Riemann-Hilbert analysis for a Nikishin system

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Abstract. In this paper we give the asymptotic behaviour of type I multiple orthogonal polynomials for a Nikishin system of order two with two disjoint intervals. We use the Riemann-Hilbert problem for multiple orthogonal polynomials and steepest descent analysis for oscillatory Riemann-Hilbert problems to obtain the asymptotic behaviour in all the relevant regions of the complex plane.

Bibliography: 38 titles.

Keywords: multiple orthogonal polynomials, Riemann-Hilbert problem, Hermite-Padé approximations, asymptotic.

§ 1. Introduction

It is well known (see [32], Ch. 4) that the polynomials appearing in Hermite-Padé approximation satisfy a number of orthogonality relations, and these polynomials are therefore known as multiple orthogonal polynomials (polyorthogonal polynomials, Hermite-Padé polynomials). Let \( \vec{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_+^r \) be a multi-index and let \( |\vec{n}| = n_1 + \cdots + n_r \). Type I multiple orthogonal polynomials for measures \( (\mu_1, \ldots, \mu_r) \) on the real line for which all the moments exist are \( (A_{\vec{n},1}, \ldots, A_{\vec{n},r}) \), where \( \deg A_{\vec{n},j} \leq n_j - 1 \), for which

\[
\sum_{j=1}^r \int A_{\vec{n},j}(x)x^k d\mu_j(x) = 0, \quad 0 \leq k \leq |\vec{n}| - 2.
\]

The type II multiple orthogonal polynomial \( P_{\vec{n}} \) is the polynomial of degree \( \leq |\vec{n}| \) for which

\[
\int P_{\vec{n}}(x)x^k d\mu_j(x) = 0, \quad 0 \leq k \leq n_j - 1,
\]

for all \( j \) with \( 1 \leq j \leq r \). The corresponding Hermite-Padé approximation for the functions

\[
f_j(z) = \int \frac{d\mu_j(x)}{z - x}, \quad 1 \leq j \leq r,
\]
for type I is that there exists a polynomial $B_{\vec{n}}$ such that
\[ \sum_{j=1}^{r} A_{\vec{n},j}(z)f_j(z) - B_{\vec{n}}(z) = O(z^{-|\vec{n}|}), \quad z \to \infty, \]
and for type II Hermite-Padé approximation there are $r$ polynomials $Q_{\vec{n},j}$ such that
\[ P_{\vec{n}}(z)f_j(z) - Q_{\vec{n},j}(z) = O(z^{-n_j-1}), \quad z \to \infty, \]
for $1 \leq j \leq r$. The existence of these multiple orthogonal polynomials is easy to justify, see below. However, their uniqueness, in general, is not guaranteed and one needs extra conditions on the system of measures $(\mu_1, \ldots, \mu_r)$, apart from the existence of all the moments. Two systems of measures for which all multi-indices have unique solutions are Angelesco systems (the measures $\mu_j$ are supported on disjoint intervals) and Nikishin systems (the measures $\mu_j$ are supported on the same interval, but their Radon-Nikodym derivatives can be described in terms of a measure on a disjoint interval; see further for a more precise definition for $r = 2$).

Nikishin systems were introduced in 1980 by Nikishin [30], who claimed that multi-indices $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{Z}_+^r \setminus \{0\}$ for which $n_1 \geq n_2 \geq \cdots \geq n_r$ are normal; that is, the corresponding multiple orthogonal polynomials exhibit maximum degree. Driver and Stahl ([15], p. 171) proved that all the multi-indices of a Nikishin system of order two are normal, so that a Nikishin system of order 2 is perfect. Bustamante and López [10] had all the ingredients to prove this but did not state it or deduce it in their paper. Recently Fidalgo Prieto and López Lagomasino [16] proved that every Nikishin system of order $r \geq 2$ is perfect.

We will be investigating multiple orthogonal polynomials for a Nikishin system of order two. In particular, we will consider a Nikishin system of two positive measures $(\mu_1, \mu_2)$ on an interval $[a, b]$, for which
\[ d\mu_2(x) = w(x)\, d\mu_1(x), \quad w(x) = \int_c^d \frac{d\sigma(t)}{x-t}, \quad (1.1) \]
where $\sigma$ is a positive measure on $[c, d]$ and the intervals $[a, b]$ and $[c, d]$ are disjoint. We will assume (without loss of generality) that $c < d < a < b$, so that the interval $[c, d]$ is to the left of $[a, b]$ and hence the function $w$ in (1.1) is positive on $[a, b]$. Furthermore, we assume that $\mu_1$ and $\sigma$ are absolutely continuous (with respect to Lebesgue measure), with
\[ d\mu_1(x) = w_1(x)\, dx, \quad w_1(x) = (x-a)^{\alpha}(b-x)^{\beta}h_1(x), \quad x \in [a, b], \quad (1.2) \]
and
\[ d\sigma(t) = w_2(t)\, dt, \quad w_2(t) = (t-c)^{\gamma}(d-t)^{\delta}h_2(t), \quad t \in [c, d], \quad (1.3) \]
where $h_1$ is analytic in a neighbourhood $\Omega_1$ of $[a, b]$ and $h_2$ is analytic in a neighbourhood $\Omega_2$ of $[c, d]$, $h_1$ and $h_2$ have no zeros at the endpoints of the intervals and $\alpha, \beta, \gamma, \delta > -1$.

The asymptotic behaviour of the ratio of two neighbouring multiple orthogonal polynomials for Nikishin systems has been investigated previously, in [4], [5], [26] and [17]. In this paper we wish to obtain strong asymptotics, that is, the asymptotics of individual polynomials, uniformly in the complex plane using
the Riemann-Hilbert approach. Using a different method, Aptekarev [1] gave the strong asymptotic behaviour of (type II) multiple orthogonal polynomials of a general Nikishin system \( r \geq 2 \) for diagonal sequences \( \vec{n} = (n, n, \ldots, n), \ n \in \mathbb{Z}_+ \).

The Riemann-Hilbert problem for multiple orthogonal polynomials was formulated in [37]. The authors gave the first few transformations of the Riemann-Hilbert problem for Nikishin systems, but they did not perform the steepest descent analysis to get the full asymptotic behaviour of the multiple orthogonal polynomials. Foulquié Moreno [18] showed how to set up the Riemann-Hilbert problem for a generalized Nikishin system (see [19]), but did not work out the steepest descent analysis either. For an Angelesco system the Riemann-Hilbert analysis was worked out in [9], but their analysis is incomplete since they did not include the local analysis near the endpoints of the intervals (the local parametrices). The Riemann-Hilbert analysis for a system of measures (or Markov functions) generated by graphs was done in [6]. In fact, the diagonal case \( m = n \) for type II multiple orthogonal polynomials is contained in [6] and they used very much the same Riemann-Hilbert technique as we do in this paper. A full analysis of the Riemann-Hilbert problem for particular examples of multiple orthogonal polynomials is given in [8] for multiple Hermite polynomials behaving like an Angelesco system, and in [28] for multiple Laguerre polynomials which behave like a Nikishin system. Two new phenomena in the steepest descent analysis of these Riemann-Hilbert problems had already been demonstrated in [2], [7] and [29]: the global opening of the lenses and the transformation based on the generalized Nikishin equilibrium potentials. The Riemann-Hilbert analysis for ray sequences of indices, where \( n/m \to \gamma \), was done recently in [38] (for an Angelesco system) and [3] (for Frobenius-Padé approximants). There is also great interest in the asymptotics of type I Nikishin systems with complex singular points, see [34], [22] and [23] and the references there.

As we mentioned before, there are two types of multiple orthogonal polynomials (and Hermite-Padé approximants). In this paper we will mainly focus on type I multiple orthogonal polynomials, and the main result will be the asymptotic behaviour of the type I multiple orthogonal polynomials, which will be given in §8. In §9 we will work out the asymptotic behaviour of the type II multiple orthogonal polynomials.

Since we are only dealing with \( r = 2 \), we can simplify the notation. Type I multiple orthogonal polynomials for multi-index \( (n, m) \) of the Nikishin system \( (\mu_1, \mu_2) \) are given as a vector of two polynomials \( (A_{n,m}, B_{n,m}) \neq (0, 0) \), where \( \deg A_{n,m} \leq n - 1 \) and \( \deg B_{n,m} \leq m - 1 \), for which

\[
\int_a^b (A_{n,m}(x) + w(x)B_{n,m}(x)) x^k w_1(x) \, dx = 0, \quad 0 \leq k \leq n + m - 2. \tag{1.4}
\]

A type II multiple orthogonal polynomial \( P_{n,m} \) for multi-index \( (n, m) \) is a polynomial of degree \( \leq n + m \), not identically equal to zero, for which

\[
\int_a^b x^k P_{n,m}(x)w_1(x) \, dx = 0, \quad 0 \leq k \leq n - 1,
\]

and

\[
\int_a^b x^k P_{n,m}(x)w(x)w_1(x) \, dx = 0, \quad 0 \leq k \leq m - 1.
\]
Establishing the existence of \((A_{n,m}, B_{n,m})\) and \(P_{n,m}\) reduces (for each type) to solving a homogeneous system, on the coefficients of the polynomials, with one more equation than unknowns. So nontrivial solutions are guaranteed. Since \((\mu_1, \mu_2)\) is a perfect system, we know that for each \((n, m)\) any solution of one type or the other must verify that \(\deg A_{n,m} = n - 1\), \(\deg B_{n,m} = m - 1\) and \(\deg P_{n,m} = n + m\). Other immediate consequences of perfectness is that \((A_{n,m}, B_{n,m})\) and \(P_{n,m}\) are defined uniquely except for constant factors and

\[
\int_a^b (A_{n,m}(x) + w(x)B_{n,m}(x))x^{n+m-1}w_1(x) \, dx = \kappa_{n,m} \neq 0,
\]

\[
\int_a^b x^n P_{n,m}(x)w_1(x) \, dx \neq 0 \quad \text{and} \quad \int_a^b x^m P_{n,m}(x)w(x)w_1(x) \, dx \neq 0.
\]

In the rest of the paper we normalize \((A_{n,m}, B_{n,m})\) so that \(\kappa_{n,m} = 1\) and normalize \(P_{n,m}\) to be monic.

The perfectness of a Nikishin system of order 2 such as ours is a consequence of the following (extended) AT-property (see [30] for the original definition). For any pair of polynomials \((p, q) \neq (0, 0)\), \(\deg p \leq n - 1\) and \(\deg q \leq m - 1\), with real coefficients and any \((n, m)\) the linear form \(p + qw\) has at most \(n + m - 1\) zeros in \(\mathbb{C} \setminus [c, d]\). For completeness we include a proof.

Let \(n \geq m\) and assume that \(p + qw\) has at least \(n + m\) zeros in \(\mathbb{C} \setminus [c, d]\). Since the coefficients of \((p, q)\) are real and \(w\) is symmetric with respect to \(\mathbb{R}\), the zeros of \(p + qw\) come in conjugate pairs. Therefore, there exists a polynomial \(W_{n,m}\), \(\deg W_{n,m} \geq n + m\), with real coefficients and zeros in \(\mathbb{C} \setminus [c, d]\) such that \(x^k(p + qw)/W_{n,m}, k = 0, \ldots, m - 1\), is holomorphic in \(\mathbb{C} \setminus [c, d]\) and has a zero of order \(\geq 2\) at infinity. Take a contour \(\Gamma\) going round \([c, d]\) once in the positive direction and separating it from \(\infty\), with \([c, d]\) inside \(\Gamma\) and \(\infty\) and all the zeros of \(W_{n,m}\) outside. Using Cauchy’s theorem, the definition of \(w\), Fubini’s theorem and Cauchy’s integral formula it follows that

\[
0 = \int_{\Gamma} \frac{z^k(p + qw)(z)}{W_{n,m}(z)} \, dz = \int_{\Gamma} \frac{z^k q(z) w(z)}{W_{n,m}(z)} \, dz = \int_c^d \frac{x^k q(x)}{W_{n,m}(x)} \, d\sigma(x),
\]

\(k = 0, \ldots, m - 1\).

