Algorithms for affine Kac-Moody algebras

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Abstract. Weyl groups are ubiquitous, and efficient algorithms for them — especially for the exceptional algebras — are clearly desirable. In this letter we provide several of these, addressing practical concerns arising naturally for instance in computational aspects of the study of affine algebras or Wess-Zumino-Witten (WZW) conformal field theories. We also discuss the efficiency and numerical accuracy of these algorithms.

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1. Introduction

This letter contains several formulas and algorithms involving the (affine) Weyl groups, which play central roles in the author's present work. Though some are surely known to other experts, some are new, and they are all collected (and analysed) here to benefit also the researchers and students less experienced with these matters. An indication of the need for good algorithms is provided by the number of papers (see e.g. [1–4]) on it.

One place the finite Weyl groups arise is in the modular transformation [5]

\[ \chi_{\lambda}(-1/\tau) = \sum_{\mu \in P^k_+} S_{\lambda \mu} \chi_{\mu}(\tau) \quad (1a) \]

of affine characters, where the sum is over the level \( k \) integrable highest weights \( \mu \in P^k_+ \) for the given affine Kac-Moody algebra \( X_r^{(1)} \), for the \( k \) satisfying \( \lambda \in P^k_+ \). The level \( k \) is a nonnegative integer, and the finite set \( P^k_+ \) is defined in terms of the colabels \( a_i^+ \) by

\[ P^k_+ = \{ (\lambda_1, \ldots, \lambda_r) \mid a_1^+ \lambda_1 + \cdots + a_r^+ \lambda_r \leq k, \; \lambda_i \in \mathbb{Z}_{\geq 0} \} \quad (1b) \]

We identify the \( r \)-tuple with the combination \( \lambda_1 \Lambda_1 + \cdots + \lambda_r \Lambda_r \) in terms of the fundamental weights \( \Lambda_i \). An expression for this unitary and symmetric matrix \( S_{\lambda \mu} \) is [5]

\[ S_{\lambda \mu} = \kappa^{-r/2} \sum_{w \in W} (\det w) \exp[-2\pi i \frac{w(\lambda + \rho) \cdot (\mu + \rho)}{\kappa}] \quad (1c) \]
where $\rho = (1, \ldots, 1)$ and $\kappa = k + h^\vee$. The colabels $a_i^\vee$, the finite Weyl group $W$, the number $s = i|\Delta_+|/\sqrt{\|Q^\vee*/Q^\vee\|}$, the dot product, and the dual Coxeter number $h^\vee$, will be explicitly given below for each algebra.

The challenge raised by several researchers is to find an effective way of computing this matrix $S$. For instance, in modular invariant classifications [6], literally millions of $S$ entries typically must be computed for each algebra. The difficulty is that the Weyl group $W$ — though finite — can be very large and unwieldy. This is particularly acute for the exceptional $X_r$: the Weyl groups of $G_2, F_4, E_6, E_7, E_8$ respectively have $12, 1152, 51840, 2903040$, and $696729600$ elements. We will show that this sum (1c) is easy to evaluate for the classical $X_r^{(1)}$, and so a way to evaluate it for the exceptional ones would be to find a large classical Weyl subgroup. Our algorithm involves computing $O((\log|W|)^2)$ trigonometric functions and performing $O((\log|W|)^3)$ arithmetic operations.

One of the many places in which the matrix $S$ appears is Verlinde’s formula [7] for the fusion coefficients $N^\nu_{\lambda\mu}$. Write 0 for the weight $(0, 0, \ldots, 0) \in P^k_+$. Then

$$N^\nu_{\lambda\mu} := \sum_{\gamma \in P^k_+} \frac{S_{\lambda\gamma} S_{\mu\gamma} S^*_{\nu\gamma}}{S^*_{0\gamma}}$$  \hspace{2cm} (2a)

Verlinde’s formula is not of much direct computational value: to leading power in $k$, it has

$$\|P^k_+\| \approx \frac{k^r}{r! a_1^\vee \cdots a_r^\vee}$$  \hspace{2cm} (2b)

terms. This also gives the size of $S$. More effective is the Kac-Walton formula [8,9], which expresses the fusion coefficients in terms of $X_r$ tensor product coefficients $T^\nu_{\lambda\mu} = \text{mult}_\nu(L_\lambda \otimes L_\mu)$:

$$N^\nu_{\lambda\mu} = \sum_{\tilde{w} \in \tilde{W}} (\det \tilde{w}) T_{\lambda\mu}^\nu$$  \hspace{2cm} (2c)

where $\tilde{W}$ is the affine Weyl group; its action on the weights $P^k_+$ will be discussed in §2.2.

Background material on simple Lie and affine algebras can be found in [8,10–12]. For an account of the rich role of the matrix $S$ and fusion rules in rational conformal field theory and elsewhere, see e.g. [13,14] and references therein.

In this paper we focus on three tasks: effective formulas for computing $S$ entries; fast algorithms for finding the preferred affine Weyl orbit representative $[\lambda] = \tilde{w}.\lambda \in P^k_+$; and super-fast formulas for obtaining the parity $\epsilon(\lambda) = \det(\tilde{w})$ for that representative, when the weight $\tilde{w}.\lambda$ itself is not needed. In Section 4 we sketch some applications.

2. Review and statement of the problems

Introduce the notation $P_\mathbb{Z}$ for all $\sum_i \lambda_i \Lambda_i = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$. It inherits the usual dot product of $P^k_+$. When the level $k$ is understood, we’ll write

$$\lambda_0 = k - \sum_i a_i^\vee \lambda_i$$
Also, we’ll abbreviate ‘algebra $X_r^{(1)}$ at level $k$’ by ‘$X_{r,k}$’. We will write ‘$[y]$’ for the largest integer not greater than $y$ — e.g. $[\pi] = 3 = [3]$. As always, $\kappa = k + h^\vee$.

2.1. Nice weights and useful symmetries. If either $\lambda$ or $\mu$ is a ‘nice’ weight, then significant simplifications to (1c) can be made. In this letter we are interested rather in generic weights $\lambda, \mu$ in (1c) and so we will not devote much space reviewing these.

The denominator identity of Lie algebras permits us to express the sum (1c), whenever $\lambda$ is a multiple of $\rho$, as a product over the positive roots:

$$S_{m,\rho,\mu} = \kappa^{-r/2} |s| \prod_{\alpha \in \Delta_+} 2 \sin(\pi \frac{(m+1)(\mu + \rho) \cdot \alpha}{\kappa}) \quad (3)$$

The most important choice is $m = 0$, which yields a product formula for the quantum-dimensions $S_{\lambda,0}/S_{0,0}$. For a rank $r$ Lie algebra $X_r$, the Weyl group has approximate size $r! \approx \sqrt{2\pi r(r/e)^r}$, while the number of positive roots grows like $r^2$, so (3) is a significant simplification.

Remarkably, we will find that all entries of $S$ are about as accessible as $S_{\lambda,0}$, in particular, via equations with complexity growing like $r^3$.

An important observation from (1c) is that the ratio

$$\frac{S_{\lambda,\rho}}{S_{0,\mu}} = \text{ch}_\lambda(\exp[-2\pi i \frac{\mu + \rho}{\kappa}]) \quad (4a)$$

is a finite Lie group character evaluated at an element of finite order [3]. Effective expressions exist for some $\text{ch}_\lambda$, implying that the corresponding entries $S_{\lambda,\mu}$ can be computed quickly. The weight multiplicities $\text{mult}_{L(\lambda)}(\beta)$ for simple algebras of small rank and small weights are given explicitly in [15]. When $L(\lambda)$ has few dominant weights $\beta$, and each of these has a simple Weyl orbit, then (4a) can be evaluated explicitly. For instance, for $X_r = A_r$, when $\lambda$ is a fundamental weight $\Lambda_m$, we get

$$\frac{S_{\lambda,\rho}}{S_{0,\mu}} = \exp[2\pi i m t(\mu + \rho)/(r + 1)\kappa] \sum_{1 \leq i_1 < \cdots < i_m \leq r + 1} x_{i_1} \cdots x_{i_m} \quad (4b)$$

where $t(\nu) = \sum_{i=1}^{r} i\nu_i$ and $x_i = \exp[-2\pi i \mu^+[i]/\kappa]$ for orthogonal coordinates $\nu^+[i] = \sum_{j=1}^{r} (\nu_j + 1)$.

Another application of (4a) is for the use of branching rules in replacing $S_{\lambda,\mu}/S_{0,\mu}$ for $X_{r}^{(1)}$ with a sum (over $\lambda'$) of $S'_{\lambda',\mu'}/S'_{0,\mu'}$ for a smaller algebra $X_{s}^{(1)}$ of equal rank. E.g. the embedding $D_8 \subset E_8$ has the branching rule $(\Lambda_2) + (\Lambda_3) = (\Lambda_4)$ (see e.g. [30]), and hence the ratio $S_{\lambda,\mu'}/S_{0,\mu}$ equals the sum $S'_{\lambda,\mu'}/S'_{0,\mu'} + S'_{\lambda',\mu'}/S'_{0,\mu'}$ for an appropriate choice of $\mu'$ (depending only on $\mu$). Later we will work this out explicitly, for both $D_4 \subset F_4$ and $A_2 \subset G_2$.

