Creep rupture of viscoelastic fiber bundles

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We study the creep rupture of bundles of viscoelastic fibers occurring under uniaxial constant tensile loading. A novel fiber bundle model is introduced which combines the viscoelastic constitutive behaviour and the strain controlled breaking of fibers. Analytical and numerical calculations showed that above a critical external load the deformation of the system monotonically increases in time resulting in global failure at a finite time $t_f$, while below the critical load the deformation tends to a constant value giving rise to an infinite lifetime. Our studies revealed that the nature of the transition between the two regimes, i.e. the behaviour of $t_f$ at the critical load $\sigma_c$, strongly depends on the range of load sharing: for global load sharing $t_f$ has a power law divergence at $\sigma_c$ with a universal exponent of 0.5, however, for local load sharing the transition becomes abrupt: at the critical load $t_f$ jumps to a finite value, analogous to second and first order phase transitions, respectively. The acoustic response of the bundle during creep is also studied.

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Fiber reinforced composites are of great technological importance due to their very good performance under extreme circumstances. Under high steady stresses these fiber composites may exhibit time dependent failure called creep rupture, which limits their life time and consequently has a high impact on the applicability of these materials for construction elements. Both natural fiber composites like wood \cite{1} and various types of fiber reinforced composites \cite{2,3,4} show creep rupture phenomena, which have attracted continuous theoretical and experimental interest over the past years. The underlying microscopic failure mechanism of creep rupture is very complex depending on several characteristics of the specific types of materials, and is far from being well understood. From technological point of view, one of the most important aspects of creep rupture is the statistics of life time (or time to failure) as a function of the external steady load, however, only a limited number of systematic experimental works is available for fiber reinforced composites \cite{5,6,7}, more information has been accumulated about natural fiber composites \cite{8,9,10}. In Ref. \cite{10} a theoretical model of creep rupture of brittle matrix composites reinforced with time dependent fibers was worked out, where the fibers are assumed to have a finite life time under a constant load in the spirit of the classical model of Coleman \cite{11}. For natural fiber composites a so-called damage accumulation model has been developed, which simply assumes that the time derivative of the accumulated damage depends exponentially on the external load history of the specimen \cite{12}.

In the present letter we study the creep rupture of fiber composites where the fibers have viscoelastic behaviour and the microscopic damage mechanism leading to creep rupture is the strain dependent breaking of fibers under the time evolution of the deformation of the system. Creep failure tests are usually performed under uniaxial tensile loading when the specimen is subjected either to a constant load $\sigma_o$ or to an increasing load (ramp-loading) and the time evolution of the damage process is followed by recording the strain $\varepsilon$ of the specimen and the acoustic signals emitted by microscopic failure events. In the present study we focus on the general aspects of creep rupture, i.e. the behaviour of the life time of the bundle as a function of the external load, its dependence on the range of load redistribution, furthermore, general aspects of the acoustic response of the bundle are considered without fitting the theoretical results to any specific materials.

In order to work out a theoretical description of creep failure of viscoelastic fiber composites we improve the classical fiber bundle model \cite{10,11} which has proven very successful in the study of fracture of disordered materials \cite{13,14}. Our model consists of $N$ parallel fibers having viscoelastic constitutive behaviour. For simplicity, the pure viscoelastic behaviour of fibers is modeled by a Kelvin-Voigt element which consists of a spring and a dashpot in parallel and results in the constitutive equation

$$\sigma_o = \beta \dot{\varepsilon} + E \varepsilon,$$

(1)

where $\beta$ denotes the damping coefficient, and $E$ the Young modulus of fibers, respectively. Eq. (1) provides the time dependent deformation $\varepsilon(t)$ of a fiber at a fixed external load $\sigma_o$.

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\[ \varepsilon(t) = \frac{\sigma_0}{E} \left[ 1 - e^{-Et/\beta} \right] + \varepsilon_0 e^{-Et/\beta}, \]

where \( \varepsilon_0 \) denotes the initial strain at \( t = 0 \). It can be seen that \( \varepsilon(t) \) converges to \( \sigma_0/E \) for \( t \to \infty \), which implies that the asymptotic strain fulfills Hook’s law.

