Anomalous behavior of the Kramers rate at bifurcations in classical field theories

Nils Berglund1 and Barbara Gentz2

1 Université d’Orléans, Laboratoire Mapmo, CNRS, UMR 6628, Fédération Denis Poisson, FR 2964, Bâtiment de Mathématiques, B.P. 6759 45067, Orléans Cedex 2, France
2 Faculty of Mathematics, University of Bielefeld, PO Box 10 01 31, 33501 Bielefeld, Germany

E-mail: nils.berglund@univ-orleans.fr and gentz@math.uni-bielefeld.de

Received 23 October 2008, in final form 10 December 2008
Published 6 January 2009
Online at stacks.iop.org/JPhysA/42/052001

Abstract
We consider a Ginzburg–Landau partial differential equation in a bounded interval, perturbed by weak spatio-temporal noise. As the interval length increases, a transition between activation regimes occurs, in which the classical Kramers rate diverges (Maier and Stein 2001 Phys. Rev. Lett. 87 270601). We determine a corrected Kramers formula at the transition point, yielding a finite, though noise-dependent, rate prefactor, confirming a conjecture by Maier and Stein (2003 SPIE Proc. vol 5114 pp 67–78). For both periodic and Neumann boundary conditions, we obtain explicit expressions for the prefactor in terms of Bessel and error functions.

PACS numbers: 05.40.-a, 05.45.Yv, 11.10.Wx, 75.60.Jk

1. Introduction
Weak noise acting on spatially extended systems can cause a wide range of interesting phenomena. In particular, it can induce rare transitions between states which would be otherwise invariant, e.g. nucleation of one phase within another [Lan67], micromagnetic domain reversal [Nee49, Bra93, BNR00], pattern nucleation in electroconvection [CH93], instabilities in metallic nanowires [BSS05] and many others. The rate of such transitions for weak noise intensity $\epsilon$ is in general governed by the Kramers law $\Gamma \sim \Gamma_0 \exp\{-\Delta W/\epsilon\}$, where the activation energy $\Delta W$ is the energy difference between stable and transition states and the rate prefactor $\Gamma_0$ is related to second derivatives of the system’s energy functional at these states [Eyr35, Kra40].

In a series of recent works [MS01, MS03, Ste04], Maier and Stein studied transition rates in a Ginzburg–Landau partial differential equation on a finite interval, perturbed by space-

Throughout this communication, the notation $a \asymp b$ indicates that $\lim_{\epsilon \to 0} a/b = 1$. 

1751-8113/09/052001+09$30.00 © 2009 IOP Publishing Ltd Printed in the UK
time white noise. They discovered the striking fact that as the interval length approaches a critical value, which depends on the boundary conditions (b.c.), the rate prefactor $\Gamma_0$ diverges. Although this divergence is reminiscent of the behavior of certain thermodynamic quantities at phase transitions, it has a different origin [Ste05]: it is due to the fact that the Kramers law only takes into account the effect of quadratic terms in the energy functional on thermal fluctuations, while at the critical length some quadratic terms vanish due to a bifurcation, and higher order terms come into play.

Maier and Stein conjectured [MS01] that the actual rate prefactor at the bifurcation point behaves like $\Gamma_0 \simeq C \varepsilon^{-\alpha}$, for some constants $C, \alpha > 0$. Until recently, Kramers rate theory was not sufficiently sharp to allow for the computation of these constants. Based on a new approach by Bovier et al [BEGK04], we developed a method allowing us to compute rate prefactors for potentials with nonquadratic transition states [BG08b]. The aim of this communication is to illustrate the method by determining the constants $C$ and $\alpha$ in the case of the Ginzburg–Landau equation.

