Symmetric vibrations of a liquid in a vessel with a separator and an elastic bottom

D A Goncharov* and A A Pozhalostin**
Bauman Moscow State Technical University, Moscow, Russia
E-mail: *goncharov@bmstu.ru, **a.pozhalostin@mail.ru

Abstract. The paper considers the problem of small axisymmetric vibrations of an ideal fluid filling a vessel with rigid walls and an elastic bottom. The liquid is divided into two layers by an elastic septum. The elastic baffle and the vessel elastic bottom are modeled by elastic membranes. The Neumann boundary-value problem is posed for the fluid. The equations of motion of the membranes are integrated with boundary conditions.

1. Introduction
The problem originates in practice. In rocket technology, various intra-tank structures are widely used to launch a propulsion systems under the space flight conditions [1]. These designs are significantly complicated for the frequency analysis. At the same time, the frequency analysis is very important, since the elastic tank, the propulsion path of the rocket engine, and the engine itself form a closed system with feedback in which self-oscillations are possible [2].

Since these phenomena and their influence on the spacecraft and launch vehicle operation are of great importance, the symmetric liquid oscillations in vessels of various shapes and configurations were studied in numerous works [3–8]. Some generalizations are given in the monograph [9].

2. Formulation of the problem
We consider small axisymmetric oscillations of an ideal fluid filling a vessel of cylindrical shape of radius $R$ with rigid walls. The vessel is divided by an elastic membrane. The bottom of the vessel is also an elastic membrane. We introduce cylindrical coordinates $O_1x_1y_1r$ and $O_2x_2y_2r$ with origins at the respective centers of the partition and the bottom. The liquid fills the areas $\vartheta_1$ and $\vartheta_2$. The motion of the fluid is assumed to be potential with velocity potential $\Phi_i = \Phi_i(x_i, r, t)$.

In the areas $\vartheta_1$ and $\vartheta_2$, the Laplace equation has the form

$$\nabla^2 \Phi_i = 0, \quad i = 1, 2,$$

where $\nabla$ is the Hamilton operator and $\Phi_i$ is the velocity potential in the area $\vartheta_i$. On the free surface, we pose the linearized boundary condition in simplified form

$$\Phi_1|_{x_1=h_1} = 0.$$

If the inertia of membranes is neglected, then their equations of motion take the following forms for the separating and bottom membranes:

\[ \nabla^2 \dot{w}_1 = \frac{\varrho}{\tau} \left( \frac{\partial^2 \Phi_1}{\partial t^2} \right)_{x_1=0} - \frac{\partial^2 \Phi_2}{\partial t^2} \bigg|_{x_2=h_2} , \quad w_1 \bigg|_{r=R} = 0, \quad w_1 \bigg|_{r=0} < \infty, \]  

(3)

\[ \nabla^2 \dot{w}_2 = \frac{\varrho}{\tau} \frac{\partial^2 \Phi_2}{\partial t^2} \bigg|_{x_2=0} , \quad w_2 \bigg|_{r=R} = 0, \quad w_2 \bigg|_{r=0} < \infty, \]  

(4)

where \( w_1 \) and \( w_2 \) are deflections of the separating and bottom membranes, \( \dot{w}_1 \) and \( \dot{w}_2 \) are velocities of the membrane deflections, and \( \tau \) is the tension of the membranes.

The kinematic conditions are also satisfied. In particular, the condition that the liquid velocities are the same in the areas \( \vartheta_1 \) and \( \vartheta_2 \):

\[ \frac{\partial \Phi_1}{\partial x_1} \bigg|_{x_1=0} = \frac{\partial \Phi_2}{\partial x_2} \bigg|_{x_2=h_2}, \]  

(5)

the condition that the fluid speed is equal to the rate of deflection of the separating membrane:

\[ \frac{\partial \Phi_2}{\partial x_2} \bigg|_{x_2=h_2} = \dot{w}_1, \quad \frac{\partial \Phi_1}{\partial x_1} \bigg|_{x_1=0} = \dot{w}_1, \]  

(6)

and the condition that the fluid speed is equal to the rate of deflection of the bottom membrane:

\[ \frac{\partial \Phi_2}{\partial x_2} \bigg|_{x_2=0} = \dot{w}_2. \]  

(7)

In addition, the non-flow conditions are satisfied:

\[ \frac{\partial \Phi_1}{\partial r} \bigg|_{r=R} = 0, \quad \frac{\partial \Phi_2}{\partial r} \bigg|_{r=R} = 0, \]  

(8)

