SET-VALUED SORTING INDEX AND JOINT EQUIDISTRIBUTIONS

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Abstract. Recently Petersen defined a new Mahonian index sor over the symmetric group \( S_n \) and proved that \((\text{inv}, \text{rmin})\) and \((\text{sor}, \text{cyc})\) have the same joint distribution. Foata and Han proved that the pairs of set-valued statistics \((\text{Cyc}, \text{Rmil}), (\text{Cyc}, \text{Lmap})\), \((\text{Rmil}, \text{Lmap})\) have the same joint distribution over \( S_n \).

In this paper we introduce the set-valued statistics \( \text{Inv} \), \( \text{Lmil} \), \( \text{Sor} \) and \( \text{Lmic}_1 \) and generalize simultaneously results of Petersen and Foata-Han and find many equidistributed triples of set-valued statistics and quadruples of statistics.

1. Introduction

Let \( S_n \) be the symmetric group of \([n] := \{1, 2, \ldots, n\}\). For \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in S_n \), define the inversion statistic \( \text{inv} \) by

\[
\text{inv}(\sigma) := \# \{(i, j) : i < j \text{ and } \sigma_i < \sigma_j\},
\]

and the cycle statistic \( \text{cyc}(\sigma) \) by

\[
\text{cyc}(\sigma) := \text{the number of cycles in the cycle decomposition of } \sigma.
\]

We say a permutation statistic over \( S_n \) is Mahonian if it is equidistributed with \( \text{inv} \), and is Stirling if with \( \text{cyc} \). For example, it is well known that the right to left minimum statistic \( \text{rmin} \), defined by

\[
\text{rmin}(\sigma) := \# \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j > i\},
\]

is Stirling.

Recently Petersen found a new Mahonian statistic \( \text{sor} \), called the sorting index (see Section 2 for definition), and proved the following:

**Theorem 1.1** (Petersen [8]). The pairs of statistics \((\text{inv}, \text{rmin})\) and \((\text{sor}, \text{cyc})\) have the same joint distribution over \( S_n \). Also, we have

\[
\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} x^{\text{rmin}(\sigma)} = \sum_{\sigma \in S_n} q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} = x \prod_{r=2}^{n} (x + [r]_q - 1),
\]

where \([r]_q := 1 + q + q^2 + \cdots + q^{r-1}\).

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A combinatorial proof is found by Chen et al. [2] via a bijection \( \phi : S_n \to S_n \) such that 

\[(\mathrm{inv}, \mathrm{rmin})\sigma = (\mathrm{sor}, \mathrm{cyc})\phi(\sigma),\]

where \((\mathrm{inv}, \mathrm{rmin})\sigma\) means \((\mathrm{inv}(\sigma), \mathrm{rmin}(\sigma))\). The bijection \( \phi \) turns out to be of the form

\[\phi = B(\sigma)^{-1} \circ A(\sigma),\]

a composition of \(B\)-code and \(A\)-code introduced by Foata and Han [6]. By defining the set-values statistics right-to-left minimum letters \(\mathrm{Rmil}\), left-to-right maximum places \(\mathrm{Lmap}\) and the cycle set \(\mathrm{Cyc}\) respectively by

\[\mathrm{Rmil}(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j > i\},\]

\[\mathrm{Lmap}(\sigma) := \{i : \sigma_i > \sigma_j \text{ for all } i > j\},\]

and

\[\mathrm{Cyc}(\sigma) := \{\text{the smallest number in each cycle of the cycle decomposition}\},\]

Foata and Han derived the following set-valued joint equidistribution results.

**Theorem 1.2** (Foata, Han [6]). The followings hold.

1. For \(\sigma \in S_n\), we have

\[(\mathrm{Rmil}, \mathrm{Lmap})\sigma = (\mathrm{Cyc}, \mathrm{Lmap})\phi(\sigma).\]

2. The set-valued statistics \((\mathrm{Cyc}, \mathrm{Rmil})\), \((\mathrm{Cyc}, \mathrm{Lmap})\), \((\mathrm{Rmil}, \mathrm{Lmap})\) are symmetric and joint equidistributed over \(S_n\).

The motivation of this work is to generalize above two theorems, to see if there is a set-valued version of Petersen’s result or other pairs of set-valued statistics having the same distribution à la Foata and Han. It turns out that we can have them both.

Throughout the paper a statistic is set-valued if and only if the first letter is in capital. By introducing new set-valued statistics \(\mathrm{Inv}, \mathrm{Lmil}, \mathrm{Sor}, \mathrm{Lmic}_1\) (see Section 2 for definitions) and corresponding ordinary statistics \(\mathrm{lmin}, \mathrm{lmic}_1\), our first main theorem extends simultaneously both Petersen and Foata-Han’s results.

