Perturbations of self-adjoint operators in semifinite von Neumann algebras: Kato-Rosenblum theorem

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Abstract. In the paper, we prove an analogue of the Kato-Rosenblum theorem in a semifinite von Neumann algebra. Let $M$ be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on a Hilbert space $H$ and let $\tau$ be a faithful normal semifinite tracial weight of $M$. Suppose that $H$ and $H_1$ are self-adjoint operators affiliated with $M$. We show that if $H - H_1$ is in $M \cap L^1(M, \tau)$, then the norm absolutely continuous parts of $H$ and $H_1$ are unitarily equivalent. This implies that the real part of a non-normal hyponormal operator in $M$ is not a perturbation by $M \cap L^p(M, \tau)$ of a diagonal operator. Meanwhile, for $n \geq 2$ and $1 \leq p < n$, by modifying Voiculescu’s invariant we give examples of commuting $n$-tuples of self-adjoint operators in $M$ that are not arbitrarily small perturbations of commuting diagonal operators modulo $M \cap L^p(M, \tau)$.

1. Introduction

This paper is a continuation of the investigation, which we began in [11], of diagonalizations of self-adjoint operators modulo norm ideals in semifinite von Neumann algebras.

Let $H_0$ be a complex separable infinite dimensional Hilbert space. Denote by $B(H_0)$ the set of bounded linear operators on $H_0$. The Weyl-von Neumann theorem [32, 14] states that a self-adjoint operator in $B(H_0)$ is a sum of a diagonal operator and an arbitrarily small Hilbert-Schmidt operator. A result by Kuroda in [10] implies that a self-adjoint operator in $B(H_0)$ is a sum of a diagonal operator and an arbitrarily small Schatten $p$-class operator with $p > 1$. Berg and Sikonia independently showed in [3] and [19] that a normal operator in $B(H_0)$ is a compact perturbation of a diagonal operator. In [22], Voiculescu proved a surprising result by showing that a normal operator in $B(H_0)$ is a diagonal operator plus an arbitrarily small Hilbert-Schmidt operator. This result of Voiculescu has recently been generalized in [11] to semifinite von Neumann algebras with separable predual. It is worth noting that Kuroda’s result in [10] was also extended to countably decomposable, properly infinite, semifinite von Neumann algebras in [11].

In the case of perturbations by trace class operators, the influential Kato-Rosenblum theorem (see [6] and [18]) provides an obstruction to diagonalizations, modulo the trace class, of self-adjoint operators in $B(H_0)$. More specifically, if $H$ and $H_1$ are densely defined self-adjoint operators on $H_0$ such that $H - H_1$ is in the trace class, then the Kato-Rosenblum theorem asserts that the absolutely continuous parts of $H$ and $H_1$ are unitarily equivalent. Thus, if a self-adjoint operator $H$ in $B(H_0)$ has a nonzero absolutely continuous spectrum, then $H$ can not be a sum of a diagonal operator and a trace class operator.

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The purpose of this paper is to provide a version of the Kato-Rosenblum theorem in a semifinite von Neumann algebra. (For general knowledge about von Neumann algebras, the reader is referred to [4, 5].) Let \( \mathcal{H} \) be a complex infinite dimensional Hilbert space and let \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) be a countably decomposable, properly infinite von Neumann algebra with a faithful normal tracial weight \( \tau \). A quick example (see Example 2.4.2) shows the existence of a self-adjoint operator \( A \in \mathcal{M} \) satisfying that \( A \) has a nonzero absolutely continuous spectrum and \( A \) is also a sum of a diagonal operator and an arbitrarily small operator in \( \mathcal{M} \cap L^1(\mathcal{M}, \tau) \). Thus we should not expect that a direct generalization of the Kato-Rosenblum theorem still holds in a general semifinite von Neumann algebra \( \mathcal{M} \).

Before stating the results of the paper, we recall the following notation. Let \((\mathcal{X}, \| \cdot \|)\) be a Banach space. A mapping \( f : \mathbb{R} \to \mathcal{X} \) is locally absolutely continuous if, for all \( a < b \) and every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \sum_i \| f(b_i) - f(a_i) \| < \epsilon \) for every finite collection \( \{(a_i, b_i)\} \) of disjoint intervals in \([a, b]\) with \( \sum_i (b_i - a_i) < \delta \).

In this paper, we introduce a notion of norm absolutely continuous projections with respect to a self-adjoint operator \( H \) affiliated with \( \mathcal{M} \). Suppose \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) is the spectral resolution of the identity for \( H \) in \( \mathcal{M} \). A projection \( P \) in \( \mathcal{M} \) is called a norm absolutely continuous projection with respect to \( H \) if the mapping \( \lambda \mapsto PE(\lambda)P \) from \( \mathbb{R} \) into \( \mathcal{M} \) is locally absolutely continuous (see Definition 5.2.1). It is shown in Proposition 5.3.4 that, in the case of \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \), if \( x \in \mathcal{H} \) and \( x \otimes x \) is the rank one projection associated with \( x \), then \( x \otimes x \) is a norm absolutely continuous projection with respect to \( H \) if and only if the vector \( x \) is absolutely continuous with respect to \( H \).

For a self-adjoint operator \( H \) affiliated with \( \mathcal{M} \), we define the norm absolutely continuous support \( P_{ac}^\infty(H) \) of \( H \) to be the union of these norm absolutely continuous projections with respect to \( H \) (see Definition 5.2.1). When \( H \in \mathcal{M} \) is bounded, the following criterion gives a characterization of \( P_{ac}^\infty(H) \) in terms of hyponormal operators in \( \mathcal{M} \).

**Corollary 5.3.2** \( H \) is a self-adjoint element in \( \mathcal{M} \) with \( P_{ac}^\infty(H) \neq 0 \) if and only if \( H \) is the real part of a non-normal hyponormal operator \( T \) in \( \mathcal{M} \).

Now we are ready to state our analogue of the Kato-Rosenblum theorem for a semifinite von Neumann algebra.

**Theorem 5.2.5** Suppose \( H \) and \( H_1 \) are self-adjoint operators affiliated with \( \mathcal{M} \) such that \( H_1 - H \) is in \( \mathcal{M} \cap L^1(\mathcal{M}, \tau) \). Then

\[
W_+ \triangleq \operatorname{sot-lim}_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H) \text{ exists in } \mathcal{M}.
\]

Moreover,

(i) \( W_+ W_+^* = P_{ac}^\infty(H) \) and \( W_+ W_+^* = P_{ac}^\infty(H_1) \);

(ii) \( W_+ HW_+^* = H_1 P_{ac}^\infty(H_1) \).

A direct consequence of Theorem 5.2.5 is the next result.

**Proposition 5.3.4** If \( H \) is a self-adjoint element in \( \mathcal{M} \) such that \( P_{ac}^\infty(H) \neq 0 \), then there exists no self-adjoint diagonal operator \( K \) in \( \mathcal{M} \) satisfying \( H - K \in L^1(\mathcal{M}, \tau) \). In particular, if \( H \) is the real part of a non-normal hyponormal operator in \( \mathcal{M} \), then there exists no self-adjoint diagonal operator \( K \) in \( \mathcal{M} \) satisfying \( H - K \in L^1(\mathcal{M}, \tau) \).
We are also able to obtain an analogue of the Kuroda-Birman theorem for a semifinite von Neumann algebra as follows.

**Theorem 5.4.2** Suppose $H$ and $H_1$ are self-adjoint operators affiliated with $\mathcal{M}$ such that
\[(H_1 + i)^{-1} - (H + i)^{-1} \in \mathcal{M} \cap L^1(\mathcal{M}, \tau).\]
Then
\[W_+ \triangleq \text{sot-}\lim_{t \to \infty} e^{itH_1}e^{-itH}P_{ac}^\infty(H) \text{ exists in } \mathcal{M}.

Moreover,
1. $W_+W_+^* = P_{ac}^\infty(H)$ and $W_+W_+^* = P_{ac}^\infty(H_1)$;
2. $W_+HW_+^* = H_1P_{ac}^\infty(H_1)$.

For a commuting $n$-tuple of self-adjoint operators in $\mathcal{B}(\mathcal{H}_0)$ as $n \geq 2$, the simultaneous diagonalization theory has been extensively investigated in [2, 3, 21-31, 33-37]. In this paper, we consider obstructions to simultaneous diagonalization of self-adjoint operators in a countably decomposable, proper infinite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial weight $\tau$. By modifying Voiculescu’s invariant, in Example 6.2.2 we exhibit an example of an $n$-tuple of commuting self-adjoint operators in $\mathcal{M}$ that is not an arbitrarily small perturbation of commuting diagonal operators modulo $\mathcal{M} \cap L^p(\mathcal{M}, \tau)$ for all $1 \leq p < n$.

The present paper has six sections. In section 2, we prepare related notation, definitions and lemmas. We recall the concept of absolutely continuous spectrum and give an example of a purely absolutely continuous self-adjoint operator in a semifinite von Neumann algebras that is an arbitrarily small max-$\{\| \cdot \|, \| \cdot \|_1\}$-norm perturbation of a diagonal operator. In section 3, we introduce a smooth condition for a densely defined self-adjoint operator. Under this condition, we are able to give the point-wise convergence of generalized wave operators. Norm absolutely continuous projections with respect to a self-adjoint operator $H$ affiliated with $\mathcal{M}$ are introduced in section 4. Section 5 is devoted to show an analogue of the Kato-Rosenblum Theorem in semifinite von Neumann algebras. We also provide an analogue of the Kuroda-Birman Theorem in a semifinite von Neumann algebra. Section 6 provides examples of $n$-tuple of self-adjoint operators in $\mathcal{M}$ that can not be an arbitrary small perturbations of commuting diagonal operators modulo $\mathcal{M} \cap L^p(\mathcal{M}, \tau)$ for all $1 \leq p < n$.

2. Preliminaries and Notation

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$.

2.1. Semifinite von Neumann algebra. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight $\tau$ (see Definition 7.5.1 in [5] for the details). Let
\[\mathcal{F}(\mathcal{M}, \tau) = \{AEB : E = E^* = E^2 \in \mathcal{M} \text{ with } \tau(E) < \infty \text{ and } A, B \in \mathcal{M}\}\]
be the set of finite rank operators in $(\mathcal{M}, \tau)$.

The following result is well-known. For the purpose of completeness, we include its proof here.
Lemma 2.1.1. Let $\mathcal{M} \subseteq \mathcal{B(H)}$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight $\tau$.

(i) There exists a sequence $\{P_n\}_{n \in \mathbb{N}}$ of orthogonal projections in $\mathcal{M}$ such that $\tau(P_n) < \infty$ for each $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} P_n = I$ (convergence is in strong operator topology).

(ii) There exists a sequence $\{x_m\}_{m \in \mathbb{N}}$ of vectors in $\mathcal{H}$ such that

$$\tau(X^*X) = \sum_{m} \langle X^*X_{xm}, x_m \rangle, \quad \forall X \in \mathcal{M}.$$ 

Moreover, the linear span of the set $\{A'x_m : A' \in \mathcal{M}'$ and $m \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $\mathcal{B(H)}$.

Proof. From Proposition 8.5.2 in [5], if $P$ is a nonzero projection in $\mathcal{M}$, then there exists a sub-projection $P_0$ of $P$ such that $0 < \tau(P_0) < \infty$. Now by Zorn’s lemma, there exists a family $\{P_\lambda\}_{\lambda \in \Lambda}$ of orthogonal projections in $\mathcal{M}$ such that $\sum_{\lambda \in \Lambda} P_\lambda = I$ and $\tau(P_\lambda) < \infty$ for each $\lambda \in \Lambda$. From the fact that $\mathcal{M}$ is countably decomposable, it follows that $\Lambda$ is countable. This ends the proof of (i).

