On correlation functions in $J\bar{T}$-deformed CFTs

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Received 31 January 2019, revised 8 March 2019
Accepted for publication 12 March 2019
Published 4 April 2019

Abstract

The $J\bar{T}$ deformation, built from the components of the stress tensor and of a $U(1)$ current, is a universal irrelevant deformation of two-dimensional CFTs that preserves the left-moving conformal symmetry, while breaking locality on the right-moving side. Operators in the $J\bar{T}$-deformed CFT are naturally labeled by the left-moving position and right-moving momentum and transform in representations of the one-dimensional extended conformal group. We derive an all-orders formula for the spectrum of conformal dimensions and charges of the deformed CFT, which we cross-check at leading order using conformal perturbation theory. We also compute the linear corrections to the one-dimensional OPE coefficients and comment on the extent to which the correlation functions in $J\bar{T}$-deformed CFTs can be obtained from field-dependent coordinate transformations.

Keywords: integrable deformations, asymptotic fragility, dipole CFTs

1. Introduction

There has been plenty of recent interest in deformations of two-dimensional QFTs by irrelevant operators constructed from bilinears of conserved currents [1], of which the so-called $TT$ deformation [1, 2] is the best studied example [3–16]. Remarkably, this deformation is solvable in a certain sense and the deformed spectrum is determined from that of the original QFT through a universal formula. $TT$ deformed CFTs have also found several very interesting applications in string theory [17–22] and in holography [23–30].

Another remarkable property of $TT$ deformed QFTs is that they are believed to be UV complete, albeit non-local. The reason for this claim is that certain on-shell quantities, such as the S-matrix, appears to be well-defined up to arbitrarily high energies, at least for one sign of the deformation parameter [3, 31, 32]. The status of off-shell quantities such as correlation functions is less clear but, given the proposal that the $TT$ deformation produces a theory of
quantum gravity in two dimensions [5, 33], one should perhaps not expect local observables to exist beyond perturbation theory.

It is interesting to ask about the ultraviolet behaviour of other deformations in the Smirnov–Zamolodchikov class [1] and to find an appropriate set of local or non-local observables. In this article, we will address this question for the case of the $JT$ deformation of two-dimensional CFTs [34], which is comparable to $TT$ in what regards its universality, but different from it in an interesting way and possibly more tractable. The $JT$ deformation is defined by adding to the action a composite operator constructed from a chiral $U(1)$ current $J$ and the right-moving stress tensor $T$

$$\frac{\partial}{\partial \mu} S(\mu) = \int d^2z \langle JT \rangle_{\mu}$$

(1.1)

where the subscript indicates that the deformation is always performed using the current and stress tensor of the deformed theory. Just like $TT$, the $JT$ deformation is solvable. The finite-size spectrum was first derived in [34] for the case of vanishing chiral anomaly and in section 6 of [35] for the general case. The holographic dictionary for $JT$ deformed CFTs was derived in [36] and the string-theoretical realization of a single-trace version of $JT$ was studied in [35, 37]; The modular properties of the partition function of $JT$ deformed CFTs were studied in [38]. Other related work is [39].

At the CFT point, the $JT$ operator has dimension $(1, 2)$, so it is marginal from the point of view of the left conformal symmetry and irrelevant from the point of view of the right one. As a consequence of adding this operator to the CFT action, the deformed theory becomes non-local on the right, but still remains local and conformal on the left, at least at leading order in conformal perturbation theory. Given this structure, it is natural to label operators by their left-moving position $z$ and right-moving momentum $p$. Assuming the $SL(2, \mathbb{R})_L$ symmetry is unbroken at higher orders in the deformation parameter, we can view the $JT$-deformed CFT as a one-dimensional CFT—obtained by dimensional reduction along the null direction $\bar{z}$—where operators come in continuous towers labeled by $\bar{p}$. Since the left-moving global symmetries $SL(2, \mathbb{R})_L \times U(1)_J$ admit the usual infinite-dimensional extension, these operators will transform in Virasoro$_L \times U(1)$ Kač–Moody representations.

Thus, a natural set of observables in $JT$-deformed CFTs are the correlation functions of the operators $\mathcal{O}_{\mu \bar{p}}(z)$. These one-dimensional correlation functions are determined in terms of the usual conformal data: operator dimensions, charges, and the OPE coefficients

$$h_\mu(\mu \bar{p}) \quad q_\mu(\mu \bar{p}) \quad C_{ijk}(\mu \bar{p}) \quad (1.2)$$

Due to the irrelevant deformation, these data can now depend on the dimensionless combination $\mu \bar{p}$, where $\mu$ is the deformation parameter, a null vector with dimension of (length)$^{-1}$. This structure is virtually identical with that previously studied in the context of non-relativistic holography [40]; the main difference between $JT$ and the deforming operator in those works is that $JT$ is double-trace and has universal correlation functions with the operators in the undeformed CFT. As a consequence, the above conformal data are related to the corresponding data in the original CFT$_2$ in a universal way.

Note that from the point of view of the one-dimensional conformal data (1.2), the combination $\mu \bar{p}$ need not be small; indeed, below we present exact expressions for $h_\mu(\mu \bar{p})$ and $q_\mu(\mu \bar{p})$ that have no obvious pathology for large $\mu \bar{p}$. Thus, $JT$-deformed CFTs appear to be

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well-defined well above the scale set by the irrelevant operator, providing yet another potential example of a two-dimensional QFT that is UV complete without possessing a usual UV fixed point. Even though the non-locality of $JT$-deformed CFTs makes it natural to use the momentum space representation for the right-movers, note that the same amount of data is contained in the correlation functions of the $O_{\rho}(z)$ operators as in the original position space representation; by contrast, in $TT^\ast$ deformed QFTs only on-shell observables are believed to be well-defined. Thus, the $JT$ deformation leads to a slightly different notion of asymptotic fragility [32] than $TT$. The ultraviolet behaviour of $JT$-deformed CFTs is most likely to resemble that of lightlike dipole theories, a set of non-local field theories based on a star product $[41–43]$. The conformal data (1.2) can in principle be computed up to any desired order in the deformation using conformal perturbation theory. Two questions of interest are whether: (i) we can provide closed-form expressions for these data that are exact in the $\mu \bar{\rho}$ parameter and (ii) whether the resulting expressions hint towards the existence of an additional structure that may allow one to specify all correlation functions in the deformed CFT in terms of those in the original CFT, and possibly also provide a geometric interpretation of the correlators directly in right-moving position space. That such an interpretation may exist is suggested by the holographic analysis of [36], who found that the expectation value of the stress tensor was related by a field-dependent coordinate transformation to the expectation value in the undeformed CFT. Similar field-dependent coordinate transformations appear ubiquitously in the $TT^\ast$ context $[5, 33]$, and it would be very interesting to understand their meaning at a precise quantum level.

In this article, we partially answer (i) in the affirmative, by providing an exact expression for the deformed conformal dimensions and charges

$$h(\mu) = h + \frac{\mu q}{2\pi} \bar{\rho} + \frac{\mu^2 k}{16\pi^2} \bar{\rho}^2, \quad q(\mu) = q + \frac{\mu k}{4\pi} \bar{\rho}$$

(1.3)

where $h, q$ represent the left conformal dimension and charge in the undeformed CFT and $k$ is the coefficient of the chiral anomaly. These expressions are obtained by applying an infinite boost to the known spectrum of $JT$-deformed CFTs on the cylinder and, as we show, they agree perfectly with leading order conformal perturbation theory calculations. We also show that the pattern of operator mixing implied by the above formula is consistent with the structure we obtain using conformal perturbation theory. Notice that the shifts in $h, q$ are reminiscent of spectral flow, where the spectral flow parameter depends on the right-moving momentum.

In order for (ii) to hold, the OPE coefficients in (1.2) would also need to take very special values. To check whether this occurs, we compute the linearized corrections to the three-point functions, using Euclidean conformal perturbation theory. While the structure we obtain for the deformed correlator is in perfect agreement with the one dictated by one-dimensional conformal invariance, the corrections to the OPE coefficients not only do not vanish, but in fact are nonzero for structural reasons. It is not clear to us whether this is an artifact of the particular regularization procedure we have used, which may or may not be suited for $JT$-deformed CFTs. We leave this interesting question to future work.

The organisation of this paper is as follows. In section 2, we discuss the general structure of the correlation functions, how to use conformal perturbation theory to compute the deformed correlators and to what extent the latter can be obtained by applying field-dependent coordinate transformations to correlation functions in the original CFT. In section 3, we derive the formula for the spectrum of conformal dimensions and charges from the known spectrum of $JT$-deformed CFTs on the cylinder. In section 4, we use conformal perturbation theory to check (1.3) to leading order, compute the linearized corrections to two sample three-point
functions and discuss operator mixing. We end in section 5 with a discussion and future directions. Various useful formulae for the correlators, certain integrals and Fourier transforms are relegated to the appendices.

2. General structure of the correlation functions

As explained in the introduction, JT-deformed CFTs are expected to be local and conformal on the left, but non-local on the right. The most natural way of labeling operators in such a theory is in terms of the left-moving coordinate $z$ (as local operators transform nicely under the conformal group) and the right-moving momentum $\bar{p}$. In the original CFT, these operators are given by the Fourier transform\(^2\) of the local operators with respect to the right-moving coordinate $\bar{z}$:

$$O(z, \bar{p}) = \int d\bar{z} O(z, \bar{z}) e^{-i\bar{p}\bar{z}}. \quad (2.1)$$

Rather than $O(z, \bar{p})$, we will use the notation $O_{\bar{p}}(z)$, thus viewing the Fourier-transformed operators as an infinite tower of local operators in a CFT, which carry a continuous\(^3\) $\bar{p}$ label. This representation is also natural if we view the JT-deformed CFT as a non-relativistic CFT in zero spatial dimensions, with $\bar{p}$ playing the role of a continuous particle number label.

As we turn on the deformation, the operators $O_{\bar{p}}(z)$ can acquire an anomalous left conformal dimension. Since, as discussed in [36, 40], the deformation parameter is a null vector with dimension of length, the general form of the left dimension of $O_{\bar{p}}$ will be $h(\mu\bar{p})$. The correlation functions of the operators $O_{\bar{p}}$ have the form of those in a one-dimensional CFT, but with the conformal data being in principle arbitrary functions of $\mu\bar{p}$. In the following, we review the expected general structure of the correlation functions that follows from one-dimensional conformal symmetry on the one hand and conformal perturbation theory on the other. Apart from the issue of operator mixing, this section mostly reviews the structure previously discussed in the context of non-relativistic CFTs [40].

The consequences of the $SL(2, \mathbb{R})_L$ invariance at the level of two and three point functions are reviewed in section 2.1. In section 2.2, we briefly review how the deformed correlators are computed using conformal perturbation theory, and discuss issues related to operator degeneracy, mixing and how to choose a diagonal basis of operators. In section 2.3, we briefly address the question whether the correlators in JT-deformed CFTs may have a simple structure directly in right-moving position space, and in particular whether this structure can be interpreted in terms of field-dependent coordinate transformations.

2.1. Constraints from one-dimensional conformal invariance

In this section, we review the well-known form of correlation functions in one dimensional CFTs, as dictated by $SL(2, \mathbb{R})_L$ invariance. The operators we consider carry two additional labels, the right-moving momentum $\bar{p}$ and the $U(1)_J$ charge $q(\mu\bar{p})$, which are constrained by momentum and, respectively, charge conservation. The only additional structure we require is that when $\mu = 0$, the correlators reduce to the Fourier transform of the original CFT correlators with respect to the right-moving coordinate, and that the full answer be a perturbative series in $\mu\bar{p}$.

In view of these constraints, the two-point functions take the general form

\(^2\)For the purposes of the Fourier transform, the coordinate $\bar{z}$ is assumed to be real and independent from $z$.

\(^3\)It may also be possible to discretize the $\bar{p}$ label by considering the DLCQ of JT-deformed CFTs.
\[ \langle O_{\hat{p}_1, q_1}(z_1)O_{\hat{p}_2, q_2}(z_2) \rangle = \frac{\mathcal{N}_\hat{y}(\mu, \hat{p})}{c_{12}^{2k(\mu p)}} \delta(\hat{p}_1 + \hat{p}_2) \delta_{q_1 \cdots q_2}, \quad \hat{p} \equiv \hat{p}_1 = -\hat{p}_2. \]  

(2.2)

The coefficients \( \mathcal{N}_\hat{y} \) vanish if the conformal dimensions of \( O_1 \) and \( O_2 \) are different. Throughout this article, we will choose the operator basis and normalization\(^4\) such that

\[ \mathcal{N}_\hat{y}(\mu, \hat{p}) = \mathcal{N}(\hat{p}) \delta_{ij} \]  

(2.3)

where \( \mathcal{N}(\hat{p}) \) is the Fourier transform of the right-moving part of the CFT two-point function with respect to \( \hat{z} \)

\[ \mathcal{N}(\hat{p}) = \int_{-\infty}^{\infty} dz e^{-i|z|^h} \frac{2\pi(-i)^{2h}|\hat{p}|^{2h-1}}{\Gamma(2h)} \Theta(\hat{p}) \]  

(2.4)

and \( \hat{h} \) is the right-moving dimension of the operator \( O_i \). This result agrees with previous calculations of Wightman functions in non-relativistic CFTs [44].

Once the basis has been normalized, the non-trivial dynamical data of \( JT \) deformed CFTs lies at the level of the three-point functions

\[ \langle O_{\hat{p}_1}(z_1)O_{\hat{p}_2}(z_2)O_{\hat{p}_3}(z_3) \rangle = \frac{C_{ijk}(\hat{p}_1, \hat{p}_2, \hat{p}_3)}{c_{12}^{2k(\hat{p}_1 \hat{p}_2 \hat{p}_3)}} \delta(\hat{p}_1 + \hat{p}_2 + \hat{p}_3) \]  

(2.5)

where

\[ h_{ij}^{k}(\hat{p}_1) = h_i(\mu \hat{p}_1) + h_j(\mu \hat{p}_2) - h_k(\mu \hat{p}_3) \]  

(2.6)

and the remaining two combinations are given by permutations. We have dropped for simplicity the charge labels, keeping in mind that the total charge must be zero. The coefficient \( C_{ijk}(\hat{p}_1, \hat{p}_2, \hat{p}_3) \) takes the general form

\[ C_{ijk}(\hat{p}_1, \mu) = K_{ijk}(\hat{p}_1) C_{ijk}(\mu \hat{p}_1) \]  

(2.7)

where \( K_{ijk}(\hat{p}_1) \) is a kinematic factor equal to the Fourier transform of the right-moving part of the original CFT\(_2\) three-point function and \( C_{ijk}(\mu \hat{p}_1) \) represents the corrected OPE coefficient, i.e.

\[ C_{ijk}(\mu \hat{p}_1) = C_{ijk}^{\text{CFT}} + O(\mu \hat{p}_1) \]  

(2.8)

where \( C_{ijk}^{\text{CFT}} \) is the OPE coefficient in the two-dimensional CFT (a number). The function \( K_{ijk}(\hat{p}_1) \) is defined as

\[ K_{ijk}(\hat{p}_1) = \int_{1}^{3} \prod_{k=1}^{3} dz_k e^{-\sum_{k=1}^{3} c_{12}^{a_s b_t c_z}} \equiv K(a, b, c; \hat{p}_1) \delta(\hat{p}_1 + \hat{p}_2 + \hat{p}_3) \]  

(2.9)

where \( a, b, c \) are given by

\[ a = \bar{h}_j + \bar{h}_k - \bar{h}_i, \quad b = \bar{h}_j + \bar{h}_k - \bar{h}_i, \quad c = \bar{h}_i + \bar{h}_k - \bar{h}_j. \]  

(2.10)

The momentum-conserving delta function ensures that \( K(a, b, c; \hat{p}_1) \) only depends on two of the three momenta. The Fourier transform is performed in appendix \( C \) and yields

\[^4\text{To reach such a basis, we should be able to rescale the operators by arbitrary functions of } \mu \hat{p}. \text{ While the } SL(2, \mathbb{R})_{L} \text{ symmetry, from whose point of view } \hat{p} \text{ is simply a label, allows for arbitrary such rescalings, there may in principle exist additional structures in } JT \text{ deformed CFTs—beyond those studied in this paper—that restrict what normalizations can be reached.}\]

\[ K(a, b, c; p_1, p_2) = \frac{4\pi^2(-1)^{a+b+c}}{\Gamma(a+c)\Gamma(b)} p_1^{a+c-1} p_2^{b-1} F_1 \left(1 - b, a, c + a, \frac{p_1}{p_2}\right) \]  
(2.11)

where we have chosen to represent \( K \) in terms of \( p_1 \) and \( p_2 \), both of which are assumed to be positive. Note that for \( b \) a strictly positive integer, the hypergeometric function becomes a polynomial of degree \( b - 1 \). We can of course choose to write \( K(a, b, c; p_1) \) in terms of any two of the momenta, using the conservation equation. As discussed in appendix C, depending on the signs of the momenta, the closed-form expression for \( K \) may change slightly. Throughout this article, we will exclusively use the representation above in terms of \( K(a, b, c; p_1) \).

