A Derivation of Dirac’s Equation From a Model of an Elastic Medium

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Abstract

Starting from a model of an elastic medium, we derive equations of motion that are identical in form to Dirac’s equation for a spin 1/2 particle with mass, coupled to electromagnetic and gravitational interactions. The mass and electromagnetic terms are not added by hand but emerge naturally from the formalism. A two dimensional version of this equation is derived by starting with a model in three dimensions and deriving equations for the dynamics of the lowest Fourier modes assuming one dimension to be periodic. Generalizations to higher dimensions are discussed.
I. INTRODUCTION

Dirac’s equation describes the behavior of particles with mass and spin and how they couple to the electromagnetic field. The usual form of Dirac’s equation is

\[(\imath \gamma^\mu \partial_\mu - m)\Psi(x) = 0\]

The electromagnetic field is introduced by the minimal coupling prescription\(\partial_\mu \rightarrow D_\mu\), with

\[D_\mu = \partial_\mu + \imath A_\mu(x)\]

where \(A_\mu\) is the electromagnetic vector potential. Dirac’s equation can be further coupled to gravity (at the classical level) using the prescription\(\partial_\mu \rightarrow \partial_\mu - \Gamma_\mu\)

and the equation then takes the form\(\tilde{\gamma}^\mu [\imath \partial_\mu - \imath \Gamma_\mu - A_\mu] \Psi(x) - m \Psi(x) = 0\) (1)

where \(\Gamma_\mu\) is known as the spin connection, \(A_\mu\) is the electromagnetic vector potential, and \(m\) is the mass. The gravitational coupling enters through the modified dirac matrices \(\tilde{\gamma}_\mu\) which satisfy the anticommutation relation

\[\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = I g^{\mu\nu}.\]

and the operator \(\tilde{\gamma}^\mu [\partial_\mu - \Gamma_\mu]\) is (in the absence of electromagnetic interactions) the covariant derivative for spinor fields in a curved space\(\tilde{\gamma}_\mu\).

The above form of Dirac’s equation describes the dynamics of the spinor field \(\Psi\) when coupled to the scalar fields \(A_\mu\) and gravity. There are two additional equations which describe the dynamics of \(A_\mu\) and \(g_{\mu\nu}\), these are the Einstein field equations

\[R_{\mu\nu} - \frac{1}{2} R = T_{\mu\nu}\] (2)

and Maxwell’s equations

\[\nabla_\mu F^{\mu\nu} = 4\pi e \bar{\Psi} \gamma^\nu \Psi\] (3)
The Equations (1), (2) and (3) are collectively known as the Einstein-Dirac-Maxwell equations. The subject of this paper is Equation (1). We will show that the equations of motion of an elastic solid have the same form as Equation (1) with the mass and electromagnetic term emerging naturally from the formalism.

II. ELASTICITY THEORY

The theory of elasticity is usually concerned with the infinitesimal deformations of an elastic body. We assume that the material points of a body are continuous and can be assigned a unique label \( \vec{a} \). For definiteness the elastic body can be taken to be a three dimensional object so each point of the body may be labeled with three coordinate numbers \( a_i \) with \( i = 1, 2, 3 \).

If this three dimensional elastic body is placed in a large ambient three dimensional space then the material coordinates \( a_i \) can be described by their positions in the 3-D fixed space coordinates \( x_i \) with \( i = 1, 2, 3 \). In this description the material points \( a_i(x_1, x_2, x_3) \) are functions of \( \vec{x} \). A deformation of the elastic body results in infinitesimal displacements of these material points. If before deformation a material point \( a_0 \) is located at fixed space coordinates \( x_1, x_2, x_3 \) then after deformation it will be located at some other coordinate \( x_1', x_2', x_3' \). The deformation of the medium is characterized at each point by the displacement vector

\[
\mathbf{u}_i = x'_i - x_i
\]

which measures the displacement of each point in the body after deformation.

It is the aim of this paper to take this model of an elastic medium and derive from it equations of motion that have the same form as Dirac’s equation.

We first consider the effect of a deformation on the measurement of distance. After our elastic body is deformed, the distances between its points changes as measured with the fixed space coordinates. If two points which are very close together are separated by a radius vector \( dx_i \) before deformation, these same two points are separated by a vector \( dx'_i = dx_i + du_i \). The square distance between the points before deformation is then \( ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \). Since these coincide with the material points in the undeformed state, this can be written \( ds^2 = da_1^2 + da_2^2 + da_3^2 \). The squared distance after deformation can be written \( ds'^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 = \sum_i dx_i'^2 = \sum_i(da_i + du_i)^2 \). The differential element \( du_i \) can be written as
\[ du_i = \sum_i \frac{\partial u_i}{\partial a_k} da_k, \] which gives for the distance between the points

\[
ds^2 = \sum_i \left( da_i + \sum_k \frac{\partial u_i}{\partial a_k} da_k \right) \left( da_i + \sum_l \frac{\partial u_i}{\partial a_l} da_l \right)
\]

\[
= \sum_i \left( da_i da_i + \sum_k \frac{\partial u_i}{\partial a_k} da_k da_i + \sum_l \frac{\partial u_i}{\partial a_l} da_l da_i + \sum_k \sum_l \frac{\partial u_i}{\partial a_k} \frac{\partial u_i}{\partial a_l} \right)
\]

\[
= \sum_i \sum_k \left( \delta_{ik} + \frac{\partial u_i}{\partial a_k} + \sum_l \frac{\partial u_i}{\partial a_i} \frac{\partial u_l}{\partial a_k} \right) da_k da_i
\]

\[
= \sum_{ik} \left( \delta_{ik} + 2\epsilon'_{ik} \right) da_i da_k
\]

where \( \epsilon'_{ik} \) is

\[
\epsilon'_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial a_k} + \frac{\partial u_k}{\partial a_i} + \sum_l \frac{\partial u_i}{\partial a_i} \frac{\partial u_l}{\partial a_k} \right)
\]

(4)

The quantity \( \epsilon'_{ik} \) is known as the strain tensor. It is fundamental in the theory of elasticity. In most treatments of elasticity it is assumed that the displacements \( u_i \) as well as their derivatives are infinitesimal so the last term in Equation (4) is dropped. This is an approximation that we will not make in this derivation.

