A Principle of Least Action for the First Law of Thermodynamics in Rational Mechanics

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Abstract

A new theoretical procedure is presented to study the conservation of energy equation with dissipation in continuum mechanics in 1D. This procedure is used to transform this nonlinear evolution-diffusion equation into a hyperbolic PDE; specifically, a second order quasi-linear wave equation. An immediate implication of this procedure is the formation of a least action principle for the balance of energy with dissipation. The corresponding action functional enables us to establish a complete analytical mechanics for thermomechanical systems: a Lagrangian-Hamiltonian theory, bracket formalism, and Noether’s theorem. Furthermore, we apply our procedure iteratively and produce an infinite sequence of interlocked variational principles, a variational hierarchy, where at each level or iteration the full implication of the least action principle can be again shown. Finally we offer comments on the implication of our work to field theories in general.

Keywords: Continuum mechanics, energy conservation, first law of thermodynamics, least action principle, dissipation, variational hierarchy

1 Introduction

Hamilton’s principle is, undoubtedly, one of the great insights of physics. While historically it was formulated in the context of classical mechanics [1], it has been remarkably extended to other field theories such as fluid mechanics, electromagnetism, general relativity and various quantum field theories. It is well-known however that the many consequences of this principle—such as Lagrangian mechanics, Hamilton’s equations, and Noether’s theorem—apply only to conservative systems and can not capture the irreversible effects of a general dissipative system, such as the diffusion of heat.

The purpose of this work is to construct an action functional whose stationary points satisfies the (nonlinear) conservation of energy equation with heat dissipation in one spatial dimension. In other words, we look to formulate a stationary principle for the first law of thermodynamics (as formulated in rational mechanics) analogous to Hamilton’s principle of stationary action. This will

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allow us, among other things, to find a bona fide variational principle for the coupled heat equation in classical thermoelasticity and the classic heat equation.

Consider one of the most basic equations in all of physics: the classic wave equation in some domain $B \subseteq \mathbb{R}^n$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0, \quad (\vec{x}, t) \in B \times (0, \infty) \quad (1.1)$$

where the scalar $c$ denotes the speed of propagation of the wave. The Lagrangian $\mathcal{L}$ for the above PDE is the difference between the kinetic energy and the (potential) strain energy

$$\mathcal{L} = \frac{1}{2} \int_B \left( \frac{\partial u}{\partial t} \right)^2 - c^2 |\nabla u|^2 \, dB \quad (1.2)$$

Then formally by Hamilton’s principle of least action

$$\delta \int_0^\tau \mathcal{L} \, dt = 0$$

we can recover the wave equation. In other words, the solution $u$ to (1.1) in $[0, \tau]$ corresponds to the stationary points of $\int_0^\tau \mathcal{L} \, dt$. Once $\mathcal{L}$ is defined we can rewrite the wave equation in Euler-Lagrange form

$$\frac{\partial}{\partial t} \left( \frac{\delta \mathcal{L}}{\delta u} \right) - \frac{\delta \mathcal{L}}{\delta u} = 0 \quad (1.3)$$

If function $u$ has compact support on $B$ (or decays sufficiently fast when $B = \mathbb{R}^n$) then the total energy of the system is conserved:

$$\frac{\partial}{\partial t} \mathcal{H} = 0 \quad (1.4)$$

where the total energy is given as

$$\mathcal{H} = \frac{1}{2} \int_B \left( \frac{\partial u}{\partial t} \right)^2 + c^2 |\nabla u|^2 \, dB \quad (1.5)$$

These classical results, however, do not have a counterpart for the classic heat equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0 \quad (1.6)$$

where constant $\alpha$ is the thermal diffusivity, and the function $u$ here represents the temperature field. In fact, it is shown [2] that no action exists in the form

$$\int_B L(\vec{x}, t, \partial_t u, \nabla u) \, dB \quad (1.7)$$

such that (1.7) can be deduced as the Euler-Lagrange equations of functional having the form give in equation (1.7).

As far as we can tell, the first successful attempt to include dissipative effects into the classic variational framework dates back to Rayleigh in the end of the 19th century [3]. Rayleigh introduced, in addition to the Lagrangian, a dissipation function—a positive quadratic function in the velocities—to account
for friction in the system; this allowed him to extend Lagrange’s equations of motion. In the 1950s M.A. Biot developed a variational formulation for the equations of classical thermoelasticity by means of a modified free energy (referred to as Biot’s potential) and a dissipation function. However, Biot’s variational formulation was not given in terms of a single action such as in Hamilton’s principle; rather he formulated a quasi-variational principle in which he does not consider the total variation of the dissipation function—only the product of its derivatives with the appropriate infinitesimal variation \[ 4, 5. \]

Since then many have sought to uncover variational formulations for dissipative continua. For instance in certain cases \[ 6, 7, 8, 9 \] researchers were successful in extending the classic Lagrangian and/or Hamiltonian formulations to include effects of entropy production, while falling short of constructing a unified action. Others \[ 10, 11 \] were able to formulate action integrals for evolution-diffusion equations. However, these functionals do not have the simple form of a density function as does the Lagrangian, rather they are complicated expressions given in terms of convolutions of a one parameter integral, and it is not obvious how they would fit a Lagrangian-Hamiltonian framework. Dissipation has also been incorporated into various variational schemes via the Lagrange-d’Alembert principle (see for example \[ 12, 13, 14 \]). For other possible extensions the reader is referred to \[ 15, 16 \]. Moreover, a Noether’s theorem was advanced in \[ 17 \] for the theory of nonlinear thermoelasticity without dissipation. But despite of progress in this area, a unified extension of the variational formalism of analytical mechanics to general dissipative systems remains still out of reach.

In this paper we construct a new least action principle analogous to Hamilton’s principle by calculating the rate of change in the energy flux. This allows us to write the conservation of energy equation as a second order hyperbolic PDE for the total energy of the system in one space dimension. A myriad of consequence will then follow: the hyperbolic PDE can be re-written as the Euler-Lagrange equations in a new action we denote \( \Sigma \). This produces a natural way of revealing the symmetry that exists between the balance of energy and momentum, and as such the new least action principle follows without extraneous physics assumptions. A Hamiltonian and bracket formalisms also follow from the Euler-Lagrange equations in a similar fashion to analytical mechanics. Furthermore, the symmetries leaving \( \Sigma \) invariant correspond to new conservation laws.

A noteworthy consequence of the above procedure is that it gives ground for producing a third functional, this time form the energy equation associated with \( \Sigma \) (i.e. Noether’s theorem under time invariance applied to \( \Sigma \)). In fact, we can carry this procedure indefinitely, that is we prove that our procedure is iterative giving rise to a hierarchy of variational principles each constructed from the previous iteration. We, hence, obtain an infinite number of functionals (i.e. Lagrangians) and an (infinite) iterative scheme, and advance a complete analytic mechanics at each iteration.

The paper is organized as follows. We consider isentropic systems in Section 2–5, which will set the foundation for tackling the dissipative case and constructing the variational hierarchy. We begin in Section 2 by considering the rate of change in the energy flux. We show that the total energy propagates according to a homogeneous wave equation, which can be derived as stationary points of a
new functional $\Sigma$. In Section 3, we establish a unified Lagrangian-Hamiltonian formulation through the newly construed functional $\Sigma$ and its Legendre transformation $\Pi$, respectively. Hamilton's equations for the total energy are shown to be equivalent to a bracket formulation over a new phase space that consists of the energy-power pair. We establish in Section 4 Noether’s theorem: transformations that leave functional $\Sigma$ invariant result in conservation laws. As such we examine two groups of transformations: arbitrary translations in the total energy, and space-time translations. Invariance under the first corresponds to the balance of energy. The second is the basis for constructing a second order tensor comparable to the energy-momentum tensor in classical field theory. Subsequently we obtain a number of new conservation laws, one of which governs the evolution of density function $\pi$. One of our main results is contained in Section 4: the formulation of infinite hierarchy of variational principles (i.e. least action principles). This result is made possible because the basic procedure used in constructing $\Sigma$ is iterative in nature; the Lagrangian at each iteration is formulated based on the field variables of the previous iteration. Therefore, at each iteration we produce a complete variational analysis for the relevant fields.

In Section 6 we look to apply our procedure to non-conservative systems, namely to dissipative thermoelastic material. The effects of entropy production due to heat flow manifest itself as a non-homogeneous term in the hyperbolic PDE for the total energy. Hence, a set of two coupled Euler-Lagrange equations determines the complete evolution of thermoelastic materials. Specifically, the functional $\Sigma$ for a linear thermoelastic body produces the classic energy equation in the theory of thermoelasticity, and as a special case the heat equation. Moreover, a modified Noether’s theorem is given for the dissipative case. While we can not produce conservation laws–because of the inhomogeneity in the balance of energy–we obtain auxiliary equations which are fundamental for obtaining the next iteration in the variational hierarchy. We comment, in Section 7, on the implication of our procedure to general field theories which depend on free internal variables/fields. Finally, we offer concluding remarks in Section 8.

2 Least Action Principle for Balance of Energy

In this section we show that the balance of energy equation in 1D corresponds to the stationary points of a functional $\Sigma$. We commence our construction by first considering the isentropic problem. The scheme we employ for the construction of said functional will naturally suggest a more general calculation, which we apply to the full (dissipative) problem in Section 6.

The point of departure for us is to consider the first law of thermodynamics (i.e. the balance of energy) in one spatial dimension, which in material coordinates reads

$$\rho_0 \frac{\partial I}{\partial t} = \frac{\partial}{\partial x} (vS - q) \quad \text{in } B \times [0, \tau]$$

(2.1)

Here, $B \equiv (a_1, a_2)$ represents the reference configuration, and time $\tau > 0$. The quantity $\rho_0$ is the material density; $I = e + \frac{v^2}{2}$ is the total energy: the sum of the internal energy and the kinetic energy; $S$ is the scalar-valued Piola-Kirchhoff stress; and $q$ represents the heat flux across the boundary of $B$. For simplicity,
we have considered the balance of energy in the absence of body forces and heat sources (see Remark 2 and Appendix A).

