Spin and Dualization of SU(5) Dyons

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Motivated by the dual standard model, we study the angular momentum spectrum of stable SU(5) dyons that can be transformed into purely electric states by a suitable duality rotation i.e. are dualizable. The problem reduces to solving a Diophantine equation for the holomorphic charges in each topological sector, but the solutions also have to satisfy certain constraints. We show that these equations can be solved and sets of dualizable, half-integer spin SU(5) dyons can be found, each of which corresponds to a single family of the standard model fermions. We then find two predictions of the dual standard model. First, the family of half-integer spin, dualizable dyons is accompanied by a set of dualizable, integer-spin partner states. Secondly, the dyon corresponding to the electron must necessarily contain non-trivial color internal structure. In addition, we provide other general results regarding the spectrum of dualizable dyons and their novel properties, and extend the stability analysis of SU(5) monopoles used in the dual standard model so far to discuss the stability of the half-integer spin dyons.

I. INTRODUCTION

The idea that particles may be viewed as solitons can be traced back to Skyrme [1] who introduced what is now called the Skyrme model in which a classical solution (“Skyrmion”) represents the proton. The model has proved useful in the discussion of the properties of light nuclei even though it is known that the Skyrmion does not have the constituent structure of the proton.

The recent attempts to build a dual standard model are along the lines that Skyrme developed — that is, to find a model that admits soliton analogs of the known fundamental particles. Partial success in this direction was achieved in the discovery that the topological charges of the stable magnetic monopoles in an SU(5) field theory are in one to one correspondence with the electric charges of one family of fermions of the standard model [2,3]. A possible scheme to obtain three families of identically charged magnetic monopoles was outlined in Ref. [4], though at the expense of considerably complicating the group structure of the model.

So far, a substantial shortcoming of the model (summarized in section II) has been that the monopoles emerging from the SU(5) field theory are all bosonic while the standard model particles are known to be fermionic. The issue of spin and handedness of the solitons was discussed in Ref. [3] though not resolved in the SU(5) context. The basis for the discussion was the discovery of “spin from isospin” [5-7] in which dyons can have half-integer spin even in a purely bosonic particle theory. The possibility for handedness was discussed in the context of a θ term in the action and the result that dyons can carry fractional electric charges proportional to θ [8]. Also needed was the angular momentum of dyons in presence of a θ term [6].

The success of the spin from isospin phenomenon for the dual standard model depends on the existence of half-integer spin states for all the dyons that ultimately correspond to the standard model particles. In the case of the ’t Hooft-Polyakov [10,11] monopole, spin from isospin can provide half-integer spin to the fundamental monopole but not to the monopole with twice the topological winding. In contrast, in the SU(5) case it has been shown that all the stable monopoles can be provided with electric charges to make them into half-integer spin dyons [12]. However, the particles that we observe are not ostensibly dyons. Hence it is important to show that all the half-integer spin dyons which arise in the SU(5) field theory and which will be identified with standard model fermions can be transformed by a duality rotation into purely electric charges. This is the aim of the present paper.

Here we shall take the approach that the known standard model particles are purely electric (in contrast to Refs. [8,9]), and then we would like to know whether the spin 1/2 dyonic states of the SU(5) model — that are in one-one correspondence with the standard model particles — can all be dualized into purely electric charges. In trying to answer this question, we strictly need to consider duality rotations for gauge fields transforming in representations of the unbroken symmetry group $H = [SU(3) \times SU(2) \times U(1)]/Z_6$. Such non-Abelian duality transformations are not fully understood yet. Our approach (section III) will be to assume that independent
duality rotations can be applied to the field strengths in the directions of the four commuting generators of $H$. In other words, we assume that the transformation
\[ E_i^a + iB_i^a \rightarrow e^{i\phi_a} (E_i^a + iB_i^a), \quad a = 0, 8, 3, 1 \]  \tag{1}

is valid with independent phase angles $\phi_a$ for the generators $\lambda_3$ and $\lambda_8$ of $SU(3)$, $\tau_3$ of $SU(2)$ and $Y$ of $U(1)$:
\[ \lambda_3 = \frac{1}{2} \text{diag}(1, -1, 0, 0, 0), \quad \lambda_8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2, 0, 0), \quad \tau_3 = \frac{1}{2} \text{diag}(0, 0, 0, 1, -1), \]
\[ Y = \frac{1}{2\sqrt{15}} \text{diag}(2, 2, 2, -3, -3). \]  \tag{5}
The corresponding magnetic charges $m_a$ and electric charges $q_a$ transform as
\[ (q_a + im_a) \rightarrow e^{i\phi_a} (q_a + im_a). \]  \tag{6}

We cannot rigorously justify these transformations since the $SU(3)$ non-Abelian equations of motion explicitly involve the gauge fields and not just the field strengths (in contrast to Maxwell’s equations). However, the equations of motion are indeed invariant under the transformation if only the commuting gauge field components (Cartan subalgebra) are non-vanishing. Also note that the Hamiltonian and the Euclidean action remain invariant under the transformation in eq. (5). Furthermore, if $SU(5)$ were broken down to $U(1)^4$, each of the gauge field components labelled by the index $a = 0, 8, 3, 1$ would be Abelian and then the Abelian duality transformations would correspond exactly to eq. (5).

