Model of approximation of a velocity, vorticity and pressure in an incompressible fluid

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Abstract. Generally, the temporary discretization of the Navier-Stokes equations despise the convective term, and consider a boundary ∂Ω = Γ. This paper introduces a Galerkin scheme designed to solve by means of the finite element method for the Oseen problem in three dimensions written in terms of speed, vorticity and pressure, in a viscous and incompressible fluid that flows through a porous medium, this problem is obtained from a temporary discretization of the Navier-Stokes equations and we consider a partitioned boundary ∂Ω = Γ₁ ∪ Γ₂ and disjoint, that is, the velocity is of the homogeneous Dirichlet type on Γ₁, while the tangential velocity and pressure They are of the non-homogeneous Dirichlet type on Γ₂. In the variational formulation the speed is completely decoupled, which allows you to approximate the vorticity and pressure independently. The speed is recovered from the vorticity and pressure. Galerkin’s scheme is based on Nédélec finite elements and continuous polynomials to pieces of the same order, for vorticity and pressure, respectively. Likewise, convergence rates are obtained for vorticity, speed and pressure in natural norms. Finally, a numerical example is provided that illustrates the behavior of the model.

1. Introduction

The development of appropriate numerical methods to simulate incompressible fluid flows, modeled by the Navier-Stokes equations, has been studied during the last decades, due to its applications in different applied sciences such as: biomedical engineering, aerodynamics, among others. Particularly in [5] the authors make a study based on a temporary discretization of the Navier-Stokes equations, depreciating the convective term, they obtain the Brinkman problem. From [1, 2, 5] arises the study of this work, by not neglecting the convective term, the problem is obtained written in terms of speed, vorticity and pressure, in a viscous and incompressible fluid that flows through a porous medium. This work aims to propose the Galerkin scheme to approximate the solution to the Brinkman Oseen problem using finite elements of Nédélec and continuous stubborn polynomials to approximate the vorticity and pressure, then the velocity field is recovered.

2. Framework

The development of appropriate numerical methods to simulate incompressible fluid flows in porous media, in particular can be studied by the Brinkman equations formulated in terms
of speed, vorticity and pressure [1–3]. Depending on the physical phenomenon of the fluid, it is considered a partitioned and disjoint border [4]. Recently, a finite element method for the Brinkman equations formulated in terms of vorticity, velocity and pressure is presented, decoupling speed [5]. It serves as a reference for the formulation of the Oseen problem, to approximate solutions a Galerkin scheme with finite elements of Nédelec continuous polynomials to pieces of grade \( k \geq 1 \) [6–8] is proposed. It should be borne in mind that the interest of considering in the analysis more general boundary conditions that are usually found in applications such as [9–17].

3. Mathematical model

The speed, vorticity and pressure of a viscous and incompressible fluid flowing in a porous medium can be determined by the Oseen equations commonly known as generalized Navier-Stokes [8,15,18]. Understanding that the fundamental feature of an incompressible fluid is that \( \text{div} \, u = 0 \), where \( u \) is the fluid velocity vector. The circulation of the velocity vector \( u \) being a kinematic magnitude, helps to interpret the movement of the fluid. This circulation is directly related to the existence of rotation of the particles in the fluid. The vorticity vector is defined as the rotational of the velocity vector \( \omega = \sqrt{\nu} \text{curl} \, u \), which is a magnitude to quantify the rotation in the fluid. From the linearization of the Navier-Stokes equations, the Oseen model is obtained as Equation (1).

\[
\sigma u - \nu \Delta u + \text{curl} \, u \times \beta + \nabla p = f \quad \text{in } \Omega, \quad \text{div} \, u = 0 \quad \text{in } \Omega,
\]

where \( \nu > 0 \) is the kinematic viscosity of the fluid, \( \sigma > 0 \) is the time of approximation of the phenomenon, \( \beta \) is an approximate estimate of the velocity, the force vector \( f \) not only incorporates external forces in the fluid but it stores the contributions of the field over time and the contribution obtained from linearizing the stable Navier-Stokes equations. As usual in this context, Bernoulli’s pressure \( p = P + \frac{1}{2} |u|^2 \), is introduced, where \( P \) is the actual fluid pressure, in Equation (1) \( \text{curl} \, u \) suggests the appearance of the vorticity defined as \( \omega = \sqrt{\nu} \text{curl} \, u \). The appearance of vorticity suggests the presence of tangential tensions in different places of the fluid, whereby the boundary \( \partial \Omega \) is formed by \( \Gamma_1 \cup \Gamma_2 \) disjoint, where \( \Gamma_1 \) has homogeneous Dirichlet type conditions \( u = 0 \), while in \( \Gamma_2 \) a non-homogeneous tangential velocity is presented, that is to say, \( u \times n = a \times n \) gives rise to conditions of non-homogeneous Dirichlet type after a fixed Bernoulli pressure \( p = p_0 \) on this same border. Then a model that describes the speed, vorticity and pressure of an incompressible fluid that flows over a porous medium is Equation (2).