We further assume that our constitutive laws determine a classic thermoelastic medium, that is

\[ e = e(\partial_x u, s), \quad S = \rho_0 \frac{\partial e}{\partial (\partial_x u)}, \quad \theta = \frac{\partial e}{\partial s}, \quad q = q(\partial_x \theta) \quad (2.2) \]

Here \( s \) is the entropy density, and \( \theta \) is the absolute temperature (i.e. \( \theta = \theta_0 + T \) where \( \theta_0 \) is constant temperature in the undeformed configuration). Equation (2.1) holds for solids as well as fluids. In case of fluids the pressure is given as \( S = -p \), and the internal energy \( e \) (hence the pressure) is a state function of the density \( \rho \) associated with the motion through the relation \( \frac{\rho_0}{1 + \partial_x u} = \rho \).

It is known that hyperbolic PDEs such as the wave equation possess a variational structure because they are associated with symmetric operators \([18]\). The balance of energy equation on the other hand, supplemented by Fourier’s law for heat conduction, usually results in a nonlinear parabolic PDE in the temperature field. In fact, even upon linearization, this equation does not possess an obvious variational form since it will ultimately correspond to a non-symmetric operator \([11]\). We circumvent this impediment by showing that the total energy \( I \) satisfies a wave equation.

We begin by considering the isentropic problem: \( q = 0, e = e(\partial_x u), S = S(\partial_x u), \partial_t s = 0 \). The rate of change of the energy flux \( F = vS \) reads

\[ \frac{\partial F}{\partial t} = S \frac{\partial v}{\partial t} + v \frac{\partial S}{\partial t} \]

We substitute for the conservation of momentum equation (in the absence of body forces) to obtain

\[
\frac{\partial F}{\partial t} = S \frac{1}{\rho_0} \frac{\partial S}{\partial (\partial_x u)} + v \frac{\partial S}{\partial t} \]

\[ = \frac{S}{\rho_0} \frac{\partial S}{\partial (\partial_x u)} \frac{\partial (\partial_x u)}{\partial x} + v \frac{\partial S}{\partial (\partial_x u)} \frac{\partial (\partial_x u)}{\partial t} \]

\[ = \frac{\partial S}{\partial (\partial_x u)} \frac{\partial}{\partial x} \left( e + \frac{v^2}{2} \right) \]

\[ = \frac{\partial I}{\partial x} \quad (2.3) \]

where \( c \equiv \frac{\partial S}{\partial (\partial_x u)} \) is the nonlinear elastic modulus.

\[ ^1 \text{A departure from the classic constitutive laws can result in a wave-like equation for the temperature field. See for example \([18, 20]\).} \]
Now Equation (2.3) together with (2.1) produces
\[ \rho_0 \frac{\partial^2 I}{\partial t^2} - \frac{\partial}{\partial x} \left( c \frac{\partial I}{\partial x} \right) = 0 \] (2.4)

Therefore for constant \( c \)
\[ \frac{\partial^2 I}{\partial t^2} - \frac{c}{\rho_0} \frac{\partial^2 I}{\partial x^2} = 0 \] (2.5)

We have obtained the classic wave equation for the total energy \( I \), which we choose to replace the classic balance of energy equation (2.1). The speed of propagation is well-defined and finite (for constant \( c \)) and is equal to the speed of propagation of an elastic wave \( \sqrt{c/\rho_0} \). That the total energy is governed by a wave equation should not come as a surprise. The propagating elastic wave packet has an associated energy, which propagates with the wave into the material. So as long as \( u \neq 0 \) the associated total energy propagates with the wave in the same direction.

Two immediate consequences present themselves. The first is that both equations (2.4) and (2.5) can be derived as the stationary points of an action functional, which we denote \( \Sigma \). The functional \( \Sigma \) is defined analogously to the classical Lagrangian (but in terms of \( I \))
\[ \Sigma(I) = \int_B \sigma(\partial_t I, \partial_x I) \, dx + \int_B \frac{1}{2} \left( \rho_0 \left( \frac{\partial I}{\partial t} \right)^2 - c \left( \frac{\partial I}{\partial x} \right)^2 \right) \, dx \] (2.6)

Therefore have obtained a least action principle analogous to Hamilton’s principle

**Theorem 1** (A least action principle). The actual evolution of the total energy \( I \) in \([0, \tau]\) coincides with the stationary points of the functional
\[ \int_0^\tau \Sigma \, dt \] (2.7)

**Remark 1.**

i A direct implication of Theorem 1 is that equations (2.4) and (2.5) can be put in Euler-Lagrange form, which we present in the next section.

ii The symmetric operator associated with (2.4) or (2.5) (in weak form) can be constructed as follows. Let \( V \) be some appropriate Hilbert space such that \( I \in V \) solves (2.4) or (2.5) (e.g. \( V = L^2(0, \tau; H^1_0(B)) \)), and define the operator \( A : V \times V \rightarrow \mathbb{R} \)
\[ A(I_1, I_2) = \int_0^\tau \int_B \rho_0 \frac{\partial I_1}{\partial t} \frac{\partial I_2}{\partial t} \, dx \, dt - \int_0^\tau \int_B c \frac{\partial I_1}{\partial x} \frac{\partial I_2}{\partial x} \, dx \, dt \]
with \( \partial_t I_1(\cdot, 0) = \partial_t I_2(\cdot, \tau) = 0 \) for all \( I_1, I_2 \in V \). Clearly in this construction \( A \) is symmetric.

iii While \( I \) is related to the other state variables through the constitutive laws, taking the variation of \( I \) independently is required for the stationary principle to hold. In fact, equations (2.4) and (2.5) reveal the significance of treating
the total energy as an independent quantity. In other words, the main insight we have obtained is that the total energy \( I \) is the natural field to consider if one is looking to formulate a least action principle for the first law of thermodynamics in 1D; it is the correct conduit for obtaining a symmetric operator, hence a variational principle.

Another consequence of (2.5) is that we can obtain an analogous result to conservation of energy in wave mechanics

**Corollary 1** (Constant of motion). If \( I \) satisfies (2.5) and has compact support on \( B \). Then

\[
\Pi(t) = \int_B \frac{1}{2} \left( \rho_0 \left( \frac{\partial I}{\partial t} \right)^2 + c \left( \frac{\partial I}{\partial x} \right)^2 \right) dx
\]

is a constant of motion, that is \( \Pi(t) = \Pi(0) \).

**Remark 2.** We have considered the isentropic energy equation in its simplest form, that is we have omitted the effects of body forces and heat sources. However, this doesn’t hinder the construction of functional \( \Sigma \). We consider the most general formulation for the conservation of energy equation in Appendix A, and demonstrate how \( \Sigma \) can be deduced from a calculation similar to the one in this section.

## 3 Lagrangian-Hamiltonian Formulation

In this section we formulate the Lagrangian-Hamiltonian theory corresponding to the balance of energy, now rewritten in hyperbolic form (2.4) or (2.5).

Since Theorem I is analogous to Hamilton’s principle of least action, the Euler-Lagrange equations directly follow in terms of the functional \( \Sigma \) and the total energy \( I \)

\[
\frac{d}{dt} \left( \frac{\delta \Sigma}{\delta (\partial_t I)} \right) - \frac{\delta \Sigma}{\delta I} = 0
\] (3.1)

where \( \frac{\delta}{\delta u} \) denotes the functional derivative. Clearly, the Euler-Lagrange equation is equivalent to (2.4) (or to (2.5) for constant \( c \)).

The Hamiltonian formulation corresponding to the new least action principle follows analogously to classical mechanics. We first introduce the variable \( J \) defined analogously to the momentum variable in mechanics

\[
J = \frac{\delta \Sigma}{\delta (\partial_t I)}
\] (3.2)

By taking the Legendre transformation of the function \( \sigma \), the density function of \( \Sigma \), with respect to the change of variable \((I, \partial_t I) \mapsto (I, J)\), we obtain an analogous quantity to the Hamiltonian density

\[
\pi(\nabla I, J) = J \cdot \frac{\partial I}{\partial t} = \sigma
\] (3.3)

A simple calculation gives the total quantity of \( \pi \) inside \( B \)

\[
\Pi = \int_B \pi dB = \int_B \left( \frac{1}{2\rho_0} \left( J^2 + \frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 \right) \right) dx
\] (3.4)
The physical interpretation of $J$ becomes evident once we carry-out the calculations in (3.1)

$$J = \frac{\delta \Sigma}{\delta (\partial_t I)} = \rho_0 \frac{\partial I}{\partial t}$$

The quantity $J$ is the total power density of the system. Therefore, the phase space associated with $\Pi$ specifies the energy-power pair of the system at each instant time.

Equipped with the Euler-Lagrange equation (3.1) together with functional $\Pi$, we can rewrite the nonlinear evolution equation for the total energy $I$ as Hamilton’s equations in $I$

$$\frac{\partial I}{\partial t} = \frac{\delta \Pi}{\delta J}$$

$$\frac{\partial J}{\partial t} = -\frac{\delta \Pi}{\delta I}$$

For arbitrary functionals $\Phi_1$ and $\Phi_2$ over the (infinite dimensional) phase space associated with the functional $\Pi$: $\{(I, J) \mid I = e + \frac{v^2}{2}, J = \frac{\delta \Sigma}{\delta (\partial_t I)}\}$, we can introduce the canonical Poisson bracket $\{, \}$

$$\{\Phi_1, \Phi_2\} = \int_B \frac{\delta \Phi_1}{\delta I} \frac{\delta \Phi_2}{\delta J} - \frac{\delta \Phi_2}{\delta I} \frac{\delta \Phi_1}{\delta J} dB = \{\Phi, \Pi\}$$

The evolution of an arbitrary functional $\Phi$ over the phase space is governed by $\Pi$ through the Poisson structure

$$\dot{\Phi}(I, J) = \int_B \left( \frac{\delta \Phi}{\delta I} \frac{\delta \Phi}{\delta J} \right) \cdot \left( \frac{\partial I}{\partial t}, \frac{\partial J}{\partial t} \right)^T dB = \int_B \frac{\delta \Phi}{\delta I} \frac{\delta \Pi}{\delta J} - \frac{\delta \Phi}{\delta J} \frac{\delta \Pi}{\delta I} dB = \{\Phi, \Pi\}$$

Specifically, we can rewrite Hamilton’s equations (3.3) in Poisson form

$$\frac{\partial I}{\partial t} = \{I, \Pi\}$$

$$\frac{\partial J}{\partial t} = \{J, \Pi\}$$

Finally, if we assume that $\Pi$ is strictly a function over the phase space, that is $c$ is constant, we reproduce the result of Corollary 1

$$\frac{\partial \Pi}{\partial t} = \{\Pi, \Pi\} = 0$$

4 Variational Symmetries and Conservation Laws

In this section we demonstrate that invariance of the quantity $\Sigma$ under group transformations (up to a full divergence of a field) result in conservation laws, one of which is the balance of energy. We shall state a version of Noether’s theorem $\Sigma$ convenient for our setting, and examine the consequences; namely by constructing a quantity analogous to the energy-momentum tensor in field theory.
Set $z = \{t, x\} \in B = [0, \tau] \times B$ for arbitrary $\tau \in \mathbb{R}^+$, and $\mu = 0, 1$. This notation (often used in field theory) proves more convenient in stating Noether’s theorem. Under this notation, the Euler–Lagrange equation (3.1) reads

$$\frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \right) = 0$$

(4.1)

Here we have employed Einstein’s summation convention: repeated indices are implicitly summed over. We further assume in this section that the elastic modulus $c$ is constant.