Thus, \(q\) has at least \(m\) sign changes on \((c, d)\), but this is not possible since it has degree \(\leq m - 1\). Consequently, \(q \equiv 0\) which implies that also \(p \equiv 0\). Since \((p, q) \neq (0, 0)\) we arrive at a contradiction.

The case when \(n < m\) reduces to the previous one and follows the same scheme if we take into consideration (see [35], Lemma 6.3.5) the well known fact that

\[
\frac{1}{w(z)} = \ell(z) - \int_c^d \frac{d\tilde{\sigma}(x)}{z - x} = \ell(z) - \tilde{w}(z),
\]

where \(\ell\) is a polynomial of degree \(\leq 1\) and \(\tilde{\sigma}\) is a finite positive measure on \([c, d]\). This transformation allows \((\mu_2, \mu_1)\) to also be viewed as a Nikishin system. Incidentally, the proof that general Nikishin systems (with \(r \geq 2\)) are perfect relies on the same basic ideas of the AT-property for more general linear forms involving Nikishin
systems and the reduction to the case when \( n_1 \geq \cdots \geq n_r \), but now the reduction
formulae turn out to be quite intricate. Unless otherwise stated, in the rest of the
paper we will assume that \( n \geq m \); however, taking account of (1.5) the asymptotic
formulae we obtain remain valid for sequences of multi-indices for which \( n < m \),
provided that appropriate conditions hold.

The Riemann-Hilbert problem for type I multiple orthogonal polynomials is to
find a matrix function \( X : \mathbb{C} \to \mathbb{C}^{3 \times 3} \) such that

1. \( X \) is analytic in \( \mathbb{C} \setminus [a, b] \);

2. the boundary values \( X_\pm(x) = \lim_{\varepsilon \to 0^\pm} X(x \pm i\varepsilon) \) exist for \( x \in (a, b) \) and
   satisfy
   \[
   X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -2\pi i w_1(x) & 1 & 0 \\ -2\pi i w(x) w_1(x) & 0 & 1 \end{pmatrix}, \quad x \in (a, b);
   \]

3. near infinity \( X \) has the behaviour
   \[
   X(z) = \left( I + O\left( \frac{1}{z} \right) \right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \quad z \to \infty;
   \]

4. near \( a \) and \( b \) the behaviour is
   \[
   X(z) = \begin{pmatrix} O(r_a(z)) & O(1) & O(1) \\ O(r_a(z)) & O(1) & O(1) \\ O(r_a(z)) & O(1) & O(1) \end{pmatrix}, \quad z \to a,
   \]
   and
   \[
   X(z) = \begin{pmatrix} O(r_b(z)) & O(1) & O(1) \\ O(r_b(z)) & O(1) & O(1) \\ O(r_b(z)) & O(1) & O(1) \end{pmatrix}, \quad z \to b,
   \]
   where
   \[
   r_a(z) = \begin{cases} |z - a|^\alpha, & -1 < \alpha < 0, \\ \log |z - a|, & \alpha = 0, \\ 1, & \alpha > 0, \end{cases} \quad r_b(z) = \begin{cases} |z - b|^\beta, & -1 < \beta < 0, \\ \log |z - b|, & \beta = 0, \\ 1, & \beta > 0. \end{cases}
   \]

The solution \( X(z) \) of this Riemann-Hilbert problem is

\[
\begin{pmatrix}
\int_a^b \frac{A_{n,m}(x) + w(x) B_{n,m}(x)}{z - x} w_1(x) \, dx & A_{n,m}(z) & B_{n,m}(z) \\
\frac{1}{c_1} \int_a^b \frac{A_{n+1,m}(x) + w(x) B_{n+1,m}(x)}{z - x} w_1(x) \, dx & c_1 A_{n+1,m}(z) & c_1 B_{n+1,m}(z) \\
\frac{1}{c_2} \int_a^b \frac{A_{n,m+1}(x) + w(x) B_{n,m+1}(x)}{z - x} w_1(x) \, dx & c_2 A_{n,m+1}(z) & c_2 B_{n,m+1}(z)
\end{pmatrix},
\]

where \( c_1 = c_1(n, m) \) and \( c_2 = c_2(n, m) \) are such that
\( c_1 A_{n+1,m}(z) = z^n + \text{lower order terms} \) and \( c_2 B_{n,m+1}(z) = z^m + \text{lower order terms} \).
This Riemann-Hilbert problem was first formulated in [37], but we added the condition near the endpoints a and b, which is not needed when the weights are defined on the full real line. Such endpoint conditions were first introduced in [25] for orthogonal polynomials on [−1, 1].

§ 2. First transformation

The weight function \( w \) in the second measure \( \mu_2 \) is given in (1.1) and it is a Stieltjes transform of a weight function on \([c, d]\), so that \([c, d]\) is a branch cut for \( w \). This is not yet apparent in our Riemann-Hilbert problem. Our first transformation is intended to bring these singularities into the Riemann-Hilbert problem and was suggested in [37]. We assume that \( m \leq n \) and then the transformation is

\[
U(z) = X(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \int_c^d \frac{d\sigma(t)}{z-t} & 1 \end{pmatrix}. \tag{2.1}
\]

Then \( U : \mathbb{C} \to \mathbb{C}^{3 \times 3} \) satisfies the following Riemann-Hilbert problem.

1. \( U \) is analytic on \( \mathbb{C} \setminus ([a, b] \cup [c, d]) \).
2. \( U \) has jumps on \((a, b)\) and \((c, d)\) which are given by

\[
U_+(x) = U_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -2\pi i w_1(x) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (a, b),
\]

and

\[
U_+(x) = U_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\pi i w_2(x) & 1 \end{pmatrix}, \quad x \in (c, d).
\]

3. Near infinity \( U \) has the behaviour (here we need \( m \leq n \))

\[
U(z) = \left( \mathbb{I} + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \quad z \to \infty.
\]

4. Near \( a \) and \( b \) the behaviour is

\[
U(z) = \begin{pmatrix} O(r_a(z)) & O(1) & O(1) \\ O(r_a(z)) & O(1) & O(1) \\ O(r_a(z)) & O(1) & O(1) \end{pmatrix}, \quad z \to a,
\]

\[
U(z) = \begin{pmatrix} O(r_b(z)) & O(1) & O(1) \\ O(r_b(z)) & O(1) & O(1) \\ O(r_b(z)) & O(1) & O(1) \end{pmatrix}, \quad z \to b,
\]
and near $c$ and $d$, 

$$U(z) = \begin{pmatrix} O(1) & O(r_c(z)) & O(1) \\ O(1) & O(r_c(z)) & O(1) \\ O(1) & O(r_c(z)) & O(1) \end{pmatrix}, \quad z \to c,$$

$$U(z) = \begin{pmatrix} O(1) & O(r_d(z)) & O(1) \\ O(1) & O(r_d(z)) & O(1) \\ O(1) & O(r_d(z)) & O(1) \end{pmatrix}, \quad z \to d,$$

where

$$r_c(z) = \begin{cases} |z - c|^{\gamma}, & -1 < \gamma < 0, \\ \log |z - c|, & \gamma = 0, \\ 1, & \gamma > 0, \end{cases} \quad r_d(z) = \begin{cases} |z - d|^{\delta}, & -1 < \delta < 0, \\ \log |z - d|, & \delta = 0, \\ 1, & \delta > 0. \end{cases}$$

§ 3. The vector equilibrium problem

In this section we assume that $n$ and $m = m(n)$ are related in such a way that

$$\lim_{n \to \infty} \frac{m}{n + m} = q_1, \quad 0 < q_1 < 1. \quad (3.1)$$

Since we will be working with the case when $m \leq n$, in fact $0 < q_1 \leq 1/2$. In the next section it will be required that $q_1$ be a rational number.

The asymptotic distribution of the zeros of type I (and type II) multiple orthogonal polynomials has been well studied and is given in terms of the solution of a vector equilibrium problem for two probability measures $(\nu_1, \nu_2)$, where $\nu_1$ is supported on $[a, b]$ and $\nu_2$ is supported on $[c, d]$ or a subset $[c^*, d]$ of $[c, d]$. This was first worked out by Nikishin [31] and can be found in [32], Ch. 5, §7; for a more general setting we refer to [19] and [17]. The measures $\nu_1$ and $\nu_2$ are such that $\text{supp}(\nu_1) = [a, b]$ but the support of $\nu_2$ can be a subset $[c^*, d]$ of $[c, d]$ when $q_1 < 1/2$ (see [19], §5.6). In fact if $a, b, d$ and $q_1$ are fixed, then there exists a $c^* < d$ such that $\text{supp}(\nu_2) = [c^*, d]$ when $c < c^*$ and $\text{supp}(\nu_2) = [c, d]$ when $c \geq c^*$. Denote the logarithmic potential of a measure $\nu$ by

$$U(z; \nu) = \int \log \frac{1}{|z - y|} d\nu(y).$$

When $\text{supp}(\nu_2) = [c, d]$ the variational relations for the equilibrium problem are

$$2U(x; \nu_1) - q_1 U(x; \nu_2) = \ell_1, \quad x \in [a, b], \quad (3.2)$$

$$2q_1 U(x; \nu_2) - U(x; \nu_1) = \ell_2, \quad x \in [c, d], \quad (3.3)$$

where $\ell_1, \ell_2$ are constants, and when $\text{supp}(\nu_2) = [c^*, d]$ with $c < c^*$, then

$$2U(x; \nu_1) - q_1 U(x; \nu_2) = \ell_1, \quad x \in [a, b], \quad (3.4)$$

$$2q_1 U(x; \nu_2) - U(x; \nu_1) = \ell_2, \quad x \in [c^*, d], \quad (3.5)$$

$$2q_1 U(x; \nu_2) - U(x; \nu_1) > \ell_2, \quad x \in [c, c^*). \quad (3.6)$$
It is well known that this equilibrium problem has a unique solution (see [32], Ch. 5, § 4, for example).

