ATOMS OF ROOT-CLOSED SUBMONOIDS OF $\mathbb{Z}^2$

GÜNTER LETTL

Abstract. We describe how one can explicitly obtain all atoms of an arbitrary root-closed monoid, whose quotient group is isomorphic to $\mathbb{Z}^2$. For this purpose, we solve this task for three special types of such monoids in Theorems 5 and 6 and then transfer these results to the general case. It turns out that all atoms can be obtained from the (regular) continued fraction expansion of the slopes of the bounding rays of the cone, which is spanned by the monoid.

1. Introduction

Factorization theory has a long and rich history. In the course of time one realized that many factorization problems in integral domains only depend on the multiplicative structure, so many investigations and problems were translated into the theory of monoids, i.e. commutative, cancellative semigroups with a neutral element. For an overview of the relevant literature, we refer the reader to books and conference proceedings, e.g. [27], [1], [6], [17], [13], [11].

Additive submonoids of $\mathbb{Z}^k$ found special attention since they have applications and connections to various areas of mathematics. In particular, they are of crucial importance in $K$-theory [5] and toric geometry [8].

Throughout this paper, $H$ denotes a monoid and $\mathbb{Q}(H)$ its quotient group. We will use the notations and definitions as given in [17], but use addition for the operation of $H$. Let $0_H$ denote the neutral element of $H$, and $H \subset \mathbb{Q}(H) = \{h_1 - h_2 \mid h_1, h_2 \in H\}$. Furthermore, let $\mathcal{A}(H)$ denote the set of atoms of $H$. This set is fundamental to study the arithmetic and many factorization properties of $H$. The monoid $H$ is called root-closed, if for any $x \in \mathbb{Q}(H)$ and any $n \in \mathbb{N}$ we have that $nx \in H$ implies $x \in H$.

The arithmetic of additive monoids with finitely generated quotient group attracted a lot of research interest in recent time, see for example [12], [10], [7], [15], [20], [21], [14]. Root-closed monoids with rank at most two were investigated in [9], and the arithmetic of strongly primary monoids was studied in [16]. Finally, it was shown in [26] that the theory of root-closed monoids is axiomatizable, but not finitely axiomatizable.

Our main interest will be in monoids $H$ with $\mathbb{Q}(H) \simeq \mathbb{Z}^2$, so we will tacitly assume that $H$ has rank 2 and $H \subset \mathbb{Z}^2$, and put $0_H = O = (0, 0)$. In this paper we will explicitly describe how one obtains the set of atoms $\mathcal{A}(H)$, supposing that $H$ is root-closed. It turns out that the atoms are closely related to the convergents and second convergents of the (regular) continued fraction expansions of the slopes of the rays bounding the cone of $H$. For this purpose we will recall results and notations from the theory of continued fraction expansions of real numbers in Section 2.
In Section 3 we will prove the main result for special monoids, which consist of all elements of $\mathbb{N}_0^2$, which lie between the $x$-axis and a line with positive slope $\alpha$. It turns out that the atoms of such monoids can be determined from the continued fraction expansion of $\alpha$. The monoids $M_{\alpha}^*$ (as defined in Definition 11) show up e.g. in Example 4.7 in [19]. The monoids $M_{5/2}$ and $M_5$ appear in Propositions 6.3 and 6.4 in [3]. The monoids $M_{\alpha}$ with $\alpha = (e \pm \sqrt{e^2 - 4})/2$, $3 \leq e \in \mathbb{N}$, are tightly related to the monoids in Appendix B of [24], also referred to in Example 3.4 in [21]. The atoms of those monoids were determined with some ingenious method, but this cannot be applied to arbitrary $\alpha$, as can be seen from our present results.

Finally, in Section 4 we will use the results of Section 3 to describe the atoms of any root-closed monoid $H \subset \mathbb{Z}^2$ with $Q(H) = \mathbb{Z}^2$ in Theorem 10. This answers an open problem posed by F. Gotti, Question 3.5 in [21].

2. Notations and results for continued fractions

For more detailed information and proofs about continued fraction expansions the reader may consult [1] or [24].

Each irrational $\alpha \in \mathbb{R}$ has a unique (regular) continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$ with $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$ for all $i \in \mathbb{N}$. Each rational $\alpha \in \mathbb{Q}$ has a unique continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots, a_N]$ with even $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $1 \leq i \leq N$. In the following we will tacitly assume that if $\alpha$ is rational the indices $n$ will always be bounded above such that all appearing partial quotients $a_n$ are defined.

For given $\alpha \in \mathbb{R}$ with continued fraction expansion as above, the numbers $p_n$, $q_n$ (with $n \leq N$, if $\alpha$ is rational) are given recursively by

\[
\begin{align*}
p_{-2} &= 0, \quad p_{-1} = 1, \quad p_n = a_np_{n-1} + p_{n-2} \in \mathbb{Z},
p_{-2} &= 1, \quad q_{-1} = 0, \quad q_n = a_nq_{n-1} + q_{n-2} \in \mathbb{N} \text{ for } n \geq 0.
\end{align*}
\]

Then the following relations hold:

1. for $0 \leq n$: \[ [a_0; a_1, \ldots, a_n] = \frac{p_n}{q_n}, \]

2. for $-1 \leq n$: \[ p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1} \quad \text{and for } 1 \leq n: \quad \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_nq_{n-1}}, \]

3. for $0 \leq n$: \[ p_nq_{n-2} - p_{n-2}q_n = (-1)^n a_n \quad \text{and for } 2 \leq n: \quad \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_nq_{n-2}}. \]

The rational number $p_n/q_n$ is called the $n$-th convergent to $\alpha$.