By (i), there exists a sequence $\{P_n\}_{n \in \mathbb{N}}$ of orthogonal projections in $\mathcal{M}$ such that $\tau(P_n) < \infty$ for each $n \in \mathbb{N}$ and $\sum_{n} P_n = I$. Thus $\tau(A) = \sum_{n} \tau(AP_n)$, $\forall A \in \mathcal{M}^+$, where $\mathcal{M}^+$ is the positive part of $\mathcal{M}$. As the mapping $A \mapsto \tau(AP_n)$ is a normal positive functional on $\mathcal{M}$, the existence of $\{x_m\}_{m \in \mathbb{N}}$ follows from Theorem 7.1.12 in [5]. Moreover, let $Q$ be the projection from $\mathcal{H}$ onto $\mathcal{H}_1$, the closure of the linear span of the set $\{A'x_m : A' \in \mathcal{M}'$ and $m \in \mathbb{N}\}$ in $\mathcal{H}$. Then $Q \in \mathcal{M}$ and $(I - Q)x_m = 0$ for all $m \in \mathbb{N}$. Thus $\tau(I - Q) = \sum_{m} \langle (I - Q)x_m, x_m \rangle = 0$. As $\tau$ is faithful, we conclude that $I - Q = 0$, whence $\mathcal{H}_1 = \mathcal{H}$. \qed

2.2. Noncommutative $L^p(\mathcal{M}, \tau)$. Here, we will briefly review the definition of noncommutative $L^p$-spaces associated to a semifinite von Neumann algebra. For $1 \leq p < \infty$, the mapping

$$\| \cdot \|_p : \mathcal{F} (\mathcal{M}, \tau) \to [0, \infty)$$

is defined by

$$\| A \|_p = (\tau(|A|^p))^{1/p}, \quad \forall A \in \mathcal{F} (\mathcal{M}, \tau).$$

It is a highly non-trivial fact that $\| \cdot \|_p$ is a norm on $\mathcal{F} (\mathcal{M}, \tau)$. We let $L^p(\mathcal{M}, \tau)$ be the completion of $\mathcal{F} (\mathcal{M}, \tau)$ with respect to the norm $\| \cdot \|_p$ (see [16] for more details). When $p = \infty$, we let $\| A \|_\infty = \| A \|$ for all $A \in \mathcal{M}$ and let $L^\infty (\mathcal{M}, \tau) = \mathcal{M}$.

2.3. Spectral theory for self-adjoint operators. Recall a densely defined, closed operator $A$ is affiliated with $\mathcal{M}$ if $AU' = U'A$ for all unitary operator $U'$ in $\mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $\mathcal{B(H)}$. Let $\mathcal{A} (\mathcal{M})$ be the set of all densely defined, closed operators that are affiliated with $\mathcal{M}$. Note that, from Theorem 5 in [13], $L^p(\mathcal{M}, \tau)$ can be identified as a subset of $\mathcal{A} (\mathcal{M})$ for each $1 \leq p \leq \infty$.

Let $H$ be a self-adjoint element in $\mathcal{A} (\mathcal{M})$. Then there exists a family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ of projections in $\mathcal{M}$ that is the spectral resolution of the identity for $H$ such that $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$. (In fact, each $E(\lambda)$ is a spectral projection of $H$ corresponding to the interval $(-\infty, \lambda]$. See
Theorem 5.2.6 of [5] for more details.) If \( f \) is a bounded Borel function on \( \mathbb{R} \), then \( f(H) \) is an element in the von Neumann subalgebra generated by \( \{ E(\lambda) \}_{\lambda \in \mathbb{R}} \) in \( \mathcal{M} \), satisfying

\[
\langle f(H)x, y \rangle = \int_{-\infty}^{\infty} f(\lambda) \, d\langle E(\lambda)x, y \rangle, \quad \forall \, x, y \in \mathcal{H}. \tag{2.1}
\]

In particular, for each \( t \in \mathbb{R} \), we have

\[
\langle e^{-itH}x, y \rangle = \int_{-\infty}^{\infty} e^{-it\lambda} \, d\langle E(\lambda)x, y \rangle, \quad \forall \, x, y \in \mathcal{H}. \tag{2.2}
\]

2.4. Absolutely continuous spectrum. Let \( H \) be a self-adjoint element in \( \mathcal{A}(\mathcal{M}) \) and let \( \{ E(\lambda) \}_{\lambda \in \mathbb{R}} \) be the spectral resolution of the identity for \( H \) in \( \mathcal{M} \). We let \( \mathcal{H}_{ac}(H) \) be the set of all these vectors \( x \) in \( \mathcal{H} \) such that the mapping \( \lambda \mapsto \langle E(\lambda)x, x \rangle \), with \( \lambda \in \mathbb{R} \), is a (locally) absolutely continuous function on \( \mathbb{R} \) (see [9] for details of the definition). It is known that \( \mathcal{H}_{ac}(H) \) is a closed subspace of \( \mathcal{H} \) (see Theorem X.1.5 of [9]). Let \( P_{ac}(H) \) be the projection from \( \mathcal{H} \) onto \( \mathcal{H}_{ac}(H) \). Then \( P_{ac}(H) \) is in the von Neumann subalgebra generated by \( \{ E(\lambda) \}_{\lambda \in \mathbb{R}} \) in \( \mathcal{M} \) (see Theorem X.1.6 in [9]).

The following result can be found in the proof of Theorem X.4.4 of [9].

**Lemma 2.4.1.** Let \( x \in \mathcal{H}_{ac}(H) \). If \( \Delta \) is a Borel subset of \( \mathbb{R} \) and \( \chi_{\Delta} \) is the characteristic function of \( \Delta \), then \( \chi_{\Delta}(H)x \in \mathcal{H}_{ac}(H) \) and

\[
\frac{d\langle E(\lambda) \chi_{\Delta}(H)x, x \rangle}{d\lambda} = \chi_{\Delta}(\lambda) \frac{d\langle E(\lambda)x, x \rangle}{d\lambda}, \quad \text{for } \lambda \in \mathbb{R} \text{ a.e.}
\]

We end this subsection with an example of a self-adjoint operator \( A \) in a semifinite von Neumann algebra \( \mathcal{M} \) such that \( A \) has purely absolutely continuous spectrum and \( A \) is an arbitrarily small \( \max\{\| \cdot \|, \| \cdot \|_1\} \)-norm perturbation of a diagonal operator.

**Example 2.4.2.** Let \( \mathcal{N} \) be a diffuse finite von Neumann algebra with a faithful normal tracial state \( \tau_N \) and let \( \mathcal{H}_0 \) be an infinite dimensional separable Hilbert space. Then \( \mathcal{M} = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0) \) is a semifinite von Neumann algebra with a faithful normal tracial weight \( \tau_M = \tau_N \otimes \text{Tr} \), where \( \text{Tr} \) is the canonical trace of \( \mathcal{B}(\mathcal{H}_0) \). We might further assume that \( \mathcal{M} \) acts naturally on the Hilbert space \( \mathcal{H} = L^2(\mathcal{N}, \tau_N) \otimes \mathcal{H}_0 \).

Let \( \{ E(\lambda) \}_{0 \leq \lambda \leq 1} \) be an increasing family of projections in \( \mathcal{N} \) such that \( \tau(E(\lambda)) = \lambda \) for \( 0 \leq \lambda \leq 1 \). Let \( X = \int_0^1 \lambda dE(\lambda) \) and \( A = X \otimes I_{\mathcal{B}(\mathcal{H}_0)} \). Notice we can identify \( \mathcal{N} \) with a subset of \( L^2(\mathcal{N}, \tau_N) \). For any unitary element \( u \) in \( \mathcal{N} \subseteq L^2(\mathcal{N}, \tau_N) \) and any unit vector \( y \) in \( \mathcal{H}_0 \), we have

\[
\langle (E(\lambda) \otimes I_{\mathcal{B}(\mathcal{H}_0)})(u \otimes y), u \otimes y \rangle = \tau(u^*E(\lambda)u)\|y\|^2 = \lambda, \quad \text{for } 0 \leq \lambda \leq 1.
\]

Thus the vector \( u \otimes y \) is in \( \mathcal{H}_{ac}(A) \), which shows that \( \mathcal{H}_{ac}(A) = L^2(\mathcal{N}, \tau_N) \otimes \mathcal{H}_0 \).

On the other hand, there exist a family \( \{ p_n \}_{n \in \mathbb{N}} \) of orthogonal projections in \( \mathcal{B}(\mathcal{H}_0) \) such that \( \text{Tr}(p_n) = 1 \) for each \( n \in \mathbb{N} \) and \( \sum_n p_n = I_{\mathcal{B}(\mathcal{H}_0)} \). Then \( A = X \otimes I_{\mathcal{B}(\mathcal{H}_0)} = \sum_n X \otimes p_n \). Let \( \epsilon > 0 \) be given. For each \( n \in \mathbb{N} \), by spectral theory, there exists a self-adjoint diagonal operator \( Y_n \in \mathcal{N} \) such that \( \|X - Y_n\| \leq \epsilon/2^n \). Let \( Y = \sum_n Y_n \otimes p_n \). Then \( Y \) is a self-adjoint diagonal element in \( \mathcal{M} \) such that \( \|A - Y\|, \|A - Y\|_1 \leq \epsilon \).

Thus there is a self-adjoint element \( A \) in \( \mathcal{M} \) with purely absolutely continuous spectrum such that \( A \) is an arbitrarily small \( \max\{\| \cdot \|, \| \cdot \|_1\} \)-norm perturbation of a diagonal element in \( \mathcal{M} \).
2.5. Identification operator. For $H \in \mathcal{A}(\mathcal{M})$, we denote the domain of $H$ in $\mathcal{H}$ by $D(H)$.

**Lemma 2.5.1.** Assume that $H_1$ and $H$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Let $J$ be an element in $\mathcal{M}$ such that $J^2 = D(H_1)$ and $H_1 J - J H$ extends to a bounded operator $B$ in $\mathcal{M}$. Let

$$W(t) = e^{itH_1} J e^{-itH}, \quad \text{for } t \in \mathbb{R}.$$ 

Then, for all $x, y \in \mathcal{H}$ and $s, t \in \mathbb{R}, a > 0$,

(i) the mapping $\lambda \mapsto e^{i\lambda H_1} B e^{-i\lambda H} x$ from $[s, t]$ into $\mathcal{H}$ is Bochner integrable with

$$(W(t) - W(s)) x = i \int_s^t e^{i\lambda H_1} B e^{-i\lambda H} x d\lambda,$$

(ii) $e^{i\lambda H_1} B e^{-i\lambda H} W(t) = e^{i\lambda H} W(t)$, and $e^{i\lambda H} W(t)$ is a partial isometry and $e^{i\lambda H} W(t)$ defines a bounded operator in $\mathcal{H}$. Then, for all $t, s \in \mathbb{R}$,

$$W(t) - W(s) \in \mathcal{M} \cap L^1(\mathcal{M}, \tau).$$

**Proof.** The proof can be found in Chapter X (3.21) and Chapter X (5.8) of [9] (also see [15]).

**Proposition 2.5.2.** Assume that $H_1$ and $H$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Let $J$ be an element in $\mathcal{M}$ such that $J^2 = D(H_1)$ and $H_1 J - J H$ extends to a bounded operator $B$ in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Let $W(t) = e^{itH_1} J e^{-itH}, \quad \text{for } t \in \mathbb{R}$. Then, for all $t, s \in \mathbb{R}$,

$$W(t) - W(s) \in \mathcal{M} \cap L^1(\mathcal{M}, \tau).$$

**Proof.** We need only to show that $W(t) - W(s) \in L^1(\mathcal{M}, \tau)$. By Lemma 2.1.1 there exists an increasing sequence $\{P_n\}_{n \in \mathbb{N}}$ of projections in $\mathcal{M}$ such that $\tau(P_n) < \infty$ for each $n \in \mathbb{N}$ and $P_n \to I$ in strong operator topology. Fix an $n \in \mathbb{N}$. As the mapping $A \mapsto \tau(AP_n)$ defines a normal positive linear functional on $\mathcal{M}$, from Theorem 7.1.11 in [5], there exists an orthogonal family $\{y_m\}_{m \in \mathbb{N}}$ of vectors in $\mathcal{H}$ such that

$$\sum_m \|y_m\|^2 < \infty \quad \text{and} \quad \tau(AP_n) = \sum_m \langle Ay_m, y_m \rangle, \quad \forall A \in \mathcal{M}.$$ 