If no fiber failure occurs Eq. (1) would fully describe the time evolution of the system. Motivated by the experimental observations of the acoustic response \( P(\varepsilon) \) of fiber composites during creep, we introduce a strain controlled failure criterion to incorporate damage in the model: a fiber fails during the time evolution of the system if its strain exceeds a damage threshold \( \varepsilon_d \), which is an independent identically distributed random variable of fibers with probability density \( p(\varepsilon_d) \) and cumulative distribution \( P(\varepsilon_d) = \int_{\varepsilon_d}^{\infty} p(x)dx \). Similar strain controlled breaking was recently used in Ref. [20]. Due to the validity of Hook’s law for the asymptotic strain values, the formulation of the failure criterion in terms of strain instead of stress implies that under a certain steady load the same amount of damage occurs as in the case of stress controlled failure, however, the breaking of fibers is not instantaneous but distributed over time. When a fiber fails its load has to be redistributed to the intact fibers. As the simplest approach, we assume global load sharing \( \int_0^{\varepsilon_d} p(x)dx \), i.e. after a failure event the excess load is equally distributed among the intact fibers, and hence, at a certain strain \( \varepsilon \) the load on the surviving fibers of the number \( N_s(\varepsilon) \) can be cast into the form \( \sigma(\varepsilon) = \sigma_0 N_s(\varepsilon) = \sigma_0/(1 - P(\varepsilon)) \). The time evolution of the system under a steady external load \( \sigma_0 \) is finally described by the equation

\[ \frac{\sigma_0}{1 - P(\varepsilon)} = \beta \varepsilon + E\varepsilon, \]

where the viscoelastic behaviour of fibers is coupled with the failure of fibers in a global load sharing framework.

For the behaviour of the solutions of Eq. (2) two distinct regimes can be distinguished depending on the value of the external load \( \sigma_0 \). When \( \sigma_0 \) is below a critical value \( \sigma_c \), Eq. (2) has a stationary solution \( \varepsilon_s \), which can be obtained by setting \( \dot{\varepsilon} = 0 \)

\[ \sigma_0 = E\varepsilon_s [1 - P(\varepsilon_s)]. \]

It means that until this equation can be solved for \( \varepsilon_s \) at a given external load \( \sigma_0 \), the solution \( \varepsilon(t) \) of Eq. (2) converges to \( \varepsilon_s \) when \( t \to \infty \), and no macroscopic failure occurs. However, when \( \sigma_0 \) exceeds the critical value \( \sigma_c \) no stationary solution exists, furthermore, \( \dot{\varepsilon} \) remains always positive, which implies that for \( \sigma > \sigma_c \) the strain of the system \( \varepsilon(t) \) monotonically increases until the system fails globally at a time \( t_f \).

In the regime \( \sigma_0 \leq \sigma_c \), Eq. (2) also provides the asymptotic constitutive behaviour of the fiber bundle which can be measured by controlling the external load \( \sigma_0 \) and letting the system relax to \( \varepsilon_s \). It follows from the above argument that the critical value of the load \( \sigma_c \) is the static fracture strength of the bundle which can be determined from Eq. (1) as \( \sigma_c = E\varepsilon_c [1 - P(\varepsilon_c)] \), where \( \varepsilon_c \) is the solution of the equation \( d\sigma_c/d\varepsilon_c = 0 \), as shown by Sornette [13]. Since \( \sigma_c(\varepsilon_c) \) has a maximum of the value \( \sigma_c \) at \( \varepsilon_c \), in the vicinity of \( \varepsilon_c \) it can be approximated as

\[ \sigma_c \approx \sigma_c - A(\varepsilon_c - \varepsilon)^2, \]

where the multiplication factor \( A \) depends on the probability distribution \( P \). A complete description of the system can be obtained by solving the differential equation Eq. (3). After separation of variables the integral arises

\[ t = \beta \int d\varepsilon \frac{1 - P(\varepsilon)}{\sigma_o - E\varepsilon [1 - P(\varepsilon)]} + C, \]

where the integration constant \( C \) is determined by the initial condition \( \varepsilon(t = 0) = 0 \).