The quantity

\[
g_{ik} = \delta_{i,k} + \frac{\partial u_i}{\partial a_k} + \sum_l \frac{\partial u_i}{\partial a_i} \frac{\partial u_l}{\partial a_k}
\]

(5)

\[
= \delta_{i,k} + 2\epsilon'_{ik}
\]

is the metric for our system and determines the distance between any two points.

That this metric is simply the result of a coordinate transformation from the flat space metric can be seen by writing the metric in the form

\[
g_{\mu\nu} = \begin{pmatrix}
\frac{\partial x'_1}{\partial a_1} & \frac{\partial x'_2}{\partial a_1} & \frac{\partial x'_3}{\partial a_1} \\
\frac{\partial x'_1}{\partial a_2} & \frac{\partial x'_2}{\partial a_2} & \frac{\partial x'_3}{\partial a_2} \\
\frac{\partial x'_1}{\partial a_3} & \frac{\partial x'_2}{\partial a_3} & \frac{\partial x'_3}{\partial a_3}
\end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix}
\frac{\partial x_1}{\partial a_1} & \frac{\partial x_2}{\partial a_1} & \frac{\partial x_3}{\partial a_1} \\
\frac{\partial x_1}{\partial a_2} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_3}{\partial a_2} \\
\frac{\partial x_1}{\partial a_3} & \frac{\partial x_2}{\partial a_3} & \frac{\partial x_3}{\partial a_3}
\end{pmatrix}
\]

\[= J^T I J \]

where

\[
\frac{\partial x'_\mu}{\partial x_\nu} = \delta_{\mu\nu} + \frac{\partial u_\mu}{\partial a_\nu}.
\]
and $J$ is the Jacobian of the transformation. Later in section VI we will show that the metric for the Fourier modes of our system is not a simple coordinate transformation.

The inverse matrix $(g^{ik}) = (g_{ik})^{-1}$ is given by $(g^{ik}) = (J^{-1})(J^{-1})^T$ where

$$
J^{-1} = \begin{pmatrix}
\frac{\partial a_1}{\partial x_1'} & \frac{\partial a_1}{\partial x_2'} & \frac{\partial a_1}{\partial x_3'} \\
\frac{\partial a_2}{\partial x_1'} & \frac{\partial a_2}{\partial x_2'} & \frac{\partial a_2}{\partial x_3'} \\
\frac{\partial a_3}{\partial x_1'} & \frac{\partial a_3}{\partial x_2'} & \frac{\partial a_3}{\partial x_3'}
\end{pmatrix}
$$

(6)

This yields for the inverse metric

$$
g^{ik} = \delta_{ik} - \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k}
$$

(7)

where $\epsilon_{ik}$ is defined by

$$
\epsilon_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \sum_l \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k} \right)
$$

We see that the metric components involves derivatives of the displacement vector with respect to the internal coordinates and the inverse metric involves derivatives with respect to the fixed space coordinates.

### III. EQUATIONS OF MOTION

In the following we will use the notation

$$
u_{\mu\nu} = \frac{\partial u_\mu}{\partial x_\nu}
$$

and therefore the inverse strain tensor is

$$
\epsilon_{\mu\nu} = \frac{1}{2} \left( \nu_{\mu\nu} + \nu_{\nu\mu} + \sum_\beta \nu_{\beta\mu} \nu_{\beta\nu} \right).
$$

We will use the lagrangian method to derive the equations of motion for our system. Our model consists of an elastic solid embedded in a 3 dimensional euclidean space.

In the following we work in the fixed space coordinates and take the strain energy as the lagrangian density of our system. This approach leads to the usual equations of equilibrium.
in elasticity theory. The strain energy is quadratic in the strain tensor $\epsilon^{\mu\nu}$ and can be written as

$$ E = \sum_{\mu\nu\alpha\rho} C_{\mu\nu\alpha\rho} \epsilon_{\mu\nu} \epsilon_{\alpha\rho} $$

The quantities $C_{\mu\nu\alpha\rho}$ are known as the elastic stiffness constants of the material. For an isotropic space most of the coefficients are zero and in 3 dimensions, the lagrangian density reduces to

$$ L = (\lambda + 2\mu) \left[ \epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 \right] + 2\lambda \left[ \epsilon_{11}\epsilon_{22} + \epsilon_{11}\epsilon_{33} + \epsilon_{22}\epsilon_{33} \right] + 4\mu \left[ \epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{33}^2 \right] $$

where $\lambda$ and $\mu$ are known as Lamé constants.

