Wave Dynamical Chaos in Superconducting Microwave Billiards

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Abstract

During the last few years we have studied the chaotic behavior of special Euclidian geometries, so-called billiards, from the quantum or in more general sense “wave dynamical” point of view [1–3]. Due to the equivalence between the stationary Schrödinger equation and the classical Helmholtz equation in the two-dimensional case (plain billiards), it is possible to simulate “quantum chaos” with the help of macroscopic, superconducting microwave cavities. Using this technique we investigated spectra of three billiards from the family of Pascal’s Snails (Robnik-Billiards) with a different chaoticity in each case in order to test predictions of standard stochastical models for classical chaotic systems.

1 Introduction

In the last few decades the theoretical investigation of two-dimensional Euclidian and Riemannian geometries, so-called billiards, has led to a very fruitful new discipline in nonlinear physics [4–6]. Due to the conserved energy of an ideal particle propagating inside the billiard’s boundaries with specular reflections on the walls, the plain billiard belongs to the class of Hamiltonian systems with the lowest degree of freedom in which chaos can occur. This does only depend on the given boundary shape. Because of their simplicity
two-dimensional billiards are in particular suitable to study the behavior of the particle in the corresponding quantum regime \cite{7,8} where spectral properties are completely described by the stationary Schrödinger equation inside the domain with Dirichlet boundary conditions on the walls. In this context the investigation of “quantum chaos” has become one of the most fascinating goals of theoretical physics at the end of this century \cite{10,11}.

The family of billiards described in this paper were introduced by M. Robnik in 1983 \cite{12} and were investigated from the classical and quantum point of view \cite{13}. The shape of these so-called Robnik-Billiards is known from mathematics as Pascal’s Snails. These billiards are suited for an investigation of the chaotic behavior in the classical and also in the quantum mechanical case, because by varying only one control-parameter, \( \lambda \), the system changes from an integrable regular billiard, the circle \cite{14}, through a wide range of intermediate billiards to a fully chaotic billiard, the Cardioid \cite{15}.

The following equation describes the mapping from the unit disc to one certain Pascal Snail with a special \( \lambda \): \( \omega = z + \lambda z^2 \), with \( |z| = 1 \). For \( \lambda = 0 \) one obtains the circle and for \( \lambda = 0.5 \) the Cardioid. To choose special parameters \( \lambda \) for the investigations we calculate the Poincaré surface of section for different \( \lambda \) and look for the fraction of chaotic area inside them. To examine the transition from the integrable to the chaotic case we choose the following fractions of chaotic phase space for our billiards: 55 \%, 66 \% and 100 \% which correspond to the following parameter \( \lambda \): 0.125, 0.15 and 0.3. Because the phase space shows no regular motion above \( \lambda = 0.279 \) \cite{16}, we took \( \lambda = 0.3 \) for the chaotic billiard.

2 Experiment

Due to the equivalence of the stationary Schrödinger equation for quantum systems and the corresponding Helmholtz equation for electromagnetic resonators in two dimensions, it is possible to simulate a quantum billiard of arbitrary shape with the help of a sufficiently flat macroscopic electromagnetic cavity of the same shape \cite{1,17,18}. We have studied three billiards of the shape given in Sec. I using microwave resonators made of Niobium which become superconducting (sc) below 9.2 K.

The measurements were carried out in a very stable 4K-bath-cryostat \cite{20}. The billiards were excited in a frequency range between 0 and 20 GHz. We
used capacitively coupling dipole antennas placed in small holes on the Niobium surface. Using one antenna for the excitation and either another as well as the same one for the detection of the microwave signal, we were able to measure the transmission or the reflection spectrum of the resonator by using an HP-8510B vector network analyzer. In the lower part of Fig. 1, a typical transmission spectrum at 4.2 K is shown for two different frequency ranges. The signal is given as the ratio of output power to input power on a logarithmic scale. By comparing the upper with the lower part of Fig. 1 the advantages of using sc resonators instead of normal conducting (nc) ones are clearly visible. Obviously the use of sc resonators leads to an immense improvement of detecting almost all resonances in the spectra. In the nc spectrum, due to the very broad resonances which interfere with each other, one is not able to detect all resonances in the upper frequency range. Only in the sc case one can be sure to find nearly all resonances. In the sc case resonances typically posses quality factors of up to $Q = f/\Delta f \approx 10^7$ (nc $Q \approx 10^3$) and signal-to-noise ratios of up to $S/N \approx 60$ dB (nc $S/N \approx 30$ dB). This high resolution allows easy separation of individual resonances from each other and from the background. As a consequence, all the important characteristics like the eigenfrequencies and widths could be extracted with avery high accuracy [3,21]. A detailed analysis of the original spectra yielded a total number of about 1100 resonances for each of the three measured billiards.

![Fig. 1: Comparison of the normal conducting (upper part, at 300 K) and the superconducting (lower part, at 4.2 K) transmission spectrum of the $\lambda = 0.15$ billiard for two different frequency ranges.](image-url)
3 Results and Discussion

In order to derive meaningful statistical measures for the given eigenvalue sequences it is first of all necessary to extract the smooth part of the resonator’s number of eigenmodes which is given by the generalized electromagnetic Weyl-formula [22, 23]

\[ N_{\text{Weyl}}(f) = \frac{A\pi}{c_0^2} f^2 - \frac{U}{2c_0} f + \text{const.} , \]  

where \( A \) denotes the area, \( U \) the circumference of the cavity and \( f \) the upper frequency limit of the given spectrum. The constant term contains contributions from the boundary’s curvature and from the edges of the cavity.

