DERIVATIVE FORMULA AND GRADIENT ESTIMATE FOR SDES DRIVEN BY \(\alpha\)-STABLE PROCESSES∗

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ABSTRACT. In this paper we prove a derivative formula of Bismut-Elworthy-Li’s type as well as gradient estimate for stochastic differential equations driven by \(\alpha\)-stable noises, where \(\alpha \in (0, 2)\). As an application, the strong Feller property for stochastic partial differential equations driven by subordinated cylindrical Brownian motions is presented.

1. INTRODUCTION AND MAIN RESULT

The derivative formula of diffusion semigroups about the stochastic differential equations (SDEs) on Riemannian manifolds was originally introduced by Bismut in [4], and used to derive the estimates of heat kernels and large deviation principles. His approach is based upon the Malliavin calculus. In [8], Elworthy and Li used simple martingale arguments to derive a derivative formula for a large class of diffusion semigroups on Riemannian manifolds. Nowadays, this type of formula has been proved to be a quite useful tool in various aspects such as functional inequalities, heat kernel estimates, strong Feller properties and sensitivity analysis, see [1], [2], [9], [11], [14], [20] etc.

Due to various successful applications, recently, there is great interest to establish an analogous derivative formula for jump-diffusion processes. In [6], Cass and Fritz proved a derivative formula of Bismut-Elworthy-Li’s type for SDEs with jumps and nondegenerate Brownian diffusion term. In [18], Takeuchi proved a similar formula for some pure-jump diffusions with finite moments of all orders. However, their works rule out the interesting \(\alpha\)-stable processes. In [19], Wang used the coupling method and Girsanov’s transform to prove a derivative formula for linear SDEs driven by pure-jump Lévy processes including \(\alpha\)-stable processes, where the explicit gradient estimates and heat kernel inequalities are also derived. For infinitely dimensional Ornstein-Uhlenbeck processes with cylindrical \(\alpha\)-stable processes, using the infinitely dimensional analysis, Priola and Zabczyk [15] established an explicit derivative formula in terms of the distributional density of \(\alpha\)-stable processes. Then the strong Feller property was obtained for a class of semilinear stochastic partial differential equations (SPDEs) driven by cylindrical \(\alpha\)-stable processes, where \(\alpha \in (1, 2)\).

In this paper, we aim to establish a derivative formula of Bismut-Elworthy-Li’s type for non-linear SDEs driven by \(\alpha\)-subordinated Brownian motions. Before moving on, we first recall the classical derivative formulas for diffusion semigroups. Let \(\{W_t\}_{t \geq 0}\) be a standard \(d\)-dimensional Wiener process. Consider the following SDE in \(\mathbb{R}^d\):

\[
dX_t(x) = b_t(X_t(x))dt + \sigma \cdot W_t, \quad X_0(x) = x, \tag{1.1}
\]

where \(\sigma\) is a \(d \times d\) invertible matrix, and \(b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\) satisfies that (H) \(b\) has continuous first order partial derivatives with respect to \(x\), and

\[
\|\nabla b\|_\infty < +\infty,
\]

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where $\nabla b_s(x) := (\partial_x b_s(x), \cdots, \partial_x b_s(x))$ and $\| \cdot \|_{\infty}$ denotes the uniform norm with respect to $s$ and $x$.

It is well-known that there are at least two forms of derivative formulas: for any function $f \in C_b^1(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$ (see [11, 21]),

$$\nabla_h \mathbb{E}f(X_t(x)) = \frac{1}{t} \mathbb{E} \left(f(X_t(x)) \int_0^t \sigma^{-1} [h + (t - s) \nabla_h b_s(X_s(x))]dW_s \right)$$

(1.2)

and (see [7, 8])

$$\nabla_h \mathbb{E}f(X_t(x)) = \frac{1}{t} \mathbb{E} \left(f(X_t(x)) \int_0^t \sigma^{-1} \cdot \nabla_h X_s(x)dW_s \right),$$

(1.3)

where for a function $\varphi$, $\nabla_h \varphi := \langle \nabla \varphi, h \rangle$ denotes the directional derivative along $h$, and $\nabla_h X_t(x)$ satisfies the following linear equation:

$$\nabla_h X_t(x) = h + \int_0^t \nabla b_s(X_s(x)) \cdot \nabla_h X_s(x)ds.$$ (1.4)

The difference between (1.2) and (1.3) lies in that in formula (1.2), $\nabla b$ is allowed to be polynomial growth as in [11, 20, 21], while in formula (1.3), $\nabla b$ usually needs to be bounded. Below, we shall see that formula (1.3) is crucial for us since $b$ does not appear explicitly in (1.3).

