The rate of convergence of estimate for Hurst index of fractional Brownian motion involved into stochastic differential equation✩

K. Kubiliusᵃ,∗, Y. Mishuraᵇ

ᵃVilniaus university Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania
ᵇNational Taras Shevchenko Kyiv University, Volodymyrska 64, 01601 Kiev, Ukraine

Abstract

We consider stochastic differential equation involving pathwise integral with respect to fractional Brownian motion. The estimates for the Hurst parameter are constructed according to first- and second-order quadratic variations of observed values of the solution. The rate of convergence of these estimates to the true value of a parameter is established.

Keywords: Fractional Brownian motion; stochastic differential equation; first- and second-order quadratic variations; estimates of Hurst parameter; rate of convergence.

2000 MSC: 60G22, 60H10

1. Introduction

Consider stochastic differential equation

\[ X_t = \xi + \int_0^t f(X_s) \, ds + \int_0^t g(X_s) \, dB^H_s, \quad t \in [0, T], \ T > 0, \]  \tag{1}  

where \( f \) and \( g \) are measurable functions, \( B^H \) is a fractional Brownian motion (fBm) with Hurst index \( 1/2 < H < 1 \), \( \xi \) is a random variable. It is well-known that almost all sample paths of \( B^H \) have bounded p-variations for \( p > 1/H \). Therefore it is natural to define the integral with respect to fractional Brownian motion as pathwise Riemann-Stieltjes integral (see, e.g., [1] for the original definition and [3] for the advanced results).

A solution of stochastic differential equation (1) on a given filtered probability space \( (\Omega, \mathcal{F}, P, \mathcal{F} = \{\mathcal{F}_t\}, t \in [0, T]) \), with respect to the fixed fBm \( (B^H, \mathcal{F}) \),

✩This paper was supported by Research program at International Center of Mathematics Meetings (CIRM, [http://www.cirm.univ-mrs.fr/])
∗Corresponding author
1/2 < H < 1 and with \( \mathcal{F}_0 \)-measurable initial condition \( \xi \) is an adapted to the filtration \( \mathcal{F} \) continuous process \( X = \{X_t: 0 \leq t \leq T\} \) such that \( X_0 = \xi \) a.s.,

\[
P\left( \int_0^t |f(X_s)| \, ds + \int_0^t g(X_s) \, dB_s^H \right) < \infty \quad \text{for every } 0 \leq t \leq T,
\]

and its almost all sample paths satisfy (1).

For \( 0 < \alpha \leq 1 \), \( C^{1+\alpha}(\mathbb{R}) \) denotes the set of all \( C^1 \)-functions \( g: \mathbb{R} \to \mathbb{R} \) such that

\[
\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x - y|^\alpha} < \infty.
\]

Let \( f \) be a Lipschitz function and let \( g \in C^{1+\alpha}(\mathbb{R}), \frac{1}{\alpha} < \beta \leq 1 \). Then there exists a unique solution of equation (1) with almost all sample paths in the class of all continuous functions defined on \([0, T]\) with bounded \( p \)-variation for any \( p > \frac{1}{\alpha} \) (see [5], [12], [13] and [10]). Different (but similar in many features) approach to the integration with respect to fractional Brownian motion based on the integration in Besov spaces and corresponding stochastic differential equations were studied in [15], see also [4] and [14] where the different approaches to stochastic integration and to stochastic differential equations involving fractional Brownian motion are summarized.

The main goal of the present paper is to establish the rate of convergence of two estimates of Hurst parameter to the true value of a parameter. The estimates are based on the two types of the quadratic variations of the observed solution to stochastic differential equation involving the integral with respect to fractional Brownian motion and considered on the fixed interval \([0, T]\). The paper is organized as follows: Section 2 contains some preliminary information. More precisely, subsection 2.1 describes the properties of \( p \)-variations and of the integrals with respect to the functions of bounded \( p \)-variations while subsection 2.2 contains the results on the asymptotic behavior of the normalized first- and second-order quadratic variations of fractional Brownian motion. Section 3 describes the rate of convergence of the first- and second-order quadratic variations of the solution to stochastic differential equation involving fBm. Section 4 contains the main result concerning the rate of convergence of the constructed estimates of Hurst index to its true value when the diameter of partitions of the interval \([0, T]\) tends to zero. Section 5 contains simulation results.

2. Preliminaries

2.1. The functions of bounded \( p \)-variation

First, we mention some information concerning \( p \)-variation and the functions of bounded \( p \)-variation. It is containing, e.g., in [5] and [10]. Let interval \([a, b] \subset \mathbb{R}\). Consider the following class of functions:

\[
\mathcal{W}_p([a, b]) := \{ f: [a, b] \to \mathbb{R}: v_p(f; [a, b]) < \infty \},
\]
where

\[ v_p(f; [a, b]) = \sup_{\pi} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p. \]

Here \( \pi = \{x_i: i = 0, \ldots, n\} \) stands for any finite partition of \([a, b]\) such that \(a = x_0 < x_1 < \cdots < x_n = b\). Denote \(\Pi([a, b])\) the class of such partitions. We say that function \(f\) has bounded \(p\)-variation on \([a; b]\) if \(v_p(f; [a, b]) < \infty\).

Let \(V_p(f) := V_p(f; [a, b]) = \frac{1}{p} v_p(f; [a, b]).\) Then for any fixed \(f\) we have that \(V_p(f)\) is a non-increasing function of \(p\). It means that for any \(0 < q < p\) the relation \(V_p(f) \leq V_q(f)\) holds.

Let \(a < c < b\) and let \(f \in \mathcal{W}_p([a, b])\) for some \(p \in (0, \infty)\). Then

\[
\begin{align*}
    v_p(f; [a, c]) + v_p(f; [c, b]) &\leq v_p(f; [a, b]), \\
v_p(f; [a, b]) &\leq v_p(f; [a, c]) + v_p(f; [c, b]).
\end{align*}
\]

Let \(f \in \mathcal{W}_q([a, b])\) and \(h \in \mathcal{W}_p([a, b])\), where \(p^{-1} + q^{-1} > 1\). Then the well-known Love-Young inequality states that

\[
\left| \int_a^b f \, dh - f(y) [h(b) - h(a)] \right| \leq C_{p,q} V_q(f; [a, b]) V_p(h; [a, b]),
\]

whence

\[
V_p \left( \int_a^b f \, dh; [a, b] \right) \leq C_{p,q} V_{p,\infty}(f; [a, b]) V_p(h; [a, b]).
\]

Here \(V_{p,\infty}(f; [a, b]) = V_q(f; [a, b]) + \sup_{a \leq x \leq b} |f(x)|, \ C_{p,q} = \zeta(p^{-1} + q^{-1})\) and \(\zeta(s) = \sum_{n \geq 1} n^{-s}\) is the Riemann zeta function. Further, for any \(y \in [a, b]\)

\[
\begin{align*}
    &\quad V_p \left( \int_a^y [f(x) - f(y)] \, dh(x); [a, b] \right) \leq C_{p,q} [V_q(f; [a, b]) \\
    &\quad + \sup_{a \leq x \leq b} |f(x) - f(y)|] V_p(h; [a, b]) \leq 2C_{p,q} V_q(f; [a, b]) V_p(h; [a, b]).
\end{align*}
\]

Denote \(|A| = \sup_{x \in \mathbb{R}} |A(x)|, \ |A|_\alpha = \sup_{x, y \in \mathbb{R}} \frac{|A(x) - A(y)|}{|x - y|^\alpha}.\)

