Optimal Investment in a Large Population of Competitive and Heterogeneous Agents

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February 22, 2023

Abstract This paper studies a stochastic utility maximization game under relative performance concerns in finite agent and infinite agent settings, where a continuum of agents interact through a graphon (see definition below). We consider an incomplete market model in which agents have CARA utilities, and we obtain characterizations of Nash equilibria in both the finite agent and graphon paradigms. Under modest assumptions on the denseness of the interaction graph among the agents, we establish convergence results for the Nash equilibria and optimal utilities of the finite player problem to the infinite player problem. This result is achieved as an application of a general backward propagation of chaos type result for systems of interacting forward-backward stochastic differential equations, where the interaction is heterogeneus and through the control processes, and the generator is of quadratic growth. In addition, characterizing the graphon game gives rise to a novel form of infinite dimensional forward-backward stochastic differential equation of McKean-Vlasov type, for which we provide well–posedness results. An interesting consequence of our result is the computation of the competition indifference capital, i.e., the capital making an investor indifferent between whether or not to compete.

MSC 2000 Subject Classification: 91A06, 91A13, 91A15.

Keywords: Stochastic graphon games, Propogation of chaos, FBSDE, Mckean-Vlasov equations.

1 Introduction

We consider agents investing in a common riskless bond and a vector of stocks of their own choosing. Each agent aims to maximize their utility as a function of their terminal wealth, benchmarked by the industry average. In addition, agents’ utilities are of constant absolute risk aversion type. This problem was first investigated by Espinosa and Touzi [17, 16] in the setting of a complete market with a finite number of agents. In these works, the benchmark for a particular agent was taken to be an empirical average of the other players’ terminal wealth, multiplied by a constant factor between 0 and 1 representing the sensitivity of this particular agent to their peers’ performance. Such a utility maximization problem under relative performance concerns has since been explored extensively using various techniques, see for instance [18, 19, 31, 20, 29, 24] and references therein for a small sample of works on the question. We also refer to [14, 1, 15] for more recent articles studying relative performance concerns through the lens of the forward criteria of Musiela and Zariphopoulou [39]. As in [17], [18, 19, 20] approach the problem from a purely probabilistic perspective through characterizing the game using systems of (forward) backward stochastic differential equation (F)BSDEs. In particular, Fu, Xizhi, and Zhou [20] explore an extension of the game in an incomplete market framework in the following sense: all agents invest in the same vector of stocks (with dimension $d$) and their strategies can take values in $\mathbb{R}^d$. In addition to having a common Brownian motion $W^*$ representing the market uncertainty or “common noise” in the price dynamics of all stocks, they allow each stock to be driven by a separate Brownian motion representing the “idiosyncratic noise”. These Brownian motions are i.i.d and independent of $W^*$. In this setting, the characterizing system of BSDEs has the particular feature that it is quadratic in the control variable. As first observed in the work of

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*Both authors gratefully acknowledge partial support from the NSF grants DMS-2005832 and CAREER DMS-2143861. We thank the anonymous referees and the editor for their constructive feedback as well as Stefanie Hesse for fruitful discussions.

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Frei and dos Reis [19], such systems are not always globally solvable, making the analysis of the problem in full generality particularly challenging. For instance, Frei and dos Reis [19, 18] provide specific counter examples when equilibria do not exist even in the case of a complete market where stocks are driven by idiosyncratic noise only. Frei [18] showed that multidimensional quadratic BSDEs are in general only locally solvable (i.e. the solutions exist only on small time intervals), and he provided equilibria for the original game of Espinosa and Touzi [17] using the existence of local solutions to the characterizing BSDEs. Fu, Xizhi, and Zhou [20] established existence and uniqueness of the characterizing BSDEs in an incomplete market framework where all investors invest in two stocks only and the strategies are unconstrained.

Recent developments in mean field games provide new avenues to approach the above described utility maximization game by considering the infinite population case. In fact, standard mean field games heuristics of Lasry and Lions [33], Huang, Caines, and Malhamé [26] and Carmona and Delarue [9] suggest that in a homogeneous game, that is when agents are symmetric and identical, the infinite population analogue of the game can be solved by considering a single representative player whose best response is obtained as solution of a (one-dimensional) McKean–Vlasov BSDE. More precisely, one thus expects to bypass the subtleties coming with studying multi-dimensional quadratic BSDEs by analyzing a one-dimensional McKean–Vlasov quadratic BSDE. Despite also having quadratic growth, the latter equation seems much easier to analyze (both analytically and numerically) than the former, making the mean field game paradigm particularly attractive for this game. The mean field setting was first considered by Lacker and Zariphopoulou [31] and Lacker and Soret [29] in the Markovian setting with deterministic, constant coefficients, and equilibria was derived using Hamilton–Jacobi–Bellman SPDE methods. More recently, probabilistic techniques were proposed by Fu, Xizhi, and Zhou [20] and dos Reis and Platovov [15].

In the present paper we rigorously study the link between the finite and infinite population games. Our main modeling assumptions can be summarized as follows: we consider an incomplete market in which

- agents are allowed to invest in different vectors of stocks with random coefficients, driven by idiosyncratic noise and common noise,
- agents’ strategies are constrained to be in a closed, convex set,
- agents benchmark their performance by a weighted average of other agents’ terminal wealth.

Let us elaborate on the latter and probably less studied model feature mentioned above. For a particular agent $i$, instead of having a single factor $\lambda_i$ representing their sensitivity to a plain average of the other agents’ terminal wealth, we allow this agent to have different sensitivity factors $(\lambda_{ij})_{j \neq i}$ towards each agent. This assumption is a lot more realistic in the sense that funds usually aim to out-perform a small, specific group of competitors, and are usually completely indifferent of the performance of other funds that are, for example, on a much smaller or larger scale, utilize completely different strategies, or operate in a widely different market sector. This leads to a heterogeneous game set on a (random) graph and deviates from the standard symmetric agent interaction assumption, which is arguably the main limitation of the mean field game formulation. Following the seminal works of Lovász and Szegedy [36, 37] on the convergence of graphs to the so-called graphons (see precised definition and discussion below), the natural infinite population analogue of the game we consider in the present heterogeneous setting is a utility maximization graphon game. It is worth pointing out that in addition to the methodology, the main modeling difference between the present work and [20] resides in the heterogeneous interactions among agents and the consideration of constrained strategies here. These give rise to infinite dimensional McKean–Vlasov type (F)BSDEs with quadratic generators, making the analyses more demanding and requiring new techniques.

Similar to mean field games, graphon games provide an alternative to study large scale network games that in general suffers less from the curse of dimensionality. However, unlike mean field games, agents in a graphon game are no longer anonymous, as mentioned by Carmona, Cooney, Graves, and Lauriere [11]. The benefit associated with a graphon interaction is that agents are now aware of who their neighbors are, and are allowed to possess different preference metrics towards different neighbors. As a result, when deriving the optimal strategy for a specific agent, one needs to take into account (an aggregation of) a continuum of infinitely many other agents, where the aggregation is established through the graphon. The analysis of graphon games has gained traction in recent years, mostly in the engineering community. We refer for instance to [7, 8, 21, 41]. Parise and Ozdaglar [41] was the first work to analyze equilibria for static graphon games. In [7, 8], Caines and Huang studied decentralized control for graphon mean field games, and established an $\epsilon$-Nash theory that relates the equilibria for an infinite population game to that of a finite population game. Gao, Caines, and Huang [21] explored linear quadratic...
Gaussian mean field games. Outside of the engineering community Carmona, Cooney, Graves, and Lauriere [11] studied various static graphon games. Aurell, Carmona, and Laurière [2] studied stochastic graphon games in a linear–quadratic setting. We also refer to the recent works of Lacker and Soret [30] and Bayraktar, Wu, and Zhang [5] for recent results, on generic games. The main contributions of the present work can be summarized as follows:

- We derive explicit characterization properties of the Nash equilibria in the finite and the graphon utility maximization games.
- We show that if the sensitivity matrix in the finite-agent game stems from a graphon and follows a Bernouilli distribution, then the heterogeneous finite-agent game converges to the graphon game in the sense that every sequence of Nash equilibria converges (up to a subsequence) to a graphon equilibrium along with the associated value functions.
- We prove solvability of the graphon utility maximization game.

For the characterization properties, we adopt an extension of a well–known methodology proposed by Hu et al. [25]. The convergence and existence results are more involved. Convergence is obtained as a byproduct of a general backward propagation of chaos type result which appears to be of independent interest: Consider a general system of weakly interacting FBSDEs, in which the interaction is given through a random graph (appropriately) stemming from a graphon. We prove strong convergence of the interacting particle system to a limit consisting of infinitely many coupled particles. Backward propagation of chaos type results and their link to the mean field game convergence problem was first developed in recent works by Laurière and Tangpi [34], Luo and Tangpi [38] and Possamaï and Tangpi [42]. Also note that in these works, generators are Lipschitz–continuous. Our work contributes to the theory by extending it to systems in heterogeneous interactions through the control processes, and where the generators are of quadratic growth. A case of FBSDEs with heterogeneous interactions was posted on ArXiv a week before the present work by Bayraktar, Wu, and Zhang [5]. See also Bayraktar, Chakraborty, and Wu [4] for results along the same lines for forward particle systems. The results and methods of the present work further allows us to introduce and compute the so–called competition–indifference capital, which is the capital allowing to make the investor indifferent between being concerned by their peer’s performance or not.

The paper is organized as follows. In Section 2, we first introduce the probabilistic setting and the market model, followed by the finite–agent model and the graphon model, and lastly the main result of this paper, namely the convergence of the finite-agent Nash equilibrium to the graphon Nash equilibrium. The BSDE characterizations of the finite-agent game and the graphon game are presented in Section 3.1 and Section 3.2 respectively. Section 4 is dedicated to the proofs of the existence results. In Section 5 we prove the main results, which are propagation of chaos for heterogeneous particle systems. Section 6 establishes existence and uniqueness of general graphon FBSDEs of Mckean-Vlasov type allowing to derive well–posedness of the graphon game.

2 Probabilistic setting and main results

Let us now present the probabilistic setting underpinning this work. In this section we will also describe the market model as well as the finite and infinite population games under consideration. At the end of the section we present the main results of the article.

2.1 The market model

We fix a finite time horizon $T > 0$ and integers $n, d \in \mathbb{N}$. Let $(W^i)_{i \geq 1}$ be a sequence of independent $d$-dimensional Brownian motions supported on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, this probability space supports another independent one-dimensional Brownian motion $W^*$ and independent $\mathbb{R}$–valued random variables $(\xi^i)_{i \geq 1}$. We denote by $\mathcal{F}^n := (\mathcal{F}^n_t)_{t \in [0,T]}$ the $\mathbb{P}$–completion of the natural filtration of $\{(W^i)_{1 \leq i \leq n}, W^*, \xi^1, \ldots, \xi^n\}$. Let us define spaces and norms that will be used throughout the paper. Fix a generic finite–dimensional normed vector space $(E, \|\cdot\|_E)$, let $\mathcal{G}$ be a filtration, and $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- For any $p \in [1, \infty]$, $L^p(\mathcal{E}, \mathcal{G})$ is the space of $E$-valued, $\mathcal{G}$-measurable random variables $R$ such that
  \[ \|R\|_{L^p(\mathcal{E}, \mathcal{G})} := \left( E \left[ \|R\|^p_E \right] \right)^{\frac{1}{p}} < \infty, \text{ when } p < \infty, \|R\|_{L^\infty(\mathcal{E}, \mathcal{G})} := \inf \{ \ell \geq 0 : \|R\|_E \leq \ell, \mathbb{P} \text{-a.s.} \} < \infty. \]
• For any \( p \in [1, \infty) \), \( H^p(E, \mathcal{G}) \) is the space of \( E \)-valued, \( \mathcal{G} \)-predictable processes \( Z \) such that

\[
\|Z\|_{H^p(E, \mathcal{G})} := \mathbb{E} \left[ \left( \int_0^T \|Z_s\|_{E}^p \, ds \right)^{p/2} \right] < \infty.
\]

• \( L^2(E, \mathcal{G}) \) is the space of \( E \)-valued, \( \mathcal{G} \)-predictable processes \( Z \) such that

\[
\int_0^T \|Z_s\|_{E}^2 \, ds < \infty \quad \mathbb{P}\text{-a.s.}
\]

• For any \( p \in [1, \infty] \), \( S^p(E, \mathcal{G}) \) is the space of \( E \)-valued, continuous, \( \mathcal{G} \)-adapted processes \( Y \) such that

\[
\|Y\|_{S^p(E, \mathcal{G})} := \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \|Y_t\|_{E}^p \right] \right)^{1/p} < \infty, \text{ when } p < \infty, \quad \|Y\|_{S^\infty(E, \mathcal{G})} := \left\| \sup_{t \in [0,T]} \|Y_t\|_E \right\|_{L^\infty(E, \mathcal{G}^T)} < \infty.
\]

When the probability measure in the definition of these norms is different, say another probability measure \( \mathbb{Q} \) on \((\Omega, \mathcal{F})\), we will specify this by writing \( L^p(E, \mathcal{G}, \mathbb{Q}) \), \( H^p(E, \mathcal{G}, \mathbb{Q}) \), and \( S^p(E, \mathcal{G}, \mathbb{Q}) \).

The financial market consists of \( n \) agents trading in a common risk-less bond with interest rate \( r = 0 \) and \( n \times d \) stocks. In particular, each agent trades in a \( d \)-dimensional vector of stocks \( S^t \) with price evolution following the dynamics

\[
dS^t_i = \text{diag}(S^t_i) \left( \mu^t_i \, dt + \sigma^t_i dW^t_i + \sigma^* i dW^*_t \right) \quad i = 1, \ldots, n,
\]

where we denote by \( \text{diag}(x) \) the square matrix with entries \( x \in \mathbb{R}^d \) on the diagonal and 0 everywhere else. The coefficients \( \mu^t, \sigma^t \) and \( \sigma^* \) are predictable stochastic processes assumed to be bounded. Let \( \Sigma^t_i := (\sigma^t_i, \sigma^{*i}) \). We assume throughout that for all \( i \in \{1, \ldots, N\} \), the matrix \( \Sigma^t (\Sigma^t)^T \) is uniformly elliptic, that is, \( K I_d \geq \Sigma^t (\Sigma^t)^T \geq \varepsilon I_d \mathbb{P}\text{-a.s.} \) for some constants \( K > \varepsilon > 0 \). Let us introduce the process \( \theta^t \) given by

\[
\theta^t_i := (\Sigma^t_i)^T (\Sigma^t_i)^{-1} \mu^t_i.
\]

2.2 The \( n \)-agent game

A portfolio strategy is an \( \mathbb{F}^n \)-predictable, \( \mathbb{R}^d \)-valued process \((\pi_t)_{t \in [0,T]}\), with each component representing the amount invested in the corresponding stock at time \( t \). Let \( X^i, \pi \) denote the wealth of agent \( i \) at time \( t \) when starting with the initial position \( \xi^i \) and employing the trading strategy \( \pi \), which we assume to be self-financing. Then \( X^i, \pi \) satisfies

\[
dX^i, \pi_t = \pi_t \cdot (\Sigma^t_i \theta^t_i \, dt + \sigma^t_i dW^t_i + \sigma^{*i} dW^*_t), \quad X^i, \pi_0 = \xi^i.
\]

Each agent aims at maximizing their own utility\(^1\) from the terminal wealth, and in this work we assume the utility function to be exponential. In addition, each agent is concerned with the relative performance of their peers; see e.g. [17, 19, 16] for early works on the problems. Thus, the terminal wealths are benchmarked by a weighted average of the other agents’ terminal values\(^2\): \( \frac{1}{n-1} \sum_{j \neq i} \frac{\lambda^i}{\beta^i_n} X^j, \pi^\top \). The main modeling novelty considered in the present work is the addition of the term \( \frac{\lambda^i}{\beta^i_n} \) which measures agent \( i \)'s sensitivity to agent \( j \)'s wealth. The point is that each agent will try to perform better than the average of the other agents in the market, but they are not concerned with the performance of all agents. Think for instance of hedge funds. They will typically compete with “similar” hedge funds, for instance those raising capital from same investors. Thus, \( \lambda_{ij} = 1 \) if agent \( i \) is concerned with agent \( j \)'s performance and \( \lambda_{ij} = 0 \) if not. Denote for simplicity

\[
\lambda^i_n := \frac{1}{n-1} \sum_{j \neq i} \frac{\lambda^i}{\beta^i_n} \quad \text{with} \quad \lambda^i_n := 0.
\]

\(^1\)In the non-competitive case, utility maximization has a very long history, and solvability and characterization issues are settled. See for instance [25, 22, 28, 12, 13, 43, 44, 6] for a very incomplete list of references.

\(^2\)Throughout this work we use \( \sum_{j \neq i} x^j \) as a shorthand notation for \( \sum_{j \in \{1, \ldots, n\} \setminus \{i\}} x^j \).
The terminal utility of agent $i$ takes the form

$$U_i(X_T^{i,\pi}, \sum_{j \neq i} \lambda^n_{ij}X_T^{j,\pi}) := -\exp \left\{ -\frac{1}{\eta_i} \left( X_T^{i,\pi} - \rho \sum_{j \neq i} \lambda^n_{ij}X_T^{j,\pi} \right) \right\}, \quad (2.1)$$

where $\eta_i \in (0, 1)$ measures the risk preference level for agent $i$ and $\rho$ models the interaction weight. Since we are interested in competition, we fix $\rho \in (0, 1]$ throughout the article, see e.g. Hu and Zariphopoulou [24]. Let $A_i$ denote the set of admissible strategies for agent $i$ (which we will define shortly). To avoid bulky notations, we will use the abbreviated $\pi^i$ for the rest of this section with the understanding that the strategy depends on the size $n$ of the game. The optimization problem for agent $i$ thus takes the form

$$V_i^{0, n} := V_0^{i, n}((\pi^j)_{j \neq i})$$

$$:= \sup_{\pi \in A_i} \mathbb{E} \left[ -\exp \left\{ -\frac{1}{\eta_i} \left( X_T^{i,\pi} - \rho \sum_{j \neq i} \lambda^n_{ij}X_T^{j,\pi} \right) \right\} \right]. \quad (2.2)$$

**Definition 2.1** (Admissibility). Let $A_i$ be a closed convex subset of $\mathbb{R}^d$ that we call constraint set. A strategy $\pi^i$ for player $i$ is admissible if $\pi^i \in \mathbb{H}^2(A_i, \mathbb{F}^n)$ and for every $j \in \{1, \ldots, n\}$, there is $p > 2$ such that the family

$$\left\{ e^{\frac{p}{\eta_i} \lambda_{ij}} X_t^{i,\pi^i} ; \text{ with } \tau \text{ a } \mathbb{F}^n \text{-stopping time on } [0, T] \right\}$$

is uniformly integrable. In this case we will say that $\pi^i \in A_i$.

As usual we will be interested in Nash equilibria, whose definition we recall:

**Definition 2.2.** A vector $(\tilde{\pi}^1, \tilde{\pi}^2, \ldots, \tilde{\pi}^n)$ of admissible strategies in $A_1 \times A_2 \times \cdots \times A_n$ is a Nash equilibrium if for every $i = 1, \ldots, n$, the strategy $\tilde{\pi}^i$ is a solution to the portfolio optimization problem given in Equation (2.2) with value $V_0^{i}((\tilde{\pi}^1, \ldots, \tilde{\pi}^{-i}, \tilde{\pi}^i, \ldots, \tilde{\pi}^n))$. That is, for each $i$,

$$V_0^{0, n}((\tilde{\pi}^j)_{j \neq i}) = \mathbb{E} \left[ -\exp \left\{ -\frac{1}{\eta_i} \left( X_T^{i,\tilde{\pi}^i} - \rho \sum_{j \neq i} \lambda^n_{ij}X_T^{j,\tilde{\pi}^j} \right) \right\} \right].$$

In this work we will assume that $(\lambda_{ij})_{1 \leq i, j \leq n}$ are realizations of i.i.d. random variables, which are independent of the randomness source $(W^1, W^2, \ldots, W^n)$ and results will be proved for almost any realization of the graph. Therefore, we are actually working on the product space $(\Omega \times \mathcal{D} \otimes \mathcal{F} \otimes \mathcal{G} \otimes \mathcal{P})$. We will often use $\mathbb{P}$ to simplify the exposition. The interaction parameters $(\lambda_{ij})_{1 \leq i, j \leq n}$ give rise to an undirected random graph. Notice at this point already that our setting will include Erdős-Rényi graphs and the traditional complete graph.

