CORRELATION PROPERTIES OF THE KINEMATIC SUNYAЕV-ZEL’DOVICH EFFECT AND IMPLICATIONS FOR DARK ENERGY

CARLOS HERNÁNDEZ-MONTEAGUDO,1 LICIA VERDE,1 RAUL JIMÉNEZ,1 AND DAVID N. SPERGEL2

Received 2005 October 31; accepted 2006 February 1

ABSTRACT

In the context of a cosmological study of bulk flows in the universe, we present a detailed study of the statistical properties of the kinematic Sunyaev-Zel’dovich (kSZ) effect. We first compute analytically the correlation function and the power spectrum of the projected peculiar velocities of galaxy clusters. By taking into account the spatial clustering properties of these sources, we perform a line-of-sight computation of the all-sky kSZ power spectrum and find that at large angular scales (ℓ < 10), the local bulk flow should leave a visible signature above the Poisson-like fluctuations dominant at smaller scales, while the coupling of density and velocity fluctuations should give a much smaller contribution. We conduct an analysis of the prospects of future high-resolution CMB experiments (such as ACT and SPT) for detecting the kSZ signal and extracting cosmological information and dark energy constraints from it. We present two complementary methods, one suitable for “deep and narrow” surveys such as ACT and one suitable for “wide and shallow” surveys such as SPT. Both methods can constrain the equation of state of dark energy (w) to about 5%–10% when applied to forthcoming and future surveys and can probe w in complementary redshift ranges, which could shed some light on its time evolution. This is mainly due to the high sensitivity of the peculiar velocity field to the onset of the late acceleration of the universe. We stress that this determination of w does not rely on the knowledge of cluster masses, although it relies on cluster redshifts and makes minimal assumptions about cluster physics.

Subject headings: cosmic microwave background — large-scale structure of universe

1. INTRODUCTION

The new generation of ground-based high-resolution cosmic microwave background (CMB) experiments (e.g., the Atacama Cosmology Telescope3 [ACT; Kosowsky 2003; Fowler et al. 2005] and the South Pole Telescope4 [SPT; Ruhl et al. 2004]) are designed to scan the microwave sky with very high sensitivity and arcminute resolution. Their main goal is the study of the thermal Sunyaev-Zel’dovich (tSZ) effect (Sunyaev & Zeldovich 1980): the change of frequency of CMB photons due to inverse Compton scattering by hot electrons. Such hot electron plasmas are known to be found in clusters of galaxies and should also be present in larger structures, such as filaments and superclusters of galaxies. This scattering translates into a redshift-independent distortion of the CMB blackbody spectrum, making the tSZ effect an ideal tool to probe the baryon distribution on the large scales of our universe at different cosmic epochs. However, this is not the only effect of an electron plasma on the CMB radiation. If a cloud of electrons is moving with some bulk velocity with respect to the CMB frame, then Thomson scattering by these electrons will imprint new (Doppler induced) temperature fluctuations on the CMB photons. This is the so-called kinematic Sunyaev-Zel’dovich (kSZ) effect (Sunyaev & Zel’dovich 1972), which is spectrally indistinguishable from the intrinsic CMB temperature fluctuations.

Although the kSZ effect is typically an order of magnitude smaller than the tSZ effect in clusters of galaxies (and for this reason much harder to detect), it encodes precious cosmological information since it depends on the peculiar velocity field. Indeed, kSZ measurements can yield valuable information about the large-scale velocity field, the evolution of the dark matter potential, and the growth of fluctuations. The study of large-scale velocity fields (cosmic flows) was an active research area in the 1990s. There were numerous attempts to measure bulk flows using the large-scale distribution of galaxies and their peculiar velocities and to place constraints on the matter power spectrum or the universe matter density (see the reviews of Strauss & Willick [1995], Courteau & Dekel [2001], and references therein). However, it became clear that these measurements had to be corrected for systematic errors, such as the biases introduced when calibrating the distances of the galaxies under study, or the nonlinear components of the velocities of those objects. With kSZ observations, by using clusters as tracers of the velocity fields, one is more confident in probing larger (less nonlinear) scales.

While there have been no kSZ detections to date, upper limits on the peculiar velocities of individual clusters have been reported by Benson et al. (2003). Such a difficult measurement could in principle be hampered by other effects such as nonlinearities and the complicated physics of the intracluster gas. Nagai et al. (2003) have shown that the kSZ effect is not diluted by the internal velocity dispersion in the intracluster gas. Ma & Fry (2002) calculated the temperature fluctuations produced by the kSZ effect in the nonlinear regime using the halo model. Benson et al. (2003) showed that the signal-to-noise ratio (S/N) of a kSZ measurement should be distance-independent and suggested combining signals from different redshifts. Holder (2002) and Aghanim et al. (2005) discuss how to extract the kSZ signal from maps, while Schäfer et al. (2004) use N-body simulations to build templates of kSZ maps in the context of the Planck mission, and Kashlinsky & Atrio-Barandela (2000) study the possibility of extracting the kSZ dipole from CMB surveys covering a

1 Department of Physics and Astronomy, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104; carloshm@astro.upenn.edu, lverde@physics.upenn.edu, raulj@physics.upenn.edu.
2 Department of Astrophysical Sciences, Peyton Hall, Princeton University, Princeton, NJ 08540; dnw@astro.princeton.edu.
3 See http://www.hep.upenn.edu/act.
4 See http://spt.uchicago.edu.
large fraction of the sky. Zhang et al. (2004) propose to constrain reionization histories by looking at the kSZ power spectrum at small scales. Due to its weak signal, indirect detection of the kSZ has been proposed through cross-correlation techniques with weak lensing (Doré et al. 2004) or old galaxies (DeDeo et al. 2005). As we show below, a measurement of the kSZ effect would be very valuable, since it would not only allow us to measure bulk flows of clusters of galaxies and test the predictions of the standard model, but also provide additional constraints on cosmological parameters, especially on the equation of state of dark energy.

In this paper we compute the correlation function and the power spectrum of the kSZ effect. This requires modeling of the peculiar velocity field and the cluster population. We explore the prospects for future CMB experiments for measuring the kSZ correlation function and find that ACT-like experiments should be able to detect the kSZ-induced CMB variance at high (≈ 12) significance levels. Further, we study the dependence of the kSZ correlation function on the cosmological parameters and show that it can be used to measure the equation of state of dark energy (w) if the redshifts of the clusters of galaxies detected in CMB surveys are available. We find that with the SALT (Southern African Large Telescope) follow-up of ACT data or with a wider but shallower (SPT-like) survey, the w-parameter can be constrained with an accuracy of 8% for an ACT scan of 400 deg². This error should scale inversely with the square root of the covered area and hence becomes 5% for a 1000 deg² area. These determinations of w do not rely on the knowledge of cluster masses, but do rely on cluster redshifts and reasonable assumptions about cluster physics.

Unless otherwise stated, throughout this work we assume a LCDM cosmological model (Spergel et al. 2003) with \( \Omega_m = 0.3 \), \( \Omega_k = 0.7 \), \( h = 0.72 \), and \( \sigma_8 = 0.88 \). The paper is organized as follows. In § 2 we study the statistical properties of the projected peculiar velocity field and provide an analytical expression for the correlation function of projected velocities. In § 3 we compare the kSZ and tSZ effects, and discuss the strategy for enhancing the probabilities of detecting the former. In § 4 we compute the correlation function and the power spectrum of the kSZ, both when we consider only a given set of clusters present in a survey and when we consider the whole celestial sphere. In § 5 we outline two methods for estimating the kSZ effect in future CMB cluster surveys, and in § 6 we explore the dependence of kSZ measurements on cosmological parameters, with particular emphasis on w. We conclude in § 7.

2. THE CORRELATION FUNCTION OF LINE-OF-SIGHT LINEAR PECULIAR VELOCITIES

While the measurement of the kSZ effect of individual clusters is difficult (e.g., Aghanim et al. 2001; Benson et al. 2003), in this paper we address the prospects for statistical detection of peculiar velocities in future CMB surveys. This requires the knowledge of the ensemble properties of the velocity field traced by the galaxy cluster population, to which we devote the current section. For clarity and future reference, a statistical description of the linear velocity field and related quantities is given in Appendix A.

Throughout this paper, we assume that the measured velocity field obeys linear theory. However, as noted by Colberg et al. (2000), this is not completely fulfilled by cluster velocities, since clusters are peculiar tracers of the large-scale matter distribution and show biased velocities compared to the expectations provided by linear theory. Colberg et al. (2000) found that this bias was typically a 30%–40% effect. Although several attempts have been made to model this boost in terms of the underlying density field (Sheth & Diaferio 2001; Hamana et al. 2003), in subsequent sections we account for it by simply increasing cluster velocities by a factor \( b_v = 1.3 \) (Sheth & Diaferio 2001). The goal of this paper is to present a theoretical calculation of the detectability of the signal and the forecasted S/N; detailed comparison with numerical simulations are left to future work. Full treatment can be implemented only with the help of numerical simulations matched to a given observing program (see Peal 2006 for a recent study). We must stress that although some modeling of nonlinear effects must be included in this study, clusters are the largest virialized structures known in the universe and probe much larger scales than galaxies. Therefore we must expect them to be significantly better tracers of the linear velocity field. Furthermore, as we show below, the peculiar velocity estimator (the kSZ effect) does not depend on distance, which avoids the need to use redshift-independent distance indicators, as opposed to other peculiar velocity estimators.

On large, linear scales, density and peculiar velocities are related through the continuity equation: \( \partial \delta / \partial t = - \mathbf{k} \cdot \mathbf{v}_i / a \), where \( a \) and \( k \) are the scale factor and comoving Fourier mode, respectively. The peculiar velocity of a cluster, as probed by its kSZ effect, can be interpreted as the linear peculiar velocity field smoothed on comoving scale \( R \), which corresponds to the cluster’s mass \( M \) via

\[
R = \left( \frac{3M}{4\pi \bar{\rho}} \right)^{1/3},
\]

where \( \bar{\rho} \) is the background matter density. The kSZ effect is sensitive to the line-of-sight component of the velocity, but under the assumption that the velocity field is Gaussian and isotropic (which should be satisfied in the linear regime), the three spatial components of the velocity field must be statistically independent. Moreover, the power spectrum must completely determine the statistical properties of the velocity field. Thus, in a given cosmological model, the linear velocity field power spectrum (which in turn is related to the matter power spectrum) should univocally determine the angular correlation function (and angular power spectrum) of the line-of-sight cluster velocities.