**Definition 1.** With the above notations we define for $-2 \leq n$ and $0 \leq i \leq a_{n+2}$

\[
\begin{align*}
p_{n,i} &= p_n + iq_{n+1} \quad \text{and} \quad q_{n,i} = q_n + iq_{n+1}.
\end{align*}
\]

For $1 \leq i \leq a_{n+2} - 1$ the rational numbers $\frac{p_{n,i}}{q_{n,i}}$ are called the second convergents to $\alpha$.

(Not that $q_{n,i} = 0$ if and only if $(n, i) = (-1, 0)$.)

\footnote{For $n = -2$ and negative $\alpha$ take $a_0 \leq i \leq 0$.}
Remark. The second convergents (in German: “Nebennäherungsbrüche”) are called by Khinchin [24] “intermediate fractions”.

We will often make use of the following well-known lemma (see e.g. Lemma 1.41 in [4]):

Lemma 2. Let \( x_1, x_2 \in \mathbb{N} \) and \( y_1, y_2 \in \mathbb{Z} \). Then
\[
\frac{y_1}{x_1} < \frac{y_2}{x_2} \implies \frac{y_1}{x_1} < \frac{y_1 + y_2}{x_1 + x_2} < \frac{y_2}{x_2}.
\]

Obviously, “\( \leq \)” in the left inequality implies twice “\( \leq \)” in the right hand side of the above implication.

Lemma 3. With the notations of Definition [7] we have:

\( a) \) for all \(-2 \leq n \) and \( 0 \leq i \leq a_{n+2} \) : \( \gcd(p_{n,i}, q_{n,i}) = 1 \).

\( b) \) for all even \( n \geq 0 \)
\[
\frac{p_n}{q_n} = \frac{p_{n,0}}{q_{n,0}} < \frac{p_{n,1}}{q_{n,1}} < \frac{p_{n,2}}{q_{n,2}} < \cdots < \frac{p_{n,a_{n+2}}}{q_{n,a_{n+2}}} = \frac{p_{n+2}}{q_{n+2}} \leq \alpha.
\]

If \( \alpha > 0 \) we additionally have
\[
0 = \frac{p_{-2}}{q_{-2}} = \frac{p_{-2,0}}{q_{-2,0}} < \cdots < \frac{p_{-2,i}}{q_{-2,i}} = i < \cdots < \frac{p_{-2,a_0}}{q_{-2,a_0}} = \frac{p_0}{q_0} = a_0.
\]

Proof. \( a) \) Using Definition [1] and (2) yields \( p_{n,i} q_{n,0} - q_{n,i} p_{n+1} = \pm 1 \).

\( b) \) For even \( n \geq 0 \) we have \( \frac{p_n}{q_n} < \alpha < \frac{p_{n+1}}{q_{n+1}} \). Using Lemma 2 we obtain
\[
\frac{p_n}{q_n} < \frac{p_{n,1}}{q_{n,1}} < \frac{p_{n+1}}{q_{n+1}} \text{ and inductively } \frac{p_{n,i}}{q_{n,i}} < \frac{p_{n,i+1}}{q_{n,i+1}} < \frac{p_{n+1}}{q_{n+1}} \text{ for all } 1 \leq i < a_{n+2}.
\]
The case \( \alpha > 0 \) and \( n = -2 \) follows directly from the definitions. \( \square \)

The above results can be found e.g. in [24], Ch. I.4.

3. The monoid \( M_\alpha \) and its relatives

Now we investigate monoids which consist of all elements of \( \mathbb{N}_0^2 \), which lie between the \( x \)-axis and a line with positive slope \( \alpha \). We will distinguish whether the points of \( \mathbb{N}_0^2 \) on the bounding rays belong to the monoid or not, and so obtain four cases. In Theorems 5 and 6 we will explicitly give the atoms of three typical types of such monoids. The results depend on whether \( \alpha \) is irrational or rational, and in the latter case also whether the points on the line with slope \( \alpha \) belong to the monoid or not. From these results we will deduce a description of the atoms for \( M_{\alpha,>0} \) and \( M_{\alpha,>0}^\circ \) in Theorem 7, and finally in section 4 for any root-closed monoid \( H \) with \( \mathcal{Q}(H) = \mathbb{Z}^2 \).
Definition 4. For $0 < \alpha \in \mathbb{R}$ we define the following special monoids
\[
M_\alpha = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq y \leq \alpha x\},
\]
\[
M_\alpha^0 = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq y < \alpha x\} \cup \{O\},
\]
\[
M_{\alpha, > 0} = \{(x, y) \in \mathbb{Z}^2 \mid 0 < y \leq \alpha x\} \cup \{O\},
\]
\[
M_{\alpha, > 0}^0 = \{(x, y) \in \mathbb{Z}^2 \mid 0 < y < \alpha x\} \cup \{O\},
\]
where $O = (0, 0)$, as before.

Remark. All the monoids of Definition 4 and more generally, any submonoid of $\mathbb{N}_0^d$, are FF-monoids, i.e. each element is the sum of finitely many atoms. This is Proposition 4.2 in [20], and the proof is very immediate.

The atoms of $M_\alpha$ are explicitly given in the following

Theorem 5. Let $0 < \alpha \in \mathbb{R}$ and $\alpha = [a_0; a_1, a_2, \ldots]$ be its continued fraction expansion. If $\alpha \in \mathbb{Q}$, we choose the expansion $\alpha = [a_0; a_1, a_2, \ldots, a_N]$ with even $N \geq 0$.

