MULTI-CRITERIA DECISION MAKING METHOD BASED ON BONFERRONI MEAN AGGREGATION OPERATORS OF COMPLEX INTUITIONISTIC FUZZY NUMBERS

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Abstract. Complex intuitionistic fuzzy sets (CIFSs), characterized by complex-valued grades of membership and non-membership, are a generalization of standard intuitionistic fuzzy (IF) sets that better speak to time-periodic issues and handle two-dimensional data in a solitary set. Under this environment, in this article, various mean-type operators, namely complex IF Bonferroni means (CIFBM) and complex IF weighted Bonferroni mean (CIFWBM) are presented along with their properties and numerous particular cases of CIFBM are discussed. Further, using the presented operators a decision-making approach is developed and is illustrated with the help of a practical example. Also, the reliability of the developed methodology is investigated with the aid of validity test criteria and the example results are compared with prevailing methods based on operators.

1. Introduction. Multicriteria decision making (MCDM) is the relevant connection of the decision making science whose intention is to recognize the optimal alternative(s) from the set of achievable ones. In practical decision making (DM), the person obliges to furnish the evaluation of the furnished alternatives by different types of evaluation conditions such as crisp numbers, intervals, etc. However, in many doubtful statuses, it is regularly challenging for the decision-maker to supply their inclinations in terms of crisp numbers. In real-life conditions, decision-makers no lasting paid with the numerical values stated in terms of crisp numbers due to numerous deductions such as inappropriate knowledge, shortage of experience, the human factor, complex background and the intricacy itself. Due to the non-availability of their distribution function, the decision making is quite conscious and hence it may be operated as a positive imprecise figure among zero and one rather of a fixed real number. Since the evaluation of the objects depends on the character to character and the environmental conditions, therefore, it is not tolerable to discover a fixed number, which attests to an overall appraisal of the object in all requirements.

Traditionally, the greatest of the data collected is either in the form of an unpredictable view, which may commence a fallacious judgment during the method. To

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this consequence, both probabilistic and non-probabilistic approaches are employed to manage the portion of ambiguity during the analysis. Probabilistic or stochastic programming is based on the distribution function which is nondecreasing while the fuzzy programming appropriates the notion of membership function which is not necessarily so. Also, fuzzy commits to the uncertainty in the ideas while stochastic reflect randomness. Thus, the probabilistic theory, based on the binary state postulates, and hence the conclusions based on it do not regularly contribute beneficial information to the practitioners due to the weakness of being able to manage only quantitative information. To master these complexities, methodologies based on the fuzzy set (FS) theory [49] are being managed in the examination for producing the basic event uncertainty, in which a membership degree with range \([0,1]\) is attributed to each element. However, following their appearance, a large-scale group of its continuations such as intuitionistic FS (IFS) [6], interval-valued IFS (IVIFS) [5], linguistic interval-valued IFS [15], complex FS (CFS) [35], complex IFS (CIFS) [2], complex IVIFS (CIVIFS) [16] are allotted by the researchers. Under the utilization of such theories, decision-makers advance their DM criteria following the critical condition whether it is human thought or pattern identification. In these theories, each object is assessed with the two levels namely “membership degree” (MD) and “non-membership degree” (NMD) such that their calculated sum does not transcend one.

Recently, decision making (DM) has been one of the sizzling legends in the study which comprises the following three phases:

1) Accumulate the data on a proper order to express the knowledge.
2) Obtain the overall decision value of the target by aggregating the collected attribute preferences.
3) Order the objectives to find the accurate one(s).

In phase (1), the common crucial responsibility related to the extraction of the information to represent the object is marked. The different kinds of the environment are assessed by the scholar to rates the given information such as FSs, IFSs, IVIFSs, CFS, CIFS, CIVIFSs, etc. In phase (2), the collective information during the above phase is labeled and join the various values either by using the aggregation operators (AOs) or information measures. Among them, prominent consideration has been given by the scholars on it by defining the various kinds of AOs, which is nothing but a mathematical function that can combine the \(n\)-array information into a single one. Finally, in-phase (3), the combined values acquired from the foregoing phases are ordered with suitable measures and hence get the optimal one(s).

In the literature, the most generally used conditions to access the erudition are IFSs where the data is concentrated in terms of MD and NMD, such that their sum is bound to one. However, the information fusion rule which includes a diversity of tools to manage the information, while the most common look during any MCDM process is: how to aggregate the information. In recent years, the topic of AO has fascinated a lot of concentration and enhance sizzling subjects in the problems of MCDM. From the many existing AOs, we discretion the existing AOs into two aspects, i.e., the operation aspect and the functional aspects, which are reported within the succeeding prospects to solve the MCDM problems.

1) The operations aspect: Many AOs are offered by the researchers based on the operational laws. For instance, some scholars [43, 44] gave some essential averaging and geometric (AG) operators. Additional than it, some extended AOs by using interactive operations [10, 12], Hamacher norms [13, 25] are composed
by the scholars for solving the decision-making problems (DMPs) by using IF information. He et al. [24] presented some interactive AOs for IFSs. Wang and Liu [39] presented Einstein norm AOs for intuitionistic fuzzy numbers (IFNs). Goyal et al. [23] exhibited some geometric AOs for IFNs. Ye [48] displayed hybrid averaging and geometric AOs with the parameter of the decision-maker attitude character. Zhou and Xu [50] analyzed the extreme weighted geometric AOs while Chen and Chang [8] presented an AO based on the transformation technique of IFNs. Garg [11] extended interactive Hamacher t-norm based AOs for IFNs. Kaur and Garg [27] presented some generalized AOs using t-norms for cubic IFNs. All the above presented AOs are based on the Archimedean t-norm and t-conorm family.

2) The function aspect: The above AOs are based only on the operational laws by considering that all the characteristics during the aggregation are autonomous of each other. However, there are certain real-life problems in which some degree of interrelationship survives among the arguments and it is vital to take into statement this interdependence during the aggregation process to make an optimal decision. Such AOs are dependent on the function aspect which mainly categorized into power, Bonferroni mean (BM) [7], Maclaurin Symmetric mean (MSM) [32] etc. For instance, in [41], the power weighted averaging operator is treated under the IFS environment. Xu and Yager [42] discussed the idea of BM for the IFNs. Yager [45] discusses their importance and the generalization. Verma [38] introduced new generalized BM operators for IFNs. Kaur and Garg [28] put forward BM operator under cubic IFS environment. Li et al. [31] presented the generalized BM operators for IFNs. Garg and Arora [14] investigated BM operators for intuitionistic fuzzy soft set theory. The major advantage of the BM operator is to examine two disputes at a time and hence it is remarkable to estimate the analysis rather than the operational based AOs.

With the growth of choice complexity, the exact DMPs are ordinarily caused by the interacting aspect. At the identical interval, some authorities may have a substantial own inclination in the manner of evaluating the problem. From the preceding comprehensive studies and focused environment, it is recognized that their similar approaches are restricted in access and concurrently fail to give their judgment on the problem which is changing over the given time phase such as medical diagnosis, biometric recognition, etc. To deal with the periodicity as well as quantification of the data, Ramot et al. [35] built the concept of complex FS (CFS) by expanding the domain of MDs from real set to unit-disc complex plane. To decorate and scrutinize this set, authors in [34, 35] studied their features such as complement, intersection, union. Dick et al. [9] impersonated the connection between CFS and Pythagorean FS [46]. The wide mixture of the utilization of CFS was given by Yazdanbakhsh and Dick [47]. After their prosperous application, Alkouri and Salleh [2] extended the concept of CFS to complex IFS (CIFS) by consolidating the NMDs in it over the unit-disc complex plane. Authors in [3, 30] defined some measures and relations for CIFSs. Ali and Smarandache [1] showed the complex neutrosophic set. In the aspect of the MCDM problem, researchers have impersonated some AOs and information measures for CIFS for to solve the DMPs. For instance, in the AOs domain, Rani and Garg [37] given power AOs while in [19, 21], authors had performed AOs based on the operational laws. Based on the Archimedean t-norm operations, authors in [18] examined the generalized BM operators for CIFSs. In [33], the authors addressed the operations for the complex neutrosophic soft sets.
Recently, authors in [17] exhibited some compensative forms of AOs to aggregate the CIFSs information. However, to rank the given numbers, different kinds of measures have been outlined in the literature such as distance measure [36], correlation [20]. For some other kinds of measures such as inclusion, similarity, divergence, etc., we refer to read the article [22] under the CIFS environment.

The CIFS is a generalization of the IFS holding both the MDs and NMDs on the complex argument plane. Under it, the amplitude term provides the amount of belongingness while the phase term describes the periodicity of an object. These phase terms identify the CIFS from the regular IFS theory. In IFS theory, the circumstance of periodicity is neglected and hence there is a specific loss of information. To avoid, a factor of it is added into the analysis. To further elucidate the notion of phase terms, consider a certain firm that requires to acquire cars from the carmakers concerning specialties such as (i) Models and (ii) Production dates of cars. Since every year, the carmakers assemble the same models of cars with insignificant changes and differences, therefore, due to the modifications made, people believe their levels and reviews for the new model. Hence, the composition date of the car also acts a vital role during the buying or decision. Consequently, such a problem estimated a two-dimensional one which can’t be fashioned together in the existing FSs or IFSs environment. Moreover, to accomplish such kinds of the problem under IFS environment, suddenly there is a need to examine two or more IFSs by the decision-makers and then execute it, which reaches to the events increasing the execution time, and the number of calculations during solving the problem. On the other hand, CIFS is a more loyal description of such queries in which both dimensions examine as a single set. Thus, CIFS is a more reliable design of the data than the existing ones. The supremacy analysis of the suggested model over current ones is given in Table 1.

