The essential norm of multiplication operators on $L^p(\mu)$

Jürgen Voigt

Abstract

We show that the formula for the essential norm of a multiplication operator on $L^p$, for $1 < p < \infty$, also holds for $p = 1$. We also provide a proof for the formula which works simultaneously for all $p \in [1, \infty)$.

MSC 2010: 47B38, 46E30, 46B42.

Keywords: Multiplication operator, $L^p$-space, compact operator, essential norm.

1 Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, and let $1 \leq p < \infty$. For $u \in L^\infty(\mu)$ let $M_u$ be the bounded multiplication operator on $L^p(\mu)$ defined by

$$M_u f := uf \quad (f \in L^p(\mu)).$$

Compactness properties of multiplication operators in various function spaces have been investigated in several papers; see [14], [3], [1], [6], [10], [4], [12], [3], [11]. It is only in the recent paper [2] that the essential norm

$$\|M_u\|_e := \inf \{|\|M_u + K\|; K \in \mathcal{K}(L^p(\mu))\},$$

(1.1)

where $\mathcal{K}(L^p(\mu))$ denotes the space of compact operators on $L^p(\mu)$, has been determined, for $1 < p < \infty$. (The essential norm $\|M_u\|_e$ is the quotient norm in the Calkin algebra.) In order to describe this result we recall that the measure space can be decomposed as a disjoint union $\Omega = \Omega_d \cup \Omega_a$, where $\Omega_d, \Omega_a \in \mathcal{A}$, the restriction $\mu_d$ of $\mu$ to $\Omega_d$ is a diffuse measure, and the restriction $\mu_a$ of $\mu$ to $\Omega_a$ is (purely) atomic. The property of being diffuse means that for every measurable subset $A$ of $\Omega_d$ with $\mu_d(A) > 0$ there exists a measurable subset $A'$ of $A$ such that $0 < \mu_d(A') < \mu_d(A)$. And the atomic part $\Omega_a$ is the union of a disjoint sequence $(B_n)_{n \in \mathbb{N}}$ of measurable sets, where each $B_n$ is an atom, which means that any measurable subset $B'$ of $B_n$ has measure $\mu_a(B') \in \{0, \mu_a(B_n)\}$. With this notation, the essential norm of $M_u$ is given by

$$\|M_u\|_e = \max\{\|u|_{\Omega_d}\|_\infty, \limsup_{n \to \infty} |u(B_n)|\}.$$
(By \( u(B_n) \) we denote the a.e.-value of \( u \) on \( B_n \); if \( \mu_u(B_n) = 0 \) we choose \( u(B_n) := 0 \.) The proof of (1.2) given in [2, Theorem 4.1] does not carry over to the case \( p = 1 \).

In Section 2 we show that (1.2) also holds for \( p = 1 \). In Sections 3 and 4 we provide a second – quite different – proof, which works simultaneously for all \( p \in [1, \infty) \).

2 The essential norm of \( M_u \)

Let \((\Omega, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space, and let \( \Omega = \Omega_d \cup \Omega_a \) and \( \Omega_a = \bigcup_{n \in \mathbb{N}} B_n \) be as described above.

2.1 Theorem. Let \( u \in L_\infty(\mu) \), and let \( M_u \) be the multiplication operator associated with \( u \) on \( L_1(\mu) \). Then \( \|M_u\|_e \) is given by (1.2).

Proof. (i) For the inequality ‘\( \leq \)’ in (1.2) we refer to [2, first part of the proof of Theorem 4.1].