Related to this, for fixed $\gamma \in P_+$, the ratios (4a) form a one-dimensional representation of the fusion ring. That is,

$$\frac{S_{\lambda,\gamma}}{S_{\mu,\gamma}} = \sum_{\nu \in P_+} N_{\lambda,\mu}^{\nu} \frac{S_{\nu,\gamma}}{S_{0,\gamma}} \quad (5)$$
Using (2c), we can obtain S entries for ‘larger’ weights from those of smaller weights.

Considerable simplifications [3] to the calculation of \( \chi_\lambda(\exp[-2\pi i \frac{\rho}{k}]) \) occur when some Dynkin labels \( \mu_i \) of \( \mu \) are 0. While this is not of direct value to us (because of the ‘+ρ’ in (4a)), it can be very useful as an approximation when the level \( k \) is large (see e.g. [6]).

Incidentally, the matrix S obeys several symmetries. Best known are those related to the symmetries of the extended Dynkin diagrams (see e.g. [12]). Thus determining one S entry automatically yields several others. For example, consider \( X_r = A_r \). Define an involution \( C \) by \( C(\lambda_1, \ldots, \lambda_r) = (\lambda_r, \ldots, \lambda_2, \lambda_1) \), and an order-(\( r + 1 \)) map \( J \) by \( J(\lambda_1, \ldots, \lambda_r) = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1}) \) where \( \lambda_0 = k - \sum_{i=1}^{r} \lambda_i \). Then both \( C \) and \( J \) are permutations on \( P_+^k \), corresponding to the symmetries of the extended Dynkin diagram of \( A_r \).

The various powers \( J^i \) are the simple-currents of \( A_r^{(1)} \). We have \( S_{C\lambda,\mu} = S_{\lambda,\mu}^* \) and

\[
S_{J\lambda,\mu} = \exp[2\pi i t(\mu)/(r+1)] S_{\lambda,\mu}
\]

(6)

Similar equations hold for the other algebras (see e.g. [12]).

Another symmetry — more complicated but more powerful — is the Galois symmetry [16] discussed in §4.4 below. Making it more accessible is a big motivation for the paper.

For the classical algebras, another symmetry (rank-level duality [17]) of the S matrix tells us that small level acts like small rank. In particular, the S matrix for \( \widehat{sl(n)}_k \) and \( \widehat{sl(k)}_n \) are closely related, as are \( \widehat{so(n)}_k \) and \( \widehat{so(k)}_n \), and \( C_{r,k} \) and \( C_{k,r} \).

Finally, simplifications (called fixed-point factorisation [18]) occur whenever \( \lambda \) or \( \mu \) is a fixed-point of a (nontrivial) simple-current — the simple-currents \( J^i \) for \( A_r^{(1)} \) are given above. If \( \varphi \) is fixed by \( J^d \), then for any weight of \( \lambda \) of \( \widehat{sl(n)}_k \) we can write \( S_{\varphi,\lambda} \) as a product of \( n/d \) S-entries for \( \widehat{sl(d)}_{kd/n} \). For a very simple example, take \( \widehat{sl(n)}_k \) when \( n \) divides \( k \): the unique \( J \)-fixed-point is \( \varphi = (\frac{k}{n}, \ldots, \frac{k}{n}) \), and \( S_{\lambda,\varphi} = \pm (\frac{n}{k})^{(n-1)/2} \) or 0, depending on \( \lambda \).

Incidentally, an analogue of fixed-point factorisation works for the symmetries (called conjugations) of the unextended Dynkin diagrams. For example, consider the \( Z_3 \)-symmetry triality of \( D_4 \): a weight \( \mu \) invariant under it obeys \( \mu_1 = \mu_3 = \mu_4 \), and the \( D_4^{(1)} \) ratio \( S_{\lambda\mu}/S_{0\mu} \) for such a \( \mu \) equals a sum \( \sum_{\lambda} S'_{\lambda'\mu'}/S'_{0\mu'} \) of \( G_2^{(1)} \) ratios, where \( \otimes(\lambda') = (\lambda) \) are \( G_2 \subset D_4 \) branching rules (see e.g. [30]), and \( \mu' = (\mu_2, 3\mu_1 + 2) \in P_+^{k+2}(G_2) \). Applications of these fixed-point factorisations is given in [31].

In practice the remarks of this subsection are quite effective. For example, consider \( A_{3,6} \). There are 7056 S entries: any involving \( J^0, J^1(1,1,1), J^3(2,2,2), J^4 A_1, J^4(1,0,1), J^3(1,1,0), C^i J^2(2,0,0), \) or any \( J^2 \)-fixed point, are immediately calculated from the above. Finally, exploiting the Galois symmetry (see §4.4 below), we reduce the calculation of the 7056 entries \( S_{\lambda\mu} \) to precisely 1 less pleasant calculation: that of \( S_{(3,0,0),(3,0,0)} \). Of course, for large \( k \) and \( r \), most weights won’t be ‘nice’ in this sense, and another approach is required.

2.2. Calculating Weyl orbits. The affine Weyl group \( \widehat{W} \) is generated by the reflections \( r_i \) through simple roots: explicitly, for any \( \lambda \in P_\mathbb{Z} \), the Dynkin labels of \( r_i \lambda \) are

\[
(r_i \lambda)_j = \lambda_j - \lambda_i A_{ji}
\]

(7a)
where \(A_{ji}\) are entries of the Cartan matrix of \(X_1^{(1)}\). It is important that \(\hat{W}\) can be expressed as the semidirect product of the finite Weyl group with the coroot lattice interpreted as an additive group — i.e. each \(\hat{w}\) can be uniquely identified with a pair \((w, \alpha)\), for \(w \in W\) and \(\alpha \in Q^\vee\), and \(\hat{w}\lambda = w(\lambda) + \kappa\alpha\). It is often more convenient to express this action using the notation

\[
\hat{w}.\lambda := \hat{w}(\lambda + \rho) - \rho \quad (7b)
\]

Given any level \(k\) weight \(\lambda = (\lambda_1, \ldots, \lambda_r) \in P_\mathbb{Z}\), there are two possibilities: either \(\lambda + \rho\) is fixed by some affine Weyl reflection \(\hat{w}\), i.e. \(\hat{w}.\lambda = \lambda\), and we shall call \(\lambda\) null; or the orbit \(\hat{W}.\lambda\) intersects the fundamental alcove \(P_+^k\) in precisely one point:

\[
[\lambda] := \hat{w}.\lambda \in P_+^k \quad (7c)
\]

Define the parity \(\epsilon(\lambda)\) to be 0 or \(\det(w)\), depending on whether or not \(\lambda\) is null. This preferred orbit representative \([\lambda]\), and/or the parity \(\epsilon(\lambda)\), are often desired — see §4.

A very useful fact is that \(\lambda\) will be null if \(\lambda_i = -1\) for any \(i = 0, 1, \ldots, r\).

A method for finding \([\lambda]\) and \(\epsilon(\lambda)\) is proposed in [12]. Put \(\epsilon = +1\) and \(\mu = \lambda\).

1. If \(\mu_i \geq 0\) for all \(i = 0, 1, \ldots, r\), then \([\lambda] = \mu\) and \(\epsilon(\lambda) = \epsilon\).
2. Otherwise, let \(0 \leq i \leq r\) be the smallest index for which \(\mu_i < 0\). Replace \(\mu\) with \(r_i.\mu\), and \(\epsilon\) with \(-\epsilon\). Goto (1).

It does not appear to be known yet whether their method will always terminate, although in practice it seems to. We give an alternative next section.

3. The formulas and algorithms

The essence of the following \(S\) matrix formulas is the observation that an alternating sum over the symmetric group also occurs in determinants. The point is that determinants can be evaluated very effectively, using Gaussian elimination, and is easy to implement on a computer. By comparison, determinants are much more accessible than permanents. This suggests that we look no further than (1c) for an effective algorithm.

The classical \(S\) matrices all reduce to evaluating one or two determinants. Our strategy for the exceptional algebras is to find a classical Weyl group of small index. For example \(W(D_8)\) is contained in \(W(E_8)\) with index 135. Once we find coset representatives for the 135 cosets in \(W(D_8)\setminus W(E_8)\), then the \(E_8\) \(S\) matrix would be the alternating sum of 135 \(D_8\) \(S\) matrix entries. Equally important, these coset representatives also permit effective algorithms to find \([\lambda]\) and \(\epsilon(\lambda)\), as we will see.

A priori, it could be hard to find these coset representatives, but here the task is made elementary by the following result [19] (there’s a typo in their definition of \(D_\Psi\):

**Theorem.** Let \(\Psi\) be a subsystem of a root system \(\Phi\). Choose simple roots \(\alpha \in J\) for \(\Psi\) which are positive roots \(\Phi^+\) in \(\Phi\) (this is always possible). Define \(D_\Psi = \{w \in W(\Phi) | w(J) \subset \Phi^+\}\). Then every element of \(W(\Phi)\) can be uniquely expressed in the form \(dw'\) where \(d \in D_\Psi\) and \(w' \in W(\Psi)\). Furthermore, \(d\) is the unique element of minimal length in the coset \(dW(\Psi)\).
In particular, the set of all $d^{-1}$, for $d \in D\Psi$, constitute the desired coset representatives for $W(\Psi)\backslash W(\Phi)$. We chose the largest classical subsystem possible for each of the exceptional root systems. The results are explained in the following subsections.

