ON A MAXIMUM PRINCIPLE AND ITS APPLICATION TO LOGARITHMICALLY CRITICAL BOUSSINESQ SYSTEM

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Abstract. In this paper we study a transport-diffusion model with some logarithmic dissipations. We look for two kinds of estimates. The first one is a maximum principle whose proof is based on Askey theorem concerning characteristic functions and some tools from the theory of $C_0$-semigroups. The second one is a smoothing effect based on some results from harmonic analysis and sub-Markovian operators. As an application we prove the global well-posedness for the two-dimensional Euler-Boussinesq system where the dissipation occurs only on the temperature equation and has the form $|D|^\alpha \log(e+|D|)$, with $\alpha \in [0, 1/2]$. This result improves the critical dissipation ($\alpha = 0$) needed for global well-posedness which was discussed in [15].

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The first goal of this paper is to study some mathematical problems related to the following transport-diffusion model with logarithmic dissipations

\[
\begin{aligned}
\partial_t \theta + v \cdot \nabla \theta + \kappa \frac{|D|^{\beta}}{\log^\alpha (\lambda + |D|)} \theta &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
\text{div } v &= 0 \\
\theta_{|t=0} &= \theta_0.
\end{aligned}
\]

Here, the unknown is the scalar function \(\theta\), the velocity \(v\) is a time-dependent vector field with zero divergence and \(\theta_0\) is the initial datum. The parameter \(\kappa \geq 0\), \(\lambda > 1\) and \(\alpha, \beta \in \mathbb{R}\). The operator \(\frac{|D|^{\beta}}{\log^\alpha (\lambda + |D|)}\) is defined through its Fourier transform

\[
\mathcal{F}\left(\frac{|D|^{\beta}}{\log^\alpha (\lambda + |D|)} f\right)(\xi) = \frac{|\xi|^{\beta}}{\log^\alpha (\lambda + |\xi|)} \left(\mathcal{F} f\right)(\xi).
\]

We will discuss along this paper some quantitative properties for this model, especially two kinds of information will be established: maximum principle and some smoothing effects. We notice that the special case of the equation (1) corresponding to \(\alpha = 0\) and \(\beta \in [0, 2]\) appears naturally in some fluid models like quasi-geostrophic equations or Boussinesq systems. In this context A. Córdoba and D. Córdoba [8] established the \textit{a priori} \(L^p\) estimates: for \(p \in [1, \infty]\) and \(t \geq 0\)

\[
\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.
\]

We remark that the proof in the case \(p = +\infty\) can be obtained from the following representation of the fractional Laplacian \(|D|^{\beta}|D|^{\beta}\),

\[
|D|^{\beta} f(x) = c_d \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d + \beta}} dy.
\]

Indeed, one can check that if a continuous function reaches its maximum at some point \(x_0\) then \(|D|^{\beta} f(x_0) \geq 0\) and hence we conclude as for the heat equation. Our first main result is a generalization of the result of [8] to (1)

**Theorem 1.1.** Let \(\kappa \geq 0, d \in \{2, 3\}, \beta \in [0, 1], \alpha \geq 0, \lambda \geq e^{\frac{3 + 2\alpha}{d}}\) and \(p \in [1, \infty]\). Then any smooth solution of (1) satisfies

\[
\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.
\]

**Remark 1.2.** The restriction on the parameter \(\beta\) is technical and we believe that the above theorem remains true for \(\beta \in [1, 2]\) and \(\alpha > 0\).

Let us discuss the proof in the special case of \(v \equiv 0\). The equation (1) is reduced to the fractional heat equation

\[
\partial_t \theta + \kappa \mathcal{L} \theta = 0 \quad \text{with} \quad \mathcal{L} := \frac{|D|^{\beta}}{\log^\alpha (\lambda + |D|)}.
\]

The solution is explicitly given by the convolution formula

\[
\theta(t, x) = K_t * \theta_0(x) \quad \text{with} \quad \widehat{K_t}(\xi) = e^{-t \frac{|\xi|^{\beta}}{\log^\alpha (\lambda + |\xi|)}}.
\]

We will show that the family \((K_t)_{t \geq 0}\) is a convolution semigroup of probabilities which means that \(\mathcal{L}\) is the generator of a Lévy semigroup. Consequently, this family is a \(C_0\)-semigroup of contractions on \(L^p\) for every \(p \in [1, \infty]\). The important step in the proof
is to get the positivity of the kernel $K_t$. For this purpose we use Askey’s criterion for characteristic functions, see Theorem 3.4. We point out that the restrictions on the dimension $d$ and the values of $\beta$ are due to the use of this criterion. Now to deal with the full transport-diffusion equation (1) we use some results from the theory of $C_0-$semigroups of contractions.

The second estimate that we intend to establish is a generalized Bernstein inequality. Before stating the result we recall that for $q \in \mathbb{N}$ the operator $\Delta_q$ is the frequency localization around a ring of size $2^q$, see next section for more details. Now our result reads as follows,

**Theorem 1.3.** Let $d \in \{1, 2, 3\}, \beta \in [0, 1], \alpha \geq 0, \lambda \geq e^{\frac{342\alpha}{\beta}}$ and $p \in ]1, \infty[$. Then we have for $q \in \mathbb{N}$ and $f \in S(\mathbb{R}^d)$,

$$2^{q\beta}(q + 1)^{-\alpha}||\Delta_q f||_p^p \leq C \int_{\mathbb{R}^d} \left( \frac{|D|^\beta}{\log^\alpha(\lambda + |D|)} \Delta_q f \right) |\Delta_q f|^p - 2 \Delta_q f dx,$$

where $C$ is a constant depending on $p, \alpha, \beta$ and $\lambda$.

The proof relies on some tools from the theory of Lévy operators or more generally sub-Markovians operators combined with some results from harmonic analysis.

**Remarks 1.4.**

1. When $\alpha = 0$ then the above inequality is valid for all $\beta \in [0, 2]$. The case $\beta = 2$ was discussed in [9, 23]. The remaining case $\beta \in [0, 2[$ was treated by Miao et al. in [7] but only for $p \geq 2$.

2. The proof for the case $p = 2$ is an easy consequence of Plancherel identity and does not require any assumption on the parameters $\alpha, \beta$ and $\lambda$.

The second part of this paper is concerned with an application of Theorems 1.1 and 1.3 to the following Boussinesq model with general dissipation

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla \pi = \theta e_2, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\
\partial_t \theta + v \cdot \nabla \theta + \kappa \mathcal{L} \theta = 0 \\
\text{div} v = 0 \\
v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0.
\end{cases}$$

Here, the velocity field $v$ is given by $v = (v^1, v^2)$, the pressure $\pi$ and the temperature $\theta$ are scalar functions. The force term $\theta e_2$ in the velocity equation, with $e_2$ the vector $(0, 1)$, models the effect of the gravity on the fluid motion. The operator $\mathcal{L}$ whose form may vary is used to take into account anomalous diffusion in the fluid motion. From mathematical point of view, the question of global well-posedness for the inviscid model, corresponding to $\kappa = 0$, is extremely hard to deal with. We point out that the classical theory of symmetric hyperbolic quasi-linear systems can be applied for this system and thus we can get the local well-posedness for smooth initial data. The significant quantity that one need to bound in order to get the global existence is the $L^\infty$-norm of the vorticity defined by $\omega = \text{curl} v = \partial_1 v^2 - \partial_2 v^1$. Now we observe from the first equation of (3) that $\omega$ solves the equation

$$\partial_t \omega + v \cdot \nabla \omega = \partial_1 \theta.$$

The main difficulty encountered for the global existence is due to the lack of strong dissipation in the temperature equation: we don’t see how to estimate in a suitable way the quantity $\int_0^T ||\partial_1 \theta||_{L^\infty}$. However, the situation in the viscous case, $\kappa > 0$ and
\[ \mathcal{L} = -\Delta, \] is well-understood and the question of global existence is solved recently in a series of papers. In [5], Chae proved the global existence and uniqueness for initial data \((v_0, \theta_0) \in H^s \times H^s,\) with \(s > 2,\) see also [17]. This result was improved by the author and Keraani in [13] to initial data \(v_0 \in B_{p,1}^{2\frac{r}{p}+1} \) and \(\theta_0 \in B_{p,1}^{-\frac{1}{2}\frac{r}{p}} \cap L^r, r > 2.\) The global existence of Yudovich solutions for this system was treated in [10]. We also mention that in [11], Danchin and Paicu constructed global strong solutions for a dissipative term \(L \theta \) needed for global existence. In [16] the author and Zerguine proved the global well-posedness when \(\alpha \in ]1, 2[.\) The proof relies on the fact that the dissipation is sufficiently strong to counterbalance the possible amplification of the vorticity due to \(\partial_t \theta.\) However the case \(\alpha = 1\) is not reached by the method and this value of \(\alpha\) is called critical in the sense that the dissipation and the amplification of the vorticity due to \(\partial_t \theta\) have the same order.

In [15] we prove that there is some hidden structure leading to global existence in the critical case. More precisely, we introduced the mixed quantity \(\Gamma = \omega + \frac{\partial_1}{|D|} \theta\) which satisfies the equation

\[ \partial_t \Gamma + v \cdot \nabla \Gamma = -[R, v \cdot \nabla] \theta, \quad \text{with} \quad R := \frac{\partial_1}{|D|}. \]

As a matter of fact, the problem in the framework of Lebesgue spaces is reduced to the estimate the commutator between the advection \(v \cdot \nabla\) and Riesz transform \(R\) which is homogenous of degree zero. Since Riesz transform is a Calderón-Zygmund operator then using in a suitable way the smoothing effects for \(\theta\) we can get a global estimate of \(\|\omega(t)\|_{L^p}\). One can then use this information to control more strong norms of the vorticity like \(\|\omega(t)\|_{L^\infty}\) or \(\|\omega(t)\|_{B_{\infty,1}^0}\).

Our goal here is to relax the critical dissipation needed for global well-posedness by some logarithmic factor. More precisely we will study the logarithmically critical Boussinesq model

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla \pi &= \theta e_2 \\
\partial_t \theta + v \cdot \nabla \theta + \frac{|D|}{\log^a(\lambda + |D|)} \theta &= 0 \\
\text{div } v &= 0 \\
v|_{t=0} &= v^0, \quad \theta|_{t=0} = \theta^0.
\end{aligned}
\]

Before stating our result we will need some new definitions. First, we define the logarithmic Riesz transform \(R_\alpha\) by \(R_\alpha = \frac{\partial_1}{|D|} \log^a(\lambda + |D|).\) Second, for a given \(\alpha \in \mathbb{R}\) we define the function spaces \(\{X_p\}_{1 \leq p \leq \infty}\) by

\[ u \in X_p \Leftrightarrow \|u\|_{X^p} := \|u\|_{B_{\infty,1}^0 \cap L^p} + \|R_\alpha u\|_{B_{\infty,1}^0 \cap L^p} < \infty. \]

Our result reads as follows (see section 2 for the definitions and the basic properties of Besov spaces).

**Theorem 1.5.** Let \(\alpha \in [0, \frac{1}{2}], \lambda \geq e^4\) and \(p \in ]2, \infty[.\) Let \(v_0 \in B_{\infty,1}^1 \cap \dot{W}^{1,p}\) be a divergence free vector-field of \(\mathbb{R}^2\) and \(\theta_0 \in X_p.\) Then there exists a unique global solution \((v, \theta)\) to the
Remarks

1.6 differential calculus combined with Theorems 1.1 and 1.3. To estimate the commutator in the framework of Lebesgue spaces we use the para-
system (5)

\[ R \]

Difficulties. We define

\[ \text{The proof shares the same ideas as the case } \alpha = 0 \text{ treated in [15] but with more technical difficulties. We define} \]

\[ \mathcal{R}_\alpha = \frac{\partial}{\partial t} \log^\alpha(\lambda + |D|) \text{ and } \Gamma = \omega + \mathcal{R}_\alpha \theta. \]

Then we get

\[ \partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}_\alpha, v \cdot \nabla] \theta. \]

To estimate the commutator in the framework of Lebesgue spaces we use the para-
differential calculus combined with Theorems 1.1 and 1.3.

Remarks 1.6. (1) We point out that for global well-posedness to the generalized
Navier-Stokes system in dimension three, Tao proved in a recent paper [26] that
we can improve the dissipation \(|D|^{s/2} \) to \( \frac{|D|^{s/2}}{\log(2+|D|)} \).

(2) The space \( \mathcal{X}_p \) is less regular than the space \( B_{\infty,1}^s \cap B_{p,1}^s, \forall \varepsilon > 0 \). More precisely, we will see in Proposition 4.3 that \( B_{\infty,1}^s \cap B_{p,1}^s \hookrightarrow \mathcal{X}_p \).

(3) If we take \( \theta = 0 \) then the system (5) is reduced to the two-dimensional Euler
system. It is well known that this system is globally well-posed in \( H^s \) for \( s > 2 \).

The main tool for global existence is the BKM criterion [2] ensuring that the
development of finite-time singularities for Kato’s solutions is related to the blowup
of the \( L^\infty \) norm of the vorticity near the maximal time existence. In [29] Vishik
extended the global existence of strong solutions to initial data belonging to Besov
spaces \( B_{p,1}^{1+2/p} \). Notice that these spaces have the same scale as Lipschitz functions
and in this sense they are called critical and it is not at all clear whether BKM
criterion can be used in this context.

(4) Since \( B_{r,1}^{1+2/r} \hookrightarrow B_{\infty,1}^s \cap \dot{W}^1p \) for all \( r \in [1, +\infty[ \) and \( p > \max\{r, 2\} \), then the space
of initial velocity in our theorem contains all the critical spaces \( B_{p,1}^{1+2/p} \) except the
biggest one, that is \( B_{\infty,1}^s \). For the limiting case we have been able to prove the
global existence only up to the extra assumption \( \nabla v_0 \in L^p \) for some \( p \in ]2, +\infty[ \). The
reason behind this extra assumption is the fact that to obtain a global \( L^\infty \) bound
for the vorticity we need before to establish an \( L^p \) estimate for some \( p \in ]2, +\infty[ \) and
it is not clear how to get rid of this condition.

(5) Since \( \nabla v \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty) \) then we can propagate all the higher regularities: critical
(for example \( v_0 \in B_{p,1}^{1+2/p} \) with \( p < \infty \)) and sub-critical (for example \( v_0 \in H^s, \)
with \( s > 2 \)).

2. Notations and preliminaries

2.1. Notations. Throughout this paper we will use the following notations.

- For any positive \( A \) and \( B \) the notation \( A \lesssim B \) means that there exist a positive harmless
constant \( C \) such that \( A \leq CB \).
- For any tempered distribution \( u \) both \( \hat{u} \) and \( \mathcal{F}u \) denote the Fourier transform of \( u \).
- For every \( p \in [1, \infty[ \), \( \| \cdot \|_{L^p} \) denotes the norm in the Lebesgue space \( L^p \).
- The norm in the mixed space time Lebesgue space \( L^p((0,T], L^r(\mathbb{R}^d)) \) is denoted by \( \| \cdot \|_{L^p_T L^r} \)
(with the obvious generalization to \( \| \cdot \|_{L^p_T \mathcal{X}} \) for any normed space \( \mathcal{X} \)).
For any pair of operators $P$ and $Q$ on some Banach space $X$, the commutator $[P, Q]$ is given by $PQ - QP$.

For $p \in [1, \infty]$, we denote by $\dot{W}^{1,p}$ the space of distributions $u$ such that $\nabla u \in L^p$.

### 2.2. Functional spaces. Let us introduce the so-called Littlewood-Paley decomposition and the corresponding cut-off operators. There exists two radial positive functions $\chi \in D(\mathbb{R}^d)$ and $\varphi \in D(\mathbb{R}^d \setminus \{0\})$ such that

i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall q \geq 1, \ supp \ \chi \cap supp \ \varphi(2^{-q}) = \emptyset$

ii) $supp \ \varphi(2^{-j.}) \cap supp \ \varphi(2^{-k.}) = \emptyset$, if $|j - k| \geq 2$.

