UNIVERSAL MEAGER $F_\sigma$-SETS IN LOCALLY COMPACT MANIFOLDS

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Abstract. In each manifold $M$ modeled on a finite or infinite dimensional cube $[0,1]^n$, $n \leq \omega$, we construct a meager $F_\sigma$-subset $X \subset M$ which is universal meager in the sense that for each meager subset $A \subset M$ there is a homeomorphism $h : M \to M$ such that $h(A) \subset X$. We also prove that any two universal meager $F_\sigma$-sets in $M$ are ambiently homeomorphic.

In this paper we shall construct and characterize universal meager $F_\sigma$-sets in $\mathbb{I}^n$-manifolds.

A meager subset $A$ of a topological space $X$ is called universal meager if for each meager subset $B \subset X$ there is a homeomorphism $h : X \to X$ such that $h(B) \subset A$. So, each universal meager subset of $X$ contains homeomorphic copies of all other meager subsets of $X$.

In fact, the notion of a universal meager set is a special case of a more general notion of a $\mathcal{K}$-universal set for some family $\mathcal{K}$ of subsets of a topological space $X$. Namely, we define a set $U \in \mathcal{K}$ to be $\mathcal{K}$-universal if for each set $K \in \mathcal{K}$ there is a homeomorphism $h : X \to X$ such that $h(K) \subset U$.

$\mathcal{K}$-Universal sets for various classes $\mathcal{K}$ often appear in topology. A classical example of such set is the Sierpiński Carpet $M_2^2$, known to be a $\mathcal{K}$-universal set for the family $\mathcal{K}$ of all (closed) nowhere dense subsets of the square $\mathbb{I}^2 = [0,1]^2$ (see [14]). The Sierpiński Carpet $M_2^2$ is one of the Menger cubes $M_k^n$, which are $\mathcal{K}$-universal for the family $\mathcal{K}$ of all $k$-dimensional compact subsets of the $n$-dimensional cube $\mathbb{I}^n$ (see [13], [8, §4.1]). An analogue of the Sierpiński Carpet exists also in the Hilbert cube $\mathbb{I}^\omega$, which contains a $\mathcal{Z}_0$-universal set for the family $\mathcal{Z}_0$ of closed nowhere dense subsets of $\mathbb{I}^\omega$ (see [3]).

Many $\mathcal{K}$-universal spaces arise in infinite-dimensional topology. For example, the pseudo-boundary $B(\mathbb{I}^\omega) = [0,1]^{\omega} \setminus (0,1)\omega$ of the Hilbert cube $\mathbb{I}^\omega$ is known to be $\sigma \mathcal{Z}_\omega$-universal for the family $\sigma \mathcal{Z}_\omega$ of $\sigma \mathcal{Z}_\omega$-subsets of $\mathbb{I}^\omega$.

What is surprising, up to an ambient homeomorphism, $B(\mathbb{I}^\omega)$ is a unique $\sigma \mathcal{Z}_\omega$-universal set in $\mathbb{I}^\omega$. In this paper we shall show that such a uniqueness theorem also holds for $\sigma \mathcal{Z}_0$-universal subsets in the Hilbert cube $\mathbb{I}^\omega$.

Let us recall the definition of the families $\sigma \mathcal{Z}_\omega$ and $\sigma \mathcal{Z}_0$. They consist of $\sigma \mathcal{Z}_\omega$-sets and $\sigma \mathcal{Z}_0$-sets, respectively.

A closed subset $A \subset X$ of a topological space $X$ is called a $\mathcal{Z}_n$-set in $X$ for a (finite or infinite) number $n \leq \omega$ if the set $\{f \in C(\mathbb{I}^n, X) : f(\mathbb{I}^n) \cap A = \emptyset\}$ is dense in the space $C(\mathbb{I}^n, X)$ of all continuous functions $f : \mathbb{I}^n \to X$, endowed with the compact-open topology. Here by $\mathbb{I} = [0,1]$ we denote the unit interval and by $\mathbb{I}^n$ the $n$-dimensional cube. For $n = \omega$ the space $\mathbb{I}^\omega = \mathbb{I}^\omega$ is the Hilbert cube.

A subset $A \subset X$ is called a $\sigma \mathcal{Z}_n$-set in $X$ if $A$ can be written as the union $A = \bigcup_{k \in \omega} A_k$ of countably many $\mathcal{Z}_n$-sets $A_k \subset X$. Let us observe that a subset $A \subset X$ is a $\mathcal{Z}_0$-set in $X$ if and only if it is closed and nowhere dense in $X$, and $A$ is a $\sigma \mathcal{Z}_0$-set if and only if $A$ is a meager $F_\sigma$-set in $X$.

For a topological space $X$ by $\mathcal{Z}_n$ and $\sigma \mathcal{Z}_n$ we denote the families of $\mathcal{Z}_n$-sets and $\sigma \mathcal{Z}_n$-sets in $X$, respectively.

