THE ALUFFI ALGEBRA OF A HYPERSURFACE WITH ISOLATED SINGULARITY

ABBAS NASROLLAH NEJAD

Abstract. The Aluffi algebra is algebraic definition of characteristic cycles of a hypersurface in intersection theory. In this paper we focus on the Aluffi algebra of quasi-homogeneous and locally Eulerian hypersurface with isolated singularities. We prove that the Jacobian ideal of an affine hypersurface with isolated singularities is of linear type if and only if it is locally Eulerian. We show that the gradient ideal of a projective hypersurface is of linear type if and only if the corresponding affine curve in the affine chart associated to singular points is locally Eulerian. We prove that the gradient ideal of the Nodal and Cuspidal projective plane curves are of linear type.

Introduction

In [9] P. Aluffi introduced a graded algebra to describe the characteristic cycle of a hypersurface parallel to well known conormal cycle in intersection theory. Let $Y \subseteq X \subseteq M$, where $X$ is a hypersurface in the smooth variety $M$ and $Y$ is singular subscheme of $X$. Aluffi constructed a graded $O_X$-algebra, $q\text{Sym}_{X \subset M}(J_{Y,X})$ which he called quasi-symmetric algebra, defined by

$$q\text{Sym}_{X \subset M}(J_{Y,X}) := \text{Sym}_{O_X}(J_{Y,X}) \otimes_{\text{Sym}_{O_M}(J_{Y,M})} \mathcal{R}_{O_M}(J_{Y,M}),$$

where $\text{Sym}_{O_M}(J_{Y,X})$ is the symmetric algebra of the ideal sheaf $J_{Y,X}$ of $Y$ in $X$ and $\mathcal{R}_{O_M}(J_{Y,M})$ is the Rees algebra of the ideal sheaf $J_{Y,M}$ of $Y$ in $M$. Aluffi proved that the characteristic cycle of $X$ is

$$\text{Ch}(X) = (-1)^{\dim X} [\text{Proj}(q\text{Sym}_{X \subset M}(J_{Y,M}))].$$

One of the Aluffi’s main result [9, Theorem 4.4] is that the (dual) Chern-Mather and the (dual) Schwartz-MacPherson classes of $X$ are (up to sing) the shadows of the cycles $[\text{Proj}(\mathcal{R}_{O_X}(J_{Y,X}))]$ and $[\text{Proj}(q\text{Sym}_{X \subset M}(J_{Y,M}))]$ in $\mathbb{P}(T^*M|_X)$.

Inspired by this construction, the author and A. Simis have explored its algebraic flavors, naming it the Aluffi algebra of a pair of ideals [9]( See also [8]). Let $R$ be a commutative ring and $J \subset I$ a pair of ideals in $R$. The (embedded) Aluffi algebra of $I/J$ is

$$\mathcal{A}_{R/J}(I/J) = \text{Sym}_{R/J}(I/J) \otimes_{\text{Sym}_R(I)} \mathcal{R}_R(I).$$

The Aluffi algebra is algebraic version of characteristic cycle of a hypersurface. As seen in ([9, Theorem 1.8 and Proposition 2.11]), in order that the Aluffi algebra

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have a certain expected behavior over a regular ambient ring, the ideal \( J \) had better be principal. This may be taken as a motivation for this work. Another source for motivation, as mentioned above, is the original work of Aluffi, where he has inquired into the structure of the algebra in the case that the ideal \( J = (f) \) is generated by equation of a reduced hypersurface.

Let \( R = k[x_1, \ldots, x_n] \) be the polynomial ring in \( n \geq 2 \) variables over an algebraically closed field \( k \) of characteristic zero and \( f \in R \) a reduced polynomial which defines the hypersurface \( X = V(f) \). We regard the non-singular hypersurfaces as uninteresting case and assume that \( X \) is singular. The singular subscheme of the hypersurface \( X \) is described by the ideal \( I(f) = (f, J(f)) \), the so called Jacobian ideal of \( f \) where \( J(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \) is the gradient ideal of \( f \).

An ideal \( I \) in a commutative ring \( A \) is called of linear type if the symmetric and the Rees algebra of \( I \) are isomorphic. By definition of the Aluffi algebra, if \( I(f) \) is of linear type then the Aluffi algebra of \( I(f)/(f) \) is isomorphic with the symmetric algebra. Then by formula (1) for computing characteristic cycle of the hypersurface \( X = V(f) \), we need just compute cycles of the naive blowup. In particular, for Schwartz-MacPherson classes of the hypersurface \( X \) we need shadows of the cycles \([\text{Proj} \,(\text{Sym}_{R/(f)}(I(f)/(f)))]\). By the results of Hunecke and Rossi [6] we have more information about the symmetric algebra, especially the minimal associated primes.

The principal question which motivated this paper is: when the Jacobian ideal of a hypersurface is of linear type?

Assume that \( f \) is a reduced homogeneous polynomial and the projective hypersurface \( X = V(f) \subset \mathbb{P}^{n-1}_k \) has isolated singularities. There is a central structural backstage for the Aluffi algebra and linear type property of \( I(f) \) [9, Corollary 3.2 and Theorem 3.3]. In fact, the gradient ideal \( J(f) \) is of linear type if and only if the coordinates of the vector fields of \( \mathbb{P}^{n-1}_k \) vanishing on \( f \) generate an irrelevant ideal in \( R \).

