ON THE SIMULTANEOUS DIVISIBILITY OF CLASS NUMBERS OF TRIPLES OF IMAGINARY QUADRATIC FIELDS

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Abstract. Let \( k \geq 1 \) be a cube-free integer with \( k \equiv 1 \pmod{9} \) and \( \gcd(k, 7 \cdot 571) = 1 \). In this paper, we prove the existence of infinitely many triples of imaginary quadratic fields \( \mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}) \) and \( \mathbb{Q}(\sqrt{d+k^2}) \) with \( d \in \mathbb{Z} \) such that the class number of each of them is divisible by 3. This affirmatively answers a weaker version of a conjecture of Iizuka [3].

1. Introduction

For a number field \( K \), we denote the ideal class group of \( K \) by \( \text{Cl}_K \) and the class number by \( h_K \). The divisibility of class numbers of quadratic fields is one of the central themes in algebraic number theory. The first known result in this direction is due to Ankeny and Chowla [1], Nagell [10] and many others.

Theorem 1. (see [1], [10]) Let \( n \geq 2 \) be an integer. Then there exist infinitely many imaginary quadratic fields whose class numbers are divisible by \( n \).

Later, Weinberger [14], Yamamoto [15] and several other mathematicians have proved the analogue of Theorem 1 for real quadratic fields. It is interesting to consider the simultaneous divisibility of class numbers of tuples of quadratic fields. In this paper, we address this problem for triples of imaginary quadratic fields. Before coming to our main theorem, we recall some basic definitions and relevant results in the literature.

We first start with the \( p \)-rank of a finite abelian group, for a prime number \( p \).

Definition 1. Let \( p \) be a prime number and let \( G \) be a finite abelian group, written additively. Then \( G/pG \) is a finite-dimensional vector space over the field \( \mathbb{Z}/p\mathbb{Z} \) and the dimension of \( G/pG \) over \( \mathbb{Z}/p\mathbb{Z} \) is called the \( p \)-rank of \( G \), often denoted by \( \text{rk}_p(G) \).

Since the class group \( \text{Cl}_K \) of a number field \( K \) is a finite abelian group, it makes sense to study the \( p \)-ranks of this group. We note that if \( \text{rk}_p(\text{Cl}_K) \geq 1 \), then \( p \mid h_K \).

For \( p = 3 \), Scholz [12] proved the following “reflection” principle for the 3-ranks of the class groups of quadratic fields.

Theorem 2. [12] Let \( d > 0 \) be a square-free integer. Let \( r \) and \( s \) be the 3-ranks of the class groups of \( \mathbb{Q}(\sqrt{d}) \) and \( \mathbb{Q}(\sqrt{-3d}) \), respectively. Then

\[ r \leq s \leq r + 1. \]

In particular, if 3 divides \( h_{\mathbb{Q}(\sqrt{d})} \), then so is for \( h_{\mathbb{Q}(\sqrt{-3d})} \).

We observe that, Theorem 2 together with the fact that there exist infinitely many real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) for which \( 3 \mid h_{\mathbb{Q}(\sqrt{d})} \), implies the existence of infinitely many pairs...
of real and imaginary quadratic fields whose class numbers are both divisible by 3. It is natural to ask whether one can replace the integer 3 in $\mathbb{Q}(\sqrt{-3d})$ by any other integer. Recently, Komatsu [8] addressed this question and proved the following theorem.

**Theorem 3.** [8] Let $m$ be a non-zero integer. Then there exist infinitely many distinct pairs of quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{md})$, with $d > 0$, such that $3 \mid h_{\mathbb{Q}(\sqrt{d})}$ and $3 \mid h_{\mathbb{Q}(\sqrt{md})}$.

We note that, in Theorem 3, either both $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{md})$ are real or both are imaginary, according as $m > 0$ or $m < 0$. Later, Komatsu improved Theorem 3 to the pairs of imaginary quadratic fields with class numbers divisible by any integer $n \geq 2$. More precisely, he proved the following theorem.

**Theorem 4.** [9] Let $m \geq 2$ and $n \geq 2$ be integers. Then there exist infinitely many distinct pairs of imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{md})$ such that $n \mid h_{\mathbb{Q}(\sqrt{d})}$ and $n \mid h_{\mathbb{Q}(\sqrt{md})}$.

