Relations between growth of entire functions
and behavior of its Taylor coefficients.

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Abstract.

We derive in the closed and unimprovable form the bilateral non-asymptotic relations between growth of entire functions and decay rate at infinity of its Taylor coefficients. We investigate the functions of one as well as of several complex variables.

We will apply the convex analysis: Young - Fenchel (Legendre) transform, Young inequality, saddle - point method etc.

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1 Notations, definitions, statement of problem, previous results.

Let \( f = f(z) \) be entire (analytical) complex valued function defined on the whole complex plane:

\[
f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k = c_k[f],
\]

i.e. such that the radius of convergence of the Taylor (power) series (1) is equal to infinity:

\[
\lim_{n \to \infty} n^{1/\rho} |c_n| = 0.
\]

Recall that the so-called maximal function, or equally maximal majorant \( M_f(r) = M(r), \ r \in (0, \infty) \) for the source one is defined as follows

\[
M_f(r) \overset{\text{def}}{=} \max_{|z| \leq r} |f(z)| = \max_{|z| = r} |f(z)|. \tag{2}
\]

It is the one of the classical problem from the theory of the entire functions: establish the relations between the asymptotical behavior as \( n \to \infty \) the coefficients \( c_n = c_n[f] \) and the asymptotical behavior for the maximal function \( M_f(r) = M(r) \) as \( r \to \infty \).

See for example the classical monographs [3], [10]; where are described also some important applications, for instance in the theory of distributions of zeros of entire functions and in the functional analysis, in particular in the theory of operators.

As regards for the modern work we mention an article [5].

Let us bring some notions from the classical theory. The order \( \rho[f] \) of the function \( f = f(z) \) may be calculated by the following relations

\[
\rho[f] = \lim_{n \to \infty} \left\{ \frac{n \ln n}{\ln |c_n|} \right\}. \tag{3}
\]

Correspondingly, the type \( \beta[f] \) is equal to

\[
\beta[f] = \lim_{n \to \infty} \left\{ n^{1/\rho[f]} \sqrt[\rho[f]]{|c_n|} \right\}. \tag{4}
\]

Our target in this short report is to establish the non-asymptotical exact bilateral estimations between the coefficients of the considered function and its maximal majorant.

They have a very simple, closed and very general form.
We extend obtained results in the last section onto the entire functions of several complex variables.

We derive as a consequence the natural conditions for the possible coincidence of these estimates: upper and lower ones.

Recall also one important for us auxiliary facts from the theory of convex function. The so-called Young-Fenchel, or Legendre transform \( g^*(y), \ y \in R \) for the given numerical valued function \( g = g(x) \) having the convex non-empty domain of definition \( \text{Dom}[g] \) is defined as follows

\[
g^*(y) \overset{\text{def}}{=} \sup_{x \in \text{Dom}[g]} (xy - g(x)). \quad (5)
\]

The function \( g^*(\cdot) \) named as customary Young conjugate, or simple conjugate to the source one \( g(\cdot) \).

Many examples and properties of this transform with applications e.g. to the theory of Orlicz spaces may be found in the classical monographs [8], [15], [16], [17], [19].

The famous theorem of Fenchel-Moreau tell us that if the function \( g = g(x) \) is convex and continuous defined on the convex set, then

\[
g^{**}(x) = g(x), \quad (6)
\]

see [17], chapters 2,3; [19].

Recall also the following important Young’s inequality

\[
xy \leq g(x) + g^*(y), \ x, y \in R; \quad (7)
\]

or more generally

\[
xy \leq g(\gamma x) + g^*(y/\gamma), \ \gamma = \text{const} > 0, \ x, y \in R. \quad (8)
\]

2 Main results: upper and lower non-asymptotic estimates.

**Upper estimate.**

We mean to derive the non-asymptotic upper estimate of the Taylors coefficients for the function \( f : c_0[f] \) via its maximal function \( M_f(r) \).

Introduce the function

\[
\Lambda(v) \overset{\text{def}}{=} \ln M_f(e^v), \ v \in R. \quad (9)
\]
However, it is sufficient for us hereinafter to suppose at last instead (9) only an unilateral restriction

\[ M_f(r) \leq \exp \left( \Lambda(\ln r) \right), \quad r > 0, \tag{10} \]

for certain function \( \Lambda = \Lambda(v), \in R. \) The restrictions on this function will be clarified below.

