Toward fault-tolerant quantum computation without concatenation

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April 1, 2022

Abstract

It has been known that quantum error correction via concatenated codes can be done with exponentially small failure rate if the error rate for physical qubits is below a certain accuracy threshold. Other, unconcatenated codes with their own attractive features—improved accuracy threshold, local operations—have also been studied. By iteratively distilling a certain two-qubit entangled state it is shown how to perform an encoded Toffoli gate, important for universal computation, on CSS codes that are either unconcatenated or, for a range of very large block sizes, singly concatenated.

1 Codes and computation

At the end of the long tunnel of experimental quantum computing there is the light of accuracy thresholds provided by quantum error correcting codes. The lengthy computations necessary for efficient factorization and simulation of quantum systems are all of a sudden possible if error rates for qubits are reduced below certain critical values in a sufficiently parallel quantum computer. The quantum codes which give rise to this intriguing phase transition work on the same basic principle as their classical precursors—they keep information secure by using more physical (qu)bits per logical (qu)bit of the code. What is desired of quantum codes is that, as this redundancy is increased, there should be exponential improvement in the storage/computation fidelity of the code. Once the basic problem of correcting fully quantum errors was solved by the advent of quantum codes, a general recipe was obtained for expanding a given few-qubit code to achieve these exponential fidelity gains. This recipe generates a new higher level code by mimicking the old code, but with the role of physical qubits played by logical qubits of the old code. Detailed methods of error correction and computation have been obtained for versions of this recipe with an arbitrary number of iterations or “concatenations”—even when faults might occur in error correction processes themselves [1]–[7].

However, different frameworks for fault-tolerant error correction, e.g. topological quantum codes [8]–[11], might prove superior, and more general methods
for universal computation are desirable. Here, it is shown how to achieve universal fault-tolerant computation by construction of a Toffoli (C-C-NOT) gate on encoded qubits, provided the code possesses a normalizer operator (C-NOT) which factors over the qubits in a block.

Most known quantum error correcting codes can be defined by a set of operators, the stabilizer, each of which fixes every codeword. For a number of stabilizer codes capable of simple operations, like a bit flip $X_a$ or phase flip $Z_a$ on logical qubits $a, b, \ldots$, it is also known how to perform any “normalizer” operation—i.e. one that can be built from a sequence of operations involving only the C-NOT $X_{ab}$, $\pi/2$ phase shift, and Hadamard rotation $R_a$. Normalizer operations alone, however, are insufficient for universal quantum computation; a quantum computer with only normalizer operations can be simulated in polynomial time by a classical machine. A genuine quantum computer is realized either by the addition of a non-trivial one or two-qubit gate, like a single qubit rotation by an irrational multiple of $\pi$, or of a three-qubit gate like the Toffoli. Here, Shor’s procedure [1] for performing a Toffoli given the ancilla state $|\psi_3\rangle \equiv |000\rangle + |001\rangle + |010\rangle + |100\rangle$, (1)

will be rephrased and then a new method will be given for preparing this state, which does not rely on any concatenated structure within the code itself.

## 2 Toffoli gate from $|\psi_3\rangle$

It turns out one can perform a Toffoli gate on three qubits $ABC$ given normalizer operations and three ancilla qubits $abc$ prepared in the state $|\psi_3\rangle$. First consider the following construction, which uses one ancilla bit $c$. Letting $c$ start as $|0\rangle$, suppose one could perform a “majority vote” on $ABc$ so that, for example, $|010\rangle \rightarrow |000\rangle$ and $|110\rangle \rightarrow |111\rangle$. Equivalently, one might majority vote, but only carry out the effect on $c$, leaving $A$ and $B$ unchanged, so $|010\rangle \rightarrow |010\rangle$ and $|110\rangle \rightarrow |111\rangle$. Now just C-NOT $c$ into $C$ and disentangle $c$ from $ABC$. The result is exactly a Toffoli on $ABC$.

To majority vote on $ABc$, one measures $Z_A Z_B$ and $Z_B Z_c$. If both measurement results are +1, $ABc$ are already unanimous. Otherwise, the measurement results will reveal which bit is the odd-one-out. Unfortunately, these measurements have also revealed information about the initial state $AB$, in general collapsing it, inconsistent with the desired Toffoli gate, a linear operation. The solution is to perform a majority vote not directly on $ABc$ but on three ancilla qubits, which are first entangled with $AB$. Here is where $|\psi_3\rangle$ enters.

