The Regularity of Some Vector-Valued Variational Inequalities with Gradient Constraints

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Abstract. We prove the optimal regularity for some class of vector-valued variational inequalities with gradient constraints. We also give a new proof for the optimal regularity of some scalar variational inequalities with gradient constraints. In addition, we prove that some class of variational inequalities with gradient constraints are equivalent to an obstacle problem, both in the scalar case and in the vector-valued case.

1. Introduction. Let \( U \subset \mathbb{R}^n \) be an open bounded set. Suppose \( K \subset \mathbb{R}^n \) is a balanced (symmetric with respect to the origin) compact convex set whose interior contains 0. Also suppose that \( \eta \in \mathbb{R}^N \) is a fixed nonzero vector. Consider the following problem of minimizing

\[
I(v) := \int_U |Dv|^2 - \eta \cdot v \, dx
\]

over

\[
K_1 := \{ v = (v^1, \cdots, v^N) \in H^1_0(U; \mathbb{R}^N) \mid \|Dv\|_{2,K} \leq 1 \text{ a.e.} \},
\]

Where

\[
\|A\|_{2,K} := \sup_{z \neq 0} \frac{|Az|}{\gamma_K(z)}
\]

for an \( N \times n \) matrix \( A \), and \( \gamma_K \) is the norm associated to \( K \) defined by

\[
\gamma_K(x) := \inf\{ \lambda > 0 \mid x \in \lambda K \}.
\]

As \( K_1 \) is a closed convex set and \( I \) is coercive, bounded and weakly sequentially lower semicontinuous, this problem has a unique solution \( u \). We will show that under some extra assumptions on \( K \)

\[
u \in C^{1,1}_{\text{loc}}(U; \mathbb{R}^N).
\]

This problem is a generalization to the vector-valued case of the elastic-plastic torsion problem, which is the problem of minimizing

\[
J_\eta(v) := \int_U |Du|^2 - \eta v \, dx
\]

2000 Mathematics Subject Classification. Primary: 35J88, 35B65; Secondary: 35R35, 49J40.

Key words and phrases. Vector-valued variational inequalities, equivalence of variational inequalities, gradient constraint, regularity, obstacle problem.
for some $\eta > 0$, over

$$\{ v \in H^1_0(U) \mid |Dv| \leq 1 \text{ a.e.} \}.$$  

The regularity of the elastic-plastic torsion problem has been studied by Brezis and Stampacchia [2], and Caffarelli and Riviere [3]. There has been several extensions of their results to more general scalar problems with gradient constraints. See for example [4, 6, 8, 9, 14]. To the best of author’s knowledge, the only work on the regularity of vector-valued problems with gradient constraints is [12].

Our approach is to show that the above vector-valued problem is reducible to the scalar problem of minimizing $J_1$ over

$$\{ v \in H^1_0(U) \mid |\eta Dv| \in K^0 \text{ a.e.} \},$$

where $K^0$ is the polar of $K$ (See section 2). Then we show that this scalar problem is equivalent to a double obstacle problem with only Lipschitz obstacles. At the end, we generalize the proof of Caffarelli and Riviere [3], to obtain the optimal regularity.

In the process described above, we also show that our vector-valued problem with gradient constraint is equivalent to a vector-valued obstacle problem. This result, which is the first result of its kind as far as the author knows, is a generalization to the vector-valued case of the equivalence between the elastic-plastic torsion problem and an obstacle problem, proved by Brezis and Sibony [1]. Later Treu and Vornicescu [13] proved that the equivalence holds for a larger class of scalar variational inequalities with gradient constraints. We will further generalize their result.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are convex functions. Consider the problem of minimizing

$$J(v) := \int_U f(Dv(x)) + g(v(x)) \, dx$$

over

$$W_K := \{ v \in u_0 + W^{1,p}_0(U) \mid Dv(x) \in K \text{ a.e.} \},$$

where $u_0 \in W^{1,p}(U)$. We will show that under appropriate assumptions, the minimizer of $J$ over $W_K$ is the same as its minimizer over

$$W_{u^-,u^+} := \{ v \in u_0 + W^{1,p}_0(U) \mid u^-(x) \leq v(x) \leq u^+(x) \text{ a.e.} \},$$

for some suitable functions $u^-, u^+$. The difference of our result with that of Treu and Vornicescu [13] is that we allow $f, g$ to be only convex, and $K$ to have empty interior. Some of our results has been proved using different means by Mariconda and Treu [10].

2. The equivalence in the scalar case. Suppose $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin. Let $J$, $W_K$, and $W_{u^-,u^+}$ be as above. We assume that on $W^{1,p}(U)$, $J$ is finite, coercive, and sequentially weakly lower semicontinuous. These assumptions are satisfied if in addition to the convexity of $f, g$, we impose some growth condition on them, and require $\partial U$ to be mildly regular, for example Lipschitz (If $u_0 \equiv 0$ we do not need any assumption about $\partial U$). Therefore, $J$ attains its minimum on any nonempty closed convex subset of $W^{1,p}(U)$.

Furthermore, we assume that $u_0$ is Lipschitz, and

$$Du_0 \in K \quad \text{ a.e.}$$

Thus in particular, $W_K$ is nonempty.
Definition 2.1. The gauge of $K$ is a convex function defined by
\[ \gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}, \]
and its polar is the convex set
\[ K^\circ := \{x \mid x \cdot k \leq 1 \text{ for all } k \in K\}. \]
We recall that for all $x, y \in \mathbb{R}^n$, we have
\[ x \cdot y \leq \gamma_K(x)\gamma_K^*(y). \]
Its proof can be found in [11]. Also, when $K$ is balanced, $K^\circ$ is balanced too, and $\gamma_K, \gamma_K^*$ are both norms on $\mathbb{R}^n$.

Now, let us find $u^\pm \in W_K$ such that for all $u \in W_K$ we have $u^- \leq u \leq u^+$. Let $u^\pm$ be respectively the unique minimizers of $J^\pm(v) = \int_U \pm v(x) \, dx$ over $W_K$. We show that they have the desired property. We need the following lemma.

Lemma 2.2. Suppose $u$ is a function in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ with $Du \in K$ a.e.. Then
\[ u(y) - u(x) \leq \gamma_K^*(y - x) \]
for all $x, y$.

Proof. Consider the mollifications
\[ u_\varepsilon(x) := (\eta_\varepsilon * u)(x) := \int_{B_\varepsilon(x)} \eta_\varepsilon(x - y)u(y) \, dy, \]
where $\eta_\varepsilon$ is a nonnegative smooth function with support in $B_\varepsilon(0)$, and $\int_{B_\varepsilon(0)} \eta_\varepsilon \, dx = 1$. Then we know that $u_\varepsilon$ converges to $u$ a.e., and $Du_\varepsilon = \eta_\varepsilon * Du$. Hence
\[ \gamma_K(Du_\varepsilon(x)) \leq \int_{B_\varepsilon(x)} \gamma_K(\eta_\varepsilon(x - y)Du(y)) \, dy \]
\[ = \int_{B_\varepsilon(x)} \eta_\varepsilon(x - y)\gamma_K(Du(y)) \, dy \leq 1, \]
where we used Jensen’s inequality in the first inequality. Thus
\[ u_\varepsilon(y) - u_\varepsilon(x) = \int_0^1 Du_\varepsilon(x + t(y - x)) \cdot (y - x) \, dt \]
\[ \leq \int_0^1 \gamma_K(Du_\varepsilon(x + t(y - x)))\gamma_K^*(y - x) \, dt \leq \gamma_K^*(y - x). \]
Now we can let $\varepsilon \to 0$ to obtain
\[ u(y) - u(x) \leq \gamma_K^*(y - x) \quad \text{for a.e. } x, y. \]
We can redefine $u$ on the measure zero set where this relation fails, in a similar way that we extend Lipschitz functions to the closure of their domains. The extension will satisfy this relation everywhere. 

