The large $N$ limit from the lattice

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Abstract. A numerical study of the string tension and of the masses of the lowest-lying glueballs is performed in SU($N$) gauge theories for $2 \leq N \leq 8$ in D=3+1 and $2 \leq N \leq 6$ in D=2+1. It is shown that for the string tension a smooth $N \to \infty$ limit exists that depends only on the 't Hooft coupling $\lambda = g^2 N$. An extrapolation of the masses of the lightest glueballs to $N = \infty$ using a power series in $1/N^2$ shows that the leading correction to the infinite $N$ value accounts for finite $N$ effects for $N$ at least as small as 3 and all the way down to $N = 2$ in many cases. $k$-string tension ratios and possible issues connected with correlation functions at large $N$ are also discussed.

1 Introduction

A better understanding of the large $N$ limit of SU($N$) gauge theories is an important prerequisite for advances in several branches of particle physics. It has been shown that accurate numerical results for observables in SU($\infty$) can be attained by extrapolating to $N = \infty$ values obtained at finite $N$ [1, 2, 3]. The extrapolation is based on the diagrammatic expectation that observables at finite $N$ differ from the corresponding observables at $N = \infty$ by corrections that are described by a power series in $1/N^2$ [4]. In this work we will present results for the string tension, $k$-string tension ratios and glueball masses in 3+1 [5] and 2+1 dimensions [6].

2 The method

We have studied numerically on the lattice SU($N$) gauge theories for $2 \leq N \leq 8$ in 3+1 dimensions and $2 \leq N \leq 6$ in 2+1 dimensions. For any given $N$ we have used the Wilson action

$$S = \beta \sum_{i,\mu,\nu} \left( 1 - \frac{1}{2N} \text{Tr} \left( U_{\mu\nu}(i) + U_{\mu\nu}^\dagger(i) \right) \right),$$

(1)

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Figure 1. In our scheme, a smeared link is obtained by adding to the original link (a) the parallel transported nearest neighbour links (b) weighted by a coefficient $\alpha$ and the next to nearest neighbour links parallel transported along all possible paths on the elementary cube (c) weighted by another coefficient $\beta$. For a judicious choice of $\alpha$ and $\beta$ it is possible to build operators with a 98% overlap on the physical states.

where the $U_{\mu\nu}(i)$ are path ordered products of SU($N$) link variables along the elementary lattice squares (plaquettes) and $\beta = 2N/(g^2a^{4-D})$, with $g$ the coupling of the theory, $D$ the dimensionality of the system and $a$ the lattice spacing. The values of $\beta$ used in our study have been chosen above the bulk phase transition point [2].

Masses can be extracted from the exponential decay of correlation functions built with traces of path ordered products of functions of link variables. In order to have a significant overlap with physical states, these functions must be smooth on the characteristic distance scale of those states. Traditionally, iterated smearing [7] and blocking [8] are used to achieve this result. In our work in D=3+1 we have used a combination of improved smearing (see Fig. 1) and blocking, while simple blocking has been used in D=2+1. For each state of interest, a set of operators carrying the required quantum numbers has been built out of smeared and blocked variables and the corresponding correlation matrix has been computed. Masses have been obtained by a variational calculation performed on this matrix. More details on the numerical methods we have used can be found in [5, 6].

3 Correlators and large $N$

There is a potential problem with the application of the method outlined in the previous section at large $N$. In fact, from general arguments, as $N \to \infty$ we expect factorisation of correlators, which implies that the connected part of the 2-point function (from which we extract masses) is $\mathcal{O}(1/N^2)$. At large enough $N$ then this signal might become invisible. However, what determines the accuracy of the calculation is not the absolute magnitude of the signal, but the ratio signal over noise. Large $N$ arguments [5] show that this ratio is independent of $N$, and this expectation is confirmed by our data. Thus, despite the fact that the signal disappears as $1/N^2$, we are able to compute correlators with an accuracy that is independent of $N$. 
4 String tension

The string tension $\sigma$ can be obtained from torelon correlation functions \cite{2}. Diagrammatic arguments suggest that equivalent physics across SU($N$) gauge groups is obtained at fixed $\lambda = g^2 N$. Since $\lambda$ depends on the scale at which the physics is probed, equivalent $\lambda$ are expected at the same physical scale, or equivalently the same scale is identified by a given $\lambda$ at any large enough value of $N$. Although this is not an ideal choice, on the lattice we can easily verify this statement on length scales of the order of $a$. At this scale lattice artifacts resulting from a naive definition of the lattice coupling can spoil the comparison. To limit the effects of those artifacts we have used the mean field improved
coupling $\beta_I = \beta / < U_{\mu\nu} >$, where $< U_{\mu\nu} >$ is the average plaquette. Fig. 2 shows a plot of $a\sqrt{\sigma}$ as a function of $\lambda_I = 2N^2 / \beta_I$ in $D=3+1$. This plot suggests that a smooth large $N$ limit for $\lambda_I$ exists at all values of $a\sqrt{\sigma}$.