The method goes as follows. Notice that the function \( A_{n,m}(z) + w(z)B_{n,m}(z) \) has exactly \( n + m - 1 \) simple zeros on \((a, b)\) at points \(y_1, \ldots, y_{n+m-1}\), which depend on the multi-index \((n, m)\), and no other zeros in \(\mathbb{C} \setminus [c, d]\). In fact, the AT-property entails that this linear form can have at most \(n + m - 1\) zeros in \(\mathbb{C} \setminus [c, d]\) whereas (1.4) entails that this linear form can have at most \(n + m - 1\) simple zeros on these points, then from (1.4)

\[
\int_a^b H_{n,m}(x) A_{n,m}(x) + w(x)B_{n,m}(x) x^k w_1(x) \, dx = 0, \quad k = 0, \ldots, n + m - 2.
\]

That is, \(H_{n,m}\) is the monic orthogonal polynomial of degree \(n + m - 1\) on \([a, b]\) for the (varying) measure

\[
\pm \frac{A_{n,m}(x) + w(x)B_{n,m}(x)}{H_{n,m}(x)} \, d\mu_1(x)
\]

(notice that \((A_{n,m} + wB_{n,m})/H_{n,m}\) has constant sign on \([a, b]\)). The sign \(\pm\) is such that the measure is positive on \([a, b]\). If \(n \geq m\), then \(A_{n,m}(z) + w(z)B_{n,m}(z) = O(z^{n-1})\) as \(z \to \infty\), so that \((A_{n,m}(z) + w(z)B_{n,m}(z))/H_{n,m}(z) = O(z^{-m})\) as \(z \to \infty\). Hence, by Cauchy’s theorem (for the exterior of \([c, d]\)),

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{A_{n,m}(z) + w(z)B_{n,m}(z)}{H_{n,m}(z)} z^k \, dz = 0, \quad 0 \leq k \leq m - 2,
\]

whenever \(\Gamma\) is a closed contour encircling \([c, d]\). If we take the contour in such a way that it stays away from \([a, b]\), then

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{A_{n,m}(z)}{H_{n,m}(z)} z^k \, dz = 0,
\]

since the integrand is analytic on and inside \(\Gamma\). Changing the order of integration

and using (1.1) then gives

\[
\int_c^d B_{n,m}(x) x^k \frac{d\sigma(x)}{H_{n,m}(x)} = 0, \quad 0 \leq k \leq m - 2,
\]

so that \(B_{n,m}\) is an orthogonal polynomial of degree \(m - 1\) on \([c, d]\) for the (varying) measure \((-1)^{n+m-1}d\sigma(x)/H_{n,m}(x)\), where the sign makes the measure positive on \([c, d]\). Furthermore,

\[
\frac{A_{n,m}(x) + w(x)B_{n,m}(x)}{H_{n,m}(x)} = \int_c^d \frac{B_{n,m}(t)}{x - t} \frac{d\sigma(t)}{H_{n,m}(t)}, \quad x \notin [c, d],
\]

because we can write the left-hand side using Cauchy’s theorem as

\[
\frac{A_{n,m}(x) + w(x)B_{n,m}(x)}{H_{n,m}(x)} = -\frac{1}{2\pi i} \int_{\Gamma} \frac{A_{n,m}(z) + w(z)B_{n,m}(z)}{H_{n,m}(z)} \frac{dz}{z-x},
\]
where $\Gamma$ is a contour going counterclockwise around $[c, d]$ but not around $x$. Note that

$$\frac{1}{2\pi i} \int_{\Gamma} A_{n,m}(z) \frac{dz}{H_{n,m}(z) (z-x)} = 0,$$

since the integrand is analytic on and inside $\Gamma$. Consequently, using (1.1) and interchanging the order of integration, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma} w(z) B_{n,m}(z) \frac{dz}{H_{n,m}(z) (z-x)} = \int_{c}^{d} d\sigma(t) \frac{1}{2\pi i} \int_{\Gamma} B_{n,m}(z) \frac{dz}{H_{n,m}(z) (z-x)(z-t)}$$

$$= -\int_{c}^{d} B_{n,m}(t) d\sigma(t) \frac{1}{H_{n,m}(t)} x-t,$$

from which (3.9) follows.

Due to (3.7) and (3.8) the vector equilibrium problem corresponds to combining the equilibrium condition (3.2) for the asymptotic zero distribution $\nu_1$ of $H_{n,m}$, which has external field

$$\lim_{n,m \to \infty} -\frac{1}{n+m} \log \frac{|A_{n,m}(x) + w(x) B_{n,m}(x)|}{|H_{n,m}(x)|} = 0,$$

with the equilibrium condition (3.3) for the asymptotic distribution $\nu_2$ of the zeros of $B_{n,m}$, which has external field

$$\lim_{m \to \infty} \frac{1}{m} \log |H_{n,m}(x)| = 0,$$

Clearly, the external field on $[c, d]$ is $-U(x; \nu_1)/q_1$ (up to an additive constant). This gives the variational equation (3.3) or (3.5), (3.6). For the external field on $[a, b]$ we use (3.9) and the orthogonality of $B_{n,m}$ for the measure $d\sigma(t)/H_{n,m}(t)$ to find

$$\frac{|A_{n,m}(x) + w(x) B_{n,m}(x)|}{|H_{n,m}(x)|} = \frac{1}{|B_{n,m}(x)|} \int_{c}^{d} B_{n,m}^2(t) \frac{d\sigma(t)}{|H_{n,m}(t)|}, \quad x > d,$$

so that (up to an additive constant) the external field on $[a, b]$ is

$$\lim_{n,m \to \infty} \frac{1}{n+m} \log |B_{n,m}(x)| = -q_1 U(x; \nu_2).$$

This gives the variational equation (3.2) or (3.4). For the case where supp($\nu_2$) = $[c^*, d]$ with $c < c^* < d$ we have the variational condition (3.5) on $[c^*, d]$, which has to be supplemented with the inequality (3.6) on $[c, c^*)$. For details see [32], Ch. 5, §7.

§ 4. Normalizing the Riemann-Hilbert problem

The next transformation of the Riemann-Hilbert problem is to normalize it at infinity, but in such a way that we get nice jumps on the intervals. This transformation will give the main term in the asymptotics outside the intervals. For this
we now introduce \( g \)-functions, which are the complex potentials of the measures \( \nu_1 \) and \( \nu_2 \):

\[
g_1(x) = \int_a^b \log(x - t) \, d\nu_1(t) \quad \text{and} \quad g_2(x) = \int_c^d \log(x - t) \, d\nu_2(t),
\]

or, when \( \operatorname{supp}(\nu_2) = [c^*, d] \),

\[
g_2(x) = \int_{c^*}^d \log(x - t) \, d\nu_2(t).
\]

For the logarithm we choose the branch cut on the negative real line. Observe that for \( x \in \mathbb{R} \)

\[
g_{1,2}^{\pm}(x) = \begin{cases} 
-\mathcal{U}(x; \nu_1), & x > b, \\
-\mathcal{U}(x; \nu_1) \pm i\pi, & x < a, \\
-\mathcal{U}(x; \nu_1) \pm i\pi \varphi_1(x), & a \leq x \leq b,
\end{cases}
\]

and similarly

\[
g_{1,2}^{\pm}(x) = \begin{cases} 
-\mathcal{U}(x; \nu_2), & x > d, \\
-\mathcal{U}(x; \nu_2) \pm i\pi, & x < c \text{ (or } x < c^*) , \\
-\mathcal{U}(x; \nu_2) \pm i\pi \varphi_2(x), & c \text{ (or } c^*) \leq x \leq d,
\end{cases}
\]

We now introduce the second transformation

\[
V(z) = LU(z) \begin{pmatrix} e^{(n+m)g_1(z)} & 0 & 0 \\
0 & e^{-(n+m)g_1(z)+mg_2(z)} & 0 \\
0 & 0 & e^{-mg_2(z)} \end{pmatrix} L^{-1},
\]

where

\[
L = L(n, m) = \begin{pmatrix} e^{-\frac{n+m}{3}(2\ell_1+\ell_2)} & 0 & 0 \\
0 & e^{\frac{n+m}{3}(\ell_1-\ell_2)} & 0 \\
0 & 0 & e^{\frac{n+m}{3}(\ell_1+2\ell_2)} \end{pmatrix}.
\]

This transformation normalizes the behaviour for \( z \to \infty \):

\[
V(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty,
\]

this is because of \( g_i(z) = \log z + O(1/z) \) for \( i = 1, 2 \) as \( z \to \infty \). The functions \( e^{(n+m)g_1(z)} \) and \( e^{mg_2(z)} \) have jumps on the intervals \((a, b)\) and \((c, d)\) or \((c^*, d)\), respectively, but are otherwise analytic (the jumps over the branch cuts \(( -\infty, a ]\) and \(( -\infty, c ] \) or \(( -\infty, c^* ]\) disappear by taking the exponential), hence \( V \) is analytic on \( \mathbb{C} \setminus ([a, b] \cup [c, d]) \). The price for normalizing the Riemann-Hilbert problem is that the jumps will be more complicated, but our choice of \( g \)-functions using the
equilibrium measures \((\nu_1, \nu_2)\) gives oscillatory jumps on the intervals. Indeed, the jump over \((a, b)\) now is
\[
V_+(x) = V_-(x) \begin{pmatrix} e^{(n+m)[g_1^+(x)-g_1^-(x)]} & 0 & 0 \\ -v_1(x)e^{(n+m)[g_1^+(x)+g_1^-(x)+\ell_1]-mg_2(x)} & e^{-(n+m)[g_1^+(x)-g_1^-(x)]} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
and over \((c, d)\) we have
\[
V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{m[g_2^+(x)-g_2^-(x)]} & 0 \\ 0 & -v_2(x)e^{-(n+m)[g_1(x)-\ell_2]+m[g_2^+(x)+g_2^-(x)]} & e^{-m[g_2^+(x)-g_2^-(x)]} \end{pmatrix}.
\]
Here we have used the functions
\[
\begin{align*}
v_1(x) &= 2\pi iw_1(x) = 2\pi i(x-a)^\alpha(b-x)^\beta h_1(x), \\
v_2(x) &= 2\pi iw_2(x) = 2\pi i(x-c)^\gamma(d-x)^\delta h_2(x)
\end{align*}
\]
to simplify the notation. Recall from (4.2) that for \(x \in (a, b)\)
\[
\begin{align*}
g_1^+(x) - g_1^-(x) &= 2\pi i\varphi_1(x), \\
g_1^+(x) + g_2^-(x) &= -2U(x; \nu_1),
\end{align*}
\]
and for \(x \in (c, d)\) (4.3) gives
\[
\begin{align*}
g_2^+(x) - g_2^-(x) &= 2\pi i\varphi_2(x), \\
g_2^+(x) + g_2^-(x) &= -2U(x; \nu_2),
\end{align*}
\]
so that the jump for \(V\) on \((a, b)\) is
\[
V_+(x) = V_-(x) \begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -v_1(x)e^{-2(n+m)U(x;\nu_1)+mU(x;\nu_2)+(n+m)\ell_1} & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
and on \((c, d)\) the jump is
\[
V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi im\varphi_2(x)} & 0 \\ 0 & -v_2(x)e^{(n+m)U(x;\nu_1)-2mU(x;\nu_2)+(n+m)\ell_2} & e^{-2\pi im\varphi_2(x)} \end{pmatrix}.
\]
If we take \(q_1\) rational and \((n, m)\) so that
\[
q_1 = \frac{m}{n+m}, \quad (4.6)
\]
and if \(\text{supp}(\nu_2) = [c, d]\) then the variational equations (3.2) and (3.3) imply that the jumps simplify to
\[
V_+(x) = V_-(x) \begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -v_1(x) & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (a, b), \quad (4.7)
\]
and
\[ V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i \varphi_2(x)} & 0 \\ 0 & -v_2(x) & e^{-2\pi i \varphi_2(x)} \end{pmatrix}, \quad x \in (c, d). \quad (4.8) \]

If \( \text{supp}(\nu_2) = [c^*, d] \) with \( c < c^* < d \), then (3.5) and (3.6) give
\[ V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i \varphi_2(x)} & 0 \\ 0 & -v_2(x) & e^{-2\pi i \varphi_2(x)} \end{pmatrix}, \quad x \in (c^*, d), \quad (4.9) \]

and
\[ V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v_2(x)e^{(n+m)\Phi(x)} & 1 \end{pmatrix}, \quad x \in [c, c^*), \quad (4.10) \]

where \( \Phi(x) = U(x; \nu_1) - 2q_1 U(x; \nu_2) + \ell_2 < 0 \) on \([c, c^*)\). Since \( \varphi_1 \) and \( \varphi_2 \) are real positive functions, these jumps are oscillatory on the intervals \((a, b)\) and \((c, d)\) or \((c^*, d)\), respectively. This is what we wanted to achieve, since it allows us to use the method of steepest descent for oscillatory Riemann-Hilbert problems, which was introduced by Deift and Zhou (see [12] and [13]). Observe that the jumps (4.7) and (4.8) are essentially reduced to \((2 \times 2)\)-jumps, which will make our work easier because we can rely on some of the work done earlier (see [11] and [25], for example).