Lemma 2.3. Each function in $W_K$ is Lipschitz continuous. Also, $W_K$ is bounded in $L^\infty(U)$ and in $W^{1,p}(U)$.

Proof. To see this, let $u \in W_K$. Then $u = u_0 + v$ where $v \in W^{1,p}_0(U)$. Thus
\[ |Dv| = |Du - Du_0| < 2R \]
for some $R > 0$. Now we can extend $v$ by zero to all of $\mathbb{R}^n$, and the extension will satisfy the same gradient bound. Therefore by arguments similar to the previous lemma, we can see that the extension of $v$, and hence $v$ itself, is Lipschitz with
Lipschitz constant $2R$. Using the fact that $v$ is zero on the boundary, this also implies that $\|v\|_{L^\infty} \leq 2RD$, where $D$ is the diameter of $U$. The result for $u$ follows easily, noting that $w_0$ is Lipschitz.

Now as $\|Du\|_{L^\infty} < C$ for some constant $C$ independent of $u$, we have $\|Du\|_{L^p} < C$ since $U$ is bounded. Noting that all $u \in W_K$ have the same boundary value, we get by Poincare inequality $\|u\|_{W^{1,p}} < C$.

Now we can see that $J^\pm$ are bounded on $W_K$. As $J^\pm$ are linear, they are weakly continuous. Furthermore $W_K$ is convex, closed and bounded in $W^{1,p}(U)$. Hence $W_K$ is compact with respect to sequential weak convergence. These imply that $J^\pm$ have minimizers over $W_K$. The uniqueness and the fact that $u^- \leq u^+$ a.e. on $U$, follows from a similar argument to the proof of the next lemma.

**Lemma 2.4.** We have

$$W_K \subset W_{u^-,u^+}.$$  

**Proof.** Suppose $u \in W_K$, then $J^\pm(u^\pm) \leq J^\pm(u)$. Thus

$$\int_U u^\pm dx \leq \int_U u dx,$$

so

$$\int_U u^- dx \leq \int_U u dx \leq \int_U u^+ dx.$$

Suppose to the contrary that, for example, the set $E := \{x \mid u(x) > u^+(x)\}$ has positive measure. Consider the function

$$w(x) := \max(u,u^+) = \begin{cases} u^+(x) & x \notin E \\ u(x) & x \in E. \end{cases}$$

The derivative of $w$ is

$$Dw(x) = \begin{cases} Du^+(x) & x \notin E \\ Du(x) & x \in E \end{cases} \text{ for a.e. } x.$$  

Therefore we have $Dw(x) \in K$ a.e.. But

$$J^+(w) = -\int_U w^+ dx < -\int_U u^+ dx = J^+(u^+),$$

which is a contradiction. □

The following characterization of $u^\pm$ will be used later. Here $d_{K^\circ}$ is the metric associated to the norm $\gamma_{K^\circ}$.

**Theorem 2.5.** Suppose $K$ is a balanced compact convex set whose interior contains 0, and $w_0$ equals a constant $c$ everywhere. Then

$$u^\pm(x) = c \pm d_{K^\circ}(x,\partial U).$$

**Proof.** It is enough to show that $c \pm d_{K^\circ}(x,\partial U)$ are the minimizers of $J^\pm$. The fact that $c \pm d_{K^\circ}(x,\partial U)$ belong to $W_K$ is equivalent to the fact that $d_{K^\circ}(x,\partial U)$ is in $W^{1,p}_0(U)$ and its derivative has $\gamma_{K}$ norm less than one. But $d_{K^\circ}(x,\partial U)$ is a Lipschitz function that vanishes on the boundary of $U$. It also satisfies

$$d_{K^\circ}(x,\partial U) - d_{K^\circ}(y,\partial U) \leq \gamma_{K^\circ}(x - y).$$

As proved in [13], this last property implies that the $\gamma_{K}$ norm of the derivative of $d_{K^\circ}(x,\partial U)$ is less than or equal to 1 a.e..
Now similarly to the proof of Lemma 2.2, we can show that

\[ |v(x) - c| \leq d_{K^*}(x, \partial U) \]

for all \( v \in W_K \). Therefore \( c \pm d_{K^*}(x, \partial U) \) minimize \( J^\pm \) over \( W_K \).

\[ \square \]

The following theorem is the generalization of the result of Treu and Vornicescu [13]. We removed the assumptions on the derivatives of \( g \), and allowed \( K \) to have empty interior.

**Theorem 2.6.** Suppose \( K \) is a compact convex set containing \( 0 \), and \( u_0 \) is the restriction to \( U \) of a function in \( W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) with gradient a.e. in \( K \). Also, suppose \( f, g \) are convex and at least one of them is strictly convex. Then the minimizer of

\[ J(v) = \int_U f(Dv(x)) + g(v(x)) \, dx \]

over \( W_{u^- - u^+} \) is the same as its minimizer over \( W_K \).

**Proof.** Note that the convexity assumptions on \( f, g \) imply that the minimizer of \( J \) over any nonempty convex closed set is unique. Also the assumption on \( u_0 \) implies \( u_0(y) - u_0(x) \leq \gamma_{K^*}(y - x) \) for all \( x, y \), by Lemma 2.2. Let the minimizer of \( J \) over \( W_{u^- - u^+} \) be \( u \). As \( W_K \subset W_{u^- - u^+} \), it is enough to show that \( u \in W_K \).

First assume that \( 0 \) is in the interior of \( K \), and \( g \) is \( C^1 \) with strictly increasing derivative.

Similarly to [13], using \( u_0 \) we can extend \( u^\pm \) and \( u \) to all of \( \mathbb{R}^n \) in a way that the gradient of \( u^\pm \) is still in \( K \). Fix a nonzero vector \( h \in \mathbb{R}^n \), and define

\[
\begin{align*}
u^+_h(x) &:= \max\{u(x + h) - \gamma_{K^*}(h), u(x)\} \\
u^-_h(x) &:= \min\{u(x - h) + \gamma_{K^*}(h), u(x)\},
\end{align*}
\]

and

\[
\begin{align*}
E^+ &:= \{ x \in \mathbb{R}^n \mid u^+_h(x) = u(x + h) - \gamma_{K^*}(h) > u(x)\} \\
E^- &:= \{ x \in \mathbb{R}^n \mid u^-_h(x) = u(x - h) + \gamma_{K^*}(h) < u(x)\}.
\end{align*}
\]

The following assertions are easy to check

i) \( u^+_h \in W_{u^- - u^+} \).

ii) \( E^+ \setminus U \) have measure zero.

iii) \( E^+ = E^- - h \).

