A REFINEMENT OF THE BINOMIAL DISTRIBUTION USING THE QUANTUM BINOMIAL THEOREM

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Abstract. $q$-analogs of special functions, including hypergeometric functions, play a central role in mathematics and have numerous applications in physics. In the theory of probability, $q$-analogs of various probability distributions have been introduced over the years, including the binomial distribution. Here, I propose a new refinement of the binomial distribution by way of the quantum binomial theorem (also known as the noncommutative $q$-binomial theorem), where the $q$ is a formal variable in which information related to the sequence of successes and failures in the underlying binomial experiment is encoded in its exponent.

1. Background and motivation

Many of the standard mathematical objects used in probability and statistics (e.g. factorials, the gamma function, the exponential function, the beta function, the binomial coefficients, etc.) have well-known $q$-analogs that play a central rôle in the theory of special functions (see Andrews et al., 1999) and basic hypergeometric series (see Gasper and Rahman, 2004). A $q$-analogue of a mathematical object $A$ is a function $f(q)$ such that $f(1) = A$ or, failing that, at least $\lim_{q \to 1} f(q) = A$ and in some imprecise sense, $f(q)$ retains some of the remarkable properties possessed by $A$.

A number of $q$-analogs of the binomial distribution have been introduced over the years, (see Dunkl, 1981; Kemp, 1987; Kemp and Kemp, 1991; Sicong, 1994; Kemp, 2002; Kim, 2012). All of these are based on the standard commutative version of the $q$-binomial theorem (see, e.g., Andrews et al., 1999, p. 488, Theorem 10.2.1).

Here we propose a refinement of the binomial distribution that draws its inspiration from the quantum binomial theorem, also known as the noncommutative $q$-binomial theorem (Potter, 1950; Schützenberger, 1953). It retains the ordinary binomial experiment setting (counting successes in $n$ independent Bernoulli trials), but the inclusion of the additional parameter $q$ encodes additional combinatorial information. Unlike the $q$-analogs of various probability distributions well known in the literature, where $q$ can meaningfully assume a numerical value within a specified range, the $q$ presented herein is strictly a formal variable with associated combinatorial
information encoded in its exponent. The classical binomial probability distribution is recovered when the formal variable $q$ is set equal to unity in our construction.

In section 2, we recall some basic facts about integer partitions, state and prove a generalization of the quantum binomial theorem, and then derive the quantum binomial theorem and the classical binomial theorem as corollaries. In section 3, the binomial probability distribution is discussed, and a refinement inspired by the quantum binomial theorem is motivated. In section 4, the exponent of $q$ from the previous section is interpreted in its new role as a random variable. In section 5, our refined binomial distribution is given, where we now consider a joint probability distribution of two random variables describing the original setting of $n$ independent Bernoulli trials with constant probability of success $\pi$ on each trial. Marginal and conditional distributions are derived along with various moments. In section 6, some concluding words are offered, suggesting possibilities for further research.

2. Introduction

Let $\mathcal{P}_{k,m} := \{(\lambda_1, \lambda_2, \ldots, \lambda_k) : m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ and each $\lambda_j \in \mathbb{Z}\}$. Thus $\mathcal{P}_{k,m}$ can be thought of as the set of all partitions with at most $k$ parts, none of which is greater than $m$. If a given partition has strictly less than $k$ parts, we simply pad on the right with zeros. For example,

$$\mathcal{P}_{3,2} = \{(0,0,0), (1,0,0), (1,1,0), (2,0,0), (1,1,1), (2,1,0), (2,1,1), (2,2,0), (2,2,1), (2,2,2)\}.$$ 

Equivalently, $\mathcal{P}_{k,m}$ can be visualized as the set of all Ferrers diagrams that fit inside a rectangle $k$ units high and $m$ units wide (see Andrews, 1976, p. 6 ff). If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we let $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and call this the size (or weight) of $\lambda$. Notice that $\mathcal{P}_{k,m}$ consists of partitions with sizes from 0 to $mk$ inclusive. Later, we will need to consider only those partitions of size $t$ in $\mathcal{P}_{k,m}$, and note this set as $\mathcal{P}_{k,m}(t)$.

Let $x$ and $y$ be indeterminates that do not commute under multiplication. For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_{k,n-k}$, let $Q^\lambda$ denote the operator that permutes the factors of $x^{n-k}y^k$ by the product of transpositions

$$Q^\lambda := \prod_{j=1}^{k} (n-k+j, n-k+j-\lambda_j).$$

The transposition, written in cycle notation, $(i,j)$ applied to $x^{n-k}y^k$ means we swap the $i$th and $j$th factors of $x^{n-k}y^k$. The transpositions are not in general disjoint and therefore their product is not commutative. We
interpret the order of factors in (2.1) as the $j = 1$ factor is applied first (rightmost), the $j = 2$ factor is applied second (immediately to the left of the $j = 1$ factor), etc.