**Proposition 2.1.** We propose under condition (10)

\[ |c_n| \leq \exp \left( -\Lambda^*(n) \right), \quad n = 1, 2, \ldots \tag{11} \]

**Proof.** We start from the simple estimate

\[ |c_n| \leq \frac{M_f(r)}{r^n}, \quad r > 0. \]

Therefore

\[ |c_n| \leq \exp \left( -n \ln r + \ln M_f(r) \right) = \exp \left( -nv + \Lambda(v) \right) = \]

\[ \exp \left( -(nv - \Lambda(v)) \right), \quad v = \ln r \in (-\infty, \infty). \]

Since the last inequality is true for arbitrary value \( v, \) one can take the minimum over \( v : \)

\[ |c_n| \leq \exp \left( -\sup_{v \in R} (nv - \Lambda(v)) \right) = \exp \left( -\Lambda^*(n) \right), \]

Q.E.D.

**LOWER ESTIMATE.**

We intent to derive in this subsection the non - asymptotic lower estimate of the Taylors coefficients \( c_n[f] = c_n \) for the function \( f : c_n[f] \) via its maximal function \( M_f(r). \)

Equivalently: we want to obtain the upper estimate for the maximal function \( M_f(r) \) through its series of Taylor coefficients.

The lower estimate is more complicated. We will follow the authors on an article [7]. Suppose

\[ |c_n| \leq \exp \left( -Q(n) \right) \tag{12} \]

for certain increasing to infinity function \( Q = Q(z), \quad z \geq 1. \) We have for all the sufficient greatest values \( r \geq e \)
\[ M_f(r) \leq \sum_{n=0}^{\infty} \exp(nv - Q(n)) =: R(v) R_Q(v), \ v = \ln r. \] (13)

It follows from the theory of the saddle-point method, see e.g. [4], chapters 1,2, that for some finite positive constant \( C = C[Q] \in (1, \infty) \)

\[ R_Q(v) \leq \exp\left( \sup_n (Cnv - Q(n)) \right) = \]

\[ \exp(Q^*(Cv)) = \exp(Q^*(C \ln r)), \ r \geq e. \] (14)

If in addition the function \( Q = Q(v) \) is in turn Young conjugate to some continuous and convex one, say \( G(\cdot) : Q = G^* \), then by virtue of Theorem of Fenchel-Moreau

\[ M_f(r) \leq C_0[G] \exp( G(C \ln r) ), \ r \geq e. \] (15)

In particular, the function \( G(\cdot) \) may coincides with the introduced before function \( \Lambda^*(\cdot) \); and we conclude in this case

\[ M_f(r) \leq C_0[\Lambda] \exp( \Lambda(C \ln r) ), \ r \geq e. \] (16)

Let us bring the strong proof of (15), of course, under appropriate conditions. Define the following function

\[ K_Q(\epsilon) := \sum_{n=0}^{\infty} \exp(-\epsilon Q^*(n)), \ \epsilon \in (0,1). \]

It is proved in particular in [7] that if \( \exists \epsilon \in (0,1) \Rightarrow K(\epsilon) < \infty, \) then

\[ R(v) \leq K_Q(\epsilon) \exp \left( (1 - \epsilon)Q^{**} \left( \frac{v}{1-\epsilon} \right) \right). \]

Further, as long as

\[ nv \leq Q^*((1 - \epsilon) v) + Q^{**}(n/(1 - \epsilon)), \ \epsilon \in (0,1). \]

we have

\[ R(v) \leq U_Q(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right), \]

where in addition to the (14)

\[ U(\epsilon) = U_Q(\epsilon) := \sum_{n=0}^{\infty} \exp(Q^*((1 - \epsilon) n) - Q^*(n)). \]

To summarize:

**Proposition 2.2.** Denote
\[
Y_Q(\epsilon) = Y(\epsilon) := \min(U(\epsilon), K(\epsilon)), \ \epsilon \in (0, 1).
\]

We assert
\[
R_Q(v) \leq Y_Q(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right), \ \epsilon \in (0, 1). \tag{17}
\]

Of course,
\[
R_Q(v) \leq \inf_{\epsilon \in (0,1)} \{ Y_Q(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right) \}. \tag{18}
\]

As a consequence:

**Proposition 2.3.** If
\[
\exists \ \epsilon \in (0, 1) \ \Rightarrow Y_Q(\epsilon) < \infty, \tag{19}
\]

then the estimations (14) holds true. If in addition the function \( Q = Q(v) \) is continuous and convex, then (16) is valid, as well.