Let \(F\) be a Lipschitz function and let \(G \in C^{1+\alpha}(\mathbb{R})\) with \(0 < \alpha \leq 1\) and \(1 \leq p < 1 + \alpha\). Then for any \(h \in \mathcal{W}_p([a, b])\)

\[
\begin{align*}
    V_{p,\infty}(F(h); [a, b]) &\leq LV_p(h; [a, b]) + \sup_{a \leq x \leq b} |F(h(x)) - F(h(a))| + |F(h(a))| \\
    &\leq 2LV_p(h; [a, b]) + |F(h(a))|, \quad \text{(5)}
\end{align*}
\]

\[
\begin{align*}
    V_{p/\alpha,\infty}(G(h); [a, b]) &\leq V_{p,\infty}(G(h); [a, b]) \leq 2|G|_{\infty} V_p(h; [a, b]) + |G(h(a))|, \quad \text{(6)}
\end{align*}
\]
Finally, assume that

\[ V_{p/\alpha, \infty}(G'(h); [a, b]) \]

\[ \leq |G'|_\alpha V_p^\alpha(h; [a, b]) + \sup_{a \leq x \leq b} |G'(h(x)) - G'(h(a))| + |G'(h(a))| \]

\[ \leq 2|G'|_\alpha V_p^\alpha(h; [a, b]) + |G'(h(a))|. \quad (7) \]

Let \( f \in W_p([a, b]) \) and \( p_1 > p > 0 \). Then

\[ V_{p_1}(f; [a, b]) \leq \text{Osc}(f; [a, b])^{(p_1-p)/p_1} V_{p_{p_1}}(f; [a, b]), \quad (8) \]

where \( \text{Osc}(f; [a, b]) = \sup\{|f(x) - f(y)| : x, y \in [a, b]\} \).

Take functions \( f_1, f_2 \in W_p([a, b]), 0 < p < \infty \). Then \( f_1, f_2 \in W_p([a, b]) \) and

\[ V_p(f_1f_2; [a, b]) \leq V_{p, \infty}(f_1f_2; [a, b]) \leq C_p V_{p, \infty}(f_1; [a, b]) V_{p, \infty}(f_2; [a, b]). \quad (9) \]

Let \( f_1 \in W_q([a, b]) \) and \( f_2 \in W_p([a, b]) \). Then it follows from Young’s version of Hölder’s inequality that for any partition \( \pi \in \Pi([a, b]) \) and for any \( p^{-1} + q^{-1} \geq 1 \)

\[ \sum_i V_q(f_1; [x_{i-1}, x_i]) V_p(f_2; [x_{i-1}, x_i]) \leq V_q(f_1; [a, b]) V_p(f_2; [a, b]). \quad (10) \]

Second, we state some facts from the theory of Riemann-Stieltjes integration. Let \( f \in W_q([a, b]) \) and \( h \in W_p([a, b]) \) with \( 0 < p < \infty, q > 0, 1/p + 1/q > 1 \). Let symbol \((R)\) stands for the Riemann integration, and \((RS)\) stands for Riemann-Stieltjes integration. Then integral \((RS) \int_a^b f \, dh\) exists under the additional assumption that \( f \) and \( h \) have no common discontinuities.

**Proposition 1.** Let \( f : [a, b] \to \mathbb{R} \) be such function that for some \( 1 \leq p < 2 \) \( f \in CW_p([a, b]) \). Also, let \( F : \mathbb{R} \to \mathbb{R} \) be a differentiable function with locally Lipschitz derivative \( F' \). Then composition \( F'(f) \) is Riemann-Stieltjes integrable with respect to \( f \) and

\[ F(f(b)) - F(f(a)) = (RS) \int_a^b F'(f(x)) \, df(x). \]

Furthermore, the following substitution rule holds.

**Proposition 2.** Let \( f_1, f_2 \) and \( f_3 \) be functions from \( CW_p([a, b]), 1 \leq p < 2 \). Then

\[ (RS) \int_a^b f_1(x) \, df_3(x) = (RS) \int_a^x f_2(y) \, df_3(y) = (RS) \int_a^b f_1(x)f_2(x) \, df_3(x). \]

Finally, assume that

\[ F_1(x) = (R) \int_a^x f_1(y) \, dy \quad \text{and} \quad F_2(x) = (RS) \int_a^x f_2(y) \, df_3(y), \]
where $f_1$ is continuous function, $f_2, f_3 \in CW_p([a, b])$ for some $1 \leq p < 2$, and $Q$ is a differentiable function with locally Lipschitz derivative $q$. It follows from Propositions 2 and 3 that

$$Q(F_1(x) + F_2(x)) - Q(0) = \int_a^x q(F_1(y) + F_2(y)) d(F_1(y) + F_2(y))$$

$$= \int_a^x q(F_1(y) + F_2(y))f_1(y) dy + \int_a^x q(F_1(y) + F_2(y))f_2(y) df_3(y). \quad (11)$$

### 2.2. Asymptotic property of the first- and second-order quadratic variations of fractional Brownian motion

Consider the fractional Brownian motion (fBm) $B^H = \{B^H_t, t \in [0, T]\}$ with Hurst index $H \in (\frac{1}{2}, 1)$. Its sample paths are almost all Hölder up to order $H$. Moreover, for any $0 < \gamma < H$ we have that $L^{H, \gamma}_T := \sup_{s \neq t \in [0, T]} \frac{|B^H_s - B^H_t|}{|s - t|^{1 + \gamma}}$ is finite a.s. and even more, $E(L^{H, \gamma}_T)^k < \infty$ for any $k \geq 1$. The following estimate for the $p$-variation of fBm is evident:

$$V^p_T(B^H; [s, t]) \leq L^{H, 1/p}_T (t - s)^{1/p}, \quad (12)$$

where $s < t \leq T$, $p > 1/H$.

Let $\pi_n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_n = T\}$, $T > 0$, be a sequence of uniform partitions of interval $[0, T]$ with $t^n_k \sim \frac{kt}{n}$ for all $n \in \mathbb{N}$ and all $k \in \{0, \ldots, n\}$, and let $X$ be some real-valued stochastic process defined on the interval $[0, T]$.

**Definition 3.** The normalized first- and second-order quadratic variations of $X$ taking along the partitions $(\pi_n)_{n \in \mathbb{N}}$ and corresponding to the value $1/2 < H < 1$ are defined as

$$V^{(1)}_n(X, 2) = n^{2H - 1} \sum_{k=1}^{n} (\Delta^{(1)}_{k, n} X)^2, \quad \Delta^{(1)}_{k, n} X = X(t^n_k) - X(t^n_{k-1}),$$

and

$$V^{(2)}_n(X, 2) = n^{2H - 1} \sum_{k=1}^{n-1} (\Delta^{(2)}_{k, n} X)^2, \quad \Delta^{(2)}_{k, n} X = X(t^n_{k+1}) - 2X(t^n_k) + X(t^n_{k-1}).$$

For simplicity, we shall omit index $n$ for points $t^n_k$ of partitions $\pi_n$.

It is known (see, e.g., Gladyshev [7]) that $V^{(1)}_n(B^H, 2) \rightarrow T$ a.s. as $n \rightarrow \infty$. Also, it was proved in Benasi et al. [3] and Istas et al. [9] that $V^{(2)}_n(B^H, 2) \rightarrow (4 - 2^{2H})T$ a.s. as $n \rightarrow \infty$.

Denote

$$V^{(1)}_n(B^H, 2)_t = n^{2H - 1} \sum_{k=1}^{r(t)} (\Delta^{(1)}_{k, n} B^H)^2,$$

$$V^{(2)}_n(B^H, 2)_t = n^{2H - 1} \sum_{k=1}^{r(t)-1} (\Delta^{(2)}_{k, n} B^H)^2, \quad t \in [0, T].$$
Lemma 4. (Lévy-Octaviani inequality) Let \( X_1, \ldots, X_n \) be independent random variables. Then for any fixed \( t, s \)
\[
\rho \left( \sum_{j=1}^n X_j \right) > t + s \leq 1 - \max_{1 \leq i \leq n} \rho \left( \sum_{j=1}^n X_j \right) > s \].
\]
The following asymptotic property holds for the first- and second-order quadratic variations of fractional Brownian motion:
\[
\sup_{t \leq T} \left| V_n^{(i)}(B^H, 2)_t - \mathbf{E} V_n^{(i)}(B^H, 2)_t \right| = \mathcal{O}(n^{-1/2} \ln^{1/2} n) \quad \text{a.s.}, \quad i = 1, 2.
\]

Theorem 5. The following asymptotic property holds for the first- and second-order quadratic variations of fractional Brownian motion:
\[
\sup_{t \leq T} \left| V_n^{(i)}(B^H, 2)_t - \mathbf{E} V_n^{(i)}(B^H, 2)_t \right| = \mathcal{O}(1) \quad \text{a.s.}, \quad i = 1, 2.
\]

Remark 6. It follows from (13) and (14) that \( \sup_{t \leq T} \left| V_n^{(i)}(B^H, 2)_t \right| = \mathcal{O}(1) \).