In linear theory, the velocity dispersion smoothed over spheres of comoving radius \( R \) (corresponding to a given cluster mass \( M \)) is given by

\[
\sigma_{v_{\perp}}^2(R, z) = \left[ H(z) \frac{dD_s}{dz} \right]^2 \int dk k^2 \frac{P_m(k)}{2\pi^2k^2} W(kR)^2, \tag{2}
\]

where \( W(kR) \) is the Fourier transform of the top-hat window function, \( H(z) \) is the Hubble function, \( D_s(z) \) is the linear growth factor, and \( P_m(k) \) is the present-day linear matter power spectrum [the power spectra at any redshift will be denoted as \( P_m(z, k) \equiv D_s^2(z)P_m(k) \)]. Hence the power spectrum of the velocities is

\[
P_{v_{\perp}}(k) = \left[ H(z) \frac{dD_s}{dz} \right]^2 \frac{P_m(k)}{k^2} = D_v^2 \frac{P_m(k)}{k^2}, \tag{3}
\]

where \( D_s \equiv H(z)dD_s/dz \) is the velocity growth factor. When computing \( D_s \) for different dark energy models, we used the analytical fit provided by Linder (2005):

\[
g(a) = \exp \int_0^a d \log a \left\{ \frac{\Omega_m H_0^2}{a^2 H^2(a)} - 1 \right\}, \tag{4}
\]
where \( g(a) \equiv D_s(a)/a \) gives the deviation of the growth factor from that of a critical (\( \Omega_m = 1 \)) universe, and \( \gamma \) is given by

\[
\gamma = 0.55 + \beta [1 + w(z = 1)],
\]

with \( \beta = 0.05 \) if \( w > -1 \) and \( \beta = 0.02 \) otherwise.

The growth of the velocity perturbations with redshift may provide useful cosmological constraints, such as constraints on the equation of state of dark energy (DeDeo et al. 2005). This is illustrated in Figure 1, where we show the redshift evolution of the velocity growth factor for three different cosmological models: a \( \Lambda \)CDM model (\( \Omega_L = 0.7, \Omega_m = 0.3; \text{ thick solid line} \)), a flat universe with \( \Omega_m = 0.3 \) and dark energy equation of state parameter \( w = -0.6 \) (\( \text{dotted line} \)), and another flat model with \( \Omega_m = 0.3 \) and \( w = -\frac{1}{3} \) (\( \text{dashed line} \)).

Due to the \( k^2 \) factor in the denominator of equation (3), the signal is weighted by the largest scales, making this probe relatively insensitive to the smoothing scale and therefore to clusters mass. Indeed, if the dependence of \( \sigma_v \) versus mass is approximated by a power law, then one finds that for the concordance model, \( \sigma_v \propto M^{-0.13} \).

Having this in mind, we compute here the angular correlation function of the line-of-sight cluster velocities. Note that Peel (2006) takes a different approach to this calculation. Assuming that we can measure the line-of-sight component of the peculiar velocity of a cluster, we compute the quantity

\[
C_{v,v}(\theta_{12}) = \left\langle |r(x_1) \cdot n_1||r(x_2) \cdot n_2| \right\rangle,
\]

where \( n_1 \) and \( n_2 \) denote two different directions in the sky, “connecting” the observer to the cluster positions \( x_1 \) and \( x_2 \), and \( \theta_{12} \) denotes the angle between \( n_1 \) and \( n_2 \). We refer the reader to Appendix B for the detailed derivation, and here we report the final expression for this correlation function:

\[
C_{v,v}(\theta_{12}) = \sum_{\text{even } l} \frac{2l + 1}{4\pi} \cos \theta_{12} \times \left( \frac{2}{\pi} F_l \right) \int k^2 dk P_{v,v}(k) W(k R_1) W(k R_2) \times j_l[k(x_1 - x_2 \cos \theta_{12})] j_l(k x_2 \sin \theta_{12}).
\]

In this equation, the factor \( F_l \) is given by

\[
F_l = \frac{(l - 1)!!}{2^{l/2}(l/2)^l} \cos \left( \frac{l \pi}{2} \right),
\]

and \( x_1 \) and \( x_2 \) are the (comoving) distances to the clusters (without loss of generality we have used the convention that \( x_1 \geq x_2 \)). Here \( j_l(x) \) denote the spherical Bessel functions, and the summation must take place only over even values of \( l \); \( R_1 \) and \( R_2 \) refer to the linear scales corresponding to the masses of each cluster. Note that we recover equation (2) in the limit of \( \theta_{12} \to 0 \) and \( x_1 \to x_2 \).

Figure 2a shows the behavior of \( C_{v,v} \) versus \( \theta_{12} \) in the concordance \( \Lambda \)CDM model for a couple of \( 10^{14} M_\odot \) clusters when they are both placed at \( z = 0.005 \) (solid line), when both are placed at \( z = 0.1 \) (dashed line), and when one cluster is at \( z = 0.1 \) and the other at \( z = 1 \) (dotted line). In the first (clearly unrealistic) case, the clusters are so close to us that both are moving in the same bulk flow with respect to the CMB frame, giving rise to the dipolar pattern shown by the solid line. In the case in which both clusters are at \( z \sim 0.1 \) (dashed line), their correlation properties are strongly dependent on \( \theta_{12} \), since this angle defines the distance between the clusters. For small angular separation, the clusters are still relatively nearby and hence their peculiar velocities are correlated, but this correlation dies as the separation of the clusters increases. The angular distance at which the correlation drops to half its value at zero separation is \( \sim 10^8 \), corresponding to roughly \( 40 h^{-1} \) Mpc. Finally, if clusters are very far apart from each other (dotted line), their peculiar velocities are not correlated. (Note that in this work we understand by “bulk flow” the coherent motion of clusters in scales of \( \sim 40 h^{-1} \) Mpc, and this definition differs from the one given in Atrio-Barandela et al. (2004)).

An alternative way to present the correlation properties of the projected peculiar velocities of the clusters is the velocity field angular power spectrum. In Appendix B we invert \( C_{v,v}(\theta_{12}) \) into its angular power spectrum \( C_l^{vv} \), i.e.,

\[
C_{v,v}(\theta_{12}) = \sum_{l} C_l^{vv} P_l(\cos \theta_{12}).
\]

We find that

\[
C_l^{vv} = \frac{4\pi}{2l + 1} \left[ B_{l-1} + (l + 1) B_{l+1} \right],
\]

where the \( B_l \) coefficients are defined by

\[
B_l \equiv 4\pi \int \frac{k^2 dk}{(2\pi)^3} P_{v,v}(k) j_l(k x_1) j_l(k x_2).
\]

Figure 2b displays the power spectra for two cases: two very nearby clusters (both at \( z = 0.005 \); solid line), showing an almost
dipolar pattern, and two relatively far away clusters (both at \( z = 0.1 \); dashed line), in which case the power is transferred to higher multipoles. The power spectrum for the clusters placed at \( z = 0.005 \) and \( z = 0.1 \) is zero.

Here we have characterized the projected peculiar velocity field at cluster scales. Next, we address the study of the kSZ effect and its comparison with the tSZ effect.

3. THE SUNYAEV-ZEL’DOVICH EFFECTS

3.1. The Kinematic Sunyaev-Zel’dovich Effect

The kinematic Sunyaev-Zel’dovich effect describes the Doppler kick that CMB photons experience when they encounter a moving cloud of electrons. Since this is simple Thomson scattering, there is no change in the photon frequency, and hence it leaves no spectral signatures in the CMB blackbody spectrum. Therefore, this effect is solely determined by the number density of free electrons and their relative velocity in the CMB frame. An observer will only be sensitive to the radial component of the electron peculiar velocity, so the expression for the change in brightness temperature becomes (Sunyaev & Zel’dovich 1972)

\[
\frac{\delta T_{\text{KSZ}}}{T_0} = \int dl \, n_e(l) \sigma_T \left( \frac{v \cdot H}{c} \right) \equiv \tau \left( \frac{v \cdot H}{c} \right).
\]  

Here we have assumed that the peculiar velocity is the same for all electrons; \( \tau \) stands for the optical depth, and \( n \) is a unitary vector giving the direction of observation. This process must take place in two contexts: (1) When the intergalactic medium becomes ionized by the high-energy photons emitted by the first stars, the inhomogeneities in the electron velocity and density distributions generate a kSZ signal, which is known as the Ostriker-Vishniac effect. Despite the large size of ionized structures encountered by the CMB photons, the electron density contrast is relatively small, and furthermore, we do not know where in CMB maps to look for this signal because we do not know the location of the ionized bubbles that formed during reionization. The amplitudes and angular scales at which this effect should be visible are model-dependent but can be as high as a few \( \mu \)K in the multipole range \( l > 2000 \) (Santos et al. 2003). (2) Clusters of galaxies at lower redshift leave a more easily detectable signal. Their high electron density can give rise to values of \( \tau \) as high as \( 10^{-3} \) to \( 10^{-2} \), and their peculiar velocities should be close to 300 km s\(^{-1} \) at \( z = 0 \), which together can produce temperature fluctuations on the order of 1–10 \( \mu \)K. The clusters’ high optical depth is also responsible for large spectral distortions of the CMB generated through the thermal Sunyaev-Zel’dovich (tSZ) effect. The tSZ effect is typically an order of magnitude larger than the kSZ effect and introduces frequency-dependent brightness temperature fluctuations that, in the nonrelativistic limit, change sign at 218 GHz. Therefore, by combining observations in bands at frequencies lower and higher than this cross frequency, it is possible to obtain the cluster position and to characterize the tSZ cluster signal. Once the cluster position is identified, its kSZ contribution should be accessible at 218 GHz. However, as noted by Sehgal et al. (2005) and earlier by Holder (2004), even with measurements in three different frequencies it may not be possible to obtain a clean estimate of the kSZ effect for a single cluster. We show below that we are not interested in a very accurate kSZ estimate for a given cluster, but in unbiased estimates for our entire cluster sample.

In the next subsection we make a detailed comparison of the amplitude of the kSZ and tSZ effects.

3.2. Comparison of the kSZ and the tSZ Effects in Clusters of Galaxies

In what follows we describe the galaxy cluster population by adopting the model presented in Verde et al. (2002). This model is based on the spherical collapse description of galaxy clusters (Gunn & Gott 1972) and assumes that clusters are isothermal and that their gas acquires the virial temperature of the halo. The halo mass and redshift distribution is approximated by the formalism presented in Sheth & Tormen (1999). We refer to Verde et al. (2002) for further details on this modeling.

Both the kSZ and tSZ effects can be written as integrals of some function \( K(r) \) along the line of sight crossing the cluster, weighted by the electron density:

\[
\frac{\delta T}{T_0} = g(x) \int dl \, \sigma_T n_e(l) K(l). 
\]  

For the nonrelativistic tSZ, \( g(x) = x \coth(x/2) - 4, x \equiv h \nu/k_B T_0 \) is the nondimensional frequency in terms of the CMB monopole...
For the majority of clusters, the kSZ flux will be a few times larger than the tSZ. In this case, the line of sight and velocity dispersion of the clusters, respectively, must be different in each case: while clusters are hotter at earlier epochs, the tSZ effect is minimal. At the same time, the radial peculiar velocity dispersion of the clusters, \( \sigma_v \sim 1/(1+z)^{1/2} \), decreases very slowly with mass (\( \sim M^{-0.13} \)), so at fixed redshift we must expect the ratio kSZ/tSZ to be greater for low-mass clusters, whereas the tSZ flux at 222 GHz, which we observe in the literature, we present results for.