2. Model

Consider a one-dimensional classical field $\phi(x,t)$, subjected to the quartic double-well potential energy function

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2,$$

(2.1)

to diffusion and to weak space-time white noise. Its evolution is given by the stochastic partial differential equation (SPDE):

$$\partial_t \phi(x,t) = \partial_{xx} \phi(x,t) + \phi(x,t) - \phi(x,t)^3 + \sqrt{2\varepsilon} \xi(x,t),$$

(2.2)

where $\xi(x,t)$ denotes space-time Gaussian white noise, i.e. formally,

$$\mathbb{E}\{\xi(x_1,t_1)\xi(x_2,t_2)\} = \delta(x_1 - x_2)\delta(t_1 - t_2).$$

(2.3)

Here we consider the case of a bounded interval $x \in [0, L]$, and either periodic or Neumann b.c. with a zero flux, i.e. $\partial_x \phi(0,t) = \partial_x \phi(L,t) = 0$.

Note that $\xi(x,t)$ can be rigorously defined by independent white noises acting on each Fourier mode. For periodic b.c., this leads to setting

$$\phi(x,t) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \phi_k(t) e^{2\pi ikx/L},$$

(2.4)

and substituting in (2.2). The resulting system of stochastic differential equations (SDEs) is given by

$$\dot{\phi}_k = -\lambda_k \phi_k - \frac{1}{L} \sum_{k_1,k_2,k_3} \phi_{k_1} \phi_{k_2} \phi_{k_3} + \sqrt{2\varepsilon} W_t^{(k)},$$

(2.5)

where $\lambda_k = -1 + (2\pi k/L)^2$, and $W_t^{(k)}$ are by definition independent Wiener processes (see, for instance, [Jet86] for a discussion of the equivalence of different approaches to SPDEs). In the case of Neumann b.c., setting

$$\phi(x,t) = \frac{1}{\sqrt{L}} \phi_0(t) + \frac{\sqrt{2}}{L} \sum_{k=1}^{\infty} \phi_k(t) \cos(\pi kx/L)$$

(2.6)

yields a similar system of SDEs.
With the SPDE (2.2), we associate the energy functional
\[ \mathcal{H}[\varphi] = \int_0^L \left[ \frac{1}{2} (\varphi'(x))^2 + V(\varphi(x)) \right] \, dx. \] (2.7)

For both periodic and Neumann b.c., the uniform configurations \( \varphi_{\pm} \equiv \pm 1 \) are stable stationary configurations of the system without noise. Both are minima of the energy functional, of energy \( \mathcal{H}[\varphi_{\pm}] = -L/4 \). In terms of the Fourier coefficients, for periodic b.c., the potential energy is given by
\[ \mathcal{H}[\varphi] = \hat{\mathcal{H}}[\{\phi_k\}] = \frac{1}{2} \sum_{k=-\infty}^{\infty} \lambda_k |\phi_k|^2 + \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}, \] (2.8)
and a similar relation can be obtained for Neumann b.c.

The value of the activation energy \( \Delta W \) for this model is well known [FJL82, MS01]. The Kramers rate prefactor \( \Gamma_0 \), however, has only been determined for parameters \( L \) in certain ranges, excluding bifurcation values of the model [MS03].

3. Transition states and activation energy

The activation energy is the potential energy difference between the initial stable state \( \varphi_- \) and the transition state \( \varphi_t \). The latter is defined as the configuration of highest energy one cannot avoid reaching, when continuously deforming \( \varphi_- \) to \( \varphi_+ \) while keeping the energy as low as possible. The transition state is a stationary state of the energy functional, that is, it satisfies \( \varphi''_t(x) = -\varphi_t(x) + \varphi_t(x)^3 \). In addition, the Hessian operator \( \delta^2 \mathcal{H}/\delta \varphi^2 \) must have a single negative eigenvalue at \( \varphi_t \). The corresponding eigenfunction specifies the direction in which the most probable transition path approaches the transition state.

The shape of \( \varphi_t \) depends on whether the bifurcation parameter \( L \) is smaller or larger than a critical value, the latter depending on the chosen b.c. [MS01].

