q \geq 0 \) and \( Q \neq 0 \), the mountain pass value for problem (1.1) is not a critical value of the corresponding functional, all the arguments based on the concentration compactness arguments \([24, 25]\) can not be used to obtain an existence result of solutions for (1.1). So far, there are very few existence results of solutions for problem (1.1). To our best knowledge, the first existence result for problem (1.1) is from \([4]\) where Benci and Cerami proved that if \( \|Q\|_{L^{\frac{N}{N-2}}(\mathbb{R}^N)} \) is suitably small, problem (1.1) has a solution whose energy is in the interval \((\frac{1}{N}S^2, \frac{2}{N}S^\frac{N}{2})\), where \( S \) is the best Sobolev constant in the embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N) \). Note that the assumption \( Q \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \) excludes the case \( Q \geq c_0 > 0 \). Later, in order to cancel this restriction, in \([9]\) Chen, Wei and Yan has proved problem (1.1) has infinitely many non-radial solutions where \( Q(y) \) is a radial bounded positive function, and \( r^2Q(r) \) has a local maximum point or a local minimum point \( r_0 > 0 \) with \( Q(r_0) > 0 \). Assuming that \( Q(y) \) satisfies a weaker symmetric condition, Peng, Wang and Yan in \([26]\) has shown that problem (1.1) has infinitely many solutions by combining the finite dimensional reduction argument and local Pohozaev identities, where the concentrated points of the bubble solutions include a saddle point of a function involving the potential function \( Q(y) \). Very recently, He, Wang and Wang in \([19]\) have proved the non-degeneracy of the bubble solutions constructed in \([26]\) by some local Pohozaev identities and also construct a new type of bubble solutions for problem (1.1).
As far as we know, the results on the uniqueness of solutions which have the concentration phenomena are few. But it is a very profound topic to study the uniqueness of solutions. In this aspect, the first result is the uniqueness of solutions concentrating at one point for Dirichlet problems with critical nonlinearity on bounded domains given by Glangetas in [16]. When the right nonlinearity of equation (1.1) is subcritical, by calculating the number of single-bump solutions to the equation, Grossi [17] proved that there are one solution concentrating at any non-degenerate critical point of $Q(y)$. Later, using topological degree, Cao and Heniz [7] gained the uniqueness of multi-bump solutions to the subcritical problem. For more other work about the uniqueness of solutions with the concentration phenomena, one can refer to [8, 11, 18].

In this paper, motivated by [11, 18, 26], we aim to study the existence, the local uniqueness and the periodicity of the bubble solutions for equation (1.1), where we mainly want to study the impact of the linear term $Q(y)u$ in equation (1.1) to them. Here we call the solutions are local unique if two sequences of solutions blow up at the same set. This uniqueness implies certain kind of symmetry. We will prove that the two solutions are the same by obtaining some useful estimates and applying some local Pohozaev identities.

We assume that $Q(y)$ satisfies the following conditions:

$(Q_1)$ $0 \leq \inf_{y \in \mathbb{R}^N} Q(y) < \sup_{y \in \mathbb{R}^N} Q(y) < \infty$;

$(Q_2)$ $Q \in C^1(\mathbb{R}^N)$ is 1-periodic in $y_1$;

$(Q_3)$ $Q(y)$ has a local minimum value at 0 with $Q(0) = 0$ and there exists some real number $\beta \in (N - 4, N - 2)$ such that for all $|y|$ small, it holds

$$Q(y) = a|y|^\beta + O(|y|^{\beta+1}) = 0 + a|y|^\beta + O(|y|^{\beta+1}),$$

where $a > 0$.

To state the main results of this paper, we need to introduce some notations first. It is well known that the functions

$$U_{x,\mu} = (N(N-2))^{\frac{N-2}{4}} \left( \frac{\mu}{1 + \mu^2|y-x|^2} \right)^{\frac{N-2}{2}}, \; \mu > 0, \; x \in \mathbb{R}^N$$

are the only solutions to the problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, \; u > 0, \; \text{in } \mathbb{R}^N. \quad (1.4)$$

Denote $x_j = (jL, 0, 0, \cdots, 0), j = 0, \pm 1, \pm 2, \ldots, \pm m$, where $L > 0$ is a large integer. We are going to construct a bubbling solution, concentrating at $x_j, j = 0, \pm 1, \pm 2, \ldots, \pm m$. For this purpose, we take $x_{jL},$ which is close to $x_j, \mu_L > 0$ large and denote $W_{x_{jL},\mu_L} = \xi(y-x_{jL})U_{x_{jL},\mu_L},$ $W_{x,\mu_L} = \sum_{j=-m}^{m} W_{x_{jL},\mu_L}$, where $\xi \in C^\infty_0(B_2(0))$ satisfying $\xi(|y|) = \xi(|y|)$, $\xi = 1$ in $B_1(0)$, and $\xi = 0$ in $\mathbb{R}^N \setminus B_2(0)$. Applying the cut-off function $\xi$ above can not only help us deal with the slow decay of the bubble $U$ when the space dimension is not big but also make some calculations much simpler.

Here we introduce the following norms which capture the decay property of the perturbation term
\[ \|\phi\|_* = \sup_{x \in \mathbb{R}^N} \left( \sum_{j=-m}^{m} \frac{\mu_L^N}{(1 + \mu_L|y - x_jL|)^{\frac{N}{2} + \tau}} \right) |\phi(y)|, \]
\[ \|f\|_{**} = \sup_{x \in \mathbb{R}^N} \left( \sum_{j=-m}^{m} \frac{\mu_L^{N+2}}{(1 + \mu_L|y - x_jL|)^{\frac{N}{2} + \tau}} \right) |f(y)|, \]

where \( \tau = 1 + \vartheta \) and \( \vartheta \in (0, \min\left(\frac{N-4}{2}, \frac{\beta}{2}\right)) \) is a fixed small constant.

The existence result in our paper is the following.

**Theorem 1.1.** Suppose that \( Q(y) \geq 0 \) is bounded and belongs to \( C^1 \). If \( Q(y) \) satisfies the assumptions \((Q_1)-(Q_3)\) and \( N \geq 5 \), then there is an integer \( L_0 > 0 \), such that for any integer \( L \geq L_0 \) and for any positive integer \( m \) large enough which is independent of \( L \), \( (1.1) \) has a solution \( u_L \) of the form

\[ u_L = W_{x,\mu_L}(y) + \varphi_L = \sum_{j=-m}^{m} \xi(y - x_jL)U_{x_jL,\mu_L} + \varphi_L \quad (1.5) \]

for some \( x_jL \) and \( \mu_L \), with

\[ x_jL = x_j + o_L(1) \text{ for all } j, \quad (1.6) \]

and

\[ \mu_L = L^{\frac{N-4}{2}}(\tilde{B} + o_L(1)), \quad (1.7) \]

for some constant \( \tilde{B} > 0 \) and

\[ |\varphi_L(y)| = o_L(1) \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y - x_jL|)^{\frac{N}{2} + \tau}}, \quad (1.8) \]

where \( o_L(1) \to 0 \) as \( L \to +\infty \).

**Remark 1.2.** In fact, with careful estimates of \( u_L \) in the weighted \( L^\infty \) spaces, similar to the proof of Theorem 2 in [23], we can prove that \( u_L \) converges locally in \( \mathbb{R}^N \) to a solution \( u \) with the form \( u = \sum_{j=-\infty}^{\infty} \xi(y - x_jL)U_{x_jL,\mu_L} \) as \( m \to \infty \) independent of \( L \).

Now we also give a nonexistence result of the bubbling solutions for equation \( (1.1) \).

**Theorem 1.3.** Assume that \( Q(y) = Q(x_j) + a|y - x_j|^\beta + O(|y - x_j|^{\beta+1}), x_j = (jL, 0, ..., 0), j = 0, \pm 1, ..., \pm m \) and \( Q(x_j) > 0 \), where \( a > 0, \beta \in (N - 4, N - 2) \), \( L, m \) are integers and \( L \) is large enough, \( N \geq 5 \). Then there has no \( m \)-bubbles solutions \( u_L \) of the form \((1.5)\) satisfying \((1.6), (1.7)\), and

\[ \|\varphi_L(y)\|_* = o_L\left(\frac{1}{\mu_L^2}\right). \quad (1.9) \]

**Remark 1.4.** To obtain Theorem 1.3 \((1.8)\) is not enough. And we need more accurate estimate of \( \varphi_L \) i.e. \((1.9)\).
Remark 1.5. From the non-existence result of Theorem 1.3, we know that the assumption $Q(x_j) = 0, j = 0, \pm 1, ..., \pm m$ is natural and necessary.

To discuss the symmetric properties of the solutions obtained in Theorem 1.1, we proceed with the following local uniqueness result for the bubbling solutions of (1.1).

Our main result is as follows.

Theorem 1.6. Under the same assumptions as in Theorem 1.1 and $N > 6$, if $u^{(1)}_L$ and $u^{(2)}_L$ are two sequence of solutions of problem (1.1), which have the form (1.5) satisfying (1.6), (1.7) and (1.8), then $u^{(1)}_L = u^{(2)}_L$ provided $L > 0$ is large enough.

Remark 1.7. Different from the existence result of Theorem 1.1, we assume that $N > 6$ in Theorem 1.6. This is because that the estimate of $\|\varphi_L\|_{s}$ is not good enough caused by the appearance of the linear term $Q(y)u$ in equation (1.1). And we need to require $N > 6$ in the proof of (3.9). Therefore, when $N = 5, 6$, our methods are not suitable and the local uniqueness and the periodicity of the bubbling solutions are still open.

Remark 1.8. In the proof of Theorem 1.6 in order to obtain some precise estimates involved by the linear term $Q(y)u$ in equation (1.1), we choose the radius $\delta$ of a small ball centered at some $x_{jL}$ satisfying $\delta = \mu_L^{-\theta}$, where $\theta > 0$ (defined in (3.3)).

Our results here shows that concentration of the solution results in the local uniqueness. That is, if two sequence of solutions blow up at the same set, they must coincide.

Theorem 1.6 can be used to study some other properties of the bubbling solutions. A direct consequence of this result is the following periodic property of the solutions.

Theorem 1.9. Under the same assumption as in Theorem 1.1 and $N > 6$, and $u_L$ is a solution of (1.1), which have the form (1.5) satisfying (1.6), (1.7) and (1.8), then $u_L$ is periodic in $y_1$ which has infinite bumps as stated in Remark 1.2, provided that $L > 0$ is large enough and $m$ goes to infinity independent of $L$.

We would like to stress that the novelty and the main difficulty in the proofs of all the Theorems above are the techniques used to obtain some new and delicate estimates involved by the linear term $Q(y)u$ in problem (1.1).

The structure of our paper is as follows. In Section 2, we prove the existence and nonexistence of bubbling solutions by the finite dimensional reduction method and a contradiction argument respectively. In Section 3, applying some local Pohozaev identities, we prove the local uniqueness of the bubbling solutions which can induce the periodicity. Some technical and important estimates are put in Appendices A, B and C.

2. EXISTENCE AND NONEXISTENCE OF SOLUTIONS WITH INFINITELY MANY BUBBLES

In this section, we will prove Theorems 1.1 and 1.3. To prove Theorem 1.1, we mainly use the finite dimensional reduction method. And we use a contradiction argument to show Theorem 1.3.
2.1. Linearization and the finite-dimensional problem. We first construct a bubbling solutions blowing up at finite points. We define the function spaces $X$ and $Y$ as follows: $\phi \in X$ if $\|\phi\|_* < +\infty$, while $f \in Y$ if $\|f\|_* < +\infty$. Set

$$Z_{ij} = \xi(y - x_{iL}) \frac{\partial W_{x_{iL},\mu_L}}{\partial x_{iL}}, Z_{i,N+1} = \frac{\partial W_{x_{iL},\mu_L}}{\partial \mu_L}, i = 0, \pm 1, \ldots, \pm m, \quad j = 1,2,\ldots,N$$

and

$$H_m := \left\{ \phi : \phi \in X, \int_{\mathbb{R}^N} \phi W^{2^*-2}_{x_{iL},\mu_L} Z_{i,j} = 0, i = 0, \pm 1, \ldots, \pm m, \quad j = 1,\ldots,N+1 \right\}. \quad (2.1)$$

Denote $W_{x,\mu} = \sum_{j=-m}^{m} \xi(y - x_{iL}) U_{x_{iL},\mu}, \varphi \in H_m$. We intend to find a solution of the form $W_{x,\mu} + \varphi$ for problem (1.1) with $\|\varphi\|_*$ small enough. To achieve this goal, we first prove that for fixed $(x, \mu_L)$, there exists a smooth function $\varphi \in H_m$ such that

$$-\Delta (W_{x,\mu} + \varphi) + Q(y)(W_{x,\mu} + \varphi) = (W_{x,\mu} + \varphi)^{2^*-1} + \sum_{i=-m}^{m} \sum_{j=1}^{N+1} c_{ij} W^{2^*-2}_{x_{iL},\mu_L} Z_{ij}; \quad (2.2)$$

for some constants $c_{ij}$. Then, we show the existence of $(x, \mu_L)$ such that

$$\int_{\mathbb{R}^N} -\Delta (W_{x,\mu} + \varphi) Z_{ij} + Q(y)(W_{x,\mu} + \varphi) Z_{ij} - \int_{\mathbb{R}^N} (W_{x,\mu} + \varphi)^{2^*-1} Z_{ij} = 0. \quad (2.3)$$

With this $(x, \mu_L)$, it is easy to prove that all $c_{ij}$ must be zero.

**Part I: the reduction.** In this part, for fixed $(x, \mu_L)$ we find $\varphi(x, \mu_L)$, such that $C^1$ in $(x, \mu_L)$ and (2.2) holds. Indeed, we will apply the contraction mapping theorem to prove the following result.

**Proposition 2.1.** Under the assumptions of Theorem 1.1, if $L > 0$ is sufficiently large, (2.2) admits a unique solution $\varphi_L \in H_m$ such that $\|\varphi_L\|_* \leq C \left( \frac{1}{\mu_L} \right)^{\frac{N}{2} - \frac{2}{\tau} + m}$, $|c_l| \leq C \left( \frac{1}{\mu_L} \right)^{\frac{N}{2} - \frac{2}{\tau} + m}$. Moreover, $\varphi_L$ is $C^1$ in $(x, \mu_L)$.

