A BOUND ON GRASSMANNIAN CODES

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ABSTRACT. We give a new asymptotic upper bound on the size of a code in the Grassmannian space. The bound is better than the upper bounds known previously in the entire range of distances except very large values.

1. Introduction. Let be the Grassmann manifold, i.e., the set of \(k\)-planes passing through the origin in \(\mathbb{R}^n\). Our focus is the packing problem in \(G_{k,n}\), i.e., the problem of estimating the number of planes whose pairwise distances are bounded below by some given value \(\delta\), for a suitably defined distance function \(d(p,q)\). This problem has attracted attention in the recent years for several reasons. As a coding-theoretic (geometric) problem, it is a natural generalization of the coding problem for the projective space \(P \mathbb{R}^{n-1}\) and a closely related case of the sphere in \(\mathbb{R}^n\), both having long history in coding theory [4]. This problem arises also in engineering applications related to transmission of signals with multiple antennas in wireless environment [1]. Finally, [9] introduced a construction of Grassmannian packings which is closely related to the construction of quantum stabilizer codes, another subject of interest in recent years.

There are several possibilities to define a metric on \(G_{k,n}\) [5]. We consider the so-called chordal metric (projection 2-norm in the terminology of [5]), which can be defined in two equivalent ways. By a well-known fact [6], given two planes \(p, q \in G_{k,n}\) one can define \(k\) principal angles between them. This is done recursively as follows: take unit vectors \(x_1 \in p, y_1 \in q\) with the maximum possible angular separation and denote this angle by \(\theta_1\). In step \(i = 2, \ldots, k\), take the unit vectors \(x_i \in p, x_i \perp \langle x_1, \ldots, x_{i-1}\rangle\) and \(y_i \in q, y_i \perp \langle y_1, \ldots, y_{i-1}\rangle\) with the maximum possible angle between them and denote this angle by \(\theta_i\). In this way we obtain the set of principal angles \(0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \pi/2\); moreover, \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) form orthonormal bases in \(p\) and \(q\), respectively.

Let \(\sin \theta = (\sin \theta_1, \ldots, \sin \theta_k)\). For a matrix (vector) \(P\) let \(\|P\| = \sqrt{\sum_{i,j} P_{ij}^2}\) denote its Euclidean 2-norm. Define the chordal distance between \(p\) and \(q\) as follows: \(d(p,q) = \|\sin \theta\|\). It turns out [4] that the Grassmannian space with the chordal metric affords an isometric embedding in a sphere \(S_r\) of radius \(r = \sqrt{k(n-k)/n}\) in \(\mathbb{R}^{(n-1)(n+2)/2}\). To describe it, let \(A_p\) be a “generator matrix” of \(p\), i.e., a \(k \times n\) matrix whose rows form an orthonormal basis of \(p\). Then the orthogonal projection from \(\mathbb{R}^n\) on \(p\) can be written as \(\Pi_p = A_p^t A_p\). Define a map \(\Phi : G_{k,n} \rightarrow S_r\) as \(\Phi(p) = \Pi_p - k/n I_n\) (the plane is mapped to the traceless part of the projection on it). For any \(p\), the norm of \(\Phi(p)\) equals \(\|\Pi_p - k/n I_n\| = r\). The main result of [4] is that the mapping \(\Phi\) is an isometry in the sense that

\[
d^2(p,q) = 1/2\|\Pi_p - \Pi_q\|^2.
\]

We call a collection of \(M\) points in \(G_{k,n}\) with pairwise distances at least \(\delta\) an \((M, \delta)\) code in the Grassmannian space and call \(\delta\) the distance of the code. By [11] such a code gives rise to an \((M, \sqrt{2}\delta)\) code \(C \subset S_r\), so any upper bound on the distance of \(C\) gives an estimate on the distance of \(G_{k,n}\). In particular, by

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Rankin’s bounds \( R \) for any \((M, \delta)\) code \( G \),

\[ \delta \leq \left\{ \begin{array}{ll} \frac{k(n-k)}{n} & \text{if } M \leq n(n+1)/2, \\ \frac{k(n-k)}{n} & \text{if } M > n(n+1)/2. \end{array} \right. \]

These bound are tight in the sense that there exist codes that meet them with equality \([9, 4]\). However, in the majority of cases, particularly, for codes of the large size, direct application of bounds on spherical codes to codes in \( G_{k,n} \) gives poor results because the image of \( G_{k,n} \) on \( S_r \) forms a very sparse subset of it.