**Periodic b.c.** For \( L \leq 2\pi \), the transition state is the identically zero function, which has energy zero. The activation barrier has thus the value \( \Delta W = L/4 \).

For \( L > 2\pi \), there is a continuous one-parameter family of transition states, of the so-called instanton shape, given in terms of Jacobi’s elliptic sine by
\[ \varphi_{\text{inst}, \varphi}(x) = \sqrt{\frac{2m}{m+1}} \sin \left( x \sqrt{\frac{1}{m+1}} + \varphi, m \right). \] (3.1)

Here, \( \varphi \) is an arbitrary phase shift and \( m \in [0, 1] \) is a parameter related to \( L \) by
\[ 4\sqrt{m+1}K(m) = L, \] (3.2)
where \( K(m) \) denotes the complete elliptic integral of the first kind. Note that \( m \to 0^+ \) as \( L \) approaches the critical length \( 2\pi \) from above. Computing the energy of any instanton transition state (3.1), one gets [MS01] the activation barrier
\[ \Delta W = \mathcal{H}[\varphi_{\text{inst}}] - \mathcal{H}[\varphi_-] = \frac{1}{3\sqrt{1+m}} \left[ 8E(m) - \frac{(1-m)(3m+5)}{1+m} K(m) \right], \] (3.3)
where \( E(m) \) denotes the complete elliptic integral of the second kind.

**Neumann b.c.** In this case, the identically zero solution forms the transition state for all \( L \leq \pi \), so that the activation barrier has again the value \( \Delta W = L/4 \).

For \( L > \pi \), there are two transition states of an instanton shape, given by
\[ \varphi_{\text{inst}, \pm}(x) = \pm \sqrt{\frac{2m}{m+1}} \sin \left( x \sqrt{\frac{1}{m+1}} + K(m), m \right). \] (3.4)
where the parameter $m \in [0, 1]$ is now related to $L$ by
\[ 2 \sqrt{m + 1} K(m) = L. \] (3.5)
In this case, we have $m \to 0^+$ as $L$ approaches the critical length $\pi$ from above. The activation energy is simply half the activation energy (3.3) of the periodic case [MS01].

4. Rate prefactor

The rate prefactor $\Gamma_0$ is usually computed by the Kramers formula [Eyr35, Kra40]
\[ \Gamma_0 \simeq \frac{1}{2\pi} \sqrt{\left| \frac{\det \Lambda_s}{\det \Lambda_t} \right| |\lambda_{t,0}|}. \] (4.1)
Here $\Lambda_s \equiv \partial^2 H/\partial \phi^2[\phi_{-}]$ denotes the linearized evolution operator at the stable state $\phi_{-}$. $\Lambda_t \equiv \partial^2 H/\partial \phi^2[\phi_{t}]$ denotes the linearized evolution operator at the transition state $\phi_{t}$ and $\lambda_{t,0}$ denotes the single negative eigenvalue of $\Lambda_t$.

For instance, for Neumann b.c. and $L \leq \pi$, the eigenvalues of $\Lambda_t = -d^2/dx^2 + 1$ are given by $\lambda_k = -1 + (\pi k/L)^2$, $k = 0, 1, 2, \ldots$, while the eigenvalues of $\Lambda_s = -d^2/dx^2 - 2$ are of the form $\eta_k = 2 + (\pi k/L)^2$. It follows [MS03] that the rate prefactor is given by
\[ \Gamma_0 \simeq \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{2 + (\pi k/L)^2}{|-1 + (\pi k/L)^2|} = \frac{1}{2^{3/4}\pi} \sqrt{\sinh(\sqrt{2}L) \sin L}. \] (4.2)
The striking point is that this prefactor diverges, like $(\pi - L)^{-1/2}$ as $L$ approaches the critical value $\pi$. In fact, this is due to the Kramers formula (4.1) not being valid in cases of vanishing $\det \Lambda_t$. To confirm Maier and Stein’s conjecture that the rate prefactor at $L = \pi$ behaves like $C \varepsilon^{-\alpha}$ and determine the constants $C$ and $\alpha$, we have to derive a corrected Kramers formula valid in such cases. This can be done [BG08b] by extending a technique initially developed by Bovier et al [BEGK04], which we outline now.