While adopting the transformation in eq. (1) as the duality rotation, we also discuss the case when this may not be true. If, for example, we restrict ourselves to $\phi_0 = \phi_8$, we can show that the half-integer spin states in the even winding topological sectors must necessarily carry both magnetic and electric $SU(3)$ charge.

In the case when the $\phi_a$ are independent, we find an infinite set of dualizable, half-integer $SU(5)$ dyon states that are in one to one correspondence with the standard model particles. To arrive at this conclusion we need to solve constrained quadratic Diophantine equations that can be definite or indefinite. Such equations have been considered at least since 600 A.D. by Bhaskara and Brahmagupta and techniques to solve them can be found in number theory text books (eg. [13]). We shall describe some of the equations and their solutions in Appendix B.

The infinity of solutions is unlikely to be of any direct physical relevance. The reason is that we are interested only in the lowest energy state in any given topological sector since, presumably, the higher energy states are unstable to decay into the lowest energy state. However, the energy of a dyon is not known at strong coupling — which is the relevant regime for making contact with the standard model — and so there is no sure way of determining the lowest energy states. The best that we can do at present is to assume a BPS form for the energy [16,17] (see also the monopole reviews in Ref. [18]) in which the energy of a dyon is proportional to the magnitude of its charge:
\[ E_{BPS} \propto \sqrt{q_a^2 + m_a^2} \]  \tag{7}

where a sum over the index $a$ is understood. This form of the energy does not apply to the dual standard model where the monopoles may be close to being BPS but are not exactly BPS [4], and neither does it apply to the standard model particles. The purpose of considering eq. (7) is simply that it enables us to find the lowest energy dyons in the weak coupling, near BPS limit.

If we assume eq. (7) for the energy, then for any given value of the magnetic charge $m_a$, it would pick out the state $q_a = 0$ as the state with the lowest energy. These purely magnetic states would have zero spin (see below). The situation is more interesting when we include a $\theta$ term in the $SU(5)$ action because then the electric charge contains a contribution from the $\theta$ term. In that case, the lowest energy state can indeed have half integer spin. The hope then would be that for a certain value of $\theta$, of the phases $\phi_a$, and of the coupling constant $g$, one would obtain a complete family of spin half dyons which would be the lowest energy states. However, we show that this hope is not realized due to the monopole with topological winding $n = 6$. In this topological class, the state with the lowest BPS energy necessarily has integer spin.

Ideally we would like to work with the energy of a dyon at strong coupling and then determine the lightest states for given parameters. This would require understanding the quantum properties of magnetic monopoles — a subject that has been under intense research over the last two decades. Remarkable progress has been achieved in the understanding of monopoles at strong coupling in the supersymmetric case [19] but several tantalizing issues remain open especially in the non-supersymmetric setting (eg. [20]). An issue that is central to particle-soliton duality is the group representation in which the monopoles transform when they are considered as particles. Goddard-Nuyts-Olive conjectured that monopoles transform in a representation of a dual symmetry group [21]. Bais and Schroers [22,23] find that a richer structure is applicable to non-Abelian monopoles, since they carry “holomorphic” charges in addition to a topological charge. (This will be important to us in Sec. 7.) In the $SU(5)$ model, Lepora has provided strong evidence that the monopoles transform in the fundamental representation of the dual symmetry group $(SU(3) \times SU(2) \times U(1))$
based on the transformation properties of the monopoles under rigid gauge transformations. This evidence seems to support the concept of a dual standard model. Further support comes from Lepora’s calculation of the value of the weak mixing angle $\theta_w$ in the context of the $SU(5)$ dual standard model. Lepora finds $\sin^2 \theta_w = 0.22$ which is in good agreement with experiment at a few GeV. However, the relevance of the few GeV scale to the dual standard model has not yet been investigated. Naively it seems that this should be the scale at which the monopole-like structure of elementary particles becomes relevant. Then it is possible that phenomenological considerations already impose strong constraints on the idea of the dual standard model. It would be very interesting to pursue this idea further.

II. REVIEW OF DUAL STANDARD MODEL

Consider the symmetry breaking

$$G = SU(5) \rightarrow H = [SU(3) \times SU(2) \times U(1)]/Z_6.$$  \hspace{1cm} (8)

The magnetic monopoles in this symmetry breaking are labelled by their $SU(3)$, $SU(2)$ and $U(1)$ magnetic charges,

$$M = (m_0, m_8, m_3, m_1) = \left( \begin{array}{c} n_8 \\
\sqrt{3}g \\
\frac{1}{2} \sqrt{3}n_\lambda \\
3n_1 \end{array} \right) \hspace{1cm} (9)$$

where,

$$n_8 = n + 3k \hspace{0.5cm} n_3 = n + 2l \hspace{0.5cm} n_1 = n.$$ \hspace{1cm} (10)

Here, $k$ and $l$ are arbitrary integers since the $\lambda_8$ (of $SU(3)$) and $\tau_3$ (of $SU(2)$) charges are only defined modulo 3 and 2 respectively.