For example, with $n - k = 5, k = 3$, we have

$$Q^{(3,1,0)} x^5 y^3 = (8, 8)(7, 6)(6, 3)(xxxxyyy)$$

$$= (8, 8)(7, 6)(xyxyxyy)$$

$$= (8, 8)(xyxyxyy)$$

$$= xyxyxyy.$$

For a bijection between the partition $\lambda \in \mathcal{P}_{k,n-k}$ and the permutation of $x^{n-k}y^k$ represented by $Q^\lambda x^{n-k}y^k$, see Andrews (1976, p. 40, Theorem 3.5).

**Theorem 2.1** (generalized quantum binomial theorem). For non-commuting indeterminates $x$ and $y$ and the operator $Q^\lambda$ defined in (2.1),

$$Q^\lambda x^{n-k}y^k.$$  

Note that the operator $Q^\lambda$ acts as a generalization of the formal expression $q^{\lambda}$. Before proving Theorem 2.1 we will provide some context and motivation. If we replace the operator $Q^\lambda$ with the $|\lambda|$th power of the indeterminate $q$ (note that $q$ commutes with both $x$ and $y$), we obtain the quantum binomial theorem, usually attributed to M. P. Schützenberger (1953), but note also an essentially equivalent result due to H. S. A. Potter (1950). To state this theorem, we need the usual $q$-binomial coefficient, also known as the Gaussian polynomial:

$$[n \atop k]_q := \frac{(1 - q^{n-k+1})(1 - q^{n-k+2}) \cdots (1 - q^n)}{(1-q)(1-q^2) \cdots (1-q^k)},$$

for $0 \leq k \leq n$, and 0 otherwise. It is well known that (2.3) is a polynomial in $q$ of degree $k(n - k)$, satisfies $q$-analogs of the Pascal triangle recurrence, and is the generating function for the function that counts the number of members of $\mathcal{P}_{k,n-k}$ (see Andrews 1976, Chapter 3) of a given size:

$$[n \atop k]_q = \sum_{\lambda \in \mathcal{P}_{k,n-k}} q^{\lambda} = \sum_{j=0}^{k(n-k)} \# \mathcal{P}_{k,n-k}(j)q^j,$$

where $\# \mathcal{P}_{k,m}(j)$ is the number of partitions of size $j$ into at most $k$ parts, with each part at most $m$.

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The name “quantum binomial theorem” dates back to at least Beattie et al. (2002). Elsewhere in the literature (see, e.g., Andrews et al. (1999, p. 485)) this result is called the “noncommutative $q$-binomial theorem”. Other authors just call it the “$q$-binomial theorem,” but this risks possible confusion with other (commutative) results also known by that name.
Recall that the Potter–Schützenberger quantum binomial theorem may be stated as follows:

**Corollary 2.2 (Quantum binomial theorem).** If \( yx = qxy \), and \( n \) is a nonnegative integer, then

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} y^k.
\]

To obtain Corollary 2.2 from Theorem 2.1, we replace the operator \( Q^\lambda \) with the formal variable \( q|\lambda| \) and use (2.4).

Of course, the \( q = 1 \) case of Corollary 2.2 (so that multiplication of \( x \) and \( y \) is now commutative) is the classical binomial theorem:

**Corollary 2.3 (classical binomial theorem).** For nonnegative integer \( n \),

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

**Proof of Theorem 2.1.** Observe that Eq. (2.4) provides the key link between Theorem 2.1 and Corollary 2.2; in fact the first equality of (2.4) is all we need to establish Theorem 2.1 given Corollary 2.2, the result of Potter–Schützenberger.

\[\square\]

### 3. On the Binomial Distribution

In the binomial experiment, we have \( n \) independent Bernoulli trials where the probability of a “success” on each trial is some fixed value \( \pi, 0 < \pi < 1 \), and the probability of a “failure” is \( 1 - \pi \).

**Remark 3.1.** We acknowledge immediately that there is an unfortunate conflict between the standard notations of probability and that of the theory of \( q \)-series. Accordingly, here we avoid using \( p \) for the probability of a success a Bernoulli trial (since \( p(n) \) is a standard notation for the number of partitions of \( n \)) and avoid \( q = 1 - p \) for the probability of a failure on a Bernoulli trial, as “\( q \)” is used in the sense of \( q \)-series.