**Theorem 1.3.** We have:

1. For \(\sigma \in S_n\), the following holds:

\[(\mathrm{Inv}, \mathrm{Rmil}, \mathrm{Lmap}, \mathrm{Lmil})\sigma = (\mathrm{Sor}, \mathrm{Cyc}, \mathrm{Lmap}, \mathrm{Lmic}_1)\phi(\sigma).\]

2. The quadruple statistics \((\mathrm{inv}, \mathrm{rmin}, \mathrm{lmax}, \mathrm{lmin})\) and \((\mathrm{sor}, \mathrm{cyc}, \mathrm{lmax}, \mathrm{lmic}_1)\) have the same joint distribution over \(S_n\), and

\[
\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} x^{\mathrm{rmin}(\sigma)} y^{\mathrm{lmin}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{sor}(\sigma)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{lmic}_1(\sigma)}
\]

\[= xy \prod_{r=2}^{n} (x + [r]_q + y q^{r-1} - 1 - q^{r-1}).\]
Theorem 1.3 generalizes Theorem 1.1 and Theorem 1.2 (1). We will see that Theorem 1.2 (2) will be generalized in a later theorem.

By abuse of terminology, a set-valued statistic is called Mahonian (or Stirling) if the corresponding ordinary statistic is so. Our second result is to find triples of set-valued Stirling statistics having the same joint distribution as \((Rmil, Lmap, Lmil)\) and \((\text{Cyc}, Lmic_1, Lmap)\). By switching between certain set-valued statistics by applying inverse, reverse, or complement operations on permutations, we obtain 8 more (and 4 partial) triples of set-valued statistics having the same joint distribution. See Theorem 4.2 for the complete list. From these we obtain two sets of symmetric and equidistributed pairs of set-valued statistics, the first of which includes those three pairs in Theorem 1.2 (2).

The third part of the work is to derive quadruples of statistics which are joint equidistributed with \((\text{inv}, rmin, lmax, lmin)\) and \((\text{сор}, \text{cyc}, lmic_1, lmax)\). Note that the first statistic is Mahonian and the others are Stirling. Again by switching among statistics, in Theorem 5.2 we derive 10 more (and 11 partial) quadruples of statistics having the same joint distribution.

The rest of the paper is organized as follows. Definitions and preliminary results will be put in Section 2. In Section 3 we prove Theorem 1.3. Section 4 is devoted to triples of set-valued statistics, and Section 5 to quadruples of ordinary statistics.

2. Preliminary results

2.1. A-code and B-code. We first introduce the A- and B-code of Foata and Han [6], which are the key tools of this paper. Given \(\sigma \in S_n\), define its Lehmer code [7] by

\[
\text{Leh}(\sigma) := (\ell_1, \ell_2, \ldots, \ell_n),
\]

where \(\ell_i = |\{j : 1 \leq j \leq i, \sigma_j \leq \sigma_i\}|\). Let \(L_n := \{(\ell_1, \ell_2, \ldots, \ell_n) : 1 \leq \ell_i \leq i \text{ for } 1 \leq i \leq n\}\). It is clear that \(\text{Leh} : S_n \to L_n\) is a bijection. The A-code of a permutation \(\sigma\) is defined by

\[
A(\sigma) := \text{Leh}(\sigma^{-1}).
\]

For example, let \(\sigma = 2413765\). Then \(\sigma^{-1} = 3142765\) and \(A(\sigma) = (1, 1, 3, 2, 5, 5, 5)\).

The B-code of \(\sigma\) is defined in the following way. For each \(i = 1, \ldots, n\), let \(k_i \geq 1\) be the smallest integer such that \(\sigma^{-k_i}(i) \leq i\). Define

\[
B(\sigma) := (b_1, b_2, \ldots, b_n) \text{ with } b_i = \sigma^{-k_i}(i).
\]

Equivalently, \(B(\sigma)\) can be determined from the cycle decomposition of \(\sigma\). Assume that \(i\) appears in a cycle \(c\). If \(i\) is the smallest element of \(c\), then set \(b_i = i\); otherwise, choose \(b_i\) to be the first element \(j\) in \(c\) with respect to the reverse direction such that \(j < i\). For example, let \(\sigma = 2431756 = (124)(3)(576)\). Then \(B(\sigma) = (1, 1, 3, 2, 5, 5, 5)\). By definition it is easy to see that B-code is a bijection from \(S_n\) to \(L_n\).

2.2. The set-valued statistic \(Lmic_1\). Define the left-to-right minimum letters statistic \(Lmil\), the set-valued left-to-right minimum places statistic \(Lmip\) and the left to right minimum statistic \(lmin\) respectively by

\[
Lmil(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j < i\},
\]

\[
Lmip(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j < i\},
\]

\[
lmin(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for all } j < i\}.
\]
Let \( L_{\text{mip}}(\sigma) := \{ i : \sigma_i < \sigma_j \text{ for all } j < i \} \),

and

\[ l_{\text{min}}(\sigma) := \# L_{\text{mil}}(\sigma) \quad \text{(or } \# L_{\text{mip}}(\sigma) \text{)}. \]

It is easy to see that \( L_{\text{mil}}(\sigma) = L_{\text{mip}}(\sigma-1) \).