The total number of eigenmodes \( N(f) \), up to a certain frequency \( f \), i.e. the spectral staircase function, contains in addition a fluctuating part

\[ N(f) = N_{\text{Weyl}}(f) + N_{\text{Fluc}}(f) = \sum_i \Theta(f - f_i) = \sum_{f > f_i} 1 + \sum_{f < f_i} \frac{1}{2} . \]  

To determine the spectral fluctuations, the smooth part of the spectral staircase has to be eliminated. For this a special staircase function (see Eq. (3)) was constructed and a second order polynomial, Eq. (1), was fitted to it.

In order to perform a statistical analysis of the given eigenvalue spectra independently of the special sizes of the resonators, the spectra were first unfolded i.e. from the measured sequence of eigenfrequencies \( \{f_1, \ldots, f_i, f_{i+1}, \ldots\} \) the spacings \( s_i = (f_{i+1} - f_i)/\bar{s} \) between adjacent eigenmodes were obtained by calculating the local average \( \bar{s} \) from the fit of Eq. (1). The proper normalization of the measured spacings of eigenmodes then yielded the desired nearest neighbour spacing distribution \( P(s) \) (NND), the probability for a certain spacing \( s \). On the left side of Fig. 2 the three nearest neighbour spacing distributions of the measured resonators are shown.

Furthermore, to obtain a quantitative criterion concerning the degree of chaoticity in the system the spectra were analyzed in terms of a statistical description introduced in a model of Berry and Robnik [24] which interpolates between the two limiting cases of pure Poissonian and pure GOE behavior for a classically regular or chaotic system respectively. The model introduces a mixing-parameter \( q \) which is directly related to the relative chaotic fraction of the invariant Liouville measure of the underlying classical phase space in
Fig. 2: a) Nearest neighbour spacing distributions for the three measured configurations. The dashed lines show the best fits using the model of Berry-Robnik on the data, given as histograms. Also the two limiting cases (Poisson and GOE) are displayed. The inserts of this figure show the shape of the investigated billiards. b) Number variance $\Sigma^2$ for the three billiards. The shaded bands display the data including error bars and the dashed curves represent the fitted Berry-Robnik distributions corresponding to the given mixing-parameters. For the billiard with $\lambda = 0.3$ the data and the fitted curve fall together.

which the motion takes place, see also Sec. [4]. Using this ansatz one obtains the mixing-parameters for the three measured cavities: $q = 0.59 \pm 0.03$ for $\lambda = 0.125$, $q = 0.63 \pm 0.03$ for $\lambda = 0.15$ and $q = 0.99 \pm 0.03$ for $\lambda = 0.3$.

To uncover correlations between nonadjacent resonances, one has to use a statistical measure which is sensitive on larger scales. As an example we used the number variance $\Sigma^2$ [8]. The $\Sigma^2$-statistics describes the variance of the number of levels $n(L)$ in a given range of length $L$ around the mean for this interval, which is due to the unfolding equal to $L$,

$$\Sigma^2(L) = \left( \langle n(L) - \langle n(L) \rangle_{L} \rangle_{L} \right)^2 = \langle n^2(L) \rangle_{L} - L^2 .$$

On the right side of Fig. [2] the number variances for the three measured billiards are displayed, again together with the two limiting cases for Poissonian and the GOE behavior. Using the Berry-Robnik model yields the
following mixing-parameters: $q = 0.57 \pm 0.05$ for $\lambda = 0.125$, $q = 0.65 \pm 0.04$ for $\lambda = 0.15$ and $q = 1.00 \pm 0.05$ for $\lambda = 0.3$. Comparing short range (NND) and long range correlations ($\Sigma^2$) one observes a good agreement of the mixing parameters. The same is true if one compares the classical mixing-parameter from Sec. [1] with the spectral ones.

Beside the comparison between the classical and the spectral mixing-parameter, there is another method which allows to identify relations between the classical and the quantum system. Taking the Fourier Transform of the fluctuating part of the level density, given through Eq. (2), one obtains the length spectrum of the resonator [11]. Here the peaks correspond to the periodic orbits (PO) of the classical billiard. The heights of these peaks are a measure for their stability. On the right side of Fig. 3 the Fourier spectrum of the fluctuating part obtained from the $\lambda = 0.3$ billiard is displayed. On the left side of this figure the shortest POs of the classical billiard are sketched and related to the length scale in the Fourier spectrum by labels.

![Diagram of periodic orbits and Fourier spectrum](image)

Fig. 3: On the left side the shortest periodic orbits for the classical billiard ($\lambda = 0.3$) are sketched. On the right side the Fourier Transform of the fluctuating part of the level density is shown. The encircled numbers above the peaks label the classical periodic orbits.

We have examined the behavior of systems which are located inbetween integrable and chaotic characteristics. We have also shown the necessity of using superconducting microwave billiards for a proper examination of the wave dynamical system in order to get the heighest possible resolution.
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