Option Pricing with Transaction Costs under the Subdiffusive Mixed Fractional Brownian Motion

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Abstract. This paper probes into the issue of option pricing with transaction costs under the subdiffusive mixed fractional Brownian motion. Under reasonable economic assumptions, and by applying the strategy of the mean-self-financing delta hedging in the discrete-time setting, the generalized European call option pricing formula is further developed to capture the certain property of financial time series and better observe the law of finance market.

1. Introduction
Since 1973, the Black-Scholes (BS) model [1] has become the most popular method for option pricing till now, in which the price of an asset \( X_t \) follows a geometric Brownian motion (GBM)

\[
X_t = X_0 e^{\mu t + \sigma B(t)}, \quad X_0 > 0,
\]

where \( B(t) \) is the standard Brownian motion. However, the empirical research shows that the BS model fail to exhibit the long-range dependence property. That is why many researchers including Cutland et al. [2] have proposed and studied the fractional Black-Scholes model, in which the price of an asset \( X_t \) follows the fractional Brownian motion (FBM)

\[
X_t = X_0 e^{\mu t + \sigma B_H(t)}, \quad X_0 > 0,
\]

where \( B_H(t) \) is the fractional Brownian motion with index \( H \in (1/2,1) \).

Unfortunately, this model has also some deficiencies. For instance, the papers [3,4] has shown that such model admits arbitrage, for which we recall that intuitively. The existence of an arbitrage is a sign of the lack of equilibrium in the market. In other words, no real market equilibrium can exist in the long run if there are arbitrages.

Cheridito [5] presented the stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process \( X_t \) defined by

\[
X_t = X_0 e^{\mu t + \sigma B(t) + \sigma_H B_H(t)}, \quad X_0 > 0,
\]

where \( \sigma B(t) + \sigma_H B_H(t) \) is the mixed fractional Brownian motion(MFBM) of parameters \( \sigma, \sigma_H \) and \( H \). Many authors [6-8] studied pricing problem of option and warrant under this model.

In spite of many obvious advantages of the MFBM model, but it cannot capture the characteristic features of constant prices, which can be observed on emerging markets where the number of transactions is low. Notably, these constant periods of financial processes are similar in nature to the trapping events in which the subdiffusive particle gets immobilized.

In the paper [9], Magdziarz applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion(SGBM) \( X_\alpha(t) = X(T_\alpha(t)) \), where \( X(t) \) is the GBM, \( T_\alpha(t), 0 < \alpha < 1 \) is the inverse \( \alpha \) -stable subordinator. To see more models which describe such characteristic behaviour, one
can refer to the relevant papers [10-14]. Gu et al. [10] handled the problem of discrete time option pricing by the fractional subdiffusive Black-Scholes model, whose price of underlying stock is $X_t = X_0 e^{\sigma B(t) + \sigma_B B_H(t)}$, $X_0 > 0$, where $B_H(t)$ is the FBM.

This paper, inspired by the work of Wang [7] and Gu [10], to capture the above mentioned property of financial time series, considers the subdiffusive mixed fractional Brownian BS model with transaction costs, whose price of underlying stock is $X_t = X_0 e^{\sigma B(t) + \sigma_B B_H(t) + \sigma_B B_H(t)}$, $X_0 > 0$, where $T_a(t), 0 < \alpha < 1$ is the inverse $\alpha$-stable subordinator. $T_a(t)$ is assumed to be independent of $B(t)$ and $B_H(t)$, and $B(t)$ are independent of $B_H(t)$.

The rest of the paper is organized as follows. In Section 2, we deduce some properties of the subdiffusive mixed fractional Brownian motion. In Section 3, by using the delta hedging strategy, the option pricing problem is studied in the presence of transaction costs and an option pricing formula is deduced under the environment of the subdiffusive mixed fractional Brownian motion environment. The conclusion is set out in Section 5.

2. Subdiffusive Mixed Fractional Brownian Motion

First, this section provides the definition of the subdiffusive mixed fractional Brownian motion.