\[
\sigma u + \sqrt{\nu} \text{curl} \, \omega + \nu^{1/2} \omega \times \beta + \nabla p = f \quad \text{in } \Omega,
\]

\[
\omega - \sqrt{\nu} \text{curl} \, u = 0 \quad \text{in } \Omega,
\]

\[
\text{div} \, u = 0 \quad \text{in } \Omega,
\]

\[
u^{1/2} u = 0 \quad \text{over } \Gamma_1,
\]

\[
u^{1/2} u \times n = a \times n \quad \text{over } \Gamma_2,
\]

\[
p = p_0 \quad \text{over } \Gamma_2,
\]

where \( n \) is the normal unit vector over \( \partial \Omega \) and \( a \) is an arbitrary vector.

3.1. Variational formulation

The variational formulation also known as weak formulation of Equation (2) in terms of vorticity and pressure is determined when multiplying by appropriate test functions \( (\theta, q) \in Z \times Q \) (Equation (3)).

\[
A((\omega, p), (\theta, q)) = F(\theta, q) \quad \forall (\theta, q) \in Z \times Q.
\]
From Equation (3) the bilinear form, $A : (\mathbb{Z} \times \mathbb{Q}) \times (\mathbb{Z} \times \mathbb{Q}) \rightarrow \mathbb{R}$, and the lineal functional $F : \mathbb{Z} \times \mathbb{Q} \rightarrow \mathbb{R}$, are defined by Equation (4).

$$A((\omega, p), (\theta, q)) := \sigma \int_{\Omega} \omega \cdot \theta + \int_{\Omega} (\sqrt{\nu} \text{curl} \omega + \nabla p) \cdot (\sqrt{\nu} \text{curl} \theta + \nabla q)$$

$$+ \nu^{-1/2} \int_{\Omega} (\omega \times \beta) \cdot (\sqrt{\nu} \text{curl} \theta + \nabla q),$$

$$F(\theta, q) := \int_{\Omega} f \cdot (\sqrt{\nu} \text{curl} \theta + \nabla q) + \sqrt{\nu} \sigma \langle a \times n, \theta \rangle_{\Gamma_2}. \tag{4}$$

4. Approximation of the solution

In this section, we introduce the Galerkin of problem in Equation (3), using the formalism of the Nédélec polynomials to approximate the vorticity, to approximate the velocity and pressure, continuous polynomials are used in pieces. In addition, an estimator for the error calculation is described and the velocity field is approximated.

4.1. Finite elements

Being $\{T_h(\Omega)\}_{h>0}$ a family of partitions on a regular basis of the polyhedral region $\Omega$, by triangles $T$ in $\mathbb{R}^2$ or tetrahedral $T$ in $\mathbb{R}^3$ of diameter $h_T$, with size of the mesh $h := \max\{h_T : T \in T_h(\Omega)\}$, is followed, given an integer $k \geq 1$ and a subspace $S \in \mathbb{R}^3$, $\mathbb{P}_k(S)$ denotes the space of degree polynomials at most $k$, defined on $S$. In addition, for all $T \in T_h(\Omega)$ the local space of Nédélec $\mathbb{N}_k(T) := \mathbb{P}_{k-1}(T) \oplus \mathbb{P}_{k-1} \times x$ is introduced for vorticity and continuous polynomials to pieces for pressure, where $x \in \mathbb{R}^3$. The following subspaces of $\mathbb{Z}$ and $\mathbb{Q}$ are defined (Equation (5)).

$$\mathbb{Z}_h := \{\theta_h \in \mathbb{Z} : \theta_h|_{T} \in \mathbb{N}_k(T) \forall T \in T_h(\Omega)\} \tag{5}$$

$$\mathbb{Q}_h := \{q_h \in \mathbb{Q} : q_h|_{T} \in \mathbb{P}_k(T) \forall T \in T_h(\Omega)\}.$$

Its discrete formulation is analogous to the formulation of Equation (3). Find $(\omega_h, p_h) \in \mathbb{Z}_h \times \mathbb{Q}_h$ (Equation (6)).