(ii) For the proof of ‘\( \geq \)’ we first note that in the infimum of the formula (1.1) (where in the present step we treat the general case \( p \in [1, \infty) \)) one does not need all compact operators, but it is sufficient to consider operators leaving \( L_p(\Omega_d) \) and \( L_p(\Omega_a) \) invariant. Indeed, let \( P_d \) and \( P_a \) denote the canonical projections from \( L_p(\mu) \) onto \( L_p(\Omega_d, \mu_d) \) and \( L_p(\Omega_a, \mu_a) \), respectively. Then for any bounded operator \( S \) on \( L_p(\mu) \) one has \( \|(P_d - P_a)S(P_d - P_a)\| \leq \|S\| \), and because of

\[
P_dSP_d + P_aSP_a = \frac{1}{2}((P_d + P_a)S(P_d + P_a) + (P_d - P_a)S(P_d - P_a))
\]

one obtains \( \|P_dSP_d + P_aSP_a\| \leq \|S\| \). In view of \( P_dM_aP_d + P_aM_aP_a = M_a \), this yields

\[
\|M_u + P_dKP_d + P_aKP_a\| \leq \|M_a + K\|
\]

for all compact operators, and \( P_dKP_d + P_aKP_a \) is a compact operator leaving \( L_p(\Omega_d) \) and \( L_p(\Omega_a) \) invariant. As a consequence one also concludes that it is sufficient to prove the inequality ‘\( \geq \)’ separately for diffuse and atomic measure spaces. For the remainder of the proof we now return to the case \( p = 1 \).

(iii) In this part of the proof we show ‘\( \geq \)’ for the case that \( \Omega = \Omega_d \), i.e. that \( \mu \) is a diffuse measure. The case \( u = 0 \) being trivial, assume that \( \|u\|_\infty > 0 \) and let \( 0 < \varepsilon < \|u\|_\infty \). Then there exists a descending sequence \((A_n)_{n \in \mathbb{N}}\) in \( \mathcal{A} \) such that \( 0 < \mu(A_n) \to 0 \) as \( n \to \infty \) and \( |u|_{A_n} \gg \|u\|_\infty - \varepsilon \) for all \( n \in \mathbb{N} \); without restriction \( \mu(A_1) < \infty \). For \( n \in \mathbb{N} \) put

\[
f_n := \frac{1}{\mu(A_n)} 1_{A_n},
\]

where \( 1_{A_n} \) denotes the indicator function of the set \( A_n \). Let \( K \in \mathcal{K}(L_1(\mu)) \). Because \((f_n)_{n \in \mathbb{N}}\) is a bounded sequence, the compactness of \( K \) implies that there
exists a subsequence \((f_n)_{j \in \mathbb{N}}\) such that the sequence \((Kf_n)\) is convergent; by passing to a subsequence, we can assume that \((Kf_n)\) is already convergent. Then there exists \(n \in \mathbb{N}\) such that \(\|Kf_n - Kf_m\| \leq \varepsilon\) for all \(m \geq n\). Choose \(m \geq n\) large enough to obtain additionally \(\frac{1}{\mu(A_m)} \geq 2\frac{1}{\mu(A_n)}\). Then one has

\[
Kf_n - Kf_m = \frac{1}{\mu(A_n)} 1_{A_n} - \frac{1}{\mu(A_m)} 1_{A_m} = \frac{1}{\mu(A_n)} 1_{A_n} \setminus A_m - \left( \frac{1}{\mu(A_m)} - \frac{1}{\mu(A_n)} \right) 1_{A_m},
\]

\[
|f_n - f_m| \geq f_n, \|f_n - f_m\|_1 \geq \|f_n\|_1 = 1;
\]

\[
\|(M_u + K)(f_n - f_m)\|_1 \geq \|M_u(f_n - f_m)\|_1 - \|K(f_n - f_m)\|_1 \geq (\|u\|_\infty - \varepsilon) \|f_n - f_m\|_1 - \varepsilon \geq (\|u\|_\infty - 2\varepsilon) \|f_n - f_m\|_1,
\]

\[
\|M_u + K\| \geq \|u\|_\infty - 2\varepsilon.
\]

As this holds for all \(\varepsilon \in (0, \|u\|_\infty)\), we obtain \(\|M_u + K\| \geq \|u\|_\infty\).