In this work, first we focus on the structure of the Aluffi algebra of an affine hypersurface, especially the linear type property of the Jacobian and gradient ideals. The outline of the paper is as follow: In section 1, we study the structure of the Aluffi algebra for quasi-homogeneous hypersurfaces. In Proposition 1.1, we prove that the Aluffi algebra of a quasi-homogeneous hypersurface is isomorphic with the symmetric algebra, Aluffi algebra is Cohen-Macaualay and we determine the minimal associated primes. An affine hypersurface \( X = V(f) \subset A^n_k \) with isolated singularity is called locally Eulerian if \( f \in J(f)_{\mathfrak{m}} \) for every \( \mathfrak{m} \in \text{Sing}(X) \). One of the main results is Theorem 1.4. We prove that \( X \) is locally Eulerian if and only if \( I(f) \) is locally a complete intersection at singular points which equivalent that \( I(f) \) is of linear type.

In the sequel, we consider the affine plane curve \( X = V(x^a + x^c y^d + y^b) \subset A^2_k \). We determine when \( X \) is locally Eulerian in terms of powers \((a, c, d, b) \in k^4\).

In section 2, We study linear type property of a reduced projective hypersurface. We say that a reduced projective hypersurface \( X = V(f) \subset \mathbb{P}^n_k \) is of gradient linear type if \( J(f) \) is of linear type. In Corollary 2.2, we show that \( X \) is of gradient linear type if and only if the corresponding affine hypersurface in the affine chart associated to a singular point is locally Eulerian. By using this results in Theorem 2.6, we prove
that projective plane curves with only $A_k$ singularities with $k \geq 1$ are of gradient linear type.

1. **Affine Hypersurfaces**

Throughout this section $k$ is an algebraically closed field of characteristic zero and $R = k[x_1, \ldots, x_n]$ the polynomial ring over $k$ with $n \geq 2$. Let $f \in R$ be a reduced polynomial which defines a reduced affine hypersurface $X = V(f) \subseteq A^n_k$. Let

$$J(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \subset R,$$

the so-called the gradient ideal of $f$. The singular locus of $X$ is defined by the Jacobian ideal $I(f) = (f, J(f))$. We say that $X$ has isolated singularity if the singular locus of $X$ consist of finitely many points. If $X$ has only isolated singularities then $\text{ht } I(f) = n$. The Milnor number $\mu(f)$ and Tjurina number $\tau(f)$ of $f$ at a singular point $m$ are the $k$-vector space dimension of the local Milnor algebra $R_m/J(f)_m$ and the local Tjurina algebra $R_m/I(f)_m$, respectively.

**Remark 1.** If $f \in R$ is a reduced polynomial which defines a hypersurface with isolated singularities. Then by [7, Lemma 4] the codimension of $J(f)_m$ is $n$. Since $R_m$ is a local Cohen-Macaulay ring, it follows that the partial derivatives $\partial f/\partial x_i$ localized at singular point $m$ form an $R_m$-regular sequence. In particular, $R_m/J(f)_m$ is Artinian Gorenstein ring.

In the sequel, we study the structure of the Aluffi algebra and the Jacobian ideal for quasi-homogeneous and locally Eulerian hypersurfaces.

1.1. **Quasi-homogeneous hypersurface.** Recall that a polynomial $f \in R$ is quasi-homogeneous of degree $d$ with weight $r_i$ if it is satisfies:

$$f = \sum_{i=1}^{n} (r_i/d) x_i \partial f/\partial x_i. $$

A reduced hypersurface $X \subseteq A^n_k$ is called quasi-homogeneous if the polynomial defining $X$ is quasi-homogeneous. We have the following central backstage for the Aluffi algebra of quasi-homogeneous hypersurface with isolated singularities.

**Proposition 1.1.** Let $X = V(f) \subseteq A^n_k$ be a reduced quasi-homogeneous hypersurface with isolated singularities. Then

(a) The Aluffi algebra of $I(f)/(f)$ is isomorphic with the symmetric algebra. In particular the Aluffi algebra has the following representation

$$\mathcal{A}_{R/(f)}(I(f)/(f)) \simeq R[T]/(f, \sum_{i=1}^{n} (r_i/d) x_i T_i, \mathcal{L})$$

where $\mathcal{L} = (\partial f/\partial x_i T_j - \partial f/\partial x_j T_i : 1 \leq i < j \leq n)$.

(b) The Aluffi algebra of $I(f)/(f)$ is Cohen-Macaulay.

