Realistic parameters of a continuous superradiant laser based on Sr-88

Mikkel Tang\textsuperscript{1,2}, Stefan A. Schäffer\textsuperscript{2}, and Jörg H. Müller\textsuperscript{1}
\textsuperscript{1}Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen, Denmark
\textsuperscript{2}Van der Waals-Zeeman Institute, Institute of Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands

The prospects of superradiant lasing on the 7.5 kHz wide \textsuperscript{1}S\textsubscript{0}-\textsuperscript{3}P\textsubscript{1} transition in \textsuperscript{88}Sr is explored by using numerical simulations of two systems based on realistic experimental numbers [1,2]. One system uses the idea of demonstrating continuous superradiance in a simple, hot atom beam with high flux [3], and the other system is based on using ultra-cold atoms in a dipole guide. We find that the hot beam system achieves lasing above \(3 \times 10^{12}\) atoms/s. It is capable of outputting hundreds of nW and suppressing cavity noise by a factor of 20-30. The second order Doppler shift causes a shift in the lasing frequency on the order of 500 Hz. For the cold atom beam we find that the output power is on the order of hundreds of pW, however the second order Doppler shift can be neglected, and cavity noise can be suppressed on the order of a factor 600.

I. INTRODUCTION

The precision of frequency references based on optical cavities is limited by thermal fluctuations in the mirrors [4]. This motivates the development of frequency references based on superradiant lasers with reduced sensitivity to cavity noise [5]. These operate using atomic transitions which are much more narrow than the cavity linewidth, so that the spectral properties of the atoms dominate over the resonator. Alkali-earth atoms such as strontium are a promising source due to a level structure allowing for efficient laser cooling and the availability of narrow, forbidden transitions. So far pulsed [6,11] and quasi-continuous superradiant lasing [9] has been demonstrated extensively, both experimentally and theoretically. Achieving continuous superradiant lasing experimentally is an ongoing challenge, but theoretical studies have both considered continuous superradiance in ultracold atom systems [5,12,17], and more recently in simpler systems with a high flux of hot atoms [3,18,20]. The aim of pushing towards the development of continuous superradiance has also sparked development in sources of atom beams [11,2] which can meet the technical requirements required for lasing on very narrow transitions.

Here we investigate the prospects of continuous superradiant lasing on the \textsuperscript{1}S\textsubscript{0}-\textsuperscript{3}P\textsubscript{1} intercombination line in \textsuperscript{88}Sr based on two concrete atomic beam sources developed at UvA [2,21]. We aim to capture all the vital physical effects present in a realistic physical system and bridging the gap to more idealized theoretical descriptions which often omit e.g. details about phase-space distributions, thermal effects, optical forces and realistic energy-level schemes for the sake of simplicity and computational advantages. We consider a cold atom system based on the source in [2], and a hot atom system [21] based on a recent theoretical proposal [3]. We use numerical simulations to investigate these two systems, the requirements to overcome the lasing threshold, and the influence of cavity fluctuations. In Section II we present a simple cooperativity model for the cold atom system which highlights the parameter regime that is necessary to obtain lasing. In Section III we introduce the full theoretical model used to study superradiant lasing quantitatively in the cold and hot atom systems. In Section IV and V we present the expected lasing dynamics and relevant physical effects for the cold and hot beam systems, respectively.

II. SYSTEM AND COOPERATIVITY MODEL

The systems we describe here consist of \textsuperscript{88}Sr atoms propagating through a cavity while initially in the excited \textsuperscript{3}P\textsubscript{1} state. In this section we will consider the cold-atom system. To achieve superradiant lasing, the atoms must preferentially emit into the cavity mode, and this process competes with emission into the environment and decoherence effects. In terms of the collective cooperativity we can write this as:

\[ C_0 \Gamma N \gg \Gamma_{\text{decoh}} \]  \hfill (1)