The topological sector is only determined by the integer $n_1$ which gives the topological winding number ($\Pi_2(G/H) = Z$). The integers $n_8$ and $n_3$ are related to the “holomorphic” charges which are discussed in Ref. [23,24,26] and which are not topological. In [26], Murray derived constraints that the sum of the topological and holomorphic charges has to be greater than or equal to zero. The holomorphic charges are the diagonal entries of the magnetic charge matrix which in this $SU(5)$ case is

$$2M = 2g[m_0 \lambda_3 + m_8 \lambda_8 + m_3 \tau_3 + m_1 Y]$$

$$= \text{diag}(\frac{n_8 - n_1}{3}, \frac{n_8 - n_1}{3}, \frac{-2n_8 - n_1}{3}, \frac{n_3 + n_1}{2}, \frac{-n_3 + n_1}{2}).$$ \hspace{1cm} (11)

Murray’s constraints [26] are then that the first three entries of the charge matrix must be non-negative and the last two entries must be greater than or equal to minus the topological charge (our $n_1$). For $n_1 \leq 0$ this leads to:

$$-n_1 \geq 2k \geq 0, \hspace{0.5cm} -n_1 \geq l \geq 0.$$ \hspace{1cm} (12)

(For positive values of $n_1$, these inequalities would be reversed.)

The physical origin of Murray’s constraints are, however, not clear. As the ingredients in the derivation of the constraints only involve the structure of the gauge field theory, it is likely that any configuration that violates the conditions is gauge equivalent to one that does satisfy the conditions. This is borne out by considering the trivial case with $n_1 = 0$. However, as we shall see below, the integer $k$ is crucially important in determining the spin of a dyon: there are values of $k$ that violate the constraints but which give rise to angular momentum that cannot be achieved by states satisfying the constraints. For this reason, we will assume Murray’s constraints provided there is no state that violates them and which has a different value (integer versus half-integer) of the angular momentum.

A stability analysis of the non-BPS monopoles in any topological sector shows that only the $\pm n = 1, 2, 3, 4, 6$ monopoles are stable. (This result assumes a range of parameters in the $SU(5)$ potential.) A comparison with the standard model particles shows that these monopoles are in one to one correspondence as depicted in Table I.

| $n$ | $n_8/3$ | $n_3/2$ | $n_1/6$ |
|-----|---------|---------|---------|
| +1  | 1/3     | 1/2     | 1/6     |
| -2  | 1/3     | 0       | -1/3    |
| -3  | 0       | 1/2     | -1/2    |
| +4  | 1/3     | 0       | 2/3     |
| -6  | 0       | 0       | -1      |

The spin of the $SU(5)$ monopoles is provided by bound states of quanta of a fundamental scalar field. (The existence of such bound states will depend on the details of the $SU(5)$ potential. Here we will simply assume that the bound states exist.) This is the “spin from isospin”

*In addition, we have not been able to resolve the different constraints that result if we order the charge matrix differently. If the charge matrix is organised to have the $SU(2)$ block first and then the $SU(3)$ block, the constraints in eq. (13) are reversed and also $2k$ must be replaced by $k$. Note that Murray’s constraints are not related to the 3-fold color degeneracy and 2-fold spin degeneracy in the choice of $SU(3)$ and $SU(2)$ generators.
idea [3][4] extended to $SU(5)$ monopoles [13][27]. To determine whether the spin is integer or half-integer, one needs to calculate the angular momentum in the gauge fields of the dyon. It is given by

$$J = - \sum_a q_a m_a$$  

where the index $a$ runs over 0, 8, 3, 1 and labels the two $SU(3)$ charges, one $SU(2)$ charge and the hypercharge. The $m_a$ have been defined in eq. [8] and the $q_a$ are the electric charges present in the state under consideration.

The fundamental scalar field of $SU(5)$ has five components and we can consider dyonic states with any number of quanta of these five components. Let us label the components by the index $h$, then the four different electric charges on a single quanta of each of the five components can be written as:

$$e^h_0 = \frac{g}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^h_8 = \frac{g}{2\sqrt{3}} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$e^h_3 = \frac{g}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad e^h_1 = \frac{g}{2\sqrt{15}} \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \\ -3 \end{pmatrix},$$  

(These assignments are obtained by considering the corresponding Noether charges.) To clarify the meaning of these charge assignments consider the example in which we have one quanta of the first component ($h = 1$) of the fundamental scalar field. This quanta will have $q_0 = g/2$, $q_8 = g/2\sqrt{3}$, $q_3 = 0$ and $q_1 = g/\sqrt{15}$. Similarly we can work out the charges on any of the other four ($h = 2, 3, 4, 5$) scalar field components. If we now consider $N_h$ quanta of the component $h$, then the total electric charge is:

$$Q = (q_0, q_8, q_3, q_1),$$  

with

$$q_0 = \frac{g}{2}(N_1 - N_2)$$  

$$q_8 = \frac{g}{2\sqrt{3}}(N_1 + N_2 - 2N_3)$$  

$$q_3 = \frac{g}{2}(N_4 - N_5)$$  

$$q_1 = \frac{g}{2\sqrt{15}}[2(N_1 + N_2 + N_3) - 3(N_4 + N_5)].$$  

Let us now define

$$M_0 \equiv -(N_1 - N_3)$$  

$$M_8 \equiv -(N_1 + N_2 - 2N_3)$$  

$$M_3 \equiv -(N_4 - N_5)$$  

$$M_1 \equiv -3(N_4 + N_5) + 2(N_1 + N_2 + N_3).$$  

Since the $N_h$ are integers, so are the $M_a$. Solving the above equations gives the $N_h$ in terms of the $M_a$:

$$N_2 = N_1 + M_0$$  

$$N_3 = N_1 + \frac{M_8 + M_0}{2}$$  

$$N_4 = N_1 + \frac{M_0 - M_5}{2} + \frac{M_8 - M_1}{6}$$  

$$N_5 = N_1 + \frac{M_0 + M_5}{2} + \frac{M_8 - M_1}{6}.$$  

Now since the $N_h$ are integers, we have the following two constraints on the $M_a$:

$$\frac{M_8 + M_0}{2} = \text{integer}$$  

$$\frac{M_0 - M_5}{2} + \frac{M_8 - M_1}{6} = \text{integer}.$$  

The second constraint can be combined with the first to put it in a more useful form:

$$\frac{M_8}{3} + \frac{M_3}{2} + \frac{M_1}{6} = \text{integer}.$$  

Then the angular momentum from eq. (13) with eqs. (14) and (8) is found to be:

$$J = \frac{1}{2} \left[ \frac{M_8 n_8}{3} + \frac{M_3 n_3}{2} + \frac{M_1 n_1}{6} \right].$$  

In Ref. [13] it was shown that we can have $J = 1/2$ for every value of $n$ for suitable values of the electric charges $M_a$ (which will be different on the different monopoles). Note that $J$ is only the angular momentum in the long range gauge fields and does not contain other possible contributions such as orbital angular momentum and spin of the gauge particles. These extra contributions can only change the angular momentum by an integer and cannot change a half-integer angular momentum state to one that has integer angular momentum (or vice versa).

Next let us consider the addition of an $SU(5)$ $\theta$ term. In terms of the gauge fields corresponding to the diagonal generators, the additional piece of the Lagrangian is:
\[ L_0 = \kappa \left[ C_{\mu \nu}^3 \tilde{G}^{\mu \nu \delta} + C_{\mu \nu}^8 \tilde{G}^{\mu \nu \delta} + W_{\mu \nu}^3 \tilde{W}^{\mu \nu \delta} + Y_{\mu \nu} Y^{\mu \nu} \right] \]

(32)

where,

\[ \kappa = \frac{g^2 \theta}{16 \pi^2} . \]

(33)

The addition of such a term does not alter the expression for the angular momentum of the dyons given in eq. (3) but it does affect the values of the electric charges in eqs. (28)(30). (In the case of SU(2) monopoles, the effect of a \( \theta \) term on the electric charge has been discussed in Ref. \cite{8}) and on the angular momentum of dyons in Ref. \cite{14}.)

The new expressions for the electric charges on the dyons are:

\[ q_0 = \frac{g}{2} M_0 \]

(34)

\[ q_8 = -\frac{g}{2 \sqrt{3}} M_8 + \frac{g m_8}{\sqrt{3}} \frac{\theta}{2 \pi} \]

(35)

\[ q_3 = -\frac{g}{2} M_3 + \frac{g m_3}{2} \frac{\theta}{2 \pi} \]

(36)

\[ q_1 = \frac{g}{2 \sqrt{15}} M_1 - \frac{g m_1}{2} \sqrt{\frac{5}{3}} \frac{\theta}{2 \pi} . \]

(37)

It is straightforward to check that a shift of \( \theta \) by \( 2\pi \) can be compensated for by shifts of the \( M_\alpha \) that satisfy eqs. (28) and (30), thus verifying that the spectrum of states is invariant under \( \theta \to \theta + 2\pi j \) for any integer \( j \).

We now want to know if a complete set (i.e. all topological sectors occurring in Table 1) of the half-integer spin dyons can be made to be purely electric by performing a suitable duality rotation (see Fig. 1).

![Diagram of SU(5) dyons](image_url)

**FIG. 1.** The SU(5) dyons have four magnetic and electric charges and so they can be depicted as points on the four mq-planes (only three planes have been shown for convenience). We depict the bosonic states by filled circles and the fermionic states by unfilled circles. The question we would like to address is whether a complete family of half-integer spin dyons lies on straight lines passing through the origin i.e. if a rotation of the four sets of axes can give us a full family of purely electric half-integer spin states.