(iv) It remains to show that ‘\(\geq\)’ holds in the case that \(\Omega = \Omega_a\), i.e. that \(\mu\) is an atomic measure. If \(\mu(B_n) \neq 0\) only for finitely many \(n \in \mathbb{N}\), then \(\limsup_{n \to \infty} \|u(B_n)\| = 0\), and the assertion is trivial. Assume that this is not the case, without restriction \(\mu(B_n) \neq 0\) for all \(n \in \mathbb{N}\). For \(n \in \mathbb{N}\) let \(P_n\) be the canonical projection from \(L_1(\mu)\) onto \(L_1(B_n)\), i.e. \(P_n f := 1_{B_n} f (f \in L_1(\mu))\), and put \(Q_n := I - \sum_{j=1}^n P_j\). Iterating the procedure applied in step (ii) above one concludes that for all bounded operators \(S\) on \(L_1(\mu)\) and all \(n \in \mathbb{N}\) one obtains

\[
\|\sum_{j=1}^n P_j S P_j + Q_n S Q_n\| \leq \|S\|.
\]

Given a compact operator \(K \in \mathcal{K}(L_1(\mu))\) we note that \(\|Q_n K\| \to 0\) as \(n \to \infty\). This holds because for any \(g \in L_1(\mu)\) one has \(\|Q_n g\| \to 0\), and from the relative compactness of \(K(B_{L_1(\mu)}[0,1])\) (where \(B_{L_1(\mu)}[0,1]\) denotes the closed unit ball of \(L_1(\mu)\)) together with the equicontinuity of the sequence \((Q_n)\) one concludes that

\[
\|Q_n K\| = \sup_{\|f\|_1 \leq 1} \|Q_n K f\| = \sup_{g \in K(B_{L_1(\mu)}[0,1])} \|Q_n g\| \to 0 \quad (n \to \infty).
\]

In particular, we conclude that \(\|P_n K P_n\| \leq \|(Q_{n-1} - Q_n) K\| \to 0\) \((n \to \infty)\). Note that, for \(n \in \mathbb{N}\), there exists \(d_n \in \mathbb{K}\) such that \(\|P_n K P_n\| = |d_n|\) and \(P_n K P_n f = d_n P_n f\) for all \(f \in L_1(\mu)\). Hence the multiplication operator \(D_K\), given by \(L_1(\mu) \ni f \mapsto \sum_{n \in \mathbb{N}} d_n P_n f \in L_1(\mu)\), is a compact operator.

Now we estimate

\[
\|M_u + K\| \geq \|\sum_{j=1}^n P_j (M_u + K) P_j + Q_n (M_u + K) Q_n\| \quad (2.1)
\]

\[
= \|M_u + D_K + Q_n (K - D_K) Q_n\| \geq \|M_u + D_K\| - \|Q_n (K - D_K) Q_n\|.
\]

From the argument given above we obtain \(\|Q_n (K - D_K) Q_n\| \leq \|Q_n (K - D_K)\| \to 0\) \((n \to \infty)\), and from \((2.1)\) we conclude that \(\|M_u + K\| \geq \|M_u + D_K\|\). This shows
that
\[ \|M_u\|_e = \inf \{ \|M_u + D\|; D \text{ compact multiplication operator} \} \]
\[ = \inf \{ \sup_{n \in \mathbb{N}} |u(n) + d_n|; (d_n)_{n \in \mathbb{N}} \text{ null sequence} \} \]
\[ = \limsup_{n \to \infty} |u(n)|. \]

2.2 Remark. Step (iv) of our proof applies also to \( p \in (1, \infty) \) and is an alternative to the last part of [2; proof of Theorem 4.1]. The idea of our proof is that \( \|M_u + K\| \) can be estimated from below by \( \|M_u + D\| \) for a suitable compact multiplication operator \( D \).