For every $v \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1}v = \chi(D)v; \ \forall q \in \mathbb{N}, \ \Delta_q v = \varphi(2^{-q}D)v \quad \text{and} \quad S_q = \sum_{j=-1}^{q-1} \Delta_j.$$  

The homogeneous operators are defined by

$$\dot{\Delta}_qv = \varphi(2^{-q}D)v, \quad \dot{S}_q v = \sum_{j \leq q-1} \dot{\Delta}_j v, \quad \forall q \in \mathbb{Z}.$$  

From [4] we split the product $uv$ into three parts:

$$uv = T_uv + T_vu + R(u, v),$$  

with

$$T_uv = \sum_q S_{q-1}u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \dot{\Delta}_q v \quad \text{and} \quad \dot{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$  

For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B_{p,r}^s$ as the set of tempered distributions $u$ such that

$$\|u\|_{B_{p,r}^s} := \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{r'} < +\infty.$$  

The homogeneous Besov space $\dot{B}_{p,r}^s$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}_{p,r}^s} := \left(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\right)_{r'} < +\infty.$$  

For $s, s' \in \mathbb{R}$ and $p, r \in [1, \infty]$ we define the generalized Besov space $B_{p,r}^{s,s'}$ as the set of tempered distributions $u$ such that

$$\|u\|_{B_{p,r}^{s,s'}} := \left(2^{qs}(\|q| + 1)^{s'}\|\Delta_q u\|_{L^p}\right)_{r'} < \infty.$$  

Let $T > 0$ and $p \geq 1$, we denote by $L^p_T B_{p,r}^{s,s'}$ the space of distributions $u$ such that

$$\|u\|_{L^p_T B_{p,r}^{s,s'}} := \left\|\left(2^{qs}(\|q| + 1)^{s'}\|\Delta_q u\|_{L^p}\right)_{r'}\right\|_{L^p_T} < +\infty.$$  

We say that $u$ belongs to the space $\tilde{L}^p_T B_{p,r}^{s,s'}$ if

$$\|u\|_{\tilde{L}^p_T B_{p,r}^{s,s'}} := \left(2^{qs}(\|q| + 1)^{s'}\|\Delta_q u\|_{L^p_T L^p}\right)_{r'} < +\infty.$$  


By a direct application of the Minkowski inequality, we have the following links between these spaces. Let $\varepsilon > 0$, then

$$L^p_{\rho} B^s_{p,r} \hookrightarrow \tilde{L}^p_{T} B^s_{p,r} \hookrightarrow L^p_{T} B^s_{p,r},$$

if $r \geq \rho$, and

$$L^p_{T} B^{s+\varepsilon}_{p,r} \hookrightarrow \tilde{L}^p_{T} B^s_{p,r} \hookrightarrow L^p_{T} B^s_{p,r},$$

if $\rho \geq r$.

We will make continuous use of Bernstein inequalities (see [6] for instance).

**Lemma 2.1.** There exists a constant $C$ such that for $q, k \in \mathbb{N}$, $1 \leq a \leq b$ and for $f \in L^a(\mathbb{R}^d)$,

$$\sup_{|\alpha| = k} \| \partial^\alpha S_q f \|_{L^b} \leq C k 2^q (k + d(\frac{1}{2} - \frac{1}{a})) \| S_q f \|_{L^a},$$

$$C^{-k} 2^q \| \Delta_q f \|_{L^a} \leq \sup_{|\alpha| = k} \| \partial^\alpha \Delta_q f \|_{L^a} \leq C k 2^q \| \Delta_q f \|_{L^a}.$$

## 3. Maximum Principle

Our task is to establish some useful estimates for the following equation generalizing (1)

$$\begin{cases}
\partial_t \theta + v \cdot \nabla \theta + \frac{|D|^\beta}{\log^{\alpha} (\lambda + |D|)} \theta = f \\
div v = 0 \\
\theta|_{t=0} = \theta_0,
\end{cases}
$$

(6)

Two special problems will be studied: the first one deals with $L^p$ estimates that give in particular Theorem 1.1. However the second one consists in establishing some logarithmic estimates in Besov spaces with index regularity zero. The first main result of this section generalizes Theorem 1.1.

**Theorem 3.1.** Let $p \in [1, \infty], \beta \in [0, 1], \alpha \geq 0$ and $\lambda \geq e^{\frac{3+2\alpha}{\beta}}$. Then any smooth solution of (6) satisfies

$$\| \theta(t) \|_{L^p} \leq \| \theta_0 \|_{L^p} + \int_0^t \| f(\tau) \|_{L^p} d\tau.$$

The proof will be done in two steps. The first one is to valid the result for the free fractional heat equation. More precisely we will establish that the semigroup $e^{t\mathcal{L}}$, with

$$\mathcal{L} := \frac{|D|^\beta}{\log^{\alpha} (\lambda + |D|)},$$

is a contraction in Lebesgue spaces $L^p$, for every $p \in [1, \infty]$ of course under suitable conditions on the parameters $\alpha, \beta, \lambda$. This problem is reduced to show that $\| K_t \|_{L^1} \leq 1$. This is equivalent to $K_t \in L^1$ and $K_t \geq 0$. As we will see, to get the integrability of the kernel we do not need any restriction on the value of our parameters. Nevertheless, the positivity of $K_t$ requires some restrictions which are detailed in Theorem 3.1. The second step is to establish the $L^p$ estimate for the system (6) and for this purpose we use some results about Lévy operators or more generally sub-Markovians operators.
3.1. **Definite positive functions.** As we will see there is a strong connection between the positivity of the kernel $K_t$ introduced above and the notion of definite positive functions. We will first gather some well-known properties about definite positive functions and recall some useful criteria for characteristic functions. Second and as an application we will show that the kernel $K_t$ is positive under suitable conditions on the involved parameters.

**Definition 3.2.** Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a complex-valued function. We say that $f$ is definite positive if only if for every integer $n \in \mathbb{N}^*$ and every set of points $\{x_j, j = 1, \ldots, n\}$ of $\mathbb{R}^d$ the matrix $(f(x_j - x_k))_{1 \leq j,k \leq n}$ is positive Hermitian, that is, for every complex numbers $\xi_1, \ldots, \xi_n$ we have

$$\sum_{j,k=1}^{n} f(x_j - x_k) \xi_j \bar{\xi}_k \geq 0.$$ 

We will give some results about definite positive functions.

1. From the definition, every definite positive function $f$ satisfies $f(0) \geq 0$, $f(-x) = f(x)$, $|f(x)| \leq f(0)$.

2. The continuity of a definite positive function $f$ at zero gives the continuity everywhere. More precisely we have $|f(x) - f(y)| \leq 2f(0)(f(0) - f(x-y))$.

3. The sum of two definite positive functions is also definite positive and according to Shur’s lemma the product of two definite positive functions is also definite positive and therefore the class of definite positive functions is a convex cone closed under multiplication.

4. Let $\mu$ be a finite positive measure then its Fourier-Stieltjes transform is given by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} d\mu(x).$$

It is easy to see that $\hat{\mu}$ is a definite positive function. Indeed

$$\sum_{j,k=1}^{n} \hat{\mu}(x_j - x_k) \xi_j \bar{\xi}_k = \int_{\mathbb{R}^d} \left( \sum_{j,k=1}^{n} e^{-ix \cdot x_j} \xi_j e^{ix \cdot x_k} \bar{\xi}_k \right) d\mu(x)$$

$$= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} e^{-ix \cdot x_j} \xi_j \right|^2 d\mu(x) \geq 0.$$ 

The converse of the last point (4) is stated by the following result due to Bochner, see for instance Theorem 19 in [3].

**Theorem 3.3** (Bochner’s theorem). Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a continuous definite positive function, then $f$ is the Fourier transform of a finite positive Borel measure.

Hereafter we will focus on the class of radial definite positive functions. First we say that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is radial if $f(x) = F(|x|)$ with $F : [0, +\infty) \rightarrow \mathbb{C}$. There are some criteria for radial functions to be definite positive. For example in dimension one the celebrated criterion of Pólya [24] states that if $F : [0, +\infty) \rightarrow \mathbb{R}$ is continuous and convex with $F(0) = 1$
and \( \lim_{r \to +\infty} F(r) = 0 \) then \( f(x) = F(|x|) \) is definite positive. This criterion was extended in higher dimensions by numerous authors [1, 12, 28]. We will restrict ourselves to the following one due to Askey [1].

**Theorem 3.4 (Askey).** Let \( d \in \mathbb{N}, F : [0, +\infty) \to \mathbb{R} \) be a continuous function such that

1. \( F(0) = 1 \),
2. the function \( r \mapsto (-1)^d F^{(d)}(r) \) exists and is convex on \([0, +\infty)\),
3. \( \lim_{r \to +\infty} F(r) = \lim_{r \to +\infty} F^{(d)}(r) = 0 \).

Then for every \( k \in \{1, 2, \ldots, 2d + 1\} \) the function \( x \mapsto F(|x|) \) is the Fourier transform of a probability measure on \( \mathbb{R}^k \).

**Remark 3.5.** As an application of Askey’s theorem we have that \( x \mapsto e^{-t|x|^\beta} \) is definite positive for all \( t > 0, \beta \in [0, 1] \) and \( d \in \mathbb{N} \). Indeed, the function \( F(r) = e^{-tr^\beta} \) is completely monotone, that is, \((-1)^k F^{(k)}(r) \geq 0, \forall r > 0, k \in \mathbb{N}\). Although the case \( \beta \in [1, 2] \) can not be reached by this criterion the result is still true.

We will now see that the perturbation of the above function by a logarithmic damping is also definite positive. More precisely, we have

**Proposition 3.6.** Let \( \alpha, t \in [0, +\infty], \beta \in [0, 1], \lambda \geq e^{3+2\alpha} \) and define \( f : \mathbb{R}^d \to \mathbb{R} \) by

\[
f(x) = e^{-t|\log x|^{\beta}}.
\]

Then \( f \) is a definite positive function for \( d \in \{1, 2, 3\} \).

**Remarks 3.7.**

1. It is possible that the above result remains true for higher dimension \( d \geq 4 \) but we avoid to deal with this more computational case. We think also that the radial function associated to \( f \) is completely monotone.
2. The lower bound of \( \lambda \) is not optimal by our method. In fact we can obtain more precise bound but this seems to be irrelevant.

**Proof.** We write \( f(x) = F(|x|) \) with

\[
F(r) = e^{-t \phi(r)} \quad \text{and} \quad \phi(r) = \frac{r^\beta}{\log^\alpha (\lambda + r)}.
\]

The function \( F \) is smooth on \([0, \infty]\) and assumptions (1) and (3) of Theorem 3.4 are satisfied. It follows that the function \( f \) is definite positive for \( d \in \{1, 2, 3\} \) if

\[
F^{(3)}(r) \leq 0.
\]

Easy computations give for \( r > 0 \),

\[
F^{(3)}(r) = \left[ -t \phi^{(3)}(r) + 3t^2 \phi'(r)\phi^{(2)}(r) - t^3 (\phi'(r))^3 \right] F(r).
\]

We will prove that

\[
\phi'(r) \geq 0, \phi^{(2)}(r) \leq 0 \quad \text{and} \quad \phi^{(3)}(r) \geq 0.
\]
This is sufficient to get $F^{(3)}(r) \leq 0, \forall r > 0$. The first derivative of $\phi$ is given by

$$
\phi'(r) = \frac{\beta r^{\beta-1}}{\log^\alpha(\lambda + r)} - \frac{\alpha r^\beta}{(\lambda + r) \log^{\alpha+1}(\lambda + r)}
$$

$$
= \frac{r^{\beta-1}}{(\lambda + r) \log^{\alpha+1}(\lambda + r)} \left( \beta \lambda \log(\lambda + r) + r(\beta \log(\lambda + r) - \alpha) \right).
$$

We see that if $\lambda$ satisfies

$$(7) \quad \lambda \geq e^\frac{a}{\beta}$$

then $\phi'(r) \geq 0$. For the second derivative of $\phi$ we obtain

$$
\phi^{(2)}(r) = -\frac{\beta(1 - \beta) r^{\beta-2}}{\log^\alpha(\lambda + r)} - \frac{2\alpha \beta r^{\beta-1}}{(\lambda + r) \log^{\alpha+1}(\lambda + r)} + \frac{\alpha r^\beta}{(\lambda + r)^2 \log^{\alpha+1}(\lambda + r)} + \frac{\alpha(\alpha + 1) r^\beta}{(\lambda + r)^2 \log^{\alpha+2}(\lambda + r)}
$$

$$
= \frac{r^{\beta-2}}{\log^\alpha(\lambda + r)} \left[ -\beta(1 - \beta) - \frac{2\alpha \beta r}{(\lambda + r) \log(\lambda + r)} + \frac{\alpha r^2}{(\lambda + r)^2 \log(\lambda + r)} + \frac{\alpha(\alpha + 1) r^2}{(\lambda + r)^2 \log^2(\lambda + r)} \right].
$$

Since $\frac{r^2}{(\lambda + r)^2} \leq \frac{r}{\lambda + r} \leq 1$, then

$$
\phi^{(2)}(r) \leq \frac{r^{\beta-2}}{\log^\alpha(\lambda + r)} \left[ (1 - \beta) \left( -\beta + \frac{2\alpha}{\log(\lambda + r)} \right) - \frac{2\alpha \beta r}{(\lambda + r) \log(\lambda + r)} \left( 1 - \frac{\alpha + 1}{\log(\lambda + r)} \right) \right]
$$

$$
\leq \frac{r^{\beta-2}}{\log^\alpha(\lambda + r)} \left[ (1 - \beta) \left( -\beta + \frac{2\alpha}{\log \lambda} \right) - \frac{\alpha r}{(\lambda + r) \log(\lambda + r)} \left( 1 - \frac{\alpha + 1}{\log \lambda} \right) \right].
$$

Now we choose $\lambda$ such that

$$
-\beta + \frac{2\alpha}{\log \lambda} \leq 0 \quad \text{and} \quad 1 - \frac{\alpha + 1}{\log \lambda} \geq 0
$$

which is true despite $\lambda$ satisfies

$$(8) \quad \max(e^{\frac{2\alpha}{\beta}}, e^{\alpha+1}) \leq \lambda.
$$

Under this assumption we get

$$
\phi^{(2)}(r) \leq 0, \forall r > 0.
$$
Similarly we have

\[
\phi^{(3)}(r) = \alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-\alpha - 3}(\lambda + r)}{(\lambda + r)^2} \left[ 3\lambda\beta\log(\lambda + r) + r\left(3\beta\log(\lambda + r) - (2 + \alpha)\right)\right]
\]

\[
+ \alpha r^{\beta - 2}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[r^2\left(-3(1 + \alpha) + (-3\beta^2 + 6\beta - 2)\log(\lambda + r)\right)\right]
\]

\[
+ \log(\lambda + r)\left(\lambda\beta(9 - 6\beta)r + 3\lambda^2\beta(1 - \beta)\right)\]

\[
+ (2 - \beta)(1 - \beta)\beta r^{\beta - 3}\log^{-\alpha}(\lambda + r)
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]

It is easy to see that \(I_3\) and \(I_4\) are nonnegative. On the other hand we have

\[
I_1 + I_2 = 3\lambda\beta\alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-\alpha - 2}(\lambda + r)}{(\lambda + r)^2}
\]

\[
+ \alpha r^{\beta - 2}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[-3(1 + \alpha) + (\alpha + 1)(\lambda + r)\left(3\beta - \frac{2 + \alpha}{\log(\lambda + r)}\right)\right]
\]

\[
+ (-3\beta^2 + 6\beta - 2)\log(\lambda + r)
\]

Since \(-3\beta^2 + 6\beta - 2 \geq -2\), for \(\beta \in [0, 1]\) and \(-\frac{\log x}{x} \geq -\frac{\log \lambda}{\lambda}\), \(\forall x \geq \lambda \geq e\), then

\[
I_1 + I_2 \geq \alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[-3 + (\lambda + r)\left(3\beta - \frac{2 + \alpha}{\log(\lambda + r)}\right)\right]
\]

\[
\geq \alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[3\beta - \frac{3}{\lambda} - \frac{2 + \alpha}{\log(\lambda + r)}\right].
\]

We can check that

\[
\log \lambda \leq \lambda \quad \text{and} \quad \log^2 \lambda \leq \lambda, \forall \lambda \geq e.
\]

Thus

\[
I_1 + I_2 \geq \alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[3\beta - \frac{1}{\log(5 + \alpha + \frac{2}{\alpha + 1})}\right]
\]

\[
\geq \alpha(\alpha + 1)r^{\beta - 1}\frac{\log^{-2-\alpha}(\lambda + r)}{(\lambda + r)^2}\left[3\beta - \frac{7 + \alpha}{\log(\lambda)}\right].
\]

We choose \(\lambda\) such that

\[
3\beta - \frac{7 + \alpha}{\log(\lambda)} \geq 0.
\]

It follows that \(I_1 + I_2\) is nonnegative if

(9) \[\lambda \geq e^{\frac{7 + \alpha}{3\beta}}.\]

Remark that the assumptions (7), (8) and (9) are satisfies under the condition

\[
\lambda \geq e^{\frac{3 + 2\alpha}{7 + \alpha}}.
\]
Finally, we get: \( \forall \alpha \geq 0, \beta \in ]0, 1], \lambda \geq e^{\frac{3+2\alpha}{\beta}}, \)
\[ \forall r > 0, \quad \phi^{(3)}(r) \geq 0. \]

This achieves the proof. \( \square \)

More precise informations about the kernel \( K_t \) will be listed in the following lemma.