Let us conclude this subsection by introducing some more notation that will be used in the paper. Given a vector $y = (y^1, \ldots, y^n)$, we put

$$y^i := \sum_{j \neq i} \lambda^n_{ij}y^j,$$

the weighted average of the vector $y$ (taking out $y^i$). Let $X_t^{i,\pi^i}$ be a short hand notation for $X_t^{i,\pi^i}$ and given a Nash equilibrium, $(\tilde{\pi}^1, \tilde{\pi}^2, \ldots, \tilde{\pi}^n)$, denote $\tilde{X}_t^j := \sum_{j \neq i} \lambda^n_{ij}X_t^{j,\tilde{\pi}^j}$ the weighted average of the portfolio values for agents $j \neq i$ when they all use the Nash equilibrium strategy $\tilde{\pi}^j$. These notation will be used in the statement of the main results.

### 2.3 The graphon game

Let $I = [0, 1]$ denote the unit interval. Intuitively, in the context of an infinite-player graphon game, we will label by $u \in I$ a given agent amid a continuum. The following probabilistic setup models the infinite population game.

Let $B_I$ be the Borel $\sigma$-field of $I$, and $\mu_I$ be the Lebesgue measure on $I$. Given a probability space $(I, \mathcal{I}, \mu_I)$ extending the usual Lebesgue measure space $(I, B_I, \mu_I)$, and the sample space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a rich Fubini extension $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \mu \otimes \mathbb{P})$ of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \mu \otimes \mathbb{P})$. Unfamiliar readers can consult Sun [46] for a self-contained presentation of the theory of rich Fubini extensions. Let $C([0, T]; \mathbb{R}^d)$ denote the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$. By [46], we can construct $\mathcal{I} \otimes \mathcal{F}$-measurable processes $(W, \xi) : I \times \Omega \rightarrow C([0, T], \mathbb{R}^d) \times \mathbb{R}$.
with essentially pairwise independent (e.p.i.)\(^3\), and identically distributed random variables \((W^u, \xi^u)_{u \in I}\), such that for each \(u \in I\), the process \(W^u = (W^u_t)_{0 \leq t \leq T}\) is a \(d\)-dimensional Brownian motion supported on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and \(\xi^u\) represents the starting wealth of agent \(u\). Suppose that in addition to \((W^u)_{u \in I}\), the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) supports the independent one-dimensional Brownian motion \(W^*\).

**Remark 2.3.** By [46, Lemma 2.3], we have the usual Fubini property on the rich product space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \mu \otimes \mathbb{P})\), i.e., we are free to exchange order of integrations. That is, given a measurable and integrable function \(f\) on \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \mu \otimes \mathbb{P})\) we can write

\[
\int_{I \times \Omega} f(u, \omega) \mu \otimes \mathbb{P}(d\omega, du) = \int_I \mathbb{E}[f(u)] \mu(du) = \mathbb{E} \left[ \int_I f(u) \mu(du) \right].
\]

This will be used often in the proof without further mention of [46, Lemma 2.3]. Moreover, we will write

\[
\mu(du) \equiv du
\]

to lighten the notation.

Let \(\mathbb{F}^u\) denote the completion of the filtration generated by \((W^u, W^*, \xi^u)\), and let \(\mathbb{F}\) denote the completion of filtration generated by \(((W^u)_{u \in I}, W^*, (\xi^u)_{u \in I})\). As above, we assume to be given a continuum of stocks \(S^u\) with dynamics

\[
dS^u_t = \text{diag}(S^u_t)(\mu^u_t dt + \sigma^u_t dW^u_t + \sigma^*_u dW^*_t), \quad u \in I
\]

so that the wealth process for agent \(u\) when employing strategy \(\pi\) follows the dynamics

\[
dx^u_t = \pi_t \cdot (\Sigma^u_t \theta^u_t dt + \sigma^u_t dW^u_t + \sigma^*_u dW^*_t), \quad X^u_0 = \xi^u
\]

where \(\Sigma, \theta, \sigma\) and \(\sigma^*\) are \(\mathcal{B}([0, T]) \otimes \mathcal{F}\)-measurable stochastic processes on \([0, T] \times I \times \Omega\), bounded uniformly in \(u \in I\), with

\[
\Sigma^u := (\sigma^u_t, \sigma^*_u) \quad \text{and} \quad \theta^u := \Sigma^u_t (\Sigma^u_t)\mathbb{F}^u_t) \mathbb{I}_t^{-1} \mu^u_t,
\]

with \(\Sigma^u_t, \Sigma^u_t\mathbb{F}^u_t)\mathbb{I}_t\) assumed to be uniformly elliptic, and where for almost every \(u \in I\), \(\sigma^u, \sigma^*u\) and \(\mu^u\) are \(\mathbb{F}^u\)-predictable.

We finally assume that \((\sigma^u)_{u \in I}, (\sigma^*u)_{u \in I}\) and \((\mu^u)_{u \in I}\) are e.p.i. and identically distributed.

**Definition 2.4.** A strategy profile is a family \((\pi^u)_{u \in I}\) of \(\mathbb{F}^u\)-progressive processes taking values in \(\mathbb{R}^d\) and such that \((u, t, \omega) \rightarrow \pi^u\) is \(\mathcal{I} \otimes \mathcal{B}([0, T]) \otimes \mathcal{F}\)-measurable.

Let the mapping \(\eta : I \rightarrow (0, 1)\) be \(\mathcal{I}\)-measurable and bounded away from zero uniformly in \(u\). Assume that the agent \(u\) is an exponential utility maximizer with risk aversion parameter \(\eta^u\) and is additionally concerned with the performance of their peers. The interaction among the continuum of agents will be modeled by a graphon, which is a symmetric and measurable function

\[
G : I \times I \rightarrow I.
\]

Throughout the paper, we fix a graphon \(G\). The utility function for a representative agent \(u\) is similar in form to that of (2.1). In particular, let \(\mathbb{F}^* := (\mathcal{F}^*_t)_{t \in [0, T]}\) denote the \(\mathbb{F}\)-completion of the filtration generated by \(W^*\).

Given \(u \in I\), consider the utility maximization problem

\[
V^u_G = V^u_G((\pi^u)_{\pi^u \neq u}) := \sup_{\pi^u \in \mathcal{A}^G} \mathbb{E} \left[ -\exp \left( -\frac{1}{\eta^u} \left( X^u_T, \pi^u - \mathbb{E} \left[ \rho \int_{I} X^u_T, \pi^u G(u, v) dv | \mathcal{F}^*_T \right] \right) \right) \right]. \tag{2.4}
\]

The set of admissible strategies \(\mathcal{A}^G\) in the infinite population game is defined as:

**Definition 2.5.** Let \(u \in I\) and let \(A_u\) be a closed convex subset of \(\mathbb{R}^d\). A strategy profile \((\pi^u)_{u \in I}\) is admissible if \(\mu\)-almost every \(u \in I\), it holds \(\pi^u \in \mathbb{H}^2(A^u, \mathbb{F}^u)\) and \(\int_I \|\pi^u\|_{\mathbb{H}^2(A^u, \mathbb{F}^u)} du < \infty\).

Taking inspiration from the theory of mean field games, see e.g. Carmona and Delarue [10, 9] or Lasry and Lions [33], we are interested in graphon Nash equilibria defined as follows:

**Definition 2.6.** A family of admissible strategy profiles \((\tilde{\pi}^u)_{u \in I}\) is called a graphon Nash equilibrium if for \(\mu\)-almost every \(u \in I\) the strategy \(\tilde{\pi}^u\) is optimal for (2.4) with \((\pi^v)_{v \neq u}\) replaced by \((\tilde{\pi}^v)_{v \neq u}\). That is,

\[
V^u_G((\tilde{\pi}^v)_{v \neq u}) := \mathbb{E} \left[ -\exp \left( -\frac{1}{\eta^u} \left( X^u_T, \tilde{\pi}^u - \mathbb{E} \left[ \rho \int_{I} X^u_T, \tilde{\pi}^u G(u, v) dv | \mathcal{F}^*_T \right] \right) \right) \right].
\]

\(^3\)Here, following [46, Definition 2.7], essentially pairwise independent means that for \(\mu\)-almost every \(u \in I\) and \(\mu\)-almost every \(v \in I\), the processes \((W^u, \xi^u)\) and \((W^v, \xi^v)\) are independent.
2.4 Main results

Let us now present the main results of this work. These are essentially the existence of graphon games, the convergence of the finite population game to the graphon game and a new notion of competition–indifference capital.

2.4.1 Existence of the graphon game

We will begin with the solvability of the graphon utility maximization problem. Existing results on well–posedness of graphon games largely focus on linear quadratic games or static games, see e.g. Aurell, Carmona, and Laurière [2] and Carmona, Cooney, Graves, and Lauriere [11]; we also refer to the more recent works by Lacker and Soret [30] and Bayraktar, Wu, and Zhang [5] for more general settings. Moreover, the case of games with common noise has remained untouched. The existence result given here relies on general solvability of graphon BSDEs and FBSDEs discussed in the final section of the paper.

Theorem 2.7. Assume that \( \xi^u \in L^2(\mu \otimes P) \). Then the following hold:

(i) If \( A^u = R^d \) for all \( u \in I \), and \( \rho \) satisfies \( \rho < e^{-(2\|\theta\|_\infty + \frac{1}{2}T)(2\|\Sigma\|_\infty \vee \|\theta\|_\infty)^{-1}} \), then the graphon game admits a graphon Nash equilibrium.

(ii) If \( \sigma^{*u} = 0 \) for all \( u \in I \), then the graphon game admits a graphon Nash equilibrium.

In the existence Theorem 2.7 above, we consider two cases. The first one is the common noise case. Here, we make the simplifying assumption that the strategies are unconstrained. This is a standard assumption in the literature. We additionally require the competition parameter \( \delta \) to be sufficiently small. The second case (ii) is the non–common noise case. Here, existence is obtained in full generality.

2.4.2 Convergence

The second result states that as \( n \to \infty \), the \( n \)-agent problem converges in the strong sense to the graphon problem, of course given some link between the sensitivity parameters \( (\lambda_{ij})_{1 \leq i,j \leq n} \) of the \( n \)-agent problem and its counterpart \( G(u,v) \) in the graphon problem. Essentially, we will assume below that \( (\lambda_{ij})_{1 \leq i,j \leq n} \) forms the adjacency matrix of a (random) graph converging in an appropriate sense to the graph represented by the graphon \( G \). See for instance Lovász [36] and Lovász and Szegedy [37] for extensive accounts on convergence of graphs an link between graphs and graphons. Here we remind the readers the definitions of cut metric, which will be used in the main results.

Definition 2.8. The cut norm for a graphon \( G \) is defined by

\[
\|G\|_{\square} := \sup_{E,E' \in B_i} \left| \int_{E \times E'} G(u,v) dudv \right|,
\]

and the corresponding cut metric for two graphons \( G_1 \) and \( G_2 \) is defined by \( d_{\square}(G_1,G_2) := \|G_1 - G_2\|_{\square} \).

Although \( \| \cdot \|_{\square} \) is not exactly a norm, we can make it one by identifying graphons which agree almost everywhere. We will also consider the usual \( L^2 \) norm on graphons, which is defined as

\[
\|G\|_2 := \left( \int_{I \times I} |G(u,v)|^2 du dv \right)^{1/2}.
\]

Thus, we make the following assumptions:

Condition 2.9. (1) There is a sequence \((\beta_n)_{n \geq 1}\) in \( \mathbb{R}_+ \) such that \( \lim_{n \to \infty} n \beta_n^2 = \infty \);

(2) there exists a sequence of graphons \((G_n)_{n \geq 1}\) such that:

2a) the graphons \( G_n \) are step functions, i.e. they satisfy

\[
G_n(u,v) = G_n \left( \left\lfloor \frac{nu}{n} \right\rfloor, \left\lfloor \frac{nv}{n} \right\rfloor \right) \text{ for } (u,v) \in I \times I,
\]

and it holds

\[
n \|G_n - G\|_2 \xrightarrow{n \to \infty} 0,
\]
(2b) \( \lambda_{ij} = \lambda_{ji} = \text{Bernoulli}(\beta_n G_n(\frac{i}{n}, \frac{j}{n})) \) independently for \( 1 \leq i, j \leq n \), and independently of \((\xi^u, \sigma^u, \theta^u, \sigma^u, \eta^u)_{u \in I}, W^*, \) and \((W^u)_{u \in I} \).

The graphons \( G_n \) introduced above are called step graphons, given that they are piecewise constant. The conditions (1) and (2) are the important modeling conditions. By [36, Theorem 11.22], (2b) says that the graph on which the finite population game is written converges (in the cut metric) to an infinite population graph. (2b) implicitly implies that \( \beta_n G_n(\frac{i}{n}, \frac{j}{n}) \in [0, 1] \), and means that the finite population graph is a simple graph with weights \((0, 1)\) depending on the outcome of a "coin toss". The parameter \( \beta_n \) can be seen as a density parameter on the graph, our condition (1) allows the graph to become more and more sparse as \( n \) becomes large. In fact, we have in mind the situation \( \lim_{n \to \infty} \beta_n = 0 \).

Before stating the results, we start by putting the \( n \)-agent problem and the graphon problem in the same probabilistic setting.

**Remark 2.10.** Let us re-brand the sequence of \( d \)-dimensional Brownian motions \((W^i)_{i \in \{1, \ldots, n\}}\) from Section 2.2 by \((\hat{W}^i)_{i \in \{1, \ldots, n\}}\), so that the completion of the natural filtration generated by \((\hat{W}^i)_{i \in \{1, \ldots, n\}}\) and \( W^* \) is now a subset of \( \mathbb{F} \). Consequently, all indices \( i \in \mathbb{N} \) that appeared in Section 2.2 should be interpreted as \( \frac{i}{n} \). The coefficients for the price evolution in the \( n \)-agent game, namely, \((\sigma^i, \sigma^*, \theta^i)_{i \in \{1, \ldots, n\}}\), which are now \((\sigma^\frac{i}{n}, \sigma^*, \theta^\frac{i}{n})_{i \in \{1, \ldots, n\}}\) after this re-branding, should obey the same conditions imposed upon \((\sigma^u, \sigma^u, \theta^u)_{u \in I}\), as stated in Section 2.3. To avoid unnecessarily complicated notations, we will keep the original indexing in the following sections. This re-branding will come up again in the proofs of the main convergence theorem.

The following is the main contribution of this work. It provides convergence of the heterogeneous \( n \)-player game to the graphon game.

**Theorem 2.11.** Let Condition 2.9 be satisfied, assume that \( \mathbb{E}[\frac{2p}{\sqrt{n}}|\xi^1|] < \infty \) for all \((i, n)\) and \( \xi^u \in L^2(\mu \otimes \mathbb{P}) \). Further assume that one of the following two conditions is satisfied:

(i) \( A^u = \mathbb{R}^d \) for all \( u \in I \) and \( \delta \) satisfies \( \delta < e^{-(2\|\theta\|_\infty+\frac{1}{2})T} (2\|\Sigma\|_\infty \vee \|\theta\|_\infty)^{-1} \).

(ii) \( \sigma^{*u} = 0 \) for all \( u \in I \).

If the \( n \)-agent problem (2.2) admits a Nash equilibrium \((\tilde{\pi}^i, n)_{i \in \{1, \ldots, n\}}\), then for each \( i \), the control \( \tilde{\pi}^i, n \) converges to \( \pi^* \) for some \( u \) and a graphon Nash equilibrium \((\hat{\pi}^u)_{u \in I} \) in the sense that, up to a subsequence,

\[
||\tilde{\pi}^i, n - \pi^*||^2 \xrightarrow{n \to \infty} 0 \quad \text{and} \quad |V^i, n((\pi^j, n)_{j \neq i}) - V^i, G((\hat{\pi}^u)_{u \in I})| \xrightarrow{n \to \infty} 0 \quad \mathbb{P} \otimes dt \text{ a.s.} \quad (2.5)
\]

This result will follow as a consequence of a general propagation of chaos result for (quadratic) FBSDEs in non–homogeneous interaction. These propagation of chaos results seem to be first of the kind, we devote Section 5 to these results.

Before going any further, let us present an example where the above result becomes easy in that propagation of chaos is not needed, at least granted our characterization results to come in Remark 3.3 and Corollary 3.5. This example deals with the case of a market with constant coefficients, and it will further motivate the analysis of random coefficients case done in this paper.

**Proposition 2.12.** Assume that for all \( u \in I \), \( A^u = \mathbb{R}^d \), \( \sigma^{*u} = 0 \) and \( \sigma^u, \mu^u \) are deterministic measurable functions of time. Let us consider a slight modification of the utility maximization problem (2.2): \( \lambda_{ii} \neq 0 \), i.e., agent \( i \) takes into account a weighted average of all agents’ terminal wealth as their benchmark. Under this modification, the utility maximization problem for agent \( i \) now reads

\[
V^i, n := V^i, n((\pi^j)_{j \neq i}) = \sup_{\pi \in \mathbb{R}^d} \mathbb{E}
\left[
-\exp\left\{-\frac{1}{\eta} \left(X_{T}^{i, n} - \frac{\rho}{n\beta_n} \sum_{j=1}^{n} \lambda_{ij} X_{T}^{j, \pi^j}ight)\right\}\right].
\]

Then, for all \( n \in \mathbb{N} \) there is an \( n \)-Nash equilibrium \((\tilde{\pi}^i, n)_{i \in \{1, \ldots, n\}}\) satisfying

\[
\sigma^i, t, n = \frac{n\beta_n}{n\beta_n - \rho\lambda_{ii}} \eta^t \forall (n, i) \in \mathbb{N}^* \times \{1, \ldots, n\} \text{ and a.s. } t,
\]
Furthermore, there is a graphon Nash equilibrium \((\tilde{\pi}^n_u)_{u \in \mathcal{I}}\) satisfying
\[
\sigma^u_t \tilde{\pi}^n_t = \eta^n \theta^n_t \quad \text{a.e } (u, t) \in I \times [0, T].
\]
In particular, \(\tilde{\pi}^{1,n}\) and \(\tilde{\pi}^u\) are deterministic and it holds
\[
\left\| \sigma^u_t \tilde{\pi}^{1,n} - \sigma^u_t \tilde{\pi}^u_t \right\| \leq \frac{\rho \lambda_{ii}}{n \beta n} \| \eta^i \theta^i \|_{\infty} \quad \forall (n, i) \in \mathbb{N}^* \times \{1, \ldots, n\} \text{ and a.s. } t. \quad (2.7)
\]

In addition to providing an easy way to prove convergence result, Proposition 2.12 is interesting in that it shows that in the present heterogeneous game, when the coefficients are constant, the Nash equilibrium (both in the finite and the graphon games) are constant as well, at least up to the randomness of the graph. This is in line with the homogeneous case studied by Lacker and Zariphopoulou [31] using PDE techniques and Espinosa and Touzi [17] using BSDE techniques.

### 2.4.3 Competition–indifferent capital

To conclude this section on the presentation of our main result, we use the rich literature on exponential utility maximization to assess the effect of competition on an individual investor. As said repeatedly, our results build on characterizations of the Nash and graphon equilibriums by system of (F)BSDEs, or of McKean–Vlasov type equations. And as is well–known in the numerical simulation literature, efficient simulation of the control process is much harder than that of the value process. One might wonder whether appropriately choosing the initial capital could make the investor indifferent between being concerned with the relative performance of their peers or not. That is, denoting\(^4\)
\[
J^{i,n}(\xi^i, F) := \sup_{\pi \in \mathcal{A}^i} \mathbb{E} \left[ - \exp \left( - \frac{1}{\eta^i} (X^{i,\pi}_T - F) \right) \right] \quad \text{where } X^{i,\pi}_0 = \xi^i,
\]
we would like to compute \(p^{i,n}\) such that
\[
J^{i,n}(\xi^i - p^{i,n}, 0) = J^{i,n}(\xi^i, \rho \sum_{j \neq i} \lambda_{ij}^{n} X^{j,i^{i^n}} T)
\]
(2.8)

where \((\tilde{\pi}^{i,n})_{i \in \{1, \ldots, n\}}\) is a Nash equilibrium. This is precisely the (spirit of the) utility indifference pricing of Hodges and Neuberger [23]. In the infinite population game, this indifference capital takes the form
\[
J^u(\xi^u - p^u, 0) = J^u(\xi^u, \rho \mathbb{E} \left[ \int_{X^{u,\pi}_T} X^{u,\pi}_T G(u, v) \ dF_T \right])
\]
(2.9)

with
\[
J^u(\xi^u, F) := \sup_{\pi \in \mathcal{A}^u} \mathbb{E} \left[ - \exp \left( - \frac{1}{\eta^u} (X^{u,\pi}_T - F) \right) \right] \quad \text{where } X^{u,\pi}_0 = \xi^u.
\]

We thus have the following corollary:

**Corollary 2.13.** Under the conditions of Theorem 3.1, the competition–indifferent capital \(p^{i,n}\) is given by
\[
p^{i,n} = \eta^i \log \left( \frac{\gamma^{i,n}}{\gamma_0} \right)
\]
where \(\gamma^{i,n}\) is the value process of the system (3.2) and \((\gamma, \zeta, \zeta^u)\) solves the BSDE
\[
\gamma_t = \int_t^T \left( - \left( \left( \zeta^u_s \right)^2 + \frac{1}{\eta^u} \left( I - P^u_s \right) \left( \frac{\zeta^u_s}{\zeta^u_{ss}} \right)^2 \right) \right) \ ds - \int_t^T \zeta_s \cdot dW_s - \int_t^T \zeta_s dW^*_s
\]

Moreover, if the conditions of Theorem 2.11 are satisfied, then we have
\[
|p^{i,n} - p^{i\pi}| \xrightarrow{n \to \infty} 0
\]
where \(p^u\) is the competition–indifferent capital of player \(u\) in the graphon game.