3. Solution of the boundary-value problem

We consider the potential for the liquid velocities in the form

\[ \Phi_i = \Phi_{i0} e^{i\omega t}, \]  

(9)

where \( i \) is the imaginary unit, \( \omega \) is the natural frequency, and \( t \) denotes the time. Satisfying the Laplace equation, the non-flow conditions, and the condition on the free surface, we obtain expressions for the potentials of fluid velocities in the form

\[ \Phi_{10} = B_{10} \left( \frac{x_1}{h_1} - 1 \right) + \sum_{i=1}^{\infty} C_{1i} J_0 \left( \xi_i \frac{r}{R} \right) \left[ \cosh \left( \xi_i \frac{x_1}{R} \right) - \coth \left( \xi_i \frac{h_1}{R} \right) \sinh \left( \xi_i \frac{x_1}{R} \right) \right], \]  

(10)

\[ \Phi_{20} = A_{20} + B_{20} \frac{x_2}{p_2} + \sum_{i=1}^{\infty} J_0 \left( \xi_i \frac{r}{R} \right) \left[ C_{2i} \cosh \left( \xi_i \frac{x_2}{R} \right) + C_{3i} \sinh \left( \xi_i \frac{x_2}{R} \right) \right]. \]  

(11)

Here \( B_0, A_{20}, B_{20}, C_{1i}, C_{2i}, C_{3i} \) are constants of integration, \( J_0(\xi_i dr/dR) \) is the Bessel function of the first kind of order zero, and \( \xi_i \) is the root of equation \( J'_0(\xi) = 0 \) obtained from the non-flow
The coefficients are determined from the following relations

\[ \nabla^2 \ddot{w}_1 = \frac{\varrho \omega^2}{r} \left\{ - \left\{ A_{20} + B_{20} + \sum_{i=1}^{\infty} J_0 \left( \xi_i \frac{r}{R} \right) \left[ C_{2i} \cosh \left( \xi_i \frac{x_2}{R} \right) + C_{3i} \sinh \left( \xi_i \frac{x_2}{R} \right) \right] \right\} - B_0 + \sum_{i=1}^{\infty} C_{1i} J_0 \left( \xi_i \frac{r}{R} \right) \right\}, \]  

(12)

\[ \nabla^2 \ddot{w}_2 = \frac{\varrho \omega^2}{r} \left\{ \left[ A_{20} + \sum_{i=1}^{\infty} C_{2i} J_0 \left( \xi_i \frac{r}{R} \right) \right] \right\}. \]  

(13)

Integrating the above equations, we obtain the following expressions for the rates of deflection of the separate and bottom membrane, respectively

\[ \dot{w}_1 = \frac{\varrho \omega^2}{r} \left\{ C_{10} - (A_{20} + B_{20}) \frac{r^2}{4} + \sum_{i=1}^{\infty} \frac{R^2}{\xi_i^2} \left[ C_{2i} \cosh \left( \frac{\xi_i x_2}{R} \right) + C_{3i} \sinh \left( \frac{\xi_i x_2}{R} \right) \right] - B_0 + \sum_{i=1}^{\infty} \frac{R^2}{\xi_i} C_{1i} J_0 \left( \frac{\xi_i r}{R} \right) \right\}, \]  

(14)

\[ \dot{w}_2 = C_0 + \frac{\varrho \omega^2}{r} A_{20} \frac{r^2}{4} + \frac{\varrho \omega^2}{r} \sum_{i=1}^{\infty} \frac{R^2}{\xi_i} C_{2i} J_0 \left( \frac{\xi_i r}{R} \right). \]  

(15)

To complete the system of arbitrary constants of integration in expressions (14) and (15), we use the following procedure [10]. We represent \( r^2/4 \) as a series in Bessel eigenfunctions

\[ \frac{r^2}{4} = \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) + \beta_0. \]  

(16)

The coefficients are determined from the following relations

\[ \int_0^R \frac{r^3}{4} dr = \int_0^R \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) \frac{r}{dr} + \int_0^R \frac{r}{dr} \beta_0 dr, \]  

(17)

\[ \int_0^R \frac{r^3}{4} J_0 \left( \frac{\xi_i r}{R} \right) \frac{r}{dr} = \int_0^R \sum_{i=1}^{\infty} \beta_{1i} J_0^2 \left( \frac{\xi_i r}{R} \right) \frac{r^2}{dr} + \int_0^R \frac{r^2}{dr} \beta_0 J_0^2 \left( \frac{\xi_i r}{R} \right) \frac{r}{dr}. \]  

(18)

