A generalized Weierstrass representation of Lorentzian surfaces in $\mathbb{R}^{2,2}$ and applications

VICTOR PATTY

Instituto de Física y Matemáticas, U.M.S.N.H., 58000, Morelia, Michoacán, México
E-mail address: victorp@ifm.umich.mx

Abstract

We give a generalized Weierstrass formula for a Lorentz surface conformally immersed in the four-dimensional space $\mathbb{R}^{2,2}$ using spinors and Lorentz numbers. We also study the immersions of a Lorentzian surface in the Anti-de Sitter space (a pseudo-sphere in $\mathbb{R}^{2,2}$): we give a new spinor representation formula and deduce the conformal description of a flat Lorentzian surface in that space.

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1 Introduction and results

Let $\mathbb{R}^{2,2}$ be the space $\mathbb{R}^4$ endowed with the metric of signature $(2,2)$

$$\langle \cdot, \cdot \rangle := -dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2.$$  

A surface $M \subset \mathbb{R}^{2,2}$ is said to be Lorentzian if the metric $\langle \cdot, \cdot \rangle$ induces on $M$ a Lorentzian metric, i.e. a metric of signature $(1,1)$; if we consider the conformal class of the Lorentzian metric, we obtain a Lorentz surface, that is a surface which can be parameterized by open subsets $U \subset A$, where

$$A := \{u + \sigma v \mid u, v \in \mathbb{R}, \sigma \notin \mathbb{R}, \sigma^2 = 1\}$$

is the real algebra of the Lorentz numbers (see details in Appendix A). This parameterization is analogous to the parameterization of Riemann surfaces by complex numbers. In a conformal parameter $a := u + \sigma v : U \subset A \to M$, we define $\widehat{a} := u - \sigma v$,

$$\partial_a := \frac{1}{2} (\partial_u + \sigma \partial_v) \quad \text{and} \quad \partial_\widehat{a} := \frac{1}{2} (\partial_u - \sigma \partial_v),$$

with dual 1-forms $da := du + \sigma dv$ and $d\widehat{a} := du - \sigma dv$. We also define the real and imaginary parts of $a = u + \sigma v \in A$ by

$$\Re(a) := \frac{a + \widehat{a}}{2} = u \quad \text{and} \quad \Im(a) := \frac{a - \widehat{a}}{2} = \sigma v,$$

and we write $|a|^2 := a\widehat{a}$, for all $a \in A$.

The first result of this paper is a generalized Weierstrass formula for a Lorentz surface conformally immersed into $\mathbb{R}^{2,2}$. This result extends to the Lorentzian case the generalized Weierstrass formula for a Riemannian surface in four-dimensional spaces given by B.G. Konopelchenko in [10, 11].
Theorem 1. We consider \( \phi_1, \phi_2, \psi_1, \psi_2 : A \rightarrow A \) and \( p, q : A \rightarrow \mathbb{R} \) smooth functions such that
\[
\partial_a \phi_\alpha = -p\psi_\alpha, \quad \alpha = 1, 2, \\
\partial_\hat{a} \psi_\alpha = -q\phi_\alpha, \quad \alpha = 1, 2,
\]
and \( |\psi_2 \phi_1 - \psi_1 \phi_2|^2 \neq 0 \). The following formulas
\[
F_0 + F_1 = -\int_\gamma \left( \phi_1 \hat{\psi}_1 da + \hat{\phi}_1 \phi_1 d\hat{a} \right), \\
F_0 - F_1 = \int_\gamma \left( \phi_2 \hat{\psi}_2 da + \hat{\phi}_2 \phi_2 d\hat{a} \right), \\
F_2 + \sigma F_3 = \int_\gamma \left( \psi_1 \hat{\phi}_2 da + \hat{\psi}_2 \phi_1 d\hat{a} \right), \\
F_2 - \sigma F_3 = \int_\gamma \left( \psi_2 \hat{\phi}_1 da + \hat{\psi}_1 \phi_2 d\hat{a} \right),
\]
where \( \gamma \) is an arbitrary path between a fixed point and a variable point in \( A \), define the conformal immersion of a Lorentz surface into \( \mathbb{R}^{2,2} \)
\[
F = (F_0, F_1, F_2, F_3) : A \rightarrow \mathbb{R}^{2,2},
\]
The induced metric of the Lorentz surface is of the form
\[
g = -|\psi_2 \phi_1 - \psi_1 \phi_2|^2 d\hat{a} d\hat{a},
\]
and the Lorentzian norm of the mean curvature vector of the immersion is
\[
|\vec{H}|^2 = \frac{pq}{|\psi_2 \phi_1 - \psi_1 \phi_2|^2}.
\]
Conversely, the isometric immersion of a simply-connected Lorentzian surface \((M, g)\) into \( \mathbb{R}^{2,2} \) may be described in that way.

We note that Equations (1) imply that
\[
\partial_a (\hat{\psi}_\alpha \phi_\alpha) = \partial_\hat{a} (\psi_\alpha \hat{\phi}_\alpha), \quad \alpha = 1, 2,
\]
and the formulas in (2) do not depend on the path \( \gamma \): the coordinates \( F_k : A \rightarrow \mathbb{R}^{2,2} \) \((k = 0, 1, 2, 3)\) are uniquely defined up to constants. This result essentially relies on a spinor representation theorem of a Lorentzian surface in \( \mathbb{R}^{2,2} \): an isometric immersion of a Lorentz surface in \( \mathbb{R}^{2,2} \) with mean curvature vector \( \vec{H} \in E \) (\( E \) is the normal bundle on \( M \)) is equivalent to a normalized spinor field \( \varphi \in \Gamma(S \Sigma E \otimes \Sigma M) \) solution of the Dirac equation \( D \varphi = \vec{H} \cdot \varphi \) (see Section 3). In this context, the maps \( \phi_1, \phi_2, \psi_1 \) and \( \psi_2 \) appear to be the components of \( \varphi \) in a convenient spinorial frame, and the Dirac equation is equivalent to (1).

As a consequence of Theorem 1 we obtain a Weierstrass representation of a minimal Lorentzian surface in \( \mathbb{R}^{2,2} \) which extends the classical Weierstrass representation of a minimal Lorentz surface in \( \mathbb{R}^{2,1} \) given by J. Konderak in [9]. A representation of a minimal Lorentzian surface was also given by M.P. Dussan and M. Magid in [4]. Here we make explicit the dependence of the representation on the components of a spinor field, and the fact that this representation is a special case of a general spinor representation formula (Theorem 1).
Corollary 1. Let $\psi_1, \psi_2, \hat{\phi}_1, \hat{\phi}_2 : A \to A$ be conformal maps. The formulas

\[
\begin{align*}
F_0 &= \Re \left[ \int_{\Gamma} \left( -\psi_1 \hat{\phi}_1 + \psi_2 \hat{\phi}_2 \right) \, da \right] \\
F_1 &= \Re \left[ \int_{\Gamma} \left( -\psi_1 \hat{\phi}_1 - \psi_2 \hat{\phi}_2 \right) \, da \right] \\
F_2 &= \Re \left[ \int_{\Gamma} \left( \psi_2 \hat{\phi}_1 + \psi_1 \hat{\phi}_2 \right) \, da \right] \\
F_3 &= \Im \left[ \int_{\Gamma} \left( -\psi_2 \hat{\phi}_1 + \psi_1 \hat{\phi}_2 \right) \, da \right],
\end{align*}
\]

define a conformal minimal immersion of a Lorentz surface into $\mathbb{R}^{2,2}$.

Conversely, the isometric immersion of a minimal simply-connected Lorentzian surface $(M, g)$ into $\mathbb{R}^{2,2}$ may be described in that way.

We also study the isometric immersions of a Lorentzian surface in the pseudo-spheres of $\mathbb{R}^{2,2}$: we consider the three-dimensional Anti-de Sitter space

\[
\mathbb{H}^{2,1} := \{ x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1 \}
\]

(also called the three-dimensional pseudo-hyperbolic space), of constant negative curvature $-1$, and the three-dimensional pseudo-sphere with index 2

\[
\mathbb{S}^{1,2} := \{ x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = 1 \}
\]

of constant positive curvature 1. We obtain a new spinor representation of a Lorentzian surface in $\mathbb{H}^{2,1}$ and in $\mathbb{S}^{1,2}$: an isometric immersion of a Lorentzian surface in $\mathbb{H}^{2,1}$ or in $\mathbb{S}^{1,2}$ with mean curvature $H$ is equivalent to a normalized spinor field $\psi \in \Gamma(\Sigma M)$ solution of the Dirac equation

\[
D_M \psi = H \psi + \bar{\psi} \quad \text{or} \quad D_M \psi = -i H \psi + i \bar{\psi}
\]

respectively. These spinor representations are different to the characterizations obtained by M.A. Lawn and J. Roth in [12], where two spinor fields are needed. As a consequence of this spinor representation we obtain a correspondence between a minimal Lorentzian surface in the pseudo-spheres of $\mathbb{R}^{2,2}$ and a Lorentzian surface with constant mean curvature one in the three-dimensional Minkowski spaces of $\mathbb{R}^{2,2}$. This transformation is a generalization of a classical transformation for surfaces in $\mathbb{S}^3$, described by H.B. Lawson in [13].

We then deduce the structure of a flat Lorentzian surface in the pseudo spheres of $\mathbb{R}^{2,2}$. If $M_2(A)$ stands for the set of the $2 \times 2$ matrices with entries belonging to $A$, we can define the usual determinant on $M_2(A)$ and set

\[
Sl_2(A) := \{ B \in M_2(A) \mid \det B = 1 \}.
\]

We define the set of the $2 \times 2$ Hermitian matrices with coefficients in $A$ by

\[
\text{Herm}_2(A) := \{ B \in M_2(A) \mid B = B^* \},
\]

where $B^*$ is the conjugate transpose of $B$ (see Section 6 for details). There exists an identification

\[
\mathbb{R}^{2,2} \simeq (\text{Herm}_2(A), -\det),
\]

such that the pseudo-spheres are described by

\[
\mathbb{H}^{2,1} \simeq \{ BB^* \mid B \in Sl_2(A) \} \quad \text{and} \quad \mathbb{S}^{1,2} \simeq \left\{ B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B^* \mid B \in Sl_2(A) \right\}.
\]

The result is the following:

\footnote{See Appendix A for the definition of conformal maps and 1–forms on Lorentz surfaces.}
Theorem 2. Let \( M \) be an oriented simply-connected Lorentz surface, \( B : M \rightarrow SL_2(A) \) be a conformal immersion such that there exist \( \theta, \omega \) conformal 1-forms that satisfy
\[
B^{-1}dB = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}.
\]
Assume that \(|\theta|^2 \neq |\omega|^2\) (resp. \(|\theta|^2 \neq -|\omega|^2\)). Then
\[
F = BB^* : M \rightarrow \mathbb{H}^{2,1} \quad \text{(resp. } F = B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B^* : M \rightarrow \mathbb{S}^{1,2})
\]
defines with the induced metric, a flat isometric immersion.

