THE GENERALIZED QUADRATIC COVARIATION FOR FRACTIONAL BROWNIAN MOTION WITH HURST INDEX LESS THAN 1/2

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Abstract. Let $B^H$ be a fractional Brownian motion with Hurst index $0 < H < 1/2$. In this paper we study the generalized quadratic covariation $[f(B^H), B^H]^W_t$ defined by

$$[f(B^H), B^H]^W_t = \lim_{\varepsilon \to 0} \frac{2H}{\varepsilon^{2H}} \int_0^t \left\{ f(B^H_{s+\varepsilon}) - f(B^H_s) \right\} (B^H_{s+\varepsilon} - B^H_s)^{2H-1} ds,$$

where the limit is uniform in probability and $x \mapsto f(x)$ is a deterministic function. We construct a Banach space $H$ of measurable functions such that the generalized quadratic covariation exists in $L^2$ and the Bouleau-Yor identity takes the form

$$[f(B^H), B^H]^W_t = - \int_\mathbb{R} f(x) \mathcal{L}^H(dx, t)$$

provided $f \in H$, where $\mathcal{L}^H(x, t)$ is the weighted local time of $B^H$. This allows us to write the fractional Itô formula for absolutely continuous functions with derivative belonging to $H$. These are also extended to the time-dependent case.

1. INTRODUCTION

Given $H \in (0, 1)$, a fractional Brownian motion (fBm) with Hurst index $H$ is a mean zero Gaussian process $B^H = \{B^H_t, 0 \leq t \leq T\}$ such that

$$E\left[B^H_t B^H_s\right] = \frac{1}{2} \left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s \geq 0$. For $H = 1/2$, $B^H$ coincides with the standard Brownian motion $B$. $B^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $B^H$. As a Gaussian process, one can construct the stochastic calculus of variations with respect to $B^H$. Some surveys and complete literatures for fBm could be found in Biagini et al [2], Decreusefond...
and Üstünel [6], Gradinaru et al [14, 15], Hu [18], Mishura [19] and Nualart [23]. It is well-known that the usual quadratic variation $[B^H, B^H]_t = 0$ for $2H > 1$ and $[B^H, B^H]_t = \infty$ for $2H < 1$, where

$$[B^H, B^H]_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t (B^H_{s+\varepsilon} - B^H_s)^2 ds$$

in probability. Clearly, we have also

$$[B^H, B^H]_t = \lim_{n \to \infty} \sum_{j=1}^n \left( B^H_{jt/n} - B^H_{(j-1)t/n} \right)^2,$$

where the limit is uniform in probability. This is inconvenience to some studies and applications for fBm. We need to find a substitution tool. Recently, Gradinaru et al [14] (see also [15] and the references therein) have introduced some substitution tools and studied some fine problems. They introduced firstly an Itô formula with respect to a symmetric-Stratonovich integral, which is closer to the spirit of Riemann sums limits, and defined a class of high order integrals having an interest by themselves. On the other hand, inspired by Gradinaru-Nourdin [12, 13] and Nourdin et al [21, 22], as the substitution tool of the quadratic variation, Yan et al [27] considered the generalized quadratic covariation, and proved its existence for $\frac{1}{2} < H < 1$ (Thanks to the suggestions of some Scholars we use the present appellation).

**Definition 1.1.** Let $0 < H < 1$ and let $f$ be a measurable function on $\mathbb{R}$. The limit

$$\lim_{\varepsilon \downarrow 0} \frac{2H}{\varepsilon} \int_0^t \left\{ f(B^H_{s+\varepsilon}) - f(B^H_s) \right\} (B^H_{s+\varepsilon} - B^H_s)s^{2H-1} ds$$

is called the generalized quadratic covariation of $f(B^H)$ and $B^H$, denoted by $[f(B^H), B^H]_{t}^{(W)}$, provided the limit exists uniformly in probability.

In particular, we have

$$[B^H, B^H]_{t}^{(W)} = t^{2H}$$

for all $0 < H < 1$. If $H = \frac{1}{2}$, the generalized quadratic covariation coincides with the usual quadratic covariation of Brownian motion $B$. For $\frac{1}{2} < H < 1$, Yan et al [28] showed the generalized quadratic covariation can also be defined as

$$[f(B^H), B^H]_{t}^{(W)} = 2H \lim_{\|\pi_n\| \to 0} \sum_{t_j \in \pi_n} (\Lambda_j)^{2H-1} \left\{ f(B^H_{t_j}) - f(B^H_{t_{j-1}}) \right\} (B^H_{t_j} - B^H_{t_{j-1}}),$$

provided the limit exists uniformly in probability, where $\pi_n = \{ 0 = t_0 < t_1 < \cdots < t_n = t \}$ denotes an arbitrary partition of the interval $[0, t]$ with $\|\pi_n\| = \sup_j (t_j - t_{j-1}) \to 0$, and $\Lambda_j = \frac{t_j - t_{j-1}}{t_{j+1} - t_{j-1}}, j = 1, 2, \ldots, n$. Moreover, by applying the time reversal $\tilde{B}^H_t = B^H_{T-t}$ on $[0, T]$ and the integral

$$\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t),$$

Yan et al [28] constructed a Banach space $\mathbb{B}_H$ of measurable functions such that the generalized quadratic covariation $[f(B^H), B^H]_{t}^{(W)}$ exists in $L^2$ if $f \in \mathbb{B}_H$, where

$$\mathcal{L}^H(x, t) = 2H \int_0^t \delta(B^H_s - x)s^{2H-1} ds.$$
is the weighted local time of fBm $B^H$. However, when $0 < H < \frac{1}{2}$ the method used in Yan et al [27, 28] is inefficacy. In the present paper, we shall consider the generalized quadratic covariation with $0 < H < \frac{1}{2}$. Our start point is to consider the decomposition

$$
\frac{1}{\varepsilon^{2H}} \int_0^t \{ f(B^H_{s+\varepsilon}) - f(B^H_s) \} (B^H_{s+\varepsilon} - B^H_s) ds^{2H} = \frac{1}{\varepsilon^{2H}} \int_0^t f(B^H_s) (B^H_{s+\varepsilon} - B^H_s) ds^{2H} - \frac{1}{\varepsilon^{2H}} \int_0^t f(B^H_s) (B^H_{s+\varepsilon} - B^H_s) ds^{2H}.
$$

(1.3)

Clearly, if the modulus in expression (1.3) is $\varepsilon$, the decomposition is meaningless in general. For example, for $f(x) = x$ we have

$$
\frac{1}{\varepsilon} \int_0^t E \left[ B^H_s (B^H_{s+\varepsilon} - B^H_s) \right] ds^{2H} = \frac{1}{\varepsilon} \int_0^t \frac{1}{2} [(s+\varepsilon)^{2H} - s^{2H} - \varepsilon^{2H}] ds^{2H} \rightarrow -\infty,
$$

as $\varepsilon \downarrow 0$. However,

$$
\frac{1}{\varepsilon^{2H}} \int_0^t E B^H_s (B^H_{s+\varepsilon} - B^H_s) ds^{2H} = \frac{1}{\varepsilon^{2H}} \int_0^t \frac{1}{2} [s^{2H} + \varepsilon^{2H} - (s+\varepsilon)^{2H}] ds^{2H} \rightarrow \frac{1}{2} 2^{2H},
$$

as $\varepsilon \downarrow 0$. Thus, for $0 < H < \frac{1}{2}$ we can consider the decomposition (1.3). By estimating the two terms of the right hand side in the decomposition (1.3), respectively, we can construct a Banach space $\mathcal{H}$ of measurable functions $f$ on $\mathbb{R}$ such that $\|f\|_{\mathcal{H}} < \infty$, where

$$
\|f\|_{\mathcal{H}} = \sqrt{\int_0^T \int_\mathbb{R} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi s^{1-H}}}} + \sqrt{\int_0^T \int_\mathbb{R} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{dx ds}{\sqrt{2\pi (T-s)^{1-H}}}}.
$$

We show that generalized quadratic covariation $[f(B^H), B^H_t]^{(W)}$ exists in $L^2$ for all $t \in [0, T]$ if $f \in \mathcal{H}$. This allows us to write Itô’s formula for absolutely continuous functions with derivative belonging to $\mathcal{H}$ and to give the Bouleau-Yor identity. It is important to note that the decomposition (1.3) is inefficacy for $\frac{1}{2} < H < 1$.