Now we turn to the case of SDEs driven by $\alpha$-stable processes. For $\alpha \in (0, 2)$, let $\{S_t\}_{t \geq 0}$ be an independent $\alpha/2$-stable subordinator, i.e., an increasing $\mathbb{R}$-valued process with stationary independent increments, and

$$\mathbb{E} e^{iu S_t} = e^{t |u|^{\alpha/2}}.$$ (1.5)

It is well-known that the subordinated Brownian motion $\{W_t\}_{t \geq 0}$ is an $\alpha$-stable process (cf. [3, 17]). Let us consider the following SDE in $\mathbb{R}^d$ driven by $W_t$:

$$dX_t(x) = b_t(X_t(x))dt + \sigma \cdot dW_t, \quad X_0(x) = x.$$ (1.5)

The generator of Markov process $\{X_t(x), t \geq 0, x \in \mathbb{R}^d\}$ is given by

$$\mathcal{L} f(x) := \text{P.V.} \int_{\mathbb{R}^d} (f(x + sy) - f(x)) \frac{dy}{|y|^{d+\alpha}} + b_t(x) \cdot \nabla f(x),$$

where P.V. stands for the Cauchy principal value. Notice that if $b_t(x) = b_0(x)$ is time homogeneous, then $u_t(x) := \mathbb{E} f(X_t(x))$ solves the following integro-differential equation (cf. [6, 22]):

$$\partial_t u_t(x) = \mathcal{L} u_t(x), \quad u_0 = f.$$ (1.6)

The main aim of the paper is to derive a formula for $\nabla \mathbb{E} f(X_t(x))$ as stated follows:

**Theorem 1.1.** Under (H), for any function $f \in C_b^1(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, we have

$$\nabla_h \mathbb{E}f(X_t(x)) = \mathbb{E} \left(\frac{1}{S_t} f(X_t(x)) \int_0^t \sigma^{-1} \cdot \nabla_h X_s(x)dW_s \right),$$

(1.6)

where $\nabla_h X_t(x)$ is determined by equation (1.4). In particular, for any $\alpha \in (0, 2)$ and $p \in (1, \infty)$, there exists a constant $C = C(\alpha, p) > 0$ such that for all $t > 0$,

$$|\nabla \mathbb{E}f(X_t(x))| \leq C \|\sigma^{-1}\|_{\mathbb{R}^d} \|f\|_p t^{\frac{1}{2}} \left(\mathbb{E} \|f(X_t(x))\|^p \right)^{\frac{1}{2}},$$

(1.7)

where $\|\sigma^{-1}\| := \sup_{|x|=1} |\sigma^{-1} x|$ and $\| \cdot \|$ denotes the Euclidian norm.

**Remark 1.2.** From equation (1.4), it is easy to see that $s \mapsto \nabla_h X_s(x)$ is a bounded and continuous $\sigma(W_s : r \leq s)$-adapted process. Thus, the stochastic integral in (1.6) makes sense.
We now introduce the main idea of proving this theorem. Let $\mathcal{W}$ be the space of all continuous functions from $[0, \infty)$ to $\mathbb{R}^d$ vanishing at starting point 0, which is endowed with the locally uniform convergence topology and the Wiener measure $\mu_{\mathcal{W}}$ so that the coordinate process

$$W_t(w) = w_t$$

is a standard $d$-dimensional Brownian motion. Let $S$ be the space of all increasing and càdlàg functions from $(0, \infty)$ to $(0, \infty)$ with $\lim_{t \downarrow 0} \ell_t = 0$, which is endowed with the Skorohod metric and the probability measure $\mu_S$ so that the coordinate process

