Acoustic radiation force and torque on an absorbing compressible particle in an inviscid fluid

Glauber T. Silva$^{1}$

$^1$Physical Acoustics Group, Instituto de Física, Universidade Federal de Alagoas, Maceió, AL 57072-900, Brazil.

Exact formulas of the acoustic radiation force and torque exerted by an arbitrary time-harmonic wave on an absorbing compressible particle that is suspended in an inviscid fluid are presented. It is considered that the particle diameter is much smaller than the incident wavelength, i.e. the so-called Rayleigh scattering limit. Moreover, the particle absorption assumed here is due to the attenuation of compressional waves only. Shear wave propagation inside and outside the particle is neglected, since the inner and outer viscous boundary layer of the particle are supposed to be much smaller than the particle radius. The obtained radiation force formulas are used to establish the trapping conditions of a particle by a single-beam acoustical tweezer based on a spherically focused ultrasound beam. In this case, it is shown that the particle absorption has a pivotal role in single-beam trapping. Furthermore, the axial radiation torque induced by a Bessel vortex beam is calculated. It is found that only the first-order Bessel vortex beam can generate the radiation torque on a small particle. In addition, numerical evaluation of the radiation force and torque exerted on a benzene and an olive oil droplet suspended in water are presented and discussed.

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I. INTRODUCTION

An increasing interest on acoustic radiation force has arisen as a result of the concept of acoustical tweezers was developed. An acoustical tweezer can be accomplished by means of an ultrasound standing wave or a focused beam. Furthermore, acoustophoretic devices are also relies on the acoustic radiation force actuating on micro-sized particles. A key point in developing and enhancing the use of acoustical tweezers and acoustophoretic devices is to understand how an employed acoustic wave generates radiation force and torque on a small particle (i.e. a particle whose radius is much smaller than the incident wavelength).

The first study on the radiation force exerted on spherical particle was presented in a seminal work by King. An extension of this work considering an incident spherical wave was provided by Emblenton. It was noticed that the particle can be either attracted to or repulsed by the wave source depending on their relative distance. Effects of particle compressibility on the radiation force was accounted by Yosioka et al. Subsequently, Gorkov derived, based on a fluid dynamics approach, a general radiation force formula exerted on a particle by a plane wave and any stationary acoustic wave. Nyborg has also considered the radiation force exerted by a spherical wave on a rigid sphere. He showed that the traveling part of the spherical wave can be described with Gorkov’s theory by adding a correction term. Wu et al. analyzed the radiation force produced by a Gaussian beam and a focused piston on an on-axis particle. Recently, Marston obtained the radiation force generated by the interaction of a zero-order Bessel beam and an on-axis particle. He gave the conditions to produce a negative force (i.e. opposite to the beam’s propagation direction) on the particle with a Bessel beam. Aazarpeyvand theoretically study the radiation force on an on-axis porous aluminum sphere. In this case, the increase of porosity degrades the radiation force on a small particle.

Commonly, acoustic particle trapping is described through Gorkov’s theory. In turn, this theory is valid when the Stokes’ boundary layer around the particle satisfies $\delta_0 = \sqrt{2\pi\nu_0/\omega} \ll a$, where $a$ is the particle radius, $\nu_0$ is the kinematic viscosity of the host fluid and $\omega$ is the angular frequency of the wave. When $\delta_0 \sim a$, effects of the host fluid viscosity become relevant and the radiation force may considerably deviate from from inviscid theories. It should be further noticed that if the particle is made of a viscous fluid, a shear wave may develop inside it. The propagation of this wave can be confined within an internal viscous bondary layer, which is given by $\delta_1 = 2\nu_1/\omega$, where $\nu_1$ is the kinematic viscosity of the particle. Here, we will consider particles for which $\delta_1 \ll a$. Thus, shear wave propagation effects on the radiation force and torque will be neglected.