**Lemma 3.8.** Let \( \lambda \geq 2 \) and denote by \( K_t \) the element of \( \mathcal{S}'(\mathbb{R}^d) \) such that
\[
\widehat{K_t}(\xi) = e^{-t \log \alpha(\lambda + |\xi|)}.
\]
Then we have the following properties.
\( \lambda \)

1. For \( (t, \alpha, \beta) \in ]0, \infty[ \times ]0, \infty[ \) the function \( K_t \) belongs to \( \mathcal{L}^1 \cap C_0 \).
2. For \( d \in \{1, 2, 3\}, (t, \alpha, \beta) \in ]0, \infty[ \times ]0, \infty[ \times ]0, 1] \) and \( \lambda \geq e^{\frac{3+2\alpha}{\beta}} \), we have
\[
K_t(x) \geq 0, \forall x \in \mathbb{R}_+ \quad \text{and} \quad \|K_t\|_{\mathcal{L}^1} = 1.
\]

**Proof.** (1) By definition we have
\[
K_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t \log \alpha(\lambda + |\xi|)} e^{i \cdot x \cdot \xi} d\xi.
\]
Let \( \mu \geq 0 \), then integrating by parts we get
\[
|x|^\mu x_j^d K_t(x) = (-2i\pi)^{-d} \int_{\mathbb{R}^d} \partial_j^d \left( e^{-t \log \alpha(\lambda + |\xi|)} \right) |x|^\mu e^{i \cdot x \cdot \xi} d\xi.
\]
On the other hand we have
\[
|x|^\mu e^{i \cdot x \cdot \xi} = |D|^\mu e^{i \cdot x \cdot \xi},
\]
here \( |D| \) is a fractional derivative on the variable \( \xi \). Thus we get
\[
|x|^\mu x_j^d K_t(x) = (-2i\pi)^{-d} \int_{\mathbb{R}^d} |D|^\mu \partial_j^d \left( e^{-t \log \alpha(\lambda + |\xi|)} \right) e^{i \cdot x \cdot \xi} d\xi.
\]
Now we use the following representation for \( |D|^\mu \) when \( \mu \in ]0, 2] \)
\[
|D|^\mu f(x) = C_{\mu, \nu} \int_{\mathbb{R}^d} \frac{f(x) - f(x - y)}{|y|^{d+\mu}} dy.
\]
It follows that
\[
|x|^\mu |x_j^d K_t(x)| \leq C_{\mu, \nu} \int_{\mathbb{R}^{2d}} \frac{|\mathcal{K}_\nu(\xi) - \mathcal{K}_\nu(\xi - y)|}{|y|^{d+\mu}} dy d\xi
\]
with
\[
\mathcal{K}_\nu(\xi) := \partial_j^d \left( e^{-t \log \alpha(\lambda + |\xi|)} \right).
\]
Now we decompose the integral into two parts
\[
\int_{\mathbb{R}^{2d}} \frac{|\mathcal{K}_\nu(\xi) - \mathcal{K}_\nu(\xi - y)|}{|y|^{d+\mu}} dy d\xi = \int_{|y| \geq \frac{|\xi|}{2}} \frac{|\mathcal{K}_\nu(\xi) - \mathcal{K}_\nu(\xi - y)|}{|y|^{d+\mu}} dy d\xi
\]
\[
+ \int_{|y| \leq \frac{|\xi|}{2}} \frac{|\mathcal{K}_\nu(\xi) - \mathcal{K}_\nu(\xi - y)|}{|y|^{d+\mu}} dy d\xi
\]
\[
= I_1 + I_2.
\]
To estimate the first term we use the following estimate that can be obtained by straightforward computations

\[ |K_j(\xi)| \leq C_{t,\alpha,\beta} \left| \frac{\xi}{\log^{\alpha}(\lambda + |\xi|)} \right|^{\beta - d} e^{-t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}} \]

\[ \leq C_{t,\alpha,\beta} |\xi|^{\beta - d} e^{-\frac{1}{2} t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}}. \]

Hence we get under the assumption \( \mu \in [0, \beta] \),

\[ I_1 \leq C_{t,\alpha,\beta} \int_{|\xi| \leq 2|y|} \frac{1}{|y|^{d+\mu}} \left( |\xi|^{\beta - d} e^{-\frac{1}{2} t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}} + |\xi|^{\beta - d} e^{-\frac{1}{2} t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}} |y| \right) dx dy \]

\[ \leq C_{t,\alpha,\beta} \int_{|\xi| \leq 3|y|} \frac{1}{|y|^{d+\mu}} |\xi|^{\beta - d} e^{-\frac{1}{2} t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}} dx dy \]

\[ \leq C_{t,\alpha,\beta} \int_{\mathbb{R}^d} \frac{1}{|\xi|^{d+\mu}} e^{-\frac{1}{2} t \frac{|\xi|^{\beta}}{\log^{\alpha}(\lambda + |\xi|)}} d\xi \]

\[ \leq C_{t,\alpha,\beta}. \]

To estimate the second term we use the mean-value Theorem

\[ |K_j(\xi) - K_j(\xi - y)| \leq |y| \sup_{\eta \in [\xi - y, \xi]} |\nabla K_j(\eta)|. \]

On the other hand we have

\[ |\nabla K_j(\eta)| \leq C_{t,\alpha,\beta} |\eta|^{\beta - d - 1} e^{-\frac{1}{2} t \frac{|\eta|^{\beta}}{\log^{\alpha}(\lambda + |\eta|)}}. \]

Now since \( |y| \leq \frac{1}{2} |\xi| \) then for \( \eta \in [\xi - y, \xi] \) we have

\[ \frac{1}{2} |\xi| \leq |\eta| \leq \frac{5}{2} |\xi|. \]

This yields

\[ |K_j(\xi) - K_j(\xi - y)| \leq C_t |y||\xi|^{\beta - d - 1} e^{-C_t |\xi|^{\beta}}. \]

Therefore we find for \( \mu \in [0, [\beta]], 0, 1 \],

\[ I_2 \leq C_{t,\alpha,\beta} \int_{|y| \leq \frac{1}{2} |\xi|} \frac{1}{|y|^{d+\mu - 1}} |\xi|^{\beta - d - 1} e^{-C_t |\xi|^{\beta}} dy d\xi \]

\[ \leq C_{t,\alpha,\beta} \int_{\mathbb{R}^d} \frac{1}{|\xi|^{d+\mu - \beta}} e^{-C_t |\xi|^{\beta}} d\xi \]

\[ \leq C_{t,\alpha,\beta}. \]

Finally we get

\[ j = 1, \ldots, d, \ |x|^\mu |x_j|^d |K_j(x)| \leq C_{t,\alpha,\beta}. \]

Since \( K_t \in C_0 \) then

\[ (1 + |x|^{d+\mu}) |K_t(x)| \leq C_t. \]

This proves that \( K_t \in L^1(\mathbb{R}^d) \).

(2) Using Theorem 3.4 and Proposition 3.6 we get \( K_t \geq 0 \). Since \( K_t \in L^1 \) then

\[ \|K_t\|_{L^1} = K_t(0) = 1. \]
Now we define the propagator $e^{-\frac{\beta |D|^\beta}{\log^{\alpha}(\lambda+|D|)}}$ by convolution
\[ e^{-\frac{\beta |D|^\beta}{\log^{\alpha}(\lambda+|D|)}} f = K_t * f. \]

We have the following result.

**Corollary 3.9.** Let $\alpha \geq 0$, $\beta \in [0, 1]$, $\lambda \geq e^{\frac{3+2\alpha}{\beta}}$ and $p \in [1, \infty]$. Then
\[ \|e^{-\frac{\beta |D|^\beta}{\log^{\alpha}(\lambda+|D|)}} f\|_{L^p} \leq \|f\|_{L^p}, \quad \forall f \in L^p. \]

**Proof.** From the classical convolution inequalities combined with Lemma 3.8 we get
\[ \|e^{-\frac{\beta |D|^\beta}{\log^{\alpha}(\lambda+|D|)}} f\|_{L^p} \leq \|K_t\|_{L^1} \|f\|_{L^p} \leq \|f\|_{L^p}. \]

\[ \square \]

### 3.2. Structure of the semigroup \((e^{-\frac{\beta |D|^\beta}{\log^{\alpha}(\lambda+|D|)}})_{t \geq 0}\).

We will first recall the notions of \(C_0\)-semigroup and sub-Markovian generators. First we introduce the notion of strongly continuous semigroup.

**Definition 3.10.** Let \(X\) be a Banach space and \((T_t)_{t \geq 0}\) be a family of bounded operators from \(X\) into \(X\). This family is called a strongly continuous semigroup on \(X\) or a \(C_0\)-semigroup if

1. \(T_0 = \text{Id}\),
2. for every \(t, s \geq 0\), \(T_{t+s} = T_t T_s\),
3. for every \(x \in X\), \(\lim_{t \to 0^+} \|T_t x - x\| = 0\).

If in addition the semigroup satisfies the estimate
\[ \|T_t\|_{\mathcal{L}(X)} \leq 1, \]
then it is called a \(C_0\)-semigroup of contractions.

For a given \(C_0\)-semigroup of contractions \((T_t)_{t \geq 0}\) we define its domain \(\mathcal{D}(A)\) by
\[ \mathcal{D}(A) := \left\{ f \in X; \lim_{t \to 0^+} \frac{T_t f - f}{t} \text{ exists in } X \right\}, \]

\[ Af = \lim_{t \to 0^+} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(A). \]

It is well-known that the operator \(A\) is densely defined: its domain \(\mathcal{D}(A)\) is dense in \(X\). We introduce now the special case of sub-Markovian semigroups.

**Definition 3.11.** Let \(X = L^p(\mathbb{R}^d)\), with \(p \in [1, \infty]\) and \(d \in \mathbb{N}^*\). Let \((T_t)_{t \geq 0}\) be a \(C_0\)-semigroup of contractions on \(X\). It is said a sub-Markovian semigroup if

1. If \(f \in X\), \(f(x) \geq 0\), a.e. then for every \(t \geq 0\), \(T_t f(x) \geq 0\), a.e.,
2. If \(f \in X\), \(|f| \leq 1\) then for every \(t \geq 0\), \(|T_t f| \leq 1\).

Denote by \(L^p_+ := \left\{ f \in L^p; f(x) \geq 0, \text{a.e} \right\}\). Then we have the following classical result.
Theorem 3.12 (Beurling-Deny theorem). Let $A$ be a nonnegative self-adjoint operator of $L^2$. Then we have the equivalence between

1. $\forall t > 0, f \in L^2_+ \Rightarrow e^{-tA} f \in L^2_+$. 
2. $f \in \mathcal{D}(A^{1/2}) \Rightarrow |f| \in \mathcal{D}(A^{1/2})$ and $\|A^{1/2}|f|\|_{L^2} \leq \|A^{1/2} f\|_{L^2}$

Now we will establish the following result.

Proposition 3.13. Let $d \in \{1, 2, 3\}$, $p \in [1, \infty], \alpha \geq 0, \beta \in [0, 1]$ and $\lambda \geq e^{3+2\alpha}$. Define $\mathcal{L} := \frac{|D|^2}{\log^2 (\lambda + |D|)}$, then

1. The family $(e^{-t\mathcal{L}})_{t \geq 0}$ defines a $C_0$-semigroup of contractions in $L^P(\mathbb{R}^d)$.
2. The family $(e^{-t\mathcal{L}})_{t \geq 0}$ defines a sub-Markovian semigroup in $L^p(\mathbb{R}^d)$.
3. The operator $(e^{-t\mathcal{L}})_{t \geq 0}$ satisfies the Beurling-Deny theorem described in Theorem 3.12.

Proof. (1) For $f \in L^P$ we define the action of the semigroup to this function by

$$e^{-t\mathcal{L}} f(x) = K_t \ast f(x),$$

where $\hat{K_t}(\xi) = e^{-t\log^2 (\lambda + |\xi|)}$. From Corollary 3.9 we have that the semigroup maps $L^P$ to itself for every $p \in [1, \infty]$ and

$$\|K_t \ast f\|_{L^P} \leq \|f\|_{L^P}.$$ 

The points (1) and (2) of the Definition 3.10 are easy to check. It remains to prove the third point concerning the strong continuity of the semigroup. Since $\|K_t\|_{L^1} = 1$ and $K_t \geq 0$, then for $\eta > 0$ we have

$$K_t \ast f(x) - f(x) = \int_{\mathbb{R}^d} K_t(y)(f(x - y) - f(x))dy$$

$$= \int_{|y| \leq \eta} K_t(y)(f(x - y) - f(x))dy$$

$$+ \int_{|y| \geq \eta} K_t(y)(f(x - y) - f(x))dy$$

$$= I_1(x) + I_2(x).$$

The first term is estimated as follows

$$\|I_1\|_{L^P} \leq \int_{|y| \leq \eta} K_t(y)\|f(\cdot - y) - f(\cdot)\|_{L^P}dy$$

$$\leq \sup_{|y| \leq \eta} \|f(\cdot - y) - f(\cdot)\|_{L^P}.$$ 

For the second term we write

$$\|I_2\|_{L^P} \leq 2\|f\|_{L^P} \int_{|y| \geq \eta} K_t(y)dy.$$ 

Combining these estimates we get

$$\|K_t \ast f - f\|_{L^P} \leq \sup_{|y| \leq \eta} \|f(\cdot - y) - f(\cdot)\|_{L^P} + 2\|f\|_{L^P} \int_{|y| \geq \eta} K_t(y)dy.$$
It is well-know that for every \( p \in [1, \infty] \) we have
\[
\lim_{\eta \to 0^+} \sup_{|y| \leq \eta} \| f(\cdot - y) - f(\cdot) \|_{L^p} = 0.
\]
Thus for a given \( \varepsilon > 0 \) we can find \( \eta > 0 \) small enough such that
\[
\sup_{|y| \leq \eta} \| f(\cdot - y) - f(\cdot) \|_{L^p} \leq \varepsilon.
\]
Now to conclude the proof it suffices to prove that
\[
\lim_{t \to 0^+} \int_{|y| \geq \eta} K_t(y) dy = 0.
\]
This assertion is a consequence of the following result
\[
K_t \xrightarrow{t \to 0^+} \delta_0.
\]
To prove the last one we write for \( \phi \in \mathcal{S} \),
\[
\langle K_t, \phi \rangle = \frac{1}{(2\pi)^d} \langle \hat{K}_t, \hat{\phi} \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} \hat{\phi}(\xi) d\xi.
\]
We can use now Lebesgue theorem and the inversion Fourier transform leading to
\[
\lim_{t \to 0^+} \langle K_t, \phi \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) d\xi = \phi(0).
\]
Finally we get that \( (K_t \ast)_{t \geq 0} \) defines a \( C_0 \)-semigroup of contractions for every \( p \in [1, \infty] \).

(2) From the Definition 3.11 and the first part of Proposition 3.13 it remains to show that

(1') For \( f \in L^p \) with \( f(x) \geq 0 \), a.e. we have \( e^{-tL}f(x) \geq 0 \)

(2') For \( f \in L^p \) with \( |f(x)| \leq 1 \), a.e. we have \( |e^{-tL}f(x)| \leq 1 \)

The proof is a direct consequence of the explicit formula
\[
e^{-tL}f(x) = K_t \ast f(x),
\]
where according to Lemma 3.8 we have \( K_t \geq 0 \) and \( \|K_t\|_{L^1} = 1 \).

(3) It is not hard to see that the operator \( \frac{|D|^\beta}{\log^\gamma (\lambda + |D|)} \) is a nonnegative self-adjoint operator of \( L^2 \). This operator satisfies the first condition of Theorem 3.12 since the kernel \( K_t \) is positive. \( \square \)

The following result gives in particular Theorem 3.1.

