A SPIKY BALL

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ABSTRACT. The Illumination Problem may be phrased as the problem of covering a convex body in Euclidean n-space by a minimum number of translates of its interior. By a probabilistic argument, we show that, arbitrarily close to the Euclidean ball, there is a centrally symmetric convex body of illumination number exponentially large in the dimension.

1. Introduction

For two sets $K$ and $L$ in $\mathbb{R}^n$, let $N(K, L)$ denote the translative covering number of $K$ by $L$, that is, the minimum number of translates of $L$ that cover $K$.

Let $K$ be a convex body (that is, a compact, convex set with non-empty interior) in $\mathbb{R}^n$. Following Hadwiger [Had60], we say that a point $p \in \mathbb{R}^n \setminus K$ illuminates a boundary point $b \in \text{bd } K$, if the ray $\{p + \lambda(b - p) : \lambda > 0\}$ emanating from $p$ and passing through $b$ intersects the interior of $K$. Boltyanski [Bol60] gave the following slightly different definition. A direction $u \in S^{n-1}$ is said to illuminate $K$ at a boundary point $b \in \text{bd } K$ if the ray $\{b + \lambda u : \lambda > 0\}$ intersects the interior of $K$. It is easy to see that the minimum number of directions that illuminate each boundary point of $K$ is equal to the minimum number of points that illuminate each boundary point of $K$. This number is called the illumination number $i(K)$ of $K$.

We call a set of the form $\lambda K + v$ a smaller positive homothet of $K$ if $0 < \lambda < 1$ and $v \in \mathbb{R}^n$. Gohberg and Markus asked how large the minimum number of smaller positive homothets of $K$ covering $K$ can be. It is not hard to see that this number is equal to $N(K, \text{int } K)$. It is also easy to see that $i(K) = N(K, \text{int } K)$.

Any smooth convex body (ie., a convex body with a unique support hyperplane at each boundary point) in $\mathbb{R}^n$ is illuminated by $n + 1$ directions. Indeed, for a smooth body, the set of directions illuminating a given boundary point is an open hemisphere of $S^{n-1}$, and one can find $n + 1$ points (eg., take the vertices of a regular simplex) in $S^{n-1}$ with
the property that every open hemisphere contains at least one of the points. Thus, these \( n + 1 \) points in \( S^{n-1} \) (i.e., directions) illuminate any smooth convex body in \( \mathbb{R}^n \) (cf. \cite{BMS97} for details).

On the other hand, the illumination number of the cube is \( 2^n \), since no two vertices of the cube share an illumination direction. Even though we do not discuss it, it would be difficult to omit mentioning the Gohberg–Markus–Levi–Boltyanski–Hadwiger Conjecture (or, Illumination Conjecture), according to which for any convex body \( K \) in \( \mathbb{R}^n \), we have \( i(K) = 2^n \), where equality is attained only when \( K \) is an affine image of the cube.

For more background on the problem of illumination, see \cite{Bez06, Bez10, BMP05, MS99}. In Chapter VI. of \cite{BMS97}, one can find a proof of the equivalence of the four definitions of \( i(K) \) given above.

The Euclidean ball is a smooth convex body, and hence, is of illumination number \( n + 1 \). Theorem 1.1 shows that, arbitrarily close to the Euclidean ball, there is a convex body of much larger illumination number.

We denote the closed Euclidean unit ball in \( \mathbb{R}^n \) centered at the origin \( o \) by \( B^n \), and its boundary, the sphere by \( S^{n-1} \).

**Theorem 1.1.** Let \( 1 < D < 1.116 \) be given. Then for any sufficiently large dimension \( n \), there is an \( o \)-symmetric convex body \( K \) in \( \mathbb{R}^n \), with illumination number

\[
(1) \quad i(K) = N(K, \text{int } K) \geq 0.05D^n,
\]

for which

\[
(2) \quad \frac{1}{D} B^n \subset K \subset B^n.
\]

We will use a probabilistic construction to find \( K \). We are not aware of any lower bound for the Illumination Problem that was obtained by a probabilistic argument.