Proof. We can consider \( V_n^{(i)}(B^H, 2) \) as the square of the Euclidean norm of the \( n \)-dimensional Gaussian vector \( X_n \) with the components
\[
n^{H - 1/2} \Delta_k B^H, \quad 1 \leq k \leq n - (i - 1).
\]
Obviously, one can get a new \( n \)-dimensional Gaussian vector \( \tilde{X}_n \) with independent components applying the linear transformation to \( X_n \). It means that there exist nonnegative real numbers \( (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \) and such \( n - (i - 1) \)-dimensional Gaussian vector \( Y_n \) with independent Gaussian \( \mathcal{N}(0, 1) \)-components that
\[
V_n^{(i)}(B^H, 2) = \sum_{j=1}^{n-(i-1)} \lambda_{j,n} (Y_n^{(j)})^2.
\]
The numbers \( (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \) are the eigenvalues of the symmetric \( n-(i-1) \times n-(i-1) \)-matrix
\[
(n^{2H-1} \mathbf{E} [\Delta_k B^H \Delta_k B^H])_{1 \leq j, k \leq n-(i-1)}.
\]
Now we can apply the Hanson and Wright’s inequality (see Hanson et al. [8] or Begyn [1]), and it yields that for \( \varepsilon > 0 \)
\[
P \left( \left| \sum_{j=k}^{n-(i-1)} \lambda_{j,n}^{(i)} (Y_n^{(j)})^2 - 1 \right| \geq \varepsilon \right) \leq 2 \exp \left( - \min \left[ \frac{C_1 \varepsilon}{\lambda_{k,n}^{(i)}}, \frac{C_2 \varepsilon^2}{\sum_{j=k}^{n-(i-1)} (\lambda_{j,n}^{(i)})^2} \right] \right),
\]
\[
\leq 2 \exp \left( - \min \left[ \frac{C_1 \varepsilon}{\lambda_{k,n}^{(i)}}, \frac{C_2 \varepsilon^2}{\sum_{j=1}^{n-(i-1)} (\lambda_{j,n}^{(i)})^2} \right] \right),
\]

(15)
where $C_1, C_2$ are nonnegative constants, $\lambda^*(i)_{k,n} = \max_{1 \leq j \leq n-(i-1)} \lambda^*_j, \lambda^*_n = \max_{1 \leq j \leq n-(i-1)} \lambda^*_j$.

The evident equality holds:

$$\sum_{j=1}^{n-(i-1)} \lambda^*_j = \mathbb{E}V_n^{(i)}(B^H, 2).$$

Furthermore, it follows from (13) that the sequence $\mathbb{E}V_n^{(i)}(B^H, 2), n \geq 1$ is bounded. So, the sums $\sum_{j=1}^{n-(i-1)} \lambda^*_j$ are bounded as well. It is easy to check that

$$\sum_{j=1}^{n-(i-1)} (\lambda^*_j)^2 \leq \lambda^*_n \sum_{j=1}^{n-(i-1)} \lambda^*_j.$$ 

Therefore for any $0 < \varepsilon \leq 1$ the inequality (15) can be rewritten as

$$P\left(\left|\sum_{j=i}^{n-(i-1)} \lambda^*_j [(Y^*_j)^2 - 1] \right| \geq \varepsilon \right) \leq 2 \exp\left(-\frac{K\varepsilon^2}{\lambda^*_n}\right),$$

where $K$ is a positive constant.

Now we use Lévy-Octaviani inequality (see Lemma 4) and evident inequality

$$\frac{x}{1 - x} \leq 2x \quad \text{for} \quad 0 < x \leq 1/2$$

to obtain the bound

$$P\left(\max_{1 \leq k \leq n-(i-1)} \left|\sum_{j=1}^{k} \lambda^*_j [(Y^*_j)^2 - 1] \right| > 2\varepsilon \right) \leq 2 \exp\left(-\frac{K\varepsilon^2}{\lambda^*_n}\right) \leq 4 \exp\left(-\frac{K\varepsilon^2}{\lambda^*_n}\right),$$

assuming that

$$\exp\left(-\frac{K\varepsilon^2}{\lambda^*_n}\right) \leq 1/4 \quad \text{and} \quad 0 < \varepsilon \leq 1.$$ 

So, for the values of parameters mentioned above,

$$P\left(n^{2H-1} \max_{1 \leq k \leq n-(i-1)} \left|\sum_{j=1}^{k} (\Delta^*_j B^H)^2 - \sum_{j=1}^{k} \mathbb{E}(\Delta^*_j B^H)^2 \right| > 2\varepsilon \right) \leq 4 \exp\left(-\frac{K\varepsilon^2}{\lambda^*_n}\right).$$

Furthermore,

$$\lambda^*_n \leq Kn^{2H-1} \max_{1 \leq k \leq n-(i-1)} \sum_{j=1}^{n-(i-1)} |d^*_j|,$$
where $d_{nk}^{(i)} = \mathbb{E}\Delta_{j,n}^{(i)}B^H\Delta_{k,n}^{(i)}B^H$. From Gladyshev [7] and Begyn [1] we get

$$\lambda_n^{(i)} \leq Cn^{-1}. \quad (17)$$

Now we set

$$\varepsilon_n^2 = \frac{2C}{K}n^{-1}\ln n$$

and conclude that

$$P\left(\max_{1 \leq k \leq n-(i-1)}\left|\sum_{j=1}^{k} (\Delta_{j,n}^{(i)}B^H)^2 - \sum_{j=1}^{k} \mathbb{E}(\Delta_{j,n}^{(i)}B^H)^2 \right| > 2\varepsilon_n\right) \leq 4\exp\left(-2\ln n\right) = \frac{4}{n^2}.$$ 

It means that

$$\sum_{n=0}^{\infty} P\left(\max_{1 \leq k \leq n-(i-1)}\left|\sum_{j=1}^{k} (\Delta_{j,n}^{(i)}B^H)^2 - \sum_{j=1}^{k} \mathbb{E}(\Delta_{j,n}^{(i)}B^H)^2 \right| > 2\varepsilon_n\right) < \infty.$$ 

Finally, we get the statement of the present theorem from the Borel-Cantelli lemma and the evident equality

$$\sup_{t \leq T} |V_n^{(i)}(B^H, 2)_t - \mathbb{E}V_n^{(i)}(B^H, 2)_t| = n^{2H-1}\max_{1 \leq k \leq n-(i-1)}\left|\sum_{j=1}^{k} (\Delta_{j,n}^{(i)}B^H)^2 - \sum_{j=1}^{k} \mathbb{E}(\Delta_{j,n}^{(i)}B^H)^2 \right|.$$ 

\[\blacksquare\]

3. The rate of convergence of the first- and second-order quadratic variations of the solution of stochastic differential equation

First, we formulate the following result from [1] about convergence of first- and second-order quadratic variation.

