Comment on “Hidden truncation hyperbolic distributions, finite mixtures thereof, and their application for clustering” by Murray, Browne, and McNicholas

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Abstract

We comment on the paper of Murray, Browne, and McNicholas (2017), who proposed mixtures of skew distributions, which they termed hidden truncation hyperbolic (HTH). They recently made a clarification (Murray, Browne, McNicholas, 2019) concerning their claim that the so-called CFUST distribution is a special case of the HTH distribution. However, there are also some other matters in the original version of the paper that are in need of clarification as discussed here.

1 Introduction

In the paper Murray, Browne, and McNicholas (2017), herewith referred to as MBM, consideration was given to mixtures of skew distributions belonging to the family of distributions that were termed hidden truncation hyperbolic (HTH). One matter in MBM that has since been clarified by the authors (Murray et al., 2019) is that the so-called CFUST distribution is a special case of the HTH distribution. However, there are also some other matters in the original version of the paper that are in need of clarification.

In particular, we wish to point out that the comparison in MBM with two other skew mixture models (called classical and SDB skew-t mixtures in MBM) did not include mixtures of canonical fundamental skew t (CFUST) distributions. As explained in Lee and McLachlan (2014), an attractive feature of the CFUST distribution is that it includes the two distributions corresponding to the classical and SDB skew t distributions as special cases, and it can be fitted with little additional effort; see also Lee and McLachlan (2015, 2016). We would also point out that Lee and McLachlan (2014, 2016) have provided the EM equations for the fitting of a mixture of CFUST distributions. Moreover, Lee and McLachlan (2015, 2018) have given an R package for the fitting of a mixture of CFUST distributions.

It is not as if the CFUST distribution is a special case of the HTH distribution as noted above. It is to be demonstrated here that the HTH distribution belongs to the family...
of skew distributions that have the form of the canonical fundamental skew symmetric generalized hyperbolic (CFUSSGH) distribution.

We shall report here results showing that mixtures of CFUST distributions outperform mixtures of HTH distributions for the two real data sets considered in MBM.

On another point on the comparison in MBM, the near-zero value reported for the ARI of one of the models in the comparisons of MBM is due to algorithmic failure rather than the claimed inability of the model to fit the data.

Before we consider further the aforementioned points, we define the skew distributions relevant to this paper.

2 Skew distributions

To establish some notation, we let \( \mathbf{Y} \) denote a \( p \)-dimensional random vector, \( \mathbf{I}_p \) be the \( p \times p \) identity matrix, and \( \mathbf{0} \) be a vector/matrix of zeros of appropriate size. The notation \( |\mathbf{y}| \) implies taking the absolute value of each element of \( \mathbf{y} \). Also, we let \( \phi_r(\mathbf{y}; \mu, \Sigma) \) and \( \Phi_r(\mathbf{y}; \mu, \Sigma) \) denote the \( r \)-dimensional multivariate normal density and (cumulative) normal distribution function, respectively, with mean \( \mu \) and covariance \( \Sigma \).

The skew distributions to be considered here belong to the class of canonical fundamental skew symmetric (CFUSS) distributions proposed by Arellano-Valle and Genton (2005). The density of members of the class of CFUSS distributions can be expressed as

\[
f(\mathbf{y}; \theta) = 2^r f_p(\mathbf{y}; \theta) Q_r(\mathbf{y}; \theta),
\]

where \( f_p(\mathbf{y}; \theta) \) is a symmetric density on \( \mathbb{R}^p \), \( Q_r(\mathbf{y}; \theta) \) is a skewing function that maps \( \mathbf{y} \) into the unit interval, and \( \theta \) is the vector containing the parameters of \( \mathbf{Y} \). Let \( \mathbf{U} \) be a \( r \times 1 \) random vector, where \( \mathbf{Y} \) and \( \mathbf{U} \) follow a joint distribution such that \( \mathbf{Y} \) has marginal density \( f_p(\mathbf{y}; \theta) \) and \( Q_r(\mathbf{y}; \theta) = P(\mathbf{U} > 0 \mid \mathbf{Y} = \mathbf{y}) \). If the latent random vector \( \mathbf{U} \) (the vector of skewing variables) has its canonical distribution (that is, with mean \( \mathbf{0} \) and scale matrix \( \mathbf{I}_r \)), we obtain the canonical form of \( [\Pi] \), namely the CFUSS distribution. The class of CFUSS distributions encapsulates many existing distributions, including those to be considered here. We now proceed to define those members.

