Holography as a highly efficient RG flow: Part 1

Nicolas Behr,\textsuperscript{a,b} Stanislav Kuperstein\textsuperscript{c} and Ayan Mukhopadhyay\textsuperscript{c,d,e}

\textsuperscript{a}LFCS, University of Edinburgh, Informatics Forum, 10 Crichton Street, Edinburgh, EH8 9AB, Scotland, UK
\textsuperscript{b}Maxwell Institute for Mathematical Sciences, Edinburgh, UK
\textsuperscript{c}Institut de Physique Théorique, CEA Saclay, F-91191 Gif-sur-Yvette, France
\textsuperscript{d}Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France
\textsuperscript{e}Crete Center for Theoretical Physics (CCTP) and Crete Center for Quantum Complexity and Nanotechnology (CCQCN), University of Crete, P.O. Box 2208, 71003, Heraklion, Greece

E-mail: Nicolas.Behr@gmx.de, stanislav.kuperstein@gmail.com, ayan@physics.uoc.gr

Abstract: We investigate how the holographic correspondence can be reconstructed as a special RG flow in a strongly interacting large $N$ field theory. We firstly define a \textit{highly efficient RG flow} as one in which the cut-off in momentum space can be adjusted as a functional of the elementary fields, and of the external sources and of the background metric in order to be compatible with the following requirement: the Ward identities for single-trace operators involving conservation of energy, momentum and global charges must preserve the same form at every scale. In order to absorb the contributions of the multi-trace operators to these effective Ward identities, the external sources and the background metric need to be redefined at each scale, and thus they become dynamical as in the dual gravity equations. We give a schematic construction of such highly efficient RG flows using appropriate collective variables, leaving a more explicit construction in certain limits to the second part of this work. We find that all highly efficient RG flows that can be mapped to classical gravity equations have an additional \textit{lifted Weyl symmetry}, which is related to the ultraviolet Weyl symmetry, and which also has complete information about the gauge fixing of the diffeomorphism symmetry of the equivalent classical gravity equations. We present strong arguments for our claim that the presence of the lifted Weyl symmetry along with the requirement that the infrared end point can be characterised by a finite number of parameters, are sufficient conditions for a highly efficient RG flow to have a precise dual classical gravity description.

Keywords: Holographic principle, ads/cft correspondence, local renormalisation group flow, emergent gravity

\textsuperscript{1}Present address of A.M. since October 1, 2014
1 Introduction

The holographic correspondence implies that many features of a large class of strongly interacting many–body quantum systems can be understood via a dual classical gravity theory in one higher dimensional spacetime with appropriate asymptotic symmetries. Furthermore, the dual classical gravity theory typically should involve only a finite number of fields.

Since it has been first conjectured for the $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory [1], a large body of supporting evidence has been gathered, albeit in a limited
number of examples, most of which can be embedded in string theory. It is a widespread belief among experts that the correspondence should be more general. This is supported by the expectation that the emergent extra radial dimension in the dual classical gravity theory encodes a special kind of renormalisation group (RG) flow for most of the degrees of freedom in a large class of strongly interacting quantum many-body systems.

In the first part of this work, which is presented in this paper, we propose general principles for constructing the special RG flow in the (strongly interacting large \(N\)) field theory that leads to an emergent classical gravity theory in one higher dimension implying the traditional holographic correspondence \([2, 3]\), at least in a special dynamical sector. In the second part of this work \([4]\) to appear soon, we will explicitly construct the RG flow in a special limit, that satisfies our required principles and we will moreover demonstrate that that it also reproduces the dual classical gravity equations. However, we will succeed only partially to show that our explicit construction is uniquely determined by the general principles for the RG flow to be discussed here. If we can prove that our general principles can determine the RG flow construction to be presented in \([4]\) (and schematically discussed here) uniquely, then we will be able to completely reconstruct the holographic correspondence as a special RG flow in the field theory, at least in certain limits. Along the way, we will gather new insights into how the holographic correspondence may fail even in the strongly interacting large \(N\) limit, and perhaps how it can be generalised.

The general picture that emerges is as follows. In a strongly interacting large \(N\) field theory (which we define in a very general way in Section 2.1), it is possible to define a highly efficient RG flow under which the Ward identities for single-trace operators (meaning operators which generate the entire operator algebra) related to conservation of energy, momentum and global charges take the same form at each scale, with appropriate redefinitions of the external sources and background metric. Such redefinitions serve to absorb mixing of single-trace operators with multi-trace operators under the RG flow in a precise manner, as partly anticipated earlier in \([5–7]\). A class of such highly efficient RG flows can be mapped to classical gravity equations in one higher dimension. We find this map in an ultraviolet expansion in the inverse of the cut-off scale in a conformal field theory.

The explicit construction of highly efficient RG flows in the field theory can be achieved by a generalised coarse-graining procedure, in which the cut-off in momentum space that leaves out Fourier modes bigger than the cut–off scale is smoothened by a functional of the elementary fields and the external sources themselves in a very precise way. This coarse-graining will be described schematically in Section 2.1, and will be achieved via use of appropriate collective variables. In the following publication \([4]\), we will describe an

\[1\]Ward identities remain invariant under Wilsonian RG flow. Here we consider more generalized RG flows where the cut-off becomes a functional of the fields and the external sources themselves. In such cases, the Ward identities will be badly broken by the RG flow. An even simpler example is as follows. Consider a projection to a subsector of slowly varying modes \((k \ll \Lambda)\) via a projection operator \(P(\Lambda)\). The projected energy-momentum tensor \(t^{\mu\nu}(\Lambda) = P(\Lambda)t^{\mu\nu}P^\dagger(\Lambda)\) then does not satisfy the conservation equation \(\partial_\mu t^{\mu\nu}(\Lambda) = 0\) simply because the slowly varying modes will receive driving forces from the faster modes projected out, via exchange of energy and momentum. The highly efficient RG flow requires the broken Ward identity to take a very specific structural form, namely that \(t^{\mu\nu}(\Lambda)\) satisfies the standard conservation equation \(\nabla_\nu t^{\mu\nu}(\Lambda) = 0\) in a fictitious effective scale-dependent metric \(g_{\mu\nu}(\Lambda)\).
explicit construction of the highly efficient RG flow in the limit when the dynamics in the field theory is captured by an exact asymptotic hydrodynamic expansion, and show that it indeed reproduces dual classical gravity equations.

The dual classical gravity theory to which this RG flow maps to is determined uniquely by requiring the absence of any infrared Landau pole like singularity. In fact, unless the field theory is special, such a RG flow may not even map to a classical gravity theory (in the sense of mapping to regular well behaved spacetimes which are solutions of this classical gravity theory) even in the strongly interacting large $N$ limit. However, we will be able to demonstrate based on some previous results that when the highly efficient RG flow maps to classical gravity equations, the latter are uniquely determined by the requirement of infrared regularity, which more precisely means that the end point of the RG flow can be characterised by a finite number of parameters, as we will explain later.

Furthermore, we will show here that an additional necessary condition (aside from having a good infrared behaviour as discussed above) for the highly efficient RG flow to map to a classical gravity theory is that it should have a symmetry of a special kind. This asymptotically reduces to the Weyl symmetry in the ultraviolet, but is valid at all scales, so we will call it lifted Weyl symmetry. At any scale, the lifted Weyl symmetry is a simple but specific combination of a translation in scale, a Weyl transformation and a diffeomorphism along the field-theory coordinates for the effective scale-dependent metric and background sources. This transformation of the effective scale-dependent metric (which does not appear in the RG flow equation but is rather constructed out of it) implies a unique transformation for the energy-momentum tensor and other single-trace operators. The specific combination of translation in scale, Weyl transformation and the $d$–diffeomorphism is determined uniquely by the gauge-fixing of diffeomorphism symmetry in the dual gravity equations, and takes the same form for any dual classical gravity theory. Thus remarkably, the lifted Weyl symmetry of the RG flow carries complete information of the gauge-fixing of diffeomorphisms in the dual gravity equations.

An open question that remains is whether the lifted Weyl symmetry along with infrared regularity as mentioned above are sufficient conditions to ensure that a highly efficient RG flow should map to an appropriate classical gravity theory in one higher dimension. However, we argue strongly that this is indeed the case. An even deeper question is whether in the absence of such a lifted Weyl symmetry we can obtain highly efficient RG flows in certain strongly interacting large $N$ field theories, which are well defined in the infrared, although they do not map to any classical gravity theory with full diffeomorphism invariance. We leave such investigations for the future.

Our work borrows key insights from [5–7], particularly that the dynamics of the sources in the dual gravity theory can be reconstructed by absorbing the mixing of the single-trace operators with the multi-trace operators under the RG flow in a strongly interacting large $N$ field theory. In this work we have been able to propose general principles for how this is to be done precisely in order to reconstruct the dual gravity theory. However, our approach differs significantly from [7], in which a construction has been proposed where the Wilsonian RG flow is projected onto the subspace spanned by single-trace operators. Our approach involves generalising the Wilsonian RG flow itself by making the cut-off in momentum
space dependent on the elementary fields and sources (via the use of appropriate collective
variables). Our approach also has deep relations with many other recent works in the
literature. In order to make the discussion more precise, we will postpone our comments
on these connections to Section 2 and Section 5.

Although for most of this paper we focus on the limit where the dynamics on the
gravity side can be consistently truncated to pure gravity, in Section 5 we will argue
that the three principles discussed here should be sufficient to reconstruct the holographic
correspondence as a RG flow more generally in the strongly interacting and large \( N \) limit.
A full demonstration is also left to a future work.

The organisation of this paper is as follows. In Section 2, we give a precise definition
of the highly efficient RG flow, depict schematically how it can be constructed as a specific
kind of coarse graining, and discuss various fundamental issues which arise while mapping it
to a dual classical gravity theory. This section is designed partly to give precise definitions
and introduce new concepts, and partly to give a mental map of the rest of the work
(including the second part \([4]\)) and a quick summary of various conceptual issues, which
we deal with more completely later. Along the way, we also discuss some related works in
the literature.

In Section 3, we show how the gravity equations can be obtained from a highly efficient
RG flow and vice versa, in an ultraviolet expansion. In Section 4, we demonstrate how such
highly efficient RG flows inherit a special symmetry, which is a specific type of lift of the
ultraviolet Weyl symmetry to an arbitrary scale, and which also has complete information
about the gauge fixing of the diffeomorphism symmetry in the corresponding classical
gravity equations.

In Section 5, we give an outlook and discuss some open issues. We point out connec-
tions with some other approaches for reconstructing holography using aspects of quantum
information. We further point out investigations which require our immediate attention.
We emphasise directions which can lead to a deeper understanding of quantum gravity,
and which can lead to wider generalisations and applications of the holographic principle.
The appendices give supporting details.

2 Introducing the concept of the highly efficient RG flow and how it
leads to emergence of gravity

2.1 Definition and construction of the highly efficient RG flow

It has been emphasised recently [5] that one of the main requirements for a conformal field
theory (CFT) to have a holographic classical gravity dual with finitely few fields, is that the
spectrum of scaling dimensions of operators should have a large gap in a parametric sense.
This means that the anomalous dimensions of most operators will become arbitrarily large
when the parameters, such as the coupling constants, are appropriately tuned. In fact, this
constitutes an abstract definition of the strongly interacting limit.

The operators which survive this limit must form an algebra generated by a few ele-
ments, which are the energy momentum tensor \( t^\mu_\nu \), conserved currents \( j_\mu \) and a few order
parameters \( O \) required to describe various kinds of spontaneous symmetry breaking in
different phases. These elements, which generate the full algebra of operators, form the generalisation of the single-trace operators which survive the strongly interacting limit.\footnote{In a supersymmetric theory, the number of relevant single-trace operators could be infinite even in the strongly interacting limit, as many of them, particularly the chiral primaries, receive protection from getting large anomalous dimensions by supersymmetry. In a non-supersymmetric theory, there is no analogue of the chiral primary operators. Thus it can be expected that the relevant single-trace operators are the conserved currents for various global symmetries, or order parameters relevant for spontaneous breaking of global symmetries.}

It is also necessary to have a large $N$ limit in which the expectation values of the single-trace operators should factorise in all states. For instance,\footnote{Of course the expectation value of $t_{\mu \nu}$ cannot be arbitrary, and this is reflected by the fact that the solutions of gravity can also have bad naked singularities. Only appropriate regular solutions of classical gravity, corresponding to specific expectation values of $t_{\mu \nu}$, will be dual to field theory states. We will discuss more on this later.}

\[ \langle t_{\mu \nu} t_{\rho \sigma} \rangle = \langle t_{\mu \nu} \rangle \langle t_{\rho \sigma} \rangle + O \left( \frac{1}{N^2} \right) \text{ etc.} \quad (2.1) \]

Typically this parameter $N^2$ is related to the number of elementary constituents of the field theory. By the holographic dictionary, it is mapped to the inverse of Newton’s gravitational constant of the one higher dimensional dual gravity theory, which is thus small in the large $N$ limit, implying that the quantum corrections in the gravity theory can be neglected.

We assume that such a strongly interacting large $N$ limit exists in the field theory under study. In addition, we assume that there exists a sector of states where the expectation values of all other operators except $t_{\mu \nu}$ are either state-independent, or are algebraic functionals of the expectation value of $t_{\mu \nu}$ and its spacetime derivatives. Indeed, if the holographic duality exists, such a sector must exist as well, which follows from the fact that the dual classical gravity theory can always be consistently truncated to just pure gravity with a negative cosmological constant. As fields on the gravity side are dual to operators in the field theory, and because only the graviton has independent dynamics of its own here, it follows that the energy-momentum tensor $t_{\mu \nu}$ can only have an independent expectation value in this sector of states dual to pure gravity.\footnote{In this work, we focus on reconstructing the holographic correspondence as a special RG flow in this pure gravity sector in the strongly interacting large $N$ limit. The crucial point is that all the assumptions stated above, are necessary but not sufficient for the holographic correspondence to exist. Later in Section 5, we discuss why the general principles we find in the pure gravity sector should be sufficient for the reconstruction of the holographic correspondence more generally in the strongly interacting large $N$ limit.}

Under the assumptions stated above, it follows that we can write the RG flow of $t_{\mu \nu}$ in any state in the stated sector as a classical equation of the form:

\[ \frac{\partial t_{\mu \nu}(\Lambda)}{\partial \Lambda} = \text{a non-linear functional of } t_{\mu \nu}(\Lambda) \text{ and its spacetime derivatives.} \quad (2.2) \]

The classicality of the above equation, in the sense that products of $t_{\mu \nu}(\Lambda)$ are just ordinary products, follows from the large $N$ factorisation (2.1) which is valid in all states. This
equation can be thought of as a Legendre transform of the beta functions which give the flow of (space-time dependent) couplings in a local RG flow as defined by Osborn [8]. In the case of $t_{\mu}^{\nu}$, the dual coupling is the background metric which is usually taken to be fixed up to an overall scaling in a RG flow. In this case it is more natural to study the RG flow of the operator $t_{\mu}^{\nu}(\Lambda)$ itself.

Furthermore, in a CFT there is no intrinsic scale. Therefore the RG flow equation should depend explicitly on $t_{\mu}^{\nu}(\Lambda)$ and $\Lambda$ only. There will be exceptions only for log($\Lambda$/ΛIR)-like terms which assumes a (state-dependent) scale ΛIR, but such terms will be completely determined by the conformal anomaly, in particular the central charges. In any given state, there should be a (state-dependent) scale ΛIR such that $t_{\mu}^{\nu}(\Lambda)$ can be identified with $t_{\mu}^{\nu}s$ the original microscopic operator sans any coarse-graining. The latter satisfies the standard Ward identity

$$\partial_{\mu}t_{\mu}^{\nu} = 0,$$

which follows from local energy-momentum conservation.

The above statements are very general. However, for the sake of illustration, let us assume that there exists a precise form of coarse-graining in the strongly interacting large $N$ CFT, under which the RG flow takes this specific form in $d = 4$:

$$\frac{\partial t_{\mu}^{\nu}(\Lambda)}{\partial \Lambda} = \frac{1}{\Lambda^3} \left( \frac{1}{2} \Box t_{\mu}^{\nu}(\Lambda) - \frac{7}{128} \eta^{\alpha\beta}(\Lambda) t_{\alpha}^{\beta}(\Lambda) - \frac{1}{32} \Box t_{\mu}^{\nu}(\Lambda) \right) + \frac{1}{\Lambda^5} \log \Lambda \cdot \frac{1}{12} \cdot \Box^2 t_{\mu}^{\nu}(\Lambda) + O\left( \frac{1}{\Lambda^7} \log \Lambda \right).$$

As we have assumed we are in the infinitely strongly interacting limit, so the coefficients of the expansion multiplying various operators appearing at arbitrary orders $\Lambda^{-n}$ are pure numbers, which otherwise would have been functions of the coupling constants of the CFT. Moreover in the log($\Lambda$/ΛIR) term we have suppressed the ΛIR which is a state-dependent scale – it is implicit. We will eventually derive this RG flow equation in Section 3.2 and show that it reproduces Einstein’s equations in a precise way. At present, let us consider this equation as an example for illustrating the definition of the highly efficient RG flow.

The $(1/\Lambda^5)$ log $\Lambda$ term in this equation is completely determined by the conformal anomaly.

In case of the above special RG flow, we can define the following fictitious effective scale-dependent metric $g_{\mu\nu}(\Lambda)$,

$$g_{\mu\nu}(\Lambda) = \eta_{\mu\nu} + \frac{1}{\Lambda^2} \cdot \frac{1}{4} \eta_{\mu\rho} t_{\rho}^{\alpha}(\Lambda) + \frac{1}{2} \cdot \frac{1}{\Lambda^6} \cdot \frac{1}{24} \eta_{\mu\alpha} \Box t_{\rho}(\Lambda) + \frac{1}{\Lambda^8} \cdot \frac{1}{32} \eta_{\mu\rho} t_{\rho}^{\alpha}(\Lambda) + \frac{7}{384} \eta_{\mu\nu} t_{\alpha}^{\beta}(\Lambda) t_{\rho}^{\beta}(\Lambda) + \frac{11}{1536} \eta_{\mu\rho} \Box^2 t_{\rho}(\Lambda) - \frac{1}{\Lambda^8} \log \Lambda \cdot \frac{1}{512} \cdot \eta_{\mu\rho} \Box^2 t_{\rho}(\Lambda) + O\left( \frac{1}{\Lambda^{10}} \log \Lambda \right).$$

- 6 -
such that $t^\mu_{\nu}(\Lambda)$ satisfies
\[
\nabla_{(\Lambda)} t^\mu_{\nu}(\Lambda) = 0
\] (2.6)
at all scales, with $\nabla_{(\Lambda)}$ being the covariant derivative constructed from $g_{\mu\nu}(\Lambda)$.$^4$ This equation holds at all scales as a consequence of Eq. (2.4) depicting the RG flow, and the standard Ward identity (2.3) for energy-momentum conservation which is satisfied asymptotically.

This is a necessary, if not sufficient, condition for the RG flow (2.4) to be re-interpreted as a classical gravity equation in one higher dimensional space-time. To see this, one needs to identify $g_{\mu\nu}(\Lambda)$ and $t^\mu_{\nu}(\Lambda)$ with the induced metric and the quasi-local energy-momentum tensor (a.k.a. the Brown-York tensor in pure Einstein’s gravity), respectively, on the hypersurface $r = \Lambda^{-1}$ (up to precise corrections to be discussed later in Section 3.1) after taking the expectation value of $t^\mu_{\nu}(\Lambda)$ in (2.5) and assuming large $N$ factorisation. Here, $r$ denotes the emergent radial coordinate in the theory of gravity, defined via a very specific kind of gauge fixing of the $(d+1)$-dimensional diffeomorphism symmetry. Indeed, the emergence of the $(d+1)$-dimensional diffeomorphism symmetry in the theory of gravity is related to the fact that the dynamical equation for scale evolution, namely (2.4), leads to so-called momentum constraints of general relativity, namely (2.6), which is satisfied at all scales, and which reduces to the standard Ward identity for energy-momentum conservation (2.3) asymptotically.

The RG flow (2.4) constitutes an example of what we will call a highly efficient RG flow. One of the principal criteria that defines such a RG flow is that there exists a form of coarse-graining such that the Ward identities for local conservation of energy, momentum and global charges (involving the single-trace operators, namely the energy-momentum tensor and conserved currents) should take the same form at each scale, but with the sources and background metric redefined in a scale-dependent manner, such that they can absorb the mixing of the single-trace operators with the multi-trace operators under the resultant RG flow.$^5$ If indeed there exists a coarse-graining which results in the RG flow (2.4), then it is possible to keep the effective operator equation for energy-momentum conservation in the same form at each scale, as in (2.6), but with the background metric redefined as in (2.5) now written in flat Minkowski space.

$^4$Up to given orders, this means that $t^\mu_{\nu}(\Lambda)$ satisfies the following effective Ward identity
\[
\partial_{\mu} t^\mu_{\nu}(\Lambda) = \left( \frac{1}{16} \partial_{\rho} t^\rho_{\sigma}(\Lambda) t^\sigma_{\nu}(\Lambda) \right) - \left( \frac{1}{8} t^\mu_{\nu}(\Lambda) \partial_{\alpha} \text{Tr} t^\alpha(\Lambda) \right) + \\
+ \frac{1}{16} \left( \frac{1}{48} t^\alpha_{\beta}(\Lambda) \partial_{\sigma} t^\sigma_{\nu}(\Lambda) - \frac{1}{48} t^\alpha_{\nu}(\Lambda) \partial_{\alpha} \text{Tr} t^\alpha(\Lambda) \right) + \\
+ \mathcal{O} \left( \frac{1}{\Lambda^8} \right),
\]
(2.7)
now written in flat Minkowski space.

$^5$The Ward identity for conformal invariance, which results in $\text{Tr} t^\alpha = 0$ asymptotically, cannot be included in the list of operator equations whose form must be preserved at each scale — this is because the coarse-graining scale itself explicitly breaks conformal symmetry. Nevertheless it plays an important role in determining consistency relations in the RG flow as discovered in [8]. We will discuss this issue in Section 2.3.
at each scale, such that it absorbs the contributions coming from the multi-trace operators (as explicitly appearing in Eq. (2.7) for instance).

Although we have assumed so far that the field theory lives in flat Minkowski space, the highly efficient RG flow can be constructed for any background metric. In Section 4.2, we will generalise the RG flow equation (2.4) for conformally flat background metric.

The observation that the sources and background metric for single-trace operators become dynamical, as is the case in the dual gravity theory in the holographic correspondence, by absorbing the mixing of single-trace operators with multi-trace operators under the RG flow, has been emphasised before in [5–7]. However, here we are able to propose concrete principles (of which one is as stated above) which can be used to construct the RG flow that leads to the emergence of dual gravity. In our approach here, we will focus more on the Heisenberg picture, rather than on the path integral point of view.