4. THE POWER SPECTRUM OF THE KINEMATIC SUNYAEEV-ZEL’DOVICH EFFECT

4.1. Model of the Cluster Population

We next study the kSZ signal generated by the entire population of clusters of galaxies by computing its two-second-order momenta, i.e., the correlation function and the power spectrum. For this, it is first necessary to have a model to describe the population of galaxy clusters in the universe. We adopt the hierarchical scenario, in which small-scale overdensities in the universe become nonlinear and collapse first, and then merge and give rise to larger nonlinear structures. The abundance of halos of a given mass at a given cosmic epoch or redshift is given by the clusters’ mass function. We adopt the Sheth & Tormen (hereafter ST) mass function (Sheth & Tormen 1999), which is denoted by \( \bar{\rho}(M, z) \equiv dN/M; dV(z)dM \) and provides the average number density of halos of masses between \( M \) and \( M + dM \) at redshift \( z \). This must not be confused with \( n(M, x) \), which is the actual number of halos in that mass range at position \( x \). The latter can be understood as a random variable, the former as its mean. As it will be useful below, we first compute (and derive below) the mean number of halos present in two volume elements centered at \( x_1 \) and \( x_2 \):

\[
\langle n(M_1, \mathbf{x}_1)n(M_2, \mathbf{x}_2) \rangle = \bar{n}(M_1, z_1)\bar{n}(M_2, z_2) + \delta_D(M_1 - M_2)\delta_D^2(x_1 - x_2)\bar{n}(M_1, z_1) + \langle \Delta[n(M_1, \mathbf{x}_1)]\Delta[n(M_2, \mathbf{x}_2)] \rangle.
\] (14)

In this equation, \( z_1 \) and \( z_2 \) are the redshifts corresponding to positions \( x_1 \) and \( x_2 \), respectively. The first term on the right-hand side of the equation is merely a constant but will have its relevance, because it will couple with the velocity field, as we see below. The next term containing the Dirac deltas accounts for the (assumed) Poissonian statistics ruling the (discrete) number density of sources and will be referred to as the “Poissonian term. In the third term, \( \Delta[n(M_1, \mathbf{x}_1)] \) stands for the deviation with respect to the average halo number density due to the environment, i.e., due to large-scale overdensities, which condition the halo clustering. Therefore, this third term describes the spatial clustering of halos, which is a biased tracer of the spatial clustering of matter. In the extended Press-Schechter approach it can be shown that the power spectrum and the correlation function of halos and underlying matter are merely proportional to each other over a wide range of scales. This is commonly expressed by a “bias factor” (Mo & White 1996), so that

\[
\bar{\xi}_{hh}(r) = b^2(M, z)\xi_m(r), \quad P_{hh}(k) = b^2(M, z)P_m(k).
\] (15)

Here \( \bar{\xi}_{hh}, \xi_m, P_{hh}(k), P_m(k) \) stand for the halo-halo and matter correlation functions and power spectra, respectively. Therefore, the third term on the right-hand side of equation (14) equals \( \bar{n}(M_1, z_1)\bar{n}(M_2, z_2)b(M_1, z_1)b(M_2, z_2)\xi_m(x_1 - x_2) \).
A parallel approach consists of writing the halo mass function as a function of some linear-scale matter overdensity \( \delta \equiv (\rho - \bar{\rho})/\bar{\rho} \). The number of halos at \( x \) is then approximated as
\[
n(M, x) = \bar{n}(M, z) + \eta(M, x) + \delta \bar{n}(M, z) + O(\delta^2),
\]
where \( \eta(M, x) \) is a random variable that introduces the Poissonian behavior of the source counts, so that \( \langle \eta(M_1, x_1)\eta(M_2, x_2) \rangle = \bar{n}(M_1, z)\bar{n}(M_2, z) \). In the extended Press-Schechter formalism, it turns out that \( \partial\bar{n}(M, z)/\partial\delta \big|_{\delta=0} \) coincides with the bias factor \( b \) in clusters of galaxies as an integral along the line of sight. This is due to the fact that clustering all higher order powers of \( x \) with the bias factor \( b \) in equation (16) we have taken \( \delta \) to be in the linear regime, but since \( \xi_{bb}(r) \approx b^2 \xi_{mm}(r) \) down to scales comparable to the halo size, we use this formalism down to halo scales.

4.2. A Line-of-Sight Approach for the kSZ Effect

We next write the temperature anisotropies induced by the kSZ effect in clusters of galaxies as an integral along the line of sight \( n \):
\[
\frac{\Delta T_{\text{kSZ}}}{T_0}(n) = \int_0^{r_m} dr \int d\delta \chi \left( -\frac{v_r \cdot n}{c} \right) W_{\text{gas}}^{\text{G}(r-r_j)}.
\]
Here \( r \) is the comoving radius integrated to the last scattering surface, and the sum over the index \( j \) represents a sum over all clusters; \( \chi \) denotes the opacity in the center of the cluster (\( \chi = a \sigma T n_{e,1} \) with \( n_{e,1} \), the central electron number density and \( a \) the scale factor), and the window function \( W_{\text{gas}}^{\text{G}(r-r_j)} \) denotes the gas profile of the cluster. Although we should adopt some realistic shape for this profile (Komatsu & Seljak 2001), we have adopted a simple Gaussian window with scale radius equal to the virial radius of the cluster. This is justified since, in the Fourier domain, at scales larger than the cluster size, the window function is merely equal to the volume occupied by the gas, regardless of the shape of the gas profile. This step simplifies our computations significantly and does make the accuracy in the relatively large scales (bulk flow scales) in which we are interested. Note that even if the sum is made over all clusters, only the clusters being intersected by the line of sight will contribute to the integral. This sum can be rewritten first as an integral and then as a convolution,
\[
\frac{\Delta T_{\text{kSZ}}}{T_0}(n) = \int_0^{r_m} dr \int dM \left\{ \int dy \delta(M, z) \left[ -\frac{v(M, y) \cdot n}{c} \right] \right\}
\times n(M, y)W_{\text{gas}}^{\text{G}(M, r-y)}
\]
\[
= \int_0^{r_m} dr \int dM \left\{ \int dy \delta(M, z) \left[ -\frac{v(M, y) \cdot n}{c} \right] \right\} n(M)
\times W_{\text{gas}}^{\text{G}(M, z)}(r).
\]
Note that central optical depth and the window function have an intrinsic dependence on redshift. This is due to the fact that clusters formed at higher redshift tend to be more concentrated. The symbol \( * \) denotes here convolution in real space.

4.3. The All-Sky Correlation Function and Power Spectrum of the kSZ Effect

If two different lines of sight \( n_1 \) and \( n_2 \) are now combined to estimate the angular correlation function, then one obtains that
\[
\left\langle \frac{\Delta T_{\text{kSZ}}}{T_0}(n_1)\frac{\Delta T_{\text{kSZ}}}{T_0}(n_2) \right\rangle
\propto \langle n(M_1, r_1) [v(M_1, r_1) \cdot n_1] n(M_2, r_2) [v(M_2, r_2) \cdot n_2] \rangle.
\]

Next we plug in equation (16) and make use of the cumulant expansion theorem. We must note as well that Poissonian fluctuations will be assumed to be independent of \( \delta \) and that for Gaussian statistics, the three- and four-point functions are zero, as are \( \langle v \rangle \) and \( \langle \delta \rangle \). Therefore we are left with a sum of products of two-point functions of the form
\[
\bar{n}(M_1, z_1) \sigma_{x,1}^2(M_1, z_1)
+ \bar{n}(M_1, z_1) \bar{n}(M_2, z_2) \langle [v(M_1, r_1) \cdot n_1] [v(M_2, r_2) \cdot n_2] \rangle
+ \frac{\partial \bar{n}(M_1, z_1)}{\partial \delta} \big|_{\delta=0} \frac{\partial \bar{n}(M_2, z_2)}{\partial \delta} \big|_{\delta=0}
\times \left\langle \left[ \delta(r_1) \delta(r_2) \right] \left[ [v(M_1, r_1) \cdot n_1] [v(M_2, r_2) \cdot n_2] \right] \right\rangle
+ \frac{\partial \delta(r_1)}{\partial \delta(r_2)} \left[ [v(M_1, r_1) \cdot n_1] \right] \left[ [v(M_2, r_2) \cdot n_2] \right]
+ \frac{\partial \delta(r_1)}{\partial \delta(r_2)} \left[ [v(M_1, r_1) \cdot n_1] \right] \left[ [v(M_2, r_2) \cdot n_2] \right].
\]

Note that the last term is a constant. We refer again to Appendix A, where the cross terms \( \langle v \cdot \delta \rangle \) are studied. In Appendix C we provide an explicit computation of the power spectra arising from each of the terms considered in equation (20). Since the last term introduces no anisotropy, the kSZ power spectrum is the sum of four contributions: a Poisson term, a term proportional to the velocity-velocity correlation \( (vv) \) term, a term proportional to the product of the velocity-velocity correlation and the density-density correlation \( (v - dd) \), and a term proportional to the density-density squared \( (dd) \). Therefore,
\[
C_t = C_t^{P} + C_t^{vv} + C_t^{vd - dd} + C_t^{dv - vd}.
\]

Figure 4 displays each of the terms in equation (21); the thick solid line shows the Poisson term, whereas the thick dashed line corresponds to the \( vv \) term. Although the latter decreases rapidly with increasing \( l \), at large scales it dominates over all other terms, reflecting the presence of the local bulk flow. In an attempt to simplify the expressions for these two terms given in Appendix C, we have found the approximate integral
\[
C_t^{P} \approx \int dz \frac{dV(z)}{dz} dM \bar{n}(M, z) \sigma_{x,1}^2(M, z) \left[ \frac{\gamma(z)}{\pi} \right]^2
\]
for the Poisson term, and
\[
C_t^{vv} \approx \int dz \frac{dV(z)}{dz} P_{vv} \left[ \frac{z}{r(z)} \right] \times \left[ \int dM \frac{1}{4\pi} \bar{n}(M, z) \gamma(z) \right]^2
\]
Note that \( n \) designates a direction in the sky, whereas \( \sigma \) refers to the number density of galaxy clusters.
for the $vv$ term. Here $y_l(M, z)$ is the Fourier transform of the cluster profile in the sphere,

$$y_l = \left(\frac{2}{\sqrt{2\pi}}\right)^2 \exp\left[-\frac{l(l+1)\theta_j^2}{2}\right],$$

and $P_{v,v}(k, z)$ is the $k$- and $z$-dependent velocity power spectrum; $\theta_j$ is the angular size of the cluster virial radius. The $vv$ term can be understood as a projection of the three-dimensional velocity power spectrum. Filled circles for the Poisson term and open diamonds for the $vv$ term provide a comparison of these approximations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals. Although the amplitudes and slopes are not too dissimilar, the approximated integrations with the exact integrals.