(c) The minimal primes of the Aluffi algebra on $R(I(f))$ are

- The minimal prime ideals of $R_{R/(f)}(I(f)/(f))$ all of the form $\sum_{t \geq 0}(g) \cap I(f)^t$ for a prime factors $g$ of $f$. 

\begin{itemize}
  \item The extended ideal \((x)\mathcal{R}_R(I(f))\).
\end{itemize}

Proof. (a) Since \(f\) is quasi-homogeneous it follows that \(I(f)\) is generated by \(R\)-
regular sequence. Hence \(I(f)\) is of linear type and the Aluffi algebra is isomorphic with symmetric algebra. By the definition of the Aluffi algebra one has

\[ \mathcal{A}_{R}(I(f)/(f)) \simeq \text{Sym}_{R}(I(f)/(f)) \simeq \text{Sym}_{R}(I(f))/(f, \bar{f})\text{Sym}_{R}(I(f)), \]

where \(\bar{f}\) is degree one of \(f\) in the symmetric algebra. By using a representation of the symmetric algebra

\[ \text{Sym}_{R}(I(f)) \simeq R[T]/I_1([T_1 \ldots T_n].\phi), \]

where \(I_1([T_1 \ldots T_n].\phi)\) is ideal generated by 1-minors of product of \(1 \times n\) variable matrix \([T_1 \ldots T_n]\) with the first syzygy matrix \(\phi\) of \(I(f)\). Since \(I(f)\) is generated by a regular sequence then \(I(f)\) only have Koszul syzygies. Then all together, we get such representation.

(b) We apply the Criterion [5, Theorem 10.1]. We have to verify the following conditions:

(I) \(\mu((I(f)/(f))_{P}(f)) \leq \text{ht}(P)/(f)) + 1 = \text{ht}(P), \) for every prime ideal \(P\) contain \(I(f)\).

(II) \(\text{depth}(H_i)_{P}(f) \geq \text{ht}(P) - \mu((I(f)/(f))_{P}(f)) + i - 1, \) for every prime ideal \(I(f) \subset P\) and every \(i\) such that \(0 \leq i \leq \mu((I(f)/(f))_{P}(f))\) where \(H_i\) denotes the \(i\)th Koszul homology module of the partial derivatives on \(R/(f)\).

The prime ideals contain \(I(f)\) are maximal ideal which correspond with singular points. Since \(I(f)\) is generated by \(n\) elements, then in local ring \(R/(f)_{P}(f)\), we have \(\mu((I(f)/(f))_{P}(f)) = \mu(I(f)_{P}) = n\) and \(\text{ht}((I(f)/(f))_{P}(f)) = \text{ht}(P/(f)) = n - 1\). Then the condition (I), (II) trivially verified.

(c) This ia an immediate translation of [9, Propositions 2.8 and 1.9(ii)]. \(\square\)

**Example 1.2.** Let \(f \in R = k[x, y]\) be a reduced quasi-homogeneous polynomial. By [11, Proposition 2.5], \(f\) has the following form:

\[ f = ax^p + by^q + \sum_{1 \leq r < s} c_r x^{(e-s)q} y^{sp}, \]

where \(p, q, s \in \mathbb{N}_{>0}\) such that \(\gcd(p, q) = 1\). The hypersurface defined by \(f\) has singularities if and only if one of the following cases hold:

1. \(a, b \neq 0\) and \(s \geq 2\).
2. \(a, b = 0\) and \(s = 2\).

Then the Aluffi algebra of a quasi-homogeneous plane curve \(X = V(f) \subseteq \mathbb{A}_k^2\) is Cohen-Macaulay and isomorphic with the symmetric algebra.

1.2. **Locally Eulerian hypersurface.** We start with the definition of locally Euler-
ian hypersurface.

**Definition 1.3.** A hypersurface \(X = V(f) \subseteq \mathbb{A}_k^n\) with isolated singularities is called locally Eulerian if \(f \in J(f)_m\) for every maximal ideal \(m\) correspond with singular points.
Clearly, any quasi-homogeneous hypersurface is locally Eulerian but the converse is not holds, for example the plane curve defined by $f(x, y) = xy + x^3 + y^3 \in k[x, y]$ which has one singular point at origin is locally Eulerian, but it is not quasi-homogeneous. If an affine hypersurface $X = V(f)$ is locally Eulerian then the Milnor number of $f$ at each singular point is equal to Tjurina number.

In what follow we say that a reduced affine hypersurface $X = V(f) \subseteq \mathbb{A}^n_k$ is of Jacobian linear type if the Jacobian ideal $I(f)$ is of linear type. The nonsingular and quasi-homogeneous affine hypersurfaces are of Jacobian linear type. In the following result, we characterize Jacobian linear type affine hypersurfaces with isolated singularities.

**Theorem 1.4.** Let $X = V(f) \subseteq \mathbb{A}^n_k$ be a hypersurface with isolated singularities. The following are equivalent:

(a) The hypersurface $X$ is locally Eulerian.
(b) The Jacobian ideal $I(f)$ is locally a complete intersection at singular points.
(c) The hypersurface $X$ is of Jacobian linear type.

