Clapeyron equations and fitting formula of the coexistence curve in the extended phase space of the charged AdS black holes

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In this paper, we first review the equal area laws and Clapeyron equations in the extended phase space of the charged AdS black holes. With different fixed parameters, the Maxwell’s equal area law not only hold in $P - V$ (pressure-thermodynamic volume) oscillatory line, but also in $Q - \Phi$ (charge-electric potential) and $T - S$ (temperature-entropy) oscillatory lines. The classical Clapeyron equation also obtains its generalizations that two extra equations are found. Moreover, we present the fitting formula of the coexistence curve that the small and large charged black holes coexist. The result shows that the fitting formula is charge independent in the reduced parameter space for any dimension of spacetime. Using such analytic expression of the coexistence curve, we find that the Clapeyron equations are highly consistent with the calculated values. The fitting formula is useful for further study on the thermodynamic property of the system varying along the coexistence curve.

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I. INTRODUCTION

Black hole thermodynamics has been an exciting and challenging topic in theoretical physics. Especially, inspired by the AdS/CFT correspondence [1–3], the AdS black hole thermodynamics has attracted much attention. It was first noted that in AdS space, there exists Hawking-Page phase transition between stable large black hole and thermal gas [4], which can be explained as the confinement/deconfinement phase transition of gauge field [4, 5].

Further study on charged AdS black hole states that there is a first-order phase transition between small and large black holes in the canonical ensemble [4, 7]. With the increasing of the temperature, the phase transition point terminates at a critical point, where the first-order phase transition becomes a second one. The behavior of such phase transition is quite similar to the van der Waals (VdW) fluid.

Most recently, the investigation of the AdS black hole thermodynamics has been generalized to the extended phase space, where the cosmological constant is treated as a dynamical pressure, and its conjugate quantity is the thermodynamic volume of the black hole [8–13]. Phase transition in the charged AdS black hole was reexamined in Ref. [14]. It was found that the black hole system and VdW system share the same $P - V$ oscillatory behavior, critical exponents, and scaling laws. Thus, the precise analogy between the charged AdS black hole and VdW system was established. Later, the study of the AdS black hole phase transition in extended phase space was generalized to different AdS black hole backgrounds [15–40]. Among these study, it was shown that besides the small/large black hole first-order phase transition reminiscent of the liquid/gas transition of the VdW fluid, there also appear some new interesting phenomena, such as reentrant phase transitions of multicomponent liquids, multiple first-order solid/liquid/gas phase transitions, and liquid/gas phase transitions of the VdW type. For example, the $d \geq 6$ dimensional single spinning vacuum Kerr-AdS black holes demonstrate the peculiar behavior of the large/small/large black hole phase transitions [19], and the multiply rotating Kerr-AdS black hole system displays a small/intermediate/large black hole phase transition with one tricritical and two critical points in some range of the parameters [21]. The doubly-spinning Myers-Perry black hole also has the similar reentrant phase transition structure [22]. Such reentrant phase transition is also observed in the modified gravity theory, such as Born-Infeld gravity [15], Gauss-Bonnet gravity [28], and Lovelock gravity [35].

On the other hand, understanding the thermodynamic properties of a black hole system along or across the coexistence curve is also an interesting subject. The authors of Ref. [41] calculated the quasinormal modes of massless
scalar perturbations around the small and large four-dimensional charged AdS black holes. And the result shows that the quasinormal modes can be a dynamic probe of the thermodynamic phase transition.

Toward understanding the thermodynamic properties of the charged AdS black hole system along or across the coexistence curve, one expects an analytic formula of the coexistence curve. However, there is only numerical calculation of it. In this paper, we reconsider the small/large black hole phase transition and present the fitting formula of the coexistence curve for the \( d = 4 - 10 \) dimensional charged AdS black holes. It is shown that in the reduced parameter space, the fitting formula is independent of charge. Along the coexistence curve, the Clapeyron equations are also reexamined. Besides the classical one, other additional two equations are obtained in the extended phase space. Based on the fitting formula of the coexistence curve, these Clapeyron equations are numerically checked.

