SINGULARITIES OF RICCI FLOW AND DIFFEOMORPHISMS

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Abstract. Comparing and recognizing metrics can be extraordinarily difficult because of the group of diffeomorphisms. Two metrics, that could even be the same, could look completely different in different coordinates. This is the gauge problem. The general gauge problem is extremely subtle for non-compact spaces. Often it can be avoided if one uses some additional structure of the particular situation. However, in many problems there is no additional structure. Instead we solve the gauge problem directly in great generality.

The techniques and ideas apply to many problems. We use them to solve a well-known open problem in Ricci flow: Strong rigidity of cylinders. Strong rigidity is an illustration of a shrinker principle that uniqueness radiates out from a compact set. It implies that if one tangent flow at a future singular point is a cylinder, then all tangent flows are.

We solve the gauge problem by solving a nonlinear system of PDEs. The PDE produces a diffeomorphism that fixes an appropriate gauge in the spirit of the slice theorem for group actions. We then show optimal bounds for the displacement function of the diffeomorphism.

Strong rigidity relies on gauge fixing and several other new ideas. One of these is “propagation of almost splitting”, another is quadratic rigidity in the right gauge, and a third is an optimal polynomial growth bound for PDEs that holds in great generality.

0. Introduction

Suppose we have two weighted manifolds \((M_i, g_i, f_i)\) for \(i = 1, 2\) satisfying some PDE. Assume that on a large, but compact set, the manifolds \(M_i\), metrics \(g_i\) and weights \(e^{-f_i}\) almost agree after identification by a diffeomorphism.

- Is there a diffeomorphism so that the metrics and weights are the same everywhere?

This is a common problem in many questions. A major obstacle for understanding this is the infinite dimensional gauge group of diffeomorphisms:

- Two metrics, that could even be the same, could look very different in different coordinates.

In some situations the gauge problem can be avoided if there is some additional structure. A classical example is the Killing-Hopf theorem that classifies constant curvature metrics. This classification uses that the curvature tensor is constant to construct a “canonical” isometry between the two spaces. In general, the gauge problem can be solved when there is

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\[^{1}\text{"Gauge theory is a term which has connotations of being a fearsomely complicated part of mathematics - for instance, playing an important role in quantum field theory, general relativity, geometric PDE, and so forth. But the underlying concept is really quite simple: a gauge is nothing more than a coordinate system that varies depending on location... By fixing a gauge (thus breaking or spending the gauge symmetry), the model becomes something easier to analyse mathematically.... Deciding exactly how to fix a gauge (or whether one should spend the gauge symmetry at all) is a key question in the analysis of gauge theories, and one that often requires the input of geometric ideas and intuition into that analysis." \[\text{Tt.}\]}\]
strong asymptotic decay and circumvented when the space is characterized in a coordinate-free way, such as a large symmetry group, the vanishing of a special tensor, or a strong curvature condition.

In the problems we will be interested in, the manifold will be non-compact and we will not have any special structure. Thus, we will be forced to deal with the gauge problem head on. We do this by solving a nonlinear PDE to get a diffeomorphism that fixes the gauge in the spirit of the slice theorem for group actions. Since the manifold is non-compact, we need strong bounds for the displacement function of the diffeomorphism. This works in great generality, giving new tools with broad applications.

0.1. Where do questions like these arise? Problems about identifying spaces occur in many different situations. The one we are interested in is from Ricci flow. A one parameter family \((M, g(t))\) of manifolds flows by the Ricci flow if \(g_t = -2 \text{Ric}_{g(t)}\), where \(\text{Ric}_{g(t)}\) is the Ricci curvature of the evolving metric \(g(t)\) and \(g_t\) is the time derivative of the metric.\(^2\)

The key to understand Ricci flow is the singularities that form. The simplest singularity is a homothetically shrinking sphere that becomes extinct at a point. The product of a sphere with \(\mathbb{R}\) gives a shrinking cylinder. This singularity is called a neck pinch. It is more complicated than the spherical extinction. In dimension three, spherical extinctions and neck pinches are essentially the only singularities. Adding another \(\mathbb{R}\) factor gives a cylinder with a two-dimensional Euclidean factor; this singularity is the so-called bubble sheet that is only recently partially understood. With each additional \(\mathbb{R}\) factor, the singularities become more complicated and the sets where they occur are larger.

A triple \((M, g, f)\) of a manifold \(M\), metric \(g\) and function \(f\) is a gradient shrinking Ricci soliton (or *shrinker*) if

\[
\text{Ric} + \text{Hess}_f = \frac{1}{2} g.
\]

Shrinkers give special solutions of the Ricci flow that evolve by rescaling up to diffeomorphism and are singularity models. They arise as time-slices of limits of rescalings (magnifications) of the flow around a fixed future singular point in space-time. Such limits are said to be tangent flows at the singularity. Even when \(M\) is compact, the shrinker is typically non-compact and the convergence is on compact subsets. Shrinkers also arise in other important ways, such as blowdowns from \(-\infty\) for ancient flows. Ancient flows are flows that have existed for all prior times; every tangent flow is ancient. Shrinkers are the key singularities in Ricci flow and will be our focus.

Among shrinkers, cylinders are particularly important; they are the most prevalent. This is because the Almgren-Federer-White dimension reduction, cf. \([W, KL2, BaK1]\), divides the singular set into strata whose dimension is the dimension of translation-invariance of the blowup. For Ricci flow, this suggests:

- Top strata of the singular set corresponds to points where the blowup is \(\mathbb{R}^{n-2} \times N^2\).
- The next strata consists of points where the blowup is \(\mathbb{R}^{n-3} \times N^3\).

\(^2\)The gauge group is known to cause difficulties in Ricci flow. The invariance under the group makes the equation degenerate so standard parabolic techniques do not apply. The Ricci-DeTurck flow deals with this by fixing an arbitrary initial gauge and then solving coupled equations for evolving metrics and gauges to get a parabolic PDE. The arbitrary initial choice of gauge makes this unsuitable for the problems we are interested in since the gauge has to be right to compare two solutions.
The $N$’s are themselves shrinkers and have been classified in low dimensions by Cao-Chen-Zhu, Hamilton, Ivey, Naber, Ni-Wallach, Perelman, [CaCZ]. In dimensions two and three, they are $N^2 = S^2$ or $\mathbb{RP}^2$ and $N^3 = S^3$ or $S^2 \times \mathbb{R}$ (plus quotients). The classification in dimension three relies on an equation for the 2-tensor $\text{Ric}$ that fails in higher dimensions where there is no similar classification. In fact, there are large families of shrinkers in higher dimensions. Combining dimension reduction with the classification in low dimensions, we see that the most prevalent singularities are:

$$S^2 \times \mathbb{R}^{n-2} \text{ followed by } S^3 \times \mathbb{R}^{n-3} \text{ (and quotients).}$$

As one approaches a singularity in the flow and magnifies, one would like to know which singularity it is. Since most singularities are non-compact yet the evolving manifolds are closed, one only sees a compact piece of the singularity at each time as one approaches it. The next theorem recognizes the most prevalent singularities from just a compact piece.

**Theorem 0.2.** Cylindrical shrinkers $S^\ell \times \mathbb{R}^{n-\ell}$ are strongly rigid for any $\ell$.

*Strong rigidity means* that if another shrinker is close enough on a large compact set, then it must agree. The theorem holds for products of $\mathbb{R}^{n-\ell}$ with quotients of $S^\ell$ and a large class of other positive Einstein manifolds; see Section 5 for details. An important difficulty is that there are nontrivial infinitesimal variations, i.e., in the kernel of the linearized operator (and not generated by diffeomorphisms). One consequence of Theorem 0.2 is that the infinitesimal variations are not integrable.

Uniqueness is important in many areas of geometry, PDE, and general relativity. Unlike here, one typically makes global assumptions - e.g., symmetries, curvature conditions, or asymptotics at infinity. In most problems in geometric PDEs, it would be unthinkable to control an entire solution from just knowing roughly how it looks on a compact set. This is exactly what we do here. If one knew exactly how it looked like on a compact set, it would be much less surprising and essentially follow from unique continuation. The surprising thing here is that we only assume closeness and only on a compact set and this is enough to characterize the shrinker. This is an illustration of a shrinker principle which roughly says that “uniqueness radiates outwards”. Nothing like this is true for Einstein manifolds (or steady solitons), where gravitational instantons contain arbitrarily large arbitrarily Euclidean regions. The shrinker principle was originally discovered in mean curvature flow [CIM, CM2]. It has been conjectured since that something similar holds for Ricci flow, but the gauge group has been one of the major obstacles. In mean curvature flow, the gauge is circumvented using extrinsic coordinates.

Tangent flows are limits of a subsequence of rescalings at the singularity. A priori different subsequences might give different limits. Using Theorem 0.2, we get the following uniqueness:

**Theorem 0.3.** For a Ricci flow, if one tangent flow at a point in space-time is a cylinder, then all other tangent flows at that point are also cylinders.

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3In GR uniqueness and stability of solutions to Einstein’s equations are fundamental problems and the gauge group causes well-known difficulties. Unlike here, in GR space-time is assumed to have strong asymptotic decay. The two central difficulties in stability of black holes are determining the final state (uniqueness) and proving convergence. Convergence can only be established relative to a coordinate system which cannot be a priori fixed but has instead to be constructed dynamically. This is often referred to as “the famous problem of gauge determination”. For the uniqueness of the final state, the gauge group can be circumvented when the space can be characterized by the vanishing of a special tensor like the Mars-Simon tensor.
Unlike most results in Ricci flow, these results hold for every $n$ and $\ell$. Increasing the dimension of the Euclidean factor is a subtle problem (e.g. surgery, cylindrical estimates, and $k$-convexity estimates only allow small Euclidean factors). For general $n$ and $\ell$, cylinders do not have a coordinate-free characterization. This is a major part of the difficulty.

At singularities where the tangent flows are compact shrinkers, the singularities are isolated in space-time. For compact shrinkers, rigidity was proven in dimension three by Hamilton and by Huisken for higher dimensional spheres. Even in the compact case, rigidity fails in general; see [Bs, Ho, BGK, Ca] and [Kö, SZ].

Rigidity for necks $S^{n-1} \times \mathbb{R}$ was proven independently by Li-Wang [LW2]. They are able to circumvent the gauge problem using that their Euclidean factor is a line. They do that, in part, using tensors with special properties on the product of a sphere with a line to prove asymptotic structure and approximate symmetry. Once they have this, they are able to use again that their Euclidean factor is a line to adapt Brendle’s symmetry improvement [Br1, Br2] to get $O(n)$ symmetry and, finally, Kotschwar’s classification of rotationally symmetric shrinkers [K].

0.2. What is needed for rigidity? We need to show that if two shrinkers are close on a large but compact set, then there is a global diffeomorphism between them that preserves the metric and weight. The two shrinkers are not assumed to be globally diffeomorphic, so we must build the global map starting from the map between compact pieces. This is done in stages, first building the initial map out to a larger scale so that it still roughly preserves the metric and weight (Theorem 6.1). This comes at the cost of a loss in the estimates: the metrics and weights will not be as close on the larger set as they were initially. This loss means that this process cannot be repeated indefinitely. To overcome this, we make a change of gauge to recover the loss and get even better estimates on the larger scale (Theorem 9.1). Together, Theorems 6.1 and 9.1 can be iterated to get better and better estimates on larger and larger scales, eventually giving the strong rigidity. Estimates proving polynomial losses will be played off against exponential gains.

There are four key ingredients in the proof of strong rigidity. All of them are new.

1. Gauge fixing.
2. New polynomial growth estimates for PDEs.
3. Propagation of almost splitting.
4. Quadratic rigidity in the right gauge.

We will use the new polynomial growth estimates as ingredients in both (1) and (3).

0.3. Gauge fixing. Fix $(M, g, f)$. We are given a diffeomorphism from a large compact set in $M$ to a second weighted space.

- The pull-back metric and weight are $g + h$ and $e^{-f-k}$.
- $h$ and $k$ are small on the compact set.

Composing with a diffeomorphism on $M$ gives a different $h$ and $k$. We want to mod out by this group action, by choosing a diffeomorphism so that the new $h$ is orthogonal to the group action. This is gauge fixing.

One of the most interesting results of transformation groups is the existence of slices. A slice for the action of a group on a manifold is a submanifold which is transverse to the
Ebin and Palais proved the existence of a slice for the diffeomorphism group of a compact manifold acting on the space of all Riemannian metrics. The slice can be thought of as the gauge fixing on the compact manifold.

In our setting, $M$ is noncompact and gauge fixing is choosing a diffeomorphism $\Phi$ on $M$ so $\tilde{h} = \Phi^* (g + h) - g$ is orthogonal to the group action. Orthogonality corresponds to
\[
\text{div}_f \tilde{h} = 0,
\tag{0.4}
\]
where $\text{div}_f (h) = \text{div} (h) - h(\nabla f, \cdot)$. The equation (0.4) is a nonlinear PDE for $\Phi$. Terms involving $\text{div}_f$ come up again and again, so many quantities simplify in this gauge and having them drop out as they do when $\text{div}_f \tilde{h} = 0$ makes things possible to analyze.

We construct the diffeomorphism $\Phi$ that solves (0.4) using an iteration scheme for the linearized operator $\mathcal{P}$ on vector fields $Y$. Using optimal polynomial bounds on $\mathcal{P}$, we show sharp polynomial bounds for the displacement function of $\Phi$
\[
x \to \text{dist}_g(x, \Phi(x)).
\]
For applications, it is crucial that we only assume closeness on a compact set and, in particular, a priori the two shrinkers do not need to be diffeomorphic. This means that we cannot fix the gauge at the outset. Instead we need to apply our gauge fixing procedure iteratively to fix the gauge on larger and larger scales as we move outward and show closeness on larger and larger scales. To pull this off requires very strong estimates for the displacement. Our optimal estimates show that the displacement of the gauge fixing diffeomorphism grows at a sharp polynomial rate. These results are very general and apply to all shrinkers.

On a shrinker $(M, g, f)$, the natural gaussian $L^2 = L^2(e^{-f})$ norm is given by $\|u\|^2_{L^2(e^{-f})} = \int_M u^2 e^{-f}$. Diffeomorphisms near the identity are infinitesimally generated by integrating a vector field $X$. The infinitesimal change of the metric is given by the Lie derivative of the metric with respect to $X$. This is equal to $-\frac{1}{2} \text{div}_f^* X$, where $\text{div}_f^*$ is the operator adjoint of $\text{div}_f$ with respect to the gaussian inner product. Thus, if we define the operator $\mathcal{P}$ by
\[
\mathcal{P}X = \text{div}_f \circ \text{div}_f^* X,
\tag{0.5}
\]
then the linearization of (0.4) is to find a vector field $Y$ with
\[
\mathcal{P}Y = \frac{1}{2} \text{div}_f h.
\tag{0.6}
\]
A detailed analysis of $\mathcal{P}$ and its properties plays an important role in the gauge fixing.

Solutions of (0.6) are unique once we require that $Y$ is orthogonal to the kernel of $\mathcal{P}$. The kernel is the Killing fields. We will solve (0.6) on any shrinker (Theorem 4.15) and show via $L^2$ methods that $\|Y\|_{W^{1,2}(e^{-f})} \leq \|\text{div}_f h\|_{L^2(e^{-f})}$. Given the non-compactness, $L^2$ estimates are not sufficient to implement the iteration scheme and we need stronger polynomial estimates.  

If the group is compact and Lie and the space is completely regular, Mostow proved, as a generalization of works of Gleason, Koszul, Montgomery, Yang and others, that there is a slice through every point. If the group is not compact but Lie and if the space is a Cartan space, then Palais proves the same result.

The $L^2$ theory for $\mathcal{P}$ shares formal similarities with Hörmander’s influential $L^2 \bar{\partial}$ method in several complex variables. In the $L^2 \bar{\partial}$ method, one solves the Poisson equation $\bar{\partial}u = F$, with estimates, where $\bar{\partial}F = 0$. To do so, one introduces the adjoint of $\bar{\partial}$ with respect to a weight. Hörmander’s idea for the weight came from Carleman’s method for proving unique continuation of a PDE. Here we solve $\mathcal{P}Y = F$, where $F = \frac{1}{2} \text{div}_f h$ is orthogonal to the kernel of $\text{div}_f^*$. Hörmander’s method gives weighted $L^2$ bounds for
The problems are magnified by that initial closeness is only on a given compact set. As one builds out to get closeness on larger sets, one needs at each step to adjust the entire diffeomorphism so the normalization is zero on larger and larger sets.

The operator $\mathcal{P}$ is related to the generalized Ornstein-Uhlenbeck operator $\mathcal{L} = \Delta - \nabla \nabla f$. Given a vector field $X$ on a shrinker, the operators $\mathcal{L}$ and $\mathcal{P}$ commute and are related by

$$-2 \mathcal{P} X = \nabla \text{div}_f X + \mathcal{L} X + \frac{1}{2} X$$

(Proposition 2.3 and Lemma 2.1). The unweighted version of $\mathcal{P}$ was used implicitly by Bochner to show that closed manifolds with negative Ricci curvature have no Killing fields and later by Bochner and Yano to show that the isometry group is finite. The unweighted operator also arises in general relativity and fluid dynamics. The weighted operator $\mathcal{P}$ appears to have been largely overlooked. The relationship between $\mathcal{P}$ and the unweighted version mirrors the relationship between the Ornstein-Uhlenbeck operator and the Laplacian.

0.4. Optimal growth bounds. Laplace discovered that on the line eigenfunctions of $\mathcal{L} u = u'' - \frac{x^2}{4} u'$ in the gaussian $L^2$ space are polynomials whose degree is exactly twice the eigenvalue. These polynomials were later rediscovered twice. First by Chebyshev and a few years later by Hermite. They are known as the Hermite polynomials and the eigenvalue equation as the Hermite equation. They play an important role in diverse fields.

On the line, the space $L^2(e^{-\frac{x^2}{4}})$ allows extremely rapid growth, so it is surprising that the $L^2(e^{-\frac{x^2}{4}})$ eigenfunctions grow just polynomially. The standard proofs of this use the special structure of Euclidean space that do not extend to manifolds without making very strong assumptions. However, we will prove that this polynomial growth holds for a wide class of manifolds, metrics and weights. In many settings one has an $n$-dimensional Riemannian manifold $(M,g)$ with two nonnegative functions $f$ and $S$ that satisfy

$$\Delta f + S = \frac{n}{2},$$
$$|\nabla f|^2 + S = f,$$

and where $f$ is proper and $C^n$. Two important examples are shrinkers in both Ricci flow and mean curvature flow (MCF). In Ricci flow, $S$ is scalar curvature, while $f = \frac{|x|^2}{4}$ and $S = |H|^2$ in MCF, where $H$ is the mean curvature (see, e.g., [Hu], [CM1], [CM9]).

Theorem 0.10. If (0.8) and (0.9) hold and a tensor $u \in L^2(e^{-f})$ satisfies $\mathcal{L} u = -\lambda u$, then $u$ grows polynomially of degree at most $2\lambda$.

This and a corresponding Poisson version give powerful new tools with many applications, including in the proofs of propagation of almost splitting and gauge fixing.

Combining Theorem 0.10 with the following gives optimal growth bound for eigenvector fields of $\mathcal{P}$ on any Ricci shrinker (note that $\mathcal{P}$ and $\mathcal{L}$ have opposite signs):

Theorem 0.11. On any shrinker, any eigenvector field $Y$ for $\mathcal{P}$ with eigenvalue $-\lambda$ can be written as the $L^2(e^{-f})$-orthogonal sum of two eigenvector fields for $\mathcal{L}$. One is $\text{div}_f$-free with eigenvalue $2\lambda + \frac{1}{2}$ and the other is $\frac{2}{2\lambda+1} \nabla \text{div}_f Y$ and has eigenvalue $\lambda$.\[\bar{\partial}\] similar to our weighted bounds for $\mathcal{P}$. To introduce a second weight to capture the growth à la Carleman and Hörmander is less natural here. Instead, we go a different route to prove stronger bounds.
These growth estimates hold in remarkable generality and without any assumptions on asymptotic decay. This is surprising and in contrast to most other situations, like unique continuation, that require very strong geometric assumptions on the space. A typical starting point for growth estimates is a Pohozaev identity or commutator estimate that comes from a dilation, or approximate dilation, structure. We have none of these here in this general setting. In contrast, we rely on a miraculous cancellation for just the right quantity.

0.5. Propagation of almost splitting. One of the important new ingredients is that a Ricci shrinker close to a product $N \times \mathbb{R}^{n-\ell}$ on a large scale remains close on a fixed larger scale. The idea is that the initial closeness will imply that $\mathcal{L}$ has eigenvalues that are exponentially close to $\frac{1}{2}$. The drift Bochner formula on a shrinker implies that every eigenvalue is at least $\frac{1}{2}$ with equality only when it splits. We show that being close to $\frac{1}{2}$ gives that the hessian is almost zero in $L^2(e^{-f})$, which is very strong when the weight $e^{-f}$ is close to one but says almost nothing further out. The crucial point is that our polynomial growth estimates imply that the hessian can grow only polynomially, so the very small initial bound gives bounds much further out. Thus, the gradients of these eigenfunctions give the desired almost parallel vector fields and almost splitting. This is very much a Ricci flow fact that does not have an analog in MCF where there is no corresponding description of the bottom of the spectrum.

Once we have this metric almost splitting, we show that it also almost splits as shrinker on the larger scale. Namely, the cross-sections are close to $N$ and the potential $f$ is well-approximated by $\frac{|x|^2}{4}$. However, there is a loss in the estimates - it may look less cylindrical on the larger scale - that makes this impossible to iterate on its own.

0.6. Quadratic rigidity. The propagation of almost splitting and gauge fixing give that the shrinker is close to a cylinder on a large set via a diffeomorphism that fixes the gauge. The last of the four key ingredients is an estimate for the difference in metrics that is small enough to be iterated. For this, it is essential that the gauge be right, or else it just isn’t true. The closeness cannot be seen via linear analysis. However, we show that there is a second order rigidity that gives the estimate; we call this quadratic rigidity.

To explain the estimate, let $(M, g, f)$ be the cylinder and $(M, g+h, f+k)$ the shrinker that is close on a large compact set. We need bounds on $h$ and $k$ that can be iterated. The linearization of the shrinker equation is

\begin{equation}
\frac{1}{2} L h + \text{Hess} \frac{1}{2} \text{Tr}(h)-k + \text{div}^*_f \text{div}_f h.
\end{equation}

This linearization was derived by Cao-Hamilton-Ilmanen in their calculation of the second variation operator for Perelman’s entropy. The operator $L$ acts on 2-tensors by $L h = \mathcal{L} h + 2 R(h)$ and $R(h)$ is the natural action of the Riemann tensor.

Since $(M, g+h, f+k)$ is also a shrinker, \eqref{0.12} must be at least quadratic in $(h, k)$. The last two terms in \eqref{0.12} are gauge terms - i.e., in the image of $\text{div}^*_f$ and there is no reason for these - or $h$ - to be small if not in right gauge. In the right gauge, $h$ satisfies the Jacobi equation $L h = 0$ up to higher order terms. This does not force $h$ to be small since cylinders have non-trivial Jacobi fields that could potentially integrate to give nearby shrinkers. However, it will give that $h$ is a Jacobi field to first order. The Jacobi field is described by a quadratic Hermite polynomial $u$, so $|h|$ is $|u|$ up to higher order. The second variation of the shrinker
**equation** in the direction of the Jacobi field is given by the tensor

\[ -2 |\nabla u|^2 \text{Ric} - 2 S u \text{Hess}_u - S \nabla u \otimes \nabla u, \]

where \( S \) is scalar curvature. The second order Taylor expansion will imply that (0.13) vanishes to at least third order in \( h \), so the quadratic expression (0.13) is at least cubic in \( u \). When \( u \) is small, this implies that \( u \) and \( h \) vanish; we will have extra error terms so will get that \( h \) is exponentially small, giving the improvement that we needed to iterate.

0.7. **Further applications.** Rigidity and uniqueness of blowups are fundamental questions in regularity theory with many applications. In mean curvature flow, they play a major role in understanding the singular set, proving optimal regularity, understanding solitons, classifying ancient solutions, and understanding low entropy flows. In MCF, cylinders are rigid by [CIM, CM10] and cylindrical blowups unique by [CM2, CM9].

One of the central problems in many areas of dynamical systems, ergodic theory, PDEs and geometry is to understand the dynamics of a flow near singularities. Such as classifying nearby singularities, determining whether flows have unique limits or oscillate, and identifying dynamically stable solutions that attract nearby flows. These questions are more complicated in the presence of a gauge group. The techniques introduced here open a door for understanding dynamical properties for Ricci flows nearby. By further developing these techniques, we show uniqueness of blowups for Ricci flow.

Theorem 0.2 was first announced on July 6, 2018 at the Stanford conference “Minimal Surfaces and Mean Curvature Flow”.

**1. Elliptic systems on tensor bundles and their commutators**

In this section, the triple \((M,g,f)\) is a manifold with Riemannian metric \( g \) and a function \( f \). Given a constant \( \kappa \), define the symmetric 2-tensor

\[ \phi = \kappa g - \text{Ric} - \text{Hess}_f. \]

The triple \((M,g,f)\) is a gradient Ricci soliton when \( \phi = 0 \); it is shrinking for \( \kappa = 1/2 \), steady for \( \kappa = 0 \), and expanding for \( \kappa = -1/2 \); see [H, Ch, Ca, ChL, ChLN, CRF, KL1, P, T]. Later, we will take \( \kappa = 1/2 \) and focus on shrinking solitons. For now, we leave \( \kappa \) as a variable as the results here apply to all three cases.

We recall some basic properties of \( \mathcal{L} \). First, \( \mathcal{L} \) is self-adjoint for the weighted \( L^2 = L^2(e^{-f}) \) norm \( \int_M (\cdot) e^{-f} \) and \( \mathcal{L} = -\nabla^* \nabla \) where \( \nabla^* \) is the adjoint of \( \nabla \) with respect to the weighted \( L^2 \) norm. When \( V \) is a vector field and \( u \) is a function with compact support, then integration by parts gives

\[ \int \langle \nabla u, V \rangle e^{-f} = -\int u \text{div} (V e^{-f}) = -\int u \left( \text{div} V - \langle V, \nabla f \rangle \right) e^{-f}. \]

Motivated by this, define \( \text{div}_f \) on vector fields by \( \text{div}_f V = -\nabla^* V = \text{div} V - \langle V, \nabla f \rangle \).

Let \( R_{ijkl} \) be the full Riemann curvature tensor in an orthonormal frame, so

\[ R_{ijkl} = \langle R(e_i,e_j) e_k, e_l \rangle = \langle \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{e_i} \nabla_{e_j} e_k + \nabla_{[e_i,e_j]} e_k, e_l \rangle. \]
The sign convention is that $\text{Ric}_{ij} = R_{kikj}$, where, by convention, we sum over the repeated index $k$. Define the operator $L$ on a 2-tensor $B$ in an orthonormal frame by

$$L B_{ij} = \mathcal{L} B_{ij} + 2 R_{\ell i kj} B_{\ell k}.$$  

Since $\mathcal{L} g = 0$ (as the metric is parallel), we see that $L g_{ij} = 2 \text{Ric}_{ij}$.

The next result gives Simons-type differential equations for the Ricci and scalar curvature, $\text{Ric}$ and $S$, in terms of the drift operators $L$ and $\mathcal{L}$ and the tensor $\phi$.

**Theorem 1.5.** We have

$$L \text{Ric}_{ij} = 2 \kappa \text{Ric}_{ij} + 2 R_{kijn} \phi_{nk} - \phi_{ij,kk} + \phi_{jk,ki} + \phi_{ik,kj},$$

$$\mathcal{L} S = 2 \kappa S - 2 |\text{Ric}|^2 - 2 \text{Ric}_{kn} \phi_{nk} - 2 \Delta \phi_{kk} + 2 \phi_{ik,ki}.$$

When $\phi = 0$, Theorem 1.5 recovers well-known identities for gradient Ricci solitons (cf. [CaZ], [H], [T]). However, the theorem applies to *any* metric $g$ and weight $e^{-\phi}$. Allowing $\phi \neq 0$ is important in analyzing Ricci flow near a singularity. Furthermore, even for solitons, it is useful to allow $\phi \neq 0$ when “cutting off” a non-compact solution.

### 1.1. Bochner formulas and commutators.

To keep notation short, let $f_{i_{1}\cdots i_{k}}$ denote the $(k-1)$-st covariant derivative of $\nabla f$ evaluated on $(e_{i_{1}}, \ldots, e_{i_{k}})$, where $e_{i_{k}}$ goes into the slot for the last derivative. In the calculations below, we work at a point $p$ in an orthonormal frame $e_{i}$ with $\nabla e_{i} e_{j} = 0$ at $p$. We will use subscripts on a bracket to denote the ordinary directional derivative. For example, $f_{ijk} = (\nabla_{e_{k}} \text{Hess}_{f})(e_{i}, e_{j})$ and

$$f_{ij} = \nabla_{e_{i}} (\text{Hess}_{f}(e_{i}, e_{j})) = f_{ijk} + \text{Hess}_{f}(\nabla_{e_{k}} e_{i}, e_{j}) + \text{Hess}_{f}(e_{i}, \nabla_{e_{k}} e_{j}),$$

where the last equality is the Leibniz rule. Thus, at $p$ we have $f_{ijk} = (f_{ij})_{k}$. We use corresponding notation for tensors, with a comma to separate the derivatives from the original indices. Thus, if $Y$ is a vector field, then $Y_{i} = \langle Y, e_{i} \rangle$ and $(Y_{i})_{j} = Y_{ij} + \langle Y, \nabla_{e_{j}} e_{i} \rangle$. The next lemma computes the commutator of $\nabla$ and $\mathcal{L}$, i.e., the drift Bochner formula, [BE], [L].

