THE NON-ADS/NON-CFT CORRESPONDENCE, OR THREE DIFFERENT PATHS TO QCD

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Abstract
In these lecture notes from the 2002 Cargese summer school we review the progress that has been made towards finding a string theory for QCD (or for pure (super)Yang-Mills theory) following the discovery of the AdS/CFT correspondence. We start with a brief review of the AdS/CFT correspondence and a general discussion of its application to the construction of a string theory for QCD. We then discuss in detail two possible paths towards a QCD string theory, one which uses a mass deformation of the $\mathcal{N} = 4$ super Yang-Mills theory (the Polchinski-Strassler background) and the other using a compactification of “little string theory” on $S^2$ (the Maldacena-Nunez solution). A third approach (the Klebanov-Strassler solution) is described in other lectures of this school. We briefly assess the advantages and disadvantages of all three approaches.

1. INTRODUCTION

The most remarkable discovery of the last few years in theoretical particle physics is the realization that certain string theories are identical to certain field theories. More generally, a large class of gravitational theories are precisely equivalent to non-gravitational theories; we will be interested here in the special case which is the equivalence between string theories and gauge theories.

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An equivalence of this type was previously suspected on general grounds from both directions:

**Gauge theory to string theory**: Historically, string theory originated from an attempt to describe the strong nuclear interactions, motivated by the linear confining force between quarks and anti-quarks (as if there is a string connecting them) and by the appearance of Regge trajectories in the meson and hadron spectra. Now, we know that these interactions can be described by a gauge theory (QCD), but this gauge theory seems to behave (at least qualitatively) like a string theory at low energies, where the running gauge coupling constant is large. A more quantitative reason to expect gauge theories to be related to string theories comes from ’t Hooft’s analysis of the large $N$ limit of $SU(N)$ gauge theories [1]. ’t Hooft showed that if one takes $N \to \infty$ with fixed $\lambda_{YM} \equiv g_{YM}^2 N$, then (in a double-line notation) the Feynman diagram expansion arranges itself according to the minimal genus $g$ of the surface which the diagrams can be drawn on, and every amplitude can be written in the form $\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda_{YM})$ with some functions $f_g$ arising from genus $g$ diagrams. This is the same expansion as in a string theory whose string coupling constant is $g_s \simeq 1/N$. The leading contribution in the large $N$ limit comes from “planar diagrams” with $g = 0$. ’t Hooft conjectured that for large $\lambda_{YM}$ the Feynman diagrams become dense and form smooth two dimensional surfaces which may be identified with string worldsheets.

**Gravity to field theory**: String theory is a theory of quantum gravity. It was argued by ’t Hooft and Susskind (see Bousso’s lectures at this school) that the number of degrees of freedom in a theory of quantum gravity on some manifold scales as the area of the boundary of the manifold rather than as the volume of the manifold. This suggests that theories of quantum gravity may be described as non-gravitational theories living on the boundary of space-time; in some cases this description could involve local field theories.

The first explicit realization of these general expectations was the AdS/CFT correspondence, and we will start by briefly reviewing this correspondence. The correspondence allows us to use string theory and gravity to learn about the strong coupling limits of specific field theories, such as the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The standard model includes in it QCD, which is an $SU(3)$ gauge theory which becomes strongly coupled at low energies, and it would be very interesting to find a good description of this theory in the regime where the original variables of QCD (the quarks and gluons) become strongly coupled. As discussed above, one expects such a description to involve string theory (at least for large $N$, but the stringy description could
perhaps be useful even for $N = 3$). In these lectures we will discuss how we may be able to get such a description from generalizations of the AdS/CFT correspondence, in several different ways. All these ways will lead to a description of the large $N$ limit of a pure Yang-Mills theory (sometimes with $\mathcal{N} = 1$ supersymmetry), but this is presumably a necessary first step towards constructing a string theory for QCD, and the theories we will discuss have the same qualitative features as QCD (exhibiting phenomena like confinement and chiral symmetry breaking). We will encounter some obstacles on all the different paths towards a string theory of QCD, but the obstacles seem to be technical rather than problems of principle, so hopefully it will be possible to overcome them in the future.

2. A BRIEF REVIEW OF THE ADS/CFT CORRESPONDENCE

The first concrete example (above two dimensions) to be found of the general relation between large $N$ gauge theories and string theories is the AdS/CFT correspondence [2, 3, 4]. In general this relates a theory of quantum gravity on $d + 1$ dimensional anti-de Sitter (AdS) space (times some compact manifold) to a $d$-dimensional conformal field theory (CFT). In the case of type IIB string theory compactified on 5-dimensional AdS space $AdS_5$ (times a compact 5-dimensional manifold) the dual theory is usually a gauge theory.

The simplest example of the AdS/CFT correspondence is the duality between type IIB string theory compactified on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ supersymmetric four dimensional Yang-Mills theory with gauge group $SU(N)$. The parameters of string theory in this background\footnote{It is often said that string theory has no continuous parameters since all its apparent parameters are actually fluctuating fields. For instance, the string coupling constant is related to the vacuum expectation value of the dilaton field. However, while this is true in flat space, it is not true in backgrounds like AdS space, where the fluctuations of the asymptotic value of fields like the dilaton are non-normalizable, so the asymptotic values are frozen and serve as parameters for the string theory.} are the string coupling constant $g_s$ and the radius of curvature $R$ of the $AdS_5$ and of the $S^5$, measured in units of the string tension $1/2\pi\alpha'$; from these two parameters one can determine (using the equations of motion) the (quantized) flux of the 5-form field of type IIB string theory on the $S^5$. The parameters of the Yang-Mills theory are the Yang-Mills coupling constant $g_{YM}$ (which does not depend on the scale in this theory) and the number of colors $N$. The two sets of parameters are related by

$$4\pi g_s = g_{YM}^2 \quad (1.1)$$
These two relations imply that the flux of the 5-form field on the \(S^5\) is equal to \(N\) (in the units in which it is quantized to be an integer).

The AdS/CFT correspondence is believed to be an exact equivalence, in the sense that all observables are supposed to be equal on both sides (when the parameters are identified according to equations (1.1), (1.2)). Some of the evidence for this equivalence is reviewed in [5]. It is often said that the AdS/CFT correspondence is a strong/weak coupling duality (like S-duality), but this is not precise since each side of the correspondence is really labeled by two parameters. As described above, at the leading order in \(1/N\) we keep only planar diagrams in the field theory, and these can be described by a perturbation theory in the parameter \(\lambda_{YM} = g_{YM}^2 N\); on the other hand at leading order in \(g_s\) in the string theory, we have in the string worldsheet sigma model an expansion in the space-time curvature \(\alpha'/R^2\). Equation (1.2) tells us that these two expansions are indeed related by a strong/weak coupling duality. On the other hand, the large \(N\) genus expansion in the field theory is governed by \(1/N\), and the string theory genus expansion is governed by \(g_s\), and equation (1.1) tells us that for fixed \(\lambda_{YM}\) these two expansions are actually the same. Thus, if we could somehow perform exact computations in \(\lambda_{YM}\) in the field theory, or in \(\alpha'\) in the string theory, then the AdS/CFT correspondence would turn into a perturbative duality where we could compare results from both sides order by order in \(g_s\) (or in \(1/N\)). An example of this was recently found in the plane-wave limit of \(AdS_5 \times S^5\) [6], where the string theory worldsheet sigma model is exactly solvable [7].