Another important phenomenon in the acoustic radiation force is the absorption of longitudinal waves inside the particle. Radiation force on an absorbing particle was first discussed by Westervelt. He established a relation between the radiation force on a sphere due to a plane wave with the scattering and the absorption cross-sections. A more extensive analysis of this problem was performed by Löfstedt et al. in which it is shown that absorption enhances the radiation force. So far, the radiation force theory in the Rayleigh scattering limit involve a symmetry consideration between the particle and the incident wave, i.e. the on-axis configuration. Moreover, Zhang et al. developed a geometric interpretation of the radiation force exerted by a zero-order Bessel beam on an on-axis sphere in terms of the absorption, scattering and extinct cross-sections. A similar study on a broader class of acoustic beams was also conducted.It is believed that the concept of acoustical tweezers was introduced by Westervelt in 1969. The first study on the radiation force exerted on a single particle by a focused Gaussian beam was performed by Lofstedt et al.

a) Electronic address: glauber@pq.cnpq.br
The interaction of acoustic wave with an absorbing particle may induce a radiation torque. Heffner et al.\cite{22} showed that an axial acoustic radiation torque can be developed on an absorbing target by a paraxial vortex beam. Moreover, the axial radiation torque on an axisymmetric object suspended in an inviscid fluid was theoretically studied by Zhang et al.\cite{23}. A general model for the three-dimensional radiation torque exerted by an acoustic beam of arbitrary wavefront was provided by Silva et al.\cite{24}.

The purpose of this article is to provide exact formulas of the acoustic radiation force and torque generated by the scattered pressure is expanded in a partial-wave series in spherical coordinates \((r, \theta, \varphi)\) with respect to the particle center. Accordingly\cite{25,26}:

\[
p_s = p_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} s_n a_n^{m} h_n^{(1)}(kr) Y_n^m(\theta, \varphi),
\]

where \(p_0\) is peak pressure magnitude of the incident wave, \(k = \omega/c_0\) is the wavenumber, \(h_n^{(1)}(kr)\) the \(n\)-th order spherical Hankel function of first-type, and \(Y_n^m(\theta, \varphi)\) is the \(n\)-th order and \(m\)-th degree spherical harmonic. The quantities \(a_n^{m}\) and \(s_n\) are the beam-shape and the scaled scattering coefficients, respectively.

The beam-shape coefficients are the weights of the partial-waves in the incident beam expansion. They will be determined in terms of the incident pressure and particle velocity up to the quadrupole approximation in Appendix A. Note that \(a_n^{m} = 0\) if \(n < 0\) or \(|m| > n\).

The scale scattering coefficient is obtained by applying the continuity condition of pressure and particle velocity across the particle’s surface at \(r = a\). These conditions lead to following coefficient:

\[
s_n = -\det \left[ \begin{array}{cc} j_n(k_1 a) & j_n(k_2 a) \\ j'_n(k_1 a) & j'_n(k_2 a) \end{array} \right] \det \left[ \begin{array}{cc} h_n^{(1)}(k_1 a) & j_n(k_2 a) \\ h_n^{(1)}(k_2 a) & j_n(k_1 a) \end{array} \right]^{-1},
\]

where \(k_1\) in the inner wavenumber of the particle, \(\gamma = \rho_0 k_1/(\rho_1 k)\), and the prime symbol indicates differentiation. In the Rayleigh scattering regime, the particle is much smaller than the incident wavelength or \(ka \ll 1\).

Our analysis is limited to weak-absorption, i.e. \(\alpha \omega^3 \ll \omega/c_1\) is taken. We may neglect shear wave propagation effects inside the particle, when the inner viscous boundary layer is much smaller than the particle radius, \(\delta_i/a \ll 1\). Thus, we assume as an upper-limit \(\delta_i = 0.1a\). Since we have established that \(ka \leq 0.3\) and given that \(\delta_i = \sqrt{2\nu_i/\omega}\), then the particle size factor \(ka\) should satisfy

\[
10\sqrt{2\nu_i/\omega} \leq ka \leq 0.3.
\]