Definition 2.1. The subdiffusive mixed fractional Brownian motion of parameters $\sigma, \sigma_H, \alpha$ and $H$ is a linear combination of subdiffusive Brownian motion and subdiffusive fractional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$ for any $t \in \mathbb{R}$, by

$$M^{\mu_H}_{T_a(t)} = \sigma B(T_a(t)) + \sigma_H B_H(T_a(t)),$$

where $B(t)$ is a Brownian motion, $B_H(t)$ is an independent fractional Brownian motion with Hurst parameter $H \in (0,1)$, $\sigma$ and $\sigma_H$ are two real constants such that $(\sigma, \sigma_H) \neq (0,0)$, $T_a(t)$ is the inverse $\alpha$-stable subordinator independent of $B(t)$ and $B_H(t)$, $0 < \alpha < 1$, $\mathcal{F}_t = \sigma[B(s); s \leq t] \lor \sigma[B(s); s \leq t]$. The inverse $\alpha$-stable subordinator is defined as $T_a(t) = \inf\{t > 0 : U_a(t) > t\}, 0 < \alpha < 1$, where $U_a(t)$ is the $\alpha$-stable subordinator [9] with Laplace transform $E(e^{-u U_a(t)}) = e^{-u^\alpha}$. Therefore the Laplace transform of $\{T_a(t)\}$ can be represented as $E[e^{-u T_a(t)}] = E_a(-u^\alpha)$, where $E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}$ is the Mittag-Leffler function [15]. $U_a(t)$ is a pure-jump process, for every jump of $U_a(t)$, there is a corresponding flat period of its inverse $T_a(t)$, these heavy-tailed flat periods of $T_a(t)$ represent long waiting-times in which the subdiffusive particle gets immobilized in the trap.

When $\alpha \uparrow 1$, $T_a(t)$ reduces to the "objective time" $t$.

Let $Z_a(t) = B(T_a(t))$ and $Z_{a,H}(t) = B_H(T_a(t))$, then by (1) $M^{\mu_H}_{T_a(t)} = \sigma Z_a(t) + \sigma_H Z_{a,H}(t)$.

The process $T_a(t)$ is $\alpha$-self-similar, $B(t)$ is $1/2$-self-similar and $B_H(t)$ is $H$-self-similar, therefore $Z_a(t)$ is $\alpha/2$-self-similar, and $Z_{a,H}(t)$ is $\alpha H$-self-similar. Then we have

$$M^{\mu_H}_{T_a(t)}(\sigma, \sigma_H) \triangleq M^{\mu_H}_{T_a(t)}(\sigma^{\alpha^2}, \sigma_H^{\alpha H}),$$

where $\triangleq$ means "to have the same law".

The next lemma is about the increments of the inverse $\alpha$-stable subordinator and the subdiffusive mixed fractional Brownian motion.

Lemma 2.1. For $n \in \mathbb{N}, 0 \leq s \leq t < \infty$, we have

$$M^{\mu_H}_{T_a(t)}(\sigma, \sigma_H, \mu_H, \mu_H) \triangleq M^{\mu_H}_{T_a(t)}(\sigma^{\alpha^2}, \sigma_H^{\alpha H}),$$

where $\triangleq$ means "to have the same law".
\[ E[|T_a(t) - T_a(s)|^n] \leq \frac{n!}{\Gamma^n(\alpha + 1)} (t - s)^{\alpha n}, \quad (2) \]

\[ E[|M_{T_a(t)}^{H} - M_{T_a(s)}^{H}|^n] \leq (n - 1)!! \left( |\sigma|^n \left[ \frac{n!}{\Gamma^n(\alpha + 1)} \right]^2 (t - s)^{\alpha n} + |\sigma_H|^n \left[ \frac{n!}{\Gamma^n(\alpha + 1)} \right]^H (t - s)^{\alpha Hn} \right). \quad (3) \]

Furthermore, for each \( \varepsilon \in (0, \alpha / 2) \), \( \Delta T_a(t) = o((\Delta t)^{\alpha - \varepsilon}) \) and \( \Delta M_{T_a(t)}^{H} = o((\Delta t)^{\alpha / 2 - \varepsilon}) \).