This paper is organized as follows. In Sec. II we investigate the equal area laws and Clapeyron equations in the extended phase space for the charged AdS black hole. In Sec. III we present the fitting formula of the coexistence curve and check the Clapeyron equations. We will summarize our results in Sec. IV.

II. EQUAL AREA LAWS AND CLAPEYRON EQUATIONS

Recently, there is a great interest on studying the phase transition with treating the cosmological constant as a dynamical pressure, i.e.,

\[ P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}. \] (1)

With such new interpretation, the cosmological constant can be regarded as a thermodynamic variables. In this extended phase space, the black hole mass \( M \) is considered as the enthalpy \( H \equiv M \) rather than the internal energy of the gravitational system \([42]\), and the Smarr relation for a \( d \)-dimensional charged AdS black hole is modified as

\[ (d-3)H = (d-2)TS - 2PV + (d-3)Q\Phi. \] (2)

Differentiating it, we get the first law

\[ dH = TdS + VdP + \Phi dQ, \] (3)

with \( V \) being the thermodynamic volume

\[ V = \left( \frac{\partial H}{\partial P} \right)_{S,Q}. \] (4)

Furthermore, the Gibbs free energy can be obtained by Legendre transformations as

\[ G = H - TS, \] \[ dG = -SDT + \Phi dQ + VdP. \] (6)

For a specific spacetime background, one can clearly get its Gibbs free energy. Considering that the thermodynamic system prefers a low Gibbs free energy, we then can obtain its phase transition information. Now the study of phase transition has been applied to a number of AdS black holes, and richer phase structures are displayed. Many AdS black hole systems exhibit the small/large black hole phase transition, which is remarkable as the VdW liquid/gas phase transition. The state equation for them can be expressed as

\[ P = \frac{T}{v} + \mathcal{O} \left( \frac{1}{v^2} \right). \] (7)

Analogous to the VdW fluid, here \( v \) can be interpreted as the specific volume of the black hole fluid. Study shows that both of them share the similar physical properties, such as the \( P - v \) (pressure-specific volume) oscillatory behavior, the critical exponents, and scaling laws near the critical point. Moreover the reentrant phase transitions were also found in the black hole system.

In general, the phase diagram or the coexistence curve can be obtained by analyzing the characteristic swallow tail behavior of the Gibbs free energy, and it is extensively used in most of the related papers. On the other hand, it can also be obtained by constructing the Maxwell’s equal area law. However, after an simple check, we arrive the result that the phase diagrams obtained from the Gibbs free energy and equal area law in \( P - v \) behavior do not consistent with each other. After a simple check, we argue that the problem originates from the confusion of the thermodynamic volume \( V \) and specific volume \( v \). For the VdW fluid, the equal area law is effective for both thermodynamic and specific volumes, because \( V = Nv \) with \( N \) a fixed constant denoting the total number of the fluid molecules. While for the black hole case, such linear relation does not hold anymore. Thus, considering Eq. (6), the Maxwell’s equal area law is only effective for the thermodynamic volume \( V \) in a black hole system.
Before discussing the equal area law, we plot a sketch picture of the phase diagram in Fig. 1. The red line denotes the coexistence curve above and below which are two different system phases. Each point located on the curve corresponds to two different phases while with the same value of Gibbs free energy. For example, states A and E are related with one system state, while states A’ and E’ with another state. Among them, states A (E) and A’ (E’) have the same Gibbs free energy.

Considering that the system transforms state A to A’, i.e., \( G_A = G_{A'} \), we have
\[
-SDT + \Phi dQ + V dP = 0 \tag{8}
\]

Now, we integrate it:
\[
-\int_{T_A}^{T_{A'}} SdT + \int_{Q_A}^{Q_{A'}} \Phi dQ + \int_{P_A}^{P_{A'}} V dP = 0 \tag{9}
\]

**Case I:** charge \( Q \) and temperature \( T \) fixed. For this case, we consider the isotherm in the \( P-V \) plane. Therefore, the first and second terms in Eq. (9) vanish, and we get
\[
\int_{P_A}^{P_{A'}} V dP = 0 \tag{10}
\]