The usual Lagrange equations,

$$ \sum_{\nu} \frac{d}{dx_{\nu}} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) - \frac{\partial L}{\partial u_{\rho}} = 0, $$

apply with each component of the displacement vector treated as an independent field variable. Since our Lagrangian contains no terms in the field $u_{\rho}$, Lagrange’s equations reduce to

$$ \sum_{\nu} \frac{d}{dx_{\nu}} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) = 0. $$

The quantity

$$ V_{\rho} = \sum_{\nu} \frac{d}{dx_{\nu}} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) $$

is a vector and as such can always be written as the sum of the gradient of a scalar and the curl of a vector or

$$ \vec{V} = \nabla \phi + \nabla \times \vec{A}. $$

From this decomposition we can immediately conclude,

$$ \nabla^2 \phi = 0 $$

We see therefore that the scalar quantity $\phi$ in the medium obeys Laplace’s equation.

**A. Physical Interpretation of $\phi$**

To understand the physical origin of $\phi$ we derive its form in the usual infinitesimal theory of elasticity. The advantage of the infinitesimal theory is that an explicit form of the vector $\vec{V}$ may be obtained. In the infinitesimal theory of elasticity the strain components $u_{\mu\nu}$ are
assumed to be small quantities and therefore the quadratic terms in the strain tensor are dropped and the strain tensor reduces to

\[
\epsilon_{\mu\nu} = \frac{1}{2} (u_{\mu\nu} + u_{\nu\mu})
\]

Using the above Lagrangian we obtain the explicit form

\[
V_\rho = \sum_\nu \frac{d}{dx_\nu} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) = (2\mu + 2\lambda) \frac{\partial \sigma}{\partial x_\rho} + 2\mu \nabla^2 u_\rho = 0,
\]

where \(\sigma = u_{11} + u_{22} + u_{33} \equiv \nabla \cdot \vec{u}\).

Finally taking the divergence of \(\vec{V}\) yields

\[
\nabla^2 \sigma = 0
\]

From this we see that the scalar in the infinitesimal theory is the divergence of the strain field \(\sigma = \nabla \cdot \vec{u}\). It is an invariant with respect to change of coordinates and in general varies from point to point in the medium. This exercise exhibits the physical origin of \(\phi\) which to lowest order in the strain components is the divergence of the strain field.

In this work however will not make the infinitesimal approximation and we will work with the scalar \(\phi\) and not \(\sigma\). In most of what follows, the exact form of \(\phi\) is not important. It is only important that such a quantity exists and obeys Laplace’s equation.

**B. Internal Coordinates**

The central results of this work will be given in sections \(\S V\) and \(\S VI\) where we will take one of our internal coordinates to be periodic and we will Fourier transform all quantities in that coordinate. We therefore need to translate the equations of motion \(\nabla^2 \phi = 0\) from the fixed space coordinates to the internal coordinates. For clarity, in the remainder of this text we change notation slightly and write the internal coordinates not as \(a_i\) but as \(x'_i\) and the fixed space coordinates will be unprimed and denoted \(x_i\). Now using \(u_i = x'_i - x_i\) we can write

\[
\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j}
\]

\[
= \sum_j \left( \frac{\partial x_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial}{\partial x'_j}
\]

\[
= \sum_j \left( \delta_{ij} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial}{\partial x'_j}
\]

(10)
Equation (10) relates derivatives in the fixed space coordinates $x_i$ to derivatives in the material coordinates $x'_i$. As mentioned earlier, in the standard treatment of elastic solids the displacements $u_i$ as well as their derivatives are assumed to be infinitesimal and so the second term in Equation (10) is dropped and there is no distinction made between the $x_i$ and the $x'_i$ coordinates. In this paper we will keep the nonlinear terms in Equation (10) when changing coordinates. Hence we will make a distinction between the two sets of coordinates and this will be pivotal in the derivations to follow.

We will now demonstrate that Laplace’s equation (9) implies Dirac’s equation.

IV. CARTAN’S SPINORS

The concept of Spinors was introduced by Eli Cartan in 1913\textsuperscript{14}. In Cartan’s original formulation spinors were motivated by studying isotropic vectors which are vectors of zero length. In three dimensions the equation of an isotropic vector is

$$x_1^2 + x_2^2 + x_3^2 = 0 \tag{11}$$

for complex quantities $x_i$. A closed form solution to this equation is realized as

$$x_1 = \xi_0^2 - \xi_1^2, \quad x_2 = i(\xi_0^2 + \xi_1^2), \quad \text{and} \quad x_3 = -2\xi_0\xi_1 \tag{12}$$

where the two quantities $\xi_i$ are

$$\xi_0 = \pm \sqrt{\frac{x_1 - ix_2}{2}} \quad \text{and} \quad \xi_1 = \pm \sqrt{\frac{-x_1 - ix_2}{2}}.$$  

That the two component object $\xi = (\xi_0, \xi_1)$ is a spinor\textsuperscript{14} can be seen by considering a rotation on the quantities $v_1 = x_1 - ix_2$, and $v_2 = -x_1 - ix_2$. If $v_i$ is rotated by an angle $\alpha$,

$$v_i \rightarrow v_i \exp(i\alpha)$$

then the spinor component $\xi_0$ is rotated by $\alpha/2$. It is clear that the spinor is not periodic in $2\pi$ but in $4\pi$. A quantity of this type is a spinor and any equation of the form (11) has a spinor solution.