**Lemma 1.9.** If $Y$ is a vector field, then $Y_{i,j,k} - Y_{i,k,j} = R_{kijn} Y_{n}$. In particular, $u_{ijk} - u_{ikj} = R_{kijn} u_{n}$ and we get the drift Bochner formulas

$$\mathcal{L} \nabla u = \nabla \mathcal{L} u + (\text{Ric} + \text{Hess}_{f})(\nabla u, \cdot) = \nabla \mathcal{L} u + \kappa \nabla u - \phi(\nabla u, \cdot),$$

$$\mathcal{L} \nabla u = \mathcal{L} \nabla u + \kappa \nabla u - \phi(\nabla u, \cdot).$$

**Proof.** The first claim is essentially the definition of $R$. The second claim follows immediately with $Y = \nabla u$. Next, using the second claim, we have

$$L \nabla u_{i} = u_{ij} - u_{ij} f_{j} + f_{i} - u_{ij} f_{j} = u_{jji} + R_{jik} u_{k} - u_{i,j} f_{j} = u_{jji} + R_{ik} u_{k} - (u_{k} f_{k})_{i} + u_{k} f_{ik}$$

$$= (\nabla \mathcal{L} u_{i} + (\text{Ric}_{ik} + f_{ik}) u_{k} = (\nabla \mathcal{L} u_{i}) + \kappa u_{i} - \phi_{ik} u_{k}.$$  

Finally, (1.11) follows by taking the adjoint of (1.10) since $(\mathcal{L} \nabla)^{*} = -\nabla \mathcal{L}$ and we have $(\nabla (\mathcal{L} + \kappa))^{*} = -(\mathcal{L} + \kappa) \nabla \mathcal{L}$. 

We compute the gradient and Hessian of $S$ (cf. [ChLN], [PW] for solitons):
Lemma 1.13. The gradient and Hessian of $S$ are given by

\begin{align}
\frac{1}{2} S_i &= \nabla^j \text{Ric}_{ij} = \text{Ric}_{ik} f_k - \phi_{kk,i} + \phi_{ik,k}, \\
\frac{1}{2} S_{ij} &= -\phi_{ik,kj} - f_{kikj} = \text{Ric}_{ik,j} f_k + \text{Ric}_{ik} f_{kj} - \phi_{kk,ij} + \phi_{ik,kj}.
\end{align}

Proof. The first equality is a standard consequence of the contracted second Bianchi identity

\begin{equation}
\text{Ric}_{kn,i} + R_{kijn,j} - \text{Ric}_{in,k} = 0.
\end{equation}

Use the first equality, take the divergence of (1.1) and use Lemma 1.9 to get

\begin{equation}
-\text{Ric}_{ij,j} = \phi_{ij,j} + f_{jij} = \phi_{ij,j} + f_{jji} + \text{Ric}_{ij} f_j = \phi_{ij,j} + \text{Ric}_{ij} f_j + (\kappa n - \phi_{jj} - S) i.
\end{equation}

This gives the second equality. The second claim follows from taking the derivative of (1.17). Taking the derivative of the first claim gives the third claim.

Corollary 1.18. We have $\frac{1}{2} \left( S + |\nabla f|^2 - 2 \kappa f \right)_i = -\phi_{ik} f_k - \phi_{kk,i} + \phi_{ik,k}$ and

\begin{equation}
\frac{1}{2} \left( \mathcal{L} f + 2 \kappa f \right)_i = \phi_{ik} f_k + \frac{1}{2} \phi_{kk,i} - \phi_{ik,k}.
\end{equation}

Proof. Substituting the definition of $\phi$ in the first claim in Lemma 1.13 gives

\begin{equation}
\frac{1}{2} S_i = \text{Ric}_{ik} f_k - \phi_{kk,i} + \phi_{ik,k} = \kappa f_i - f_{ik} f_k - \phi_{ik} f_k - \phi_{kk,i} + \phi_{ik,k}.
\end{equation}

This gives the first claim. The second claim follows from the first and $\Delta f = n \kappa - S - \text{Tr} \phi$.

We next compute the Laplacian of $\text{Hess} f$.

Lemma 1.21. We have the following formula

\begin{equation}
f_{ijkk} = -\text{Ric}_{ji,k} f_k + 2 R_{kjin} f_{nk} + \phi_{kk,ji} - \phi_{jk,ki} - \phi_{ik,kj}.
\end{equation}

Proof. Working at $p$, Lemma 1.9 gives that

\begin{equation}
f_{ijkk} = (f_{ijk}) k = (f_{ik,j} + R_{kjin} f_n) k = f_{ik,j} + R_{kjin,k} f_n + R_{kjin} f_{nk}.
\end{equation}

The Ricci identity gives

\begin{equation}
f_{ijk} = f_{kikj} + R_{kijn} f_{ni} + R_{kjin} f_{kn} = f_{kikj} + \text{Ric}_{jn} f_{ni} + R_{kjin} f_{kn}.
\end{equation}

Using this in (1.23) gives

\begin{equation}
f_{ijk} = f_{kikj} + R_{kijn} f_{ni} + R_{kjin} f_{kn} + R_{kjin,k} f_n + R_{kjin} f_{nk}.
\end{equation}

Using the second and third claims in Lemma 1.13 (and symmetry of $S_{ij}$) gives

\begin{equation}
\text{Ric}_{jk,i} f_k + \text{Ric}_{jk} f_{ki} - \phi_{kk,ji} + \phi_{jk,ki} = -\phi_{ik,kj} - f_{kikj}.
\end{equation}

Applying this to the first two terms on the right in (1.25), we get

\begin{equation}
f_{ijkk} = -\text{Ric}_{jk,i} f_k + \phi_{kk,ji} - \phi_{jk,ki} - \phi_{ik,kj} + R_{kjin,k} f_n + 2 R_{kjin} f_{nk}
= (-\text{Ric}_{jk,i} + R_{mijk,m}) f_k + \phi_{kk,ji} - \phi_{jk,ki} - \phi_{ik,kj} + 2 R_{kjin} f_{nk}.
\end{equation}

The lemma follows from this and using the trace (1.16) of the second Bianchi identity to rewrite the first term on the right as $-\text{Ric}_{jk,i} + R_{mijk,m} = -\text{Ric}_{ij,k}$.
Proof of Theorem 1.32. Use the definition of $\phi$ to write $\mathcal{L} \text{Ric} = -\Delta \phi - \Delta \text{Hess}_f - \nabla \nabla f \text{Ric}$. Using Lemma 1.21 we get
\begin{equation}
(1.33) \quad \mathcal{L} \text{Ric} = -\phi_{ij,kk} - f_{ij,kk} - \text{Ric}_{ij,k} f_k = -2 R_{kjin} f_{nk} - \phi_{ij,kk} - \phi_{kk,ji} + \phi_{jk,ki} + \phi_{ik,kj} .
\end{equation}
The definition of $\phi$ gives $f_{nk} = -\phi_{nk} + \kappa g_{nk} - \text{Ric}_{nk}$. Thus, we have
\begin{equation}
(1.29) \quad R_{kjin} f_{nk} = -R_{kjin} \phi_{nk} - R_{kjin} \text{Ric}_{nk} - \kappa \text{Ric}_{ij} .
\end{equation}
Substituting this gives
\begin{equation}
(1.30) \quad (\mathcal{L} \text{Ric})_{ij} = 2 \kappa \text{Ric}_{ij} + 2 R_{kjin} \text{Ric}_{nk} + 2 R_{kjin} \phi_{nk} - \phi_{ij,kk} - \phi_{kk,ji} + \phi_{jk,ki} + \phi_{ik,kj} ,
\end{equation}
and (1.6) follows from substituting the definition of $L$. The second claim follows from taking the trace over $i = j$ in (1.6). \hfill \square

1.2. The $f$-divergence and its adjoint. As in [CaZ], define the $f$-divergence of a symmetric 2-tensor $h$ to be the vector field $(\text{div}_f h)(e_i) = e^j (e^{-f} h_{ij})_j = h_{ij,j} - f_j h_{ij}$. The second equality in Lemma 1.13 gives $(\text{div}_f \text{Ric})(e_i) = \text{Ric}_{ik,k} - \text{Ric}_{ik} f_k = -\phi_{kk,i} + \phi_{ik,k}$, so that $\text{div}_f \text{Ric} = 0$ on a soliton. The adjoint $\text{div}_f^*$ of $\text{div}_f$ is given on a vector field $Y$ by
\begin{equation}
(1.31) \quad (\text{div}_f^* Y)(e_i, e_j) = -\frac{1}{2} (\nabla_i Y_j + \nabla_j Y_i) .
\end{equation}
Namely, if $\int (|Y|^2 + |\nabla Y|^2 + |h|^2 + |\nabla h|^2) e^{-f} < \infty$, then $\int (h, \text{div}_f^* Y) e^{-f} = \int (Y, \text{div}_f h) e^{-f}$. Note that $\text{div}_f^*$ applied to a gradient gives $\text{div}_f^* \nabla v = -\text{Hess}_v$. Thus, if $\text{div}_f h = 0$, then $h$ is orthogonal to any Hessian and, more generally, to variations coming from diffeomorphisms since $-2 \text{div}_f^* Y$ is the Lie derivative of the metric in the direction of $Y$.

The next theorem computes the commutator of $\mathcal{L}$ with $\text{div}_f$ and $\text{div}_f^*$. As a consequence, $L$ preserves the image of $\text{div}_f^*$ when $(M, g, f)$ is a gradient Ricci soliton.

Theorem 1.32. If $V$ is a vector field and $h$ is a symmetric two-tensor, then
\begin{equation}
L \text{div}_f^* (V) = \text{div}_f^* (\mathcal{L} V + \kappa V) + \phi_{jn} V_{i,n} + \phi_{in} V_{j,n} - \frac{V_n}{2} (2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}) ,
\end{equation}
(1.33) \quad \text{div} L h = (\mathcal{L} + \kappa) \text{div}_f h - 2 h_{nj} (\text{div}_f \phi)_{j} - 2 h_{in,j} \phi_{ij} - \frac{h_{ij}}{2} (2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}) .

Corollary 1.34. If $(M, g, f)$ is a gradient Ricci soliton, then $L \text{Hess}_u = \text{Hess}_{(2 \kappa u + \mathcal{L} u)}$.

Proof. Theorem 1.32 with $V = \nabla u$ and $\phi = 0$ gives that $-L \text{Hess}_u = \text{div}_f^* (\mathcal{L} \nabla u + \kappa \nabla u)$. This and the drift Bochner formula $\mathcal{L} \nabla u = \nabla \mathcal{L} u + \kappa \nabla u$, see (1.10), gives the claim. \hfill \square

The next lemma will be used in the proof of Theorem 1.32.

Lemma 1.35. We have that $\text{div}_f R_{kjin} = R_{kjin,k} - R_{nijk} f_k = \phi_{ji,n} - \phi_{jn,i}$. Moreover, if $V$ is vector field, then
\begin{equation}
(1.36) \quad V_{i,jk} = V_{i,kj} + \text{Ric}_{jn} V_{i,n} + R_{nijk} f_k V_n + 2 R_{kjin} V_{n,k} + V_n (\phi_{ji,n} - \phi_{jn,i}) .
\end{equation}
Proof. The trace of the second Bianchi identity gives that $R_{kjin,k} = \text{Ric}_{jn,i} - \text{Ric}_{ji,n}$. Using that $\text{Ric} = \kappa g - \phi - \text{Hess}_f$, $g$ is parallel, and then using Lemma 1.19 gives
\begin{equation}
(1.37) \quad R_{kjin,k} = \text{Ric}_{jn,i} - \text{Ric}_{ji,n} = \phi_{ji,n} - \phi_{jn,i} + f_{ji,n} - f_{jn,i} = \phi_{ji,n} - \phi_{jn,i} + R_{nijk} f_k .
\end{equation}
This gives the first claim. Lemma 1.39 gives $V_{i,jk} = V_{i,kj} + R_{kijn} V_n$. Working at a point where $\nabla_i e_j(p) = 0$ for the orthonormal frame $e_i$, differentiating gives

$$V_{i,jkk} = (V_{i,kj} + R_{kijn} V_n)_k = V_{i,kjk} + R_{kjin,k} V_n + R_{kjin} V_{n,k}.$$  

The Ricci identity gives

$$V_{i,kjk} = V_{i,kkj} + R_{kjkn} V_{i,n} + R_{kjin} V_{n,k} = V_{i,kkj} + R_{jkn} V_{i,n} + R_{kjin} V_{n,k}.$$  

Using this gives $V_{i,jkk} = V_{i,kkj} + R_{jkn} V_{i,n} + R_{kjin,k} V_n + 2 R_{kjin} V_{n,k}$. The second claim follows from this and the first claim. 

**Corollary 1.40.** If $V$ is a vector field, then

$$(\mathcal{L} \nabla V)_{i,j} = (\nabla \mathcal{L} V)_{i,j} + \kappa V_{i,j} + 2 R_{kijn} V_{n,k} - \phi_{jn} V_{i,n} + V_n(\phi_{ji,n} - \phi_{jn,i}).$$  

**Proof.** We will work at a point where $\nabla_i e_j(p) = 0$ and $g_{ij} = \delta_{ij}$. Lemma 1.35 gives that

$$(\Delta \nabla V)_{i,j} = (\nabla \Delta V)_{i,j} + \text{Ric}_{jn} V_{i,n} + R_{nijn} f_k V_n + 2 R_{kijn} V_{n,k} + V_n(\phi_{ji,n} - \phi_{jn,i}).$$  

On the other hand, $(\nabla_j \nabla f V)_i = f_{jn} V_{i,n} + f_n V_{i,nj}$ so we get

$$(\Delta \nabla V)_{i,j} = (\nabla \mathcal{L} V)_{i,j} + f_{jn} V_{i,n} + f_n V_{i,nj} + \text{Ric}_{jn} V_{i,n} + R_{nijn} f_k V_n + 2 R_{kijn} V_{n,k} + V_n(\phi_{ji,n} - \phi_{jn,i}).$$  

Using that $\text{Ric} + \text{Hess}_f = \kappa g - \phi$ and, by Lemma 1.39 that $V_{i,jn} = V_{i,nj} + R_{nijn} V_k$, we get

$$(\mathcal{L} \nabla V)_{i,j} = (\nabla \mathcal{L} V)_{i,j} + \kappa V_{i,j} - \phi_{jn} V_{i,n} + 2 R_{kijn} V_{n,k} + V_n(\phi_{ji,n} - \phi_{jn,i}).$$  

**Proof of Theorem 1.32.** Set $W_{ij} = \text{div}^f V$. Since $-2 W_{ij} = V_{i,j} + V_{j,i}$, Lemma 1.35 gives

$$-2 \Delta W_{ij} = -2 W_{i,jk} = V_{i,kkj} + V_{j,kki} + \text{Ric}_{jn} V_{i,n} + R_{nijn} f_k V_n + 2 R_{kijn} V_{n,k}$$

$$+ \text{Ric}_{in} V_{j,n} + R_{nijn} f_k V_n + 2 R_{kijn} V_{n,k} + V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$

Relabeling indices, using the symmetries of $R$, and using the definition of $W$ gives

$$R_{kijn} V_{n,k} + R_{kijn} V_{n,k} = R_{kijn} V_{n,k} + R_{nijn} f_k V_n + 2 R_{kijn} V_{n,k} + V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$

Using this and $V_{i,kkj} + V_{j,kki} = -2 \text{div}^f \Delta V$, we can rewrite (1.42) as

$$2 \text{div}^f \Delta V = 2 \Delta W + \text{Ric}_{jn} V_{i,n} - 4 R_{kijn} W_{k,n} + f_k (R_{nijn} + R_{njik}) V_n + \text{Ric}_{jn} V_{i,n}$$

$$+ V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$

Using that $V_{i,jk} = V_{i,kj} + R_{kijn} V_n$ by Lemma 1.39, the first derivative term is

$$-2 \text{div}^f (\nabla \nabla f V) = (V_{i,kj} f_k)_j + (V_{j,kj} f_k)_i = V_{i,kj} f_k + V_{i,kj} f_k + V_{j,ki} f_k + V_{j,kj} f_k$$

$$= (V_{i,jk} - R_{kijn} V_n) f_k + V_{i,kj} f_k + (V_{j,ik} - R_{kijn} V_n) f_k + V_{j,ki} f_k.$$  

Adding the last two equations, we get

$$2 \text{div}^f \mathcal{L} V = 2 LW + (\text{Ric}_{jn} + f_{jn}) V_{i,n} + (\text{Ric}_{in} + f_{in}) V_{j,n}$$

$$+ f_k (R_{nijn} + R_{njik} - R_{kijn} - R_{kijn}) V_n + V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$

The first term on the last line vanishes because of the symmetries of the curvature tensor. Using this and $\text{Ric} + \text{Hess}_f = \kappa g - \phi$, we get

$$2 \text{div}^f \mathcal{L} V = 2 LW + \kappa (V_{i,j} + V_{j,i}) - \phi_{jn} V_{i,n} - \phi_{in} V_{j,n} + V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$  

$$= 2 (L - \kappa) \text{div}^f V - \phi_{jn} V_{i,n} - \phi_{in} V_{j,n} + V_n(2 \phi_{ji,n} - \phi_{jn,i} - \phi_{in,j}).$$
This gives the first claim. The second claim follows from taking the adjoint of the first claim and using that \((L \text{div} f)^* = \text{div}_f L\) and \((\text{div} f (\mathcal{L} + \kappa))^* = (\mathcal{L} + \kappa) \text{ div}_f\). \(\square\)

1.3. Solitons. For a soliton, \((S + |\nabla f|^2 - 2 \kappa f)\) is constant (cf. [ChLN] or Corollary 1.18) and it is customary to subtract a constant from \(f\) so that

\[
S + |\nabla f|^2 = 2 \kappa f .
\]

Combining this with the trace of the soliton equation gives that \(\mathcal{L} f = n \kappa - 2 \kappa f\). If \(\kappa = \frac{1}{2}\) (i.e., a shrinker), then \(S \geq 0\) by [Ch] and we have that

\[
\frac{1}{4} (r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4} (r(x) + c_2)^2 ,
\]

where \(r(x)\) is the distance to a fixed point \(x_0\).

The simplest shrinker is the Gaussian soliton \((R^n, \delta_{ij}, \frac{1}{2} |x|^2)\), followed by cylinders \(S^\ell \times R^{n-\ell}\) with the product metric where the sphere has Ricci curvature \(\frac{1}{2}\) and \(f = \frac{|x|^2}{4} + \frac{\ell}{2}\). There are also shrinkers asymptotic to cones; see, e.g., [FK, KW]. The next lemma gives a concentration for vector fields on any shrinker; cf. lemma 1.5 of [CM9]. We think of this as a concentration because of the asymptotics \(1.49\) of \(f\).

**Lemma 1.50.** If \(M\) is a shrinker and \(Y\) is any vector field or function in \(W^{1,2}\), then

\[
\int |Y|^2 ((|\nabla f|^2 - n) e^{-f}) \leq \int |Y|^2 (f - n) e^{-f} \leq 4 \int |\nabla Y|^2 e^{-f} .
\]

**Proof.** Since \(f\) is normalized so that \(\mathcal{L} f + f = \frac{n}{2}\), we have that

\[
\int |Y|^2 \left( f - \frac{n}{2} \right) e^{-f} = 2 \int \langle \nabla Y, Y \rangle e^{-f} \leq 2 \int |\nabla Y|^2 e^{-f} + \frac{1}{2} \int |Y|^2 |\nabla f|^2 e^{-f} .
\]

We get \(1.51\) since \(f \geq |\nabla f|^2\) by \(1.48\). \(\square\)

We will use the following elementary interpolation inequality:

**Lemma 1.52.** Given any shrinker, if \(Y, \mathcal{L} Y \in L^2\), then \(Y \in W^{1,2}\), \(\text{div}_f Y \in L^2\), and \(\|\nabla Y\|^2_{L^2} \leq 2 \|Y\|_{L^2} \|\mathcal{L} Y\|_{L^2}\). If in addition \(\text{div}_f \mathcal{L} Y \in L^2\), then \(\text{div}_f Y \in W^{1,2}\). Finally, if the sectional curvature is bounded, then \(\|\nabla^2 Y\|^2_{L^2} \leq \|\mathcal{L} Y\|^2_{L^2} + C \|\nabla Y\|^2_{L^2}\).

**Proof.** Let \(\eta\) be a cutoff function with \(|\eta| \leq 1\). The Cauchy-Schwarz inequality, integration by parts and an absorbing inequality give (with \(\| \cdot \| = \| \cdot \|_{L^2}\))

\[
\|\mathcal{L} Y\| \|Y\| \geq - \int \langle \mathcal{L} Y, \eta^2 Y \rangle e^{-f} = \int \left( \eta^2 \nabla Y^2 + \langle \nabla^2 Y, \frac{\nabla Y^2}{2} \rangle \right) e^{-f} \geq \frac{\|\eta \nabla Y\|^2}{2} - 2 \|\nabla \eta Y\|^2 .
\]

Taking a sequence of \(\eta\)'s converging to one and applying the dominated convergence theorem gives that \(Y \in W^{1,2}\) and \(\|\nabla Y\|^2_{L^2} \leq 2 \|Y\|_{L^2} \|\mathcal{L} Y\|_{L^2}\). By this and Lemma 1.50 \(\text{div}_f Y \in L^2\).

Since Lemma 1.9 gives that \(\mathcal{L} \text{div}_f Y = \text{div}_f \mathcal{L} Y - \frac{1}{2} \text{div}_f Y\), it follows that \(\mathcal{L} \text{div}_f Y \in L^2\) if \(\text{div}_f \mathcal{L} Y \in L^2\). The first part of the lemma now gives that \(\text{div}_f Y \in W^{1,2}\). Finally, integrating Corollary 1.40 gives a \(W^{2,2}\) bound when the sectional curvature is bounded. \(\square\)

We will need a \(W^{1,2}\) localization result for eigenfunctions:

...
Lemma 1.53. If $v \in W^{2,2}$ satisfies $\mathcal{L} v = -\mu v$ and $\|v\|_{L^2} = 1$, then
\begin{equation}
\frac{s^2}{4} \int_{s^2} \left\{ v^2 + |\nabla v|^2 \right\} e^{-f} \leq 4 \mu^2 + (n+2) \mu + n. \tag{1.54}
\end{equation}

Proof. Integrating $\frac{1}{2} \mathcal{L} v^2 = |\nabla v|^2 - \mu v^2$ and the drift Bochner formula $\frac{1}{2} \mathcal{L} |\nabla v|^2 = |\text{Hess}_v|^2 + (\frac{1}{2} - \mu) |\nabla v|^2$ gives
\begin{equation}
\|\nabla v\|_{L^2}^2 = \mu \quad \text{and} \quad \|\text{Hess}_v\|_{L^2}^2 = \left( \frac{1}{2} - \mu \right) \mu. \tag{1.55}
\end{equation}
Applying Lemma 1.50 to $v$ and to $\nabla v$ and adding these inequalities gives the claim. \hfill \square

2. Diffeomorphisms and the $\mathcal{P}$ operator

A key tool in this paper for dealing with the infinite dimensional gauge group is a natural second order system operator $\mathcal{P}$ that seems to have been largely overlooked. This operator is defined on vector fields and given by composing $\text{div}_f^*$ with its adjoint $\text{div}_f$ so $\mathcal{P} = \text{div}_f \circ \text{div}_f^*$. In one dimension, $\mathcal{P} = -\mathcal{L}$, but in higher dimensions $\mathcal{P}$ and $\mathcal{L}$ are very different.

A vector field $Y$ is a Killing field if the Lie derivative of the metric with respect to $Y$ is zero, i.e., $\text{div}_f^* Y = 0$. Since a Killing field is determined by its value and first derivative at a point, the space of Killing fields is finite dimensional. Integration by parts shows that if $Y \in W^{1,2}$ and $\mathcal{P} Y \in L^2$, then $\int \langle Y, \mathcal{P} Y \rangle e^{-f} = \|\text{div}_f^*(Y)\|_{L^2}^2$. Thus, the $L^2$ kernel of $\mathcal{P}$ is the space $\mathcal{K}_\mathcal{P}$ of $L^2$ Killing fields.

2.1. Basic properties of $\mathcal{P}$. In this subsection, we prove the basic properties of $\mathcal{P}$. Many of the results are valid on any manifold and for any function $f$. The results are strongest for gradient Ricci solitons - shrinking, steady or expanding; $f$ is normalized so $\mathcal{L} f = n \kappa - 2 \kappa f$ and $\kappa = \pm \frac{1}{2}$, 0. The next lemma relates $\mathcal{P}$ and $\mathcal{L}$.

Lemma 2.1. Given a vector field $Y$, we have
\begin{equation}
-2 \mathcal{P} Y = \nabla \text{div}_f Y + \mathcal{L} Y + \text{Hess}_f(Y, Y) + \text{Ric}(Y, Y). \tag{2.2}
\end{equation}

Thus, for a Ricci soliton, $-2 \mathcal{P} Y = \nabla \text{div}_f Y + \mathcal{L} Y + \kappa Y$.

Proof. Fix a point $p$ and let $e_i$ be an orthonormal frame with $\nabla_{e_i} e_j(p) = 0$. Set $h = \text{div}_f^* Y$, so that $-2 h(e_i, e_j) = \langle \nabla_{e_i} Y, e_j \rangle + \langle \nabla_{e_j} Y, e_i \rangle$. Working at $p$, we have
\begin{equation}
-2 (\text{div}_f h)(e_i) = \nabla_{e_i} \langle \nabla_{e_i} Y, e_j \rangle + \nabla_{e_j} \langle \nabla_{e_j} Y, e_i \rangle - \langle \nabla_{e_i} Y, \nabla f \rangle - \langle \nabla_{e_j} Y, \nabla f \rangle = \langle \nabla_{e_j} \nabla_{e_i} Y, e_j \rangle + \langle \mathcal{L} Y, e_i \rangle - e_i \langle Y, \nabla f \rangle + \langle Y, \nabla_{e_i} \nabla f \rangle. \tag{2.3}
\end{equation}

Commuting the covariant derivatives introduces a curvature term, giving
\begin{equation}
-2 (\text{div}_f h)(e_i) = \langle \nabla_{e_i} \nabla_{e_j} Y, e_j \rangle + \text{Ric}(e_i, Y) + \langle \mathcal{L} Y, e_i \rangle - e_i \langle Y, \nabla f \rangle + \text{Hess}_f(e_i, Y). \hfill \square
\end{equation}

We will next show that on any gradient Ricci soliton $\mathcal{L}$ and $\mathcal{P}$ commute.

Proposition 2.4. For a gradient Ricci soliton and any vector field $V$, $\mathcal{L} \mathcal{P} V = \mathcal{P} \mathcal{L} V$ and $\mathcal{P} \nabla \text{div}_f(V) = \nabla \text{div}_f(\mathcal{P} V)$. 

Proof. By Theorem 1.32 \[ \text{div}_f L \text{div}^*_f (V) = \text{div}_f \text{div}^*_f (\mathcal{L} V + \kappa V) = \mathcal{P} (\mathcal{L} V + \kappa V). \] Moreover, Theorem 1.32 with \( h = \text{div}^*_f V \) gives
\[ \text{div}_f L \text{div}^*_f (V) = (\mathcal{L} + \kappa) \text{div}_f \text{div}^*_f V = (\mathcal{L} + \kappa) \mathcal{P} V. \]
Combining these two equations and cancelling terms gives the first claim. The second follows from the first together with Lemma 2.1.

The next result characterizes \( \mathcal{P} \) locally on all vector fields.

**Proposition 2.6.** The operators \( \mathcal{L} \) and \( \mathcal{P} \) are self-adjoint. Moreover,
\[ -\mathcal{P} V = \begin{cases} \mathcal{L} V = \nabla \mathcal{L} u + \text{Hess}_f (\cdot, \nabla u) + \text{Ric} (\cdot, \nabla u) & \text{if } V = \nabla u; \\ \frac{1}{2} [\mathcal{L} V + \text{Hess}_f (\cdot, V) + \text{Ric} (\cdot, V)] & \text{if } \text{div}_f (V) = 0. \end{cases} \]
For a Ricci soliton, \( \mathcal{L} \) and \( \mathcal{P} \) preserve this orthogonal decomposition.