In the light of Theorem 3 and Theorem 4, it is interesting to ask the following question.

**Question 1.** Let $m \geq 1$ and $n \geq 2$ be integers. Do there exist infinitely many pairs of quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{d + m})$ such that $n$ divides both $h_{\mathbb{Q}(\sqrt{d})}$ and $h_{\mathbb{Q}(\sqrt{d + m})}$?

In [5], Iizuka, Konomi and Nakano addressed Question 1 for $n = 3, 5$ and 7. The statement of their result is as follows.

**Theorem 5.** [5] Let $n \in \{3, 5, 7\}$ and let $m_1, m_2, n_1$ and $n_2$ be rational numbers with $m_1m_2 \neq 0$. Then there exist infinitely many pairs of quadratic fields $K_1 = \mathbb{Q}(\sqrt{m_1d + n_1})$ and $K_2 = \mathbb{Q}(\sqrt{m_2d + n_2})$ with $d \in \mathbb{Q}$ such that $n \mid h_{K_1}$ and $n \mid h_{K_2}$.

Note that in Theorem 5, the hypothesis is $d \in \mathbb{Q}$. If $n_1 = n_2 = 0$, then we can take $d \in \mathbb{Z}$. But if either $n_1 \neq 0$ or $n_2 \neq 0$, then Theorem 5 does not necessarily hold for $d \in \mathbb{Z}$. However, $d$ can be chosen to be an integer for some particular values of $m_1, m_2, n_1, n_2$ and $n$. In [3], Iizuka considered Question 1 for $m = 1$ and $n = 3$ and proved the following.

**Theorem 6.** [3] There exist infinitely many pairs of imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{d + 1})$ with $d \in \mathbb{Z}$ such that $3 \mid h_{\mathbb{Q}(\sqrt{d})}$ and $3 \mid h_{\mathbb{Q}(\sqrt{d + 1})}$.

Motivated from Theorem 6, Iizuka formulated the following conjecture in [3].

**Conjecture 1.** [3] Let $m \geq 1$ be an integer and let $\ell \geq 3$ be a prime number. Then there exists infinitely many tuples $\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d + 1}), \ldots, \mathbb{Q}(\sqrt{d + m})$ of quadratic fields such that $\ell$ divides the class numbers of all them.

2. **Statement of main theorem**

In this paper, we extend Theorem 6 for triples of imaginary quadratic fields. This addresses a weaker version of Conjecture 1 for $\ell = 3$. To the best of our knowledge, this is the first result in the direction of the simultaneous divisibility of class numbers of quadratic fields, taken three at a time. The precise statement of our main theorem is as follows.
Theorem 7. Let $k \geq 1$ be a cube-free integer such that $k \equiv 1 \pmod{9}$ and $\gcd(k, 7 \cdot 571) = 1$. Then there exist infinitely many triples of imaginary quadratic fields $\mathbb{Q} (\sqrt{d})$, $\mathbb{Q} (\sqrt{d+1})$ and $\mathbb{Q} (\sqrt{d+k^2})$ with $d \in \mathbb{Z}$ such that 3 divides $h_{\mathbb{Q} (\sqrt{d})}$, $h_{\mathbb{Q} (\sqrt{d+1})}$ and $h_{\mathbb{Q} (\sqrt{d+k^2})}$.

3. Preliminaries

We begin with a result due to Siegel regarding the solutions of an equation in $S$-integers. We first define the “$S$-integers” in a number field as follows.

Definition 2. [13] For a number field $K$, let $S$ be a finite set of valuations on $K$, containing all the archimedean valuations. Then

$$R_S = \{ \alpha \in K : v(\alpha) \geq 0 \text{ for all } v \notin S \}$$

is called the set of $S$-integers.

The case $K = \mathbb{Q}$ will be of utmost importance to us. For that, we recall Ostrowski’s theorem for $\mathbb{Q}$ as follows.

Theorem 8. (cf. [6]) A non-archimedean valuation of $\mathbb{Q}$ is equivalent to a $p$-adic valuation for some prime number $p$. An archimedean valuation of $\mathbb{Q}$ is equivalent to the usual absolute value.

Remark 1. For $K = \mathbb{Q}$ and $S = \{ | \cdot | \}$, we see that $R_S = \mathbb{Z}$.