**Table 1. Comparison of CIFS model with existing models in literature**

| Model                          | Uncertainty | Falsity | Hesitation | Periodicity | Ability to represent two-dimensional information |
|-------------------------------|-------------|---------|------------|-------------|---------------------------------------------------|
| Fuzzy set                     | ✓           | x       | x          | x           | x                                                 |
| Interval-valued fuzzy set     | ✓           | x       | x          | x           | x                                                 |
| Intuitionistic fuzzy set      | ✓           | ✓       | x          | x           | x                                                 |
| Interval-valued intuitionistic fuzzy set | ✓ | ✓       | ✓          | x           | x                                                 |
| Complex fuzzy set             | ✓           | x       | x          | ✓           | ✓                                                 |
| Interval-valued complex fuzzy set | ✓ | x       | x          | ✓           | ✓                                                 |
| Complex intuitionistic fuzzy set | ✓ | ✓       | ✓          | ✓           | ✓                                                 |

In the modern decision-making environment, several parameters are dependent on each other. In other words, the interrelationship between the objects is occurred frequently such as cost, efficiency, reliability, etc., which directly impact the decision process and hence it is essential to take into consideration. So, it is important to consider it into the analysis to make more reasonable results. Also, due to complexities occurs in the decision-making process, it is necessary to design a more versatile AO for CIFSs. Furthermore, at the same time, the CIFSs can express the data with a wider range and handle the two-dimensional information at the same stage. Based on the aforementioned analysis, we can derive the following results:

1) The CIFSs is the successful way to deal with the information than IFSs and their representation is more flexible and broader for solving DMPs.
2) The existing algebraic AOs have considered the independent nature of each attribute during the process, while it is common that many factors play a vital role during the aggregation. Thus, a concept of BM which has a prominent characteristic to aggregate the different values by considering the interrelation between the pairs has been discussed.

3) The structure of BM gives two flexible parameters $p$ and $q$, which reflect the attitude character towards the decision-making process.

Based on the aforementioned comprehensive analysis and motivated by the benefits of CIFS, it is important and useful to present the BM operators to complex intuitionistic fuzzy numbers (CIFNs) and apply them to solve the MCDM problems. To address it, we present a framework in which a novel complex intuitionistic fuzzy Bonferroni mean (CIFBM) operator, as well as complex intuitionistic fuzzy weighted Bonferroni mean (CIFWBM) operator, are defined to aggregate the preferences of the decision-makers. Also, several features of these operators are examined in detail. In the proposed AOs, multi-features of the attributes are used in pairs to aggregate the information by taking two interrelations components at the same time. The applicability of the developed algorithm is explained with some numerical examples and confirmed by contrasting their results with numerous prevailing methods. It is worth noting that the proposed AOs and the algorithm have these advantages:

1) the scope of expressing the vague information is more flexible and wider.

2) various existing AOs under CIFS and IFS conditions are special cases of the stated operators.

3) these hold interrelation among the two numbers of inputs.

4) by varying the parameters $p$ and $q$ in the presented operators, a decision-maker may choose their optimal decision as per their needs and hence the proposed algorithm has more achievable and suitable to solve DMPs.

The rest of the paper is structured as follows. In Section 2, we describe the basic concept of CIFS and their relative’s symbols. In section 3, we developed two new operators denoted by CIFBM and CIFWBM in conjunction with discussing their properties and numerous specific cases of CIFBM operators. Section 4 describes an approach based on the developed operators under CIF theory and illustrates with a numerical example. The conclusion of the work is summarized in Section 5.

2. Preliminaries. In this section, some basic overview of CIFS are presented over the universal set $\mathcal{U}$.

**Definition 2.1.** [6] An IFS, $\mathcal{H}$ on $\mathcal{U}$, is defined as

$$\mathcal{H} = \{(x, \mu_\mathcal{H}(x), \nu_\mathcal{H}(x)) \mid x \in \mathcal{U}\},$$

(1)

where $\mu_\mathcal{H}, \nu_\mathcal{H} : \mathcal{U} \to [0, 1]$ such that for all $x$, $0 \leq \mu_\mathcal{H} + \nu_\mathcal{H} \leq 1$. A pair of $\mathcal{H} = (\mu, \nu)$ is called intuitionistic fuzzy number (IFN) [44].

**Definition 2.2.** [44] The score and accuracy functions are defined as

$$S(\mathcal{H}) = \mu - \nu$$

(2)

and

$$H(\mathcal{H}) = \mu + \nu$$

(3)

respectively. Further, a rule to compute a relation between two IFNs $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \succ \mathcal{H}_2$, if either $S(\mathcal{H}_1) > S(\mathcal{H}_2)$ or $S(\mathcal{H}_1) = S(\mathcal{H}_2)$ and $H(\mathcal{H}_1) > H(\mathcal{H}_2)$. Here, ‘$\succ$’ stands for “preferred to”.
Definition 2.3. [35] A CFS “$\mathcal{H}$” is defined as:

$$\mathcal{H} = \{(x, \mu_{\mathcal{H}}(x)) : x \in \mathcal{U}\}$$

(4)

where $\mu_{\mathcal{H}} : \mathcal{U} \to \{b : b \in \mathcal{C}, |b| \leq 1\}$ defined as $\mu_{\mathcal{H}}(x) = r_{\mathcal{H}}(x)e^{i2\pi(w_{r_{\mathcal{H}}}(x))} \forall x \in \mathcal{U}$ where $i = \sqrt{-1}$, $0 \leq r_{\mathcal{H}}(x), w_{r_{\mathcal{H}}}(x) \leq 1$.

Definition 2.4. [2] A CIFSN $\mathcal{H}$ is defined as:

$$\mathcal{H} = \{(x, \mu_{\mathcal{H}}(x), \nu_{\mathcal{H}}(x)) : x \in \mathcal{U}\}$$

(5)

where $\mu_{\mathcal{H}}, \nu_{\mathcal{H}} : \mathcal{U} \to \{b : b \in \mathcal{C}, |b| \leq 1\}$ defined as $\mu_{\mathcal{H}}(x) = r_{\mathcal{H}}(x)e^{i2\pi(w_{r_{\mathcal{H}}}(x))}$ and $\nu_{\mathcal{H}}(x) = k_{\mathcal{H}}(x)e^{i2\pi(w_{k_{\mathcal{H}}}(x))}$, where $0 \leq r_{\mathcal{H}}(x), k_{\mathcal{H}}(x), w_{r_{\mathcal{H}}}(x), w_{k_{\mathcal{H}}}(x), r_{\mathcal{H}}(x) + k_{\mathcal{H}}(x), w_{r_{\mathcal{H}}}(x) + w_{k_{\mathcal{H}}}(x) \leq 1$. A pair of them are denoted as $\mathcal{H} = ((r, w_r), (k, w_k))$ and named as complex IFN (CIFN), where $0 \leq r, k, r + k, w_r, w_k, w_r + w_k \leq 1$.

Definition 2.5. [2] For two CIFNs $\mathcal{H} = ((r_{\mathcal{H}}, w_{r_{\mathcal{H}}}), (k_{\mathcal{H}}, w_{k_{\mathcal{H}}}))$ and $\mathcal{V} = ((r_{\mathcal{V}}, w_{r_{\mathcal{V}}}), (k_{\mathcal{V}}, w_{k_{\mathcal{V}}}))$, some relations between them are defined as:

i) $\mathcal{H} \subseteq \mathcal{V}$ if $r_{\mathcal{H}} \leq r_{\mathcal{V}}, k_{\mathcal{H}} \geq k_{\mathcal{V}}$ and $w_{r_{\mathcal{H}}} \leq w_{r_{\mathcal{V}}}, w_{k_{\mathcal{H}}} \geq w_{k_{\mathcal{V}}}$;

ii) $\mathcal{H} = \mathcal{V} \Leftrightarrow \mathcal{H} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \mathcal{H}$;

iii) $\mathcal{H} = (k_{\mathcal{H}}, w_{k_{\mathcal{H}}}), (r_{\mathcal{H}}, w_{r_{\mathcal{H}}})$.

