Lecture on Langlands Functoriality Conjecture

Jae-Hyun Yang

Inha University
jhyang@inha.ac.kr

Department of Mathematics
Kyoto University
Kyoto, Japan
June 30 (Tue), 2009
1. Langlands Functoriality Conjecture (briefly LFC)

\(F = \) number field
\(G = \) connected, quasi-split reductive \(/F\).

For each place \(v\) of \(F\),

\(F_v = \) Completion of \(F\) at \(v\),
\(o_v = \) Ring of integers of \(F_v\),
\(p_v = \) Maximal ideal of \(o_v\),
\(q_v = \) Order of residue field \(k_v = o_v/p_v\)
\(A = A_F = \) Ring of adeles of \(F\).

For each place \(v\) of \(F\), we let \(G_v = G(F_v)\).
Let

\[ \Omega = \otimes_v \Omega_v : \text{Cusp Auto Repn of } \mathcal{G}(\mathbb{A}) \]

Let \( S \) be a finite set of places including all archimedean ones such that both \( \Omega_v \) and \( G_v \) are unramified for any place \( v \notin S \). Then for each \( v \notin S \), \( \Omega_v \) determines uniquely a semi-simple conjugacy class

\[ c(\Omega_v) \subset L_{G_v} \]

in the \( L \)-group \( L_{G_v} \) of \( G_v \) as a group defined over \( F_v \). We note that there exists a natural homomorphism

\[ \xi_v : L_{G_v} \rightarrow L_G. \]

For a finite dimensional repn \( r \) of \( L_G \), putting \( r_v = r \circ \xi_v \), the local Langlands \( L \)-function \( L(s, \Omega_v, r_v) \) associated to \( \Omega_v \) and \( r_v \) is defined to be

\[ L(s, \Omega_v, r_v) = \det \left( I - r_v(c(\Omega_v)) q_v^{-s} \right)^{-1}. \]
We set
\[ L_S(s, \Omega, r) = \prod_{v \not\in S} L(s, \Omega_v, r_v). \]

Langlands [1970] proved that \( L_S(s, \Omega, r) \) converges absolutely for sufficiently large \( \text{Re}(s) > 0 \) and defines a holomorphic function there. Furthermore he proposed the following question.

**Conjecture (Langlands [1970]).** \( L_S(s, \Omega, r) \) has a meromorphic continuation to the whole complex plane and satisfies a standard functional equation.

F. Shahidi (cf. [1988] [1990]) gave a partial answer to the above conjecture using the so-called *Langlands-Shahidi method.*
**Special Case** $G = GL(n)$:

[1] (Godement-Jacquet[1972], Jacquet[1979])

Let $n \geq 1$ be a positive integer. Assume $\pi = \otimes_v \pi_v$ be a nontrivial cuspidal representation of $GL(n, \mathbb{A}_F)$. Then $L(s, \pi) := L(s, \pi, \text{st})$ is entire. Moreover, for any finite set $S$ of places of $F$, the incomplete $L$-functions

$$L_S(s, \pi) = \prod_{v \notin S} L(s, \pi_v)$$

is holomorphic and non-zero in $\text{Re}(s) > w + 1$ if $\pi$ has weight $w$. And we have a functional equation

$$L(w + 1 - s, \pi^\vee) = \varepsilon(s, \pi) L(s, \pi)$$

with

$$\varepsilon(s, \pi) = W(\pi) N_{\pi}^{(w+1)/2-s}.$$ 

Here $\pi^\vee$ denotes the contragredient of $\pi$, $N_{\pi}$ denotes the norm of the conductor $\mathcal{N}_{\pi}$ of $\pi$, and $W(\pi)$ is the root number of $\pi$. When $n = 1$, this result is due to **Hecke**.
Given any $m$-tuple of cuspidal representations $\pi, \cdots, \pi_m$ of $GL(n_1, \mathbb{A}_F), \cdots, GL(n_m, \mathbb{A}_F)$ respectively, there exists an irreducible automorphic representation

$$\pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_m \text{ (isobaric auto reprn)}$$

of $GL(n, \mathbb{A}_F)$, $n = \sum_{j=1}^{m} n_j$, which is unique up to equivalence such that for any finite set $S$ of places,

$$L_S(s, \boxplus_{j=1}^{m} \pi_j) = \prod_{j=1}^{m} L_S(s, \pi_j).$$

Here $\boxplus$ denotes the Langlands sum that comes from his theory of Eisenstein series.