**Proposition 3.14.** Let \( A \) be a generator of a \( C_0 \)-semigroup of contractions, then

(1) Let \( p \in [1, \infty] \) and \( u \in D(A) \). then
\[
\int_{\mathbb{R}^d} Au |u|^{p-1} \text{sign } u \, dx \leq 0.
\]
(2) Let $\theta$ be a smooth solution of the equation

$$\partial_t \theta + v \cdot \nabla \theta - A\theta = f$$

where $v$ is a smooth vector-field with zero divergence and $f$ a smooth function. Then for every $p \in [1, \infty]$

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau.$$

Proof. (1) We introduce the operation $[h, g]$ between two functions by

$$[h, g] = \|g\|_{L^p}^2 \int_{\mathbb{R}^2} h(x) |g(x)|^{p-1} \text{sign} g(x) dx.$$

Now, we define the function $\psi : [0, \infty[ \rightarrow \mathbb{R}$ by

$$\psi(t) = \left[ e^{tA} u, u \right].$$

We have $\psi(0) = \|u\|_{L^p}^2$ and from Hölder inequality combined with the fact that the operator $e^{tA}$ is a contraction on $L^p$ we get

$$\psi(t) \leq \|e^{tA} u\|_{L^p} \|u\|_{L^p} \leq \|u\|_{L^p}^2.$$

Thus we find $\psi(t) \leq \psi(0), \forall t \geq 0$. Therefore we get $\lim_{t \to 0^+} \frac{\psi(t) - \psi(0)}{t} \leq 0$. This gives

$$\int_{\mathbb{R}^2} Au(x)|u(x)|^{p-1} \text{sign} u(x) dx \leq 0.$$

2) Let $p \in [1, \infty]$ then multiplying the equation (6) by $|\theta|^{p-1} \text{sign} \theta$ and integrating by parts using $\text{div} v = 0$ we get

$$\frac{1}{p} \frac{d}{dt} \|\theta(t)\|_{L^p}^p + \int_{\mathbb{R}^2} |A\theta(x)\theta(x)|^{p-1} \text{sign} \theta(x) dx \leq \|f(t)\|_{L^p} \|\theta(t)\|_{L^p}^{p-1}.$$

Using Proposition 3.14 we find

$$\frac{1}{p} \frac{d}{dt} \|\theta(t)\|_{L^p}^p \leq \|f(t)\|_{L^p} \|\theta(t)\|_{L^p}^{p-1}.$$

By simplifying

$$\frac{d}{dt} \|\theta(t)\|_{L^p} \leq \|f(t)\|_{L^p}.$$

Integrating in time we get for $p \in [1, \infty[$

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

Since the estimates are uniform on the parameter $p$ then we can get the limit case $p = +\infty$. □
3.3. Logarithmic estimate. Let us now move to the last part of this section which deals with some logarithmic estimates generalizing the results of [29, 13]. First we recall the following result of propagation of Besov regularities.

**Proposition 3.15.** Let $\kappa \geq 0$ and $A$ be a $C_0$ semigroup of contractions on $L^m(\mathbb{R}^d)$ for every $m \in [1, \infty]$. We assume that for every $q \in \mathbb{N} \cup \{-1\}$, the operator $\Delta_q$ and $A$ commute on a dense subset of $L^p$. Let $(p,r) \in [1, \infty]^2$, $s \in [-1, 1[$ and $\theta$ be a smooth solution of

$$\partial_t \theta + v \cdot \nabla \theta - \kappa A \theta = f.$$ 

Then we have

$$\|\theta\|_{L^\infty_t L^p_{s,r}} \lesssim e^{CV(t)} \left( \|\theta_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)} \|f(\tau)\|_{B^s_{p,r}} d\tau \right),$$

where $V(t) = \|\nabla v\|_{L^1L^\infty}$ and $C$ a constant depending only on $s$ and $d$.

**Proof.** We set $\theta_q := \Delta_q \theta$ then by localizing in frequency the equation of $\theta$ we get

$$\partial_t \theta_q + v \cdot \nabla \theta_q - \kappa A \theta_q = -[\Delta_q, v \cdot \nabla] \theta + f_q.$$ 

Using Proposition 3.14 we get

$$\|\theta_q(t)\|_{L^p} \leq \|\theta_q(0)\|_{L^p} + \int_0^t \|[\Delta_q, v \cdot \nabla] \theta(\tau)\|_{L^p} d\tau + \int_0^t \|f_q(\tau)\|_{L^p} d\tau.$$ 

On the other hand we have the classical commutator estimate, see [6]

$$\|[\Delta_q, v \cdot \nabla] \theta\|_{L^p} \leq C 2^{-qs} c_q \|\nabla v\|_{L^\infty} \|\theta\|_{B^s_{p,r}}, \quad \|(c_q)\|_{L^r} = 1.$$ 

Thus

$$\|\theta(t)\|_{B^s_{p,r}} \leq \|\theta_0\|_{B^s_{p,r}} + C \int_0^t \|\nabla v\|_{L^\infty} \|\theta\|_{B^s_{p,r}} + \int_0^t \|f(\tau)\|_{B^s_{p,r}} d\tau.$$ 

It suffices now to use Gronwall inequality.

Now we will show that for the index regularity $s = 0$ we can obtain a better estimate with a linear growth on the norm of the velocity.

**Proposition 3.16.** Let $v$ be a smooth divergence free vector-field on $\mathbb{R}^d$. Let $\kappa \geq 0$ and $A$ be a generator of $C_0$-semigroup of contractions on $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty]$. We assume that for every $q \in \mathbb{N}$, the operators $\Delta_q$ and $A$ commute on a dense subset of $L^p$. Let $\theta$ be a smooth solution of

$$\partial_t \theta + v \cdot \nabla \theta - \kappa A \theta = f.$$ 

Then we have for every $p \in [1, \infty]$

$$\|\theta\|_{L^\infty_t L^p_{0,1}} \leq C \left( \|\theta_0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}} \right) \left( 1 + \int_0^t \|\nabla v\|_{L^\infty} d\tau \right),$$

where the constant $C$ does not depend on $p$ and $\kappa$.

**Proof.** We mention that the result is first proved in [29] for the case $\kappa = 0$ by using the special structure of the transport equation. In [14] Keraani and the author generalized Vishik’s result for a transport-diffusion equation where the dissipation term has the form $-\kappa \Delta \theta$. The method described in [14] can be easily adapted here for our model.
Let $q \in \mathbb{N} \cup \{-1\}$ and denote by $\bar{\theta}_q$ the unique global solution of the initial value problem

$$
\begin{cases}
\partial_t \bar{\theta}_q + v \cdot \nabla \bar{\theta}_q - \kappa A \bar{\theta}_q = \Delta_q f, \\
\bar{\theta}_q(t=0) = \Delta_q \theta^0.
\end{cases}
$$

(10)

Using Proposition 3.15 with $s = \pm \frac{1}{2}$ we get

$$
\|\bar{\theta}_q\|_{L_t^\infty B^\pm \frac{1}{2}_p,\infty} \lesssim (\|\Delta_q \theta^0\|_{B^\pm \frac{1}{2}_p,\infty} + \|\Delta_q f\|_{L_t^1 B^\pm \frac{1}{2}_p,\infty}) e^{CV(t)},
$$

where $V(t) = \|\nabla v\|_{L_t^1 L^\infty}$. Combined with the definition of Besov spaces this yields for $j, q \geq -1$

$$\|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p} \lesssim 2^{-\frac{j}{2}} \|\Delta_q \theta^0\|_{L^p} + \|\Delta_q f\|_{L_t^1 L^p} e^{CV(t)}.
$$

(11)

By linearity and again the definition of Besov spaces we have

$$
\|\theta\|_{L_t^\infty B^0_{p,1},L^p} \leq \sum_{|j-q| \geq N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p} + \sum_{|j-q| < N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p},
$$

(12)

where $N \in \mathbb{N}$ is to be chosen later. To deal with the first sum we use (11)

$$
\sum_{|j-q| \geq N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p} \lesssim 2^{-N/2} \sum_{q \geq -1} (\|\Delta_q \theta^0\|_{L^p} + \|\Delta_q f\|_{L_t^1 L^p}) e^{CV(t)}
\lesssim 2^{-N/2} (\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L_t^1 B^0_{p,1}}) e^{CV(t)}.
$$

We now turn to the second sum in the right-hand side of (12). It is clear that

$$
\sum_{|j-q| < N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p} \lesssim \sum_{|j-q| < N} \|\bar{\theta}_q\|_{L_t^\infty L^p}.
$$

Applying Proposition 3.15 to the system (10) yields

$$
\|\bar{\theta}_q\|_{L_t^\infty L^p} \leq \|\Delta_q \theta^0\|_{L^p} + \|\Delta_q f\|_{L_t^1 L^p}.
$$

It follows that

$$
\sum_{|j-q| < N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^p} \lesssim N (\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L_t^1 B^0_{p,1}}).
$$

The outcome is the following

$$
\|\theta\|_{L_t^\infty B^0_{p,1},L^p} \lesssim (\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L_t^1 B^0_{p,1}}) \left(2^{-N/2} e^{CV(t)} + N\right).
$$

Choosing

$$
N = \left\lfloor \frac{2CV(t)}{\log 2} \right\rfloor + 1,
$$

we get the desired result.

\[ \square \]

Combining Propositions 3.17 and 3.13 we get,
Corollary 3.17. Let \( v \) be a smooth divergence free vector-field on \( \mathbb{R}^d \), with \( d \in \{2, 3\} \). Let \( \kappa, \alpha \geq 0, \beta \in ]0, 1[, \lambda \geq e^{\frac{3+2\alpha}{\beta}} \), \( p \in [1, \infty] \) and \( \theta \) be a smooth solution of
\[
\partial_t \theta + v \cdot \nabla \theta + \kappa |D|^\beta \log^{-\alpha}(\lambda + |D|) \theta = f.
\]
Then we have
\[
\|\theta\|_{L^\infty_t B^p_{0,1}} \leq C(\|\theta_0\|_{B^p_{0,1}} + \|f\|_{L^1_t B^p_{0,1}})(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau),
\]
where the constant \( C \) depends only on \( \lambda \) and \( \alpha \).

4. Proof of Theorem 1.3

4.1. Bernstein inequality. This section is devoted to the generalization of the classical Bernstein inequality described in Lemma 2.1 for more general operators.

Proposition 4.1. Let \( \alpha \in \mathbb{R}, \beta > 0 \) and \( \lambda \geq 2 \). Then there exists a constant \( C \) such that for every \( f \in S(\mathbb{R}^d) \) and for every \( q \geq -1 \) and \( p \in [1, \infty] \) we have
\[
\left\| \Delta_q \left( \frac{|D|^\beta}{\log^\alpha(\lambda + |D|)} f \right) \right\|_{L^p} \leq C2^{q\beta}(|q| + 1)^{-\alpha} \|\Delta_q f\|_{L^p}.
\]
Moreover
\[
\left\| S_q \left( \frac{|D|^\beta}{\log^\alpha(\lambda + |D|)} f \right) \right\|_{L^p} \leq C2^{q\beta}(|q| + 1)^{-\alpha} \|S_q f\|_{L^p}.
\]

Remark 4.2. The first result of Proposition 4.1 remains true for more general situation where \( q \in \mathbb{N} \) and the operator \( |D|^\beta \) is replaced by \( a(D) \) with \( a(\xi) \) a homogeneous distribution of order \( \beta \in \mathbb{R} \) that is \( a \in C^\infty(\mathbb{R} \setminus \{0\}) \) and for every \( \gamma \in \mathbb{N}^d \)
\[
|\partial^{\gamma}_\xi a(\xi)| \leq C|\xi|^{|\beta| - |\gamma|}.
\]

Proof. Case \( q \in \mathbb{N} \). It is easy to see that
\[
\Delta_q \left( \frac{|D|^\beta}{\log^\alpha(\lambda + |D|)} f \right) = K_q \ast \Delta_q f,
\]
with
\[
\tilde{K}_q(\xi) = \frac{\tilde{\phi}(2^{-q}\xi)|\xi|^\beta}{\log^\alpha(\lambda + |\xi|)}
\]
and \( \tilde{\phi} \) is a smooth function supported in the ring \( \{\frac{1}{2} \leq |x| \leq 3\} \) and taking the value 1 on the support of the function \( \phi \) introduced in section 2. By Fourier inversion formula and change of variables we get
\[
K_q(x) = c_d \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{\tilde{\phi}(2^{-q}\xi)|\xi|^\beta}{\log^\alpha(\lambda + |\xi|)} d\xi = c_d2^{q\beta}2^{qd} \int_{\mathbb{R}^d} e^{2^{q}i x \cdot \xi} \frac{\tilde{\phi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda + 2^q|\xi|)} d\xi := c_d2^{q\beta}2^{qd} \tilde{K}_q(2^q|x|),
\]
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with

\[ \tilde{K}_q(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{\tilde{\psi}(\xi)|\xi|^\beta}{\log^\alpha(\lambda + 2^q|\xi|)} d\xi. \]

Obviously we have

\[ \|\tilde{K}_q\|_{L^1} = c_d 2^{q\beta} \|\tilde{K}_q\|_{L^1}. \]

Hence to prove Proposition 4.1 it suffices to establish

(13) \[ \|\tilde{K}_q\|_{L^1} \leq C(q + 1)^{-\alpha}. \]

From the definition of \( \tilde{K}_q \) we see that

\[ \tilde{K}_q(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda + 2^q|\xi|)} d\xi \]

where \( \tilde{\psi} \) belongs to Schwartz class and supported in \( \{ \frac{1}{4} \leq |x| \leq 3 \} \). By integration by parts we get for \( j \in \{1, 2, ..., d\} \)

\[ x_j^{d+1} \tilde{K}_q(x) = (-i)^{d+1} \int_{\frac{1}{4} \leq |\xi| \leq 3} e^{ix \cdot \xi} \partial_{\xi_j}^{d+1} \left( \frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda + 2^q|\xi|)} \right) d\xi. \]

Now we claim that

\[ \left| \partial_{\xi_j}^{d+1} \left( \frac{\tilde{\psi}(\xi)}{\log^\alpha(\lambda + 2^q|\xi|)} \right) \right| \leq C_{\lambda, \alpha, d} \frac{g(\xi)}{\log^\alpha(\lambda + 2^q)}, \]

where \( g \in \mathcal{S}(\mathbb{R}^d) \). This is an easy consequence of Leibniz formula and the following fact

\[ \left| \partial_{\xi_j}^l \left( \frac{1}{\log^\alpha(\lambda + 2^q|\xi|)} \right) \right| \leq \sum_{l,k=1}^n c_{l,k} \left( \frac{2^q}{\lambda + 2^q|\xi|} \right)^l \frac{1}{\log^\alpha(\lambda + 2^q|\xi|)} \]

\[ \leq \frac{C_{\lambda, \alpha, n}}{\log^\alpha(\lambda + 2^q)}, \text{ for } \frac{1}{4} \leq |\xi| \leq 2. \]

Thus we get for \( j \in \{1, ..., d\} \)

\[ |x_j|^{d+1} |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q), \forall x \in \mathbb{R}^d. \]

It follows that

\[ |x|^{d+1} |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q), \forall x \in \mathbb{R}^d. \]

It is easy to see that \( \tilde{K}_q \) is continuous and

\[ |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q) \]

Consequently,

\[ |\tilde{K}_q(x)| \leq C \log^{-\alpha}(\lambda + 2^q)(1 + |x|)^{-d-1}, \forall x \in \mathbb{R}^d. \]

This yields

\[ \|\tilde{K}_q\|_{L^1} \leq C \log^{-\alpha}(\lambda + 2^q) \leq C(q + 1)^{-\alpha}. \]

This concludes the proof of the first case \( q \in \mathbb{N} \).
Case $q = -1$. We can write in this case the kernel $K_{-1}$ as

$$K_{-1}(x) = \int_{\mathbb{R}^d} e^{i x \cdot \xi} \frac{\tilde{\chi}(\xi) |\xi|^\beta}{\log^\alpha (\lambda + |\xi|)} d\xi$$

$$= \int_{\mathbb{R}^d} e^{i x \cdot \xi} \chi_1(\xi) d\xi,$$

where $\tilde{\chi}$ is a smooth compactly supported function taking the value 1 on the support of the function $\chi$ introduced in section 2. The function $\chi_1$ is given by $\chi_1(\xi) = \frac{\tilde{\chi}(\xi) |\xi|^\beta}{\log^\alpha (\lambda + |\xi|)}$. We can see by easy computations that $\tilde{\chi}$ is smooth outside zero and satisfies for every $\gamma \in \mathbb{N}^d$,

$$|\partial^\gamma \tilde{\chi}(\xi)| \leq C |\xi|^{\beta - |\gamma|}, \quad \forall \xi \neq 0.$$

Using Mikhlin-Hörmander theorem we get

$$|K_{-1}(x)| \leq C |x|^{-d-\beta}.$$

Since $K_{-1}$ is continuous at zero then we have $|K_{-1}(x)| \leq C(1 + |x|)^{-d-\beta}$. This proves that $K_{-1} \in L^1$.