A characterization of $\sigma \mathcal{Z}_\omega$-universal sets in the Hilbert cube is quite simple and can be easily derived from the $\mathcal{Z}$-Set Unknotting Theorem 11.1 from [7].

Proposition 1. A subset $A \subset \mathbb{I}^\omega$ is $\mathcal{Z}_\omega$-universal in $\mathbb{I}^\omega$ if and only if $A$ is a $\mathcal{Z}_\omega$-set in $\mathbb{I}^\omega$, containing a topological copy of the Hilbert cube $\mathbb{I}^\omega$.

A characterization of $\sigma \mathcal{Z}_\omega$-universal sets in the Hilbert cube also is well-known and can be given in many different terms (skeletoid of Bessaga-Pelczyński [4], capssets of Anderson [1], [6], absorptive sets of West [10], pseudoboundaries of Geoghegan and Summerhill [11], [12]). For our purposes the most appropriate approach is that of West [10] and Geoghegan and Summerhill [12]. To formulate this approach, we need to recall some notation.

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Let $\mathcal{U}, \mathcal{V}$ be two families of sets of a topological space $X$. Put

$$\mathcal{U} \cap \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V}, U \cap V \neq \emptyset \}$$

and

$$\mathcal{U} \cup \mathcal{V} = \{ U \cup V : U \in \mathcal{U}, V \in \mathcal{V}, U \cap V \neq \emptyset \}.$$ 

We shall write $\mathcal{U} \prec \mathcal{V}$ and say that $\mathcal{U}$ refines $\mathcal{V}$ if each set $U \in \mathcal{U}$ is contained in some set $V \in \mathcal{V}$. Let $\text{St}(\mathcal{U}, \mathcal{V}) = \{ \text{St}(U, V) : U \in \mathcal{U} \}$ where $\text{St}(U, V) = \bigcup \{ V \in \mathcal{V} : U \cap V \neq \emptyset \}$. Put $\text{St}(\mathcal{U}) = \text{St}(\mathcal{U}, \mathcal{U})$ and $\text{St}^{n+1}(\mathcal{U}) = \text{St}(\text{St}^n(\mathcal{U}))$ for each $n > 0$. We shall say that two maps $f, g : Z \to X$ are $\mathcal{U}$-near and denote it by $(f, g) \prec \mathcal{U}$ if the family $(f, g) = \{ (f(z), g(z)) : z \in Z \}$ refines the family $\mathcal{U} \cup \{ \{ x \} : x \in X \}$. For a family $\mathcal{F}$ of subsets of a metric space $(X, d)$ we put $\text{mesh}(\mathcal{F}) = \sup_{x \in X} \text{diam}(F)$.

Let $\mathcal{K}$ be a family of closed subsets of a Polish space $X$ and $\sigma \mathcal{K} = \{ \bigcup_{n \in \omega} A_n : A_n \in \mathcal{K}, n \in \omega \}$. We shall say that $\mathcal{K}$ is topologically invariant if $\mathcal{K} = \{ h(K) : K \in \mathcal{K} \}$ for each homeomorphism $h : X \to X$.

A subset $B \subset X$ is called $\mathcal{K}$-absorptive in $X$ if $B \in \sigma \mathcal{K}$ and for each set $K \subset \mathcal{K}$, open set $V \subset X$, and open cover $\mathcal{U}$ of $V$ there is a homeomorphism $h : V \to V$ such that $h(K) \subset B \cap V$ and $(h, \text{id}) \prec \mathcal{U}$. An important observation is that each set $A \in \sigma \mathcal{K}$ containing a $\mathcal{K}$-absorptive subset of $X$ is also $\mathcal{K}$-absorptive.

The following powerful uniqueness theorem was proved by West [16] and Geoghegan and Summerhill [12, 2.5].

**Theorem 1** (Uniqueness Theorem for $\mathcal{K}$-absorptive sets). Let $\mathcal{K}$ be a topologically invariant family of closed subsets of a Polish space $X$. Then any two $\mathcal{K}$-absorptive sets $B, B' \subset X$ are ambiently homeomorphic. More precisely, for any open set $V \subset X$ and any open cover $\mathcal{U}$ of $V$ there is a homeomorphism $h : V \to V$ such that $h(V \cap B) = V \cap B'$ and $h$ is $\mathcal{U}$-near to the identity map of $V$.

Two subsets $A, B$ of a topological space $X$ are called ambently homeomorphic if there is a homeomorphism $h : X \to X$ such that $h(A) = B$. This happens if and only if the pairs $(X, A)$ and $(X, B)$ are homeomorphic. We shall say that two pairs $(X, A)$ and $(Y, B)$ of topological spaces $A \subset X$ and $B \subset Y$ are homeomorphic if there is a homeomorphism $h : X \to Y$ such that $h(A) = B$. In this case we say that $h : (X, A) \to (Y, B)$ is a homeomorphism of pairs.

