Imaging defects in an elastic waveguide using time-dependent surface data

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Abstract. We are interested here in applying the Linear Sampling Method to the context of Non Destructive Testing for waveguides. Specifically the Linear Sampling Method \cite{1} in its modal form \cite{2} is adapted to image defects in an elastic waveguide from realistic scattering data, that is data coming from sources and receivers on the surface of the waveguide in the time domain, as it has already been done in the acoustic case \cite{16}. The obtained method is applied to artificial data and to experimental data in the special case of back-scattering.

Introduction

This paper presents a Non Destructive Testing method to locate and reconstruct the shape of some defects in an elastic waveguide using guided modes. Since the sensor is supposed to be far from the defect, the data consist in the scattered waves associated with the propagating modes only. The usual methods rely on a small number of propagating modes (at low frequency) \cite{3, 4, 5, 6}, which allow the tracking of each of these modes. In addition, these methods depend on the type of defect which is expected. Our method is based on a mathematical approach, namely the Linear Sampling Method (LSM) \cite{1} and more precisely in a modal formulation \cite{2}, which takes into account the scattered fields associated with a large number of propagating modes (at high frequency). The LSM consists, for each sampling point \( z \) describing a grid, to check if some appropriate and analytically known test function depending on \( z \) belongs or not to the range of a linear operator, the definition of which is based on the data. In the case of a positive answer, the point \( z \) belongs to the defect, which is a practical way of retrieving its boundary. An important feature of the LSM is that it is independent of the number of components and of the type of the defect.

This method has been widely studied since its introduction \cite{7, 8, 9, 13} but its application has been restricted to artificial data except for electromagnetism, see \cite{12}, and most of the time in the frequency regime (for time domain data, see \cite{10, 11}). We propose here to adapt the LSM to realistic data in the non destructive testing framework, that is data in the time domain measured on the surface of the inspected waveguide.

Our strategy is the following. Firstly, we transform the time domain data into multi-frequency data with the help of the Fourier transform. Secondly, at each frequency we transform the surface data into a novel set of ideal data which are suitable for the modal formulation of the LSM. Thirdly, we combine all the images given by the LSM at each frequency in order to obtain the best possible defect identification. In addition, the optimization of the sources and receivers...
is possible thanks to this strategy. The Linear Sampling Method in the frequency domain for an elastic waveguide is already justified for a Dirichlet obstacle in [2] and for cracks in [14]. The justification for a Neumann obstacle would follow the same lines and the justification for a penetrable obstacle would be very close to that of [15] for the periodic acoustic waveguide. Hence, this article is focused on the application of the LSM in the presence of real data (in particular, surface data in the time domain) and on how the substantial issues generated by those real data can be fixed.

The outline of this paper is what follows: the LSM in a modal formulation is briefly summarized in section 1 while the case of data on the surface in the frequency domain is explained in section 2. In section 3 the adaptation of the method to time domain data is shown and an application of the method to artificial data is done. Finally the experimental setting is explained in section 4 and the corresponding imaging results are presented.

1. The Linear Sampling Method: a modal formulation
Let the space dimension be 2. We consider a waveguide $W = \Sigma \times \mathbb{R}$ of transverse section $\Sigma$ and boundary $\Gamma$. This waveguide is made of an isotropic material of density $\rho$ and Lamé constants $(\lambda, \mu)$. Let be denoted $u$ the displacement, $\sigma$ the stress tensor associated with $u$ following

$$\sigma(u) = \lambda(\text{div} u) I + \mu(\nabla u + \nabla u^T),$$

and let us decompose

$$u = \begin{pmatrix} u_S \\ u_3 \end{pmatrix}, \quad \sigma \cdot e_3 = \begin{pmatrix} t_S \\ -t_3 \end{pmatrix},$$

where the subscripts $S$ and 3 denote the components of a vector along the transverse section and along the axis, respectively. We introduce the mixed variables $X, Y$ defined by:

$$X = \begin{pmatrix} t_S \\ u_3 \end{pmatrix}, \quad Y = \begin{pmatrix} u_S \\ t_3 \end{pmatrix}.$$