**TO SUMMARIZE.**

**A.** Recall that
\[
\ln M_f \left( e^v \right) \leq \Lambda(v), \ v \in R,
\]

therefore
\[
|c_n[f]| \leq \exp \left( -\Lambda^* (n) \right), \ n = 1, 2,\ldots. \tag{20}
\]

**B.** Conversely, let the inequality (20) be given for certain non-negative continuous convex function \( \Lambda = \Lambda(v), \ v \in R \) for which
\[
\exists \epsilon_0 \in (0, 1) \ \Rightarrow S_0 = S(\epsilon_0) \overset{def}{=} Y_{\Lambda^*(\epsilon_0)} < \infty.
\]

Then
\[
M_f(r) \leq S(\epsilon_0) \exp \left\{ \Lambda \left( \frac{\ln r}{1 - \epsilon_0} \right) \right\}, \ r \geq e. \tag{21}
\]

Briefly: under formulated above conditions
\[
M_f(r) \leq e^{\Lambda(\ln r)} \ \Rightarrow |c_n| \leq e^{-\Lambda^*(n)}, \ r \geq e; \tag{22}
\]

\[
|c_n| \leq e^{-\Lambda^*(n)} \Rightarrow M_f(r) \leq S_0 \ e^{\Lambda(\ln r/(1-\epsilon_0))}. \tag{23}
\]

**C. ”Tauberian” theorem.**
Let us impose an additional restriction on the function $\Lambda(\cdot)$:

$$\exists \gamma = \gamma(\Lambda, \epsilon_0) = \text{const} < \infty \Rightarrow \Lambda\left(\frac{v}{1 - \epsilon_0}\right) \leq \gamma \Lambda(v), \ v \geq 1. \quad (24)$$

It follows immediately from the relations (22) and (23) the following assertion.

**Theorem 2.1.** We conclude under formulated above conditions

$$\lim_{r \to \infty} \frac{\ln M_f(r)}{\Lambda(\ln r)} = \lim_{n \to \infty} \frac{|\ln 1/c_n|}{\Lambda^*(n)}. \quad (25)$$

More precisely: if there exists the left-hand side of (25), then there exists also the right-hand one and they are equal; the converse proposition is also true: if there exists the right-hand side of (25), then there exists also the left-hand one and they are equal.

### 3 Examples

**Auxiliary fact.** Introduce a following family of regular varying functions

$$\phi_{m,L}(\lambda) \defeq \frac{1}{m} \lambda^m L(\lambda), \ \lambda \geq 1. \quad (26)$$

Here $m = \text{const} > 1$, and define as ordinary $m' := m/(m-1)$ and $L = L(\lambda), \ \lambda \geq 1$ is positive continuous *slowly varying* as $\lambda \to \infty$ function. It is known that as $x \to \infty, \ x \geq 1$

$$\phi_{m,L}^*(x) \sim (m')^{-1} x^{m'} L^{-1/(m-1)} \left(x^{1/(m-1)}\right), \quad (27)$$

see [18], pp. 40 - 44; [7]. For instance, if $\phi_m(\lambda) = m^{-1} |\lambda|^m, \ \lambda \in R$, then

$$\phi_m^*(x) = (m')^{-1} |x|^{m'}, \ x \in R.$$  

More generally, if

$$\psi_{m,q}(\lambda) = C_1 \lambda^m [\ln \lambda]^q, \ \lambda \geq e, \ m = \text{const} > 1, \ q = \text{const} \geq 0,$$

then

$$\psi_{m,q}^*(x) \sim C_2 (m, q) x^{m'} [\ln x]^{-q/(m-1)}, \ x \geq e.$$  

**Example 3.1.** Suppose that for some entire function $f = f(z)$

$$\ln M_f(r) \sim C_3 (m) [\ln r]^m, \ r \geq e, \ m = \text{const} > 1. \quad (28)$$

Then
\[ c_n[f] \leq \exp \left\{ -C_4(m) n^{m'} \right\}, \ n \geq 0; \] (29)

and conversely proposition is also true: from the estimation (29) follows the inequality (28).