For a point \( u \in S^{n-1} \), and \( 0 < \varphi < \pi/2 \), let \( C(u, \varphi) = \{ v \in S^{n-1} : \varphi(u, v) \leq \varphi \} \) denote the spherical cap centered at \( u \) of angular radius \( \varphi \). We denote the normalized Lebesgue measure (that is, the Haar probability measure on \( S^{n-1} \)) of \( C(u, \varphi) \) by \( \Omega_{n-1}(\varphi) \).

In Theorem 1.2, we give an upper bound for the illumination number for bodies close to the Euclidean ball. It follows from \cite{BK09} but, for the sake of completeness, we will sketch a proof.

**Theorem 1.2.** Let \( K \) be a convex body in \( \mathbb{R}^n \) such that \( \frac{1}{D} B^n \subset K \subset B^n \) for some \( D > 1 \). Then the illumination number of \( K \) is at most

\[
(3) \quad i(K) \leq \frac{n \ln n + n \ln \ln n + 5n}{\Omega_{n-1}(\alpha)},
\]

where \( \alpha = \arcsin(1/D) \).
By combining Theorem 1.2 with the estimate (5) on $\Omega_{n-1}$, one can see that (1) is asymptotically sharp, that is, the base $D$ cannot be improved.

Next, we consider an application of Theorem 1.1. Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$, and denote its gauge function by $\|p\|_K = \inf\{\lambda > 0 : p \in \lambda K\}$, for any $p \in \mathbb{R}^n$. We use $\text{vert } P$ to denote the set of vertices of the polytope $P$. The illumination parameter, introduced by K. Bezdek [Bez06], is defined as

$$\text{ill}(K) = \inf \left\{ \sum_{p \in \text{vert } P} \|p\|_K \mid P \text{ a polytope such that vert } P \text{ illuminates } K \right\}.$$  \hspace{1cm} (4)

The vertex index of $K$, introduced by K. Bezdek and Litvak [BL07], is

$$\text{vein}(K) = \inf \left\{ \sum_{p \in \text{vert } P} \|p\|_K \mid P \text{ a polytope such that } K \subseteq P \right\}.$$  \hspace{1cm} (5)

Clearly, $\text{ill}(K) \geq \text{vein}(K)$ for any centrally symmetric body $K$, and they are equal for smooth bodies. It is shown in [BL07] that $\text{vein}(B^n)$ is of order $n^{3/2}$ (see also [GL12]).

By (2), for the body $K$ constructed in Theorem 1.1 we have that $\text{vein}(K)$ is of order $n^{3/2}$, while $\text{ill}(K) \geq i(K)$ is exponentially large.

Thus, as an application of Theorem 1.1, we obtain that $\text{ill}(K)$ and $\text{vein}(K)$ are very far from each other for some $K$.

2. Preliminaries

We will rely heavily on the following estimates of $\Omega_n$ by Böröczky and Wintsche [BW03].

**Lemma 2.1** (Böröczky – Wintsche [BW03]). Let $0 < \varphi < \pi/2$.

(4) $\Omega_n(\varphi) > \frac{\sin^n \varphi}{\sqrt{2\pi(n+1)}}$.

(5) $\Omega_n(\varphi) < \frac{\sin^n \varphi}{\sqrt{2\pi n \cos \varphi}}$, if $\varphi \leq \arccos \frac{1}{\sqrt{n+1}}$.

(6) $\Omega_n(t \varphi) < t^n \Omega_n(\varphi)$, if $1 < t < \frac{\pi}{2\varphi}$.

The following is known as Jordan’s inequality:

(7) $\frac{2x}{\pi} \leq \sin x$, for $x \in [0, \pi/2]$.

3. Construction of a Spiky Ball

We work in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$ to obtain slightly simpler formulas. We describe a probabilistic construction of $K \subseteq \mathbb{R}^{n+1}$ which is close to the Euclidean ball and has a large illumination number. We use
Figure 1. Event $E_1$: when $X_j$ falls on the dotted cap (the arc with arrows at its endpoints) or on its reflection about the origin.

the standard notation $[N]$ for the set $\{1, \ldots, N\}$, and $|A|$ denotes the cardinality of a set $A$.