Where \(C_0\) is the single-atom cooperativity, \(N\) is the number of atoms, \(\Gamma\) is the decay rate of the energy level and \(\Gamma_{\text{decoh}}\) is the decoherence rate. In terms of \(\Gamma\), the cavity linewidth \(\kappa\) and the atom-cavity coupling \(g\), we have \(C_0 = 4g^2/\Gamma\kappa\). In a realistic scenario the cavity mode is a standing wave and has a Gaussian intensity distribution, giving a spatial dependence \(g(r,z)\). The atomic beam will have a Gaussian density profile in the plane perpendicular to its propagation axis characterized by standard deviation \(\sigma_y\) in the dimension that is also perpendicular to the cavity axis, leading to an overlap integral with the cavity mode. Thus if we have a beam of excited Sr atoms propagating at a speed \(v\), we can rewrite the lasing condition in terms of experimental parameters as:
The first fraction is constant for a given transition frequency $\omega$. In the second fraction we see that the cooperativity increases linearly with the flux, and decreases if we increase the mode volume (cavity length $L$ or waist $W$), linewidth (loss rate) and propagation velocity ($v$, which affects the density for a fixed flux). The term in the square root is a scaling between 0 and 1 related to the overlap of the atomic beam with the cavity: If the cavity waist is much larger than the atomic beam it yields 1, but if it is much smaller than the beam it approaches 0. The final term $\Gamma/\Gamma_{\text{decoh}}$ can approach 1/2 (for the lowest possible decoherence rate), but given a finite temperature $T$ and a repumping rate from the ground state the term will generally be smaller. If the lasing condition is fulfilled and the atom-cavity overlap is good, the output power from the cavity can approach the limit set by energy conservation of $P_{\text{max}} = \hbar \omega \Phi$. If repumping is included at the rate $\eta$ and we assume only atoms within the cavity waist participate in lasing, this limit increases to approximately $P_{\text{max}} = \hbar \omega \Phi(1 + 2\nu W/v)$.

Using a dipole guide enables confinement of atoms down to tens of micrometers, combined with propagation velocities on the order of 10 cm/s and mK temperatures. High atomic flux in such a system has been shown to be possible in [2]. Though we considered the cold atom system here, we can also note the most significant qualitative differences of the hot beam system: in this system the thermal decoherence rate and much higher propagation velocities lead to a much higher flux requirement to achieve lasing, and the output power when lasing cannot exceed $\hbar \omega \Phi$ due to the lack of repumping. However this value is still orders of magnitude higher than $P_{\text{max}}$ for the cold beam system due to the much higher flux.

III. THEORETICAL MODEL

Our model is based on a Tavis-Cummings Hamiltonian, including an array of filter cavities [11] [10] for extracting spectral information:

$$H = \hbar \omega_c a + \sum_{j=1}^{N} \hbar \omega_c \sigma_{e,c}^{j} + \sum_{k=1}^{N_f} \hbar \omega_{f,k} f_k^{\dagger} f_k$$
$$+ \frac{1}{2} \nu \eta \left( \alpha e^{-i\omega_{d}t} + a^{\dagger} e^{i\omega_{d}t} \right)$$
$$+ \sum_{j=1}^{N} \hbar g_{e,j}^{f} \left( \sigma_{e,c}^{j} + \sigma_{e,g}^{j} \right) (a + a^{\dagger})$$
$$+ \sum_{k=1}^{N_f} \hbar g_{f} (a + a^{\dagger}) \left( f_k + f_{k}^{\dagger} \right).$$

Here $\omega_c$ is the resonance frequency of the cavity (i=c), stationary atom (i=c), driving laser (i=d) or $k$’th filter cavity (i=f), $a$ ($f_k$) is the ($k$’th filter) cavity field annihilation operator and $\sigma_{e,g}^{j}$ is the spin operator of the $j$’th atom, $N$ is the number of atoms in the simulation and $N_f$ is the number of filter cavities chosen to obtain a given spectral resolution. The $j$’th atom couples to the cavity at a rate $g_{e,j}^{f}$ which depends on its position at a given time. Each filter cavity is coupled to the main cavity with a tiny constant $g_{f}$, such that physical back-action on the main cavity is neglected. $\eta$ is the rate of a driving laser that is used to initiate the lasing dynamics at the start of a simulation, and subsequently set to 0. This Hamiltonian is used to derive a set of differential equations for the atom and cavity states in first order mean-field theory, which are numerically integrated over time, while the propagation of each individual atom is treated classically. The rate of coupling to the environment (spontaneous emission and cavity leakage) is derived via Lindblad operators. This model has been presented in [10] [11] where it has been compared to experiments and used to describe pulsed lasing dynamics in the mK regime.

In contrast to previous applications of the model, we here describe a system where atoms continually enter and exit a cavity. Thus the number of atoms continually fluctuates: based on the atom flux a number of new atoms are periodically introduced, and existing atoms are deleted if they move significantly out of the cavity mode. Furthermore we here assume atoms are pumped incoherently, and we therefore include a classical driving laser to initiate the lasing dynamics - in reality this occurs due to quantum fluctuations, which are not included in 1st order mean field theory. This drive is turned off when we evaluate the steady-state parameters.

