On the conservativeness and the recurrence of symmetric jump-diffusions

Jun Masamune a, Toshihiro Uemura b, Jian Wang c,*

a Department of Mathematics and Statistics, Penn State, Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA
b Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita-shi, Osaka 564-8680, Japan
c School of Mathematics and Computer Science, Fujian Normal University, 350007, Fuzhou, PR China

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Abstract

Sufficient conditions for a symmetric jump-diffusion process to be conservative and recurrent are given in terms of the volume of the state space and the jump kernel of the process. A number of examples are presented to illustrate the optimality of these conditions; in particular, the situation is allowed to be that the state space is topologically disconnected but the particles can jump from a connected component to the other components.

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* Corresponding author.
E-mail addresses: jum35@psu.edu (J. Masamune), t-uemura@kansai-u.ac.jp (T. Uemura), jianwang@fjnu.edu.cn (J. Wang).

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1. Introduction and main results

Let \((X, d, m)\) be a metric measure space. We assume that every metric ball \(B(x, r) = \{z \in X: d(x, z) < r\}\) centered at \(x \in X\) with radius \(r > 0\) is pre-compact, and the measure \(m\) is a Radon measure with full support. In particular, \(X\) is locally compact and separable. Let \((\mathcal{E}, \mathcal{F})\) be a regular symmetric Dirichlet form in \(L^2(X; m)\). We denote the extended Dirichlet space of \((\mathcal{E}, \mathcal{F})\) by \(\mathcal{F}_e\), and a quasi-continuous version of \(u \in \mathcal{F}_e\) by \(\tilde{u}\). According to the Beurling–Deny theorem, see, e.g., [8, Theorem 3.2.1 and Lemma 4.5.4], we can express \((\mathcal{E}, \mathcal{F})\) as follows

\[
\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int\int_{x \neq y} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y)) J(dx, dy)
\]

\[
+ \int_{X} \tilde{u}(x)\tilde{v}(x) k(dx)
\]

for any \(u, v \in \mathcal{F}_e\),

where \((\mathcal{E}^{(c)}, C_0(X) \cap \mathcal{F})\) is a strongly-local symmetric form and \(C_0(X)\) is the space of all real-valued continuous functions on \(X\) with compact support; \(J\) is a symmetric positive Radon measure on the product space \(X \times X\) off the diagonal \(\{(x, x): x \in X\}\); and \(k\) is a positive Radon measure on \(X\).

Let \(\mu_{\langle \cdot, \cdot \rangle}\) be a bounded signed measure, see [8, Lemma 3.2.3], such that

\[
\mathcal{E}^{(c)}(u, v) = \frac{1}{2} \mu_{\langle u, v \rangle}(X) = \frac{1}{2} \int_{X} \mu_{\langle u, v \rangle}(dx)
\]

for \(u, v \in \mathcal{F}_e\).

Throughout the paper, we assume the following set (A) of conditions:

(A-1) The killing measure \(k\) does not appear; that is, the corresponding process is no killing inside.

(A-2) For each \(u, v \in \mathcal{F}_e\), the measure \(\mu_{\langle u, v \rangle}\) is absolutely continuous with respect to \(m\). We denote the corresponding Radon–Nikodym density by \(\Gamma^{(c)}(u, v)\); namely,

\[
\mu_{\langle u, v \rangle}(dx) = \Gamma^{(c)}(u, v)(x) m(dx).
\]

(A-3) The jump measure \(J\) has a symmetric kernel \(j(x, dy)\) over \(X \times \mathcal{B}(X)\) such that

\[
J(dx, dy) = j(x, dy) m(dx) (= j(y, dx) m(dy) = J(dy, dx)).
\]
For $u, v \in \mathcal{F}_e$, define
\[
\Gamma^{(j)}(u,v)(x) = \int_{x \neq y} \left( \tilde{u}(x) - \tilde{u}(y) \right) \left( \tilde{v}(x) - \tilde{v}(y) \right) j(x, dy),
\]
and
\[
\mathcal{E}^{(j)}(u,v) = \int \Gamma^{(j)}(u,v)(x) m(dx).
\]
Therefore, the form $\mathcal{E}$ has the following expression for any $u,v \in \mathcal{F}_e$:
\[
\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \mathcal{E}^{(j)}(u,v)
= \frac{1}{2} \int \Gamma^{(c)}(u,v)(x) m(dx) + \int \Gamma^{(j)}(u,v)(x) m(dx)
= \frac{1}{2} \int \Gamma^{(c)}(u,v)(x) m(dx) + \int \int_{x \neq y} \left( \tilde{u}(x) - \tilde{u}(y) \right) \left( \tilde{v}(x) - \tilde{v}(y) \right) j(x, dy) m(dx).
\]

Let $\psi_K$ be the distance function from a compact set $K$ of $X$, i.e., $\psi_K(\cdot) = \inf_{y \in K} d(\cdot, y)$. For every $r > 0$, we denote $B(K, r) = \{ x \in X : \psi_K < r \}$ and its closure $\{ x \in X : \psi_K \leq r \}$ by $\overline{B}(K, r)$. Clearly, $B(K, r)$ is pre-compact. Let $\mathcal{F}_{loc}$ be the set of measurable functions $u$ such that for each relatively compact open set $G$ of $X$ there exists $w \in \mathcal{F}$ which satisfies that $u|_G = w|_G$ $m$-a.e. Additionally, we assume the following set (M) of conditions so that both $\mathcal{E}^{(c)}$ and $\mathcal{E}^{(j)}$ are compatible with the distance $d$:

(M-1) $\psi_K \in \mathcal{F}_{loc}$ for every compact set $K \subset X$,
(M-2) $M_c := \text{ess sup}_{x \in X^{(c)}} \Gamma^{(c)}(d, d)(x) < \infty$,
(M-3) $M_j := \text{ess sup}_{x \in X^{(j)}} \int_{x \neq y} (1 \wedge d^2(x, y)) j(x, dy) < \infty$,

where $X^{(c)} = \{ x \in X : \Gamma^{(c)} \neq 0 \}$ and $X^{(j)} = \{ x \in X : \Gamma^{(j)} \neq 0 \}$.

There are many classical examples of symmetric diffusions or symmetric pure jump processes whose Dirichlet form satisfies conditions (A) and (M): for instance, strongly-local Dirichlet forms on a metric measure space, whose distance is the Carnot–Carathéodory distance associated with the Dirichlet form. This includes canonical Dirichlet forms on Riemannian manifolds, CR manifolds, sub-Riemannian manifolds, and weighted manifolds; divergence type operators with bounded coefficients on Euclidean spaces; the sum of squares of vector fields satisfying Hörmader’s condition, the quantum graphs, and pre-fractals. Other examples are symmetric $\alpha$-stable Lévy processes with $\alpha \in (0, 2)$ on Euclidean spaces, and symmetric random walks on graphs.

Let $A$ be the generator of $(\mathcal{E}, \mathcal{F})$ in $L^2(X; m)$. We denote the associated semigroup and the resolvent by $(T_t)_{t \geq 0} = (e^{tA})_{t \geq 0}$ and $G = \int_0^\infty T_t dt$, respectively. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called conservative if
\[
T_t 1 \equiv 1, \quad m\text{-a.e. for any } t > 0
\]
and recurrent if 

$$Gf(x) \equiv 0 \text{ or } \infty \quad \text{for any } f \in L^1_{\text{loc}}(X; m) \text{ and } m\text{-a.e. } x \in X.$$  

It is a classical result that Brownian motion on $\mathbb{R}^n$ is conservative for any $n \geq 1$ and is recurrent if and only if $n = 1, 2$. This result has been generalized to the Wiener process of complete Riemannian manifolds, and one of the most important discoveries is that a certain bound of the volume at infinity – rather than the dimension – implies these properties. This fact was first found by M.P. Gaffney [10] for the conservativeness, and it has been refined by various methods in [1,23,36,17,5,14]. Especially, R. Azencott [1] and A. Grigor’yan [14] demonstrated that the conservativeness may fail without a condition on the curvature or volume. On the other hand, the recurrence of the Wiener process of Riemannian manifolds or jump processes has been investigated by several authors in [4,22,38,11,12,28]. Furthermore, K.-T. Sturm [35] extended the theory to a general strongly-local regular Dirichlet form on a metric measure space equipped with the Carnot–Carathéodory distance.

Recently, there has been a tremendous amount of work devoted to the conservation property of a non-local Dirichlet form; for instance, the physical Laplacian on an infinite graph [7,6,39–41,24,18–20] and non-local Dirichlet forms [26,15,33]; however, as far as the authors know, there is only one result by Z.-Q. Chen and T. Kumagai [3] for the Dirichlet form which has both the strongly-local and non-local terms. Due to its nature, the associated process is called a jump-diffusion process.

Our first main purpose is to investigate the conservative property of a jump-diffusion process. For any $x \in X$ and $r > 0$, the volume of $B(x, r)$ is denoted by $V(x, r)$.

**Theorem 1.1.** If 

$$\liminf_{r \to \infty} \frac{\ln V(x_0, r)}{r \ln r} < \infty, \quad (1.1)$$  

for some $x_0 \in X$, then $(\mathcal{E}, \mathcal{F})$ is conservative.