**Proof** The estimate (2) can be found in [9]. Now we prove (3), by using the stationarity and the mixed-self-similarity of the increments of the mixed fractional Brownian motion in [6], and we obtain

\[
E \left[ |M_{T_a(t)}^{H} - M_{T_a(s)}^{H}|^n \right] = E \left[ |M_{T_a(t)}^{H} - M_{T_a(s)}^{H}|^n \right]
\]

\[
= E \left[ |M_{T_a(t)}^{H} - M_{T_a(s)}^{H}|^n \right] = E \left[ |\sigma H(T_a(t) - T_a(s))(\alpha n)|^n \right]
\]

Since \( T_a(t), B(t) \) and \( B_H(t) \) are independent each other, for \( n \in \mathbb{N} \), we get that

\[
E \left[ |\sigma H(T_a(t) - T_a(s))(\alpha n)|^n \right] \leq E \left[ |\sigma H| \right]^n \left[ B_H(1)(T_a(t) - T_a(s))(\alpha n) \right]^n
\]

which implies \( \Delta T_a(t) = o((\Delta t)^{\alpha - \varepsilon}) \) and \( \Delta M_{T_a(t)}^{H} = o((\Delta t)^{\alpha / 2 - \varepsilon}) \). The proof is completed.

**Lemma 2.2.** Let \( \alpha \in (2 / 3, 1) \) and \( n \in \mathbb{N} \). Then for any positive numbers \( a \neq 1 \) and \( T \), there exist the \( n \)-th moments of \( a^{\sigma Z_a(t) + \sigma_n Z_a(t)} \). Moreover, the moments of any order of \( a^{\sigma Z_a(t) + \sigma_n Z_a(t)} \) are uniformly bounded with respect to \( t \in [0, T] \).

**Proof** Let \( Y = a^{\sigma Z_a(t) + \sigma_n Z_a(t)} \), then \( Y^n = e^{n \ln a^{\ln a^{\sigma Z_a(t) + \sigma_n Z_a(t)}}} \), and

\[
E[Y^n] = E[a^{n \ln a^{\ln a^{\sigma Z_a(t) + \sigma_n Z_a(t)}}}] = \sum_{k=0}^{\infty} \frac{(n \ln a)^k}{k!} E[|\sigma Z_a(t) + \sigma_n Z_a(t)|]^k
\]

\[
\leq \sum_{k=0}^{\infty} (n \ln a) \left| \frac{(n \ln a)^k}{k!} \right|^2 \left( \frac{(k - 1)!}{k!} \right)^2 + \sum_{k=0}^{\infty} (n \ln a) \left| \frac{(n \ln a)^k}{k!} \right|^H \left( \frac{(k - 1)!}{k!} \right)^H
\]

Since

\[
\frac{a_{k+2}}{a_k} = \frac{(n \ln a) \left| \frac{(n \ln a)^k}{k!} \right|^2}{k + 2} (k + 1)^2 (k + 1) \left( \frac{(k + 1)!}{(k + 2)!} \right)^2 \left( \frac{(k + 1)!}{(k + 2)!} \right)^2 \rightarrow 0 (k \rightarrow \infty).
\]

Similarly, we have \( b_{k+2} / b_k \rightarrow 0 (k \rightarrow \infty) \). This means \( E[Y^n] < +\infty \). From the above proof, it is obvious that \( E[Y^n(t)] \) is uniformly bounded on any compact subset of \([0, +\infty)\). The proof is completed.
3. Option Pricing with Transaction Costs

In the presence of transaction, Leland [16] first examined option replication in the discrete time setting, and pose the modified replicating strategy, which depends upon the level of transaction costs and upon the revision interval, as well as upon the option to be replicated and the environment factors. In this section we will derive the pricing formula for the European call option of the subdiffusive mixed fractional BS model under the other usual assumptions in the Black-scholes model but with the following assumptions:

(i) A $(D,S)$-market with a riskless bond $D_t$ and a stock $S_t$, where $D_t = D_0e^{rt}$, and

$$S_t = S_0e^{aT_{a}(t)+\sigma_{a}(t)+\sigma_{a,H}(t)}, \quad S_0 > 0,$$

where $\alpha \in (2/3,1)$, $\mu, \sigma, \sigma_H > 0$, $H \in (1/2,1)$, $Z_a(t) = B(T_a(t)), Z_{a,H}(t) = B_H(T_a(t)), T_a(t)$ is the inverse $\alpha$-stable subordinator independent of the Brownian motion $B(t)$ and fractional Brownian motion $B_{H}(t), B(t)$ are independent of the $B_{H}(t)$, both defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$, where $P$ is the real world probability measure, $\mathcal{F}_0$ is trial, $\mathcal{F} = \mathcal{F}_t$ for some $T \in (0, +\infty)$, and $(\mathcal{F}_t)_{t \in [0,T]}$ denotes the $P$-augmentation of the filtration generated by $(B(\tau), B_{H}(\tau), \tau \leq t)$.