The non-trivial data that we are interested in computing is \( h_i(\mu p) \) and \( C_{ijk}(\mu p) \). Higher-point correlation functions can in principle be built by repeated application of the OPE. These correlators satisfy bootstrap relations, which impose constraints on the \( \mu p \)-dependent dimensions and OPE coefficients that should be obeyed order by order in the deformation parameter \( \mu \).

2.2. Building the correlators using conformal perturbation theory

In this section, we review the standard procedure to construct correlation functions in the deformed CFT in terms of correlators in the undeformed one. This procedure will be used in section 4 to compute various two and three-point functions of interest.

Even though the \( JT \) deformation is defined in terms of the instantaneous \( U(1) \) current and stress tensor, let us for now pretend, for simplicity, that the deforming operator is just the \( JT \) operator in the undeformed CFT. This assumption does not affect the low-order calculations we perform in this paper; we comment on the differences from the actual Smirnov–Zamolodchikov \( JT \) deformation at the end of this section. The correlation functions in such a \( JT \) deformed CFT are defined in terms of the original correlators via

\[ \langle O_1 O_2 \ldots O_n \rangle_{\mu E} = \langle O_1 O_2 \ldots O_n \rangle \subset \subset e^{\mu E \int \mathcal{O}^{T^\text{CFT}}} \]  
(2.12)

where \( \mu E \) is related to the coefficient \( \mu \) appearing in the action as

\[ \mu E = \frac{i \mu}{4 \pi^2} \]  
(2.13)

and we are using the conventions\(^5\) of [34, 36].

To compute the deformed correlation functions to any order in the perturbation parameter \( \mu E \), one simply needs to bring down powers of \( JT \) from the exponent, evaluate the corresponding correlators in the original CFT and then integrate. We denote the corresponding \( n \)th order correction as \( \delta_{\mu E} \):

\[ \delta_{\mu E} \langle O_1 O_2 \ldots O_n \rangle_{\mu E} \equiv \frac{\mu E}{k!} \left\langle O_1 O_2 \ldots O_n \left( \int J^T \right)^k \right\rangle_{\text{CFT}} \]  
(2.14)

\(^5\)In these conventions, \( \Delta S_E = -\mu \int d^4x dz dz J_\mu T_{\mu \nu} \). Passing to Euclidean signature, we have

\[ i \Delta S_E \rightarrow -i \mu \int d^2z J_z^T T_{zz} = \mu E \int d^2z J^T = -\Delta S_E \]

where the relation between \( T_{zz} \) and \( J_z \) used in [36] and the usual \( \bar{T}, J \) used in the CFT literature is

\[ T_{zz} = -\bar{T}_z, J_z = \frac{1}{\pi} \]  

The Euclidean measure differs by a factor of \( i \) from the conventions of [36]. The relation (2.13) follows from the above.
Since the correlation functions of any number of $J$ and $T$ insertions with the $O_i$ are determined by the Ward identities, the integrands are given by a tractable formula at any order in perturbation theory, which depends on the original CFT data in a universal way.

The next step is to perform the integrals, which are in general divergent. In this paper, we perform the integrals in Euclidean position space and use dimensional regularization to keep track of the divergent terms. Since the physical data of the deformed CFT depend on $\bar{p}$, we subsequently perform a Fourier transform with respect to the right-moving coordinate to rewrite the operators in the $(z, \bar{p})$ basis. We can then absorb the divergences by defining the renormalized operators

$$O_{i, \bar{p}}^{\text{ren}}(z) = O_{i, \bar{p}}(z) + \sum_{n=1}^{\infty} c_n \mu^n (i\bar{p})^n O_{i, \bar{p}}(z).$$

(2.15)

The divergent part of the $c_n$ is chosen to cancel the divergences, while their finite part is scheme-dependent and can be fixed by choosing the normalization of the two-point function, for example (2.3). Once the $c_n$ have been fixed, the correlation functions of the renormalized operators are finite and meaningful to any order in conformal perturbation theory. The anomalous dimensions can be read off as usual from the coefficients of the $\ln z$ pieces in the two-point function.

### 2.2.1. Building a diagonal operator basis.

As is well known, operators that have the same conformal dimension can mix in conformal perturbation theory. In the following, we would like to discuss operator degeneracy in $JT$-deformed CFTs in the $(z, \bar{p})$ representation, how degenerate operators mix in conformal perturbation theory and how to obtain a diagonal basis of the form (2.3) in the deformed theory. For simplicity, we assume that the spectrum of conformal dimensions of the original CFT is non-degenerate, except for the unavoidable degeneracies associated with the extended Virasoro $L \times$ Virasoro $R \times U(1)$ Kač–Moody symmetry.

Let us start by discussing the original CFT spectrum in the $(z, \bar{p})$ representation. The Fourier transform (2.1) will mix an $SL(2, \mathbb{R})_R$ primary with its descendants, but will not mix different $SL(2, \mathbb{R})_R$ representations with each other. This implies that for each Virasoro$_L \times$ Virasoro$_R$ primary in the original CFT, we will obtain one Virasoro$_L$ primary in the $(z, \bar{p})$ representation.

Since the original CFT spectrum is usually specified in terms of Virasoro$_R$ representations, we should first decompose these representations into a tower of global conformal primaries of the schematic form

$$O \quad (\bar{T}O) \quad (\bar{T}^2O) \ldots$$

(2.16)

each of which will give rise to a continuous set of operators labeled by the momentum $\bar{p}$. All these operators have the same left-moving dimension $h$. Thus, we find that in the original CFT in the $(z, \bar{p})$ representation, there is an infinite number of degenerate operators for each left conformal dimension, which can in principle mix in conformal perturbation theory.

To understand their mixing, note first that it can only occur between operators with the same $\bar{p}$ quantum number. Furthermore, since the correlation functions of the deforming operator are obtained from the original correlators by a simple application of the Ward identities, two operators whose dimensions are initially different will not mix with each other at any
order in conformal perturbation theory. Thus, we only need to study the mixing of the operators in the tower (2.16) among themselves, at fixed $\bar{p}$.

We now make use of the formula (1.3) for the exact deformed spectrum, which will be derived in the next section. This formula shows that while the initial degeneracy between operators of the same $h$ and different $\bar{p}$ is lifted, the degeneracy between the infinite number of operators appearing in the tower (2.16) at fixed $\bar{p}$ is not. We would now like to understand how to build a diagonal basis for these operators order by order in conformal perturbation theory.

If we choose the original CFT operators in (2.16) to be $\text{SL}(2, \mathbb{R})_R$ primaries, then the initial operator basis is diagonal

$$\langle O_i O_j^\dagger \rangle = \delta_{ij} \quad (2.17)$$

where we have absorbed for simplicity the normalizations (2.4) into the individual operators and dropped the $p$ and $q$ labels. To first order the deformation, the corrections to the two-point function take the form

$$\delta_1 \langle O_i O_j^\dagger \rangle = \mu M_{ij} + \mu N_{ij} \ln z \quad (2.18)$$

where $M, N$ are hermitean matrices. To rediagonalize the basis, we define the new operators

$$O'_i = S_{ij} O_j = (S_0 + \mu S_1 + \ldots) O_j \quad (2.19)$$

which should satisfy

$$\langle O'_i O'_j^\dagger \rangle = (1 - 2\mu \gamma_i \ln z) \delta_{ij} \quad (2.20)$$

The $\gamma_i$ denote the linearized corrections to the anomalous dimensions, which can be packaged into a diagonal matrix $\Gamma$. From this, we find that the first few $S_n$ should satisfy

$$S_0 S_0^\dagger = I, \quad S_0 M S_0^\dagger + S_1 S_0^\dagger + S_0 S_1^\dagger = 0, \quad S_0 N S_0^\dagger = -2\Gamma \quad (2.21)$$

The first relation tells us that $S_0$ is a unitary matrix. If the perturbation did break the initial degeneracy, then the last equation would fix $S_0$ to be some particular unitary matrix, built from the eigenvectors of the matrix $N$. However, since we know that all anomalous dimensions will be the same, $\Gamma \propto I$, from which it follows that $N \propto I$. Thus, unlike in the case of non-degenerate perturbation theory, we are still free to choose any $S_0$. A natural choice is $S_0 = I$, from which there follows that by choosing

$$(S_1 + S_1^\dagger) = -M \quad (2.22)$$

we can make the basis diagonal to this order. Since this equation only fixes the sum of $S_1$ and $S_1^\dagger$, we are free to choose these matrices to be lower diagonal, for example. It is easy to convince oneself that the same choice can be made at higher orders in conformal perturbation theory.

Taking the operator basis to be $O_{\ell p} = \{O_p, (\bar{T}O)_p, (\bar{T}^2O)_p \ldots\}$, this implies that we can always reach a basis of operators $O'_{\ell p}$ that have diagonal correlators via a transformation of the form

$$(\bar{T}^nO)'_p = (\bar{T}^nO)_p + \sum_{k=0}^{n} \sum_{m=0}^{\infty} c_{m,k} \ell^m p^{m+2k} (\bar{T}^{n-k}O)_p \quad (2.23)$$

i.e. in order to diagonalize the deformed two-point functions of an operator that was a level $2n$ descendant from the point of view of the original Virasoro$_R$ symmetry, we only need to

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8 This implies in particular that there should be no logarithmic divergences in the off-diagonal correlators, a fact that we will check explicitly in several examples in section 4.
compute its mixing with descendants of equal or lower level. This makes the process of basis diagonalization for these operators iteratively tractable.

So far, we have only discussed how the various Virasoro \( L \) primaries mix with each other. However, in the original CFT the Virasoro \( L \) symmetry was accompanied by an affine left-moving \( U(1)_j \) symmetry. When the original Virasoro \( \times \) Virasoro \( \times \) Kac–Moody blocks are decomposed with respect to Virasoro \( L \) \( \times \) SL(2, \( \mathbb{R} \))\( _R \) representations, they give rise to a double tower of operators of the schematic form

\[
\begin{array}{cccc}
\mathcal{O} & T\mathcal{O} & T^2\mathcal{O} & \ldots \\
J\mathcal{O} & TJ\mathcal{O} & T^2J\mathcal{O} & \ldots \\
J^2\mathcal{O} & TJ^2\mathcal{O} & T^2J^2\mathcal{O} & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

(2.24)

The operators on the second row have the holomorphic dimension increased by one and the same charge as the ones on the first row, while operators on the third row have the holomorphic dimension increased by two and the same charge, etc. In general, the operators on the lower rows mix with holomorphic derivatives of the rows above. Since their charges are the same, all the operators in the above table acquire the same anomalous dimension at fixed \( \bar{p} \). Consequently, it seems rather clear that after the deformation the columns will again combine into representations of the Virasoro–Kac–Moody algebra, which is the full left-moving symmetry group that is expected to survive.

2.2.2. Remarks about the deforming operator. At the beginning of this section, we made the simplifying assumption that the deforming operator is the \( JT \) operator of the undeformed CFT to all orders in the deformation. We would now like to make a few comments on the relation between this deformation and the one introduced by [1]. While the discussion to follow does not concern the leading order conformal perturbation theory calculations that we perform in section 4, these issues will eventually need to be addressed at higher orders in perturbation theory.

The Smirnov–Zamolodchikov deformation built from a \( U(1) \) current and the right-moving stress tensor is defined as

\[
(JT)_{\text{SZ}} \equiv (JT - J\Theta)_\mu
\]

(2.25)

where \( \Theta = Tz + \bar{Tz} \) and all the currents are defined in the deformed theory. Our first simplifying assumption with respect to (2.25) is that \( J \) is chiral, which means that \( \bar{J} = 0 \) as an operator and thus the last term can be dropped. In particular, we imagine that the original CFT has a purely chiral spectrum of charges, which is a rather strong assumption\(^9\) We additionally assume that there are no contributions from contact terms associated to the \( \bar{J} \) term that could yield finite results upon integration.

The current that stays chiral along the flow differs from the original CFT current at second order in conformal perturbation theory\(^{10}\) [35]

\[
J \rightarrow J' = J - 2\pi^2 \mu^2 \kappa k : JT : + \ldots
\]

(2.26)

Note this correction is not of the form (2.23); as we explain in section 4.3, the reason this does not follow the general pattern we have argued for is that the current is a purely holomorphic

\(^9\) It may also be possible to start from a generic CFT with a non-chiral spectrum and only use the chiral component of the current, which is separately conserved, to define the deforming operator.
operator. The same argument implies that higher order corrections of the form $\mu^2 m : J T^m :$ may also be necessary. The corrections to this operator, which is expected to stay holomorphic to all orders in perturbation theory, can be systematically computed using the method we exemplify in section 4.3.

On the other hand, the form of the perturbative corrections to the antiholomorphic stress tensor is somewhat unusual. In [35], the correction to $\partial T$ that follows from the OPE with the deforming operator was computed to linear order in $\mu$, with the result

$$\partial T = -2\pi \mu E J \partial \bar{T}.$$ (2.27)

The solution for $\bar{T}$ is, formally

$$\bar{T} \to \bar{T}' = \bar{T} + 2\pi \mu E \int J \partial \bar{T} + \ldots$$ (2.28)

which resembles the expansion to linear order of an operator of the form $\bar{T}(\bar{z} + 2\pi \mu E \int J)$. Similar expressions were obtained in the examples of [34] and in holography [36]. The problem with this correction is that it is non-local from the point of view of the left-movers11, so it does not particularly fit into our current framework, which keeps locality on the left-moving side manifest. Fortunately, this non-local correction does not affect the integrated operator; indeed, the integral picks out the $p = 0$ component of the operator (2.25), but since the current is chiral, this translates into the $p = 0$ component of the corrected stress tensor, for which the correction (2.28) vanishes.

Let us remark that, in principle, one can discuss two different notions of $\bar{T}$ operator in the deformed theory. One operator corresponds to the $\bar{z}$ component of the Noether current associated with right-moving translations, $T_{\text{Noether}}$, which in general becomes non-local when the deformation is turned on. This is the operator that is supposed to appear in (2.25). It is not clear how to define this operator abstractly, i.e. without making reference to a Lagrangian. The second option is to consider the left primary operator that starts out as $T_p$ in the undeformed CFT, i.e. as the Fourier transform of the right-moving stress tensor, and then is deformed, while preserving the primary condition. As we show in section 4, $T_p$ behaves just like any other operator in the theory: in particular, it receives an anomalous dimension and charge given by (3.19) and it mixes according to (2.23). The relation between $T_p$ and $T_{\text{Noether}}$ is rather unclear, except at the level of the zero mode. Indeed, the zero mode $T_{p_{\text{zero}}}$ does not acquire a left anomalous dimension and is not corrected in perturbation theory, and thus could still be playing the role of right-moving global translations generator.