$$S_t(\ell) := \ell_t$$

is an $\alpha/2$-stable subordinator $S_t$ (cf. [3, 17]). Consider the following product probability space

$$(\Omega, \mathcal{F}, P) := (\mathcal{W} \times S, \mathcal{B}(\mathcal{W}) \times \mathcal{B}(S), \mu_{\mathcal{W}} \times \mu_S),$$

and define

$$L_t(w, \ell) := w_{\ell_t}.$$ 

Then $\{L_t\}_{t \geq 0}$ is an $\alpha$-stable process on $(\Omega, \mathcal{F}, P)$. We shall use the following two natural filtrations associated to the Lévy process $L_t$ and the Brownian motion $W_t$:

$$\mathcal{F}_t := \sigma\{L_s(w, \ell) : s \leq t\}, \quad \mathcal{F}_t^\mathcal{W} := \sigma\{W_s(w) : s \leq t\}.$$ 

In particular, we can regard the solution $X_t(x)$ of SDE (1.5) as an $(\mathcal{F}_t)$-adapted functional on $\Omega$, and therefore,

$$\mathbb{E}f(X_t(x)) = \int_S \int_{\mathcal{W}} f\big(X_t(x); w_{\ell_t}\big)\mu_{\mathcal{W}}(dw)\mu_S(d\ell).$$

For $\ell \in S$, let $X^\ell_t(x)$ solve the following SDE:

$$dX^\ell_t(x) = b_\ell(X^\ell_t(x))dt + \sigma \cdot dW_{\ell_t}, \quad X^\ell_0(x) = x. \quad (1.8)$$

Now, our task is to establish a formula for $\nabla_{\ell_t}\mathbb{E}f(X^\ell_t(x))$. This is not obvious since $t \mapsto W_{\ell_t}$ is not continuous and the classical Bismut-Elworthy-Li formula can not be used directly.

The remainder of this paper is organized as follows: In Section 2, we shall prove a formula for $\nabla_{\ell_t}\mathbb{E}f(X^\ell_t(x))$ by suitable approximation for $\ell_t$. In Section 3, we prove Theorem 1.1. In Section 4, we prove the strong Feller property for nonlinear SPDEs driven by subordinated cylindrical Brownian motions.

2. Derivative formula of SDEs under nonrandom time changed

In this section, we fix an $\ell \in S$ and consider SDE (1.3). If there is no special declaration, all expectations are taken on the Wiener space $(\mathcal{W}, \mathcal{B}(\mathcal{W}), \mu_{\mathcal{W}})$. First of all, notice that $t \mapsto W_{\ell_t}$ is a Gaussian process with zero means and independent increments. In particular,

$$W_{\ell_t}$$

is a càdlàg $\mathcal{F}_t^\mathcal{W}$-martingale.

Thus, under $\textbf{(H)}$, it is well-known that for each $x \in \mathbb{R}^d$, SDE (1.8) admits a unique càdlàg $(\mathcal{F}_t^\mathcal{W})$-adapted solution $X^\ell_t(x)$ (cf. [16, p.249, Theorem 6]).

The main aim of this section is to establish the following formula:

**Theorem 2.1.** Under $\textbf{(H)}$, for any function $f \in C^1_b(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, we have

$$\nabla_h \mathbb{E}f(X^\ell_t(x)) = \mathbb{E}\left(\frac{1}{\ell_t} f(X^\ell_t(x)) \int_0^{\ell_t} \sigma^{-1} \cdot \nabla_h X^\ell_s(x) dW_{\ell_s}\right), \quad (2.1)$$

where $\nabla_h X^\ell_t(x)$ is determined by the following linear equation:

$$\nabla_h X^\ell_t(x) = h + \int_0^{\ell_t} \nabla b_\ell(X^\ell_s(x)) \cdot \nabla_h X^\ell_s(x) ds. \quad (2.2)$$
For proving this formula, we shall use the time changed argument to transform SDE (1.8) into an SDE driven by standard Brownian motions, and then use the classical Bismut-Elworthy-Li formula (1.3). For this aim, for $\varepsilon \in (0, 1)$, we define

$$\ell^\varepsilon_t := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \ell_s ds + \varepsilon t = \int_0^1 \ell_{\varepsilon s+t} ds + \varepsilon t. \quad (2.3)$$