§ 5. Opening the lenses

A simple calculation shows that we can factorize the jump matrix in (4.7) as
\[
\begin{pmatrix} (\Phi_1^+)^{n+m} & 0 & 0 \\ -v_1 & (\Phi_1^-)^{n+m} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{v_1} & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{v_1} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where we have used \( \Phi_1^\pm = \exp(\pm 2\pi i \varphi_1) \). In a similar way, the jump matrix in (4.8) and (4.9) factorizes as
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\Phi_2^+)^m & 0 \\ 0 & -v_2 & (\Phi_2^-)^m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{v_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
where $\Phi_2^\pm = \exp(\pm 2\pi i \varphi)$. Therefore, instead of making one jump over $(a, b)$ with the jump matrix in (4.7) we can make three jumps over $(a, b)$, each with one of the matrices in the matrix factorization. The two outer matrices in the matrix factorization contain oscillatory terms. By opening a lens with $[a, b]$ in the middle of the lens (see Figure 1) we can make jumps using these outer matrices in the factorization over the lips of the lenses, but then the Riemann-Hilbert problem changes inside the lens, where only ‘part’ of the jump in (4.7) is done. The new Riemann-Hilbert matrix is

$$S(z) = \begin{cases} V(z) & \text{outside the lens for } [a, b], \\ V(z) \begin{pmatrix} 1 & \Phi_1^{-(n+m)} & 0 \\ 0 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{inside the lens, upper part,} \\ V(z) \begin{pmatrix} 1 & -\Phi_1^{-(n+m)} & 0 \\ 0 & v_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{inside the lens, lower part.} \end{cases}$$

Figure 1. Riemann-Hilbert problem for $S$.

This can be done when the function $h_1$ (in the function $w_1$) is analytic in a region that contains the lens and $\Phi_1$ can be extended analytically from $(a, b)$ to a region that contains the lens in such a way that $\lim_{\varepsilon \to 0^+} \Phi_1(x \pm i\varepsilon) = \Phi_1^\pm(x)$. We also need to modify the Riemann-Hilbert problem near the lens for $[c, d]$: if $\text{supp}(\nu_2) = [c, d]$ then

$$S(z) = \begin{cases} V(z) & \text{outside the lens for } [c, d], \\ V(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Phi_2^{-m} \frac{v_2}{v_2} \\ 0 & 0 & 1 \end{pmatrix} & \text{inside the lens, upper part,} \\ V(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^{-m} \frac{v_2}{v_2} \\ 0 & 0 & 1 \end{pmatrix} & \text{inside the lens, lower part,} \end{cases}$$

which can be done if $h_2$ (in the function $v_2$) is analytic in a region that contains the lens and $\Phi_2$ can be extended analytically from $(c, d)$ to a region that contains
the lens in such a way that \( \lim_{\varepsilon \to 0^+} \Phi_2(x \pm i\varepsilon) = \Phi_2^\pm(x) \). The jumps for the Riemann-Hilbert matrix \( S \) over the six contours in Figure 1 are then given by

\[
S_+(z) = \begin{cases}
    S_-(z) \begin{pmatrix} 1 & -\Phi_1^{-(n+m)} & 0 \\ 0 & v_1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma^{a,b}_+ \cup \Sigma^{a,b}_- \\
    S_-(z) \begin{pmatrix} 0 & 1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (a, b),
\end{cases}
\]

and

\[
S_+(z) = \begin{cases}
    S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^{-m} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma^{c,d}_+ \cup \Sigma^{c,d}_- \\
    S_-(z) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -v_2 & 0 \end{pmatrix}, & z \in (c, d).
\end{cases}
\]

If \( \text{supp}(\nu_2) = [c^*, d] \) with \( c < c^* < d \), then we only open the lens over \((c^*, d)\) (see Figure 2) and

\[
S_+(z) = \begin{cases}
    S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^{-m} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma^{c^*,d}_+ \cup \Sigma^{c^*,d}_- \\
    S_-(z) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -v_2 & 0 \end{pmatrix}, & z \in (c^*, d),
\end{cases}
\]

We already know that \( \Phi < 0 \) on \([c, c^*)\). We claim that \( |\Phi_1| > 1 \) on the lips \( \Sigma^{a,b}_\pm \) of the lens for \([a, b]\) (except at \(a\) and \(b\)) and that \( |\Phi_2| > 1 \) on the lips \( \Sigma^{c,d}_\pm \) or \( \Sigma^{c^*,d}_\pm \) of
the lens for \([c,d]\) or \([c^*,d]\) (except at the endpoints), provided that these lenses are thin enough. The function \(\Phi^+_1\), which is defined on \([a,b]\) by \(\Phi^+_1(x) = \exp(2\pi i \varphi_1(x))\), with \(\varphi_1\) given by (4.2), can be extended to the function \(\Phi_1(z) = \exp(2\pi i \varphi_1(z))\) with \(\text{Im} \ z > 0\), where \(\varphi_1\) is given by

\[
\varphi_1(z) = \int_z^b \nu'_1(t) \, dt,
\]

where \(\nu'_1(t) = m_1(t)(t-a)^{-1/2}(b-t)^{-1/2}\) is the density of the measure \(\nu_1\), which is absolutely continuous with square root singularities at the endpoints, and with \(m_1\) a positive analytic function on \([a,b]\). Write \(\varphi_1(z) = u(x,y) + iv(x,y)\), with \(z = x + iy\); then on the interval \([a,b]\) we have \(u(x,0) = \int_x^b \nu'_1(t) \, dt\) and \(v(x,0) = 0\), so that \(\partial u / \partial x = -\nu'_1(x) < 0\) for \(x \in (a,b)\). The Cauchy-Riemann equations then imply that \(\partial v / \partial y < 0\), so that \(v(x,y)\) is a decreasing function of \(y\), and hence \(v(x,y) < 0\) for \(x \in (a,b)\) when \(y > 0\) and close to 0. Then \(|\Phi_1(z)| = \exp(-2\pi v(x,y)) > 1\) for \(y > 0\) and close to zero and hence \(|\Phi_1| > 1\) on \(\Sigma^{a,b}_+\), away from \(a\) and \(b\). In a similar way we have \(\Phi^{-1}_1(x) = \exp(-2\pi i \varphi_1(x))\) on \([a,b]\), so in the lower complex plane \(|\Phi_1(z)| = \exp(2\pi v(x,y))\), with \(y < 0\) and close to zero. Since \(v(x,y)\) is a decreasing function of \(y\) and \(v(x,0) = 0\), we have \(v(x,y) > 0\) for \(y < 0\) and small enough, meaning that \(|\Phi_1| > 1\) on \(\Sigma^{a,b}_-\), away from the points \(a\) and \(b\). The same reasoning can be given to show that \(|\Phi_2| > 1\) on the lips \(\Sigma^{c,d}_\pm\) or \(\Sigma^{c^*,d}_\pm\) away from the points \(c\) (or \(c^*\)) and \(d\).

§ 6. The global parametrix

The global parametrix is the solution of the Riemann-Hilbert problem for \(S\) if we ignore the jumps on the lips of the lenses, which for \(n,m \to \infty\) converge to the identity matrix. So we look for a \((3 \times 3)\)-matrix \(N\) which is analytic in \(\mathbb{C} \setminus ([a,b] \cup [c,d])\) with jumps

\[
N_+(x) = N_-(x) \begin{pmatrix} 0 & \frac{1}{v_1} & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (a,b),
\]

and

\[
N_+(x) = N_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{v_2} \\ 0 & -v_2 & 0 \end{pmatrix}, \quad x \in (c,d).
\]

The case \(\text{supp}(\nu_2) = [c^*,d]\) is similar and one only needs to change \(c\) to \(c^*\).

We solve this in two steps, as was done, for example, in [9] or [14] for an Angelesco system. First, we need the Szegő functions for \((v_1,v_2)\) for the geometry of the Riemann surface \(\mathcal{R}\) with three sheets \(\mathcal{R}_0, \mathcal{R}_1\) and \(\mathcal{R}_2\), which is of genus 0 and has branch points \(a, b, c\) and \(d\). The interval \([a,b]\) on the first sheet \(\mathcal{R}_0\) is connected in the usual way to the interval \([a,b]\) on \(\mathcal{R}_1\), and \([c,d]\) on the third sheet \(\mathcal{R}_2\) is connected to \([c,d]\) on the second sheet \(\mathcal{R}_1\) (see Figure 3). These Szegő functions
are analytic functions $D_0$, $D_1$ and $D_2$ on $\mathbb{C}\setminus([a, b] \cup [c, d])$ which satisfy the boundary conditions
\begin{align}
\begin{cases}
D_1^+(x) = v_1(x)D_0^-(x), \\
D_1^-(x) = v_1(x)D_0^+(x), \\
D_2^+(x) = D_2^-(x),
\end{cases}
\quad x \in [a, b], 
\tag{6.1}
\end{align}
and
\begin{align}
\begin{cases}
D_0^+(x) = D_0^-(x), \\
D_2^+(x) = v_2(x)D_1^-(x), \\
D_2^-(x) = v_2(x)D_1^+(x),
\end{cases}
\quad x \in [c, d],
\tag{6.2}
\end{align}
and for which the limit for $z \to \infty$ does not vanish: $D_0(\infty) \neq 0$, $D_1(\infty) \neq 0$ and $D_2(\infty) \neq 0$.

Then, with these functions we can define the matrix
\begin{equation}
N_0(z) = \begin{pmatrix} D_0(\infty) & 0 & 0 \\ 0 & D_1(\infty) & 0 \\ 0 & 0 & D_2(\infty) \end{pmatrix}^{-1} N(z) \begin{pmatrix} D_0(z) & 0 & 0 \\ 0 & D_1(z) & 0 \\ 0 & 0 & D_2(z) \end{pmatrix},
\tag{6.3}
\end{equation}
and this has the same behaviour as $N$ when $z \to \infty$, but has a simpler jump on the intervals:
\begin{equation}
N_0^+(x) = N_0^-(x) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (a, b),
\tag{6.4}
\end{equation}
and
\begin{equation}
N_0^+(x) = N_0^-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad x \in (c, d).
\tag{6.5}
\end{equation}

The second step consists in finding an expression for $N_0$.

For both steps we need a conformal mapping $\psi: \mathfrak{R} \to \overline{\mathbb{C}}$, $\psi_j(x) = y$, $j = 0, 1, 2$, between the Riemann surface $\mathfrak{R}$ (see Figure 3) and the extended complex plane $\overline{\mathbb{C}}$ (see Figure 4).