Note that the case \( m \leq 1 \) is trivial: the function \( f(z) \) is polynomial, see [10], chapter 1, sections 2 - 4. This implies that

\[ \exists N \in \{ 2, 3, \ldots \} \ \forall n \geq N \Rightarrow c_n = 0. \]

**Example 3.2.** Suppose that for some entire function \( f = f(z) \)

\[ \ln M_f(r) \sim C_4 r^\rho, \ r \geq 1, \ \rho = \text{const} > 0, \ C_4 = \text{const} \in (0, \infty), \] (30)
a classical case, [10], chapter 1, sections 1 - 5. Then

\[ |c_n| \leq \left[ \frac{n}{C_4 \rho} \right]^{-n/\rho} e^{n/\rho}, \] (31)

and conversely proposition is also true: from the estimation (31) follows the inequality (30).

More generally, the relation of the form

\[ \ln M_f(r) \sim \rho^{-1} r^\rho \ln^n r, \ r \to \infty \]
is quite equivalent to the following equality: as \( n \to \infty \)

\[ \ln \left( \frac{1}{|c_n|} \right) \sim \rho^{-1} n \ln n + \gamma n \ln \ln n/\rho - \frac{n}{\rho}. \]

**Example 3.3.** We propose that for arbitrary entire function \( f = f(z) \) the following relations are equivalent:

\[ \exists C_5, C_6 \in (0, \infty) \Rightarrow \ln M_f(r) \leq C_5 e^{C_6 r}, \ r \geq 0, \] (32)

and

\[ \exists C_7 \in (0, \infty) \Rightarrow |c_n| \leq C_7 (\ln n)^{-n}, \ n \geq 3. \] (33)
4 Generalization on the entire (holomorphic) functions of several complex variables.

It is no hard to generalize the obtained results on the case of the analytical functions $f = f(z)$ of several complex variables.

Let us introduce first all some (ordinary) used notations. The (finite) dimension of a considered problem will be denoted by $d; d = 2, 3, \ldots$. Correspondingly the multivariate variable $z$ consists on the $d$ independent complex variables

$$z = z = \{z_1, z_2, \ldots, z_d\}.$$  

Ordinary vector notations:

$$k = k = \{k_1, k_2, \ldots, k_d\}, \quad k_j = 0, 1, 2, \ldots; \quad |k| := \sum_{j=1}^{d} k_j;$$

$$z^k = z^k = \prod_{j=1}^{d} z_j^{k_j};$$

$$r = r = \{r_1, r_2, \ldots, r_d\} \in R^d, \quad r_j \geq 0;$$

$$v = v = \bar{v}(r) := \exp(r) = \{e^{r_1}, e^{r_2}, \ldots, e^{r_d}\} \in R^d,$$

$$\iff \ln \bar{v} = \ln v = \{ln v_j\}, \quad j = 1, 2, \ldots, d.$$  

The multivariate Young - Fenchel transform $g^*(y)$, $y \in R^d$ for the given numerical valued function $g = g(x)$, $x \in R^d$ having the convex non - empty domain of definition $\text{Dom}[g]$ is defined as usually

$$g^*(y) \overset{\text{def}}{=} \sup_{x \in \text{Dom}[g]} \left( (x, y) - g(x) \right), \quad (34)$$

where $(x, y)$ denotes the inner, or scalar product of the two $d$ - dimensional vectors $x, y$. As above, if the function $g = g(x)$ is continuous and convex, $g^{**} = g$. The famous Young inequality has a form

$$(x, y) \leq g(\gamma x) + g^*(y/\gamma), \quad \gamma = \text{const} > 0, \quad x, y \in R^d. \quad (35)$$

Further, the analytical function $f = f(z)$ has a form

$$f(\bar{z}) = f(z) = \sum_{k} c_k z^k = \sum_{k} c_k \bar{z}^k, \quad (36)$$

where the numbers $\{c_k\} = \{c_k\}$ are the Taylor's coefficients for $f = f(z)$ and the series in (36) converges for all the complex vectors $\bar{z}$.  