Conversely, an isometric immersion of a simply-connected flat Lorentzian surface \((M, g)\) in \( \mathbb{H}^{2,1} \) (resp. in \( \mathbb{S}^{1,2} \)) may be described as above.

This conformal description extends to the Lorentzian case the main result of [6] concerning surfaces in the three-dimensional hyperbolic space. As a consequence of this theorem we show that a flat Lorentzian surface in Anti-de Sitter space is locally the product of two curves in \( SL_2(\mathbb{R}) \) (see Section 6.2).

We quote the following related papers. A direct extension of the Weierstrass representation to generic nonminimal surfaces in \( \mathbb{R}^3 \) have been given by K. Kenmotsu in [8]; a different (but equivalent) extension was proposed by B.G. Konopelchenko using two complex functions and one real function satisfying a linear system similar to (1) (see [11, Section 1], and the references therein). Following this approach, a generalized formula for surfaces in \( \mathbb{R}^{4-r} \) \((r = 0, 1, 2)\) was described by B.G. Konopelchenko in [11] (see also [10]). Finally, a conformal description of a surface in the three-dimensional hyperbolic space was given by J.A. Gálvez, A. Martínez y F. Milán in [6, 7], and by P. Bayard using spinors in dimension 4 in [1, Theorem 5].

The outline of the paper is as follows: in Section 2 we describe the Clifford algebra of \( \mathbb{R}^{2,2} \) and its spinor representation using quaternions and Lorentz numbers, in Section 3 we recall the spinor representation of a Lorentzian surface in \( \mathbb{R}^{2,2} \) ([2]). In Section 4 we prove the generalized Weierstrass formula (Theorem 1); we also deduce a generalized formula for a Lorentzian surface in the three-dimensional Minkowski space \( \mathbb{R}^{2,1} \), analogous to the case of surfaces in \( \mathbb{R}^3 \). In Section 5 we deduce the spinor representation of a Lorentzian surface in the pseudo-spheres of \( \mathbb{R}^{2,2} \), and finally we obtain a conformal description of a flat Lorentzian surface in the pseudo-spheres (Theorem 2) in Section 6. An appendix on Lorentz structures ends the paper.

2 Clifford algebra of \( \mathbb{R}^{2,2} \), spinorial group and their representation

We recall here the main results concerning the Clifford algebras and spinors of \( \mathbb{R}^{2,2} \) using Lorentz numbers and quaternions. Details may be found in [2].

We consider the complexified Lorentz numbers
\[
\mathcal{A}_C := A \otimes \mathbb{C} \simeq \{ u +\sigma v : \ u, v \in \mathbb{C} \},
\]
and the quaternions with coefficients in \( \mathcal{A}_C \)
\[
\mathbb{H}^{\mathcal{A}_C} := \{ \zeta_0 I + \zeta_1 I + \zeta_2 J + \zeta_3 K : \ \zeta_0, \zeta_1, \zeta_2, \zeta_3 \in \mathcal{A}_C \},
\]
where \( I, J \) and \( K \) are such that
\[
I^2 = J^2 = K^2 = -I, \quad IJ = -JI = K.
\]
If \( \zeta = \zeta_0 I + \zeta_1 I + \zeta_2 J + \zeta_3 K \) belongs to \( \mathbb{H}^{\mathcal{A}_C} \), we define its conjugate by
\[
\overline{\zeta} := \zeta_0 I - \zeta_1 I - \zeta_2 J - \zeta_3 K,
\]
and writing \( \hat{a} := u - \sigma v \) for \( a = u + \sigma v \in \mathcal{A}_C \), we set
\[
\hat{\zeta} := \hat{\zeta}_0 I + \hat{\zeta}_1 I + \hat{\zeta}_2 J + \hat{\zeta}_3 K.
\]
2.1 Clifford map and spin representation

If $\mathbb{H}^{4c}(2)$ stands for the set of the $2 \times 2$ matrices with entries belonging to $\mathbb{H}^{4c}$, the map

$$\gamma : \mathbb{R}^{2,2} \rightarrow \mathbb{H}^{4c}(2)$$

$$(x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} 0 & \sigma i x_0 I + x_1 I + i x_2 J + x_3 K \\ -\sigma i x_0 I + x_1 I + i x_2 J + x_3 K \end{pmatrix}$$

is a Clifford map, that is satisfies

$$\gamma(x)^2 = -(x, x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $x \in \mathbb{R}^{2,2}$, and thus identifies

$$\text{Cl}(2, 2) \simeq \{ (p q \hat{q}) : p \in \mathbb{H}_0, q \in \mathbb{H}_1 \}, \quad (9)$$

where

$$\mathbb{H}_0 := \{ p_0 I + i p_1 I + p_2 J + ip_3 K : p_0, p_1, p_2, p_3 \in \mathcal{A} \}$$

and

$$\mathbb{H}_1 := \{ iq_0 I + q_1 I + iq_2 J + q_3 K : q_0, q_1, q_2, q_3 \in \mathcal{A} \}.$$ 

Using (9), the sub-algebra of elements of even degree is

$$\text{Cl}_0(2, 2) \simeq \left\{ \begin{pmatrix} p & 0 \\ 0 & \hat{p} \end{pmatrix} : p \in \mathbb{H}_0 \right\} \simeq \mathbb{H}_0$$

and the set of elements of odd degree is

$$\text{Cl}_1(2, 2) \simeq \left\{ \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} : q \in \mathbb{H}_1 \right\} \simeq \mathbb{H}_1.$$ 

Let us consider the map

$$H : \mathbb{H}^{4c} \times \mathbb{H}^{4c} \rightarrow \mathcal{A}_c$$

$$(\zeta, \zeta') \mapsto \frac{1}{2} (\zeta \overline{\zeta'} + \zeta' \overline{\zeta}) = \zeta_0 \zeta'_0 + \zeta_1 \zeta'_1 + \zeta_2 \zeta'_2 + \zeta_3 \zeta'_3$$

where $\zeta = \zeta_0 I + \zeta_1 I + \zeta_2 J + \zeta_3 K$ and $\zeta' = \zeta'_0 I + \zeta'_1 I + \zeta'_2 J + \zeta'_3 K$. It is $\mathcal{A}_c$-bilinear and symmetric.

The restriction of this map to $\mathbb{H}_0$ permits us to define the spin group

$$\text{Spin}(2, 2) := \{ p \in \mathbb{H}_0 : H(p, p) = p_0^2 - p_1^2 + p_2^2 - p_3^2 = 1 \} \subset \text{Cl}_0(2, 2).$$

Now, if we consider the identification

$$\mathbb{R}^{2,2} \simeq \{ \sigma i x_0 I + x_1 I + i x_2 J + x_3 K : x_0, x_1, x_2, x_3 \in \mathbb{R} \} \simeq \{ q \in \mathbb{H}_1 : q = -\hat{q} \},$$

we get the double cover

$$\Phi : \text{Spin}(2, 2) \rightarrow \text{SO}(2, 2)$$

$$p \mapsto (q \in \mathbb{R}^{2,2} \mapsto pq\hat{p}^{-1} \in \mathbb{R}^{2,2}).$$

Here and below $\text{SO}(2, 2)$ stands for the component of the identity of the orthogonal group $O(2, 2)$ (see [17]).
In this section, we recall the principal theorem concerning the spin or representation of a Lorentzian

2.2 Spinors under the splitting

We now consider the splitting

\[ \Sigma^+ \]

the volume element

\[ g \]

where \( \Sigma \) represented as follows:

\[ \rho: Cl(2, 2) \rightarrow End(\mathbb{H}_0) \]

so that the spinorial representation of \( Spin(2, 2) \) simply reads

\[ \rho|_{Spin(2, 2)}: Spin(2, 2) \rightarrow End_{\mathbb{C}}(\mathbb{H}_0) \]

\[ p \rightarrow (\xi \in \mathbb{H}_0 \mapsto p\xi \in \mathbb{H}_0). \]

Since \( \rho(\sigma I)^2 = id_{\mathbb{H}_0} \), this representation splits into

\[ \mathbb{H}_0 = \Sigma^+ \oplus \Sigma^- , \]

where \( \Sigma^+: \{ \xi \in \mathbb{H}_0 : \sigma\xi = \xi \} \) and \( \Sigma^- : \{ \xi \in \mathbb{H}_0 : \sigma\xi = -\xi \} \). Note that \( \sigma I \in \mathbb{H}_0 \) represents the volume element \( e_0 \cdot e_1 \cdot e_2 \cdot e_3 \), which thus acts as \( +id \) on \( \Sigma^+ \) and as \( -id \) on \( \Sigma^- \).

2.2 Spinors under the splitting \( \mathbb{R}^{2,2} = \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \)

We now consider the splitting \( \mathbb{R}^{2,2} = \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \) and the corresponding natural inclusion

\[ SO(1, 1) \times SO(1, 1) \subset SO(2, 2). \]

Using the definition (10) of \( \Phi \), it is easy to get

\[ \Phi^{-1}(SO(1, 1) \times SO(1, 1)) = \{ \pm (\cosh(a) + i \sinh(a)I) : a \in A \} =: S^1_A \subset Spin(2, 2), \]

where, for all \( a = \frac{1+\sigma}{2}(u+v) + \frac{1-\sigma}{2}(u-v) \in A \), the \( A \)-valued hyperbolic sin and cosin functions are such that

\[ \cosh(a) = \frac{1+\sigma}{2} \cosh(u+v) + \frac{1-\sigma}{2} \cosh(u-v) \quad \text{and} \quad \sinh(a) = \frac{1+\sigma}{2} \sinh(u+v) + \frac{1-\sigma}{2} \sinh(u-v). \]

The transformation \( \Phi(\pm (\cosh(a) + i \sinh(a)I)) \) of \( \mathbb{R}^{2,2} \) consists of a Lorentz rotation of angle \( -2v \) in the first factor \( \mathbb{R}^{1,1} \) and of angle \( -2u \) in the second factor \( \mathbb{R}^{1,1} \). We thus have the double cover

\[ \Phi : S^1_A \rightarrow SO(1, 1) \times SO(1, 1); \]

we moreover have an isomorphism

\[ S^1_A \simeq Spin'(1, 1) \times_{\mathbb{Z}_2} Spin''(1, 1), \]

where \( Spin'(1, 1) \) and \( Spin''(1, 1) \) are two copies of the group \( Spin(1, 1) \) (for details see [2]).

