Non-Hermitian topological states in 2D line-graph lattices: evolving triple exceptional points on reciprocal line graphs

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Abstract

Non-Hermitian (NH) topological states, such as the doubly-degenerate nodes dubbed as exceptional points (EPs) in Bloch band structure of 2D lattices driven by gain and loss, have attracted much recent interest. We demonstrate theoretically that in the three-site edge-centered lattices, i.e. the so-called line-graph lattices, such as kagome lattice which is a line graph of hexagonal lattice, there exist three types of triply-degenerate EPs evolving intriguingly on another set of line graphs in the reciprocal space. A single TEP (STEP) with $\pm 1/3$ topological charge moves faithfully along the edges of reciprocal line graphs with varying gain and loss, while two STEPs merge distinctively into one unconventional orthogonal double TEP (DTEP) with $\pm 2/3$ charge at the vertices, which is characterized with two ordinary self-orthogonal eigenfunctions but one surprising ‘orthogonal’ eigenfunction. Differently, in a modified line-graph lattice with an off-edge-center site, the ordinary coalesced state of DTEPs emerges with three identical self-orthogonal eigenfunctions. Such NH states and their evolution can be generally realized in various artificial systems, such as photonic and sonic crystals, where light and sonic vortex beams with different fractional twisting can be found. Our findings shed new light on fundamental understanding of gapless topological states in NH systems in terms of creation and evolution of high-order EPs, and open up new research directions to further link line graph and flow network theory coupled with topological physics, especially under non-equilibrium gain/loss conditions.

1. Introduction

Non-Hermitian (NH) lattices with onsite gain and loss have been shown to exhibit various topological phenomena. For example, in a rhombic lattice under driving, a Dirac point splits into a pair of doubly-degenerate exceptional points (EPs), each characterized with degenerate eigenvalues and identical self-orthogonal eigenvectors [1–7]. The topology of an EP is characterized by a nonzero topological invariant

$$\nu = - \frac{1}{2\pi} \oint_{\text{EP}} \nabla_k \arg [E_1(k) - E_2(k)] \cdot dk = \pm \frac{1}{2},$$

where integration is carried out on a $k$-space loop enclosing the EP. $E_1(k)$ and $E_2(k)$ are two complex eigenvalues at each $k$ point. The invariant represents a winding number, associated with the energy dispersion of NH band structures around the EP, i.e. the vorticity around the EP. The vorticity manifests also a topological charge $\nu$ with the physical implication of polarization [1]. It has been demonstrated that [2] the nonzero $\nu$ implies the existence of band degeneracy within the region enclosed by the loop, and an EP with half topological charge $\nu = \pm 1/2$ can be described by the Riemann surface $f = z^{1/2}$ ($z$ is a complex number) with a nodal point being doubly degenerated. A pair of EPs are connected by a bulk Fermi arc.
(BFA) with the degenerate real part but nondegenerate imaginary part of eigenvalues [1], which manifests itself in creating vortex beam with fractional charge. The EPs with higher degeneracy are used to create vortex beam beyond ±1/2 charge for enhancing sensitivity of NH detectors [8–13]. Other gapless NH topological states include exceptional lines, rings, and surfaces [14–22]. Furthermore, gapped NH topological states have been demonstrated for potential applications in making robust single-color laser based on NH Haldane cavity [23–26]. The second-order NH topological insulators can afford sound guides in sonic crystals [27, 28]. Therefore, imposing non-Hermiticity to a Hermitian system is an effective way to induce novel topological states, which is of both scientific and technological interests.