Potential theory. For simplicity, first consider the case of $d$-dimensional Brownian motion $W^*_x$, starting at a point $x \in \mathbb{R}^d$. Given a set $A \subset \mathbb{R}^d$, the expected value $w_A(x) = \mathbb{E}[\tau^*_x A]$ of the first time $\tau^*_x A$ the Brownian path hits $A$ is known [Dyn65] to satisfy the boundary value problem
\[ \Delta w_A(x) = 1 \quad x \in A^c, \]
\[ w_A(x) = 0 \quad x \in A. \] (4.3)
The solution can be written as
\[ w_A(x) = \int_{A^c} G_A^*(x, y) \, dy, \] (4.4)
where $G_A^*$ denotes the associated Green’s function, satisfying $\Delta_s G_A^*(x, y) = \delta(x - y)$ and the b.c. (for instance, $G_{\mathbb{R}^d}(x, y) = 1/(4\pi \|x - y\|)$). Similarly, let $h_{A, B}(x) = \mathbb{P}\{\tau^*_x A < \tau^*_y B\}$ denote the probability that the Brownian path starting in $x$ hits the set $A$ before hitting the set $B$. It satisfies the boundary value problem
\[ \Delta h_{A, B}(x) = 0 \quad x \in (A \cup B)^c, \]
\[ h_{A, B}(x) = 1 \quad x \in A, \] (4.5)
\[ h_{A, B}(x) = 0 \quad x \in B. \]
This, however, is also the equation satisfied by the electric potential of a capacitor, with conductors $A$ and $B$ at respective potentials 1 and 0. If $\rho_{A,B}(x)$ denotes the surface charge density on the two conductors, we can write

$$h_{A,B}(x) = \int_{\partial A} G_{B^c}(x,y) \rho_{A,B}(y) \, dy.$$  \hfill (4.6)

The capacity of the capacitor is simply the total charge accumulated on one conductor, divided by the potential difference, which equals 1:

$$\text{cap}_A(B) = \int_{\partial A} \rho_{A,B}(y) \, dy.$$  \hfill (4.7)

The key observation is the following. Let $h(x)$ be a ball of radius $\varepsilon$ around $x$, and consider the integral $\int_C w_A(z) \rho_{C,A}(z) \, dz$. On one hand, using expression (4.4) of $w_A$, symmetry of Green’s function and then (4.6), one sees that this integral is equal to $\int_{\partial A} h_{C,A}(y) \, dy$. On the other hand, as $w_A$ does not vary much on the small ball $C$ [BEGK04], we can replace $w_A(z)$ by $w_A(x)$, and the remaining integral is just the capacity. This yields the relation

$$\Gamma^{-1} = E[t_A^+|1] = w_A(x) \simeq \frac{\int_{\partial A} h_{B_1,1}(z) \, dz}{\text{cap}_{B_1,1}(A)}.$$  \hfill (4.8)

The interest of this relation lies in the fact that capacities can be estimated by a variational principle. Indeed, the capacity for unit potential difference is equal to the total energy of the electric field,

$$\text{cap}_A(B) = \int_{(A \cup B)^c} \| \nabla h_{A,B}(x) \|^2 \, dx = \inf_h \int_{(A \cup B)^c} \| \nabla h(x) \|^2 \, dx,$$  \hfill (4.9)

where the infimum is taken over all twice differentiable functions satisfying the b.c. in (4.5).

