On deformations of metric Lie superalgebras

Yong Yang\textsuperscript{a,b}

\textsuperscript{a}School of Mathematics and Statistics, Northeast Normal University, Changchun, China; \textsuperscript{b}Doctoral School of Physics, University of Pécs, Pécs, Hungary

\textbf{ABSTRACT}

In this paper we study metric deformations of indecomposable complex metric Lie superalgebras of dimensions \( \leq 6 \). We consider formal deformations obtained by even cocycles, because the odd ones cannot be used for constructing formal deformations.

\textbf{ARTICLE HISTORY}

Received 18 September 2020
Revised 7 November 2021
Communicated by K. Misra

\textbf{KEYWORDS}

Cohomology; invariant bilinear form; (metric) deformation; (metric) Lie superalgebra

2020 MATHEMATICS SUBJECT CLASSIFICATION
14D15; 17B56

\section{1. Introduction}

Lie algebras endowed with an invariant bilinear form are important objects in Lie theory and mathematical physics. A Lie algebra with a non-degenerate, symmetric, invariant bilinear form is called a metric (or quadratic) Lie algebra. Examples of metric Lie algebras are semi-simple Lie algebras with an invariant bilinear form given by the Killing form, due to the Cartan’s criterion. However, there are still many metric solvable Lie algebras, although their Killing forms are degenerate. Thus, it seems not easy to classify metric Lie algebras, even in low-dimensional cases. Recently, deformations of low-dimensional metric Lie algebras over the field \( \mathbb{C} \) and \( \mathbb{R} \) of dimension \( \leq 6 \) was given by Fialowski and Penkava [13].

In the study of metric Lie algebras, an effective method to construct them is by double extension, which can be regarded as a combination of central extension and semi-direct product, introduced by Kac [18] for solvable Lie algebras. Metric Lie algebras were described inductively, based on double extensions, by Medina and Revoy [21] in indecomposable, non-simple case, and by Favre and Santharoubane [10] in non-trivial center case. Another interesting method is \( T^* \)-extension, which can be regarded as a semi-direct product of a Lie algebra and its dual space by means of the coadjoint representation, introduced by Bordemann [5]. Bordemann proved that every finite-dimensional nilpotent metric Lie algebra of even dimension can by obtained by a \( T^* \)-extension. However, \( T^* \)-extension does not exhaust all possibilities for constructing metric Lie algebras of even dimension. In fact, for the case of dimension \( \leq 6 \), there is only one class of non-Abelian metric Lie algebras, which can be obtained by a \( T^* \)-extension [13].

H. Benamor and S. Benayadi generalized the notion of double extension to metric Lie superalgebras by considering a supersymmetric invariant bilinear form and proved that every non-simple
indecomposable metric Lie superalgebra with 2-dimensional odd part is a double extension of a one-dimensional or semi-simple Lie algebra [2]. It also holds for the metric Lie superalgebra with 2-dimensional even part [9]. In the past years the study of metric Lie superalgebras became intensive. Many classes of metric Lie superalgebras have been studied [1, 3, 9]. Different from what happens in the Lie case, the Killing form is not always non-degenerate on a semi-simple Lie superalgebra. So it becomes more difficult to classify metric Lie superalgebras, even for semi-simple cases. An interesting fact, that any indecomposable non-simple metric Lie superalgebra of dimension $\leq 6$ is a double extension of a 1-dimensional Lie algebra. A classification of indecomposable metric Lie superalgebras of dimension $\leq 6$ was obtained in [8]. However, there are no results for deformations of metric Lie superalgebras even in low-dimensional cases. For a metric Lie superalgebra, an interesting question is which deformations are metric.

The aim of this paper is to study metric deformations of indecomposable metric Lie superalgebras of dimension $\leq 6$. In Section 2, we recall the basic definitions and results for metric Lie superalgebras. In Section 3, we recall the cohomology and deformation theory for Lie superalgebras. In Section 4, we compute metric deformations of indecomposable metric Lie superalgebras of dimension $\leq 6$, and show which ones are metric among these deformations. Throughout the paper, the ground field is supposed to be the complex field $\mathbb{C}$.

2. Metric Lie superalgebra

In this section, we recall the basic facts for metric Lie superalgebras from [2, 9]. We also introduce the notion of double extension, and give some examples.

2.1. Structure of metric Lie superalgebras

Recall that a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded algebra whose multiplication $[,]$ satisfies the skew-supersymmetry and super Jacobi identity, i.e.

\[
(-1)^{|x||x|}[x, [y, z]] + (-1)^{|y||z|}[y, [x, z]] + (-1)^{|z||y|}[z, [x, y]] = 0,
\]

for any $x, y, z$ in $\mathfrak{g}$ [19, 22]. For a homogeneous element $x \in \mathfrak{g}_0 \cup \mathfrak{g}_1$, write $|x|$ for the $\mathbb{Z}_2$-degree of $x$. Call an ideal of a Lie superalgebra a $\mathbb{Z}_2$-graded ideal. A homomorphism between superalgebras is such, that preserves $\mathbb{Z}_2$-grading. The definition of solvable Lie superalgebras is the same as for Lie algebras. A Lie superalgebra $\mathfrak{g}$ is called simple (semi-simple) if $\mathfrak{g}$ is not Abelian and does not contain nontrivial (solvable) ideals. Recall that a bilinear form $B$ on a Lie superalgebra $\mathfrak{g}$ is called invariant if

\[
B([x, y], z) = B(x, [y, z])
\]

for any $x, y, z$ in $\mathfrak{g}$. Due to the Cartan’s criterion, there always exists an invariant bilinear form on a semi-simple Lie algebra. Unfortunately, this is not true for semi-simple Lie superalgebras. The following definition of metric Lie superalgebras can be viewed as a generalization of semi-simple Lie algebras to Lie superalgebras.

**Definition 2.1.** [2, Definition 2.2] A metric (or quadratic) Lie superalgebra $(\mathfrak{g}, B)$ is a Lie superalgebra $\mathfrak{g}$ with a non-degenerate, supersymmetric, even, and $\mathfrak{g}$-invariant bilinear form $B$. In this case, $B$ is called an invariant scalar product on $\mathfrak{g}$.

**Definition 2.2.** [9, Definition 1.15] Two metric Lie superalgebras $(\mathfrak{g}, B)$ and $(\mathfrak{g}', B')$ are called isometrically isomorphic (or $i$-isomorphic) if there exists a Lie superalgebra isomorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ satisfying $B'(f(x), f(y)) = B(x, y)$ for all $x, y \in \mathfrak{g}$. In this case, we write $\mathfrak{g} \simeq \mathfrak{g}'$. 
The even part of a metric Lie superalgebra is a metric Lie algebra, which is a Lie algebra with a non-degenerate, symmetric, invariant bilinear form. In fact, the characterization for metric Lie superalgebras can be reduced to their metric Lie algebras by the following Lemma.

**Proposition 2.3.** [2, Proposition 2.9] A Lie superalgebra \( g \) is metric if and only if \( g_0 \) is a metric Lie algebra with respect to an invariant scalar product \( B_0 \) and there exists a skew-symmetric non-degenerate bilinear form \( B_1 \) on \( g_1 \), such that for all \( x, y \in g_1, z \in g_0 \),

1. \( B_0([x, y], z) = B_1(x, [y, z]) \),
2. \( (g_0\text{-invariant}) B_1([z, x], y) = -B_1(x, [z, y]) \).

**Corollary 2.4.** In the above Proposition, \( \dim g_1 \) is even.

Let \( (g, B) \) be a metric Lie superalgebra. An ideal \( I \) of \( g \) is called non-degenerate, if \( B|_{I \times I} \) is non-degenerate. Obviously, if \( I \) is a non-degenerate ideal, then \( (I, B|_{I \times I}) \) is also a metric Lie superalgebra. It is known that any semi-simple Lie algebra can be decomposed into the direct sum of its simple ideals. The following Proposition is analogous to what happens in the semi-simple Lie algebra case.

**Proposition 2.5.** [2, Proposition 2.6] Let \( (g, B) \) be a metric Lie superalgebra. Then

\[
g = \bigoplus_{i=1}^{r} g_i,
\]

such that, for all \( 1 \leq i \leq r \),

1. \( g_i \) is a non-degenerate ideal of \( g \).
2. \( g_i \) contains no nontrivial non-degenerate ideal of \( g \).
3. \( B(g_i, g_i) = 0 \) for all \( i \neq j \).

**Definition 2.6.** In the above Proposition, if \( r = 1, g \) is called indecomposable. Otherwise, \( g \) is called decomposable.

**Lemma 2.7.** For a metric Lie superalgebra \( g \), if \( \dim g_0 \leq 1 \), then \( g \) is Abelian.

**Proof.** It is sufficient to prove the Lemma in the 1-dimensional even part case. Suppose that \( (g, B) \) is a metric Lie superalgebra, spanned by

\( \{e_1 \mid e_2, ..., e_n\} \).

From the invariance of \( B \), we obtain

\[
B(e_1, [g_1, g_1]) = B([e_1, g_1], g_1).
\]

Thus \([g_1, g_1] = 0\) if and only if \([e_1, g_1] = 0\). Suppose that \([e_i, e_j] = k_{ij}e_1\) for any \( 2 \leq i, j \leq n \). In order to prove that \( g \) is Abelian, it is sufficient to prove \( k_{ij} = 0 \). At first, we claim that \( k_{ii} = 0 \).