First, we consider the following linear problem

$$\begin{cases}
-\Delta \varphi + Q(y) \varphi - (2^* - 1) W_{x,\mu}^{2^*-2} \varphi = h + \sum_{i=-m}^{m} \sum_{j=1}^{N+1} c_{ij} W_{x_{iL},\mu_L}^{2^*-2} Z_{ij}, \\
\varphi \in H_m, \quad \sum_{j=-m}^{m} \int_{\mathbb{R}^N} W_{x_{iL},\mu_L}^{2^*-2} Z_{jj} \varphi = 0, l = 1,2,\ldots,N+1.
\end{cases} \quad (2.4)$$

for some real number $c_{ij}$, where $h$ is a function in $Y$.

**Lemma 2.2.** Suppose that $\varphi_L$ solves (2.4) for $h = h_L$. If $\|h_L\|_*$ goes to zero as $L$ goes to infinity, then so does $\|\varphi_L\|_*$.

**Proof.** We argue by a contradiction argument. Suppose that there exist $L \to +\infty$, and $\varphi_L$ solving (2.4) for $h = h_L$, $\mu = \mu_L$ with $\|h_L\|_* \to 0$ and $\|\varphi_L\|_* \geq C > 0$. We may assume that $\|\varphi_L\|_* = 1$. For simplicity, we drop the subscript $L$. 

Since $Q(y)$ is non-negative, we have
\[ |\varphi(y)| \leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} W_{\lambda L}^{2-2} |\varphi(z)| dz \]
\[ + C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \left(|h| + \sum_{j=1}^{N+1} \sum_{i=-m}^{m} c_{ij} W_{x_i L}^{2-2} Z_{ij} |\right) dz. \]

As in [26], using Lemmas B.2 and B.3, we can prove
\[ \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} W_{\lambda L}^{2-2} |\varphi| dz \leq C \|\varphi\|_{\ast} \sum_{j=-m}^{m} \frac{n_{-2}}{\lambda L} \frac{1}{(1 + \lambda L|y - x_j L|)^{N-2 + \tau}}, \]
\[ \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} |h(z)| dz \leq C \|h\|_{\ast} \sum_{j=-m}^{m} \frac{n_{-2}}{\lambda L} \frac{1}{(1 + \lambda L|y - x_j L|)^{N-2 + \tau}}, \]

and
\[ \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \sum_{j=-m}^{m} W_{x_j L}^{2-2} Z_{ij}^2 dz \leq C \sum_{j=-m}^{m} \frac{n_{-2} + n_{l}}{\lambda L} \frac{1}{(1 + \lambda L|y - x_j L|)^{N-2 + \tau}}, \]
where $n_{j} = 1, j = 0, 1, 2, \ldots, N$, $n_{N+1} = -1$, $\tau, \theta$ are positive and small enough.

To estimate $c_{ij}$, $i = 0, 1, 2, \ldots, \pm m$, $j = 1, 2, \ldots, N + 1$, multiplying (2.4) by $Z_{ij}$ and integrating, we see that $c_{ij}$ satisfies
\[ \sum_{j=-m}^{m} \sum_{i=-m}^{m} \int_{\mathbb{R}^N} c_{ij} W_{x_i L}^{2-2} Z_{ij}^2 = \langle -\Delta \varphi + Q(y)\varphi - (2^* - 1) W_{x_i L}^{2-2} \varphi, Z_{ij} \rangle - \langle h, Z_{ij} \rangle. \quad (2.5) \]

It follows from Lemma B.1 that
\[ \|\langle h, Z_{ij} \rangle\| \leq C n_{L} \|h\|_{\ast}. \quad (2.6) \]

By direct computation, we have
\[ |\langle Q(y)\varphi, Z_{il} \rangle| \leq C \|\varphi\|_{\ast} \int_{\mathbb{R}^N} \frac{\xi}{(1 + \lambda L|y - x_i L|)^{N-2}} \sum_{j=-m}^{m} \frac{1}{(1 + \lambda L|y - x_j L|)^{N-2 + \tau}} \]
\[ = \frac{\mu_{L}}{\mu_{1+\tau}} \|\varphi\|_{\ast}. \quad (2.7) \]

On the other hand, we have
\[ |\langle -\Delta \varphi - (2^* - 1) W_{x_i L}^{2-2} \varphi, Z_{il} \rangle| = O\left(\frac{n_{L} \|\varphi\|_{\ast}}{\mu_{1+\tau}}\right). \quad (2.8) \]

Combining (2.6)-(2.8), we have
\[ \langle -\Delta \varphi + Q(y)\varphi - (2^* - 1) W_{x_i L}^{2-2} \varphi, Z_{il} \rangle - \langle h, Z_{il} \rangle = O\left(\frac{n_{L} \|\varphi\|_{\ast}}{\mu_{1+\tau}} + \|h\|_{\ast}\right). \]
It is easy to check that
\[
\sum_{j=-m}^{m} \langle W_{x_j, \mu_L}^{2^* - 2} Z_{jh}, Z_{il} \rangle = (c + o(1)) \delta_{hL} \mu_L^{2n}
\]  
(2.9)
for some constant \(c > 0\).

Now inserting (2.8) and (2.9) into (2.5), we find
\[
c_{il} = \frac{1}{\mu_L^{2^*}} (o(\|\varphi\|_* + o(\|h\|_{**})))
\]
so
\[
\|\varphi\|_* \leq c \left( o(1) + \|h\|_{**} + \sum_{j=-m}^{m} \frac{1}{(1+\mu_L|y-x_jL|)^{\frac{N}{2}+\rho}} \right).
\]  
(2.10)

We can finish the proof of this lemma by using (2.10) as in [18].

From Lemma 2.2, applying the same argument as in the proof of Proposition 4.1 in [12]. We can prove the following result.

**Lemma 2.3.** There exist \(L_0\) and a constant \(C > 0\) independent of \(L\), such that for \(L > L_0\) and all \(h \in L^\infty(\mathbb{R}^N)\), problem (2.4) has a unique solution \(\varphi \equiv \mathcal{L}(h)\). Moreover
\[
\|\mathcal{L}(h)\|_* \leq C\|h\|_{**}, \quad |c_i| \leq \frac{C}{\mu_L} \|h\|_{**}.
\]

Now we consider
\[
\begin{cases}
-\Delta(W_{x, \mu_L} + \varphi) + Q(y)(W_{x, \mu_L} + \varphi) = (W_{x, \mu_L} + \varphi)^{2^* - 1} + \sum_{i=-m}^{m} \sum_{j=1}^{N+1} c_{ij} W_{x_i, \mu_L}^{2^* - 2} Z_{ij}, \\
\varphi \in H_m, \int_{\mathbb{R}^N} \sum_{i=-m}^{m} W_{x_i, \mu_L}^{2^* - 2} Z_{ij} \varphi = 0, j = 1, 2, \ldots, N + 1.
\end{cases}
\]  
(2.11)

Rewrite as
\[
-\Delta \varphi + Q(y)\varphi - (2^* - 1)W_{x, \mu_L}^{2^* - 2}\varphi = N_L(\varphi) + l_L + \sum_{i=-m}^{m} \sum_{j=1}^{N+1} c_{ij} W_{x_i, \mu_L}^{2^* - 2} Z_{ij},
\]  
(2.12)
where
\[
N_L(\varphi) = (W_{x, \mu_L} + \varphi)^{2^* - 1} - W_{x, \mu_L}^{2^* - 1} - (2^* - 1)W_{x, \mu_L}^{2^* - 2}\varphi,
\]
and
\[
l_L = (W_{x, \mu_L}^{2^* - 1} - \sum_{j=-m}^{m} \xi(y - x_jL) U_{x_j, \mu_L}^{2^* - 1}) - Q(y)W_{x, \mu_L} + \sum_{j=-m}^{m} U_{x_j, \mu_L} \Delta \xi + 2 \nabla \xi \nabla (\sum_{j=-m}^{m} U_{x_j, \mu_L}).
\]
In order to apply the contraction mapping theorem to prove that (2.12) is uniquely solvable, we have to estimate $N_L(\varphi)$ and $l_L$ respectively.

**Lemma 2.4.** If $N \geq 5$, then
\[
\|N_L(\varphi)\|_{**} \leq C\|\varphi\|_{*} \min(2^{*}-1,2), \quad \|l_L\|_{**} \leq \frac{C}{\mu_L^{\frac{N-2}{2}-\tau}}.
\]

**Proof.** Since the proofs are just the same as that of Lemmas A.5 and A.6 in Appendix A, we omit it.

**Proof of Proposition 2.1.** First set
\[
\mathcal{N} = \left\{ w : w \in C(\mathbb{R}^N) \cap H_m, \|w\|_{*} \leq \frac{1}{\mu_L^{\frac{N-2}{2}-\tau-\epsilon}}, \int_{\mathbb{R}^N} \sum_{j=-m}^{m} W_{x_j,\mu_L}^{2^{*}-2} Z_j w = 0 \right\},
\]
where $\epsilon > 0$ small and $l = 1, 2, \cdots, N+1$.

Then (2.12) is equivalent to
\[
\varphi = A(\varphi) =: \mathcal{L}(N_L(\varphi)) + \mathcal{L}(l_L),
\]
where $\mathcal{L}$ is defined in Lemma 2.2. We will prove that $A$ is a contraction map from $\mathcal{N}$ to $\mathcal{N}$. We only consider the case $N \geq 6$ since that of $N = 5$ is similar.

Firstly, we have
\[
\|A(\varphi)\|_{*} \leq C(\|N_L(\varphi)\|_{**} + \|l_L\|_{**}) \leq C \left( \|\varphi\|_{*} \min(2^{*}-1,2) + \left( \frac{1}{\mu_L} \right)^{\frac{N-2}{2}} \right) \leq \frac{C}{\mu_L^{\frac{N-2}{2}-\tau-\epsilon}}.
\]

Hence, $A$ maps $\mathcal{N}$ to $\mathcal{N}$.

On the other hand, we see
\[
\|A(\varphi_1) - A(\varphi_2)\|_{*} \leq C\|N_L(\varphi_1) - N_L(\varphi_2)\|_{**}.
\]

It is easy to check that if $N \geq 6$, then
\[
|N_L(\varphi_1) - N_L(\varphi_2)| \leq |N_L'(\varphi_1 + \theta\varphi_2)| |\varphi_1 - \varphi_2| \leq C(|\varphi_1|^{2^{*}-2} + |\varphi_2|^{2^{*}-2}) |\varphi_1 - \varphi_2|
\]
\[
\leq C(\|\varphi_1\|_2^{2^{*}-2} + \|\varphi_2\|_2^{2^{*}-2}) \|\varphi_1 - \varphi_2\|_{*} \left( \sum_{j=-m}^{m} \frac{\mu_L^{\frac{N-2}{2}}}{(1 + \mu_L |y - x_j|)^{\frac{N-2}{2}+\tau}} \right)^{2^{*}-1}
\]
\[
\leq C(\|\varphi_1\|_2^{2^{*}-2} + \|\varphi_2\|_2^{2^{*}-2}) \|\varphi_1 - \varphi_2\|_{*} \sum_{j=-m}^{m} \frac{\mu_L^{\frac{N-2}{2}}}{(1 + \mu_L |y - x_j|)^{\frac{N-2}{2}+\tau}}.
\]

Hence
\[
\|A(\varphi_1) - A(\varphi_2)\|_{*} \leq C(\|\varphi_1\|_2^{2^{*}-2} + \|\varphi_2\|_2^{2^{*}-2}) \|\varphi_1 - \varphi_2\|_{*} \leq \frac{1}{2} \|\varphi_1 - \varphi_2\|_{*}.
\]

Therefore, $A$ is a contraction map.

Now by the contraction mapping theorem, there exists a unique $\varphi \in \mathcal{N}$ such that
\[
\|\varphi\|_{*} \leq \|\mathcal{L}(N_L(\varphi))\|_{*} + \|\mathcal{L}(l_L)\|_{*} \leq C(\|N_L(\varphi)\|_{**} + \|l_L\|_{**}) \leq C \left( \frac{1}{\mu_L} \right)^{\frac{N-2}{2}-\tau}.
\]
Moreover, we get the estimate of $c_l$ from Lemma 2.3. □

Part II: the finite-dimensional problem

It follows from Lemmas A.7 and A.8, Proposition 2.1, (2.3) is equivalent to

$$|x_j - x_{jL}| = O\left(\frac{1}{\mu_L^2}\right),$$

and

$$\sum_{i \neq j} \mu^{-\beta+2} \frac{C_4}{|x_i - x_{jL}|^{N-2}} = O\left(\frac{1}{\mu_L^{\beta+3}}\right).$$

2.2. Proof of the existence result.

Proof of Theorem 1.1. We need to solve (2.13) and (2.14). Note that

$$\frac{1}{|x_i - x_j|^{N-2}} = \frac{1}{|i - j|^{N-2} L^{N-2}} := d_{ij}, \quad i \neq j.$$

So, we can use Lemma D.1 in [18] to obtain the result. □

2.3. Proof of the nonexistence result. In this subsection, we mainly use a contradiction argument to prove Theorem 1.3. To obtain a contradiction, we need to obtain some technical and careful estimates.

