The Spectrum of the Kazakov-Migdal Model

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Abstract

Gross has found an exact expression for the density of eigenvalues in the simplest version of the Kazakov-Migdal model of induced QCD. In this paper we compute the spectrum of small fluctuations around Gross’s semi-circular solution. By solving Migdal’s wave equation we find a string-like spectrum which, in four dimensions, corresponds to the infinite tower of mesons in strong coupling lattice QCD with adjoint matter. In one dimension our formula reproduces correctly the well known spectrum of the hermitean matrix model with a harmonic oscillator potential. We comment on the relevance of our results to the possibility of the model describing extended objects in more than one dimension.
Introduction

The original Kazakov-Migdal (KM) model of induced QCD [1] consists of a scalar in the adjoint representation of $SU(N)$, covariantly coupled to gauge fields on a hypercubic lattice of spacing $a$. There is no kinetic term for the gauge fields and the action of the model is

$$S = N \sum_x \left[ \text{tr} U(\Phi(x)) - \text{tr} \sum_\mu \Phi(x) U_\mu(x) \Phi(x + \mu a) U_\mu^\dagger(x) \right]. \quad (1)$$

The hope was that this model could perhaps induce QCD in the sense that a distance much larger than $a$ a kinetic term for the gauge fields would be generated through scalar interactions at distances on the order of the cutoff $a$. By carefully tuning the parameters of the potential it might then be possible to reach an ‘asymptotic freedom domain’ in which the continuum limit could be taken. The mass of the scalar, which is kept heavy in this limit, would act as an effective cutoff for the resulting continuum (QCD?) theory.

The beauty of this idea of course lies in the fact that the model given in Eq. (1) is analytically tractable in the limit of an infinite number of colors $N$ [2]. The model is especially simple in the case of a purely quadratic potential and initially there was clearly no a priori reason for why the objective of inducing QCD, if possible at all, could not be achieved by just tuning the mass of the scalar. This simplest of all possibilities is by now ruled out. The reason is as follows: D. Gross [3] has found an exact solution to the saddle point equations describing the translationally invariant eigenvalue distribution of the $\Phi$-field in the $N = \infty$ limit. This solution has no continuum limit in four dimensions. It rather describes the large $N$, infinite coupling (remember, there is no plaquette term for the gauge fields in (1)) limit of lattice QCD with adjoint matter. This in by itself is not sufficient to prove ‘non-induction’ for there could be other saddle points. However computer simulations of the $SU(2)$ theory [4] also show that there is no critical point in the case of a purely gaussian potential. Assuming that the large $N$ limit is smooth the theorem is proved. Incidentally, the simulations also showed that by adding a quartic term a critical point could be reached. The nature of the resulting continuum theory is not clear at the moment. Migdal [5] has made considerable progress in the investigation of the theory with a general potential but lately he has concentrated his efforts on the study of an extended version of Eq. (1), the so called ‘mixed model’ of induced QCD [6].

In the mixed model [6] $n_f \ll N$ heavy fermions in the fundamental representation are added to the action in order to break the additional local symmetries [8] of the model, in particular the local $Z_N$ symmetry $U_\mu(x) \to Z_\mu(x) U_\mu(x)$ first discussed by Kogan et al. [7]. As pointed out by these authors, due to this symmetry the Wilson loop cannot acquire an expectation value. The adjoint loop of course does, but it is always screened and cannot serve as an order parameter for confinement. It is Migdal’s hope that there is a phase of the mixed model in which the center symmetry is broken and Wilson loops show an area law.