The guided modes are the solutions with separated variables of

$$\begin{cases} \text{div}\sigma(u) + \rho \omega^2 u = 0 & \text{in } W \\ \sigma(u)\nu = 0 & \text{on } \Gamma, \end{cases}$$

where $\nu$ is the exterior normal to $W$. They are given for $n \in \mathbb{N}$ by

$$u_n^\pm(x) = \begin{pmatrix} u_{nS}^\pm(xS) \\ \pm u_{n3}^\pm(xS) \end{pmatrix} e^{\pm i \beta_n x_3}, \quad \begin{pmatrix} X_n^\pm(x) \\ Y_n^\pm(x) \end{pmatrix} = \begin{pmatrix} \pm X_n(xS) \\ Y_n(xS) \end{pmatrix} e^{\pm i \beta_n x_3},$$

where $X_n$ and $Y_n$ are the $X$ and $Y$ extensions of the transverse functions ($u_{nS}^S$, $u_{n3}^S$). They satisfy $(X_n, Y_m)_\Sigma = \delta_{mn}$, where $(\cdot, \cdot)_\Sigma$ is a scalar product over $L^2(\Sigma)$ without complex conjugation.

For a given frequency $\omega$, $\beta_n$ is real for only a finite number of guided modes, which are named propagating modes. The other ones are either inhomogeneous or evanescent. Because we only consider far fields, those modes will not be taken into account. The assumption is then made that any elastic field, written in the $(X, Y)$ variables, can be decomposed as follows:

$$X|_\Sigma = \sum_n (X, X_n)_\Sigma X_n, \quad Y|_\Sigma = \sum_n (X_n, Y)_\Sigma Y_n.$$
We then consider a defect $D$ inside the waveguide which lies between two sections $\Sigma_{\pm}$ and denote
\[ \Omega = W \setminus D. \]
The scattered field $u_n^{\pm}$ associated to the incident propagating mode $u_n^{\pm}$ is solution of the following forward problem for a given frequency $\omega$:
\[
\begin{cases}
\text{div}(u_n^{\pm}) + \rho \omega^2 u_n^{\pm} = 0 & \text{in } \Omega \\
\sigma(u_n^{\pm}) \nu = 0 & \text{on } \Gamma \\
u_n^{\pm} = -u_n^{\pm} & \text{on } \partial D
\end{cases}
\]
with $(RC)$ a radiation condition.

The data in this case are the components of the scattering matrix $S$, namely the projections $S_{mn}^{\pm}$, respectively $S_{mn}^{-\pm}$, along the $X_m$ on the two sections $\Sigma_\pm$ of the $Y$ extensions $Y_n^{\pm}$ of the scattered fields $u_n^{\pm}$, respectively $Y_n^{-\pm}$ of the scattered fields $u_n^{\pm}$. By only considering the propagating modes, the number of lines and columns of the scattering matrix is limited to $2P$, $P$ being the number of propagating modes in each direction. The Linear Sampling Method consists in solving the following system for all sampling points $z = (z_S, z_3)$:
\[
\begin{align*}
\sum_{n=0}^{P-1} U_{mn}^{++} h_n^- + U_{mn}^{--} h_n^+ &= e^{i\beta_m(R+z_3)} Y_m(z_S) \cdot p, \\
\sum_{n=0}^{P-1} U_{mn}^{+-} h_n^- + U_{mn}^{--} h_n^+ &= e^{i\beta_m(R-z_3)} Y_m(z_S) \cdot p,
\end{align*}
\]
where
\[
U := \begin{pmatrix} U^{+-} & U^{--} \\ U^{++} & U^{--} \end{pmatrix} = \begin{pmatrix} S^{--} & S^{-+} \\ S^{++} & S^{-+} \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} =: SK,
\]
$K$ is the $P \times P$ diagonal matrix the components of which are $e^{i\beta_nR}/2i\beta_n$, $\pm R$ is respectively the $z_3$ coordinate of the section $\Sigma_{\pm}$ and $p$ is a polarisation parameter. If, roughly speaking, a solution $H = (h^-, h^+)$, with $h^\pm = (h_1^\pm, \ldots, h_P^\pm)$, is found, then $z \in D$ according to a classical result related to the LSM [2]. It is then possible to define a characteristic function of the defect over a sampling grid $G$, given by $\psi(z, \omega) = \|H\|_2^{-1}$, and so to obtain an image of the defect.

Let us remark that if every modes were considered in (1) (not only the propagating ones), the problem would be ill-posed as the near-field operator, from which the left-hand side of (1) is obtained, is compact [2]. The limitation to propagating modes can be viewed as a physical regularization of the near-field equation [13].