**Theorem 7.** Consider stochastic differential equation (1), where function $f$ is Lipschitz and $g \in C^{1+\alpha}$ for some $0 < \alpha < 1$. Let $X$ be its solution. Then

$$V_n^{(i)}(X, 2) \to \ell^{(i)} \int_0^T g^2(X(t)) \, dt \quad \text{a.s. as } n \to \infty, \quad (18)$$

where

$$\ell^{(i)} = \begin{cases} 1 & \text{for } i = 1, \\ (4 - 2^{2H}) & \text{for } i = 2. \end{cases}$$

Second, we prove the following auxiliary result.
Lemma 8. Let $X$ be a solution of stochastic differential equation. Define a step-wise process $X^\pi$ that is a discretization of process $X$:

$$X^\pi_t = \begin{cases} X(t_k) & \text{for } t \in [t_k, t_{k+1}), \ k = 0, 1, \ldots, n - 2, \\ X(t_{n-1}) & \text{for } t \in [t_{n-1}, t_n]. \end{cases}$$

Then for any $p > \frac{1}{H}$ we have that

$$\sup_{t \leq T} |X^\pi_t - X_t| = O(n^{-1/p}).$$

**Proof.** Consider $t \in [t_k, t_{k+1})$. We get immediately from the Love-Young inequality (2) that

$$|X^\pi_t - X_t| = \left| \int_{\rho^n(t)}^t f(X_s) \, ds + \int_{\rho^n(t)}^t g(X_s) \, dB^H_s \right|$$

$$\leq T n^{-1} \sup_{t_k \leq t \leq t_{k+1}} |f(X_t)| + C_{p,p} V_{p,\infty}(g(X);[t_k, t_{k+1}])V_p(B^H;[t_k, t_{k+1}]).$$

Further,

$$\sup_{t_k \leq t \leq t_{k+1}} |f(X_t)| \leq \sup_{t \leq T} |f(X_t) - f(\xi)| + |f(\xi)| \leq L V_p(X;[0, T]) + |f(\xi)|, \quad (19)$$

where $L$ is a Lipschitz constant for $f$, and

$$V_{p,\infty}(g(X);[t_k, t_{k+1}]) \leq |g'_\infty| V_p(X;[t_k, t_{k+1}]) + \sup_{t \leq T} |g(X_t) - g(\xi)| + |g(\xi)|$$

$$\leq 2 |g'_\infty| V_p(X;[0, T]) + |g(\xi)|. \quad (20)$$

We get the statement of the lemma from (19), (20) and inequality (12).

Now we prove the main result of this section which specifies the rate of convergence in Theorem 7.

**Theorem 9.** Let the conditions of Theorem 7 hold and, in addition, $\alpha > \frac{1}{H} - 1$.

Then

$$V^{(i)}_n(X, 2) - c^{(i)} \int_0^T g^2(X(t)) \, dt = O(n^{-1/4} \ln^{1/4} n). \quad (21)$$

**Proof.** Decompose the left-hand side of (21) into three parts:

$$I^{(i)}_n := V^{(i)}_n(X, 2) - c^{(i)} \int_0^T g^2(X(t)) \, dt = I^{(1,i)}_n + I^{(2,i)}_n + I^{(3,i)}_n,$$
where

\[
I_{n}^{(1,i)} = n^{2H-1} \sum_{k=1}^{n-(i-1)} (\Delta_{k,n}^{(i)}X)^2 - \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) (\Delta_{k,n}^{(i)}B_H)^2,
\]

\[
I_{n}^{(2,i)} = n^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) (\Delta_{k,n}^{(i)}B_H)^2
\]

\[
- \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \mathbb{E}(\Delta_{k,n}^{(i)}B_H)^2,
\]

\[
I_{n}^{(3,i)} = n^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \mathbb{E}(\Delta_{k,n}^{(i)}B_H)^2 - c(i) \int_0^T g^2(X_s) ds,
\]

and \(X_k = X(t_k)\). We start with the most simple term \(I_{n}^{(3,i)}\) and get immediately, similarly to bounds contained in (6), that for any \(p > \frac{1}{H}\)

\[
|I_{n}^{(3,i)}| \leq C(i) \sum_{k=1}^{n-(i-1)} \int_{t_{k-1+(i-1)}}^{t_{k+(i-1)}} |g^2(X(t_{k-1+(i-1)})) - g^2(X_s)| ds
\]

\[
\leq 2c(i)T|g'|_\infty \sup_{t \leq T} |X_T - X_t| \cdot \left[ |g'|_\infty V_p(X; [0, T]) + |g(\xi)| \right].
\]

In order to estimate \(I_{n}^{(2,i)}\), denote

\[
S_t^{(i)} = n^{2H-1} \sum_{k=1}^{r(t)-(i-1)} (\Delta_{k,n}^{(i)}B_H)^2, \quad t \in [0, T], \quad i = 1, 2.
\]

Then

\[
n^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) (\Delta_{k,n}^{(i)}B_H)^2 = \int_0^T g^2(X_t) dS_t^{(i)}
\]

and

\[
n^{2H-1} \sum_{k=1}^{n-(i-1)} g^2(X_{k-1+(i-1)}) \left[ (\Delta_{k,n}^{(i)}B_H)^2 - \mathbb{E}(\Delta_{k,n}^{(i)}B_H)^2 \right]
\]

\[
= \int_0^T g^2(X_t) d[S_t^{(i)} - \mathbb{E}S_t^{(i)}].
\]
Note that $1/p + 1/2 > 1$ for $1/p < 2$. Therefore, we obtain from the Love-Young inequality and from (5)–(9) that

$$|I_n^{(2,1)}| = \left| \int_0^T g^2(X_t) \left[ S_t^{(i)} - ES_t^{(i)} \right] \right| \leq C_{p,2} V_{p,\infty} \left( g^2(X); [0, T] \right) V_{2} \left( S_t^{(i)} - ES_t^{(i)}; [0, T] \right) \leq C_{p,2} \left\{ \text{Osc} \left( S_t^{(i)} - ES_t^{(i)}; [0, T] \right) \right\}^{1/2} V_{p,\infty} \left( g^2(X); [0, T] \right) \times V_1^{1/2} \left( S_t^{(i)} - ES_t^{(i)}; [0, T] \right) \leq 2C_{p,2} \left( \sup_{t \leq T} |S_t^{(i)} - ES_t^{(i)}| \right)^{1/2} V_{2} \left( g(X); [0, T] \right) \times \left[ n^{2H-1} \sum_{k=1}^{n-1} (\Delta_{k,n}^{(i)} B^H)^2 + c^{(i)} T \right]^{1/2}.$$

It follows from Theorem 5, Remark 6, and (6) that the rate of convergence of $I_n^{(2,1)}$ is $O(n^{-1/4} \ln^{1/4} n)$.

It still remains to estimate $I_n^{(1,1)}$. Consider only $i = 2$, the proof for $i = 1$ is similar.

Denote

$$J_k^1 = \int_{t_k}^{t_{k+1}} [f(X_s) - f(X_k)] ds - \int_{t_{k-1}}^{t_k} [f(X_s) - f(X_k)] ds,$$

$$J_k^2 = \int_{t_{k-1}}^{t_k} \left( g(X_k) - g(X_s) - \int_s^t g'(X_k) f(X_k) du - \int_s^t g'(X_k) g(X_k) dB_H^u \right) dB_s^H,$$

$$J_k^3 = \int_{t_k}^{t_{k+1}} \left( g(X_k) - g(X_s) - \int_s^t g'(X_k) f(X_k) du - \int_t^s g'(X_k) g(X_k) dB_H^u \right) dB_s^H,$$

$$J_k^4 = g'(X_k) f(X_k) \left( \int_{t_k}^{t_{k+1}} (s - t_k) dB_H^s + \int_{t_k}^{t_{k+1}} (t_k - s) dB_H^s \right),$$

$$J_k^5 = \frac{1}{2} g'(X_k) g(X_k) \left( (\Delta_{k,n}^{(1)} B^H)^2 + (\Delta_{k+1,n}^{(1)} B^H)^2 \right),$$

$$J_k^6 = g(X_k) \Delta_{k,n}^{(2)} B^H.$$

Equalities

$$\int_{t_{k-1}}^{t_k} \left( \int_s^t g'(X_k) g(X_k) dB_H^u \right) dB_s^H = \frac{1}{2} g'(X_k) g(X_k) (\Delta_{k,n}^{(1)} B^H)^2,$$

$$\int_{t_k}^{t_{k+1}} \left( \int_s^t g'(X_k) g(X_k) dB_H^u \right) dB_s^H = \frac{1}{2} g'(X_k) g(X_k) (\Delta_{k+1,n}^{(1)} B^H)^2.$$