According to the following corollary of Theorem 1, each $\mathcal{K}$-absorptive set is $\sigma \mathcal{K}$-universal.

**Corollary 1.** Let $\mathcal{K}$ be a topologically invariant family of closed subsets of a Polish space. If a $\mathcal{K}$-absorptive set $B$ in $X$ exists, then a subset $A \subset X$ is $\sigma \mathcal{K}$-universal in $X$ if and only if $A$ is $\mathcal{K}$-absorptive.

**Proof.** Assume that a subset $A$ of $X$ is $\mathcal{K}$-absorptive. The definition implies that $A \in \sigma \mathcal{K}$. To show that $A$ is $\sigma \mathcal{K}$-universal, fix any subset $K \in \mathcal{K}$. The definition of a $\mathcal{K}$-absorptive set implies that the union $A \cup K$ is $\mathcal{K}$-absorptive. By the Uniqueness Theorem 1 there is a homeomorphism of pairs $h : (X, A \cup K) \to (X, A)$. This homeomorphism embeds the set $K$ into $A$, witnessing that the $\mathcal{K}$-absorptive set $A$ is $\sigma \mathcal{K}$-universal.

Now assume that a set $A \subset X$ is $\sigma \mathcal{K}$-universal. Since the $\mathcal{K}$-absorptive set $B$ belongs to the family $\sigma \mathcal{K}$, there is a homeomorphism $h : X \to X$ such that $h(B) \subset A$. The topological invariance of the class $\mathcal{K}$ implies that the set $h(B)$ is $\mathcal{K}$-absorptive, and so is the set $A \cup h(B)$. \hfill $\square$

Corollary 1 reduces the problem of studying $\sigma \mathcal{K}$-universal sets in a Polish space $X$ to studying $\mathcal{K}$-absorptive sets in $X$ (under the assumption that a $\mathcal{K}$-absorptive set in $X$ exists). The problem of the existence of $\mathcal{K}$-absorptive sets was considered in several papers. In particular, Geoghegan and Summerhill [12] proved that each Euclidean space $\mathbb{R}^n$ contains a $\mathbb{Z}_0$-absorptive set and such a set is unique up to ambient homeomorphism.

Unfortunately, the methods of constructing $\mathbb{Z}_0$-absorptive sets in Euclidean spaces used in [12] does not work in case of the Hilbert cube or Hilbert cube manifolds (in spite of the fact that the paper [12] was written to demonstrate applications of methods of infinite-dimensional topology in the theory of finite-dimensional manifolds). Known results on $\mathbb{Z}_\omega$-absorptive sets in the Hilbert cube $\mathbb{I}^\omega$ and $\mathbb{Z}_0$-absorptive sets in Euclidean spaces allow us to make the following:

**Conjecture 1.** The Hilbert cube contains a $\mathbb{Z}_n$-absorptive set for every $n \leq \omega$.

This conjecture is true for $n = \omega$ as witnessed by the pseudoboundary $B(\mathbb{I}^\omega) = \mathbb{I}^\omega \setminus (0,1)^\omega$ of $\mathbb{I}^\omega$ which is a $\mathbb{Z}_\omega$-absorptive set in $\mathbb{I}^\omega$. In this paper we shall confirm Conjecture 1 for $n = 0$. In fact, our proof works not only for the Hilbert cube but also for any $\mathbb{I}^k$-manifold of finite or infinite dimension. By a manifold modeled on a space $E$ (briefly, an $E$-manifold) we understand any paracompact space $M$ admitting a cover by open subsets homeomorphic to open subspaces of the model space $E$. In this paper we consider only manifolds modeled on (finite or infinite dimensional) cubes $\mathbb{I}^n$, $n \leq \omega$. So, from now on, by a manifold we shall understand an
\[\mathbb{R}^n\text{-manifold for some } 0 < n \leq \omega. \text{ If a manifold } X \text{ is finite-dimensional, then its boundary } \partial X \text{ consists of all points } x \in X \text{ which do not have neighborhoods homeomorphic to Euclidean spaces. If } X \text{ is a Hilbert cube manifold, then we put } \partial X = \emptyset.\]

Our approach to constructing \(Z_0\)-absorptive sets in manifolds is based on the notion of a tame \(G_\delta\)-set which is interesting by itself, see [2]. First we recall some definitions.

A family \(\mathcal{F}\) of subsets of a topological space \(X\) is called \emph{vanishing} if for each open cover \(\mathcal{U}\) of \(X\) the family \(\mathcal{F}' = \{F \in \mathcal{F} : \forall U \in \mathcal{U}, F \not\subset U\}\) is locally finite in \(X\). It is easy to see that a countable family \(\mathcal{F} = \{F_n\}_{n<\omega}\) of subsets of a compact metric space \((X,d)\) is vanishing if and only if \(\lim_{n \to \infty} \text{diam}(F_n) = 0\).