Given some arbitrary state of $ABC$, prepare $abc$ as $|\psi_3\rangle$ and perform the following operations: (I) C-NOT $A$ into $a$ and $B$ into $b$, and (II) majority vote on $abc$ (by measuring $Z_a Z_b$ and $Z_b Z_c$ and flipping the odd-bit-out if necessary). Suppose, for example, the measurement results from (II) are $Z_a Z_b = -1$ and $Z_b Z_c = +1$. All but 8 terms will be collapsed away of the total $2^3 \times 4 = 32$ terms in the initial 6-qubit state. These 8 terms, as they undergo (I) and (II),
are (suppressing bra-ket notation):

|     | I              | II            |
|-----|----------------|---------------|
|     | 00C₀100        | 00C₀100       |
|     | 01C₁001        | 01C₁011       |
|     | 10C₂000        | 10C₂000       |
|     | 11C₃010        | 11C₃010       |

where \( C_i = 0, 1 \). Note that all of the 8 possible bit values for \( ABC \) are equally represented, so that all information in the initial superposition of \( ABC \) is preserved (albeit decoherently). Now C-NOT \( \text{c} \) into \( C \). From the above table, this will flip \( C \) iff \( AB \) are 01—not iff \( AB \) are 11, as desired for the Toffoli. Applying \( \dot{X}_{BC} \) then gives the desired result. Finally, \( ABC \) must be disentangled from the ancillas \( abc \) to restore the coherence of the original state. This is accomplished by applying \( \dot{X}_{ab} \) and \( \dot{X}_{ac} \) and then measuring \( X_a \). If the result is +1, \( ABC \) are disentangled. If \(-1\), a phase error on the \( AB = 01 \) term has been introduced; it may be corrected by applying \( X_A \dot{Z}_{AB} X_A \), where \( \dot{Z}_{AB} \equiv R_B \dot{X}_{AB} R_B \) is the controlled-phase (C-PHASE) gate.

Had the measurement results for \( Z_a Z_b \) and \( Z_a Z_c \) been other than \(-1\) and \(+1\) respectively, as in the above example, it is straightforward to determine what gates must be applied in place of \( \dot{X}_{BC} \) and \( X_A \dot{Z}_{AB} X_A \).

The Toffoli now just requires preparation of the three-qubit state \( |\psi_3\rangle \). First observe that if one can prepare

\[
|\psi_2\rangle \equiv |00\rangle + |01\rangle + |10\rangle,
\]

\( |\psi_3\rangle \) may be obtained by preparing four qubits \( a b c d \) in the state \( |\psi_2\rangle |\psi_2\rangle \), measuring \( Z_b Z_c \), and performing a few simple normalizer operations. In particular, the measurement result \(-1\) gives the state

\[
|0010\rangle + |0100\rangle + |0101\rangle + |1010\rangle,
\]

which can be turned into \( |\psi_3\rangle |1\rangle \) by applying the C-NOTs: \( \dot{X}_{ac}, \dot{X}_{db}, \dot{X}_{ad}, \dot{X}_{bd} \), and \( \dot{X}_{cd} \) in that order.

3 \( |\psi_2\rangle \) from \( \rho(\alpha_i) \)

Let us define \( \rho(\alpha_1, \alpha_2, \alpha_3) \) as the (unnormalized) mixed state

\[
\begin{bmatrix}
1 & \alpha_1 \\
\alpha_2 & \alpha_3
\end{bmatrix}
\]