By the super Jacobi identity, for any \( 2 \leq i \leq n \) we have \([e_i, e_i], e_i] = 0 \). If there exists \( i_0 \) such that \( k_{i_0i_0} \neq 0 \), then \([e_1, e_{i_0}] = 0 \). From

\[
B([e_1, e_{i_0}], e_{i_0}) = B(e_1, [e_{i_0}, e_{i_0}]) = 0
\]

we obtain that \( B(e_1, e_i) = 0 \). It is a contradiction to the non-degenerate property of \( B \). In addition, we obtain that for \( 2 \leq i \neq j \leq n \), \([e_i, e_j], e_i] = 0 \) from the super Jacobi identity of \( e_i, e_j \) and \( e_j \). If there exist \( i_1, j_1 \) such that \( k_{i_1j_1} \neq 0 \), then we have \([e_1, e_{j_1}] = 0 \). From

\[
B([e_1, e_{j_1}], e_{j_1}) = B(e_1, [e_{j_1}, e_{j_1}]) = 0
\]

we obtain that \( B(e_1, e_i) = 0 \). It is a contradiction to the non-degenerate property of \( B \). The proof is complete. 

\( \square \)
2.2. Double extension for metric Lie superalgebras

We introduce the theory of double extensions in this section, which is an important method to construct metric Lie superalgebras. For more details, the reader is referred to [2, 8, 9, 13, 23].

Definition 2.8. [19, Chapter I, 1.4] Let \( \mathfrak{g} \) be a Lie superalgebra and \( D \) a homogeneous linear transformation of \( \mathfrak{g} \). \( D \) is called a \textit{derivation} of \( \mathfrak{g} \) if

\[
D([x,y]) = [D(x),y] + (-1)^{|D||x|}[x,D(y)], \quad x,y \in \mathfrak{g}.
\]

Denote \( \text{Der}(\mathfrak{g}) \) the derivation space of \( \mathfrak{g} \) with respect to \( B \).

Definition 2.9. [2, Definition 3.4] Let \( \mathfrak{g} \) be a Lie superalgebra and \( B \) a bilinear form on \( \mathfrak{g} \). Let \( D \) be a derivation of \( \mathfrak{g} \). \( D \) is called \textit{skew-supersymmetric} with respect to \( B \), if

\[
B(D(x),y) = -(-1)^{|D||x|}B(x,D(y)), \quad x,y \in \mathfrak{g}.
\]

Denote by \( \text{Der}(\mathfrak{g},B) \) the skew-supersymmetric derivation space of \( \mathfrak{g} \) with respect to \( B \).

Remark 2.10. By the above definitions, \( \text{Der}(\mathfrak{g}) \) and \( \text{Der}(\mathfrak{g},B) \) are both Lie-super subalgebras of the general linear superalgebra \( gl(\mathfrak{g}) \).

Theorem 2.2.1. [2, Theorem 1] Let \((\mathfrak{g}_1,B_1)\) be a metric Lie superalgebra, \( \mathfrak{g}_2 \) a Lie superalgebra and \( \psi : \mathfrak{g}_2 \to \text{Der}(\mathfrak{g}_1,B_1) \subseteq \text{Der}(\mathfrak{g}_1) \) a morphism of Lie superalgebras. Let \( \varphi \) be the linear mapping from \( \mathfrak{g}_1 \times \mathfrak{g}_1 \) to \( \mathfrak{g}_2 \), defined by

\[
\varphi(x,y)(z) = (-1)^{(|x|+|y|)|z|}B_1(\psi(z)(x),y), \quad x,y \in \mathfrak{g}_1, z \in \mathfrak{g}_2.
\]

Let \( \pi \) be the coadjoint representation of \( \mathfrak{g}_2 \). Then the vector space \( \mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^* \) with the products

\[
[x_2,y_2]_\mathfrak{g} = [x_2,y_2]_{\mathfrak{g}_2}, \quad [x_2,y_1]_\mathfrak{g} = \psi(x_2)y_1, \quad [x_2,f]_\mathfrak{g} = \pi(x_2)f,
\]

where \( x_1,y_1 \in \mathfrak{g}_1, x_2,y_2 \in \mathfrak{g}_2, f \in \mathfrak{g}_2^* \), is a Lie superalgebra. Moreover, if \( B_2 \) is an even, supersymmetric, invariant (not necessary non-degenerate) bilinear form on \( \mathfrak{g}_2 \), then the bilinear form \( B_0 \), defined on \( \mathfrak{g} \) by

\[
B_0(x_2,y_2) = B_2(x_2,y_2), \quad B_0(f,y_2) = f(y_2), \quad B_0(x_1,y_1) = B_1(x_1,y_1),
\]

where \( x_1,y_1 \in \mathfrak{g}_1, x_2,y_2 \in \mathfrak{g}_2, f \in \mathfrak{g}_2^* \), is an invariant scalar product on \( \mathfrak{g} \).

Definition 2.11. In the above Theorem, the metric Lie superalgebra \((\mathfrak{g},B_0)\) is called the \textit{double extension} of \((\mathfrak{g}_1,B_1)\) by \( \mathfrak{g}_2 \) by means of \( \psi \). In particular, if \( \mathfrak{g}_2 \) is a 1-dimensional Lie algebra, \((\mathfrak{g},B_0)\) is called the \textit{1-dimensional double extension} of \((\mathfrak{g}_1,B_1)\) by means of \( \psi \).

Corollary 2.12. Suppose \((\mathfrak{g}_1,B_1)\) is a metric Lie superalgebra and \( D \in \text{Der}_0(\mathfrak{g}_1,B_1) \). Let \( B_{CD} \) be an invariant symmetric (not necessary non-degenerate) bilinear form on \( \mathbb{C}D = \langle D \rangle \). Then the 1-dimensional double extension \((\mathfrak{g} = \mathbb{C}D \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*,B_0)\) of \((\mathfrak{g}_1,B_1)\) by means of \( D \) is given as follows:

1. The Lie super brackets on \( \mathfrak{g} \) are given by
   \[
   [x,y]_\mathfrak{g} = [x,y]_{\mathfrak{g}_1} + B_1(D(x),y)D^*, \quad [D,x]_\mathfrak{g} = D(x), \quad x,y \in \mathfrak{g}_1.
   \]

2. The invariant scalar product \( B_0 \) is given by
   \[
   B_0(D,D) = B_{CD}(D,D), \quad B_0(D^*,D) = 1, \quad B_0(x,y) = B_1(x,y), \quad x,y \in \mathfrak{g}_1.
   \]

Proof. It follows from Theorem 2.2.1. \(\square\)
Double extension is an important method to construct examples of metric Lie superalgebras. In particular, the following result allows us to inductively classify metric Lie algebras by double extension.

**Theorem 2.2.2.** [21, Theorem 1] A non-simple metric Lie algebra, which is indecomposable, is a double extension of some metric Lie algebra by a 1-dimensional Lie algebra or a simple Lie algebra.

The situation for metric Lie superalgebras is more complicated. It is still an open question whether every indecomposable non-simple metric Lie superalgebra \((\mathfrak{g}, B)\) can be obtained by double extension. However, in the cases \(\dim \mathfrak{g}_0 = 2\) and \(\dim \mathfrak{g}_1 = 2\), the answer is positive.

**Theorem 2.2.3.** [2, Theorem 3] Any indecomposable non-simple metric Lie superalgebra \((\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, B)\) with \(\dim \mathfrak{g}_i = 2\) is a double extension by a 1-dimensional or semi-simple Lie algebra.

**Proposition 2.13.** [9, Proposition 4.8] Let \(\mathfrak{g}\) be a non-Abelian metric Lie superalgebra with 2-dimensional even part. Then \(\mathfrak{g}\) is a double extension of the symplectic space \(\mathfrak{g}_1\) (regarded as an Abelian Lie superalgebra) by a 1-dimensional Lie algebra.