Lemma 1. (Siegel’s theorem, [13]) Let $K$ be a number field and let $S$ be a finite set of valuations on $K$, containing all the archimedean valuations. Let $f(X) \in K[X]$ be a polynomial of degree $d \geq 3$ with distinct roots in the algebraic closure $\overline{K}$ of $K$. Then the equation $y^2 = f(x)$ has only finitely many solutions $x, y \in R_S$.

We recall the definition of an infinite prime in a number field $K$ as follows.

Definition 3. [2] Let $K$ be a number field of degree $d$. An embedding $\sigma : K \to \mathbb{C}$ is called an infinite prime of $K$. The infinite prime $\sigma$ is said to be real, if $\sigma(K) \subseteq \mathbb{R}$. Otherwise, $\sigma$ is said to be complex.

Next, we define the ramification of an infinite prime.

Definition 4. [2] Let $L/K$ be an extension of number fields and let $\sigma$ be an infinite prime of $K$. Then $\sigma$ is said to be ramified in $L$, if $\sigma$ is real but it has an extension to $L$ which is complex. If $\sigma$ is not ramified, then it is said to be unramified in $L$.

Remark 2. For an extension of number fields $L/K$, by Definition [4], it is clear that a complex infinite prime of $K$ is always unramified in $L$. Also, if $L$ is a totally real number field, then any infinite prime of $K$ is unramified in $L$.

The necessary ingredients behind the proof of our main theorem come from the class field theory. For that, we introduce the Hilbert class field of a number field in the following proposition.
Lemma 2. For a given number field $K$, there exists a unique maximal, abelian, unramified extension $H(K)$ of $K$. Moreover, we have the following isomorphism of groups

$$\text{Gal}(H(K)/K) \simeq \text{Cl}_K.$$ 

In particular, $|\text{Gal}(H(K)/K)| = h_K$.

Definition 5. The number field $H(K)$ in Proposition 1 is called the Hilbert class field of the number field $K$.

Remark 3. Let $f(X) \in \mathbb{Z}[X]$ be an irreducible polynomial of degree 3 with discriminant $D(f)$. Assume that $D(f)$ is not a perfect square and let $E$ be the splitting field of $f$ over $\mathbb{Q}$. Then $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{D(f)}) \subseteq E$ with $[\mathbb{Q}(\sqrt[3]{D(f)}) : \mathbb{Q}] = 2$ and $[E : \mathbb{Q}(\sqrt[3]{D(f)})] = 3$. Also, since $E$ is the splitting field of $f$ over $\mathbb{Q}$, the field $E$ is Galois over both $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{D(f)})$. If $E$ is an unramified extension of $\mathbb{Q}(\sqrt[3]{D(f)})$, then by Proposition 1, we have $E \subseteq H(\mathbb{Q}(\sqrt[3]{D(f)}))$ and consequently, $[E : \mathbb{Q}(\sqrt[3]{D(f)})]$ divides $[H(\mathbb{Q}(\sqrt[3]{D(f)})) : \mathbb{Q}(\sqrt[3]{D(f)})]$. In other words, 3 divides $h_{\mathbb{Q}(\sqrt[3]{D(f)})}$.

The next lemma has been stated in [7] without proof. For the sake of completeness, we provide the proof here.

Lemma 2. Let $f(X) \in \mathbb{Z}[X]$ be a cubic irreducible polynomial and let $E$ be the splitting field of $f$ over $\mathbb{Q}$. Assume that $D(f)$ is not a perfect square and let $F = \mathbb{Q}(\sqrt[3]{D(f)})$. For a prime number $p$, let $\mathfrak{p}_F$ be a prime ideal in $\mathcal{O}_F$ lying above $p$. Let $\alpha$ be a root of $f$ and let $K = \mathbb{Q}(\alpha)$. Then $\mathfrak{p}_F$ is ramified in $E$ if and only if $p$ is totally ramified in $K$.

Proof. Let $\mathfrak{p}_E$ be a prime ideal in $\mathcal{O}_E$ lying above $\mathfrak{p}_F$ and let $\mathfrak{p}_K = \mathfrak{p}_E \cap K$.

