A Note on Free Bosonic Vertex Algebra and its Conformal Vectors

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Abstract: We classify all the Heisenberg and conformal vectors and determine the full automorphism group of the free bosonic vertex algebra without gradation. To describe it we introduce a notion of inner automorphisms of a vertex algebra.

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0 Introduction

The notion of vertex algebras and their conformal vectors was invented by Borcherds in [B1]. It is indeed a mathematical axiomatization of a certain class of chiral algebras appearing in two dimensional quantum field theory, and the existence of conformal vector is nothing but peculiar feature, Virasoro symmetry of the energy–momentum tensor, of two dimensional conformal field theory initiated by Belavin–Polyakov–Zamolodchikov [BPZ]. There are a huge number of studies of such structure both in mathematics and physics, e.g., [B1], [FLM], [P2], [G], [DL], [Li], [LZ].

Now let us focus on the simplest nontrivial example of vertex algebras, the free bosonic vertex algebra, whose underlying vector space is the polynomial ring with countably many indeterminates identified with the Fock space of the Heisenberg algebra of charge 0. In spite of simplicity, the vertex algebra structure is already complicated and involves lots of nontrivial features.

In particular, there are infinitely many choices of conformal vectors depending on one parameter, giving different conformal theories, called the Feigin–Fuks modules when regarded as Virasoro modules. Our naive interest is whether they exhaust all the conformal vectors or not. If we assume that a conformal vector should belong to the degree 2 subspace with respect to the natural gradation, then it is easily seen that they exhaust, as Lian observed in a general framework [Lian]. He also determined the automorphism group that preserves the gradation. However, if we abandon such restriction concerning the gradation, there might be other possibility of conformal vectors and automorphisms.

In this note, we classify all the conformal vectors of the free bosonic vertex algebra without any restriction on the gradation. The key observation is that a commutative vector of the vertex algebra, a vector $c$ satisfying $[Y(c, y), Y(c, z)] = 0$, is proportional to the vacuum vector. Such uniqueness of the commutative vectors restricts the possibility of the conformal vectors: they do not have components of degree greater than two. Then the result is that a conformal vector $v$ is described as

$$Y(v, z) = \frac{1}{2} \mu^2 I(z) + \mu \alpha(z) + \frac{1}{2} \bar{z} \alpha(z) \alpha(z) \bar{z} + \lambda \partial \alpha(z), \quad (\lambda, \mu \in \mathbb{C}),$$

where $I(z) = \text{id}$ is the identity field and $\alpha(z)$ is the Heisenberg field which generates the vertex algebra. This shows that a conformal vector is transformed by an inner automorphism to a unique conformal vector in the degree
two subspace. We also classify all the Heisenberg vectors, and determine the full automorphism group: it is a semidirect product of the automorphism group that preserves the gradation, which is isomorphic to $\mathbb{Z}_2$, and the inner automorphism group isomorphic to the additive group $\mathbb{C}$.

Now, we mention the contents of this note. In the first section, we recall definitions and basic properties of vertex algebras. This section includes Borcherds’ axioms, local fields, operator product expansion, Goddard’s axioms, existence theorem, gradation, conformal vectors and automorphisms. The terminologies used in this note are slightly different from the literatures, so we give these definitions precisely and refer differences of the meanings. In particular, we introduce the notion of inner automorphisms suitable for our purpose.

The second section is a summary of the free bosonic vertex algebra, which can also be viewed as giving examples of the notions described in the first section. We first review the notion of the Heisenberg algebra and Heisenberg fields, and describe Wick’s theorem which calculates the operator product of fields constructed from Heisenberg fields. Next we consider the Fock representations $\mathcal{F}_r, (r \in \mathbb{C})$, of the Heisenberg algebra and we provide $\mathcal{F}_0$, the vacuum representation, with a natural structure of vertex algebra. The operations of vertex algebra $\mathcal{F}_0$ can be computed by Wick’s theorem. We also describe the conformal vectors and the standard gradation.

In section 3, we will state, prove and apply our main results on the classification of conformal vectors. We first prepare the notion of commutative vectors of a vertex algebra and state the uniqueness of them in the case of the free bosonic vertex algebra. We then use this result to classify Heisenberg, Virasoro, and conformal vectors of the vertex algebra. We determine the full automorphism group using this classification. We also mention the complete reducibility of this vertex algebra as an $\mathfrak{sl}_2$–module. We finally prove the above mentioned uniqueness of commutative vectors by a calculation using Wick’s theorem.

Final section is devoted to further discussion on general vertex algebras, where our method of classifying all conformal vectors in the free bosonic vertex algebra is not valid. In this section, we will point out the difficulties in the general case and give some examples.

Throughout this note, we will always work over the complex number field $\mathbb{C}$. We denote by $\mathbb{N}$ the set of all non–negative integers.
1 Vertex Algebras

In this section, we will summarize the axioms and properties of vertex algebras to fix the notations and give some basic ideas on them. Most of the contents given in this section are taken from literatures (\[B1\], \[B2\], \[FLM\], \[G\], \[Li\]).

In Subsections 1.1–1.5 we review some definitions and basic results, while in 1.6 we introduce a notion of inner automorphisms which is a new feature. For the detail of the contents of Subsections 1.1–1.3, we refer the reader to \[MN\], in which we will discuss alternative descriptions and proofs of fundamental results based on the locality of fields.

1.1 Borcherds’ Axioms

A vertex algebra, in the formulation by Borcherds \[B2\], is a \(\mathbb{C}\)-vector space \(V\) equipped with countably many bilinear binary operations

\[
(a, b) \mapsto a_{(n)} b, \quad (n \in \mathbb{Z})
\]

and an element \(|I\rangle \in V\) satisfying the following axioms:

(B1) For any pair of elements \(a, b \in V\), there exists an (positive) integer \(n_0\) such that

\[
a_{(n)} b = 0 \quad \text{for all} \quad n \geq n_0.
\]

(B2) (Borcherds identity) For each triple of elements \(a, b, c \in V\) and integers \(k, \ell, m\),

\[
\sum_{i=0}^{\infty} \binom{k}{i} (a_{(m+i)} b)_{(k+\ell-i)} c = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (a_{(k+m-i)} b_{(\ell+i)} c) - (-1)^m b_{(\ell+m-i)} (a_{(k+i)} c).
\]

(B3) For any \(a \in V\)

\[
a_{(n)} |I\rangle = \begin{cases} 
0 & (n \geq 0) \\
   a & (n = -1).
\end{cases}
\]

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The element $|I\rangle$ is called the vacuum vector\(^2\) and the identity (2) is called the Borcherds identity\(^3\).

We define $T \in \text{End} V$ by $Ta = a(-2)|I\rangle$, which we call the translation of the vertex algebra $V$. From the axioms (B1), (B2) and (B3), we derive the following formulae:

\begin{align*}
(4) \quad T |I\rangle &= 0, \\
(5) \quad a(n)|I\rangle &= \begin{cases} 0 & (n \geq 0) \\ T(-n-1)a & (n \leq -1), \end{cases} \\
(6) \quad |I\rangle_{(n)}a &= \begin{cases} 0 & (n \neq -1) \\ a & (n = -1), \end{cases} \\
(7) \quad (Ta)_{(n)}b &= -na_{(n-1)}b, \\
(8) \quad a_{(n)}(Tb) &= T(a_{(n)}b) + na_{(n-1)}b, \\
(9) \quad b_{(n)}a &= (-1)^{n+1}a_{(n)}b + \sum_{i=1}^{\infty} (-1)^{n+i+1}T^{(i)}(a_{(n+i)}b)
\end{align*}

where $T^{(n)} = T^n/n!$. In particular, combining (7) and (8), we see that $T$ is a derivation for all the binary operations. We also have, as particular cases of Borcherds identity,

\begin{align*}
(10) \quad [a_{(k)}, b_{(\ell)}] &= \sum_{i=0}^{\infty} \binom{k}{i} (a_{(i)}b)_{(k+i-\ell)}, \\
(11) \quad (a_{(m)}b)_{(\ell)} &= \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (a_{(m-i)}b_{(\ell+i)} - (-1)^m b_{(\ell+m-i)}a_{(i)}).
\end{align*}

**Note 1.1.** Originally in [B1], Borcherds took (1), (5), (6), (9) and (11) as the set of axioms of vertex algebras.

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\(^2\)Perhaps it is better to call this the identity vector because it corresponds to the identity operator under the state–field correspondence (cf. Subsection 1.3).