An open subset \(B\) of an \(\mathbb{R}^n\)-manifold \(X\) is called a \emph{tame open ball} in \(X\) if its closure \(\overline{B}\) has on open neighborhood \(O(\overline{B})\) in \(X\) such that the pair \((O(\overline{B}),B)\) is homeomorphic to the pair \((\mathbb{R}^n,\mathbb{I}^n)\) if \(n < \omega\) and to the pair \((\mathbb{I}^n \times [0,\infty),\mathbb{I}^n \times [0,1])\) if \(n = \omega\). Tame balls form a neighborhood base at each point \(x \in X\), which does not belong to the boundary \(\partial X\) of \(X\) (this is trivial for \(n < \omega\) and follows from Theorem 12.2 of [2] for \(n = \omega\)).

A subset \(U\) of a manifold \(X\) is called a \emph{tame open set in} \(X\) if \(U = \bigcup \mathcal{U}\) for some vanishing family \(\mathcal{U}\) of tame open balls having pairwise disjoint closures in \(X\). Observe that the family \(\mathcal{U}\) is unique and coincides with the family \(\mathcal{C}(U)\) of connected components of the set \(U\). By \(\mathcal{C}(U) = \{C : C \in \mathcal{C}(U)\}\) we will denote the family of the closures of the connected components of \(U\) in \(X\).

A subset \(G \subset X\) is called a \emph{tame} \(G_\delta\)-\emph{set} in \(X\) if \(U = \bigcap_{n<\omega} U_n\) for some decreasing sequence \((U_n)_{n<\omega}\) of tame open sets such that the family \(\mathcal{C} = \bigcup_{n<\omega} \mathcal{C}(U_n)\) is vanishing and for every \(n \in \omega\) the family \(\mathcal{C}(U_{n+1})\) refines the family \(\mathcal{C}(U_n)\) of connected components of \(U_n\).

Tame open and tame \(G_\delta\)-sets can be equivalently defined via tame families of tame open balls. A family \(U\) of non-empty open subsets of a topological space \(X\) is called \emph{tame if} \(U\) is vanishing and for any distinct sets \(U,V \in \mathcal{U}\) one of three possibilities hold: either \(U \cap V = \emptyset\) or \(U \subset V\) or \(V \subset U\). For a family \(U\) of subsets of a set \(X\) by

\[\bigcup^\infty \mathcal{U} = \bigcap \{ \bigcup (\mathcal{U} \setminus \mathcal{F}) : \mathcal{F} \text{ is a finite subfamily of } \mathcal{U} \}\]

we denote the set of all points \(x \in X\) which belong to infinite number of sets \(U \in \mathcal{U}\).

**Proposition 2.** A subset \(T\) of a manifold \(X\) is tame open (resp. tame \(G_\delta\)) if and only if \(T = \bigcup T\) (resp. \(T = \bigcup^\infty T\)) for a suitable tame family \(T\) of tame open balls in \(X\).

**Proof.** The “only if” part follows directly from the definition of a tame open (resp. tame \(G_\delta\)) set. To prove the “if” part, assume that \(T\) is a tame family of tame open balls in \(X\). Endow the family \(T\) with a partial order \(\leq\) defined by the reverse inclusion criterion, that is \(U \leq V\) if and only if \(U \supset V\). The vanishing property of \(T\) guarantees that for each set \(U \in T\) the set \(\downarrow U = \{V \in T : V \subseteq U\}\) is finite. This allows us to define the ordinal \(\text{rank}(U)\) letting \(\text{rank}(U) = |\downarrow U|\). For each \(n \in \omega\) let \(T_n = \{U \in T : \text{rank}(U) = n + 1\}\). It follows from the definition of a tame family that the union \(U_n = \bigcup T_n\) is a tame open set and \(U_n \subset U_{n-1}\), where \(U_{-1} = X\). In particular, the union \(\bigcup T = U_0\) is tame open set in \(X\) and the set \(T = \bigcup^\infty T = \bigcap_{n<\omega} U_n\) is a tame \(G_\delta\)-set in \(X\).

The classes of dense tame open sets and dense tame \(G_\delta\)-sets have the following cofinality property.

**Proposition 3.**

1. Each open subset of a manifold \(X\) contains a dense tame open set.
2. Each \(G_\delta\)-subset of a manifold contains a dense tame \(G_\delta\)-set.