The first obvious question is: How can such a form of coarse-graining be constructed in the field theory, such that it results in a highly efficient RG flow equation, as for instance (2.4) for the coarse-grained operator $t^\mu_\nu(\Lambda)$? In the second part of this work [4], we will explicitly construct this coarse-graining in a special limit which amounts to a generalisation of the Wilsonian RG flow. In the following, we will describe our coarse-graining procedure schematically, which will be more generally applicable in the strongly interacting large $N$ limit.\(^6\)

Let $X^A(x)$ be a set of functions and $\gamma^I$ a set of numerical constants, which can parametrise the expectation value of the microscopic operator $t^\mu_\nu(\infty)$ in a sector of states. By definition, these functions $X^A(x)$ are thus not operators by themselves, but rather they parametrise the expectation value of the microscopic operator $t^\mu_\nu(\infty)$, which can be depicted concisely in the form of the functional $\langle t^\mu_\nu(\infty) \rangle [X^A(x), \gamma^I]$, with the numerical constants $\gamma^I$’s determined by the field theory under consideration. The Heisenberg equations of motion for the operator $t^\mu_\nu(\infty)$ including the Ward identity (2.3), thus imply specific ordinary partial differential equations that determine the evolution of the functions $X^A(x)$ in space and time.\(^7\)

The first step in the coarse-graining is to define coarse-grained functions $X^A(\Lambda, x)$ in a way that generalises simply cutting off Fourier modes $k^2 > \Lambda^2$. Let $\tilde{X}^A(k)$ denote Fourier transforms of $X^A(x)$. Then $X^A(\Lambda, x)$ can be defined recursively via

$$X^A(\Lambda, x) = \int d^d k \, \Theta \left( 1 - \frac{k^2}{\Lambda^2} \right) F^A \left( X^A(\Lambda, x), \frac{k}{\Lambda_{\text{IR}}} \right) e^{ik \cdot x} \tilde{X}^A(k), \quad (2.8)$$

where $F^A \left( X^A(\Lambda, x), k/\Lambda_{\text{IR}} \right)$ can be thought of as an operator that smoothens out the effect of the sharp cut-off in momentum space, but being a functional of the coarse-grained variables $X^A(\Lambda, x)$ themselves. Furthermore,

$$F^A \left( X^A(\Lambda, x), \frac{k}{\Lambda_{\text{IR}}} \right) = 1 + O(\Lambda^{-1}) \quad \text{as} \quad \Lambda \to \infty \quad (2.9)$$

\(^6\)Our coarse-graining algorithm has some resemblances in spirit with the recent construction in [9].

\(^7\)Beyond the hydrodynamic limit, of course we need more than the Ward identity (2.3) to determine the evolution of $t^\mu_\nu(\infty)$. 

---

- 8 -
in order for \( X^A(\Lambda, x) \) to coincide with \( X^A(x) \) in this limit. As before, we can claim that for every state, there exists a (state-dependent) scale \( \Lambda_{IR} \), such that for \( \Lambda \gg \Lambda_{IR} \) the functional \( F^A \left( X^A(\Lambda, x), k/\Lambda_{IR} \right) \) can be expanded in powers of \( \Lambda^{-1} \), in a fashion similar to (2.4). This scale \( \Lambda_{IR} \) is the natural scale associated with the cut-off smoothening functional \( F^A \left( X^A(\Lambda, x), k/\Lambda_{IR} \right) \).

The purpose of obtaining these coarse-grained functions \( X^A(\Lambda, x) \) is that it can define the coarse-grained operator \( t^\mu_\nu(\Lambda) \) via the functional \( \langle t^\mu_\nu(\Lambda) \rangle \left[ X^A(\Lambda, x), \gamma^I(\Lambda) \right] \), which is the same as the one which defines \( t^\mu_\nu \) via \( \langle t^\mu_\nu \rangle \left[ X^A(x), \gamma^I \right] \) in terms of \( X^A(x) \) and \( \gamma^I \), but with \( \gamma^I \) replaced by new scale-dependent numerical constants \( \gamma^I(\Lambda) \). The reader may immediately note that this can only be possible in the strongly interacting large \( N \) limit, where (i) \( t^\mu_\nu(\Lambda) \) mixes only with multi-trace operators which are products of itself and its spacetime derivatives, and (ii) the expectation values of the multi-trace operators factorise. Otherwise we would have required more functions other than \( X^A(\Lambda, x) \) to parametrise the expectation values of other single-trace operators which could have mixed with \( t^\mu_\nu(\Lambda) \) under the RG flow, and also more numerical constants than \( \gamma^I(\Lambda) \) for taking into account that the expectation values of products of \( t^\mu_\nu(\Lambda) \) and its spacetime derivatives do not factorise.

To complete the definition of \( t^\mu_\nu(\Lambda) \), we need to give

- criteria for choosing the cut-off smoothening functional \( F^A \left( X^A(\Lambda, x), k/\Lambda_{IR} \right) \) in (2.8), and
- criteria for choosing the numerical constants \( \gamma^I(\Lambda) \).

In order to have a highly efficient RG flow, we need the following criteria:

- \( \langle t^\mu_\nu(\Lambda) \rangle \) will satisfy an RG flow equation which takes the form
  \[
  \frac{\partial \langle t^\mu_\nu(\Lambda) \rangle}{\partial \Lambda} = \text{a non-linear functional of} \ \langle t^\mu_\nu(\Lambda) \rangle \ \text{and its spacetime derivatives only,}
  \]
  after necessary redefinition of the scale and coordinates (a more precise statement will be given in Section 3.2),
- we can construct a metric \( g_{\mu\nu}(\Lambda) \left[ \langle t^\mu_\nu(\Lambda) \rangle \right] \) at each scale such that \( \langle t^\mu_\nu(\Lambda) \rangle \) satisfies
  \[
  \nabla_{(\Lambda)\mu} \langle t^\mu_\nu(\Lambda) \rangle = 0.
  \]

The first criterion is itself highly constraining, because any random scale-evolution of \( X^A(\Lambda, x) \) and \( \gamma^I(\Lambda) \) cannot be rewritten in the form of scale-evolution of \( \langle t^\mu_\nu(\Lambda) \rangle \) where \( X^A(\Lambda, x) \) and \( \gamma^I(\Lambda) \) do not appear explicitly. This is essential, because otherwise the scale evolution of the expectation value of the operator, namely \( \langle t^\mu_\nu(\Lambda) \rangle \), cannot be lifted to the scale evolution of the operator \( t^\mu_\nu(\Lambda) \) itself. Indeed, in the large \( N \) limit, where factorisation of expectation values applies, this lift becomes trivial when the first criterion is satisfied – the RG flow equation for the operator \( t^\mu_\nu(\Lambda) \) can simply be obtained by replacing \( \langle t^\mu_\nu(\Lambda) \rangle \) with \( t^\mu_\nu(\Lambda) \) in the RG flow equation for \( \langle t^\mu_\nu(\Lambda) \rangle \). However, to bring the RG flow equation in the form that depends on \( t^\mu_\nu(\Lambda) \) and \( \Lambda \) only explicitly, we may need to redefine the scale, coordinates, and the parametric functions \( X^A(\Lambda) \) and variables \( \gamma^I(\Lambda) \). A more precise statement of this criterion will be given in the beginning of Section 3.2. The
second criterion, as discussed before, is the principle defining feature of a highly efficient RG flow.

One of our main results to appear in the second part of this work \cite{4} is a precise implementation of a coarse-graining operation (which is schematically as described above) that results in a highly efficient RG flow for $t^\mu_\nu(\Lambda)$ satisfying our criteria, in the hydrodynamic limit, where the functions $X^A(\Lambda, x)$ are the scale-dependent hydrodynamic variables, namely the velocity $u^\mu(\Lambda, x)$ and the temperature $T(\Lambda, x)$, and the numerical constants $\gamma^I(\Lambda)$ are the scale-dependent transport coefficients.\(^8\) This coarse-graining will reproduce the fluid/gravity correspondence \cite{10–16}, which is the other name for holography in the hydrodynamic limit. Indeed, in this limit, we will be able to sum over the entire series expansion in powers of $\Lambda^{-1}$, but at a fixed order in derivatives. At each scale, the effective conservation equation (2.6) reduces to an effective hydrodynamic equation, which is to be understood as an asymptotic series involving the derivative expansion, parametrised by scale-dependent transport coefficients, and constructed in an effective scale-dependent metric $g_{\mu\nu}(\Lambda)$. In the hydrodynamic limit, this effective Ward identity itself gives the full spacetime dynamics of $t^\mu_\nu(\Lambda)$.

We will continue now to formulate here the other principles which the highly efficient RG flow should satisfy in order to reproduce dual gravity. In \cite{4}, as mentioned above, we will construct an explicit coarse-graining in the hydrodynamic limit which satisfy all these principles and indeed reproduces dual gravity.\(^9\)

2.2 Issues regarding the invertible map between the gravity equations and the RG flow

A fundamental conundrum appears in any attempt to think of classical gravity as a RG flow of a field theory living in one lower dimension. If we think of the radial coordinate $r$ as the inverse of the scale, then the evolution of the variables of gravity in the radial direction is second order,\(^10\) while the scale evolution in the RG flow of a field theory is first order.

Indeed, the highly efficient RG flow equation, as for instance (2.4), is a first order evolution in flat Minkowski space, which is the background (non-dynamical) metric in

\(^8\)At first, defining a RG flow in the hydrodynamic limit may sound like an oxymoron, because the hydrodynamic limit itself is the low energy limit of near-equilibrium dynamics. However, by the hydrodynamic limit, we do not mean the limit of simple Navier-Stokes hydrodynamics, but rather the limit of exact asymptotic hydrodynamic expansion which gives an asymptotic series for the space-time dependent behaviour of any observable during the late time stage of thermal equilibration. This asymptotic expansion in principle involves arbitrarily large number of transport coefficients, which give higher derivative corrections to Navier-Stokes dynamics. The highly efficient RG flow in this case, results in a different asymptotic series, which is similar in form, but can reproduce the same time dependent behaviour for any observable with a fewer number of parameters, namely transport coefficients, when defined at an appropriate scale.

\(^9\)In order to reproduce the full pure gravity sector, we require to go beyond hydrodynamics, as the full dynamics of pure gravity reproduces non-hydrodynamic aspects of non-equilibrium dynamics in the field theory (like thermalisation) via quasi-normal modes. We will also sketch the route for derivation of holography as a RG flow beyond hydrodynamics using a generalisation of Israel-Stewart theory \cite{17–19} for parametrising general non-equilibrium dynamics.

\(^10\)Higher derivative corrections to Einstein’s equations should be viewed in the perturbative framework as in string theory. These corrections thus do not change the second order evolution of Einstein’s equations to higher order.
which the field theory lives. Nevertheless, a dynamical (meaning scale-dependent) metric
\(g_{\mu\nu}(\Lambda)\) emerges as a functional of \(t^\mu_\nu(\Lambda)\), as in (2.5), in order for the effective Ward identity (2.6) to preserve the same form at all scales.

It is firstly not clear that a highly efficient RG flow, in which the Ward identities take the same effective form at all scales with appropriately redefined sources and background metric, necessarily implies a map to a dual theory of classical gravity. In fact, we will soon discuss the necessity of appending additional principles to ensure this is the case. In any event, it is intuitively rather obvious that mapping to a classical gravity theory is the most natural way to proceed — by identifying \(t^\mu_\nu(\Lambda)\) with the renormalised (in the sense to be discussed below) quasi-local stress tensor and \(g_{\mu\nu}(\Lambda)\) with the appropriately scaled induced metric on a hypersurface \(r = \Lambda^{-1}\) (with \(r\) being the radial coordinate of an arbitrary \((d+1)\)-dimensional coordinate system), and the effective Ward identity (2.6) with the so-called momentum constraints in an appropriate classical theory of gravity. Nevertheless, it is a deep question if there could be other ways to construct sensible highly efficient RG flows in which infrared behaviour can be captured by fewer effective parameters, but which cannot be mapped to any classical gravity theory with \((d + 1)\)-dimensional bulk diffeomorphism symmetry. We leave this investigation for the future.

The dynamical equations of (pure) gravity can be written as a set of coupled first order
equations in the following schematic form:

\[
t^\mu_\nu = f_1 \left( g^{\mu\rho} \frac{\partial g_{\rho\nu}}{\partial r} \right) + \text{gravitational counterterms},
\]

\[
\frac{\partial t^\mu_\nu}{\partial r} = f_2 \left( t^\mu_\nu, g_{\mu\nu} \right).
\]

(2.10)

The above form of the equations of classical gravity originates from the Arnowitt-Deser-Misner (ADM) formalism of general relativity [20] coupled with the notion of holographic renormalisation [21–25], which is required to obtain a finite \(t^\mu_\nu_{\infty}\) asymptotically via the addition of gravitational counterterms. We will discuss this more explicitly in Section 3.1. We have also suppressed the auxiliary ADM variables, namely the pseudo-lapse and the pseudo-shift functions (related to \(G_{rr}\) and \(G_{r\mu}\) components of the \((d+1)\)-dimensional metric), which are specified by the gauge fixing of diffeomorphism symmetry in the bulk.

Aside from these dynamical equations, we have additionally the momentum constraints, which as advertised are equivalent to (2.6), and also a Hamiltonian constraint which gives \(\text{Tr } t\) on each hypersurface \(r = \text{constant}\). Of course, these constraints are satisfied by the dynamical evolution once they are satisfied at any given value of \(r\). Asymptotically they reduce to the standard Ward identities \(\partial_{\mu} t^{\mu}_{\nu \infty} = 0\) and \(\text{Tr } t^{\infty} = 0\) respectively.

In order to pass from the equations of gravity (2.10) to the RG flow evolution (2.4) in the non-dynamical flat Minkowski metric, we need to eliminate \(g_{\mu\nu}\) and yet obtain a first order equation for \(t^\mu_\nu\) after identifying \(r\) with \(\Lambda^{-1}\). We succeed to do this in an expansion in \(\Lambda^{-1}\) in the Fefferman–Graham gauge in Section 3.2, and then in any arbitrary gauge which is infinitesimally far away from the Fefferman-Graham gauge in Section 4.1. If a highly efficient RG flow can be mapped to classical gravity equations in a specific gauge choice for fixing \((d + 1)\)-diffeomorphism symmetry, then there should exist another
highly efficient RG flow as well which maps to the same classical gravity equations in a different gauge, simply because for any gauge fixing we obtain momentum constraints which give the effective Ward identities for energy-momentum conservation in the form of (2.6) at any scale. This map will be shown to be is reversible here for any gauge fixing of \((d + 1)\)-diffeomorphism symmetry once we fix our choice of the background metric for the field theory, meaning that we will be be able to also go back from the RG flow equation (2.4) to the corresponding classical gravity equations (2.10).

Once again in the hydrodynamic limit, we will be able to do much better. In the second part of this work [4], we will construct the invertible map between the high efficient RG flow and gravity summing over all orders in \(\Lambda^{-1}\) at a fixed order in derivatives in the hydrodynamic limit. To accomplish this task we will gain assistance from the methods of [26].

A crucial question arises: Can this map between a highly efficient RG flow and a dual gravity theory fail? Also, even if such a map exists, which classical gravity theory does the highly efficient RG flow map to?

Based on the results of [26] and on this work, we will show in [4] that

- Although a highly efficient RG flow which can be mapped to a classical gravity theory can be constructed for any CFT in the strongly interacting large \(N\) limit using the schematic procedure described before, it hits a sort of Landau pole singularity at a finite scale \(\Lambda_{IR}\) unless the CFT is special and the dual classical gravity theory is uniquely chosen.

- In the latter case the RG flow ends at a fixed point typically at a finite (state-dependent) scale \(\Lambda_{IR}\) under appropriate infrared rescaling – in other words the end point can be characterised by a finite number of parameters.

In the hydrodynamic limit, the infrared fixed point corresponds to incompressible non-relativistic Navier-Stokes equations. In fact this infrared limit itself determines the ultraviolet (UV) data, namely the transport coefficients of the UV fluid, via the first order RG flow [26]. These transport coefficients are exactly the same as those predicted by the well-known fluid-gravity correspondence [10–16] by explicitly solving the gravity equations and requiring that the future horizon has no naked singularity. Thus remarkably the first order RG flow itself has complete information about properties of the emergent spacetime from dual classical gravity equations, including its regularity.

Even after gauge-fixing, apparently we have another ambiguity in the map between the RG flow and the classical gravity theory – this is related to the choice of gravitational counterterms in the definition of the renormalized \(t^\mu_\nu(r)\) on the gravitational side, which

\[\text{The observation that gravity leads to a natural Wilsonian RG flow of the dual fluid via the fluid-gravity correspondence has been first made in [27]. There have been many developments based on this observation, which is not directly related to the focus of our discussion here, namely reinterpreting holography as a special RG flow. In fact, to the best of our knowledge, the exact field-theoretic reconstruction of this geometric RG flow of the fluid obtained from gravity has not been attempted before. In the context of asymptotically AdS spacetime, some structural aspects of [26] have been anticipated partially in [28] (see also [29] for a closely related approach).} \]
should be identified with the coarse-grained $t^{\mu}_{\nu} (\Lambda)$ on the field theory side at the cut-off scale given by $r = \Lambda^{-1}$. A finite number of these counterterms can be fixed by requiring UV finiteness as observed in [21–25], however there are infinitely many which cannot be determined from this alone. Once again, the results of [26] strongly indicate that even these gravitational counterterms are uniquely fixed by requiring the existence of a well defined infrared fixed point that can be characterised by finite number of parameters.

In Section 3.1, we will further show that different choices of gravitational counterterms, which assumes different choices of boundary conditions at the cut-off scale, are all equivalent on-shell, and they result in the same RG flow. In Section 3.1, we will also be able to establish a unique relation between $g_{\mu\nu} (\Lambda)$ and $t^{\mu}_{\nu} (\Lambda)$, and the ADM variables of gravity, for any arbitrary choice of gauge fixing of diffeomorphism symmetry in the classical gravity equations.

The upshot is that:

- The map between highly efficient RG flow and a dual classical gravity theory can fail (in the sense of not having a well defined end point characterised by finite number of parameters) unless the corresponding field theory is special (as for example it has to have right transport coefficients).

- When the map exists, it is unique for any choice of gauge fixing of the bulk diffeomorphism symmetry.

The possibility is still open that we can have well defined highly efficient RG flows in strongly interacting large $N$ field theories, which do not map to any classical gravity theory. We wish to examine this issue further in the future.

2.3 On the emergence of bulk diffeomorphism symmetry

As mentioned in the Introduction, when the RG flow can be mapped to a classical gravity theory, it inherits an additional symmetry, which is related to the ultraviolet Weyl symmetry, and originates from the bulk diffeomorphism symmetry (or rather how it is gauge–fixed) in the dual classical gravity equations. This could be the key for understanding how bulk diffeomorphism symmetry emerges in the holographic correspondence.

In order to see this symmetry, we first need to address an ambiguity related to the construction of $g_{\mu\nu} (\Lambda)$ in the highly efficient RG flow, namely whether it is unique.

Note on the gravitational side, we construct $t^{\mu}_{\nu} (\Lambda)$ from $g_{\mu\nu} (\Lambda)$, as evident from the first equation in (2.10), and it is unique only up to gravitational counterterms. As mentioned in the previous subsection, these ambiguities on the gravity side are removed by requiring UV finiteness and a sensible IR fixed point (after possibly necessary rescalings).

On the field theory side, as evident from (2.5), we do the reverse: we construct $g_{\mu\nu} (\Lambda)$ from $t^{\mu}_{\nu} (\Lambda)$, whose scale evolution takes place in the non-dynamical flat Minkowski metric. The principle for its construction is that $t^{\mu}_{\nu} (\Lambda)$ should satisfy the effective Ward identity, which takes the scale–invariant form (2.6). If the highly efficient RG flow can be mapped

12This ambiguity however does not affect $t^{\mu}_{\nu} (\infty)$ seriously, leaving us with a well defined dictionary for stating the traditional holographic correspondence.
to classical gravity, this $g_{\mu\nu}(\Lambda)$ is unique only up to a multiplicative function $f(\Lambda_{IR}/\Lambda)$, because

$$\bar{g}_{\mu\nu}(\Lambda) = f\left(\frac{\Lambda_{IR}}{\Lambda}\right) \cdot g_{\mu\nu}(\Lambda) \text{ and } \Lambda_{IR} = \text{constant}, \text{ then } \bar{\nabla}_{(\Lambda)} = \nabla_{(\Lambda)}. \quad (2.11)$$

This is not related to the issue of counterterms on the gravity side.

The above ambiguity is the only one that arises if the highly efficient RG flow can be mapped to classical gravity equations as we will show in Section 3.1. This can be removed by requiring that as $\Lambda \to \infty$, $g_{\mu\nu}(\Lambda)$ must reduce to the actual non-dynamical metric in which the field theory is living, and that it is a state-independent functional of $t^{\mu\nu}(\Lambda)$. Indeed in a CFT, there can be no scale which can appear in all states. To put it differently, then $g_{\mu\nu}(\Lambda)$ should be an explicit functional of $\Lambda$ and $t^{\mu\nu}(\Lambda)$ only.\(^{13}\) This $g_{\mu\nu}(\Lambda)$ with no explicit dependence on any additional infrared scale, as in (2.5), is unique for any highly efficient RG flow that maps to classical gravity equations.

The additional symmetry in the RG flow is related to the residual gauge symmetry of the corresponding classical gravity relations, which survives the gauge fixing, and which reduces to Weyl transformations asymptotically, thus being a specific kind of large diffeomorphisms\(^{14}\). This residual gauge symmetry is well known for the Fefferman-Graham gauge, and we will also construct it explicitly in any gauge fixing which is infinitesimally far away from the Fefferman-Graham gauge in Section 4.3. In case of the Fefferman-Graham gauge, these are known as Penrose-Brown-Henneaux (PBH) transformations \([30, 31]\) in the literature. They have also been used in the holographic correspondence as a method for obtaining the anomalous Weyl transformation for $t^{\mu\nu} \propto [25, 32, 33]$, and furthermore in the context of the holographic renormalisation group in relation to the study of Osborn-like Weyl consistency conditions \([34, 35]\).

Given that there is a unique $g_{\mu\nu}(\Lambda)$ for any highly efficient RG flow that maps to classical gravity equations, we are now in a position to state the symmetry transformation for the RG flow for any gauge fixing of bulk diffeomorphisms in the corresponding gravity equations. We state this explicitly for the Fefferman-Graham gauge below, leaving the statement for arbitrary gauge fixing to Section 4.3, where we will also see that the symmetry transformation itself has complete information about the gauge fixing of the bulk diffeomorphism symmetry.

In case of the Fefferman-Graham gauge, the (infinitesimal form of) PBH transformations amounts to a correlated combination of a local scale transformation and of a $d$—dimensional diffeomorphism, determined parametrically by a single function $\delta \sigma(x)$ as

\(^{13}\)This automatically implies a similar requirement for the RG flow equation for $t^{\mu\nu}(\Lambda)$, as satisfied by Eq. (2.4) for instance. Then it also follows that in the ultraviolet expansion of $g_{\mu\nu}(\Lambda)$, as in Eq. (2.5), the coefficients of $\Lambda^{-n}$ are local polynomials of $t^{\mu\nu}(\Lambda)$ and its spacetime derivatives only, \textit{and with the same dimensionality as $\Lambda^m$}. Allowing for another scale $\Lambda_{IR}$ to appear explicitly makes room for multi-trace operators constructed from $t^{\mu\nu}(\Lambda)$ and with \textit{arbitrarily large dimensions appearing in the coefficient of $\Lambda^{-n}$}.