This paper is organized as follows. In Section 2 we present some preliminaries for fBm. In Section 3, we establish some technical estimates associated with fractional Brownian motion with $0 < H < \frac{1}{2}$. In Section 4 we prove the existence of the generalized quadratic covariation. We construct the Banach space $\mathcal{H}$ such that the generalized quadratic covariation $[f(B^H), B^H_t]^{(W)}$ exists in $L^2$ for $f \in \mathcal{H}$. As an application we show that the Itô type formula (Föllmer-Protter-Shirayev’s formula)

$$
F(B^H) = F(0) + \int_0^t f(B^H_s) dB^H_s + \frac{1}{2} [f(B^H), B^H_t]^{(W)}
$$

holds, where $F$ is an absolutely continuous function with the derivative $F' = f \in \mathcal{H}$. In Section 5 we introduce the integral of the form

$$
\int_\mathbb{R} f(x) \mathcal{Z}^H(dx, t),
$$

(1.4)
where $x \mapsto f(x)$ is a deterministic function. We show that the integral (1.4) exists in $L^2$, and the Bouleau-Yor identity takes the form
\[
[f(B^H), B^H]_t^{(W)} = -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)
\]
provided $f \in \mathcal{H}$. Moreover, by applying the integral (1.4) we show that (1.1) and (1.2) coincide for $0 < H < 1$ when $f \in \mathcal{H}$. In Section 6 we consider the time-dependent case, and define the local time of $B^H$ with $0 < H < 1$ on a continuous curve.

2. Preliminaries

In this section, we briefly recall some basic definitions and results of fBm. For more aspects on these material we refer to Biagini et al [2], Hu [18], Mishura [19], Nualart [23] and the references therein. Throughout this paper we assume that $0 < H < 1$ is arbitrary but fixed and let $B^H = \{B^H_t, 0 \leq t \leq T\}$ be a one-dimensional fBm with Hurst index $H$ defined on $(\Omega, \mathcal{F}, P)$. Let $(S)^*$ be the Hida space of stochastic distributions and let $\diamond$ denote the Wick product on $(S)^*$. Then $t \mapsto B^H_t$ is differentiable in $(S)^*$. Denote
\[
W^{(H)}_t = \frac{dB^H_t}{dt} \in (S)^*.
\]
We call $W^{(H)}$ the fractional white noise. For $u : \mathbb{R}_+ \rightarrow (S)^*$, in a white noise setting we define its Wick-Itô-Skorohod (WIS) stochastic integral with respect to $B^H$ by
\[
\int_0^t u_s dB^H_s := \int_0^t u_s \diamond W^{(H)}_s ds,
\]
whenever the last integral exists as an integral in $(S)^*$. We call these fractional Itô integrals, because these integrals share some properties of the classical Itô integral. The integral is closed in $L^2$, and moreover, for any $f \in C^{2,1}(\mathbb{R} \times [0, +\infty))$ the following Itô type formula holds:
\[
f(B^H_t, t) = f(0, 0) + \int_0^t \frac{\partial}{\partial x} f(B^H_s, s) dB^H_s
\]
\[
+ \int_0^t \frac{\partial}{\partial s} f(B^H_s, s) ds + H \int_0^t \frac{\partial^2}{\partial x^2} f(B^H_s, s) s^{2H-1} ds.
\]
The fBm $B^H$ has a local time $\mathcal{L}(x, t)$ continuous in $(x, t) \in \mathbb{R} \times [0, \infty)$ which satisfies the occupation formula (see Geman-Horowitz [11])
\[
\int_0^t \phi(B^H_s, s) ds = \int_\mathbb{R} dx \int_0^t \phi(x, s) \mathcal{L}^H(x, ds)
\]
for every continuous and bounded function $\phi(x, t) : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, and such that
\[
\mathcal{L}^H(x, t) = \int_0^t \delta(B^H_s - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(s \in [0, t], |B^H_s - x| < \epsilon),
\]
where $\lambda$ denotes Lebesgue measure and $\delta(x)$ is the Dirac delta function. Define the so-call weighted local time $\mathcal{L}^H(x, t)$ of $B^H$ at $x$ as follows
\[
\mathcal{L}^H(x, t) = 2H \int_0^t s^{2H-1} \mathcal{L}^H(x, ds) = 2H \int_0^t \delta(B^H_s - x) s^{2H-1} ds.
\]
Then the Tanaka formula

\[ |B_t^H - x| = |x| + \int_0^t \text{sign}(B_s^H - x)dB_s^H + \mathcal{L}^H(x,t) \]

holds.

For \( H \in (0,1) \) we define the operator \( M \) on \( L^2(\mathbb{R}) \) as follows (see Chapter 4 in Biagini et al [2] and Elliott-Van der Hoek [8]):

\[
Mf(x) = -\frac{\beta_H}{2H} d\frac{d}{dx} \int_\mathbb{R} \frac{(s-x)}{|s-x|^{\frac{3}{2}-H}} f(s) ds, \quad f \in L^2(\mathbb{R}),
\]

where \( \beta_H \) is a normalizing constant. In particular, for \( H = \frac{1}{2} \) we have \( Mf(x) = f(x) \), and for \( 0 < H < \frac{1}{2} \) we have

\[
Mf(x) = \beta_H \int_\mathbb{R} \frac{f(x-s) - f(x)}{|s|^{\frac{3}{2}-H}} ds.
\]

As an example let us recall \( M_{1[a,b]}(x) \), i.e., \( Mf \) when \( f \) is the indicator function of an interval \([a,b]\) with \( a < b \). By Elliott-Van der Hoek [8], \( M_{1[a,b]}(x) \) can be calculated explicitly as

\[
M_{1[a,b]}(x) = \frac{\sqrt{\Gamma(2H+1)} \sin(\pi H)}{2\Gamma(H+\frac{1}{2}) \cos\left(\frac{\pi}{2}(H+\frac{1}{2})\right)} \left( \frac{b-x}{|b-x|^{\frac{3}{2}-H}} - \frac{a-x}{|a-x|^{\frac{3}{2}-H}} \right).
\]

By using the operator \( M \) we can give the relation between fractional and classical white noise (see Chapter 4 in Biagini et al [2])

\[ W_t^{(H)} = MW_t, \]

which leads to

\[
\int_0^T u_t dB_t^H = \int_\mathbb{R} M (u 1_{[0,T]})_t \delta B_t,
\]

where \( u \) is an adapted process and \( \int_\mathbb{R} v_t \delta B_t \) denotes the Skorohod integral with respect to Brownian motion \( B \) defined by

\[
\int_\mathbb{R} v_t \delta B_t := \int_\mathbb{R} v_t \diamond W_t dt.
\]

Let \( D_t^{(H)} \) denotes the Hida-Malliavin derivative with respect to \( B^H \). In the classical case \( (H = 1/2) \) we use the notation \( D_t \) for the corresponding Hida-Malliavin derivative (for further details, see Nualart [23] and Biagini et al [2]). We have

\[ D_t F = MD_t^{(H)} F \]

and

\[
E \left[ F \int_0^T u_s dB_s^H \right] = E \left[ \int_\mathbb{R} (Mu 1_{[0,T]})_s (MD_s^{(H)} F) ds \right]
\]

for \( F \in L^2(P) \).
3. Some basic estimates

In this section we will establish some technical estimates associated with fractional Brownian motion with \(0 < H < \frac{1}{2}\). For simplicity throughout this paper we let \(C\) stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to \(c\).

**Lemma 3.1.** For all \(t, s \in [0, T]\), \(t \geq s\) and \(0 < H < 1\) we have

\[
\frac{1}{2} (2 - 2^H) s^{2H} (t - s)^{2H} \leq t^{2H} s^{2H} - \mu^2 \leq 2 s^{2H} (t - s)^{2H},
\]

where \(\mu = E(B_t^H B_s^H)\).