in the basis \( \{ |\psi_2\rangle, |11\rangle \} \), where \( |\alpha_3| < 1 \). It turns out, in the continuum of states \( \rho(\alpha_i) \), there is nothing special about \( |\psi_2\rangle \), obtained as \( \alpha_i \rightarrow 0 \). Being able to prepare any one state \( \rho(\alpha_i) \) with \( |\alpha_3| < 1 \) is sufficient to prepare \( |\psi_2\rangle \), hence to prepare \( |\psi_3\rangle \) and construct a Toffoli gate.
The state $|\psi_2\rangle$ is prepared by combining two copies of $\rho(\alpha_i)$ through measurement to obtain a new mixed state which is closer to $|\psi_2\rangle$ than before, and combining two of these to get one still closer, etc., progressively distilling $|\psi_2\rangle$ from the initial states. To start, prepare qubits $a\ b\ c\ d$ in the state $\rho_0 \otimes \rho_0$, where $\rho_0 = \rho(\alpha_i)$, and measure $Z_aZ_c$ and $Z_bZ_d$. Suppose the results are $+1$ and $+1$. Now perform $X_{ac}$ and $X_{bd}$ to disentangle $c\ d$. For pure states, this whole process would give $|\psi_2\rangle|\psi_2\rangle \rightarrow |\psi_2\rangle|00\rangle$ and $|11\rangle|11\rangle \rightarrow |11\rangle|00\rangle$, while either of the initial states $|\psi_2\rangle|11\rangle$ or $|11\rangle|\psi_2\rangle$ are inconsistent with the assumed measurement results. In terms of mixed states, this means $\rho_0 \otimes \rho_0 \rightarrow \rho_1 \otimes |00\rangle\langle 00|$ where $\rho_1$ is
\[
\begin{pmatrix}
1 & \alpha_1^2 \\
\alpha_2^2 & \alpha_3^2 \\
\alpha_4^2 & \alpha_3^2 \\
\alpha_2^2 & \alpha_1^2
\end{pmatrix}
\]
which is exactly $\rho(\alpha_2^2)$. Prepare another $\rho_1$ from two new $\rho_0$ states, and combine the two $\rho_1$ states by again measuring $Z_aZ_c$ and $Z_bZ_d$. Supposing the results are again $+1$ and $+1$, $c\ d$ are disentangled, leaving $a\ b$ in the state $\rho_2 = \rho(\alpha_i^4)$. Continuing this process through $N$ levels gives $\rho_N = \rho(\alpha_i^{2N})$. The whole procedure may be pictured as a tree of $\rho_L$ states, joining in pairs from level $L = 0$ to $L = N$ (see Fig. 1). The recursiveness is reminiscent of concatenated codes, but here the complexity appears in the auxiliary distillation process, not in the code itself.

The fidelity in preparing $|\psi_2\rangle$ is
\[
1 - \epsilon \equiv \frac{\text{tr}(\rho_N|\psi_2\rangle\langle \psi_2|)}{\text{tr}(\rho_N)} = \frac{3}{3 + \alpha_3^{2N}}
\]
very close to 1 if $|\alpha_3| < 1$. The number of (logical) qubits used to achieve this fidelity is $\sim 2^N$, which by (3) is $\sim \log \epsilon / \log |\alpha_3|$. This is the same kind of polylog scaling desired from the code itself (referring to the scaling of block size with desired failure rate $\epsilon$). Finding the number of operations on encoded qubits necessary to prepare $|\psi_2\rangle$ is not as easy, since the assumption that all $Z_aZ_b$, $Z_cZ_d$ measurement outcomes are $+1,+1$ requires repetition of the procedure a number of times before one expects such to occur.

To prepare a single $\rho_0$ state prepare two $\rho_{L-1}$ states and then combine them by measurements. If the measurement results are not $+1,+1$, just discard these states and keep trying. (This is not an optimal procedure, but it will suffice.) Therefore, if the chances of any one attempt succeeding are $P(L)$, the expected number of logical operations $G(L)$ necessary to prepare $\rho_L$ is $\sim 2G(L-1)/P(L)$. This assumes high confidence in the one pair of measurement results $+1,+1$, which should be the case since $a\ b\ c\ d$ are logical qubits. But even if there is a significant probability $\epsilon_m \gg \epsilon$ for any one measurement result to be in error, the distillation procedure can be made robust. Once a $+1,+1$ result is obtained, just repeat the measurements a number of times and accept the state only if, say, a majority of the results are $+1,+1$. To get $1 - \epsilon$ confidence in the measurement outcome, one must repeat $\sim \log \epsilon / \log \epsilon_m$ times. This implies
\[
G(L) \approx \frac{2}{P(L)} G(L-1) + \frac{\log \epsilon}{\log \epsilon_m}.
\]
It is not hard to see that $P(L)$ must increase with $L$, since this recursion relation implies that either $|\psi_2\rangle$ will quickly begin to dominate successive $\rho_L$ states, in which case $P(L) \to 1/3$, or $|11\rangle$ will dominate and $P(L) \to 1$. Both of these values are larger than $P(1)$, which can be calculated as a function of $|\alpha_3| < 1$ but is always bounded from below by 1/4. Iterating (3) with this bound gives

$$G(N) \sim 8^N \frac{\log \epsilon}{\log \epsilon_m} \sim \frac{(\log \epsilon)^4}{(\log |\alpha_3|)^3 \log \epsilon_m}. \quad (4)$$