---

\(^4\)In the definition of \(J\), when \(F = 0\) we take the elements of \(\mathcal{A}^i\) to be \(F^i\)–progressive, since in this case the agent is not concerned with the performance (and thus investments) of other market participants.
The gist here is that $p^i \pi_n$ is given in terms of the value process of a system of BSDEs, so that an investor starting with capital $\xi^i - p^i \pi_n$ (only) needs to simulate the control process of a one-dimensional BSDE in order to compute the optimal trading strategy.

**Proof of Corollary 2.13.** The proof starts with the general duality result of Delbaen, Grandits, Rheinländer, Samperi, Schweizer, and Stricker [13] which asserts that

$$
\sup_{\pi \in \mathcal{A}^i} \mathbb{E} \left[ - \exp \left( - \frac{1}{\eta^i} \left( X_T^{i,\pi} - \rho \sum_{j \neq i}^n \lambda_{i,j}^n X_T^{j,\tilde{\pi}_i} \right) \right) \right] = - \exp \left( \frac{1}{\eta^i} \sup_{Q \in \mathcal{Q}} \left( \mathbb{E}_Q \left[ \rho \sum_{j \neq i}^n \lambda_{i,j}^n X_T^{j,\tilde{\pi}_i} \right] - \xi^i - \eta^i H(Q/\mathbb{P}) \right) \right)
$$

where $H(Q/\mathbb{P})$ is the relative entropy given by

$$
H(Q/\mathbb{P}) := \begin{cases} 
\mathbb{E}_Q \left[ \log \left( \log \frac{dQ}{d\mathbb{P}} \right) \right] & \text{if } Q \ll \mathbb{P} \\
+\infty & \text{else}
\end{cases}
$$

and $\mathcal{Q}$ is the set of probability measures $Q$ that are absolutely continuous with respect to $\mathbb{P}$, such that the stock price processes are $\mathcal{Q}$–local martingales and $H(Q/\mathbb{P}) < \infty$. Applying this result to both sides of Equation (2.8) yields

$$
p^i \pi_n = \sup_{Q \in \mathcal{Q}} \left( \mathbb{E}_Q \left[ \rho \sum_{j \neq i}^n \lambda_{i,j}^n X_T^{j,\tilde{\pi}_i} \right] - \xi^i - \eta^i H(Q/\mathbb{P}) \right) - \sup_{Q \in \mathcal{Q}} \left( - \xi^i - \eta^i H(Q/\mathbb{P}) \right)
$$

where the latter equality follows by Theorem 3.1 and [25, Theorem 7]. The above argument also shows that $p^i = \eta^i \log(\gamma^i_\alpha / \gamma^i_0)$ where $Y^u$ satisfies Equation (3.7) and $(\gamma^i, \zeta^i, \zeta^\alpha)$ solves

$$
\gamma^i_t = \int_t^T \left( \left( \frac{\zeta^i_s}{\zeta^\alpha_s} \right) \cdot \theta^u_s - \frac{\eta^u_s}{2} \left| \theta^u_s \right|^2 + \frac{1}{2\eta^u_s} \left( \left( \frac{\zeta^i_s}{\zeta^\alpha_s} + \eta^u_s \theta^u_s \right) \right)^2 \right) ds - \int_t^T \zeta^i_s \cdot dW^u_s - \int_t^T \zeta^\alpha_s \cdot dW^u_s.
$$

The convergence statement therefore follows from Theorem 2.11.

The rest of the paper is dedicated to the proofs of the convergence and existence results.

### 3 Characterizations of the utility maximization games

This section provides characterizations of the Nash equilibria of the two games presented above in terms of solutions of backward SDEs. These characterizations will play a key role in the proofs of our main results.

#### 3.1 FBSDE characterization of the $n$-agent problem

The following theorem provides a FBSDE characterization for the $n$-agent utility maximization problem (2.2). In particular, it expresses the Nash equilibrium and the associated utilities as functions of solutions to a system of (quadratic) FBSDEs. This is an extension of the main result of Espinosa and Touzi [17] to the case where both common noise and idiosyncratic noise is considered. In the statement below and throughout the paper, we denote by $P^i(\zeta)$ the projection of a vector $\zeta$ onto the constraint set $\Sigma^i A^i$. Also recall the notation $X^i_T := \sum_{j \neq i}^n \lambda_{i,j}^n X^j_T$.

**Theorem 3.1.** Assume that $\mathbb{E} \left[ e^{\frac{2}{\eta^i_n} |\tilde{\pi}^i|^2} \right] < \infty$. If the $n$–player game admits a Nash equilibrium $(\tilde{\pi}^i \pi_n)_{i \in \{1, \ldots, n\}}$, then it holds

$$
\tilde{\pi}^i \pi_n = \left( \Sigma^i \Sigma^i_T \right)^{-1} \Sigma^i P^i \left( \begin{pmatrix} \tilde{\pi}^i \pi_n \\ \eta^i \theta^i \end{pmatrix} \right) dt \otimes \mathbb{P} \text{–a.s. and } V_{0}^n((\tilde{\pi}^j \pi_n)_{j \neq i}) = -e^{-\frac{1}{\eta^i_n} |\tilde{\pi}^i|^2} \forall i \in \{1, \ldots, n\}
$$

\[ (3.1) \]
with \((X^i, \gamma^i, \zeta^i, \zeta^*) \in \mathcal{S}^1(\mathbb{R}, \mathbb{F}^n) \times \mathcal{S}^1(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2_{\text{loc}}(\mathbb{R}^d, \mathbb{F}^n) \times \mathbb{H}^2_{\text{loc}}(\mathbb{R}, \mathbb{F}^n)\) for all \((i, j) \in \{1, \ldots, n\}^2\) solving the FBSDE

\[
d\gamma^i_t = \left( \left( \frac{c^{ii}_i}{\sigma^i_i^2} \right) \cdot \theta^i_t + \frac{\eta^i_t}{2} \theta^i_t \right)^2 - \frac{1}{2\eta^i} \sum_{j \neq i} \frac{1}{|\zeta^j|^2} \gamma^j_t \left( I - P^i_t \right) \left( \left( \frac{c^{ii}_i}{\sigma^i_i^2} \right) + \eta^i_t \theta^i_t \right)^2 dt + \sum_{j=1}^n \zeta^j \cdot dW^j_t + \zeta^i_t dW^*_t, \quad \mathbb{P}\text{-a.s.}
\]

\[
\gamma^i_T = \rho \tilde{\xi} - \xi^i
\]

\[
dX^i_t = \tilde{\pi}^i_t \cdot \left( \left( \frac{\sigma^i_i}{\sigma^i_i^2} \right) \theta^i_t dt + \sigma^i_t dW^i_t + \sigma^i_t dW^*_t \right), \quad X^i_0 = \xi^i.
\]

The reader might wonder why our characterizing equation is a multidimensional coupled FBSDE in contrast to BSDEs usually derived in the literature, see for instance [17, 19]. We can achieve a characterization by a BSDE to BSDE usually derived in the literature, see for instance [17, 19]. We can achieve a characterization by a BSDE (3.4) given in the next corollary.

**Corollary 3.2.** Assume that

\[
\sum_{j \neq i} \lambda^i_{ij} \in [0, 1]. \tag{3.3}
\]

If the \(n\)-player game admits a Nash equilibrium \((\tilde{\pi}^i, \pi^i)_i \in \{1, \ldots, n\}\), then it holds

\[
\tilde{\pi}^i_t \cdot \psi_t (\tilde{\xi}^i) = -e^{-\frac{1}{2}(\zeta^i - \rho^i \tilde{\xi} - \xi^i)}
\]

with \((Y^i, Z^i, \psi^i) \in \mathcal{S}^1(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2_{\text{loc}}(\mathbb{R}, \mathbb{F}^n) \times \mathbb{H}^2_{\text{loc}}(\mathbb{R}, \mathbb{F}^n)\) for all \((i, j) \in \{1, \ldots, n\}^2\) solving the following \(n\)-dimensional BSDEs:

\[
Y^i_t = \int_t^T \left( -\frac{\eta^i_t}{2} |\theta^i_t|^2 + \frac{1}{2\eta^i} \gamma^j_t \left( I - P^i_t \right) \left( \left( \frac{c^{ii}_i}{\sigma^i_i^2} \right) + \eta^i_t \theta^i_t \right)^2 \right) ds \quad \mathbb{P}\text{-a.s.}, t \in [0, T].
\]

where for every fixed \(t \in [0, T]\), \(\psi_t \equiv \psi(\cdot, \cdot)\) is the inverse of the mapping \(\Phi(\cdot, \cdot) : \mathbb{R}^n \to \mathbb{R}^n\) given by

\[
\Phi_t^i(\zeta^i, \zeta^*) = \zeta^i + \sum_{j \neq i} \rho^i \lambda^i_{ij} \sigma^j \cdot (\Sigma^j_i)^{-1} \rho^i P^j_t \left( \left( \frac{c^{jj}_j}{\sigma^j_j^2} \right) + \eta^j_t \theta^j_t \right) \quad \text{for all } (\zeta^i, \zeta^*) \in \mathbb{R}^{nd} \times \mathbb{R}^n,
\]

where with abuse of notation, \(\Phi_t^i\) maps from \(\mathbb{R}^n\) to \(\mathbb{R}^n\) up to fixing a single trajectory of \((\Sigma^j_if_t)_{j \in \{1, \ldots, n\}}\) and \((\theta^i_j)_{j \in \{1, \ldots, n\}}\). Furthermore, for \(n \geq 3\), \(\psi_t\) is Lipschitz–continuous with a constant depending on \(n\).

Observe that the dimension of the domain of the function \(\psi\) depends on \(n\). Thus, \(\psi_t\) will undoubtedly present a major obstacle when studying the limit of the game as \(n \to \infty\). For instance, in the infinite population game this decoupling procedure does not seem to work. Furthermore, the condition 3.3 will also present an obstacle to the fact that we would like to consider the limit of the game on a relatively sparse graph. To avoid the above difficulties while studying the limit, we will rather work with the FBSDE (3.2).

**Remark 3.3.** In the absence of the common noise \(W^*\) (i.e. when \(\sigma^{**} = 0\) for all \(u \in I\)), the complications associated with \(\psi_t\) discussed above vanish. In fact, the system of BSDEs in Corollary 3.2 takes the much simpler form

\[
Y^i_t = \int_t^T \left( -\frac{\eta^i_t}{2} |\theta^i_t|^2 - Z^i_t \cdot \theta^i_t + \frac{1}{2\eta^i} \gamma^j_t \left( I - P^i_t \right) \left( Z^i_t + \eta^i_t \theta^i_t \right)^2 \right) ds \quad \mathbb{P}\text{-a.s.}, t \in [0, T]
\]

and the equilibrium strategy now takes the form

\[
\tilde{\pi}^i_t = (\sigma^i)^{-1} P^i_t (Z^i_t + \eta^i_t \theta^i_t), \quad \mathbb{P} \otimes \text{dt–a.s.} \tag{3.6}
\]
3.2 FBSDE characterization of the graphon problem

Similar to the n-player game just discussed, we will also derive (F)BSDE characterizations of the graphon game. This time, the characterization obtained is with respect to a system of (infinitely many) McKean–Vlasov (F)BSDEs. We will call these equations graphon (F)BSDEs to stress the fact that the dependence between the equations occurs through the graphon G. As above, we use the notation $P^u_t(\zeta)$ for the projection of a vector $\zeta$ onto the constraint set $\Sigma_t^u A^u$.

**Proposition 3.4.** Assume that $\xi^u_0 \in L^2(\mu \otimes P)$, that the following graphon FBSDE admits a solution $(X^u, Y^u, Z^u, \pi^u) \in S^2(\mathbb{R}, \mathbb{F}^u) \times S^2(\mathbb{R}, \mathbb{F}^u) \times H^2(\mathbb{R}^d, \mathbb{F}^u) \times H^2(\mathbb{R}, \mathbb{F}^u)$ such that $(u, t, \omega) \mapsto X^u_t$ is measurable:

$$
\begin{align*}
\frac{dX^u_t}{dt} &= \pi^u_t \cdot \{\Sigma^u_t \theta^u_t dt + \sigma^u_t dW^u_t + \sigma^*_u dW^*_t\} \\
\frac{dY^u_t}{dt} &= \left(\frac{Z^u_t}{Z^*_u} \cdot \theta^u_t + \frac{\sigma^*_u}{2} |\theta^u_t|^2 - \frac{1}{2\eta^u} \left( I - P^u_t \right) \left( \frac{Z^u_t}{Z^*_u} + \eta^u \theta^u_t \right) \right)^2 dt + Z^u_t \cdot dW^u_t + Z^*_u dW^*_t \quad \mu \otimes \mathbb{P} \text{-a.s.} \\
Y^u_T &= \mathbb{E} \left[ \int_{I} \rho(X^u_T - \xi^u) G(u, v) dv \right] \\
X^u_0 &= \xi^u, \quad \pi^u_T = \left( \Sigma^u_T \Sigma^u_T^\top \right)^{-1} \Sigma^u_T P^u_t \left( \frac{Z^u_t}{Z^*_u} + \eta^u \theta^u_t \right) \frac{dt}{\mu \otimes \mathbb{P} \text{-a.s.}} \quad (3.7)
\end{align*}
$$

Then, the graphon game described in (2.4) admits a graphon Nash equilibrium $(\pi^u_{t})_{u \in I}$ such that for almost every $u \in I$ it holds

$$
V^u_{0, G} = - \exp \left( - \frac{1}{\eta^u} \left( \xi^u - \int_{I} \rho \mathbb{E} [\xi^v] G(u, v) dv - Y^u_0 \right) \right) \quad (3.8)
$$

and

$$
\pi^u_t = \left( \Sigma^u_t \Sigma^u_t^\top \right)^{-1} \Sigma^u_t P^u_t \left( \frac{Z^u_t}{Z^*_u} + \eta^u \theta^u_t \right) \frac{dt}{\mu \otimes \mathbb{P} \text{-a.s.}} \quad (3.9)
$$

The above result characterizes the graphon game with common noise in the sense that solvability of the game reduces to solvability of the system (3.7). Moreover, the value function as well as the equilibrium strategies in the infinite population game are given explicitly in terms of solutions of (3.7). In the case where there is no common noise, i.e. $\sigma^*_u = 0$ for almost all $u \in I$, the above result simplifies as follows:

**Corollary 3.5.** Assume that the graphon BSDE

$$
\begin{align*}
\frac{dY^u_t}{dt} &= \left( \frac{\eta^u}{2} |\theta^u_t|^2 + Z^u_t \cdot \theta^u_t - \frac{1}{2\eta^u} \left( I - P^u_t \right) (Z^u_t + \eta^u \theta^u_t)^2 \right) dt \\
&\quad + Z^u_t \cdot dW^u_t, \quad \mu \otimes \mathbb{P} \text{-a.s.,} \quad t \in [0, T], \quad Y^u_T = 0
\end{align*}
$$

admits a solution $(Y^u, Z^u)_{u \in I}$ such that $(u, t, \omega) \mapsto Z^u_t$ is measurable and $(Y^u, Z^u) \in S^2(\mathbb{R}, \mathbb{F}^u) \times H^2(\mathbb{R}^d, \mathbb{F}^u)$ for almost every $u \in I$. Then the graphon game described in Equation (2.4) admits a graphon Nash equilibrium $(\pi^u_{t})_{u \in I}$ such that for almost every $u \in I$ it holds

$$
\pi^u_t = (\sigma^*_u)^{-1} P^u_t (Z^u_t + \eta^u \theta^u_t) \frac{dt}{\mu \otimes \mathbb{P} \text{-a.s.}} \quad \text{and} \quad V^u_{0, G} = - \exp \left( - \frac{1}{\eta^u} \left( \xi^u - \int_{I} \mathbb{E} [\rho \xi^v] G(u, v) dv - Y^u_0 \right) \right) \quad (3.11)
$$

4 Proofs of existence and characterization results

The proof of Theorem 2.11 will be based on general propagation of chaos results that will be given in Section 5, and the existence Theorem 2.7 is a consequence of existence of graphon BSDEs discussed in the final section of the paper where we present existence results for graphon (F)BSDEs.
4.1 Proof of the existence Theorem 2.7

We will distinguish two cases: The case with common noise and the case without.

(i) Case with common noise: In this case, when $A^u = \mathbb{R}^d$ for all $u$, the FBSDE (3.7) becomes

\[
\begin{align*}
\frac{dX^u}{dt} &= b^u(t, Z^u_t, Z^u_t)dt + h^u_1(t, Z^u_t, Z^u_t)dW^u_t + h^u_2(t, Z^u_t, Z^u_t)dW^v_t, \\
\frac{dY^u}{dt} &= -g^u(t, Z^u_t, Z^u_t)dt + Z^u_t \cdot dW^u_t + Z^u_t \cdot dW^v_t, \\
Y^u_T &= \mathbb{E}\left[ \int_t^T \rho(X^u_T - \xi^u)G(u, v)dv \right], \quad X^u_0 = \xi^u,
\end{align*}
\]

with

\[
\begin{align*}
b^u(t, z, z^*) &= \left( \Sigma^u \Sigma^u \right)^{-1} \Sigma^u \left( \begin{array}{c} z \\ \eta^u \theta_t \end{array} \right) \Sigma^u \theta_t, \\
h^u_1(t, z, z^*) &= \left( \Sigma^u \Sigma^u \right)^{-1} \Sigma^u \left( \begin{array}{c} z \\ \eta^u \theta_t \end{array} \right) \cdot \sigma_t, \\
h^u_2(t, z, z^*) &= \left( \Sigma^u \Sigma^u \right)^{-1} \Sigma^u \left( \begin{array}{c} z \\ \eta^u \theta_t \end{array} \right) \cdot \sigma_t.
\end{align*}
\]

In particular, given that the processes $\Sigma^u, \mu^u$ are bounded, the coefficients of this equation satisfy the conditions of Proposition 6.1. Thus, it follows that Equation (3.7) admits a unique square integrable solution. Therefore, the result follows from Proposition 3.4.

(ii) Case without common noise: When $\sigma^* = 0$, the proof is similar. In fact, it follows by Proposition 6.2 that the graphon BSDE (3.10) admits a unique solution such that $(Y^u, Z^u) \in \mathbb{S}_\infty(\mathbb{P}^u, \mathbb{R}^d) \times \mathbb{H}^1_{BMO}(\mathbb{P}^u, \mathbb{R}^d)$ for almost every $u \in I$ with $(u, t, \omega) \mapsto Z^u_t$ measurable and $\sup_u \|Z^u\|_{\mathbb{H}^1(\mathbb{P}^u, \mathbb{R}^d)} < \infty$. Then, the result follows by Corollary 3.5.