9
The maximal function \( M(r) = M_f(r) = M_f(\vec{r}) \) for \( f = f(z) \) at the point \( r \in R^d_+ \) is defined as before

\[
M_f(\vec{r}) \overset{\text{def}}{=} \max_{|z_j| \leq r_j} |f(\vec{z})| = \max_{|z_j| = r_j} |f(z)|.
\]

(37)

Define also the function

\[
\Lambda(v) = \Lambda(\vec{v}) \overset{\text{def}}{=} \ln M(\vec{e}v) = \ln M_f(\vec{e}v), \ v \in R^d.
\]

(38)

**Upper estimate.**

We apply the following estimate

\[
|c_k| \leq \frac{M_f(\vec{r})}{\vec{r}^k}, \ \vec{r} \in R^d_+,
\]

see, e.g., [6], page 15, formula 1.4.3. Following,

\[
|c_k| \leq \exp \left( -((k,v) - \Lambda(v)) \right);
\]

and as before

\[
|c_k^\perp| \leq \exp \left( -\sup_{v \in R}((k,v) - \Lambda(v)) \right),
\]

Proposition 4.1.

\[
|c_k^\perp| \leq \exp \left( -\Lambda^*(\vec{k}) \right).
\]

(39)

**Lower estimate.**

The lower estimate is quite alike ones obtained in the second section in the one-dimensional case, i.e. when \( d = 1 \); we will apply also the methods explained in [7]. Suppose

\[
|c_k| = |c_k^\perp| \leq \exp \left( -Q(\vec{k}) \right) = \exp \left( -Q(k) \right)
\]

(40)

for certain increasing to infinity relative the all variables \( k_j, j = 1, 2, \ldots, d \) function \( Q = Q(z), \ z = \vec{z} = \{z_j\}, \ z_j \geq 0 \). We have for all the sufficient greatest values \( r_j \geq e \)

\[
M_f(\vec{r}) \leq \sum_{k \geq 6} \exp ( (k,v) - Q(k) ) =: R(v) = R(\vec{v}), \ \vec{v} = \ln \vec{r}.
\]

(41)
Define the following function

\[ K(\epsilon) := \sum_{\vec{k} \geq \vec{0}} \exp \left( -\epsilon Q^*(\vec{k}) \right), \quad \epsilon \in (0, 1). \]

It is proved in particular in [7] that if \( \exists \epsilon \in (0, 1) \Rightarrow K(\epsilon) < \infty, \) then

\[ R(v) \leq K(\epsilon) \exp \left( (1 - \epsilon) Q^{**} \left( \frac{v}{1 - \epsilon} \right) \right). \]

Further, as long as

\[ (k, v) \leq Q^*((1 - \epsilon) v) + Q^{**}(k/(1 - \epsilon)), \quad \epsilon \in (0, 1), \]

we have

\[ R(v) \leq U(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right), \]

where

\[ U(\epsilon) := \sum_{\vec{k} \geq \vec{0}} \exp(Q^*((1 - \epsilon) \vec{k}) - Q^*(\vec{k})). \]

To summarize:

**Proposition 4.2.** Denote

\[ Y(\epsilon) := \min(U(\epsilon), K(\epsilon)), \quad \epsilon \in (0, 1). \]

We assert

\[ R(v) \leq Y(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right), \quad \epsilon \in (0, 1). \] \hspace{1cm} (42)

Of course,

\[ R(v) \leq \inf_{\epsilon \in (0, 1)} \{ Y(\epsilon) \exp \left( Q^{**}(v/(1 - \epsilon)) \right) \}. \] \hspace{1cm} (43)

As a consequence:

**Proposition 4.3.** If

\[ \exists \epsilon \in (0, 1) \Rightarrow Y(\epsilon) < \infty, \] \hspace{1cm} (44)

then the following estimation

\[ M_f(\vec{r}) \leq \exp( Q^{**}(C \ln r) ), \quad r_j \geq e. \] \hspace{1cm} (45)

holds true.