Let $\pi$ be an automorphic reprn of $GL(n, \mathbb{A}_F)$. Then the order of pole at $s = 1$ of $L_S(s, \pi \times \pi^\vee)$ is 1 if and only if $\pi$ is cuspidal.
Local Langlands Conjecture
(briefly, LLC)

Let $G$ be a connected reductive group over a
local field $k$. Let $\mathcal{G}_k(G)$ be the set of all admissible homomorphisms $\phi : W'_k \to L^G$ modulo inner automorphisms by elements of $L^G0$, and let $\prod(G(k))$ be the set of all equivalence classes of irreducible admissible representations of $G(k)$. Then there is a surjective map

$$\prod(G(k)) \to \mathcal{G}_k(G)$$

with finite fibres which partitions $\prod(G(k))$ into disjoint finite sets $\prod_\phi(G(k))$, called $L$-packets satisfying suitable properties. Here

$$W'_k = W_k \ltimes \mathbb{G}_a$$

(semidirect product)
denotes the Weil-Deligne group ($W_k$ is the Weil group of $k$).
Remark: (a) If $k$ is archimedean, i.e., $k = \mathbb{R}$ or $\mathbb{C}$, [LLC] was solved by Langlands (1973).

(b) In case $k$ is non-archimedean, Kazhdan and Lusztig (1987) had shown how to parametrize those irreducible admissible representations of $G(k)$ having an Iwahori fixed vector in terms of admissible homomorphisms of $W'_k$.

(c) For a local field $k$ of positive characteristic $p > 0$, [LLC] was established by Laumon, Rapoport and Stuhler (1993).

(d) In case $G = GL(n)$ for a non-archimedean local field $k$, [LLC] was established by Harris and Taylor (2001), and by Henniart (2000). In both cases, the correspondence was established at the level of a correspondence between irreducible Galois representations and supercuspidal representations.
(e) Let $k$ be a non-archimedean local field of characteristic 0 and let $G = SO(2n + 1)$ the split special orthogonal group over $k$. In this case, Jiang and Soudry (2003) gave a parametrization of generic supercuspidal representations of $SO(2n + 1)$ in terms of admissible homomorphisms of $W'_k$. More precisely, there is a unique bijection of the set of conjugacy classes of all admissible, completely reducible, multiplicity-free, symplectic complex representations $\phi : W'_k \rightarrow LSO(2n + 1) = Sp(2n, \mathbb{C})$ onto the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n + 1, k)$.

We recall that $(\pi, V)$ is said to be supercuspidal if the matrix coefficient

$$f_{v,v^*}(g) := (\pi(g)v, v^*), \quad g \in G$$

has compact support on $G$ modulo the center $Z$ of $G$, where $v \in V$ and $v^* \in V^*$ (the dual space of $V$) or equivalently if the Jacquet module $V_N = 0$ for any unipotent radical $N$.  

8
For $\pi \in \Pi_{\phi}(G')$ with $\phi \in \mathcal{G}_k(G')$, if $r$ is a finite dimensional complex representation of $LG$, we define the $L$- and $\varepsilon$-factors

$$L(s, \pi, r) = L(s, r \circ \phi)$$

and

$$\varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \phi, \psi),$$

where $L(s, r \circ \phi)$ is the Artin-Weil $L$-function.

**Remark:** The representations in the $L$-packet $\Pi_{\phi}$ are parametrized by the component group

$$C_{\phi} := S_{\phi}/Z_L S_{\phi}^0,$$

where $S_{\phi}$ is the centralizer of the image of $\phi$ in $LG$, $S_{\phi}^0$ is the identity component of $S_{\phi}$, and $Z_L$ is the center of $LG$. 
LFC. Let $F$ be a number field, and let $H$ and $G$ be connected reductive groups over $F$ such that $G$ is quasi-split over $F$. Suppose

$$\sigma : \mathcal{L}H \to \mathcal{L}G$$

is an $L$-homomorphism. Then for any automorphic repn $\pi = \bigotimes_v \pi_v$ of $H(\mathbb{A})$, there exists an automorphic repn $\Pi = \bigotimes_v \Pi_v$ of $G(\mathbb{A})$ such that

$$c(\Pi_v) = \sigma(c(\pi_v)), \quad v \notin S(\pi) \cup S(\Pi), \quad (1)$$

where

$$S(\pi) (resp. S(\Pi)) = \{ v \mid \pi_v (resp. \Pi_v) \text{ is ramified} \}.$$

We note that (1) is equivalent to the condition

$$L_S(s, \Pi, r) = L_S(s, \pi, r \circ \sigma), \quad S = S(\pi) \cup S(\Pi)$$

for every finite dim complex repn $r$ of $\mathcal{L}G$. 