Definition 2.6. [19, 37] For CIFNs $\mathcal{H}_1 = ((r_1, w_{r_1}), (k_1, w_{k_1}))$, $\mathcal{H}_2 = ((r_2, w_{r_2}), (k_2, w_{k_2}))$ and a real number $\lambda > 0$, the algebraic operations are given as:

(i) $\mathcal{H}_1 \oplus \mathcal{H}_2 = \left(\left(1 - \frac{2}{i=1} (1 - r_i), 1 - \frac{2}{i=1} (1 - w_{r_i})\right), \left(\frac{2}{i=1} k_i, \frac{2}{i=1} w_{k_i}\right)\right)$

(ii) $\mathcal{H}_1 \otimes \mathcal{H}_2 = \left(\left(\frac{2}{i=1} r_i, \frac{2}{i=1} w_{r_i}\right), \left(1 - \frac{2}{i=1} (1 - k_i), 1 - \frac{2}{i=1} (1 - w_{k_i})\right)\right)$

(iii) $\lambda \mathcal{H}_1 = \left(\left(1 - (1 - r_1)^\lambda, 1 - (1 - w_{r_1})^\lambda\right), \left(k_1^\lambda, w_{k_1}^\lambda\right)\right)$

(iv) $\mathcal{H}_1^\lambda = \left(\left(r_1^\lambda, w_{r_1}^\lambda\right), \left(1 - (1 - k_1)^\lambda, 1 - (1 - w_{k_1})^\lambda\right)\right)$

and are all CIFNs.

Definition 2.7. [19, 37] The score and accuracy functions of CIFN $\mathcal{H} = ((r, w_r), (k, w_k))$ are stated as

$$S(\mathcal{H}) = (r - k) + (w_r - w_k)$$

(6)

and

$$H(\mathcal{H}) = (r + k) + (w_r + w_k)$$

(7)

respectively.

Further, CIFN $\mathcal{H}$ is better than another CIFN $\mathcal{V}$ represented by $\mathcal{H} \succ \mathcal{V}$ if either $S(\mathcal{H}) > S(\mathcal{V})$ or $S(\mathcal{H}) = S(\mathcal{V})$ and $H(\mathcal{H}) > H(\mathcal{V})$.

Definition 2.8. [7] For positive real numbers $a_t$ ($t = 1, 2, \ldots, n$) and $p, q$, BM operator is defined as

$$B^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{t,s=1, t \neq s}^n a_t^p a_s^q\right)^{\frac{1}{p+q}}$$

(8)
Definition 2.9. [42] The BM operators for IFNs $H_t$, ($t = 1, 2, \ldots, n$) are defined as

$$\text{IFBM}^{p,q}_{H}(H_1, H_2, \ldots, H_n) = \left( \frac{1}{n(n-1)} \left( \bigoplus_{t,s=1, t \neq s}^{n} (H_p^t \otimes H_q^s) \right)^{\frac{1}{p+q}} \right)$$

and

$$\text{IFWBM}^{p,q}_{H}(H_1, H_2, \ldots, H_n) = \left( \frac{1}{n(n-1)} \left( \bigoplus_{t,s=1, t \neq s}^{n} (\xi_t H_t)^p \otimes (\xi_s H_s)^q \right)^{\frac{1}{p+q}} \right)$$

where $\xi_t$ is the weight associated with IFNs $H_t$ for each $t$.

3. Proposed complex intuitionistic fuzzy Bonferroni mean operators. In this section, by keeping the features of CIFNs and advantages of the BM operators, we develop some series of BM operators for a collection of CIFNs, denoted by $\Gamma$.

Definition 3.1. A CIFBM operator is a map CIFBM$^{p,q}_{\Gamma} : \Gamma^n \rightarrow \Gamma$ defined by

$$\text{CIFBM}^{p,q}_{\Gamma}(H_1, H_2, \ldots, H_n) = \left( \frac{1}{n(n-1)} \left( \bigoplus_{t,s=1, t \neq s}^{n} (H_p^t \otimes H_q^s) \right)^{\frac{1}{p+q}} \right)$$

where $p, q > 0$ are real numbers.

Theorem 3.2. For "$n" CIFNs $H_t = ((r_t, w_{r_t}), (k_t, w_{k_t}))$, the value obtained by Definition 3.1 is also CIFN and given by

$$\text{CIFBM}^{p,q}_{\Gamma}(H_1, H_2, \ldots, H_n) = ((R, w_R), (K, w_K))$$

where $R = \left( 1 - \prod_{t,s=1, t \neq s}^{n} (1 - r_t^p r_s^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$, 

$K = 1 - \left( 1 - \prod_{t,s=1, t \neq s}^{n} (1 - (1 - k_t^p (1 - k_s)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$, 

$w_R = \left( 1 - \prod_{t,s=1, t \neq s}^{n} (1 - w_{r_t}^p w_{r_s}^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$, 

and 

$w_K = 1 - \left( 1 - \prod_{t,s=1, t \neq s}^{n} (1 - (1 - w_{k_t}^p (1 - w_{k_s})^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$

Proof. It is given in Appendix.

Remark 1. If we set $w_{r_t}, w_{k_t} = 0$ for $t = 1, 2, \ldots, n$ then proposed CIFBM operator becomes IFBM operator [42]. Therefore, the proposed CIFBM is an extension of existing IFBM operator.

The demonstration of the above operator is given with a numerical example as below.
Example 3.1. Consider an expert who wants to buy a new model car by keeping
given three major attributes namely mileage ($\mathcal{H}_1$), safety ($\mathcal{H}_2$) and reliability ($\mathcal{H}_3$). To access it, an expert gives their preferences towards the assessment of a car by considering the model of the car as well as its production date. This two kinds of information are noted with the help of CIFNs by an expert and their preferences are noted as $\mathcal{H}_1 = ((0.6, 0.3), (0.1, 0.2)), \mathcal{H}_2 = ((0.5, 0.4), (0.2, 0.3)), \mathcal{H}_3 = ((0.7, 0.5), (0.1, 0.1))$. Since each attribute depends on each other and hence applying the proposed BM AOs corresponding to the values of $p = q = 1$ (for simplicity of calculations), we have

$$
\prod_{t,s=1}^{3} (1 - r_t r_s) \prod_{p,q=1}^{k} (1 - w_r w_s) = \left(1 - 0.6 \times 0.5\right)^\frac{1}{2} \times \left(1 - 0.6 \times 0.7\right)^\frac{1}{2} \times \left(1 - 0.5 \times 0.6\right)^\frac{1}{2}
\times \left(1 - 0.5 \times 0.7\right)^\frac{1}{2} \times \left(1 - 0.7 \times 0.6\right)^\frac{1}{2} \times \left(1 - 0.7 \times 0.5\right)^\frac{1}{2}
= 0.6414
$$

Similarly, we get

$$
\prod_{t,s=1}^{3} \left(1 - (1 - k_t)(1 - k_s)\right) = 0.2460,
\prod_{p,q=1}^{k} \left(1 - w_r w_s\right) = 0.8427,
\prod_{t,s=1}^{3} \left(1 - (1 - w_t)(1 - w_s)\right) = 0.3572.
$$

Thus,

$$
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)
= \left(\left(1 - 0.6414\right)\left(1 - 0.8427\right), 1 - 0.2460, 1 - 0.3572\right)
= \left((0.5988, 0.3966), (0.1317, 0.1983)\right)
$$

which is again CIFN. This aggregated result suggests that the selected model of the car under the given three attributes is acceptable with the feature of the model is 59.88\% while rejection is 39.66\%. Similarly, the aggregated observation has been obtained corresponding to the feature of the production date.

From the proposed CIFBM, it is concluded that it satisfy certain properties.

Property 3.1. (Idempotency:) For $\mathcal{H}_t = \mathcal{H}_0$ for all $t$, we have

$$
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2 \ldots \mathcal{H}_n) = \mathcal{H}_0
$$

Proof. Let $\mathcal{H}_t = \mathcal{H}_0$ for all $t$. Then, by using Theorem 3.2, we get

$$
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)
= \left(\frac{1}{n(n - 1)} \left(\bigoplus_{t,s=1}^{n} \left(\mathcal{H}_0^p \otimes \mathcal{H}_0^q\right)\right)\right)\n= \left(\frac{1}{n(n - 1)} \left(\bigoplus_{t,s=1}^{n} \mathcal{H}_0^{p+q}\right)\right)\n= \left(\mathcal{H}_0^{p+q}\right)\n= \mathcal{H}_0.
$$

Hence the result. \qed
Property 3.2. (Monotonicity:) For any two collection of CIFNs $\mathcal{H}_t = \left((r_{H_t}, w_{r_{H_t}}), (k_{H_t}, w_{k_{H_t}})\right)$ and $\mathcal{V}_t = \left((r_{V_t}, w_{r_{V_t}}), (k_{V_t}, w_{k_{V_t}})\right)$ satisfying $r_{H_t} \leq r_{V_t}$, $k_{H_t} \geq k_{V_t}$, $w_{r_{H_t}} \leq w_{r_{V_t}}$ and $w_{k_{H_t}} \geq w_{k_{V_t}} \forall t$, we have
\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) \subseteq \text{CIFBM}^{p,q}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n).
\]

Proof. See Appendix.

Property 3.3. (Boundedness:) For CIFNs $\mathcal{H}_t = \left((r_t, w_{r_t}), (k_t, w_{k_t})\right)$, ($t = 1, 2, \ldots, n$), we have
\[
\mathcal{H}^{-} \subseteq \text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2 \ldots \mathcal{H}_n) \subseteq \mathcal{H}^{+}
\]
where $\mathcal{H}^{-} = \left((\min_t r_t), (\min_t w_{r_t}), (\max_t k_t), (\max_t w_{k_t})\right)$ and $\mathcal{H}^{+} = \left((\max_t r_t), (\max_t w_{r_t}), (\min_t k_t), (\min_t w_{k_t})\right)$.