The matching of all observables related to local operators can be formulated via the equivalence of the partition function of the conformal field theory with the partition function of string theory. We use the metric \(ds^2 = (dz^2 + d\vec{x}^2)/z^2\) for AdS space\(^2\), in which the boundary of the space, spanned by the \(\vec{x}\) coordinates, is at \(z = 0\). If we denote by \(O_i(x)\) the gauge-invariant local operators in the CFT, with scaling dimension \(\Delta_i\), and by \(\chi_i(x, z)\) the corresponding string theory fields in AdS space, whose mass is related to the operator dimensions by

\[
R^2 m_{\chi_i}^2 = \Delta_i (\Delta_i - 4),
\]

\(^2\)These coordinates cover the Euclidean AdS space if we choose a Euclidean signature for the \(\vec{x}\) coordinates, or a Poincaré patch of the Lorentzian AdS space if we choose a Lorentzian signature for the \(\vec{x}\) coordinates.
then we can write the equivalence as

\[ Z[\lambda_i] \equiv \langle e^{i \int d^4x \lambda_i(x)O_i(x)} \rangle_{\text{CFT}} = Z_{\text{IIB}}[\lim_{z \to 0}(\chi_i(x, z)z^{\Delta_i - 4}) = \lambda_i(x)]. \]

(1.4)

Here we introduced arbitrary sources \( \lambda_i(x) \) for every operator, and on the right-hand side we have the string theory partition function with particular boundary conditions on the boundary of AdS space. The correlation functions of local operators in the theory are given by derivatives of this expression with respect to \( \lambda_i(x) \) (at \( \lambda_i = 0 \)).

In fact, the relation (1.4) only holds for a particular class of operators called “single-trace operators”, which in the gauge theory may be written as a trace of a product of adjoint representation fields; the generalization of this formula to include also “multiple-trace operators” which cannot be written in this way is discussed in [8, 9, 10] and is less well-understood on the string theory side.

A particular interesting class of operators in the \( \mathcal{N} = 4 \) SYM theory includes the chiral primary operators, which are in short representations of the \( \mathcal{N} = 4 \) superconformal algebra such that their dimensions cannot receive any quantum corrections. Some of these are mapped to type IIB supergravity fields, and for large field theory coupling \( \lambda_{YM} \) we can compute their correlation functions on the right-hand side of (1.4) in the supergravity approximation (since \( \alpha'/R^2 \) is small). In this approximation, the boundary conditions corresponding to putting in delta-function sources for the fields (as we usually do in computing correlation functions) are imposed by bulk-to-boundary propagators in the supergravity computation. More generally, we have in the string theory (at least at weak string coupling) a vertex operator \( V_i(x, \sigma) \) for each single-trace operator \( O_i(x) \) (where \( \sigma \) is a coordinate on the string worldsheet), and we can identify \( O_i(x) \) with \( \int d^2\sigma V_i(x, \sigma) \) in the sense that inserting the former into the field theory path integral is the same as inserting the latter into the string theory path integral. In this way we can map all single-trace local observables between the two sides of the correspondence.

This mapping of operators is precisely known only for the chiral primary operators (and for some related states with large R-charges, as described in Minwalla’s lectures). Denoting the field content of the \( \mathcal{N} = 4 \) SYM theory by \( A_\mu \) for the gauge field, \( \lambda^a \) (\( a = 1, 2, 3, 4 \)) for the fermions and \( \phi^i \) (\( i = 1, 2, 3, 4, 5, 6 \)) for the scalar fields (all these fields are in the adjoint representation of \( SU(N) \)), the chiral primary operators of the \( \mathcal{N} = 4 \) superconformal algebra are given by \( \text{tr}(\phi^{i_1} \phi^{i_2} \cdots \phi^{i_k}) \) (\( k = 2, 3, \cdots, N \)) where the \( SO(6)_R \) indices \( i_j \) are contracted in a symmetric traceless manner. All other single-trace chiral operators in the theory are descendants of these (they can be obtained from the operators...
above by acting on them with the supercharges of the theory), and their spectrum agrees precisely (for \( k \ll N \)) with that of the single-particle states of type IIB supergravity compactified on \( AdS_5 \times S^5 \). This precise matching means that all other operators in the SYM theory must be matched to other fields of type IIB string theory which are not type IIB supergravity fields. All these fields have a mass at least of the order of the string scale. Using (1.3) and (1.2) this means that the dimension of the corresponding operators grows at least as fast as \((g_{YM}^2 N)^{1/4}\) in the limit of large \( N \) and large \( g_{YM}^2 N \).

Many tests and generalizations of the correspondence have been made in the last five years, which I will not go into here – some of them are described in [5] (where more details of the correspondence may also be found).

3. FROM ADS/CFT TO QCD-LIKE THEORIES

As described above, the AdS/CFT correspondence allows us (among other things) to learn about the strong coupling behavior of the \( \mathcal{N} = 4 \) SYM theory (for instance, to compute the anomalous dimensions of certain operators in the strong coupling limit) from type IIB string theory (or supergravity). This field theory is quite different from QCD – in particular it is scale-invariant (both classically and quantum mechanically), while QCD is weakly coupled at high energies but strongly coupled at low energies. For describing nature it would be more interesting to understand various strong coupling properties of QCD, such as confinement, chiral symmetry breaking, the formation of a mass gap, the spectrum of bound states like mesons and baryons, and so on. All these properties are strong coupling effects that cannot be computed in QCD perturbation theory. However, as described in the introduction above, there is no reason why it shouldn’t be possible to learn about these properties by using a dual string theory (with weak string coupling in the large \( N \) limit), analogous to the one appearing in the AdS/CFT correspondence reviewed in the previous section.

In the previous section we saw that the string theory dual of the \( \mathcal{N} = 4 \) SYM theory had a limit where it was well approximated by supergravity. On the other hand, a string theory background describing QCD (or pure Yang-Mills theory) must be strongly curved. One argument for this is that asymptotically free theories like QCD are weakly coupled at high energy scales (compared to some characteristic scale \( \Lambda_{QCD} \)), and we saw above that weak coupling tends to be related to large curvatures; in particular it is hard to imagine how we could reproduce perturbative QCD
amplitudes at high energies from a supergravity theory. A more general argument is that the spectrum of QCD-like theories, including gauge-invariant particles like mesons and glueballs, comes (approximately) in Regge trajectories (recall that this was one of the original reasons to suspect that they are related to string theories). The Regge trajectory including particles of spin $J$ starts at a mass of order $M_J \simeq \sqrt{J} \Lambda_{\text{QCD}}$.

On the other hand, a weakly curved superstring theory background has a supergravity approximation at low energies, and supergravity theories include only particles of spin less than or equal to 2, so in such theories there is a large mass gap between the spectrum of particles with $J \leq 2$ and with $J > 2$ (the low-energy theory is generally a ten dimensional supergravity theory, which gives rise via a Kaluza-Klein reduction to many particles of spin 2 or lower in four dimensions).

Thus, it seems that to approach QCD we will eventually need to understand string theory in highly curved backgrounds – this has been one of the main obstacles to formulating a string theory of QCD. As described above, the coupling between strings is not an obstacle, since in the large $N$ limit we expect to find a dual string theory with a string coupling constant of order $g_s \simeq 1/N$ (but this could also become a problem when we eventually take $N = 3$).

Unfortunately, progress in studying strongly curved backgrounds in string theory has been rather slow (except for some special cases, like WZW models, which are well understood). Thus, so far almost all computations in string theory duals to QCD-like theories have been done in supergravity approximations. As described above, such approximations will never be quantitatively similar to QCD. However, they can be qualitatively similar to QCD, and can exhibit similar properties like confinement and chiral symmetry breaking. As we will see, they could even be continuously related to theories like pure Yang-Mills theory by changing parameters – for some range of parameters of the theory we could have a good supergravity approximation (and have a large ratio between $M_{J=3}$ and $M_{J=2}$) while in a different range of parameters the same theory could go over to pure Yang-Mills theory (with a small ratio between $M_{J=3}$ and $M_{J=2}$). Thus, it is interesting to study such “toy models of QCD” even if we do not currently know how to control their behavior in the regime where they approach QCD, and this will be the main subject of these lecture notes.