**Example 1.** We construct in detail the 1-dimensional double extensions of \(\mathbb{C}_{0|2}\). Let \(\mathbb{C}_{0|2} = \text{span}\{h_1, h_2\}\) be the symplectic space with the symplectic form given by

\[
B_{\mathbb{C}_{0|2}} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

Denote by \(\text{sp}(\mathbb{C}_{0|2}, B_{\mathbb{C}_{0|2}})\) the 3-dimensional symplectic Lie algebra, which consists of all linear transformations of \(\mathbb{C}_{0|2}\) which are skew-symmetric with respect to \(B_{\mathbb{C}_{0|2}}\) [17]. Then we have \(\text{Der}_0(\mathbb{C}_{0|2}, B_{\mathbb{C}_{0|2}}) = \text{sp}(\mathbb{C}_{0|2}, B_{\mathbb{C}_{0|2}})\), spanned by

\[
D_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

**Case 1:** \(k_1 = k_2 = k_3 = 0\). Then \(\mathfrak{g}\) is the Abelian Lie superalgebra \(\mathbb{C}_{2|2}\).

**Case 2:** \(k_1 = 0\), \(k_2, k_3 \neq 0\). We choose a basis of \(\mathfrak{g}\) as follows:

\[
e_1 = -2k_3D^*, \quad e_2 = \frac{1}{\sqrt{k_2k_3}}D, \quad e_3 = h_1 + \frac{k_3}{k_2}h_2, \quad e_4 = h_1 - \frac{k_3}{k_2}h_2.
\]

Then the brackets are given by

\[
[e_3, e_4] = e_1, \quad [e_2, e_3] = e_5, \quad [e_2, e_4] = -e_4.
\]

**Case 3:** \(k_1 = k_2 = 0\), \(k_3 \neq 0\). We choose a basis of \(\mathfrak{g}\) as follows:

\[
e_1 = -D^*, \quad e_2 = D, \quad e_3 = \sqrt{k_3}h_2, \quad e_4 = \frac{1}{\sqrt{k_3}}h_1.
\]

Then the brackets are given by

\[
[e_4, e_3] = e_1, \quad [e_2, e_4] = e_3.
\]
Case 4: $k_1 = k_3 = 0$, $k_2 \neq 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = k_2 D^*, \quad e_2 = D, \quad e_3 = k_2 h_1, \quad e_4 = h_2.$$ 

Then the brackets are given by

$$[e_4, e_4] = e_1, \quad [e_2, e_4] = e_3.$$ 

Case 5: $k_1 \neq 0$, $k_2 = k_3 = 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = k_1 D^*, \quad e_2 = \frac{1}{k_1} D, \quad e_3 = h_1, \quad e_4 = h_2.$$ 

Then the brackets are given by

$$[e_3, e_4] = e_1, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = -e_4.$$ 

Case 6: $k_1 \neq 0$, $k_2 = 0$, $k_3 \neq 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = \frac{2k_1^2}{k_3} D^*, \quad e_2 = \frac{1}{k_1} D, \quad e_3 = \frac{2k_1}{k_3} h_1 + h_2, \quad e_4 = h_2.$$ 

Then the brackets are given by

$$[e_3, e_4] = e_1, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = -e_4.$$ 

Case 7: $k_1, k_2 \neq 0$, $k_3 = 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = -\frac{2k_1^2}{k_2} D^*, \quad e_2 = \frac{1}{k_1} D, \quad e_3 = h_1, \quad e_4 = h_1 - \frac{2k_1}{k_2} h_2.$$ 

Then the brackets are given by

$$[e_3, e_4] = e_1, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = -e_4.$$ 

Case 8: $k_1, k_2, k_3 \neq 0$, $k_1^2 + k_2 k_3 = 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = k_2 D^*, \quad e_2 = \frac{1}{k_1} D, \quad e_3 = h_1 - \frac{k_1}{k_2} h_2, \quad e_4 = h_2.$$ 

Then the brackets are given by

$$[e_4, e_4] = e_1, \quad [e_2, e_4] = e_3.$$ 

Case 9: $k_1, k_2, k_3 \neq 0$, $k_1^2 + k_2 k_3 \neq 0$. We choose a basis of $\mathfrak{g}$ as follows:

$$e_1 = -2 \frac{k_1^2 + k_2 k_3}{k_2} D^*, \quad e_2 = \frac{1}{\sqrt{k_1^2 + k_2 k_3}} D,$$

$$e_3 = h_1 - \frac{k_1 - \sqrt{k_1^2 + k_2 k_3}}{k_2} h_2, \quad e_4 = h_1 - \frac{k_1 + \sqrt{k_1^2 + k_2 k_3}}{k_2} h_2.$$ 

Then the brackets are given by

$$[e_3, e_4] = e_1, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = -e_4.$$ 

We summarize the results in the following Proposition.

**Proposition 2.14.** [9, Proposition 3.3] Suppose that $\mathfrak{g}$ is a $2|2$-dimensional non-Abelian metric Lie superalgebra. Then $\mathfrak{g}$ is isomorphic to the following ones:

1. $\mathfrak{g}_{2|2}^1$:
   $$[e_4, e_4] = e_1, \quad [e_2, e_4] = e_3.$$
2. $\mathfrak{g}_{2|2}^2$:
   $$[e_3, e_4] = e_1, \quad [e_2, e_3] = e_3, \quad [e_2, e_4] = -e_4.$$
Proof. It follows from Theorem 2.2.3 or Proposition 2.13.

3. Cohomology and deformations

In this section, we recall the definition of cohomology for Lie superalgebras. For more details, the reader is referred to [4]. Suppose \( g \) is a Lie superalgebra and \( M \) is a \( g \)-module. For \( n \geq 0 \), the space of \( n \) cochains \( C^n(g, M) \) is defined by

\[
C^n(g, M) = \text{Hom}(\wedge^n g, M).
\]

The coboundary operator \( d : C^n(g, M) \to C^{n+1}(g, M) \) is defined by

\[
d(f)(x_0, ..., x_n) = \sum_{i=0}^{n} (-1)^{b_i + |x_i|/|f|} x_i \cdot f(x_0, ..., \hat{x}_i, ..., x_n)
\]
\[
+ \sum_{0 \leq p < q \leq n} (-1)^{c_{p,q}} f([x_p, x_q], x_0, ..., \hat{x}_p, ..., \hat{x}_q, ..., x_n),
\]

where

\[
b_i = i + |x_i|(|x_0| + \cdots + |x_{i-1}|),
\]
\[
c_{p,q} = p + q + (|x_p| + |x_q|)(|x_0| + \cdots + |x_{p-1}|) + |x_q|(|x_{p+1}| + \cdots + |x_{q-1}|),
\]

and \( \wedge \) denotes an omitted term. A standard fact is that \( d^2 = 0 \). We can define the \( n \)-cohomology of \( g \) with coefficients in \( M \) by

\[
H^n(g, M) = Z^n(g, M)/B^n(g, M),
\]

where

\[
Z^n(g, M) = \text{Ker}(d : C^n(g, M) \to C^{n+1}(g, M)),
\]
\[
B^n(g, M) = \text{Im}(d : C^{n-1}(g, M) \to C^n(g, M)).
\]

A cochain is called a cocycle (resp. coboundary) if it is in \( Z^n(g, M) \) (resp. \( B^n(g, M) \)). The cohomology theory has many applications in mathematics and physics [4, 6, 14, 22]. One of the most important applications of cohomology is computing formal deformations, for which we need the even part of the second cohomology with adjoint coefficients \( H^2(g, g) \).

Definition 3.1. A formal 1-parameter deformation of a Lie superalgebra \((g, [, ])\) (for associative algebras, see [15]) is a family of Lie superalgebra structures on the \( \mathbb{C}[t] \)-module \( g[t] = g \otimes \mathbb{C}[t] \) such that

\[
[\cdot, \cdot]_t = [\cdot, \cdot] + \sum_{i=1}^{\infty} t^i \phi_i(\cdot, \cdot)
\]

where each \( \phi_i \) is in \( C^2(g, g)_0 \).

Remark 1. By definition, a formal 1-parameter deformation of \( g \) is a family of Lie superalgebra structures on \( g \), parameterized by \( \mathbb{C}[t] \). For general theory, readers are referred to [4, 11, 12, 14].

Definition 3.2. If the super Jacobi identity for \([\cdot, \cdot]_t\) holds up to order \( n \), then \([\cdot, \cdot]_t\) is called a deformation of order \( n \). In particular, if the super Jacobi identity for \([\cdot, \cdot]_t\) is satisfied only up to the \( t \)-term, then \([\cdot, \cdot]_t\) is called a first order deformation or infinitesimal deformation.

Remark 2. By definition, a deformation of order \( n \) is a family of Lie superalgebra structures on \( g \), parameterized by \( \mathbb{C}[t]/(t^{n+1}) \).
Definition 3.3. Suppose that \([.,.]_t = \sum t^i \phi_i\) and \([.,.]_t' = \sum t^i \phi_i'\) are two formal 1-parameter deformations of \(\mathfrak{g}\). The deformations \([.,.]_t\) and \([.,.]_t'\) are called equivalent if there exists a linear isomorphism \(\hat{\psi}_t = \text{id}_\mathfrak{g} + \psi_1 t + \psi_2 t^2 + \cdots\), where \(\psi_i\) is in \(C^1(\mathfrak{g}, \mathfrak{g})_0\), such that

\[
\hat{\psi}_t([x,y]) = [\hat{\psi}_t(x), \hat{\psi}_t(y)], \quad \text{for } x, y \in \mathfrak{g}.
\]

Definition 3.4. A formal 1-parameter deformation is called trivial, if it is equivalent to the original bracket. If every formal 1-parameter deformation of \(\mathfrak{g}\) is trivial, then \(\mathfrak{g}\) is called rigid.