Laplace’s equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \phi = 0$$

8
can be viewed as an isotropic vector in the following way. The components of the vector are the partial derivative operators \( \frac{\partial}{\partial x_i} \) acting on the quantity \( \phi \). As long as the partial derivatives are restricted to acting on the scalar field \( \phi \) it has a spinor solution given by

\[
\hat{\xi}_0^2 = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) = \frac{\partial}{\partial z_0}
\]

and

\[
\hat{\xi}_1^2 = -\frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) = \frac{\partial}{\partial z_1}
\]

where

\[ z_0 = x_1 + ix_2 \quad \text{and} \quad z_1 = -x_1 + ix_2 \]

and the “hat” notation indicates that the quantities \( \hat{\xi} \) are operators. The equations

\[
\hat{\xi}_0^2 = \frac{\partial}{\partial z_0}
\]

and

\[
\hat{\xi}_1^2 = \frac{\partial}{\partial z_1}
\]

are equations of fractional derivatives of order 1/2 denoted \( \hat{\xi}_0 = D_{z_0}^{1/2} \) and \( \hat{\xi}_1 = D_{z_1}^{1/2} \). Fractional derivatives have the property that

\[
D_{z_0}^{1/2} D_{z_1}^{1/2} = \frac{\partial}{\partial z}
\]

and solutions for these fractional derivatives can be written

\[
D_{\frac{z}{2}}^{\frac{1}{2}} \phi = \frac{1}{\Gamma \left( \frac{1}{2} \right)} \frac{\partial}{\partial z} \int_0^z (z - t)^{-\frac{1}{2}} \phi(t) dt
\]

The exact form for these fractional derivatives however, is not important here. The important thing to note is that a solution to Laplace’s equation can be written in terms of spinors which are fractional derivatives.

If we assume that the fractional derivatives \( \hat{\xi}_0 \) and \( \hat{\xi}_1 \) commute then we also have

\[
(\hat{\xi}_0 \hat{\xi}_1)^2 = \hat{\xi}_0 \hat{\xi}_0 \hat{\xi}_1 \hat{\xi}_1
\]

\[
= -\frac{\partial}{\partial z_0} \frac{\partial}{\partial z_1}
\]

\[
= -\frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2} \right) \left( \frac{\partial^2}{\partial x_3^2} + \frac{\partial}{\partial x_2} \right)
\]

\[
= -\frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)
\]

\[
= \frac{1}{4} \frac{\partial^2}{\partial x_1^2}
\]
Using this result combined with Equations (13) and (14) we may write for the components of our vector

\[ \frac{\partial}{\partial x_1} = -2\hat{\xi}_0\hat{\xi}_1 \]  
\[ \frac{\partial}{\partial x_2} = i(\hat{\xi}_0^2 + \hat{\xi}_1^2) \]  
and

\[ \frac{\partial}{\partial x_3} = \hat{\xi}_0^2 - \hat{\xi}_1^2. \]

This result gives the explicit solution of our vector quantities \( \frac{\partial}{\partial x_i} \) in terms of the spinor quantities \( \xi_i \).

### A. Matrix Form

It can be readily verified that our spinors satisfy the following equations

\[
\begin{bmatrix}
\hat{\xi}_0 \frac{\partial}{\partial x_1} + \hat{\xi}_1 \left( \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_2} \right) \\
\hat{\xi}_0 \left( \frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_2} \right) - \hat{\xi}_1 \frac{\partial}{\partial x_1}
\end{bmatrix}
\phi = 0
\]

and in matrix form

\[
\begin{pmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1}
\end{pmatrix}
\begin{pmatrix}
\hat{\xi}_0 \\
\hat{\xi}_1
\end{pmatrix}
\phi = 0 \tag{19}
\]

The matrix

\[
X = \begin{pmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1}
\end{pmatrix}
\]

is equal to the dot product of the vector \( \frac{\partial}{\partial x_\mu} \equiv \frac{\partial}{\partial x_\mu} \) with the Pauli spin matrices

\[
X = \frac{\partial}{\partial x_1} \gamma^1 + \frac{\partial}{\partial x_2} \gamma^2 + \frac{\partial}{\partial x_3} \gamma^3
\]

where

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

are the Pauli matrices.
So Equation (19) can be written

\[ \sum_{\mu=1}^{3} \partial_{\mu} \gamma^{\mu} \xi = 0. \]  

(20)

where we have used the notation \( \xi \equiv \hat{\xi} \phi \). This equation has the form of Dirac’s equation in 3 dimensions. It describes a spin 1/2 particle of zero mass that is free of interactions.

B. Relation to the Dirac Decomposition

The fact that Laplace’ equation and Dirac’s equation are related is not new. However the decomposition used here is not the same as that used by Dirac. In the usual method, starting with the dirac equation \( (i\gamma^{\mu} \partial_{\mu})\Psi = 0 \) and operating with \(-i\gamma^{\mu} \partial_{\mu}\) yields Laplace’s equation for each component of the spinor field. In other words this method results in not one Laplace’s equation but several (one for each component of the spinor). Conversely if one starts with Laplace’s equation and tries to recover Dirac’s equation one must start with 2 independent scalars (4 in the usual 4 dimensional case) in order to derive the two component spinor equation (20).

What has been demonstrated in the preceding sections is that starting with only one scalar quantity satisfying Laplace’s equation Dirac’s equation for a two component spinor may be derived. Furthermore any medium (such as an elastic solid) that has a single scalar that satisfies Laplace’s equation must have a spinor that satisfies Dirac’s equation and such a derivation necessitates the use of fractional derivatives.

The form of Equation (20) is relevant for a massless, non-interacting spin 1/2 particle. We will now demonstrate that if one of our internal coordinates is taken to be periodic a mass term as well as gravitational and electromagnetic interaction terms appear in Dirac’s equation.