Above we described the AdS/CFT correspondence for the case of the $\mathcal{N} = 4$ SYM theory. In order to approach QCD we want to find string theory duals for non-conformal theories which have less (or no) supersymmetry. There are three general ways that have been used so far to obtain such theories from the AdS/CFT correspondence:
The last method leads to "duality cascades", and it is described in Klebanov’s talks in this school. In these talks I will focus on the first two methods in sections (4.) and (5.), respectively, and briefly discuss the third method in section (6.).

4. DEFORMATIONS OF THE $\mathcal{N} = 4$ SYM THEORY

4.1 DEFORMATIONS IN FIELD THEORY

Since the $\mathcal{N} = 4$ SYM theory provides the simplest example of the AdS/CFT correspondence, the simplest way to use the first method is to study deformations of this theory. The $\mathcal{N} = 4$ theory includes four massless adjoint fermions $\lambda^a$ and six massless adjoint scalars $\phi^i$, and we would like to get rid of them in order to remain with the pure Yang-Mills theory. As mentioned above, this theory is already quite similar to QCD. Adding a small number of quark flavors is not expected to significantly change the large $N$ limit of the theory, but it adds some complications, so we will be content here with finding a dual for pure Yang-Mills theory.

So, we would like to deform the $\mathcal{N} = 4$ theory by mass terms for the "extra" fields, doing something like

$$L_{SYM} \rightarrow L_{SYM} + \text{tr}(M^2 \phi^i \phi^i + (\tilde{m}_{ab} \lambda^a \lambda^b + \text{c.c.})).$$

The problem with this deformation is that the first term we wrote down, $\text{tr}(\phi^i \phi^i)$, is not a chiral operator (it is the “trace” part of $\text{tr}(\phi^i \phi^j)$, and it sits in the so-called Konishi multiplet). This not only means that we cannot study it using supergravity (since it is not in the supergravity spectrum, which includes only chiral operators), but also that for large $g_{YM}^2 N$ this operator acquires a large anomalous dimension (as described above), and the deformation by this operator becomes irrelevant (so it does not make sense as a quantum field theory). We can write down chiral scalar mass operators of the form $(M^2)_{ij} \text{tr}(\phi^i \phi^j)$ with traceless $(M^2)_{ij}$, but then the sum of the eigenvalues of $(M^2)_{ij}$ must vanish, so
some of the eigenvalues must be negative, leading to instabilities and to a breaking of the gauge symmetry (which we do not desire).

Thus, the only mass deformation which makes sense as a leading deformation is of the form $\tilde{m}_{ab} \text{tr}(\lambda^a \lambda^b) + \text{c.c.}$. The mass matrix can always be diagonalized so we can generally write this as

$$m_1 \text{tr}(\lambda^1 \lambda^1) + m_2 \text{tr}(\lambda^2 \lambda^2) + m_3 \text{tr}(\lambda^3 \lambda^3) + m_4 \text{tr}(\lambda^4 \lambda^4) + \text{c.c.} \quad (1.6)$$

This involves chiral operators $\text{tr}(\lambda^a \lambda^b)$ (arising from the action of two supercharges on the chiral primary operator $\text{tr}(\phi^i \phi^j - \frac{1}{6} \delta^{ij} \phi_k \phi_k)$), of (protected) scaling dimension $\Delta = 3$, so we can study this deformation (at least in principle) in the supergravity approximation. We expect that when we perform such a deformation scalar masses will be induced quantum mechanically (by loop corrections, by operator mixings or by RG flow) at order $m^2$, and we could also turn on tree-level scalar masses of order $m^2$ proportional to the chiral scalar mass operator. But, we should still require that the masses squared of all the scalars are positive so that we have a theory which does not spontaneously break the gauge symmetry.

One way to ensure this stability property is to retain some of the supersymmetry of the original theory. This requires that one of the fermions must remain massless, $m_4 = 0$, so that it can be in the same supersymmetry multiplet as the gauge field. If $m_4 = 0$ we can always choose the second order scalar masses such that the theory preserves supersymmetry, and then we can describe the deformation by a superpotential (in $\mathcal{N} = 1$ superspace notation). In $\mathcal{N} = 1$ language the $\mathcal{N} = 4$ theory includes one vector multiplet and three chiral multiplets $\Phi_i$ in the adjoint representation, and we can describe the theory after the mass deformation by the superpotential

$$W = \frac{1}{g_{YM}^2} \text{tr}(2\sqrt{2}\Phi_1 [\Phi_2, \Phi_3] + m_1 \Phi_1^2 + m_2 \Phi_2^2 + m_3 \Phi_3^2). \quad (1.7)$$

At leading order in the deformation this is exactly the fermion mass term we wrote above (with $m_4 = 0$, where we normalize the kinetic terms of all the fields to be proportional to $1/g_{YM}^2$), and at second order the theory with the superpotential (1.7) includes particular scalar mass terms which ensure that an $\mathcal{N} = 1$ subgroup of the original $\mathcal{N} = 4$ supersymmetry remains unbroken. This ensures that even after we take quantum corrections into account, the scalar masses squared will always
be non-negative. If we do not preserve supersymmetry this may or may not happen.\(^3\)

One simple possibility is to give a supersymmetry-preserving mass only to a single chiral multiplet by taking only \(m_1\) to be non-zero. In this case the mass deformation is believed to lead at low energies to a strongly coupled superconformal field theory \(^{12, 13}\) with a superpotential of the form \(W \sim \lambda \text{tr}([\Phi_2, \Phi_3]^2)\) (obtained by integrating out \(\Phi_1\) in (1.7)). The supergravity dual of this theory is known (though the solution describing the full mass deformation has only been computed numerically and not explicitly \(^{14, 15}\)), and \(\lambda\) is an exactly marginal operator in the low-energy conformal theory which maps to the dilaton in its string theory dual\(^4\). If we turn on only two fermion masses (one way to do this is to choose \(|m_1| = |m_2|\), in which case the deformation actually preserves \(\mathcal{N} = 2\) supersymmetry) we find at low energies a theory with a Coulomb branch along which a vacuum expectation value (VEV) for \(\Phi_3\) breaks the gauge symmetry to \(U(1)^{N-1}\). The supergravity dual of this deformation is known at some special points in its moduli space (called “enhançon points”) \(^{16, 17, 18, 19}\). Both for one non-zero mass and for two non-zero masses the low-energy behavior is very different from that of QCD so we will not discuss it here.