**Proof.** We have already seen that both \( \mathcal{L} \) and \( \mathcal{P} \) are self-adjoint and that there is an orthogonal decomposition of vector fields into gradient of functions and those with \( \text{div}_f = 0 \). To compute \( \mathcal{P} \nabla u \), use Lemma 2.1 and Lemma 1.9 to get
\[ -2 \mathcal{P} \nabla u = \nabla \text{div}_f \nabla u + \mathcal{L} \nabla u + \text{Hess}_f (\cdot, \nabla u) + \text{Ric} (\cdot, \nabla u) \]
\[ = 2 \mathcal{L} \nabla u = 2 \nabla \mathcal{L} u + 2 [\text{Hess}_f (\cdot, \nabla u) + \text{Ric} (\cdot, \nabla u)]. \]
In particular, for a Ricci soliton \( \mathcal{L} \) and \( \mathcal{P} \) preserves the subspace of vector fields that are gradient of functions. Next, if \( \text{div}_f (V) = 0 \), then Lemma 2.1 gives that \( -2 \mathcal{P} V = \mathcal{L} V + \text{Hess}_f (\cdot, V) + \text{Ric} (\cdot, \nabla u) \), implying that \( -\mathcal{P} V \) is as claimed. Finally, for a Ricci soliton, if \( \text{div}_f (V) = 0 \), then it follows from 1.11 that \( \text{div}_f (\mathcal{L} V) = 0 \), and thus \( \text{div}_f (\mathcal{P} V) = 0 \). This shows that for a Ricci soliton both \( \mathcal{L} \) and \( \mathcal{P} \) preserves the orthogonal splitting.

**Lemma 2.8.** On any gradient Ricci soliton for any vector field \( Y \)
\[ (\mathcal{L} + \kappa) \text{div}_f Y = -\text{div}_f (\mathcal{P} Y), \]
\[ \mathcal{L} \nabla \text{div}_f (Y) = -\nabla \text{div}_f (\mathcal{P} Y), \]
\[ (L - \kappa) \text{Hess}_{\text{div}_f (Y)} = -\text{Hess}_{\text{div}_f (\mathcal{P} Y)}, \]
\[ L \text{div}^*_f (Y) = \text{Hess}_{\text{div}_f (Y)} - 2 \text{div}^*_f (\mathcal{P} Y). \]

**Proof.** Lemma 1.9 together with Lemma 2.1 gives that
\[ \mathcal{L} \text{div}_f Y = \text{div}_f (\mathcal{L} - \kappa) Y = -2 \text{div}_f (\mathcal{P} Y) - \mathcal{L} \text{div}_f (Y) - 2 \kappa \text{div}_f (Y). \]
Thus, (2.9). By Lemma 1.9 \( \mathcal{L} \nabla u = \nabla \mathcal{L} u + \kappa \nabla u \), so (2.9) gives (2.10). Combining (2.9) with Corollary 1.34 gives (2.11). Applying \( \text{div}^*_f \) to Lemma 2.1 gives
\[ -2 \text{div}^*_f (\mathcal{P} Y) = \text{div}^*_f (\mathcal{L} Y) - \text{Hess}_{\text{div}_f (Y)} + \kappa \text{div}^*_f (Y). \]
Theorem 1.32 gives that \( L \text{div}^*_f Y = \text{div}^*_f (\mathcal{L} + \kappa) Y \) and, thus, (2.12).

**Lemma 2.15.** For any gradient Ricci soliton if \( Y, \mathcal{P} Y \in L^2 \), then \( \text{div}_f (Y), \nabla Y \in L^2 \) and
\[ \| \nabla Y \|_{L^2} + \| \text{div}_f Y \|_{L^2} \leq 2 \| Y \|_{L^2} \| (2 \mathcal{P} + \kappa) Y \|_{L^2}. \]
Proof. Since \( Y, \mathcal{P} Y \in L^2 \), so is \( (2 \mathcal{P} + \kappa) Y = -\mathcal{L} Y - \nabla \text{div}_f Y \) (by Lemma 2.1). If \( 0 \leq \eta \leq 1 \) has compact support, then the Cauchy-Schwarz inequality and integration by parts give
\[
\| (2 \mathcal{P} + \kappa) Y \|_{L^2} \| Y \|_{L^2} \geq -\int \eta^2 \langle \mathcal{L} Y, Y \rangle e^{-f} - \int \eta^2 \langle \nabla \text{div}_f Y, Y \rangle e^{-f} \]
\[
= \| \eta \nabla Y \|_{L^2}^2 + \| \eta \text{div}_f Y \|_{L^2}^2 + 2 \int \eta \text{div}_f Y \langle \nabla \eta, Y \rangle e^{-f} + 2 \int \eta \langle \nabla \nabla \eta Y, Y \rangle e^{-f}.
\]
Using an absorbing inequality on the last two terms, then taking \( \eta \to 1 \) and applying the monotone convergence theorem gives the lemma. \( \square \)

Lemma 2.17. For any gradient Ricci soliton if \( Y \) is a weak solution of \( (\mathcal{P} - \lambda) Y = V_0 \), where \( V_0 \) is smooth and \( Y, \text{div}_f Y \in L^2_\text{loc} \), then \( Y \) is smooth.

Proof. Given any smooth \( X \) with compact support, we have \( \int \langle X, V_0 \rangle e^{-f} = \int \langle (\mathcal{P} - \lambda) X, Y \rangle e^{-f} \).
If \( X = \nabla u \) where \( u \in C^\infty_c \), then \( \mathcal{P} \nabla u = \nabla (\mathcal{L} + \kappa) u \) by Proposition 2.6 so
\[
- \int u \text{div}_f (V_0) e^{-f} = \int \langle \nabla u, V_0 \rangle e^{-f} = \int \langle (\mathcal{P} - \lambda) \nabla u, Y \rangle e^{-f} = \int \langle \nabla (\mathcal{L} + \kappa - \lambda) u, Y \rangle e^{-f}
\]
\[
= - \int \langle \mathcal{L} + \kappa - \lambda \rangle u \text{div}_f (Y) e^{-f}.
\]
The last equality used that \( \text{div}_f (Y) \in L^2_\text{loc} \). It follows that \( \text{div}_f (Y) \) is an \( L^2_\text{loc} \) weak solution to \((\mathcal{L} + \kappa - \lambda) \text{div}_f (Y) = \text{div}_f (V_0) \). Since \( V_0 \) is smooth, elliptic regularity gives that \( \text{div}_f (Y) \) is also smooth. Since \( -2 \mathcal{P} = (\mathcal{L} + \kappa) + \nabla \text{div}_f \) by Lemma 2.1 we have
\[
2 \int \langle X, V_0 \rangle e^{-f} = - \int \{ \langle (\mathcal{L} + \kappa + 2 \lambda) X, Y \rangle + \langle X, \nabla \text{div}_f Y \rangle \} e^{-f}.
\]
It follows that \( Y \) is an \( L^2_\text{loc} \) weak solution to \((\mathcal{L} + \kappa + 2 \lambda) Y = \nabla \text{div}_f (Y) - 2 V_0 \). Since the right-hand side is smooth, elliptic regularity gives that so is \( Y \). \( \square \)

3. Optimal growth bounds

In this section, we will prove the optimal growth bound Theorem 0.10. Throughout this section \((M, g, f)\) will be assumed to satisfy (0.8) and (0.9). This applies to all shrinkers in both Ricci flow and MCF, but is much more general than that.
Since \( |\nabla \sqrt{f}| \leq \frac{1}{2} \) by (0.9), the function \( b = 2 \sqrt{f} \) satisfies \( |\nabla b| \leq 1 \) as in [CaZd], cf. [CZZh1]. Throughout, \( \lambda > 0 \) is a constant and \( u \) is a tensor. We will often assume that
\[
\langle \mathcal{L} u, u \rangle \geq -\lambda |u|^2;
\]
this includes eigentensors with \( \mathcal{L} u = -\lambda u \). To understand the growth of \( u \), we will study a weighted average of \( |u|^2 \) on level sets of \( b \)
\[
I(r) = r^{1-n} \int_{b=r} |u|^2 |\nabla b|.
\]
This is defined at regular values of \( b \), but extends continuously to all values to be differentiable a.e. and absolutely continuous. The weight \( |\nabla b| \) will play a crucial role (cf. [CM7] [CM12]}
The growth of $I$ will be bounded above in terms of the solid integral
\begin{equation}
D(r) = r^{2-n} e^{f_r} \int_{b<r} \left( |\nabla u|^2 + \langle L u, u \rangle \right) e^{-f}.
\end{equation}

The frequency $U = \frac{D}{r}$ is defined when $I$ is positive and will measure the growth of $\log I$.

The next theorem is the precise version of Theorem 0.10. It shows that an $L^2$ tensor satisfying (3.1) has frequency bounded by $2\lambda$ and, accordingly, it grows at most polynomially at this rate. This may seem surprising since the weight $e^{-f}$ decays rapidly, so the $L^2$ condition a priori allows extremely rapid growth. The theorem holds very generally and does not assume any cone or dilation structure.

**Theorem 3.4.** Suppose $u, L u \in L^2$, (0.8), (0.9), (3.1) hold, and $u$ does not vanish identically outside a compact set. Given $\epsilon > 0$, there exists $R = R(n, \lambda, \epsilon)$ so if $r > R$, then
\begin{equation}
U(r) \leq 2\lambda \left( 1 + \frac{4\lambda + 2n - 4 + \epsilon}{r^2} \right)
\end{equation}
and for all $r_2 > r_1 > R$ and $c = 2\lambda (4\lambda + 2n - 4 + \epsilon)$
\begin{equation}
I(r_2) \leq I(r_1) \left( \frac{r_2}{r_1} \right)^{4\lambda} e^{c(r_1^2 - r_2^2)}.
\end{equation}

This is sharp for the Ornstein-Uhlenbeck operator on $\mathbb{R}^n$ where the $L^2$ eigenfunctions are Hermite polynomials with degree twice the eigenvalue. The upper bound (3.5) is sharp not just in the $2\lambda$ in front, but in all the other constants as well as can be seen from the Hermite polynomials. The $R$ in Theorem 3.4 does not depend on $f, M$ or $S$. The theorem still holds if (0.8), (0.9), and (3.1) hold outside of a compact set. Moreover, it holds with obvious changes when the constant $n$ in (0.8) is replaced by any other constant. Finally, note that $u$ cannot vanish on an open set if $u$ has unique continuation, e.g. if $L u = -\lambda u$.

There is a long history of studying the growth of solutions to differential equations, inequalities, and systems. At a very rough level, there are two main techniques. The first, exemplified in the work of Carleman and Hörmander, is to consider weighted $L^2$ norms with growing weights. The second, seen for instance in the work of Hadamard and Almgren, is to study the growth of spherical maxima or averages. Almgren’s frequency has been used to show unique continuation and structure of the nodal sets; prior to this, the main tool in unique continuation was Carleman estimates that still is the primary technique. Almgren’s frequency bounds relied on scaling for $\mathbb{R}^n$; cf. CM12, CM3.

As an application, polynomially growing “special functions” are dense in $L^2$. This gives manifold versions of some very classical problems in analysis. Whereas Weierstrass’s approximation theorem shows that polynomials are dense among continuous functions on any compact interval, the classical Bernstein problem, [LM], dating back to 1924, asks if polynomials are dense on $\mathbb{R}$ in the weighted $L^p(e^{-f} \, dx)$ space if $f$ is assumed to grow sufficiently fast at infinity. On the line, the Hermite polynomials are dense in $L^2(e^{-|x|^2} \, dx)$ and Carleson (and implicitly Izumi-Kawata) showed that polynomials are dense in $L^p(e^{-|x|^n} \, dx)$ if and only if $\alpha \geq 1$. A similar problem in several complex variables is the completeness problem, going back to Carleman in 1923, about density of polynomials in weighted $L^2$ spaces of holomorphic functions; [BFW].
For the applications to $\mathcal{P}$, we will need a more general Poisson version where $u$ satisfies
\begin{equation}
\langle \mathcal{L} u, u \rangle \geq -\lambda |u|^2 - \psi,
\end{equation}
where $\psi$ is a nonnegative function. Define the quantity $J$ by
\begin{equation}
J(r) = \int_{b<r} b^{2-n} \psi.
\end{equation}
The next theorem gives polynomial growth in terms of $\lambda$ and $J$.

**Theorem 3.9.** If $u, \mathcal{L} u \in L^2$, (0.8), (0.9), (3.7) hold, $\delta \in (0, 2)$ and $r_2 > r_1 \geq R(\lambda, n, \delta)$, then
\begin{equation}
I(r_2) \leq \left( \frac{r_2}{r_1} \right)^{4\lambda + \delta} \left\{ I(r_1) + \frac{20 \sup J}{4 \lambda + \delta} \right\}.
\end{equation}

One application will be to gradient shrinking Ricci solitons. The standard drift Bochner formula gives that if $\mathcal{L} v = -\left( \frac{1}{2} + \lambda \right) v$, then $\mathcal{L} \nabla v = -\lambda \nabla v$ and (3.1) applies to $u = \nabla v$:

**Corollary 3.11.** If $(M, g, f)$ is a gradient shrinking soliton, then (3.5) and (3.6) hold if $u = \nabla v$ where $v$ is an eigenfunction with eigenvalue $\lambda + \frac{1}{2}$.

The papers [Be, CM11] developed frequencies for conical and cylindrical MCF shrinkers (cf. [Wa]). These results were perturbative in that they assumed the existence of an exhaustion function $r$ that behaves like Euclidean distance up to higher order. For instance, in [Be], it was assumed that $||\nabla r| - 1| = O(r^{-4})$ and $|\text{Hess}_r x - 2g| = O(r^{-2})$. Theorems 3.4, 3.9 in contrast, hold very generally, including for all shrinkers in both Ricci flow and MCF and make no use of any approximate conical structure. This is a crucial new point. A much weaker version of Theorem 3.4, that was not relative, was proven in [CM9] in the special case of MCF.

### 3.1. The level sets of $b$ and the properties of $I$ and $D$.

We will define $D(r)$ and $I(r)$ as solid integrals over sub-level sets $\{ b < r \}$ of a proper $C^n$ function $b$. For these functions to be continuous, we must show that level sets of $b$ have measure zero. This is (2) in the next lemma: (1) will be used to prove absolute continuity, while (3) will be used to show that $I > 0$. Since $b$ is $C^n$, Sard’s theorem gives that almost every level set is regular.

**Lemma 3.12.** Suppose $f : M \to \mathbb{R}$ is a proper function with $\mathcal{L} f = \frac{n}{2} - f$. Let $\mathcal{C}$ denote the set of critical points of $f$ and $\mathcal{H}_r$ the boundary of $\{ f > \frac{n}{2} \}$. We get for $r > \sqrt{2n}$ that:

1. The critical set $\mathcal{C}$ in $\{ f > \frac{n}{2} \}$ is locally contained in a smooth $(n-1)$-manifold.
2. Each level set $\{ f = c \}$ for $c \geq \frac{n}{2}$ has $\mathcal{H}^n(\{ f = c \}) = 0$.
3. The regular set $\mathcal{R}_r = \mathcal{H}_r \setminus \mathcal{C}$ is dense in $\mathcal{H}_r$.

The nodal sets of eigenfunctions have a great deal of structure, but the value zero is special and many properties do not hold for non-zero values. In fact, it is possible to have a level set that is entirely critical, as occurs at the local extrema for the radial eigenfunction $J_0(|x|)$ on $\mathbb{R}^2$ where $J_0$ is the Bessel function of the first kind. However, by (3), this does not occur for the subset $\mathcal{H}_r$ of $\{ f = r \}$ that is the boundary of $\{ f > r \}$.
\textbf{Proof of Lemma 3.12.} Note first that $L f < 0$ on $\{ f > \frac{\alpha}{2} \}$ and, thus, $\Delta f < 0$ on $C \cap \{ f > \frac{\alpha}{2} \}$. Working in a neighborhood of a critical point we can therefore choose a coordinate system $\{ x_i \}$ so that $\partial_{x_i}^2 f < -1$. If $x \in C$, then $\partial_{x_i} f(x) = 0$ and thus by the implicit function theorem we can choose a new coordinate system in a neighborhood of $x$ so that in those coordinates $\{ \partial_{x_i} f = 0 \} \subset \{ y_1 = 0 \}$ and so that $\partial_{x_i}$ is transverse to $\{ y_1 = 0 \}$. We therefore have that (nearby) $C \subset \{ \partial_{x_i} f = 0 \} \subset \{ y_1 = 0 \}$. This gives (1).

For $c > \frac{n}{2}$, claim (2) follows from (1) since $\{ f = c \} \setminus C$ is a countable union of $(n-1)$-manifolds. The borderline case $c = \frac{n}{2}$ in (2) follows from \cite{HHL}.

We turn next to (3). Note first that at $x = (x_1, \ldots, x_n) \in C$ if we let $h(s) = f(s, x_2, \ldots, x_n)$, then $h'(x_1) = 0$ and $h''(x_1) < 0$ so $h$ has a strict local maximum at $x_1$. In particular, any neighborhood of any $x \in C \cap \{ f > \frac{\alpha}{2} \}$ intersects $\{ f < f(x) \}$. Suppose now that the conclusion (3) fails; so suppose that there exists $x \in H_r$ and a neighborhood $O$ so that $O \cap H_r \subset C$. It follows that $O \cap H_r \subset \{ y_1 = 0 \}$. Since $O \cap H_r$ separates the two non-empty sets $O \cap \{ f > \frac{\alpha^2}{4} \}$ and $O \cap \{ \frac{\alpha^2}{4} > f \}$ and $O \cap H_r$ is contained in $\{ y_1 = 0 \}$ it follows that $O \cap H_r = O \cap \{ y_1 = 0 \}$ and after possibly changing the orientation of $y_1$ we may assume that $O \cap \{ y_1 > 0 \} \subset \{ f > \frac{\alpha^2}{4} \}$ and $O \cap \{ y_1 < 0 \} \subset \{ f < \frac{\alpha^2}{4} \}$. This, however, contradicts that at $x$ we have that $\partial_{x_i}^2 f < 0$ and $\partial_{x_i}$ is transverse to the level set $\{ y_1 = 0 \}$ so both $O \cap \{ y_1 > 0 \}$ and $O \cap \{ y_1 < 0 \}$ contains points where $f < f(x) = \frac{\alpha^2}{4}$.

The functions $I(r), D(r)$ and $U(r)$ may not be differentiable everywhere, but they will be absolutely continuous and differentiable a.e. A function $Q(r)$ is \textit{absolutely continuous} on an interval $\mathcal{I}$ if for every $\epsilon > 0$, there exists $\delta > 0$ so that if $U_\alpha(r_\alpha, R_\alpha)$ is a finite disjoint union of intervals in $\mathcal{I}$ with $\sum (R_\alpha - r_\alpha) < \delta$, then we have $\sum |Q(R_\alpha) - Q(r_\alpha)| < \epsilon$. Absolutely continuous functions are precisely the ones where the fundamental theorem of calculus holds (\cite{E}, page 165): $Q$ is absolutely continuous if and only if it is continuous, differentiable a.e., the derivative is in $L^1$, and for every $r_1 < r_2$

\begin{equation}
Q(r_2) - Q(r_1) = \int_{r_1}^{r_2} Q'(t) \, dt.
\end{equation}

We will use the following standard fact: If $Q_1$ and $Q_2$ are absolutely continuous and $W : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz on the range of $(Q_1, Q_2)$, then $W(Q_1, Q_2)$ is absolutely continuous.

\textbf{Lemma 3.14.} Suppose that $b$ is a proper $C^\infty$ function and $H^n(|\nabla b| = 0) = 0$ in $\{ b \geq r_0 \}$ for some fixed $r_0$. If $g$ is a bounded function and $Q(r) = \int_{r_0 < b < r} g$, then $Q$ is absolutely continuous and $Q'(r) = \int_{b=r} g |\nabla b| \sqrt{V(b)} \ a.e.$

\textbf{Proof.} By separately considering the positive and negative parts of $g$, it suffices to assume that $g \geq 0$ is bounded. Define a sequence of functions $Q_i$ by

\begin{equation}
Q_i(r) = \int_{r_0 < b < r} \frac{g |\nabla b|}{|\nabla b| + i^{-1}}.
\end{equation}

The functions $\frac{g |\nabla b|}{|\nabla b| + i^{-1}}$ are bounded above by $g$ everywhere and converge to the bounded function $g$ a.e. (since $H^n(|\nabla b| = 0) = 0$), so $\lim_{i \to \infty} Q_i(r) = Q(r)$ by the dominated convergence theorem. Define functions $q_i(t)$ and $q(t)$ at regular values $t$ of $b$ by

\begin{equation}
q_i(t) = \int_{b=t} \frac{g}{|\nabla b| + i^{-1}} \quad \text{and} \quad q(t) = \int_{b=t} \frac{g}{|\nabla b|}.
\end{equation}
Since $b$ is $C^m$, Sard’s theorem (3.4.3 in [F]) gives that a.e. $t$ is a regular value of $b$ and, thus, these functions are defined a.e. The co-area formula ([F], page 243) gives that

$$\text{(3.17)} \quad Q_i(r) = \int_0^r q_i(t) \, dt.$$

The sequence $q_i$ is monotonically increasing with $q_i \leq q_{i+1} \leq \cdots < q$. Moreover, $q_i$ converges to $q$ a.e. The monotone convergence theorem gives that

$$\text{(3.18)} \quad \lim_{i \to \infty} \int_0^r q_i(t) \, dt = \int_0^r q(t) \, dt.$$

Combining this with (3.17) and $\lim_{i \to \infty} Q_i(r) = Q(r)$ gives the lemma. \qed

3.1.1. Absolute continuity of $I$ and $D$. In the remainder of this section, we specialize to $M$ non-compact and $f$ satisfying (0.8) and (0.9) and $b = 2 \sqrt{f}$. It follows that

$$\text{(3.19)} \quad |\nabla b|^2 = 1 - \frac{4S}{b^2} \leq 1,$$

$$\text{(3.20)} \quad b \Delta b = n - |\nabla b|^2 - 2S.$$

Since $f$ is nonnegative and proper, then so is $b$ and, thus, the level sets of $b$ are compact. Furthermore, Lemma 3.12 applies and, thus, so does Lemma 3.14.

The definition (3.2) of $I(r)$ at regular values of $b$ will be extended continuously to all values next. To do this, choose a regular value $r_0 < 2 \sqrt{2n}$ of $b$ and set

$$\text{(3.21)} \quad I(r) = \int_{r_0 < b < r} b^{1-n} \left\{ \langle \nabla |u|^2, \nabla b \rangle + \frac{|u|^2}{b^2} 2S (2n - b^2) \right\} + \int_{b=r_0} |u|^2 |\nabla b|.$$

The reason for stopping the integral at $b = r_0$ is that $b^{1-n}$ and $Sb^{-2-n}$ might not be integrable in the interior if $\min b = 0$.

Lemma 3.22. At regular values $r$ of $b$, the definitions (3.2) and (3.21) of $I(r)$ agree and

$$\text{(3.22)} \quad D(r) = \frac{r^{2-n}}{2} \int_{b=r} \langle \nabla |u|^2, \frac{\nabla b}{|\nabla b|} \rangle.$$

Proof. To see that (3.2) and (3.21) agree at regular values, observe that the unit normal to the level set $b = r$ is given, at regular points, by $n = \frac{\nabla b}{|\nabla b|}$, so we can rewrite (3.2)

$$\begin{align*}
\int_{b=r} |u|^2 |\nabla b| &= \int_{b<r} \text{div} (|u|^2 b^{1-n} \nabla b) \\
&= \int_{b<r} b^{1-n} \left\{ \langle \nabla |u|^2, \nabla b \rangle + |u|^2 \left( \Delta b - \frac{(n-1) |\nabla b|^2}{b} \right) \right\}.
\end{align*}$$

By (3.19) and (3.20), we have that $b \Delta b = n - |\nabla b|^2 - 2S$ and $|\nabla b|^2 = 1 - \frac{4S}{b^2}$ and, thus,

$$\text{(3.23)} \quad b \left( \Delta b - \frac{(n-1) |\nabla b|^2}{b} \right) = n (1 - |\nabla b|^2) - 2S = \frac{2S}{b^2} (2n - b^2).$$

Substituting this into (3.24) gives (3.21). The divergence theorem gives

$$\text{(3.24)} \quad \int_{b=r} \langle \nabla |u|^2, \frac{\nabla b}{|\nabla b|} \rangle = e^{\frac{q^2}{2}} \int_{b<r} \text{div} \left( |u|^2 e^{-f} \right) = e^{\frac{q^2}{2}} \int_{b<r} \mathcal{L} |u|^2 e^{-f}.$$
Multiplying this by \( \frac{r^{2-n}}{2} \) gives (3.23).

**Lemma 3.27.** Both \( I(r) \) and \( D(r) \) are absolutely continuous with derivatives given a.e. by

\[
I'(r) = r^{1-n} \int_{b=r}^\infty \langle \nabla |u|^2, \nabla b \rangle + (2n r^{-2} - 1) r^{1-n} \int_{b=r}^\infty \frac{2 \mathcal{S}|u|^2}{r |\nabla b|},
\]

and

\[
D'(r) = \frac{2-n}{r} D + \frac{r^{2-n}}{2} \int_{b=r}^\infty \mathcal{L} |u|^2 |\nabla b|.
\]

Where \( I \) is positive \( \log I \) is absolutely continuous and the derivative is given a.e. by

\[
r (\log I)'(r) = 2U + (2n r^{-2} - 1) \frac{2r^{1-n}}{I} \int_{b=r}^\infty \frac{S |u|^2}{|\nabla b|}.
\]

Furthermore, \( (\log I)' \leq 2U/r \) a.e. when \( r \geq \sqrt{2n} \).

**Proof.** The continuity of \( I(r) \) (as defined in (3.21)) and \( D(r) \) follows from the dominated convergence theorem since \( \mathcal{H}^n \{b = r\} = 0 \) by (2) in Lemma 3.12. Furthermore, Lemma 3.14 applies to both \( I \) and \( D \) and, thus, both are absolutely continuous and \( I' \) is given a.e. by (3.28) and \( D' \) is given a.e. by (3.29). Equation (3.30) follows from (3.23) and (3.28). Since \( S \geq 0 \), we see that \( (\log I)' = \frac{D'}{I} \leq \frac{2U}{r} \) for \( r \geq \sqrt{2n} \). \( \square \)

### 3.2. Positivity of \( I(r) \)

We show next that \( I(r) > 0 \) when \( r \) is sufficiently large:

**Proposition 3.31.** If \( u, \mathcal{L}u \in L^2 \) and (3.1) holds, then either

- (A) \( I(r) > 0 \) for every \( r > 2 \sqrt{n + 4 \lambda} \), or
- (B) \( u \) vanishes identically outside of a compact set.

An immediate consequence of (A) in Proposition 3.31 is that \( U(r) \) is well-defined and absolutely continuous for \( r > 2 \sqrt{n + 4 \lambda} \), and \( U' \) is given a.e. by

\[
U'(r) = \frac{D'}{I} - \frac{D I'}{I^2}.
\]

The next elementary lemma shows that \( |u| \in W^{1,2} \) and \( |u| |\nabla f| \in L^2 \) if \( u, \mathcal{L}u \in L^2 \) (cf. [CZ92, CM99]).

**Lemma 3.33.** If \( u, \mathcal{L}u \in L^2 \), then \( |\nabla |u||, |\nabla u|, |u| \sqrt{f}, \) and \( |u| |\nabla f| \) are all in \( L^2 \).

**Proof.** By the Kato inequality and (0.9), \( |\nabla |u|| \leq |\nabla u| \) and \( |\nabla f|^2 \leq f \). Thus, it suffices to prove that \( |\nabla u|, |u| \sqrt{f} \in L^2 \). We show first that \( |\nabla u| \in L^2 \). Let \( \eta \) be a compactly supported function with \( |\eta|, |\nabla \eta| \leq 1 \). Since \( \mathcal{L}|u|^2 = 2 |\nabla u|^2 + 2 \langle u, \mathcal{L}u \rangle \), applying the divergence theorem to \( \eta^2 |\nabla u|^2 e^{-f} \) gives

\[
\int \eta^2 |\nabla u|^2 e^{-f} \leq \|u\|_{L^2} \|\mathcal{L} u\|_{L^2} + 2 \int |\eta| |\nabla \eta| |u| |\nabla |u|| e^{-f}.
\]

Using \( |\nabla |u|| \leq |\nabla u| \) and the absorbing inequality \( 2 |\eta| |u| |\nabla u| \leq 2 |u|^2 + \frac{1}{2} \eta^2 |\nabla u|^2 \), we can absorb the \( |\nabla |u|| \) term and then apply the monotone convergence theorem for a sequence of \( \eta \)’s going to one everywhere gives that \( |\nabla u| \in L^2 \). To see that \( |u| \sqrt{f} \in L^2 \), apply the divergence theorem to \( \eta^2 |u|^2 \nabla f e^{-f} \) and use that \( \mathcal{L} f = \frac{n}{2} - f \) to get

\[
\int \eta^2 |u|^2 \left( f - \frac{n}{2} \right) e^{-f} \leq 2 \int \{ \eta^2 |u| |\nabla u|| |\nabla f| + |\eta| |\nabla \eta| |u|^2 |\nabla f| \} e^{-f}.
\]
First, we apply this with \( (3.40) \), applying the dominated convergence theorem to \((3.39)\) gives a Lipschitz. Moreover, \( \nu > 0 \) exists an \( \epsilon > 0 \) such that if \( s \) be any regular value sufficiently close to \( r \), then the level set \( b = s \) is a smooth hyper-surface and \( I(s) \geq \nu > 0 \). The claim follows.

**Proof of Proposition 3.34** Suppose that \( A \) fails and, thus, \( I(r) = 0 \) for some \( r > 2 \sqrt{n+4} \). By Corollary 3.36, we know that \( |u| = 0 \) on \( \mathcal{H}_r = \partial\{b > r\} \). Assume \( B \) also fails and choose a connected component \( \Omega \) of \( \{ |u| > 0 \} \) with \( \Omega \subset \{b > r\} \). This will lead to a contradiction.

