SOME NEW RESULTS ON NEGATIVE POLYNOMIAL PELL’S EQUATION

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ABSTRACT. We consider the negative polynomial Pell’s equation $P^2(X) - D(X)Q^2(X) = -1$, where $D(X) \in \mathbb{Z}[X]$ be some fixed, monic, square-free, even degree polynomials. In this paper, we investigate the existence of polynomial solutions $P(X), Q(X)$ with integer coefficients.

1. INTRODUCTION

The classical Pell’s equation is

$$x^2 - Dy^2 = 1, \quad (1.1)$$

where $D$ is a square-free positive integer. Solving a Pell’s equation for integers $x$ and $y$ is one of the classical problems in number theory. In 1768, Lagrange proved that the equation $(1.1)$ has infinitely many solutions ([4, vol. XXIII, p. 272], [5, vol. XXIV, p. 236]). In fact, a classical result says that there exists a non-trivial solution $(x_0, y_0)$ is called a fundamental solution such that any other solution takes the form $(x_0 + y_0\sqrt{D})^n$, $n \in \mathbb{Z}$.

On the other hand, the problem of solving a negative Pell’s equation

$$x^2 - Dy^2 = -1, \quad (1.2)$$

where $D$ is a square-free integer and $x, y$ are integer solutions. There is no solution for equation $(1.2)$ if $D$ is a negative integer and the length of the period in the continued fraction expansion of $\sqrt{D}$ is even. Nevertheless, if the length of the period in the continued fraction expansion of $\sqrt{D}$ is odd, then $(1.2)$ has infinitely many integer solutions [12, Theorem 7.26]. Further, negative Pell’s equation is not solvable for $D$ with prime divisor congruent to 3 mod 4 or $D$ is divisible by 4. Moreover, É. Fouvry and J. Klûners [2] gave the upper and lower bounds for the long-lasting conjecture on the asymptotic formulae for the number of square-free integers $D$ for which fundamental solution of the equation $(1.2)$ has norm $-1$. Recently, the bound was further improved by P. Koymans and C. Pagano [3].

Similarly, we can consider the polynomial Pell’s equation

$$P^2(X) - D(X)Q^2(X) = \pm 1, \quad (1.3)$$

where $D(X)$ is a given fixed, square-free polynomial with integer coefficients and $P(X), Q(X)$ are its integer polynomial solutions.
In 1976, Nathanson [8] proved that when \( D(X) = X^2 + d \in \mathbb{Z}[X] \), the equation \( P^2(X) - D(X)Q^2(X) = 1 \) is solvable in \( \mathbb{Z}[X] \) if and only if \( d = \pm 1, \pm 2 \). Moreover, such a polynomial solutions can be expressed in terms of Chebyshev polynomials [10]. In 2004, A. Dubickas and J. Steuding [1] extended Nathanson’s result for polynomials of the form \( D(X) = X^{2k} + d \in \mathbb{Z}[X] \), \( k \in \mathbb{N} \). More precisely, they proved that the equation \( P^2(X) - (X^{2k} + d)Q^2(X) = 1 \) is solvable in \( \mathbb{Z}[X] \) if and only if \( d \in \{ \pm 1, \pm 2 \} \).

Despite many results on positive polynomial Pell’s equation, there is no notable work on negative polynomial Pell’s equation, where \( D(X) \) be a fixed, even degree, square-free polynomial with integer coefficients and \( P(X), Q(X) \) are its integer polynomial solutions. More precisely, we prove the following theorem:

**Theorem 1.1.** Let \( d \) be an integer with \( d \neq \pm 1, \pm 2 \). Then the negative polynomial Pell’s equation,
\[
P^2(X) - (X^2 + d)Q^2(X) = -1
\]
has no non-trivial solutions over \( \mathbb{Z}[i] \).

**Theorem 1.2.** The equation (1.5) has non-trivial polynomial solutions over \( \mathbb{Z} \) if and only if \( d = 1 \).