To end this discussion, let us point out that there also exists a second notion of $J T$ operator, which is different from that proposed in [1]. Rather than first constructing the $J$ and $T_{\text{Noether}}$ operators in the deformed theory and then using the OPE to define the composite operator $J T$, one can instead ask for the left primary operator that is the deformation of the double-trace operator $J T$ in the original CFT. As we explained earlier in this section, this operator is given in terms of the original CFT ones by a relation of the form (2.23). When integrated, this $J T$ operator differs from the integral of (2.25) at $\mathcal{O}(\mu^3)$, but it equals the integrated $J T$ operator of the undeformed CFT to all orders in $\mu$, as can be seen from the fact that all corrections to

10 The relation between the conventions of [35] and ours is $\mu_{\text{bare}} = -2\mu_E$. We also reinstated the factors of $k$.

11 It may be possible to interpret this correction as the perturbative expansion of the vertex operator $e^{q \bar{T}}$, where $\varphi$ is the chiral boson associated with the current $J$ via $k \partial \varphi = J$ and $q_T = \mu k p/4\pi$ is the charge of the stress tensor (3.19) in the deformed theory. We have not investigated the implications of such an interpretation.
it vanish when \( \bar{p} = 0 \). It is this \( JT \) operator whose correlation functions we will compute in section 4.

The correct identification of the deforming operator is clearly very important in obtaining a UV-complete theory\(^{12}\). One relatively simple test that the deforming operator should pass is whether it can reproduce the prediction (1.3) for the conformal dimensions and charges. This should provide a non-trivial check of the deforming operator already at cubic order in \( \mu \).

### 2.3. Relation to field-dependent coordinate transformations

So far, in using the \((z, \bar{p})\) basis for labeling operators, we have only assumed the minimum amount of structure of \( J\bar{T}\)-deformed CFTs, which follows from the symmetries of the problem. The holographic analysis of [36] suggests however that there may be additional structure to the correlation functions that can be seen directly in (right-moving) position space. Concretely, [36] found that the holographic one-point functions of the stress tensor were related to their CFT counterparts via a simple field-dependent coordinate transformation of the form

\[
x^+ \to x^+, \quad x^- \to x^- - \mu \int x^+ \, \langle J_+ (x^+) \rangle
\]

where the integrand is the classical expectation value of \( J_+ \) in the heavy state dual to a classical background and \( x^\pm \) - which correspond to \( z, \bar{z} \) — are the Lorentzian lightlike coordinates. Very similar operator-dependent coordinate transformations, where the integrals are performed over null lines, have been extensively used in the \( T\bar{T} \) context [5].

One can therefore ask whether a relation of the form

\[
\langle O_1(z_1, \bar{z}_1) \ldots O_n(z_n, \bar{z}_n) \rangle \mu = \langle O_1(z_1, \bar{z}_1 - \alpha \mu \int z_1 \, d\bar{z}_1 \, J(z_1)) \ldots O_n(z_n, \bar{z}_n - \alpha \mu \int z_n \, d\bar{z}_n \, J(z_n)) \rangle_{\text{CFT}}
\]

(2.30)

holds, for some precise interpretation of the field-dependent coordinate transformation (\( \alpha \) is a numerical factor depending on the current normalization). There are many possibilities \textit{a priori} for what this interpretation could be: the integral could be over the operator \( J \) or over its classical expectation value sourced at the various insertions, the integration contour could be in the complex \( z \) plane or along a Lorentzian null line, etc. The goal of this subsection is to use our knowledge of correlation functions in \( JT \) deformed CFTs to check whether they admit an interpretation in terms of field-dependent coordinate transformations and, if so, how exactly these should be defined. We will restrict our attention to two and three-point functions, which are fixed in terms of the conformal dimensions (1.3) and the OPE coefficients.

Consider first the two-point function, with normalization given by (2.4)

\[
\langle O_{p}(z_1) \bar{O}_{-p}(z_2) \rangle_\mu = \frac{N(p)}{z_{12}^{2|\mu|}} e^{-\frac{(\mu^{2}q + \ln z_{12} - \frac{z_{12}^{2}}{2\pi i})^{2}}{2(\pi i)^{2}}
\]

(2.31)

where we have used the explicit formula (1.3) for the deformed dimensions. Let us first assume, for simplicity, that \( k = 0 \). Fourier transforming both sides with respect to \( \bar{z}_1 \), we obtain

\[
\langle O(z_1, \bar{z}_1) \bar{O}(z_2, \bar{z}_2) \rangle_\mu = \frac{1}{z_{12}^{2\mu}(z_{12} + i\mu q/\pi \ln z_{12})^{2\mu}}
\]

(2.32)

\(^{12}\) Assuming that our conjecture about the UV-completeness of \( JT \)-deformed CFTs is correct.
The shift in \( \tilde{z} \) can be interpreted as a field dependent coordinate transformation of the form (2.30)

\[
\tilde{z} \to \tilde{z} - \frac{i\mu}{2\pi} \int dz' J(z')
\]

where \( J \) is a classical field sourced at the locations of the two operators, i.e. it satisfies

\[
\partial J(z) = 2\pi \sum_i q_i \delta(z - z_i) = \partial \left( \frac{q}{z - z_1} - \frac{q}{z - z_2} \right).
\]

If we now compute the \( J \) integral with a short distance cutoff at \( z_1 + \delta \) and \( z_2 - \delta \), we find exactly the shift in (2.32). Notice that, due to the anomalous dimension, we are led to an ‘Euclidean’ interpretation of the coordinate transformation. This interpretation should be contrasted with the more intrinsically ‘Lorentzian’ formulae found e.g. in [5], in analogy to which one would interpret the integral over \( J_\pi \) in (2.29) as the total charge to the past of the line of constant \( x^\mu \), leading to \( \Theta \)-function shifts of the right-moving coordinate and no anomalous dimension.

When the anomaly term is taken into account (\( k \neq 0 \)), there is no nice interpretation of the two-point function in terms of a classical field. It is however easy to notice that if we replace the operators on the right-hand side of (2.30) by a formal expansion

\[
\mathcal{O} \left( z\tilde{z} - \frac{i\mu}{2\pi} \int dz' J(z') \right) \equiv \tilde{\mathcal{O}}(z, \tilde{z}) = \mathcal{O}(z, \tilde{z}) - \frac{i\mu}{2\pi} \int dz' J(z') \partial \mathcal{O}(z, \tilde{z}) - \frac{\mu^2}{8\pi^2} \int :J\partial^2 \mathcal{O} : + \ldots
\]

then compute the formal correlator

\[
\langle \tilde{\mathcal{O}}(z_1, \tilde{z}_1)\tilde{\mathcal{O}}(z_2, \tilde{z}_2) \rangle_{\text{CFT}} = \left( 1 + \frac{iq}{2\pi} \ln z_{12} (\tilde{\mathcal{O}}_1 - \tilde{\mathcal{O}}_2) - \frac{\mu^2 k}{8\pi^2} \ln z_{12} \mathcal{O}_1 \tilde{\mathcal{O}}_2 - \frac{\mu^2 q^2}{8\pi^2} \ln^2 z_{12} (\mathcal{O}_1 - \mathcal{O}_2)^2 + \ldots \right) 
\]

\[
\times \langle \mathcal{O}(z_1, \tilde{z}_1)\mathcal{O}(z_2, \tilde{z}_2) \rangle_{\text{CFT}}
\]

and Fourier transform, it will reproduce the normalized two-point function (2.31). We conclude that the coordinate transformation we need to reproduce the full two-point function involves an integral over the operator \( J \), and that the shift by this operator-valued quantity needs to be carefully defined. It would be very interesting to ascertain whether similar subtleties play a role in the \( TT \) deformation.

The next step is to check whether the three-point functions also obey a relation of this sort. It is easy to check that the dimension shifts in (2.5), including the contribution of the anomaly, will be reproduced by the formal correlator

\[
\langle \tilde{\mathcal{O}}(z_1, \tilde{z}_1)\tilde{\mathcal{O}}(z_2, \tilde{z}_2)\tilde{\mathcal{O}}(z_3, \tilde{z}_3) \rangle_{\text{CFT}}
\]

provided the correction to the OPE coefficients (2.8) vanishes to all orders in conformal perturbation theory. If this turns out to be the case, then the \( JT \) deformation can also be interpreted as a spectral flow transformation, where the flow parameter depends on \( \tilde{p} \). However, as we will show in section 4.2, the corrections to the OPE coefficients do not vanish, at least for the (standard) way in which we have chosen to deal with the UV divergences. Thus, our

13 It is intriguing to note that if we plug into (2.30) the Lorentzian coordinate transformation that shifts \( \tilde{z} = x^- \) by the charge to the past of \( x^+ \), where \( x^- \) is viewed as (null) time, we obtain a structure that is identical to the dipole star product [43].

14 The Fourier transform of (2.31) for \( k \neq 0 \) is a \( F_1 \) hypergeometric function whose argument is \( \frac{z^e(x^+ q / \mu \ln z)}{\mu \mu^e \ln e} \).
current results do not support an interpretation of general correlators in \( JT \)-deformed CFTs in terms of operator-dependent coordinate transformations.

3. The spectrum of \( JT \) deformed CFTs on the plane

In this section, we derive the spectrum of conformal dimensions of \( JT \)-deformed CFTs on the plane, starting from the known spectrum of energies on the cylinder \([34–36]\).

In a two-dimensional CFT, the two spectra are simply related by the exponential map from the cylinder to the plane, \( z_{pl} = \exp(2\pi iz_{cyl}/R) \) and its complex conjugate, which are conformal transformations and thus symmetries of the theory. For a \( JT \)-deformed CFT, only the holomorphic part of the exponential map is a symmetry of the action, as the latter is only invariant under \( SL(2, \mathbb{R})_L \times U(1)_R \). This symmetry is broken to \( U(1)_L \times U(1)_R \) when placing the theory on a cylinder, due to the simultaneous action of the cylinder identifications on \( z_{cyl} \) and \( \bar{z}_{cyl} \). The idea of this section is to first find the spectrum of \( JT \) deformed CFTs on an infinitely boosted cylinder, on which the \( SL(2, \mathbb{R})_L \) symmetry is restored, and then use the holomorphic exponential map to recover the conformal dimensions.

We start this section by briefly reviewing the spectrum of \( JT \)-deformed CFTs on the cylinder, and then perform an infinite boost to obtain a prediction for the exact spectrum on the plane.

3.1. Review of the spectrum on the cylinder

We consider a \( JT \)-deformed CFT on a cylinder of circumference \( R \), i.e. with the spatial coordinate identified as \( \varphi \sim \varphi + R \). The change in energy levels and chiral \( U(1) \) charge as the deformation parameter is varied is

\[
\frac{\partial E}{\partial \mu} = 2R \langle J_\varphi \rangle \langle T_{zz} \rangle = -Q \left( \frac{\partial E}{\partial R} + \frac{P}{R} \right) \tag{3.1}
\]

\[
\frac{\partial Q}{\partial \mu} = \frac{k}{4\pi} R \langle T_{\bar{z}\bar{z}} \rangle = -\frac{k}{4\pi} R \frac{\partial}{\partial R} E_R \tag{3.2}
\]

where \( P \) is the total momentum (quantized in units of \( 1/R \)), \( E_R = \frac{1}{2}(E - P) \) is the right-moving energy, and we are using the conventions of \([36]\). Using the fact that the only dimensionless parameter in the problem is \( \mu R \), and so \( E_R = R^{-1} f(\mu R) \), the above equations can be written as \([35]\]

\[
\partial_\mu E_R = -Q \partial_R E_R, \quad \partial_\mu Q = \frac{k}{4\pi} \left( E_R + \mu \partial_\mu E_R \right) \tag{3.3}
\]

and imply that

\[
E_R - \frac{2\pi Q^2}{kR} = \text{const.}, \quad Q = Q_0 + \frac{\mu k}{4\pi} E_R. \tag{3.4}
\]

Plugging in the expression for \( Q \), the first equation becomes

\[
E_R - \frac{2\pi}{kR} \left( Q_0 + \frac{\mu k}{4\pi} E_R \right)^2 = \frac{2\pi}{kR} \left( \bar{h} - c \frac{Q_0^2}{24} - \frac{Q_0^2}{k} \right) \tag{3.5}
\]
where \( \bar{h} \) denotes the right-moving conformal dimension associated to the corresponding state in the undeformed CFT. The solution for the right-moving energy in terms of the original CFT data is

\[
E_R = \frac{4\pi}{\mu^2 k} \left( R - \mu Q_0 - \sqrt{(R - \mu Q_0)^2 - \mu^2 k \left( \bar{h} - \frac{c}{24} \right)} \right). \tag{3.6}
\]

The expression for \( E_L \) follows from momentum conservation

\[
E_L = E_R + \frac{2\pi}{R} (h - \bar{h}). \tag{3.7}
\]

### 3.2. The spectrum on an infinitely boosted cylinder

Let \( x^\pm = \varphi^\pm t \) be the coordinates on the cylinder, which above are identified as \( x^\pm \sim x^\pm + R \). We would like to derive the spectrum on a boosted cylinder, with the identifications

\[
\tilde{x}^+ \sim \tilde{x}^+ + \tilde{R}_+, \quad \tilde{x}^- \sim \tilde{x}^- + \tilde{R}_-. \tag{3.8}
\]

where \( \tilde{R}_+ \) is finite and \( \tilde{R}_- \to \infty \). The spectrum of \( J\tilde{T} \) deformed CFTs on a space with these identifications can be related to the spectrum on the usual cylinder via a boost

\[
\tilde{x}^\pm \to x^\pm = e^{\pm \gamma} \tilde{x}^\pm
\]

with the boost parameter chosen such that

\[
\tilde{R}_+ e^{\gamma} = \tilde{R}_- e^{-\gamma} = R. \tag{3.10}
\]

The various quantities before and after the boost are related as

\[
\mu = \tilde{\mu} e^{-\gamma}, \quad E_L = e^{-\gamma} \tilde{E}_L, \quad E_R = e^{\gamma} \tilde{E}_R
\]

where the transformation law for \( \mu \) follows from the fact that it is a constant null vector with a lower ‘+’ component. \( E_{L,R} \) are the energies on the unboosted cylinder, given by (3.6) and (3.7).