V. TRANSFORMATION TO INTERNAL COORDINATES

In section VI we will take the \( x'_{3} \) coordinate to be periodic and we will derive equations for the Fourier components of our fields. Since the elastic solid is assumed to be periodic in the internal coordinates we need to translate our equations of motion from fixed space coordinates to internal coordinates. Using Equation (10) we can rewrite Equation (20), as
\[\sum_{\mu=1}^{3} \gamma^{\mu} \left( \partial_{\mu} + \sum_{\nu} \frac{\partial u_{\mu}}{\partial x'_{\nu}} \partial_{\nu} \right) \xi = 0 \] (21)

or

\[\sum_{\mu=1}^{3} \gamma'^{\mu} \partial'_{\mu} \xi = 0\]

where \(\partial'_{\mu} = \partial/\partial x'_{\mu}\) and \(\gamma'^{\mu}\) is given by

\[\gamma'^{\mu} = \gamma^{\mu} + \sum_{\alpha} \frac{\partial u_{\mu}}{\partial x_{\alpha}} \gamma^{\alpha}. \] (22)

The anticommutator of these matrices is

\[\{\gamma'^{\mu}, \gamma'^{\nu}\} = \{\gamma^{\mu} + \sum_{\alpha} \frac{\partial u_{\mu}}{\partial x_{\alpha}} \gamma^{\alpha}, \gamma^{\nu} + \sum_{\beta} \frac{\partial u_{\nu}}{\partial x_{\beta}} \gamma^{\beta}\}\]

\[= \{\gamma^{\mu}, \gamma^{\nu}\} + \sum_{\beta} u_{\nu\beta} \{\gamma^{\mu}, \gamma^{\beta}\} + \sum_{\alpha} u_{\mu\alpha} \{\gamma^{\alpha}, \gamma^{\nu}\} + \sum_{\alpha} \sum_{\beta} u_{\mu\alpha} u_{\nu\beta} \{\gamma^{\alpha}, \gamma^{\beta}\}\]

\[= \delta_{\mu\nu} + \sum_{\beta} u_{\nu\beta} \delta_{\mu\beta} + \sum_{\alpha} u_{\mu\alpha} \delta_{\alpha\nu} + \sum_{\alpha} \sum_{\beta} u_{\mu\alpha} u_{\nu\beta} \delta_{\alpha\beta}\]

\[= \delta_{\mu\nu} + 2u_{\mu\nu} + \sum_{\alpha} u_{\mu\alpha} u_{\nu\alpha}\]

\[\equiv g^{\mu\nu}\]

This shows that the gamma matrices have the form of the usual Dirac matrices in a curved space\(^3\). To further develop the form of Equation (21) we have to transform the spinor properties of \(\xi\). As currently written \(\xi\) is a spinor with respect to the \(x_i\) coordinates not the \(x'_{\mu}\) coordinates. To transform its spinor properties we use a similarity transformation and write \(\xi = S \tilde{\xi}\) where \(S\) is a similarity transformation that takes our spinor in \(x_{\mu}\) to a spinor in \(x'_{\mu}\). We will not attempt to give an explicit form for \(S\). We simply assume (similar to reference\(^3\)) that this transformation can be effected by a real similarity transformation.

We then have

\[\partial'_{\mu} \xi = (\partial'_{\mu} S) \tilde{\xi} + S \partial'_{\mu} \tilde{\xi}.\]

Equation (21) then becomes

\[\gamma'_{\mu} [S \partial'_{\mu} \tilde{\xi} + (\partial'_{\mu} S) \tilde{\xi}] = 0\]

\[= \gamma'_{\mu} S[\partial'_{\mu} \tilde{\xi} + S^{-1}(\partial'_{\mu} S) \tilde{\xi}]\]

\[= S^{-1} \gamma'_{\mu} S[\partial'_{\mu} \tilde{\xi} + S^{-1}(\partial'_{\mu} S) \tilde{\xi}]\]

Using \((\partial'_{\mu} S^{-1})S = -S^{-1}(\partial'_{\mu} S)\). This can finally be written as

\[\tilde{\gamma}_{\mu} [\partial'_{\mu} - \Gamma_{\mu}] \tilde{\xi} = 0\] (23)
where $\Gamma_\mu = (\partial_\mu S^{-1})S$ and $\tilde{\gamma}_\mu = S^{-1}\gamma'_\mu S$. Equation (23) has the form of the Einstein-Dirac equation in 3 dimensions for a free particle of zero mass. The quantity $\partial_\mu - \Gamma_\mu$ is the covariant derivative for an object with spin in a curved space. In order to make this identification, the field $\Gamma_\mu$ must satisfy the additional equation

$$\frac{\partial \tilde{\gamma}_\mu}{\partial x^\nu} + \tilde{\gamma}_\beta \Gamma^\mu_{\beta \nu} - \Gamma_\nu \tilde{\gamma}_\mu + \tilde{\gamma}_\mu \Gamma_\nu = 0 \tag{24}$$

where $\Gamma^\mu_{\beta \nu}$ is the usual Christoffel symbol. We will now show that this equation does hold for this form of $\Gamma$.