This result was obtained for a non-local Dirichlet form in [15, Theorem 1.1], where the left-hand side of (1.1) is required to be less than 1/2. Let us explain the significance of removing the constant 1/2 by comparing the uniqueness class with the conservation property. Let $\mathcal{U}$ be the set of the solutions to the Cauchy problem of the heat equation with zero initial data. If any $u \in \mathcal{U}$ is identically 0, then $\mathcal{U}$ is called a uniqueness class. Under an integrability assumption, determining the uniqueness class implies the conservativeness of Riemannian manifolds [13], Dirichlet forms [35], and graphs [20]. In fact, A. Grigor’yan [13] and K.-T. Sturm [35] established the sharp conservation test for complete Riemannian manifolds and strongly-local Dirichlet forms, respectively, in this way. However, X. Huang [20, Section 3.3] constructed an example of a graph, which verifies that the constant 1/2 is indeed needed for the uniqueness class. Therefore, Theorem 1.1 together with Huang’s example demonstrates that the uniqueness class condition is really stronger than the conservation property for a graph.

Next, we turn to the recurrence. For any $x \in X$ and $r > 0$, the volumes of the closed ball $\overline{B}(x, r)$ intersected with $X^{(c)}$ and $X^{(j)}$ are denoted by $V^{(c)}(x, r)$ and $V^{(j)}(x, r)$, respectively. For $r > 0$, define
\[ \omega(r) = \sup_{x \in X^{(j)}} \int_{x \neq y} (d(x, y) \wedge r)^2 j(x, dy). \]

Our second main result is

**Theorem 1.2.** If

\[
\liminf_{r \to \infty} \frac{1}{r^2} \left[ V^{(c)}(x_0, r) + V^{(j)}(x_0, r)\omega(r) \right] < \infty, \tag{1.2}
\]

for some \( x_0 \in X \), then \((E, F)\) is recurrent.

Theorem 1.2 was proven in the case of the Wiener process (namely, the process does not jump) on a complete Riemannian manifold by S.Y. Cheng and S.T. Yau [4]. Theorem 1.2 is sharp for an isotropic symmetric \( \alpha \)-stable Lévy process on \( \mathbb{R}^n \), see, e.g., [30, Corollary 37.17 and Theorem 37.18] or Example 5.2 in Section 5. Here, let us mention that [30, Corollary 37.17 and Theorem 37.18] are derived from the characteristic functions of the associated processes, see [32] for the recent development on this topic; while Theorem 1.2 is based on the theory of Dirichlet forms.

This paper is organized as follows. Section 2 is devoted to the preliminaries. Here we establish an integral-derivation type property for a Dirichlet form of jump-process type, which is a technical key to prove the conservation property. The main results, Theorems 1.1 and 1.2, are proved in Sections 3 and 4, respectively. Finally, in Section 5 we present some examples of symmetric jump-diffusions to illustrate the power of our main theorems.

2. Preliminaries: the integral-derivation property

In this section, we first prepare the preliminaries and then proceed to establish an integral-derivation type property for a Dirichlet form with jump-diffusion type. This will be used to prove the conservation property in the next section.

We begin with the following quite elementary fact.

**Lemma 2.1.** If \( u \in \mathcal{F}_{\text{loc}} \cap L^\infty \) has compact support, where \( L^\infty = L^\infty(X) \) is the space of real-valued bounded measurable functions on \( X \), then \( u \in \mathcal{F} \cap L^\infty \).

**Proof.** Suppose that \( \text{supp} \ u \subset K \) with a compact set \( K \). Let \( \eta \in \mathcal{F} \cap L^\infty \) agree with \( u \) on \( B(K, 1) \). Because of the regularity and the fact that the constant function belongs to \( \mathcal{F}_{\text{loc}} \), see the remark in [8, p. 117], there is a function \( \chi \in \mathcal{F} \cap L^\infty \) such that \( \chi|_K = 1 \) and \( \text{supp} \ \chi \subset B(K, 1) \). Since \( \eta \chi \in \mathcal{F} \) and \( u = \eta \chi \), the statement follows. \( \Box \)

For the sake of simplicity, hereafter we denote \( \Gamma[\cdot] = \Gamma(\cdot, \cdot) \), \( \mathcal{E}[\cdot] = \mathcal{E}(\cdot, \cdot) \), etc. We say that the jump range of \( \mathcal{E} \) or \( \mathcal{E}^{(j)} \) is uniformly bounded, if there exists a constant \( a > 0 \) such that \( \text{supp}(j(x, \cdot)) \subset B(x, a) \) for every \( x \in X \).

**Lemma 2.2.** Suppose that the jump range of \( \mathcal{E} \) is uniformly bounded. If \( u \in \mathcal{F}_{\text{loc}} \cap L^\infty \) is constant outside a compact set, then for any \( v \in \mathcal{F} \cap L^\infty \), \( uv \in \mathcal{F} \cap L^\infty \).
Proof. Let $K \subset X$ be a compact set such that $u$ is constant outside it. Consider the sequence of cut-off functions $(\chi_l)_{l \in \mathbb{N}}$, where for $l \geq 1$,

$$\chi_l = \left(2 - l^{-1} \psi \right) \wedge 1_+.$$ 

By Lemma 2.1, the function $\chi_l$ belongs to $\mathcal{F}$ for any $l \geq 1$. Obviously, $\chi_l = 1$ on $B(K, l)$ and $\text{supp}(\chi_l) \subset B(K, 2l)$.

We set for any $l \geq 1$, $v_l = uv|_{B(K, l)}$. Since $u \in \mathcal{F}_{\text{loc}} \cap L^\infty$ and $v \in \mathcal{F} \cap L^\infty$, $v_l$ belongs to $\mathcal{F}_{\text{loc}} \cap L^\infty$ and has compact support. Hence, Lemma 2.1 shows that $v_l \in \mathcal{F}$ for any $l \geq 1$.

Next, we claim that the sequence $(v_l)_{l \geq 1}$ is $\mathcal{E}$-Cauchy. Set $\chi_{l,l'} = \chi_l - \chi_{l'}$ for $l, l' \geq 1$. Since the jump range of $\mathcal{E}$ is uniformly bounded, for large enough $l$ and $l'$, $\mathcal{E}[v_l - v_{l'}] = \mathcal{E}[(\chi_l - \chi_{l'})uv] = \kappa \cdot \mathcal{E}[\chi_{l,l'}v],$

where $\kappa = u|_{K^c}$. By [8, Lemma 3.2.5], 

$$\mathcal{E}^{(c)}[\chi_{l,l'}v] \leq 2 \int v^2 \Gamma^{(c)}[\chi_{l,l'}] \, dm + 2 \int \chi_{l,l'}^2 \Gamma^{(c)}[v] \, dm.$$ 

Because of (M) and the chain rule of the strongly-local Dirichlet form, see, e.g., [35, p. 190], $\Gamma^{(c)}[\chi_{l,l'}] \to 0$ as $l, l' \to \infty$. This together with the fact $\chi_{l,l'} \to 0$ as $l, l' \to \infty$ yields that $\mathcal{E}^{(c)}[\chi_{l,l'}v]$ tends to zero as $l, l' \to \infty$.

On the other hand,

$$\mathcal{E}^{(j)}[\chi_{l,l'}v] \leq 2 \int v^2(x) \int (\chi_{l,l'}(x) - \chi_{l,l'}(y))^2 j(x, dy) \, m(dx) + 2 \int \chi_{l,l'}^2(y)(v(x) - v(y))^2 j(x, dy) \, m(dx) =: (I) + (II).$$

For any $x \in X$,

$$\int (\chi_{l,l'}(x) - \chi_{l,l'}(y))^2 j(x, dy) = \int ((\chi_l(x) - \chi_l(y)) - (\chi_{l'}(x) - \chi_{l'}(y)))^2 j(x, dy) \leq 2 \int (\chi_l(x) - \chi_l(y))^2 j(x, dy) + 2 \int (\chi_{l'}(x) - \chi_{l'}(y))^2 j(x, dy) \leq 2(l^{-2} + l'^{-2}) \int d(x, y)^2 j(x, dy).$$

Combining the fact that $\text{supp}(j(x, dy)) \subset B(x, a)$ for all $x \in X$ and some $a > 0$ with the assumption (M), the last term in the right-hand side of the equation above is dominated by

$$2(1 + a^2)M_j(l^{-2} + l'^{-2}),$$

where $M_j$ is a constant depending on $j$. Thus,

$$\mathcal{E}^{(j)}[\chi_{l,l'}v] \leq 2(1 + a^2)M_j(l^{-2} + l'^{-2}),$$ 

which implies $\mathcal{E}^{(j)}[\chi_{l,l'}v] \to 0$ as $l, l' \to \infty$.

Finally, we need to show that $\mathcal{E}[v_l - v_{l'}] \to 0$ as $l, l' \to \infty$. By the chain rule of the strongly-local Dirichlet form, see, e.g., [35, p. 190],

$$\mathcal{E}[v_l - v_{l'}] \leq \int v^2 \Gamma[\chi_{l,l'}] \, dm + \int \chi_{l,l'}^2 \Gamma[v] \, dm.$$ 

Because of (M) and the chain rule of the strongly-local Dirichlet form, see, e.g., [35, p. 190], $\Gamma[\chi_{l,l'}] \to 0$ as $l, l' \to \infty$. This together with the fact $\chi_{l,l'} \to 0$ as $l, l' \to \infty$ yields that $\mathcal{E}[v_l - v_{l'}]$ tends to zero as $l, l' \to \infty$. Therefore, $(v_l)_{l \geq 1}$ is an $\mathcal{E}$-Cauchy sequence.
which tends to 0 as \( l, l' \to \infty \). Hence \((I) \to 0\) as \( l, l' \to \infty \). Since \( \chi_{l,l'} \to 0 \), \( m \)-a.e. as \( l, l' \to \infty \), and so the desired claim follows.