Note that $G(N)$ is the total number of logical operations, but these can be done in parallel so that the actual distillation time is $\sim N \log \epsilon / \log \epsilon_m \sim \log(\log \epsilon) \log \epsilon / \log \epsilon_m$. The point is that even with the demand of a definite sequence of measurement results, time requirements still scale polylogarithmically with $\epsilon$. The crucial fact leading to this scaling is that the probability for getting the measurement results $+1,+1$ in combining two $\rho_L$ states is finite as $L \to \infty$. Thus one can prepare $|\psi_2\rangle$, $|\psi_3\rangle$, and execute a Toffoli gate if one can prepare one of the mixed states $\rho(\alpha_i)$ with $|\alpha_3| < 1$.

There are multiple ways of obtaining a state $\rho(|\alpha_3| < 1)$ for codes which are not too large. In fact, Shor’s own procedure for preparing $|\psi_3\rangle$ can do so. An alternative method will be presented here, applicable to codes possessing a non-trivial normalizer operation (here C-NOT) that is transversal, so the encoded operation factors into a number of independent operations on physical qubits. The method works by performing a very noisy measurement of the C-NOT operator.

## 4 Noisy measurement of C-NOT

Were it possible to measure C-NOT with high fidelity, one could easily prepare $|\psi_2\rangle = \rho(\alpha_i = 0)$. It turns out imperfect measurement of C-NOT is still capable of yielding $\rho(|\alpha_3| < 1)$.

For reference, the eigenstates of the C-NOT operator $X_{ab}$ are $|00\rangle$, $|01\rangle$, and $|10\rangle + |11\rangle$ with eigenvalue +1, and $|10\rangle - |11\rangle$ with eigenvalue −1. Let us first describe a fault-intolerant measurement procedure, that is, one which permits a single error to spread rampantly throughout a block. Prepare one physical ancilla bit $c_0$ as $|0\rangle$ and apply a certain three-bit gate $U_{a_i, b_i, c_0}$ bitwise over physical bits $a_i$ and $b_i$ in the blocks encoding $a$ and $b$ (but always using the bit $c_0$). $U$ is shown in Fig. 3. The first Hadamard rotation causes the Toffoli to flip $c_0$ just if $a_ib_i$ start in the $-1$ eigenstate $|10\rangle - |11\rangle$ of $X_{a_ib_i}$, and the second Hadamard undoes the effect on $b_i$.

Apply $U$ bitwise over $a\ b$ and measure $Z_{c_0}$. The result $Z_{c_0} = \pm 1$ is equivalent to the result that $X_{ab} = \prod_i X_{a_ib_i} = \pm 1$, so one has effectively measured $X_{ab}$. To understand this process in detail, expand the initial state of $a\ b$ in eigenstates.
of the operators $\hat{X}_{a,b,i}$ for $i = 1, \ldots, n$:

$$\sum_{x} C_{x} |x\rangle = \left( \sum_{w(x) = 0} + \sum_{w(x) = 1} \right) C_{x} |x\rangle$$

where $|x\rangle = |x_1 \cdots x_n\rangle$ and each $|x_i\rangle$ is one of the four eigenstates ($x_i = 1, 2, 3, 4$) of $\hat{X}_{a,b,i}$. The right hand sum is rearranged to segregate strings of even and odd weight. The weight function $w(x)$ equals the number (mod 2) of “4”s occurring in the string $x$, $x_i = 4$ corresponding to the $-1$ eigenstate $|10\rangle - |11\rangle$ of $\hat{X}_{a,b,i}$. Using the transversality of C-NOT and the definition of $w(x)$, one finds $\hat{X}_{ab} |x\rangle = (-1)^{w(x)} |x\rangle$. Thus the sum over strings with $w(x) = 0$ is the projection onto the +1 eigenspace of $\hat{X}_{ab}$, and the sum with $w(x) = 1$ is the projection onto the $-1$ eigenspace. It follows that the action of $U = \prod_i U_{a,b_i,c_i}$ is

$$U |x\rangle_{ab} |0\rangle_{c_0} = |x\rangle_{ab} |w(x)\rangle_{c_0},$$

which means measuring $Z_{c_0}$ is equivalent to measuring $\hat{X}_{ab}$.