In the present paper we will have nothing more to say about the question of whether and how QCD can be induced. Rather we will extend the very pretty work of Ref. [3] on the original Gaussian KM model by calculating the spectrum of small fluctuations around...
the Gross saddle point. We believe that there are many reasons for why this is interesting. In four dimensions our result gives the masses of the mesons of the infinite coupling lattice gauge theory to leading order in $1/N$. The large $N$, strong coupling spectrum of the theory with fundamental fermionic matter has been known for over a decade now [3] and the adjoint scalar spectrum nicely complements this old work. Furthermore, the model in Eq. (1) can be viewed as a gauged matrix model. Hermitean matrix models in $D \leq 1$ dimensions are intimately related to discretized versions of the Polyakov string [1] and as such describe 2D gravity coupled to matter fields. In particular, a case of great current interest is the one dimensional case, describing $c = 1$ matter coupled to 2D gravity. In more than one dimension ordinary matrix models become very complicated [9] and one might hope that the gauged version in Eq. (1) might be useful as theory of extended objects in more than one dimension. It has also recently been suggested [10] that the KM model can be viewed in $D$ dimensions as the high temperature limit of the $(D + 1)$ dimensional Wilson action. If this interesting conjecture is indeed correct then our result in three dimensions should describe the spectrum of fluctuations of the 4D-Polyakov line around one of it’s $Z_N$ minima at infinite temperature. Finally, our calculation can serve as a testing ground for the methods developed by Migdal. In some sense the present problem represents the ‘hydrogen atom’ of induced QCD; we can actually work out the eigenvalues and the eigenfunctions of Migdal’s wave equation in closed form. In the process we also found a term in the wave equation which had been omitted in Ref. [11].

The Gross Solution

In this section we will quickly review Gross’s solution of induced QCD with a quadratic potential. In the process we will also establish notation. We will follow Migdal’s approach [11] to the problem and express the action in Eq. (1) in terms of the density of eigenvalues

$$\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i).$$  \hfill (2)

This is easily done by first integrating out the gauge fields using the Itzykson-Zuber integral [12], which produces

$$S = N \sum_{x,i} U(\lambda_i(x)) - \sum_{x,i \neq j} \ln |\lambda_i(x) - \lambda_j(x)| - \sum_{x,\mu} \ln[I(\lambda(x), \lambda(x + \mu a)].$$ \hfill (3)

where the logarithm comes from two factors of the Vandermond determinant and $I$ denotes the Itzykson-Zuber determinant. Now using Eq. (2) the action can be written in terms of a $x$-dependent density as

$$S[\rho] = N^2 \sum_x \int d\lambda \rho_x(\lambda)U(\lambda) - N^2 \sum_x \int d\lambda \rho_x(\lambda) \int d\lambda' \rho_x(\lambda') \ln |\lambda - \lambda'| - \sum_{x,\mu} \ln[I(\rho_x, \rho_{x+\mu a}].$$ \hfill (4)

Note that the effective action for $\rho$ will also receive contributions from a Jacobian due to the change of variables from the eigenvalues to the density [11]. Since everything we have to say is independent of this term however we will just drop it here.
An equation for the eigenvalue distribution $\rho_x$ can be found by looking for stationary points of the effective action in Eq. (4). In performing infinitesimal variations of $\rho_x$ one must be careful though not to change the normalization of $\rho_x$ in Eq. (2). This can be achieved by taking variations of the form

$$\delta \rho_x(\lambda) = -\frac{1}{N} \frac{d\psi(\lambda)}{d\lambda}$$

or equivalently

$$\frac{\delta}{\delta \psi(\lambda)} = \frac{1}{N} \frac{d}{d\lambda} \frac{\delta}{\delta \rho(\lambda)}$$

where $\psi(\lambda)$ vanishes at the end point of the support of $\rho(\lambda)$. The saddle point equation for (a translationally invariant) $\rho$ follows by setting the first variation of the effective action $\frac{\delta S}{\delta \psi(\lambda)}$ equal to zero. One obtains, denoting the logarithmic derivative of the Itzykson-Zuber determinant by $F(\lambda)$,

$$F(\lambda) = \frac{-2 \text{Re} V'(\lambda) + U'(\lambda)}{2D}$$

where

$$V'(\lambda) = \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}.$$  

This equation by itself does not determine $\rho$ since $F$ is unknown. The function $F(\lambda)$ however satisfies a set of Schwinger-Dyson equations which were derived by Migdal in Ref. [2]. Using these equations one finally ends up with an integral equation for $\text{Re} V'(\lambda)$ whose imaginary part determines $\rho$:

$$\text{Re} V'(\lambda) = P \int \frac{d\lambda'}{\pi} \arctan \frac{\pi \rho(\lambda')}{\lambda - R(\lambda')}$$

where

$$R(\lambda) = \frac{D-1}{D} \text{Re} V'(\lambda) + \frac{U'(\lambda)}{2D}.$$  

In one dimension the gauge field can be gauged away and using $U(\Phi) = \frac{1}{2} m^2 \Phi^2$ the action in Eq. (1) is just that of a free scalar field. In this case it is well known that $\rho$ is semicircular in shape, i.e.

$$\rho(\lambda) = \rho_0(\lambda) = \frac{1}{\pi} \sqrt{\mu - \frac{\mu^2 \lambda^2}{4}}$$

and $\mu = \sqrt{m^4 - 4}$. Gross [3] realized that the semicircular form (11) for $\rho$ actually solves Eqs. (9,10) in any number of dimensions. He showed that in $D$ dimensions

$$\mu_{\pm} = \frac{m^2(D-1) \pm D \sqrt{m^4 - 4(2D - 1)}}{2D - 1}$$
and also computed the free energy:

\[ F = \frac{1}{2} \frac{m^2}{2\mu} + \frac{1}{2} \ln \mu - \frac{D}{2} \left[ 1 + \frac{4}{\mu^2} - 1 - \ln \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4}{\mu^2}} \right) \right] \] (13)

The following things are important to note: \( \mu_+ \) is a minimum of \( F \) and vanishes at \( m^2 = 2D \) in \( D \leq 1 \) implying that in this case the continuum limit can be taken. In \( D > 1 \) on the other hand, \( \mu_+ \) never vanishes and instead \( \mu_- \to 0 \) as \( m^2 \to 2D^- \). However the saddle point at \( \mu = \mu_- \) is a local maximum which, as we shall see, has important consequences for the spectrum of the theory.

To summarize this section: For a quadratic potential the distribution of eigenvalues is semicircular in shape in any number of dimensions. In one or less dimensions a continuum limit can be constructed and we expect a physical spectrum since one is perturbing around a local minimum of the free energy. In more than one dimensions on the other hand one expects tachyons in the continuum limit. The spectrum of fluctuations around \( \mu_+ \) will describe the spectrum of mesons made out of adjoint matter in the strong coupling lattice theory. We will now go on to show all this explicitly.

**Migdal’s Wave Equation**

In order to get at the spectrum of the theory in leading order in \( \frac{1}{N} \) one must work out the effective action describing the fluctuations around an extremum of the action (4). This has been done by Migdal [11]. Writing

\[ \delta \rho_x(\lambda) = \rho_x(\lambda) + \delta \rho_x(\lambda) \] (14)

with \( \delta \rho \) as in Eq. (5) and \( \rho_x(\lambda) \) a solution of (7), one obtains

\[ S_2[\rho] = -\sum_x \int d\lambda \int d\lambda' \left[ \frac{1}{2} \eta(\lambda, \lambda') \psi_x(\lambda) \psi_{x+a\mu}(\lambda') + \frac{1}{(\lambda - \lambda')^2} + D\sigma(\lambda, \lambda') \psi_x(\lambda) \psi_x(\lambda') \right] . \] (15)

In the case at hand here we have \( \rho_x(\lambda) = \rho_0(\lambda) \). Now using plane waves, \( \psi_x(\lambda) = \psi(\lambda)e^{iP_\lambda} \), one immediately obtains Migdal’s wave equation for the particle spectrum:

\[ \Omega^2 \int d\lambda' \eta(\lambda, \lambda') \psi(\lambda') = -\int d\lambda' \left[ \frac{1}{(\lambda - \lambda')^2} + D\sigma(\lambda, \lambda') \right] \psi(\lambda') . \] (16)