![Diagram showing ramification of primes.](image)

Suppose that $\mathfrak{p}_E$ is ramified in $E$. That is, the ramification index $e(\mathfrak{p}_E|\mathfrak{p}_F) > 1$. Since $E$ is a Galois extension of $F$, we have

$$3 = e(\mathfrak{p}_E|\mathfrak{p}_F) \cdot f(\mathfrak{p}_E|\mathfrak{p}_F) \cdot g,$$

where $f(\mathfrak{p}_E|\mathfrak{p}_F)$ is the inertial degree and $g$ is the number of prime ideals in $\mathcal{O}_E$ lying above $\mathfrak{p}_F$. Since $e(\mathfrak{p}_E|\mathfrak{p}_F) > 1$, we obtain $e(\mathfrak{p}_E|\mathfrak{p}_F) = 3$. Now, by the multiplicativity of the ramification indices, we have

$$e(\mathfrak{p}_E|p) = e(\mathfrak{p}_E|\mathfrak{p}_F) \cdot e(\mathfrak{p}_F|p) = e(\mathfrak{p}_E|\mathfrak{p}_K) \cdot e(\mathfrak{p}_K|p).$$
This implies that 3 divides \(e(\varphi_E|\varphi_K) \cdot e(\varphi_K|p)\). Since \(e(\varphi_E|\varphi_K) \leq [E : K] = 2\), we see that either 3 divides \(2 \cdot e(\varphi_K|p)\) or 3 divides \(e(\varphi_K|p)\). Thus \(3 \mid e(\varphi_K|p) \leq [K : \mathbb{Q}] = 3\) and consequently, \(e(\varphi_K|p) = 3\). Hence \(p\) is totally ramified in \(K\).

Conversely, assume that \(p\) is totally ramified in \(K\). We want to prove that \(e(\varphi_E|\varphi_F) > 1\). If not, then \(e(\varphi_E|\varphi_F) = 1\) and

\[
(1) \quad e(\varphi_E|p) = e(\varphi_E|\varphi_F) \cdot e(\varphi_F|p) = e(\varphi_F|p) \leq [F : \mathbb{Q}] = 2.
\]

Since \(e(\varphi_K|p) \leq e(\varphi_E|p)\), by (1), we get \(e(\varphi_K|p) \leq 2\). This contradicts the fact that \(p\) is totally ramified in \(K\). This completes the proof of the lemma.

**Lemma 3.** Let \(f(X) \in \mathbb{Z}[X]\) be a cubic irreducible polynomial and let \(E\) be the splitting field of \(f\) over \(\mathbb{Q}\). Assume that \(D(f)\) is not a perfect square and let \(F = \mathbb{Q}(\sqrt{D(f)})\). Then any infinite prime of \(F\) is unramified in \(E\).

**Proof.** Let \(\sigma\) be an infinite prime of \(F\).

Case 1. \(D(f) < 0\). Then \(F\) is an imaginary quadratic field and hence by Remark 2 we have that \(\sigma\) is unramified in \(E\).

Case 2. \(D(f) > 0\). Then \(F\) is a real quadratic field. Since \(f\) is a cubic, irreducible polynomial, it has a real root, say \(a\) and let \(K = \mathbb{Q}(\alpha)\). Since \([F : \mathbb{Q}] = 2\) and \([K : \mathbb{Q}] = 3\), we have \([FK : \mathbb{Q}] = 6\). Since \(FK\) is a subfield of \(E\) and \([E : \mathbb{Q}] = 6 = [FK : \mathbb{Q}]\), we conclude that \(E = FK\). As both \(K\) and \(F\) are subfields of \(\mathbb{R}\), we conclude that \(E \subseteq \mathbb{R}\). Since \(E\) is a finite Galois extension of \(\mathbb{Q}\) with \(E \subseteq \mathbb{R}\), we conclude that \(E\) is totally real. Thus by Remark 2 we get that \(\sigma\) is unramified in \(E\).

Since \(\sigma\) is an arbitrary infinite prime of \(F\), we conclude that any infinite prime of \(F\) is unramified in \(E\). This completes the proof of the lemma.

For a prime number \(p\) and a non-zero integer \(n\), the integer \(v_p(n)\) is defined to be the unique integer \(m\) such that \(p^m \mid n\) but \(p^{m+1} \nmid n\). The next lemma is a consequence of a theorem in [11] and is presented in [7] as follows.