We will be concerned with asymptotic bounds on \( M \) for a given value of the code distance \( \delta \). Let \( R = R(\delta) = \lim \sup_{n \to \infty} (1/n) \ln M \) be the largest possible rate of a sequence of codes with distance \( \delta \) in \( G_{k,n} \). It is possible to compute the volume bounds on \( R \) analogous to the Gilbert-Varshamov and Hamming bounds of coding theory \([10]\). Namely, it is proved in \([3]\) that for all \( 0 \leq \delta \leq \sqrt{k} \),

\[ R_{GV}(\delta) \leq R \leq R_{H}(\delta), \]

where

\[ R_{GV}(\delta) = -k \ln(\delta/\sqrt{k}), \]

\[ R_{H}(\delta) = -k \ln \left( \sqrt{1 - \sqrt{1 - \delta^2/2k}} \right). \]

The upper bound was subsequently improved in \([2]\) relying on Delsarte’s linear programming method in the form developed in \([7]\). The result of \([2]\) is as follows:

\[ R \leq R_{LP}(\delta) := k[(1 + s) \ln(1 + s) - s \ln s], \]

where \( s = (k/2)(\sqrt{k} - 1) \). This bound coincides with the result of \([7]\) for \( k = 1 \) (codes in the projective space) and can be viewed as its generalization. For \( k = 1 \), \( R_{LP}(\delta) < R_{H}(\delta) \) for all \( 0 < \delta \leq 1 \). However, for greater \( k \) the inequality \( R_{LP}(\delta) < R_{H}(\delta) \) holds only for \( \delta \) close to \( \sqrt{k} \) and thus the linear programming bound provides a better estimate of \( R \) only for large values of the distance. For instance, for \( k = 2, 3 \) the crossing point is \( \delta = 0.74, 1.31 \) respectively. We note that \( R_{LP}(\sqrt{k}) = 0 \) showing that the lower bound \( R_{GV} \) is tight for \( \delta = \sqrt{k} \). In this note we establish an improved upper bound stated in the following theorem.

**Theorem 1.**

\[ R \leq R_{R}(\delta) := -k \ln \left( \sqrt{1 - \sqrt{1 - \delta^2/k}} \right). \]
Clearly \( R_R(\delta) < R_H(\delta) \) for all \( \delta \in (0, \sqrt{r}) \) and \( R_R(\sqrt{r}) = 0 \). Moreover, \( R_R(\delta) < R_{LP}(\delta) \) for most values of \( \delta \) except for values in a small neighborhood of \( \sqrt{r} \). The intersection point \( \delta^* \) of the curves \( R_R \) and \( R_{LP} \) is given in the following table.

| \( k \) | 2 | 3 | 4 | 5 | 10 |
|---|---|---|---|---|---|
| \( \delta^* \) | 1.37 | 1.717 | 1.992 | 2.231 | 3.161 |

The behavior of the bounds for \( k = 3 \) is shown in the figure.

2. Proof. The proof combines the isometric embedding of \( G_{k,n} \) in \( S_r \) with an application of Blichfeldt’s density method similar to the arguments of Rankin \[8\]. The intuition behind this method is as follows. Consider an \( (M, \delta) \) code \( G \subset G_{k,n} \). Denote by \( B_{\delta} = B_{\delta}(x) \) a metric ball in \( G_{k,n} \) with center at \( x \). Open balls of radius \( \delta/2 \) centered at code points do not intersect, so no point of \( G_{k,n} \) can be contained in more than one such ball. The idea is to extend the radius \( \delta/2 \) to some radius \( \rho \) so that while one point can belong to several balls, we can control the way the balls intersect and use some type of the volume argument to derive an upper bound on \( M \). This idea, first suggested by Blichfeldt, can be viewed as a precursor to the well-known Elias bound of coding theory (see e.g., \[10\] p.61).