The interesting case for us will be the case where all three of the supersymmetry-preserving masses are non-zero. Naively this gives a mass to all the fields except for the \(\mathcal{N} = 1\) vector multiplet, so for energies \(E \ll m_i\) we expect to remain with the pure \(\mathcal{N} = 1\) SYM theory. This theory is quite similar to QCD—it also confines, exhibits a mass gap of the order of a strong coupling scale \(\Lambda_{\text{SYM}}\) and chiral symmetry breaking, and so on. Unfortunately, this naive expectation is not always realized. We can easily compute the characteristic scale \(\Lambda_{\text{SYM}}\) of this “low-energy” SYM theory, defined by \(\Lambda_{\text{SYM}}^b = \mu \exp(-8\pi^2/g_{\text{YM}}^2(\mu))\) where \(\mu\) is the renormalization scale and \(b\) is the one-loop beta function, by matching the running coupling constant at the mass scales of the chiral multiplets. We find \(\Lambda_{\text{SYM}}^3 = m_1m_2m_3 \exp(-8\pi^2/g_{\text{YM}}^2N)\) where \(g_{\text{YM}}^2\) is the gauge coupling constant of the high-energy \(\mathcal{N} = 4\) SYM theory. This means that for large \(g_{\text{YM}}^2N\), where we might expect to have a dual supergravity description (at least we know that we have one for energies above the mass scales), \(\Lambda_{\text{SYM}}\) is of the same order as the \(m_i\), so there is no separation of scales between the fields \(\Phi_i\) that

\(^3\)After these talks were given the case of four equal fermion masses was analyzed in \(^{11}\), and some evidence was given for the claim that this case does not give rise to any tachyonic scalar modes. However, the solution presented there appears to be singular.

\(^4\)Note that this is an example of a theory where having small \(g_sN\) in the string theory does not lead to a weakly coupled theory on the field theory side of the duality.
we are trying to get rid of and the states of the “low-energy” $\mathcal{N} = 1$ SYM theory, since the lowest mass states of this theory are expected to have masses of order $\Lambda_{\text{SYM}}$. In order to get such a separation we must take $g_{YM}^2 N \ll 1$. This leads to $\Lambda_{\text{SYM}} \ll m_i$, but then we do not have a good supergravity approximation and we must deal with a strongly curved string theory background. This is in agreement with our general discussion in the previous section.

When all three masses are non-zero it turns out that the theory has many discrete supersymmetric vacua. Classically the vacua are given, as in any other supersymmetric gauge theory, by solving the F-term equations $dW/d\Phi_i = 0$ and dividing the space of solutions by the complexified gauge group, which in our case is $SL(N, C)$ (this ensures that the D-term equations are also satisfied). Taking all the masses to be equal for simplicity (the generalization to arbitrary masses is straightforward), the F-term equations are

$$[\Phi_i, \Phi_j] = -\frac{m}{\sqrt{2}} \epsilon_{ijk} \Phi_k.$$

Since this is the defining equation for a representation of the $SU(2)$ Lie algebra (up to a rescaling of the $\Phi_i$’s), this is solved by any three matrices which are an $N$-dimensional representation of $SU(2)$. Any such representation is a sum of irreducible representations of dimensions $n_j$, and we can write the classical solutions for the matrices $\Phi_i$ in a block-diagonal form involving blocks of size $n_j \times n_j$ (with $\sum_j n_j = N$). Only three of the six real matrices $\phi^i$ are non-zero in these vacua, and the choice of which three is determined by the phase of the mass $m$.

In most of these vacua the gauge group is not completely broken, so classically there is no mass gap. The only vacuum where the gauge group is completely broken classically is the one corresponding to the $N$-dimensional representation of $SU(2)$, which is called the “Higgs vacuum”. The opposite case is when we have $N$ copies of the 1-dimensional representation of $SU(2)$, and then $\langle \Phi_i \rangle = 0$. Classically this leaves the full $SU(N)$ group unbroken. Quantum mechanically we expect in this vacuum to get at low energies a behavior like that of the pure $\mathcal{N} = 1$ SYM theory (as described above, this expectation is valid for small $g_{YM}^2 N$), which confines (so this is called the “confining vacuum”) and has $N$ different vacua (differentiated by the “gluino condensate” $\langle \text{tr}(\lambda \lambda) \rangle$). An intermediate case is the vacuum corresponding to $p$ copies of the $q$-dimensional representation of $SU(2)$ (with $pq = N$). Classically in such a vacuum the gauge group is broken to $SU(p)$, and the low-energy gauge theory is the $\mathcal{N} = 1$ SYM theory with gauge group $SU(p)$, so quantum mechanically we expect this vacuum to have a mass
gap and split into $p$ discrete vacua [20]. All other vacua, involving irreducible representations of different dimensions, have classically unbroken $U(1)$ gauge groups, so we expect that they will not have a mass gap even in the quantum theory (since the $U(1)$ vector multiplets, which are free at low energies, are expected to remain massless).

In field theory, much is known about the behavior of this theory in its different vacua. For example, one can exactly compute the vacuum expectation values of various chiral operators like $\text{tr}(\Phi^k)$ and $\text{tr}(\lambda\lambda)$ [21, 22, 23, 24]. In the non-supersymmetric case we expect to get a similar qualitative behavior of the theory (if the deformation is stable), except that the degeneracy between the different vacua is expected to be lifted (it is not clear whether the “confining vacua”, which are related to the pure Yang-Mills theory, are the ones with the lowest vacuum energy).

4.2 DEFORMATIONS IN STRING THEORY

What is the string theory dual of the mass-deformed field theory in all the different vacua described above? In order to compute the correlation functions of the $\mathcal{N} = 4$ SYM theory one uses the relation (1.4) only for infinitesimal $\lambda_i(x)$. However, this relation is valid also for finite $\lambda_i(x)$. Thus, deformations of the $\mathcal{N} = 4$ SYM theory by $\int d^4 x \lambda_i(x) O_i(x)$ correspond to solutions of string theory with an asymptotically AdS background in which $\langle \chi_i(x, z) \rangle \simeq \lambda_i(x) z^\Delta_i - 4$, or equivalently by adding to the worldsheet action a term of the form $\int d^4 x \lambda_i(x) \int d^2 \sigma V_i(x, \sigma)$ (instead of just inserting this combination into correlation functions). We are interested in Lorentz-invariant deformations for which $\lambda_i(x)$ is independent of $x$.

In the supergravity approximation, in order to describe a particular deformation we need to find a solution to type IIB supergravity with the appropriate boundary conditions. For supersymmetric solutions (which we are mainly interested in here) this can actually be done by solving only first order equations (second order equations are generally needed for non-supersymmetric solutions). When we decompose the ten dimensional type IIB supergravity multiplet on $AdS_5 \times S^5$ into multiplets of five dimensional $\mathcal{N} = 8$ supergravity, the field $\chi$ appearing in the mass deformation belongs to the lowest multiplet, which is the five dimensional graviton multiplet (containing also the massless graviton). It is believed that in order to find solutions involving the fields in this multiplet one can first solve the five dimensional supergravity equations of motion and then lift the solution to ten dimensions (this is called “consistent truncation”), and solving these equations is not very difficult. However, it turns out [25] that doing this leads to a family of singular
solutions (which become singular at the interior of AdS space), parameterized by different (continuous) values for $\langle \text{tr}(\lambda \lambda) \rangle$ (where $\lambda$ is the adjoint fermion which classically remains massless), while from the field theory discussion above we expect to find a discrete family of solutions, with particular known values for $\langle \text{tr}(\lambda \lambda) \rangle$ in each discrete vacuum. How can we resolve this mismatch? Since we find singular solutions it is clear that the supergravity approximation breaks down, so we have to go beyond this approximation.

Some hints for how we should go beyond supergravity are provided by the following facts:

(i) The supergravity field $\chi$ corresponding to the mass operator $\text{tr}(\lambda^a \lambda^a)$, which gets a vacuum expectation value when we perform the mass deformation, is a particular linear combination of the components of the NS-NS and R-R 2-form fields with indices in the $S^5$. For small values of $z$ (near the boundary) this expectation value grows as $\chi \propto z$ as we move into the interior of AdS space, suggesting that in order to desingularize the solution we may need to insert a source for these 2-form fields (though the total charge of the 3-form field strengths vanishes, so there is no necessity for doing this). Such a source would be a D5-brane or an NS5-brane wrapped on an $S^2$ inside the $S^5$, filling the four dimensional space $R^4$ which the gauge theory lives on, and localized in the radial coordinate $z$.