We will be interested in the limit \( \gamma \to \infty \) with \( \tilde{\mu}, \tilde{R}_+ \) and the right-moving energy/momentum \( \tilde{E}_R \) fixed. The expression for \( \tilde{E}_L \) in terms of these fixed quantities is

\[
\tilde{E}_L = e^{\gamma} \left( \tilde{E}_R + \frac{2\pi(h - \bar{h})}{R} \right) = \frac{2\pi (h - c/24)}{R_+} + e^{2\gamma} \left( \tilde{E}_R - \frac{2\pi(h - c/24) e^{-2\gamma}}{R_+} \right). \tag{3.12}
\]

The only way that \( \tilde{E}_L \) could be finite is if \( \tilde{E}_R \) cancels against \( 2\pi(h - c/24) e^{-2\gamma}/R_+ \) with precision \( e^{-2\gamma} \). Since \( \tilde{E}_R \) itself is finite, we conclude that we should scale

\[
\bar{h} - \frac{c}{24} = \tilde{h}_0 e^{2\gamma} \tag{3.13}
\]

with \( \tilde{h}_0 \) fixed as \( \gamma \to \infty \). The expansion of \( \tilde{E}_R \) in this limit, using the replacements (3.10), (3.11) and (3.13), is given by

\[
\tilde{E}_R = \frac{4\pi e^{-\gamma}}{\tilde{\mu}^2 k e^{-2\gamma}} \left( \tilde{R}_+ e^{-\gamma} - \tilde{\mu} Q_0 e^{-\gamma} - \sqrt{(\tilde{R}_+ e^{-\gamma} - \tilde{\mu} Q_0 e^{-\gamma})^2 - \tilde{\mu}^2 k \tilde{h}_0} \right)
\]

\[= \frac{2\pi \tilde{h}_0}{R_+} + \frac{2\pi e^{-2\gamma}}{R_+} \left( \tilde{\mu} Q_0 \frac{\tilde{h}_0}{R_+} + \frac{\tilde{\mu}^2 k}{4} \left( \frac{\tilde{h}_0}{R_+} \right)^2 \right) + O(e^{-4\gamma}). \tag{3.14}
\]
We see that $\tilde{E}_R$ equals indeed the finite quantity $\frac{2\pi h_0}{R_+}$ within $e^{-2\gamma}$ precision. The $O(e^{-4\gamma})$ term will not survive the $\gamma \to \infty$ limit inside $\tilde{E}_L$, so we can neglect it. Defining the right-moving momentum

$$\tilde{p} \equiv \tilde{E}_R = \frac{2\pi h_0}{R_+}$$

we find that

$$\tilde{E}_L = \frac{2\pi(h - c/24)}{R_+} + \frac{2\pi}{R_+} \left( \tilde{\mu} Q_0 \frac{\tilde{p}}{2\pi} + \frac{\tilde{\mu}^2 k}{4\cdot4\pi^2 \tilde{p}^2} \right).$$

Using the usual map from the cylinder to the plane, which maps energies to conformal dimensions, the anomalous left dimensions we find are

$$h(\mu) = h + \frac{\mu}{2\pi} q\tilde{p} + \frac{\mu^2 k}{16\pi^2 \tilde{p}^2}$$

where we have dropped the tilde from $\mu$ and replaced the initial charge $Q_0$ by $q$. The expression for the charge in the deformed theory is given by (3.4), which in our new notation reads

$$q(\mu) = q + \frac{\mu k}{4\pi} \tilde{p}.$$ 

The above expressions correspond to an exact formula for the conformal dimensions and charges in $J\bar{T}$-deformed CFTs as a function of the deformation parameter. As expected, these data only depend on the combination $\mu \tilde{p}$ and, unlike the spectrum on the cylinder, show no obvious pathology as $\mu$ becomes large. Remarkably, these exact expressions terminate at $O(\mu^2)$. The combination $\hat{h} \equiv h(\mu) - q(\mu)^2/k$ is independent of $\mu$, in perfect agreement with the argument presented in [35], and hints towards a possible interpretation of the $J\bar{T}$ deformation as spectral flow.

A simple consequence of the above formulae is that purely holomorphic quantities do not acquire an anomalous dimension, as they have $\tilde{p} = 0$; however, initially purely antiholomorphic quantities, such as the right-moving stress tensor, receives a non-trivial left-moving anomalous dimension and charge

$$h_T(\mu) = \frac{\mu^2 k \tilde{p}^2}{16\pi^2}, \quad q_T(\mu) = \frac{\mu k \tilde{p}}{4\pi}.$$

In the following section, we will check these formulae to leading order in conformal perturbation theory, finding perfect agreement.

### 4. Conformal perturbation theory calculations

In this section, we present a selection of explicit conformal perturbation theory calculations, which follow the outline of section 2.2. First, we compute the leading order correction to the dimension and charge of an operator and show that it is in perfect agreement with the result of the previous section. Next, we compute the linearized correction to a set of three-point functions. Finally, we discuss the issue of operator mixing and perform several checks of the general structure discussed in section 2.2.
4.1. Leading order correction to the dimension and the charge

To find the leading order shift in the conformal dimension and the charge, we need to compute the correlators \(\langle O O \rangle\) and \(\langle J O O \rangle\) to linearized or second order in \(\mu E\). The anomalous dimension can be read off from the coefficient of the \(\ln z\) term in the two-point function, while the charge is read off from the correlator with the current. The deforming operator we use is just the \(J T\) operator of the original CFT, since the corrections to it only become relevant at cubic order and higher.

4.1.1. Anomalous dimension of a generic operator at linear order. To first order in \(\mu E\), the correction to the two-point function of a generic operator is\(^{15}\)

\[
\delta_1 \langle O(z_1, \bar z_1)O^\dagger(z_2, \bar z_2) \rangle = \mu E \int d^2z_3 \langle O_1 O_2^\dagger : JT_3 : \rangle = \frac{\mu E q \hbar}{z_{12}^{\alpha - 1} z_{23}^{\beta - 1}} \int \frac{dz_3 d\bar z_3}{z_{23} z_{13} z_{23} z_{13}}.
\]

The integral in (4.1) can be written as \(\partial_1, \partial_2 \mathcal{I}\) of the basic integral

\[
\mathcal{I}(z_1, z_2) = \int \frac{d^2z_3}{|z_{13}|^2 |z_{23}|^2} = \frac{4\pi}{|z_{12}|^2} \left( \frac{2}{\epsilon} + \ln |z_{12}|^2 + \gamma + \ln \pi + \mathcal{O}(\epsilon) \right)
\]

which is computed in appendix B, equation (B.4), using dimensional regularization with \(d = 2 + \epsilon\). The first order change in the correlator thus reads

\[
\delta_1 \langle O(z_1, \bar z_1)O^\dagger(z_2, \bar z_2) \rangle = -\frac{8\pi \mu E q \hbar}{z_{12}^{\alpha - 1} z_{23}^{\beta - 1}} \left( \frac{2}{\epsilon} + \ln |z_{12}|^2 + \gamma + \ln \pi - \frac{3}{2} + \mathcal{O}(\epsilon) \right).
\]

It is now useful to write the operators in terms of the \((z, \bar p)\) variables by first analytically continuing \(z, \bar z\) to real values and then performing the Fourier transform with respect to \(\bar z_1, \bar z_2\). We find

\[
\delta_1 \langle O_p(z_1)O^\dagger_{-p}(z_2) \rangle = \frac{4\pi i \mu E \bar p}{z_{12}^{\alpha - 1} z_{23}^{\beta - 1}} \left( \frac{2}{\epsilon} + \ln z_{12} + \gamma + \ln \pi - \frac{3}{2} + \psi(2\hbar + 1) - \ln |\bar p| + \frac{i\pi}{2} + \mathcal{O}(\epsilon) \right)
\]

\[
\times \langle O_p(z_1)O^\dagger_{-p}(z_2) \rangle_0
\]

where we used the Fourier transforms listed in appendix C. Next, we define the renormalized operator

\[
O_{p \text{ren}}(z) = O_p(z) + i\mu E \bar p (c^O_{\epsilon}(L') \cdot O_p(z))
\]

where \(L\) is a length scale that compensates the shift by \(\epsilon\) in the dimension of \(\mu E\). If \(c^O_{\epsilon}\) is constant, this simply corresponds to a redefinition \(O_{\text{ren}} = O + \mu E c\epsilon L' \partial L O\) in position space. The coefficient \(c^O_{\epsilon}\) is chosen to absorb the \(2/\epsilon\) divergence, as well as any finite terms needed to yield some particular choice of normalization for the two-point function. Choosing the normalization (2.3), the associated coefficient is given by

\[
c^O_{\epsilon} = -2\pi q \left( \frac{2}{\epsilon} + \gamma + \ln \pi - \frac{3}{2} + \psi(2\hbar + 1) - \ln |\bar p|L + \frac{i\pi}{2} \right) = -c^O_{\epsilon}^\dagger.
\]

Absorbing the \(\ln |\bar p|\) term in the normalization is rather unusual from the point of view of position space interpretation of the operator. However, since here \(\bar p\) is simply a label for the
operator and the theory is non-local on the right, one may argue that such a redefinition should be allowed. Using the relation \( \mu_E = \frac{1}{4\pi^2} \) between the Euclidean and Lorentzian deformation parameters, the two-point function of the renormalized operators reads

\[
\delta_1\langle O_p^{\text{ren}}(z_1)O_{-p}^{\text{ren}}(z_2) \rangle = \left( 1 - \frac{q\mu E}{\pi} \ln \frac{z_{12}}{L} \right) \langle O_p(z_1)O_{-p}(z_2) \rangle_{\text{CFT}}. \tag{4.7}
\]

The term in parenthesis corresponds to an anomalous dimension of \( \delta_1 h = \frac{q\mu E}{2\pi} \). This perfectly matches the prediction of section 3 for the linear correction to the dimension of a charged operator.

4.1.2. Charge shift of a generic operator at linear order. We would now like to reproduce the linear correction (3.18) to the charge of an operator from a conformal perturbation theory calculation. For this, we need to evaluate the three-point function (4.8) to the linear correction (3.18) to the charge of an operator from a conformal perturbation theory calculation. For this, we need to evaluate the three-point function

\[
\delta_1\langle O_1O_j^{\text{ren}}(z_1) \rangle = \left( \frac{q}{z_{13}} - \frac{q}{z_{12}} \right) \delta_1\langle O_1O_j \rangle + \left( \frac{\delta_1 q O_1 z_{31}}{z_{32}} + \frac{\delta_1 q_{O_1} z_{31}}{z_{32}} \right) \langle O_1O_j \rangle_{\text{CFT}} \tag{4.8}
\]

where \( \delta_1\langle O_1O_j \rangle \) was computed in (4.3) and \( \delta_1 q O_1 \) and \( \delta_1 q_{O_1} \) represent the first-order correction to the charge of \( O_1 \) and respectively \( O_j^{\text{ren}} \).

The linearized correction to the three-point function is given by

\[
\delta_1\langle O_1O_j^{\text{ren}}(z_1) \rangle = \mu_E \int d^2z_4 \langle O_1O_j^{\text{ren}}(z_1) : JT_4 : \rangle = \frac{\mu_E h}{z_{12}z_{32}} \left( k \int \frac{d^2z_4}{z_{14}z_{24}z_{34}z_{43}} + q^2 \frac{z_3^2}{z_{13}z_{23}} \int \frac{d^2z_4}{z_{14}z_{24}z_{34}z_{43}} \right) \tag{4.9}
\]

The first integral can be performed by taking sums and derivatives of the basic integral

\[
I_\gamma = \int \frac{d^2z_4}{z_{14}z_{24}} = -2\pi \left( \frac{2}{\epsilon} + \ln |z_{14}|^2 + \gamma_E + \ln \pi + O(\epsilon) \right) \equiv -2\pi (\ln |z_{14}|^2 + \epsilon) \tag{4.10}
\]

and is calculated in the appendix B, equation (B.8) as an example. The second integral is \( \partial_{\bar{z}_1}\partial_{\bar{z}_2}I(z_1, z_2) \), as before. Thus, the correlator evaluates to

\[
\delta_1\langle O_1O_j^{\text{ren}}(z_1) \rangle = \frac{4\pi \mu E h}{z_{12}z_{32}} \left[ k \int \frac{d^2z_4}{z_{14}z_{24}z_{34}z_{43}} - 2q^2 \left( \frac{1}{z_{31}} - \frac{1}{z_{32}} \right) \left( \frac{2}{\epsilon} + \ln |z_{14}|^2 + \gamma + \ln \pi - \frac{3}{2} \right) \right]. \tag{4.11}
\]

Next, we Fourier transform with respect to \( \bar{z}_{1,2} \). Since the current is purely holomorphic, \( \tilde{p}_3 = 0 \), so by the momentum conservation equation \( O \) and \( O_j^{\text{ren}} \) have equal and opposite momenta, \( p \) and \( -p \). After introducing the renormalized operators (4.5) and trading \( \mu_E \) for \( \mu \), we find

\[
\delta_1\langle O_p^{\text{ren}}(z_1)O_{-p}^{\text{ren}}(z_2) \rangle = \left( 1 - \frac{1}{z_{31}} - \frac{1}{z_{32}} \right) \left[ q \delta_1\langle O_p^{\text{ren}}(z_1)O_{-p}^{\text{ren}}(z_2) \rangle + \frac{\mu kp}{4\pi} \langle O_p(z_1)O_{-p}(z_2) \rangle_{\text{CFT}} \right]. \tag{4.12}
\]

This correlator is precisely of the general form (4.8), where the second term corresponds to an \( O(\mu) \) shift in the charge

\[
\delta_1 q O = \frac{kp}{4\pi} \tag{4.13}
\]

which is equal and opposite for \( O \) and \( O_j^{\text{ren}} \). Notice that holomorphy of \( J \) is essential for maintaining charge conservation.
4.1.3. Second order charge shift of a generic operator. The change in the $\langle J \bar{O} O \rangle$ correlator to second order in the deformation is given by

$$\delta_2 (J_3 \bar{O}_1 O_2^{1}) = \frac{\mu^2}{2} \langle J_3 O_1 \bar{O}_2^{1} \int d^2 z_4 : J T_4 : \int d^2 z_5 : J T_5 : \rangle. \quad (4.14)$$

We can use the OPE of $J_3$ with the various other insertions to rewrite this correlator as

$$\delta_2 (J_3 \bar{O}_1 O_2^{1}) = q \left( \frac{1}{z_{31}} - \frac{1}{z_{32}} \right) \delta_2 (O_1 \bar{O}_2^{1}) + \frac{k\mu E}{2} \int d^2 z_4 \zeta_{34} \delta_1 (O_1 \bar{O}_2^{1} T_4). \quad (4.15)$$

The last term can be evaluated by using the OPE for $T_4$

$$\int d^2 z_4 \zeta_{34} \delta_1 (O_1 \bar{O}_2^{1} T_4) = \int d^2 z_4 \left( \hat{h} \frac{1}{z_{41}} + \frac{1}{z_{42}} \partial_{z_1} + \frac{1}{z_{42}} \partial_{z_2} \right) \delta_1 (O_1 \bar{O}_2^{1})$$

$$+ \mu E \int d^2 z_4 d^2 z_5 \left[ \frac{c}{2z_{45}} \delta_1 (O_1 \bar{O}_2^{1} J_3)_{\text{CFT}} + \left( \frac{2}{z_{45}} + \frac{1}{z_{45}} \partial_{z_5} \right) \delta_1 (O_1 \bar{O}_2^{1} J_3 T_5)_{\text{CFT}} \right]$$

$$= \frac{2\pi}{z_{13}} \delta_1 (\partial \bar{O}_1 O_2^{1}) + \frac{2\pi}{z_{23}} \delta_1 (\partial \bar{O}_1 \partial O_2^{1})_{n-1} + \frac{\pi^2 \mu E}{3} \partial_5^2 (O_1 \bar{O}_2^{1} J_3)_{\text{CFT}} + 4\pi^2 \mu E (O_1 \bar{O}_2^{1} J_3 T_5)_{\text{CFT}} \quad (4.16)$$

where in the last line we have discarded the contact terms proportional to $\hat{h}$. The end result is

$$\delta_2 (J_3 \bar{O}_1 O_2^{1}) = q \left( \frac{1}{z_{31}} - \frac{1}{z_{32}} \right) \delta_2 (O_1 \bar{O}_2^{1}) - \frac{\pi k\mu E}{z_{31}} \delta_2 (\partial \bar{O}_1 O_2^{1}) - \frac{\pi k\mu E}{z_{32}} \delta_2 (\partial \bar{O}_1 \partial O_2^{1})$$

$$+ \frac{\pi^2 k^2 \mu^2 E}{6} \partial_5^2 (O_1 \bar{O}_2^{1} J_3)_{\text{CFT}} + 2\pi^2 k\mu E (O_1 \bar{O}_2^{1} J_3 T_5)_{\text{CFT}}. \quad (4.17)$$

Replacing $\partial_i \to i\bar{p}_i$, we immediately find the charge shifts we expect; in particular, the second order correction to the charge vanishes, in agreement with (3.18). The first term on the second line is just a contact term, whereas the last term should be moved to the left-hand side and represents the quadratic correction the current $J_{\text{corr}} = J - 2\pi^2 \mu^2 k : J T : \text{derived in (4.64)}$. Notice that once we know the corrections to the deforming operator and to the chiral current at higher orders in $\mu$, the above calculation can be easily extended to higher orders to give a recursive formula for $\delta_n (\langle J \bar{O} O \rangle)$ in terms of the corrections at lower orders in conformal perturbation theory.

