Petersson scalar products and $L$-functions arising from modular forms

Shigeaki Tsuyumine

Abstract
Let $f(z) = \sum_{n=0}^{\infty} a_n e(nz)$, $g(z) = \sum_{n=0}^{\infty} b_n e(nz)$ ($e(z) = e^{2\pi \sqrt{-1} z}$) be holomorphic modular forms for $\Gamma_0(N)$ of integral weight or half integral weight, where their weights or characters are not necessarily equal to each other. We show that $L(s; f, g) := \sum_{n=1}^{\infty} a_n b_n n^{-s}$ extends meromorphically to the whole $s$ plane, and that it satisfies some functional equation. Some residues of the $L$-function or some special values are expressed in terms of the Petersson scalar product. Applications to quadratic forms are included.

Keywords  L-function · Modular form · Petersson scalar product

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The Rankin–Selberg method developed by Rankin [12] and by Selberg [14] gives us $L$-functions from cuspidal automorphic forms by taking the scalar products of them with real analytic Eisenstein series of weight 0. Some of the analytic properties the $L$-functions inherit from the Eisenstein series are functional equations or positions of possible poles. Zagier [16] applied this important method to the automorphic forms on $\text{SL}_2(\mathbb{Z})$ which are not cuspidal. Research in this direction has also been made by Gupta [4], Mizuno [6] and Chiera [2].

In the present paper, we consider mainly modular forms on

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$
Let \( z = x + \sqrt{-1}y \in \mathfrak{H} \) where \( \mathfrak{H} \) denotes the complex upper half plane. Instead of Eisenstein series of weight 0 we employ Eisenstein series of integral weight \( k \)

\[
y^s E_{k,\chi}(z, s) := 2^{-1}y^s \sum_{c,d} \chi(d)(cz+d)^{-k}|cz+d|^{-2s} \quad (s \in \mathbb{C})
\]

with a Dirichlet character \( \chi \) modulo \( N \) where \( c, d \) runs over the set of second rows of matrices in \( \Gamma_0(N) \), or Eisenstein series of half integral weight (see Sect. 3). We show that the following \( L \)-functions inherit analytic properties from these Eisenstein series.

Let \( f(z) = \sum_{n=0}^\infty a_n e(nz) \), \( g(z) = \sum_{n=0}^\infty a_n e(nz) \) with \( e(z) = e^{2\pi \sqrt{-1}z} \) be modular forms for \( \Gamma_0(N) \) of integral or half integral weight, with weights and characters not necessarily equal to each other. Define

\[
L(s; f, g) := \sum_{n=1}^\infty \frac{a_n b_n}{n^s}, \tag{1}
\]

which converges for \( s \) with sufficiently large \( \Re s \). Let \( f, g \) be of weight \( l, l' \) respectively, and suppose that \( l \geq l' \). We show that \( L(s; f, g) \) extends meromorphically to the whole complex plane, and has the functional equation under \( s \mapsto -(l-l')+1-s \) (Corollaries 5.5, 6.2). If \( l = l' \) and \( f, g \) have the same character, then

\[
\langle f, g \rangle_{\Gamma_0(N)} = c \text{Res}_{s=l} L(s; f, g)
\]

for a suitable constant \( c \) where \( \langle f, g \rangle_{\Gamma_0(N)} \) denotes the Petersson scalar product (see Sect. 1), and \( \text{Res}_{s=l} \) denotes the residue at \( s = l \). This was proved in Petersson [10], Satz 6 for cusps forms \( f, g \) of integral weight. If either weights of \( f, g \) or characters are distinct, then \( L(l-1; f, g) \) is written in terms of the scalar product involving \( f, g \) and a suitable Eisenstein series (Corollaries 5.5, 6.2).

A Dirichlet character \( \chi \) is called even or odd according as \( \chi(-1) \) is 1 or \(-1\). Let

\[
\theta(z) := \sum_{n=-\infty}^{\infty} e(n^2z) = 1 + 2 \sum_{n=1}^{\infty} e(n^2z)
\]

be the theta series, and let \( \theta_\chi(z) := \sum_{n=-\infty}^{\infty} \chi(n)e(n^2z) \) be its twist by an even Dirichlet character \( \chi \). For an odd \( \chi \), put \( \Theta_\chi(z) := \sum_{n=-\infty}^{\infty} \chi(n)me(n^2z) \), which is a cusp form of weight 3/2. Then, the Riemann zeta function and the Dirichlet \( L \)-functions appear as \( L(s; \theta, \theta) = 4\xi(2s) \), \( L(s; \theta_\chi, \theta) = 4L(2s, \chi) \) for \( \chi \) even, \( L(s; \Theta_\chi, \theta) = 4L(2s-1, \chi) \) for \( \chi \) odd, and we may expect that many other interesting \( L \)-functions appear by this method. We have the following application. Let \( Q, Q' \) be positive definite integral quadratic with \( 2l, 2l' \) variables \( (l, l' \in \frac{1}{2}\mathbb{N}, l \geq l') \). Let \( r_Q(n), r_{Q'}(n) \) be the numbers of integral representations of \( n \) by \( Q, Q' \), respectively. Then, the Dirichlet series \( \sum_{n=1}^{\infty} r_Q(n)r_{Q'}(n)n^{-s} \) extends meromorphically to the whole \( s \) plane. Indeed if \( f(z), g(z) \) are the theta series associated with \( Q, Q' \) respectively, then \( L(s; f, g) = \sum_{n=1}^{\infty} r_Q(n)r_{Q'}(n)n^{-s} \). Then it satisfies the functional equation under \( l-1+s \mapsto l'-s \). Further the asymptotic value of
$X^{-l-l'+1} \sum_{0<n\leq X} r_Q(n) r_Q(n')$ as $X \to \infty$ is obtained in terms of the 0-th Fourier coefficients of $f, g$ at cusps and the constant terms of Eisenstein series appropriately taken (Sect. 8).

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Let us fix our notation. Let $N \in \mathbb{N}$. We denote by $(\mathbb{Z}/N)^*$, the group of Dirichlet characters modulo $N$. The identity element of $(\mathbb{Z}/N)^*$ is denoted by $1_N$, where $1_N(n)$ is 1 or 0 according as $n$ is coprime to $N$ or not. In particular, $1_1$ always takes the value 1, which we denote simply by 1. We denote by $\tilde{\chi}$ the conductor of $\chi$, and denote by $\chi$ the primitive character associated with $\tilde{\chi}$. For a prime $p$, $v_p(n)$ denotes the $p$-adic valuation of $n \in \mathbb{Z}$. We put

$$\epsilon_{\chi} := \prod_{p \mid f_{\chi}, \chi(p) = 0} p, \quad \epsilon'_{\chi} := \prod_{p \mid f_{\chi}, \chi(p) = 0} p^2.$$  

For a prime $p | N$, $\{\chi\}_p \in (\mathbb{Z}/p^{v_p(N)})^*$ denotes the $p$-part of $\chi$, and there holds an equality $\chi = \prod p | N(\chi)_p$. For $D$ a discriminant of a quadratic number field, $\chi_D$ is defined to be the Kronecker–Jacobi–Legendre symbol, and for $D = 1$, $\chi_1$ is defined to be 1. We extend this notation as follows. Let $a \in \mathbb{Z}$, $\neq 0$. Then, we define $\chi_a = \chi_{D(a)} 1_a$ for the discriminant $D(a)$ of the quadratic number field $Q(\sqrt{a})$. When $a$ is odd, $\chi_a$ denotes $\chi_a$ if $a \equiv 1$ (mod 4), $\chi_a$ if $a \equiv -1$ (mod 4). Let $\mu(n), \varphi(n)$ for $n \in \mathbb{N}$ denote as usual, the Möbius function, the Euler function, respectively.

Let $\mathfrak{K}$ be the fundamental domain of the group $\text{SL}_2(\mathbb{Z})$; $\mathfrak{K} = \{z = x + \sqrt{-1}y \in \mathbb{C} | |x| \leq 1/2, |z| \geq 1\}$. Let $\Gamma$ be a congruence subgroup. The fundamental domain of $\Gamma$ is obtained by $\mathfrak{K}(\Gamma) := \bigcup A \mathfrak{K}$ where $A$ runs over a set of left representatives of $\text{SL}_2(\mathbb{Z})$ modulo $\Gamma$. We denote by $\mathfrak{K}(N)$, the fundamental domain $\mathfrak{K}(\Gamma_0(N))$ of $\Gamma_0(N)$. We denote by $\mathcal{M}_{s_1,s_2}(\Gamma)$ for $s_1, s_2 \in \mathbb{C}$, the space of real analytic functions on $\mathfrak{K}$ which satisfy

$$f(Az) = (cz + d)^{s_1} (\overline{cz + d})^{s_2} f(z) \quad \left( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right)$$

with $-\pi < \arg(cz + d) \leq \pi$ and $\arg(\overline{cz + d}) = -\arg(cz + d)$. The imaginary part of $z$ is in $\mathcal{M}_{-1,-1}(\Gamma)$. We define

$$f|_A(z) := (cz + d)^{-s_1} (\overline{cz + d})^{-s_2} f(Az) \quad \left( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

with $A \in \text{GL}_2(\mathbb{R})$, $\det A > 0$. For two congruence subgroups $\Gamma_1, \Gamma_2$ with $\Gamma_1 \supset \Gamma_2$, the trace of $f \in \mathcal{M}_{s_1,s_2}(\Gamma_2)$ is defined by $\text{tr}_{\Gamma_1/\Gamma_2}(f) := \sum_A f|_A$ where $A$ runs over the set of left representatives of $\Gamma_1$ modulo $\Gamma_2$. The trace of $f$ is in $\mathcal{M}_{s_1,s_2}(\Gamma_1)$. For $k \in \mathbb{Z}$, $s \in \mathbb{C}$ and for a Dirichlet character $\chi$ modulo $N$ with the same parity as $k$, $\mathcal{M}_{k+s,s}(N, \chi)$ denotes the space of elements $f \in \mathcal{M}_{k+s,s}(\Gamma_1(N))$ satisfying

$$f(Az) = \chi(d)(cz + d)^k |cz + d|^{2s} f(z) \quad (A \in \Gamma_0(N)).$$
Let \( j(A, z) := \theta(Az)/\theta(z) \) \((A \in \text{SL}_2(\mathbb{Z}))\). For \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \), \( j(A, z) = 1 \)
if \( c = 0 \), and \( j(A, z) = \chi(c) t_d^{-1} |cz + d|^{1/2} \) if \( c > 0 \) where \( t_d \) is 1 or \( \sqrt{-1} \) according as \( d \equiv 1 \) modulo 4 or \( d \equiv 3 \) and where \( -\pi/2 < \arg(cz + d)^{1/2} \leq \pi/2 \).
On the group \( \Gamma_0(4) \), \( j(A, z) \) gives the automorphy factor for modular forms of weight 1/2.

Let \( 4 | N \). For a Dirichlet character \( \chi \) modulo \( N \) with the same parity as \( k \), \( M_{k+1/2+s,s}(N, \chi) \) denotes the space of elements \( f \in M_{k+1/2+s,s}(\Gamma_1(N)) \) satisfying

\[
f(Az) = \chi(d) j(A, z)(cz + d)^k |cz + d|^{2s} f(z) \quad \left( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right).
\]

We denote \( M_l(N, \chi) \) for \( l \in (1/2)\mathbb{N} \), the space of holomorphic modular forms in \( M_l,0(N, \chi) \). If \( \chi = 1_N \), we drop \( \chi \) from the notations \( M_l(N, \chi) \) or \( M_{l,0}(N, \chi) \).

## 1 Petersson scalar product

In the section, we define the Petersson scalar product of modular forms which are not necessarily holomorphic. Further, we obtain a formula between the scalar product and a special value of the \( L \)-function \((1)\) of holomorphic modular forms of the same weight and with the same character.

Let \( \Gamma \) be a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Let \( r \) be a cusp of \( \Gamma \), and let \( A_r \) be a matrix so that

\[
A_r(\sqrt{-1} \infty) = r \quad (A_r \in \text{SL}_2(\mathbb{Z})).
\]

Let \( w(r) = w^{(r)}(\Gamma) \) be the width of a cusp \( r \) of \( \Gamma \), namely, \( w^{(r)} \) is the least natural number so that \( A_r \begin{pmatrix} \pm 1 & w^{(r)} \\ 0 & \pm 1 \end{pmatrix} A_r^{-1} \in \Gamma \). Let \( f, g \) be real analytic modular forms for \( \Gamma \) of weight \( l \in \frac{1}{2} \mathbb{Z}, \geq 0 \) with same character so that \( y^l f g \) has the Fourier expansions at each cusp \( r \) in the form

\[
(y^l f g) |_{A_r}(z) = P^{(r)}_{y^l f g}(y) + \sum_{n=-\infty}^{\infty} u_n^{(r)}(y) e(nx/w^{(r)}), \quad P^{(r)}_{y^l f g}(y) = \sum_j c_j^{(r)} y_j^{(r)}
\]

where \( u_n^{(r)}(y) \) is a rapidly decreasing function as \( y \to \infty \) and where the constant term \( P^{(r)}_{y^l f g}(y) \) with respect to \( x \), is a finite linear combination of powers of \( y \) with \( v_j^{(r)}, c_j^{(r)} \in \mathbb{C} \). For \( T > 1 \), we put

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Petersson scalar products and \( L \)-functions arising from modular forms

\[ F(\Gamma) \sqrt{-1} = T \]

\[ \begin{align*}
Q_{y f \tilde{g}}^{(r)}(T) & := w^{(r)} \left\{ c^{(r)}_1 \log T + \int_0^T \sum_{\Im v_j \geq 1, v_j^{(r)} \neq 1} c^{(r)}_j y^{v_j^{(r)} - 2} dy \right\} \\
& = w^{(r)} \left\{ c^{(r)}_1 \log T + \sum_{\Im v_j \geq 1, v_j^{(r)} \neq 1} c^{(r)}_j \frac{T^{v_j^{(r)} - 1}}{v_j^{(r)} - 1} \right\} \quad (4)
\end{align*} \]

For \( T > 1 \), let \( \mathcal{F}_T(\Gamma) \) denote the domain obtained from \( \mathcal{F}(\Gamma) \) by cutting off neighborhoods of all cusps \( r \) along the lines \( \Im(A_r^{-1} z) = T \) (see Figure 1) where \( \Im \) means the imaginary part. Then, we define the Petersson scalar product of \( f \) and \( g \) by

\[ \langle f, g \rangle_{\Gamma} := \lim_{T \to \infty} \left( \int_{\mathcal{F}_T(\Gamma)} y^l f(z) \bar{g}(z) \frac{dx \, dy}{y^2} - \sum_r Q_{y f \tilde{g}}^{(r)}(T) \right), \quad (5) \]

\( r \) running over the set of the representatives of cusps of \( \Gamma \), or equivalently,

\[ \langle f, g \rangle_{\Gamma} = \int_{\mathcal{F}_T(\Gamma)} y^l f(z) \bar{g}(z) \frac{dx \, dy}{y^2} + \sum_r \left( \int_T^\infty \int_0^{w^{(r)}} (y^l f(z))(A_r(z) - P_{y f \tilde{g}}^{(r)}(y)) dx \, dy - Q_{y f \tilde{g}}^{(r)}(T) \right), \]

where the right hand side is independent of \( T > 1 \).

Suppose that \( f, g \) are holomorphic with \( l \in \mathbb{Z} \), \( > 0 \). Let \( a_0^{(r)}, b_0^{(r)} \) be the 0-th Fourier coefficients at a cusp \( r \), of \( f, g \), respectively. Then \( P_{y f \tilde{g}}^{(r)}(y) \) is equal to...
Let \( f(z) = \sum_{n=0}^{\infty} a_n e(nz/w) \), \( g(z) = \sum_{n=0}^{\infty} b_n e(nz/w) \) be the Fourier expansions with \( w = w_{\sqrt{-1}} \infty \) the width of the cusp \( \sqrt{-1} \infty \). We define the Dirichlet series associated with them by \( L(s; f, g) := \sum_{n=1}^{\infty} a_n b_n n^{-s} \), namely we make the same definition as (1) ignoring \( w \).

Replacing \( \Gamma \) by \( \pm \Gamma \) if necessary, we may assume that \( \pm 1 \in \Gamma \) where \( 1 \) denotes the identity matrix. We define the Eisenstein series \( E_{\Gamma, r}(z, s) := y^s \sum_{(c, d) \in A_{\Gamma^{-1}} \Gamma} |cz + d|^{-2s} \) where \( (c, d) \) run over the set of the second rows of matrices in \( A_{\Gamma^{-1}} \Gamma \) with \( c > 0 \), or with \( c = 0, d > 0 \). It converges absolutely and uniformly on any compact subset of \( \mathcal{H} \) for \( \Re(s) > 1 \), and gives a function in \( \mathcal{M}_{0,0}(\Gamma) \). As functions of \( s \), it extends meromorphically to the whole plane, and satisfies a functional equation under \( s \mapsto 1 - s \) (cf. Kubota [5]). We denote \( E_{\Gamma, r}(z, s) \) by \( E_{\Gamma}(z, s) \) when \( r = \sqrt{-1} \infty \). The constant term of the Fourier expansion of \( E_{\Gamma, r}(z, s) \) with respect to \( x \), at the cusp \( r \) is in the form \( y^s + \xi(s; r)y^{1-s} \), and at a cusp \( r' \) not equivalent to \( r \), in the form \( \xi(r')(s; r)y^{1-s} \). Here, both \( \xi(s; r) \) and \( \xi(r')(s; r) \) are involving the Riemann zeta function or partial zeta functions, and they are meromorphic functions on the \( s \)-plane and holomorphic on the domain \( \Re(s) > 1 \). They have poles of order 1 at \( s = 1 \), and \( \text{Res}_{s=1} E_{\Gamma, r}(z, s) = \text{Res}_{s=1} \xi(s; r) = \text{Res}_{s=1} \xi(r')(s; r) \).

Let \( l \in 1/2 \mathbb{Z} \geq 3/2 \). An alternative definition of the scalar product is

\[
(f, g)_\Gamma := \int_{\mathcal{H}(\Gamma)} \left\{ y^l f(z) \bar{g}(z) - \sum_r a_r^{(r)} \bar{b}_0^{(r)} E_{\Gamma, r}(z, l) \right\} \frac{dx \, dy}{y^2},
\]

where the integral is well-defined since the integrand is decreasing at all the cusps, indeed it is \( O(y^{-l-1}) \) as \( y \to \infty \) at each cusp (Zagier [16], Sect. 5, Pasol and Popa [8,9]).

We denote the Laplacian by

\[
\Delta := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 4y^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}
\]

where \( \frac{\partial}{\partial z} = 2^{-1} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right) \), \( \frac{\partial}{\partial \bar{z}} = 2^{-1} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right) \). It is an \( \text{SL}_2(\mathbb{R}) \) invariant differential operator on \( \mathcal{H} \). As is well known, the Eisenstein series \( E_{\Gamma, r}(z, s) \) is an eigenfunction of \( \Delta \), in fact, we have \( \Delta(E_{\Gamma, r}(z, s)) = s(s-1)E_{\Gamma, r}(z, s) \).

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Theorem 1.1 Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$ with the width $w_\Gamma$ of the cusp $\sqrt{-1}\infty$. Let $f, g$ be holomorphic modular forms for $\Gamma$ of weight $l \in \frac{1}{2}\mathbb{Z}, \geq 3/2$ or $l = 1/2$ with the same character. Then $L(s; f, g)$ extends meromorphically to the whole s-plane, and satisfies a functional equation under $l - 1 + s \mapsto l - s$ which comes from the functional equation of $E_\Gamma(z, s)$ under $s \mapsto 1 - s$, and

$$
(f, g)_\Gamma = (4\pi)^{-l} w^{l+1}_\Gamma \Gamma(l) C^{-1} \text{Res}_{s=1} L(s; f, g),
$$

where $C = \text{Res}_{s=1} E_\Gamma(z, s) = \text{Res}_{s=1} \xi(s)$, $\xi(s)$ being so that the constant term with respect to $x$, of $E_\Gamma(z, s)$ is $y^s + \xi(s)y^{1-s}$.