In Eqs. (15,16) we have introduced Migdal’s notation for the derivatives of \( F(\lambda) \), i.e.

\[ \sigma(\lambda, \lambda') = \frac{d^2}{d\lambda' d\rho_x(\lambda')} \delta F(\lambda) \] (17)

\[ ^1 \text{It is interesting to note that the the free energy obtained by Kawamoto and Smit} \] [8] \[ \text{can be brought into the same form if one makes the curious substitution } \lambda = -\frac{4}{\mu^2} \text{ and } M = \frac{im^2}{8}, \text{ where } \lambda \text{ is the square of the meson field and } M \text{ is the bare quark mass.} \]
\[ \eta(\lambda, \lambda') = \frac{d}{d\lambda'} \delta F(\lambda) \delta \rho_{x+a\mu}(\lambda'). \]  

Also, in Eq. (16) we have defined \( \Omega^2 = \sum_\mu \cos(P_\mu). \) Migdal [11] has derived Integral equations for the functions \( \eta \) and \( \sigma. \) We have checked his derivation of these equations and agree up to an additional term in the equation for \( \sigma. \) The origin of this term is explained in Appendix 1. Using the correct equations for \( \eta \) and \( \sigma \) from the Appendix we obtain, denoting \( \frac{d}{d\lambda'} \psi(\lambda') \equiv H(\lambda') \) and integrating Eq. (16) over \( \lambda, \) the following equation:

\[ \Omega^2 \int d\lambda' H(\lambda') \frac{\lambda - \lambda_0}{\lambda' - \lambda_0} = \int d\lambda \int d\lambda' H(\lambda') K(\lambda_0, \lambda) \left[ \frac{(1 - D)}{\lambda - \lambda'} + D \frac{G(\lambda_0, \lambda')}{G(\lambda_0, \lambda)} \frac{1}{\lambda - \lambda'} \right] \]

\[ - D \int d\lambda' H(\lambda') G(\lambda_0, \lambda'). \]  

Seventeen comments are in order. First, all the integrals in Eq. (19) are defined in the principal value sense. Second, Migdal’s functions \( K(\lambda_0, \lambda) \) and \( G(\lambda_0, \lambda) \) are given by

\[ K(\lambda_0, \lambda) = \frac{\rho(\lambda)}{\pi^2 \rho^2(\lambda) + (\lambda_0 - R(\lambda))^2} \]  

and

\[ G(\lambda_0, \lambda) = \frac{1}{\lambda_0 - R(\lambda)} \Re \exp \left[ + \int \frac{d\lambda'}{\pi(\lambda' - \lambda)} \arctan \frac{\pi \rho(\lambda')}{\lambda_0 - R(\lambda')} \right]. \]

Third, the range of the integrals is over the support of \( \rho(\lambda) \) which in our case is the interval \([- \sqrt{\mu}, \sqrt{\mu}]. \) Fourth, the last term in Eq. (19) is the one absent in Migdal’s paper [11].

In principal the calculation of the spectrum of fluctuations around the Gross saddle point is now straightforward: Compute \( G(\lambda_0, \lambda) \) using it’s definition Eq. (21), do the \( \lambda \)-integral in (19) and finally solve the resulting (singular) integral equation. However there are several subtleties in the calculation which we thought make it worthwhile to present the calculation of \( G(\lambda_0, \lambda) \) in some detail. This is done in Appendix 2. The interested reader will be able to check all the other integrals relevant here by using the method used in the Appendix. Suffice it to say here, that boundary conditions are important. For example, the branch of the ‘arctan’ in Eqs. (19,21) is chosen by a boundary condition at infinity [2]. Hence, in the calculation \( \lambda_0 \) must be taken outside the support of \( \rho_0. \)