4.1.4. Anomalous dimension of the right-moving stress tensor. One of the predictions of our formula (1.3) for the spectrum is that the right-moving stress tensor, which starts out as a purely antiholomorphic operator, acquires a left-moving conformal dimension and charge given by (3.19). In the following, we reproduce this prediction to second order in $\mu$.

Since the linear correction to the $\langle \bar{T} T \rangle$ two-point function vanishes, we need to evaluate

$$\delta_2 (\bar{T}_1 T_2) = \frac{\mu^2 E}{2} \int d^2 z_3 d^2 z_4 \langle \bar{T}_1 T_2 : J T_3 : J T_4 : \rangle. \quad (4.18)$$

The integrand factorizes into a purely holomorphic part, which equals the current two-point function $k/2z_{34}^2$, and a purely antiholomorphic part that corresponds to the correlation function of four stress tensors. The latter is given in appendix A, equation (A.10). The integral then reads
\[
\delta_2(T_1 T_2) = \frac{\mu^2 k}{4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^2} \left[ c \left( \frac{1}{z_{12}^2 z_{13}^2 z_{14}^2 z_{24}^2} + \frac{1}{z_{12}^2 z_{13}^2 z_{14}^2 z_{24}^2} + \frac{1}{z_{12}^2 z_{13}^2 z_{14}^2 z_{24}^2} \right) + \frac{c^2}{4} \left( \frac{1}{z_{12}^2 z_{13}^2} + \frac{1}{z_{12}^2 z_{13}^2} + \frac{1}{z_{12}^2 z_{13}^2} \right) \right].
\]

(4.19)

It is simplest to express the end result in terms of the basic integral (4.10)

\[
\delta_2(T_1 T_2) = \frac{\mu^2 k}{4} \left( 2 c \cdot \frac{12 \pi I_{12} + 44 \pi^2}{z_{12}^2} + c^2 \cdot \frac{16 \pi (I_{12} + 3 \pi)}{z_{12}^2} + 2 \cdot \frac{c^2}{4} \cdot \frac{40 \pi^2}{3 x_{12}^2} \right).
\]

(4.20)

The provenance of each term from (4.19) should be clear. Passing to momentum space, we find

\[
\delta_2(T_p(z_1) T_{-p}(z_2)) = -\frac{k \mu^2 p^2}{8 \pi^2} \left( \ln z_{12} + c_c - \frac{17}{12} - \frac{c}{z_{12}} + \psi(6) - \ln |p| + \frac{\imath}{2} \right) \langle T_p(z_1) T_{-p}(z_2) \rangle_{\text{CFT}}.
\]

(4.21)

From the log term, we can read off the holomorphic anomalous dimension of the stress tensor

\[
h_T = \frac{k \mu^2 p^2}{16 \pi^2}
\]

(4.22)

which precisely agrees with our expectations. Defining the renormalized operators

\[
T'_p(z) = T_p(z) - c_T \mu^2 p^2 \bar{T}_p
\]

(4.23)

with \(c_T = -\frac{k}{2 \pi} (c_c - \frac{17}{12} - \frac{7}{12} + \psi(6) + \frac{\imath}{2} - \ln |p|)\), the two-point function can be put in the form (2.3).

4.1.5. Charge shift of the right-moving stress tensor. The charge shift of the antiholomorphic stress tensor is visible already at first order in the perturbation. The integral that we need to perform is

\[
\delta_1(J_3 \bar{T}_1 T_2) = \mu E (J_3 \bar{T}_1 T_2) \int d^2 z_4 : J T_4 : = \frac{\mu E k c}{2 z_{12}} \int \frac{d^2 z_4}{z_{34}^2 z_{14}^2 z_{24}^2} = \frac{\mu E k c}{2 z_{12}} \cdot \frac{4 \pi}{z_{31}} \left( \frac{1}{z_{31}} - \frac{1}{z_{32}} \right).
\]

(4.24)

Passing to momentum space, we find

\[
\delta_1(J(z_3) T_p(z_1) T_{-p}(z_2)) = \frac{\mu k p}{4 \pi} \left( \frac{1}{z_{31}} - \frac{1}{z_{32}} \right) \langle T_p(z_1) T_{-p}(z_2) \rangle_{\text{CFT}}
\]

(4.25)

and thus \(T\) acquires the expected change of \(\mu k p / (4 \pi)\). Each of the \(T\) insertions in the correlator above carries an equal and opposite charge, due to the fact that \(J\) has zero right-moving momentum.

The \(O(\mu^2)\) contribution to the \(\langle J \bar{T} T \rangle\) correlator vanishes trivially, which is consistent with the fact that the charge does not receive any \(O(\mu^2)\) correction. Indeed,

\[
\delta_2(J_3 \bar{T}_1 T_2) = 0 = \frac{z_{12}}{z_{31} z_{32}} \left( q \delta_1 \bar{T}_1 T_2 \right) + \delta_1 q_T \delta_1 \langle T_1 T_2 \rangle + \delta_2 q_T \langle T_1 T_2 \rangle_{\text{CFT}} + 2 \pi^2 q_T^2 k (J \bar{T}_3 : T_1 T_2 :_{\text{CFT}}).
\]

(4.26)

The first term on the right-hand side vanishes because \(T\) is originally neutral, the second because the linearized correction to the \(\langle \bar{T} T \rangle\) two-point function is zero and the fourth because this particular correlator vanishes in the original CFT. We thus deduce that \(\delta_2 q_T = 0\), in agreement with (3.19).
4.2. Linear correction to the three-point functions

In this subsection, we would like to (i) check that the structure of the three-point function computed to first order in conformal perturbation theory agrees with the general form proposed in section 2 and (ii) check whether the correction to the OPE coefficients is non-zero at linear order in \( \mu \). The computations are performed for two different three-point functions: \( \langle TTO \rangle \) and the correlator of three generic charged operators.

4.2.1. Linearized correction to \( \langle TTO \rangle \). Given the general form of the three-point functions (2.5), the answer that we expect to obtain is the expansion to linear order in \( \mu \) of

\[
\langle T_3 O_1 O_2 \rangle_{1\text{ren}} = \frac{K(2\bar{h} - 2, 2, 2)}{\mu_q(p_1 + p_2) \bar{z}_{13}^{\frac{d}{2}} - \mu_q(p_1 + p_2) \bar{z}_{12}^{\frac{d}{2}} - \mu_q(p_1 - p_2) \bar{z}_{23}^{\frac{d}{2}}} \tag{4.27}
\]

where we have omitted the \( \mathcal{O}(\mu^2) \) corrections to the dimensions, which will not contribute, and also dropped the momentum-conserving delta function. The OPE coefficient has the expansion

\[
C_{\mathcal{T}O_1}(\mu p_i) = h + \mathcal{O}(\mu^2). \tag{4.28}
\]

We would now like to reproduce this form from a conformal perturbation theory calculation. The linearized correction to the bare correlator is

\[
\delta_1 \langle T_3 O_1 O_2 \rangle = \mu_E \langle T_3 O_1 O_2 \rangle \int d^2 z_4 : JT_4 : \]

\[
= \frac{\mu E}{z_{12}^{\frac{d}{2}} z_{23}^{\frac{d}{2}}} \left[ \frac{\bar{h}^2}{z_{13}^{\frac{d}{2}}} \int \frac{d^2 z_4}{z_{14} z_{24} z_{12}^{\frac{d}{2}}} + \frac{2 \bar{h}^2}{z_{13}^{\frac{d}{2}}} \int \frac{d^2 z_4}{z_{14} z_{24}^{\frac{d}{2}}} \right] \tag{4.29}
\]

where \( I_{ij} \) is as usual given by (4.10). This result can be split into three parts: one part proportional to \( \ln z_{ij} \), another part, with an equal coefficient, multiplying \( \ln z_{ij} \) and a third part that only involves powers of \( z_{ij} \). After performing the Fourier transform with respect to the \( z_{ij} \), the first part should give the shifts in the powers of \( z_{ij} \) in (4.27), while the remaining two parts will combine, together with the renormalization of the operators, to give the linearized correction to the OPE coefficient (4.28).

4.2.2. Holomorphic logarithmic terms. To check that the coefficients of the \( \ln z_{ij} \) terms agree with (4.27), we need to collect the full coefficient of each logarithmic term, Fourier transform it with respect to the \( z_{ij} \), and then divide by the zeroth order contribution to the right-moving three-point function, which is \( hK(2h - 2, 2, 2) \). We find:

\[
\ln z_{13} = -\frac{4\pi \mu_E q h z_{13} + z_{23}}{z_{12}^{\frac{d}{2}} z_{23}^{\frac{d}{2}} z_{13}^{\frac{d}{2}}} \Rightarrow -\frac{4\pi \mu_E q h}{z_{12}^{\frac{d}{2}} z_{23}^{\frac{d}{2}} z_{13}^{\frac{d}{2}}} \left[ K(2h - 2, 2, 3) + K(2h - 2, 3, 2) \right]
\]

\[
= \frac{2\pi \mu_E q}{z_{12}^{\frac{d}{2}}} (\bar{p}_1 + \bar{p}_2) \times hK(2h - 2, 2, 2). \tag{4.30}
\]
Plugging in $\mu_E = i\mu/(4\pi^2)$ and using the explicit expression (2.11) for $K$, we find that the correction to the exponent of $z_{13}$ in the denominator is

$$\frac{i\mu E}{2\pi} (p_1 + p_2)$$  \hspace{1cm} (4.31)

which perfectly matches our expectation. The coefficient of $\ln z_{32}$ is just minus the above, so the correction to the exponent of $z_{32}$ is

$$-\frac{i\mu E}{2\pi} (\bar{p}_1 + \bar{p}_2)$$  \hspace{1cm} (4.32)

which again matches what we expect. Finally, the coefficient of

$$\ln z_{12} : \frac{4i\mu L h}{z_{12}^{28} z_{12}^{12} z_{12}^{23}} (z_{13}^2 + \bar{z}_{23}^2 - 2\bar{h} z_{13} \bar{z}_{23})$$  \hspace{1cm} (4.33)

leading to the following correction to the exponent of $z_{12}$

$$2h \rightarrow 2h + \frac{i\mu E}{2\pi} (\bar{p}_1 - \bar{p}_2)$$  \hspace{1cm} (4.34)

gain in perfect agreement with (4.27).

4.2.3. Correction to the OPE coefficients. To compute the correction to the OPE coefficients, we need to work in terms of the renormalized operators (4.5)

$$O^{\text{ren}}_p(z) = O_p(z) - 2\pi i\mu_E \bar{p} \left( d_e - \ln \bar{p} \right) O_p(z)$$  \hspace{1cm} (4.35)

where we have defined the constant $d_e = \frac{1}{2} + \gamma + \ln \pi - \frac{1}{2} + \ln L + \psi(2h + 1) + \frac{13}{2}$. The leading correction to the three-point function of renormalized operators takes the form

$$\delta_1 \left( \langle T_{p_3} (z_3) O^{\text{ren}}_{p_1} (z_1) O^{\text{ren}}_{p_2} (z_2) \rangle \right) = \delta_1 \langle T_3 O_1 O_2 \rangle + 2\pi i\mu_E \langle \bar{p}_1 \ln \bar{p}_1 - \bar{p}_2 \ln \bar{p}_2 + (\bar{p}_2 - \bar{p}_1) d_e \rangle \langle T_3 O_1 O_2 \rangle_{\text{CFT}}$$  \hspace{1cm} (4.36)

where we have assumed there is no renormalization of $T$ to this order. All correlators above should be written in the $(z, \bar{z})$ representation. The linearized correction $\delta_1 \langle T_3 O_1 O_2 \rangle_{\text{CFT}}$ is given in (4.29). Its holomorphic and antiholomorphic logarithmic pieces can be read off from (4.30) and (4.33), while the part only involving power laws reads

$$\frac{i\mu E}{2\pi} \left[ \frac{12\pi h^2 z_{12}}{z_{12}^{28} z_{12}^{12} z_{12}^{23}} - \frac{4\pi h z_{12}^3}{z_{12}^{28} z_{12}^{12} z_{12}^{23}} + \frac{\pi c}{3} \left( \frac{1}{z_{13}} - \frac{1}{z_{23}} \right) + c_e \cdot \frac{4\pi h z_{12}}{z_{12}^{28} z_{12}^{12} z_{12}^{23}} (z_{12}^2 - 2(\bar{h} - 1) z_{13} \bar{z}_{23}) \right].$$  \hspace{1cm} (4.37)

It is not hard to check that all the $1/\epsilon$ divergences cancel in the renormalized correlator. Our task now is to compute the Fourier transform of the antiholomorphic log pieces and of (4.37) and plug them into the renormalized correlator (4.36). Note that the Fourier transforms of the logarithms will produce terms proportional to $\ln \bar{p}_i$. The resulting expression for the correction to the OPE coefficient is not particularly illuminating, so we do not present it here; it is however easy to see that it does not vanish, since: (i) the terms proportional to $c_q$ in (4.37) have nothing against which to cancel and (ii) the $\ln \bar{p}_i$ terms do not cancel among each other, as we now show.

To compute the Fourier transform of $\ln \bar{z}_g$, we differentiate $-K(a, b, c)$ with respect to the parameters $a, b, c$. For example,

$$\frac{4i\mu E h (z_{13} + \bar{z}_{23})}{z_{12}^{28} z_{12}^{12} z_{12}^{23}} \ln z_{13} \rightarrow \frac{4i\mu E h}{z_{12}^{12}} [\partial_h K(2\bar{h} - 2, 3, 2) + \partial_{\bar{h}} K(2\bar{h} - 2, 2, 3)].$$  \hspace{1cm} (4.38)
Since in the above formula \( h = 2, 3 \), the hypergeometric function (2.11) entering \( K(a, b, c) \) has one negative argument, so it reduces to a polynomial. The part proportional to \( \ln \hat{p}_i \) is given by
\[
\frac{-2\pi i \mu e_q}{\xi_{12}} (p_1 + p_2) \ln \hat{p}_1 \hbar K(2\hbar - 2, 2, 2).
\] (4.39)

The Fourier transform of the term proportional to \( \ln \hat{p}_{12} \) can be obtained in a similar fashion
\[
\frac{4\pi \mu e_q \hbar}{\xi_{12}^2 \xi_{12}^2 - \xi_{13}^2 \xi_{23}^2} (z_{13}^2 + z_{23}^2 - 2h z_{13} z_{23}) \ln z_{12} \rightarrow - \frac{2\pi i \mu e_q h}{\xi_{12}} (p_1 - p_2) \hbar K(2\hbar - 2, 2, 2) \ln \hat{p}_1.
\] Notice that the term proportional to \( \hbar \ln \hat{p}_1 \) cancels between these two contributions. Even though in principle, our procedure gives the exact answer for the Fourier transform, the \( \ln \hat{p}_i \) dependence can be simply derived from that of the \( \hat{p} \)-dependent prefactors that multiply the hypergeometric function in (2.11).