Remark 1.2 The method used in the following proof is found in Chiera [2] in which Theorem 1.1 in the case $l = 1/2$ is proved. He also uses the method to compute the scalar products of Eisenstein series of integral weight.

Proof Let $l \geq 3/2$. Let $F(z)$ be an automorphic form so that $F$ and $\Delta(F)$ are both integrable on $\mathfrak{f}(\Gamma)$. We compute the integral of $\Delta(F)$ over the fundamental domain as in Kubota [5, Sect. 2.3]. Then

$$
\int_{\mathfrak{f}(\Gamma)} \Delta(F) \frac{dx dy}{y^2} = \int_{\mathfrak{f}(\Gamma)} \left( \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} \right) F(z) dx dy = \int_{\partial(\mathfrak{f}(\Gamma))} y \frac{\delta}{\delta n} F(z) \frac{dl}{y}
$$

by Green’s theorem where $\partial(\mathfrak{f}(\Gamma))$ is the boundary of $\mathfrak{f}(\Gamma)$, $\delta/\delta n$ is the outer normal derivative, and $dl$ is the euclidean arc length. We note that $y \delta/\delta n$ and $dl/y$ are invariant under $SL_2(\mathbb{R})$. But on the lines of the boundary, arcs or pieces of arcs which are paired under $\Gamma$, the integral cancels, and $\int_{\mathfrak{f}(\Gamma)} \Delta(F) \frac{dx dy}{y^2}$ vanishes. We apply this to $F(z) := y^l f(z)\overline{g}(z) - \sum_r a_0^{(r)} b_{0}^{(r)} E_{\Gamma, r}(z, l)$, and employ the method developed by Chiera [2].

We have equality $\Delta(y^l f \overline{g}) = Q(f, g) + l(l - 1) y^l f \overline{g}$ with

$$
Q(f, g) := 4y^{l+2} \frac{\delta^2 f}{\delta z^2} \overline{g} + 2\sqrt{-1} y^{l+1} \left( \frac{\delta f}{\delta z} \overline{g} - f \frac{\delta g}{\delta z} \right)
$$

([2] Proposition 2.1). Then $\Delta(F) = Q(f, g) + l(l - 1) F$, and hence

$$
\langle f, g \rangle_{\Gamma} = -\frac{1}{l(l - 1)} \int_{\mathfrak{f}(\Gamma)} Q(f, g) \frac{dx dy}{y^2}
$$

by the above result. Since $Q(f, g)$ is an automorphic function rapidly decreasing at each cusp, to evaluate the integral we follow the argument due to Petersson [10] in which the Rankin–Selberg method (Rankin [12], Selberg [14]) is used.
By a standard unfolding trick, we have for $\Re s > 1$,
\[
\int_{\bar{\mathcal{S}}(\Gamma)} Q(f, g) E^\Gamma_{123}(z, s) \frac{dx\,dy}{y^2} = \int_0^\infty \int_0^{w_\Gamma} Q(f, g) dx\,dy
\]
\[
= w_\Gamma \int_0^\infty \left[ \sum_{n=1}^{\infty} a_n \bar{b}_n e^{-4\pi ny/w_\Gamma} \left( (4\pi/w_\Gamma)^4 n^2 y^{1+s} - (8\pi l/w_\Gamma) n y^{1+s} \right) \right] dy
\]
\[
= -(4\pi)^{-l+1} l^s y^{l+s} (l-s) \Gamma(l+s) L(l-1+s; f, g).
\]

The extreme left-hand side has a meromorphic continuation to whole of the $s$-plane since $E^\Gamma_{123}(z, s)$ has a meromorphic continuation and $Q(f, g)$ is rapidly decreasing at cusps. Hence, $L(l-1+s; f, g)$ has also, and the functional equation of $E^\Gamma_{123}(z, s)$ gives that of $L(l-1+s; f, g)$. We note that this part holds true also for $l = 1$.

Evaluating the residues at $s = 1$, we have $-CL(l-1)(f, g)_\Gamma = -(4\pi)^{-l} l^1 (L(l-1)\Gamma(l+1)\text{Res}_{s=1} L(l-1+s; f, g)$, which gives (6) since $l \neq 1$.

When $l = 1$, the equality (8) does not make sense, and (6) does not hold in general (see Remark 5.6 later).

# 2 Eisenstein series of integral weight

For the remainder of this paper we consider modular forms on $\Gamma_0(N)$. In this section, we study analytic property of real analytic Eisenstein series on $\Gamma_0(N)$ of weight $(k+s, s)$ with $k \in \mathbb{Z}$, $\geq 0$. A generalization of Theorem 1.1 is given using Eisenstein series of weight 0.

Let $l \in \frac{1}{2} \mathbb{Z}, \geq 0$. For a real number $m$ and for $s \in \mathbb{C}$ with $l + 2\Re s > 1$ we put
\[
w_{-m}(y, l, s) := e(m) \int_{-\infty}^\infty e(mt) \frac{e(mx)}{(z+t)^{1+l+2s}} dt
\]
with $z = x + \sqrt{-y} \in \mathbb{H}$, which satisfies $w_m(ny, l, s) = n^{-l+1-2s} w_{nm}(y, l, s)$ ($n \in \mathbb{N}$). Then, $w_{-m}(y, l, s)$ is equal to
\[
e^{-\frac{1}{4}} \cdot (2\pi)^{-l+1} \Gamma(l-1+2s) \Gamma(s)^{-1} \Gamma(l+s)^{-1} (m = 0),
\]
\[
e^{-\frac{1}{4}} l^{-1} \Gamma(s)^{-1} W_{-\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - s} (4\pi my) (m > 0),
\]
\[
e^{-\frac{1}{4}} |m|^{-1} \Gamma(s)^{-1} W_{\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - s} (4\pi |m| y) (m < 0)
\]
which meromorphically extend to the whole complex $s$-plane, where $W_{-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2} - s}(y)$ denote the Whittaker functions of $y$. We have $w_{-m}(y, l, 0) = 0$ for $m \geq 0$ except for $(l, m) = (1, 0)$. If $m < 0$, then $w_{-m}(y, 0, 0) = 0$, and $w_{-m}(y, l, 0) = \mathbb{C}$ Springer
\[(2\pi)^l e^{-\frac{1}{2}l} \Gamma(l)^{-1} |m|^{-l-1} e^{2\pi my} \] for \(l > 0\). By the Poisson summation formula, we have the Fourier expansion

\[
\sum_{m=-\infty}^{\infty} (z + m)^{-l} |z + m|^{-2s} = \sum_{n=-\infty}^{\infty} w_n(y, l, s) e(nx). \quad (9)
\]

The right-hand side extends meromorphically to the whole complex \(s\)-plane.

Let \(\chi\) be a Dirichlet character modulo \(N\), and let \(\tilde{\chi}\) be the primitive character associated with \(\chi\). Let \(\mathcal{I}_Z\) denote the characteristic function of \(Z\) on \(Q\). The Gauss sum \(\tau(\tilde{\chi})\) is defined to be \(\sum_{i:Z/\tilde{\chi}} \chi(i) e(i/\tilde{\chi})\), where \(i : Z/\tilde{\chi}\) implies that \(i\) runs through a set of representatives of \(Z\) modulo \(\tilde{\chi}\). For a Dirichlet character \(\chi\) not necessarily primitive, the sum \(\sum_{i:Z/N} \chi(i) e(i/N) (m \in Z)\) does not vanish only when \(v_p(m) = v_p(N) - 1\) for \(p|\tilde{\chi}\), and \(v_p(m)\) is equal to \(v_p(N) - 1\) or \(v_p(N)\) for \(p|\tilde{\chi}\)\(^{-1}\) where \(\epsilon_\chi\) is as in the introduction. In such a case, if \(R\) denotes the product of \(p|\tilde{\chi}\)\(^{-1}\) with \(v_p(m) = v_p(N) - 1\), then the sum is equal to \(\tau(\tilde{\chi}) \mu(R)\varphi(N) / \varphi(\tilde{\chi}) R\tilde{\chi}(m R \tilde{\chi} N^{-1})\). Altogether, we can write

\[
\sum_{i:Z/N} \chi(i) e(i/N) = \tau(\tilde{\chi}) \sum_{0 < R|\tilde{\chi}\tilde{\chi}^{-1}} \frac{\mu(R)\varphi(N)}{\varphi(\tilde{\chi}) R\tilde{\chi}} \tilde{\chi}(R) \tilde{\chi}(1_R \mathcal{I}_Z) (m \tilde{\chi} R^{-1}), \quad (10)
\]

where in the summation of the right-hand side, at most one term survives for each \(m\) in \(Z\).

As a set of representatives of cusps of \(\Gamma_0(N)\), we take

\[
\mathcal{C}_0(N) := \left\{ \frac{i}{M} \mid 0 < M \leq N, \ M|N, \ i \in S_M(N) \right\}, \quad (11)
\]

where \(S_M(N)\) can be chosen to be a set of cardinality \(\varphi((M, N/M))\) of values \(0 \leq i < M\) coprime to \(M\). A cusp \(0\) is considered to be \(0/1\). We note that \(1/N\) is equivalent to \(\sqrt{-1}\infty\), and we denote it also by \(\sqrt{-1}\infty\) as a cusp. Each rational number \(r\) is equivalent to only one element of \(\mathcal{C}_0(N)\) under the action of \(\Gamma_0(N)\). If \(r\) is equivalent to \(i/M\), then we put

\[
M^{(r)} := M, \quad w^{(r)} := N/(M^{(r)})^2, \quad (12)
\]

where \(w^{(r)}\) is the width of a cusp \(r\).

Let \(k \in Z, k \geq 0\). We assume that \(N \geq 3\) if \(k\) is odd. Let \(M\) be a fixed positive divisor of \(N\), and let \(c_0, d_0 \in Z\) with \((c_0, N/M) = 1, (d_0, M) = 1\). For \(s \in C\), we define Eisenstein series for \(k + 2\delta s > 2\), by

\[
G_k(z, s; c_0, d_0; M, N) := \sum_{c \equiv c_0 \ (N/M)} \sum_{d \in M^{-1} d_0 + Z} (cz + d)^{-k} |cz + d|^{-2s},
\]
\[ E_k(z, s; c_0, d_0; M, N) := M^{-k-2s} \sum'_{c|c_0 \ (N/M) \atop d \in M^{-1}d_0+\mathbb{Z} \atop (Mc, Md)=1} (cz + d)^{-k}|cz + d|^{-2s}, \]

where \( \sum' \) implies that the term corresponding to \( c = d = 0 \) is omitted in the summation.

Let \( \rho \in (\mathbb{Z}/M)^* \), \( \rho' \in (\mathbb{Z}/(N/M))^* \) with \( N/M = \varepsilon_{\rho'} \) so that \( \rho \rho' \) has the same parity as \( k \). We define Eisenstein series

\[
G_{k, \rho, M}(z, s) := \frac{\Gamma(k + s)}{(-2\sqrt{-1}\pi)^k \tau(\overline{\rho})} \sum_{c|c_0 : (\mathbb{Z}/(N/M))^*} \sum_{d_0 : (\mathbb{Z}/M)^*} \overline{\rho}(d_0) \rho'(c_0) G_k(z, s; c_0, d_0; M, N),
\]

\[
E_{k, \rho, M}(z, s) := 2^{-1} \sum_{c|c_0 : (\mathbb{Z}/(N/M))^*} \sum_{d_0 : (\mathbb{Z}/M)^*} \overline{\rho}(d_0) \rho'(c_0) E_k(z, s; c_0, d_0; M, N)
\]

\[
= 2^{-1} \sum_{(c, d) = 1 \atop c \equiv 0 (\text{mod } M)} \overline{\rho}(d) \rho' \left( c/M \right) (cz + d)^{-k}|cz + d|^{-2s}
\]

\[
= \sum_{A \in \Gamma_0(N) \setminus \Gamma_0(M)} \overline{\rho}(d_A) \rho'(c_A/M) 1|A(z, s) \quad \left( A = \begin{pmatrix} * & * \\ c_A & d_A \end{pmatrix} \right)
\]

for \( \Gamma_0(N) \) with character \( \rho \rho' \), where \( (\mathbb{Z}/M)^* \) implies that \( d_0 \) runs over the complete set of representatives.

We omit \( M \) from \( G_{k, \rho, M} \) and \( E_{k, \rho, M} \) when \( M = \varepsilon_{\rho} \). We also omit \( \rho \) or \( \rho' \) if \( \rho = 1 \) or \( \rho' = 1 \). There holds equalities,

\[
G_{k, \rho, M}(z, s) = (\sqrt{-1}\pi)^{-k} \rho'(-1) 2^{-k+1} M^{k+2s} f_{\rho}^{-1} \tau(\overline{\rho}) \Gamma(k+s)L(k+2s, \overline{\rho} \rho') E_{k, \rho, M}(z, s),
\]

(13)

noting that \( \tau(\overline{\rho}) \tau(\overline{\rho}) = \rho(-1)f_{\rho} \), and if \( M = \varepsilon_{\rho} \), then \( M^{k/2+s} E_{k, \rho, M}(z, s)|_{SN} = \rho'(-1) (N/M)^{k/2+s} E_{k, \rho, M}(z, s) \).

The Eisenstein series \( G_{k, \rho, M}(z, s) \) as functions of \( s \) extend meromorphically to the whole complex plane. The constant term of \( E_{k, \rho, M}(z, s) \) with respect to \( x \) at each cusp, is in the form \( c + \xi(s)y^{-k+1-2s} \) with a constant \( c \) and a meromorphic function \( \xi(s) \) of \( s \), which follows from (9). The Fourier expansion of \( G_{k, \rho, M}(z, s) \) is given by
by using (10) where \( \delta_{M,N}, \delta_{f',1} \) denote the Kronecker delta. The Fourier expansion of 
\( E_{k,\rho,M}(z,s) \) is obtained from (13) and (14). For later use we write down the constant term of the Fourier expansion with respect to \( x \), of \( y^s E_{k,1,1,N}(z,s) \) for even \( k \) at each cusp \( r \in \mathbb{Q} \). Let us denote by \( y^s + \xi(r)(s)y^{-k+1-s} \), the constant term at a cusp \( r \) equivalent to \( 1/N \), and denote by \( \xi(r)(s)y^{-k+1-s} \), the constant term at a cusp \( r \) not equivalent to \( 1/N \). Then

\[
\xi(r)(s) = \frac{(-1)^{k/2} \pi^{1/2} \Gamma(k - 1 + 2s)}{2^{k - 2 + 2s} \Gamma(s) \Gamma(k + s) N M(r)^{k - 1 + 2s} \varphi(N/M(r))} \times \frac{\xi(k + 1 + 2s) \prod_{p|N/M(r)} (1 - p^{-1 - 2s})}{\xi(k + 2s) \prod_{p|N} (1 - p^{-k - 2s})},
\]

(15)

where \( M(r) \) is as in (12). In particular if \( k = 0 \), then

\[
\xi(r)(s) = \frac{\pi^{1/2} \Gamma(s - 1/2)}{\Gamma(s)} \frac{M(r)^{-2s} \varphi(N) \xi(-1 + 2s) \prod_{p|N/M(r)} (1 - p^{1 - 2s})}{\xi(2s) \prod_{p|N} (1 - p^{-2s})},
\]

(16)

all of which have poles of order 1 at \( s = 1 \) with same residue \( 3\pi^{-1/2} \Gamma_0(1) : \Gamma_0(N) \)^{-1} where \( [\Gamma_0(1) : \Gamma_0(N)] = N^{-1} \prod_{p|N} (1 + p^{-1})^{-1} \). There holds an equality

\[
\text{tr}_{\Gamma_0(1)/\Gamma_0(N)}(y^s E_{0,1,1,N}(z,s)) = y^s E_0(z,s) = E_{\Gamma_0(1)}(z,s).
\]

For a square free \( S \in \mathbb{N} \) and for a character \( \chi \), let us define an operator on the function on \( \mathfrak{h} \) by

\[
\Lambda_{S,s,\chi} f(z) := \sum_{0 < R | S} \mu(S/R) \chi(S/R) R^s f(Rz).
\]

Then, from the Fourier expansion (14), we see

\[
G_{k,\rho,M}(z,s) = (M e_{\rho}^{-1})^{k + 2s} \Lambda_{\epsilon_{\rho}^{-1},k + 2s,\rho} G_{k,\rho}(M e_{\rho}^{-1} z,s).
\]

(17)

Now, we assume that \( \rho' \) is primitive and \( N/M = f_{\rho'} \). If \( \rho \) is also primitive, then it follows from (14), the functional equation

\[
y^{-k+1-s} G_{k,\rho}(z,-k+1-s) =
\]
\[ \pi^{-k+1-2s}y^s G_{k,\rho}^\rho(z, s). \] Then, for \( \rho \) not necessarily primitive, we have the functional equations by (17), as

\[
y^{-k+1-s} G_{k,\rho, M}(z, -k+1-s) = \pi^{-k+1-2s} \mathbf{M} \mathbf{c}_{\rho}^{-1} y^s \mathbb{A}_{\mathbf{c}_{\rho}^{-1}, 1, \bar{\rho}} G_{k,\rho}^\rho(\mathbf{M} \mathbf{c}_{\rho}^{-1} z, s),
\]

\[
y^{-k+1-s} E_{k,\rho, M}(z, -k+1-s) = \frac{(-1)^k \pi^{-k+1-2s} (M | \rho)'(k+1) (\rho)' \Gamma(k+s) L(k+2s, \bar{\rho} \rho')}{\mathbf{c}_{\rho}^{-1} \tau(\bar{\rho}) \Gamma(1-s) L(-k+2-2s, \bar{\rho} \rho')} \times y^s \mathbb{A}_{\mathbf{c}_{\rho}^{-1}, 1, \bar{\rho}} E_{k,\rho}^\rho(\mathbf{M} \mathbf{c}_{\rho}^{-1} z, s). \tag{18}
\]

Let \( \rho' = 1 \). For \( R|\mathbf{c}_{\rho}^{-1} \rho' \), we have an equality \( E_{k}^\rho(N R \mathbf{c}_{\rho}^{-1} z, s) = (N | \mathbf{c}_{\rho}^{-1})^{-k-2s} z^{-k}|z|^{-2s} \tilde{\eta}(\mathbf{c}_{\rho}^{-1} R^{-1}) \sum_{P|\mathbf{c}_{\rho}^{-1} R^{-1}} \tau(P) P^{k+2s} E_{k, \rho, \mathbf{c}_{\rho}^{-1}, \rho} (-\frac{1}{N \mathbf{c}_{\rho}^{-1}}, s) \). From this equality, and a functional equation for \( L(-k+2-2s, \bar{\rho}) = L(-k+2-2s, \bar{\rho}) \prod_{p|\mathbf{c}_{\rho}^{-1}} (1 - \bar{\rho}(p) p^{k+2-2s}) \) and from (18), we obtain

\[
y^{-k+1-s} E_{k,\rho, N}(z, -k+1-s) = \pi^{-1/2} z^{-k} U_{k,\rho}(s) \sum_{0 < P|\mathbf{c}_{\rho}^{-1}} U_{k,\rho, P}(s)(3(-\frac{1}{N \mathbf{c}_{\rho}^{-1}})) s E_{k, \rho, \mathbf{c}_{\rho}^{-1}, \rho} \left(-\frac{1}{N \mathbf{c}_{\rho}^{-1}}, s\right) \tag{19}
\]

for \( \rho \in (\mathbb{Z}/N)^* \) with the same parity as \( k \) and with

\[
U_{k,\rho}(s) := \frac{(\sqrt{-1})^k N^{-1+s} \mathbf{c}_{\rho}^{-1} | 2 \Gamma(s) \Gamma(k+s) L(k+2s, \bar{\rho})}{\Gamma((k-1)/2+s) \Gamma(k/2+s) L(k+1+2s, \bar{\rho}) \prod_{p|\mathbf{c}_{\rho}^{-1}} (1 - \bar{\rho}(p) p^{k+2-2s})},
\]

\[
U_{k,\rho, P}(s) := \prod_{p|P} (1 - \bar{\rho}(p) p^{k+2s}) \phi(\mathbf{c}_{\rho}^{-1} P^{-1}). \tag{20}
\]

As in Sect. 1, we obtain the following using a similar argument two theorems, making use of the Eisenstein series of weight 0.