**Proof.** We prove (a) $\iff$ (b). By Remark 1, (a) implies (b). Assume that (b) holds. Let $P \in \mathbb{A}^n_k$ be a singular point of $X$. By a linear change of coordinates we may assume that $P$ is origin of $\mathbb{A}^n_k$. Denote by $m$ the maximal ideal correspond with origin. The ring $R_m$ is local so that the Nakayama’s lemma emphasize that the $n$ generators of $I(f)_m$ may be found among $f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$. Assume that $I(f)_m = (f, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)_m$. We have the surjective $k$-algebra homomorphism

$$R_m/J(f)_m \twoheadrightarrow R_m/I(f)_m.$$ 

Thus the Milnor number $\mu(f)$ at singular point $m$ is great or equal than to its Tjurina number $\tau(f)$. Now consider the ideal $(f, x_1)$ which is height 2 complete intersection ideal with an isolated singularity at origin of $\mathbb{A}^n_k$. Using the Lê-Greuel formula for the Milnor number of an isolated complete intersection singularity (see [13] and [3]), we have

$$\mu(f) + \mu(f, x_1) = \dim_k R_m/(f, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)_m.$$ 

Therefore with above surjection one has

$$\mu(f) \geq \tau(f) = \mu(f) + \mu(f, x_1).$$ 

Thus $\mu(f, x_1) = 0$ and this implies that $V(f, x_1) \subseteq \mathbb{A}^n_k$ is non-singular at $m$ which is a contradiction. Hence $I(f)_m = J(f)_m$ and (a) holds.

(b) $\implies$ (c). This is a general fact since linear type property is local and $R/I(f)$ is Artinain Gornestain local ring at each singular point. Now assume that (b) holds. By [5, Proposition 2.4] for each singular point $m$, one has

$$\text{ht } I(f)_m \leq \mu(I(f)_m) \leq \text{ht } m = \text{ht } I(f) \leq \text{ht } I(f)_m,$$

which shows that $I(f)$ is locally a complete intersection. $\square$

**Example 1.5.** Let $f = x^4 - x^2y^2 + y^5$. The affine curve $X = V(f) \subseteq \mathbb{A}^2_k$ has one singular point at origin. We can check easily that

$$((\partial f/\partial x, \partial f/\partial y) :_{k[x,y]} f) = (5y^2 - y, 5xy - x, 10x^2 + y).$$
Hence \( f \) can not be locally Eulerian. A computation in [2] yields that the defining ideal of the Rees algebra of \( I(f) \) contains a messy polynomial of degree 2 in \( T_i \)'s. This quadratic polynomial is responsible for the Jacobian ideal failing to be generated by analytically independent elements. The symmetric algebra of \( I(f) \) is reduced and Cohen-Macaulay but the Rees algebra is not Cohen-Macaulay.

**Corollary 1.6.** Let \( X = V(f) \subseteq \mathbb{A}^n_k \) be a locally Eulerian affine hypersurface with isolated singularities then

(a) The Aluffi algebra of \( I(f)/f \) is isomorphic with the symmetric algebra of \( I(f)/f \). In Particular:

\[
\mathcal{A}_{R/(f)}(I(f)/f) \simeq R[T_2, \ldots, T_n]/(f, I_1([T_2 \ldots T_n], \phi)),
\]

where \( \phi \) is a matrix obtained by deleting first row of the first syzygy matrix of \( I(f) \).

(b) The Aluffi algebra of \( I(f)/f \) is Cohen-Macaulay.

**Proof.** (a). The first assertion follows from the Theorem 1.4. Since \( \tilde{f} = T_i \) then such presentation follows form the proof of Proposition 1.1(a). For (b), we can use the same argument as in the Proposition 1.1(b). \( \square \)

**Example 1.7.** Let \( X = V(x^a + x^cy^d + y^b) \) be a family of plane curves in \( \mathbb{A}^2_k \) with \( a, b, c, d \in \mathbb{N}_0 \). An easy calculation in plane curve theory show that in the following cases \( X \) is locally Eulerian singular plane curve:

1. \( c \geq a \) and \( d \geq b \).
2. \( a, b \geq 2, \ c = a - 1 \) and \( d = b - 1 \).
3. \( a, b \geq 2, \ c = 1 \) and \( d = 1 \).
4. \( a, b \geq 3, \ c = 1 \) and \( d \leq \frac{b+1}{2} \).
5. \( a, b \geq 3, \ d = 1 \) and \( c \leq \frac{a+1}{2} \).
6. \( a, b \geq 3, \ c = a - 1 \) and \( d \geq \frac{b-1}{2} \).
7. \( a, b \geq 3, \ d = b - 1 \) and \( c \geq \frac{a-1}{2} \).
8. \( a = 2 \) or \( b = 2 \).

Now consider the following regions in \( k^2 \):

(I) \( \{(c, d) \mid 2 \leq c \leq a - 2, \ 2 \leq d \leq b - 2 \} \).

(II) \( \{(c, d) \mid c = 1, \ d > \frac{b+1}{2} \} \cup \{(c, d) \mid d = b - 1, \ c < \frac{a-1}{2} \} \).

(III) \( \{(c, d) \mid d = 1, \ c > \frac{a-1}{2} \} \cup \{(c, d) \mid c = a - 1, \ d < \frac{b-1}{2} \} \).

Assume that \( a, b \geq 3 \) and \( (c, d) \) belongs to one of the above regions. Then if \( X \) is locally Eulerian then it is quasi-homogeneous. Therefore, if \( a, b \geq 3 \) and \( (c, d) \) belongs to regions (I), (II), (III) then \( X \) is not locally Eulerian.