\(^3\)This identity is nothing but the Cauchy–Jacobi identity of Frenkel–Lepowsky–Meurman [FLM, (8.8.29) and (8.8.41)], while the special case (11) is due to Borcherds [B1]; we here follow the terminology in [K].
Now let us introduce the generating function
\[ Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]. \]

Then the axioms are restated as follows:
(Y1) For any pair of elements \( a, b \in V \), the series
\[ Y(a, z)b = \sum_{n \in \mathbb{Z}} (a_n b) z^{-n-1} \]
is a Laurent series with only finitely many terms of negative degree.

(Y2) (Jacobi–identity)
\[
\delta(z, y, x)Y(Y(a, x)b, z)c = \delta(x, y, z)Y(a, y)Y(b, z)c + \delta(-x, z, y)Y(b, z)Y(a, y)c
\]
where\(^4\)
\[
\delta(x, y, z) = \sum_{n \in \mathbb{Z}} \sum_{i \in \mathbb{N}} (-1)^i \binom{n}{i} x^{-n-1} y^{n-i} z^i.
\]

(Y3) \( Y(a, z)|I \) is a formal power series in \( z \) with the constant term \( a \).

The properties (5)–(9) are expressed as

(12) \[ Y(a, z)|I \rangle = e^zT a, \]
(13) \[ Y(|I \rangle, z) = \text{id}_V, \]
(14) \[ Y(Ta, z) = \partial_z Y(a, z), \]
(15) \[ [T, Y(a, z)] = \partial_z Y(a, z), \]
(16) \[ Y(a, z)b = e^zTY(b, -z)a, \]
respectively.

1.2 Local Fields and Operator Product Expansion

Let \( M \) be a \( \mathbb{C} \)–vector space and \( A(z) \) a Laurent series with coefficients in \( \text{End } M \):
\[ A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \in (\text{End } M)[[z, z^{-1}]]. \]\(^4\)

\(^4\)It is denoted by \( x^{-1}\delta(\frac{x-z}{x}) \) in [FLM] and subsequent papers.
Here each $A_{(n)}$ is called a *Fourier mode* of $A(z)$. The $A(z)$ is called a *field* on $M$ if the series

$$A(z)u = \sum_{n \in \mathbb{Z}} (A_{(n)}u)z^{-n-1}$$

has only finitely many terms of negative degree for any $u \in V$. If $A(z)$ is a field, then so is

$$\partial A(z) = \sum_{n \in \mathbb{Z}} (-n - 1)A_{(n)}z^{-n-2}.$$

We set

$$A(z)_+ = \sum_{n \geq 0} A_{(n)}z^{-n-1}, \quad A(z)_- = \sum_{n < 0} A_{(n)}z^{-n-1}.$$

If $A(z)$ is a field and $B(z)$ is another one, then the *normally ordered product* defined by

$$A(z)_+ B(z)_+ = A(z)_- B(z) + B(z)A(z)_+$$

gives an element of $(\text{End } M)[[z, z^{-1}]]$, which is also a field. Further we understand the nested normally ordered product by

$$A(z)_+ A(z)_+ = A(z)_- \cdot A(z)_+ + A(z)_- A(z)_+\cdot A(z)_+.$$

This construction is generalized as follows: For each integer $n \in \mathbb{Z}$, define

$$A(z)_{(n)}B(z) = \text{Res}_{y=0} A(y)B(z)(y-z)^n|_{|y|>|z|} - \text{Res}_{y=0} B(z)A(y)(y-z)^n|_{|y|<|z|}$$

which we call the *residual n–th product* $\footnote{This is simply called the n–th product in literatures. However, in order to distinguish it from the abstract operations of a vertex algebra, we prefer to add the adjective residual.}$. Here the symbols $(y-z)^n|_{|y|>|z|}$ and $(y-z)^n|_{|z|>|y|}$ denote the elements of $\mathbb{C}[[y, y^{-1}, z, z^{-1}]]$ obtained by expanding the rational function $(y-z)^n$ into a series convergent in the regions $|y| > |z|$ and $|z| > |y|$ respectively:

$$(y-z)^n|_{|y|>|z|} = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} y^{n-i}z^i,$$

$$(y-z)^n|_{|z|>|y|} = \sum_{i=0}^{\infty} (-1)^{n+i} \binom{n}{i} y^i z^{n-i}. $$
Then we have

\[ A(z)_{(n)} B(z) = \partial^{(-n-1)} A(z) B(z) \circ \]

for \( n < 0 \) and in particular \( A(z)_{(-1)} B(z) = \partial A(z) B(z) \circ \). If \( A(z) \) and \( B(z) \) are fields, then also are \( A(z)_{(n)} B(z) \) and we have

\[ \partial A(z)_{(n)} B(z) = -n A(z)_{(n-1)} B(z). \]

Now two fields \( A(z) \) and \( B(z) \) are called (mutually) local \(^6\) if there exists an (positive) integer \( m \) such that

\[ (y - z)^m A(y) B(z) = (y - z)^m B(z) A(y), \]

namely,

\[ \sum_{i=0}^{m} (-1)^m \binom{m}{i} (A_{(k-i)} B_{(\ell+i)} - B_{(\ell-i)} A_{(k+i)}) = 0 \]

for any \( k, \ell \in \mathbb{Z} \) (cf. \([\text{Li}]\)). In this case we also say that \( A(z) \) is local with \( B(z) \). If \( A(z) \) and \( B(z) \) are local, then so are \( \partial A(z) \) and \( B(z) \). Further, if \( A(z), B(z) \) and \( C(z) \) are pairwise local, then \( A(z) B(z) \circ \) and \( C(z) \) are local (cf. \([\text{Li}]\)).

If \( A(z) \) and \( B(z) \) are local, then

\[ A(z)_{(m)} B(z) = 0 \quad \text{for sufficiently large } m, \]

and we have

\[ A(y) B(z) = \sum_{j=0}^{m-1} \left. \frac{A(z)_{(j)} B(z)}{(y - z)^{j+1}} \right|_{|y| > |z|} + \partial A(y) B(z) \circ \]

and

\[ B(z) A(y) = \sum_{j=0}^{m-1} \left. \frac{A(z)_{(j)} B(z)}{(y - z)^{j+1}} \right|_{|y| < |z|} + \partial A(y) B(z) \circ. \]

\(^6\)The locality is sometimes called commutativity. The term commutativity is also used to mean the identity (7) in literature. The locality in \([\text{LZ}]\) is different from ours, while the commutativity there coincides with our locality. Here we followed the terminology in \([\text{Li}]\) and \([\text{K}]\).
Here the normally ordered product $\hat{\cdot} A(y)B(z)\hat{\cdot}$ is defined by

$$
\hat{\cdot} A(y)B(z)\hat{\cdot} = A(y) - B(z) + B(z)A(y) + .
$$

We abbreviate them into a single expression

$$
A(y)B(z) \sim \sum_{j=0}^{m-1} A(z)(j)B(z)\frac{1}{(y-z)^{j+1}}.
$$

which is called the operator product expansion (OPE). It has sufficient information as far as commutation relations of Fourier modes are concerned.

In general we understand the symbol $\sim$ as the equivalence relation in $(\text{End } M)[[y, y^{-1}, z, z^{-1}]]$ defined as

$$
K(y, z) \sim L(y, z)
$$

if and only if $K(y, z)u - L(y, z)u$ has only finitely many terms of negative degree in $y$ and $z$ for any $u \in M$. Then we have

$$
\hat{\cdot} A(y)B(z)\hat{\cdot} \sim \sum_{i=0}^{n-1} \partial^{(i)} A(z)(y-z)^{i-n}.
$$

which holds for expansion both in the region $|y| > |z|$ and $|y| < |z|$. This is understood to be obtained by formal substitution of Taylor’s formula:

$$
A(y) = \sum_{i=0}^{\infty} \partial^{(i)} A(z)(y-z)^i
$$

though this does not hold as an equality of formal Laurent series.