**Theorem 2.1** Let \( f, g \) be holomorphic modulars forms for \( \Gamma_0(N) \) of weight \( l \in \frac{1}{2} \mathbb{Z}, l \geq 1/2 \) with characters. We assume that \( f, g \in \mathcal{M}_{l,1}(N, \rho) \) with \( \rho \in (\mathbb{Z}/N)^* \). Then, \( L(s; f, g) \) converges for \( s \) with \( \Re s > \max\{2l-1, 1/2\} \), and extends meromorphically to the whole \( s \)-plane.

(i) Let \( \rho = 1_N \). Then

\[
\langle f, g \rangle_{\Gamma_0(N)} = 3^{-1} 4^{-l} \pi^{-l+1} \Gamma(l) N \prod_{p|N} (1 + p^{-1}) \text{Res}_{s=1} L(s; f, g) \tag{21}
\]

except for \( l = 1 \).

(ii) Let \( \tilde{f} \tilde{g}(z) := (f \tilde{g})|S_N(z) \in \mathcal{M}_{l,1}(N, \bar{\rho}) \) with \( S_N := \left( \begin{array}{cc} 0 & -N^{-1/2} \\ N^{1/2} & 0 \end{array} \right) \). For \( M \in \mathbb{N} \) with \( \mathbf{c}_M | M \), let \( \text{tr}_{\Gamma_0(N)/\Gamma_0(M)}(\rho(f \tilde{g})) := \sum_{A: \Gamma_0(N)/\Gamma_0(M)} \rho(d_A) \tilde{f} \tilde{g}|A(z), d_A \) being the \( (2, 2) \) entry of \( A \). If \( \text{tr}_{\Gamma_0(N)/\Gamma_0(M)}(\rho(f \tilde{g})) \) for \( M \) has \( \sum_{n=0}^{\infty} c_n^{(M)} e^{-4\pi n y} \)
as the constant term of its Fourier expansion with respect to $x$, then we put
$L(s; \text{tr}_{\Gamma_0(N)/\Gamma_0(M), \rho}(\widetilde{f}_{\mathcal{G}})) := \sum_{n=1}^{\infty} c_n^{(M)} n^{-s}$. Then we have a functional equation

$$L(l - s; f, g) = \frac{2^{-4s} \pi^{1/2} \Gamma(l - 1 + s) \Gamma(l - 1 - s)}{\Gamma(l - s) \Gamma(-1/2 + s) \Gamma(-1/2 - s)} \prod_{p|\rho, |\rho|^{-1}} (1 - \rho^2(p) p^{2-2s}) \times \sum_{p|\rho, |\rho|^{-1}} (1 - \rho^2(p) p^{2s}) \varphi(\epsilon_{\rho})^{-1} P^{-1} L(l - 1 + s; \text{tr}_{\Gamma_0(N)/\Gamma_0(P_{1, p}, \rho}(\widetilde{f}_{\mathcal{G}})).$$

**Proof** Let $Q(f, g)$ be as in (7). Then $Q(f, g)$ is a real analytic automorphic form with character $\rho$, and $Q(f, g)^{s} E_{0, \mathcal{F}, N}(z, s)$ is an automorphic form with trivial character. Then, $\int_{\mathcal{F}(\mathcal{F})} Q(f, g)^{s} E_{0, \mathcal{F}, N}(z, s) \frac{dx dy}{y^2}$ is well-defined, and it has a meromorphic continuation to whole of the $s$-plane since $Q(f, g)$ is rapidly decreasing at cusps. The assertion (i) follows from that $E_{\Gamma_0(N)}(z, s)$ as well as $\xi^{(r)}(s)$ of (16), has the residue $N^{-1} \prod_{p|N} (1 + p^{-1})^{-1}$ at $s = 1$. We prove (ii). As in the proof of Theorem 1.1, the integral is shown to be equal to $-4\pi^{-1/2} (l - s) \Gamma(l + s) L(l - 1 + s; f, g)$ by a standard unfolding trick for $s$ with $\Re s$ large enough. Then,

$$-(4\pi)^{-1/2} (l - 1 + s) \Gamma(l + 1 - s) L(l - s; f, g)$$

$$= \int_{\mathcal{F}(\mathcal{F})} Q(f, g)(z) y^{1-s} E_{0, \mathcal{F}, N}(z, 1 - s) \frac{dx dy}{y^2}$$

$$= \pi^{-1/2} U_{0, \mathcal{F}, N}(s) \sum_{0 < p, |p|^{-1}} U_{0, \mathcal{F}, N}(s) \int_{\mathcal{F}(\mathcal{F})} Q(f, g)(z) \left( \frac{1}{Nz} \right)^s$$

$$\times E_{0, \mathcal{F}, N}(z, s) \frac{dx dy}{y^2}$$

$$= \pi^{-1/2} U_{0, \mathcal{F}, N}(s) \sum_{0 < p, |p|^{-1}} U_{0, \mathcal{F}, N}(s) \int_{\mathcal{F}(\mathcal{F})} Q(f|S_{N}, g|S_{N})(z) y^s$$

$$\times E_{0, \mathcal{F}, N}(z, s) \frac{dx dy}{y^2}$$

$$= -\pi^{-1/2} U_{0, \mathcal{F}, N}(s) \sum_{0 < p, |p|^{-1}} U_{0, \mathcal{F}, N}(s)$$

$$\times (4\pi)^{-1/2} (l - s) \Gamma(l + s) L(l - 1 + s; \text{tr}_{\Gamma_0(N)/\Gamma_0(P_{1, p}, \rho}(\widetilde{f}_{\mathcal{G}})).$$

This shows the functional equation.

We note that the function $E_{0, \rho, N}(z, s)$ of $s$ is holomorphic on the real axis with $s \geq 1$ for even $\rho \neq 1_N$. Thus, the formula of type (21) is not obtained in this case. Theorem 2.1 are generalized in Corollaries 5.5 and 6.2.
The scalar product defined in Petersson [10] satisfies the equality \( \langle f, g \rangle_{\Gamma_0(N)} = (f|S_N, g|S_N)_{\Gamma_0(N)} \) for holomorphic cusp forms \( f, g \) for \( \Gamma_0(N) \) of the same weight and with same character where \( S_N \) is as in Theorem 2.1. Let \( f, g \) be real analytic modular forms satisfying (3). If for all cusps \( r \), \( Q^{(r)}_{yf \overline{g}}(T) \) has only terms given by definite integrals form 0 to \( \infty \), namely, no \( Q^{(r)}_{yf \overline{g}}(T) \) has a term containing \( \log T \), then equality also holds for the scalar product (5). However it does not hold in general.

**Lemma 2.2** Let \( f, g \) be as above. Then, \( \langle f|S_N, g|S_N \rangle_{\Gamma_0(N)} = \langle f, g \rangle_{\Gamma_0(N)} - \sum_{r \in C_0(N)} w^{(r)}c_1^{(r)} \log \frac{N}{M^{(r)2}} \) with \( M^{(r)} \), \( w^{(r)} \) in (12) and with \( c_1^{(r)} \) in (3).

**Proof** Let \( T = (T^{(r)})_{r \in C_0(N)} \) with \( T^{(r)} > N \), and let \( \mathfrak{F}(N) \) denote the domain obtained from \( \mathfrak{F}(N) \) by cutting off neighborhoods of cusps \( r \) along the lines \( \mathfrak{A}_r^{-1}z = T^{(r)} \) \( (r \in C_0(N)) \). As easily seen, the equality

\[
\langle f, g \rangle_{\Gamma} = \lim_{T^{(r)} \to \infty \atop r \in C_0(N)} \left( \int_{\mathfrak{F}(N)} y^j f(z) \overline{g}(z) \frac{dx \, dy}{y^2} - \sum_r Q^{(r)}_{yf \overline{g}}(T^{(r)}) \right)
\]

holds in the notation of (5).

Now, we consider a cusp in the form \( i/M \, (M|N, \, (i, M) = 1) \). The matrix \( S_N \) maps \( \mathfrak{F}(N) \) onto a fundamental domain of \( \Gamma_0(N) \), and hence \( S_N \mathfrak{F}(N) \) can be decomposed into a finite number of pieces so that the union of their suitable translations by matrices in \( \Gamma_0(N) \), is equal to \( \mathfrak{F}(N) \). Let \( \phi_{S_N} \) denote the map of \( \mathfrak{F}(N) \) onto itself obtained in this manner. Then, \( \phi_{S_N}^2 \) is the identity map of \( \mathfrak{F}(N) \). The map \( \phi_{S_N} \) can be naturally extended to \( \mathfrak{F}(N) \cup C_0(N) \), and then a cusp in the form \( i/M \, (M|N) \) in \( C_0(N) \) is mapped to a cusp in the form \( j/(N/M) \) and vice versa. If we take \( T \) so that \( T^{(j/M)} = (N/M^2)T \), then \( \phi_{S_N}(\mathfrak{F}(N)) = \mathfrak{F}(T) \). Hence, \( \int_{\mathfrak{F}(N)} y^j f(z) \overline{g}(z) \frac{dx \, dy}{y^2} = \int_{\mathfrak{F}(T)} y^j f(z) \overline{g}(z) \frac{dx \, dy}{y^2} \)

for sufficiently large \( T \).

Since \( ((y^j f \overline{g})|S_N)|_{A_{j/(N/M)}}(z) = ((y^j f \overline{g})|S_N)|_{A_{j/M}}((N/M^2)z) \), there is an equality \( P_{y^j f \overline{g}}^{(j/M)}(y) = P_{y^j f \overline{g}}^{(j/M)}((N/M^2)y) \). Let \( \tilde{P}_{y^j f \overline{g}}^{(j/M)}(y) := \sum_{y_j \neq 1, v_j \neq 1} c_{v_j}^{(j/M)} \times y_j^{(j/M)} \). Then, noticing that \( (N/M^2)w^{(j/M)} = w^{(j/M)} \), we have

\[
Q_{y^j f \overline{g}}^{(j/(N/M))}(T) = w^{(j/(N/M))} \left\{ c_1^{(j/M)} (N/M^2) \log T + \int_0^T \tilde{P}_{y^j f \overline{g}}^{(j/M)}((N/M^2)y) y^{-2} \, dy \right\}
\]

\[
= w^{(j/M)} \left\{ c_1^{(j/M)} \log T + \int_0^{(N/M^2)^T} \tilde{P}_{y^j f \overline{g}}^{(j/M)}(y) y^{-2} \, dy \right\}
\]

\[
= Q_{y^j f \overline{g}}^{(j/M)}((N/M^2)T) - w^{(j/M)} c_1^{(j/M)} \log(N/M^2).
\]

It follows that

\[\square \]
In this section, we study analytic property of real analytic Eisenstein series for $\Gamma_0(N)$ of weight $(k + 1/2 + s, s)$ with $k \in \mathbb{Z}, s \geq 0$, or equivalently $y^s$ times it of half integral weight $k + 1/2$.

Let $N, M \in \mathbb{N}$ with $4|N, M|N$ so that $4|M$ or $4|(N/M)$. Let $\rho \in (\mathbb{Z}/M)^*, \rho' \in (\mathbb{Z}/(N/M))^*$ where

$$2|\nu_p(M\rho^{-1}) (p|M) \text{ for which } \{\rho\}_p \text{ is trivial or real},$$

$$N/M = \epsilon'_{\rho'}.$$  \hfill (22)

$\epsilon'_{\rho'}$ being as in the introduction. For $k \in \mathbb{Z}, s \geq 0$ with the same parity as $\rho\rho'$, we define an Eisenstein series of weight $(k + 1/2 + s, s)$ for $\Gamma_0(N)$ with character $\rho\rho'$ as follows. If $4|M$, then

$$E^{\rho'}_{k+1/2, \rho, M}(z, s) := \sum_{c,d} \overline{\rho}(d)\rho'(c/M)\chi_c(d)t_d(cz+d)^{-k-1/2}|cz+d|^{-2s},$$

and if $2 \nmid M$, then

$$E^{\rho'}_{k+1/2, \rho, M}(z, s) := \sum_{c,d} \overline{\rho}(d)\rho'(c/M)\chi_c^\vee(d)t_d^{-1}(cz+d)^{-k-1/2}|cz+d|^{-2s},$$

where $c, d$ run over the set of the second rows of matrices in $A_{1/M}^{-1}\Gamma_0(N)$ with $c > 0$, or with $c = 0$ and $d > 0$, $A_{1/M}$ being as in (2). More precisely $c, d$ satisfy the condition that $c > 0, M|c, (c/M, N/M) = 1, d \in \mathbb{Z}, (d, M) = 1, (c, d) = 1$ where $c = 0, d = 1$ is added if $M = N$. We drop the notation $M$ from $E^{\rho'}_{k+1/2, \rho, M}(z, s)$ if $M = \epsilon_{\rho}$, and drop $\rho$ or $\rho'$ if $\rho = 1$ or $\rho' = 1$. The Eisenstein series $E^{\rho'}_{k+1/2, \rho, M}(z, s)$ as a function of $s$ extends meromorphically to the whole complex plane. The constant term of $E^{\rho'}_{k+1/2, \rho, M}(z, s)$ with respect to $x$ at each cusp, is in the form

$$c_0 + \xi_0(s)y^{-k-1/2-2s}$$  \hfill (23)

with a constant $c_0$ and a meromorphic function $\xi_0(s)$ of $s$, which follows from (9) where $c_0$ and $\xi_0(s)$ could be 0.
Let \( \tilde{\rho} \) be a primitive Dirichlet character. We put
\[
\rho := \tilde{\rho}1_2, \quad N := \text{lcm}(4, f_\rho) \tag{24}
\]
and we consider \( \rho \) as a character in \((\mathbb{Z}/N)^\times\). If \( 2|f_{\tilde{\rho}} \), then the equality \( \rho = \tilde{\rho} \) holds. Obviously, \( \rho, N \) such as (24) satisfy the condition (22) with \( M = N \). Then
\[
E_{k+1/2, \rho}(z, s) = 1 + \sum_{c=0(\text{mod}N), c > 0} (\overline{\chi_c}(d) t_d (cz + d)^{-k+1/2} |cz + d|^{-2s}). \tag{25}
\]
For \( k \geq 2 \), \( E_{k+1/2, \rho}(z, 0) \) is a holomorphic in \( z \) and, it is in \( \mathbb{M}_{k+1/2}(N, \rho) \). The Eisenstein series has the Fourier expansion for \( s \in \mathbb{C} \) with \( 2s + k + 1/2 > 2 \),
\[
1 + c_{k,s,\rho,N}(0) w_0(y, k+1/2, s) + \sum_{n \neq 0} c_{k,s,\rho,N}(n) w_n(y, k+1/2, s) e(nx) \tag{26}
\]
with
\[
c_{k,s,\rho,N}(n) := \sum_{m=0(N), m > 0} m^{-k+1/2-2s} \sum_{i; (\mathbb{Z}/m)^\times} (\overline{\chi_m}(i) t_i e(ni/m) \quad (n \in \mathbb{Z}) \text{ by (9)}.
\]
Let \( \rho_2 := \{\rho\}_2 \), and let \( \rho_e \) be the product of complex \( \{\rho\}_p \) (2 \( \neq \) \( p \mid N \)), and \( \rho_r \) be the product of real \( \{\rho\}_p \) (2 \( \neq \) \( p \mid N \)), so that
\[
\rho = \rho_2 \rho_e \rho_r. \tag{27}
\]
Further, we put \( \rho_{2e} := \rho_2 \rho_e \).

We compute the constant terms of Fourier expansions with respect to \( x \) at cusps in \( C_0(N) \) of (11), of some specific Eisenstein series of half integral weight. They are useful to investigate the functional equations of Eisenstein series.

Let \( 4|N, \rho \in (\mathbb{Z}/N)^\times \) be as in (22) with \( M = N \). Then
\[
E_{k+1/2, \rho, 2^mN}(z, s) = E_{k+1/2, \rho, N}(2^m z, s) \quad (m \geq 0),
\]
\[
E_{k+1/2, \rho, 2^m+1N}(z, s) = E_{k+1/2, \rho, N}(2^m+1 z, s) \quad (m \geq 0),
\]
\[
E_{k+1/2, \rho, \chi p, p^2N}(z, s) = E_{k+1/2, \rho, \chi p, pN}(pz, s) \quad (p \nmid N),
\]
\[
E_{k+1/2, \rho, \chi p, N}(p^2z, s) \quad (p \mid N).
\]

For \( \rho' \in (\mathbb{Z}/N') \) satisfying (22) with \( \rho = \rho' \) and \( M = N' \), the above equalities imply that Eisenstein series \( E_{k+1/2, \rho', N'}(z, s) \) is written in the form \( E_{k+1/2, \rho', N'}(mz, s) \) for some natural number \( m \) and for \( \rho, N \) so that
\[
\rho = \tilde{\rho}1_2, \quad N = \text{lcm}(4, f_\rho), \text{ and } \rho_2 = 1_2, \chi_4 \text{ or } \rho_2 \text{ is complex}, \tag{28}
\]
where \( \tilde{\rho} \) is a primitive character and where \( \rho_2 \) is as in (27).

**Lemma 3.1** Let \( \rho, \tilde{\rho}, N \) be as in (28), and let \( \rho = \rho_2 \rho_e \rho_r = \rho_{2e} \rho_r \) be as in (27). Let \( k \) be a nonnegative integer with the same parity as \( \rho \).
(i) Suppose that $\rho_2$ is real. Then $N = 4f_{\rho_2}(\rho_2)\pmod{4}$, and $E_{k+1/2,\rho}(z, s)$ vanishes at a cusp $i/(2m)$ $(i, 2m) = 1$ for any odd $m$.

(ii) Suppose that $\rho_2$ is complex, which is irreducible by our assumption. Then, $E_{k+1/2,\rho}(z, s)$ vanishes at a cusp $i/(2-1f_{\rho_2}m)$ $(i, 2f_{\rho_2}m) = 1$ for any odd $m$.

(iii) Let $M | N$. Suppose that $M$ has a prime divisor $p$ so that $\rho | p$ is complex. Then, $E_{k+1/2,\rho}(z, s)$ has the constant term (23) at a cusp $i/M$ $(i, M) = 1$ with $\xi_0(s) = 0$.

**Proof** (i) Put $B_n = \left(\begin{array}{cc} -1+2imn & -t^2_n \\ 4m^2n & -1-2imn \end{array}\right) \in \text{SL}_2(\mathbb{Z})$, which stabilizes the cusp $i/(2m)$. 