Using the notations of the previous section, we put for all even $n \geq -2$ and $0 \leq i \leq a_{n+2}$ (and $n \leq N - 2$, if $\alpha$ is rational)
\[
A_{n,i} = (q_{n,i}, p_{n,i}).
\]

a) If $\alpha$ is irrational, then the set of atoms of $M_\alpha$ is given by
\[
\mathcal{A}(M_\alpha) = \{A_{n,i} \mid -2 \leq n \text{ even and } 0 \leq i \leq a_{n+2}\}.
\]

b) If $\alpha$ is rational, then $M_\alpha$ is a finitely generated Krull monoid with $1 + a_0 + a_2 + a_4 + \cdots + a_N$ atoms, and its set of atoms is given by
\[
\mathcal{A}(M_\alpha) = \{A_{n,i} \mid -2 \leq n \leq N - 2, \text{ n even and } 0 \leq i \leq a_{n+2}\}.
\]

Remark. Note that in the above description of the atoms we have $A_{n,a_{n+2}} = A_{n+2,0}$, which is convenient for the proof. Up to this ambiguity, all elements in the above description of $\mathcal{A}(M_\alpha)$ are pairwise different, as follows immediately from Lemma 3b).

For the monoids $M_\alpha^0$ we have to consider only rational $\alpha$, since for irrational $\alpha$ one has $M_\alpha^0 = M_\alpha$.

Theorem 6. Let $0 < \alpha \in \mathbb{Q}$ and $\alpha = [a_0, a_1, a_2, \ldots, a_N]$ be its continued fraction expansion with even $N \geq 0$. Using the notations of Theorem 5, we put for $k \in \mathbb{N}$
\[
A_k = A_{N-2,a_{N-1}} + kA_{N,0} = (q_{N-2,a_{N-1}} + kq_N, p_{N-2,a_{N-1}} + kp_N) \quad \text{and} \quad \mathcal{A}' = \{A_k \mid k \in \mathbb{N}\}.
\]

Then the set of atoms of $M_\alpha^0$ is given by
\[
\mathcal{A}(M_\alpha^0) = \mathcal{A}' \cup \mathcal{A}'',
\]
where
\[
\mathcal{A}'' = \{A_{n,i} \mid -2 \leq n \leq N - 2, \text{ n even, } 0 \leq i \leq a_{n+2} \text{ and } (n, i) \neq (N - 2, a_N)\}.
\]

For the monoids $M_{\alpha, > 0}$ and $M_{\alpha, > 0}^0$ we use the results of the previous theorems to obtain the following description of the atoms:
Lemma 8. Let \( \alpha \in \mathbb{R} \). Then for the monoids \( M_{\alpha,>0} \) and \( M_{\alpha, >0}^o \) their sets of atoms are given by

\[
\mathcal{A}(M_{\alpha,>0}) = (\mathcal{A}(M_{\alpha}) \setminus \{(1,0)\}) \cup \{(n,1) | n \in \mathbb{N} \text{ with } 1 \leq \alpha n\}
\]

and

\[
\mathcal{A}(M_{\alpha, >0}^o) = (\mathcal{A}(M_{\alpha}^o) \setminus \{(1,0)\}) \cup \{(n,1) | n \in \mathbb{N} \text{ with } 1 < \alpha n\}.
\]

To characterize the atoms of \( M_{\alpha} \) and \( M_{\alpha}^o \) the following lemma is very useful:

Lemma 8. Let \( \alpha \in \mathbb{R} \) and put \( H = M_{\alpha} \) or \( H = M_{\alpha}^o \). Then \( A = (x_0, y_0) \in H \setminus \{O\} \) is an atom of \( H \) if and only if either

i) \( y_0 = 0 \) and \( A = (1,0) \) or

ii) \( y_0 > 0 \) and for all \( h = (x,y) \in H \) with \( h \not\in \{O, A\} \) and \( 0 < y \leq y_0 \) we have: \( \frac{y}{x} < \frac{y_0}{x_0} \).

Remarks.

1. Since for all \( O \neq h = (x,y) \in H \) we have \( \frac{y}{x} \leq \alpha \), this lemma reveals the intimate connection between atoms of \( H \) and “best approximations of \( \alpha \)” (from below), cp. [24], p. 21.

2. Geometrically, condition ii) means that the triangle with corners \( O, A \) and \((y_0/\alpha, y_0)\) contains no elements of \( H \) up to \( O \) and \( A \).

Proof of Lemma 8. Since \((1,0)\) is the only atom of \( H \) contained in \( H \cap (\mathbb{Z} \times \{0\}) \), we may assume for the proof that \( y_0 > 0 \).

Let \( A = (x_0, y_0) \) be an atom with \( y_0 > 0 \), and suppose that there exists some \( h = (x,y) \in H \setminus \{O,A\} \) with \( 0 < y \leq y_0 \) and \( \frac{y}{x} \geq \frac{y_0}{x_0} \). Then also \( x_0 \geq x \), and we put \( x' = x_0 - x \geq 0 \), \( y' = y_0 - y \geq 0 \) and \( h' = (x',y') \). Since \( h \neq A \), we have \( h' \neq O \). From

\[
A = h + h'
\]

we obtain with Lemma 2 \( \frac{y'}{x'} \leq \frac{y_0}{x_0} \leq \frac{y}{x} \leq \alpha \), so \( O \neq h' \in H \), and \( (6) \) is a contradiction that \( A \) is an atom.

Now suppose that \( A \) is not an atom. Then we have \( h = (x,y), h' = (x', y') \in H \setminus \{O\} \) with \( A = h + h' \). Without loss of generality we may assume that \( \frac{y'}{x'} \leq \frac{y}{x} \). With Lemma 2 we obtain \( \frac{y'}{x'} \leq \frac{y_0}{x_0} \leq \frac{y}{x} \). Since \( y \leq y_0 \), \( h \) contradicts ii) in the statement of the lemma. \( \square \)

Proof of Theorem 7. We will give a joint proof for a) and b).

For \( 0 \leq i \leq a_0 \) we have \( A_{-2,i} = (1,i) \), and these are exactly all atoms of \( M_{\alpha} \) with \( x \leq 1 \).