IV. LASING FROM A GUIDED ATOM BEAM

The flux demonstrated in [2] is $\Phi = 3 \times 10^7$ atoms/s, but later improvements to the experimental system have shown that up to $\Phi = 3 \times 10^8$ atoms/s is realistic [22]. We find that to reach the most interesting regime
FIG. 1. Illustration of the cold beam system, viewed along the cavity axis (a) and the repumping axis (b). Each atom is color coded by its state according to panel (c) and (d), where the shelving and pumping scheme is illustrated. The atoms are initially shelved in \( ^3P_0 \) and \( ^3P_2 \). Within the cavity they are pumped to \( ^3P_1 m_J=0 \) and contribute to lasing. An optical lattice in the cavity causes the atom beam to be focused towards the center of the cavity along the \( y \)-axis, while confining the atoms’ positions along the cavity axis. (c) Pumping scheme to store atoms in long-lived states before entering the cavity and after lasing. (d) Pumping scheme to transfer atoms to \( ^3P_1 m_J=0 \) for the lasing process, once they are in the cavity.

for superradiant lasing it is necessary to continuously repump the atoms. A repumping scheme which has been used to study quasi-continuous superradiant lasing in a MOT has been demonstrated in [9]. Since many recoils are imparted during each repumping cycle, the atoms can heat up significantly using this scheme. This makes it necessary to shelve the atoms in long-lived states until their position within the Gaussian mode provides a sufficiently high coupling rate for them to emit primarily into the cavity rather than the environment. We find that an optical lattice within the cavity mode further helps to reduce the effects of heating on the lasing characteristics. Based on this we consider the system depicted in Fig. 1. Here the atoms start in long-lived states and are continuously repumped within the cavity. To describe this the two-level model is extended to include \( ^3P_0 \) and the Zeeman sublevels of \( ^3P_1, ^3P_2 \) and \( ^3S_1 \), yielding a set of equations for the evolution of atom and cavity states given in Appendix A. Each atom is treated separately with its own position, velocity, coupling rate \( g_J \) to the cavity field, and incoherent driving rates from the repumping lasers. The initial atom velocities are based on a radial temperature of 890 nK and 29 \( \mu \)K along the propagation axis [2], and the distribution in the propagation direction is also treated as Gaussian, with a mean velocity of 0.084 m/s. At these low velocities, repumping can change the velocity of an atom significantly: an average of 21 photon recoils are imparted to bring an atom from \( ^1S_0 \) to \( ^3P_1 m_J = 0 \) (10 from repumping lasers and 11 from decays). To minimize heating along the cavity axis we therefore assume the repumping lasers are orthogonal to the cavity axis, and simulate heating from repumping by integrating the change in the \( ^3P_1 \) population due to repumping for each atom. Every time this value crosses an integer, it corresponds to a full repumping cycle, and 10 random recoils are then imparted to the atom along the repumping laser axis, and 11 recoils with random directions. In addition we include the forces of a dipole guide on the atoms (we assume a 913 nm guide beam with a waist of 165 \( \mu \)m and power of 15 W) and an intra-cavity optical lattice (10 W intra-cavity power), chosen to be near the magic wavelength of 913.9 nm. These forces are calculated on each atom depending on their position and internal state. Collisions between atoms are not taken into account.

The performance of the system can be evaluated in terms of the cavity pulling coefficient \( c_{pull} = \Delta \omega / \Delta \omega_c \); if this is 1, the cavity fully determines the lasing frequency, and if it is 0, the lasing frequency is determined fully by the atom transition, and the laser is maximally insensitive to cavity length fluctuations. In addition the output power is an important parameter, since a power in the pW regime can be demanding to detect and use as a frequency reference. A third parameter that is experimentally important is the cavity length. A short cavity length gives a stronger atom-cavity coupling, which means the finesse can be lowered to achieve the same lasing threshold, enabling one to operate further in the bad cavity regime. However a short cavity can also be more technically demanding e.g. for reaching magic wavelength conditions with the optical lattice, and due to finer control requirements for the piezo-voltage to keep fluctuations in the resonance frequency acceptable.

Assuming a cavity length of 25 mm and waist radius of 50 \( \mu \)m we find that the optimal cavity linewidth will be on the order of tens of MHz, and here we choose 20 MHz to obtain a lasing threshold below \( 10^7 \) atoms/s. The expected output power is shown in Fig. 2 for two different repumping rates. We find that the 10 W optical lattice reduces the threshold flux required for lasing by approximately a factor 2, which we attribute to locking each atom’s position along the cavity axis, preventing the coupling from changing sign. With this power the lattice potential depth is hundreds of \( \mu \)K at the center.
of the cavity waist, depending on the atom state. The radial forces of the lattice are also significant; atoms will be focused towards the waist, where they experience a stronger coupling, but will also pass through the cavity more quickly and have less time to emit photons because of this acceleration. However, the radial forces of the lattice can also temporarily hold on to some atoms that would otherwise have passed through or escaped after a few recoils from repumping. Over-all the steady-state number of atoms within the cavity is reduced slightly compared to the case with no lattice.