Let $W(t) - W(s) = W|W(t) - W(s)|$ be the polar decomposition of $W(t) - W(s)$ in $\mathcal{M}$, where $W$ is a partial isometry and $|W(t) - W(s)|$ is a positive operator in $\mathcal{M}$. Then, from Lemma 2.5.1 it induces that

$$\tau(|W(t) - W(s)| P_n) = \sum_m \langle |W(t) - W(s)| y_m, y_m \rangle = \sum_m \langle (W(t) - W(s)) y_m, W y_m \rangle$$

$$= i \sum_m \int_s^t \langle e^{i\lambda H_1} B e^{-i\lambda H} y_m, W y_m \rangle d\lambda. \quad (2.3)$$

Observe that

$$\sum_m \langle e^{i\lambda H_1} B e^{-i\lambda H} y_m, W y_m \rangle \leq \sum_m \langle B \|y_m\|^2 \rangle < \|B\| \cdot \sum_m \|y_m\|^2 < \infty. \quad (2.4)$$
Combining (2.3), (2.4) and applying the Lebesgue Dominating Theorem,
\[
\tau(|W(t) - W(s)|P_n) = i \int_s^t \sum_m \langle e^{i\lambda H} B e^{-i\lambda H} y_m, W y_m \rangle d\lambda \\
= i \int_s^t \sum_m \langle W e^{i\lambda H} B e^{-i\lambda H} y_m, y_m \rangle d\lambda \\
= i \int_s^t \tau(W e^{i\lambda H} B e^{-i\lambda H} P_n) d\lambda.
\]
This implies that, for all \( n \in \mathbb{N} \),
\[
|\tau(|W(t) - W(s)|P_n)| \leq \int_s^t |\tau(W e^{i\lambda H} B e^{-i\lambda H} P_n)| d\lambda \leq (t - s)\|B\|_1.
\]
Since \( \tau \) is a normal weight of \( \mathcal{M} \) and \( P_n \to I \) in strong operator topology, we conclude that
\[
\|W(t) - W(s)\|_1 = \tau(|W(t) - W(s)|) = \sup_n \tau(|W(t) - W(s)|P_n) \leq (t - s)\|B\|_1.
\]
This ends the proof of the proposition. \( \square \)

3. \( \mathcal{M} \)-\( H \)-smoothness in Semifinite von Neumann Algebras

Let \( \mathcal{H} \) be a complex Hilbert space. Let \( \mathcal{M} \) be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on \( \mathcal{H} \) and \( \tau \) a faithful normal semifinite tracial weight of \( \mathcal{M} \). Let \( \mathcal{A}(\mathcal{M}) \) be the set of densely defined, closed operators that are affiliated with \( \mathcal{M} \).

3.1. A smooth condition. The following definition will be crucial when showing the existence of wave operators in semifinite von Neumann algebras.

**Definition 3.1.1.** Let \( H \) be a self-adjoint element in \( \mathcal{A}(\mathcal{M}) \). A pair \((A, x)\) is said to be \( \mathcal{M} \)-\( H \)-smooth if
\[
(i) \ A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau) \text{ and } x \in \mathcal{H}; \\
(ii) \text{there exists a positive constant } c \text{ such that} \\
\int_{\mathbb{R}} \|ASe^{-i\lambda H}x\|^2 d\lambda \leq c^2 \|S\|^2, \quad \forall S \in \mathcal{M}.
\]

3.2. Point-wise convergence of wave operator.

**Proposition 3.2.1.** Let \( H \) be a self-adjoint element in \( \mathcal{A}(\mathcal{M}) \). Assume \( B \in \mathcal{M} \cap L^1(\mathcal{M}, \tau) \) and \( x \in \mathcal{H} \) such that \( (|B|^{1/2}, x) \) is \( \mathcal{M} \)-\( H \)-smooth. Then there exists a positive constant \( c \) such that, for all \( s, t \in \mathbb{R}, a > 0 \) and \( S \in \mathcal{M} \),
\[
|\langle \int_0^a e^{i(\lambda+t)H} SBe^{-i(\lambda+s)H} x \ d\lambda, x \rangle| \leq c\|S\| \left( \int_s^t \|B|^{1/2} e^{-i\lambda H} x\|^2 d\lambda \right)^{1/2}
\]
and
\[
|\langle \int_0^a e^{i(\lambda+t)H} B^*Se^{-i(\lambda+s)H} x \ d\lambda, x \rangle| \leq c\|S\| \left( \int_s^t \|B|^{1/2} e^{-i\lambda H} x\|^2 d\lambda \right)^{1/2}.
\]
Proof. Let $s, t \in \mathbb{R}, a > 0$ be given. As $(|B|^{1/2}, x)$ is $\mathcal{M}$-$H$-smooth, there exists a positive constant $c$ such that
\[
\int_{\mathbb{R}} \| |B|^{1/2} S e^{-i\lambda H} x \|^2 d\lambda \leq c^2 \| S \|^2, \quad \forall S \in \mathcal{M}.
\]
(3.1)
Assume that $B = W|B|$ is the polar decomposition of $B$ in $\mathcal{M}$, where $W$ is a partial isometry in $\mathcal{M}$ and $|B|$ is a positive operator in $\mathcal{M}$. We have
\[
\left| \int_{0}^{a} \langle e^{i(\lambda + t)H} S B e^{-i(\lambda + t)H} x, x \rangle d\lambda \right|
\]
\[
\leq \int_{0}^{a} \left| \langle |B|^{1/2} e^{-i(\lambda + t)H} x, |B|^{1/2} W^* S^* e^{-i(\lambda + t)H} x \rangle \right| d\lambda
\]
\[
\leq \int_{0}^{a} \left\| |B|^{1/2} W^* S^* e^{-i(\lambda + t)H} x \right\| \left\| |B|^{1/2} e^{-i(\lambda + t)H} x \right\| d\lambda
\]
\[
\leq \left( \int_{0}^{a} \left\| |B|^{1/2} W^* S^* e^{-i(\lambda + t)H} x \right\|^2 d\lambda \right)^{1/2} \left( \int_{0}^{a} \left\| |B|^{1/2} e^{-i(\lambda + t)H} x \right\|^2 d\lambda \right)^{1/2}
\]
\[
= \left( \int_{t}^{a+t} \left\| |B|^{1/2} W^* S^* e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2} \left( \int_{s}^{a+s} \left\| |B|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2}
\]
\[
\leq \left( \int_{s}^{\infty} \left\| |B|^{1/2} W^* S^* e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2} \left( \int_{s}^{\infty} \left\| |B|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2}
\]
\[
\leq c \| S \| \left( \int_{s}^{\infty} \left\| |B|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2}. \quad \text{(by (3.1))}
\]
Similarly, we have
\[
\left| \int_{0}^{a} \langle e^{i(\lambda + t)H} B^* S e^{-i(\lambda + t)H} x, x \rangle d\lambda \right| \leq c \| S \| \left( \int_{t}^{\infty} \left\| |B|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \right)^{1/2}.
\]

\[
\text{Lemma 3.2.2. Suppose $H$ and $H_1$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Assume $J$ is in $\mathcal{M}$ such that $J \mathcal{D}(H) \subseteq \mathcal{D}(H_1)$ and the closure of $H_1 J - JH$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Let $W(t) = e^{-it H_1} J e^{-it H}$ for each $t \in \mathbb{R}$.}
\]
If $x$ is a vector in $\mathcal{H}$ such that $(|H_1 J - JH|^{1/2}, x)$ is $\mathcal{M}$-$H$-smooth, then
\[
\lim_{a \to \infty} \|(W(t) - W(s)) e^{-ia H} x\| = 0, \quad \text{for all } t > s.
\]

Proof. Note that $(|H_1 J - JH|^{1/2}, x)$ is $\mathcal{M}$-$H$-smooth. There exists a positive number $c$ such that
\[
\int_{\mathbb{R}} \left\| |H_1 J - JH|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \leq c^2.
\]
(3.2)
From Lemma 2.5.1,

\[ \left\| (W(t) - W(s)) e^{-iaH} x \right\| \leq \int_s^t \left\| e^{i\lambda H} (H_1 J - J H) e^{-i\lambda H} x \right\| d\lambda \]

\[ \leq \int_s^t \left\| H_1 J - J H \right\|^{1/2} \left\| H_1 J - J H \right\|^{1/2} e^{-i\lambda H} x \right\| d\lambda \]

\[ \leq \left\| H_1 J - J H \right\|^{1/2} \left( \int_s^t \left\| H_1 J - J H \right\|^{1/2} e^{-i\lambda H} x \right\|^2 d\lambda \]

\[ \leq \left\| H_1 J - J H \right\| \left( \int_s^t \left\| H_1 J - J H \right\| e^{-i\lambda H} x \right\|^2 d\lambda \]

From (3.2) we have

\[ \lim_{a \to \infty} \left\| (W(t) - W(s)) e^{-iaH} x \right\| = 0. \]

The proof of the next result follows a strategy by Pearson in [15].

**Proposition 3.2.3.** Suppose $H$ and $H_1$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Assume $J$ is in $\mathcal{M}$ such that $J^* \mathcal{D}(H) \subseteq \mathcal{D}(H_1)$ and the closure of $H_1 J - J H$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Let $W(t) = e^{iH_1 t} J e^{-itH}$ for each $t \in \mathbb{R}$.

If $x$ is a vector in $\mathcal{H}$ such that $\left( \left| H_1 J - J H \right|^{1/2}, x \right)$ is $\mathcal{M}$-$H$-smooth, then $W(t) x$ converges in $\mathcal{H}$ as $t \to \infty$.

**Proof.** To prove the result, it suffices to show that for every $\epsilon > 0$ there exists an $N > 0$ such that, if $t > s > N$, then $\left\| (W(t) - W(s)) x \right\| < \epsilon$.

Denote by $B$ the closure of $H_1 J - J H$. Thus $B$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Let $\epsilon > 0$ be given. From Lemma 2.5.1, for any $t, s, a > 0$, we have

\[ \langle (W(t) - W(s)) e^{iaH} W(t)^* (W(t) - W(s) e^{-iaH}) x, x \rangle \]

\[ = i \int_0^a \langle e^{i(\lambda + t)H} (B^* J - B e^{-i(s-t)H_1} J - J^* B + J^* e^{-i(t-s)H_1} B) e^{-i(\lambda + s)H} x, x \rangle d\lambda. \]

As $\left( \left| B \right|^{1/2}, x \right)$ is $\mathcal{M}$-$H$-smooth, by Proposition 3.2.1 there exists an $N_1 > 0$ such that for all $t, s > N_1$ and all $a > 0$, we have

\[ \left\| \langle (W(t) - W(s)) e^{iaH} W(t)^* (W(t) - W(s) e^{-iaH}) x, x \rangle \right\| < \frac{\epsilon}{4}. \] (3.3)

For each $t, s > N_1$, from Lemma 3.2.2 it follows that

\[ \left\| \langle (W(t) - W(s)) e^{-iaH} x, x \rangle \right\| < \frac{\epsilon}{4}, \quad \text{when } a \text{ is large enough.} \] (3.4)

Thus, from (3.3) and (3.4), we conclude that, for $t, s > N_1$,

\[ \left\| \langle (W(t) - W(s)) x, x \rangle \right\| < \frac{\epsilon}{2}. \] (3.5)
Similarly, there exists an \( N_2 > 0 \) such that, when \( t, s > N_2 \), we have
\[
|(\langle W(s)(W(t) - W(s))x, x \rangle) < \frac{\epsilon}{2}.
\]
Now, (3.5) and (3.6) imply that, for all \( t, s > \max\{N_1, N_2\} \),
\[
\|(W(t) - W(s))x\| < \epsilon,
\]
which ends the proof of the proposition. \( \square \)

4. Norm Absolutely Continuous Projections in Semifinite von Neumann algebras

Let \( \mathcal{H} \) be a complex Hilbert space. Let \( \mathcal{M} \) be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on \( \mathcal{H} \) and \( \tau \) a faithful normal semifinite tracial weight of \( \mathcal{M} \). Let \( \mathcal{A}(\mathcal{M}) \) be the set of densely defined, closed operators affiliated with \( \mathcal{M} \).