(ii) The stock pays no dividends of other distributions and all securities are perfectly divisible. There are no penalties to short selling. It is possible to borrow any fraction of the price of a security to buy it or to hold the security, at the short-term interest rate. There are the same valuation policy as in the BS model.

(iii) Transaction costs are proportional to the value of the transaction in the underlying. Let $k$ denote the round trip transaction cost per unit dollar of transaction. Suppose $V$ shares are bought ($V > 0$) or sold ($V < 0$) at the price of $S_t$, then the transaction cost is given by $k/2|V|S_t$ in either buying or selling, where $k$ is a constant. Moreover, trading takes place only at discrete intervals.

(iv) The option value is replicated by the replicating portfolio with $X_1(t)$ units of the riskless bond and $X_2(t)$ units of the underlying asset. The value of the option must equal to the value of the replicating portfolio to reduce (but not to avoid) arbitrage opportunities and be consistent with economic equilibrium.

(v) The hedge portfolio has an expected return equal to that from an option. The portfolio is revised every $\Delta t$ and hedging takes place at equidistant time points with rebalancing intervals of (equal) length $\Delta t$, where $\Delta t$ is a finite and fixed, small timestep.

Let $C = C(t, S_t)$ be the value of a European call option on the stock $S_t$ at time $t$ with expiration date $T$ and exercise price $X$ and the boundary conditions:

$$C = C(T, S_T) = (S_T - X)^+, C(t, T) = 0, C(t, 0) = 0, C(t, S_t) \rightarrow S_t, as \quad S_t \rightarrow +\infty.$$ 

Under the above assumptions we have the following theorem.

**Theorem 3.1.** Let $C = C(t, S_t)$ be the value of a European call option on the stock $S_t$ satisfied (4) and the trading takes place discretely with rebalancing intervals of length $\Delta t$. Then under the above assumption, $C$ is given by

$$C(t, S_t) = S_t N(d_1) - X e^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln(S_t / X) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}}, d_2 = d_1 - \sigma \sqrt{T-t},$$

$$d_1 = \frac{\ln(S_t / X) + (r + \tilde{\sigma}^2 / 2)(T-t)}{\tilde{\sigma} \sqrt{T-t}}, d_2 = d_1 - \tilde{\sigma} \sqrt{T-t},$$
\[ \sigma = \left[ A^{\alpha}_{t,H}(t, \Delta t) + kA^{\beta}_{t,H}(t, \Delta t) \right]^2 (\Delta t)^{-\frac{1}{2}}, \]

\[ A^{\alpha}_{t,H}(t, \Delta t) = \sqrt{\frac{2}{\pi}} \left( \sigma^2 E[\Delta T_a(t)] + \sigma_H^2 E[(\Delta T_a(t))^{2H}] \right), \]

\[ A^{\beta}_{t,H}(t, \Delta t) = \sigma^2 E[\Delta T_a(t)] + \sigma_H^2 E[(\Delta T_a(t))^{2H}], \]

and \( N(\cdot) \) is the value of the cumulative normal density function.

