QUANTUM APPROACH ON LINEAR COMBINATION OF HARMONIC UNIVALENT MAPPINGS

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Abstract. In this paper using \( q \)-calculus operator we obtain some sufficient conditions on \( f_1 \) and \( f_2 \) so that their linear combination \( f = tf_1 + (1-t)f_2, \ t \in [0, 1] \), is univalent and convex in the direction of the real axis. Some examples are also illustrated to support our main results.

Keywords: \( q \)-derivative operator; harmonic univalent functions; Linear combination.

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1. Introduction

The theory of \( q \)-calculus has motivated the researchers due to its applications in the field of physical sciences, specially in quantum physics. Jackson \cite{10,11} was the first to give some applications of \( q \)-calculus by introducing the \( q \)-analogues of derivative and integral. Jackson’s \( q \)-derivative operator \( \partial_q \) on a function \( h \) analytic in \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) is defined for \( 0 < q < 1 \), by

\[
\partial_q h(z) = \left\{ \begin{array}{ll}
h(z) - \frac{h(qz)}{(1-q)z} & z \neq 0, \\
h'(0) & z = 0.
\end{array} \right.
\]

For a power function \( h(z) = z^k, \ k \in \mathbb{N} = \{1, 2, 3, \ldots \} \),

\[
\partial_q h(z) = \partial_q (z^k) = [k]_q z^{k-1},
\]

\[
\int_0^z t^n dt_q = z(1 - q) \sum_{k=0}^{\infty} q^k (zq^k)^n = \frac{z^{n+1}}{(1+1)_q}.
\]

where \([k]_q\) is the \( q \)-integer number \( k \) defined by

\[
[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + q^2 + \ldots q^{k-1},
\]

For any non-negative integer \( k \) the \( q \)-number factorial is defined by

\[
[k]_q! = [1]_q[2]_q[3]_q \ldots [k]_q \quad ([0]_q! = 1).
\]
For more detailed study see \[1\]. Clearly, \( \lim_{q \to 1^-} |k|_q = k \) and \( \lim_{q \to 1^-} \partial_q h(z) = h'(z) \). Research work in connection with function theory and \( q \)-calculus was first introduced by Ismail et al. \[9\]. Recently, \( q \)-calculus is involved in the theory of analytic functions in the work \[7, 8, 15\]. But research on \( q \)-calculus in connection with harmonic functions is fairly new and not much published (one may find papers \[12, 16, 17, 19, 20, 22, 25\]).

Let \( \mathcal{H} \) denotes the class of complex-valued functions \( f = u + iv \) which are harmonic in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), where \( u \) and \( v \) are real-valued harmonic functions in \( \mathbb{D} \). Functions \( f \in \mathcal{H} \) can also be expressed as \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \), called the analytic and co-analytic parts of \( f \), respectively. The Jacobian of \( f = h + \overline{g} \in \mathcal{H} \) is given by \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \).

According to the Lewy's Theorem, every harmonic function \( f = h + \overline{g} \in \mathcal{H} \) is locally univalent and sense preserving in \( \mathbb{D} \) if and only if \( J_f(z) > 0 \) in \( \mathbb{D} \) which is equivalent to the existence of an analytic function \( \omega(z) = g'(z)/h'(z) \) in \( \mathbb{D} \) such that \[ |\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}. \]

The function \( \omega \) is called the dilatation of \( f \). By requiring harmonic functions to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principal, and zeros being isolated (see for detail \[4\]). The class of all univalent, sense preserving harmonic functions \( f = h + \overline{g} \in \mathcal{H} \), with the normalized conditions \( h(0) = 0 = g(0) \) and \( h'(0) = 1 \) is denoted by \( S_{\mathcal{H}} \). If the function \( f = h + \overline{g} \in S_{\mathcal{H}} \), then \( h \) and \( g \) are of the form

\[
 h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|b_1| < 1; z \in \mathbb{D}).
\]

A subclass of functions \( f = h + \overline{g} \in S_{\mathcal{H}} \) with the condition \( g'(0) = 0 \) (or \( \omega_f(0) = 0 \)) is denoted by \( S_{\mathcal{H}}^0 \). Further, the subclasses of functions \( f \) in \( S_{\mathcal{H}} (S_{\mathcal{H}}^0) \), denoted by \( K_{\mathcal{H}} \) (\( K_{\mathcal{H}}^0 \)), consists of functions \( f \) that map the unit disk \( \mathbb{D} \) onto a convex region.