It is checked that the automorphy factor of $E_{k+1/2,\rho}(z, s)$ has the value in $\pm\sqrt{-1}$ at $z = i/(2m)$ for $B_{1/\rho_2}\in\Gamma_0(N)$ noting that $-1-2im\bar{f}_{\rho_2}\rho_2 \equiv 1$ (mod 4), namely $\ell(-1-2im\bar{f}_{\rho_2}\rho_2) = 1$, and $\lim_{z\to i/(2m)}(4m^2f_{\rho_2}\rho_2\bar{z}-1-2im\bar{f}_{\rho_2}\rho_2)^{1/2} = \sqrt{-1}$. Then, the constant term of the Fourier expansion of $E_{k+1/2,\rho}(z, s)$ at the cusp $i/(2m)$ vanishes.

(ii) We note that $16|f_{\rho_2}$. Put $B_n = \left(\begin{array}{cc} -1+2imn & -t^2_n \\ 2^2imn & -1-2imn \end{array}\right) \in \text{SL}_2(\mathbb{Z})$, which stabilizes the cusp $i/(2-1f_{\rho_2}m)$. Take $f_{\rho_2}(> 0)$ as $n$, then, $\lim_{z\to i/(2m)}(4m^2n-1-2imn)^{1/2} = (-1)^k\chi(2-1f_{\rho_2}m)\chi(-1-2f_{\rho_2}m) = \chi_n(-1) = 1$, and $-1-2f_{\rho_2}m \equiv 3$ (mod 4). Hence the automorphy factor of $E_{k+1/2,\rho}(z, s)$ has the value $(-1)^k\rho(-1-2f_{\rho_2}m)$ at $z = i/(2-1f_{\rho_2}m)$ for $B_n \in \Gamma_0(N)$. Then $(-1)^k\rho(-1-2f_{\rho_2}m) = \rho_2(1-2f_{\rho_2}m) = -1$. Then the constant term of the Fourier expansion at the cusp $i/(2-1f_{\rho_2}m)$ vanishes. Though Lemma makes no mention of the case $\rho_2 = \chi \pm \delta$, this proof is effective.

(iii) At first, we assume that there is a prime divisor $p_0$ of $M$ with $1 \leq p_0 < v_{p_0}(N)$ so that $\rho | p_0$ is complex. When $p_0 = 2$, we may assume by the assertion (ii), that $v_2(2) < v_2(N) - 2$. Put $B_n = \left(\begin{array}{cc} -1+2imn & -t^2_n \\ M^2 & -1-iMn \end{array}\right) \in \text{SL}_2(\mathbb{Z})$, which stabilizes the cusp $i/M$. We take $n > 0$ so that $N|M^2n, v_p(Mn) \geq v_p(N)$ $(p | N, p \neq p_0)$ and that $v_{p_0}(N) < v_{p_0}(N)$ if $p_0 \neq 2$, and that $v_2(Mn) < v_2(N) - 1$ if $p_0 = 2$. Then, the automorphy factor of $E_{k+1/2,\rho}(z, s)$ does not takes a real value at $i/M$ for $B_n \in \Gamma_0(N)$, in particular it does not take the value 1 and hence the constant term of the Fourier expansion at $i/M$ vanishes.

Now, we may assume that $(N, N/M) = 1$ and the cusp $i/M$ is $1/M$ (see (11)). Let $A_{1/M} = \left(\begin{array}{c} 1 \\ \frac{b_0}{M} \end{array}\right) \in \text{SL}_2(\mathbb{Z})$. Put $u(c, d) := (cz + d)^{k-1/2}|cz + d|^{-2s}$ for short. Then

$$E_{k+1/2,\rho}(z, s)|_{A_{1/M}} = u(M, d_0) + \sum_{N|c>0, (c,d)=1, c+dM>0} (\overline{\rho}\rho)(d)u(c + dM, cb_0 + dd_0)$$
\[-(-1)^k \sqrt{-1} \sum_{N|c>0,(c,d)=1}^{N} (\overline{\rho} \chi_c)(d) \zeta_d u((-c+dM),-cb_0-dd_0)\]
\[= \sum_{N|c>0,(c,d)=1}^{N} (\overline{\rho} \chi_c)(d) \zeta_d u(c+dM,cb_0+dd_0). \quad (29)\]

We fix \(c+dM>0\). Then, (29) has the partial sum \(\sum_{n=-\infty}^{\infty} \overline{\rho}(d-N \frac{N}{M} n) \chi_{c+Nn}(d-N \frac{N}{M} n) \times \zeta_{d-N \frac{N}{M} n} u(c+dM,cb_0+dd_0-N \frac{N}{M} n)\), whose constant term containing \(y^{-k-1/2-2s}\) is equal to

\[2^{-1}(1+\sqrt{-1}) \sum_{n=0}^{N} \left\{ (\overline{\rho} \chi_{c+Md})(d-N \frac{N}{M} n) - (\overline{\rho} \chi_{(c+Md)} \chi_{-4})(d-N \frac{N}{M} n) \sqrt{-1} \right\} \times w(N^{-1}(c+dM),k+1/2,s).\]

This is 0, and hence the Fourier expansion of (29) does not have the constant term containing \(y^{-k-1/2-2s}\).

The nonzero constant term of the Fourier expansion at each cusp , of the Eisenstein series (25) is obtained similarly as in the preceding section. We state this as a lemma.

**Lemma 3.2.** Let \(\rho, \overline{\rho}, N\) be as in (28), and let \(\rho = \rho_2 \rho_c \rho_r = \rho_2 e \rho_r\) be as in (27). Let \(k \in \mathbb{Z}, \geq 0\) be so that \(k\) and \(\rho\) have the same parity. Put

\[U_{k+1/2,\rho}(s) := e(-2^{-2}(k+1/2))2^{-k-1/2-2s}(\int_{\rho_c} \int_{\rho_r})^{-1}\pi \times \frac{\Gamma(k-1/2+2s)}{\Gamma(s)\Gamma(k+1/2+s)} \frac{L(2k-1+4s,\overline{\rho}^2 1_2)}{L(2k+4s,\overline{\rho}^2 1_2)},\]

and put for \(R|4\int_{\rho_r}\)

\[U_{k+1/2,\rho,R}(s) := R^{-k+1/2-2s} \prod_{p|R} \left\{ (p-1)(1-\overline{\rho}_e(p))^2 p^{-2k+1/2-2s} \right\}^{-1}. \]

(i) Suppose that \(\rho_2\) is real. Then \(N = 4\int_{\rho_c} \int_{\rho_r}\) and \(E_{k+1/2,\rho}(z,s) \in M_{k+1/2+s,s}(N, \rho)\) has 0 as the constant terms of the Fourier expansion with respect to \(x\) at cusps in \(C_0(N)\) except \(1/N,1/P,1/(4P)\) with \(P|\rho_r\). The constant term of \(E_{k+1/2,\rho}(z,s)\) at a cusp \(1/P\) is \(U_{k+1/2,\rho}(s)U_{k+1/2,\rho,P}(s) y^{-k+1/2-2s}\), and the constant term at a cusp \(1/(4P)\) is \(1+\rho_2(-1)\chi_{-4}(P)^{-1}U_{k+1/2,\rho}(s)U_{k+1/2,\rho,A4P}(s) y^{-k+1/2-2s}\) where there is the additional term 1 if \(\int_{\rho_c} = 1\) and \(P = \int_{\rho_r}\). If \(\int_{\rho_c} > 1\), then the constant term at a cusp \(1/N\) is 1.

(ii) Suppose that \(\rho_2\) is complex. Then \(N = \int_{\rho_2e} \int_{\rho_r}\) and \(E_{k+1/2,\rho}(z,s)\) has 0 as the constant terms at cusps in \(C_0(N)\) except \(1/N,1/P\) with \(P|\rho_r\). The constant term at a cusp \(1/N\) of \(E_{k+1/2,\rho}(z,s)\) is 1, and the constant term at a cusp \(1/P\) is \(2^{2} \int_{\rho_2} U_{k+1/2,\rho}(s)U_{k+1/2,\rho,P}(s) y^{-k+1/2-2s}\).

\(\square\) Springer
For $\rho, N$ as in \((28)\), the Eisenstein series $E_{k+1/2}^{\rho}(z, s) = \sum_{c>0, (c, N)=1} \rho(c) 
abla_{c}^{-1}(c z + d)^{-k-1/2}|c z + d|^{-2s}$ satisfies equalities $E_{k+1/2, \rho}(z, s)|_{c} = \begin{pmatrix} 0 & -1/N \\ 1 & 0 \end{pmatrix}$.

Then the constant term of $E_{k+1/2}^{\rho}(z, s)$, and $E_{k+1/2, \rho}(z, s)|_{c}$ is in the form $\sqrt{-T} E_{k+1/2, \rho}(z, s)$.

**Lemma 3.3** Let $\rho, N, k$ be as in Lemma 3.2.

(i) Let $P \parallel \rho_{e}$ and let $\varepsilon \in \{\pm 1\}$ be so that $\rho = \rho_{2} \chi_{P} \chi_{(e A_{\parallel P})}$, namely $\varepsilon = \chi_{-4}(f_{\rho_{e}}/P)$. Then, the constant term of $E_{k+1/2, \chi}^{\rho}(z, s) \in \mathbf{M}_{k+1/2+s, s}(N, \rho)$ at a cusp $1/ P$ is the form \((23)\) with $c_{0} = (-1)^{k-1} \sqrt{-T} \chi_{-4}(P)\rho_{e}$, and the constant term at other cusps in $C_{0}(N)$ are in the form \((23)\) with $c_{0} = 0$. The constant terms \((23)\) with $\xi_{i}(s) \neq 0$ possibly appear only at cusps $i/M \in C_{0}(N)$ $(i, M = 1)$ where $f_{\rho_{e}}|M$ if $\rho_{2}$ is real, and $f_{\rho_{e}}|M$ if $\rho_{2}$ is complex.

(ii) Suppose that $\rho_{2} = 1$ or $\chi_{-4}$. Let $\rho = \rho_{2} \chi_{A_{p}} \chi_{(e A_{\parallel P})}$, namely $\varepsilon = \chi_{-4}(P)$. Then the constant term of $E_{k+1/2, \rho_{2} \chi_{A_{p}}}^{\rho}(z, s) \in \mathbf{M}_{k+1/2+s, s}(N, \rho)$ at a cusp $1/ (4P)$ is in the form \((23)\) with $c_{0} = (-1)^{k} \rho_{2}(-1) \chi_{-4}(P)$, and the constant terms at other cusps in $C_{0}(N)$ are in the form \((23)\) with $c_{0} = 0$. The constant terms \((23)\) with $\xi_{0}(s) \neq 0$ possibly appear only at cusps $i/M \in C_{0}(N)$ $(i, M = 1)$ with $f_{\rho_{e}}|M$.

**Proof** (i) Let $b_{0}, d_{0}$ be so that $A_{1/P} = \begin{pmatrix} 1 & b_{0} \\ P & d_{0} \end{pmatrix} \in \text{SL}_{2}(\mathbb{Z})$. Put $Q := f_{\rho_{e}}/P$. Then

\[
E_{k+1/2, \rho_{2} \chi_{A_{p}}}^{\rho_{e}}(z, s)|_{A_{1/P}} = (P z + d_{0})^{-k-1/2} |P z + d_{0}|^{-2s} E_{k+1/2, \rho_{2} \chi_{A_{p}}}^{\rho_{e}} \left(\frac{z + b_{0}}{P z + d_{0}}, s\right)
\]

\[
= (-1)^{k} \chi_{-4}(P) \sum_{c \in \mathbb{Z}, (c, P/N) = 1} \chi_{P}^{-1}(d)(\rho_{2} \chi_{A_{p}})(c/P) \left(\frac{d}{|c|}\right) c^{-1} \times ((c + d P)z + cb_{0} + dd_{0})^{-k-1/2} (c + d P)z + cb_{0} + dd_{0}|^{-2s},
\]

which implies that $c_{0} = (-1)^{k} \chi_{-4}(P)\rho_{e}$ in \((23)\) at the cusp $1/ P$. In the series $E_{k+1/2, \rho_{2} \chi_{A_{p}}}^{\rho_{e}}(z, s)|_{A_{i/M}}$ corresponding to \((30)\) at all other cusps $i/M \in C_{0}(N)$, the coefficients $c$ of $z$ in $(cz + d)^{-k-1/2}$ are always nonzero, and hence $c_{0} = 0$.

Arguing as in the proof of Lemma 3.1 (iii) shows that $\xi_{0}(s)$ in the constant term \((23)\) of the Fourier expansion at $i/M = 0$ unless $M$ does not satisfy the condition of the lemma.

The assertion (ii) is proved similarly.
4 Theta series as Eisenstein series

In this section, we mainly consider Eisenstein series of weight less than 3/2 with character $\rho$. Their relations between theta series are given when $N = 4, \rho = \chi_{-4}^k$.

**Proposition 4.1** Assume that $k \in \mathbb{Z}, k \geq 1, \rho \in (\mathbb{Z}/N)^*$ have the same parity. If $k \geq 2$, or if $k = 1$ and $\rho$ is a complex character, then $E_{k+1/2,\rho, N}(z, 0)$ is in $M_{k+1/2}(N, \rho)$.

**Proof** If $k \geq 2$, then the series $E_{k+1/2,\rho, N}(z, 0)$ converges absolutely and uniformly on any compact subset of $\mathfrak{H}$, and hence it is holomorphic and it is in $M_{k+1/2}(N, \rho)$.

Let $k = 1$, and $\rho$ be complex. We may assume that $\rho, N$ satisfy (24), since for $\rho' \in (\mathbb{Z}/N')^*$ satisfying (22) with $\rho = \rho'$ and $M = N', E_{k+1/2, \rho', N'}(z, 0)$ is written as $E_{k+1/2, \rho', N}(mz, 0)$ for some $m \in \mathbb{N}$ as stated before Lemma 3.1. Then $c_{1,s,\rho, N}(0) = 0$, and hence the constant term of the Fourier expansion (26) is 1. For $n > 0$, it is checked that the $n$-th Fourier coefficient involves $L(1, \rho \chi)$ with a real character $\chi$ depending on $n$. Since $\rho$ is complex, $L(1, \rho \chi)$ is finite for any real $\chi$. As $s \rightarrow 0, w_n(y, k + 1/2, s)$ tends to 0 for $n \leq 0$, the expansion (26) gives a holomorphic function in $z$ at $s = 0$. $\square$

In the rest of this section, we consider the Eisenstein series with $N$ of (24) exclusively in the case that $\rho$ is real. Then $N = \epsilon_1^\prime$. Though our main objective here is the case $k \leq 1$, we do not restrict $k$ to be $\leq 1$. Let $\rho = \rho_2\rho_1$ as in (27), and put $c^{(2)}_{k, s, \rho, N}(0) := 2^{-1}(1 + \sqrt{-1})\sum_{l = \nu_2(N)} \rho_1(2^l)2^{-(k+1/2+2s)}(\sum_{i: (\mathbb{Z}/2^l)}(\rho_2 \chi_{-2}^l)^{(i)}) - \sqrt{-1}\sum_{i: (\mathbb{Z}/2^l)}(\rho_2 \chi_{-2}^l)^{(i)})$, and $c^{(r)}_{k, s, \rho, N}(0) := \sum_m \rho_{2e}(m)m^{-k-1/2-2s}m^{-1}(\sum_{i: (\mathbb{Z}/m)}(\rho \chi_{-4}^k)^{(i)})$, where $m$ runs the set of all positive multiples of $f_{\rho_1}$ whose radicals are $f_{\rho_1}$. Let us put

$$\delta_{k+1/2, \rho}(z, s) := E_{k+1/2, \rho}(z, s) - c^{(2)}_{k, 0, \rho, N}(0)c^{(r)}_{k, 0, \rho, N}(0)E_{k+1/2}^\rho(z, s).$$

If $k \geq 2$, then $E_{k+1/2, \rho}(z, 0)$ and $E_{k+1/2}^\rho(z, 0)$ are holomorphic in $z$, and so is $\delta_{k+1/2, \rho}(z, 0)$. For $k = 0, 1$, neither $E_{k+1/2, \rho}(z, 0)$ nor $E_{k+1/2}^\rho(z, 0)$ is holomorphic. For $k = 1$, it is checked by computing the Fourier coefficients that $\delta_{1/2, \rho}(z, 0)$ is holomorphic, which is found in Pei [11].

We describe theta series $\theta(z), \theta(z)^3, \theta(z)^5, \theta(z)^7$ as Eisenstein series. Let $k = 0$. Then the direct computation shows that $\delta_{1/2, 1_2}(z, 0)$ is equal to $\theta(z)$. As is shown, $\delta_{k+1/2, \chi_{-4}^k}(z, 0)$ is a holomorphic modular form for $k \geq 1$, and it is in $M_{k+1/2}(4, \chi_{-4}^k)$ with the value 1 at $\sqrt{-1}\infty$ and with the value $2^{-k-1/2}e(\frac{6k-1}{8})$ at 1. Since the subspaces of $M_{k+1/2}(4, \chi_{-4}^k)$ consisting of the such modular forms are one dimensional for $k = 0, 1, 2, 3$ and since $\theta(z)^{2k+1}$'s satisfy the same condition, we have the following proposition.

**Proposition 4.2** (1) Let $N, \rho$ be as in (24). Then $\delta_{k+1/2, \rho, N}(z, 0)$ is a holomorphic modular form in $M_{k+1/2}(N, \rho)$ for $k \geq 1$ with the same parity as $\rho$.

(2) If $k = 0, 1, 2, 3$, then

$$\theta(z)^{2k+1} = \delta_{k+1/2, \chi_{-4}^k}(z, 0).$$

$\square$
Remark 4.3 The equality (31) holds for any $k \geq 0$ up to cusp forms. As for even powers of $\theta(z)$, the following equalities hold up to cusp forms; $\theta(z)^2 = E_{1, \chi-4}(z, 0)$, and $\theta(z)^{2k} = E_{k, \chi-4}(z, 0) + 2^{-k} \sqrt{1} k \frac{E_k^{\chi-4}(z, 0)}{(z, 0)}$ for $k > 1$ odd. For $k \geq 2$, even, $\theta(z)^{2k} = \{(-1)^{k/2} 2^{-k} E_{k, 12}(z, 0) - (-1)^{k/2} k \frac{E_{k, 12, 2}(z, 0)}{(z, 0)} + E_{k, 12, 4}(z, 0)\}$. These equalities are exact if the weight is less than or equal to 4.

Corollary 4.4 All the holomorphic Eisenstein series of weight 1/2 on $\Gamma_1(N)$ are linear combinations of $\sum_{i: i/\omega} \omega(i) E_{1/2, 4}(i z + i/\omega, 0)$ where $\omega$ are primitive real analytic Eisenstein series, have analytic continuation to the whole complex $\tau$-plane with the same parity as $\chi$, and where $t$ are natural numbers with $4(t/\omega)^2 t |N$.

Proof Let $\chi$ be a square of $\omega$. If both of $\omega$ and $\chi$ are primitive, then an equality $\sum_{i: i/\omega} \omega(i) E_{1/2, 4}(i z + i/\omega, 0)$ holds. By Serre and Stark [13] Corollary 1 to Theorem A, the subspace in $M_{1/2}(\Gamma_1(N))$ consisting of Eisenstein series is spanned by \( \theta(z)^2 = E_{1, \chi-4}(z, 0) \), \( \theta(z)^{2k} = E_{k, \chi-4}(z, 0) + 2^{-k} \sqrt{1} k \frac{E_k^{\chi-4}(z, 0)}{(z, 0)} \) for $k > 1$ odd. For $k \geq 2$, even, $\theta(z)^{2k} = \{(-1)^{k/2} 2^{-k} E_{k, 12}(z, 0) - (-1)^{k/2} k \frac{E_{k, 12, 2}(z, 0)}{(z, 0)} + E_{k, 12, 4}(z, 0)\}$. These equalities are exact if the weight is less than or equal to 4.