If \( Y \) is the random variable that counts the number of successes encountered during the \( n \) independent Bernoulli trials, then we say \( Y \) is a binomial random variable with parameters \( n \) and \( \pi \), writing \( Y \sim \text{Bin}(n, \pi) \) for short. Note that

\[
P(Y = k) = \binom{n}{k} (1 - \pi)^{n-k} \pi^k,
\]

for \( k = 0, 1, 2, \ldots, n \), and that

\[
\sum_{k=0}^{n} P(Y = k) = 1
\]

follows immediately by taking \( x = 1 - \pi \) and \( y = \pi \) in (2.6).
Notice, however, that if we apply the same interpretation of $x$ and $y$ (i.e. probabilities of failure and success respectively) to the context of Theorem 2.1 then we are using the extra information preserved to track each possible sequence of $n$ Bernoulli trials. For example, consider the $n = 4$ case. There is a one-to-one correspondence between terms generated by the left member of (2.2) and those generated by the right member as follows, where F denotes failure and S denotes success, grouped by the values of $Y = 0, 1, 2, 3, 4$:

| Outcome | LHS summand | corresponding RHS summand |
|---------|-------------|--------------------------|
| FFFF    | $xxxx$      | $Q^4 x^4 = (4) x^4$      |
| FFFS    | $xxxy$      | $Q^{(0)} x^2 y = (4, 4) x^2 y$ |
| FFSF    | $xyxx$      | $Q^{(1)} x^3 y = (4, 3) x^3 y$ |
| FSFF    | $xyxx$      | $Q^{(2)} x^3 y = (4, 2) x^3 y$ |
| SFFF    | $yxxx$      | $Q^{(3)} x^3 y = (4, 1) x^3 y$ |
| FFSS    | $xxyy$      | $Q^{(0,0)} x^2 y^2 = (4, 4)(3, 3) x^2 y^2$ |
| FSFS    | $xyxy$      | $Q^{(1,0)} x^2 y^2 = (4, 4)(3, 2) x^2 y^2$ |
| SFFS    | $yyxx$      | $Q^{(2,0)} x^2 y^2 = (4, 4)(3, 1) x^2 y^2$ |
| SFSF    | $yyxy$      | $Q^{(1,1)} x^2 y^2 = (4, 3)(3, 1) x^2 y^2$ |
| SSFF    | $yxyx$      | $Q^{(2,1)} x^2 y^2 = (4, 3)(3, 1) x^2 y^2$ |
| FSSS    | $xyyy$      | $Q^{(0,0,0)} x y^3 = (4, 4)(3, 3)(2, 2) x y^3$ |
| SFSS    | $yyxy$      | $Q^{(1,0,0)} x y^3 = (4, 4)(3, 3)(2, 1) x y^3$ |
| SSFS    | $yyxy$      | $Q^{(1,1,0)} x y^3 = (4, 4)(3, 2)(2, 1) x y^3$ |
| SSSF    | $yyyy$      | $Q^{(1,1,1)} x y^3 = (4, 3)(3, 2)(2, 1) x y^3$ |
| SSSS    | $yyyy$      | $Q^{(0,0,0,0)} y^4 = (4, 4)(3, 3)(2, 2)(1, 1) y^4$ |

If we consider the binomial experiment from the perspective of Corollary 2.2 we have more information than in the ordinary binomial distribution, but not always enough to uniquely identify each summand in the right member of (2.5) with a specific sequence of successes and failures. (Notice, e.g., in the table below that outcomes SFSS and FSSF both contribute a factor of $q^2 x^2 y^2$ to the sum, and thus cannot be distinguished at this level of refinement. To remedy this, we proposed Theorem 2.1.)

Once again, consider the $n = 4$ case in detail:
| Y | Outcome | LHS summand | RHS summand |
|---|---------|-------------|-------------|
| 0 | FFFF | $xxx$ | $x^4$ |
| 1 | FFFS | $xxxy$ | $x^3y$ |
|   | FFSF | $xxyx$ | $qx^3y$ |
|   | FSFF | $xyxx$ | $q^2x^3y$ |
|   | SFFF | $yxxx$ | $q^3x^3y$ |
| 2 | FFSS | $xxyy$ | $x^2y^2$ |
|   | FSFS | $xyxy$ | $qx^2y^2$ |
|   | SFFS | $yxyy$ | $q^2x^2y^2$ |
|   | SFSF | $xyyx$ | $q^3x^2y^2$ |
|   | SSFF | $yyxx$ | $q^4x^2y^2$ |
| 3 | FSSS | $xyyy$ | $xy^3$ |
|   | SFSS | $yxyy$ | $qxy^3$ |
|   | SSFS | $yyxy$ | $q^2xy^3$ |
|   | SSSF | $yyxx$ | $q^3xy^3$ |
| 4 | SSSS | $yyyy$ | $y^4$ |

**Remark 3.2.** Note that the noncommutivity of $x$ and $y$ is essential here: the noncommutivity is that which allows us track information relating to the where the successes and failures occur in the underlying binomial experiment.