Now for any \( 0 < \lambda < 1 \) we have \( (i = 1, \cdots, m) \)

\[
\begin{align*}
J(u + \lambda(u^+_h - u) - J(u) &\leq \int_{E^+} f(Du(x) + \lambda(Du(x + h) - Du(x))) \\
&- f(Du(x)) + g(u(x) + \lambda(u(x + h) - \gamma_{K^*}(h) - u(x))) - g(u(x)) \, dx \geq 0,
\end{align*}
\]

and

\[
\begin{align*}
J(u + \lambda(u^-_h - u) - J(u) &\leq \int_{E^-} f(Du(x) + \lambda(Du(x - h) - Du(x))) \\
&- f(Du(x)) + g(u(x) + \lambda(u(x - h) + \gamma_{K^*}(h) - u(x))) - g(u(x)) \, dx \geq 0.
\end{align*}
\]

By changing the variable from \( x \) to \( x + h \) in the last integral, we get

\[
\begin{align*}
\int_{E^+} f(Du(x + h) + \lambda(Du(x) - Du(x + h))) &- f(Du(x + h)) \\
&+ g(u(x + h) + \lambda(u(x) + \gamma_{K^*}(h) - u(x + h))) - g(u(x + h)) \, dx \geq 0.
\end{align*}
\]
Adding this to the first integral and using the convexity of $f$, we have
\[
\int_{E^+} g(u(x + h) + \lambda(u(x) + \gamma_{K^*}(h) - u(x + h))) - g(u(x + h)) + g(u(x) + \lambda(u(x + h) - \gamma_{K^*}(h) - u(x))) - g(u(x)) \, dx \geq 0.
\]
We divide this inequality by $\lambda > 0$ and take the limit as $\lambda \to 0$. Then, as $g$ is $C^1$ and $u$ is bounded, by the Dominated Convergence Theorem we get
\[
\int_{E^+} [g'(u(x + h)) - g'(u(x))](u(x) - u(x + h) + \gamma_{K^*}(h)) \, dx \geq 0.
\]
But on $E^+$, $u(x) - u(x + h) + \gamma_{K^*}(h) < 0$. Also $g'$ is strictly increasing and therefore $g'(u(x + h)) - g'(u(x)) > 0$. Hence $E^+$ must have measure zero. This means that for a.e. $x \in \mathbb{R}^n$
\[
u(x + h) - u(x) \leq \gamma_{K^*}(h).
\]
Taking $h \to 0$ (through a countable sequence) we get
\[
D_h u(x) \leq \gamma_{K^*}(h),
\]
for a.e. $x$. This implies $\gamma_K(Du(x)) \leq 1$, which is equivalent to $u \in W_K$.

Now suppose that we only have $0 \in K$. Let
\[
K_i := \{x + y \mid x \in K, |y| \leq \frac{1}{i}\} = \{z \mid d(z, K) \leq \frac{1}{i}\}.
\]
Then $\{K_i\}$ is a decreasing family of compact convex sets containing $K$ with $0 \in \text{int } K_i$. Therefore $\{W_{K_i}\}$ is also a decreasing family containing $W_K$. Let $u_i^{\pm}$ be the corresponding obstacles to $W_{K_i}$. Then we have $u_i^{\pm} \geq u^+$ and $u_i^{-} \leq u^-$. Also $u_i^{\pm}$ decreases with $i$, and $u_i^{-}$ increases with $i$. Thus $\{W_{u_i^{-},u_i^{+}}\}$ is a decreasing family too and contains $W_{u^{-},u^{+}}$.

Let $u_i$ be the minimizer of $J$ over $W_{K_i}$. We have $D u_0 \in K \subset K_i$. Therefore we can apply the previous argument and we have $J(u_i) \leq J(v)$ for all $v \in W_{u_i^{-},u_i^{+}}$. Now as $u_i$’s are all in $W_{K_i}$, we have $\|u_i\|_{W^{1,p}} < C$ for some universal $C$.

Therefore there is a subsequence of $u_i$’s, where we denote it by $u_{i_k}$, which converges weakly in $u_0 + W^{1,p}_{0}(U)$ to $u$. By weak lower semicontinuity of $J$ we get $J(u) \leq \liminf J(u_{i_k}) \leq J(v)$ for all $v \in W_{u^{-},u^{+}}$. Thus to finish the proof we only need to show that $u \in W_K$. To see this note that the sequence $u_{i_k}$ is eventually in each $W_{K_{i_k}}$ and as these are closed convex sets they are weakly closed, hence $u \in W_{K_{i_k}}$ for all $k$. This means $d(Du, K) \leq \frac{1}{i_k}$ a.e. Thus $d(Du, K) = 0$ a.e., and by closedness of $K$ we get the desired result.

Next suppose that $g$ is only convex. Consider the mollifications $g_* := \eta_* \ast g$, where $\eta_*$ is the standard mollifier. First let us show that $g_*$ is convex too. We have
\[
g_*(\lambda x + (1 - \lambda)y) = \int \eta_*(z)g(\lambda x + (1 - \lambda)y - z) \, dz
\leq \int \eta_*(z)[\lambda g(x - z) + (1 - \lambda)g(y - z)] \, dz
\leq \lambda g_*(x) + (1 - \lambda)g_*(y).
\]
Now let
\[
J_i(v) := \int f(Dv) + g_*(v) + \frac{1}{2} v^2 \, dx.
\]
Then since $g_*(v) + \frac{1}{2} v^2$ is a smooth strictly convex function, it has strictly increasing derivative. Let $u_i$ be the minimizer of $J_i$ over $W_K$. Then by the above we have
\( J_i(u_i) \leq J_i(v) \) for all \( v \in W_{u_i,u_i}^- \). As the \( u_i \)'s are in \( W_K \), and \( W_K \) is bounded in \( W^{1,p} \), we can say that there is a subsequence of \( u_i \), which we continue to denote it by \( u_i \), that converges weakly to \( u \in W_K \).

Since \( g_1 \) uniformly converges to \( g \) on compact sets, and for \( v \in W_{u_i,u_i}^- \) we have \( \|v\|_{L^\infty} < C \) for some constant \( C \) independent of \( v \), we have for \( i \) large enough and independent of \( v \)

\[
|J_i(v) - J(v)| \leq \int_U |g_1(v) - g(v)| + \frac{1}{i}v^2 \, dx < \delta.
\]

Hence \( J(u_i) \leq J(v) + 2\delta \). Then by weak lower semicontinuity of \( J \) we have \( J(u) \leq \liminf J(u_i) \leq J(v) + 2\delta \). Since \( \delta \) is arbitrary we get that \( u \) is the minimizer of \( J \) over \( W_{u_i,u_i}^- \) as required.

**Remark 1.** We can also prove a version of this theorem when \( 0 \notin K \), by translating \( K \). But we need to have a bound on the distance of \( K \) and the origin.

### 3. The equivalence in the vector-valued case

Suppose \( K \subset \mathbb{R}^n \) is a balanced compact convex set whose interior contains 0. Also suppose that \( \eta \in \mathbb{R}^N \) is a fixed nonzero vector. Consider the following problems of minimizing

\[
I(v) := \int_U |Dv|^2 - \eta \cdot v \, dx
\]

over

\[
K_1 := \{ v = (v^1, \cdots, v^N) \in H^1_0(U; \mathbb{R}^N) \mid \|Dv\|_{2,K} \leq 1 \text{ a.e.} \},
\]

and over

\[
K_2 := \{ v = (v^1, \cdots, v^N) \in H^1_0(U; \mathbb{R}^N) \mid |v(x)| \leq d_K(x, \partial U) \text{ a.e.} \}.
\]

Where

\[
\|A\|_{2,K} := \sup_{z \neq 0} \frac{|Az|}{\gamma_K(z)}
\]

for an \( n \times n \) matrix \( A \), and \( \gamma_K, d_K \) are respectively the norm associated to \( K \) and the metric of that norm. We show that these problems are equivalent.