Along the coexistence curve, we know that the two states have the same pressure, i.e., \( P_{A'} = P_A \). Here we show the isotherm in the \( V-P \) plane in Fig. 2(a). It is clear that there exists oscillatory behavior. Then the integral in Eq. (10) can be reexpressed as
\[
\left( \int_{P_A}^{P_{D}} V dP + \int_{P_D}^{P_C} V dP \right) + \left( \int_{P_C}^{P_B} V dP + \int_{P_B}^{P_{A'}} V dP \right) = 0 \tag{11}
\]

The pressure \( P_C = P_A = P_{A'} \) is equivalent to that of the phase transition pressure. From Fig. 2(a) it is clear that the first bracket in (11) denotes the area of region II and the second one the negative area of region I. Thus, we get
\[
\text{Area(I)} = \text{Area(II)} \tag{12}
\]

This is in fact the Maxwell’s equal area law, and we are familiar with it showed in the \( P-V \) plane (see Fig. 2(b)). Therefore, we see that this equal area law holds during the phase transition. On the other hand, we can also using such area law to find the value of the phase transition parameter. Let the temperature \( T \) take all possible values, we will obtain the phase diagram for each charge \( Q \) using such area law. It is also worthwhile to note that this equal area law is effective for the the thermodynamic volume \( V \) rather the specific volume \( v \).

**Case II:** charge \( Q \) and pressure \( P \) fixed. In the above case, we see that the equal area law is held in the \( P-V \) plane with \( Q \) and \( T \) fixed. There is an equivalent choice that \( Q \) and \( P \) is fixed. Under such choice, one can carry out the
above analysis, the equal area law will be found in the $T - S$ plane. This generalized area law was first studied in Refs. [13, 14].

**Case III:** pressure $P$ and temperature $T$ fixed. During the phase transition, Eq. (9) reduces to

$$\int_Q A' Q \Phi dQ = 0,$$

(13)

which can be reexpressed as

$$\int_{Q_A}^{Q_{A'}} Q(\Phi) d\Phi = (\Phi_{A'} - \Phi_A) Q(\Phi_A),$$

(14)

where $Q_A = Q_{A'}$ is considered. This equation is just another expression of the equal area law.

Here, it is clear that the Maxwell’s equal area law for the charged AdS black hole gets a generalization. And the three area laws are consistent with each other.

**B. Phase transition pressure**

Here we still consider that the black hole system passes from state $A$ to $A'$. The Gibbs free energies related to the two states are

$$G_A = T_A S_A + \Phi_A Q_A - 2P_A V_A,$$

(15)

$$G_{A'} = T_{A'} S_{A'} + \Phi_{A'} Q_{A'} - 2P_{A'} V_{A'}.$$  

(16)

Taking charge $Q$ fixed and noting $T_A = T_{A'}, P_A = P_{A'}$, we have

$$\frac{S_A - S_{A'}}{2} T + \frac{\Phi_A - \Phi_{A'}}{2} Q - (V_A - V_{A'}) P_A = 0.$$  

(17)

Then the phase transition pressure can be expressed as

$$P_A = \frac{Q}{2} \frac{\Delta \Phi}{\Delta V} + \frac{T_A}{2} \frac{\Delta S}{\Delta V}.$$  

(18)

Similar calculation can be applied to the case of fixed $\Phi$, which corresponds to the grand canonical ensemble, and Eq. (18) becomes

$$P_A = \frac{\Phi}{2} \frac{\Delta Q}{\Delta V} + \frac{T_A}{2} \frac{\Delta S}{\Delta V}.$$  

(19)
C. Clapeyron equations

Clapeyron equation is a useful equation to study the liquid/gas phase transition of the VdW fluid. Here we will review it and make a generalization to the charged black hole system among the small/large black hole phase transition.