2.1 CFUSN distribution

The so-called canonical fundamental skew normal (CFUSN) distribution is a location-scale variant of the canonical fundamental skew normal distribution in Arellano-Valle and Genton (2005). If the \( p \times 1 \) random vector \( \mathbf{Y} \) has a CFUSN distribution, its density is given by

\[
f_{\text{CFUSN}}(\mathbf{y}; \mu, \Sigma, \Delta) = 2^r \phi_p(\mathbf{y}; \mu, \Omega) \Phi_r(c(\mathbf{y}); 0, \Lambda),
\]

where

\[
\Lambda = \mathbf{I}_r - \Delta^T \Omega^{-1} \Delta,
\]

\[
c(\mathbf{y}) = \Delta^T \Omega^{-1} (\mathbf{y} - \mu),
\]

\[
\Omega = \Sigma + \Delta \Delta^T,
\]

and \( \Delta \) is a \( p \times r \) matrix of skewness parameters with \( pr \) free parameters. This density is invariant under permutations of the columns of \( \Delta \), but this does not affect the number of free parameters.

2
The convolution-type characterization of this distribution is given by

\[ Y = \mu + \Delta |U_0| + U_1, \]  

(3)

where

\[
\begin{bmatrix}
U_0 \\
U_1
\end{bmatrix} \sim N_{p+r}(0, \begin{bmatrix}
I_r & 0 \\
0 & \Sigma
\end{bmatrix}).
\]  

(4)

### 2.2 CFUST distribution

The canonical fundamental skew t (CFUST) distribution can be characterized by

\[ Y = \mu + \Delta |U_0| + U_1 \]  

(5)

where, conditional on \( W = w \),

\[
\begin{bmatrix}
U_0 \\
U_1
\end{bmatrix} \sim N_{p+r}(0, w \begin{bmatrix}
I_r & 0 \\
0 & \Sigma
\end{bmatrix})
\]  

(6)

and \( W \) is distributed according to the inverse gamma distribution \( IG(\nu_2, \nu_2) \).

If we take the matrix \( \Delta \) of skewness parameters in the formulation (5) to be a \( p \)-dimensional vector (that is, \( r = 1 \)), then we obtain the skew t-density as proposed by Azzalini and Capitanio (2003). Lee and McLachlan (2013a) referred to this distribution as the restricted multivariate skew t (rMST) distribution since restriction to a single skewing variable implies that it is restricted to modelling skewness in only one direction in the feature space. Sahu, Dey, and Branco (2003) considered the model where the matrix \( \Delta \) of skewness parameters was taken to be diagonal (so \( r = p \)). Lee and McLachlan (2013a) termed this distribution the unrestricted skew multivariate normal distribution to distinguish it from the rMST distribution, although they noted that it did not embed the rMST distribution since the feature-specific skewing effects are taken to be uncorrelated (McLachlan and Lee, 2016). The rMST and uMST distributions are referred to as the classical and SDB skew t-distributions in MBM. As they are special cases of the CFUST distribution with the skewness matrix \( \Delta \) having \( r = 1 \) and being diagonal, respectively, we can denote them as CFUST(r=1) and CFUST(diag), respectively.

Although the CFUST(diag) distribution can model skewness in multiple directions, the assumption is that they are parallel to the axes of the feature space. Consequently, Lee and McLachlan (2014, 2015, 2016, 2018) developed the methodology and algorithms for the fitting of mixtures of CFUST distributions with arbitrary skewness matrices so that they can handle skewness in multiple directions that are not necessarily parallel to the axes of the feature space.

By letting the degrees of freedom go to infinity in the formulation (6), we obtain a similar formulation for the restricted multivariate skew normal CFUSN(r=1) and unrestricted multivariate skew normal CFUSN(diag) distributions.

### 2.3 Scale mixture of CFUSN distribution

A Scale Mixture of the CFUSN distribution (SMCFUSN) can be defined by the stochastic representation

\[ Y = \mu + W^\frac{1}{2} Y_0, \]  

(7)
where $Y_0$ follows a central CFUSN distribution and $W$ is a positive (univariate) random variable independent of $Y_0$. Thus, conditional on $W = w$, the density of $Y$ has a CFUSN distribution with scale matrix $w\Sigma$. The marginal density of $Y$ is given by

$$f_{SMCFUSN}(y; \mu, \Sigma, \Delta; F_\zeta) = 2^r \int_0^\infty \phi_p(y; \mu, w\Omega) \Phi_r \left( \frac{1}{\sqrt{w}} \Delta^T \Omega^{-1}(y - \mu); 0, \Lambda \right) dF_\zeta(w), \quad (8)$$

where $F_\zeta$ denotes the distribution function of $W$ indexed by the parameter $\zeta$. We shall use the notation $Y \sim SMCFUSN_{p,r}(\mu, \Sigma, \Delta; F_\zeta)$ if the density of $Y$ can be expressed in the form of (8).