10
2. Examples of Functoriality

Example 1. Let $H = \{1\}$ and $G = GL(n)$. Then

$$LH = \Gamma_F = Gal(\bar{F}/F), \quad LG = GL(n, \mathbb{C}) \rtimes \Gamma_F.$$ 

We set

$$\bar{\sigma} : Gal(K/F) \longrightarrow GL(n, \mathbb{C}) \longrightarrow PGL(n, \mathbb{C}).$$

(1) If $n = 1$, $F^\times \cong W_F^{ab}$. This is the so-called Artin Reciprocity Law. In this case, $\text{Im } \bar{\sigma}$ is cyclic or dihedral (E. Artin, E. Hecke).

(2) The case $n = 2$. The image $\text{Im } \bar{\sigma}$ is classified as follows:

(2-a) $\text{Im } \bar{\sigma}$ is cyclic or dihedral (Artin).
(2-b) $\text{Im } \bar{\sigma} = A_4$ is tetrahedral (R. Langlands; Base Change for $GL(2)$ [1980]).
(2-c) $\text{Im } \bar{\sigma} = S_4$ is octahedral (J. Tunnell [1981]).
(2-d) $\text{Im} \tilde{\sigma} = A_5$ is icosahedral (R. Taylor et al. [2003] and C. Khare [2005]).

(3) The case $\text{Gal}(K/F)$ is \textbf{nilpotent} and $K/F$ cyclic of prime degree for $n \geq 3$ (L. Clozel and J. Arthur: \textbf{BC} for $GL(n)$ [1989]).

\textbf{Example 2.} (Cogdell, Kim, P.-S, Shahidi [2004])

(1) $H = Sp(2n)$, $G = GL(2n + 1)$ and $\sigma$ is an embedding.

(2) $H = SO(2n + 1)$, $G = GL(2n)$ and $\sigma$ is an embedding.

(c) $H = SO(2n)$, $G = GL(2n)$ and $\sigma$ is an embedding.

All the above classical groups are of the \textit{split} form. For an irreducible, \textbf{generic} cuspidal repn $\pi$ of $H(\mathbb{A})$, there is a \textit{weak} functorial lift of $\pi$. The proof is based on \textbf{Converse Theorem} of Cogdell and Piateski-Shapiro.
Let $\pi = \otimes_v \pi_v$ be an irreducible admissible reprn of $GL(n, \mathbb{A})$ and $\tau = \otimes_v \tau_n$ be a cuspidal reprn of $GL(m, \mathbb{A})$ with $m < n$. Let $\psi$ be a fixed non-trivial continuous additive character of $A$ which is trivial on $F$. We define formally

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v)$$

and

$$\varepsilon(s, \pi \times \tau) = \prod_v \varepsilon(s, \pi_v \times \tau_v, \psi_v).$$

The $L$-function $L(s, \pi \times \tau)$ is said to be nice if it satisfies the following properties (N1)-(N-3):

(N1) $L(s, \pi \times \tau)$ and $L(s, \pi^\vee \times \tau^\vee)$ has A.C. to the whole complex plane;
(N2) $L(s, \pi \times \tau)$ and $L(s, \pi^\vee \times \tau^\vee)$ are bounded in vertical strips of finite width;
(N3) These entire functions satisfy the standard functional equation

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \pi^\vee \times \tau^\vee)$$
Converse Theorem (Cogdell and Piatetski-Shapiro [1994, 1999]).

Let \( \pi \) be an irred admissible repn of \( GL(n) \) whose central character is trivial on \( F^* \) and whose \( L \)-function \( L(s, \pi) \) converges absolutely in some half plane. Let \( S \) be a finite set of finite places. Assume that \( L(s, \pi \times \tau) \) is \textbf{nice} for every cuspidal repn \( \tau \) of \( GL(m) \) for \( 1 \leq m \leq n - 2 \), which is unramified at the places in \( S \). Then there is an auto repn \( \pi' \) of \( GL(n) \) such that \( \pi_v \cong \pi'_v \) for all \( v \notin S \).