To give the extreme example, we read off from Table 2 that the $W(D_8)\backslash W(E_8)$ coset representative we call $c_{135}$ is the following composition of simple reflections:

$$r_7 r_6 r_5 r_4 r_3 r_2 r_1 r_8 r_5 r_4 r_3 r_2 r_6 r_5 r_4 r_3 r_5 r_7 r_4 r_6 r_5 r_8.$$ 

It acts on a weight $\lambda \in P(E_8)$ as follows: write $\nu = c_{135} \cdot \lambda$, then using notation defined in §3.E8 we get

$$\left( \begin{array}{c}
\nu^+[1] \\
\vdots \\
\nu^+[8]
\end{array} \right) = \frac{1}{4} \left( \begin{array}{cccccccc}
-1 & 1 & 1 & 1 & 1 & 1 & -3 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -3 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 3
\end{array} \right) \left( \begin{array}{c}
\lambda^+[1] \\
\vdots \\
\lambda^+[8]
\end{array} \right)$$

Root system data is explicitly given in e.g. [10,12]. For convenience we reproduce it in the following subsections.

The best way to determine the parity $\epsilon(\lambda)$ of §2.2 seems to be to use (3):

$$\epsilon(\lambda) = \prod_{\alpha > 0} \text{sign}\{\sin(\frac{\pi}{\kappa} (\lambda + \rho) \cdot \alpha)\}$$  \hspace{1cm} (8)$$

The (easy) proof that (8) holds is that the product of sine’s can be expressed as alternating sums over the Weyl group, thanks to equation (3), and that when evaluated at any weight $\lambda \in P^+_k$ they yield a positive value (proportional to the quantum-dimension). Equation (8) has been kicking around for years — see e.g. the authors earlier work, and also [20]. Later in this section we make significant improvements to (8) for all algebras except $A_r^{(1)}$.

We often require ordering lists of numbers (lists whose length $N$ is approximately the rank $r$ of the algebra). Straightforward ordering algorithms have a computing time growing like $N^2$, but more sophisticated algorithms have order $N \log N$, and so are relevant for large rank. See [21] for details.

Many of our formulas involve taking determinants of $N \times N$ matrices, where $N$ approximately equals the rank. Using Gaussian elimination, this involves approximately $N^3$ operations. We will briefly discuss the error analysis in §4.1 — see also [22].

Some of the following (namely the $S$ matrix for the classical algebras) has appeared in [17], and B. Schellekens has used but not published similar $S$ matrix formulas (see his webpage at http://www.nikhef.nl/~t58/kac.html). Determinant formulas for the characters of the classical Lie algebras go back to Weyl and are discussed in §24.2 of [23]. Other approaches to computing generic $S$ entries can be found in [1,2,4].

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3.A. The A-series. The colabels $a_i^\vee$ for $X_r = A_r = \text{sl}(r+1)$ all equal 1. The scale factor in (1b) is $s = i^{(r+1)/2}/\sqrt{r+1}$, and $\kappa = k+r+1$. To the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ associate the orthogonal coordinates $\lambda[\ell] = \sum_{i=\ell}^r \lambda_i$ for $\ell = 1, \ldots, r+1$, and write $t(\lambda) = \sum_{\ell=1}^r \ell \lambda_\ell$ and $\lambda^+[j] := (\lambda + \rho)[j] = r + 1 - j + \lambda[j]$. The finite Weyl group is the symmetric group $S_{r+1}$, acting on $\lambda$ by permuting the components $\lambda[\ell]$. The affine Weyl group also acts on a level $k$ weight $\lambda$ by translation: $\lambda \mapsto \lambda + k\alpha$ for any $\alpha$ belonging to the $A_r$ root lattice, i.e. $\alpha = (c_1, \ldots, c_{r+1}) \in \mathbb{Z}^{r+1}$, $\sum_i c_i = 0$. Then

$$S_{\lambda \mu} = \kappa^{-r/2} s \exp[2\pi i \frac{t(\lambda + \rho) t(\mu + \rho)}{\kappa (r+1)}] \det(\exp[-2\pi i \frac{\lambda^+[i] \mu^+[j]}{\kappa}])_{1 \leq i, j \leq r+1} \quad (A.1)$$

‘det’ in (A.1) denotes the determinant of the $(r+1) \times (r+1)$ matrix whose $(i, j)$-th entry is provided.

The fastest algorithm known to this author for finding the orbit representative $[\lambda]$ of (7c) is as follows.

(i) Write $x_\ell = \lambda^+[\ell] = r + 1 - \ell + \sum_{i=\ell}^r \lambda_i$ for $\ell = 1, 2, \ldots, r$ and put $x_{r+1} = 0$.
(ii) Reorder these so that $x_1 > x_2 > \cdots > x_r > x_{r+1} \geq 0$. If $x_1 - x_{r+1} < \kappa$, then done; goto step (iv).
(iii) Otherwise for each $i = 1, \ldots, (r+1)/2$ put $m_i = [(x_i - x_{r+2-i})/\kappa]$; if $m_i$ is even let $x_i = x_i - m_i \kappa/2$ and $x_{r+2-i} = x_{r+2-i} + m_i \kappa/2$, while if $m_i$ is odd let $x_i = x_{r+2-i} - (m_i + 1) \kappa/2$ and $x_{r+2-i} = x_i + (m_i + 1) \kappa/2$. Return to step (ii).
(iv) The Dynkin labels $[\lambda]_i$ of the desired weight are

$$[\lambda]_i = y_i - y_{i+1} - 1$$

The weight $\lambda$ is null iff this algorithm breaks down: i.e. two (or more) $x_i$’s are equal.

The parity $\epsilon(\lambda)$ is the product of the signs of the permutations in step (ii), together with $(-1)^m$’s from step (iii). An alternative is provided in [24].

Equation (8) becomes

$$\epsilon(\lambda) = \text{sign} \prod_{1 \leq i < j \leq r+1} \sin(\pi \frac{\lambda^+[i] - \lambda^+[j]}{\kappa}) \quad (A.2)$$

3.B. The B-series. For $X_r = B_r = \text{so}(2r+1)$, $a_i^\vee = a_i^\vee = 1$ and all other colabels equal 2. The scale factor is $s = i^r/2$, and $\kappa = k+2r-1$. To the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ associate the orthogonal coordinates $\lambda[\ell] = \sum_{i=\ell}^{r-1} \lambda_i + \lambda_r/2$ for $\ell = 1, \ldots, r$ (put $\sum_{i=r}^{r-1} \lambda_i = 0$) and write $\lambda^+[j] := (\lambda + \rho)[j] = r + 1 - j + \lambda[j]$. The finite Weyl group is a group of order $2^r \cdot r!$, acting on $\lambda$ by arbitrarily permuting and changing signs of the components $\lambda[\ell]$. The affine Weyl group also acts on a level $k$ weight $\lambda$ by translation: $\lambda \mapsto \lambda + k\alpha$ for any $\alpha$ belonging to the $D_r$ root lattice, i.e. $\alpha = (c_1, \ldots, c_r) \in \mathbb{Z}^r$ with $\sum_i c_i$ is even. Then

$$S_{\lambda \mu} = \kappa^{-r/2} 2^{r-1} i^{r^2-r} \det(\sin(2\pi \frac{\lambda^+[i] \mu^+[j]}{\kappa}))_{1 \leq i, j \leq r} \quad (B.1)$$

The best algorithm known to this author for finding the orbit representative $[\lambda]$ is:
(i) Write \( x_\ell = \lambda^+[\ell] = r - \ell + \frac{1}{2} + \sum_{i=\ell}^{r-1} \lambda_i + \lambda_r/2 \) for \( \ell = 1, 2, \ldots, r \).

(ii) By adding the appropriate multiples of \( \kappa \) to each \( x_\ell \), find the unique numbers \( y_1, \ldots, y_r \) such that \( x_\ell \equiv y_\ell \mod \kappa \) and \(-\kappa/2 < y_\ell \leq \kappa/2\). Let \( m \) be the total number of \( \kappa \) added: i.e. if \( x_i = y_i + m_i \kappa \), then \( m = \sum_{i=1}^r m_i \). We need to know later whether \( m \) is even or odd.

(iii) Replace each \( y_i \) with its absolute value. Reorder these so that \( y_1 > y_2 > \cdots > y_r > 0 \).

(iv) Then the Dynkin labels \([\lambda]\) of the desired weight are

\[
[\lambda]_i = y_i - y_{i+1} - 1, \quad \text{for } 1 \leq i < r, \quad \text{and } [\lambda]_r = 2y_r - 1
\]

The weight \( \lambda \) is null iff two of the \( y_i \)'s in (iii) are equal, or if at least one \( y_i \) equals 0. The parity \( \epsilon(\lambda) \) is \((-1)^m \) times the sign of the product \( \prod y_i \) in (ii), times the sign of the permutation in (iii).

Equation (8) becomes (note the decoupling of the \( i \)th and \( j \)th terms here)

\[
\epsilon(\lambda) = \text{sign} \prod_{1 \leq i \leq r} \sin(\pi \frac{\lambda^+[i]}{\kappa}) \prod_{1 \leq i < j \leq r} \{\cos(2\pi \frac{\lambda^+[i]}{\kappa}) - \cos(2\pi \frac{\lambda^+[j]}{\kappa})\}
\]

where we used the identity \( 2 \sin(x - y) \sin(x + y) = \cos(2y) - \cos(2x) \).