(see Proposition 11) imply

$$\Delta_{k,n}^{(2)} X = \sum_{i=1}^{n} J_k^i.$$
Taking into account Lipschitz property of $f$ and Lemma 8, we can conclude that

$$|f(X_t) - f(X_k)| \leq L \sup_{t \in T} |X^x_t - X^x_t| = O(n^{-1/p}).$$

Therefore

$$\sum_{k=1}^{n-1} (J^1_k)^2 \leq 2Tn^{-1} \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} [f(X_s) - f(X_k)]^2 ds$$

$$+ 2Tn^{-1} \sum_{k=1}^{n-1} \int_{t_k}^{t_{k-1}} [f(X_s) - f(X_k)]^2 ds$$

$$\leq 4Tn^{-1}L^2 \left( \sup_{t \in T} |X^x_t - X^x_t| \right)^2$$

$$= O(n^{1-2/p}).$$

Consider $J^2_k$. It follows from equality (11) that for any fixed $t \in [t_{k-1}, t_k]$,

$$g(X_k) - g(X_t) = \int_t^{t_k} g'(X_s)f(X_s) ds + \int_t^{t_k} g'(X_s)g(X_s) dB^H_s. \quad (22)$$

Substituting equality (22) into $J^2_k$ we get

$$|J^2_k| \leq \left| \int_{t_{k-1}}^{t_k} \int_{s}^{t_k} [g'(X_u)f(X_u) - g'(X_k)f(X_k)] du dB^H_s \right|$$

$$+ \left| \int_{t_{k-1}}^{t_k} \int_{s}^{t_k} [g'(X_u)g(X_u) - g'(X_k)g(X_k)] dB^H_s dB^H_s \right|. \quad (23)$$

Transforming identically the first term in the right-hand side of (23) and applying to it Love-Young inequality (2), we conclude that for any $p > \frac{1}{1 + \alpha}$,

$$\left| \int_{t_{k-1}}^{t_k} \int_{s}^{t_k} [g'(X_u)f(X_u) - g'(X_k)f(X_k)] du dB^H_s \right|$$

$$\leq C_{p,1}V_1 \left( \int_{t_{k-1}}^{t_k} [g'(X_u)f(X_u) - g'(X_k)f(X_k)] du; [t_{k-1}, t_k] \right) V_p(B^H; [t_{k-1}, t_k])$$

$$\leq C_{p,1}V_p(B^H; [t_{k-1}, t_k]) \int_{t_{k-1}}^{t_k} |g'(X_u)f(X_u) - g'(X_k)f(X_k)| du. \quad (24)$$

Henceforth we consider the following interval of the values of $p$: $\frac{1}{1 + \alpha} < p < 1 + \alpha$. Then it follows from inequality (2) that the second term in the right-hand side
Applying inequalities (2) and (12), we obtain the following bound for the first ∑ of (23) admits the bound:

\[ \left| \int_{t_{k-1}}^{t_k} \int_{s}^{t_k} \left[ g'(X_u)g(X_u) - g'(X_k)g(X_k) \right] dB_u^H \, dB_s^H \right| \]

\[ \leq C_{p,p/\alpha} V_{p/\alpha} \left( \int_{t_{k-1}}^{t_k} \int_{s}^{t_k} \left[ g'(X_u)g(X_u) - g'(X_k)g(X_k) \right] dB_u^H \, [t_{k-1}, t_k] \right) \]

\[ \times V_p(B^H; [t_{k-1}, t_k]) \]

\[ \leq 2C_{p,p/\alpha}^2 V_{p/\alpha} \left( g'(X_u)g(X_u); [t_{k-1}, t_k] \right) V_{p}^2(B^H; [t_{k-1}, t_k]). \] (25)

We conclude from (23)–(25) that

\[ |J_k^2| \leq T n^{-1} C_{p,1} V_{p/\alpha} \left( g'(X) f(X); [t_k, t_{k+1}] \right) V_{p} (B^H; [t_k, t_{k+1}]) \]

\[ + 2C_{p,p/\alpha}^2 V_{p/\alpha} \left( g'(X) g(X); [t_k, t_{k+1}] \right) V_{p}^2(B^H; [t_k, t_{k+1}]). \]

Applying inequalities (9) and (10) we immediately obtain that

\[ \sum_{k=1}^{n} \left( J_k^2 \right)^2 \leq 2T^2 C_{p,1} n^{-2} \max_{0 \leq k \leq n-1} \left[ V_{p/\alpha} \left( g'(X) f(X); [t_k, t_{k+1}] \right) V_{p} (B^H; [t_k, t_{k+1}]) \right] \]

\[ \times V_{p/\alpha} \left( g'(X) g(X); [0, T] \right) V_{p} (B^H; [0, T]) \]

\[ + 4C_{p,p/\alpha}^4 \max_{0 \leq k \leq n-1} \left[ V_{p/\alpha} \left( g'(X) g(X); [t_k, t_{k+1}] \right) V_{p}^3(B^H; [t_k, t_{k+1}]) \right] \]

\[ \leq 2T^2 C_{p,1} n^{-2} \max_{0 \leq k \leq n-1} \left[ V_{p} (B^H; [t_k, t_{k+1}]) \right] V_{p/\alpha, \infty}^2 \left( g'(X); [0, T] \right) \]

\[ \times V_{p, \infty} (f(X); [0, T]) V_{p} (B^H; [0, T]) \]

\[ + 4C_{p,p/\alpha}^4 \max_{0 \leq k \leq n-1} \left[ V_{p}^3(B^H; [t_k, t_{k+1}]) \right] \]

\[ \times V_{p/\alpha, \infty}^2 \left( g'(X); [0, T] \right) V_{p/\alpha, \infty} (g(X); [0, T]) V_{p} (B^H; [0, T]). \]

It follows from the inequalities (5)–(7) that the values of the variations

\[ V_{p, \infty} (f(X); [0, T]), \quad V_{p/\alpha, \infty} (g(X); [0, T]), \quad \text{and} \quad V_{p/\alpha, \infty} (g'(X); [0, T]) \]

are finite. Therefore we get from (12) that

\[ \sum_{k=1}^{n} \left( J_k^2 \right)^2 = \mathcal{O}(n^{-2-1/p}) + \mathcal{O}(n^{-3/p}) = \mathcal{O}(n^{-3/p}). \]

The similar reasonings lead to the similar bound for \( J_k^3 \), and we conclude that

\[ \sum_{k=0}^{n-1} \left| J_k^2 + J_k^3 \right|^2 = \mathcal{O}(n^{-3/p}). \]

Consider \( J_k^3 \). It consists of two terms that can be estimated in a similar way. Applying inequalities (2) and (12), we obtain the following bound for the first
term:

\[
\sum_{k=1}^{n-1} \left[ g'(X_k) f(X_k) \right]^2 \left( \int_{t_k}^{t_{k+1}} (s - t_k) \, dB_s^H \right)^2
\]

\[
\leq C^2_{p,1} \sum_{k=1}^{n-1} \left[ g'(X_k) f(X_k) \right]^2 (t_{k+1} - t_k)^2 V^2_p(B^H; [t_k, t_{k+1}])
\]

\[
\leq n^{-2} T^2 C^2_{p,1} \max_{1 \leq k \leq n-1} \left[ g'(X_k) f(X_k) V_p(B^H; [t_k, t_{k+1}]) \right]^2 = O(n^{-1-2/p}).
\]

As a consequence,

\[
\sum_{k=1}^{n-1} (J^1_k)^2 = O(n^{-1-2/p}).
\]