### III. DUALIZATION OF HALF-INTEGER SPIN DYONS

The duality rotation phase angles \( \phi_\alpha \) (see eq. (1)) required to make a dyon into a purely electric object are given by the inverse tangent of the ratios of its magnetic and electric charges. Therefore

\[ \tan \phi_0 = 0 , \quad \text{(if } M_0 \neq 0) \]

(38)

\[ \tan \phi_8 = -\frac{2}{g^2 M_8 / n_8 - 2 \theta / 2 \pi} \]

(39)

\[ \tan \phi_3 = -\frac{1}{g^2 M_3 / n_3 - \theta / 2 \pi} \]

(40)

\[ \tan \phi_1 = -\frac{5}{g^2 M_1 / n_1 - 5 \theta / 2 \pi}. \]

(41)

Note that the \( M_\alpha \) are integers and denote the electric charges on the dyons and hence can depend on the winding number \( n \). Also the integers \( m_\alpha \) clearly depend on \( n \).

For the dyons to be dualizable, we want that the duality phase angles be independent of \( n \). Hence we require that \( \alpha_\alpha \) be independent of \( n \) where

\[ M_8 = n_8 \alpha_8 \]

(42)

\[ M_3 = n_3 \alpha_3 \]

(43)

\[ M_1 = n_1 \alpha_1 \]

(44)

The \( \alpha_\alpha \) are independent of \( n \) and hence by considering the dyon with \( n_1 = 1 \) we find that \( \alpha_1 \) must be an integer. The constraint in eq. (12) shows that we must also take \( n_8 = 1 \) and \( n_3 = 1 \) for \( n_1 = 1 \) and so all the \( \alpha_\alpha \) must be taken to be integers\(^\dagger\). Furthermore, there is a constraint that the \( \alpha_\alpha \) must satisfy, coming from the constraint eq. (30) when combined with eq. (10) and setting \( n = 1 \):

\[ \frac{\alpha_8}{3} + \frac{\alpha_3}{2} + \frac{\alpha_1}{6} = \text{integer} . \]

(45)

In terms of the \( \alpha_\alpha \), the angular momentum is given by

\[ 2J_n = \left[ \frac{\alpha_8}{3} + \frac{\alpha_3}{2} + \frac{\alpha_1}{6} \right] n^2 \]

\[ + 2[n(\alpha_8 k_n + \alpha_3 l_n) + \alpha_3 l_n^2] + 3 \alpha_8 k_n^2 . \]

(46)

\(^\dagger\) Following the discussion after eq. (12), if we relax the constraint to allow \( n_8 = -2 \) for \( n_1 = 1 \), half-integer values of \( \alpha_8 \) could still yield integer values of \( M_8 \). However, in Appendix A we show that half-integer values of \( \alpha_8 \) cannot yield a family of spin half dyons and so we will restrict our discussion to integer \( \alpha_8 \).
For the whole family of dyons (±n = 1, 2, 3, 4, 6) to have half-integer spin, we need the right-hand side of eq. (46) to be odd for each member. First consider the n = 2 monopole. The first term on the right-hand side is clearly even in this case. The second term is also even since the αa are integers. So 2J2 is odd if and only if 3α8k2 2 is odd. Now suppose that αα and k2 are chosen so that 3α8k2 2 is odd. Then all the other dyons in the dualizable family will have half-integer spin if we set kα = ±k2 when the first term on the right-hand side of eq. (46) is even, and kα = 0 when this term is odd.

Two explicit examples satisfying the constraint in eq. (45) are:

\[ \alpha_8 = 1, \quad \alpha_3 = -1, \quad \alpha_1 = 1. \]  
\[ \alpha_8 = 1, \quad \alpha_3 = 0, \quad \alpha_1 = 4. \]

For the first example, the first term on the right-hand side of eq. (46) vanishes and therefore 2Jn is odd provided k2 2 = odd for all n. Hence a whole family of dyons has half-integer spin and is dualizable. In fact, there are an infinite number of solutions (αα) that have this property. This can be seen by noting that a shift of each of the αα by any fixed even integer also leads to a solution that satisfies the constraints and preserves the half-integer angular momentum.

The dualizable 2Jn = 1 dyon states for a fixed set of αα correspond to solutions of the Diophantine equation:

\[ 2α_8n_8^2 + 3α_3n_3^2 = 6 - α_1n_1^2. \]

In Appendix B we show that for the αα in eq. (47) there are an infinite number of dualizable dyonic states in every topological sector that have half-integer spin. This conclusion is expected to be valid whenever some of the αα’s differ in their signs, leading to indefinite (hyperbolic) Diophantine equations. If all the αα have the same sign, we expect there to be a finite set (possibly empty) of solutions. In view of the constraints in eq. (12), the infinite set of states is not of physical interest. Besides we only expect the lightest of the states for any given winding and angular momentum to be stable.

IV. GENERAL RESULTS

1) There are infinitely many solutions to the constraints leading to a dualizable family of half-integer spin dyons.

This has been shown above in the paragraph following eq. (48).

2) Each member of the family of dualizable half-integer spin dyons has an integer spin partner that is also dualizable.

To see this conclusion, note that if for a certain n one has 2Jn = odd, then the state with kα → kα ± 1 has (eq. (46))

\[ 2J_n (2J_n) = 2J_n + \text{even integer} + 3α_8. \]

However, 3α8 has to be odd since 2J2 = even + 3α8k2 2 and this has to be odd for the n = 2 monopole to have half-integer spin. Hence the state with the shifted value of kα has integer spin.