Proof of Theorem 1.3. For simplicity of notations, here we still denote $W_{x,\mu_L} = \sum_{j=-m}^{m} \xi(y-x_{jL})U_{x_{jL},\mu_L}$. Assume that problem (1.1) has a solution $u_{x,\mu_L}$, that is

$$-\Delta(W_{x,\mu_L} + \varphi_L) + Q(y)(W_{x,\mu_L} + \varphi_L) = (W_{x,\mu_L} + \varphi_L)^{2^*-1}.$$

Since $U_{x_{jL},\mu_L}$ satisfies $-\Delta\left(\sum_{j=-m}^{m} U_{x_{jL},\mu_L}\right) = \sum_{j=-m}^{m} U_{x_{jL},\mu_L}^{2^*-1}$, we can rewrite (2.15) as

$$\sum_{j=-m}^{m} Q(x_j)U_{x_{jL},\mu_L}$$

$$= - \sum_{j=-m}^{m} (Q(y) - Q(x_j))U_{x_{jL},\mu_L} + \sum_{j=-m}^{m} Q(y)(1 - \xi(y-x_{jL}))U_{x_{jL},\mu_L}$$

$$- (-\Delta \varphi_L + Q(y)\varphi_L) + \left(W_{x,\mu_L} + \varphi_L\right)^{2^*-1} - \sum_{j=-m}^{m} \xi(y-x_{jL})U_{x_{jL},\mu_L}^{2^*-1}$$

$$+ \sum_{j=-m}^{m} \Delta \xi(y-x_{jL})U_{x_{jL},\mu_L} + 2 \sum_{j=-m}^{m} \nabla \xi(y-x_{jL}) \cdot \nabla U_{x_{jL},\mu_L}.$$ (2.16)
Multiply $U_{x_1, \mu_1}$ on both sides of equation (2.16) and integrate over $\mathbb{R}^N$. Applying integrating by parts, we obtain

$$
\int_{\mathbb{R}^N} \sum_{j=-m}^{m} Q(x_j) U_{x_1, \mu_1} U_{x_1, \mu_1} = -\int_{\mathbb{R}^N} \sum_{j=-m}^{m} (Q(y) - Q(x_j)) U_{x_1, \mu_1} U_{x_1, \mu_1} + \int_{\mathbb{R}^N} \sum_{j=-m}^{m} Q(y) \left(1 - \xi(y - x_j L)\right) U_{x_1, \mu_1} U_{x_1, \mu_1}
$$

$$
- \int_{\mathbb{R}^N} \left(-\Delta \varphi_L + Q(y) \varphi_L\right) U_{x_1, \mu_1} + \int_{\mathbb{R}^N} \left((W_{x_1, \mu_1} + \varphi_L)^{2^* - 1} - \sum_{j=-m}^{m} \xi(y - x_j L) U_{x_j}^{2^* - 1}\right) U_{x_1, \mu_1}
$$

$$
+ \int_{\mathbb{R}^N} \sum_{j=-m}^{m} \Delta \xi(y - x_j L) U_{x_1, \mu_1} U_{x_1, \mu_1} + 2 \int_{\mathbb{R}^N} \sum_{j=-m}^{m} \nabla \xi(y - x_j L) \cdot \nabla U_{x_1, \mu_1} U_{x_1, \mu_1}.
$$

Now we estimate every term of (2.17) respectively. First, we have

$$
\int_{\mathbb{R}^N} \sum_{j=-m}^{m} Q(x_j) U_{x_1, \mu_1} U_{x_1, \mu_1} = \int_{\mathbb{R}^N} Q(x_1) U_{x_1, \mu_1}^2 + \sum_{j \neq 1} \int_{\mathbb{R}^N} Q(x_j) U_{x_1, \mu_1} U_{x_1, \mu_1} = \frac{1}{\mu_L^2} \int_{\mathbb{R}^N} Q(x_1) U^2 + o\left(\frac{1}{\mu_L^2}\right).
$$

From the assumption of $Q(y)$, we have

$$
- \int_{\mathbb{R}^N} \sum_{j=-m}^{m} (Q(y) - Q(x_j)) U_{x_1, \mu_1} U_{x_1, \mu_1}
$$

$$
= - \int_{\mathbb{R}^N} (Q(y) - Q(x_i)) U_{x_1, \mu_1}^2 - \int_{\mathbb{R}^N} \sum_{j \neq i} (Q(y) - Q(x_j)) U_{x_1, \mu_1} U_{x_1, \mu_1}
$$

$$
= - \int_{B_{\frac{1}{\mu_L}}(x_i)} a|x - x_i|^{\beta} + O(|x - x_i|^{\beta + 1}) U_{x_1, \mu_1}^2 - \int_{\mathbb{R}^N \setminus B_{\frac{1}{\mu_L}}(x_i)} (Q(y) - Q(x_i)) U_{x_1, \mu_1}^2 + o\left(\frac{1}{\mu_L^2}\right)
$$

$$
= o\left(\frac{1}{\mu_L^2}\right),
$$

where $\epsilon > 0$ is small. Also, we have

$$
\int_{\mathbb{R}^N} \sum_{j=-m}^{m} Q(y) \left(1 - \xi(y - x_j L)\right) U_{x_1, \mu_1} U_{x_1, \mu_1}
$$

$$
= \int_{\mathbb{R}^N} Q(y) \left(1 - \xi(y - x_1 L)\right) U_{x_1, \mu_1}^2 + \sum_{j \neq 1} \int_{\mathbb{R}^N} Q(y) \left(1 - \xi(y - x_j L)\right) U_{x_1, \mu_1} U_{x_1, \mu_1}.
\[
\int_{\mathbb{R}^3 \setminus B_1(x_{iL})} Q(y)(1 - \xi(y - x_{iL}))U_{x_{iL},\mu_L}^2 + o\left(\frac{1}{\mu_L^2}\right) = o\left(\frac{1}{\mu_L^2}\right).
\]

Applying integrating by parts and by \(\|\varphi_L\|_* = o\left(\frac{1}{\mu_L^2}\right)\), we have

\[
\int_{\mathbb{R}^N} \Delta \varphi_L U_{x_{iL},\mu_L} = - \int_{\mathbb{R}^N} U_{x_{iL},\mu_L}^{2^*-1} \varphi_L \\
\leq C\|\varphi_L\|_* \int_{\mathbb{R}^N} \sum_{j=-m}^m \mu_L^{N-2} \left(1 + \mu_L|y - x_j|\right)^{-\frac{N-2}{2} + \tau} \left(1 + \mu_L^2|y - x_{iL}|^{N-2}\right)^{2^*-1} \\
\leq C\|\varphi_L\|_* = o\left(\frac{1}{\mu_L^2}\right).
\]

Similarly, we can estimate

\[
- \int_{\mathbb{R}^N} Q(y) \varphi_L U_{x_{iL},\mu_L} \\
\leq C\|\varphi_L\|_* \int_{\mathbb{R}^N} \sum_{j=-m}^m \mu_L^{N-2} \left(1 + \mu_L|y - x_j|\right)^{-\frac{N-2}{2} + \tau} \left(1 + \mu_L^2|y - x_{iL}|^{N-2}\right)^{2^*-1} \\
\leq C\frac{\|\varphi_L\|_*}{\mu_L^{N-2}} = o\left(\frac{1}{\mu_L^2}\right).
\]

Noting that if \(|y - x_{kL}| \leq 2\), then for any \(k \neq j\) and \(L\) large enough, there holds

\(|y - x_{jL}| \geq |x_k - x_j| - (|x_{kL} - x_k| + |x_{jL} - x_j|) - |y - x_{kL}| \geq |k - j|L - 2 + o_L(1) \geq \frac{|k - j|L}{2} > 2,\)

which implies that \(\xi(y - x_{kL})\) and \(\xi(y - x_{jL})\) can not be equal to non-zero at the same time. Therefore, we have

\[
\int_{\mathbb{R}^N} \left((W_{x_{iL}} + \varphi_L)^{2^*-1} - \sum_{j=-m}^m \xi(y - x_{jL})U_{x_{jL}}^{2^*-1}\right)U_{x_{iL},\mu_L} \\
= \int_{\mathbb{R}^N} \left((W_{x_{iL}} + \varphi_L)^{2^*-1} - W_{x_{iL}}^{2^*-1}\right)U_{x_{iL},\mu_L} + \int_{\mathbb{R}^N} \left(W_{x_{iL}}^{2^*-1} - \sum_{j=-m}^m \xi(y - x_{jL})U_{x_{jL}}^{2^*-1}\right)U_{x_{iL},\mu_L} \\
= \int_{\mathbb{R}^N} \left((W_{x_{iL}} + \varphi_L)^{2^*-1} - W_{x_{iL}}^{2^*-1}\right)U_{x_{iL},\mu_L} + \sum_{j=-m}^m \int_{\mathbb{R}^N} (\xi_j^{2^*-1} - \xi_j)U_{x_{jL},\mu_L}^{2^*-1} U_{x_{iL},\mu_L} \\
=: \Gamma_1 + \Gamma_2.
\]
where for simplicity we denote $\xi_j(y) =: \xi(y - x_{jL})$. When $N \geq 6$ and $1 < 2^* - 1 \leq 2$, we have

$$
\Gamma_1 \leq C \int_{\mathbb{R}^N} \left( W_{x_i, \mu_L}^{2^*-2} |\varphi_L| + |\varphi_L|^{2^*-1} \right) U_{x_i, \mu_L}
$$

$$
\leq C \|\varphi_L\|_{*} \int_{\mathbb{R}^N} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L^2 |y - x_{jL}|^{2^*/2})^{2^*/4}} \right)^{2^*-2} \sum_{j=-m}^{m} \frac{\mu_L^N}{(1 + \mu_L |y - x_{jL}|^{2^*/2})^{2^*/4}} \left( 1 + \frac{1}{(1 + \mu_L^2 |y - x_{iL}|^{2^*/2})^{N/2}} \right)
$$

$$
+ C \|\varphi_L\|_{*}^{2^*-1} \int_{\mathbb{R}^N} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L |y - x_{jL}|^{2^*/2})^{2^*/4}} \right)^{2^*-1} \frac{\mu_L^N}{(1 + \mu_L^2 |y - x_{iL}|^{2^*/2})^{N/2}}
$$

$$
= o \left( \frac{1}{\mu_L^2} \right), \quad (2.24)
$$

and

$$
\Gamma_2 = \sum_{j=-m}^{m} \int_{B(x_{jL}) \setminus B(x_{iL})} \left( \xi_j^{2^*-1} - \xi_j \right) U_{x_i, \mu_L}^{2^*-1} U_{x_j, \mu_L}
$$

$$
= \int_{B(x_{iL}) \setminus B(x_{jL})} \left( \xi_i^{2^*-1} - \xi_i \right) U_{x_i, \mu_L}^{2^*-1} U_{x_j, \mu_L} + \sum_{j \neq i} \int_{B(x_{jL}) \setminus B(x_{iL})} \left( \xi_j^{2^*-1} - \xi_j \right) U_{x_i, \mu_L}^{2^*-1} U_{x_j, \mu_L}
$$

$$
= O \left( \frac{1}{\mu_L^2} \right) + O \left( \sum_{j \neq i} \int_{B(\mu_L(0)) \setminus B(\mu_L(0))} \frac{1}{(1 + |y|^2)^{N+2}/2} \frac{1}{(1 + |y - \mu_L(x_{iL} - x_{jL})|^2)^{N/2}} \right)
$$

$$
= O \left( \frac{1}{\mu_L^2} \right) + O \left( \frac{1}{\mu_L^{2N-2}} \right) = o \left( \frac{1}{\mu_L^2} \right). \quad (2.25)
$$

Similarly, when $N = 5$ and $2^* - 1 = \frac{7}{3} > 2$, we have

$$
\Gamma_1 \leq C \int_{\mathbb{R}^5} \left( W_{x_i, \mu_L}^{\frac{5}{4}} |\varphi_L| + W_{x_i, \mu_L}^{\frac{5}{4}} |\varphi_L|^2 + |\varphi_L|^{\frac{5}{4}} \right) U_{x_i, \mu_L}
$$

$$
\leq C \|\varphi_L\|_{*} \int_{\mathbb{R}^5} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L^2 |y - x_{jL}|^{2^*/2})^{2^*/4}} \right)^{\frac{3}{4}} \sum_{j=-m}^{m} \frac{\mu_L^5}{(1 + \mu_L |y - x_{jL}|^{2^*/2})^{2^*/4}} \left( 1 + \frac{1}{(1 + \mu_L^2 |y - x_{iL}|^{2^*/2})^{N/2}} \right)
$$

$$
+ C \|\varphi_L\|_{*}^{2^*-1} \int_{\mathbb{R}^5} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L^2 |y - x_{jL}|^{2^*/2})^{2^*/4}} \right)^{2^*-1} \frac{\mu_L^5}{(1 + \mu_L^2 |y - x_{iL}|^{2^*/2})^{N/2}}
$$

$$
+ C \|\varphi_L\|_{*}^{\frac{7}{4}} \int_{\mathbb{R}^5} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L |y - x_{jL}|^{2^*/2})^{2^*/4}} \right)^{\frac{7}{4}} \frac{\mu_L^5}{(1 + \mu_L^2 |y - x_{iL}|^{2^*/2})^{N/2}}
$$

$$
= o \left( \frac{1}{\mu_L^2} \right), \quad (2.26)
$$
and
\[
\Gamma_2 = \sum_{j=-m}^{m} \int_{B_2(x_jL)\setminus B_1(x_jL)} (\xi_j^2 - \xi_i) U_{x_jL,\mu_L} U_{x_iL,\mu_L} \\
= \int_{B_2(x_jL)\setminus B_1(x_jL)} (\xi_j^2 - \xi_i) U_{x_jL,\mu_L} + \sum_{j \neq i} \int_{B_2(x_jL)\setminus B_1(x_jL)} (\xi_j^2 - \xi_j) U_{x_jL,\mu_L} U_{x_iL,\mu_L} \\
= O \left( \frac{1}{\mu_L^2} \right) + O \left( \sum_{j \neq i} \int_{B_{2\mu_L}(0)\setminus B_{\mu_L}(0)} \frac{1}{(1+|y|^2)^{N-2}} \frac{1}{(1+|y-\mu_L(x_iL-x_jL)|^2)^{\frac{N}{2}}} \right) \\
= O \left( \frac{1}{\mu_L^2} \right) + O \left( \frac{1}{\mu_L^5} \right) = o \left( \frac{1}{\mu_L^2} \right). \tag{2.27}
\]