(ii) In the “Higgs vacuum” the gauge group is completely broken classically, and (at weak coupling) we can choose this breaking to occur at a scale much larger than the strong coupling scale $\Lambda_{\text{SYM}}$ of the theory, so we expect quantum corrections in this vacuum to be small. This vacuum corresponds to the VEVs of the $\Phi_i$ being given by the $N$-dimensional irreducible representation of $SU(2)$. In this representation we have $\sum_i |\Phi_i|^2 \simeq |m|^2 N^2$ (as a matrix equation, with the identity matrix implied on the right-hand side), so it looks like the eigenvalues of the VEVs of the $\Phi_i$ sit on a sphere in the moduli space with radius $|m|N$ (recall that only three of the six real fields $\phi^i$ are non-vanishing, so this is an $S^2$ and not an $S^5$). For large $N$ the eigenvalues almost completely cover this sphere, though it should be noted that since the $\Phi_i$ do not commute this geometrical description is slightly naive (the vacuum expectation values really sit on a “fuzzy sphere” and not on a sphere). Now, if we think of the $\Phi_i$’s as corresponding to the positions of D3-branes (which is how they started life in the “derivation” of the AdS/CFT correspondence before we took the near-horizon limit), then such
vacuum expectation values correspond to a configuration where
the \( N \) D3-branes are blown up into a spherical D5-brane (this is
often called “Myers’ dielectric effect” [26, 27]). This suggests again
that (at least) the “Higgs vacuum” could be related to a spherical
D5-brane wrapped on an \( S^2 \) inside the \( S^5 \).

The arguments above motivate looking for solutions which are not
just described by supergravity but which involve also 5-branes wrapped
on 2-spheres inside the \( S^5 \). Polchinski and Strassler [28] found that
non-singular solutions of this type indeed exist with the appropriate
boundary conditions. The solutions are not known explicitly, but their
form is known asymptotically near the boundary of AdS and near the
5-branes, and the 5-branes are stable in these backgrounds (and even
exhibit a classical mass gap if there is only a single 5-brane) even though
topologically they could collapse to a point since they carry no total
5-brane charge. It is believed that these asymptotic solutions can be
pieced together into non-singular backgrounds of supergravity including
5-branes (which may be viewed as specific sources for the supergravity
fields). I will not describe here the details of these solutions. A special
case of the Polchinski-Strassler solutions involves a single D5-brane, so
it is natural to identify it with the “Higgs vacuum” of the gauge theory
following the discussion above. This D5-brane carries \( N \) units of gauge
field flux on the \( S^2 \), \( \int_{S^2} F \simeq N \), which implies (using the \( F \wedge C_4 \) coupling
in the Wess-Zumino term of the D5-brane action) that it carries \( N \) units
of D3-brane charge. Thus, one might say (motivated by the discussion
above) that the original \( N \) D3-branes have blown up into a D5-brane
in this configuration, though this is somewhat misleading since the D3-

branes were not there in \( AdS_5 \times S^5 \) before we did the deformation\(^5\).
The D5-brane sits at some particular value of \( z_0 \) where it is stable (\( z_0 \) is
proportional to \( 1/\alpha' mN \)), and since it carries \( N \) units of charge of the
R-R 5-form field \( F_5 \), this means that the total 3-brane charge \( \int_{S^5} F_5 \),
which equals \( N \) near the boundary of AdS, decreases as we increase
\( z \). For \( z \gg z_0 \) there is no longer any 5-form flux, and the solution in
fact goes over to flat space there. So, this solution interpolates between
\( AdS_5 \times S^5 \) for small \( z \) and \( R^{10} \) for large \( z \) (with a 5-brane in the middle).

Similar solutions were found to exist also with \( p \) D5-branes, each of
which carries \( q \) units of D3-brane charge (with \( pq = N \)) and sitting at
\( z_0 \simeq 1/\alpha' mq \). It is natural to identify these with the vacua of the gauge
theory in which the gauge symmetry is classically broken to \( SU(p) \) – in

\(^5\)One could try to study these deformations in the full D3-brane background before taking
the near-horizon limit, but it is not known how to do this.
fact we can explicitly see this $SU(p)$ as the classical gauge symmetry on the (overlapping) D5-branes. Both in the field theory and in the string theory this $SU(p)$ gauge group is expected to confine, and each such vacuum is expected to split into $p$ distinct vacua, but this is not visible in the classical string theory.

The solutions described above are weakly coupled everywhere if we have $g_s p \ll 1$, and they are weakly curved far away from the D5-branes for $g_s N \gg 1$ (the regions near the D5-branes can be described well by open+closed string theory as usual). If we continue the logic of the previous paragraph, we would find that the $\langle \Phi \rangle = 0$ “confining” vacuum should be described by $N$ coincident D5-branes. However, such a collection of D5-branes would have a large back-reaction on the background (since $g_s N$ is assumed to be large) and lead to large curvatures, so we cannot use supergravity to analyze it or even to check if such a solution exists or not.

Luckily, by using S-duality we can find a different description of the same vacuum (more precisely, of one of the $N$ vacua coming from the classical $\langle \Phi \rangle = 0$ vacuum). From a direct field theory analysis one can argue that the $SL(2,\mathbb{Z})$ S-duality symmetry of the $\mathcal{N} = 4$ SYM theory permutes the different vacua, and in particular the S-duality transformation $g_{YM} \to 4\pi/g_{YM}$ (for zero theta angle) exchanges the “Higgs vacuum” (where the $\Phi_i$ which are electrically charged degrees of freedom condense, so we expect magnetically charged degrees of freedom to confine as in the Meissner effect in superconductors) with one of the “confining” $\langle \Phi \rangle = 0$ vacua (where we expect the electrically charged particles to be confined, and magnetically charged particles to condense) [20]. So, we can try to describe the $\langle \Phi \rangle = 0$ vacuum by performing an S-duality transformation on the solution that we found for the “Higgs vacuum”. Recall that the AdS/CFT correspondence maps the electromagnetic S-duality of the $\mathcal{N} = 4$ SYM theory to the S-duality of the type IIB string theory. The S-duality transformation of the “Higgs vacuum” gives a background with a single NS5-brane and no D5-branes, and it turns out that such a solution indeed exists and gives a good description for the “confining vacuum”, in which (for large $N$ and $g_s N$) supergravity and string perturbation theory are good approximations except in the vicinity of the NS5-brane. Similarly, other 5-brane configurations can be found to describe all the other vacua of the mass-deformed theory. Whenever we find more than one possible description for a vacuum (as we found above for the “confining vacuum”), they do not have an overlapping range of validity, so at most one solution can be trusted for describing a particular vacuum at a particular value of the coupling constant.
The description of the various vacua in terms of branes wrapping $S^2$’s was recently confirmed directly in the field theory by Dorey and Sinkovics in [24]. They computed the vacuum expectation values of the chiral operators $\text{tr}(\Phi^k)$ in the various vacua at large $g_{YM}^2 N$ (this is possible since exact expressions are known for the VEVs of chiral operators in the mass-deformed SYM theory), extracted from them the distributions of the eigenvalues of the $\Phi_i$ matrices, and showed that indeed they lie on spheres in the range of parameters where we have solutions in which the 5-form charge is carried by 5-branes wrapping spheres.