Furthermore, note that under our assumptions \( \sup_{s \in [0,T]} |g(X_s)| < \infty \) and \( \sup_{s \in [0,T]} |g'(X_s)| < \infty \) a.s. Therefore we have for the first term in \( J^1_k \) that

\[
\sum_{k=1}^{n-1} \left[ g'(X_k) g(X_k) \left( \Delta^{(1)}_{k+1,n} B^H \right) \right]^2 \leq \max_{1 \leq k \leq n-1} \left[ g'(X_k) g(X_k) \right]^2 \sum_{k=0}^{n-1} \left( \Delta^{(1)}_{k+1,n} B^H \right)^4
\]

\[
= O(n^{-1-4/p}).
\]

The second term is bounded in a similar way, and we conclude that

\[
\sum_{k=1}^{n-1} (J^5_k)^2 = O(n^{-1-4/p}).
\]

Thus

\[
\left( \sum_{l=1}^{5} J^1_l \right)^2 = O(n^{-1-2/p} \lor n^{-3/p} \lor n^{-1-4/p}) = O(n^{-1-4/p}).
\]

At last,

\[
n^{2H-1} \sum_{k=1}^{n-1} \left[ \Delta^{(2)}_{k,n} X - g(X_k) \Delta^{(2)}_{k,n} B^H \right]^2 = O(n^{-1-4/p+2H-1})
\]

\[
= O(n^{-4/p+2H})
\]

for any \( \frac{1}{H} < p < 1 + \alpha \). Set \( 1/p = H - \varepsilon \) for \( \varepsilon < (H/2 - 1/16) \land (H - \frac{1}{1+\alpha}) \). Then

\[
V_n^{(2)}(X, 2) - \varepsilon(2) \int_0^T g^2(X_s) \, ds
\]

\[
= O(n^{-4/p+2H}) + O(n^{-1/4 \ln^{1/4} n}) + O(n^{-1/p})
\]

\[
= O(n^{-2H+4\varepsilon}) + O(n^{-1/4 \ln^{1/4} n}) + O(n^{-H+\varepsilon})
\]

\[
= O(n^{-1/4 \ln^{1/4} n}).
\]
4. The rate of convergence of estimators of Hurst index

Consider the following statistics:

\[ R^{(i)}_n = \frac{\sum_{k=1}^{2n^{-1}} (\Delta_{k,2n}^{(i)} X)^2}{\sum_{k=1}^{n^{-1}} (\Delta_{k,n}^{(i)} X)^2} \]

and construct the following estimate of Hurst index \( H \):

\[ \hat{H}^{(i)}_n = \left( \frac{1}{2} - \frac{1}{2 \ln 2} \ln R^{(i)}_n \right) \mathbf{1}_{\tilde{C}_n}, \]

where

\[ \tilde{C}_n = \left\{ 2^{-1} \left( 1 - 2n^{-1/4} (\ln n)^{1/4+\beta} \right) \leq R^{(i)}_n \leq 1 + 2n^{-1/4} (\ln n)^{1/4+\beta} \right\}, \quad \beta > 0. \]

Further, introduce the following notation: \( g^{(i)}(T) = c^{(i)} \int_0^T g^2(X_s) ds \).

**Theorem 10.** Let conditions of Theorem 4 hold with \( \alpha > \frac{1}{H} - 1 \). Also, let \( X \) be a solution of (1) and assume that random variable \( g^{(i)}(T) \) is separated from zero: there exists a constant \( c_0 > 0 \) such that \( g^{(i)}(T) \geq c_0 \) a.s. Then \( \hat{H}^{(i)}_n \) is a strongly consistent estimator of the Hurst index \( H \) and the following rate of convergence holds:

\[ |\hat{H}^{(i)}_n - H| = O \left( n^{-1/4} (\ln n)^{1/4+\beta} \right) \quad \text{a.s.,} \]

for any \( \beta > 0 \).

**Proof.** Consider a sequence \( 1 > \delta_n \downarrow 0 \) as \( n \to \infty \). It will be specified later on. Introduce the events

\[ C_n = \left\{ \frac{1}{2} (1 - \delta_n) \leq R^{(i)}_n \leq 1 + \delta_n \right\}. \]

Also, introduce the notations

\[ A^{(i)}_n = V^{(i)}_{2n}(X,2) \quad \text{and} \quad B^{(i)}_n = V^{(i)}(X,2) \]

and note that \( 2^{2H-1} R^{(i)}_n = \frac{A^{(i)}_n}{B^{(i)}_n} \). Then

\[ C_n = \left\{ 2^{2H-2} (1 - \delta_n) \leq \frac{A^{(i)}_n}{B^{(i)}_n} \leq 2^{2H-1} (1 + \delta_n) \right\}, \]

and estimate \( \hat{H}^{(i)}_n \) has a form

\[ \hat{H}^{(i)}_n = \left( \frac{1}{2} - \frac{1}{2 \ln 2} \ln R^{(i)}_n \right) \mathbf{1}_{C_n}. \]
It is easy to see that \( \overline{C}_n := \Omega \setminus C_n \) has a form

\[
\overline{C}_n = \left\{ \frac{A_n^{(i)}}{B_n^{(i)}} < 2^{2H-2}(1-\delta_n) \right\} \cup \left\{ \frac{A_n^{(i)}}{B_n^{(i)}} > 2^{2H-1}(1+\delta_n) \right\} \\
\subset \left\{ \left| \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right| > \delta_n \right\} .
\] (28)

Then

\[
\hat{H}_n^{(i)} = H1_{C_n} - \frac{1}{2 \ln 2} \ln \frac{(2n)^{2H-1}V_{2n}^{(i)}(X, 2)}{n^{2H-1}V_n^{(i)}(X, 2)} 1_{C_n} \\
= H1_{C_n} - \frac{1}{2 \ln 2} \ln \frac{A_n^{(i)}}{B_n^{(i)}} 1_{C_n} .
\]

The latter representation implies that

\[
\left| \hat{H}_n^{(i)} - H \right| \leq H1 \left\{ \left| \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right| > \delta_n \right\} + \frac{1}{2 \ln 2} \left| \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right| \left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1+\delta_n \right\} \\
- \left( \frac{1}{2 \ln 2} \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \left\{ 2^{2H-2}(1-\delta_n) \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n \right\} \\
+ \left( \frac{1}{2 \ln 2} \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \left\{ 1+\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 2^{2H-1}(1+\delta_n) \right\} := \sum_{i=1}^{4} L_n^i .
\] (29)

In what follows we need an elementary inequalities: \( -\ln(1-x) \leq 2 \ln(1+x) \leq 2x \) provided that \( 0 \leq x \leq 1/2 \).

Consider \( L_n^2 \). We divide it in two parts. As to the first part, it is obvious that

\[
\left( \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \mathbf{1}\left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \right\} = \left( \ln \left[ 1 - \left( 1 - \frac{A_n^{(i)}}{B_n^{(i)}} \right) \right] \right) \mathbf{1}\left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \right\} ,
\]

and

\[
1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \quad \text{implies that} \quad 0 < 1 - \frac{A_n^{(i)}}{B_n^{(i)}} \leq \delta_n .
\]

Applying inequality \( -\ln(1-x) \leq 2x, 0 \leq x \leq 1/2 \), we deduce that for \( \delta_n \leq 1/2 \)

\[
\left( -\ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \mathbf{1}\left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \right\} \leq 2 \left( 1 - \frac{A_n^{(i)}}{B_n^{(i)}} \right) \mathbf{1}\left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \right\} \leq 2\delta_n \mathbf{1}\left\{ 1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1 \right\} .
\]

As to the second part,

\[
\left( \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \mathbf{1}\left\{ 1 \leq \frac{A_n^{(i)}}{B_n^{(i)}} \leq 1+\delta_n \right\} = \left( \ln \left[ 1 + \left( \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right) \right] \right) \mathbf{1}\left\{ 1 \leq \frac{A_n^{(i)}}{B_n^{(i)}} \leq 1+\delta_n \right\} \\
\leq \left( \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right) \mathbf{1}\left\{ 1 \leq \frac{A_n^{(i)}}{B_n^{(i)}} \leq 1+\delta_n \right\} \leq \delta_n \mathbf{1}\left\{ 1 \leq \frac{A_n^{(i)}}{B_n^{(i)}} \leq 1+\delta_n \right\} .
\]