2. Projective hypersurfaces

Let \( k \) be an algebraically closed filed of characteristic zero and \( R = k[x_1, \ldots, x_{n+1}] \) the polynomial ring with standard grading over \( k \). Let \( f \in R \) be a reduced homogeneous polynomial of degree \( d \geq 2 \). By the Euler formula the singular subscheme of the reduced projective hypersurface \( X = V(f) \subseteq \mathbb{P}^n_k \) is defined by the gradient ideal \( J(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_{n+1}) \). A reduced projective hypersurface \( X = V(f) \subseteq \mathbb{P}^n_k \)
is called of \textit{gradient linear type} if the gradient ideal \(J(f)\) is of linear type. It is well
know that if \(X\) is non-singular then \(J(f)\) is irrelevant, i.e., generated by regular
sequence, and hence is of linear type. Therefore, every non-singular projective hypsersurface is of gradient linear type.

Let \(X = V(f) \subseteq \mathbb{P}^n_k\) be a reduced projective hypersurface defined by \(f \in R\)
with \(\deg(f) = d \geq 2\). Assume that \(P \in \mathbb{P}^n_k\) is a singular point of \(X\). By a
projective transformation we may assume that \(P = [0 : 0 : \ldots : 0 : 1]\). The ideal
\(\mathfrak{p} = (x_1, \ldots , x_n)\) is prime ideal correspond with the point \(P\). Consider the affine
chart \(U_{x_{n+1}} = \mathbb{A}^n_k\) with coordinate ring \(A = k[x_1/x_{n+1}, \ldots , x_n/x_{n+1}] = k[T_1, \ldots , T_n]\).
The equation of \(f\) in this affine chart is \(F(T_1, \ldots , T_n) := f(x_1, \ldots , x_n, 1)\). Hence the
Jacobian ideal of \(F\) is
\[
I(F) = (F, \partial F/\partial T_1, \ldots , \partial F/\partial T_n).
\]
Note that the origin of \(\mathbb{A}^n_k\) is the singular point of the affine hypersurface \(V(F) \subseteq \mathbb{A}^n_k\).
Denote by \(q = (T_1, \ldots , T_n)\) the ideal correspond with singular origin.

**Proposition 2.1.** By assumption and notation as above. The following are equivalent.

(a) The affine Jacobian ideal \(I(F)\) is a complete intersection locally at \(q\).

(b) The gradient ideal \(J(f)\) is a complete intersection locally at \(p\).

**Proof.** Let us make some remarks before we address the proof. Setting \(f_i := \partial f/\partial x_i\)
and \(F_{T_i} := \partial F/\partial T_i\). Denote by \(J(f)\) dehomogenization of the ideal \(J(f)\) with
respect to \(x_{n+1}\). We have
\[
J(f) = (f_1, \ldots , f_n, J_{x_{n+1}}).
\]
Clearly, \(F_{T_i} = f_i\), for \(i = 1, \ldots , n\). Using the Euler formula and dehomogenizing
it with respect to \(x_{n+1}\) we get
\[
F = \sum_{i=1}^n T_i F_{T_i} + f_{x_{n+1}}
\]
This relation implies that \(I(F) = J(f)\). Assume that (a) holds. By Theorem
1.4, the polynomial \(F\) is locally Eulerian at singular point \(q\). Hence in the ring \(A_q\)
we can write \(F = \sum_{i=1}^n G_i F_{T_i}\) where \(G_i \in A_q \subseteq R_p\). Using relation (2) we have
\[
f_{x_{n+1}} \in (f_{x_1}, \ldots , f_{x_n})\text{ locally at }q. \quad \text{Then in } R_p \text{ one has}
\]
\[
f_{x_{n+1}} = (f_{x_{n+1}})_{\text{hom}} \in (f_{x_1}, \ldots , f_{x_n})_{\text{hom}} = ((f_{x_1}, \ldots , f_{x_n}) : x_{n+1}^{\infty}).
\]
Hence there exists \(a \in \mathbb{N}\) such that \(x_{n+1}^{a+1} f_{x_{n+1}} \in (f_{x_1}, \ldots , f_{x_n})\) locally at \(p\). Since \(x_{n+1}\)
is unit in \(R_p\) we get \(f_{x_{n+1}} \in (f_{x_1}, \ldots , f_{x_n})\). Conversely, If \(J(f) \subseteq R_p\) is a complete
intersection, the Nakayama’s lemma emphasize that the \(n\) generators of \(J(f)\) can be find among \(f_{x_1}, \ldots , f_{x_n} f_{x_{n+1}}\). We claim that \(J(f)_p = (f_{x_1}, \ldots , f_{x_n})\). Otherwise,
Assume that \(J(f)_p = (f_{x_2}, \ldots , f_{x_{n+1}})\). Then in \(R_p\) one has \(f_{x_1} = \sum_{i=2}^{n+1} g_i f_{x_i}\) with \(g_i \in R_p\). By dehomogenization with respect to \(x_{n+1}\) and using formula (2), we
get that \(F_{T_1}\) locally belongs to \(F, F_{T_2}, \ldots , F_{T_n})\) which is impossible by the proof of
Theorem 1.4. Therefore, by formula (2) and \(I(F) = J(f)\) we get the assertion. □
Corollary 2.2. Let $X = V(f) \subset \mathbb{P}^n_k$ be a reduced projective hypersurface with isolated singularities. The following are equivalent:

1. The hypersurface $X$ is of gradient linear type.
2. Locally at each singular prime the gradient ideal $J(f)$ is a complete intersection.
3. For each singular prime $p$ such that $x_i \notin p$, the Jacobian ideal $I(F)$ in the affine coordinate ring $A = k[x_1/x_1, \ldots, x_{n+1}/x_1]$ is a locally complete intersection at $q$ where $F = f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1})$ and $q$ is maximal ideal associated to $p$ in $A$.