\textbf{Example 3.} (Tensor Product)

(1) \( GL(2) \times GL(2) \rightarrow GL(4) \)  
   (D. Ramakrishnan [2000])

(2) \( GL(2) \times GL(3) \rightarrow GL(6) \)  
   (Kim-Shahidi [2002]).
Example 4. (Asgari-Shahidi [2005])

\[ H = \text{GSpin}(m), \quad G = \text{GL}(N) \] with \( N = m \) or \( 2^{\left\lfloor \frac{m}{2} \right\rfloor} \) and \( \sigma \) is an embedding. We note that

\[
L^H H^0 = \begin{cases} 
GSO(m) & \text{if } m \text{ is even} \\
\text{GSp}(2^{\left\lfloor \frac{m}{2} \right\rfloor}) & \text{if } m \text{ is odd}.
\end{cases}
\]

\( \text{LF} \) holds for an irreducible generic cuspidal repn of \( H(\mathbb{A}) \).

Example 5. (Jiang-Soudry [2003])

If \( H = \text{SO}(2n + 1) \) and \( G = \text{GL}(2n) \), the functorial lift from \( H \) to \( G \) is injective up to isomorphism.
Example 6. (Symmetric Power Product for $GL(2)$).
Let $H = GL(2)$ and $G = GL(m + 1)$. Then we have an $L$-homomorphism

$$\text{Sym}^m : GL(2, \mathbb{C}) \to GL(m + 1, \mathbb{C}).$$

(1) $m = 2$ (Gelbart-Jacquet [1978]).
(2) $m = 3$ (Kim-Shahidi [2002])
(3) $m = 4$ (Kim [2003]).

It was observed by Langlands that the functoriality for $\text{Sym}^m$ for all $m \geq 1$ implies the Ramanujan Conjecture for Maass forms (also the Selberg Conjecture) and the Sato-Tate Conjecture.
Example 7. (Exterior Square for $GL(4)$)

$$GL(4) \rightarrow GL(6)$$

(Kim [2003])

Example 8. (P.-S. Chan and Y. Flicker [2007])

- $E/F$: a quadratic ext of a number field $F$
- $F'/F$: a cyclic ext of an odd degree

Then there is a **BC functorial lifting** from $U(3, E/F)$ to $U(3, F'E/F')$.

**Tools of Proof:**
- Trace Formula
- Known results on BC lifting from $U(3, E/F)$ to $GL(3, E)$ and on BC lifting for $GL(n)$. 

17
3. Applications of Functoriality

[1] The Ramanujan Conjecture

(1-1) **R.C for holomorphic cusp forms**

Let

\[ f(\tau) = \sum_{n \geq 1} a_n e^{2\pi i n \tau} \quad (a_1 = 1) \]

be a holomorphic cusp new eigenform for \( \Gamma_0(N) \). Then by P. Deligne [1974], R.C. is valid for \( f \), in other words, for \( p \nmid N \),

\[ |a_p| \leq 2 p^{\frac{k-1}{2}}. \]

(1-2) **R.C. for Maass cusp forms**

Let

\[ f(x + iy) = \sum_{n \neq 0} a_n K_{s/2}(2\pi |n|y)(|n|y)^{1/2} e^{2\pi inx} \]

be the Fourier expansion of a Maass cusp form \( f \) for \( \Gamma_0(N) \) such that

\[ \Delta f = \frac{1}{4} (1 - s^2) f, \quad s \in \mathbb{C}, \]
where

\[ \Delta = -y^2 \left( \partial_x^2 + \partial_y^2 \right) \]  

Laplacian.

For \( p \nmid N \), we write

\[ a_p = p^{-1/2} \left( \alpha_p + \alpha_p^{-1} \right), \quad \alpha_p \in \mathbb{C}. \]

**Ramanujan Conj:**  \( |\alpha_p| = 1 \) for all \( p \nmid N \).

**Theorem (Kim-Sarnak [2003]).** Let \( N = 1 \). For any prime \( p \),

\[ p^{-\frac{7}{64}} \leq |\alpha_p| \leq p^{\frac{7}{64}}. \quad (2) \]