By the local nondeterminacy of fBm we can prove the lemma. Here, we shall use an elementary method to prove it. We shall use the following inequalities:

\[
(1 + x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha
\]

\[
(2 - 2^\alpha)x^\alpha (1 - x)^\alpha \leq (1 - x)^\alpha - (1 - x^\alpha) \leq x^\alpha (1 - x)^\alpha
\]

with \(0 \leq x, \alpha \leq 1\). The inequality (3.2) is a calculus exercise, and it is stronger than the well known (Bernoulli) inequality

\[
(1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha,
\]

because \(2^\alpha - 1 \leq \alpha\) for all \(0 \leq \alpha \leq 1\). The inequalities (3.3) are the improvement of the classical inequality

\[
1 - x^\alpha \leq (1 - x)^\alpha.
\]

The right inequality in (3.3) follows from the fact

\[
(1 - x)^\alpha (1 - x^\alpha) \leq 1 - x^\alpha.
\]

For the left inequality in (3.3), by (3.2) we have

\[
1 = (1 - x + x)^\alpha \leq (1 - x)^\alpha \lor x^\alpha + (2^\alpha - 1) [(1 - x)^\alpha \land x^\alpha]
\]

for \(0 \leq x \leq 1\), where \(x \lor y = \max\{x, y\}\) and \(x \land y = \min\{x, y\}\), which deduces

\[
(1 - x)^\alpha - (1 - x^\alpha) \geq (2 - 2^\alpha)(1 - x)^\alpha \land x^\alpha
\]

\[
\geq (2 - 2^\alpha)(1 - x)^\alpha x^\alpha.
\]

**Proof of 3.1.** Take \(s = xt, 0 \leq x \leq 1\). Then we can rewrite \(\rho_{r,s} := t^{2H} s^{2H} - \mu^2\) as

\[
\rho_{r,s} = t^{4H} \left\{ x^{2H} - \frac{1}{4} \left[ 1 + x^{2H} - (1 - x)^{2H} \right]^2 \right\}
\]

\[
\equiv t^{4H} G(x).
\]

In order to show the lemma we claim that

\[
\frac{1}{2} (2 - 2^H) x^{2H} (1 - x)^{2H} \leq G(x) \leq 2 x^{2H} (1 - x)^{2H}
\]
for all \( x \in [0, 1] \). We have
\[
G(x) = x^{2H} - \frac{1}{4} \left[ 1 + x^{2H} - (1 - x)^{2H} \right]^2 \\
= \frac{1}{4} \left\{ 2x^H - (1 + x^{2H} - (1 - x)^{2H}) \right\} \left\{ 2x^H + (1 + x^{2H} - (1 - x)^{2H}) \right\} \\
= \frac{1}{4} \left\{ (1 - x)^{2H} - (1 - x^H)^2 \right\} \left\{ 2x^H + x^{2H} + 1 - (1 - x)^{2H} \right\}.
\]

Thus, (3.4) follows from (3.3) and the facts
\[
(1 - x)^H \leq (1 - x)^H + (1 - x^H) \leq 2(1 - x)^H, \\
2x^H \leq 2x^H + x^{2H} + 1 - (1 - x)^{2H} \leq 4x^H.
\]
This completes the proof. \( \square \)

**Lemma 3.2.** For all \( t, s \in [0, T] \), \( t \geq s \) and \( 0 < H < \frac{1}{2} \) we have
\[
\frac{1}{2} \left| (t - s)^{2H} - \mu \right| \leq (t - s)^{2H}, \\
and
\frac{1}{2} \left| (2 - 2^H)(s^H)(t - s)^{2H} - \mu \right| \leq \frac{1}{2} \left( \frac{s}{t} \right)^{2H} (t - s)^{2H},
\]
where \( \mu = E(B_t^H B_s^H) \).

**Proof.** The inequalities (3.5) follow from
\[
t^{2H} - \mu = t^{2H} - \frac{1}{2} \left( t^{2H} + s^{2H} - (t - s)^{2H} \right) \\
= \frac{1}{2} \left( t^{2H} - s^{2H} \right) + \frac{1}{2} (t - s)^{2H}.
\]
In order to show that (3.6), we have
\[
s^{2H} - \mu = s^{2H} - \frac{1}{2} \left( t^{2H} + s^{2H} - (t - s)^{2H} \right) \\
= \frac{1}{2} t^{2H} \left\{ \left( 1 - s^t \right)^{2H} - \left( 1 - \left( \frac{s}{t} \right)^{2H} \right) \right\}.
\]
Thus, the inequalities (3.6) follow from (3.3). This completes the proof. \( \square \)

**Lemma 3.3.** For \( 0 < H < \frac{1}{2} \) we have
\[
| E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right] | \leq C_H \frac{(t - s)^{2H}(t' - s')^{2H}}{(s - t')^{2H}}
\]
for all \( 0 < s' < t' < s < t \).
Moreover, the estimate (3.7) holds also for all $0 < s' < s < t'< t$. In fact we have
\[
(t' - s)^{2H} = (t' - s)^{2H} (t' - s)^{2H} \leq (t - s)^{2H} (t' - s')^{2H},
\]
\[
(t - t')^{2H} (t' - s)^{2H} \leq (t - s)^{2H} (t' - s')^{2H},
\]
\[
(s - s')^{2H} (t' - s)^{2H} \leq (t' - s')^{2H} (t - s)^{2H},
\]
\[
(t - s')^{2H} = \{(t - s) + (s - s')\}^{2H} \leq (t - s)^{2H} + (s - s')^{2H}
\]
\[
= \frac{(t - s)^{2H} (t - s')^{2H} + (s - s')^{2H} (t' - s)^{2H}}{(t' - s)^{2H}}
\]
\[
\leq 2 \frac{(t - s)^{2H} (t' - s')^{2H}}{(t' - s)^{2H}},
\]
which gives
\[
|E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] | = \frac{1}{2} \left\{ |t - s'|^{2H} + |s - t'|^{2H} - |t - t'|^{2H} - |s - s'|^{2H} \right\}
\]
\[
\leq 3 \frac{(t - s)^{2H} (t' - s')^{2H}}{(t' - s)^{2H}}.
\]

Proof of (3.7). For $0 < s' < t' < s < t \leq T$ we define the function $x \mapsto G_{s,t}(x)$ on $[s', t']$ by
\[
G_{s,t}(x) = (s - x)^{2H} - (t - x)^{2H}.
\]
Thanks to mean value theorem, we see that there are $\xi \in (s', t')$ and $\eta \in (s, t)$ such that
\[
2E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] = G_{s,t}(t') - G_{s,t}(s')
\]
\[
= 2H (t' - s') \left[ (t - \xi)^{2H-1} - (s - \xi)^{2H-1} \right]
\]
\[
= 2H (2H - 1) (t' - s') (t - s) (\eta - \xi)^{2H-2} \leq 0,
\]
which gives
\[
(3.8) \quad |E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] | \leq \frac{(t' - s')(t - s)}{(s - t')^{2-2H}}.
\]

On the other hand, noting that
\[
\frac{|E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] |}{(t - s)^{H} (t' - s')^{H}} \leq 1,
\]
we see that
\[
\frac{|E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] |}{(t - s)^{H} (t' - s')^{H}} \leq \left( \frac{|E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] |}{(t - s)^{H} (t' - s')^{H}} \right)^{\alpha}
\]
for all $\alpha \in [0, 1]$. Combining this with (3.8), we get
\[
|E \left[ (B_t^H - B_s^H) (B_{t'}^H - B_{s'}^H) \right] | \leq \frac{(t - s)^{(1-\alpha)H + \alpha} (t' - s')^{(1-\alpha)H + \alpha}}{(s - t')^{\alpha(2-2H)}},
\]
and the lemma follows by taking $\alpha = H/(1 - H)$. □
Lemma 3.4. For $0 < H < \frac{1}{2}$ we have
\[
\begin{align*}
|E[B_t^H(B_t^H - B_s^H)]| & \leq (t - s)^{2H}, \\
|E[B_t^H(B_s^H - B_r^H)]| & \leq (s - r)^{2H}, \\
|E[B_r^H(B_t^H - B_s^H)]| & \leq (t - s)^{2H}
\end{align*}
\]
for all $t > s > r > 0$.

Let $\varphi(x, y)$ be the density function of $(B_s^H, B_r^H)$ $(s > r > 0)$. That is
\[
\varphi(x, y) = \frac{1}{2\pi \rho} \exp \left\{ - \frac{1}{2\rho^2} \left( r^{2H} x^2 - 2\mu xy + s^{2H} y^2 \right) \right\},
\]
where $\mu = E(B_s^H B_r^H)$ and $\rho^2 = r^{2H} s^{2H} - \mu^2$.

Lemma 3.5. Let $f \in C^1(\mathbb{R})$ admit compact support. Then we have
\[
|E[f'(B_s^H)f'(B_r^H)]| \leq \frac{C_H s^H}{r^{H(s-r)}2^{2H}} \left( E\left[|f(B_s^H)|^2\right] E\left[|f(B_r^H)|^2\right] \right)^{1/2}
\]
for all $s > r > 0$ and $0 < H < \frac{1}{2}$.