By Lemma 3.33, \( |u|, |u| |\nabla f|, |\nabla u| \) and \( |\nabla u| \) are all in \( L^2 \). For each \( j \), let \( \eta_j : \mathbb{R} \to [0, \infty) \) be a smooth function with \( 0 \leq \eta_j \leq 4 \) and

\[
(3.37) \quad \eta_j(x) = \begin{cases} x & \text{for } \frac{1}{j} \leq x, \\ 0 & \text{for } x \leq \frac{1}{2j}. \end{cases}
\]

Let \( \chi \) be the characteristic function of \( M \), i.e., \( \chi \) is one on \( \Omega \) and zero otherwise, and define \( v_j = \eta_j(|u|) \chi \). Note that each \( v_j \) is smooth on all of \( M \) and \( v_j \in W^{1,2} \) since \( v \) is and \( \eta_j \) is Lipschitz. Moreover, \( v_j \) has support in \( \{ b > r \} \) since \( \Omega \subset \{ b > r \} \).

Let \( V \) be a vector field with \( V \in L^2 \) and \( v_j (\text{div} V - \langle V, \nabla f \rangle) \in L^1 \). Given \( \eta \) with compact support and \( |\eta|, |\nabla \eta| \leq 1 \), applying the divergence theorem to \( \eta v_j V e^{-f} \) gives

\[
(3.38) \quad \int \eta \, (\langle \nabla v_j, V \rangle + v_j \, (\text{div} V - \langle V, \nabla f \rangle)) \, e^{-f} = - \int v_j \langle V, \nabla \eta \rangle \, e^{-f}.
\]

Taking a sequence of \( \eta \)'s converging to one, the dominated convergence theorem gives

\[
(3.39) \quad \int (\langle \nabla v_j, V \rangle + v_j \, (\text{div} V - \langle V, \nabla f \rangle)) \, e^{-f} = 0.
\]

By the Lipschitz bound on \( \eta_j \) and the Kato inequality, \( |v_j| \leq 4 |u| \) and \( |\nabla v_j| \leq 4 |\nabla u| \). Furthermore, \( v_j \to |u| \chi \) and \( \nabla v_j \to \chi \nabla |u| \) a.e. (since \( \nabla |u| = 0 \) a.e. on \( \{ |u| = 0 \} \)). Thus, applying the dominated convergence theorem to (3.39) gives

\[
(3.40) \quad \int \Omega (|\nabla |u||, V) + |u| \, (\text{div} V - \langle V, \nabla f \rangle) \, e^{-f} = 0.
\]

First, we apply this with \( V = |\nabla u| \) and then use (3.1) and \( |\nabla u| \leq |\nabla u| \) to get

\[
(3.41) \quad 0 = \int \Omega (|\nabla u|^2 + \langle u, \mathcal{L} u \rangle) \, e^{-f} \geq \int \Omega (|\nabla u|^2 - \lambda |u|^2) \, e^{-f}.
\]

For the second application of (3.40), take \( V = |u| \nabla f \) and use \( \mathcal{L} f = \frac{n}{2} - f \) to get

\[
0 = \int \Omega \left\{ 2 \langle |u| \nabla |u|, \nabla f \rangle + |u|^2 \mathcal{L} f \right\} \, e^{-f} = \int \Omega \left\{ 2 \langle |u| \nabla |u|, \nabla f \rangle + |u|^2 \left( \frac{n}{2} - f \right) \right\} \, e^{-f}.
\]
Since $|\nabla f|^2 \leq f$, the absorbing inequality $2 \langle |u| \nabla |u|, \nabla f \rangle \leq 2 |\nabla |u||^2 + \frac{1}{2} |u|^2 |\nabla f|^2$ gives

$$
\int_{\Omega} |u|^2 \left( f - n \right) e^{-f} \leq 4 \int_{\Omega} |\nabla |u||^2 e^{-f} \leq 4 \lambda \int_{\Omega} |u|^2 e^{-f},
$$

where the last inequality is (3.41). Since $|u| > 0$ and $f = \frac{b^2}{4} > \frac{r^2}{4}$ on $\Omega$, we get that $(\frac{r^2}{4} - n - 4 \lambda) \leq 0$. This is the desired contradiction since $r > 2 \sqrt{n + 4 \lambda}$. \hfill \Box

### 3.3. Growth estimates

Assume now that $u$ satisfies (3.1). We will use that, by Lemma 3.27, $\log I$, $\log D$, and $\log U$ are absolutely continuous as long as $I, D > 0$. One challenge for controlling the growth of $D$ and $I$ is that $D'$ and $I'$ have terms involving $S$, with the wrong sign in one case and a variable sign in the other. The terms will be played off each other and we will be able to control the right combination; this miraculous cancelation makes it work.

**Proposition 3.43.** If $r$ is a regular value of $b$ and $D(r), I(r) > 0$, then

$$
(3.44) \quad r (\log D)' \geq 2 - n + \frac{r^2}{2} + U - \frac{\lambda r^2}{U} - \frac{4 \lambda}{U} r^{1-n} \int_{b=r} S |u|^2 |\nabla b|,
$$

$$
(3.45) \quad r (\log U)'(r) \geq 2 - n + \frac{r^2}{2} - U - \frac{\lambda r^2}{U} + \left( 1 - \frac{2 \lambda}{r^2} - \frac{2 n}{r^2} \right) \frac{2 r^{1-n}}{I} \int_{b=r} S |u|^2 |\nabla b|.
$$

**Proof.** Lemma 3.27 and (3.1) give

$$
D'(r) = \frac{2-n}{r} D + \frac{r^2}{2} D + \frac{r^2-2}{2} \int_{b=r} \mathcal{L} |u|^2 |\nabla b| \geq \frac{2-n}{r} D + \frac{r}{2} D + r^{2-n} \int_{b=r} \left( |\nabla u|^2 - \lambda |u|^2 \right) |\nabla b|.
$$

Since $4 S = b^2 - b^2 |\nabla b|^2$, we get that

$$
(3.46) \quad r (\log D)'(r) \geq 2 - n + \frac{r^2}{2} - \frac{\lambda r^2}{D} + \frac{r^{3-n}}{D} \int_{b=r} |\nabla u|^2 |\nabla b| - \frac{4 \lambda r^{1-n}}{D} \int_{b=r} S |u|^2 |\nabla b|.
$$

Note that by the Cauchy-Schwarz inequality

$$
(3.47) \quad D^2(r) = \left( \frac{r^{2-n}}{2} \int_{b=r} \langle \nabla |u|^2, \frac{\nabla b}{|\nabla b|} \rangle \right)^2 \leq I(r) r^{3-n} \int_{b=r} |\nabla u|^2 |\nabla b|.
$$

Dividing this by $I(r)$ gives $U D \leq r^{3-n} \int_{b=r} |\nabla u|^2 |\nabla b|$. Using this in (3.46) gives (3.44). Combining (3.44) and (3.30) gives (3.35). \hfill \Box

An immediate consequence of the proposition is the following:

**Corollary 3.48.** If $r$ is a regular value with $U(r) > 2 \lambda$ and $r > \sqrt{\frac{2n}{1 - \frac{\lambda}{U}}}$, then

$$
(3.49) \quad (\log U)' \geq \frac{2 - n - U}{r} + r \left( \frac{1}{2} - \frac{\lambda}{U} \right).
$$

We use this to show that if $U$ goes strictly above $2 \lambda$, then it grows quadratically; this does not assume that $u \in L^2$ and, indeed, it is impossible when $u, \mathcal{L} u \in L^2$.

**Theorem 3.50.** Given $\delta > 0$, there exists $R > \sqrt{2n}$ so that if $U(r_0) > (2 + \delta) \lambda$ for some $r_0 \geq R$, then $U(r) \geq \frac{1}{2} r^2 - r$ for every $r$ sufficiently large.
Proof. If \( U(r) > (2 + \delta) \lambda \) for a regular value \( r > \sqrt{(4n (2 + \delta)/\delta)} \), then Corollary 3.48 gives
\[
(\log U)'(r) \geq \frac{2 - n - U}{r} + \frac{\delta r}{2(2 + \delta)}.
\]
It follows that if \((2 + \delta) \lambda < U < \frac{\delta r^2}{5(2 + \delta)}\) and \( r > \sqrt{(4n (2 + \delta)/\delta)} \), then
\[
(\log U)'(r) > \frac{\delta r}{2(2 + \delta)} - \frac{n}{r} - \frac{\delta r}{5(2 + \delta)} > \frac{\delta r}{2 + \delta} \left( \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \right) = \frac{\delta r}{20(2 + \delta)}.
\]
This implies \( U \) is increasing on this interval and that there exists an \( R > 0 \) and \( c > 0 \) such that \( U(r) \geq cr^2 \) for \( r > R \). Thus, by Corollary 3.48 if \( \frac{r^2}{2} - r > U \) for \( r > R \), then
\[
(\log U)' \geq \frac{2 - n}{r} + \frac{1}{c r}.
\]
This forces \( U \) to grow exponentially to the top of this range, eventually giving the claim. \( \square \)

Proof. (of Theorem 3.4). Since \((\log I)' \leq 2 U/r \) for \( r > \sqrt{2n} \) by Lemma 3.27, the growth bound \( (3.6) \) will follow from the bound \( (3.5) \) on \( U \). We first show for any \( \delta > 0 \) that
\[
U(r) \leq 2 \lambda + \delta
\]
for all \( r \) sufficiently large. We will argue by contradiction, so suppose that \( (3.54) \) fails for some \( r \) sufficiently large. Theorem 3.50 gives that \( U \geq \frac{r^2}{2} - r \) for all sufficiently large \( r \). It follows that \( K(r) = D(r) - 4 \lambda I(r) \) is positive for all large \( r \). At a regular value \( r > 2 \sqrt{n} \), Proposition 3.43 and Lemma 3.27 give
\[
r K' \geq \left( 2 - n + \frac{r^2}{2} + U - 8 \lambda \right) D - \lambda r^2 I + \left[ 8 \left( 1 - \frac{2n}{r^2} \right) - 4 \right] \lambda r^{1-n} \int_{b=r}^{S |u|^2 |\nabla b|}
\]
\[
(3.55) \geq (2 - n + r^2 - r - 8 \lambda) D - \lambda r^2 I
\]
\[
\geq (2 - n + r^2 - r - 8 \lambda) K + 4 \lambda \left( 2 - n + \frac{3r^2}{4} - r - 8 \lambda \right) I.
\]
Thus, for \( r \) large, we have \((\log K)' \geq \frac{3}{4} r \). Integrating this gives for \( t > s > R \)
\[
D(t) \geq K(t) \geq K(s) e^{\frac{3r^2}{4} s^2 r^2}.
\]
This implies that
\[
2 \int_{b \leq t} (|\nabla u|^2 + \langle \mathcal{L} u, u \rangle) e^{-\frac{|u|^2}{2}} = \int_{b \leq t} \mathcal{L} |u|^2 e^{-\frac{|u|^2}{2}} = 2 e^{-\frac{r^2}{2}} t^{n-2} D(t) \to \infty \text{ as } t \to \infty.
\]
This is a contradiction since \( \mathcal{L} u \in L^2 \) and \( u \in W^{1, 2} \) by Lemma 3.38 so \( (3.54) \) holds.

We turn to the sharper bound \( (3.55) \); we can assume that \( \lambda > 0 \) since otherwise \( u \) is parallel since \( u, \mathcal{L} u \in L^2 \). The proof is by contradiction, so suppose that \( r \geq R \) satisfies
\[
2 \lambda + \delta \geq U(r) \geq 2 \lambda \left( 1 + \frac{\mu}{r^2} \right),
\]
where \( \mu \in \mathbb{R} \) will be chosen below. At any \( r \) satisfying \( (3.57) \), we have
\[
r^2 \geq \frac{r^2 - U \mu^2}{U} \geq \frac{r^2}{2} \left( 1 - \frac{2 \lambda}{U} \right) \geq \frac{\lambda \mu}{U} \geq \frac{\lambda \mu}{2 \lambda + \delta}.
\]
Together with (3.45), this gives at regular values that
\[
(3.59) \quad r (\log U)'(r) \geq 2 - n + \frac{\lambda \mu - n - 2}{2} - 2 \lambda - \delta + \left( \frac{\lambda \mu - n - 2}{2} + \delta \right) \frac{4 r^{1-n}}{1} \int_{b=r} S |u|^2 |\nabla b|.
\]
Assuming that \( \mu > \left( 2 + \frac{\delta}{\lambda} \right) n \) so the last term is nonnegative, we have
\[
(3.60) \quad r (\log U)'(r) \geq \frac{\lambda \mu - (2 \lambda + \delta)^2 - (n-2)(2 \lambda + \delta)}{2 \lambda + \delta}.
\]
If \( \mu > 4 \lambda + 2n - 4 \), then this is strictly positive for \( \delta > 0 \) sufficiently small, forcing \( U \) to grow out of the range (3.57), giving the desired contradiction. \( \square \)

3.3.1. Examples. We will next consider examples which show that Theorem 3.4 is surprisingly sharp. Not only is the threshold \( 2\lambda \) sharp, but even the next order term is sharp. If \( u = b^2 - 2n \), then \( \mathcal{L} u = -u \), so that \( \lambda = 1 \), and (3.23) gives
\[
(3.61) \quad D(r) = \frac{r^{2-n}}{2} \int_{b=r} \langle \nabla (b^2 - 2n)^2, \overline{\nabla b} \rangle = 2 r^{3-n} (r^2 - 2n) \int_{b=r} |\nabla b| = \frac{2 r^2 I(r)}{r^2 - 2n}.
\]
Therefore, we see that the frequency \( U = \frac{D}{T} \) satisfies
\[
(3.62) \quad U(r) = \frac{2 r^2}{r^2 - 2n} = 2 \left( 1 + \frac{2n}{r^2} + O(r^{-4}) \right) = 2 \lambda \left( 1 + \frac{4 \lambda + 2n - 4}{r^2} + O(r^{-4}) \right).
\]
Next, let \( M = \mathbb{R}, f = \frac{x^2}{4} \), and \( \mathcal{L} \) be the Ornstein-Uhlenbeck operator. The degree \( m \) Hermite polynomial has \( \lambda = \frac{m}{2} \) and is given by \( x^m - m(m-1) x^{m-2} + O(x^{m-4}) \), so that
\[
(3.63) \quad I(r) = 2 \left( r^2 - 2n \right) (m-1) r^{2(m-2)} + O(r^{2(m-2)}).
\]
It follows that
\[
(3.64) \quad 2 U(r) = \frac{r I'}{T} = 2 m \frac{r^{2m} - 2 (m-1)^2 r^{2(m-1)} + O(r^{2(m-2)})}{r^{2m} - 2 m (m-1) r^{2(m-1)} + O(r^{2(m-2)})}.
\]
Thus, we have \( U(r) = m \left( 1 + 2 (m-1) r^{-2} + O(r^{-4}) \right) = 2 \lambda \left( 1 + (4 \lambda - 2) r^{-2} + O(r^{-4}) \right) \).

3.4. Poisson equation. Suppose that \( u \) satisfies \( \langle \mathcal{L} u, u \rangle \geq -\lambda |u|^2 - \psi \), where \( \lambda \geq 0 \) is a constant and \( \psi \geq 0 \) is a function. By Lemma 3.14, \( J \) from (3.8) is absolutely continuous and \( J' \) is given a.e. by
\[
(3.65) \quad J' = r^{2-n} \int_{b=r} \frac{\psi}{|\nabla b|}.
\]
We will use the following immediate analog of Proposition 3.43 (with the additional term in \( D' \) (cf. (3.46)), resulting in \( J' \) terms in (3.67), (3.68)).

Lemma 3.66. If \( r \) is a regular value of \( b \) and \( D(r), I(r) > 0 \), then
\[
(3.67) \quad r (\log D)' \geq 2 - n + \frac{r^2}{2} + U - \frac{\lambda r^2}{U} - \frac{4 \lambda r^{1-n}}{U I} \int_{b=r} S |u|^2 |\nabla b| - \frac{r}{D} J',
\]
\[
(3.68) \quad r (\log U)' \geq 2 - n + \frac{r^2}{2} - U - \frac{\lambda r^2}{U} + \left( 1 - \frac{2 \lambda}{U} - \frac{r}{r^2} \right) \frac{2 r^{1-n}}{I} \int_{b=r} S |u|^2 |\nabla b| - \frac{r}{D} J'.
\]
**Lemma 3.69.** Given \( \delta \in (0,2) \), set \( K = D - (2\lambda + \delta/2) I \). There exists \( r_0(\lambda, \delta, n) \), so that if \( r \geq r_0 \) is a regular value with \( K(r) > 0 \), then

\[
(3.70) \quad r K' \geq \frac{2\lambda r^2}{4\lambda + \delta} K + \left[ U + 2 - n + \frac{\delta r^2}{2(4\lambda + \delta)} - (4\lambda + \delta) \right] D - r J'.
\]

**Proof.** By (3.67) and (3.30), we have

\[
(3.71) \quad r D' \geq \left( 2 - n + \frac{r^2}{2} + U \right) D - \lambda r^2 I - 4\lambda r^{1-n} \int_{b=r} S u^2 |\nabla b| - r J',
\]

\[
(3.72) \quad r (2\lambda + \delta/2) I' = (4\lambda + \delta) D + \left( \frac{2n}{r^2} - 1 \right) (4\lambda + \delta) r^{1-n} \int_{b=r} S u^2 |\nabla b|.
\]

Since \( S \geq 0 \) and \([4\lambda + \delta)(1 - 2n r^{-2}) - 4\lambda] \geq 0\) for \( r \geq r_0(\lambda, \delta, n) \), it follows that

\[
(3.73) \quad r K' \geq \left[ \left( 2 - n + \frac{r^2}{2} + U \right) - (4\lambda + \delta) \right] D - \lambda r^2 I - r J'.
\]

Since \([D - 2\lambda I] = \frac{4\lambda}{4\lambda + \delta} K + \frac{\delta}{4\lambda + \delta} D \), this gives the claim. \(\square\)

**Proof of Theorem 3.39.** Set \( J_0 = \sup J \). We will show that

\[
(3.74) \quad K(r) \leq 10 J_0 \text{ for all } r > R(\lambda, \delta, n).
\]

Once we have (3.74), we use (3.30) to get that

\[
(3.75) \quad r J' \leq 2 D \leq (4\lambda + \delta) I + 20 J_0.
\]

Equivalently, \((r^{-4\lambda + \delta}) I') \leq 20 r^{-4\lambda + \delta - 1} J_0\). Integrating this gives (3.10).

We will prove (3.74) by contradiction, so suppose instead that \( K(r_0) > 10 J_0 \) for some large \( r_0 \). At any regular value \( r \) with \( K(r) > 0 \), we have \( D(r) > 0 \), thus, also \( I(r) > 0 \) by Lemma 3.22 and \( \dot{U}(r) > 2\lambda + \delta/2 > 0 \). Lemma 3.69 then implies that if \( r \) is large enough and \( K > 0 \), then \( K' \geq -J' \). Integrating this from \( r_0 \) gives that \( K(r) \geq 9 J_0 \) for all \( r \geq r_0 \) and, thus, also that \( D, I > 0 \) and \( U > (2\lambda + \delta/2) > 0 \). In particular, (3.68) gives

\[
(3.76) \quad (\log U)' \geq \frac{2 - n - U}{r} + \frac{\lambda r}{2} - \frac{J'}{D} \geq \frac{2 - n - U}{r} + \frac{r}{2} - \frac{\lambda r}{U} - \frac{J'}{9 J_0}.
\]

Suppose first \( U(r) < \frac{\delta r^2}{4(4\lambda + \delta)} \) for every larger \( r \), then (3.76) would give

\[
(3.77) \quad (\log U)' \geq \frac{2 - n}{r} - \frac{\delta r}{4(4\lambda + \delta)} + \frac{r}{2} - \frac{2\lambda r}{4\lambda + \delta} - \frac{J'}{9 J_0} = \frac{2 - n}{r} - \frac{J'}{9 J_0} + \frac{\delta r}{4(4\lambda + \delta)}.
\]

Integrating this contradicts the upper bound on \( U \), so we conclude that there is a large \( r \) where \( U \geq \frac{\delta r^2}{4(4\lambda + \delta)} \). Next, at any large \( r \) where \( \frac{\delta r^2}{8(4\lambda + \delta)} \leq U(r) \leq \frac{r^2}{2} - r \), then (3.76) gives

\[
(3.78) \quad (\log U)' \geq 1 + \frac{2 - n}{r} - \frac{8\lambda (4\lambda + \delta)}{\delta r} - \frac{J'}{9 J_0},
\]

forcing \( U \) to grow exponentially and, thus, eventually overtake the quadratic upper bound. Thus, we get \( R_1 \) large so that for all \( r \geq R_1 \) we have \( U > \frac{r^2}{2} - r - \frac{1}{2} \) (the last term comes
from integrating $\frac{\rho - \rho_0}{\rho_0}$. Using this lower bound for $U$ in Lemma 3.69 gives

\begin{equation}
(K + J)' \geq \frac{2 \lambda r}{4 \lambda + \delta} K + \left[ \left( \frac{r^2}{2} - r - \frac{1}{9} \right) + 2 - n + \frac{\delta r^2}{2 (4 \lambda + \delta)} \right] \frac{K}{r}
\end{equation}

\begin{equation}
= \left( r - 1 + \frac{2 - n - 1/9 - (4 \lambda + \delta)}{r} \right) \frac{8 r}{9} K \geq \frac{4 r}{5} (K + J),
\end{equation}

where the last inequality used $K + J \leq K + J_0 \leq \frac{10}{9} K$. Integrating gives that $K + J$ grows at least like $r^{4 \lambda + 2 \delta}$. This contradicts that $u \in W^{1,2}, \mathcal{L} u \in L^2$ as in the proof of Theorem 3.4. □

We will also prove an effective growth bound similar in spirit to Hadamard’s three circles theorem, [LJ, N]. Roughly, this shows that if $u$ is very small on a scale $r_1$ and bounded at larger scale $R$, then $u$ stays small out to scale $R - 1$.

**Proposition 3.80.** Given $\lambda > 0$ and $\delta \in (0, 2 \lambda)$, there exists $r_0$ so that if $r_0 \leq r_1 < R$, $u$ satisfies (3.1) on $\{r_1 \leq b \leq R\}$ and $D(R) \leq e^{\frac{2 R - 1}{6}} I(r_1)$, then for all $r \in [r_1, R - 1]$

\begin{equation}
I(r) \leq \left( \frac{r}{r_1} \right)^{4 \lambda + 2 \delta} \left[ 1 + \frac{1}{(2 \lambda + \delta)} \right] I(r_1).
\end{equation}

**Proof.** By Lemma 3.69 with $J = 0$, if $r \geq r_0 = r_0(\lambda, \delta, n)$ and $K(r) > 0$, then $K' \geq \frac{5}{3} K$ and, thus, $e^{-\frac{r^2}{2 \lambda}} K(r)$ is monotone non-decreasing. If $r \in [r_1, R - 1]$ with $K(r) > I(r_1)$, then $D(r) > K(r) > 0$ and, thus, also $I(r) > 0$ by Lemma 3.22. Moreover,

\begin{equation}
D(R) > K(R) \geq e^{\frac{r^2}{2 \lambda}} K(r) \geq e^{\frac{r^2}{2 \lambda} - 2} I(r_1) \geq e^{\frac{2 R - 1}{6}} I(r_1).
\end{equation}

This contradicts $D(R) \leq e^{\frac{2 R - 1}{6}} I(r_1)$, so $K(r) \leq I(r_1)$ for all $r \in (r_1, R - 1)$ and, thus,

\begin{equation}
D(r) = K(r) + (2 \lambda + \delta) I(r) \leq I(r_1) + (2 \lambda + \delta) I(r).
\end{equation}

Combining this with the bound on $I'$ from Lemma 3.22 gives

\begin{equation}
(r^{-4 \lambda + 2 \delta} I(r))' \leq -4 (2 \lambda + 2 \delta) r^{-4 \lambda + 2 \delta} I + 2 r^{-4 \lambda + 2 \delta} - 1 D \leq 2 r^{-4 \lambda + 2 \delta} - 1 I(r_1).
\end{equation}

Integrating from $r_1$ to $r \leq R - 1$ gives the claim. □

4. Growth of eigenvector fields for $\mathcal{P}$

We use the relationship between $\mathcal{P}$ and $\mathcal{L}$ to solve the Poisson equation $\mathcal{P} Y = \frac{1}{2} \text{div}_f h$ in Theorem 4.11 and to get strong bounds for $Y$ in Theorem 4.11 using also the previous section. The next theorem proves similar growth bounds for eigenvector fields for $\mathcal{P}$ which are generalizations of Killing fields.

**Theorem 4.1.** For any shrinker $(M, g, f)$, if $Y \in L^2, \mathcal{P} Y = \lambda Y$ and $Z \equiv Y + \frac{2}{2 \lambda + 1} \nabla \text{div}_f(Y)$, then $\text{div}_f(Z) = 0$ and for any $\delta > 0$ and $r_2 > r_1 > R = R(\lambda, n, \delta)$

\begin{equation}
I_{\text{div}_f(Y)}(r_2) \leq \left( \frac{r_2}{r_1} \right)^{4 \lambda + \delta} I_{\text{div}_f(Y)}(r_1),
\end{equation}

\begin{equation}
I_Z(r_2) \leq \left( \frac{r_2}{r_1} \right)^{8 \lambda + 2 \delta} I_Z(r_1).
\end{equation}
Each of these growth bounds is sharp and so is the requirement that $Y \in L^2$. Combining them bounds $Y$. As a corollary, $L^2$ Killing fields on a shrinker grow at most linearly.

**Corollary 4.4.** On any shrinker, for any $L^2$ Killing field $Y$, $\nabla \text{div}_f(Y)$ is parallel and if $Z = Y + 2 \nabla \text{div}_f(Y)$, then $\text{div}_f(Z) = 0$ and for any $\delta > 0$ and $r_2 > r_1 > R = R(n, \delta)$

$$I_Z(r_2) \leq \left( \frac{r_2}{r_1} \right)^{2+\delta} I_Z(r_1).$$

(4.5)

It is easy to see that this is sharp; on the two dimensional Gaussian soliton $Y = x_2 e_1 - x_1 e_2$ is a Killing field with $\text{div}_f(Y) = 0$ that grows linearly.

4.1. **Growth bounds for $\mathcal{P}$.** We will need bounds for vector fields given in terms of $\mathcal{P}$, but the results of the previous section are for $\mathcal{L}$. The next two results use the relation between $\mathcal{P}$ and $\mathcal{L}$ to bridge this gap. The next proposition immediately implies Theorem 0.11.

**Proposition 4.6.** On any gradient Ricci soliton if $Y$ is vector field with $\mathcal{P}Y - \lambda Y = V$ (where $\lambda \neq -\kappa$) and $Z = Y + \frac{1}{\lambda + \kappa} \nabla \text{div}_f(Y)$, then

$$\mathcal{L} + \lambda \nabla \text{div}_f(Y) = -\nabla \text{div}_f(V),$$

(4.7)

$$\mathcal{L} + 2\lambda + \kappa Z = -2V - \frac{1}{\lambda + \kappa} \nabla \text{div}_f(V).$$

(4.8)

Moreover, if $Y, V, \text{div}_f(V) \in L^2$, then $\nabla \text{div}_f(Y), Z \in L^2$.

If $V = 0$, then $\text{div}_f(Z) = 0$ and if also $Y \in L^2$, then $\|Y\|^2 = \|Z\|^2 + (\lambda + \kappa)^{-2} \|\nabla \text{div}_f(Y)\|^2$.

**Proof.** We will show the proposition when $V = 0$; the general case follows similarly. By (2.9)

$$(\mathcal{L} + \lambda) \nabla \text{div}_f(Y) = \mathcal{L} \text{div}_f(Y) - \text{div}_f(\mathcal{P}Y) - \kappa \text{div}_f(Y) = -((\lambda + \kappa) \text{div}_f(Y).$$

(4.9)

From this, $\text{div}_f(Z) = 0$ follows. Equation (4.7) follows from (2.10). To see (4.8) let $c = \frac{1}{\lambda + \kappa}$ so (2.10) and Lemma 2.15 give

$$\mathcal{L} Z = \mathcal{L} Y - c\lambda \nabla \text{div}_f(Y) = -2\mathcal{P} Y - \lambda Y - \nabla \text{div}_f(Y) - c\lambda \nabla \text{div}_f(Y)$$

(4.10)

$$= -(2\lambda + \kappa) \left( Y + \frac{c\lambda + 1}{2\lambda + \kappa} \nabla \text{div}_f(Y) \right) = -(2\lambda + \kappa) Z.$$

By Lemma 2.15, $\text{div}_f Y \in L^2$, so (2.9) gives that $\mathcal{L} \text{div}_f Y = \kappa \text{div}_f Y - \text{div}(\mathcal{P}Y) \in L^2$. Lemma 1.52 now gives that $\nabla \text{div}_f Y \in L^2$ and, thus, also $Z \in L^2$. Since $\text{div}_f(Z) = 0$ and $Z \in L^2$, $Z$ is automatically orthogonal to gradients of all $W^{1,2}$ functions and thus, in particular, to $\nabla \text{div}_f(Y)$. Therefore, Pythagoras gives the last claim.