It is the best known estimate so far. The above theorem is a consequence of some results in Langlands functoriality. By the estimate (2), the eigenvalue \( \lambda \) of a Maass form \( f \) is given by

\[ \lambda \geq \frac{1}{4} - \left( \frac{7}{64} \right)^2 \approx 0.238037 \cdots. \quad (3) \]
[2] The Sato-Tate Conjecture

Suppose $f$ is a holomorphic cusp new eigenform for $\Gamma_0(N)$ as above or
Suppose $f$ is a Maass cusp form with the assumption of the validity of the Ramanujan Conjecture.
We set
$$\alpha_p = e^{i\theta_p}, \quad \theta_p \in [0, \pi].$$

**Sato-Tate Conjecture.** The sequence $\{\theta_p\}$ is equi-distributed (or uniformly distributed) with respect to the measure
$$\frac{2}{\pi} \sin^2 t \, dt$$
on $[0, \pi]$. In other words, for any real numbers $\alpha$ and $\beta$ with $0 \leq \alpha < \beta \leq \pi$,
$$\int_{\alpha}^{\beta} \frac{2}{\pi} \sin^2 t \, dt = \lim_{X \to \infty} \frac{|\{p \leq X \, | \, p \nmid N, \, \alpha < \theta_p < \beta\}|}{|\{p \, | \, p \leq X, \, p \nmid N\}|}. $$
Recently L. Clozel, M. Harris and R. Taylor [2006] proved the Sato-Tate conjecture for elliptic curves with a certain condition.

**Theorem [Clozel-Harris-Taylor].** Let $E/\mathbb{Q}$ be an elliptic curve of non-CM type. Assume $j(E) \notin \mathbb{Z}$. Then the Sato-Tate conjecture is valid for $E$.

**Ideas of Proof.** By the modularity of $E$, the $L$-function $L(s, E)$ is entire. For a prime $p$, we put

$$a_p = p + 1 - |E(\mathbb{F}_p)|.$$

Then

$$L(s, E) = \prod_p L_p(s, E),$$

where

$$L_p(s, E) = \begin{cases} (1 - a_pp^{-s})^{-1} & \text{if } p \text{ is bad}, \\ (1 - a_pp^{-s} + p^{1-2s})^{-1} & \text{if } p \text{ is good}. \end{cases}$$
For a good prime $p$, Hasse’s inequality gives

$$|a_p| \leq 2p^{1/2}.$$ 

We set

$$a_p^* = \frac{a_p}{2p^{1/2}} \in [-1, 1]$$

We can also write

$$a_p^* = \cos \theta_p \quad \text{for a unique } \theta_p \in [0, \pi].$$

In this special case (the holomorphic cusp eigenform of weight 2 corresponding to $E$), the Sato-Tate conjecture can be rephrased as follows:

**Sato-Tate Conjecture.** Suppose $E/\mathbb{Q}$ has no complex multiplication. Then the sequence $\{a_p^*\}$ (resp. $\{\theta_p\}$) is equidistributed in $[-1,1]$ (resp. $[0,\pi]$) with respect to the probability measure

$$\frac{2}{\pi} \sqrt{1 - t^2} \, dt \quad \text{(resp. } \frac{2}{\pi} \sin^2 \theta \, d\theta \text{)}$$
Let $\alpha_p$ and $\beta_p$ be complex numbers such that

$$\alpha_p + \beta_p = a_p \quad \text{and} \quad \alpha_p \beta_p = p.$$ 

For a good prime $p$, we see that

$$L_p(s, E) = \left[ (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}) \right]^{-1}.$$

For $n \in \mathbb{Z}^+$ and a good prime $p$, we put

$$L_p(s, E, \text{Sym}^n) = \left[ \prod_{i=0}^{n} (1 - \alpha_p^i \beta_p^{n-i} p^{-s}) \right]^{-1}.$$

There is also a definition of $L_p(s, E, \text{Sym}^n)$ for a bad prime $p$.

Define

$$L(s, E, \text{Sym}^n) = \prod_p L_p(s, E, \text{Sym}^n).$$

If $n = 1$, $L(s, E, \text{Sym})$ is entire and nonvanishing on $\text{Re}(s) = 1$ (R. Rankin).

If $n = 2$, $L(s, E, \text{Sym}^2)$ is entire (G. Shimura [$F = \mathbb{Q} : 1975$], Gelbart-Jacquet [$F =$any number field: 1978]).
If \( n = 3 \), \( L(s, E, \text{Sym}^3) \) is entire (Kim-Shahidi [2002]).