Finally, since \( \psi_l \to \psi, m \)-a.e. as \( l \to \infty \), \( \psi \in \mathcal{F} \). Thus this together with the fact \( uv \in L^2 \) and [8, Theorem 1.5.2(iii)] yields that \( uv \in \mathcal{F} \). \( \square \)

The following is the integral-derivation property for our Dirichlet form.

**Lemma 2.3.** Suppose that the jump range of \( \mathcal{E} \) is uniformly bounded. If \( u \in \mathcal{F} \cap L^\infty \) and \( \phi \in \mathcal{F}_{loc} \cap L^\infty \) is constant outside a compact set, then

\[
\mathcal{E}(u, \phi) = \int u \Gamma (u, \phi) \, dm + \int \phi \Gamma [u] \, dm,
\]

where \( \Gamma = \frac{1}{2} (\Gamma (c) + \Gamma (j)) \).

**Proof.** According to Lemma 2.2, \( u\phi \in \mathcal{F} \). By the derivation property of \( \mathcal{E}^{(c)} \), see, e.g., [8, Lemma 3.2.5 and the note on p. 117],

\[
\int \Gamma^{(c)} (u, \phi) \, dm = \int u \Gamma^{(c)} (u, \phi) \, dm + \int \phi \Gamma^{(c)} [u] \, dm.
\]

Next, by the integral property of a non-local Dirichlet form, see [27, Proposition 2.2], we have

\[
\int \Gamma^{(j)} (u, \phi) \, dm = \int u \Gamma^{(j)} (u, \phi) \, dm + \int \phi \Gamma^{(j)} [u] \, dm.
\]

Combining the two identities, we obtain (2.3). \( \square \)

### 3. Proof of Theorem 1.1: the conservation property

The aim of this section is to prove Theorem 1.1. For any \( a > 0 \), consider a symmetric form \((\mathcal{E}^{(j,a)}, \mathcal{F})\) defined by

\[
\mathcal{E}^{(j,a)} [u] = \int \int (u(x) - u(y))^2 \mathbf{1}_{d(x,y) \leq a} \, f(x, dy) \, m(dx) \quad \text{for } u \in \mathcal{F}.
\]

Under the condition (M), \((\mathcal{E}^{(j,a)} + \mathcal{E}^{(c)}, \mathcal{F})\) is a regular Dirichlet form, and it is conservative if and only if so is \((\mathcal{E}, \mathcal{F})\), see [31, Section 4] and [26, Section 3]. Clearly, \((\mathcal{E}^{(j,a)}, \mathcal{F})\) has uniformly bounded range. Therefore, in order to prove the conservation property, we may and do assume that \( \mathcal{E} \) has uniformly bounded jump range. More precisely, we suppose that there exists a constant \( a > 0 \) such that

\[
j(x, dy) = \mathbf{1}_{B(x,a)} (y) \, j(x, dy) \quad \text{for all } x \in X.
\]

Our proof is basically the Davies method [5], which was used also in [15]; however, we are able to get a better result because of the choice of \( a \). In this section, the constant \( a \) will be

\[
a = a(x_0, m) := \left[ \lim_{r \to \infty} \frac{\log V(x_0, r)}{r \log r} + 9 \right]^{-1},
\]

(3.4)
where \( x_0 \in X \) is the reference point in Theorem 1.1. For \( f \in C_0(X) \) with \( f \geq 0 \), set

\[
\psi(x) = d(x, \text{supp}(f))
\]

and

\[
\phi(x) = e^{\alpha \psi(x)},
\]

where \( \alpha > 0 \) is a constant determined later. Note that if \( n \geq 1 \) and \( x \in X \) satisfy

\[
n \geq a^{-1}[4a + 2d(x_0, \text{supp}(f))] \quad \text{and} \quad (n - 2)a \leq d(x, x_0) \leq (n + 1)a,
\]

then

\[
\psi(x) \geq d(x, x_0) - d(x_0, \text{supp}(f)) \geq (n - 2)a - d(x_0, \text{supp}(f)) \geq an/2,
\]

and so

\[
\phi(x) = e^{\alpha \psi(x)} \geq e^{an/2}.
\]

(3.5)

For the function \( f \) above and any \( t \geq 0 \), we denote \( u_t = T_t f \). Since \((T_t)_{t \geq 0}\) is analytic, \( u_t \) belongs to the domain of the \( L^2 \)-generator \( A \) of \((E, \mathcal{F})\); in particular, \( u_t \in \mathcal{F} \cap L^\infty \) for any \( t > 0 \).

The following lemma provides the key estimate.

Lemma 3.1. Using the notations above, for any \( t \geq 0 \),

\[
\int_0^t \int \phi \Gamma [u_s] \, dm \, ds \leq 2e^{\gamma t} \| \phi^{1/2} u_t \|_2^2,
\]

(3.6)

where \( \gamma = \alpha^2 (e^{2\alpha a} + 1)M/2 \) and \( M = M_c \lor M_j \).

Proof. In the following, we denote the norm and the inner product of \( L^2(X; m) \) by \( \| \cdot \|_2 \) and \( \langle \cdot, \cdot \rangle \), respectively. For any \( n \geq 1 \), set

\[
\phi_n(x) = e^{\alpha (\psi(x) \land n)}.
\]

Since \( \psi \in \mathcal{F}_{\text{loc}} \), we may apply an argument in [8, pp. 116–117] to deduce that \( \phi_n \in \mathcal{F}_{\text{loc}} \) for every \( n \geq 1 \). Taking into account that \( \psi \in L^\infty \) is constant outside a compact set, Lemma 2.2 shows that for every \( t > 0 \) and \( n \geq 1 \), \( u_t \phi_n \in \mathcal{F} \). Therefore, by Lemma 2.3, for all \( t > 0 \),

\[
\frac{1}{2} \frac{d}{dt} \| \phi_n^{1/2} u_t \|_2^2 = \langle u_t, \phi_n u_t \rangle = -E(u_t, \phi_n u_t)
\]

\[
= -\int \phi_n \Gamma [u_t] \, dm - \int u_t \Gamma (u_t, \phi_n) \, dm
\]

\[
\leq -\int \phi_n \Gamma [u_t] \, dm + \left| \int u_t \Gamma (u_t, \phi_n) \, dm \right|,
\]
where $\dot{u}_t = \frac{d}{dt}u_t$. This is,
\[
\int \phi_n \Gamma[u_t] \, dm \leq \left| \int u_t \Gamma(u_t, \phi_n) \, dm \right| - \frac{1}{2} \frac{d}{dt} \left\| \phi_n^{1/2} u_t \right\|_2^2.
\] (3.7)

Next, we estimate the first term on the right side of this equation. For every $x \in X$, according to the Cauchy–Schwarz inequality,
\[
\left| \Gamma^{(j)}(u_t, \phi_n)(x) \right| = \left| \int (u_t(x) - u_t(y)) (\phi_n(x) - \phi_n(y)) j(x, dy) \right|
\leq \sqrt{\int (u_t(x) - u_t(y))^2 j(x, dy) \int (\phi_n(x) - \phi_n(y))^2 j(x, dy)}
= \sqrt{\Gamma^{(j)}[u_t](x) \Gamma^{(j)}[\phi_n](x)}.
\]

By the Cauchy–Schwarz inequality again,
\[
\left| \int u_t \Gamma^{(j)}(u_t, \phi_n) \, dm \right| \leq \int \phi_n^{1/2} \sqrt{\Gamma^{(j)}[u_t] \phi_n^{1/2}} \sqrt{u_t^2 \Gamma^{(j)}[\phi_n]} \, dm
\leq \sqrt{\int \phi_n \Gamma^{(j)}[u_t] \, dm \int \phi_n^{-1} u_t^2 \Gamma^{(j)}[\phi_n] \, dm}.
\]

Since
\[
|e^{\alpha r} - 1| \leq \alpha e^{\alpha a} |r| \quad \text{for any } r \in (0, a],
\]
it follows that
\[
|\phi_n(x) - \phi_n(y)| \leq \alpha e^{\alpha a} \phi_n(x) d(x, y) \quad \text{for any } x, y \in X \text{ with } d(x, y) \leq a,
\]
and so
\[
\Gamma^{(j)}[\phi_n](x) \leq (\alpha e^{\alpha a} \phi(x))^2 \int d^2(x, y) j(x, dy) \quad \text{for every } x \in X.
\]

Since $\text{supp}(j(x, dy)) \subset B(x, a)$ for any $x \in X$ and some constant $a \in (0, 1)$, we get
\[
\int \phi_n^{-1} u_t^2 \Gamma^{(j)}[\phi_n] \, dm \leq \alpha^2 e^{2\alpha a} \int \phi_n(x) u_t^2(x) \int d(x, y)^2 j(x, dy) m(dx)
\leq \alpha^2 e^{2\alpha a} \int \phi_n(x) u_t^2(x) \int (d(x, y) \wedge a)^2 j(x, dy) m(dx)
\leq M_j \alpha^2 e^{2\alpha a} \int \phi_n u_t^2 \, dm.
\]

Therefore, for any $\lambda > 0$,
\[ \left| \int u_t \Gamma^{(j)}(u_t, \phi_n) \, dm \right| \leq \sqrt{M_j} \int \phi_n \Gamma^{(j)}[u_t] \, dm \sqrt{\alpha^2 e^{2\alpha_a} \int \phi_n u_t^2 \, dm} \]

\[ \leq \frac{M_j}{2\lambda} \int \phi_n \Gamma^{(j)}[u_t] \, dm + \frac{\lambda \alpha^2 e^{2\alpha_a}}{2} \int \phi_n u_t^2 \, dm \]

\[ = \frac{M_j}{2\lambda} \int \phi_n \Gamma^{(j)}[u_t] \, dm + \frac{\lambda \alpha^2 e^{2\alpha_a}}{2} \| \phi_n^{1/2} u_t \|_2^2, \]

where in the last inequality we have used the fact that \( 2\xi \eta \leq \lambda^{-1} \xi^2 + \lambda \eta^2 \) for any \( \xi, \eta \geq 0 \) and \( \lambda > 0 \).