Finally, let \( \rho_1 \) and \( \rho_2 \) be the spinorial representations of \( Spin'(1, 1) \) and \( Spin''(1, 1) \) respectively, the representation

\[ Spin'(1, 1) \times Spin''(1, 1) \rightarrow End_{\mathbb{C}}(\mathbb{H}_0) \]

\[ (g_1, g_2) \mapsto \rho(g) : \xi \mapsto g\xi, \]

where \( g = g_1 g_2 \in S^1_A \), is equivalent to the representation \( \rho_1 \otimes \rho_2 \).

3 Previous work on the spinor representation of a Lorentzian surface in \( \mathbb{R}^{2,2} \)

In this section, we recall the principal theorem concerning the spinor representation of a Lorentzian surface immersed in \( \mathbb{R}^{2,2} \), and some fundamental results of the paper [2].
3.1 Twisted spinor bundle

Let $(M, g)$ be a Lorentzian surface and $E$ a bundle of rank 2 on $M$, equipped with a fibre Lorentzian metric and a compatible connection; we assume that $M$ and $E$ are oriented (in space and in time), with given spin structures. We set $\Sigma := \Sigma E \otimes \Sigma M$, the tensor product of the spinor bundles $\Sigma E$ and $\Sigma M$ constructed from $E$ and $TM$. If we denote by $Q_E$ and $Q_M$ the $SO(1, 1)$ principal bundles of the oriented and orthonormal frames of $E$ and $TM$, by $\tilde{Q}_E \to Q_E$ and $\tilde{Q}_M \to Q_M$ the given spin structures on $E$ and $TM$, and by $p_E : \tilde{Q}_E \to M$ and $p_M : \tilde{Q}_M \to M$ the natural projections, we define the principal bundle over $M$

$$\tilde{Q} := \tilde{Q}_E \times_M \tilde{Q}_M = \{(\tilde{s}_1, \tilde{s}_2) \in \tilde{Q}_E \times \tilde{Q}_M : p_E(\tilde{s}_1) = p_M(\tilde{s}_2)\}.$$

Since the representation $\rho$ introduced in the previous section is equivalent to the representation $\rho_1 \otimes \rho_2$ of the structure group $Spin'(1, 1) \times Spin''(1, 1)$, the bundle $\Sigma$ is the vector bundle associated to $\tilde{Q}$ and to the representation $\rho$, that is

$$\Sigma = \tilde{Q} \times \mathbb{H}_0/\rho.$$

Since the $\mathcal{A}$-bilinear map $H$ defined on $\mathbb{H}_0$ is $Spin(2, 2)$--invariant, $\Sigma$ is also equipped with a $\mathcal{A}$-bilinear map $H$. We may also define a $\mathbb{H}_1$-valued scalar product on $\Sigma$ by

$$\langle \langle \varphi, \varphi' \rangle \rangle := \sigma i \overline{\varphi} \xi,$$

(11)

where $\xi$ and $\xi'$ are $\mathbb{H}_0$ are respectively the components of $\varphi$ and $\varphi'$ in some local section of $\tilde{Q}$. This scalar product is $\mathcal{A}$-bilinear, and satisfies the following properties: for all $\varphi, \varphi' \in \Sigma$ and for all $X \in E \oplus TM$

$$\langle \langle \varphi, \varphi' \rangle \rangle = \overline{\langle \langle \varphi', \varphi \rangle \rangle} \quad \text{and} \quad \langle \langle X \cdot \varphi, \varphi' \rangle \rangle = -\langle \langle \varphi, X \cdot \varphi' \rangle \rangle.$$

(12)

**Notation.** We will use the next notation: if $\tilde{s} \in \tilde{Q}$ is a given spinorial frame, the brackets $[\cdot]$ will denote the coordinates in $\mathbb{H}_0$ of the spinor fields in the frame $\tilde{s}$, that is, for $\varphi \in \Sigma$,

$$\varphi \simeq [\tilde{s}, [\varphi]] \in \Sigma \simeq \tilde{Q} \times \mathbb{H}_0/\rho.$$

We will use the brackets to denote the coordinates in $\tilde{s}$ of the elements of the Clifford algebra $Cl(E \oplus TM) : X \in Cl_0(E \oplus TM)$ and $Y \in Cl_1(E \oplus TM)$ will be respectively represented by $[X] \in \mathbb{H}_0$ and $[Y] \in \mathbb{H}_1$ such that, in $\tilde{s}$,

$$X \simeq \begin{pmatrix} [X] & 0 \\ 0 & [\bar{X}] \end{pmatrix} \quad \text{and} \quad Y \simeq \begin{pmatrix} 0 & [Y] \\ [\bar{Y}] & 0 \end{pmatrix}.$$  

Note that

$$[X \cdot \varphi] = [X][\varphi] \quad \text{and} \quad [Y \cdot \varphi] = \sigma i [Y][\varphi]$$

and that, in a spinorial frame $\tilde{s} \in \tilde{Q}$ such that $\pi(\tilde{s}) = (e_0, e_1, e_2, e_3)$, where $\pi : \tilde{Q} \to Q_1 \times_M Q_2$ in the natural projection onto the bundle of the orthonormal frames of $E \oplus TM$ adapted to the splitting, $e_0, e_1, e_2$ and $e_3 \in Cl_1(E \oplus TM)$ are respectively represented by $\sigma i 1, i I, i J$ and $K \in \mathbb{H}_1$ (recall the Clifford map $\gamma$ at the beginning of Section 2.1).

3.2 Spinor representation of a Lorentzian surface in $\mathbb{R}^{2,2}$

We keep the notation of the previous section, and recall that $\Sigma = \Sigma E \otimes \Sigma M$ is equipped with a natural connection

$$\nabla := \nabla^{\Sigma E} \otimes id_{\Sigma M} + id_{\Sigma E} \otimes \nabla^{\Sigma M},$$

the tensor product of the spinor connections on $\Sigma E$ and on $\Sigma M$, and also with a natural action of the Clifford bundle

$$Cl(E \oplus TM) \simeq Cl(E) \otimes Cl(M);$$
see [2]. This permits to define the Dirac operator acting on \( \Gamma(\Sigma) \) by

\[
D\varphi := -e_2 \cdot \nabla e_2 \varphi + e_3 \cdot \nabla e_3 \varphi
\]

where \((e_2, e_3)\) is an orthogonal basis tangent to \( M \) such that \(|e_2|^2 = -1\) and \(|e_3|^2 = 1\). We have the following theorem:

**Theorem 3.** [2, Theorem 1] Suppose that \((M, g)\) is moreover simply connected, and let \( \vec{H} \) be a section of \( E \). The following statements are equivalent.

1. There is a spinor field \( \varphi \in \Gamma(\Sigma) \) with \( H(\varphi, \varphi) = 1 \), solution of the Dirac equation

\[
D\varphi = \vec{H} \cdot \varphi.
\]

2. There is a spinor field \( \varphi \in \Gamma(\Sigma) \) with \( H(\varphi, \varphi) = 1 \), solution of

\[
\nabla_X \varphi = -\frac{1}{2} \sum_{j=2}^{3} \epsilon_j e_j \cdot B(X, e_j) \cdot \varphi
\]

where \( \epsilon_j = g(e_j, e_j) \) and \( B : TM \times TM \to E \) is bilinear symmetric with \( \frac{1}{2} tr_g B = \vec{H} \).

3. There is an isometric immersion \( F : M \to \mathbb{R}^{2,2} \) with normal bundle \( E \) and mean curvature vector \( \vec{H} \).

Moreover, \( F = \int \xi \), where \( \xi \) is the closed 1-form on \( M \) with values in \( \mathbb{R}^{2,2} \) defined by

\[
\xi(X) := \langle (X \cdot \varphi, \varphi) \rangle \quad \in \quad \mathbb{R}^{2,2} \subset \mathbb{H}_1
\]

for all \( X \in TM \).

**Remark 1.** The map \( X \in E \mapsto \langle (X \cdot \varphi, \varphi) \rangle \in \mathbb{R}^{2,2} \) identifies \( E \) with the normal bundle of the immersion; it preserves the metrics, the connections and the second fundamental forms; see [2, Theorem 2].

Applications of this spinor representation formula in Sections 4, 5 and 6 will rely on the following simple observation: assume that \( F_0 : M \to \mathbb{R}^{2,2} \) is an isometric immersion and consider \( \varphi = \sigma I|_{M} \) the restriction to \( M \) of the constant spinor field \( \sigma I \) of \( \mathbb{R}^{2,2} \); if

\[
F = \int \xi, \quad \xi(X) = \langle (X \cdot \varphi, \varphi) \rangle
\]

is the immersion given in the theorem, then \( F \simeq F_0 \). This is in fact trivial since

\[
\xi(X) = \langle (X \cdot \varphi, \varphi) \rangle = -[\varphi][X][\bar{\varphi}] = [X] \simeq X
\]

in a spinorial frame \( \bar{s} \) of \( \mathbb{R}^{2,2} \) which is above the canonical basis (in such a frame \( [\varphi] = \pm \sigma I \)). The representation formula (15), when written in moving frames adapted to the immersion, will give non trivial formulas.

### 4 Weierstrass representation of a Lorentzian surface in \( \mathbb{R}^{2,2} \)

In this section, we prove the generalized Weierstrass formula for a Lorentz surface conformally immersed into \( \mathbb{R}^{2,2} \) (Theorem 1). As a consequence of this formula, we deduce a generalized formula for a Lorentz surface conformally immersed in the three-dimensional Minkowski space \( \mathbb{R}^{2,1} \), analogous to the case of surfaces in \( \mathbb{R}^3 \) (see [10, Section 2]); in particular, we obtain the classical Weierstrass representation of a minimal Lorentz surface in \( \mathbb{R}^{2,1} \) given by J. Konderak [9].
Before proving Theorem 1 we note the following: the immersion $F : M \rightarrow \mathbb{R}^{2,2}$ of the spinor representation theorem (Theorem 3) is given by

$$F = \int \xi = \left( \int \xi_0, \int \xi_1, \int \xi_2, \int \xi_3 \right).$$

This formula generalizes the Weierstrass representation: we consider $\sigma : TM \rightarrow TM$ the Lorentz structure on $M$ induced by the conformal class of the metric (see Appendix A for the definition), and let $\alpha_0, \alpha_1, \alpha_2, \alpha_3 : TM \rightarrow \mathcal{A}$ be the linear forms defined by

$$\alpha_k(X) := \xi_k(X) + \sigma \xi_k(\sigma X), \quad k = 0, 1, 2, 3.$$  

**Proposition 4.1.** $M$ is a minimal Lorentzian surface (i.e. $\vec{H} = 0$) if and only if $\alpha_0, \alpha_1, \alpha_2$ and $\alpha_3$ are conformal 1-forms.