**Proof.** Let \(X\) be a manifold and \(d\) be a metric generating the topology of \(X\).

1. Given an open set \(V \subset X\) and an open cover \(\mathcal{U}\) of \(V\) we shall construct a tame open set \(W \subset X\) such that \(W\) is dense in \(V\) and the family \(\mathcal{C}(W)\) refines the cover \(\mathcal{U}\). Replacing \(V\) by \(V \setminus \partial X\), we can assume that the set \(V\) does not intersect the boundary \(\partial X\) of \(X\). Replacing the set \(V\) by \(V \setminus \{v\}\) for some point \(v \in V\), we can additionally assume that the set \(V\) is not compact. We can also assume that \(V = \bigcup \mathcal{U}\). Without loss of generality, the manifold \(X\) is connected and hence separable. So, we can fix a countable dense subset \(\{x_n\}_{n<\omega}\) in \(V\). By induction we can construct an increasing number sequence \((n_k)_{k<\omega}\) and a sequence \(B_k\) of tame open balls in \(X\) such that for each \(k \in \omega\) the following conditions hold:

   1. \(n_k\) is the smallest number \(n\) such that \(x_n \notin \bigcup_{i<k} \overline{B_i}\);
   2. \(B_k\) is a tame open ball such that \(x_n \in B_k\), the closure \(\overline{B_k}\) of \(B_k\) in \(X\) has diameter < \(2^{-k}\) and is contained in \(U \setminus \bigcup_{i<k} \overline{B_i}\) for some set \(U \in \mathcal{U}\).
It is easy to check that $W = \bigcup_{k \in \omega} B_k$ is a required dense tame open set in $V$ with $\mathcal{C}(W) = \{B_k\}_{k \in \omega} \prec \mathcal{U}$.

2. Fix an arbitrary $G_\delta$-set $G$ in $X$ and write it as the intersection $G = \bigcap_{n \in \omega} U_n$ of a decreasing sequence $(U_n)_{n \in \omega}$ of open sets in $X$. By the (proof of the) preceding item, we can construct inductively a decreasing sequence $(V_n)_{n \in \omega}$ of tame open sets in $X$ such that for every $n \in \omega$ we get

- $\text{mesh} \mathcal{C}(V_n) < 2^{-n}$,
- $\bigcap_{n \in \omega} V_n \subset V_{n - 1} \cap U_n$, and
- $V_n$ is dense in $V_{n - 1} \cap U_n$.

Here we assume that $V_{-1} = X$. It follows that $V = \bigcap_{n \in \omega} \mathcal{C}(V_n)$ is a tame family of tame open balls whose limit set $\bigcup_{n \in \omega} V_n$ is a required dense tame $G_\delta$-set in $G$. □

It is easy to see that any two tame open balls in a connected $\mathbb{P}$-manifold are ambiently homeomorphic. A similar fact also holds for dense tame open sets. Generalizing earlier results of Whyburn [17] and Cannon [5], Banakh and Repovš in [3] Corollary 2.8 proved the following Uniqueness Theorem for dense tame open sets.

**Theorem 2** (Uniqueness Theorem for Dense Tame Open Sets in Manifolds). Any two dense tame open sets $U, U' \subset X$ of a manifold $X$ are ambiently homeomorphic. Moreover, for each open cover $\mathcal{U}$ of $X$ there is a homeomorphism $h : (X, U) \to (X, U')$ such that $(h, \text{id}) \sim \text{St}(\mathcal{C}(U), \mathcal{U}) \vee \text{St}(\mathcal{C}(U'), \mathcal{U})$. This theorem will be our main tool in the proof of the following Uniqueness Theorem for dense tame $G_\delta$-sets.

**Theorem 3** (Uniqueness Theorem for Dense Tame $G_\delta$-Sets in Manifolds). Any two dense tame $G_\delta$-sets $G, G'$ in a manifold $X$ are ambiently homeomorphic. Moreover, for each open cover $\mathcal{U}$ of $X$ there is a homeomorphism $h : (X, G) \to (X, G')$ such that $(h, \text{id}) \prec \mathcal{U}$.

**Proof.** Fix a bounded complete metric $d$ generating the topology of the manifold $X$. By [3] 8.1.10, the metric $d$ can be chosen so that the cover $\{B(x, 1) : x \in X\}$ by closed balls of radius 1 refines the cover $\mathcal{U}$. In this case any two functions $f, g : X \to X$ with $d(f, g) = \sup_{x \in X} d(f(x), g(x)) \leq 1$ are $\mathcal{U}$-near.

Represent the tame $G_\delta$-sets $G$ and $G'$ as the limit sets $G = \bigcup_n G$ and $G' = \bigcup_n G'$ of suitable tame families $\mathcal{G}$ and $\mathcal{G}'$ of tame open balls in $X$. For every $n \in \omega$ let $G_n = \{U \in G : |\{V \in \mathcal{G} : V \supset U\}| \geq n\}$ and $G'_n = \{U \in G' : |\{V \in \mathcal{G}' : V \supset U\}| \geq n\}$. It follows that $G = \bigcap_{n \in \omega} G_n$ and $G' = \bigcap_{n \in \omega} G'_n$.