**Noether’s Theorem**

We introduce a one-parameter smooth transformation $\lambda \in [0, \infty) \mapsto I_\lambda = I(z_\mu; \lambda)$ with $I_\lambda |_{\lambda=0} = I$. Similarly, we define $\sigma_\lambda = \sigma(\partial_\mu I_\lambda)$.

We say that function $\sigma$ is invariant under the one-parameter group of transformations $\lambda \mapsto I_\lambda$ if

$$\frac{d}{d\lambda} \int_{B_T} \sigma_\lambda |_{\lambda=0} \ d^2z = \int_{B_T} \frac{\partial K_\mu}{\partial z_\mu} d^2z$$

(4.2)

for some (possibly zero) four-vector field $\vec{K} = \vec{K}(I, \partial \mu I)$.

**Theorem 2** (Noether’s theorem for $\sigma$). Assume $\sigma$ is invariant under the one-parameter group of transformations $\lambda \mapsto I_\lambda$, then the Euler–Lagrange system corresponding to $\sigma$ admits the conservation law

$$\frac{\partial P_\mu}{\partial z_\mu} = 0$$

(4.3)

where the conserved current $P_\mu$ is defined as

$$P_\mu = \frac{\partial \sigma}{\partial (\partial_\mu I)} \cdot \frac{\partial I_\lambda}{\partial \lambda} |_{\lambda=0} - K_\mu$$

(4.4)

Proof. We begin by computing the LHS of (4.2)

$$\frac{d}{d\lambda} \int_{B_T} \sigma_\lambda |_{\lambda=0} \ d^2z = \int_{B_T} \frac{\partial \sigma}{\partial (\partial_\mu I)} \cdot \frac{\partial (\partial_\mu I_\lambda)}{\partial \lambda} |_{\lambda=0} \ d^2z$$

$$= -\int_{B_T} \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \right) \cdot \frac{\partial I_\lambda}{\partial \lambda} |_{\lambda=0} \ d^2z + \int_{B_T} \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \cdot \frac{\partial I_\lambda}{\partial \lambda} |_{\lambda=0} \right) d^2z$$

(4.5)

where we have used the fact the $I$ satisfies the Euler–Lagrange equations (4.1).

By comparing equations (4.2) and (4.5), we obtain the conservation law for the current $P_\mu$. 

\[ \square \]
Remark 3.

i In terms of coordinates \((t, x)\), the conservation law \((4.3)\) can be rewritten as
\[
\frac{\partial P_0}{\partial t} + \frac{\partial P_1}{\partial x} = 0 \quad (4.6)
\]

ii For simplicity we have chosen \(\lambda\) to be a scalar parameter. Theorem 1 is equally valid for a \(\mu\)-dimensional parameter \(\hat{\lambda}\) \([17]\).

iii A more basic diffeomorphism can be defined with respect to the independent variable, that is, \(\lambda \rightarrow z_\lambda^\mu \equiv z_\mu(\lambda)\). Then the group of transformations \(I_\lambda\) is defined in terms of \(z_\lambda^\mu\), namely, \(I_\lambda \equiv I(z_\lambda^\mu)\). Therefore, invariance of \(\sigma\) with respect to the diffeomorphism \(\lambda \rightarrow z_\mu(\lambda)\) still produces the result of Theorem 1 \([21]\).

An immediate implication of Theorem 2 is that the conservation of energy, written in hyperbolic form \((2.4)\), can be derived as a conservation law corresponding to invariance under energy translations.

To see this, we define the family of energy translations \(I_\lambda = I + \lambda \hat{T}\) for some constant energy scalar \(\hat{T}\). Then, under this group of transformations, we have
\[
\sigma_\lambda = \sigma(\partial_\mu I_\lambda) = \sigma(\partial_\mu I) = \sigma
\]
Therefore, \(\sigma\) is invariant under the transformation \(\lambda \rightarrow I_\lambda\); in other words, equation \((4.2)\) is satisfied with \(\dot{K} = 0\). The conserved current \(P_\mu\), in this case, reads
\[
P_\mu = \frac{\partial \sigma}{\partial (\partial_\mu I)} \cdot \hat{T} \quad (4.7)
\]
and the corresponding conservation law holds
\[
\frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \right) = 0 \quad (4.8)
\]
This is nothing but the balance of energy equation written in Euler-Lagrange form \((4.1)\).

Another consequence of Theorem 2 is the global conservation law for total charge \(Q\). Integrating both sides of \((4.6)\) over the some large region (interval) \(\Omega\) yields:
\[
\int_\Omega \frac{\partial P_0}{\partial t} d\Omega = - \int_\Omega \frac{\partial P_1}{\partial x} d\Omega = - \int_{\partial \Omega} P_1 \mid_{\partial \Omega}
\]
By assuming \(P_1\) has compact support on \(\Omega\), we conclude that the total charge \(Q \equiv \int_\Omega P_0 d\Omega\) is conserved
\[
\frac{d}{dt} Q = 0 \quad (4.9)
\]
Energy-Momentum Tensor

Among the conservation laws associated with a field Lagrangian, those that are derived from the energy-momentum tensor are the most significant from a physics perspective. Here again we can construct a quantity analogous to that of classical field theory, namely the energy-momentum tensor, and obtain a host of conservation laws as a result of Theorem 2.

We begin the construction by considering the following space-time translation (see Remark 2-ii)
\[ z^\mu_\eta \pm \lambda_\mu = z^\mu - \lambda^\eta \delta^\mu_\eta \] (4.10)
where \( \mu, \eta = 0, 1 \).

Under this perturbation the field \( I \) satisfying the Euler-Lagrange equations can be written
\[ I_\lambda = I(z_\mu - \lambda_\mu) = I(z_\mu) + \lambda_\mu \frac{\partial I_\Lambda}{\partial z_\mu} |_{\lambda=0} + o(|\lambda_\mu|) \] (4.11)
Next, we show \( \sigma_\lambda \) satisfies (4.2)
Proposition 1. Function \( \sigma_\lambda \) is invariant under the space-time translation (4.10).
Proof. Function \( \sigma_\lambda \) is defined as \( \sigma_\lambda(\partial_\alpha I_\lambda) \). Expanding this expression in \( \lambda_\mu \) yields
\[ \sigma_\lambda = \sigma(\partial_\alpha I) + \lambda_\mu \frac{d}{d\lambda_\mu} \sigma |_{\lambda=0} + o(|\lambda_\mu|) \]
\[ = \sigma + \lambda_\mu \frac{\partial \sigma}{\partial (\partial_\alpha I)} \frac{\partial (\partial_\alpha I)}{\partial z_\mu} |_{\lambda=0} + o(|\lambda_\mu|) \]
\[ = \sigma + \lambda_\mu \frac{\partial \sigma}{\partial z_\mu} + o(|\lambda_\mu|) \] (4.12)
where \( \alpha, \mu = 0, 1 \). Hence
\[ \frac{d}{d\lambda_\eta} \sigma_\lambda |_{\lambda=0} = \frac{\partial}{\partial z_\mu} (\delta_\mu_\eta \sigma) \] (4.13)
Since we have assumed a diffeomorphism with respect to a two dimensional parameter \( \lambda_\mu \), the vector field in (4.2) is augmented to a second order tensor. Therefore, according to (4.13), we take \( K_\mu_\eta = \delta_\mu_\eta \sigma \) to satisfy the condition (4.12).

As a result of Proposition 1 (together with Theorem 2), we have the following system of conservation laws
\[ \frac{\partial T_{\mu_\eta}}{\partial z_\mu} = 0 \] (4.14)
where the tensor \( T_{\mu_\eta} \), defined by
\[ T_{\mu_\eta} = \frac{\partial \sigma}{\partial (\partial_\alpha I)} \cdot \frac{\partial I}{\partial z_\eta} - \delta_\mu_\eta \sigma \] (4.15)

\(^2\)We shall suppress the index \( \eta, \mu \) on \( \lambda \) if the parameter \( \lambda \) appears as a subscript.
is a quantity analogous to the energy-momentum tensor in classical (and quantum) field theory. The system of conservation laws (4.14) are accompanied, again, by the conservation of their global counterparts \( \int_{\Omega} T_{\alpha 0} d\Omega \), and \( \int_{\Omega} T_{01} d\Omega \) as discussed earlier. Moreover, as one expects,

\[
T_{00} = \frac{\partial \sigma}{\partial (\partial_0 I)} \cdot \frac{\partial I}{\partial z_0} - \sigma = \rho_0 \frac{\partial I}{\partial t} \frac{\partial I}{\partial \alpha} - \sigma = \pi
\]

and the evolution of \( \pi \) is governed by

\[
\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial I}{\partial t} \frac{\partial \sigma}{\partial (\partial_x I)} \right) = 0 \quad (4.16)
\]

Finally, we can conclude that \( T_{\mu \eta} \) is symmetric by writing \( \sigma \) as

\[
\sigma = \frac{1}{2} B_{\mu \eta} \frac{\partial I}{\partial z_\mu} \frac{\partial I}{\partial z_\eta} \quad (4.17)
\]

where

\[
B_{\mu \eta} = \begin{bmatrix} \rho_0 & 0 \\ 0 & -c \end{bmatrix} \quad (4.18)
\]

Since \( B_{\mu \eta} \) is symmetric, the tensor \( T_{\mu \eta} \) in this case:

\[
T_{\mu \eta} = B_{\mu \alpha} \frac{\partial I}{\partial z_\alpha} \frac{\partial I}{\partial z_\eta} - \delta_{\mu \eta} \sigma \quad (4.19)
\]

must also be symmetric.