Let $N$ be a fixed positive integer, to be given later. We pick $N$ points, $X_1, \ldots, X_N$ independently and uniformly on the Euclidean unit sphere $S^n$ of $\mathbb{R}^{n+1}$. Let

$$K = \text{conv} \left( \{\pm X_i : i \in [N]\} \cup \frac{1}{D} B^{n+1} \right).$$

Clearly, $K$ is $o$-symmetric and $\frac{1}{D} B^{n+1} \subset K \subset B^{n+1}$. We need to bound the illumination number of $K$ from below. Let $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ be such that $\sin \alpha = 1/D$.

We define two “bad” events, $E_1$ and $E_2$. Let $E_1$ be the event that there are $i \neq j \in [N]$ with $\angle(X_i, X_j) < \pi - 2\alpha$ or $\angle(-X_i, X_j) < \pi - 2\alpha$. We observe that if $E_1$ does not occur, then for all $i \in [N]$ we have

$$i(K) \geq \frac{2N}{T}. $$

However, it is difficult to bound the probability of $E_2$. Thus, we will replace $E_2$ by a “more finite” condition $E_2'$ as follows.

We fix a $\delta > 0$. We call a set $\Lambda \subset S^n$ a $\delta$-net (could also be called a metric $\delta$-net) if $\bigcup_{v \in \Lambda} C(v, \delta) = S^n$, that is, if the caps of radius $\delta$ centered at the points of $\Lambda$ cover the sphere. By [1], the measure of a
cap of radius \( \delta \) is larger than \( \frac{\sin^n(\delta)}{3\sqrt{n}} \). Thus, Theorem 1 of [Rog63] yields that, there is a covering of the sphere by at most \( n^2 / \sin^n(\delta) \) caps of radius \( \delta \). That is, there is a \( \delta \)-net \( \Lambda \) of size at most \( |\Lambda| \leq n^2 / \sin^n(\delta) \).

Let \( p = 2\Omega_n(\alpha + \delta) \). Let \( \Theta > 1 \) be fixed, and set \( T = N\Theta p \). We define the event \( E_2' \) as follows: there is a direction \( v \in \Lambda \) with \( |C(v, \alpha + \delta) \cap \{ \pm X_i : i \in [N] \}| > N\Theta p \). Clearly, if \( E_2 \) occurs, then so does \( E_2' \).

Thus, we have

\[
\text{(10) } \left( \text{not}(E_1) \text{ and not}(E_2') \right) \text{ implies } i(K) \geq \frac{2}{(\Theta p)}.
\]

(10) (not(\(E_1\)) and not(\(E_2'\))) implies i(\(K\)) \(\geq\) 2/((\(\Theta p\))).

Now, we need to set our parameters such that the event (not(\(E_1\)) and not(\(E_2'\))) is of positive probability and \(2/(\Theta p)\) is exponentially large in the dimension.

Clearly,

\[
\text{(11) } \mathbb{P}(E_1) \leq N^2\Omega_n(\pi - 2\alpha).
\]

Consider a fixed \( v \in \Lambda \). When \( X_i \) is picked randomly, the probability that \( v \) is contained in \( C(X_i, \alpha + \delta) \) or in \( C(-X_i, \alpha + \delta) \) is \( p \) (recall that \( p = 2\Omega_n(\alpha + \delta) \)). Thus, the probability that \( v \) is contained in more than \( N\Theta p \) caps of the form \( C(\pm X_i, \alpha + \delta) \) is \( \mathbb{P}(\xi > N\Theta p) \), where \( \xi \) is a binomial random variable of distribution \( \text{Binom}(N, p) \). Thus,

\[
\text{(12) } \mathbb{P}(E_2') \leq \frac{n^2}{\sin^n(\delta)} \mathbb{P}(\xi > N\Theta p) \text{ with } \xi \sim \text{Binom}(N, p).
\]