Hence the dual standard model predicts bosonic partners of all the standard model fermions. Unlike in the case of supersymmetry, the masses of the partners do not have to be degenerate.

3) The n = 6 dyon with the least BPS energy has integer spin.

The energy of a BPS dyon is given by eq. (7),

\[ EBPS = c\sqrt{q_a^2 + m_a^2} \]

where c is a proportionality constant. The state with the lowest energy is the one with the smallest electric and magnetic charge. For the n = 6 dyon, this is the state with n8 = 0 = n3 since then, both the electric and magnetic charges in the SU(3) and SU(2) sectors vanish. Now using eq. (33) together with (44) we see that this state has integer spin.

A general statement of this kind cannot be made for dyons with other windings since they necessarily have non-vanishing SU(3) and/or SU(2) magnetic charge. However it is not difficult to determine which spin state among the dualizable dyons has the least BPS energy. First note that dualizability implies qa ≈ ma ≈ na. This relation does not hold for a = 0 where we have qa = −ga/2 and M0 is constrained by eq. (23). Therefore, for fixed values of the αα, θ and for small values of g (when the electric charge contributions are subdominant), the least BPS energy state is one that has the minimum values of n8 2 and n3 2. For A ≡ α8/3 + α3/2 + α1/6 = odd, this ensures that the n = 1, −2, 4 states with half-integer spin have lower BPS energy than the corresponding integer spin states. However, for the n = −3 half-integer spin state to have lower energy than the integer spin state in the case of small g, we need A = even because only then the n8 = 0 (k3 = 1) state has half-integer spin.
It is worthwhile pointing out the role of the \( \theta \) term in these considerations. The lowest BPS energy states for a non-zero \( \theta \) angle will occur for non-zero values of the \( \alpha \). If \( \theta \) were zero, the states with the least energy would be those with vanishing electric charges (since \( \alpha = 0 \) would minimize the BPS energy) and hence, with zero spin.

4) The \( n = 2, 4, 6 \) half-integer spin dualizable dyons carry \( \lambda_3 \) electric charge i.e. \( M_0 \neq 0 \).

To see this, note that eq. (31) implies that \( 3M_3 + M_1 \)
is even. Therefore both \( M_3 \) and \( M_1 \) are even or both are odd. For the even \( n \) dyons, \( M_1 = \alpha_1 n_1 \) is even. Hence \( M_3 \) is also even for even \( n \). Now from the angular momentum formula eq. (1) and the relations in eq. (10) we get

\[
2J_n = \left[ \frac{M_6}{3} + \frac{M_3}{2} + \frac{M_1}{6} \right] n + M_8k_n + M_3l_n \quad (51)
\]

Therefore, taking eq. (31) into account, we see that \( 2J_n \)
is even for even \( n \) if \( M_4 \) is even. Hence to obtain an odd value for \( 2J_n \) (i.e. half-integer spin), we must necessarily set \( M_8 \) to be odd. Next we use the constraint in eq. (28) which shows that \( M_0 \) has to be odd and, in particular, has to be non-zero. Therefore these half-integer states necessarily carry \( \lambda_3 \) electric charge.

A consequence of this conclusion is that the two \( SU(3) \)
duality rotation phase angles \( \phi_0 \) and \( \phi_8 \) cannot be equal. If non-Abelian duality rotations can only be applied with \( \phi_0 = \phi_8 \) then the dual standard model would only work if the particles transforming non-trivially under \( SU(3) \) carry magnetic charge.

5) The \( n = 6 \) half-integer spin dualizable dyon must have \( n_8 \neq 0 \).

Inserting \( n = 6 \) in eq. (40) shows that we must necessarily have \( k_8 = \text{odd} \) to get half-integer spin. Therefore \( n_8 = n + 3k_6 \) is necessarily non-vanishing and the \( n = 6 \) half-integer spin state carries \( SU(3) \) gluonic structure.

Similarly if \( \alpha_8 \) and \( \alpha_8/3 + \alpha_3/2 + \alpha_1/6 \) are odd integers, then \( k_3 \) has to be even for the \( n = 3 \) monopole to have half-integer spin. Then \( n_8 \neq 0 \) and this monopole also carries gluonic structure.

V. STABILITY OF HALF-INTEGER SPIN DYONS

The monopoles in any topological sector have two decay channels. Firstly, the monopoles can emit scalar and vector particles and change their values of \( k \) and \( l \). Secondly, a monopole can fragment into two monopoles of smaller magnetic charge. We have to show that neither of these instabilities apply to the states that we would like to interpret as standard model particles.

The first instability will not apply to the lowest lying half-integer spin state in any given topological sector and so we need only concern ourselves with the second instability.

Next we show that the dyons with topological winding \( n > 6 \) are all unstable to fragmentation into dyons with \( n = 6 \) and something else.