Proof. By definition of $\mathcal{H}^{-}$ and $\mathcal{H}^{+}$, we get $\mathcal{H}^{-} \subseteq \mathcal{H}_t \subseteq \mathcal{H}^{+} \forall t$. Further, by Properties 3.1 & 3.2, we have
\[
\text{CIFBM}^{p,q}(\mathcal{H}^{-}, \mathcal{H}^{-}, \ldots, \mathcal{H}^{-}) \subseteq \text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2 \ldots \mathcal{H}_n) \subseteq \text{CIFBM}^{p,q}(\mathcal{H}^{+}, \mathcal{H}^{+}, \ldots, \mathcal{H}^{+}).
\]
which implies that
\[
\mathcal{H}^{-} \subseteq \text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2 \ldots \mathcal{H}_n) \subseteq \mathcal{H}^{+}
\]

Property 3.4. (Commutativity:) For a collection of CIFNs $\mathcal{H}_t$ and its arrangement $\mathcal{H}_t^{\prime}$, we have
\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \text{CIFBM}^{p,q}(\mathcal{H}_1^{\prime}, \mathcal{H}_2^{\prime}, \ldots, \mathcal{H}_n^{\prime})
\]

Proof. For any arrangement $\mathcal{H}_t$ of $\mathcal{H}_t$, we have
\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{t,s=1}^{n} \left(\mathcal{H}_t^{p} \otimes \mathcal{H}_s^{q}\right)\right)\right)^{\frac{1}{p+q}}
\]
\[
= \left(\frac{1}{n(n-1)} \left(\bigoplus_{t,s=1}^{n} \left(\mathcal{H}_t^{p} \otimes \mathcal{H}_s^{q}\right)\right)\right)^{\frac{1}{p+q}}
\]
\[
= \text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)
\]
Hence the result.

By varying $p$ and $q$ in the proposed operator, we obtain some interesting cases, which are stated as below.
1) When \( q \rightarrow 0 \), Eq. (12) reduces to generalized CIF mean as:

\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \lim_{q \rightarrow 0} \left( \left( \prod_{t=1}^{n} (1 - r_t^{q})^{\frac{n}{(n-1)}} \right)^{\frac{1}{p}} \right)
\]

\[
= \left( \frac{1}{n} \bigoplus_{t=1}^{n} \mathcal{H}_t^2 \right)^{\frac{1}{p}}
\]

2) When \( p = 2 \) and \( q \rightarrow 0 \), Eq. (12) reduces to

\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \left( \prod_{t=1}^{n} (1 - r_t^{q})^{\frac{1}{n}} \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{1}{n} \bigoplus_{t=1}^{n} \mathcal{H}_t^2 \right)^{\frac{1}{2}}
\]

3) When \( p = 1 \) and \( q \rightarrow 0 \), Eq. (12) becomes

\[
\text{CIFBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \left( \prod_{t=1}^{n} (1 - r_t) \right)^{\frac{1}{n}}
\]

\[
= \left( \prod_{t=1}^{n} (1 - w_t) \right)^{\frac{1}{n}}
\]
When \( p = q = 1 \), Eq. (12) reduces to

\[
\text{CIFBM}^{p,q}(H_1, H_2, \ldots, H_n)
= \left( \left( \prod_{t,s=1 \atop t \neq s}^n (1 - r_{ts})^{-\frac{1}{2}} \right) \prod_{t=1}^n (1 - \left( 1 - k_t \right) \left( 1 - k_s \right))^{-\frac{1}{2}} \right)
\]

However, from the above stated CIFBM operator, it is noted that this operator cannot consider the importance factor of the different attributes into the analysis. Thus, this operator is limited in access in a modern decision-making problems. To consider the attribute weights during the aggregation also, we will propose CIFWBM operator.

**Definition 3.3.** Let \( H_t = (r_t, w_{r_t}), (k_t, w_{k_t}) \) be a set of \( n \) CIFNs with \( p, q \geq 0 \), then the CIFWBM operator, which is a mapping from \( \text{CIFWBM}^{p,q} : \Gamma^n \rightarrow \Gamma \), is defined as

\[
\text{CIFWBM}^{p,q}(H_1, H_2, \ldots, H_n) = \left( \left( \prod_{t,s=1 \atop t \neq s}^n (1 - r_{ts})^{-\frac{1}{2}} \right) \prod_{t=1}^n (1 - \left( 1 - k_t \right) \left( 1 - k_s \right))^{-\frac{1}{2}} \right) \left( \prod_{t=1}^n (1 - \left( 1 - w_{r_t} \right) \left( 1 - w_{k_t} \right))^{-\frac{1}{2}} \right)
\]

where \( \xi_t > 0 \) is the weighted vector of the different attributes which satisfies \( \sum_{t=1}^n \xi_t = 1 \).

Based on the operational laws of the given CIFNs, we can derive the aggregated results from Eq. (13) as follows.
Theorem 3.4. The aggregated value acquired from Eq. (13) for \( n \) CIFNs \( \mathcal{H}_t = ((r_t, w_{r_t}), (k_t, w_{k_t})) \), is still a CIFN and can be represented as

\[
\text{CIFWBM}^{p,q}(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)
\]

\[
= \left(1 - \left(1 - \sum_{t,s=1}^{n} \left(1 - (1 - r_t)\xi_t^p (1 - r_s)\xi_s^q\right)\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)
\]

Remark 2. It has been observed that if we set the phase angle corresponding to each CIFN zero, i.e., \( w_{r_t}, w_{k_t} = 0 \) for \( t = 1, 2, \ldots, n \) then, proposed CIFWBM operator reduces to Weighted IF Bonferroni mean(WIFBM) operator [42]. Therefore, the proposed CIFWBM operator is more generalized than the prevailing WIFBM operator.

4. Proposed MCDM method based on the operator. This section deals with a new method to solve the MCDM method based on the proposed operator.

4.1. Proposed method. Consider a MCDM problem which consists of \( m \) alternatives \( \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m \) evaluated under \( n \) criteria \( C_1, C_2, \ldots, C_n \). Let \( \xi_v > 0 \) and \( \sum_{v=1}^{n} \xi_v = 1 \) be the weight vector of \( C_v \). These given alternatives are evaluated by an expert under CIFS environment and give their preferences in terms of CIFNs represented as \( \mathcal{H}_{uv} = ((r_{uv}, w_{r_{uv}}), (k_{uv}, w_{k_{uv}})) \) where \( u = 1, 2, \ldots, m \) and \( v = 1, 2, \ldots n \) with \( 0 \leq r_{uv}, k_{uv} \leq 1 \) such that \( 0 \leq r_{uv} + k_{uv} \leq 1 \) and \( 0 \leq w_{r_{uv}}, w_{k_{uv}} \leq 1 \) such that \( 0 \leq w_{r_{uv}} + w_{k_{uv}} \leq 1 \). Then, the following steps have been summarized for dealing with MCDM problems under CIF information.

Step 1: Arrange the information in terms of CIF matrix \( \mathcal{D} = (\mathcal{H}_{uv})_{m \times n} \), as follows:

\[
\begin{pmatrix}
C_1 & C_2 & \cdots & C_n \\
\mathcal{H}_1 & \mathcal{H}_{11} & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1n} \\
\mathcal{H}_2 & \mathcal{H}_{21} & \mathcal{H}_{22} & \cdots & \mathcal{H}_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{H}_m & \mathcal{H}_{m1} & \mathcal{H}_{m2} & \cdots & \mathcal{H}_{mn}
\end{pmatrix}
\]
Step 2: Aggregate the CIFNs $\mathcal{H}_{uv}$ for each alternative $\mathcal{H}_u (u = 1, 2, \ldots, m)$ into collective one $\mathcal{H}_u = ((r_u, w_{ru}), (k_u, w_{ku}))$ by using the proposed CIFWBM operator for arbitrary real numbers $p, q$.

Step 3: For the computed number $\mathcal{H}_u$, compute the score values as:

$$S(\mathcal{H}_u) = r_u - k_u + w_{ru} - w_{ku}.$$  \hspace{1cm} (15)

If for arbitrary two indices, score values are equal then compute the accuracy values as

$$H(\mathcal{H}_u) = r_u + k_u + w_{ru} + w_{ku}.$$ \hspace{1cm} (16)

Step 4: Applying the Definition 2.7 to rank $\mathcal{H}_u$.