A way to obtain this mapping was described in [27]. Using an affine transformation, if necessary, without loss of generality we can assume that $[a, b] = [1, \lambda]$ and
[c, d] = [−µ, −1], λ, µ > 0. In [27], Theorem 3.1, it was proved that the rational function $H(y)$ below establishes a one-to-one correspondence between $\mathcal{C}$ and $\mathcal{R}$. Here,

$$H(y) = x = h + y + \frac{Ay}{1 - y} + \frac{By}{1 + y}, \quad (6.6)$$

with constants $A$, $B$ and $h$ given by

$$A = \frac{1}{4}(1 - \hat{b})(1 - \hat{\alpha})(1 - \hat{a})(1 - \hat{\beta}),$$
$$B = \frac{1}{4}(1 + \hat{b})(1 + \hat{\alpha})(1 + \hat{a})(1 + \hat{\beta}),$$
$$h = \frac{1}{4}(\hat{a} + \alpha)\left(2\hat{a}\alpha - \frac{(\hat{a} - \alpha)^2}{1 - \hat{a}\hat{\alpha}}\right),$$

where $\hat{\beta}$, $\hat{\alpha}$, $\hat{a}$ and $\hat{b}$ ($\hat{\beta} < -1 < \hat{\alpha} < \hat{a} < 1 < \hat{b}$) are the critical points of $H$, which are uniquely determined as solutions of some algebraic equations depending solely on $\mu$ and $\lambda$. Specifically, $\hat{\beta}$ and $\hat{b}$ are the solutions of the quadratic equation

$$x^2 + (\hat{a} + \hat{\alpha})x + \frac{(\hat{a} - \hat{\alpha})^2}{1 - \hat{a}\hat{\alpha}} - 3 = 0,$$

whereas $\hat{\alpha}$ and $\hat{a}$ are the unique solutions of the algebraic system

$$(\mu - \lambda)(\hat{a} - \hat{\alpha})^3 = 2(\hat{a} + \hat{\alpha})[(9 - \hat{a}\hat{\alpha})(1 - \hat{a}\hat{\alpha}) - (\hat{a} - \hat{\alpha})^2],$$
$$(\mu + \lambda)^2(\hat{a} - \hat{\alpha})^6 = 4(3 + \hat{a}\hat{\alpha})^2(1 - \hat{a}\hat{\alpha})[((\hat{a} + \hat{\alpha})^2 + 12)(1 - \hat{a}\hat{\alpha}) - 4(\hat{a} - \hat{\alpha})^2].$$

Note that we have used the notation $\hat{a}$, $\hat{\alpha}$, $\hat{b}$ and $\hat{\beta}$ because $a$ and $b$ are already being used for the endpoints of the interval $[a, b]$, and $\alpha$ and $\beta$ for the exponents in $w_1$ (see (1.2)). The $\hat{a}$, $\hat{b}$, $\hat{\alpha}$ and $\hat{\beta}$ correspond to $a$, $b$, $\alpha$ and $\beta$ in [27], respectively. (When $\mu = \lambda$, that is, if the intervals $[a, b]$ and $[c, d]$ have equal length, these equations reduce substantially and can be solved exactly using radicals, see [27].)
In other words, we can take $\psi$ as the solution of the cubic equation
\[ y^3 - (x + A - B - h)y^2 - (1 + A + B)y + x - h = 0. \]

Let $\psi_0, \psi_1$ and $\psi_2$ be the branches of $\psi$ corresponding to $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Set $\widetilde{\mathcal{R}}_j = \psi(\mathcal{R}_j)$, $j = 0, 1, 2$ (see Figure 3). We have $\psi(\infty(0)) = \psi_0(\infty) = 1$, $\psi(\infty(1)) = \psi_1(\infty) = \infty$ and $\psi(\infty(2)) = \psi_2(\infty) = -1$. We orient the closed curves $\partial \mathcal{R}_0$ and $\partial \mathcal{R}_2$ so that the left (+) sides induced by the orientation coincide with the regions $\mathcal{R}_0$ and $\mathcal{R}_2$.

We will find $D = \text{diag}(D_0, D_1, D_2)$ that is holomorphic in $\overline{\mathbb{C}} \setminus ([-\mu, -1] \cup [1, \lambda])$ and satisfies (6.1) and (6.2). We seek the functions $D_j(y) = \tilde{D}(\psi_j(x))$, with $\psi_j(x) = y$, $j = 0, 1, 2$, for some function $\tilde{D}$ verifying:

i) $\tilde{D} \in H(\overline{\mathbb{C}} \setminus (\partial \mathcal{R}_0 \cup \partial \mathcal{R}_2))$ and is nowhere zero;

ii) $\tilde{D}_-(x) = v_1(H(x))\tilde{D}_+(x)$, $x \in \partial \mathcal{R}_0$;

iii) $\tilde{D}_+(x) = v_2(H(x))\tilde{D}_-(x)$, $x \in \partial \mathcal{R}_2$;

where $H(y)$ is defined in (6.6). This is consistent with the orientation and (6.1) and (6.2).

Applying a branch of the logarithm to ii) and iii) we see that finding $\tilde{D}$ reduces to solving a scalar additive Riemann-Hilbert problem with boundary conditions
\[
\begin{align*}
\log \tilde{D}_-(x) &= \log v_1(H(x)) + \log \tilde{D}_+(x), \quad x \in \partial \mathcal{R}_0, \\
\log \tilde{D}_+(x) &= \log v_2(H(x)) + \log \tilde{D}_-(x), \quad x \in \partial \mathcal{R}_2,
\end{align*}
\]

Using the Sokhotsky-Plemelj formula, we find that
\[ \tilde{D}(w) = C \exp\left(\frac{1}{2\pi i} \int_{\partial \mathcal{R}_2} \frac{\log v_2(H(x))}{x-w} \, dx - \frac{1}{2\pi i} \int_{\partial \mathcal{R}_0} \frac{\log v_1(H(x))}{x-w} \, dx\right), \]

where $C$ is an arbitrary constant and integration is performed according to the orientation selected. Hence, defining
\[ D_j(z) = C \exp\left(\frac{1}{2\pi i} \int_{\partial \mathcal{R}_2} \frac{\log v_2(H(x))}{x-\psi_j(z)} \, dx - \frac{1}{2\pi i} \int_{\partial \mathcal{R}_0} \frac{\log v_1(H(x))}{x-\psi_j(z)} \, dx\right) \] (6.7)

for $j = 0, 1, 2$, we obtain a solution of (6.1) and (6.2). Moreover, notice that $D_0D_1D_2$ can be extended to an entire function on $\overline{\mathbb{C}}$. Indeed, the boundary conditions imply that it is analytic except possibly at $\{-\mu, -1, 1, \lambda\}$. At these points the conditions on $v_j, j = 1, 2$ imply that there it behaves like $O(1)$. It is also bounded at infinity; therefore, it is constant. We can take $C$ so that $D_0D_1D_2 \equiv 1$.

Now we find $N_0$ in the form
\[ N_0(z) = \begin{pmatrix} N_1(\psi_0(z)) & N_1(\psi_1(z)) & N_1(\psi_2(z)) \\ N_2(\psi_0(z)) & N_2(\psi_1(z)) & N_2(\psi_2(z)) \\ N_3(\psi_0(z)) & N_3(\psi_1(z)) & N_3(\psi_2(z)) \end{pmatrix}, \] (6.8)

where $N_1$, $N_2$ and $N_3$ are appropriate algebraic functions defined on $\overline{\mathbb{C}}$. Set
\[
\begin{align*}
\Gamma_1^+ &= \psi_{0,+}([1, \lambda]), & \Gamma_1^- &= \psi_{0,-}([-1, \lambda]), \\
\Gamma_2^+ &= \psi_{1,+}([-\mu, -1]), & \Gamma_2^- &= \psi_{1,-}([-\mu, -1]).
\end{align*}
\]
We have \( \partial \mathcal{R}_0 = \Gamma_1^+ \cup \Gamma_1^- \) and \( \partial \mathcal{R}_2 = \Gamma_2^+ \cup \Gamma_2^- \). It is not difficult to verify that \( \Gamma_1^- \) and \( \Gamma_2^+ \) lie in the upper half plane whereas \( \Gamma_1^+ \) and \( \Gamma_2^- \) lie in the lower half-plane (see Figure 4). The boundary conditions (6.4) and (6.5) imply that

\[
N_{j,+}(x) = -N_{j,-}(x), \quad x \in \Gamma_1^+ \cup \Gamma_2^+, \quad N_{j,+}(x) = N_{j,-}(x), \quad x \in \Gamma_1^- \cup \Gamma_2^-,
\]

\( j = 1, 2, 3 \).

Recall that \( \beta, \alpha, \hat{a} \) and \( \hat{b} \) are the critical points of \( H \), which has simple poles at \(-1, 1 \) and \( \infty \). The condition which the matrix function \( N_0 \) should verify at infinity reduces to requiring that \( N_j(\tau_i) = \delta_{i,j} \) where \( \tau_i \) stands for \(-1, 1 \) or \( \infty \). Define

\[
r(z) = \sqrt{(z - \beta)(z - \alpha)(z - \hat{a})(z - \hat{b})}
\]

with a branch cut along \( \Gamma_1^+ \cup \Gamma_2^+ \) which behaves as \( z^2 + O(z) \) as \( z \to \infty \). Define

\[
N_1(z) = r_1 \frac{z + 1}{r(z)}, \quad N_2(z) = \frac{z^2 - 1}{r(z)} \quad \text{and} \quad N_3(z) = r_3 \frac{z - 1}{r(z)}, \quad (6.9)
\]

where \( r_1 \) and \( r_3 \) are constants selected so that \( N_1(1) = 1 = N_3(-1) \). Taking \( N_0 \) as in (6.8) relations (6.4) and (6.5) can be verified directly. Moreover, \( N_0(z) = I + O(1/z), \ z \to \infty \). From (6.4) and (6.5) it follows that \( \det N_0(z) \) is analytic in \( \mathbb{C} \setminus \{-\mu, -1, 1, \lambda\} \). The behaviour of \( \det N_0(z) \) in a neighbourhood of any one of these extreme points, say \( \zeta \), is at worst like \( O(|z - \zeta|^{-1/2}) \), \( z \to \zeta \), so these singularities are removable and \( \det N_0(z) \) is an entire function, and because of its behaviour at \( \infty \) it is constant and equal to 1.

\[\text{§ 7. Parametrices around the endpoints}\]

Unfortunately, the jumps for \( S \) on the lips \( \Sigma_{a,b}^+ \) and \( \Sigma_{c,d}^+ \) or \( \Sigma_{c^*,d}^+ \) do not tend uniformly to the identity matrix. The uniformity is violated near the endpoints \( a, \ b, \ c \) (or \( c^* \)) and \( d \) of the intervals. We need to make a local analysis near each of these endpoints and construct a parametrix that describes the local behaviour near such a point, and which matches the global parametrix outside a neighbourhood. We will only do this for the point \( b \), but the analysis is similar for the other three points.

![Figure 5. Parametrix around b.](image)
The idea is to approximate the Riemann-Hilbert problem for \( S \) inside a curve \( \Gamma_b \) around \( b \) (see Figure 5) by a model Riemann-Hilbert problem for a matrix \( P_b \) which matches the global parametrix \( N \) on \( \Gamma_b \) with an error \( O(1/n) \). Such a local parametrix was constructed earlier for orthogonal polynomials on \([-1, 1]\) with Jacobi type weights in [25], and this is for a \( 2 \times 2 \) Riemann-Hilbert problem. Observe that all the jumps for \( S \) inside \( \Gamma_b \) are of the form

\[
\begin{pmatrix}
    J & 0 \\
    0 & 1
\end{pmatrix},
\]

where \( J \) is a \((2 \times 2)\)-matrix and 0 is a row (column) vector containing zeros, so basically the Riemann-Hilbert problem for \( S \) near \( b \) behaves like a \( 2 \times 2 \) Riemann-Hilbert problem, so that we can use the construction from [25] with some modifications.