3.C. The C-series. The colabels for \( X_r = C_r = \text{sp}(2r) \) all equal 1. The scale factor is \( s = i^2/2r^2 \), and \( \kappa = k + r + 1 \). To the weight \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) associate the orthogonal coordinates \( \lambda[\ell] = \sum_{i=\ell}^{r} \lambda_i \) for \( \ell = 1, \ldots, r \), and write \( \lambda^+[\ell] = (\lambda + \rho)[\ell] = r + 1 - \ell + \lambda[\ell] \).

The finite Weyl group is a group of order \( 2^r \cdot r! \), acting on \( \lambda \) by arbitrarily permuting and changing signs of the components \( \lambda[\ell] \). The affine Weyl group also acts on a level \( k \) weight \( \lambda \) by translation: \( \lambda \mapsto \lambda + k\alpha \) for any \( \alpha \) belonging to \( (2\mathbb{Z}) \oplus \cdots \oplus (2\mathbb{Z}) \), i.e. \( \alpha = (c_1, \ldots, c_r) \in \mathbb{Z}^r \), where each \( c_i \) is even. Then

\[
S_{\lambda\mu} = (2/\kappa)^{r/2} i^{r^2 - r} \det(\sin(\pi \frac{\lambda^+[i] \mu^+[j]}{\kappa}))_{1 \leq i, j \leq r}
\]

The best algorithm known to this author for finding the orbit representative \([\lambda]\) is:

(i) Write \( x_\ell = \lambda^+[\ell] = r + 1 - \ell + \sum_{i=\ell}^{r} \lambda_i \).

(ii) By adding the appropriate multiples of \( 2\kappa \) to each \( x_\ell \), find the unique numbers \( y_1, \ldots, y_r \) such that \( x_\ell \equiv y_\ell \mod 2\kappa \) and \(-\kappa < y_\ell \leq \kappa \).

(iii) Replace each \( y_i \) with its absolute value. Reorder these so that \( \kappa \geq y_1 > y_2 > \cdots > y_r > 0 \).

(iv) Then the Dynkin labels \([\lambda]\) of the desired weight are (put \( y_{r+1} = 0 \))

\[
[\lambda]_i = y_i - y_{i+1} - 1
\]

The weight \( \lambda \) is null iff some \( y_i = 0 \), or two \( y_i \)'s are equal. The parity \( \epsilon(\lambda) \) is the sign of the product \( \prod y_i \) in (ii), times the sign of the permutation in (iii).
Equation (8) becomes (note the decoupling of the \( i \)th and \( j \)th terms here)

\[
\epsilon(\lambda) = \text{sign} \left( \prod_{1 \leq i \leq r} \sin(\pi \frac{\lambda^+[i]}{\kappa}) \prod_{1 \leq i < j \leq r} \left\{ \cos(\pi \frac{\lambda^+[j]}{\kappa}) - \cos(\pi \frac{\lambda^+[i]}{\kappa}) \right\} \right) \\
(C.2)
\]

3.D. The D-series. For \( X_r = D_r = \text{so}(2r) \), \( a_\gamma^\prime = a_{\gamma-1}^\prime = a_\gamma^\prime = 1 \) and all other colabels are 2. The scale factor in (1b) is \( s = i^{r(r-1)}/2 \) and \( \kappa = k + 2r - 2 \). To the weight \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) associate the orthogonal coordinates \( \lambda[\ell] = \sum_{i=\ell}^{r-1} \lambda_i + \frac{\lambda_{r-\ell+1}}{2} \) for \( \ell = 1, \ldots, r \) (\( \sum_{i=r}^{r-1} = 0 \)), and define \( \lambda^+[\ell] = (\lambda + \rho)[\ell] = r - \ell + \lambda[\ell] \). The finite Weyl group has order \( r!2^{r-1} \), acting on \( \lambda \) by permuting the components \( \lambda[\ell] \) and changing an even number of signs. The action of the affine Weyl group includes the translations \( \lambda \mapsto \lambda + k\alpha \) for any \( \alpha \) belonging to the \( D_r \) root lattice, i.e. \( \alpha = (c_1, \ldots, c_r) \in \mathbb{Z}^r \) where \( \sum_i c_i \) is even. Then

\[
S_{\lambda\mu} = \kappa^{-r/2}2^{r-2}i^{r(r-1)} \left\{ \det(\cos(2\pi \frac{\lambda^+[i] \mu^+[j]}{\kappa})) \right\}_{1 \leq i,j \leq r} \\
+ (-i)^r \det(\sin(2\pi \frac{\lambda^+[i] \mu^+[j]}{\kappa}))_{1 \leq i,j \leq r} \right\} \\
(D.1)
\]

The best algorithm known to this author for finding the orbit representative \([\lambda]\) is:

(i) Write \( x_\ell = \lambda^+[\ell] = r - \ell + \sum_{i=\ell}^{r-1} \lambda_i + \frac{\lambda_{r-\ell+1}}{2} \).

(ii) By adding the appropriate multiples of \( \kappa \) to each \( x_\ell \), find the unique numbers \( y_1, \ldots, y_r \) such that \( x_\ell \equiv y_\ell \pmod{\kappa} \) and \( -\kappa/2 < y_\ell \leq \kappa/2 \). Let \( m \) be the total number of \( \kappa \) added: i.e. if \( x_i = y_i + m_i\kappa \), then \( m = \sum_i m_i \). We need to know later whether \( m \) is even or odd.

(iii) Let \( m' = \pm 1 \) be the sign of \( \prod_i y_i \) (take \( m' = 1 \) if the product is 0). Replace each \( y_i \) with its absolute value \( |y_i| \). Reorder these so that \( \kappa/2 \geq y_1 > y_2 > \cdots > y_r \geq 0 \).

(iv) Replace \( y_r \) with \( m'y_r = \pm y_r \). If \( m \) is odd, replace \( y_1 \) with \( \kappa - y_1 \) and \( y_r \) with \( -y_r \).

(v) Then the Dynkin labels \([\lambda]_i\) of the desired weight are

\[
[\lambda]_i = y_i - y_{i+1} - 1 \quad 1 \leq i \leq r - 1, \quad [\lambda]_r = y_r + y_{r-1} - 1
\]

The weight is null iff two \( y_i \)'s in (iii) are equal. The parity \( \epsilon(\lambda) \) is the sign of the permutation in (iii).

Equation (8) becomes (note the decoupling of the \( i \)th and \( j \)th terms here)

\[
\epsilon(\lambda) = \text{sign} \left( \prod_{1 \leq i < j \leq r} \left\{ \cos(2\pi \frac{\lambda^+[j]}{\kappa}) - \cos(2\pi \frac{\lambda^+[i]}{\kappa}) \right\} \right) \\
(D.2)
\]

Curiously, this sign has the interpretation as the sign of the permutation which orders the numbers \( \cos(2\pi \lambda^+[i]/\kappa) \) in increasing order.
3.E6. The algebra $E_6$. The colabels for $E_6$ are $1,2,3,2,1,2$, respectively. The scale factor in (1b) is $s = 1/\sqrt{3}$, and $\kappa = k + 12$. To the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_6)$ associate the orthogonal coordinates (interpret $\sum_{i=2}^{1} \lambda_i$ as 0)

$$
\lambda[\ell] = \frac{\lambda_6 - \lambda_2}{2} + \sum_{i=2}^{\ell} \lambda_i \text{ for } \ell \leq 5, \quad \lambda[6] = \frac{5\lambda_2 + 3\lambda_6}{2} + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 2\lambda_1
$$

The 27 coset representatives for $W(D_5)\backslash W(E_6)$ are given in Table 1, recursively defined in terms of the reflections $r_i, i = 1, \ldots, 6$, through the simple roots $\alpha_i$. Then

$$
S_{\lambda\mu} = \frac{16}{3\kappa^3} \sum_{\ell=1}^{27} (\det c_\ell) \{ \det(\cos(2\pi \frac{(c_\ell, \lambda)^+[i] \mu^+[j]}{\kappa}))_{1 \leq i, j \leq 5}

- i \det(\sin(2\pi \frac{(c_\ell, \lambda)^+[i] \mu^+[j]}{\kappa}))_{1 \leq i, j \leq 5})
$$

(E6.1)

Recall that the numbers $(c_\ell, \lambda)^+[i]$ are the orthogonal coordinates of the weight $c_\ell(\lambda + \rho)$.

The orbit representative $[\lambda]$ in (7c) can be found as follows.

(i) Choose one of the 27 coset representatives $c_\ell$. Write $z_i := (c_\ell, \lambda)^+[6-i], i = 1, \ldots, 5,$ and $z_6 = (c_\ell, \lambda)^+[6]$, for the orthogonal coordinates (reordered) of $c_\ell(\lambda + \rho)$. (The ‘$[6-i]$’ is chosen to make the first five indices here consistent with those of $D_5$.)

(ii) Add an appropriate multiple $3n\kappa/2$ of $3\kappa/2$ to $z_6$ to get $0 \leq x_6 < 3\kappa/2$; replace $x_1 = n\kappa/2 + z_1$, and $x_i = -n\kappa/2 + z_i$ for $2 \leq i \leq 5$. Use the $D_5$ algorithm for the present value of $\kappa$ to get $|y_5| < y_4 < y_3 < y_2 < y_1$ and $y_1 + y_2 < \kappa$.