$$A((\omega_h, p_h), (\theta_h, q_h)) = F(\theta_h, q_h) \quad \forall (\theta_h, q_h) \in \mathbb{Z}_h \times \mathbb{Q}_h. \tag{6}$$

From Equation (6) the bilinear form $A : (\mathbb{Z}_h \times \mathbb{Q}_h) \times (\mathbb{Z}_h \times \mathbb{Q}_h) \rightarrow \mathbb{R}$ and lineal functional $F : \mathbb{Z}_h \times \mathbb{Q}_h \rightarrow \mathbb{R}$, is defined by Equation (7).

$$A((\omega_h, p_h), (\theta_h, q_h)) := \sigma \int_{\Omega} \omega_h \cdot \theta_h + \int_{\Omega} (\sqrt{\nu} \text{curl} \omega_h + \nabla p_h) \cdot (\sqrt{\nu} \text{curl} \theta_h + \nabla q_h)$$

$$+ \nu^{-1/2} \int_{\Omega} (\omega_h \times \beta) \cdot (\sqrt{\nu} \text{curl} \theta_h + \nabla q_h), \tag{7}$$

$$F(\theta_h, q_h) := \int_{\Omega} f \cdot (\sqrt{\nu} \text{curl} \theta_h + \nabla q_h) + \sqrt{\nu} \sigma \langle a \times n, \theta_h \rangle_{\Gamma_2}.$$

**Theorem 1.** Assuming that $\frac{2||\beta||_{\infty, \Omega}}{\nu \sigma} < 1$, then the discrete problem in Equation (6) is well placed.

**Proof.** It is known that in Equation (7) corresponds to a system of linear and square equations. Indeed, considering $f = 0$ and $a = 0$, with $(\theta_h, q_h) = (\omega_h, p_h)$, we have the Equation (8).
\[ \sigma \| \omega_h \|^2_{0, \Omega} + \| \nu \text{curl} \omega_h + \nabla p_h \|^2_{0, \Omega} + \nu^{-1/2} \int_{\Omega} (\omega_h \times \beta) \cdot (\nu \text{curl} \omega_h + \nabla p_h) = 0. \] (8)

Form Equation (8) consider \( \omega_h = 0 \) and \( p_h = 0 \), the existence, uniqueness of solution of Equation (6).

### 4.2. Error estimation

Error estimates are presented in standard \( L^2 \) for vorticity and pressure. For \( s > 1/2 \), the global interpolant of Nédélec \( R_h : H^s(\text{curl}; \Omega) \cap Z \rightarrow Z_h \) is introduced: which is defined in [19] and satisfies the following estimate for vorticity:

**Lemma 1.** For all \( \omega \in H^s(\text{curl}; \Omega), s \in (1/2, k] \) there is a constant \( C > 0 \), independent of \( h \), so that we have the Equation (9).

\[ \| \omega - R_h \omega \|_Z \leq C h^s \| \omega \|_{H^s(\text{curl}; \Omega)}. \] (9)

Equation (9) is important for estimating vorticity. For \( s > 1/2 \), the Lagrange interpolant \( \Pi_h : H^{1+s}(\Omega) \cap Q \rightarrow Q_h \) is introduced, which satisfies the following estimate for pressure (Lemma 2).

**Lemma 2.** For all \( q \in H^{1+s}(\Omega), s \in (1/2, k] \) there is a constant \( C > 0 \) independent of \( h \), so that we have the Equation (10).

\[ \| q - \Pi_h q \|_Q \leq C h^s \| q \|_{H^{1+s}(\Omega)}. \] (10)

Equation (9) and Equation (10) are important for estimating vorticity. Writing the following error equation (Equation (11)):

\[ \mathcal{A}(\omega - \omega_h, p - p_h, (\theta_h, q_h)) = 0, \quad \forall (\theta_h, q_h) \in Z_h \times Q_h. \] (11)

**Lemma 3.** The following subspaces are considered \( H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \), there is a constant \( \alpha > 0 \), so that we have the Equation (12).

\[ \| q \|_{0, \Omega} \leq \alpha \left( \| \theta \|_{0, \Omega} + \| \sqrt{\nu} \text{curl} \theta + \nabla q \|_{0, \Omega} \right). \] (12)

**Proof.** Indeed if it is considered that Equation (13).

\[ \| q \|_{0, \Omega} \leq \| \nabla q \|_{-1, \Omega} := \sup_{v \in H_0^1(\Omega)} \left( \frac{\langle v, \nabla q \rangle_{0, \Omega}}{\| v \|_{1, \Omega}} \right) = \sup_{v \in H_0^1(\Omega)} \left( \frac{\langle v, \nabla q + \sqrt{\nu} \text{curl} \theta - \sqrt{\nu} \text{curl} \theta \rangle_{0, \Omega}}{\| v \|_{1, \Omega}} \right). \] (13)

The inequality of Equation (12) is obtained. The following lemma is of great importance for the order of convergence of vorticity and pressure (Lemma 4).