In $D=2+1$ the lattice spacing $a$ can be traded with the coupling $g^2$, which now has dimensions of mass. Our data are displayed in Fig. 3. From diagrammatic arguments, we expect that up to $O(1/N^2)$ corrections, in the continuum limit $\sqrt{\sigma}/g^2 \propto N$. A fit gives

$$\frac{\sqrt{\sigma}}{g^2 N} = 0.19755(35) - \frac{0.120(3)}{N^2}. \quad (2)$$

**Figure 4.** Ratios of the tensions of strings of various $\mathcal{N}$-alities over the tension of the fundamental string. The scale on the $y$ axis has been chosen in such a way that the values $R = 1$ and $R = 2$ always correspond respectively to Casimir scaling and to the sine formula. Those values are indicated by the horizontal lines in the plot.

### 5 $k$-strings

Strings connecting sources in a representation of rank (or $\mathcal{N}$-ality) $k < N$ of SU($N$) with their antisources are known in literature as $k$-strings. Since strings associated to sources of different $\mathcal{N}$-ality can not mix, at fixed $N$ there are $\text{int}[N/2]$ stable $k$-strings. The values of their tensions are determined by the underlying dynamics of colour confinement.

$k$-string tensions can be extracted from correlators of multi-wrapping torelons and powers of them that under the centre transformation $e^{2\pi i/N}$ acquire the phase $e^{2\pi ik/N}$. Our numerical calculations show that the ratios $R$ of tensions of strings connecting sources of $\mathcal{N}$-ality $k$ and $N - k$ over the tension of the fundamental string lay between the Casimir scaling ($R = k(N - k)/(N - 1)$) and the sine formula ($R = \sin(k\pi/N)/\sin(\pi/N)$) predictions in $D=3+1$ [5, 9] (see also [10]).
where a better agreement with the sine formula is claimed), while in D=2+1 Casimir scaling seems to be favoured [9]. Although the corresponding numerical values are very close, those formulae hide very different scenarios: while the sine formula can be expanded in a series that contains only even powers of 1/N, the first term in the expansion of the Casimir prediction is proportional to 1/N, and this is hardly accommodated by diagrammatic arguments [11, 12]. Even if both formulae at small N are expected to receive sizeable corrections that undermine the numerical comparison, the question of the power dependence of the leading correction in a 1/N expansion still stands. Current numerical data are not accurate enough to provide a reliable answer to this problem.

6 Glueball masses

Glueball masses are extracted from correlators of products of smeared and blocked links along contractible paths. The transformation properties of the paths under rotation, parity and charge conjugation determine the quantum numbers of the states. We have fitted our data with the ansatz

\[ m_N/g^2 N = d_0 + d_1/N^2 \]  

in D=2+1 and

\[ m_N/a \sqrt{\sigma} = d_0 + d_1/N^2 \]  

in D=3+1. The results of our fits are reported in Table 1. In D=2+1, the ansatz works for all the states all the way down to \( N = 2 \). In D=3+1 we draw similar conclusions for the \( 0^{++} \) glueball, while a correction \( \mathcal{O}(1/N^2) \) turns out not to be enough to describe the SU(2) numerical data for the \( 0^{++*} \) and the \( 2^{++} \) states. Note that while both in 2+1 and 3+1 dimensions we can extract accurate values for the adimensional masses in the \( N \to \infty \) limit, the values of the coefficients of

| state | \( d_0 \) | \( d_1 \) | \( d_0 \) | \( d_1 \) |
|-------|---------|---------|---------|---------|
| \( 0^{++} \) | 0.8116(36) | -0.090(28) | 3.28(8) | 2.1(1.1) |
| \( 0^{++*} \) | 1.227(9) | -0.343(82) | 5.93(17) | 2.7(2.0) |
| \( 0^{++*} \) | 1.65(4) | -2.2(7) | – | – |
| \( 0^{--} \) | 1.176(14) | 0.14(20) | – | – |
| \( 0^{--*} \) | 1.535(28) | -0.35(35) | – | – |
| \( 2^{++} \) | 1.359(12) | -0.22(8) | 4.78(14) | 0.3(1.7) |
| \( 2^{++*} \) | 1.822(62) | -3.9(1.3) | – | – |
| \( 2^{--} \) | 1.615(33) | -0.10(42) | – | – |

Table 1. Fit parameters for the extrapolation to the \( N = \infty \) limit of masses of glueballs in D=2+1 according to formula (3) and in D=3+1 according to formula (4). The quantum numbers of the states are indicated with the standard notation \( J^P C \) and higher excitations in a given channel are denoted with a corresponding number of \( * \).
the leading correction is determined unambiguously only in the D=2+1 case. In D=3+1 we can however estimate their order of magnitude. Our results support the conclusion that at finite $N$ the deviation from the $N \to \infty$ value is a genuine $O(1/N^2)$ correction.

7 Conclusions

Our investigation shows that a smooth large $N$ limit exists for several physical observables and the physics at finite $N$ is correctly described by the leading expected correction all the way down to at least $N = 3$. In particular, we have discussed our results for the masses of the lightest glueballs and for the string tension in both D=2+1 and D=3+1. Unlike early works on this subject, in most cases we were able to provide accurate and reliable numerical estimates for the $N = \infty$ value and the leading correction. An interesting problem is whether for some observables the approach to the $N = \infty$ limit can be described by a $1/N$ correction. On the light of present results, this possibility can not be ruled out for ratios of $k$-string tensions.

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