Finally, we estimate
\[
\int_{\mathbb{R}^N} \sum_{j=-m}^{m} \Delta \xi (y-x_jL) U_{x_jL,\mu_L} U_{x_iL,\mu_L} \\
\leq C \int_{B_2(x_jL)\setminus B_1(x_jL)} \sum_{j=-m}^{m} U_{x_jL,\mu_L} U_{x_iL,\mu_L} \\
\leq C \int_{B_2(x_iL)\setminus B_1(x_iL)} U_{x_iL,\mu_L}^2 + \sum_{j \neq i} \int_{\mathbb{R}^N} U_{x_jL,\mu_L} U_{x_iL,\mu_L} \\
\leq \frac{C}{\mu_L^2} \int_{B_{2\mu_L}(0)\setminus B_{\mu_L}(0)} \frac{1}{(1+|y|^2)^{N-2}} + o \left( \frac{1}{\mu_L^2} \right) \\
= o \left( \frac{1}{\mu_L^2} \right). \tag{2.28}
\]

Similar to (2.29), there holds
\[
2 \int_{\mathbb{R}^N} \sum_{j=-m}^{m} \nabla \xi (y-x_jL) \cdot \nabla U_{x_jL,\mu_L} U_{x_iL,\mu_L} = o \left( \frac{1}{\mu_L^2} \right). \tag{2.29}
\]

From (2.17) to (2.29), we obtain
\[
o \left( \frac{1}{\mu_L^2} \right) = \int_{\mathbb{R}^N} \sum_{j=-m}^{m} Q(x_j) U_{x_jL,\mu_L} U_{x_iL,\mu_L} = \frac{1}{\mu_L^2} \int_{\mathbb{R}^N} Q(x_i) U^2 + o \left( \frac{1}{\mu_L^2} \right).
\]

Observing that \( \int_{\mathbb{R}^N} Q(x_i) U^2 \neq 0 \) since \( Q(x_i) > 0 \), we get a contradiction.

\[\square\]

3. LOCAL UNIQUENESS AND PERIODICITY

3.1. **Some estimates of the bubbling solutions.** Let \( u_L \) be a solution of (1.1), which have the form (1.5) satisfying (1.6), (1.7) and (1.8). In this section, we will estimate \( \mu_L \) and

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|x_jL - x_j|. The following two Pohozaev identities play an important role in these estimates:

\[-\int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial y_i} \frac{\partial u_L}{\partial y_i} + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla u_L|^2 \nu_i + \frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 \nu_i - \frac{1}{2} \int_{B_\delta(x_jL)} \frac{\partial Q}{\partial y_i} u_L^2 = \frac{1}{2^*} \int_{\partial B_\delta(x_jL)} u_L^{2^*} \nu_i, \tag{3.1}\]

and

\[-\int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial y_i} (y - x_jL, \nabla u_L) + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla u_L|^2 (y - x_jL, \nu) + \frac{2 - N}{2} \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial y_i} u_L

+ \frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 (\nu, y - x_jL) - \frac{1}{2} \int_{\partial B_\delta(x_jL)} u_L^2 (\nabla Q, y - x_jL) - \int_{B_\delta(x_jL)} Q(y) u_L^2

= \frac{1}{2^*} \int_{\partial B_\delta(x_jL)} u_L^{2^*} (\nu, y - x_jL), \tag{3.2}\]

where \(\nu\) is the outward unit normal of \(\partial B_\delta(x_jL)\).

We would like to point out that in order to estimate \(\frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 \nu_i\) in (3.1) and \(\frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 (\nu, y - x_jL)\) in (3.2) more accurately, different from \([11, 18]\) in the sequel we take \(\delta = \mu_L^{-\theta}\) (\(\theta > 0\)), where

\[\theta = \max\left\{\frac{\beta + 4 - N}{\beta + 1 - 2\tau}, 1\right\}. \tag{3.3}\]

**Lemma 3.1.** Relation (3.1) is equivalent to

\[\int_{B_\delta(x_jL)} \frac{\partial Q(y)}{\partial y_i} U_{x_jL,\mu_jL}^2 = O\left(\frac{1}{\mu_L^{N/2} L^{N-2}} + \frac{1}{\mu_L^{\beta+2}} + \max |x_iL - x_i|^{2\beta-1}\right). \tag{3.4}\]

**Proof.** First, we have

\[\int_{\partial B_\delta(x_jL)} u_L^2 \nu_i = O\left(\frac{1}{\mu_L^{N}}\right) \tag{3.5}\]

and

\[\int_{\partial B_\delta(x_jL)} Q(x) u_L^2 \nu_i = O\left(\frac{1}{\mu_L^{\beta+2}}\right). \tag{3.6}\]

whose proof we put in Appendix [C].

On the other hand, note that

\[U_{x_jL,\mu L}(y) = \frac{(N(N - 2))^{\frac{N-2}{4}}}{\mu_L^{\frac{N-2}{4}}} \frac{1}{|y - x_jL|^{N-2}} + O\left(\frac{1}{\mu_L^{\frac{N-2}{2}}}\right), \quad y \in \partial B_\delta(x_jL).\]

We can deduce from Corollary [3.2]

\[\frac{1}{N(N - 2)^{\frac{N-2}{4}}} \left(- \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial y_i} \frac{\partial u_L}{\partial y_i} + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla u_L|^2 \nu_i\right)\]
Moreover, from
\[ \int_{\partial B_3(x_{jL})} \frac{1}{\mu_L^{N-2}} \frac{\partial^2}{\partial y_i^2} \partial_{|y-x_jL|^{N-2}} \frac{1}{\partial y_i} \partial_{|y-x_jL|^{N-2}} + \frac{1}{2\mu_L^{N-2}} \int_{\partial B_3(x_{jL})} |\nabla \frac{1}{|y-x_jL|^{N-2}}| \nu_i \]
\[ + O\left( \frac{1}{\mu_L^{N-2}L^{N-2}} + \frac{1}{\mu_L^N} + \frac{\max_i |x_{iL} - x_i|^{\beta}}{\mu_L^{\frac{N-2}{2} + \tau}} + \frac{\max_i |x_{iL} - x_i|^{2\beta}}{\mu_L^{2\tau}} \right) \]
\[ = O\left( \frac{1}{\mu_L^{N-2}L^{N-2}} + \frac{1}{\mu_L^N} + \frac{\max_i |x_{iL} - x_i|^{\beta}}{\mu_L^{\frac{N-2}{2} + \tau}} + \frac{\max_i |x_{iL} - x_i|^{2\beta}}{\mu_L^{2\tau}} \right). \tag{3.7} \]

Moreover, from
\[ \int_{B_3(x_{jL})} U_{x_{jL},\mu L} \sum_{i \neq j} U_{x_{iL},\mu L} + \left( \sum_{i \neq j} U_{x_{iL},\mu L} \right)^2 = O\left( \frac{1}{\mu_L^{N-2}L^{N-2}} \right), \tag{3.8} \]
and
\[ \int_{B_3(x_{jL})} \partial Q u_{x_{jL},\mu L} = \int_{B_3(x_{jL})} \partial Q \left( U_{x_{jL},\mu L} + \sum_{k \neq j} \xi(y - x_{kL})U_{x_{kL},\mu L} + \varphi_L \right)^2 \]
\[ = \int_{B_3(x_{jL})} \partial Q U_{x_{jL},\mu L}^2 + C\|\varphi_L\| \left( \frac{1}{\mu_L^{\beta+1}} + |x_{jL} - x_j|^{\beta-1} \right) + \frac{1}{\mu_L^{N-2}L^{N-2}} \]
\[ = \int_{B_3(x_{jL})} \partial Q U_{x_{jL},\mu L}^2 + C\left( \frac{1}{\mu_L^{\beta+2}} + \max_j |x_{jL} - x_j|^{2\beta-1} \right) + \frac{1}{\mu_L^{N-2}L^{N-2}}. \tag{3.9} \]

Combining (3.7)-(3.9), we obtain
\[ \int_{B_3(x_{jL})} \partial Q U_{x_{jL},\mu L}^2 = O\left( \frac{1}{\mu_L^{N-2}L^{N-2}} + \frac{1}{\mu_L^{\beta+2}} + \max |x_{iL} - x_i|^{2\beta-1} \right). \]

\[ \Box \]

**Lemma 3.2.** Relation (3.2) is equivalent to
\[ \frac{1}{\mu_L^{\beta+2}} = \sum_{i \neq j} \frac{B}{\mu_L^{N-2}|x_{iL} - x_jL|^{N-2}} + O\left( \frac{1}{\mu_L^N} + \frac{\max |x_{iL} - x_i|}{\mu_L^{\beta+1}} + \frac{\max_i |x_{iL} - x_i|^{\beta-1}}{\mu_L^\beta} + \frac{1}{\mu_L^{\beta+3}} \right), \tag{3.10} \]
where \( B > 0 \) is a constant.

**Proof.** Noting that for \( y \in \partial B_3(x_{jL}) \), we have
\[ u_L(y) = D_L(y) + O\left( \frac{1}{\mu_L^{N-2}} + \frac{\max |x_{iL} - x_i|^{\beta}}{\mu_L^\tau} \right) \]
\[ =: (N(N-2))^{N-2} \sum_{i=-m}^m \frac{1}{\mu_L^N |y - x_{iL}|^{N-2}} + O\left( \frac{1}{\mu_L^{N-2}} + \frac{\max |x_{iL} - x_i|^{\beta}}{\mu_L^\tau} \right). \]
We can deduce
\[- \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} (y - x_jL, \nabla u_L) + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla u_L|^2 (y - x_jL, \nu) + \frac{2 - N}{2} \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} u_L \]
\[= - \int_{\partial B_\delta(x_jL)} \frac{\partial D_L}{\partial \nu} (y - x_jL, \nabla D_L) + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla D_L|^2 (y - x_jL, \nu) + \frac{2 - N}{2} \int_{\partial B_\delta(x_jL)} \frac{\partial D_L}{\partial \nu} D_L + O\left( \frac{1}{\mu_L^N} + \max_i \frac{|x_{iL} - x_i|^\beta}{\mu_L^{N-2+\tau}} + \max_i \frac{|x_{iL} - x_i|^{2\beta}}{\mu_L^{2\tau}} \right).\]

On the other hand, it is easy to check that
\[\int_{\partial B_\delta(x_jL)} \frac{1}{(y - x_jL)^{N-2}} (y - x_jL, \nabla \frac{1}{y - x_jL}) + \frac{1}{2} \int_{\partial B_\delta(x_jL)} \nabla \frac{1}{y - x_jL} (y - x_jL, \nu) \]
\[+ \frac{2 - N}{2} \int_{\partial B_\delta(x_jL)} \frac{1}{(y - x_jL)^{N-2}} \frac{1}{y - x_jL} = 0.\]

So we find
\[- \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} (y - x_jL, \nabla u_L) + \frac{1}{2} \int_{\partial B_\delta(x_jL)} |\nabla u_L|^2 (y - x_jL, \nu) + \frac{2 - N}{2} \int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} u_L \]
\[= B' \sum_{i \neq j} \frac{1}{\mu_L^{N-2} |x_{iL} - x_j|^{N-2}} + O\left( \frac{1}{\mu_L^N} + \frac{\max_i |x_{iL} - x_i|^\beta}{\mu_L^{N-2+\tau}} + \frac{\max_i |x_{iL} - x_i|^{2\beta}}{\mu_L^{2\tau}} \right),\]
where $B' > 0$ is a constant.

Similar to (3.11), we can also obtain
\[\int_{\partial B_\delta(x_jL)} Q(y) u_L^2 (y - x_jL, \nu) = O\left( \frac{1}{\mu_L^\beta} \right).\]

On the other hand, we have
\[\int_{B_\delta(x_jL)} u_L^2 (\nabla Q(y), y - x_jL) \]
\[= a \int_{\mathbb{R}^N} |\nabla y - x_jL|^\beta (y - x_jL) u_L^2 + O\left( \frac{\max_i |x_{iL} - x_i|}{\mu_L^{\beta+1}} + \frac{\max_i |x_{iL} - x_i|^{\beta-1}}{\mu_L^3} + \frac{1}{\mu_L^{\beta+3}} \right) \]
\[= a \frac{\beta}{\mu_L^{\beta+2}} \int_{\mathbb{R}^N} |y|^{\beta} u_L^2 + O\left( \frac{\max_i |x_{iL} - x_i|}{\mu_L^{\beta+1}} + \frac{\max_i |x_{iL} - x_i|^{\beta-1}}{\mu_L^3} + \frac{1}{\mu_L^{\beta+3}} \right).\]

Since $Q(y)$ is positive and $Q(y) = a|x - x_j|^\beta + O(|x - x_j|^{\beta+1})$ in $B_\delta(x_jL)$, similar to (3.13) we obtain
\[\int_{B_\delta(x_jL)} Q(y) u_L^2 = O\left( \frac{1}{\mu_L^{\beta+2}} \right).\]
Finally we estimate the right hand side of (3.2)
\[ - \int_{\partial B_\delta(x_jL)} u_L^2 \langle y - x_jL, \nu \rangle = O\left( \frac{1}{\mu_L^N} \right). \]  
(3.15)
Thus, the result follows from (3.11)-(3.14).
\[ \square \]

**Proposition 3.3.** There holds
\[ |x_jL - x_j| = O\left( \frac{1}{\mu_L^2} \right). \]

**Proof.** It follows from (3.1) that
\[ \frac{1}{2} \int_{B_\delta(x_jL)} \frac{\partial Q}{\partial y_i} u_L^2 = -\int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} \frac{\partial u_L}{\partial y_i} + \frac{1}{2} \int_{B_\delta(x_jL)} |\nabla u_L|^2 \nu_i + \frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 \nu_i - \frac{1}{2} \int_{\partial B_\delta(x_jL)} u_L^2 \nu_i. \]
(3.16)
Similar to Lemma 3.1, we obtain
\[ -\int_{\partial B_\delta(x_jL)} \frac{\partial u_L}{\partial \nu} \frac{\partial u_L}{\partial y_i} + \frac{1}{2} \int_{\Omega} |\nabla u_L|^2 \nu_i + \frac{1}{2} \int_{\partial B_\delta(x_jL)} Q(y) u_L^2 \nu_i - \frac{1}{2} \int_{\partial B_\delta(x_jL)} u_L^2 \nu_i \]
\[ = \frac{1}{\mu_L^{N-2}} |x_iL - x_jL|^{N-2} + O\left( \frac{1}{\mu_L^{\beta+2}} \right). \]
(3.17)
On the other hand, from Lemma A.4, we have
\[ \int_{B_\delta(x_jL)} \frac{\partial Q}{\partial y_i} u_L^2 = \frac{a\beta}{\mu_L^{\beta+1}} \int_{B_\delta(x_jL)} |\mu_L^{-1} y + x_jL - x_j|^\beta |\mu_L^{-1} y + x_jL - x_j| \mu_L U_{0,1}^2 \]
\[ + O\left( \frac{1}{\mu_L^{\beta+2}} \right). \]
(3.18)
It follows from (3.17) and (3.18) that
\[ \int_{B_\delta(x_jL)} |y + x_0|^\beta |y_i + x_{0,i}| \mu_L U_{0,1}^2 = o(1), \]
which yields \( x_0 = 0. \)

Observing that \( \int_{B_\delta(x_jL)} |y_i|^\beta |y_i U_{0,1}^2 = 0, \) we get
\[ \int_{B_\delta(x_jL)} \frac{\partial Q}{\partial y_i} u_L^2 = \frac{a\beta}{\mu_L^{\beta+1}} \int_{B_\delta(x_jL)} |y|^\beta |y U_{0,1}^2 \mu_L (x_jL - x_j)_i + O\left( \frac{1}{\mu_L^{\beta+2}} \right). \]
So we find
\[ \mu_L (x_jL - x_j)_i = O\left( \frac{1}{\mu_L} \right), \]
which shows

$$|x_j - x_j| = O\left(\frac{1}{\mu_L^2}\right).$$

□

Remark 3.4. Applying Lemma 3.2 and Proposition 3.3, we can prove that to obtain a bubbling solution such that the center of different bubbles are separated from each other, the constant $\beta$ in assumption $(Q_3)$ must satisfy $\beta > N - 4$. In fact, if $\beta \leq N - 4$, then (3.10) yields

$$\frac{1}{\mu_L^{\beta+2}} = O\left(\frac{1}{\mu_L^{N-2}L^{N-2}} + \frac{1}{\mu_L^{\beta+3}}\right),$$

which is impossible if $\beta \leq N - 4$.