**Definition 1.** [5] **Harmonic right half-plane mappings**

A mapping \( f = h + \overline{g} \) is said to be a harmonic right half-plane mapping if it maps \( \mathbb{D} \) onto a right-half plane

\[
 H_0 = \{ w : \Re(w) > -1/2 \}.
\]

For

\[
 h(z) + g(z) = \frac{z}{1 - z} \quad \text{and} \quad \omega(z) = g'(z) = -z \frac{h'(z)}{h'(z)}
\]

\[
 h'(z) + g'(z) = \frac{1}{(1-z)^2} \quad \text{and} \quad g'(z) = -zh'(z)
\]

we get

\[
 h'(z) = \frac{1}{(1-z)^3} \quad \text{and} \quad g'(z) = \frac{-z}{(1-z)^3}
\]

which on integration and normalization gives

\[
 h(z) = \frac{z - \frac{z^2}{2}}{(1-z)^2} \quad \text{and} \quad g(z) = \frac{-\frac{z^2}{2}}{(1-z)^2}.
\]
Thus the mapping \( f = h + \overline{g} \) given by
\[
f = \frac{z - \frac{z^2}{1 - z^2}}{(1 - z)^2} - \frac{\overline{z^2}}{(1 - z)^2}.
\]

(1.1)

The class of right half-plane mappings \( f \in S^0(H_0) \) that map \( \mathbb{D} \) onto \( f(\mathbb{D}) = H_0 = \{ w : \Re(w) > -1/2 \} \), and such a mapping clearly assumes the form
\[ h(z) + g(z) = \frac{z}{1 - z}. \]

We now recall fundamental result, called shearing theorem, due to Clunie and Sheil-Small as follows:

**Theorem 1.** [3] A locally univalent harmonic function \( f = h + \overline{g} \) in \( \mathbb{D} \) is a univalent mapping of real axis \( \mathbb{D} \) onto a domain convex in the direction of real axis if and only if \( h - g \) is an analytic univalent mapping of \( \mathbb{D} \) onto a domain convex in the direction of real axis.

In this paper using quantum approach we find the new harmonic functions. Such that, if we have an analytic function of the form

**Example 1.** \( h(z) - g(z) = z - \frac{1}{2}z^2 \) and \( \omega_q(z) = \frac{\partial_q h(z)}{\partial_q g(z)} = \frac{2}{2q}z \)
\[
\partial_q(h(z)) - \partial_q(g(z)) = 1 - \frac{2}{2q}z, \quad \partial_q(g(z)) = \frac{2}{2q}z\partial_q(h(z))
\]
we get
\[ \partial_q(h(z)) = 1 \]
which on \( q \)-integration and normalization gives
\[ h(z) = z, \quad g(z) = \frac{2}{4}z^2 \]
Thus the mapping \( f = h + \overline{g} \) given by
\[ f = z - \frac{2}{4}z^2 \]
On \( q \to 1^- \), we get the original function \( f = z - \frac{1}{2}z^2 \)

**Definition 2.** Harmonic \( q \) right half-plane mappings:
If we have an analytic function of the form
\[ h(z) + g(z) = \frac{z}{1 - z} \] and \( \omega_q(z) = \frac{z^2 - 2z + qz}{1 - qz} \).
Then
\[
\partial_q(h(z)) + \partial_q(g(z)) = \frac{1}{(1 - z)(1 - qz)}
\]
we get
\[ \partial_q(h(z)) = \frac{1}{(1 - z)^3} \]
which on \( q \)-integration and normalization gives
\[ h(z) = \sum_{n=0}^{\infty} \frac{(n + 1)(n + 2)}{2n + 1} \frac{1}{q} z^{n+1} \]
Using this we get
\[ g(z) = \sum_{n=0}^{\infty} \frac{2[n+1]_q - (n+1)(n+2)}{2[n+1]_q} z^{n+1} \]