Motivated by the similarity in appearance between the pmf for $Y \sim \text{Bin}(n, \pi)$,

$$P(Y = k) = \binom{n}{k} (1 - \pi)^{n-k} \pi^k,$$

which occurs as the generic summand in right member of (2.6) with $x = 1 - \pi$ and $y = \pi$, and the expression

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q (1 - \pi)^{n-k} \pi^k,$$

we examine whether the latter (which is the generic summand in the right member of (2.5) with $x$ replaced by $1 - \pi$ and $y$ replaced by $\pi$) can be utilized to generalize the binomial distribution in some useful sense.

Let us informally define a "$q$-generalized probability mass function" $P_q(Y = k)$ as one where the sum over its support is in general not 1, but rather a $q$-analogue of 1, i.e. a function of $q$ that evaluates to 1 when $q = 1$. Additionally, although we have already emphasized the role of $q$ as a formal variable, we could safely impose the additional condition that $P_q(Y = k) \geq 0$ for real $q$, $0 < q < 1$, to mimic the classical condition that a probability mass function must be nonnegative.
For the case at hand, we wish to have
\[
P_q(Y = k) = \binom{n}{k}_q (1 - \pi)^{n-k} \pi^k.
\]

Again, the expression \(P_q(Y = k)\) fails to retain the property from the classical \(q = 1\) case that summing over the support yields unity. In fact,
\[
\sum_{k=0}^{n} P_q(Y = k) = \sum_{k=0}^{n} \binom{n}{k}_q (1 - \pi)^{n-k} \pi^k
\]
is a polynomial \(s(q)\) in \(q\) with the property that \(s(1) = 1\). Also,
\[
\binom{n}{k}_q (1 - \pi)^{n-k} \pi^k \geq 0
\]
for all \(0 < q < 1\), although we prefer to not evaluate the preceding expression for different values of \(q\), with the exception of \(q = 1\), and then only when we wish to pass from the proposed \(q\)-generalization to the classical case.

For example, in the \(n = 4\) case,
\[
P_q(Y = 2) = (1 - \pi)^2 \pi^2 (1 + q + 2q^2 + q^3 + q^4).
\]

If we set \(q = 1\), we recover the fact that if \(Y \sim \text{Bin}(4, \pi)\), \(P(Y = 2) = 6(1 - \pi)^2 \pi^2\). However, if we leave the \(q\) unevaluated, the polynomial \(P_q(Y = 2)\) effectively segregates the \(\binom{4}{2} = 6\) outcomes with \(k = 2\) successes and \(n - k = 2\) failures into \(k(n - k) + 1 = 5\) subcategories according to where the \(n - k = 2\) failures occur in relation to the \(k = 2\) successes in the sequence of \(n = 4\) trials. This phenomenon will be explored further in the next section.

Remark 3.3. Another connection with the \(q\)-series literature is as follows. The Rogers–Szegő polynomials \(H_n(q; z)\), (see, e.g., Andrews (1976, pp. 49–50)), a family of polynomials orthogonal on the unit circle, are defined as
\[
H_n(q; z) := \sum_{k=0}^{n} \binom{n}{k}_q z^k.
\]

Thus the sum over the support of the \(q\)-generalized pmf is
\[
\sum_{k=0}^{n} P_q(Y = k) = \sum_{k=0}^{n} \binom{n}{k}_q (1 - \pi)^{n-k} \pi^k = (1 - \pi)^n H_n \left( q; \frac{\pi}{1 - \pi} \right).
\]

An anonymous referee pointed out that by dividing through by the right member of (3.4), one would obtain a legitimate pmf, i.e.

\[
P(Y = k) = \frac{\binom{n}{k}_q (1 - \pi)^{n-k} \pi^k}{(1 - \pi)^n H_n \left( q; \frac{\pi}{1 - \pi} \right)} = \binom{n}{k}_q (1 - \pi)^k H_n \left( q; \frac{\pi}{(1 - \pi)} \right),
\]

and then it must be the case that
\[
\sum_{k=0}^{n} P(Y = k) = 1.
\]
However, the referee went on to observe that "unfortunately, this simple rectification ruined the entire model; the random variable Y does not count, anymore, the number of successes in a sequence of n independent Bernoulli trials, with constant success probability."

4. Interpretation of the exponent of q

A precise interpretation of the term $q^t x^{n-k} y^k$ may be given as follows: consider an outcome of a binomial experiment with n trials and k successes. Let $s_j$ count the number of failures that occur after the $j$th success, and let

$$t := \sum_{j=1}^{k} s_j.$$ 

This outcome will be represented in the right member of (2.3) by the term $q^t x^{n-k} y^k$. We can think of $t = \log_q q^t$ as a weighted count of failures that occur after successes.