The Fourier transform of the term proportional to \( \ln \hat{z}_{12} \) is
\[
\frac{4\pi \mu e_q g \hbar(z_{13}^2 + z_{23}^2)}{\xi_{12}^2 \xi_{12}^2 - \xi_{13}^2 \xi_{23}^2} \ln z_{12} \rightarrow - \frac{4\pi \mu e_q g \hbar}{\xi_{12}} [\partial_b K(2h - 2, 3, 2) + \partial_b K(2h - 2, 2, 3)]
\] (4.40)
is slightly more involved, since the derivative with respect to \( b \) is taken before setting \( b = 2, 3 \). However, in appendix C we show that the derivative of the hypergeometric with respect to \( b \) does not produce a \( \ln \hat{p} \) term. Thus, the only logarithmic terms come from the \( \hat{p}_2^{b-1} \) prefactor in (2.11), and we obtain
\[
\frac{2\pi i \mu e_q}{\xi_{12}} (p_1 + p_2) \ln \hat{p}_2 \hbar K(2h - 2, 2, 2).
\] (4.41)

Summing up the \( \ln \hat{p}_i \) contributions in (4.36), we find
\[
\delta(T_{p_i}(z_i)O_{p_i}^{\text{ren}}(z_i)O_{p_i}^{\text{ren}})(z_i))_{\ln \hat{p}} = \frac{2\pi i \mu e_q}{\xi_{12}} [(p_1 + p_2) \ln \hat{p}_2 - 2 \hat{p}_1 \ln \hat{p}_1 + (\hat{p}_1 \ln \hat{p}_1 - \hat{p}_2 \ln \hat{p}_2)]
\] \times \hbar K(2h - 2, 2, 2) = \frac{-2\pi i \mu e_q}{\xi_{12}} \hat{p}_1 \ln \frac{p_1}{p_2} (T_{p_i}(z_i)O_{p_i}^{\text{ren}}(z_i))_{\text{CFT}}.
\] (4.42)

Thus, we find that the correction to the OPE coefficient is non-zero for structural reasons, which follow from the particular way in which we have regulated the UV divergences of the integrated correlator. In addition, there are power law terms that show no particular structure, including the ones proportional to \( c_q \) that we mentioned earlier.

4.2.4. Linearized corrections to a generic three-point function. Let us now compute the linearized corrections to a correlation function of three arbitrary charged operators, with charges \( q_{1,2,3} \) (which must sum to zero) and right-moving dimensions \( h_{1,2,3} \). The correlator that we need to integrate is
\[
\langle O_1 O_2 O_3 : J^4 \rangle := \left( \frac{q_1}{c_1} + \frac{q_2}{c_2} + \frac{q_3}{c_3} \right) \left( \frac{\bar{h}_1 \bar{z}_{12} \bar{z}_{13}}{z_{14} z_{24} z_{34}} - \frac{\bar{h}_2 \bar{z}_{12} \bar{z}_{23}}{z_{14} z_{24} z_{34}} + \frac{\bar{h}_3 \bar{z}_{13} \bar{z}_{23}}{z_{14} z_{24} z_{34}} \right) \langle O_1 O_2 O_3 \rangle_{\text{CFT}}.
\] (4.43)

Note that it is sufficient to compute the terms proportional to \( q_1 \), the rest follow by permutation symmetry. This contribution is given by
where the integral evaluates to
\[
\int \frac{d^2 z_4}{z_41 z_42 z_43} = \frac{1}{z_{13}} \left( \frac{I_{13}}{z_{13}} - \frac{I_{12}}{z_{12}} \right).
\] (4.45)

The full contribution proportional to \( q_1 \) then is
\[
2\pi q_1 \mu_E \left[ \frac{\bar{z}_{13}(\bar{h}_1 - \bar{h}_2) + \bar{h}_2(\bar{z}_{12} - \bar{z}_{23})(\ln |z_{12}|^2 + c_i) + (\bar{h}_1 - \bar{h}_2)(\bar{z}_{12} - \bar{h}_3(\bar{z}_{13} + \bar{z}_{23}))(\ln |z_{13}|^2 + c_i)}{\bar{z}_{12} \bar{z}_{23}} + \bar{h}_3 \bar{z}_{12} + \bar{h}_2 \bar{z}_{13} + \bar{h}_1(\bar{z}_{12} + \bar{z}_{23}) \right] \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\text{CFT}}.
\] (4.46)

As already mentioned, the contributions proportional to \( q_2 \) and \( q_3 \) are obtained via permutations. Let us now check that the coefficients of the \( \ln z_{ij} \) terms are consistent with the shifts in the anomalous dimensions. Explicitly, the coefficient of
\[
\ln z_{13} : 2\pi q_1 \mu_E \left( (\bar{h}_1 - \bar{h}_2)K(a - 1, b + 1, c + 1) - \bar{h}_3 K(a, b, c + 1) - \bar{h}_3 K(a, b + 1, c) \right) + 2\pi q_1 \mu_E \left( (\bar{h}_2 - \bar{h}_3)K(a + 1, b - 1, c + 1) + h_1 K(a + 1, b, c) + h_1 K(a, b + 1, c) \right)
= 2\pi i \mu_E [q_1 (\bar{p}_1 + \bar{p}_2) - q_3 \bar{p}_3] K(a, b, c)
\] (4.47)

where \( a, b, c \) are given by (2.10) and the simplification is a consequence of hypergeometric identities. Thus, the correction to the exponent of \( z_{13} \) takes the form
\[
\delta h_{1,1,2} = \frac{\mu}{2\pi} (q_1 \bar{p}_1 + q_3 \bar{p}_3 - q_2 \bar{p}_2).
\] (4.48)

This should be compared to the expected
\[
\delta h_1 (h_1 + h_3 - h_2) = \frac{\mu}{2\pi} (q_1 \bar{p}_1 + q_3 \bar{p}_3 - q_2 \bar{p}_2).
\] (4.49)

The two expressions are indeed equal after using the charge conservation equation to write \( q_2 = -q_1 + q_3 \), and momentum conservation to eliminate \( \bar{p}_3 \). Thus, the coefficient of the \( \ln z_{13} \) term exactly agrees with the prediction based on one-dimensional conformal symmetry. The terms proportional to \( \ln z_{12} \) and \( \ln z_{23} \) can also be shown to perfectly match the CFT prediction.

As far as the corrections to the OPE coefficients are concerned, we can easily show that they do not vanish, by again focussing on the terms proportional to \( \ln \bar{p}_1 \). Concretely, the linearized three-point function of renormalized correlators is given by
\[
\delta_1 (\mathcal{O}_{\bar{p}_1}^{\text{ren}} \mathcal{O}_{\bar{p}_2}^{\text{ren}} \mathcal{O}_{\bar{p}_3}^{\text{ren}}) = \delta_1 \langle \mathcal{O}_{\bar{p}_1} \mathcal{O}_{\bar{p}_2} \mathcal{O}_{\bar{p}_3} \rangle - \frac{\mu}{2\pi} (q_1 \bar{p}_1 \ln \bar{p}_1 + q_3 \bar{p}_3 \ln \bar{p}_3 - d_c (q_1 \bar{p}_1 + q_3 \bar{p}_3 + q_2 \bar{p}_2)) \langle \mathcal{O}_{\bar{p}_1} \mathcal{O}_{\bar{p}_2} \mathcal{O}_{\bar{p}_3} \rangle_{\text{CFT}}.
\] (4.50)

Concentrating on the pieces proportional to \( \ln \bar{p}_1 \), we find
\[
\delta_1 (\mathcal{O}_{\bar{p}_1}^{\text{ren}} \mathcal{O}_{\bar{p}_2}^{\text{ren}} \mathcal{O}_{\bar{p}_3}^{\text{ren}})_{\ln \bar{p}_1} = \frac{\mu}{2\pi} \left( q_1 \bar{p}_1 \ln \frac{\bar{p}_1}{\bar{p}_2} + q_3 \bar{p}_3 \ln \frac{\bar{p}_2}{\bar{p}_3} \right) \mathcal{O}_{\bar{p}_1} \mathcal{O}_{\bar{p}_2} \mathcal{O}_{\bar{p}_3} \rangle_{\text{CFT}}
\] (4.51)

which again is non-vanishing. Note this correction agrees with (4.42) if \( q_3 = 0 \).
4.3. Operator mixing and basis diagonalization

In this subsection, we bring supporting evidence for the general picture of operator mixing presented in section 2. In particular, we check explicitly that there are no logarithmic terms in sample off-diagonal correlators, as those would imply a breaking of the operator degeneracy. We pay special attention to the mixing of the current and its Virasoro descendants.

4.3.1. Mixing between $T\bar{O}$ and $\bar{O}$. As explained in section 2, in order to study operator mixing we need to first decompose the original Virasoro representations into global $SL(2,\mathbb{R})_R$ ones. At level zero, we have the primary operator $O$, whereas at level two, there is a new $SL(2,\mathbb{R})_R$ primary

$$(\bar{T}O) =: \bar{T}O : = \frac{3}{2(2h + 1)} \bar{\partial}^2 O.$$ (4.52)

The two-point function of this operator is

$$\langle (\bar{T}O)_1 (\bar{T}O)_2 \rangle = \frac{N_{\bar{T}O}}{z_{12}^{h-1} z_{23}^{h-1}} , \quad N_{\bar{T}O} = \frac{c}{2} + \frac{h(8h - 5)}{2h + 1}$$ (4.53)

and $(\bar{T}O)$ is orthogonal to $O$. At linear order in $\mu$, the two operators mix

$$\delta_1 (\langle \bar{T}O \rangle_1 ) = \mu E \int d^2 z_3 : J T_3 :$$

The integral does not produce a log, which implies that the matrix $N$ in (2.18) does not acquire an off-diagonal term, in agreement with our prediction that operators will stay degenerate to $O(\mu)$. To show that $(\bar{T}O)$ and $O$ stay degenerate to linear order, we compute the anomalous dimension of $(\bar{T}O)$

$$\delta_1 (\langle \bar{T}O \rangle_1 ) = \mu E \int d^2 z_3 : (\bar{T}O)_1 (\bar{T}O)_2 : = \mu E \int d^2 z_3 \frac{q(h + 2) N_{\bar{T}O}}{z_{12}^{h-1} z_{13}^{h-1} z_{23}^{h-1}}.$$ (4.55)

Comparing with (4.1), we note that the integrand is identical to that of an operator of dimension $h + 2$ with the unconventional normalization (4.53). Consequently, upon Fourier transform the anomalous dimension will be the same as that of $O$. In order to make the $\{O, (\bar{T}O), \ldots\}$ basis diagonal to order $\mu$, we simply need to redefine

$$(\bar{T}O) \rightarrow (\bar{T}O) + \frac{\pi q E N_{\bar{T}O}}{6h(h + 1)(2h + 1)} \bar{\partial}^2 O.$$ (4.56)

Since $O$ and $(\bar{T}O)$ are orthogonal in the undeformed CFT, the anomalous dimension of $(\bar{T}O)$ is unaffected by this shift.

This analysis can be in principle continued to arbitrarily high order. It would be interesting to have an all-orders proof that the structure we obtain is the one described in section 2.2.

4.3.2. Mixing between $T$ and $\bar{T}^2$. We now repeat the above analysis for the operators starting off as $T$ and $(T)^2$, both of which are Virasoro$_R$ descendants of the identity operator. In particular, we show that these operators follow exactly the same mixing pattern as generic operators in the deformed CFT.

While $T$ is an $SL(2,\mathbb{R})$ primary in the undeformed CFT, the level four primary $(T)^2$ is given by
\[(T^2) \equiv \bar{\mathcal{T}} \mathcal{T} : = \frac{3}{10} \partial^2 \mathcal{T}. \tag{4.57}\]

The normalization of the \((T^2)\) two-point function is
\[\langle \langle \hat{T}^2 \rangle \hat{T}^2 \rangle \rangle = \frac{\mathcal{N}_{T^2}}{c_{12}}, \quad \mathcal{N}_{T^2} = \frac{e(5c + 22)}{10}. \tag{4.58}\]

Let us now compute the mixing between \(T\) and \((T^2)\), which receives the first non-trivial contribution at \(O(\mu^2)\)
\[\delta_2 \langle (T^2) \rangle = \frac{\mu_2^2}{2} \int d^2 z_1 d^2 z_2 \left\langle \left( : \bar{\mathcal{T}} \mathcal{T}_1 : - \frac{3}{10} \partial_1^2 \mathcal{T}_1 \right) \bar{\mathcal{T}}_2 : J \mathcal{T}_3 : J \mathcal{T}_4 : \right\rangle. \tag{4.59}\]

The above correlator of stress tensors takes a surprisingly simple form, since the \(\bar{z}_i\) dependence of the integrand is the same in the coefficient proportional to \(c\) as in that proportional to \(c^2\). More precisely, we have
\[\delta_2 \langle (T^2) \rangle = \frac{c(\hat{c} + \frac{42}{3}) k \mu_2^2}{4} \int d^2 z_1 d^2 z_2 \left( \frac{1}{\ell_{12} \ell_{13} \ell_{14} \ell_{24}} + \frac{1}{\ell_{13} \ell_{12} \ell_{45} \ell_{24}} + \frac{1}{\ell_{14} \ell_{12} \ell_{35} \ell_{23}} \right). \tag{4.60}\]

These integrals do not give rise to logarithms, as expected. The polynomial term can be absorbed into a redefinition of \((T^2)\) by \(\mu_2^2 \partial^2 \mathcal{T}\), which is precisely of the form \((2.23)\) we argued for. This confirms that at least to this order, \(\bar{\mathcal{T}}\) follows the same mixing pattern as generic operators in the CFT. Note however that, as already mentioned in section 2.2, the operator \(T_\hat{p}\) studied here is not the same as the Noether current associated to right-moving translations.

### 4.3.3. Mixing between \(J\) and \(J T\)

The argument of section 2.2, which we checked above in examples, would lead us to think that we never need to consider mixing between \(O\) and higher powers of \(T^2\) when diagonalizing the operator basis. While this is generally true, there exist exceptions if the operator in question is purely holomorphic, as for example in the case of the chiral current \(J\).

The two-point function of the current first receives corrections at \(O(\mu^2)\), but the correction is a pure contact term\(^{16}\)
\[\delta_2 \langle J_1 J_2 \rangle = \frac{\mu_2^2}{2} \langle J_1 J_2 \rangle \int d^2 z_3 : J \mathcal{T}_3 : \int d^2 z_4 : J \mathcal{T}_4 : \rangle = - \frac{\pi^2 \mu_2^2 k^2}{6} \bar{\partial}_1 \partial_1 \delta(\bar{z}_1) \delta(z_1) \tag{4.61}\]

which we ignore (it can also be removed by a shift of \(J\) by \(\mu_2^2 \partial^2 J\)). Consequently, \(J\) does not acquire an anomalous dimension to this order, which is consistent with the fact that it has \(\bar{\mathcal{T}} = 0\).

The current \(J\) does mix non-trivially with \(J \bar{\mathcal{T}}\) at second order in \(\mu\), as can be seen by computing
\[\delta_2 \langle J_1 : J \bar{\mathcal{T}}_2 : \rangle = \frac{\mu_2^2}{2} \langle J_1 : J \bar{\mathcal{T}}_2 : \rangle \int d^2 z_3 : J \mathcal{T}_3 : \int d^2 z_4 : J \mathcal{T}_4 : \rangle \tag{4.62}\]
with the result

\(^{16}\) It is interesting to compare this calculation with its counterpart in the \(J J\) deformation case, where the correction to the level is finite.
\[ \delta_2(J_1 : JT_2 :) = \frac{\mu^2 k^2 c}{8} \int \frac{d^2 z_3 d^2 z_4}{z^{12} z^{23} z^{24} z^{34}} \left( \frac{1}{z^{12} z^{34}} + \frac{1}{z^{13} z^{24}} + \frac{1}{z^{14} z^{23}} \right) = \frac{\mu^2 \pi^2 k^2 c}{2 z^{12} z^{13} z^{14} z^{23} z^{24} z^{34}}. \] (4.63)

Note that we cannot reset this off-diagonal term to zero by shifting \( JT \) by \( \mu^2 \partial \mu \), because the correlation function of the shift with \( J \) is a pure contact term in the original CFT. Thus, the only way to render the basis diagonal is by instead redefining \( J \) as

\[ J \rightarrow J' = J - 2 \mu^2 \pi^2 k : JT : + \ldots \] (4.64)

The same shift in the current was obtained in [35], by requiring that the current stay chiral in the deformed theory. Indeed, acting with \( \partial_\mu \) on (4.63) yields a non-zero answer, which means that the current was no longer holomorphic before we shifted it.