Remark 3. Suppose that \([.,.]_t = \sum t^i \phi_i\) and \([.,.]_t' = \sum t^i \phi_i'\) are two formal 1-parameter deformations of \(\mathfrak{g}\). The skew-supersymmetry for \([.,.]_t\) follows from the fact, that each \(\phi_i\) is a cochain. The bracket \([.,.]_t\) is \(\mathbb{Z}_2\)-graded, because each \(\phi_i\) is even. The super Jacobi identity implies that \(\phi_i\) is indeed an even cocycle. More generally, if \(\phi_1\) vanishes identically, the first non-vanishing \(\phi_i\) will be an even cocycle. If \([.,.]_t = \sum t^i \phi_i\) and \([.,.]_t' = \sum t^i \phi_i'\) are equivalent, then there exists a linear isomorphism \(\hat{\psi}_t = \text{id}_\mathfrak{g} + \psi_1 t + \psi_2 t^2 + \cdots\). Comparing the two sides, we obtain that

\[
\phi_1 - \phi_1' = d(\psi_1).
\]

It means that every equivalence class of deformations defines uniquely a cohomology class of the even part of the 2-cohomology. We call the cocycle \(\phi_1\) the infinitesimal part of the deformation. On the other hand, if every even 2-cocycle of \(\mathfrak{g}\) is a coboundary, then \(\mathfrak{g}\) only has trivial deformations.

Theorem 3.0.4. The even 2-cohomology classes are in 1-1 correspondence with nonequivalent infinitesimal deformations.

Theorem 3.0.5. If \(H^2(\mathfrak{g}, \mathfrak{g})_0 = 0\), then \(\mathfrak{g}\) is rigid.

Given a cohomology class \([x]\), a natural question is whether there exists a formal 1-parameter deformation with the infinitesimal part being a representative of \([x]\). In order to answer this question, we introduce the following definition.

Definition 3.5. [16] A complex \(C = (C^n, d)^\infty_{n=0}\) with an operation \([.,.]\) is called a Lie \(\mathbb{Z}_2\)-graded superalgebra if

\[
[x,y] = -(\mathbb{1})^{-|x||y|+pq}[y,x],
\]

\[
d([x,y]) = [d(x), y] + (-1)^p [x, d(y)],
\]

\[
(-1)^{|x||z|+pq}[x, [y, z]] + (-1)^p [y, [x, z]] + (-1)^{|x||y|+pq}[z, [x, y]] = 0,
\]

for any \(x \in C^n, y \in C^l, z \in C^m\).

For \(x \in C^p(\mathfrak{g}, \mathfrak{g}), \beta \in C^q(\mathfrak{g}, \mathfrak{g})\), the product \(x \beta \in C^{p+q-1}(\mathfrak{g}, \mathfrak{g})\) is defined by

\[
(x \beta)(x_1, \ldots, x_{p+q-1}) = \sum_{1 \leq i_1 \leq \cdots \leq i_{p-1} \leq p+q-1} (-1)^{a_{i_1,\ldots,i_{p-1}}} \beta(x_{i_1}, \ldots, x_{i_{p-1}}, x_p)
\]

where

\[
a_{i_1,\ldots,i_{p-1}} = \sum_{j=1}^{p-1} (i_s - s + a_{i_1,\ldots,i_{p-1}}),
\]

\[
a_{i_1,\ldots,i_{p-1}} = |x_{i_1}| \left( |\beta| + \sum_{t \in \{1,\ldots,i_{p-1}\} \setminus \{i_1,\ldots,i_{p-1}\}} |x_t| \right).
\]

Define the bracket operation \([x, \beta] = x \beta - (-1)^{|x||\beta|+(p-1)(q-1)} \beta x\).
Theorem 3.0.6. [16] If we denote \( C^q = C^{q+1}(g, g) \) and \( \mathcal{H}^q = H^{q+1}(g, g) \), then the above bracket makes \( C = \oplus_q C^q \) and \( \mathcal{H} = \oplus_q \mathcal{H}^q \) Lie \( \mathbb{Z} \)-graded superalgebras.

Lemma 3.6. [16, Lemma 3.3.1] The 2-cochain sequence \( \{\phi_n\}_{n=1}^{\infty} \) defines a formal deformation of \( g \) if and only if the elements \( \phi_n \) in \( C^2(\mathfrak{g}, \mathfrak{g})_0 \) satisfy the equations

\[
\text{d}(\phi_n) + \frac{1}{2} \sum_{i+j=n \atop i,j \geq 0} [\phi_i, \phi_j] = 0, \quad \text{for any } n \geq 1.
\]

Remark 4. Suppose that \( \phi_1 \) is a cocycle. If \( [\phi_1, \phi_1] = 0 \), then a formal deformation with the infinitesimal part \( \phi_1 \) can be given by taking \( \phi_i = 0 \) for all \( i \geq 2 \).

Definition 3.7. A formal 1-parameter deformation is called

1. metric deformation if it defines a metric Lie superalgebra;
2. jump deformation if for any non-zero value of the parameter \( t \) near the origin, it gives isomorphic algebra (which is of course different from the original one);
3. smooth deformation if for any different non-zero values of the parameter near the origin, it defines non-isomorphic algebras (by symmetry sometimes there can be coincidences).

Definition 3.8. An infinitesimal deformation of \( \mathfrak{g} \) is called

1. real, if no higher order terms in \( t \) appear in the super Jacobi identity;
2. metric, if it defines a metric Lie superalgebra on \( \mathfrak{g} \), parameterized by \( \mathbb{C}[\![t]\!]/(t^2) \).

Remark 5. By definition, an infinitesimal deformation is real if and only if the bracket of its infinitesimal part is zero.

Remark 6. Note that the corresponding infinitesimal deformation of a metric deformation is a metric infinitesimal deformation, which allows us to construct a metric deformation by starting with a metric infinitesimal deformation. In particular, if this metric infinitesimal deformation is real (i.e., it defines a metric Lie superalgebra), then it gives a metric deformation without any further obstruction (for more examples, see [11]).

Consider now two special cases, namely Abelian Lie superalgebras and simple Lie superalgebras with non-degenerate Killing forms. Note that an Abelian Lie superalgebra with an even-dimensional odd part is always metric by Proposition 2.3. However, Abelian Lie superalgebras deform everywhere, since the coboundary operator is zero. Another special case is of simple Lie superalgebras with non-degenerate Killing form. They only have trivial deformations because of the triviality of the cohomology, according to the following Theorem.

Theorem 3.0.7. [20, Theorem 3] Let \( \mathfrak{g} \) be a semi-simple Lie superalgebra over a field of characteristic zero. Suppose that \( M \) is a \( \mathfrak{g} \)-module, defined by a nontrivial irreducible representation \( \rho \), and the super trace form \( T(\rho(X)\rho(Y)) \), corresponding to this representation, is non-degenerate. Then \( H^n(\mathfrak{g}, M) = 0 \) for any \( n \).

Remark 7. A basic fact is that every semi-simple Lie algebra is rigid [7, Theorem 24.1]. Although it is not the case for semi-simple Lie superalgebras, any finite-dimensional Lie superalgebra with non-degenerate Killing form can be decomposed into a direct sum of a semi-simple Lie algebra and classical simple Lie superalgebras [19, 22].
4. Low-dimensional metric Lie superalgebras and their deformations

From now on, we suppose that \( g_{m|n} \) is a \( m|n \)-dimensional non-Abelian metric Lie superalgebra, spanned by

\[
\{ e_1, \ldots, e_m \mid e_{m+1}, \ldots, e_{m+n} \},
\]

where \( |e_1| = \cdots = |e_m| = 0 \) and \( |e_{m+1}| = \cdots = |e_{m+n}| = 1 \). Define \( e_{ij}^k \in C^2(g_{m|n}, g_{m|n}) \) by \( e_{ij}^k : (e_i, e_j) \mapsto e_k, \ 1 \leq i, j, k \leq m + n \).

The classification and metric deformations of the metric Lie algebras of dimension \( \leq 6 \) have been studied in [13]. In particular, there are only \( sl(2, \mathbb{C}) \) and the diamond Lie algebra \( b \) which are indecomposable metric Lie algebras of dimension \( \leq 4 \).

**Lemma 4.1.** Suppose that \( g \) is a metric Lie superalgebra.

1. If \( g_0 \) is 2-dimensional, then \( g_0 \cong \mathbb{C}_{2|0} \).
2. If \( g_0 \) is 3-dimensional, then \( g_0 \cong \mathbb{C}_{3|0} \) or \( sl(2, \mathbb{C}) \).
3. If \( g_0 \) is 4-dimensional, then \( g_0 \cong \mathbb{C}_{4|0}, sl(2, \mathbb{C}) \oplus \mathbb{C} \) or \( b \).

Suppose that \( g \) is a non-Abelian metric Lie superalgebra of dimension \( \leq 6 \), which has a non-zero odd part. Then \( \dim g_0 = 2 \) or \( \dim g_1 = 2 \) by Corollary 2.4 and Lemma 2.7. Consequently, all indecomposable metric Lie superalgebras, which have non-zero odd parts, have been classified by 1-dimensional double extension in [8]. By Corollary 2.4 and Lemma 2.7, these metric Lie superalgebras are 2|2, 3|2, 4|2 and 2|4 dimensional. In this section, we study nontrivial metric deformations of these metric Lie superalgebras case by case, and we also point out jump and smooth deformations among them.

4.1. 2|2-dimensional metric Lie superalgebras

There are two indecomposable metric Lie superalgebras: \( g^1_{2|2} \) and \( g^2_{2|2} \) with nontrivial brackets [8]:

1. \( g^1_{2|2} : [e_4, e_4] = e_1, [e_2, e_4] = e_5, \)
2. \( g^2_{2|2} : [e_3, e_4] = e_1, [e_2, e_3] = e_5, [e_2, e_4] = -e_4. \)

We have already obtained the classification by double extension in Proposition 2.14.