Formally this idea is developed as follows. Under the mapping \( \Phi : G_{k,n} \rightarrow S_r \) an \( (M, \delta) \) code \( G \) is mapped to a spherical code \( A \) with minimum angular distance \( 2\alpha \), where \( \delta = \sqrt{2r} \sin \alpha \). Let \( \beta \) be the angle given by \( \sin \beta = \sqrt{2} \sin \alpha \) and let \( \rho = \sqrt{2r} \sin \beta/2 \). We compute

\[
\rho = r \sqrt{1 - \sin^2 \beta} = r \sqrt{1 - \frac{\delta^2}{r^2}}.
\]

Let \( p \in G \) be fixed and let \( q \in G_{k,n} \) be a point (plane) whose principal angles to \( p \) are given by \( \theta = (\theta_1, \ldots, \theta_k) \). Let \( d = \| \sin \theta \| \) be the value of the distance between \( p \) and \( q \). Consider the function on \( G_{k,n} \) defined by

\[
\sigma_p(q) = \begin{cases} 
\frac{2 \cos \beta}{r^2 \sin^2 \beta} (\rho^2 - d^2) & \text{if } d \leq \rho \\
0 & \text{if } d > \rho.
\end{cases}
\]

In other words, \( \sigma_p(q) \) can be viewed as a “density” defined on the metric ball \( B_\rho \subset G_{k,n} \) with center at a point \( p \in G \) and radius \( \rho \). It depends only on the distance to the center (is spherically symmetric).

Let us project the sphere \( S_r \) radially on the unit sphere in \( S \in \mathbb{R}^{(n-1)(n+2)/2} \) and denote the image of the code \( A \) by \( C \). Applying \( \Phi \) followed by the projection to the ball \( B_\rho \) with center at \( p \) transforms it into a cap on \( S \) with angular radius \( \beta \) and center at \( x = (1/r) \Phi(p) \) on the surface of the sphere. The linear radius of the cap equals \( P = 2 \sin \beta/2 \). Letting \( q \) be a plane at distance \( d \) from \( p \), we observe that the distance between \( x \) and \( z = \Phi(q) \) equals \( s = \sqrt{2d/r} \). The function \( \tau \) induces a function \( \sigma \) on this cap defined with respect to \( x \) by

\[
\sigma_x(z) = \frac{\cos \beta}{\sin^2 \beta} \left( P^2 - s^2 \right)
\]

for \( s \leq P \) and \( \sigma_x(z) = 0 \) otherwise. A point \( z \) can belong to several caps with centers at points of the code \( C \). The following lemma, whose proof is included for completeness, is due to \[8\].

**Lemma 2.** For any point \( z \in S \), its total density satisfies

\[
\sum_{x \in C} \sigma_x(z) \leq 1.
\]

**Proof:** Let \( C \subset S \) be a code with distance \( \tilde{\delta} \) and let \( z \in S \) be a point. Denote by \( x_1, \ldots, x_m \in C \) the code points whose distance to \( z \) is at most \( P \) and let \( d_1, \ldots, d_m \) be the values of these distances. We have

\[
\frac{1}{2} m(m-1) \tilde{\delta}^2 \leq \frac{1}{2} \sum_{i=1}^m \sum_{j,k=1}^m (x_{ij} - x_{ik})^2 = \sum_{i=1}^m \left\{ m \sum_j x_{ji}^2 - \left( \sum_k x_{ki} \right)^2 \right\}
\]
\[
\sum_{i=1}^{n} \left\{ m \sum_{j=1}^{m} x_{j,i}^2 - \left( \sum_{j=1}^{m} x_{j,i} \right)^2 \right\} = m \sum_{j} (1 - x_{j,1})^2 - \left( m - \sum_{j} x_{j,1} \right)^2 + \sum_{i=2}^{n} \left( m \sum_{j} x_{j,i}^2 - \left( \sum_{j} x_{j,i} \right)^2 \right)
\]