4.1. Norm absolutely continuous projections.

**Definition 4.1.1.** Let \( H \) be a self-adjoint element in \( \mathcal{A}(\mathcal{M}) \) and let \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) be the spectral resolution of the identity for \( H \) in \( \mathcal{M} \). We define \( \mathcal{P}_{ac}^\infty(H) \) to be the collection of those projections \( P \) in \( \mathcal{M} \) such that

the mapping \( \lambda \mapsto PE(\lambda)P \) from \( \lambda \in \mathbb{R} \) into \( \mathcal{M} \) is locally absolutely continuous, i.e. for all \( a, b \in \mathbb{R} \) with \( a < b \) and every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \sum_i \|PE(b_i)P - PE(a_i)P\| < \epsilon \) for every finite collection \( \{(a_i, b_i)\} \) of disjoint intervals in \( [a, b] \) with \( \sum_i (b_i - a_i) < \delta \).

A projection \( P \) in \( \mathcal{P}_{ac}^\infty(H) \) is called a norm absolutely continuous projection with respect to \( H \).

**Remark 4.1.2.** Definition 4.1.1 is closely related to the concept of \( H \)-smoothness introduced by Kato in [7] for \( \mathcal{B}(\mathcal{H}) \), where \( \mathcal{H} \) is a complex separable Hilbert space. From one of equivalent definitions of \( H \)-smoothness in Theorem 5.1 of [7], it is not hard to see that if a projection \( P \) is \( H \)-smooth, then \( P \in \mathcal{P}_{ac}^\infty(H) \).

In the case when \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \), the following proposition relates \( \mathcal{P}_{ac}^\infty(H) \) to \( \mathcal{H}_{ac}(H) \).

**Proposition 4.1.3.** Let \( \mathcal{H} \) be a complex infinite dimensional separable Hilbert space and let \( \mathcal{B}(\mathcal{H}) \) be the set of all bounded linear operators on \( \mathcal{H} \). Assume \( H \) is a densely defined self-adjoint operator on \( \mathcal{H} \) (so \( H \in \mathcal{A}(\mathcal{B}(\mathcal{H})) \)). Then a vector \( x \) is in \( \mathcal{H}_{ac}(H) \) if and only if the rank one projection \( x \otimes x \) is in \( \mathcal{P}_{ac}^\infty(H) \), where \( x \otimes x \) is defined by \( (x \otimes x)y = \langle y, x \rangle x \) for all \( y \in \mathcal{H} \).

**Proof.** Let \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) be the spectral resolution of the identity for \( H \) in \( \mathcal{B}(\mathcal{H}) \). Let \( x \) be a vector in \( \mathcal{H} \). For all \( \lambda \in \mathbb{R} \), we have
\[
(x \otimes x)E(\lambda)(x \otimes x) = \langle E(\lambda)x, x \rangle (x \otimes x).
\]
Thus, \( x \) is in \( \mathcal{H}_{ac}(H) \) if and only if \( x \otimes x \) is in \( \mathcal{P}_{ac}^\infty(H) \). \( \square \)

Next example shows there exists self-adjoint operators in \( \mathcal{M} \) with nonzero norm absolutely continuous projections.
Example 4.1.4. Let $\mathcal{N}$ be a diffuse finite von Neumann algebra with a faithful normal tracial state $\tau_\mathcal{N}$ and let $\mathcal{H}_0$ be an infinite dimensional separable Hilbert space. Then $\mathcal{M} = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0)$ is a semifinite von Neumann algebra with a faithful normal tracial weight $\tau_\mathcal{M} = \tau_\mathcal{N} \otimes Tr$, where $Tr$ is the canonical trace of $\mathcal{B}(\mathcal{H}_0)$. We might further assume that $\mathcal{M}$ acts naturally on the Hilbert space $L^2(\mathcal{N}, \tau_\mathcal{N}) \otimes \mathcal{H}_0$.

Let $X$ be a densely defined, self-adjoint operator with purely absolutely continuous spectrum on $\mathcal{H}_0$. Then $I_N \otimes X$ is a densely defined, self-adjoint operator affiliated with $\mathcal{M}$. For each vector $y \in \mathcal{H}_0$, denote by $Q_y$ the rank one projection $y \otimes y$ in $\mathcal{B}(\mathcal{H}_0)$. We claim that $I_N \otimes Q_y \in \mathcal{P}_\infty(I_N \otimes X)$. In fact, if $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $X$ in $\mathcal{B}(\mathcal{H}_0)$, then $\{I_N \otimes E(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $I_N \otimes X$ in $N \otimes \mathcal{B}(\mathcal{H}_0) = \mathcal{M}$. Note $\|(I_N \otimes Q_y)(I_N \otimes E(\lambda) - I_N \otimes E(\mu))(I_N \otimes Q_y)\| = \|Q_y(E(\lambda) - E(\mu))Q_y\| = \langle (E(\lambda) - E(\mu))y, y \rangle \|y\|^2$ for all $\lambda > \mu$. We have that $I_N \otimes Q_y \in \mathcal{P}_\infty(I \otimes X)$.

It is easy to see the following statement.

Lemma 4.1.5. Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. If $P \in \mathcal{P}_\infty(H)$, then $P \leq P_{ac}(H)$, where $P_{ac}(H)$ is the projection from $\mathcal{H}$ onto $\mathcal{H}_{ac}(H)$.

Definition 4.1.6. Suppose that $P \in \mathcal{P}_\infty(H)$. For each interval $[a, b]$, we define

$$V_{[a,b]}(P) = \sup \left\{ \sum_{i=1}^m \|PE_{\lambda_i}P - PE_{\lambda_{i-1}}P\| : a = \lambda_0 < \lambda_1 < \cdots < \lambda_m = b \text{ is a partition of } [a, b] \right\}$$

and

$$\Psi(\lambda) = \begin{cases} V_{[0,\lambda]}(P) & \text{ if } \lambda \geq 0 \\ V_{[\lambda,0]}(P) & \text{ if } \lambda < 0 \end{cases}$$

Lemma 4.1.7. $\Psi$ is locally absolutely continuous on $\mathbb{R}$ and $\Psi'$ exists almost everywhere.

Proof. It can be verified directly (also see Proposition 1.2.1 in [1]).

4.2. Cut-off function $\omega_n$.

Definition 4.2.1. Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Assume that $P \in \mathcal{P}_\infty(H)$ and $\Psi$ are as in Definition 4.1.1 and Definition 4.1.6. For each $n \in \mathbb{N}$, we define,

$$\omega_n(\lambda) = \begin{cases} 1 & \text{ if } |\Psi'(\lambda)| \leq n \text{ and } |\lambda| \leq n \\ 0 & \text{ otherwise} \end{cases}$$

Lemma 4.2.2. Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Assume $P \in \mathcal{P}_\infty(H)$ and $x \in \mathcal{H}$. For each $n \in \mathbb{N}$, let $\omega_n(\lambda)$ be as in Definition 4.2.1. Let

$$\omega_n(H) = \int \omega_n(\lambda)dE(\lambda).$$

Then

$$\omega_n(H) \to I \text{ in strong operator topology, as } n \to \infty.$$ 

Proof. Let $\Delta_n = \{ \lambda \in \mathbb{R} : \omega_n(\lambda) = 1 \}$. Observe that $\mathbb{R} \setminus (\cup_n \Delta_n)$ is a measure zero set. Thus $\omega_n(H) \to I$ in strong operator topology, as $n \to \infty$. 

□
Lemma 4.2.3. Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Assume $P \in \mathcal{P}_a(H)$ and $x \in \mathcal{H}$. For each $n \in \mathbb{N}$, let $\omega_n(\lambda)$ be as in Definition 4.2.1. The following statements are true.

(i) For real numbers $a, b$ with $a < b$, the mapping $\lambda \mapsto P E(\lambda)\omega_n(H)x$ from $[a, b]$ into $\mathcal{H}$ is absolutely continuous.

(ii) We have

$$\frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda}$$

exists in $\mathcal{H}$ for $\lambda \in \mathbb{R}$ a.e.

and the mapping

$$\lambda \mapsto \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda}$$

from $\mathbb{R}$ into $\mathcal{H}$ is locally Bochner integrable.

(iii) We have

$$\left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\| \leq \omega_n(\lambda) \sqrt{n} \cdot \sqrt{\frac{d(E(\lambda)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda}}$$

for $\lambda \in \mathbb{R}$ a.e.

(iv) We have

$$\int_{\mathbb{R}} \left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\| d\lambda < \infty$$

and

$$\int_{\mathbb{R}} \left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\|^2 d\lambda < n \|x\|^2.$$

(v) The mapping

$$\lambda \mapsto e^{-it\lambda} \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda}$$

is in $L^1(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{H})$ for each $t \in \mathbb{R}$, with

$$P e^{-iHt} \omega_n(H)x = \int_{\mathbb{R}} e^{-it\lambda} \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} d\lambda.$$

(vi) We have

$$\int_{\mathbb{R}} \left\| P e^{-i\lambda H} \omega_n(H)x \right\|^2 d\lambda = \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\|^2 d\lambda < \frac{n}{2\pi} \|x\|^2.$$

Proof. (i) Recall $P_{ac}(H)$ is the projection from $\mathcal{H}$ onto $\mathcal{H}_{ac}(H)$. Notice that $P_{ac}(H)$ commutes with $E(\lambda)$. Moreover, from Lemma 4.1.5, $P = PP_{ac}(H)$. Assume that $\{(a_i, b_i)\}$ is a
finite family of disjoint intervals in $[a, b]$. Then

$$
\sum_i \|P(E(b_i) - E(a_i))\omega_n(H)x\| = \sum_i \|P(E(b_i) - E(a_i))P_{ac}(H)\omega_n(H)x\|
$$

$$
\leq \sum_i \|P(E(b_i) - E(a_i))\| \|(E(b_i) - E(a_i))P_{ac}(H)\omega_n(H)x\|
$$

$$
\leq \left( \sum_i \|P(E(b_i) - E(a_i))\|^2 \right)^{1/2} \left( \sum_i \|(E(b_i) - E(a_i))P_{ac}(H)\omega_n(H)x\|^2 \right)^{1/2}
$$

$$
= \left( \sum_i \|P(E(b_i) - E(a_i))P\| \right)^{1/2} \cdot \left( \sum_i \langle(E(b_i) - E(a_i))P_{ac}(H)\omega_n(H)x, (P_{ac}(H)\omega_n(H)x)\rangle \right)^{1/2}.
$$

Now the result follows from the fact that $P_{ac}(H)\omega_n(H)x \in H_{ac}(H)$ and $P \in P_{ac}^\infty(H).

(ii) The first statement follows from (i) and the Radon-Nikodym Property of the Hilbert space $H$ (see Definition 1.2.5 in [1]). The second statement follows (i) and the first statement (see Proposition 1.2.3 in [1]).