4.1.1 Proof of Proposition 2.12

Under the same assumptions given, the systems of FBSDEs (3.2) characterizing the $n$-agent optimization problem simplify to the following

\[
\begin{align*}
d\gamma^i_t &= \left( \zeta^{ij} \cdot \theta^i + \frac{n}{2} |\theta^i|^2 - \frac{1}{2n} \sum_{j \neq i} n_i |c_{ij}^t|^2 \right) dt + \sum_{j=1}^n c_{ij}^t \cdot dW^j_t, \quad \mathbb{P}\text{-a.s., } t \in [0, T] \\
\gamma^i_T &= \rho(X^i_T - \bar{\xi}) = \rho \sum_{j=1}^n \lambda^i_{ij} \int_0^T \pi^{i,n} \cdot \sigma_t^j \theta^j_t dt + dW^j_t, \\
dX^i_t &= \pi^{i,n} \cdot \sigma_t^j \{ \theta^j_t dt + dW^j_t \}, \quad X^i_0 = \xi^i,
\end{align*}
\]

with the equilibrium strategies given by

\[
\sigma^{i,i,n} = \zeta^{ij} + \eta^i \theta^i, \quad \mathbb{P} \otimes dt\text{-a.s.}
\]

Let $Y^i_t = \gamma^i_t - \rho \sum_{j=1}^n \lambda^i_{ij} \int_0^t \pi^{i,n} \cdot \sigma_t^j \theta^j_s ds + dW^j_s$ (recall that $\lambda^i_{ij} = \lambda_{ij}/n_i \beta_n$). Then we have $Y^i_T = 0$ and we can re-write the FBSDEs (4.2) as

\[
\begin{align*}
Y^i_t &= \int_t^T \left( - \zeta^{ij} \cdot \theta^j_t - \frac{n}{2} |\theta^j|^2 + \frac{1}{2n} \sum_{j \neq i} n_i |c_{ij}^t|^2 + \rho \sum_{j=1}^n \lambda^i_{ij} (\zeta^{jj}_s + \eta^j \theta^j_s) \theta^j_s \right) ds - \sum_{j=1}^n \int_t^T \left( c_{ij}^s - \rho \lambda^i_{ij} (\zeta^{jj}_s + \eta^j \theta^j_s) \right) dW^j_s.
\end{align*}
\]

Observe that choosing

\[
\begin{align*}
\zeta^{ij}_t &= \frac{\rho \lambda_{ij}}{1 - \rho \lambda_{ij}} \eta^j \theta^j_t, \\
\zeta^{ii}_t &= \frac{\rho \lambda_{ii}}{1 - \rho \lambda_{ii}} \eta^i \theta^i_t
\end{align*}
\]

make the stochastic integral in the above BSDE vanish, leaving $Y^i_t$ a deterministic process. Thus

\[
\begin{align*}
Y^i_t &= \int_t^T \left( - \zeta^{ii}_s \cdot \theta^i_t - \frac{n}{2} |\theta^i|^2 + \frac{1}{2n} \sum_{j \neq i} n_i |c_{ij}^s|^2 + \sum_{j=1}^n \lambda^i_{ij} (\zeta^{jj}_s + \eta^j \theta^j_s) \theta^j_s \right) ds, \\
\zeta^{ii}_t &= \frac{\rho \lambda_{ii}}{1 - \rho \lambda_{ii}} \eta^i \theta^i_t, \quad \zeta^{jj}_t = \frac{\rho \lambda_{jj}}{1 - \rho \lambda_{jj}} \eta^j \theta^j_t \text{ for } i \neq j.
\end{align*}
\]
is a solution to the above BSDE.

Similarly, the BSDE (3.10) characterizing the graphon game simplifies to

$$Y_t^u = \int_t^T \left(-\frac{\eta u}{2} |\theta_s|^2 - Z_t^u \cdot \theta_s^u + \mathbb{E} \left[ \rho \int_j^T (Z_s^u + \eta u \theta_s^u) \cdot \theta_s^u G(u, v) dv \right] \right) ds - \int_t^T Z_s^u \cdot dW_s^u,$$

with the equilibrium strategy given by

$$\sigma_t^u \pi_t^u = Z_t^u + \eta u \theta_t^u.$$

Using a change of measure argument, we can rewrite (4.3) as

$$Y_t^i = \int_t^T \left(-\frac{\eta u}{2} |\theta_s|^2 + \mathbb{E} \left[ \rho \int_j^T (Z_s^u + \eta u \theta_s^u) \cdot \theta_s^u G(u, v) dv \right] \right) ds - \int_t^T Z_s^u \cdot (dW_s^u + \theta_s^u ds),$$

where $W_s^u, Q$ is a standard Brownian motion under a new measure $Q$ such that $\frac{dQ}{d\mathbb{P}} = e^{\int_0^t -\eta u - \frac{\eta u}{2} |\theta_s|^2 ds}$. Noting that $Z_t^u \in \mathbb{H}^2(\mathbb{R}^d, \mathbb{F}^u)$, taking conditional expectation with respect to $\mathbb{F}_t^u$ on both sides, we can now conclude that

$$\begin{cases} Y_t^i = \int_t^T \left(-\frac{\eta u}{2} |\theta_s|^2 + \mathbb{E} \left[ \rho \int_j^T |\theta_s|^2 G(u, v) dv \right] \right) ds, \\
Z_t^u = 0, \quad \sigma_t^u \pi_t^u = \eta u \theta_t^u \end{cases}$$

is a solution to the BSDE (4.3). The convergence results (2.7) thus follows from the boundedness of $\eta$ and $\theta_t$.

### 4.2 Proofs for Section 3.1

We now present the proof of the characterization result for the $n$-player game. This section consists of the proof for Theorem 3.1 and two auxiliary lemmas: Lemma 4.1 and Lemma 4.2.

**Proof of Theorem 3.1.** Assume that $(\tilde{\pi}^i)_{i \in \{1, \ldots, n\}}$ is a Nash equilibrium of the problem (2.2). First note that our assumptions on $\sigma_t^i$, $\sigma_t^{i_i}$, and $\mu_t^i$ imply that $\tilde{X}_t^i \in L^2(\mathbb{R}, \mathbb{F}_t^i)$. Let $\mathcal{T}$ be the set of all $\mathbb{F}^n$-stopping times in $[0, T]$. Define the following family of random variables:

$$\mathcal{J}^{i, \pi}(\tau) := \mathbb{E} \left[ -e^{-\frac{\rho}{\eta} \int_0^\tau \pi_s \cdot (\Sigma_s^i \theta_s^i ds + \sigma_s^i dW_s^i + \sigma_s^* dW_s^*) - \rho(\tilde{X}_\tau^i - \tilde{X}_0^i) \right] | \mathcal{F}_\tau^i,$$

and let

$$\mathcal{V}^i(\tau) := \operatorname{ess sup}_{\pi \in \mathcal{A}^i} \mathcal{J}^{i, \pi}(\tau) \text{ for all } \tau \in \mathcal{T}, \text{ so that } \mathcal{V}^i(0) = e^{\frac{\rho}{\eta} \epsilon i - \rho \tilde{X}_0^i} \mathcal{V}^{i, n}(\pi^i) \text{ for } \pi^i \neq i.$$

Now let

$$\beta_t^{i, \pi} := e^{\frac{\rho}{\eta} \int_0^t \pi_s \cdot (\Sigma_s^i \theta_s^i ds + \sigma_s^i dW_s^i + \sigma_s^* dW_s^*)}.$$

Then it can be checked as in the proof of [17, Lemma 4.13] that for all $\pi \in \mathcal{A}_i$,

$$\beta_t^{i, \pi} \mathcal{V}^i(\tau_1) \geq \mathbb{E} \left[ \beta_t^{i, \pi} \mathcal{V}^i(\tau_2) | \mathcal{F}_{\tau_2}^i \right] \text{ for all stopping times } \tau_1 \leq \tau_2, \text{ (4.4)}$$

and by [27, Proposition I.3.14], the process $\mathcal{V}^i$ has a càdlàg modification again denoted $(\mathcal{V}^i_t)_{t \in [0, T]}$. Moreover, this process also satisfies (4.4), so that for any $\pi \in \mathcal{A}_i$, the process $\beta_t^{i, \pi} \mathcal{V}^i$ is a $\mathbb{P}$-supermartingale. Now the definition of Nash equilibrium implies that $\tilde{\pi}^i$ is optimal for agent $i$. In other words,

$$\mathcal{V}^i_0 = \sup_{\pi \in \mathcal{A}_i} \mathbb{E} \left[ -e^{-\frac{\rho}{\eta} \int_0^T (X_t^i - \xi^i - \rho(\tilde{X}_T^i - \tilde{X}_0^i))} \right] = \mathbb{E} \left[ -e^{-\frac{\rho}{\eta} \int_0^T (X_t^i - \xi^i - \rho(\tilde{X}_T^i - \tilde{X}_0^i))} \right].$$

The above implies that $\beta_t^{i, \pi} \mathcal{V}^i$ is a $\mathbb{P}$-martingale, where we write $\beta_t^{i, \pi}$ as a shorthand notation for $\beta_t^{i, \tilde{\pi}^i}$. Denote $\check{M}^i := \beta_t^{i, \pi} \mathcal{V}^i$. We now proceed to show that the adapted and continuous process

$$\gamma_t^i = X_t^{\tilde{\pi}^i} - \xi^i + \eta^i \ln(-\check{M}_t^i), \quad t \in [0, T]$$

(4.5)
solves a BSDE. Note already that by definition of $V_t^i$ and $\tilde{M}_t^i$, we have

$$V_{0,n}^i((\tilde{x}_{j,n})_{j\neq i}) = -e^{-\frac{1}{\eta}(\xi^i - \tilde{\xi}^i - \gamma^i)}.$$  

This proves the representation of $V_{0,n}^i((\tilde{x}_{j,n})_{j\neq i})$.

We first need to check that $\gamma^i$ is indeed in $S^1(\mathbb{R}, \mathbb{F}^n)$. On the one hand, using Jensen’s inequality, we have

$$\frac{1}{\eta} \mathbb{E}[X_T^i - \xi^i - \rho(X_T^i - \tilde{\xi}^i)|\mathcal{F}_t^n] \leq \ln(-\tilde{M}_t^i). \quad (4.6)$$

On the other hand, by definition of $V_t^i$, we have $-\tilde{M}_t^i = -\beta_t^{i,\pi}V_t^i \leq \beta_t^{i,\pi}\mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]$. Thus, using the inequality $\ln(x) \leq x$ we have

$$\ln(-\tilde{M}_t^i) \leq \ln(\beta_t^{i,\pi}) + \ln\left(\mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]\right) \leq \frac{1}{\eta}\int_0^t \tilde{\pi}^{i,n} \cdot (\Sigma_u \theta_u^i du + \sigma_u^i dW_u^i + \sigma_u^{i*}dW_u^{i*}) + \mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]. \quad (4.7)$$

Now, combining this with Equation (4.6) and the definition of $\beta_t^{i,\pi}$, we obtain

$$\mathbb{E}\left[\sup_{t \in [0,T]} \ln(-\tilde{M}_t^i)\right] \leq \mathbb{E}\left[\sup_{t \in [0,T]} \frac{1}{\eta} \mathbb{E}[||X_T^i - \xi^i - \rho(X_T^i - \tilde{\xi}^i)|||\mathcal{F}_t^n]\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]\right] \leq \frac{1}{\eta} \mathbb{E}\left[\sup_{t \in [0,T]} \mathbb{E}[||X_T^i|||\mathcal{F}_t^n]\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \|X_T^i\|\right] + 2\mathbb{E}[\xi^i]$$

$$+ \mathbb{E}\left[\sup_{t \in [0,T]}\mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]\right].$$

It is then sufficient to bound the last term. By Jensen’s and Hölder’s inequalities we have

$$\mathbb{E}\left[\sup_{t \in [0,T]} \mathbb{E}[e^{\frac{1}{\eta}(X_T^i - \tilde{\xi}^i)}|\mathcal{F}_t^n]\right] \leq \frac{1}{n-1} \sum_{j \neq i} \mathbb{E}\left[\mathbb{E}[e^{\frac{1}{\eta}(X_T^j - \tilde{\xi}^j)}]|\mathcal{F}_t^n]\right] \leq \frac{1}{n-1} \sum_{j \neq i} \mathbb{E}\left[\mathbb{E}[e^{\frac{1}{\eta}(X_T^j - \tilde{\xi}^j)}]|\mathcal{F}_t^n]\right].$$

By the admissibility condition on $\pi$, it follows that $\ln(-\tilde{M}_t^i) \in S^1(\mathbb{R}, \mathbb{F}^n)$. It thus follows that $\gamma^i \in S^1(\mathbb{R}, \mathbb{F}^n)$ for every $i \in \{1, \ldots, n\}$. For an arbitrary $\pi \in \mathcal{A}_i$, define

$$M_t^{i,\pi} := e^{-\frac{1}{\eta}(X_T^i - \tilde{\xi}^i - \gamma^i)} = \tilde{M}_t e^{-\frac{1}{\eta}(X_T^i - X_T^{i,n})}.$$  

It follows from the same argument as in [17, Theorem 4.7 2(b)] that $M_t^{i,\pi}$ is a supermartingale. Now by Equation (4.5), Doob–Meyer decomposition and Itô’s formula, there is $(\zeta^i, \zeta^{i*}) \in \mathbb{H}_{loc}^2(\mathbb{R}^{nd}, \mathbb{F}) \times \mathbb{H}_{loc}^2(\mathbb{R}^d, \mathbb{F})$ such that

$$d\gamma^i_t = -b^i_t dt + \sum_{j=1}^n \zeta_t^{ij} \cdot dW_j^i + \zeta_t^{i*}dW_t^{i*}.$$  

We will proceed by first computing $b^i, \gamma^i$ and $\tilde{\pi}^i$, and next deriving the BSDEs satisfied by $(\gamma^i, \zeta^i, \zeta^{i*})$.

By Itô’s formula, we have

$$-de^{-\frac{1}{\eta}(X_T^i - \tilde{\xi}^i - \gamma^i)} = e^{-\frac{1}{\eta}(X_T^i - \tilde{\xi}^i - \gamma^i)} \cdot \left\{ \frac{1}{\eta^2} \left( \sigma_t^{i,\pi_t^i} \cdot dW_t^i + \sum_{j=1}^n \sigma_t^{ij} \cdot dW_t^j + (\sigma_t^{i*} \cdot \pi_t^i + \zeta_t^{i*}) dW_t^{i*} \right) + \frac{1}{\eta}(b_t^i + \pi_t^i \cdot \Sigma_t^i \theta_t^i) dt + \frac{1}{\eta^2} \left( \sigma_t^{i,\pi_t^i} \cdot \zeta_t^i + \sigma_t^{i*} \pi_t^i \zeta_t^{i*} \right) dt \right\} - \frac{1}{2}\left( \sigma_t^{i,\pi_t^i} |^2 + |\sigma_t^{i*} \pi_t^i|^2 + \sum_{j=1}^n |\zeta_t^{ij}|^2 + |\zeta_t^{i*}|^2 \right) dt.$$  

(4.8)
Using the supermartingale property of $\Gamma^i$, the martingale property of $\tilde{\Gamma}^i$, together with (4.8), keeping in mind that $\Sigma^i := (\sigma^i, \sigma^i_\pi)$, and writing $\zeta^i := (\tilde{\zeta}^i, \zeta^i)$, we get

$$b^i_t \leq \frac{1}{2\eta^i} |\Sigma^i_{\pi}^T \pi^i_t - (\zeta^i_t + \eta^i \theta^i_t)|^2 + \frac{1}{2\eta^i} \sum_{j \neq i} n |\zeta^{ij}_t|^2 - \frac{\eta^i}{2} |\theta^i_t|^2 - \zeta^i_t \cdot \theta^i_t, \quad (4.9)$$

and

$$b^i_t = \frac{1}{2\eta^i} |\Sigma^i_{\pi}^T \pi^i_t - (\zeta^i_t + \eta^i \theta^i_t)|^2 + \frac{1}{2\eta^i} \sum_{j \neq i} n |\zeta^{ij}_t|^2 - \frac{\eta^i}{2} |\theta^i_t|^2 - \zeta^i_t \cdot \theta^i_t.$$ 

Thus, $\pi^i_{\pi}^{\pi^n}$ minimizes the function (in $\pi^i$) on the right hand side of (4.9). Therefore, we can express $\pi^i_{\pi}^{\pi^n}$ and $b^i_t$ as follow:

$$\pi^i_{\pi} = \left(\Sigma^i_{\pi} \Sigma^i_{\pi}^T\right)^{-1} \Sigma^i_{\pi} P^i_t (\zeta^i_t + \eta^i \theta^i_t), \quad (4.10)$$

$$b^i_t = \frac{1}{2\eta^i} \text{dist} (\zeta^i_t + \eta^i \theta^i_t, \Sigma^i_{\pi} A^i)^2 + \frac{1}{2\eta^i} \sum_{j \neq i} n |\zeta^{ij}_t|^2 - \frac{\eta^i}{2} |\theta^i_t|^2 - \zeta^i_t \cdot \theta^i_t.$$ 

Therefore, $(\gamma^i, \zeta^i, \zeta^i)$ is a mapping from $\mathbb{R} \times \mathbb{E}^2_\omega \times \mathbb{E}^2_\omega \times \mathbb{E}^2_\omega$ solves the BSDE

$$d\gamma^i_t = \left(\zeta^i_t \cdot \theta^i_t + \eta^i |\theta^i_t|^2 - \frac{\eta^i}{2} \sum_{j \neq i} n |\zeta^{ij}_t|^2 - \frac{1}{2\eta^i} (I - P^i_t) (\zeta^i_t + \eta^i \theta^i_t) \right)^2 dt$$

$$+ \sum_{j \neq i} \zeta^{ij}_t \cdot dW^j_t + \zeta^i_t dW^i_t,$$

$$(4.11)$$

$$\gamma^i_T = \rho (X^i_T - \zeta^i_T) = \rho \sum_{j \neq i} \lambda^i_{ij} \int_0^T \pi^i_{\pi} \cdot \left\{ \Sigma^i_{\pi} \theta^i ds + \sigma^i_{\pi} dW^i_s + \sigma^i_{\pi}^* dW^i_s \right\}.$$

**Proof.** (of Corollary 3.2) The proof of this corollary builds upon that of Theorem 3.1, with exactly the same notation. Define the process

$$Y^i_t := \gamma^i_t - \sum_{j \neq i} \rho \lambda^i_{ij} \int_0^T \pi^i_{\pi} \cdot \left\{ \Sigma^i_{\pi} \theta^i ds + \sigma^i_{\pi} dW^i_s + \sigma^i_{\pi}^* dW^i_s \right\}$$

as well as

$$Z^{ij}_t := \zeta^{ij}_t - \rho \lambda^i_{ij} \sigma^i_{\pi} \pi^i_{\pi}, \quad \text{and} \quad Z^i_{\pi} := \phi^i_t (\zeta^i_t) = \zeta^i_t - \sum_{j \neq i} \rho \lambda^i_{ij} \sigma^i_{\pi} \pi^i_{\pi}. \quad (4.12)$$

Here, $\phi_t$ is a mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined component-wise as above. Moreover, notice that $Z_{\pi}^i = \zeta^i_t$ since $\lambda^n_{ij} = 0$, and that $\gamma_0^i = Y^i_0$.

The processes $(Y^i, Z^{ij}, Z^i_{\pi})$ thus satisfies

$$Y^i_t = \int_t^T \left( - \eta^i |\theta^i_s|^2 - \zeta^i_s \cdot \theta^i_s + \frac{1}{2\eta^i} (I - P^i_t) (\zeta^i_t + \eta^i \theta^i_t) \right)^2 dt + \frac{1}{2\eta^i} \sum_{j \neq i} n |\zeta^{ij}_s|^2$$

$$+ \sum_{j \neq i} \rho \lambda^i_{ij} \pi^i_{\pi} \cdot \Sigma^i_{\pi} \theta^i ds - \sum_{j = 1}^n \int_t^T Z^{ij}_s \cdot dW^i_s - \int_t^T Z^i_{\pi} dW^i_s.$$ 

By Lemma 4.2, $\phi_t$ has an inverse $\psi_t$, so that $\zeta^i_t = \psi_t(Z^i_{\pi})^i$. We can thus express the equilibrium strategy for player $i$ as

$$\Sigma^i_{\pi} \pi^i_t = P^i_t \left( \phi^i_t (Z^i_{\pi})^i \right)^i + \eta_i \theta^i_t := f^i_t(Z^{ii}_t, \psi(Z^i_{\pi})), \quad t \in [0, T] \quad (4.13)$$

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and

\[ V_0^{i,n}(\pi_j)_{j \neq i} = - \alpha \psi_0(\pi_i - \rho \pi_i - \gamma_0). \]

By construction, \((Y^i, Z^i, Z^{\ast i}) \in S^1(\mathbb{R}, \mathbb{F}^n) \times H^2_{loc}(\mathbb{R}^{nd}, \mathbb{F}^n) \times H^2_{loc}(\mathbb{R}, \mathbb{F}^n)\) solves the BSDE (3.4).

\[ \square \]

**Lemma 4.1.** For any \( t \in [0, T] \), fixed \( \alpha \in \mathbb{R}^d \) and \( \beta \in \mathbb{R}^{d+1} \), the map

\[ H_{\alpha, \beta}(x) = x + \frac{1}{n-1} \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \cdot P_i \left( \left( \alpha \right) \right) + \beta \]

is a bijection on \( \mathbb{R} \) for every \( i \). Furthermore, its inverse is a contraction.

**Proof.** Fix \( t \in [0, T] \) and \( i \). \( H_{\alpha}(\cdot) \) is a bijection if and only if the map

\[ M^i(x) = y - \frac{1}{n-1} \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \cdot P_i \left( \left( \alpha \right) \right) + \beta \]

has a unique fixed point. Notice that since the projection operator is 1–Lipschitz,

\[ |M^i(x) - M^i(x')| \leq \frac{1}{n-1} \left| \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \right| |x - x'|. \quad (4.14) \]

It is thus sufficient to show that \( \left| \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \right| < 1 \). For notational convenience, let us omit all \( t \) subscripts.