We consider the replicating portfolio with \( X_1(t) \) units of the riskless bond and \( X_2(t) \) units of underlying asset. The value of the portfolio at current time \( t \) is

\[ \Pi_t = X_1(t)D_t + X_2(t)S_t. \] — (6)

Next consider the changes in \( S_t \) and \( \Pi_t \) over the discrete-time interval \( \Delta t \). After time interval \( \Delta t \), the change in the value of the underlying asset is

\[ \Delta S_t = \mu S_t \Delta T_a(t) + S_t \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) + 1/2 S_t \left[ \mu \Delta T_a(t) + \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) \right]^2 \]

\[ + S_t / 6 e^{\theta[\mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)]}, \left[ \mu \Delta T_a(t) + \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) \right]^3, \] — (7)

where \( \theta = \theta(t, \omega), \omega \in \Omega, 0 < \theta < 1. \)

Noting that \( 1/6 e^{\theta[\mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)]} \leq 1/6 e^{\theta \mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)} \)

Since

\[ E[e^{\theta \mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)}] = \sum_{n=0}^{\infty} \frac{\mu^n E[(T_a(T))^n]}{n!} \frac{\mu^n E[(T_a(T))^n]}{n!} < \infty, \] — (8)

By (8) and Lemma 2.2, we get \( E\left[ \frac{1}{6} e^{\theta[\mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)]} \right] < \infty, \) therefore

\[ (\Delta t)^{2\epsilon} \frac{1}{6} e^{\theta[\mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)]} = o((\Delta t)^{\epsilon}). \] — (9)

It follows from Lemmas 2.1 and (9) that

\[ \frac{1}{6} e^{\theta[\mu \Delta T_a(t) + \sigma Z_a(t) + \sigma_H Z_{a,H}(t)]} \left( \mu \Delta T_a(t) + \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) \right)^3 \]

\[ = (\Delta t)^{-2\epsilon} o((\Delta t)^{\epsilon}) + o((\Delta t)^{2\epsilon}), \] — (10)

Where \( \alpha \in (2 / 3,1), \) so \( o((\Delta t)^{3\alpha/2 - 4\epsilon}) = o(\Delta t). \)

From (7)–(10), we deduce

\[ \Delta S_t = \mu_s \Delta T_a(t) + S_t \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) + \frac{1}{2} S_t \left[ \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) \right]^2 + o(\Delta t), \] — (11)

which implies \( (\Delta S_t)^2 = S_t \left[ \Delta(\sigma Z_a(t) + \sigma_H Z_{a,H}(t)) \right]^2 + o(\Delta t). \)

The change in the value of the portfolio (6) is

\[ \Delta \Pi_t = X_1(t) \Delta D_t + X_2(t) \Delta S_t - \frac{k}{2} \Delta X_2(t) S_t, \] — (12)
where $\Delta D_i$ is the change in the riskless bond price, $\Delta X_i(t)$ is the change in the number of units of asset held in the portfolio. Since the timestep and asset movement are both small, from Taylor’s theorem we have

$$\Delta D_i = rD_i \Delta t + o(\Delta t), \quad (13)$$

$$\Delta C(t, S_i) = \frac{\partial C(t, S_i)}{\partial t} \Delta t + \frac{\partial C(t, S_i)}{\partial S_i} \Delta S_i + \frac{1}{2} \frac{\partial^2 C(t, S_i)}{\partial S_i^2} (\Delta S_i)^2 + o(\Delta t), \quad (14)$$

and

$$\Delta X_i(t, S_i) = \frac{\partial X_i(t, S_i)}{\partial t} \Delta t + \frac{\partial X_i(t, S_i)}{\partial S_i} \Delta S_i + \frac{1}{2} \frac{\partial^2 X_i(t, S_i)}{\partial S_i^2} (\Delta S_i)^2 + o(\Delta t) \quad (15)$$

$$= \left. \frac{\partial X_i(t, S_i)}{\partial t} \right|_t \Delta t + \frac{1}{2} \left. \frac{\partial^2 X_i(t, S_i)}{\partial S_i^2} \right|_t (\Delta S_i)^2 + o(\Delta t).$$

By assumption (v), $C = C(t, S_i)$ is replicated by the portfolio $\Pi_i$, therefore $C(t, S_i) = X_i(t) D_i + X_i(t) S_i$, and $X_i(t) = \frac{\partial C(t, S_i)}{\partial S_i}$. From (5), we can check that $\left. \frac{\partial^2 C}{\partial S_i \partial t} \right|_t$ are $o((\Delta t)^{\frac{1}{2} - \epsilon})$ (see [13]). Thus by (15), we get that the leading order of $|\Delta X_i(t)|$ is

$$|\Delta X_i(t, S_i)| = S_i \left| \frac{\partial X_i(t, S_i)}{\partial S_i} \right| \left| \Delta(\sigma Z_a(t) + \sigma H Z_{a,H}(t)) \right| + o(\Delta t). \quad (16)$$