We now expand the scaled scattering coefficients \(s_0\) and \(s_1\) in Eq. (2) for \(a \rightarrow 0\) and \(ka \ll \rho_1/\rho_0\). Using MATHEMATICA software\cite{26}, we find the first relevant terms of this expansion as

\[
s_0 = -\frac{2\alpha_0 \omega^{\nu - 1} \rho_0 c_0^2 (ka)^3}{\rho_1 c_1^3} \frac{3}{9} - f_0^2 (ka)^6 - if_0 (ka)^3,\]

\[
s_1 = -2\rho_0 \rho_1 c_1 \alpha_0 \omega^{\nu - 1} \frac{36}{5(\rho_0 + 2\rho_1)^2} (ka)^5 - f_1^2 (ka)^6 - if_1 (ka)^3 / 6,
\]

where \(f_0 = 1 - \rho_0 c_0^2 / \rho_1 c_1^2\) and \(f_1 = 2(\rho_1 - \rho_0)/(2\rho_1 + \rho_0)\) are the monopole and the dipole scattering factors, respectively. It is worthy to notice that we kept the lowest term that involves the absorption coefficient \(\alpha_0\) and those for a non-absorbing particle, which depends on \(f_0\) and \(f_1\). This will enable us to decompose the radiation force as a sum of two components, one related to the particle absorption and the other as though the particle was lossless.

II. RAYLEIGH SCATTERING

Consider an acoustic wave of angular frequency \(\omega\) propagating in an inviscid fluid of density \(\rho_0\) and speed of sound \(c_0\). The wave is scattered by a compressional fluid particle of radius \(a\), density \(\rho_1\), and speed of sound \(c_1\). The total pressure in the host fluid is \([p_i(r) + p_s(r)] e^{-i\omega t}\), where \(i\) is the imaginary unit, and \(p_i\) and \(p_s\) are the incident and the scattered pressure amplitudes, \(r\) is the position vector, and \(t\) is denotes time. Due the particle symmetry, the scattered pressure is expanded in a partial-wave series in spherical coordinates \((r, \theta, \varphi)\) with respect to the host fluid, while axial trapping also reckons on the particle absorption. In addition, we derive the radiation torque produced by a Bessel vortex beam on these droplets in the on-axis configuration. It is shown that only the first-order Bessel vortex beam can generate radiation torque in the Rayleigh scattering limit.
III. ACOUSTIC RADIATION FORCE

The linear momentum carried by the incident wave is transferred to the suspended particle generating the so-called acoustic radiation force. Based on the partial-wave expansion of the incident and the scattered fields, it has been shown that the radiation force exerted on a sphere by a time-harmonic beam with arbitrary wave-front is given by

\[ F = \frac{\pi a^2 I_0}{c_0} Y, \]

where the symbol * means complex conjugation and ‘Im’ signifies the imaginary-part of. We stress here that \( Y_x \), \( Y_y \), and \( Y_z \) are real quantities. Note that the quadrupole moment of the incident beam \( a_2^0 \) is necessary to compute the radiation force. This result has been also noticed in the axial radiation force exerted on a particle by a Bessel beam.

By substituting the equations in \((a3)\) into Eqs. \((8)\) and \((9)\), one obtains

\[ Y_x + i Y_y = \frac{i}{2\pi(ka)^2} \left[ \sqrt{\frac{2}{3}} \left( s_0 + s_1 + 2s_0s_1 \right) a_0^0 a_1^1 + (s_0^* + s_1 + 2s_0^*s_1) a_0^0 a_1^{-1} \right] \]

\[ + \frac{1}{15} \left( 2 + m \right) \left( 3 + m \right) \left( s_1 a_1^m a_2^{m+1*} + s_1^* a_1^{-m} a_2^{-m-1} \right), \]