Since states $A$ and $A'$, $E$ and $E'$ are in phase equilibrium,

$$G_A = G_{A'}, G_E = G_{E'}.$$  (20)

Thus

$$G_E - G_A = G_{E'} - G_{A'}.$$  (21)

If states $A$ ($A'$) and $E$ ($E'$) are very close, we then have

$$G_E - G_A = -S_A dT + \Phi_A dQ + V_A dP,$$  (22)

$$G_{E'} - G_{A'} = -S_{A'} dT + \Phi_{A'} dQ + V_{A'} dP.$$  (23)

By inserting them into Eq. (21), we easily get

$$-S_{A'} dT + \Phi_{A'} dQ + V_{A'} dP = -S_A dT + \Phi_A dQ + V_A dP.$$  (24)

Taking $Q$, $P$, and $T$ fixed, respectively, we obtain

$$\left(\frac{dP}{dT}\right)_Q = \left(\frac{S_{A'} - S_A}{V_{A'} - V_A}\right) = \frac{-\Delta S}{\Delta V},$$  (25)

$$\left(\frac{dQ}{dP}\right)_T = \left(\frac{V_{A'} - V_A}{\Phi_{A'} - \Phi_A}\right) = \frac{-\Delta V}{\Delta \Phi},$$  (26)

$$\left(\frac{dT}{dQ}\right)_P = \left(\frac{\Phi_{A'} - \Phi_A}{S_{A'} - S_A}\right) = \frac{-\Delta \Phi}{\Delta S}.$$  (27)

The first one is just the classical Clapeyron equation. And the other two are the generalized Clapeyron equations in the extended phase space of the charged AdS black hole system. In Ref. [45], the author has checked Eq. (25) for the black hole/AdS space phase transition in the nonrotating and rotating black hole systems. However, checking these equations during the small/large black hole phase transition needs a numerical calculation, and we will carry out it in the next section. Using these equations, one can easily check the Maxwell’s relation

$$\left(\frac{dP}{dT}\right)_Q \left(\frac{dT}{dQ}\right)_P \left(\frac{dQ}{dP}\right)_T = -1.$$  (28)

On the other hand, the pressure $P_A$ in Eq. (18) can be reexpressed as

$$P_A = -\frac{Q}{2} \left(\frac{dP}{dQ}\right)_T + \frac{T_A}{2} \left(\frac{dP}{dT}\right)_Q.$$  (29)

III. FITTING FORMULA OF THE COEXISTENCE CURVE

Here we first give a brief review of the charged AdS black hole. Its line element is given by

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega_{d-2}^2,$$  (29)

$$f(r) = 1 - \frac{m}{r^{d-3}} + \frac{q^2}{r^2 (d-3)} + \frac{r^2}{l^2}.$$  (30)

The parameters $m$ and $q$, respectively, related to the enthalpy and charge of the black hole as

$$H = \frac{d-2}{16\pi} \omega_{d-2} m,$$  (31)

$$Q = \frac{\sqrt{2}(d-2)(d-3)}{8\pi} \omega_{d-2} q,$$  (32)
FIG. 3: Phase transition and characteristic swallow tail behavior of the Gibbs free energy. “SBH”, “IBH”, and “LBH” denote the small, intermediate, and large black holes, respectively.

with $\omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ the volume of the unit $d$-sphere. The Smarr relation (2) and first law (3) are held. The black hole temperature, entropy, electric potential, and thermodynamic volume are given by

$$T = \frac{16\pi r_h^d}{4\pi(d-2)r_h} \left( P - \frac{2\pi Q^2 r_h^{d-2}}{\omega_{d-2}} \right) + (d-5)d + 6$$

$$S = \frac{\omega_d - 2r_h^{d-2}}{4}, \quad \Phi = \frac{4\pi Q r_h^{d-2}}{(d-3)\omega}, \quad V = \frac{\omega_d - 2r_h^{d-1}}{d-1}. \quad (34)$$

The Gibbs free energy $G = H - TS$ reads

$$G = \frac{\omega_d - 2}{16\pi} \left( \frac{d-3}{4} - \frac{16\pi P r_h^{d-1}}{(d-1)(d-2)} + \frac{32(2d-5)\pi^2 Q^2 r_h^{3-d}}{(d-2)(d-3)\omega_{d-2}} \right). \quad (35)$$

We plot in Fig. 3 the characteristic swallow tail behavior of the Gibbs free energy with fixed charge $Q$ and pressure $P$ in the $G - T$ plane. At point “N”, it can be seen there is vanished Gibbs free energy of the charged black hole. The AdS space also has vanished Gibbs free energy. Hence it seems that at that point the black hole phase and AdS space phase can coexist. However, the concept that “empty charged space” is unphysical. Thus we could not compare these two phase with fixed charge together, and the AdS phase will not be considered in following discussion.