4.2. Illustrative example. For demonstrating the applications of the proposed methodology, an example is illustrated as follows:

The center for development of telematics (C-DOT) was established in August 1984 as an autonomous body. Its goal was to develop telecommunication technology to meet the needs of the Indian telecommunication network. It has offices in Delhi, Banglore and Kolkata. In order to carry out senior officers’ transport requirements at Delhi, C-DOT consults a car company X to purchase cars. The car company provides information to C-DOT regarding five models of cars $\mathcal{H}_u (u = 1, 2, \ldots, 5)$ with different production dates for each model of car. C-DOT constitutes a committee of experts in order to evaluate car models and select the most optimal one. The committee of experts evaluate the model of cars on the basis of four criteria namely: $C_1$: Reliability, $C_2$: Speed, $C_3$: Durability and $C_4$: Safety. Obviously, these criteria would be affected with the changes in production date. The target of the C-DOT is to choose the most optimal model of car and the production date simultaneously. Thus the problem is two dimensional namely, model of car and production date of car. Therefore, the committee of the experts give their preferences corresponding to each alternative in terms of CIFNs as the CIF model handles two-dimensional information simultaneously. The rating values of the committee for $\mathcal{H}_1$ at $C_1$ are given as $((0.7, 0.9), (0.1, 0.1))$ which describes that the committee of the experts is 70% agreed with the suitability of $\mathcal{H}_1$ at $C_1$ and 10% disagrees. The phase term that represents the production date of car is given as: the expert is 90% satisfied with the suitability of production date of car at $C_1$ and 10% is not satisfied. In the similar manner, all data of Table 2 can be interpreted. The weight vectors corresponding to four preferences factors is $\xi = (0.4, 0.25, 0.15, 0.2)^T$. For fulfilling the required purpose, by applying proposed method, the main steps are given as:

Step 1: The committee of the experts has given their preferences to each alternative of car with respect to the four major criteria as CIFNs which are given in Table 2.

Step 2: By taking $p = q = 1$ and then, utilizing weighted BM, as given in Eq. (14), we obtain their corresponding results as

$\mathcal{H}_1 = ((0.2588, 0.2546), (0.6555, 0.6880)),$

$\mathcal{H}_2 = ((0.1801, 0.2625), (0.7171, 0.6649)),$

$\mathcal{H}_3 = ((0.1465, 0.1664), (0.7784, 0.7689)),$

$\mathcal{H}_4 = ((0.1730, 0.1743), (0.7484, 0.7179)),$

$\mathcal{H}_5 = ((0.3135, 0.2571), (0.6271, 0.6287)).$
Table 2. Input information in the form of the complex intuitionistic fuzzy decision-matrix

|   | $C_1$                  | $C_2$                  | $C_3$                  | $C_4$                  |
|---|------------------------|------------------------|------------------------|------------------------|
| $H_1$ | ((0.7,0.9),(0.1,0.1)) | ((0.8,0.5),(0.1,0.4)) | ((0.6,0.6),(0.3,0.2)) | ((0.7,0.7),(0.3,0.2)) |
| $H_2$ | ((0.7,0.6),(0.3,0.3)) | ((0.4,0.9),(0.2,0.1)) | ((0.7,0.7),(0.2,0.3)) | ((0.4,0.6),(0.3,0.1)) |
| $H_3$ | ((0.3,0.4),(0.6,0.4)) | ((0.6,0.6),(0.3,0.4)) | ((0.3,0.4),(0.5,0.6)) | ((0.7,0.7),(0.1,0.1)) |
| $H_4$ | ((0.4,0.8),(0.5,0.1)) | ((0.7,0.3),(0.3,0.3)) | ((0.6,0.5),(0.1,0.3)) | ((0.5,0.5),(0.3,0.4)) |
| $H_5$ | ((0.9,0.7),(0.1,0.2)) | ((0.7,0.7),(0.2,0.1)) | ((0.7,0.6),(0.2,0.2)) | ((0.8,0.8),(0.1,0.1)) |

Step 3: By Eq. (15), we get $S(H_1) = -0.8301$, $S(H_2) = -0.9395$, $S(H_3) = -1.2343$, $S(H_4) = -1.1190$ and $S(H_5) = -0.6852$.

Step 4: By Definition 2.7, we get ordering position as: $H_5 > H_1 > H_2 > H_4 > H_3$.

Thus, the best alternative is $H_5$.

4.3. Impact of change in $p$, $q$ values. As the above analysis is done for fixed $p$ and $q$, however, the detailed analysis on the impact of the parameters by varying the parameters $p$ and $q$ are given in Table 3. Furthermore, the impact of these individual parameters are analyzed and their variations are shown in Figure 1. It is seen from Fig. 1(a) that corresponding to $q = 1$ and for $p < 5.55$, the ordering of the alternatives becomes $H_5 > H_1 > H_2 > H_4 > H_3$ while it becomes $H_4 > H_5 > H_2 > H_1 > H_3$ when $p > 5.55$. At $p = 5.55$, we get $S(H_1) = S(H_5) = -0.4264$ and $H(H_1) = 1.8380$ and $H(H_5) = 1.8689$. Thus, the complete ordering is $H_5 > H_1 > H_2 > H_4 > H_3$ for $p = 5.55$. Similarly, for different values of $q$, we get the ordering position of the alternative which is compiled in Table 4. Further, fixing the one parameter and ranging the other will exhibits optimism and pessimism choices to the decision-makers. That is, for optimism decisions, a personage can prefer a larger value to $p$ while a smaller value to $p$ for pessimistic decision. Hence, for several choices of these parameters, the decision-maker can pick the nature of the problem.

Table 3. Ranking on changing values of $p$ and $q$

| Values of $p$ and $q$ | Score values | Ranking |
|-----------------------|--------------|---------|
|                       | $H_1$        | $H_2$   | $H_3$   | $H_4$   | $H_5$   |
| $p = 1 ; q = 1$       | -0.8301      | -0.9395 | -1.2343 | -1.1190 | -0.6852 |
| $p = 1 ; q = 2$       | -0.7549      | -0.9000 | -1.1946 | -1.0670 | -0.6387 |
| $p = 2 ; q = 2$       | -0.7566      | -0.8957 | -1.1800 | -1.0779 | -0.6358 |
| $p = 2 ; q = 3$       | -0.7017      | -0.8657 | -1.1530 | -1.0419 | -0.6009 |
| $p = 3.5 ; q = 0.1$   | -0.4369      | -0.7490 | -1.0863 | -0.8261 | -0.4493 |
| $p = 4 ; q = 0.1$     | -0.3898      | -0.7234 | -1.0694 | -0.7911 | -0.4178 |
| $p = 5 ; q = 0.5$     | -0.3927      | -0.7177 | -1.0684 | -0.7962 | -0.4108 |
| $p = 6 ; q = 1$       | -0.3966      | -0.7141 | -1.0639 | -0.8049 | -0.4050 |

On the other hand, by practicing CFWBM operator and concurrently altering the parameters $p$ and $q$ from 1 to 10, the impact on score values are examined.
through a surface plot given in Figure 2. Also, it is mentioned that by adjusting the values of \( p \) and \( q \), the optimal alternative may vary between \( \mathcal{H}_1 \) or \( \mathcal{H}_5 \) while the worst one remains \( \mathcal{H}_3 \). However, when \( p = q \), ordering position of alternatives is constantly alike which is \( \mathcal{H}_5 \succ \mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \) and contrarily, the ranking order of the alternatives \( \mathcal{H}_2, \mathcal{H}_3 \) and \( \mathcal{H}_4 \) remains same.

4.4. **Validity test.** On implementing the different MCDM approaches to the same DMPs, we may receive different results due to which there appears uncertainty regarding MCDM methods. To discuss the validity of proposed MCDM method, we inquired the criteria given by Wang and Triantaphyllo [40], as:

**Test 1:** “An MCDM method is effective if on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision-criteria, the indication of the best alternative remains same.”

**Test 2:** “An effective MCDM method should follow transitive property.”

**Test 3:** “An MCDM method is effective if on decomposing the MCDM problem into smaller problems and by applying the same MCDM method to these sub-problems for ranking the alternatives, the combined ranking of the alternatives remains same to the ranking of the original problem.”

**Figure 1.** Variation in score values with parameter \( p \) by fixing \( q \)
These criteria are tested over the proposed method as follows.

4.4.1. Test with criterion 1. In this, we select the arbitrary worst alternative $H_3$ and replace their values with new worse alternative $H'_3$ as $H'_3 = \{(0.1, 0.3), (0.7, 0.5), (0.3, 0.3), (0.5, 0.6), (0.2, 0.2), (0.6, 0.8), (0.5, 0.4), (0.2, 0.4)\}$. Based on this updated information, we implemented the proposed method and score values obtained as $S(H_3) = -0.8301$, $S(H'_3) = -0.9395$, $S(H_3) = -1.5578$, $S(H'_3) = -1.1190$ and $S(H'_3) = -0.6852$. Thus, ordering is $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H'_3$ which gives that $H_5$ is still the best alternative. Hence, the proposed method validate this criterion.

4.4.2. Test with other criteria. Under it, we split the problem into smaller parts which consists of $\{H_1, H_2, H_3, H_4 \}$, $\{H_3, H_4, H_5, H_1\}$, $\{H_4, H_5, H_1, H_2\}$ and $\{H_2, H_3, H_4, H_5\}$ alternatives. by implementing the proposed method over each sub-problems and get the ordering corresponding to them as $H_1 \succ H_2 \succ H_4 \succ H_3$, $H_5 \succ H_1 \succ H_4 \succ H_3$, $H_5 \succ H_1 \succ H_2 \succ H_4$ and $H_5 \succ H_2 \succ H_4 \succ H_3$ respectively. Thus, by combining them, we get $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ which is identical to original one. Thus, the proposed method also satisfy criteria 2 and 3.