We close this subsection with the following fundamental result due to Li.

**Theorem 1.1 ([L]).** Let $M$ be a vector space and $\mathcal{O}$ a vector space consisting of fields on $M$ which are pairwise local. If $\mathcal{O}$ is closed under residual products and contains the identity field, then the residual products gives $\mathcal{O}$ a structure of vertex algebra with the vacuum vector $I(z)$.

Li showed this theorem by establishing that the map

$$
Y : \mathcal{O} \rightarrow (\text{End } \mathcal{O})[[\zeta, \zeta^{-1}]]
A(z) \mapsto \sum_{n \in \mathbb{Z}} A(z)(n)\zeta^{-n-1}
$$

satisfies Goddard’s axioms, which we will explain in the next subsection, with $V = \mathcal{O}, |I| = I(z)$ and $T = \partial_z$. See [MN] for an alternative proof. Note that the vector space $M$ need not have a natural structure of vertex algebra.
1.3 State–Field Correspondence

Let us consider the case when $M$ is a vertex algebra $V$. For given $a, b \in V$, take $n_0 \in \mathbb{N}$ such that $a(n)b = 0$ (for all $n \geq n_0$). Then, letting $m = n_0$, we see the Borcherds identity immediately implies that $Y(a, z)$ and $Y(b, z)$ are local (cf. [DL, (7.24)]). On the other hand, we have property (14):

$$\partial_z Y(a, z) = Y(Ta, z)$$

and a particular case of (14):

$$(18) \quad Y(a_{(-1)}b, z) = Y(a, z)Y(b, z)\hat{\circ}.$$  

Therefore the vector space

$$\mathcal{V} = \{ Y(a, z) \in \text{End } V[[z, z^{-1}]] \mid a \in V \}$$

is a set of pairwise local fields closed under taking the derivative and the normally ordered product. From axiom (B2), it follows that the space $\mathcal{V}$ is naturally isomorphic to the vector space $V$. This is called the state–field correspondence. Furthermore, by successive use of (3) and formulas (14) and (18), we have

$$(19) \quad Y(a_{(j_1-1)}\cdots a_{(j_n-1)}\mid I \rangle, z) = \hat{\circ} \partial^{(j_1)}Y(a_{(j_1)}\mid I \rangle, z)\cdots \partial^{(j_n)}Y(a_{(j_n)}\mid I \rangle, z)\hat{\circ},$$

where $j_1, j_2, \ldots, j_n \in \mathbb{N}$.

Now let us consider the formula (10). This is equivalent to the OPE

$$Y(a, y)Y(b, z) \sim \sum_{j=0}^{n_0-1} \frac{Y(a_{(j)}b, z)}{(y-z)^{j+1}}.$$  

On the other hand, the formula (11) is nothing but

$$Y(a, z)_{(j)}Y(b, z) = Y(a_{(j)}b, z).$$  

This means that the vector space $\mathcal{V}$ equipped with the residual $n$–th product is a vertex algebra with the vacuum vector being the identity operator, such that the map

$$Y : V \rightarrow \mathcal{V}$$

$$a \mapsto Y(a, z)$$
is an isomorphism of vertex algebras. This is another version of state-field correspondence.

In the spirit of the locality, there is another characterization of vertex algebras essentially given by Goddard\[G\]: A vertex algebra is a vector space equipped with a linear map
\[
V \rightarrow (\text{End } V)[[z, z^{-1}]]
\]
whose image is a set of local fields, an element \(|I\rangle \in V\), and an endomorphism
\[
T : V \rightarrow V
\]
satisfying (4), (12) and (15):
\[
T|I\rangle = 0, \quad Y(a, z)|I\rangle = e^{zT}a, \quad [T, Y(a, z)] = \partial_z Y(a, z).
\]
Here we can equivalently replace (12) by (3).

For the equivalence of this definition and the one described in Subsection 1.1, we refer the reader to \[Li\], \[K\] and \[MN\].

An application is the following existence theorem due to Frenkel–Kac–Radul–Wang, which is useful in providing a vertex algebra structure on a given vector space.

**Theorem 1.2** (\[FKRW\], see also \[K\]). Let \(\{A^\lambda(z)\mid \lambda \in \Lambda\}\) be a set of pairwise local fields on a vector space \(V\) such that \(A^\lambda(z)|I\rangle\) does not have terms of negative degree for a given vector \(|I\rangle \in V\). If the set
\[
\{A_{(-j_1-1)}^{\lambda_1} \cdots A_{(-j_n-1)}^{\lambda_n}|I\rangle \mid \lambda_i \in \Lambda, j_1, \ldots, j_n \in \mathbb{N}\}
\]
spans \(V\) and there exists an endomorphism \(T \in \text{End } V\) satisfying
\[
[T, A^\lambda(z)] = \partial_z A^\lambda(z),
\]
for all \(\lambda \in \Lambda\), then \(V\) has a unique structure of vertex algebra with vacuum vector \(|I\rangle\) such that
\[
Y(A_{(-1)}^\lambda|I\rangle, z) = A^\lambda(z).
\]

Note that the uniqueness is clear, since by (13) we must have
\[
Y(A_{(-j_1-1)}^{\lambda_1} \cdots A_{(-j_n-1)}^{\lambda_n}|I\rangle, z) = \partial^{(j_1)}A^{\lambda_1}(z) \cdots \partial^{(j_n)}A^{\lambda_n}(z)_{-\lambda}.
\]
In the situation of the theorem, we say that the vertex algebra \(V\) is generated by \(\{A^\lambda(z)\mid \alpha \in \Lambda\}\).
1.4 Conformal Vectors and Gradation

Let us recall the definition of the Virasoro algebra. It is the Lie algebra

\[ \mathcal{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \mathcal{L}_n \oplus \mathbb{C} \]

with the Lie bracket defined by

\[ [L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C, \quad [C, L_m] = 0. \]

A representation of \( \mathcal{Vir} \) on which \( C \) acts by a scalar \( c \in \mathbb{C} \) is called a representation of central charge \( c \).

Let \((\pi, M)\) be a representation of \( \mathcal{Vir} \) of central charge \( c \). Consider the series

\[ L(z) = \sum_{n \in \mathbb{Z}} \pi(L_n) z^{-n-2}. \]

If \( L(z) \) is a field, then it is local with itself and has the following OPE:

\[ L(y)L(z) \sim \frac{c/2}{(y-z)^4} + \frac{2L(z)}{(y-z)^2} + \frac{\partial L(z)}{y-z}. \]

Conversely, a field \( L(z) \) with such properties defines a representation of \( \mathcal{Vir} \) of central charge \( c \) on \( M \) by its Fourier modes. We call such a field \( L(z) \) a Virasoro field. In the sequel, we omit writing \( \pi \) and denote as \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \).

Now consider the case when \( M \) is a vertex algebra \( V \). Suppose that there exists a non–zero vector \( v \in V \) such that

\[ L(z) = Y(v, z). \]

Then the OPE above is equivalent to

\[ v(n)v = \begin{cases} 0 & (n \geq 4) \\ (c/2)|I_1 & (n = 3) \\ 0 & (n = 2) \\ 2v & (n = 1) \\ Tv & (n = 0). \end{cases} \]

Such a vector \( v \in V \) is called a Virasoro vector of the vertex algebra \( V \).
Remark 1.1. Using the skew symmetry of vertex algebras, we can reduce the condition above to a weaker one

\[
v(n)v = \begin{cases} 
0 & (n \geq 4) \\
(c/2)|I| & (n = 3) \\
2v & (n = 1).
\end{cases}
\]

Now, for a vertex algebra \((V, Y, |I\rangle, T)\), a Virasoro vector \(v \in V\) is called a conformal vector\(^7\) if \(L_0 = v(1)\) is semi-simple and \(L_{-1} = v(0)\) coincides with the translation \(T\) of the vertex algebra \(V\). A vertex algebra equipped with a conformal vector is called conformal vertex algebra.