On the other hand, we apply the argument above for the local term to get that

\[ \left| \int u_t \Gamma^{(c)}(u_t, \phi_n) \, dm \right| \leq \sqrt{\int \phi_n \Gamma^{(c)}[u_t] \, dm} \sqrt{\int \phi_n^{-1} u_t^2 \Gamma^{(c)}[\phi_n] \, dm}. \]

According to the chain rule for a strongly-local Dirichlet form, see, e.g., [35, p. 190],

\[ \int \phi_n^{-1} u_t^2 \Gamma^{(c)}[\phi_n] \, dm \leq \alpha^2 \int u_t^2 \phi_n \Gamma^{(c)}[d] \, dm, \]

which along with the assumption (M) gives us

\[ \int \phi_n^{-1} u_t^2 \Gamma^{(c)}[\phi_n] \, dm \leq M_c \alpha^2 \int u_t^2 \phi_n \, dm. \]

We again follow the argument above to obtain the estimate:

\[ \left| \int u_t \Gamma^{(c)}(u_t, \phi_n) \, dm \right| \leq \frac{M_c}{2\lambda} \int \phi_n \Gamma^{(c)}[u_t] \, dm + \frac{\lambda \alpha^2}{2} \| \phi_n^{1/2} u_t \|_2^2 \quad \text{for any } \lambda > 0. \]

Combining the estimates for the non-local and strongly-local terms, we get that

\[ \left| \int u_t \Gamma(u_t, \phi_n) \, dm \right| \leq \frac{M_c}{2\lambda} \int \phi_n \Gamma[u_t] \, dm + \frac{\lambda \alpha^2 (e^{2\alpha_a} + 1)}{2} \| \phi_n^{1/2} u_t \|_2^2. \]

By applying this inequality for (3.7), we have

\[ \left(2 - \frac{M}{\lambda}\right) \int \phi_n \Gamma[u_s] \, dm \leq \lambda \alpha^2 (e^{2\alpha_a} + 1) \| \phi_n^{1/2} u_s \|_2^2 - \frac{d}{ds} \| \phi_n^{1/2} u_s \|_2^2. \quad (3.8) \]

If we integrate this with respect to \( s \) over \([0, t]\), then

\[ \left(2 - \frac{M}{\lambda}\right) \int_0^t \int \phi_n \Gamma[u_s] \, dm \]

\[ \leq \lambda \alpha^2 (e^{2\alpha_a} + 1) \int_0^t \| \phi_n^{1/2} u_s \|_2^2 \, ds - (\| \phi_n^{1/2} u_t \|_2^2 - \| \phi_n^{1/2} f \|_2^2). \quad (3.9) \]
We estimate \( \| \phi_{n/2}^1 u_s \|_2^2 \) for any \( s \leq t \) by first letting \( \lambda = M/2 \) in (3.8),

\[
\frac{d}{ds} \| \phi_{n/2}^1 u_s \|_2^2 \leq \frac{M\alpha^2(e^{2\alpha a} + 1)}{2} \| \phi_{n/2}^1 u_s \|_2^2,
\]

and then, by applying the Gronwall inequality:

\[
\| \phi_{n/2}^1 u_s \|_2^2 \leq \exp\left(\frac{M\alpha^2(e^{2\alpha a} + 1)}{2}t/2\right) \| \phi_{n/2}^1 f \|_2^2.
\]

Substituting this into (3.9), we have

\[
\left(2 - \frac{M}{\lambda}\right) \int_0^t \int \phi_n \Gamma[u_s] \, dm \, ds
\]

\[
\leq \| \phi_{n/2}^1 f \|_2^2 + \frac{2\lambda}{M} [\exp(M\alpha^2(e^{2\alpha a} + 1)t/2) - 1] \| \phi_{n/2}^1 f \|_2^2.
\]

Setting \( \lambda = M \), this becomes

\[
\int_0^t \int \phi_n \Gamma[u_s] \, dm \, ds \leq 2 \exp(M\alpha^2(e^{2\alpha a} + 1)t/2) \| \phi_{n/2}^1 f \|_2^2.
\]

The required assertion (3.6) follows by letting \( n \to \infty \). \( \square \)

We are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We adopt the notations in the proof of Lemma 3.1. Define a cut-off function \( g_n \) for any \( n \geq 1 \) as follows

\[
g_n(x) := (n - a^{-1}d(x, x_0)) \land 1.
\]

By Lemma 2.1, \( g_n \) belongs to \( \mathcal{F} \). To the end of the proof, we show that there exists a sequence \( (n_k)_{k \geq 0} \) such that \( n_k \to \infty \) as \( k \to \infty \), and for every \( t > 0 \),

\[
\int_0^t \langle \dot{u}_s, g_{n_k} \rangle \, ds \to 0 \quad \text{as} \quad k \to \infty.
\]

Indeed, we can deduce from this and the dominated convergence theorem that

\[
\langle T_t f, 1 \rangle = \lim_{k \to \infty} \langle u_t, g_{n_k} \rangle = \lim_{k \to \infty} \langle f, g_{n_k} \rangle + \lim_{k \to \infty} \int_0^t \langle \dot{u}_s, g_{n_k} \rangle \, ds = \langle f, 1 \rangle,
\]

which immediately implies the conservation property.
Since \((u_s)_{s>0}\) solves the heat equation and \(g_n \in \mathcal{F}\),

\[
\int_0^t \langle \dot{u}_s, g_n \rangle ds = -\int_0^t \mathcal{E}(u_s, g_n) ds = -\int_0^t (\mathcal{E}^{(c)}(u_s, g_n) + \mathcal{E}^{(j)}(u_s, g_n)) ds.
\]

(3.10)

First, we estimate the second term, the harder one, on the right side. For any \(t > 0\),

\[
\left| \int_0^t \mathcal{E}^{(j)}(u_s, g_n) ds \right| \leq t \left[ \int_0^t \int \sqrt{\Gamma^{(j)}[u_s]} \sqrt{\Gamma^{(j)}[g_n]} dm \right] ds
\]

\[
= t \left[ \int_0^t \int \sqrt{\phi \Gamma^{(j)}[u_s]} \sqrt{\phi^{-1} \Gamma^{(j)}[g_n]} dm \right] ds
\]

\[
\leq t \left[ \int_0^t \int \phi \Gamma^{(j)}[u_s] dm \right] \sqrt{t \int \phi^{-1} \Gamma^{(j)}[g_n] dm} ds
\]

\[
\leq \left[ \int_0^t \int \phi \Gamma^{(j)}[u_s] dm ds \right] \left[ \int_0^t \phi^{-1} \Gamma^{(j)}[g_n] dm ds \right]
\]

\[
= \left[ \int_0^t \int \phi \Gamma^{(j)}[u_s] dm ds \right] \sqrt{t \int \phi^{-1} \Gamma^{(j)}[g_n] dm}.
\]

(3.11)

where all the inequalities above follow from the Cauchy–Schwarz inequality. For any \(n > 0\), let \(A_n\) denote the following annulus associated with the constant \(a\)

\[
A_n = A_n(a) = \overline{B(x_0, (n+1)a)} \setminus B(x_0, (n-2)a).
\]