5  A Hierarchy of Variational Principles

In this section we show that the process sketched thus far is iterative in nature. At each iteration, a corresponding “Lagrangian” can be constructed. As a result, we can formulate a least action principle at the \( i \)th iteration, and all the results of Sections 2–4 can be produced once again. Moreover, the constituents of each new variational principle depend on the preceding level of analysis. Therefore, we can visualize a hierarchy comprising of an infinite number of interrelated Lagrangians and their resulting variational principles.

To clearly illustrate the procedure for obtaining the general iteration, we first consider the following. In Section 2, the rate of energy flux (in the isentropic case) \( \partial_t (\rho S) \) was computed and was shown to be proportional to the gradient of the total energy (i.e. equation (2.3)). In the same spirit, we can view the term \( \partial_t I \frac{\partial \sigma}{\partial (\partial_x I)} \) as the “energy flux” associated with \( \pi \) in (4.16). In fact, this equation is completely analogous to the isentropic balance of energy equation

\[
\rho_0 \frac{\partial I}{\partial t} \frac{\partial}{\partial x} (\rho S) = \frac{\partial I}{\partial t} \frac{\partial L}{\partial I} + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \frac{\partial L}{\partial (\partial_x I)} \right) = 0
\]
Therefore, it is natural to consider the rate of change of $\frac{\partial I}{\partial (\partial_x I)}$:

$$\frac{\partial}{\partial t} \left( \frac{\partial I}{\partial \sigma} \frac{\partial \sigma}{\partial x} \right) = \frac{\partial}{\partial t} \left( \frac{\partial I}{\partial x} \right)$$

$$= -c \frac{\partial^2 I}{\partial x \partial t} \frac{\partial I}{\partial \sigma} - c \frac{\partial^2 I}{\partial x^2} \frac{\partial I}{\partial \sigma}$$

$$= -c \frac{\partial^2 I}{\partial x \partial t} \frac{\partial I}{\partial \sigma} - c \frac{\partial^2 I}{\partial x^2} \frac{\partial I}{\partial \sigma}$$

$$= \frac{c}{\rho_0} \frac{\partial}{\partial x} \left( \frac{\rho_0}{2} \left( \frac{\partial I}{\partial t} \right)^2 \right) - \frac{c}{\rho_0} \frac{\partial}{\partial x} \left( \frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 \right)$$

$$= \frac{c}{\rho_0} \frac{\partial}{\partial x} \pi \quad (5.1)$$

Equation (5.1) together with (4.16) gives

$$\frac{\partial^2 \pi}{\partial t^2} - c \frac{\partial^2 \pi}{\partial x^2} = 0 \quad (5.2)$$

We have obtained the classic wave equation for the quantity $\pi$. Hence, by treating $\pi$ as an independent quantity—as we did with the total energy $I$—all the results proven in Sections 2–4 hold too once we replace $I$ with $\pi$, and treat $\pi$ as the new field.

We, now, present the main result of this section: the preceding calculation can be put into an iterative scheme

**Theorem 3.** Let $L^0 = \rho_0 \left( \frac{\partial u^0}{\partial t} \right)^2 - \frac{c}{2} \left( \frac{\partial u^0}{\partial x} \right)^2$ denote the classic Lagrangian with displacement field $u^0(x,t) = u(x,t)$. Assume $c = \frac{\partial^2 L}{\partial (\partial_x u)^2} > 0$ is constant. Then there exist infinitely many density functionals $L^i$, $i = 1, 2...$ satisfying the least action principle

$$\delta^i \int_0^\pi \int_B L^i dx dt = 0 \quad (5.3)$$

where $L^i = \rho_0 \left( \frac{\partial u^i}{\partial t} \right)^2 - \frac{c}{2} \left( \frac{\partial u^i}{\partial x} \right)^2$, $u^{i+1} = \rho_0 \left( \frac{\partial u^i}{\partial t} \right)^2 + \frac{c}{2} \left( \frac{\partial u^i}{\partial x} \right)^2$, and $\delta^i$ is the variation taken with respect to $u^i$, $i = 0, 1, 2...$ .

**Proof.** We prove by induction.

For $i = 0$, we obtain Hamilton’s principle of least action.

Assume theorem holds for the $i$-th iteration. We look to construct the $(i + 1)$th iteration. Assume that $u_i$ is a solution to the $i$th variational problem. First we compute the Euler-Lagrange equation at this iteration

$$\frac{d}{dt} \left( \frac{\partial L^i}{\partial (\partial_t u^i)} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L^i}{\partial (\partial_x u^i)} \right) = \rho_0 \frac{\partial^2 u^i}{\partial t^2} - c \frac{\partial^2 u^i}{\partial x^2} = 0 \quad (5.4)$$
Next consider

\[
\frac{\partial u^{i+1}}{\partial t} = \rho_0 \frac{\partial u^i}{\partial t} \frac{\partial^2 u^i}{\partial t^2} + c \frac{\partial u^i}{\partial x} \frac{\partial^2 u^i}{\partial t \partial x}
\]

\[
= c \frac{\partial u^i}{\partial t} \frac{\partial^2 u^i}{\partial x^2} + c \frac{\partial u^i}{\partial x} \frac{\partial^2 u^i}{\partial t \partial x}
\]

\[
= \frac{\partial}{\partial x} \left( \frac{\partial u^i}{\partial t} \frac{\partial u^i}{\partial x} \right)
\]

\[
= \frac{\partial}{\partial x} \left( \frac{\partial u^i}{\partial t} L^i \right)
\]

Thus, we have the local balance of energy

\[
\frac{\partial u^{i+1}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial u^i}{\partial t} L^i \right) = 0 \quad (5.5)
\]

Finally, we calculate the energy flux rate

\[
\frac{\partial}{\partial t} \left( \frac{\partial u^i}{\partial t} \frac{\partial L^i}{\partial (\partial_x u^i)} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial u^i}{\partial t} L^i \right)
\]

\[
= -c \frac{\partial^2 u^i}{\partial t^2} \frac{\partial u^i}{\partial x} - c \frac{\partial u^i}{\partial x} \frac{\partial^2 u^i}{\partial t \partial x}
\]

\[
= -c^2 \frac{\partial^2 u^i}{\partial x^2} \frac{\partial u^i}{\partial t} - c \frac{\partial u^i}{\partial t} \frac{\partial^2 u^i}{\partial t \partial x}
\]

\[
= -c \frac{\partial}{\partial x} \left( \left( \frac{\partial u^i}{\partial t} \right)^2 \right) - c^2 \frac{\partial}{\partial t} \left( \frac{\partial u^i}{\partial x} \right)^2
\]

\[
= -c \frac{\partial}{\partial x} \left( \rho_0 \left( \frac{\partial u^i}{\partial t} \right)^2 + c \left( \frac{\partial u^i}{\partial x} \right)^2 \right)
\]

\[
= -c \frac{\partial}{\partial x} u^{i+1} \quad (5.6)
\]

Combining (5.6) with (5.5) we obtain

\[
\rho_0 \frac{\partial^2 u^{i+1}}{\partial t^2} - c \frac{\partial^2 u^{i+1}}{\partial x^2} = 0 \quad (5.7)
\]

Written differently

\[
\frac{d}{dt} \left( \frac{\partial L^{i+1}}{\partial (\partial_t u^{i+1})} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L^{i+1}}{\partial (\partial_x u^{i+1})} \right) = 0 \quad (5.8)
\]

Therefore \(u^{i+1}\) is a stationary point for the \((i + 1)\)th variational problem.

**Remark 4.** A more general variational hierarchy theorem holds for non-constant \(c\). In Appendix A, we sketch the general method of obtaining the second iteration \((i.e. u^2 = \pi)\), which then can be extended to higher iterations.

Since at each iteration the corresponding scalar field \(u^i\) satisfies the wave equation (5.3), we have an infinite number of constants of motion.
Corollary 2 (Constants of motion). If \( u^i \) solves the variational problem for \( i = 0, 1, 2, ... \) and has compact support on \( B \). Then for every \( i \in \mathbb{N} \)

\[
\mathcal{H}^i(t) = \int_B \frac{1}{2} \left( \rho_0 \left( \frac{\partial u^i}{\partial t} \right)^2 + c \left( \frac{\partial u^i}{\partial x} \right)^2 \right) dx
\]

is a constant of motion, that is \( \mathcal{H}^i(t) = \mathcal{H}^i(0) \).

Since at each level of analysis we have a corresponding Lagrangian \( L^i \), we can define the \( i \)-th Hamiltonian \( H^i \) via the Legendre transform (3.2) by first defining the \( i \)-th momentum

\[
p^i = \frac{\partial L^i}{\partial (\partial_t u^i)}
\]

Then \( H^i \) will have the form

\[
H^i(p^i, \partial_x u^i) = \frac{1}{2\rho_0}(p^i)^2 + \frac{c}{2} \left( \frac{\partial u^i}{\partial x} \right)^2
\]

Clearly for \( i = 1 \), \( p^1 = J \) and \( H^1 = \pi \) given in Section 3.

Therefore, at the \( i \)-th iteration, Hamilton’s equations are given

\[
\begin{align*}
\frac{\partial u^i}{\partial t} &= \frac{\delta \mathcal{H}^i}{\delta p^i} \\
\frac{\partial p^i}{\partial t} &= \frac{\delta \mathcal{H}^i}{\delta u^i}
\end{align*}
\]

And in a similar fashion to Section 3, we can define the Poisson brackets \( \{ \cdot, \cdot \} \) for each \( i \).

Finally, we touch on the key results of Section 4. We begin by rewriting the Euler-Lagrange equations at the \( i \)-th level in the space-time coordinates \( z_\mu \):

\[
\frac{\partial}{\partial z_\mu} \left( \frac{\partial L^i}{\partial (\partial_\mu u^i)} \right) = 0
\]

where the summation is over the subscripts (lower indices) only.