Let us first prove that all indicated \( A_{n,i} \) are indeed atoms of \( M_{\alpha} \). Choose any even \( n \geq 0 \) and \( 0 \leq i \leq a_{n+2} \). From (5) we see that \( A_{n,i} \in M_{\alpha} \). Using Lemma 8 (ii), let us suppose that there exists some \( h = (x,y) \in H \setminus \{O, A_{n,i}\} \) with \( 0 < y \leq p_{n,i} \) and \( \frac{p_{n,i}}{q_{n,i}} \leq \frac{y}{x} \). By Lemma 8 (i), and since \( y \leq p_{n,i} \), \( \frac{p_{n,i}}{q_{n,i}} = \frac{y}{x} \) would imply \( h = A_{n,i} \), so we have \( \frac{p_{n,i}}{q_{n,i}} < \frac{y}{x} \).
and consequently \( q_{n,i} > x \). Since \( n \) is even (and \( n \leq N - 2 \) in case b)), we have
\[
\frac{p_{n,i}}{q_{n,i}} < \frac{y}{x} \leq \alpha < \frac{p_{n+1}}{q_{n+1}},
\]
and obtain
\[
\frac{1}{q_{n+1}q_{n,i}} = \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n,i}}{q_{n,i}} > \frac{p_{n+1}}{q_{n+1}} - \frac{y}{x} > 0.
\]
Therefore \( \frac{1}{q_{n+1}q_{n,i}} > \frac{1}{xq_{n+1}} \) and \( x > q_{n,i} \), a contradiction to the inequality obtained above.

Now let us show that \( M_\alpha \) has no other atoms than those indicated in the theorem.

First observe that in case \( b) ~ A_{N-2,a_N} = (q_N, p_N) \) is an atom of \( M_\alpha \) (with \( \frac{p_N}{q_N} = \alpha \)), and Lemma \( 8 \) shows that there exist no atoms with second component exceeding \( b \).

Let \( A = (x_0, y_0) \) be any atom of \( M_\alpha \) with \( y_0 \geq 1 \). We can find some \( n \geq -2 \) and 0 \( \leq i < a_{n+2} \) with

\[
(7) \quad p_{n,i} < y_0 \leq p_{n,i+1}.
\]

(Here we use that \( p_{n,a_{n+2}} = p_{n+2,0} \).) Assume that \( A \neq A_{n,i+1} \), so \( A, A \) and \( A_{n,i+1} \) are three different atoms, and with Lemma \( 8 ii) \) we obtain

\[
\frac{p_{n,i}}{q_{n,i}} = \frac{y_0}{x_0} < \frac{p_{n,i+1}}{q_{n,i+1}}. \tag{8}
\]

We have
\[
\frac{1}{q_{n,i+1}q_{n,i}} = \frac{p_{n,i+1}}{q_{n,i+1}} - \frac{p_{n,i}}{q_{n,i}} = \left( \frac{p_{n,i+1}}{q_{n,i+1}} - \frac{y_0}{x_0} \right) + \left( \frac{y_0}{x_0} - \frac{p_{n,i}}{q_{n,i}} \right) \geq \frac{1}{x_0q_{n,i+1}} + \frac{1}{x_0q_{n,i}},
\]

which yields \( x_0 \geq q_{n,i} + q_{n,i+1} \). Combining this inequality with (7) and (8) we obtain

\[
(q_{n,i} + q_{n,i+1})p_{n,i} \leq x_0p_{n,i} < y_0q_{n,i} \leq q_{n,i}p_{n,i+1}
\]

and finally

\[
p_{n,i}q_{n,i} < q_{n,i}p_{n,i+1} - p_{n,i}q_{n,i+1} = 1.
\]

This implies \( p_{n,i} = 0 \), i.e. \((n, i) = (0, 0)\) and \( a_0 = 0 \). So \( A_{n,i} = A_{0,0} = (1, 0) \) and \( A_{n,i+1} = A_{0,1} = (1 + a_1, 1) \), but then \( A = (x_0, 1) \) with \( x_0 > a_1 + 1 \) cannot be an atom.

Finally, in case \( b) \) the remark following the statement of Theorem \( 5 \) immediately yields the number of atoms for rational \( \alpha \). It is well known that in this case \( M_\alpha \) is a Krull monoid. \( \square \)

**Proof of Theorem** \( 2 \). We have \( \alpha = \frac{p_N}{q_N} \) and put \( B = (q_N, p_N) \notin M_\alpha^\circ \).

First we will prove the following

**Claim 1:** If \( A = (x_0, y_0) \in M_\alpha^\circ \) is an atom with \( y_0 \geq p_N \), then also \( A - B = (x_0 - q_N, y_0 - p_N) \) is an atom of \( M_\alpha^\circ \).

Since \( \alpha(x_0 - q_N) > y_0 - p_N \geq 0 \), we have \( A - B \in M_\alpha^\circ \setminus \{O\} \).

Suppose that \( A - B \) is not an atom.

**Case 1:** Assume that \( y_0 - p_N = 0 \).

Since \( (1, 0) \) is an atom, we have \( x_0 - q_N \geq 2 \) and \( A = (q_N + 1, p_N) + (x_0 - q_N - 1, 0) \) contradicts that \( A \) is an atom.