**Proof.** Let $a := u + \sigma v \in U \subset \mathcal{A} \rightarrow M$ be a conformal parameter such that the metric is given by $\lambda^2(-du^2 + dv^2)$ with $\lambda > 0$, and suppose that $(\partial_u, \partial_v)$ is positively oriented. Using the Dirac equation $D\varphi = \vec{H} \cdot \varphi$, we get

$$\vec{H} = 0 \quad \text{iff} \quad \partial_u \cdot \nabla_{\partial_u} \varphi = \partial_v \cdot \nabla_{\partial_v} \varphi. \quad (16)$$

Since the linear forms $\alpha_k (k = 0, 1, 2, 3)$ preserve the Lorentz structure (i.e. $\alpha_k(\sigma X) = \sigma \alpha_k(X)$ for all $X \in TM$), we can consider the maps $\psi_0, \psi_1, \psi_2, \psi_3 : M \rightarrow \mathcal{A}$ such that

$$\alpha_0 = \psi_0 da, \quad \alpha_1 = \psi_1 da, \quad \alpha_2 = \psi_2 da, \quad \alpha_3 = \psi_3 da;$$

more explicitly we have $\psi_k = \alpha_k(\partial_u) = \xi_k(\partial_u) + \sigma \xi_k(\partial_v)$ $(k = 0, 1, 2, 3)$. Using $\nabla_{\partial_u} \partial_u = \nabla_{\partial_v} \partial_v$, we easily get

$$\partial_u(\xi_k(\partial_u)) = \partial_v(\xi_k(\partial_v)) \quad \text{iff} \quad \partial_u \cdot \nabla_{\partial_u} \varphi = \partial_v \cdot \nabla_{\partial_v} \varphi, \quad (17)$$

whereas that $\nabla_{\partial_u} \partial_v = \nabla_{\partial_v} \partial_u$ implies

$$\partial_v(\xi_k(\partial_u)) = \partial_u(\xi_k(\partial_v)) \quad \text{iff} \quad \partial_v \cdot \nabla_{\partial_v} \varphi = \partial_u \cdot \nabla_{\partial_u} \varphi; \quad (18)$$

from (16) and (17)-(18) we have $\vec{H} = 0$ if and only if $\psi_k (k = 0, 1, 2, 3)$ satisfy $\partial_v \psi_k = \sigma \partial_u \psi_k$, i.e. if and only if $\psi_0, \psi_1, \psi_2$ and $\psi_3$ are conformal maps (see Equation (47) in the appendix). □

Thus, if $M$ is a minimal Lorentzian surface in $\mathbb{R}^{2,2}$,

$$F = \Re e \left( \int \alpha_0, \int \alpha_1, \int \alpha_2, \int \alpha_3 \right) \quad \Re e \left( \int \psi_0 da, \int \psi_1 da, \int \psi_2 da, \int \psi_3 da \right)$$

where $\psi_0, \psi_1, \psi_2$ and $\psi_3$ are conformal maps. This is the Weierstrass representation of a minimal Lorentzian surface in $\mathbb{R}^{2,2}$, which extends the classical Weierstrass representation of a Lorentz surface in $\mathbb{R}^{2,1}$ given by J. Konderak in [9, Theorem 2]. This representation was also given by M.P. Dussan and M. Magid in [4, Theorem 2.1]; in contrast, we have here a spinorial interpretation of this representation.

### 4.1 The generalized Weierstrass representation. Proof of Theorem 1.

The proof of the direct statement is obtained easily. The induced metric and the Lorentzian norm of the mean curvature vector are calculated straightforwardly. We prove the converse statement. We suppose that $(M, g)$ is a simply connected Lorentzian surface immersed in $\mathbb{R}^{2,2}$. We consider the spinor field $\varphi \in \Sigma$ solution of the Dirac equation

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1$$
obtained by the restriction to $M$ of the constant spinor $\pm \sigma I \in \mathbb{H}_0$ of $\mathbb{R}^{2, 2}$; it induces the immersion (14) (see Section 3). We consider a conformal chart $(U, a = u + \sigma v)$ such that the metric is given by
\begin{equation}
   g_{ij} = \lambda^2(-du^2 + dv^2) \quad \text{with} \quad \lambda > 0,
\end{equation}
and suppose that $(\partial_u, \partial_v)$ is positively oriented. We moreover choose an orthonormal and positively oriented basis $(e_0, e_1)$ of $E$ (normal to the surface $M$): $(e_0, e_1, \frac{\partial_u}{\lambda}, \frac{\partial_v}{\lambda})$ is adapted to the immersion $M \subset \mathbb{R}^{2, 2}$. Now, let $\beta : M \rightarrow \mathbb{R}$ be a solution of the system
\begin{equation}
   \begin{cases}
   \partial_u \beta = -2\frac{\partial_v \lambda}{\lambda} - (\nabla_{\partial_v} e_0, e_1) \\
   \partial_v \beta = -2\frac{\partial_u \lambda}{\lambda} - (\nabla_{\partial_u} e_0, e_1)
\end{cases}
\end{equation}
(by a direct computation the compatibility equation of (20), $\partial_u (\nabla_{\partial_v} e_0, e_1) = \partial_u (\nabla_{\partial_u} e_0, e_1)$ is satisfied, and thus the system (20) is solvable) and define the normal vectors
\begin{equation*}
   e_0 := \cosh \beta \ e_0 + \sinh \beta \ e_1 \quad \text{and} \quad e_1 := \sinh \beta \ e_0 + \cosh \beta \ e_1
\end{equation*}
obtained from $(e_0, e_1)$ by a Lorentzian rotation of angle $\beta$. Setting
\begin{equation*}
   e_2 := \frac{\partial_u}{\lambda} \quad \text{and} \quad e_3 := \frac{\partial_v}{\lambda},
\end{equation*}
we consider the spinorial frame $\tilde{s} \in \tilde{Q}$ such that $\pi(\tilde{s}) = (e_0, e_1, e_2, e_3) \in Q$; the coordinates of $e_0, e_1, e_2$ and $e_3$, in the spinorial frame $\tilde{s}$, are given by $\sigma i I, iJ$ and $K \in \mathbb{H}_1$ respectively. In $\tilde{s}$, the Dirac equation $D\varphi = H \cdot \varphi$ reads
\begin{equation}
   2\lambda \nabla\tilde{H} \cdot \varphi = -iJ \partial_u [\varphi] + K \partial_v [\varphi] - \frac{1}{2} \left( \frac{\partial_u \lambda}{\lambda} + \sigma (\partial_v \beta + (\nabla_{\partial_v} e_0, e_1)) \right) iJ [\varphi] + \frac{1}{2} \left( \frac{\partial_v \lambda}{\lambda} + \sigma (\partial_u \beta + (\nabla_{\partial_u} e_0, e_1)) \right) K [\varphi]
\end{equation}
\begin{equation*}
   = -iJ \partial_u [\varphi] + K \partial_v [\varphi] + \frac{1}{\lambda} \left\{ \left(-\partial_3 \lambda + \frac{\sigma}{2} \partial_v \lambda \right) iJ - \sigma \left( \partial_3 \lambda + \frac{1}{2} \partial_v \lambda \right) K \right\} [\varphi],
\end{equation*}
where we use the system (20). Writting
\begin{equation}
   \tilde{H} := h_0 e_0 + h_1 e_1 \in E \quad \text{and} \quad [\varphi] := \varphi_0 I + \varphi_1 iJ + \varphi_2 J + \varphi_3 iK \in \mathbb{H}_0
\end{equation}
the coordinates of spinor field $\varphi$ in $\tilde{s}$, we easily get the system (1) with
\begin{equation*}
   \varphi_1 = \lambda^{-\frac{1}{2}}(\varphi_3 + \sigma \varphi_2), \quad \varphi_2 = \lambda^{-\frac{1}{2}}(\varphi_0 - \varphi_1), \quad \varphi_3 = \lambda^{\frac{1}{2}}(\varphi_0 + \varphi_1), \quad \varphi_2 = \lambda^{\frac{1}{2}}(\varphi_3 + \sigma \varphi_2),
\end{equation*}
and
\begin{equation*}
   p = \lambda^{-1}(h_1 + h_0), \quad q = \lambda^3(h_1 - h_0).
\end{equation*}
Using the relation $H(\varphi, \varphi) = \varphi_0^2 - \varphi_1^2 + \varphi_2^2 - \varphi_3^2 = 1$ we get
\begin{equation*}
   |\varphi_2\varphi_1 - \varphi_1\varphi_2|^2 = \lambda^2(\varphi_0^2 - \varphi_1^2 + \varphi_2^2 - \varphi_3^2)(\varphi_0^2 - \varphi_1^2 + \varphi_2^2 - \varphi_3^2) = \lambda^2,
\end{equation*}
thus the metric $g$ (given in (19)) of $M$ satisfies (3), and the Lorentzian norm of the mean curvature vector $\tilde{H}$ (defined in (21)) satisfies
\begin{equation*}
   |\tilde{H}|^2 = -h_0^2 + h_1^2 = \lambda^{-2} pq
\end{equation*}
as is (4). Finally, if we write the induced immersion as $F = \int \xi$ with
\begin{equation*}
   \xi(X) = \langle X \cdot \varphi, \varphi \rangle := \xi_0(X) \sigma i I + \xi_1(X) iJ + \xi_2(X) J + \xi_3(X) K,
\end{equation*}
we get by a direct computation
\begin{align*}
   \xi_0 + \xi_1 &= -\left( \psi_1 \hat{\phi}_1 da + \hat{\psi}_1 \phi_1 d\hat{a} \right), & \xi_2 + \sigma \xi_3 &= \left( \psi_1 \hat{\phi}_2 da + \hat{\psi}_2 \phi_1 d\hat{a} \right), \\
   \xi_0 - \xi_1 &= \left( \psi_2 \hat{\phi}_2 da + \hat{\psi}_2 \phi_2 d\hat{a} \right), & \xi_2 - \sigma \xi_3 &= \left( \psi_2 \hat{\phi}_1 da + \hat{\psi}_1 \phi_2 d\hat{a} \right),
\end{align*}
and thus the formulas (2).
4.2 Lorentzian surfaces in $\mathbb{R}^{2,1}$

We suppose that the vector bundle $E$ is flat, i.e. is of the form $E = \mathbb{R}e_0 \oplus \mathbb{R}e_1$ where $e_0$ and $e_1$ are unit, orthogonal and parallel sections of $E$ such that $\langle e_0, e_0 \rangle = -1$ and $\langle e_1, e_1 \rangle = 1$; we moreover assume that $e_0$ is future-directed and that $(e_0, e_1)$ is positively oriented. We consider the isometric embeddings of $\mathbb{R}^{2,1}$ and $\mathbb{R}^{1,2}$ in $\mathbb{R}^{2,2} \subset \mathbb{H}_1$ given by

$$\mathbb{R}^{2,1} = (\sigma i I)^{\perp} \text{ and } \mathbb{R}^{1,2} = (I)^{\perp},$$

where $\sigma i I$ and $I$ are the first two vectors of the canonical basis of $\mathbb{R}^{2,2} \subset \mathbb{H}_1$.