For each \( i \) fixed we obtain a conserved current \( P^i_\mu \) associated with \( L^i \) by Noether’s theorem (Theorem 2):

\[
\frac{\partial P^i_\mu}{\partial z_\mu} = 0
\]

where

\[
P^i_\mu = \frac{\partial L^i}{\partial (\partial_\mu u^i)} \cdot \frac{\partial u^i_\lambda}{\partial \lambda} \big|_{\lambda=0} - K^i_\mu
\]

for \( i = 0, 1, 2, \ldots \).
Particularly, if we consider the one-parameter group of transformation \([4.10]\), we can show, as done in Section 4, that \(L^i\) is invariant under space-time translation:

\[
\frac{d}{d\lambda} L^i_\lambda \big|_{\lambda=0} = \frac{\partial}{\partial z^\mu} (\delta_{\mu\eta} L^i) = 0
\]

Therefore, the \(i\)-th energy-momentum tensor:

\[
T^{i\mu\eta} = \frac{\partial L^i}{\partial (\partial_{\mu} u^i)} \frac{\partial u^i}{\partial z^\eta} - \delta_{\mu\eta} L^i
\]

satisfies

\[
\frac{\partial T^{i\mu\eta}}{\partial z^\mu} = 0
\]

for each \(i = 0, 1, 2, \ldots\).

So in addition to the constants of motion obtained in Corollary 2, we have a host of other conservation laws (both global and local) at each emanating from Noether’s theorem. Hence, in aggregate, our hierarchy contains an infinite number of conservation laws.

6 Application to Dissipative Systems

We now turn our attention to the original problem of heat flow given by the first law of thermodynamics \([2.1]\). Our main goal here is to show that the total energy field \(I\) solves a hyperbolic PDE as well; specifically a non-homogeneous wave equation in the total energy. The dissipative effects appear as additional terms independent of the field \(I\). A more general action functional can then be constructed that includes dissipation. Therefore, a least action principle, Lagrangian-Hamiltonian formalism, and a (modified) Noether’s theorem will all follow as well in the dissipative case.

We proceed as in Section 2 and consider the rate of change in the total energy flux: \(G = vS - q\):

\[
\frac{\partial G}{\partial t} = S \frac{\partial v}{\partial t} + v \frac{\partial S}{\partial t} - \frac{\partial q}{\partial t} = \frac{1}{\rho_0} \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial t} - \frac{\partial q}{\partial t}
\]

Since the internal energy and the stress are functions of both \(\partial_x u\) and \(s\), we obtain terms in \([6.1]\) involving the entropy \(s\) in addition to those obtained in
\[
\frac{\partial G}{\partial t} = S \frac{\partial S}{\partial (\partial_x u)} \frac{\partial^2 u}{\partial x^2} + v \frac{\partial S}{\partial (\partial_x u)} \frac{\partial v}{\partial x} + \frac{\partial S}{\partial s} \frac{\partial s}{\partial t} - \frac{\partial q}{\partial t} = \frac{\partial S}{\partial (\partial_x u)} \frac{\partial e}{\partial (\partial_x u)} \frac{\partial^2 u}{\partial x^2} + v \frac{\partial v}{\partial x} + \frac{\partial S}{\partial s} \frac{\partial s}{\partial s} \frac{\partial s}{\partial t} - \frac{\partial q}{\partial s}, \quad (6.1)
\]

where we have defined
\[
\dot{D} = \frac{\partial S}{\partial s} v \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} S^2 \frac{\partial}{\partial (\partial_x u)} \left( \frac{\theta}{S} \right) - \frac{\partial q}{\partial x}, \quad (6.2)
\]

Clearly, for a conservative system \( \dot{D} \) is identically zero and we recover (2.3) as expected. We also note that \( \dot{D} \) is independent of \( I \).

Now by equation (2.1) and (6.2) we obtain
\[
\rho_0 \frac{\partial I^2}{\partial t^2} - \frac{\partial}{\partial x} \left( c \frac{\partial I}{\partial x} \right) \frac{\partial \dot{D}}{\partial x} = 0 \quad (6.4)
\]

We see immediately that (6.4) is the Euler-Lagrange equations associated with the functional
\[
\Sigma(I, \dot{I}) = \int_B \sigma(\partial_t I, \partial_x I, \dot{I}) \, dx \equiv \int_B \left( \frac{\rho_0}{2} \left( \frac{\partial I}{\partial t} \right)^2 - c \frac{\partial I}{\partial x} \right) - \dot{D} \frac{\partial I}{\partial x} \right) \, dx \quad (6.5)
\]

Therefore, the evolution of the energy component \( I \) in an arbitrary interval \([0, \tau]\) coincides with the stationary points of
\[
\int_0^\tau \Sigma dt \quad (6.6)
\]

provided the boundary conditions \( \delta I \big|_{t=0} = \delta I \big|_{t=\tau} = 0 \) are satisfied. This is the least action principle in the dissipative case.

Remark 5. The Euler-Lagrange equations for (6.3) share the same form with the conservative case since the variation is taken only with respect to the total energy \( I \)
\[
\frac{\partial}{\partial t} \left( \frac{\delta \Sigma}{\delta (\partial_t I)} \right) - \delta \Sigma = 0 \quad (6.7)
\]

Furthermore, the evolution of the dissipative material is entirely governed by a set of two coupled Euler-Lagrange equations: (6.7) coupled with
\[
\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta (\partial_t u)} \right) - \delta L = 0 \quad (6.7)
\]
where the Lagrangian $\mathcal{L}$ is

$$
\mathcal{L} = \int_\mathcal{B} \left( \frac{\rho_0}{2} \left( \frac{\partial u}{\partial t} \right)^2 - e(\partial_x u, s) \right) \, dx.
$$

Therefore, the functionals $\Sigma$ and $\mathcal{L}$ completely determine the state of the dissipative system at each instant in time, and each must be varied with respect to $u$ and $I$, respectively, to obtain the evolution equations.

In the example below, we obtain a special form for functional $\Sigma$ in the case of the heat equation. Throughout the example we assume that the reference configuration is stress free i.e. $S(\partial_x u, s) |_0 = 0$, and the temperature at the reference configuration $\theta |_0 = \theta_0$, where we have written $\Phi |_0$ to mean that the function $\Phi$ is evaluated at $\partial_x u = 0$ and $s = 0$.

**Example (heat flow in a thermoelastic medium)**

We begin first by writing $\sigma$ as the sum of two quantities: $\sigma = \sigma_h + R_h$, where we have defined

$$
\sigma_h = \frac{\rho_0}{2} \left( \frac{\partial I}{\partial t} \right)^2 + \frac{\partial q}{\partial t} \frac{\partial I}{\partial x} \quad (6.8)
$$

$$
R_h = -\frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 + \frac{\partial S}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial S}{\partial s} \frac{\partial^2 \theta}{\partial (\partial_x u) \partial s} \left( \frac{\theta}{S} \right) \frac{\partial I}{\partial x} \quad (6.9)
$$

We examine the linear theory of thermoelasticity which is characterized by “small” thermomechanical deformations. Therefore, it is appropriate to rescale the displacement field $\epsilon u$ for suitable non-dimensional positive small parameter $\epsilon$ [22], and consider the temperature $\theta$ to be everywhere close to $\theta_0$. Since both the displacement field and the temperature undergo small changes, the entropy would also be rescaled to $\epsilon s$. Linearizing the theory entails keeping terms up to the order of $\epsilon^2$ in the functional $\Sigma$, while terms up to the order of $\epsilon$ alone are considered in the Euler-Lagrange equations.

By expanding the constitutive equations for the stress and temperature around $\partial_x u = 0$ and $s = 0$, while recalling $S |_0 = 0$, we obtain

$$
S(\partial_x u, s; \epsilon) = \frac{\partial S}{\partial I} (\partial_x u) \left|_0 \epsilon \frac{\partial u}{\partial x} + \frac{\partial S}{\partial s} \right|_0 \epsilon s + O(\epsilon^2)
$$

$$
\theta(\partial_x u, s; \epsilon) = \theta_0 + \frac{\partial \theta}{\partial (\partial_x u)} \left|_0 \epsilon \frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial s} \right|_0 \epsilon s + O(\epsilon^2)
$$

By substituting the above linearizations together with the rescaled displacement and entropy fields into $R_h$, the second order approximation (for constant $c$) gives us:

$$
R_h = -\frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 + \theta_0 c \frac{\partial s}{\partial x} \frac{\partial I}{\partial x} \quad (6.12)
$$

While we have not rescaled the total energy explicitly, a simple calculation reveals that up to the order of $\epsilon$, we have $\frac{\partial I}{\partial x} = \theta_0 \frac{\partial s}{\partial x}$. Therefore, it is justified to retain the terms involving the total energy in (6.12). However, for the stationary principle to hold (see equation (6.6)), we maintain the basic form of $R_h$ in terms of total energy field $I$ and not the displacement and entropy fields.
The Euler-Lagrange equations for the system are
\[
\frac{\partial}{\partial t} \left( \frac{\delta (\Sigma_h + R_h)}{\delta I} \right) - \frac{\delta (\Sigma_h + R_h)}{\delta I} = 0
\]
which reduce to
\[
\frac{\partial}{\partial t} \left( \frac{\delta \Sigma_h}{\delta I} \right) - \frac{\delta \Sigma_h}{\delta I} = 0 \quad (6.13)
\]
However, in the linear approximation (i.e. up to the order of \( \epsilon \)) we have in fact
\[
\frac{\delta R_h}{\delta I} = 0 \quad (6.14)
\]
Therefore, in the linear approximation we get
\[
\frac{\partial}{\partial t} \left( \frac{\delta \Sigma_h}{\delta I} \right) - \frac{\delta \Sigma_h}{\delta I} = 0 \quad (6.15)
\]
Functional \( \Sigma_h \) determines the evolution of the energy for a dissipative linear thermoelastic medium. Equation (6.15) reads
\[
\frac{\partial}{\partial t} \left( \rho_0 \frac{\partial I}{\partial t} + \frac{\partial q}{\partial x} \right) = \frac{\partial}{\partial t} \left( \rho_0 \frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} \right) = 0 \quad (6.16)
\]
We have obtained the classic entropy balance as a constant of motion. Substituting for the linear constitutive law (6.11) (up to the order of \( \epsilon \)) together with Fourier’s law \( q = -k \nabla \theta \), we produce the classic evolution-diffusion equation of thermoelasticity
\[
\frac{\partial}{\partial t} \left( \rho_0 c_0 \frac{\partial \theta}{\partial t} - \rho_0 \gamma \theta_0 \frac{\partial^2 u}{\partial x^2} - k \nabla^2 \theta \right) = 0 \quad (6.17)
\]
where, \( c_0 = \frac{\partial s}{\partial \theta} \big|_0 \), \( \gamma = \frac{c_0}{\rho_0 \frac{\partial \theta}{\partial u}} \big|_0 \), and \( k \) are the heat capacity, stress-temperature modulus, and conductivity constant, respectively. If we assume the fields \( \theta \) and \( u \) have compact support in \( B \) then the classic evolution-diffusion equation of thermoelasticity can be readily recovered.