**Proof. (of Theorem 4.11)** Since $Y \in L^2$, Proposition 4.6 gives that $\nabla \text{div}_f(Y), Z \in L^2$. Equations (4.2), (4.3) now follow from (4.7), (4.8), respectively, and Theorem 3.9

**Proof. (of Corollary 4.14)** Since $Y \in K_\mathcal{P}$, $\mathcal{P}Y = 0$, $Y \in L^2$ and thus, $\text{div}_f(Y) \in W^{1,2}$ by Lemma 2.15 and Proposition 4.6. Since $\mathcal{L} \nabla \text{div}_f(Y) = 0$ by (4.7), $\nabla \text{div}_f(Y)$ is parallel. By Proposition 4.6 $\text{div}_f(Z) = 0$. The bound (4.5) follows from Theorem 4.11.
Theorem 4.11. For any shrinker, if \( Y \in L^2, (\mathcal{P} - \lambda) Y = V \) and we set \( Z = Y + \frac{2}{2\lambda + 1} \nabla \text{div}_f (Y) \), then for any \( \beta, \delta > 0 \) and \( r_2 > r_1 > R = R(\lambda, n, \delta) \)

\[
I_{\nabla \text{div}_f (Y)}(r_2) \leq \left( \frac{r_2}{r_1} \right)^{4(\lambda + \beta) + \delta} \left( I_{\nabla \text{div}_f (Y)}(r_1) + \frac{5 \int b^{2-n} |\nabla \text{div}_f (V)|^2}{\beta (4(\lambda + \beta) + \delta)} \right),
\]

(4.12)

\[
I_Z(r_2) \leq \left( \frac{r_2}{r_1} \right)^{8(\lambda + \beta) + 2 + \delta} \left( I_Z(r_1) + \frac{\int b^{2-n} (|V|^2 + |\nabla \text{div}_f (V)|^2)}{\beta (8 \lambda + 2 + 8 \beta + \delta)} \right).
\]

(4.13)

Proof. By Proposition 4.6 \( (\mathcal{L} + \lambda) \nabla \text{div}_f (Y) = -\nabla \text{div}_f (V) \), so we get

\[
\langle \mathcal{L} \nabla \text{div}_f (Y), \nabla \text{div}_f (Y) \rangle \geq -(\lambda + \beta) |\text{div}_f (V)|^2 - \left( \frac{|\nabla \text{div}_f (V)|^2}{4\beta} \right).
\]

(4.14)

Thus, Theorem 3.19 applies with \( \psi = \left[ \frac{|\nabla \text{div}_f (V)|^2}{4\beta} \right] \) to give (4.12). Similarly, Proposition 4.6 gives \( (\mathcal{L} + 2\lambda + \kappa) Z = -2 V - \frac{1}{\lambda + \kappa} \nabla \text{div}_f (V) \), so we have

\[
\langle \mathcal{L} Z, Z \rangle \geq \left( 2\lambda + \frac{1}{2} + 2 \beta \right) |Z|^2 - \frac{|V|^2}{\beta} + \left( \frac{|\nabla \text{div}_f (V)|^2}{\beta (2\lambda + 1)^2} \right). \quad \square
\]

4.2. Fredholm properties for \( \mathcal{P} \). Throughout this subsection, we assume that \( (M, g, f) \) is a shrinker and \( \mathcal{K}_\mathcal{P} \) is the space of \( L^2 \) Killing fields, i.e., the \( L^2 \) kernel of \( \mathcal{P} \).

Theorem 4.15. There exists \( C_1 \) so that if \( h \) is a smooth compactly supported symmetric 2-tensor, then there is a smooth vector field \( Y \in W^{1,2} \) with \( \text{div}_f \left( \frac{1}{2} h - \text{div}_f Y \right) = 0 \) that is \( L^2 \)-orthogonal to \( \mathcal{K}_\mathcal{P} \) and satisfies

\[
\|Y\|_{W^{1,2}} + \|\text{div}_f Y\|_{W^{1,2}} + \|\mathcal{L} Y\|_{L^2} \leq C_1 \|\text{div}_f h\|_{L^2}.
\]

(4.16)

Lemma 4.17. If \( Y \) is a vector field, then \( \|\mathcal{L} Y\|_{L^2}^2 \leq (2n + 8) \|Y\|_{W^{1,2}}^2 \). If \( Y, \mathcal{L} Y \in L^2 \), then

\[
\frac{1}{4n} \|Y\|_{W^{1,2}}^2 \leq \|\mathcal{L} Y\|_{L^2}^2 \leq \|\mathcal{L} Y\|_{L^2}^2 + 2 \|Y\|_{L^2}^2.
\]

(4.18)

The \( L^2 \) kernel of \( \mathcal{L} \) is equal to the space \( \mathcal{K}_\mathcal{L} \) of parallel vector fields and \( \mathcal{L} \) has discrete eigenvalues \( 0 \leq \mu_0 < \mu_1 < \mu_2, \cdots \to \infty \) with finite dimensional eigenspaces \( E_{\mu_i} \subset W^{1,2} \).

Proof. The first claim follows from the squared triangle inequality and Lemma 1.50

\[
\|\mathcal{L} Y\|_{L^2}^2 \leq 2 \|\Delta Y\|_{L^2}^2 + 2 \|\nabla f\|_{L^2} \|\nabla Y\|_{L^2} \leq 2 n \|\nabla^2 Y\|_{L^2}^2 + 2 n \|\nabla Y\|_{L^2}^2 + 8 \|\nabla \nabla Y\|_{L^2}^2.
\]

Suppose now that \( Y, \mathcal{L} Y \in L^2 \). Lemma 1.52 gives that \( Y \in W^{1,2} \) and

\[
\|\nabla Y\|_{L^2}^2 \leq 2 \|Y\|_{L^2} \|\mathcal{L} Y\|_{L^2} \leq \|Y\|_{L^2}^2 + \|\mathcal{L} Y\|_{L^2}^2.
\]

(4.19)

The first inequality gives that the \( L^2 \) kernel of \( \mathcal{L} \) is equal to the space \( \mathcal{K}_\mathcal{L} \) of parallel vector fields. Combining (4.19) and Lemma 1.50 gives (4.18). The estimate (4.18) implies that the inverse of \( \mathcal{L} \) is a compact symmetric operator, so the eigenvalues of \( \mathcal{L} \) go to infinity and the eigenspaces are finite dimensional (cf. the appendix in [CZ2] for functions, plus Rellich compactness for vector fields). \( \square \)

Below let \( \mu \) be an eigenvalue of \( \mathcal{L} \) and \( E_\mu = \{ V \in L^2 \mid \mathcal{L} V + \mu V = 0 \} \), the corresponding eigenspace. Recall that the convention is that the operators \( \mathcal{L} \) and \( \mathcal{P} \) have opposite sign.
Lemma 4.20. We have

(1) For each $\mu$, the map $\mathcal{P}$ maps $E_\mu$ to $E_\mu$, is self-adjoint, and has a basis of eigenvectors.
(2) If $V \in E_\mu$ and $\mathcal{P} V = \lambda V$, then $\mu - 2 \lambda \leq \frac{1}{2}$ with equality if and only $\text{div}_f V = 0$.
(3) $\mathcal{P}$ has discrete eigenvalues $\lambda_i \to \infty$ and each eigenspace is finite dimensional.

Proof. Suppose that $V \in E_\mu$. Lemma 1.52 gives that $V, \text{div}_f V \in W^{1,2}$ and, thus, $\mathcal{P} V \in L^2$ by Lemma 2.1. By Proposition 2.4 $\mathcal{L} \mathcal{P} V = \mathcal{P} \mathcal{L} V = -\mu \mathcal{P} V$. It follows that $\mathcal{P}$ maps $E_\mu$ to itself. The first claim follows from this together with that $\mathcal{P} V \in L^2$ by Lemma 2.1. Since $V, \text{div}_f V \in W^{1,2}$, the taking the inner product with $V$ and integrating gives

\[
\frac{1}{2} - \mu + 2 \lambda \] 

This gives (2). The third claim follows by combining (1), (2) and Lemma 4.17.

Proof of Theorem 4.15. If $V \in \mathcal{K}_\mathcal{P}$, then $\int \langle \text{div}_f h, V \rangle e^{-f} = \int \langle h, \text{div}_f^* V \rangle e^{-f} = 0$. Therefore, by (3) in Lemma 4.20 there exist $v_i \in \mathbb{R}$ and $L^2$-orthonormal vector fields $V_i$ so that $\mathcal{P} V_i = \lambda_i V_i$, $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lambda_i \to \infty$, $\mathcal{L} V_i = -\mu_i V_i$, and $\text{div}_f h = \sum_{i=1}^\infty a_i V_i$. Note that $\|\text{div}_f h\|^2_{L^2} = \sum_i a_i^2 < \infty$. Set $Y = \frac{1}{2} \sum_{i=1}^\infty \frac{a_i}{\lambda_i} V_i$, so $\mathcal{P} Y = \frac{1}{2} \text{div}_f h$ weakly and

\[
\|Y\|^2_{L^2} = \frac{1}{4} \sum_{i=1}^\infty \frac{a_i^2}{\lambda_i^2} \leq \frac{\|\text{div}_f h\|^2_{L^2}}{4 \lambda_1^2}.
\]

Lemma 2.15 then also gives $L^2$ bounds on $\text{div}_f Y$ and $\nabla Y$. To get the $L^2$ bounds on $\mathcal{L} Y$ (and, thus, also $\nabla \text{div}_f Y$), observe that (2) in Lemma 4.20 gives $0 \leq \mu_i \leq \frac{1}{2} + 2 \lambda_i$, so that

\[
\left(\frac{\mu_i}{\lambda_i}\right)^2 \leq \left(\frac{1}{2} + 2 \lambda_i\right)^2 \leq \frac{1}{2} + 8 \lambda_i^2 \leq \frac{1}{2} \lambda_1^2 + 8.
\]

Since $\mathcal{L} Y = -\frac{1}{2} \sum_{i=1}^\infty \frac{a_i}{\lambda_i} V_i$, we see that $\|\mathcal{L} Y\|^2_{L^2} \leq \left(\frac{1}{8 \lambda_1^2} + 2\right) \|\text{div}_f h\|^2_{L^2}$. Finally, since $\mathcal{P} Y = \frac{1}{2} \text{div}_f h$ weakly and $Y, \text{div}_f (Y) \in L^2$, Lemma 2.17 gives that $Y$ is smooth.

5. Jacobi fields and a spectral gap

We need to understand Jacobi fields on a gradient shrinking soliton $\Sigma$. A symmetric 2-tensor $h$ gives a variation of the metric. Given $h$ with $\text{div}_f h = 0$ and a function $k$, $\phi' = \frac{1}{2} L h + \text{Hess}_{\frac{1}{2} \text{Tr}_k h}$ by [CaHL]. We will say that $h$ is a Jacobi field if $L h = 0$ and $\text{div}_f h = 0$. We omit $\text{Hess}_{\frac{1}{2} \text{Tr}_k h}$ since this will be $L^2$-orthogonal to $L h$ when $\text{div}_f h = 0$.

In this section, we will assume that $\Sigma$ splits as a product

\[
\Sigma = (N^\ell, g^1) \times \mathbb{R}^{n-\ell},
\]

where $N$ is Einstein with $\text{Ric}_N = \frac{1}{2} g^1$ and $f = \frac{|x|^2}{2} + \frac{\ell}{2}$ with $x \in \mathbb{R}^{n-\ell}$.

We need to understand the spectrum of $\mathcal{L}$ on $\Sigma$. The $L^2$ eigenfunctions of $\mathcal{L}$ on $\mathbb{R}^{n-\ell}$ are polynomials with eigenvalues at $\{0, \frac{1}{2}, 1, \ldots\}$. Let $\mathcal{K}$ be the $L^2$ kernel of $\mathcal{L} + 1$ on $\mathbb{R}^{n-\ell}$. Each $v \in \mathcal{K}$ can be written $v = a_{ij} x_i x_j - 2 \text{Tr} a$ for a matrix $a_{ij}$ (see, e.g., lemma 3.26 in [CM2]). The Lichnerowicz theorem says that $\lambda_1(N) \geq \frac{\ell}{2(\ell-1)} > \frac{1}{2}$. It follows that
If $\mathcal{L} w = -w$ and $w \in L^2$ on $\Sigma$, then $w = \zeta + v$ where $v \in \mathcal{K}$ is a quadratic polynomial and $\zeta$ is a 1-eigenfunction on $N$.

There is a natural orthogonal decomposition of symmetric 2-tensors

\begin{equation}
    h = u g^1 + h_0 + h_2.
\end{equation}

Here $u$ is a function on $M$, $h_0$ is the trace-free part of the projection of $h$ to $N$, and the remainder $h_2$ satisfies $h_2(V,W) = 0$ when $V$ and $W$ are both tangent to $N$. We will see that $L$ preserves this decomposition. Since $g^1$ is parallel and $R(g^1) = \text{Ric}_N = \frac{1}{2} g^1$, we have $L(u g^1) = (u + \mathcal{L} u) g^1$. Since $R$ is zero if any of the indices is Euclidean, we see that

\begin{equation}
    L h_2 = \mathcal{L} h_2 \text{ and } (\mathcal{L} h_2)(V,W) = 0 \text{ if } V, W \text{ are both tangent to } N.
\end{equation}

Using that $\mathcal{L} g^1 = 0$ and $\langle R(h_0), g^1 \rangle = \langle h_0, R(g^1) \rangle = 0$ gives

\begin{equation}
    \langle L h_0, g^1 \rangle = 0 \text{ and } (L h_0)(V, \cdot) = 0 \text{ if } V \text{ is Euclidean}.
\end{equation}

Thus, using that $L$ preserves this orthogonal decomposition of $h$, we get

\begin{equation}
    |L h|^2 = |L h_0|^2 + \ell (u + \mathcal{L} u)^2 + |\mathcal{L} h_2|^2.
\end{equation}

The strong rigidity will hold when $\Sigma$ satisfies the condition:

(\ast) There exists $C_{N,n}$ so that if $h_0, L h_0 \in L^2(\Sigma)$, then

\begin{equation}
    \|h_0\|^2_{W^{1,2}} \leq C_{N,n} \|L h_0\|^2_{L^2}.
\end{equation}

If $v \in \mathcal{K}$, then $v g^1$ is a Jacobi field. Conversely, if $N$ satisfies (\ast), $h = u g^1 + h_0 + h_2$ with $L h = 0$, $\text{div}_f h = 0$ and $h \in L^2$, then Theorem 5.12 gives that $h_0 = h_2 = 0$ and $u$ is in $\mathcal{K}$.

We will see next that (\ast) holds for the sphere; it also holds for the families of symmetric spaces in $[\text{CalH}]$ that are linearly, but not neutrally, stable for Perelman’s $\nu$-entropy.

**Lemma 5.7.** If $N = S^{\ell}_{\sqrt{2}}$ or is any quotient of $S^{\ell}_{\sqrt{2}}$, then $N$ satisfies (\ast) with $C_{N,n} = (\ell - 1)^2$.

In fact, if $N$ has positive sectional curvature and both $\Delta + 2 R$ and $\Delta + 2 R - \frac{1}{2}$ are injective on the space of trace-free symmetric 2-tensors on $N$, then (5.6) holds.

**Proof.** If $N$ is round, then $R^N_{ikjn} = \frac{1}{2(\ell - 1)} (g_{ij} g^k_n - g^k_n g_{ij})$ and, thus, $R(h_0)_{ij} = -\frac{1}{2(\ell - 1)} (h_0)_{ij}$. Since $L = \mathcal{L} + 2 R$ and $2 R(h_0) = -\frac{1}{\ell - 1} h_0$, we get that

\begin{equation}
    \|\nabla h_0\|^2_{L^2} = -\int \langle h_0, \mathcal{L} h_0 \rangle e^{-f} = -\frac{1}{(\ell - 1)} \int |h_0|^2 e^{-f} + \int \langle h_0, \left(\frac{1}{(\ell - 1)} - \mathcal{L}\right) h_0 \rangle e^{-f}
\end{equation}

\begin{equation}
    \leq -\frac{1}{2(\ell - 1)} \|h_0\|^2_{L^2} + \frac{\ell - 1}{2} \|L h_0\|^2_{L^2},
\end{equation}

where the last inequality used the absorbing inequality $a b \leq \frac{a^2}{2(\ell - 1)} + \frac{\ell - 1}{2} b^2$. This gives (5.6).

We turn to the second claim. Since $L$ preserves the decomposition and $R$ is bounded, it has a spectral decomposition and it suffices to show that if $L h_0 = 0$ and $\|h_0\|_{L^2} < \infty$, then $h_0$ vanishes. Since $K_N > 0$, proposition 4.9 in $[\text{BK}]$ gives that the largest eigenvalue of $R$ acting on $h_0$ (at each point) is at most $\frac{1}{2} - \ell \min K_N < \frac{1}{2}$. Thus, $2 R - 1$ is a negative operator on the trace-free symmetric 2-tensors and, thus, $L - 1$ has trivial $L^2$ kernel. Let $\partial_i$ and $\partial_j$ be $\mathbb{R}^{n-\ell}$ derivatives. Since $L h_0 = 0$, we have

\begin{equation}
    0 = \nabla_{\partial_i} (L h_0) = L (\nabla_{\partial_i} h_0) - \frac{1}{2} (\nabla_{\partial_i} h_0) \text{ and } (L - 1) (\nabla_{\partial_j} \nabla_{\partial_i} h_0) = 0.
\end{equation}
Consequently, \( \nabla \partial_i \nabla \partial_i h_0 = 0 \) and, thus, \( h_0 = h_0^0 + \sum a_i x_i h_0^i \), where \( h_0^0 \) and the \( h_0^i \)'s are symmetric 2-tensors on \( N \). It follows that \( (\Delta + 2 R)(h_0^0) = 0 \) and \( (\Delta + 2 R - \frac{1}{2})(h_0^i) = 0 \). \( \square \)

We will use the next Poincaré inequality for vector fields tangent to \( N \):

**Lemma 5.10.** There is a constant \( C = C(N) \) so that if \( V \) is tangent to \( N \) and \( \|V\|_{W^{1,2}} < \infty \), then \( \|V(1 + |\nabla f|)\|_{L^2} \leq C \|\nabla V\|_{L^2} \).

**Proof.** Let \( \lambda \) be the smallest eigenvalue of \( \Delta \) acting on vector fields on \( N \). Since \( \text{Ric}_N > 0 \), \( N \) has no nontrivial harmonic one forms and thus no parallel vector fields, we have therefore \( \lambda > 0 \). Given \( x \in \mathbb{R}^{n-\ell} \), let \( V_x \) be the restriction of \( V \) to \( N_x = N \times \{x\} \). We get that

\[
\int |V|^2 e^{-f} = \int_{\mathbb{R}^{n-\ell}} \int_{N_x} |V_x|^2 e^{-f} \leq \frac{1}{\lambda} \int_{\mathbb{R}^{n-\ell}} \int_{N_x} |\nabla_N V_x|^2 e^{-f} \leq \frac{1}{\lambda} \int |\nabla V|^2 e^{-f}.
\]

The lemma follows from this and Lemma 1.50. \( \square \)

We see that \( h \) is well-approximated by a Jacobi field when \( Lh \) and \( \text{div}_f h \) are small.

**Theorem 5.12.** There exists \( C \) so that if \( h, \text{div}_f h \in W^{2,2} \), then \( \|h - v g^1\|_{W^{2,2}} \leq C \{\|Lh\|_{L^2} + \|\text{div}_f h\|_{L^2}\} \).

**Proof.** The tensor \( h \) has a (pointwise) orthogonal decomposition \( h = u g^1 + h_0 + h_2 \). We will bound \( h_0, h_2 \) and \( (u - v) \) in \( W^{1,2} \) to control \( \|h - v g^1\|_{W^{1,2}} \) in terms of \( \|Lh\|_{L^2} + \|\text{div}_f h\|_{L^2} \).

Since \( L(h - v g^1) \) can be bounded by these and the curvature of the cylinder and \( \text{Hess}_f \) are bounded, we then also get the desired bound on \( \|h - v g^1\|_{W^{2,2}} \) (cf. (3.19) in [CM2]).

Using (5.5), we have that

\[
\ell \|\mathcal{L} + 1\| \|u\|_{L^2} + \|Lh_0\|_{L^2}^2 + \|\mathcal{L}h_2\|_{L^2}^2 = \|Lh\|_{L^2}^2.
\]

The bound \( \|h_0\|_{W^{1,2}} \) follows immediately from this and (5.5). To control \( h_2 \), start by noting that there is a further orthogonal decomposition \( h_2 = h_2^N + h_2^\perp \) that is also preserved by \( \mathcal{L} \), where \( h_2^\perp \) is the purely Euclidean part. In a block decomposition for a frame, \( h_2^N \) consists of two off-diagonal parts that are transposes of each other. Applying the part of \( h_2^N \) that maps the Euclidean factor to \( N \) to each Euclidean derivative \( \partial_i \) to get a vector field tangent to \( N \) and applying Lemma 5.10 gives

\[
\|h_2^N\|_{L^2}^2 \leq C \|\nabla h_2^N\|_{L^2}^2 = C \int (-h_2^N, \mathcal{L}h_2^N) e^{-f} \leq C \|h_2^N\|_{L^2} \|\mathcal{L}h_2\|_{L^2},
\]

where the last inequality used the Cauchy-Schwarz inequality and the fact that \( \mathcal{L} \) preserves the decomposition of \( h_2 \). It follows from this and (5.14) that \( \|h_2^N\|_{W^{1,2}} \leq C \|Lh\|_{L^2} \).

It remains to bound \( h_2^\perp \) and \( (u - v) \); this is where we will also need the bound on \( \text{div}_f h \). Let \( a_{ij} \) be a symmetric constant \( (n-\ell) \) matrix so that \( \int (a - h_2^\perp) e^{-f} = 0 \). The Poincaré inequality on \( \Sigma \) gives that \( \|a - h_2^\perp\|_{W^{1,2}} \leq C \|\mathcal{L}h_2^\perp\| \leq C \|Lh\| \). Next, let \( \zeta \) be the projection of \( u \) onto the 1-eigenspace of \( N \) (if this is empty, set \( \zeta = 0 \)). The spectral gap on \( \Sigma \) gives that \( \|u - v - \zeta\|_{W^{1,2}} \leq C \|\mathcal{L} + 1\| u_{L^2} \). The desired bounds on \( h_2^\perp \) and \( (u - v) \) will follow once we bound \( |a| \) and \( \zeta \). The triangle inequality and the bounds thus far give

\[
\|\text{div}_f (a + \zeta g^1)\|_{L^2} \leq \|\text{div}_f h\|_{L^2} + \|\text{div}_f (h - a - \zeta g^1)\|_{L^2} \leq \|\text{div}_f h\|_{L^2} + C \|Lh\|_{L^2}.
\]
Since $a$ is purely Euclidean and $\text{div}_f(a + \zeta g^1) = \nabla \zeta - a(\nabla f)$, we have
\begin{equation}
\|\nabla \zeta\|_{L^2}^2 + \|a(\nabla f)\|_{L^2}^2 = \|\text{div}_f(a + \zeta g^1)\|_{L^2}^2 \leq 2 \|\text{div}_f h\|_{L^2} + C \|L h\|_{L^2}.
\end{equation}
Since $\zeta$ has eigenvalue one, this gives the desired bound $W^{1,2}$ bound on $\zeta$. It also gives the bound on $|a|$ since $\int f_j f_k e^{-f} = \frac{2}{\delta} \int e^{-f}$, so that
\begin{equation}
\|a(\nabla f)\|_{L^2}^2 = \sum_i \left( \sum_j a_{ij} f_j \right)^2 = \sum_i \int a_{ij} a_{ik} f_j f_k e^{-f} = \frac{1}{2} |a|^2 \int e^{-f}. \quad \square
\end{equation}

**Lemma 5.17.** There exists $C_n, C > 0$ depending on $n$ so if $v \in K$ on $\mathbb{R}^n$, then
\begin{equation}
|v| + (1 + |x|^2) |\text{Hess}_v| \leq C_n (1 + |x|^2) \|v\|_{L^2} \quad \text{and} \quad |\nabla v| \leq C_n |x| \|v\|_{L^2}.
\end{equation}
Furthermore, $2 \|\text{Hess}_v\|_{L^2} = \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2$ and $u = |\nabla v|^2 - \Delta |\nabla v|^2$ is in $K$ with
\begin{equation}
\int u \, |\nabla v|^2 e^{-f} = \|u\|_{L^2}^2 \geq C \|v\|_{L^2}^4.
\end{equation}

**Proof.** By lemma 3.26 in [CM2], $v = a_{ij} x_i x_j - 2 a_{ii}$ for a constant matrix $a_{ij}$. This gives (5.18) and also that $|\nabla v|^2$ is a homogeneous quadratic polynomial. Since $L v = -v$, (1.35) (using the drift Bochner formula) gives $\|\text{Hess}_v\|_{L^2}^2 = \frac{1}{2} \|\nabla v\|_{L^2}^2 = \frac{1}{2} \|v\|_{L^2}^2$.

Let $Q$ be the space of homogeneous quadratic polynomials and define the linear map $\Psi(w) = w - \Delta w$. We will show that $\Psi$ maps to $K$. For each $w \in Q$, there is a constant $c$ so that $w - c \in K$; since $K$ is orthogonal to constants, $\int (w - c) e^{-f} = 0$. Using homogeneity again, we have $r \partial_r w = 2 w$ so that $L w = \Delta w - \frac{2}{\delta} \partial_r w = \Delta w - w$. It follows that $c = \Delta w$ and, thus, $\Psi(w) \in K$. Since $\Delta w$ is a constant while $w$ is second order, $\Psi$ is one to one. Thus, since $Q$ is finite dimensional, there is a constant $C_0$ such that for any $w \in Q$
\begin{equation}
\|w\|_{L^1} \leq C_0 \|\Psi(w)\|_{L^1} = C_0 \|w - \Delta w\|_{L^1}.
\end{equation}
The equality in (5.19) uses that $K$ is orthogonal to constants. Applying (5.20) with $w = |\nabla v|^2$ and $\Psi(w) = u$ gives $\|v\|_{L^2}^2 = \|\nabla v\|_{L^1}^2 \leq C_0 \|u\|_{L^1}$. The Cauchy-Schwarz inequality then gives the inequality in (5.19). \hfill $\square$

Finally, we will need a simpler estimate for metrics on $N$:

**Lemma 5.21.** Suppose that $(\ast)$ holds. There exists $C_N$ so that if $h$ is a symmetric 2-tensor on $N$ and $\text{div}_N h = 0$, then
\begin{equation}
\|h\|_{W^{2,2}} \leq C_N \|L_N h\|_{L^2}.
\end{equation}

**Proof.** Since $L_N$ is elliptic and $N$ is compact, it suffices to show that $L_N h = 0$ and $\text{div}_N h = 0$ implies that $h = 0$. The trace-free part of $h$ vanishes by $(\ast)$, so we can assume that $h = w g^1$. Then being in the kernel of $L_N$ forces $w$ to be a 1-eigenfunction on $N$, but being divergence-free forces $w$ to be constant so $w \equiv 0$. \hfill $\square$

**Corollary 5.23.** Suppose that $(\ast)$ holds. There exists $\delta_0 > 0$ so that if $f$ is a function and $h$ is a symmetric 2-tensor on $N$ with $\text{div}_N h = 0$ and $\|h\|_{C^2} + \|\nabla f\|_{C^1} \leq \delta_0$, then
\begin{equation}
\|h\|_{W^{2,2}} + \|\nabla f\|_{W^{1,2}} \leq C \|\phi(h, f)\|_{L^2},
\end{equation}
where $\phi(h, f)$ is the shrinker quantity for $(N, g^1 + h, f)$. 
Proposition 6.2. There exist functions on the larger scale $(1 + \frac{6.5}{\beta})$ to scale Proposition 6.4. Let \(N\) be the pointwise \(C^2\) norm of \(h\). Since \(\text{div}_N h = 0\) and \(N\) is Einstein, Theorem 1.32 gives that \(L_N h\) is \(L^2\)-orthogonal to any Hessian. Combining this with (5.25) gives that

\[
(5.26) \quad \frac{1}{4} L_N h^2 + \|\text{Hess}_{\frac{1}{2}} \text{Tr} h - f\|^2 \leq 2 \|\phi(h, f)\|^2 + C \delta_0 (\|h\|_{W^{2,2}}^2 + \|\nabla f\|_{W^{1,2}}^2)
\]

Combining this with Lemma 5.21 gives the desired bound on \(h\) when \(\delta_0 > 0\) is small. \(\square\)

6. Eigenfunctions and almost parallel vector fields

The main result of this section is an extension theorem which shows that if a shrinker is close to a model shrinker on some large scale, then it remains close on a larger scale with a loss in the estimates. To explain this, let \(\Sigma = (N^\ell \times \mathbb{R}^{n-\ell}, \bar{g}, \bar{f})\) be the model gradient shrinking soliton, where \(N\) is closed and Einstein, \(x_i\) are Euclidean coordinates, and \(\bar{f} = \frac{|x|^2}{4} + \frac{\ell}{2}\).