We now generalize the Theorem 1.2 and prove the following:

**Theorem 1.3.** The negative polynomial Pell’s equation
\[
P^2(X) - (X^{2k} + d)Q^2(X) = -1
\]
where \( d \in \mathbb{Z} \) and \( k \in \mathbb{N} \) has non-trivial solutions in \( \mathbb{Z}[X] \) if and only if \( d = 1 \).

We need the following lemma to prove Theorem 1.3.

**Lemma 1.4.** Let \( D(X) \) be a polynomial in \( \mathbb{C}[X] \) with degree \( 2k \). Then the fundamental solutions \( (U(X), V(X)) \) in \( \mathbb{C}[X] \) of equation (1.4) satisfying \( \deg U(X) = 1/2 \deg D(X) \) and \( \deg V(X) = 0 \) is minimal.

**Proof.** Firstly, let us consider \( D(X) \) be a quadratic polynomial in \( \mathbb{C}[X] \). We observe that the non-trivial solutions of (1.4) exists only if \( D(X) \) has distinct roots. Let \( \gamma, \delta \) be the roots of \( D(X) \). Then we write \( D(X) = c(X - \gamma)(X - \delta), \ c \in \mathbb{C}, \gamma \neq \delta \).

We set,
\[
U(X) = \frac{2X - (\gamma + \delta)}{\sqrt{-1}(\gamma - \delta)}; V(X) = \frac{2}{\sqrt{-1}(\gamma - \delta)}.
\]

For general case, we assume the contrary. Suppose that \( \deg U(X) < 1/2 \deg D(X) \) and \( \deg V(X) > 0 \). Since \( \deg D(X) = 2 \deg P(X) - 2 \deg Q(X) \) and \( \deg P(X) \) must be at least 1 greater than the \( \deg Q(X) \).

Thus
\[
\deg D(X) = 2 \deg U(X) - 2 \deg V(X) < \deg D(X) - 2t,
\]
for some positive integer $t$. This completes the proof. \hfill \Box

2. PROOF OF THEOREM 1.1

We prove the theorem by contradiction. We first consider the equation (1.5) as a polynomial over $\mathbb{Z}[i]$. We suppose that the equation (1.5) has non-trivial solutions over $\mathbb{Z}[i]$. We choose a solution $P(X), Q(X)$ of (1.5) with $\deg P(X) > 0$ is minimal and we take a non-zero $d$ with $|d| \geq 3$. We split the proof into two cases.

Case(i): If $d \neq -\alpha^2$, $\alpha \in \mathbb{Z}[i]$, then $X^2 + d$ is irreducible over $\mathbb{Z}[i]$. We now rewrite (1.5) as,

$$ (P(X) + i)(P(X) - i) = (X^2 + d)Q^2(X). $$

Since $(X^2 + d)$ is irreducible over $\mathbb{Z}[i]$ and $\mathbb{Z}[i]$ is a unique factorization domain, it divides one of the $(P(X) + i)$ or $(P(X) - i)$. We assume that $(X^2 + d)$ divides $P(X) - i$. Therefore,

$$ P(X) - i = (X^2 + d)P_1(X), $$

where $P_1(X)$ is a polynomial over $\mathbb{Z}[i]$.

Then

$$ P(X) - i + 2i = P(X) + i = (X^2 + d)P_1(X) + 2i. $$

On substituting into the equation (2.1), we have

$$ P_1(X)(X^2 + d)P_1(X) + 2i = Q^2(X). $$

Since the greatest common divisor of $P_1(X)$ and $(X^2 + d)P_1(X) + 2i$ is 1 or 2, we must obtain at least one of the following conditions:

1. $(X^2 + d)P_1(X) + 2i = P_2^2(X)$, $P_1(X) = Q_2^2(X)$;
2. $(X^2 + d)P_1(X) + 2i = -P_2^2(X)$, $P_1(X) = -Q_2^2(X)$;
3. $(X^2 + d)P_1(X) + 2i = -iP_2^2(X)$, $P_1(X) = iQ_2^2(X)$;
4. $(X^2 + d)P_1(X) + 2i = iP_2^2(X)$, $P_1(X) = -iQ_2^2(X)$;
5. $(X^2 + d)P_1(X) + 2i = 2P_2^2(X)$, $P_1(X) = 2Q_2^2(X)$;
6. $(X^2 + d)P_1(X) + 2i = -2P_2^2(X)$, $P_1(X) = -2Q_2^2(X)$;
7. $(X^2 + d)P_1(X) + 2i = -2iP_2^2(X)$, $P_1(X) = 2iQ_2^2(X)$;
8. $(X^2 + d)P_1(X) + 2i = 2iP_2^2(X)$, $P_1(X) = -2iQ_2^2(X)$.