Let \(V\) be a conformal vertex algebra. Since \(L_0\) is semisimple, we have a direct sum decomposition

\[V = \bigoplus_{\Delta \in \mathbb{C}} V^\Delta, \quad V^\Delta = \{a \in V | L_0 a = \Delta a\}.
\]

Then, by (10) and (7), we have

\[
L_0(a_{(n)}b) = v(1)(a_{(n)}b)
= a_{(n)}(v(1)b) + (v(0)a)_{(n+1)}b + (v(1)a)_{(n)}b
= (v(1)a)_{(n)}b + a_{(n)}(v(1)b) + (Ta)_{(n+1)}b
= (L_0a)_{(n)}b + a_{(n)}(L_0b) - (n + 1)a_{(n)}b.
\]

In other words, if \(a \in V^{\Delta_1}\) and \(b \in V^{\Delta_2}\), then \(a_{(n)}b \in V^{\Delta_1+\Delta_2-n-1}\) for any \(n \in \mathbb{Z}\). We take this property as a definition of a gradation of a vertex algebra. Namely, a gradation of a vertex algebra is a direct sum decomposition

\[V = \bigoplus_{\Delta \in \mathbb{C}} V^\Delta
\]

such that

\[(V^{\Delta_1})_{(n)}(V^{\Delta_2}) \subset V^{\Delta_1+\Delta_2-n-1}\]

for any \(n \in \mathbb{Z}\).

A vertex algebra equipped with a gradation is called a graded vertex algebra. The above argument shows that a conformal vertex algebra is naturally graded by the eigenspace decomposition with respect to \(L_0\). In this case, a vector belonging to \(V^\Delta\) is said to have the conformal weight \(\Delta\).

Note 1.2. For a graded vertex algebra, the axioms are interpreted in terms of “operator valued rational functions” (see [FHL]).

\(^7\) In [FLM] the term Virasoro element is used to mean a conformal vector.
1.5 Automorphisms

Let \( V \) be a vertex algebra. An automorphism of \( V \) is an isomorphism
\[
\sigma : V \rightarrow V
\]
of vector spaces which preserves all the products:
\[
\sigma(a_{(n)}b) = \sigma(a)_{(n)}\sigma(b).
\]
In other words, an isomorphism \( \sigma \) is an automorphism if and only if
\[
\sigma Y(a, z)\sigma^{-1} = Y(\sigma(a), z)
\]
holds for any \( a \in V \).

The group of all automorphisms of the vertex algebra \( V \) is denoted by \( \text{Aut} V \). Each automorphism \( \sigma \in \text{Aut} V \) preserves the vacuum vector since
\[
\sigma(|I\rangle) = |I\rangle_{(-1)}\sigma(|I\rangle) = \sigma(\sigma^{-1}(|I\rangle)_{(-1)}|I\rangle) = \sigma(\sigma^{-1}(|I\rangle)) = |I\rangle,
\]
and commutes with the translation \( T \) since
\[
\sigma(Ta) = \sigma(a_{(-2)}|I\rangle) = \sigma(a)_{(-2)}\sigma(|I\rangle) = \sigma(a)_{(-2)}|I\rangle = T(\sigma(a)).
\]

Let us introduce a notion of inner automorphisms of a vertex algebra. Before giving the definition, we prepare the following lemma:

**Lemma 1.3.** Let \( a \) be an element of a vertex algebra \( V \). Then, for a non-negative integer \( m \), \( a_{(m)} \) is a derivation for all the products of the vertex algebra if and only if \( a_{(0)} = a_{(1)} = \cdots = a_{(m-1)} = 0 \).

The proof is easily carried out as follows. By Borcherds’ commutator formula (101), we have
\[
a_{(m)}(b_{(n)}c) = (a_{(m)}b)_{(n)}c + b_{(n)}(a_{(m)}c) + \sum_{j=0}^{m-1} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c.
\]
So if \( a(0) = a(1) = \cdots = a(m-1) = 0 \), then \( a(m) \) is a derivation for all the \( n \)-th products. Conversely, if \( a(m) \) is a derivation, we have

\[
\sum_{j=0}^{m-1} \binom{m}{j} (a(j)b)(m+n-j)c = 0.
\]

Putting \( c = |I\rangle \) and \( n = -2 \), we have \( a(m-1)b = 0 \) for any \( b \in V \). Inductively we deduce that \( a(0) = a(1) = \cdots = a(m-1) = 0 \). We call such a derivation as in the lemma an inner derivation\(^8\) of level \( m \).

Let \( a \) be an element of \( V \) with \( a(0) = a(1) = \cdots = a(m-1) = 0 \). If the exponential

\[
\sigma = \exp (a(m)) = \sum_{n=0}^{\infty} \frac{a(m)^n}{n!}
\]

of the derivation \( a(m) \) makes sense, then it give rise to an automorphism of \( V \). We call such an automorphism an inner automorphism of level \( m \). We denote

\[
\text{Inn}_{(m)} V = \langle \sigma \in \text{Aut} V \mid \sigma = \exp (a(m)) \text{ for some } a \in V \rangle.
\]

Then, since

\[
a(m) = -\frac{1}{m+1}(Ta)(m+1), \quad (m \geq 0)
\]

we have the following inclusions:

\[
\text{Inn}_{(0)} V \subseteq \text{Inn}_{(1)} V \subseteq \text{Inn}_{(2)} V \subseteq \cdots.
\]

Let us denote by \( \text{Inn} V \) the union of \( \text{Inn}_{(n)} V \). Note that all \( \text{Inn}_{(m)} V \) and \( \text{Inn} V \) are normal subgroups of \( \text{Aut} V \) since

\[
\sigma \exp (a(m))\sigma^{-1} = \exp (\sigma(a(m))
\]

for any \( \sigma \in \text{Aut} V \).

Now let us turn to the case when \( V \) is graded: \( V = \oplus_{\Delta \in \mathbb{C}} V^\Delta \). We denote the subgroup of \( \text{Aut} V \) consisting of all automorphisms that preserve the gradation by

\[
\text{Aut}^0 V = \{ \sigma \in \text{Aut} V \mid \sigma(V^\Delta) = V^\Delta \text{ for all } \Delta \in \mathbb{C} \}.
\]

\(^8\) The notion of inner derivation in \( \text{Lian} \) coincides with our inner derivation of level 0.
We set
\[ \text{Inn}^0_{(m)} V = \text{Inn}_{(m)} V \cap \text{Aut}^0 V, \quad \text{and} \quad \text{Inn}^0 V = \text{Inn} V \cap \text{Aut}^0 V. \]
Since the gradation satisfies
\[ (V^{\Delta_1})_{(m)}(V^{\Delta_2}) \subset V^{\Delta_1+\Delta_2-m-1}, \]
the inner automorphism \( \sigma = \exp(a_{(m)}) \) preserves the gradation if and only if \( a \in V^{m+1} \). Hence we have
\[ \text{Inn}^0_{(m)} V = \langle \sigma \in \text{Aut} V \mid \sigma = \exp(a_{(m)}) \text{ for some } a \in V^{m+1} \rangle. \]
We define the outer automorphism groups as
\[
\begin{align*}
\text{Out}_{(m)} V &= \text{Aut} V / \text{Inn}_{(m)} V, \\
\text{Out} V &= \text{Aut} V / \text{Inn} V, \\
\text{Out}^0_{(m)} V &= \text{Aut}^0 V / \text{Inn}^0_{(m)} V, \\
\text{Out}^0 V &= \text{Aut}^0 V / \text{Inn}^0 V
\end{align*}
\]
though they will not appear in the rest of the paper.

2 Free Bosonic Vertex Algebra

This section is devoted to give a definition and fundamental properties of the free bosonic vertex algebra, which is the main objective of this article.