(iii) Write $t = (-y_1 - y_2 - y_3 - y_4 + y_5 + y_6)/2$. If $t > 0$ and $y_1 + y_2 + \cdots + y_6 < 2\kappa$, then the Dynkin labels of the desired weight $[\lambda]$ are

$$
[\lambda]_1 = t - 1, \quad [\lambda]_6 = y_4 + y_5 - 1, \quad [\lambda]_i = y_{6-i} - y_{7-i} - 1 \quad 2 \leq i \leq 5
$$

(iv) Otherwise try the next $c_\ell$ in (i).

Equation (8) becomes

$$
\epsilon(\lambda) = \sign \prod_{1 \leq i < j \leq 5} \{ \cos(2\pi \frac{\lambda^+[i]}{\kappa}) - \cos(2\pi \frac{\lambda^+[j]}{\kappa}) \}

\times \prod_{s_i} \{ \cos(\pi \frac{s_1\lambda^+[1] + \lambda^+[5]}{\kappa}) - \cos(\pi \frac{\lambda^+[6] + s_2\lambda^+[2] + s_3\lambda^+[3] + s^\prime \lambda^+[4]}{\kappa}) \}
$$

(E6.2)

where the second product is over all 8 choices of signs $s_1, s_2, s_3 \in \{ \pm 1 \}$, and $s^\prime = s_1s_2s_3$.

3.E7. The algebra $E_7$. The colabels for $E_7$ are $1,2,3,4,3,2,2$, respectively. The scale factor in (1b) is $s = -i/\sqrt{2}$, and $\kappa = k + 18$. To the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_7)$ associate the orthogonal coordinates $\lambda[\ell] = \frac{\lambda_7 - \lambda_2}{2} + \sum_{i=2}^{\ell} \lambda_i$ for $\ell \leq 6$, (interpret $\sum_{i=2}^{1} \lambda_i$ as 0) and $\lambda[7] = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 2\lambda_7$. 

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The 63 coset representatives for $W(A_1D_6)\backslash W(E_7)$ are given in Table 1, recursively defined in terms of the reflections $r_i$, $i = 1, \ldots, 7$, through the simple roots $\alpha_i$. Then

$$S_{\lambda, \mu} = \frac{32\sqrt{2}}{\kappa^{\ell/2}} \sum_{\ell=1}^{63} (\det c_\ell) \sin\left(\pi \frac{(c_\ell, \lambda)^+[\ell]}{\kappa} \mu^+[\ell]\right)$$

$$\times \{\det(\cos(2\pi \frac{(c_\ell, \lambda)^+[i]}{\kappa} \mu^+[j])_{1 \leq i, j \leq 6} - \det(\sin(2\pi \frac{(c_\ell, \lambda)^+[i]}{\kappa} \mu^+[j])_{1 \leq i, j \leq 6})\}$$

(E7.1)

Recall that the numbers $(c_\ell, \lambda)^+[i]$ are the orthogonal coordinates of the weight $c_\ell(\lambda + \rho)$. The orbit representative $[\lambda]$ in (7c) can be found as follows.

(i) Choose one of the 63 coset representatives $c_\ell$. Write $x_i := (c_\ell, \lambda)^+[7 - i]$, $i = 1, \ldots, 6$, and $x_7 := (c_\ell, \lambda)^+[7]$, for the orthogonal coordinates of $c_\ell(\lambda + \rho)$. (The $'[7 - i]'$ is to make the indices here consistent with those of $D_6$).

(ii) By adding an appropriate multiple of $2\kappa$ to $x_7$ and replacing $x_7$ if necessary by $\kappa - x_7$, get $0 \leq y_7 < 3\kappa/2$. Similarly, using the $D_6$ movements for the given value of $\kappa$, get $0 \leq |y_6| < y_5 < y_4 < y_3 < y_2 < y_1$ and $y_1 + y_2 < \kappa$.

(iii) Write $t = (\gamma - y_1 - y_2 - y_3 - y_4 - y_5 + y_6 + y_7)/2$. If both $t > 0$ and $t \in \mathbb{Z}$, then the Dynkin labels of the desired weight $[\lambda]$ are

$$[\lambda]_1 = t - 1, \quad [\lambda]_7 = y_5 + y_6 - 1, \quad [\lambda]_i = x_{7-i} - x_{8-i} - 1 \quad 2 \leq i \leq 6$$

(iv) Otherwise replace $y_7 = \kappa - y_7$, $y_1 = y_6 - \kappa/2$, $y_6 = y_1 + \kappa/2$, and $y_i = \kappa/2 - y_{7-i}$ for $i = 2, 3, 4, 5$. Define $t$ as in (iii); if both $t > 0$ and $t \in \mathbb{Z}$ then write $[\lambda]_i$ as in (iii).

(v) Otherwise try the next $c_\ell$ in (i).

Equation (8) becomes

$$\epsilon(\lambda) = \sign\sin\left(\pi \frac{\lambda^+[7]}{\kappa}\right) \prod_{1 \leq i < j \leq 6} \{\cos(2\pi \frac{\lambda^+[i]}{\kappa}) - \cos(2\pi \frac{\lambda^+[j]}{\kappa})\}$$

$$\times \prod_{s_i} \{\cos\left(\frac{\pi}{\kappa} s_1 \lambda^+[1] + s_2 \lambda^+[2] + s_3 \lambda^+[3] + \lambda^+[4]\right) - \cos\left(\pi \frac{\lambda^+[7] + s_5 \lambda^+[5] - s' \lambda^+[6]}{\kappa}\right)\}$$

(E7.2)

where the second product is over all 16 choices of signs $s_1, s_2, s_3, s_5 \in \{\pm 1\}$, and $s' = s_1 s_2 s_3 s_5$.

3.E8. The algebra $E_8$. The colabels here are 2,3,4,5,6,4,2,3, respectively. The scale factor in (1b) is $s = 1$, and $\kappa = k + 30$. To the weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_8)$ associate the orthogonal coordinates $\lambda[\ell] = \frac{\lambda_\ell - \lambda_8}{2} + \sum_{i=8-\ell}^{6} \lambda_i$ for $1 \leq \ell \leq 7$, (interpret $\sum_{i=7}^{6} \lambda_i$ as 0) and $\lambda[8] = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 5\lambda_5 + \frac{3}{2} \lambda_6 + 2\lambda_7 + \frac{5}{2} \lambda_8$.

The 135 coset representatives for $W(D_8)\backslash W(E_8)$ are given in Table 2, recursively defined in terms of the reflections $r_i$, $i = 1, \ldots, 8$, through the simple roots $\alpha_i$. Then

$$S_{\lambda, \mu} = \frac{128}{\kappa^4} \sum_{\ell=1}^{135} (\det c_\ell) \{\det(\cos(2\pi \frac{(c_\ell, \lambda)^+[i]}{\kappa} \mu^+[j])_{1 \leq i, j \leq 8})$$

$$+ \det(\sin(2\pi \frac{(c_\ell, \lambda)^+[i]}{\kappa} \mu^+[j])_{1 \leq i, j \leq 8})\}$$

(E8.1)
Recall that the numbers \((c_\ell,\lambda)^+[i]\) are the orthogonal coordinates of the weight \(c_\ell(\lambda + \rho)\).

The orbit representative \([\lambda]\) in (7c) can be found as follows.

(i) Choose one of the 135 coset representatives \(c_\ell\). Write \(x_i := (c_\ell,\lambda)^+[9-i], i = 1, \ldots, 8\), for the orthogonal coordinates of \(c_\ell(\lambda + \rho)\) (the \('[9-i]'\) is to make the indices here consistent with those of \(D_8\)).

(ii) Using the \(D_8\) movements for the given value of \(\kappa\), get \(0 \leq |y_8| < y_7 < \cdots < y_1\) and \(y_1 + y_2 < \kappa\).

(iii) Write \(t = y_1 - y_2 - y_3 - y_4 - y_5 - y_6 - y_7 + y_8\). If \(t > 0\) then the Dynkin labels of the desired weight \([\lambda]\) are

\[
[\lambda]_7 = t/2 - 1, \quad [\lambda]_8 = y_7 + y_8 - 1, \quad [\lambda]_i = y_{i+1} - y_{i+2} - 1 \quad 1 \leq i \leq 6
\]

(iv) Otherwise replace each \(y_i\) with \(\kappa/2 - y_{9-i}\). Compute the resulting \(t\) as in (iii); if \(t\) is positive than write \([\lambda]_i\) as in (iii).

(v) Otherwise try the next coset representative \(c_\ell\) in (i).

Equation (8) becomes

\[
\epsilon(\lambda) = \text{sign} \prod_{1 \leq i < j \leq 8} \{\cos(2\pi \frac{\lambda^+[i]}{\kappa}) - \cos(2\pi \frac{\lambda^+[j]}{\kappa})\} \quad (E8.2)
\]

\[
\times \prod_{s_i} \{\cos(\pi \frac{s_1 \lambda^+[1] + s_2 \lambda^+[2] + s_3 \lambda^+[3] + \lambda^+[7]}{\kappa}) - \cos(\pi \frac{s_4 \lambda^+[4] + s_5 \lambda^+[5] + s' \lambda^+[6] + \lambda^+[8]}{\kappa})\}
\]

where the second product is over all 32 choices of signs \(s_1, s_2, s_3, s_4, s_5 \in \{\pm 1\}\), and where \(s' = s_1 s_2 s_3 s_4 s_5\).