As both \( K_1, K_2 \) are closed convex sets and \( I \) is coercive, bounded and weakly sequentially lower semicontinuous, both problems have unique solution.

**Lemma 3.1.** We have

\( K_1 \subseteq K_2 \).

**Proof.** To see this let \( v \in K_1 \). Similarly to the proof of Lemma 2.2 we obtain

\[
|v(y) - v(x)| \leq \gamma_K(y - x)
\]

for a.e. \( x, y \). Using this relation we can redefine \( v \) on a set of measure zero the same way that we extend Lipschitz functions. Therefore we can assume that \( v \) is continuous. Now as \( v \) is 0 on \( \partial U \), we can choose \( x \) to be the closest point on \( \partial U \) to \( y \) with respect to \( d_K \), and get the desired result. \( \square \)

**Lemma 3.2.** Let \( u = (u^1, \cdots, u^N) \) be the minimizer of \( I \) over \( K_2 \), and let

\[
T = (T^k) : \mathbb{R}^N \to \mathbb{R}^N
\]

be an orthogonal linear map that fixes \( \eta \). Then \( Tu \in K_2 \) and

\[
I(Tu) = I(u).
\]
Proof. To see this note that $Tu \in H^1_0(U; \mathbb{R}^N)$ and as $T$ preserves the norm, for a.e. $x$ we have

$$|Tu(x)| = |u(x)| \leq d_K(x, \partial U).$$

Furthermore as $T$ is orthogonal we have

$$|Du|^2 = \sum_i \sum_k (T^k_i D_i u')^2 = \sum_i \sum_l (D_i u')^2 = |Du|^2.$$

Hence (since $T\eta = \eta$ and $T$ is orthogonal)

$$I(Tu) = \int_U |Du|^2 - \eta \cdot Tu\, dx = \int_U |Du|^2 - T\eta \cdot Tu\, dx = \int_U |Du|^2 - \eta \cdot u\, dx = I(u).$$

\(\square\)

Theorem 3.3. We have

$$u(x) = u(x)\eta,$$

where $u$ is the minimizer of

$$J_1(v) := \int_U |Dv|^2 - v\, dx$$

over

$$K_3 := \{ v \in H^1_0(U; \mathbb{R}) \mid |v(x)| \leq \frac{1}{|\eta|} d_K(x, \partial U) \text{ a.e.} \}.$$

Proof. By the above lemma and uniqueness of the minimizer, we must have $Tu = u$ for all orthogonal linear maps $T$ that fix $\eta$. This implies that $u(x) = u(x)\eta$ for some scalar function $u$. Now we have

$$|u(x)\eta| = |u| \leq d_K(x, \partial U).$$

Hence for a.e. $x$

$$|u(x)| \leq \frac{1}{|\eta|} d_K(x, \partial U).$$

Also we have

$$D_i u = D_i u\eta.$$

Thus

$$I(u) = \int_U |\eta|^2 |Du|^2 - |\eta|^2 u\, dx = |\eta|^2 \int_U |Du|^2 - u\, dx = |\eta|^2 J_1(u).$$

It is easy to see that $u$ is the minimizer of $J_1$ over $K_3$. Because for any $w \in K_3$ we have $w\eta \in K_2$, therefore

$$J_1(u) = |\eta|^{-2} I(u\eta) = |\eta|^{-2} I(u) \leq |\eta|^{-2} I(w\eta) = J_1(w).$$

\(\square\)

Theorem 3.4. The minimizer of $I$ over $K_2$ is the same as its minimizer over $K_1$.

Proof. By the above theorem

$$u(x) = u(x)\eta,$$

where $u$ is the minimizer of $J_1$ over $K_3$. But we know that the minimizer of $J_1$ over $K_3$ is the same as its minimizer over

$$K_4 := \{ v \in H^1_0(U; \mathbb{R}) \mid \gamma_{K^*}(Dv) \leq \frac{1}{|\eta|} \text{ a.e.} \}.$$
Therefore for all \( z \in \mathbb{R}^n \), we have a.e.
\[
|Du \cdot z|^2 = \sum_i \sum_l (D_i u^l z^i)^2 = \sum_l \sum_i (D_i u^l z^i)^2 \\
= \sum_i (D_i u^l)^2 \sum_l (\eta^l)^2 = |\eta|^2 |Du \cdot z|^2 \\
\leq |\eta|^2 \gamma_K^2 (Du)^2 \gamma_K (z)^2 \leq \gamma_K (z)^2.
\]

This means that
\[
\|Du\|_{2,K} \leq 1 \quad \text{a.e.}
\]
Hence \( u \in K_1 \). Since \( K_1 \subseteq K_2 \), \( u \) is also the minimizer of \( I \) over \( K_1 \). \( \square \)

4. The optimal regularity. Let
\[
J_n(v) := \int_U \frac{1}{2} |Dv|^2 - \eta v \, dx.
\]
Suppose \( \partial U \) is Lipschitz, and \( K \subseteq \mathbb{R}^n \) is a balanced compact convex set whose interior contains 0. Let \( u \) be the minimizer of \( J_n \) over
\[
W_K := \{ v \in c + H_0^1(U) \mid \gamma_K(Dv) \leq k \ \text{a.e.}\},
\]
where \( c, k \) are constants and \( \gamma_K \) is the gauge function of \( K \). We showed that \( u \) is also the minimizer of \( J_n \) over
\[
W_{\phi,v} := \{ v \in H^1(U) \mid \phi \leq v \leq \psi \ \text{a.e.}\}.
\]

Here \( \phi(x) := c - kd_K^+(x, \partial U) \) and \( \psi(x) := c + kd_K^-(x, \partial U) \), where \( K^c \) is the polar of \( K \), and \( d_K^+ \) is the metric associated to the norm \( \gamma_K^+ \). (Note that \( \gamma_K(x) \leq k \) is equivalent to \( \frac{1}{k} x \in K \), since \( K \) is closed.)

By the above assumptions, there is \( A \geq 1 \) such that \( \gamma_K^+(x) \leq A |x| \) for all \( x \). We also need some sort of bound on the second derivative of \( \gamma_K^+ \), hence we assume that
\[
\frac{\gamma_K^+(x + h z) + \gamma_K^+(x - h z) - 2 \gamma_K^+(x)}{h^2} \leq \frac{B}{\gamma_K^+(x) - h}, \tag{1}
\]
where \( B \geq 1 \) is a constant, \( \gamma_K^+(z) = 1 \) and \( h < \gamma_K^+(x) \).

**Lemma 4.1.** The inequality (1) holds when \( \gamma_K^+ \) is the \( p \)-norm for \( 2 \leq p < \infty \). (In this case, \( K \) is the unit disk in the \( \frac{p}{p-1} \)-norm.)