**Case 2:** Assume that \( y_0 - p_N \geq 1 \).

By Lemma \( 8 ii) \) there exists \( h = (x, y) \in M_\alpha^\circ \setminus \{O, A - B\} \) with \( 0 < y \leq y_0 - p_N \) and

\[
\frac{y}{x} \geq \frac{y_0 - p_N}{x_0 - q_N}. \tag{9}
\]
From this one easily obtains $0 < x \leq x_0 - q_N$. Using Lemma 2 and (9) we obtain
\[ \frac{p_N}{q_N} > \frac{y}{x} \geq \frac{y_0 - p_N}{x_0 - q_N} \geq \frac{y_0 - p_N - y}{x_0 - q_N - x}, \]
which shows that $O \neq A - B - h \in M_\alpha^o$. Again employing Lemma 2 we get
\[ \frac{p_N}{q_N} > \frac{y_0 - y}{x_0 - x} \geq \frac{y_0 - p_N - y}{x_0 - q_N - x}. \]
From (9) and (10) we obtain
\[ \frac{p_N + p_{n,i}}{q_N + q_{n,i+1}} > \frac{p_N + p_{n,i}}{q_N + q_{n,i}}. \]

Claim 2: For all even $-2 \leq n \leq N - 2$ and all $0 \leq i < a_{n+2}$ we have
\[ \frac{p_N + p_{n,i+1}}{q_N + q_{n,i+1}} > \frac{p_N + p_{n,i}}{q_N + q_{n,i}}. \]

The stated inequality is equivalent to $p_{n,i+1}q_N + p_Nq_{n,i} + p_{n,i+1}q_{n,i} > p_Nq_{n,i+1} + p_nq_N + p_{n,i}q_{n,i+1}$, which yields $(q_Np_{n,i} + p_Nq_{n,i+1}) + (p_{n,i+1}q_{n,i} - q_{n,i+1}p_{n,i}) > 0$.

Since $n + 1 \leq N - 1$ is odd, $\frac{p_N}{q_N} = \alpha < \frac{p_{n,i+1}}{q_{n,i+1}}$, and furthermore $\frac{p_{n,i}}{q_{n,i}} < \frac{p_{n,i+1}}{q_{n,i+1}}$, both brackets above are positive, which proves Claim 2.

Since the monoids $M_\alpha$ and $M_\alpha^o$ contain the same elements $(x, y)$ with $y < p_N$, it is immediate that the atoms within this subset coincide, and so by Theorem 5, $\mathcal{A}''$ is exactly the set of all atoms $(x, y) \in M_\alpha^o$ with $y < p_N$.

Now we search for atoms $A = (x_0, y_0) \in M_\alpha^o$ with $p_N \leq y_0 < 2p_N$. By Claim 1 they must be of the form $A = B + A_{n,i}$ with $A_{n,i} \in \mathcal{A}''$. We have
\[ A_{N-2,a_{N-1}} = (q_N - q_{N-1}, p_N - p_{N-1}). \]

Case 1: Assume that $a_N \geq 2$. One calculates $A_{N-2,a_{N-2}} = 2A_{N-2,a_{N-1}}$, so this is not an atom and $\frac{p_N - p_{N-1}}{q_N - q_{N-1}} = \frac{p_N + p_{N-2,a_{N-2}}}{q_N + q_{N-2,a_{N-2}}}$. Now Claim 2 and Lemma 8(ii) show that the only element of the form $B + A_{n,i}$ which could be an atom, is $A_1 = B + A_{N-2,a_{N-1}}$.

Case 2: Assume that $a_N = 1$. Now we calculate $A_{N-4,a_{N-3}} = (a_{N-1} + 2)A_{N-2,0}$, so $\frac{p_N - 2}{q_{N-2}} = \frac{p_N + p_{N-4,a_{N-2}-1}}{q_N + q_{N-4,a_{N-2}-1}}$, and the same ideas as above again show that the only element of the form $B + A_{n,i}$ which could be an atom, is $A_1 = B + A_{N-2,a_{N-1}}$.

Since $M_\alpha^o$ must have infinitely many atoms, Claim 1 proves with an inductive argument that $A_1$ must indeed be an atom, and that $A_k$ is the only atom of $M_\alpha^o$ with second component between $kp_N$ and $(k+1)p_N$. $\square$
Proof of Theorem 7. Since $M_{a,>0} \subset M_a$, all atoms of $M_a$ which lie in $M_{a,>0}$ are also atoms of $M_{a,>0}$. Obviously, all elements $(x, 1) \in M_{a,>0}$ are atoms, too.

Now let us prove that $M_{a,>0}$ has no further atoms. Suppose that $A = (x_0, y_0) \in M_{a,>0}$ is an atom with $y_0 \geq 2$, but not an atom of $M_a$. Then we have $O \neq h, h' \in M_a$ with $A = h + h'$. Without restriction, we can suppose that $h = (1, 0)$, and $h' = (x_0 - 1, y_0) \in M_a$, so

$$\frac{y_0}{x_0 - 1} \leq \alpha.$$  \hspace{1cm} (11)

Choose $n \in \mathbb{N}$ minimal with $\frac{1}{n} \leq \alpha$. We want to show that $A = (n, 1) + (x_0 - n, y_0 - 1)$ is a sum of 2 elements of $M_{a,>0}$, a contradiction that $A$ is an atom.

By our choice of $n$, $(n, 1) \in M_{a,>0}$. Now suppose that $\frac{y_0}{x_0 - n} > \alpha$. If $n \geq 2$, we have $\frac{1}{n-1} > \alpha$, and Lemma 2 yields $\frac{1}{n-1} + (\frac{y_0}{x_0 - n}) = \frac{y_0}{x_0 - 1} \geq \alpha$, a contradiction to (11). If $n = 1$, we directly obtain $\frac{y_0}{x_0 - 1} \leq \alpha$, so again $(x_0 - n, y_0 - 1) \in M_{a,>0}$.