To prove the second estimate we use the first result combined with the following identity $S_{q+2} \tilde{S}_q = S_q$.

$$\left\| S_q \left( \log^\alpha (\lambda + |D|) f \right) \right\|_{L^p} \leq \sum_{j=-1}^{q+1} \left\| \Delta_j \left( \frac{|D|^\beta}{\log^\alpha (\lambda + |D|)} S_q f \right) \right\|_{L^p}$$

$$\leq C \|S_q f\|_{L^p} \sum_{j=-1}^{q+1} 2^{j\beta} (|j| + 1)^{-\alpha}.$$

Since $\beta > 0$ then the last series diverges and

$$\sum_{j=-1}^{q+1} 2^{j\beta} (|j| + 1)^{-\alpha} \leq C 2^{q\beta} (|q| + 1)^{-\alpha}.$$

This can be deduced from the asymptotic behavior

$$\int_1^x e^{\beta t - \alpha t} dt \approx \frac{1}{\beta} e^{\beta x} x^{-\alpha}, \quad \text{as} \quad x \to +\infty.$$

As a consequence of Proposition 4.1 we get the following result which describes the action of the logarithmic Riesz transform $R_\alpha = \frac{\partial_1 \log^\alpha (\lambda + |D|)}{|D|}$ on Besov spaces.

**Corollary 4.3.** Let $\alpha \in \mathbb{R}, \lambda > 1$ and $p \in [1, \infty]$. Then the map

$$(\text{Id} - \Delta_{-1}) R_\alpha : B^s_{p,r} \to B^s_{p,r}$$

is continuous.
4.2. **Generalized Bernstein inequality.** The main goal of this section is to prove Theorem 1.3. Some preliminaries lemmas will be needed. The first one is a Stroock-Varopoulos inequality for sub-Markovian operators. For the proof see [21, 22].

**Theorem 4.4.** Let $p > 1$ and $A$ be a sub-Markovian generator, then we have
\[
4p^{-1} \frac{1}{p^2} \| A^\frac{1}{2} (|f|^\frac{p}{2} \text{sign } f) \|_{L^2}^2 \leq \int_{\mathbb{R}^d} (Af) |f|^{p-1} \text{sign } f \, dx \leq C_p \| A^\frac{1}{2} (|f|^\frac{p}{2} \text{sign } f) \|_{L^2}^2.
\]
Moreover the generator $A$ satisfies the first Deuring-Deny condition
\[
4p^{-1} \frac{1}{p^2} \| A^\frac{1}{2} (|f|^\frac{p}{2}) \|_{L^2}^2 \leq \int_{\mathbb{R}^d} (Af) |f|^{p-1} \text{sign } f \, dx.
\]

Combining this result with Proposition 3.13 we get,

**Corollary 4.5.** Let $p, \beta \in [0, 1], \alpha \geq 0$ and $\lambda \geq e^{\frac{3+2\alpha}{p}}$. Then we have
\[
4p^{-1} \frac{1}{p^2} \left\| \frac{|D|^\beta}{\log^2 (\lambda + |D|)} (|f|^\frac{p}{2}) \right\|_{L^2}^2 \leq \int_{\mathbb{R}^d} \left( \frac{|D|^\beta}{\log^2 (\lambda + |D|)} f \right) |f|^{p-1} \text{sign } f \, dx.
\]

We will make use of the following composition results,

**Lemma 4.6.** (1) Let $\mu \geq 1$ and $s \in [0, \mu] \cap [0, 2]$. Then
\[
\| |f|^\mu \|_{B^s_{2,2}} \leq C \| f \|_{B^s_{2,2}} \| f \|^{\mu-1}_{B^s_{2,2}}.
\]
(2) $\mu \in [0, 1], p, q \in [1, \infty]$ and $0 < s < 1 + \frac{1}{p}$. Then
\[
\| |f|^\mu \|_{B^s_{p,q}} \leq C \| f \|^{\mu}_{B^s_{p,q}}.
\]

We point out that the first estimate is a particular case of a general result due to Miao et al., see [7]. The second one is established by Sickel in [25], see also Theorem 1.4 of [19].

Next we will recall the following result proved in [7, 9, 23],

**Proposition 4.7.** Let $d \geq 1$, $\beta \in [0, 2]$ and $p \geq 2$. Then we have for $q \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^d)$,
\[
2^q \| \Delta_q f \|_{L^p}^p \leq C \int_{\mathbb{R}^d} (|D|^\beta \Delta_q f) |\Delta_q f|^{p-1} \text{sign } \Delta_q f \, dx.
\]
where $C$ depends on $p$ and $\beta$. Moreover, for $\beta = 2$ we can extend the above inequality to $p \in [1, \infty]$. Now we will restate and prove Theorem 1.3.

**Proposition 4.8.** Let $d \in \{1, 2, 3\}, \beta \in [0, 1], \alpha \geq 0, \lambda \geq e^{\frac{3+2\alpha}{p}}$ and $p > 1$. Then we have for $q \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^d)$,
\[
2^q (q+1)^{-\alpha} \| \Delta_q f \|_{L^p}^p \leq C \int_{\mathbb{R}^d} \left( \frac{|D|^\beta}{\log^2 (\lambda + |D|)} \Delta_q f \right) |\Delta_q f|^{p-1} \text{sign } \Delta_q f \, dx.
\]
where $C$ depends on $p, \alpha$ and $\lambda$. 

**Proof.** Using Corollary 4.5 it suffices to prove

\[ C^{-1}2^{q^2(q+1)-\alpha} \|\Delta_q f\|_{L^p}^p \leq \left\| \frac{|D|^{\alpha/2}}{\log^{\alpha/2}(\lambda + |D|)} (|\Delta_q f|^{1/2}) \right\|_{L^2}^p. \]

We will use an idea of Miao et al. [7]. Let \( N \in \mathbb{N} \) then we have

\[ \| |D|(f_q^{1/2})\|_{L^2} \leq \| S_N|D|(f_q^{1/2})\|_{L^2} + \| (\text{Id} - S_N)|D|(f_q^{1/2})\|_{L^2}. \]

It is clear that for \( s \geq 0 \)

\[ \|\text{Id} - S_N\| |D|(f_q^{1/2})\|_{L^2} \leq C2^{-N' s}\|f_q^{1/2}\|_{B^1_{2,2}}. \]

We have now to deal with fraction powers in Besov spaces. We will treat differently the cases \( p > 2 \) and \( p \leq 2 \).

**Case \( p > 2 \).** Combining Lemma 4.6-(1) with Bernstein inequality we get under the assumption \( 0 < s < \min(\frac{2}{p} - 1, 2) \),

\[ \|f_q^{1/2}\|_{B^1_{2,2}} \leq C\|f_q\|_{B^2_{p,2}}^{p-1}\|f_q\|_{B^1_{p,2}} \leq C2^{q(1+s)}\|f_q\|_{L^p}^{p}. \]

**Case \( 1 < p \leq 2 \).** Using Lemma 4.6-(2) and Bernstein inequality, we get for \( 0 < s < \frac{p-1}{2} \),

\[ \|f_q^{1/2}\|_{B^1_{2,2}} \leq C\|f_q\|_{B^2_{p,2}}^{p-1}\|f_q\|_{B^1_{p,2}} \leq C2^{q(1+s)}\|f_q\|_{L^p}^{p}. \]

It follows from (14) and the previous inequalities that there exists \( s_p > 0 \) such that for \( 0 < s < s_p \)

\[ \|\text{Id} - S_N\| |D|(f_q^{1/2})\|_{L^2} \leq C2^{-N' s}\|f_q^{1/2}\|_{L^p}. \]

On the other hand Proposition 4.1 gives

\[ \|S_N|D|(f_q^{1/2})\|_{L^2} \leq \left\| (|S_N|D|^{1-\alpha/2}\log^{\alpha/2}(\lambda + |D|))(\frac{|D|^{\alpha/2}}{\log^{\alpha/2}(\lambda + |D|)} (f_q^{1/2})) \right\|_{L^2} \]

\[ \leq C2^{N(1-\alpha/2)}N^{\alpha/2}\left\| \frac{|D|^{\alpha/2}}{\log^{\alpha/2}(\lambda + |D|)} (f_q^{1/2}) \right\|_{L^2}. \]

Therefore we get

\[ \| |D|(f_q^{1/2})\|_{L^2} \leq C2^{-N' s}\|f_q\|_{L^p}^{p} + C2^{N(1-\alpha/2)}N^{\alpha/2}\left\| \frac{|D|^{\alpha/2}}{\log^{\alpha/2}(\lambda + |D|)} (f_q^{1/2}) \right\|_{L^2}. \]

According to Proposition 4.7 we have for \( p \in ]1, \infty[ \)

\[ C_p^2\|f_q\|_{L^p}^{p} \leq \| |D|(f_q^{1/2})\|_{L^2}. \]

Combining both last estimates we get

\[ 2^q\|f_q\|_{L^p}^{p} \leq C2^{s(q-N)}2^q\|f_q\|_{L^p}^{p} + C2^{N(1-\alpha/2)}N^{\alpha/2}\left\| \frac{|D|^{\alpha/2}}{\log^{\alpha/2}(\lambda + |D|)} (f_q^{1/2}) \right\|_{L^2}. \]
We take \( N = q + N_0 \) such that \( C2^{-N_0} \leq \frac{1}{2} \). Then we get

\[
\|f_q\|_{L^p} \leq C2^{-\frac{\alpha}{2}}(q + 1)^{\frac{\beta}{2}} \left\| \frac{|D|^\frac{\beta}{2}}{\log^\frac{\alpha}{2}(\lambda + |D|)} (|f_q|^\frac{\beta}{2}) \right\|_{L^2}.
\]

This gives the desired result. \( \Box \)

5. Commutator estimates

We will establish in this section some commutator estimates. The following result was proved in [15].

**Lemma 5.1.** Given \((p,m) \in [1, \infty]^2\) such that \( p \geq m' \) with \( m' \) the conjugate exponent of \( m \). Let \( f, g \) and \( h \) be three functions such that \( \nabla f \in L^p, g \in L^m \) and \( xh \in L^{m'} \). Then,

\[
\|h \ast (fg) - f(h \ast g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}.
\]

Now we will prove the following lemma.

**Lemma 5.2.** Let \((a_n)_{n \in \mathbb{Z}}\) be a sequence of strictly nonnegative real numbers such that

\[
M := \max \left( \sup_{n \in \mathbb{Z}} a_n^{-1} \sum_{j \leq n} a_j, \sup_{n \in \mathbb{Z}} a_n \sum_{j \geq n} a_j^{-1} \right) < \infty.
\]

Then for every \( p \in [1, \infty] \) the linear operator \( T : \ell^p \to \ell^p \) defined by

\[
T((b_n)_{n \in \mathbb{Z}}) = \left( \sum_{j \leq n} a_j a_n^{-1} b_j \right)_{n \in \mathbb{Z}}
\]

is continuous and \( \|T\|_{\mathcal{L}(\ell^p)} \leq M \).

**Proof.** By interpolation it suffices to prove the cases \( p = 1 \) and \( p = +\infty \). Let’s start with \( p = 1 \) and denote \( b = (b_n)_{n \in \mathbb{Z}} \). Then from Fubini lemma and the hypothesis

\[
\|Tb\|_{\ell^1} \leq \sum_{n \in \mathbb{Z}} \sum_{j \leq n} a_j a_n^{-1} |b_j| \leq \sum_{j \in \mathbb{Z}} |b_j| a_j \sum_{n \geq j} a_n^{-1} \leq M \|b\|_{\ell^1}.
\]

For the case \( p = +\infty \) we write

\[
\|Tb\|_{\ell^\infty} \leq \sup_{n \in \mathbb{Z}} \sum_{j \leq n} a_j a_n^{-1} |b_j| \leq \|b\|_{\ell^\infty} \sup_{n \in \mathbb{Z}} a_n^{-1} \sum_{j \leq n} a_j \leq M \|b\|_{\ell^\infty}.
\]

This completes the proof. \( \Box \)
The goal now is to study the commutation between the following operators
\[ R_\alpha = \frac{\partial_1}{|D|} \log^\alpha(\lambda + |D|) \quad \text{and} \quad v \cdot \nabla. \]

Recall that \( B^{s,r}_{\infty,2} \) is the space given by the set of tempered distributions \( u \) such that
\[ \|u\|_{B^{s,r}_{\infty,2}} = \| (2^qs + 1)^r \|_{L^{\infty}}. \]

The main result of this section reads as follows.

**Proposition 5.3.** Let \( \alpha \in \mathbb{R}, \lambda > 1, v \) be a smooth divergence free vector-field and \( \theta \) be a smooth scalar function.

1. For every \( (p,r) \in [2,\infty]\times[1,\infty] \) there exists a constant \( C = C(p,r) \) such that
\[ \| [R_\alpha, v \cdot \nabla] \theta \|_{B^0_{p,r}} \leq C \| \nabla v \|_{L^p} (\| \theta \|_{B^{0,\alpha}_{\infty,r}} + \| \theta \|_{L^p}). \]

2. For every \( (r,\rho) \in [1,\infty]\times[1,\infty] \) and \( \epsilon > 0 \) there exists a constant \( C = C(r,\rho,\epsilon) \) such that
\[ \| [R_\alpha, v \cdot \nabla] \theta \|_{B^0_{\infty,r}} \leq C (\| \omega \|_{L^\infty} + \| \omega \|_{L^\rho}) (\| \theta \|_{B^{\epsilon,\alpha}_{\infty,r}} + \| \theta \|_{L^\rho}). \]

**Proof.** (1) We split the commutator into three parts according to Bony’s decomposition [4],
\[ [R_\alpha, v \cdot \nabla] \theta = \sum_{q \in \mathbb{N}} [R_\alpha, S_{q-1} v \cdot \nabla] \Delta_q \theta + \sum_{q \in \mathbb{N}} [R_\alpha, \Delta_q v \cdot \nabla] S_{q-1} \theta \]
\[ + \sum_{q \geq -1} [R_\alpha, \Delta_q v \cdot \nabla] \tilde{\Delta}_q \theta \]
\[ = \sum_{q \in \mathbb{N}} I_q + \sum_{q \in \mathbb{N}} II_q + \sum_{q \geq -1} III_q \]
\[ = I + II + III. \]

We start with the estimate of the first term \( I \). It is easy to see that there exists \( \tilde{\varphi} \in \mathcal{S} \) whose spectrum does not meet the origin such that
\[ I_q(x) = h_q * (S_{q-1} v \cdot \nabla \Delta_q \theta) - S_{q-1} v \cdot (h_q * \nabla \Delta_q \theta), \]
where
\[ \hat{h}_q(\xi) = i\tilde{\varphi}(2^{-q}\xi) \frac{\xi_1}{|\xi|} \log^\alpha(\lambda + |\xi|). \]

Applying Lemma 5.1 with \( m = \infty \) we get
\[ \|I_q\|_{L^p} \lesssim \|xh_q\|_{L^1} \|\nabla S_{q-1} v\|_{L^p} \|\Delta_q \nabla \theta\|_{L^\infty} \]
\[ \lesssim 2^q\|xh_q\|_{L^1} \|\Delta_q \theta\|_{L^\infty} \|\nabla v\|_{L^p}. \]

(15)

We can easily check that
\[ \|xh_q\|_{L^1} = 2^{-q}\|x\hat{h}_q\|_{L^1} \quad \text{with} \quad \hat{h}_q(\xi) = i\tilde{\varphi}(\xi) \frac{\xi_1}{|\xi|} \log^\alpha(\lambda + 2^q|\xi|). \]
We can get by a similar way to the proof of Proposition 4.1
\[ \| \tilde{h}_q \|_{L^1} \leq C(1 + |q|)^{\alpha}. \]

Thus estimate (15) becomes
\[ \| I_q \|_{L^p} \leq C(1 + |q|)^{\alpha} \| \Delta_q \theta \|_{L^\infty} \| \nabla v \|_{L^p}. \]