**Proof.** Let \( \gamma_p(x) = (\sum |x_i|^p)^{1/p} \) then for \( \gamma_p(x) \neq 0 \) we have
\[
D_1 \gamma_p(x) = |x_i|^{p-1} \text{sgn}(x_i) (\sum |x_j|^p)^{1/p-1} = \frac{|x_i|^{p-1} \text{sgn}(x_i)}{\gamma_p(x)^{p-1}},
\]
where \( \text{sgn}(x_i) \) is the sign of \( x_i \). Thus
\[
D^2_{ij} \gamma_p(x) = (p-1) |x_i|^{p-2} \delta_{ij} \frac{1}{\gamma_p(x)^{p-1}} - (p-1) |x_i|^{p-1} |x_j|^{p-1} \frac{\text{sgn}(x_i) \text{sgn}(x_j)}{\gamma_p(x)^{2p-1}}.
\]
Hence
\[
D^2_{zz} \gamma_p(x) = \sum D^2_{ij} \gamma_p(x) z_i z_j \\
= \frac{p-1}{\gamma_p(x)^{p-1}} \sum |x_i|^{p-2} z_i^2 - \frac{p-1}{\gamma_p(x)^{2p-1}} (\sum \text{sgn}(x_i) |x_i|^{p-1} z_i)^2.
\]

By Holder’s inequality we get
\[
D^2_{zz} \gamma_p(x) \leq \frac{p-1}{\gamma_p(x)^{p-1}} (\sum |x_i|^{p-2}) \frac{2^{p-1}}{p} (\sum z_i^2)^{\frac{p}{2}} = \frac{p-1}{\gamma_p(x)} \gamma_p(z)^2.
\]
Thus if $\gamma_p(z) = 1$, we have
\[
D^2_{zz} \gamma_p(x) \leq \frac{p - 1}{\gamma_p(x)}.
\]
When $\gamma_p(x) > h$, $\gamma_p$ is nonzero on the segment $L := \{x + \tau z \mid -h \leq \tau \leq h\}$; and so it is twice differentiable there. Therefore we can apply the mean value theorem to the restriction of $\gamma_p$ and its first derivative to the segment $L$. Hence we get
\[
\gamma_p(x + hz) + \gamma_p(x - hz) - 2\gamma_p(x) = \frac{hD_z \gamma_p(x + sz) - hD_z \gamma_p(x - tz)}{h^2}
\]
where $0 < s, t < h$ and $-t < t < s$. Now as $\gamma_p$ is convex, its second derivative is nonnegative definite. Hence
\[
\gamma_p(x + hz) + \gamma_p(x - hz) - 2\gamma_p(x) \leq 2D^2_{zz} \gamma_p(x + rz) \leq \frac{2(p - 1)}{\gamma_p(x + rz)} \leq \frac{2(p - 1)}{\gamma_p(x) - h}.
\]
In the last inequality we used the triangle inequality for $\gamma_p$.

Before proving the main regularity result, we need to prove a few lemmas. Let us assume that $U$ has smooth boundary, we will remove this restriction at the end.

We know that
\[
\phi(x) \leq u(x) \leq \psi(x).
\]
Let $\phi_\epsilon := \eta_\epsilon \ast \phi + \delta_\epsilon$ and $\psi_\epsilon := \eta_\epsilon \ast \psi$, where $\eta_\epsilon$ is the standard mollifier and $4kA\epsilon < \delta_\epsilon < 5kA\epsilon$ is chosen such that $\partial \{\phi_\epsilon < \psi_\epsilon\}$ is $C^\infty$ (which is possible by Sard’s Theorem). Note that
\[
\{x \in U \mid d_{k^2}(x, \partial U) > 4A\epsilon\} \subset \{x \in \bar{U} \mid \phi_\epsilon(x) \leq \psi_\epsilon(x)\} \subset \{x \in U \mid d_{k^2}(x, \partial U) > A\epsilon\}. \tag{2}
\]
Since $\psi(x) - \phi(x) = 2k d_{k^2}(x, \partial U)$, and
\[
|\psi_\epsilon(x) - \psi(x)| \leq \int_{|y| \leq \epsilon} \eta_\epsilon(y)|\psi(x - y) - \psi(x)| \, dy
\]
\[
\leq \int_{|y| \leq \epsilon} \eta_\epsilon(y)k\gamma_{k^2}(y) \, dy \leq kA\epsilon \int_{|y| \leq \epsilon} \eta_\epsilon(y) \, dy = kA\epsilon,
\]
and similarly $|\eta_\epsilon \ast \phi - \phi| \leq kA\epsilon$.

We can easily show that $\gamma_k(D\phi_\epsilon) \leq k$ and $\gamma_k(D\psi_\epsilon) \leq k$. Because of Jensen’s inequality, convexity of $\gamma_k$ and its homogeneity, we have
\[
\gamma_k(D\phi_\epsilon(x)) \leq \int \gamma_k(\eta_\epsilon(y)D\phi(x - y)) \, dy
\]
\[
= \int \eta_\epsilon(y)\gamma_k(D\phi(x - y)) \, dy \leq k \int \eta_\epsilon(y) \, dy = k.
\]

Let $U_\epsilon := \{x \in U \mid \phi_\epsilon(x) < \psi_\epsilon(x)\}$, and denote by $u_\epsilon$ the minimizer of $J_\epsilon$ over $\{v \in H^1(U_\epsilon) \mid \phi_\epsilon \leq v \leq \psi_\epsilon \text{ a.e. }\}$. Set
\[
N_\epsilon := \{x \in U_\epsilon \mid \phi_\epsilon(x) = u_\epsilon(x) < \psi_\epsilon(x)\}
\]
\[
\Lambda_1 := \{x \in U_\epsilon \mid u_\epsilon(x) = \phi_\epsilon(x)\}
\]
\[
\Lambda_2 := \{x \in U_\epsilon \mid \phi_\epsilon(x) = \psi_\epsilon(x)\}.
\]
Since \( \phi_\epsilon, \psi_\epsilon \) are smooth, \( u_\epsilon \in W^{2,p}(U_\epsilon) \) for any \( 1 < p < \infty \) (See [2]). Therefore \( N_\epsilon \) is open and \( \Lambda_1 \)'s are closed. Also we define the free boundaries \( F_i := \partial \Lambda_i \cap U_\epsilon \). Note that \( \partial N_\epsilon \) consists of \( F_1 \)'s and part of \( \partial U_\epsilon \).

**Lemma 4.2.** We have
\[
\gamma_K(Du_\epsilon) \leq k
\] (3)
on \( U_\epsilon \).

**Proof.** Since on \( \partial U_\epsilon \) we have \( u_\epsilon = \phi_\epsilon = \psi_\epsilon \) we get \( D_zu_\epsilon = D_z\phi_\epsilon = D_z\psi_\epsilon \) for any direction \( z \) tangent to \( \partial U_\epsilon \), and as \( u_\epsilon \) is between the obstacles inside \( U_\epsilon \) we have \( D_\nu \phi_\epsilon \leq D_\nu u_\epsilon \leq D_\nu \psi_\epsilon \) where \( \nu \) is the normal direction to \( \partial U_\epsilon \). Therefore \( Du_\epsilon \) is a convex combination of \( D\phi_\epsilon, D\psi_\epsilon \) and we get the bound on \( \partial U_\epsilon \) by convexity of \( \gamma_K \). The bound holds on \( \Lambda_1 \)'s (and hence on \( F_1 \)'s) obviously, as \( u_\epsilon \) equals one of the obstacles there and attains its extremum, thus \( Du_\epsilon \) equals the derivative of that obstacle there.