Another equivalent way of thinking of $t$ is as the number of inversions in a permutation of a sequence of $n - k$ x’s followed by k y’s. For example, consider the permutation $xxyxyxy$ of $x^5y^3 = xyyyyy$. As shown earlier, $Q^{(3,1,0)} x^5 y^3 = xxyxyxy$. The inversions in a permutation are pairs of a y occurring before an x: these pairs are in the third and fourth entry, the third and fifth entry, the third and seventh, and finally the sixth and seventh; four such inversions in all. The number of inversions in a given permutation corresponds to the size of the indexing partition; the size of (3,1,0) is 4. The number of permutations of $x^{n-k} y^k$ containing exactly $t$ inversions is denoted $\text{inv}(k, n-k; t)$, and by [Andrews (1976, p. 40, Theorem 3.5)],

$$\text{inv}(n-k, k; t) = \#P_{k,n-k}(t).$$

Accordingly, those who prefer permutations to partitions may wish to replace all subsequent references to “$\#P_{k,n-k}(t)$” by “inv($k-n; k; t$)”. If $S_j$ denotes the random variable that counts the number of failures after the $j$th success in a binomial experiment with $n$ independent Bernoulli trials and probability of success equal to $\pi$, let

$$T := \sum_{j=1}^{k} S_j,$$

(4.1) $$P(T = t) = \sum_{k=0}^{n} \#P_{k,n-k}(t)(1-\pi)^{n-k}\pi^k,$$

for $t = 0, 1, 2, \ldots, \lfloor n^2/4 \rfloor$; and 0, otherwise. Note that the way we calculate $P(T = t)$ is to observe the exponent on $q$, and then set $q = 1$. The order of these operation matters, because if we set $q = 1$ first, then the exponent on $q$ becomes inaccessible.
As an example, here is the previous table \((n = 4)\) case of the binomial experiment) grouped by values \(t\) of \(T\), rather than by the values of \(Y\):

| \(t\) | Outcome | LHS summand | corresponding RHS summand |
|-------|----------|--------------|--------------------------|
| 0     | FFFF     | \(xxxx\)     | \(x^4\)                  |
|       | FFFS     | \(xxxy\)     | \(x^3y\)                 |
|       | FFSS     | \(xxyy\)     | \(x^2y^2\)               |
|       | FSSS     | \(xyyy\)     | \(xy^3\)                 |
|       | SSSS     | \(yyyy\)     | \(y^4\)                  |
| 1     | FFSF     | \(xxyx\)     | \(qx^3y\)                |
|       | FSFS     | \(xyxy\)     | \(qx^2y^2\)              |
|       | SFSS     | \(yxxy\)     | \(qxyy^3\)               |
| 2     | FSFF     | \(xyxx\)     | \(q^2x^3y\)              |
|       | SFFS     | \(yxxx\)     | \(q^2x^2y^2\)            |
|       | FSSF     | \(xyyx\)     | \(q^2x^2yz^2\)           |
|       | SFFF     | \(yxxy\)     | \(q^2xy^3\)              |
| 3     | SFSF     | \(yxxy\)     | \(q^3x^2y^2\)            |
|       | SSFF     | \(yxxy\)     | \(q^3xy^3\)              |
| 4     | SSFF     | \(yxxy\)     | \(q^4x^2y^2\)            |

**Remark 4.1.** Observe that for a given value \(t\) of \(T\), the summation bounds

\[
\sum_{k=0}^{n} \#P_{k,n-k}(t)(1 - \pi)^{n-k}\pi^k
\]

may include some terms that are 0. For instance, in the preceding table we see that for \(n = 4\),

\[
P(T = 3) = \sum_{k=1}^{3} \#P_{k,n-k}(t)(1 - \pi)^{3-k}\pi^k
\]

since \(\#P_{0,4}(t) = \#P_{4,0}(t) = 0\).

The function (4.1) is a pmf since clearly \(P(T = t) \geq 0\) for all \(t\), and

\[
\sum_{t=0}^{\lfloor n^2/4 \rfloor} P(T = t) = \sum_{t=0}^{\lfloor n^2/4 \rfloor} \sum_{k=0}^{n} \#P_{k,n-k}(t)(1 - \pi)^{n-k}\pi^k
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (1 - \pi)^{n-k}\pi^k
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (1 - \pi)^{n-k}\pi^k
\]

\[
= 1.
\]
5. A refined binomial distribution

To recap, for the experiment with \( n \) independent Bernoulli trials and constant probability of success \( \pi \) on each trial, let \( Y \) count the number of successes in \( n \) trials. Then \( Y \sim \text{Bin}(n, \pi) \). In the last section, the random variable \( T \) was defined as \( T = \sum_{j=1}^{k} S_j \) where \( S_j \) denotes the random variable that counts the number of failures that occur after the \( j \)th success. Equivalently, \( T \) counts the number of inversions (success before failure pairs) in the outcome of a given instance of the \( n \) independent Bernoulli trials, each with probability of success \( \pi \). Next, we explore the joint distribution of \( Y \) and \( T \).