Combined with the trivial fact
\[ \Delta_j \sum_{q} I_q = \sum_{|j-q| \leq 4} I_q \]
this yields
\[ \| I \|_{B^{0}_{p,r}} \lesssim \left( \sum_{q \geq -1} \| I_q \|_{L^p}^r \right)^{\frac{1}{r}} \]
\[ \lesssim \| \nabla v \|_{L^p} \| \theta \|_{B^{0}_{\infty,0,r}}. \]

Let us move to the second term II. As before one writes
\[ I_q(x) = h_q \ast (\Delta_q v \cdot \nabla S_{q-1} - \Delta_q v \cdot (h_q \ast \nabla S_{q-1})), \]
and then we obtain the estimate
\[ \| I_q \|_{L^p} \lesssim 2^{-q} (1 + |q|)^{\alpha} \| \Delta_q \nabla v \|_{L^p} \| S_{q-1} \nabla \theta \|_{L^\infty} \]
\[ \lesssim \| \nabla v \|_{L^p} \sum_{j \leq q-2} 2^j (1 + |j|)^{-\alpha} ((1 + |j|)^{\alpha} \| \Delta_j \theta \|_{L^\infty}). \]

Combined with Lemma 5.2 this yields
\[ \| I \|_{B^{0}_{p,r}} \lesssim \| \nabla v \|_{L^p} \| \theta \|_{B^{0}_{\infty,0,r}}. \]

Let us now deal with the third term III. Using the fact that the divergence of \( \Delta_q v \) vanishes, then we can write III as
\[ III = \sum_{q \geq 2} R_{\alpha} \text{div}(\Delta_q v \tilde{\Delta}_q \theta) - \sum_{q \geq 2} \text{div}(\Delta_q v R_{\alpha} \tilde{\Delta}_q \theta) + \sum_{q \leq 1} |R_{\alpha}, \Delta_q v \cdot \nabla| \tilde{\Delta}_q \theta \]
\[ = J_1 + J_2 + J_3. \]

Using Remark 4.2 we get
\[ \| \Delta_j R_{\alpha} \text{div}(\Delta_q v \tilde{\Delta}_q \theta) \|_{L^p} \lesssim 2^j (1 + |j|)^{\alpha} \| \Delta_q v \|_{L^p} \| \tilde{\Delta}_q \theta \|_{L^\infty}. \]

and since \( q \geq 2 \)
\[ \| \Delta_j \text{div}(\Delta_q v R_{\alpha} \tilde{\Delta}_q \theta) \|_{L^p} \lesssim 2^j \| \Delta_q v \|_{L^p} \| R_{\alpha} \tilde{\Delta}_q \theta \|_{L^\infty} \]
\[ \lesssim 2^j (1 + |q|)^{\alpha} \| \Delta_q v \|_{L^p} \| \tilde{\Delta}_q \theta \|_{L^\infty}. \]
Therefore we get
\[ \| \Delta_j (J_1 + J_2) \|_{L^p} \lesssim \sum_{q \in \mathbb{N}} 2^j (1 + |q|)^\alpha \| \Delta_q v \|_{L^p} \| \Delta_q \theta \|_{L^\infty} \]
\[ \lesssim \| \nabla v \|_{L^p} \sum_{q \in \mathbb{N}} 2^j (1 + |q|)^\alpha \| \Delta_q \theta \|_{L^\infty}, \]
where we have again used Bernstein inequality to get the last line. It suffices now to use Lemma 5.2
\[ \| J_1 + J_2 \|_{L^{p,r}} \lesssim \| \nabla v \|_{L^p} \| \theta \|_{L^{p',r}}. \]
For the last term \( J_3 \) we can write
\[ \sum_{-1 \leq q \leq 1} [ R_\alpha, \Delta_q v \cdot \nabla \bar{\Delta}_q \theta (x) ] = \sum_{q \leq 1} [ \text{div} \, \bar{\chi} (D) R_\alpha, \Delta_q v ] \bar{\Delta}_q \theta (x), \]
where \( \bar{\chi} \) belongs to \( D (\mathbb{R}^d) \). From the proof of Proposition 4.1 we get that \( \text{div} \, \bar{\chi} (D) R_\alpha \) is a convolution operator with a kernel \( \bar{h} \) satisfying
\[ | \bar{h} (x) | \lesssim (1 + |x|)^{-d-1}. \]
Thus \( J_3 = \sum_{q \leq 1} \bar{h} * (\Delta_q v \cdot \bar{\Delta}_q \theta) - \Delta_q v \cdot (\bar{h} * \bar{\Delta}_q \theta). \)
First of all we point out that \( \Delta_j J_3 = 0 \) for \( j \geq 6 \), thus we just need to estimate the low frequencies of \( J_3 \). Noticing that \( x \bar{h} \) belongs to \( L^{p'} \) for \( p' > 1 \) then using Lemma 5.1 with \( m = p \geq 2 \) we obtain
\[ \| \Delta_j J_3 \|_{L^p} \lesssim \sum_{q \leq 1} \| x \bar{h} \|_{L^{p'}} \| \Delta_q \nabla v \|_{L^p} \| \bar{\Delta}_q \theta \|_{L^p} \]
\[ \lesssim \| \nabla v \|_{L^p} \sum_{-1 \leq q \leq 1} \| \Delta_q \theta \|_{L^p}. \]
This yields finally
\[ \| J_3 \|_{L^{p,r}} \lesssim \| \nabla v \|_{L^p} \| \theta \|_{L^{p'}}. \]
This completes the proof of the first part of Theorem 5.3.
(2) The second part can be done in the same way so we will give here just a shorten proof.
To estimate the terms I and II we use two facts: the first one is \( \| \Delta_q \nabla u \|_{L^\infty} \approx \| \Delta_q \omega \|_{L^\infty} \) for all \( q \in \mathbb{N} \). The second one is
\[ \| \nabla S_{q-1} v \|_{L^\infty} \lesssim \| \nabla \Delta_{q-1} v \|_{L^\infty} + \sum_{j=0}^{q-2} \| \Delta_j \nabla v \|_{L^\infty} \]
\[ \lesssim \| \omega \|_{L^p} + q \| \omega \|_{L^\infty}. \]
Thus (15) becomes
\[ \| I_q \|_{L^\infty} \lesssim \| \omega \|_{L^\infty} (1 + |q|)^{1+\alpha} \| \Delta_q \theta \|_{L^\infty}. \]
and by Proposition 4.3

\[ \|I\|_{\mathcal{B}^\alpha_{\infty,r}} \leq \|\omega\|_{L^\infty}\|\theta\|_{\mathcal{B}^{\alpha,0}_{\infty,r}} \]

The second term II is estimated as follows

\[ \|II\|_{\mathcal{B}^\alpha_{\infty,r}} \leq \|\omega\|_{L^\infty}\|\theta\|_{\mathcal{B}^{\alpha,0}_{\infty,r}} \]

For the remainder term we do strictly the same analysis as before except for \( J_3 \): we apply Lemma 5.1 with \( p = \infty \) and \( m = \rho \) leading to

\[ \|\Delta_j J_3\|_{L^p} \lesssim \sum_{q \leq 1} \|x\tilde{h}\|_{L^{p'}} \|\Delta_q \nabla v\|_{L^\infty} \|\tilde{\Delta}_q \theta\|_{L^{p'}} \]

\[ \lesssim \|\nabla v\|_{L^\rho} \sum_{-1 \leq q \leq 1} \|\Delta_q \theta\|_{L^{p'}} \]

\[ \lesssim \|\omega\|_{L^\rho}\|\theta\|_{L^{p'}}. \]

This ends the proof of the theorem.

\[ \square \]

6. Smoothing effects

In this section we will describe some smoothing effects for the model (6) and focus only on the case \( \beta = 1 \). Remark that we can obtain similar results for the case \( \beta \in (0,1) \).

\[ (TD) \]

\[ \begin{cases} \partial_t \theta + v \cdot \nabla \theta + \frac{|D|}{\log^\alpha(\lambda + |D|)} \theta = f \\ \theta|_{t=0} = \theta^0. \end{cases} \]

We intend to prove the following smoothing effect.

**Theorem 6.1.** Let \( \alpha \geq 0, \lambda \geq e^{3+2\alpha}, d \in \{2,3\} \) and \( v \) be a smooth divergence-free vector field of \( \mathbb{R}^d \) with vorticity \( \omega \). Then, for every \( p \in [1,\infty[ \) there exists a constant \( C \) such that

\[ \sup_{q \in \mathbb{N}} 2^q(1 + q)^{-\alpha} \|\Delta_q \theta\|_{L^p_t L^1_x} \leq C \|\theta_0\|_{L^p} + C \|\theta_0\|_{L^\infty} \|\omega\|_{L^1_t L^p_x}. \]

for every smooth solution \( \theta \) of (TD) with zero source term \( f \).

**Remark 6.2.** For the sake of simplicity we state the result of smoothing effect only for \( \beta = 1 \) but the result remains true under the hypothesis of Proposition 4.8.

**Proof.** We start with localizing in frequencies the equation: for \( q \geq -1 \) we set \( \theta_q := \Delta_q \theta \). Then

\[ \partial_t \theta_q + v \cdot \nabla \theta_q + \frac{|D|}{\log^\alpha(\lambda + |D|)} \theta_q = -[\Delta_q, v \cdot \nabla] \theta. \]

Recall that \( \theta_q \) is real function since the functions involved in the dyadic partition of the unity are radial. Then multiplying the above equation by \( |\theta_q|^{p-2} \theta_q \), integrating by parts and using Hölder inequalities we get

\[ \frac{1}{p} \frac{d}{dt} \|\theta_q\|_{L^p_t}^p + \int_{\mathbb{R}^d} \left( \frac{|D|}{\log^\alpha(\lambda + |D|)} \theta_q \right) |\theta_q|^{p-2} \theta_q dx \leq \|\theta_q\|_{L^p_t}^{p-1} \|\Delta_q, v \cdot \nabla\theta\|_{L^p_x}. \]
Using Proposition 4.8 we get for \( q \geq 0 \)

\[
c2^q(1+q)^{-\alpha}\|\theta_q\|_{L^p}^p \leq \int_{\mathbb{R}^2} \left( \frac{|D|}{\log^\alpha(\lambda + |D|)} \theta_q \right) |\theta_q|^{p-2}\theta_q \ dx,
\]

where \( c \) depends on \( p \). Inserting this estimate in the previous one we obtain

\[
\frac{1}{p} \frac{d}{dt}\|\theta_q\|_{L^p}^p + c2^q(1+q)^{-\alpha}\|\theta_q\|_{L^p} \lesssim \|\theta_q\|_{L^p}^{p-1} \|[\Delta_q, v \cdot \nabla]\theta\|_{L^p}.
\]

Thus we find

\[
\frac{d}{dt}\|\theta_q\|_{L^p} + c2^q(1+q)^{-\alpha}\|\theta_q\|_{L^p} \lesssim \|[\Delta_q, v \cdot \nabla]\theta\|_{L^p}.
\]

To estimate the right hand-side we will use the following result, see Proposition 3.3 of [15].

\[
\|[\Delta_q, v \cdot \nabla]\theta\|_{L^p} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty, \infty}}.
\]

Combined with (16) this lemma yields

\[
\frac{d}{dt}(e^{ct2^q(1+q)^{-\alpha}}\|\theta_q(t)\|_{L^p}) \lesssim e^{ct2^q(1+q)^{-\alpha}}\|\nabla v(t)\|_{L^p}\|\theta(t)\|_{B^0_{\infty, \infty}}
\]

\[
\lesssim e^{ct2^q(1+q)^{-\alpha}}\|\omega(t)\|_{L^p}\|\theta_0\|_{L^\infty}.
\]

To get the last line, we have used the conservation of the \( L^\infty \) norm of \( \theta \) and the classical fact

\[
\|\nabla v\|_{L^p} \lesssim \|\omega\|_{L^p} \quad \forall p \in [1, +\infty[.
\]

Integrating the differential inequality we get for \( q \in \mathbb{N} \)

\[
\|\theta_q(t)\|_{L^p} \lesssim \|\theta_0\|_{L^p} e^{-ct2^q(1+q)^{-\alpha}} + \|\theta_0\|_{L^\infty} \int_0^t e^{-c(t-\tau)2^q(1+q)^{-\alpha}} \|\omega(\tau)\|_{L^p} d\tau.
\]

Integrating in time yields finally

\[
2^q(1+q)^{-\alpha}\|\theta_q\|_{L^1_{L^p}} \lesssim \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau
\]

\[
\lesssim \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau,
\]

which is the desired result. \( \square \)

7. Proof of Theorem 1.5

Throughout this section we use the notation \( \Phi_k \) to denote any function of the form

\[
\Phi_k(t) = C_0 \exp(\ldots \exp(C_0 t)\ldots),
\]

where \( C_0 \) depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentionning it) of the following trivial facts

\[
\int_0^t \Phi_k(\tau) d\tau \leq \Phi_k(t) \quad \text{and} \quad \exp\left( \int_0^t \Phi_k(\tau) d\tau \right) \leq \Phi_{k+1}(t).
\]
The proof of Theorem 1.5 will be done in several steps. The first one deals with some \textit{a priori} estimates for the equations (5). In the second one we prove the uniqueness part. Finally, we will discuss the construction of the solutions at the end of this section.

7.1. \textbf{A priori estimates}. In Theorem 1.5 we deal with critical regularities and one needs to bound the Lipschitz norm of the velocity in order to get the global persistence of the initial regularities. For this purpose we will proceed in several steps: one of the main steps to bound the Lipschitz norm of the velocity in order to get the global persistence of the Riesz transforms this will not be done in a straight way. We prove before an $L^p$ estimate for the vorticity with $2 < p < \infty$.

7.1.1. \textit{L}^p\textit{-estimate of the vorticity}. We intend now to bound the $L^p$-norm of the vorticity and to describe a smoothing effect for the temperature.

\textbf{Proposition 7.1.} Let $\alpha \in [0, \frac{1}{2}]$, $\lambda \geq e^{3+2\alpha}$ and $p \in [2, \infty]$. Let $(v, \theta)$ be a solution of (5) with $\omega^0 \in L^p$, $\theta_0 \in L^p \cap L^\infty$ and $\mathcal{R}_\alpha \theta_0 \in L^p$. Then for every $\epsilon > 0$

$$\|\omega(t)\|_{L^p} + \|\theta\|_{L^1_t B^{1-\epsilon}_{p,1}} \leq \Phi_2(t).$$

\textbf{Proof.} Applying the transform $\mathcal{R}_\alpha$ to the temperature equation we get

$$\frac{\partial}{\partial t} \mathcal{R}_\alpha \theta + v \cdot \nabla \mathcal{R}_\alpha \theta + \frac{|D|}{\log^\alpha (\lambda + |D|)} \mathcal{R}_\alpha \theta = -[\mathcal{R}_\alpha, v \cdot \nabla] \theta.$$  \hspace{1cm} (17)

Since $\frac{|D|}{\log^\alpha (\lambda + |D|)} \mathcal{R}_\alpha = \partial_t$, then the function $\Gamma := \omega + \mathcal{R}_\alpha \theta$ satisfies

$$\frac{\partial}{\partial t} \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}_\alpha, v \cdot \nabla] \theta.$$  \hspace{1cm} (18)

According to the first part of Proposition 5.3 applied with $r = 2$,

$$\| [\mathcal{R}_\alpha, v \cdot \nabla] \theta \|_{L^0_{t,2}} \leq \| \nabla v \|_{L^p} (\| \theta \|_{B^{1,\alpha}_{\infty,2}} + \| \theta \|_{L^p}).$$

Using the classical embedding $B^0_{2,2} \hookrightarrow L^p$ which is true only for $p \in [2, \infty)$

$$\| [\mathcal{R}_\alpha, v \cdot \nabla] \theta \|_{L^p} \leq \| \nabla v \|_{L^p} (\| \theta \|_{B^{0,\alpha}_{\infty,2}} + \| \theta \|_{L^p}).$$

Since $\text{div} \, v = 0$ then the $L^p$ estimate applied to the transport equation (18) gives

$$\| \Gamma(t) \|_{L^p} \leq \| \Gamma^0 \|_{L^p} + \int_0^t \| [\mathcal{R}_\alpha, v \cdot \nabla] \theta(\tau) \|_{L^p} d\tau.$$