3.F4. The algebra \(F_4\). The colabels are 2,3,2,1, resp. The scale factor in (1b) is \(s = 1/2\) and \(\kappa = k + 9\). To the weight \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) associate the orthogonal coordinates

\[(\lambda[1], \lambda[2], \lambda[3], \lambda[4]) = (\lambda_1 + 2\lambda_2 + \frac{3}{2}\lambda_3 + \lambda_4, \lambda_1 + \lambda_2 + \frac{1}{2}\lambda_3, \lambda_2 + \frac{1}{2}\lambda_3, \frac{1}{2}\lambda_3)\]

Let \(r_i, i = 1, \ldots, 4\) be the reflections through the simple roots \(\alpha_i\). Define the matrix

\[
c = \frac{1}{2}\begin{pmatrix}
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1
\end{pmatrix}
\]

It is orthogonal, with determinant +1. Then

\[
S_{\lambda,\mu} = \frac{8}{\kappa^2} \{(\det(2\pi \frac{\lambda^+[i]}{\kappa}) + \det(2\pi \frac{c_\ell(\lambda)^+[i]}{\kappa} \mu^+[j]) + \det(\sin(2\pi \frac{(c^t, \lambda)^+[i]}{\kappa} \mu^+[j])))_{1 \leq i, j \leq 4}
\]

\[
\quad + \det(\sin(2\pi \frac{(c^t, \lambda)^+[i]}{\kappa} \mu^+[j]))_{1 \leq i, j \leq 4}\} \quad (F4.1)
\]
In (F4.1), \((c,\lambda)^+[i]\) denotes the \(i\)th orthogonal coordinate of the matrix product of \(c\) with the column vector with entries \((\lambda + \rho)[j]\), and \(c^t\) means the transpose(=inverse) of \(c\).

The orbit representative \([\lambda]\) in (7c) can be found as follows.

(i) Let \(x_i = \lambda^+[i]\) for \(i = 1, 2, 3, 4\). Apply the \(B_4\) algorithm (steps (ii)–(iv)) with the given value of \(\kappa\) to obtain the 4-tuple \((y_i)\), \(\kappa > y_1 > y_2 > y_3 > y_4 > 0\).

(ii) If \(y_1 > y_2 + y_3 + y_4\), proceed directly to (iii); otherwise first replace \(y_i\) with the \(i\)th coordinate of the matrix product of \(c\) with the column vector \((y_i)\), and if necessary change the sign of the resulting \(y_4\) so that it’s positive.

(iii) The desired weight \([\lambda]\) has Dynkin labels

\[
[\lambda]_1 = y_2 - y_3 - 1 \ , \ [\lambda]_2 = y_3 - y_4 - 1 \ , \ [\lambda]_3 = 2y_4 - 1 \ , \ [\lambda]_4 = y_1 - y_2 - y_3 - y_4 - 1
\]

Equation (8) becomes

\[
\epsilon(\lambda) = \text{sign} \prod_{i=1}^{4} \sin(\pi \frac{\lambda^+[i]}{\kappa}) \prod_{1 \leq i < j \leq 4} \{\cos(2\pi \frac{\lambda^+[j]}{\kappa}) - \cos(2\pi \frac{\lambda^+[i]}{\kappa})\}
\]

\[
\prod_{s_2, s_3} \{\cos(\pi \frac{\lambda^+[4]}{\kappa}) - \cos(\pi \frac{\lambda^+[1] + s_2 \lambda^+[2] + s_3 \lambda^+[3]}{\kappa})\}
\]

\[
(F4.2)
\]

where the product is over the four choices of signs \(s_2, s_3 \in \{\pm 1\}\).

The embedding \(B_4 \subset F_4\) (resp. \(D_4 \subset F_4\)) let us write the \(F_{4,k}\) quantities in terms of the more familiar \(B_{4,k+2}\) (resp. \(D_{4,k+3}\)) ones, and this can be useful (see [31]). In particular, a weight \(\lambda \in P_+^k(F_4)\) corresponds to the orbit of \(\lambda' = (\lambda_2 + \lambda_3 + \lambda_4 + 2, \lambda_1, \lambda_2, \lambda_2 + \lambda_3 + 1) \in P_+^{k+1}(D_4)\), under \(P' : \lambda'_3 \leftrightarrow \lambda'_4\) (which changes the chirality of \(D_4\) weights) and triality \(T' : \lambda_1 \rightarrow \lambda_3 \rightarrow \lambda_4 \rightarrow \lambda_1\) (we prime the \(D_4\) quantities). We get \(S_{\lambda \mu} = \sum_{i=0}^1 \sum_{j=0}^2 (-1)^i S_{\lambda'}^{\lambda'} T_{\lambda'}^{\lambda'} \mu'\mu'\)

\[
N_{\lambda \mu} = \sum_{\gamma'} \sum_{i=0}^1 \sum_{j=0}^2 (-1)^i b_{\gamma'}^{\lambda} N_{\gamma'}^{\lambda'} T_{\gamma'}^{\lambda'} \mu'\mu'
\]

where \(\oplus_{\gamma'} b_{\gamma'}^{\lambda} (\gamma') = (\lambda)\) are \(D_4 \subset F_4\) branching rules (see e.g. [30]).

3.G2. The algebra \(G_2\). The colabels here are 2,1, respectively. The scale factor in (1b) is \(s = -1/\sqrt{3}\), and \(\kappa = k+4\). To the weight \(\lambda = (\lambda_1, \lambda_2)\) associate the orthogonal coordinates \((\lambda[1], \lambda[2], \lambda[3]) = (\lambda_1 + \lambda_2, \lambda_1, -2\lambda_1 - \lambda_2)\). The finite Weyl group is the symmetric group \(S_3\), together with the sign change \(\lambda \mapsto -\lambda\). Then

\[
S_{\lambda, \mu} = \frac{-2}{\sqrt{3} \kappa} \{c(2mm' + mn' + nm' + 2nn') + c(-mm' - 2mn' - nn' + nm') \}
\]

\[
(G2.1)
\]

\[
c(-mm' + mn' - 2nm' - nn') - c(-mm' - 2mn' - 2nm' - nn') - c(2mm' + mn' + nm' - nn') - c(-mm' + mn' + nm' + 2nn')
\]

where for convenience we put \(c(x) = \cos(2\pi x/3\kappa)\), \(m = \lambda^+[1]\), \(n = \lambda^+[2]\), \(m' = \mu^+[1]\), and \(n' = \mu^+[2]\).
The orbit representative \([\lambda]\) in (7c) can be found as follows.

(i) Let \(x_i = \lambda^+[i]\).
(ii) By adding the appropriate multiples of \(\kappa\) to each \(x_i\), find the unique numbers \(0 < y_1, y_2, y_3 < \kappa\) such that \(x_i \equiv y_i \pmod{\kappa}\).
(iii) If \(y_1 + y_2 + y_3 = 2\kappa\), then replace each \(y_i\) with \(\kappa - y_i\).
(iv) Reorder the \(y_i\) so that \(y_3 > y_1 > y_2\). The Dynkin labels of the desired weight are \([\lambda]_1 = y_2 - 1\) and \([\lambda]_2 = y_1 - y_2 - 1\).

Equation (8) becomes

\[
\epsilon(\lambda) = \text{sign}\{(s(3a) s(b) s(3a + b) s(3a + 2b) s(a + b) s(6a + 3b)\} \tag{G2.2}
\]

where for convenience we write \(a = \lambda_1 + 1\) and \(b = \lambda_2 + 1\), and put \(s(x) = \sin(\pi x/3\kappa)\).

The embedding \(A_2 \subset G_2\) lets us write the \(G_{2,k}\) quantities in terms of the more familiar \(A_{2,k+1}\) ones, and this can be useful (see [31]). In particular, any weight \(\lambda \in P^{k+1}_+(G_2)\) corresponds to a pair \(\lambda' = (\lambda_1, \lambda_1 + \lambda_2 + 1), C' \lambda' = (\lambda_1 + \lambda_2 + 1, \lambda_1)\) of weights in \(P^{k+1}_+(A_2)\). We get \(S_{\lambda_1 \mu} = iS_{\lambda' \mu'} - iS_{\lambda' \mu'}\) and

\[
N_{\lambda_{\mu}} = \sum_{\gamma'} b_{\gamma'}^{\lambda} N_{\gamma' \mu'} - N_{\gamma', C' \mu'} \tag{G2.2}
\]

where \(\oplus_{\gamma'} b_{\gamma'}^{\lambda} (\gamma') = (\lambda)\) are \(A_2 \subset G_2\) branching rules (see e.g. [30]), and \(C': \mu_1 \leftrightarrow \mu_2\) is \(A_2\) charge-conjugation.

4. Further remarks

4.1. Error analysis and exact results. The error analysis of the Gaussian elimination method for computing determinants is surprisingly subtle and has been the subject of extensive study — see e.g. [22]. It depends on the size of the pivots, but very typically the error grows by a factor on the order of \(\sqrt{n}\) for an \(n \times n\) matrix. A good idea however is to make sure in your program that the pivots are never too close to 0.