**Lemma 4.** Assuming that \( \frac{2 \| \theta \|_{0, \Omega}^2}{\nu \sigma} < 1 \). For all \( (\theta, q) \in Z \times Q \), then there is \( C_1, C_2 > 0 \) so that Equation (14) and Equation (15) are presented as:

\[ C_1(\| \theta \|^2_{0, \Omega} + \| \sqrt{\nu} \text{curl} \theta + \nabla q \|^2_{0, \Omega} + \| q \|^2_{0, \Omega}) \leq \mathcal{A}(\theta, q, (\theta, q)) \] (14)

and

\[ \mathcal{A}(\theta, q, (\theta, q)) \leq C_2(\| \theta \|^2_{0, \Omega} + \| \sqrt{\nu} \text{curl} \theta + \nabla q \|^2_{0, \Omega} + \| q \|^2_{0, \Omega}). \] (15)

**Proof.** For Equation (14) using definition of absolute value, Young’s inequality and Lemma 3. Then, for Equation (15), applying triangular inequality and Young’s inequality.
Theorem 2. Convergence of vorticity and pressure. Assuming that \( \omega \in H^s(\text{curl}; \Omega) \) and \( p \in H^{1+s}(\Omega) \), for any \( s \in (1/2, k] \). Then, there is a constant \( C > 0 \), independent of \( h \), we have the Equation (16).

\[
\|\omega - \omega_h\|_{0,\Omega} + \|\sqrt{\nu \text{curl}} (\omega - \omega_h) + \nabla (p - p_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^s(\|\omega\|_{H^s(\text{curl}; \Omega)} + \|p\|_{H^{1+s}(\Omega)}).
\]  

Proof. The result is a consequence of the Equation (11) together with Lemma 4, Lemma 1 and Lemma 3 inequality of Equation (16) is concluded.

4.3. Velocity field approach
If \( \omega \in \mathbb{Z} \) and \( p \in Q \) are unique solutions of Equation (3) and \( \omega_h \in Z_h \), and \( p_h \in Q_h \) are the unique solutions of Equation (6), respectively. Thus as Equation (17).

\[
u = \sigma^{-1}(f - \sqrt{\nu \text{curl}} \omega - \nu^{-1/2} \omega \times \beta - \nabla p),
\]

\( u_h \) is defined as Equation (18).

\[
u_h := \sigma^{-1}(P_h f - \sqrt{\nu \text{curl}} \omega_h - \nu^{-1/2} \omega_h \times \beta - \nabla p_h).
\] (18)

The velocity is approximated by, \( P_h : [L^2(\Omega)]^3 \rightarrow U_h = \{v_h \in [L^2(\Omega)]^3 : v|_T \\
\in [P_{k-1}(T)]^3 \ T \in T_h(\Omega)\} \) orthogonal projection of \( L^2 \). Then \( \forall s \in (0, k] \), Then \( \forall s \in (0, k] \), it is known that Equation (19) is:

\[
u = \nu - P_h \nu \leq Ch^s \|v\|_{s,\Omega}.
\] (19)

Theorem 3. Speed Convergence. Being \( \omega \in \mathbb{Z} \) and \( p \in \mathbb{Q} \) the unique solutions of Equation (3), \( \omega_h \in Z_h \) and \( p_h \in Q_h \) the unique solutions of Equation (6), respectively. Supposing that \( \omega \in H^s(\text{curl}; \Omega), p \in H^{1+s}(\Omega) \) and \( f \in H^s(\Omega)^3 \), for some \( s \in (1/2; k] \). Then, there is a constant \( C > 0 \) independent of \( h \), so that we have Equation (20).

\[
u = \nu - u_h \leq Ch^s (\|f\|_{H^s(\Omega)} + \|\omega\|_{H^s(\text{curl}; \Omega)} + \|p\|_{H^{1+s}(\Omega)}).
\] (20)

Proof. The result of the inequality of Equation (20) is a consequence of the Equation (17), Equation (18) and Equation (19), together with the triangular inequality and Theorem 2.