We have obtained the following expression for \( G(\lambda_0, \lambda): \)

\[ G(\lambda_0, \lambda) = \frac{1}{(R - \frac{\sqrt{\mu}}{2})} \cdot \frac{(R\lambda_0 - \lambda) - \sqrt{\mu^2 \lambda_0^2 - \mu}}{(\lambda_0 - R\lambda)^2 + \mu - \mu^2 \lambda^2} \]

where \( R(\lambda) = R\lambda = \frac{1}{2D} (m^2 + \mu(D - 1)) \lambda \) is linear in \( \lambda \) for the semicircular solution. To derive our final result for the wave equation below, we have repeatedly used the important identity

\[ R^2 - \frac{\mu^2}{4} = 1 \]  

(23)
satisfied by $\mu_\pm$. Finally, for completeness we also give the results of the $\lambda$-integrals in Eq. (19). We obtained
\[ \int d\lambda K(\lambda_0, \lambda) = \frac{\mu^2 R}{4\sqrt{x^2 - 1}(x - Ry)^2 + \frac{\mu^2}{4}(1 - y^2)} \] (24)
and
\[ \int d\lambda K(\lambda_0, \lambda) \frac{G(\lambda_0, \lambda')}{G(\lambda_0, \lambda)} \frac{1}{\lambda - \lambda'} = \frac{\mu}{2} \left( 1 - \frac{\sqrt{(Rx - \frac{\mu}{2}\sqrt{x^2 - 1})^2 - 1}}{(Rx - y) - \frac{\mu}{2}\sqrt{x^2 - 1}} \right) \] (25)
where we have introduced the new variables $x = \frac{\sqrt{x^2}}{2}\lambda_0$ and $y = \frac{\sqrt{x^2}}{2}\lambda'$.

**Calculation of Spectrum**

Using the results of the previous section we obtained the following integral equation determining the spectrum of the theory to leading order in $1/N$:
\[ \Omega^2 \int_{-1}^{+1} dy \frac{h(y)}{y - x} = \int_{-1}^{+1} dy h(y) \frac{A(x) - B(x)y}{(x - Ry)^2 + \frac{\mu^2}{4}(1 - y^2)} \] (26)
where
\[ A(x) = \frac{(1 - D)M R}{2\sqrt{x^2 - 1}} - Dx \] (27)
and
\[ B(x) = \frac{(1 - D)Mx}{2\sqrt{x^2 - 1}} - DR. \] (28)
To bring the wave equation into this form we have assumed that $H(\lambda')$ depends on $\lambda'$ only through $y$, i.e. $H(\lambda') = h(y)$.