(iii) We have, from (i),

$$
\left\| \frac{P(E(\lambda) - E(\mu))\omega_n(H)x}{\lambda - \mu} \right\| \leq \frac{\| (E(\lambda) - E(\mu))P_{ac}(H)\omega_n(H)x\| \|P(E(\lambda) - E(\mu))\|}{|\lambda - \mu|}
$$

$$
= \frac{\| (E(\lambda) - E(\mu))P_{ac}(H)\omega_n(H)x\| \|P(E(\lambda) - E(\mu))P\|}{\sqrt{|\lambda - \mu|}} \cdot \left\| \frac{P(E(\lambda) - E(\mu))P}{\lambda - \mu} \right\|^{1/2}
$$

$$
\leq \sqrt{\frac{\| (E(\lambda) - E(\mu))\omega_n(H)P_{ac}(H)x, P_{ac}(H)x\|}{\lambda - \mu}} \cdot \left\| \frac{\Psi(\lambda) - \Psi(\mu)}{\lambda - \mu} \right\|^{1/2}.
$$

Hence, by the definitions of $\omega_n$ and $\Psi$, we obtain

$$
\left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\| \leq \sqrt{\frac{d(E(\lambda)\omega_n(H)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda}} \cdot \sqrt{|\Psi'(\lambda)|} \quad \text{a.e.}
$$

$$
\leq \omega_n(\lambda) \sqrt{\frac{d(E(\lambda)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda}} \cdot \sqrt{n} \quad \text{a.e.} \quad \text{(by Lemma 2.4.1)}
$$

(iv) We have

$$
\int_\mathbb{R} \left\| \frac{d(P E(\lambda)\omega_n(H)x)}{d\lambda} \right\|^2 \, d\lambda \leq \int_\mathbb{R} \omega_n(\lambda) \cdot \frac{d(E(\lambda)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda} \cdot n \, d\lambda \quad \text{(by (iii))}
$$

$$
\leq n \|P_{ac}(H)x\|^2 \leq n\|x\|^2. \quad \text{(by (2.1))}
$$
Similarly,
\[
\int_{\mathbb{R}} \left\| \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda} \right\| d\lambda \leq \int_{\mathbb{R}} \omega_n(\lambda) \sqrt{\left| \frac{d(E(\lambda)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda} \right|} \sqrt{n} d\lambda \quad \text{(by (iii))}
\]
\[
\leq \left( \int_{\mathbb{R}} \omega_n(\lambda) \cdot n d\lambda \right)^{1/2} \left( \int_{\mathbb{R}} \frac{d(E(\lambda)P_{ac}(H)x, P_{ac}(H)x)}{d\lambda} d\lambda \right)^{1/2}
\]
\[
< \infty
\]

(v) It follows from (i), (ii) and (iv) that the mapping
\[
\lambda \mapsto e^{-it\lambda} \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda}
\]
is in \(L^1(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{H})\) for each \(t \in \mathbb{R}\). Obviously, \(Pe^{-itH}\omega_n(H)x \in \mathcal{H}\). We need only to verify that, for all \(y \in \mathcal{H}\),
\[
\langle Pe^{-itH}\omega_n(H)x, y \rangle = \langle \left( \int_{\mathbb{R}} e^{-it\lambda} \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda} d\lambda \right), y \rangle.
\]
In fact,
\[
\langle \left( \int_{\mathbb{R}} e^{-it\lambda} \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda} d\lambda \right), y \rangle = \int_{\mathbb{R}} \langle e^{-it\lambda} \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda}, y \rangle d\lambda
\]
\[
= \int_{\mathbb{R}} e^{-it\lambda} \frac{d(E(\lambda)\omega_n(H)x, y)}{d\lambda} d\lambda
\]
\[
= \int_{\mathbb{R}} e^{-it\lambda} \frac{d(E(\lambda)\omega_n(H)x, Py)}{d\lambda} d\lambda
\]
\[
= \langle e^{-itH}\omega_n(H)x, Py \rangle
\]
\[
= \langle Pe^{-itH}\omega_n(H)x, y \rangle.
\]

Hence the result holds.

(vi) By Plancherel’s Theorem for Fourier transformation on Hilbert spaces (see Theorem 1.8.1 in [1]), we have
\[
\int_{\mathbb{R}} \|Pe^{-i\lambda H}\omega_n(H)x\|^2 d\lambda = \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \frac{d(PE(\lambda)\omega_n(H)x)}{d\lambda} \right\|^2 d\lambda \leq \frac{n}{2\pi} \|x\|^2. \quad \text{(by (v) and (iv))}
\]

\[\Box\]

**Proposition 4.2.4.** Suppose \(H\) is a self-adjoint element in \(\mathcal{A}(\mathcal{M})\). Assume \(P \in \mathcal{P}_{ac}^\infty(H)\) and let \(\omega_n(\lambda)\) be as in Definition 4.2.1 for each \(n \in \mathbb{N}\).
If \(A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)\), then
\[
\int_{\mathbb{R}} \|Ae^{-i\lambda H}\omega_n(H)P\|^2 d\lambda \leq \frac{n}{2\pi} \|A\|^2.
\]
Proof. Note that $e^{-i\lambda H}$ commutes with $\omega_n(H)$. As
\[\|X\|_2^2 = \tau(X^*X) = \tau(XX^*) = \|X^*\|_2^2, \quad \forall \ X \in \mathcal{M},\]
it suffices to show that
\[\int_{\mathbb{R}} \|P e^{-i\lambda H} \omega_n(H) A\|_2^2 \, d\lambda \leq \frac{n}{2\pi} \|A\|_2^2.\]
By Lemma 2.1.1, there exists a sequence $\{x_m\}_{m \in \mathbb{N}}$ of vectors in $\mathcal{H}$ such that
\[\|X\|_2^2 = \tau(X^*X) = \sum_m \langle X^* X x_m, x_m \rangle = \sum_m \|X x_m\|^2, \quad \forall \ X \in \mathcal{M}. \quad (4.1)\]
By Lemma 4.2.3 (vi), for all $m \in \mathbb{N}$,
\[\int_{\mathbb{R}} \|P e^{-i\lambda H} \omega_n(H) P x_m\|_2^2 \, d\lambda \leq \frac{n}{2\pi} \|A x_m\|^2. \quad (4.2)\]
By (4.1), we have
\[\int_{\mathbb{R}} \|P e^{-i\lambda H} \omega_n(H) A\|_2^2 \, d\lambda = \int_{\mathbb{R}} \sum_m \|P e^{-i\lambda H} \omega_n(H) A x_m\|_2^2 \, d\lambda \quad (by \ 4.1)\]
\[= \sum_m \int_{\mathbb{R}} \|P e^{-i\lambda H} \omega_n(H) A x_m\|_2^2 \, d\lambda \]
\[\leq \sum_m \frac{n}{2\pi} \|A x_m\|^2 \quad (by \ 4.2)\]
\[= \frac{n}{2\pi} \|A\|_2^2. \quad (by \ 4.1)\]
This ends the proof of the lemma. \qed

Corollary 4.2.5. Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Let $P \in \mathcal{P}_\infty(H)$ and let $\omega_n(\lambda)$ be as in Definition 4.2.7 for each $n \in \mathbb{N}$. Assume $\{x_m\}_{m \in \mathbb{N}}$ is a family of vectors in $\mathcal{H}$ such that
\[\|X\|_2^2 = \tau(X^*X) = \sum_n \langle X^* X x_m, x_m \rangle = \sum_m \|X x_m\|^2, \quad \forall \ X \in \mathcal{M}.\]
Then, for all $A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau), \ S \in \mathcal{M}$ and $m, n \in \mathbb{N}$, we have
\[\int_{\mathbb{R}} \|A S e^{-i\lambda H} \omega_n(H) P x_m\|_2^2 \, d\lambda \leq \frac{n}{2\pi} \|A\|_2^2 \|S\|^2. \]
Proof. It follows from Proposition 4.2.4 and the choice of $\{x_m\}_{m \in \mathbb{N}}$ that
\[\int_{\mathbb{R}} \|A S e^{-i\lambda H} \omega_n(H) P x_m\|_2^2 \, d\lambda \leq \int_{\mathbb{R}} \|A S e^{-i\lambda H} \omega_n(H) P\|_2^2 \, d\lambda \leq \frac{n}{2\pi} \|A S\|_2^2 \leq \frac{n}{2\pi} \|A\|_2^2 \|S\|^2. \]
Now we are ready to state the main result of this section.
**Proposition 4.2.6.** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Let $P \in \mathcal{P}_ac(H)$. Assume $\{x_m\}_{m \in \mathbb{N}}$ is a family of vectors in $\mathcal{H}$ such that
\[
\|X\|_2^2 = \tau(X^*X) = \sum_m \langle X^*X x_m, x_m \rangle = \sum_m \|X x_m\|^2, \quad \forall \ X \in \mathcal{M}.
\]
Then there exists an increasing sequence $\{Q_n\}_{n \in \mathbb{N}}$ of projections in $\mathcal{M}$ such that
(i) $Q_n$ converges to $I$ in strong operator topology as $n \to \infty$.
(ii) If $A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$ and $m, n \in \mathbb{N}$, then $(A, Q_n P x_m)$ is $\mathcal{M}$-$H$-smooth.

**Proof.** Let $Q_n = \omega_n(H)$ be as in Definition 4.2.1 for each $n \in \mathbb{N}$. Now the result follows from Lemma 4.2.2, Corollary 4.2.5 and Definition 3.1.1. \(\square\)

5. Existence of Generalized Wave Operator in Semifinite von Neumann Algebras

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight $\tau$. Let $\mathcal{A}(\mathcal{M})$ be the set of densely defined, closed operators that are affiliated with $\mathcal{M}$. Let $\mathcal{P}_ac(H)$ be the set of norm absolutely continuous projections with respect to $H$ in $\mathcal{M}$.

5.1. Generalized wave operators.

**Theorem 5.1.1.** Suppose $H$ and $H_1$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Assume $J$ is in $\mathcal{M}$ such that $J\mathcal{D}(H) \subseteq \mathcal{D}(H_1)$ and the closure of $H_1J - JH$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Let $W(t) = e^{itH_1}e^{-itH}$ for each $t \in \mathbb{R}$.

If $P \in \mathcal{P}_ac(H)$, then $W(t)P$ converges in strong operator topology in $\mathcal{M}$ as $t \to \infty$.

**Proof.** By Lemma 2.1.1, there exists a family $\{x_m\}_{m \in \mathbb{N}}$ of vectors in $\mathcal{H}$ such that
\[
\|X\|_2^2 = \tau(X^*X) = \sum_m \langle X^*X x_m, x_m \rangle = \sum_m \|X x_m\|^2, \quad \forall \ X \in \mathcal{M}.
\]
Note $P \in \mathcal{P}_ac(H)$. By Proposition 4.2.6, there exists an increasing sequence $\{Q_n\}_{n \in \mathbb{N}}$ of projections in $\mathcal{M}$ such that (a) $Q_n$ converges to $I$ in strong operator topology as $n \to \infty$; and (b) $(A, Q_n P x_m)$ is $\mathcal{M}$-$H$-smooth for all $A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$ and $m, n \in \mathbb{N}$. In particular, $(|H_1J - JH|^{1/2}, Q_n P x_m)$ is $\mathcal{M}$-$H$-smooth for all $m, n \in \mathbb{N}$. By Theorem 3.2.3, $W(t)Q_n P x_m$ converges in $\mathcal{H}$ as $t \to \infty$. Since $Q_n$ converges to $I$ in strong operator topology and $W(t)$ is uniformly bounded, $W(t)P x_m$ converges in $\mathcal{H}$ as $t \to \infty$. This further implies that $W(t)P A' x_m = A' W(t)P x_m$ converges in $\mathcal{H}$, as $t \to \infty$, for all $A' \in \mathcal{M}'$ and $m \in \mathbb{N}$. By Lemma 2.1.1, $W(t)P x$ converges in $\mathcal{H}$ for all $x \in \mathcal{H}$, whence $W(t)P$ converges in strong operator topology in $\mathcal{M}$ as $t \to \infty$. \(\square\)

**Proposition 5.1.2.** Let $H$ and $H_1$ be self-adjoint elements in $\mathcal{A}(\mathcal{M})$. Suppose that $H_1 - H$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. If $P \in \mathcal{P}_ac(H)$, then $e^{itH_1}e^{-itH}P$ converges in strong operator topology in $\mathcal{M}$ as $t \to \infty$.

**Proof.** The result is a special case of Theorem 5.1.1 when $J = I$. \(\square\)
5.2. Kato-Rosenblum Theorem in semifinite von Neumann algebras. Recall $\mathcal{P}_{ac}^\infty(H)$ is the set of norm absolutely continuous projections with respect to $H$ in $\mathcal{M}$ (see Definition 4.1.1).