Since \(\text{supp}(g_n) \subset B(x_0, na)\) and \(\text{supp}(j(x, dy)) \subset B(x, a)\) for all \(x \in X\), it holds that if \(x \notin A_n\),

\[
\Gamma^{(j)}[g_n](x) = \int (g_n(x) - g_n(y))^2 j(x, dy) = 0;
\]

if \(x \in A_n\),

\[
\Gamma^{(j)}[g_n](x) \leq a^{-2} \int d(x, y)^2 j(x, dy)
\]

\[
\leq a^{-2} \int (d(x, y) \wedge a)^2 j(x, dy)
\]

\[
\leq a^{-2} M_j,
\]
where in the last inequality we have used the fact that $0 < a < 1$. Choosing $n$ large enough so that $n \geq a^{-1}[4a + 2d(x_0, \text{supp}(f))]$, we get from (3.5) that
\[
\int_{A_n} \phi^{-1} \Gamma^{(j)}[g_n] \, dm = \int_{A_n} \phi^{-1} \Gamma^{(j)}[g_n] \, dm \\
\leq a^{-2} M_j e^{-a\alpha n/2} m(A_n).
\]
Therefore, by (3.11),
\[
\left| \int_0^t \mathcal{E}^{(j)}(u_s, g_n) \, ds \right|^2 \leq a^{-2} t M_j e^{-a\alpha n/2} m(A_n) \int_0^t \phi \Gamma^{(j)}[u_s] \, dm \, ds.
\]
In a similar way, we can prove that
\[
\left| \int_0^t \mathcal{E}^{(c)}(u_s, g_n) \, ds \right|^2 \leq a^{-2} t M_c e^{-a\alpha n/2} m(A_n) \int_0^t \phi \Gamma^{(c)}[u_s] \, dm \, ds.
\]
Therefore,
\[
\left| \int_0^t \mathcal{E}(u_s, g_n) \, ds \right|^2 \leq 2a^{-2} t M e^{-a\alpha n/2} m(A_n) \int_0^t \phi \Gamma[u_s] \, dm \, ds. \tag{3.12}
\]
We now apply (3.12) and Lemma 3.1 for (3.10) to get that
\[
\left| \int_0^t \langle \dot{u}_s, g_n \rangle \, ds \right|^2 \leq 2a^{-2} t M e^{-a\alpha n/2} m(A_n) \int_0^t \phi \Gamma[u_s] \, dm \, ds \\
\leq 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \exp \left( \frac{M \alpha^2 (e^{2\alpha a} + 1)t}{2} - \frac{\alpha n}{2} + \log m(A_n) \right). \tag{3.13}
\]
Finally, we estimate (3.13) by applying the volume assumption (1.1). Indeed, according to (1.1), there exists a sequence $(n_k)_{k \geq 1}$ such that $n_k \to \infty$ as $k \to \infty$, and for a large enough $k \geq 1$,
\[
\log m(A_{n_k}) \leq \log V(x_0, (n_k + 1)a) \\
\leq (c_3 - 1/2)((n_k + 1)a) \log((n_k + 1)a) \\
\leq ac_3 n_k \log n_k,
\]
where
\[
c_3 = \liminf_{r \to \infty} \frac{\log V(x_0, r)}{r \log r} + 1.
\]
Taking $\alpha = 4c_3 \log n_k$ and $k$ large enough such that $n_k \geq a^{-1}[4a + 2d(x_0, \text{supp}(f))]$, we estimate the right side of (3.13) to get

\[
\left| \int_0^t \langle \dot{u}_s, gn_k \rangle \, ds \right|^2 \leq 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \\
\times \exp \left( \frac{M \alpha^2 (e^{2\alpha a} + 1) t}{2} - 2ac_3 n_k \log n_k + ac_3 n_k \log n_k \right)
\]

\[
= 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \exp \left( \frac{M \alpha^2 (e^{2\alpha a} + 1) t}{2} - ac_3 n_k \log n_k \right).
\]

Since $e^{2\alpha a} = n_k^{8ac_3}$ and $8ac_3 < 1$, the inequality above implies that for any $t > 0$

\[
\lim_{k \to \infty} \int_0^t \langle \dot{u}_s, gn_k \rangle \, ds = 0.
\]

This completes the proof. $\Box$

**4. Proof of Theorem 1.2: the recurrence**

This section is devoted to the proof of the recurrence test, Theorem 1.2.

**Proof of Theorem 1.2.** Let $x_0 \in X$ be the reference point in Theorem 1.2. For $R > 2$, set

\[
\theta_R(x) = \left( \left( \frac{R - d(x, x_0)}{R - 1} \right) \wedge 1 \right) _+.
\]

Since $\theta_R$ belongs to $\mathcal{F}_{\text{loc}} \cap L^\infty$ and has compact support, by Lemma 2.1, $\theta_R$ belongs to $\mathcal{F}$. According to the condition (M) and the chain-rule for a strongly-local Dirichlet form,

\[
\mathcal{E}(\theta_R) = \int_X \Gamma(\theta_R) \, dm
\]

\[
= \left( \frac{1}{R - 1} \right)^2 \int_{B(x_0, R)} \Gamma(\theta_R) \, dm
\]

\[
\leq \frac{1}{R - 1} \left( \frac{1}{R - 1} \right)^2 V(x_0, R)
\]

\[
\leq 4Mc \frac{V(x_0, R)}{R^2}.
\]

On the other hand, we find that for any $c_1 > 2$
\[ \mathcal{E}^{(j)}[\theta_R] = \iint (\theta_R(x) - \theta_R(y))^2 j(x, dy) \, m(dx) \]
\[ \leq \frac{2}{(R - 1)^2} \int_{B(x_0, R)} \int_{B(x_0, c_1 R)} d(x, y)^2 j(x, dy) \, m(dx) \]
\[ + 2 \int_{B(x_0, R)} \int_{B(x_0, c_1 R)^c} j(x, dy) \, m(dx) \]
\[ \leq \frac{2}{(R - 1)^2} \int_{B(x_0, R)} \int_{d(x, y) \leq 2c_1 R} d(x, y)^2 j(x, dy) \, m(dx) \]
\[ + 2 \int_{B(x_0, R)} \int_{d(x, y) \geq (c_1 - 1) R} j(x, dy) \, m(dx), \]

where we used the facts that \( d(x, y) \leq R + c_1 R \leq 2c_1 R \) if \( x \in B(x_0, R) \) and \( y \in B(x_0, c_1 R) \); \( d(x, y) \geq c_1 R - R \geq R_1 \) if \( x \in B(x_0, R) \) and \( y \notin B(x_0, c_1 R) \). The last expression is bounded from above by

\[ \leq \frac{8c_1^2}{(R - 1)^2} \int_{B(x_0, R)} \iint (d(x, y) \wedge R)^2 j(x, dy) \, m(dx) \]
\[ + \frac{2}{R^2} \int_{B(x_0, R)} \iint (d(x, y) \wedge R)^2 j(x, dy) \, m(dx) \]
\[ \leq \frac{33 c_1^2}{R^2} \int_{B(x_0, R)} \iint (d(x, y) \wedge R)^2 j(x, dy) \, m(dx). \]

Therefore, under the assumption (M), we have that for \( c_2 = 4M_c + 33c_1^2 \)

\[ \mathcal{E}^c[\theta_R] \leq \frac{1}{R^2} \left[ 4M_c V^{(c)}(x_0, R) + 33c_1^2 V^{(j)}(x_0, R) \sup_{x \in X^{(i)}} \int (d(x, y) \wedge R)^2 j(x, dy) \right] \]
\[ \leq \frac{c_2}{R^2} \left[ V^{(c)}(x_0, R) + V^{(j)}(x_0, R) \sup_{x \in X^{(i)}} \int (d(x, y) \wedge R)^2 j(x, dy) \right]. \]

According to the volume condition (1.2), there exists a sequence \( (n_k)_{k \geq 0} \) such that \( n_k \to \infty \) as \( k \to \infty \), and

\[ \liminf_{k \to \infty} \mathcal{E}[\theta_{R_{n_k}}] < \infty. \]

Applying [8, Theorem 1.6.3] and [34, (1.6.1) and (1.6.1')], this completes the proof. \Box

5. Examples

In this section we present some examples to illustrate the power of Theorems 1.1 and 1.2. Throughout the section, we denote the space of real-valued Lipschitz continuous functions with
compact support on a metric space $X$ by $C^0_{\text{Lip}}(X)$. For a measure space $(X, m)$ and a quadratic form $E$ defined in $L^2(X; m)$, we denote

$$E_1[u] = \|u\|_{L^2}^2 + E[u],$$

whenever the right side makes sense. We start with the following remark for the volume test in Theorem 1.1.

**Remark 5.1.** Let $(X, d, m)$ be a complete metric measure space such that $m$ is a Radon measure with full support. Assume that there is a point $x_0 \in X$ such that

$$\sup_{r > 0} \frac{V(x_0, 2r)}{V(x_0, r)} < \infty,$$

where $V(x_0, r)$ denotes the volume of the closed ball centered at $x_0$ with radius $r > 0$. This assumption is called the *volume doubling condition* at point $x_0$, and it implies that there is a constant $\kappa > 0$ such that

$$\sup_{r > 0} \frac{V(x_0, r)}{r^\kappa} < \infty.$$ 

In particular, condition (1.1) in Theorem 1.1 is satisfied. A typical example which fulfills the volume doubling condition is a Riemannian manifold with non-negative Ricci curvature.

### 5.1. Sharpness examples

In the following example, we consider two classes of symmetric jump processes on the so-called $\kappa$-set.

**Example 5.2.** Let $(X, |\cdot|, m)$ be a closed $\kappa$-set in $\mathbb{R}^n$ with $0 < \kappa \leq n$, i.e., $|\cdot|$ is the Euclidean distance, and for all $x \in X$ and $r > 0$,

$$m(B(x, r)) \asymp r^\kappa.$$ 

Here, the symbol $\asymp$ means that the ratio of the left and the right-hand sides is pinched by two positive constants. Assume that the jump kernel $j(x, dy)$ has a density $j(x, y)$ with respect to the measure $m(dy)$ such that one of the following two conditions is satisfied with a constant $\alpha \in (0, 2)$:

(i) $j(x, y) \asymp \frac{1}{|x - y|^\kappa + \alpha} \cdot 1_{|x - y| \leq 1} + \frac{1}{|x - y|^\kappa + \beta} \cdot 1_{|x - y| > 1}$, where $0 < \beta < \infty$;

(ii) $j(x, y) \asymp \frac{1}{|x - y|^\kappa + \alpha} \cdot 1_{|x - y| \leq 1} + \frac{e^{-c|x - y|}}{|x - y|^\kappa + \alpha} \cdot 1_{|x - y| > 1}$, where $c > 0$.