If, instead of the Brownian motion, we consider the solution of a $d$-dimensional SDE $\dot{x} = -\nabla \mathcal{H}(x) + \sqrt{2} \varepsilon \tilde{W}_t$, the above steps can be repeated, provided we replace $\Delta$ by the generator $\varepsilon \Delta - \nabla \mathcal{H} \cdot \nabla$ of the equation (the generator is the adjoint of the operator appearing in the Fokker–Planck equation). The above relations remain valid, only with the Lebesgue measure replaced by the invariant measure $e^{-\varepsilon \mathcal{H}(x)}/dx$. Thus, we have

$$\Gamma = E[t_A^+|1] \simeq \frac{\text{cap}_{B_1,1}(A)}{\int_{(A \cup B)^c} h_{B_1,1}(z) \, dz} e^{-\varepsilon \mathcal{H}(z)/\varepsilon},$$  \hfill (4.10)

where the capacity can be computed via the Dirichlet form

$$\text{cap}_A(B) = \inf_{h} \int_{(A \cup B)^c} \| \nabla h(x) \|^2 e^{-\varepsilon \mathcal{H}(x)/\varepsilon} \, dx.$$  \hfill (4.11)

The denominator in (4.10) can be easily estimated by saddle-point methods, using the fact that $h_{B_1,1}$ is essentially 1 in the basin of attraction of $x$ and 0 in the basin of $A$. It is equal to leading order to $(2\pi \varepsilon)^{d/2} e^{-\varepsilon \mathcal{H}(z)/\varepsilon}/\sqrt{\det(\delta^2 \mathcal{H}/\delta x^2)}(x)$. A good upper bound of the denominator in (4.11) is obtained by inserting a sufficiently good guess for the potential $h$ in (4.11). Assume, e.g., that near a transition state at 0, the energy has the expansion

$$\mathcal{H}(x) = -\frac{1}{2} \lambda_0 |x_0|^2 + \mu(x_1) + \frac{1}{2} \sum_{j=2}^{d-1} \lambda_j x_j^2 + \cdots,$$  \hfill (4.12)

where $\mu(x_1)$ corresponds to the possibly neutral direction in which a bifurcation occurs. Choosing $h(x) = f(x_0)$, where $\varepsilon f''(x_0) - \partial_0 \mathcal{H}(x_0, 0, \ldots, 0) f'(x_0)$ with appropriate b.c., and substituting in (4.11) yield

$$\text{cap}_A(B) \leq \frac{1}{2\pi} \left( \frac{(2\pi \varepsilon)^{d/2} |\lambda_0|}{\lambda_2 \cdots \lambda_{d-1}} \right) \int_{-\infty}^{\infty} e^{-u(x_1)/\varepsilon} \, dx_1.$$  \hfill (4.13)
A matching lower bound for the capacity can be obtained by a slightly more elaborate argument; see [BG08b] for details. If \( u(x_1) = \frac{1}{2} \lambda_1 \chi_1 \), the integral has the value \( \sqrt{2 \pi \varepsilon / \chi_1} \) and we recover the usual Kramers formula. However, (4.13) applies to other cases as well, e.g. a quartic \( u(x_1) \).

We now return to the SPDE (2.2). We apply the above theory first to a finite-dimensional approximation of the system (2.2), obtained either by truncation of high wave numbers in its Fourier transform (2.5) or by replacing the system by a discrete chain [BFG07a, BFG07b], and then taking the limit. A difficulty is that the error terms will depend on the number of retained modes (see, for instance, [Liu03] for estimates on the convergence rate of the spectral approximations). Thus, the results below are for now only formal. The error terms in the capacity can, however, be controlled [BG08a].