For an integer sequence \((\ell_1, \ldots, \ell_n) \in \mathcal{L}_n\), define \( O((\ell_1, \ldots, \ell_n)) := \{ i : \ell_i = 1 \} \), the set of indices with values 1. It turns out that \( L_{\text{mip}} \) and \( L_{\text{mil}} \) correspond to Lehmer code and \( A \)-code respectively. The proof of the following lemma is directly by definition.

**Lemma 2.1.** We have

\[ L_{\text{mip}}(\sigma) = O(\text{Leh}(\sigma)) \quad \text{and} \quad L_{\text{mil}}(\sigma) = O(A(\sigma)). \]  

Hence it is natural to consider the statistic corresponding to \( B \)-code. Define the set-valued statistic \( L_{\text{mic}}_1 \) by

\[ L_{\text{mic}}_1(\sigma) := O(B(\sigma)). \]

For example, \( L_{\text{mic}}_1(579328164) = O((1, 1, 3, 3, 1, 6, 2, 6, 3)) = \{1, 2, 5\} \).

We have the following combinatorial interpretation of \( L_{\text{mic}}_1 \), which explains the somewhat awkward notation, standing for the left-to-right minimum of the shifted cycle containing 1.

**Lemma 2.2.** For \( \sigma \in \mathcal{S}_n \), write the the cycle containing 1 in the way that 1 is at the end of the cycle and denote the resulting cycle \( \vec{c} \). Then

\[ L_{\text{mic}}_1(\sigma) = L_{\text{mil}}(\vec{c}) \]

by regarding \( \vec{c} \) as a word.

**Proof.** By the definition of \( B \)-code, \( b_i = 1 \) if and only if \( i \in \vec{c} \) and all letters on the left of \( i \) in \( \vec{c} \) are larger than \( i \). In other words, \( i \) is a left-to-right minimum letter in \( \vec{c} \). Then we have \( O(B(\sigma)) = L_{\text{mil}}(\vec{c}) \).

For the running example, \( \sigma = 579328164 = (1527)(394)(68) \) and the shifted cycle is \( \vec{c} = (5271) \), hence \( L_{\text{mic}}_1(579328164) = L_{\text{mil}}(5271) = \{1, 2, 5\} \).

Also, \( L_{\text{mil}} \) and \( L_{\text{mic}}_1 \) are related via \( \phi \).

**Lemma 2.3.** We have

\[ L_{\text{mil}}(\sigma) = L_{\text{mic}}_1(\phi(\sigma)). \]

**Proof.** From [1], [2] and the definition of \( \phi \), we have

\[ L_{\text{mic}}_1(\phi(\sigma)) = O(B(\phi(\sigma))) = O(A(\sigma)) = L_{\text{mil}}(\sigma). \]

\[ \square \]

### 2.3. The set-valued statistics Inv and Sor

The goal of this subsection is to define and investigate the set-valued statistics \( \text{Sor} \) and \( \text{Inv} \). First we need the concept of the induced set.

**Definition 2.4.** Given \((\ell_1, \ell_2, \ldots, \ell_n) \in \mathcal{L}_n\), define its induced set \((\ell_1, \ell_2, \ldots, \ell_n)\) according to the following algorithm:

1. Set \( S, U \) with the initial values \( S = \{1, 2, \ldots, n\} \) and \( U = \emptyset \).
(2) For \( i \) from \( n \) down to 1 do the followings:
(a) let \( \ell'_i \) be the \( \ell_i \)-th smallest element among \( S \).
(b) add ordered pairs \((\ell'_i, j)\) into \( U \) for those \( j \in S \) with \( j > \ell'_i \). If there is no such \( j \) then skip this step.
(c) delete \( \ell'_i \) from \( S \).
(3) Define \( \langle (\ell_1, \ell_2, \ldots, \ell_n) \rangle := U \).

For example, for \((1, 1, 3, 2, 5, 5, 5) \in L_7\) we have

| \( i \) | \( \ell'_i \) | \( S \) | \( U \) |
|---|---|---|---|
| 1 | 1, 2, 3, 4, 5, 6, 7 | \( \emptyset \) | |
| 7 | 5 | 1, 2, 3, 4, *, 6, 7 | (5,6), (5,7) |
| 6 | 6 | 1, 2, 3, 4, *, *, 7 | (5,6), (5,7), (6,7) |
| 5 | 7 | 1, 2, 3, 4, *, *, * | (5,6), (5,7), (6,7) |
| 4 | 2 | 1, *, 3, 4, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4) |
| 3 | 4 | 1, *, 3, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4), (1,3) |
| 2 | 1 | *, *, 3, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4), (1,3) |
| 1 | 3 | *, *, *, *, *, *, * | (5,6), (5,7), (6,7), (2,3), (2,4), (1,3) |

Thus \( \langle (1, 1, 3, 2, 5, 5, 5) \rangle = \{(5, 6), (5, 7), (6, 7), (2, 3), (2, 4), (1, 3)\} \).