The $dd$-vv and $ds$-vd terms are shown by the thin dotted lines. We must remark that the $dd$-vd term is negative and that we are plotting its absolute value. The sum of both is given by the thick dotted line. The sum of these two terms is particularly hard to detect, since it never dominates—not at large scales (it is about a factor of 20 below the $vv$ term), nor at small scales, where it is well below the Poisson term.

4.4. The Cluster-Cluster kSZ Correlation Function

Future high-resolution multifrequency CMB experiments like ACT or SPT can provide kSZ estimates on those regions of the sky where clusters of galaxies have been identified via tSZ. Therefore one could attempt to measure the velocities’ correla-

![Figure 4](image1.png)

**Fig. 4.—** Different components of the all-sky kSZ power spectra: the thick solid line shows the Poisson term, which dominates over the other terms except in the low-$l$ limit. In this large-scale range, the $vv$ term generated by the local bulk flow is the one introducing the most power (dashed line). The filled circles and open diamonds show semianalytical approximations for the Poisson and $vv$ terms, respectively (see eqs. [22] and [23]). Of lower amplitude, the thick dotted line shows the sum of the $dd$-vv and the $ds$-vd terms (thin dotted lines; note that the latter is negative).

![Figure 5](image2.png)

**Fig. 5.—** Cluster-cluster kSZ correlation functions for a sample of clusters with masses above $2 \times 10^{14} M_\odot$. When forming cluster pairs we require pair constituents to be at similar redshifts. If we consider all redshifts simultaneously, we obtain the solid line. The dashed lines correspond to cluster-cluster correlation functions for clusters of different redshifts: the thick dashed line shows only clusters around $z \sim 0.01$, whereas the intermediate dashed line corresponds to $z \sim 0.1$ and the thin dashed line to $z \sim 1$.

Since the signal comes from clusters whose velocities are correlated, we should consider pairs close in redshift. The solid line in Figure 5 shows the correlation function measured from all galaxy clusters from $z = 0$ to $z = 4$ more massive than $2 \times 10^{14} h^{-1} M_\odot$, according to the standard ΛCDM cosmology and the ST mass function. Members of the cluster pairs must be within $\Delta z = 0.01$. Since the kSZ amplitude per cluster is typically of few tens of microkelvins, the zero-lag correlation function can be as high as ~$150$ ($\mu$K)$^2$, and drops to one half of this value at $\theta \sim 2^\circ-3^\circ$. The thick dashed line shows the correlation function for clusters located at $z \sim 0.01$. As the redshift increases ($z \sim 0.1$, intermediate dashed line; $z \sim 1$, thin dashed line), the amplitude at zero lag increases (clusters are more concentrated) and the correlation angle decreases. The solid line shows the redshift-integrated cluster-cluster correlation function; the signal is dominated by high-redshift clusters, more concentrated and more numerous per unit solid angle.

Diaferio et al. (2000) pointed out a potentially important nonlinear aspect that is not included in our modeling: in very massive superclusters, the kSZ effect shows typically a dipolar pattern, plausibly caused by the encounter of two opposite bulk flows at their common attractor’s position. Since our model predicts no dipolar pattern at scales of a few degrees, such a scenario is not
accounted for by our approach. Hence, in a realistic application the core of such overdense regions should be excised from the analyses. Such massive structures, however, are very rare and form at very late epochs, and their exclusion should not compromise the analysis presented here.

5. CAN THE KINEMATIC SUNYAEV-ZEL’DOVICH EFFECT BE MEASURED?

In this section we outline two different procedures for extracting the kSZ signal from future high-resolution and high-sensitivity CMB experiments. The procedures presented here may be suboptimal, but our aim is to quantify the relative importance of different sources of error and to roughly forecast the expected S/N for these experiments. We defer the development of an optimal procedure to future work. Here try to extract the kSZ signal in a statistical sense: while previous works (e.g., Aghanim et al. 2005) have addressed the difficulty of separating the kSZ effect from potential contaminants (tSZ, radio-source emission, infrared galaxies, CMB, etc.) in a given cluster, our approach consists of combining the signal coming from subsets of clusters in such a way that the contribution of the noise sources averages out. As we see below, the success of this procedure relies on the precision to which the average properties of the potential contaminants are known.

The first approach, which we refer to as method a, is based on measurements of the kSZ flux and its redshift evolution. Its sensitivity to cosmology increases with redshift, and for this reason it is suited for “deep and narrow” survey strategies. Here we use ACT’s specifications. Our second approach (method b) uses the ratio of kSZ- to tSZ-induced temperature anisotropies. This ratio is particularly sensitive to w at z ≤ 0.8, and since cosmic variance for the peculiar velocity field is more important at low redshifts, this method is more suitable for “wide and shallow” survey strategies. Here we use a survey with specifications similar to those of ACT, but covering 4000 deg² and thus having higher noise. These specifications are not too dissimilar from those of SPT, assuming that accurate photometric redshifts can be obtained for all SPT clusters.

5.1. Probing the kSZ Flux at High Redshift

In what follows, we use specifications for ACT 1 yr data (Gaussian beam with FWHM equal to 2', noise amplitude lower than 2 µK per beam, and a clean scanned area of 400 deg²). ACT observes in three bands: 145, 220, and 250 GHz. We concentrate on the 220 GHz channel, although some knowledge is assumed to be inferred from the other bands. For instance, the 250 GHz channel is very useful for estimating the level of infrared galaxy emission at lower frequencies. Likewise, the 145 GHz channel is critical when characterizing the tSZ flux from each cluster. For simplicity, we concentrate on the variance of the kSZ signal, but it is easy to estimate the kSZ angular correlation function introduced in § 4.4.

The strip covered by ACT should also be surveyed by SALT. Cluster detection via the tSZ effect will provide targets for optical observations. Alternatively, optical cluster identification should be possible up to z ~ 1 from SALT’s multiband imaging using algorithms such as those developed by Kim et al. (2002) and Miller et al. (2005). Hence, a direct comparison can be made with tSZ-detected cluster sample. SALT spectroscopic follow-up will enable us to obtain the cluster redshifts.

The method is as follows:

1. For every detected galaxy cluster, we take a patch of radius equal to one projected cluster virial radius. We can use the edge of significant tSZ emission at other frequencies to define this radius. We compute the mean temperature within this patch and draw a ring surrounding it, of width, say, 10% of the virial radius. We next compute the mean temperature within this ring and subtract it from the mean of the patch. This operation should remove most of the CMB contribution to the average temperature in the patch but will unavoidably leave some residuals, which will be denoted here as δTres,MB. These residuals will have two different contributions: the first coming from the inaccurate CMB subtraction, the second one being due to instrumental noise residuals,

$$\left( \langle \delta T_{\text{res,MB}}^2 \rangle \right) = \langle \delta T_{\text{subs}}^2 \rangle + \frac{N^2}{N_{\text{beams}}}.$$  (26)

Nbeams is the number of beam sizes present in the ring, and N² is the instrumental noise variance. If we denote the patch whose radius is the cluster virial radius as “region 1,” and by “region 2” refer to the ring surrounding it, we can easily prove that the first term on the right-hand side of the last equation reads

$$\langle \delta T_{\text{subs}}^2 \rangle = \frac{1}{(\Delta \Omega_1)^2} \int_{\Delta \Omega_1} \int_{\Delta \Omega_1} d\Omega_1 d\Omega_2 C(n_1, n_2)$$

$$+ \frac{1}{(\Delta \Omega_2)^2} \int_{\Delta \Omega_2} \int_{\Delta \Omega_2} d\Omega_1 d\Omega_2 C(n_1, n_2)$$

$$- \frac{2}{\Delta \Omega_1 \Delta \Omega_2} \int_{\Delta \Omega_1} \int_{\Delta \Omega_2} d\Omega_1 d\Omega_2 C(n_1, n_2).$$  (27)

ΔΩ₁ and ΔΩ₂ denote the solid angles of the patch and the ring, respectively, whereas n₁ and n₂ denote directions in the sky and C(n₁, n₂) is the CMB angular correlation function evaluated at the angle separating the directions n₁ and n₂. The rms fluctuations introduced by this residual are plotted in Figure 6; although it can be as high as a few tens of µK for nearby clusters subtending 30'-40', their effect reduces to 3–4 µK for clusters of a few arcminutes in size, which correspond to most of clusters at z > 0.3 to be detected by ACT-like CMB experiments. This contaminant will be dominant over the instrumental noise contribution but subdominant with respect to cosmic variance (as is shown below), even when a larger beam worsens the CMB residual level by a few µK.

2. We assume that by observing the tSZ amplitudes at 145 and 250 GHz, it is possible to provide an unbiased estimate of the mean tSZ contribution to the cluster patch at 220 GHz. After subtracting the tSZ and CMB component estimates, the temperature in the patch corresponding to cluster i can be written as

$$\delta T = \delta T_{\text{MB, i}} + \delta T_{\text{tSZ, i}} + N + \delta T_{\text{int, kSZ, i}} + T_{\text{tSZ, i}}.$$  (28)

δTtSZ,i denotes the tSZ residuals after subtracting the estimated tSZ amplitude, N accounts for contribution of instrumental noise to the patch average, and δTint, kSZ,i accounts for the residual contribution of internal velocities. According to Diaferio et al. (2005), we assume a typical rms for internal velocities of one-third the bulk flow velocity expected for each cluster, and random