The \((2 \times 2)\)-matrix \( \Psi \) that was considered in [25], §6, p. 365, solves the following Riemann-Hilbert problem on the system of contours \( \Sigma \Psi = \gamma_1 \cup \gamma_2 \cup \gamma_3 \), with

\[
\gamma_1 = \{ re^{2\pi i/3} : r > 0 \}, \quad \gamma_2 = (-\infty, 0] \quad \text{and} \quad \gamma_3 = \{ re^{-2\pi i/3} : r > 0 \}
\]

orientated toward the point 0:
- \( \Psi \) is analytic in \( \mathbb{C} \setminus \Sigma \Psi \);
- \( \Psi \) satisfies the jump conditions

\[
\Psi_+ (\zeta) = \Psi_-(\zeta) \begin{pmatrix}
    1 & 0 \\
    e^{\beta \pi i} & 1
\end{pmatrix}, \quad \zeta \in \gamma_1,
\]

\[
\Psi_+ (\zeta) = \Psi_-(\zeta) \begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix}, \quad \zeta \in \gamma_2,
\]

\[
\Psi_+ (\zeta) = \Psi_-(\zeta) \begin{pmatrix}
    1 & 0 \\
    e^{-\beta \pi i} & 1
\end{pmatrix}, \quad \zeta \in \gamma_3;
\]

- for \( \beta < 0 \)

\[
\Psi (\zeta) = O \left( \frac{|\zeta|^{\beta/2}}{|\zeta|^{\beta/2}} \right), \quad \zeta \to 0,
\]

for \( \beta = 0 \)

\[
\Psi (\zeta) = O \left( \frac{\log |\zeta|}{\log |\zeta|} \right), \quad \zeta \to 0,
\]

and for \( \beta > 0 \)

\[
\Psi (\zeta) = O \left( \frac{|\zeta|^{-\beta/2}}{|\zeta|^{-\beta/2}} \right), \quad \zeta \to 0, \quad |\arg \zeta| < \frac{2\pi}{3},
\]

\[
\Psi (\zeta) = O \left( \frac{|\zeta|^{-\beta/2}}{|\zeta|^{-\beta/2}} \right), \quad \zeta \to 0, \quad \frac{2\pi}{3} < |\arg \zeta| < \pi.
\]

The matrix \( \Psi \) is explicitly given in [25], Theorem 6.3,

\[
\Psi (\zeta) = \begin{pmatrix}
    I_\beta (2\zeta^{1/2}) & \frac{i}{\pi} K_\beta (2\zeta^{1/2}) \\
    2\pi i \zeta^{1/2} I_\beta (2\zeta^{1/2}) & -2\zeta^{1/2} K_\beta (2\zeta^{1/2})
\end{pmatrix}, \quad |\arg \zeta| < \frac{2\pi}{3},
\]
\[ \Psi(\zeta) = \left( \begin{array}{cccc} \frac{1}{2}H^{(1)}_{\beta}(2(-\zeta)^{1/2}) & \frac{1}{2}H^{(2)}_{\beta}(2(-\zeta)^{1/2}) \\ \pi \zeta^{1/2}(H^{(1)}_{\beta})'(2(-\zeta)^{1/2}) & \pi \zeta^{1/2}(H^{(2)}_{\beta})'(2(-\zeta)^{1/2}) \end{array} \right) \times \begin{pmatrix} e^{\beta \pi i/2} & 0 \\ 0 & e^{-\beta \pi i/2} \end{pmatrix}, \quad \frac{2\pi}{3} < \arg \zeta < \pi, \]

\[ \Psi(\zeta) = \left( \begin{array}{cccc} \frac{1}{2}H^{(2)}_{\beta}(2(-\zeta)^{1/2}) & -\frac{1}{2}H^{(1)}_{\beta}(2(-\zeta)^{1/2}) \\ -\pi \zeta^{1/2}(H^{(2)}_{\beta})'(2(-\zeta)^{1/2}) & \pi \zeta^{1/2}(H^{(1)}_{\beta})'(2(-\zeta)^{1/2}) \end{array} \right) \times \begin{pmatrix} e^{-\beta \pi i/2} & 0 \\ 0 & e^{\beta \pi i/2} \end{pmatrix}, \quad -\pi < \arg \zeta < -\frac{2\pi}{3}, \]

where \( I_\beta \) and \( K_\beta \) are modified Bessel functions (see [33], §10.25) and \( H^{(1)}_{\beta} \) and \( H^{(2)}_{\beta} \) are Hankel functions (see [33], §10.2). The asymptotic behaviour of the modified Bessel functions \( I_\beta \) and \( K_\beta \) (see [33], §10.40) shows that

\[ \Psi(\zeta) = \left( \begin{array}{cccc} \frac{1}{\sqrt{2\pi}} \zeta^{1/4} & 0 \\ 0 & 1/\sqrt{2} \end{array} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O(\zeta^{-1/2}) & i + O(\zeta^{-1/2}) \\ i + O(\zeta^{-1/2}) & 1 + O(\zeta^{-1/2}) \end{pmatrix} \]

\[ \times \begin{pmatrix} e^{\zeta^{1/2}} & 0 \\ 0 & e^{-\zeta^{1/2}} \end{pmatrix} \]  

as \( \zeta \to \infty \) in the sector \( |\arg \zeta| < 2\pi/3 \). The same asymptotic formula holds in the regions \( 2\pi/3 < |\arg \zeta| < \pi \), using the asymptotic behaviour of the Hankel functions \( H^{(1)}_{\beta} \) and \( H^{(2)}_{\beta} \) (see [33], §10.17).

We use this \((2 \times 2)\)-matrix \( \Psi \) (in fact we use \( \Psi^{-T} = (\Psi^{-1})^T \), the transpose of the inverse of \( \Psi \)) to construct the parametrix \( P_b \) around the point \( b \) as follows. We define the contours \( \Sigma_{a,b} \) inside \( \Gamma_b \) as the preimages of the rays \( \gamma_1 \) and \( \gamma_3 \) under the mapping \( \zeta = (n+m)^{1/2} g_1^2(z)/4 \), where \( g_1 \) is given in (4.1). Then the parametrix \( P_b \) is given by

\[ P_b(z) = E^b_n \begin{pmatrix} \Psi^{-T} \left( \frac{(n+m)^2 g_1^2(z)}{4} \right) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_1(z) & 0 & 0 \\ 0 & W_1(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \Phi_1^{(n+m)/2} & 0 \\ 0 & \Phi_1^{-(n+m)/2} \\ 0 & 0 & 1 \end{pmatrix}, \]  

(7.2)

with \( W_1(z) = (2\pi i (z-a)^\alpha (z-b)^\beta)^{1/2} \) and

\[ E^b_n(z) = N(z) \begin{pmatrix} B_n & 0 \\ 0 & 1 \end{pmatrix} \]

where

\[ B_n = \begin{pmatrix} \frac{1}{W_1} & 0 \\ 0 & W_1 \end{pmatrix} \frac{1}{\sqrt{\pi}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{\pi (n+m) g_1^{1/2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{\pi (n+m) g_1^{1/2}} \end{pmatrix}. \]
After some calculations and using the asymptotic behaviour (7.1), we can show that $P_b N^{-1} = I + O(1/(n + m))$ on the contour $\Gamma_b$.

The parametrix $P_a$ around $a$ can be constructed in a similar way and uses the Bessel functions of order $\alpha$. For $c$ and $d$ we proceed in the same way when $\text{supp}(\nu_2) = [c, d]$, and $c^* < c < d$ but observe that now the jumps of $S$ near $c$ and $d$ are of the form

$$
\begin{pmatrix}
1 & 0 \\
0 & \hat{J}
\end{pmatrix},
$$

where $\hat{J}$ is a $(2 \times 2)$-matrix. So the parametrices $P_c$ and $P_d$ can also be constructed using the function $\Psi$ (but with $\gamma$ or $\delta$) and $P_c$ contains the Bessel functions of order $\gamma$, whereas $P_d$ contains the Bessel functions of order $\delta$. In particular we have

$$P_d = E_d^n \left( \begin{pmatrix}
1 & 0 \\
0 & \Psi^{-T} \left( \frac{m^2 g_2^2(z)}{4} \right) 
\end{pmatrix} \right) \left( \begin{pmatrix}
1 & 0 & 0 \\
0 & W_2(z) & 0 \\
0 & 0 & 1/W_2(z)
\end{pmatrix} \right) \left( \begin{pmatrix}
1 & 0 & 0 \\
0 & \Phi_2^{m/2} & 0 \\
0 & 0 & \Phi_2^{-m/2}
\end{pmatrix} \right),
$$

where

$$W_2(z) = (2\pi i(z - c)^\gamma(z - d)^\delta)^{1/2} \quad \text{and} \quad E_d^n(z) = N(z) \left( \begin{pmatrix}
1 & 0 \\
0 & C_n
\end{pmatrix} \right)$$

with

$$C_n = \left( \begin{pmatrix}
1 & 0 \\
0 & W_2
\end{pmatrix} \right) \frac{1}{\sqrt{2}} \left( \begin{pmatrix}
1 & i \\
i & 1
\end{pmatrix} \right) \left( \begin{pmatrix}
1 & 0 \\
\sqrt{\pi m g_2^{1/2}} & 0 \\
0 & \sqrt{\pi m g_2^{1/2}}
\end{pmatrix} \right).$$

Then $P_d N^{-1} = I + O(1/m)$ on the contour $\Gamma_d$. When $\text{supp}(\nu_2) = [c^*, d]$ with $c < c^* < d$, we need another parametrix $P_{c^*}$ around $c^*$. The point $c^*$ is a soft edge and the density $\nu_2'$ vanishes near $c^*$ as $(x - c^*)^{1/2}$. Near $c^*$ we look for a local Riemann-Hilbert problem with contours as in Figure 6. The jumps on these contours are all of the form

$$\begin{pmatrix}
1 & 0 \\
0 & \hat{J}
\end{pmatrix},$$

and so locally the problem reduces to a $(2 \times 2)$-problem. It is well known that around a soft edge Airy functions can be used for the local parametrix, and the
matching on the boundary $\Gamma_{c^*}$ can be achieved in a similar way to above, but using
the asymptotic behaviour of the Airy function instead of Bessel functions. See, for
example, [8], §7, where this was done in detail. For $c = c^*$ there is a transition
from soft edge ($c < c^*$) to hard edge ($c > c^*$). We will not deal with this special
case since it requires a different parametrix in terms of Painlevé transcendents.

§ 8. Asymptotics for the type I multiple orthogonal polynomials

The final transformation is

$$R(z) = \begin{cases} 
S(z)N^{-1}(z), & z \text{ outside } \Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d, \\
S(z)P^{-1}_e(z), & z \text{ inside } \Gamma_e, \ e \in \{a, b, c \text{ or } c^*, d\}.
\end{cases} \quad (8.1)$$

The contours of the Riemann-Hilbert problem for $R$ are given in Figure 7 for the
case when $\text{supp}(\nu_2) = [c, d]$ (top picture). When $\text{supp}(\nu_2) = [c^*, d]$ (bottom picture)
we replace $c$ by $c^*$ and there is an extra horizontal contour from $c$ to $\Gamma_{c^*}$.