\[ Y_z = \frac{1}{\pi(ka)^2} \text{Im} \left[ \sqrt{\frac{1}{3}} \left( s_0 + s_1 + 2s_0s_1 \right) a_0^0 a_1^0 + \sum_{m=-1}^{1} \sqrt{\left( 2 - m \right) \left( 2 + m \right)} s_1 a_1^m a_2^{2*} \right]. \]

for which we have used the relation

\[ \rho_0 \nabla \cdot v_i^* v_i = \nabla \left( \frac{\rho_0 |v_i|^2}{2} - \frac{|p_i|^2}{2\rho_0 c_0^2} \right), \]

where \( |\cdot| \) denotes the vector norm. Substituting Eq. \((13)\) into Eq. \((10)\) yields

\[ F = -\frac{2\pi}{k^2 c_0} \text{Re} \left[ \frac{3is_1}{k} \rho_0 c_0 [v_i(0) \cdot \nabla] v_i^*(0) \right] \]

\[ + (s_0 + 2s_0 s_1^*) p_i(0) v_i^*(0) \].

This equation was also obtained in terms of the incident pressure by Sapožničko et al.

By using Eqs. \((13)\) and \((10)\) into Eq. \((15)\), we find that the radiation force can be decomposed into three contributions, namely gradient, scattering, and absorbing components. The gradient radiation force is given by

\[ F_{\text{grad}} = -\nabla U(0), \]

where

\[ U = \pi a^3 \left( f_0 \frac{|p_i|^2}{3\rho_0 c_0^2} - f_1 \frac{\rho_0 |v_i|^2}{2} \right) \]

is the radiation force potential function as obtained by Gorkov. This force does not depend on the ultrasound absorption by the particle. Moreover, the gradient radiation force is relevant whenever the incident beam has a stationary spatial distribution such as standing waves or focused beams. On the other hand, the scattering radiation force reads

\[ F_{\text{sca}} = \frac{4\pi a^2}{9c_0} \left( f_0^2 + f_0 f_1 + \frac{3f_1^2}{4} \right) (ka)^2 \mathbf{T}(0), \]
where \( I = (1/2)\text{Re}\{p_i V_i\} \) is the incident intensity averaged in time. Equation (18) generalizes the previous result obtained by Gorkov for the radiation force produced by a plane traveling wave on a non-absorbing particle. Finally, the absorbing radiation force is expressed as

\[
F_{\text{abs}} = \frac{8\pi a^3 \rho_0 \alpha \omega^{-4}}{3\rho_1 c_1} I(0). \tag{19}
\]

This force varies with frequency like the absorption itself. We see that both the scattering and the absorbing radiation forces depend on the averaged incident intensity. Therefore, they will be relevant whenever the incident wave has a traveling part, such as a plane wave, for instance. For a standing wave, the scattering and the absorption radiation forces are much smaller than their gradient counterpart.

According to Eqs. (18) and (19), the magnitude ratio of the absorbing force to the scattering force is given by

\[
\frac{\| F_{\text{abs}} \|}{\| F_{\text{sc}} \|} = \frac{6\rho_0 \alpha_{\text{abs}}}{\omega^4} \frac{a^3}{\rho_1 c_1} \frac{v^{-4}}{1 + f_0 f_1 + 4 f_1^2/4}. \tag{20}
\]

Therefore, the importance of the absorbing radiation force decreases with frequency as \( \omega^{-4} \), since \( v \leq 2 \). Though, this force increases with respect to the scattering radiation force with the particle radius to the power third. When \( v = 2 \) and the incident beam is a plane wave, we recover the result given in Eq. (52) in Ref. [18], which in turn has an apparent typographical error (an extra factor of 1/3).