3 The case \( \Omega = \Omega_{d} \), revisited

Let \((\Omega, \mathcal{A}, \mu)\) be a diffuse \( \sigma \)-finite measure space, \( p \in [1, \infty) \), and let \( u \in L_\infty(\mu) \). In this section we will present a proof of the equality
\[ \|M_u\|_e = \|u\|_\infty \] (3.1)
(i.e. (1.1) for the present special case), which might throw a new light on this property.

We recall that an operator \( S \in \mathcal{L}(L_p(\mu)) \) (the space of all bounded linear operators) is positive, \( S \in \mathcal{L}(L_p(\mu))_+ \), if \( Sf \geq 0 \) for all \( f \in L_p(\mu)_+ \). Then \( \mathcal{L}'(L_p(\mu))_+ \), defined as the linear hull of \( \mathcal{L}(L_p(\mu))_+ \), is the space of regular operators. It is a Banach lattice under the lattice operations
\[ (S \vee T)f := \sup \\{ Sg + Th; g, h \geq 0, g + h = f \} \]
\[ (S \wedge T)f := \inf \\{ Sg + Th; g, h \geq 0, g + h = f \} \quad (f \in L_p(\mu)_+) \]
(valid for real operators \( S, T \in \mathcal{L}'(L_p(\mu)) \)), the absolute value
\[ |S|f := \sup \|Sg\|; |g| \leq f \] \( (f \in L_p(\mu)_+) \),
and with the regular norm \( \|S\|_r := \|\|S\|| \). We refer to [13; Chap. 4], [8; Sec. 1.3] for more information.

As a preparation to the proof of (3.1) we need the following property, where \( q \in (1, \infty) \) denotes the exponent conjugate to \( p; \frac{1}{p} + \frac{1}{q} = 1 \).

3.1 Lemma. Let \( \eta \in L_q(\mu) \) (\( = L_p(\mu)' \)), \( g \in L_p(\mu) \), \( \eta, g \geq 0 \), \( K \in \mathcal{L}(L_p(\mu)) \) defined by
\[ Kf := \left( \int \eta f \, d\mu \right) g \quad (f \in L_p(\mu)). \]
(Note that \( K \in \mathcal{L}(L_p(\mu))_+ \subseteq \mathcal{L}'(L_p(\mu)) \).) Let \( u \in L_\infty(\mu)_+ \) be such that \( M_u \leq K \). Then \( u = 0 \).
Proof. Assume on the contrary that \( u \neq 0 \). Then there exists \( \varepsilon > 0 \) such that 
\( \mu([u \geq \varepsilon]) > 0 \) (with the notation \([u \geq \varepsilon] := \{x \in \Omega; u(x) \geq \varepsilon\}\)). Further there exists \( c > 0 \) such that \( \mu([u \geq \varepsilon] \cap [g \leq c]) > 0 \). Let \( B \in \mathcal{A}, B \subseteq [u \geq \varepsilon] \cap [g \leq c] \) with \( 0 < \mu(B) < \infty \). Then \( M_u 1_B \geq \varepsilon \) and \( K 1_B = \int_B \eta \, d\mu \leq c \int_B \eta \, d\mu \) on \( B \).

There exists \( B \) as above and such that \( \int_B \eta \, d\mu < \varepsilon/c \), and this leads to the contradiction \( K 1_B \leq c \int \eta \, d\mu < \varepsilon \leq M_u 1_B \) on \( B \).

The centre \( Z(L_p(\mu)) \) of \( \mathcal{L}(L_p(\mu)) \) is the linear hull of the order interval
\[ [-I, I] = \{ S \in \mathcal{L}(L_p(\mu)); \ -f \leq Sf \leq f \ (f \in L_p(\mu)_+) \}. \]

Then \( Z(L_p(\mu)) \subseteq L^r(L_p(\mu)) \) consists of the bounded multiplication operators and is isometrically isomorphic to \( L_\infty(\mu) \); see [9, C-I, Section 9].