This method of measurement is highly sensitive to errors; just one physical bit error can change $w(x)$ for an entire string of bits, making the measurement result erroneous. As the block size $n$ gets large, the chances of an even number of such errors occurring becomes nearly equal to the chances of an odd number occurring. Thus the measurement result tells very little about whether a +1 eigenstate or a $-1$ eigenstate of $\hat{X}_{ab}$ has been obtained. This little bit of information, however, turns out to be important for preparing $|\psi_2\rangle$. As mentioned the above procedure is fault-intolerant, since one physical bit phase error may infect $c_0$ and thus spread rampantly throughout the block. It can be made fault-tolerant by using an ancilla $c$, which is not just one bit, but a superposition of $n$ physical bits over all even weight strings (“weight” is now in the sense of counting “1”s). Such a superposition is prepared as

$$|\text{even}\rangle_c = \left( \prod_i R_{c_i} \right) (|0\cdots0\rangle_c + |1\cdots1\rangle_c).$$

The gate $U_{a,b,c_i}$ will be applied bitwise across $abc$ so that a single error in one block can at most spread to one bit in each of the other two blocks. Acting bitwise on $|x_1\rangle_{c_1}$ through $|x_n\rangle_{c_n}$, $U$ will flip a number of bits in the initial $c$ state equal (mod 2) to exactly $w(x)$. Thus

$$U |x\rangle_{ab} |\text{even}\rangle_c = \begin{cases} |x\rangle_{ab} |\text{even}\rangle_c & w(x) = 0 \\ |x\rangle_{ab} |\text{odd}\rangle_c & w(x) = 1 \end{cases} \quad (5)$$

Measuring $Z_{c_i}$ bitwise over $c$ with the result $\prod_i Z_{c_i} = \pm 1$ is now equivalent to measuring $\hat{X}_{ab}$ with the result $\hat{X}_{ab} = \pm 1$. Note that a single phase error in the $n$-bit cat state, or equivalently a bit flip in the sum over even weight strings, will change this sum into one over odd weight strings, again altering the measurement result while still projecting the state onto one of the eigenspaces of
\( \hat{X}_{ab} \). So the measurement procedure is now fault-tolerant, but the measurement result is still highly sensitive to single bit errors, giving little information about which eigenspace the state \(|\rangle_{ab}\) collapses into.

One can also perform a noisy measurement of the C-PHASE operator \( \hat{Z}_{ab} \). The action of \( \hat{Z}_{ab} \) is just to apply a minus sign if \( ab \) are in \(|11\rangle\), which is unitarily equivalent to \( \hat{X}_{ab} \) through the basis change \( R_b \). To measure \( \hat{Z}_{ab} \) first apply \( R_b \), then measure \( \hat{X}_{ab} \) by the above method, and reapply \( R_b \). These procedures may be adapted, by changing the bitwise operation \( U \), to noisy measurement of such operators as \( \hat{X}_{ab}\hat{X}_{cd} \), \( \hat{Z}_{ab}\hat{Z}_{cd} \), and \( \hat{Z}_{ab}\hat{Z}_{bc} \).

5 Preparation of \( \rho(\alpha_i) \)

First prepare two logical qubits \( ab \) as \((|0\rangle + |1\rangle)^2\) and measure \( \hat{Z}_{ab} \) by the method given above, making use of an ancilla block \( c \). If the measurement result were +1 and all qubits were error-free, one would have prepared exactly \(|\psi_2\rangle\). But this will be changed by errors (i.e. decoherence, gate errors, or measurement errors) occurring either to the bits encoding \( a \) and \( b \) or to those of the cat-like ancilla \( c \) used in the noisy measurement procedure. In fact \( c \) is especially vulnerable because it is not protected by any code at all—a single bit error anywhere in \( c \) can reverse the observed measurement result for \( \hat{Z}_{ab} \).

Depending on whether errors are unitary or decoherent, this yields a coherent or incoherent superposition of \(|\psi_2\rangle \) and \(|11\rangle\), which will be shown to be of the form \( \rho(\alpha_3 < 1) \) in either the unitary or decoherent case, hence a candidate for distillation.

Phase errors to the bits of \( ab \) cannot be transmitted to \( c \) by the above procedure, so are irrelevant. Bit flip errors to \( ab \) can be transmitted but are equivalent to bit flip errors occurring to the bits of \( c \) so all errors can be effectively regarded as occurring to \( c \) alone. Let us first consider the case of (uncorrelated) errors purely decoherent in the Pauli basis \( \sigma^m \), so that each qubit \( c_i \) suffers no error, a phase error, a bit error, or both errors—each with some fixed classical probability.