IV. ACOUSTIC RADIATION TORQUE

An acoustic beam can produce a time-averaged torque, known as radiation torque, on a particle with respect to its center. This happens due to the transferring of the angular momentum of the beam to the particle. It has been demonstrated that the radiation torque produced on an absorbing particle by an arbitrary time-harmonic wave is

\[
N = \frac{\pi a^3 I_0}{c_0} \tau \tag{21}
\]

where \( \tau \) is the dimensionless radiation torque vector. Similarly to the radiation force previously analyzed, the dimensionless radiation torque vector is given in terms of the beam-shape and the scattering coefficients. In the Rayleigh scattering limit, the Cartesian component of this vector can be calculated by keeping the monopole and the dipole scattering coefficients in [Eq. 14, 24]. Accordingly, we have

\[
\tau_x = -\frac{\mathbf{\omega}}{2} \left( -\frac{1}{2} + |s_1|^2 \right) \left( a_1^{-1} a_1^* + a_0 a_1^* \right), \tag{22}
\]

\[
\tau_z = -\frac{1}{2} \left( \frac{1}{2} + |s_1|^2 \right) \left( |a_1|^2 - |a_1^{-1}|^2 \right). \tag{23}
\]

Using the beam-shape coefficients given in [A3] into these equations, one finds

\[
\tau = -\frac{12i(\rho_0 c_0)^2}{p_0^2 (ka)^3} \left( \frac{s_1 + s_1^*}{2} + |s_1|^2 \right) |v_i(0) \times \nu_i^*(0)|. \tag{24}
\]

We emphasize that this vector has real components.

Substituting the scaled scattering coefficient given in [B1] into this equation, we obtain

\[
\tau = -\frac{12i a^2 \rho_0 c_1}{5p_0^2 (\rho_0 + 2\rho_1)^2} |v_i(0) \times \nu_i^*(0)|. \tag{25}
\]

This equation shows that no torque is produced in a non-absorbing particle suspended in a nonviscous fluid. Furthermore, the radiation torque increases with frequency as \( \omega^{+1} \).

V. SOME EXAMPLES

A. Spherically focused beam

Consider that a spherically focused transducer has diameter \( 2b \) and curvature radius \( z_0 \). The generated ultrasound beam by the transducer can be described in the paraxial approximation if the following conditions are met: \( z_0/b \gg 2.5 \) and \( kb = \sqrt{z_0}/b \). The produced ultrasound beam hits a spherical particle as depicted in Fig. 1. In the paraxial approximation, the pressure amplitude of the generated beam in the focal plane and along the axial direction are given in cylindrical coordinates \((\rho, z)\), respectively, by

\[
p_i(\rho, z_0) = \frac{i p_0 b^2}{2} \exp \left[ \frac{ik z_0 (\rho^2/z_0 + 1)}{z_0} \right] J_1 \left( \frac{k b \rho}{z_0} \right), \tag{26}
\]

\[
p_i(0, z) = \frac{p_0 e^{ikz}}{z/z_0 - 1} \left[ 1 - \exp \left( -ikb^2 z^2/z_0^2 \right) \right], \tag{27}
\]

where \( J_1 \) is the first-order Bessel function of first type.

The particle might be transversely trapped if the radiation force potential has a minimum at the beam axis.
i.e., $\vartheta = 0$. Hence, substituting Eq. (11) and (26) into Eq. (17), one encounters
\[
\lim_{\vartheta \to 0} U(\vartheta, z_0) = \frac{\pi a^3 I_0}{24 c_0 F^2} f_0, \tag{28}
\]
where $F = z_0/(2b)$ is the transducer $f$-number. Therefore, the particle will be transversely trapped if $f_0 < 0$, otherwise it will be pushed away from the beam axis.

The situation is different in the axial direction. Using Eq. (26) in Eq. (11) and substituting the result into Eq. (29), we find that the radiation force potential in the vicinity of the transducer focus point is
\[
\lim_{z \to z_0} U(0, z) = \frac{\pi a^3 I_0}{8 c_0 F^2} \left( \frac{f_0}{3} - \frac{f_1}{2} \right). \tag{29}
\]
In general, commercial transducers operate with $kb \gg 1$. Therefore in this limit Eq. (28) becomes
\[
\lim_{z \to z_0} U(0, z) = \frac{\pi a^3 I_0}{8 c_0 F^2} (f_0 - f_1). \tag{30}
\]
Thus, the radiation force potential has a minimum only when one of the following conditions is met: $f_0 < 0$ and $f_1 > 0$, or $f_0 < 0$ and $f_1 < 2 f_0/3$, or $f_0 > 0$ and $f_1 > 2 f_0/3$. Note that the last condition allows the particle to be axially trapped if $f_0 > 0$, but this implies, according to Eq. (26), that no transverse trapping occurs. Another important issue is that even if the radiation force potential has a minimum, the particle may not be trapped. This happens because the absorbing and the scattering radiation forces may push the particle away from the potential minimum.