The cavity pulling coefficients achieved for the parameters explored here are on the order of 0.002, depending on the repumping rate (see Fig. 3). Due to the lattice preventing the atoms from crossing nodes and antinodes along the cavity axis, the cavity pulling is reduced on the order of a factor 4. The repumping rate mainly affects cavity pulling by determining the decoherence rate, but a higher repumping rate also increases the rate of heating, which will make it more likely to jump between lattice sites. Increasing the atom flux has a similar effect, because the Rabi frequency of the atom-cavity interaction increases, so more photons are emitted and more repumping cycles occur for a higher flux. The slightly increasing trend of the cavity pulling with atom flux seen in Fig. 3 may be attributed to this effect. Another experimental variable that can be optimized is the repumping laser waist radius. Here a value of 20 µm is chosen, such that only a narrow band of atoms near the center of the cavity interact. If this is increased the output power increases and lasing threshold lowers further, but the atoms will heat up more, which can increase cavity pulling if more atoms hop between lattice sites. If the repumping laser waist is extended significantly beyond the cavity waist, the atom beam will start to heat up and expand before entering the cavity while mostly emitting into the environment, which can be detrimental to the performance. Finally, different cavity parameters can also be considered. If a larger cavity linewidth is chosen, the cavity pulling coefficient can be reduced further, at the expense of a higher lasing threshold flux. Alternatively a shorter cavity length can be chosen, which allows for a larger cavity linewidth to yield less cavity pulling without a reduction in finesse and thus threshold.

The spatial dependence of the dynamics are shown in Fig. 4 here evaluated for Φ = 10⁸ atoms/s and a repumping rate of w/2π = 50 kHz. In panel (a) a breakdown of the atomic states is shown as a function of position in the beam. Here we see the effect of choosing a repumping waist of 20 µm, which causes all atoms outside the cavity waist to remain shelved. In panel (b) the average number of emission events from 3⁠P₁ into the cavity (red) or environment (dashed purple) is shown as function of position. For this flux and repumping rate, we see each atom can emit upwards of 40 photons into the cavity and 15 into the environment before escaping. This demands over 1100 recoils during the repumping process, highlighting the importance of this heating mechanism. Note that atoms escaping near x=300 µm are generally the fastest and will thus not emit as many photons as those that spend a long time inside the cavity waist, emit many photons, but eventually heat up and escape along the repumping axis. A few atoms also escape from the direction they originally came from and contribute to the nonzero average value for x<100 µm. In (c) we show the spatially dependent temperature profile in each dimension. One factor influencing this profile is the lattice, which accelerates atoms towards the center of the cavity. This contributes to the bump in the middle, however if all atoms simply passed through the lattice in ¹S₀

FIG. 2. The cavity output power dependency on atom flux Φ for two different repumping rates w out of ¹S₀. A flux of 10⁸ s⁻¹ corresponds to approximately 10⁸ atoms in the cavity waist. The points are connected by splines to guide the eye. A higher repumping rate increases the lasing threshold, output power and rate of heating from repumping.

FIG. 3. The shift in lasing frequency (∆L) and calculated cavity pulling coefficient for varying flux and two different repumping rates. Most atoms are prevented from moving along the cavity axis due to the intense optical lattice. This significantly reduces the influence of the finite temperature on the cavity pulling. Single simulations are connected by splines.
FIG. 4. Spatial profile of an actively lasing cold atom beam for a flux of $\Phi = 10^8$ atoms/s and pumping rate $w = 2\pi \times 50$ kHz. (a) Histogram of the spatial distribution of states (yellow is $m_J = -1$, others are indicated in the figure). Atoms start in $1S_0$ but are pumped to the long-lived states until they reach the cavity (waist: red dotted lines). Here one pumping laser (waist: green dotted lines) enables inversion on the lasing transition. (b) The mean accumulated number of emission events in the steady-state regime of a given atom as function of their x-position, throughout their trajectory up to that point. Note some atoms escape on the left and along the repumping axis near $x = 150 \mu$m after interacting with the cavity mode and pumping lasers, while faster atoms tend to interact less and escape on the right. In this regime on the order of 30 photons can be emitted from each atom into the cavity before exiting. (c) Temperature profile along each dimension ($T_x$: propagation axis, $T_y$: repumping axis, $T_z$: cavity axis) as function of position. The temperature profile is a result of interactions with the repumping lasers and optical lattice in the cavity.

they would undergo adiabatic expansion and compression, and the final value of $T_z$ and $T_x$ would not change significantly, while $T_y$ would change due to the focusing (this is also seen in Fig. 1b). However, the fact that the atoms are repumped and change state as they pass through the cavity results in non-adiabatic dynamics, because the potential depth is different for each state. Photon recoils from the repumping process mainly heats the atoms along the y axis, but also to a smaller degree along the x and z axes due to the spontaneously emitted photons.