After excluding the AdS space phase, the system will first go along the small black hole branch with the increasing of $T$ until the point “M” is arrived, where the small and large black holes have the same Gibbs free energy, and the small and large black hole can coexist at that point. With the temperature further increasing, the system will prefer the large black hole with low Gibbs free energy than the small and intermediate black holes. So there is the small/large black hole phase transition.

In the extended phase space, the small/large black hole phase transition has been well studied in Ref. [14]. This phase transition is reminiscent of the liquid/gas phase transition of the VdW fluid. This coexistence curve has positive slope everywhere and terminates at the critical point $(P_c, T_c, V_c)$, above which the small and large phases cannot be clearly distinguished. The critical point is determined by

$$\left( \frac{\partial P}{\partial V} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial V^2} \right)_T = 0. \quad (36)$$

By solving the above two equations, we get the critical point

$$V_c = \frac{\omega_d - 2}{d-1} \left( \frac{d-2}{4} \right)^{d-1} v_c^{d-1}, \quad T_c = \frac{4(d-3)^2}{(d-2)(2d-5)\pi v_c}, \quad P_c = \frac{(d-3)^2}{(d-2)^2\pi v_c^2} \quad (37)$$

with the critical specific volume $v_c = \frac{d-2}{4} \left( q^2 (d-2)(2d-5) \right)^{1/(2d-6)}$. 

$\omega$
FIG. 4: Coexistence surface of the charged AdS black hole with \( d = 4 \). (a) \( P - Q - T \) parameter space, and (b) \( P/P_c - Q - T/T_c \) reduced parameter space.

A. \( d = 4 \)-dimensional charged AdS black hole

For the case of \( d = 4 \), the values of these critical parameters reduce to

\[
V_c = 8\sqrt{6}\pi Q^3, \quad T_c = \frac{\sqrt{6}}{18\pi Q}, \quad P_c = \frac{1}{96\pi Q^2}. \quad (38)
\]

According to the swallow tail behavior of \( G \) or the equal area law, the phase diagram in \( P - T \) plane (or the coexistence curve) can be obtained with the charge \( Q \) fixed see Ref. [14] for detail). Here we show the coexistence surface in Fig. 4(a) with the charge \( Q \) varying from 0 to 2. Above the surface, it denotes the small black hole region, and below it is the large black hole region. The small and large black holes can coexist on that surface. The boundary of the surface denotes the critical points. When the charge \( Q \) increases, we see that the critical point quickly approaches to \( T = 0 \) and \( P = 0 \). For example, \( T_c = 0.0217 \) and \( P_c = 0.0008 \) for \( Q = 2 \).

This coexistence surface has an analytic parametrization

\[
P = 3.70697 \times 10^{-9} Q^{-2} - 7.18260 \times 10^{-6} T Q^{-1} + 1.18216 T^2 + 6.32696 Q T^3 + 228.060 Q^2 T^4 - 13324.5 Q^3 T^5 + 79855 Q^4 T^6 - 2.79498 \times 10^7 Q^5 T^7 + 6.18084 \times 10^8 Q^6 T^8 - 7.66596 \times 10^9 Q^7 T^9 + 4.25474 \times 10^{10} Q^8 T^{10}. \quad (39)
\]

The fitted \( P \) values agree with the calculated values to 0.01\% of \((P, T) < (P_c, T_c)\) over \( Q \in (0, 2) \) of the fit. With this parametrization form, study of the thermodynamic properties when the black hole system goes along or crosses the coexistence surface becomes easy. The parametrization can also be expressed with the reduced pressure \( \tilde{P} \) and temperature \( \tilde{T} \)

\[
\tilde{P} = \frac{P}{P_c}, \quad \tilde{T} = \frac{T}{T_c}. \quad (40)
\]