To determine the spectrum of the theory means to determine $\Omega^2$ in Eq. (26). Due to the compactness of the integration interval we expect a discrete spectrum. To solve the equation we follow the time honored method of expansion in a complete, orthonormal set of functions. In particular, due to the range of the integration interval and the form of the integrand on the left hand side of the equation, Chebyshev polynomials are a natural set of functions to use here. Hence we make the ansatz
\[ h(y) = \sum_n c_n \frac{T_n(y)}{\sqrt{1 - y^2}}. \] (29)
Using this ansatz the integral on the left hand side of Eq. (26) can be done immediately (remembering though that $|x| > 1$ in the calculation) and one obtains
\[ \Omega^2 \int_{-1}^{+1} dy \frac{h(y)}{y - x} = -\pi \Omega^2 \sum_n c_n \frac{(x - \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}. \] (30)
\[ ^2 \text{We use the conventions of Ref. [13] for the definition of the Chebyshev polynomials.} \]
On the right hand side we first write
\begin{equation}
\frac{A(x) - B(x)y}{(x - Ry)^2 + \frac{\mu^2}{4}(1 - y^2)} = \frac{f(x)}{y - y_+} + \frac{g(x)}{y - y_-} \tag{31}
\end{equation}
where
\begin{equation}
y_\pm = Rx \pm \frac{\mu}{2}\sqrt{x^2 - 1}. \tag{32}
\end{equation}
In the last equation the identity Eq. (23) has been used. Now the integral on the right hand side of the wave equation (26) can also be done and one obtains after some algebra,
\begin{equation}
\int_{-1}^{+1} dy h(y) \frac{A(x) - B(x)y}{(x - Ry)^2 + \frac{\mu^2}{4}(1 - y^2)} =
-\frac{\pi}{2} \sum_n c_n \{(D - 1) + D\}(R + \frac{\mu}{2})^{-n} + \{D - (D - 1)\}(R + \frac{\mu}{2})^{-n}\} \frac{x - \sqrt{x^2 - 1}}{\sqrt{x - 1}}. \tag{33}
\end{equation}
Now comparing Eqs. (30) and (33) we finally obtain for the eigenfunctions and eigenvalues
\begin{equation}
h_n(y) = \frac{T_n(y)}{\sqrt{1 - y^2}} \tag{34}
\end{equation}
and
\begin{equation}
\Omega_n^2 = \frac{(D - 1)}{2}[(R + \frac{\mu}{2})^{-n} - (R + \frac{\mu}{2})^n] + \frac{D}{2}[(R + \frac{\mu}{2})^n + (R + \frac{\mu}{2})^{-n}]. \tag{35}
\end{equation}
The last two equations constitute the main result of this paper. We will discuss it’s significance in the next and final section. It is noted that the fluctuations themselves (at a given momentum $P_\mu$) are of course given by $\delta \rho(y) = -\frac{1}{N} \frac{T_n(y)}{\sqrt{1 - y^2}}$ which properly satisfies
\begin{equation}
\int_{-1}^{+1} dy \delta \rho(y) = 0 \text{ for } n = 1, 2, \cdots, \text{ but } \neq 0 \text{ for } n = 0. \text{ Therefore } h_0(y) \text{ should be excluded from the eigenfunctions.}
\end{equation}

**Discussion and Conclusion**

As expected we have found a discrete spectrum with an infinite number of states. The quantum number $n$ labels states according to their parity under $y \to -y$. The eigenfunctions $h_n(y)$ are even (odd) under this operation for $n$ even (odd). Let us begin our considerations of the spectrum now by looking at the one dimensional case in some detail. To this end it is useful to define
\begin{equation}
M = \ln(R + \frac{\mu}{2}) \tag{36}
\end{equation}
and write Eq. (35) as
\begin{equation}
\Omega_n^2 = D \cosh(nM) - (D - 1) \sinh(nM) \tag{37}
\end{equation}
In one dimension we obtain

$$\cos(P_n) = \cosh(n \cdot \arccosh(\frac{m^2}{2})) = T_n(\frac{m^2}{2})$$  \hfill (38)

The interpretation of this result is straightforward. As we mentioned before, in one dimension the gauge fields play no role and the theory is free with propagator

$$G_{ij,kl}(P) = \frac{1}{N} \frac{1}{m^2 - 2 \cos(P)} \delta_{il} \delta_{jk}.$$  \hfill (39)

Clearly, the ‘pole’ of the propagator is at $\cos(P) = \frac{m^2}{2}$ which agrees with (38) in the case $n = 1$. In position space this corresponds to an exponential falloff of the propagator with mass $M$. The states with $n > 1$ correspond to ‘mesons’ made out of $n$ $\Phi$’s. The $x$-space propagator for these objects decays exponentially with mass $n \cdot M$ which in momentum space gives Eq. (38).

In one dimension $\mu = \sqrt{m^4 - 4}$ vanishes as $m^2 \to 2^+$ and we can take the continuum limit. Writing $m^2 = 2 + m_0^2 a^2$ and $P = P_0 a$ one obtains

$$-P_0^2 \equiv E_n^2 = n^2 \cdot m_0^2.$$  \hfill (40)

Hence in the continuum limit we get a linearly rising spectrum of states of mass $n m_0$. Note, that for $m_0 = 0$ we get a massless boson in one dimension as opposed to the massless boson in $1 + 1$ (one spacelike, one timelike) obtained in the double scaling limit of the continuum model \[14\]. In less than one dimension $\mu$ still vanishes at $m^2 = 2D$ and one obtains expanding Eq. (37) at $m^2 = 2D + m_0^2 a^2$

$$E_n^2 = n \cdot m_0^2$$  \hfill (41)

in the continuum limit. Contrary to the case in one dimension, the particle masses are proportional to $\sqrt{n}$.