The CFUST distribution corresponds to taking $W$ to have the inverse gamma distribution $IG(\nu^2, \nu^2)$.

### 2.4 CFUSSGH distribution

The so-called hidden truncation hyperbolic distribution in MBM can be viewed as a member of the class of the canonical fundamental skew symmetric generalized hyperbolic (CFUSSGH) distributions (Lin, Lee, McLachlan, 2019). To see this, we suppose now that the latent variable $W$ in (7) follows a generalized inverse Gaussian (GIG) distribution (Seshadri, 1997). The GIG density can be expressed as

$$f_{GIG}(w; \psi, \chi, \lambda) = \frac{\psi \chi^{\lambda-1}}{2K_\lambda(\sqrt{\psi\chi})} e^{-\frac{\psi}{2}w + \frac{\chi}{2}w^2}, \quad (9)$$

where $W > 0$, the parameters $\psi$ and $\chi$ are positive, and $\lambda$ is a real parameter. In the above, $K_\lambda(\cdot)$ denotes the modified Bessel function of the third kind of order $\lambda$. If we put $\chi = \nu$, $\lambda = -\frac{1}{2}\nu$, and let $\psi$ tend to zero in (9), then it tends to the inverse gamma distribution $IG(\frac{1}{2}\nu, \frac{1}{2}\nu)$.

Taking the latent variable $W$ in (7) to have a GIG distribution, we obtain the CFUSSGH distribution. It has the $p$-dimensional symmetric GH (generalized hyperbolic) density $h_p(\cdot)$ and the $r$-dimensional symmetric GH distribution function $H_r(\cdot)$, corresponding to the symmetric density $f_p(\cdot)$ and the distribution function $Q_r(\cdot)$, respectively, in (1).

The symmetric GH density is given by

$$h_p(y; \mu, \Sigma, \psi, \chi, \lambda) = \left( \frac{\chi + \eta(y; \mu, \Sigma)}{\psi} \right)^{\frac{\lambda}{2}} K_{\lambda-\frac{1}{2}} \left( \sqrt{\psi \eta(y; \mu, \Sigma)} \right) \frac{\left( \frac{\chi}{\psi} \right)^{\frac{\lambda}{2}} K_\lambda(\sqrt{\psi\chi})}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{p}{2}} K_\lambda(\sqrt{\psi\chi})}, \quad (10)$$

where

$$\eta(y; \mu, \Sigma) = (y - \mu)^T \Sigma^{-1}(y - \mu).$$

It is well known that the GH distribution has an identifiability issue in that the parameter vectors $\theta = (\mu, k\Sigma, k\psi, \chi/k, \lambda)$ and $\theta^* = (\mu, \Sigma, \psi, \chi, \lambda)$ both yield the same symmetric GH distribution (10) for any $k > 0$. It is therefore not surprising that the CFUSSGH distribution also suffers from such an issue. To handle this, restrictions are imposed on some of the parameters of the CFUSSGH distribution. An example is the
HTH distribution considered in MBM where the constraint $\psi = \chi = \omega$ is used, leading to the density

$$f_{HTH}(y; \mu, \Sigma, \Delta, \omega, \lambda) = 2^r h_p(y; \mu, \Omega, \omega, \omega, \lambda) H_r \left( \Delta^T \Omega^{-1} (y - \mu) \left( \frac{\omega}{\omega + \eta} \right)^{\frac{r}{2}} ; 0, \Lambda, \gamma, \gamma, \lambda - \frac{\nu}{2} \right),$$

(11)

where $\gamma = \sqrt{\omega \left[ \omega + \eta(y; \mu, \Sigma) \right]}$.

Note that in their terminology, MBM are using ‘hidden truncation’ to describe the latent skewing variable that follows a truncated distribution in the convolution-type characterization of the CFUSSGH distribution. Another alternative is to restrict the parameters of $W$ so that, for example, $E(W) = 1$. A commonly used constraint on the GH distribution is to set $|\Sigma| = 1$. This can be applied to the CFUSSGH distribution to achieve identifiability; see also the unrestricted skew normal generalized hyperbolic (SUNGH) distribution considered by Maleki, Wraith, and Arellano-Valle (2019).