16
Consider $L_n^3$. From here we easy deduce that

$$
- \left( \frac{1}{2 \ln 2} \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \textbf{1}_{\{2^{2H-2}(1-\delta_n) \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\}} \\
\leq - \frac{1}{2 \ln 2} \left[ \ln \left(2^{2H-2}(1-\delta_n)\right) \right] \textbf{1}_{\{2^{2H-2}(1-\delta_n) \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\}} \\
\leq \left( (1-H) - \frac{\ln(1-\delta_n)}{2 \ln 2} \right) \textbf{1}_{\{2^{2H-2}(1-\delta_n) \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\}} \\
\leq \left( 1 - H + \frac{\delta_n}{\ln 2} \right) \textbf{1}_{\{2^{2H-2}(1-\delta_n) \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 1-\delta_n\}} \\
\leq (1 + 2\delta_n) \textbf{1}_{\{\frac{A_n^{(i)}}{B_n^{(i)}} > 1-\delta_n\}}.
$$

The term $L_n^4$ is estimated similarly as the second part of $L_n^2$. Thus we get

$$
\left( \ln \frac{A_n^{(i)}}{B_n^{(i)}} \right) \textbf{1}_{\{1+\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 2^{2H-1} (1+\delta_n)\}} \leq \left( \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right) \textbf{1}_{\{1+\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} < 2^{2H-1} (1+\delta_n)\}} \\
\leq (1 + 2\delta_n) \textbf{1}_{\{\frac{A_n^{(i)}}{B_n^{(i)}} > 1+\delta_n\}}.
$$

Summarizing, we conclude that

$$
|{\hat{H}}_n^{(i)} - H| \leq (1 + 2\delta_n) \textbf{1}_{\{|\frac{A_n^{(i)}}{B_n^{(i)}} - 1| > \delta_n\}} + 2\delta_n \textbf{1}_{\{1-\delta_n \leq \frac{A_n^{(i)}}{B_n^{(i)}} \leq 1+\delta_n\}} \\
\leq (1 + 2\delta_n) \textbf{1}_{\{|\frac{A_n^{(i)}}{B_n^{(i)}} - 1| > \delta_n\}} + 2\delta_n.
$$

Now, let $\beta > 0$. Note that

$$
\left\{ \left| \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right| > \delta_n \right\} \\
\subseteq \left\{ \left| \frac{A_n^{(i)}}{B_n^{(i)}} - 1 \right| > \delta_n, B_n^{(i)} \geq (\ln n)^{-\beta} \right\} \bigcup \left\{ B_n^{(i)} < (\ln n)^{-\beta} \right\} \\
= \left\{ \left| A_n^{(i)} - B_n^{(i)} \right| > \delta_n B_n^{(i)}, B_n^{(i)} \geq (\ln n)^{-\beta} \right\} \bigcup \left\{ B_n^{(i)} < (\ln n)^{-\beta} \right\} \\
\subseteq \left\{ \left| A_n^{(i)} - B_n^{(i)} \right| > \delta_n (\ln n)^{-\beta} \right\} \bigcup \left\{ B_n^{(i)} < (\ln n)^{-\beta} \right\}.
$$

Therefore

$$
|{\hat{H}}_n^{(i)} - H| \leq (1 + 2\delta_n) \textbf{1}_{\{|A_n^{(i)} - B_n^{(i)}| > \delta_n (\ln n)^{-\beta}\} \cup \{B_n^{(i)} < (\ln n)^{-\beta}\}} + 2\delta_n.
$$

It follows from (27) that

$$
|A_n^{(i)} - B_n^{(i)}| = O(n^{-1/4} \ln^{1/4} n)
$$

17
and

\[ |B_n^{(i)} - c^{(i)} \int_0^T g^2(X_s)ds| = O(n^{-1/4} \ln^{1/4} n). \]

Obviously, for any \( n > \exp\{\left(\frac{2}{c_0}\right)^{1/\beta}\} \) we have that \( g^{(i)}(T) \geq c_0 \geq 2(ln n)^{-\beta} \) a.s. and

\[ \{B_n^{(i)} < (ln n)^{-\beta}\} = \{B_n^{(i)} < (ln n)^{-\beta}, g^{(i)}(T) \geq 2(ln n)^{-\beta}\}. \]

Now, let \( \delta_n < (ln n)^{-\beta} \). Then it is not hard to deduce that

\[ \{B_n^{(i)} < (ln n)^{-\beta}, g^{(i)}(T) \geq 2(ln n)^{-\beta}\} = \{B_n^{(i)} < (ln n)^{-\beta}, g^{(i)}(T) \geq 2(ln n)^{-\beta}, B_n^{(i)} < g^{(i)}(T) - \delta_n\} \]
\[ \subset \{|B_n^{(i)} - g^{(i)}(T)| > \delta_n\}. \]

Therefore,

\[ \{B_n^{(i)} < (ln n)^{-\beta}\} \subset \{|B_n^{(i)} - g^{(i)}(T)| > \delta_n\} \]

if \( n > \exp\{\left(\frac{2}{c_0}\right)^{1/\beta}\} \).

Finally, specify \( \delta_n \). More precisely, set \( \delta_n = n^{-1/4}(ln n)^{1/4+2\beta}, \beta > 0 \). Note that \( \delta_n < (ln n)^{-\beta} \) for sufficiently large \( n \) and, moreover,

\[ \frac{O(n^{-1/4} \ln^{1/4} n)}{\delta_n(ln n)^{-\beta}} = \frac{O(n^{-1/4} \ln^{1/4} n)}{n^{-1/4}(ln n)^{1/4+\beta}} \rightarrow 0 \quad a.s. \quad as \quad n \rightarrow \infty. \]

The latter relation together with Theorem 9 imply that for any \( \omega \in \Omega' \) with \( P(\Omega') = 1 \) there exists \( n_0 = n_0(\omega) \) such that for any \( n > n_0 \)

\[ 1\{|A_n^{(i)} - B_n^{(i)}| > \delta_n(ln n)^{-\beta}\} \cup \{B_n^{(i)} < (ln n)^{-\beta}\} = 0 \quad a.s., \]

and we obtain the proof.

\[ \blacksquare \]

5. Simulation results

Consider fractional Ornstein-Uhlenbeck process that is the solution of the linear stochastic differential equation

\[ dX_t = -X_t dt + dB_t^H, \quad X_0 = 0. \]

with the step 0.05 and for increasing (in the logarithmic scale) number \( n \) of points from \( n = 10^2 \) to \( n = 10^5 \). Table 1 presents the values of the difference \( \|\hat{H}_n^{(1)} - H\| \) for the values of \( H \) from 0.55 to 0.95. We can conclude that the difference \( \|\hat{H}_n^{(1)} - H\| \) decreases rapidly in \( n \) and for fixed value of \( n \) increases in \( H \). Table 2 demonstrates that the rate of convergence agrees with Theorem 10 at least, for \( \beta = 0.95 \). Moreover, we can see from Table 3 that in the case of the linear equation the rate of convergence for \( H \in (0.5, 0.7) \) can be estimated by \( n^{-1/2}(\ln n)^{1/2} \).
Table 1: $|\hat{H}_n^{(1)} - H|$  