### B. Acoustic Bessel beam

We derive here the radiation force and torque induced by an acoustic Bessel beam on an absorbing particle located in the beam axis at $r = 0$. Assume that the Bessel beam propagates along the $z$-axis. Hence, the pressure amplitude of this beam is given, in cylindrical coordinates, by
\[
p_i = p_0 J_\ell(k r \sin \beta) e^{i(\ell \varphi + k z \cos \beta)}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \tag{31}
\]
where $J_\ell$ is the $\ell$th-order Bessel function of first type, and $\beta$ is the beam’s half-cone angle. The index $\ell$ is known as the orbital angular momentum of the beam. It is worth to notice that the radiation force exerted on a particle due to this beam with $\beta = 90^\circ$ has been experimentally performed.

Due to the beam symmetry only the axial radiation force is produced on the particle. Using Eqs. (11) and (31) we find the average incident intensity to the particle is given by $
\bar{I}(0) = I_0 \cos \beta J_\ell^2(0) e_2 = I_0 \cos \beta e_z,$ with $\ell = 0$ and $e_z$ being the unit-vector along the $z$ direction. Therefore, only the zero-order Bessel beam produces axial radiation force on the particle. Accordingly, we have that the absorbing and the scattering radiation force components produced by a zero-order Bessel beam are
\[
F_{\text{abs}} = \frac{8 \pi a^3 I_0 \rho_0 \alpha_e \omega^\nu}{3 \rho_1 c_1} \cos \beta, \tag{32}
\]
\[
F_{\text{sca}} = \frac{4 \pi a^2 I_0}{9 c_0} \left( f_0^2 + f_0 f_1 + \frac{3 f_1^2}{4} \right) (ka)^3 \cos \beta. \tag{33}
\]
The term $\cos \beta$ in Eq. (32) is in agreement to the previous result obtained in Eq. 20, 19.

Now we turn to obtain the acoustic radiation torque. After calculating the particle velocity of the Bessel beam using Eqs. (11) and (31), we obtain the axial radiation torque as
\[
\tau_z = \frac{6 \ell \alpha^2 \rho_0 \rho_1 c_1 \alpha_e \omega^\nu + 1}{5(\rho_0 + 2 \rho_1)^2 a^2} \sin^2 \beta, \quad |\ell| = 1. \tag{34}
\]
When $|\ell| \neq 1$ no radiation torque is produced on the particle in the Rayleigh scattering limit. This happens because radiation torque is induced by the dipole mode only as shown in Eqs. (22) and (23). The dipole with respect to the beam’s axis is not present in all but the first-order Bessel vortex beam. The term $\sin^2 \beta$ present in Eq. (34) also appears in the rotational velocity induced by a first-order Bessel beam on a nonabsorbing particle suspended in a viscous fluid. Note that Eq. (34) can also be obtained directly from Eq. 14, 33. Moreover, this equation follows immediately from Eq. 18, 13 (with $m = 1$ and $n = 1$ in the associated Legendre function in the notation of that paper), combined with the axial radiation torque expression given in Eq. 10, 24.