**Lemma 4.** [7] Let \(f(X) = X^3 - aX - b \in \mathbb{Z}[X]\) be an irreducible polynomial over \(\mathbb{Q}\) such that either \(v_p(a) < 2\) or \(v_p(b) < 3\) for every prime number \(p\). Suppose that the discriminant \(D(f)\) of \(f\) is not a perfect square and let \(F = \mathbb{Q}(\sqrt{D(f)})\). Let \(\alpha\) be a root of \(f\) and let \(K = \mathbb{Q}(\alpha)\). Let \(E\) be the splitting field of \(f\) over \(\mathbb{Q}\) and let \(q\) be a prime number. Then the following assertions hold.

1. If \(q \neq 3\), then \(q\) is totally ramified in \(K\) if and only if \(1 \leq v_q(b) \leq v_q(a)\).
2. \(3\) is totally ramified in \(K\) if and only if one of the following conditions holds.
   - \(i\) \(1 \leq v_3(a) \leq v_3(b)\),
   - \(ii\) \(3 \mid a, a \neq 3 \pmod{9}, 3 \nmid b\) and \(b^2 \neq a + 1 \pmod{9}\),
   - \(iii\) \(a \equiv 3 \pmod{9}, 3 \nmid b\) and \(b^2 \neq a + 1 \pmod{27}\).

Using Lemma 2 and Lemma 4, we construct two families of quadratic fields with class numbers divisible by 3 as follows.
Proposition 2. For any non-zero integer $t$ with $t \not\equiv 0 \pmod{3}$, the class number of the quadratic field $\mathbb{Q}(\sqrt{3t(3888t^2 + 108t + 1)})$ is divisible by 3.

Proof. First, we prove that $\mathbb{Q}(\sqrt{3t(3888t^2 + 108t + 1)})$ is indeed a quadratic field. That is, we need to prove that $3t(3888t^2 + 108t + 1)$ is not a perfect square for any integer $t$ with $t \not\equiv 0 \pmod{3}$. If $3t(3888t^2 + 108t + 1) = m^2$ for some integer $m$, then $3 | m$ and hence $9 | m^2$. Consequently, $9 | 3t(3888t^2 + 108t + 1)$ and hence $9 | 3t$. This is a contradiction to the hypothesis that $3 \nmid t$. Therefore, the equation $3t(3888t^2 + 108t + 1) = m^2$ has no integer solutions with $t \not\equiv 0 \pmod{3}$ and thus $\mathbb{Q}(\sqrt{3t(3888t^2 + 108t + 1)})$ is a quadratic field.

Now, consider the polynomial $f(X) = X^3 - 3 \cdot (108t + 1)X - 2 \in \mathbb{Z}[X]$. 

Claim: $f$ is irreducible over $\mathbb{Q}$.

If $f$ is reducible over $\mathbb{Q}$, then it must have a linear factor and hence a rational root, say, $\frac{a}{b}$ with $\gcd(a, b) = 1$. Then $f\left(\frac{a}{b}\right) = 0$ implies that

$$a^3 - 3 \cdot (108t + 1)ab^2 - 2b^3 = 0.$$ 

Therefore, $b | a$ and since $\gcd(a, b) = 1$, we obtain $b = \pm 1$.

Similarly, from equation (2), we have $a | 2b^3$. This, together with the fact $\gcd(a, b) = 1$, implies that $a | 2$. Hence the rational root $\frac{a}{b}$ has 4 possibilities, namely, $\pm 1, \pm 2$.

Case 1. $\frac{a}{b} = -1$. Then $f(-1) = 0$ implies $t = 0$ which contradicts our hypothesis that $t \neq 0$.

Case 2. $\frac{a}{b} = 1$. Then $f(1) = 0$ implies $3 \cdot (108t + 1) = -1$, which is impossible for any $t \in \mathbb{Z}$.

Case 3. $\frac{a}{b} = -2$. Then $f(-2) = 0$ implies $6 \cdot (108t + 1) = 10$ which is impossible for any $t \in \mathbb{Z}$.

Case 4. $\frac{a}{b} = 2$. Then $f(2) = 0$ implies $t = 0$, which is again a contradiction.

Consequently, $f$ is irreducible over $\mathbb{Q}$ and this proves the claim.