From (12), (13) and (16), we obtain

$$\Delta \Pi_i = rX_i(t) D_i \Delta t + X_i(t) \Delta S_i - \frac{k}{2} S_i^2 \left| \Delta(\sigma Z_a(t) + \sigma H Z_{a,H}(t)) \right| \left| \frac{\partial X_i(t, S_i)}{\partial S_i} \right| + o(\Delta t) \quad (17)$$

From the practical point of view, we assume that trading occurs at $t$ and $t + \Delta t$, but not in between. That means that the current stock price $S_i$ and the number of shares given by the delta hedging strategy are held constants over the rebalancing interval $[t, t + \Delta t]$. Thus by (14) and (17), we get

$$\Delta C(t, S_i) - \Delta \Pi_i = \left( \frac{\partial C(t, S_i)}{\partial t} - rX_i(t) \right) \Delta t + \left( \frac{\partial C(t, S_i)}{\partial S_i} - X_i(t) \right) \Delta S_i + \frac{1}{2} \frac{\partial^2 C(t, S_i)}{\partial S_i^2} (\Delta S_i)^2$$

$$+ \frac{k}{2} S_i^2 \left| \Delta(\sigma Z_a(t) + \sigma H Z_{a,H}(t)) \right| \left| \frac{\partial X_i(t, S_i)}{\partial S_i} \right| + o(\Delta t) \quad (18)$$

$$= \left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_i} - rC \right) \Delta t + \frac{1}{2} \left. \frac{\partial^2 C}{\partial S_i^2} \right|_t \Delta(\sigma Z_a(t) + \sigma H Z_{a,H}(t))^2$$

$$+ \frac{k}{2} S_i^2 \left| \Delta(\sigma Z_a(t) + \sigma H Z_{a,H}(t)) \right| \left| \frac{\partial^2 C}{\partial S_i^2} \right| + o(\Delta t).$$
Taking the expectation on both sides of (18), we deduce

\[
E\left[ \Delta C(t, S_t) - \Delta \Pi_t \right] = \left( \frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} - rC + \frac{1}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} \right) \Delta t + \frac{k}{2} S_t^2 E\left[ \left| \frac{\partial^2 C}{\partial S_t^2} \right| \right] o(\Delta t)
\]

\[
= \left( \frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} - rC + \frac{1}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} \right) \Delta t + \frac{k}{2} S_t^2 E\left[ \left| \frac{\partial^2 C}{\partial S_t^2} \right| \right] (\Delta t)^{-1} + o(\Delta t)
\]

(19)

Since \( T_\alpha(\tau) \) is non-decreasing, \( B(\tau) \) is 1/2-self-similar with stationary increments, \( B_\mu(\tau) \) is \( H \) -self-similar with stationary increments, and the three processes are independent from each other, we get that

\[
E[(\sigma \Delta Z_\alpha(t))^2] = \sigma^2 E[B^2(1)]E[\Delta T_\alpha(t)] = \sigma^2 E[\Delta T_\alpha(t)],
\]

(20)

\[
E[(\sigma_\mu \Delta Z_{\alpha,\mu}(t))^2] = \sigma_\mu^2 E[B^2_\mu(1)]E[\Delta T_{\alpha,\mu}(t)^{2H}] = \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t)^{2H}],
\]

(21)

From (20) and (21), we have

\[
E\left[ \left( \Delta (\sigma Z_\alpha(t) + \sigma_\mu Z_{\alpha,\mu}(t)) \right)^2 \right] = E\left[ \left( \sigma \Delta Z_\alpha(t) + \sigma_\mu \Delta Z_{\alpha,\mu}(t) \right)^2 \right]
\]

\[
= \sigma^2 E[B^2(1)(\Delta T_\alpha(t))] + \sigma_\mu^2 E[B^2_\mu(1)(\Delta T_{\alpha,\mu}(t))^{2H}]
\]

\[
= \sigma^2 E[\Delta T_\alpha(t)] + \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t))^{2H}],
\]