The CFUST distribution is not a special case of the CFUSSGH distribution. It can be obtained as a limiting case. One approach to obtain the limiting case is to put $\lambda = -\nu/2$ and to replace $\Sigma, \Delta$ and $\omega$ by $\frac{1}{k} \Sigma, \frac{1}{k} \Delta$, and $k\nu$ in the density (11) to give

$$f_{HTH}(y; \mu, \frac{1}{k} \Sigma, \frac{1}{k} \Delta, k\nu, -\frac{\nu}{2}),$$

and then to let $k$ tend to zero; see also Murray et al. (2019).

### 3 Mixtures of CFUST distributions versus mixtures of HTH distributions

To demonstrate the performance of mixtures of HTH distributions in clustering data, they were compared in MBM with mixtures of the two special cases of CFUST distributions defined in the previous section, namely the CFUST$(r=1)$ and CFUST(diag) distributions, the latter two being referred to as the classical skew $t$ and the SDM skew $t$ distribution, respectively. These three models were fitted to two real data sets referred to as the Seeds and HSCT (hematopoietic stem cell transplant) sets. However, a mixture of CFUST distributions was not fitted to these two data sets in MBM, where it was stated that “the SDB skew $t$ mixture model is regarded by some as the state of the art approach (see [36]).” The reference [36] is the paper by Lee and McLachlan (2013a) in which the CFUST distribution was not considered. In subsequent papers (Lee and McLachlan, 2014, 2015, 2016), it was explained and demonstrated how the SDB skew $t$ distribution, along with the classical skew $t$, is embedded in the CFUST distribution. The results from MBM are reproduced here in Table 1, along with our results for mixtures of CFUST distributions. The notation HTHu and HTHm in Table 1 as used in MBM refer to the HTH component distributions having $r = 1$ and $r = p$ in forming the skewness matrix $\Delta$ in the formulation of the HTH distribution as a member of the class of SMCFUSN distributions (5).

On fitting mixtures of CFUST distributions to the HSCT and Seeds data sets with $r = p$ skewing variables, we obtained higher values of the adjusted Rand Index (ARI), namely 0.991 and 0.916 relative to the values of 0.984 (0.976) and 0.877 (0.877) for mixtures of HTHm (HTHu) distributions fitted to the HSCT and Seeds data sets, respectively.
Table 1: ARI value of CFUST mixture model versus the ARI values for the mixture models in MBM

|       | CFUST | HTHu | HTHm   | Classical skew t | SDB skew t |
|-------|-------|------|--------|------------------|------------|
| HSCT  | 0.991 | 0.976| 0.984  | 0.782            | 0.890      |
| Seeds | 0.916 | 0.877| 0.877  | 0.836            | 0.009      |

In MBM, the very small value of 0.009 for the ARI for the SDB skew $t$ mixture model in Table 1 is not interpreted as reflecting the failure of the algorithm to fit the model to the data. Rather it is taken at face value with the statement that “the SDB skew-$t$ mixture performs better than the classical skew-$t$ mixture approach for the Seeds data.”

As stated in MBM, “the expected value of the ARI under random classification is zero.” Thus the reported value of 0.009 for the SDB skew $t$-mixture model is implying that this model does no better than random classification! But this is unrealistic, particularly as an ARI value of 0.836 was obtained for the classical model; such a difference between the ARI’s for these two models is highly unlikely as both embed the $t$-mixture model with just a few additional parameters to allow for any skewness in the data.

To investigate this further, we followed the procedure adopted in MBM for the fitting of this model (McNicholas, 2015). We first scaled the data so that each variable had mean zero and unit standard deviation before applying the EMMIXuskew program (Lee and McLachlan, 2013b) using the default options for the starting values for the skewness parameters. We found that the EM algorithm stopped after three iterations as essentially all the observations were being put into the one cluster. It was a consequence of the initial estimates of the skewness parameters not being scale invariant which for the Seeds data set after scaling caused problems. However, the EMMIXuskew algorithm also has optional starting strategies in addition to the default ones, such as those provided by $k$-means applied to the unscaled data and by the normal mixture model. When we used these latter options, we obtained a fit for the CFUST(diag) mixture model (that is, the SDB skew $t$ model) with an ARI of 0.84. However, given the problems that our algorithm EMMIXuskew encountered with the data as scaled in MBM, we have modified those default steps for the provision of starting values that were not scale invariant. But we recommend using our latest algorithm EMMIXcskew (Lee and McLachlan, 2015, 2018), which also has the provision to fit mixtures of CFUST skew distributions.

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