Let $U_{-1} = U'_1 = X$ and $h_{-1} : X \to X$ be the identity homeomorphism of $X$. Let also $U_{-1} = U'_1$ be a cover of $X$ by open subsets of diameter $\leq \frac{1}{2}$. For every $n \in \omega$ we shall construct a homeomorphism $h_n : X \to X$, two tame open sets $U_n, U'_n \subset X$, and open covers $\mathcal{U}_n, \mathcal{U}'_n$ of the sets $U_n, U'_n$ respectively, such that

1. $G \subset U = U_{-1} \cap \bigcup G_n$ and $\mathcal{C}(U_n) \prec \mathcal{U}_{n - 1}$;
2. $G' \subset U' = U_{-1} \cap \bigcup G'_n$ and $\mathcal{C}(U'_n) \prec \mathcal{U}'_{n - 1} \cap h_{n - 1}(U_{n - 1})$;
3. $h_n(U_{n - 1}) = U'_n$;
4. $h_n(X \setminus U_{n - 1}) = h_{n - 1}(X \setminus U_{n - 1})$;
5. $d(h_n, h_{n - 1}) < 2^{-n - 1}$ and $d(h_{n - 1}, h_{n - 1}^{-1}) < 2^{-n - 1}$;
6. $\text{mesh}(\mathcal{U}_n) < 2^{-n - 3}$, $\text{mesh}(\mathcal{U}'_n) < 2^{-n - 3}$, and $\text{St}^2(\mathcal{U}_n) < \{B(x, d(x, X \setminus U_n) / 2) : x \in U_n\}$.

Assume that for some $n \in \omega$ the open sets $U_{n - 1}, U'_{n - 1}$, open covers $\mathcal{U}_{n - 1}, \mathcal{U}'_{n - 1}$ and a homeomorphism $h_{n - 1} : (X, U_{n - 1}) \to (X, U'_{n - 1})$ satisfying the conditions (1)–(6) have been constructed. Consider the subfamilies $\mathcal{F}_n = \{U \in \mathcal{G}_n : \{U\} \prec \mathcal{U}_{n - 1}\}$ and $\mathcal{F}'_n = \{U \in \mathcal{G}'_n : \{U\} \prec \mathcal{U}'_{n - 1} \cap h_{n - 1}(U_{n - 1})\}$. The vanishing property of the tame families $\mathcal{G}$ and $\mathcal{G}'$ implies that the sets $U_n = \bigcup \mathcal{F}_n$ and $U'_n = \bigcup \mathcal{F}'_n$ satisfy the conditions (1), (2) of the inductive construction. The sets $U_n$ and $U'_n$ are tame open, being unions of the tame families $\mathcal{F}_n$ and $\mathcal{F}'_n$, respectively. Moreover, $\mathcal{C}(U_n) \prec \mathcal{U}_{n - 1}$ and $\mathcal{C}(U'_n) \prec \mathcal{U}'_{n - 1} \cap h_{n - 1}(U_{n - 1})$.

Now we shall construct a homeomorphism $h_n : (X, U_n) \to (X, U'_n)$. Since $h_{n - 1}(U_{n - 1}) = U'_{n - 1}$, each connected component $C \in \mathcal{C}(U_{n - 1})$ of the open set $U_{n - 1}$ maps onto the connected component $C' = h_{n - 1}(C) \in \mathcal{C}(U'_{n - 1})$ of the set $U'_{n - 1}$. Taking into account that each set $B \in \mathcal{C}(U_n)$ is a compact connected subset of the open set $\bigcup U'_n = U_{n - 1}$, we see that the intersection $U'_n \cap C'$ is a dense tame open set in the open set $C'$. Consequently, its image $h_{n - 1}^{-1}(U'_n \cap C')$ is a dense tame open set in the open set $C = h_{n - 1}^{-1}(C')$. By Theorem 2 there is a homeomorphism of pairs $g_C : (C, C \cap U_n) \to (C, h_{n - 1}^{-1}(C' \cap U'_n))$ which is $\mathcal{W}_C$-near to the identity map $\text{id}_C : C \to C$ for the cover $\mathcal{W}_C = \text{St}(\mathcal{C}(C \cap U_n), \mathcal{U}_{n - 1}) \vee \text{St}(\mathcal{C}(h_{n - 1}^{-1}(C' \cap U'_n)), \mathcal{U}_{n - 1})$.