![Figure 7. Riemann-Hilbert problem for the final matrix $R$.](image)

The jumps of $S$ on $[a, b]$, $[c, d]$ or $[c^*, d]$ and the lips of the lenses $\Sigma^{a,b}$ and $\Sigma^{c,d}$
or $\Sigma^{c^*,d}$ inside the curves $\Gamma_a$, $\Gamma_b$, $\Gamma_c$ or $\Gamma_{c^*}$, $\Gamma_d$ are eliminated by jumps that the
global parametrix $N$ and each of the local parametrices $P_a$, $P_b$, $P_c$ or $P_{c^*}$, $P_d$
have at those contours. On the remaining contours the jumps tend to the identity matrix $I$ uniformly,
at the rate $O(1/n)$ on $\Gamma_a$, $\Gamma_b$, $\Gamma_c$ or $\Gamma_{c^*}$, $\Gamma_d$, and exponentially
fast at the remaining lips of the lenses $\Sigma^{a,b}$ and $\Sigma^{c,d}$ or $\Sigma^{c^*,d}$. The matrix $R$
for this Riemann-Hilbert problem then converges uniformly on $\mathbb{C}$ to the identity matrix
(see, for example, [11], §7.5, or [24], Theorem 3.1)

$$\lim_{n \to \infty} \|R - I\| = 0,$$

and in fact the rate of convergence is of the same order as the rate at which the
jumps converge to the identity matrix,

$$\|R(z) - I\| = O\left(\frac{1}{n}\right). \quad (8.2)$$
Theorem 1. Let $A_{n,m}$ and $B_{n,m}$ be the type I multiple orthogonal polynomials for a Nikishin system with measures $(\mu_1, \mu_2)$ on $[a, b]$ that satisfy (1.1) and (1.2) and with a measure $\sigma$ on $[c, d]$ that satisfies (1.3). Let $(n, m)$ be multi-indices that tend to infinity but for which $m/(n + m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$.

Then, uniformly on compact subsets of $\mathbb{C} \setminus ([a, b] \cup [c, d])$

\[
A_{n,m}(z) = \left[ N_1(\psi_1(z)) + O \left( \frac{1}{n} \right) \right] \frac{D_0(\infty)}{D_1(z)} e^{(n+m)g_1(z) - mg_2(z) + (n+m)\ell_1} - \left[ N_1(\psi_2(z)) + O \left( \frac{1}{n} \right) \right] \frac{D_0(\infty)}{D_2(z)} e^{mg_2(z) + (n+m)(\ell_1 + \ell_2)} \int_c^d \frac{d\sigma(t)}{z - t}
\]

and

\[
B_{n,m}(z) = \left[ N_1(\psi_2(z)) + O \left( \frac{1}{n} \right) \right] \frac{D_0(\infty)}{D_2(z)} e^{mg_2(z) + (n+m)(\ell_1 + \ell_2)},
\]

where $g_1$ and $g_2$ are given in (4.1), $\ell_1$ and $\ell_2$ are given in (3.2) and (3.3), $N_1$ is given in (6.9), and $D_0$, $D_1$ and $D_2$ are given in (6.7). Furthermore

\[
A_{n,m}(z) + B_{n,m} \int_c^d \frac{d\sigma(t)}{z - t} = \left[ N_1(\psi_1(z)) + O \left( \frac{1}{n} \right) \right] \frac{D_0(\infty)}{D_1(z)} e^{(n+m)g_1(z) - mg_2(z) + (n+m)\ell_1}.
\]

Proof. We need to undo all the transformations from the original matrix $X$ in §1 to $R$ and then use the asymptotic behaviour (8.2) for $R$. From (2.1) we find

\[
A_{n,m}(z) = X_{1,2} = U_{1,2} - U_{1,3} \int_c^d \frac{d\sigma(t)}{z - t}
\]

and

\[
B_{n,m}(z) = X_{1,3} = U_{1,3}.
\]

From (4.4) we find

\[
U_{1,2} = V_{1,2} e^{(n+m)(\ell_1 + (n+m)g_1 - mg_2}
\]

and

\[
U_{1,3} = V_{1,3} e^{(n+m)(\ell_1 + \ell_2) + mg_2}.
\]

Since $z$ lies in a compact subset of $\mathbb{C} \setminus ([a, b] \cup [c, d])$, we will take the lenses around $[a, b]$ and $[c, d]$ and the neighbourhoods around $a$, $b$, $c$ and $d$ sufficiently small so that the compact subset is outside the system of curves in Figure 7. Then $V = S$ for the matrix $S$ in §5. Finally, from (8.1) we find that

\[
S_{1,2} = R_{1,1} N_{1,2} + R_{1,2} N_{2,2} + R_{1,3} N_{3,2}
\]

and

\[
S_{1,3} = R_{1,1} N_{1,3} + R_{1,2} N_{2,3} + R_{1,3} N_{3,3}.
\]

Then using the asymptotic behaviour in (8.2) and the expressions for $N$ from (6.3) and (6.8) we find the required asymptotic result away from $[a, b] \cup [c, d]$. 
When $\text{supp}(\nu_2) = [c^*, d]$ with $c < c^* < d$, the asymptotic formula for $A_{n,m}$ still holds on $\mathbb{C} \setminus ([a, b] \cup [c, d])$ and it is not valid on $[c, c^*]$ since $A_{n,m}(z)$ contains the function $w(z)$ from (1.1) and this function makes a jump over the interval $[c, c^*]$. The asymptotic formula for $B_{n,m}$ is true on $[c, c^* - \varepsilon]$, taking into account that $g_2^\pm(z) = -U(x; \nu_2) \pm i\pi$ there. Note however that the functions $N_1$, $D_1$ and $D_2$ are different from the case where $\text{supp}(\nu_2) = [c, d]$ because the Riemann surface is different.

The theorem is proved.

Notice that using (1.5) we can rewrite (1.4) as

$$\int_a^b (A_{n,m}(x)\bar{w}(x) + \tilde{B}_{n,m}(x)) x^k w(x)w_1(x) \, dx = 0, \quad 0 \leq k \leq n + m - 2,$$

where $\tilde{B}_{n,m} = -B_{n,m} - \ell A_{n,m}$ and $\deg(\tilde{B}_{n,m}) \leq m - 1$ when $n < m$. From here, reasoning as in §3, relations (3.7)–(3.9) can be replaced by

$$\int_a^b H_{n,m}(x) A_{n,m}(x)\bar{w}(x) + \tilde{B}_{n,m}(x) \frac{x^k w(x)w_1(x)}{H_{n,m}(x)} \, dx = 0, \quad k = 0, \ldots, n + m - 2,$$

$$\int_c^d A_{n,m}(x)x^k \frac{d\bar{\sigma}(x)}{H_{n,m}(x)} = 0, \quad 0 \leq k \leq n - 2,$$

and

$$\frac{A_{n,m}(x)\bar{w}(x) + \tilde{B}_{n,m}(x)}{H_{n,m}(x)} = \int_c^d \frac{\tilde{B}_{n,m}(t)}{x - t} \frac{d\bar{\sigma}(t)}{H_{n,m}(t)}, \quad x \notin [c, d],$$

where $H_{n,m}$ represents the same polynomial as before. Now, if we replace the constant $q_1$ in (3.2) and (3.3) by $\tilde{q}_1 = \lim_n(n/(n + m)) = 1 - q_1$, $1/2 < q_1 < 1$, and $(\tilde{\nu}_1, \tilde{\nu}_2)$ is the solution of the corresponding variational relations, then $\tilde{\nu}_1$ gives the normalized asymptotic zero distribution of the zeros of the polynomials $H_{n,m}$ (as before) but $\tilde{\nu}_2$ gives the normalized asymptotic zero distribution of the zeros of the polynomials $A_{n,m}$. Repeating §§4–7 we obtain an analogue of Theorem 1 for the case when $1/2 < q_1 < 1$. The details are left to the reader.

The asymptotic behaviour on the intervals $[a, b]$ and $[c, d]$ can be obtained in a similar way. The only difference is that we need to use the relation between $S$ and $V$ inside the lenses, and $S \neq V$ there. The asymptotic behaviour of $B_{n,m}$ on $(c, d)$ is then given by the following theorem.

**Theorem 2.** Let $B_{n,m}$ be the type I multiple orthogonal polynomials for a Nikishin system with measures $(\mu_1, \mu_2)$ on $[a, b]$ that satisfy (1.1) and (1.2), with a measure $\sigma$ on $[c, d]$ that satisfies (1.3). Let $(n, m)$ be multi-indices that tend to infinity but for which $m/(n + m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$.

Then for $\text{supp}(\nu_2) = [c, d]$, uniformly on closed subintervals of $(c, d)$

$$B_{n,m}(x) = -2 \left[ N_1(\psi_1^+(x)) + O \left( \frac{1}{n} \right) \right] \frac{D_0(\infty)}{|D_2(\infty)|} \frac{e^{-mU(x; \nu_2)}}{D_2^+(x)} \cos \left( m\pi \varphi_2(x) - \arg D_2^+(x) \right).$$

If $\text{supp}(\nu_2) = [c^*, d]$ then this asymptotic formula holds uniformly on closed intervals of $(c^*, d)$. 
Proof. Since \( x \) is now on a closed subinterval of \((c, d)\), we need to use \( S \) inside the lens around \((c, d)\). We will use the limiting values \( S_+ \) to get the behaviour of \( B_{n,m} \) on \((c, d)\). The relation between \( S \) and \( V \), as described in §5, is

\[
V_{1,3} = S_{1,3} - S_{1,2} \frac{\Phi_2^{-m}}{v_2}.
\]

We avoid the points \( c \) and \( d \) by taking the neighbourhoods around those points small enough. Then \( S \) and \( R \) are related by (8.3) and (8.4). The asymptotic behaviour of \( R \) in (8.2) then gives

\[
B_{n,m}(x) = \left[ N_1(\psi_2^+) + O\left(\frac{1}{n}\right)\right] \frac{D_0(\infty)}{D_2^+(x)} - \left[ N_1(\psi_1^+) + O\left(\frac{1}{n}\right)\right] \frac{D_0(\infty)}{D_1^+(x)v_2(x)} e^{(n+m)(\ell_1+\ell_2)+mg_2^+}.
\]

Now, recall that on \((c, d)\) we have \( v_2(x)D_1^+(x) = D_2^-(x), \psi_2^+(x) = \psi_1^-(x), N_1(\psi_1^+) = -N_1(\psi_1^-), \Phi_2^+(x) = \exp(2\pi i \varphi_2(x)) \) and \( g_2^+(x) = -U(x; \nu_2) + i\pi \varphi_2(x) \) (see (6.2), Figure 4 and (4.3)). Combining all these relations then gives the required result. If \( \text{supp}(\nu_2) = [c^*, d] \) we need to use \( S \) inside the lens around \([c^*, d]\). On \([c, c^* - \varepsilon]\) the \( B_{n,m} \) have exponential behaviour; see our remark at the end of the proof of previous theorem.

The theorem is proved.

On the interval \((a, b)\) we have the following asymptotic result.

**Theorem 3.** Let \( A_{n,m} \) and \( B_{n,m} \) be the type 1 multiple orthogonal polynomials for a Nikishin system with measures \((\mu_1, \mu_2)\) on \([a, b]\) that satisfy (1.1) and (1.2), with a measure \( \sigma \) on \([c, d]\) that satisfies (1.3). Let \((n, m)\) be multi-indices that tend to infinity but for which \( m/(n + m) = q_1 \) remains constant, with \( 0 < q_1 \leq 1/2 \).