**Theorem 4.1.1.** The algebra \( g^1_{2|2} \) has only one metric deformation, which is a jump deformation to \( g^2_{2|2} \).

**Proof.** The even part of the 2-cohomology space is 4-dimensional, spanned by the representative even cocycles:

\[
f_1 = e_1^{1,2} - \frac{1}{2} e_1^{3,4}, \quad f_2 = e_2^{1,2} + e_3^{1,3} - \frac{1}{2} e_2^{3,4}, \quad f_3 = e_3^{2,3}, \quad f_4 = e_4^{2,3} - e_1^{3,3}.\]

Let \( [\cdot, \cdot]_t \) be the metric infinitesimal deformation defined by \( \sum_{i=1}^4 a_i f_i \) \( (a_i \in \mathbb{C}) \). Then \( (g_{2|2}^{1}, [\cdot, \cdot]_t) \), parameterized by \( \mathbb{C}[[t]]/(t^2) \), is a metric Lie superalgebra with respect to an invariant scalar product \( B \). We have the even part \( ((g^1_{2|2})_0, [\cdot, \cdot]_t) \cong \mathbb{C}_{2|0} \) by Lemma 4.1 (1). So \( a_1 = a_2 = 0 \). From the invariance of \( B \), one has

\[
B([e_2, e_3], e_4) = B(e_2, [e_3, e_4]) = 0.
\]

Thus \( a_3 = 0 \). From \( [f_4, f_4] = 0 \), the cocycle \( a_4 f_4 \) \( (a_4 \neq 0) \) defines a real infinitesimal deformation isomorphic to \( g^2_{2|2} \), via the change of basis:
\[ e'_1 = -2a_4te_1, \quad e'_2 = \frac{1}{\sqrt{a_4t}}e_2, \quad e'_3 = e_3 + \sqrt{a_4t}e_4, \quad e'_4 = e_3 - \sqrt{a_4t}e_4. \]

**Theorem 4.1.2.** The algebra \( g_{2|2}^1 \) has only trivial metric deformation.

**Proof.** The even part of the 2-cohomology space is 1-dimensional, spanned by the representative even cocycle \( f = -e_1^2 + e_4^4 \). Let \( [\cdot, \cdot]_f \) be the metric infinitesimal deformation defined by \( af \) \((a \in \mathbb{C})\). By Lemma 4.1 (1), we have \( ((g_{2|2}^1)_0, [\cdot, \cdot]_f) \cong \mathbb{C}^2_{2|0} \), parameterized by \( \mathbb{C}[t]/(t^2) \), so \( a = 0 \). Therefore it has only trivial metric deformations. \( \square \)

### 4.2. 3|2-dimensional metric Lie superalgebras

There is only one indecomposable metric Lie superalgebra \( \mathfrak{osp}(1, 2) \) with nontrivial brackets [8]:
\[
\begin{align*}
[e_1, e_2] &= e_3, \quad [e_1, e_3] = -2e_1, \quad [e_2, e_3] = 2e_2, \quad [e_1, e_5] = -e_4, \quad [e_2, e_4] = -e_5, \\
[e_3, e_4] &= e_4, \quad [e_3, e_5] = -e_5, \quad [e_4, e_4] = \frac{1}{2}e_1, \quad [e_4, e_3] = \frac{1}{4}e_3, \quad [e_5, e_5] = -\frac{1}{2}e_2.
\end{align*}
\]

**Theorem 4.2.1.** The algebras \( \mathfrak{osp}(1, 2) \) has only trivial metric deformation.

**Proof.** Since the Killing form of \( \mathfrak{osp}(1, 2) \) is non-degenerate [19], \( \mathfrak{osp}(1, 2) \) is rigid by Theorem 3.0.5 and Theorem 3.0.7. \( \square \)

### 4.3. 4|2-dimensional metric Lie superalgebras

There are the following indecomposable metric Lie superalgebras: \( g_{4|2}^1 \), \( g_{4|2}^2(\lambda) \ (\lambda \neq 0) \) and \( g_{4|2}^3 \) with nontrivial brackets [8]:

1. \( g_{4|2}^1 : \]
\[
\begin{align*}
[e_1, e_2] &= e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = e_4, \\
[e_1, e_5] &= e_5, \quad [e_6, e_6] = e_4.
\end{align*}
\]

2. \( g_{4|2}^2(\lambda) \ (\lambda \neq 0) : \]
\[
\begin{align*}
[e_1, e_2] &= e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = e_4, \\
[e_1, e_5] &= \lambda e_5, \quad [e_1, e_6] = -\lambda e_6, \quad [e_5, e_6] = \lambda e_4.
\end{align*}
\]

3. \( g_{4|2}^3 : \]
\[
\begin{align*}
[e_1, e_2] &= e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = e_4, \\
[e_1, e_5] &= \frac{1}{2}e_5, \quad [e_1, e_6] = -\frac{1}{2}e_6, \quad [e_2, e_6] = e_5, \\
[e_5, e_6] &= -\frac{1}{2}e_4.
\end{align*}
\]

Here \( g_{4|2}^2(\lambda_1) \cong g_{4|2}^2(\lambda_2) \) if and only if \( \lambda_1 = \lambda_2 \). However, \( g_{4|2}^2(\lambda) \) is isomorphic to \( g_{4|2}^2(-\lambda) \), via the change of basis:
\[
\begin{align*}
e'_1 &= -e_1, \quad e'_2 = e_3, \quad e'_3 = e_2, \quad e'_4 = -e_4, \quad e'_5 = e_5, \quad e'_6 = e_6.
\end{align*}
\]

**Theorem 4.3.1.** The algebra \( g_{4|2}^1 \) has only one metric deformation, which is a smooth deformation to the family \( g_{4|2}^2(\lambda) \) around \( \lambda = 0 \).
Proof. The even part of the 2-cohomology space is 3-dimensional, spanned by the representative even cocycles:

\[ f_1 = e^{1.5}_1, \quad f_2 = e^{1.5}_6 - e^{5.5}_4, \quad f_3 = e^{1.2}_4 + e^{1.4}_4 + \frac{1}{2} e^{5.6}_4. \]

If the cocycle \( \sum_{i=1}^3 a_i f_i \) defines a metric infinitesimal deformation, then we have \( a_1 = a_3 = 0 \). From \([f_2, f_2] = 0\), the cocycle \( a_2 f_2 \) \((a_2 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{4|2}^2(\sqrt{a_2}) \), via the change of basis:

\[ e'_1 = e_1, \quad e'_2 = e_2, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = -\frac{1}{2}\sqrt{a_2 t} e_5 - \frac{1}{2} e_6, \quad e'_6 = e_5 - \sqrt{a_2 t} e_6. \]

\[ \square \]

Theorem 4.3.2. Metric deformations of the family \( g_{4|2}^2(\lambda) \) \((\lambda \neq 0)\) follow two different patterns:

1. The generic element for \( \lambda \neq \pm \frac{1}{2} \) has one metric deformation, which is a smooth metric deformation to the family \( g_{4|2}^2(\lambda) \) around itself.

2. The special element \( g_{4|2}^2(\frac{1}{2}) \cong g_{4|2}^2(-\frac{1}{2}) \) has a smooth metric deformation to the family \( g_{4|2}^2(\lambda) \) around itself and it has jump metric deformations to \( g_{4|2}^3 \) and \( \mathfrak{osp}(1,2) \oplus \mathbb{C}_{1|0} \).

Proof. Note that the case \( \lambda = 0 \) is generally excluded from this family, because \( g_{4|2}^2(0) = \mathfrak{b} \oplus \mathbb{C}_{0|2} \). Beside that, in the cases \( \lambda = \pm \frac{1}{2} \), the cohomology and deformation patterns are not generic.

1. The even part of the 2-cohomology space is 2-dimensional, spanned by the representative even cocycles:

\[ f_1 = e^{1.5}_1 - e^{1.6}_6, \quad f_2 = e^{1.2}_2 + e^{1.4}_4 + e^{1.6}_6. \]

If the cocycle \( \sum_{i=1}^2 a_i f_i \) defines a metric infinitesimal deformation, then we have \( a_2 = 0 \). From \([f_1, f_1] = 0\), the cocycle \( a_1 f_1 \) \((a_1 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{4|2}^2(\lambda + a_1 t) \), via the change of basis:

\[ e'_1 = e_1, \quad e'_2 = \lambda e_2, \quad e'_3 = \frac{1}{\lambda + a_1 t} e_3, \quad e'_4 = \frac{\lambda}{\lambda + a_1 t} e_4, \quad e'_5 = e_5, \quad e'_6 = e_6. \]

2. From the isomorphism \( g_{4|2}^2(\frac{1}{2}) \cong g_{4|2}^2(-\frac{1}{2}) \), it is sufficient to consider \( g_{4|2}^2(\frac{1}{2}) \). The even part of the 2-cohomology space is 4-dimensional, spanned by the representative even cocycles:

\[ f_1 = e^{1.5}_1 - e^{1.6}_6, \quad f_2 = e^{1.2}_2 + e^{1.4}_4 + e^{1.6}_6, \quad f_3 = e^{5.6}_5 + e^{3.6}_6, \quad f_4 = e^{3.5}_6 - e^{5.5}_2. \]

If the cocycle \( \sum_{i=1}^4 a_i f_i \) defines a metric infinitesimal deformation, then we have that \( a_i = 0, i \neq 1, \) or \( a_1 = a_2 = 0 \). We determine nontrivial metric deformation in the following cases:

Case 1: \( a_i = 0, i \neq 1 \). The cocycle \( a_1 f_1 \) \((a_1 \neq 0)\) leads a smooth metric deformation around itself (see (1)).