**Neumann b.c.** The potential energy along the normalized eigenvector in the bifurcating direction \( v_1(x) = \sqrt{2} \cos(\pi x/L) \) is

\[
\begin{align*}
    u(\phi_1) &= \mathcal{H}[\phi_1 v_1] = L \left[ \frac{1}{2} \lambda_1 \phi_1^2 + \frac{1}{8} \phi_1^4 + \cdots \right].
\end{align*}
\]

(Evaluating the integral in (4.13), we find [BG08b] that for \( L \leq \pi \) the corrected Kramers prefactor to leading order is given by

\[
\begin{align*}
    \Gamma_0 &\simeq \frac{1}{2^{3/4} \pi} \sqrt{\frac{\lambda_1}{\lambda_1 + \sqrt{3\varepsilon/4L}}} \Psi_+(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}}) \sqrt{\frac{\sinh(2L)}{\sin L}},
\end{align*}
\]

where \( \lambda_1 = -1 + (\pi/L)^2 \) and \( \Psi_+ \) is a universal scaling function, given in terms of the modified Bessel function of the second kind \( K_{1/4} \) by

\[
\begin{align*}
    \Psi_+(\alpha) &= \sqrt{\frac{\alpha(1 + \alpha)}{8\pi}} e^{\alpha^2/16} K_{1/4}(\frac{\alpha^2}{16}).
\end{align*}
\]

For \( L \ll \pi \), since \( \Psi_+(\alpha) \) tends to 1 as \( \alpha \to \infty \), we recover to leading order the rate (4.2). For \( \pi - L \) of order \( \sqrt{\varepsilon} \), however, the correction terms come into play, and the factor \( \sqrt{\lambda_1} \) in the numerator of (4.15) counteracts the divergence of the prefactor (4.2). In particular, we have

\[
\lim_{L \to \pi} \Gamma_0 \simeq \frac{\Gamma(1/4)}{2(3\pi^{3/2})^{1/4}} \sqrt{\sinh(2\pi) e^{-1/4}}.
\]

For \( L > \pi \), the rate prefactor is harder to compute, because the transition states are not uniform. The computation can nevertheless be done [MS03] with the help of a method due to Gel’fand, with the result

\[
\Gamma_0 \simeq \frac{1}{\pi} |\mu_0| \sqrt{\frac{\sinh(\sqrt{2}L)}{\sqrt{2}(1 - m)K(m) - (1 + m)E(m)}},
\]

where \( \mu_0 = 1 - \frac{1}{m^2} \sqrt{m^2 - m + 1} \) is the negative eigenvalue of \( \Lambda \) and \( m \) is related to \( L \) by (3.5). As \( L \to \pi^+ \) (that is, \( m \to 0^+ \)), this expression again diverges, namely like \( (L - \pi)^{-1/2} \).

Proceeding as above, we find [BG08b] that the corrected prefactor is obtained by multiplying (4.18) by

\[
\begin{align*}
    \frac{1}{2} \sqrt{\frac{\mu_1}{\mu_1 + \sqrt{3\varepsilon/4L}}} \Psi_-(\frac{\mu_1}{\sqrt{3\varepsilon/4L}}).
\end{align*}
\]

Here \( \Psi_- \) is again a universal scaling function, given in terms of modified Bessel functions of the first kind \( I_{1/4} \) by

\[
\begin{align*}
    \Psi_-(\alpha) &= \sqrt{\frac{\pi\alpha(1 + \alpha)}{32}} e^{-\alpha^2/64} \left[ I_{-1/4}(\frac{\alpha^2}{64}) + I_{1/4}(\frac{\alpha^2}{64}) \right].
\end{align*}
\]
which converges to 2 as $\alpha \to \infty$, and $\mu_1$ is the second eigenvalue of $\Lambda_t$. We can in fact avoid the computation of this eigenvalue. Indeed, near the bifurcation a local analysis shows that $\mu_1 = -2\lambda_1 + O(\lambda_1^2) = 3m + O(m^2)$, while further away from the bifurcation, the quotient in (4.19) is close to 1. One can thus replace $\mu_1$ by $3m$ in (4.19), only causing a multiplicative error $1 + O(\epsilon^{1/4})$. The resulting behavior of the prefactor $\Gamma_0$ as $L$ crosses the critical value $\pi$ is shown in figure 1.