Let us now go on and discuss the case $D > 1$. As we pointed out before the stable solution is $\mu = \mu_+$ in Eq. (12) which is always greater than zero. In this case the interpretation of the spectrum in Eq. (37) in four dimensions is that of the mesons in strong coupling lattice QCD with adjoint matter. Note that asymptotically in $n$, $\Omega_n^2 = (R + \mu/2)^n$. The same formula holds in the limit $D \to \infty$ for all $n$. It is interesting to see what happens if instead we use $\mu = \mu_-$. Setting $m^2 = 2D - m_0^2 a^2$ in order to approach the critical point from below, one gets

$$-P_0^2 \equiv E_n^2 = -n \cdot m_0^2$$  \hfill (42)

in the continuum limit. Although infinitely many states appear in the continuum limit, they are all tachyonic as expected. The spectrum in $D > 1$ could be improved if an upside-down quadratic potential\[15\] is used for $U(\Phi)$ in Eq. (1). The continuum limit in this case is equivalent to setting $m^2 = -2D - m_0^2 a^2$ in Eq. (37), which gives

$$E_n^2 a^2 = 2D \left((-1)^n - 1\right) + (-1)^n \cdot n \cdot m_0^2 a^2.$$  \hfill (43)

For $n$ even, we obtain $E_n = n \cdot m_0^2 > 0$ though $E_n = -\infty$ for $n$ odd.
What do the above results tell us? In one dimension we got the expected result. In more than one dimension we find an infinite tower of states which certainly suggests a ‘stringy’ interpretation of the spectrum. Makeenko [16] has written down Schwinger-Dyson equations for the functions

$$G(C_{xy}) = \langle \frac{1}{N} tr(\Phi(x)U(C_{xy})\Phi(y)U(C_{yx})) \rangle$$ \hspace{1cm} (44)$$

where $U(C_{xy})$ denotes a string of links along the path $C_{xy}$. For the quadratic potential Makeenko has obtained the solution for $G(C_{xy})$ and it would be interesting to see whether by summing his result over all paths our spectrum can be reproduced [3]. If it is an interpretation of our result in terms of the excitations of a string of flux seems natural.

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Appendix 1. Derivation of the extra term in eq.(19)

In this Appendix we show how the last term in eq.(19) is derived. In the Appendix of Ref. [11] Migdal has derived the following equation (78)

$$\int d\mu \rho(\mu)g_{\lambda}(\phi', \mu) = \mathcal{T}_\lambda(z)\int d\phi \rho(\phi) \frac{\sigma(\phi', \phi) + \frac{1}{G_{\lambda}(\phi)} \frac{d}{d\phi'} G_{\lambda}(\phi') - G_{\lambda}(\phi)}{(\lambda - R(\phi))^2 + \pi^2 \rho^2(\phi)}$$ \hspace{1cm} (45)$$

where

$$g_{\lambda}(\phi', \phi) = \frac{d}{d\phi'} \delta G_{\lambda}(\phi) .$$ \hspace{1cm} (46)$$

Setting $z \to \infty$ in the above formula, we find

$$\int d\mu \rho(\mu)g_{\lambda}(\phi', \mu) = \int d\phi \rho(\phi) \frac{\sigma(\phi', \phi) + \frac{1}{G_{\lambda}(\phi)} \frac{d}{d\phi'} G_{\lambda}(\phi') - G_{\lambda}(\phi)}{(\lambda - R(\phi))^2 + \pi^2 \rho^2(\phi)} .$$ \hspace{1cm} (47)$$