The centre \( Z(L_p(\mu)) \) is a projection band in the Banach lattice \( L^r(L_p(\mu)) \), i.e. for all \( S \in L^r(L_p(\mu)) \) there exists a (unique) decomposition \( S = S_1 + S_2 \), where \( S_1 \in Z(L_p(\mu)) \) and
\[
S_2 \in Z(L_p(\mu))^d = \{ T \in L^r(L_p(\mu)); \ |T| \wedge R = 0 \ (R \in Z(L_p(\mu)_+)) \};
\]
see [13, Chap. II, Theorem 2.10] Let \( \mathcal{P}: L^r(L_p(\mu)) \to Z(L_p(\mu)) \), \( S \mapsto S_1 \) denote the associated band projection. What we have shown in Lemma 3.1 is that \( \mathcal{P}K = 0 \) for the special (positive) rank-one operators \( K \). (Indeed, the lemma shows that \( K \) belongs to \( Z(L_p(\mu))^d \).) It is easy to see that any finite-rank operator \( K \in \mathcal{L}(L_p(\mu)) \) can be written as a linear combination of rank-one operators as in Lemma 3.1 hence \( \mathcal{P}K = 0 \) for all finite-rank operators.

3.2 Theorem. Let \( u \in L_\infty(\mu) \). Then
\[
\| M_u + K \| \geq \| M_u \| = \| u \|_\infty \quad (K \in \mathcal{K}(L_p(\mu)), \quad (3.2)
\]
and (3.1) holds.

Proof. Clearly, it suffices to show (3.2). There are two ingredients of the proof:

(i) By the very definition, \( \mathcal{P} \) is contractive with respect to the regular norm (because band projections are contractive). However, it is shown in [13, Theorem 1.4] that \( \mathcal{P} \) is also contractive with respect to the operator norm. This implies that \( \mathcal{P} \) can be extended by continuity to the closure of \( L^r(L_p(\mu)) \) in \( \mathcal{L}(L_p(\mu)) \). In particular, for the extension one obtains \( \mathcal{P}K = 0 \) for all \( K \) in the operator norm closure of the finite rank operators.

(ii) The space \( L_p(\mu) \) enjoys the approximation property, i.e. every compact operator on \( L_p(\mu) \) can be approximated in operator norm by finite rank operators. (We refer to [7, Sections 3.4 and 4.1] for the approximation property.) This implies that \( \mathcal{P}K = 0 \) for all compact operators on \( L_p(\mu) \).

Putting together these two properties we obtain
\[
\| M_u \| = \| \mathcal{P}(M_u + K) \| \leq \| M_u + K \|
\]
for all \( K \in \mathcal{K}(L_p(\mu)) \).
4 Supplement on the case \( \Omega = \Omega_a \)

We add that the case \( \Omega = \Omega_a \) can be treated analogously to the case \( \Omega = \Omega_d \) described in Section 3. Then again the centre of \( \mathcal{L}(L_p(\mu)) \) consists of the bounded multiplication operators. Lemma 3.1 is replaced by the property that multiplication operators are disjoint to positive rank-one operators \( K \) of the type

\[
K f = \int_{\Omega \setminus B_j} f \eta \, d\mu \mathbf{1}_{B_j} = \left( \sum_{k \neq j} f(B_k) \eta(B_k) \mu(B_k) \right) \mathbf{1}_{B_j}, \quad (f \in L_p(\mu)),
\]

where \( j \in \mathbb{N} \) and \( \eta \in L_q(\mu)_+ \). Indeed, if \( u \in L^\infty(\mu)_+ \) is such that \( M_u \leq K \), then clearly \( u(B_k) = 0 \) for all \( k \neq j \). But \( u(B_j) \mathbf{1}_{B_j} = M_u \mathbf{1}_{B_j} \leq K \mathbf{1}_{B_j} = 0 \); hence also \( u(B_j) = 0 \). (Recall that \( u(B_k) = 0 \) if \( \mu(B_k) = 0 \), by our convention in the Introduction.)