Phase errors in \( c \) can affect only the relative sign of terms in \(|\text{even}\rangle_c \) and \(|\text{odd}\rangle_c \) of (5), hence are extinguished once the \( Z_{c_i} \) measurements are made. Depending on whether \( c \) is attacked by an even or odd number of bit errors, the measured eigenvalue of \( \hat{Z}_{ab} \) will be inferred either rightly or wrongly from the outcome of the \( Z_{c_i} \) measurements. So given the result \( \prod_i Z_{c_i} = +1 \), an even number of bits errors will yield \(|\psi_2\rangle \) as desired; however, an odd number will yield \(|11\rangle\) unbeknownst to us.

If each \( c_i \) suffers a bit error with probability \( p_i \), the difference between the chances of an even number of bit errors and of an odd number is

\[
\left( \prod_{i=1}^{n} \sum_{x_i=0,1} (-1)^{x_i}p_i^{x_i}(1-p_i)^{1-x_i} \right) = \prod_i (1 - 2p_i). \tag{6}
\]

Given that these two probabilities sum to 1, this implies the preparation proce-
dure will yield not exactly $|\psi_2\rangle$, but the state $\rho(0, 0, \alpha_3)$ with
\[
\alpha_3 = \frac{1 - \prod_i (1 - 2p_i)}{1 + \prod_i (1 - 2p_i)} \approx 1 - 2 \prod_i (1 - 2p_i)
\]
where the last expression holds for large $n$. It thus appears that one cannot
tell whether or not $|\alpha_3| < 1$ for a given ancilla block $c$ if even a few of its
bits might have $p_i > 1/2$. This is true even though current codes themselves
are completely robust to these “defective” bits so long as their distribution is
suitably uncorrelated and infrequent at the level of the code’s threshold error
rate. Still what has to be considered for the distillation process is not a single
ancilla block $c$, giving rise to one $|\psi_2\rangle$-like state, but a sequence of such blocks,
each with its own set of defective bits and consequent value of $\alpha_3 = \alpha_3^{(m)}$, where
$m$ runs from 1 to $2^N$, the number of $|\psi_2\rangle$-like states input to the distillation
process.

The fidelity in distilling $|\psi_2\rangle$ is that given by (2) with $\alpha_3^N$ replaced by
\[
\prod_m \alpha_3^{(m)} \approx e^{-2^{N+1} \langle \prod_i (1-2p_i) \rangle}
\]
where $\langle \cdots \rangle$ is an average over the ensemble of $c$ blocks. Assuming errors are un-
correlated between different $c$ blocks, the average factorizes and the distillation fidelity is
\[
1 - \frac{1}{2^N} e^{-2^{N+1} \langle \prod_i (1-2p_i) \rangle}
\]
Here only the ensemble averaged bit flip error rates appear. Assuming the loca-
tions of defective qubits are uncorrelated between different $c$ blocks, their
$p_i > 1/2$ contributions are simply weighted out in the average. This means infrequent defective bits no longer pose a problem, owing to the distributed nature of the distillation process. Defining an average error rate $p = \langle p_i \rangle$, the above product is approximately $e^{-2pn}$. In order that the resulting fidelity be comparable to that of the code itself, which is $\sim 1 - \exp(-Kn^\beta)$ for some power $\beta$ and constant $K$, the number of physical qubits used in distillation is roughly
\[
n2^N \sim n^{1+\beta} e^{2pn},
\]
which puts a limit on the block size $n$ of the code being used, since the number of qubits used in distillation should not grow exponentially with block size. Thus $n$ cannot be larger than
\[
n \sim \frac{1}{p} \log \frac{1}{p}.
\]
(7)