5 Rankin–Selberg method

We show that the functions of $s$ obtained from the constant terms of some real analytic modular forms in $M_{l', l}(N, \rho)$ with $l, l' \in \frac{1}{2} \mathbb{Z}, \geq 0$ with $\rho \in (\mathbb{Z}/N)^*$ together with a real analytic Eisenstein series, have analytic continuation to the whole complex $s$-plane by using the unfolding trick. From this, the analytic continuation of $L$-functions (1) is proved in the case $l - l' \in \mathbb{Z}$. Further some special value of the $L$-function is written in terms of the scalar product, and the functional equation between the $L$-function (1) and the $L$-function defined similarly is derived.

Let $l \geq l' \geq 0$. Let $N \in \mathbb{N}$ be so that $4|N$ if at least one of $l, l'$ is not integral. Put $k := l - l'$ if $l - l' \in \mathbb{Z}$, and put $k := l - l' - 1/2$ otherwise, and let $\rho$ be a Dirichlet character modulo $N$ with the same parity as $k$. We assume that $\rho, N$ satisfy (22) with $M = N$ if $l - l'$ is not integral. We consider a real analytic modular form $F(z)$ in $M_{l, l'}(\Gamma_1(N))$ which satisfies $F(Az) = \overline{\rho(d)(cz + d)^l(cz + d)^{-k}} F(z)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ if $l - l' \in \mathbb{Z}$, or $F(Az) = \overline{\rho(d)(cz + d)^l(cz + d)^{-k}} (A, z)^{-1} F(z)$ if $l - l' \notin \mathbb{Z}$, $j(A, z)$ being the automorphy factor of $\theta(z)$ defined in the introduction. We assume that $F(z)$ has the Fourier expansion in the form

\[ F|A_r(z) = P_F^{(r)}(y) + \sum_{n=-\infty}^{\infty} u_n^{(r)}(y)e(nx/w^{(r)}) \text{ with } P_F^{(r)}(y) = \sum_j a_j^{(r)} y^{q_j^{(r)}} \] (32)

at each cusp $r \in C_0(N), A_r$ being as in (2), where the summation $P_F^{(r)}(y)$ is a finite sum with $a_j^{(r)}, q_j^{(r)} \in \mathbb{C}$ and all $u_n^{(r)}(y)$ are rapidly decreasing as $y \longrightarrow \infty$. We drop the notation $(r)$ from $P_F^{(r)}, a_j^{(r)}, q_j^{(r)}, u_n^{(r)}$ when $r = 1/N$ or equivalently $r = \sqrt{-1}$. We define the Rankin–Selberg transform

\[ \text{ Springer} \]
\[ R(F, s) := \int_0^\infty y^{l+s-2} \varphi(y)dy, \quad (33) \]

following Zagier [16]. In Theorem 5.2, we show that the integral converges for \( s \) with sufficiently large \( \Re s \).

We denote by \( Q(r) \) \( (r \in \mathbb{Q}) \), the set of rational numbers in \((-1/2, 1/2] \) equivalent to \( r \) under \( \Gamma_0(N) \).

**Lemma 5.1** Let \( q \) be the maximum of \( 0, \Re q_j^{(r)} \) \( (r \in \mathcal{C}_0(N)) \). Then \( |F(z)| = O(y^{-q-(l'+l)/2}) \) as \( y \to +0 \).

**Proof** The function \( \psi(z) := y^{l'+l}/\text{tr}_{\Gamma_0(N)/SL_2(\mathbb{Z})}(|F(z)|) \) is \( SL_2(\mathbb{Z}) \) invariant function. Then \( y^{-q-(l'+l)/2} \psi(z) \) is bounded on the fundamental domain \( \mathcal{F} \) of \( SL_2(\mathbb{Z}) \), and hence

\[ \left( \max_{\mathcal{F}} \Im(Az) \right)^{-q-(l'+l)/2} \psi(z) \]

is bounded on \( \mathcal{F} \). Since \( \max_{A \in SL_2(\mathbb{Z})} \Im(Az) \leq \max\{y, y^{-1}\} \), we have \( |F(z)| = O(y^{-q-(l'+l)/2}) \) as \( y \to +0 \). \( \square \)

We fix the notation for the rest of the section. We denote by \( y^s + \xi_\infty(s) \times y^{-(l'-l)+1-s} \) and \( \xi(r)(s) y^{-(l'-l)+1-s} \), the constant terms of Fourier expansions with respect to \( x \), of \( y^s E_{l-l', \rho, N}(z, s) \) at \( \sqrt{-1}\infty \) and at \( r \in \mathcal{C}_0(N) \), \( \neq 1/N \) respectively. The function \( \xi_\infty(s) \) is also denoted by \( \xi_0(s) \). For \( T > 0 \),

\[ h_+(T, s) = h_{+}^{(\sqrt{-1}\infty)}(T, s) \]
\[ = \sum_j \frac{a_j T q_j + l - 1 + s}{q_j + l - 1 + s} \left( = \int_0^T y^{l+s} P_F^{(\sqrt{-1}\infty)}(y)dy (\Re s \gg 0) \right), \]

\[ h_-(T, s) = h_{-}^{(\sqrt{-1}\infty)}(T, s) \]
\[ = \sum_j \frac{a_j T q_j + l' - s}{q_j + l' - s} \left( = - \int_T^\infty y^{l'+1-s} P_F^{(\sqrt{-1}\infty)}(y)dy (\Re s \gg 0) \right), \]

\[ h^{(r)}(T, s) \]
\[ = w^{(r)} \sum_j \frac{a_j^{(r)} T q_j^{(r)} + l' - s}{q_j^{(r)} + l' - s} \left( = -w^{(r)} \int_T^\infty y^{l'+1-s} P_F^{(r)}(y)dy (\Re s \gg 0) \right) (r \neq 1/N), \]

\[ h(T, s) = h_+(T, s) + \xi_0^{(1/N)}(s) h_-(T, s) + \sum_{r \in \mathcal{C}_0(N) - \{1/N\}} \xi^{(r)}(s) h^{(r)}(T, s) \quad (34) \]

where \( \Re s \gg 0 \) implies that \( \Re s \) is sufficiently large. If \( \Re s \ll 0 \) (sufficiently small), then \( h_+(T, s) = -\int_T^\infty y^{l+s} P_F^{(1/N)}(y)dy, h_-(T, s) = \int_T^\infty y^{l'+1-s} P_F^{(1/N)}(y)dy, h^{(r)}(T, s) = w^{(r)} \int_T^\infty y^{l'+1-s} P_F^{(r)}(y)dy \).
Theorem 5.2 Let \( l, l', k, \xi^{(r)}, h_+, h_-, h^{(r)}, F, R(F, s) \) be as above. Then, for \( s \) with \( \Re s \) sufficiently large, the integral (33) converges and \( R(F, s) \) is defined. Further \( R(F, s) \) extends meromorphically to the whole \( s \) plane, and its possible poles are poles of the Eisenstein series \( E_{l-l', \rho, N}(z, s) \) and \( s = q^r_j + l' \) (\( r \in \mathcal{C}_0(N) \)), \( s = -q^r_j - l + 1 \). We have

\[
R(F, s) = \lim_{T \to \infty} \left( \int_{\mathfrak{H}_T(N)} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{s}) \frac{dx \, dy}{y^2} - h_+(T, s) \right) \tag{35}
\]

for \( s \) with sufficiently large \( \Re s \). For \( s \) with sufficiently small \( \Re s \) at which \( E_{l-l', \rho, N}(z, s) \) is holomorphic, we have

\[
R(F, s) = \lim_{T \to \infty} \left( \int_{\mathfrak{H}_T(N)} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{s}) \frac{dx \, dy}{y^2} - \bar{\xi}^{(1/N)}(\overline{s}) h_-(T, s) - \sum_{r \in \mathcal{C}_0(N)-\{1/N\}} \bar{\xi}^{(r)}(\overline{s}) h^{(r)}(T, s) \right) \tag{36}
\]

**Proof** Let

\[
\mathcal{D} := \{ z \in \mathfrak{H} \mid |x| \leq 1/2 \}.
\]

We denote by \( \mathfrak{H}_T \) \((T > 1)\), the truncated fundamental domain \( \{ z \in \mathfrak{H} \mid |x| \geq 1, y \leq T \} \), and by \( \mathfrak{H}_T(N) \), the union \( \bigcup_A A\mathfrak{H}_T \subset \mathcal{D} \) where \( A \) runs over the representatives of \( \text{SL}_2(\mathbb{Z}) \) modulo \( \Gamma_0(N) \).

Let \( a/c \) be a rational number with \(-1/2 < a/c \leq 1/2, c > 0, (a, c) = 1 \). For \( T > 1 \), let \( S_{a/c, T} \) be the open disk of radius \((2e^2T)^{-1}\) tangent to the real axis at \( a/c \) (Fig. 2) where \( S_{1/2, T} \) is exceptionally the union of the left half of the disk tangent to the real axis at \( 1/2 \) and the right half of the disk tangent to the real axis at \(-1/2 \). The domain \( \{ z \in \mathfrak{H} \mid y > T \} \) and all \( S_{a/c, T} \) but \( S_{1/2, T} \) are mapped onto each other by matrices of \( \text{SL}_2(\mathbb{Z}) \).

Put \( \mathfrak{B}_T(n) := \{ z \in \mathfrak{H} \mid -1/2 \leq x \leq n - 1/2, y > T \} \) for \( n \in \mathbb{N} \). For \( r \in \mathcal{C}_0(N) \), let \( A_r \) be as in (2) so that it sends \( \mathfrak{B}_T(w^{(r)}) \) onto \( S_{r, T} \cap \mathfrak{H}_T(N) \). By Lemma 5.1, the integral \( \int_0^T \int_{-1/2}^{1/2} y^{l+s} F(z) \frac{dx \, dy}{y^2} \) converges absolutely and uniformly for \( s \) with \( \Re s \gg 0 \). Let \( \mathcal{D}_T = \{ z \in \mathfrak{D} \mid \Re z \leq T \} - \bigcup_{r \in \mathcal{C}_0(N)} \bigcup_{a/c \in \mathcal{Q}(r)} S_{a/b, T} \). Then, by taking suitable representatives of \( \Gamma_0(N) \) modulo \( \Gamma_\infty = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \} \), we have \( \mathcal{D} = \bigcup_{A: \Gamma_\infty \setminus \Gamma_0(N)} A\mathfrak{H}(N) \) and \( \mathcal{D}_T = \bigcup_{A: \Gamma_\infty \setminus \Gamma_0(N)} A\mathfrak{H}_T(N) \). Then,

\[
\int_{\mathfrak{H}_T(N)} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{s}) \frac{dx \, dy}{y^2} = \int_{\mathcal{D}_T} y^{l+s} F(z) \frac{dx \, dy}{y^2}. \tag{37}
\]

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Indeed if $l - l'$ is integral, then the left hand side of (37) is equal to

\[
\int_{\mathfrak{D}_T(N)} y^{l+s} F(z) \left\{ 1 + \sum_{(c,d) = 1, c > 0 \atop c \equiv 0 \bmod N} \rho(d)(cz+d)^{-k} |cz+d|^{-2s} \right\} \frac{dx \, dy}{y^2}
\]

\[
= \int_{\mathfrak{D}_T(N)} y^{l+s} F(z) \sum_{A \in \Gamma_\infty \backslash \Gamma_0(N)} \rho(d_A) 1_{A}(z, s) \frac{dx \, dy}{y^2}
\]

\[
= \int_{\bigcup_{A \in \Gamma_\infty \backslash \Gamma_0(N)} A \mathfrak{D}_T(N)} y^{l+s} F(z) \frac{dx \, dy}{y^2} = \int_{\mathfrak{D}_T} y^{l+s} F(z) \frac{dx \, dy}{y^2}.
\]

The similar argument holds also in the case that $l - l'$ is a half integer. Since $h_+(T, s) = \int_{-1/2}^{1/2} y^{l+s} P_F(y) \frac{dx \, dy}{y^2}$, from (37) we obtain

\[
\int_{\mathfrak{D}_T(N)} y^{l+s} F(z) E_{l-l', \rho, N}(z, s) \frac{dx \, dy}{y^2}
\]

\[
= \int_{\mathfrak{D} - \mathfrak{D}_T(1)} y^{l+s} F(z) \frac{dx \, dy}{y^2} - \int_{\bigcup_{r \in \mathcal{C}_0(N)} \bigcup_{a/c \in Q(r)}} \int_{S_{a/b, T}} y^{l+s} F(z) \frac{dx \, dy}{y^2}
\]

\[
= \int_{0}^{T} y^{l+s-2} u_0(y) dy + h_+(T, s) - \sum_{r \in \mathcal{C}_0(N)} \sum_{a/c \in Q(r)} \int_{S_{a/b, T}} y^{l+s} F(z) \frac{dx \, dy}{y^2}.
\]

(38)
and for \( r \in C_0(N), \neq 1/N, \)

\[
\int_{\mathfrak{S}(N) \cap S_{r,T}} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{\gamma}) \frac{dx \, dy}{y^2} = \int_{\bigcup_{a/c \in Q(r)} S_{a/h,T}} y^{l+s} F(z) \, \frac{dx \, dy}{y^2}.
\]

(39)

We have \( E_{l-l', \rho, N}(z, \overline{\gamma})|_{A_r} = O(y^{1-2s}) \) as \( y \to \infty \) for \( r \in C_0(N), \neq 1/N, \) and \( E_{l-l', \rho, N}(z, s) - 1 = O(y^{1-2s}). \) We take \( s \) so that \( \Re s > \max(q + l', q + 1). \) Then \( y^{l+s} F|_{A_r}(z) E_{l-l', \rho, N}(z, \overline{\gamma})|_{A_r} \) is integrable on \( \mathfrak{B}_T(w^{(r)}), \) and \( y^{l+s} F(z) \{ E_{l-l', \rho, N}(z, \overline{\gamma}) - 1 \} \) is integrable on \( \mathfrak{B}_T(1). \) Then the integral (39) is equal to \( \int_{\mathfrak{B}_T(w^{(r)})} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{\gamma})|_{A_r} \frac{dx \, dy}{y^2}. \) Since there holds an equality \( \bigcup_{A : \Gamma_{\infty} \setminus (\Gamma_0(N) \setminus \Gamma_{\infty})} A \mathfrak{B}_T(1) = \bigcup_{a/c \in Q(1/N)} S_{a/c,T} \) for a suitable choice of representatives, we have

\[
\int_{\mathfrak{B}_T(1)} y^{l+s} F(z) \left\{ E_{l-l', \rho, N}(z, \overline{\gamma}) - 1 \right\} \frac{dx \, dy}{y^2}
= \int_{\mathfrak{B}_T(1)} y^{l+s} F(z) \sum_{(c,d) = 1, c > 0} \sum_{c \equiv 0 (\text{mod } N)} \rho(d)(cz + d)^{-k}|cz + d|^{-2s} \frac{dx \, dy}{y^2}
= \sum_{a/c \in Q(1/N)} \int_{S_{a/c,T}} y^{l+s} F(z) \frac{dx \, dy}{y^2}.
\]

If \( \Re s \gg 0, \) then we obtain from (38),

\[
R(F, s) - \int_T \infty y^{l+s-2} u_0(y)dy = \int_0^T y^{l+s-2} u_0(y)dy
= \int_{\mathfrak{S}_T(N)} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{\gamma}) \frac{dx \, dy}{y^2}
+ \int_{\mathfrak{B}_T(1)} y^{l+s} F(z) \left\{ E_{l-l', \rho, N}(z, \overline{\gamma}) - 1 \right\} \frac{dx \, dy}{y^2}
+ \sum_{r \in C_0(N) \setminus \{1/N\}} \int_{\mathfrak{B}_T(w^{(r)})} y^{l+s} F|_{A_r}(z) E_{l-l', \rho, N}(z, \overline{\gamma})|_{A_r} \frac{dx \, dy}{y^2} - h_+(T, s).
\]

(40)

Thus integral \( \int_0^T y^{l+s-2} u_0(y)dy \) converges, and since \( y^{l+s-2} u_0(y) \) is rapidly decreasing as \( y \to \infty \) by our assumption, the integral (33) converges for \( \Re s \gg 0. \)

Noting that \( \int_T \infty y^{l+s-2} u_0(y)dy + \int_{\mathfrak{B}_T(1)} y^{l+s} F(z) \left\{ E_{l-l', \rho, N}(z, \overline{\gamma}) - 1 \right\} \frac{dx \, dy}{y^2} = \int_{\mathfrak{B}_T(1)} y^{l+s} F(z) E_{l-l', \rho, N}(z, \overline{\gamma}) - (y^{l+s} + \xi(1/N)(\overline{\gamma}) y^{l'-1-s}) P_F(y) \frac{dx \, dy}{y^2}, \) we obtain

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from (40),

\[
R(F, s) = \int_{S^1} y^{l+s} F(z) E_{l-1', \rho, N}(z, \bar{\mathfrak{s}}) \frac{dx\,dy}{y^2}
\]

\[
+ \int_T \int_{-1/2}^{1/2} \left\{ y^{l+s} F(z) E_{l-1', \rho, N}(z, \bar{\mathfrak{s}})
- (y^{l+s} + \xi((1/N)(\bar{\mathfrak{s}})) y^{l'+1-s}) P_F(y) \right\} \frac{dx\,dy}{y^2}
\]

\[
= \sum_{r \in \mathbb{C}_0(N) - \{1/N\}} \int_T \int_{-1/2}^{1/2} \left\{ y^{l+s} F(z) E_{l-1', \rho, N}(z, \bar{\mathfrak{s}})|_{A_r}
- \frac{\xi((r)(\bar{\mathfrak{s}}))}{y^{l'+1-s}} P_F^{(r)}(y) \right\} \frac{dx\,dy}{y^2}
- h(T, s)
\]

with \( h(T, s) \) of (34) since \( h(T, s) = h(\lambda(T, s)) = \int_T \int_{-1/2}^{1/2} \xi((1/N)(\bar{\mathfrak{s}})) y^{l'+1-s} P_F(y) \frac{dx\,dy}{y^2}
\]

By our assumption that the second term of (32) is rapidly decreasing as \( y \longrightarrow \infty \), and the integrands of the second and third terms of (41) are also rapidly decreasing. The equality (41) is proved for \( s \) with \( \Re s \gg 0 \). However, the right hand side is a meromorphic function on the whole \( s \) plane since the first integral is over a compact set, and the integrands of other integrals are rapidly decreasing, and \( h(T, s) \) is meromorphic on the \( s \) plane. Thus \( R(F, s) \) is a meromorphic on the \( s \) plane, and \( R(F, s) \) is holomorphic at \( s \) if both of \( E_{l-1', \rho, N}(z, s) \) and \( h(T, s) \) are holomorphic at \( s \). This shows the second statement of the theorem.

In the right hand side of (41), the second and the third terms tend to 0 as \( T \longrightarrow \infty \), and \( h(T, s) - h_{\lambda}(T, s) \) also tend to 0 as \( T \longrightarrow \infty \) for \( s \) with \( \Re s \gg 0 \). This shows (35). The equality (36) is proved similarly. \( \square \)

**Corollary 5.3**

(i) For \( s \) with sufficient large \( \Re s \), there holds equalities

\[
R(F, s) = \int_{S^1(\mathbb{N})} y^{l+s} F(z) E_{l-1', \rho, N}(z, \bar{\mathfrak{s}}) - E_0(z, s) \frac{dx\,dy}{y^2}
\]

for \( E_0(z, s) := \sum_j a_j E_{\Gamma_0(N)}(z, l+q_j+s) \in \mathcal{M}_{0,0}(N) \).