---

6 SALT’s Web site is http://www.salt.ac.za.
and so do our angle-integrated kSZ temperature anisotropies. If we denote by $N_I$ and $N_J$ the number of cluster members in bins $I$ and $J$, respectively, then the number of cluster pairs that can be formed by combining these two mass bins is given by $N_{IJ}$; $N_{I,J} = N_I N_J$ if $I \neq J$ and $N_{I,J} = N_I(N_I-1)/2$ for the same bin ($I = J$). The indexes $i$ and $j$ run for individual cluster members in each mass bin; $w_{IJ}$ is a weight factor, which we define as

$$w_{IJ} = \frac{N_I N_J}{\sigma_i \sigma_j}$$

with the subscripts $I$ and $J$ evaluating the brackets in the corresponding mass bins. The estimator of equation (29) provides a weighted measurement of the kSZ variance,

$$\left< F^2_{\text{kSZ},I,J}(n) \right> = \frac{\sum_{I \leq J} w_{IJ} (F_{\text{kSZ},I,J} F_{\text{kSZ},J,I})}{\sum_{I \leq J} w_{IJ}},$$

with $F_{\text{kSZ},I,J}$ the expected kSZ amplitude at redshift $z_i$ and mass equal to that corresponding to bin $I$. Its formal error is given by

$$\Delta^2 \left[ F^2_{\text{kSZ},I,J}(n) \right] = 2 \left[ \frac{\sum_{I \leq J} w_{IJ} (F_{\text{kSZ},I,J} F_{\text{kSZ},J,I})}{\sum_{I \leq J} w_{IJ}} \right]^2$$

$$+ \frac{\sum_{I \leq J,L} w_{I,J,L} (F_{\text{kSZ},I,J} F_{\text{kSZ},J,I,L})}{(\sum_{I \leq J} w_{IJ})^2}$$

$$+ \frac{\sum_{I \leq J,M} w_{I,J,M} (F_{\text{kSZ},I,J} F_{\text{kSZ},J,I,M})}{(\sum_{I \leq J} w_{IJ})^2}$$

$$+ \frac{1}{\sum_{I \leq J} w_{IJ}},$$

where $w_{I,J,L,M} \equiv w_{IJ} N_I/N_J$. Note that the first term on the right-hand side of this equation is not sensitive to the number of clusters within the coherence patch. Such term, containing the squared kSZ expectations for mass bins $I$ and $J$, is associated with the (assumed) intrinsic Gaussian nature of the kSZ fluctuations, and it corresponds to the cosmic variance contribution. It thus scales as the inverse of the survey area when different coherence patches are combined in the analysis. The fourth term is exclusively due to observational errors, whereas the second and third terms are hybrid: they show contributions from both the intrinsic uncertainty of the velocity field and observational errors. If $N_b$ denotes the number of mass bins and we take the weights, the cluster numbers, and the $F_{\text{kSZ}}$ values as equal for all mass bins ($\sigma_i = \sigma, N_I = N_{\text{cl}}$, and $F_{\text{kSZ},I,I} = F_{\text{kSZ},I}$ for every mass bin $I$), then it can easily be proved that the fourth term scales roughly as $\sigma^4 N_b^2/[N_b(N_b+1)]$, that is, the squared variance expected for each cluster over the total number of pairs that can be formed. The second and third terms, in this limit, yield $\sigma^2 F_{\text{kSZ}}^2/[N_b(N_b+1)]$, which scales inversely to the number of clusters. In our case, our kSZ flux estimates will be limited by the cosmic variance term. On the other hand, because of having very few objects, we considered one mass bin only.

---

**Fig. 6.—** Typical amplitude of CMB residuals remaining after attempting to approximate the CMB average within a circular patch of angular radius $\theta$, with the CMB average computed within a ring surrounding the patch of width $10\%$ the patch radius. For most of clusters located at $z > 0.3$, these residuals will typically be of $3-4\, \mu K$ amplitude.
circles/C1/C14
The quantity and the confusion noise associated with unresolved sources.
cally the nonrelativistic tSZ temperature increment expected at of these residuals (remaining after the tSZ subtraction) is typi-
contains the contribution coming from relativistic tSZ correc-
f different projected coherent regions in the sky, where
0 equal to 2
h
10
M
0
P
1
0
2
-close to the typical correlation length of a bulk flow.

\[ \frac{F^2_{\text{KSZ,1}}}{n} = \sum \frac{F^2_{\text{KSZ,1}}}{\Delta^2 F_{\text{KSZ,1}}} \sqrt{\sum \frac{1}{\Delta^2 F_{\text{KSZ,1}}}}. \]  

with an uncertainty

\[ \Delta^2 \left( \frac{F^2_{\text{KSZ,1}}}{} \right) = 1 \sqrt{\sum \frac{1}{\Delta^2 F_{\text{KSZ,1}}}}. \]  

Note that the size of the coherent patch must depend on redshift: a bulk flow extending up to 20 h
1Mpc at z = 1 subtends a degree on the sky, whereas if it is at z = 0.05, then it subtends around 8°.

We take the effective noise \( N \) to have a typical amplitude of 5 \( \mu K \) per beam, and this accounts for both instrumental noise and the confusion noise associated with unresolved sources. The quantity \( \delta T_{\text{CMB}} \) is computed as stated in equation (27) and typically contributes a few \( \mu K \). Regarding the tSZ residuals, \( \delta T_{\text{tSZ}} \) contains the contribution coming from relativistic tSZ corrections and power leakage associated with the finite spectral width of the detectors. We conservatively assume that the amplitude of these residuals (remaining after the tSZ subtraction) is typically the nonrelativistic tSZ temperature increment expected at 222 GHz. We approximate the beam as a Gaussian of FWHM equal to 2', and take 400 deg\(^2\) (\( f_{\text{sky}} \approx 10^{-2} \)) as the clean sky region covered by ACT. ACT’s sensitivity limit is set to clusters above \( 2 \times 10^{14} h^{-1} M_\odot \), and the total number of clusters above this threshold predicted in this region of the sky by our ST mass function is roughly 4400. Figure 7 shows the expected cluster number per square degree for two different mass thresholds versus redshift. The redshift bin width is taken to be \( \Delta z \approx 0.02 \), then it subtends a degree on the sky, whereas if it is at \( z = 0.05 \), then it subtends around 8°.

As a result of this section, in Figure 8 we show our expectations of ACT’s sensitivity to the kSZ variance when all clusters are grouped in the redshift bins centered at \( z = 0.2 \), \( 0.4 \), \( 0.8 \), \( 1.2 \), \( 1.6 \), and \( 2.0 \). The solid, dotted, and dashed lines correspond to a \( \Lambda \)CDM model (\( \Omega_m = 0.3 \), \( \Omega_\Lambda = 0.7 \)), a flat universe with dark energy equation of state \( w = -0.6 \), and one with \( w = -\frac{1}{3} \), respectively. We report error bars for the \( \Lambda \)CDM model: we drop the first point at \( z = 0.02 \) (which covers the redshift range \( z \in [0, 0.2] \) but is dominated by cosmic variance) and focus on the high redshift range: the S/N shows a maximum at \( z = 0.8 \), and beyond this redshift the error bars start to increase due to the lack of massive clusters.

5.2. The kSZ/tSZ Ratio \( \mathcal{R} \)

In this subsection, we address the study of the ratio of the kSZ to the tSZ effects. When referring to temperature anisotropies, the kSZ and tSZ effects are integrals weighted by the cluster electron density along the cluster diameter and are sensitive to the size and/or the concentration parameter of these objects. The ratio of the kSZ to the tSZ effects, however, should cancel these dependencies out to great extent and provide a cleaner view of the cluster’s temperature and peculiar velocity.
Following our cluster model, we investigate the behavior of the statistic $R \equiv \delta T_{kSZ}/\delta T_{tSZ}$ in the cluster population. If the evolution of the cluster temperature is well described by the spherical collapse model, we should find that $R$ is a good estimator of the dark energy equation of state $w$ at recent epochs. But since cosmic variance becomes important at low redshifts (smaller volume for a given solid angle), in this case we compute our expectations for an experiment with sky coverage close to $f_{sky} = 0.1$ (4000 deg$^2$). To compensate for the wider survey area, we assume a noise of 10 $\mu$K arcmin$^{-2}$ and that clusters up to $z = 0.8$ can be detected and resolved. We assume that the frequency coverage enables contaminant subtraction and extraction of the clusters’ tSZ signal. We also assume that follow-up observations will yield redshifts for all observed clusters. These specifications are not too dissimilar to those of the SPT telescope when combined with photometric follow-up.

We see that these are precisely the requirements needed to obtain cosmological information from $R$. However, since in this case the kSZ signal is divided by the tSZ temperature decrement (measured at, say, 145 GHz), we must be very careful with the noise contribution to the denominator of $R$. Our approach is to conservatively consider only clusters whose integrated tSZ temperature decrements are larger than 160 $\mu$K arcmin$^{-2}$ at 145 GHz. This implies that tSZ errors will be typically 5%–10% of the estimated tSZ temperature decrement, and that they can be treated perturbatively as errors in the kSZ estimation in the numerator. Therefore, our model for the estimation of $R$ in a single cluster is given by

$$ r_i = \frac{\delta T_{CMB}^{\text{res},i} + \delta T_{tSZ}^{\text{res},i} + \delta T_{tSZ}^{\text{int},i} + e T_{\text{tSZ},i} + T_{\text{tSZ},i}}{N(t_{SZ,i})}. $$

Most of the terms of this equation are defined exactly as in equation (28). The only new term is $e T_{\text{tSZ},i}$, which accounts for the extra error introduced by the uncertainty in the denominator, $T_{\text{tSZ},i}$. Here $T_{\text{tSZ},i}$ is the absolute value of the tSZ decrement of the cluster at 145 GHz, and $e$ is taken to be a normal random variable of rms 0.05 (5% error in $T_{tSZ,i}$, which results in a 5% error in $T_{tSZ,i}$). We neglect the correlation of the errors in $T_{tSZ,i}$ with those of $T_{kSZ,i}$ and take $e$ as independent of the other noise sources. We now write the analog to equation (29) as

$$ \hat{R}_i(n) \equiv \sum_{l} \frac{w_{l,j} r_{l,j}}{N_{l,j}} / \sum_{l} w_{l,j}, $$

where the weights are defined as

$$ w_{l,j} \equiv \frac{N_l N_j}{\sigma_l \sigma_j}. $$

The estimate of $\hat{R}_i(n)$ in a redshift band $z_l$ and direction $n$, together with its uncertainty $\Delta^2[\hat{R}_i(n)]$, can be obtained from equations (31) and (32) by simply replacing $\tilde{F}_{kSZ,i}(n)$ with $\tilde{R}_i(n)$.

Similarly, the expressions for $\hat{R}_i$ and $\Delta^2[\hat{R}_i]$ can be obtained from equations (33) and (34). In Figure 9 we plot the ratio $R$ at different redshifts as it would be seen by an experiment like SPT. As before, we have assumed that only clusters above $2 \times 10^{14} h^{-1} M_{\odot}$ can be seen, and grouped all clusters in the redshift bins centered at $z_{\text{bin}} = 0.02, 0.4, 0.8, 1.2, 1.6, 2.0$. The lines refer to the same cosmological models as in Figure 8. Note the similarity of this plot and Figure 1. We see that $R$ is sensitive to cosmology at much lower redshifts ($z \leq 0.8$) than $F_{kSZ}$, and for the sensitivity analysis following in the next section we have only used the first three redshift bins.