Since $t \mapsto \ell_t$ is increasing and right continuous, it follows that for each $t \geq 0$,

$$\ell^\varepsilon_t \downarrow \ell_t \quad \text{as} \quad \varepsilon \downarrow 0. \quad (2.4)$$

Moreover, $t \mapsto \ell^\varepsilon_t$ is absolutely continuous and strictly increasing. Let $\gamma^\varepsilon$ be the inverse function of $\ell^\varepsilon$, i.e.,

$$\ell_{\gamma^\varepsilon_t} = t, \quad t \geq \ell^\varepsilon_0 \quad \text{and} \quad \gamma^\varepsilon_{\ell^\varepsilon_t} = t, \quad t \geq 0. \quad (2.5)$$

By definition, $\gamma^\varepsilon_t$ is also absolutely continuous on $[\ell^\varepsilon_0, \infty)$. Let us now define

$$\ell^\varepsilon_t: X_t^\varepsilon(x), \quad t \geq \ell^\varepsilon_0.$$ 

By equation (1.8) and the change of variables, one sees that for $t \geq \ell^\varepsilon_0$,

$$Y_t^\varepsilon(x) = x + \int_0^{\gamma^\varepsilon_t} b_s(X_s^\varepsilon(x)) ds + \sigma \cdot W_t$$

$$= x + \int_0^{\gamma^\varepsilon_t} b_s(Y_s^\varepsilon(x)) \dot{\gamma}^\varepsilon_s ds + \sigma \cdot W_t.$$ 

Hence, one can use the classical Bismut-Elworthy-Li formula (see (1.3)) to derive that

$$\nabla_h \mathbb{E} f(Y_t^\varepsilon(x)) = \frac{1}{\ell^\varepsilon_t} \mathbb{E} \left( f(Y_t^\varepsilon(x)) \int_0^{\gamma^\varepsilon_t} \sigma^{-1} \cdot \nabla_h Y_s^\varepsilon(x) dW_s \right), \quad t \geq \ell^\varepsilon_0,$$

where $\nabla_h Y_t^\varepsilon(x)$ satisfies

$$\nabla_h Y_t^\varepsilon(x) = h + \int_{\ell^\varepsilon_0}^{\gamma^\varepsilon_t} \nabla b_s(Y_s^\varepsilon(x)) \nabla_h Y_s^\varepsilon(x) \dot{\gamma}^\varepsilon_s ds.$$ 

Clearly, for each $t \geq 0$,

$$Y_t^\varepsilon(x) = X_t^\varepsilon(x), \quad \nabla_h Y_t^\varepsilon(x) = \nabla_h X_t^\varepsilon(x),$$

and therefore,

$$\nabla_h \mathbb{E} f(X_t^\varepsilon(x)) = \frac{1}{\ell^\varepsilon_t} \mathbb{E} \left( f(X_t^\varepsilon(x)) \int_0^{\ell^\varepsilon_t} \sigma^{-1} \cdot \nabla_h X_s^\varepsilon(x) dW_s \right)$$

$$= \frac{1}{\ell^\varepsilon_t} \mathbb{E} \left( f(X_t^\varepsilon(x)) \int_0^{\ell^\varepsilon_t} \sigma^{-1} \cdot \nabla_h X_s^\varepsilon(x) dW_s \right). \quad (2.5)$$

Now we want to take limits for both sides of the above formula. We need several lemmas. First of all, the following lemma is easy.

**Lemma 2.2.** For any $p \geq 1$ and $t \geq 0$, we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left( \sup_{x \in \mathbb{R}^d} |X_t^\varepsilon(x) - X_t^\varepsilon(x)|^p \right) = 0, \quad (2.6)$$

and for any $x, h \in \mathbb{R}^d$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left( \sup_{x \in [0,t]} |\nabla_h X_t^\varepsilon(x) - \nabla_h X_t^\varepsilon(x)|^p \right) = 0, \quad (2.7)$$


where
\[
\nabla_h X^e_t(x) = h + \int_0^t \nabla b_s(X^e_t(x)) \cdot \nabla_h X^e_t(x) \, ds. \tag{2.8}
\]