If \( n = 4 \), \( L(s, E, \text{Sym}^4) \) is entire (Kim [2003]).

If \( n = 5, 6, 7, 8 \), \( L(s, E, \text{Sym}^n) \) extends to a meromorphic function to \( \mathbb{C} \) which is holomorphic in \( \text{Re}(s) \geq 1 \) (Kim-Shahidi [2002]).

If \( n = 9 \), \( L(s, E, \text{Sym}^9) \) extends to a meromorphic function to \( \mathbb{C} \) which is holomorphic in \( \text{Re}(s) > 1 \) and may have a pole at \( s = 1 \) (Kim-Shahidi [2002]).

\textbf{Theorem (J.-P. Serre).} Let \( E/\mathbb{Q} \) be an elliptic curve over \( \mathbb{Q} \). Assume for any positive integer \( n \in \mathbb{Z}^+ \), \( L(s, E, \text{Sym}^n) \) extends to a meromorphic function on \( \mathbb{C} \) which is holomorphic and nonvanishing in \( \text{Re}(s) \geq 1 + \frac{n}{2} \). Then the Sato-Tate conjecture is valid for \( E \).
When \( j(E) \notin \mathbb{Z} \), Clozel, Harris and Taylor proved that for any positive integer \( n \in \mathbb{Z}^+ \), \( L(s, E, \text{Sym}^n) \) extends to a meromorphic function on \( \mathbb{C} \) which is holomorphic and non-vanishing in \( \text{Re}(s) \geq 1 + \frac{n}{2} \). By Serre’s Theorem, the Sato-Tate conjecture is true for \( E \).

In order to prove this fact, when \( n \) is odd, they first proved that \( L(s, E, \text{Sym}^n) \) is potentially automorphic, in other words, \( L(s, E, \text{Sym}^n) \) is associated to a cuspidal rep of \( GL(n + 1) \) over some totally real Galois ext of \( \mathbb{Q} \). This argument involves two parts.

1. An extension of Wiles’ technique used to prove the modularity of an elliptic curve based on Galois cohomology and analysis of automorphic representations on different groups, in particular unitary groups.
(2) An extension of Taylor’s idea for proving meromorphic continuation of $L$-functions attached to 2-dimensional Galois representations.

For any even $n$, Harris, Shepherd and Taylor found a twisted form of the moduli space of certain Calabi-Yau manifolds that could be used to study $n$-dimensional Galois representations. □

**Theorem [Taylor, 2008].** Let $K$ be a totally real field and let $E/K$ an elliptic curve with multiplicative reduction at some prime. Then the Sato-Tate conjecture is true for $E$, i.e., the numbers

$$\frac{1 + p - |E(\mathbb{F}_p)|}{2p^{1/2}}$$

are equidistributed in $[-1, 1]$ w.r.t the probability measure

$$\frac{2}{\pi} \sqrt{1 - t^2} \, dt.$$
References

[1] L. Clozel, M. Harris and R. Taylor, Automorphy for some $\ell$-adic lifts of automorphic mod $\ell$ Galois representations, Publ. Math. IHES 108 (2008), 1-181.

[2] R. Taylor, Automorphy for some $\ell$-adic lifts of automorphic mod $\ell$ Galois representations. II, Publ. Math. IHES 108 (2008), 183-239.

[3] M. Harris, N. Shepherd-Barron and R. Taylor, A family of Calabi-Yau varieties and potential automorphy, to appear in Annals of Math. (2009).

[4] M. Harris, N. Shepherd-Barron and R. Taylor, Ihara’s lemma and potential automorphy, preprint (2006).
[5] **M. Harris and R. Taylor**, *The geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies 151, Princeton Univ. Press (2001).

[6] **M. Kisin**, *Moduli of finite flat group schemes, and modularity*, to appear in Annals of Math.

[7] **C. Skinner and A. Wiles**, *Base Change and a problem of Serre*, Duke Math. Journal 107 (2001), 15-25.

[8] **A. Wiles**, *Modular elliptic curves and Fermat’s last theorem*, Annals of Math. 141 (1995), 443-551.
[3] The Inverse Galois Problem

Using the functorial lifting from $SO(2n + 1)$ to $GL(2n)$, C.Khare, M. Larsen and G. Savin [2008] proved that for any prime $\ell$ and any even positive integer $n$, there are infinitely many exponents $k$ for which the finite simple group $PSp_n(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.