Periodic b.c. For $L \leq 2\pi$, the transition state is uniform, and the computations are analogous to those in the previous case. The eigenvalues at the stable and transition states are now given by $\lambda_k = -1 + (2\pi k/L)^2$ and $\eta_k = 2 + (2\pi k/L)^2$, respectively, with $k \in \mathbb{Z}$, and are thus double except for $k = 0$. This implies that the integral in (4.13) is to be replaced by a double integral over the subspace of the two bifurcating modes [BG08a]. The result is

$$
\Gamma_0 \simeq \frac{1}{2\pi} \frac{\lambda_1}{\lambda_1 + \sqrt{3\epsilon/4L}} \tilde{\Psi}_+ \left( \frac{\lambda_1}{\sqrt{3\epsilon/4L}} \right) \frac{\sinh(L/2\sqrt{2})}{\sin(L/2)}.
$$

(4.21)

where the scaling function $\tilde{\Psi}_+$ is now given in terms of the error function by

$$
\tilde{\Psi}_+(\alpha) = \sqrt{\frac{\pi}{8}} (1 + \alpha) e^{\alpha^2/8} [1 + \text{erf}(-2^{-3/2}\alpha)].
$$

(4.22)

As $\tilde{\Psi}_+$ converges to 1 as $\alpha \to \infty$, for $2\pi - L \gg \sqrt{\epsilon}$, we recover the usual Kramers prefactor, which diverges as $(2\pi - L)^{-1}$ as $L \to 2\pi^-$. However, as $L$ approaches $2\pi$, the correction terms come into play and we get

$$
\lim_{L \to 2\pi^-} \Gamma_0 \simeq \frac{\sinh(\sqrt{2\pi})}{\sqrt{3\pi}} \epsilon^{-1/2}.
$$

(4.23)

For $L > 2\pi$, we again have to deal with a non-uniform transition state $\phi_t$. An additional difficulty stems from the fact that transition states form a continuous family, so that the Hessian at $\phi_t$ always admits one vanishing eigenvalue. This eigenvalue can be removed by a regularization procedure due to McKane and Tarlie [MT95], which has been applied in the
case of an asymmetric potential in [Ste04]. The computations are similar in the symmetric case [Ste], and yield a rate prefactor per unit length

\[
\frac{\Gamma_0}{L} \sim \frac{|\mu_0|}{(2\pi)^{3/2}} \sqrt{\frac{2m(1-m) \sinh^2(L/\sqrt{2})}{(1+m)^{5/2}K(m) - \frac{1+m}{1-m}E(m)}} \varepsilon^{-1/2},
\]

with \(4\sqrt{m+1}K(m) = L\) and the same \(\mu_0\) as for Neumann b.c. The factor \(\varepsilon^{-1/2}\) reflects the fact that nucleation can occur anywhere in space [Ste04]. The prefactor now converges to a finite limit as \(L \to 2\pi^+\), which differs, however, by a factor 2 from (4.23). This apparent discrepancy is solved by applying the corrected Kramers formula, which shows that (4.24) has to be multiplied by a factor

\[
\Phi\left(3m \sqrt{3} \varepsilon/L \right), \tag{4.25}
\]

where \(\Phi(x) = \frac{1}{2} [1 + \text{erf}(x/\sqrt{2})]\). The resulting rate prefactor is indeed continuous at \(L = 2\pi\).

5. Conclusion

We have presented a new method allowing the computation of the Kramers rate prefactor in situations where the transition state undergoes a bifurcation. In contrast with the quadratic case, the prefactor is no longer independent of the noise intensity \(\varepsilon\) to leading order, but diverges like \(C\varepsilon^{-\alpha}\), where \(\alpha\) is equal to \(1/4\) times the number of vanishing eigenvalues. The constant \(C\) can in fact be computed in a full neighborhood of the bifurcation point, and involves universal functions, depending only on the type of bifurcation. A similar non-Arrhenius behavior of the prefactor has been observed in irreversible systems [MS96], but there it has an entirely different origin, namely the development of a caustic singularity in the most probable exit path.