4.5. Comparative study. To check and explain the performance of the proposed method with several of the existing studies under CIFS [3, 19, 20, 36, 37] and IFS [8, 10, 11, 12, 23, 24, 25, 39, 41, 42, 43, 44, 48, 50] environment, an experiment is conducted as follows.

4.5.1. Under CIFS environment. In this section, we have analyzed the appearance of the proposed results with all those comparisons existing under the CIFS environment. These existing theories are based on the distance measure [3, 36], power AOs [37], correlation coefficients [20] and weighted averaging AO [19] and used them on to the considered data. During the accomplishment of the approaches [3, 36], we build a reference set named as a positive ideal alternative (PIA) as $\mathcal{H}^+ = \{H_1^+, H_2^+, \ldots, H_n^+\}$ where $H_k^+ = (\max_u \{r_{uv}\}, \max_u \{w_{rv}\}, \min_u \{w_{ku}\})$. The outcomes communicating with these existing approaches along

| Figure 1(a) | 5.55 | $H(H_1) = 1.8380$, $H(H_2) = 1.8689$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Figure 1(b) | 1.593 | $H(H_1) = 1.8307$, $H(H_2) = 1.8686$ | $H_1 \succ H_5 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Figure 1(c) | 2.93 | $H(H_1) = 1.8191$, $H(H_2) = 1.8656$ | $H_3 \succ H_5 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |

Table 4. Analysis of the Figures 1(a), 1(b), 1(c) and 1(d)

| Value of $\varphi$ | Accuracy for $p = \varphi$ | Ranking of the alternatives |
|---|---|---|
| When $p < \varphi$ | When $p = \varphi$ | When $p > \varphi$ |
| Figure 1(a) | 5.55 | $H(H_1) = 1.8380$, $H(H_2) = 1.8689$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Figure 1(b) | 1.593 | $H(H_1) = 1.8307$, $H(H_2) = 1.8686$ | $H_1 \succ H_5 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Figure 1(c) | 2.93 | $H(H_1) = 1.8191$, $H(H_2) = 1.8656$ | $H_3 \succ H_5 \succ H_2 \succ H_4 \succ H_3$ | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
with the suggested method are summed in Table 5. From this examination, we presume the following information.

1) The results measured through the existing distance measures [3, 36] on to the supposed data are based on the reference set $H^+$. Based on it, we presume that the ranking order of the given alternatives is $H_5 > H_1 > H_2 > H_4 > H_3$ and it harmonizes with the present reported results. Since the best alternative persists
the same, but the computational rule of the proposed method with these methods are fully different. In such existing methods [3, 36], authors have employed the information without interacting the information with each other. Further, in it, the optimality is measured with the help of PIA, without considering the negative ideal alternative (NIA). However, it is completely sensible that larger the distance from NIA and smaller from PIA would be the excellent one and hence both PIA and NIA should together be considered into the analysis. Thus, such existing approaches are sometimes weak to make a decision when there is an impact of NIA also on the analysis. An analog to these, in the proposed operator, we consider the interacting between the pairs of the attributes and hence it is more stable.

2) By implementing the existing power AOs [37] on to the considered data, the overall score value of the given alternatives five alternatives \( \mathcal{H} \)'s are obtained as 1.1449, 0.8829, 0.3540, 0.6432, 1.2504. Thus, the best alternative is again \( \mathcal{H}_5 \) and it coincides with the presented result. The existing AOs [37] are based on the t-norm operations by taking \( t(x) = -\log(x) \) for \( 0 < x \leq 1 \) and it does not consider the interaction among the arguments. Apart from it, the importance of each attribute is determined with the support function to interact between the argument. However, in the present study, during the aggregation, each argument is aggregated by considering the interaction between the pairs of the argument.

3) By using the existing correlation coefficient measure [20] on to the considered data, we collect the ranking order of the given alternative as \( \mathcal{H}_5 \succ \mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \) and hence the best alternatives is still \( \mathcal{H}_5 \). In this method, the rating of each alternative is estimated form its ideal set \( \mathcal{H}^+ \). However, in the modern decision-making problem, it is sometimes tough to construct such an ideal setting or different decision-makers’ may have their own reference set. In such cases, the ranking order computed by using the correlation coefficient may alter their decision. However, in the presented operators, there is no role of this reference set and hence it is more generalized and reliable to decide the problems.

4) The CIFWA operator [19] is performed on the given information and received the same ranking order with the proposed one. But it is remarkable to note that the CIFWA operator is a special case of the proposed one by setting \( p = 1 \) and \( q = 0 \). In other words, we can assume that if we set \( p = 1 \) and \( q = 0 \) in the proposed CIFWBM operator then it reduces to CIFWA operator [19]. Further, from the composition of the CIFWA operator, it is observed that the information is aggregated by considering the independent nature of the argument. Thus, their method is limited in nature and can’t apply to those cases, where the different factors are dependent on nature. Finally, from the ranking order of the proposed results, it is verified that the presented work is applicable under both the cases - dependent or independent arguments. Also, it is given from the structure of the proposed CIFWBM operator that by varying the parameters \( p \) and \( q \), which reflect the attitude character towards the decision-making process, a decision-maker may have various choices to select their best alternative(s). Hence, the presented approach is more reliable and valid.

4.5.2. Under IFS environment. As IFS is the special case of the considered CIFS, so in this section, we have examined the performance of the presented results with the several existing results [8, 10, 11, 12, 23, 24, 25, 39, 41, 42, 43, 44, 48, 50] under the IFS environment. If we use the phase term of each component of CIFS as zero
Table 5. Comparative Analysis results with CIFS studies

| Method used                                      | Score values | Ranking                      |
|-------------------------------------------------|--------------|------------------------------|
| Method based on CIFWA operator [19]              |              | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Method based on CIFWPA operator [37]             |              | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Method based on Distance measure [3]             |              | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Method based on Euclidean distance measure [36]  |              | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Method based on Correlation coefficient [20]     |              | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Proposed method with $(p = 1; q = 1)$            | -0.8301, -0.9395, -1.2343, -1.1190, -0.6852 | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Proposed method with $(p = 1; q = 10)$           | -0.2174, -0.6143, -0.9993, -0.6744, -0.2686 | $H_5 > H_1 > H_2 > H_4 > H_3$ |
| Proposed method with $(p = 1; q = 0)$            | -0.6921, -0.8863, -1.1942, -1.0262, -0.6041 | $H_5 > H_1 > H_2 > H_4 > H_3$ |

Used: $t(a) = -\log(a)$ for $0 < a \leq 1$ with $t(0) = \infty$ in [19], $\alpha_1 = \beta_1 = \sigma_1 = \alpha_2 = \beta_2 = \sigma_2 = \frac{1}{3}$ in [3]

Then obviously the CIFS reduces to IFS. Under this position, we have performed the existing approaches namely, IFWBM operator [42], IFPWA operator [41], IFEWA operator [39] and other averaging or geometric operators [8, 10, 11, 12, 23, 24, 25, 43, 44, 48, 50] on to the given data set and the effects corresponding to it have been abstracted in Table 6.

The following information is noted from this table which is summarized as below.

1) The existing AOs based on the different Archimedean t-norm given in [8, 10, 11, 12, 23, 24, 25, 43, 44, 48, 50] gives the same results as that of the proposed ones. But in these operators, there is no evidence of the dependency factor among the attributes during the process. However, by inserting the values of parameters $p = 1$ and $q = 0$ or $q = 1$ and $p = 0$ in the proposed operators then we can easily obtain such AOs. Thus, all such existing AOs are the special cases of the presented one. Further, these AOs are restrained to the domain of the one-dimensional and hence don’t implemented on the problem of the two-dimensional problems. So, there is a wide scope and applicability of the proposed operators as compared to such existing operators.

2) By implementing the existing BM operator [42] under IFS restriction, as given in Eq. (10), to the given information then we get the overall score values of the alternatives as $-0.3968$, $-0.5370$, $-0.6319$, $-0.5754$ and $-0.3136$ for $H$’s with $p = q = 1$. The final ranking obtained through it is $H_5 > H_1 > H_2 > H_4 > H_3$ and the best alternative remains the same. However, the advantages of the proposed BM operator over the IFWBM are that the existing one considers only the one-dimensional information and hence there is an adequate loss of information during the process. On the other hand, the presented operator can check the CIFS as well as IFS information and hence we conclude that the existing IFWBM operator is one of the special cases of the proposed CIFWBM operator.