The \( JT \) operator does, on the other hand, behave as a generic operator from the tower (2.16), as one expects from the fact that it is not holomorphic. Its two-point function is first corrected at \( O(\mu^2) \)

\[ \delta_2(\delta J_1 : JT_2 :) = \frac{\mu^2 k^2}{8} \int \frac{d^2 z_3 d^2 z_4}{z^{12} z^{23} z^{24} z^{34}} \left( \frac{1}{z^{12} z^{34}} + \frac{1}{z^{13} z^{24}} + \frac{1}{z^{14} z^{23}} \right) \langle T_1 T_2 T_3 T_4 \rangle. \] (4.65)

The integral over the first term is identical to the one performed in (4.19) and produces the correct coefficient of the logarithmic divergence to correspond to an anomalous dimension \( \mu^2 k p^2 / 16 \pi^2 \). The integrals over the other two terms are equal to each other and we need to check that they do not lead to any new logarithmic divergences. This is easily checked for the terms corrected to \( c^2 \) in (A.12). We are left to evaluate

\[ 2 : \frac{\mu^2 k^2}{8} \int \frac{d^2 z_3 d^2 z_4}{z^{12} z^{23} z^{24} z^{34}} \left( \frac{1}{z^{12} z^{13} z^{24} z^{23} z^{14} z^{23} z^{12} z^{24} z^{34}} + \frac{1}{z^{12} z^{13} z^{14} z^{23} z^{24} z^{34}} + \frac{1}{z^{12} z^{13} z^{14} z^{23} z^{24} z^{34}} \right). \] (4.66)

Even though the first two integrals are logarithmically divergent, their divergences cancel exactly against each other.

The conclusion of this analysis is that in order to keep a diagonal operator basis, \( J \) needs to be corrected at \( O(\mu^2) \) as in (4.64). The form of the correction is a consequence of the holomorphy of the current. Higher order corrections of the form \( \mu^{2n} \langle \bar{J} \bar{T}^n \rangle \) would also be necessary if the corresponding \( \delta_{2n}(J : \bar{J} \bar{T}^n :) \) matrix elements are non-zero. On the other hand, since the operator \( JT \) is not holomorphic, its matrix elements with the other \( \bar{J} \bar{T}^n \) can be rediagonalized via a basis change of the form (2.23), i.e. it is only (multiplicatively) renormalized by derivatives of itself. It would be interesting to better understand the relation between this operator and the one defined via the OPE of the deformed currents.

4.3.4. Mixing between \( \bar{T} \) and \( JT \). This is again an interesting case of mixing. Already at linear order in \( \mu \), we find a non-zero answer

\[ \delta_1(\delta T_1 : JT_2 :) = \mu_E \langle \delta T_1 : JT_2 : \rangle = \int \frac{d^2 z_3}{z^{12} z^{23} z^{24} z^{34}} \left( \frac{1}{z^{12} z^{13} z^{24} z^{23} z^{14} z^{23} z^{12} z^{24} z^{34}} \right) = -\frac{2 \pi \mu_E k c}{z^{12} z^{13} z^{24} z^{23} z^{34}}. \] (4.67)

Since the \( \partial_\mu \) derivative of the above does not vanish, it reflects the fact that the stress tensor is no longer holomorphic at first order in \( \mu_E \). This agrees with the results of [35], where \( \partial_\mu \bar{T} = -2 \pi \mu_E \partial \mu J T \).

Naively, (4.67) suggests that a non-local redefinition of the stress tensor, such as (2.28), is needed in order to keep the basis diagonal. However, such a redefinition is not allowed within our framework, where locality in the \( z \) direction should always be manifest. Additionally, it is
easy to check that such a shift would affect the anomalous dimension of $T$, which would no longer match (3.19).

An interpretation of this non-zero matrix element that does fit within our framework is the following. As mentioned in section 2, the operators in the $JT$-deformed CFT belong to left-moving Virasoro–Kać–Moody representations. The Kać–Moody descendants may be written in terms of Virasoro$_\ell$ primaries, e.g. at level one we have

$$ (JO) =: JO := -\frac{q}{2\hbar}\partial O $$ (4.68)

which has vanishing inner product with $O$. Note that now the bracket notation stands for a Virasoro, rather than a global, primary. For the case of the stress tensor, the corresponding combination in the deformed theory should be

$$ (JT) =: J\bar{T} := -\frac{q_{\bar{T}}}{2\hbar}\partial \bar{T} =: J\bar{T} := -\frac{2\pi}{\mu p}\partial \bar{T}. $$ (4.69)

Thus, the expansion of this operator around $\mu = 0$ looks singular, if we choose the coefficient of the $JO$ term to be one. By our general recipe, this primary operator should be orthogonal to the primary constructed from $\bar{T}$. It is not hard to check that this is indeed the case: if we compute $\langle \partial \bar{T} \bar{T} \rangle$ in conformal perturbation theory using (4.20), the first non-vanishing contribution is at second order in $\mu$, which perfectly cancels the contribution (4.67) from the first term. We conclude that the non-zero two-point function of $J\bar{T}$ with $\bar{T}$ is simply due to the fact that $J\bar{T}$ is not a Virasoro primary, but it needs to be shifted by $\partial \bar{T}$ in order to become one, with a coefficient that is divergent as $\mu \to 0$. If we do not mind working with Virasoro descendants, then these singular-looking combinations need not be considered.

### 5. Discussion

In this article, we have made the first steps towards the specification of correlation functions in $JT$-deformed CFTs in terms of the conformal data of the original CFT. In particular, we have proposed an exact formula for the spectrum of conformal dimensions and charges of $JT$-deformed CFTs and checked it to leading order in conformal perturbation theory. We have also computed the OPE coefficients to linear order in the perturbation and addressed the issue of operator mixing.

To make further progress, there are several technical and conceptual issues that need to be resolved. First, one needs a better understanding of the deforming operator, and in particular of the two currents that compose it. While the chiral current $J$ is well-defined and relatively straightforward to construct order by order in conformal perturbation theory, it is not completely clear what the definition of $T$ should be in the deformed CFT. As we pointed out, the CFT antiholomorphic stress tensor gives rise to at least two different notions of $T$ operator in the deformed theory: the primary operator $T_p$ discussed in this article, with anomalous dimension and charge given by (3.19), and the $\bar{z}$ component of the Noether current associated to right-moving translations, which depends on the non-local combination $\bar{z} - \mu \int J(z)$ and does not yet have a definition outside the Lagrangian framework. It would be interesting to understand how these two notions relate to each other (even though the difference between them is irrelevant at the level of the integrated operator, which projects on their coinciding $p = 0$ component). It would also be interesting to understand the difference between the two notions of $JT$ operator that we discussed. Another issue that one should address is whether contact terms, which we have completely ignored in our analysis, may play a role in computing the
deformed correlators. Finally, it would be very interesting if one could obtain flow equations for the correlation functions as $\mu$ is varied, along the lines \cite{8} followed for the case of $TT$, which may allow us to access the correlation functions at finite $\mu$.

Another important technical issue concerns the choice of ultraviolet regulator for the divergences of the integrated correlators. Usually, the UV regulator is chosen to respect as many symmetries of the problem as possible. Our choice of a standard Lorentz invariant regulator, however, does not present any particular advantage, since the symmetry it is designed to respect is explicitly broken by the deformation. While it is encouraging that the one-dimensional conformal structure of the deformed correlators agrees with the general expectations, the rather unappealing form of the OPE coefficients we have found is a direct consequence of having used a Lorentz-invariant ultraviolet regulator. We cannot help wondering whether a different regulator may be more natural in $JT$-deformed CFTs, especially since these theories do not possess a usual UV fixed point as far as the right-movers are concerned, and may be sensitive to the particular choice we make. The question of suitable ultraviolet regulators can in principle also be studied in the context of dipole theories \cite{41}, whose UV structure is expected to be similar to that of $JT$-deformed CFTs and for which Lagrangian and integrability-based methods are also available \cite{47}. Apart from these ultraviolet issues, note that one also needs to worry about possible infrared problems due to the presence of a continuous spectrum of operators, which extends all the way to zero dimension.

Once these issues are resolved, there are many interesting features of $JT$-deformed CFTs that one can explore, and possible new structures to uncover. For example, if it is found that the OPE coefficients do not receive corrections in perturbation theory, then $JT$-deformed CFTs can be interpreted in terms of momentum-dependent spectral flow and operator-dependent coordinate transformations. These would not only provide a more geometric picture for the deformation directly in right-moving position space, but they would also allow one to specify all correlation functions in $JT$-deformed CFTs by applying a simple operation to the correlation functions of the original CFT.

A very interesting question is whether $JT$-deformed CFTs possess hidden symmetries associated to the right-moving Virasoro symmetry of the original CFT. There are two different indications that this structure may continue to exist: first, the asymptotic symmetry analysis of \cite{36} found that the right-moving translational symmetry $U(1)_R$ was enhanced to a full right-moving Virasoro symmetry that was field-dependent; second, the analysis of the current paper shows that operators that were originally part of the same Virasoro$_R$ highest weight representation acquire the same anomalous dimension in the deformed theory at fixed $\tilde{p}$, and thus stay degenerate. It would be very interesting to investigate whether these degenerate operators could still be related by a larger symmetry. For this, one needs to understand how the OPE coefficients of the various $SL(2,\mathbb{R})_R$ primaries in (2.16) are modified relative to each other. A complementary approach would be to understand whether a current $T_{\text{Noether}}$ associated to the right-moving symmetries can still be defined in the deformed CFT.

A natural question is whether the results of this paper can be reproduced from a holographic calculation. For this, one needs to extend the holographic dictionary proposed in \cite{36}, which was restricted to the study of the currents, to include general propagating degrees of freedom. In particular, it would be interesting to reproduce the shift (1.3) in the operator dimensions and charges from holography, check our prediction for the OPE coefficients and check how the primary operator $\tilde{T}_p$ we discussed fits within the holographic framework.

Finally, let us note that $JT$-deformed CFTs provide the first concrete example of a dipole CFT$_3$, a type of quantum field theory loosely defined as a deformation of a two-dimensional CFT by a set of irrelevant operators of dimensions $(1, n)$, whose coefficients are finely tuned so that the resulting theory is UV-complete, though non-local. That many such dipole CFTs
should exist is indicated by the ubiquity of warped AdS$_3$ backgrounds—the spacetimes holographically dual to such field theories—in string theory [48–50], but so far no general field-theoretical procedure for defining them has been proposed. On the other hand their study, as argued in [49], is directly relevant to the holographic description of general extremal black holes [51].

A central ingredient in the microscopic description of extremal black holes is the presence of a right-moving Virasoro symmetry, and our hope is that the study of $\bar{J}T$-deformed CFTs can shed light on this interesting issue. One simple calculation suggested by our analysis is to study operator degeneracy in dipole CFTs and determine whether the original degeneracy associated to the right-moving Virasoro symmetry is lifted by the deformation. If it is not, then this may have interesting implications for the survival of a right-moving Virasoro symmetry. More generally, one can try to set up a bootstrap programme for dipole CFTs, which should hold order by order in the deformation parameter, and in which $\bar{J}T$-deformed CFTs could be used as a simple concrete example. One can hope that such a programme would allow one to classify which CFTs admit a dipole deformation and find the general properties of the resulting spectra.

Acknowledgments

The author would like to thank Camille Aron, Costas Bachas, Brando Bellazzini, Pierre Heidmann, David Kutasov, Ruben Monten, Miguel Paulos, Slava Rychkov, Andrew Strominger, Marika Taylor, Jan Troost and Xi Yin for interesting discussions, and Adam Bzowski for collaboration in the early stages of this project. This research was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958, as well as by the ERC Starting Grant 679278 Emergent-BH and the Swedish Research Council grant number 2015-05333.

Appendix A. List of basic correlators

In this appendix, for easy reference, we collect the list of basic correlators that are used in the main text. They usually involve a primary operator $\mathcal{O}$ of dimension $(h, \bar{h})$ and charge $q$ and/or the currents $J, T$ and $\bar{T}$. The notation $\mathcal{O}_i$ stands for $\mathcal{O}_i(z_i, \bar{z}_i)$.

The correlation functions that involve the currents are easily determined by repeated use of the Ward identities

\[
\langle J(\z) \mathcal{O}_1 \cdots \mathcal{O}_n J_{n+1} \cdots J_{n+m} \rangle = \sum_{i=1}^{n} \frac{q_i}{z - z_i} \langle \mathcal{O}_1 \cdots \mathcal{O}_n J_{n+1} \cdots J_{n+m} \rangle \\
+ \sum_{j=1}^{m} \frac{k/2}{(z - z_{n+j})^2} \langle \mathcal{O}_1 \cdots \mathcal{O}_n J_{n+1} \cdots J_{n+j} \cdots J_{n+m} \rangle
\]

(A.1)

where the notation $\bar{J}$ means that the corresponding term has been ommitted. The stress tensor Ward identity reads

\[
\langle T(\z) \mathcal{O}_1 \cdots \mathcal{O}_n T_{n+1} \cdots T_{n+m} \rangle = \sum_{i=1}^{m+n} \left( \frac{\delta_i}{(z - z_i)T} + \frac{1}{z - z_i} \delta_i \right) \langle \mathcal{O}_1 \cdots \mathcal{O}_n T_{n+1} \cdots T_{n+m} \rangle \\
+ \sum_{j=1}^{m} \frac{c/2}{(z - z_{n+j})^4} \langle \mathcal{O}_1 \cdots \mathcal{O}_n T_{n+1} \cdots T_{n+j} \cdots T_{n+m} \rangle
\]

(A.2)
where in the first sum \( h_i = 2 \) for \( i > n \). The Ward identity for the antiholomorphic stress tensor is identical, upon the replacements \( T \to \bar{T} \), \( h \to \bar{h} \) and \( z \to \bar{z} \).

A.1. Correlation functions involving a generic operator

The two-point function of a generic primary operator \( \mathcal{O} \) of dimension \((h, \bar{h})\) and charge \( q \) is

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{1}{z_{12}^{2\bar{h}-1}} \frac{1}{z_{12}^{2h}}. \tag{A.3}
\]

Using the above Ward identities, we obtain

\[
\langle \mathcal{O}_1 \mathcal{O}_2 J_3 T_4 \rangle = \frac{q}{z_{12}^{2\bar{h}-1}z_{12}^{23}} \cdot \frac{\bar{h}}{z_{12}^{2h}z_{12}^{23}} \tag{A.4}
\]

\[
\langle \mathcal{O}_1 \mathcal{O}_2 J_3 J_4 T_4 \rangle = \left( \frac{k}{z_{12}^{24}} + \frac{(q \bar{z}_{12})^2}{z_{12}^{23}z_{12}^{24}} \right) \frac{1}{z_{12}^{2h}z_{12}^{23}} \cdot \frac{\bar{h}}{z_{12}^{2h}z_{12}^{24}} \tag{A.5}
\]

\[
\langle \mathcal{O}_1 \mathcal{O}_2 T_3 J_4 T_4 \rangle = \left( \frac{h_1^2}{z_{12}^{23}} + \frac{z_{12}^2 z_{24}}{z_{12}^{23}z_{24}} \right) \frac{q}{z_{12}^{2h}z_{12}^{23}} \cdot \frac{\bar{h}}{z_{12}^{2h}z_{12}^{24}} + \frac{c}{z_{12}^{2h}} \tag{A.6}
\]

\[
\langle \mathcal{O}_1 \mathcal{O}_2 J_3 J_4 J_4 T_4 \rangle = \left( \frac{q_1}{z_{41}} + \frac{q_2}{z_{42}} + \frac{q_3}{z_{43}} \right) \left( \frac{\bar{h}_1 z_{12} z_{13}}{z_{12}^{24}z_{24} z_{24} z_{43}} - \frac{\bar{h}_2 z_{12} z_{23}}{z_{12}^{24}z_{24} z_{24} z_{24}} + \frac{h_3 z_{12} z_{23}}{z_{12}^{24}z_{24} z_{24} z_{24}} \right) \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle. \tag{A.7}
\]

A.2. Correlation functions of the currents

Correlation functions of \( J \)'s are simply given by Wick contractions of the basic correlator

\[
\langle J_1 J_2 \rangle = \frac{k}{z_{12}^{24}}. \tag{A.10}
\]

The first few correlation functions of the stress tensor are given by

\[
\langle T_1 T_2 \rangle = \frac{c}{z_{12}^{24}} \quad \langle T_1 T_2 T_3 \rangle = \frac{c}{z_{12}^{24} z_{12}^{23}}. \tag{A.11}
\]

The four-point function of four stress tensors is

\[
\langle T_1 T_2 T_3 T_4 \rangle = \frac{c^2}{4} \left( \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} \right) + c \left( \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} + \frac{1}{z_{12}^{24} z_{12}^{24} z_{12}^{24}} \right). \tag{A.12}
\]

Correlation functions of double-trace operators : \( AB \) : are obtained by keeping the \( \mathcal{O}(c^0) \) term in the expansion of \( \langle A(z)B(0) \ldots \rangle \).
Appendix B. Useful integrals

There are two basic integrals that appear recurrently in our calculations.