As $P_2(X)$ is a polynomial over $\mathbb{Z}[i]$. We substitute $X = \sqrt{-d}$ in conditions (1) – (8) and we see that the following possibilities are admissible: $(r + s\sqrt{-d})^2 = \pm 2i$ or $(r +
\(s \sqrt{-d})^2 = \pm 2\) or \((r + s \sqrt{-d})^2 = \pm i\) or \((r + s \sqrt{-d})^2 = \pm 1\) for some \(r, s \in \mathbb{Z}[i]\). We need the following arguments to sort out the impossible conditions.

We first consider that \((r + s \sqrt{-d})^2 = \pm 2i\) and \((r + s \sqrt{-d})^2 = \pm i\). Substituting \(r = x + iy, s = u + iv,\) where \(x, y, u, v \in \mathbb{Z}\), we have

\[
(x + iy)^2 - (u + iv)^2d + 2i((x + iy)(u + iv))\sqrt{d} = \pm 2i, \pm i.
\]

On equating real and imaginary parts, we get

\[
x^2 - y^2 - (u^2 - v^2)d - 2\sqrt{d}(xy + yu) = 0, \quad x^2 - y^2 - (u^2 - v^2)d = 0.
\]

By our choice of \(d\), equation (2.2) can be separated as rational and irrational parts,

\[
x^2 - y^2 = 1, \pm 1/2.
\]

This could be possible only when \(d\) is a perfect square or \(d = \pm 1\). This ends in a contradiction.

Now we will explore the equation \((r + s \sqrt{-d})^2 = \pm 2\). As we proceed before, we equate real and imaginary parts and we obtain

\[
x^2 - y^2 - (u^2 - v^2)d - 2\sqrt{d}(xy + yu) = \pm 2,
\]

\[
xy - uv = \pm 1, \pm 1/2.
\]

Again we repeat the same procedure as separating rational and irrational parts,

\[
x^2 - y^2 = 1, \pm 1/2, \quad x^2 - y^2 = 0.
\]

By solving the simultaneous equations (2.7) and (2.9), we get either \(y = 0\) or \(u^2 + v^2 = 0\). We first assume that \(y = 0\) and \(x \neq 0\) then \(u = v = 0\). Therefore \(x = \pm \sqrt{2}\) or \(\pm i\sqrt{2}\). Since \(x\) is an integer, both can not be possible. On the other hand, if we assume both \(x\) and \(y\) are zero, then \(uv = 0\) (by using (2.8)). Again a contradiction. Hence we conclude that \(y\) should be a non-zero and \(u^2 + v^2 = 0\). Here the only possibility is \(u = v = 0\). Thus we end with \(x = 0\) (by using (2.5)) and the values of \(y\) are \(\pm \sqrt{2}\) or \(\pm i\sqrt{2}\). This is again a contradiction.

Now we take \((r + s \sqrt{-d})^2 = \pm 1\). As we done in the previous arguments, we first deal with the equation,

\[
x^2 - y^2 - (u^2 - v^2)d = 1.
\]

There are two cases either \(y = 0\) or \(u^2 + v^2 = 0\) (by using (2.7) and (2.9)). At first, we suppose to consider both \(x\) and \(y\) are zero. Then we obtain \(uv = 0\) (by using (2.8)). So
we omit it. If we assume \( y = 0 \) and \( x \neq 0 \), then \( u = v \). Thus \( x = \pm 1 \) and the value of \( r \) is \( \pm 1 \). On the other side, if \( u^2 + v^2 = 0 \) then \( u = v = 0 \). Therefore value of \( s = 0 \).