In this case, the expressions for the deflections rates of the separating and bottom membranes become

\[ \ddot{w}_1 = \frac{\varrho \omega^2}{r} \left\{ C_{10} - (A_{20} + B_{20}) \frac{r^2}{4} + \sum_{i=1}^{\infty} \frac{R^2}{\xi_i^2} \left[ C_{2i} \cosh \left( \frac{\xi_i h_2}{R} \right) + C_{3i} \sinh \left( \frac{\xi_i h_2}{R} \right) \right] J_0 \left( \frac{\xi_i r}{R} \right) - B_0 \left( \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) + \beta_0 \right) + \sum_{i=1}^{\infty} \frac{R^2}{\xi_i^2} C_{1i} J_0 \left( \frac{\xi_i r}{R} \right) \right\}, \]  

(19)

\[ \ddot{w}_2 = C_0 + A_{20} \frac{\varrho \omega^2}{r} \left\{ \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) + \beta_0 \right\} + \frac{\varrho \omega^2}{r} \sum_{i=1}^{\infty} \frac{R^2}{\xi_i^2} C_{2i} J_0 \left( \frac{\xi_i r}{R} \right). \]  

(20)
With the last expression taken account, the kinematic conditions can be written as

\[
\frac{\mu^2}{\tau} \left\{ C_{10} - (A_{20} + B_{20}) \left[ \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) + \beta_0 \right] 
+ \sum_{i=1}^{\infty} \frac{R^2}{x_i^2} C_{2i} \cosh \left( \frac{\xi_i h_2}{R} \right) + C_{3i} \sinh \left( \frac{\xi_i h_2}{R} \right) \right] J_0 \left( \frac{\xi_i r}{R} \right) 
- B_0 \left\{ \sum_{i=1}^{\infty} \beta_{1i} J_0 \left( \frac{\xi_i r}{R} \right) + \beta_0 \right\} + \sum_{i=1}^{\infty} \frac{R^2}{x_i^2} C_{1i} J_0 \left( \frac{\xi_i r}{R} \right) \right\} 
= \frac{B_{20}}{h_2} + \sum_{i=1}^{\infty} J_0 \left( \frac{\xi_i r}{R} \right) \left[ C_{2i} \sinh \left( \frac{\xi_i h_2}{R} \right) + C_{3i} \cosh \left( \frac{\xi_i h_2}{R} \right) \right] \frac{\xi_i r}{R}. \tag{21}
\]

In addition, in view of (5), we have

\[
\frac{B_0}{h_1} + \sum_{i=1}^{\infty} C_{1i} J_0 \left( \frac{\xi_i r}{R} \right) \frac{\xi_i}{R} \coth \left( \frac{\xi_i h_1}{R} \right) 
= \frac{B_{20}}{h_2} + \sum_{i=1}^{\infty} J_0 \left( \frac{\xi_i r}{R} \right) \left[ C_{2i} \sinh \left( \frac{\xi_i h_2}{R} \right) + C_{3i} \cosh \left( \frac{\xi_i h_2}{R} \right) \right]. \tag{23}
\]

Thus, we obtained a system of equations that allows us to close the system of arbitrary constants. In particular, we have

\[
B_{20} = B_0 \frac{h_1}{h_2}, \quad C_0 = \frac{B_0}{h_1} - A_{20} \beta_0, \quad C_{10} = \frac{B_0}{h_1} + A_{20} \beta_0 + B_0 \beta_0,
\]

\[
C_{1i} = \frac{1}{\coth(\xi_i h_1/R)} \left[ C_{2i} \sinh \left( \frac{\xi_i h_2}{R} \right) + C_{3i} \cosh \left( \frac{\xi_i h_2}{R} \right) \right],
\]

\[
C_{2i} = a_{10}^{(i)} B_0 + a_{20}^{(i)} A_{20}, \quad C_{3i} = b_{10}^{(i)} B_0 + b_{20}^{(i)} A_{20},
\]

where, in turn,

\[
a_{10}^{(i)} = -\frac{\varepsilon_{2i}}{\varepsilon_{1i}}, \quad a_{20}^{(i)} = -\frac{\varepsilon_{1i}}{\varepsilon_{1i}}, \quad b_{20}^{(i)} = \frac{\omega^2}{\tau} \beta_{1i} \frac{R}{\xi_i} + \frac{\omega^2}{\tau} \beta_{1i} \frac{R^3}{\xi_i^3} a_{20}^{(i)}, \quad b_{10}^{(i)} = \frac{\omega^2}{\tau} \beta_{1i} \frac{R^2}{\xi_i^2} a_{10}^{(i)}.
\]
Here