The proof for $M_{a,>0}^\text{c}$ is completely analogous. \hfill \square

4. Arbitrary root-closed submonoids of $\mathbb{Z}^2$

Suppose for the moment that $H$ is a monoid with quotient group $Q(H) = \mathbb{Z}^s \subset \mathbb{R}^s$ for some $s \in \mathbb{N}$, and put

$$\text{cone}(H) = \left\{ \sum_{i=1}^{n} \lambda_i h_i \mid n \in \mathbb{N}, h_i \in H, 0 \leq \lambda_i \in \mathbb{R} \right\},$$

the convex cone (with apex $O$) spanned by $H$. Then several properties of $H$ can be characterized in geometric terms (see e.g. [25], [18], [23], [22]). We summarize some of these results in the following proposition. For the definition of those properties, which are not explicitly needed in the present paper, the reader is referred to [17].

**Proposition 9.** Let $s \in \mathbb{N}$ and $H$ be a monoid with quotient group $Q(H) = \mathbb{Z}^s \subset \mathbb{R}^s$. Then we have:

- **a)** $H$ is root-closed if and only if $H = \text{cone}(H) \cap \mathbb{Z}^s$.

- **b)** $H$ is completely integrally closed if and only if $H = \overline{\text{cone}(H)} \cap \mathbb{Z}^s$ (the bar denotes the topological closure).

- **c)** $H$ is a Krull monoid if and only if $\text{cone}(H)$ is a closed polytopal cone.

- **d)** $H$ is primary and reduced if and only if $\text{cone}(H) \setminus \{O\}$ is open and $\text{cone}(H) \not= \mathbb{R}^s$.

Now let us return to the case $s = 2$. Then Proposition 9a shows that any root-closed monoid $H$ with $Q(H) = \mathbb{Z}^2$ can be obtained from some convex cone $C \subset \mathbb{R}^2$ with apex $O$, which spans $\mathbb{R}^2$, by $H = C \cap \mathbb{Z}^2$. To make the results about the atoms of $H$ more clearly and avoid an abundance of possible cases, we will use automorphisms $\varphi$ of the group $\mathbb{Z}^2$ to transform $C$ into special positions. Note that then the monoids $H$ and $\varphi(H)$ are isomorphic to each other, and we have $A(\varphi(H)) = \varphi(A(H))$.

As is well known, any automorphism $\varphi$ of the group $\mathbb{Z}^2$ is given by $\varphi : \binom{x}{y} \mapsto A \binom{x}{y}$ with some $A = \binom{a}{b} \in \text{GL}_2(\mathbb{Z})$, and we will tacitly extend this to a linear map of $\mathbb{R}^2$, again called $\varphi$. In particular, let $\sigma_x$ denote the reflection at the $x$-axis, given by $A = \binom{1}{0}$. \hfill \square
Our main result is the following

**Theorem 10.** Let $H \subset \mathbb{Z}^2 \subset \mathbb{R}^2$ be a root-closed monoid with quotient group $Q(H) = \mathbb{Z}^2$, and let $C \subset \mathbb{R}^2$ be a convex cone with apex $O$ and $H = C \cap \mathbb{Z}^2$. Then $H$ is isomorphic to one of the monoids $H'$ of the following list, depending on the geometric form of $C$:

A. $C = \mathbb{R}^2$. Then $H = H' = \mathbb{Z}^2$ is a group.

B. $C$ is a halfspace of $\mathbb{R}^2$.

B1. $H' = \mathbb{Z} \times \mathbb{N}_0$: this is a Krull monoid with divisor theory (even a factorial monoid). Its group of units is $\mathbb{Z} \times \{0\}$, and there is one class of associated atoms, namely $\mathbb{Z} \times \{1\}$.

B2. $H' = (\mathbb{Z} \times \mathbb{N}) \cup (\mathbb{N}_0 \times \{0\})$: one has $\mathcal{A}(H') = \{(1, 0)\}$, and no $(x, y) \in H'$ with $y > 0$ is an atom or a sum of atoms.

B3. $H' = (\mathbb{Z} \times \mathbb{N}) \cup \{(0, 0)\}$: this is a primary BF-monoid, but not an FF-monoid. It is half-factorial, and one has $\mathcal{A}(H') = \{(n, 1) \mid n \in \mathbb{Z}\}$.

B4. $H' = \{(x, y) \in \mathbb{Z}^2 \mid y \leq \alpha x\}$ with some $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$: one has $\mathcal{A}(H') = \{\}$.

C. $C$ is bounded by two rays with a positive angle less than $\pi$.

C1. $H' \in \{M_\alpha, M_\alpha^0, M_{\alpha,0}, M_{\alpha,0}^0\}$ with some $0 < \alpha \in \mathbb{R}$: the properties and atoms of these monoids are given in Theorems 2, 3 and 4.

C2. $H' = H_1 \cup \sigma_\delta(H_2)$, where $H_1 \in \{M_\alpha, M_\alpha^0\}$ and $H_2 \in \{M_\beta, M_\beta^0\}$ with some $0 < \alpha, \beta \in \mathbb{R}$: then $\mathcal{A}(H') = \mathcal{A}(H_1) \cup \sigma_\delta(\mathcal{A}(H_2))$, and the properties of $H'$ depend on $H_1$ and $H_2$.

**Remark.** The fact that cone$(H) \subset \mathbb{R}^s$ is bounded by (at most) two rays for dimension $s = 2$ makes the characterization result of Theorem 10 feasible. A generalization to dimensions $s \geq 3$ requires completely different ideas and methods.

**Proof.** Let $C \subset \mathbb{R}^2$ be any convex cone with apex $O$, which spans $\mathbb{R}^2$, and put $H = C \cap \mathbb{Z}^2$.