\(^{14}\)This means these diffeomorphisms are non-trivial even at the boundary of the $(d + 1)$-dimensional spacetime and change boundary data which corresponds to UV observables in the field theory.
follows:
\[ \Lambda = \tilde{\Lambda} \cdot (1 - \delta \sigma(\tilde{x})) , \]
\[ x^\mu = \tilde{x}^\mu + \chi^\mu(\tilde{x}, \tilde{\Lambda}) , \quad \text{with} \quad \chi^\mu(\tilde{x}, \tilde{\Lambda}) = \int_{\tilde{\Lambda}}^{\infty} \frac{d\tilde{\Lambda}}{\tilde{\Lambda}^3} \cdot g^{\mu\nu}(\tilde{\Lambda}, \tilde{x}) \cdot \frac{\partial}{\partial \tilde{x}^\nu} \delta \sigma(\tilde{x}) , \quad (2.12) \]

with \( g^{\mu\nu} \) being the inverse of \( g_{\mu\nu} \). The transformation of \( g_{\mu\nu} \) is then given by
\[ \tilde{g}_{\mu\nu}(\tilde{\Lambda}, \tilde{x}) = g_{\mu\nu}(\tilde{\Lambda}, \tilde{x}) - \tilde{\Lambda} \cdot \delta \sigma(\tilde{x}) \cdot \frac{\partial g_{\mu\nu}(\tilde{\Lambda}, \tilde{x})}{\partial \tilde{\Lambda}} - 2 \cdot \delta \sigma(\tilde{x}) \cdot g_{\mu\nu}(\tilde{\Lambda}, \tilde{x}) + \mathcal{L}_\chi g_{\mu\nu}(\tilde{\Lambda}, \tilde{x}) , \quad (2.13) \]

with \( \mathcal{L}_\chi \) being the Lie-derivative along \( \chi^\mu \). At \( \Lambda = \infty \), this transformation reduces to a Weyl transformation by the factor \( \delta \sigma \). \footnote{The corresponding large \((d+1)\)-dimensional diffeomorphism on the gravity side can be obtained simply by replacing \( \Lambda \) by the radial coordinate \( r \) using \( r = \Lambda^{-1} \).}

The issue is that we need to state the symmetry transformation of the highly efficient RG flow in terms of the transformation of \( t^{\mu\nu}(\Lambda) \) alone – indeed the equation for its evolution, as in \((2.4)\), is in the fixed non-dynamical background metric of the field theory and \( g_{\mu\nu}(\Lambda) \) does not appear in this equation even implicitly. Nevertheless, when the highly efficient RG flow can be mapped to classical gravity equations, the relation between \( g_{\mu\nu}(\Lambda) \) and \( t^{\mu\nu}(\Lambda) \) is invertible, meaning that there exists a \textit{unique} \( t^{\mu\nu}(\Lambda) \) for any \( g_{\mu\nu}(\Lambda) \), as we will show in Section 3.1. This implies that the transformation \((2.13)\) for \( g_{\mu\nu}(\Lambda) \) implies a \textit{unique} transformation for \( t^{\mu\nu}(\Lambda) \), which asymptotically reduces to a Weyl transformation for \( t^{\mu\nu}_\infty \). The latter will be constructed explicitly in Section 4.1. It will be shown in Section 4.2 that the transformation of \( t^{\mu\nu}(\Lambda) \), which follows from the transformation of \( g_{\mu\nu}(\Lambda) \), is a symmetry of the highly efficient RG flow equation, as for instance \((2.4)\), when the latter is generalised for an arbitrary conformally flat space background. Once again, this symmetry also exists for the highly efficient RG flow equation corresponding to other kinds of gauge fixings (which are infinitesimally far away from Fefferman–Graham gauge), and furthermore the gauge fixing in the bulk can be extracted from the symmetry transformation (see Section 4.3).

In case of the Fefferman-Graham gauge, when the symmetry of the corresponding highly efficient RG flow is stated in terms of \( g_{\mu\nu}(\Lambda) \) instead of \( t^{\mu\nu}(\Lambda) \), as in \((2.13)\), it is the \textit{same for any classical gravity theory}. Thus it is a universal feature for any highly efficient RG flow which can be mapped to classical gravity.

We suspect that the presence of this symmetry is not only a necessary, but also a sufficient condition (aside from the requirement of good infrared behaviour which in turn requires right ultraviolet data) for the highly efficient RG flow to be equivalent to appropriate classical gravity equations. Our expectation is supported by the observation that the extra symmetry is needed to generate the Hamiltonian constraint of the dual gravity equations in the ADM formalism. Thus together with the first principle for preservation of the form of Ward identities related to local conservation of symmetries (which is connected to the momentum constraints), it suggestively implies emergence of \((d+1)\)-dimensional diffeomorphism invariance of the dual classical gravity theory, and the content of the latter

\footnote{This is evident from the ultraviolet expansion of \( g_{\mu\nu}(\Lambda) \) as in \((2.5)\).}
is then determined by requiring that the infrared end point can be characterised by finite number of parameters. We will discuss this more in Section 5.

However, we have not been able to prove the sufficiency of these criteria to imply holographic correspondence here. One of the main obstacles for this proof is that it is not clear whether there is a unique $g_{\mu\nu}(\Lambda)$ in a highly efficient RG flow if we do not assume that the RG flow maps to appropriate classical gravity equations, even after imposing that (i) $g_{\mu\nu}(\Lambda)$ reduces to the background metric of the CFT at $\Lambda = \infty$, and (ii) it depends only on $\Lambda$ and $t^\mu(x)$ explicitly. Once this issue is understood, we can study whether the lifted Weyl symmetry of the highly efficient RG flow mentioned above structurally constrains it uniquely. Nevertheless, this lifted Weyl symmetry along with the requirement for sensible infrared end point behaviour could indeed give us all the necessary and sufficient criteria to reconstruct the holographic correspondence as a highly efficient RG flow. In particular, the lifted Weyl symmetry could be sufficient to imply the emergence of $(d+1)$-dimensional diffeomorphism symmetry in the dual gravity theory from a highly efficient RG flow.

3 Gravity equations from a highly efficient RG flow in an ultraviolet expansion

3.1 Geometric variables and the RG flow

The RG flow in a field theory in $d$ dimensions can be geometrically represented very efficiently via a hypersurface foliation $\Sigma_r$ and a congruence of curves $C_x$ in a $(d+1)$-dimensional spacetime.

The boundary of this higher dimensional spacetime represents the UV fixed point of the field theory. Let us foliate this spacetime by non-intersecting spacelike hypersurfaces $\Sigma_r$, which are labeled uniquely by the radial coordinate $r$ (i.e. there exists a function $r$ such that each hypersurface in the foliation is given by $r = constant$, and the normal to each hypersurface is spacelike). The radial coordinate $r$ can be identified with the scale $\Lambda$ using $r = \Lambda^{-1}$, without any loss of generality – the boundary then corresponds to $r = 0$, where the UV data should be specified.

Let us now consider a congruence of curves $C_x$, such that each curve in $C_x$ intersects each hypersurface in $\Sigma_r$ only once, and furthermore these curves do not intersect with each other. In such a case, we can label each curve in $C_x$ uniquely by a $d$-dimensional spacetime coordinate $x$. In a QFT, the local operators are also labeled by the $d$-dimensional spacetime coordinates $x$. The RG flow of the QFT local operators then occurs geometrically along $C_x$, each point in the curve representing a scale $\Lambda$, corresponding to the intersecting hypersurface in the foliation $\Sigma_r$. For this reason, we may call $x$ the field-theory coordinates.

ADM variables: If the RG flow maps to a classical theory of gravity as sketched in Section 2.2, then the above pictorial depiction of the RG flow via $(\Sigma_r, C_x)$ gains a real significance. As stated above, there exists a natural $(d+1)$-dimensional coordinate system associated with $(\Sigma_r, C_x)$ which is given by the radial coordinate $r$ and the field-theory coordinates $x$. In this coordinate system, the $(d+1)$- dimensional metric assumes the
general ADM form [20]:
\[
\begin{align*}
\text{d}s^2 &= \alpha^2(r, x)\text{d}r^2 + \gamma_{\mu\nu}(r, x) \left( \text{d}x^\mu + \beta^\mu(r, x)\text{d}r \right) \left( \text{d}x^\nu + \beta^\nu(r, x)\text{d}r \right).
\end{align*}
\]

(3.1)

It can be shown that $\gamma_{\mu\nu}(r, x)$ is the induced metric on the hypersurface $\Sigma_r$ and constitutes the dynamical variable in general relativity. The other variables $\alpha(r, x)$ and $\beta^\mu(r, x)$ are not dynamical, since their radial derivatives do not appear in the equations of gravity, and we call them the pseudo-lapse function and the pseudo-shift vector respectively.\(^{17}\) They play a role similar to Lagrange multipliers in enforcing a vector-like momentum constraint and a scalar-like Hamiltonian constraint for the data on each hypersurface $\Sigma_r$. Indeed, a choice of $(d+1)$–dimensional coordinate system is related to a specific method of gauge fixing of the diffeomorphism symmetry, and hence specific conditions that determine $\alpha(r, x)$ and $\beta^\mu(r, x)$.\(^{18}\)

Furthermore, as is well known, in order to describe the RG flow in a CFT, we need asymptotically anti-de Sitter (AdS) spacetimes corresponding to the states in the CFT. The asymptotic isometries of the spacetime maps to the conformal group $SO(d, 2)$ of the $d$–dimensional CFT, and is needed for reproducing the conformal Ward identities of the CFT via equations of gravity [21–23].

An example of the choice of gauge-fixing of bulk diffeomorphisms is the Fefferman-Graham gauge, in which any $(d+1)$–dimensional asymptotically AdS spacetime metric assumes the form
\[
\begin{align*}
\text{d}s^2 &= \frac{l^2}{r^2} \left( \text{d}r^2 + g_{\mu\nu}(r, x)\text{d}x^\mu\text{d}x^\nu \right),
\end{align*}
\]

meaning that
\[
\begin{align*}
\alpha &= \frac{l}{r}, \quad \beta^\mu = 0, \quad \text{and} \quad \gamma_{\mu\nu} = \frac{l^2}{r^2} \cdot g_{\mu\nu}.
\end{align*}
\]

(3.3)

The constant $l$ is the radius of the asymptotic AdS region of the spacetime. The corresponding coordinates and $(\Sigma_r, C_x)$ are well defined for sufficiently small $r$ in any asymptotically AdS spacetime, meaning for sufficiently large $\Lambda$ in the corresponding highly efficient RG flow. This is therefore very suitable for studying the ultraviolet expansion of the RG flow, as we will see soon.

The momentum constraints associated with each hypersurface in $\Sigma_r$ imply that there should be an appropriate quasi-local energy-momentum tensor $T_{\mu\nu}^{\text{ql}}$ [36] which is a functional of $\alpha$, $\beta^\mu$, $\gamma_{\mu\nu}$ and $\partial \gamma_{\mu\nu}/\partial r$ and is conserved, i.e. it satisfies,
\[
\nabla_{(\gamma)\mu} T_{\mu\nu}^{\text{ql}} = 0.
\]

(3.4)

\(^{17}\)For timelike hypersurface foliation, $\alpha$ is called the lapse function, and $\beta^\mu$ is called the shift vector. Strictly speaking, the statement that $\alpha$ and $\beta^\mu$ are auxiliary variables is true in Einstein’s gravity only. In the presence of higher derivative corrections to Einstein’s gravity, this statement is true only if the latter are treated perturbatively.

\(^{18}\)Strictly speaking, these statements are true for a tubelike region in the bulk spacetime, ending on a sufficiently small patch on the boundary $r = 0$. It should be interesting to investigate, how the absence of any global spacelike hypersurface foliation reflects structurally in the RG flow equation, as for instance eq. (2.4).
In case of Einstein’s equations, this $T_{\mu}^{\nu}q^{l}$ is the well-known Brown-York tensor, as given by [36]

$$T_{\mu}^{\nu}q^{l} = \frac{1}{8\pi G_N} \gamma^{\mu\rho} (K_{\rho\nu} - K_{\gamma\rho\nu}), \quad \text{(3.5)}$$

where $G_N$ is the $(d+1)$-dimensional Newton’s gravitational constant, $K_{\mu\nu}$ is the extrinsic curvature of the hypersurface defined via

$$K_{\mu\nu} = -\frac{1}{2\alpha} \left( \frac{\partial \gamma_{\mu\nu}}{\partial r} - \nabla_{(\gamma)} \beta_{\mu} - \nabla_{(\gamma)} \beta_{\nu} \right), \quad \text{(3.6)}$$

with $\beta_{\rho} = \gamma_{\rho\mu} \beta^{\mu}$, and $K = K_{\mu\nu} \gamma^{\mu\nu}$. As $\alpha$ and $\beta^{\mu}$ are non-dynamical, and are determined by the gauge-fixing conditions (associated with the coordinate system $r, x^{\mu}$ and corresponding $(\Sigma_{r}, C_{x})$), $T_{\mu}^{\nu}q^{l}$ should be regarded as a functional of the induced metric $\gamma_{\mu\nu}$.

It is intuitively obvious that in order to map classical gravity equations to a highly efficient RG flow, we need to identify $\gamma_{\mu\nu}$ with $g_{\mu\nu}$ and $T_{\mu}^{\nu}q^{l}$ with $t_{\mu}^{\nu}$ at $r = \Lambda^{-1}$. This turns out to be a bit naive, because $\gamma_{\mu\nu}$ and $T_{\mu}^{\nu}q^{l}$ both blow up in the UV, meaning at $r = 0$. This blow up indeed has an immense physical significance, allowing us to map Weyl transformations of the field theory data to specific diffeomorphisms in the corresponding $(d + 1)$-dimensional spacetime geometry. We will discuss this aspect in more detail in Section 4.2 and Section 4.3.

The (dis)appearance of the parameter $l$: The parameter $l$, which is the radius of the asymptotic AdS region of the $(d + 1)$-dimensional spacetime, by itself has no meaning in the CFT. It is basically the unit of measurement for mass/length/time on the gravitational side – dimensionless quantities in these units correspond to parameters in the CFT. The asymptotic radius $l$ is determined by the (negative) cosmological constant in the $(d + 1)$-dimensional gravity.

The other parameters of the classical gravity theory are of two kinds:

- the overall factor $1/(16\pi G_N)$ which multiplies the $(d+1)$-dimensional gravity action, and
- the relative coefficients giving higher derivative corrections to Einstein’s gravity, involving the $(d + 1)$-dimensional Riemann tensor and its covariant derivatives – without loss of generality it can be assumed these relative coefficients should take the form of numerical constant times $a_{(i)}^{(n)}$, for a suitable $n$ and with $a_{(i)}^{(n)}$ being a parameter with mass dimension $-2$.

Only in the large $N$ limit, where the factorisation (2.1) applies, the highly efficient RG flow equation, like (2.4), can be thought of as a classical equation that can be mapped to classical gravity. Thus the factor $N^2$ should be related to a parameter that controls quantum corrections on the gravity side, which is $1/(16\pi G_N)$ – the corresponding dimensionless

\[\text{[19]}\]
parameter is \( l^{d-1}/(16\pi G_N) \). Therefore, we expect\(^{20}\)

\[
N^2 \approx \frac{l^{d-1}}{16\pi G_N}.
\] (3.8)

Furthermore, we expect that only in the strongly interacting limit, where the algebra of operators is generated by a finite number of single-trace operators, the RG flow can be mapped to a two-derivative gravity theory. In the special case of pure gravity, this is precisely Einstein’s equation with a negative cosmological constant. In this limit, the map between RG flow and classical gravity is invertible, as expected. The higher derivative corrections to Einstein’s gravity still render an invertible map to the RG flow, albeit only if they are treated perturbatively, as we will see in the next subsection. Conversely, the map is non-invertible as expected, if these corrections are not treated perturbatively.\(^{21}\) We thus expect that the dimensionless parameter \( \alpha'(i)/l^2 \) giving corrections to Einstein’s gravity is related to a function of the (dimensionless) coupling constants \( g_{(A)} \) in the CFT, whose value is small in the strongly interacting limit. In other words, we expect that

\[
\frac{\alpha'(i)}{l^2} \approx f(g_{(A)}), \quad \text{such that } f \to 0 \text{ as } g_{(A)} \to \infty.
\] (3.9)

We have assumed here that the CFT has a weakly coupled quasi-particle like description when all \( g_{(A)} \)'s are small.\(^{22}\)

We want to establish a relation between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \), and between \( T_{\mu\nu}^{ql} \) and \( t_{\mu\nu} \), such that \( l \) does not appear explicitly in the RG flow equation, but only in the dimensionless combinations \( l^{d-1}/(16\pi G_N) \) and \( \alpha'(i)/l^2 \), which as discussed are related directly to parameters in the CFT.

**Relationship between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \):** Let us begin with establishing the relation between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \). Firstly, in order that the momentum constraint of gravity, namely (3.4) maps to the criterion (2.6) of the highly efficient RG flow, we require

\[
\nabla_{(\gamma)} = \nabla_{(g)},
\] (3.11)

\(^{20}\)For example, in \( \mathcal{N} = 4 \) SYM holography, the precise relation between \( N \) and the *five-dimensional* Newton’s constant is \([1]\)

\[
\frac{N^2}{8\pi^2} \approx \frac{l^3}{16\pi G_N},
\] (3.7)

where \( N \) is the rank of the \( SU(N) \) gauge group.

\(^{21}\)On the CFT side, this is reflected by the fact that we need to involve more single-trace operators (and hence also multi-trace operators formed out of them) in the RG flow. We can restore the invertibility on the gravity side by including more fields in gravity, which will be dual to these additional single-trace operators. In the large \( N \) limit, due to factorisation of expectation values, the RG flow will still map to classical gravity with these additional fields.

\(^{22}\)In \( \mathcal{N} = 4 \) SYM, there is only one coupling constant, which is the ’t Hooft coupling \( \lambda \). This is related holographically to the string tension \( \alpha' \) that gives higher derivative corrections to Einstein’s gravity via \([1]\)

\[
\frac{\alpha'}{l^2} = \sqrt{\frac{1}{\lambda}}.
\] (3.10)
which then implies

\[ g_{\mu\nu} = f \left( \frac{r}{l} \right) \cdot \gamma_{\mu\nu} \]  

(3.12)

at \( r = \Lambda^{-1} \), if the relationship between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \) is to be state-independent. As mentioned in Section 2.3, state-independence is a guiding principle behind the construction of \( g_{\mu\nu}(\Lambda) \) from the highly efficient RG flow equation. This equation, as for instance in the case of eq. (2.4), is itself state-independent, meaning that it depends only on \( \Lambda \) and \( t^\mu \nu \) explicitly, and not on any other intrinsic scale or variable. Thus the state-independence of the relation between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \) should also be imposed.

The factor \( f(r/l) \) cannot also depend on the auxiliary ADM variables \( \alpha \) and \( \beta^\mu \), because the latter, generally speaking, may depend both on \( r \) and \( x^\mu \) depending on the gauge fixing. Thus in particular \( f(r/l) \) does not not depend on the choice of gauge-fixing of \((d+1)\)-diffeomorphisms.\(^{23}\)

In any asymptotically \( AdS \) spacetime, \( \gamma_{\mu\nu} \) diverges as \( l^2/r^2 \) near \( r = 0 \). Furthermore, there is no other kind of subleading divergence. In order that \( g_{\mu\nu} \) is finite in the UV and does not depend explicitly on the parameter \( l \), the proportionality factor \( f(r/l) \) in (3.12) must be \( r^2/l^2 \), meaning

\[ g_{\mu\nu} = \frac{r^2}{l^2} \cdot \gamma_{\mu\nu} \]  

(3.13)

at \( r = \Lambda^{-1} \). To see that \( g_{\mu\nu} \) as obtained via the above relation, does not depend on \( l \) explicitly, we need to study gravitational dynamics, which we will discuss in the next subsection.

We recall from the discussion in Section 2.1 that at \( \Lambda = \infty \) (or equivalently at \( r = 0 \)), we require that \( g_{\mu\nu} \) in the highly efficient RG flow should coincide with the background metric on which the CFT lives (which we choose to be \( \eta_{\mu\nu} \) for this section). Then the relation (3.13) also implies the traditional rule of holographic correspondence, namely that

\[ g^{(b)}_{\mu\nu} = \lim_{r \to 0} \frac{r^2}{l^2} \cdot \gamma_{\mu\nu} \]  

(3.14)

coincides with the background metric on which the CFT lives. In the literature, \( g^{(b)}_{\mu\nu} \) is often called the boundary metric of the \((d+1)\)-dimensional spacetime.

In the Fefferman-Graham gauge (3.3), the relation (3.13) reduces to the simple form

\[ g_{\mu\nu}(\Lambda) = g_{\mu\nu}(r = \Lambda^{-1}). \]  

(3.15)

Notably, the \((d+1)\)-dimensional spacetime corresponding to the vacuum of the CFT in flat Minkowski space is pure \( AdS_{d+1} \). In the Fefferman-Graham coordinates, \( g_{\mu\nu} = \eta_{\mu\nu} \) for all values of \( r \) in pure \( AdS_{d+1} \). This implies that \( g_{\mu\nu} \) remains \( \eta_{\mu\nu} \) at all scales in the RG flow. This is (almost) expected because \( \langle t^\mu \nu \rangle \) vanishes in the vacuum, and so should \( \langle t^\mu \nu(\Lambda) \rangle \). However, away from the Fefferman-Graham gauge, as we will see in the Section

\(^{23}\)Once again, in this argument we assume that the relation between \( \gamma_{\mu\nu} \) and \( g_{\mu\nu} \) can be stated in terms of ADM variables alone, so that it takes the same form in all gauges. This also follows from the requirement that state-dependent variables (that characterise the corresponding gravity solution), do not enter explicitly in this relation.
4.3, $g_{\mu\nu}$ typically does evolve even in the vacuum, but only via a Weyl transformation (under which the vacuum remains invariant) and a $d-$dimensional diffeomorphism. This is the only possible form of evolution of $g_{\mu\nu}$ in the vacuum under a highly efficient RG flow.