Proof. Elementary calculation shows that
\[
\int_{\mathbb{R}^2} f^2(y)(x - \frac{\mu}{r^{2H}} y)^2 \varphi(x, y) dxdy = \frac{\rho^2}{r^{2H}} \int_{\mathbb{R}} f^2(y) \frac{1}{\sqrt{2\pi \rho}} e^{-\frac{y^2}{2\rho^2}} dy = \frac{\rho^2}{r^{2H}} E\left[|f(B_r^H)|^2\right],
\]
which implies that
\[
\frac{1}{\rho^2} \int_{\mathbb{R}^2} |f(x)f(y)(s^{2H}y - \mu x)(r^{2H}x - \mu y)| \varphi(x, y) dxdy \\
\leq \frac{r^{H} s^{H}}{\rho^2} \left( E\left[|f(B_s^H)|^2\right] E\left[|f(B_r^H)|^2\right] \right)^{1/2}
\]
\[
\leq \frac{C_H s^H}{r^{H(s-r)}2^{2H}} \left( E\left[|f(B_s^H)|^2\right] E\left[|f(B_r^H)|^2\right] \right)^{1/2}
\]
by Lemma 3.1. It follows that
\[
|E[f'(B_s^H)f'(B_r^H)]| = \int_{\mathbb{R}^2} f(x)f(y) \left( \frac{\partial^2}{\partial x \partial y} \right) \varphi(x, y) dxdy \\
= \int_{\mathbb{R}^2} f(x)f(y) \left\{ \frac{1}{\rho^2} (s^{2H}y - \mu x)(r^{2H}x - \mu y) + \frac{\mu}{\rho^2} \right\} \varphi(x, y) dxdy \\
\leq \frac{C_H s^H}{r^{H(s-r)}2^{2H}} \left( E\left[|f(B_s^H)|^2\right] E\left[|f(B_r^H)|^2\right] \right)^{1/2}.
\]
This completes the proof.

Lemma 3.6. Let $f \in C^2(\mathbb{R})$ admit compact support. Then we have
\[
|E[f''(B_s^H)f(B_r^H)]| \leq \frac{C_H}{(s-r)^{2H}} \left( E\left[|f(B_s^H)|^2\right] E\left[|f(B_r^H)|^2\right] \right)^{1/2}
\]
for all $s > r > 0$ and $0 < H < \frac{1}{2}$. 
Proof. A straightforward calculation shows that

$$\int_{\mathbb{R}^2} f^2(y)(x - \frac{\mu}{r^{2H}}y)^4 \varphi(x, y) dxdy = \frac{3\rho^4}{r^{4H}} \int_{\mathbb{R}} f^2(y) \frac{1}{\sqrt{2\pi r^{2H}}} e^{-\frac{x^2}{2r^{2H}}} dy,$$

which deduces

$$\frac{1}{\rho^4} \int_{\mathbb{R}^2} f(x)f(y)(r^{2H} x - \mu y)^2 \varphi(x, y)dxdy \leq \frac{C_H}{(s - r)^{2H}} \sqrt{E[|f(B^H_s)|^2] E[|f(B^H_r)|^2]}$$

by Cauchy’s inequality and Lemma 3.1. It follows that

$$|E[f''(B^H_s)f(B^H_r)]| = \left| \int_{\mathbb{R}^2} f(x)f(y) \frac{\partial^2}{\partial x^2} \varphi(x, y) dxdy \right|$$

$$= \left| \int_{\mathbb{R}^2} f(x)f(y) \left\{ \frac{1}{\rho^4}(r^{2H} x - \mu y)^2 - \frac{r^{2H}}{\rho^2} \right\} \varphi(x, y)dxdy \right|$$

$$\leq \frac{C_H}{(s - r)^{2H}} \left( E[|f(B^H_s)|^2] E[|f(B^H_r)|^2] \right)^{1/2}.$$

This completes the proof. \(\square\)

4. Existence of the generalized quadratic covariation

In this section, for \(0 < H < \frac{1}{2}\) we study the existence of the generalized quadratic covariation. Denote

$$J_\varepsilon(f, t) := \frac{1}{\varepsilon^{2H}} \int_0^t \{ f(B_{s+\varepsilon}^H) - f(B_s^H) \} (B_{s+\varepsilon}^H - B_s^H) ds^{2H}$$

for \(\varepsilon > 0\) and \(t \geq 0\). Recall that the generalized quadratic covariation \([f(B^H), B^H]_t^{(W)}\) is defined as

$$[f(B^H), B^H]_t^{(W)} := \lim_{\varepsilon \downarrow 0} J_\varepsilon(f, t),$$

provided the limit exists uniformly in probability. Clearly, we have (see, for example, Klein and Giné [16])

$$[B^H, B^H]_t^{(W)} = t^{2H}$$

for all \(t \geq 0\). In fact, one can easily prove that

$$E \left| \frac{1}{\varepsilon^{2H}} \int_0^t (B_{s+\varepsilon}^H - B_s^H)^2 ds - t^{2H} \right|^2$$

$$= \frac{1}{\varepsilon^{2H}} \int_0^t \int_0^t E \left[ (B_{r+\varepsilon}^H - B_r^H)^2(B_{s+\varepsilon}^H - B_s^H)^2 \right] ds^{2H} dr^{2H} - t^{4H}$$

$$\rightarrow 0$$

for \(t \geq 0\), as \(\varepsilon \downarrow 0\).
Consider the decomposition

\[
\frac{1}{\varepsilon^{2H}} \int_0^t \left\{ f(B^H_{s+\varepsilon}) - f(B^H_s) \right\} (B^H_{s+\varepsilon} - B^H_s) ds^{2H} = \frac{1}{\varepsilon^{2H}} \int_0^t f(B^H_{s+\varepsilon})(B^H_{s+\varepsilon} - B^H_s) ds^{2H} - \frac{1}{\varepsilon^{2H}} \int_0^t f(B^H_s)(B^H_{s+\varepsilon} - B^H_s) ds^{2H}
\]

\(\equiv I^+ \varepsilon (f, t) - I^- \varepsilon (f, t),\) 

and define the set \(\mathcal{H} = \{ f : \text{measurable functions on } \mathbb{R} \text{ such that } \|f\|_\mathcal{H} < \infty \},\) where

\[
\|f\|_\mathcal{H} := \sqrt{\int_0^T \int_\mathbb{R} |f(x)|^2 e^{-\frac{x^2}{2s^{1-H}}} \frac{dxds}{\sqrt{2\pi s^{1-H}}} + \sqrt{\int_0^T \int_\mathbb{R} |f(x)|^2 e^{-\frac{x^2}{2s^{1-H}}} \frac{dxds}{\sqrt{2\pi (T-s)^{1-H}}}}.
\]

Then, \(\mathcal{H}\) is a Banach space and the set \(\mathcal{E}\) of elementary functions of the form

\[
f_\Delta (x) = \sum_i f_i 1_{(x_{i-1}, x_i]} (x)
\]

is dense in \(\mathcal{H}\), where \(\{x_i, 0 \leq i \leq l\}\) is an finite sequence of real numbers such that \(x_i < x_{i+1}\). Moreover, \(\mathcal{H}\) contains the sets \(\mathcal{H}_\gamma, \gamma > 2\), of measurable functions \(f\) such that

\[
\int_0^T \int_\mathbb{R} |f(x)|^\gamma e^{-\frac{x^2}{2s^{1-H}}} \frac{dxds}{\sqrt{2\pi s^{1-H}}} < \infty.
\]

Our main object of this section is to explain and prove the following theorem.

**Theorem 4.1.** Let \(0 < H < \frac{1}{2}\) and \(f \in \mathcal{H}\). Then the generalized quadratic covariation \([f(B^H), B^H]_t^{(W)}\) exists and

\[
E \left| [f(B^H), B^H]_t^{(W)} \right|^2 \leq C_H \|f\|_\mathcal{H}^2.
\]

We split the proof into several lemmas, and for simplicity throughout this paper we let \(T = 1\).