Now, we prove the following Proposition.

Proposition 3.5. It holds

$$\mu_L = L^{\frac{N-2}{3-N+4}} \left(\bar{B} + \frac{1}{\mu_L}\right),$$

for some constant $\bar{B} > 0$.

Proof. By Proposition 3.3 $|x_j - x_j| = O\left(\frac{1}{\mu_L}\right)$, we find

$$\frac{1}{|x_i - x_j|^{N-2}} = \frac{1}{(|x_i - x_j| + O\left(\frac{1}{\mu_L}\right))^{N-2}} = \frac{1}{|x_i - x_j|^{N-2}} \left(1 + \frac{1}{\mu_L^2}\right).$$

As a result, we see that (3.10) is equivalent to

$$\frac{1}{\mu_L^{\beta+2}} = B \sum_{i \neq j} \frac{1}{\mu_L^{N-2} |x_i - x_j|^{N-2}} + O\left(\frac{1}{\mu_L^{\beta+3}} + \frac{1}{\mu_L^N}\right). \quad (3.19)$$

We can easily deduce from (3.19) that

$$c_0 L^{\frac{N-2}{3-N+4}} \leq \mu_L \leq c_1 L^{\frac{N-2}{3-N+4}}.$$

Let $\frac{a_j}{\mu_L} = \frac{a_{jL}}{L^{\frac{(3-N+4)}{2}}}$. Then $0 < c_0 \leq a_{jL} \leq c_1 < +\infty$, and from Lemma C.1, we obtain

$$a_{jL}^\beta = \bar{B} \sum_{i \neq j} \frac{a_{iL}}{|i - j|^{N-2}} + O\left(\frac{1}{\mu_L}\right).$$

Thus, we complete the result. □
3.2. **Local uniqueness.** Assume that problem (1.1) has two different solutions \( u_L^{(1)} \) and \( u_L^{(2)} \), which blow up at \( x_j, j = 0, \pm 1, \pm 2, \ldots, \pm m \). For \( l = 1, 2 \), we will use \( x_j^{(l)} \) and \( \mu_L^{(l)} \) to denote the center and the height of the bubbles appearing in \( u_L^{(l)} \), respectively.

Let
\[
\eta_L = \frac{u_L^{(1)} - u_L^{(2)}}{\|u_L^{(1)} - u_L^{(2)}\|_*}.
\] (3.20)

Then, \( \eta_L \) satisfies \( \|\eta_L\|_* = 1 \) and
\[
- \Delta \eta_L + Q(y)\eta_L = f(y, u_L^{(1)}, u_L^{(2)}),
\] (3.21)
where
\[
f(y, u_L^{(1)}, u_L^{(2)}) = \frac{1}{\|u_L^{(1)} - u_L^{(2)}\|_*} \left( (u_L^{(1)})^{2^* - 1} - (u_L^{(2)})^{2^* - 1} \right).
\] (3.22)

Write
\[
f(y, u_L^{(1)}, u_L^{(2)}) = c_L(y)\eta_L(y),
\] (3.23)
where
\[
c_L(y) = (2^* - 1) \int_0^1 (tu_L^{(1)}(y) + (1 - t)u_L^{(2)}(y))^{2^* - 2} dt.
\] (3.24)

It follows from Propositions 3.3 and 3.5 that
\[
U_{x_j^{(1)}, \mu_L^{(1)}} - U_{x_j^{(2)}, \mu_L^{(2)}} = O \left( |x_j^{(1)} - x_j^{(2)}| \|U_{x_j^{(1)}, \mu_L^{(1)}} - \nabla U_{x_j^{(1)}, \mu_L^{(1)}}| + |\mu_L^{(1)} - \mu_L^{(2)}| \|\partial_\mu U_{x_j^{(1)}, \mu_L^{(1)}}\| \right)
\]
\[
= O \left( \frac{1}{\mu_L} U_{x_j^{(1)}, \mu_L^{(1)}} \right),
\]
which gives
\[
u_L^{(1)} - \nu_L^{(2)} = O \left( \frac{1}{\mu_L} \sum_{i=-m}^{m} U_{x_i^{(1)}, \mu_L^{(1)}} + |\varphi_L^{(1)}| + |\varphi_L^{(2)}| \right).
\] (3.25)

Thus, we have proved
\[
c_L(y) = (2^* - 1) U_{x_j^{(1)}, \mu_L^{(1)}}^{2^* - 2} + O \left( \left( \frac{1}{\mu_L} U_{x_j^{(1)}, \mu_L^{(1)}} \right)
\right.
\]
\[
+ \frac{1}{\mu_L^{N - 2}} \|\varphi_L^{(1)}\| + \|\varphi_L^{(2)}\| )^{2^* - 2}, \quad y \in B_\delta(x_j^{(1)}).
\] (3.26)

It is not difficult to deduce from Lemma A.3 that
\[
|c_L(y)| \leq C(W_{x_j^{(1)}, \mu_L^{(1)}}^{2^* - 2}(y) + W_{x_j^{(2)}, \mu_L^{(2)}}^{2^* - 2}(y)).
\]

Hence Lemma A.3 implies that there exists a large \( R > 0 \) such that
\[
|\eta_L(y)| = o(1), \quad y \in \mathbb{R}^N \setminus \cup_j B_{R(\mu_L^{(1)})^{-1}}(x_j^{(1)}).
\]

To obtain a contradiction, we only need to check that \( |\eta_L(y)| = o(1) \) in \( \cup_j B_{R(\mu_L^{(1)})^{-1}}(x_j^{(1)}) \), which will be achieved by using the Pochoznev identities in the small ball \( B_{\delta}(x_j^{(1)}) \).
Let
\[ \tilde{\eta}_{Lj}(y) = \left( \frac{1}{\mu_{(1)}^{L}} \right)^{2N-2} \eta_{L}(\frac{1}{\mu_{(1)}^{L}}y + x_{jL}^{(1)}). \] (3.27)

**Lemma 3.6.** It holds
\[ \tilde{\eta}_{Lj}(y) \to \sum_{l=0}^{N} b_{jl} \psi_{l}(y), \quad \text{as} \quad L \to \infty, \] (3.28)
uniformly in \( C^{1}(B_{R}(0)) \) for any \( R > 0 \), where \( b_{jl}, l = 0, \cdots, N, \) are some constants, and
\[ \psi_{0} = \frac{\partial U_{0,1}}{\partial \mu_{L}} \bigg|_{\mu_{L}=1}, \quad \psi_{j} = \frac{\partial U_{0,1}}{\partial y_{j}}, \quad j = 1, \cdots, N. \] (3.29)

**Proof.** In view of \( |\tilde{\eta}_{L,j}| \leq C \) in any compact subset of \( \mathbb{R}^{N} \), we may assume that \( \tilde{\eta}_{L,j} \to \eta_{j} \) in \( C_{loc}(\mathbb{R}^{N}) \). Then it follows from the elliptic regularity theory and \( (3.21) \) and \( (3.26) \) that \( \eta_{j} \) satisfies
\[ -\Delta \eta_{j} = (2^{*} - 1)U_{0,1}^{2^{*}-2}\eta_{j}, \quad \text{in} \quad \mathbb{R}^{N}, \] (3.30)
which combining with the non-degeneracy of \( U_{0,1} \) gives \( \eta_{j} = \sum_{l=0}^{N} b_{jl} \psi_{l}. \) \( \square \)

Let \( G(y,x) = C_{N}|y - x|^{2-N} \) be the corresponding Green's function of \(-\Delta \) in \( \mathbb{R}^{N} \), where \( C_{N} = (N(N - 2)\omega_{N-1})^{-1}. \)

**Lemma 3.7.** There holds
\[ |\eta_{L}(x)| \leq \sum_{j=-m}^{m} \sum_{|\alpha|=0}^{1} A_{j,L,\alpha} \partial^{\alpha}G(x_{jL}^{(1)},x) + O\left( \frac{1}{\mu_{L}^{2\frac{N-2}{N+2}}} \right), \] (3.31)
\[ := \sum_{j=-m}^{m} F_{jL}(x) + O\left( \frac{1}{\mu_{L}^{2\frac{N-2}{N+2}}} \right), \quad \text{in} \quad C^{1}(\mathbb{R}^{N} \setminus \bigcup_{j=-m}^{m} B_{2\delta}(x_{jL}^{(1)}) \bigcup \),
where \( \delta > 0 \) is any small constant, and the constants \( A_{j,L,\alpha} \) satisfy the following estimates:
\[ A_{j,L,0} = \int_{B_{\sigma}(x_{jL}^{(1)})} f(y,u_{L}^{(1)}(y),u_{L}^{(2)}(y)) dy = o\left( \frac{1}{\mu_{L}^{2\frac{N-2}{N+2}}} \right), \] (3.32)
\[ A_{j,L,\alpha} = O\left( \frac{1}{\mu_{L}^{2\frac{N-2}{N+2} + |\alpha|}} \right), \quad |\alpha| \geq 1. \] (3.33)

**Proof.** Denote \( f_{L}^{*}(y) = f(y,u_{L}^{(1)}(y),u_{L}^{(2)}(y)). \) We have
\[ |\eta_{L}(x)| = \int_{\mathbb{R}^{N}} G(y,x)f_{L}^{*}(y) dy - \int_{\mathbb{R}^{N}} G(y,x)Q(y)|\eta_{L}(y)| dy \] (3.34)
Note that

\[
\int_{\mathbb{R}^N} G(y, x) f^*_L(y) \, dy = \int_{\bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) f^*_L(y) \, dy + \int_{\mathbb{R}^N \setminus \bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) f^*_L(y) \, dy
\]

\[
= \int_{\bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(x_{jL}^{(1)}, x) f^*_L(y) + \sum_{j=-m}^m \int_{B_{\delta}(x_{jL}^{(1)})} G(y, x) - G(x_{jL}^{(1)}, x) f^*_L(y) \, dy
\]

\[
+ \int_{\mathbb{R}^N \setminus \bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) f^*_L(y) \, dy
\]

\[
= \sum_{j=-m}^m \frac{1}{A_{j,L,\alpha}} \partial^\alpha G(x_{jL}^{(1)}, x) + \sum_{j=-m}^m O \left( \int_{B_{\delta}(x_{jL}^{(1)})} |y - x_{jL}^{(1)}|^2 |f^*_L(y)| \, dy \right)
\]

\[
+ \int_{\mathbb{R}^N \setminus \bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) f^*_L(y) \, dy.
\]

(3.35)

And

\[
\int_{\mathbb{R}^N} G(y, x) Q(y) |\eta_L(x)| \, dy
\]

\[
= \int_{\bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) |y - x_{jL}^{(1)}| |\eta_L| \, dy + \int_{\mathbb{R}^N \setminus \bigcup_{j=-m}^m B_{\delta}(x_{jL}^{(1)})} G(y, x) |y - x_{jL}^{(1)}| |\eta_L| \, dy
\]

\[
= O \left( \frac{1}{\mu_L^{\beta+2}} \right).
\]

(3.36)

For \( y \in \mathbb{R}^N \setminus \bigcup_{j} B_{\delta}(x_{jL}^{(1)}) \), noting that \( \tau = \frac{N-2}{2} - \vartheta \) for \( \vartheta > 0 \) small, similar to [13], we find

\[
|f^*_L(y)| \leq C \mu_L^{-\tau} \left( \frac{1}{\mu_L^2} + (\mu_L^{-\tau} |\varphi_L^{(1)}|)^{2^* - 2} \right) \left( \sum_{j=-m}^m \frac{1}{|y - x_{jL}^{(1)}|^{\frac{N-2}{4} + \vartheta}} \right)^{2^{*} - 1}
\]

\[
\leq C \mu_L^{-\frac{N+2}{2} + \vartheta} \sum_{j=-m}^m \frac{1}{|y - x_{jL}^{(1)}|^{\frac{N-2}{4} + \vartheta}}.
\]

Thus, we have

\[
\int_{\mathbb{R}^N \setminus \bigcup_{j} B_{\delta}(x_{jL}^{(1)})} G(y, x) f^*_L(y) \, dy \leq C \mu_L^{-\frac{N+2}{2} + \vartheta} \sum_{j=-m}^m \frac{1}{|x - x_{jL}^{(1)}|^{\frac{N+2}{4} + \vartheta}} \leq \frac{C}{\mu_L^{\frac{N+2}{2} - \vartheta}}.
\]