Taking into account that $\mathcal{C}(C \cap U_n) \prec \mathcal{C}(U_n) \prec \mathcal{U}_{n - 1}$ and $\mathcal{C}(h_{n - 1}^{-1}(U'_n \cap C')) \prec \mathcal{C}(h_{n - 1}^{-1}(U'_n)) = h_{n - 1}(\mathcal{C}(U'_n)) \prec h_{n - 1}(h_{n - 1}(U_{n - 1})) = U_{n - 1}$,
we conclude that
\[ \mathcal{W}_C = St(\mathcal{C}(C \cap U_n), U_{n-1}) \cup St(\mathcal{C}(h_n^{-1}(C' \cup U')), U_{n-1}) \cup St(U_{n-1}, U_{n-1}) \cup St(U_{n-1}, U_{n-1}) = \]
\[ = St(U_{n-1}) \cup St(U_{n-1}) \cup St^2(U_{n-1}) \cup \{B(x, d(X \setminus U_{n-1})/2) : x \in U_{n-1}\}. \]

Now the vanishing property of the family \( C(U_{n-1}) \) implies that the map \( g_n : X \to X \) defined by
\[ g_n(x) = \begin{cases} 
  x & \text{if } x \notin U_{n-1}, \\
  g_C & \text{if } x \in C \in C(U_{n-1}) 
\end{cases} \]
is a homeomorphism of \( X \) such that \( (g_n, id) \prec \mathcal{C}(U_{n-1}) \) and \( (g_n, id) \prec \mathcal{C}(U_{n-1}) \). Then \( h_n = h_n \circ g_n \) is a homeomorphism of \( X \) satisfying the conditions (3) and (4) of the inductive construction.

To prove the condition (5) we shall consider separately the cases of \( n = 0 \) and \( n > 0 \). If \( n = 0 \), then \( h_0 = g_0 \) and hence \( (h_0, h_{-1}) = (g_0, id) \prec \mathcal{C}(U_{n-1}) \). Then \( h_n = h_n \circ g_n \) is a homeomorphism of \( X \) with \( A \setminus h_n \) \( U \prec \mathcal{C}(U_{n-1}) \). By Proposition 3, the dense tame \( h \) of \( X \setminus h_n \) \( U \prec \mathcal{C}(U_{n-1}) \) implies \( (h_n, h_{-1}) \circ U \prec \mathcal{C}(U_{n-1}) \) \( U_{n-2} \prec \mathcal{C}(U_{n-1}) \). Therefore, the condition (5) holds.

Finally, using the paracompactness of the metrizable spaces \( U_n \) and \( U'_n \) choose two open covers \( U_n \) and \( U'_n \) of \( U_n \) and \( U'_n \) satisfying the condition (6).

After completing the inductive construction, we obtain a sequence of homeomorphisms \( h_n : (X, U_n) \to (X, U'_n), n \in \omega \). The condition (5) guarantees that the limit map \( h = \lim_{n \to \infty} h_n \) is a well-defined homeomorphism of \( X \) such that \( d(h, id) \leq 1 \). Moreover, the conditions (1) and (3) imply
\[ h(G) = h\left( \bigcap_{n \in \omega} U_n \right) = \bigcap_{n \in \omega} h(U_n) = \bigcap_{n \in \omega} U'_n = G'. \]

By the choice of the metric \( d \), the inequality \( d(h, id) \leq 1 \) implies \( (h, id) \prec U \). So, \( h : (X, G) \to (X, G')\) is a required homeomorphism of pairs with \( (h, id) \prec U \).

Now we are able to prove a characterization of \( \sigma Z_0 \)-universal sets in manifolds.

**Theorem 4** (Characterization of \( \sigma Z_0 \)-Universal Sets in Manifolds). For a subset \( A \) of a manifold \( X \) the following conditions are equivalent:

1. \( A \) is \( \sigma Z_0 \)-universal in \( X \);
2. \( A \) is \( Z_0 \)-absorptive in \( X \);
3. the complement \( X \setminus A \) is a dense tame \( G_\delta \)-set in \( X \).

**Proof.** We shall prove the equivalences (3) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (1). Let \( d \) be a metric generating the topology of the manifold \( X \).

To prove that (3) \( \Rightarrow \) (2), assume that the complement \( X \setminus A \) is a dense tame \( G_\delta \)-set in \( X \). To prove that \( A \) is \( Z_0 \)-absorptive, fix any open set \( V \subset X \), an open cover \( U \) of \( V \) and a closed nowhere dense subset \( K \subset X \). We lose no generality assuming that \( U \prec \{B(x, d(x, X \setminus V)/2) : x \in V\} \). Since \( V \setminus (A \cup K) \) is a dense \( G_\delta \)-set in \( V \), we can apply Proposition 3 to find a dense tame \( G_\delta \)-set \( G \subset V \setminus (A \cup K) \). The characterization of tame \( G_\delta \)-sets given in Proposition 2 implies that the intersection \( V \cap (X \setminus A) = V \cap A \) is a dense tame \( G_\delta \)-set in \( V \). By Theorem 3 there is a homeomorphism of pairs \( h : (V, G) \to (V, V \setminus A) \) such that \( (h, id) \prec U \). Since \( U \prec \{B(x, d(x, X \setminus V)/2) : x \in V\} \), the homeomorphism \( h \) of \( V \) extends to a homeomorphism \( \tilde{h} : X \to X \) such that \( \tilde{h}(X \setminus V) = \text{id} \). Observing that \( \tilde{h}(V \cap K) \subset \tilde{h}(V \setminus G) \subset V \cap A \), we see that the set \( A \) is \( Z_0 \)-absorptive.