(ii) Let the notations be the same as in the theorem. For \( s \) with \( \Re s \ll 0 \) at which \( E_{l-1', \rho, N}(z, s) \) is holomorphic, there holds

\[
R(F, s) = \int_{S^1(\mathbb{N})} y^{l+s} F(z) E_{l-1', \rho, N}(z, \bar{\mathfrak{s}}) \frac{dy}{y^2} - E_0(z, s) \frac{dx\,dy}{y^2}
\]

for \( E_0(z, s) := \xi((1/N)(\bar{\mathfrak{s}})) \sum_j a_j E_{\Gamma_0(N)}(z, q_j+l'+3-s) + \sum_{r \in \mathbb{C}_0(N) - \{1/N\}} \sum_j \xi((r)(\bar{\mathfrak{s}})) \times E_{\Gamma_0(N), r}(z, q_j^{(r)}+l'+3-s) \in \mathcal{M}_{0,0}(N) \).

(iii) Assume that \( E_{l-1', \rho, N}(z, s) \) is holomorphic in \( s \) at \( s = s_0 \). Let \( \tilde{h}(T, s) \) denote the sum of terms of (34) so that for \( s = s_0 \) the powers of \( T \) are nonzero and their real
parts are nonnegative. Let \( \frac{C_0}{s-s_0} T^{s-s_0} - \frac{\alpha(s)}{s-s_0} T^{s+s_0} \) be the sum of terms of \((34)\) so that the powers of \( T \) for \( s = s_0 \) are 0, where \( C_0 \) is a constant. Then

\[
\left( R(F, s) + \frac{C_0 - \alpha(s)}{s-s_0} - \frac{d}{ds} \alpha(s_0) \right) \bigg|_{s=s_0} = \lim_{T \to \infty} \left( \int_{\mathcal{F}_T(N)} y^{l+s} F(z) \overline{E}_{l-l', \rho, N(z, \bar{z})} \frac{dx dy}{y^2} - (C_0 + \alpha(s_0)) \log T - \tilde{h}(T, s_0) \right).
\]

(43)

If \( C_0 = \alpha(s) = 0 \), then \( R(F, s) \) is holomorphic at \( s = s_0 \).

**Proof**  
(i) We put \( H(z, s) := y^{l+s} F(z) \overline{E}_{l-l', \rho, N(z, \bar{z})} - \sum_j a_j E_{\Gamma_0(N)}(z, l + q_j + s) \). Then \( H(z, s) = O(y^{l'+1+q-s}) \), \( H(z, s)|_{A_y} = O(y^{l'+1+q-s}) \) as \( y \to \infty \) where \( q \) is as in Lemma 5.1. Hence the integral of the right hand side of \((42)\) converges for \( \Re s \gg 0 \). We apply the argument in the theorem to \( E_{\Gamma_0(N)}(z, l + q_j + s) \). Since the term \( u_0(y) \) of \( E_{\Gamma_0(N)}(z, l + q_j + s) \) is 0, we obtain from \((40)\),

\[
0 = \int_{\mathcal{F}_T(N)} E_{\Gamma_0(N)}(z, l+q_j+s) \frac{dx dy}{y^2} + \int_{\mathcal{B}_T(1)} \left\{ E_{\Gamma_0(N)}(z, l+q_j+s) - y^{l+q_j+s} \right\} \frac{dx dy}{y^2}
+
\sum_{r \in \mathcal{C}_0(N) - \{1/N\}} \int_{\mathcal{B}_T(w^{(r)})} E_{\Gamma_0(N)}(z, l+q_j+s)|_{A_r} \frac{dx dy}{y^2} - \frac{y^{l+1+s}}{q_j+l+1+s}.
\]

Then by this and by the equality \((40)\), \( R(F, s) \) is equal to

\[
\int_{\mathcal{F}_T(N)} H(z, s) \frac{dx dy}{y^2} + \int_{T}^{\infty} \int_{-1/2}^{1/2} \{ H(z, s) + Q(y, s) \} \frac{dx dy}{y^2}
+
\sum_{r \in \mathcal{C}_0(N) - \{1/N\}} \int_{T}^{\infty} \int_{-1/2}^{w^{(r)}-1/2} H(z, s)|_{A_r} \frac{dx dy}{y^2} + Q'(T, s)
\]

(44)

where \( Q(y, s) \) is a linear combination of powers of \( y \), and \( Q'(T, s) \) is a linear combination of powers of \( T \). It is easily checked that the real parts of powers of terms in \( Q(y, s) \) or in \( Q'(T, s) \) are all negative if \( \Re s \gg 0 \). The first term of \((44)\) tends to the right hand side of \((42)\) as \( T \to \infty \) and the other terms tend to 0. This shows the equality \((42)\).

The assertion (ii) is proved similarly.

(iii) In \((41)\), the first integral is holomorphic in \( s \) at \( s = s_0 \) since it is over the compact set, and the other integrals are also since integrands are rapidly decreasing. The coefficients of \( h(T, s) \) of \((41)\) in \( T \) as well as \( \alpha(s) \) are holomorphic at \( s = s_0 \) from our assumption that \( E_{l-l', \rho, N}(z, \bar{z}) \) is holomorphic at \( s = s_0 \). If \( C_0 = \alpha(s) = 0 \), then \( R(F, s) \) is obviously holomorphic at \( s = s_0 \). By \((41)\), we can write \( R(F, s) \) as \( R(F, s) = \int_{\mathcal{F}_T(N)} y^{l+s} F(z) \overline{E}_{l-l', \rho, N(z, \bar{z})} \frac{dx dy}{y^2} + g_T(z, s) - h(T, s) \) for a holomorphic function \( g_T(z, s) \) of \( s \) rapidly decreasing as \( T \to \infty \) which is a sum of integrals in \((41)\) other than the first one. If we put \( n(T, s) := h(T, s) - \frac{C_0 T^{s-s_0}}{s-s_0} + \frac{\alpha(s)}{s-s_0} \) then
By (19) and by Corollary 5.3 (ii), we have
\[
\frac{\alpha(s)T^{-s+s_0}}{s-s_0} - \widetilde{h}(T, s),
\]
then \(\lim_{T \to \infty} n(T, s_0) = 0\) since the powers of \(T\) of terms in \(n(T, s_0)\) have negative real parts. By the Taylor expansions \(T^{s-s_0} = 1 + (s - s_0) \log T + O((s - s_0)^2)\), \(T^{-s+s_0} = 1 - (s - s_0) \log T + O((s - s_0)^2)\) at \(s = s_0\), we have \(\{R(F, s) + C_0 - \alpha(s_0) - \frac{d}{ds} \alpha(s_0)\}|_{s=s_0} = \int_{\mathfrak{B}(N)} y^l z^{-k} F(z) \frac{dz}{y^2} + g_T(z, s_0) - (C_0 + \alpha(s_0)) \log T - \widetilde{h}(T, s_0) - n(T, s_0)\). Taking the limit as \(T \to \infty\) of the right hand side, the equality (43) follows.

**Theorem 5.4** Assume that \(F(z)\) of Theorem 5.2 is in \(\mathcal{M}_{l,l-k}(N, \rho)\) with \(k \in \mathbb{Z}, \geq 0\) and with \(\rho \in (\mathbb{Z}/N)^*\) where \(\rho\) and \(k\) have the same parity. Put \(\widetilde{F}(z) = F|_{S_N}(z)\) in \(\mathcal{M}_{l,l-k}(N, \rho)\). Then \(R(F, s)\) defined in (33) has the meromorphic continuation to the whole complex plane, and satisfies the functional equation

\[
R(F, -k + 1 - s) = \pi^{-1/2} U_{k, \rho}(s) \sum_{0 < P | \epsilon \rho, \rho^{-1}} U_{k, \rho, p}(s) R(\text{tr} \Gamma_0(N)/\Gamma_0(\epsilon \rho, \rho), \rho)(\widetilde{F}, s),
\]

where \(U_{k, \rho}(s)\) and \(U_{k, \rho, p}(s)\) are as in (20).

**Proof** By (19) and by Corollary 5.3 (ii), we have

\[
(-1)^k \pi^{1/2} U_{k, \rho}(s)^{-1} R(F, -k + 1 - s)
\]

\[
= \int_{\mathfrak{B}(N)} y^l z^{-k} \sum_{0 < P | \epsilon \rho, \rho^{-1}} \prod_{p | P} (1 - \widehat{\rho}(p) p^{k+2s}) \varphi(\epsilon \rho, \rho^{-1} P^{-1})
\]

\[
\times \Im \left( \frac{-1}{Nz} \right) E_{k, \rho, 1, \rho, P, \rho} \left( \frac{-1}{Nz}, \frac{1}{z} \right) - \mathcal{E}_0(z, s) \left( \frac{dy}{y^2} \right)
\]

for a linear combination \(\mathcal{E}_0(z, s)\) of Eisenstein series of weight \((0, 0)\) if \(\Re z \gg 0\) and \(\mathcal{E}_{k, \rho, N}(z, s)\) is holomorphic at \(s\). The integrand of (45) takes the value 0 at each cusp, and hence its transformation by the matrix \(S_N\) also takes the value 0 at each cusp. Then (45) is equal to

\[
\int_{\mathfrak{B}(N)} \{(-1)^k y^l z^{-k} \widetilde{F}(z) \sum_{0 < P | \epsilon \rho, \rho^{-1}} \prod_{p | P} (1 - \widehat{\rho}(p) p^{k+2s})
\]

\[
\times \varphi(\epsilon \rho, \rho^{-1} P^{-1}) \frac{E_{k, \rho, 1, \rho, P, \rho}(z, \frac{1}{z}) - \mathcal{E}_0(z, s)(z, s)}{y^2} \} dx dy
\]

where \(\mathcal{E}_0(z, s) = \mathcal{E}_0(-\frac{1}{Nz}, s)\). Hence

\[
(-1)^k \pi^{1/2} U_{k, \rho}(s)^{-1} R(F, -k + 1 - s)
\]

\[
= (-1)^k \sum_{0 < P | \epsilon \rho, \rho^{-1}} \prod_{p | P} (1 - \widehat{\rho}(p) p^{k+2s}) \varphi(\epsilon \rho, \rho^{-1} P^{-1}) R(\text{tr} \Gamma_0(N)/\Gamma_0(\epsilon \rho, \rho))(\widetilde{F}, s),
\]
which shows our assertion.

\[ \Box \]

**Corollary 5.5** Let \( k \in \mathbb{Z}, k \geq 0 \) and let \( l \in \frac{1}{2}\mathbb{Z}, l > 0, l \geq k \). Let \( f, g \) be holomorphic modulars forms for \( \Gamma_0(N) \) of weight \( l \) and of weight \( l - k \) with characters, respectively. We assume that \( f \notin \mathcal{M}_{l, l-k}(N, \rho) \) for \( \rho \in (\mathbb{Z}/N)^* \) with the same parity as \( k \).

(i) Then \( L(s; f, g) \) defined in (1) converges at least if \( \Re s > \max\{2l - k - 1, 1/2\} \), and extends meromorphically to the whole \( s \)-plane. Further it has a functional equation under \( l-k-s \mapsto l-1+s \) which is similar to the functional equation in Theorem 2.1.

(ii) Assume that \( \rho \) is primitive with \( N = f, k \geq 1 \). Let \( P_{\gamma}^{(r)}(y) \) be as in (3). If \( P_{\gamma}^{(\sqrt{-1} \infty)}(y) \) does not have a term containing \( y \) to the power of \( 1 \), and if \( P_{\gamma}^{(r)}(y) \) does not have a term containing \( y^k \) for any \( r \in \mathcal{C}_0(\mathcal{N}) \), then there holds

\[ (f(z), g(z)E_{k, \rho, N}(z, 0))_{\Gamma_0(N)} = (4\pi)^{-l+1} \Gamma(l-1)\Gamma(l-1; f, g) \quad (46) \]

with the scalar product defined in (5).

If \( C_0 \) is a coefficient of \( y \) in \( P_{\gamma}^{(\sqrt{-1} \infty)}(y) \), and if \( \alpha(s) \) is the coefficient of \( y^k \) in

\[ \sum_{r \in \mathcal{C}_0(\mathcal{N})} \xi^{(r)}(\gamma)w^{(r)}P_{\gamma}^{(r)}(y) \]

then Eq. (46) holds replacing the right hand side by

\[ ([4\pi]^{l-1+1-s}\Gamma(l-1+s)L(l-1+s; f, g) + s^{-1}(C_0 - \alpha(0)) - \frac{d}{ds}\alpha(0))|_{s=0}. \]

(iii) Let \( k = 0, \rho = 1_N \). If \( P_{\gamma}^{(\sqrt{-1} \infty)}(y) \) does not have a nonzero constant term, and if \( P_{\gamma}^{(r)}(y) \) does not have a term containing \( y \) to the power of \( 1 \) for any \( r \in \mathcal{C}_0(\mathcal{N}) \), then the equality (21) holds.

If \( C_0 \) is a constant term of \( P_{\gamma}^{(\sqrt{-1} \infty)}(y) \), and if \( \alpha(s) \) is the coefficient of \( y \) in

\[ \sum_{r \in \mathcal{C}_0(\mathcal{N})} \xi^{(r)}(\gamma)w^{(r)}P_{\gamma}^{(r)}(y) \]

then an equality

\[ \langle f, g \rangle_{\Gamma_0(N)} - C_0 + \frac{1}{2}s^2 = \text{Res}_{s=1} \frac{\Gamma(s)N\prod_{p|N}(1 + \frac{1}{p})}{3 \cdot 4s \pi s^{-1}}L(s; f, g) \]

holds.

**Proof** (i) We take \( f(z)\gamma(z) \) as \( F(z) \) in the theorem and then \( R(F, s) = (4\pi)^{-l+1-s} \Gamma(l-1+s)L(l-1+s; f, g) \). The other assertion follows from the theorem.

(ii) Our assumption implies that the Eisenstein series \( E_{k, \rho}(z, s) \) is holomorphic in \( s \) at \( s = 0 \). Then the assertion follows from the identity (43) for \( s_0 = 0 \). Indeed \( (C_0 + \alpha(0))\log T + \tilde{h}(T, 0) \) of the right hand side of (43) is equal to

\[ \sum_{r \in \mathcal{C}_0(\mathcal{N})} Q^{(r)}_{\gamma}E_{k, \rho}(z, 0)(T), \]

by (5), the right hand side of (43) equals

\[ \langle f(z), g(z)E_{k, \rho, N}(z, 0) \rangle_{\Gamma_0(N)}. \]

(iii) The Eisenstein series \( y^4E_{0, 1_N, N}(z, s) \) has a simple pole at \( s = 1 \) with residue \( C = 3(\pi N\prod_{p|N}(1 + p^{-1}))^{-1} \), and \( C = \text{Res}_{s=1}\xi^{(r)}(s) \) for all cusp \( r \in \mathcal{C}_0(\mathcal{N}) \). So

\[ (s-1)\xi^{(r)}(s) \]

tends to \( C \) as \( s \to 1 \). By (41),

\[ (s-1)R(f, s) \]
\[ = \int_{f \in \mathcal{F}(N)} y^{l+s} f(z)g(z)(s-1)E_{0,1,N}(z, z) \frac{dx\,dy}{y^2} + g_T(z, s) - (s-1)h(T, s), \]

(47)

where \( g_T(z, s) \) is \( s - 1 \) times the sum of all integrals but the first one in (41). Then, \( g_T(z, s) \) is holomorphic in \( s \) at \( s = 1 \) and \( g_T(z, 1) \) is rapidly decreasing as \( T \to \infty \). The integral of the right side of (47) tends to \( C \int_{f \in \mathcal{F}(N)} y^{l} f(z)(z) \frac{dx\,dy}{y^2} \) as \( s \to 1 \) since \((s-1)E_{0,1,N}(z, z)\) uniformly convergent to \( C \) on the compact set \( \mathcal{F}(N) \). If \( C_0 = 0 \) and \( \alpha(s) = 0 \), then we see from (34) that \((s-1)h(T, s)\) tends to \( C \sum \int_{f \in \mathcal{F}(N)} y^{l} f(z)(z) \frac{dx\,dy}{y^2} \) \( + n(T) \) where the powers of \( T \) of terms in \( n(T) \) have only negative real parts. Then the right hand side of (47) turns out to be \( C \left( \int_{f \in \mathcal{F}(N)} y^{l} f(z)(z) \frac{dx\,dy}{y^2} \right) + g_T(z, 1) + n(T) \). Taking the limit as \( T \to \infty \), we obtain (21).

Suppose that \( C_0 = 0 \) or \( \alpha(s) \neq 0 \). Then \((s-1)h(T, s)\) has a term \( C_0 T^{-1+s} - \alpha(s)T^{1-s} \) additionally. Let \( a \) be the coefficient of \( y \) in \( \sum f(y) \). Then \( \alpha(s) \) has \( aC \) as the residue at \( s = 1 \), since all \( \xi^{(r)}(s) \) have the common residues \( C \). Let \( \alpha(s) = \frac{aC}{s-1} + C_0 + O(s-1) \) be the Laurent expansion at \( s = 1 \) with \( c_0 = \frac{1}{2} \frac{d^2}{ds^2}(s-1)\alpha(s) \). Then \( C_0 T^{-1+s} - \alpha(s)T^{1-s} = -\frac{aC}{s-1} + C_0 + aC \log T - C_0 + O(s-1) \). Hence \((s-1)h(T, s) + \frac{aC}{s-1} - C_0 + C_0 \) tends to \( C \sum \int_{f \in \mathcal{F}(N)} y^{l} f(z)(z) \frac{dx\,dy}{y^2} \) \( + n(T) \) as \( s \to 1 \), and the same argument as above leads to an equality

\[
\frac{\Gamma(l-1+s)N \prod_{p|N}(1 + \frac{1}{p})}{3 \cdot 4^{l-1+s} \pi^{l-2+s}} \Gamma(l-1+s; f, g) = a(s-1)^{-2} + \left\{ (f, g) \Gamma_0(N) - C_0 + \frac{1}{2} \frac{d^2}{ds^2}(s-1)\alpha(s) \right\} (s-1)^{-1} + O(1)
\]

which holds near \( s = 1 \). The last assertion of (iii) follows from this. \( \square \)

**Remark 5.6** Let \( f, g \in M_1(N, \rho) \), and let \( a_0^{(r)}, b_0^{(r)} \) be the 0-th Fourier coefficients of \( f, g \) at a cusp \( r \), respectively. Let \( a := \sum_{r \in \mathcal{C}(N)} a_0^{(r)} b_0^{(r)} w^{(r)} \) and \( \alpha(s) := \sum_{r \in \mathcal{C}(N)} a_0^{(r)} b_0^{(r)} w^{(r)} \xi^{(r)}(s) \) where \( w^{(r)} \) is as in (12), and \( \xi^{(r)}(s) \) is as in (16). Corollary 5.5 (iii) and its proof imply that

\[
\frac{\Gamma(s)N \prod_{p|N}(1 + \frac{1}{p})}{3 \cdot 4^{l-1+s} \pi^{l-2+s}} \Gamma(s; f, g) = a(s-1)^{-2} + \left\{ (f, g) \Gamma_0(N) + \frac{1}{2} \frac{d^2}{ds^2}(s-1)\alpha(s) \right\} (s-1)^{-1} + O(1)
\]

near \( s = 1 \). Hence if \( a \neq 0 \), then \( L(s; f, g) \) has a pole of order 2 at \( s = 1 \). If \( a = 0 \), then (21) holds by adding \( \frac{1}{2} \frac{d^2}{ds^2}(s-1)\alpha(s) \) to the left hand side, and if a product \( fg \) is a cusp form, then \( \alpha(s) = 0 \) and (21) holds.
6 The case of half integral weight

In this section, the analytic continuation of $L$-function (1) is proved in the case $l - l' \in \frac{1}{2} \mathbb{Z}$. The properties of the $L$-function (1) shown in the preceding section are proved also in the half integral weight case.