### VI. NUMERICAL RESULTS AND DISCUSSION

To compute the radiation force and torque we consider two different particles (liquid droplets) suspended in water ($\rho_0 = 1000 \text{ kg/m}^3$, $c_0 = 1480 \text{ m/s}$, and $\nu_0 = 10^{-6} \text{ m}^2/\text{s}$) at room temperature. One droplet is formed with benzene36–37 ($\rho_1 = 870 \text{ kg/m}^3$, $c_1 = 1295 \text{ m/s}$, $\alpha_3 = 2.21 \times 10^{-14} \text{ Np} \cdot \text{MHz}^2/\text{m}$, $\nu = 2$, and $\nu_1 = 6.94 \times 10^{-7} \text{ m}^2/\text{s}$) and the other is made of olive oil38 ($\rho_1 = 915.8 \text{ kg/m}^3$, $c_1 = 1464 \text{ m/s}$, $\alpha_2 = 4.10 \times 10^{-14} \text{ Np} \cdot \text{MHz}^2/\text{m}$, $\nu = 2$, and $\nu_1 = 1.00 \times 10^{-4} \text{ m}^2/\text{s}$). The droplets were chosen because they are immiscible in water. The compressibility and the density contrast factors for the benzene and the olive oil droplets are $f_0 = -0.5, 0.07$ and $f_1 = -0.09, 0.06$, respectively.

Consider that a spherically focused transducer with diameter $2b = 50 \text{ mm}$ and curvature radius $z_0 = 70 \text{ mm}$ ($F = 1.4$) generates an ultrasound beam of frequency 1 MHz. The peak pressure of the ultrasound beam is assumed to be $p_0 = 10 \text{ kPa}$. We chose these parameters because such transducer could be readily manufactured for experimental arrangements. Moreover, the generated ultrasound beam can be described in the paraxial approximation. At 1 MHz frequency, the inner viscous boundary layer of the benzene and the olive oil droplets are, respectively, $\delta_1 = 0.47, 5.65 \mu\text{m}$. The outer boundary
FIG. 2. (Color online) The transverse radiation force generated by the focused transducer on the benzene and the olive oil droplets suspended in water. The normalized transverse radial distance is \( \tilde{\rho} = \rho/W_{-3 \text{dB}} \). The transducer parameters are \( f \)-number \( F = 1.4 \), operation frequency 1 MHz, focal distance \( z_0 = 70 \) mm, and peak pressure magnitude is \( p_0 = 10 \) kPa. The droplets have radii \( a = 58.8 \) \( \mu \)m with \( ka = 0.25 \).

layer is \( \delta_0 = 0.56 \) \( \mu \)m. Thus considering \( ka = 0.25 \) with \( a = 58.8 \) \( \mu \)m, the inequality in (25) is satisfied. Hence, we may neglect the inner and the outer shear wave propagation effects for both droplets, since \( \delta_0, \delta_1 \ll a \).

In Fig. 2 we show the transverse radiation force due to the focused transducer on the benzene and the olive oil droplets. The force varies with the normalized transverse radial distance \( \tilde{\rho} = \rho/W_{-3 \text{dB}} \), where \( W_{-3 \text{dB}} = 3.24F/k \) is 3 dB-width of the focal spot. The benzene droplet is transversely trapped at the transducer focus. In contrast, this position is unstable for the olive oil droplet and no entrapment occurs. The maximum magnitude of the transverse radiation force are 1.03 and 0.16 nN for the benzene and the olive oil droplets, respectively.

The axial radiation force exerted by the ultrasound focused beam on the benzene droplet as a function of \( \tilde{z} = z/z_0 \) is depicted in Fig. 3. The droplet can be trapped in the nearfield at approximately \( \tilde{z} = 0.46, 0.55, 0.68 \), and at the transducer focal distance \( \tilde{z} = 1.023 \). These points corresponds to the minima of the radiation force potential \( U \). The maximum radiation force magnitude is \( F_z = 177.2 \) pN and it occurs at \( \tilde{z} = 0.86 \). It should be noticed that the radiation force on the benzene droplet is mostly due to its gradient component. Furthermore, theoretical aspects of nearfield particle trapping was also discussed in Ref. [39].