6. KINEMATIC SUNYAЕV-ZЕL’ДОВИЧ SENSITIVITY TO MEASURING COSMOLOGICAL PARAMETERS

We can now explore the dependence of the two statistics introduced in the previous section on cosmological parameters. We define the $\chi^2$ as

$$ \chi^2 = \sum_{l} \frac{(Q_{kSZ,i} - Q_{kSZ,i}^2)}{\Delta^2(Q_{kSZ,i})^2}, $$

where $Q_{kSZ,i}$ can refer to either the angle-averaged kSZ anisotropy ($F$) or the kSZ/tSZ ratio ($R$). The likelihood is thus $\mathcal{L} \propto \exp(-\frac{1}{2} \chi^2)$. We estimate errors using the Fisher matrix approach. In principle the parameters that enter in the analysis are $\Omega_m, w, \Gamma$, the fraction of cluster mass in the intracluster medium $f_{ICM}$, the present-day normalization of the matter density fluctuations $\sigma_8$, and the reduced Hubble constant $h$. In practice both methods are insensitive to $f_{ICM}, \sigma_8$, and $h$ separately for a fixed number of detected clusters, but but are sensitive to their combination in the form of an overall amplitude of the kSZ signal, which we assign to an amplitude parameter $A$ alone.
We assume that $A$ is redshift-independent (or that its scaling in redshift can be constrained) and consider a 20% uncertainty in this normalization, owing to 10% uncertainty in $\sigma_8 F_{\text{ICM}}$, and $h$ each, over which we marginalize. We remark, however, that our results will be practically insensitive to the uncertainty in $A$.

In Figure 10 we show the resulting constraints in the $\Omega_m$-$w$ plane. Clearly, the $F_{\text{KSZ}}$ method is more sensitive to $w$ than to $R$, and their directions of degeneracy are also different, but both are distinct from the directions of degeneracy corresponding to estimators based on large-scale structure. Indeed, these estimators are restricted to the low-redshift universe, and hence their sensitivity to $w$ is very limited, giving rise to a degeneracy direction almost parallel to the $w$-axis (see, e.g., Fig. 11 of Eisenstein et al. [2005] or Fig. 13 of Tegmark et al. [2004]). The constraint on $w$, marginalized over $\Omega_m$, is $\sim 12\%$ for $F_{\text{KSZ}}$ and $\sim 60\%$ for $R$. As the degeneracy direction in each case is different from that corresponding to CMB temperature measurements, a combination with analyses of CMB temperature data can break the degeneracy. When considering WMAP (Wilkinson Microwave Anisotropy Probe) first-year data, the marginalized error on $w$ reduces to $\sim 8\%$ and $\sim 12\%$ for $F_{\text{KSZ}}$ and $R$ methods, respectively. As mentioned above, these errors are dominated by the cosmic variance term present in equation (32). Therefore, by increasing the sky covered by the experiments, they should decrease as $1/\sqrt{f_{\text{sky}}}$.

These results have been obtained after using an ST mass function and assuming that the cluster density was uniform (i.e., we...
have considered no biases). The error amplitudes and the orientation of the ellipses in Figure 10 depend strongly on the number of cluster pairs that can be formed in each redshift bin, particularly at the high-z end. If the actual sensitivity of future CMB experiments is such that the lower limit on detectable clusters can be relaxed, then the resulting number of cluster pairs that can be formed would increase considerably, and this would result in an increased sensitivity to \( w \). Indeed, simulations show that ACT should be sensitive to galaxy groups and low-mass galaxy clusters, so we believe the limit \( 2 \times 10^{14} h^{-1} M_\odot \) to be very conservative for ACT’s noise level.

We note that a priori, the two methods are nicely complementary in their redshift ranges sensitive to \( w \), as shown in Figure 11. We find that the \( P_{12}^{\mathrm{kSZ}} \) estimation (method \( a \); thick line) is sensitive to \( w \) at high redshifts \((z > 0.5)\), whereas \( R \) for different dark energy models differs at low redshift and converges at \( z \approx 1 \). The sensitivity of this method is localized mainly at low redshift (thin line).

After parameterizing the evolution of \( w \) as \( w(a) = w_0 + w_a(1 - a) \), with \( a = 1/(1 + z) \) the scale factor, we conduct a Fisher matrix analysis considering the parameter set \((\Omega_m, w_0, f_{\mathrm{ICM}}, w_a)\) for the \( P_{12}^{\mathrm{kSZ}} \) method. The marginalization in the \( w_0-w_a \) plane is given in Figure 12, which shows that this method should have some residual sensitivity to \( w_a \) (a typical error of \( \sim 1.75 \) in \( w_a \)). Note that this error should be improved further after combining the two methods proposed here, provided that they are sensitive to different redshift ranges.

7. CONCLUSIONS

We have studied bulk flows in the large-scale structure in the context of the kSZ effect and future high-sensitivity and high-resolution CMB experiments such as ACT. Since the kSZ effect is only sensitive to the projected peculiar velocities of electron clouds, we have focused our analysis on the angular correlation of radial peculiar velocities in galaxy clusters. We have provided an analytical expression for the angular correlation function of projected peculiar velocities in the linear regime and have interpreted it in the context of local and distant bulk flows. We have also presented an expression for the power spectrum of projected linear peculiar velocities.

We have investigated in which redshift and for which cluster mass ranges large-scale bulk flows should be more easily detectable in future kSZ surveys, and computed the overall effect of the entire galaxy cluster population on the CMB sky. We have shown that the main contribution comes from Poissonian/random fluctuations of the number of clusters along the line of sight, especially at small angular scales. However, we find that the local bulk flow generates a signal that should be dominant at the very large angular scales of the quadrupole and octupole. Other terms associated with the coupling of velocity with density fluctuations give smaller contributions. We have calculated the kSZ signal for the cluster sample accessible by forthcoming experiments and considered different sources of contamination that may limit our capacity to distinguish the kSZ signal from other components.

We have presented two approaches to measure the kSZ signal and exploit its dependence on cosmological parameters such as the equation of state of dark energy \((w)\). The first method is based on measurements of the kSZ flux and its redshift evolution. Its sensitivity to cosmology increases with redshift. For this reason it is suited for “deep and narrow” survey strategies. A data set such as that provided by ACT 2 yr observations with SALT follow-up is well suited for an application of this method. ACT can detect the kSZ signal with an S/N of \( \sim 12 \) at \( z \approx 1 \).

The second approach uses the ratio of kSZ- to tSZ-induced temperature anisotropies. The cosmology dependence is strongest at low redshift, and hence this method is suitable for “wide and shallow” survey strategies. In this case we have considered an ACT-like experiment with larger sky coverage (1/10 of the sky) and increased instrumental noise. These are specifications similar to those of SPT with at least three frequency bands and combined with redshift determinations of the detected clusters. In this case the kSZ effect can be detected with an S/N of \( \sim 30 \).

These methods can yield constraints on cosmological parameters, and in particular can constrain the equation of state for dark energy at the 10% level. The two methods are nicely complementary as they measure dark energy in different redshift ranges, opening up the possibility of constraining dark energy redshift evolution.

We are indebted to Ravi Sheth for help and enlightening discussions on cluster spatial correlations and nonlinear velocities. L. V. thanks Mark Devlin for discussions. We thank Simon DeDeo for comments and discussions. This research is supported in part by grant NSF AST 04-08698 to the Atacama Cosmology Telescope. C. H. M. and L. V. are supported by NASA grants ADP03-0000-0092 and ADP04-0000-0093. The work of R. J. is supported by NSF grants AST 02-06031, AST 04-08698, PIRE-0507768, and NASA grant NNG05GG01G. D. N. S. is supported by NSF grant PIRE-0507768 and through the NASA ATP programs and the WMAP project.

APPENDIX A

COUPLING THE LINEAR DENSITY AND VELOCITY FIELDS

In linear theory, the density contrast \( \delta(x) \equiv \rho(x) - \bar{\rho} / \bar{\rho} \) is still much smaller than unity. This allows linearizing the evolution equations and neglecting all nonlinear orders, which makes the evolution of each Fourier mode \( \delta_k \) independent of the other modes.

In a Friedmann-Robertson-Walker (FRW) universe, the perturbed continuity equation reads

\[
\frac{\partial \delta(x)}{\partial t} + \frac{\nabla \nu(x)}{a} = 0, \tag{A1}
\]

which can be rewritten in the Fourier domain as

\[
\frac{\partial \delta_k}{\partial t} = -\frac{i}{a} k \cdot \nu_k. \tag{A2}
\]
with \( a = 1/(1+z) \) the scale factor. Throughout this paper, the velocities are proper peculiar velocities. We must note that the expansion of the velocities in terms of their Fourier modes is given by

\[
v(x) = \int \frac{dk}{(2\pi)^3} v_k \exp(-ik \cdot x)
\]  

(A3)

and that, due to isotropy, the different components are independent. The statistical properties of each of them must be, however, identical.

Coming back to equation (A2), if we denote by \( D_\delta(z) \) the growth factor of the density perturbations, then we can express the component of \( v_k \) parallel to \( \mathbf{k} \) (denoted here by \( v_k \)) as

\[
v_k = iH(z) \frac{dD_\delta}{dz} \frac{\delta_k}{k};
\]

(A4)

\( H \) stands for the Hubble function and \( z \) for redshift.

The power spectrum for any component of the Fourier velocity mode hence reads

\[
P_{vv}(k) = \left[ H(z) \frac{dD_\delta}{dz} \right]^2 P_m(k) \frac{k^2}{k^2}.
\]

(A5)

If we now combine \( \delta_k \) and \( v_q \), we obtain

\[
\langle \delta_k v_q^* \rangle = (2\pi)^3 \delta^D(k-q) P_{\delta v}(k) = (2\pi)^3 \delta^D(k-q) iH(z) \frac{dD_\delta}{dz} P_m(k).
\]

(A6)

Note that this is an imaginary quantity. In this work, we find this power spectrum either squared or in a convolution with itself, giving rise to negative power and indicating anticorrelation.

Finally, we compute the power spectrum of the quantity

\[
\phi(x, n_1) \equiv \delta(x) [v(x) \cdot n_1] = \sum_i \delta(x) [v_i(x)n_i^1],
\]

(A7)

where \( n_1 \) stands for the unitary vector connecting an observer with the position \( x \) and the sum is over the three spatial components. Since a product in real space involves a convolution in Fourier space, the Fourier mode of \( \phi \) reads

\[
\phi_{\mathbf{k}, n_1} = \sum_i \int \frac{dq}{(2\pi)^3} \delta^D_{q_{i1}} (k-q^i_{1})^1.
\]

(A8)

Having this in mind, the average product of two Fourier modes of \( \phi \) when looking in two different directions \( n_1 \) and \( n_2 \) is

\[
\langle \phi_{\mathbf{k}, n_1} \phi_{\mathbf{q}, n_2}^* \rangle = (2\pi)^3 \delta^D(k-q) \left( \cos \theta_{12} [P_{\delta \epsilon} * P_{\delta v}](k) + (\cos \theta_{12} + \sin \theta_{12})[P_{\delta v} * P_{\delta v}](k) \right) + \langle (2\pi)^3 \delta^D(k)(2\pi)^3 \delta^D(q) \int \frac{du}{(2\pi)^3} P_{\delta v}(u) \rangle^2;
\]

(A9)

\( \theta_{12} = \arccos(n_1 \cdot n_2) \) is the angle separating \( n_1 \) and \( n_2 \). Without introducing any loss of generality in equation (A9), we have taken \( n_1 = (0, 0, 1) \) and \( n_2 = (0, \sin \theta_{12}, \cos \theta_{12}) \). The symbol \( * \) denotes convolution and is present in the first two terms in brackets. The third term is nonzero only when \( k = q = 0 \).