Now we review the sorting index \( \text{sor} \) of Petersen [8]. Given \( \sigma \), decompose it uniquely into the product of transpositions \( \sigma = (i_1 j_1)(i_2 j_2) \cdots (i_k j_k) \) with \( j_1 < j_2 < \cdots < j_k \) and \( i_r < j_r \) for \( 1 \leq r \leq k \), and then define

\[
\text{sor}(\sigma) := \sum_{r=1}^{k} (j_r - i_r).
\]

For example, since \( \sigma = 2431765 = (12)(24)(56)(57) \) we have \( \text{sor}(\sigma) = (2 - 1) + (4 - 2) + (6 - 5) + (7 - 5) = 6 \). In other words, \( \text{sor}(\sigma) \) measures the total distance of the letters needed to move during the bubble-sorting process. In this example, we have

\[
2431765 \xrightarrow{(57)} 2431657 \xrightarrow{(67)} 2431567 \xrightarrow{(24)} 2134567 \xrightarrow{(12)} 1234567.
\]

In [2] it is proved that \( \text{inv}(\sigma) = \text{sor}(\phi(\sigma)) \) and

\[
\text{sor}(\sigma) = \sum_{i=1}^{n} (i - b_i), \tag{3}
\]

which clarifies the relation between sorting index and the \( B \)-code \((b_1, b_2, \ldots, b_n)\) of \( \sigma \). Observe that in the step (2)(b) of 2.4 we add exactly \( (i - \ell_i) \) ordered pairs into \( U \) for each \( i \), hence from [3] it makes sense to define the set-valued statistic sorting set \( \text{Sor} \) by

\[
\text{Sor}(\sigma) := \langle B(\sigma) \rangle. \tag{4}
\]

As for the Inv, since \( \text{inv}(\sigma) := \# \{(i, j) : i < j \text{ and } \sigma_i < \sigma_j \} \), it is natural to define the set-valued statistic inversion set \( \text{Inv} \) by

\[
\text{Inv}(\sigma) := \{(i, j) : i < j \text{ and } \sigma_i < \sigma_j \}.
\]

Similar to the relation between \( \text{Sor} \) and \( B \)-code, we have the following:
Proposition 2.5. For $\sigma \in S_n$, we have
\[ \text{Inv}(\sigma) = \langle A(\sigma) \rangle. \] (5)

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$ be the permutation with $A(\sigma) = (a_1, \ldots, a_n)$. Since $A(\sigma) = \text{Leh}(\sigma^{-1})$, hence $\sigma_n^{-1} = a_n$, $\sigma_{n-1}^{-1} = (a_{n-1})$-th smallest element in $[n] \setminus \{\sigma_n^{-1}\}$. In general, for $1 < i < n$ we have
\[ \sigma_{n-i}^{-1} = (a_{n-i})$-th smallest element in $[n] \setminus \{\sigma_n^{-1}, \ldots, \sigma_{n-i+1}^{-1}\}$. 

Thus $\sigma$ can be rebuilt from $A(\sigma)$ as follows. At the initial stage there are $n$ vacancies from left to right. For $i$ from $n$ down to 1, we recursively put letter $i$ into the $a_i$-th vacancy from the left. The resulting permutation is exactly $\sigma$. For example, if $A(\sigma) = (1, 1, 3, 2, 5, 5, 5)$, then $\sigma$ can be recovered in the following way:
\[
\begin{align*}
\cdots & \rightarrow \cdots 7 \rightarrow \cdots 76 \rightarrow \cdots 765 \rightarrow \cdots 4765 \rightarrow \cdots 43765 \rightarrow 243765 \rightarrow 2143765
\end{align*}
\]

Now observe that in each step above, the position we choose for the letter $i$ is exactly $\ell_i'$ in Definition 2.4. In other words, we can synchronize the rebuilding of $\sigma$ and the construction of the induced set $\langle A(\sigma) \rangle$. Moreover, note that the ordered pair $(\ell_i', j)$ is added to $U$ if and only if it is an inversion of $\sigma$, for the letter $i$ must be larger than $\sigma_j$. Thus it must have $\text{Inv}(\sigma) = \langle A(\sigma) \rangle$. □

And finally there is a set version of $\text{inv}(\sigma) = \text{sor}(\phi(\sigma))$:

Lemma 2.6. We have
\[ \text{Inv}(\sigma) = \text{Sor}(\phi(\sigma)). \]