**Definition 5.2.1.** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Define
\[ P_{ac}^\infty(H) = \vee\{P : P \in \mathcal{P}_ac^\infty(H)\}. \]
Such $P_{ac}^\infty(H)$ is called the norm absolutely continuous support of $H$ in $\mathcal{M}$.

**Lemma 5.2.2.** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Then $P_{ac}^\infty(H) \leq P_{ac}(H)$. Furthermore, if $\mathcal{H}$ is separable and $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then $P_{ac}^\infty(H) = P_{ac}(H)$.

**Proof.** The first statement follows from Lemma 4.1.5 and Definition 5.2.1. The second statement follows from a combination of the first statement and Proposition 4.1.3. □

**Lemma 5.2.3.** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Let $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ be the spectral resolution of the identity for $H$ in $\mathcal{M}$. If $S \in \mathcal{M}$ satisfies that the mapping $\lambda \mapsto S^*E(\lambda)S$ from $\mathbb{R}$ into $\mathcal{M}$ is locally absolutely continuous, then $R(S)$, the range projection of $S$ in $\mathcal{M}$, is a subprojection of $P_{ac}^\infty(H)$.

**Proof.** Let $S = |S^*|W$ be a polar decomposition of $S$ in $\mathcal{M}$ where $W$ is a partial isometry and $|S^*|$ is a positive operator in $\mathcal{M}$. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function such that $f_n(\lambda) = 1/\lambda$ when $1/n < \lambda < n$ and 0 otherwise. It is not hard to check that $|S^*| \cdot f_n(|S^*|)$ is a projection in $\mathcal{M}$ satisfying the mapping $\lambda \mapsto |S^*|f_n(|S^*|)E(\lambda)|S^*|f_n(|S^*|) = f_n(|S^*|)WS^*E(\lambda)SW^*f_n(|S^*|)$ from $\mathbb{R}$ into $\mathcal{M}$ is locally absolutely continuous. Therefore, $|S^*|f_n(|S^*|) \in \mathcal{P}_ac^\infty(H)$ for each $n \in \mathbb{N}$. Notice, when $n \to \infty$, $|S^*| \cdot f_n(|S^*|) \to R(S)$ in strong operator topology. Now we conclude that $R(S)$ is a subprojection of $P_{ac}^\infty(H)$. □

**Proposition 5.2.4.** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. If $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $H$ in $\mathcal{M}$ and $A$ is the von Neumann subalgebra generated by $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ in $\mathcal{M}$, then $P_{ac}^\infty(H)$ is in $A' \cap \mathcal{M}$.

**Proof.** Let $P \in \mathcal{P}_ac^\infty(H)$. Let $a \in \mathbb{R}$. It is not hard to verify that the mapping $\lambda \mapsto PE(a)E(\lambda)E(a)P$ from $\mathbb{R}$ into $\mathcal{M}$ is locally absolutely continuous. Hence, from Lemma 5.2.3, $R(E(a)P)$, the range projection of $E(a)P$ in $\mathcal{M}$, is a subprojection of $P_{ac}^\infty(H)$. It follows that $E(a)P = P_{ac}^\infty(H)E(a)P$. This implies that $E(a)P_{ac}^\infty(H) = P_{ac}^\infty(H)E(a)P_{ac}^\infty(H)$, whence $E(a)P_{ac}^\infty(H) = P_{ac}^\infty(H)E(a)$. Therefore $P_{ac}^\infty(H)$ is in $A' \cap \mathcal{M}$. □

The following is an analogue of Kato-Rosenblum Theorem in semifinite von Neumann algebras.

**Theorem 5.2.5.** Suppose $H$ and $H_1$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$ such that $H_1 - H$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Then
\[ W_+ \triangleq \text{sot- lim }_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H) \text{ exists in } \mathcal{M}. \]
Moreover,
(i) $W*W_+ = P_{ac}^\infty(H) \leq P_{ac}(H)$ and $W_+W* = P_{ac}^\infty(H_1) \leq P_{ac}(H_1)$;
(ii) $W_+HW_+ = H_1 P_{ac}^\infty(H_1)$. 

And it is not hard to check that \( W_+ \) is a partial isometry in \( \mathcal{M} \). Therefore
\[
W_+^* W_+ = P_{ac}^\infty(H) \leq P_{ac}(H).
\] (5.2)

Moreover, from Proposition 5.2.4 we have
\[
W_+ e^{-isH} = \left( \text{sot-lim}_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H) \right) e^{-isH} = \text{sot-lim}_{t \to \infty} \left( e^{itH_1} e^{-itH} P_{ac}^\infty(H) \right) e^{-isH_1} \left( \text{sot-lim}_{t \to \infty} e^{i(t+s)H_1} e^{-i(t+s)H} P_{ac}^\infty(H) \right) = e^{-isH_1} W_+, \quad \forall \, s \in \mathbb{R}.
\] (5.3)

Let \( \{E(\lambda)\}_{\lambda \in \mathbb{R}} \) and \( \{F(\lambda)\}_{\lambda \in \mathbb{R}} \) be the spectral resolutions of the identity for \( H \) and \( H_1 \) respectively, in \( \mathcal{M} \). Then, for all \( x, y \in \mathcal{H} \) and \( s \in \mathbb{R} \),
\[
\langle W_+ e^{-isH} x, y \rangle = \langle e^{-isH} x, W_+^* y \rangle = \langle e^{-isH_1} W_+ x, y \rangle\quad \text{(by (5.3))}
\]
\[
= \int_{\mathbb{R}} e^{-is\lambda} d \langle E(\lambda)x, W_+^* y \rangle = \int_{\mathbb{R}} e^{-is\lambda} d \langle F(\lambda)W_+ x, y \rangle.\quad \text{(by (2.1))}
\]

By the uniqueness of Fourier-Stieltjes transform, we have
\[
\langle E(\lambda)x, W_+^* y \rangle = \langle F(\lambda)W_+ x, y \rangle, \quad \forall \, x, y \in \mathcal{H}, \forall \, \lambda \in \mathbb{R}.
\]

Thus
\[
W_+ E(\lambda) = F(\lambda)W_+, \quad \forall \, \lambda \in \mathbb{R}.
\] (5.4)

Let \( P \in \mathcal{P}_{ac}(H) \). Then
\[
(W_+ P)^* F(\lambda)(W_+ P) = PW_+^* F(\lambda)W_+ P = PW_+^* W_+ E(\lambda) P = P E(\lambda) P. \quad \text{(by (5.4) and (5.2))}
\]

This implies that the mapping \( \lambda \mapsto (W_+ P)^* F(\lambda)(W_+ P) \) from \( \mathbb{R} \) into \( \mathcal{M} \) is locally absolutely continuous. By Lemma 5.2.3 we get that \( R(W_+ P) \), the range projection of \( W_+ P \) in \( \mathcal{M} \), is a subprojection of \( P_{ac}^\infty(H_1) \). So, we obtain that \( R(W_+) \leq P_{ac}^\infty(H_1) \). Therefore,
\[
W_+ W_+^* \leq P_{ac}^\infty(H_1). \quad (5.5)
\]

Similarly, as \( H - H_1 \in \mathcal{M} \cap L^1(\mathcal{M}, \tau) \), we let \( V_+ = \text{sot-lim}_{t \to \infty} e^{itH} e^{-itH_1} P_{ac}^\infty(H_1) \). Then
\[
V_+^* V_+ = P_{ac}^\infty(H_1) \quad \text{and} \quad V_+ V_+^* \leq P_{ac}^\infty(H). \quad (5.6)
\]

We claim that
\[
\lim_{t \to \infty} \| (I - P_{ac}^\infty(H)) e^{-itH_1} P_{ac}^\infty(H_1) x \| = 0, \quad \forall \, x \in \mathcal{H}.
\] (5.7)
In fact, we have, for all $x \in \mathcal{H}$,
\begin{align*}
\lim_{t \to \infty} \| (I - P_{\text{ac}}^\infty(H)) e^{-itH} P_{\text{ac}}^\infty(H_1)x \| &= \lim_{t \to \infty} \| (I - P_{\text{ac}}^\infty(H)) e^{itH} e^{-itH} P_{\text{ac}}^\infty(H_1)x \| \\
&= \| (I - P_{\text{ac}}^\infty(H)) V_+x \| = 0. \quad \text{(by definition of } V_+ \text{ and } (5.6))
\end{align*}

Furthermore,
\begin{align*}
W_+ V_+ &= \left( \text{sot-} \lim_{t \to \infty} e^{itH} e^{-itH} P_{\text{ac}}^\infty(H) \right) \cdot \left( \text{sot-} \lim_{t \to \infty} e^{itH} e^{-itH} P_{\text{ac}}^\infty(H_1) \right) \\
&= \text{sot-} \lim_{t \to \infty} \left( e^{itH} P_{\text{ac}}^\infty(H) e^{-itH} P_{\text{ac}}^\infty(H_1) \right) \\
&= P_{\text{ac}}^\infty(H_1) + \text{sot-} \lim_{t \to \infty} \left( e^{itH} (I - P_{\text{ac}}^\infty(H)) e^{-itH} P_{\text{ac}}^\infty(H_1) \right) \\
&= P_{\text{ac}}^\infty(H_1). \quad \text{(by } (5.7))
\end{align*}

Thus $R(W_+) = P_{\text{ac}}^\infty(H_1)$, where $R(W_+)$ is the range projection of $W_+$ in $\mathcal{M}$. Combining with $(5.5)$, we conclude that
\begin{equation}
W_+ W_+^* = P_{\text{ac}}^\infty(H_1) \leq P_{\text{ac}}(H_1). \tag{5.8}
\end{equation}

Now from $(5.2)$, $(5.3)$, and $(5.8)$, it follows that
\begin{equation}
W_+ e^{itH} W_+^* = e^{itH} P_{\text{ac}}^\infty(H_1), \quad \forall t \in \mathbb{R}. \tag{5.9}
\end{equation}

By Stone’s Theorem (see Theorem 5.6.36 in [5]), we obtain from $(5.9)$ that
\begin{equation}
W_+ H W_+^* = H_1 P_{\text{ac}}^\infty(H_1). \tag{5.10}
\end{equation}

The proof is now complete from $(5.1)$, $(5.2)$, $(5.8)$, and $(5.10)$. \qed

**Remark 5.2.6.** Similarly it can be shown that, if $H$ and $H_1$ are self-adjoint elements in $\mathcal{A}(\mathcal{M})$ such that $H_1 - H \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$, then
\begin{equation*}
W_- \triangleq \text{sot-} \lim_{t \to -\infty} e^{itH} e^{-itH} P_{\text{ac}}^\infty(H) \quad \text{exists in } \mathcal{M},
\end{equation*}
and
\begin{equation*}
W_-^* W_- = P_{\text{ac}}^\infty(H) \leq P_{\text{ac}}(H) \quad \text{and} \quad W_- W_-^* = P_{\text{ac}}^\infty(H_1) \leq P_{\text{ac}}(H_1).
\end{equation*}

**Remark 5.2.7.** By Lemma $(5.2.3)$, Theorem $(5.2.8)$ and Remark $(5.2.6)$ imply the classical Kato-Rosenblum Theorem for self-adjoint operators in $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is separable.

**Example 5.2.8.** Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state. Let $\mathcal{H}_0 = L^2(\mathbb{R}^3, \mu)$, where $\mu$ is the Lebesgue measure on $\mathbb{R}^3$. Let $\mathcal{M} = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0)$, acting naturally on the Hilbert space $\mathcal{H} = L^2(\mathcal{N}, \tau) \otimes \mathcal{H}_0$, be a semifinite von Neumann algebra. Let $\Delta$ be the Laplacian operator on $L^2(\mathbb{R}^3, \mu)$. Then $-(I_{\mathcal{N}} \otimes \Delta)$ is a densely defined, self-adjoint operator in $\mathcal{A}(\mathcal{M})$. As $-\Delta$ is spectrally absolutely continuous on $L^2(\mathbb{R}^3, \mu)$ (see Section X.3.4 in [9]), Proposition $(4.1.3)$ shows that $P_{\text{ac}}^\infty(-(I_{\mathcal{N}} \otimes \Delta)) = I_\mathcal{M}$. 