5.1. The joint distribution of \( Y \) and \( T \). Having defined the random variables \( Y \) and \( T \), let us now consider their joint distribution.

First, a table of the \( n = 4 \) case displaying the outcomes and corresponding probabilities for all values of \( Y \) and \( T \) in the support:

| \( Y \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | FFFF | — | 0 | 0 | 0 |
| 1 | FFS | FFSF | FSFF | SFFF | — |
| 2 | FFSS | FSFS | SFFS, FSSF | SSSF | SSFF |
| 3 | FSSS | SFSS | SFSF | SSF | — |
| 4 | SSSS | — | — | — | — |
| \( \pi^4 \) | 0 | 0 | 0 | 0 |

(5.1) \[ P(Y = k, T = t) = \left( \text{coëff of } q^t \text{ in } \binom{n}{k} q \right) (1 - \pi)^{n-k} \pi^k \]

for \( k = 0, 1, 2, \ldots, n \) and for each \( k, t = 0, 1, \ldots, k(n-k) \); and 0 otherwise.

It is immediate that \( P(Y = k, T = t) \geq 0 \) for all \( k \) and \( t \). Also,

\[
\sum_{k=0}^{n} \sum_{t=0}^{k(n-k)} P(Y = k, T = t) = \sum_{k=0}^{n} \sum_{t=0}^{k(n-k)} \# \mathcal{P}_{k,n-k}(t)(1 - \pi)^{n-k} \pi^k \\
= \sum_{k=0}^{n} (1 - \pi)^{n-k} \pi^k \sum_{t=0}^{k(n-k)} \# \mathcal{P}_{k,n-k}(t)
\]
Thus (5.1) defines a joint pmf.

The marginal pmf of $T$ is given in (4.1), and the marginal pmf of $Y \sim \text{Bin}(n, \pi)$.

5.2. A summation lemma. In order to derive various moments, we will need to take certain weighted sums of the coefficients of $\left[ \begin{array}{c} n \\ k \end{array} \right] q^j$. These will be proved in the following lemma.

**Lemma 5.1.** Let $n$ and $k$ be fixed nonnegative integers. Then

\[
\sum_{j \geq 0} \left( j \cdot \#P_{k,n-k}(j) \right) = \frac{n}{2} \left( \frac{n-2}{k-1} \right),
\]

\[
\sum_{j \geq 0} \left( j^2 \cdot \#P_{k,n-k}(j) \right) = \frac{n}{k} \frac{k(n-k)}{12} \left( n + 1 + 3k(n-k) \right),
\]

where we follow the convention that $\left( \begin{array}{c} n \\ -1 \end{array} \right) = \left( \begin{array}{c} n \\ n+1 \end{array} \right) = 0$ for all $n$.

**Proof.** Recall that $\#P_{k,n-k}(j)$ is given by the coefficient of $q^j$ in $\left[ \begin{array}{c} n \\ k \end{array} \right] q^j$. The desired sum in (5.2) is therefore equal to

\[
\frac{d}{dq} \left[ \begin{array}{c} n \\ k \end{array} \right] q^j \Bigg|_{q=1}.
\]

Letting

\[
f(q) = \left[ \begin{array}{c} n \\ k \end{array} \right] q = \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q^j},
\]

and proceeding by logarithmic differentiation:

\[
\log f(q) = \sum_{j=1}^{k} \left( \log(1 - q^{n-k+j}) - \log(1 - q^j) \right),
\]

thus

\[
\frac{d}{dq} f(q) = \left[ \begin{array}{c} n \\ k \end{array} \right] q^j \sum_{j=1}^{k} \left( \frac{jq^j}{1 - q^j} - \frac{(n-k+j)q^{n-k+j-1}}{1 - q^{n-k+j}} \right).
\]

To find $f'(1)$, put the $j$ term over a common denominator, apply L'Hôpital’s rule to find the limit as $q \to 1$, and finally obtain

\[
f'(1) = \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{j=1}^{k} \frac{n-k}{2} = \frac{n}{k} \frac{(n-k)k}{2} = \frac{n}{2} \left( \frac{n-2}{k-1} \right).
\]
Next, the desired sum in (5.3) is equal to
\[
\frac{d}{dq} \left( \frac{d}{dq} \left( \frac{n}{k} \right) \right) \bigg|_{q=1}.
\]
The details of the derivation are straightforward, as in (5.2), but even more tedious to do by hand; and therefore omitted. □