**Lemma 4.1.** Let \(0 < H < \frac{1}{2}\) and let \(f\) be an infinitely differentiable function with compact support. We then have

\[
E \left| I^- \varepsilon (f, t) \right|^2 \leq C_H \|f\|_\mathcal{H}^2,
\]

\[
E \left| I^+ \varepsilon (f, t) \right|^2 \leq C_H \|f\|_\mathcal{H}^2
\]

for all \(0 < \varepsilon \leq 1\).
Proof. We need only to obtain the first estimate. It follows from (2.6) that

\[
E \left[ f(B^H_s) f(B^H_r) (B^H_{s+\epsilon} - B^H_s) (B^H_{r+\epsilon} - B^H_r) \right] \\
= E \left[ f(B^H_s) f(B^H_r) (B^H_{s+\epsilon} - B^H_s) \int_r^{r+\epsilon} dB_t^H \right] \\
= E \int_\mathbb{R} M_{1[r,r+\epsilon]}(l) MD_t^H f(B^H_s) f(B^H_r) (B^H_{s+\epsilon} - B^H_s) dl \\
= \int_\mathbb{R} M_{1[r,r+\epsilon]}(l) M_{1[0,s]}(l) E \left[ f'(B^H_s) f(B^H_r) (B^H_{s+\epsilon} - B^H_s) \right] dl \\
+ \int_\mathbb{R} M_{1[r,r+\epsilon]}(l) M_{1[0,r]}(l) E \left[ f'(B^H_s) f'(B^H_r) (B^H_{s+\epsilon} - B^H_s) \right] dl \\
+ \int_\mathbb{R} M_{1[r,r+\epsilon]}(l) M_{1[s,s+\epsilon]}(l) E \left[ f(B^H_s) f(B^H_r) \right] dl \\
= E \left[ B^H_s (B^H_{r+\epsilon} - B^H_r) \right] E \left[ f'(B^H_s) f(B^H_r) (B^H_{s+\epsilon} - B^H_s) \right] \\
+ E \left[ B^H_r (B^H_{r+\epsilon} - B^H_r) \right] E \left[ f(B^H_s) f'(B^H_r) (B^H_{s+\epsilon} - B^H_s) \right] \\
+ E \left[ (B^H_{r+\epsilon} - B^H_r) (B^H_{s+\epsilon} - B^H_s) \right] E \left[ f(B^H_r) f(B^H_r) \right] \\
\equiv \Psi_\epsilon(s,r,1) + \Psi_\epsilon(s,r,2) + \Psi_\epsilon(s,r,3).
\]

In order to end the proof we claim now that

\[
(4.7) \quad \frac{1}{\epsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\epsilon(s,r,k) ds 2^H dr 2^H \right| \leq C_H \|f\|_{2^H}^2, \quad k = 1, 2, 3,
\]

for all \( \epsilon > 0 \) small enough. Some elementary calculus can show that, for all \( 0 < \epsilon \leq 1 \)

\[
\int_\epsilon^1 E \left[ |f(B^H_s)|^2 \right] s^{2H-1} ds \int_0^{s-\epsilon} \frac{dr}{r^{1-2H}(s-\epsilon-r)^{2H}} \\
= \int_\epsilon^1 E \left[ |f(B^H_s)|^2 \right] s^{2H-1} ds \int_0^{s-\epsilon} \frac{dr}{r^{1-2H}(s-\epsilon-r)^{2H}} \\
= \int_\epsilon^1 s^{2H-1} E \left[ |f(B^H_s)|^2 \right] ds \left( \int_0^1 \frac{dx}{x^{1-2H}(1-x)^{2H}} \right),
\]

\[
\int_\epsilon^1 E \left[ |f(B^H_s)|^2 \right] s^{2H-1} ds \int_{s-\epsilon}^s \frac{dr}{r^{1-2H}(r+\epsilon-s)^{2H}} \\
\leq \int_\epsilon^1 E \left[ |f(B^H_s)|^2 \right] ds \int_{s-\epsilon}^s \frac{dx}{x^{2-4H}(s-x)^{2H}} \\
= \int_\epsilon^1 E \left[ |f(B^H_s)|^2 \right] ds \int_1^{s-\epsilon} \frac{dx}{x^{2-4H}(x-\epsilon)^{2H}} \\
\leq \int_0^1 E \left[ |f(B^H_s)|^2 \right] ds \left( \int_1^{+\infty} \frac{dx}{x^{2-4H}(x-1)^{2H}} \right),
\]
where the estimate (4.8) follows from the monotonicity of the function $\varepsilon$ with $\varepsilon$ and 

\begin{equation}
(4.9) \leq C_H \int_0^1 \int_0^1 |f(x)|^2 e^{-\frac{x^2}{2\pi x^H}} ds \, dx
\end{equation}

where the estimate (4.8) follows from the monotonicity of the function 

$\varepsilon \mapsto \int_0^\varepsilon \frac{s^{2H-1}}{(e-s)^H} e^{-\frac{x^2}{2\pi x^H}} ds$

with $\varepsilon \in [0, 1]$. It follows that

\begin{equation}
\frac{1}{\varepsilon^{4H}} \left| \int_0^1 \int_0^1 \Psi_\varepsilon(s, r, 3) ds \, dr \right|
\end{equation}

\begin{align*}
\leq \frac{H}{\varepsilon^{4H}} & \int_0^1 \int_0^1 \left| E \left[ (B^H_{s+\varepsilon} - B^H_{r+\varepsilon})(B^H_{s+\varepsilon} - B^H_{r+\varepsilon}) \right] \right| \\
& \cdot \left| \left\{ E \left[ f^2(B^H_{r+\varepsilon}) \right] + E \left[ f^2(B^H_{s+\varepsilon}) \right] \right\} (sr)^{2H-1} ds dr \\
= \frac{H}{\varepsilon^{4H}} & \int_0^1 \int_0^1 \left| E \left[ (B^H_{s+\varepsilon} - B^H_{r+\varepsilon})(B^H_{s+\varepsilon} - B^H_{r+\varepsilon}) \right] \right| E \left[ f^2(B^H_{s+\varepsilon}) \right] (sr)^{2H-1} ds dr \\
\leq H & \int_\varepsilon^1 \int_0^1 \left| f(B^H_{s+\varepsilon}) \right|^2 s^{2H-1} ds \int_0^{s-\varepsilon} \frac{dr}{r^{1-2H}(s-\varepsilon-r)^{2H}} \\
& + H \int_0^1 \int_\varepsilon^1 \left| f(B^H_{s+\varepsilon}) \right|^2 s^{2H-1} ds \int_0^{s-\varepsilon} \frac{dr}{r^{1-2H}(r+\varepsilon-s)^{2H}} \\
& + H \int_0^\varepsilon \int_0^1 \left| f(B^H_{s+\varepsilon}) \right|^2 s^{2H-1} ds \int_0^{r^{2H-1}} \frac{dr}{(r+\varepsilon-s)^{2H}} \\
\leq C_H \| f \|_{W^2}^2
\end{align*}

for all $0 < \varepsilon \leq 1$.

Now, let us obtain the estimate (4.7) for $k = 1$. By (2.6) we see that

$\Psi_\varepsilon(s, r, 1) = E \left[ B^H_{s+\varepsilon}(B^H_{r+\varepsilon} - B^H_{r}) \right] E \left[ f'(B^H_{s})f(B^H_{s+\varepsilon} - B^H_{s}) \right]$

$= E \left[ B^H_{s+\varepsilon}(B^H_{r+\varepsilon} - B^H_{r}) \right] E \left[ B^H_{s+\varepsilon}(B^H_{s+\varepsilon} - B^H_{s}) \right] E \left[ f'(B^H_{s})f(B^H_{r}) \right]$

$+ E \left[ B^H_{s+\varepsilon}(B^H_{r+\varepsilon} - B^H_{r}) \right] E \left[ B^H_{r}(B^H_{s+\varepsilon} - B^H_{s}) \right] E \left[ f'(B^H_{s})f'(B^H_{r}) \right]$

$\equiv \Psi_\varepsilon(s, r, 1, 1) + \Psi_\varepsilon(s, r, 1, 2)$.

Together Lemma 3.5, Lemma 3.6, Lemma 3.4 and the fact

\begin{equation}
E \left[ f^2(B^H_{s}) \right] = \int_{s^2}^{s^2} f^2(x) \frac{1}{\sqrt{2\pi x^H}} e^{-\frac{x^2}{2\pi x^H}} dx
\end{equation}

\begin{align*}
\leq \frac{s^H}{r^H} & \int_{s^2}^{s^2} f^2(x) \frac{1}{\sqrt{2\pi s^H}} e^{-\frac{s^2}{2\pi s^H}} dx = \frac{s^H}{r^H} E \left[ f^2(B^H_{s}) \right]
\end{align*}
with $s \geq r > 0$ lead to
\[
\frac{1}{\varepsilon^{2H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s,r,1,1) ds^{2H} dr^{2H} \right| \leq \int_0^t \int_0^t \left| E \left[ f''(B^H_s) f(B^H_r) \right] \right| ds^{2H} dr^{2H} \\
\leq C_H \int_0^t \int_0^s \frac{1}{(s-r)^{2H}} E|f(B^H_s) f(B^H_r)| ds^{2H} dr^{2H} \\
\leq C_H \int_0^t \int_0^s E[f^2(B^H_s)] ds^{2H} \int_0^s \frac{s^{H/2}}{(s-r)^{2H+H/2}} dr^{2H} \\
\leq C_H \|f\|^2_{\mathcal{H}},
\]
and
\[
\frac{1}{\varepsilon^{2H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s,r,1,2) ds^{2H} dr^{2H} \right| \leq \int_0^t \int_0^t \left| E \left[ f'(B^H_s) f'(B^H_r) \right] \right| ds^{2H} dr^{2H} \\
\leq C_H \int_0^t \int_0^s \frac{s^{H}}{r^{H}(s-r)^{2H}} E|f(B^H_s) f(B^H_r)| rs^{H} dr^{2H} \\
\leq C_H \|f\|^2_{\mathcal{H}}
\]
for all $\varepsilon > 0$ and $t \geq 0$. Thus, we get
\[
\frac{1}{\varepsilon^{4H}} \left| \int_0^t \int_0^t \Psi_\varepsilon(s,r,1) ds^{2H} dr^{2H} \right| \leq C_H \|f\|^2_{\mathcal{H}}.
\]
Similarly, we can also obtain the estimate \((4.7)\) for $k = 2$, and the lemma follows.