**A. Spin Connection**

To show that Equation (24) holds we consider the equation $\partial_\nu \tilde{\gamma} = 0$ where the vector $\tilde{\gamma}$ is

$$\tilde{\gamma} = \sum_{\mu=1}^{3} \gamma^\mu e^*_\mu$$

and $e^*_\mu$ is a unit vector in the $x_\mu$ direction. Since $\tilde{\gamma}$ is a vector, then the quantity $\partial_\nu \tilde{\gamma} = 0$ is a tensor equation. Therefore, in the primed coordinate system we can immediately write

$$\sum_{\mu=1}^{3} \left( \partial'_{\nu} \gamma'^{\mu} + \gamma'^{\mu} \Gamma^\mu_{\beta \nu} \right) e^*_\mu = 0$$

where $\gamma'^{\mu} = \gamma^\mu + \sum_\alpha \frac{\partial u_\alpha}{\partial x_\alpha} \gamma^\alpha$ is the expression of $\gamma^\mu$ in the primed coordinate system. Using $\gamma'^{\mu} = S\tilde{\gamma}^\mu S^{-1}$, we have

$$\sum_{\mu=1}^{3} \left( \partial'_{\nu} (S\tilde{\gamma}^\mu S^{-1}) + (S\tilde{\gamma}^\beta S^{-1}) \Gamma^\mu_{\beta \nu} \right) e^*_\mu = 0$$

or

$$\sum_{\mu=1}^{3} \left( (\partial'_{\nu} S) \tilde{\gamma}^\mu S^{-1} + S(\partial'_{\nu} \tilde{\gamma}^\mu) S^{-1} + S\tilde{\gamma}^\mu (\partial'_{\nu} S^{-1}) + (S\tilde{\gamma}^\beta S^{-1}) \Gamma^\mu_{\beta \nu} \right) e^*_\mu = 0.$$  

Multiplying by $S^{-1}$ on the left and $S$ on the right yields

$$\sum_{\mu=1}^{3} \left( S^{-1}(\partial'_{\nu} S) \tilde{\gamma}^\mu + (\partial'_{\nu} \tilde{\gamma}^\mu) + \tilde{\gamma}^\mu (\partial'_{\nu} S^{-1}) S + \tilde{\gamma}^\beta \Gamma^\mu_{\beta \nu} \right) e^*_\mu = 0$$

Finally, using $\Gamma_\mu = (\partial_\mu S^{-1})S$ and again noting that $\partial_\mu S^{-1}S = -S^{-1}\partial_\mu S$ we have,

$$\tilde{\gamma}^\mu \Gamma_\mu - \Gamma_\mu \tilde{\gamma}^\mu + \left( \partial'_{\nu} \tilde{\gamma}^\mu + \tilde{\gamma}^\beta \Gamma^\mu_{\beta \nu} \right) e^*_\mu = 0$$

We have just demonstrated that in the internal coordinates, the equations of motion of an elastic medium have the same form as the free-field Einstein-Dirac equation for a massless particle in three dimensions.
B. Physical Content

Thus far all of the transformations that have been obtained are “trivial” in the sense that they only result due to changing coordinates from the unprimed coordinates \(x_\mu\) to the primed coordinates \(x'_\mu\). Changes of coordinates of course do not result in any new physical content. In particular the metric derived in Equation (5) does not lead to a curved space. The Riemann curvature tensor calculated from Equation (5) is identically zero. Likewise the spin connection \(\Gamma_\mu \) is due solely to a gauge transformation \(\xi \rightarrow S \xi'\) and as such contains no physical content since it can be removed by transforming \(\xi' \rightarrow S^{-1} \xi\).

What we will demonstrate in the following sections is that for a system where one coordinate is periodic, the resulting 2 dimensional quantities are NOT trivial. In other words the metric that determines the dynamics of the Fourier components of \(\xi\) does in fact lead to a curved space and the spin connection cannot be removed by a gauge transformation. Furthermore the introduction of the fourier components will generate extra terms in Equation (21) that imply a series of equations relevant for particles with mass coupled to fields that can be associated with electromagnetism. We will show that in the low energy approximation (ie a system in which only the lowest few modes are present) the equations of motion are identical in form to Equation (1).

VI. INTERACTING PARTICLES WITH MASS

In this section we again consider a three dimensional elastic solid but we take the third internal dimension to be compact with the topology of a circle. All variables then become periodic functions of \(x'_3\) and can be Fourier transformed.

In preparation for Fourier Transforming we isolate the terms involving \(x'_3\) and rewrite Equation (21) as,

\[
2 \sum_{\mu=1}^{2} \gamma^\mu \left( \partial'_\mu + \sum_{\nu=1}^{2} \frac{\partial u_\nu}{\partial x_\mu} \partial'_\nu + \frac{\partial u_3}{\partial x_\mu} \partial'_3 \right) \xi + \gamma^3 \left( \partial'_3 + \sum_{\nu=1}^{2} \frac{\partial u_\nu}{\partial x_3} \partial'_\nu + \frac{\partial u_3}{\partial x_3} \partial'_3 \right) \xi \quad (25)
\]

We first transform the partial derivatives of the \(u_\nu\) in equation (25) to obtain

\[
u_{\nu\mu} \equiv \frac{\partial u_\nu}{\partial x_\mu} = \sum_k u_{\nu\mu,k} e^{ikx'_3}
\]

where \(u_{\nu\mu,k}\) is the \(k^{th}\) Fourier mode of \(\partial u_\nu/\partial x_\mu\) and \(k = 2 \pi i/a\) with \(a\) the length of the
circle formed by the elastic solid in the $x_3'$ direction and $i$ is an integer. Equation (25) now becomes,

$$
\sum_k e^{ikx_3'} \left[ \sum_{\mu=1}^2 \gamma^\mu \left( \partial_\mu \delta_{k,0} + \sum_{\nu=1}^2 u_{\nu \mu, k} \partial_\nu' + u_{3 \mu, k} \partial_3' \right) \xi + \gamma^3 \left( \partial_3' \delta_{k,0} + \sum_{\nu=1}^2 u_{3 \nu, k} \partial_\nu' + u_{33, k} \partial_3' \right) \xi \right] = 0
$$