Before ending this appendix, it is worthwhile to make some remarks on the redshift and mass dependence of the power spectra we have computed. The redshift dependence is explicit via the growth factors \( D_\delta \) and \( D_v \). Now, if we are interested in the peculiar velocity of a given cluster, then the (linear) velocity field must be averaged in a sphere of dimensions corresponding to the cluster mass. Something similar can be said about the density field: when studying the dependence of the number of halos on the environment density, the field \( \delta \) must be smoothed on scales that a priori are dependent on the mass of the halos whose number density we are studying. In this scenario, all Fourier modes of the density and the velocity should be multiplied by the window functions corresponding to the scales within which we are averaging. This introduces a dependence on the masses of the clusters under study in \( P_{\delta \epsilon}, P_{\delta v}, \) and \( P_{\delta v} \).

APPENDIX B

THE ANGULAR CORRELATION FUNCTION OF THE PROJECTED VELOCITIES

Let \( i \) and \( j \) be two components of the Fourier mode of the peculiar velocity field, so that

\[
\langle v_i^j(v_q^q)^* \rangle = (2\pi)^3 \delta^D(k-q) \delta^K_{ij} P_{vv}(k).
\]

(B1)
Using this, we can write the average product of projected velocities as

$$
\langle |\mathbf{r}(x_1) \cdot \mathbf{n}_1| |\mathbf{r}(x_2) \cdot \mathbf{n}_2| \rangle = \int \frac{k^2 dk}{(2\pi)^3} P_{\nu}(k) W(kR_1) W(kR_2) \int d\phi d\cos \theta \cos \theta_{12} \exp \{-ik[x_1 \cos \theta - x_2(\sin \theta \cos \phi \sin \theta_{12} + \cos \theta_{12} \cos \theta)]\}. \tag{B2}
$$

In this equation, the polar axis for the \( k \) integration has been taken along the direction given by \( \mathbf{n}_1 \); \( \theta \) and \( \phi \) are the polar and azimuthal angles of \( k \), and \( \theta_{12} \) is the angle separating the two directions of observation, \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \); \( W(kR) \) is the Fourier transform of the top-hat window function at scale \( R \). If we now make use of the Rayleigh expansion of the plane wave, i.e.,

$$
\exp(-ik \cdot \mathbf{x}) = \sum_{l} (-i)^l (2l+1) j_l(kx) P_l(\mu) \tag{B3}
$$

(where \( \mu \) is the cosine of the angle between \( k \) and \( x \) and \( P_l \) are Legendre polynomials), and also the theorem of addition of Legendre functions (see, e.g., Gradsteyn & Ryzhik 1965, eq. [8.794]), then we end up with

$$
\langle |\mathbf{r}(x_1) \cdot \mathbf{n}_1| |\mathbf{r}(x_2) \cdot \mathbf{n}_2| \rangle = \sum_{\text{even } l} \frac{2l+1}{4\pi} \cos \theta_{12} \left( \frac{2}{\pi} \mathcal{F}_l \right) \int k^2 dk P_{\nu}(k) W(kR_1) W(kR_2) j_l[k(x_1 - x_2 \cos \theta_{12})] j_l(kx_2 \sin \theta_{12}), \tag{B4}
$$

where the factor \( \mathcal{F}_l \) is given by

$$
\mathcal{F}_l = \frac{(l-1)!!}{2^{l/2}(l/2)!} \cos \left( \frac{l\pi}{2} \right), \tag{B5}
$$

\( j_l(x) \) are the spherical Bessel functions, and the summation must take place only over even values of \( l \). Note that in the limit of \( \theta_{12} \rightarrow 0 \) and \( x_1 \rightarrow x_2 \), this expression becomes equation (2).

At the same time, we can rewrite equation (B2) as an expansion on Legendre polynomials. Indeed, if we denote by \( x_1 \) and \( x_2 \) the position vectors of the clusters, we can use the Rayleigh expansion of the plane wave for \( \exp(ik \cdot x_1) \) and \( \exp(-ik \cdot x_2) \), and write

$$
\langle |\mathbf{r}(x_1) \cdot \mathbf{n}_1| |\mathbf{r}(x_2) \cdot \mathbf{n}_2| \rangle = \sum_{l,l'} (2l+1)(2l'+1)(-i)^{l-l'} \int \frac{dk}{(2\pi)^3} P_{\nu}(k) \cos \theta_{12} j_l(kx_1) j_l(kx_2) P_{l'}(\mu_{x_1,x_2}), \tag{B6}
$$

with \( \mu_{x_1,x_2} \) the cosine of the angle formed by vectors \( x \) and \( y \). Note also that \( \mu_{x_1,x_2} = \mu_{x_2,x_1} = \cos \theta_{12} \). Next we apply the addition theorem of Legendre functions on a spherical triangle formed by \( \mathbf{n}_1, \mathbf{n}_2, \) and \( \mathbf{k} \). As before, we take the polar axis of \( \mathbf{k} \) to be aligned with \( x_1 \):

$$
\langle |\mathbf{r}(x_1) \cdot \mathbf{n}_1| |\mathbf{r}(x_2) \cdot \mathbf{n}_2| \rangle = \sum_{l,l'} (2l+1)(2l'+1)(-i)^{l-l'} \int \frac{dk}{(2\pi)^3} P_{\nu}(k) \cos \theta_{12} j_l(kx_1) j_l(kx_2) P_{l'}(\mu_{x_1,x_2})
\times \left\{ P_{l'}(\mu_{x_1,x_2}) P_{l'}(\mu_{x_2,x_1}) + 2 \sum_{m=1}^{l'} P_{l'}^m(\mu_{x_1,x_2}) P_{l'}^m(\mu_{x_2,x_1}) \cos[m(\phi_1 - \phi_2)] \right\}, \tag{B7}
$$

Because \( dk = k^2 dk \sin \theta d\theta d\phi_1 \), the integration in the azimuthal angle cancels the sum over \( m \) in the curly brackets. Finally, the product \( \mu_{x_1,x_2} P_{l'}(\mu_{x_1,x_2}) \) can be rewritten, via a Legendre recurrence relation, as a linear combination of \( P_{l' - 1}(\mu_{x_1,x_2}) \) and \( P_{l' + 1}(\mu_{x_1,x_2}) \). After putting all this together, we find

$$
\langle |\mathbf{r}(x_1) \cdot \mathbf{n}_1| |\mathbf{r}(x_2) \cdot \mathbf{n}_2| \rangle = \sum_{l} \frac{2l+1}{4\pi} C_{l}^{ee} P_l(\cos \theta_{12}), \tag{B8}
$$

where the power spectrum multipoles \( C_{l}^{ee} \) are given by

$$
C_{l}^{ee} = \frac{4\pi}{2l+1} \left[ lB_{l-1} + (l+1)B_{l+1} \right] \tag{B9}
$$

and the \( B_l \) are defined as

$$
B_l = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P_{\nu}(k) j_l(kx_1) j_l(kx_2). \tag{B10}
$$
Using equation (18), we can write the average product of the kSZ temperature anisotropies along two directions \( n_1 \) and \( n_2 \) as

\[
\left\langle \frac{\delta T_{\text{kSZ}}}{T_0} (n_1) \frac{\delta T_{\text{kSZ}}}{T_0} (n_2) \right\rangle = \int dr_1 dr_2 dm_1 dm_2 dy_1 dy_2 \hat{\tau}(m_1) \hat{\tau}(m_2) W^{\text{gas}}(y_1 - r_1) W^{\text{gas}}(y_2 - r_2) \left\langle n(M_1, y_1) \left\{ \frac{v(y_1)}{c} \cdot n_1 \right\} n(M_1, y_2) \left\{ \frac{v(y_2)}{c} \cdot n_2 \right\} \right\rangle. \tag{C1}
\]

Now we recall the number for the model of halos of equation (16) to rewrite the ensemble average in equation (C1) as

\[
\left\langle n(M_1, y_1) \left\{ \frac{v(y_1)}{c} \cdot n_1 \right\} n(M_1, y_2) \left\{ \frac{v(y_2)}{c} \cdot n_2 \right\} + \hat{n}(M_1, z_1) \hat{n}(M_2, z_2) \left\langle \frac{\partial \hat{n}(M_1, z_1)}{\partial \delta} \right\rangle_{\delta=0} \left\langle \frac{\partial \hat{n}(M_2, z_2)}{\partial \delta} \right\rangle_{\delta=0} \left\langle \frac{\phi(y_1, n_1) \phi(y_2, n_2)}{c^2} \right\rangle \right\rangle. \tag{C2}
\]

The first term is the Poisson term and is zero unless both \( n_1 \) and \( n_2 \) are looking at the same cluster. The second term is the velocity-velocity (vv) term, and the third contains the coupling of density and velocity studied in Appendix A (see eq. [A7]). Plugging equation (C2) into equation (C1) and writing the integrands in terms of integrals in the Fourier domain, one finds

\[
\left\langle \frac{\delta T_{\text{kSZ}}}{T_0} (n_1) \frac{\delta T_{\text{kSZ}}}{T_0} (n_2) \right\rangle = \int dr_1 dr_2 dm_1 dm_2 \frac{dk}{(2\pi)^3} \frac{dq}{(2\pi)^3} \exp[-\text{i}k \cdot (r_1 - r_2)] \hat{\tau}(m_1) \hat{\tau}(m_2) W^{\text{gas}}(W^{\text{gas}})^\dagger \left\langle n(M_1, z_1) \frac{\partial \hat{n}(M_1, z_1)}{\partial \delta} \right\rangle \left\langle n(M_2, z_2) \frac{\partial \hat{n}(M_2, z_2)}{\partial \delta} \right\rangle \left\langle \frac{\phi(y_1, n_1) \phi(y_2, n_2)}{c^2} \right\rangle. \tag{C3}
\]