By a Chernoff-type inequality, (cf. p. 64 of [MU05]),

\[
\text{(13) } \mathbb{P}(\xi > N\Theta p) < 2^{-N\Theta p}, \text{ for any } \Theta \geq 6.
\]

Consider the following three inequalities.

\[
\text{(14) } N \leq \left( \frac{1}{4\Omega_n(\pi - 2\alpha)} \right)^{1/2},
\]

\[
\text{(15) } \frac{n^2}{\sin^n(\delta)} 2^{-\Theta N p} \leq \frac{1}{4},
\]

\[
\text{(16) } 6 \leq \Theta.
\]

Combining (10), (11), (12) and (13), we obtain the following. If there are \( N \in \mathbb{Z}^+, \delta > 0 \) and \( \Theta \geq 0 \) (all depending on \( n \)) such that the three inequalities (14), (15) and (16) hold, then there is a \( K \subset \mathbb{R}^{n+1} \) \( \alpha \)-symmetric convex body with \( i(K) \geq 2/(\Theta p) \), where \( p = 2\Omega_n(\alpha + \delta) \). In fact, in this case, our construction yields such a \( K \) with probability at least 1/2.

Now, (15) holds if \( \Theta N p > 2n \log_2 \frac{1}{\sin \delta} \). Thus, an integer \( N \) satisfying (14) and (15) exists if

\[
4n \log_2 \frac{1}{\sin \delta} \leq \Theta p \left( \frac{1}{4\Omega_n(\pi - 2\alpha)} \right)^{1/2},
\]
which we rewrite as
\[ \frac{1}{\theta p} \leq \frac{1}{8n(\Omega_n(\pi - 2\alpha))^{1/2} \log_2 \frac{1}{\sin \delta}}. \]

By (7), we can replace it by the following stronger inequality:
\[ (17) \quad \frac{1}{\theta p} \leq \frac{1}{24n(\Omega_n(\pi - 2\alpha))^{1/2} \log_2 (1/\delta)}. \]

On the other hand, by substituting the value of \( p \), we see that (16) is equivalent to
\[ (18) \quad \frac{1}{\theta p} \leq \frac{1}{12\Omega_n(\alpha + \delta)}. \]

Finally, let \( \delta = \frac{\alpha}{n} \).
Since \( 1 < D = \frac{1}{\sin \alpha} < 1.116 \), we have that \( 1.11 < \alpha < \pi/2 \), and thus, \( \sin^2(\alpha + \delta) > \sin(\pi - 2\alpha) \). Now, by Lemma \[2.1 \] \[18 \] is a stronger inequality than \( (17) \). Thus, so far we have that if we can satisfy \( (18) \), then the proof is complete.
By (6), we have that \( (18) \) holds, if
\[ (19) \quad \frac{1}{\theta p} \leq \frac{1}{36\Omega_n(\alpha)}. \]

By (5), it holds for \( \frac{1}{\theta p} = \frac{1}{36} D^n \). Since \( i(K) \geq 2/(\Theta p) \), this finishes the proof of Theorem \[1.1 \].

Remark 3.1. The body \( K \) is not a polytope. However, the construction can easily be modified to obtain a polytope. One simply replaces the ball of radius \( 1/D \) by a sufficiently dense finite subset \( A \) of this ball in the definition of \( K \) as follows: \( K = \text{conv}(\{\pm X_i : i \in [N]\} \cup A) \).

Proof of Theorem \[1.2 \]. Since \( \frac{1}{D} B^n \subset K \subset B^n \), it follows that for any boundary point \( b \) of \( K \), the set of directions (as a subset of \( S^{n-1} \)) that illuminate \( K \) at \( b \) contains an open spherical cap of radius \( \alpha = \arcsin(1/D) \). Thus, any subset \( A \) of \( S^{n-1} \) that pierces each such cap illuminates \( K \). However, finding such \( A \) is equivalent to finding a covering of \( S^{n-1} \) by caps of radius \( \alpha \). Such a covering of the desired size exists by \[ \text{Rog63} \] (see also \[ \text{BW03} \]).

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