Recently, Gradinaru-Nourdin \cite{12} introduced the following perfect result:

**Theorem A** (Theorem 2.1 in Gradinaru–Nourdin \cite{12}). Assume that $H \in (0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying
\[
|f(x) - f(y)| \leq C|x - y|^a + (1 + x^2 + y^2)^b, \quad (C > 0, 0 < a \leq 1, b > 0),
\]
for all $x, y \in \mathbb{R}$, and let $\{Y_t : t \geq 0\}$ be a continuous stochastic process. Then, as $\varepsilon \to 0$,
\[
\int_0^t Y_s f \left( \frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon^{H}} \right) ds \to E[f(N)] \int_0^t Y_s ds,
\]
almost surely, uniformly in $t$ on each compact interval, where $N$ is a standard Gaussian random variable.

According to the theorem above we get the next lemma.

**Lemma 4.2.** Let $0 < H < 1$ and $f \in C(\mathbb{R})$. We then have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{4H}} \int_0^t f(B^H_s)(B^H_{s+\varepsilon} - B^H_s)^2 ds^{2H} = \int_0^t f(B^H_s) ds^{2H}
\]
almost surely, for all $t \geq 0$.

As a direct consequence of Lemma \ref{4.2}, for $f \in C^1(\mathbb{R})$ we have
\[
[f(B^H), B^H]_t^{(W)} = 2H \int_0^t f'(B^H_s)s^{2H-1} ds
\]
for all \(0 < H < 1\). In fact, the Hölder continuity of fBm \(B^H\) yields
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t \sigma (B^H_{s+\varepsilon} - B^H_s) (B^H_{s+\varepsilon} - B^H_s)^2 ds^{2H} = 0
\]
amtually surely. It follows that
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t \{ f(B^H_{s+\varepsilon}) - f(B^H_s) \} (B^H_{s+\varepsilon} - B^H_s) ds^{2H}
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t f'(B^H_s) (B^H_{s+\varepsilon} - B^H_s)^2 ds^{2H} = \int_0^t f'(B^H_s) ds^{2H}
\]
amtually surely.

Now we can show our main result.

Proof of Theorem 4.1. Given \(f \in \mathcal{H}\). If \(f \in C^1(\mathbb{R})\), then the theorem follows from the identity (4.13) and the following estimate:
\[
E \left( \int_0^t f'(B^H_s) s^{2H-1} ds \right)^2 = \int_0^t \int_0^t E \left[ f'(B^H_s) f'(B^H_t) \right] (s-r)^{2H-1} ds dr
\leq CH \int_0^t \int_0^s \frac{s^{2H-1}}{r^{1-H} (s-r)^{2H}} E \left[ f^2(B^H_r) \right] ds dr
\leq CH \int_0^t s^{2H-1} E \left[ f^2(B^H_s) \right] ds \leq CH \| f \|^2_{\mathcal{H}}
\]
by Lemma 3.5 and (4.9). Let now \(f \notin C^\infty_0(\mathbb{R})\).

Consider the function \(\zeta\) on \(\mathbb{R}\) by
\[
\zeta(x) := \begin{cases} 
\frac{1}{c e^{(x-1)^2/2 - 1}}, & x \in (0,2), \\
0, & \text{otherwise}, 
\end{cases}
\]
where \(c\) is a normalizing constant such that \(\int_{\mathbb{R}} \zeta(x) dx = 1\). Define the so-called mollifiers
\[
\zeta_n(x) := n \zeta(nx), \quad n = 1, 2, \ldots
\]
and the sequence of smooth functions
\[
f_n(x) := \int_{\mathbb{R}} f(x-y) \zeta_n(y) dy = \int_0^2 f(x - \frac{y}{n}) \zeta(y) dy, \quad n = 1, 2, \ldots
\]
for all \(x \in \mathbb{R}\). Then \(\{f_n\} \subset C^\infty(\mathbb{R}) \cap \mathcal{H}\) and \(f_n\) converges to \(f\) in \(\mathcal{H}\), as \(n\) tends to infinity.

On the other hand, by Lemma 4.11 we have
\[
P(\{J_{t_1}(f,t) - J_{t_2}(f,t) \geq \delta\} \leq P \left( J_{t_1}(f-f_n,t) \geq \frac{\delta}{3} \right) + P \left( J_{t_2}(f-f_n,t) \geq \frac{\delta}{3} \right)
+ P \left( |J_{t_1}(f_n,t) - J_{t_2}(f_n,t)| \geq \frac{\delta}{3} \right)
\leq \frac{C_H}{\delta^2} \| f - f_n \|_{\mathcal{H}}^2 + P \left( |J_{t_1}(f_n,t) - J_{t_2}(f_n,t)| \geq \frac{\delta}{3} \right)
\]
for all $n$ and $\delta, \varepsilon_1, \varepsilon_2 > 0$. Combining this with
\[
\lim_{\varepsilon \downarrow 0} J_\varepsilon(f_n, t) = [f_n(B^H), B^H]_t^{(W)} = 2H \int_0^t f_n'(B^H_s) s^{2H-1} ds, \quad n \geq 1
\]
in probability, we show that the generalized quadratic covariation $[f(B^H), B^H]_t^{(W)}$ exists for $f \in \mathcal{K}$. Thus, the estimate (4.14) follows from Lemma 4.1. This completes the proof. □

**Corollary 4.1.** Let $f, f_1, f_2, \ldots \in \mathcal{K}$. If $f_n \to f$ in $\mathcal{K}$, as $n$ tends to infinity, then we have
\[
[f_n(B^H), B^H]_t^{(W)} \to [f(B^H), B^H]_t^{(W)}
\]
in $L^2$ as $n \to \infty$.

**Proof.** The convergence follows from
\[
E \left| [f_n(B^H), B^H]_t^{(W)} - [f(B^H), B^H]_t^{(W)} \right|^2 \leq C_H \|f_n - f\|_{\mathcal{K}}^2 \to 0,
\]
as $n$ tends to infinity. □

By using the above result, we immediately get an extension of Itô formula stated as follows.

**Theorem 4.2.** Let $0 < H < \frac{1}{2}$ and let $f \in \mathcal{K}$ be left continuous. If $F$ is an absolutely continuous function with the derivative $F' = f$, then the following Itô type formula holds:
\[
(4.17) \quad F(B^H) = F(0) + \int_0^t f(B^H_s) dB^H_s + \frac{1}{2} \left[ f(B^H), B^H \right]_t^{(W)}.
\]

Clearly, this is an analogue of Föllmer-Protter-Shiryayev’s formula (see Eisenbaum [7], Föllmer et al [10], Moret–Nualart [20], Russo–Vallois [26], and the references therein). It is an improvement in terms of the hypothesis on $f$ and it is also quite interesting itself.