If we assume that the functions $\phi_1, \phi_2, \psi_1, \psi_2$ in Theorem 1 satisfy $\psi_2 = \pm \hat{\phi}_1$ and $\phi_2 = \pm \hat{\psi}_1$ (and consequently that $p = q$), then $F_0 \equiv 0$, and we thus obtain the following generalized Weierstrass representation for a Lorentzian surface into $\mathbb{R}^{2,1}$:

**Corollary 2.** Let $\phi_2, \psi_2 : \mathcal{A} \rightarrow \mathcal{A}$ and $p : \mathcal{A} \rightarrow \mathbb{R}$ be smooth functions such that

$$\partial_s \phi_2 = -p \psi_2$$

$$\partial_t \psi_2 = -p \phi_2,$$

and $|\phi_2|^2 \neq |\psi_2|^2$. The formulas

$$F_1 = -\int_{\Gamma} \left( \psi_2 \hat{\phi}_2 da + \hat{\psi}_2 \phi_2 d\hat{a} \right),$$

$$F_2 + \sigma F_3 = \pm \int_{\Gamma} \left( \hat{\phi}_2^2 da + \hat{\psi}_2^2 d\hat{a} \right),$$

$$F_2 - \sigma F_3 = \pm \int_{\Gamma} \left( \psi_2^2 da + \phi_2^2 d\hat{a} \right),$$

define a conformal immersion $F = (F_1, F_2, F_3) : \mathcal{A} \rightarrow \mathbb{R}^{2,1}$, whose induced metric is given by

$$g = -(|\phi_2|^2 - |\psi_2|^2)^2 d\hat{a} d\hat{a},$$

and the mean curvature satisfies

$$H^2 = \frac{p^2}{(|\phi_2|^2 - |\psi_2|^2)^2}.$$

Conversely, a simply-connected Lorentzian surface in $\mathbb{R}^{2,1}$ may be described in that way.

Writing this generalized formula in the coordinates $(s, t)$ given by $a = \frac{1+\sigma}{2} s + \frac{1-\sigma}{2} t$ (see (48) in the appendix), we obtain exactly the Weierstrass representation formula described by S. Lee in [14, Theorem 2]. On the other hand, using this generalized formula in $\mathbb{R}^{2,1}$, in the case $p = 0$, we obtain the classical Weierstrass representation of a minimal Lorentz surface into $\mathbb{R}^{2,1}$: indeed, the formulas

$$F_1 = \Re e \int_{\Gamma} \frac{1}{2} \chi_1 \chi_2 da,$$

$$F_2 = \Re e \int_{\Gamma} \left( \chi_1^2 + \chi_2^2 \right) da,$$

$$F_3 = \Im m \int_{\Gamma} \left( \chi_1^2 - \chi_2^2 \right) da,$$

(22)

where $\chi_1 = \hat{\phi}_2$ and $\chi_2 = \hat{\psi}_2$ are conformal maps, define a minimal conformal immersion of a Lorentz surface into $\mathbb{R}^{2,1}$. A similar representation was already given by J. Konderak in [9, Theorem 4]: if we suppose that $\chi_1 \chi_1 \neq 0$ and define

$$\Phi := \chi_1^2 da \quad \text{and} \quad g := \frac{\chi_2^2}{\chi_1^2},$$

and
then \( \Phi \) is a conformal 1–form such that \( \Phi \hat{\Phi} > 0, \ 1 - g \hat{g} \neq 0 \), and the formulas in (22) may be written in the form

\[
(F_1, F_2, F_3) = \left( \Re \int_\Gamma \frac{1}{2} g \Phi, \Re \int_\Gamma (1 + g^2) \Phi, \Re \int_\Gamma \sigma (1 - g^2) \Phi \right),
\]

which is exactly the representation obtained by Konderak in [9].

**Remark 2.** Similarly, the reduction \( \psi_2 = \mp \hat{\psi}_1 \) and \( \phi_2 = \pm \hat{\phi}_1 \) implies the generalized Weierstrass representation for a Lorentzian surface into \( \mathbb{R}^{1,2} \) (see details in [16]).

5 Spinor representation of a Lorentzian surface in pseudospheres of \( \mathbb{R}^{2,2} \)

The aim of this section is to deduce spinor representations for isometric immersions of a Lorentzian surface in the pseudo-spheres of \( \mathbb{R}^{2,2} \); we obtain characterizations which are different to the characterizations given by M.A. Lawn and J. Roth in [12].

Keeping the notation of Section 2, the map \( (\xi, q) \in \mathbb{H}_0 \times \mathbb{H}_1 \mapsto \sigma i \xi \ q \in \mathbb{H}_0 \) defines a linear action of \( \mathbb{H}_1 \) by the multiplication on the right on \( \mathbb{H}_0 \); the spinor bundle \( \Sigma \) is thus also naturally equipped with a linear right-action of \( \mathbb{H}_1 \):

\[
(\varphi, q) \mapsto \varphi \cdot q.
\]

With respect to this structure the Clifford action satisfies

\[
X \cdot (\varphi \cdot q) = -(X \cdot \varphi) \cdot \hat{q}
\]

for all \( \varphi \in \Sigma, \ X \in E \oplus TM \) and \( q \in \mathbb{H}_1 \).

We suppose that \( E = \mathbb{R} e_0 \oplus \mathbb{R} e_1 \) where \( e_0 \) and \( e_1 \) are unit, orthogonal and parallel sections of \( E \) such that \( \langle e_0, e_0 \rangle = -\langle e_1, e_1 \rangle = -1 \); we moreover assume that \( e_0 \) is future-directed and that \( (e_0, e_1) \) is positively oriented. Let \( \tilde{H} \) be a section of \( E \) and \( \varphi \in \Gamma(\Sigma) \) be a solution of

\[
D \varphi = \tilde{H} \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1. \quad \text{(23)}
\]

According to the spinor representation theorem (Theorem 3), the spinor field \( \varphi \) defines an isometric immersion \( M \hookrightarrow \mathbb{R}^{2,2} \) (unique, up to translations), with normal bundle \( E \) and mean curvature vector \( \tilde{H} \). We give a characterization of the isometric immersion in the pseudo-spheres \( \mathbb{H}^{2,1} \) and \( \mathbb{S}^{1,2} \) (defined in (6)-(7) respectively), up to translations, in terms of the spinor field \( \varphi \).

**Proposition 5.1.** 1. Consider the function \( F = \langle (e_0 \cdot \varphi, \varphi) \rangle \), and suppose that

\[
\tilde{H} = e_0 + H e_1 \quad \text{and} \quad dF(X) = \langle (X \cdot \varphi, \varphi) \rangle \quad \text{(24)}
\]

for all \( X \in TM \). Then the isometric immersion \( M \hookrightarrow \mathbb{R}^{2,2} \) belongs to \( \mathbb{H}^{2,1} \).

2. Consider the function \( F = \langle (-e_1 \cdot \varphi, \varphi) \rangle \), and suppose that

\[
\tilde{H} = -H e_0 + e_1 \quad \text{and} \quad dF(X) = \langle (X \cdot \varphi, \varphi) \rangle \quad \text{(25)}
\]

for all \( X \in TM \). Then the isometric immersion \( M \hookrightarrow \mathbb{R}^{2,2} \) belongs to \( \mathbb{S}^{1,2} \).

Reciprocally, if \( M \hookrightarrow \mathbb{R}^{2,2} \) belongs to \( \mathbb{H}^{2,1} \) (resp. to \( \mathbb{S}^{1,2} \)), then (24) (resp. (25)) holds for some unit, orthogonal and parallel sections \( (e_0, e_1) \) of \( E \).

**Proof.** Assuming that (24) holds, the function \( F \) is a primitive of the 1–form \( \xi(X) = \langle (X \cdot \varphi, \varphi) \rangle \), and is thus the isometric immersion defined by \( \varphi \) (uniquely defined, up to translations); since the norm of \( \langle (e_0 \cdot \varphi, \varphi) \rangle \in \mathbb{R}^{2,2} \subset \mathbb{H}_1 \) coincides with the norm of \( e_0 \), and is thus constant equal to \(-1\), the immersion belongs to \( \mathbb{H}^{2,1} \). For the converse statement, we choose \( (e_0, e_1) \) such that \( \langle (e_0 \cdot \varphi, \varphi) \rangle \) is normal to \( \mathbb{H}^{2,1} \) in \( \mathbb{R}^{2,2} \). Writing the spinors in a frame \( \hat{s} \) adapted to \( (e_0, e_1, e_2, e_3) \), we easily deduce (24) since \( \langle (e_0 \cdot \varphi, \varphi) \rangle \) is the immersion. The proof for the case of the pseudo-sphere \( \mathbb{S}^{1,2} \) is analogous. \( \square \)
Remark 3. The isometric immersion in \( \mathbb{R}^{2,1} \) or in \( \mathbb{R}^{1,2} \), in terms of the spinor field \( \varphi \), is characterized by the following conditions:

\[
\hat{H} = H e_1 \quad \text{and} \quad e_0 \cdot \varphi = \varphi \tag{26}
\]
or

\[
\hat{H} = H e_0 \quad \text{and} \quad e_1 \cdot \varphi = -\varphi \cdot I, \tag{27}
\]

respectively; see details in [2, Proposition 2.4].

Now, we assume that \( M \subset Q(c) \subset \mathbb{R}^{2,2} \), with \( c = \pm 1 \), where \( Q(-1) \) is the Anti-de Sitter space \( \mathbb{H}^{2,1} \) and \( Q(+1) \) is the pseudo-sphere \( \mathbb{S}^{1,2} \), and consider \( e_0 \) and \( e_1 \) timelike and spacelike unit vector fields such that

\[
\mathbb{R}^{2,2} = \mathbb{R} e_{1\pm} \oplus \perp TQ(c) \quad \text{and} \quad TQ(c) = \mathbb{R} e_{1\pm} \oplus \perp TM.
\]

The intrinsic spinors of \( M \) identify with the spinors of \( Q(c) \) restricted to \( M \), which in turn identify with the positive spinors of \( \mathbb{R}^{2,2} \) restricted to \( M \) (Propositions 5.2 and 5.4 below), which, together with Proposition 5.1, will give the spinor representation of a Lorentzian surface in \( \mathbb{H}^{2,1} \) and in \( \mathbb{S}^{1,2} \) by means of spinors of \( \Sigma M \) only. We examine separately the case of a surface in the Anti-de Sitter space \( \mathbb{H}^{2,1} \), and in the pseudo-sphere \( \mathbb{S}^{1,2} \).