3) If we perform IFPWA operator [41] to the considered information under the IFS restriction then their results are still the same as the proposed one. But the computational procedure to achieve the best alternative is different. For instance, the IFPWA operator aggregates the information with the help of support function and considering the independency nature between the argument. On the
Table 6. Comparative Analysis results with IFS studies

| Method used | Score values | Ranking |
|-------------|--------------|---------|
| Xu and Yager [42] method based on IFWBM operator ($p = 1; q = 1$) | -0.3968 -0.5370 -0.6319 -0.5754 -0.3136 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Xu [41] method based on IFPWA operator | 0.5653 0.3332 0.1484 0.2441 0.6839 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Wang and Liu [29] method based on IFFWA operator | 0.5670 0.3276 0.1183 0.2181 0.6871 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Xu and Yager [44] method based on IFWG operator | 0.5314 0.2826 -0.0179 0.1466 0.6536 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Xu [43] method based on IFWA operator | 0.5701 0.3351 0.1432 0.2301 0.6898 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Garg [10] method based on IFEWGIA operator | 0.6563 0.4787 0.0142 0.2849 0.7193 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| He et al. [24] method based on IFGIA method | 0.6844 0.4768 -0.0085 0.2849 0.7193 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Huang [25] method based on IFHWA operator | 0.5658 0.3241 0.1064 0.2127 0.6860 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Garg [11] method | 0.4307 0.1603 0.0710 0.0594 0.6375 | $H_5 \succ H_1 \succ H_2 \succ H_3 \succ H_4$ |
| Chen and Chang [8] method | 0.4339 0.1804 0.1064 0.0845 0.6435 | $H_5 \succ H_1 \succ H_2 \succ H_3 \succ H_4$ |
| Goyal et al. [23] method | 0.7882 0.6623 0.3109 0.4510 0.8604 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Garg [12] method | 0.4316 0.1669 0.0809 0.0743 0.6392 | $H_5 \succ H_1 \succ H_2 \succ H_3 \succ H_4$ |
| Ye [48] method | 0.5506 0.3084 0.0596 0.1876 0.6715 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| Zhou and Xu [50] method | 0.5668 0.3924 0.3288 0.3776 0.6079 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| The proposed CIFWBM operator ($p = 1; q = 1$) | -1.3968 -1.5370 -1.6319 -1.5754 -1.3136 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |
| The proposed CIFWBM operator ($p = 1; q = 0$) | -1.3485 -1.5108 -1.6119 -1.5700 -1.2525 | $H_5 \succ H_1 \succ H_2 \succ H_4 \succ H_3$ |

Abbreviations. IFWA: Intuitionistic fuzzy weighted averaging; IFWG: Intuitionistic fuzzy weighted geometric; IFPWA: intuitionistic fuzzy power weighted averaging; IFWWBM: intuitionistic fuzzy weighted Bonferroni mean; IFGIA: intuitionistic fuzzy geometric interactive averaging; IFEWGIA: intuitionistic fuzzy Einstein weighted geometric interactive averaging; IFHWA: intuitionistic fuzzy Hamacher weighted averaging; CIFWBM: complex intuitionistic fuzzy weighted Bonferroni mean.

other hand, the presented operator considers the direct interaction between the pairs of the argument during the analysis with the parameters $p$ and $q$.

Therefore, based on the above comparative analysis, we can conclude that the presented operator is more generalized and valid as compared to the several existing operators under the CIFS as well as IFS restriction.

In addition to the above discussion and the comparative analysis, we present some salient features of the proposed operator with the several existing ones [3, 19, 36, 37, 39, 42, 41] in terms of their characteristic comparison. The results calculated from this measurement are tabulated in Table 7, which explains that the proposed MCDM method and hence operators are more beneficial than the others. In other words, we can say that the novel proposed method is more flexible and suitable to solve the decision-making problems and hence the proposed method outperforms over the existing methods.

5. Conclusion. In this paper, a concept of Bonferroni mean is granted for the CIFS environment to fuse the information and hence impersonate a decision-making approach to solve the MCDM problems. CIFS is an expansion of the IFS where the
range of the membership functions is extended from real set to the complex plane over the unit disc. The benefit of the set is to add the phase terms for representing the periodicity of the data. Keeping these advantages in mind, the study offers some new weighted averaging operator named as CIFBM and CIFWBM for CIFNs. The various features and the special cases of its are considered by varying the parameters $p, q$ associated with the operators. To deal with more control over the study, a new MCDM method based on the proposed approach is presented and described with a numerical example. The applicability, as well as the superiority of the method, is explained with a comparative study as well as a validity test. Also, by varying the parameters $p, q$ associated with the method can give the various choices to the decision-makers to select their optimal task. In future research, we shall examine to solve some other problems such as supplier selection, risk assessment, etc. Besides, we shall provide some more generalized interactive AOs by utilizing the diverse fuzzy environment [4, 26, 29].

**Appendix.** Proof of Theorem 3.2:

**Proof.** Using the Definition 2.6, it clearly follows that, the value acquired after applying CIFBM operator remains CIFN. Now, we will show that Eq. (12) holds. For positive real numbers $p, q$ and CIFNs $H_t$, we have:

$$H_t^p = \left( (r_t^p, w_t^p), (1 - k_t^p, 1 - w_{k_t}^p) \right)$$

$$H_s^q = \left( (r_s^q, w_s^q), (1 - k_s^q, 1 - w_{k_s}^q) \right)$$

and

$$H_t^p \otimes H_s^q = \left( (r_t^p r_s^q, w_t^p w_s^q), \frac{1 - (1 - k_t^p)(1 - k_s^q)}{1 - (1 - w_{k_t}^p)(1 - w_{k_s}^q)} \right)$$

In order to prove Eq. (12), firstly we shall prove the following Eq. (17)

$$\oplus_{\alpha, \beta = 1}^n \left( H_t^p \otimes H_s^q \right) = \left( \prod_{\alpha, \beta = 1}^n (1 - r_t^\alpha r_s^\beta), \prod_{\alpha, \beta = 1}^n \left( \frac{1 - (1 - k_t^\alpha)(1 - k_s^\beta)}{1 - (1 - w_{k_t}^\alpha)(1 - w_{k_s}^\beta)} \right) \right)$$

(17)

by principle of mathematical induction on $n$. 

---

**Table 7. The characteristic comparison of different approaches**

| Method | Captures interrelationship among arguments | Ability to capture information using complex numbers | Ability to handle two-dimensional information | Ability to integrate Information | Flexible according to decision-maker’s preferences |
|--------|------------------------------------------|-----------------------------------------------------|-----------------------------------------------|---------------------------------|-----------------------------------------------|
| In [37] | × | ✓ | ✓ | ✓ | × |
| In [19] | × | ✓ | ✓ | ✓ | √ |
| In [3] | × | ✓ | ✓ | × | ✓ |
| In [36] | × | ✓ | ✓ | × | √ |
| In [42] | ✓ | × | ✓ | ✓ | ✓ |
| In [41] | × | × | × | ✓ | × |
| In [39] | × | × | × | ✓ | ✓ |
| The proposed approach | ✓ | ✓ | ✓ | ✓ | ✓ |
1) For $n = 2$, we have
\[
\bigoplus_{t,s=1 \atop t \neq s}^2 \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) \\
= (\mathcal{H}_1^p \otimes \mathcal{H}_2^q) \oplus (\mathcal{H}_2^p \otimes \mathcal{H}_1^q) \\
= \left( r_1^p, w_{r_1}^p, w_{r_1}^q, 1 - (1 - k_1)^p (1 - k_2)^q, 1 - (1 - w_{k_1})^p (1 - w_{k_2})^q \right) \\
\oplus \left( r_2^p, w_{r_2}^p, w_{r_2}^q, 1 - (1 - k_2)^p (1 - k_1)^q, 1 - (1 - w_{k_2})^p (1 - w_{k_1})^q \right)
\]
\[
= \left( 1 - \sum_{t,s=1 \atop t \neq s}^2 (1 - r_t^p r_s^q), \prod_{t,s=1 \atop t \neq s}^2 (1 - (1 - k_t)^p (1 - k_s)^q), \right) \\
= \left( 1 - \sum_{t,s=1 \atop t \neq s}^2 (1 - w_t^p w_s^q), \prod_{t,s=1 \atop t \neq s}^2 (1 - (1 - w_{k_t})^p (1 - w_{k_s})^q) \right)
\]
which is true.