Proof. From (4), (5) and the definition of $\phi$, we have
\[ \text{Sor}(\phi(\sigma)) = \langle B(\phi(\sigma)) \rangle = \langle A(\sigma) \rangle = \text{Inv}(\sigma). \]

□

3. Proof of Theorem 1.3

Proof of Theorem 1.3. (1) is obtained by combining Lemma 2.3, 2.6 and Theorem 1.2. The first statement of (2) is directly from (1) and in the following we look at the generating function. For $n \geq 1$ let
\[ F_n(q, x, y) := \sum_{\sigma \in S_n} q^{\text{sor}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{lmic}_1(\sigma)}. \]

It is clear that $F_1(q, x, y) = xy$. We claim that for $n \geq 2$ one has
\[ F_n(q, x, y) = xy \prod_{r=2}^{n} (x + [r]_q + yq^{r-1} - 1 - q^{r-1}). \]

Let $t_{ij}$ denote the transposition $(ij)$ and let $\eta_1 = 1, \eta_2 = 1 + t_{12}$ and
\[ \eta_j = 1 + \sum_{i<j} t_{ij} \]
for $j \geq 3$. Petersen \cite{Petersen} showed that
\[
\eta_1 \eta_2 \cdots \eta_n = \sum_{\sigma \in S_n} \sigma. \tag{6}
\]
Now define the linear map $\Theta : S_n \to \mathbb{Z}[q, x, y]$ by $\Theta(\sigma) := q^{sor(\sigma)} x^{cyc(\sigma)} y^{lmic_1(\sigma)}$. Hence by (6) it suffices to show that
\[
\Theta(\eta_1 \eta_2 \cdots \eta_n) = xy \prod_{r=2}^{n} (x + \lceil r \rceil q + y q^{r-1} - 1 - q^{r-1}).
\]
It is easy to see that $\Theta(\eta_1) = xy$ and $\Theta(\eta_1 \eta_2) = xy(x + yq)$. Let $n \geq 3$. We proceed by induction. Suppose
\[
\Theta(\eta_1 \eta_2 \cdots \eta_{n-1}) = xy \prod_{r=2}^{n-1} (x + \lceil r \rceil q + y q^{r-1} - 1 - q^{r-1}).
\]
Take $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in S_{n-1}$. It can be embedded in $S_n$ as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1} n$. Let $\sigma' := \sigma t_{i,n}$ for some $1 \leq i \leq n$ and it is clear that $sor(\sigma') = sor(\sigma) + (n - i)$.

Assume that the cycle decomposition of $\sigma$ is $c_1 \cdots c_m$ for some $m$ and $c_i$ contains the letter $i$. Hence $\sigma' = c_1 \cdots c_m(n)$ if $i = n$ and $c_1 \cdots c'_i \cdots c_m$ if $i \neq n$, where $c'_i = (\ldots, i, n, \sigma(i), \ldots)$.

Thus
\[
cyc(\sigma') = \begin{cases} 
cyc(\sigma) + 1 & \text{if } i = n, \\
cyc(\sigma) & \text{otherwise}; 
\end{cases}
\]
and
\[
lmic_1(\sigma') = \begin{cases} 
lmic_1(\sigma) + 1 & \text{if } i = 1, \\
\text{lmic}_1(\sigma) & \text{otherwise}.
\end{cases}
\]

So we have
\[
\Theta(\sigma \cdot \eta_n) = \Theta(\sigma t_{i,n} + \sigma t_{n-1,n} + \cdots + \sigma t_{1,n}) = \Theta(\sigma)(x + q + q^2 + \cdots + q^{n-2} + y^n q^{n-1}),
\]
and therefore
\[
\Theta(\eta_1 \eta_2 \cdots \eta_n) = \Theta \left( \sum_{\sigma \in S_n, \sigma(n) = n} \sigma \cdot \eta_n \right) = \sum_{\sigma \in S_n, \sigma(n) = n} \Theta(\sigma \cdot \eta_n)
= (x + q + q^2 + \cdots + q^{n-2} + y^n q^{n-1}) \sum_{\sigma \in S_{n-1}} \Theta(\sigma)
= (x + \lceil n \rceil q + y q^{n-1} - 1 - q^{n-1}) \Theta(\eta_1 \eta_2 \cdots \eta_{n-1}).
\]
The proof is then completed by induction. \hfill \square
4. Set-valued joint equidistribution

In this section we seek for more set-valued statistics having the same joint distribution as \((R_{mil}, L_{mil}, L_{map})\) and \((C_{yc}, L_{mic}, L_{map})\). In the introduction we have defined \(R_{mil}\) (right-to-left minimum letters) and \(L_{map}\) (left-to-right maximum places) while in Section 2.2 \(L_{mil}\) (left-to-right minimum letters) and \(L_{mip}\) (left-to-right minimum places). Similarly we can defined set-valued statistics \(R_{mip}, L_{mal}, R_{map}\) and \(R_{mal}\) by

\[
R_{mip}(\sigma) := \{i : \sigma_i < \sigma_j \text{ for all } j > i\},
\]

\[
L_{mal}(\sigma) := \{\sigma_i : \sigma_i > \sigma_j \text{ for all } j < i\},
\]

\[
R_{map}(\sigma) := \{i : i > \sigma_j \text{ for all } j > i\},
\]

\[
R_{mal}(\sigma) := \{i : \sigma_i > \sigma_j \text{ for all } j > i\}.
\]

The idea is quite simple: we look at the relations between these statistics by performing operations of “inverse”, “complement”, or “reverse” on permutations.