To prove that (2) \( \Rightarrow \) (3), assume that the set \( A \) is \( Z_0 \)-absorptive. By Proposition 3 the dense \( G_\delta \)-set \( X \setminus A \) contains a dense tame \( G_\delta \)-set \( G \) in \( X \). Since \( A \subset X \setminus G \), the set \( X \setminus G \in \sigma Z_0 \) is \( Z_0 \)-absorptive. By the Uniqueness Theorem 3 there is a homeomorphism of pairs \( h : (X, A) \to (X, X \setminus G) \). Then \( X \setminus A = h(G) \) is a dense tame \( G_\delta \)-set in \( X \), which completes the proof of the implication (2) \( \Rightarrow \) (3).

By Proposition 3 \( X \) contains a dense tame \( G_\delta \)-set \( G \) and by the implication (3) \( \Rightarrow \) (2) proved above the complement \( X \setminus G \) is \( Z_0 \)-absorptive. Now Corollary 1 yields the equivalence (2) \( \Leftrightarrow \) (1).

**Theorem 3** implies:

**Corollary 2.** Each dense \( G_\delta \)-subset of a dense tame \( G_\delta \)-set in a manifold is tame.
We finish this paper by some open problems. It is clear that each tame $G_δ$-set in a manifold is zero-dimensional. However, not each zero-dimensional dense $G_δ$-subset of the Hilbert cube $I^ω$ is tame.

Proposition 4. For any dense $G_δ$-set $G ⊂ I$ the countable product $G^ω$ is not a tame $G_δ$-set in $Γ^ω$.

Proof. Assuming that $G^ω$ is tame, we can find a dense tame open set $T ⊂ I^ω$ containing $G^ω$. By Theorem 1.4 of [3], the complement $S = I^ω \setminus T$ is homeomorphic to the Hilbert cube and the boundary $B \setminus B$ of each tame open ball $B ∈ C(T)$ in $I^ω$ is a $Z_ω$-set in $S$. Let $pr_n : Γ^ω → I$, $n ∈ ω$, denote the projection of the Hilbert cube $I^ω$ onto the $n$th coordinate. Since $I^ω \setminus T ⊂ I^ω \setminus pr_n⁻¹(I \setminus G)$, Baire Theorem yields a non-empty open subset $W ⊂ S$ such that $W ⊂ pr_n⁻¹(I \setminus G)$ for some $n ∈ ω$. Since $S$ is homeomorphic to the Hilbert cube, we can assume that the set $W$ is connected and hence is contained in $pr_n⁻¹(t)$ for some point $t ∈ I \setminus G$. Since the union $Δ = \bigcup_{B ∈ C(U)} B \setminus B$ is a $σZ_ω$-set in $S$, we can chose a point $x_0 ∈ W \setminus Δ$. Choose an open neighborhood $U$ of $x_0$ in $I^ω$ such that $U \cap S ⊂ W$ and $U \setminus pr_n⁻¹(t)$ has at most two connected components.

Since the family $C(T)$ is vanishing and $T = \bigcup C(T)$ is dense in $I^ω$, there are three pairwise distinct tame open balls $B_1, B_2, B_3 ∈ C(T)$ such that $B_i \cup B_j \cup B_k ⊂ U$. Since the set $U \setminus pr_n⁻¹(t)$ has at most two connected components, there are two distinct indices $1 ≤ i, j ≤ 3$ such that the balls $B_i$ and $B_j$ meet the same connected component $V$ of $U \setminus pr_n⁻¹(t)$. Since $B_i \setminus B_j ⊂ U \cap S ⊂ pr_n⁻¹(t)$, the set $V \cap B_i$ is closed-and-open in the connected set $V$ and hence coincides with $V$. So, $V ⊂ B_i$. By the same reason, $V ⊂ B_j$, which is not possible as the balls $B_i$ and $B_j$ are disjoint.

Problem 1. Can the countable power $G^ω$ of a dense $G_δ$-set $G ⊂ I$ be covered by countably many dense tame $G_δ$-sets?

By Smirnov’s result [5, 5.2.B], the Hilbert cube $I^ω$ can be covered by $ℵ_1$ zero-dimensional $G_δ$-sets.

Problem 2. What is the smallest cardinality of a cover of the Hilbert cube $I^ω$ by tame $G_δ$-sets? Is it equal to $ℵ_1$? (By Theorem 1.6 of [2] this cardinality does not exceed $\text{add}(M)$, the additivity of the ideal $M$ of meager subsets on the real line.)

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