Thus \( h \) and \( g \) of the form
\[ h(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2[n+1]_q} z^{n+1} \quad g(z) = \sum_{n=0}^{\infty} \frac{2[n+1]_q - (n+1)(n+2)}{2[n+1]_q} z^{n+1} \]

On \( q \to 1^- \), we get the mapping defined in (1.1).

**Lemma 1.** Let \( \Omega \subset \mathbb{C} \) be a domain convex in the direction of the real axis. Also let \( p \) be a real-valued continuous function in \( \Omega \). Then the mapping \( \omega \mapsto \omega + p(\omega) \) is univalent in \( \Omega \) if and only if it is locally univalent. If it is univalent, then its range is convex in the direction of the real axis.

**Lemma 2.** Let \( f \) be analytic function in \( \mathbb{D} \) with \( f(0) = 0 \) and \( f'(0) \neq 0 \). Suppose also that
\[ \varphi(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})} \quad (\theta \in \mathbb{R}; z \in \mathbb{D}) . \]

If
\[ \Re \left( \frac{zf'(z)}{\varphi(z)} \right) > 0 \quad (z \in \mathbb{D}) , \]

then \( f \) is convex in the direction of real axis.

Dorff and Rolf [5] applied another way of constructing a univalent harmonic map by taking two suitable harmonic maps \( f_1 \) and \( f_2 \) with same dilatations, whose linear combination \( f_3 = tf_1 + (1-t)f_2 \), \( t \in [0,1] \) is univalent and convex in the direction of the imaginary axis. Wang et al. [26] derived several sufficient conditions on harmonic univalent functions \( f_1 \) and \( f_2 \) so that their linear combination \( f = tf_1 + (1-t)f_2 \), \( t \in [0,1] \), is univalent and convex in the direction of the real axis. More results on the linear combination \( f \) of \( f_1 \) and \( f_2 \) may also be found in [6,13,21,23,24,26] etc. (also see the references cited in these). In this paper using quantum approach we find sufficient conditions on \( f_1 \) and \( f_2 \) so that their linear combination \( f = tf_1 + (1-t)f_2 \), \( t \in [0,1] \), is univalent and convex in the direction of the real axis. Some examples are also illustrated to support our main results.

2. MAIN RESULTS

First, we give \( q \)-analogue of Lemma 2

**Lemma 3.** Let the function \( f : \mathbb{D} \to \mathbb{C} \) be an analytic function with \( f(0) = 0 \) and \( \partial_q f(0) \neq 0 \). Suppose that
\[ \lim_{q \to 1^-} \varphi_q(z) = \varphi(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})} \quad (\theta \in \mathbb{R}; z \in \mathbb{D}) . \]  \tag{2.1}

If the function \( f \) satisfy
\[ \Re \left( \frac{z\partial_q(f(z))}{\varphi_q(z)} \right) > 0 \quad (z \in \mathbb{D}) , \]  \tag{2.2}

then the function \( f \) is convex in the direction of real axis.
Theorem 2. Let for \( j = 1, 2, f_j = h_j + \overline{f_j} \in S_H \) with
\[
F_j = h_j(z) - g_j(z).
\]
If \( \omega_{q_1}(z) = \omega_{q_2}(z) \) and satisfy the condition \( \Re \left( \frac{z \partial_q (F_j(z))}{\varphi_q(z)} \right) > 0 \) for some function \( \varphi_q(z) \) given by (2.2), then \( f_3 = tf_1 + (1-t)f_2, t \in [0, 1] \) is univalent and convex in the direction of real axis.