Let \(\delta_0 > 0\) be a fixed constant that is sufficiently small and fix \(\alpha \in (0, 1)\). We consider two notions of closeness for a gradient shrinking \((M, g, f)\):

\((\ast_R)\) There is a diffeomorphism \(\Phi_R\) from a subset of \(\Sigma\) to \(M\) onto \(\{b < R\}\) so that

\[
|\bar{g} - \Phi^* g|^2 + |\bar{f} - f \circ \Phi^*|^2 \leq \delta_0 e^{f - \frac{\bar{f}}{2}} \text{ and } \|\bar{g} - \Phi^* g\|_{C^{2,\alpha}} + \|\bar{f} - f \circ \Phi^*\|_{C^{2,\alpha}} \leq \delta_0.
\]

\((\ddagger_R)\) There is a diffeomorphism \(\Psi_R\) from a subset of \(\Sigma\) to \(M\) that is onto \(\{b < R\}\) so that

\[
\|\bar{g} - \Psi^* g\|_{C^{3,\alpha}} + \|\bar{f} - f \circ \Psi^*\|_{C^{3,\alpha}} \leq e^{-k_1}\text{ and, furthermore, for each } \ell\text{ there is a constant } C_{\ell} \text{ so that the } C^\ell\text{ norms are bounded by } C_{\ell} R^\ell.
\]

Note that \((\ast_R)\) gives stronger bounds on the region where \(\bar{f}\) is small.

The next proposition, which relies on the growth bounds, uses \((\ast_R)\) to get almost linear functions on the larger scale \((1 + \beta) R\). This is the key ingredient in Theorem 6.1.

Proposition 6.2. There exist \(C, m, R_1, \beta_1 > 0\) so that if \((\ast_R)\) holds and \(R \geq R_1\), then \((\ddagger_{(1+\beta)} R)\) holds.

The next proposition, which relies on the growth bounds, uses \((\ast_R)\) to get almost linear functions on the larger scale \((1 + \beta) R\). This is the key ingredient in Theorem 6.1.

Proposition 6.2. There exist \(C, m, R_1, \beta_1 > 0\) so that if \((\ast_R)\) holds and \(R \geq R_1\), then we get \(n - \ell\) functions \(u_i\) so that \(\int u_i e^{-f} = 0\) and on \(\{b < (1 + \beta_1) R\}\)

\[
(6.3) \quad |\delta_{ij} - \langle \nabla u_i, \nabla u_j \rangle| + \|\text{Hess}_{u_i}\|_{C^2} + |2 \langle \nabla u_i, \nabla f \rangle - u_i| \leq C R^m e^{-\frac{\delta_0}{10}}.
\]

Furthermore, for each \(\ell\), there exists \(c_\ell\) so that \(|u_i|_{C^\ell} \leq c_\ell R^\ell\) on \(\{b < (1 + \beta_1) R\}\).

6.1. Pseudo-locality. Applying pseudo-locality to the flow generated by \((M, g, f)\) gives estimates forward in time for the flow and, thus, estimates on the shrinker on a larger scale. Let \(c_n\) be the Euclidean isoperimetric constant and define \((M, g, f)\) to be \((\delta, r_0)\)-Euclidean to scale \(R\) if \(|\partial \Omega|_{n} \geq (1 - \delta) c_n |\Omega|_{n-1}\) for every \(\Omega \subset \{b \leq R\}\) with \(\text{diam } \Omega \leq r_0\).

Proposition 6.4. There exist \(\delta_0 > 0\), \(R_0\) and \(C_0\) so that for any \(r_0 \in (0, 1)\), we get \(\alpha_0 = \alpha_0(r_0, n) > 0\) so that if \((M, g, f)\) is \((\delta, r_0)\)-Euclidean out to scale \(R \geq R_0\), then

\[
(6.5) \quad \sup_{\{b \leq (1 + \alpha_0) R\}} |R| \leq C_0 r_0^{-2}.
\]
Once we have the $R$ bound, then the Shi estimates, [S], give corresponding bounds on $\nabla R$, etc. There are also versions of this estimate for expanding solitons, where the estimate forward in time for the flow corresponds to estimates for the expander on smaller scales.

We will use Li-Wang’s version for shrinkers (theorem 25 in [LW1]) of Perelman’s pseudo-locality (cf. 10.1 in [P] or 30.1 on page 2658 of [KL1]): There exist $\delta, \epsilon > 0$ with the following property. Suppose that $(M, (x_0, -1), g(t))$ is a smooth pointed Ricci flow for $t \in [-1, \epsilon r_0^2 - 1]$. If $S \geq -r_0^{-2}$ on $B_{r_0}(x_0) \times \{t = -1\}$ and $B_{r_0}(x_0)$ is $(\delta, r_0)$-Euclidean at time $-1$, then

\begin{equation}
|R|(x, t) < (t + 1)^{-1} + (\epsilon r_0)^{-2}
\end{equation}

whenever $-1 < t < (\epsilon r_0)^2 - 1$ and $\text{dist}(x, x_0) \leq \epsilon r_0$.

**Proof of Proposition 6.4.** For each $x \in M$, let $\gamma_x$ be the integral curve given by $\gamma_x(-1) = x$ and $\gamma_x'(t) = -\frac{1}{t} \nabla f \circ \gamma_x(t)$. Here, the gradient is computed with respect to the fixed metric $g$. Define $\Phi(x, t) = \gamma_x(t)$ so that $\Phi(x, -1) = x$ and $\partial_t \Phi = \frac{1}{t} \nabla f \circ \Phi$. Working in the background metric $g$, we have

\begin{equation}
\partial_t f(\Phi(x, t)) = \langle \nabla f(\Phi(x, t)), \frac{1}{t} \nabla f(\Phi(x, t)) \rangle = \frac{1}{t} |\nabla f|^2 \circ \Phi(x, t).
\end{equation}

We will show that there exists $C$ depending on $B_1 \subset M$ so that if $\alpha > 0$ and $y$ is a point with $S \leq C_S$ on $\{f \leq (1 + \alpha) f(y)\}$, then for $t \in (-1, 0)$

\begin{equation}
 f(\Phi(y, t)) \geq \min \left\{ (1 + \alpha) f(y), \frac{f(y) - C_S}{-t} \right\}.
\end{equation}

By (6.7) and (1.48), $\partial_t f(\Phi(y, t)) = \frac{1}{t} (f - S) \circ \Phi(y, t)$. Rewriting this as $\partial_t (-t f) = -S$ and integrating from $-1$ to $t$ gives $-(\sup S)(1 + t) \leq -t f(\Phi(y, t)) - f(y) \leq 0$, where the supremum of $S$ is taken over the curve $\Phi(y, s)$ for $-1 \leq s \leq t$. Combining this and monotonicity of $f$ along the flow line gives (6.8).

It is well-known that $g(x, t) = -t \Phi^* g(y)$ is a Ricci flow. By assumption, the set $\{b \leq R\}$ is $(\delta, r_0)$-Euclidean at time $-1$ and has $S \geq 0$ by [Cr]. Thus, if $x \in M$ is any point with $B_{r_0}(x) \subset \{b \leq R\}$, then pseudo-locality (6.6) gives $|R_{g(x, -1 + r_0^2)}| \leq C r_0^{-2}$. It follows from (6.8) that this curvature bound for the evolving metric is equivalent to a curvature bound for $R_g$ some fixed factor further out. Here there is an additive loss because of the last term that can be absorbed as long as $R$ is large enough. \qed

#### 6.2. Spectral estimates.

We show that if $(\ast_R)$ holds, then $M$ has $(n - \ell)$ $L^2$ eigenfunctions with eigenvalues exponentially close to $\frac{1}{2}$. By [HN] and [CZ11] (cf. [AN BE FLL]), $\mathcal{L}$ has discrete spectrum $\mu_0 = 0 < \frac{1}{2} \leq \mu_1 \leq \cdots \to \infty$ on $M$ and the eigenfunctions are in $W^{1,2}$.

**Lemma 6.9.** There exists $C$ so that if $(\ast_R)$ holds, then there are $(n - \ell)$ $L^2$-orthonormal functions $v_i$ on $M$ with $\mathcal{L} v_i + \mu_i v_i = 0$, $\mu_i \geq \frac{1}{2}$, and

\begin{equation}
\|\text{Hess}_{v_i}\|^2_{L^2} + \left( \mu_i - \frac{1}{2} \right) \leq C e^{-\frac{R^2}{8}}.
\end{equation}

We will use the low eigenfunctions on $\Sigma$ as test functions to get an upper bounds for the low eigenvalues on $M$. The next lemma recalls the properties of the low eigenvalues on $\Sigma$. 

Lemma 6.11. There exist \( \bar{c}, C \) so that \( \bar{u}_0 = \bar{c} \) and \( \bar{u}_i = \frac{\bar{c}}{\sqrt{2}} x_i \), for \( 1 \leq i \leq n - \ell \) satisfy
\[
\left| \delta_{ij} - \int_{b<r} \bar{u}_i \bar{u}_j e^{-f} \right| \leq C r^{n-\ell} e^{-\frac{r^2}{8}} \text{ for all } i, j, \quad \tag{6.12}
\]
\[
\frac{1}{2} \left| \delta_{ij} - \int_{b<r} \langle \nabla \bar{u}_i, \nabla \bar{u}_j \rangle e^{-f} \right| \leq C r^{n-\ell} e^{-\frac{r^2}{8}} \text{ for } i, j \geq 1. \quad \tag{6.13}
\]

Proof. Choose \( \bar{c} \) depending on \( \ell, n, \text{Vol}(N) \) so that \( \int_{B^1} e^{u} e^{-f} = 1 \). Since \( L(x, x_j) = 2 \delta_{ij} - x_i x_j \), it follows that \( \bar{u}_0 = \bar{c} \) and \( \bar{u}_i = \frac{\bar{c}}{\sqrt{2}} x_i \), for \( 1 \leq i \leq n - \ell \) are \( L^2 \)-orthonormal. To estimate the “tails” of these integrals, observe that
\[
\int_{b \geq s} (1 + |x|^2) e^{-f} \leq \text{Vol}(N) \text{Vol}(S^{n-\ell-1}) e^{-\frac{4}{\ell}} \int_{s \geq 2} r^{n-\ell-1} (1 + r^2) e^{-\frac{r^2}{8}}. \quad \tag{6.14}
\]

Proof of Lemma 6.9. Define a cutoff function on \( M \) to be zero for \( \{ b > R \} \), one for \( \{ b < R - 1 \} \), and with \( \eta = R - b \) in between. Note that \( |\nabla \eta| \leq 1 \). Let \( \bar{u}_i \) be as in Lemma 6.11 and set \( u_i = \eta \bar{u}_i \circ \Phi_{R}^{-1} \). Define symmetric matrices \( a_{ij} \) and \( b_{ij} \) by
\[
a_{ij} = \int u_i u_j e^{-f} \quad \text{and} \quad b_{ij} = \int \langle \nabla u_i, \nabla u_j \rangle e^{-f}. \quad \tag{6.15}
\]

Lemma 6.11 and the change of variable formula give
\[
|\delta_{ij} - a_{ij}| \leq C R^n e^{-\frac{(R-1)^2}{4}} + \int_{\{ b < R - 1 \}} |u_i u_j| \left| e^{-f} \frac{dv_g}{dv_y} - e^{-f} \right| dv_y, \quad \tag{6.16}
\]
where we use the shorthand \( f \equiv f \circ \Phi_{R}^{-1} \) and similarly for \( dv_y \). The triangle inequality gives
\[
|e^{-f} \frac{dv_g}{dv_y} - e^{-f}| \leq \left| 1 - \sqrt{\det(\bar{g}^{-1} g)} \right| e^{-f} + |e^{-f} - 1| e^{-f}. \quad \tag{6.17}
\]

Therefore, using \((*)_R\), we get \( |e^{-f} \frac{dv_g}{dv_y} - e^{-f}| \leq C e^{-\frac{f}{2} - \frac{R^2}{8}} \). Combining this with (6.15) gives
\[
|\delta_{ij} - a_{ij}| \leq C e^{-\frac{R^2}{8}}. \quad \text{Thus, } a_{ij} \text{ is invertible, the inverse } a^{ij} \text{ has the same estimate, and we get } c_{ij} \text{ so that the } v_i = c_{ij} u_j \text{’s are } L^2(M) \text{ orthonormal. It follows that } \delta_{ij} = c_{ik} a_{km} c_{jm} \text{ and, hence, } c^T c = a^{-1}. \quad \tag{6.18}
\]

The variational characterization of eigenvalues gives
\[
\sum_{i=0}^{n} \mu_i \leq \sum_i \|
abla v_i\|^2_{L^2(M)} = \sum_{i,j,k} c_{ij} b_{jk} c_{ik} = \sum_{i,j} b_{ij} (c^T c)_{ij} = \sum_{i,j} b_{ij} a^{ij}. \quad \tag{6.19}
\]

Set \( \bar{b}_{00} = 0 \) and \( \bar{b}_{ij} = \frac{1}{2} \delta_{ij} \) for \( 1 \leq i + j \). Note that \( \sum_{i,j} \delta_{ij} \bar{b}_{ij} = \frac{n-\ell}{2} \). Arguing as above gives \( |\bar{b}_{ij} - b_{ij}| \leq C e^{-\frac{R^2}{8}} \). Using this and \( |\delta_{ij} - a^{ij}| \leq C e^{-\frac{R^2}{8}} \) in (6.17) gives
\[
\sum_{i=0}^{n-\ell} \mu_i \leq \sum_{i,j} a^{ij} b_{ij} \leq \frac{n-\ell}{2} + C e^{-\frac{R^2}{8}}. \quad \tag{6.20}
\]

By (1.55), the drift Bochner formula gives \( \|\text{Hess}_v\|^2_{L^2} = (\mu - \frac{1}{2}) \|
abla v\|^2_{L^2} \). It follows that \( \mu_0 = 0 \) and \( \mu_i \geq \frac{1}{2} \) for \( i \geq 1 \). Combining this with (6.18) gives (6.10). \( \square \)
Proof of Proposition 6.2. Let $v_1, \ldots, v_{n-\ell}$ from Lemma 6.3 and set $I_i = r_1^{-n} \int_{b \leq r} v_i^2 |\nabla b|$. Since $\mu_i < 1$, Theorem 3.4 gives $r_0 = r_0(n)$ so that

$$I_i(r_2) \leq 2 \left( \frac{r_2}{r_1} \right)^4 I_i(r_1) \text{ if } r_0 < r_1 < r_2. \tag{6.19}$$

The bound (6.19) used the complete shrinker $M$. The rest of the argument focuses on the region where we have a priori bounds. Proposition 6.4 gives $\alpha_0 > 0$ and $C_0$ so that $|R| + |\nabla R| \leq C_0$ on $\{b < (1 + \alpha_0) R\}$. This bound Ric and $S$ and, thus, gives a positive lower bound for $|\nabla b|$. Therefore, (6.19) gives polynomial bounds for the ordinary $L^2$ norm (i.e., without $|\nabla b|$ or $e^{-f}$) on $\{b < (1 + \alpha_0) R\}$. This and elliptic estimates for the eigenvalue equation bound $Hess_{v_i}$ and $\nabla Hess_{v_i}$ on $\{b < (1 + \alpha_0) R - R^{-1}\}$.

Moreover, local elliptic estimates and the $L^2$ Hessian bound in (6.10) give that for each fixed $r << R$ we have $I_{Hess_{v_i}}(r) \leq C r e^{-\frac{R^2}{8}}$. Therefore, we can apply proposition 3.80 to get $m$ and $C$ so that $I_{Hess_{v_i}}(r) \leq C R m e^{-\frac{R^2}{8}}$ for $r \leq (1 + \alpha_0) R - R^{-1} - 1$. Using elliptic estimates on scale $R^{-1}$ again, we conclude that on $\{b < (1 + \alpha_0) R - 2\}$

$$|Hess_{v_i}|^2 + R^{-2} |Hess_{v_i}|^2 \leq C R^{m+n} e^{-\frac{R^2}{16}}. \tag{6.22}$$

Since $\mu_i < 1$, Lemma 1.33 gives for each $s$ that

$$\frac{s^2}{4} \int_{b \geq s} \{v_i^2 + |\nabla v_i|^2\} e^{-f} \leq 4 \mu_i^2 + (n + 2) \mu_i + n < 2 n + 6. \tag{6.23}$$

It follows from (6.23) that there is a fixed $s$ and constant $q_0 > 0$ (independent of $R$) so that the matrix $Q_{ij} = \int_{b \geq s} \langle \nabla u_j, \nabla u_j \rangle e^{-f}$ is invertible with $|Q| + |Q^{-1}| < q_0$. Note that (6.22) and the fundamental theorem of calculus imply that $\langle \nabla u_i, \nabla u_j \rangle$ is exponentially close to being constant on $\{b < (1 + \alpha_0) R - 2\}$. Therefore, we can choose a bounded linear transformation $\tilde{Q}$ so that $u_i = \tilde{Q}_{ij} v_j$ satisfy $\int u_i e^{-f} = 0$ and

$$\sup_{\{b < (1 + \alpha_0) R - 2\}} |Q_{ij} - \langle \nabla u_i, \nabla u_j \rangle| \leq C R^{m'} e^{-\frac{R^2}{16}}. \tag{6.24}$$

This gives the first bound in (6.3) and the next two bounds follow from (6.22) since $\tilde{Q}$ is bounded. The last bound follows similarly, using also that the $\mu_i$ close to $\frac{1}{2}$. \hfill \Box

Proof of Theorem 6.7. We will use freely below that $|R| + |\nabla R| \leq C_0$ on $\{b < (1 + \alpha_0) R\}$ by Proposition 6.4. Proposition 6.2 gives $n - \ell$ “almost linear” functions $u_i$ satisfying (6.3). Using the bounds on $\nabla^2 u_i$ and $\nabla^3 u_i$ in (6.3), the definition of $R$ gives

$$|R(\nabla u_i, \cdot, \cdot)| \leq C R^m e^{-\frac{R^2}{16}}. \tag{6.25}$$
Tracing this gives $|\text{Ric}(\nabla u_i, \cdot)| \leq C R^m e^{-\frac{\ell_2}{\ell}}$. Set $\tilde{f} = \frac{1}{2} + \frac{1}{4} \sum u_i^2$, so that (6.3) gives that $\tilde{f} - \frac{1}{2} - |\nabla f|^2$ is “exponentially small”, i.e., bounded by $C R^m e^{-\frac{\ell_2}{\ell}}$ for some $C$ and $m$. Since $|\nabla f|^2 + S = f$ (by (1.47)) and $S$ is bounded, we see that $|\tilde{f} - f| \leq C$.

Let $N_0 = \{ u_1 = \cdots = u_{n-\ell} = 0 \}$ be the intersection of the zero sets and $f_0$ the restriction of $f$ to $N_0$. Since $|\tilde{f} - f| \leq C$, $f_0 \leq \frac{1}{2} + C$ and, thus, $\tilde{f} \leq C'$ on $N_0$. It follows that $N_0$ is a smooth $\ell$-dimensional submanifold, the $\nabla u_i$’s span its normal space, and $|\nabla^\perp f|$ is exponentially small on $N_0$ by (6.3). Moreover, the level sets of the map $(u_1, \ldots, u_{n-\ell})$ foliate $b < R$, so $N_0$ is connected and diffeomorphic to $N$.

Since $M$ is fixed close to the model $\Sigma$ on a fixed central ball, $\text{Ric}$ has a block decomposition with an $\ell \times \ell$ block close to $\frac{1}{2} g^1$ and a complementary block that almost vanishes. Thus, by (6.26), the span of the $\nabla u_i$’s is almost orthogonal to $\Phi_R(N)$ and $N_0$ is locally a graph with small gradient over $\Phi_R(N)$. Using the slice theorem, fix a diffeomorphism $\Psi : N \to N_0$ with $\text{div}_N (\Psi^* g_0 - g^1) = 0$ and with $(\Psi^* g_0 - g^1)$ fixed $C^2$ small (cf. theorem 3.6 in [V], 3.1 in [CIT]).

Let $e_j$ be an orthonormal tangent frame for $N_0$. Using (6.3), we see that the second fundamental form $A$ of $N_0$ satisfies

$$|\langle A(e_j, e_k), \nabla u_i \rangle| = |\langle e_k, \nabla e_j \nabla u_i \rangle| \leq C R^m e^{-\frac{\ell_2}{\ell}}. \tag{6.26}$$

Combining this, the Gauss equation, and (6.26), we see that the Ricci curvature $\text{Ric}^0$ of $N_0$ and the Hessian $\text{Hess}^0_{f_0}$ satisfy

$$|\text{Ric}^0(e_j, e_k) - \text{Ric}(e_j, e_k)| + |\text{Hess}^0_{f_0}(e_j, e_k) - \text{Hess}_f(e_j, e_k)| \leq C R^m e^{-\frac{\ell_2}{\ell}}. \tag{6.27}$$

Since $\text{Ric}$ vanishes exponentially in the normal directions by (6.25), we get the same bound for the difference $|S^0 - S|$ of the scalar curvatures. It follows that the shrinker quantity $\phi_0$ on $N_0$ satisfies

$$|\phi_0| + \|\nabla f_0\|^2 + S^0 - f_0 \leq C R^m e^{-\frac{\ell_2}{\ell}}. \tag{6.28}$$

Corollary 5.23 now gives that $|\tilde{\theta}|_{W^{2,2}}$ and $\|\nabla f_0\|_{W^{1,2}}$ are exponentially small. Furthermore, since $\text{div}_N \tilde{\theta} = 0$, the equation for $\text{Ric}_{N_0}$ is elliptic in $\tilde{\theta}$; elliptic estimates give uniform bounds for the $C^k$ norms of $\tilde{\theta}$. Standard interpolation between these uniform bounds and the exponentially small $W^{2,2}$ bounds then gives the desired $C^{3,\alpha}$ exponential bounds.

We must now extend the estimates off of $N$ and $N_0$. Note that $|f - \tilde{f}| \leq C R^m e^{-\frac{\ell_2}{\ell}}$ on $N_0$. Differentiating gives

$$\nabla \nabla u_j \left( f - \tilde{f} \right) = \langle \nabla u_j, \nabla f \rangle - \frac{1}{2} u_j + \frac{1}{2} u_i \left( \langle \nabla u_j, \nabla u_i \rangle - \delta_{ij} \right). \tag{6.29}$$

This is exponentially small by (6.3) and, by integrating up, so is $f - \tilde{f}$.

Define the map $H : N_0 \times \mathbb{R}^{n-\ell} \to M$ by letting $H(q, x_1, \ldots, x_{n-\ell})$ be the time one flow starting at $q$ along the vector field $\sum x_i \nabla u_i$. Now, set $\Psi(p, x_1, \ldots, x_{n-\ell}) = H(\Psi_0(p), x_1, \ldots, x_{n-\ell})$. Write $x = r y$ where $y \in S^{n-1}$ and observe that $H_r = \langle y, \nabla u \circ H \rangle$ and this is exponentially $\ldots$
parallel. It follows that $H$ is exponentially close to a local isometry and, thus, also a local diffeomorphism. Similarly, $u_i \circ H - x_i$ is exponentially small and, thus, $H$ is proper. Since $H$ is a proper local diffeomorphism between complete connected spaces and has pull-back metric bounded from below, [GW] implies that $H$ has the path-lifting property. Using that the image and target are topologically $N$ times a Euclidean space, we see that $H$ is a diffeomorphism from a subset of $N \times \mathbb{R}^{n-\ell}$ onto $\{ b < (1 + \alpha_0) R \}$.

Since pseudo-locality gives uniform curvature bounds, the drift eigenfunction equation has uniform bounds on scale $\frac{1}{b}$. Elliptic estimates on this scale give higher derivative bounds on the eigenfunctions and, thus, on $\Psi$. \hfill \Box

7. Variations of Geometric Quantities

The main result of this section is the formula (7.3) for $\phi''$ in the direction of a Jacobi field. Let $g(t) = g + t \cdot h$ and $f(t) = f + t \cdot k$ be families of metrics and functions. We will work in a frame $\{ e_i \}$ of coordinate vector fields independent of $t$.

Cao-Hamilton-Ilmanen, [CaHI] (cf. [CaZ]), computed the first variation of $\phi$

\begin{equation}
(7.1) \quad \phi'(0) = \frac{1}{2} L h + \text{Hess}(\frac{1}{2} \text{Tr}(h) - k) + \text{div}_l \text{div}_f h.
\end{equation}

Thus, the Jacobi fields on $\Sigma = N \times \mathbb{R}^{n-\ell}$ consist of $h = u g^l$ and $k = \frac{\ell}{2} u$ with $u \in \mathcal{K}$.

**Proposition 7.2.** If $h = u g^l$ and $k = \frac{\ell}{2} u$ on $\Sigma$ where $u$ depends only on $\mathbb{R}^{n-\ell}$, then

\begin{equation}
(7.3) \quad 2 \phi''_{ij}(0) = -2 |\nabla u|^2 g^{ij} - 2 \ell u u_{ij} - \ell u_i u_j.
\end{equation}

Formally, Proposition (7.2) and Lemma 5.17 say that the Jacobi fields are not integrable since $\int (\phi'', (|\nabla u|^2 - \Delta |\nabla u|^2) g^{ij}) e^{-f} = -\ell \int (|\nabla u|^2 - \Delta |\nabla u|^2) |\nabla u|^2 e^{-f} \leq -C \|u\|^4_{L^2}$ is strictly negative, but $\phi$, and thus $\phi''$, vanish on a one-parameter family of shrinkers.

7.1. First variations. In this subsection, we collect well-known first variation formulas (see, e.g., [T] or [CaZ]) for reference; these results do not use the product structure on $\Sigma$.

**Proposition 7.4.** The variations of $S, \text{Ric}$ and $\text{R}$ are given by

\begin{equation}
(7.5) \quad S' = -\langle h, \text{Ric} \rangle + \text{div}^2 h - \Delta \text{Tr} h,
\end{equation}

\begin{equation}
(7.6) \quad 2 \text{Ric}'_{ij} = -\Delta h + h_{ik} \text{Ric}^k_j + \text{Ric}^k_i h_{kj} - 2 R(h) - \text{Hess}_{\text{Tr} h} + \nabla \text{div} h + (\nabla \text{div} h)^T,
\end{equation}

\begin{equation}
(7.7) \quad 2 R'_{ijkn} = R_{ijkn} e^l_n - R_{ijmn} h_{kn} - R_{ijnk} e^l_n - R_{jnki} e^l_n - R_{jkni} e^l_n - R_{nkij} e^l_n.
\end{equation}

Here $\text{div} h$ is the divergence\footnote{Note the different sign convention from [T].} of $h$ given by $(\text{div} h)_i = h_{ij,j}$ and $(\nabla \text{div} h)^T$ is the transpose.

By definition, $\nabla e_i e_j = \Gamma^k_{ij} e_k$, where $\Gamma^k_{ij} = \frac{1}{2} g^{kn} (e_i (g_{mj}) + e_j (g_{mi}) - e_m (g_{ji}))$ is the Christoffel symbol. Since these are coordinate vector fields, we have $\Gamma^k_{ij} = \Gamma^k_{ji}$. Even though $\nabla (\cdot)$ is not a tensor (it is not tensorial in the upper slot), the derivative is a tensor.

**Lemma 7.8.** At a point where $g_{ab} = \delta_{ab}$ and $e_c (g_{ab}) = 0$ at $t = 0$, we have

\begin{equation}
(7.9) \quad (\nabla e_i e_j)' = \sum_k C^k_{ij} e_k \quad \text{where} \quad C^k_{ij} = \frac{1}{2} (h_{k,j,i} + h_{k,i,j} - h_{j,i,k}),
\end{equation}

\begin{equation}
(7.10) \quad (\nabla e_i \nabla e_j)' = \frac{1}{2} \{ h_{k,j,in} + h_{k,i,jn} - h_{j,i,kn} \} e_k.
\end{equation}
Lemma 7.11. If \( u \) is a one-parameter family of functions, then at \( t = 0 \) at a point where \( g_{ij} = \delta_{ij} \) and \( e_c(g_{ab}) = 0 \) we have \( (\text{Hess}_u)' = \text{Hess}_{u'} - C^k_{ij} u_k \).

Lemma 7.12. At \( t = 0 \) at a point where \( g_{ij} = \delta_{ij} \) and \( e_c(g_{ab}) = 0 \), we have

\[
(7.13) \quad e_m (b^i_{ij})' = [b_{ij}, m] + b_{mj} C^m_{mi} + b_{in} C^m_{mj},
\]

\[
e_n [e_m ([b_{ij}]')] = (b_{ij,mn}) + (\nabla b)(C^p_{mi} e_p, e_j, e_m) + (\nabla b)(e_i, C^p_{mj} e_p, e_m) + (\nabla b)(e_i, e_j, C^p_{nm} e_p)
\]

\[
(7.14) \quad + (\nabla b)(C^p_{mi} e_p, e_j, e_n) + b([\nabla_{e_n} \nabla_{e_m} e_i]', e_j) + b'(\nabla_{e_n} \nabla_{e_m} e_i, e_j)
\]

\[
+ (\nabla b)(e_i, [\nabla_{e_n} \nabla_{e_m} e_j']) + b'(e_i, \nabla_{e_n} \nabla_{e_m} e_j).
\]

Lemma 7.15. The derivative of \( \phi = \kappa - \text{Ric} - \text{Hess}_f \) is

\[
(7.16) \quad \phi'_{ij} = \kappa h_{ij} + \frac{1}{2} (L h)_{ij} + \left( \frac{1}{2} \text{Tr} (h) - k \right)_{ij}
\]

\[- \frac{1}{2} \left( g^{kn} h_{jk,ni} + g^{kn} h_{ik,nj} + \text{Ric}^k_i h_{jk} + \text{Ric}^k_j h_{ik} \right) + \frac{1}{2} \left( h_{jn,i} + h_{in,j} \right) g^{nm} f_m. \]

Lemma 7.17. If \( (M, g, f) \) is a gradient shrinking soliton, then

\[
(\text{div}_f^* \text{div}_f h)(e_i, e_j) = \kappa h_{ij} - \frac{g^{kn}}{2} \{ h_{jk,ni} - f_n h_{kj,i} + h_{ik,nj} - f_n h_{ik,j} + \text{Ric}_{ni} h_{kj} + \text{Ric}_{nj} h_{ik} \}.
\]

Combining Lemma 7.15 and Lemma 7.17 recovers the first variation (7.1) for \( \phi \).