For $u, v \in C^0_{\text{Lip}}(X)$, define

$$E(u, v) = \int\int_{x \neq y} (u(x) - u(y))(v(x) - v(y)) j(x, y) m(dx) m(dy).$$
Let $F$ be the closure of $C^\text{Lip}_0(X)$ with respect to the $\sqrt{E_1}$-norm. The symmetric form $(\mathcal{E}, F)$ is a regular Dirichlet form in $L^2(X, m)$, see, e.g., [37]. According to Theorems 1.1 and 1.2, the Dirichlet form $(\mathcal{E}, F)$ is conservative, and it is recurrent if additionally $0 < \kappa \leq \beta + 2$ and $0 < \kappa \leq 2$ for the cases (i) and (ii), respectively.

**Remark 5.3.** Example 5.2 is motivated by recent developments for layered stable processes [16] and tempering stable processes [29]. In particular, in case (i) if $\beta = \alpha$, then the associated Hunt process is called a stable-like process [2].

### 5.2. Disconnected space

The following example shows that the state space may be topologically disconnected, and the particles jump between different connected components and it behaves as a jump-diffusion inside a connected component.

**Example 5.4.** Let $X = \bigcup_{i \in \mathbb{Z}} X_i$, where for each $i \in \mathbb{Z}$, $X_i = \{(x_i, i) \in \mathbb{R}^{n+1}; \ x_i \in \mathbb{R}^n\}$. Any point $x$ in $X$ can be expressed uniquely as $x = (x_i, i)$ with $x_i \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and we denote the associated projections by $p : X \to \mathbb{R}^n$ and $q : X \to \mathbb{Z}$. For any $x, y \in X$, the distance $d$ is given by

$$d(x, y) = |p(x) - p(y)| + |q(x) - q(y)|,$$

where $|\cdot|$ is the Euclidean distance. Let $m(dx) = \sum_{i \in \mathbb{Z}} m_i(dx_i)$ be a measure on $X$ such that for each $i \geq 1$, $m_i(dx_i) = \Psi(x_i) dx_i$ is a measure on $X_i$, where $\Psi \in C(\mathbb{R}^n)$ is a positive function, and $dx_i$ is the $n$-dimensional Lebesgue measure. Clearly, $m$ is a Radon measure on $X$. The state space is the triple $(X, d, m)$.

For any $u \in C^\text{Lip}_0(X)$, define

$$\mathcal{E}[u] = \mathcal{E}^{(c)}[u] + \mathcal{E}^{(j)}[u],$$

where

$$\mathcal{E}^{(c)}[u] = \int_X |\nabla u|^2 dm,$$

$$\mathcal{E}^{(j)}[u] = \int_X \int_{x \neq y} (u(x) - u(y))^2 j(x, y) m(dx) m(dy),$$

and

$$j(x, y) \asymp \frac{d(x, y)^{-(n+\alpha)} \mathbb{1}_{[d(x, y)<1]} + d(x, y)^{-(n+\beta+1)} \mathbb{1}_{[d(x, y)\geq 1]}}{\Psi(p(x)) + \Psi(p(y))}, \quad x, y \in X$$

with some constants $0 < \alpha < 2$ and $\beta > 0$. Let $\mathcal{F}$ be the closure of $C^\text{Lip}_0(X)$ with respect to the $\sqrt{E_1}$-norm. Since for any $x \in X$
\[
\int_{x \neq y} (1 \land d(x, y)^2) j(x, y) m(dy)
\leq \int_{0 < d(x, y) < 1} d(x, y)^{-(n+\alpha-2)} \Psi(p(y)) \, dp(y) + \int_{d(x, y) \geq 1} d(x, y)^{-(n+\beta+1)} \Psi(p(y)) \, dp(y)
\leq \int_{0 < d(x, y) < 1} d(x, y)^{-(n+\alpha-2)} \, dp(y) + 2 \sum_{k \geq 0} \int_{|p(x) - p(y)| \geq k+1} |p(x) - p(y)|^{-(n+\beta+1)} \, dp(y),
\]
which is bounded from above by some absolute constant \(c > 0\), it follows form the proof of [37] that \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form in \(L^2(X, m)\).

According to the arguments above, we can easily claim that the condition (M) is satisfied. Therefore, by Theorem 1.1, if there is a constant \(c > 0\) such that for \(r > 0\) large enough

\[
\sum_{0 \leq k \leq [r]} \int_{B(0, [r] - k)} \Psi(z) \, dz \leq r^c,
\]
where \(dz\) is the \(n\)-dimensional Euclidean measure and \([r]\) is the least integer such that \([r] \geq r\), then the Dirichlet form \((\mathcal{E}, \mathcal{F})\) is conservative. For instance, (5.14) is satisfied, if \(\Psi(x) \leq |x|^\beta |\ln |x|| \) for \(|x|\) large enough.

For the recurrence, we additionally assume that there are two constants \(c_0, c_1 > 0\) such that

\[
j(x, y) \leq \frac{\mathbb{1}_{[d(x, y) \leq c_0]} \cdot d(x, y)^{1+\alpha}}{d(x, y)^{1+\alpha}}
\]
and

\[
\Psi(x) \leq c_1 |x|^{1-n} \quad \text{for } |x| \text{ large enough.}
\]
Condition (5.16) will imply that for any point \(x_0 \in X\),

\[
\liminf_{r \to \infty} \frac{V(x_0, r)}{r^2} \leq 2 \liminf_{r \to \infty} \frac{1}{r^2} \sum_{0 \leq k \leq [r]} \int_{B(x_0, [r] - k)} \Psi(x) \, dx < \infty.
\]
Next, by (5.15), there is a constant \(c_2 > 0\) depending only on the dimension such that

\[
\omega(r) \leq \sup_{x \in X} \int_{X} d(x, y)^2 j(x, y) \Psi(p(y)) \, dp(y)
\leq c_1 \sup_{x \in X} \int_{d(x, y) \leq c_0} d(x, y)^{1-\alpha} |p(y)|^{1-n} \, dp(y)
\leq 2c_1c_2 \sum_{0 \leq k \leq [c_0]} \int_{0}^{[c_0]-k} r^{1-\alpha} \, dr < \infty.
\]
Therefore, \((\mathcal{E}, \mathcal{F})\) is recurrent by Theorem 1.2.
5.3. Volume tests

The first volume test for non-local Dirichlet forms to be conservative was obtained in [26, Main Result], and then refined in [15, Theorem 1.1]. It is easy to construct an example, which is not covered by these tests but by Theorem 1.1. Here, we illustrate this by using a weighted Euclidean space as well as a model manifold.

Example 5.5. Let \((\mathbb{R}, |\cdot|, m)\) be a weighted Euclidean space, where \(|\cdot|\) is the Euclidean distance and the measure is \(m(dx) = e^{2\lambda|x|} dx\) for some \(\lambda > 0\). For \(u \in C^\text{Lip}_0(\mathbb{R})\), define

\[
\mathcal{E}[u] = \int \int_{x \neq y} (u(x) - u(y))^2 j(x, y) m(dx) m(dy),
\]

where

\[
j(x, y) = \left(e^{-\lambda(|x|+|y|)}\right)I_{\{|x-y| \leq 1\}}.
\]

Let \(F\) be the closure of \(C^\text{Lip}_0(\mathbb{R})\) with respect to the \(\sqrt{\mathcal{E}}\)-norm. The symmetric form \((\mathcal{E}, F)\) becomes a regular Dirichlet form in \(L^2(\mathbb{R}, m)\), see, e.g., [37]. Let \(j(x, dy) = j(x, y) m(dy)\). It holds that

\[
\sup_{x \in \mathbb{R}} \int \left(1 \wedge |x - y|^2\right) j(x, dy) = \sup_{x \in \mathbb{R}} \int_{\{|y-x| \leq 1\}} |x - y|^2 j(x, y) m(dy)
\]

\[
= \sup_{x \in \mathbb{R}} e^{-\lambda|x|} \int_{\{|z| \leq 1\}} z^2 e^{\lambda|x-z|} dz
\]

\[
\leq \int_{\{|z| \leq 1\}} z^2 e^{\lambda|z|} dz < \infty.
\]

On the other hand, it is easy to see that in this example (1.1) is also satisfied. Therefore, according to Theorem 1.1, the Dirichlet form \((\mathcal{E}, F)\) is conservative.

However, since \(x \mapsto e^{-r|x|} \notin L^1(\mathbb{R}, m)\) for any \(r \leq 2\lambda\), this example is not covered by [26, Main Result].