First notice that \( \Sigma^i \Sigma^i = \sigma^i \sigma^i + \sigma^i \sigma^{i*} \). Using the Sherman-Morrison formula, we have that

\[ \left( \Sigma^i \Sigma^i \right)^{-1} = \sigma^{-1} - \frac{\sigma^{-1} \sigma^{-1} \sigma^{i*} \sigma^{i*}}{1 + \sigma^i \sigma^{-1} \sigma^{i*} \sigma^{i*}}, \]

and

\[ \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i = \left[ \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \sigma^i \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \sigma^i \right] = \left[ \left( 1 - \frac{\sigma^i \sigma^i \sigma^{-1} \sigma^{i*}}{1 + \sigma^i \sigma^{-1} \sigma^{i*} \sigma^{i*}} \right) \sigma^i \sigma^{-1} \sigma^{i*} \right]. \]

Thus

\[ \left| \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \right| = \frac{\sigma^i \sigma^{-1} \sigma^{-1} \sigma^{i*}}{1 + \sigma^i \sigma^{-1} \sigma^{i*} \sigma^{i*}} < 1, \]

where the last line follows from the fact that \( \sigma^i \) is uniformly elliptic for every \( i \). We now proceed to show that the inverse of \( H_{\alpha} \), which we denote by \( H_{\alpha}^{-1} \), is a contraction. For \( x \neq x' \), we have

\[ |x - x' + \frac{1}{n-1} \sigma^i \left( \sum_i \Sigma_i \right)^{-1} \Sigma_i \cdot \left( P_i \left( \left( \alpha \right) \right) + \beta \right) |^2 = \frac{\sigma^i \sigma^{-1} \sigma^{i*} \sigma^{i*}}{1 + \sigma^i \sigma^{-1} \sigma^{i*} \sigma^{i*}} \]

\[ \geq |x - x'|^2, \]

since the property of projection onto closed convex sets implies that the second term is nonnegative. \[ \square \]

**Lemma 4.2.** Consider the map \( \Phi_i : \mathbb{R}^d \to \mathbb{R}^n \) introduced in the statement of Theorem 3.1 and defined component-wise below as

\[ \Phi_i^j(\zeta_i^*) = \zeta_i^* - \sum_{j \neq i} \lambda_{ij} \sigma_{j*} \cdot \left( \Sigma_i \Sigma_i^* \right)^{-1} \Sigma_i P_i \left( \left( \alpha \right) \right) + \eta \theta_i. \]

Under the assumption that \( \sum_{j \neq i} \lambda_{ij} \in [0, 1] \), for \( t \in [0, T] \), \( \Phi_i \) is a bijection on \( \mathbb{R}^n \) and has an inverse that we denote by \( \Psi_i \). Furthermore, \( \Psi_i \) is measurable and Lipschitz-continuous with a constant that depends only on \( n \) when \( n \geq 3 \).
Proof. Omit all $t$ subscripts for notational convenience. Let $Z^*$ and $\zeta^*$ denote the column vectors $(Z_1^{*\top},\ldots,Z_n^{*\top})^\top$ and $(\zeta_1^{*\top},\ldots,\zeta_n^{*\top})^\top$ respectively. Further, let $\begin{bmatrix} \sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right) \end{bmatrix}$ denote the column vector with the $j$-th component equal to $\sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right)$. By Equation (4.12), we have

$$Z^* = \zeta^* - \frac{1}{n-1} \Lambda \begin{bmatrix} \sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right) \end{bmatrix},$$

where $\Lambda$ is the matrix $(\lambda_{ij})_{0 \leq i,j \leq n}$ and

$$\frac{1}{n-1} \bar{1}^i (\Lambda + I) \begin{bmatrix} \sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right) \end{bmatrix} + Z^{i*} = \mathcal{H}_Z^{i*,n,\theta^i}(\zeta^*)$$

where $\bar{1}^i$ denote the $n$-dimensional vector with 1 at the $i$-th position and 0’s in all other positions. By Lemma 4.1, $\mathcal{H}_Z^{i*,n,\theta^i}(\zeta^*)$ is invertible. Using Equation (4.12) again we have

$$\zeta^{i*} = Z^{i*} + \sum_{j \neq i}^n \lambda_{ij}^n \sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j\left(\begin{array}{c} \bar{Z} \nn \\ \sum_{j \neq i}^n \bar{Z} \end{array}\right) \begin{bmatrix} P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right) \end{bmatrix} + \sum_{j \neq i}^n \lambda_{ij}^n \sigma^{j*} \cdot (\Sigma^j \Sigma^j)^{-1} \Sigma^j\left(\begin{array}{c} \bar{Z} \nn \eta_j \theta_j \end{array}\right)$$

$$:= N^{i*,Z^*}(\zeta^*).$$

(4.17)

We then proceed to showing that $N^{i*,Z^*}(\zeta^*)$ has a unique fixed point. Notice that for $x \neq y$, following the inequality in (4.15),

$$|\mathcal{H}_{Z^{jj},\eta^j}(x) - \mathcal{H}_{Z^{jj},\eta^j}(y)|^2 \geq \left(1 + \frac{1}{n-1}\right)^2 \left|\left(P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right) - P_j\left(\left(Z_{jj}^j\right)^* + \eta_j \theta_j\right)\right) \right|^2,$$

Thus for fixed $Z^{jj}$ and $\eta^j \theta^j$, the map $P_j\left(\left(H_{Z^{jj},\eta^j}(\cdot)\right)^{Z_{jj}} + \eta_j \theta_j\right)$ is $(1/ (1 + \frac{1}{n-1}))$-Lipschitz. For $X,Y \in \mathbb{R}^d$ and $X \neq Y$,

$$|N^{i*,Z^*}(X) - N^{i*,Z^*}(Y)| \leq \sum_{j \neq i}^n \lambda_{ij}^n |\sigma^{j*}| \sum_{k \neq j}^n \lambda_{ik}^n |(X - Y)_k| + \sum_{j \neq i}^n \lambda_{ij}^n |\sigma^{j*}| |X - Y| / n$$

$$\leq \frac{1}{n-1} |X - Y|,$$

where the last inequality follows since $\sum_{j \neq i}^n \lambda_{ij}^n \in [0,1]$ for all $i$, and $|\sigma^{j*}| < 1$ for all $j$ (see proof of lemma 4.1). We can now conclude that for $n \geq 3$, $N^{i*,Z^*}$ admits a unique fixed point which we denote by $\varphi(Z^*)$, and that $\zeta^* = \psi(Z^*)$ is the unique solution to Equation (4.16).

Finally we proceed to prove that $\psi$ is Lipschitz with a constant that depends only on $n$ when $n \geq 3$. From (4.17), we have that for all $i$ and $n \geq 3$,

$$|\varphi(Z_1^i)^j - \varphi(Z_2^i)^j| \leq |(Z_1^i)^j - (Z_2^i)^j| + \frac{2|\varphi(Z_1^i)^j - \varphi(Z_2^i)^j|}{n(n-1)} + \frac{1}{n \leq j \leq n} \left|(Z_1^i)^j - (Z_2^i)^j\right|.$$

Then we have $\sup_{1 \leq j \leq n}|\varphi(Z_1^i)^j - \varphi(Z_2^i)^j| \leq \frac{n-1}{n^2} \sup_{1 \leq i \leq n}|(Z_1^i)^j - (Z_2^i)^j|$. Therefore, the function $\psi_t$ is Borel measurable. □
4.3 Proofs for Section 3.2

We now prove results pertaining to the characterization of the infinite population game. These are direct consequences of the work of Hu et al. [25].

Proof of Proposition 3.4. Let \((X^u, Y^u, Z^u, Z^{u*})_{u \in I}\) solve Equation (3.7) with \((X^u, Y^u, Z^u, Z^{u*}) \in S^2(\mathbb{R}, F^u) \times S^2(\mathbb{R}, F^u) \times H^2(\mathbb{R}, F^u) \times H^2(\mathbb{R}, F^u)\). Then, for almost every \(u \in I\) the processes \((Y^u, Z^u, Z^{u*})\) solves the BSDE

\[
\begin{cases}
\mathrm{d}Y_t^u = Z_t^u \cdot \mathrm{d}W_t^u + Z_t^{u*} \cdot \mathrm{d}W_t^* + \left(\frac{\eta^u}{\sqrt{T}} \theta_t^u |^2 + \left(\frac{Z_t^u}{Z_t^{u*}}\right) \cdot \theta_t^u - \frac{1}{\sqrt{T}} \left(\left(\frac{Z_t^u}{Z_t^{u*}}\right) + \eta^u \theta_t^u\right)^2\right) \, \mathrm{d}t.
\end{cases}
\]

with \(F := \mathbb{E}\left[\rho \int_I (X^u_T - \xi^u) G(u, v) \, \mathrm{d}v\right]\). Thus, it follows from [25, Theorem 7] that \(\tilde{\pi}^u\) given by (3.9) is an optimal strategy for the utility maximization problem (2.4) while the value function satisfies (3.8). By linear growth of the projection operator, it follows that \(\tilde{\pi}^u \in H^2(A^u, F^u)\) for almost every \(u \in I\). And by measurability of \(Z^u\), we have that \(\tilde{\pi}^u\) is measurable. \(\square\)

Remark 4.3. [25, Theorem 7] assumes bounded terminal condition \(F\), but examining the proof reveals that the boundedness assumption is needed only to guarantee existence of the BSDE and BMO property of \(Z^u \cdot \mathrm{d}W^u + Z^{u*} \cdot \mathrm{d}W^*\).

Proof of Corollary 3.5. Let \((Y^u, Z^u)_{u \in I}\) solve Equation (3.10) and introduce the processes

\[
\gamma^u_t := Y^u_t + \int_0^t \mathbb{E}\left[\rho \int_I P^v_s (Z^v_s + \eta^v \theta^v_s) \cdot \theta^v_s G(u, v) \, \mathrm{d}v\right] \, \mathrm{d}s.
\]

Then, \((\gamma^u, Z^u)_{u}\) satisfies

\[
\mathrm{d}\gamma^u_t = \left(-\frac{\eta^u}{2} |\theta^u_t|^2 - Z_t^u \cdot \theta^u_t + \frac{1}{2\eta^u} (I - P^u_t) (Z_t^u + \eta^u \theta^u_t)^2\right) \, \mathrm{d}t - Z_t^u \cdot \mathrm{d}W^u_t + \mu \otimes \mathbb{P}\text{-a.s.}, \ t \in [0, T],
\]

and it follows by Fubini theorem and the martingale property that

\[
\gamma^u_T = Y^u_T + \mathbb{E}\left[\rho (X^u_T - \xi^u) G(u, v) \, \mathrm{d}v\right].
\]

In particular, \(\gamma^u_T = \mathbb{E}\left[\rho (X^u_T - \xi^u) G(u, v) \, \mathrm{d}v\right]\). Thus, by [25, Theorem 7], the value function of the utility maximization problem (2.4) (when \(\sigma^* = 0\)) satisfies (3.11) and the process \(\tilde{\pi}^u\) given by (3.11) is an optimal strategy that is square–integrable. In the present case, we even have that

\[
\left\{\exp\left(-\frac{1}{\eta^u} X^u_T\right), \tau \mathbb{F}^u\text{-stopping times}\right\}
\]

is uniformly integrable. In particular, \((\tilde{\pi}^u)_{u \in I}\) is admissible. \(\square\)

5 General backward propagation of chaos theorem: proof of Theorem 2.11

In this section we present backward propagation of chaos results that are central in the proof of our main convergence result. We will start by proving the case with common noise and then we will come back to the case without common noise. The two proofs are similar, but the case with common noise is slightly more involved because the representing backward particle system if fully coupled with a forward process.
5.1 Proof of Theorem 2.11: The common noise case

Consider an interacting particle system \((X^{i,n}, Y^{i,n}, Z^{j,n}, Z^{k,n})\) with the processes \((Y^{1,n}, Y^{n}, \ldots, Y^{n})\) evolving backward in time, and \((X^{1,n}, X^{n,n}, \ldots, X^{n,n})\) evolving forward in time and characterizing the Nash equilibrium, i.e. such that

\[
\pi^{i,n}_t = \left( \sum_{i,n} \right)^{-1} \sum_{i,n} P_t \left( \left( Z^{i,n}_t, Z^{k,n}_t \right) + \eta_t \theta_i \right) dt \otimes P \text{- a.s. and } V^{i,n}_t((\pi^{i,n}_j)_{j \neq i}) = -e^{-\frac{1}{\rho}}(\xi^i - \rho^i - V^{i,n}_t) \quad \forall i \in \{1, \ldots, n\},
\]

see Theorem 3.1. We can find functions \(h, g\) such that they satisfy the following FBSDEs:

\[
\begin{align*}
\frac{dX^{i,n}_t}{dt} &= h^i(t, Z^{i,n}_t, Z^{k,n}_t) \left( \theta_t + \sigma_t^{i,n} \xi_t + \sigma_t^{i,n} \xi_t \right), \quad X^{i,n}_0 = \xi^i. \\
\frac{dY^{i,n}_t}{dt} &= -g^i(t, Z^{i,n}_t, Z^{k,n}_t) \left( \xi_t + \sigma_t^{i,n} \xi_t + \sigma_t^{i,n} \xi_t \right) + \sum_{j \neq i} Z^{i,j,n}_t \xi_t + Z^{k,i,n}_t \xi_t \\
\frac{dY^{i,n}_t}{dt} &= \rho \sum_{j \neq i} \lambda_{ij}^n (X^{j,n}_t - X^{i,n}_t).
\end{align*}
\]

Observe that due to the graph \((\lambda_{ij})_{1 \leq i, j \leq n}\), the particles in the above system are not indistinguishable as in the homogeneous case considered by Lauret and Tangpi [34, 35] and Possamaï and Tangpi [42]. Our goal here is to show that as the number of particles in the system approaches infinity, the above particle system converges to the infinite particle system \((X^u, Y^u, Z^u, Z^{*,u})_{0 \leq u \leq 1}\) given by

\[
\begin{align*}
\frac{dX^u_t}{dt} &= h^u(t, Z^u_t, Z^{*,u}_t) \left( \theta_t + \sigma_t^u \xi_t + \sigma_t^u \xi_t \right), \quad X^u_0 = \xi^u. \\
\frac{dY^u_t}{dt} &= -g^u(t, Z^u_t, Z^{*,u}_t) \left( \xi_t + \sigma_t^u \xi_t + \sigma_t^u \xi_t \right) + \sum_{i,j} Z^{i,j,n}_t \xi_t + Z^{k,i,n}_t \xi_t \\
\frac{dY^u_t}{dt} &= \rho \sum_{i,j} \lambda_{ij}^u (X^j_t - X^i_t).
\end{align*}
\]

As above, this system is understood in the sense that the mapping \((u, t, \omega) \rightarrow (X^u_t, Y^u_t, Z^u_t, Z^{*,u}_t)\) is measurable and for almost every \(u \in I\), we have \((X^u, Y^u, Z^u, Z^{*,u}) \in S^2(\mathbb{R}^d) \times S^2(\mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d)\). In particular, if we consider a specific particle \(u = \frac{1}{n}\) in the continuum, we will show that \((Y^{i,n}_t, Z^{i,n}_t, Z^{k,i,n}_t)\) and \((Y^u_t, Z^u_t, Z^{*,u}_t)\) are “close” when \(n \rightarrow \infty\). We will consider the following assumption on the coefficients of the FBSDEs:

**Condition 5.1.** \(h^u : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) and \(g^u : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) are two functions such that there exist nonnegative constants \(\ell_g, \ell_h\) so that for almost every \(u \in I\), it holds

\[
|h^u(t, z, z^*) - h^u(t, z', z'^*)| \leq \ell_h(\|z - z\| + \|z^* - z^*\|) \quad \text{and} \quad \|h^u(t, x, z^*)\|_\infty \leq \ell_h(1 + \|z\| + \|z^*\|)
\]

and

\[
|g^u(t, z, z^*) - g^u(t, z', z'^*)| \leq \ell_g(\|z - z\| + \|z^* - z^*\|)
\]

for all \((t, z, z^*, z'^*) \in [0, T] \times (\mathbb{R}^d)^2 \times \mathbb{R}^d\).

**Remark 5.2.** Recall that we use the same probability setting as described in Remark 2.10. In other words, the indices in (5.2) should be considered as \(\frac{1}{n}\). Further recall the link between \(\lambda_{ij}\) and \(\beta_n > 0\) and the graphon \(G\) is made in Condition 2.9.

Theorem 2.11(i) is then a direct corollary of the following theorem:

**Theorem 5.3.** Assume that Conditions 2.9 and 5.1 are satisfied. Further assume that the FBSDE (5.2) and (5.3) admit respective solutions \((X^{i,n}_t, Y^{i,n}_t, Z^{j,n}_t, Z^{k,n}_t)_{(i,j) \in \{1, \ldots, n\}^2}\) and \((X^u, Y^u, Z^u, Z^{*,u})_{u \in I}\) such that \((X^{i,n}_t, Y^{i,n}_t, Z^{j,n}_t, Z^{k,n}_t) \in S^2(\mathbb{R}^d, \mathbb{R}^d) \times S^2(\mathbb{R}^d, \mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d)\) for every \(i, j\) and \((X^u, Y^u, Z^u, Z^{*,u}) \in S^2(\mathbb{R}^d, \mathbb{R}^d) \times S^2(\mathbb{R}^d, \mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d) \times H^2(\mathbb{R}^d, \mathbb{R}^d)\) for almost every \(u \in I\). Then for every \(i \in \mathbb{N}^\ast\), it holds

\[
|Y^{i,n}_t - Y^{*,u}_t| \xrightarrow{n \to \infty} 0.
\]

Moreover, up to a subsequence, it holds

\[
E(\|Z^{*,u}_t - Z^{i,n}_t - Z^{*,u}_t\| + \|Z^{*,u}_t - Z^{*,u}_t\|) \rightarrow 0 \quad \text{for almost every } t \in [0, T].
\]
Proof. Using Condition 5.1, in light of Remark 2.3 and the definition of $\mathcal{F}_t^*$, we have that for almost all $(t, v) \in [0, T] \times I$,

$$
\mathbb{E}\left[ \int_0^t X_v^e G(u, v)dv \bigg| \mathcal{F}_t^* \right] = \int_0^t \mathbb{E}\left[ X_v^e G(u, v)dv \bigg| \mathcal{F}_t^* \right] = \int_0^t \mathbb{E}\left[ h^v(s, Z_s^v, Z_s^{v*})\sigma_s^v G(u, v)dv \bigg| \mathcal{F}_t^* \right] ds + \mathbb{E}\left[ \int_0^t h^v(s, Z_s^v, Z_s^{v*})\sigma_s^v G(u, v) dv \bigg| \mathcal{F}_t^* \right] dW_s^v
$$

where the first equality uses the fact that $X_v^e$ is independent of $W^*$, and the second equality follows from [32, Lemma B.1]. Let us now introduce the “shifted” processes

$$
Z^v_s := Z_s^v - \mathbb{E}\left[ \rho \int_t^s h^v(t, Z_t^v, Z_t^{v*})\sigma_t^v G(u, v)dv \bigg| \mathcal{F}_t^* \right], \quad Z^u_s := Z_t^v,
$$

and

$$
\mathcal{Y}_t^v := Y_t^v - \rho \left( \mathbb{E}\left[ \int_t^v X_v^e G(u, v)dv \bigg| \mathcal{F}_t^* \right] - \mathbb{E}\left[ X_t^e G(u, v)dv \bigg| \mathcal{F}_t^* \right] \right)
$$

so that using Equation (5.3), the processes $(\mathcal{Y}_t^v, Z_t^v, Z_t^{v*})$ satisfy

$$
\mathcal{Y}_t^v = \int_t^v g^v(s, Z_s^v, Z_s^{v*}) + \mathbb{E}\left[ \rho \int_s^v h^v(Z_u^v, Z_u^{v*})\sigma_u^v G(u, v)dv \bigg| \mathcal{F}_t^* \right] ds - \int_t^v Z_u^v dW_u^v - \int_t^v Z_u^{v*} dW_u^v.
$$

Observe that the drift term is not written with respect to the newly defined $(\mathcal{Y}_t^v, Z_t^v, Z_t^{v*})$, but rather with respect to the original $(\mathcal{Y}_t^v, Z_t^v, Z_t^{v*})$. Similarly, for the preliminaries, consider

$$
Z_t^{i,n} := Z_t^{i,n} - \rho \sum_{j \neq i}^{n} \lambda_{ij} h^j(t, Z_t^{i,j,n}, Z_t^{j,i,n}), \quad Z_t^{j,i,n} := Z_t^{j,i,n} - \rho \lambda_{ij} \sigma_t^i h^j(t, Z_t^{i,j,n}, Z_t^{j,i,n}),
$$

and

$$
\mathcal{Y}_t^{i,n} := Y_t^{i,n} - \rho \sum_{j \neq i}^{n} \lambda_{ij} (X_t^{j,n} - X_t^{i,n}),
$$

so that using Equation (5.2), the processes $(\mathcal{Y}_t^{i,n}, Z_t^{i,n}, Z_t^{j,i,n})$ satisfy

$$
\mathcal{Y}_t^{i,n} = \int_t^v g^i(s, Z_s^{i,n}, Z_s^{j,i,n}) + \frac{1}{2} \sum_{j \neq i}^{n} \|Z_s^{j,i,n}\|^2 + \rho \sum_{j \neq i}^{n} \lambda_{ij} h^j(s, Z_s^{j,i,n}, Z_s^{j,i,n}) ds - \int_t^v Z_s^{i,n} dW_s^j - \int_t^v Z_s^{j,i,n} dW_s^i.
$$

To further simplify the notation, let us put

$$
\begin{align*}
\Delta \mathcal{Y}_t^{i,n} &:= \mathcal{Y}_t^{i,n} - \mathcal{Y}_t^{\frac{i}{n}} \\
\Delta Z_t^{i,n} &:= Z_t^{i,n} - Z_t^{\frac{i}{n}} \quad \text{and} \quad \Delta Z_t^{j,i,n} = Z_t^{j,i,n} - \delta_{i=j} Z_t^{\frac{i}{n}} \\
\end{align*}
$$

Let $t \in [0, T]$ be fixed. We now define the sequence of stopping times $\tau_k$ such that for every positive $k$,

$$
\tau_k := \inf \left\{ s \geq t : \sup_{r \in [t,s]} \| \Delta \mathcal{Y}_r^{i,n} \|^2 + \int_t^s \sum_{j=1}^{n} (\| \Delta Z_r^{j,i,n} \|^2 + \| \Delta Z_r^{j,j,n} \|^2 + \| \Delta Z_r^{i,j,n} \|^2 + \| h^j(r, Z_r^{j,j,n}, Z_r^{j,i,n}) \|^2) dr \geq k \right\} \wedge T.
$$

5Here, $\delta_{i=j}$ is an indicator function for $\{ i = j \}$. 