Acknowledgments

We would like to thank Dan Stein for helpful advice and for sharing unpublished computations on the periodic-b.c. case. BG was supported by CRC 701 ‘Spectral Structures and Topological Methods in Mathematics’.

References

[BEGK04] Bovier A, Eckhoff M, Gayrard V and Klein M 2004 Metastability in reversible diffusion processes: I. Sharp asymptotics for capacities and exit times J. Eur. Math. Soc. 6 399–424
[BFG07a] Berglund N, Fernandez B and Gentz B 2007 Metastability in interacting nonlinear stochastic differential equations: I. From weak coupling to synchronization Nonlinearity 20 2551–81
[BFG07b] Berglund N, Fernandez B and Gentz B 2007 Metastability in interacting nonlinear stochastic differential equations: II. Large-\(N\) behaviour Nonlinearity 20 2583–614
[BG08a] Berglund N and Gentz B 2008 in preparation
[BG08b] Berglund N and Gentz B 2008 The Eyring–Kramers law for potentials with nonquadratic saddles arXiv:0807.1681
[BMR00] Brown G, Novotny M A and Rikvold P A 2000 Micromagnetic Simulations of Thermally Activated Magnetization Reversal of Nanoscale Magnets vol 87 (New York: AIP) pp 4792–4
[Bra93] Braun H-B 1993 Thermally activated magnetization reversal in elongated ferromagnetic particles Phys. Rev. Lett. 71 3557–60
[BSS05] Burki I, Stafford C A and Stein D L 2005 Theory of metastability in simple metal nanowires Phys. Rev. Lett. 95 090601
[CH93] Cross M C and Hohenberg P C 1993 Pattern formation outside of equilibrium Rev. Mod. Phys. 65 851–1112

[Dyn65] Dynkin E B 1965 Markov Processes vols 1, 2 (New York: Academic)

[Eyr35] Eyring H 1935 The activated complex in chemical reactions J. Chem. Phys. 3 107–15

[FJL82] Faris W G and Jona-Lasinio G 1982 Large fluctuations for a nonlinear heat equation with noise J. Phys. A: Math. Gen. 15 3025–55

[Jet86] Jetschke G 1986 On the equivalence of different approaches to stochastic partial differential equations Math. Nachr. 128 315–29

[Kra40] Kramers H A 1940 Brownian motion in a field of force and the diffusion model of chemical reactions Physica 7 284–304

[Lan67] Langer J S 1967 Theory of the condensation point Ann. Phys. 41 108–47

[Liu03] Liu D 2003 Convergence of the spectral method for stochastic Ginzburg–Landau equation driven by space-time white noise Commun. Math. Sci. 1 361–75

[MS96] Maier R S and Stein D L 1996 A scaling theory of bifurcations in the symmetric weak-noise escape problem J. Stat. Phys. 83 291–357

[MS01] Maier R S and Stein D L 2001 Droplet nucleation and domain wall motion in a bounded interval Phys. Rev. Lett. 87 270601–1

[MS03] Maier R S and Stein D L 2003 The effects of weak spatiotemporal noise on a bistable one-dimensional system Noise in Complex Systems and Stochastic Dynamics (SPIE Proc. Series vol 5114) ed L Schimanski-Geier, D Abbott, A Neimann and C Van den Broeck pp 67–78

[MT95] McKane A J and Tarlie M B 1995 Regularization of functional determinants using boundary conditions J. Phys. A: Math. Gen. 28 6931–42

[Nee49] Néel L 1949 Théorie du trainage magnétique des ferro-magnétiques en grains fins avec application aux terres cuites Ann. Géophys. 5 99–136

[Ste] Stein D L private communication

[Ste04] Stein D L 2004 Critical behavior of the Kramers escape rate in asymmetric classical field theories J. Stat. Phys. 114 1537–56

[Ste05] Stein D L 2005 Large fluctuations, classical activation, quantum tunneling, and phase transitions Braz. J. Phys. 35 242–52