Example 5.6 (Model manifolds). (See, e.g., [14].) Let \((S^n, g)\) be the \(n\)-dimensional unit sphere with \(n \geq 1\). A model manifold \(M = (0, +\infty) \times S^n\) is a Riemannian manifold with Riemannian tensor

\[
dr^2 + \sigma^2(r)g
\]

where \(\sigma\) is a locally-Lipschitz continuous positive function on \([0, +\infty)\) such that \(\sigma(0) = 0\) and \(\sigma'(0) = 0\). Thanks to these two conditions, the manifold \(M\) is geodesically complete, and so it satisfies the assumption for the state space as explained in Introduction. Let \(dm = \omega_n \sigma^n(r) dr\) be a measure on \(M\), where \(\omega_n\) is the volume of \(S^n\).
For any $u \in C^{1,\text{Lip}}_0 (M)$, define
\[ E[u] = E^{(c)}[u] + E^{(j)}[u], \]
where
\[ E^{(c)}[u] = \int_M |\nabla u|^2 \, dm, \]
\[ E^{(j)}[u] := \iint_{M \times M \setminus \text{diag}} (u(x) - u(y))^2 \, j(x, y) \, m(dy) \, m(dx) \]
and
\[ j(x, y) = \left[ \frac{1_{d(x,y) < 1}}{\sigma(r(x)) \sigma(r(y))} \right]^n. \]

Let $\mathcal{F}$ be the closure of $C^{1,\text{Lip}}_0 (M)$ with respect to the $\sqrt{E}$-norm. It is easy to check that the symmetric form $(E, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, m)$.

By [9], it is known that (M-2) is satisfied. On the other hand, since
\[ \sup_{x, y \in M} j(x, y) \sigma^n(r(y)) \leq 1, \]
we obtain that
\[ M_j = \sup_{x \in M} \int_M (1 \wedge d(x, y)^2) \, j(x, dy) \leq \sup_{x \in M} \int_M d(x, y)^2 \, j(x, y) \, m(dy) \leq \sup_{x \in M} \int_{d(x, y) < 1} d(x, y)^2 \omega_n \, dy \leq \omega_n. \]
Therefore, (M-3) is also satisfied. Since (M-1) clearly follows, we can apply our main theorem. For example, if $\sigma$ satisfies
\[ \sigma(r) \asymp \left[ r^r (1 + \ln r) \lor 1 \right]^{1/n}, \]
then for any fixed $x_0 \in M$,
\[ r^{r/2} < V(x_0, r) < 2r^r \quad \text{for large } r > 0. \]
Therefore, $(\mathcal{E}, \mathcal{F})$ is conservative by Theorem 1.1. We note that this model manifold $M$ does not satisfy the volume tests in [26,15].
5.4. A mixed-type Laplacian on graphs

A graph admits natural different “Laplacians”; namely, a physical Laplacian, a combinatorial Laplacian, and a quantum Laplacian. The former two are non-local operators, and the last one is a local operator. The combinatorial Laplacian is bounded, and so the corresponding process always is conservative. The conservativeness of the process associated with the physical Laplacian was studied in [6,7,39,40,15]. The conservativeness and recurrence of the process generated by the quantum Laplacian was studied in [35]. In the following example, we consider the sum of a physical Laplacian and a quantum Laplacian, and study its conservativeness.

Let $X = (V, E)$ be a locally finite graph, where $V$ and $E$ are the sets of vertices and edges, respectively. Let $\mu$ be a positive function on $X$, and $\omega: X \times X \to [0, \infty)$ be a symmetric non-negative function, such that $\omega(x, y) = 0$ whenever $x = y$ for $x, y \in X$ or at least one of $x$ and $y$ does not belong to $V$. Now, we recall the standard adapted distance $d$ in [15]. For any $x, y \in X$, $x \sim y$ means that $x, y$ are neighbors; that is, $(x, y) \in E$. For all $x, y \in V$ with $x \sim y$, define

$$\sigma(x, y) = \min \left\{ \frac{1}{\sqrt{\deg(x)}}, \frac{1}{\sqrt{\deg(y)}}, 1 \right\},$$

where

$$\deg(x) = \frac{1}{\mu(x)} \sum_{y: y \sim x} \omega(x, y).$$

It naturally induces a metric $d$ on $V$ as

$$d(x, y) = \inf \left\{ \sum_{i=0}^{n-1} \sigma(x_i, x_i + 1): x_0, \ldots, x_n \text{ is a chain connecting } x \text{ and } y \right\}.$$

The metric $d$ can be extended to $X$ by linear interpolation. We assume that the lengths of all edges $e \in E$ are uniformly bounded from below by a positive constant. This implies that $(X, d)$ is a metrically complete space; in particular, our assumption on the space is satisfied.

We further assume that each edge $e \in E$ is isometric to an interval of $\mathbb{R}$, which yields the measure $dx$ on $e$. The space $(X, d)$ is a metric graph. Consider the following measure $m$ on $X$:

$$m := \delta_E \phi \, dx + \delta_V \mu,$$

where $\phi$ is a continuous positive function on $E$.

For $u \in C^{\text{Lip}}_0(X)$, define

$$\mathcal{E}[u] := \mathcal{E}^{(c)}[u] + \mathcal{E}^{(j)}[u],$$

where

$$\mathcal{E}^{(c)}[u] = \int_E \left( \frac{\partial u}{\partial x} \right)^2 \, dm,$$
and
\[ \mathcal{E}^{(j)}[u] = \sum_{x,y \in V} (u(x) - u(y))^2 \omega(x, y). \]

The generators associated with \( \mathcal{E}^{(c)} \) and \( \mathcal{E}^{(j)} \) are called the quantum graph, see, e.g. [25] and the physical Laplacian, respectively. Let \( \mathcal{F} \) be the closure of \( C^0_{\text{lip}}(X) \) with respect to the \( \sqrt{\mathcal{E}_1} \)-norm. We have

**Lemma 5.7.** The form \( (\mathcal{E}, \mathcal{F}) \) is a regular Dirichlet form.

**Proof.** First, we claim that \( C^0_{\text{lip}}(X) \) is dense in \( L^2(X; m) \). Let \( x_0 \) be a fixed point in \( V \). For any \( u \in L^2(X; m) \) and any \( \epsilon > 0 \), choose \( R > 0 \) so large that there is a function \( v_\epsilon \in C^\infty_0(B(R) \cap E) \) which satisfies
\[ \| v_\epsilon - u \|_{L^2(E; dx)} < \epsilon, \]
and that the function \( w_\epsilon = \mathbb{1}_{B(R)}u \) satisfies that
\[ \| w_\epsilon - u \|_{L^2(V; \mu)} < \epsilon, \]
where \( B(R) := B(x_0, R) \). Set \( \tilde{u}_\epsilon = \delta_E v_\epsilon + \delta_V w_\epsilon \). For any \( x \in B(R) \) and \( e \in E \) with \( x \sim e \) (i.e., \( x \in e \)), let \( \delta = \delta(x, e) \) be a positive number such that \( \delta < |e|/2 \), and modify \( \tilde{u}_\epsilon \) on \( e \cap B(x, \delta) \) so that \( \tilde{u}_\epsilon \) is linear and continuous on \( e \cap B(x, \delta) \). Furthermore, since \( B(R) \cap V \) is finite, by the Hopf–Rinow type property of locally finite graphs [21], we are able to do this modification for any \( x \in B(R) \cap V \) and any \( e \in E \) with \( x \sim e \). Consequently, we obtain a sequence of functions \( u^\delta_n \in C^0_{\text{lip}}(B(R)) \) which converges to \( u \) in \( L^2(X; m) \) as \( \delta, \epsilon \to 0 \). The required claim is proved.

Next, we verify that \( (\mathcal{E}, C^0_{\text{lip}}(X)) \) is closable. Let \( (u_n)_{n \geq 1} \subset C^0_{\text{lip}}(X) \) be an \( \mathcal{E}_1 \)-Cauchy sequence such that \( u_n \to 0 \) in \( L^2(X; m) \) as \( n \to \infty \). One can easily prove that \( \mathcal{E}^{(c)}[u_n|_E] \to 0 \) as \( n \to \infty \), since \( \mathcal{E}^{(c)} \) is equivalent to the Dirichlet integral of an open interval. Moreover, if \( v \in C^0_{\text{lip}}(X) \), then
\[ \mathcal{E}^{(j)}(u_n|_V, v|_V) = \sum_{x,y \in V} (u_n(x) - u_n(y))(v(x) - v(y))\omega(x, y) \to 0 \quad \text{as } n \to \infty. \]

Therefore, the desired claim follows and we denote the closure of \( (\mathcal{E}, C^0_{\text{lip}}(X)) \) by \( (\mathcal{E}, \mathcal{F}) \).

The Markov property of \( (\mathcal{E}, \mathcal{F}) \) follows immediately from the definition of \( \mathcal{E} \). Finally, since \( C_0 \cap \mathcal{F} \) is both dense in \( C_0 \) and \( \mathcal{F} \) with respect to the sup-norm and the \( \mathcal{E}_1 \)-norm, respectively, \( (\mathcal{E}, \mathcal{F}) \) is regular. \( \square \)

It is easy to see that the conditions (M-1) and (M-2) are satisfied since \( X^{(c)} = E \). Moreover, since \( \mathcal{E}^{(j)} \) can be expressed as
\[ \mathcal{E}^{(j)}[u] = \iint_{X \times X} (u(x) - u(y))^2 \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy) m(dx), \]
the associated jump kernel \( j \) and \( \Gamma_j \) have the forms
\[
j(x, dy) = \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy)
\]
and
\[
\Gamma_j [u](x) = \int_X (u(x) - u(y))^2 \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy)
\text{ for any } x \in X.
\]

Clearly, (M-3) is satisfied. Therefore the Dirichlet form \((E, F)\) satisfies the condition (M).