7.2. Computing \( \phi''(0) \): Proof of Proposition 7.2.

Lemma 7.18. At a point where \( g_{ij} = \delta_{ij} \) and \( e_c(g_{ab}) = 0 \) at \( t = 0 \),

\[
(7.19) \quad 2 \left[ (\nabla_{e_n} \nabla_{e_m} e_i)' \right]_k = h_{ki,nn} + h_{kn,i} f_n - h_{in,k} f_n + (\text{div}_f h)_{k,i} - (\text{div}_f h)_{i,k}.
\]

Proof. Taking the trace in the second claim in Lemma 7.8 at \( t = 0 \) gives

\[
(7.20) \quad 2 \left[ (\nabla_{e_n} \nabla_{e_m} e_i)' \right]_k = h_{kn,in} + h_{ki,nn} - h_{ni,nn}.
\]

The Ricci identity and \( (\text{div}_f h)_{k,i} = h_{kn,ni} - f_k h_{kn,i} - f_{ki} h_{kn} \) gives

\[
h_{kn,in} = h_{kn,ni} + R_{nikm} h_{mn} + R_{ninm} h_{mk} = h_{kn,ni} - [R(h)]_{ki} + \text{Ric}_{in} h_{mk}
\]

\[
(7.21) \quad = (\text{div}_f h)_{k,i} + h_{kn,i} f_n + h_{kn} f_{mi} - [R(h)]_{ki} + \text{Ric}_{in} h_{mk}.
\]

Using the shrinker equation, this becomes \( h_{kn,ni} = (\text{div}_f h)_{k,i} + h_{kn,i} f_n - [R(h)]_{ki} + \frac{1}{2} h_{ik} \). The last two terms are symmetric in \( i \) and \( k \), so we get

\[
(7.22) \quad h_{kn,in} - h_{ni,kn} = (\text{div}_f h)_{k,i} - (\text{div}_f h)_{i,k} + h_{kn,i} f_n - h_{in,k} f_n.
\]

Substituting this into (7.20) gives the lemma. \( \square \)

In the remainder of this section, all results will be stated at a point where \( g_{ij} = \delta_{ij} \) so that there is no difference between upper and lower indices.
Corollary 7.23. If \( h' = 0 \) at \( t = 0 \), then we have at \( t = 0 \) that

\[
- [\mathcal{L} h_{ij}]' = h_{mn} h_{ij, mn} + 2 h_{pj,m} c_{mi}^p + 2 h_{ip,m} c_{mj}^p + h_{ij,p} \left( k - \frac{1}{2} (\text{Tr} h) \right)_p
\]

\[
+ \frac{1}{2} h_{mj} \mathcal{L} h_{mi} + \frac{1}{2} h_{mi} \mathcal{L} h_{mj} + h_{ij,p} (\text{div} f)_p
\]

\[
+ \frac{1}{2} (h_{mi} (\text{div} f)_m,j - (\text{div} f)_j,m) + h_{mj} (\text{div} f)_m,i - (\text{div} f)_i,m) .
\]

If, in addition, \( h = u g^1 \) and \( k = \frac{\ell}{2} u \) where \( u \) depends only on \( \mathbb{R}^{n-\ell} \), then

\[
[\mathcal{L} h]' = -[2 |\nabla u|^2 + u \mathcal{L} u] g^1 .
\]

Proof. We will work at a point where \( g_{ij} = \delta_{ij} \) and \( e_c(g_{ab}) = 0 \) at \( t = 0 \). By definition, we have \( [\Delta h](e_i, e_j) = g^{mn} h_{ij, mn} \) and, thus, that at \( t = 0 \) at this point

\[
[\Delta h]'_{ij} = -h_{mn} h_{ij, mn} + (h_{ij, mn})' .
\]

Since \( h' = 0 \), Lemma 7.12 gives that

\[
-(h_{ij, mn})' = h_{pj,m} c_{mi}^p + h_{ip,m} c_{mj}^p + \frac{1}{2} (\text{div} f)_p + h_{mn} f_m + \frac{1}{2} (\text{Tr} h) \right)_p \text{ by (7.9) in (7.26) gives}
\]

\[
-(h_{ij, mn})' = 2 h_{pj,m} c_{mi}^p + 2 h_{ip,m} c_{mj}^p + h_{ij,p} \left( (\text{div} f)_p + h_{pm} f_m - \frac{1}{2} (\text{Tr} h) \right)_p
\]

\[
+ \frac{1}{2} h_{nj} \{h_{mn,i} f_m + h_{ni,mn} - h_{im,n} f_m + (\text{div} f)_n,i - (\text{div} f)_i,n \}
\]

\[
+ \frac{1}{2} h_{ni} \{h_{mn,j} f_m + h_{nj,mn} - h_{jn,n} f_m + (\text{div} f)_n,j - (\text{div} f)_j,n \} .
\]

For the drift term, we have

\[
(\nabla \nabla f h_{ij})' = (g^{nm} f_n h_{ij,m})' = -h_{nm} f_n h_{ij,m} + k_m h_{ij,n} + f_n (h_{ij,n})' .
\]

Since \( h' = 0 \), Lemma 7.12 gives that \( (h_{ij,n})' = -h_{mj} c_{ni}^m - h_{mi} c_{nj}^m \), so we get

\[
(\nabla \nabla f h_{ij})' = -h_{nm} f_n h_{ij,m} + k_m h_{ij,n} - f_n \left( h_{mj} c_{ni}^m + h_{mi} c_{nj}^m \right) .
\]

Combining, canceling terms and using that \( h_{mn,i} - h_{in,m} = 2 c_{m}^m - h_{mi,n} \) gives the first claim.

To get the second claim, we plug in \( h = u g^1 \) and \( k = \frac{\ell}{2} u = \frac{1}{2} \text{Tr} h \) into the first claim. With these choices, \( \text{div} f h = 0 \) so the last term on the second line and the entire third line drop out immediately. Using (7.9), \( \nabla g^1 = 0 \), and the fact that \( g^1 \) is nonzero only on the first factor \( N \), while \( u \) depends only on \( \mathbb{R}^{n-\ell} \), the second claim follows.
Corollary 7.31. If $h' = 0$ at $t = 0$, then we have at $t = 0$ that

$$-[g^{kn} h_{jk,ni}]' = h_{kn} h_{jk,ni} + h_{pm,m} C_{ij}^{p} + h_{jp,m} C_{im}^{p} + h_{jm,p} C_{im}^{p} + h_{pm,i} C_{mj}^{p}$$

(7.32)

$$+ h_{jp,i} \left( (\text{div} f h)_p + h_{pm} f_m - \frac{1}{2} (\text{Tr} h)_p \right) + \frac{h_{pm}}{2} \left( h_{pm,ji} + h_{pj,mi} - h_{mj,pi} \right)$$

$$+ \left\{ (\text{div}_f h)_p,i + h_{pm,i} f_m + h_{pm} f_{mi} - \frac{1}{2} (\text{Tr} h)_p \right\} h_{jp}.$$ 

If, in addition, $h = u g^1$ where $u$ depends only on $\mathbb{R}^{n-\ell}$, then $[g^{kn} h_{jk,ni}]' = -\frac{\ell}{2} (u_i u_j + u u_{ij}).$

Proof. Working at a point as before, we have $[g^{kn} h_{jk,ni}]' = -h_{kn} h_{jk,ni} + [h_{jk,ki}]'$, so we must compute $[h_{jk,ki}]'$. Since $h' = 0$, Lemma 7.12 gives that

$$- (h_{jm,mi})' = (\nabla h)(C_{ij}^{p} e_p, e_m, e_m) + (\nabla h)(e_j, C_{im}^{p} e_p, e_m) + (\nabla h)(e_j, e_m, C_{im}^{p} e_p)$$

(7.33)

$$+ (\nabla h)(C_{mj}^{p} e_p, e_m, e_i) + h([\nabla e_i \nabla e_m]') e_m + (\nabla h)(e_j, C_{mm}^{p} e_p, e_i) + h(e_j, [\nabla e_i \nabla e_m]').$$

Lemma 7.8 gives that $(\nabla e_i \nabla e_m)' = \frac{1}{2} \left\{ h_{pj,im} + h_{pi,jm} - h_{ji,pm} \right\} e_p$, so we have

$$h([\nabla e_i \nabla e_m]') e_m = \frac{h_{pm}}{2} \left\{ h_{pm,ji} + h_{pj,mi} - h_{mj,pi} \right\},$$

$$h(e_j, [\nabla e_i \nabla e_m]') = \left\{ (\text{div} f h)_p,i + h_{pm,i} f_m + h_{pm} f_{mi} - \frac{1}{2} (\text{Tr} h)_p \right\} h_{jp}.$$

Using these and the formula for $C_{mm}^{p}$ gives the first claim. If $h = u g^1$, then most terms drop out immediately and the definition (7.9) for $C_{ij}^{p}$ gives

$$2 \left[ g^{kn} h_{jk,ni} \right]' = -2 (h_{jp,m} + h_{jm,p}) C_{im}^{p} - 2 h_{pm,i} C_{mj}^{p} - h_{pm,ji} h_{pm} = -\ell u_i u_j - \ell u u_{ij}. \quad \square$$

Proof of Proposition 7.2. We will work at a point using coordinates where $g_{ij} = \delta_{ij}$ at $t = 0$. Using that $h' = 0$ and differentiating Lemma 7.13 at $t = 0$ gives

$$2 \phi''_{ij}(0) = [(\mathcal{L} h)_{ij}']' + 2 [R(h)]' + \left[(\text{Tr} h) - 2 k\right]_{ij}'.$$

(7.35)

$$- \left( g^{kn} h_{jk,ni} + g^{kn} h_{ik,ij} \right)' - \left( [\text{Ric}_i^{k}]' h_{jk} + [\text{Ric}_j^{k}]' h_{ik} \right)'$$

$$+ (h_{jn,i} + h_{in,j}) f_n - (h_{jn,i} + h_{in,j}) h_{nm} f_m + (h_{jn,i} + h_{in,j}) h_{kn}.$$ 

We will compute each term next. The third claim in Proposition 7.4 gives at $t = 0$ that

$$2 [R(h)]' = 2 [R_{ikjn} g^{kp} g^{nm} h_{pm}]' = 2 R_{ikjn} h_{kn} - 2 R_{ikjn} h_{kp} h_{pm} - 2 R_{ikjn} h_{nm} h_{km}$$

(7.36)

$$\{ R_{ikjn} h_{kn} - R_{ikjn} h_{kj} + h_{in,kj} - h_{kn,ji} - h_{ik,ni} - h_{ij,nk} \} h_{kn} - 4 R_{ikjn} h_{kp} h_{pn}.$$ 

Since $h = u g^1$ where $u$ depends only on $\mathbb{R}^{n-\ell}$ and $R_{ikjn} = \frac{1}{2(\ell-1)} (g_{ij} g_{kn} - g_{in} g_{kj})$, we have

$$2 [R(h)]' = \left\{ 2 u R_{ikjn} g_{kn} - g_{kn,uj} \right\} u g^1_{kn} - 4 u^2 R_{ikjn} g_{kn} = -u^2 g^1_{ij} - \ell u u_{ij}. $$

The second claim in Proposition 7.4 gives at $t = 0$ that

$$2 (\text{Ric}_i^{k})' = 2 (\text{Ric}_ip g^{kp})' = 2 (\text{Ric}_ip)' g^{kp} - g^{ip}_1 h_{kp}$$

(7.37)

$$= -\Delta h + \frac{1}{2} h_{in} g^{1}_{kn} + \frac{1}{2} g^{1}_{in} h_{kn} - 2 R(h) - \text{Hess}_h - \nabla \text{div} h + (\nabla \text{div} h)^T - g^{1}_{ip} h_{kp}. $$
Therefore, since \( h = u g^1 \), we get that \( 2 \left( \text{Ric}^k \right)' h_{jk} = -u(\Delta u) g^1 - u^2 g^1 \). Since \([\text{Tr}(h)]' = [g_{ij} h_{ij}]' = -|h|^2\) and \( k' = 0 \), Lemma 7.11 gives that

\[
(7.38) \quad (\text{Hess}_{(\text{Tr}(h)-2k)})' = \text{Hess}_{-|h|^2} - \frac{1}{2} (h_{nj,i} + h_{ni,j} - h_{ji,n}) (\text{Tr}(h) - 2k) n.
\]

Since \( h = u g^1 \) and \( k = \ell u \), this becomes \( (\text{Hess}_{(\text{Tr}(h)-2k)})' = \text{Hess}_{-|h|^2} = -\ell \text{Hess}_{u^2} \). The first claim in Lemma 7.12 and the definition (7.39) of \( C_{ij}^k \) give

\[
(7.39) \quad [h_{in,j}]' = -\frac{1}{2} h_{ip} (h_{pj,n} + h_{pn,j} - h_{jn,p}) - \frac{1}{2} h_{np} (h_{pj,i} + h_{pi,j} - h_{ji,p}).
\]

Using that \( h = u g^1 \) gives \([h_{in,j}]' f_n = -\frac{1}{2} h_{ip} h_{pj,n} f_n = -\frac{1}{2} u u_n f_n g_{1j}^1 \). We now use these calculations in (7.35), together with Corollary 7.23 for the \([\mathcal{L} h]' \) term and Corollary 7.31 for the first terms in the middle line. This gives

\[
(7.40) \quad 2 \phi_{ij}''(0) = -2 |\nabla u|^2 + u \mathcal{L} u |g^1 - (u^2 g^1 + \ell u u_{ij}) - \ell \text{Hess}_{u^2} + \ell (u u_{ij} + u u_{ij})
\]

\[+ (u \Delta u + u^2) g^1 - u u_n f_n g_{1}^1 = -2 |\nabla u|^2 g^1 - 2 \ell u u_{ij} - \ell u_i u_j. \quad \square
\]

8. Second order stability of \( N \times \mathbb{R}^{n-\ell} \)

In this section \( \Sigma = N^\ell \times \mathbb{R}^{n-\ell} \) and \( g^1 \) is an Einstein metric on \( N \) with \( \text{Ric} = \frac{1}{2} g^1 \) and satisfying \((\ast)\). Given a nearby metric \( \bar{g} + h \) and potential \( \bar{f} + k \), let \( u g^1 \) be the orthogonal projection of \( h \) onto \( \mathcal{K} g^1 \) and write \( h = u g^1 + \bar{h} \) and \( k = \ell u + \bar{\psi} \). Bars denote quantities relative to \( \bar{g} \); e.g., \( \overline{\text{Ric}} \) is the Ricci tensor for \( \bar{g} \).

The main result of this section shows that \( \Sigma \) has a local rigidity: If \( (h, k) \) is small, then it can be bounded in terms of the failure \( \phi \) to be a gradient shrinking soliton, with two caveats. First, we need to bound \( \text{div}_{\bar{f}} \) to control the gauge. Second, even if \( h = 0 \), \( k \) could be linear, corresponding to a translation along the axis of \( \Sigma \). To mod out for this, we must bound the “center of mass” vector

\[
(8.1) \quad \mathcal{B}(h, k) = \int x_i \left( k - \frac{1}{2} \text{Tr}_g h \right) e^{-\bar{f}} = \frac{1}{2} \int (\partial x_i, \overline{\nabla} \left( k - \frac{1}{2} \text{Tr}_g h \right) ) e^{-\bar{f}}.
\]

The next theorem uses a first order Taylor expansion to show that the Jacobi field \( u g^1 \) dominates the error term \( \bar{h} \) and then uses the second order expansion to estimate \( \|u\|_{L^2} \).

**Theorem 8.2.** There exist \( C, \delta > 0 \) so that if \( \|h\|_{C^2} + \|\overline{\nabla} k\|_{C^1} \leq \delta \), then for any \( \epsilon > 0 \)

\[
\|h\|_{W^{2,2}}^2 + \|\overline{\nabla} \psi\|^2_{W^{1,2}} \leq C \left\{ \|\phi\|_{L^2}^2 + \|\text{div}_{\bar{f}} h\|^2_{W^{1,2}} + \|\mathcal{B}(h, k)\|^2 + \|u\|_{L^2} \right\},
\]

\[
\|u\|_{L^2}^2 \leq C \left\{ \|u\|_{L^2}^3 + \|\phi(1 + |x|^2)\|_{L^1} \right\} + C \epsilon \left\{ \|\phi\|_{L^2}^{2-\epsilon} + \|\mathcal{B}(h, k)\|^{2-\epsilon} + \|\text{div}_{\bar{f}} h\|_{W^{1,2}}^{2-\epsilon} \right\}.
\]

When \( \phi, \text{div}_{\bar{f}} h \) and \( \mathcal{B}(h, k) \) vanish globally, we get:

**Corollary 8.3.** There exists \( \delta > 0 \) so that if \( \phi = 0, \text{div}_{\bar{f}} h = 0, \mathcal{B}(h, k) = 0 \) and \( \|h\|_{C^2} + \|\overline{\nabla} k\|_{C^1} \leq \delta \), then \( h = 0 \) and \( k = 0 \).

**Proof.** The second claim in Theorem 8.2 gives that \( \|u\|_{L^2}^2 \leq C \|u\|_{L^2}^2 \). Since \( \|u\|_{L^2} \leq \|h\|_{L^2} \), \( u \) vanishes if \( \|h\|_{C^2} \) is small. Once \( u \equiv 0 \), then the first claim in the theorem gives that \( \bar{h} = 0 \) and \( \overline{\nabla} \psi = 0 \). It follows that \( \overline{\nabla} k = 0 \). Combining this with the normalizations \( S + |\nabla f|^2 = f \) and \( \bar{S} + |\nabla \bar{f}|^2 = \bar{f} \), we conclude that \( k = 0 \). \( \square \)
8.1. **Pointwise Taylor expansion of $\phi$.** The estimates in this subsection Taylor expand near $\Sigma$ and, as such, assume that $h$, $k$ and $v$ are small at the point where we compute.

**Lemma 8.4.** There is a smooth map $\Psi$ so that $\text{Ric} = \Psi(h, \nabla h, \nabla \nabla h)$. Furthermore, $\text{Hess}_{f^2+h} = \text{Hess}_f + \text{Hess}_h - \left( \Gamma_i^m \right) e_n(\bar{f} + k)$.

**Proof.** The Christoffel symbols of $g = \bar{g} + h$ are given by

$$ (8.5) \quad \Gamma_{ij}^m = \frac{1}{2} (\bar{g} + h)^{pm} (e_i(\bar{g} + h)_{jm} + e_j(\bar{g} + h)_{mi} - e_m(\bar{g} + h)_{ij}) . $$

Note that $e_i h_{pm} = h_{pm, i} + \Gamma_i^m h_{nm} + \Gamma_i^m h_{pn}$ where $h_{pm, i}$ is the covariant derivative of $h$ (with respect to $\bar{g}$). Thus, $\Gamma$ is a smooth function of $h$ and $\nabla h$. The curvature tensor $R^i_{jkm}$ of $\bar{g} + h$ is the sum of linear terms in the derivative of $\Gamma$ and quadratic terms in $\Gamma$, giving the first claim. The last claim follows from $\text{Hess}_v(e_i, e_j) = e_i(e_j(v)) - \Gamma_{ij}^n w_n$. \hfill \Box

Define the one-parameter families of 2-tensors $H(t) = \text{Hess}_{f^2+tk}(\bar{g} + th)$ to be the Hessian of $\bar{f} + t k$ computed with respect to the metric $g(t) = \bar{g} + t h$; define $\phi(t)$ similarly.

**Lemma 8.6.** There exists $C$ so that

$$ (8.7) \quad |H(1) - H(0) - H'(0)| \leq C |\nabla h| \left( |h| |\nabla \bar{f}| + |\nabla k| \right) , $$

$$ (8.8) \quad \left| H(1) - H(0) - H'(0) - \frac{1}{2} H''(0) \right| \leq C |h| |\nabla h| \left( |h| |\nabla \bar{f}| + |\nabla k| \right) . $$

**Proof.** Let $(\Gamma^t)_{ij}^k$ be the Christoffel symbols for the metric $\bar{g} + th$. We will bound the $t$ derivatives of $\Gamma^t$ for $t \in [0, 1]$. Since the difference of Christoffel symbols is a tensor, we can do this at a point using coordinates where $e_c(g_{ab}) = 0$ and $\bar{G} = 0$, so that

$$ (8.9) \quad 2 (\Gamma^t)_{ij}^k = t (\bar{g} + th)^{km} (e_i(h_{jm}) + e_j(h_{mi}) - e_m(h_{ij})) . $$

Differentiating this expression, we see that

$$ (8.10) \quad |\partial_t \Gamma^t| \leq C |\nabla h|, \quad |\partial_t^2 \Gamma^t| \leq C |h| |\nabla h| and \quad |\partial_t^3 \Gamma^t| \leq C |h|^2 |\nabla h| . $$

The last claim in Lemma 8.4 gives

$$ H_{ij}(t) - H_{ij}(0) = t \text{Hess}_k + (\bar{G} - \Gamma^t)^{p}_{ij} (\bar{f}_p + t k_p) . $$

Differentiating gives that

$$ H_{ij}' = \text{Hess}_k + (\bar{G} - \Gamma^t)^{p}_{ij} k_p - (\partial_t \Gamma^t)^{p}_{ij} (\bar{f}_p + t k_p), $$

$$ (8.11) \quad H_{ij}'' = -2 (\partial_t \Gamma^t)^{p}_{ij} k_p - (\partial_t^2 \Gamma^t)^{p}_{ij} (\bar{f}_p + t k_p), $$

$$ (8.12) \quad H_{ij}''' = -3 (\partial_t^3 \Gamma^t)^{p}_{ij} k_p - (\partial_t^4 \Gamma^t)^{p}_{ij} (\bar{f}_p + t k_p) . $$

Thus, we get $|H''| \leq C |\nabla h|(|\nabla k| + |h| |\nabla \bar{f}|)$ and $|H'''| \leq C |h| |\nabla h|(|\nabla k| + |h| |\nabla \bar{f}|)$.

To keep notation short, set $[h]_1 = |h| + |\nabla h|$ and $[h]_2 = |h| + |\nabla h| + |\nabla^2 h|.$

**Corollary 8.13.** We have $|\phi'(0) + \frac{1}{2} \phi''(0)| \leq |\phi(1)| + C [h]_2^3 + C |h| |\nabla h| \left( |h| |\nabla \bar{f}| + |\nabla k| \right)$.

**Proof.** Lemma 8.4 and the chain rule give $|\text{Ric}(1) - \text{Ric}(0) - \text{Ric}'(0) - \frac{1}{2} \text{Ric}''(0)| \leq C [h]_2^3$. Combining this with the second bound in Lemma 8.6 gives the claim. \hfill \Box
Proposition 8.14. There exists $C$ so that if $h = u g^1 + \hat{h}$ and $k = \frac{\ell}{2} u + \psi$ where $u$ depends only on $R^{n-\ell}$, then
\[ 2 \phi''_i(0) + 2 |\nabla u|^2 g^1 + 2 \ell u_{ij} + \ell u_i u_j \leq C [u]_2 \hat{h} + C \hat{h}_2 + C |\nabla \tilde{f}| ([u]_1 \hat{h}_1 + \hat{h}_2^2) + C |\nabla \psi| (|\nabla u| + |\nabla \hat{h}|) . \]
(8.15)

Proof. We divide $\phi(t)$ into two pieces, $\phi_0(t) = \frac{1}{2} (g + t h) - \text{Ric}_{g+th}$ and the Hessian part $H(t)$. Similarly, let $\phi_u(t) = \phi_{u,0}(t) - H_u(t)$ be the variation of $\phi$ in the direction $(u g^1, \frac{\ell}{2} u)$. Proposition 7.2 gives
\[ 2 (\phi_u)''_i(0) = -2 |\nabla u|^2 g^1 - 2 \ell u_{ij} - \ell u_i u_j . \]
By Lemma 8.4 and the chain rule, $\phi''_0(0)$ is quadratic in $(h, \nabla h, \nabla^2 h)$ and, thus,
\[ |\phi''_0(0) - \phi''_u(0)| \leq C [u]_2 \hat{h} + C \hat{h}_2^2 . \]
(8.17)

On the other hand, (8.11) plus (8.10) imply that
\[ |H''(0) - H''_u(0)| \leq C \left[ |u| |\nabla \hat{h}| + \hat{h} \left( |\nabla u| + |\nabla \hat{h}| \right) \right] |\nabla \tilde{f}| + C |\nabla \psi| (|\nabla u| + |\nabla \hat{h}|) + C |\nabla u| |\nabla \hat{h}| . \]
The proposition follows by combining this with (8.10) and (8.17). □

8.2. Integral estimates. We turn now to integral estimates. Suppose that $h$, $k$ and $u$ are as in Theorem 8.2. Even though $h$ and $k$ are small, $u$ grows quadratically so the Taylor expansion is not valid for $x$ large.

Lemma 8.18. We have $\|u\|^2_{L^2} \leq C \left\{ \|\phi(1) + [h]^3_2 + [\hat{h}]_2^2 + |\nabla \psi|^2 \right\} (1 + |x|^2) \|_{L^1}$.

Proof. Lemma 5.17 gives that $[u]_2 \leq C (1 + |x|^2) \|u\|_{L^2}$, so $u$ remains small as long as $|x|^2 \leq \frac{e}{\|u\|_{L^2}^2}$. By (7.1), the first variation of $\phi$ in a direction $(h, k)$ is given by
\[ \phi'(0) = \frac{1}{2} L h + \text{Hess} (\frac{1}{2} \text{Tr} (h) - k) + \text{div}_f \text{div} \tilde{f} h . \]
To simplify the equations, let $E$ denote the point-wise error function
\[ \mathcal{E} \equiv [u]_2 \hat{h} + [\hat{h}]_2^2 + |x| ([u]_1 \hat{h}_1 + \hat{h}_2^2) + |\nabla \psi| (|\nabla u| + |\nabla \hat{h}|) . \]
With this notation, Proposition 8.14 gives $C$ so that on the set $|x|^2 \leq \frac{e}{\|u\|_{L^2}^2}$
\[ 2 \phi''(0) + 2 |\nabla u|^2 g^1 + 2 \ell u_{ij} + \ell u_i u_j \leq C \mathcal{E} . \]
(8.21)

By Lemma 5.17, $v = |\nabla u|^2 - \Delta |\nabla u|^2 \in \mathcal{K}$. Note that $v g^1$ is point-wise orthogonal to $u u_{ij}$ and $u_i u_j$. Since $\text{div}_f (v g^1) = 0$, it is $L^2$-orthogonal to $\text{Hess} (\frac{1}{2} \text{Tr} (h) - k)$ and $\text{div}_f \text{div} \tilde{f} h$. Furthermore, since $L$ is symmetric and $L (v g^1) = 0$, $v g^1$ is also $L^2$-orthogonal to $L h = L \hat{h}$. Taking the $L^2$ inner product of $\phi'(0) + \frac{1}{2} \phi''(0)$ with $v g^1$ and using (8.19) and (8.21) gives
\[ \left\{ \int (\phi'(0) + \frac{1}{2} \phi''(0), v g^1) e^{-f} + \frac{\ell}{2} \int |\nabla u|^2 v e^{-f} \right\} \leq \hat{e} + C \int \mathcal{E} \|v\| e^{-f} , \]
(8.22)
where $\hat{e}$ is an upper bound for the “tail of the integral” where $|x|^2 \geq c \|u\|^{-1}_{L^2}$. Using the triangle inequality to bound the second term in (8.22) and using that $\hat{e}$ is much smaller than $\|u\|^{4}_{L^2}$, Lemma 8.17 gives

\begin{equation}
\|u\|_{L^2} \leq C \left\| \phi'(0) + \frac{1}{2} \phi''(0) \right\|_1 + C \| \mathcal{E} v \|_{L^1}.
\end{equation}

Lemma 8.17 gives $|v| \leq C (1 + |x|^2) \|u\|_{L^2}^2$. Using this in (8.23) and dividing by $\|u\|_{L^2}^2$ gives

\begin{equation}
\|u\|_{L^2}^2 \leq C \left\| \phi'(0) + \frac{1}{2} \phi''(0) \left( 1 + |x|^2 \right) \right\|_1 + C \| \mathcal{E} (1 + |x|^2) \|_{L^1}.
\end{equation}

Corollary 8.13 gives $|\phi'(0) + \frac{1}{2} \phi''(0)| \leq |\phi(1)| + C [h]_3^2 + C [h]_1^2 (|h| |x| + |\nabla k|)$. Using this and $|\nabla k| \leq |\nabla \psi| + \frac{1}{2} |\nabla u|$, we get

\begin{equation}
\|u\|_{L^2} \leq C \left\{ |\phi(1)| + \mathcal{E} + [h]_2^2 + [h]_1^2 (|h| |x| + |\nabla u| + |\nabla \psi|) \right\} \left( 1 + |x|^2 \right) \|_{L^1}.
\end{equation}

Substituting in (8.20), we can use an absorbing inequality on the $u$ terms (and Lemma 5.17) to get the claim.