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5.3. Real part of hyponormal operators. When $H$ is bounded, the following analogue of Kato-Putnam criterion in [8] and [17] is sometimes useful to determine whether $P_{ac}^\infty(H)$ is zero or not.

**Proposition 5.3.1.** Assume that $H$ is a self-adjoint element in $\mathcal{M}$. Then the following statements are equivalent.

(i) $P_{ac}^\infty(H) \neq 0$.
(ii) There exist a self-adjoint element $K \in \mathcal{M}$ and a nonzero positive element $L \in \mathcal{M}$ such that

$$i(HK - KH) = L.$$ 

**Proof.** (i)$\Rightarrow$(ii) As $P_{ac}^\infty(H) \neq 0$, there exists a nonzero projection $P \in \mathcal{P}_{ac}^\infty(H)$. Let $\omega_n(\cdot)$ be as in Definition 4.2.1. By Lemma 4.2.3 (vi),

$$\int_\mathbb{R} \|P e^{-i\lambda H} \omega_n(H)x\|^2 d\lambda \leq \frac{n}{2\pi} \|x\|^2,$$

for all $x \in \mathcal{H}$. Since $e^{-i\lambda H}$ and $\omega_n(H)$ commute, by Theorem 2.1 in [8],

$$\sup_{\lambda > \mu} \frac{\|P \omega_n(H) (E(\lambda) - E(\mu))\|^2}{\lambda - \mu} < \infty,$$

where $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of the identity for $H$ in $\mathcal{M}$. Hence

$$\sup_{\lambda > \mu} \frac{\|\omega_n(H) P \omega_n(H) (E(\lambda) - E(\mu))\|^2}{\lambda - \mu} \leq \sup_{\lambda > \mu} \frac{\|P \omega_n(H) (E(\lambda) - E(\mu))\|^2}{\lambda - \mu} < \infty.$$

By Lemma 4.2.2, we might assume that $\omega_n(H) P \omega_n(H) \neq 0$ for a large $n \in \mathbb{N}$. Again from Theorem 2.1 in [8], $\omega_n(H) P \omega_n(H)$ is a nonzero positive $H$-smooth operator in $\mathcal{M}$. Let

$$L = (\omega_n(H) P \omega_n(H))^2.$$ 

Theorem 3.2 of [8] shows that

$$K = -\int_0^{\infty} e^{i\lambda H} L e^{-i\lambda H} d\lambda \quad \text{(convergence is in strong operator topology)},$$

exists in $\mathcal{M}$ and Lemma X.5.2 in [9] implies that $i(HK - KH) = L$.

(ii)$\Rightarrow$(i) Assume that $K, L$ are elements in $\mathcal{M}$ with desired properties. From Theorem 6.2 in [8], it follows that $L^{1/2}$ is $H$-smooth. Now Theorem 2.1 in [8] implies that the mapping $\lambda \mapsto L^{1/2} E(\lambda)L^{1/2}$ from $\mathbb{R}$ into $\mathcal{M}$ is Lipschitz continuous (so locally absolutely continuous). By Lemma 5.2.3, the range projection of $L^{1/2}$ is a nonzero subprojection of $P_{ac}^\infty(H)$. Therefore $P_{ac}^\infty(H) \neq 0$, which ends the proof of the proposition. \[\square\]

Recall that an operator $T$ in $\mathcal{B}(\mathcal{H})$ is hyponormal if $T^* T - TT^*$ is positive. The following result could be compared to Corollary VI.3.3 in [12].

**Corollary 5.3.2.** $H$ is a self-adjoint element in $\mathcal{M}$ with $P_{ac}^\infty(H) \neq 0$ if and only if $H$ is the real part of a non-normal hyponormal operator $T$ in $\mathcal{M}$.

**Proof.** It is a direct consequence of Proposition 5.3.1. \[\square\]
Remark 5.3.3. Proposition 5.3.1 may be used to construct new examples of self-adjoint operators with nonzero norm absolutely continuous projections in $\mathcal{M}$. For example, let $H, K$ and $L$ be as above. Assume $K_1$ is a self-adjoint element in $\mathcal{M}$ such that $K_1 K = K_1^2$. Then $i((H + K_1)K - K(H + K_1)) = L$. Hence, $H + K_1$ is a self-adjoint element in $\mathcal{M}$ with nonzero norm absolutely continuous projections.

Proposition 5.3.4. If $H$ is a self-adjoint element in $\mathcal{M}$ such that $P_{\text{ac}}^\infty(H) \neq 0$, then there exists no self-adjoint diagonal operator $K$ in $\mathcal{M}$ satisfying $H - K \in L^1(\mathcal{M}, \tau)$. In particular, if $H$ is the real part of a non-normal hyponormal operator in $\mathcal{M}$, then there exists no self-adjoint diagonal operator $K$ in $\mathcal{M}$ satisfying $H - K \in L^1(\mathcal{M}, \tau)$.

Proof. Note that, if $K$ is a self-adjoint diagonal operator in $\mathcal{M}$, then $P_{\text{ac}}^\infty(K) \leq P_{\text{ac}}(K) = 0$. Now that result is a direct consequence of Theorem 5.2.5 and Corollary 5.3.2. □

5.4. Kuroda-Birman Theorem in semifinite von Neumann algebras.

Lemma 5.4.1. Suppose $H$ is a self-adjoint operator in $\mathcal{A}(\mathcal{M})$.

(i) If $A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$, then $\text{sot-} \lim_{t \to \infty} (A e^{-itH} P_{\text{ac}}^\infty(H)) = 0$.

(ii) If $B \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$, then $\text{sot-} \lim_{t \to \infty} (B e^{-itH} P_{\text{ac}}^\infty(H)) = 0$.

Proof. (i) Let $A \in \mathcal{M} \cap L^2(\mathcal{M}, \tau)$. If suffices to show that, if $P \in P_{\text{ac}}^\infty(H)$, then $\text{sot-} \lim_{t \to \infty} (A e^{-itH} P) = 0$.

By Lemma 2.1.1, there exists a family $\{x_m\}_{m \in \mathbb{N}}$ of vectors in $\mathcal{H}$ such that
\[ \|X\|_2^2 = \tau(X^* X) = \sum_m \langle X^* X x_m, x_m \rangle = \sum_m \|X x_m\|^2, \quad \forall X \in \mathcal{M}. \quad (5.11) \]

Assume $P \in P_{\text{ac}}^\infty(H)$. Let $\omega_n(\cdot)$ be as in Definition 4.2.1. By Lemma 4.2.3 (v) and the Riemann-Lebesgue Lemma (also see Theorem 1.8.1 of [1]),
\[ \lim_{t \to \pm \infty} \|P e^{-itH} \omega_n(H) x\| = 0, \quad \text{for all } x \in \mathcal{H}. \]

From Lemma 4.2.2 it follows that
\[ \lim_{t \to \pm \infty} \|P e^{-itH} x\| = 0, \quad \text{for all } x \in \mathcal{H}. \quad (5.12) \]

We claim that
\[ \lim_{t \to \pm \infty} \|A e^{-itH} P\|_2 = \lim_{t \to \pm \infty} \|P e^{-itH} A^*\|_2 = 0. \quad (5.13) \]

In fact, by (5.11), we have
\[ \|P e^{-itH} A^*\|_2^2 = \sum_m \|P e^{-itH} A^* x_m\|^2. \quad (5.14) \]

It induces from (5.12) that
\[ \lim_{t \to \pm \infty} \|P e^{-itH} A^* x_m\| = 0, \quad \text{for all } m \in \mathbb{N}. \quad (5.15) \]
Note that
\[\| P e^{-itH} A^* x_m \| \leq \| A^* x_m \| \quad \text{and} \quad \sum_m \| P e^{-itH} A^* x_m \|^2 \leq \sum_m \| A^* x_m \|^2 = \| A^* \|^2 < \infty. \quad (5.16)\]

By Lebesgue Dominating Convergence Theorem, from (5.14), (5.15) and (5.16), we conclude that
\[\lim_{t \to \pm \infty} \| A e^{-itH} P \|_2 = \lim_{t \to \mp \infty} \| P e^{-itH} A^* \|_2 = 0.\]

I.e. (5.13) holds.

From (5.11) and (5.13), it follows that
\[\lim_{t \to \infty} \| A e^{-itH} P x_m \| \leq \lim_{t \to \infty} \| A e^{-itH} P \| = 0, \quad \forall \ m \in \mathbb{N}.\]

Furthermore,
\[\lim_{t \to \infty} \| A e^{-itH} P A' x_m \| = \lim_{t \to \infty} \| A'(A e^{-itH} P x_m) \| = 0, \quad \forall \ A' \in \mathcal{M}', \text{ and } \ m \in \mathbb{N}.\]

By Lemma 2.1.1,
\[\lim_{t \to \infty} \| A e^{-itH} P x \| = 0, \quad \forall \ x \in \mathcal{H},\]
i.e.
\[sot- \lim_{t \to \infty} (A e^{-itH} P) = 0.\]

This ends the proof of (i).

(ii) follows from the fact that \( M \cap L^1(\mathcal{M}, \tau) \subseteq M \cap L^2(\mathcal{M}, \tau). \)

The following is an analogue of Kuroda-Birman Theorem in a semifinite von Neumann algebra.

**Theorem 5.4.2.** Suppose \( H \) and \( H_1 \) are self-adjoint elements in \( \mathcal{A}(\mathcal{M}) \) such that
\[(H_1 + i)^{-1} - (H + i)^{-1} \in \mathcal{M} \cap L^1(\mathcal{M}, \tau).\]
Then
\[W_+ \triangleq sot- \lim_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty (H) \quad \text{exists in} \ \mathcal{M}.\]

Moreover,
\[(i) \ W_+ W_+ = P_{ac}^\infty (H) \leq P_{ac} (H) \text{ and } W_+ W_+^* = P_{ac}^\infty (H_1) \leq P_{ac} (H_1).\]
\[(ii) \ W_+ H P_{ac}^\infty (H) W_+^* = H_1 P_{ac}^\infty (H_1).\]

**Proof.** Firstly, we will show that
\[W_+ \triangleq sot- \lim_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty (H) \quad \text{exists in} \ \mathcal{M}.\]

In fact, we let \( J = (H_1 + i)^{-1} (H + i)^{-1} \) and \( B = -(H_1 + i)^{-1} + (H + i)^{-1}. \) Then \( J \mathcal{D}(H) \subseteq \mathcal{D}(H_1) \) and
\[H_1 J - J H = (H_1 + i - i) J - J (H + i - i) = B \in \mathcal{M} \cap L^1(\mathcal{M}, \tau).\]