In Fig. 4 we present the axial radiation force exerted on the olive oil droplet by the focused beam with \( \tilde{z} = z/z_0 \). In this case, the droplet can be trapped only in the nearfield at \( \tilde{z} = 0.43, 0.72 \). The maximum magnitude of the axial radiation force \( F_z = 24.6 \) pN occurs at \( \tilde{z} = 0.93 \). It can be seen that the absorption radiation force is dominant part in the total radiation force acting on the olive oil droplet.

Now we present in Fig. 5 the axial radiation torque exerted by a first-order Bessel vortex beam on the benzene and the olive oil droplets as a function of the half-cone angle \( \beta \). The peak pressure magnitude of the beam is \( p_0 = 10 \) kPa. The olive oil droplet develops a larger radiation torque than that generated on the benzene droplet for all values of the half-cone angle.

VII. SUMMARY AND CONCLUSION

Exact formulas of the radiation force and torque were provided for any time-harmonic beam interacting with
an absorbing small particle (in the Rayleigh scattering limit) suspended in an inviscid fluid. Internal shear viscous effects were not considered since we assumed that the inner viscous boundary layer is much smaller than the particle radius. Using the developed radiation force theory, the stability of axial and transverse entrapment of a benzene and an olive oil droplet suspended in water by a spherically focused ultrasound beam was analyzed. Moreover, the radiation torque caused by a first-order Bessel vortex beam on the benzene and olive oil droplets in the on-axis configuration was computed.

In conclusion, the developed closed-form expressions for the radiation force and torque on an absorbing particle might be useful in the analysis of trapping stability of single-beam acoustical tweezers. In a future work, we will take into consideration shear wave propagation effects inside the absorbing particle on the acoustic radiation force and torque.

**APPENDIX A: BEAM-SHAPE COEFFICIENTS**

The beam-shape coefficient of a pressure field $p$ is given by

$$a_n^m = \int \frac{p(kR, \theta, \varphi)}{\rho_0 c_0} Y_n^m(\theta, \varphi) d\Omega, \quad n \geq 0, |m| \leq n, \quad (A1)$$

where $\rho_0$ is the peak pressure magnitude, $d\Omega$ is the differential solid angle, and $R$ is the radius of a control spherical region in which the incident beam propagates. Expanding the pressure around the point up to the second-order approximation, yields

$$p(\mathbf{r}) = p(0) + i\rho_0 c_0 \mathbf{r} \cdot \mathbf{v}(0) + \frac{i\rho_0 c_0 k}{2} [\mathbf{r} \cdot \nabla \mathbf{v}(0)], \quad (A2)$$

where $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z \enspace (e_i, i = x, y, z$ are the Cartesian unit-vectors). Substituting Eq. (A2) into Eq. (A1) along with $r = R(\sin \theta \cos \varphi \mathbf{e}_x + \sin \theta \sin \varphi \mathbf{e}_y + \cos \theta \mathbf{e}_z)$ and $R \to 0$, we obtain the beam-shape coefficients up to the quadrupole approximation as

$$a_0^0 = \frac{\sqrt{4\pi}}{\rho_0} p(0),$$
$$a_1^\pm = \frac{\rho_0 c_0}{\rho_0} \sqrt{6\pi}|\pm iv_x(0) + v_y(0)|,$$
$$a_1^0 = \frac{2i\rho_0 c_0}{\rho_0} \sqrt{3v_z(0)},$$
$$a_2^\pm = \frac{\rho_0 c_0}{k\rho_0} \sqrt{30\pi} \left[i \left( \frac{\partial v_x(0)}{\partial x} - \frac{\partial v_y(0)}{\partial y} \right) \pm 2 \frac{\partial v_z(0)}{\partial y} \right], \quad (A3)$$
$$a_2^0 = \frac{2i\rho_0 c_0}{k\rho_0} \sqrt{30\pi} \left[-i \left( \frac{\partial v_x(0)}{\partial z} + \frac{\partial v_y(0)}{\partial z} \right) \right].$$

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