Case 2: \( a_1 = a_2 = a_3 = 0 \). The cocycle \( a_4 f_4 \) \((a_4 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{4|2}^3 \), via the change of basis:

\[ e'_1 = -e_1, \quad e'_2 = e_2, \quad e'_3 = -a_4 t e_2, \quad e'_4 = a_4 t e_4, \quad e'_5 = a_4 t e_6, \quad e'_6 = e_5. \]

Case 3: \( a_1 = a_2 = a_3 = 0 \). The cocycle \( a_4 f_5 \) \((a_3 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{4|2}^3 \), via the change of basis:

\[ e'_1 = e_1, \quad e'_2 = e_2, \quad e'_3 = a_3 t e_3, \quad e'_4 = a_3 t e_4, \quad e'_5 = a_3 t e_5, \quad e'_6 = e_6. \]

Case 4: \( a_1 = a_2 = 0, a_3 a_4 \neq 0 \). The infinitesimal deformation, defined by the cocycle \( a_3 f_3 + a_4 f_4 \), is not real, since \([a_3 f_3 + a_4 f_4, a_3 f_3 + a_4 f_4] \neq 0\). However, it can be extended to a metric
deformation. An example can be given by taking
\[ \phi_1 = a_3f_3 + a_4f_4, \quad \phi_2 = 2a_3a_4e_1^{2,3} + a_3a_4e_1^{5,6}. \]
Then we have
\[ d(\phi_2) = -\frac{1}{2} [\phi_1, \phi_1], \quad [\phi_1, \phi_2] = [\phi_2, \phi_2] = 0. \]
Moreover, the bracket
\[ [., .]_t^1 = [., .] + t\phi_1(., ) + t^2\phi_2(., ) \]
defines a metric Lie superalgebra isomorphic to \( \mathfrak{osp}(1, 2) \oplus \mathbb{C}_{1|0} \), via the change of basis:
\[
\begin{align*}
    e_1' &= \frac{1}{a_3} e_2, \\
    e_2' &= \frac{1}{a_4} e_3, \\
    e_3' &= \frac{1}{a_3a_4} e_4 + 2e_1, \\
    e_4' &= \frac{1}{a_3a_4} e_5, \\
    e_5' &= \frac{1}{a_3a_4} e_6.
\end{align*}
\]

**Theorem 4.3.3.** The algebra \( \mathfrak{g}_{4|2}^3 \) has only one metric deformation, which is a jump deformation to \( \mathfrak{osp}(1, 2) \oplus \mathbb{C}_{1|0} \).

**Proof.** The even part of the 2-cohomology space is 2-dimensional, spanned by the representative even cocycles:
\[
    f_1 = e_2^{1,2} + e_4^{1,4} + e_5^{1,5}, \quad f_2 = e_1^{2,3} + \frac{1}{2} e_6^{3,5} - \frac{1}{2} e_2^{4,5} + \frac{1}{2} e_1^{5,6}.
\]
If the cocycle \( \sum_{i=1}^2 a_i f_i \) defines a metric infinitesimal deformation, then we have \( a_1 = 0 \). From \( [f_1, f_2] = 0 \), the cocycle \( a_2 f_2 \) \( (a_2 \neq 0) \) defines a real infinitesimal deformation isomorphic to \( \mathfrak{osp}(1, 2) \oplus \mathbb{C}_{1|0} \), via the change of basis:
\[
    e_1' = -e_2, \quad e_2' = -\frac{2}{a_2} e_3, \quad e_3' = \frac{2}{a_2} e_4 + 2e_1, \quad e_4' = e_5, \quad e_5' = \frac{1}{\sqrt{a_2}} e_6, \quad e_6' = \frac{1}{\sqrt{a_2}} e_5.
\]

**Conclusion 1.** In dimension \( 4|2 \), except for \( \mathfrak{g}_{4|2}^3 (\frac{1}{2}) \), every metric infinitesimal deformation is real. We summarize the metric deformation picture of \( 4|2 \)-dimensional indecomposable metric Lie superalgebras in the table given below.

| Algebra | dim \( H^2_0 \) | Jump deformation | Smooth deformation |
|---------|-----------------|------------------|--------------------|
| \( \mathfrak{g}_{4|2}^1 \) | 3 | - | \( \mathfrak{g}_{4|2}^2 (0) = \mathfrak{b} \oplus \mathbb{C}_{0|2} \) |
| \( \mathfrak{g}_{4|2}^2 (\lambda), \lambda \neq 0, \pm \frac{1}{2} \) | 2 | - | \( \mathfrak{g}_{4|2}^2 (\lambda) \) |
| \( \mathfrak{g}_{4|2}^3 (\pm \frac{1}{2}) \) | 4 | \( \mathfrak{g}_{4|2}^3, \mathfrak{osp}(1, 2) \oplus \mathbb{C}_{1|0} \) \( \mathfrak{g}_{4|2}^2 (\pm \frac{1}{2}) \) |
| \( \mathfrak{g}_{4|2}^4 \) | 2 | \( \mathfrak{osp}(1, 2) \oplus \mathbb{C}_{1|0} \) | - |

In particular, we get the picture of jump metric deformations as follows:
where the down arrows show jump metric deformations.

### 4.4. 2|$4$-dimensional metric Lie superalgebras

Up to now, in the literature there were the following nonisomorphic indecomposable metric Lie superalgebras: $g^1_{2|4}$, $g^2_{2|4}$, $g^3_{2|4}(\lambda)$ with $\lambda \neq 0$, and $g^4_{2|4}$, with nontrivial brackets [8]:

1. $g^1_{2|4}$:
   
   $[e_2, e_4] = e_3$, $[e_2, e_5] = -e_6$, $[e_4, e_5] = e_1$,

2. $g^2_{2|4}$:
   
   
   $[e_2, e_4] = e_4$, $[e_2, e_5] = e_3$, $[e_2, e_6] = -e_6$,
   $[e_5, e_5] = [e_4, e_6] = e_1$,

3. $g^3_{2|4}(\lambda)$:
   
   $[e_2, e_3] = e_3$, $[e_2, e_4] = \lambda e_4$, $[e_2, e_5] = -e_5$,
   $[e_2, e_6] = -\lambda e_6$, $[e_3, e_5] = e_1$, $[e_4, e_6] = \lambda e_1$,

4. $g^4_{2|4}$:
   
   $[e_2, e_3] = e_3$, $[e_2, e_4] = e_3 + e_4$, $[e_2, e_5] = -e_5 - e_6$,
   $[e_2, e_6] = -e_6$, $[e_3, e_3] = [e_4, e_5] = [e_4, e_6] = e_1$.

Here $g^3_{2|4}(\lambda) \overset{i}{\simeq} g^3_{2|4}(-\lambda) \overset{i}{\simeq} g^3_{2|4}(\lambda^{-1})$.

In addition, the algebra $g^4_{2|4}$ is also referred to $g^5_{6, 7}$ in [8], although there is a mistake of the multiplication of the algebra $g^5_{6, 7}$ in that paper.

When computing metric deformations, we found a new metric Lie superalgebra $g^5_{2|4}$, which is not isomorphic to any of the above, and is missing from the classification in [8]. This superalgebra we denoted by $g^5_{2|4}$, and it has the following nonzero superbrackets:

$g^5_{2|4}$:

$[e_2, e_3] = e_5$, $[e_2, e_4] = e_3$, $[e_2, e_5] = -e_6$,

$[e_3, e_3] = -e_1$, $[e_4, e_5] = e_1$.

Two invariant scalar products on this algebra can be given by

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Remark 4.2.** This is an advantage of using deformation theory in classification problems.

So we have the following Theorem.

**Theorem 4.4.1.** There are four nonisomorphic metric Lie superalgebras and a family of metric Lie superalgebras in dimension 2|$4$: $g^1_{2|4}$, $g^2_{2|4}$, $g^3_{2|4}(\lambda)$ with $\lambda \neq 0$, $g^4_{2|4}$ and $g^5_{2|4}$.
Theorem 4.4.2. The algebra $\mathfrak{g}^1_{2|4}$ has jump metric deformations to $\mathfrak{g}^2_{2|4} \oplus \mathbb{C}_{0|2}$, $\mathfrak{g}^3_{2|4}$, $\mathfrak{g}^3_{2|4}(1)$, $\mathfrak{g}^3_{2|4}(\sqrt{-1})$, $\mathfrak{g}^4_{2|4}$, $\mathfrak{g}^5_{2|4}$ and it has smooth metric deformations to the family $\mathfrak{g}^3_{2|4}(\lambda)$ around $\lambda = 0, 1, \sqrt{-1}$.