To state our main result in this subsection, we need some notations. Denote by \( \rho \) the graph distance extended to \( X \), and by \( B_{\rho}(x_0, R) \) the associated ball at \( x_0 \in V \) with radius \( R > 0 \). For any \( n \in \mathbb{N} \), let \( S_{\rho}(x_0, n) \) be the “boundary” \( B_{\rho}(x_0, n) \setminus B_{\rho}(x_0, n - 1) \).

**Proposition 5.8.** If \( \mu \) is the counting measure and there are a point \( x_0 \in V \) and a constant \( C > 0 \) such that
\[
m(S_{\rho}(x_0, n)) \leq Cn^2 \text{ for all large enough } n \in \mathbb{N}, \tag{5.17}
\]
then \((E, F)\) is conservative.

**Proof.** The condition (5.17) implies that for any \( x \in V \),
\[
d(x_0, x) \geq \delta \log \rho(x_0, x), \tag{5.18}
\]
where \( \delta > 0 \) is a constant depending only on \( C \) in (5.17) (see [15]). Let \( xx' \) be the edge with boundary \( \{x, x'\} \). Let \( y \in X \) and \( x, x' \in V \) such that \( y \in xx' \). Without loss of generality, we assume that \( \rho(x_0, y) \leq \rho(x_0, x') \). By using (5.18), the triangle inequality and the fact that \( d(x, x') \leq \rho(x, x') = 1 \), we find that
\[
\rho(x_0, y) \leq e^{d(x_0, x')/\delta} \leq e^{1/\delta} e^{d(x_0, x)/\delta}.
\]
Since \( d(x_0, y) \geq d(x_0, x) \wedge d(x_0, x') \), we obtain that there is a constant \( c > 0 \) such that
\[
\rho(x_0, y) \leq ce^{d(x_0, y)/\delta} \text{ for any } y \in X.
\]

It follows that there exists a constant \( b > 0 \) such that
\[
m(B_d(x_0, r)) \leq m(B_{\rho}(x_0, ce^{r/\delta})) \leq \exp(br) \text{ for all large enough } r > 0.
\]

Therefore, \((E, F)\) is conservative by Theorem 1. \( \Box \)

**Remark 5.9.** By an example of R. Wojciechowski [41], the boundary volume growth of quadratic rate (5.17) is sharp. The second part of Proposition 5.8 was obtained in [15] for a physical Laplacian on a graph.

On the other hand, it is easy to check that the condition (5.17) is satisfied, if there is a constant \( C > 0 \) such that
(1) $\mu(S_\rho(x_0, n)) \leq Cn^2$ for all large enough $n \in \mathbb{N}$,

(2) $\phi(x) \leq C\rho(x_0, x)^{-2}$ for every $x \in X$.

Indeed, the first condition implies that there are at most $(Cn^2)^2$-many edges in $S_\rho(x_0, n)$ connecting vertices in $S_\rho(n)$ and $S_\rho(n-1)$. The second condition then implies that there is a constant $c > 0$ such that

$$m(S_\rho(x_0, n) \cap E) \leq \frac{C^2n^4}{(n-1)^2} \leq cn^2$$

for all large enough $n$.

This together with the first condition yields (5.17).

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References

[1] R. Azencott, Behavior of diffusion semi-groups at infinity, Bull. Soc. Math. 102 (1974) 192–240.
[2] Z.-Q. Chen, T. Kumagai, Heat kernel estimates for stable-like processes on $d$-sets, Stochastic Process. Appl. 108 (2003) 27–62.
[3] Z.-Q. Chen, T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps, Rev. Mat. Iberoam. 26 (2010) 551–589.
[4] S.Y. Cheng, S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975) 333–354.
[5] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property, J. Anal. Math. 58 (1992) 99–119.
[6] J. Dodziuk, Elliptic operators on infinite graphs, in: B. Bavnbek, S. Klimek, M. Lesch, W. Zhang (Eds.), Analysis, Geometry and Topology of Elliptic Operators, World Sci. Publ., Hackensack, NJ, 2006, pp. 353–368.
[7] J. Dodziuk, V. Mathai, Kato’s inequality and asymptotic spectral properties for discrete magnetic Laplacians, in: J. Jorgenson, L. Walling (Eds.), The Ubiquitous Heat Kernel, in: Contemp. Math., vol. 398, Amer. Math. Soc., Providence, RI, 2006, pp. 69–81.
[8] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin, 1994.
[9] M.P. Gaffney, A special Stokes’s theorem for complete Riemannian manifolds, Ann. of Math. 60 (1954) 140–145.
[10] M.P. Gaffney, The conservation property of the heat equation on Riemannian manifolds, Comm. Pure Appl. Math. 12 (1959) 1–11.
[11] A. Grigor’yan, On the existence of a Green function on a manifold, Uspekhi Mat. Nauk 38 (1983) 161–162 (in Russian); Russian Math. Surveys 38 (1983) 190–191 (in English).
[12] A. Grigor’yan, On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds, Uspekhi Mat. Nauk 128 (1985) 354–363 (in Russian); Math. USSR Sb. 56 (1987) 349–358 (in English).
[13] A. Grigor’yan, On stochastically complete manifolds, Dokl. Akad. Nauk SSSR 290 (1986) 534–537 (in Russian); Soviet Math. Dokl. 34 (1987) 310–313 (in English).
[14] A. Grigor’yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. 36 (1999) 135–249.
[15] A. Grigor’yan, X.-P. Huang, J. Masamune, On stochastic completeness of jump processes, Math. Z. 271 (2012) 1211–1239.
[16] C. Houdré, R. Kawai, On layered stable processes, Bernoulli 13 (2007) 261–287.
[17] E.P. Hsu, Heat semigroup on a complete Riemannian manifold, Ann. Probab. 17 (1989) 1248–1254.
[18] X. Huang, Stochastic incompleteness for graphs and weak Omori–Yau maximum principle, J. Math. Anal. Appl. 379 (2011) 764–782.
[19] X. Huang, On stochastic completeness of weighted graphs, Dissertation, Bielefeld University, 2011.
[20] X. Huang, On uniqueness class for a heat equation on graphs, J. Math. Anal. Appl. 393 (2012) 377–388.
[21] X. Huang, M. Keller, J. Masamune, R. Wojciechowski, A note on self-adjoint extensions of the Laplacian on weighted graphs, arXiv:1208.6358.
[22] L. Karp, Subharmonic functions, harmonic mappings and isometric immersions, in: S.T. Yau (Ed.), Seminar on Differential Geometry, in: Ann. of Math. Stud., vol. 102, Princeton, 1982, pp. 133–142.
[23] L. Karp, P. Li, The heat equation on complete Riemannian manifolds, unpublished manuscript, 1983.
[24] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, J. Reine Angew. Math. 666 (2012) 189–223.
[25] P. Kuchment, Quantum graphs. I. Some basic structures, Waves Random Media 14 (2004) 107–128.
[26] J. Masamune, T. Uemura, Conservation property of symmetric jump processes, Ann. Inst. H. Poincaré Probab. Stat. 47 (2011) 650–662.
[27] J. Masamune, T. Uemura, $L^p$-Liouville property for nonlocal operator, Math. Nachr. 284 (2011) 2249–2267.
[28] H. Okura, Capacity inequalities and recurrence criteria for symmetric Markov processes of pure jump type, in: S. Watanabe, et al. (Eds.), Probability Theory and Mathematical Statistics, World Sci. Publ., River Edge, NJ, 1996, pp. 387–395.
[29] J. Rosinski, Tempering stable processes, Stochastic Process. Appl. 117 (2007) 677–707.
[30] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.
[31] R.L. Schilling, T. Uemura, On the Feller property of Dirichlet forms generated by pseudo-differential operator, Tohoku Math. J. 59 (2007) 401–422.
[32] R.L. Schilling, J. Wang, Some theorems on Feller processes: transience, local times and ultracontractivity, Trans. Amer. Math. Soc., in press.
[33] Y. Shiozawa, Conservation property of symmetric jump-diffusion processes, Forum Math., in press.
[34] M. Silverstein, Symmetric Markov Processes, Lecture Notes in Math., vol. 426, Springer, Berlin, 1974.
[35] K.-T. Sturm, Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and Lp-Liouville properties, J. Reine Angew. Math. 456 (1994) 173–196.
[36] M. Takeda, On a martingale method for symmetric diffusion processes and its applications, Osaka J. Math. 26 (1989) 605–623.
[37] T. Uemura, On symmetric stable-like processes: Some path properties and generators, J. Theoret. Probab. 17 (2004) 541–555.
[38] N.Th. Varopoulos, Potential theory and diffusion of Riemannian manifolds, in: W. Beckner, A.P. Calderón, R. Fefferman, P.W. Jones (Eds.), Conference on Harmonic Analysis in Honor of Antoni Zygmund, in: Wadsworth Math. Ser., Wadsworth, 1983, pp. 821–837.
[39] A. Weber, Analysis of the physical Laplacian and the heat flow on a locally finite graph, J. Math. Anal. Appl. 370 (2010) 146–158.
[40] R. Wojciechowski, Heat kernel and essential spectrum of infinite graphs, Indiana Univ. Math. J. 58 (2009) 1419–1441.
[41] R. Wojciechowski, Stochastically incomplete manifolds and graphs, in: D. Lenz, F. Sobieszczky, W. Woess (Eds.), Boundaries and Spectra of Random Walks, in: Progr. Probab., vol. 64, Birkhäuser, Kathrein, 2009, pp. 165–181.