Proof. Since, for \( j = 1, 2, \omega_{q_j}(z) = \frac{\partial_q (g_j(z))}{\partial_q (h_j(z))} \), with \( |\omega_{q_j}(z)| < 1 \). If \( \omega_{q_1}(z) = \omega_{q_2}(z) \) then
\[
\omega_{q_3}(z) = \frac{t \partial_q (g_1(z)) + (1-t) \partial_q (g_2(z))}{t \partial_q (h_1(z)) + (1-t) \partial_q (h_2(z))}
= \frac{t \omega_{q_1}(z) \partial_q (h_1(z)) + (1-t) \omega_{q_1}(z) \partial_q (h_2(z))}{t \partial_q (h_1(z)) + (1-t) \partial_q (h_2(z))}
= |\omega_{q_1}(z)| < 1.
\]
To show \( f_3 \) is convex in the direction of real axis, we have to show by Lemma 3.
\[
\Re \left( \frac{z (\partial_q (h_j(z)) - \partial_q (g_j(z)))}{\varphi_q(z)} \right) > 0
\]
Now
\[
\Re \left( \frac{z (\partial_q (h_j(z)) - \partial_q (g_j(z)))}{\varphi_q(z)} \right)
= \Re \left( \frac{z}{\varphi_q(z)} (t \partial_q (h_1(z)) - \partial_q (g_1(z))) + (1-t) (\partial_q (h_2(z)) - \partial_q (g_2(z))) \right)
= t \Re \left( \frac{z}{\varphi_q(z)} (\partial_q (h_1(z))) + (1-t) \Re \left( \frac{z}{\varphi_q(z)} (\partial_q (h_2(z))) \right) \right)
> 0.
\]

A generalization of Theorem 2 may be given as follows:

Corollary 1. Let for \( j = 1, 2, ..., n, f_j = h_j + \overline{f_j} \in S_H \) with the condition (2.3).
If \( \omega_{q_1}(z) = \omega_{q_2}(z) = ... = \omega_{q_n}(z) \) and satisfy the condition \( \Re \left( \frac{z \partial_q (F_j(z))}{\varphi_q(z)} \right) > 0 \) for some function \( \varphi_q(z) \) given by (2.2). Then \( F = \sum_{j=1}^{n} t_j f_j, t_j \in [0, 1] \) such that \( \sum_{j=1}^{n} t_j = 1 \) is univalent and convex in the direction of real axis.

Theorem 3. Let for \( j = 1, 2, f_j = h_j + \overline{f_j} \in S_H \) be convex in the direction of the real axis. If \( \Re \left\{ (1 - \omega_{q_1} \overline{\omega_{q_2}}) \partial_q (h_1(z)) \partial_q (h_2(z)) \right\} \geq 0 \), then \( f_3 = tf_1 + (1-t)f_2, t \in [0, 1] \) is convex in the direction of real axis.

Proof. Since, for \( j = 1, 2, \omega_{q_j}(z) = \frac{\partial_q (g_j(z))}{\partial_q (h_j(z))} \), with \( |\omega_{q_j}(z)| < 1 \). We get
\[
|\omega_{q_3}(z)| = \left| \frac{t \partial_q (g_1(z)) + (1-t) \partial_q (g_2(z))}{t \partial_q (h_1(z)) + (1-t) \partial_q (h_2(z))} \right|
= \left| \frac{t \omega_{q_1}(z) \partial_q (h_1(z)) + (1-t) \omega_{q_1}(z) \partial_q (h_2(z))}{t \partial_q (h_1(z)) + (1-t) \partial_q (h_2(z))} \right|
\]
To show $|\omega_q| < 1$. We have to prove
\[
|t\partial_q(h_1(z)) + (1 - t)\partial_q(h_2(z))|^2 - |t\omega_q, \partial_q(h_1(z)) + (1 - t)\omega_q, \partial_q(h_2(z))|^2
\]
\[
= (t\partial_q(h_1(z)) + (1 - t)\partial_q(h_2(z))) (t\partial_q(h_1(z)) + (1 - t)\partial_q(h_2(z)))
\]
\[
- (t\omega_q, \partial_q(h_1(z)) + (1 - t)\omega_q, \partial_q(h_2(z))) (t\omega_q, \partial_q(h_1(z)) + (1 - t)\omega_q, \partial_q(h_2(z))
\]
\[
= t^2 (1 - |\omega_q|^2) |\partial_q(h_1(z))|^2 + (1 - t)^2 (1 - |\omega_q|^2) |\partial_q(h_2(z))|^2
\]
\[
+ 2t(1 - t)\Re \left\{ (1 - \omega_q, \overline{\omega_q}) \partial_q(h_1(z))\overline{\partial_q(h_2(z))} \right\}
\]
\[
> 0
\]