2.1 Heisenberg Algebra

Let us recall the definition of the Heisenberg algebra. Consider the Lie algebra
\[ \mathcal{H} = (\bigoplus_{n \in \mathbb{Z}} \mathbb{C}H_n) \oplus \mathbb{C}K \]
with the Lie bracket defined by
\[ [H_m, H_n] = m\delta_{m+n,0}K, \quad [H_m, K] = 0. \]
Let $V$ be a representation of $\mathcal{H}$ on which $K$ acts by a non-zero scalar. Then, by rescaling $H_n$, we may assume that $K$ acts by 1. Such a representation is nothing but a module of the associative algebra $A$:

$$A = U(\mathcal{H})/U(\mathcal{H})(K - 1),$$

where $U(\mathcal{H})$ is the universal enveloping algebra of $\mathcal{H}$. Let us denote the generators of $A$ corresponding to $H_n$ by $\alpha_n$. Then $A$ is the associative algebra generated by $\{\alpha_n | n \in \mathbb{Z}\}$ with fundamental relations

$$\alpha_m\alpha_n - \alpha_n\alpha_m = m\delta_{m+n,0} \quad (m, n \in \mathbb{Z}).$$

The algebra $A$ is called the Heisenberg algebra.

Let $(\pi, M)$ be a representation of $A$. Consider the series

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \pi(\alpha_n)z^{-n-1}.$$

If $\alpha(z)$ is a field, then it is local with itself and has the following OPE

$$\alpha(y)\alpha(z) \sim \frac{1}{(y-z)^2}.$$

Conversely, a field with such properties defines a $A$–module on $M$. We call such a field a (normalized) Heisenberg field. In the sequel, we omit writing $\pi$.

**Note 2.1.** Since the Lie algebra $\mathcal{H}$ is the affinization of the abelian Lie algebra $u(1)$:

$$\mathcal{H} \cong (\mathbb{C}[t, t^{-1}] \otimes u(1)) \oplus \mathbb{C} K,$$

a Heisenberg field is called a $U(1)$–current in physics.

Now consider the case when $M$ is a vertex algebra $V$. Suppose that there exists an element $h \in V$ such that

$$\alpha(z) = Y(h, z)$$

is a Heisenberg field. Then the OPE above is equivalent to

$$h_{(n)}h = \begin{cases} 0 & (n \geq 2) \\ |I\rangle & (n = 1) \\ 0 & (n = 0). \end{cases}$$

---

9The term Heisenberg algebra is used to mean the Lie algebra $\mathcal{H}$ in some literatures.
We call such an element $h \in V$ a (normalized) Heisenberg vector of the vertex algebra $V$.

Let $M$ be a vector space and $\alpha(z)$ a Heisenberg field. Consider the space $O$ of all the fields which are obtained from $\alpha(z)$ and $I(z)$ by successive use of normally ordered product and derivative. Let us first consider the OPE of fields $\partial^{(p)} \alpha(y)$ and $\partial^{(q)} \alpha(z)$ for $p, q \in \mathbb{N}$. It is given by

$$
\partial^{(p)} \alpha(y) \partial^{(q)} \alpha(z) \sim \partial^{(p)} \partial^{(q)} \frac{1}{(y-z)^2} = (-1)^p \frac{(p+q+1)!}{p!q!} \frac{1}{(y-z)^{p+q+2}}.
$$

The RHS is denoted as

$$
\langle \partial^{(p)} \alpha(y) \partial^{(q)} \alpha(z) \rangle = (-1)^p \frac{(p+q+1)!}{p!q!} \frac{1}{(y-z)^{p+q+2}}
$$

and is called the contraction. In particular, this does not vanish for any $p, q \in \mathbb{N}$.

Now consider the general case: The OPE of two fields in $O$ is given by the following theorem, together with the formula (17).

**Theorem 2.1 (Wick’s theorem).** We have

$$
\langle \circ \partial^{(p_1)} \alpha(y) \cdots \partial^{(p_m)} \alpha(y) \circ \partial^{(q_1)} \alpha(z) \cdots \partial^{(q_n)} \alpha(z) \circ \rangle
$$

$$
= \sum_{d=0}^{\max(m,n)} \frac{1}{d!} \sum_{\phi : [1,d] \to [1,m], \psi : [1,d] \to [1,n]} \prod_{i=1}^{d} \langle \partial^{(p_{\phi(i)})} \alpha(y) \partial^{(q_{\psi(i)})} \alpha(z) \rangle \times
$$

$$
\times \circ \prod_{j \in [1,d] \setminus \text{Im } \phi} \partial^{(p_j)} \alpha(y) \prod_{j \in [1,d] \setminus \text{Im } \psi} \partial^{(q_j)} \alpha(z) \circ
$$

where the second summation is over all injective maps $\phi$ and $\psi$. 

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For example, we have
\[
\left(\circ \alpha(y)\alpha(z)\circ\right)\left(\circ \alpha(y)\alpha(z)\circ\right)
\]
\[
= \circ \alpha(y)\alpha(y)\alpha(z)\alpha(z)\circ + 4\frac{1}{(y-z)^2}\circ \alpha(y)\alpha(z)\circ + 2\frac{1}{(y-z)^4}
\]
\[
\sim 2\frac{1}{(y-z)^4} + 4\frac{1}{(y-z)^2}\circ \alpha(z)\alpha(z)\circ + 4\frac{1}{y-z}\circ \partial \alpha(z)\alpha(z)\circ,
\]
\[
\partial \alpha(y)\left(\circ \alpha(z)\alpha(z)\circ\right)
\]
\[
= \circ \partial \alpha(y)\alpha(z)\alpha(z)\circ - 4\frac{1}{(y-z)^3}\alpha(z) \sim -4\frac{1}{(y-z)^3}\alpha(z),
\]
\[
(\circ \alpha(y)\alpha(y)\circ)\partial \alpha(z)
\]
\[
= \circ \alpha(y)\alpha(y)\partial \alpha(z)\circ + 4\frac{1}{(y-z)^3}\alpha(y)
\]
\[
\sim 4\frac{1}{(y-z)^3}\alpha(z) + 4\frac{1}{(y-z)^2}\partial \alpha(z) + 4\frac{1}{(y-z)^2}\partial^{(2)} \alpha(z),
\]
\[
\partial \alpha(y)\partial \alpha(z) = \circ \partial \alpha(y)\partial \alpha(z)\circ - 6\frac{1}{(y-z)^4} \sim -6\frac{1}{(y-z)^4}.
\]

2.2 Fock Representation

Let \( \mathbb{C}[x_1, x_2, \ldots] \) be the polynomial ring in countably many variables. We define the degree of a polynomial by setting

\[
\text{deg}(x_{i_1} \cdots x_{i_k}) = i_1 + \cdots + i_k
\]

for a monomial \( x_{i_1} \cdots x_{i_k} \).

Now, for each complex number \( r \in \mathbb{C} \), consider the \( \mathcal{A} \)-module

\[
\pi_r : \mathcal{A} \longrightarrow \text{End} \mathcal{F}_r, \quad \mathcal{F}_r = \mathbb{C}[x_1, x_2, \ldots]
\]

defined by

\[
\alpha_n \longmapsto \begin{cases} 
  n \frac{\partial}{\partial x_n} & (n > 0) \\
  r & (n = 0) \\
  x_n & (n < 0).
\end{cases}
\]
This is an irreducible representation of \( \mathcal{A} \) and is called the Fock representation of charge \( r \). The unit 1 of the ring \( \mathbb{C}[x_1, x_2, \ldots] \) satisfies

\[
\alpha_n 1 = \begin{cases} 
0 & (n > 0) \\
r & (n = 0) \\
x_{-n} & (n < 0).
\end{cases}
\]

and the space \( \mathcal{F}_r \) is generated by 1 as an \( \mathcal{A} \)-module. For the charge \( r = 0 \), the Fock representation \( \mathcal{F}_0 \) is called the vacuum representation and plays the special role: \( V = \mathcal{F}_0 \) has a natural structure of vertex algebra and \( M = \mathcal{F}_r \) are its modules.