On the other hand, it is well recognized that all the 2D edge-center lattices can be viewed as line-graph lattices, since they can be constructed as line graphs in mathematical graph theory. A line graph is made by connecting the centers of edges that share a common vertex of the original graph. Interestingly, the line-graph lattices have been shown to exhibit exotic properties associated with topology and many-body interactions, such as magnetism and superconductivity [29–32]. One prevailing theorem is that the topological construction of line graph underlines a necessary condition for the existence of topological flat bands in the line-graph lattices, such as the well-known kagome lattice, where destructive interference of lattice wavefunctions leads to formation of highly localized compact plaquette quantum states [33, 34]. Then, an intriguing question arises: what happens when non-Hermiticity is introduced in the line-graph lattices? How the band topology will evolve with gain and loss, especially noting that an NH line graph may be linked to a non-equilibrium flow network in graph theory.

In this article, the band structure and topology of a class of 2D NH line-graph lattices, with three sites having gain and loss, are explored. Most strikingly, three types of triply degenerate EPs (TEPs) are found to always exist on another set of line graphs in the reciprocal space. By continuously varying gain and loss, single TEPs (STEPs) with ±1/3 charge move faithfully along the edges of reciprocal line graphs. When two STEPs meet at the vertices of the reciprocal line graph, they merge distinctively into an unconventional orthogonal double TEP (DTEP) with ±2/3 charge, featured with two identical ordinary self-orthogonal eigenfunctions but another surprising orthogonal eigenfunction. Differently, in the modified off-center line-graph lattices, in which one of the vertices of line graph is chosen along the edge but at an off-center position of the original graph, an ordinary coalesced state of DTEPs appear at the vertices, with three degenerate eigenvalues and identical self-orthogonal eigenfunctions. The intriguing evolution of TEPs in a reciprocal line graph and the existence of orthogonal DTEP (ODTEP) signify a set of unique NH topological properties of line-graph lattices.

### 2. TEPs in NH kagome lattice

We first consider a kagome lattice with gain and loss $\gamma_n$ ($n = 1, 2, 3$) at $n$th site, as shown in figure 1(a). The NH kagome Hamiltonian is

$$H = 2t \begin{pmatrix} 0 & \cos k_1 & \cos k_2 \\ \cos k_1 & 0 & \cos k_3 \\ \cos k_2 & \cos k_3 & 0 \end{pmatrix} + i \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \tag{2}$$

where $k_n = k \cdot a_n$, and $t$ is the nearest-neighbor (NN) hopping integral. The balanced gain and loss $\gamma_1 = 0, \gamma_2 = -\gamma_3 = \gamma$ is adopted to study the evolution of band structure. When $\gamma = 0$, the original Hermitian kagome model exhibits two Dirac bands touched with a bottom flat band (figures 1(b) and (c)), where the Dirac (degenerate) point at $K$ ($\Gamma$) is labeled as DP1 (DP2). As shown in figures 1(d) and (e), when $\gamma$ increases to 0.5$t$, DP1 splits into a pair of two-fold EPs (labeled as EP1), whose band shape in low energy can be described by the complex function $f = z^{1/2}$ containing all topologies of nodal points EP1 (see figure S1 [https://stacks.iop.org/NJP/23/123038/mmedia] in supplemental material). The EP1 pair, with a topological charge $\nu = \pm 1/2$ on each EP (computed by equation (1)), is connected by a BFA with the degenerate real but nondegenerate imaginary part of eigenvalues, which illustrates the nontrivial topology of the EPs. Meanwhile, DP2 splits into four EPs (labeled as EP2) which are connected by two crossing BFAs.

When $\gamma$ increases further to 2$t$, EP1 ($\nu = -1/2$) and EP2 ($\nu = -1/2$) merge together to form a TEP node, as shown in figures 1(f)–(h). To distinguish the exceptionality of the TEP node, we calculated the Petermann factor [8, 35]