- Functoriality and the inverse Galois problem, Compositio Math. 144 (2008), 541-564.

Furthermore, in their recent paper [2008] they extended their earlier work to prove that for a positive integer $t$, assuming that $t$ is even if $\ell = 3$ in the first case (1) below, the following statements (1)-(3) hold:

(1) Let $\ell$ be a prime. Then there exists an integer $k$ divisible by $t$ such that the simple group $G_2(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.
(2) Let $\ell$ be an odd prime. Then there exists an integer $k$ divisible by $t$ such that the simple finite group $SO_{2n+1}(\mathbb{F}_\ell^k)^{\text{der}}$ or the finite classical group $SO_{2n+1}(\mathbb{F}_\ell^k)$ appears as a Galois group over $\mathbb{Q}$.

(3) If $\ell \equiv 3, 5 \pmod{8}$ and $\ell$ is a prime, then there exists an integer $k$ divisible by $t$ such that the simple finite group $SO_{2n+1}(\mathbb{F}_\ell^k)^{\text{der}}$ appears as a Galois group over $\mathbb{Q}$.

The construction of Galois groups in (1)-(3) is based on the functorial lift from $Sp(2n)$ to $GL(2n+1)$, and the backward lift from $GL(2n+1)$ to $Sp(2n)$ plus the theta lift from $G_2$ to $Sp(6)$.

- Functoriality and the inverse Galois problem II: Groups of type $B_n$ and $G_2$, arXiv:0807.0861v1 (2008).
[4] Cuspidality and Irreducibility

**Question:** When is a functorial lift $\Pi$ of $\pi$ cuspidal? Give a criterion that $\Pi$ is cuspidal.

We have partial results for the special cases.

**Example 1.** Let $\pi$ be a cuspidal rep of $GL(2)$. Then the following properties are known:

1. $\pi$ is dihedral iff $\text{Sym}^2(\pi)$ is not cuspidal. (G-J [1978]).
2. Assume that $\pi$ is not dihedral and not monomial. Then $\pi$ is tetrahedral if and only if $\text{Sym}^3(\pi)$ is not cuspidal. (K-S [2002]).
3. Assume that $\pi$ is not dihedral and not monomial. Then $\pi$ is not octahedral if and only if $\text{Sym}^4(\pi)$ is cuspidal. (Kim [2003]).

By (2) and (3), $\pi$ is octahedral if and only if $\text{Sym}^3(\pi)$ is cuspidal but $\text{Sym}^4(\pi)$ is not cuspidal.
**Definitions:**

(1) $\pi$ is called **monomial** if $\pi \simeq \pi \otimes \eta$ for some idele class character $\eta$ of $F$.

(2) $\pi$ is said to be **solvable polyhedral** if it is dihedral, tetrahedral or octahedral.

(3) $\pi$ is said to be **self-dual** if $\pi \simeq \pi^\vee$.

(4) $\pi$ is said to be **essentially self-dual** if $\pi^\vee \simeq \pi \otimes \chi$ for some idele class character $\chi$.

(5) $\pi$ is said to be **almost self-dual** if $\pi^\vee \simeq \pi \otimes |\cdot|^t$, where $\xi$ is a finite order character and $t \in \mathbb{C}$.

(6) According to Langlands, the archimedean component $\pi_\infty$ is associated to an $n$-dimensional representation $\sigma_\infty(\pi)$ of the real Weil group $W_\mathbb{R}$. We have a canonical exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow W_\mathbb{R} \longrightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$
which represents the unique nontrivial extension of $\text{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^*$. One has a decomposition

$$\sigma_\infty(\pi)|_{\mathbb{C}^*} \simeq \bigoplus_{j=1}^{n} \chi_j,$$

(4) where each $\chi_j$ is a quasi-character of $\mathbb{C}^*$. $\pi$ is said to be algebraic if each $\chi_j$ is algebraic, that is, there are integers $p_j, q_j$ such that

$$\chi_j(z) = z^{p_j} \bar{z}^{q_j}, \quad z \in \mathbb{C}^*.$$

We say that $\pi$ is regular if the multiplicity of each $\chi_j$ is one, and that $\pi$ is semi-regular if the multiplicity of each $\chi_j$ is at most two.