Proof. (1) $\Rightarrow$ (2). The same argument as in the proof of Theorem 1.4. (2) $\Rightarrow$ (1). The gradient ideal $J(f)$ is almost complete intersection, depth $(R/J(f)) \geq \dim R/J(f) - 1 = 0$ and by assumption $J(f)_p$ is a complete intersection at singular primes, then Theorem 5.2 in [12] complete the proof. The equivalence of (2) and (3) follow form Lemma 2.1.

Remark 2. By [9, Corollary 3.2], the conditions of the Theorem (2.2) are equivalent to the statement that the coordinates of the vector fields of $\mathbb{P}^n_k$ vanishing on $f$ generate an irrelevant ideal.

Example 2.3. Let $X$ be an irreducible cubic surface in $\mathbb{P}^3_k$ with 4 singular point of type node. We show that $X$ is of gradient linear type. By a projective transformation we may assume that the singular point located at coordinate points:

$$P_1 = (1 : 0 : 0 : 0), \quad P_2 = (0 : 1 : 0 : 0), \quad P_3 = (0 : 0 : 1 : 0), \quad P_4 = (0 : 0 : 0 : 1).$$

We should find a polynomial $f$ such that $X = V(f) \subset \mathbb{P}^3_k$.

Let $f \in k[x, y, z, w]$ be a linear combination of all monomials of degree 3 in $k[x, y, z, w]$ and passes through points $P_1, P_2, P_3, P_4$. Then monomials $x^3, y^3, z^3, w^3$ must be absent from $f$. Since $P_1$ is singular point of $X$ then all coefficient of linear terms in $y, z, w$ must vanish, i.e., $y^2, z^2, wx^2$. Similarly, for singular points $P_2, P_3, P_4$ we have respectively $xy^2, zy^2, wy^2, xz^2, yz^2, wz^2$ and $xw^2, yw^2, zw^2$ terms disappearing too. Then the remaining equation is

$$f = a_1xyz + a_2xyw + a_3xzw + a_4yzw.$$  

Note that $a_1, a_2, a_3, a_4 \neq 0$, otherwise $f$ reduces, we can assume that $a_1 = a_2 = a_3 = a_4 = 1$ by scaling $x, y, z, w$. One get $f = xyz + xyw + xzw + yzw$. The surface $X$ is called Cayley’s nodal cubic surface. By symmetry and using the same notation, we show that the affine surface $F = xy + xz + yz + xyz$ is locally Eulerian at singular point $p = (x, y, z)$. The gradient ideal of $F$ is generated by

$$F_x = y + z(1 + y), \quad F_y = z + x(1 + z), \quad F_z = x + y(1 + x).$$

We claim that $J(F)_p = (p)_p$ which implies that $F$ is locally Eulerian. One has

$$x(1 + (1 + x)(1 + y)(1 + z)) = F_x - (1 + x)F_x + (1 + x)(1 + y)F_y,$$

$$y(1 + (1 + x)(1 + y)(1 + z)) = F_y - (1 + y)F_y + (1 + x)(1 + y)F_z,$$

$$z(1 + (1 + x)(1 + y)(1 + z)) = F_z - (1 + z)F_z + (1 + y)(1 + z)F_x.$$
Since \(1 + (1 + x)(1 + y)(1 + z) \notin \mathfrak{p}\) then the claim holds. These calculations also shows that \(\mu(f) = \tau(f) = 1\) at singular point \(\mathfrak{p}\).

**Question 2.4.** Is any irreducible singular cubic surface in \(\mathbb{P}_k^3\) of gradient linear type?

**Example 2.5.** Let \(f = x^4z - x^2y^2z + y^5\). The projective curve \(X = V(f) \subseteq \mathbb{P}_k^2\) has one singular point at \(P = [0 : 0 : 1]\) with one tangent of multiplicity 2 and two remaining distinct tangents. By Example 1.5, the gradient ideal \(J(f)\) is not of linear type.

### 2.1. Nodal and Cuspidal projective plane curves.

Let \(X = V(f) \subseteq \mathbb{P}_k^2\) be a projective plane curves defined by a reduced polynomial \(f \in k[x, y, z]\) of degree \(d \geq 2\). If \(X\) is singular then singularities of \(X\) are isolated. Let \(P \in \mathbb{P}_k^2\) be a singular point of \(X\). By a projective transformation we may assume that \(P = [0 : 0 : 1]\). Write \(F(x, y) := f(x, y, 1)\), since \(P\) is singular point then the multiplicity of \(f\) at \(P\), \(m_P(f) \geq 2\). Hence \(F(x, y) = F_2 + \ldots + F_d\) where \(F_i \in k[x, y]\) is a homogeneous polynomial of degree \(i\). The simplest case is \(m_P(f) = 2\). For \(F_2 = ax^2 + bxy + cy^2\) there are two cases to be considered. If \(b^2 \neq 4ac\), then \(F_2\) has two distinct zeros in \(\mathbb{P}_k^1\) that is, \(X\) has two distinct tangents at \(P\). In this case, the singular point \(P\) is called ordinary double point or ordinary node. We can change coordinates so that \(F_2 = xy\). Then the affine equation of ordinary node is given by

\[
F(x, y) = xy + h(x, y),
\]

where \(m_P(h) \geq 3\).