We will use the following Poincaré inequality:

**Lemma 8.26.** There exists $C$ so if $V \in W^{1,2}$ is a vector field on $\Sigma$, then $\|V(1 + |\nabla \tilde{f}|)\|_{L^2} \leq C \|\nabla V\|_{L^2} + C \sum_{i=1}^{n-\ell} \left| \int \langle \partial_{x_i}, V \rangle e^{-\tilde{f}} \right|$.

**Proof.** Let $T = \sum a_i \partial_{x_i}$ be the constant $\mathbb{R}^{n-\ell}$ vector field with $\int \langle \partial_{x_i}, V - T \rangle e^{-\tilde{f}} = 0$. Using Lemma 5.10 to control the projection to $N$ and the Poincaré inequality on $N \times \mathbb{R}^{n-\ell}$ to control the Euclidean part of $N$, we get that $\|V - T\|_{L^2} \leq C \|\nabla V\|_{L^2}$. Combining this with Lemma 1.50 gives the claim.

**Lemma 8.27.** Given $m, \epsilon \in (0, 1/2)$ and $p, q > 0$, there exists $c = c(m, p, q, \epsilon)$ so that if $\eta$ is any function on $\mathbb{R}^m$ with $|\eta| \leq 1 + |x|^q$, then $\int \eta^2 |x|^p e^{-\frac{|x|^2}{4 \epsilon}} \leq c \|\eta\|_{L^2}^{2-\epsilon}$.

**Proof.** For $\epsilon \in (0, 1/2)$, we have $\eta^\epsilon \leq 1 + |x|^q$ and, thus the Hölder inequality gives

\begin{equation}
\|\eta^2 |x|^p\|_{L^1} \leq \|\eta^{2-\epsilon} (1 + |x|^q) |x|^p\|_{L^1} \leq \|\eta^{2-\epsilon}\|_{L^{\frac{2}{2-\epsilon}}} \|1 + |x|^q) |x|^p\|_{L^\frac{2}{2-\epsilon}} = c_{m, p, q, \epsilon} \|\eta\|_{L^2}^{2-\epsilon}.
\end{equation}

**Proof of Theorem 8.24.** By Lemma 8.4 and the chain rule, $|\text{Ric}(1) - \text{Ric}(0) - \text{Ric}'(0)| \leq C [h]_2^2$. Combining this with the first bound in Lemma 8.6 we get

\begin{equation}
|\phi(1) - \phi(0) - \phi'(0)| \leq C ( [h]_2^2 + |\nabla h| |h| |\nabla \tilde{f}| + |\nabla h| |\nabla k|).
\end{equation}

Using that $\phi(0) = 0$ and $\phi'(0)$ is given by (7.2), we get that

\begin{equation}
\frac{1}{2} L h + \text{Hess}_w \leq \left| \frac{1}{2} L h + \text{Hess}_w \right| \leq \left| \frac{1}{2} L h + \text{Hess}_w \right| + C \left( [h]_2^2 + |\nabla h| |h| |\nabla \tilde{f}| + |\nabla h| |\nabla k| \right),
\end{equation}

where $w = \frac{1}{2} \text{Tr} h - k$. Subtract a linear function from $w$ to get $\bar{w}$ with $\int \bar{w} e^{-\tilde{f}} = \int x_i \bar{w} e^{-\tilde{f}} = 0$. Obviously, $\text{Hess}_w = \text{Hess}_{\bar{w}}$. Self-adjointness of $L$ and Corollary 1.34 give

\begin{equation}
\int \langle L h, \text{Hess}_w \rangle e^{-\tilde{f}} = \int \langle h, L \text{Hess}_w \rangle e^{-\tilde{f}} = \int \langle h, \text{Hess}(L+1) \bar{w} \rangle e^{-\tilde{f}}
\end{equation}

\begin{equation}
= -\int \langle \text{div}_f h, \nabla (L+1) \bar{w} \rangle e^{-\tilde{f}} = \int \text{div}_f (\text{div}_f h) (L+1) \bar{w} e^{-\tilde{f}}.
\end{equation}
Putting the last two equations together, we get that
\[
\frac{1}{4} \| L h \|_{L^2}^2 + \| \text{Hess}_w \|_{L^2}^2 \leq 2 \| \phi(1) \|_{L^2}^2 + 2 \| \nabla \text{div}_f h \|_{L^2}^2 + \| (\mathcal{L} + 1) \tilde{w} \|_{L^2} \| \text{div}_f (\text{div}_f h) \|_{L^2}
\]
(8.31)
\[+ C \int (|h|^2_2 + |\nabla h|^2_2 |\nabla f|^2 + |\nabla h|^2_2 |\nabla k|^2_2) e^{-f}.
\]

The second term on the last line is bounded by the first by Lemma 1.50 while the third term is bounded the first and |\nabla h|^2_2 |\nabla w|^2_2|. Lemma 8.26 gives
(8.32)
\[\| \tilde{w} \|_{W^{2,2}} \leq C \| \text{Hess}_w \|_{L^2} \text{ and } \| \nabla w \|_{W^{1,2}} \leq C (\| \text{Hess}_w \|_{L^2} + |\mathcal{B}(h, k)|) .
\]
We use (8.32) and the absorbing inequality to bound the \( \| (\mathcal{L} + 1) \tilde{w} \|_{L^2} \| \text{div}_f (\text{div}_f h) \|_{L^2} \) term by a small constant times \( \| \text{Hess}_w \|_{L^2}^2 \) plus a multiple of \( \| \text{div}_f h \|_{W^{2,1}}^2 \). Thus, we get
\[\| L h \|_{L^2}^2 + \| \nabla w \|_{W^{1,2}}^2 \leq C \left\{ \| \phi(1) \|_{L^2}^2 + \| \text{div}_f h \|_{W^{1,2}}^2 + |\mathcal{B}(h, k)|^2 \right\} + C \int (|h|^2_2 + |\nabla h|^2_2 |\nabla w|^2_2) e^{-f}.
\]
As long as sup |\nabla h| \leq \delta_0 for some \( \delta_0 > 0 \) small enough (depending on \( n \)), (8.32) allows us to absorb the sup |\nabla h|^2_2 \int |\nabla w|^2_2 e^{-f} term on the left to get
(8.33)
\[\| L h \|_{L^2}^2 + \| \nabla w \|_{W^{1,2}}^2 \leq C \| \phi(1) \|_{L^2}^2 + C \| \text{div}_f h \|_{W^{1,2}}^2 + C |\mathcal{B}(h, k)|^2 + C \int [h]^3 e^{-f}.
\]
Since \( N \) satisfies (\( \ast \)), Theorem 5.12 gives \( C \) so that
(8.34)
\[\| \hat{h} \|_{W^{2,2}}^2 = \| h - u g^1 \|_{W^{2,2}}^2 \leq C \| L h \|_{L^2}^2 + C \| \text{div}_f h \|_{L^2}^2 .
\]
Combining this with (8.33) and using that \( \psi = \frac{1}{2} \text{Tr} \hat{h} - w \) gives
(8.35)
\[\| \hat{h} \|_{W^{2,2}}^2 + \| \nabla \psi \|_{W^{1,2}}^2 \leq C \| \text{div}_f h \|_{W^{1,2}}^2 + C \| \phi(1) \|_{L^2}^2 + C |\mathcal{B}(h, k)|^2 + C \int [h]^3 e^{-f}.
\]
We still need to get better bounds on the \([h]^3 \) term. Lemma 5.17 gives a constant \( C_n \) so that
(8.36)
\[\| u \|_{L^2} \leq C_n \| u \|_{L^2}(1 + |x|^2) \leq C_n \| h \|_{L^2}(1 + |x|^2) .
\]
The triangle inequality \([h] \leq \hat{h} + (u g^1)^2 \), the absorbing inequality, and (8.36) give
\[\int [h]^3 e^{-f} \leq C \| \hat{h} \|_{L^2} \| u \|_{L^2}^2 + C \| \hat{h} \|_{L^2}^2 \leq \frac{1}{2} \int [h]^3 e^{-f} + C \| u \|_{L^2}^4 + C (\text{sup} |\hat{h}|^2) \| \hat{h} \|_{W^{2,2}}^2 .
\]
As long as \([h] \leq \hat{h} \|_{L^2} \leq [u] \), we can use this in (8.35) and absorb the last term on the right to replace \( \int [h]^3 e^{-f} \) with \([u]_{L^2}^4 \). This completes the proof of the first claim.

We turn now to the second claim. For this, we will use an elementary inequality using that \([h] \leq 1 \) and \([h] \leq [\hat{h}] + [u] \)
(8.37)
\[\| u \|_{L^2}^2 \leq C \| \phi(1) (1 + |x|^2) \|_{L^1} + C \left\{ \| u \|_{L^2} + [\hat{h}] + [\nabla \psi] \right\} (1 + |x|^2) \|_{L^1} .
\]
Lemma 5.17 gives that \( \| (u_{L^2}^3 + [u]^3_{L^3}) (1 + |x|^2) \|_{L^1} \leq C (\| u \|_{L^2}^3 + \| u \|_{L^2}^4) \). From this, the first claim and Lemma 8.27 we get \( C \) and \( C_\epsilon \)
\[\| u \|_{L^2}^2 \leq C (\| u \|_{L^2}^2 + \| u \|_{L^2}^4) + C_\epsilon \left\{ \| \phi(1) \|_{L^2}^\epsilon + |\mathcal{B}(h, k)|^{2-\epsilon} + \| \text{div}_f h \|_{W^{1,2}}^2 \right\} . \]
9. The action of the diffeomorphism group

The main result of this section is the following “improvement” estimate, proving that a shrinker which is close to a model on some large scale is even closer on smaller scales:

**Theorem 9.1.** Given \( \theta < 1 \), there exists \( R_1 \) so that if \((\tilde{t}_R)\) and \( R > R_1 \), then \((\overline{\epsilon}_R)\) holds.

Theorem 9.1 is the last ingredient needed to prove the strong rigidity Theorem 0.2. Before doing so, we will state a more general result:

**Theorem 9.2.** Let \( N^\ell \) satisfy \((\ast)\) in Section 6 and let \( \Sigma = N \times \mathbb{R}^{n-\ell} \) be a shrinker with potential \( f_\Sigma = \frac{|x|^2}{4} + \frac{\ell}{2} \). There exists an \( R = R(n) \) such that if \((M^n, g, f)\) is another shrinker and \( \{f_\Sigma \leq R\} \cap \Sigma \) is close to \( \{f \leq R\} \subset M \) in the smooth topology and \( f_\Sigma \) and \( f \) are close on this set, then \((M, g, f)\) is identical to \( \Sigma \) after a diffeomorphism.

**Proof of Theorems 0.2, 9.2.** Repeatedly applying Theorems 6.1 and 9.1 gives maps \( \Psi_{R_i} \) satisfying \((\tilde{t}_R)\) with \( R_i \to \infty \). The maps are uniformly Lipschitz on compact subsets since the \( \Psi_{R_i} \)'s are almost isometries and, since \( f \) and \( \bar{f} \) are proper, the Arzela-Ascoli theorem gives a uniformly convergent subsequence and a limiting proper map \( \Psi \). As \( R_i \to \infty \), the Lipschitz constants go to one and we conclude that \( \Psi \) preserves both the metric and the potential. \( \square \)

The challenge for proving Theorem 9.1 is that Theorem 8.2 requires bounds on \( \text{div}_f h \) and \( B(h, k) \) that are stronger than what comes out of \((\tilde{t}_R)\). This is a gauge problem: these quantities are only small in the right coordinates, and this is true even if the shrinker is isometric to the model \( \Sigma \). We will use the \( \mathcal{P} \) operator to find the right coordinates.

Suppose \( h(x) \) is a 2-tensor and \( k \) a function on \( \Sigma \), \( V(x) \) is a vector field, \( \Phi(x, t) \) is the family of diffeomorphisms given by integrating \( V \), and \( g(x, t) = \Phi^*(\tilde{g} + h) \) is the pull-back metric. We are interested in how \( B(h, k) \) and \( \text{div}_f h \) change under the diffeomorphism.

**Lemma 9.3.** We have that \( g'(x, 0) = -2 \text{div}^*_f V + V^k_i h_{jk} + V^k_i h_{ij,k} + V^k_i h_{ik} \).

**Proof.** By definition, \( g_{ij}(x, t) = g(\Phi(x, t)) (d\Phi(x, t)(\partial_i), d\Phi(x, t)(\partial_j)) = \Phi^*_i \Phi^*_j g_{kl} \circ \Phi \). Thus, \( g'_{ij} = (\partial_k g_{ij}) V^k + g_{ij} \partial_i V^k + g_{ij} \partial_j V^k \). Observe that for any symmetric \((0, 2)\) tensor \( b_{ij} \)

\[
(\partial_k b_{ij}) V^k + b_{ij} \partial_i V^k + b_{ij} \partial_j V^k = V^k_i b_{jk} + V^k_i b_{ij,k} + V^k_i b_{ik}.
\]

The lemma follows from writing \( g = \tilde{g} + h \) and applying (9.3) to both \( \tilde{g} \) and \( h \); applying it to \( \tilde{g} \) gives the leading order term \(-2 \text{div}^*_f V \) (the covariant derivative of \( \tilde{g} \) drops out) and \( h \) contributes the second order terms \( V^k_i h_{jk} + V^k_i h_{ij,k} + V^k_i h_{ik} \).

The next lemma computes the derivative of the center of mass \( B \); it is easy to see that the second derivative of \( B \) is bounded by \( C \|V\|^2_{C^2} \), where \( C \) depends on bounds for \( h \) and \( k \).

**Lemma 9.5.** The derivative \( \mathcal{F}(V) \) of \( B \) in the direction \( V \) satisfies

\[
\left| \mathcal{F}(V) - \int \langle \partial_x, V \rangle e^{-f} \right| \leq \int |V| \left( |x| |\nabla (k - \frac{1}{2} \text{Tr}_g h)| + |\text{div}_f (x_i h)| \right) e^{-f}.
\]

**Proof.** Since \( g' = -2 \text{div}^*_f V + V^k_i h_{jk} + V^k_i h_{ij,k} + V^k_i h_{ik} \) and \( f' = \langle \nabla f, V \rangle \), it follows that

\[
\left( k - \frac{1}{2} \text{Tr}_g h \right)' = \langle \nabla f, V \rangle + \text{Tr}_g \text{div}^*_f V - \langle \nabla V, h \rangle - \frac{1}{2} \langle V, \nabla \text{Tr}_g h \rangle.
\]
Using this to take the derivative of $B_i(h, k) = \int x_i \left( k - \frac{1}{2} \text{Tr}_g h \right) e^{-\bar{f}}$ and then integrate by parts on $x_i \text{Tr}_g \text{div}^g_f V = \langle x_i \bar{g}, \text{div}^g_f V \rangle$ and $-\langle x_i h, \nabla V \rangle = \langle x_i h, \text{div}_f^g V \rangle$ to get

$$\mathcal{F}^i(V) = \int \left( \langle x_i \nabla f, V \rangle + \langle \text{div}_f (x_i \bar{g}), V \rangle + \langle V, \text{div}_f (x_i h) \rangle - \frac{1}{2} \langle V, x_i \nabla \text{Tr}_g h \rangle \right) e^{-\bar{f}},$$

$$= \int \langle x_i \nabla (k - \frac{1}{2} \text{Tr}_g h) + \text{div}_f (x_i h) + \partial_{x_i}, V \rangle e^{-\bar{f}},$$

where the last equality used $\text{div}_f (x_i \bar{g}) = \partial_{x_i} - x_i \nabla \bar{f}$ and $f - \bar{f} = k$. \Box

The next lemma linearizes the effect on $\text{div}_f h$; to first order, we want $\mathcal{P} V = \frac{1}{2} \text{div}_f h$.

**Lemma 9.9.** Given $q \geq 0$, there exist $m_q$ and $C_q$ so that if $h$ has support in $\{ \bar{b} \leq R \}$ and $V$ is a vector field with support in $\{ \bar{b} \leq R \}$ with $|V| \leq R^{-1}$, then

$$\|\text{div}_f (\Phi^* g - \bar{g}) - \text{div}_f h + 2 \mathcal{P} V \|_{C^q} \leq C_q R^{m_q} \| V \|_{C^{q+2}} \left\{ (1 + \|h\|_{C^{q+2}}) \|V\|_{C^{q+2}} + \|h\|_{C^{q+2}} \right\}.$$

**Proof.** We write $\Phi^* g(x) = g(x, 1) + g'(x, 0) + \int_0^1 \int_0^t g''(x, s) ds \, dt$ and then apply $\text{div}_f$ to this. The second derivative of $g$ is quadratic in $V$ and its first two derivatives with coefficients that depend on up to two derivatives of $h$. Lemma 9.3 gives $g'(x, 0) = -2 \text{div}_f V + V^k h_{jk} + V^k h_{ij,k} + V^k h_{ik}$. Taking $\text{div}_f$ of this gives

$$-2 \mathcal{P} V = \text{div}_f (g'_{ij}) - V^k h_{ij,k} - V^k (\text{div}_f h)_{jk} - V^k h_{ij,k} - V^k h_{ij,k} - V^k h_{ik} \bar{f}_j$$

(9.10)

$$- (\mathcal{L} V)^k h_{ik} - V^k h_{ik,j}.$$ \Box

**Proposition 9.11.** Given $q \geq 0$, there exist $m_q$ and $C_q$ so that if $h$ has support in $\{ \bar{b} \leq R \}$, then there is a vector field $V$ with $\text{div}_f \left( h - 2 \text{div}_f^g V \right) = 0$, $\int \langle \partial_{x_i}, V \rangle e^{-\bar{f}} = -B(h, k)$ and

$$\sup_{b \leq R} \|V\|_{C^{q+2}}^2 \leq C_q R^{m_q} \|\text{div}_f h\|_{C^q}^2 + C_q |B(h, k)|^2.$$

**Proof.** Theorem 4.12 gives $Y$ with $\|Y\|_{W^{1,2}} + \|\text{div}_f Y\|_{W^{1,2}} + \|\mathcal{L} Y\|_{L^2} \leq C' \|\text{div}_f h\|_{L^2}$ and $2 \mathcal{P} Y = \text{div}_f h$. Using this and Theorem 4.11 gives $m_1, m'_1$ and $C'$ so that for $r \leq 3 R$

$$I_{Y}(r) \leq C' r^{m'_1} \left( \|\text{div}_f h\|_{L^2}^2 + \int B^{2-n} \left( |\text{div}_f h|^2 + |\mathcal{L} \text{div}_f h|^2 \right) \right) \leq C R^{m_1} \|\text{div}_f h\|_{C^2}^2,$$

where we integrated over an annulus to bound $I_Y$ on a fixed scale in terms of $\|\text{div}_f h\|_{L^2}$. Since $|\nabla b|$ is essentially one, this gives *unweighted* $L^2$ bounds on $Y$. Since the operator $\mathcal{P}$ is an elliptic system in the sense of [DN], Schauder estimates on balls of radius $R^{-1}$ give

$$\sup_{b \leq 3 R - R^{-1}} \|Y\|_{C^{q+2}}^2 \leq C'_q R^{m_q} \left( \|\mathcal{P} Y\|_{C^q}^2 + \int_{b \leq 3 R} |Y|^2 \right) \leq C q R^{m_q} \|\text{div}_f h\|_{C^q}^2.$$

By construction, $Y$ is $L^2$-orthogonal to all Killing fields. We have the freedom to add a Killing field $T$ to $Y$ and still have $\mathcal{P}(Y + T) = \mathcal{P} Y$. Thus, we can choose a translation $T = \sum a_j \partial_{x_j}$ so that $\int \langle \partial_{x_i}, V \rangle e^{-\bar{f}} = -B(h, k)$ and $|T| \leq C |B(h, k)|$. \Box
We construct the diffeomorphism \( \Phi \) iteratively, adjusting by a diffeomorphism at each stage that solves the linearized equation. We do this globally by multiplying by a cutoff function and repeat the process until the quadratic error is comparable to the error from the cutoff. We then apply Theorem 8.2 to get exponentially small \( L^2 \) estimates on \( h \) and \( \nabla k \) in these coordinates. Since the construction gives polynomially growing higher order bounds, interpolation converts the exponential \( L^2 \) estimates into exponential point-wise estimates.

**Proof of Theorem 9.11.** Fix a smooth cutoff function \( \eta \) with support in \( \bar{b} \leq R - 1 \) and that is one on \( \bar{b} \leq R - 1 \). Set \( h_0 = \eta(\Psi_R g - \bar{g}) \) and \( k_0 = \eta(f \circ \Psi_R - \bar{f}) \), so that

\[
|h_0|^2_{C^{3,\alpha}} + |k_0|^2_{C^{3,\alpha}} \leq C e^{-\frac{R^2}{16}}
\]

and \( |h_0|_{C^6} + |k_0|_{C^6} \leq C R^\ell \). In particular, \( \| \text{div}_f h_0 \|^2_{C^{2,\alpha}} \leq C R^2 e^{-\frac{R^2}{16}} \). Interpolating Schauder norms (cf. the appendix in [CM2]), gives for each \( q > 3 \) and \( \theta < 1 \) that

\[
\|h_0\|^2_{C^{q,\alpha}} \leq C_{q,\theta} R^{m_q \theta} e^{-\frac{R^2}{16}}.
\]

Proposition 9.11 gives \( V_0 \) with \( \text{div}_f (h_0 - 2 \text{div}_f V_0) = 0 \), \( \int \langle \partial_x i, V_0 \rangle e^{-f} = -\mathcal{B}(h_0, k_0) \) and

\[
\sup_{\bar{b} < 2 R} \| V_0 \|_{C^{q+2,\alpha}}^2 \leq C_q R^{m_q} e^{-\frac{\theta q R^2}{16}},
\]

where the constant \( \theta_q \) can be chosen close to one at the cost of increasing \( C_q \) and \( m_q \). The vector field \( \eta V_0 \) has the same bounds, agrees with \( V_0 \) in \( \{ \bar{b} < R - 1 \} \) and has support in \( \{ \bar{b} \leq R \} \). It’s time one flow \( \Phi_0 \) moves points by at most \( C R^m e^{-\frac{R^2}{16}} \) and \( h_1 = \eta(\Phi_0^*(\bar{g} + h) - \bar{g}) \) and \( k_1 = \eta(f \circ \Phi_0 - \bar{f}) \) satisfy \( (9.14) \) with \( C R^2 \) in place of \( C \). Lemma 9.9 gives that

\[
|\mathcal{B}(h_1, k_1)|^2 \leq C R^m e^{-\frac{R^2}{8}} + C R^2 e^{-\frac{(R-2)^2}{4}}.
\]

To bound \( \| \text{div}_f h_1 \|_{C^{2,\alpha}} \), use Lemma 9.9 to get

\[
\sup_{\bar{b} < R - 2} \| \text{div}_f h_1 \|_{C^{2,\alpha}}^2 \leq C R^m e^{-\frac{R^2}{8}} + C R^2 e^{-\frac{(R-2)^2}{4}}.
\]

After repeating this, the quadratic error will be on the same scale as the contribution from the cutoff and the new diffeomorphism \( \Phi \) gives that \( h = \eta(\Phi^*(g) - \bar{g}) \) and \( k = \eta(f \circ \Phi - \bar{f}) \) satisfy \( \|h\|^2_{C^{3,\alpha}} + \|k\|^2_{C^{3,\alpha}} \leq C R^m e^{-\frac{R^2}{32}} \), and

\[
\| \text{div}_f h \|_{C^{2,\alpha}}^2 \{ \bar{b} < R - 3 \} + |\mathcal{B}(h, k)|^2 \leq e^{-\frac{(R-3)^2}{4}}.
\]

Furthermore, since the new metric satisfies the shrinker equation on \( \{ \bar{b} < R - 3 \} \) and \( h \) and \( k \) are bounded everywhere, it follows that

\[
\| \phi(1)(1 + |x|^2) \|_{L^1} + \| \phi(1) \|^2_{L^2} \leq C e^{-\frac{(R-3)^2}{4}}.
\]

Let \( u g \) be the orthogonal projection of \( h \) onto \( K g^1 \) and write \( \eta h = u g^1 + \hat{h} \) and \( \eta k = \frac{\ell}{2} u + \psi \). Given \( \epsilon > 0 \), Theorem 8.2 gives constants \( C \) and \( C_{\epsilon} \) so that

\[
\| \hat{h} \|^2_{W^{1,2}} + \| \nabla \psi \|^2_{W^{1,2}} \leq C \left\{ \| \phi(1) \|^2_{L^2} + \| \text{div}_f h \|^2_{W^{1,2}} + |\mathcal{B}(h, k)|^2 + \| u \|^2_{L^2} \right\},
\]

\[
\| u \|^2_{L^2} \leq C \left\{ \| u \|^2_{L^2} + \| \phi(1)(1 + |x|^2) \|_{L^1} \right\} + C_{\epsilon} \left\{ \| \phi \|^2_{L^2} + |\mathcal{B}(h, k)|^{2-\epsilon} + \| \text{div}_f h \|^2_{W^{1,2}} \right\}.
\]
Since projection cannot increase the norm, we have \( \|u g^j\|_{L^2} \leq \|h\|_{L^2}^2 \) and, thus, we can absorb the \( \|u\|_{L^2}^2 \) term in the second equation and use (9.18) and (9.19) to get

\[
\|u\|_{L^2}^2 \leq C e^{-\frac{(R - 9)^2}{4}} + C e^{-\frac{(1-\epsilon)(R - 9)^2}{4}}.
\]

Using the bounds on \( \text{div}_f (h) \) and \( \mathcal{B}(h,k) \) from (9.18) and on \( \phi(1) \) that comes from that \( M \) is a shrinker up to the scale where we cut-off, it follows that \( \|u\|_{L^2}^2 \leq C e^{-\frac{(1-\epsilon)(R - 9)^2}{4}} \)

Using this in the first bound from Theorem 8.2 gives exponentially small bounds on \( \|\hat{h}\|_{W^{2,2}} \) and \( \|\nabla \psi\|_{W^{2,2}} \). Interpolation (cf. the appendix in [CM2]) then gives pointwise exponential bounds. Finally, to see that \( k \) itself, and not just \( \nabla k \), is small, we use the normalizations \( S + |\nabla f|^2 = f \) and \( \bar{S} + |\nabla \bar{f}|^2 = \bar{f} \).

We will next use strong rigidity of Theorem 0.2 to prove that if one tangent flow is a cylinder, then every tangent flow is. To make this precise, let \( \bar{g}(t) \) a Ricci flow on \( M \times [T,0) \) that has a singularity at \( t = 0 \) where the conclusions of theorem 1.4 in [MM] hold; this includes closed manifolds with type-I singularities.

**Theorem 9.21.** If \( M, \bar{g} \) is a Ricci flow as above and one tangent flow at a point is a cylinder, then every tangent flow at that point is a cylinder (with the same \( t \)).

**Proof.** As in [MM], by solving the conjugate heat equation, continuously rescaling and reparameterizing the Ricci flow gives a solution \( (M, g(t), f(t)) \) of the rescaled Ricci flow equation where a sequence of times converges to a cylinder \( \Sigma \). The curvature bound assumed in [MM] (see (1.2) there) and the Shi estimates, [S], bound all derivatives of the flow.

We will argue by contradiction. Suppose instead that \( t_j, t'_j \) are sequences going to infinity with \( t_j < t'_j < t_{j+1} < t'_{j+1} \ldots \) and so that

1. \( (M, g(t_j), f(t_j)) \) converges to \( \Sigma \).
2. \( (M, g(t'_j), f(t'_j)) \) converges to a different shrinker.

Theorem 9.2 gives an \( R \) so that if \( \hat{t}_{1R} \) (relative to \( \Sigma \)) holds for a shrinker, then the shrinker agrees identically with \( \Sigma \) (up to a diffeomorphism).

By (1), we have that \( \hat{t}_{2R} \) holds for every \( t_j \) sufficiently large. On the other hand, (2) implies that \( \hat{t}_{2R} \) must fail for \( t'_j \) sufficiently large. Since \( g \) and \( f \) vary continuously in \( t \), there must be a first \( s_j \in (t_j, t'_j) \) where \( \hat{t}_{2R} \) fails. In particular, using also that we have uniform higher derivative bounds, we see that \( \hat{t}_{R} \) holds at \( s_j \). Theorem 1.4 in [MM] gives that a subsequence of the \( s_j \)'s gives a limiting shrinker \( (M, \bar{g}, \bar{f}) \), where the convergence is smooth on compact subsets. On the one hand, this limit must be different from \( \Sigma \) since \( \hat{t}_{2R} \) fails at every \( s_j \). On the other hand, \( \hat{t}_{R} \) holds for the limiting shrinker, so Theorem 0.2 implies that it agrees with \( \Sigma \) giving the desired contradiction.

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