Observe that $(\tau_k)_k$ depends on $i$ and $n$, but this dependence will be omitted to simplify notation. Since $(X^{i,n}, Y^{n,i}) \in S^2(\mathbb{R}) \times S^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ and $(X^u, Y^u, Z^u, Z^u) \in S^2(\mathbb{R}) \times S^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ it follows that for each $n$ and $i$, $\tau_k$ converges to $T \mathbb{P}$-a.s. as $k \to \infty$. Furthermore, \begin{equation}
abla_s^{i,n} := \rho \sum_{j \neq i} \lambda^i_j h^s_j (s, Z^s_j, Z^s_j) \cdot \theta^s_j - \rho \mathbb{E} \left[ \int_t^T h^s(s, Z^s_v, Z^s_v) \cdot \theta^s_v G(i, v) dv | F_T^s \right], \end{equation}
and
\begin{equation}
\nabla^{i,n}_s := \rho \sum_{j \neq i} \lambda^i_j h^s_j (s, Z^s_j, Z^s_j) \cdot \sigma^s_j - \rho \mathbb{E} \left[ \int_t^T h^s(s, Z^s_v, Z^s_v) \cdot \sigma^s_v G(i, v) dv | F_T^s \right]. \end{equation}

Now, applying Itô’s formula to $|\Delta Y^{i,n}_s|^2$, we get
\begin{equation}
|\Delta Y^{i,n}_s|^2 + \int_t^T \left( \sum_{j=1}^n \|\Delta Z^{ij,n}_s\|^2 + \|\Delta Z^{si,n}_s\|^2 \right) ds
= |\Delta Y^{i,n}_s|^2 + \int_t^T 2\Delta Y^{i,n}_s \left( g^i(s, Z^{ii,n}_s, Z^{si,n}_s) - g^i(s, Z^s_j, Z^s_j) + \sum_{j \neq i} \|\Delta Z^{ij,n}_s\|^2 \right) ds
+ \int_t^T 2\Delta Y^{i,n}_s \rho \sum_{j \neq i} \lambda^i_j \theta^s_j \left(h^j(s, Z^{ij,n}_s, Z^{sj,n}_s) - h^s(s, Z^s_j, Z^s_j) \right) ds
+ \int_t^T 2\Delta Y^{i,n}_s \nabla^{i,n}_s ds - \sum_{j=1}^n \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{ij,n}_s \cdot dW^j - \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{si,n}_s dW^s. \end{equation}

Recall that $\Delta Z^{ij,n} = \Delta Z^{ij,n} - \frac{\rho}{n} \sum_{j=1}^n \lambda^i_j \sigma^j h^j(Z^{ij,n}, Z^{sj,n})$ for $i \neq j$ and $\Delta Z^{ii,n} = \Delta Z^{ii,n}$. Equation (5.8) now takes the form
\begin{equation}
|\Delta Y^{i,n}_t|^2 + \int_t^T \left( \sum_{j=1}^n \|\Delta Z^{ij,n}_s\|^2 + \|\Delta Z^{si,n}_s\|^2 \right) ds
= |\Delta Y^{i,n}_t|^2 + \int_t^T 2\Delta Y^{i,n}_s \left( g^i(s, Z^{ii,n}_s, Z^{si,n}_s) - g^i(s, Z^s_j, Z^s_j) \right) ds
+ \int_t^T 2\Delta Y^{i,n}_s \rho \sum_{j \neq i} \lambda^i_j \theta^s_j \left(h^j(s, Z^{ij,n}_s, Z^{sj,n}_s) - h^s(s, Z^s_j, Z^s_j) \right) ds
+ \int_t^T 2\Delta Y^{i,n}_s \nabla^{i,n}_s ds + \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{ij,n}_s ds + \sum_{j \neq i} \rho \lambda^i_j \int_t^T 2\Delta Y^{i,n}_s h^j(s, Z^{ij,n}_s, Z^{sj,n}_s) \sigma^j \cdot \Delta Z^{ij,n}_s ds
- \sum_{j \neq i} \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{ij,n}_s \cdot (dW^j - \Delta Z^{ij,n}_s ds) - \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{ii,n}_s \cdot dW^i - \int_t^T 2\Delta Y^{i,n}_s \Delta Z^{si,n}_s dW^s. \end{equation}

Let $Q$ be the probability measure with density
\begin{equation}
\frac{dQ}{d\mathbb{P}} = \exp \left( \sum_{j \neq i} \int_t^T \Delta Z^{ij,n}_s \cdot dW^j - \frac{1}{2} \sum_{j \neq i} \int_t^T \|\Delta Z^{ij,n}_s\|^2 ds \right).
\end{equation}

The probability measure $Q$ depends on $i$ and $n$, but its density has second moment bounded by a constant $C_k$ depending on $k$, but not on $i$ and $n$. Taking conditional expectation under $Q$ with respect to $F^i_T$ in (5.9), we
obtain the following:

\[
| \Delta Y_{t,n}^{i,n} |^2 + E^Q \left[ \int_t^{T_k} \left( \sum_{j=1}^{n} \| \Delta Z_{s}^{i,j,n} \|^2 + \| \Delta Z_{s}^{i,n} \|^2 \right) ds \right]_t^{F_n}
\]

\[
\leq E^Q \left[ | \Delta Y_{t,n}^{i,n} |^2 | F_{t,n} \right] + E^Q \left[ \int_t^{T_k} \frac{2\varepsilon}{\beta_n} | \Delta Y_{s,n}^{i,n} |^2 + \varepsilon | \Delta Y_{s,n}^{i,n} |^2 + \varepsilon | \Delta Y_{s,n}^{i,n} |^2 ds \right]_t^{F_n}
\]

\[
+ E^Q \left[ \int_t^{T_k} 2 \Delta Y_{s,n}^{i,n} \Gamma_{s,n}^i ds \right]_t^{F_n} + E^Q \left[ \int_t^{T_k} 2 \Delta Y_{s,n}^{i,n} \rho_{s,n} \Gamma_{s,n}^i ds \right]_t^{F_n}
\]

\[
+ C \varepsilon E^Q \left[ \sum_{j \neq i} \int_t^{T_k} 2 \rho_{s,n} \Delta Y_{s,n}^{i,n} | | h_j(s, Z_{s,n}^{i,j,n}, Z_{s,n}^{s,j,n}) \sigma_{s,n}^{i,j} \cdot \Delta Z_{s,n}^{i,j,n} | ds \right]_t^{F_n}.
\]

Using \( \varepsilon \varepsilon^2 \leq \beta_n \) and by definition of the stopping time \( \tau_k \), this estimate can be simplified to

\[
| \Delta Y_{t,n}^{i,n} |^2 + (1 - \varepsilon) E^Q \left[ \int_t^{T_k} \left( \sum_{j=1}^{n} \| \Delta Z_{s}^{i,j,n} \|^2 + \| \Delta Z_{s}^{i,n} \|^2 \right) ds \right]_t^{F_n}
\]

\[
\leq E^Q \left[ | \Delta Y_{t,n}^{i,n} |^2 | F_{t,n} \right] + E^Q \left[ \int_t^{T_k} \left( 1 + \frac{2\varepsilon^2}{\beta_n} \right) | \Delta Y_{s,n}^{i,n} |^2 ds \right]_t^{F_n}
\]

\[
+ E^Q \left[ \int_t^{T_k} 2 | \Delta Y_{s,n}^{i,n} | | \Gamma_{s,n}^i | + | \Gamma_{s,n}^{i,n} | ds \right]_t^{F_n} + \frac{C_{\rho,\sigma,h,T,k}}{(n-1)\beta_n}
\]

\[
+ C \varepsilon E^Q \left[ \sum_{j \neq i} \int_t^{T_k} 2 \rho_{s,n} \Delta Y_{s,n}^{i,n} | | h_j(s, Z_{s,n}^{i,j,n}, Z_{s,n}^{s,j,n}) \sigma_{s,n}^{i,j} \cdot \Delta Z_{s,n}^{i,j,n} | ds \right]_t^{F_n}.
\]

Applying Young’s inequality and recalling the definition of \( \tau_k \), the last term above can be estimated as

\[
E^Q \left[ \sum_{j \neq i} \int_t^{T_k} 2 \rho_{s,n} \Delta Y_{s,n}^{i,n} | | h_j(s, Z_{s,n}^{i,j,n}, Z_{s,n}^{s,j,n}) \sigma_{s,n}^{i,j} \cdot \Delta Z_{s,n}^{i,j,n} | ds \right]_t^{F_n}
\]

\[
\leq \frac{\rho_{s,n}}{(n-1)\beta_n} E^Q \left[ \sup_{t \leq s \leq \tau_k} | \Delta Y_{s,n}^{i,n} | \left( \int_t^{T_k} \sum_{j \neq i} \| h_j(s, Z_{s,n}^{i,j,n}, Z_{s,n}^{s,j,n}) \|^2 ds + \sum_{j \neq i} \int_t^{T_k} \| \Delta Z_{s,n}^{i,j,n} \|^2 ds \right) \right]_t^{F_n}
\]

\[
\leq \frac{C_{\rho,\sigma,h,T,k}}{(n-1)\beta_n}.
\]

Thus, choosing \( \varepsilon < 1 \) and subsequently using in (5.10) Gronwall’s inequality, taking expectation with respect to \( \mathbb{P} \), Cauchy–Schwarz inequality and Doob’s inequality, we are left with

\[
E \left[ | \Delta Y_{t}^{i,n} |^2 \right] \leq E \left[ E^Q \left[ | \Delta Y_{\tau_k}^{i,n} |^2 | F_{t} \right] \right] + C_{k,T} \varepsilon E \left[ \left( \frac{dQ}{d\mathbb{P}} \right)^4 \right]_{\frac{1}{2}} \left[ E \left[ \int_0^T | \Gamma_{s}^{i,n} |^2 + | \Gamma_{s,n}^{i,n} |^2 ds \right] \right]_{\frac{1}{2}} + C_{\rho,\sigma,h,T,k} \frac{1}{(n-1)\beta_n}.
\]

Observe that using again Cauchy–Schwarz we have

\[
E \left[ E^Q \left[ | \Delta Y_{\tau_k}^{i,n} |^2 | F_{t} \right] \right] \leq C E \left[ \left( \frac{dQ}{d\mathbb{P}} \right)^2 \right] E \left[ | \Delta Y_{\tau_k}^{i,n} |^4 \right].
\]

To proceed, first notice that for every \( n \), we have that \( (| \Delta Y_{\tau_k}^{i,n} |^4)_{k \geq 1} \) converges to 0 in probability as \( k \rightarrow \infty \) almost surely, since \( \tau_k \) converges to \( T \) \( P \)-a.s. There thus exists a fast sub-sequence \( \Delta Y_{\tau_{k,m}}^{i,n} \) such that

\[
\mathbb{P}( | \Delta Y_{\tau_{k,m}}^{i,n} |^4 > \varepsilon ) \leq \frac{e^{-k^2}}{m}.
\]
Therefore, for every $\varepsilon > 0$ we have
\[
\mathbb{E}[(\Delta Y_{t_k,m}^{i,n})^4] = \mathbb{E}[(\Delta Y_{t_k,m}^{i,n})^4 \delta(\Delta Y_{t_k,m}^{i,n} \leq \varepsilon)] + \mathbb{E}[(\Delta Y_{t_k,m}^{i,n})^4 \delta(\Delta Y_{t_k,m}^{i,n} > \varepsilon)] \leq \varepsilon + k^2 \cdot \frac{e^{-k^3}}{m}.
\]

Our definition of $\tau_k$ implies that all moments of $\frac{dQ}{dP}$ are upper bounded by $e^{Ck^2}$ for some constant $C > 0$ independent of $k$. Thus, coming back to (5.12) we continue the estimation as
\[
\mathbb{E}[\Delta Y_{\tau_{k,m}}^{i,n}]^2 \leq Ck^2 \left( \varepsilon + k^2 \frac{e^{-k^3}}{m} \right) + C_{p,\rho,\theta,T,k} \mathbb{E} \left[ \int_0^T |\Gamma_{s,n}^{*i,n}|^2 ds \right] + C_{k,T} \mathbb{E} \left[ \int_0^T |\Gamma_{s,n}^{*i,n}|^2 ds \right].
\]

Applying Lemma 5.4, first fix $k$ and let $n \to \infty$, followed by letting $m \to \infty$ and $\varepsilon \to 0$, we conclude that
\[
\mathbb{E}[\Delta Y_{\tau_{k,m}}^{i,n}]^2 \xrightarrow{n \to \infty} 0.
\]

In particular, starting with $t = 0$, it follows that the sequence $(\Delta Y_0^{i,n})_{n \geq 1}$ converges to zero, and since $\Delta Y_0^{i,n} = \Delta Y_{\tau_{k,m}}^{i,n}$, we obtain Equation (5.4).

Let us now turn to the convergence of the control processes. By Equation (5.10), Equation (5.14), Cauchy–Schwarz inequality and the above estimates we have
\[
\mathbb{E} \left[ \int_t^{\tau_{k,m}} \|\Delta Z_{s,n}^{i,n}\| + \|\Delta Z_{s,n}^{*i,n}\| \right] \, ds \leq T \mathbb{E} \left[ \left( \frac{dQ}{dP} \right)^2 \right]^{\frac{1}{2}} \mathbb{E} \left[ \int_t^{\tau_{k,m}} \|\Delta Z_{s,n}^{i,n}\|^2 + \|\Delta Z_{s,n}^{*i,n}\|^2 \right]^{\frac{1}{2}}
\]
\[
\leq T \cdot Ck^2 \left( \varepsilon + k^2 \frac{e^{-k^3}}{m} \right) + C_{h,\theta,T,k} \mathbb{E} \left[ \int_0^T |\Gamma_{s,n}^{*i,n}|^2 ds \right]^{\frac{1}{2}} + C_{k,T} \mathbb{E} \left[ \int_0^T |\Gamma_{s,n}^{*i,n}|^2 ds \right]^{\frac{1}{2}}.
\]

Using Lemma 5.4, first fix $k$ and let $n \to \infty$, followed by letting $m \to \infty$ and $\varepsilon \to 0$, we conclude that, up to a subsequence, it holds
\[
\lim_{m \to \infty} \lim_{n \to \infty} \|\Delta Z_{t}^{i,n}\| + \|\Delta Z_{t}^{*i,n}\| \xrightarrow{n \to \infty} 0 \quad \mathbb{P}\text{-a.s.}; \text{ for a.e. } s \in [t,T].
\]

Since $\mathbb{P}(\tau_{k,m} \geq t) = 1$, this shows that
\[
\|\Delta Z_{t}^{i,n}\| + \|\Delta Z_{t}^{*i,n}\| \xrightarrow{n \to \infty} 0 \quad \mathbb{P}\text{-a.s.}
\]

By the identity $\Delta Z_{s,n}^{i,n} = \Delta Z_{s}^{i,n}$, we have thus obtained that $(\Delta Z_{s,n}^{i,n})_{n \geq 1}$ converges to zero. For the convergence of $\Delta Z_{s,n}^{*i,n}$, observe that
\[
|\Delta Z_{s,n}^{*i,n}| \leq |\Delta Z_{s,n}^{i,n}| + |\Gamma_{s,n}^{*i,n}| + \rho \sum_{j \neq i} \lambda_{ij} |\sigma_{s,n}^{ij}| \left| h^j(s, Z_{s}^{j,n}, Z_{s}^{i,n}) - h^i(s, Z_{s}^{i,n}, Z_{s}^{i,n}) \right|
\]
\[
\leq |\Delta Z_{s,n}^{i,n}| + |\Gamma_{s,n}^{*i,n}| + \rho \|\sigma_{s,n}^{i} \|_{\infty} \mathbb{E} \left[ \int_0^T |\Delta Z_{s,n}^{*i,n}| \right] + |\Delta Z_{s,n}^{*i,n}|
\]
\[
\leq |\Delta Z_{s,n}^{i,n}| + |\Gamma_{s,n}^{*i,n}| + \rho \|\sigma_{s,n}^{i} \|_{\infty} \mathbb{E} \left[ \int_0^T |\Delta Z_{s,n}^{*i,n}| \right] + |\Delta Z_{s,n}^{*i,n}|
\]

Thus,
\[
\mathbb{E} \left[ \int_0^{\tau_{k,m}} |\Delta Z_{s,n}^{*i,n}| \, ds \right] \leq \mathbb{E} \left[ \int_0^{\tau_{k,m}} |\Delta Z_{s,n}^{i,n}|^2 + |\Gamma_{s,n}^{*i,n}|^2 \, ds \right] + C_{p,\rho,\theta,h,k} \frac{1}{(n-1)\beta_n}.
\]

Therefore, arguing as above and using again Lemma 5.4 we have that, up to a subsequence,
\[
\|\Delta Z_{s,n}^{*i,n}\| \xrightarrow{n \to \infty} 0 \quad \mathbb{P}\text{-a.s.}
\]

Therefore, Equation (5.5) follows by dominated convergence. This concludes the proof.
Lemma 5.4. Under the conditions of Theorem 5.3, it holds
\[ E \left[ \int_0^T |\Gamma_s^{i,n}|^2 ds \right] + E \left[ \int_0^T |\Gamma_s^{i,n}|^2 ds \right] \xrightarrow{n \to \infty} 0 \quad \text{for every } i \in \mathbb{N}^*.
\]