\[ \varepsilon_{11} = \frac{\varrho \omega^2 R^2}{\tau} \cot \left( \frac{\xi_i h_2}{R} \right) + \frac{\varrho \omega^4 R^5}{\tau^2} \sinh \left( \frac{\xi_i h_2}{R} \right) + \frac{\varrho \omega^2 R^4}{\tau} \sin \left( \frac{\xi_i h_2}{R} \right) \tan \left( \frac{\xi_i h_2}{R} \right) + \frac{\varrho \omega^2 R^4}{\tau} \cosh \left( \frac{\xi_i h_2}{R} \right) \tan \left( \frac{\xi_i h_2}{R} \right) - \frac{\varrho \omega^2 R^3}{\tau} \sinh \left( \frac{\xi_i h_1}{R} \right) \tan \left( \frac{\xi_i h_1}{R} \right), \]

\[ \varepsilon_{21} = -\frac{\varrho \omega^2}{\tau} \beta_{11} + \frac{\varrho \omega^2}{\tau} \frac{R^3}{\xi_i} \beta_{11} \sin \left( \frac{\xi_i h_2}{R} \right) + \frac{\varrho \omega^2}{\tau} \frac{R^5}{\xi_i} \beta_{11} \cosh \left( \frac{\xi_i h_2}{R} \right), \]

\[ \varepsilon_{31} = -\frac{\varrho \omega^2}{\tau} \beta_{11} - \frac{\varrho \omega^2}{\tau} \frac{R^2}{\xi_i} \beta_{11}. \]

A closed system of arbitrary constants makes it possible to construct the system

\[
\begin{align*}
& a_{11} B_0 + a_{12} A_{20} = 0 \quad \text{from } \dot{w}_1|_{r=R} = 0, \\
& a_{21} B_0 + a_{22} A_{20} = 0 \quad \text{from } \dot{w}_2|_{r=R} = 0.
\end{align*}
\]

(24)

From the this system we can obtain the frequency equation in the form

\[ a_{11} a_{22} - a_{12} a_{21} = 0. \]

(25)

Conclusions

Thus, we showed that, adopting a number of simplifying assumptions, it is possible to obtain an analytical solution of the problem of fluid oscillations in a vessel with an elastic bottom and a separating membrane. In further studies, it will be possible to modify the condition on the free surface, since the above solution method works for various conditions on the free surface. In addition, it is possible to consider membranes without neglecting their inertia, but then the expressions become much more complicated. The frequency equation is very important for studying various technical systems, since it allows modeling of real systems by using mechanical analogs and laboratory experiments.

References

[1] Bagrov V V, Kurpatenkov A V, and Polyaev V M 1997 Capillary Systems for Fluid Extraction from Tanks of Spacecrafts (Moscow: UNPTs Energomash) [in Russian]
[2] Mikishev G N and Rabinovich B I 1971 Dynamics of Thin-Walled Structures with Compartments Containing Liquid (Moscow: Mashinostroyenie) [in Russian]
[3] Okhotsinskii D E 1956 Theory of motion of a body with cavities partially filled with liquid Prikl. Mat. Mekh. 20 (1) 3–20
[4] Chernousko F L 1966 On free oscillations of a viscous fluid in a vessel J. Appl. Math. Mech. 30 (5) 990–1003
[5] Bauer H F 1981 Hydroelastische schwingungen einer oberflächen spannungsstruktur in einem sateliten behälter Z. Flugwiss. Weltraumforsch. 5 (5) 303–13
[6] Kononov Yu N and Tatarenko E A 2006 Free vibrations of elastic membranes that separate a multilayer liquid in a cylindrical vessel with an elastic bottom Dynamich. sist. No 21 7–13
[7] Kravtsov A V and Sekerzh-Zeni’kovich S Ya 2004 On free vibrations of a low-viscosity liquid in a vessel partially filled with a porous medium Zh. Vych. Mat. Mat. Fiz. 44 (4) 746–51
[8] Bakreev V I, Sturova I V, and Chebotnikov A V 2014 Seiche oscillations in a reservoir filled with a double-layer fluid Fluid Dynamics 49 (3) 395–402
[9] Ibrahim R A 2005 Liquid Sloshing Dynamics: Theory and Applications (Cambridge: Cambridge Univ. Press)
[10] Pozhalostin A A and Goncharov D A 2015 Free axisymmetric oscillations of a two-layer liquid with an elastic separator between layers Russ. Aeronaut. 58 (1) 37–41