V. LASING FROM A HOT ATOM BEAM

A promising source for superradiant lasing is a thermal beam of atoms emanating directly from an oven, cooled on the $1S_0-1P_1$ transition in a 2D molasses to bring the radial temperature to the mK regime. This concept has been explored theoretically in [3, 18–20]. Here we investigate this proposal with the physical constraints of a real system operating on the $1S_0-3P_1$ transition of $^{88}$Sr, illustrated in Fig. 5. We assume the atoms can start in the excited state $3P_1$ or ground state $1S_0$ and the atoms are treated as a two-level system. Due to the high number of atoms we use a clustering approach for the hot beam system, treating atoms in groups of 100 where each atom in a group has the same position, velocity and internal state. The atom velocities are drawn from Gaussian distributions based on a temperature of 3.6 mK orthogonally to the propagation axis, and from a thermal beam distribution in the propagation direction with a most probable velocity of 400 or 450 m/s. At these velocities the second order Doppler shift becomes significant, and is therefore included in the transition frequency of each atom, typically being on the order of -500 Hz.
We can expect that the cavity pulling can be reduced by preventing atoms from moving across the cavity axis, as the optical lattice did in the cold beam system. The effect of this motion has been studied in the hot beam regime, where a bistable regime was found when atoms collectively move across half a wavelength during transit\cite{20}. Thus to reduce the influence of atoms that move further than half a wavelength while traversing the cavity waist, we include a velocity selection stage \cite{1} in our simulations. In this scheme the atoms are initially shelved in a long-lived stage in a velocity-selective manner, according to the criterium \( |v_z| > \lambda \times v_{px}/4W \), where \( v_z \) is the velocity along the cavity axis, and \( v_{px} \) is the most probable velocity in the propagation direction. The atoms that move too quickly along the cavity axis thus remain in \(^1\text{S}_0\). These atoms are subsequently shifted in momentum space using a resonant laser on the \(^1\text{S}_0-^3\text{P}_1\) transition such that they do not interact with the cavity photons. This requires a Doppler shift significantly greater than the power broadening of the lasing transition due to the intracavity field, which a few m/s is sufficient to achieve in the regime considered here. Since this is done by a laser along the cavity (z) axis, the selection is imperfect, as some atoms may move slow enough along z to not be selected, but sufficiently slow along x that they still cross half a wavelength. The velocity selection is not simulated in detail here, but the criterium is merely used to determine which atoms start in \(^3\text{P}_1\) or \(^3\text{S}_0\).

Since multiple stages of loading atoms into magneto-optical traps and guided beams are not necessary, a much higher atomic flux can be obtained through the cavity than in the cold beam system, thus expected values are on the order of \(10^{12}-10^{13}\) atoms/s\cite{1}. For this system, using a cavity length of 25 mm, linewidth of \(2\pi \times 57\) MHz, we find a lasing threshold near \(\Phi = 3 \times 10^{12}\) atoms/s and on the order of 1 \(\mu\)W at \(10^{13}\) atoms/s (see Fig. 6).

As for the cold beam system we also evaluate the system performance in terms of cavity pulling. Due to the 2nd order Doppler shifts, cavity pulling will occur relative to a frequency that is shifted depending on the 2nd order Doppler shifts of the atoms in the ensemble. As the atoms interact differently depending on their velocity, and this interaction also depends on the atom flux, the exact ensemble resonance can depend on the 2nd order Doppler shifts and flux in a nontrivial way, but it will be close to the most probable 2nd order Doppler shift. Thus for the hot beam system we define the pulling coefficient \(c_{pull} = (\Delta_L - \Delta_D)/(\Delta_{ce} - \Delta_D) \approx (\Delta_L - \Delta_D)/\Delta_{ce} \), where \(\Delta_L\) is the shift in lasing frequency and \(\Delta_D\) is the shift in the atom ensemble resonance caused by the 2nd order Doppler shifts. The approximation holds for \(\Delta_{ce} \gg \Delta_D\). The cavity pulling characteristics are shown in Fig. 7 for a fixed cavity detuning of \(2\pi \times 100\) kHz. Here we find cavity pulling coefficients in the range of about 0.03 to 0.05. We see that the velocity selection scheme reduces cavity pulling on the order of 10\%, though in a system where the cutoff velocity is considered a free parameter one could reduce this further at higher flux values to reduce cavity pulling further, at the cost of a lower output power. In the simulations with \(v_{px} = 450\) m/s the velocity selection scheme has a smaller impact, as more atoms follow the selection criterium.