The creep rupture of the viscoelastic bundle can be interpreted so that for \( \sigma_0 \leq \sigma_c \) the life time (or the time to failure) of the bundle is infinite \( t_f = \infty \), while above the critical load \( \sigma_0 > \sigma_c \), the global failure occurs at a finite time \( t_f \), which can be determined by evaluating the integral Eq. (2) over the whole domain of definition of \( P(\varepsilon) \). From the theoretical and experimental point of view it is very important how \( t_f \) depends on the external load above \( \sigma_c \). When \( \sigma_0 \) is in the vicinity of \( \sigma_c \), i.e. \( \sigma_0 = \sigma_c + \Delta \sigma_0 \), where \( \Delta \sigma_0 \ll \sigma_c \), it can be expected that the curve of \( \varepsilon(t) \) falls very close to \( \varepsilon_c \) for a very long time and the breaking of the system occurs suddenly. Hence, the total time to failure, i.e. the integral in Eq. (2), is dominated by the region close to \( \varepsilon_c \) when \( \Delta \sigma_0 \) is small. Making use of the power series expansion Eq. (3) the integral in Eq. (2) can be rewritten as

\[ t_f \approx \beta \int d\varepsilon \frac{1 - P(\varepsilon)}{\Delta \sigma_0 - A(\varepsilon_c - \varepsilon)^2}, \]

which has to be evaluated over a small \( \varepsilon \) interval in the vicinity of \( \varepsilon_c \). After performing the integration it follows

\[ t_f \approx (\sigma_0 - \sigma_c)^{-1/2}, \quad \text{for} \quad \sigma_0 > \sigma_c. \]

Thus, \( t_f \) has a power law divergence at \( \sigma_c \) with a universal exponent \(-\frac{1}{2}\) independent of the specific form of the disorder distribution \( p(\varepsilon) \).

For the purpose of explicit calculations we considered the case of a uniform distribution of the damage thresholds between 0 and a maximum value \( \varepsilon_m \), thus, \( p(\varepsilon_d) = 1/\varepsilon_m \) and \( P(\varepsilon_d) = \varepsilon_d/\varepsilon_m \). The stationary solution, the critical load and the corresponding critical strain can be obtained as \( \sigma_c = E\varepsilon_c [1 - \varepsilon_c/\varepsilon_m] \), \( \sigma_c = E\varepsilon_m/4 \), \( \varepsilon_c = \varepsilon_m/2 \) respectively. Finally, the solution of the integral Eq. (2) taking the initial condition also into account can be cast into the implicit form

\[ t = -\frac{\beta}{2E} \left\{ \frac{1}{1 - \frac{\sigma_0}{E\varepsilon_m}} \ln \left[ \frac{\varepsilon_m - \varepsilon}{\varepsilon_m} \right] + \frac{2\varepsilon}{\varepsilon_m} \right\} \]

\[ + \frac{\ln \left( \frac{1 + \sqrt{1 - \frac{4\sigma_0}{E\varepsilon_m}}}{2(1 - \frac{4\sigma_0}{E\varepsilon_m})} \right)}{\sigma_0\varepsilon_m}. \]
for $\sigma_o < \sigma_c$ (below the critical point), and

$$ t = \frac{\beta}{E} \left\{ -\arctg \frac{1}{\sqrt{\frac{4\sigma_o}{E\varepsilon_m} - 1}} \right\} \left[ \arctg \frac{2\varepsilon_m - 1}{\sqrt{\frac{4\sigma_o}{E\varepsilon_m} - 1}} \right] $$

(10)

$$ -\arctg \frac{1}{\sqrt{\frac{4\sigma_o}{E\varepsilon_m} - 1}} \right\} - \frac{1}{2} \ln \frac{E\varepsilon^2 - E\varepsilon_m\varepsilon + \sigma_o\varepsilon_m}{\sigma_o\varepsilon_m} \right\} $$

for $\sigma_o > \sigma_c$ (above the critical point). The behaviour of this analytic solution is illustrated in Fig. 1 for several different values of $\sigma_o$.

FIG. 1. The analytic solution $\varepsilon(t)$ given by Eqs. (10) for several values of $\sigma_o$ below and above $\sigma_c$. The critical strain $\varepsilon_c$ and the time to failure $t_f$ for one example are indicated.

The time to failure $t_f$ can be determined by setting $\varepsilon = \varepsilon_m$ in Eq. (10), which results in the form

$$ t_f \approx \frac{\beta\pi}{2} \sqrt{\frac{\varepsilon_m}{E}} (\sigma_o - \sigma_c)^{-1/2}, \quad (11) $$

in accordance with the above general arguments.

A further important general property of $E(t)$ that can be deduced from Eqs. (10) is that at the time to failure $t_f$ the deformation rate $d\varepsilon/dt$ diverges. For disorder distributions $P(\varepsilon)$ defined in a finite interval the exponent is universal $d\varepsilon/dt \approx (t_f - t)^{-1/2}$.