Applying Theorem 3.1 to (17) yields

$$\| \mathcal{R}_\alpha \theta(t) \|_{L^p} \leq \| \mathcal{R}_\alpha \theta_0 \|_{L^p} + \int_0^t \| [\mathcal{R}_\alpha, v \cdot \nabla] \theta(\tau) \|_{L^p} d\tau.$$

We set $f(t) := \| \omega(t) \|_{L^p} + \| \mathcal{R}_\alpha \theta(t) \|_{L^p}$. Then from the previous estimates we get

$$f(t) \lesssim \| \Gamma^0 \|_{L^p} + \| \mathcal{R}_\alpha \theta_0 \|_{L^p} + \int_0^t \| \nabla v(\tau) \|_{L^p} (\| \theta(\tau) \|_{B^{1,\alpha}_{\infty,2}} + \| \theta \|_{L^p}) d\tau$$

$$\lesssim f(0) + \int_0^t f(\tau) (\| \theta(\tau) \|_{B^{1,\alpha}_{\infty,2}} + \| \theta_0 \|_{L^p}) d\tau.$$
We have used here two estimates: the Calderón-Zygmund estimate: for \( p \in (1, \infty) \)
\[
\|\nabla v\|_{L^p} \leq C \|\omega\|_{L^p}.
\]
The second one is \( \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} \) described in Theorem 3.1.
According to Gronwall lemma we get
\[
(19) \quad f(t) \lesssim f(0)e^{C\|\theta_0\|_{L^p}t} e^{C\|\theta\|_{L_1^p L_{\infty}^{0,\alpha}}}. \]
Let \( N \in \mathbb{N} \), then Bernstein inequalities and Theorem 3.1 give
\[
\|\theta\|_{L_1^p L_{\infty}^{0,\alpha}} \leq \|t\| \left( \sum_{q < N} (1 + |q|)^{2\alpha} \|\Delta_q \theta\|_{L_1^p L_{\infty}^{2}} \right)^{\frac{1}{2}} + \|(Id - S_N)\theta\|_{L_1^p B_{\infty,1}^{0,\alpha}}.
\]
Using Theorem 6.1 then for \( p > 2 \) and \( 0 < \epsilon < 1 - \frac{2}{p} \) we obtain
\[
\sum_{q \geq N} (1 + |q|)^{\alpha} 2^{\frac{q}{p}} \|\Delta_q \theta\|_{L_1^p L^p} \lesssim \sum_{q \geq N} (1 + |q|)^{\alpha} 2^{\alpha q} 2^{(\frac{q}{p} - 1)} \left( \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \|\omega\|_{L_1^p L^p} \right)
\]
\[
\lesssim \sum_{q \geq N} 2^{q(\frac{3}{2} + \epsilon - 1)} \left( \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \|\omega\|_{L_1^p L^p} \right)
\]
\[
\lesssim \|\theta_0\|_{L^p} + \|\theta_0\|_{L^\infty} \|\omega\|_{L_1^p L^p}.
\]
Consequently,
\[
\|\theta\|_{L_1^p B_{\infty,2}^{0,\alpha}} \lesssim N^{\frac{3}{2} + \alpha} \|\theta_0\|_{L^{\infty} t} + \|\theta_0\|_{L^p} + 2^{N(1+\epsilon + \frac{2}{p})} \|\theta_0\|_{L^\infty} \|\omega\|_{L_1^p L^p}.
\]
We choose \( N \) as follows
\[
N = \left\lfloor \log \left( e + \|\omega\|_{L_1^p L^p} \right) / (1 - \epsilon - 2/p) \log 2 \right\rfloor.
\]
This yields
\[
\|\theta\|_{L_1^p B_{\infty,2}^{0,\alpha}} \lesssim \|\theta_0\|_{L^\infty \cap L^p} + \|\theta_0\|_{L^\infty} t \log^{\frac{3}{2} + \alpha} \left( e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau \right).
\]
Combining this estimate with (19) we get
\[
\|\theta\|_{L_1^p B_{\infty,2}^{0,\alpha}} \lesssim \|\theta_0\|_{L^\infty \cap L^p} + \|\theta_0\|_{L^\infty} t \log^{\frac{3}{2} + \alpha} \left( e + C f(0)e^{C\|\theta_0\|_{L^p}t} e^{C\|\theta\|_{L_1^p B_{\infty,2}^{0,\alpha}}^2} \right)
\]
\[
\lesssim C_0 \log^{\frac{3}{2} + \alpha} (e + f(0)) (1 + t^{\frac{3}{2} + \alpha}) + C\|\theta_0\|_{L^\infty} t \|\theta\|_{L_1^p B_{\infty,2}^{0,\alpha}}^{\frac{1}{2} + \alpha},
\]
where \( C_0 \) is a constant depending on \( \|\theta_0\|_{L^p \cap L^\infty} \).

Case 1: \( \alpha < \frac{1}{2} \). We will use the following lemma.
Lemma 7.2. Let $a, b > 0$ and $\alpha \in [0, 1]$. Let $x \in \mathbb{R}_+$ be a solution of the inequality
\[(*) \quad x \leq a + bx^\alpha.\]
Then there exists $C := C(\alpha)$ such that
\[x \leq C_\alpha(a + b^{1-\alpha}).\]

Proof. We set $y = a^{-1}x$. Then the inequality $(*)$ becomes
\[y \leq 1 + ba^{-1}y^\alpha.\]
We will look for a number $\mu > 0$ such that $y \leq e^{\mu}$. Then it suffices to find $\mu$ such that
\[1 + ba^{-1}e^{\mu\alpha} \leq e^{\mu}.\]
It suffices also to find $\mu$ such that
\[(1 + ba^{-1})e^{\mu\alpha} = e^{\mu}.\]
This gives $e^{\mu} = (1 + ba^{-1})^{1-\alpha}$. Now we can use the inequality: for every $t, s \geq 0$
\[(t + s)^{1-\alpha} \leq C_\alpha(t^{1-\alpha} + s^{1-\alpha}).\]
It follows that
\[y \leq C_\alpha(1 + a^{-1}b^{1-\alpha}).\]
This yields
\[x \leq C_\alpha(a + b^{1-\alpha})\]
\[\square\]
Applying this lemma to (20) we get for every $t \in \mathbb{R}_+$
\[\|\theta\|_{L^1_t B^{0,\alpha}_{\infty,2}} \leq C_0(t^{\frac{1}{2}} + t^{\frac{2-2\alpha}{2}}) \leq C_0(1 + t^{\frac{2-2\alpha}{2}}) \leq \Phi_1(t).\]
(21)
It follows from (19)
\[f(t) \leq C_0e^{C_0t^{1-\alpha}} \leq \Phi_2(t).\]
(22)
Applying Theorem 6.1 and (22) we get for every $\epsilon > 0, q \in \mathbb{N}$
\[2^q(1 + |q|)^{-\alpha}\|\Delta_q \theta\|_{L^1_t L^p} \leq C_0e^{C_0t^{1-\alpha}} \leq \Phi_2(t).\]
(23)
Case 2: $\alpha = \frac{1}{2}$. The estimate (20) becomes
\[\|\theta\|_{L^1_t B^{0,1/2}_{\infty,2}} \leq C_0 \log(e + f(0))(1 + t^2) + C\|\theta_0\|_{L^\infty}t\|\theta\|_{L^1_t B^{0,1/2}_{\infty,2}}\]
with $C_0$ a constant depending on $\|\theta_0\|_{L^p \cap L^\infty}$. Hence if we choose $t$ small enough such that
\[C\|\theta_0\|_{L^\infty}t = \frac{1}{2},\]
\[2(1 + |q|)^{-\alpha}\|\Delta_q \theta\|_{L^1_t L^p} \leq C_0e^{C_0t^{1-\alpha}} \leq \Phi_2(t).\]
(23)
then
\[ \| \theta \|_{L^1_t B_{\infty,2}^0} \leq C_0 \log(e + f(0)). \]

From (19) we get that
\[ f(t) \leq C_0(e + f(0))^{C_0}. \]

Now let \( t \) be a given positive time and choose a partition \((t_i)_{i=1}^N\) of \([0, t]\) such that
\[ (25) \quad C \| \theta_0 \|_{L^\infty}(t_{i+1} - t_i) \approx \frac{1}{2}. \]

Set \( a_i := \int_{t_i}^{t_{i+1}} \| \theta(\tau) \|_{B_{\infty,2}^{0, \frac{1}{2}}} d\tau \) and \( b_i = f(t_i) \). Thus reproducing similar computations to (20) yields
\[ a_i \leq C_0 \log(e + b_i)(1 + (t_{i+1} - t_i)^2) + C \| \theta_0 \|_{L^\infty}(t_{i+1} - t_i)a_i. \]

Hence we get
\[ (26) \quad a_i \leq C_0 \log(e + b_i). \]

The analogous estimate to (19) is
\[ b_{i+1} \leq b_i e^{C(t_{i+1} - t_i)} \| \theta_0 \|_{L^p} e^{C a_i} \leq C_0 b_i e^{C a_i}. \]

Combining (26) and (27) yields
\[ b_{i+1} \leq C_0(e + b_i) C_0. \]

By induction we can prove that for every \( i \in \{1, \ldots, N\} \) we have
\[ b_i \leq C_0 e^{C_0 i} \]
and consequently from (26)
\[ a_i \leq C_0 e^{C_0 i}. \]

It follows that
\[ \| \theta \|_{L^1_t B_{\infty,2}^{0, \frac{1}{2}}} = \sum_{i=1}^{N} a_i \leq C_0 e^{C_0 N} \leq C_0 e^{C_0 t}. \]

We have used in the last inequality the fact that
\[ \sum_{i=1}^{N} \leq N \approx C_0 t \]
which is a consequence of (25). We have also obtained
\[ f(t) \leq C_0 e^{C_0 t}. \]

It is not hard to see that from (23) one can obtain that for every \( s < 1 \)
\[ (28) \quad \| \theta \|_{L^1_t B_{p,1}^s} \leq \| \theta \|_{L^1_t B_{p,\infty}^{0,0}} \leq \Phi_2(t). \]

This ends the proof of Proposition 7.1.
Remark 7.3. Combining (28) with Bernstein inequalities and the fact that \( p > 2 \) this yields
\[
\| \theta \|_{L^p_t B^s_{p,1}} \leq \Phi_2(t),
\]
for every \( \epsilon < 1 - \frac{2}{p} \).

\[ \square \]

7.1.2. \( L^\infty \)-bound of the vorticity. We will prove the following result.

**Proposition 7.4.** Let \( \alpha \in [0, \frac{1}{2}] \), \( \lambda \geq e^{3+2\alpha} \), \( p \in ]2, \infty[ \) and \((v, \theta)\) be a smooth solution of the system (5) such that \( \omega^0, \theta_0, R\theta_0 \in L^p \cap L^\infty \). Then we have
\[
\| \omega(t) \|_{L^\infty} + \| R\theta(t) \|_{L^\infty} \leq \Phi_3(t)
\]
and
\[
\| v(t) \|_{L^\infty} \leq \Phi_4(t).
\]

**Proof.** Proof of (30). By using the maximum principle for the transport equation (18), we get
\[
\| \Gamma(t) \|_{L^\infty} \leq \| \Gamma^0 \|_{L^\infty} + \int_0^t \| [R\theta, v \cdot \nabla] \theta(\tau) \|_{L^\infty} d\tau.
\]
Since the function \( R\theta \) satisfies the equation
\[
(\partial_t + v \cdot \nabla + |D| \log^\circ (\lambda + |D|)) R\theta = -[R\theta, v \cdot \nabla] \theta,
\]
then using Theorem 3.1 we get
\[
\| R\theta(t) \|_{L^\infty} \leq \| R\theta(t) \|_{L^\infty} + \int_0^t \| [R\theta, v \cdot \nabla] \theta(\tau) \|_{L^\infty} d\tau.
\]
Thus we obtain
\[
\| \Gamma(t) \|_{L^\infty} + \| R\theta(t) \|_{L^\infty} \leq \| \Gamma^0 \|_{L^\infty} + \| R\theta_0 \|_{L^\infty} + 2 \int_0^t \| [R\theta, v \cdot \nabla] \theta(\tau) \|_{L^\infty} d\tau
\]
= \[
C_0 + \int_0^t \| [R\theta, v \cdot \nabla] \theta(\tau) \|_{B^s_{p,1}} d\tau.
\]
It follows from Theorem 3.1, Proposition 5.3-(2) and Proposition 7.1
\[
\| \omega(t) \|_{L^\infty} + \| R\theta(t) \|_{L^\infty} \leq C_0 + \int_0^t \| \omega(\tau) \|_{L^\infty \cap L^p} (\| \theta(\tau) \|_{B^s_{p,1}} + \| \theta(\tau) \|_{L^p}) d\tau
\]
\[
\leq C_0 + \| \omega \|_{L^\infty L^p}(\| \theta \|_{L^1_t B^s_{p,1}} + t\| \theta_0 \|_{L^p})
\]
\[
+ \int_0^t \| \omega(\tau) \|_{L^\infty}(\| \theta(\tau) \|_{B^s_{p,1}} + \| \theta_0 \|_{L^p}) d\tau.
\]
Let \( 0 < \epsilon < 1 - \frac{2}{p} \) then using (29) we get
\[
\| \omega(t) \|_{L^\infty} + \| R\theta(t) \|_{L^\infty} \leq \Phi_2(t) + \int_0^t \| \omega(\tau) \|_{L^\infty}(\| \theta(\tau) \|_{B^s_{p,1}} + \| \theta_0 \|_{L^p}) d\tau.
\]
Therefore we obtain by Gronwall lemma and (29)
\[ \|\omega(t)\|_{L^\infty} + \|\mathcal{R}_\alpha \theta(t)\|_{L^\infty} \leq \Phi_3(t). \]

**Proof of (31).** Let \( N \in \mathbb{N} \) to be chosen later. Using the fact that \( \|\tilde{\Delta}_q v\|_{L^\infty} \approx 2^{-q} \|\tilde{\Delta}_q \omega\|_{L^\infty} \), we then get
\[ \|v(t)\|_{L^\infty} \leq \|\chi(2^N|D|)v(t)\|_{L^\infty} + \sum_{q \leq -N} 2^{-q} \|\tilde{\Delta}_q \omega(t)\|_{L^\infty} \]
\[ \leq \|\chi(2^N|D|)v(t)\|_{L^\infty} + 2^N \|\omega(t)\|_{L^\infty}. \]

Applying the frequency localizing operator to the velocity equation we get
\[ \chi(2^N|D|)v = \chi(2^N|D|)v_0 + \int_0^t \mathcal{P}\chi(2^N|D|)\theta(\tau)d\tau + \int_0^t \mathcal{P}\chi(2^N|D|)\text{div}(v \otimes v)(\tau)d\tau, \]
where \( \mathcal{P} \) stands for Leray projector. From Lemma 2.1, Calderón-Zygmund estimate and the uniform boundness of \( \chi(2^N|D|) \) we get
\[ \int_0^t \|\chi(2^N|D|)\mathcal{P}\theta(\tau)\|_{L^\infty}d\tau \lesssim 2^{-N^2/\alpha} \int_0^t \|\theta(\tau)\|_{L^p}d\tau \]
\[ \lesssim t\|\theta_0\|_{L^p}. \]

Using Proposition 3.9-(2) we find
\[ \int_0^t \|\mathcal{P}\chi(2^N|D|)\text{div}(v \otimes v)(\tau)\|_{L^\infty}d\tau \lesssim 2^N \int_0^t \|v(\tau)\|_{L^p}^2d\tau. \]

The outcome is
\[ \|v(t)\|_{L^\infty} \lesssim \|v_0\|_{L^\infty} + t\|\theta_0\|_{L^p} + 2^{-N} \int_0^t \|v(\tau)\|_{L^p}^2d\tau + 2^N \|\omega(t)\|_{L^\infty} \]
\[ \lesssim 2^{-N} \int_0^t \|v(\tau)\|_{L^\infty}^2d\tau + 2^N \Phi_3(t) \]
Choosing judiciously \( N \) we find
\[ \|v(t)\|_{L^\infty} \leq \Phi_3(t) \left( 1 + \left( \int_0^t \|v(\tau)\|_{L^\infty}^2d\tau \right)^{1/2} \right). \]

From Gronwall lemma we get
\[ \|v(t)\|_{L^\infty} \leq \Phi_4(t). \]

\( \square \)

7.1.3. **Lipschitz bound of the velocity.** Now we will establish the following result.