**Back-scattering case** The back-scattering case is a special case where only one side of the area to image, for example its left side, is accessible. It means that only data corresponding to sources and measurements on this side are available. In this case, the modal LSM system reduces to
\[
\sum_{n=0}^{P-1} U_{mn}^{+-} h_n^- = e^{i\beta_m(R+z_3)} Y_m(z_S) \cdot p.
\]

2. The case of surface solicitations and measurements
The method shown above needs data within the waveguide, which is not realistic in the context of Non Destructive Evaluation. We write $\Gamma = \Gamma_0 \cup \Gamma_d$, $d$ being the height of the waveguide and $\Gamma_0$ and $\Gamma_d$ being respectively its lower and upper boundary. A family of source functions
Our data are the components of a measurement matrix \( M \) defined as follows:

\[
M = \begin{pmatrix}
M^{++} & M^{-+} \\
M^{++} & M^{--}
\end{pmatrix},
\]

where

\[
M^{ij}_{ij} = \int_{\mathbb{R}} f^{\pm}_{i}(x_3) \cdot u^{s}(x_S, x_3) \, dx_3
\]

with \( u^{s} \) the solution of (2) for \( g^{j}_{-} \) and

\[
M^{ij}_{ij} = \int_{\mathbb{R}} f^{\pm}_{i}(x_3) \cdot u^{s}(x_S, x_3) \, dx_3
\]

with \( u^{s} \) the solution of (2) for \( g^{j}_{+} \).

The corresponding scattered field \( u^{s} \) is \( u - u^{i} \), where \( u^{i} \) solves the same problem (2) as \( u \) in \( W \) without the boundary condition \( u = 0 \) on \( \partial D \). To simplify the notations, a family of measurement functions \( (f^{\pm}_{i}(x))_{0 \leq i \leq M-1} = (f(x - x^{\pm}_{i}))_{0 \leq i \leq M-1} \) is also defined for an even and compactly supported function \( f \) defined on \( \mathbb{R} \) and for the same family of points \( (x^{\pm}_{i})_{0 \leq i \leq M-1} \).

The condition number of the emission \( E \) and reception \( \mathcal{R} \) matrices can be rigorously analyzed and optimized [16, 18]: it strongly depends on the number \( 2M \) of sources and receivers and on the minimal distance \( \delta \) between them. Inverting the system (3) enables us to compute \( \mathcal{U} \) and then to apply the modal LSM as in section 1. The above method can be extended to the 3D case by considering one or several lines of source and measurement points.
Back-scattering case  Again in this case only one fourth of the data is available, which means that only one block of the measurement matrix is considered, namely $M^{+-}$. The equation to solve is then

$$M^{+-} = -RU^{+-}E^T,$$

the inversion of which enables us to solve the modal LSM in the back-scattering case.

3. The case of data in the time domain
In the time domain, we consider the following problem:

$$
\begin{cases}
\rho \partial_{tt}u - \text{div}\sigma(u) = 0 & \text{in } \Omega \times (0, +\infty) \\
\sigma(u)\nu = g^\pm(x)\chi(t) & \text{on } \Gamma \times (0, +\infty) \\
u = 0 & \text{on } \partial D \times (0, +\infty) \\
u = \partial_t u = 0 & \text{on } \Omega \times \{0\}.
\end{cases}
$$

The data consist of the corresponding scattered fields measured at the same points as before in the time interval $(0, +\infty)$. Here, the function $\chi(t)$ is suitably chosen so that the frequencies for which the group velocity vanishes are avoided. By applying a Fourier transform to our data we recover, up to a $\hat{\chi}(\omega)$ factor, $\hat{\chi}$ being the Fourier transform of $\chi$, the previous system (2) at a given frequency $\omega$. We now have multi-frequencies data, which allows us to image the defect with a better accuracy than in the frequency domain. More precisely, as explained in [16], the following combination is chosen:

$$
\Psi(z) = \left(\int_{\omega_-}^{\omega_+} \max_{z' \in G} |\psi(z', \omega)|^2 d\omega\right)^{-1/2},
$$

where $\omega_-$ and $\omega_+$ are such that $\text{supp}\, \hat{\chi} \subset (\omega_-, \omega_+)$ and $G$ is the sampling grid.