\section{VI. CONCLUSION}

We find that the hot beam approach can realistically produce output power of hundreds of nW to \(\mu\)W on the \(^1\text{S}_0-^3\text{P}_1\) transition in \(^{88}\text{Sr}\), and the hot beam system investigated here is capable of suppressing cavity noise by a factor of 20. The exact lasing frequency is shifted by approximately 500 Hz due to the 2nd order Doppler shift. On the other hand, this effect is negligible in the cold-atom system. This system is significantly more complex and relies on repumping of atoms within the optical cavity to provide a power on the order of hundreds of pW. However this system could realistically suppress cavity noise by a factor of 600 with the help of an intra-cavity lattice to reduce the thermal effects from repumping. Both approaches using the \(^3\text{P}_1\) state in Sr to generate
The lasing frequency shift at \( \Delta \pi = 100 \text{kHz} \) (kHz) is illustrated for the two velocities. The cavity pulling coefficients are below 0.06, and we find the velocity selection scheme reduces cavity pulling on the order of 10 \% in this regime. The simulations are connected by splines.

Superradiant lasing are promising as reference laser candidates that can provide output power levels high enough to be easily detectable, and are thus advantageous over much more narrow clock lines in this respect.

**ACKNOWLEDGMENTS**

The authors would like to thank Shayne Bennetts and Florian Schreck for useful discussions, and the rest of the iqClock group at UvA for their ideas and collaboration on the hot beam project. This project has received funding from the European Union’s (EU) Horizon 2020 research and innovation programme under grant agreement No 820404 (iqClock project), the USOQS project (17FUN03) under the EMPIR initiative, and the Q-Clocks project under the European Comission’s QuantERA initiative. It was additionally supported by a research grant 17558 from VILLUM FONDEN. SAS would like to thank the Independent Research Fund Denmark under project No. 0131-00023B.

**APPENDIX A: MODEL EQUATIONS**

The expectation value of the cavity photon number is \( \langle n \rangle \approx \langle a \rangle^\dagger \langle a \rangle \), where the expectation value of the lowering operator evolves according to:

\[
\dot{\langle a \rangle} = \left( i \Delta_{cd} - \frac{\kappa}{2} \right) \langle a \rangle - i \frac{\eta}{2} \sum_{j=1}^{N} g_j \langle \sigma_{ge}^j \rangle
\]

The detuning with respect to a driving laser (chosen as the rotating reference frame at \( \omega_d \)) is \( \Delta_{cd} = \omega_c - \omega_d \), the cavity linewidth is \( \kappa \) and driving laser parameter \( \eta \). This equation couples to the atomic coherences, which for the \( j \)’th atom evolves according to:

\[
\dot{\langle \sigma_{ge}^j \rangle} = -\left( i \Delta_{cd} + \frac{\Gamma_{eg} + w_{ge}^j}{2} \right) \langle \sigma_{ge}^j \rangle + ig_j \left( \langle \sigma_{ee}^j \rangle - \langle \sigma_{gg}^j \rangle \right) \langle a \rangle
\]

Where \( \Delta_{cd} = \omega_c - \omega_d \), and \( \omega_c \) is the atomic transition frequency. \( \Gamma = 2\pi \times 7.5 \text{kHz} \) is the exited state decay rate and \( w_j \) is the repumping rate out of \( ^1S_0 \). For each atom these equations are coupled to the equations for the 13 relevant atomic levels, which are internally coupled by the pumping scheme and decay paths. In the following equations, the subscripts correspond to: \( gg = ^1S_0, ee = ^3P_1 \ m_j = 0, ii = ^3P_1 \ m_j = -1, yy = ^3P_1 \ m_j = 1, ub = ^3S_1 \ m_j = -1, uc = ^3S_1 \ m_j = 0, ud = ^3S_1 \ m_j = 1, nm = ^3P_0, ma = ^3P_2 \ m_j = -2, mb = ^3P_2 \ m_j = -1, mc = ^3P_2 \ m_j = 0, md = ^3P_2 \ m_j = 1, me = ^3P_2 \ m_j = 2 \). For \( ^1S_0 \) and \( ^3P_1 \) we have:

\[
\begin{align*}
\langle \sigma_{gg}^j \rangle &= ig_j \left( \langle \sigma_{eg}^j \rangle \langle a \rangle - \langle \sigma_{ge}^j \rangle \langle a \rangle^\dagger \right) \\
&- w_{gi}^j \langle \sigma_{gg}^j \rangle + \Gamma_{eg} \left( \langle \sigma_{ii}^j \rangle + \langle \sigma_{ee}^j \rangle + \langle \sigma_{gg}^j \rangle \right) \\
\langle \sigma_{ei}^j \rangle &= ig_j \left( \langle \sigma_{ge}^j \rangle \langle a \rangle^\dagger - \langle \sigma_{eg}^j \rangle \langle a \rangle \right) \\
&- \Gamma_{eg} \langle \sigma_{ee}^j \rangle + \frac{\Gamma_{ue}}{2} \left( \langle \sigma_{ub}^j \rangle + \Gamma \langle \sigma_{ud}^j \rangle \right) \\
\langle \sigma_{ii}^j \rangle &= w_{gi}^j \langle \sigma_{gg}^j \rangle - \left( w_{ii}^j + \Gamma_{eg} \right) \langle \sigma_{ii}^j \rangle \\
&+ \frac{\Gamma_{ue}}{2} \left( \langle \sigma_{ub}^j \rangle + \sigma_{uc}^j \right) \\
\langle \sigma_{gg}^j \rangle &= - \left( w_{yu}^j + \Gamma_{eg} \right) \langle \sigma_{gg}^j \rangle + \frac{\Gamma_{ue}}{2} \left( \langle \sigma_{uc}^j \rangle + \langle \sigma_{ud}^j \rangle \right)
\end{align*}
\]
For $^3P_0$ and $^3P_2$ we have:

$$\langle \dot{\sigma}_{nn} \rangle = -w_{nu}^j \langle \sigma_{nn}^j \rangle + \Gamma_u \left( \langle \sigma_{ub}^j \rangle + \langle \sigma_{uc}^j \rangle + \langle \sigma_{ud}^j \rangle \right)$$

$$\langle \dot{\sigma}_{ma}^j \rangle = -w_{mu}^j \langle \sigma_{ma}^j \rangle + \frac{6}{10} \Gamma_{um} \langle \sigma_{ub}^j \rangle$$

$$\langle \dot{\sigma}_{mb}^j \rangle = -w_{mu}^j \langle \sigma_{mb}^j \rangle + \frac{3}{10} \Gamma_{um} \left( \langle \sigma_{ub}^j \rangle + \langle \sigma_{uc}^j \rangle \right)$$

$$\langle \dot{\sigma}_{mc}^j \rangle = -w_{mu}^j \langle \sigma_{mc}^j \rangle + \Gamma_{um} \left( \frac{1}{10} \langle \sigma_{ub}^j \rangle + \frac{4}{10} \langle \sigma_{uc}^j \rangle + \frac{1}{10} \langle \sigma_{ud}^j \rangle \right)$$

$$\langle \dot{\sigma}_{md}^j \rangle = -w_{mu}^j \langle \sigma_{md}^j \rangle + \frac{3}{10} \Gamma_{um} \left( \langle \sigma_{uc}^j \rangle + \langle \sigma_{ud}^j \rangle \right)$$

$$\langle \dot{\sigma}_{mc}^j \rangle = -w_{mu}^j \langle \sigma_{mc}^j \rangle + \frac{6}{10} \Gamma_{um} \langle \sigma_{ud}^j \rangle$$  \(7\)

And for $^3S_1$ we have:

$$\langle \dot{\sigma}_{ub}^j \rangle = \frac{w_{nu}^j}{3} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2} \langle \sigma_{ii}^j \rangle$$

$$+ w_{mu}^j \left( \frac{1}{2} \langle \sigma_{mb}^j \rangle + \frac{1}{6} \langle \sigma_{mc}^j \rangle \right) - \Gamma_u \langle \sigma_{ub}^j \rangle$$

$$\langle \dot{\sigma}_{uc}^j \rangle = \frac{w_{nu}^j}{3} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2} \langle \sigma_{ii}^j \rangle$$

$$+ \frac{w_{mu}^j}{2} \left( \frac{1}{2} \langle \sigma_{mb}^j \rangle + \frac{4}{6} \langle \sigma_{mc}^j \rangle + \frac{1}{2} \langle \sigma_{md}^j \rangle \right) - \Gamma_u \langle \sigma_{uc}^j \rangle$$

$$\langle \dot{\sigma}_{ud}^j \rangle = \frac{w_{nu}^j}{3} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2} \langle \sigma_{ii}^j \rangle$$

$$+ \frac{w_{mu}^j}{2} \left( \frac{1}{6} \langle \sigma_{mc}^j \rangle + \frac{1}{2} \langle \sigma_{md}^j \rangle + \langle \sigma_{me}^j \rangle \right) - \Gamma_u \langle \sigma_{ud}^j \rangle$$  \(8\)