$$PF_n = \frac{|\langle \psi_{nL}^R | \psi_{nL}^R \rangle \langle \psi_{nR}^L | \psi_{nR}^L \rangle|}{|\langle \psi_{nL}^R | \psi_{nR}^L \rangle|^2}, \tag{3}$$
The STEP exhibits a fractional topological charge \( \nu' = \frac{1}{3} \) for the states with eigenvalues \( E_n \) around the node, where \( n = 1, 2, 3 \) is band index. In equation (3), the left eigenfunction \( |\psi_n^L\rangle \) and right eigenfunction \( |\psi_n^R\rangle \) are solved from Schrödinger equations \( H |\psi_n^L\rangle = E_n |\psi_n^L\rangle \) and \( H |\psi_n^R\rangle = E_n |\psi_n^R\rangle \) with the NH Hamiltonian \( H \), respectively. As shown in figure 1(b), approaching the TEP nodal point, the factors \( P_{\nu, n} \) of three energy branches become divergent \( P_{\nu, n} \to +\infty \), indicating the left and right eigenfunctions are self-orthogonal, i.e., \( \langle \psi_n^L | \psi_n^R \rangle \to 0 \). The self-orthogonality indicates that the nodal point is an EP with triple degeneracy \([8]\), i.e. STEP. To characterize its topology, we define the winding number (similar to equation (1)) of eigenvalues enclosing the EP as a topological invariant

\[
\nu' = \frac{1}{2\pi} \oint_{\text{TEP}} \frac{1}{E_n} \left\{ \nabla_k \arg \left[ +E_1(k) - E_2(k) - E_3(k) \right] + \nabla_k \arg \left[ -E_1(k) + E_2(k) - E_3(k) \right] + \nabla_k \arg \left[ -E_1(k) - E_2(k) + E_3(k) \right] \right\} \, dk
\]

The STEP exhibits a fractional topological charge \( \nu' = \frac{1}{3} \), consistent with its low-energy band shape described by the complex function \( f = \pm (z^*)^{1/3} \). The STEPs are connected by two BFAs originating from the degenerate real part of eigenvalues on two lower and higher bands, respectively. Meanwhile, two EP1 \( (\nu = +1/2) \) and two EP2 \( (\nu = +1/2) \) merge together at M point, forming a TEP node, as shown in figures 1(f) and (i). The node is found to have a low-energy band shape \( f = \pm z^{2/3} \) and exhibit a topological charge \( \nu' = +2/3 \) which is twice of that for an STEP. Approaching the node, all three branches exhibit divergent PF (figure 1(j)), indicating also the node is exceptional (a known signature of EPS [8]). However, although this exceptional nodal point has three degenerate eigenvalues, surprisingly, only two eigenfunctions are found self-orthogonal as expected, while the remaining third eigenfunction is orthogonal to the first two. Given this unusual form of orthogonality and its doubled topological charge, the nodal point is dubbed as ODTEP. Meanwhile, as shown in figure 1(f), the BFAs, which connect all the TEPs, develop into a network, where the ODTEP locates at the nodes of network; the continuous BFAs for high-order TEPs are in sharp contrast to the separate BFAs of doubly-degenerate EPs in figure 1(d).

Next, we study the evolution of TEPs by continuously varying the gain and loss. To create TEPs, the gain and loss should satisfy

\[
12t^2 = (\gamma_1 + \gamma_2 + \gamma_3)^2 - 3(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3),
\]

as shown by the cylindrical surface in figure 2(a) (see details in supplemental material), in the parameter space of \( \gamma_n \). We first analyze the case of balanced gain and loss \( \gamma_1 + \gamma_2 + \gamma_3 = 0 \). When the gain and loss follow \( \gamma_1 = t, \gamma_2 = (-1 + \sqrt{15}) t/2, \gamma_3 = (-1 + \sqrt{15}) t/2 \), there exist STEPs with \( \nu' = +1/3 \) \((-1/3)\) on
Figure 2. The evolution of TEPs in an NH kagome lattice. (a) The gain and loss $\gamma_1, \gamma_2, \gamma_3$ for producing TEPs. The light blue cylindrical surface and blue lines represent the gain and loss for realizing STEP and ODTEP, respectively. Balanced gain and loss $\gamma_1 + \gamma_2 + \gamma_3 = 0$ are identified by the red circle. (b)–(d) The $k$-space location and BFA (blue and red lines, evolved from figure 1(f) by changing gain/loss) of TEPs induced by balanced gain and loss marked as A, B, and C in (a). Solid (dash) gray lines show the reciprocal kagome lattice lines (first Brillouin zone). The circle and square dots represent STEPs and ODTEPs, respectively. Plus and minus signs represent respectively the topological charge of TEPs: STEPs, $\nu' = +1/3$; ODTEPs, $\nu' = -2/3$. The unbalanced gain and loss will induce the same type of TEP evolution. To create the ODTEP at reciprocal kagome sites (i.e. vertices of reciprocal line graphs) at $\cos k_1 = \cos k_2 = 0$ and $\cos k_3 = 0$, the required gain and loss is found to be