Finally, we consider the equation
\[
x^2 - y^2 - (u^2 - v^2)d = -1.
\]
Again by the same procedure as we deal with the equation (2.10), we end with \( y = \pm 1 \) and \( u = v = x = 0 \). Thus \( r = \pm i \), \( s = 0 \). Among eight conditions, only (7) and (8) are possible. We now rewrite the condition (7) as,
\[
P_2^2(X) - (X^2 + d)(iQ_2(X))^2 = -1
\]
and condition (8) as,
\[
(iP_2(X))^2 - (X^2 + d)Q_2^2(X) = -1.
\]

But in both equations (2.11) and (2.12), \( 2 \deg(P_2(X)) = 2 + \deg(P_1(X)) = \deg(P(X)) \). It leads to a contradiction on minimality of \( \deg(P(X)) \). Therefore equation (1.5) has no non-trivial solutions, if \( d \neq \pm 1, \pm 2 \) and also \( d \neq -\alpha^2 \), \( \alpha \in \mathbb{Z}[i] \).

**Case(ii):** Let \( d = -\alpha^2 \), \( \alpha \) be a non-unit in \( \mathbb{Z}[i] \) and \( N(\alpha) > 2 \). The constant term of the solution polynomials \( P(X) \) and \( Q(X) \) are \( \pm i \), 0 respectively. Suppose that \( P(0) = i \). Then \( P(X) = i + XP_1(X) \) and \( Q(X) = XQ_1(X) \). We substitute \( P(X) \), \( Q(X) \) into equation (1.5) and we obtain,
\[
P_1(X)(XP_1(X) + 2i) = X(X^2 - \alpha^2)Q_1^2(X).
\]
Since \( P_1(X) \) is a polynomial without constant term, we write \( P_1(X) = XP_2(X) \). Now we rewrite (2.13) as,
\[
P_2(X)(X^2P_2(X) + 2i) = (X^2 - \alpha^2)Q_1^2(X).
\]
We suppose that \( X \pm \alpha \) divides \( X^2P_2(X) + 2i \). Then we put \( X = \mp \alpha \) and we get \( \alpha^2P_2(\mp \alpha) = -2i \). Thus \( \alpha^2 \) divides \( 2i \). Since \( N(\alpha) > 2 \), this is not possible. Therefore both \( X + \alpha \) and \( X - \alpha \) should divide \( P_2(X) \). We can say \( P_2(X) = (X^2 - \alpha^2)P_3(X) \). On substituting in (2.14), we obtain,
\[
P_3(X)(X^2(X^2 - \alpha^2)P_3(X) + 2i) = Q_1^2(X).
\]
The greatest common divisor of \( P_3(X) \) and \( X^2(X^2 - \alpha^2)P_3(X) + 2i = 1 \) or 2. Again we repeat the same procedure as in case(i). This completes the proof of Theorem 1.1.

3. CONTINUED FRACTION EXPANSION OF \( \sqrt{D(X)} \)

In this section, we use the same technique which is used for irrational \( \sqrt{D} \) in [9]. The continued fraction expansion of \( \sqrt{D(X)} \) is of the form
\[
[a_0(X), a_1(X), a_2(X), \ldots, a_{r-1}(X), 2a_0(X)].
\]
with convergents $H_n(X)/K_n(X)$ and $a_i(X)$ be a non-constant polynomial in $\mathbb{Z}[X]$. Let $r$ be the length of the shortest period in the continued fraction expansion of $\sqrt{D(X)}$. We define

$$\zeta_0(X) = \frac{M_0(X) + \sqrt{D(X)}}{N_0(X)}$$

with $N_0(X) = 1$ and $M_0(X) = 0$.