While the field theory analysis we made relied strongly on supersymmetry, the construction of the solutions described above does not depend on supersymmetry, and one can find qualitatively similar solutions also in some non-supersymmetric cases, when all four fermion masses are non-zero. At least when the supersymmetry-breaking mass is small, one can argue that these solutions are still stable, since the original supersymmetric solutions had a mass gap (see below), and adding a small additional mass deformation cannot destabilize the background when the new mass is much smaller than the mass gap. Of course, supersymmetry breaking generically removes the degeneracy between the different vacua, so we expect that most of the solutions of [28] should be metastable (classically stable but quantum mechanically unstable) after a small supersymmetry breaking deformation, and one of them should be stable.

4.3 QCD-LIKE PROPERTIES OF THE POLCHINISKI-STRASSLER THEORY

As we discussed above, the regime where the solution of [28] is well-described by supergravity is quite different from the regime where it is equivalent (at low energies) to a pure $\mathcal{N} = 1$ SYM theory. However, the two regimes are continuously related (by changing $g_s$) so one expects to find the same qualitative properties in this background as one has in SYM, and this is indeed the case.

First, it is believed that the SYM theory dynamically generates a mass gap of the order of the QCD scale $\Lambda_{SYM}$. One can show that the background of [28] also has a mass gap. For the bulk fields this requires solving their equations of motion (in the presence of the 5-branes), looking for non-singular solutions whose dependence on the $R^4$ directions is of the form $e^{ik \cdot x}$, and showing that the spectrum of the $3+1$ dimensional $m^2 = k^2$ is strictly positive – one can argue that this is indeed the case in the solution of [28] even without knowing the explicit solution. Sim-
ilarly, one has to solve the equations of motion for the fields living on the branes and show that they also have a mass gap. In the background with $p$ overlapping 5-branes we classically have a $U(p)$ gauge theory in $5 + 1$ dimensions living on the 5-branes, compactified on $R^4 \times S^2$. It turns out that the $U(1)$ factor is actually unphysical (it can be gauged away; when there is more than one group of overlapping 5-branes only the sum of their $U(1)$ fields may be gauged away), so one is left with an $SU(p)$ gauge theory, and one can show that the classical massless spectrum on the 5-branes is that of a $3 + 1$ dimensional $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $SU(p)$. In particular, for the “confining vacuum” and the “Higgs vacuum” where $p = 1$ there are already classically no massless fields on the 5-branes, so the full background has a mass gap. For the vacua with $p > 1$ we expect that quantum effects on the 5-branes will generate a mass gap for the $SU(p)$ gauge theory just like they do on the gauge theory side of the correspondence, so we will have a mass gap also in this case but it is not directly visible in the supergravity (plus 5-branes) approximation. In vacua with more than one group of overlapping 5-branes some $U(1)$ gauge multiplets remain massless, and these configurations correspond to the vacua without a mass gap in the field theory.

Another qualitative property of the quantum SYM theory is confinement of fields charged (electrically) under the gauge group, which is expected to be related to condensation and screening of magnetically charged particles. As discussed above, we expect to see this property in the “confining vacua”, while we expect to see screening of electrically charged particles in the “Higgs vacuum”. This property is usually tested by putting in an external quark-anti-quark pair and computing the force between them (or, equivalently, the Wilson loop operator). It is easy to see that in the “Higgs vacuum” one indeed gets screening. External electrically charged particles are described by fundamental strings ending on the boundary of (asymptotically) AdS space [29, 30], and these strings can end on the D5-brane, so the force between charges at different positions in $R^4$ decays at least like the force between charged particles in a $3 + 1$ dimensional $U(1)$ gauge theory. On the other hand, in the “confining vacuum” whose description involves an NS5-brane, the string cannot end anywhere, but must stretch between the external particles, so the potential energy turns out to be proportional to the distance between the external particles as expected for confinement. For magnetically charged sources, which are described by D-strings ending on the boundary, the situation is the opposite, as expected from general considerations. The behavior in other vacua is also as expected [31].
Other QCD-like properties of the solution of [28] were also analyzed, such as the existence of a baryon vertex, instantons, condensates analogous to the gluino condensate of the pure SYM theory, etc.

Let us end this section by summarizing the advantages and disadvantages of this solution as a string theory for QCD. In the limit $g_s N \to 0$ the solution provides (at energies much smaller than $m$) a string theory for pure SYM theory (or, when we turn on supersymmetry-breaking masses, for pure YM theory), as desired. As expected from general considerations, this limit involves large curvatures. In the case of [28] it also involves a rather complicated sigma model, including RR fields, whose form is not known exactly even in the supergravity approximation. Moreover, in the case of the “confining vacuum” (which is most closely related to the pure SYM theory) the solution involves NS5-branes, which are not under control in perturbative string theory. So, while these solutions provide implicit string theories for pure SYM theory, it seems very difficult to analyze them in the limit which is interesting for QCD. On the plus side, the construction of [28] easily generalizes to the non-supersymmetric case, so if one understands the supersymmetric case (and uses it to construct a string theory for pure SYM theory) it should be possible to easily find also a string theory for the (phenomenologically more interesting) pure Yang-Mills theory.

5. COMPACTIFICATION SCENARIOS

Two main compactification scenarios have been extensively discussed in the literature (though many others are also possible). Both of them start from maximally supersymmetric configurations, such that only the compactification breaks supersymmetry to four dimensional $\mathcal{N} = 1$ or $\mathcal{N} = 0$ supersymmetry. The first scenario [32] involves the six dimensional conformal field theory, arising as the low-energy limit on M5-branes, compactified on $S^1 \times S^1$, such that the fermions have anti-periodic boundary conditions around one of the circles. In an appropriate decoupling limit this gives the pure non-supersymmetric $3 + 1$ dimensional Yang-Mills theory. The second scenario [33] involves the “little string theory”, coming from the decoupling limit of type IIB NS5-branes, compactified on an $S^2$ inside a Calabi-Yau manifold. In an appropriate decoupling limit this gives the $3 + 1$ dimensional $\mathcal{N} = 1$ SYM theory. We will focus on the second scenario here, because it has the advantage (which we did not have in the previous section) of having a limit where it is weakly coupled and weakly curved everywhere (with no branes), and it does not involve any Ramond-Ramond fields, so in this limit this scenario is amenable to standard string theory computa-
tions. However, a disadvantage of this scenario is that the high-energy theory is much more complicated than in the previous section (where it was just the $\mathcal{N} = 4$ SYM theory).

5.1 “LITTLE STRING THEORIES”

Let us begin by reviewing the decoupling limit of NS5-branes in general, before they are compactified. Since in type IIB string theory NS5-branes are S-dual to D5-branes, their decoupling limit should be the same. In general, to obtain a decoupled theory on D-branes we need to take the Planck length $l_p$ to zero so as to remove the gravitational interactions with the bulk modes, but we want to keep the brane Yang-Mills coupling constant $g_{YM}$ fixed so as to remain with a non-trivial theory on the brane. Recall that $l_p^8 \propto g_s^2(\alpha')^4$, and that for Dp-branes $g_{YM}^2 \propto g_s(\alpha')^{(p-3)/2}$. Thus, for $p = 3$ the decoupling limit takes $\alpha' \to 0$ and keeps $g_s$ constant, for $p < 3$ we need to take both $g_s$ and $\alpha'$ to zero while keeping $g_{YM}$ constant, while for $p = 4, 5$ we need to take $g_s$ to infinity and $\alpha'$ to zero, again keeping $g_{YM}$ constant (there is no decoupling limit for $p > 5$). Since we are taking $g_s$ to be very large we cannot trust the original string theory description, but in the type IIB case we can use an S-dual description; note that S-duality does not change $l_p$ and $g_{YM}$ (which are physically measurable by low-energy interactions).