Then all the critical points are pushed to \((\tilde{P}, \tilde{T}) = (1, 1)\). The coexistence surface in the reduced parameter space is plotted in Fig. 4(b). The parametrization form of it is changed to

\[
\tilde{P} = 1.11800 \times 10^{-6} - 0.000094 \tilde{T} + 0.668965 \tilde{T}^2 + 0.155087 \tilde{T}^3 + 0.242149 \tilde{T}^4 - 0.612825 \tilde{T}^5 + 1.59111 \tilde{T}^6 - 2.41198 \tilde{T}^7 + 2.31044 \tilde{T}^8 - 1.24127 \tilde{T}^9 + 0.298419 \tilde{T}^{10}. \quad (41)
\]

Here, we are surprised that the coexistence surface is independent of the charge in the reduced parameter space. This may because the value of the charge is in a small value range, i.e., \( Q \in (0, 2) \). And the parametrization formula can be regarded as the result obtained in the small charge limit. We plot the numerical values (the discrete points) and the fitting formula (solid line) of the coexistence curve in Fig. 4 for \( Q = 0.2 \) and \( 1.9 \). It is quite clear that the numerical values and the fitting formula are in highly matched with each other. Then the curve seems to be charge independent.
determined by one of the following four conditions: (I) \( \frac{\partial}{\partial T} P, Q = 0 \), (II) \( \partial v, P \), (III) \( \partial_s T \), (IV) \( \partial Q \). The numerical check in high temperature is listed in Table I, which shows that the relative deviations \( |\Delta_1| \) and \( |\Delta_2| \) are of \( 10^{-8} \) and \( 10^{-7} \), respectively. The deviations \( |\Delta_1| \) and \( |\Delta_2| \) are also showed in Fig. 7 for \( T/T_c \), respectively.

In Ref. [45], the author pointed out that the Clapeyron equation (25) is held for the black hole/AdS phase transition in the nonrotating and rotating black hole systems. In the following, we will numerically check Eqs. (25)-(28) for the small/large black hole phase transition using the analytic parametrization of the coexistence surface. In order to make a clear comparison, we define two relative deviations

\[
\Delta_1 = \frac{\left( \frac{dP}{dT} \right)_Q - \frac{\Delta S}{\Delta V} }{\frac{dP}{dT} }, \quad \Delta_2 = \frac{\left( \frac{dQ}{dT} \right)_T - \left( - \frac{\Delta V}{\Delta S} \right) }{\frac{dQ}{dT} },
\]

where \( \Delta_1 \) measures the deviation of \( \frac{\Delta S}{\Delta V} \) from \( \left( \frac{dP}{dT} \right)_Q \), and \( \Delta_2 \) the deviation of \( - \frac{\Delta V}{\Delta S} \) from \( \left( \frac{dQ}{dT} \right)_T \). The numerical check in high temperature is listed in Table I which shows that the relative deviations \( |\Delta_1| \) and \( |\Delta_2| \) are of \( 10^{-8} \) and \( 10^{-7} \), respectively. The deviations \( |\Delta_1| \) and \( |\Delta_2| \) are also showed in Fig. 7 for \( T = 0.1 \) and \( Q = 0.19 \), respectively. With fixed \( T \), \( |\Delta_1| \) shows an oscillatory behavior. At small charge \( Q \), \( |\Delta_1| \) can approach 0.08% and quickly decreases to below 0.003% with \( Q \) increasing. Near the critical value of \( Q \), \( |\Delta_2| \) blows up, which is caused by the fitting method. \( |\Delta_2| \) also shares the oscillatory behavior. At low \( T/T_c \), it is about of 0.07%. While it decreases to 0.002% at high

FIG. 5: Coexistence curve of charged AdS black hole with fixed charge \( Q \). The solid line is plotted with the fitting formula, and the numerical result is shown with the dots.
FIG. 6: Area law for the charged AdS black hole. (a) \( Q = 0.8, T / T_c = 0.95 \). Phase transition point at \( P / P_c = 0.8744 \). (b) \( Q = 0.8, T / T_c = 0.95 \). Phase transition point at \( P / P_c = 0.8744 \). The two areas have a difference of 0.0023. (c) \( Q = 0.8, P / P_c = 0.95 \). Phase transition point at \( T / T_c = 0.9809 \). (d) \( P = 0.005, T = 0.054 \). Phase transition point at \( Q = 0.7641 \).