**Relationship between $T^\mu_{\nu,\text{ql}}$ and $t^\mu_{\nu}$**: The next step is to establish the relation between $T^\mu_{\nu,\text{ql}}$ on the gravity side with $t^\mu_{\nu}$ of the highly efficient RG flow at $r = \Lambda^{-1}$. Once again, in order to map the momentum constraint of gravity (3.4) to the defining criterion (2.6) of the highly efficient RG flow (given that the relation between $\gamma_{\mu\nu}$ and $g_{\mu\nu}$ is as in (3.13)), the relationship between $T^\mu_{\nu,\text{ql}}$ and $t^\mu_{\nu}$ should be of the form

$$t^\mu_{\nu} = h \left( \frac{r}{l} \right) \cdot \left( T^\mu_{\nu,\text{ql}} + T^\mu_{\nu,\text{ct}} \right),$$

where $T^\mu_{\nu,\text{ct}}$ denotes general counterterms which are *conserved identically*, meaning that they satisfy $\nabla_{(\gamma)}^\mu T^\mu_{\nu,\text{ct}} = 0$ when considered individually, via Bianchi-like identities. The latter thus assumes the general form:

$$T^\mu_{\nu,\text{ct}} = -\frac{1}{8\pi G_N} \left[ C_{(0)} \left( \frac{r}{l} \right) \cdot \frac{1}{l} \cdot \delta^\mu_{\nu} + C_{(2)} \left( \frac{r}{l} \right) \cdot \frac{C_{(2)}}{l^2} \right] \cdot \left( R^\mu_{\nu,\gamma} - \frac{1}{2} R[\gamma] \delta^\mu_{\nu} \right) + \cdots.$$

We note that each term in $T^\mu_{\nu,\text{ct}}$ above, namely the one proportional to $\delta^\mu_{\nu}$ and the one proportional to the Einstein tensor constructed out of $\gamma_{\mu\nu}$, are the unique terms up to two derivatives, which are conserved identically. In order to be conserved identically (in other words to be conserved even off-shell), we require the terms in $T^\mu_{\nu,\text{ct}}$ to be functionals of $\gamma_{\mu\nu}$ only, up to proportionality factors that are functions of $r$ only (i.e. independent of $x^\mu$). Therefore, these terms cannot depend on the auxiliary ADM variables, namely $\alpha$ and $\beta^\mu$. Up to any given order in derivatives, there are only a finite number of such terms, and there is a well-known procedure to construct them.\(^{24}\)

This argument for construction of counterterms does not rely on any choice of boundary condition in gravity at the cut-off scale (hypersurface) as in the case of traditional holographic renormalisation [21–25] (which assumes the Dirichlet boundary condition at any cut-off scale). The basic feature that only terms which are conserved identically can appear in the counterterms is simply a consequence of the fact that $T^\mu_{\nu,\text{ql}}$ is the only non-trivial tensor which is conserved via classical gravity equations of motion in all solutions of gravity. However, $T^\mu_{\nu,\text{ql}}$ is not conserved identically, implying that it is the only non-trivial tensor that is conserved only on-shell in an arbitrary solution. Thus assuming that the relationship between $T^\mu_{\nu,\text{ql}}$ and $t^\mu_{\nu}$ is independent of the state in the CFT, no tensor

\(^{24}\)This procedure is to take all possible diffeomorphism invariant Lagrangian densities (like $R$) up to fixed order in derivatives, and then construct the corresponding equations of motion (like the Einstein tensor for $R$). The diffeomorphism invariance of the Lagrangian implies the Bianchi identities.
other than $T^\mu\nu,^{al}$ which is not conserved identically can appear in (3.16). Therefore, the counterterms in $T^\mu\nu,^{ct}$ have to be conserved identically.\footnote{This argument implies that even if we make a specific choice of a specific boundary condition at the cut-off to construct a counterterm tensor $T^\mu\nu,^{ct}$ which respects the corresponding variational principle, it has to take the form (3.17) on-shell.}

We can determine the $r/l$ dependence in the various coefficient functions $h(r/l)$ or $C(i)(r/l, \alpha'(i)/l^2)$ in (3.16) and (3.17) by requiring that $t^\mu\nu$ does not depend on $l$ explicitly. Furthermore, it is also assumed that these functions are independent of the gauge fixing of bulk diffeomorphisms, for exactly the same reasons for which we have assumed that $f(r/l)$ in (3.12) must be gauge-independent while establishing relationship between $\gamma_{\mu\nu}$ and $g_{\mu\nu}$ (recall footnote 23). All these criteria imply that

$$h \left( \frac{r}{l} \right) = \left( \frac{l}{r} \right)^d, \quad C(i) \left( \frac{r}{l}, \frac{\alpha'(i)}{l^2} \right) = C(i) \left( \frac{\alpha'(i)}{l^2} \right) \text{ and thus independent of } \frac{r}{l}. \quad (3.18)$$

A finite number of these constants $C(i)(\alpha'(i)/l^2)$ can be determined by requiring that $t^\mu\nu(\Lambda = \infty) = \langle t^\mu\nu \rangle$ is finite \cite{21,22,23,24}. We will state another principle for determining all these constants below.

Since the form of $t^\mu\nu$ as given by (3.16) and (3.17) is gauge-independent, we can check whether the explicit form satisfies all the discussed criteria conveniently in Fefferman-Graham gauge. In the Section 4.1, we will then see explicitly that the above features are also preserved under infinitesimal gauge transformations. Let us examine the case of Einstein’s equations first. Here, the counterterm coefficients $C(i)$ should be just pure universal numbers (independent of the underlying CFT, which is infinitely strongly interacting).

In the Fefferman-Graham gauge (3.3), we obtain

$$K_{\mu\nu} = \frac{l}{r} \left( g_{\mu\nu} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial r} \right). \quad (3.19)$$

Let us also define $z^\mu\nu$ as

$$z^\mu\nu = g^{\mu\rho} \frac{\partial g_{\rho\nu}}{\partial r}. \quad (3.20)$$

We observe that (3.13) or equivalently (3.15) implies

$$R^\mu\nu\rho\sigma[\gamma] = R^\mu\nu\rho\sigma[g].$$

Using the above observation, we can show that (3.5), (3.16), (3.17), (3.18) and (3.19) imply that in the case of Einstein’s gravity

$$t^\mu\nu = \frac{l^{d-1}}{16\pi G_N} \left[ \frac{1}{r^{d-1}} \cdot \left( z^\mu\nu - \langle \text{Tr} \eta \rangle \delta^\mu\nu \right) + 2 \cdot \frac{1}{r^d} \cdot \left( d - 1 - C(0) \right) \cdot \delta^\mu\nu - \right.$$

$$\left. - 2 \cdot \frac{1}{r^{d-2}} \cdot C(2) \cdot \left( R^\mu\nu[g] - \frac{1}{2} R[g] \delta^\mu\nu \right) + \cdots \right], \quad (3.21)$$

at $r = \Lambda^{-1}$. As mentioned earlier (and as we will also prove in the next subsection), $g_{\mu\nu}$ does not depend on $l$ explicitly (or even implicitly in the case of Einstein’s gravity). Then it
follows from the above that $t^{\mu \nu}$ indeed depends on $l$ only through the dimensionless combination $t^{d-1}/(16\pi G_N)$ in Einstein’s gravity, as claimed. Furthermore, the terms not shown, are subleading in the ultraviolet expansion, in which it can be systematically expanded, as we will see in the next subsection. We will demonstrate in addition that (3.18) remains applicable for higher derivative gravity, but in this case, $g_{\mu \nu}$ and $C_{(i)}$s will also depend on the parameters $\alpha'_{(i)}/l^2$.

In order that $t^{\mu \nu}$ is finite at $\Lambda = \infty$, we need

\begin{equation}
C_{(0)} = d - 1, \quad C_{(2)} = -\frac{1}{d - 2}, \quad \text{etc.} \tag{3.22}
\end{equation}

for $d > 2$ in Einstein’s gravity. These coefficients can be readily determined by substituting the on-shell ultraviolet expansion of $g_{\mu \nu}$ (which is also known as the Fefferman-Graham expansion [37] in gravity) into (3.21), as will be presented in the next subsection.26 Since $t^{\mu \nu}$ has only a finite number of $\Lambda^n$ (or equivalently $r^{-n}$) UV divergences, only a finite number of counterterm coefficients $C_{(i)}$ can be determined from ultraviolet finiteness.

Nevertheless, there is strong evidence from [26] that requiring a sensible IR fixed point of the first order RG flow, particularly in the hydrodynamic limit, determines all counterterm coefficients $C_{(i)}$, at least in Einstein’s gravity, certainly including those which are otherwise fixed by UV finiteness. Furthermore, the requirement regarding the IR fixed point of the RG flow is equivalent to demanding the absence of naked singularities in the corresponding spacetime. We can argue that such a sensible IR fixed point will also determine all $C_{(i)}$ even in higher derivative gravity, because higher derivative corrections, which capture perturbative departure from infinite strong interaction limit, cannot change the qualitative nature of the IR fixed point. With this assumption, Eqs. (3.16), (3.17) and (3.18) establish a unique relationship between $t^{\mu \nu}$ and $T^{\mu \rho \nu \lambda}$, for any choice of gauge fixing of bulk diffeomorphisms in the classical gravity theory.

In fact, the requirement of a sensible infrared fixed point of the first order RG flow also determines the possible initial values of $t^{\mu \nu}$ in the UV, meaning $\langle t^{\mu \nu} \rangle$ [26]. This is expected, because only those gravity solutions which have no bad naked singularities can correspond to field theory states – the possible values of $\langle t^{\mu \nu} \rangle$ are exactly those which reproduce regular gravity solutions. This also implies that for a given field theory, we can map the RG flow only to a specific dual gravity theory.27 Furthermore, the fact that the

---

26 In order to determine $C_{(2)}$, we need to put a curved boundary metric. This produces a $1/r^{d-2}$ in $t^{\mu \nu}$ unless $C_{(2)}$ is properly adjusted. This is relevant even when the boundary metric is conformally flat, as we will see in Section 4.2.

27 One can determine the dual classical gravity theory, meaning the perturbative higher derivative corrections to Einstein’s gravity as follows. Say up to $n^{th}$ order in strong coupling expansion we have $k$ possible higher derivative corrections to Einstein’s gravity with $k$ unknown numerical coefficients. Of course $k > n$. We can simply choose $k$ transport coefficients and see what their values should be in order that the dual gravity solutions have regular future horizons. These $k$ transport coefficients are each evaluated up to $n^{th}$ order in the expansion, and then matched to the field theory data up to the same orders in the expansion. Using this we can determine the $k$ unknown coefficients in the dual gravity theory. Note two field theories up to a given order in the expansion can have the same dual classical gravity theory, but the latter is uniquely determined by infrared regularity of the geometry, or equivalently the infrared fixed point of the RG flow, when the holographic correspondence exists (recall our discussion in Section 2.2).
requirement of the sensible infrared fixed point also determines even those coefficients \( C_{(i)} \)
which as in (3.22) are otherwise fixed by UV finiteness is also not unexpected: As the RG
flow is of first order, the IR conditions should be sufficient to determine the UV fixed point
as well.

**Effect of the conformal anomaly:** It is convenient to rescale \( t^\mu_\nu \) by the factor \((16\pi G_N)/l^{d-1}\),
i.e. to perform the replacement

\[
t^\mu_\nu \rightarrow \frac{16\pi G_N}{l^{d-1}} t^\mu_\nu. \tag{3.23}
\]

This amounts to rescaling by a factor of \( \propto 1/N^2 \), and this rescaled \( t^\mu_\nu \) is finite in the large
\( N \) limit. In \( N = 4 \) SYM holography, this rescaling factor is precisely \( 8\pi^2/N^2 \).

The conformal anomaly has no effect on the transformation of \( t^\mu_\nu \rightarrow \infty \) in flat Minkowski
space background, but not surprisingly it has an effect on the RG flow. In order to see
this, let us assume minimalistic counter-terms required to cancel UV divergences only which
appears for an arbitrary curved background metric (i.e. for an arbitrary boundary metric).
Of course, this need not necessarily be justified, since other non-minimalistic counter-terms
may be required to have a sensible infrared limit. However one way to see this will be to go
to four derivatives and beyond in the hydrodynamic expansion and repeat the calculations
in [26] to see for which values of the counterterm coefficients the RG flow ends at the
incompressible non-relativistic Navier-Stokes fixed point. This is beyond the scope of this
paper. Therefore we will simply proceed with the assumption that the non-minimalist
counterterms vanish.

In this case, in \( d = 4 \) and when the RG flow maps to Einstein’s gravity, we can write
\[25\]^{28}

\[
t^\mu_\nu = t^\mu_\nu \text{bare} + t^\mu_\nu \text{ct}(1) + t^\mu_\nu \text{ct}(2) + t^\mu_\nu \text{ct}(a), \tag{3.24}
\]

where

\[
t^\mu_\nu \text{bare} = -2 \frac{l}{r^4} \gamma^{\mu\rho} (K_{\rho\nu} - K \gamma_{\rho\nu}), \tag{3.25}
\]

\[
t^\mu_\nu \text{ct}(1) = -6 \frac{1}{r^4} \delta^\mu_\nu, \tag{3.26}
\]

\[
t^\mu_\nu \text{ct}(2) = \frac{l^2}{r^4} \left( R^\mu_\nu[\gamma] - \frac{1}{2} R[\gamma] \delta^\mu_\nu \right), \tag{3.27}
\]

\[
t^\mu_\nu \text{ct}(a) = \frac{l^4}{r^4} \log r \left( \frac{1}{8} R^\mu_{\alpha \beta \gamma \delta}[\gamma] R^{\alpha \beta \gamma \delta}[\gamma] - \frac{1}{48} \nabla^\mu \nabla_\nu R[\gamma] + \frac{1}{16} \nabla^2 R^\mu_\nu[\gamma] - \frac{1}{24} R[\gamma] R^\mu_\nu[\gamma] + \frac{1}{96} R^2[\gamma] - \frac{1}{32} R_{\alpha \beta}[\gamma] R^{\alpha \beta}[\gamma] - \frac{1}{96} \nabla^2 R[\gamma] \right) \delta^\mu_\nu \right). \tag{3.28}
\]

\[28\]In comparing our equation with this reference, we must keep in mind that the latter uses the convention
for definition of the Riemann tensor in which AdS has constant positive curvature.
In Fefferman-Graham gauge the above reads

\[
    t_{\mu \nu} = \frac{1}{r^3} \cdot (z^\mu_{\nu} - (\text{Tr} \, z) \delta^\mu_{\nu}) + \frac{1}{r^2} \left( R^\mu_{\nu}[g] - \frac{1}{2} R[g] \delta^\mu_{\nu} \right) +
    - \log r \left( \frac{1}{8} R^\mu_{\alpha \beta \gamma} [g] R^\alpha_{\beta \gamma} [g] - \frac{1}{48} \nabla^\mu \nabla_\nu R[g] + \frac{1}{16} \nabla^2 R^\mu_{\nu} [g] - \frac{1}{24} R[g] R^\mu_{\nu} [g] + \right.
    \left. + \left( \frac{1}{96} R^2 [g] - \frac{1}{32} R^\alpha_{\beta \gamma} [g] R^\alpha_{\beta \gamma} [g] - \frac{1}{96} \nabla^2 R[g] \right) \delta^\mu_{\nu} \right). \tag{3.29}
\]

The log counterterm \( t_{\mu \nu}^{\text{ct}(a)} \) is completely fixed by the conformal anomaly, in particular the central charges.

As we will see soon, only one possible non-minimalist counter-term, namely \( \nabla^2 R^\mu_{\nu} \) without the log \( r \) pre-factor, can affect the highly efficient RG flow equation (2.4) up to \( O(1/\Lambda^7 \log \Lambda) \).

### 3.2 The highly efficient RG flow from gravity equations and vice versa

**Why prioritise the Fefferman-Graham gauge:** The Fefferman-Graham gauge plays a pivotal role in mapping the highly efficient RG flow to classical gravity equations, and vice versa. This is because only when the classical gravity equations are in this gauge, the corresponding highly efficient RG flow equation, as in (2.4) involves only \( t_{\mu \nu}(\Lambda) \) and \( \Lambda \) explicitly.

The RG flow corresponding to other choices of gauge fixing of \((d + 1)\)-dimensional diffeomorphisms can be found by performing the following arbitrary (infinitesimal) transformations of \( \Lambda \) and \( x^\mu \):

\[
    \Lambda = \tilde{\Lambda} + P(\tilde{\Lambda}, \tilde{x}), \quad x^\mu = \tilde{x}^\mu + \chi^\mu(\tilde{\Lambda}, \tilde{x}). \tag{3.30}
\]

That the above corresponds to an arbitrary \((d + 1)\)-dimensional diffeomorphism can be readily seen by substituting \( \Lambda \) with \( r^{-1} \). Under these transformations, \( t^\mu_{\nu}(\Lambda) \) and \( g_{\mu \nu}(\Lambda) \) undergoes non-trivial transformations, as will be presented in Section 4.1. In any event, the upshot is that the arbitrary functions \( P \) and \( \chi^\mu \) appearing in these transformations also appear in the RG flow equation.

Generally speaking, the functions \( P \) and \( \chi^\mu \) should be functionals of the functions \( X^A(\Lambda, x) \) and variables \( \gamma^I(\Lambda) \) which parametrise \( \langle t^\mu_{\nu}(\Lambda) \rangle \), in the manner discussed in Section 2.1. This is required so that the (approximate) solutions in different gauges found in a certain perturbation expansion are valid in the same regime (meaning that they solve the equations of motion up to the same level of approximation). This aspect will be discussed in further detail in our following publication [4].

In fact when \( X^A(\Lambda) \) are defined via the coarse-graining (2.8), they also define \( t^\mu_{\nu}(\Lambda) \), as discussed in Section 2.1. The resultant RG flow equation will be not only a functional of \( t^\mu_{\nu}(\Lambda) \) and \( \Lambda \), but will also depend on the parametric functions \( X^A(\Lambda) \) and \( \gamma^I(\Lambda) \) explicitly in a generic case.

Nevertheless, these explicit dependencies on \( X^A(\Lambda) \) and \( \gamma^I(\Lambda) \) can be removed by transforming back to Fefferman-Graham gauge when the RG flow maps to classical gravity...
equations. As $X^A(\Lambda)$ and $\gamma^I(\Lambda)$ parameterise $\langle t^\mu{}^\nu(\Lambda) \rangle$ in a specific way, by knowing the action of $(d + 1)$–dimensional diffeomorphisms on the latter, we also find out the transformation of $X^A(\Lambda)$ and $\gamma^I(\Lambda)$ under $(d + 1)$–dimensional diffeomorphisms. Thus knowing the unique $(d + 1)$–dimensional diffeomorphism that transforms to Fefferman-Graham gauge, we can bring back the RG flow equation to a form that depends only on $t^\mu{}^\nu(\Lambda)$ and $\Lambda$ explicitly.

In fact without assuming that the RG flow maps to classical gravity equations, we can perform the following general steps for a RG flow equation obtained from the generalised coarse-graining (2.8):

- transform the scale $\Lambda$ and the $d$–dimensional coordinates in an arbitrary manner as in (3.30),
- assign how $t^\mu{}^\nu(\Lambda)$ and $g^{}_{\mu\nu}(\Lambda)$ change under these transformations such that the effective Ward identity for energy-momentum conservation preserves the form (2.6) at each scale,
- use the transformation of $t^\mu{}^\nu(\Lambda)$ to find those of $X^A(\Lambda)$ and $\gamma^I(\Lambda)$, and finally
- lift it to a modification of the coarse-graining function $F^A$ in (2.8).

If all the above steps can be performed for a specific transformation (3.30), such that we get a RG flow equation of the type of (2.4) that depends only on the operator $t^\mu{}^\nu(\Lambda)$ and the scale $\Lambda$ explicitly, then we can indeed claim that the RG flow equation represents a coarse-graining of an operator in the field theory, and is indeed an instance of a highly efficient RG flow. This is the more precise version of the defining criterion of a highly efficient RG flow that follows from a precise coarse-graining procedure in the field theory, as mentioned in Section 2.1.

When the highly efficient RG flow equation maps to classical gravity equation, the above steps can always be performed, as we will see in Section 4.1 and (regarding the uplift to the coarse-graining parameters) in our forthcoming publication [4], where we will use the geometry of bulk diffeomorphisms to carry out these transformations.

In what follows in the remaining pat of this subsection, we focus on mapping the classical gravity equation in the Fefferman-Graham gauge to highly efficient RG flow, and showing that the map is invertible.

**Mapping in the UV expansion:** Assuming that $g^{}_{\mu\nu}(\Lambda)$ depends only on $t^\mu{}^\nu(\Lambda)$ and $\Lambda$ explicitly one can do an UV expansion, which is valid when $\langle t^\mu{}^\nu(\Lambda) \rangle / \Lambda^d \ll 1$, and when similarly spacetime derivatives of $\langle t^\mu{}^\nu(\Lambda) \rangle$ are also small. These conditions should be valid for any state in the CFT for sufficiently large $\Lambda$, meaning for $\Lambda \gg \Lambda_{IR}$, with $\Lambda_{IR}$ being a suitable (state-dependent) infrared scale.

It is actually convenient to first write this expansion in terms of $t^\mu{}^\nu{}^\infty$, because then it matches with the well-known asymptotic Fefferman-Graham expansion [37] of $g^{}_{\mu\nu}$ in the $(d + 1)$–dimensional metric (3.2), which is identified with $g^{}_{\mu\nu}$ at $r = \Lambda^{-1}$ as in (3.15). As
in the flat Minkowski metric, $t^{\mu\nu}_\infty$ satisfies $\partial_\mu t^{\mu\nu}_\infty = 0$ and $\text{Tr} t^\infty = 0$ in a CFT, the most general expansion takes the form:

$$
g_{\mu\nu} = \eta_{\mu\nu} + c_4 \left( \frac{\alpha(i)}{l^2} \right) \cdot \frac{1}{\Lambda^4} \cdot \eta_{\mu\nu} t^{\alpha\beta}_\infty + c_6 \left( \frac{\alpha(i)}{l^2} \right) \cdot \frac{1}{\Lambda^8} \cdot \eta_{\mu\nu} \square t^{\mu\nu}_\infty +$$

$$+ \frac{1}{\Lambda^8} \cdot \left( c_{8a} \left( \frac{\alpha(i)}{l^2} \right) \cdot \eta_{\mu\nu} t^{\alpha\rho}_\infty t^{\rho\mu}_\infty + c_{8b} \left( \frac{\alpha(i)}{l^2} \right) \cdot \eta_{\mu\nu} \cdot t^{\alpha\beta}_\infty t^{\rho\mu}_\infty + c_{8c} \left( \frac{\alpha(i)}{l^2} \right) \cdot \eta_{\mu\nu} \square^2 t^{\mu\nu}_\infty \right) +$$

$$+ \mathcal{O}\left( \frac{1}{\Lambda^{10}} \right), \quad (3.31)$$

in $d = 4$. We have assumed that the conformal anomaly does not introduce any log term, which will be vindicated by mapping the RG flow to classical gravity equation. A similar expansion in $d = 4$ for $t^{\mu\nu}(\Lambda)$ reads

$$
t^{\mu\nu}_\infty = t^{\mu\nu}_\infty + b_2 \left( \frac{\alpha(i)}{l^2} \right) \cdot \frac{1}{\Lambda^2} \cdot \square t^{\mu\nu}_\infty +$$

$$+ \frac{1}{\Lambda^4} \left( b_{4a} \left( \frac{\alpha(i)}{l^2} \right) \cdot t^{\mu\rho}_\infty t^{\rho\mu}_\infty + b_{4b} \left( \frac{\alpha(i)}{l^2} \right) \cdot \delta^{\mu\nu} \cdot t^{\alpha\beta}_\infty t^{\rho\mu}_\infty + b_{4c} \left( \frac{\alpha(i)}{l^2} \right) \cdot \square^2 t^{\mu\nu}_\infty \right) +$$

$$+ \frac{1}{\Lambda^4} \log \Lambda \cdot b_{4c} \left( \frac{\alpha(i)}{l^2} \right) \cdot \square^2 t^{\mu\nu}_\infty + \mathcal{O}\left( \frac{1}{\Lambda^{6} \log \Lambda} \right). \quad (3.32)$$

Above we have assumed that the conformal anomaly restricts the coefficient of $(1/\Lambda^4) \log \Lambda$ term to proportional to $\square^2 t^{\mu\nu}_\infty$ only disregarding other possible contributions which have the same dimensions. This assumption will be vindicated by mapping to classical gravity. It would be interesting once again to find a fundamental justification, perhaps using generalisation of Osborn’s Weyl-consistency conditions. We will discuss it more in Section 5.

The coefficients of these expansions, are functions of $\alpha(i)/l^2$, which as noted in the previous subsection, are in turn functions of the coupling constants $g_{(A)}$ of the CFT. In the infinitely strongly interacting limit, when the gravitational dynamics is given by Einstein’s equation with a negative cosmological constant, these coefficients are pure universal numbers which are independent of the underlying CFT.

Let us first study the case when the highly efficient RG flow maps to Einstein’s equation with a negative cosmological constant. It is convenient to start from gravity equations first and then derive the corresponding highly efficient RG flow, and then do the inverse mapping.