In figure 1, the reconstruction is done using multiple frequencies which correspond to a number of propagating modes $P$ ranging from 8 to 14. More precisely, the height of the waveguide is $d = 1$ mm and the material constants are $\lambda = 121$ GPa, $\mu = 80$ GPa and $\rho = 7900$ kg.m$^{-3}$. No noise was added in this case as it has already been seen in the acoustic case [16] that the method is robust regarding to noise. Furthermore, some noise is naturally present in the experimental data considered in the next section.

4. Application to experimental data
The method has been tested for data measured on a steel plate (see figure 2), the parameters of which are $\rho = 7926$ kg.m$^{-3}$, $d = 12$ mm, $\lambda = 110.75$ GPa and $\mu = 82.435$ GPa.

In order to use a 2D waveguide model for the plate, it is needed that the defects and sensor are considered as invariant according to one direction, namely the $x_2$ direction. Therefore, the plate can be separated in 3 different areas:

- a zone without defect allowing the measurement of the incident field;
- a zone containing a cylindrical notch of rectangular section, its dimension being 0.25 mm in the $x_3$ direction and 2 mm in the $x_1$ one;
- a zone containing a vertically centered cylindrical hole of circular section, its diameter being of 2 mm.

A single linear multi-element piezoelectric sensor was used to obtain the data. More precisely, each of the transducer was successively used as a source while all of them were used as measurement points. Its main characteristics are $\delta = 0.8$ mm (element width of 0.55 mm...
and inter-element space of 0.25 mm), $M=128$ and its central frequency is of 2 MHz, which means that $P \approx 20$. Its element’s bandwidth at -6 dB is of 1.2 MHz and the sampling frequency used is of 10 MHz. Its elements have a width of 18 mm enabling us to use the 2D approximation. The fact that only one sensor was used implies that the case under consideration is the back-scattering one.

In the following images, $D$ will denote the distance between the sensor and the defect that is being imaged according to the $x_3$ direction. In each of the images, the sensor is on the upper left corner, outside of the image (see Figure 2 (b) for an example). In Figure 3, the cylindrical hole is imaged for various values of $D$. It can be seen that the defect is well retrieved for every value of $D$, even though the resolution slightly decreases with the distance. It can be explained by some attenuation that is not taken into account in our model. The resolution of the defect is good according to the section and degraded a little along the axis, which is expected in the back-scattering configuration [13]. The same results are obtained for the cylindrical notch (see Figure 4).

**Conclusion and perspectives**

The method presented here allow the imaging of defects inside 2D elastic waveguides, that is a section of a plate, from surface data. Further testings need to be done regarding its application to other kind of materials, for example multi-layered waveguides, and three-dimensional waveguides such as rails or pipes.
Figure 3: Identification of the cylindrical hole for $D = 35$, $D = 55$, $D = 95$ and $D = 150$ (log scale, axes unit: mm).

Figure 4: Identification of the cylindrical notch for $D = 25$ and $D = 45$ (log scale, axes unit: mm).

References
[1] Colton D and Kirsch A 1996 Inverse Problems 12 383-393
[2] Bourgeois L, Le Louëër F and Lunéville E 2011 Inverse Problems 27
[3] Huthwaite P 2014 Proceedings of the Royal Society of London 470
[4] Fletcher S, Lowe M, Rataspepp M and Brett C 2012 Journal of Nondestructive Evaluation 31 56-64
[5] Rao J, Rataspepp M, Fan Z 2016 IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control 63 737-745
[6] Rodriguez S, Deschamps M, Castaings M and Ducasse E 2014 Ultrasonics 54 1880-1890
[7] Charalambopoulos A, Gintides D, Kiriaki K 2003 Inverse Problems 19 549-561
[8] Haddar H and Monk P 2002 Inverse Problems 18 891-906
[9] Borcea L and Nguyen D 2016 Inverse Probl. Imaging 10 915-941
[10] Chen Q, Haddar H, Lechleiter A, Monk P 2010 Inverse Problems 26
[11] Monk P and Selgas V 2016 Inverse Problems 32
[12] Catapano I, Crocco L, D’Urso M and Isernia T 2009 Inverse Problems 25
[13] Bourgeois L and Lunéville E 2008 Inverse Problems 24
[14] Bourgeois L and Lunéville E 2013 Inverse Problems 29
[15] Bourgeois L and Fliss S 2014 Inverse Problems 30
[16] Baronian V, Bourgeois L and Recoquillay A 2016 Wave Motion 66 66-87
[17] Recoquillay A 2018 PhD thesis
[18] Baronian V, Bourgeois L, Chapuis B and Recoquillay A 2018 Inverse Problems