For \(b^2 = 4ac\), \(F_2\) has a double zero, so \(X\) has only one tangent at \(P\). In this case, \(P\) is called cusp. We can change coordinates so that \(F_2 = y^2\). Then the affine equation of cusp is

\[
F(x, y) = y^2 + h(x, y)
\]

where \(m_P(h) \geq 3\). Note that in two above cases, the intersection multiplicity \(\text{mult}_P(X, L) \geq 3\) where \(L\) is the tangent line at \(P\). If \(\text{mult}_P(X, L) = 3\) then \(P\) is called simple ordinary node and simple cusp.

A projective plane curve \(X \subseteq \mathbb{P}_k^2\) is called Nodal if its singular points are node and is called Cuspidal if its singular points are cusp.

**Theorem 2.6.** Nodal and Cuspidal curves are of gradient linear type.

**Proof.** Let \(X \subseteq \mathbb{P}_k^n\) stands for a projective curve defined by reduced homogeneous polynomial \(f \in R = k[x, y, z]\) of degree \(d \geq 3\). Let \(P \in \mathbb{P}_k^2\) be a singular point of \(X\). Denote by \(q \in R\) the prime ideal of \(P\). By projective transformation we may assume that \(P = [0 : 0 : 1]\) and then \(q = (x, y)\). We apply Corollary 2.2, so that we prove the affine equation of nodal and cuspidal curves are locally Eulerian at singular point \(q = (x, y)\).

The affine equation of nodal curve is \(F(x, y) = xy + h(x, y)\), where \(h(x, y)\) has initial degree at least 3. The Jacobian ideal of \(F\) is

\[
J(F) = (x + \partial h/\partial x, y + \partial h/\partial y).
\]
We can write
\[ \frac{\partial h}{\partial x} = yG_1(x, y) + G_2(x), \quad \frac{\partial h}{\partial y} = xL_1(x, y) + L_2(y), \]
where \( G_1(0, 0) = L_1(0, 0) = 0 \) and \( G_2 \in k[x], L_2 \in k[y] \) have initial degree at least 2. Thus
\[ J(F) = (y(1 + G_1(x, y)) + G_2(x), x(1 + L(x, y)) + L_2(y)). \]
Note that \( 1 + G_1 \) and \( 1 + L_1 \) are unit locally at \( q \). Thus the gradient ideal locally at \( q \) is generated by polynomials \( x - \alpha L_2(y), y - \beta G_2(x) \), with \( \alpha, \beta \) unit and \( G, L \) of initial degree at least 2. We claim that in \( k[x, y]_q \):
\[ (x - \alpha L(y), y - \beta G(x))_q = (x, y)_q. \]
We can write \( L(y) = y\ell(y) \) and \( G(x) = x\varphi(x) \). We have
\[ x(1 - \alpha \beta \ell(y)\varphi(x)) = (x - \alpha y\ell(y)) + \alpha \ell(y)(y - \beta x\varphi(x)) \]
and
\[ y(1 - \alpha \beta \ell(y)\varphi(x)) = (y - \beta x\varphi(x)) + \beta \varphi(x)(x - \alpha y\ell(y)). \]
Since the element \( 1 - \alpha \beta \ell(y)\varphi(x) \) is unit locally at \( q \) this proves the claim. Therefore, \( F \) is locally Eulerian.

For a cuspidal curve the affine equation is given by \( F(x, y) = y^2 + h(x, y) \), where \( h(x, y) \) has initial degree at least 3. The tangent line at point \( P \) is \( L = V(y) \) and we assume that \( \text{mult}_F(X \cap L) = 3 \). This condition means that \( y \) is not divisor of \( F \)
\[ F_3 = d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^33. \]
Then \( d_1 \neq 0 \) and we may assume that \( d_1 = -1 \). This gives
\[ F(x, y) = y^2 - x^3 + yg(x, y) + h(x, y), \]
where \( g(x, y) \) is homogeneous of degree 2 and the initial degree of \( h(x, y) \) is at least 4, as the affine equation of a simple cusp. The Jacobian ideal of \( F \) is generated by
\[ \frac{\partial F}{\partial x} = -3x^2 + yg/\partial x + \partial h/\partial x, \]
\[ \frac{\partial F}{\partial y} = 2y + g(x, y) + yg/\partial y + \partial h/\partial y. \]
We can write
\[ \frac{\partial h}{\partial x} = x^2G_1(x, y) + xG_2(y) + G_3(y), \quad \frac{\partial h}{\partial y} = yH_1(x, y) + H_2(x), \]
where \( G_1(0, 0) = H_1(0, 0) = 0 \), the initial degree of \( G_2(y) \) is at least 2 and the initial degree of \( G_3(y) \) and \( H_2(x) \) is at least 3. Using these relation and renaming \( G_i \) and \( H_i \), one has
\[ \frac{\partial F}{\partial x} = -3x^2 + x^2G_1(x, y) + xG_2(y) + G_3(y), \]
\[ \frac{\partial F}{\partial y} = 2y + yH_1(x, y) + H_2(x), \]
where \( G_1(0, 0) = H_1(0, 0) = 0 \), the initial degree of \( G_2(y) \) is at least 1, the initial degrees of \( G_3(y) \) and \( H_2(x) \) is at least 2.
We have
\[ J(F) = (x^2(-3 + G_1(x, y)) + xG_2(y) + G_3(y), y(2 + H_1(x, y)) + H_2(x)). \]
Note that the elements $-3 + G_1(x, y)$, $2 + H_1(x, y)$ are unit locally at $q$. Thus the gradient ideal ideal $J(F)$ locally at $q$ is generated by $x^2 - \alpha(xG_2(y) + G_3(y))$, $y - \beta H_2(x)$. We claim that