As mentioned before, the asymptotic radius $l$ is related to the $(d + 1)$–dimensional cosmological constant $\Lambda_{cc}$, and we use the standard convention where it is given by

$$
\Lambda_{cc} = - \frac{d(d - 1)}{2l^2}. \quad (3.33)
$$
With this convention \((d + 1)\)-dimensional Einstein’s equation for \(g_{\mu\nu}\) in the Fefferman-
Graham metric (3.2) reduces to (see [25] for example):\(^{29}\)
\[
\frac{\partial}{\partial r} z^\mu - \frac{d - 1}{r} z^\mu + \text{Tr} z \left( \frac{1}{2} z^\mu - \frac{1}{r} \delta^\mu_\nu \right) = 2 R^\mu_\nu, \quad (3.34)
\]
\[
\nabla_\mu (z^\mu - (\text{Tr} z) \delta^\mu_\nu) = 0, \quad (3.35)
\]
\[
\frac{\partial}{\partial r} \text{Tr} z - \frac{1}{r} \text{Tr} z + \frac{1}{2} \text{Tr} z^2 = 0, \quad (3.36)
\]
where \(z^\mu_\nu\) is as defined in (3.20).\(^{30}\) Crucially, as claimed before, the constant \(l\) does not
appear in these equations which determine \(g_{\mu\nu}\). Thus \(g_{\mu\nu}\) does not depend on \(l\) as claimed
in the previous subsection.

The vector equation (3.35) is the momentum constraint, and it is easy to see from
(3.21) that it implies the conservation of \(t^\mu_\nu\). The scalar equation (3.36) asymptotically
implies \(\text{Tr} t^\infty = 0\) when the boundary metric is flat Minkowski space. Of course, both these
equations are constraints – once they are satisfied at any hypersurface \(r = \text{constant}\), the
dynamical radial evolution given by (3.34) preserves them for all values of \(r\).

Substituting the Anstaz (3.31) in (3.34), and replacing \(\Lambda\) by \(r^{-1}\), we can determine all
the coefficients in the UV expansion. We obtain [38]\(^{31}\)
\[
c_4 = \frac{1}{4}, \quad c_6 = -\frac{1}{48}, \quad c_8a = \frac{1}{32}, \quad c_8b = -\frac{1}{384}, \quad c_8c = \frac{1}{1536}, \quad \text{etc.} \quad (3.37)
\]
in \(d = 4\). Actually only the coefficient \(c_8b\) in the entire UV expansion (in \(d = 4\)) cannot be
determined from the tensor equation (3.34), to do this we need to use the scalar constraint
(3.36) too. Indeed, from this point of view an appropriate linear combination of the tensor
equation and the scalar equation times \(\delta^\mu_\nu\) can be regarded as the radial dynamical equation
[38], which can determine all the coefficients in the UV expansion (3.31), including \(c_8b\).

It is clear from (3.31) that schematically
\[
R \approx \frac{1}{\Lambda^4} \cdot \Box t + \mathcal{O} \left( \frac{1}{\Lambda^6} \right), \quad \nabla^2 R \approx \frac{1}{\Lambda^4} \cdot \Box^2 t + \mathcal{O} \left( \frac{1}{\Lambda^6} \right),
\]
\[
R^2 \approx \frac{1}{\Lambda^8} \cdot (\Box t)^2 + \mathcal{O} \left( \frac{1}{\Lambda^{10}} \right) \quad \text{etc.} \quad (3.38)
\]
when the boundary metric is flat Minkowski space. Therefore it is clear that only a finite
number of geometric counter-terms in \(t^\mu_\nu\), as defined by (3.21), can contribute up to a
fixed order in \(\Lambda^{-n}\) in the UV expansion. In order to determine all terms in (3.32) up
to order \(\Lambda^{-6} \log \Lambda\), the three counterterms (namely the one proportional to \(\delta^\mu_\nu\), the one
proportional to the Einstein tensor and the log counterterm) are sufficient, except only for
the coefficient \(b_{4c}\), which can be affected by a non-minimalist \(\nabla^2 R\) type of counterterm that

\(^{29}\) We are back to old notations where \(\nabla\) is the covariant derivative constructed from \(g_{\mu\nu}\) and \(R^\mu_\nu\) is the
Ricci tensor also obtained from \(g_{\mu\nu}\).

\(^{30}\) We are using conventions where the Riemann curvature \(R^A_{BCD}\) is defined via \(R^A_{BCD} = \partial_c \Gamma^A_{BD} + \Gamma^A_{CE} \Gamma^E_{BD} - (C \leftrightarrow D)\). This differs from the convention of [25] by a minus sign.

\(^{31}\) In order to compare with [38], we need to further rescale \(t^\nu_\nu\) here by a factor of 4. In this reference, the rescaling factor as in (3.23) for \(t^\nu_\nu\) was chosen to be \(l^3/(4\pi G_N)\) instead of \(l^3/(16\pi G_N)\).
may appear without the log $r$ pre-factor. For the purposes of illustration of the mapping between highly efficient RG flow and classical gravity equations, we will not consider such a possible counterterm here, although requiring a sensible IR fixed point may demand such a counterterm to exist with a precise coefficient, as discussed in the previous subsection.\footnote{In order to determine this counterterm, one may consider the hydrodynamic limit and check if the RG flow ends at the incompressible non-relativistic Navier-Stokes fixed point. To do this one needs to go to the fourth order in the derivative expansion.}

The log counterterm determines the coefficient $\tilde{b}_4c$.

Substituting the Ansatz for the UV expansions (3.31) and (3.32) in the defining relation for $t^\mu_\nu$ given by (3.29), replacing $\Lambda$ by $1/r$ and the coefficients of the UV expansion of $g_{\mu\nu}$ in (3.31) with already determined values as in (3.37), we can determine the coefficients of the UV expansion of $t^\mu_\nu$ in (3.32). We obtain

\[ b_2 = -\frac{1}{4}, \quad b_{4a} = 0, \quad b_{4b} = \frac{1}{16}, \quad b_{4c} = \frac{1}{64}, \quad \tilde{b}_{4c} = -\frac{1}{128}, \quad \text{etc.} \]  

(3.39)

in $d = 4$. The relation between $t^\mu_\nu$ and $t^\mu_\nu^\infty$ can be inverted with the coefficients as above. This inverted relation is:

\[
t^\mu_\nu^\infty = t^\mu_\nu + \frac{1}{\Lambda^2} \frac{1}{4} \Delta t^\mu_\nu + \frac{1}{\Lambda^4} \left( -\frac{1}{16} \delta^\alpha_\mu \cdot t^\beta_\nu \right) t^\alpha_\beta \right) + \frac{1}{\Lambda^4} \log \Lambda \left( \frac{1}{128} \Delta ^2 t^\mu_\nu \right) + O \left( \frac{1}{\Lambda^6} \log \Lambda \right).
\]

(3.40)

Finally using the above and the original UV expansion (3.32) with the known values of coefficients as in (3.39) again, we obtain the highly efficient RG flow equation (2.4).

Also when the inverted UV expansion (3.40) is substituted in (3.31) with the known values of coefficients as in (3.37), the UV expansion of $g_{\mu\nu}$ given by (2.5) as a functional of $t^\mu_\nu$ instead of $t^\mu_\nu^\infty$ is obtained. We thus derive the highly efficient RG flow equation (2.4) and the associated $g_{\mu\nu}$ as in (2.5) from the classical gravity equations.

The obvious question is if we can do the reverse, meaning starting from the highly efficient RG flow (2.4) can we go back to the classical gravity equations.

Firstly the highly efficient RG flow equation (2.4) can be readily solved in the UV expansion and the result is (3.32) with coefficients as in (3.39). Of course, (2.4) is a first order equation, so the only integration constant needed is $t^\mu_\nu$ at $\Lambda = \infty$, which is $t^\mu_\nu^\infty$.

The RG flow equation (2.4) is in flat space, and so is its solution, given by the UV expansion (3.32) with coefficients as in (3.39). The question is, whether there is a unique $g_{\mu\nu}$ at each $\Lambda$ so that $\nabla_{(\Lambda)}^\mu t^\mu_\nu = 0$ is satisfied, and also such that $g_{\mu\nu}$ takes a state-independent form (meaning that it depends only on $t^\mu_\nu$ and $\Lambda$ explicitly) and that it satisfies a classical gravity equation in the Fefferman-Graham gauge?

This is indeed the case, provided we also require that the relationship between $t^\mu_\nu$ and $g_{\mu\nu}$ in the highly efficient RG flow, and the ADM variables of gravity (in an arbitrary gauge) is such that, in addition to satisfying $\nabla_{(\Lambda)}^\mu t^\mu_\nu = 0$ at each scale and being state-independent in form, it is also such that,

- it can be stated in a gauge-invariant manner,
• $l$ does not appear in $t^\mu_\nu$ and $g_{\mu\nu}$ explicitly, and

• it guarantees that the infrared end point of the RG flow of $t^\mu_\nu$ can be characterised by a finite number of parameters.

We recall from the previous subsection that, we can indeed construct a unique $g_{\mu\nu}$ and a unique $t^\mu_\nu$ from the ADM variables on-shell using the above criteria. The first two additional criteria listed above completely determine the structural form of the relationship, and the last criteria is needed to fix the geometric counter-terms uniquely. The upshot is that we can also construct automatically other highly efficient RG flows, which correspond to the gravity equations in the other gauges. This we will do in Section 4.1 via $(d + 1)$-dimensional diffeomorphism transformations.

In order to obtain the gravity equations in Fefferman-Graham gauge, from the highly efficient RG flow equation (2.4), we can proceed as follows. Substituting the Ansatz for the UV expansion of $g_{\mu\nu}$ as given by (3.31) in the defining relation for $t^\mu_\nu$ given by (3.21), and using the known coefficients (3.39) in the UV expansion (3.32) of the $t^\mu_\nu$ which solves the RG flow equation (2.4), we can solve the unknown coefficients in the UV expansion for $g_{\mu\nu}$. We thus obtains (3.37). It is the same solution for $g_{\mu\nu}$ as can be directly obtained from Einstein’s equations. However, here instead of using Einstein’s equations we have used the defining relation of $t^\mu_\nu$ given by (3.21), to construct $g_{\mu\nu}$ out of $t^\mu_\nu$. Once again, this is a first order equation which has a unique solution given that at $\Lambda = \infty$, $g_{\mu\nu}$ corresponds to the actual background metric of the field theory, namely $\eta_{\mu\nu}$.

Thus we are able to show that the highly efficient RG flow (2.4) itself encodes complete information about Einstein’s equations. Then $g_{\mu\nu}$ obtained from it via the above procedure is unique and satisfies Einstein’s equations. Furthermore, it is also the unique solution of Einstein’s equations given the boundary metric (or the background metric of the dual field theory) is $\eta_{\mu\nu}$ and that $t^\mu_\nu\sim$ is the value of the right hand side of its defining equation (3.21) when evaluated in the limit $\Lambda = \infty$.

Another non-trivial claim, as made before, is that the first order highly efficient RG flow (2.4) is not only equivalent to Einstein’s equations, but also makes sense only for those field theories which have right UV data specifying $t^\mu_\nu\sim$ that leads to dual solutions of Einstein’s gravity without naked singularities. In such case, the infrared end point of the RG flow can be characterised by a finite number of parameters. Indeed the above feature that a sensible RG flow is dual to a spacetime without naked singularities, is required to establish the complete equivalence between highly efficient RG flow equation (2.4) and Einstein’s equations given by (3.34), (3.35) and (3.36). Without this feature we would have required to actually construct the $(d + 1)$-dimensional spacetime metric to see if the corresponding solution in gravity is also sensible. The latter would have been strange from the field theory point of view as $g_{\mu\nu}$ (and hence also the $(d + 1)$-dimensional spacetime metric) appears as a fictional construct in order to preserve the form of the Ward identities.

**Higher derivative gravity:** The entire discussion can also be repeated for higher derivative corrections to Einstein’s equations, provided these can be treated perturbatively in powers of the inverse CFT coupling constants as in $g^{-n}_{(A)}$, with $n > 0$, signalling systematic
departure from infinitely strongly interacting limit. We recall these coupling constants \( g_{(A)} \) map to the relative coefficients of the higher derivative corrections (to Einstein’s equations) multiplied by appropriate powers of \( l \), which are specified by the dimensionless parameters \( \alpha'/(\pi l^2) \). Thus at a fixed order in \( g_{(A)}^n \), we need to take into account only a finite number of higher derivative corrections in gravity. These higher derivative corrections should be such that the UV expansions (3.31) and (3.32) of \( g_{\mu\nu} \) and \( t^{\mu\nu} \) respectively are not modified, although their coefficients are. These corrections can then be expanded systematically in powers of the inverse CFT coupling constants.

**Hamilton-Jacobi like formulation:** An alternative approach to construction of the invertible map between the highly efficient RG flow and the dual classical gravity equations is to use the Hamilton-Jacobi like formulation of the classical gravity equations. In the Fefferman-Graham gauge, this has been done in [26].

In order to obtain this, we need to first invert the relation between \( z^{\mu\nu} \) and \( t^{\mu\nu} \) as given by (3.21). After some simple linear algebra, we get

\[
z^{\mu\nu} = r^{d-1} \left( t^{\mu\nu} - \frac{\text{Tr} t}{d-1} \delta^{\mu\nu} \right) - \frac{2r}{d-2} \left( R^{\mu\nu} - \frac{R}{2(d-1)} \delta^{\mu\nu} \right) + \ldots \quad (3.41)
\]

The inverted relation above can be readily substituted in the tensor equation (3.34) to obtain an equation for radial evolution of \( t^{\mu\nu} \). At first substituting (3.41) in (3.34) (for \( d > 2 \)), we obtain

\[
\frac{\partial t^{\mu\nu}}{\partial r} - \frac{2r^{2-d}}{d-2} \frac{\partial R^{\mu\nu}}{\partial r} - \frac{r^{d-1}}{2(d-1)} \left( \text{Tr} t + r^{2-d} R \right) \left( t^{\mu\nu} - \frac{2r^{2-d}}{d-2} R^{\mu\nu} \right) + \\
+ \frac{1}{d-1} \left( - \frac{\partial \text{Tr} t}{\partial r} + \frac{r^{2-d}}{d-2} \frac{\partial R}{\partial r} + \frac{\text{Tr} t}{r} + \\
+ \frac{r^{d-1}}{2(d-1)} \left( \text{Tr} t + r^{2-d} R \right) \left( \text{Tr} t - \frac{r^{2-d}}{d-2} R \right) \right) \delta^{\mu\nu} + \ldots = 0 \quad (3.42)
\]

Above we have included all terms which are relevant up to third order in derivatives. Furthermore using the identities,

\[
\frac{\partial T^{\mu\nu}_{\rho\sigma}}{\partial r} = \frac{1}{2} \left( \nabla_{\nu} z^{\mu}_{\rho} + \nabla_{\rho} z^{\nu}_{\mu} - \nabla^{\mu} z_{\rho\sigma} \right), \quad (3.43)
\]

\[
\frac{\partial R^{\mu\nu}_{\rho\sigma}}{\partial r} = \frac{1}{2} \left( \nabla_{\rho} \nabla_{\nu} z^{\mu}_{\sigma} - \nabla_{\sigma} \nabla_{\nu} z^{\mu}_{\rho} - \nabla_{\rho} \nabla^{\mu} z_{\nu\sigma} + \nabla_{\sigma} \nabla^{\mu} z_{\nu\rho} \right) + \frac{1}{2} \left( R^{\mu\nu}_{\rho\sigma} z^{\kappa}_{\sigma} - R^{\kappa}_{\nu\rho\sigma} z^{\mu}_{\kappa} \right), \quad (3.44)
\]

and iteratively substituting \( z^{\mu}_{\nu} \) in terms of \( t^{\mu}_{\nu} \) using (3.41), we obtain the second equation of the Hamilton-Jacobi like form as schematically given by (2.10).

Obviously, this equation for radial evolution of \( t^{\mu}_{\nu} \) is not a highly efficient RG flow unless we can eliminate \( g_{\mu\nu} \). Indeed one can substitute the Anstaz for the UV expansions of \( g_{\mu\nu} \) and \( t^{\mu}_{\nu} \) given by (3.31) and (3.32) respectively in (3.41) and (3.42), to solve for all
the unknown coefficients as well. The equation (3.42) then reduces to the flat space highly efficient RG flow (2.4) in the UV expansion, where $g_{\mu\nu}$ has disappeared.

The Hamilton-Jacobi type formulation is not only useful to understand how the highly efficient RG flow looks like when it maps to classical gravity equations in other gauges and to see its symmetries (we will need to use it in Appendices A and D to derive the action of $(d+1)$–diffeomorphisms on the Fefferman-Graham RG flow), but it also useful if we want to find the highly efficient RG flow summed over all orders in $\Lambda^{-1}$ but up to fixed order in derivatives. The procedure for doing this in the hydrodynamic limit has been found in [26], where $g_{\mu\nu}$ was eliminated to find the RG flow equation to all orders in $\Lambda^{-1}$ up to fixed order in derivatives. This is crucial for finding the infrared fixed point order by order in derivatives. We will come back to this point in our forthcoming publication [4].

4 Bulk diffeomorphisms and the lifted Weyl symmetry of the RG flow

4.1 The action of bulk diffeomorphisms on the Fefferman-Graham RG flow

When the highly efficient RG flow maps to classical gravity equations, bulk diffeomorphisms are the most general transformations which preserve the form of the Ward identity for local conservation of energy and momentum in an appropriately redefined background. Here we study the action of general infinitesimal bulk diffeomorphisms which take the RG flow away from the Fefferman-Graham gauge in the corresponding gravity equations.

The $(d+1)$–dimensional metric itself does not appear in the RG flow equation as in (2.4). Nevertheless when the highly efficient RG flow maps to classical gravity equations, there is an unique relation between ADM variables parametrising the $(d+1)$–dimensional metric and $t^{\mu\nu}$, at any arbitrary scale (identified with the inverse radial coordinate), and any arbitrary gauge choice on the gravity side. In case of Einstein’s gravity, this relation is given via (3.5), (3.6), (3.16), (3.17) and (3.18), along with the further requirement of sensible IR limit which determines all geometric counter-term coefficients. Thus the change of ADM variables under $(d+1)$–diffeomorphisms induce a unique transformation for $t^{\mu\nu}$ at any arbitrary scale.

As mentioned before, at least in the Fefferman-Graham gauge one can invert the relation and obtain the ADM variables of the $(d+1)$–dimensional metric from $t^{\mu\nu}$. The inversion is possible in any other gauge also if we can decode the gauge fixing of diffeomorphisms in the corresponding gravity equations from the structural property of the RG flow – we will return to this issue in the next subsection.

Under an arbitrary infinitesimal bulk diffeomorphism transformation given by Eq.(3.30), but now written in the equivalent form:

\[ r = \tilde{r} + \rho(\tilde{r}, \tilde{x}), \quad x^{\mu} = \tilde{x}^{\mu} + \chi^{\mu}(\tilde{r}, \tilde{x}) \]  

(4.1)
about the Fefferman-Graham gauge, the transformed ADM variables take the form:

$$\tilde{\alpha} = \frac{l}{r} \left( 1 - \frac{\rho}{r} + \frac{\partial \rho}{\partial \tilde{r}} \right),$$

$$\tilde{\beta}^\mu = \frac{\partial \chi^\mu}{\partial \tilde{r}} + g^{\mu \nu} \frac{\partial \rho}{\partial \tilde{x}^\nu},$$

$$\tilde{\gamma}^\mu_{\nu} = \frac{l^2}{r^2} \left( g^\mu_{\nu} + \rho \frac{\partial g^\mu_{\nu}}{\partial \tilde{r}} - 2 \frac{\rho}{r} g^\mu_{\nu} + \mathcal{L}_\chi g^\mu_{\nu} \right).$$  \hspace{1cm} (4.2)

Note, for the sake for brevity from now on, we will avoid explicitly writing that the variables on both sides of the equations relating old and new variables after diffeomorphisms are functions of the new coordinates $\tilde{r}$ and $\tilde{x}$ as above. In fact, transforming to new variables while comparing both the new and old variables at the new coordinate point, forms the core of the definition of the diffeomorphism transformation.

The relation between $\gamma^\mu_{\nu}$ and the effective metric $g^\mu_{\nu}$ of the highly efficient RG flow is unique as discussed before and is given by (3.13) (we recall in Fefferman-Graham gauge we simply have $g^\mu_{\nu} = \tilde{g}^\mu_{\nu}$). This relation along with (4.2) readily implies that

$$\tilde{g}^\mu_{\nu} = g^\mu_{\nu} - \rho \tilde{\Lambda}^2 \frac{\partial g^\mu_{\nu}}{\partial \tilde{\Lambda}} - 2 \rho \tilde{\Lambda} g^\mu_{\nu} + \mathcal{L}_\chi g^\mu_{\nu},$$  \hspace{1cm} (4.3)

after $\tilde{\eta}$ is identified with $\tilde{\Lambda}^{-1}$. The above form is remarkably simple. The transformation of the effective metric is just a combination of (i) a translation in scale by $-\rho \tilde{\Lambda}^2$, (ii) a Weyl transformation by $\rho \tilde{\Lambda}$, and (iii) a d-dimensional diffeomorphism by $\chi^\mu$, at any arbitrary scale. Note this simple form arises if we make an infinitesimal bulk diffeomorphism transformation only about the Fefferman-Graham gauge and not any other gauge.

A crucial remark is that the transformation (4.3) of the effective metric $g^\mu_{\nu}$ under bulk diffeomorphisms is true for any arbitrary classical theory of gravity corresponding to the highly efficient RG flow as this arises from pure kinematics of the transformation (4.2) of the ADM variables. Furthermore, as the transformation (4.2) of ADM variables also uniquely determine the transformation of $t^\mu_{\nu}$, the latter has a hidden universal structure as well, although it depends on the relation of $t^\mu_{\nu}$ to the ADM variables in the specific classical gravity theory corresponding to the RG flow also.

We do not want the diffeomorphism to change boundary data in particular the boundary metric. Therefore, $\rho$ should have an asymptotic expansion of the form

$$\rho = r^2 \rho(2) + r^3 \rho(3) + \cdots.$$  \hspace{1cm} (4.4)

Similarly requiring that $\beta$ vanishes asymptotically, implies the following asymptotic expansion of $\chi^\mu$:

$$\chi^\mu = r^2 \chi^\mu(2) + \cdots.$$  \hspace{1cm} (4.5)

Let us now obtain the transformation of $t^\mu_{\nu}$ when the RG flow maps to Einstein’s gravity. Firstly, using the defining relation (3.6) for the extrinsic curvature and (4.2), we can obtain its transformation which takes the form

$$\tilde{K}^\mu_{\nu} = K^\mu_{\nu} + \rho \frac{\partial K^\mu_{\nu}}{\partial \tilde{r}} + \mathcal{L}_\chi K^\mu_{\nu} + \frac{l}{r} \nabla_\mu \nabla_\nu \rho.$$  \hspace{1cm} (4.6)
where $\nabla_\mu$ is the covariant derivative constructed from the effective scale-dependent metric $g_{\mu\nu}$. For more steps in the derivation of the above, please see Appendix A. Above $\nabla$ is the covariant derivative constructed from $g$.