Let us denote the dyonic states by their magnetic and electric charges as follows:

\[
|n_8, n_3, n_1; M_8, M_3, M_1 >
\]

Then we want to show that the decay process:

\[
|n_8, n_3, n_1; M_8, M_3, M_1 > \rightarrow |n_8, n_3, n_1 - 6; M_8 - p_8, M_3 - p_3, M_1 - p_1 > + |0, 0, 6; p_8, p_3, p_1 > \quad (52)
\]
is energetically favorable. The two states on the right-hand side interact by the U(1) magnetic interactions and we know that this is repulsive. The electric interactions are small compared to the magnetic interactions at weak coupling by a factor \( g^4 \) and so we ignore them for the present. (Later we will check that the decay would proceed even with the electric interactions taken into account.) Hence it is clear that this decay process is energetically favorable. What is not so clear is if the process is allowed by angular momentum conservation. (The magnetic and electric charges are conserved in eq. (52).) This is what we will now check.

The angular momentum of the states on the right-hand side can be written as (eq. (31)):

\[
2J_{rhs} = 2J_{lhs} + 2p_3 - M_1 - \frac{p_8n_8}{3} - \frac{p_3n_3}{2} - \frac{p_1n_1}{6} \quad (53)
\]

upto the addition of an integer (which may be carried off in orbital angular momentum etc.). For angular momentum conservation — meaning that half-integer initial angular momentum should go to half-integer final angular momentum and similarly for integer angular momentum — we therefore need

\[
-M_1 - \frac{p_8n_8}{3} - \frac{p_3n_3}{2} - \frac{p_1n_1}{6} = \text{even integer} \quad (54)
\]

A solution is simply given by \( p_8 = 0 = p_3, p_1 = 6\alpha_1 \) because then the left-hand side is even. With these values of the \( p_a \), the electric interactions are also purely U(1) and repulsive. This shows that the decay process is not forbidden by angular momentum conservation and hence can occur for purely energetic reasons which we know favor it.

A similar stability analysis goes through for the \( n = 5 \) dyon. Consider the decay process:

\[
|n_8, n_3, 5; M_8, M_3, M_1 > \rightarrow |0, n_3, 3; p_8, M_3 - p_3, p_1 > + |n_8, 0, 2; M_8 - p_8, p_3, M_1 - p_1 > . \quad (55)
\]
This is energetically favored since the two dyons on the right-hand side interact only via the U(1) magnetic interaction which is repulsive. Next we need to check if the decay is allowed by angular momentum conservation.

Using the formula for the angular momentum (eq. [22]), we find:

$$2J_{\text{rs}} = 2J_{\text{hs}} - \left[ \frac{n_8p_8}{3} + \frac{n_3p_3}{2} - \frac{p_1}{6} + \frac{M_1}{2} \right].$$ (56)

So the decay will be allowed provided:

$$\frac{n_8p_8}{3} + \frac{n_3p_3}{2} - \frac{p_1}{6} + \frac{M_1}{2} = \text{even integer}. $$ (57)

This is clearly so if we choose $p_8 = 0 = p_3$ and $p_1 = 3\alpha_1$. (Recall that $M_1 = 5\alpha_1$ for the initial state to be dualizable.) With this choice of $p_3$, once again the electric interactions are purely U(1) and repulsive. Hence the $n = 5$ dyon is unstable.

This tells us that the dyons with $|n| = 5$ and $|n| \geq 7$ are unstable, exactly as found for the monopoles in [22]. The $\pm n = 1, 2, 3, 4, 6$ dyons will still be stable because the fragmentation is completely governed (in the weak coupling limit) by the magnetic interactions [22]. Therefore the spectrum of stable half-integer spin dyons also agrees with the standard model fermions.

VI. DISCUSSION

Our general results can be found in Sec. [16]. The main conclusion is that it is possible to find a family of dyons each member of which has half-integer spin and the family as a whole can be dualized into purely electric states (subject to the discussion of duality rotations given in the introduction). In addition, there are two new features that have emerged and that may be considered as predictions of the dual standard model. The first is that each of the half-integer spin dyons has a bosonic partner. In the dualized theory, these states would appear as bosonic partners of the known standard model fermions. Since the bosonic partners are not due to an imposed symmetry (eg. supersymmetry), there is no reason to expect them to be degenerate in mass with their fermionic partners. The second new feature is that some of the half-integer spin dyons (in particular the $n = 6$ dyon) may have non-vanishing values of $n_8$ and $n_3$ even though the minimum allowed values of these quantum numbers may be zero. For example, in the $n = 6$ case, the minimum values are $n_8 = 0 = n_3$, yet to get half-integer spin it is necessary to have $n_8 \neq 0$ (see Sec. [15]). An interpretation of this result is that since these monopoles with $n_8 = n + 3k_n \neq 0$ carry the same topological charge as the monopole with winding number $n$ but with $n_8 = 0$, they too must transform in the fundamental representation of the dual symmetry group [24]. However, the value of $k_n$ is another charge associated with the monopole (related to the “holomorphic” charge in [22]).