After the S-duality we find for $p = 5$ that the same decoupling limit can be described by starting with $N$ NS5-branes in type IIB string theory and taking $g_s \to 0$ while keeping $\alpha'$ constant (in this S-dual description; note that on NS5-branes $g_{YM}^2 \propto \alpha'$).

This limit [34] is called a “little string theory” (LST) since it involves a string tension parameter $\alpha'$ which is kept constant in the decoupling limit but it does not involve a massless graviton; see [35] for a review of these theories. The resulting decoupled theories are not local field theories. For instance, they have (at high energies) a Hagedorn-like density of states, $S \propto E$, as in free string theories, they have a continuous spectrum of single-particle states (above a mass gap $M_0 \simeq 1/\sqrt{N\alpha'}$), and when compactified on circles they have a T-duality symmetry. All these properties are most easily seen from the holographic dual of the “little string theories” [36]. This is just the near-horizon limit of the string theory background of $N$ NS5-branes [37]. The metric of this near-horizon limit in the string frame is simply

$$ds^2 = dx_{R^{5,1}}^2 + N\alpha' d\rho^2 + N\alpha' ds_{S^3}^2,$$

and there is also a linear dilaton in the $\rho$ direction, $g_s = g_0 e^{-\rho}$, and $N$ units of NS-NS 3-form flux on the $S^3$ (such that the sigma model on the $S^3$ is simply the $SU(2)$ Wess-Zumino-Witten model at level $N$).
From the S-duality to D5-branes it is clear that at low energies the “little string theory” reduces to a 5 + 1 dimensional SYM theory (with $\mathcal{N} = (1, 1)$ supersymmetry) with gauge group $SU(N)$ and with coupling constant $g_{YM}^2 \propto \alpha'$. This theory is not renormalizable so it requires some high-energy completion, which is provided by the LST. The holographic description is problematic because the string coupling diverges as we take $\rho \to -\infty$. It is not surprising that the holographic description is not under full control, since the low-energy theory (at energies below $1/\sqrt{N\alpha'}$) is weakly coupled, and it is hard to imagine how weakly coupled SYM computations could emerge from a simple string theory background. This problem will be less severe after we compactify (there are also ways to get around it in six dimensions).

5.2 COMPACTIFIED “LITTLE STRING THEORIES”

Next, we want to reduce the “little string theory” to a 3 + 1 dimensional field theory. Compactifying on a torus $T^2$ would give at low energies an $\mathcal{N} = 4$ theory in 3 + 1 dimensions, so in order to reduce the amount of supersymmetry we need to compactify on an $S^2$; when we compactify NS5-branes on a 2-cycle with this topology which is embedded inside a Calabi-Yau manifold the configuration preserves precisely 3 + 1 dimensional $\mathcal{N} = 1$ supersymmetry. At low energies we find the 5 + 1 dimensional SYM theory compactified on an $S^2$, with a particular embedding of the spin connection of the $S^2$ inside the $SO(4)$ R-symmetry of the 5 + 1 dimensional SYM theory (which follows from the way that the $S^2$ is embedded in the Calabi-Yau manifold). This turns out to give a mass of the order of the inverse radius of the $S^2$ to the four scalar fields in the 5 + 1 dimensional vector multiplets, and to most of the fermionic fields as well. Classically, the only remaining massless degrees of freedom are those of a 3 + 1 dimensional $\mathcal{N} = 1$ SYM theory (namely, $SU(N)$ gauge fields and a fermion field in the adjoint representation). There are two scales where additional degrees of freedom appear. One is the scale $1/R_{S^2}$ which comes from the Kaluza-Klein reduction on the $S^2$, and the other is the scale $1/\sqrt{N\alpha'}$ where additional degrees of freedom of the LST become important (beyond its massless modes discussed above).

By the usual matching of coupling constants, the four dimensional coupling constant at the scale $1/R_{S^2}$ is $g_{YM}^2(1/R_{S^2}) \approx \alpha'/R_{S^2}^2$. Below this scale we can use the running coupling of the $\mathcal{N} = 1$ SYM theory, and we find that the strong coupling scale of the theory is given by

$$\Lambda_{SYM} \approx \frac{\exp(-8\pi^2 R_{S^2}^2/3N\alpha')}{R_{S^2}}, \tag{1.10}$$
In order to get a decoupling of the SYM theory from all the other degrees of freedom we need $\Lambda_{\text{SYM}} \ll 1/R_S^2$ and $\Lambda_{\text{SYM}} \ll 1/\sqrt{N\alpha'}$, and we see that this requires $N\alpha' \ll R_S^2$, so the radius of the sphere should obey $R_S \gg \sqrt{N\alpha'}$. In this limit we obtain at low energies (of order $\Lambda_{\text{SYM}}$) the pure SYM theory. However, from the general considerations we described in section (3.) we expect that this limit will not correspond to a weakly coupled and weakly curved string theory dual. We will see that this is indeed the case.

5.3 THE STRING THEORY DUAL

The discussion above suggests that the compactified theory has one dimensionless parameter, $R_S^2/\alpha'$, in addition to $N$ and to the string scale $\alpha'$ (which were parameters already in the six dimensional theory). The holographic dual of this compactification was found (as a solution to supergravity, which should correspond to a solution to type IIB string theory as well) by Maldacena and Nunez in [33], by modifying and reinterpreting a solution from [38].

To write the solution, let us parameterize the $S^3$ of (1.9) by $\psi, \tilde{\theta}$ and $\phi$, and define a basis of 1-forms by $w^1 + iw^2 = e^{-i\psi}(d\tilde{\theta} + i\sin(\tilde{\theta})d\phi)$, $w^3 = d\psi + \cos(\tilde{\theta})d\phi$ such that the volume form on $S^3$ is $w^1 \wedge w^2 \wedge w^3$. Let us parameterize the $S^2$ by $ds_{S^2}^2 = d\theta^2 + \sin^2(\theta)d\varphi^2$; note that these coordinates parameterize the $S^2$ which the dual “little string theory” is compactified on, but in our ten dimensional solution it is not necessarily the minimal 2-cycle at every value of the radial coordinate$^6$. And, let us define the three components of an $SU(2)$ gauge field on $S^2$ by

$$A^1 = \frac{1}{2} a(\rho)d\theta, \quad A^2 = \frac{1}{2} a(\rho)\sin(\theta)d\varphi, \quad A^3 = \frac{1}{2} \cos(\theta)d\varphi,$$

(1.11)

where $a(\rho) \equiv 2\rho/\sinh(2\rho)$, and denote the corresponding $SU(2)$ field strength by $F^a$. Then, the solution involves the string frame metric

$$ds_{str}^2 = dx_{R^3,1}^2 + N\alpha'[d\rho^2 + e^{2\psi(\rho)}(d\tilde{\theta}^2 + \sin^2(\theta)d\varphi^2) + \frac{1}{4} \sum_a (w^a - A^a)^2],$$

(1.12)

where

$$e^{2\psi(\rho)} = \rho \coth(2\rho) - \frac{\rho^2}{\sinh^2(2\rho)} - \frac{1}{4}.$$

(1.13)

$^6$This issue was discussed most clearly in [39] which was published after these lectures were given.
The dilaton is given by

\[ g_s^2 = e^{2h_0} \frac{2e^{g(\rho)}}{\sinh(2\rho)}, \]

and the NS-NS 3-form is given by

\[ H = N[-\frac{1}{4}(w^1-A^1)\wedge(w^2-A^2)\wedge(w^3-A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a-A^a)]. \]