Proof. We will consider only the term \( \Gamma_{i,n}^{i,n} \); the term \( \Gamma_{i,n}^{i,n} \) is dealt with similarly. Using Condition 2.9, especially that \( \lambda_{ij} \) are i.i.d. and independent of \((\Omega, \mathcal{F}, \mathbb{P})\) and \((W^1, \ldots, W^n, W^*)\), we have
\[
E|\Gamma_{i,n}^{i,n}|^2 = E\left[ \frac{1}{n-1} \sum_{j \neq i} \lambda_{ij} h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} - E\left[ \int_T h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, v) dv | \mathcal{F}_T \right] \right]^2
\leq 2E\left[ \frac{1}{n-1} \sum_{j \neq i} \left( \frac{\lambda_{ij}}{\beta_n} h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} - h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, \dot{j} n^{-1}) \right)^2 \right]
\]
\[
+ 2E\left[ \frac{1}{n-1} \sum_{j \neq i} h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, \dot{j} n^{-1}) - E\left[ \int_T h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, v) dv | \mathcal{F}_T \right] \right]^2
\leq \frac{C_0}{(n-1)^2 \beta_n^2} \text{Var}(\lambda_{ij}) \sum_{j \neq i} |h^j(s, Z_s^{V}, Z_s^{\ast V})|^2
\]
\[
+ 4E\left[ \frac{1}{n} \sum_{j \neq i} h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, \dot{j} n^{-1}) - E\left[ \frac{1}{n} \sum_{j = 1}^n h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, \dot{j} n^{-1}) | \mathcal{F}_T \right] \right]^2
\]
\[
+ E\left[ \frac{1}{n} \sum_{j = 1}^n h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, \dot{j} n^{-1}) | \mathcal{F}_T \right] - E\left[ \int_T h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G(\dot{i} n^{-1}, v) dv | \mathcal{F}_T \right]^2
\]
\]
Using that the step function \( F_s^n(u) \) given by
\[
F_s^n(u) := \sum_{j = 1}^n h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} \delta_{\{u \in (\dot{i} n^{-1}, \dot{i+1} n^{-1})\}}
\]
approximates the function \( F_s : v \mapsto h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} \) in \( L^2(I, B(I), \mu) \), we have
\[
E|\Gamma_{i,n}^{i,n}|^2 \leq \frac{C_0}{(n-1)^2 \beta_n^2} \text{Var}(\lambda_{ij}) \sum_{j \neq i} |h^j(s, Z_s^{V}, Z_s^{\ast V})|^2
\]
\[
+ \frac{4}{(n-1)^2} \sum_{j \neq i} E \left[ \left( h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} - E[h^j(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} | \mathcal{F}_T] \right)^2 \right]
\]
\[
+ E \left[ \int F_s^n(u) G_n(\dot{i} n^{-1}, v) dv - \int h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G_n(\dot{i} n^{-1}, v) dv \right]^2
\]
\[
+ E \left[ \int h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G_n(\dot{i} n^{-1}, v) dv - \int h^v(s, Z_s^{V}, Z_s^{\ast V}) \cdot \theta_s^{\ast} G_n(\dot{i} n^{-1}, v) dv \right]^2
\]
\[
\leq \frac{C_0}{(n-1)^2 \beta_n^2} \sum_{j \neq i} (\lambda_{ij} - E[\lambda_{ij}]) |h^j(s, Z_s^{V}, Z_s^{\ast V})|^2 + \frac{C_0}{(n-1)^2} \sum_{j \neq i} E \left( h^j(s, Z_s^{V}, Z_s^{\ast V}) \right)^2
\]
\[
+ \| F_s^n - F_s \|_{L^2(I, B(I), \mu)} + C_0 E \left[ \int \left( h^v(s, Z_s^{V}, Z_s^{\ast V}) \right)^2 \left( G_n(\dot{i} n^{-1}, v) - G(\dot{i} n^{-1}, v) \right)^2 dv \right].
\]
Because the Lipschitz constants of \( h^u \) and \( g^u \) do not depend on \( u \), standard FBSDE estimates show that
sup_{u \in I} \|(Z^u, Z^{*u})\|_{L^2(\mathbb{R}^{d+1}, \mathbb{P}^u)} < \infty$. Hence, integrating on both sides above and using Condition 5.1, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{T} |\Gamma_{s}^{i,n}|^2 ds \right] \leq \left( \frac{C_{h, \theta, T}}{(n-1)\beta_n^2} + \frac{C_{\theta}}{(n-1)} \right) \left( \sup_{u \in I} \|(Z^u, Z^{*u})\|_{L^2(\mathbb{R}^{d+1}, \mathbb{P}^u)} + 1 \right) + C_T \|F_s\|_{L^2(I, B(I), \mu)} \nonumber \\
+ \ell_h^2 C_{\theta} \left( \sup_{u \in I} \|(Z^u, Z^{*u})\|_{L^2(\mathbb{R}^{d+1}, \mathbb{P}^u)} + 1 \right) \sum_{i=1}^{n} \int_{t}^{u} \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right)^2 dv 
onumber \\
\leq \frac{C_{h, \theta, T, Z}}{(n-1)\beta_n^2} + \frac{C_{\theta, Z}}{(n-1)} + C_T \|F_s\|_{L^2(I, B(I), \mu)} \nonumber \\
+ \ell_h^2 C_{\theta, Z} \int_{I} \left( G_n(\frac{nu}{n}, v) - G(\frac{nu}{n}, v) \right)^2 dv du 
\leq \frac{C_{h, \theta, T, Z}}{(n-1)\beta_n^2} + \frac{C_{\theta, Z}}{(n-1)} + C_T \|F_s\|_{L^2(I, B(I), \mu)} + \ell_h^2 C_{\theta, Z} \|G_n - G\|_{2}^2. \quad (5.16)
\]

Therefore, since \( n\|G_n - G\|_{2}^2 \to 0 \), it follows that for each \( i \), we have

\[
\mathbb{E} \left[ \int_{0}^{T} |\Gamma_{s}^{i,n}|^2 ds \right] \to 0.
\]

\( \square \)

Remark 5.5. If the function \( h \) is bounded (which is the case when the graphon equilibrium \((\tilde{\pi}^u)_{u \in I}\) is bounded), it is enough to require that \( n\|G_n - G\|_{\|} \to 0 \), which is weaker than \( L^2 \)-convergence. This is due to the fact that the last term in Equation (5.15) can be estimated as

\[
\mathbb{E} \left[ \left( \int_{I} h^v(s, Z_s^v, Z_s^{\ast v}) \cdot \theta^u_s \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right) dv \right)^2 \right] \leq C_{h, \theta} \left| \int_{I} \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right) dv \right|^2.
\]

Taking the average, we obtain the following estimation

\[
\left( \frac{1}{n} \sum_{i=1}^{n} \left| \int_{I} G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) dv \right|^2 \right) = \left( \int_{I} \left| \int_{I} G_n(\frac{nu}{n}, v) - G(\frac{nu}{n}, v) dv \right| du \right) \leq 4\|G_n - G\|_{2}^2, \quad (5.17)
\]

where \( \|G\| \) is the so-called operator norm given by

\[
\|G\| := \sup_{\|h\|_{\infty} \leq 1} \int_{I} \int_{I} h(u)G(u, v) dv du.
\]

It follows from Lovász [36, Lemma 8.11] that the \( \|G\|_{\|} \) and \( \|G\| \) are equivalent norms. Therefore the last term in (5.17) can be replaced by \( \ell_h^2 C_{\theta, Z} \|G_n - G\|_{2}^2 \).

5.2 Proof of Theorem 2.11: The non–common noise case

Let us not present the proof of Theorem 2.11.(ii). Throughout this subsection, we assume \( \sigma^{*u} = 0 \) for all \( u \in I \).

By Theorem 3.1 and Remark 3.3, the Nash equilibrium \((\tilde{\pi}^{i,n})_{i \in \{1,\ldots,n\}}\) is characterized by the BSDE 3.5. That is, it holds

\[
\tilde{\pi}^{i,n} = (\sigma^u)_{t}^{-1} P^u_t (Z^u_t + \eta^u \theta^u_t) \quad \text{and} \quad V^{i,n}_0 ((\tilde{\pi}^{j,n})_{j \neq i}) = -e^{-\frac{1}{\eta^u} (\xi^u - \pi^u)} \text{ }\mathbb{P} \otimes dt \text{-a.s.}
\]

with \((Y^{i,n}, Z^{ij,n})_{(i,j) \in \{1,\ldots,n\}^2}\) solving BSDE 3.5. Moreover, by Corollary 3.5 and Proposition 6.2, there is a graphon equilibrium \((\tilde{\pi}^u)_{u \in I}\) such that

\[
\tilde{\pi}^u = (\sigma^u)_{t}^{-1} P^u_t (Z^u_t + \eta^u \theta^u_t) \text{ } dt \otimes \mu \otimes \mathbb{P} \text{-a.s. and} \quad V^{u,G}_0 = -\exp \left( -\frac{1}{\eta^u} (\xi^u - \int_{I} \mathbb{E}[\rho^u G(u, v) dv - Y^u_0]) \right)
\]

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with \((Y^u, Z^u)_{u \in I}\) solving Equation (3.10). It thus suffices to show that
\[
|Y_0^i,n - Y_0^i,\hat{\pi}|^2 + |Z_{t}^{i,n} - Z_{t}^{i,\hat{\pi}}|^2 \longrightarrow 0 \quad \text{as} \quad n \to \infty
\]

Let us put \(\Delta Y^{i,n} := Y^{i,n} - Y^{\hat{\pi}}\) and \(\Delta Z^{i,j,n} := Z^{i,j,n} - Z^{j,i,n}\). Let \(t \in [0, T]\) be fixed and consider the stopping time
\[
\tau_k := \inf \left\{ s \geq t : \sup_{t \leq r \leq s} |\Delta Y^{i,n}_r|^2 + \int_t^s \sum_{j=1}^n \|P_j(Z^{i,j,n}_r + \eta^j \theta^j)|^2 + \|\Delta Z^{j,i,n}_r\|^2 ds \geq k \right\} \wedge T.
\]

Observe that for each \(i, n\) the sequence \((\tau_k)_{k \geq 1}\) converges to \(T\) \(P\)-a.s. Applying Itô’s formula to \(e^{\kappa t}(\Delta Y^{i,n})^2\) for some \(\kappa > 0\) to be chosen, we have
\[
e^{\kappa t}(\Delta Y^{i,n}_t)^2 = e^{\kappa \tau_k}(\Delta Y^{i,n}_{\tau_k})^2 + \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \left\{ \theta^i_s \cdot \Delta Z^{i,i,n}_s + \frac{1}{2\eta^i} \left( |(I - P^i_s)(Z^{i,i,n}_s + \eta^i \theta^i)|^2 - |(I - P^\pi_s)(Z^{i,i,n}_s + \eta^\pi \theta^\pi)|^2 \right) \right\}
\]
\[
+ \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \sum_{j \neq i} |Z^{i,j,n}_s + \sigma^j \lambda^j \rho P_j(Z^{j,j,n}_s + \eta^j \theta^j)|^2 ds
\]
\[
+ \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \rho \left\{ \sum_{j \neq i} \lambda^j \rho P_j(Z^{j,j,n}_s + \eta^j \theta^j) \theta^j_s - \mathbb{E} \left[ \int_0^s (P^v_s + \eta^v \theta^v_s) \theta^v_s G(u, v) dv \right] \right\} ds
\]
\[
- \int_0^{\tau_k} \kappa e^{\kappa s} \Delta Y^{i,n}_s \Delta Z^{i,j,n}_s dW^j_s.
\]

Let us introduce the measure \(Q\) with density
\[
\frac{dQ}{dP} = \mathbb{E} \left( \int_0^{\tau_k} (\theta^i_s + \frac{1}{2\eta^i} \gamma_s(Z^{i,i,n}_s, Z^{\pi}_s)) dW^i_s + \sum_{j \neq i} \int_0^{\tau_k} \left( \sum_{j \neq i} \theta^j_s P_j(Z^{j,j,n}_s + \eta^j \theta^j) \theta^j_s - \mathbb{E} \left[ \int_0^s (P^v_s + \eta^v \theta^v_s) \theta^v_s G(u, v) dv \right] \right) dW^j_s \right),
\]
where \(\gamma_s\) is the (linearly growing) function such that
\[
|(I - P^i_s)(Z^{i,i,n}_s + \eta^i \theta^i)|^2 - |(I - P^\pi_s)(Z^{i,i,n}_s + \eta^\pi \theta^\pi)|^2 = \gamma_s(Z^{i,i,n}_s, Z^{\pi}_s) \Delta Z^{i,i,n}_s.
\]

This follows by Lipschitz–continuity of the projection operator since \(A^j\) is convex (also recall the rebranding \(i \equiv \frac{i}{n}\)). Thus, by Girsanov’s theorem, the BMO martingale property of \(Z^{\pi}\), and square integrability of \(Z^{i,j,n}\) we have
\[
e^{\kappa t}(\Delta Y^{i,n}_t)^2 = \mathbb{E} \left[ e^{\kappa \tau_k}(\Delta Y^{i,n}_{\tau_k})^2 + \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \sum_{j \neq i} \rho \frac{\lambda^2_{ij}}{n^2 \beta^2_n} \|\sigma^j_s\|^2 \|P_j(Z^{j,j,n}_s + \eta^j \theta^j)|^2 ds | \mathcal{F}^n_s \right]
\]
\[
+ \mathbb{E} \left[ \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \rho \left\{ \frac{1}{n^2 \beta^2_n} \sum_{j \neq i} \lambda^j \rho P_j(Z^{j,j,n}_s + \eta^j \theta^j) \theta^j_s - \mathbb{E} \left[ \int_0^s (P^v_s + \eta^v \theta^v_s) \theta^v_s G(u, v) dv \right] \right\} ds
\]
\[
+ \int_0^{\tau_k} 2e^{\kappa s} \Delta Y^{i,n}_s \rho \Gamma^{i,n}_s - \kappa e^{\kappa s}(\Delta Y^{i,n}_t)^2 - \sum_{j=1}^n e^{\kappa s} \|\Delta Z^{i,j,n}_s\|^2 ds | \mathcal{F}^n_s \right]
\]

where \(\Gamma^{i,n}_s\) is the process given by
\[
\Gamma^{i,n}_s := \frac{1}{n^2 \beta^2_n} \sum_{j \neq i} \lambda^j \rho P_j(Z^{\pi}_s + \eta^\pi \theta^\pi) \theta^\pi_s - \mathbb{E} \left[ \int_0^s (P^v_s + \eta^v \theta^v_s) \theta^v_s G(u, v) dv \right].
\]

Using Lipschitz–continuity of the projection operator and boundedness of \(\Sigma\), we continue the estimation as
\[
e^{\kappa t}(\Delta Y^{i,n}_t)^2 + \mathbb{E} \left[ \sum_{j=1}^n \int_0^{\tau_k} e^{\kappa s} \|\Delta Z^{i,j,n}_s\|^2 ds | \mathcal{F}^n_t \right] = \mathbb{E} \left[ e^{\kappa \tau_k}(\Delta Y^{i,n}_{\tau_k})^2 + \frac{C_{k,\rho,\sigma}}{n^2 \beta^2_n} + \int_0^{\tau_k} e^{\kappa s} \left( \frac{C_{\theta,\sigma,\rho}}{\varepsilon} - \kappa \right) |\Delta Y^{i,n}_s|^2 ds | \mathcal{F}^n_t \right]
\]
\[
+ \varepsilon \mathbb{E} \left[ \int_0^{\tau_k} e^{\kappa s} \left( \frac{1}{n^2 \beta^2_n} \sum_{j \neq i} \lambda^j \|\Delta Z^{i,j,n}_s\|^2 + e^{\kappa s} \|\Gamma^{i,n}_s\|^2 ds | \mathcal{F}^n_t \right),
\]

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Therefore, using triangular inequality with some \( \varepsilon > 0 \). Choosing \( \kappa > 0 \) large enough, and using Cauchy–Schwarz inequality, it follows that

\[
\mathbb{E}^Q\left[e^{\kappa t}(\Delta Y_{t,n}^{i,n})^2 + \sum_{j=1}^{n} \int_{t}^{T_k} e^{\kappa s} \|\Delta Z_{s,n}^{i,j,n}\|^2 \, ds \right] \leq \mathbb{E}^Q\left[e^{\kappa \tau_k}(\Delta Y_{\tau_k}^{i,n})^2 + \frac{C_{k,\rho}}{n^2 \beta_n^2} \right]
\]

\[
+ \varepsilon \mathbb{E}^Q\left[ \int_{t}^{T_k} e^{\kappa s} \frac{1}{n \beta_n^2} \left( \sum_{j \neq i}^{n} \lambda_{ij}^2 \right) \frac{1}{n} \sum_{j \neq i}^{n} \|\Delta Z_{s,n}^{i,j,n}\|^2 + e^{\kappa s} ||\Gamma_{s,n}^i||^2 \, ds \right]
\]

\[
\leq \mathbb{E}^Q\left[e^{\kappa \tau_k}(\Delta Y_{\tau_k}^{i,n})^2 + \frac{C_{k,\rho}}{n^2 \beta_n^2} + \frac{C_{k,\kappa}}{n \beta_n^2} + \varepsilon \mathbb{E}^Q\left[ \int_{t}^{T_k} e^{\kappa s} ||\Gamma_{s,n}^i||^2 \, ds \right] \right],
\]

where we used the fact that \( \lambda_{ij} \) is independent of \( W^1, \ldots, W^n \) and \( \mathbb{E}^\mathbb{P}[\lambda_{ij}^2] \leq \beta_n \), and definition of the stopping time \( \tau_k \). Because \( (\Delta Y_{\tau_k}^{i,n})_{k \geq 1} \) converges to 0 in \( \mathbb{P} \)-probability and thus in \( \mathbb{Q} \)-probability for each \( n \), we can find a fast sub-\( \lim \) \( \Delta Y_{\tau_k,n}^{i,n} \) such that

\[
\mathbb{Q}(\|\Delta Y_{\tau_k,n}^{i,n}\| \geq \varepsilon) \leq \frac{e^{-k^2}}{m}.
\]

Thus, for every \( \varepsilon > 0 \), we have

\[
\mathbb{E}^Q(\|\Delta Y_{\tau_k,n}^{i,n}\|^2) \leq \varepsilon + k \frac{e^{-k^2}}{m}.
\]

Hence, using again definition of \( \tau_k \),

\[
\mathbb{E}^Q\left[e^{\kappa t}(\Delta Y_{t,n}^{i,n})^2 + \sum_{j=1}^{n} \int_{t}^{T_k} e^{\kappa s} \|\Delta Z_{s,n}^{i,j,n}\|^2 \, ds \right] \leq \varepsilon + k \frac{e^{-k^2}}{m} + \frac{C_{k,\rho,\sigma}}{n \beta_n} + \varepsilon \mathbb{E}^Q\left[ \int_{t}^{T_k} e^{\kappa s} ||\Gamma_{s,n}^i||^2 \, ds \right].
\]

Using Cauchy–Schwarz inequality, we further have

\[
\mathbb{E}\left[e^{\kappa t} |\Delta Y_{t,n}^{i,n}| + \int_{t}^{T_k} e^{\kappa s} \|\Delta Z_{s,n}^{i,j,n}\| \, ds \right] \leq 2T \mathbb{E}\left[ \left( \frac{dQ}{d\mathbb{P}} \right)^2 \right]^{1/2} \mathbb{E}^Q\left[e^{\kappa t}(\Delta Y_{t,n}^{i,n})^2 + \int_{t}^{T_k} e^{\kappa s} \|\Delta Z_{s,n}^{i,j,n}\|^2 \, ds \right]^{1/2}
\]

\[
\leq C_k \left( \varepsilon + k \frac{e^{-k^2}}{m} + \frac{C_{k,\rho,\sigma}}{n \beta_n} + \varepsilon \mathbb{E}^Q\left[ \int_{t}^{T_k} e^{\kappa s} ||\Gamma_{s,n}^i||^2 \, ds \right] \right)^{1/2}.
\]

We will show below that for each \( k \) fixed

\[
\mathbb{E}^Q\left[ \int_{t}^{T_k} ||\Gamma_{s,n}^i||^2 \, ds \right] \overset{m,n \to \infty}{\longrightarrow} 0. \quad (5.18)
\]

Thus, first taking the limit in \( n \), then in \( m \) and and then letting \( \varepsilon \to 0 \), it follows that

\[
\mathbb{E}\left[e^{\kappa t} |\Delta Y_{t,n}^{i,n}| + \int_{t}^{T_k} e^{\kappa s} \|\Delta Z_{s,n}^{i,j,n}\| \, ds \right] \overset{m,n \to \infty}{\longrightarrow} 0.
\]

We thus obtain that \( \Delta Y_{t,n}^{i,n} \to 0 \) as \( n \to \infty \) and, up to a subsequence, \( \|\Delta Z_{s,n}^{i,j,n}\| \delta_{\{s \leq \tau_k,n\}} \overset{m,n \to \infty}{\longrightarrow} 0 \) for almost every \( s \in [t, T] \) \( \mathbb{P} \)-a.s. for a.e. \( s \in [t, T] \).

In particular, because \( \mathbb{P}(\tau_{k,n} \geq 1) = 1 \), \( \Delta Z_{s,n}^{i,j,n} \to 0 \) \( \mathbb{P} \)-a.s. as \( n \to \infty \).