From eq.(59) in Ref. [11]

$$\int d\mu \rho_\pi(\mu)G_{\lambda}(\mu) = \int d\nu \rho_{\pi + a\mu}(\nu) \frac{\lambda - \nu}{\lambda - \nu} ,$$ \hspace{1cm} (48)$$

we obtain

$$\int d\mu \rho(\mu)g_{\lambda}(\phi', \mu) = -\frac{d}{d\phi'} G_{\lambda}(\phi') ,$$ \hspace{1cm} (49)$$

3In one dimension one just obtains the square of the x-space propagator for a free particle by this procedure, i.e one only obtains the lightest ‘meson’.
which differs from eq.(77) in Ref. [11]. Inserting (49) into (47), we finally find equation for \( \sigma(\phi', \phi) \)

\[
0 = \frac{d}{d\phi'} G_\lambda(\phi') + \int d\phi \rho(\phi) \frac{(\sigma(\phi', \phi) + \frac{1}{G_\lambda(\phi)} \frac{d}{d\phi'} G_\lambda(\phi') - G_\lambda(\phi)}{(\lambda - R(\phi))^2 + \pi^2 \rho^2(\phi))}.
\]  

(50)

**Appendix 2. Calculation of \( G(\lambda_0, \lambda) \)**

The definition of \( G(\lambda_0, \lambda) \) is

\[
G(\lambda_0, \lambda) = \frac{1}{\lambda_0 - R(\lambda)} \text{Re} \left[ \exp \left[ \int \frac{d\lambda'}{\pi(\lambda' - \lambda)} \arctan \frac{\pi \rho(\lambda')}{\lambda_0 - R(\lambda')} \right] \right]
\]

(51)

where \( R(\lambda) = R\lambda \) for the semicircular solution \( \pi \rho(\lambda) = \sqrt{\mu - \mu^2 \lambda^2 / 4} \). We first calculate the following integral

\[
J \equiv \int \frac{d\lambda'}{\pi(\lambda' - \lambda)} \arctan \frac{\sqrt{\mu - \mu^2 \lambda'^2 / 4}}{\lambda_0 - R\lambda'}. 
\]

(52)

Integration by parts together with the obvious rescaling of variables gives

\[
J = \frac{\mu x}{\pi} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}} \ln(y - z) \left[ \frac{f_+}{y - y_+} + \frac{f_-}{y - y_-} \right]
\]

(53)

where \( x = \sqrt{\mu / 2} \lambda_0, \ z = \sqrt{\mu / 2} \lambda, \ y = \sqrt{\mu / 2} \lambda', \ f_\pm = \pm \frac{y_\pm - R / x}{y_+ - y_-}, \) and \( y_\pm \) are given in Eq. (32) in the text. Using the following properties for Chebyshev polynomials \( T_n(y) \)

\[
\ln(y - z) = -T_0(y) \ln(2t_z) - 2 \sum_{n=1}^{\infty} T_n(y) \frac{(t_z)^n}{n}
\]

(54)

\[
\frac{1}{y - y_\pm} = -\frac{2t_\pm}{1 - t_\pm^2} \left( T_0(y) + 2 \sum_{n=1}^{\infty} T_n(t_\pm)^n \right)
\]

(55)

\[
\int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}} T_n(y) \cdot T_m(y) = \delta_{nm} \times \begin{cases} \pi / 2 & n \neq 0 \\ \pi & n = 0 \end{cases}
\]

(56)

where \( t_z = z - i \sqrt{1 - z^2} \) and \( t_\pm = (R \mp \mu / 2)(x - \sqrt{x^2 - 1}) \), we perform the integral in Eq. (53) and obtain

\[
J = -\ln \left[ \frac{1 - t_z t_+}{1 - t_z t_-} \right].
\]

(57)

From the above result \( G \) becomes

\[
G(\lambda_0, \lambda) = \frac{1}{\lambda_0 - R\lambda} \text{Re} \left[ \frac{1 - t_z t_+}{1 - t_z t_-} \right],
\]

(58)

which finally gives eq.(22) in the text.
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