To obtain the bound for \( N_\epsilon \) note that for any vector \( z \) with \( \gamma_{K^*}(z) = 1 \) we have
\[
|D_zu_\epsilon| = |z \cdot Du_\epsilon| \leq \gamma_{K^*}(z) \gamma_K(Du_\epsilon) \leq k
\]
on \( \partial N_\epsilon \), and as \( D_zu_\epsilon \) is harmonic in \( N_\epsilon \) we get \( |D_zu_\epsilon| \leq k \) in \( N_\epsilon \) by maximum principle. The result follows from \( \gamma_K(Du_\epsilon) = \sup_{\gamma_{K^*}(z) = 1} |D_zu_\epsilon| \).

The local behavior of the free boundaries is the same as the case of one obstacle problem as obstacles do not touch inside \( U_\epsilon \). We need the following lemma from [5].

**Lemma 4.3.** The free boundary has measure zero. Furthermore for any direction \( z \) we have
\begin{itemize}
  \item[(i)] if \( y \in N_\epsilon \) approaches \( x \in F_1 \), then \( \liminf_{y \to x} D_{zz}^2(u_\epsilon - \phi_\epsilon)(y) \geq 0 \).
  \item[(ii)] If \( y \in N_\epsilon \) approaches \( x \in F_2 \), then \( \liminf_{y \to x} D_{zz}^2(\psi_\epsilon - u_\epsilon)(y) \geq 0 \).
\end{itemize}

This enables us to prove the following.

**Lemma 4.4.** For any direction \( z \) with \( |z| = 1 \), we have
\[
D_{zz}^2\phi_\epsilon(x) \geq -\frac{kA^2B}{d_{K^*}(x, \partial U) - \epsilon},
\]
\[
D_{zz}^2\psi_\epsilon(x) \leq \frac{kA^2B}{d_{K^*}(x, \partial U) - \epsilon}
\]
for all \( x \in U \) with \( d_{K^*}(x, \partial U) > \epsilon \).

**Proof.** First we assume \( \gamma_{K^*}(z) = 1 \). Let \( x_0 \in U \) then
\[
\psi(x_0) = c + k\gamma_{K^*}(x_0, \partial U) = c + k\gamma_{K^*}(x_0 - y_0)
\]
for some \( y_0 \in \partial U \). Set \( \gamma(x) = c + k\gamma_{K^*}(x - y_0) \). Then \( \psi(x) \leq \gamma(x) \) and \( \psi(x_0) = \gamma(x_0) \). Now for \( h < \gamma_{K^*}(x_0 - y_0) \) we have
\[
\Delta_{h,z}^2\psi(x_0) := \frac{\psi(x_0 + hz) + \psi(x_0 - hz) - 2\psi(x_0)}{h^2} \leq \Delta_{h,z}^2\gamma(x_0).
\]
By our assumption (1)
\[
\Delta_{h,z}^2\gamma(x_0) \leq \frac{kB}{\gamma_{K^*}(x_0 - y_0) - h} = \frac{kB}{d_{K^*}(x_0, \partial U) - h}.
\]
Hence \( \Delta_{h,z}^2\psi(x) \leq \frac{kB}{d_{K^*}(x, \partial U) - h} \) for \( d_{K^*}(x, \partial U) > h \).
Now for $d_{K^e}(x, \partial U) > h + \epsilon$, we have
\[
\Delta^2_{h,z} \psi_\epsilon(x) = \int_{|y|<\epsilon} \eta(y) \Delta^2_{h,z} \psi(x-y) \, dy \leq \int_{|y|<\epsilon} \eta(y) \frac{kB}{d_{K^e}(x-y, \partial U) - \epsilon - h} \, dy
\]
\[
\leq \int_{|y|<\epsilon} \eta(y) \frac{kB}{d_{K^e}(x, \partial U) - \epsilon - h} \, dy = \frac{kB}{d_{K^e}(x, \partial U) - \epsilon - h}.
\]
Here we used the fact that
\[
d_{K^e}(x-y, \partial U) \geq d_{K^e}(x, \partial U) - \gamma_{K^e}(y) \geq d_{K^e}(x, \partial U) - A|y| > d_{K^e}(x, \partial U) - \epsilon - h.
\]
Taking $\epsilon \to 0$, we get for $d_{K^e}(x, \partial U) > \epsilon$
\[
D^2_{zz} \psi_\epsilon(x) \leq \frac{kB}{d_{K^e}(x, \partial U) - \epsilon}.
\]
Now if we take $|z| = 1$ and apply the above result to $w = \frac{z}{\gamma_{K^e}(z)}$, we get
\[
D^2_{zz} \psi_\epsilon(x) = (\gamma_{K^e}(z))^2 D^2_{ww} \psi_\epsilon(x) \leq A^2 D^2_{ww} \psi_\epsilon(x) \leq \frac{kA^2B}{d_{K^e}(x, \partial U) - \epsilon},
\]
as $\gamma_{K^e}(z) \leq A$ and $D^2 \psi_\epsilon$ is nonnegative since $\psi$ is convex. The inequality for $\phi_\epsilon$ follows from $D^2 \phi_\epsilon = -D^2 \psi_\epsilon$.

**Lemma 4.5.** For any direction $z$ with $|z| = 1$
\[
|D^2_{zz} u_\epsilon(x)| \leq C(n) \left[ |\eta| + \frac{kA^2B}{d_{K^e}(x, \partial U) - \epsilon} + \frac{A^2|z|}{(d_{K^e}(x, \partial U) - \epsilon)^2} \right]
\]
for a.e. $x \in U_\epsilon$, where $C(n)$ is a constant depending only on the dimension $n$.