5.1 Lorentzian surfaces in \( \mathbb{H}^{2,1} \)

We can define a scalar product on \( \mathbb{C}^2 \) by setting:

\[
\left\langle \begin{pmatrix} a + ib \\ c + id \end{pmatrix}, \begin{pmatrix} a' + ib' \\ c' + id' \end{pmatrix} \right\rangle := \frac{ad' + a'd - bc' - b'c}{2};
\]

it is of signature \((2,2)\). This scalar product is \( \text{Spin}(1,1) \)-invariant and thus induces a scalar product \( \langle \cdot, \cdot \rangle \) on the spinor bundle \( \Sigma M \). It satisfies the following properties: for all \( \psi, \psi' \in \Sigma M \) and all \( X \in TM \),

\[
\langle \psi, \psi' \rangle = \langle \psi', \psi \rangle \quad \text{and} \quad \langle X \cdot_M \psi, \psi' \rangle = -\langle \psi, X \cdot_M \psi' \rangle. \tag{28}
\]

This is the scalar product on \( \Sigma M \) that we use in this section (and in this section only). We moreover define \( |\psi|^2 := \langle \psi, \psi \rangle \).

Proposition 5.2. There is an identification

\[
\Sigma M \xrightarrow{\sim} \Sigma^+_M \quad \psi \mapsto \psi^*
\]

\( \mathbb{C} \)-linear, and such that, for all \( X \in TM \) and all \( \psi \in \Sigma M \), \( (\nabla_X \psi)^* = \nabla_X \psi^* \), the Clifford actions are linked by

\[
(X \cdot_M \psi)^* = X \cdot e_1 \cdot \psi^* \tag{29}
\]

and

\[
H(\psi^*, \psi^*) = \frac{1 + \sigma}{2} |\psi|^2. \tag{30}
\]

The detailed proof is given in [16]. Using this identification, the intrinsic Dirac operator on \( M \) defined by

\[
D_M \psi = -e_2 \cdot \nabla e_2 \psi + e_3 \cdot \nabla e_3 \psi
\]

is linked to \( D \) by

\[
(D_M \psi)^* = -e_1 \cdot D \psi^*.
\]

We suppose that \( \varphi \in \Gamma(\Sigma) \) is a solution of the equation \((23)\) such that \((24)\) holds (the immersion belongs to \( \mathbb{H}^{2,1} \)), and we choose \( \psi \in \Gamma(\Sigma M) \) such that \( \psi^* = \varphi^+ \) (note that \( \psi \neq 0 \), since \( H(\varphi, \varphi) = 1 \)); it satisfies

\[
(D_M \psi)^* = -e_1 \cdot D \psi^* = -e_1 \cdot \tilde{H} \cdot \psi^* = -e_1 \cdot (e_0 + He_1) \cdot \psi^* = H \psi^* + \overline{\psi}.
\]
since $\varphi$ solves the equation $\nabla_X \varphi = \eta(X) \cdot \varphi$, where $\eta(X) = -\frac{1}{2} \sum_{j=2}^3 \epsilon_j e_j \cdot B(X, e_j)$ (Theorem 3), the second equality in (24) is equivalent to $[\eta(X)] - [\eta(X)] = [X]\sigma i$, from (28), (29) and (30) we obtain
\[
d(|\psi|^2)(X) = 2(\nabla_X \psi, \psi) = 4\Re H(\nabla_X \psi^*, \psi^*) = 4\Re H(\eta(X) \cdot \psi^*, \psi^*) = -\langle X \cdot M \bar{\psi}, \psi \rangle;
\]
since $\langle X \cdot M \bar{\psi}, \psi \rangle = 0$ (by definition of $\langle \cdot, \cdot \rangle$) $|\psi|^2$ is constant equal to 1, and we thus get
\[
D_M \psi = H \psi + \bar{\psi} \quad \text{and} \quad |\psi|^2 = 1. \tag{31}
\]

Reciprocally, let $(M, g)$ be a Lorentzian surface and $H : M \to \mathbb{R}$ a given differentiable function, and suppose that $\psi \in \Gamma(\Sigma M)$ satisfies the Dirac equation (31). We define $\varphi^+ := \psi^* \in \Sigma^+$ and $\bar{H} := e_0 + H e_1$, where $e_0, e_1$ are orthogonal and parallel sections of $E$ with $\langle e_0, e_0 \rangle = -\langle e_1, e_1 \rangle = -1$, and such that $(e_0, e_1)$ is positively oriented. Using (31) and (30) we obtain
\[
D \varphi^+ = \bar{H} \cdot \varphi^+ \quad \text{and} \quad H(\varphi^+, \varphi^+) = \frac{1+\sigma}{2}.
\]

**Proposition 5.3.** Let $\psi \in \Gamma(\Sigma M)$ be a solution of the equation (31). There exists a spinor field $\varphi \in \Gamma(\Sigma)$ solution of
\[
D \varphi = \bar{H} \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1,
\]
with $\varphi^+ = \psi^*$ and such that the immersion defined by $\varphi$ is given by $F = \langle (e_0 \cdot \varphi, \varphi) \rangle$. In particular $F(M)$ belongs to $\mathbb{H}^{2,1}$.

**Proof.** We need to find $\varphi^-$ solution of the system
\[
\begin{cases}
F_1 = \langle (e_0 \cdot \varphi^-, \varphi^+) \rangle \\
dF_1(X) = \langle (X \cdot \varphi^-, \varphi^+) \rangle
\end{cases}
\]
with $\langle F_1, F_1 \rangle = -\frac{1}{2}$; this system is equivalent to
\[
\varphi^- = -e_0 \cdot (\varphi^+ \bullet F_1)
\]
with $H(\varphi^-, \varphi^-) = \frac{1-\sigma}{2}$, here $F_1 : M \to \frac{1+\sigma}{2} \mathbb{H}_1$ solves the equation in $\frac{1+\sigma}{2} \mathbb{H}_1$
\[
dF_1(X) = \omega(X) F_1 \tag{32}
\]
where
\[
\omega(X) = -\sigma i \langle (X \cdot e_0 \cdot \varphi^+ \cdot \varphi^+) \rangle.
\]
By a direct computation, the compatibility equation of (32),
\[
d\omega(X, Y) = \omega(X) \omega(Y) - \omega(Y) \omega(X),
\]
is satisfied, and thus the equation (32) is solvable. \qed

A solution of (31) is thus equivalent to an isometric immersion in the Anti-de Sitter space $\mathbb{H}^{2,1}$. We thus obtain a spinorial characterization of an isometric immersion of a Lorentzian surface in $\mathbb{H}^{2,1}$, which is simpler than the characterization given by M.A. Lawn and J. Roth in [12], where two spinor fields are needed. Finally, this spinorial characterization is similar to the spinor representation of surfaces in the three-dimensional hyperbolic space given by B. Morel in [15], and by P. Bayard in [1].

**Remark 4.** Let $M$ be a minimal Lorentzian surface in $\mathbb{H}^{2,1}$; the immersion $M \subset \mathbb{H}^{2,1}$ is represented by a solution $\varphi \in \Gamma(\Sigma)$ of
\[
D \varphi = e_0 \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1. \tag{33}
\]
The spinor field

\[ \psi := \varphi^+ + e_1 \cdot (\varphi^+ \cdot I) \in \Sigma \]

satisfies (27) and thus induces an isometric immersion \( M \to \mathbb{R}^{1,2} \) with constant mean curvature \( H \equiv 1 \); we thus get a natural transformation sending a minimal Lorentzian surface in \( \mathbb{H}^{2,1} \) to a Lorentzian surface in \( \mathbb{R}^{1,2} \) with constant mean curvature 1. This is analogous to a classical transformation for surfaces in \( S^3 \), described by H.B. Lawson in [13], by T. Friedrich using spinors in dimension 3 in [5, Remark 1], and by P. Bayard, M.A. Lawn and J. Roth using spinors in dimension 4 in [3, Remark 4].

5.2 Lorentzian surfaces in \( S^{1,2} \)

We consider here the following scalar product on \( \Sigma M \), given in coordinates by

\[ \langle (a + ib), (a' + ib') \rangle = \frac{ac' + a'b + bd' + b'd}{2}; \]

it is of signature (2, 2). Moreover, for all \( \psi, \psi' \in \Sigma M \) and all \( X \in TM \) we have:

\[ \langle \psi, \psi' \rangle = \langle \psi', \psi \rangle \quad \text{and} \quad \langle X \cdot_M \psi, \psi' \rangle = \langle \psi, X \cdot_M \psi' \rangle. \]

We moreover write \( |\psi|^2 := \langle \psi, \psi \rangle \) and still denote by \( i \) the complex structure on \( \Sigma \) and on \( \Sigma M \).

**Proposition 5.4.** There is an identification

\[ \Sigma M \sim \Sigma^+_M \]

\[ \psi \mapsto \psi^* \]

\( \mathbb{C} \)-linear, and such that, for all \( X \in TM \) and all \( \psi \in \Sigma M \), \( (\nabla_X \psi)^* = \nabla_X \psi^* \), the Clifford actions are linked by \( (X \cdot_M \psi)^* = i e_0 \cdot X \cdot \psi^* \), and

\[ H(\psi^*, \psi^*) = -\frac{1 + \sigma}{2} |\psi|^2. \]  

(34)

The detailed proof is given in [16]. Using this identification, we have

\[ (D_M \psi)^* = i e_0 \cdot D \psi^* \]

for all \( \psi \in \Sigma M \). If we suppose that \( \varphi \) is a solution of (23), we can choose \( \psi \neq 0 \in \Sigma M \) such that \( \psi^* = \varphi^+ \); moreover, if (25) holds, \( \psi \) satisfies

\[ (D_M \psi)^* = i e_0 \cdot \tilde{H} \cdot \psi^* = i e_0 \cdot (-He_0 + e_1) \cdot \psi^* = -i H \psi^* + i \overline{\psi}, \]

and, using (25) and (34) we deduce

\[ D_M \psi = -i H \psi + i \overline{\psi} \quad \text{and} \quad |\psi|^2 = -1. \]  

(35)

Reciprocally, let \( (M, g) \) be a Lorentzian surface and \( H : M \to \mathbb{R} \) a given differentiable function, and suppose that \( \psi \in \Gamma(\Sigma M) \) satisfies (35). We define \( \tilde{H} := -He_0 + e_1 \),

\[ \varphi^+ := \psi^* \quad \text{and} \quad \varphi^- := -e_1 \cdot (\varphi^+ \cdot F_1) \]

where \( e_0, e_1 \) are orthogonal and parallel sections of \( E \) with \( \langle e_0, e_0 \rangle = -\langle e_1, e_1 \rangle = -1 \), such that \( (e_0, e_1) \) is positively oriented, and where (with a construction analogous to the construction in Proposition 5.3) \( F_1 \) solves the equation

\[ dF_1(X) = \omega(X) F_1 \quad \text{with} \quad \omega(X) = -\sigma i \langle X \cdot e_1 \cdot \varphi^+, \varphi^+ \rangle. \]
The spinor field \( \varphi := \varphi^+ + \varphi^- \in \Sigma \) satisfies the equation (23) and the isometric immersion \( F \) induced by \( \varphi \) is given by \[
F = F_1 - \hat{F}_1 = \langle (-e_1 \cdot \varphi, \varphi) \rangle \subset S^{1,2}.
\]

A solution of (35) is thus equivalent to an isometric immersion of a Lorentzian surface in \( S^{1,2} \). Here again, we obtain a spinor characterization of an isometric immersion of a Lorentzian surface in the pseudo-sphere \( S^{1,2} \), which is simpler than the characterization obtained by M.A. Lawn and J. Roth in [12] where two spinor fields are involved.