Proof. The even part of the 2-cohomology space is 9-dimensional, spanned by the representative even cocycles:

$$f_1 = e_2^{2,3}, f_2 = e_2^{3,5}, f_3 = e_4^{3,4}, f_4 = e_6^{2,6}, f_5 = e_1^{1,2} + e_4^{1,6}, f_6 = e_2^{2,3} - e_3^{3,3}, f_7 = e_4^{3,4} + e_1^{4,6}, f_8 = e_4^{4,6} + e_1^{6,6}, f_9 = e_4^{4,3} - e_5^{5,6} + e_1^{3,6}.$$ 

If the cocycle $\sum_{i=1}^9 a_if_i$ defines a metric infinitesimal deformation, then we have $a_1 = a_2 = a_3 = a_5 = 0, a_7 = -a_4$ or $a_3 = a_7 = 0, a_4 = -a_1, a_6 = a_2x, a_8 = a_3x, a_9 = a_1x$ for some $x \neq 0$. From the brackets, we get

$$\begin{align*}
[a_4(f_4 - f_7) + a_6f_6 + a_8f_8 + a_9f_9, a_4(f_4 - f_7) + a_6f_6 + a_8f_8 + a_9f_9] &= 0, \\
[a_1(f_1 - f_4 + x f_9) + a_2(f_2 + x f_6) + a_3(f_3 + x f_8), a_1(f_1 - f_4 + x f_9) + a_2(f_2 + x f_6) + a_3(f_3 + x f_8)] &= 0,
\end{align*}$$

and all metric infinitesimal deformations are real deformations. The proof is complete by a similar argument as in the proof of Theorem 4.3.2. \hfill \Box

Theorem 4.4.3. The algebra $\mathfrak{g}^2_{2|4}$ has only one metric deformation, which is a smooth metric deformation to the family $\mathfrak{g}^3_{2|4}(\lambda)$ around $\lambda = 0$.

Proof. The even part of the 2-cohomology space is 3-dimensional, spanned by the representative even cocycles:

$$f_1 = e_2^{2,3}, f_2 = e_2^{3,5} - e_3^{3,3}, f_3 = e_1^{1,2} - e_4^{2,6} - \frac{1}{2} e_5^{3,5}.$$ 

If the cocycle $\sum_{i=1}^3 a_if_i$ defines a metric infinitesimal deformation, then we have $a_1 = a_3 = 0$. From $[f_2, f_2] = 0$, the cocycle $a_2f_2 (a_2 \neq 0)$ defines a real infinitesimal deformation isomorphic to $\mathfrak{g}^3_{2|4}(\sqrt{-\alpha_2})$, via the change of basis:

$$e_1' = -2a_2te_1, \quad e_2' = \frac{1}{\sqrt{\alpha_2}} e_2, \quad e_3' = e_3 + \sqrt{\alpha_2} e_5, \quad e_4' = -2\sqrt{\alpha_2} e_4, \quad e_5' = e_5 - \sqrt{\alpha_2} e_5, \quad e_6' = e_6.$$ 

\hfill \Box

Theorem 4.4.4. Metric deformations of the family $\mathfrak{g}^3_{2|4}(\lambda)$ ($\lambda \neq 0$) are the following:

1. The generic element for $\lambda \neq \pm 1$ has only one metric deformation, which is a smooth deformation to the family $\mathfrak{g}^3_{2|4}(\lambda)$ around itself.
2. The special element $\mathfrak{g}^3_{2|4}(1) \cong \mathfrak{g}^3_{2|4}(-1)$ has a smooth metric deformation to the family $\mathfrak{g}^3_{2|4}(\lambda)$ around itself and it has a jump metric deformation to $\mathfrak{g}^4_{2|4}$.

Proof. Note that the case $\lambda = 0$ is generally excluded from this family, because $\mathfrak{g}^3_{2|4}(0) = \mathfrak{g}^2_{2|4} \oplus \mathbb{C}_{0|2}$. Besides that, in the cases $\lambda = \pm 1$, the cohomology and deformation patterns are not generic.

1. The even part of the 2-cohomology space is 2-dimensional, spanned by the representative even cocycles:

$$f_1 = e_2^{2,3} - e_5^{2,5}, f_2 = e_1^{2,5} - e_5^{2,5} - e_6^{2,6}.$$
If the cocycle \( \sum_{i=1}^{2} a_i f_i \) defines a metric infinitesimal deformation, then we have \( a_2 = 0 \). From \([f_1, f_1] = 0\), the cocycle \( a_1 f_1 \) \((a_1 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^2 \left( \frac{j}{1 + \lambda a_i} \right) \), via the change of basis:

\[
e_1' = (1 + a_1 t) e_1, \quad e_2' = \frac{1}{1 + a_1 t} e_2, \quad e_3' = (1 + a_1 t) e_3, \quad e_4' = e_4, \quad e_5' = e_5, \quad e_6' = e_6.
\]

(2) From the \( i \)-isomorphism \( g_{3|4}^3(1) \wedge g_{3|4}^3(1) \), it is sufficient to consider \( g_{3|4}^3(1) \). The even part of the 2-cohomology space is 4-dimensional, spanned by the representative even cocycles:

\[
f_1 = e_3^2 - e_5^2, \quad f_2 = e_1^1 - e_5^2 - e_6^3, \quad f_3 = e_3^4 - e_6^3, \quad f_4 = e_4^2 - e_5^3.
\]

If the cocycle \( \sum_{i=1}^{4} a_i f_i \) defines a metric infinitesimal deformation, then we have that \( a_2 = a_3 a_4 = 0 \). We determine nontrivial metric deformation in the following cases:

Case 1: \( a_1 \neq 0, \quad a_3 = 0 \). The cocycle \( a_1 f_1 + a_4 f_4 \) \((a_1 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^3(1 + a_1 t) \), via the change of basis:

\[
e_1' = e_1, \quad e_2' = e_2, \quad e_3' = e_3 + \frac{a_1}{a_4} e_4, \quad e_4' = e_5 - \frac{a_4}{a_1} e_5, \quad e_6' = (1 + a_1 t) e_5.
\]

Case 2: \( a_1 a_3 \neq 0, \quad a_4 = 0 \). The cocycle \( a_1 f_1 + a_4 f_3 \) \((a_1 a_3 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^3(1 + a_1 t) \), via the change of basis:

\[
e_1' = e_1, \quad e_2' = e_2, \quad e_3' = e_4 - \frac{a_3}{a_1} e_3, \quad e_4' = e_5, \quad e_5' = e_6, \quad e_6' = (1 + a_1 t) \left( e_5 + \frac{a_3}{a_1} e_6 \right).
\]

Case 3: \( a_1 = a_3 = 0, \quad a_4 \neq 0 \). The cocycle \( a_1 f_4 \) \((a_4 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^4(1 + a_4 t) \), via the change of basis:

\[
e_1' = a_4 t e_1, \quad e_2' = e_2, \quad e_3' = a_4 t e_3, \quad e_4' = e_4, \quad e_5' = e_5 + a_4 t e_5, \quad e_6' = a_4 t e_5.
\]

Case 4: \( a_1 = a_4 = 0, \quad a_3 \neq 0 \). The cocycle \( a_3 f_3 \) \((a_3 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^3(1 \lambda) \), via the change of basis:

\[
e_1' = a_3 t e_1, \quad e_2' = e_2, \quad e_3' = a_3 t e_3, \quad e_4' = e_4, \quad e_5' = e_5 + a_3 t e_6, \quad e_6' = a_3 t e_6.
\]

\[\blacksquare\]

**Theorem 4.4.5.** The algebra \( g_{3|4}^4(1 \lambda) \) has only one metric deformation, which is a smooth deformation to the family \( g_{3|4}^3(1 \lambda) \) around \( \lambda = 1 \).

**Proof.** The even part of the 2-cohomology space is 2-dimensional, spanned by the representative even cocycles:

\[
f_1 = e_3^2 - e_5^2, \quad f_2 = e_1^1 - 2e_6^3 + e_4^3 + e_4^3.
\]

If the cocycle \( \sum_{i=1}^{2} a_i f_i \) defines a metric infinitesimal deformation, then we have \( a_2 = 0 \). From \([f_1, f_1] = 0\), the cocycle \( a_1 f_1 \) \((a_1 \neq 0)\) defines a real infinitesimal deformation isomorphic to \( g_{3|4}^3 \left( \frac{1 - a_1 t}{1 + a_1 t} \right) \), via the change of basis:

\[
e_1' = -2\sqrt{a_1 t} (1 + a_1 t) e_1, \quad e_2' = \frac{1}{\sqrt{a_1 t} + 1} e_2, \quad e_3' = e_3 + \sqrt{a_1 t} e_4, \quad e_4' = -e_4 + \sqrt{a_1 t} e_5, \quad e_5' = e_5.
\]

\[\blacksquare\]

**Theorem 4.4.6.** The algebra \( g_{3|4}^3(1 \lambda) \) has jump metric deformations to \( g_{3|4}^2 \) and \( g_{3|4}^3(\sqrt{-1}) \), and it has a smooth metric deformation to the family \( g_{3|4}^3(1 \lambda) \) around \( \lambda = \sqrt{-1} \).
Proof. The even part of the 2-cohomology space is 4-dimensional, spanned by the representative even cocycles:

$$f_1 = e_1^{1,4} + \frac{1}{2} e_1^{3,5} + \frac{3}{2} e_1^{4,6}, \quad f_2 = e_1^{2,4} + e_1^{4,6}, \quad f_3 = e_1^{2,5} + e_1^{5,5}, \quad f_4 = e_1^{2,6} + e_1^{6,6}.$$