**Proof of Theorem 4.2.** If $F \in C^2(\mathbb{R})$, then this is Itô’s formula since
\[
\left[ f(B^H), B^H \right]_t^{(W)} = 2H \int_0^t f'(B^H_s) s^{2H-1} ds.
\]
For $F \not\in C^2(\mathbb{R})$, by a localization argument we may assume that the function $f$ is uniformly bounded. In fact, for any $k \geq 0$ we may consider the set
\[
\Omega_k = \left\{ \sup_{0 \leq t \leq T} |B^H_t| < k \right\}
\]
and let $f^{[k]}$ be a measurable function such that $f^{[k]} = f$ on $[-k, k]$ and such that $f^{[k]}$ vanishes outside. Then $f^{[k]}$ is uniformly bounded and $f^{[k]} \in \mathcal{K}$ for every $k \geq 0$. Set $\frac{d}{ds} F^{[k]} = f^{[k]}$ and $F^{[k]} = F$ on $[-k, k]$. If the theorem is true for all uniformly bounded functions on $\mathcal{K}$, then we get the desired formula
\[
F^{[k]}(B^H_t) = F^{[k]}(0) + \int_0^t f^{[k]}(B^H_s) dB^H_s + \frac{1}{2} \left[ f^{[k]}(B^H), B^H \right]_t^{(W)}
\]
on the set $\Omega_k$. Letting $k$ tend to infinity we deduce the Itô formula (4.17) for all $f \in \mathcal{K}$ being left continuous and locally bounded.
Let now $F' = f \in \mathcal{H}$ be uniformly bounded and left continuous. For any positive integer $n$ we define

$$F_n(x) := \int_{\mathbb{R}} F(x - y) \zeta_n(y) dy, \quad x \in \mathbb{R},$$

where $\zeta_n, n \geq 1$ are the mollifiers defined by (4.15). Then $F_n \in C^\infty(\mathbb{R})$ for all $n \geq 1$ and the Itô formula

$$F_n(B^H_t) = F_n(0) + \int_0^t f_n(B^H_s) dB^H_s + H \int_0^t f'_n(B^H_s) s^{2H-1} ds$$

holds for all $n \geq 1$, where $f_n = F'_n$. Moreover using Lebesgue’s dominated convergence theorem, one can prove that as $n \to \infty$, for each $x$,

$$F_n(x) \to F(x), \quad f_n(x) \to f(x),$$

and $\{f_n\} \subset \mathcal{H}, f_n \to f$ in $\mathcal{H}$, as $n$ tends to infinity. It follows that

$$2H \int_0^t f'_n(B^H_s) s^{2H-1} ds = [f_n(B^H), B^H]_t^{(W)} \to [f(B^H), B^H]_t^{(W)}$$

in $L^2$ by Corollary 4.1 as $n$ tends to infinity. It follows that

$$\int_0^t f_n(B^H_s) dB^H_s = F_n(B^H_t) - F_n(0) - \frac{1}{2} [f_n(B^H), B^H]_t^{(W)} \to F(B^H_t) - F(0) - \frac{1}{2} [f(B^H), B^H]_t^{(W)}$$

in $L^2$, as $n$ tends to infinity. This completes the proof since the integral is closed in $L^2$. □

5. INTEGRATION WITH RESPECT TO THE LOCAL TIME

In this section we assume that $0 < H < \frac{1}{2}$ and study the integral

$$\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx,t),$$

where $f$ is a deterministic function and

$$\mathcal{L}^H(x,t) = 2H \int_0^t \delta(B^H_s - x) s^{2H-1} ds$$

is the weighted local time of fBm $B^H$. Recall that the quadratic covariation $[f(B), B]$ of Brownian motion $B$ can be characterized as

$$[f(B), B]_t = -\int_{\mathbb{R}} f(x) \mathcal{L}^B(dx,t),$$

where $f$ is locally square integrable and $\mathcal{L}^B(x,t)$ is the local time of Brownian motion. This is called the Bouleau-Yor identity. More works for this can be found in Bouleau-Yor \[3\], Eisenbaum \[7\], Föllmer et al \[10\], Feng–Zhao \[9\], Peskir \[24\], Rogers–Walsh \[25\], Yang–Yan \[29\], and the references therein. However, this is not true for fractional Brownian motion. For $\frac{1}{2} < H < 1$, Yan et al \[28\] obtained the following Bouleau-Yor identity:

$$[f(B^H), B^H]_t^{(W)} = -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx,t).$$

In this section we show that the identity above also holds for $0 < H < \frac{1}{2}$. 
Take \( F(x) = (x - a)^+ - (x - b)^+ \). Then \( F \) is absolutely continuous with the derivative 
\[
F' = 1_{(a,b)} \in \mathcal{H}
\]
being left continuous and bounded, and the Itô formula (4.17) yields
\[
\left[ 1_{(a,b)}(B^H_s), B^H_t \right]_t = 2F(B^H_t) - 2F(0) - 2 \int_0^t 1_{(a,b)}(B^H_s)dB^H_s
\]
for all \( t \in [0, 1] \). Thus, the linearity property of generalized quadratic covariation deduces the following result.

**Lemma 5.1.** For any \( f_\triangle(x) = \sum_j f_j 1_{(a_{j-1},a_j)}(x) \in \mathcal{E} \), the integral
\[
\int_{\mathbb{R}} f_\triangle(x) \mathcal{L}^H(dx, t) := \sum_j f_j \left[ \mathcal{L}^H(a_j, t) - \mathcal{L}^H(a_{j-1}, t) \right]
\]
exists and
\[
(5.2) \quad \int_{\mathbb{R}} f_\triangle(x) \mathcal{L}^H(dx, t) = -\left[ f_\triangle(B^H), B^H \right]_t
\]
for all \( t \in [0, 1] \).

Thanks to the density of \( \mathcal{E} \) in \( \mathcal{H} \), we can then extend the definition of integration with respect to \( x \mapsto \mathcal{L}^H(x, t) \) to the elements of \( \mathcal{H} \) in the following manner:
\[
\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t) := \lim_{n \to \infty} \int_{\mathbb{R}} f_\triangle,n(x) \mathcal{L}^H(dx, t)
\]
in \( L^2 \) for \( f \in \mathcal{H} \) provided \( f_\triangle,n \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, where \( \{f_\triangle,n\} \subset \mathcal{E} \). The limit obtained does not depend on the choice of the sequence \( \{f_\triangle,n\} \) and represents the integral of \( f \) with respect to \( \mathcal{L}^H \). Together this and Corollary 4.1 lead to the Bouleau-Yor identity
\[
(5.3) \quad \left[ f(B^H), B^H \right]_t = -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)
\]
for all \( t \in [0, 1] \).

**Corollary 5.1.** Let \( 0 < H < \frac{1}{2} \) and let \( f, f_1, f_2, \ldots \in \mathcal{H} \). If \( f_n \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, we then have
\[
\int_{\mathbb{R}} f_n(x) \mathcal{L}^H(dx, t) \to \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)
\]
in \( L^2 \), as \( n \) tends to infinity.

According to Theorem 4.2, we get an analogue of Bouleau-Yor’s formula.

**Corollary 5.2.** Let \( 0 < H < \frac{1}{2} \) and let \( f \in \mathcal{H} \) be left continuous. If \( F \) is an absolutely continuous function with the derivative \( F' = f \), then the following Itô type formula holds:
\[
(5.4) \quad F(B^H_t) = F(0) + \int_0^t f(B^H_s)dB^H_s - \frac{1}{2} \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t).
\]
Recall that if $F$ is the difference of two convex functions, then $F$ is an absolutely continuous function with derivative of bounded variation. Thus, the Itô-Tanaka formula
\[
F(B_t^H) = F(0) + \int_0^t F'(B_s^H)dB_s^H + \frac{1}{2} \int_\mathbb{R} \mathcal{L}^H(x,t)F''(dx)
\]
holds. This is given by Coutin et al [4] (see also Hu et al [17]).

**Remark 1.** By the proof similar to Lemma 3.1 in Gradinaru–Nourdin [12], one can obtain the following convergence (see also Gradinaru–Nourdin [13]):

\[
(5.5) \lim_{n \to \infty} \sum_{j=1}^{n} (\Lambda_j)^{2H-1} g(B_{t_j}^H)(B_{t_j}^H - B_{t_{j-1}}^H)^2 = \int_0^t g(B_s^H)s^{2H-1}ds
\]

almost surely, where $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ denotes an arbitrary partition of the interval $[0, t]$ with $\|\pi_n\| = \sup_j (t_j - t_{j-1}) \to 0$, $\Lambda_j = \frac{t_j - t_{j-1}}{t_j - t_{j-1}}$ and $g \in C(\mathbb{R})$. Thus, similar to proof of Theorem 4.1 we can show that the convergence
\[
2H \lim_{n \to \infty} \sum_{j=1}^{n} (\Lambda_j)^{2H-1} \{f(B_{t_j}^H) - f(B_{t_{j-1}}^H)\}(B_{t_j}^H - B_{t_{j-1}}^H) = - \int_\mathbb{R} f(x)\mathcal{L}^H(dx, t)
\]
holds, which deduces
\[
2H \lim_{n \to \infty} \sum_{j=1}^{n} (\Lambda_j)^{2H-1} \{f(B_{t_j}^H) - f(B_{t_{j-1}}^H)\}(B_{t_j}^H - B_{t_{j-1}}^H) = [f(B^H), B^H]_t^W,
\]
where $f \in \mathcal{H}$ and the limits are uniform in probability.