(7) An isobaric automorphic representation $\pi$ of $GL(n, \mathbb{A})$ is called odd if for every one-dimen’l representation $\xi$ of $W_{\mathbb{R}}$ occurring in $\sigma_\infty(\pi)$, the following condition

$$|m_+(\pi, \xi) - m_-(\pi, \xi)| \leq 1,$$

where $m_+(\pi, \xi)$ (resp. $m_-(\pi, \xi)$) denotes the multiplicity of the eigenvalue $+1$ (resp. $-1$) under the action $\mathbb{R}^*/\mathbb{R}^+_+$ on the $\xi$-isotypic component of $\sigma_\infty(\pi)$. 33
Example 2 (D. Ramakrishnan and S. Wang).

Let $\pi$ and $\theta$ be cuspidal reps of $GL(2)$ and $GL(3)$ respectively over a number field $F$. Then the functorial lift $\pi \boxtimes \theta$ on $GL(6)/F$ is cuspidal unless one of the following satisfies

(a) $\pi$ is dihedral, and $\theta$ is a twist of $Ad(\pi)$;

(b) $\pi$ is dihedral, $L(s, \pi) = L(s, \chi)$ for an idele class character $\chi$ of a cubic, non-normal ext $K$ of $F$, and the BC $\pi_K$ is Eisensteinian.

Remark. The $L$-function $L(s, \pi \boxtimes \theta)$ is equal to the Rankin-Selberg $L$-function $L(s, \pi \times \theta)$. By Example 1, $\pi \boxtimes \theta$ is cuspidal if $\pi$ is not dihedral and $\theta$ is not a twist of $Ad(\pi)$. 
Example 3 (D. Ramakrishnan and S. Wang).

Let $\pi$ and $\theta$ be cuspidal reps of $GL(2)$ and $GL(3)$ respectively over a number field $F$. Assume that $\pi$ is not of solvable polyhedral type, and $\theta$ not essentially self-dual. Then we have the following

(a) If $\theta$ does not admit any self twist, the functorial lift $\pi \boxtimes \theta$ is cuspidal without any self twist.

(b) If $\theta$ is not of solvable type, $\pi \boxtimes \theta$ is cuspidal and not of solvable type.
Example 4 (D. Ramakrishnan [2007]).

Let $\pi$ be a cuspidal rep of $GL(2)/F$ with central character $\omega$. Assume that $\pi$ is not solvable polyhedral and also that $\text{Sym}^m(\pi)$ is automorphic for all $m \geq 1$. Then

(a) $\text{Sym}^5(\pi)$ is cuspidal;

(b) $\text{Sym}^6(\pi)$ is not cuspidal if and only if we have

$$\text{Sym}^5(\pi) \cong \text{Ad}(\tau) \boxtimes \pi \otimes \omega^{-4}$$

for a cuspidal rep $\tau$ of $GL(2)/F$;

(c) If $\text{Sym}^6(\pi)$ is cuspidal, then so is $\text{Sym}^m(\pi)$ for all $m \geq 1$;

(d) If $F = \mathbb{Q}$ and $\pi$ is a holomorphic, non-CM new eigenform of weight $k \geq 2$, then $\text{Sym}^m(\pi)$ is cuspidal for all $m \geq 1$. 
Example 5 (D. Ramakrishnan [2008]).

Let $n \leq 5$ and $\ell$ be a prime. Suppose a continuous $\ell$-adic representation $\rho_\ell$ of the absolute Galois group $G_\mathbb{Q}$ is associated to an isobaric automorphic representation $\pi$ of $GL(n, \mathbb{A})$. Assume the following conditions (a),(b) and (c):

(a) $\rho_\ell$ is irreducible.

(b) $\pi$ is odd if $n \geq 3$.

(c) $\pi$ is semi-regular if $n = 4$, and regular if $n = 5$.

Then $\pi$ is cuspidal.
4. Methods to Tackle LFC

- **Trace Formula**
  
  [J. Arthur, R. Langlands, ...]

- **Converse Theorem**
  
  [Cogdell, Piatetski-Shapiro (1929-2009), ...]

- **Endoscopy**
  
  [R. Langlands, D. Shelstad, R. Kottwitz, ...]

- **Base Change**
  
  [R. Langlands, J. Arthur, L. Clozel, Y. Flicker, ...]

- **Theta Correspondence**
  
  [R. Howe, J.-S. Li, ...]

- **Motif and Shimura Varieties**
New Method (Idea) ???
Thank You Very Much !!!