(22)

and (see [13])

\[
E\left[ \left| \Delta (\sigma Z_\alpha(t) + \sigma_\mu Z_{\alpha,\mu}(t)) \right| \right] = \frac{2}{\sqrt{\pi}} E\left[ (\sigma \Delta Z_\alpha(t) + \sigma_\mu \Delta Z_{\alpha,\mu}(t))^2 \right]
\]

\[
= \frac{2}{\sqrt{\pi}} \left( E[\sigma^2 \Delta T_\alpha(t)] + \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t))^{2H}] \right)
\]

(23)

From (19), (22), (23) and assumption (v) , we get

\[
E\left[ \Delta C(t, S_t) - \Delta \Pi_t \right] = \left( \frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} - rC + \frac{1}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} \right) E\left[ \sigma^2 \Delta T_\alpha(t) + \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t))^{2H}] \right] + \frac{k}{2} S_t^2 \sqrt{\frac{2}{\pi}} E\left[ \left| \frac{\partial^2 C}{\partial S_t^2} \right| \right] (\Delta t)^{-1} + o(\Delta t)
\]

\[
+ \frac{k}{2} \sigma_\mu^2 \sqrt{\frac{2}{\pi}} E\left[ \left| \frac{\partial^2 C}{\partial S_t^2} \right| \right] (\Delta t)^{-1} + o(\Delta t) = 0
\]

which is the mean self-finacing delta strategy in a discrete-time setting.

Let

\[
A^{\alpha, H}_t(\Delta t) = \sqrt{\frac{2}{\pi}} \left( E[\sigma^2 \Delta T_\alpha(t)] + \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t))^{2H}] \right),
\]

(\( A^{\mu, H}_t(\Delta t) = E[\sigma^2 \Delta T_\mu(t)] + \sigma_\mu^2 E[(\Delta T_{\alpha,\mu}(t))^{2H}] \))

(23)

So we assume that
\[ \frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{1}{2} A_{\sigma,H}^2(t, \Delta t) \frac{\partial^2 C}{\partial S_t^2} + \frac{k}{2} A_{\sigma,H}^2(t, \Delta t)(\Delta t)^{-1} S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0 \]  

Let \( \tilde{\sigma} = \left[ A_{\sigma,H}^2(t, \Delta t) + kA_{\sigma,H}^4(t, \Delta t) \text{sign} \left( \frac{\partial^2 C}{\partial S_t^2} \right) \right]^{1/2} (\Delta t)^{-1/2} \), where \( \text{sign} \left( \frac{\partial^2 C}{\partial S_t^2} \right) \) represents the sign of \( \frac{\partial^2 C}{\partial S_t^2} \). It is known that \( \frac{\partial^2 C}{\partial S_t^2} \) is always positive for the simple European call option in the absence of transaction costs, if we postulate the same behaviour of \( \frac{\partial^2 C}{\partial S_t^2} \) here, then

\[ \tilde{\sigma}^2 = \left[ A_{\sigma,H}^2(t, \Delta t) + kA_{\sigma,H}^4(t, \Delta t) \right] (\Delta t)^{-1} \]

Therefore we may rewrite (24) in the form which the Black-Scholes equation

\[ \frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \tilde{\sigma}^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0 \]  

Moreover, from (25) we obtain

\[ C(t, S_t) = S_t N(d_1) - X e^{-r(T-t)} N(d_2), \]

where \( d_1 = \frac{\ln(S_t / X) + (r + \tilde{\sigma}^2 / 2)(T-t)}{\tilde{\sigma} \sqrt{T-t}} \), \( d_2 = d_1 - \tilde{\sigma} \sqrt{T-t} \).

The proof is completed.

**Remark 3.1.** if \( \sigma = 0 \), then \( A_{\sigma,H}^4(t, \Delta t) = \sqrt{\frac{2}{\pi}} \sigma_H^2 E[(\Delta T_{\sigma}(t))^{2H}] \), \( A_{\sigma,H}^2(t, \Delta t) = \sigma_H^2 E[(\Delta T_{\sigma}(t))^{2H}] \).

That is consistent with the result in [10].