Due to the large decay rate from $^3S_1$ the dynamics are simplified by setting $\langle \sigma_{ub}^j \rangle = \langle \sigma_{uc}^j \rangle = \langle \sigma_{ud}^j \rangle = 0$, giving:

$$\langle \dot{\sigma}_{ub}^j \rangle = \frac{w_{nu}^j}{3\Gamma_u} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2\Gamma_u} \langle \sigma_{ii}^j \rangle$$

$$+ \frac{w_{mu}^j}{\Gamma_u} \left( \frac{1}{2} \langle \sigma_{mb}^j \rangle + \frac{1}{6} \langle \sigma_{mc}^j \rangle + \frac{1}{2} \langle \sigma_{md}^j \rangle \right)$$

$$\langle \dot{\sigma}_{uc}^j \rangle = \frac{w_{nu}^j}{3\Gamma_u} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2\Gamma_u} \langle \sigma_{ii}^j \rangle + \frac{w_{mu}^j}{\Gamma_u} \left( \frac{1}{2} \langle \sigma_{mb}^j \rangle + \frac{4}{6} \langle \sigma_{mc}^j \rangle + \frac{1}{2} \langle \sigma_{md}^j \rangle \right)$$

$$\langle \dot{\sigma}_{ud}^j \rangle = \frac{w_{nu}^j}{3\Gamma_u} \langle \sigma_{nn}^j \rangle + \frac{w_{mu}^j}{2\Gamma_u} \langle \sigma_{ii}^j \rangle + \frac{w_{mu}^j}{\Gamma_u} \left( \frac{1}{6} \langle \sigma_{mc}^j \rangle + \frac{1}{2} \langle \sigma_{md}^j \rangle + \langle \sigma_{me}^j \rangle \right)$$  \(9\)
[12] G. A. Kazakov, T. Schumm, Active optical frequency standard using sequential coupling of atomic ensembles, Phys. Rev. A 87, 013821 (2013)
[13] G. A. Kazakov, T. Schumm, Active optical frequency standards using cold atoms: Perspectives and challenges, EFTF, Neuchatel, pp. 411-414 (2014)
[14] Y. Wang, and J. Chen, Superradiant Laser with Ultra-Narrow Linewidth Based on $^{40}\text{Ca}$, Chinese Phys. Lett. 29, 073202 (2012)
[15] J. Chen, X. Chen, Optical lattice laser, Proceedings of the 2005 IEEE International Frequency Control Symposium and Exposition (2005)
[16] K. Debnath, Y. Zhang, and K. Mølmer, Lasing in the superradiant crossover regime, Phys. Rev. A 98, 063837 (2018)
[17] Y. Zhang, C. Shan, and K. Mølmer, Ultranarrow Superradiant Lasing by Dark Atom-Photon Dressed States, Phys. Rev. Lett. 126, 123602 (2021)
[18] S. B. Jäger, H. Liu, J. Cooper, T. L. Nicholson, and M. J. Holland, Superradiant emission of a thermal atomic beam into an optical cavity, Phys. Rev. A 104, 033711 (2021)
[19] S. B. Jäger, H. Liu, J. Cooper, and M. J. Holland, Collective emission of an atomic beam into an off-resonant cavity mode, Phys. Rev. A 104, 053705 (2021)
[20] S. B. Jäger, H. Liu, A. Shankar, J. Cooper, and M. J. Holland, Regular and bistable steady-state superradiant phases of an atomic beam traversing an optical cavity, Phys. Rev. A 103, 013720 (2021)
[21] S. Zhou, F. Famà, C. B. Silva, M. Tang, Z. Zhang, S. A. Schäffer, C.-C. Chen, B. Pasquiou, S. Bennetts, and F. Schreck, Continuous superradiant laser based on a simple hot atomic beam, in 52nd Annual Meeting of the APS Division of Atomic, Molecular and Optical Physics, Volume 66, Number 6 (DAMOP 2021), [https://meetings.aps.org/Meeting/DAMOP21/Session/S03.4](https://meetings.aps.org/Meeting/DAMOP21/Session/S03.4)
[22] Private communications with Shayne Bennetts