Because our quantities \(S_{\lambda_1 \mu}\) are all cyclotomic integers (up to a global rescaling) in some field \(\mathbb{Q}[\exp(2\pi i/n)]\), the obvious way to make all calculations exact is to do it over the polynomials \(p(x)\) with integer coefficients. The desired numerical value would then simply be \(p(\exp(2\pi i/n))\). These polynomials add, multiply etc in the usual way, but we regard two polynomials as equal if they differ by a multiple of the cyclotomic polynomial \(\phi_n(x)\). The cyclotomic polynomial is the polynomial of smallest positive degree, with integer coefficients, which has \(\exp(2\pi i/n)\) as a root. For example, \(\phi_2(x) = x^{p-1} + x^{p-2} + \cdots + 1\) and \(\phi_n(x) = \phi_{p_1\cdots p_n}(x^{n/p_1\cdots p_n})\) where \(n = p_1^{e_1} \cdots p_n^{e_n}\). A manifestly integral algorithm for computing any \(\phi_n\) is provided in [25]. Because \(\phi_n\) has degree \(\varphi(n) = \prod_i (p_i^{e_i} - 1)\), we can require each of our polynomials \(p(x)\) to be of degree less than \(\varphi(n)\).

But there is an alternate approach, which is perhaps a little simpler. We represent each number by an integer polynomial \(p(x)\), and always reduce any exponents modulo \(n\) (i.e. identify \(x^n\) and 1, \(x^{n+1}\) and \(x\), etc). When \(n\) is even, we can if we like reduce
exponents modulo $n/2$, by identifying $x^{n/2}$ with $-1$, etc. We equate two polynomials

$$p(x), q(x) \in \mathbb{Z}[x]$$

if

$$|p(\exp[2\pi ik/n]) - q(\exp[2\pi ik/n])| < 0.5$$

holds for all $k$, $1 \leq k < n$, coprime to $n$. Of course, ‘$(\exp[2\pi ik/n])^n$’ is evaluated as

$$\cos(2\pi km/n) + i \sin(2\pi km/n),$$

to decrease error. The point is that if an algebraic integer $z$ (such as $z = p(\exp[2\pi ik/n]) - q(\exp[2\pi ik/n])$ and all of its Galois associates $\sigma(z)$ have modulus $< 1$, then $z$ must equal 0. The reason is that $\prod_\sigma(-\sigma(z)) \in \mathbb{Z}$ is the constant term in the minimal polynomial of $z$.

### 4.2. Character formulas

The expressions in §3 for $S$ don’t depend on the integrality of the weight $\mu$, and thus taking the appropriate ratio (4a) gives determinant expressions for finite-dimensional Lie group characters.

### 4.3. Fusion coefficients

Recall the Kac-Walton formula (2c). Consider any $\lambda, \mu \in P^+$. A tensor product coefficient $T^\nu_{\lambda\mu}$ will contribute $\epsilon(\nu) T^\nu_{\lambda\mu}$ to the fusion coefficient $N^{[\nu]}_{\lambda\mu}$ (recall that $\epsilon(\nu)$ and $[\nu]$ are computed in §3). In particular, any null weight $\nu$ in the tensor product of $\lambda$ and $\mu$ can be ignored. When the level is sufficiently high (specifically, $\lambda_0 + \mu_0 \geq k$), the fusion product of $\lambda$ and $\mu$ will equal their tensor product.

### 4.4. Galois action

By (1c) we see that the entries of $S$ lie in a cyclotomic field $\mathbb{Q}[\xi_N]$ for some root of unity $\xi_N = \exp[2\pi i/N]$. This simply means that each $S_{\lambda\mu}$ can be written as a polynomial $p_{\lambda\mu}(\xi_N)$ in $\xi_N$ with rational coefficients. For $A_r, B_r, \ldots, G_2$, respectively, we can take

$$N = 4(r + 1)k, 4k, 4k, 4k, 12k, 8k, 2k, 2k, 6k$$

but usually this is larger than necessary (see [18]).

Take any Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$. The group $\text{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$ is isomorphic to the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$ of numbers coprime to $N$, so to $\sigma$ we can assign an integer $\ell$; explicitly, $\sigma = \sigma_\ell$ takes the number $p_{\lambda\mu}(\xi_N)$ to $p_{\lambda\mu}(\xi_N^\ell)$. For example, $\ell = -1$ corresponds to complex conjugation. To the Galois automorphism $\sigma$ is a sign $\epsilon_\sigma : P^+_k \to \{\pm\}$ and a permutation $\lambda \mapsto \sigma(\lambda)$ of $P^+_k$, such that [16]

$$\sigma(S_{\lambda\mu}) = \epsilon_\sigma(\lambda) S_{\sigma(\lambda),\mu} = \epsilon_\sigma(\mu) S_{\lambda,\sigma(\mu)}$$

(9a)

For example, $\ell = -1$ corresponds to ‘charge-conjugation’ $\lambda \mapsto C\lambda$ and parity $\epsilon_{-1}(\lambda) = +1$.

Some properties and formulas for $\epsilon_\sigma(\lambda)$ for $A_{r,k}$ are given in [20]. For arbitrary $\sigma = \sigma_\ell$, the permutation $\lambda \mapsto \sigma(\lambda)$ and parity $\epsilon_\sigma(\lambda)$ are computed by

$$\sigma(\lambda) = [\ell(\lambda + \rho) - \rho]$$

(9b)

$$\epsilon_\sigma(\lambda) = \epsilon_\ell \epsilon(\ell(\lambda + \rho) - \rho)$$

(9c)

A major motivation for writing this paper was to make this Galois action more accessible. In particular, efficient and explicit algorithms for computing $[\nu]$ were given in §3 for each algebra, and by far the fastest algorithms known to this author for computing $\epsilon(\nu)$ use the simplifications to (8) given explicitly in each subsection of §3. The sign
$\epsilon'_\ell$ is a number-theoretic parity related to quadratic residues, and rarely is needed in the applications as it is independent of the weights. It can be computed as follows [16]:

Write $\kappa^{r/2} s$ in the form $R \sqrt{M}$ where $R \in \mathbb{Q}$ and $M \in \mathbb{Z}$ is square-free. Then $\epsilon'_\ell$ equals the Jacobi symbol $(\frac{M}{\ell}) \in \{\pm 1\}$. This is quickly computed, using standard properties of the Jacobi symbol (see e.g. [26]), e.g. quadratic reciprocity and factorisation.

For example, consider again $A_{3,6}$. Here, $N = 40$ works, so for the representatives $\ell \in (\mathbb{Z}/40\mathbb{Z})^\times$ we can take $\ell = \pm 1, \pm 3, \pm 7, \pm 9, \pm 11, \pm 17, \pm 19$. Then, respectively,

$$
\begin{align*}
\sigma_\ell(0) &= 0, (2, 2, 2), (2, 0, 2), (0, 6, 0), (0, 6, 0), (2, 2, 2), (0, 0, 0) \\
\epsilon(\ell \rho - \rho) &= 1, 1, 1, 1, 1, 1 \\
\epsilon'_\ell &= 1, 1, -1, -1, -1, -1, -1
\end{align*}
$$

4.5. Modular invariants. The 1-loop partition function of WZW CFT looks like

$$
Z = \sum_{\lambda, \mu \in P^k} M_{\lambda \mu} \chi_\lambda \chi^*_\mu
$$

where (among other things) each $M_{\lambda \mu} \in \mathbb{Z}_{\geq 0}$ and $MS = SM$. Hitting $MS = SM$ with a Galois automorphism $\sigma$ implies from (9a) that

$$
M_{\sigma \lambda, \sigma \mu} = M_{\lambda \mu} \quad (10a)
$$

$$
M_{\lambda \mu} \neq 0 \quad \Rightarrow \quad \epsilon_\sigma(\lambda) = \epsilon_\sigma(\mu) \quad (10b)
$$

valid for all $\sigma$. The ‘parity rule’ (10b) turns out to be especially powerful. Efficient algorithms to compute $S$ entries and parities are vital in this context and was the primary motivation for writing this paper. See [6] for more details and examples.

4.6. Branching rules. When $X_r \subset \tilde{X}_s$, and the central charge $c$ of $X_r,k$ equals that of $\tilde{X}_s,\ell$, then we say that we have a conformal embedding. In this case, the level $\ell$ modules $L(\lambda)$ of $\tilde{X}^{(1)}_s$ can be decomposed into a finite direct sum $\oplus_i L(\lambda^{(i)})$ of level $k$ modules of $X^{(1)}_r$. These decompositions are called branching rules.

All conformal embeddings are known [27], and most of their branching rules are known (see e.g. [28] and references therein), but some branching rules don’t seem to appear explicitly in the literature, or only appear conjecturally.

Branching rules can be read off from the appropriate modular invariants (see e.g. [29]). In [6] we use the previous algorithms and formulae to obtain branching rules for the conformal embeddings. Mostly this merely provides an independent check; however it also fills some gaps in the literature.