We use Lemma 1 and Theorem 1 which shows that, $F_j := h_j - g_j$ is univalent in $\Delta$ and $\Omega_j = F_j(\Delta)$ is convex in the direction of the real axis for each $j = 1, 2$. We may write $f_j = F_j + 2\Re(g_j)$ and
\[
f_j \left(F_j^{-1}(w)\right) = w + 2\Re \left(g_j \left(F_j^{-1}(w)\right)\right) = w + q_j(w)
\]
is univalent in $\Omega_j$ for each $j = 1, 2$, where $q_j(w)$ is real valued continuous function. Let $F := h - g$ and $\Omega = F(\Delta)$. Then
\[
f \left(F^{-1}(w)\right) = tf_1 \left(F_1^{-1}(w)\right) + (1 - t) f_2 \left(F_2^{-1}(w)\right)
\]
\[
= t(w + q_1(w)) + (1 - t) (w + q_2(w))
\]
\[
= w + tq_1(w) + (1 - t) q_2(w)
\]
\[
= w + q(w)
\]
is univalent in $\Omega$ which by Lemma 1 is convex in the direction of the real axis. □

3. Examples

In this section, we give some examples based on Theorem 2 and Theorem 3 to verify our main results.

**Example 2.** Let for $j = 1, 2, f_j = h_j + \overline{g_j} \in S_H$ with

\[
F_j = h_j(z) - g_j(z) = \frac{z}{1 - z}
\]

(3.1)

If $\omega_{q_1}(z) = \omega_{q_2}(z)$ and $f_3 = tf_1 + (1 - t)f_2$, $t \in [0, 1]$ is univalent and convex in the direction of real axis. By Theorem 2 we have
\[
\Re \left( \frac{z(\partial_q(h_1(z)) - \partial_q(g_1(z)))}{\varphi_q(z)} \right) > 0
\]

Since, $\partial_q(h_1(z)) - \partial_q(g_1(z)) = \frac{1}{(1 - z)(1 - z)}$, $\partial_q(h_2(z)) - \partial_q(g_2(z)) = \frac{1}{(1 - z)(1 - z)}$ and $\varphi_q(z) = \frac{1}{(1 - z)(1 - q z)}$.

**Example 3.** $f_1 = z - \frac{|z|^2}{2} \overline{z}$ and $f_2 = z + \frac{|z|^2}{2} \overline{z}$ and $f_3 = tf_1 + (1 - t)f_2$, $t \in [0, 1]$ with $\omega_{q_1}(z) = -\frac{|z|^2}{2} \overline{z}$ and $\omega_{q_2}(z) = \frac{|z|^2}{2} \overline{z}$

$|\omega_{q_1}(z)| \leq \frac{2}{3} |z| + (1 - t) \frac{3}{4} |z|^2 < \frac{2}{3} |z| + (1 - t) \frac{3}{4} |z| < \frac{2}{3} |z| + (1 - t) \frac{3}{4} |z| < 1$

and $\Re \left\{ (1 - \omega_{q_1, \overline{q_2}}) \partial_q(h_1(z))\overline{\partial_q(h_2(z))} \right\} = \Re \left( 1 + \frac{2}{3} |z| \overline{z} \right) > 0$. 

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