Next we transform the spinor (noting that it is periodic in $4\pi a$),

$$\xi = \sum_q \xi_{q/2} e^{i \frac{q}{2} x_3'}$$

with $q = 2\pi j/a$ and $j$ an integer. This yields,

$$
\sum_k \sum_q e^{i x_3'(k+q/2)} \left[ \sum_{\mu=1}^2 \gamma^\mu \left( \partial_\mu' \delta_{k,0} + \sum_{\nu=1}^2 u_{\nu \mu, k} \partial_\nu' + i(q/2)u_{3 \mu, k} \right) \xi_{q/2} + \gamma^3 \left( i(q/2) \delta_{k,0} + \sum_{\nu=1}^2 u_{3 \nu, k} \partial_\nu' + i(q/2)u_{33, k} \right) \xi_{q/2} \right] = 0.
$$

This equation is independently true for each distinct value of $k + q/2 = m/2$ or $2k + q = m$ with $k, q, m$ an integer. Writing $q = m - 2k$ yields finally,

$$
\sum_k \left[ \sum_{\mu=1}^2 \gamma^\mu \left( \partial_\mu' \delta_{k,0} + \sum_{\nu=1}^2 u_{\nu \mu, k} \partial_\nu' + i(m-2k)u_{3 \mu, k} \right) \xi_{(m-2k)/2} + \gamma^3 \left( i(m-2k) \delta_{k,0} + \sum_{\nu=1}^2 u_{3 \nu, k} \partial_\nu' + i(m-2k)u_{33, k} \right) \xi_{(m-2k)/2} \right] = 0
$$

This is an infinite series of equations describing the dynamics of the fields $\xi_m$. This set of equations describes the dynamics of our elastic solid and contains the same information as Laplace’s equation.

So far no approximations have been made. In the next section we will demonstrate that if only the lowest modes are present, this reduces to an equation that is identical in form to Equation (1).

A. Spectrum of Lowest modes

We now consider a theory in which only the lowest few modes in Equation (26) are present. We therefore keep only the $m = 0, \pm 1/2$ modes we obtain the following 3 equations,

$$
\sum_{\mu=1}^2 \gamma^\mu \left( \partial_\mu' + \sum_{\nu=1}^2 u_{\nu \mu, 0} \partial_\nu' \right) \xi_0 + \sum_{\nu=1}^2 u_{3 \nu, 0} \partial_\nu' \gamma^3 \xi_0 = 0
$$

15
\[ \sum_{\mu=1}^{2} \gamma^\mu \left( \partial'_\mu + \sum_{\nu=1}^{2} u_{\nu \mu,0} \partial'_\nu + im_{1/2}u_{3 \mu,0} \right) \xi_{1/2} + \gamma^3 m_{1/2}(1 + u_{33,0}) \xi_{1/2} \]
\[ + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,0} \partial'_\nu \xi_{1/2} + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,1} \partial'_\nu \xi_{-1/2} = 0 \] (28)

\[ \sum_{\mu=1}^{2} \gamma^\mu \left( \partial'_\mu + \sum_{\nu=1}^{2} u_{\nu \mu,0} \partial'_\nu + im_{-1/2}u_{3 \mu,0} \right) \xi_{-1/2} + \gamma^3 m_{-1/2}(1 + u_{33,0}) \xi_{-1/2} \]
\[ + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,0} \partial'_\nu \xi_{-1/2} + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,1} \partial'_\nu \xi_{1/2} = 0 \] (29)

where \( m_i = 2\pi i/a \) denotes the Fourier mode with \( i \) a half integer. These equations describe the dynamics of three fields \( \xi_0 \) and the coupled fields \( \xi_{1/2} \) and \( \xi_{-1/2} \). The first equation \((m = 0 \text{ mode})\) describes the dynamics of a massless, free particle. We will not attempt to identify this mode with any physical particle but we simply note that in this approximation this equation is completely uncoupled from the \( m = \pm 1/2 \) modes and therefore its dynamics are independent and have no affect on these other modes.

We now examine the equations describing \( \xi_{1/2} \) and \( \xi_{-1/2} \). These two equations can be combined by noting that for real fields, \( u_{\mu \nu, k} = u_{\mu \nu, -k}^* \). The \( m = \pm 1/2 \) modes can now be combined into the single equation

\[ \sum_{\mu=1}^{2} \gamma^\mu \left( \partial'_\mu + \sum_{\nu=1}^{2} u_{\nu \mu,0} \partial'_\nu + im_{1/2}u_{3 \mu,0} \right) \Psi + \gamma^3 m_{1/2}(1 + u_{33,0}) \Psi \]
\[ + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,0} \partial'_\nu \Psi + \gamma^3 \sum_{\nu=1}^{2} u_{\nu 3,1} \partial'_\nu \Psi^* = 0, \] (30)

where \( \Psi = \xi_{1/2} + \xi_{-1/2}^* \). To put this equation into a more recognizable form we multiply Equation (30) by \( \gamma^3 \) from the left and define

\[ \gamma'^\mu = \gamma^3 \gamma^\mu + \sum_{\beta=1}^{2} \gamma^3 \gamma^\beta u_{\mu \beta,0}. \] (31)

These matrices with \( \mu = 1, 2 \) and \( \nu = 1, 2 \) satisfy the anticommutation relations

\[ \{ \gamma'^\mu, \gamma'^\nu \} = \delta_{\mu \nu} + (u_{\mu \nu,0} + u_{\nu \mu,0}) + \sum_{\beta=1}^{2} u_{\mu \beta,0} u_{\nu \beta,0}. \]

If we insist that our new matrices satisfy \( \{ \gamma^\mu, \gamma^\nu \} = g^{\mu \nu} \) then we are led to define

\[ g^{\mu \nu} \equiv \delta_{\mu \nu} + (u_{\mu \nu,0} + u_{\nu \mu,0}) + \sum_{\beta=1}^{2} u_{\mu \beta,0} u_{\nu \beta,0}. \] (32)
This is the metric for our two dimensional subspace and it does not have the form of a simple coordinate transformation on a flat space metric like that of section 11.