The consequence is that, for a compact operator \( K \), its projection \( \mathcal{P}K \) onto the centre is the compact operator \( D_K \) (described in part (iv) of the proof of Theorem 2.1). This holds because for a compact operator \( K \) and \( n \in \mathbb{N} \), the finite rank operator \( (I - Q_n)K \) (with the notation of the proof of Theorem 2.1 part (iv)) can be decomposed as the multiplication operator \( (I - Q_n)D_K \) and a linear combination of rank-one operators of the type (4.1). As \( (I - Q_n)K \to K \) \((n \to \infty)\) in \( \mathcal{L}(L_p(\mu)) \) and the band projection \( \mathcal{P} \) onto the centre is contractive with respect to the operator norm, one concludes that

\[
\mathcal{P}K = \lim_{n \to \infty} \mathcal{P}(I - Q_n)K = \lim_{n \to \infty}(I - Q_n)D_K = D_K.
\]

Hence instead of (2.1) one obtains \( \|M_u + K\| \geq \|\mathcal{P}(M_u + K)\| = \|M_u + D_K\| \), and the proof can be finished as in Section 2.

References

[1] P. Bala, A. Gupta, N. Bhatia: Multiplication operators on Orlicz-Lorentz sequence spaces. Int. J. Math. Anal. (Ruse) 7, no. 30, 1461–1469 (2013).

[2] R. E. Castillo, Y. A. Lemus-Abril and J. C. Ramos-Fernández: Essential norm estimates for multiplication operators on \( L_p(\mu) \) spaces. Afr. Mat. 32, no. 7-8, 1595–1603 (2021).

[3] R. E. Castillo, H. Rafeiro, J. C. Ramos-Fernández, M. Salas-Brown: Multiplication operator on Köthe spaces: measure of non-compactness and closed range. Bull. Malays. Math. Sci. Soc. 42, no. 4, 1523–1534 (2019).

[4] R. E. Castillo, J. C. Ramos-Fernández, M. Salas-Brown: The essential norm of multiplication operators on Lorentz sequence spaces. Real Anal. Exchange 41, no. 1, 245–251 (2015/2016).

[5] H. Hudzik, R. Kumar, R. Kumar: Matrix multiplication operators on Banach function spaces. Proc. Indian Acad. Sci. (Math. Sci.) 116, no. 1, 71–81 (2006).
Essential norm of multiplication operators

[6] B. S. Komal, S. Pandoh, K. Raj: Multiplication operators on Cesàro sequence spaces. Demonstratio Math. 49, no. 4, 430–436 (2016).

[7] R. E. Megginson: An introduction to Banach space theory. Springer-Verlag, New York, 1998.

[8] P. Meyer-Nieberg: Banach lattices. Springer-Verlag, Berlin, 1991.

[9] R. Nagel (ed.): One-parameter semigroups of positive operators. Lecture Notes in Mathematics 1184, Springer-Verlag, Berlin, 1986.

[10] K. Raj, C. Sharma, S. Pandoh: Multiplication operators on Cesàro-Orlicz sequence spaces. Fasc. Math. 57, 137–145 (2016).

[11] J. C. Ramos-Fernández, M. Rivera-Sarmiento, M. Salas-Brown: On the essential norm of multiplications operators acting on Cesàro sequence spaces. J. Funct. Spaces 2019. Art. ID 5069610, 5 pages (2019).

[12] J. C. Ramos-Fernández, M. Salas-Brown: On multiplication operators acting on Köthe sequence spaces. Afr. Mat. 28, no. 3-4, 661–667 (2017).

[13] H. H. Schaefer: Banach lattices and positive operators. Springer-Verlag, New York, 1974.

[14] H. Takagi: Compact weighted composition operators on $L^p$. Proc. Amer. Math. Soc. 116, no. 2, 505–511 (1992).

[15] J. Voigt: The projection onto the center of operators in a Banach lattice. Math. Z. 199, 115–117 (1988).

Jürgen Voigt
Technische Universität Dresden
Fakultät Mathematik
01062 Dresden, Germany
juergen.voigt@tu-dresden.de