Now, the discriminant of the polynomial $f$ is

$$D(f) = -4 \cdot [-3 \cdot (108t + 1)]^3 - 27 \cdot (-2)^2 = 2^2 \cdot 3^5 \cdot t \cdot (3888t^2 + 108t + 1)$$

and let

$$F = \mathbb{Q}(\sqrt{D(f)}) = \mathbb{Q}(\sqrt{3t(3888t^2 + 108t + 1)}).$$

Let $\alpha$ be a root of $f$ and let $K = \mathbb{Q}(\alpha)$. For a prime number $p$, using Lemma 4 we see that $p$ is not totally ramified in $F$. Therefore, using Lemma 2 and Lemma 3 we conclude that the splitting field $E$ of $f$ over $\mathbb{Q}$ is an unramified extension of $F$. Thus by Remark 3 we get that the class number of $\mathbb{Q}(\sqrt{3t(3888t^2 + 108t + 1)})$ is divisible by 3. \hfill \Box

Following the similar line of argument, we provide another family of quadratic fields with class numbers divisible by 3 in the next proposition.

Proposition 3. Let $t \geq 1$ be an integer. Then the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-2916t^2})$ is divisible by 3.
Proof. We first note that $1 - 2916t^3$ is not a perfect square for any integer $t \geq 1$, for otherwise, $1 - 2916t^3 = -m^2$ for some integer $m \geq 1$ implies that $m^2 \equiv -1 \pmod{3}$, which is impossible since $-1$ is not a quadratic residue $\pmod{3}$. Thus $\mathbb{Q}(\sqrt{1 - 2916t^3})$ is indeed a quadratic field.

Consider the polynomial $f(X) = X^3 - 27tX - 1 \in \mathbb{Z}[X]$. Since $f$ does not have any rational root, we conclude that $f$ is irreducible over $\mathbb{Q}$.

Consider the discriminant

$$D(f) = -4 \cdot (-27t)^3 - 27(-1)^2 = 27 \cdot (2916t^3 - 1).$$

Let $F = \mathbb{Q}(\sqrt{D(f)}) = \mathbb{Q}(\sqrt{3 \cdot (2916t^3 - 1)})$. Since $\gcd(3, 2916t^3 - 1) = 1$, the integer $3 \cdot (2916t^3 - 1)$ is not a perfect square. Therefore, $\mathbb{Q}(\sqrt{D(f)})$ is a quadratic field.

Let $\alpha$ be a root of $f$ and let $K = \mathbb{Q}(\alpha)$. Let $p$ be a prime number. Using Lemma 4 we see that $p$ is not totally ramified in $K$ and therefore, by Lemma 2 and Lemma 3 we conclude that the splitting field $E$ of $f$ is an unramified extension of $F$. Thus 3 divides the class number of the real quadratic field $\mathbb{Q}(\sqrt{3 \cdot (2916t^3 - 1)})$. By Theorem 2 we conclude that the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3 \cdot 3 \cdot (2916t^3 - 1)}) = \mathbb{Q}(\sqrt{1 - 2916t^3})$ is also divisible by 3.

□

4. Proof of Theorem 7

Let $k \geq 1$ be a cube-free integer such that $k \equiv 1 \pmod{9}$ and $\gcd(k, 7 \cdot 571) = 1$. For an integer $t \geq 1$, consider the polynomial $f_t(X) = X^3 - 27tX - k \in \mathbb{Z}[X]$. If $f_t$ is reducible over $\mathbb{Q}$ for some $t$, then it must have an integer root $n_0$ such that $n_0 \mid k$. Since $n_0$ has finitely many choices, there can be at most finitely many values of $t$ for which $f_t$ is reducible over $\mathbb{Q}$. Let $t_0$ be an integer such that $f_t$ is irreducible for all integers $t > |t_0|$.

Now, consider the discriminant $D(f_t) = 27 \cdot (2916t^3 - k^2)$ which is a polynomial in $t$. Since $k \neq 0$, $D(f_t)$ has distinct roots in $\mathbb{Q}$. Therefore, by Lemma 4 $D(f_t)$ is a perfect square only for finitely many integers $t$. Let $t'_0$ be an integer such that $D(f_t)$ is not a perfect square for all integers $t > |t'_0|$. We put $T = \max\{|t_0|, |t'_0|\} + 1$.