| $H$ | $n$ points | $n=100$ | $n=250$ | $n=1000$ | $n=2500$ | $n=10^4$ | $n=2.5 \cdot 10^4$ | $n=10^5$ | $n=2.5 \cdot 10^5$ | $n=10^6$ |
|-----|-------------|---------|---------|-----------|-----------|---------|----------------|---------|----------------|---------|
| 0.55 | 0.08401 | 0.05488 | 0.02124 | 0.011467 | 0.00777 | 0.00549 | 0.00195 | 0.00160 | 0.00079 |
| 0.6 | 0.07216 | 0.04145 | 0.02213 | 0.01286 | 0.00683 | 0.00466 | 0.00214 | 0.00137 | 0.00069 |
| 0.65 | 0.07761 | 0.04811 | 0.01972 | 0.01296 | 0.00626 | 0.00414 | 0.00210 | 0.00144 | 0.00066 |
| 0.7 | 0.05364 | 0.03403 | 0.02023 | 0.01291 | 0.00608 | 0.00414 | 0.00210 | 0.00144 | 0.00066 |
| 0.75 | 0.06485 | 0.03798 | 0.02187 | 0.011147 | 0.00707 | 0.00424 | 0.00211 | 0.00140 | 0.00065 |
| 0.8 | 0.05938 | 0.03884 | 0.02040 | 0.01307 | 0.00791 | 0.00414 | 0.00211 | 0.00140 | 0.00068 |
| 0.85 | 0.04666 | 0.03577 | 0.02105 | 0.01684 | 0.01011 | 0.00707 | 0.00414 | 0.00211 | 0.00120 |
| 0.9 | 0.06311 | 0.04762 | 0.03037 | 0.02338 | 0.01667 | 0.01352 | 0.00984 | 0.00801 | 0.00659 |
| 0.95 | 0.06219 | 0.04763 | 0.03488 | 0.02907 | 0.01136 | 0.00549 | 0.00195 | 0.00160 | 0.00079 |

Table 2: $|\hat{H}_n^{(1)} - H| \cdot n^{0.25} (\ln n)^{-0.3}$  

| $H$ | $n$ points | $n=100$ | $n=250$ | $n=1000$ | $n=2500$ | $n=10^4$ | $n=2.5 \cdot 10^4$ | $n=10^5$ | $n=2.5 \cdot 10^5$ | $n=10^6$ |
|-----|-------------|---------|---------|-----------|-----------|---------|----------------|---------|----------------|---------|
| 0.55 | 0.16802 | 0.13070 | 0.06689 | 0.05596 | 0.03991 | 0.03449 | 0.01663 | 0.01681 | 0.01136 |
| 0.6 | 0.14433 | 0.09873 | 0.06969 | 0.04906 | 0.03506 | 0.02923 | 0.01828 | 0.01435 | 0.00994 |
| 0.65 | 0.15322 | 0.11158 | 0.06212 | 0.04942 | 0.03215 | 0.02662 | 0.01793 | 0.01510 | 0.00954 |
| 0.7 | 0.10727 | 0.08104 | 0.06370 | 0.04652 | 0.03125 | 0.02142 | 0.01565 | 0.01311 | 0.00893 |
| 0.75 | 0.12970 | 0.09044 | 0.06888 | 0.04376 | 0.03631 | 0.02664 | 0.01904 | 0.01471 | 0.01189 |
| 0.8 | 0.11877 | 0.09250 | 0.06426 | 0.04987 | 0.04066 | 0.03313 | 0.02590 | 0.02382 | 0.01723 |
| 0.85 | 0.09331 | 0.08519 | 0.06628 | 0.04425 | 0.05193 | 0.04726 | 0.03636 | 0.03031 | 0.03580 |
| 0.9 | 0.12622 | 0.11055 | 0.09563 | 0.08920 | 0.08563 | 0.08485 | 0.08410 | 0.08413 | 0.08612 |
| 0.95 | 0.12438 | 0.11344 | 0.10984 | 0.11089 | 0.11789 | 0.12669 | 0.14009 | 0.15206 | 0.17450 |

Table 3: $|\hat{H}_n^{(1)} - H| \cdot n^{0.5} (\ln n)^{-0.5}$  

| $H$ | $n$ points | $n=100$ | $n=250$ | $n=1000$ | $n=2500$ | $n=10^4$ | $n=2.5 \cdot 10^4$ | $n=10^5$ | $n=2.5 \cdot 10^5$ | $n=10^6$ |
|-----|-------------|---------|---------|-----------|-----------|---------|----------------|---------|----------------|---------|
| 0.55 | 0.59403 | 0.56033 | 0.38778 | 0.39788 | 0.38848 | 0.41148 | 0.27528 | 0.34462 | 0.32254 |
| 0.6 | 0.51028 | 0.49873 | 0.40902 | 0.34882 | 0.34127 | 0.35107 | 0.30255 | 0.29419 | 0.28218 |
| 0.65 | 0.54879 | 0.49122 | 0.36011 | 0.35142 | 0.31289 | 0.31248 | 0.29686 | 0.30600 | 0.27064 |
| 0.7 | 0.37926 | 0.34744 | 0.36932 | 0.33076 | 0.30417 | 0.25725 | 0.25904 | 0.26879 | 0.26241 |
| 0.75 | 0.45857 | 0.38776 | 0.39935 | 0.31166 | 0.35343 | 0.31994 | 0.29864 | 0.30156 | 0.33742 |
| 0.8 | 0.41990 | 0.39658 | 0.37253 | 0.35462 | 0.33573 | 0.39783 | 0.42871 | 0.48905 | 0.48905 |
| 0.85 | 0.32990 | 0.35623 | 0.38428 | 0.45683 | 0.50541 | 0.56751 | 0.72214 | 0.82624 | 1.01618 |
| 0.9 | 0.44627 | 0.47396 | 0.55444 | 0.63424 | 0.83341 | 1.01898 | 1.39207 | 1.72453 | 2.44410 |
| 0.95 | 0.43976 | 0.48637 | 0.63677 | 0.78848 | 1.14744 | 1.52138 | 2.31880 | 3.11678 | 4.95257 |

References

[1] A. Bégyn, Quadratic variations along irregular subdivisions for Gaussian processes, *Electronic Journal of Probability*, 10 691–717 (2005).
[2] A. Bégyn, Generalized Quadratic Variations of Gaussian Processes: Limit Theorems and Applications to Fractional Processes, *Dissertation* (2006).

[3] Benassi, A., Cohen, S., Istas, J., Jaffard, S. Identification of filtered white noises, *Stochastic Processes and their Applications* 75 31-49 (1998).

[4] Biagini F., Hu, Y., Øksendal, B., Zhang, T. Stochastic calculus for fractional Brownian motion and applications, Springer (2008)

[5] Dudley, R.M., Norvaša, R., Concrete functional calculus. Springer Monographs in Mathematics. New York, Springer (2011)

[6] Dudley R.M., 1999. Picard iteration and $p$-variation: the work of Lyons (1994), Mini-proceedings: Workshop on Product Integrals and Pathwise Integration, MaPhySto, Aarhus.

[7] E. G. Gladyshev, A new Limit theorem for stochastic processes with Gaussian increments, *Theory Probab. Appl.*, 6(1), p. 52-61, (1961).

[8] D.L. Hanson and F.T. Wright, A bound on tail probabilities for quadratic forms in independent random variables. *Ann. Math. Statist.*, 42, 1079-1083 (1971).

[9] J. Istas, G. Lang. Quadratic variations and estimation of the local H older index of a Gaussian process. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 33, p. 407–436 (1997).

[10] K. Kubilius, The existence and uniqueness of the solution of the integral equations driven by fractional Brownian motion, *Liet. mat. rink.*, 40(spec. nr.), 104–110 (2000).

[11] K. Kubilius, D. Melichov, Quadratic variations and estimation of the Hurst index of the solution of SDE driven by a fractional Brownian Motion, *Lithuanian Mathematical Journal*, 50(4), 401–417 (2010).

[12] Lyons, T., Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young, *Mathematical Research Letters* 1, 451-464 (1994).

[13] T. Lyons, M. Caruana, T. Lévy, Differential equations driven by rough paths, *Ecole d'Eté de Probabilités de Saint-Flour XXXIV2004*, (J. Picard, ed.), Lecture Notes in Math., vol. 1908, Springer, Berlin (2007)

[14] Mishura, Y. Stochastic calculus for fractional Brownian motion and related processes. Lectures Notes Math., 1929, Springer (2008)

[15] D. Nualart , A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.* 53(1) 55–81 (2002).

[16] Young, L.C., 1936. An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.* 67, 251–282.