**Remark 5.** Let \( M \) be a minimal Lorentzian surface in \( S^{1,2} \), the immersion \( M \subset S^{1,2} \) is represented by a solution \( \varphi \in \Gamma(\Sigma) \) of
\[
D\varphi = e_1 \cdot \varphi, \quad H(\varphi, \varphi) = 1. \tag{36}
\]
The spinor field
\[
\hat{\varphi} := \varphi^+ + c_0 \cdot \varphi^+ \in \Sigma
\]
satisfies (26) and thus induces an isometric immersion \( M \hookrightarrow \mathbb{R}^{2,1} \) with constant mean curvature \( H \equiv 1 \). We thus get a natural transformation sending a minimal Lorentzian surface in \( S^{1,2} \) to a Lorentzian surface in \( \mathbb{R}^{2,1} \) with constant mean curvature 1.

### 6 Flat Lorentzian surfaces in pseudo-spheres of \( \mathbb{R}^{2,2} \)

In this section, we obtain the conformal description of a flat Lorentzian surface in the Anti-de Sitter space \( \mathbb{H}^{2,1} \), and in the pseudo-sphere \( S^{1,2} \) (proof of Theorem 2). This conformal description extends to the Lorentzian case the representation of the flat surfaces in the three-dimensional hyperbolic and de Sitter spaces given by J.A. Gálvez, A. Martínez and F. Milán in [6, 7]. We then obtain the local description of a flat Lorentzian surface in \( \mathbb{H}^{2,1} \) (resp. in \( S^{1,2} \)) as a product of two curves in \( \mathbb{H}^{2,1} \) (resp. in \( S^{1,2} \)); this description extends the representation of the flat surfaces in \( S^3 \) as a product of two curves given by Bianchi (see [18]).

Keeping the notation of Section 2, we consider the isomorphism of algebras
\[
A_0 : \mathbb{H}_0 \longrightarrow M_2(\mathcal{A})
\]
\[
p = p_0 I + i p_1 I + p_2 J + i p_3 K \longmapsto \begin{pmatrix} p_0 - \sigma p_1 & p_2 - \sigma p_3 \\ -p_2 - \sigma p_3 & p_0 + \sigma p_1 \end{pmatrix};
\]
it is such that
\[
H(p, p) = \det A_0(p) \quad \text{and} \quad A_0 \left( \hat{p} \right) = A_0(p)^* \tag{37}
\]
for all \( p \in \mathbb{H}_0 \). We also consider the isomorphism of vector spaces
\[
A_1 : \mathbb{H}_1 \longrightarrow M_2(\mathcal{A})
\]
\[
q = iq_0 I + q_1 I + iq_2 J + q_3 K \longmapsto \begin{pmatrix} -q_1 - \sigma q_0 & -q_3 - \sigma q_2 \\ q_3 + \sigma q_2 & q_1 - \sigma q_0 \end{pmatrix}; \tag{38}
\]
it satisfies
\[
H(q, q) = -\det A_1(q) \quad \text{and} \quad A_1 \left( \hat{q} \right) = -A_1(q)^* \tag{39}
\]
for all \( q \in \mathbb{H}_1 \). By a direct computation, for all \( p, p' \in \mathbb{H}_0 \) we have
\[
A_1(\sigma I p p') = -A_0(p)A_0(p') \quad \text{and} \quad A_1(p I p') = A_0(p) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A_0(p'). \tag{40}
\]
Using (38) and (39), we get
\[
\mathbb{R}^{2,2} = \left\{ \xi \in \mathbb{H}_1 \mid \hat{\xi} = -\xi \right\} \simeq \text{Herm}_2(\mathcal{A}), \tag{41}
\]
where the metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^{2,2}$ identifies with $-\det$ defined on $\text{Herm}_2(A)$. Moreover, the Anti-de Sitter space (defined in (6)) is described by

$$\mathbb{H}^{2,1} \simeq \{ BB^* | B \in S(l_2(A)) \} \subset \text{Herm}_2(A)$$

and the pseudo-sphere (defined in (7)) by

$$\mathbb{S}^{1,2} \simeq \left\{ B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B^* | B \in S(l_2(A)) \right\} \subset \text{Herm}_2(A).$$

Indeed, from (6) and (41) we have $\mathbb{H}^{2,1} \simeq \{ C \in \text{Herm}_2(A) \mid \det C = 1 \}$, and thus

$$C \in \mathbb{H}^{2,1} \iff C = \frac{1+\sigma}{2} C_1 + \frac{1-\sigma}{2} C_1^t,$$

where $C_1 \in S(l_2(\mathbb{R})$; setting $B := \frac{1+\sigma}{2} C_1 + \frac{1-\sigma}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get $C = BB^*$ and $B \in S(l_2(A))$. The argument for the case of the pseudo-sphere $\mathbb{S}^{1,2}$ is analogous.

We consider $(M, g)$ a simply-connected Lorentzian surface and suppose that the vector bundles $TM$ and $E$ are flat; with the notation of Section 3, the spinorial connection on the bundle $\hat{Q}$ is flat, and $\hat{Q}$ admits a parallel local section $\hat{s}$; since $M$ is simply connected, the section $\hat{s}$ is in fact globally defined. We consider $\varphi \in \Gamma(\Sigma)$ as in the second statement of Theorem 3, and set $[\varphi] : M \to \text{Spin}(2,2)$ the coordinates of $\varphi$ in $\hat{s}$: the equation (13) reads

$$d[\varphi] = [\eta][\varphi],$$

where $\eta(\cdot) = -\frac{1}{2} \sum_{j=2}^3 e_j e_j \cdot B(\cdot, e_j)$ is such that $[\eta] \in \mathcal{A}J \oplus iAK \subset \mathbb{H}_0$. We moreover assume that the Gauss map

$$G : M \to \mathcal{Q} := \{ u_1 \cdot u_2 \mid u_1, u_2 \in \mathbb{R}^{2,2}, -|u_1|^2 = |u_2|^2 = 1 \} \subset Cl_0(2, 2) \simeq \mathbb{H}_0$$

($\mathcal{Q}$ identifies to the Grassmannian of the oriented Lorentzian planes in $\mathbb{R}^{2,2}$) of the immersion defined by $\varphi$ is regular; since $TM$ and $E$ are flat, for all $x \in M$, $dG_x(T_xM) \subset \mathbb{H}_0$ is stable by multiplication by $\sigma I \in \mathbb{H}_0$, and we thus define the unique Lorentz structure $\sigma$ on $M$ given by

$$\sigma u := dG_x^{-1}(\sigma dG_x(u)), \quad \forall u \in TM.$$

This Lorentz structure on $M$ is such that $G$ is a conformal map: the multiplication by $\sigma I$ on $\mathbb{H}_0$ induces a natural Lorentz structure on $\mathbb{H}_0$ and therefore on $\mathcal{Q}$, and on $\text{Spin}(2,2)$. We thus get that $[\varphi] : M \to \text{Spin}(2,2)$ is in fact a conformal map; see details in [2, Section 3].

### 6.1 Proof of Theorem 2

The proof of the direct statement is obtained easily: in the first case the fact that $F = BB^*$ defines a flat immersion in $\mathbb{H}^{2,1}$ may be proved by a direct computation; the induced metric and the shape operator are given by

$$g = (\theta + \overline{\theta}) (\omega + \overline{\omega}) \quad \text{and} \quad S = B \begin{pmatrix} 0 & \theta - \overline{\theta} \\ -\omega + \overline{\omega} & 0 \end{pmatrix} B^*,$$

thus the Gauss equation (see [17, pag. 107]) implies the result. The proof in the case of $\mathbb{S}^{1,2}$ is analogous.

Reciprocally, we suppose that there exists a flat isometric immersion $F : (M, g) \to \mathbb{H}^{2,1}$ (resp. $\mathbb{S}^{1,2}$). Using the natural isometric embedding $\mathbb{H}^{2,1} \hookrightarrow \mathbb{R}^{2,2}$ (resp. $\mathbb{S}^{1,2} \hookrightarrow \mathbb{R}^{2,2}$), we get a flat immersion $M \to \mathbb{R}^{2,2}$ with flat normal bundle and regular Gauss map, and we can consider the Lorentz structure on $M$ such that the Gauss map is conformal. We denote by $E$ its normal bundle,
\( \vec{H} \in \Gamma(E) \) its mean curvature vector field and \( \Sigma := M \times \mathbb{H}_0 \) the spinor bundle of \( \mathbb{R}^{2,2} \) restricted to \( M \). The immersion \( F \) is given by

\[
F = \int \xi, \quad \text{where} \quad \xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle,
\]

for some spinor field \( \varphi \in \Gamma(\Sigma) \) solution of \( D\varphi = \vec{H} \cdot \varphi \) and such that \( H(\varphi, \varphi) = 1 \) (the spinor field \( \varphi \) is the restriction to \( M \) of the constant spinor field \( \sigma I \) or \( -\sigma I \in \mathbb{H}_0 \)). We examine separately the case of a Lorentzian surface in the Anti-de Sitter space \( \mathbb{H}^{2,1} \), and in the pseudo-sphere \( S^{1,2} \):

**Flat Lorentzian surfaces in \( \mathbb{H}^{2,1} \).** In this case, using Proposition 5.1, 1. we have

\[
F = \langle \langle e_0 \cdot \varphi, \varphi \rangle \rangle,
\]

where \( e_0 \in \Gamma(E) \) is the future-directed vector which is normal to \( \mathbb{H}^{2,1} \) in \( \mathbb{R}^{2,2} \). We choose a parallel frame \( \hat{s} \in \Gamma(Q) \) adapted to \( e_0 \), i.e. such that \( e_0 \) is the first vector of \( \pi(\hat{s}) \in \Gamma(Q_1 \times_M Q_2) : \) in \( \hat{s} \), using (40), (43) reads

\[
F = -\sigma i [\varphi] [\varphi] \simeq -A_1(\sigma \iota_{[\varphi]} [\varphi]) = A_0([\varphi]) A_0([\varphi])
\]