**Remark 3.2.** if \( \alpha \uparrow 1 \), then \( T_{\alpha}(t) \) degenerates to \( t \) and \( Z_{\alpha}(t) \) to the standard BM \( B(t) \), \( Z_{\alpha,H}(t) \) to the FBM \( B_H(t) \). So \( A_{\sigma,H}^4(t, \Delta t) = \sqrt{\frac{2}{\pi}} (\sigma^2 \Delta t + \sigma_H^2 (\Delta t)^{2H}) \), \( A_{\sigma,H}^2(t, \Delta t) = \sigma^2 \Delta t + \sigma_H^2 (\Delta t)^{2H} \), that is consistent with the result in [7].

**Remark 3.3.** if \( \alpha \uparrow 1 \) and \( \sigma_{H} = 0 \), the result is consistent with the case of classic BS model with transaction costs in [16].

### 4. Conclusion

The pricing of European options is a hot issue in the world. Scholars have studied it under different assumptions and obtained many pricing models, for example, Brownian motion, fractional Brownian motion, mixed fractional Brownian motion, and so on. Based on the above research, scholars propose to use more general self-similar process as random model. This requirement has been put forward in many important theoretical problems about this kind of process. However, there are few researches on this kind of self-similar Gaussian process. The main reason is the complexity of the dependent structure of self-similar Gaussian processes without stationary increments. The subdiffusive mechanism of trapping events is self-similar Gaussian process and can describe properly financial data exhibiting periods of constant values. In order to capture the actual situation of the financial market, this paper addresses the problem of pricing European option under the environment of the subdiffusive mixed fractional Brownian motion, and develop the more general BS model with transaction costs, thus extending the work in [7,10,16].

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References
[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy 81 (1973) pp 637-654.
[2] N.J.Cutland, P.E.Kopp, and W.Willinger, Stock price returns and the Joseph effect: a fractional version of the Black-Sholes model, Seminaer on Stochastic Analysis, Random Fields and Applications, Progress in Probability, 36(1995) pp 327-351.
[3] P.Cheridito, Arbitrage in fractional Brownian motion models, Finance Stoch,7(2003)pp 533-553.
[4] L.C.G. Rogers, Arbitrage with fractional Brownian motion, Mathematical Finance 7(1)(1997) pp 95-105.
[5] P. Cheridito, Mixed fractional Brownian motion, Bernoulli 7(2001) pp 913-934.
[6] M.Zili, On the mixed fractional Brownian motion, J.Appl.Math.Stoch. Annal. (2006) pp 1-9.
[7] X. Wang, E. Zhu, M. Tang, H. Yan, Scaling and long-range dependence in option pricing II: pricing European option with transaction costs under the mixed Brownian-fractional Brownian model, Physica A 389(2010) pp 445-451.
[8] F. Shokrollahi, A. Kihcman M. Magdziarz, Pricing European options and currency options by time changed mixed fractional Brownian motion with transaction costs,1(2016) pp 1-22.
[9] M. Magdziarz, Stochastic representation of subdiffusion processes with time-dependent drift, Stochastic Processes and their Applications 119(2009) pp 3238-3252.
[10] H. Gu, J. Liang, Y. Zhang, Time-changed geometric fractional Brownian motion and option pricing with transaction costs, Physica A 391(2012) pp 3971-3977.
[11] S. Orzel, A. Wylomanska, Calibration of the subdiffusive arithmetic Brownian motion with tempered stable waiting-times, Journal of Statistical Physics 143(2011) pp 447-454.
[12] J. Janczura, S. Orzel, A. Wylomanska, Subordinated $\alpha$-stable Ornstein-Uhlenbeck process as a tool of financial data description, Physica A 390(2011) pp 4379-4387.
[13] A. Janicki, A. Weron, Can one see $\alpha$-stable variables and processes, Statistical Science (1994) pp 109-126.
[14] M. Magdziarz, Stochastic representation of subdiffusion processes with time-dependent drift, Stochastic Processes and their Applications 119(2009) pp 3238-3252.
[15] S. G. Samko, A. A. Kilbas, D.I. Maritchev, Integrals and derivatives of the fractional order and some of their applications, Gordon and Breach Science Publishers, Amsterdam, 1993.
[16] H. E. Leland, Option pricing and replication with transaction costs, Finance 40(1985) pp 1283-1301.