**Remark 2.1.** Let us give another description of Fock representation \( \mathcal{F}_r \). We recall that \( \mathcal{A} \) has the triangular decomposition:

\[
\mathcal{A} = \mathcal{A}_- \otimes \mathcal{A}_0 \otimes \mathcal{A}_+
\]

where \( \mathcal{A}_- \), \( \mathcal{A}_0 \) and \( \mathcal{A}_+ \) are the commutative subalgebras generated by \( \{\alpha_n | n < 0\} \), \( \{\alpha_0\} \) and \( \{\alpha_n | n > 0\} \) respectively. For each \( r \in \mathbb{C} \), let \( \mathbb{C}_r = \mathbb{C}|r\rangle \) be the one dimensional representation of \( \mathcal{A}_0 \otimes \mathcal{A}_+ \) defined by

\[
\alpha_n |r\rangle = \begin{cases} 
0 & (n > 0) \\
|r| & (n = 0).
\end{cases}
\]

Consider the induced module

\[
\mathcal{A} \otimes_{\mathcal{A}_0 \otimes \mathcal{A}_+} \mathbb{C}_r.
\]

Then this is in fact an irreducible \( \mathcal{A} \)-module which is isomorphic to the Fock representation \( \mathcal{F}_r \), where \( |r\rangle \) is identified with the unit 1 \( \in \mathcal{F}_r \).

### 2.3 Free Bosonic Vertex Algebra

Let \( V = \mathcal{F}_0 \) be the vacuum representation of Heisenberg algebra \( \mathcal{A} \). Consider the series

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}.
\]

Then this is a Heisenberg field and we have

\[
\alpha(z) 1 = x_1 + \sum_{n \geq 1} x_{n+1} z^n.
\]
Under this situation, let us seek for a vertex algebra structure on $V$ so that we have

$$\alpha(z) = Y(x_1, z)$$

and $|I\rangle = 1$. Note that the constants $c \in \mathbb{C}[x_1, x_2, \ldots]$ are the only elements which have the property that $\alpha(z)c$ has no term of negative degree. Now, such a structure is unique if it exists, because $V$ is spanned by

$$S = \{\alpha_{-i_1 -1} \cdots \alpha_{-i_n -1} | n \in \mathbb{N}, i_1, i_2, \ldots, i_n \in \mathbb{N}\}.$$ 

Note that $\alpha_{-i_1 -1} \cdots \alpha_{-i_n -1} = x_{i_1} \cdots x_{i_n}$. Thus we must have

$$(21) \quad Y(x_{i_1} \cdots x_{i_n}, z) = \partial^{(i_1)}(z) \cdots \partial^{(i_n)}(z) \circ \alpha(z).$$

To apply the existence theorem, we define $T$ by

$$Tx_n = nx_{n+1}$$

on the generators. We can uniquely extend this to a derivation of an associative algebra $V$, since $V = \mathbb{C}[x_1, x_2, \ldots]$ is a polynomial ring. Indeed, the operator

$$T = \sum_{n=1}^{\infty} nx_{n+1} \frac{\partial}{\partial x_n}$$

has the desired property. Then it is easy to check that

$$T|I\rangle = 0, \quad [T, \alpha(z)] = \partial_z \alpha(z)$$

and the existence theorem 1.2 shows that (21) gives rise to a vertex algebra structure on $V$. We call this vertex algebra $V = \mathcal{F}_0$ the free bosonic vertex algebra.

Another way of constructing the free bosonic vertex algebra is as follows: Take any $M = \mathcal{F}_r (r \in \mathbb{C})$ and consider the space $\mathcal{O}$ spanned by all the fields obtained from $\alpha(z)$ and $I(z)$ by successive use of normally ordered product and derivative. Then the space $\mathcal{O}$ is closed under the residual products so that it has a structure of vertex algebra by Li’s theorem. Moreover, it has a structure of $\mathcal{A}$-module via

$$\mathcal{A} \rightarrow \text{End } \mathcal{O}$$

$$\alpha_n \mapsto \alpha(z)_{(n)}$$
and the identity field $I(z)$ satisfies 
\[ \alpha(z)_n I(z) = 0 \quad (n \geq 0). \]
Therefore, by the construction of Remark 2.1, we have a unique $\mathcal{A}$–module map

\[ \mathcal{F}_0 \rightarrow \mathcal{O} \]

which sends $|0\rangle$ to $I(z)$. This map is an isomorphism because $\mathcal{O}$ is generated by $I(z)$ as an $\mathcal{A}$–module and $\mathcal{F}_0$ is irreducible. So we can introduce a structure of vertex algebra on $\mathcal{F}_0$ via this isomorphism, which coincides with the one described above.

Now, Wick’s theorem 2.1 and the formula (17) enable us to calculate the residual products of fields in $\mathcal{O}$, so that we can calculate the binary operations of the free bosonic vertex algebra through the state–field correspondence.

### 2.4 Conformal Vectors and Standard Gradation

Let $M$ be a vector space and $\alpha(z)$ a Heisenberg field on $M$. Then, by (20), the field

\[ L(z) = \frac{1}{2} \alpha(z) \alpha(z) + \lambda \partial \alpha(z), \quad (\lambda \in \mathbb{C}) \]

satisfies

\[ L(y)L(z) \sim \frac{1/2 - 6\lambda^2}{(y-z)^4} + \frac{2L(z)}{(y-z)^2} + \frac{\partial L(z)}{y-z}. \]

Namely, the field $L(z)$ is a Virasoro field of central charge $1 - 12\lambda^2$.

**Note 2.2.** For $M = \mathcal{F}_r$, the Vir–module obtained by $L(z)$ is called the Feigin–Fuks module and possesses an important position in mathematical physics.

In case when $M$ is the free bosonic vertex algebra $V = \mathcal{F}_0$, we have

\[ L(z) = Y \left( \frac{1}{2} x_1^2 + \lambda x_2, z \right) \]

and $L_{-1} = T$, where $T$ is the translation of the vertex algebra $V$ and

\[ L_0 = \sum_{n=1}^{\infty} nx_n \frac{\partial}{\partial x_n}, \]

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which is semi–simple. Therefore,

\[ v = \frac{1}{2}x_1^2 + \lambda x_2, \quad (\lambda \in \mathbb{C}) \]

is a conformal vector of \( V \). We call the gradation obtained by \( L_0 \) the standard gradation of \( V \). The corresponding decomposition is \( V = \bigoplus_{n=0}^{\infty} V^n \), where

\[ V^n = \bigoplus_{i_1 + \cdots + i_k = n} \mathbb{C} x_{i_1} \cdots x_{i_k}. \]

Namely, the space \( V^n \) consists of (homogeneous) polynomials of degree \( n \) in our convention. So the degree is a synonym of the conformal weight\(^{10} \) with respect to the conformal vector \(^{(22)} \).

3 Classification of Conformal Vectors of Free Bosonic Vertex Algebra

This section is the main contribution of the present work. We will classify all the conformal vectors without any restriction on its degree with respect to the standard gradation. The main result is that the conformal vectors given in Subsection 2.4 exhaust all the conformal vectors up to the action by inner automorphisms.

3.1 Commutative Vectors

We begin by introducing the notion of commutative vectors, which will become fundamental in our consideration. Let \( M \) be a vector space and \( C(z) \) a field on \( M \). We say that \( C(z) \) is a commutative field if

\[ C(y)C(z) = C(z)C(y), \]

which is equivalent to the OPE

\[ C(y)C(z) \sim 0 \]

\(^{10} \) However, since we will be dealing with the classification of conformal vectors and the notion of conformal weight depends on the choice of conformal vector, we shall use the term degree instead of the conformal weight.
under the assumption that $C(y)$ is local with itself. It is also equivalent to the condition that the Fourier modes $C_n$, $n \in \mathbb{Z}$ form a representation of commutative algebra (more precisely, of the polynomial ring)

$$C_mC_n = C_nC_m, \quad m, n \in \mathbb{Z}.$$ 

When $M$ is a vertex algebra $V$ and we have a vector $c \in V$ such that

$$C(z) = Y(c, z),$$

is a commutative field, then we call such $c \in V$ a **commutative vector**. Namely, an element $c \in V$ is a commutative vector if and only if

$$c_n c = 0, \quad (n \geq 0).$$

For example, the vacuum vector is clearly a commutative vector.