**Proof.** Since $u_\epsilon \in W^{2,p}(U_\epsilon)$ we have $D^2_{zz} u_\epsilon = D^2_{zz} \phi_\epsilon$ a.e. on $\Lambda_1$. Also in a $U_\epsilon$-neighborhood of $\Lambda_1$ we have $-\Delta u_\epsilon \geq \eta$ a.e., since $u_\epsilon$ solves the variational inequality there. Thus for a.e. $x \in \Lambda_1$
\[
-\frac{kA^2B}{d_{K^e}(x, \partial U) - \epsilon} \leq D^2_{zz} \phi_\epsilon(x) = D^2_{zz} u_\epsilon(x) = \Delta u_\epsilon(x) - \sum D^2_{zi} u_\epsilon(x)
\]
\[
\leq -\eta - \sum D^2_{zi} \phi_\epsilon(x) \leq |\eta| + \frac{(n-1)kA^2B}{d_{K^e}(x, \partial U) - \epsilon},
\]
where $\{z, z_i\}$ form an orthonormal system. Note that for $x \in U_\epsilon$ we have
\[
d_{K^e}(x, \partial U) > \epsilon.
\]
Similarly, using $\psi_\epsilon$ we obtain that for a.e. $x \in \Lambda_2$
\[
-|\eta| - \frac{(n-1)kA^2B}{d_{K^e}(x, \partial U) - \epsilon} \leq D^2_{zz} \phi_\epsilon(x) = D^2_{zz} \psi_\epsilon(x) \leq \frac{kA^2B}{d_{K^e}(x, \partial U) - \epsilon}.
\]
It only remains to obtain the bound on $N_\epsilon$. We do this using the maximum principle, since $D^2_{zz} u_\epsilon$ is harmonic in $N_\epsilon$. Therefore we need to estimate $D^2_{zz} u_\epsilon$ near $F_1$ and $\partial U_\epsilon$. First, for $x \in F_1$ and $y \in N_\epsilon$, we have by continuity of $D^2 \phi_\epsilon$
\[
\liminf_{y \to x} D^2_{zz} u_\epsilon(y) \geq \liminf_{y \to x} D^2_{zz} \phi_\epsilon(y) = D^2_{zz} \phi_\epsilon(x) \geq -\frac{kA^2B}{d_{K^e}(x, \partial U) - \epsilon}.
\]
This is true for the $z_i$ directions too. Also
\[
\limsup_{y \to x} (D^2_{zz} u_\epsilon(y) + \sum D^2_{zi} u_\epsilon(y)) = \limsup_{y \to x} \Delta u_\epsilon(y) = -\eta.
\]
Thus
\[
\limsup_{y \to x} D_{zz}^2 u_\varepsilon(y) \leq \limsup_{y \to x}(D_{zz}^2 u_\varepsilon(y) + \sum D_{zi,z}^2 u_\varepsilon(y) - \sum \liminf_{y \to x} D_{zi,z}^2 u_\varepsilon(y)) = -\eta - \sum \liminf_{y \to x} D_{zi,z}^2 u_\varepsilon(y) \leq |\eta| + (n - 1)kA^2B \frac{d_{K^*}}{\partial U} - \varepsilon .
\]
Similarly on $F_2$ we have
\[
-|\eta| - \frac{(n - 1)kA^2B}{d_{K^*}} \leq \liminf_{y \to x} D_{zz}^2 u_\varepsilon(y) \leq \limsup_{y \to x} D_{zz}^2 u_\varepsilon(y) \leq \frac{kA^2B}{d_{K^*}}.
\]
Next we show that
\[
|D_{zz}^2 u_\varepsilon(x)| \leq C(n)|\eta| + \frac{kAB}{r} + \frac{|c|}{r^2} \quad x \in N_\varepsilon, \quad d_{K^*}(x, \partial U_\varepsilon) = Ar,
\]
for fixed and small $r$ and $\varepsilon < r/16$. Note that
\[
d_{K^*}(B_{r/2}(x), \partial U_\varepsilon) > Ar - Ar/2 = Ar/2.
\]
Fix $x_0 \in N_\varepsilon$ with $d_{K^*}(x_0, \partial U_\varepsilon) = Ar$, and consider the function
\[
v_\varepsilon(y) := u_\varepsilon(x_0 + ry), \quad y \in B_1(0).
\]
Then by known bounds on $u_\varepsilon$ we have in $B_{1/2}(0)$
\[
|v_\varepsilon| \leq |c| + 6Ak\varepsilon + 3Ar/2 < |c| + 2Ar
\]
\[
\gamma_K(Dv_\varepsilon) \leq rk.
\]
Also for a.e. $y \in B_{1/2}(0)$ we have
\[
|\Delta v_\varepsilon(y)| \leq n\varepsilon^2(|\eta| + \frac{(n - 1)kA^2B}{d_{K^*}(x_0 + ry, \partial U) - \varepsilon})
\]
\[
\leq n(|\eta| + \frac{n - 1)kA^2B}{Ar/2 - \varepsilon})r^2 < n(|\eta| + \frac{16(n - 1)kAB}{7r})r^2 .
\]
Since $\Delta u_\varepsilon = -\eta$ in $N_\varepsilon$ and it is bounded on $\Lambda_i$’s (and free boundaries have measure zero). Choose $\sigma \in C^\infty_0(B_{1/2}(0))$ such that $\sigma = 1$ in $B_{1/4}(0)$. Then in $B_{1/2}(0)$
\[
|\Delta(\sigma v_\varepsilon)| = |(\Delta\sigma)v_\varepsilon + 2D\sigma \cdot Dv_\varepsilon + \sigma \Delta v_\varepsilon| \leq C(n)(|\eta| + \frac{kAB}{r})r^2 + |c| .
\]
By Calderon-Zygmund estimate (see [7]) it follows
\[
|\sigma v_\varepsilon|_{W^{2,p}(B_{1/2}(0))} \leq C(n, p)(|\eta| + \frac{kAB}{r})r^2 + |c| ,
\]
for any $1 < p < \infty$ (note that the boundary term is zero). In particular
\[
|D_{ij}^2 v_\varepsilon|_{L^p(B_{1/4}(0))} \leq C(n, p)(|\eta| + \frac{kAB}{r})r^2 + |c| .
\]
We want to extend this to $p = \infty$.
Let $\tau \in C^\infty_0(B_{1/4}(0))$ with $\tau = 1$ in $B_{1/8}(0)$. Consider the open set
\[
N := \{ y \mid x_0 + ry \in N_\varepsilon \}.
\]
In $N$ we have $\Delta v_\varepsilon = -\eta^2$. Thus (note that $v_\varepsilon$ is smooth in $N$)
\[
\Delta D_{zz}^2(\tau v_\varepsilon) = D_zh,
\]
where
\[ h := D_z \Delta (\tau v) = D_z((\Delta \sigma)v + 2D\sigma \cdot Dv + \sigma \Delta v) \]
\[ = D_z((\Delta \sigma)v + 2D\sigma \cdot Dv - \sigma \eta r^2). \]

Using the above estimates we find that
\[ |h|_{L^p(N)} \leq C(n,p)[(|\eta| + kABr)^2 + |c|]. \]
Now take
\[ V(y) = \begin{cases} \alpha_n |y|^{2-n} & n \geq 3 \\ \alpha_2 \log |y| & n = 2 \end{cases} \]
to be the fundamental solution of \(-\Delta\). Then
\[ g(y) = -\int_N \frac{\partial V(y-w)}{\partial z} h(w) \, dw \]
satisfies
\[ \Delta g = \frac{\partial h}{\partial z} \]
in \(N\). By the bound on \(h\) we find that
\[ |g|_{L^\infty(N)} \leq C(n)[(|\eta| + kABr)^2 + |c|], \]
since for \(p > n\), \(\frac{\partial V}{\partial z}\) is in \(L^q\) where \(q\) is the dual exponent of \(p\). The function \(D^2_{zz}(\tau v) - g\) is then harmonic in \(N \cap B_{1/4}(0)\). The boundary of this set consists of part of \(\partial B_{1/4}(0)\) in which \(\tau = 0\) and \(g\) is bounded, and another part inside \(B_{1/4}(0)\) where corresponds to the free boundaries and both \(g\) and \(D^2_{zz}(\tau v)\) are bounded there by the above bounds. Therefore by the maximum principle we get
\[ |D^2_{zz}v_\epsilon(0)| \leq C(n)[(|\eta| + kABr)^2 + |c|]. \]
Hence
\[ |D^2_{zz}u_\epsilon(x_0)| \leq C(n)[|\eta| + \frac{kAB}{r} + \frac{|c|}{r^2}] \]
\[ = C(n)[|\eta| + \frac{kA^2B}{d_{K^*(x_0, \partial U_\epsilon)}^2} + \frac{A^2|c|}{(d_{K^*(x_0, \partial U_\epsilon)}^2)^2}], \]
The proof of the lemma is complete once we notice that for \(x \in U_\epsilon\)
\[ d_{K^*(x, \partial U_\epsilon)} \geq d_{K^*(x, \partial U)} - A\epsilon. \]
\[ \square \]

Now we can prove our main regularity result. Note that by Theorem 3.3, we also get the regularity for the vector-valued case.