$$\begin{cases} \gamma_2 = \gamma_1 \pm 2t \\
\gamma_3 = \gamma_1 \mp 2t \\
\gamma_3 = \gamma_1 \pm 4t \end{cases}$$

which are shown by six straight lines on the cylinder in figure 2(a). The rest of gain and loss on the cylinder produces STEPs along the reciprocal kagome lines. Overall, the varying gain and loss according to equation (5) will move the STEPs along the reciprocal kagome lines, and merge two STEPs into one ODTEP at reciprocal kagome sites. Furthermore, if lattice hopping beyond NN is included, this evolution pattern of TEPs remains intact; the only difference is that the reciprocal lattice lines become curved (see figure S2 in supplemental material).
3. TEPs in NH line-graph lattices

Next, we investigate whether the above TEPs and corresponding evolution exist in other related 2D crystal lattices. For a general NH Hamiltonian

\[
H = \begin{pmatrix}
0 & m_1(k) & m_2(k) \\
m_1^*(k) & 0 & m_3(k) \\
m_2^*(k) & m_3^*(k) & 0
\end{pmatrix} + i \begin{pmatrix}
\gamma_1 & 0 & 0 \\
0 & \gamma_2 & 0 \\
0 & 0 & \gamma_3
\end{pmatrix},
\]

(6)

to have a TEP eigenvalue, the condition Re(m_1 m_3^* m_2) = 0 should be satisfied (see the derivation and figure S3 in supplemental material); meanwhile, the balanced gain and loss must satisfy \(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3 + |m_1|^2 + |m_2|^2 + |m_3|^2 = 0\), and \(\gamma_1 |m_1|^2 + \gamma_2 |m_2|^2 + \gamma_3 |m_3|^2 + \gamma_1 \gamma_2 \gamma_3 = 0\). Similar to the TEPs satisfying \(\cos k_1 \cdot \cos k_2 \cdot \cos k_3 = 0\) in the kagome lattice, for the three eigenvalues to stay along the reciprocal lattice lines in \(k\) space, the relation Re(m_1 m_3^* m_2) \(\propto \cos k_1 \cdot \cos k_2 \cdot \cos k_3\) is required, which has an obvious solution of \(m_1 \propto \cos k_1, m_2 \propto \cos k_2,\) and \(m_3 \propto \cos k_3\). This indicates that the lattice is generally a line graph of any hexagonal parallelograms (hexagons with parallel sides), which represents actually an edge-center lattice, in which \(m_1 = 2t_1 \cos k_1, m_2 = 2t_2 \cos k_2\) and \(m_3 = 2t_3 \cos k_3\), with hopping integrals \(t_1, t_2, t_3\) between site 1 and 2 (1 and 3, 2 and 3), as shown in figure 3(a). Thus, all the line-graph lattices with gain and loss, at least of this class, support TEPs which are located on another reciprocal line graph in \(k\) space (figure 3(b)). Varying gain and loss will effectively induce a transition from one ODTEP to a reciprocal line-graph lattice site to two STEPs on the neighboring reciprocal lattice lines, and vice versa.