**Theorem 6.1** Assume that $F(z)$ of Theorem 5.2 is in $\mathcal{M}_{l,l-k-1/2}(N, \rho)$ with $k \in \mathbb{Z}, \geq 0$ and with $\rho \in (\mathbb{Z}/N)^*$ where $\rho$ and $k$ have the same parity and where $\rho, N$ satisfy (22) with $M = N$. Then, $R(F, s)$ has the meromorphic continuation to the whole complex plane. Its possible poles are poles of the Eisenstein series $E_{k+1/2, \rho, N}(z, s)$ and $s = q_j^{(r)} + l - k - 1/2 (r \in \mathbb{C}(0(N)), s = -q_j + l + 1$.

**Corollary 6.2** Let $k \in \mathbb{Z}, \geq 0$ and let $l \in \frac{1}{2} \mathbb{Z}, l \geq k + 1/2$. Let $f, g$ be holomorphic modulars forms for $\Gamma_0(N)$ of weight $l$ and of weight $l - k - 1/2$ with characters, respectively. We assume that $f_{\mathbb{R}} \in \mathcal{M}_{l,l-k-1/2}(N, \rho)$ for $\rho \in (\mathbb{Z}/N)^*$ with the same parity as $k$ and where $\rho, N$ satisfy (22) with $M = N$.

(i) Then $L(s; f, g)$ defined in (1) converges at least if $\Re s > \max(2l-k-1/2, 1/2)$, and extends meromorphically to the whole $s$-plane.

(ii) If $\rho, N$ are as in (24) with $\rho_r = 1$, or $k \geq 1$, Let $P^{(r)}(y_l f^c \mathbb{R})$ be as in (3). If $P^{(r)}(y_l f^c \mathbb{R})$ does not have a term containing $y$ to the power of $1$, and if $P^{(r)}(y_l f^c \mathbb{R})$ does not have a term containing $y^{k+1/2}$ for any $r \in C_0(N)$, then there holds

$$\langle f(z), g(z)E_{k+1/2, \rho}(z, 0)\rangle_{\Gamma_0(N)} = (4\pi)^{-l+1}\Gamma(l-1)l(l-1; f, g)$$

(48)

with the scalar product defined in (5). Suppose otherwise. Let $y^{s} + \xi(1/N)(s)y^{-k+1/2-s}$ be the constant term of the Fourier expansion of $y^{s}E_{k+1/2, \rho}(z, s)$ at a cusp $\sqrt{-1}\infty$, and let $\xi^{(r)}(s)y^{-k+1/2-s}$ be the constant term at a cusp $r \in C_0(N), \neq 1/N$. If $C_0$ is a coefficient of $y$ in $P^{(r)}(y_l f^c \mathbb{R})$, and if $\alpha(s)$ is the coefficient of $y^{k}$ in $
abla_{r \in C_0(N)}(s)\psi^{(r)}(s)w^{(r)}P^{(r)}(y_l f^c \mathbb{R})$, then the equation (48) holds replacing the right hand side by $[(4\pi)^{-l+1-s}\Gamma(l-1+s)L(l-1+s; f, g) + s^{-1}(C_0-\alpha(0)) - \frac{d}{ds}\alpha(0)]|_{s=0}$.

**Proof** (i) We take $f(z)_{\mathbb{R}}(z)$ as $F(z)$ in the theorem. Then $R(F, s) = (4\pi)^{-l+1-s}\Gamma(l-1+s)L(l-1+s; f, g)$, and the assertion follows from the theorem.

(ii) The Eisenstein series $E_{k+1/2, \rho}(z, s)$ is holomorphic in $s$ at $s = 0$ under our assumption. Then the proof is the same as in the proof of Corollary 5.5 (ii). □

The functional equation for $R(F, s)$ in Theorem 6.1 under $l - k - 1/2 - s \mapsto l - 1 + s$, or the functional equation for $L(s; f, g)$ in Corollary 6.2 is obtained similarly as in the case of integral weight. They are somewhat complicated, and are omitted in the present paper (see [15]).

7 Applications—Scalar products

**Proposition 7.1** Let $l \in \frac{1}{2} \mathbb{Z}, \geq 0$. $N \in \mathbb{N}$ where $N \geq 3$ if $l$ is odd, and $4|N$ if $l$ is not integral. Let $\rho \in (\mathbb{Z}/N)^*$ be so that $\rho$ has the same parity as $l$ if $l \in \mathbb{Z}$, and that $\rho$
has the same parity as $l - 1/2$ and $\rho, N$ satisfy (22) with $M = N$ if $l \notin \mathbb{Z}$. Assume that $E_{l, \rho, N}(z, s)$ is holomorphic in $s$ at $s = 0$. Denote by $y^s + \xi(1/N)(s)y^{-l+1-s}$, the constant term of Fourier expansion with respect to $x$, of the Eisenstein series $y^sE_{l, \rho, N}(z, s)$ at a cusp $\sqrt{-1}\infty$, and by $\xi(r)(s)y^{-l+1-s}$, the constant term at a cusp $r \in \mathcal{C}_0(N)$.

Let $F(z)$ be in $\mathcal{M}_{l,0}(N, \rho)$ which has the Fourier expansion (32) at each cusp $r$ with $u_0(y) = 0$. Denote by $b$, the coefficient of $y^{-l+1}$ in $P_F^{1/L}(y)$, and by $c(r)$, the (absolutely) constant term in $P_F^r(y)$ for $r \in \mathcal{C}_0(N)$. Then $b = \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\bar{w}(r)(0)$, and

$$\langle F(z), E_{l, \rho, N}(z, 0) \rangle_{\mathcal{C}_0(N)} = - \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\frac{d}{ds}\bar{w}(r)(0).$$

(49)

**Proof** Since $u_0(y) = 0$, the equality (41) turns out to be

$$\int_{\mathfrak{H}_T(N)} y^{l+s}F(z)E_{l, \rho, N}(z, \mathfrak{h}) \frac{dx\,dy}{y^2} = -g_T(z, s) + h(T, s)$$

(50)

where $g_T(z, s)$ is the sum of the integrals of the right hand side of (41) except the first one, and $h(T, s)$ is as in (34) with $l' = l$. The function $g_T(z, s)$ is holomorphic in $s$ at $s = 0$.

For $\Re s \gg 0$, we have $\int_0^T y^l b y^{-l+1} \cdot y^s \frac{dy}{y^s} = b s^{-1}T^s$, and $\int_T^\infty y^l c(r) \cdot \frac{\xi(r)(\mathfrak{h})}{s} y^{-l+1-s} \frac{dy}{y^s} = c(r)\xi(r)(\mathfrak{h}) s^{-1}T^{-s}$. Since near $s = 0$ there holds $bs^{-1}T^s = \frac{b}{s} + b \log T + O(s)$ and $c(r)\xi(r)(\mathfrak{h}) s^{-1}T^{-s} = e^{(r)}\xi(r)(0) + c(r)\frac{d}{ds}\bar{w}(r)(0) - c(r)\bar{w}(r)(0) \log T + O(s)$, we have

$$h(T, s) = s^{-1} \left\{ b - \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\bar{w}(r)(0) \right\} + \sum_{r \in \mathcal{C}_0(N)} Q^{(r)} y^l F_{l, \rho, N}(z, 0)(T)$$

$$- \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\frac{d}{ds}\bar{w}(r)(0) + n(T) + O(s)$$

near $s = 0$ where $Q^{(r)} y^l F_{l, \rho, N}(z, 0)(T)$ is as in (4) and $n(T)$ is a finite sum of terms in which the real parts of powers of $T$ are all negative. In (50), the right hand side is holomorphic in $s$ at $s = 0$ since it is the integral over a compact set, and $g_T(z, s)$ is also holomorphic in $s$. Then $h_T(s)$ is holomorphic, and in particular $b - \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\bar{w}(r)(0) = 0$. Then

$$\int_{\mathfrak{H}_T(N)} y^{l+s}F(z)E_{l, \rho, N}(z, \mathfrak{h}) \frac{dx\,dy}{y^2} - \sum_{r \in \mathcal{C}_0(N)} Q^{(r)} y^l F_{l, \rho, N}(z, 0)(T)$$

$$= -g_T(z, s) - \sum_{r \in \mathcal{C}_0(N)} c(r)w(r)\frac{d}{ds}\bar{w}(r)(0) + n(T) + o(s).$$
At $s = 0$ taking the limit as $T \longrightarrow \infty$, the left hand side tends to 
$\langle F(z), E_{l, \rho, N}(z, 0) \rangle_{\Gamma_0(N)}$ by (5) and we obtain the equality (49). 
\[ \Box \]

Let $\rho$ be a character with $N = \varepsilon_\rho$ and with the same parity as $k$. From (14) and (13), the 0-th Fourier coefficients of $E_{k, \rho}(z, s), E_{k}^\rho(z, s)$ are give by

\[ 1 + \delta_{\varepsilon_\rho, 1} \left( -\sqrt{-1} \right)^k \frac{\pi \varphi(\varepsilon_\rho)}{2^{k-2+2s} \varepsilon_\rho^{k+2s}} \frac{\Gamma(k-1+2s)}{\Gamma(s) \Gamma(k+s)} \frac{\xi(k-1+2s)}{\xi(k+2s)} \prod_{p|\varepsilon_\rho} (1-p^{-k+2s}) y^{-k+1-2s}, \]

(51)

\[ \delta_{\varepsilon_\rho, 1} + \left( -\sqrt{-1} \right)^k \frac{\Gamma(k-1+2s)}{\Gamma(s) \Gamma(k+s)} \frac{L(k-1+2s, \rho)}{L(k+2s, \rho)} y^{-k+1-2s}, \]

(52)

respectively, and

\[ \{ y^s E_{k, \rho}(z, s) \} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow e^{-k-s} \{ y^s E_{k}^\rho(z, s) \} \text{ as } z \rightarrow \infty. \]

\[ \{ y^s E_k^\rho(z, s) \} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow (-1)^k e^s \{ y^s E_{k, \rho}(z, s) \} \text{ as } z \rightarrow \infty. \]

We compute some scalar products using Proposition 7.1.

(I) The case of integral weight and $f_\rho = 1$.

Then $k$ is even. Using the notation of Proposition 7.1, $y^s E_{k, \rho}(z, s)$ has

\[ \xi^{(1/\varepsilon_\rho)}(s) = \frac{(-1)^{k/2} \pi \varphi(\varepsilon_\rho)}{2^{k-2+2s} \varepsilon_\rho^{k+2s}} \frac{\Gamma(k-1+2s)}{\Gamma(s) \Gamma(k+s)} \frac{\xi(k-1+2s)}{\xi(k+2s)} \prod_{p|\varepsilon_\rho} (1-p^{-k+2s}) \]

by (51), and

\[ \xi^{(0)}(s) = \frac{(-1)^{k/2} \pi}{2^{k-2+2s} \varepsilon_\rho} \frac{\Gamma(k-1+2s)}{\Gamma(s) \Gamma(k+s)} \times \frac{\xi(k-1+2s) \prod_{p|\varepsilon_\rho} (1-p^{-k+1-2s})}{\xi(k+2s) \prod_{p|\varepsilon_\rho} (1-p^{-k+2s})} \]

by (52) and by the transformation law written below (52). In the case $\varepsilon_\rho = 1$, namely, 
$\rho = 1$, we have $d^2 \xi(0) = -\frac{1}{2} \xi(k = 0), 3\pi^{-1} \{-1+4 \log 2+2 \log \pi + 24 \frac{d}{ds} \xi(s) |_{s = -1} \}$ 
$k = 2, \frac{(-1)^{k/2} \pi \xi(k-1)}{2^{k-1-2s} \xi(k)} (k \geq 4)$. Then, by Proposition 7.1,

\[ \langle E_k(z, 0), E_k(z, 0) \rangle_{\Gamma_0(1)} = -\frac{d}{ds} \xi(0) \]
The Eisenstein series $E_0(z, 0)$ is a constant 1, and the above formula shows $\pi/3 = \langle E_k(z, 0), E_k(z, 0) \rangle_{\Gamma_0(1)} = \int_{\Gamma_0(1) \backslash \mathcal{H}} \frac{dx dy}{y^2}$, namely, the volume of the fundamental domain of $\text{SL}_2(\mathbb{Z})$ is $\pi/3$, which is a well-known fact. The values of the scalar products $\langle E_k(z, 0), E_k(z, 0) \rangle_{\Gamma_0(1)}$ for $k \geq 4$ are already obtained in Chiera [2], and our result coincides with his.

Let $\epsilon_\rho > 1$. When $k = 0$, $y^s E_{0, \rho}(z, s)$ is holomorphic in $s$ at $s = 0$ only if $\epsilon_\rho$ is a prime, say $\epsilon_\rho = p$. Then, $\langle E_0, E_0, E_0 \rangle_{\Gamma_0(p)} = -\frac{d}{ds} \pi(1/p)(0) = -6^{-1} \pi(p - 1) + \pi(p - 1)(3 \log p)^{-1} \{1 - 2 \log 2 - \log \pi - 12 \frac{d}{ds} \xi(s)\}_{s = 1}$. An Eisenstein series $E_0(0, 0)$ has $c^{(0)} = 1$ and $c^{(r)} = 0$ ($r \in C_0(p), \neq 0$) in the notation of Proposition 7.1. Then, $\langle E_0^p, E_0, E_0 \rangle_{\Gamma_0(p)} = -\epsilon_\rho \frac{d}{ds} \xi(0)(0) = 6^{-1}(1 + p)\pi + (3 \log p)^{-1}(1 - p)\pi[1 - 2 \log 2 - \log \pi - 12 \frac{d}{ds} \xi(s)\}_{s = 1}$. Let $\epsilon_\rho > 1$ be square free, and let $k = 2$. Then, $\langle E_2, E_0, E_0 \rangle_{\Gamma_0(2, \rho)} = 3 \epsilon_\rho^{-2} \pi(\epsilon_\rho)\pi^{-1} \{1 - p^{-2} - 2s\}_{s = 1} + \prod_{p \mid \epsilon_\rho} (1 - p^{-2})^{-1} \{1 - 4 \log 2 - 2 \log \pi + \log \epsilon_\rho - 24 \frac{d}{ds} \xi(s)\}_{s = 1}\}$. For $k \geq 4$ even,

$$\langle E_{k, \rho}(z, 0), E_{k, \rho}(z, 0) \rangle_{\Gamma_0(\epsilon_\rho)} = \frac{(1 - \epsilon_\rho)^{k/2} \pi \psi(\epsilon_\rho)\xi(k - 1)}{2k - 2(k - 1) \prod_{p \mid \epsilon_\rho} (p^{k - 1}\xi(k))},$$

$$\langle E_{0, \rho}(z, 0), E_{k, \rho}(z, 0) \rangle_{\Gamma_0(\epsilon_\rho)} = \frac{(1 - \epsilon_\rho)^{k/2} \pi \xi(k - 1) \prod_{p \mid \epsilon_\rho} (1 - p^{-k})}{2k - 2(k - 1) \xi(k) \prod_{p \mid \epsilon_\rho} (1 - p^{-k})}.$$
Petersson scalar products and L-functions arising from modular forms

\[ \{2 \log 2 + 2 \frac{d}{ds} \log \frac{L(1 + s, \rho)}{L(s, \rho)} \}_{s=0} \] (53)

For primitive \( \rho \), the equality \( E_{1, \rho}(z, 0) = \frac{\tau(\rho)L(1, \rho)}{\zeta(1)} \) holds, and the scalar product \( \langle E_1^0(z, 0), E_{1, \rho}(z, 0) \rangle_{\Gamma_0(f_{\rho})} \) can be obtained from (53).

We compute \( \langle E_k^\rho(z, 0), E_k, \rho(z, 0) \rangle_{\Gamma_0(e_{\rho})} \) for \( k \geq 2 \) and for \( \rho \) not necessarily primitive. We take \( E_k^\rho(z, 0) \) as \( F(z) \) in Proposition 7.1. Then, \( c^{(0)} = (-1)^k \) and \( c^{(r)} = 0 \) (\( r \in C_0(e_{\rho}), \neq 0 \)), and \( \langle E_k^\rho(z, 0), E_k, \rho(z, 0) \rangle_{\Gamma_0(e_{\rho})} = (-1)^k \epsilon \rho \frac{d}{ds} \xi^{(0)}(0) \) where \( y^s E_k, \rho(z, s) \) has \( \frac{(-\sqrt{-1})^k \pi \Gamma(k-1+2s) L(k-1+2s, \rho)}{\rho(\rho) \Gamma(k+s) L(k+s, \rho)} \) as \( \xi^{(0)}(s) \). Then,

\[ \langle E_k^\rho(z, 0), E_k, \rho(z, 0) \rangle_{\Gamma_0(e_{\rho})} = -\frac{(-\sqrt{-1})^k \pi L(k-1, \rho)}{2k-2(k-1)L(k, \rho)} \] (54)

If \( k = 0 \) and \( \rho \) is primitive, then the same computation leads to \( \langle E_0^\rho(z, 0), E_0, \rho(z, 0) \rangle_{\Gamma_0(f_{\rho})} = 4\pi \tau(\rho)^{-1} L(-1, \rho) L(1, \rho)^{-1} - \log(\pi/|f_{\rho}|) + \frac{d}{ds} \log \Gamma(s)|_{s=1/2} + L(-1, \rho)^{-1} \frac{d}{ds} L(s, \rho)|_{s=-1} + L(1, \rho)^{-1} \frac{d}{ds} L(s, \rho)|_{s=1} \).

(III) The case of half integral weight.