For \(\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n\), let \(\sigma^{-1}\) denote its inverse,

\[
\sigma^c := (\sigma_n, \sigma_n-1, \ldots, \sigma_1)
\]

its reverse and

\[
\sigma^c := (n+1-\sigma_1, n+1-\sigma_2, \ldots, n+1-\sigma_n)
\]

its complement. For example, if \(\sigma = 364152\), then \(\sigma^{-1} = 461352\), \(\sigma^r = 251463\) and \(\sigma^c = 413625\). It is clear that the mappings \(i, r, c : S_n \rightarrow S_n\), defined by \(i(\sigma) := \sigma^{-1}\), \(r(\sigma) := \sigma^r\) and \(c(\sigma) := \sigma^c\), are bijections.

Given two set-valued statistics \(\text{Stat}_1, \text{Stat}_2\) and one bijection \(\chi : S_n \rightarrow S_n\), we say

\[\text{Stat}_1 \xrightarrow{\chi} \text{Stat}_2\]

if \(\text{Stat}_1(\sigma) = \text{Stat}_2(\chi(\sigma))\) for all \(\sigma \in S_n\). Also we define \(\text{Stat}^* := \{n + 1 - i : i \in \text{Stat}\}\) for a set-valued statistic \(\text{Stat}\), if applicable.

It turns out that these eight set-valued statistics are related via the mappings \(i, r\) and \(c\). The proof of the following proposition is straightforward and is omitted.

**Proposition 4.1.** The relations between the eight set-valued statistics \(L_{mil}, L_{mip}, R_{mil}, R_{mip}, L_{mal}, L_{map}, R_{map}\) and \(R_{mal}\) are illustrated by the graph (with edges solid or dotted and labeled by \(i, r\) or \(c\)) in Figure 1.
Theorem 4.2. The following triples of set-valued statistics have the same joint distribution over $\mathcal{S}_n$. A dash means the statistic is omitted.

(1) A solid edge between $\text{Stat}_1$ and $\text{Stat}_2$ means $\text{Stat}_1(\sigma) = \text{Stat}_2(\chi(\sigma))$ and $\text{Stat}_2(\sigma) = \text{Stat}_1(\chi(\sigma))$ for $\sigma \in \mathcal{S}_n$, with $\chi = i, r$ or $c$ as labeled.

(2) A dotted edge between $\text{Stat}_1$ and $\text{Stat}_2$ means $\text{Stat}_1(\sigma) = \text{Stat}_2^i(\chi(\sigma))$ and $\text{Stat}_2(\sigma) = \text{Stat}_1^i(\chi(\sigma))$ for $\sigma \in \mathcal{S}_n$, with $\chi = i, r$ or $c$ as labeled.

For examples, we have $\text{Rmap} \xrightarrow{\sim} \text{Rmal}$, $\text{Rmap} \xrightarrow{\tau} \text{Lmap}^*$, and $\text{Rmap} \xleftarrow{\sim} \text{Rmip}$.

We come to the main result of this section.

Proof. By Proposition 4.1, we have

(1) $\text{Rmil}, \text{Lmil}, \text{Lmap}$
(2) $\text{Cyc}, \text{Lmic}_1, \text{Lmap}$
(3) $\text{Lmap}, \text{Lmip}, \text{Rmil}$
(4) $\text{Rmil}^*, \text{Rmap}^*, \text{Lmal}^*$
(5) $\text{Lmal}^*, \text{Rmal}^*, \text{Rmip}^*$
(6) $\text{Lmil}, \text{Rmil}, \text{Rmap}^*$
(7) $\text{Rmal}^*, \text{Lmal}^*, \text{Lmip}$
(8) $\text{Rmap}^*, \text{Rmip}^*, \text{Lmil}$
(9) $\text{Lmip}, \text{Lmap}, \text{Rmal}^*$
(10) $\text{Lmap}, \text{Cyc}$
(11) $\text{Lmic}_1, \text{Cyc}$
(12) $\text{Cyc}$
(13) $\text{Rmil}, \text{Cyc}$

Therefore, (3) to (9) are joint equidistributed with (1). Moreover, by Theorem 1.3, we have

(1) $\xrightarrow{\phi}$ (2) $\xrightarrow{\text{i roc}}$ (4) $\xrightarrow{\text{roc}}$ (5) $\xrightarrow{\tau}$ (6),
(1) $\xleftarrow{\sim}$ (7),
(1) $\xrightarrow{\text{i roc}}$ (8),
(1) $\xrightarrow{\text{i roc}}$ (9).