In the absence of mechanical processes, that is \( u = 0 \), equation (6.17) reduces to the heat equation. In fact, in this case \( c = 0 \), \( \tilde{D} = -\partial_t q \), which gives \( R_h = 0 \). The heat equation is, therefore, also obtained from the functional \( \Sigma_h \). This concludes our example.

The Hamiltonian and bracket formalisms for dissipative systems follow similarly as in Section 3. Define the Legendre transform of \( \sigma \):
\[
\pi(\nabla I, J, \dot{D}) = J \cdot \dot{I} - \sigma \quad (6.18)
\]
which gives
\[
H = \int_B \left( \frac{1}{2\rho_0} J^2 + \frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 + \dot{D} \frac{\partial I}{\partial x} \right) dx \quad (6.19)
\]
and the evolution equations can be re-written once again in terms of $\Pi$:

\[ \dot{I} = \frac{\delta \Pi}{\delta J} \] (6.20a)

\[ \dot{J} = -\frac{\delta \Pi}{\delta I} \] (6.20b)
or in terms of the Poisson bracket:

\[ \dot{I} = \{ I, \Pi \} \] (6.21a)

\[ \dot{J} = \{ J, \Pi \} \] (6.21b)

where the $\{ \cdot, \cdot \}$ is defined (3.6) and the phase space is given in Section 3.

Lastly, we consider some variational symmetries. If we consider one-parameter transformation in terms of the total energy: $\lambda \rightarrow I_\lambda = I(z_\mu; \lambda)$ with $\sigma_\lambda = \sigma(\partial_\mu I_\lambda, \dot{D})$, then it is not hard to see that Noether’s theorem, as presented in Section 4, holds. Hence, the simple transformation: $I_\lambda = I + \lambda \dot{I}$ leaves $\sigma$ invariant and so, by equation (4.3)–(4.4), we obtain

\[ \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \right) = 0 \] which is nothing other than the energy balance in hyperbolic form (6.4).

We examine, once again, the basic space-time translations (for simplicity we consider $c$ constant—see Appendix A for the non-constant $c$)

\[ \lambda \rightarrow z_\mu^{\lambda} = z_\mu - \lambda_\mu = z_\mu - \lambda_\eta \delta^\eta_\mu \] (6.22)
The question here is whether or not $\sigma$ remains invariant under the action of this transformation now that $\sigma$ depends not only on the parameterized field $I_\lambda$ but also on $\dot{D}_\lambda$. In principle we could have also required that function $\sigma$ be invariant under the transformation $\lambda \rightarrow \dot{D}_\lambda$. However, for the conservation law (4.4) to hold, the calculation in (4.5) must be justified. This is not true for arbitrary smooth transformations $\lambda \rightarrow \dot{D}_\lambda$, even for the basic space-time translations (6.22). This fact will result, as is shown below, in non-homogeneous conservation laws, which are still valuable in the context of constructing a variational hierarchy (see Section 7 and Appendix A).

Under space-time translations, functions $I$ and $\dot{D}$ satisfying the Euler-Lagrange equations become

\[ I_\lambda = I(z_\mu - \lambda_\mu) = I(z_\mu) + \lambda_\mu \frac{\partial I_\lambda}{\partial z_\mu} \big|_{\lambda=0} + O(|\lambda_\mu|^2) \] (6.23a)

\[ \dot{D}_\lambda = \dot{D}(z_\mu - \lambda_\mu) = \dot{D}(z_\mu) + \lambda_\mu \frac{\partial \dot{D}_\lambda}{\partial z_\mu} \big|_{\lambda=0} + O(|\lambda_\mu|^2) \] (6.23b)

\(^3\)Notice that according to our definition $\sigma_\lambda = \sigma(\partial_\mu I_\lambda, \dot{D})$ we must have $\frac{\partial \dot{D}}{\partial \lambda} = 0$, and so the calculation in (4.5) is still valid.
Next, we show $\sigma_\lambda$ satisfies (4.2).

**Proposition 2.** Function $\sigma$ is invariant under the space-time translation (4.2).

**Proof.** Function $\sigma_\lambda$ is defined as $\sigma(\partial_\alpha I_\lambda, \dot{D}_\lambda)$. Expanding this expression in $\lambda_\mu$ yields

\[
\sigma_\lambda = \sigma(\partial_\alpha I, \dot{D}) + \lambda_\mu \frac{d}{d\lambda_\mu} \sigma_\lambda |_{\lambda=0} + O(|\lambda_\mu|^2)
\]

\[
= \sigma + \lambda_\mu \left( \frac{\partial \sigma}{\partial (\partial_\alpha I)} \frac{\partial (\partial_\alpha I)}{\partial \lambda_\mu} + \frac{\partial \sigma}{\partial \dot{D}} \frac{\partial \dot{D}}{\partial \lambda_\mu} \right) + O(|\lambda_\mu|^2)
\]

\[
= \sigma + \lambda_\mu \frac{\partial \sigma}{\partial z_\mu} + O(|\lambda_\mu|^2)
\]

(6.24)

Hence,

\[
\frac{d}{d\lambda_\eta} \sigma_\lambda |_{\lambda=0} = \frac{\partial}{\partial \lambda_\eta} (\delta_{\mu\eta}\sigma)
\]

(6.25)

Here again since we have assumed a diffeomorphism with respect to a two dimensional parameter $\lambda_\eta$, the vector field in (4.2) is augmented to a second order tensor. Therefore, according to (6.25), we take $K_{\mu\eta} = \delta_{\mu\eta}\sigma$ to satisfy the condition (4.2).

However, we can not apply Noether’s theorem to conclude the conservation laws associated with the energy-momentum tensor since the LHS of (6.25) does not result in (4.3) in the presence of $\dot{D}_\lambda$. Nevertheless, we can establish a non-homogeneous version of equation (4.3), that is with additional terms on the RHS.

We start out calculation with

\[
\frac{d}{d\lambda_\eta} \sigma_\lambda |_{\lambda=0} = \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0} + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0} + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0}
\]

\[
= \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0} + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0} + \frac{\partial}{\partial \lambda_\eta} \left( \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) |_{\lambda=0}
\]

(6.26)

Equations (6.25) and (6.26) give us the following equation for the energy momentum tensor $T_{\mu\eta}$ (defined in (4.15))

\[
\frac{\partial T_{\mu\eta}}{\partial z_\mu} = -\frac{\partial \sigma}{\partial \dot{D}} \frac{\partial \dot{D}}{\partial \eta}
\]

(6.27)

In particular, the quantity $T_{00} = \pi$ is governed by

\[
\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial I}{\partial t} \frac{\partial \sigma}{\partial \dot{D}} \frac{\partial \dot{D}}{\partial \dot{x} \partial \eta} \right) = \frac{\partial I}{\partial \dot{x} \partial \dot{t}}
\]

(6.28)

It is also clear that the tensor $T_{\mu\eta}$ is no longer symmetric.
The quantity $\pi$ in the dissipative case is no longer governed by a conservation law as evident by equation (6.28). Nevertheless, a similar procedure outlined in the calculations (6.1) and (6.2) applied to the flux $\partial I/\partial t \partial \sigma/\partial (\partial I)$ can produce the functional $L^2$ for the dissipative case. So while we no longer have conservation laws given by Noether’s theorem in the presence of dissipation, we still manage to retain an infinite number of variational principles together with their corresponding Lagrangian-Hamiltonian formalism associated with functional $L^i$ (see Remark 4).

7 Comments on Field Theory

A closer look at the results above reveals a direction for obtaining more complex field theories from simpler ones. It is well known that in classical field theories, a Lagrangian $L$ is specified to capture the dynamics of some field $\phi$. Usually $L$ depends on $\phi$ and its derivatives. The question we are interested in is: “if we were to introduce a field $\Gamma$–independent of $\phi$–into our Lagrangian, how can we determine the evolution of this additional degree of freedom?”

Before we tackle this question in its general form, we turn to the interplay between the theories of elasticity and thermoelasticity for insight. For elasticity, the field $\phi$ is identified with the displacement $u$, and the Lagrangian density $L$ is given as

$$L = L(\partial_t u, \partial_x u) = \frac{\rho_0}{2} \left( \frac{\partial u}{\partial t} \right)^2 - e(\partial_x u) \quad (7.1)$$

where the internal energy $e$ can be identified with is the stored strain energy $W$. The evolution of the displacement field $u$ is governed by the balance of momentum, which is equivalent to the Euler-Lagrange equations for $L$ with respect to $u$.

In the theory of thermoelasticity we add the dependence of the entropy density $s$ to the internal energy $e$, hence $L = L(\partial_t u, \partial_x u, s)$, and assume the presence of heat flow through the boundary given by the flux $q$. For example, in linear thermoelasticity $e$ is a quadratic form in $\partial_x u$ and $s$ as implied in Example in Section 6. The balance of momentum is still equivalent to the Euler-Lagrange equations of $L$ with respect to $u$. However, the evolution of $s$ is not governed by the Euler-Lagrange equations of $L$ with respect to $s$, but rather by a physical principle: the first law of thermodynamics (7.1). However, as we know from Noether’s theorem the conservation of energy can be deduced through time symmetry invariance (i.e. the conservation of the zeroth component of the energy-momentum tensor, which corresponds to the total energy or Hamiltonian). This conservation law reads

$$\rho_0 \frac{\partial I}{\partial t} = \frac{\partial}{\partial x} (v S) \quad (7.2)$$

While both $I$ and $S$ depend on the field $s$, we still do not completely recover the first law of thermodynamics; we do not obtain the heat flux $q$ which is the requisite source of dissipation. In fact, in the absence of dissipation, equation (7.2) is equivalent to $\partial_s s = 0$.