By Lemma 3.1 and by direct computation, it is shown that \( y^s E_{k+1/2, \chi^k_{-4}}(z, s) \in \mathcal{M}_{k+1/2, 0}(4, \chi^k_{-4}) \) has

\[ \xi^{(1/4)}(s) = \frac{(-1)^k(k+1/2)}{2k+4s - 1} \pi \Gamma(k-1/2+2s) \Gamma(s) \Gamma(k+1/2+s) \xi(2k-1+4s), \]

\[ \xi^{(1/2)}(s) = 0, \]

\[ \xi^{(0)}(s) = \frac{(-\sqrt{-1})k(1-\sqrt{-1})(2k-1+4s)}{2k+2s} \pi \Gamma(k-1/2+2s) \Gamma(s) \Gamma(k+1/2+s) \xi(2k-1+4s). \]

The 0-the Fourier coefficient of \( E_{k+1/2, \chi^k_{-4}}(z, 0) \) is \( 1 - \frac{\pi}{6 \log 2} y^{1/2} \) for \( k = 0 \), \( 1 - \pi^{-1} y^{-1/2} \) for \( k = 1 \), and 1 for \( k \geq 2 \). Then, by Proposition 7.1,

\[ \langle E_{k+1/2, \chi^k_{-4}}(z, 0), E_{k+1/2, \chi^k_{-4}}(z, 0) \rangle_{\Gamma_0(4)} = -\frac{d}{ds} \xi^{(1/4)}(0) = \begin{cases} \frac{-2}{3 \log 2} \{ -2 + 5 \log 2 + 2 \log \pi + 24 \frac{d}{ds} \xi(s)|_{s=-1} \} & (k = 0), \\ \frac{2}{3 \log 2} \{ 3 - 20 \log 2 - 6 \log \pi - 72 \frac{d}{ds} \xi(s)|_{s=-1} \} & (k = 1), \\ \frac{(-1)^k(k+1/2)^2 - k^2 + 2 \pi \xi(2k-1)}{2k^2 - 1}(2k - 1) \xi(2k) & (k \geq 2). \end{cases} \] (54)

Since \( E_{k+1/2, \chi^k_{-4}}(z, 0) \) is equal to (54) where there is the additional term \( 2\pi/3 \) (\( k = 0 \),...
(4 \log 2)/\pi \ (k = 1) \ by \ Lemma \ 2.2. \ Again \ by \ Proposition \ 7.1,

\begin{align*}
\langle E_{k+1/2}^\chi (z, 0), E_{k+1/2}^\chi (z, 0) \rangle_{\Gamma_0(4)} &= \frac{1}{2^k \pi^{k+1}} \frac{1}{\pi (2k-1) \pi (2k)} \left( \frac{2 \pi \Gamma(l+1)\Gamma(l'+1)}{\Gamma(l+l'+1)} \right) \\
&= \left\{ \begin{array}{ll}
(1-\frac{1}{2\pi}) \frac{1}{\pi} [2 \log 2 + 2 \log \pi + 24 \frac{d}{ds} \xi (s)_{s=-1}] & (k = 0), \\
\frac{3\pi}{4(1+\sqrt{1})} [3 - 8 \log 2 - 6 \log \pi - 72 \frac{d}{ds} \xi (s)_{s=-1}] & (k = 1), \\
\frac{-3}{(2k-1)(2k-1) \pi} & (k \geq 2),
\end{array} \right.
\end{align*}

and

\begin{align*}
\langle \phi_{k+1/2}^\chi (z, 0), \phi_{k+1/2}^\chi (z, 0) \rangle_{\Gamma_0(4)} &= \left\{ \begin{array}{ll}
\frac{2\pi}{\pi} & (k = 0), \\
-12(\log 2)/\pi & (k = 1), \\
\frac{-12 [k/(k+1) + 1]}{\pi (2k-1) \pi (2k)} & (k \geq 2).
\end{array} \right.
\end{align*}

In particular \langle \theta, \theta \rangle_{\Gamma_0(4)} = \frac{2\pi}{\pi}, \langle \theta^3, \theta^3 \rangle_{\Gamma_0(4)} = -(12 \log 2)/\pi, \langle \theta^5, \theta^5 \rangle_{\Gamma_0(4)} = -\frac{2 \cdot 3^3 \cdot 5 \cdot 7 \cdot (3)}{2 \cdot 3 \cdot 7^2 \pi^3}, \langle \theta^3, \theta^3 \rangle_{\Gamma_0(4)} = \frac{3^4 \cdot 5 \cdot 7 \cdot (3)}{2 \cdot 3 \cdot 7^2 \pi^3}. \ From \ \langle \theta, \theta \rangle_{\Gamma_0(4)} = \frac{2\pi}{\pi}, \ we \ see \ that \ L(s; \theta, \theta) \ has \ the \ residue \ 2 \ at \ s = 1/2 \ by \ (21), \ however, \ this \ is \ obvious \ because \ L(s; \theta, \theta) = 4\xi(2s - 1).

8 Applications—L-functions

Proposition 8.1 \ Let \ f, g \ be \ holomorphic \ modular \ forms \ for \ \Gamma_0(N) \ of \ weight \ l, l' \in \frac{1}{2}N \ (l \geq l', \ l + l' > 1) \ respectively \ with \ \bar{f}g = M_{l,l'}(N, \rho) \ for \ \rho \in (\mathbb{Z}/N)^* \ if \ N \ geq 3 \ if \ l - l'  \ is \ odd, \ and \ that \ 4|N \ if \ l - l' \notin \mathbb{Z}. \ Assume \ also \ that \ \rho \ has \ the \ same \ parity \ as \ l - l' \ or \ l - l' - 1/2, \ depending \ on \ whether \ l - l' \ is \ integral \ or \ not. \ Further \ assume \ that \ \rho, N \ satisfy (22) \ with \ M = \bar{N} \ if \ l - l' \notin \mathbb{Z}. \ Let a_0^{(r)}, b_0^{(r)} \ be \ the \ 0-th \ Fourier \ coefficients \ of \ f, g, \ respectively, \ at \ a \ cusp \ r \ in \ \mathcal{C}_0(N). \ Denote \ \frac{y^5 + \xi(1/N)(s)}{y^{-l+l'+1-s}}, \ the \ constant \ term \ of \ \ \xi(s)_{s=-1}, \ the \ constant \ term \ of \ Fourier \ expansion \ with \ respect \ to \ x, \ of \ \xi(s)_{s=-1}, \ the \ constant \ term \ at \ a \ cusp \ \mathcal{C}_0(N). \ Let \ m \in \mathbb{Z} \ be \ the \ order \ of \ a \ pole \ of \ \xi(s)_{s=-1}, \ the \ constant \ term \ at \ a \ cusp \ \mathcal{C}_0(N). \ Let \ m < N \ be \ the \ order \ of \ a \ pole \ \xi(s)_{s=-1}, \ the \ constant \ term \ at \ a \ cusp \ \mathcal{C}_0(N). \ Let \ m < N \ be \ the \ order \ of \ a \ pole \ \xi(s)_{s=-1}, \ the \ constant \ term \ at \ a \ cusp \ \mathcal{C}_0(N). \ Then \ \lim_{s \to 0} s^{m+1} L(l+l'-1+s; f, \gamma) = (4\pi)^{l+l'-1} \Gamma(l+l'-1) \sum_{r \in \mathcal{C}_0(N)} \frac{w(r)}{a_0^{(r)} \bar{b_0^{(r)}} \xi_{-m}}.
\end{align*}

In particular if \ E_{l-l', \rho, N}(z, s) \ is \ holomorphic \ at \ s = l', \ then

\begin{align*}
\text{Res}_{s=l+l'-1} L(s; f, g) &= (4\pi)^{l+l'-1} \Gamma(l+l'-1) \sum_{r \in \mathcal{C}_0(N)} \frac{w(r)}{a_0^{(r)} \bar{b_0^{(r)}} \xi_{l'}}.
\end{align*}
If $E_{l-l', \rho, N}(z, s)$ is holomorphic on a domain $\Re s \geq l'$, then $L(s; f, g)$ has the only possible pole on the domain at $s = l'$.

**Proof** By (41), we have

$$
(4\pi)^{-l+1-s} \Gamma(l - 1 + s)(s - l')^m L(l - 1 + s; f, g) = (s - l')^m R(f \overline{g}, s)
$$

$$
= \int_{gT(N)} \delta^{l+s} f(z) g(z) (s - l')^m E_{0, 1, N}(z, \overline{s}) \frac{ds \, dv}{s^2} + gT(z, s) - (s - l')^m h(T, s)
$$

(57)

where $gT(z, s)$ is $(s - l')^m$ times the sum of all integrals but the first one in (41). Then $gT(z, s)$ is holomorphic in $s$ at $s = l'$, and

$$
h(T, s) = \frac{a_0^{(1/N)} \overline{b_0^{(1/N)}}}{l - 1 + s} T^{l-1+s} + \sum_{r \in \mathcal{C}(0(N))} w^{(r)} a_0^{(r)} \overline{b_0^{(r)}} \xi^{(r)}(s) T^r - s.
$$

Then, integral of (57) is holomorphic in $s$ at $s = l'$, and $h(T, s) = -\sum_r w^{(r)} a_0^{(r)} \overline{b_0^{(r)}} \xi^{(r)}(s-l')^{-m-1} + O((s-l')^{-m})$ around $s = l'$. Then replacing $l' + s$ by $s$, we obtain (55).

The $L$-series $L(s; f, g)$ for $f, g$ in Proposition 8.1 converges for $s > l + l' - 1$. Hence if $L(s; f, g)$ has a pole at $s = l + l' - 1$, then it is the rightmost pole.

**Theorem 8.2** Let $f, g$ be as in Proposition 8.1. Let $f(z) = \sum_{n=0}^{\infty} a_n e(nz)$, $g(z) = \sum_{n=0}^{\infty} b_n e(nz)$ be the Fourier expansions. Assume that (i) there is a nonzero constant $c$ for which $c a_n \overline{b_n}$ $(n \geq 1)$ are all real and non-negative, (ii) $E_{l-l', \rho, N}(z, s)$ is holomorphic at $s = l'$, and (iii) the right hand side of (56), which we denote by $C$, is not zero. Then $\sum_{0 < n \leq X} a_n \overline{b_n} \sim C X^{l+l'-1} \log^{12} X$ as $X \to \infty$.

**Proof** We just apply the Wiener–Ikehara theorem to the $L$-function $L(s; f, g)$ at the rightmost pole $s = l + l' - 1$.

**Corollary 8.3** (i) Let $k \geq 0$ be even and let $l \in \frac{1}{2} \mathbb{N}$ with $l > \max(k, 1/2)$. Let $f, g$ be holomorphic modular forms for $\Gamma_0(N)$ of weight $l$, $l - k$, respectively, and with same character. Let $a_0^{(r)}, b_0^{(r)}$ denote the 0-th Fourier coefficients at a cusp $r \in \mathcal{C}(0(N))$ of $f, g$ respectively. Then for $l = 1$ and $k = 0$, an equality $\lim_{s \to 0} s^2 L(1+s; f, g) = \prod_{p \mid N} (1+p)^{-1} \sum_{r \in \mathcal{C}(0(N))} a_0^{(r)} \overline{b_0^{(r)}} (M^{(r)}; N)^{2l-1}$ holds with $M^{(r)}$ in (12). If $l \geq 3/2$, then $L(s; f, g)$ has a possible pole of order 1 at $s = 2l - k - 1$ with the residue

$$
\operatorname{Res}_{s=2l-k-1} L(s; f, g) = \frac{(-1)^{k/2} 2^{l-k} \pi^{2l-k} \varphi(N) \xi(2l - k - 1)}{\Gamma(l - k) \Gamma(l) \xi(2l - k - 1) \prod_{p \mid N} (1 - p^{-2l+k})}
$$

$$
\times \sum_{r \in \mathcal{C}(0(N))} a_0^{(r)} \overline{b_0^{(r)}} \prod_{p \mid (N/M^{(r)})} (1 - p^{-2l+k+1}) (M^{(r)}; N)^{2l-2l-k+1}.
$$

(58)
(ii) Let \( f(z) = \sum_{n=0}^{\infty} a_n e(nz) \) be a holomorphic modular form for \( \Gamma_0(N) \) of weight \( l \in \frac{1}{2} \mathbb{N}, l \geq 3/2 \) with any character. Let \( a_0^{(r)} \) be the 0-th Fourier coefficients at cusps \( r \in \mathcal{C}_0(N) \) where \( a_0^{(1/N)} = a_0 \). We assume that at least one of \( a_0^{(r)} \) is not zero. Then

\[
\sum_{0 < n \leq X} |a_n|^2 \sim \frac{2^{2l} \pi^{2l} \varphi(N) \zeta(2l-1)}{\Gamma(l)^2 \zeta(2l) \prod_{p|N} (1 - p^{-2l})} \times \sum_{r \in \mathcal{C}_0(N)} |a_0^{(r)}|^2 \prod_{p|(N/M^{(r)})} (1 - p^{-2l+1}) X^{2l-1}.
\]

(iii) Let \( Q, Q' \) be positive definite integral quadratic forms of \( 2l, 2l - 2k \) variables respectively with \( l \in \frac{1}{2} \mathbb{N}, \geq 3/2, 2|k| \geq 0 \). Let \( N \) be the maximum of the levels of \( Q, Q' \). Let \( a_0^{(r)} \), \( d_0^{(r)} \) be the 0-th Fourier coefficients at \( r \in \mathcal{C}_0(N) \), of theta series associated with \( Q, Q' \), respectively. Then if the discriminants \( d_Q, d_{Q'} \) are equal to each other up to square factors, then \( \sum_{0 < n \leq X} r_Q(n) r_{Q'}(n) \sim C X^{2l-1-k}/(2l - k - 1) \) where \( C \) denotes the right-hand side of (58).

**Proof** (i) The assertion follows from (56), (55) where \( \xi^{(r)}(s) \) is given in (15).

As for (ii) and (iii), the assertions follow from Theorem, since \( |a_n|^2 \geq 0 \) in the case (ii), and since \( r_Q(n) r_{Q'}(n) \geq 0 \) in the case (iii).

For \( k > 0 \), let \( r_k(n) \) denote the number of representations of \( n \) as a sum of \( k \) squares. Then \( \theta(z)^k = 1 + \sum_{n=1}^{\infty} r_k(n) e(nz) \), which is a modular form for \( \Gamma_0(4) \) with the value 1 at \( \sqrt{-1} \infty \), the value \( 2^{-k/2} e(-k/8) \) at 0, the value 0 at 1/2. We have \( L(s; \theta^k \overline{\theta}^k) = \sum_{n=1}^{\infty} r_k(n) r_{k^*}(n) n^{-s} \). Then by applying Corollary 8.3 (ii) to \( L(s; \theta^k \overline{\theta}^k) \), we obtain

\[
\sum_{0 < n \leq X} r_k(n)^2 \sim \frac{\pi^k \zeta(k - 1)}{\Gamma(k/2)^2 \zeta(k)(1 - 2^{-k})} X^{k-1} k - 1
\]

for \( k \geq 3 \), which is proved in Müller [7] (see also Borwein and Choi [1], Choi, Kumchev and Osburn [3]). More generally we obtain from Corollary 8.3 (iii),

\[
\sum_{0 < n \leq X} r_k(n) r_{k-4m}(n) \sim \frac{[1 - (1 - (-1)^m) 2^{-k+2m+1}] \pi^k 2^m \zeta(k - 2m - 1)}{\Gamma(k/2 - 2m) \Gamma(k/2) \zeta(k - 2m)(1 - 2^{-k+2m})} X^{k-2m-1} k - 2m - 1
\]

for \( k \in \mathbb{N}, m \in \mathbb{Z}, \geq 0, k > \max\{4m, 2\} \). The estimate of this kind is obtained for many other quadratic forms.

For modular forms on \( \Gamma_0(4) \), we have the following:

**Corollary 8.4** (i) Let \( f(z) = \sum_{n=0}^{\infty} a_n e(nz), g(z) = \sum_{n=0}^{\infty} b_n e(nz) \) be holomorphic modular forms for \( \Gamma_0(4) \) of weight \( l \in \frac{1}{2} \mathbb{N} \) and \( l - k > 0 \) respectively with odd \( k \geq 1 \). Let \( a_0^{(r)} \), \( b_0^{(r)} \) denote the 0-th Fourier coefficients at a cusp \( r \in \mathcal{C}_0(4) \), of \( f, g \).
respectively. If $a_0^{(0)} b_0^{(0)} \neq 0$, then $L(s; f, g)$ has a simple pole at $s = 2l - k - 1$ with residue

$$
\text{Res}_{s=2l-k-1} L(s; f, g) = \frac{\sqrt{-1}^k 2^{2l-k} \pi^{2l-k} a_0^{(0)} b_0^{(0)}}{\Gamma(l-k) \Gamma(l)} L(2l - k - 1, \chi_{-4}).
$$

(59)

Further we suppose that there is a nonzero constant $c$ so that $c a_n \overline{b_n}$ ($n \geq 1$) are all non-negative. Then $\sum_{0 < n \leq x} a_n \overline{b_n} \sim C X^{2l-k-1}/(2l - k - 1)$ where $C$ is the left hand side of (59).

(ii) Let $k \in \mathbb{Z}$, $\geq 0$, and let $l \in \frac{1}{2} \mathbb{N}$ with $l > k + 1/2$. Let $f, g$ be holomorphic modular forms for $\Gamma_0(4)$ with automorphy factors $j(x, z)^{2l}$, $j(x, z)^{2l-2k-1}$ respectively where $j$ denotes the automorphy factor of $\theta$ as in Introduction. Then if $l \geq 3/2$, then $L(s; f, g)$ has a possible pole of order 1 at $s = 2l - k - 3/2$ with the residue

$$
\text{Res}_{s=2l-k-3/2} L(s; f, g) = \frac{(-1)^{k(k+1)/2} 2^{2l-k-3} \pi^{2l-k-1/2} \xi(4l - 2k - 3)}{(2^{4l-2k-2} - 1) \Gamma(l - k - 1/2) \Gamma(l) \xi(4l - 2k - 2)} \times \{a_0^{(1/4)} b_0^{(1/4)} + (1+(-1)^k \sqrt{-1}) 2^3 (2^{4l-2k-3} - 1) a_0^{(0)} b_0^{(0)}\}.
$$

(60)

If $L(s; f, g)$ has the pole and if there is a nonzero constant $c$ so that $c a_n \overline{b_n}$ ($n \geq 1$) are all non-negative, then $\sum_{0 < n \leq x} a_n \overline{b_n} \sim C X^{2l-k-3/2}/(2l - k - 3/2)$ where $C$ is the left hand side of (60).

**Proof** (i) Any modular form for $\Gamma_0(4)$ of odd weight has character $\chi_{-4}$. We apply Theorem to $f, g$ and $\eta^s E_{k, \chi_{-4}}(z, s)$ where $\eta^s E_{k, \chi_{-4}}(z, s)$ has $\xi^{(1/4)}(s) = \xi^{(1/2)}(s) = 0$, $\xi^{(0)}(s) = (-\sqrt{-1})^k \pi \Gamma(k+1+2s) L(1-k-2s, \chi_{-4})$). Then, the assertion follows from Proposition 8.1 and Theorem 8.2.

(ii) We apply Proposition 8.1 and Theorem 8.2. to $f, g$ and $E_{k+1/2, \chi_{-4}}(z, s)$ where $\xi^{(r)}(s)$ is given in Sect. 7, (III).

\[\square\]

By Corollary 8.4 we have

$$
\sum_{0 < n \leq X} r_k(n) r_{k-2m}(n) \sim \frac{\pi^{k-m} L(k-m-1, \chi_{-4}) X^{k-m-1}}{\Gamma(k/2-m) \Gamma(k/2) L(k-m, \chi_{-4})} k-m-1
$$

for $m \equiv 1 \pmod{2}$, $k > 2m$, and

$$
\sum_{0 < n \leq X} r_k(n) r_{k-m}(n) \sim \frac{1 + \chi_8(m) 2^{k+m-3/2} - 2^{2k+m+2} \pi^{k-m/2} \xi(2k-m-2) X^{k-m-1/2}}{\Gamma((k-m)/2) \Gamma(k/2) \xi(2k-m-1)(1 - 2^{2k+m+1})} k-m-1
$$

for $m \equiv 1 \pmod{2}$ and $k \geq 3/2$, $k > m$. 

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The $L$-function $L(s; \theta^k \bar{\theta}^k) = \sum_{n=1}^{\infty} r_k(n) n^{-s}$ has a pole also at $s = k/2$. Their residues up to 8 are $2 (k = 1)$, $2^2 \{ \log 2 + \frac{d}{ds} \log \frac{L(1+s, \chi_{-4})}{L(s, \chi_{-4})} \}_{s=0} (k = 2)$, $-2^5 3 \pi^{-1} \log 2 (k = 3)$, $2^{-1} \{ 3 - 8 \pi \log 2 - 6 \log \pi - 72 \frac{d}{ds} \xi(s) \}_{s=-1} (k = 4)$, $-2^7 3 \cdot 7^{-2} \pi^{-2} \zeta(3) (k = 5)$, $2 \pi^{-2} L(2, \chi_{-4}) (k = 6)$, $2^8 3^3 7 \cdot 31 - 2 \pi^{-3} \zeta(5) (k = 7)$ and $2^{-5} 3^{-1} \cdot 2153 \zeta(3) (k = 8)$.

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