Equation (30) can now be rewritten as

$$\sum_{\mu=1}^{2} \gamma^\mu \left( \partial'_\mu + m_{1/2} u_{3\mu,0} \right) \Psi - m_{1/2} u_{3\mu,0} \sum_{\beta=1}^{2} \gamma^\beta u_{\mu\beta,0} \Psi + m_{1/2}(1 + u_{33,0}) \xi_{1/2}$$

$$+ \sum_{\nu=1}^{2} u_{\nu3,0} \partial'_\nu \Psi + \sum_{\nu=1}^{2} u_{\nu3,1} \partial'_\nu \Psi^* = 0. \quad (33)$$

As we did for Equation (25), in going from the x to the x' coordinates, we assume that the spinor properties of $\xi_{1/2}$ may be transformed using a real similarity transformation and writing $\xi_{1/2} = S \tilde{\xi}_{1/2}$. Transforming $\Psi$ in this way and multiplying on the left by $S^{-1}$ gives us the following form for the $m = \pm 1/2$ modes,

$$\sum_{\mu=1}^{2} S^{-1} \gamma^\mu S \left( \partial'_\mu + m_{1/2} u_{3\mu,0} + S^{-1} \partial'_\mu S \right) \tilde{\Psi} - m_{1/2} u_{3\mu,0} \sum_{\beta=1}^{2} S^{-1} \gamma^\beta S u_{\mu\beta,0} \tilde{\Psi}$$

$$+ m_{1/2}(1 + u_{33,0}) \tilde{\Psi} + \sum_{\nu=1}^{2} u_{\nu3,0} (\partial'_\nu + S^{-1} \partial'_\nu S) \tilde{\Psi} + \sum_{\nu=1}^{2} u_{\nu3,1} (\partial'_\nu + S^{-1} \partial'_\nu S) \Psi^* \quad (34)$$

We now examine each quantity in Equation (34). As before, we identify $\tilde{\gamma}^\mu = S^{-1} \gamma^\mu S$ with the transformed gamma matrix and $\Gamma_\mu = (\partial'_\mu S^{-1}) S$ with the spin connection. We also would like to identify the quantity $A_\mu = m_{1/2} u_{3\mu,0}$ in the first term with the electromagnetic potential and the third term in Equation (34) as a mass term with $m = m_{1/2}(1 + u_{33,0})$ which implies that the field $u_{33,0}$ provides a mass for our $\Psi$ particle. Let us further assume that the quantities $u_{\mu\nu}$ are small compared to unity so that the second term may be neglected as being of order $u_{\mu\nu}^2$ (ie we are now assuming that our medium undergoes only small deformations). If these identifications are made we can write Equation (34) in the final form

$$\sum_{\mu=1}^{2} \tilde{\gamma}^\mu \left( i \partial'_\mu + i \Gamma_\mu - A_\mu \right) \tilde{\Psi} - m \tilde{\Psi}$$

$$+ i \sum_{\nu=1}^{2} u_{\nu3,0} (\partial'_\nu - \Gamma_\nu) \tilde{\Psi} + i \sum_{\nu=1}^{2} u_{\nu3,1} (\partial'_\nu - \Gamma_\nu) \Psi^* \quad (35)$$

Notice the formal similarity of this equation to Equation (1). The first two terms have exactly the form of Dirac’s equation for a spin 1/2 particle of mass $m$ in curved space interacting with the electromagnetic vector potential $A_\mu$. Note that the mass term and the electromagnetic potentials were not added by hand but emerged naturally from the formalism. The nature of the last two terms in Equation (35) are unknown. They don’t appear in the usual statement of Dirac’s equation and their implications are unknown.
Equation (35) is the central result of this work. We have as yet not shown that the dynamics of the fields $A_\mu$ are consistent with their identification as the electromagnetic vector potential. To truly claim that the quantity $u_{\mu 3.0}$ is the electromagnetic potential it must be shown to satisfy Maxwell’s equations. We believe however that the formal correspondence between Equation (35) and Dirac’s equation is significant in its own right and will not, in this paper, pursue the question of whether Maxwell’s equation or the Einstein Field equations are satisfied. Before concluding we note that although our derivation assumed that we were working in three dimensional space, the formalism extends to any number of dimensions. The major difference is that in the three dimensional case, we were able to find an explicit solution for the components of a spinor in terms the components of the vector $\partial_\mu$. An explicit solution might not exist in general. Nevertheless it can be shown that the quadratic form in Laplace’s equation implies the existence of a multicomponent spinor $\xi$ satisfying a dirac-like equation in any dimension.

VII. CONCLUSIONS

We have taken a model of an elastic medium and derived an equation of motion that has the same form as Dirac’s equation in the presence of electromagnetism and gravity. We derived our equation by using the formalism of Cartan to reduce the quadratic form of Laplace’s equation to the linear form of Dirac’s equation. We further assumed that one coordinate was compact and upon Fourier transforming this coordinate we obtained, in a natural way, a mass term and an electromagnetic interaction term in the equations of motion.

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