For the free bosonic vertex algebra $V = \mathcal{F}_0$, we have the following.

**Theorem 3.1 (Uniqueness of commutative vectors).** Let $V = \mathcal{F}_0$ be the free bosonic vertex algebra. Then a vector $c \in V$ is a commutative vector if and only if it is a scalar multiple of the vacuum vector.

The proof, which is based on Wick’s theorem, will be given in Subsection 3.4.

### 3.2 Classification of Heisenberg and Virasoro Vectors

As an application of the last theorem, we will give the classification of Heisenberg and Virasoro vectors of the free bosonic vertex algebra $V = \mathcal{F}_0$.

Let us first consider the case of Heisenberg vectors. By the definition, an element $h \in V$ is a Heisenberg vector if and only if

$$h_n h = \begin{cases} 0 & (n \geq 2) \\ |I| & (n = 1). \end{cases}$$

The condition $h_0 h = 0$ follows from the skew symmetry as in Remark [4]. Let us decompose $h$ into the homogeneous components according to the standard gradation:

$$h = h^0 + h^1 + \cdots + h^d, \quad h^d \neq 0.$$
Suppose $d \geq 2$. Then $h^d_{(n)}h^d$ is the component of $h_{(n)}h$ of degree $2d - n - 1$ and we have

$$h^d_{(n)}h^d = 0 \quad (n \geq 0).$$

This implies $h^d \in \mathbb{C}$, which is a contradiction. Therefore, $h$ does not have a component of degree greater than one and $h$ must be written as

$$h = a + bx_1 \quad (a, b \in \mathbb{C}).$$

Substitute this expression in the definition of Heisenberg vectors and compare the coefficients. Then we see:

**Theorem 3.2 (Classification of Heisenberg vectors).** An element of $h$ of free bosonic vertex algebra is a Heisenberg vector if and only if $h$ is written as

$$h = \mu \pm x_1, \quad i.e., \quad Y(h, z) = \mu I(z) \pm \alpha(z)$$

for some scalar $\mu \in \mathbb{C}$.

Now let us turn to the case of Virasoro vectors. The strategy is the same as above. For a Virasoro vector $v \in V$ with homogeneous decomposition

$$v = v^0 + v^1 + v^2 + v^3 + \cdots + v^d, \quad v^d \neq 0,$$

suppose $d \geq 3$, then

$$v^d_{(n)}v^d = 0 \quad (n \geq 0)$$

and $v^d$ must be proportional to 1, thus we have a contradiction. So $d \leq 2$ and

$$v = a + bx_1 + cx_1^2 + dx_2 \quad (a, b, c, d \in \mathbb{C}).$$

and by substituting it into the definition of Virasoro vectors, we have a condition for $v$ to be a Virasoro vector:

$$b^2 = 2a, \quad 4bc = 2b, \quad 4c^2 = 2c, \quad \text{and} \quad 4cd = 2d.$$
Theorem 3.3 (Classification of Virasoro vectors). An element $v$ of free bosonic vertex algebra $V$ is a Virasoro vector if and only if $v$ is written as

$$v = \frac{1}{2} \mu^2 + \mu x_1 + \frac{1}{2} x_1^2 + \lambda x_2,$$

i.e.,

$$Y(v, z) = \frac{1}{2} \mu^2 I(z) + \mu \alpha(z) + \frac{1}{2} \alpha(z) \alpha(z) + \lambda \partial \alpha(z)$$

for some scalar $\lambda, \mu \in \mathbb{C}$.

Now for a Virasoro vector $v = \frac{1}{2} \mu^2 + \mu x_1 + \frac{1}{2} x_1^2 + \lambda x_2$, we have

$$L_{-1} := v(0) = T, \quad L_0 := v(1) = \mu \frac{\partial}{\partial x_1} + \sum_{i=1}^{\infty} i x_i \frac{\partial}{\partial x_i}.$$ 

Such $L_0$ is always semisimple and thus

Corollary 3.4 (Classification of conformal vectors). All the Virasoro vectors in the free bosonic vertex algebra are conformal vectors and are given by the theorem above.

Remark 3.1. If $\mu \neq 0$, then the gradation defined by the conformal vector $v = \frac{1}{2} \mu^2 + \mu x_1 + \frac{1}{2} x_1^2 + \lambda x_2$ differs from the standard one.

3.3 Applications

Let us determine the automorphism group of the free bosonic vertex algebra $V = \mathbb{C}[x_1, x_2, \ldots]$. First, since $V$ is generated by $x_1$, we have:

Lemma 3.5. An automorphism $\sigma$ of the free bosonic vertex algebra $V$ is uniquely determined by $\sigma(x_1)$.

Then because an automorphism $\sigma$ maps a Heisenberg vector to another, we have

$$\sigma(x_1) = a \pm x_1$$

and conversely, by the above lemma, such an automorphism is unique if it exists. Now since $\alpha_1$ is a derivation which decreases the degree by 1,

$$\tau_a = \exp(a \alpha_1) = \exp \left( a \frac{\partial}{\partial x_1} \right), \quad a \in \mathbb{C}$$

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does converge and gives an inner automorphism of $V$. On the other hand, it is easy to see that

$$\iota : x_1 \mapsto -x_1$$

uniquely extends to an automorphism of $V$ which preserves the standard gradation and is not inner. Since they satisfy

$$\tau_a(x_1) = a + x_1, \quad \iota \tau_a = a - x_1,$$

we have

$$\text{Aut} V = \{\tau_a, \iota \tau_a \mid a \in \mathbb{C}\}.$$ 

Therefore

$$\text{Inn} V = \text{Inn}(1) V = \{\tau_a \mid a \in \mathbb{C}\} \cong \mathbb{C}, \quad \text{Aut}^0 V = \{\text{id}, \iota\} \cong \mathbb{Z}/2\mathbb{Z}.$$ 

Further we have

$$\iota \tau_a \iota^{-1} = \tau_{-a}$$

and $\text{Inn} V \cap \text{Aut}^0 V = \{\text{id}\}$. Summarizing, we have:

**Theorem 3.6.** The automorphism group of the free bosonic vertex algebra is a semi–direct product

$$\text{Aut} V = \text{Inn} V \rtimes \text{Aut}^0 V$$

where $\text{Inn} V = \{\tau_a \mid a \in \mathbb{C}\} \cong \mathbb{C}$ and $\text{Aut}^0 V = \{\text{id}, \iota\} \cong \mathbb{Z}/2\mathbb{Z}$.

Now by the classification given in Subsection 3.2, a conformal vector is written as

$$v = \frac{1}{2} \mu^2 + \mu x_1 + \frac{1}{2} x_1^2 + \lambda x_2$$

for some scalars $\lambda, \mu \in \mathbb{C}$. An inner automorphism $\tau_a, a \in \mathbb{C}$ maps $v$ to a vector of degree 2 if and only if $a = -\mu$ and in this case

$$\sigma(v) = \frac{1}{2} x_1^2 + \lambda x_2.$$ 

This is a conformal vector described in Subsection 2.4 and we have:
Corollary 3.7. Any conformal vector of the free bosonic vertex algebra is transformed by an inner automorphism to a unique conformal vector of the form

\[ v = \frac{1}{2} x_1^2 + \lambda x_2 \]

for some \( \lambda \in \mathbb{C} \).