5.3. Various moments.

**Theorem 5.2.** The first two raw moments of the random variable \( T \) defined by the pmf (4.1) are given by

\[
E(T) = \left( \frac{n}{2} \right) \pi (1 - \pi)
\]

and

\[
E(T^2) = \left( \frac{n}{2} \right) \pi (1 - \pi) \left( \frac{2n-1}{3} + \left( \frac{n-2}{2} \right) \pi (1 - \pi) \right),
\]

and thus the variance

\[
V(T) = \left( \frac{n}{2} \right) \pi (1 - \pi) \left( \frac{2n-1}{3} - \pi (1 - \pi)(2n-3) \right).
\]

**Proof.**

\[
E(T) = \sum_{t=0}^{\lfloor n^2/4 \rfloor} t \ P(T = t)
\]

\[
= \sum_{t=0}^{\lfloor n^2/4 \rfloor} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{t=0}^{\lfloor n/2 \rfloor} t \ #P_{k,n-k}(t) (1 - \pi)^{n-k} \pi^k
\]

\[
= \sum_{k=0}^{n} (1 - \pi)^{n-k} \pi^k \sum_{t=0}^{\lfloor n/2 \rfloor} \ #P_{k,n-k}(t)
\]

\[
= \sum_{k=0}^{n} \left( \frac{n}{2} \right) \binom{n-2}{k-1} (1 - \pi)^{n-k} \pi^k \quad \text{(by (5.2))}
\]

\[
= \left( \frac{n}{2} \right) \pi (1 - \pi) \sum_{k=1}^{n-2} \binom{n-2}{k-1} (1 - \pi)^{n-k-1} \pi^{k-1}
\]

\[
= \left( \frac{n}{2} \right) \pi (1 - \pi).
\]

\[
E(T^2) = \sum_{t=0}^{\lfloor n^2/4 \rfloor} t^2 \ P(T = t)
\]

\[
= \sum_{t=0}^{\lfloor n^2/4 \rfloor} \sum_{k=0}^{\lfloor n/2 \rfloor} t^2 \ #P_{k,n-k}(t) (1 - \pi)^{n-k} \pi^k
\]
A REFINEMENT OF THE BINOMIAL DISTRIBUTION

\begin{align*}
&= \sum_{k=0}^{n} \binom{n}{k} \frac{k(n-k)}{12} \frac{n+1+3k(n-k)}{(n-k) \pi^{n-k} \pi^{k}} \quad \text{(by (5.3))} \\
&= \left( \binom{n}{2} \pi(1-\pi) \left( \frac{2n-1}{3} + \binom{n-2}{k} \pi(1-\pi) \right) \right),
\end{align*}

where the last equality follows by hypergeometric summation (see, e.g. Petkovšek et al., 1996).

One can derive \( \text{Var}(T) \) in the usual way as \( \text{E}(T^2) - (\text{E}(T))^2 \). \qed

**Theorem 5.3.** For the distribution defined by joint pmf (5.1),

\begin{align*}
\text{E}(Y T) &= \left( \binom{n}{2} \pi(1-\pi) \left( \pi(n-2) + 1 \right) \right), \\
\text{Cov}(Y, T) &= \left( \binom{n}{2} \pi(1-\pi)(1-2\pi) \right).
\end{align*}

Proof.

\begin{align*}
\text{E}(Y T) &= \sum_{k=0}^{n} \sum_{t=0}^{k(n-k)} k t \# \mathcal{P}_{k,n-k}(t)(1-\pi)^{n-k} \pi^{k} \\
&= \sum_{k=0}^{n} k(1-\pi)^{n-k} \pi^{k} \sum_{t=0}^{k(n-k)} t \# \mathcal{P}_{k,n-k}(t) \\
&= \sum_{k=0}^{n} k(1-\pi)^{n-k} \pi^{k} \binom{n}{2} \frac{n-2}{k-1} \quad \text{(by (5.2))} \\
&= \left( \binom{n}{2} \pi(1-\pi)(1-2\pi + n\pi) \right) \quad \text{(by hypergeometric summation).}
\end{align*}

Then \( \text{Cov}(Y, T) = \text{E}(Y T) - \text{E}(Y)\text{E}(T) \). \qed

5.4. The conditional distribution of \( T \) given \( Y \). Letting \( n \) and \( k \) be given, the pmf of the conditional distribution of \( T \) given \( Y = k \) is given by

\[ P(T = t \mid Y = k) = \binom{n}{k}^{-1} \cdot \text{coeff of } q^t \text{ in } \binom{n}{k}. \]