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Table 1. Coset representatives for $W(D_5) \setminus W(E_6)$ and $W(A_1D_6) \setminus W(E_7)$

| length | $c_i$ for $E_6$ | $c_i$ for $E_7$ |
|--------|-----------------|-----------------|
| 0      | $c_1 = I$       | $c_1 = I$       |
| 1      | $c_2 = r_1$     | $c_2 = r_1$     |
| 2      | $c_3 = c_2r_2$  | $c_3 = c_2r_2$  |
| 3      | $c_4 = c_3r_3$  | $c_4 = c_3r_3$  |
| 4      | $c_5 = c_4r_4$, $c_6 = c_4r_6$ | $c_5 = c_4r_4$, $c_6 = c_4r_7$ |
| 5      | $c_7 = c_5r_5$, $c_8 = c_5r_6$ | $c_7 = c_5r_5$, $c_8 = c_5r_7$ |
| 6      | $c_9 = c_7r_6$, $c_{10} = c_8r_3$ | $c_9 = c_7r_6$, $c_{10} = c_7r_7$, $c_{11} = c_8r_3$ |
| 7      | $c_{11} = c_9r_3$, $c_{12} = c_{10}r_2$ | $c_{12} = c_9r_7$, $c_{13} = c_{10}r_3$, $c_{14} = c_{11}r_2$ |
| 8      | $c_{13} = c_{11}r_2$, $c_{14} = c_{11}r_4$, $c_{15} = c_{12}r_1$ | $c_{15} = c_{12}r_3$, $c_{16} = c_{13}r_2$, $c_{17} = c_{13}r_4$, $c_{18} = c_{14}r_1$ |
| 9      | $c_{16} = c_{13}r_1$, $c_{17} = c_{13}r_4$ | $c_{19} = c_{15}r_2$, $c_{20} = c_{15}r_4$, $c_{21} = c_{16}r_1$, $c_{22} = c_{16}r_4$ |
| 10     | $c_{18} = c_{16}r_4$, $c_{19} = c_{17}r_3$ | $c_{23} = c_{19}r_1$, $c_{24} = c_{19}r_4$, $c_{25} = c_{20}r_5$, $c_{26} = c_{21}r_4$, $c_{27} = c_{22}r_3$ |
| 11     | $c_{20} = c_{18}r_3$, $c_{21} = c_{19}r_6$ | $c_{28} = c_{23}r_4$, $c_{29} = c_{24}r_3$, $c_{30} = c_{24}r_5$, $c_{31} = c_{26}r_3$, $c_{32} = c_{27}r_7$ |
| 12     | $c_{22} = c_{20}r_2$, $c_{23} = c_{20}r_6$ | $c_{33} = c_{28}r_3$, $c_{34} = c_{28}r_5$, $c_{35} = c_{29}r_5$, $c_{36} = c_{29}r_7$, $c_{37} = c_{31}r_2$, $c_{38} = c_{31}r_7$ |
| 13     | $c_{24} = c_{22}r_6$ | $c_{39} = c_{33}r_2$, $c_{40} = c_{33}r_5$, $c_{41} = c_{33}r_7$, $c_{42} = c_{35}r_4$, $c_{43} = c_{35}r_7$, $c_{44} = c_{37}r_7$ |
| 14     | $c_{25} = c_{24}r_3$ | $c_{45} = c_{39}r_5$, $c_{46} = c_{39}r_7$, $c_{47} = c_{40}r_4$, $c_{48} = c_{40}r_7$, $c_{49} = c_{42}r_7$, $c_{50} = c_{44}r_3$ |
| 15     | $c_{26} = c_{25}r_4$ | $c_{51} = c_{45}r_4$, $c_{52} = c_{45}r_7$, $c_{53} = c_{46}r_3$, $c_{54} = c_{47}r_7$, $c_{55} = c_{49}r_3$, $c_{56} = c_{50}r_4$ |
| 16     | $c_{27} = c_{26}r_5$ | $c_{57} = c_{51}r_3$, $c_{58} = c_{51}r_7$, $c_{59} = c_{52}r_3$, $c_{60} = c_{53}r_4$, $c_{61} = c_{54}r_3$, $c_{62} = c_{55}r_2$, $c_{63} = c_{56}r_5$ |

The coset representatives $c_j$ are given recursively in terms of the simple reflections $r_i$. 
Table 2. Coset representatives for $W(D_8) \backslash W(E_8)$

| length | $c_i$ for $E_8$ |
|--------|-----------------|
| 0      | $c_1 = I$       |
| 1      | $c_2 = r_7$     |
| 2      | $c_3 = c_2r_6$  |
| 3      | $c_4 = c_3r_5$  |
| 4      | $c_5 = c_4r_4$, $c_6 = c_4r_8$ |
| 5      | $c_7 = c_5r_3$, $c_8 = c_5r_8$ |
| 6      | $c_9 = c_7r_2$, $c_{10} = c_7r_8$, $c_{11} = c_8r_5$ |
| 7      | $c_{12} = c_9r_1$, $c_{13} = c_9r_8$, $c_{14} = c_{10}r_5$, $c_{15} = c_{11}r_6$ |
| 8      | $c_{16} = c_{12}r_8$, $c_{17} = c_{13}r_5$, $c_{18} = c_{14}r_4$, $c_{19} = c_{14}r_6$, $c_{20} = c_{15}r_7$ |
| 9      | $c_{21} = c_{16}r_5$, $c_{22} = c_{17}r_4$, $c_{23} = c_{17}r_6$, $c_{24} = c_{18}r_6$, $c_{25} = c_{19}r_7$ |
| 10     | $c_{26} = c_{21}r_4$, $c_{27} = c_{21}r_6$, $c_{28} = c_{22}r_3$, $c_{29} = c_{22}r_6$, $c_{30} = c_{23}r_7$, $c_{31} = c_{24}r_5$, $c_{32} = c_{24}r_7$ |
| 11     | $c_{33} = c_{26}r_3$, $c_{34} = c_{26}r_6$, $c_{35} = c_{27}r_7$, $c_{36} = c_{28}r_6$, $c_{37} = c_{29}r_5$, $c_{38} = c_{29}r_7$, $c_{39} = c_{31}r_7$, $c_{40} = c_{31}r_8$ |
| 12     | $c_{41} = c_{33}r_2$, $c_{42} = c_{33}r_6$, $c_{43} = c_{34}r_5$, $c_{44} = c_{34}r_7$, $c_{45} = c_{36}r_5$, $c_{46} = c_{36}r_7$, $c_{47} = c_{37}r_7$, $c_{48} = c_{37}r_8$, $c_{49} = c_{39}r_6$, $c_{50} = c_{39}r_8$ |
| 13     | $c_{51} = c_{41}r_6$, $c_{52} = c_{42}r_5$, $c_{53} = c_{42}r_7$, $c_{54} = c_{43}r_7$, $c_{55} = c_{43}r_8$, $c_{56} = c_{45}r_4$, $c_{57} = c_{45}r_7$, $c_{58} = c_{45}r_8$, $c_{59} = c_{47}r_6$, $c_{60} = c_{47}r_8$, $c_{61} = c_{49}r_8$ |
| 14     | $c_{62} = c_{51}r_5$, $c_{63} = c_{51}r_7$, $c_{64} = c_{52}r_4$, $c_{65} = c_{52}r_7$, $c_{66} = c_{52}r_8$, $c_{67} = c_{54}r_6$, $c_{68} = c_{54}r_8$, $c_{69} = c_{56}r_7$, $c_{70} = c_{56}r_8$, $c_{71} = c_{57}r_6$, $c_{72} = c_{57}r_8$, $c_{73} = c_{59}r_8$, $c_{74} = c_{61}r_5$ |
| 15     | $c_{75} = c_{62}r_4$, $c_{76} = c_{62}r_7$, $c_{77} = c_{62}r_8$, $c_{78} = c_{64}r_7$, $c_{79} = c_{64}r_8$, $c_{80} = c_{65}r_6$, $c_{81} = c_{65}r_8$, $c_{82} = c_{67}r_8$, $c_{83} = c_{69}r_6$, $c_{84} = c_{69}r_8$, $c_{85} = c_{70}r_5$, $c_{86} = c_{71}r_8$, $c_{87} = c_{73}r_5$, $c_{88} = c_{74}r_4$ |
| 16     | $c_{89} = c_{75}r_3$, $c_{90} = c_{75}r_7$, $c_{91} = c_{75}r_8$, $c_{92} = c_{76}r_6$, $c_{93} = c_{76}r_8$, $c_{94} = c_{78}r_6$, $c_{95} = c_{78}r_7$, $c_{96} = c_{79}r_5$, $c_{97} = c_{80}r_8$, $c_{98} = c_{82}r_5$, $c_{99} = c_{83}r_5$, $c_{100} = c_{83}r_8$, $c_{101} = c_{84}r_5$, $c_{102} = c_{85}r_6$, $c_{103} = c_{86}r_5$, $c_{104} = c_{87}r_4$, $c_{105} = c_{88}r_3$ |
| 17     | $c_{106} = c_{89}r_7$, $c_{107} = c_{89}r_8$, $c_{108} = c_{90}r_7$, $c_{109} = c_{90}r_8$, $c_{110} = c_{91}r_5$, $c_{111} = c_{92}r_8$, $c_{112} = c_{94}r_5$, $c_{113} = c_{94}r_8$, $c_{114} = c_{95}r_5$, $c_{115} = c_{95}r_6$, $c_{116} = c_{97}r_5$, $c_{117} = c_{98}r_4$ |
| 18     | $c_{118} = c_{106}r_7$, $c_{119} = c_{106}r_8$, $c_{120} = c_{107}r_5$, $c_{121} = c_{108}r_7$, $c_{122} = c_{108}r_8$, $c_{123} = c_{109}r_5$, $c_{124} = c_{110}r_6$, $c_{125} = c_{111}r_5$ |
| 19     | $c_{126} = c_{118}r_5$, $c_{127} = c_{118}r_8$, $c_{128} = c_{119}r_5$, $c_{129} = c_{120}r_4$, $c_{130} = c_{120}r_6$ |
| 20     | $c_{131} = c_{126}r_4$, $c_{132} = c_{128}r_4$, $c_{133} = c_{129}r_6$ |
| 21     | $c_{134} = c_{133}r_5$ |
| 22     | $c_{135} = c_{134}r_8$ |

The coset representatives $c_j$ are given recursively in terms of the simple reflections $r_i$. 