The integration on \( y_1 \) in the Poisson term generates a Dirac delta on \(- \mathbf{q} \rightarrow \mathbf{q} \), whereas in the other two terms this Dirac delta arises naturally when one computes the ensemble average product of the Fourier modes of \( \epsilon \) and \( \phi \). Hence, the integral on \( \mathbf{q} \) disappears and every integral ends up with a term of the form \( \text{exp}[\text{i} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \). We introduce now the Rayleigh expansion, yielding

\[
\text{exp}[\text{i} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] = \sum_{l'} (2l' + 1)(2l' + 1)(-i)^{l-l'} j_{l'}(r_1) j_{l'}(r_2) \frac{\mu_{l',l}}{\mu_{l,l}}, \tag{C4}
\]

and we make use (as in eq. [B7] in Appendix B) of the addition theorem of Legendre polynomials to express those Legendre polynomials having as argument \( \mu_{l',l} \) as a sum of Legendre functions of \( \mu_{l',l} \) and \( \mu_{l,l} \). Here, as before, we have aligned the polar axis of \( \hat{\mathbf{k}} \) along \( \mathbf{r}_1 \) or \( \mathbf{n}_1 \). Also in this case, the integral of the azimuthal angle of \( \hat{\mathbf{k}} \) cancels the contribution of all Legendre functions having \( m \neq 0 \), so that the end one is left with

\[
\left\langle \frac{\delta T_{\text{kSZ}}}{T_0} (n_1) \frac{\delta T_{\text{kSZ}}}{T_0} (n_2) \right\rangle = \sum_{l' l} 2l' + 1 \frac{\mu_{l',l}}{\mu_{l,l}} \left\langle \frac{(4\pi)^2 k^2 dk}{(2\pi)^3} \int dr_1 dr_2 dm_1 dm_2 \hat{\tau}(m_1) \hat{\tau}(m_2) W^{\text{gas}}(W^{\text{gas}})^\dagger (M_1) \frac{\partial \hat{n}(M_1, z_1)}{\partial \delta} \left\langle \frac{\phi(y_1, n_1)}{c^2} \right\rangle \left\langle \frac{\phi(y_2, n_2)}{c^2} \right\rangle \right\rangle_{\delta=0} \left\langle \frac{\partial \hat{n}(M_2, z_2)}{\partial \delta} \right\rangle_{\delta=0} \left\langle \frac{\phi(y_1, n_1)}{c^2} \right\rangle \left\langle \frac{\phi(y_2, n_2)}{c^2} \right\rangle_{\delta=0} \left\langle \frac{\phi(y_1, n_1)}{c^2} \right\rangle \left\langle \frac{\phi(y_2, n_2)}{c^2} \right\rangle.
\]

As in Appendix A, we have taken \( n_1 = (0, 0, 1) \) and \( n_2 = (0, \sin \theta_12, \cos \theta_12) \). Since different velocity components are not correlated and we are only sensitive to the radial projection of the velocity, a \( \cos \theta_12 \) dependence appears in the vv term. For exactly the same reasons, we obtained a \( \cos \theta_12 \) and a \( \cos \theta_12 + \sin \theta_12 \) dependence when computing the power spectrum of \( \phi \) in Appendix A. Note that out of the three terms we found in that computation, we have dropped the last one because it is constant and introduces no anisotropy. Since the power spectra multipoles (C) are projections on Legendre polynomials, such projection must be applied on \( \cos \theta_12 \) and \( \cos \theta_12 + \sin \theta_12 \). The projection matrix for \( \cos \theta_12 \) will be identical to that given by equation (B9) in Appendix B. For \( \cos \theta_12 + \sin \theta_12 \), it must be computed numerically.
Summarizing, the power spectra corresponding to each of the terms considered here can be understood as a transformation of some vectors $c_i^X$ (where $X$ stands for Poisson, $vv$, $dd-vv$, or $dv-vd$) by some linear applications $A_{i,j}^X$:

$$C_i^X = \sum_{l'} A_{i,l'}^X c_{l'}^X. \quad (C6)$$

For the Poisson term, $A_{i,j}^{P_{\text{Poisson}}}$ is the identity, whereas for the $vv$ and $dd-vv$ terms we find

$$A_{i,j}^{v,v,dd-vv} = 4\pi \left[ \frac{1}{(2l - 1)^2} + \frac{1}{(2l + 3)^2} \right], \quad \text{with } \delta_{i,j} \text{ the Kronecker delta for } i \text{ and } j.$$  \quad (C7)

The projection matrix for the $dv-vd$ term reads

$$A_{i,j}^{d,v} = 2\pi \int_{-1}^{+1} d\mu \left( \mu + \sqrt{1 - \mu^2} \right) P_l(\mu) P_{l'}(\mu). \quad \text{(C8)}$$

The $c_i^X$ vectors for the Poisson, $vv$, $dd-vv$, and $dv-vd$ terms are as follows:

1. The Poisson term:

$$c_i^{P_{\text{Poisson}}} = \frac{2}{\pi} \int k^2 dk dM \left[ \Delta_i^P(k, \ M) \right]^2, \quad \text{(C9)}$$

with $\Delta_i^P(k, \ M)$ being

$$\Delta_i^P(k, \ M) = \int dr \hat{\sigma}_{vv} [n(M, z)]^{1/2} j_l(kr) W^\text{gas}_k.$$  \quad \text{(C10)}

2. The $vv$ term:

$$c_i^{vv} = \frac{2}{\pi} \int k^2 dk \left[ \Delta_i^{vv}(k) \right]^2, \quad \text{(C11)}$$

with $[\Delta_i^{vv}(k)]^2$ given by

$$[\Delta_i^{vv}(k)]^2 = \int dr_1 dr_2 dM_1 dM_2 \hat{\sigma}_{vv}(M_1, z_1) \hat{\sigma}_{vv}(M_2, z_2) n(M_1, z_1) n(M_2, z_2)$$

$$\times W^\text{gas}_k(M_1, z_1) W^\text{gas}_k(M_2, z_2) P_{vl}(M_1, M_2, z_1, z_2, k) j_l(kr_1) j_l(kr_2). \quad \text{(C12)}$$

3. The $dd-vv$ term:

$$c_i^{dd-vv} = \frac{2}{\pi} \int k^2 dk \left( \Delta_i^{dd-vv} \right)^2, \quad \text{(C13)}$$

with

$$[\Delta_i^{dd-vv}(k)]^2 = \int dr_1 dr_2 dM_1 dM_2 \hat{\sigma}_{vv}(M_1, z_1) \hat{\sigma}_{vv}(M_2, z_2) \left. \left( \frac{\partial \tilde{\sigma}(M_1, z_1)}{\partial \delta} \right) \right|_{\delta=0}$$

$$\times \left. \left( \frac{\partial \tilde{\sigma}(M_2, z_2)}{\partial \delta} \right) \right|_{\delta=0}$$

$$\times W^\text{gas}_k(M_1, z_1) W^\text{gas}_k(M_2, z_2) [P_{dd} * P_{vl}](M_1, M_2, z_1, z_2, k) j_l(kr_1) j_l(kr_2). \quad \text{(C14)}$$

4. The $dv-vd$ term:

$$c_i^{dv-vd} = \frac{2}{\pi} \int k^2 dk \left( \Delta_i^{dv-vd} \right)^2, \quad \text{(C15)}$$

with

$$[\Delta_i^{dv-vd}(k)]^2 = \int dr_1 dr_2 dM_1 dM_2 \hat{\sigma}_{vv}(M_1, z_1) \hat{\sigma}_{vv}(M_2, z_2) \left. \left( \frac{\partial \tilde{\sigma}(M_1, z_1)}{\partial \delta} \right) \right|_{\delta=0}$$

$$\times \left. \left( \frac{\partial \tilde{\sigma}(M_2, z_2)}{\partial \delta} \right) \right|_{\delta=0}$$

$$\times W^\text{gas}_k(M_1, z_1) W^\text{gas}_k(M_2, z_2) [P_{dd} * P_{vl}](M_1, M_2, z_1, z_2, k) j_l(kr_1) j_l(kr_2). \quad \text{(C16)}$$
REFERENCES

Aghanim, N., Górski, K. M., & Puget, J.-L. 2001, A&A, 374, 1
Aghanim, N., Hansen, S. H., & Lagache, G. 2005, A&A, 439, 901
Atrio-Barandela, F., Kashlinsky, A., & Mücke, J. P. 2004, ApJ, 601, L111
Benson, B. A., et al. 2003, ApJ, 592, 674
Colberg, J. M., et al. 2000, MNRAS, 319, 209
Courteau, S., & Dekel, A. 2001, in ASP Conf. Ser. 245, Astrophysical Ages and Times Scales, ed. T. von Hippel, C. Simpson, & N. Manset (San Francisco: ASP), 584
D'Eo, S., Spergel, D. N., & Trac, H. 2005, ApJ, submitted (astro-ph/0511060)
Diaferio, A., Sunyaev, R. A., & Nusser, A. 2000, ApJ, 533, L71
Diaferio, A., et al. 2005, MNRAS, 356, 1477
Doré O., Hennawi, J. F., & Spergel, D. N. 2004, ApJ, 606, 46
Eisenstein, D. J., et al. 2005, ApJ, 633, 560
Fowler, J. W., et al. 2005, ApJS, 156, 1
Gradshteyn, I. S., & Ryzhik, I. M. 1965, Table of Integrals, Series, and Products (New York: Academic Press)
Gunn, J. E., & Gott, J. R. I. 1972, ApJ, 176, 1
Hamana, T., Kayo, I., Yoshida, N., Suto, Y., & Jing, Y. P. 2003, MNRAS, 343, 1312
Holder, G. P. 2002, ApJ, 578, L1
———. 2004, ApJ, 602, 18
Kashlinsky, A., & Atrio–Barandela, F. 2000, ApJ, 536, L67
Kim, R. S. J., et al. 2002, AJ, 123, 20
Komatsu, E., & Seljak, U. 2001, MNRAS, 327, 1353
Kosowsky, A. 2003, NewA Rev., 47, 939
Linder, E. V. 2005, Phys. Rev. D, 72, 043529
Ma, C.-P., & Fry, J. N. 2002, Phys. Rev. Lett., 88, 211301
Miller, C. J., et al. 2005, AJ, 130, 968
Mo, H. J., & White, S. D. M. 1996, MNRAS, 282, 347
Nagai, D., Kravtsov, A. V., & Kosowsky, A. 2003, ApJ, 587, 524
Peel, A. C. 2006, MNRAS, 365, 1191
Ruhl, J., et al. 2004, Proc. SPIE, 5498, 11
Santos, M. G., Cooray, A., Haiman, Z., Knox, L., & Ma, C.-P. 2003, ApJ, 598, 756
Schäfer, B. M., Pfrommer, C., Bartelmann, M., Springel, V., & Hernquist, L. 2004, MNRAS, submitted (astro-ph/0407089)
Sehgal, N., Kosowsky, A., & Holder, G. 2005, ApJ, 635, 22
Sheth, R. K., & Diaferio, A. 2001, MNRAS, 322, 901
Sheth, R. K., & Tormen, G. 1999, MNRAS, 308, 119
Spergel, D. N., et al. 2003, ApJS, 148, 175
Strauss, M. A., & Willick, J. A. 1995, Phys. Rep., 261, 271
Sunyaev, R. A., & Zel’dovich, Ya. B. 1972, Comments Astrophys. Space Phys., 4, 173
———. 1980, ARA&A, 18, 537
Tegmark, M., et al. 2004, Phys. Rev. D, 69, 103501
Verde, L., Haiman, Z., & Spergel, D. N. 2002, ApJ, 581, 5
Zhang, P., Pen, U.-L., & Trac, H. 2004, MNRAS, 347, 1224