The transformation of $t^\mu \nu$ has to be found in an expansion, either in the UV expansion in $\Lambda^{-1}$ or in the derivative expansion (the latter is an expansion in derivatives of field-theory coordinates only of course). Let us use the derivative expansion for the purpose of illustration. Assuming that $\rho$ and $\chi^\mu$ start at zeroth order in derivatives, the Einstein tensor counter-term in $t^\mu \nu$ does contribute to the transformation under bulk diffeomorphisms up to this order, but higher order counter-terms do not. The result as readily obtained from (4.2) and (4.6) for infinitesimal diffeomorphisms about Fefferman-Graham gauge in $d > 2$ is:

$$\tilde{t}^\mu \nu = \rho_{\mu\nu} + \frac{\partial}{\partial \hat{r}} t^\mu \nu + d \hat{r}^\mu \nu + \mathcal{L}_\chi t^\mu \nu +$$

$$+ \frac{\hat{r}}{d-2} \left( \nabla^\mu t^\nu \nabla_\alpha \rho + \nabla_\nu t^\mu \alpha \nabla^\alpha \rho + t^\alpha \nu \nabla^\alpha \rho + t^\alpha \nu \nabla^\alpha \nabla^\rho - 2 \nabla^\alpha t^\mu \nu \nabla_\alpha \rho - t^\mu \nu \nabla^2 \rho - \delta^\mu \nu t^\alpha \beta \nabla_\alpha \nabla^\beta \right) + \mathcal{O}(\nabla^3).$$

(4.7)

It is to be understood that $\nabla$ is the covariant derivative formed from $g$ expressed as a function of the new coordinates. Above all indices have been lowered and raised with the inverse scale-dependent effective metric $g$ and its inverse respectively. The derivation of the above has been presented in Appendix A. As $g$ explicitly appears in the transformation of $t$ above, from the point of view of the RG flow equation, we must re-express $g$ in terms of $t$ as in (2.5) and substitute it in the above equation, to obtain the transformation of $t$ in terms of $t$ alone. This however in the general case can only be done in an UV expansion, meaning in a power series expansion in $r$. Furthermore we also need to substitute the explicit form of $\partial t / \partial r$ as in (2.4) which we also know in the UV expansion in the general case only. In the subsequent publication [4], we will be able to give an explicit form summing over all orders in $r$ in the hydrodynamic limit.

Nevertheless we can do a double expansion in $r$ and derivatives. We can readily substitute (2.4) and (2.5) in (4.7) and after replacing $\hat{r}$ by $\Lambda^{-1}$ we obtain

$$\tilde{t}^\mu \nu = t^\mu \nu + \left( - \frac{1}{\Lambda^3} \rho_{(2)} \cdot \frac{1}{2} \cdot \Box t^\mu \nu + d \left( \frac{1}{\Lambda^3} \rho_{(2)} + \frac{1}{\Lambda^2} \rho_{(3)} + \frac{1}{\Lambda} \rho_{(4)} \right) \right) t^\mu \nu$$

$$+ \frac{1}{\Lambda^2} \mathcal{L}_\chi(2)^\mu \nu + \frac{1}{\Lambda^3} \mathcal{L}_\chi(3)^\mu \nu +$$

$$+ \frac{1}{(d-2)\Lambda^3} \left( \eta^\beta \partial_\beta \alpha t^\alpha \nu \partial_\alpha \rho_{(2)} + \partial_\nu t^\mu \alpha \partial_\beta \rho_{(2)} \eta^\alpha \beta + t^\alpha \partial_\nu \partial_\beta \rho_{(2)} \eta^\alpha \beta + t^\alpha \nu \partial_\beta \partial_\alpha \rho_{(2)} \eta^\beta \mu - 2 \partial_\alpha t^\mu \nu \partial_\beta \rho_{(2)} \eta^\alpha \beta - t^\mu \nu \partial_\alpha \partial_\beta \rho_{(2)} \eta^\alpha \beta - \delta^\mu \nu \partial_\alpha \partial_\gamma \rho_{(2)} \eta^\beta \gamma \right) +$$

$$+ \mathcal{O} \left( \frac{1}{\Lambda^4} \right) + \mathcal{O} \left( \partial^2 \right),$$

(4.8)
when the field theory lives in flat Minkowski space. We note $\partial_{\mu}$ etc. is a short form of $\partial/\partial \tilde{x}^{\mu}$, etc. the ordinary partial derivative with respect to the new coordinate.

Remarkably, if we look for terms up to first order in derivatives only, we see the same pattern as in case of the transformation of $g_{\mu\nu}$ as in Eq. (4.3) – namely it is a combination of (i) a translation in scale by $-\rho \tilde{\Lambda}^2$, (ii) a Weyl transformation by $\rho \tilde{\Lambda}$, and (iii) a $d$-dimensional diffeomorphism by $\chi^{\mu}$, at any arbitrary scale. This simple intuitive form up to first order in derivatives holds for infinitesimal diffeomorphism transformations about Fefferman-Graham gauge only.

In order to find the new RG flow equation, we need to apply (3.30) with the transformation (4.7) for $t^{\mu}{\nu}$ to the Fefferman-Graham RG flow equation, namely (2.4). It is clear then that the new RG flow equation, corresponding to the new gauge, depends on $\rho$ and $\chi^{\mu}$ explicitly.

The crucial point is that the equations that determine $\rho$ and $\chi^{\mu}$ in terms of new gauge conditions are first order. Therefore, requiring that they do not change the boundary metric, meaning that they disappear in the UV keeping the field theory background metric unchanged, determine them uniquely. Thus there exists unique $\rho$ and $\chi^{\mu}$ which transforms the Fefferman-Graham gauge to the new gauge provided that the transformation is trivial in the UV. Therefore, if we can understand which gauge fixing in classical gravity the new RG flow equation corresponds to, we can simply apply the reverse transformation and bring the new equation into Fefferman-Graham gauge where $t^{\mu}{\nu}$ and $\Lambda$ only appear explicitly. Thus we are led to the problem of understanding the link between the structural property of the RG flow equation and the corresponding gauge fixing on the gravity side. We turn to this in Section 4.3.

As of now, we conclude that the new RG flow equations obtained from arbitrary bulk diffeomorphism transformations are all legitimate, in the sense that they preserve the form of the Ward identities in an appropriately redefined background metric at each scale, and with an appropriate change of variables they can be brought to the form where they depend only on $t^{\mu}{\nu}$ and $\Lambda$ explicitly.

### 4.2 The lifted Weyl symmetry

A CFT on a curved background is Weyl invariant up to the Weyl anomaly. By the holographic duality, Weyl transformations at the boundary can be lifted to specific diffeomorphisms in the $(d+1)$-dimensional geometry. These specific diffeomorphisms actually depend on the geometry which solves classical gravity equations and hence the dual field theory state, however their dependence on the $(d+1)$-dimensional metric has a form which is independent of the geometry and hence the dual state.

These diffeomorphisms are actually the set of diffeomorphisms which preserve the asymptotic AdS nature of the $(d+1)$-dimensional metric but yet do not vanish at the boundary, where it manifests as a pure Weyl transformation. As the asymptotic AdS boundary condition implies that the metric can be written in the Fefferman-Graham form

---

\[^{33}\text{For an explicit example see [38] where the change of coordinates from the Fefferman-Graham gauge to Eddington-Finkelstein gauge for special class of solutions have been worked out.}\]
near \( r = 0 \), these diffeomorphisms should preserve the asymptotic Fefferman-Graham form of the metric.

Clearly, the general transformation of ADM variables (4.2) away from the Fefferman-Graham gauge show that the Fefferman-Graham gauge is preserved, meaning that \( \tilde{\alpha} \) remains \( 1/\tilde{r} \) and \( \tilde{\beta}^\mu \) still vanishes, provided

\[
\rho = \tilde{r} \, \delta \sigma(\tilde{x}), \quad \chi^\mu = - \int_0^{\tilde{r}} d\tilde{r} \, \tilde{g}^{\mu \nu}(\tilde{r}, \tilde{x}) \frac{\partial \delta \sigma(\tilde{x})}{\partial \tilde{x}^\nu},
\]

(4.9)

It is clear that \( \chi^\mu \) depends on the specific geometry concerned through \( g_{\mu \nu} \), but its dependence on \( g_{\mu \nu} \) takes the above state-independent form. These are specific forms of so-called PBH transformations as mentioned before. When rewritten after replacing the radial coordinate with the inverse of the scale, the above transformations take the form (2.12). The transformation depends on one parameter only, namely \( \delta \sigma(x) \) which is a function of the field theory coordinates, and is nothing but the Weyl transformation parameter in the UV.

It is also to be noted that although these transformations preserve the Fefferman-Graham gauge, these are not isometries in a generic geometry. Indeed the effective metric \( \tilde{g}_{\mu \nu} \) still transforms non-trivially according to (4.3). It is also easy to see explicitly from here that at the boundary, meaning at \( \Lambda = \infty \), the transformation reduces to a Weyl rescaling of the background metric by \( \delta \sigma \). In particular, if the boundary metric is conformally flat meaning \( \eta_{\mu \nu} e^{2\sigma(x)} \), the new boundary metric is also conformally flat but with \( \sigma \) replaced by \( \sigma + \delta \sigma \). In this case, the asymptotic forms of \( \rho, \chi^\mu \) are

\[
\rho = \frac{1}{\Lambda} \, \delta \sigma, \quad \chi^\mu = -\frac{1}{2 \Lambda^2} e^{-2\sigma} \eta^{\mu \nu} \frac{\partial}{\partial \tilde{x}^\mu} \delta \sigma + O \left( \frac{1}{\Lambda^3} \right).
\]

(4.10)

The asymptotic form of transformation of \( g_{\mu \nu} \) as follows from (4.3) is then

\[
\tilde{g}_{\mu \nu} = g_{\mu \nu} - 2 \delta \sigma g_{\mu \nu} + 2 \frac{1}{\Lambda^2} \delta \sigma g_{(2)\mu \nu} - \frac{1}{2 \Lambda^2} \mathcal{L} e^{-2\sigma} \eta^{\rho \beta} \partial_\rho \delta \sigma (\eta_{\mu \nu} e^{2\sigma}) + O \left( \frac{1}{\Lambda^3} \right).
\]

(4.11)

It is clear that the leading term in the transformation above is a Weyl transformation by \( \delta \sigma \) as claimed before. It is worth recalling that all variables on both sides of the above equation are functions of \( \tilde{\Lambda} \), the new scale and \( \tilde{x} \), the new field theory coordinates. Also \( \partial_\beta \) stands for \( \partial / \partial \tilde{x}^\beta \), the partial derivative with respect to the new coordinate etc. Above \( g_{(2)\mu \nu} \) which appears in one of the the sub-leading terms in the asymptotic expansion of \( g_{\mu \nu} \) (and arises from \( (\partial / \partial \Lambda) g \)) can be obtained directly from Einstein’s equations. It is obtained in Appendix C and explicitly is

\[
g_{(2)\mu \nu} = -(\partial_\mu \sigma)(\partial_\nu \sigma) + \partial_\mu \partial_\nu \sigma + \frac{1}{2} \eta_{\mu \nu} \eta^{\alpha \beta}(\partial_\alpha \sigma)(\partial_\beta \sigma).
\]

(4.12)

Thus it a functional of the background metric.

Let us first show these transformations (2.12) along with the transformation of \( t^\mu \nu \) given by (4.7) are a symmetry of the highly efficient RG flow equation corresponding to the Fefferman-Graham gauge (note that the effective metric \( g \) never enters this equation so its transformation is irrelevant for the symmetry). In order to see the symmetry, we need
to generalise the RG flow equation (2.4) from the flat Minkowski space background to a (non-dynamical) conformally flat space background $\eta_{\mu\nu}e^{2\sigma(x)}$. We will call this the lifted Weyl symmetry.

The derivation of the highly efficient RG flow in the background of conformally flat space $\eta_{\mu\nu}e^{2\sigma}$ which maps to Einstein’s gravity equations can be obtained following the methodology of Section 3.2. However, the calculations are highly involved and can be simplified with the help of some tricks and useful identities. The derivation is presented in Appendix C. The result in $d = 4$ is:

$$
\frac{\partial}{\partial \Lambda} t_{\mu\nu}(\Lambda) = -\frac{1}{\Lambda^3} \left( 2t_{(2)\mu\nu}^* + \frac{1}{2} \left( - e^{-2\sigma} \eta^{\alpha\beta} \nabla_\alpha \nabla_\beta \left( t_{\mu\nu}(\Lambda) - t_{(0)\mu\nu}^* \right) + 4 \left( t_{\mu\beta}(\Lambda) - t_{(0)\beta}^* \right) e^{-2\sigma} \eta^{\beta\gamma} g_{(2)\gamma\nu} + + 6 \delta^\mu_{\nu} \left( t_{\gamma\beta}(\Lambda) - t_{(0)\beta}^* \right) e^{-2\sigma} \eta^{\beta\delta} g_{(2)\gamma\delta} \right) + t_{(2a)\mu\nu} + 2 \frac{1}{\Lambda^3} \log \Lambda t_{(2a)\mu\nu}^* + O \left( \frac{1}{\Lambda^5} \log \Lambda \right) \right). \quad (4.13)
$$

Above $\nabla^{(\sigma)}$ denotes the covariant derivative constructed from the background metric $\eta_{\mu\nu}e^{2\sigma}$. The variables $t_{(0)\mu\nu}^*$, $t_{(2)\mu\nu}^*$ and $t_{(2a)\mu\nu}^*$ are independent of $\Lambda$, are functionals of the background metric $\eta_{\mu\nu}e^{2\sigma}$ just like $g_{(2)\mu\nu}$ (which is given by (4.12)), and they vanish when $\sigma$ goes to zero, i.e. when the background metric reduces to flat Minkowski space. These functions are given explicitly in Appendix C. In that case (4.13) reduces to the flat Minkowski space highly efficient RG flow equation (2.4) as evident from the leading term. The background dependent term at order $(1/\Lambda^3) \log \Lambda$ appearing in (4.13) is completely determined by the conformal anomaly.

The transformation (4.9) preserves the Fefferman-Graham gauge and also the form of the boundary metric $\eta_{\mu\nu}e^{2\sigma}$ although replacing $\sigma$ by $\sigma + \delta \sigma$. Since the Fefferman-Graham gauge is preserved, the relation between $t$ and $g$ given by (3.29) also remains the same after the transformation. This then readily implies that the resultant highly efficient RG flow equation (4.13) should remain the same after the transformation, but with $\sigma$ replaced by $\sigma + \delta \sigma$. Therefore (4.9) should be a symmetry of the transformation provided we use (3.29) to find the transformation of $t_{\mu\nu}$ from the transformation of $g_{\mu\nu}$, the latter being given by (2.13).

The derivation of the transformation of $t_{\mu\nu}$ from the transformation of $g_{\mu\nu}$ (which is
given by (2.13) using the defining relation (3.29) is given in Appendix D. The result is:

\[
\tilde{t}^\mu_\nu = t^\mu_\nu + \left( \frac{1}{\Lambda^2} e^{-2 \sigma} \delta^\sigma_\nu \frac{1}{2} \left( - \eta^{\sigma\beta} \nabla^\sigma_\alpha \nabla^\sigma_\beta t^\mu_\nu + 4 t^\mu_\beta \eta^{\beta\gamma} g(2)_{\gamma\nu} + 6 \delta^\mu_\nu t^\gamma_\beta \eta^{\beta\delta} g(2)_{\gamma\delta} \right) + 4 \delta^\sigma t^\mu_\nu - \frac{1}{2} \frac{1}{\Lambda^2} \mathcal{L}_\epsilon^{-2 \sigma} \eta^{\sigma\beta} \partial_\sigma \partial_\nu \right) t^\mu_\nu + \frac{e^{-2 \sigma}}{2\Lambda^2} \left( \eta^{\mu\beta} \partial_\beta t^\alpha_\nu \partial_\alpha \delta^\nu_\delta + \partial_\beta t^\mu_\alpha \partial_\alpha \delta^\nu_\delta \eta^{\alpha\beta} + t^\mu_\alpha \partial_\alpha \partial_\beta \delta^\nu_\delta \eta^{\alpha\beta} + t^\nu_\beta \partial_\beta \partial_\alpha \delta^\mu_\alpha \eta^{\beta\nu} - 2 \partial_\alpha t^\mu_\nu \partial_\beta \delta^\alpha_\delta \eta^{\alpha\beta} - t^\nu_\beta \partial_\beta \partial_\alpha \delta^\mu_\alpha \eta^{\alpha\beta} - \delta^\mu_\nu \partial_\beta \partial_\alpha \delta^\nu_\delta \eta^{\beta\alpha} + 4 \partial_\alpha \partial_\nu \delta^\alpha_\delta \eta^{\alpha\beta} + \partial_\nu t^\alpha_\beta \partial_\alpha \partial_\gamma \delta^\beta_\mu \eta^{\gamma\beta} \right) + \mathcal{O} \left( \frac{1}{\Lambda^4} \right) \right) + \mathcal{O}(\delta^3),
\]

(4.14)

up to given orders in derivatives and $\Lambda^{-1}$. The highly efficient RG flow equation (4.13) in conformally flat space background is invariant under the transformations given by (2.12) and the above equation, up to the orders in $\Lambda^{-1}$ shown when restricted to second order in derivatives.

The transformation of $t^\mu_\nu$ is complicated as evident from (4.14), and in the general case can only be found in an asymptotic expansion in a specific theory of classical gravity. Nevertheless, it follows uniquely from the transformation of $g_{\mu\nu}$ as given by (2.13), which is simple, exact and holds for an arbitrary classical theory of gravity. Therefore (2.13) should be thought of as more fundamental in defining the lifted Weyl symmetry of the highly efficient RG flow corresponding to the Fefferman-Graham gauge fixing of the dual classical gravity equations.

In the following subsection, we will find the lifted Weyl symmetry of the highly efficient RG flow corresponding to an arbitrary gauge fixing of diffeomorphisms of the dual classical gravity equations.

### 4.3 Deciphering the gauge fixing of diffeomorphisms from the lifted Weyl symmetry

We will prove here that corresponding to every choice of gauge fixing of the $(d + 1)$–diffeomorphism symmetry in the classical gravity equations, the dual highly efficient RG flow inherits a lifted Weyl symmetry under a unique group of transformations, which reduces to Weyl transformations in the UV. The gauge fixing of the $(d + 1)$–diffeomorphism symmetry in classical gravity can be completely deciphered from the lifted Weyl symmetry of the corresponding highly efficient RG flow. The latter can also be defined in a manner which is also completely independent of the classical gravity theory to which the highly efficient RG flow corresponds to. Therefore, the lifted Weyl symmetry corresponding to the unique gauge fixing of diffeomorphisms, is an universal feature of a highly efficient RG flow which map to a classical gravity theory.

Firstly, let us choose a gauge which is infinitesimally different from the Fefferman-Graham one. This can be specified by the conditions which determine $\alpha$ and $\beta^\mu$ uniquely.
From the first order equations (4.2) relating the new $\alpha$ and $\beta^\mu$ to those in the case of Fefferman-Graham gauge, we can then determine the infinitesimal diffeomorphism transformations, meaning $\rho$ and $\chi^\mu$, which take us from the Fefferman-Graham gauge to the new gauge. Since in (4.2), we have only first order radial derivatives of $\rho$ and $\chi^\mu$, given the boundary conditions that $\rho = \mathcal{O}(r^2)$ and $\chi^\mu = \mathcal{O}(r^2)$ so that they do not modify boundary (UV) data, we get unique solutions for $\rho$ and $\chi^\mu$. Let us call this unique transformation $\mathcal{G}$.

The lifted Weyl symmetry transformation corresponding to a Weyl transformation by $\delta\sigma$ in the UV, and under which the new highly efficient RG flow equation is invariant, is $\mathcal{G}\mathcal{P}^{b\sigma}\mathcal{G}^{-1}$, where $\mathcal{P}^{b\sigma}$ corresponds to the PBH transformation (4.9). It can be readily seen as follows. Firstly with $\mathcal{G}^{-1}$ we can transform from the new gauge back to Fefferman-Graham gauge. Note this does not modify the UV data and the background metric. Then with $\mathcal{P}^{b\sigma}$, we can do the unique large diffeomorphism transformation which modifies the UV data and the background metric by a $\delta\sigma$ Weyl transformation, and keeps the Fefferman-Graham highly efficient RG flow equation invariant. Finally, we can go back to the new gauge by the unique transformation $\mathcal{G}$ while keeping the UV data same, meaning that we come back to the new gauge but with UV data and background metric Weyl transformed by $\delta\sigma$. Clearly $\mathcal{G}\mathcal{P}^{b\sigma}\mathcal{G}^{-1}$ should be the new and unique transformation under which the new highly efficient RG flow equation in conformally flat metric background is invariant, since after the transformation, the RG flow corresponds to the same gauge fixing of diffeomorphisms of the dual classical gravity equations, although the UV data and the conformally flat background metric are Weyl transformed by $\delta\sigma$.

The transformation of $g_{\mu\nu}$ then follows from (4.3) and (2.13) and this results in unique transformation of $t^\mu_\nu$ as discussed in the previous subsections. Nevertheless the more fundamental definition of the lifted Weyl symmetry is via the transformation of $g_{\mu\nu}$ as it takes the same form in any classical theory of gravity. From the transformation of $g_{\mu\nu}$ one can read off the diffeomorphism $\mathcal{G}$ that relates the Fefferman-Graham gauge to the new gauge, given that we can find the new lifted Weyl symmetry $\mathcal{G}\mathcal{P}^{b\sigma}\mathcal{G}^{-1}$ and we know $\mathcal{P}^{b\sigma}$ explicitly. Thus from the lifted Weyl symmetry we can decipher the corresponding gauge fixing of diffeomorphisms in the dual classical gravity equations. This correspondence is independent of the classical gravity theory to which the highly efficient RG flows map to, as the transformation of $g_{\mu\nu}$ is independent of the classical gravity theory.

5 Conclusions — Outlook and Open Issues

We have shown here that provided we can construct a highly efficient RG flow in the strongly interacting and large $N$ limit which

- preserves the form of Ward identities for conservation of energy and momentum in appropriately redefined background metrics,
- has a sensible infrared limit, and
- has a special kind of lifted Weyl symmetry that reduces to Weyl transformation in the UV and is determined uniquely by the transformation of the scale-dependent background metric,
then it maps to classical gravity equations corresponding to a specific gauge fixing of
diffeomorphisms on the gravity side (that can be deciphered from the lifted Weyl symmetry
of the RG flow equation). It can be argued that these conditions are also sufficient to imply
the existence of the holographic duality, meaning that the highly efficient RG flow can be
mapped to appropriate classical gravity equations. The proof of the sufficiency of these
conditions will imply a complete reconstruction of the holographic duality as a special kind
of RG flow in the quantum field theories, at least in the pure gravity sector.

The argument for the sufficiency of these conditions is as follows. Roughly, the first cri-
teration upholds the momentum constraints and the third criterion upholds the Hamiltonian
constraint. Then together these should imply the emergence of \((d + 1)\)-dimensional dif-
feomorphism symmetry with the (inverse) scale becoming the emergent radial coordinate.
The second criterion ensures that we map to the right theory of classical gravity, while it
also fixes the geometric counterterms that ensures that corresponding to each gauge fixing
of \((d + 1)\)-dimensional diffeomorphism symmetry on the gravity side, there exists a unique
highly efficient RG flow.

Some of the difficulties of this proof has been mentioned in Section 2.3. Firstly, we are
able to show that we have a unique scale-dependent metric in a highly efficient RG flow only
if it maps to classical gravity. So we need to prove first that the scale-dependent metric is
uniquely determined by the three criteria mentioned. Secondly, we need to show that the
relationship between scale-dependent energy-momentum tensor and the scale-dependent
metric is invertible when the background metric on which the field theory lives is specified.
It also can be shown to be true when the highly efficient RG flow maps to classical gravity
– it is not clear if it follows from the mentioned criteria. Although this is critical for the
lifted Weyl symmetry itself to exist, the reverse need not be true.

One way to prove the sufficiency of the three criteria for implying the holographic
 correspondence could be to show that these endow the RG flow with a unique simplectic
structure which can be mapped to Dirac’s surface deformation algebra \([39, 40]\) that captures
the action of diffeomorphisms on geometric variables defined on a hypersurface. For an
early work in this direction see \([41]\). We leave the details of this proof to a future work.