If the cocycle $\sum_{i=1}^{4} a_i f_i$ defines a metric infinitesimal deformation, then we have $a_1 = a_2 = 0$. We determine nontrivial metric deformation in the following cases:

Case 1: $a_4 \neq 0$. Set $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, such that $a_4 t x^2 - a_3 t x^2 + 1 = 0$ and $\beta = \sqrt{\alpha^2 a_3 t - 1}$. The cocycle $a_3 f_3 + a_4 f_4$ ($a_4 \neq 0$) defines a real infinitesimal deformation isomorphic to $\mathfrak{g}_{2|4}^3(\beta)$, via the change of basis:

$$e'_1 = 2(a_3 t x^2 - 2) e_1, \quad e'_2 = \alpha e_2, \quad e'_3 = e_3 - a_4 t x^3 e_4 + \alpha e_5 - x^2 e_6,$$
$$e'_4 = -\beta^3 e_3 + a_4 t x^3 e_4 - \alpha \beta^2 e_5 + x^2 \beta e_6, \quad e'_5 = e_5 + x^3 a_4 t e_4 - \alpha e_5 - x^2 e_6,$$
$$e'_6 = e_3 + x^3 a_4 t e_4 - \alpha e_5 - x^2 e_6.$$

In particular, if $a_3 = 0$, we get a jump metric deformation to $\mathfrak{g}_{2|4}^3(\sqrt{-1})$. If $a_3 \neq 0$, we get a smooth metric deformation to the family $\mathfrak{g}_{2|4}^3(\lambda)$ around $\lambda = \sqrt{-1}$.

Case 2: $a_4 = 0$, $a_3 \neq 0$. The cocycle $a_3 f_3$ ($a_3 \neq 0$) defines a real infinitesimal deformation isomorphic to $\mathfrak{g}_{2|4}^3$, via the change of basis:

$$e'_1 = -2e_1, \quad e'_2 = \frac{1}{\sqrt{a_3 t}} e_2, \quad e'_3 = -\frac{\sqrt{2}}{a_3 t} e_6, \quad e'_4 = e_3 + \frac{1}{\sqrt{a_3 t}} e_5 - \frac{1}{a_3 t} e_6,$$
$$e'_5 = \sqrt{\frac{2}{a_3 t}} e_5 - \sqrt{2 a_3 t e_4}, \quad e'_6 = e_3 - \frac{1}{\sqrt{a_3 t}} e_5 - \frac{1}{a_3 t} e_6.$$

\[ \square \]

Conclusion 2. In dimension $2|4$, every metric infinitesimal deformation is real. We summarize the metric deformation picture of $2|4$-dimensional indecomposable metric Lie superalgebras in the table given below.

| Algebra   | dim $H_0^2$ | Jump deformation                          | Smooth deformation                  |
|-----------|-------------|-------------------------------------------|-------------------------------------|
| $\mathfrak{g}_{2|4}^1$ | 9           | $\mathfrak{g}_{2|4}^3(0), \mathfrak{g}_{2|4}^3(\pm 1)$, $\mathfrak{g}_{2|4}^3(\sqrt{-1})$, $\mathfrak{g}_{2|4}^3$ | $\mathfrak{g}_{2|4}^3(0), \mathfrak{g}_{2|4}^3(\pm 1), \mathfrak{g}_{2|4}^3(\sqrt{-1})$ |
| $\mathfrak{g}_{2|4}^2$ | 3           | $-$                                        | $\mathfrak{g}_{2|4}^3(0) = \mathfrak{g}_{2|2}^2 \oplus \mathbb{C}_{0|2}$ |
| $\mathfrak{g}_{2|4}^3(\lambda), \lambda \neq 0, \pm 1$ | 2           | $-$                                        | $\mathfrak{g}_{2|4}^3(\lambda)$ |
| $\mathfrak{g}_{2|4}^3(1) \cong \mathfrak{g}_{2|4}^3(-1)$ | 4           | $\mathfrak{g}_{2|4}^4$                    | $\mathfrak{g}_{2|4}^3(\pm 1)$ |
| $\mathfrak{g}_{2|4}^4$ | 2           | $-$                                        | $\mathfrak{g}_{2|4}^3(\pm 1)$ |
| $\mathfrak{g}_{2|4}^5$ | 4           | $\mathfrak{g}_{2|4}^2, \mathfrak{g}_{2|4}^3(\sqrt{-1})$ | $\mathfrak{g}_{2|4}^3(\sqrt{-1})$ |

In particular, we get the picture of jump metric deformations as follows:
where the down arrows show jump metric deformations.

**Acknowledgments**

The author thanks the Doctoral School of Physics, University of Pécs for support, and Prof. Alice Fialowski for giving the problem and regular useful discussions. At last, the author thanks the referee for the suggestions which improved the presentation.

**Funding**

The author is supported by the CSC (No.201906170132) and the NSF of China (11771176).

**References**

[1] Albuquerque, H., Barreiro, E., Benayadi, S. (2009). Quadratic Lie superalgebras with a reductive even part. *J. Pure Appl. Algebra* 213(5):724–731. DOI: 10.1016/j.jpaa.2008.09.016.

[2] Benamor, H., Benayadi, S. (1999). Double extension of quadratic Lie superalgebras. *Commun. Algebra* 27(1):67–88. DOI: 10.1080/00927879908826421.

[3] Benayadi, S. (2000). Quadratic Lie superalgebras with the completely reducible action of the even part on the odd part. *J. Algebra* 223(1):344–366. DOI: 10.1006/jabr.1999.8067.

[4] Binegar, B. (1986). Cohomology and deformations of Lie superalgebras. *Lett. Math. Phys.* 12(4):301–308. DOI: 10.1007/BF00402663.

[5] Bordemann, M. (1997). Nondegenerate invariant bilinear forms on nonassociative algebras. *Acta Math. Univ. Comenian.* 66(2):151–201.

[6] Bouarroudj, S., Grozman, P., Lebedev, A., Leites, D. (2010). Divided power (co)homology, presentations of simple finite dimensional modular Lie superalgebras with Cartan matrix. *Homology, Homotopy Appl.* 12(1):237–278. DOI: 10.4310/HHA.2010.v12.n1.a13.

[7] Chevalley, C., Eilenberg, S. (1948). Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.* 63(1):85–124. DOI: 10.1090/S0002-9947-1948-0024908-8.

[8] Duong, M. T. (2014). A classification of solvable quadratic and odd quadratic Lie superalgebras in low dimensions. *Rev. Un. Mat. Argentina.* 55(1):119–138.

[9] Duong, M. T., Ushirobira, R. (2014). Singular quadratic Lie superalgebras. *J. Algebra* 407:372–412. DOI: 10.1016/j.jalgebra.2014.02.034.

[10] Favre, G., Santharoubane, L. (1987). Symmetric, invariant, non-degenerate bilinear form on a Lie algebra. *J. Algebra* 105(2):451–464. DOI: 10.1016/0021-8693(87)90209-2.

[11] Fialowski, A. (1988). An example of formal deformations of Lie algebras. *Deform. Theory Algebra Appl.* 2(375–401.

[12] Fialowski, A., Fuchs, D. B. (1999). Construction of miniversal deformations of Lie algebras. *J. Funct. Anal.* 161(1):76–110. DOI: 10.1006/jfan.1998.3349.

[13] Fialowski, A., Penkava, M. (2021). On the cohomology of Lie algebras with an invariant inner product. *Int. Journal of Math.* arXiv:2009.08426.

[14] Fuchs, D. B. (1986). *Cohomology of Infinite-Dimensional Lie Algebras. Contemporary Soviet Mathematics.* New York, NY: Consultants Bureau.
[15] Gerstenhaber, M. (1963). The cohomology structure of an associative ring. *Ann. Math.* 78(2):267–288. DOI: 10.2307/1970343.

[16] HijligenbergNico, V. D. (1993). Computations and applications of Lie superalgebra cohomology (Thesis). University of Twente, the Netherlands, p. 158.

[17] Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory.* New York, NY: Springer.

[18] Kac, V. G. (1984). Infinite dimensional Lie algebras (An introduction). *Progress Math.* 44:1–252.

[19] Kac, V. G. (1977). A sketch of Lie superalgebra theory. *Commun. Math. Phys.* 53(1):31–64. DOI: 10.1007/BF01609166.

[20] Leites, D. A. (1975). Cohomologies of Lie superalgebras. *Funktsional. Anal. i Prilozhen.* 9(4):75–76.

[21] Medina, A., Revoy, P. (1985). Algèbres de Lie et produit scalaire invariant (French). *Ann. Sci. École Norm. Sup.* 18(3):553–561. DOI: 10.24033/asens.1496.

[22] Musson, I. M. (2013). *Lie Superalgebras and Enveloping Algebras.* Providence, RI: American Mathematical Society.

[23] Ovando, G. P. (2016). Lie algebras with ad-invariant metrics (A survey-guide). *Rend. Semin. Mat. Univ. Politec. Tori.* 74(1):243–268.