B.1. The $I$ integral

\[ I(z_1, z_2) \equiv \int \frac{d^2z_3}{|z_{13}|^2|z_{23}|^2}. \]  \hspace{1cm} (B.1)

The integral can be performed by introducing a Schwinger parameter

\[ I(z_1, z_2) = \int_0^1 du \int_0^\infty \frac{d^2z_3}{(u|z_{13}|^2 + (1-u)|z_{23}|^2)^2} = \int_0^1 du \int_0^\infty \frac{d^2z_3}{(|z_3|^2 + u(1-u)|z_{12}|^2)^2}. \]  \hspace{1cm} (B.2)

Since the result is divergent, we evaluate it using dimensional regularization, i.e. we replace $d^2z_3 \to d^d z_3$. Then

\[
I(z_1, z_2) = 2V_{sr-1} \int_0^1 du \int_0^\infty \frac{d^d \rho}{(\rho^2 + u(1-u)|z_{12}|^2)^2} \\
= V_{sr-1} \Gamma \left( \frac{d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) (d-4) \int_0^1 du [u(1-u)]^{\frac{d-2}{2}} \\
= 2\pi^\frac{d}{2} \Gamma \left( 2 - \frac{d}{2} \right) B \left( \frac{d}{2} - 1, \frac{d}{2} - 1 \right) |z_{12}|^{d-4} \]  \hspace{1cm} (B.3)

where the volume of $S^{d-1}$ is $V_{sr-1} = 2\pi^{d/2}/\Gamma(d/2)$. Taking $d = 2 + \epsilon$ and expanding we find

\[ I(z_1, z_2) = 2 \frac{2\pi}{|z_{12}|^2} |z_{12}|^2 + \gamma + \ln \pi + \frac{\epsilon}{4} (\ln |z_{12}|^2 + \gamma + \ln \pi)^2 - \frac{\pi^2}{24} + O(\epsilon). \]  \hspace{1cm} (B.4)

In the main text, we will only need the expansion of this integral up to $O(\epsilon)$.

B.2. The $I_{ij}$ integral

Another useful integral is

\[ I_{ij} \equiv \int \frac{d^2z_4}{|z_{4i}|^2}. \]  \hspace{1cm} (B.5)

For definiteness, we compute $I_{12}$

\[ I_{12} = \int \frac{d^2z_4}{|z_{4i}|^2|z_{42}|^2} = \int_0^1 du \int \frac{d^2z_4}{u|z_{41}|^2 + (1-u)|z_{42}|^2} = \int_0^1 du \int \frac{d^2z_4}{|z_4^2 + u(1-u)|z_{12}|^2|^2}. \]

Changing again the dimension to $d$ we find

\[ I_{12} = 2V_{sr-1} \int_0^1 du \int_0^\infty \frac{d^d \rho}{(\rho^2 + u(1-u)|z_{12}|^2)^2} \\
= 2\pi^\frac{d}{2} \frac{\Gamma \left( \frac{d}{2} \right) \Gamma \left( 1 - \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - 1 \right)} \\
= 2\pi^\frac{d}{2} |z_{12}|^{d-2} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(d - 1)} = -2\pi \left( \gamma + \ln |z_{12}|^2 + \gamma_e + \ln \pi + O(\epsilon) \right). \]  \hspace{1cm} (B.6)
B.3. Derivatives of $I_{ij}$

Differentiating this result, we can obtain a list of simple integrals

$$\int \frac{d^2 z_4}{z_4^2 z_{14}^2} = \frac{2\pi}{z_{12}}, \quad \int \frac{d^2 z_4}{z_{13}^2 z_{14}^2} = 4\pi^2 \delta^2(z_{13})$$

(B.7)

where we use $\frac{\partial}{\partial z} = \frac{\partial}{\partial z} = 2\pi \delta(z) \delta'(z)$. All integrals in the main text are computed by taking differences and derivatives of this basic integral. For example, the integral appearing in (4.9) is calculated as

$$\int \frac{d^2 z_4}{z_{13}^2 z_{14}^2 z_{24}} = \partial_z \partial_{z_2} \frac{1}{z_{12}} \int \frac{d^2 z_4}{z_{24}^2 z_{14}^2} \left( \frac{1}{z_{24}} - \frac{1}{z_{14}} \right) = -\frac{4\pi^2}{z_{12}^3} \left( \frac{1}{z_{13}} - \frac{1}{z_{23}} \right).$$

(B.8)

It is also easy to check that $I(z_1, z_2) = \frac{2}{|z_1 z_2|} I_{12}$, without needing to evaluate the integrals.

Appendix C. Fourier transforms

C.1. Basic Fourier transforms

We start with the Fourier transform of expressions of the type $z^{-a}$ and $z^{-a} \ln |z|$, where $a$ is an arbitrary real number (the case of $a$ integer sometimes needs separate treatment). We mostly follow the treatment in [46]. To compute the Fourier transform of $z^{-a}$, we consider instead $z^{-a} \Theta(z)$. We further regulate it as

$$\mathcal{F}_z \left( \frac{\Theta(z)}{z^a} \right) = \lim_{t \to 0} \int_0^\infty \frac{dz}{z^a} e^{-i(t+ip)z} = \lim_{t \to 0} \int_0^\infty \left( t + ip \right)^{a-1} \Gamma(1-a) = \frac{\pi e^{i(a-1)} \sin \pi a \Gamma(a)}{\sin \pi a \Gamma(a)} \equiv \frac{\pi}{\sin \pi a \Gamma(a)} \Theta(p).$$

(C.1)

To find the Fourier transform of $z^{-a}$ alone we must evaluate

$$\int_{-\infty}^0 \frac{dz}{z^a} e^{-i\omega z} = (-1)^{-a} \int_0^\infty \frac{dz}{z^a} e^{i\omega z}. \quad \text{(C.2)}$$

Notice that $z^{-a}$ has a branch cut along the real negative axis, and we must specify where we place it. We choose to place it below the real axis by the replacement $z \to z + i\epsilon$. In that case

$$\int_{-\infty}^\infty \frac{dz}{z^a} e^{-i\omega z} = \left( e^{\frac{i}{2}(a-1)} \text{sgn} \, p + e^{-i\pi a} e^{-\frac{i}{2}(a-1)} \text{sgn} \, p \right) \frac{\pi p^{a-1}}{\sin \pi a \Gamma(a)} = 2\pi \frac{(-i)^a |p|^{a-1}}{\Gamma(a)} \Theta(p).$$

(C.3)

The fact that the Fourier transform of the Wightman function vanishes for negative frequencies is well known, which follows from the fact that for $p < 0$ we should close the integration contour in the upper half plane. Since the branch cut is below the real axis, the integral can be deformed to zero. Thus, we have

$$\mathcal{F}_z \left( \frac{1}{z^a} \right) = 2\pi \frac{(-i)^a |p|^{a-1}}{\Gamma(a)} \Theta(p). \quad \text{(C.4)}$$

The Fourier transform of $z^{-a} \ln z$ can be obtained by differentiation with respect to $a$, obtaining

$$\mathcal{F}_z \left( \frac{\ln z}{z^a} \right) = 2\pi \frac{(-i)^a |p|^{a-1}}{\Gamma(a)} \left( \frac{\psi(a) - \ln |p| + i\pi}{2} \right) \Theta(p). \quad \text{(C.5)}$$
C.2. Fourier transform of the three-point function

The Fourier transform of the three-point function reads

$$K(a, b, c) \delta(p_1 + p_2 + p_3) = \int \prod_{i=1}^{3} d^4x \frac{e^{-i\phi_{13}}}{e^{i_2x_{23}}} = \int d^4x d^4y e^{-i\phi_{13} + i\phi_{23}} x^a y^b (x + y)^c \delta(p_1 + p_2 + p_3).$$  \hfill (C.6)

The integral can be performed by introducing a Schwinger parameter $\alpha$ \cite{45]

$$K(a, b, c) = \frac{\Gamma(a + c)}{\Gamma(a)\Gamma(c)} \int_0^1 du u^{a-1}(1 - u)^{c-1} \int d^4x d^4y \frac{e^{-i\phi_{13} + i\phi_{23}}}{(x + (1 - u)y)^{a+b+c}}$$

$$= \frac{4\pi^2(-i)^{a+b+c+1}\Theta(\bar{p}_1)}{\Gamma(a)\Gamma(b)\Gamma(c)} \times |p_1|^{a+c-1} \int_0^1 du u^{-1}(1 - u)^{c-1} |p_2 + u\bar{p}_1|^{b-1} \Theta(-p_2 - u\bar{p}_3).$$  \hfill (C.7)

In the above, we have given $y = z_3$ and $x' = z_1z_2 + (1 - u)z_3$ a small positive imaginary part, corresponding to an ordering $\text{Im} t_1 < \text{Im} t_2 < \text{Im} t_3$. The remaining integral is easy to perform if we assume e.g. that $p_2 > 0$ (for $p_2 < -|p_1|$ the integrand vanishes), obtaining\footnote{We have used the integral representation of the hypergeometric function}

$$K(a, b, c) = \frac{4\pi^2(-i)^{a+b+c}}{\Gamma(a+c)} \bar{p}_1^{a+c-1} |p_2|^{b-1} 2F_1 \left[1 - b, a + c, -\frac{\bar{p}_1}{p_2} \right].$$  \hfill (C.9)

Of course, this result can be written in term of any two of the momenta, using the conservation equation, though its range of validity is for $p_1+2 > 0$. We can obtain other representations of the three-point function that are valid in different ranges. For example, if we introduce the Schwinger parameter in a slightly different way, we arrive at

$$K(a, b, c) = \frac{\Gamma(b + c)}{\Gamma(b)\Gamma(c)} \int_0^1 du u^{b-1}(1 - u)^{c-1} \int d^4x d^4y x^a y^b (x + (1 - u)y)^c \theta(\bar{p}_1)$$

$$= \frac{4\pi^2(-i)^{a+b+c}}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a+b+c} \times |p_3|^{b+c-1} \int_0^1 du u^{-1}(1 - u)^{c-1} |p_2 + u\bar{p}_3|^{a-1} \Theta(-p_2 - u\bar{p}_3).$$  \hfill (C.10)

If $p_2 < 0$, then the theta function can be dropped and the integral evaluates to

$$K(a, b, c) = \frac{4\pi^2(-i)^{a+b+c}}{\Gamma(a+c)} |p_1|^{b+c-1} |p_2|^{a-1} 2F_1 \left[1 - a, b + c, -\frac{\bar{p}_3}{p_2} \right]$$

$$= \frac{4\pi^2(-i)^{a+b+c}}{\Gamma(a+c)} |p_1|^{b+c-1} |p_2|^{a-1} 2F_1 \left[1 - a, b+c, -\frac{\bar{p}_3}{p_1} \right]$$  \hfill (C.11)

where in the last line we used the transformation of the hypergeometric,

$$2F_1(a, b, c, z) = (1 - z)^{-a} 2F_1 \left[ a, c - b, c, \frac{z}{z - 1} \right].$$  \hfill (C.12)

This expression, which can be obtained by a simple permutation of the arguments and parameters in (C.9), is valid for $p_{2,3} < 0$. It can also be shown that if we insist that $p_2 > 0$ in (C.10), then we are back to the expression (C.9). Thus, depending on the signs of the momenta, we obtain different closed-form expressions for $K(a, b, c)$.
C.3. Derivatives of the hypergeometric function

The way we compute the Fourier transform of power law terms multiplied by $\ln \tilde{z}_{ij}$ is by taking derivatives of the Fourier transform $K(a,b,c)$ corresponding to the power law with respect to the parameters $a, b, c$. The expression for $K(a,b,c)$ involves a hypergeometric function, and taking derivatives with respect to the parameters is usually cumbersome; however, the expression for the hypergeometric $\binom{m}{n} \equiv m$ simplifies drastically when $b$ is small integer, since the hypergeometric function becomes a polynomial of degree $b - 1 = m$

$$2F_1(-m, \beta, \gamma; z) = \sum_{n=0}^{m} \frac{(\beta)_n \Gamma(m + 1)}{(\gamma)_n \Gamma(m + 1 - n)} (-z)^n \frac{1}{n!}. \quad (C.13)$$

While taking derivatives with respect to $a$ and $c$ is straightforward, the Fourier transform of the $\ln \tilde{z}_{21}$ term involves differentiating with respect to $b$ and only then setting $b$ to be an integer. In the following, we explicitly evaluate the derivative of the hypergeometric with respect to a parameter, which only then is set to be an integer

$$\partial_z (2F_1(a, \beta, \gamma; z)) |_{a = -m} = \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \frac{(\beta)_n \Gamma(m + 1)}{(\gamma)_n \Gamma(m - \epsilon + 1 - n)} (-z)^n \Gamma(n + \epsilon) \psi(n + \epsilon - m)$$

where $\psi$ is the digamma function, which satisfies

$$\psi(n + \epsilon - m) = \sum_{k=0}^{n} \frac{1}{k + m + \epsilon}. \quad (C.15)$$

The expansion on the right-hand side can thus be split into two sums, one for $n \leq m$, and one for $n > m$

$$\text{RHS} = \sum_{n=0}^{m} \frac{(\beta)_n \Gamma(m + 1)}{(\gamma)_n \Gamma(m + 1 - n)} (-z)^n \sum_{l=0}^{n-1} \frac{1}{l - m} + \lim_{\epsilon \to 0} \sum_{n=m+1}^{\infty} \frac{(\beta)_n \Gamma(m + 1)}{(\gamma)_n \Gamma(m - \epsilon + 1 - n)} (-z)^n \sum_{l=0}^{n-1} \frac{1}{l - m + \epsilon}$$

$$= \sum_{n=0}^{m} \frac{(\beta)_n \Gamma(m + 1)}{(\gamma)_n \Gamma(m + 1 - n)} (-z)^n \sum_{l=0}^{n-1} \frac{1}{l - m} + (-1)^m m! z^{m+1} \sum_{k=0}^{\infty} \frac{(\beta)_{m+1+k} k!}{(\gamma)_{m+1+k} (m + 1 + k)!} \quad (C.16)$$

where only the terms with $l = m$ contribute in the latter sum. Letting $n = m + 1 + k$ in the last term, the sum can be converted into a hypergeometric function, and equals

$$z^{m+1} \frac{(-1)^m \beta_{m+1}}{(m+1) \gamma_{m+1}} 3F_2(1, 1, 1 + m + \beta; m + 2, 1 + m + \gamma; z). \quad (C.17)$$

Notice that neither this, nor the finite sum in the first term, exhibits a logarithmic behaviour.

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