Next, to further examine the robustness of the above line-graph TEPs, we relax the condition of constructing line graphs. A change is made to construct a modified line-graph 2D lattice with one off-center site: instead of choosing edge centers of the original graph as vertices of the line graph, we choose some off-edge-center points as the vertices, to form a 2D lattice with one site deviating from the center of ‘edge’, as shown in figure 4(a). The vectors between NN sites are \(b_1 = -a_1, b_2 \neq -a_2,\) and \(b_3 \neq -a_3\), and the corresponding Hamiltonian is

\[
H = \begin{pmatrix}
0 & 2t_1 \cos k_1 & t_2 e^{ik_2} + t_2' e^{-ik_2'} \\
2t_1 \cos k_1 & 0 & t_3 e^{ik_3} + t_3' e^{-ik_3'} \\
t_2 e^{-ik_2} + t_2' e^{ik_2'} & t_3 e^{-ik_3} + t_3' e^{ik_3'} & 0
\end{pmatrix} + i \begin{pmatrix}
\gamma_1 & 0 & 0 \\
0 & \gamma_2 & 0 \\
0 & 0 & \gamma_3
\end{pmatrix},
\]

(7)

where \(k_n = k \cdot a_n, k'_n = k \cdot b_n\), and \(t_n (t_n')\) is hopping integral along \(a_n (b_n)\). Based on Re(m_1 m_3^* m_2) = 0, the presence of TEPs requires Re \([(2t_1 \cos k_1) (t_2 e^{-ik_2} + t_2' e^{ik_2'}) (t_3 e^{-ik_3} + t_3' e^{ik_3'}) = 0\), which depends on hopping integrals.

To show TEPs, a specific model is arbitrarily chosen with lattice vectors \(a = (1, 0)a, b = \frac{\sqrt{3}}{2} (\cos 75^\circ, \sin 75^\circ) a\) and the NN vector \(b_2 = -2a_2\), and hopping integrals have an exponential decay with distance \(t_n = t_1 \exp \left(\frac{|a_n|}{|a|}\right)\). STEPs are found to exist on the \(k\)-space lines satisfying \(\cos k_1 = 0\), or Re \([(t_2 e^{-ik_2} + t_2' e^{ik_2'}) (t_3 e^{ik_3} + t_3' e^{-ik_3'}) = 0\). Compared with reciprocal line-graph TEP paths in NH.

Figure 3. The evolution of TEPs in the 2D NH line-graph lattices. (a) An arbitrary line-graph lattice (black lines) with three sites having gain and loss \(\gamma_1, \gamma_2,\) and \(\gamma_3\), respectively. The gray dashed and red solid lines show the original lattice and primitive cell, respectively. (b) The reciprocal line graph (black lines) of the line-graph lattice in (a), marking all the possible locations of TEPs in \(k\) space.
Figure 4. TEPs in a 2D NH modified line-graph lattice with an off-center site. (a) An arbitrary off-center lattice with three sites having gain and loss $\gamma_1, \gamma_2$, and $\gamma_3$, respectively. Site 3 deviates from the edge center. The primitive cell is shown by the red parallelogram. (b) The blue curves represent all possible locations of TEPs, forming the modified reciprocal line graph. (c) The band structure of conventional self-orthogonal DTEP with divergent PF (d) at the vertex $k_D = (\pi, -\pi \tan 75^\circ)$ in (b). The color in (c) denotes the imaginary part of eigenvalues.