We also need to show an actual construction of the coarse graining in the field theory
which can reproduce the highly efficient RG flow that satisfies the three criteria. The
general procedure for achieving this has been sketched in Section 2.1, where we have shown
how the cut-off in momentum space should be promoted to a functional of the elementary
fields and the external sources by using appropriate collective variables, to result in a RG
flow that satisfy the three criteria. An explicit construction in the hydrodynamic limit will
appear in the second part of this work \([4]\).

Here we conclude by delving briefly on a few issues which should be investigated in
the near future.

**Beyond pure gravity:** We have focused on the pure gravity sector here. However
it can be argued that the three criteria should also suffice beyond pure gravity. This
argument is based on the observation that once the first and third criteria with appropriate
generalisations lead to emergence of \((d + 1)\)-dimensional diffeomorphism symmetry in the
dual gravity theory, the content of the latter will be determined completely by the infrared regularity condition.

Consider the case, when we have an additional scalar operator $O$, with its dual source $J$. The scale-dependent Ward identity which maps to the momentum constraints of gravity should take the form

$$\nabla_{(\Lambda)}\mu^{\mu}_{\nu}(\Lambda) = O(\Lambda) \partial_{\nu} J(\Lambda).\quad (5.1)$$

Furthermore, the lifted Weyl symmetry can be defined via the transformation of the source $J(\Lambda)$ which should take the form

$$\tilde{J} = J - \delta\sigma \tilde{\Lambda} \frac{\partial}{\partial \tilde{\Lambda}} J + (d - \Delta) \delta\sigma J + \mathcal{L}_{\chi} J,$$

where $\chi$ is given by (4.9). The question is, whether these criteria can define both $J(\Lambda)$ and $O(\Lambda)$ uniquely in terms of variables in gravity, defined on hypersurfaces $r = \Lambda^{-1}$ for a given choice of gauge fixing of bulk diffeomorphisms. It would be interesting to compare our approach with the effective function approach of [42] in this case. We leave this investigation for the future.

**Connection with entanglement renormalisation:** Recently it has been claimed that an entanglement renormalisation scheme [43, 44] in which quantum information is coarse-grained in an efficient manner should reproduce many features of holographic duality [45, 46]. It has been shown in [45] that the entanglement renormalisation schemes indeed reproduces many features of the Ryu-Takayanagi prescription [47] for the holographic entanglement entropy qualitatively. On the other hand, our approach here is concerned directly with reproducing classical gravity equations from the RG flow. Nevertheless, a connection of our approach with entanglement renormalisation should exist. This can be argued, because it has been recently shown that the Ryu-Takayanagi formula for the holographic entanglement entropy, along with a few field-theoretic identities should imply Einstein’s equations in the dual gravity theory at least for linearised fluctuations about the vacuum [48, 49], which should be directly reproduced by our approach.

The question is, whether our three criteria for defining the RG flow that reproduces holographic duality, are also related to efficient ways of coarse-graining quantum information. If this is the case, then we can find a more elegant formulation of the highly efficient RG flow in terms of quantum information theory. On the other hand, our criteria can sharpen the identification of the precise entanglement renormalisation scheme which can reproduce holographic duality. As a spin-off, we can learn more about efficient ways of characterising and manipulating quantum information.

**Quantum gravity:** One of the obvious directions of investigation is going beyond large $N$ which implies that the holographic duality should now map the CFT to quantum gravity. Indeed recently some interesting paradoxes have been raised regarding how quantum gravity can reproduce commutation relations in CFTs [50, 51]. Our approach naturally leads to a definition of quantum gravity, as the effective scale-dependent metric which emerges from the highly efficient RG flow as in (2.5) is naturally a quantum object – it behaves
classically only when the large $N$ factorisation applies in the sense that its radial flow is a classical equation. Thus extending our framework beyond large $N$ can directly tell us how quantum gravity emerges from CFTs in one lower dimension as special RG flows. We hope to shed light on other outstanding paradoxes and open questions in quantum gravity like those raised in [52–54] using our approach of elucidating the holographic correspondence.

**Establishing a more general semi-holographic framework:** It is worth investigating how the framework of highly efficient RG flow can generalise the holographic duality into a broader effective framework for systems with strongly interacting degrees of freedom. Consider the case of an asymptotically free theory like QCD. It may be possible to define a RG flow in the large $N$ limit that satisfy all the three criteria in the UV where the theory behaves like a free field theory – however in absence of a gap in the scaling dimensions of the single-trace operators, the dual gravity theory will have infinitely many fields resembling a higher spin gravity theory. In the infrared however, most of the single-trace operators will have large anomalous scaling dimensions due to strong coupling, so that the most of the dual fields in gravity will become massive and decouple. Thus in the infrared we will find a more familiar dual theory of gravity with finite number of fields.

This suggests we can perhaps create a more general framework where we can do ordinary perturbative physics in the UV, but in order to obtain IR dynamics we can employ a dual classical gravity theory with few fields whose boundary conditions should be determined self-consistently by the perturbative UV physics. This general framework is known as semi-holography. We propose to investigate a fundamental derivation of this effective framework on the lines described above. The insights obtained from [55–60] should be highly valuable in this regard.

The semi-holographic framework has been consistently applied recently for non-Fermi liquids [61, 62] and the quark-gluon plasma [63]. In particular, it has been shown that the semi-holographic framework can lead to a generalisation of Landau’s theory for non-Fermi liquids in certain limits, such that the low energy phenomenology can be described via few effective couplings at the Fermi surface [62]. Similarly it has been argued that the semi-holographic framework can describe the physics of quark-gluon plasma using a few effective parameters [63]. However, it remains to be shown that the semi-holographic framework can generally lead to an effective theory for realistic systems with strongly interacting degrees of freedom, in the sense that a few parameters can capture all phenomena at relevant scales of observation. This is why a fundamental derivation of the semi-holographic framework is necessary. This can open the door to quantitative and more general applications of the holographic duality to hadronic and nuclear physics, and strongly correlated materials.

**Acknowledgments**

We thank B. P. Dolan for collaboration during the initial stages of this project. N.B. and A.M. acknowledge the “Research in Groups” grant sponsored by ESPRC of UK and managed by ICMS, Edinburgh, which kickstarted their mutual collaboration. A.M. thanks ICMS and Heriot-Watt University for hospitality at Edinburgh, where a large portion of
the work in this project has been done. The research of A.M. involving a large part of this work has also been supported by the LABEX P2IO, the ANR contract 05-BLAN-NT09-573739, the ERC Advanced Grant 226371 and the ITN programme PITN-GA-2009-237920. Presently, the research of A.M. is supported in part by European Union’s Seventh Framework Programme under grant agreements (FP7-REGPOT-2012-2013-1) no 316165, the EU-Greece program “Thales” MIS 375734, and is also co-financed by the European Union (European Social Fund, ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) under “Funding of proposals that have received a positive evaluation in the 3rd and 4th Call of ERC Grant Schemes”. S.K. is supported in part by the ANR grant 08-JCJC-0001-0 and the ERC Starting Grants 240210-String-QCD-BH and 259133-ObservableString.

A Derivation of transformations under bulk diffeomorphisms in derivative expansion

We first note that the transformation of the induced metric $\gamma_{\mu\nu}$ as given by (4.2) can be written compactly in the form:

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} + \rho \frac{\partial \gamma_{\mu\nu}}{\partial \tilde{r}} + \mathcal{L}_\chi \gamma_{\mu\nu}. \tag{A.1}$$

One can readily prove the following result

$$\frac{\partial}{\partial \tilde{r}} \mathcal{L}_\chi A_{\mu_1 \ldots \mu_n}^{\nu_1 \ldots \nu_n} = \mathcal{L}_\chi \frac{\partial}{\partial \tilde{r}} A_{\mu_1 \ldots \mu_n}^{\nu_1 \ldots \nu_n} + \mathcal{L}_\chi \frac{\partial}{\partial \tilde{r}} \frac{1}{2} A_{\mu_1 \ldots \mu_n}^{\nu_1 \ldots \nu_n}, \tag{A.2}$$

for an arbitrary tensor $A_{\mu_1 \ldots \mu_n}^{\nu_1 \ldots \nu_n}$. With the above rules, one can substitute (4.2) in the definition of the extrinsic curvature (3.6), we readily obtain the transformation of the latter as in (4.6).

It is also not hard to see that the transformation of the inverse induced metric is given by

$$\tilde{\gamma}^{\mu\nu} = \gamma^{\mu\nu} + \rho \frac{\partial \gamma^{\mu\nu}}{\partial \tilde{r}} + \mathcal{L}_\chi \gamma^{\mu\nu}. \tag{A.3}$$

Let us define the bare energy-momentum tensor as $t_{\mu\nu}^{\text{bare}}$ as

$$t_{\mu\nu}^{\text{bare}} = -2 \frac{1}{r^d} \gamma^{\mu\rho} (K_{\rho\nu} - \mathcal{K}_{\rho\nu}). \tag{A.4}$$

in case of Einstein’s gravity in a arbitrary $d$. The transformation of $t_{\mu\nu}^{\text{bare}}$ is then as follows

$$\tilde{t}_{\mu\nu}^{\text{bare}} = t_{\mu\nu}^{\text{bare}} + \rho \frac{\partial}{\partial \tilde{r}} t_{\mu\nu}^{\text{bare}} + \frac{d}{r} t_{\mu\nu}^{\text{bare}} + \mathcal{L}_\chi t_{\mu\nu}^{\text{bare}} - \frac{2}{r^{d-4}} \left( \nabla^\mu \nabla^\nu \rho - \nabla^2 \rho \delta^{\mu\nu} \right). \tag{A.5}$$

We readily see that the last term after transformation diverges at $r = 0$ (at the UV) if $\rho$ vanishes as $r$ asymptotically (which is allowed as it does not shift the boundary away from $r = 0$) – this must be cancelled by the transformation of the counter-term. We will see this is indeed the case. Above all indices have been lowered/raised using $g/its inverse and $\nabla$ is the covariant derivative constructed from $g$. 

- 43 -
The first counter-term required to cancel the on-shell volume divergence in $t^\mu_\nu \text{bare}$ is

$$t^\mu_\nu \text{ct}(1) = -2(d-1)\frac{1}{r^d}\delta^\mu_\nu. \quad (A.6)$$

This does not transform under bulk diffeomorphism at all, simply because in the new coordinates we have to replace $r$ by $\tilde{r}$ (note we are comparing $t^\mu_\nu \text{ct}(1)(\tilde{r},\tilde{x})$ with $\tilde{t}^\mu_\nu \text{ct}(1)(\tilde{r},\tilde{x})$ and they are verily the same as $\delta^\mu_\nu$ is invariant). Nevertheless, $\mathcal{L}_x t^\mu_\nu \text{ct}(1) = 0$ and $(\rho(\partial/\partial r) + d(\rho/r)) t^\mu_\nu \text{ct}(1) = 0$, so we can as well write for later convenience that

$$\tilde{t}^\mu_\nu \text{ct}(1) = t^\mu_\nu \text{ct}(1) + \rho \frac{\partial}{\partial \tilde{r}} t^\mu_\nu \text{ct}(1) + \frac{d\rho}{\tilde{r}} t^\mu_\nu \text{ct}(1) + \mathcal{L}_x t^\mu_\nu \text{ct}(1). \quad (A.7)$$

The second counter-term required to cancel the subleading on-shell divergence in $t^\mu_\nu \text{bare}$ is

$$t^\mu_\nu \text{ct}(2) = \frac{2}{d-2} \frac{l^2}{r^d} \left( R^\mu_\nu[\gamma] - \frac{1}{2} R[\gamma] \delta^\mu_\nu \right). \quad (A.8)$$

Identically, for any arbitrary transformation,

$$\delta R_{\nu\sigma}[\gamma] = \frac{1}{2} \nabla_\mu \left( \nabla_\nu \gamma^\mu_\rho \delta_{\rho\sigma} + \nabla_\sigma \gamma^\mu_\rho \delta_{\rho\nu} - \nabla_\rho \gamma^\mu_\nu \delta_{\rho\sigma} \right) - \frac{1}{2} \nabla_\sigma \nabla_\rho \gamma^{\alpha\beta} \delta_{\alpha\beta}. \quad (A.9)$$

To find the transformation under bulk diffeomorphism we simply need to substitute (A.1) above. Doing this, employing the vector Einstein equation (3.35) and after some further rearrangements, we get

$$\delta R_{\nu\sigma}[\gamma] = \rho \frac{\partial}{\partial \tilde{r}} R_{\nu\sigma}[\gamma] + \mathcal{L}_x R_{\nu\sigma}[\gamma] +$$

$$+ \frac{1}{2} \left( \nabla_\nu z^\mu_\sigma \nabla_\mu \rho + \nabla_\sigma z^\mu_\nu \nabla_\mu \rho + z^\mu_\nu \nabla_\mu \nabla_\sigma \rho - \nabla^2 \rho \right) +$$

$$+ \frac{1}{\tilde{r}} (d-2) \nabla_\nu \nabla_\sigma \rho + \frac{1}{\tilde{r}} \nabla_\rho \gamma^{\alpha\beta} g_{\nu\sigma}. \quad (A.10)$$

Above $z^\mu_\nu$ is as defined in (3.20) and $z_{\mu\nu} = g_{\mu\rho} z^\rho_\nu$. This implies that

$$\tilde{t}^\mu_\nu \text{ct}(2) = t^\mu_\nu \text{ct}(2) + \rho \frac{\partial}{\partial \tilde{r}} t^\mu_\nu \text{ct}(2) + \frac{d\rho}{\tilde{r}} t^\mu_\nu \text{ct}(2) + \mathcal{L}_x t^\mu_\nu \text{ct}(2) +$$

$$+ \frac{1}{d-2} \frac{1}{\tilde{r}^{d-2}} \left( \nabla^\mu \nabla_\alpha \nabla_\rho + \nabla_\nu z^\mu_\alpha \nabla^\alpha \rho + z^\mu_\alpha \nabla_\nu \nabla^\alpha \rho + z^\alpha_\nu \nabla^\mu \nabla_\alpha \rho - \nabla^2 \rho \right) +$$

$$+ \frac{1}{\tilde{r}^{d-1}} \left( \nabla^\mu \nabla_\nu \rho - \nabla^2 \rho \delta^\mu_\nu \right). \quad (A.11)$$

The terms in the last line above are precisely those needed to cancel the divergent terms in the transformation of $t^\mu_\nu \text{bare}$ as in (A.5).
We recall that
\[ t^\mu_\nu = t^\mu_\nu \text{bare} + t^\mu_\nu \text{ct}(1) + t^\mu_\nu \text{ct}(2) + O(\nabla^3). \] (A.12)

Therefore combining (A.5), (A.7) and (A.11) we obtain
\[
\tilde{t}^\mu_\nu = t^\mu_\nu + \rho \frac{\partial}{\partial \rho} t^\mu_\nu + \frac{d \rho}{r} t^\mu_\nu + \mathcal{L}_\chi t^\mu_\nu + \frac{1}{d - 2 + \frac{1}{r^2}} \left( \nabla^\mu \nabla_\alpha \nabla_\rho + \nabla_\nu z^\mu_\alpha \nabla_\rho + z^\mu_\alpha \nabla_\nu \nabla_\rho + z^\alpha_\nu \nabla_\mu \nabla_\alpha \rho - 2 \nabla^\alpha z^\mu_\nu \nabla_\alpha \rho - z^\mu_\nu \nabla_\rho \nabla_\alpha - \delta^\mu_\nu \nabla_\alpha \nabla_\beta - \right. \\
\left. - \text{Tr} \, z \, \nabla^\mu \nabla_\nu \rho + \text{Tr} \, z \, \nabla_\rho \delta^\mu_\nu \right) + O(\nabla^3). \] (A.13)

After using the inverted relation between \( z^\mu_\nu \) and \( t^\mu_\nu \) using (3.11), and retaining terms up to two derivatives, we obtain (4.7) for \( d > 2 \) as claimed before when the boundary metric is flat Minkowski space.

**B Useful identities**

Let us assume that
\[ g_{\mu\nu} = g_{(0)\mu\nu}(x) + r^2 g_{(2)\mu\nu}(x) + r^4 g_{(4)\mu\nu}(x) + O(r^6). \] (B.1)

The exact forms of \( g_{(2)\mu\nu} \) and \( g_{(4)\mu\nu} \) are not important for the moment. The expansion for the Levi-Civita connection is:
\[ \Gamma^\mu_{\nu\rho}[g] = \Gamma^\mu_{(0)\nu\rho}(x) + r^2 \Gamma^\mu_{(2)\nu\rho}(x) + r^4 \Gamma^\mu_{(4)\nu\rho}(x) + O(r^6), \] (B.2)

with
\[
\begin{align*}
\Gamma^\mu_{(0)\nu\rho} &= \Gamma^\mu_{\nu\rho}[g_{(0)}], \\
\Gamma^\mu_{(2)\nu\rho} &= \frac{1}{2} g^\nu_{(0)} \left( \nabla_{(0)\nu} g_{(2)\rho} \sigma + \nabla_{(0)\rho} g_{(2)\sigma} \nu - \nabla_{(0)\sigma} g_{(2)\nu} \right), \\
\Gamma^\mu_{(4)\nu\rho} &= \frac{1}{2} g^\nu_{(0)} \left( \nabla_{(0)\nu} g_{(4)\rho} \sigma + \nabla_{(0)\rho} g_{(4)\sigma} \nu - \nabla_{(0)\sigma} g_{(4)\nu} \right) - \frac{1}{2} g^\mu_{(0)} g_{(2)\alpha\beta} g^\beta_{(0)} \left( \nabla_{(0)\nu} g_{(2)\sigma} + \nabla_{(0)\rho} g_{(2)\sigma} + \nabla_{(0)\sigma} g_{(2)\nu} \right),
\end{align*}
\] (B.3)

where \( \nabla_{(0)} \) is the covariant derivative built from \( g_{(0)} \). It follows that
\[ R^\mu_{\nu\rho\sigma}[g] = R^\mu_{(0)\nu\rho\sigma}(x) + r^2 R^\mu_{(2)\nu\rho\sigma}(x) + r^4 R^\mu_{(4)\nu\rho\sigma}(x) + O(r^6), \] (B.4)

with
\[
\begin{align*}
R^\mu_{(0)\nu\rho\sigma} &= R^\mu_{\nu\rho\sigma}[g_{(0)}], \\
R^\mu_{(2)\nu\rho\sigma} &= \nabla_{(0)\rho} \Gamma^\mu_{(2)\nu\sigma} - \nabla_{(0)\sigma} \Gamma^\mu_{(2)\nu\rho}, \\
R^\mu_{(4)\nu\rho\sigma} &= \nabla_{(0)\rho} \Gamma^\mu_{(4)\nu\sigma} - \nabla_{(0)\sigma} \Gamma^\mu_{(4)\nu\rho} + \Gamma^\mu_{(2)\rho\alpha} \Gamma^\alpha_{(2)\nu\sigma} - \Gamma^\mu_{(2)\sigma\alpha} \Gamma^\alpha_{(2)\nu\rho},
\end{align*}
\] (B.5)
Clearly, the similar coefficients in the expansion for Ricci tensor are,

\[ R_{\mu\nu}[g] = R_{(0)\mu\nu}^\alpha(x) + r^2 R_{(2)\mu\nu}^\alpha(x) + r^4 R_{(4)\mu\nu}^\alpha(x) + \mathcal{O}(r^6), \]  

and that for the Ricci scalar are:

\[
R[g] = R_{(0)} + r^2 \left( R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} \right) + \\
+ r^4 \left( R_{(4)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} + \\
+ R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} \left( g_{(2)\alpha\gamma} g_{(0)}^{\gamma\delta} - g_{(4)\alpha\beta} \right) g_{(0)}^{\beta\nu} \right) + \\
+ \mathcal{O}(r^6).
\]  

(C.6)

\[ R_{\mu\nu}[g] = R_{(0)\mu\nu}^\alpha(x) + r^2 R_{(2)\mu\nu}^\alpha(x) + r^4 R_{(4)\mu\nu}^\alpha(x) + \mathcal{O}(r^6), \]

and that for the Ricci scalar are:

\[
R[g] = R_{(0)} + r^2 \left( R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} \right) + \\
+ r^4 \left( R_{(4)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} + \\
+ R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} \left( g_{(2)\alpha\gamma} g_{(0)}^{\gamma\delta} - g_{(4)\alpha\beta} \right) g_{(0)}^{\beta\nu} \right) + \\
+ \mathcal{O}(r^6).
\]  

(C.7)

\[ R_{\mu\nu}[g] = R_{(0)\mu\nu}^\alpha(x) + r^2 R_{(2)\mu\nu}^\alpha(x) + r^4 R_{(4)\mu\nu}^\alpha(x) + \mathcal{O}(r^6), \]

and that for the Ricci scalar are:

\[
R[g] = R_{(0)} + r^2 \left( R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} \right) + \\
+ r^4 \left( R_{(4)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} + \\
+ R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} \left( g_{(2)\alpha\gamma} g_{(0)}^{\gamma\delta} - g_{(4)\alpha\beta} \right) g_{(0)}^{\beta\nu} \right) + \\
+ \mathcal{O}(r^6).
\]  

(C.7)

\[ R_{\mu\nu}[g] = R_{(0)\mu\nu}^\alpha(x) + r^2 R_{(2)\mu\nu}^\alpha(x) + r^4 R_{(4)\mu\nu}^\alpha(x) + \mathcal{O}(r^6), \]

and that for the Ricci scalar are:

\[
R[g] = R_{(0)} + r^2 \left( R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} \right) + \\
+ r^4 \left( R_{(4)\mu\nu}^\alpha g_{(0)}^{\mu\nu} - R_{(2)\mu\nu}^\alpha g_{(0)}^{\mu\alpha} g_{(0)}^{\beta\nu} + \\
+ R_{(0)\mu\nu} g_{(0)}^{\mu\alpha} \left( g_{(2)\alpha\gamma} g_{(0)}^{\gamma\delta} - g_{(4)\alpha\beta} \right) g_{(0)}^{\beta\nu} \right) + \\
+ \mathcal{O}(r^6).
\]  

(C.7)

C Derivation of highly efficient RG flow equation in conformally flat space

We can solve Einstein’s equations perturbatively in $r$ asymptotically when the boundary metric is conformally flat, meaning it is $\eta_{\mu\nu} e^{2\sigma(x)}$ exactly as we have done in Section 3.2 when the boundary metric chosen to be flat Minkowski space. However as the conformally flat metric $\eta_{\mu\nu} e^{2\sigma(x)}$ has a curvature, the subleading term in $g_{\mu\nu}$ is $\mathcal{O}(r^2)$ and not $\mathcal{O}(r^4)$ as in $d = 4$.