5. Numerical results and discussion
The simulations are based on a FreeFem++ code [20], in conjunction with the UMFPACK linear resolution solver [21], where \( (\omega, p) \in \mathbb{Z} \times \mathbb{Q} \) and \( (\omega_h, p_h) \in Z_h \times Q_h \), are solutions from of Equation (3) and Equation (6), respectively, and individual errors are defined as Equation (21).

\[
u = \nu - \omega_h \|_{0,\Omega} , \quad (p) = \|p - p_h\|_{0,\Omega} , \quad (u) = \|u - u_h\|_{0,\Omega}.
\] (21)

On the other hand, the convergence rates is Equation (22).

\[
u = \frac{\log(e(\omega)/\hat{\epsilon}(\omega))}{\log(h/h')} , \quad (p) = \frac{\log(e(\nu)/\hat{\epsilon}(\nu))}{\log(h/h')} , \quad (u) = \frac{\log(e(\nu)/\hat{\epsilon}(\nu))}{\log(h/h')}.
\] (22)

where \( h \) and \( h' \) are two consecutive meshes and \( e \) and \( \hat{\epsilon} \) are the calculated errors. 2D results for the Equation (2), are presented, considering the following condition \( \frac{2|\beta|}{\nu \sigma} < 1 \).
Example 1. Consider the domain $\Omega = (0, 1)^2$, the coefficients $\sigma = 100$ and $\nu = 0.1$. Taking adequate data for $f$, so that the exact solution for the Equation (2) is given by the functions: with $\beta = u$ (Equation (23)).

$$u = \begin{pmatrix} \sin(\pi x)^2 \sin(\pi y) \\ \sin(2\pi x) \cos(\pi y) \end{pmatrix}, \quad p = x^4 - y^4$$

$$\omega = \sqrt{\nu \pi}(2 \cos(2\pi x) \cos(\pi y) - \pi \sin(\pi x)^2 \cos(\pi y))$$.

In the Figure 1, the graphs of exact and approximate solutions are presented and it is observed that there is a good suitability of the approximate with respect to the exact one.

![Figure 1](image)

**Figure 1.** (a) represents the approximate and exact solution for vorticity, (b) represents the approximate and exact solution for velocity and (c) represents the approximate and exact solution for the pressure.

Table 1 shows results in the $L^2$ standard, obtaining a superconvergence for vorticity and pressure, while the velocity has an order of optimal convergence.

| $h$    | $e(\omega)$ | $r(\omega)$ | $e(p)$ | $r(p)$ | $e(u)$ | $r(u)$ |
|--------|--------------|--------------|--------|--------|--------|--------|
| 0.372678 | 0.199886     | 1.455126     | 0.068622 | 0.691176 | 0.235615 | 0.766333 |
| 0.190086 | 0.045383     | 2.202206     | 0.017156 | 2.059064 | 0.116289 | 1.048851 |
| 0.093856 | 0.009024     | 2.288756     | 0.003782 | 2.142812 | 0.055203 | 1.055751 |
| 0.047891 | 0.002190     | 2.104440     | 0.000834 | 2.246188 | 0.027126 | 1.056022 |
| 0.026245 | 0.000573     | 2.228707     | 0.000219 | 2.224742 | 0.013663 | 1.140230 |
| 0.013382 | 0.000140     | 2.094855     | 0.00051 | 2.172746 | 0.006770 | 1.042499 |
In [5] the Brinkman problem formulated in terms of vorticity, velocity and pressure is studied, obtaining optimum convergence rates in natural norms. In [4], they consider a partitioned and disjoint border unlike [5], they obtain optimal convergence rates. However, for the Oseen problem, Nédélec finite elements and continuous polynomials were introduced into pieces of order \(k \geq 1\) as in [5]. Thus, a superconvergence was obtained for the vorticity and pressure shown in Table 1, while for the speed optimal convergence was obtained. Therefore, there is an optimal convergence for velocity as in [4,5], while for vorticity and pressure a superconvergence unlike in [4,5] who obtained an optimal convergence.

6. Conclusion
The existence and uniqueness of the problem is demonstrated for the discrete level, in addition there is convergence of the discrete scheme for vorticity and pressure using particular finite elements, in this case Nédélec and continuous polynomials to pieces. The most relevant fact is that a superconvergence for vorticity and pressure is obtained, that is, this information is retrieved more quickly. The velocity field is recovered with optimal convergence.

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