We come away with two conclusions from the above observations. The first is that additional physics is needed when determining the evolution of the entropy.
s (i.e the first law). Secondly, a variational analysis for the field $u$ alone can neither produce the physics of the problem nor the correct equation governing the evolution of $s$, even though it correctly predicts the balance of momentum equation for thermoelasticity. In order to obtain the energy equation for thermoelasticity we need to introduce a more sophisticated variational principle obtained (ultimately) from the classic Hamilton’s principle, after we have formulated the first law of thermodynamics. In other words, to find a variational derivation for the equations of thermoelasticity, that is to determine the evolution of $u$ and $s$ in a dissipative material, we need two functionals: $L$ and $\Sigma$ (or $L$ and $\sigma$).

The addition of a new independent field $\Gamma$ to the Lagrangian $L = L^0$, in addition to the fields $u$ and $s$, necessarily implies that $L^1 = L^1(\partial_x u, s, \Gamma)$. However, we need an additional equation to close the system; this equation must be, by Theorem 3, the Euler-Lagrange equations for $L^2$ with respect to $u^2$ (including the additional non-homogeneous terms coming from Noether’s theorem for $L^1$). And while the three coupled Euler-Lagrange equations for $L^0, L^1, L^2$ are written in terms of unknowns $u^0, u^1, u^2$, the constitutive laws reveal that in actuality we have three coupled PDEs in $u, s, \Gamma$. Yet, the addition of $\Gamma$ can not be too arbitrary, it must be accompanied by some physical principle that can be added symbolically to Noether’s theorem for $L^1$ or equivalently to the Euler-Lagrange equations for $L^2$. This is necessary. A physical field can not be solely governed by a mathematical formalism. It is the field’s distinct response when coming into contact with physical reality that distinguishes it, and is manifested in some characterizing law or equation. The variational hierarchy has the flexibility to accommodate these physical principles as extra terms in the Lagrangian $L^0$ and Noether’s theorem, and at the same time allows for a formalism to produce the correct number of equations for any number of additional fields.

We can go a step further. Given a general field $\phi$ (not necessarily the displacement field) and a Lagrangian $L = L^0 = L^0(\phi, \partial_\mu \phi)$ in the form

$$L^0 = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - W(\phi) \quad (7.3)$$

then we can apply the same procedure outlined in the proof of Theorem 3 (and Appendix A) to obtain $L^1$ since we have the following additional equation (by Noether’s theorem) [24]:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial t} \frac{\partial L}{\partial (\partial_x \phi)} \right) = 0 \quad (7.4)$$

where $W$ is some nonlinear function in $\phi$ and $h = \partial_t \phi \frac{\partial L}{\partial (\partial_x \phi)} - L$. So while we have focused in this work on $\phi = u$, the same theoretical procedure can be applied to field theories with Lagrangian having the form written in equation (7.3). That is to say, equation (7.4) can be written in hyperbolic form, which allows for the construction of the next Lagrangian $L^1$ in the hierarchy.

The addition of an some other independent field $\Gamma$ to the Lagrangian $L^0$ entails that the evolution of $\Gamma$ is governed by the coupled Euler-Lagrange for $L^0$ and $L^1$ with respect to $\phi$ and $h$, respectively. Again, it is not enough to add $\Gamma$ to
\( L^0 \) alone, a physical law governing \( \Gamma \) must also be specified, which can enter the formalism through Noether’s theorem for \( L^0 \). In general, it seems we need as many Lagrangian densities \( L^i \)’s as there are independents fields.

Finally, it should be emphasized that the Lagrangians \( L^i, i = 1, 2, \ldots \) are auxiliary functions, in the sense that they are predetermined once the original Lagrangian \( L = L^0 \) is specified. Therefore, the new physics associated with an additional field is not only mediated by a certain physical principle, but also by \( L^0 \). The variational hierarchy provides us with a systematic approach to fit all new degrees of freedom into one framework.

8 Conclusion

In 1969 C. Truesdell wrote [24]: “The difference [between mechanics and thermodynamics] is that thermodynamics never grew up”. While the theory of rational thermodynamics has been developed significantly since the late 60s, largely due to Truesdell himself, a variational formulation akin to Hamilton’s principle had still remained out of reach. In this context, we view our work as an effort to establish a foundation for the variational treatment of the first law of thermodynamics (in one space dimension), which aims to be analogous to mechanics. This is only the first step. To claim a comprehensive variational theory of thermodynamics is to present the second law of thermodynamics in terms of some variation of the functional \( \Sigma \). This is an area for future research. However, the fact that a least action principle exists for the first law of thermodynamics is an indication that \( \Sigma \) is fundamental for both a conservative and dissipative systems.

We also showed that infinite new variational principles follow from the classic Hamilton’s principle by recasting the conservation of energy equation, obtained by Noether’s theorem, into hyperbolic form. This hierarchical structure has a simple iterative form for isentropic (conservative) systems, and is more involved for dissipative and forced systems. We also commented on the physics implication of the variational hierarchy. We established a systematic approach for increasing the complexity of field theories provided that the original Lagrangian \( L^0 \) has the form in equation (7.3). The addition of a new degree of freedom is governed by two factors: the first is given by the next iteration in the hierarchy, and the second is by some physical principle similar to heat transfer in the first law of thermodynamics. The implications considered here will be a topic of further study.

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A Appendix

Throughout the paper we have asserted that the scheme for constructing the functional $\Sigma$ can accommodate external sources and the procedure is similar to that of Section 6. We, therefore, consider the most general case of the first law of thermodynamics applied to a continuum, that is the non-linear conservation of energy equation with body forces $b(x,t)$, heat sources $r(x,t)$, and heat flux $q(\partial_\theta)$ [25]

$$\rho_0 \frac{\partial I}{\partial t} = \frac{\partial}{\partial x} (vS - q) + \rho_0 vb + \rho_0 r \quad \text{in } B \times [0,\tau] \quad (A.1)$$

where the constitutive equations are given in (2.2).

In addition to the construction of $L^1 = \Sigma$ for the above problem (together with its corresponding Hamiltonian formulation and Noether’s Theorem), we also demonstrate the construction of $L^2$ in this context.

By the calculations in (6.1) and (6.2) we have

$$\frac{\partial}{\partial t} (vS - q) = c \frac{\partial I}{\partial x} + \dot{\dot{D}} \quad (A.2)$$

The above two equations thus give

$$\rho_0 \frac{\partial^2 I}{\partial t^2} = \frac{\partial}{\partial x} \left( c \frac{\partial I}{\partial x} + \dot{\dot{D}} \right) - \dot{B} \quad (A.3)$$

where we have defined $\dot{B} = -\rho_0 \dot{\dot{D}} (vb - r)$.

As such we choose functional $\Sigma$ to be

$$\Sigma = \int_B \left( \frac{\rho_0}{2} \left( \frac{\partial I}{\partial t} \right)^2 - c \left( \frac{\partial I}{\partial x} \right)^2 - \dot{\dot{D}} \frac{\partial I}{\partial x} - \dot{B} I \right) dx \quad (A.4)$$

Hence, equation (A.3) is equivalent to the standard Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\delta \Sigma}{\delta (\partial_\tau I)} \right) - \frac{\delta \Sigma}{\delta I} = 0 \quad (A.5)$$

The Hamiltonian density $\pi$ can be fashioned similarly to Sections 3 & 6. Indeed, we can combine $\dot{B}$ and $\partial_x \dot{D}$ into one term and simply apply the procedure in Section 6 to obtain

$$\Pi = \int_B \left( \frac{1}{2\rho_0} J^2 + \frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 + \dot{\dot{D}} \frac{\partial I}{\partial x} + \dot{B} I \right) dx \quad (A.6)$$

and

$$\frac{\partial I}{\partial t} = \frac{\delta \Pi}{\delta J} \quad (A.7a)$$

$$\frac{\partial J}{\partial t} = -\frac{\delta \Pi}{\delta I} \quad (A.7b)$$

where $\pi$ is defined as in equation (6.18).
Equation (A.3) (or equivalently equation (A.5)) follows from Noether's theorem if $\sigma$ remains invariant with respect to $\lambda \rightarrow I_\lambda = I + \lambda T$. However, the application of Noether's theorem is not straightforward with respect to space-time translations (A.10) because of the presence of $\dot{D}$ (as we have seen in Section 6), and now $\dot{B}$. We explore this below.

Under space-time translations (A.10) we obtain two more equations in addition to (A.2):

$$c_\lambda = c(z_\mu - \lambda_\mu) = c(z_\mu) + \lambda_\mu \frac{\partial c_\lambda}{\partial z_\mu} |_{\lambda=0} + O(|\lambda_\mu|^2) \quad (A.8a)$$

$$B_\lambda = B(z_\mu - \lambda_\mu) = B(z_\mu) + \lambda_\mu \frac{\partial B_\lambda}{\partial z_\mu} |_{\lambda=0} + O(|\lambda_\mu|^2) \quad (A.8b)$$

A similar calculation to the proof Proposition (2) reveals that $\sigma_\lambda = \sigma(I_\lambda, \partial_\mu I_\lambda, c_\lambda, \dot{D}_\lambda, \dot{B}_\lambda)$ obeys

$$\frac{d}{d\lambda_\eta} \sigma_\lambda |_{\lambda=0} = \frac{\partial}{\partial z_\mu} (\delta_{\mu\eta} \sigma) \quad (A.9)$$

On the other hand

$$\frac{d}{d\lambda_\eta} \sigma_\lambda |_{\lambda=0} = \frac{\partial \sigma}{\partial I_\lambda} \frac{\partial I_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial (\partial_\mu I_\lambda)} \frac{\partial (\partial_\mu I_\lambda)}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial \sigma_\lambda} \frac{\partial \sigma_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial \dot{D}_\lambda} \frac{\partial \dot{D}_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial \dot{B}_\lambda} \frac{\partial \dot{B}_\lambda}{\partial \lambda_\eta} |_{\lambda=0}$$

$$= \left( \frac{\partial \sigma}{\partial I} - \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \right) \right) \frac{\partial I_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \frac{\partial \dot{I}_\lambda}{\partial \lambda_\eta} |_{\lambda=0} \right) + \frac{\partial \sigma}{\partial \sigma_\lambda} \frac{\partial \sigma_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial \dot{D}_\lambda} \frac{\partial \dot{D}_\lambda}{\partial \lambda_\eta} |_{\lambda=0} + \frac{\partial \sigma}{\partial \dot{B}_\lambda} \frac{\partial \dot{B}_\lambda}{\partial \lambda_\eta} |_{\lambda=0}$$

$$= \frac{\partial}{\partial z_\mu} \left( \frac{\partial \sigma}{\partial (\partial_\mu I)} \frac{\partial I_\lambda}{\partial \lambda_\eta} \right) + \frac{\partial \sigma}{\partial \sigma_\lambda} \frac{\partial \sigma_\lambda}{\partial \lambda_\eta} + \frac{\partial \sigma}{\partial \dot{D}_\lambda} \frac{\partial \dot{D}_\lambda}{\partial \lambda_\eta} + \frac{\partial \sigma}{\partial \dot{B}_\lambda} \frac{\partial \dot{B}_\lambda}{\partial \lambda_\eta} \quad (A.10)$$