Finally, (2) $\xrightarrow{\sim}$ (12) and (10) $\xrightarrow{\sim}$ (13) follow from the fact that $\text{Cyc}(\sigma) = \text{Cyc}(\sigma^{-1})$, and the proof is completed.

From the theorem we can read off the following many pairs of set-valued statistics which are symmetric and joint equidistributed. Note that (1) includes those pairs of Foata and Han.

Corollary 4.3. In each of the following items, the pairs of set-valued statistics are symmetric and joint equidistributed over $\mathcal{S}_n$.

(1) $(\text{Rmil}, \text{Lmap})$, $(\text{Rmil}^d, \text{Lmal}^d)$, $(\text{Rmal}^d, \text{Lmip})$, $(\text{Rmap}^d, \text{Lmil})$,
(2) $(\text{Cyc}, \text{Rmil})$, $(\text{Cyc}, \text{Lmal})$.

5. JOINT EQUIDISTRIBUTED QUADRUPELS

In this section we look at the ordinary number-valued statistics. The goal is to find quadruples of statistics joint equidistributed with $(\text{inv}, \text{rmin}, \text{lmax}, \text{lmin})$ and $(\text{sort}, \text{cyc}, \text{lmax}, \text{lmic}_1)$. Most of the materials in this section are well known. Our contribution is to relate them with the $(\text{sort}, \text{cyc}, \text{lmax}, \text{lmic}_1)$ and derive the generating functions with respect to the first three statistics.

The statistics $\text{rmin}, \text{rmax}, \text{lmin}, \text{lmax}$ are defined in the obvious way. We may take more familiar Mahonian statistics into consideration. Let

$$\text{Des}(\sigma) := \{i : \sigma_i > \sigma_{i+1}\}$$
be the descent set of $\sigma$. It is well known that the statistics major $\maj$, inverse major $\imaj$, reverse major $\rmaj$, charge $\chg$, and cocharge $\cochg$, defined by
\[
\maj(\sigma) := \sum_{i \in \Des} i,
\]
\[
\imaj(\sigma) := \maj(\sigma^{-1}), \quad \rmaj(\sigma) := \maj(\sigma^{\text{roc}}),
\]
\[
\chg(\sigma) := \sum_{i \in \Des(\sigma^{-1})} (n - i), \quad \cochg(\sigma) := \sum_{i \in \Des(\sigma^{-1})} (n - i),
\]
are all Mahonian \cite{10}. Again, given two statistics $\stat_1, \stat_2$ and one bijection $\chi : \mathcal{S}_n \rightarrow \mathcal{S}_n$, the notation $\stat_1 \xrightarrow{\chi} \stat_2$ means $\stat_1(\sigma) = \stat_2(\chi(\sigma))$ for all $\sigma \in \mathcal{S}_n$. The equidistribution of $\maj$ and $\inv$ can be proved combinatorially from the celebrated fundamental bijection $\psi : \mathcal{S}_n \rightarrow \mathcal{S}_n$ of Foata \cite{11}, that is, one has
\[
\maj \xrightarrow{\psi} \inv.
\]
Moreover, a close look of $\psi$ will also show that
\[
(r\max, r\min) \xrightarrow{\psi} (r\max, r\min).
\]

We now perform the “inverse-reverse-complement” trick on these statistics.

Lemma 5.1. We have
\[
(1) \ maj \xrightarrow{i} \imaj, \quad (2) \ maj \xrightarrow{\text{roc}} \rmaj, \quad (3) \ rmaj \xrightarrow{1} \chg, \quad (4) \ chg \xrightarrow{r} \cochg.
\]

Proof. (1), (2) and (4) are obvious. For (3), if $i \in \Des(\sigma^{\text{roc}})$, then $(n + 1) - \sigma_i^r < (n + 1) - \sigma_{i+1}^r$ and thus $\sigma_{n-i} > \sigma_{n-i+1}$, which implies that $n - i \in \Des(\sigma)$. Hence we have
\[
\rmaj(\sigma) = \maj(\sigma^{\text{roc}}) = \sum_{i \in \Des(\sigma^{\text{roc}})} i = \sum_{i \in \Des(\sigma)} (n - i) = \chg(\sigma^{-1}),
\]
as desired. \qed

We are ready to state the main result of this section.