**Proposition 7.5.** Let \( \alpha \in \left[ 0, \frac{1}{2} \right], \lambda \geq e^{3+2\alpha}, p \in \left] 2, \infty \right[ \) and \( (v, \theta) \) be a smooth solution of the system (5) with \( \omega^0, \theta_0, \mathcal{R}_\alpha \theta_0 \in B^{\alpha,1}_{\infty,1} \cap L^p \). Then
\[ \|\mathcal{R}_\alpha \theta(t)\|_{B^{\alpha,1}_{\infty,1}} + \|\omega(t)\|_{B^{\alpha,1}_{\infty,1}} + \|v(t)\|_{B^{\alpha,1}_{\infty,1}} \leq \Phi_4(t). \]
Proof. Applying Corollary 3.17 to the equations (17) and (17), we obtain

$$||\Gamma(t)||_{B^0_{\infty,1}} + ||R_\alpha \theta(t)||_{B^0_{\infty,1}} \leq \left( C_0 + ||[R_\alpha, v \cdot \nabla] \theta||_{L^1_tB^0_{\infty,1}} \right) \left( 1 + ||\nabla v||_{L^1_tL^\infty} \right).$$

Thanks to Theorem 5.3, Propositions 7.4, 7.1 and (29) we get

$$||[R_\alpha, v \cdot \nabla] \theta||_{L^1_tB^0_{\infty,1}} \lesssim \int_0^t (||\omega(\tau)||_{L^\infty} + ||\omega(\tau)||_{L^p}) (||\theta(\tau)||_{B^0_{\infty,1}} + ||\theta(\tau)||_{L^p}) d\tau \lesssim \Phi_3(t).$$

By easy computations we get

$$||\nabla v||_{L^\infty} \leq \sum_{q \in \mathbb{N}} ||\Delta_q \nabla v||_{L^\infty} \lesssim ||\omega||_{L^p} + \sum_{q \in \mathbb{N}} ||\Delta_q \omega||_{L^\infty} \lesssim \Phi_2(t) + ||\omega(t)||_{B^0_{\infty,1}}.$$ \hspace{1cm} (33)

Putting together (32) and (33) leads to

$$||\omega(t)||_{B^0_{\infty,1}} \leq ||\Gamma(t)||_{B^0_{\infty,1}} + ||R_\alpha \theta(t)||_{B^0_{\infty,1}} \leq \Phi_3(t) \left( 1 + \int_0^t ||\omega(\tau)||_{B^0_{\infty,1}} d\tau \right).$$

Thus we obtain from Gronwall inequality

$$||\omega(t)||_{B^0_{\infty,1}} + ||R_\alpha \theta(t)||_{B^0_{\infty,1}} \leq \Phi_3(t).$$ \hspace{1cm} (34)

Coming back to (33) we get

$$||\nabla v(t)||_{L^\infty} \leq \Phi_4(t).$$

Let us move to the estimate of $v$ in the space $B^1_{\infty,1}$. By definition we have

$$||v(t)||_{B^1_{\infty,1}} \lesssim ||v(t)||_{L^\infty} + ||\omega(t)||_{B^0_{\infty,1}}.$$ \hspace{1cm}

Combined with (31) and (34) this yields

$$||v(t)||_{B^1_{\infty,1}} \leq \Phi(t).$$

The proof of Proposition 7.5 is now achieved. \hfill \Box

7.2. Uniqueness. We will show that the Boussinesq system (5) has a unique solution in the following function space

$$\mathcal{E}_T = (L^\infty_T B^0_{\infty,1} \cap L^1_T B^1_{\infty,1}) \times (L^\infty_T L^p \cap \widetilde{L}^1_T B^{1,-\alpha}_{p,\infty}), \quad 2 < p < \infty.$$ \hspace{1cm}

Let $(v^1, \theta^1)$ and $(v^2, \theta^2)$ be two solutions of (5) belonging to the space $\mathcal{E}_T$ and denote $v = v^2 - v^1$, $\theta = \theta^2 - \theta^1$.

Then we get

$$\begin{aligned}
\partial_t v + v^2 \cdot \nabla v &= -\nabla \pi - v \cdot \nabla v^1 + \theta e_2 \\
\partial_t \theta + v^2 \cdot \nabla \theta + \frac{|D|}{\log^* (|D| + |\theta|)} \theta &= -v \cdot \nabla \theta^1 \\
v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0.
\end{aligned}$$
According to Proposition 3.15 we have
\[
\|v(t)\|_{B^0_{\infty,1}} \leq C e^{CV(t)} \left( \|v_0\|_{B^0_{\infty,1}} + \|\nabla \pi\|_{L^1_t B^0_{\infty,1}} + \|v \cdot \nabla u^1\|_{L^1_t B^0_{\infty,1}} + \|\theta\|_{L^1_t B^0_{\infty,1}} \right),
\]
with \( V(t) = \|\nabla v^1\|_{L^1_t L^\infty} \). Straightforward computations using the incompressibility of the flows gives
\[
\nabla \pi = -\nabla \Delta^{-1} \text{div} (v \cdot \nabla (v^1 + v^2)) + \nabla \Delta^{-1} \partial_2 \theta = I + II.
\]
To estimate the first term of the RHS we use the definition
\[
\|I\|_{B^0_{\infty,1}} \lesssim \| (\nabla \Delta^{-1} \text{div} \Delta_\infty (v \otimes (v^1 + v^2)) \|_{L^\infty} + \|v \cdot \nabla (v^1 + v^2)\|_{B^1_{\infty,1}} \]
From Proposition 3.1-(2) of [15] and Besov embeddings we have
\[
\| (\nabla \Delta^{-1} \text{div} \Delta_\infty (v \otimes (v^1 + v^2)) \|_{L^\infty} \lesssim \|v \otimes (v^1 + v^2)\|_{L^\infty} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^0_{\infty,1}}.
\]
Using the incompressibility of \( v \) and using Bony’s decomposition one can easily obtain
\[
\|v \cdot \nabla (v^1 + v^2)\|_{B^0_{\infty,1}} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^1_{\infty,1}}.
\]
Putting together these estimates yields
\[
(35) \quad \|I\|_{B^0_{\infty,1}} \lesssim \|v\|_{B^0_{\infty,1}} \|v^1 + v^2\|_{B^1_{\infty,1}}.
\]
Let us now show how to estimate the second term \( II \). By using Besov embeddings and Calderón-Zygmund estimate we get
\[
\|II\|_{B^0_{\infty,1}} \lesssim \|\nabla \Delta^{-1} \partial_2 \theta\|_{B^2_{p,1}} \lesssim \|\theta\|_{B^2_{p,1}}.
\]
Combining this estimate with (35) yields
\[
\|v(t)\|_{B^0_{\infty,1}} \lesssim e^{CV(t)} \left( \|v_0\|_{B^0_{\infty,1}} + \int_0^t \|v(\tau)\|_{B^0_{\infty,1}} \left[ 1 + \|v^1, v^2(\tau)\|_{B^1_{\infty,1}}^2 \right] d\tau \right)
\]
\[
+ e^{CV(t)} \|\theta\|_{L^1_t B^2_{p,1}},
\]
where \( V(t) := \|(v^1, v^2)\|_{L^1_t B^1_{\infty,1}} \).
Now we have to estimate \( \|\theta\|_{L^1_t B^2_{p,1}} \). By applying \( \Delta_q \) to the equation of \( \theta \) and arguing similarly to the proof of Theorem 6.1 we obtain for \( q \in \mathbb{N} \)
\[
\|\theta_q(t)\|_{L^p} \lesssim e^{-c2^q(1+q)^{-\alpha}} \|\theta_0^q\|_{L^p} + \int_0^t e^{-c2^q(1+q)^{-\alpha}(t-\tau)} \|\Delta_q(v \cdot \nabla \theta^1)(\tau)\|_{L^p} d\tau
\]
\[
+ \int_0^t e^{-c2^q(1+q)^{-\alpha}(t-\tau)} \|v^2 \cdot \nabla, \Delta_q \theta(\tau)\|_{L^p} d\tau.
\]
Remark, first, that an obvious Hölder inequality yields that for every \( \varepsilon \in [0, 1] \) there exists an absolute constant \( C \) such that
\[
\int_0^t e^{-c \tau} 2^{q(1+q)-\alpha} \, d\tau \leq C t^{1-\varepsilon} 2^{-q\varepsilon} (1 + q)^{\alpha \varepsilon}, \quad \forall \, t \geq 0.
\]
Using this fact and integrating in time
\[
2^q \| \theta_q \|_{L^1_1 L^p} \lesssim (q + 1)^{\alpha} 2^{q(1-\frac{1}{p}+\frac{\varepsilon}{q})} \| \theta_q^0 \|_{L^p}
+ \frac{t^{1-\varepsilon}(q + 1)^{\alpha \varepsilon} 2^{q(\frac{1}{p}-\varepsilon+\frac{\varepsilon}{q})}}{\frac{q}{p} - \varepsilon - \varepsilon} \int_0^t \left( \| \Delta_q (v \cdot \nabla \theta^1) (\tau) \|_{L^p} + \| [v^2 \cdot \nabla, \Delta_q \theta] (\tau) \|_{L^p} \right) d\tau
\]
(37)
\[
= \frac{(q + 1)^{\alpha} 2^{q(1-\frac{1}{p}+\frac{\varepsilon}{q})}}{\frac{q}{p} - \varepsilon} \| \theta_q^0 \|_{L^p} + I_q(t) + II_q(t).
\]
Using Bony’s decomposition we get easily
\[
\left\| \Delta_q (v \cdot \nabla \theta^1) (t) \right\|_{L^p} \lesssim \| v(t) \|_{L^\infty} \sum_{j \leq q+2} 2^j \left\| \Delta_j \theta^1 (t) \right\|_{L^p}
+ 2^q \| v(t) \|_{L^\infty} \sum_{j \geq q-4} \| \Delta_j \theta^1 (t) \|_{L^p}
\]
\[
\lesssim \| v(t) \|_{L^\infty} \sum_{j \leq q+2} (1 + |j|)^{-\alpha} (2^j (1 + |j|)^{-\alpha}) \left\| \Delta_j \theta^1 (t) \right\|_{L^p}
+ \| v(t) \|_{L^\infty} \sum_{j \geq q-4} 2^{q-j} (1 + |j|)^{-\alpha} (2^j (1 + |j|)^{-\alpha}) \left\| \Delta_j \theta^1 (t) \right\|_{L^p}.
\]
Integrating in time we get
\[
I_q(t) \lesssim \frac{t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} 2^{q \left( \frac{2}{p} - \varepsilon \right)} (q + 1)^{1+\alpha (1+\varepsilon)} \| \theta^1 \|_{L^1_1 B^1_{\frac{1}{p}, \infty}^{-\alpha}}}{\frac{q}{p} - \varepsilon - \varepsilon}
+ \frac{t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta^1 \|_{L^1_1 B^1_{\frac{1}{p}, \infty}^{-\alpha}} 2^{q \left( \frac{2}{p} - \varepsilon + \frac{\varepsilon}{q} \right)} (q + 1)^{1+\alpha (1+\varepsilon)}}{\frac{q}{p} - \varepsilon - \varepsilon} \sum_{j \geq q-4} 2^{-j} (1 + |j|)^{\alpha}
\]
(38)
\[
\lesssim \frac{t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} 2^{q \left( \frac{2}{p} - \varepsilon \right)} (q + 1)^{1+\alpha (1+\varepsilon)} \| \theta^1 \|_{L^1_1 B^1_{\frac{1}{p}, \infty}^{-\alpha}}}{\frac{q}{p} - \varepsilon - \varepsilon}.
\]
To estimate the term \( II_q \) we use the following classical commutator ( since \( 2/p < 1 \), see [6]
\[
\| [v^2 \cdot \nabla, \Delta_q \theta] \|_{L^p} \lesssim 2^{-\frac{q}{p}} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta \|_{L^1_1 B^2_{p, 1}^{-\frac{2}{p}}}.
\]
Thus we obtain,
\[
II_q(t) \lesssim \frac{t^{1-\varepsilon} (q + 1)^{\alpha \varepsilon} 2^{-q \varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta \|_{L^1_1 B^2_{p, 1}^{-\frac{2}{p}}}.
\]
We choose \( \varepsilon \in [0, 1] \) such that \( \frac{2}{p} - \varepsilon < 0 \), which is possible since \( p > 2 \). Then combining (37), (38) and (39) we get
\[
\| \theta \|_{L^1_1 B^2_{p, 1}^{-\frac{2}{p}}} \lesssim \| \theta_0 \|_{L^p} + t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta^1 \|_{L^1_1 B^1_{\frac{1}{p}, \infty}^{-\alpha}} + t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta \|_{L^1_1 B^2_{p, 1}^{-\frac{2}{p}}}.
\]
It follows that there exists small \( \delta > 0 \) such that for \( t \in [0, \delta] \)
\[
\| \theta \|_{L^1_1 B^2_{p, 1}^{-\frac{2}{p}}} \lesssim \| \theta_0 \|_{L^p} + t^{1-\varepsilon} \| v \|_{L^\infty_{L^\infty} L^\infty} \| \theta^1 \|_{L^1_1 B^1_{\frac{1}{p}, \infty}^{-\alpha}}.
\]
Plugging this estimate into (36) we find
\[ \|v\|_{L^\infty_t B^{0}_{\infty,1}} \lesssim e^{CV(t)} (\|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p} + t\|v\|_{L^\infty_t B^0_{\infty,1}} + t\|v\|_{L^\infty_t L^\infty}\|\theta^1\|_{L^1_p B^{2,\infty}_{p,1}}). \]

If \( \delta \) is sufficiently small then we get for \( t \in [0, \delta] \)
\[ (40) \quad \|v\|_{L^\infty_t B^{0}_{\infty,1}} \lesssim \|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p}. \]
This gives in turn
\[ (41) \quad \|\theta\|_{L^1_t B^{2}_{p,1}} \lesssim \|v_0\|_{B^0_{\infty,1}} + \|\theta_0\|_{L^p}. \]
This gives in particular the uniqueness on \([0, \delta]\). Iterating this argument yields the uniqueness in \([0, T]\).

7.3. Existence. We consider the following system
\[
(B_n) \quad \begin{cases} 
\partial_t v_n + v_n \cdot \nabla v_n + \nabla \pi_n = \theta_n \epsilon_n \\
\partial_t \theta_n + v_n \cdot \nabla \theta_n + \frac{|D|}{\log^\lambda(1+|D|)} \theta_n = 0 \\
\text{div} v_n = 0 \\
v_n|_{t=0} = S_n v^0, \quad \theta_n|_{t=0} = S_n \theta^0.
\end{cases}
\]

By using the same method as [13] we can prove that this system has a unique local smooth solution \((v_n, \theta_n)\). The global existence of these solutions is governed by the following criterion: we can push the construction beyond the time \( T \) if the quantity \( \|\nabla v_n\|_{L^1_t L^\infty} \) is finite. Now from the \textit{a priori} estimates the Lipschitz norm can not blow up in finite time and then the solution \((v_n, \theta_n)\) is globally defined. Once again from the \textit{a priori} estimates we have for \( 2 < p < \infty \)
\[ \|v_n\|_{L^\infty_t B^0_{\infty,1}} + \|\omega_n\|_{L^\infty_t L^p} + \|\theta_n\|_{L^\infty_t X_p} \leq \Phi_4(T). \]

The space \( X_p \) was introduced before the statement of Theorem 1.5. It follows that up to an extraction the sequence \((v_n, \theta_n)\) is weakly convergent to \((v, \theta)\) belonging to \( L^\infty_t B^{1}_{\infty,1} \times L^\infty_t X_p \), with \( \omega \in L^\infty_t L^p \). For \((n, m) \in \mathbb{N}^2 \) we set \( v_{n,m} = v_n - v_m \) and \( \theta_{n,m} = \theta_n - \theta_m \) then according to the estimate (40) and (41) we get for \( T = \delta \)
\[ \|v_{n,m}\|_{L^\infty_t B^0_{\infty,1}} + \|\theta_{n,m}\|_{L^1_t B^{2}_{p,1}} \lesssim \|S_n v_0 - S_m v_0\|_{B^0_{\infty,1}} + \|S_n \theta_0 - S_m \theta_0\|_{L^p}. \]

This shows that \((v_n, \theta_n)\) is a Cauchy sequence in the Banach space \( L^\infty_t B^0_{\infty,1} \times L^1_t B^{2}_{p,1} \) and then it converges strongly to \((v, \theta)\). This allows to pass to the limit in the system \((B_n)\) and then we get that \((v, \theta)\) is a solution of the Boussinesq system (5).

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