In general, we define

$$a_i(X) = [\zeta_i(X)],$$
$$\zeta_i(X) = \frac{M_i(X) + \sqrt{D(X)}}{N_i(X)},$$
$$M_{i+1}(X) = a_i(X)N_i(X) - M_i(X),$$
$$N_{i+1}(X) = \frac{D(X) - M_{i+1}^2(X)}{N_i(X)},$$

where $[.]$ denotes the rational part of the polynomial in terms of $X$. Since $r$ be the length of the period, we write $\zeta_0 = \zeta_r = \zeta_{2r} = \ldots$. Thus for all $j \geq 0$ we write

$$\frac{M_{jr}(X) + \sqrt{D(X)}}{N_{jr}(X)} = \zeta_{jr}(X) = \zeta_0(X) = \frac{M_0(X) + \sqrt{D(X)}}{N_0(X)}.$$

**Theorem 3.1.** If $D(X)$ is a square-free polynomial in $\mathbb{Z}[X]$ with period length $r$, then $H_n(X)^2 - D(X)K_n^2(X) = (-1)^{n-1}N_{n+1}(X)$.

**Proof.** The well-known classical result ([6, Theorem 7.3]) says that,

$$\zeta_0(X) = [a_0(X), a_1(X), a_2(X), \ldots, a_n(X), \zeta_{n+1}(X)]$$
$$= \zeta_{n+1}(X)H_n(X) + H_{n-1}(X)$$
$$\zeta_{n+1}(X)K_n(X) + K_{n-1}(X)$$
$$= \left( \frac{M_{n+1}(X) + \sqrt{D(X)}}{N_{n+1}(X)} \right) H_n(X) + H_{n-1}(X)$$
$$\left( \frac{M_{n+1}(X) + \sqrt{D(X)}}{N_{n+1}(X)} \right) K_n(X) + K_{n-1}(X)$$

$$\sqrt{D(X)} = \left( \frac{M_{n+1}(X) + \sqrt{D(X)}}{N_{n+1}(X)} \right) H_n(X) + H_{n-1}(X)N_{n+1}(X)$$
$$\left( \frac{M_{n+1}(X) + \sqrt{D(X)}}{N_{n+1}(X)} \right) K_n(X) + K_{n-1}(X)N_{n+1}(X).$$

We separate it as rational and irrational part and equate each part into zero.

$$-M_{n+1}(X)H_n(X) + K_n(X)D(X) - H_{n-1}(X)N_{n+1}(X) = 0 \quad (3.1)$$
$$M_{n+1}(X)K_n(X) + N_{n+1}(X)K_{n-1}(X) - H_n(X) = 0. \quad (3.2)$$
We eliminate $M_{n+1}(X)$ from above equations (3.1) and (3.2). Then we write

$$H^2_n(X) - D(X)K^2_n(X) = (H_n(X)K_{n-1}(X) - K_n(X)H_{n-1}(X))N_{n+1}(X)$$

Then by using the result $H_n(X)K_{n-1}(X) - K_n(X)H_{n-1}(X) = (-1)^{n-1}$ [9, Theorem 7.5], we obtain

$$H^2_n(X) - D(X)K^2_n(X) = (-1)^n N_{n+1}(X). \tag{3.3}$$

This completes the proof. \[\square\]

**Corollary 3.2.** Let $r$ be the length of the period in the continued fraction expansion of $\sqrt{D(X)}$. Then we have for $n \geq 0$ equation (3.3) becomes,

$$H^2_{nr-1}(X) - D(X)K^2_{nr-1}(X) = (-1)^{nr} N_{nr}(X) = (-1)^{nr}.$$  

Proof. We replace $n$ by $nr-1$ in equation (3.3).

$$H^2_{nr-1}(X) - D(X)K^2_{nr-1}(X) = (-1)^{nr} N_{nr}(X) = (-1)^{nr} N_0(X) = (-1)^{nr}.$$ \[\square\]

4. **THE ABC CONJECTURE FOR POLYNOMIALS (W. W. STOTHERS AND R. C. MASON)**

W. W. Stothers [14] and R. C. Mason [7] independently proved the $ABC$ conjecture for polynomials.

Let $n_0(P(X))$ denotes the number of distinct complex zeros of a polynomial $P(X)$ (which does not vanish identically). If $A, B, C$ are coprime polynomials over $\mathbb{C}$, not all constant polynomials satisfy $A + B = C$ then

$$\max\{\deg A, \deg B, \deg C\} < n_0(ABC). \tag{4.1}$$

In 1984, J. H. Silverman [11] using Riemann-Hurwitz formula and gave the different proof. In 2000, N. Snyder [13] gave the slight alternate proof of the Stothers-Mason theorem. The connection between the inequality (4.1) and Fermat’s last theorem for polynomials can be found in Lang’s survey article [6]. The $ABC$ conjecture for polynomials have notable applications to polynomial Pell’s equation. The following lemma is inspired by the result in [11, Theorem 1].