**Theorem 4.6.** Suppose \(u\) is the minimizer of \(J_\eta\) over \(W_K\). Then \(u \in W^{2,\infty}_{\text{loc}}(U)\), and
\[ |D^2u(x)| \leq C(n)[|\eta| + \frac{kA^2B}{d_{K^*(x, \partial U)}^2} + \frac{A^2|c|}{(d_{K^*(x, \partial U)}^2)^2}], \]
for a.e. \(x \in U\). Here, \(C(n)\) is a constant depending only on the dimension \(n\).
Proof. Choose a decreasing sequence $\epsilon_i \to 0$ such that $U_{\epsilon_i} \subset U_{\epsilon_{i+1}}$ (This is possible by (2)). For convenience we use $U_i, u_i, \phi_i, \psi_i$ instead of $U_{\epsilon_i}, u_{\epsilon_i}, \phi_{\epsilon_i}, \psi_{\epsilon_i}$. Consider $u_i|U_i$ for $i > 1$. By (4), (3) and the fact that $\phi_i \to \phi$, $\psi_i \to \psi$ uniformly, we know that $u_i, Du_i, D^2u_i$'s are uniformly bounded. Thus $\|u_i\|_{W^{2,\infty}(U_1)}$ is bounded independent of $i$. Now as $\partial U_1$ is smooth, we can say that $\|u_i\|_{C^{1,1}(\overline{U_1})}$ is bounded independent of $i$. Therefore there is a subsequence of $u_i$'s, which we denote by $u_{i,1}$, that weakly star converges in $W^{2,\infty}(U_1)$ to a function $\tilde{u}_1$. In addition, by equicontinuity we can assume that $u_{i,1}, Du_{i,1}$ uniformly converge to $\tilde{u}_1, D\tilde{u}_1$. Now we can repeat this process with $u_{i,1}|U_i$ and get a function $\tilde{u}_2$ in $W^{2,\infty}(U_2)$ which agrees with $\tilde{u}_1$ on $U_1$. Continuing this way with subsequences $u_{i,j}$, we can finally construct a function $\tilde{u}$ in $W^{2,\infty}_{\text{loc}}(U)$. It is obvious that $\gamma_K(D\tilde{u}) \leq k$ and $\phi \leq \tilde{u} \leq \psi$, thus $\tilde{u}$ belongs to $W_K$.

Now we want to show that $\tilde{u}$ is the minimizer of $J_\eta$ over $W_K$. This is equivalent to (see [5])
\[
\int_U D\tilde{u} \cdot D(v - \tilde{u}) - \eta(v - \tilde{u}) \, dx \geq 0
\] (6)
for all $v \in W_K \subset W_{\phi,\psi}$. First suppose that $v > \phi$ on $U$, and $v = \psi$ on $\{d_{K^*}(x, \partial U) \leq \delta\}$ for some $\delta > 0$. Let $v_i := \eta_i \ast v$. Then for $i$ large enough we have $\phi_i \leq v_i \leq \psi_i$ on $U_i$ (Note that $v_i$ attains the correct boundary values on $\partial U_i$). Hence we have
\[
\int_{U_i} Du_i \cdot D(v_i - u_i) - \eta(v_i - u_i) \, dx \geq 0.
\]
By taking the limit (through a diagonal sequence of the sequences constructed in the previous paragraph) and using the Dominated Convergence Theorem, we get (6) for this special $v$. It is easy to see that an arbitrary function in $W_K$ can be approximated by such special $v$'s, so we get (6). Thus, by the uniqueness of the minimizer we have $u = \tilde{u}$.

Now take $x \in U$ and suppose $d_{K^*}(x, \partial U) = r$. Let
\[B_{s,K^*}(x) := \{y \mid \gamma_{K^*}(y - x) < s\}.\]
Then by the weak star convergence proved above we have
\[
\|D^2u\|_{L^\infty(B_{r-x,K^*}(x))} \leq \liminf \|D^2u\|_{L^\infty(B_{r-x,K^*}(x))} \leq C(n)[\eta] + \frac{kA^2B}{s} + \frac{A^2|c|}{s^2}.
\]
Letting $s \to r$ we get the desired bound (5) for a.e. $x$. Since otherwise there is a set $A$ with positive measure such that for $x \in A$
\[
|D^2u(x)| > \|D^2u\|_{L^\infty(B_{s,K^*}(x))},
\]
for all $\delta \leq \delta_x$. We can derive a contradiction easily, by looking at a Lebesgue point of $A$, noting that one of the sets $A_k := \{x \in A \mid \delta_x > \frac{1}{k}\}$ must have positive measure.

Now suppose $\partial U$ is not smooth. We approximate $U$ by a shrinking sequence $U_i^*$ of larger domains with smooth boundaries (This is easy to do, since $\partial U$ is Lipschitz and can be approximated by smooth hypersurfaces locally). Let $u_i^*$ be the minimizer of $J_\eta$ over
\[
\{v \in c + H_0^1(U_i^*) \mid \gamma_{K^*}(Dv) \leq k \text{ a.e.}\}.
\]
Since $u_i^*, Du_i^*$'s are uniformly bounded on $U$, and $\partial U$ is Lipschitz, a subsequence of $u_i^*$'s, which we still denote it by $u_i^*$, uniformly converges to a function $u^*$ on $U$. Furthermore, for any $V \subset U$, we can see similar to the above that a subsequence
of \( u_i^* \)'s is weakly star convergent in \( W^{2,\infty}(V) \). Hence \( u^* \in W^{2,\infty}_{\text{loc}}(U) \). We can also assume that they converge strongly to \( u^* \) in \( W^{1,\infty}(V) \). Thus for a further subsequence, which we still denote it by \( u_i^* \), \( Du_i^* \) converges to \( Du^* \) a.e. in \( V \). As a result we have \( u^* \in W^{1,\infty}(U) \).

We can also assume that they converge strongly to \( u^* \) in \( W^{1,\infty}(V) \). Thus for a further subsequence, which we still denote it by \( u_i^* \), \( Du_i^* \) converges to \( Du^* \) a.e. in \( V \).

As a result we have
\[
\int_{U_1^*} \frac{1}{2} |Du_i^*|^2 - \eta u_i^* \, dx \leq \int_{U_1^*} \frac{1}{2} |Du|^2 - \eta u \, dx.
\]

By taking the limit as \( i \to \infty \) (again, through a diagonal sequence of the sequences constructed in the previous paragraph corresponding to an expanding family \( V_j \) that covers \( U \)), we get \( J_\eta(u^*) \leq J_\eta(u) \) by the Dominated Convergence Theorem. Since the other inequality is satisfied too, we have \( J_\eta(u^*) = J_\eta(u) \). The uniqueness of the minimizer implies that \( u^* = u \). We can argue as before to show that \( u \) satisfies the bound (5) too.

Acknowledgments. I would like to express my gratitude to Lawrence C. Evans and Nicolai Reshetikhin for their invaluable help with this research.

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Received April 2015; revised October 2015.

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