By Theorem 5.1.1 and the definition of \( P_{ac}^\infty (H), \)
\[sot- \lim_{t \to \infty} e^{itH_1} J e^{-itH} P_{ac}^\infty (H) \quad \text{exists in} \ \mathcal{M}. \quad (5.17)\]
Proposition 5.2.4 implies that \((H + i)^{-1}\) commutes with \(P_{ac}^\infty(H)\). Combining with (5.17), we conclude
\[
\lim_{t \to \infty} e^{itH_1}(H + i)^{-1} e^{-itH} P_{ac}^\infty(H)x \quad \text{exists in } \mathcal{H} \text{ for all } x \in \mathcal{D}(H).
\]
Note that \(\mathcal{D}(H)\) is dense in \(\mathcal{H}\). We have
\[
\lim_{t \to \infty} e^{itH_1}(H + i)^{-1} e^{-itH} P_{ac}^\infty(H)x \quad \text{exists in } \mathcal{H} \text{ for all } x \in \mathcal{H}.
\]
(5.18)
Combining (5.18), Lemma 5.4.1 with the fact that \((H_1 + i)^{-1} = (H + i)^{-1} - B\), we get
\[
\lim_{t \to \infty} e^{itH_1}(H + i)^{-1} e^{-itH} P_{ac}^\infty(H)x = \lim_{t \to \infty} e^{itH_1} \left((H_1 + i)^{-1} + B\right) e^{-itH} P_{ac}^\infty(H)x
\]
\[
= \lim_{t \to \infty} e^{itH_1}(H_1 + i)^{-1} e^{-itH} P_{ac}^\infty(H)x \quad \text{exists in } \mathcal{H} \text{ for all } x \in \mathcal{H}.
\]
(5.19)
As \((H + i)^{-1}\) commutes with \(P_{ac}^\infty(H)\) and \(\mathcal{D}(H)\) is dense in \(\mathcal{H}\), it follows from (5.19) that
\[
\lim_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H)x \quad \text{exists in } \mathcal{H} \text{ for all } x \in \mathcal{H},
\]
i.e.
\[
W_+ \triangleq \text{sot-} \lim_{t \to \infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H) \quad \text{exists in } \mathcal{M}.
\]
Secondly, since the proof of \(W_+^*W_+ = P_{ac}^\infty(H), W_+W_+^* = P_{ac}^\infty(H_1)\) and \(W_+HW_+^* = H_1P_{ac}^\infty(H_1)\) is similar to the one in Theorem 5.2.5, it is skipped. □

**Remark 5.4.3.** Similarly it can be shown that, if \(H\) and \(H_1\) are densely defined, self-adjoint elements in \(\mathcal{A}(\mathcal{M})\) such that \((H_1 + i)^{-1} - (H + i)^{-1} \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)\), then
\[
W_- \triangleq \text{sot-} \lim_{t \to -\infty} e^{itH_1} e^{-itH} P_{ac}^\infty(H) \quad \text{exists in } \mathcal{M},
\]
and
\[
W_-^*W_- = P_{ac}^\infty(H) \leq P_{ac}(H) \quad \text{and} \quad W_-W_-^* = P_{ac}^\infty(H_1) \leq P_{ac}(H_1).
\]

6. Small Perturbation of Bounded Self-adjoint Operators

Let \(\mathcal{M}\) be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on a Hilbert space \(\mathcal{H}\). Let \(\tau\) be a faithful normal semifinite tracial weight of \(\mathcal{M}\).

**6.1. Voiculescu’s constant.** Let \(\mathcal{K}_\Phi(\mathcal{M}, \tau)\) be a norm ideal of \(\mathcal{M}\) (see Definition 2.1.1 in [11] for its definition). Recall \(\mathcal{F}(\mathcal{M}, \tau)\) is the set of finite rank operators in \((\mathcal{M}, \tau)\). Then \(\mathcal{K}_\Phi^0(\mathcal{M}, \tau)\) is the completion of \(\mathcal{F}(\mathcal{M}, \tau)\) with respect to the norm \(\Phi\) in \(\mathcal{K}_\Phi(\mathcal{M}, \tau)\). We further let \(\mathcal{K}_\Phi^0(\mathcal{M}, \tau)^+\) be the unit ball of positive elements in \(\mathcal{K}_\Phi^0(\mathcal{M}, \tau)\).

Following Voiculescu’s Definition in [24], we introduce the following concept.

**Definition 6.1.1.** Let \(n \in \mathbb{N}\) and \(X_1, \ldots, X_n\) be an \(n\)-tuple of elements in \(\mathcal{M}\). We define
\[
\mathcal{K}_\Phi(X_1, \ldots, X_n; \mathcal{M}, \tau) = \liminf_{A \in \mathcal{K}_\Phi^0(\mathcal{M}, \tau)^+} \left(\max_{1 \leq i \leq n} \Phi(AX_i - X_iA)\right),
\]
where the \(\liminf\)’s are taken with respect to the natural orders on \(\mathcal{K}_\Phi^0(\mathcal{M}, \tau)^+\).
When $\mathcal{K}_\Phi(M, \tau)$ is $M \cap L^p(M, \tau)$ with
\[ \Phi(X) = \max\{\|X\|, \|X\|_p\}, \quad \forall X \in M, \]
for some $1 \leq p < \infty$, we will denote $\mathcal{K}_\Phi(X_1, \ldots, X_n; M, \tau)$, $\mathcal{K}_\Phi(M, \tau)$ by $\mathcal{K}_p(X_1, \ldots, X_n; M, \tau)$, $\mathcal{K}_p(M, \tau)$ respectively.

The next lemma is a direct consequence of preceding definition.

**Lemma 6.1.2.** Let $M$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight $\tau$. Let $\mathcal{K}_\Phi(M, \tau)$ be a norm ideal of $M$. Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be an $n$-tuple of elements in $M$.

If, for any $\epsilon > 0$, there exists a family of commuting diagonal operators $D_1, \ldots, D_n$ in $M$ such that $\max_{1 \leq i \leq n} \Phi(X_i - D_i) < \epsilon$, then $\mathcal{K}_\Phi(X_1, \ldots, X_n) = 0$.

**6.2. An Example.** If $M_1$ is a von Neumann subalgebra of $M$ such that the restriction of $\tau$ to $M_1$ is semifinite, then there exists a faithful normal trace-preserving conditional expectation $\mathcal{E}$ from $M$ onto $M_1$ (see Definition IX.4.1 of [20]). For each $1 \leq p < \infty$, the conditional expectation $\mathcal{E}$ induces a contraction, still denoted by $\mathcal{E}$, from $L^p(M, \tau)$ onto $L^p(M_1, \tau)$ satisfying $\mathcal{E}(AXB) = A\mathcal{E}(X)B$ for all $A, B \in M_1$ and $X \in L^p(M, \tau)$ (see [16]).

**Proposition 6.2.1.** Let $M$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal semifinite tracial weight $\tau$. Suppose $M_1$ is a von Neumann subalgebra of $M$ such that the restriction of $\tau$ to $M_1$ is semifinite. If $X_1, \ldots, X_n$ is an $n$-tuple of elements in $M_1$, then
\[ \mathcal{K}_\Phi(X_1, \ldots, X_n; M_1, \tau) = \mathcal{K}_\Phi(X_1, \ldots, X_n; M, \tau), \quad \forall 1 \leq p < \infty. \]

**Proof.** Let $1 \leq p < \infty$. Let $\mathcal{K}_p(M, \tau) = M \cap L^p(M, \tau)$ be equipped with a $\max\{\|\cdot\|, \|\cdot\|_p\}$-norm. We should note that $\mathcal{K}_0^p(M, \tau) = \mathcal{K}_p(M, \tau)$ in this case.

It is obvious that
\[ \mathcal{K}_p(X_1, \ldots, X_n; M_1, \tau) \geq \mathcal{K}_p(X_1, \ldots, X_n; M, \tau). \]

We need only to show that
\[ \mathcal{K}_p(X_1, \ldots, X_n; M_1, \tau) \leq \mathcal{K}_p(X_1, \ldots, X_n; M, \tau). \]

Let $\epsilon > 0$ be given. By definition, there exists an increasing sequence $\{A_m\}_{m \in \mathbb{N}}$ in $\mathcal{K}_p(M, \tau)$ such that $A_m$ converges to $I$ in strong operator topology and
\[ \max\{\|A_mX_i - X_iA_m\|, \|A_mX_i - X_iA_m\|_p\} < \mathcal{K}_p(X_1, \ldots, X_n; M, \tau) + \epsilon, \]
for all $1 \leq i \leq n$ and $m \in \mathbb{N}$.

As the restriction of $\tau$ to $M_1$ is semifinite, there exist a faithful normal trace-preserving conditional expectation $\mathcal{E}$ from $M$ onto $M_1$ and an induced contraction $\mathcal{E}$ from $L^p(M, \tau)$ onto $L^p(M_1, \tau)$. Therefore, $\{\mathcal{E}(A_m)\}_{m \in \mathbb{N}}$ is an increasing sequence in $M_1$ such that $\{\mathcal{E}(A_m)\}_{m \in \mathbb{N}}$ converges to $I$ in weak operator topology (so, in strong operator topology) and
\[ \mathcal{K}_p(X_1, \ldots, X_n; M_1, \tau) + \epsilon > \max\{\|\mathcal{E}(A_mX_i - X_iA_m)\|, \|\mathcal{E}(A_mX_i - X_iA_m)\|_p\} \]
\[ = \max\{\|\mathcal{E}(A_m)X_i - X_i\mathcal{E}(A_m)\|, \|\mathcal{E}(A_m)X_i - X_i\mathcal{E}(A_m)\|_p\}. \]
Note that $0 \leq \mathcal{E}(A_m) \leq I$ and $\|\mathcal{E}(A_m)\|_p \leq \|A_m\|_p < \infty$. Thus, $\mathcal{E}(A_m) \in \mathcal{K}_p(\mathcal{M}, \tau)^+$. By definition, we have

$$\mathcal{H}_p(X_1, \ldots, X_n; \mathcal{M}_1, \tau) \leq \lim inf_m \max \{\|A_mX_i - X_iA_m\|, \|A_mX_i - X_iA_m\|_p\}$$

$$\leq \mathcal{H}_p(X_1, \ldots, X_n; \mathcal{M}, \tau) + \epsilon.$$ 

As $\epsilon$ is arbitrary, we have

$$\mathcal{H}_p(X_1, \ldots, X_n; \mathcal{M}_1, \tau) \leq \mathcal{H}_p(X_1, \ldots, X_n; \mathcal{M}, \tau).$$

This completes the proof of the proposition. \hfill \Box

**Example 6.2.2.** Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state $\tau_\mathcal{N}$ and let $\mathcal{H}_0$ be an infinite dimensional separable Hilbert space. Then $\mathcal{M} = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0)$ is a semifinite von Neumann algebra with a faithful normal tracial weight $\tau_\mathcal{M} = \tau_\mathcal{N} \otimes Tr$, where $Tr$ is the canonical trace of $\mathcal{B}(\mathcal{H}_0)$. We might further assume that $\mathcal{M}$ acts naturally on the Hilbert space $\mathcal{H} = L^2(\mathcal{N}, \tau_\mathcal{N}) \otimes \mathcal{H}_0$. Obviously, $I_\mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0)$ is a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of $\tau_\mathcal{M}$ on $I_\mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0)$ is semifinite.

Let $n \geq 2$ be a positive integer. By Proposition 4.1 in [22], there exists an $n$-tuple $X_1, \ldots, X_n$ of commuting self-adjoint elements in $\mathcal{B}(\mathcal{H}_0)$ such that

$$k_p(X_1, \ldots, X_n) > 0, \quad \forall 1 \leq p < n,$$

where $k_p(X_1, \ldots, X_n)$ is a constant defined in section 1 of [22]. By Proposition 1.1 in [27] and Definition 6.1.2,

$$\mathcal{H}_p(X_1, \ldots, X_n; \mathcal{B}(\mathcal{H}_0), Tr) = k_p(X_1, \ldots, X_n), \quad \forall 1 \leq p < n.$$ 

By Proposition 6.2.1,

$$\mathcal{H}_p(I_\mathcal{N} \otimes X_1, \ldots, I_\mathcal{N} \otimes X_n; \mathcal{M}, \tau) = \mathcal{H}_p(I_\mathcal{N} \otimes X_1, \ldots, I_\mathcal{N} \otimes X_n; I_\mathcal{N} \otimes \mathcal{B}(\mathcal{H}_0), 1 \otimes Tr)$$

$$= \mathcal{H}_p(X_1, \ldots, X_n; \mathcal{B}(\mathcal{H}_0), Tr)$$

$$= k_p(X_1, \ldots, X_n)$$

$$> 0, \quad \forall 1 \leq p < n.$$ 

By Lemma 6.1.2, $I_\mathcal{N} \otimes X_1, \ldots, I_\mathcal{N} \otimes X_n$ is a family of commuting self-adjoint elements in $\mathcal{M}$ that are not small perturbations of commuting diagonal operators modulo $\mathcal{M} \cap L^p(\mathcal{M}, \tau)$ for all $1 \leq p < n$.

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