Combining with the above results, we can conclude that the Clapeyron equations (25) and (26) are held among the small/large black hole phase transition of the charged AdS black hole. The third Clapeyron equation (27) will naturally hold when considering the Maxwell’s relation \( \left( \frac{dP}{dT} \right)_Q \left( \frac{dT}{dQ} \right)_P \left( \frac{dQ}{dT} \right)_P = -1 \). Using Eqs. (42) and (43), the pressure calculated from Eq. (28) is exactly consistent with Eq. (39).

In summary, by adopting the numerically check and the fitting formula of the coexistence curve, we show that Eqs. (25)-(28) derived above are held in the small/large black hole phase transition of the charged AdS black hole.

**B. \( d > 4 \)-dimensional charged AdS black hole**

Similar to the 4-dimensional charged AdS black hole, the small/large black hole phase transition also exists in higher-dimensional spacetime. The critical point is given in Eq. (37). After the numerical checking, the Clapeyron equations also hold. The parametrization form of the coexistence surface can also be obtained with fitting the numerical data:

\[
\tilde{P} = \sum_{i=0}^{10} a_i \tilde{T}^i, \tag{45}
\]
FIG. 7: Behaviors of the deviations $|\Delta_1|$ and $|\Delta_2|$.

TABLE I: Values of the thermodynamic quantities and deviations $|\Delta_1|$ and $|\Delta_2|$ at the phase transition points in high temperature. $r^L_h$ and $r^S_h$ correspond to the radius of the large and small black hole.

TABLE II: Values of the coefficients $a_i$ ($i = 0 \rightarrow 10$) in the fitting formula of the coexistence curves.

IV. DISCUSSIONS AND CONCLUSIONS

In this paper, we studied the thermodynamic properties of the charged AdS black hole along the coexistence curve of the small and large black holes phase. We first reviewed the classical $P-V$ oscillatory behavior that implies phase transition taking place. By means of the Maxwell’s equal area law, the unphysical branch of the system can be
excluded, and the phase transition point can also be determined. For the charged AdS black hole, we showed that such $P - V$ oscillatory behavior is also obvious in the $Q - \Phi$ and $T - S$ planes. For each oscillatory line, the equal area law is useful to find the phase transition point, which is exactly consistent with analyzing the Gibbs free energy. It is worthwhile to point out that the oscillatory behavior is also appeared in the $P - v$ plane. However, its equal area law fails to produce the phase transition point, which reminds us the difference between the thermodynamic and specific volumes.

Employing the first law, we get three Clapeyron equations at the coexistence curve. The first one is the classical Clapeyron equation with fixed charge, and the rest two are its generalizations with, respectively, fixed pressure and temperature.

Before carrying out the numerical check of the Clapeyron equations, we recalculated the coexistence surface and present the fitting formula for it for charge $Q \in (0.05, 2)$. It was shown that the formula is charge independent in the reduced parameter space. And this is a universal property for $d = 4 - 10$. A natural speculation is that such property holds for any $d$. As a consequence, we understand that these fitting formulas are effective in the small charge limit. However, after converting to $P$ and $T$, the coexistence surface is charge dependent. Making use of the fitting formula, we checked that these Clapeyron equations exactly hold when the black hole system varies along the coexistence surface. For example, at low temperature, the deviation is of $0.08\%$, and it quickly decreases to $0.003\%$ for high temperature. A simple check shows that our fitting formula agree with the calculated values to $0.01\%$ at low reduce temperature $\tilde{T}$, and to $10^{-6}$ at high $\tilde{T}$.

At last, we would like to make a few comments. Recent study mainly focuses on finding the VdW type phase transition, reentrant phase transitions or thermodynamic properties near the critical point. While only a few papers concern the property of the black hole system going along or crossing the coexistence curve. We argue that studying the physics near the coexistence curve may help us to reveal the information of the microscopic structure of a thermodynamic system, which is still unclear from the gravity side. So we hope that study of the phase transition could provide us some information of the microscopic structure of a gravitational system from the thermodynamic viewpoint. On the other hand, we presented in this paper the highly accurate formulas of the coexistence surfaces (see Eqs. 41 and 45 as well as table II). Further research can be carried out based on them.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), arXiv:hep-th/9711200.