**Theorem 5.4.** The conditional expectation

\[ \text{E}(T \mid Y = k) = \frac{k(n-k)}{2} \]

and the conditional variance

\[ \text{V}(T \mid Y = k) = \frac{k(n-k)(n+1)}{12}. \]

Proof.

\[ E(T \mid Y = k) = \binom{n}{k}^{-1} \sum_{t=0}^{\lfloor n^2/4 \rfloor} t \cdot \# \mathcal{P}_{k,n-k}(t) \]
\[ (n \choose k) \frac{1}{k!} \cdot \left. \frac{d}{dq} \left( \frac{q^n}{k} \right) \right|_{q=1} = (n \choose k) \left( \frac{n}{2} \right) \left( \frac{n-2}{k-1} \right) \]

\[ = \frac{k(n-k)}{2} \]

\[ E(T^2 \mid Y = k) = \left( n \choose k \right)^{-1} \sum_{t=0}^{\lfloor n^2/4 \rfloor} t^2 \cdot \#P_{k,n-k}(t) \]

\[ = \frac{k(n-k)}{12} \left( n + 1 + 3k(n-k) \right). \]

5.5. The conditional distribution of \( Y \) given \( T \). Letting \( n \) and \( t \) be given, the pmf of the conditional distribution of \( Y \) given \( T = t \) is given by

\[
P(Y = k \mid T = t) = \frac{\#P_{k,n-k}(t) \pi^k (1 - \pi)^{n-k}}{\sum_j \#P_{j,n-j}(t) \pi^j (1 - \pi)^{n-j}} = \frac{\#P_{k,n-k}(t) \theta^k}{\sum_j \#P_{j,n-j}(t) \theta^j},
\]

where

\[
\theta = \frac{\pi}{1 - \pi}.
\]

It is thus clear that the conditional distribution of \( Y \) given \( T = t \) is considerably messier than the expressions encountered thus far. For example, in the \( n = 4 \) case,

\[
P(Y = 2 \mid T = 3) = \frac{(1 - \pi)^2 \pi^2}{(1 - \pi)^3 \pi + (1 - \pi)^2 \pi^2 + (1 - \pi) \pi^3} = \frac{(1 - \pi) \pi}{(1 - \pi)^2 + (1 - \pi) \pi + \pi^2} = \frac{(1 - \pi) \pi}{1 - \pi + \pi^2},
\]

where the algebraic simplifications undertaken here do not generalize to arbitrary \( n, k, \) and \( t \).

An expression for the \( r \)th raw moment of \( Y \) given \( T = t \),

\[
E(Y^r \mid T = t) = \sum_k k^r \frac{\#P_{k,n-k}(t) \pi^k (1 - \pi)^{n-k}}{\sum_j \#P_{j,n-j}(t) \pi^j (1 - \pi)^{n-j}}
\]

\[ = \frac{\sum_k k^r \cdot \#P_{k,n-k}(t) \theta^k}{\sum_j \#P_{j,n-j}(t) \theta^j}, \]

with \( \theta \) as in (5.6), is therefore not particularly enlightening.
5.6. Example/Application. One way to think of our random variable $T$ is as a measure of homogeneity in the sense described as follows. Consider the classic example of tossing a fair coin $n = 15$ times. Suppose that 6 of these tosses come up heads. From the classical binomial distribution we know that there are $\binom{15}{6} = 5005$ different 15-tuples that consist of 6 heads and 9 tails. But if we further allow the $t$ to encode additional information about the where the 6 heads and 9 tails appear in the tuple, we may observe the following. The possible values of $t$ in this example are 0 through 54 inclusive, because the degree of the Gaussian polynomial $\binom{n}{k}_q$ is $k(n - k) = 6(15 - 6) = 54$ and none of the coefficients of $q$ in the Gaussian polynomial vanish. By Theorem [5.4] the mean value of $T$ given 6 successes is 27. A value of $T$ close to 27 indicates that the heads and tails are very well mixed together, i.e. are rather homogeneous. On the other hand, value of $T$ close to 0 indicates that nearly all of the tails occurred up front, with nearly all of the heads near the end. A value of $T$ close to the maximum (54 in this example) indicate the opposite: that nearly all of the heads occurred in the early trials and nearly all of the tails occurred in the later trials.

6. Conclusion

In this paper, we consider interpreting the Potter–Schützenberger quantum binomial theorem, and a generalization thereof, as providing a refinement of the binomial probability distribution, in which additional information is preserved about the sequence of successes and failures in the binomial experiment.

It seems plausible that other discrete probability distributions could be refined in an analogous way. This possibility will be explored in future work.

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