Asymptotically, at large values of \( \rho \), the solution is similar to the six dimensional case (1.9) but with an \( R^4 \times S^2 \) replacing the \( R^6 \), and with the volume of the \( S^2 \) growing linearly with the radial coordinate \( \rho \). As \( \rho \to 0 \), a 2-cycle shrinks smoothly to zero size, while the size of the \( S^3 \) remains constant. So, the region near \( \rho = 0 \) looks like \( R^7 \times S^3 \), and the full solution (1.12) interpolates between \( R^7 \times S^3 \) near \( \rho = 0 \) and \( R^4 \times S^2 \times S^3 \times R_{\text{linear dilaton}} \) for large \( \rho \). The dilaton, which decreases linearly as \( \rho \to \infty \), goes to a constant value \( g_s = e^{h_0} \) at the minimal radial position \( \rho = 0 \), which is the maximal value of the string coupling in this solution. This maximal string coupling is the additional dimensionless parameter of this solution (compared to the six dimensional solution (1.9)). By examining the asymptotic value of the volume of the \( S^2 \), one can show that \( h_0 \propto R^2_{S^2}/N\alpha' \). Thus, as expected, when this ratio is large (as required for the SYM limit) we have strong coupling at small values of \( \rho \), and we do not have a good string theory description of the limit where we get the SYM theory. However, for small \( h_0 \) the solution has weak coupling, weak curvatures and no Ramond-Ramond fields, so it is an “ideal” holographic background in which computations can easily be performed in string theory (or in a supergravity approximation), and it is continuously related to the pure SYM theory.

5.4 QCD-LIKE PROPERTIES OF THE MALDACENA-NÚÑEZ THEORY

Even in the regime of small \( h_0 \), where we can trust supergravity (and weakly coupled string theory), we can see that the theory has many properties similar to the ones we expect in the pure SYM theory:

- Mass gap – we can analyze the low-energy spectrum using supergravity and see that the theory has a mass gap. Of course, in the

\[ \text{For very strong coupling we can do an S-duality transformation and study the S-dual background, but then the string coupling diverges for large } \rho \text{ and there is an intermediate regime where the string coupling is of order one so neither description is valid. For large enough } h_0 \text{ the supergravity approximation also breaks down and large curvatures arise for small } \rho. \]
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regime where we can trust supergravity the QCD scale $\Lambda_{SYM}$ is of the same order as the Kaluza-Klein scale, and the mass gap is also of this order, so it does not make sense to compare the quantitative results to the SYM theory.

- Confinement – we can compute the quark-anti-quark force at large distances and see that the potential it is linear. In this case, since we are talking about a gauge theory originating from NS5-branes, the external charged particles are the ends of D-strings, so the tension of the confining string (the “QCD string”) is that of a D-string sitting at $\rho = 0$ and stretching in the $R^3 \times 1$ directions. For large $h_0$ we can use the S-dual description for small $\rho$, and in that regime the confining string tension is that of a fundamental string sitting at $\rho = 0$ in the S-dual background.

- Chiral symmetry breaking – one can show that the asymptotic metric at large $\rho$ has a $U(1)_R$ symmetry which can be identified with the $U(1)_R$ symmetry rotating the gluino in the pure SYM theory [33, 40]. In field theory this symmetry is broken by the chiral anomaly to $Z_{2N}$, and it is then spontaneously broken further to $Z_2$ by the gluino condensate $\langle \text{tr} (\lambda \lambda) \rangle$. In supergravity, this symmetry breaking turns out to be visible already at the classical level [41, 42]. In the asymptotic region of large $\rho$ the $U(1)_R$ symmetry is spontaneously broken to $Z_{2N}$ by the 3-form field (1.15), and in the region of small $\rho$ this symmetry is broken further to $Z_2$.

- Many other QCD-like properties are also visible in this background, like instantons, baryons, domain walls, the gluino condensate, and so on [33, 43, 44], but we will not discuss them in detail here.

6. THE KLEBANOV-STRASSLER BACKGROUND

A third type of background which leads in an appropriate limit to the pure SYM theory was presented in [45]. I will not discuss it in detail here, since it is discussed in detail in Klebanov’s talks at this school. I will focus here only on the similarities and differences between this approach to QCD and the other approaches which were described in detail above.

Like the Maldacena-Nuñez background described in the previous section, the solution of [45] also gives a smooth supergravity background, it has a parameter which can be continuously varied to give (in some limit) the pure $\mathcal{N} = 1$ SYM theory (though of course the supergravity approximation breaks down in this limit), and it has the same nice
QCD-like properties discussed in the previous subsection (including chiral symmetry breaking). An advantage of the solution of [45] is that one does not encounter strong string coupling in the decoupling limit, but only large curvatures (unlike the previous case, and like the Polchinski-Strassler case if we ignore the presence of the NS5-brane there). An important difference between the solutions is that the solution of [45] involves Ramond-Ramond fields, so it is more difficult to analyze on the worldsheet; however, note that as discussed above, if we want to use the solution of [33] in the regime where it approaches QCD then we need to use S-duality and then this solution also has RR fields. Another difference is that the high-energy properties of the Klebanov-Strassler solution are not completely clear – it appears to correspond to a new type of high-energy behavior which had not previously been encountered, called a “duality cascade” in [45]. Unlike the previous case that we discussed, here the theory remains 3+1 dimensional at high energies, but the number of degrees of freedom appears to grow logarithmically with the energy scale.

7. SUMMARY

At this point (summer 2002) we have several holographic duals to theories which at low energies include 3 + 1 dimensional \( SU(N) \) Yang-Mills theory (or the \( \mathcal{N} = 1 \) SYM theory), and involve also various additional massive fields. All of these dual theories have a parameter which interpolates between a regime where we can control the string theory side of the duality (we have weak coupling and weak curvatures) and another regime where the (S)YM fields decouple from the massive degrees of freedom. However, the qualitative properties of the theory do not depend on this parameter – in the supersymmetric case one can argue that no phase transitions occur as this parameter is changed, and it is likely that this is true also in some non-supersymmetric cases. So, one can verify various qualitative properties of Yang-Mills theory using these holographic duals, such as confinement, the formation of a mass gap, chiral symmetry breaking, etc. On the other hand, quantitative properties of the theory, like the ratio between the confining string tension and the square of the mass gap, depend non-trivially on the interpolating parameter, so they cannot be computed without obtaining some control over string theory in the strongly curved (and sometimes also strongly coupled) regime.

These constructions give us a proof in principle that (super) Yang-Mills theory has a dual string theory description, involving the usual type IIB string theory (rather than exotic new string theories), and formally we can define this dual string theory by a limiting procedure.
However, currently we do not know how to control this limit, or how to
directly define the limiting string theory without all the extra degrees of
freedom. Of course, we expect that the three different paths described
above should all lead to the same string theory in the appropriate
decoupling limit, but it is not clear how this actually happens (perhaps
there are several dual descriptions of the limiting string theory).

In these talks I focussed on the supersymmetric case. Going to non-
supersymmetric Yang-Mills theory seems to be straightforward in the
first scenario I described (the Polchinski-Strassler background), but it
is more difficult in the other two scenarios. In the scenario of [33] one
can perhaps do this by supersymmetry-breaking deformations along the
lines of [46]; in the scenario of [45] this may also be possible but the
rules for performing deformations there are not completely clear since
we do not fully understand the high-energy theory. I discussed here only
the case of $SU(N)$ gauge theories, but it is not difficult to generalize all
the solutions found above also to the case of $SO(N)$ and $USp(N)$ gauge
groups [28, 47, 40, 45]. I also focussed in these talks exclusively on the
pure (super)Yang-Mills case, although for QCD we obviously want to
add quark flavors as well. A recent general discussion on some possible
ways to do this may be found in [48].

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