Therefore, the second order tensor $T_{\mu\nu} = \frac{\partial \sigma}{\partial (\partial_\nu I)} \frac{\partial \sigma}{\partial \lambda_\eta} - \sigma \delta_{\mu\eta}$ satisfies the equation

$$\frac{\partial T_{\mu\nu}}{\partial z_\mu} = F_\eta = 0 \quad (A.11)$$

where $F_\eta = \frac{\partial \sigma}{\partial \sigma_\lambda} \frac{\partial \sigma_\lambda}{\partial \lambda_\eta} + \frac{\partial \sigma}{\partial \dot{D}_\lambda} \frac{\partial \dot{D}_\lambda}{\partial \lambda_\eta} + \frac{\partial \sigma}{\partial \dot{B}_\lambda} \frac{\partial \dot{B}_\lambda}{\partial \lambda_\eta} \frac{\partial \sigma}{\partial \dot{b}_\lambda} \frac{\partial \dot{b}_\lambda}{\partial \dot{b}_\eta}$.

And so $T_{00} = \pi$ satisfies

$$\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial I}{\partial \sigma} + \frac{\partial \sigma}{\partial \dot{b}} \frac{\partial \dot{b}}{\partial \sigma} \right) = F_0 \quad (A.12)$$

The presence of $F_0$ does not impede the construction of $L^2$ (i.e. the second iteration). We proceed as we did in the beginning of Section 5, and compute the rate of change of the flux in (A.12):

$$\frac{\partial}{\partial t} \left( \frac{\partial I}{\partial \sigma} + \frac{\partial \sigma}{\partial \dot{b}} \frac{\partial \dot{b}}{\partial \sigma} \right) = -\frac{\partial}{\partial t} \left( \frac{\partial I}{\partial \sigma} \left( \frac{\partial I}{\partial \sigma} + \dot{D} \right) \right)$$

$$= -\frac{\partial^2 I}{\partial t^2} \left( \dot{D} + \frac{\partial I}{\partial \sigma} \right) - \frac{\partial I}{\partial t} \left( \frac{\partial c \partial I}{\partial x \partial t} + \frac{\partial^2 I}{\partial x^2 \partial t} + \frac{\partial \dot{D}}{\partial t} \right)$$

26
By using the evolution equation for the total energy (A.3) and rearranging the terms we obtain

$$\frac{\partial}{\partial t} \left( \frac{\partial I}{\partial t} \frac{\partial I}{\partial \sigma} \right) = -\frac{c}{\rho_0} \frac{\partial}{\partial x} (\pi) + Q$$  \hspace{1cm} (A.13)

where 

$$Q = \frac{1}{\rho_0} \partial_x (c \partial_x I) \dot{D} - \frac{c}{\rho_0} \partial_x c (\partial_x I)^2 \frac{\partial}{\partial \rho_0} \partial_x I \dot{D} + \frac{\partial}{\partial \rho_0} \partial_x I \dot{B} + \frac{c}{\rho_0} \partial_x (1 \dot{B}) - \partial_t \partial_x c \partial_x I - \partial_t \dot{D} \dot{\partial}_x I.$$

Equations (A.13) and (A.12) give us

$$\rho_0 \frac{\partial^2 \pi}{\partial t^2} - \frac{\partial}{\partial x} \left( c \frac{\partial \pi}{\partial x} \right) = \rho_0 \left( \frac{\partial F_0}{\partial t} - \frac{\partial Q}{\partial x} \right)$$  \hspace{1cm} (A.14)

One possible Lagrangian \( L^2 \) to derive (A.12) from is

$$L^2 = \frac{\rho_0}{2} \left( \frac{\partial \pi}{\partial t} \right)^2 - \frac{c}{2} \left( \frac{\partial \pi}{\partial x} \right)^2 - \dot{C} \pi$$  \hspace{1cm} (A.15)

where \( \dot{C} = -\rho_0 (\partial_t F_0 - \partial_x Q) \).

The Lagrangian \( L^2 \) is the basis for formulating the rest of the variational framework in the second iteration. While, as we see, the detailed expressions become increasingly more complex, they can be organized in an expected pattern as we move up from one iteration to the other.

**B  Appendix**

For a linear elastic medium the displacement \( u \) satisfies the conservation of linear momentum:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial S}{\partial x} = \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right)$$  \hspace{1cm} (B.1)

In the presence of thermal effects (B.1) can be extended to

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right) + \gamma \frac{\partial \theta}{\partial x}$$  \hspace{1cm} (B.2)

where \( \gamma \) is the stress-temperature modulus.

By comparing equations (B.1) and (B.2) with (2.4) and (6.4), respectively, a one-to-one correspondence between the balance of momentum and energy can be established. We summarize this correspondence in the table below.
Table 1: A one-to-one correspondence is illustrated between the balance of momentum and energy in linear elastic medium with the presence of thermal effects. For simplicity we have taken constant $\gamma = 1$.

| Identity            | Conservation of Momentum | Conservation of Energy |
|---------------------|--------------------------|------------------------|
| **Density Function**| $L = \frac{\rho_0}{2} \left( \frac{\partial u}{\partial t} \right)^2 - \frac{c}{2} \left( \frac{\partial u}{\partial x} \right)^2 - \theta \frac{\partial u}{\partial x}$ | $\sigma = \frac{\rho_0}{2} \left( \frac{\partial I}{\partial t} \right)^2 - \frac{c}{2} \left( \frac{\partial I}{\partial x} \right)^2 - \dot{\theta} \frac{\partial I}{\partial x}$ |
| **Governing Equation** | $\rho_0 \frac{\partial^2 u}{\partial t^2} - c \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} = 0$ | $\rho_0 \frac{\partial^2 I}{\partial t^2} - c \frac{\partial^2 I}{\partial x^2} - \frac{\partial \dot{D}}{\partial x} = 0$ |
| **Action Functional** | $\int_0^\tau \int_B L(\partial_t u, \partial_x u, \theta) dx dt$ | $\int_0^\tau \int_B \sigma(\partial_t I, \partial_x I, \dot{\theta}) dx dt$ |

The above table establishes the following correspondence

\begin{align*}
  u & \longleftrightarrow I & \text{(B.3a)} \\
  \theta & \longleftrightarrow \dot{D} & \text{(B.3b)}
\end{align*}
References

[1] Gregory RD. Classical mechanics. Cambridge University Press; 2006.

[2] Berdichevsky V. Variational principles of continuum mechanics: I. Fundamentals. Springer Science & Business Media; 2009.

[3] Rayleigh L. Theory of Sound. vol. 1 (reprinted 1945 by Dover, New York). Macmillan, London; 1945.

[4] Biot MA. Thermoelasticity and irreversible thermodynamics. Journal of Applied Physics. 1956;27(3):240–253.

[5] Biot MA. Variational principles in heat transfer: a unified Lagrangian analysis of dissipative phenomena. BIOT (MA) NEW YORK; 1970.

[6] Ottiger HC. Beyond equilibrium thermodynamics. John Wiley & Sons; 2005.

[7] Kaufman AN. Dissipative Hamiltonian systems: a unifying principle. Physics Letters A. 1984;100(8):419–422.

[8] Morrison PJ. A paradigm for joined Hamiltonian and dissipative systems. Physica D: Nonlinear Phenomena. 1986;18(1-3):410–419.

[9] Said H. A Lagrangian–Hamiltonian unified formalism for a class of dissipative systems. Mathematics and Mechanics of Solids. 2019;24(4):1221–1240.

[10] Gurtin M. Variational principles for linear initial-value problems. Quarterly of Applied Mathematics. 1964;22(3):252–256.

[11] Yang Q, Stainier L, Ortiz M. A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids. Journal of the Mechanics and Physics of Solids. 2006;54(2):401–424.

[12] Kane C, Marsden JE, Ortiz M, West M. Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems. International Journal for Numerical Methods in Engineering. 2000;49(10):1295–1325.

[13] Bloch A, Krishnaprasad P, Marsden JE, Ratiu TS. The Euler-Poincaré equations and double bracket dissipation. Communications in mathematical physics. 1996;175(1):1–42.

[14] Gay-Balmaz F, Yoshimura H. A Lagrangian variational formulation for nonequilibrium thermodynamics. Part II: continuum systems. Journal of Geometry and Physics. 2017;111:194–212.

[15] Galley CR. Classical Mechanics of Nonconservative Systems. Phys Rev Lett. 2013 Apr;110:174301. Available from: https://link.aps.org/doi/10.1103/PhysRevLett.110.174301

[16] Riewe F. Mechanics with fractional derivatives. Physical Review E. 1997;55(3):3581.
[17] Kalpakides V, Maugin G. Canonical formulation and conservation laws of thermoelasticity without dissipation. Reports on Mathematical Physics. 2004;53(3):371–391.

[18] Finlayson BA. The method of weighted residuals and variational principles. vol. 73. SIAM; 2013.

[19] Green A, Naghdi P. Thermoelasticity without energy dissipation. Journal of elasticity. 1993;31(3):189–208.

[20] Straughan B. Heat waves. vol. 177. Springer Science & Business Media; 2011.

[21] Giaquinta M, Modica G. Mathematical analysis: foundations and advanced techniques for functions of several variables. Springer Science & Business Media; 2011.

[22] Dal Maso G, Negri M, Percivale D. Linearized elasticity as Γ-limit of finite elasticity. Set-Valued Analysis. 2002;10(2-3):165–183.

[23] Smoller J. Shock waves and reaction—diffusion equations. vol. 258. Springer Science & Business Media; 2012.

[24] Marsden JE, Hughes TJ. Mathematical foundations of elasticity. Courier Corporation; 1994.

[25] Dafermos C. Hyperbolic conservation laws in continuum physics. vol. 3. Springer; 2005.