Then, uniformly on closed subintervals of \((a, b)\)

\[
A_{n,m}(x) + B_{n,m}(x) \int_c^d \frac{d\sigma(t)}{x - t} = 2 \left[ N_1(\psi_1^+) + O\left(\frac{1}{n}\right)\right] \frac{D_0(\infty)}{|D_1^+(x)|} e^{(n+m)U(x; \nu_1)} \cos((n + m)\varphi_1(x) - \arg D_1^+(x)).
\]

**Proof.** It follows from (2.1) that

\[
U_{1,2} = A_{n,m}(x) + B_{n,m}(x) \int_c^d \frac{d\sigma(t)}{x - t},
\]

hence we need to find the asymptotic behaviour of \( U_{1,2} \). We will investigate this inside the lens around \([a, b]\) and away from the endpoints \( a \) and \( b \) and only investigate the limiting values from above. The transformations (4.4) and the relation between \( S \) and \( V \) show that

\[
U_{1,2} = \left( S_{1,2} - S_{1,1} \frac{\Phi_1^{-(n+m)}}{v_1} \right) e^{(n+m)\ell_1 + (n+m)g_1 - mg_2}.
\]
Since \( S = RN \), we then can use the asymptotic behaviour (8.2) to find

\[
U_{1,2} = \left[ N_1(\psi_1^+) + O\left(\frac{1}{n}\right) \right] \frac{D_0(\infty)}{D_1^+(x)} - \left[ N_1(\psi_0^+) + O\left(\frac{1}{n}\right) \right] \frac{D_0(\infty)}{D_0^+(x)v_1(x)} \left( \Phi^+_1 - (n+m) \right) e^{(n+m)\ell_1 + (n+m)g_1} - mg_2.
\]

On \((a,b)\), by (6.1) we obtain \( D_0^+(x)v_1(x) = D_1^-(x) \), and from Figure 4 we see that \( \psi_0^1(x) = \psi_1^1(x) \) and \( N_1(\psi_1^+) = -N_1(\psi_1^-) \). Furthermore, \( \Phi^+_1(x) = \exp(2\pi i \phi_1(x)) \), 
\( g_1^1(x) = -U(x; \nu_1) + i\pi \varphi_1(x) \) and \( g_1(x) = -U(x; \nu_2) \). Combining all this and using the variational relation (3.2) gives the required result.

The theorem is proved.

We can also obtain the asymptotic behaviour of \( B_{n,m} \) around the endpoints \( c \) and \( d \) using the fact that \( S = RP_c \) or \( S = RP_d \) and then using the parametrix \( P_c \) or \( P_d \) given in (7.3). This will give asymptotics in terms of Bessel functions \( J_\gamma \) or \( J_\beta \). When \( \text{supp} (\nu_2) = [c^*, d] \), the asymptotic behaviour near \( c^* \) will be in terms of the Airy function. In a similar way we can also obtain the asymptotic behaviour of the function \( A_{n,m} + B_{n,m} \) around the endpoints \( a \) and \( b \) using the parametrix \( P_a \) and \( P_b \) in (7.2), resulting in a formula involving Bessel functions \( J_\alpha \) or \( J_\beta \). We do not give the resulting formulae but leave this to the reader who is willing to do the necessary calculations.

§ 9. Asymptotics for the type II multiple orthogonal polynomials

So far we only considered the type I multiple orthogonal polynomials \( A_{n,m} \) and \( B_{n,m} \). However, we can also obtain the asymptotic behaviour of the type II multiple orthogonal polynomials \( P_{n,m} \) because there is a simple relation between the Riemann-Hilbert problem for type I and type II (see [37], Theorem 4.1, or [20], Theorem 23.8.3), namely, the matrix \( X^{-T} \) has the form

\[
\begin{pmatrix}
P_{n,m}(z) & \int_a^b \frac{P_{n,m}(t)w_1(t)}{t - z} \, dt & \int_a^b \frac{P_{n,m}(t)w(t)w_1(t)}{t - z} \, dt \\
-\gamma_1 P_{n-1,m}(z) & -\gamma_1 \int_a^b \frac{P_{n-1,m}(t)w_1(t)}{t - z} \, dt & -\gamma_1 \int_a^b \frac{P_{n-1,m}(t)w(t)w_1(t)}{t - z} \, dt \\
-\gamma_2 P_{n,m-1}(z) & -\gamma_2 \int_a^b \frac{P_{n,m-1}(t)w_1(t)}{t - z} \, dt & -\gamma_2 \int_a^b \frac{P_{n,m-1}(t)w(t)w_1(t)}{t - z} \, dt 
\end{pmatrix},
\]

where

\[
\frac{1}{\gamma_1} = \int_a^b t^{n-1} P_{n-1,m}(t)w_1(t) \, dt \quad \text{and} \quad \frac{1}{\gamma_2} = \int_a^b t^{m-1} P_{n,m-1}(t)w(t)w_1(t) \, dt.
\]

So in order to find the asymptotic behaviour of \( P_{n,m}(z) \), we need to investigate \( X^{-T} = (X^{-1})^T \), that is, the transpose of the inverse of \( X \). Note that

\[
P_{n,m}(z) = (X^{-T})_{1,1} = (U^{-T})_{1,1} = (V^{-T})_{1,1} e^{(n+m)g_1(z)},
\]

so that we only need to investigate the \((1,1)\)-entry of \( V^{-T} \). This gives the following.
Theorem 4. Let $P_{n,m}$ be the type II multiple orthogonal polynomials for a Nikishin system with measures $(\mu_1, \mu_2)$ on $[a, b]$ that satisfy (1.1) and (1.2), with a measure $\sigma$ on $[c, d]$ that satisfies (1.3). Let $(n, m)$ be multi-indices that tend to infinity but for which $m/(n + m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$.

Then, uniformly on compact subsets of $\mathbb{C} \setminus [a, b]$

$$P_{n,m}(z) = \frac{D_0(z)}{D_0(\infty)} \left[ N_1(\psi_0(z)) + O\left(\frac{1}{n}\right) \right] e^{(n+m)g_1(z)}, \quad (9.1)$$

where $g_1$ is given in (4.1). For $x$ on closed subintervals of $(a, b)$,

$$P_{n,m}(x) = 2i \frac{D_0^+(x)}{D_0(\infty)} \left[ N_1(\psi_0^+(x)) + O\left(\frac{1}{n}\right) \right] \sin((n + m)\varphi_1 + \arg D_0^+(x)). \quad (9.2)$$

Proof. Since we are looking at a compact subset of $\mathbb{C} \setminus [a, b]$, we only need to investigate $V$ outside the lens around $[a, b]$ and the neighbourhoods around $a$ and $b$. There we see that $S = V$ and $S = RN$, so that

$$P_{n,m}(z) = [(RN)^{-T}]_{1,1} e^{(n+m)g_1(z)}.$$

The asymptotic behaviour of $R$ in (8.2) gives $RN = N + O(1/n)$; hence

$$[(RN)^{-T}]_{1,1} = (N^{-T})_{1,1} + O\left(\frac{1}{n}\right).$$

Use (6.3) to write $N^{-T}$ in terms of $N_0^{-T}$ and observe that $N_0$ and $N_0^{-T}$ obey the same Riemann-Hilbert problem, since the jumps $J$ of $N_0$ satisfy $J^{-T} = J$ and $N_0$ tends to the identity matrix as $z \to \infty$. This gives the required asymptotic formula (9.1).

For $x$ on a closed subinterval of $(a, b)$ we need to use the behaviour of $V$ inside the lens around $[a, b]$ but away from the endpoints $a$ and $b$. We will use the limit from the upper half plane. There

$$V^{-T} = S^{-T} \begin{pmatrix} 1 & 0 & 0 \\ \Phi_1^{-(n+m)} & 1 & 0 \\ v_1 & 0 & 0 \end{pmatrix},$$

so that

$$(V^{-T})_{1,1} = (S^{-T})_{1,1} + (S^{-T})_{1,2} \frac{\Phi_1^{-(n+m)}}{v_1}. \quad (9.3)$$

Furthermore, we have $S = RN$, and the asymptotic behaviour of $R$ gives $S = N + O(1/n)$. From $N_0^{-T} = N_0$ we then find that

$$P_{n,m}(x) = \left[ N_1(\psi_0^+ + O\left(\frac{1}{n}\right) \frac{D_0^+(x)}{D_0(\infty)} + \frac{N_1(\psi_0^+ + O\left(\frac{1}{n}\right) \frac{D_0^+(x)}{D_0(\infty)} \Phi_1^{-(n+m)} - v_1(x)}{v_1(x)}} e^{(n+m)g_1^+(x)}. \quad (9.4)$$
On \((a, b)\) we have \(D_+^1(x) = v_1(x)D_0^-(x)\) (see (6.1)), \(g_+^1(x) = -U(x; \nu_1) + i\pi \varphi_1(x)\) (see (4.2)) and \(\Psi_+^1(x) = \exp(2\pi i \varphi_1(x))\). Furthermore, \(\psi_0^+ (x) = \psi_1^- (x)\) and \(N_1(\psi_1^+) = -N_1(\psi_1^-)\) so that the required formula (9.2) follows.

The theorem is proved.

Note that this asymptotic formula does not contain the constants \(\ell_1\) or \(\ell_2\). This is because \(P_{n,m}(z)\) is a monic polynomial.

§ 10. Concluding remarks

In this paper we have used the Riemann-Hilbert problem and the Deift-Zhou steepest descent method for oscillatory Riemann-Hilbert problems to obtain the asymptotics of the type I and type II multiple orthogonal polynomials for a Nikishin system of order two. This Riemann-Hilbert problem uses \((3 \times 3)\)-matrix functions and we showed that many steps in the Riemann-Hilbert problem can be reduced to a \(2 \times 2\) Riemann-Hilbert problem when the two intervals \([a, b]\) and \([c, d]\) are disjoint and not touching. The only steps where the \((3 \times 3)\)-character of the problem is important is when we normalize the problem in §4 using the solution of the vector equilibrium problem with Nikishin interaction from §3, and the construction of the global parametrix in §6. If the intervals \([a, b]\) and \([c, d]\) are touching, then the construction of the parametrix around the common point \(a = d\) also requires a local \(3 \times 3\) Riemann-Hilbert problem, but it is not clear what such a parametrix should contain. We believe it will be somewhat like the local parametrix which was used in [14] around the common point of the two intervals in an Angelesco system, or the parametrix used in [7] for the critical case \(a = 1/\sqrt{2}\) in that paper, but it will not be quite the same parametrix because in an Angelesco system the two intervals are repelling, whereas in a Nikishin system the two intervals are attracting. Also the critical case \(c = c^*\) is not considered in this paper. We believe the parametrix around the endpoint \(c\) will be in terms of Painlevé XXXIV, as was the case for the Angelesco case (see [38]) and similar situations in random matrix theory (see [21]) and for the asymptotics for orthogonal polynomials (see [36], §7.2). This is rather technical, so we decided not to deal with it in this paper.

Acknowledgements. This work was initiated during a visit of the second author to Universidad Carlos III de Madrid and he is grateful for the support of the Departamento de Matemáticas of UC3M. We are grateful to the referees for giving extra useful references and for pointing out that a soft edge \(c^*\) is possible for the support of \(\nu_2\).

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