In order to obtain information about the gradual breaking of fibers during the creep process, in the experiments the acoustic signal emitted by breaking events in a short time interval is investigated. In our fiber bundle model the number of fibers $N_b(t)$ which have been broken up to time $t$ can be determined as $N_b(t) = NP(\varepsilon(t))$, and hence, its derivative provides the quantity

$$ \frac{1}{N} \frac{\partial N_b}{\partial t} = \frac{d\varepsilon}{d\varepsilon} \frac{d\varepsilon}{dt} = \frac{p(\varepsilon)}{\beta} E\varepsilon \left[ \frac{\sigma_o}{E[1 - P(\varepsilon)]} - 1 \right], \quad (12) $$

which is a measure for the acoustic response. The behaviour of Eq. (12) for the uniform distribution is illustrated in Fig. 2, where it can be observed that the acoustic activity, i.e., fiber breaking, practically disappears in the plateau region of $\varepsilon(t)$ (compare to Fig. 1), however, it diverges at $t_f$ due to the diverging deformation rate.

Since during a creep test $\varepsilon(t)$ is monitored from which $d\varepsilon/dt$ can be calculated, furthermore, $\partial N_b/\partial t$ is measured by means of acoustic emission techniques, Eq. (12) makes possible to determine experimentally the distribution of the failure thresholds $p(\varepsilon_d)$.

To complement the predictions of the analytic approach Monte Carlo simulations of the failure process have been performed using global load sharing (GLS) and local load sharing (LLS) for the stress redistribution. The GLS simulation of the creep failure process of a bundle of $N$ fibers proceeds as follows: (i) random breaking thresholds $\varepsilon_i$, $i = 1, \ldots, N$ were chosen according to a probability distribution $p$, then the thresholds were put into increasing order. (ii) Since the fibers break one-by-one, the actual load on the fibers after the failure of $i$ fibers is $\sigma_i = \sigma_o N/(N - i)$ where $i = 0, \ldots, N - 1$, and the time between the breaking of the $i$th and $i+1$th fibers reads as $t_i = \frac{-\beta}{E} \ln \left( \frac{\varepsilon_{i+1} - \sigma_i}{\varepsilon_i - \sigma_i} \right)$.

Finally, the time as a function of $\varepsilon$ can be obtained as $t(\varepsilon_i) = \sum_{j=0}^{i} t_j(\varepsilon_j)$ from which $\varepsilon_i(t)$ can be determined. The time to failure $t_f$ of a finite bundle is defined as the time of the failure of the last fiber. To test the validity of the power law behaviour of $t_f$ given by Eq. (8) simulations were performed with various distributions in the
framework of GLS. The results are presented in Fig. 3 where an excellent agreement of the simulations and the analytic results can be observed. The macroscopic strain of the system $\varepsilon(t)$ and the acoustic response obtained by simulations was also found to be in agreement with the analytic results.

To clarify how the damage process and the behaviour of $t_f$ is affected by the range of interaction among fibers, i.e. by the range of load sharing we performed simulations with LLS on a square lattice of $200 \times 200$ sites, redistributing the load of the failed fiber on its nearest neighbors. The critical load $\sigma_c$ was first determined as the static fracture strength of a dry fiber bundle with LLS assuming perfectly elastic behaviour for the fibers. Comparing the results of the LLS simulations to the global load sharing results it was observed that above $\sigma_c$ the failure of the viscoelastic bundle occurs much more abruptly than in the case of GLS.

Summarizing, we studied the creep rupture of viscoelastic fiber composites by enhancing the classical fiber bundle model. Varying the external load, two regimes of the creep process were revealed characterized by an infinite life time below, and by a finite one above the critical load. The transition between the two regimes was found to be continuous for global load sharing, while it is abrupt for localized load redistribution, analogous to a second and first order phase transition, respectively.

Varying $\sigma$ as a control parameter the two regimes of the creep rupture process are characterized by an infinite life time below $\sigma_c$ and by a finite one above the critical point. The nature of the transition between the regimes in the global and local load sharing models can be characterized by studying $1/t_f$ as a function of the control parameter $\sigma$. In Fig. 4 it can be observed that below the critical point, when no global failure occurs, $1/t_f$ is zero, while above $\sigma_c$ it takes a finite value for both LLS and GLS. However, the behaviour of $1/t_f$ in the vicinity of $\sigma_c$ is completely different in the two cases, for GLS the transition is continuous, while for LLS $1/t_f$ has a finite jump, analogously to a second and first order phase transition, respectively.

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