### 6. The time-dependent case

In this section we consider the time-dependent case. For a measurable function $f$ on $\mathbb{R} \times \mathbb{R}_+$ we define the generalized quadratic covariation $[f(B^H, \cdot), B^H]_t^W$ of $f(B^H, \cdot)$ and $B^H$ as follows

\[
(6.1) \quad [f(B^H, \cdot), B^H]_t^W := \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^{2H}} \int_0^t \{ f(B_{s+\epsilon}^H, s+\epsilon) - f(B_s^H, s) \} (B_{s+\epsilon}^H - B_s^H)ds^{2H}
\]

for $t \in [0, T]$, provided the limit exists uniformly in probability. We prove the existence of the quadratic covariation.

Consider the set $\mathcal{H}_*$ of measurable functions $f$ on $\mathbb{R} \times \mathbb{R}_+$ such that the function $t \mapsto f(\cdot, t)$ is continuous and $\|f\|_{\mathcal{H}_*} < +\infty$, where

\[
\|f\|_{\mathcal{H}_*} = \sqrt{\int_0^T \int_\mathbb{R} |f(x, s)|^2 e^{-\frac{x^2}{2s^{1-H}}} \frac{dxds}{\sqrt{2\pi s^{1-H}}} + \int_0^T \int_\mathbb{R} |f(x, s)|^2 e^{-\frac{x^2}{2s^{1-H}}} \frac{dxds}{\sqrt{2\pi (T-s)^{1-H}}}}
\]

with $\varphi_s(x) = \frac{1}{\sqrt{2\pi s^{1-H}}} e^{-\frac{x^2}{2s^{1-H}}}$. Then $\mathcal{H}_*$ is a Banach space and the set $\mathcal{E}_*$ of elementary functions of the form

\[
(6.2) \quad f_{\Delta}(x, t) = \sum_{i,j} f_{ij}1_{[x_{i-1}, x_i]}(x)1_{(s_{j-1}, s_j)}(t)
\]
is dense in $\mathcal{H}$, where $\{x_i, 0 \leq i \leq n\}$ is an finite sequence of real numbers such that $x_i < x_{i+1}, \{s_j, 0 \leq j \leq m\}$ is a subdivision of $[0, T]$ and $(f_{ij})$ is a matrix of order $n \times m$. Moreover, $\mathcal{H}$ contains the set $\mathcal{H}_{\ast \gamma}$ with $\gamma > 2$ of measurable functions $f$ on $\mathbb{R}$ such that
\[ \int_0^T \int_\mathbb{R} |f(x, s)|^2 e^{-\frac{x^2}{2s^{4-H}}} \, dx \, ds < \infty. \]
As a corollary of Theorem A, we have
\[ \lim_{\epsilon \to 0} \frac{1}{2\pi H} \int_0^t s^{2H-1} g(B^H_s, s)(B^H_{s+\epsilon} - B^H_s)^2 \, ds = \int_0^t g(B^H_s, s)s^{2H-1} \, ds \]
a almost surely, for all $t \geq 0$ if $g$ is continuous. This proves the following identity:
\[ [f(B^H, \cdot), B^H]_t^{(W)} = 2H \int_0^t \frac{\partial f}{\partial x}(B^H_s, s)s^{2H-1} \, ds \]
for all $t \geq 0$, provided $f \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$. Thus, similar to proof of Theorem 4.1 one can obtain the next theorem.

**Theorem 6.1.** Let $0 < H < \frac{1}{2}$. If $f \in \mathcal{H}$, then the generalized quadratic covariation $[f(B^H, \cdot), B^H]_t^{(W)}$ exists and
\[ E \left| [f(B^H, \cdot), B^H]_t^{(W)} \right|^2 \leq C_H \|f\|^2_{\mathcal{H}} \]
for all $t \in [0, T]$.

By using the above result, we immediately get an extension of Itô formula stated as follows.

**Theorem 6.2.** Let $0 < H < \frac{1}{2}$ and let $F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$. Suppose that the function $\frac{\partial}{\partial x} F = f \in \mathcal{H}$. Then the Itô type formula
\[ F(B^H_t, t) = F(0, 0) + \int_0^t f(B^H_s, s)dB^H_s + \int_0^t \frac{\partial}{\partial t} F(B^H_s, s)ds + \frac{1}{2} [f(B^H, \cdot), B^H]_t^{(W)} \]
holds.

**Proof.** Similar to the proof of Theorem 4.2 we can use smoothing procedure to prove our result. The main different key point is the following approximation:
\[ F_n(x, s) := \int \int_{\mathbb{R}^2} F(x - y, s - r)\zeta_n(y)\zeta_n(r) \, dy \, dr, \quad n \geq 1, \]
where $\zeta_n, n \geq 1$ are the mollifiers defined by (4.15). \hfill \Box

We next consider the integral
\[ \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds), \]
where $f$ is a deterministic function. For elementary function $f_\Delta \in \mathcal{E}$ of the form (6.2) we define integration with respect to local time $\mathcal{L}^H$ as follows
\[ \int_0^t \int_{\mathbb{R}} f_\Delta(x, s) \mathcal{L}^H(dx, ds) := \sum_{i,j} f_{ij} \left[ \mathcal{L}^H(x_i, s_j) - \mathcal{L}^H(x_{i-1}, s_j) - \mathcal{L}^H(x_i, s_{j-1}) + \mathcal{L}^H(x_{i-1}, s_{j-1}) \right], \]
for all $t \in [0,T]$. Notice that
\[
\mathcal{L}^H(x_i, s_j) - \mathcal{L}^H(x_i, s_{j-1}) - \mathcal{L}^H(x_{i-1}, s_j) + \mathcal{L}^H(x_{i-1}, s_{j-1}) \\
= \left[ \mathcal{L}^H(x_i, s_j) - \mathcal{L}^H(x_{i-1}, s_j) \right] - \left[ \mathcal{L}^H(x_i, s_{j-1}) - \mathcal{L}^H(x_{i-1}, s_{j-1}) \right] \\
= - \left[ 1_{(x_{i-1},x_i]}(B^H), B^H \right]_{s_j} + \left[ 1_{(x_{i-1},x_i]}(B^H), B^H \right]_{s_{j-1}} \\
= - \left[ 1_{(x_{i-1},x_i]}(B^H)t_{(s_{j-1},s_j)}(\cdot), B^H \right]_t
\]
for all $i,j$. We get the identity
\[
(6.7) \quad \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds) = - \left[ f(B^H, \cdot), B^H \right]_t
\]
for all $t \in [0,T]$. Moreover, for $f \in \mathcal{H}$ we can define
\[
\int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds) := \lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}} f_{\triangle,n}(x, s) \mathcal{L}^H(dx, ds), \quad \text{in } L^2
\]
for all $t \in [0,1]$ if $f_{\triangle,n} \rightarrow f$ in $\mathcal{H}$, as $n$ tends to infinity, where $\{f_{\triangle,n}\} \subset \mathcal{E}$. 

**Theorem 6.3.** Let $0 < H < \frac{1}{2}$ and $f \in \mathcal{H}$. Then the integral (6.6) exists in $L^2$ and the Bouleau-Yor identity takes the form
\[
(6.8) \quad \left[ f(B^H, \cdot), B^H \right]_t = - \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds)
\]
for all $t \in [0,T]$.

**Corollary 6.1.** Let $0 < H < \frac{1}{2}$, $F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$ and $\frac{\partial}{\partial x} F = f \in \mathcal{H}$. Then the Itô type formula
\[
F(B^H_t, t) = F(0,0) + \int_0^t f(B^H_s, s)dB^H_s \\
+ \int_0^t \frac{\partial}{\partial t} F(B^H_s, s)ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{L}^H(dx, ds)
\]
holds.

Finally, let us consider the weighted local time of fBm $B^H$ with $0 < H < \frac{1}{2}$ on a continuous curve. Let $a(t)$ denote a continuous function on $[0,T]$. Then the function
\[
f_a(x, s) = 1_{(-\infty,a(s))}(x)
\]
belongs to $\mathcal{H}$, and the integral
\[
\int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds)
\]
and the generalized quadratic covariation $\left[ f_a(B^H, \cdot), B^H \right]_t$ exist in $L^2$. By the idea due to Eisenbaum [7] and Föllmer *et al.* [10], as an example, we can show that the process
\[
\int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds), \quad t \geq 0
\]
is increasing and continuous. Thus, we can define the weighted local time of $B^H$ with $0 < H < \frac{1}{2}$ at a continuous curve $t \mapsto a(t)$ by setting

$$\mathcal{L}^H(a(\cdot), t)) = \int_0^t \int_{\mathbb{R}} f_a(x, s) \mathcal{L}^H(dx, ds).$$

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