To do this, it is useful to expand around the exact solution which has no term in the Fefferman-Graham expansion beyond $\mathcal{O}(r^4)$. This corresponds to the general case when the bulk metric is conformally flat just as the boundary metric. This exact solution is \[ g^*_{\mu\nu} = g_{(0)\mu\nu}(x) + r^2 g_{(2)\mu\nu}(x) + r^4 g^*_{(4)\mu\nu}(x), \]  

(C.1)

with

\[ g^*_{(4)\mu\nu} = \frac{1}{4} g_{(2)\mu\alpha} g_{(0)}^{\alpha\beta} g_{(0)}^{\beta\nu}. \]  

(C.2)

The leading term $g_{(0)\mu\nu}$ being the boundary metric should be

\[ g_{(0)\mu\nu} = e^{2\sigma(x)} \eta_{\mu\nu}. \]  

(C.3)

The subleading term $g_{(2)\mu\nu}$ takes the form

\[ g_{(2)\mu\nu} = -\frac{1}{2} \left( R_{(0)\mu\nu} - \frac{1}{6} R(0) g_{(0)\mu\nu} \right), \]  

(C.4)

for an arbitrary solution (we have used the notation of Appendix B). More explicitly,

\[ g_{(2)\mu\nu} = -\left( \partial_{\mu}\sigma \right) \left( \partial_{\nu}\sigma \right) + \partial_{\mu}\partial_{\nu}\sigma + \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \left( \partial_{\alpha}\sigma \right) \left( \partial_{\beta}\sigma \right). \]  

(C.5)

In order to find the general solution, we can write without loss of generality that

\[ g_{(4)\mu\nu}(x) = g^*_{(4)\mu\nu}(x) + a_{\mu\nu}(x). \]  

(C.6)
In this case, the general solution takes the form
\[
g_{\mu\nu} = g_{(0)\mu\nu}(x) + r^2 g_{(2)\mu\nu}(x) + r^4 \left( g_{(4)\mu\nu}^*(x) + a_{\mu\nu}(x) \right) + r^6 g_{(6)\mu\nu}(x) + \mathcal{O}(r^8). \tag{C.7}
\]

This procedure of perturbing around the exact solution (C.1) to find the \(g_{(6)\mu\nu}\) for a general solution with a flat boundary metric is advantageous, because on simple dimensional grounds \(a_{\mu\nu}\) along with its derivatives can appear only linearly in \(g_{(6)\mu\nu}\). Thus we can simply expand (3.34) linearly about the exact solution (C.1) and then truncate the solution at the leading order to find \(g_{(6)\mu\nu}\). In turn this will give us the missing term in the general highly efficient RG flow equation in conformally flat space.

In order to satisfy the vector and scalar constraints, namely (3.35) and (3.36) we require
\[
a_{\mu\nu} g_{(0)\mu\nu} = 0, \quad g_{(0)}^{\alpha\beta} \nabla_{(0)\alpha} a_{\beta\mu} = 0, \tag{C.8}
\]

where \(\nabla_{(0)}\) is constructed from \(g_{(0)\mu\nu}\). Using the various identities in Appendix B, we obtain
\[
g_{(6)\mu\nu} = -\frac{1}{12} \left( g_{(0)}^{\alpha\beta} \nabla_{(0)\alpha} a_{\beta\mu} \right) + \frac{2}{3} \left( a_{\mu\alpha} g_{(0)}^{\alpha\beta} g_{(2)\beta\nu} + g_{(2)\mu\alpha} g_{(0)}^{\alpha\beta} a_{\beta\nu} \right) + \frac{1}{6} a_{\mu\nu} g_{(2)\alpha\beta} g_{(0)}^{\alpha\beta} - \frac{1}{6} g_{(0)\mu\nu} g_{(0)}^{\alpha\gamma} g_{(0)}^{\beta\delta} g_{(2)\gamma\delta}. \tag{C.9}
\]

Substituting the above in (3.29) and employing the identities in Appendix B once again we can obtain the coefficients of the expansion
\[
t^\mu_\nu = t^\mu_\nu_{(0)\mu\nu}(x) + r^2 t^\mu_\nu_{(2)\mu\nu}(x) + r^2 \log r t^\mu_\nu_{(2a)\mu\nu}(x) + \mathcal{O}(r^4 \log r) \tag{C.10}
\]
of \(t^\mu_\nu\). Note the term proportional to \(\log r\) in (3.29) can also be shown to be proportional to the \(r^4 \log r\) in the asymptotic expansion of \(g_{\mu\nu}\) which vanishes when the boundary metric is conformally flat (as in the exact solution (C.1)). Therefore this term does not contribute at the leading order, meaning at order \(\log r\), and this is why there is no \(\log r\) term in (C.10). However it does contribute at order \(r^2 \log r\) giving rise to \(t^\mu_{(2a)\nu}\) term in (C.10).

The leading term in (C.10) is
\[
t^\mu_\nu_{(0)\mu\nu} = t^\mu_\nu_{(0)\mu\nu} = t^\mu_\nu_{(0)\mu\nu}^* + 4 g_{(0)}^{\mu\alpha} a_{\alpha\nu}, \tag{C.11}
\]

with
\[
t^\mu_{(0)\mu\nu}^* = -g_{(0)}^{\mu\alpha} g_{(2)\alpha\beta} g_{(2)\gamma\nu} + g_{(2)\alpha\nu} g_{(0)}^{\mu\alpha} g_{(2)\beta\gamma} + \frac{1}{2} \delta^\mu_\nu \left( g_{(0)}^{\alpha\gamma} g_{(2)\alpha\beta} g_{(2)\beta\gamma} - \left( g_{(0)}^{\alpha\beta} g_{(2)\alpha\beta} \right)^2 \right). \tag{C.12}
\]

The term \(t^\mu_{(0)\mu\nu}^*\) equals \(t^\mu_{(0)\mu\nu}^\infty\) for the exact solution (C.1) and is entirely a functional of the boundary metric. The subleading term in (C.10) is
\[
t^\mu_{(2)\mu\nu} = t^\mu_{(2)\mu\nu}^* + \hat{t}^\mu_{(2)\nu}, \tag{C.13}
\]
where

\[ t_{(2)\nu}^\mu = g_{(0)}^{\mu\alpha} g(2)_{\alpha\beta} g_{(0)}^{\beta\gamma} g(2)_{\gamma\delta} g_{(0)}^{\delta\nu} - \frac{1}{2} g_{(0)}^{\mu\alpha} g(2)_{\alpha\beta} g_{(0)}^{\beta\gamma} g(2)_{\gamma\nu} g_{(0)}^{\delta\epsilon} - \frac{1}{2} g_{(0)}^{\mu\alpha} g(2)_{\alpha\nu} g_{(0)}^{\beta\delta} g_{(0)}^{\gamma}\gamma g_{(0)}^{\delta\epsilon} + \frac{1}{2} \delta_{\nu}^{\epsilon} \left( - g_{(0)}^{\alpha\beta} g_{(0)}^{\beta\gamma} g_{(0)}^{\gamma\delta} g_{(0)}^{\delta\epsilon} + \left( g_{(0)}^{\alpha\gamma} g_{(0)}^{\beta\delta} g_{(0)}^{\gamma\delta} g_{(0)}^{\epsilon\phi} \right) \right), \] (C.14)

is an explicit functional of the boundary metric and

\[ \tilde{t}_{(2)\nu}^\mu = - g_{(0)}^{\alpha\beta} \nabla(0)_{\alpha} \nabla(0)_{\beta} (g_{(0)}^{\mu\rho} a_{\rho\nu}) + 4 g_{(0)}^{\mu\alpha} a_{\alpha\beta} g_{(0)}^{\beta\gamma} g_{(0)}^{\gamma\nu} + 6 \delta_{\mu}^{\beta} g_{(0)}^{\alpha\beta} g_{(0)}^{\beta\gamma} g_{(0)}^{\gamma\nu}. \] (C.15)

Also

\[ t_{(2a)\nu}^\mu = r^2 \text{ term in the expansion of} \]

\[ \left( \frac{1}{8} R_{\alpha\beta}^{\mu\nu} |g| R_{\alpha\beta}^{\mu\nu} |g| - \frac{1}{48} \nabla^{\mu} \nabla_{\nu} R[|g|] + \frac{1}{16} \nabla^{2} R_{\mu}^{\nu} [|g|] - \frac{1}{24} R[|g|] R_{\mu}^{\nu} [|g|] + \right. \]
\[ \left. + \left( \frac{1}{96} R_{\mu}^{\nu} [g] - \frac{1}{32} R_{\alpha\beta}^{\mu\nu} R_{\alpha\beta}^{\mu\nu} |g| - \frac{1}{96} \nabla^{2} R[|g|] \right) \delta_{\nu}^{\mu} \right). \] (C.16)

which is thus an explicit functional of the boundary metric as well. This can be readily obtained from the compact but explicit expansions of \( \Gamma_{\nu\rho}^{\mu}[|g|] \), \( R_{\nu\rho\sigma}^{\mu}[|g|] \) etc. obtained in Appendix B. The exact form of this is a lengthy expression which is not too illuminating, so we skip giving a more detailed expression here.

Using (C.10), (C.11), (C.12), (C.13), (C.14), (C.15) and (C.16), we can readily invert the relation between \( a_{\mu\nu} \) and \( t_{\nu}^\mu \). We obtain,

\[ \eta^{\mu\alpha} e^{-2\sigma} a_{\alpha\nu} = \frac{1}{4} \left( t_{\nu}^\mu - t_{(0)\nu}^\mu \right) - r^2 \left( t_{(2)\nu}^\mu \right) - \frac{1}{4} \left( - g_{(0)}^{\alpha\beta} \nabla(0)_{\alpha} \nabla(0)_{\beta} \left( t_{\nu}^\mu - t_{(0)\nu}^\mu \right) + \right. \]
\[ \left. + 4 \left( t_{\beta}^\mu - t_{(0)\beta}^\mu \right) g_{(0)}^{\beta\gamma} g_{(0)}^{\gamma\nu} + \right. \]
\[ \left. + 6 \delta_{\nu}^{\beta} \left( t_{\beta}^\gamma - t_{(0)\beta}^\gamma \right) g_{(0)}^{\beta\delta} g_{(0)}^{\gamma\delta} \right) \]
\[ - r^2 \log r t_{(2a)\nu} + O(r^4 \log r). \] (C.17)

Differentiating (C.10) and using the above inverted relation we finally obtain the highly
efficient RG flow equation for $t^\mu_\nu$, which is as follows:

$$\frac{\partial}{\partial r} t^\mu_\nu = r \left( 2 t^\mu_\nu + \frac{1}{2} \left( -g^{\alpha\beta}_{(0)} \nabla_{(0)\alpha} \nabla_{(0)\beta} \left( t^\mu_\nu - t^\mu_\nu + 4t^\mu_\beta g^\gamma_\beta g^{(2)\gamma}_{(0)} + 6\delta^\mu_\nu t^\gamma_\beta g^\delta_\gamma g^{(2)\delta}_{(0)} \right) + \right. \right.$$

$$+ \left. \left. 2t^\nu_{(2\alpha\nu)} \right) + 2r \log r t^\mu_\nu + O(r^3 \log r) \right). \quad (C.18)$$

Replacing $r$ by $\Lambda^{-1}$ above we obtain (4.13). We have also replaced $g^\mu_\nu$ by its more explicit form $e^{-2\sigma} \eta^\mu_\nu$, and changed notations by replacing $\nabla_{(0)}$ by $\nabla^{(\sigma)}$.

**D Derivation of transformations under bulk diffeomorphisms in UV expansion**

Here we proceed by assuming that the boundary metric is conformally flat, meaning it is $\eta_{\mu\nu} e^{2\sigma(x)}$, and that $d = 4$. We would be interested to derive the transformation of $t^\mu_\nu$ up to $O(r^4 \log r)$. Furthermore, we will restrict ourselves to two derivatives.

Assuming minimalist counterterms (those required for cancellations of UV divergences only), only the first two counterterms and the log counterterm can contribute to the transformation of $t^\mu_\nu$ up to $O(r^4 \log r)$. We can thus proceed exactly as in Appendix A and obtain the transformation by expanding (A.13) up to required orders. Note all the results in Appendix A are very general, and only in the last step, meaning from going to (A.13) to (4.7), we imposed that the boundary metric is flat Minkowski space. On top of this we need to add the transformation of the log counterterm. However this log counterterm only contributes at sixth order in derivatives, so we can ignore its effect if we restrict ourselves to two derivatives.

Doing the UV (asymptotic) expansion of $z^\mu_\nu$, $\rho$, $\chi^\mu$, $g_{\mu\nu}$ and $\nabla$ etc. in (A.13) we get

$$\tilde{t}^\mu_\nu = t^\mu_\nu + \left( r^2 \delta^\mu_\nu \frac{1}{2} \left( -g^{\alpha\beta}_{(0)} \nabla_{(0)\alpha} \nabla_{(0)\beta} t^\mu_\nu + 4t^\mu_\beta g^\gamma_\beta g^{(2)\gamma}_{(0)} + 6\delta^\mu_\nu t^\gamma_\beta g^\delta_\gamma g^{(2)\delta}_{(0)} \right) + \right.$$

$$+ \left. \left. 2t^\nu_{(2\alpha\nu)} \right) + 2r \log r t^\mu_\nu + O(r^4 \log r) \right). \quad (D.1)$$
up to fourth order in derivatives and given orders in the asymptotic expansion. Replacing \( r \) by \( \Lambda^{-1} \) above we obtain (4.14). We have also replaced \( g_{\mu\nu}^{(0)} \) by its more explicit form \( e^{-2\sigma} \eta_{\mu\nu} \), and changed notations by replacing \( \nabla_{(0)} \) by \( \nabla^{(\sigma)} \).

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Int. J. Theor. Phys.* 38 (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th].

[2] S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* B428 (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* 2 (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[4] N. Behr and A. Mukhopadhyay, *Holography as highly efficient RG flow: Part 2*. to appear soon.

[5] I. Heemskerk and J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” *JHEP* 1106 (2011) 031, arXiv:1010.1264 [hep-th].

[6] T. Faulkner, H. Liu, and M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” *JHEP* 1108 (2011) 051, arXiv:1010.4036 [hep-th].

[7] S.-S. Lee, “Quantum Renormalization Group and Holography,” *JHEP* 1401 (2014) 076, arXiv:1305.3908 [hep-th].

[8] H. Osborn, “Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories,” *Nucl. Phys.* B363 (1991) 486–526.

[9] C. Agon, V. Balasubramanian, S. Kasko, and A. Lawrence, “Coarse Grained Quantum Dynamics,” arXiv:1412.3148 [hep-th].

[10] G. Policastro, D. T. Son, and A. O. Starinets, “From AdS / CFT correspondence to hydrodynamics,” *JHEP* 0209 (2002) 043, arXiv:hep-th/0205052 [hep-th].

[11] G. Policastro, D. T. Son, and A. O. Starinets, “From AdS / CFT correspondence to hydrodynamics. 2. Sound waves,” *JHEP* 0212 (2002) 054, arXiv:hep-th/0210220 [hep-th].

[12] R. A. Janik and R. B. Peschanski, “Asymptotic perfect fluid dynamics as a consequence of AdS/CFT,” *Phys. Rev.* D73 (2006) 045013, arXiv:hep-th/0512162 [hep-th].

[13] R. A. Janik, “Viscous plasma evolution from gravity using AdS/CFT,” *Phys. Rev. Lett.* 98 (2007) 022302, arXiv:hep-th/0610144 [hep-th].

[14] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, “Relativistic viscous hydrodynamics, conformal invariance, and holography,” *JHEP* 0804 (2008) 100, arXiv:0712.2451 [hep-th].

[15] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” *JHEP* 0802 (2008) 045, arXiv:0712.2456 [hep-th].

[16] M. Natsuume and T. Okamura, “Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality,” *Phys. Rev.* D77 (2008) 066014, arXiv:0712.2916 [hep-th].

[17] W. Israel and J. Stewart, “Transient relativistic thermodynamics and kinetic theory,” *Annals Phys.* 118 (1979) 341–372.
[18] R. Iyer and A. Mukhopadhyay, “An AdS/CFT Connection between Boltzmann and Einstein,” Phys. Rev. D81 (2010) 086005, arXiv:0907.1156 [hep-th].

[19] R. Iyer and A. Mukhopadhyay, “Homogeneous Relaxation at Strong Coupling from Gravity,” Phys. Rev. D84 (2011) 126013, arXiv:1103.1814 [hep-th].

[20] R. L. Arnowitt, S. Deser, and C. W. Misner, “The Dynamics of general relativity,” Gen. Rel. Grav. 40 (2008) 1997–2027, arXiv:gr-qc/0405109 [gr-qc].

[21] M. Henningson and K. Skenderis, “The Holographic Weyl anomaly,” JHEP 9807 (1998) 023, arXiv:hep-th/9806087 [hep-th].

[22] M. Henningson and K. Skenderis, “Holography and the Weyl anomaly,” Fortsch. Phys. 48 (2000) 125–128, arXiv:hep-th/9902129 [hep-th].

[23] V. Balasubramanian and P. Kraus, “A Stress tensor for Anti-de Sitter gravity,” Commun. Math. Phys. 208 (1999) 413–428, arXiv:hep-th/9902129 [hep-th].

[24] J. de Boer, E. P. Verlinde, and H. L. Verlinde, “On the holographic renormalization group,” JHEP 0008 (2000) 003, arXiv:hep-th/9912012 [hep-th].

[25] S. de Haro, S. N. Solodukhin, and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. 217 (2001) 595–622, arXiv:hep-th/0002230 [hep-th].

[26] S. Kuperstein and A. Mukhopadhyay, “Spacetime emergence via holographic RG flow from incompressible Navier-Stokes at the horizon,” JHEP 1311 (2013) 086, arXiv:1307.1367 [hep-th].

[27] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, “Wilsonian Approach to Fluid/Gravity Duality,” JHEP 1103 (2011) 141, arXiv:1006.1902 [hep-th].

[28] S. Kuperstein and A. Mukhopadhyay, “The unconditional RG flow of the relativistic holographic fluid,” JHEP 1111 (2011) 130, arXiv:1105.4530 [hep-th].

[29] D. Brattan, J. Camps, R. Loganayagam, and M. Rangamani, “CFT dual of the AdS Dirichlet problem: Fluid/Gravity on cut-off surfaces,” JHEP 1112 (2011) 090, arXiv:1106.2577 [hep-th].

[30] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. 104 (1986) 207–226.

[31] R. Penrose and W. Rindler, Spinors and spacetime, vol. 2, Spinor and Twistor Methods in Space-Time Geometry. Cambridge University Press, 1986; Chaper 9.

[32] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz, “Diffeomorphisms and holographic anomalies,” Class. Quant. Grav. 17 (2000) 1129–1138, arXiv:hep-th/9910267 [hep-th].

[33] A. Schwimmer and S. Theisen, “Diffeomorphisms, anomalies and the Fefferman-Graham ambiguity,” JHEP 0008 (2000) 032, arXiv:hep-th/0008082 [hep-th].

[34] J. Erdmenger, “A Field theoretical interpretation of the holographic renormalization group,” Phys. Rev. D64 (2001) 085012, arXiv:hep-th/0103229 [hep-th].

[35] Y. Nakayama, “Consistency of local renormalization group in d = 3,” Nucl. Phys. B879 (2014) 37–64, arXiv:1307.8048 [hep-th].
[36] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D 47 (Feb, 1993) 1407–1419. http://link.aps.org/doi/10.1103/PhysRevD.47.1407.

[37] C. Fefferman and C. R. Graham, Conformal Invariants, in The Mathematical Heritage of Élie Cartan (Lyon, 1984). Astérisque, 1985, Numero Hors Serie, 95-116.

[38] R. K. Gupta and A. Mukhopadhyay, “On the universal hydrodynamics of strongly coupled CFTs with gravity duals,” JHEP 0903 (2009) 067, arXiv:0810.4851 [hep-th].

[39] P. Dirac, “The Hamiltonian form of field dynamics,” Can. J. Math. 3 (1951) 1–23.

[40] C. Teitelboim, “How commutators of constraints reflect the space-time structure,” Annals Phys. 79 (1973) 542–557.

[41] B. P. Dolan, “A Geometrical interpretation of renormalization group flow,” Int. J. Mod. Phys. A9 (1994) 1261–1286.

[42] E. Kiritsis, W. Li, and F. Nitti, “Holographic RG flow and the Quantum Effective Action,” Fortsch. Phys. 62 (2014) 389–454, arXiv:1401.0888 [hep-th].

[43] G. Vidal, “Entanglement Renormalization,” Phys. Rev. Lett. 99 (Nov, 2007) 220405. http://link.aps.org/doi/10.1103/PhysRevLett.99.220405.

[44] G. Vidal, Entanglement Renormalization: an introduction, chapter of the book “Understanding Quantum Phase Transitions”, edited by Lincoln D. Carr. Taylor and Francis, Boca Raton, 2010. arXiv:0912.1651.

[45] B. Swingle, “Entanglement renormalization and holography,” Phys. Rev. D 86 (Sep, 2012) 065007. http://link.aps.org/doi/10.1103/PhysRevD.86.065007.

[46] M. Nozaki, S. Ryu, and T. Takayanagi, “Holographic Geometry of Entanglement Renormalization in Quantum Field Theories,” JHEP 1210 (2012) 193, arXiv:1208.3469 [hep-th].

[47] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96 (2006) 181602, arXiv:hep-th/0603001 [hep-th].

[48] N. Lashkari, M. B. McDermott, and M. Van Raamsdonk, “Gravitational dynamics from entanglement ‘thermodynamics’,” JHEP 1404 (2014) 195, arXiv:1308.3716 [hep-th].

[49] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, and M. Van Raamsdonk, “Gravitation from Entanglement in Holographic CFTs,” JHEP 1403 (2014) 051, arXiv:1312.7856 [hep-th].

[50] A. Almheiri, X. Dong, and D. Harlow, “Bulk Locality and Quantum Error Correction in AdS/CFT,” arXiv:1411.7041 [hep-th].

[51] E. Mintun, J. Polchinski, and V. Rosenhaus, “Bulk-Boundary Duality, Gauge Invariance, and Quantum Error Correction,” arXiv:1501.06577 [hep-th].

[52] S. Hawking, “Breakdown of Predictability in Gravitational Collapse,” Phys. Rev. D14 (1976) 2460–2473.

[53] S. D. Mathur, “The Information paradox: A Pedagogical introduction,” Class. Quant. Grav. 26 (2009) 224001, arXiv:0909.1038 [hep-th].

[54] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, “Black Holes: Complementarity or Firewalls?”, JHEP 1302 (2013) 062, arXiv:1207.3123 [hep-th].
[55] R. d. M. Koch, A. Jevicki, K. Jin, and J. P. Rodrigues, “$AdS_4/CFT_3$ Construction from Collective Fields,” *Phys. Rev.* **D83** (2011) 025006, arXiv:1008.0633 [hep-th].

[56] M. R. Douglas, L. Mazzucato, and S. S. Razamat, “Holographic dual of free field theory,” *Phys. Rev.* **D83** (2011) 071701, arXiv:1011.4926 [hep-th].

[57] R. Gopakumar, “What is the Simplest Gauge-String Duality?,” arXiv:1104.2386 [hep-th].

[58] I. Sachs, “Higher spin versus renormalization group equations,” *Phys. Rev.* **D90** no. 8, (2014) 085003, arXiv:1306.6654 [hep-th].

[59] R. G. Leigh, O. Parrikar, and A. B. Weiss, “Exact renormalization group and higher-spin holography,” *Phys. Rev.* **D91** no. 2, (2015) 026002, arXiv:1407.4574 [hep-th].

[60] E. Mintun and J. Polchinski, “Higher Spin Holography, RG, and the Light Cone,” arXiv:1411.3151 [hep-th].

[61] T. Faulkner and J. Polchinski, “Semi-Holographic Fermi Liquids,” *JHEP* **1106** (2011) 012, arXiv:1001.5049 [hep-th].

[62] A. Mukhopadhyay and G. Policastro, “Phenomenological Characterization of Semiholographic Non-Fermi Liquids,” *Phys.Rev.Lett.* **111** no. 22, (2013) 221602, arXiv:1306.3941 [hep-th].

[63] E. Iancu and A. Mukhopadhyay, “A semi-holographic model for heavy-ion collisions,” arXiv:1410.6448 [hep-th].

[64] K. Skenderis and S. N. Solodukhin, “Quantum effective action from the AdS / CFT correspondence,” *Phys.Lett.* **B472** (2000) 316–322, arXiv:hep-th/9910023 [hep-th].