(3.37)
Similarly, by Lemma A.3 we obtain

$$\int_{B_\delta(x_{jL}^{(1)})} |y - x_{jL}^{(1)}|^2 |f_L^*| \, dy$$

$$\leq C \int_{B_\delta(x_{jL}^{(1)})} |y - x_{jL}^{(1)}|^2 \frac{(\mu_L^{(1)})^{N+2}}{(1 + \mu_L^{(1)}) |y - x_{jL}^{(1)}|^{\frac{N+2}{2} + \tau}} \, dy \leq C$$

$$\leq \frac{1}{(\mu_L^{(1)})^{N-\frac{2}{2}}} \int_{B_{\frac{1}{2}}(x_{jL}^{(1)})} |y - x_{jL}^{(1)}|^{N-\frac{2}{2} - \tau(2^* - 1)} \leq \frac{1}{\mu_L^{(1)}}$$

(3.38)

Inserting (3.35)-(3.38) into (3.34), we obtain (3.31). Similarly, we can prove that (3.31) holds in $C^1(\mathbb{R}^N \cup \bigcup_{m=-M}^M B_2(x_{jL}^{(1)}))$. It remains to estimate $A_{j,L,\alpha}$.

$$A_{j,L,0} = \int_{B_\delta(x_{jL}^{(1)})} f(y, u_L^{(1)}(y), u_L^{(2)}(y)) \, dy$$

$$= \frac{1}{(\mu_L^{(1)})^{N-\frac{2}{2}}} \int_{B_R(0)} \frac{1}{\mu_L^{(1)}} f_L^* \left( \frac{1}{\mu_L^{(1)}} y + x_{jL}^{(1)} \right) \, dy$$

$$+ \frac{1}{(\mu_L^{(1)})^{N-\frac{2}{2}}} \left( \int_{B_{\frac{1}{2}}(x_{jL}^{(1)})} \frac{1}{|y|^{(N-\frac{2}{2} - \tau)2^* - 1}} \, dy \right)$$

$$= \frac{1}{(\mu_L^{(1)})^{N-\frac{2}{2}}} \left( 2^* - 1 \right) \int_{\mathbb{R}^N} U_{0,1}^{2^* - 2} \sum_{l=0}^N b_l \psi_l + o(1) \right) = o \left( \frac{1}{\mu_L} \right).$$

(3.39)

If $|\alpha| \geq 1$, then

$$|A_{j,L,\alpha}| \leq C \int_{B_\delta(x_{jL}^{(1)})} |y - x_{jL}^{(1)}|^{|\alpha|} |f(y, u_L^{(1)}(y), u_L^{(2)}(y))| = O \left( \frac{1}{\mu_L^{N-\frac{2}{2} + |\alpha|}} \right).$$

(3.40)

Thus, we complete the proof.

Using (3.1) and (3.2), we can deduce the following identities:

$$- \int_{\partial B_{\delta}(x_{jL}^{(1)})} \frac{\partial u_L^{(1)}}{\partial \nu} \frac{\partial u_L^{(1)}}{\partial y_i} - \int_{\partial B_{\delta}(x_{jL}^{(1)})} \frac{\partial u_L^{(2)}}{\partial y_i} \frac{\partial u_L^{(1)}}{\partial \nu} + \frac{1}{2} \int_{\partial B_{\delta}(x_{jL}^{(1)})} \langle \nabla (u_L^{(1)} + u_L^{(2)}), \nabla \eta_L \rangle \nu_i$$

$$= \int_{\partial B_{\delta}(x_{jL}^{(1)})} C_L(y) \eta_L \nu_i + \int_{\partial B_{\delta}(x_{jL}^{(1)})} Q(x)(u_L^{(1)} + u_L^{(2)}) \eta_L \nu_i - \int_{\partial B_{\delta}(x_{jL}^{(1)})} \frac{\partial Q}{\partial \eta_i} (u_L^{(1)} + u_L^{(2)}) \eta_L,$$
and
\[-\int_{\partial B_\delta(x_{jL}^{(1)})} \frac{\partial \eta_L}{\partial \nu} (y - x_{jL}^{(1)}, \nabla u_{L}^{(1)}) - \int_{\partial B_\delta(x_{jL}^{(1)})} \frac{\partial u_{L}^{(2)}}{\partial \nu} (y - x_{jL}^{(1)}, \nabla \eta_L)\]
\[+ \frac{1}{2} \int_{\partial B_\delta(x_{jL}^{(1)})} \langle \nabla (u_{L}^{(1)} + u_{L}^{(2)}), \nabla \eta_L \rangle (y - x_{jL}^{(1)}, \nu) + \frac{2 - N}{2} \int_{\partial B_\delta(x_{jL}^{(1)})} \frac{\partial \eta_L}{\partial \nu} (y - x_{L}^{(1)}, \nu)\]
\[+ \frac{2 - N}{2} \int_{\partial B_\delta(x_{jL}^{(1)})} \frac{\partial u_{L}^{(2)}}{\partial \nu} \eta_L + \int_{\partial B_\delta(x_{jL}^{(1)})} Q(y) (u_{L}^{(1)} + u_{L}^{(2)}) \eta_L (y - x_{jL}^{(1)}, \nu)\]
\[= \int_{B_\delta(x_{jL}^{(1)})} Q(y) (u_{L}^{(1)} + u_{L}^{(2)}) \eta_L + \int_{B_\delta(x_{jL}^{(1)})} (u_{L}^{(1)} + u_{L}^{(2)}) \eta_L \langle \nabla Q, x - x_{jL}^{(1)} \rangle\]
\[+ \int_{\partial B_\delta(x_{jL}^{(1)})} C_L(y) \eta_L (y - x_{jL}^{(1)}, \nu),\]

where $C_L(y) = \int_0^1 (tu_{L}^{(1)} + (1 - t)u_{L}^{(2)})^{2r-1} dt$ and $d > 0$ is a small constant.

Similar to (3.26), we can deduce

$$C_L(y) = U_{jL}^{2r-1} + O\left(\left(\frac{1}{\mu L} U_{jL}^{(1)} + \frac{1}{\mu L^{2r}} L^{N-2}\right) + |\varphi_L^{(1)}| + |\varphi_L^{(2)}|^{2r-1}\right), \quad y \in B_\delta(x_{jL}^{(1)}).$$

(3.43)

To estimate the boundary terms in (3.41) and (3.42), we need the following estimates which can be deduced from Proposition 3.1 and Lemma 3.7.

$u_{L}^{(l)}(y) = \sum_{j=-m}^{m} \left(\frac{N(N-2)}{\mu L^{N-2}} \frac{1}{|y - x_{jL}|^{N-2}} + O\left(\frac{1}{\mu L^{N-2}}\right)\right), \quad y \in \partial B_\delta(x_{jL}), \quad l = 1, 2.$

(3.44)

and

$$\eta_L(y) = F_{jL}(y) + o\left(\frac{1}{\mu L^{N-2}}\right), \quad y \in \partial B_\delta(x_{jL}).$$

Proof of Theorem 1.6. Step 1. We prove $b_{jL} = 0, j = 1, \cdots, N$. We need to estimate each term in (3.41). From (3.43), we obtain

$$\int_{\partial B_\delta(x_{jL}^{(1)})} C_L(y) \eta_L \nu_i = O\left(\frac{1}{\mu L} b_{ji} \int_{\mathbb{R}^N} U_{0,1} \psi_i + o(1)\right),$$

(3.45)

and

$$\int_{B_\delta(x_{jL}^{(1)})} \frac{\partial Q(y)}{\partial y_i} D_L(y) \eta_L = \frac{\beta}{\mu L^{\beta-1+N/2}} \int_{B_{\mu L}(0)} |y|^{\beta-2} y_i D_L\left(\frac{y}{\mu_{jL}} + x_{jL}^{(1)} \eta_L\right)\]
\[= \frac{\beta}{\mu L^{\beta-1+2}} \int_{B_{\mu L}(0)} |y|^{\beta-2} y_i U_{0,1} \sum_{k=0}^{N} b_{jL} \psi_k + O\left(\frac{1}{\mu L^{\beta+2}}\right)
\[= \frac{1}{\mu L^{\beta+1}} b_{ji} \int_{\mathbb{R}^N} |y|^{\beta-2} y_i U_{0,1} \psi_i + o(1) + O\left(\frac{1}{\mu L^{\beta+2}}\right),
\]

(3.46)
and
\[ \int_{\partial B_s(x^{(1)}_{jL})} Q(y)(u_L^{(1)} + u_L^{(2)})\eta_L v_i = O\left(\frac{1}{\mu_L^{\beta+2}}\right). \] (3.47)

Combining (3.45), (3.46) and (3.47), we are led to
\[ \text{RHS of (3.41)} = \frac{1}{\mu_L^{\beta+1}} b_{j,i} \int_{\mathbb{R}^N} |y_i|^\beta y_i U_{0,1} \psi_i. \] (3.48)

To estimate the left hand side of (3.41), using (3.44), we have
\[ \text{LHS of (3.41)} = o\left(\frac{1}{\mu_L^{N-2} L^{N-2}} + \frac{1}{\mu_L^{N-1}}\right). \] (3.49)
So (3.48) and (3.49) yield
\[ b_{j,i} \int_{\mathbb{R}^N} |y|^\beta y_i U_{0,1} \psi_i + o(1) = (\mu_L^{(1)})^{\beta+1} o\left(\frac{1}{\mu_L^{N-2} L^{N-2}} + \frac{1}{\mu_L^{N-1}}\right) = o(1), \] (3.50)
which implies \( b_{j,i} = 0, i = 1, \ldots, N. \)

**Step 2.** We prove \( b_{j,0} = 0. \) It follows from Lemma 2.6 and (3.44) that
\[ \text{LHS of (3.42)} = \int_{\partial B_s(x^{(1)}_{jL})} \frac{\partial \eta^*_L}{\partial \nu} (x - x^{(1)}_{jL}, \nabla) + \int_{\partial B_s(x^{(1)}_{jL})} \frac{1}{|x - x^{(1)}_{jL}|^{N-2}} \langle x - x^{(1)}_{jL}, \nabla \eta^*_L \rangle \] (3.51)
\[ + \frac{1}{2} \int_{\partial B_s(x^{(1)}_{jL})} \nabla \frac{1}{|x - x^{(1)}_{jL}|^{N-2}} \nabla \eta^*_L \rangle \] (3.52)
If we replace \( \eta^*_L \) by \( \frac{\partial \eta^*_L}{\partial |x-x_{jL}|^{N-2}} \), then it is easy to check that the sum of all the integrals is zero.

Therefore
\[ \text{LHS of (3.42)} = \frac{1}{\mu_L^{N-2} L^{N-2}} + O\left(\frac{1}{\mu_L^N}\right). \] (3.53)

Combining (3.48) and (3.52), we are led to
\[ b_{j,0} \int_{\mathbb{R}^N} |y|^\beta U_{0,1}^{2\beta-1} \psi_0 = o(1). \] (3.54)
This gives \( b_{j,0} = 0. \)

At the end of this section, we prove Theorem 1.9.
Proof of Theorem 1.9: To prove that \( u_L \) is periodic in \( y_1 \), we let

\[ v_L(y) = u_L(y_1 - L, y_2, \cdots, y_N). \]

From the expression of \( u_L \) in (1.5), we know that the blow up set of the solution of the form (1.5) is

\[ \Xi = \{ x_j = (jL, 0, \ldots, 0); j = 0, \pm 1, \pm 2, \ldots, \pm m \}. \]

Then \( v_L \) is a bubbling solution whose blow-up set is the same as that of \( u_L \). By the local uniqueness, \( v_L = u_L \), i.e. \( u_L(y_1 - L, y_2, \cdots, y_N) = u_L(y_1, y_2, \cdots, y_N) \), which implies that \( u_L(y) \) is periodic in \( y_1 \).

\( \square \)

Appendix A. Some basic estimates

In this section, we give some technical estimates. Throughout Appendices A, B, C, we will use the same notations as before and we also use the same \( C \) to denote different constants unless otherwise stated.

First we give some known estimates which can be found in [27, 18].

Lemma A.1. (Lemma B.1, [27]) Let \( x_i, x_j \in \mathbb{R}^N, x_i \neq x_j, i \neq j \). It holds

\[
\frac{1}{(1 + |y - x_i|)^\alpha(1 + |y - x_j|)^\beta} \leq \frac{C}{(1 + |x_i - x_j|)^\sigma} \left( \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} + \frac{1}{(1 + |y - x_i|)^{\alpha + \beta - \sigma}} \right),
\]

where \( \alpha \) and \( \beta \) are some positive constants, \( 0 < \sigma \leq \min(\alpha, \beta) \).

Lemma A.2. (Lemma B.1, [27]) For any constant \( 0 < \sigma < N - 2 \), there exists a constant such that

\[
\int_{\mathbb{R}^N} \frac{dz}{|y - z|^{N-2}(1 + |z|)^{2+\sigma}} \leq \frac{C}{(1 + |y|)^\sigma}.
\]

Lemma A.3. (Lemma A.3, [18]) For any \( \gamma > 1 \), there exists a constant \( C \), such that

\[
\sum_j \frac{1}{(1 + \mu_L|y - x_j|)^\gamma} \leq \frac{C}{(1 + \mu_L|y - x_i|)^\gamma}, \quad y \in B_i := B_1(x_i).
\]

Let \( u_L \) be a solution of (1.1) with the form

\[
u_L = W_{x, \mu_L} + \varphi_L, \quad W_{x, \mu_L} = \sum_{j=-m}^{m} \xi(y - x_jL)U_{x_jL, \mu_L} =: \sum_{j=-m}^{m} W_{x_jL, \mu_L}, \tag{A.2}\]

satisfying (1.6), (1.7) and (1.8). It is easy to see that \( \varphi_L \) satisfies the following equation:

\[
-\Delta \varphi_L - Q(y)\varphi_L - (2^* - 1)W_{x, \mu_L}^{2^* - 2}\varphi_L = N_L(\varphi_L) + l_L, \tag{A.3}\]

where

\[
l_L = (W_{x, \mu_L})^{2^* - 1} - \sum_{j=-m}^{m} \xi(y - x_jL)U_{x_jL, \mu_L}^{2^* - 1} - Q(y)W_{x, \mu_L}.
\]
\[
+ \sum_{j=-m}^{m} U_{x_j,\mu L} \Delta \xi + 2 \nabla \xi \nabla \left( \sum_{j=-m}^{m} U_{x_j,\mu L} \right),
\]
(A.4)

and

\[
N_L(\varphi_L) = (W_{x,\mu L} + \varphi_L)^{2^*-1} - W_{x,\mu L}^{2^*-1} - (2^* - 1)W_{x,\mu L}^{2^*} \varphi_L.
\]
(A.5)

By assumption (1.8), we have \( \|\varphi_L\|_* \to 0 \) as \( L \to +\infty \).