Finally let us mention about the complete reducibility of the vertex algebra \( V \) as a representation of \( sl_2(\mathbb{C}) \) through the homomorphism

\[
\begin{align*}
sl_2(\mathbb{C}) & \longrightarrow \text{Vir} \\
E & \mapsto -L_1 \\
H & \mapsto -2L_0 \\
F & \mapsto L_{-1}
\end{align*}
\]

where \( \{E, H, F\} \) is the standard basis of \( sl_2(\mathbb{C}) \) and the \( \text{Vir} \) module structure on \( V \) is given by a conformal vector \( v \). Then the weight space decomposition is

\[ V = \bigoplus_{k \in \mathbb{N}} V_{-2k}, \quad V_{-2k} = \{ a \in V \mid Ha = -2ka \}. \]

By applying an inner automorphism, we may assume that the conformal vector is written as

\[ v = \frac{1}{2} x_1^2 + \lambda x_2. \]

Then the gradation is the standard one and in particular,

\[ V_0 = \mathbb{C}1, \quad V_{-2} = \mathbb{C}x_1. \]

Therefore, if \( L_1 x_1 = 0 \), then we have a direct sum decomposition

\[ V = V_0 \oplus (\bigoplus_{k>0} V_{-2k}) \]

and \( V \) is completely reducible since the latter factor is a direct sum of Verma modules with negative highest weights. Otherwise, the submodule generated by \( x_1 \) is isomorphic to \( M^*(0) \), the dual Verma module with highest weight 0, and \( V \) cannot be completely reducible. Therefore \( V \) is completely reducible if and only if \( L_1 x_1 = 0 \), namely, \( v = \frac{1}{2} x_1^2 \).
Proposition 3.8. A conformal vector $v$ gives a completely reducible $sl_2$–module structure on $V$ if and only if $v$ is the image of the conformal vector $\frac{1}{2}x_1^2$ by an inner automorphism.

Since $\frac{1}{2}x_1^2$ is the unique conformal vector which is fixed by involution $\iota$, we have the following observation.

Corollary 3.9. A conformal vector $v$ of the free bosonic vertex algebra $V$ gives a completely reducible $sl_2$–module structure on $V$ if and only if $v$ is fixed by some non–trivial automorphism of $V$.

3.4 Proof of Theorem 3.1

Before proceeding to the proof, let us prepare the notion of partitions. A partition is a sequence of non–negative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

such that

$$\lambda_1 \geq \lambda_2 \geq \ldots \quad \text{and} \quad \lambda_n = 0 \quad \text{for sufficiently large} \quad n.$$  

Let $P$ denotes the set of all partitions. Set

$$\ell(\lambda) = \max\{n| \lambda_n \neq 0\}, \quad |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}.$$  

They are called the length and the size of $\lambda$ respectively. We denote by $\phi$ the partition with all the entries being zero. Then $|\phi| = \ell(\phi) = 0$. Let us denote $P_\ell$ the set of partitions of length $\ell$.

Let $V = \mathbb{C}[x_1, x_2, \ldots]$ be the free bosonic vertex algebra. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, set

$$x_\lambda = x_{\lambda_1}x_{\lambda_2} \cdots x_{\lambda_{\ell(\lambda)}}, \quad x_{\phi} = 1.$$  

Then the set of $P$ of partitions indexes a basis of $V$:

$$V = \oplus_{\lambda \in P} \mathbb{C}x_\lambda.$$  

Now, let $c$ be a commutative vector and

$$c = \sum_{\lambda \in P} c_\lambda x_\lambda \quad (c_\lambda \in \mathbb{C})$$  

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be its representation by the basis. Then to prove Theorem 3.1, i.e., to show that \( c \) is proportional to 1, it suffices to see that
\[
c_\lambda = 0 \quad (\lambda \neq 0).
\]

For the purpose, decompose \( c \) according to the length and take the longest part of them:
\[
\sum_{\lambda \in P, \ell(\lambda) = \ell} c_\lambda x_\lambda
\]
where \( \ell = \max\{\ell(\lambda) | c_\lambda \neq 0\} \). Introduce the total order defined on the set \( P_\ell \) by
\[
\lambda > \mu \quad \text{if} \quad \lambda_1 = \mu_1, \ldots, \lambda_{i-1} = \mu_{i-1}, \lambda_i > \mu_i \quad \text{for some} \quad i
\]
and take the greatest one \( \nu \) in the set \( \{\lambda \in P_\ell | c_\lambda \neq 0\} \). Now suppose \( \ell \geq 1 \) and let us derive a contradiction.

By Wick’s theorem, we have
\[
Y(c, y)Y(c, z) \sim \sum_{|\lambda| = |\mu| = \ell, \lambda, \mu \leq \nu} c_\lambda c_\mu \left( \partial^{(\lambda_i - 1)} \alpha(y) \partial^{(\mu_j - 1)} \alpha(z) \right) \times \nonumber
\]
\[
x_\lambda x_\mu \sum_{i=1}^{\ell} \hat{x}_{\lambda_i} \cdots \hat{x}_{\lambda_i} Y(x_{\mu_1} \cdots \hat{x}_{\mu_j} \cdots x_{\mu_\ell}, y)Y(x_\mu_1 \cdots \hat{x}_{\mu_j} \cdots x_{\mu_\ell}, z)
\]
\[
+ \text{(shorter length terms)}.
\]

In the summation of the longest terms, the ones with the highest possible pole at \( y = z \) is
\[
\sum_{|\lambda| = |\mu| = \ell, \ell, \mu \leq \nu} c_\lambda c_\mu K_{\lambda\mu} \frac{1}{(y - z)^{\lambda_1 + \mu_1}} Y(x_{\lambda_2} \cdots x_{\lambda_i} x_{\mu_2} \cdots x_{\mu_\ell}, z)
\]
where \( K_{\lambda\mu} \) is a non-zero scalar. Further the greatest term in this summation is
\[
K_{\nu\nu} c_\nu^2 \frac{1}{(y - z)^{2\nu_1}} Y((x_{\nu_1})^2 \cdots (x_{\nu_\ell})^2, z).
\]

By the assumption that \( Y(c, y)Y(c, z) \sim 0 \), we must have
\[
c_\nu^2 = 0.
\]

This contradicts the choice of \( \nu \) and we conclude that \( \ell = 0 \). Therefore a commutative vector \( c \in V \) must be a scalar in \( V = \mathbb{C}[x_1, x_2, \ldots] \).


4 Discussion

In this paper, we have classified Heisenberg vectors, Virasoro vectors, conformal vectors and automorphisms of the free bosonic vertex algebra. All the results obtained here depend on the fact that a commutative vector of the vertex algebra is proportional to the vacuum vector (Theorem 3.1). Indeed, this reduced the classification of Heisenberg and Virasoro vectors to those of degree less than or equal to one and two respectively.

However, such uniqueness of commutative vectors is no more true for general vertex algebra. For example, let $L$ be an integral lattice and $V_L$ be the vertex algebra associated with $L$. Then, for any $\gamma \in L$ such that $(\gamma|\gamma) \geq 0$, the vector $1 \otimes e^{\gamma}$ is a commutative vector.

Correspondingly, the failure of the uniqueness of commutative vectors complicates the classification of Heisenberg vectors, Virasoro vectors and so on. In fact, the vertex algebra $V_L$ for a positive definite $L$, for example, has a conformal vector having arbitrarily high degree with respect to the standard gradation: let $v$ be a conformal vector of degree 2 and take $\gamma \in L$ such that $(\gamma, \gamma) \geq 4$. Then the vector $v + T(1 \otimes e^{\gamma})$ is a conformal vector with degree $\frac{1}{2}(\gamma, \gamma) + 1$, which can be arbitrarily large. Moreover, this vector is obtained from $v$ by the inner automorphism $\exp (\kappa (1 \otimes e^{\gamma}(0)))$ where $\kappa = 2/(2 - (\gamma|\gamma))$ which does converge. In particular, this shows that the group of inner automorphisms is quite large in this case.

The last construction is generalized as follows. Let $v$ be a conformal vector of a vertex algebra and $u$ be a primary vector with conformal weight $\Delta$ with respect to $v$, i.e.,

$$v(1)u = \Delta u, \quad \text{and} \quad v(n)u = 0, \quad (n \geq 2),$$

such that $u$ is a commutative vector. Then the vector $v + Tu$ is always a Virasoro vector, and if $\Delta \neq 1$, then it is obtained from $v$ by the inner automorphism $\exp (\kappa u(0))$, $\kappa = 1/(1 - \Delta)$ if it makes sense.

These examples illustrate the complicated situation for general vertex algebras.

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