2) Assume Eq. (17) holds for $n = m$. Then,
\[
\bigoplus_{t,s=1 \atop t \neq s}^m \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) = \left( 1 - \prod_{t,s=1 \atop t \neq s}^m (1 - r_t^p r_s^q), \prod_{t,s=1 \atop t \neq s}^m (1 - (1 - k_t)^p (1 - k_s)^q) \right) = \left( 1 - \prod_{t,s=1 \atop t \neq s}^m (1 - w_t^p w_s^q), \prod_{t,s=1 \atop t \neq s}^m (1 - (1 - w_{k_t})^p (1 - w_{k_s})^q) \right)
\]
then for $n = m + 1$, we have
\[
\bigoplus_{t,s=1 \atop t \neq s}^{m+1} \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) \\
= \left( \bigoplus_{t,s=1 \atop t \neq s}^m \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) \right) \oplus \left( \bigoplus_{t=1}^m \left( \mathcal{H}_t^p \otimes \mathcal{H}_{m+1}^q \right) \right) \oplus \left( \bigoplus_{s=1}^m \left( \mathcal{H}_{m+1}^p \otimes \mathcal{H}_s^q \right) \right)
\]
Now, we shall prove the following Eq. (20)
\[
\bigoplus_{t=1}^m \left( \mathcal{H}_t^p \otimes \mathcal{H}_{m+1}^q \right) \\
= \left( 1 - \prod_{t=1}^m (1 - r_t^p r_{m+1}^q), \prod_{t=1}^m (1 - (1 - k_t)^p (1 - k_{m+1})^q) \right) \\
= \left( 1 - \prod_{t=1}^m (1 - w_t^p w_{m+1}^q), \prod_{t=1}^m (1 - (1 - w_{k_t})^p (1 - w_{k_{m+1}})^q) \right)
\]
by induction on $m$ as:
2a:) For $m = 2$:

\[
\bigoplus_{t=1}^{2} \left( \mathcal{H}_t^{p} \otimes \mathcal{H}_{m+1}^{q} \right)
\]

\[
= \left( \mathcal{H}_1^{p} \otimes \mathcal{H}_{m+1}^{q} \right) \oplus \left( \mathcal{H}_2^{p} \otimes \mathcal{H}_{m+1}^{q} \right)
\]

\[
= \begin{pmatrix}
\left( r_1^{p} r_{m+1}^{q} \right), & \left( \prod_{t=1}^{2} 1 - (1 - k_t)^p (1 - k_{m+1})^q \right),
\left( w_1^{p} w_{m+1}^{q} \right), & \left( \prod_{t=1}^{2} 1 - (1 - w_{k_t})^p (1 - w_{k_{m+1}})^q \right)
\end{pmatrix}
\]

\[
\oplus \begin{pmatrix}
\left( r_2^{p} r_{m+1}^{q} \right), & \left( \prod_{t=1}^{2} 1 - (1 - k_t)^p (1 - k_{m+1})^q \right),
\left( w_2^{p} w_{m+1}^{q} \right), & \left( \prod_{t=1}^{2} 1 - (1 - w_{k_t})^p (1 - w_{k_{m+1}})^q \right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\prod_{t=1}^{2} 1 - \left( r_t^{p} r_{m+1}^{q} \right), & \prod_{t=1}^{2} 1 - (1 - k_t)^p (1 - k_{m+1})^q,
\prod_{t=1}^{2} 1 - \left( w_t^{p} w_{m+1}^{q} \right), & \prod_{t=1}^{2} 1 - (1 - w_{k_t})^p (1 - w_{k_{m+1}})^q
\end{pmatrix}
\]

Thus, Eq. (20) is true when $m = 2$.

2b:) Consider that Eq. (20) is holds for $m = m_0$,

\[
\bigoplus_{t=1}^{m_0} \left( \mathcal{H}_t^{p} \otimes \mathcal{H}_{m_0+1}^{q} \right)
\]

\[
= \begin{pmatrix}
\prod_{t=1}^{m_0} 1 - \left( r_t^{p} r_{m_0+1}^{q} \right), & \prod_{t=1}^{m_0} 1 - (1 - k_t)^p (1 - k_{m_0+1})^q,
\prod_{t=1}^{m_0} 1 - \left( w_t^{p} w_{m_0+1}^{q} \right), & \prod_{t=1}^{m_0} 1 - (1 - w_{k_t})^p (1 - w_{k_{m_0+1}})^q
\end{pmatrix}
\]

then for $m = m_0 + 1$, we have

\[
\bigoplus_{t=1}^{m_0+1} \left( \mathcal{H}_t^{p} \otimes \mathcal{H}_{m_0+2}^{q} \right)
\]

\[
= \begin{pmatrix}
\prod_{t=1}^{m_0} 1 - \left( r_t^{p} r_{m_0+2}^{q} \right), & \prod_{t=1}^{m_0} 1 - (1 - k_t)^p (1 - k_{m_0+2})^q,
\prod_{t=1}^{m_0} 1 - \left( w_t^{p} w_{m_0+2}^{q} \right), & \prod_{t=1}^{m_0} 1 - (1 - w_{k_t})^p (1 - w_{k_{m_0+2}})^q
\end{pmatrix}
\]

\[
\oplus \begin{pmatrix}
\left( r_{m_0+1}^{p} r_{m_0+2}^{q} \right), & \left( \prod_{t=1}^{m_0} 1 - (1 - k_{m_0+1})^p (1 - k_{m_0+2})^q \right),
\left( w_{m_0+1}^{p} w_{m_0+2}^{q} \right), & \left( \prod_{t=1}^{m_0} 1 - (1 - w_{k_{m_0+1}})^p (1 - w_{k_{m_0+2}})^q \right)
\end{pmatrix}
\]
Thus, Eq. (17) is true, when \( n = m_0 + 1 \) and hence, Eq. (20) is true \( \forall m \).

Similarly,

\[
\bigoplus_{s=1}^{m+1} \left( \mathcal{H}_{m+1}^p \otimes \mathcal{H}_s^q \right) = \left( \prod_{t,s=1 \atop t \neq s}^m \left( 1 - r_t^p r_s^q \right), \prod_{t,s=1 \atop t \neq s}^m \left( 1 - (1 - k_t)^p (1 - k_s)^q \right) \right)
\]

Further, using the Eqs. (18), (20) and (21) in Eq. (19) we obtain:

\[
\bigoplus_{t,s=1 \atop t \neq s}^{m+1} \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) = \left( \prod_{t,s=1 \atop t \neq s}^m \left( 1 - r_t^p r_s^q \right), \prod_{t,s=1 \atop t \neq s}^m \left( 1 - (1 - k_t)^p (1 - k_s)^q \right) \right)
\]

Thus, Eq. (17) is true, when \( n = m + 1 \) and hence, Eq. (17) holds for all \( n \in \mathbb{Z}^+ \).
Now, by Definition 2.6,

$$\frac{1}{n(n-1)} \left( \bigoplus_{t,s=1 \atop t \neq s}^{n} \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) \right)$$

$$= \left( \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - r_t^p r_s^q \right) \right) \left( \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - (1-k_t)^p (1-k_s)^q \right) \right)$$

and by Definition 3.1, we get

$$\text{CIFBM}^p(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = \left( \frac{1}{n(n-1)} \left( \bigoplus_{t,s=1 \atop t \neq s}^{n} \left( \mathcal{H}_t^p \otimes \mathcal{H}_s^q \right) \right) \right)^{\frac{1}{p+q}}$$

which completes the proof. □

Proof of Property 3.2:

Proof. Let $\delta = \text{CIFBM}^p(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) = (r_s, w_r, (k_s, w_k))$ and $\beta = \text{CIFBM}^q(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n) = (r_t, w_t, (k_t, w_k))$. Since $r_{\mathcal{H}_t} \leq r_{\mathcal{V}_t}$, it implies $r_{\mathcal{H}_t} r_{\mathcal{H}_s}^q \leq r_{\mathcal{V}_t} r_{\mathcal{V}_s}^q$ for $t, s = 1, 2, \ldots, n$.

$$\Rightarrow \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - r_{\mathcal{H}_t} r_{\mathcal{H}_s}^q \right) \leq \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - r_{\mathcal{V}_t} r_{\mathcal{V}_s}^q \right)$$

$$\Rightarrow \left( 1 - \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - r_{\mathcal{H}_t} r_{\mathcal{H}_s}^q \right) \right) \geq \left( 1 - \prod_{t,s=1 \atop t \neq s}^{n} \left( 1 - r_{\mathcal{V}_t} r_{\mathcal{V}_s}^q \right) \right)$$

$$\Rightarrow r_{\delta} \leq r_{\beta}$$
Also, $k_{H_i} \geq k_{V_i}$ which gives that \((1 - k_{H_i})^p (1 - k_{H_i})^q \leq (1 - k_{V_i})^p (1 - k_{V_i})^q\) for \(t, s = 1, 2, \ldots, n\)

\[
\Rightarrow \prod_{t, s=1}^{n} \left(1 - (1 - k_{H_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \geq \prod_{t, s=1}^{n} \left(1 - (1 - k_{V_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \\
\Rightarrow \left(1 - \prod_{t, s=1}^{n} \left(1 - (1 - k_{V_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \right)^{\frac{1}{n(1-1)}} \\
\leq \left(1 - \prod_{t, s=1}^{n} \left(1 - (1 - k_{V_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \right)^{\frac{1}{n(1-1)}} \\
\Rightarrow 1 - \left(1 - \prod_{t, s=1}^{n} \left(1 - (1 - k_{V_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \right)^{\frac{1}{n(1-1)}} \\
\geq 1 - \left(1 - \prod_{t, s=1}^{n} \left(1 - (1 - k_{V_i})^p (1 - k_{V_i})^q \right)^{\frac{1}{n(1-1)}} \right)^{\frac{1}{n(1-1)}} \\
\Rightarrow k_{k} \geq k_{\beta}
\]

Similarly, we can prove that $w_{r_{\beta}} \leq w_{r_{\alpha}}$ and $w_{k_{\beta}} \geq w_{k_{\alpha}}$. Hence, by Definition 2.5, we have $\text{CIFBM}^{p,q}(H_1, \ldots, H_n) \subseteq \text{CIFBM}^{p,q}(V_1, \ldots, V_n)$.

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