Consider the following simultaneous congruences.

\[
\begin{align*}
(3) \quad & \left\{ \begin{array}{l}
 x \equiv 2 \pmod{9}; \\
 x \equiv 1 \pmod{k}.
\end{array} \right.
\end{align*}
\]

Since $\gcd(k, 9) = 1$, by the Chinese remainder theorem, there exists a unique solution $x_0 \pmod{9k}$ of equation (3).

Let

$$\mathcal{N} = \{n \in \mathbb{Z} : n \equiv x_0 \pmod{9k} \text{ and } n > \max\{T, k\}\}.$$

For $n \in \mathcal{N}$, we have

\[
(4) \quad 27 \cdot n \cdot (3888n^2 + 108n + 1) \equiv 3^3 \cdot 7 \cdot 571 \not\equiv 0 \pmod{k}.
\]

Now, for $n \in \mathcal{N}$, let $t_n = n \cdot (3888n^2 + 108n + 1)$ and we consider the polynomial $f_{t_n}(X) = X^3 - 27 \cdot n \cdot (3888n^2 + 108n + 1)X - k$ over $\mathbb{Q}$. Since $t_n > T$, it follows that $f_{t_n}$ is irreducible over $\mathbb{Q}$ and $D(f_{t_n})$ is not a perfect square. Now, using (4), Lemma 2 Lemma
and Lemma 4 we conclude that the splitting field \( E \) of \( f_{3n} \) over \( \mathbb{Q} \) is an unramified extension over \( \mathbb{Q}(\sqrt{D(f_{3n})}) \). Therefore, 3 divides the class number of \( \mathbb{Q}(\sqrt{D(f_{3n})}) \).

Note that, \( \mathbb{Q}(\sqrt{D(f_{3n})}) = \mathbb{Q}(\sqrt{27 \cdot (2916t_n^2 - k^2)}) = \mathbb{Q}(\sqrt{3 \cdot (2916t_n^3 - k^2)}) \) is a real quadratic field. Consequently, Theorem 2 yields that 3 divides the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-3 \cdot 3 \cdot (2916t_n^3 - k^2)}) = \mathbb{Q}(\sqrt{k^2 - 2916t_n^3}) \).

Also, Proposition 3 implies that 3 divides the class number of the real quadratic field \( \mathbb{Q}(\sqrt{3t_n}) \). Again from Theorem 2 we obtain that 3 divides the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-3 \cdot 3t_n}) = \mathbb{Q}(\sqrt{-t_n}) \). Using the fact that \( \mathbb{Q}(\sqrt{-2916t_n^3}) = \mathbb{Q}(\sqrt{-t_n}) \) and Proposition 3 we conclude that 3 divides the class numbers of \( \mathbb{Q}(\sqrt{-2916t_n^3}) \), \( \mathbb{Q}(\sqrt{-2916t_n^3 + 1}) \) and \( \mathbb{Q}(\sqrt{-2916t_n^3 + k^2}) \).

Finally, it only remains to prove that the family

\[
\mathcal{Q} = \{ \mathbb{Q}(\sqrt{-t_n}) : n \in \mathcal{N} \}
\]

is infinite. We prove this using the following standard argument using the ramifications of primes in quadratic fields.

If possible, suppose that \( \mathcal{Q} \) is a finite set and let \( D \) be the product of the discriminants of the quadratic fields in the finite set \( \mathcal{Q} \). Hence a prime number \( \ell \) is ramified in some \( \mathbb{Q}(\sqrt{-t_n}) \in \mathcal{Q} \) if and only if \( \ell | D \). Using Dirichlet’s theorem for primes in arithmetic progressions, there exist infinitely many prime numbers \( p \equiv x_0 \pmod{9k} \). In other words, there are infinitely many primes in \( \mathcal{N} \). We choose a prime number \( q \in \mathcal{N} \) such that \( q | D \).

Then \( q \) is unramified in \( \mathbb{Q}(\sqrt{-t_n}) \) for every \( n \in \mathcal{N} \), which is a contradiction to the fact that \( q \) is ramified in \( \mathbb{Q}(\sqrt{-q \cdot (3888q^2 + 108q + 1)}) = \mathbb{Q}(\sqrt{-t_q}) \in \mathcal{Q} \). Therefore, the family \( \mathcal{Q} \) is infinite and this completes the proof of Theorem 7. \( \square \)

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