Let us now come back to (5.18). Since the random variables \( (Z^n)_{n \in I} \) are e.p.i., it follows by the exact law of large numbers, see Sun [46, Corollary 3.10] that it holds

\[
\|\Gamma_{s,n}^i\| \leq \left\| \frac{1}{n \beta_n} \sum_{j \neq i}^{n} \lambda_{ij} P_s^{\frac{1}{n} \left( Z_s^j + \eta_s^j \theta_s^j \right) \theta_s^j} - \int_{I} P_s^n(Z_s^j + \eta_s^j \theta_s^j) \theta_s^j G_n(\frac{i}{n}, \frac{j}{n}) \, dv \right\|.
\]

Therefore, using triangular inequality and the fact that

\[
\int_{I} P_s^n(v) G_n(\frac{i}{n}, v) (v) \, dv = \frac{1}{n} \sum_{j=1}^{n} P_s^{\frac{1}{n} \left( Z_s^j + \eta_s^j \theta_s^j \right) \theta_s^j} G_n(\frac{i}{n}, \frac{j}{n})
\]
with
\[ F^n_s(u) := \sum_{j=1}^n P^\xi_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s \delta_{(u \in (\frac{j}{n}, \frac{j+1}{n}))}, \]
it follows that
\[ \|\Gamma_{s}^{i,n}\| \leq \left\| \frac{1}{n \beta_n} \sum_{j=1}^n \lambda_j P^\xi_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s - \frac{1}{n} \sum_{j=1}^n P^\xi_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s G_n(\frac{i}{n}, \frac{j}{n}) \right\|
+ \left\| \int_I F^n_s(v)G_n(\frac{i}{n}, v)(v) dv - \int_I F^n_s(v)G(\frac{i}{n}, v)(v) dv \right\|
+ \left\| \int_I P^n_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s G(\frac{i}{n}, v) dv \right\|. \]

Proceeding as in the proof of Lemma 5.4, we have
\[
\mathbb{E}^Q \left[ \int_0^{T_n} \|\Gamma_{s}^{i,n}\|^2 ds \right] \leq \frac{\text{var}(\lambda_{ij})}{n \beta_n} \|\theta\|_{\infty} \left( \|Z^\xi \cdot W^\xi\|_{\text{BMO}} + C_{\theta, \eta} \right) + \mathbb{E}^Q \left[ \int_0^{T_n} \|F^n_s(v)\|^2 dv \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right)^2 \right]
+ \mathbb{E}^Q \left[ \int_0^{T_n} \left( \int_I \|F^n_s(v) - P^n_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s\|^2 dv \right)^2 \right] \leq C_{\theta, \eta}.
\]

Since the intervals \((\frac{i}{n}, \frac{i+1}{n})\) form a partition of \(I\), and using linear growth of the projection operator, it follows that
\[ \|F^n_s(u)\|^2 \leq \sum_{j=1}^n \|Z^\xi_s\|^2 \|\delta_{u \in (\frac{i}{n}, \frac{i+1}{n})}\| + C_{\theta, \eta}. \]
Thus, using the inequality \(\|\cdot\|_{\mathbb{H}^2(\mathbb{R}^d, \mathbb{P})} \leq \|\cdot\|_{\mathbb{H}^2(\mathbb{R}^d, \mathbb{P})},\) that the BMO norm does not depend on the underlying measure and the fact that \(\sup_{u \in I} \|Z^\xi\|_{\mathbb{H}^2(\mathbb{R}^d, \mathbb{P})} < \infty\), we have
\[
\mathbb{E}^Q \left[ \int_0^T \|F^n_s(u)\|^2 ds \right] \leq \sum_{j=1}^n \|Z^\xi_s\|^2 \|\delta_{u \in (\frac{i}{n}, \frac{i+1}{n})}\| + C_{\theta, \eta} \leq C.
\]

Hence, we have
\[
\mathbb{E}^Q \left[ \int_0^{T_n} \|\Gamma_{s}^{i,n}\|^2 ds \right] \leq C_{\theta, \eta} + C \int_I \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right)^2 dv + C_k \mathbb{E} \left[ \int_0^T \|F^n_s - F_s\|_{L^2(\mu)}^2 ds \right]^{1/2},
\]
where \(F_n(v) := P^n_s(Z^\xi_s + \eta^\xi \theta^\xi_s) \theta^\xi_s).\) Since the sequence of step functions \(F^n\) converges to \(F\) in \(L^2(I, \mu)\), it follows by dominated convergence that, fixing \(k\), we have (5.18). The convergence to zero of the term \(\int_I \left( G_n(\frac{i}{n}, v) - G(\frac{i}{n}, v) \right)^2 dv\) is proved as in at the end of the proof of Lemma 5.4. This concludes the proof.

6 Wellposedness of graphon McKean–Vlasov BSDEs and FBSDEs

We conclude the article with two existence results for graphon McKean–Vlasov (F)BSDEs used in the proof of existence of graphon equilibria. In the ensuing statements and proofs, we will use the space \(\mathcal{S}(\mathbb{F}, \mathbb{P}, I)\) defined as the space of families of processes \((Y^u)_{u \in I}\) such that \((u, \omega) \mapsto Y^u\) is \(\mathcal{I} \otimes \mathbb{P}\)-measurable and for almost every \(u\), it holds \(Y^u \in \mathcal{S}(\mathbb{F}^u, \mathbb{R}^d)\). This space is equipped with the norm
\[ \|Y\|_{\mathcal{S}(\mathbb{F}, \mathbb{P}, I)} := \int_I \|Y^u\|_{\mathcal{S}(\mathbb{F}^u, \mathbb{R}^d)} du \]
which makes it a Banach space. We similarly define $\mathbb{H}(\mathbb{F}, \mathbb{R}^d, I)$. We further denote by $\mathbb{H}_{\text{BMO}}(\mathbb{F}^u, \mathbb{R}^d)$ the space of $\mathbb{F}^u$-predictable processes $Z$ with values in $\mathbb{R}^d$ such that the process $\int ZdW^u$ is a $(\mathbb{F}, \mathbb{F}^u)$–BMO martingale. The space $\mathbb{H}_{\text{BMO}}(\mathbb{F}, \mathbb{R}^d, I)$ is defined analogously to $S^p(\mathbb{F}, \mathbb{R}, I)$ with the norm
\[ \|Z\|_{\mathbb{H}_{\text{BMO}}(\mathbb{F}, \mathbb{R}^d, I)} := \int_I \|Z^u\|_{\mathbb{H}_{\text{BMO}}(\mathbb{F}^u, \mathbb{R}^d)} \, du. \]

### 6.1 Graphon McKean–Vlasov FBSDEs

We start by proving the existence of the graphon McKean–Vlasov FBSDEs with Lipschitz coefficients. Observe that this is a system involving a continuum of coupled equations, where the coupling is due to the graphon term.

**Proposition 6.1.** Assume that the functions $g : I \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$; $b, h : I \times [0, T] \times \Omega \times \mathbb{R}^{d+1} \to \mathbb{R}$ and $h_1 : I \times [0, T] \times \Omega \times \mathbb{R}^{d+1} \to \mathbb{R}^d$ are Borel measurable and Lipschitz–continuous in the sense that
\[
|g^u(t, z, z^*) - g^u(t, \tilde{z}, \tilde{z}^*)| \leq \ell_g(\|z - \tilde{z}\| + |z^* - \tilde{z}^*|)
\]
\[
|b^u(t, z, z^*) - b^u(t, \tilde{z}, \tilde{z}^*)| + \|h_1^u(t, z, z^*) - h_2^u(t, \tilde{z}, \tilde{z}^*)\| + |h_2^u(t, z, z^*) - h_2^u(t, \tilde{z}, \tilde{z}^*)| \leq \ell_h(\|z - \tilde{z}\| + |z^* - \tilde{z}^*|)
\]

for some $\ell_g, \ell_h > 0$ and $(t, z, \tilde{z}, z^*, \tilde{z}^*) \in [0, T] \times (\mathbb{R}^d)^2 \times \mathbb{R}^d$, and $\int_0^T |g^u(t, 0, 0)| \, dt < \infty$. Further assume that we are given a family $(\xi^u)_{u \in I}$ such that $\xi^u \in L^2(\mathcal{F}(I) \otimes \mathcal{F}_0^u, \mu \otimes \mathbb{P})$. Then, if $\rho < \frac{1}{2\kappa} e^{-(2\kappa^2 + 1)T}$, the graphon system
\[
\begin{aligned}
\frac{dX^u_t}{dt} &= b^u(t, Z^u_t, \xi^u_t) \, dt + h_1^u(t, Z^u_t, \xi^u_t) \, dW^u_t + h_2^u(t, Z^u_t, \xi^u_t) \, dW^*_t, \quad X^u_0 = \xi^u, \\
\frac{dY^u_t}{dt} &= -g^u(t, Y^u_t, \xi^u_t) \, dt + Z^u_t \, dW^u_t + Z^*_t \, dW^*_t, \\
Y^u_T &= \mathbb{E} \left[ \rho \int_I X^u_T G(u, v) \, dv \big| \mathcal{F}_T \right]
\end{aligned}
\]  

admits a unique solution $(X^u, Y^u, Z^u, Z^*_u)_{u \in I} \in \mathbb{S}^2(\mathbb{F}, I) \times \mathbb{S}^2(\mathbb{F}, I) \times \mathbb{H}^2(\mathbb{F}, I) \times \mathbb{H}^2(\mathbb{F}, I)$ for almost every $u$. Thus, the function
\[
\Psi((z^u, z^*_u)_{u \in I}) := (Z^u, Z^*_u)_{u \in I}
\]
maps the Banach space $\mathbb{H}^2(\mathbb{F}, I) \times \mathbb{H}^2(\mathbb{F}, I)$ into itself. It remains to show that $\Psi$ admits a unique fixed point.

Let $(z^u, z^*_u)_{u \in I}, (\tilde{z}^u, \tilde{z}^*_u)_{u \in I} \in \mathbb{H}^2(\mathbb{F}, I) \times \mathbb{H}^2(\mathbb{F}, I)$ be given. Put $\Psi((z^u, z^*_u)_{u \in I}) = (Z^u, Z^*_u)_{u \in I}$ and $\Psi((\tilde{z}^u, \tilde{z}^*_u)_{u \in I}) = (\tilde{Z}^u, \tilde{Z}^*_u)_{u \in I}$ such that $(X^u, Y^u, Z^u, Z^*_u)_{u \in I}$ and $(\tilde{X}^u, \tilde{Y}^u, \tilde{Z}^u, \tilde{Z}^*_u)_{u \in I}$ satisfy (6.2). Let us introduce the shorthand notation $\Delta X^u := X^u - \tilde{X}^u$, $\Delta Y^u := Y^u - \tilde{Y}^u$, $\Delta Z^u := Z^u - \tilde{Z}^u$ and $\Delta Z^*_u := Z^*_u - \tilde{Z}^*_u$.

Given some constant $\kappa > 0$, we apply Itô’s formula to $e^{\kappa t} |\Delta Y^u|^2$ to obtain
\[
e^{\kappa t} |\Delta Y^u|^2 \leq e^{\kappa T} \rho^2 \mathbb{E} \left[ \int_t^T \|\Delta X^u_T\|^2 G(u, v) \, dv \big| \mathcal{F}_T \right] + \int_t^T e^{\kappa s} \left(2 \rho^2 - \kappa \right) |\Delta Y^u|^2 \, ds + (\varepsilon - 1) \int_t^T e^{\kappa s}(\|\Delta Z^u_s\|^2 + |\Delta Z^*_s|^2) \, ds
\]
\[- \int_t^T e^{\kappa s} \Delta Y^u_s \Delta Z^u_s \, ds - \int_t^T e^{\kappa s} \Delta Y^*_u \Delta Z^*_u \, ds.\]

Taking expectation on both sides and choosing $\kappa = 2\rho^2 / \varepsilon$, we have
\[
\mathbb{E} \left[ e^{\kappa t} |\Delta Y^u|^2 + (1 - \varepsilon) \int_t^T e^{\kappa s} |\Delta Z^u|^2 + e^{\kappa s} |\Delta Z^*_u|^2 \, ds \right] \leq e^{\kappa T} \rho^2 \int_I \mathbb{E} \left[ \|\Delta X^u_T\|^2 \right] \, dv.
\]
On the other hand, applying Itô’s formula to \(e^{\epsilon t}\|\Delta X^u_t\|^2\) and using Lipschitz–continuity of \(b^u, h^u_1\) and \(h^u_2\), we have

\[
\mathbb{E}[e^{\epsilon t}\|\Delta X^u_t\|^2] \leq \mathbb{E}\left[\int_0^t 2\epsilon e^{\kappa s}\ell_h((z^u_s - z^u_{s-}) + |z^{s\ast}_u - z^u_{s-}|)\|\Delta X^u_s\| + \ell_h^2 e^{\kappa s}((z^u_s - z^u_{s-}) + |z^{s\ast}_u - z^u_{s-}|)^2)ds\right] \\
+ \kappa \mathbb{E}\left[\int_0^t e^{\kappa s}\|\Delta X^u_s\|^2 ds\right] \\
\leq \mathbb{E}\left[(1 + \kappa)\int_0^t e^{\kappa s}\|\Delta X^u_s\|^2 ds\right] + 2\ell_h^2 e^{(\kappa + 1)T} \mathbb{E}\left[\int_0^t e^{\kappa s}((z^u_s - z^u_{s-}) + |z^{s\ast}_u - z^u_{s-}|)^2)ds\right],
\]

where the last inequality follows from Young’s inequality. Thus, by Gronwall’s inequality, we have

\[
\mathbb{E}[e^{\epsilon t}\|\Delta X^u_t\|^2] \leq 2\ell_h^2 e^{(\kappa + 1)T} \mathbb{E}\left[\int_0^t e^{\kappa s}((z^u_s - z^u_{s-}) + |z^{s\ast}_u - z^u_{s-}|)^2)ds\right].
\]

Thus, if \(\epsilon = 1/2\), we have

\[
\int_I \left[\mathbb{E}\left[\int_I e^{\kappa s}\|\Delta Z^u_s\|^2 + e^{\kappa s}|\Delta Z^{s\ast}_u|^2 ds\right] du \leq 4\ell_h^2 e^{(\kappa + 1)T} \rho^2 \int_I \mathbb{E}\left[\int_0^T e^{\kappa s}((z^u_s - z^u_{s-}) + |z^{s\ast}_u - z^u_{s-}|)^2)ds\right] du.
\]

Thus, by the choice of \(\rho\) and the Banach fixed point theorem, the mapping \(\Psi\) admits a unique fixed point, implying that the graphon FBSDE (6.1) admits a unique solution in \(\mathbb{S}^2(\mathbb{F}, \mathbb{R}, I) \times \mathbb{S}^2(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}^2(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}^2(\mathbb{F}, \mathbb{R}, I)\).

## 6.2 Graphon McKean–Vlasov BSDE

Let us now turn to the wellposedness of graphon McKean–Vlasov FBSDEs with Lipschitz–continuous coefficients.

**Proposition 6.2.** Assume that the functions \(g : I \times [0, T] \times \mathbb{R}^d \to \mathbb{R}\) and \(f : I \times [0, T] \times \mathbb{R}^d \to \mathbb{R}\) are Borel–measurable and satisfy the locally Lipschitz and Lipschitz–continuity conditions

\[
|g^u(t, z) - g^u(t, z')| \leq \ell_g(||z|| + ||z'||)||z - z'|| \quad \text{and} \quad |g^u(t, z)| \leq \ell_g(1 + ||z||^2)
\]

and

\[
|f^u(t, z) - f^u(t, z')| \leq \ell_f||z - z'|| \quad \text{and} \quad |f^u(t, z)| \leq \ell_f(1 + ||z||)
\]

for some constants \(\ell_g, \ell_f > 0\) and every \((t, z, z') \in [0, T] \times (\mathbb{R}^d)^2\) and almost all \(u \in I\). Further assume that we are given \(\mathcal{F}_t^u\)–measurable random variables \(F^u\) such that \((u, \omega) \mapsto F^u\) is measurable and uniformly bounded. Then, the graphon system

\[
Y^u_t = F^u + \int_t^T (g^u(s, Z^u_s) + \mathbb{E}[f^u(s, Z^u_s)])dG(u, v) ds - \int_t^T Z^u_s dW^u_s
\]

admits a unique solution \((Y^u, Z^u)_{u \in I}\) such that we have \((Y^u, Z^u)_{u \in I} \in \mathbb{S}^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}^{\text{BMO}}(\mathbb{F}, \mathbb{R}^d, I)\) and \(\sup_{u \in I} \|Z^u\|_{\mathbb{H}^{\text{BMO}}(\mathbb{F}, \mathbb{R}^d)} < \infty\).

**Proof.** Let \((y^u, z^u)_{u \in I} \in \mathbb{S}^\infty(\mathbb{F}, \mathbb{R}^d, I) \times \mathbb{H}^{\text{BMO}}(\mathbb{F}, \mathbb{R}^d, I)\) be given and consider the (decoupled) quadratic BSDEs

\[
Y^u_t = F^u + \int_t^T (g^u(s, Z^u_s) + \mathbb{E}[f^u(s, z^u_s)])dG(u, v) ds - \int_t^T Z^u_s dW^u_s. \tag{6.4}
\]

It follows by Hu et al. [25] that for almost every \(u \in I\), this equation admits a unique solution \((Y^u, Z^u) \in \mathbb{S}^\infty(\mathbb{F}, \mathbb{R}) \times \mathbb{H}^{\text{BMO}}(\mathbb{F}, \mathbb{R}^d)\). Moreover, it follows by the arguments of Stricker and Yor that \((u, t, \omega) \mapsto (Y^u_t, Z^u_t)\) is measurable. Thus, the function

\[
\Psi((y^u, z^u)_{u \in I}) := (Y^u, Z^u)_{u \in I}
\]

is well–defined and maps the Banach space \(\mathbb{S}^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}^{\text{BMO}}(\mathbb{F}, \mathbb{R}^d, I)\) into itself. It therefore remains to show that this mapping admits a unique fixed point.
Let \((y^u, z^u)_{u \in I}, (\bar{y}^u, \bar{z}^u)_{u \in I} \in S^\infty(\mathbb{F}, \mathbb{R}, I) \times \mathbb{H}^{BMO}(\mathbb{F}, \mathbb{R}^d, I)\) be given and put \(\Psi((y^u, z^u)_{u \in I}) = (Y^u, Z^u)_{u \in I}\) and \(\Psi((\bar{y}^u, \bar{z}^u)_{u \in I}) = (\bar{Y}^u, \bar{Z}^u)_{u \in I}\). Let \(\kappa > 0\) be a constant to be determined and let \(\tau\) be an \(\mathbb{F}^n\)–stopping time. Apply Itô’s formula to \(e^{\kappa t}|\Delta Y^u_t|^2 = e^{\kappa t}|Y^u_t - \bar{Y}^u_t|^2\) to obtain

\[
e^{\kappa \tau} |\Delta Y^u_{\tau}|^2 = \int_\tau^T 2e^{\kappa s} |\Delta Y^u_s|^2 ds + \int_\tau^T e^{\kappa s} |\Delta Z^u_s|^2 ds - \int_\tau^T 2e^{\kappa s} \Delta Y^u_s \Delta Z^u_s dW_s^u
\]

where we used the short hand notation \(\Delta Z^u := Z^u - \bar{Z}^u\) and \(\Delta z^u := z^u - \bar{z}^u\), and \(W^u\) is a Brownian motion under the probability measure

\[
\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}\left( \int_0^\tau \beta^u(s, Z^u_s, \bar{Z}^u_s) dW^u_s \right)
\]

with \(\beta\) being a linearly growing function such that \(g^u(s, z) - g^u(s, \bar{z}) = \beta^u(s, z, \bar{z}) \cdot (z - \bar{z})\). Choose \(\kappa\) such that \(\kappa > \frac{1}{\varepsilon}\). Taking conditional expectation on both sides yields

\[
e^{\kappa \tau} |\Delta Y^u_{\tau}|^2 + \mathbb{E}^\mathbb{Q}\left[ \int_\tau^T e^{\kappa s} |\Delta Z^u_s|^2 ds \mid \mathcal{F}_\tau^u \right] \leq \varepsilon \mathcal{E}_2^2 \int_\tau^T \mathbb{E}^\mathbb{Q}\left[ |\Delta Z^u_s|^2 \mid \mathcal{F}_\tau^u \right] ds.
\]

Taking the supremum over \(\tau\) and integrating on both sides in \(u\) therefore gives

\[
\|\Delta Y\|^2_{S^\infty(\mathbb{F}, I)} + \|\Delta Z\|^2_{\mathbb{H}^{BMO}(\mathbb{F}, I)} \leq \varepsilon \mathcal{E}_2^2 \|\delta z\|^2_{\mathbb{H}^{BMO}(\mathbb{F}, I)},
\]

and where we used the fact that the BMO norm does not depend on the underlying probability measure and \(\|\cdot\|_{\mathbb{H}^{BMO}(\mathbb{F}, I)} \leq \|\cdot\|_{\mathbb{H}^{BMO}(\mathbb{F}, I)}\). Choosing \(\varepsilon > 0\) small enough allows to conclude that \(\Psi\) is a contraction, and thus it follows by the Banach fixed point theorem that \(\Psi\) admits a unique fix point in \(S^\infty(\mathbb{F}, I) \times \mathbb{H}^2_{BMO}(\mathbb{F}, I)\). □

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