**Lemma A.4.** It holds

\[
u_L \leq C \sum_{j=-m}^{m} \frac{\mu^2_{L_j}}{(1 + \mu \|y - x_j\|)^{N-2}},
\]
(A.6)

and

\[
\varphi_L \leq C \sum_{j=-m}^{m} \frac{\mu^2_{L_j}}{(1 + \mu \|y - x_j\|)^{N-2}}.
\]
(A.7)

**Proof.** Since the proof is the same as that of Lemma A.3 in [11], here we omit it.

We now estimate \( N_L(\varphi_L) \) and \( l_L \).

**Lemma A.5.** If \( N \geq 5 \), then

\[
\|N_L(\varphi_L)\|_{**} \leq C \|\varphi_L\|_{*}^{\min(2^* - 1, 2)}.
\]

**Proof.** Considering that the proof is just the same as that of Lemma 2.4 in [26], we omit it.

**Lemma A.6.** If \( N \geq 5 \), then

\[
\|l_L\|_{**} \leq \frac{C}{\mu_{L_j}^{2^{*-2}}} + C \max_j |x_j - x_j^L|^2.
\]

**Proof.** Recall that from (A.4)

\[
l_L = \left[ (W_{x,\mu L})^{2^*-1} - \sum_{j=-m}^{m} \xi U_{x_j,\mu L}^{2^*-1} \right] - Q(y)W_{x,\mu L}
+ \sum_{j=-m}^{m} U_{x_j,\mu L} \Delta \xi (y - x_j) + 2 \nabla \left( \sum_{j=-m}^{m} U_{x_j,\mu L} \nabla \xi (y - x_j) \right)
= : J_1 + J_2 + J_3 + J_4.
\]

Denote \( \Omega_i = \{ y \in \mathbb{R}^N \text{ such that } |y - x_i| \leq |y - x_j| \text{ for all } j \neq i \} \).

Assuming \( y \in \Omega_i \), then

\[
|J_1| \leq C \frac{\mu_{L_j}^2}{(1 + \mu \|y - x_i\|)^{2}}, \sum_{j \neq i} \frac{\mu_{L_j}^{N-2}}{(1 + \mu \|y - x_j\|)^{N-2}}
\]
\[+ C \left( \sum_{j \neq i} \frac{\mu_L^2}{(1 + \mu_L|y - x_{jL}|)^{N-2}} \right)^{2^*-1}.
\]

Since
\[|y - x_{jL}| \geq |y - x_{iL}|, \forall y \in \Omega,
\]
we find
\[
\frac{\mu_L^2}{(1 + \mu_L|y - x_{jL}|)^N} \leq \frac{\mu_L^2}{(1 + \mu_L|y - x_{jL}|)^N} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}}.
\]

On the other hand
\[
\frac{N-2}{\mu_L^2} \leq \frac{N-2}{\mu_L^2} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}},
\]
we deduce
\[
\left( \sum_{i \neq j} \frac{\mu_L^2}{(1 + \mu_L|y - x_{jL}|)^{N-2}} \right)^{2^*-1} \leq \frac{N+2}{\mu_L^2} \leq \frac{N+2}{\mu_L^2} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}}.
\]

Thus, we obtain
\[
\|J_1\|_{*} \leq C \sum_{i \neq j} \frac{1}{(1 + \mu_L|x_{jL} - x_{iL}|)^{N-2} \mu_L^{N-2}} \leq \frac{C}{(\mu_L)|x_{jL} - x_{iL}|^{N-2} \mu_L^{N-2}}.
\]

Now, we estimate \(J_2\). When \(|y - x_{jL}| \geq 1\), we have
\[
|J_2| \leq C \sum_{j = -m}^{m} \frac{N-2}{\mu_L^2} \leq \frac{N+2}{\mu_L^2} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}} \leq \frac{1}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}}.
\]

When \(|y - x_{jL}| \leq 1\), we have
\[
|J_2| \leq C \sum_{j = -m}^{m} \frac{\mu_L^2 \xi(y - x_{jL})}{(1 + \mu_L|y - x_{jL}|)^{N-2}} \leq \frac{\mu_L^2 \xi(y - x_{jL})}{(1 + \mu_L|y - x_{jL}|)^{N-2}} \leq \frac{\mu_L^2 (|x_{jL} - x_j|^\beta + |y - x_{jL}|^\beta)}{(1 + \mu_L|y - x_{jL}|)^{N-2} \mu_L^{N-2}}.
\]
\[ \leq C \sum_{j=-m}^{m} \frac{\mu_{L}^{\frac{N+2}{2}}}{(1 + \mu_{L}|y - x_{jL}|)^{\frac{N+2}{2}+\tau}} \left( \frac{|x_{jL} - x_j|^\beta + |y - x_{jL}|^\beta}{(1 + \mu_{L}|y - x_j|)^{\frac{N+2}{2}-\tau}} \right) \]

\[ \leq \left( \frac{C}{\mu_{L}^{\frac{N+2}{2}}} + C \max_j |x_{jL} - x_j|^\beta \right) \sum_{j=-m}^{m} \frac{\mu_{L}^{\frac{N+2}{2}}}{(1 + \mu_{L}|y - x_j|)^{\frac{N+2}{2}+\tau}}. \]  

(A.9)

since \( \tau < \frac{N-2}{2}, \beta < \frac{N-2}{2} + \tau > 0 \) and

\[ \frac{|y - x_{jL}|^\beta}{(1 + \mu_{L}|y - x_j|)^{\frac{N-2}{2}}} \leq \frac{C}{\mu_{L}^{\frac{N-2}{2}}} \]

Hence it follows from (A.8) and (A.9) that

\[ \|J_2\|_\ast \ast \leq \left( \frac{C}{\mu_{L}^{\frac{N-2}{2}}} + C \max_j |x_{jL} - x_j|^\beta \right). \]

Also, we can estimate directly

\[ |J_3| \leq C \sum_{j=-m}^{m} \frac{\mu_{L}^{\frac{N+2}{2}} |\Delta \xi(y - x_{jL})|}{(1 + \mu_{L}|y - x_{jL}|)^{N-2}} \]

\[ \leq C \sum_{j=-m}^{m} \frac{\mu_{L}^{\frac{N+2}{2}} |\Delta \xi(y - x_{jL})|}{(1 + \mu_{L}|y - x_{jL}|)^{\frac{N+2}{2}+\tau}} \frac{\mu_{L}^{\frac{N+2}{2}}}{(1 + \mu_{L}|y - x_j|)^{\frac{N+2}{2}-\tau}} \mu_{L}^{2 \frac{N-2}{2}} \frac{1}{\mu_{L}^{\frac{N-2}{2}}} \]

\[ \leq \sum_{j=-m}^{m} \frac{\mu_{L}^{\frac{N+2}{2}}}{(1 + \mu_{L}|y - x_{jL}|)^{\frac{N+2}{2}+\tau}} \frac{1}{\mu_{L}^{\frac{N-2}{2}}} \],

where we use the following fact that

\[ \frac{1}{(1 + \mu_{L}|y - x_{jL}|)^\beta} \sim \frac{C}{\mu_{L}^{\beta + 3}}, \text{ when } 1 \leq |y - x_{jL}| \leq 2. \]

Hence, we have

\[ \|J_3\|_\ast \ast \leq C \left( \frac{1}{\mu_{L}^{\frac{N-2}{2}}} \right). \]  

(A.10)

Similar to (A.10), we can estimate \( \|J_4\|_\ast \ast \leq C \left( \frac{1}{\mu_{L}^{\frac{N-2}{2}}} \right). \) Thus, combining all the estimates above, we complete the proof. \( \square \)

**Lemma A.7.** If \( N \geq 5 \), then

\[ \frac{\partial I}{\partial \mu_{L}} = m \left( - \frac{B_1}{\mu_{L}^{\beta + 3}} + \sum_{j \neq 0} \frac{B_2}{\mu_{L}^{N-1}|x_{0L} - x_{jL}|^{N-2}} + o \left( \frac{1}{\mu_{L}^{\beta + 3}} \right) \right) \]

where \( B_1, B_2 \) are constants, and

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Q(y)u^2) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*}. \]
Lemma A.8. If $N \geq 5$, then
\[
\frac{\partial I}{\partial y_i} = m \left( - \frac{C_1(x_j L - x_j)}{\mu_L^{\beta+2}} + \sum_{i \neq j} \frac{C_2}{\mu_L^{N-2}|x_i L - x_j L|^{N-2}} + o\left( \frac{1}{\mu_L^{\beta+2}} \right) \right), \quad i = 1, 2, ..., N.
\]
where $C_1, C_2$ are constants.

Proof. Since
\[
\int_{\mathbb{R}^N} Q(y) U_m \frac{\partial U_m}{\partial y_i} = - \frac{1}{2} \int_{\mathbb{R}^N} \frac{\partial Q(y)}{\partial y_i} U_m^2
\]
\[
= - \frac{\beta}{2} \int_{\mathbb{R}^N} a \beta \mu_L^{-1} y_i + (x_j L - x_j) \beta^{-2} (\mu_L^{-1} y_i + (x_j L - x_j) y_i) \left| U_m^2 \right|
\]
\[
+ O \left( \frac{1}{\mu_L^{\beta+2+c}} + |x_j L - x_j|^{\beta+2+c} \right)
\]
\[
= amC \left( \frac{1}{\mu_L^{\beta+2+c}} + |x_j L - x_j|^{\beta+2+c} \right), \quad (A.15)
\]
combining (A.15) and by the similar argument as that of Lemma A.7, we can complete the proof. \qed
Proposition B.1. Assume that $N \geq 5$. There holds

$$\| \varphi_L \|_* \leq C \left( \frac{1}{\mu_L^{N-2}} + \max_i |x_i|^{\beta} \right).$$

Proof. Since

$$-\Delta \varphi_L + Q(y) \varphi_L - (2^* - 1) W^{2^* - 2}_{x, \mu_L} \varphi_L = N_L(\varphi_L) + l_L,$$

noting that $Q(y) \geq 0$, we have

$$|\varphi_L| \leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} W^{2^* - 2}_{x, \mu_L} |\varphi_L| dz + \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} (|N(\varphi_L)| + |l_L|).$$  (B.4)
We can check easily 
\[
\left| \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} (N_L(\varphi_L) + l_L) \right|
\]
\[
\leq C (\|\varphi_L\|_{min}^{2^*-1,2} + \|l_L\|_{**}) \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|z-x_jL|)^{N-2 + \tau}}
\]
\[
\leq C (\|\varphi_L\|_{min}^{2^*-1,2} + \|l_L\|_{**}) \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|z-x_jL|)^{N-2 + \tau}}.
\]

(B.5)

Suppose \(N \geq 6\), we have
\[
W_{x,\mu_L}^{2^*-2} \leq C \sum_{j=-m}^{m} \frac{\mu_L^2}{(1 + \mu_L|z-x_jL|)^4}.
\]

If \(N = 5\), then by discrete Hölder inequality we have
\[
W_{x,\mu_L}^{2^*-2} \leq C \mu_L^2 \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L|z-x_jL|)^3} \right)^{4 \over 7}
\]
\[
\leq C \mu_L^2 \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L|z-x_jL|)^{1 - \tau}} \left( \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L|z-x_jL|)^{\tau}} \right)^{1 \over 7}
\]
\[
\leq C \mu_L^2 \sum_{j=-m}^{m} \frac{1}{(1 + \mu_L|z-x_jL|)^{1 - \tau}}.
\]

So, we obtain that for \(N \geq 6\),
\[
\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{x,\mu_L}^{2^*-2} \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|z-x_jL|)^{N-2 + \tau}} dz
\]
\[
\leq \mu_L^{N-2} \sum_{j=-m}^{m} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1 + \mu_L|z-x_jL|)^{N-2 + \tau + 1}} dz
\]
\[
+ \mu_L^{N-2} \sum_{j=-m}^{m} \sum_{i \neq j} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1 + \mu_L|z-x_jL|)^{1}} \frac{1}{(1 + \mu_L|z-x_iL|)^{N-2 + \tau}} dz
\]
\[
\leq C \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y-x_jL|)^{2 + N-2}},
\]

which implies that if \(N \geq 6\),
\[
\left| \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} W_{x,\mu_L}^{2^*-2} \varphi_L dz \right| \leq \|\varphi_L\|_{\ast} \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y-x_jL|)^{2 + N-2}}
\]

(B.6)
Also, by the similar argument we can show that for $N = 5$, there is a $\varsigma > 0$ such that
\[
\left| \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} W_{x,\mu L}^{2, -2} \varphi_L dz \right| \leq C \| \varphi_L \|_* \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y - x_jL|)^{N/2 + \tau + \varsigma}}. \tag{B.7}
\]

Therefore, it follows from (B.4) to (B.7) that
\[
|\varphi_L| \leq C \left( \| \varphi_L \|_*^{\min(2^*, 1, 2)} + \| l_* \|_{**} \right) \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y - x_jL|)^{N/2 + \tau}}
+ C \| \varphi_L \|_* \sum_{j=-m}^{m} \frac{\mu_L^{N-2}}{(1 + \mu_L|y - x_jL|)^{N/2 + \tau + \varsigma}}. \tag{B.8}
\]

where $\varsigma > 0$ is a constant.