A. If $C = \mathbb{R}^2$, we have $H = \mathbb{Z}^2$, which is a group.

B. Suppose that $C$ is a halfspace of $\mathbb{R}^2$. These monoids are studied in great detail in Section 5 of [9].

If the boundary of $C$ is a line defined over $\mathbb{Q}$, it is given by $\{(x, y) \in \mathbb{R}^2 \mid qx - py = 0\}$, where $(p, q) \in \mathbb{Z}^2 \setminus \{O\}$ with $\gcd(p, q) = 1$. So one can find $(a, b) \in \mathbb{Z}^2$ with $pb + qa = 1$. If $(-a, b) \in H$, we consider the automorphism $\varphi$ of $\mathbb{Z}^2$ given by $A = \begin{pmatrix} b & a \\ -q & p \end{pmatrix}$, otherwise we have $(a, -b) \in H$ and define $\varphi$ by $A = \begin{pmatrix} b & a \\ q & -p \end{pmatrix}$. In any case, $\varphi(C)$ is the upper half-plane, and depending whether both, one or none bounding rays belong to $C$ (and eventually use the reflection at the $y$-axis), we conclude that $H$ is isomorphic to (exactly) one of monoids $H'$ as given in B1 - B3.

The properties of these monoids are well known, see e.g. [9], or the reader may check them easily. Let us just state, that in case B3 any $(x, y) \in H$ with $y > 0$ has infinitely many different representations as a sum of atoms, but each consists of exactly $y$ summands (so $H$ is half-factorial).

\[\text{By Proposition } 9.a \text{ such } C \text{ always exists, take e.g. } C = \text{cone}(H).\]
Now suppose that the boundary of $C$ is an irrational line (i.e. is not defined over $\mathbb{Q}$). If $C$ is not contained in $\{(x, y) \in \mathbb{Z}^2 \mid y \leq \alpha x\}$ for some $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$, we apply $i$ times the rotation $\tau$, given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with some suitable $i \in \{1, 2, 3\}$ to obtain $H'$ as given in B4.

Now let $H'$ be given as in B4, and we will prove that $H'$ has no atoms. Take any $O \neq h = (x, y) \in H'$, so $y < \alpha x$. By Lemma 3 and [2] we have for all convergents $\frac{p_n}{q_n}$ of $\alpha$ with even $n$

$$0 < \alpha q_n - p_n < \frac{1}{q_{n+1}},$$

so we can find some even $n$ with $0 < \alpha q_n - p_n < \frac{1}{q_{n+1}} < \alpha y - y$. One can easily check that $h' = (q_n, p_n)$ and $h'' = h - h'$ are both non-zero elements of $H'$, thus $h = h' + h''$ shows that $h$ is not an atom.

C. Now suppose that $C$ is bounded by two rays with a positive angle less than $\pi$.

If (at least) one of the rays is defined over $\mathbb{Q}$, we can apply the same ideas as in the first part of the proof of B to find an automorphism $\varphi$ of $\mathbb{Z}^2$ such that $\varphi(C)$ has the positive $x$-axis as one bounding ray and the other ray lies in the upper half plane. If this other ray does not have a positive slope, we can find some $n \in \mathbb{N}$ with $(-n, 1) \in \mathbb{C}$ and apply the automorphism $\varphi'$ given by $\begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$ to obtain $\varphi'(\varphi(H)) = \{M_\alpha, M_\alpha^{n}, M_{\alpha, >0}, M_{\alpha, >0}^{n}\}$ with some $0 < \alpha \in \mathbb{R}$.

If (at least) one of the bounding rays of $C$ is irrational, we consider the convergents to the slope of that ray. We can find some (large enough) $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ with $(q, p) = \varepsilon(q_n, p_n) \notin C$, such that $\{(x, y) \in \mathbb{R}^2 \mid px - qy = 0\} \cap C = \{O\}$ and $(q', p') = \varepsilon(q_{n+1}, p_{n+1})$ is an interior point of $C$. Note that we have $pq' - p'q = \delta \in \{1, -1\}$. Now we apply the automorphism $\varphi$ of $\mathbb{Z}^2$ given by $\delta \begin{pmatrix} p & -q \\ -p & q \end{pmatrix}$, which maps the above line, which meets $C$ only in $\{O\}$, to the $y$-axis, and $(q', p')$ to $(1, 0)$, which is an interior point of $\varphi(C)$. Thus $\varphi(H)$ equals some $H'$ as given in C2.

Now let $H'$ be given as in C2. We finally have to show that $\mathcal{A}(H') = \mathcal{A}(H_1) \cup \sigma_x(\mathcal{A}(H_2))$. Clearly, each atom of $H'$ must be also an atom of $H_1$ or of $\sigma_x(H_2)$.

Now suppose that $(x, y) \in \mathcal{A}(H_1)$ is not an atom of $H'$. Then for $i = 1, 2$ we have $O \neq (x_i, y_i) \in H'$, not both in $H_1$, with $(x, y) = (x_1, y_1) + (x_2, y_2)$ and $x_1, x_2 \geq 0$. Supposing $y_2 < 0$, we have $y = y_1 + y_2 \geq 0$ and obtain $(x, y) = (x_1, y_1 + y_2) + (x_2, 0)$ as a sum of two elements of $H_1 \setminus \{O\}$, a contradiction.

The proof for $(x, y) \in \sigma_x(\mathcal{A}(H_2))$ is analogous. \qed

Remarks.

1. Note that in Theorem M in cases B4, C1 and C2 there are infinitely many monoids $H'$ which are isomorphic to the given $H$.

Furthermore, cases C1 and C2 do not exclude each other, but both are necessary. If both slopes of the bounding rays of $C$ are irrational, $H$ cannot be isomorphic to any $H'$ of case C1. Since in case C2 any monoid $H'$ has at least 3 atoms, $H$ cannot be isomorphic to such $H'$ if the complete integral closure of $H$ has only 2 atoms, i.e. is factorial.