(44)

where \([\varphi] \in \mathbb{H}_0\) represents the spinor field \( \varphi \) in \( \hat{s} \). Thus, setting \( B := A_0([\varphi]) \) and using (37) we have that \( B \) belongs to \( Sl_2(A) \) (since \( H(\varphi, \varphi) = 1 \)) and \( B^* = A_0([\varphi]) \). From (44) we thus get \( F \simeq BB^* \). Using (42) we finally obtain

\[
B^{-1}dB = A_0([\varphi] d[\varphi]) = -A_0(d[\varphi] [\varphi]) = -A_0(\eta_1 J + i \eta_2 K) = \left( \begin{array}{cc} 0 & -\eta_1 + \sigma \eta_2 \\ \eta_1 + \sigma \eta_2 & 0 \end{array} \right),
\]

where \( \eta_1 \) and \( \eta_2 \) are 1-forms on \( M \) with values in \( A \). With respect to the Lorentz structure induced on \( M \) (by the Gauss map), \( B : M \rightarrow Sl_2(A) \) is a conformal map (since \([\varphi] : M \rightarrow Spin(2,2) \subset \mathbb{H}_0 \) is a conformal map and \( A_0 \) is \( A \)-linear). Remark 6 implies that \( \theta := -\eta_1 + \sigma \eta_2 \) and \( \omega := \eta_1 + \sigma \eta_2 \) are conformal 1-forms, and, \( dF \) injective reads \( |\theta|^2 \neq |\omega|^2 \).

**Flat Lorentzian surfaces in \( S^{1,2} \).** In this case, the immersion is given by

\[
F = \langle \langle e_1 \cdot \varphi, \varphi \rangle \rangle,
\]

(45)

where \( e_1 \in \Gamma(E) \) is a normal vector to \( S^{1,2} \) (see Proposition 5.1, 2.). We choose a parallel frame \( \hat{s} \in \Gamma(Q) \) adapted to \( e_1 \), i.e. such that \( e_1 \) is the second vector of \( \pi(\hat{s}) \in \Gamma(Q_1 \times_M Q_2) : \) in \( \hat{s} \), using (40), (45) reads

\[
F = [\varphi] I[\varphi] \simeq A_1([\varphi] I[\varphi]) = A_0([\varphi]) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) A_0([\varphi])
\]

(46)

where \([\varphi] \in \mathbb{H}_0\) represents \( \varphi \) in \( \hat{s} \). Setting \( B := A_0([\varphi]) \) as above, we have \( B \in Sl_2(A) \) and \( B^* = A_0([\varphi]) \); from (46) we thus get \( F \simeq B \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) B^* \). In this case, \( dF \) injective reads \( |\theta|^2 \neq |\omega|^2 \).

### 6.2 Flat Lorentzian surfaces as a product of curves

As a consequence of Theorem 2, we obtain easily the local description of a flat Lorentzian surface in the Anti-de Sitter space \( \mathbb{H}^{2,1} \) or in the pseudo-sphere \( S^{1,2} \) as a product of two curves.

We note that every matrix in \( Herm_2(A) \) can be written as \( \frac{1}{2} \sigma C + \frac{1}{2} \sigma C^t \) with \( C \in M_2(\mathbb{R}) \), and thus, we can identify \( Herm_2(A) \simeq M_2(\mathbb{R}) \); under this identification we have \( \mathbb{H}^{2,1} \simeq Sl_2(\mathbb{R}) \).

**Corollary 3.** A flat Lorentzian surface in \( \mathbb{H}^{2,1} \) (resp. in \( S^{1,2} \)) may be written (locally) as a product of two curves in \( \mathbb{H}^{2,1} \) (resp. in \( S^{1,2} \)).
Proof. We prove only the case of a Lorentzian surface in $\mathbb{H}^{2,1}$. In the coordinates $(s,t)$ defined in (48), the conformal immersion $B : U \subset M \to Sl_2(\mathbb{R})$ of Theorem 2 is given by

$$B(s,t) = \frac{1+\sigma}{2}B_1(s) + \frac{1-\sigma}{2}B_2(t),$$

where $B_1, B_2 \in Sl_2(\mathbb{R})$; identifying

$$B_1(s) \simeq \frac{1+\sigma}{2}B_1(s) + \frac{1-\sigma}{2}B_1(s)^t \quad \text{and} \quad B_2(t) \simeq \frac{1+\sigma}{2}B_2(t) + \frac{1-\sigma}{2}B_2(t)^t \in \mathbb{H}^{2,1},$$

we get two curves $B_1(s), B_2(t)$ in $\mathbb{H}^{2,1}$ such that the immersion is described by

$$F = BB^* = \frac{1+\sigma}{2}B_1(s)B_2(t)^t + \frac{1-\sigma}{2}B_2(t)B_1(s)^t \simeq B_1(s)B_2(t)^t,$$

the immersion is thus a product of two curves in $\mathbb{H}^{2,1}$. \hfill \Box

A On Lorentz surfaces

A Lorentz surface is a surface $M$ together with a covering by open subsets $M = \cup_{\alpha \in S} U_{\alpha}$ and charts

$$\varphi_\alpha : U_{\alpha} \to A, \quad \alpha \in S$$

such that the transition functions

$$\varphi_\beta \circ \varphi^{-1}_\alpha : \varphi_\alpha(U_{\alpha} \cap U_\beta) \subset A \to \varphi_\beta(U_{\alpha} \cap U_\beta) \subset A, \quad \alpha, \beta \in S$$

are conformal maps in the following sense: for all $a \in \varphi_\alpha(U_{\alpha} \cap U_\beta)$ and $h \in A$,

$$d (\varphi_\beta \circ \varphi^{-1}_\alpha) (a \cdot h) = \sigma \ d (\varphi_\beta \circ \varphi^{-1}_\alpha) (a \cdot h).$$

A Lorentz structure is also equivalent to a smooth family of maps

$$\sigma_p : T_p M \to T_p M, \quad \text{with} \quad \sigma_p^2 = Id_{T_p M}, \quad \sigma_p \neq \pm Id_{T_p M}.$$ 

This definition coincides with the definition given in [19]: a Lorentz structure is equivalent to a conformal class of Lorentzian metrics on the surface, that is to a smooth family of cones in every tangent space of the surface, with distinguished lines. Indeed, the cone in $p \in M$ is

$$\text{Ker}(\sigma_p - Id_{T_p M}) \cup \text{Ker}(\sigma_p + Id_{T_p M})$$

where the sign of the eigenvalues $\pm 1$ permits to distinguish one of the lines from the other.

If $M$ is moreover oriented, we will say that the Lorentz structure is compatible with the orientation of $M$ if the charts $\varphi_\alpha : U_{\alpha} \to A$, $\alpha \in S$ preserve the orientations (the positive orientation in $A = \{ u + \sigma v \mid u, v \in \mathbb{R} \}$ is naturally given by $(\partial_u, \partial_v)$). In that case, the transition functions are conformal maps $A \to A$ preserving orientation.

Conformal maps on Lorentz surfaces. If $M$ is a Lorentz surface, a smooth map $\psi : M \to A$ (or $A^n$, or a Lorentz surface) will be said to be a conformal map if $d\psi$ preserves Lorentz structures, that is if

$$d\psi_p(\sigma_p X) = \sigma_{\psi(p)} (d\psi_p (X))$$

for all $p \in M$ and $X \in T_p M$. In a chart $a := u + \sigma v : U \subset A \to M$, a conformal map satisfies

$$\partial_v \psi = \sigma \ \partial_u \psi.$$ \hfill (47)

Writing

$$\partial_a := \frac{1}{2} (\partial_u + \sigma \partial_v), \quad \partial_b := \frac{1}{2} (\partial_u - \sigma \partial_v),$$

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and \( da := du + \sigma dv \) and \( d\tilde{a} := du - \sigma dv \), the differential \( d\psi \) of a smooth map \( \psi : M \to A \) can be written as
\[
d\psi = \partial_a \psi \, da + \partial_{\tilde{a}} \psi \, d\tilde{a},
\]
thus, the condition \( \psi \) conformal is equivalent by (47) to \( \partial_{\tilde{a}} \psi = 0 \); hence, we have \( d\psi = \psi' da \), where \( \psi' := \partial_a \psi = \partial_a \psi : M \to A \) is a smooth map.

Defining the coordinates \((s, t)\) such that
\[
u + \sigma \, v = \frac{1 + \sigma}{2} \, s + \frac{1 - \sigma}{2} \, t
\]
(\(s\) and \(t\) are parameters along the distinguished lines) and writing
\[
\psi = \frac{1 + \sigma}{2} \, \psi_1 + \frac{1 - \sigma}{2} \, \psi_2
\]
with \( \psi_1, \psi_2 \in \mathbb{R} \), (47) reads
\[
\partial_t \psi_1 = \partial_s \psi_2 = 0,
\]
and we get
\[
\psi_1 = \psi_1(s) \quad \text{and} \quad \psi_2 = \psi_2(t);
\]
a conformal map is thus equivalent to two functions of one variable.

**Conformal 1–forms on Lorentz surfaces.** If \( M \) is a Lorentz surface, a smooth 1–form \( \omega : TM \to A \) can be written (in a chart \( \sigma = u + \sigma v : U \subset A \to M \)) as
\[
\omega = P \, du + Q \, dv,
\]
where \( P, Q : M \to A \) are smooth maps. If we suppose that \( \omega \) preserves the Lorentz structure, i.e.
\[
\omega(\sigma \, X) = \sigma \, \omega(X)
\]
for all \( X \in TM \), we have \( Q = \sigma P \) and \( \omega = P \, (du + \sigma dv) = P \, da \).

We shall say that a 1–form \( \omega = P \, da \) is a conformal 1–form if \( P : M \to A \) is a conformal map. We note that a conformal 1–form is the analogous to a holomorphic 1–form in complex analysis and we obtain by a direct computation the following classical theorem of integration: let \( f : U \subset A \to A \) be a smooth map, the exterior differential of the 1–form \( f \, da \) satisfies
\[
d(f \, da) = \partial_{\tilde{a}} f \, d\tilde{a} \wedge da
\]
and \( f \) is a conformal map if and only if \( f \, da \) is a closed 1–form.

**Remark 6.** If \( \psi : M \to A \) is a conformal map, its differential \( d\psi = \psi' da \) is a conformal 1–form: indeed we have
\[
\partial_{\tilde{a}} \psi' = \partial_2 (\partial_a \psi) = \partial_a (\partial_{\tilde{a}} \psi) = 0,
\]
i.e. \( \psi' : M \to A \) is a conformal map.

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