How is the holomorphic charge manifested in the context of the dual standard model? The holomorphic charge seems to label an internal degree of freedom of the dualized dyons and, according to Bais and Schroers [22], manifests itself as a magnetic dipole moment of the dyons i.e. an electric dipole moment of the particles. Then, for example, the $n = -6$, spin half dyon necessarily has $n_8 \neq 0$ which means that it must have non-trivial SU(3) internal structure even though it transforms as an SU(3) singlet. The resolution to this apparent paradox is that the particles in the context of the dual standard model are composite objects and hence they can have internal SU(3) structure in spite of having trivial SU(3) long range interactions (as in the case of the proton). The novelty here is that the $n = -6$ dyon under discussion supposedly corresponds to the electron, implying that the electron must carry non-trivial internal SU(3) structure!

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[1] T.H.R. Skyrme, Proc. R. Soc. A247, 260 (1958); A262, 237 (1961).
[2] T. Vachaspati, Phys. Rev. Lett. 76 188 (1996).
[3] H. Liu and T. Vachaspati, Phys. Rev. D56, 1300 (1997).
[4] H. Liu, G.D. Starkman and T. Vachaspati, Phys. Rev. Lett. 78, 1223 (1997).
[5] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36 1116 (1976).
[6] P. Hasenfratz and G. ’t Hooft, Phys. Rev. Lett. 36 1119 (1976).
[7] A. S. Goldhaber, Phys. Rev. Lett. 36 1122 (1976).
[8] E. Witten, Phys. Lett. B86 283 (1979).
[9] F. Wilczek, Phys. Rev. Lett. 48 1146 (1982).
[10] G. ’t Hooft, Nucl. Phys. B79 276 (1974).
[11] A. M. Polyakov, JETP Lett. 20 194 (1974).
[12] T. Vachaspati, Phys. Lett. B427 323 (1998).
[13] H. M. Chan and S. T. Tsou, Phys. Rev. D57 2507 (1998).
[14] H. M. Chan, J. Bordes and S. T. Tsou, Int. J. Mod. Phys. A14 2173 (1999).
[15] “Number Theory”, by D. Redmound, Marcel Dekker Inc. (1996).
[16] M.K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 760 (1975).
Therefore we write \( n \) to find

\[
2p^2 - 3q^2 = 5
\]

where \( p \) and \( q \) are integers.

This is a standard problem in number theory and is related to Pell’s equation (for example, [15]). The idea of the construction is that given one solution to the equation

\[
ap^2 - bq^2 = c
\]

where \( a \), \( b \) and \( c \) are integers, and if there exists a non-trivial solution \((l, m)\) to the equation

\[
t^2 - abm^2 = 1 ,
\]

then an infinite set of solutions can be generated. (The trivial solutions are \( t^2 = 1, m = 0 \).) The construction uses the solution to the first equation, call it \( p_0, q_0 \), and the solution to the second equation, call it \( l, m \), to determine another solution:

\[
p = lp_0 + bmq_0 , \quad q = amp_0 + lq_0 .
\]

So this gives a relatively easy way to check if there are an infinite number of solutions and to generate them. Indeed for the unit winding monopole, one can check that this method generates an infinite number of spin 1/2 states. For the higher winding monopoles, we only need find one spin 1/2 solution (described below eq. (48)) and that guarantees an infinite number since the secondary equation does not care about the value of \( c \) and this is the only place where the topological winding of the monopole \((n)\) enters.

In our case we have another restriction on the solutions \( p \) and \( q \) since we require \( p = n + 3k \) and \( q = n + 2l \) where \( k \) and \( l \) are integers. However, it is easy to check that the construction still generates an infinite sequence of solutions. For the \( n = 1, 2, 4 \) cases, every alternate member of the sequence described above has the desired form. For the \( n = 3, 6 \) cases, every member has the desired form.

### APPENDIX A

Consider the possibility that \( \alpha_8 \) is a half-integer. In this case, for \( M_8 \) to be an integer, \( n_8 \) should be an even integer. Then, for even \( n \), all of \( n_8, n_3 \) and \( n_1 \) are even. Therefore we write \( n_a = \tilde{n}_a \) where \( \tilde{n}_a \) are integers and insert into the equation for the angular momentum (eq. (11)) to find

\[
2J_n = 2 \left[ \frac{1}{3} M_8 \tilde{n}_8 + M_3 \tilde{n}_3 + \frac{1}{3} M_1 \tilde{n}_1 \right]. \tag{58}
\]

Hence \( 2J_n \) is even and half-integer spin solutions do not exist. Therefore half-integer values of \( \alpha_8 \) cannot yield a family of half-integer spin dyons.

### APPENDIX B

Here we show that there are an infinite number of dyon states with \( J = 1/2 \) for the choice of \( \alpha_\alpha \) in eq. (17) (for example). This is not directly relevant to us because of eq. (12) and further physical constraints. However it is still an interesting exercise.

To see the infinity of solutions, rewrite the angular momentum constraint (eq. (31) with (12), (33), (44) and (47)) as:

\[
2n_8^2 - 3n_3^2 = 6 - n_1^2 . \tag{59}
\]

For the fundamental monopole \((n_1 = 1)\), the problem then is to find all solutions to the equation

\[
2p^2 - 3q^2 = 5
\]