Without loss of generality let \( z = (1, 0, 0, \ldots, 0) \). Since

\[
d_j^2 = (1 - x_{j,1})^2 + x_{j,2}^2 + \cdots + x_{j,n}^2 = 2(1 - x_{j,1})
\]

we obtain the inequality

\[
\frac{1}{2} m(m - 1) \delta^2 \leq m \sum_{j} d_j^2 - \left( \sum_{j} d_j^2 \right)^2 - \sum_{i=2}^{n} \left( \sum_{j} x_{j,i} \right)^2
\]

which implies

(4) \[
\left( \sum_{j} d_j^2 \right)^2 - 4m \sum_{j} d_j^2 + 2m(m - 1) \delta^2 \leq 0.
\]

Let \( \alpha_z = \sum_{j=1}^{m} \sigma x_{j}(z) \). We have

\[
\alpha_z = \frac{\cos \beta}{\sin^2 \beta} \left( mP^2 - \sum_{j} d_j^2 \right)
\]

Then

\[
\sum_{j} d_j^2 = 4m \sin^2 \beta/2 - \alpha_z \sin \beta \tan \beta = 4 \sin^2 \beta/2 \left( m - \frac{1 + \cos \beta}{2 \cos \beta} \alpha_z \right).
\]

Using this in (4) we obtain

\[
16 \sin^4 \beta/2(m - 1/2(1 + \sec \beta)\alpha_z)^2 - 16m \sin^2 \beta/2(m - 1/2(1 + \sec \beta)\alpha_z)
\]

\[
+ 2m(m - 1) \delta^2 \leq 0.
\]

This inequality reduces to \( 4m(1 - \alpha_z) \geq \alpha_z \tan^2 \beta \) which implies the claim of the lemma. □

Therefore also for any point \( q \in G_{k,n} \)

\[
\sum_{p \in C} \tau_p(q) \leq 1.
\]

Let \( m(B_\rho) \) be the total mass of the ball computed with respect to the density \( \tau \). From the last inequality we obtain

(5) \[
M m(B_\rho) \leq \text{Vol}(G_{k,n})
\]

where \( \text{Vol}(G_{k,n}) \) is the total volume of the space. Let \( \mu(B_\rho) = m(B_\rho) / \text{Vol}(G_{k,n}) \) be the normalized mass. We assume that \( k < n/2 \). The volume form on (the open part of) \( G_{k,n} \) induces a distribution on the simplex of principal angles \( \Theta = \{ (\theta_1, \ldots, \theta_k) : \pi/2 > \theta_1 > \cdots > \theta_k > 0 \} \) given by

\[
\omega_{k,n} = K(k, n) \prod_{i=1}^{k} (\sin \theta_i)^{n-2k} \prod_{1 \leq i < j \leq k} (\sin^2 \theta_i - \sin^2 \theta_j) d\theta_1 \cdots d\theta_k,
\]

where \( K(k, n) \) is a constant chosen from the normalization condition \( \int_{G_{k,n}} \omega_{k,n} = 1 \) (see, e.g., [6]). Then

\[
\mu(B_\rho) = \int_{\theta : \| \sin \theta \| \leq \rho} \tau(\| \sin \theta \|) \omega_{k,n}.
\]

Asymptotic evaluation of an integral very similar to this one was performed in [3]. We state the result in the following lemma whose proof is analogous to [3].
Lemma 3. Let $k$ be fixed and $n \to \infty$. Then $$\mu(B_{\rho}) = \left(\frac{\rho}{\sqrt{k}}\right)^{nk+o(n)}.$$ Substituting the last formula in (5) and taking logarithms we obtain $$\frac{1}{n} \ln M \leq -k \ln \frac{\rho}{\sqrt{k}} + o(1).$$ Finally, using (3) and noting that $r \to \sqrt{k}$ as $n \to \infty$, we obtain the bound of the theorem. \[\blacksquare\]

The result of Theorem 1 can be also extended to the complex Grassmannian space similarly to an extension to this case of the bounds (2) in [3]. These estimates can also be extended to the quaternionic case according to the results of [11].

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