Theorem 5.2. The following quadruple statistics are joint equidistributed over $\mathcal{S}_n$:

1. \[\text{(inv, rmin, lmin, lmax)}\]
2. \[\text{(sor, cyc, lmic}_1, \text{lmax)}\]
3. \[\text{(inv, lmax, lmin, rmin)}\]
4. \[\text{(inv, rmin, rmax, lmax)}\]
5. \[\text{(inv, lmax, rmax, rmin)}\]
6. \[\text{((n\choose2}) - \text{inv, lmin, rmin, lmax)}\]
7. \[\text{((n\choose2}) - \text{inv, rmax, lmax, lmin)}\]
8. \[\text{((n\choose2}) - \text{inv, rmax, rmin, lmin)}\]
9. \[\text{((n\choose2}) - \text{inv, lmin, lmax, rmax)}\]
10. \[\text{((n\choose2}) - \text{sor, lmax, lmic}_1, \text{cyc)}\]
11. \[\text{((n\choose2}) - \text{sor, lmic}_1, \text{cyc, -)}\]
12. \[\text{((n\choose2}) - \text{sor, -, lmax, lmic}_1)}\]
13. \[\text{((n\choose2}) - \text{sor, -, cyc, lmic}_1)}\]
14. \[\text{((n\choose2}) - \text{sor, lmic}_1, \text{lmax, -)}\]
15. \[\text{(-, cyc, -, rmin)}\]
16. \[\text{(-, rmin, -, cyc)}\]
17. \[\text{(maj, rmin, rmax, -)}\]
18. \[\text{(imaj, lmax, rmax, -)}\]
19. \[\text{(rmaj, lmax, lmin, -)}\]
20. \[\text{(chg, rmin, lmin, -)}\]
21. \[\text{(cochg, lmin, rmin, -)}\]
and the generating function with respect to the first three statistics in each quadruple is

\[ F_n(q, x, y) = xy \prod_{r=2}^{n} (x + [r]q + yq^{r-1} - 1 - q^{r-1}). \]

**Proof.** Observe that \( \text{inv}((\sigma^{-1}) = \text{inv}(\sigma) \) and \( \text{inv}((\sigma^{r}) = \text{inv}(\sigma^{r}) = \left( \frac{n}{r} \right) - \text{inv}(\sigma) \) for \( \sigma \in \mathfrak{S}_n \).

The proof is done via the following mappings.

(1) \( \phi \rightarrow (2) \), (1) \( i \rightarrow (3) \), (1) \( \text{ioc} \rightarrow (4) \), (1) \( \text{rec} \rightarrow (5) \), (1) \( \text{r} \rightarrow (6) \),

(1) \( \phi \rightarrow (7) \), (1) \( \text{ioc} \rightarrow (8) \), (1) \( \text{ioc} \rightarrow (9) \), (3) \( \phi \rightarrow (10) \), (6) \( \phi \rightarrow (11) \),

(6) \( \phi \rightarrow (12) \), (8) \( \phi \rightarrow (13) \), (9) \( \phi \rightarrow (14) \), (2) \( \phi \rightarrow (15) \), (10) \( \phi \rightarrow (16) \),

(4) \( \psi \rightarrow (17) \), (17) \( \psi \rightarrow (18) \), (17) \( \text{rec} \rightarrow (19) \), (19) \( \phi \rightarrow (20) \), (20) \( \phi \rightarrow (21) \).

\[ \square \]

From the list we can read off many pairs of symmetric and joint equidistributed statistics.

**Corollary 5.3.** In each of the following items, the pairs of statistics are symmetric and joint equidistributed over \( \mathfrak{S}_n \).

1. \((r_{\text{max}}, r_{\text{min}}), (r_{\text{max}}, l_{\text{max}}), (r_{\text{min}}, l_{\text{min}}), (l_{\text{max}}, l_{\text{min}}), (\text{cyc}, \text{lmic}_1), \) and \((\text{lmic}_1, l_{\text{max}})\).

   The generating function with respect to each pair is \( F(1, x, y) \).

2. \((r_{\text{max}}, l_{\text{min}}), (r_{\text{min}}, l_{\text{max}}), (\text{cyc}, l_{\text{max}}), \) and \((\text{cyc}, r_{\text{min}})\).

6. Concluding remarks

In this short paper we generalize simultaneously Petersen and Foata-Han’s results to more than two statistics and find many triples or quadruples of statistics having the same joint distribution over \( \mathfrak{S}_n \). In all quadruples, the first statistic is Mahonian while the others are Stirling, and we then read off many symmetric equidistributed pairs of Stirling statistics.

However, it is well known that there are pairs of Mahonian statistics with a symmetric joint distribution as well \[10\], for examples, \((\text{inv}, \text{maj})\) and \((\text{maj}, \text{imaj})\) are two of them. Hence it would be interesting to generalize our results further to include more Mahonian statistics.

On the other hand, the generating function obtained in Theorem \[13\] only involves three of the four statistics, hence a natural question is to find a four-variable generating function including \( l_{\text{max}} \) as well. We leave them to the interested reader.

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