$$(x^2 - \alpha(xG_2(y) + G_3(y)), y - \beta H_2(x))_q = (x^2, y)_q.$$ 

The condition on initial degree of $G_2, G_3, H_2$ implies the one side inclusion. For the converse, by condition on initial degree, we write

$$xG_2(y) + G_3(y) = y(xL_2(y) + L_3(y)), \quad H_2(x) = x^2Q(x).$$

We have

$$U, x^2 = x^2 - \alpha(xG_2(y) + G_3(y)) + \alpha(xL_2(y) + L_3(y))(y - \beta H_2(x))$$

$$U, y = y - \beta H_2(x) + \beta Q(x)(x^2 - \alpha(xG_2(y) + G_3(y))$$

Since the element $U = (1 - \alpha \beta Q(x)(xL_2(y) + L_3(y)))$ is unit locally at $q$, this complete the assertion.

If $\text{mult}_P(X \cap L) = k > 3$ then $y$ is not a divisor of $F_k$, homogeneous polynomial of degree $k$, then the coefficient of monomial $x^k$ is nonzero and we may assume that it is $-1$. This gives

$$F(x, y) = y^2 - x^k + yg(x, y) + h(x, y)$$

where $g(x, y)$ is homogeneous of degree $k - 1$ and the initial degree of $h(x, y)$ is at least $k + 1$. The same arguments as $k = 3$ shows that in this case the gradient ideal locally at singular point $q = (x, y)$ is generated by $(x^{k-1}, y)$ which is a complete intersection in $k[x, y]_q$. 

Corollary 2.7. Any projective cubic curve in $\mathbb{P}^2_k$ is of gradient linear type.

Proof. If $X$ is irreducible singular cubic then $X$ has at most one singular point of multiplicity 2 which is either a node or a cusp. Thus by Theorem 2.6, $X$ is of gradient linear type.

If $X$ is a reducible cubic then it is of the form an irreducible conic and a line not tangent to conic or of the form an irreducible conic and a line tangent to conic. In two cases, the reducible cubic has singular points of type node and cusp, respectively, hence is of gradient linear type.

If $X$ is a cubic which reduced to three non-concurrent line then $X$ has 3 singular points of type node. Assume that $X = V(f) \subseteq \mathbb{P}^2_k$ is a cubic which reduced to three concurrent line. Thus it is projectively equivalent to $f = (y - z)(z - x)(x - y)$. An easy calculation show that the affine curve is locally Eulerian at singular point. 

Remark 3. (1) The local plane curve calculation in Nodal case is as same as one which A. Simis in [10] used to prove that depth $(R/J(f)) = 0$.

(2) Suppose that $X = V(f) \subseteq \mathbb{P}^2_k$ is projective plane curve of multiplicity $m_p(f) = 2$ at the point $P = [0 : 0 : 1]$. The affine curve $X_z := V(F(x, y))$ is an analytic subset of $A^2_k$ and $(X_z)((0, 0))$ is an analytic set germ. If this set germ is analytically equivalent to $(V(x^{k+1} - y^2), (0, 0))$ for some $k \in \mathbb{N}$, then $P$ is called an $A_k$ singularity of $X$. By [Theorems 2.46 and 2.48][4] and the proof of Theorem 2.6, nodes singularities are $A_1$, ordinary cusps are $A_2$ and
cusps with condition $\text{mult}_P(X \cap L) = k$ are $A_k$ for $k \geq 2$, where $L$ is tangent line at $P$. The Theorem 2.6 shows that any projective plane curve only with $A_k$ singularity is of gradient linear type.

(3) If $X = V(f) \mathbb{P}^2_k$ is Nodal or Cuspidal projective curve then the Aluffi algebra of $J(f)/(f)$ is isomorphic with the symmetric algebra and it is Cohen-Macaulay [9, Theorem 3.3].

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DEPARTMENT OF MATHEMATICS INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES (IASBS) P.O.BOX 45195-1159 ZANJAN, IRAN.

SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O.Box 19395-5746, TEHRAN, IRAN

E-mail address: abbasnn@iasbs.ac.ir