The opposite case of purely unitary errors is the same in its result. Here, errors comprise a set of unitary operators which act on the $c_j$ respectively. Each such operator can be written in the Pauli basis and put in the form
\[
E_j = (A_j 1 + iB_j \sigma^z) + i\sigma^x(C_j 1 + iD_j \sigma^z),
\]
where $A_j, B_j, C_j, D_j$ are real. Assuming low error rates, so $\prod_i |A_i| \gg \prod_i |B_i|$ and likewise with $C_i$ and $D_i$ in place of $B_i$, one can show that these errors take $c$ from its prepared state $|\text{even}\rangle$ to

$$
\prod_i E_i|\text{even}\rangle = |\text{even}\rangle + i \tan(\Sigma_C)|\text{odd}\rangle,
$$

where $\Sigma_C \approx \sum_i \tan^{-1}(C_i/A_i)$. This leads to preparation of the state $|\psi_2\rangle + i \tan(\Sigma_C)|\psi_2\rangle$ in place of $|\psi_2\rangle$. But this state is precisely $\rho(\alpha_i)$ with $\alpha_{1,2} = i \tan(\Sigma_C)$ and $\alpha_3 = -\tan^2(\Sigma_C)$, which can be distilled to $|\psi_2\rangle$ if $\tan^2(\Sigma_C) < 1$.

For large $n$, $\Sigma_C$ will be very sensitive to the error amplitudes $C_i$, and in practice one would have no way of knowing whether $\tan(\Sigma_C) < 1$ or not. This is the same problem noted above in the case of pure decoherence, and it too disappears when one realizes that the distillation input is not a single state $|\psi_2\rangle + i \tan(\Sigma_C)|\psi_2\rangle$ but an ensemble of such states each generated by a different set of blocks $a b c$. The distillation fidelity is now given by (3) with $\alpha^2_n$ replaced by

$$
\prod_{m=1}^{2N} \tan^2(\Sigma_C) \approx e^{2N+1}\langle \log(\tan \Sigma_C) \rangle,
$$

where $\langle \cdots \rangle$ again averages over the ensemble of $c$ blocks. Using this, the definition of $\Sigma_C$, and expanding the logarithm gives $\langle \log(\tan \Sigma_C) \rangle$ as

$$
- \sum_{k=0}^{\infty} \frac{2}{2k+1} \prod_i \langle e^{-2(2k+1)\tan^{-1}(C_i/A_i)} \rangle,
$$

(8)

The factorization follows assuming independence of errors between different qubits in each $c$ block. Now expand the exponential in a power series. If the error distributions are all such that the bit flip amplitudes $C_i$ and $-C_i$ are equally likely to occur, the expectation values of the odd terms in the power series vanish. (If the distributions are otherwise, one might use the computational basis $\{|0\rangle, -|1\rangle\}$ instead of $\{|0\rangle, |1\rangle\}$ for the qubits in half the $c$ blocks, so that the same physical error would correspond to the bit flip amplitude $-C_i$ as often as it would to $C_i$.) The even terms may then be resummed and the expectation value in (8) becomes

$$
\langle \cos(2(2k+1)\tan^{-1}(C_i/A_i)) \rangle \equiv \cos \left( 2(2k+1)\sqrt{\langle p_i \rangle} \right),
$$

where $\langle p_i \rangle = \langle C_i^2/A_i^2 \rangle$ to lowest order in $C_i/A_i$, hence $\langle p_i \rangle$ can be taken roughly as a bit flip error rate. For small $k$ the cosine functions will be close to 1, and the product over $i$ will be greatest. As $k$ gets large, the cosines will sample their full range and the product will be highly suppressed. Therefore the cosine above can be replaced by $\exp(-2(2k+1)^2\langle p_i \rangle)$, as if $k$ were always small, giving the main contribution to the sum:

$$
\langle \log(\tan \Sigma_C) \rangle \sim - \sum_{k=0}^{\infty} \frac{2}{2k+1} e^{-2(2k+1)^2\sum_i \langle p_i \rangle} < -2e^{-2pn},
$$

9
where $p$ is again the average of $\langle p_i \rangle$ over $i = 1, \ldots, n$; this is basically the same result as obtained for purely decoherent errors. Thus again (6) gives the largest allowed block size in the regime where the resources needed for distillation scale polynomially with block size.

6 Progressive concatenation

In case higher fidelity is desired of the code than (6) allows, the above methods by themselves are insufficient and one must resort to concatenation. However, in conjunction with these methods, an unconventional, exponentially weaker form of concatenation can be used. A usual concatenated code is self-similar, the same abstract code (e.g. the 7-qubit code) being used at each level in its recursion. Here one is free to increase the block size at each level, as long as (6) is satisfied level-by-level. In these “progressive” concatenated codes, many fewer levels are necessary given a desired fidelity $1 - \epsilon$.