Assume that there exists $y \in \mathbb{R}^N \setminus \bigcup_{j=-m}^{m} B_{R\mu L^{-1}}(x_jL)$ for some large $R > 0$, such that $\| \omega \|_*$ is achieved at $y$. Then, (B.8) yields
\[
\| \varphi_L \|_* \leq C \left( \| \varphi_L \|_*^{\min(2^*, 1, 2)} + \| l_* \|_{**} \right) + o_R(1) \| \varphi_L \|_* . \tag{B.9}
\]

Noting that $\| \varphi_L \|_* \to 0$ as $L \to +\infty$, we have
\[
\| \varphi_L \|_* \leq C \| l_* \|_{**}. \tag{B.10}
\]

Suppose that $\| \varphi_L \|_*$ is achieved at $y \in B_{R\mu L^{-1}}(x_jL)$. Let
\[
\tilde{\varphi}_L(y) = \mu_L^{\frac{N-2}{2}} \varphi_L(\mu_L^{-1}y + x_jL),
\]

and
\[
\| \tilde{\varphi}_L \|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{i=-m}^{m} \frac{\mu_L^{\frac{N-2}{2}}}{(1 + \mu_L|y - \mu_L(x_iL - x_jL)|)^{N/2 + \tau}} \right)^{-1} |\tilde{\varphi}_L(y)|.
\]

Then $\| \tilde{\varphi}_L \|_*$ is achieved at some $y \in B_R(0)$.

Suppose that
\[
\| \varphi_L \|_* \geq n_L \left( \frac{1}{\mu_L^{N-2-\tau}} + C \max_i |x_iL - x_i|^{\beta} \right),
\]

for some $n_L \to \infty$. Then as $L \to +\infty$, $n_L = \frac{|\tilde{\varphi}_L \|_*}{\| \tilde{\varphi}_L \|_*}$ converges to $n \neq 0$, which satisfies
\[-\Delta \eta - (2^* - 1)U_{0,1}^{2^* - 2}\eta = 0, \quad \text{in } \mathbb{R}^N,
\]

since $\| l_* \|_{**} \leq C n_L \to 0$. This gives
\[
\eta = \alpha_0 \frac{\partial U_{0,\mu L}}{\partial \mu L} \bigg|_{\mu L = 1} + \sum_{j=1}^{N} \alpha_j \frac{\partial U_{0,1}}{\partial y_j},
\]

for some constant $\alpha_j$. 

On the other hand, we have
\[ \tilde{\varphi}_L(0) = \mu_L \frac{N-2}{2} \varphi_L(x_jL) = O\left( \frac{1}{\mu_L^{-N-2} L^{N-2}} \right), \]
\[ \nabla \tilde{\varphi}_L(0) = \mu_L^{-2} \nabla \varphi_L(x_jL) = O\left( \frac{1}{\mu_L^{-N-1} L^{N-1}} \right). \]

So, we find \( \eta(0) = 0 \) and \( \nabla \eta(0) = 0 \), which implies \( \alpha_0 = \alpha_1 = \cdots = \alpha_N = 0 \). This is a contradiction.

\[ \square \]

**Corollary B.2.** For any \( \delta > 0 \), it holds
\[ |\varphi_L|, |
\nabla \varphi_L| \leq C \left( \frac{1}{\mu_L^{\frac{N-2}{2}}} + \max_{i} |x_iL - x_i|^{\beta} \right), \quad x \in B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL). \]

**Proof.** It follows from Proposition \([B.1]\) that
\[ |\varphi_L| \leq \|\varphi_L\| \sum_{j=-m}^{m} \frac{\mu_j^{\frac{N-2}{2}}}{\mu_j L} \left( 1 + \mu_j |y - x_jL| \right)^{\frac{N-2}{2} + \tau} \]
\[ \leq C \left( \frac{1}{\mu_L^{\frac{N-2}{2} - \tau}} + \max_{i} |x_iL - x_i|^{\beta} \right), \quad x \in B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL). \]

On the other hand, using the \( \dot{L}^p \) estimates, we can deduce that for any \( p > 1 \),
\[ \|\varphi_L\|_{W^{2p}(B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL))} \leq C \|\varphi_L\|_{L^{\infty}(B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL))} \]
\[ + C \|(2^*-1)W_{x,\mu_L}^{2^*-2} \varphi_L + N_L(\varphi_L) + l_L\|_{L^{\infty}(B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL))} \]
\[ \leq C \|\varphi_L\|_{L^{\infty}(B_{2\delta}(x_jL) \setminus B_{\frac{\delta}{2}}(x_jL))} + \frac{C}{\mu_L} (\|N_L(\varphi_L)\|_{\|\|} + \|l_L\|_{\|\|}) \]
\[ \leq C \left( \frac{1}{\mu_L^{\frac{N-2}{2} - \tau}} + \max_{i} |x_iL - x_i|^{\beta} \right), \]

the results follow from Lemma \([A.6]\) and Proposition \([B.1]\). \( \square \)

**Appendix C. Proof of (3.6)**

In this section, we mainly prove (3.6) which involve some technical and precise computations.

**Proof of (3.6).** For notations of simplicity, we may denote \( d = \max_i |x_iL - x_i| \). Recall that \( \delta = \mu_L^{-\theta}, \theta \) is defined in (3.3). By the assumption \((Q_3)\), we have
\[
\int_{\partial B_{\delta}(x_jL)} Q(y) u_i^2 v_i \\
= \int_{\partial B_{\delta}(x_jL)} (a |y - x_j|^\beta + O(|y - x_j|^\beta + 1)) u_i^2 v_i
\]
and

\[ F_2 \leq C \int_{\partial B_k(x_jL)} (\delta^\beta + \delta^{\beta-1}d + \delta d^{\beta-1} + \delta^{\beta+1}) \left( \sum_{i=-m}^m \frac{\mu_L^{N-2}}{1 + \mu_L |y - x_i L|^2} \right)^2 \]

\[ =: F_{2,1} + F_{2,2} + F_{2,3} + F_{2,4} = O\left(\frac{1}{\mu_L^\beta}\right), \quad \text{(C.1)} \]
where in \([C.1]\) we use the following estimates

\[
F_{2,1} \leq C_d \beta \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \left( \sum_{i \neq j} \frac{\mu_L^{N-2}}{\mu_L^{N-2} (1 + \mu_L |x_jL - x_iL|)^{N-2}} \right)^2 \right\}
\]

\[
\leq C_d \beta \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \frac{1}{\mu_L^{N-2} L^{2(N-2)}} \right\}
\]

\[
\leq C_d \beta \left( \frac{1}{\delta^{N-2} + \mu_L^{N-2} L^{2(N-2)}} \right) \delta^{N-1}
\]

\[
= C_d \beta \left( \frac{1}{\mu_L^{\beta}} + \frac{1}{\mu_L^{N-2+(N-1)\theta} L^{2(N-2)}} \right) = O\left( \frac{1}{\mu_L^{\beta+2}} \right),
\]

\[
F_{2,2} \leq C_d \beta^{-1} \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \left( \sum_{i \neq j} \frac{\mu_L^{N-2}}{\mu_L^{N-2} (1 + \mu_L |x_jL - x_iL|)^{N-2}} \right)^2 \right\}
\]

\[
\leq C_d \beta^{-1} \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \frac{1}{\mu_L^{N-2} L^{2(N-2)}} \right\}
\]

\[
\leq C_d \beta^{-1} \left( \frac{1}{\delta^{N-2} + \mu_L^{N-2} L^{2(N-2)}} \right) \delta^{N-1}
\]

\[
= C_d \beta^{-1} \left( \frac{1}{\mu_L^{\beta-1}} + \frac{1}{\mu_L^{N-2+(N-1)\theta} L^{2(N-2)}} \right)
\]

\[
= O\left( \frac{1}{\mu_L^{\beta+2}} \right),
\]

\[
F_{2,3} \leq C_d \beta^{-1} \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \left( \sum_{i \neq j} \frac{\mu_L^{N-2}}{\mu_L^{N-2} (1 + \mu_L |x_jL - x_iL|)^{N-2}} \right)^2 \right\}
\]

\[
\leq C_d \beta^{-1} \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \frac{1}{\mu_L^{N-2} L^{2(N-2)}} \right\}
\]

\[
\leq C_d \beta^{-1} \left( \frac{1}{\delta^{N-3} + \mu_L^{N-2} L^{2(N-2)}} \right) \delta^{N-1}
\]

\[
= C_d \beta^{-1} \left( \frac{1}{\mu_L^{\beta-2}} + \frac{1}{\mu_L^{N-2+(N-2)\theta} L^{2(N-2)}} \right)
\]

\[
= O\left( \frac{1}{\mu_L^{\beta+2}} \right),
\]

and

\[
F_{2,4} \leq C_d \beta^{+1} \int_{\partial B_L(x_jL)} \left\{ \frac{\mu_L^{N-2}}{(1 + \mu_L \delta)^{2(N-2)}} + \left( \sum_{i \neq j} \frac{\mu_L^{N-2}}{\mu_L^{N-2} (1 + \mu_L |x_jL - x_iL|)^{N-2}} \right)^2 \right\}
\]

\[
\leq C_d \beta^{+1} \int_{\partial B_L(x_jL)} \left\{ \frac{1}{\mu_L^{N-2} \delta^{2(N-2)}} + \frac{1}{\mu_L^{N-2} L^{2(N-2)}} \right\}
\]

\[
\leq \delta^{+1} \left( \frac{1}{\mu_L^{N-2} \delta^{2(N-2)}} + \frac{1}{\mu_L^{N-2} L^{2(N-2)}} \right) \delta^{N-1}
\]
\[ C \left( \frac{1}{\mu_L^{N-2+(\beta+4-N)\theta}} + \frac{1}{\mu_L^{N-2+(\beta+4-N)\theta} L^2(N-2)} \right) = O \left( \frac{1}{\mu_L^{\beta+2}} \right). \]

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References

[1] A. Bahri, J. Coron, The scalar curvature problem on the standard three dimensional sphere, J. Funct. Anal., 95(1991), 106-172.
[2] T. Bartsch, T. Weth, Multiple solutions of a critical polyharmonic equation, J. Reine Angew. Math., 571(2004), 131-143.
[3] T. Bartsch, T. Weth, M. Willem, A Sobolev inequality with remainder term and critical equations on domains with topology of the domain, Calc. Var. Partial Differential Equations, 18(2003), 253-268.
[4] V. Benci, G. Cerami, Existence of positive solutions of the equation \(-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}\) in \(\mathbb{R}^N\). J. Funct. Anal., 88(1990), 90-117.
[5] H. Brezis, Y. Li, Some nonlinear elliptic equations have only constant solutions. J. Partial Differ. Equ., 19(2006), 208-217.
[6] H. Brezis, L. Peletier, Elliptic equations with critical exponent on spherical caps of \(S^3\). J. Anal. Math., 98(2006), 279-316.
[7] D. Cao, H. Heniz, Uniqueness of positive multi-bump bound states of nonlinear Schrödinger equations, Math. Z., 243(2003), 599-642.
[8] D. Cao, E. Noussair, S. Yan, Existence and uniqueness results on single-peaked solutions of a semilinear problem, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15(1998), 73-111.
[9] W. Chen, J. Wei, S. Yan, Infinitely many solutions for the Schrödinger equations in \(\mathbb{R}^N\) with critical growth, J. Differential Equations, 252(2012), 2425-2447.
[10] S. Chang, P. Yang, A perturbation result in prescribing scalar curvature on \(S^n\), Duke Math. J., 64(1991), 27-69.
[11] Y. Deng, C.-S. Lin, S. Yan, On the prescribed scalar curvature problem in \(\mathbb{R}^N\), local uniqueness and periodicity, J. Math Pure Appl., 104(2015), 1013-1044.
[12] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron’s problem. Calc. Var. Partial Differential Equations, 16(2003), 113-145.
[13] O. Druet, From one bubble to several bubbles: the low-dimensional case. J. Differential Geom., 63(2003), 399-473.
[14] W. Ding, W. Ni, On the elliptic equation \(\Delta u + Ku^{\frac{N+2}{N-2}} = 0\) and related topics, Duck. math. J., 52(1985), 485-506.
[15] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math., 34(1981), 525-598.
[16] L. Glaergetas, Uniqueness of positive solutions of a nonlinear elliptic equation involving the critical exponent, Nonlin. Anal. TMA, 20(1993), 571-603.
[17] M. Grossi, On the number of single-peak solutions of the nonlinear Schrödinger equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19(2002), 261-280.
[18] Y. Guo, S. Peng, S. Yan, Local uniqueness and periodicity induced by concentration. Proc. Lond. Math. Soc., (3) 114(2017), 1005-1043.
[19] Q. He, C. Wang, Q. Wang, A new type of bubble solutions for a Schrödinger equation with critical growth, arXiv:2101.03284.
[20] Y. Y. Li, On $-\Delta u = K(x)u^5$ in $\mathbb{R}^3$, Comm. Pure Appl. Math., 46(1993), 303-340.
[21] Y. Y. Li, Prescribing scalar curvature on $S^3, S^4$, and related problems, J. Funct. Anal., 118(1993), 43-118.
[22] Y. Y. Li, Prescribing scalar curvature on $S^n$ and related problems. Part I, J. Differential Equations, 120(1995), 319-410.
[23] Y. Y. Li, J. Wei, H. Xu, Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ on lattices in $\mathbb{R}^n$, J. Reine Angew. Math., 743(2018), 163-211.
[24] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I., Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(1984), 109-145.
[25] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II., Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(1984), 223-283.
[26] S. Peng, C. Wang, S. Yan, Construction of solutions via local Pohozaev identities, J. Funct. Anal., 274(2018), 2606-2633.
[27] J. Wei, Y. Yan, Infinitely many solutions for the prescribed scalar curvature problem on $S^N$, J. Funct. Anal., 258(2010), 3048-3081.
[28] H. Xu, Critical exponent elliptic equations: Gluing via the implicit function theorem and the moving sphere method, Ph. D. thesis, 2007.
[29] S. Yan, Concentration of solutions for the scalar curvature equation on $\mathbb{R}^N$, J. Differential Equations, 163(2000), 239-264.

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