line-graph lattices (figure 3(b)), the TEP path here is modified due to the off-center position for one of the lattice sites (figure 4(b)), which breaks down the intersection condition of $\cos k_2 = \cos k_3 = 0$, preventing two STEPs from merging into one ODTEP. Other crossing points along the TEP paths still exist to support DTEPs, which are located at the sites satisfying $k_x = (2q + 1) \pi$, and $k_y = \frac{(3-4 \cos 75^\circ)k_x + 3(2p+1)\pi}{4 \sin 75^\circ}$ with integers $p$ and $q$. As an example, a specific gain and loss is applied to create the DTEP at $k_D = (\pi, -\pi \tan 75^\circ)$, as shown in figures 4(c) and (d). The PF of three branches becomes divergent when approaching the $k_D$ point. Most noticeably, different from that in NH line-graph lattices, all three states associated with two merged STEPs are self-orthogonal, showing an ordinary coalesced state of eigenfunctions. The resulting nodal point has also a topological charge $\nu' = \pm 2/3$. Thus, it is actually a conventional self-ODTEP. One finds that although a perfect reciprocal line-graph TEP path is absent in the 2D lattice with an off-center site, there exists still a modified TEP line-graph path supporting STEPs and self-ODTEPs.

Finally, we discuss some physical phenomena and potential applications related to EPs, in particular the high-order TEPs and their evolution in NH line-graph lattices of this work. The singularity and topology of EPs in the parameter or momentum space give rise to rich physical phenomena, which have broad implications, especially in various artificial systems including microwave and optical cavities, photonic and acoustic lattices, etc. For example, the eigenvalue spectrum around doubly-degenerate EPs with deviation/perturbation $\delta k$ is $E \sim (\delta k)^{1/2}$, which is in stark contrast to the common spectrum $E \sim (\delta k)^1$ or $E \sim (\delta k)^2$ around a degenerate point in Hermitian systems. Given that $(\delta k)^{1/2} \gg (\delta k)^1 \gg (\delta k)^2$ for small $|\delta k| \ll 1$, the sensing response based on the perturbation away from an EP is greatly enhanced compared to that of Hermitian systems. One manifestation is the enhanced Sagnac effect (i.e. phase difference) between clockwise and counter-clockwise propagating light waves, which has been used to fabricate ultrasensitive gyroscope for the measurement of rotations [5, 6, 36, 37], and ultrasensitive optical microcavity for single-particle detection [38–40]. Furthermore, the sensitivity can be further boosted by higher-order EPs [8, 9] beyond the doubly degeneracy, such as the TEPs we studied here. Another interesting aspect of EPs is that it carries fractional charge, which is of general interest in entanglement of quantum states, relating to
fractional statistics. A doubly-degenerate EP is known to carry ±1/2 charge, which produces a half-twisted vector-vortex beam with a half-charge polarization [1, 2, 41]. The EPs found here in the NH kagome (i.e. line-graph) lattice can have different degrees of degeneracy, and hence can carry different fractional charges including ±1/3 and ±2/3 charge etc., which are expected to result in different degrees of beam twisting and fractional polarization.

In general, EPs have been shown to support many underlying physical phenomena, including the unidirectional transmission or reflection [42–44], topological energy transfer between different modes [45, 46], loss-induced transparency [47], lasers with reversed pump dependence or single-mode operation [48–50]. Our findings of TEPs in the new line-graph lattices, having different fractional charges, and their intriguing evolutions will certainly enrich their physical phenomena and potential applications, especially in photonics, acoustics and elastic metamaterials.

4. Conclusion

In conclusion, 2D NH line-graph lattices with three sites having gain and loss are demonstrated to support three types of TEPs (see table 1). Interestingly, all the TEPs are located on a reciprocal line graph. STEPs with ±1/3 topological charge move along the reciprocal lattice lines with varying gain and loss, while an intriguing form of ODTEPs with topological charge ±2/3 appear at the reciprocal line-graph vertices by merging two STEPs. In contrast, the conventional self-ODTEP with three coalesced states appear in the 2D NH off-center line-graph lattices. These findings not only enrich the topological states in NH systems which may exhibit novel physical properties in realistic setups, but also point to new research directions to further link graph theory with topological physics, especially under non-equilibrium conditions.

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Appendix A. Supplemental materials

Supplemental materials to this article can be found at https://stacks.iop.org/NJP/23/123038/mmedia.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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