In particular, for one error correction algorithm [11] in the context of lattice codes, some reasonable parameters are $p = p_c/10 \ll 10^{-3}$, so that $n = 1000$ is acceptable by (6). Here $\epsilon \sim (p/p_c)^n \beta$ where $\beta = \log_9 2 \approx 0.315$, which gives $\epsilon \sim 10^{-9}$. So, if the desired fidelity is below $1 - 10^{-9}$, no concatenation is necessary. Otherwise, one can begin concatenating.

Consider a single concatenation of a chosen code. Physical qubits with error rate $\epsilon_0 = p$ are arranged in code blocks of size $n_1$, and these blocks are themselves arranged in blocks of size $n_2$. The effective error rate at this higher level is just the failure rate of blocks at the lower level:

$$\epsilon_1 \sim (\epsilon_0/p_c)^{Kn_1^\beta},$$

where $p_c$, $K$, and $\beta$ come from details of the code being concatenated. The code as a whole has failure rate

$$\epsilon_2 \sim (\epsilon_1^*/p_c)^{Kn_2^\beta},$$

where $\epsilon_1^*$ includes the effect of storage errors described by $\epsilon_1$ and also gate errors associated with the operations necessary to perform error correction. Thus $\epsilon_1^*$ will have the same form as $\epsilon_1$ in (6) but with $\epsilon_0 = p$ replaced by a physical qubit error rate $\epsilon_0^*$ including the effect of these additional errors. In other words one must deal not only with a storage error threshold but also with a gate error threshold associated with the computations necessary for error correction. Of course, if one intends to use the logical qubits stored by the code for actual computations, this would be necessary anyway.

Assuming the effective error rate $\epsilon_0^*$ is still below threshold, a single concatenation of the code in the above example gives

$$\epsilon_2 = \left( (\epsilon_0^*/p_c)^{Kn_1^\beta} \right)^{Kn_2^\beta} \sim 10^{-830},$$

where the values $K = 1$, $\beta = \log_9 2$, and $\epsilon_0^* = p_c/5$ have been used. The block sizes $n_1 = 1000$ and $n_2 = 2 \cdot 10^7$ were chosen to be consistent with $n_L \sim$
(1/\epsilon_{L-1}^*) \log(1/\epsilon_{L-1}^*), i.e. the condition applied at each level. Thus, as long as one does not require a fidelity better than 1 - 10^{-830}, a single concatenation is sufficient given the above parameters.

Because the block sizes \( n_L \) may increase so rapidly, the number \( N \) of levels necessary for a desired fidelity \( 1 - \epsilon \) scales differently than in usual concatenation, in which \( N \sim \log(\log(1/\epsilon)) \equiv \log^{(2)}(1/\epsilon) \). For these progressive concatenated codes, \( N \) is determined self-consistently by \( N \sim \log^{(N)}(1/\epsilon) \). One might wonder about the asymptotic behavior of thresholds and fidelities as \( N \to \infty \), taking into account the reciprocal effects between progressive concatenation and distillation; however, this seems irrelevant given the smallness of \( \epsilon \) already at \( N = 2 \).

7 Conclusion and remarks

It has been shown how to construct a Toffoli gate, the major part of a universal gate set, for any CSS code by virtue of its possession of a normalizer operator which factorizes over bits of a code block. For the particular code considered above very high fidelity, hence large block size, may require a single concatenation, the primary effect of which would be to unify the error rate thresholds for error correction and computation. The method presented derives from the ability to perform noisy measurement of the (factorable) C-NOT operator on logical qubits. This allows the preparation of a mixed state \( \rho(|\alpha_3| < 1) \), which can then be used in a recursive scheme to distill a certain two-qubit entangled state \( |\psi_2\rangle \) of logical qubits, which in turn is easily transformed into a three-qubit state \( |\psi_3\rangle \) that enables the performance of one Toffoli gate on three separate logical qubits.

The point of this construction is to aid the cause of long quantum computations that do not rely on concatenated codes. In doing so one trades complexity within the code itself for complexity in processes auxiliary to the code with the idea that these auxiliary processes may be performed elsewhere, fostering division of labor and economy of scale.

Thanks to Daniel Gottesman, John Preskill, and Atac Imamoglu for comments on this paper. This work has been supported by ARO grant DAAG55-98-1-0366.

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Figure 1: Combining $\rho_0$ states to prepare $\rho_N$ (above, $N = 3$).

Figure 2: The operation $U$ on physical qubits $a, b, c$. 