**Lemma 4.1.** If $n_0(D(X))$, where $D(X) \in \mathbb{C}[X]$ is less than or equal to $1/2 \deg D(X)$, then the negative polynomial Pell’s equation (1.4) has no non-trivial solutions in $\mathbb{C}[X]$.

Proof. We consider $A = P^2(X)$, $B = -D(X)Q^2(X)$, $C = -1$.

We note that $\max\{\deg A, \deg B, \deg C\} = \deg B$ and $n_0(P(X)) \leq \deg P(X)$, $n_0(Q(X)) \leq \deg Q(X)$.
By using $ABC$ conjecture for polynomials, we write

$$\deg D(X)Q^2(X) < n_0(P^2(X)D(X)Q^2(X))$$
$$= n_0(P(X)D(X)Q(X))$$

$$\deg D(X) < n_0(P(X)) + n_0(D(X)) + n_0(Q(X)) - 2 \deg Q(X)$$
$$\deg D(X) < \deg P(X) - \deg Q(X) + n_0(D(X))$$
$$1/2 \deg D(X) < n_0(D(X)).$$

This completes the proof. □

5. PROOF OF THEOREM 1.3

We use the method of continued fraction expansion of $\sqrt{X^{2k} + d}$, $d \in \mathbb{Z}$. i.e.,

$$\sqrt{X^{2k} + d} = [X^k, \frac{2X^k}{d}, 2X^k].$$

By using Lemma 1.4, the fundamental solution over $\mathbb{C}$ is $(\frac{X^k}{\sqrt{d}}, \frac{1}{\sqrt{d}})$, $d \in \mathbb{Z}$. The integer polynomial solution is possible only for odd periodic lengths.

Thus

$$\left(\frac{X^k + \sqrt{X^{2k} + d}}{\sqrt{d}}\right)^{2n-1} = \frac{1}{d^{(2n-1)/2}} \left(X^k + \sqrt{X^{2k} + d}\right)^{2n-1}$$
$$= P_{2n-1}(X) + \sqrt{X^{2k} + d}Q_{2n-1}(X), n \in \mathbb{N}$$

We expand the powers as we done in the Theorem 1.2. Thus to show the existence of non-trivial solutions in $\mathbb{Z}[X]$ for the negative polynomial Pell’s equation (1.6), it is enough to show that the leading coefficient of $P_{2n-1}(X)$ is an integer. Thus the coefficient of $X^{k(2n-1)}$ in $P_{2n-1}(X)$ is

$$\frac{1}{d^{(2n-1)/2}} \left(1 + \binom{2n-1}{2} + \binom{2n-1}{4} + \ldots\right) = \frac{2^{(2n-2)}}{d^{(2n-1)/2}}.$$

The integer solutions are exist if and only if $d=1$. This completes the proof of the theorem.

The following theorems are some of other generalization of the Theorem 1.2.

Theorem 5.1. The negative polynomial Pell’s equation

$$P^2(X) - (X^{2k} + aX + b)Q^2(X) = -1, \quad (5.1)$$

where $a, b \in \mathbb{Z}$ has no non-trivial solutions in $\mathbb{Z}[X]$.

Theorem 5.2. The negative polynomial Pell’s equation

$$P^2(X) - (X^{2k} + aX^k + b)Q^2(X) = -1, \quad (5.2)$$

where $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$ has no non-trivial solutions in $\mathbb{Z}[X]$. 
Since the length of the period in the continued fraction expansions of both $\sqrt{X^{2k} + aX + b}$ and $\sqrt{X^{2k} + aX^k + b}$ is 2, then by Corollary 3.2 we can say both the negative polynomial Pell’s equations (5.1) and (5.2) have no non-trivial solutions in $\mathbb{Z}[X]$.

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