If in addition the function \( Q = Q(\vec{v}) \) is continuous and convex, then
\[ M_f(\vec{r}) \leq \exp(Q(C\ln r)), \quad r_j \geq e. \quad (46) \]

In particular, if the function \( Q(\cdot) \) coincides with the introduced before function \( \Lambda^*(\cdot) \), we conclude in this case

\[ M_f(r) \leq \exp(\Lambda(C\ln r)), \quad r_j \geq e. \quad (47) \]

**Multivariate examples.**

It is no difficult to show the exactness of obtained estimates still in the multidimensional case. It is sufficient to consider the case \( d = 2 \) and the so-called factorizable function

\[ f = f(z) = f(z_1, z_2) = f_1(z_1) \cdot f_2(z_2), \quad z = (z_1, z_2), \]

where the functions \( f_1, f_2 \) are function-examples considered in the third section. If

\[ f_1(z_1) = \sum_{k=0}^{\infty} a_k z_1^k, \quad f_2(z_2) = \sum_{l=0}^{\infty} b_l z_2^l, \]

then

\[ f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_k b_l z_1^k z_2^l = \sum_{k=l=0}^{\infty} c_{k,l} z_1^k z_2^l, \]

where \( c_{k,l} = a_k b_l \).

Obviously,

\[ M_f(r_1, r_2) = M_{f_1}(r_1) \cdot M_{f_2}(r_2) = \exp\{\Lambda_{f_1}(\ln r_1) + \Lambda_{f_2}(\ln r_2)\}. \]

As long as

\[ \sup_{\lambda, \mu \in R} [(\lambda x + \mu y) - (g(\lambda) + h(\mu))] = g^*(x) + h^*(y), \]

we conclude

\[ |c_{k,l}| \leq \exp\{-\Lambda_{f_1}^*(k) - \Lambda_{f_2}^*(l)\} \]

and so one; see theorem 3.1.
5 Concluding remarks.

A. It is no hard to generalize obtained estimations on the derivatives of the source function \( f = f(z) \), including the partial derivatives in the case of the function of several complex variables. For instance, for the function from (1) we have

\[
  f'(z) = \sum_{k=1}^{\infty} k c_k[f] z^{k-1},
\]

so that

\[
  c_k[f'] = (k + 1) c_{k+1}[f].
\]

B. The continuous version of our estimations, i.e. the Tauberian theorems for the Laplace transform are well known, see e.g. [20]. The non - asymptotical estimates may be found, e.g. in [14], pp. 27 - 37.

C. Let us list briefly a several works devoted to the applications of Tauberian Theorems in the Probability Theory: [1], [2], [9], [11], [12], [13].

Let us show some extension of obtained in these works results based on the our estimations. Let \( \xi \) be an integer values non - negative random variable (r.v.):

\[
  P(\xi = k) = c_k, \quad k = 0, 1, 2, \ldots.
\] (48)

Of course, \( c_k \geq 0 \), \( \sum_k c_k = 1 \). The so - called generating function (g.f.) \( g[\xi](z) = g[\xi](z) \) for this r.v. is as ordinary defined as follows

\[
  g[\xi](z) \overset{\text{def}}{=} \mathbb{E} z^\xi = \sum_{k=0}^{\infty} c_k z^k. \quad (49)
\]

This notion play a very important role in the probability theory, in particular, in the reliability theory, in the grand deviation theory, in the theory of queue theory etc. It is important especially for these applications the asymptotical behavior of \( g[\xi](z) \) as \( |z| \to \infty \).

One can for example apply our theorem 2.1 for the probability theory. Namely, we conclude under formulated in this theorem conditions

\[
  \lim_{r \to \infty} \frac{\ln M_g(r)}{\Lambda_P(\ln r)} = \lim_{n \to \infty} \frac{\ln 1/|c_n|}{\Lambda^*_P(n)}. \quad (50)
\]

More precisely: if there exists the left - hand side of (50), then there exists also the right - hand one and they are equal; the converse proposition is also true: if there exists the right - hand side of (50), then there exists also the left - hand one and they are equal.

Here with accordance (9)

\[
  \Lambda_P(v) \overset{\text{def}}{=} \ln M_g(e^v), \quad v \in \mathbb{R}. \quad (51)
\]
Analogous fact holds true still in the multidimensional case.

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