2. We can also use Theorem M to describe the atoms of any completely integrally closed monoid $H$ with $Q(H) = \mathbb{Z}^2$. By Proposition 3b) we may restrict to cones $C$ with $C \cap \mathbb{Z}^2 = \overline{C} \cap \mathbb{Z}^2$. So only the cases A, B1, B4, C1 with $H' = M_\alpha$ and C2 with $\alpha, \beta$ both irrational remain.
3. Instead of using automorphisms of $\mathbb{Z}^2$ one could directly obtain the atoms of $H$ from the (second) convergents to the slopes of the bounding rays of cone($H$). Then one has to modify the results of Theorems 5 and 6 accordingly, since one has to consider also negative slopes and to choose “the right side” of the approximations (i.e. either even or odd indices $n$ of the (second) convergents). This would lead to an abundance of different cases, which we wanted to avoid.

Finally, it might be interesting to note that the operation of $A \in \text{GL}_2(\mathbb{Z})$ changes the slope $\alpha$ to $\frac{a\alpha + b}{c\alpha + d}$, whose continued fraction expansion differs from that of $\alpha$ only in finitely many partial quotients.

Acknowledgment: The author wants to thank Alfred Geroldinger for many discussions and also to thank the unknown referee for his/her careful reading of the manuscript. The suggestions and comments of both of them helped very much to improve the quality of this paper.

References

[1] D.D. Anderson (ed.), Factorization in Integral Domains. Lect. Notes Pure Appl. Math., vol. 189, Marcel Dekker, 1997.
[2] L.L. Avramov, C. Gibbons and R. Wiegand, Monoids of Betti tables over graded algebras. The case of short Gorenstein algebras, 56 pp., in preparation.
[3] A. Bashir and A. Reinhart, On transfer Krull monoids, Semigroup Forum (submitted), https://arxiv.org/abs/2109.04764.
[4] J. Borwein, A. van der Poorten, J. Shallit and W. Zudilin, Neverending Fractions: An Introduction to Continued Fractions. Australian Math. Soc. Lecture Series 23, Cambridge University Press, 2014.
[5] W. Bruns and J. Gubeladze, Polytopes, rings and $K$-theory. Springer Monographs in Mathematics, Springer, New York, 2009.
[6] S.T. Chapman (ed.), Arithmetical Properties of Commutative Rings and Monoids. Lect. Notes Pure Appl. Math., vol. 241, Chapman & Hall/CRC, 2005.
[7] C. Cisto, G. Failla, C. Peterson and R. Utano, Irreducible generalized numerical semigroups and uniqueness of the Frobenius element, Semigroup Forum 99 (2019), 481–495.
[8] D.A. Cox, J.B. Little and H.K. Schenck, Toric Varieties. Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
[9] J. Coykendall and G. Oman, Factorization theory of root closed monoids of small rank, Comm. Algebra 45 (2017), 2795–2808.
[10] M. D’Anna, P.A. García-Sánchez, V. Micale and L. Tozzo, Good subsemigroups of $\mathbb{N}^n$, Int. J. Algebra Comput. 28 (2018), 179–206.
[11] A. Facchini, M. Fontana, A. Geroldinger and B. Olberding (eds.), Advances in Rings, Modules and Factorizations. Proceedings in Mathematics and Statistics, vol. 321, Springer, 2020.
[12] C. Failla, C. Peterson and R. Utano, Algorithms and basic asymptotics for generalized numerical semigroups in $\mathbb{N}^d$, Semigroup Forum 92 (2016), 460–473.
[13] M. Fontana, S. Frisch, S. Glaz, F. Tartarone and P. Zanardo (eds.), Rings, Polynomials, and Modules, Springer, Cham, 2017.
[14] E.R. García Barroso, I. García-Marco and I. Márquez-Corbella, Factorizations of the same lengths in abelian monoids, Ric. Mat. (2021), https://doi.org/10.1007/s11587-021-00562-8.
[15] P.A. García-Sánchez, C. O’Neill and G. Webb, On the computation of factorization invariants for affine semigroups, J. Algebra Appl. 18, Article ID 1950019 (2019), 21 pp.
[16] A. Geroldinger, F. Gotti and S. Tringali, On strongly primary monoids with a focus on Puiseux monoids, J. Algebra 567 (2021), 310–345.
[17] A. Geroldinger and F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory. Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, Boca Raton, 2006.
[18] A. Geroldinger, F. Halter-Koch and G. Lettl, The complete integral closure of monoids and domains II, Rend. di Mat. Roma, ser. VII, 15 (1995), 281–292.
[19] A. Geroldinger and W. Hassler, Local tameness of \( v \)-noetherian monoids, J. Pure Appl. Algebra 212 (2008), 1509–1524.
[20] F. Gotti, The system of sets of lengths and the elasticity of submonoids of a finite-rank free commutative monoid, J. Algebra Appl. 19, Article ID 2050137 (2020), 18 pp.
[21] F. Gotti, Geometric and combinatorial aspects of submonoids of a finite-rank free commutative monoid, Linear Algebra Appl. 604 (2020), 146–186.
[22] D.J. Grynkiewicz, The Characterization of Finite Elasticities, Lecture Notes in Mathematics, Springer, 2022 (to appear). (239 pp., https://arxiv.org/pdf/2012.12757.pdf)
[23] F. Kainrath and G. Lettl, Geometric notes on monoids, Semigroup Forum 61 (2000), 298–302.
[24] A. Ya. Khinchin, Continued fractions. Dover Publ., 1997.
[25] G. Lettl, Subsemigroups of Finitely Generated Groups with Divisor-Theory, Monatsh. Math. 106 (1988), 205–210.
[26] K.Alan Loper, G. Oman and N.J. Werner, The axiomatizability of the class of root closed monoids, Semigroup Forum 91 (2015), 737–740.
[27] W. Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers. Polish Scientific Publ., 1974.

Institut für Mathematik und wissenschaftliches Rechnen
Karl-Franzens-Universität
Heinrichstrasse 36
A-8010 Graz, AUSTRIA
Email address: guenter.lettl@uni-graz.at