**Proof.** For simplicity of notation, we drop the variable “\(x\)” below. From equation (1.8), we have
\[
|X^e_t - X^e_s| \leq ||\nabla b||_\infty \int_s^t |X^e_s - X^e_r| \, dr + |\sigma \cdot W_{t_r} - \sigma \cdot W_{s_r}|.
\]

By Gronwall’s inequality, we get
\[
|X^e_t - X^e_s| \leq e^{||\nabla b||_\infty t} |\sigma \cdot W_{t_r} - \sigma \cdot W_{s_r}|,
\]
which then gives (2.6) by (2.4).

As for (2.7), by (2.8) and (2.2) we have
\[
|\nabla_h X^e_t - \nabla_h X^e_s| \leq \int_s^t |\nabla b_s(X^e_r)| \cdot |\nabla_h X^e_r - \nabla_h X^e_s| \, dr + \int_s^t |\nabla b_s(X^e_r) - \nabla b_s(X^e_s)| \cdot |\nabla_h X^e_r| \, dr,
\]
which yields by Gronwall’s inequality that
\[
|\nabla_h X^e_t - \nabla_h X^e_s| \leq e^{||\nabla b||_\infty t} \int_s^t |\nabla b_s(X^e_r) - \nabla b_s(X^e_s)| \cdot |\nabla_h X^e_r| \, dr.
\]
Moreover, from equation (2.2), it is easy to see that for any \(p \geq 1\),
\[
\sup_{s \in [0, t]} \mathbb{E} |\nabla_h X^e_s|^p \leq C.
\]
Limit (2.7) now follows by the dominated convergence theorem, (2.6) and the continuity of \(x \mapsto \nabla b_s(x)\). \(\square\)

We also need the following lemma.

**Lemma 2.3.** (i) Assume that \(\xi_t\) is a bounded continuous and \((\mathcal{F}^\mathcal{W}_t)\)-adapted \(\mathbb{R}^d\)-valued process. For each \(p, T > 0\), we have
\[
\lim_{\epsilon \downarrow 0} \mathbb{E} \left| \int_0^T \xi_t \, dW^\epsilon_t - \int_0^T \xi_t \, dW_t \right|^p = 0, \tag{2.9}
\]
where \(\ell^\epsilon_0\) is defined by (2.3).

(ii) Assume that \(\xi_t\) is a left continuous and \((\mathcal{F}^\mathcal{W}_t)\)-adapted \(\mathbb{R}^d\)-valued process and satisfies that for some \(p > 0\),
\[
\mathbb{E} \left( \int_0^T |\xi_t|^2 \, d\ell_t \right)^{\frac{p}{2}} < +\infty, \quad \forall T > 0. \tag{2.10}
\]
Then there exists a constant \(C_p > 0\) such that for all \(T \geq 0\),
\[
\mathbb{E} \left( \sup_{s \in [0, T]} \left| \int_0^T \xi_t \, dW^\epsilon_t \right|^p \right) \leq C_p \mathbb{E} \left( \int_0^T |\xi_t|^2 \, d\ell_t \right)^{\frac{p}{2}}. \tag{2.11}
\]

**Proof.** Without loss of generality, we assume \(T = 1\) and \(d = 1\).

(i) For \(n \in \mathbb{N}\), set \(t_k := k/n, k = 0, 1, \ldots, n\) and define
\[
\xi^n_s = \sum_{k=0}^{n-1} \xi_{t_k} 1_{[t_k, t_{k+1})}(s).
\]
By the continuity of \(t \mapsto \xi_t\), it is clear that
\[
\lim_{n \to \infty} \sup_{s \in [0, 1]} |\xi^n_s - \xi_s| = 0. \tag{2.12}
\]
Set
\[ \eta_{n,e} := \int_0^1 \xi_s^n dW_{\ell,t}, \quad \eta_e := \int_0^1 \xi_s dW_{\ell,t}, \]
\[ \eta_n := \int_0^1 \xi_s^n dW_{\ell,t}, \quad \eta := \int_0^1 \xi_s dW_{\ell,t}. \]

Noticing that \( W_{\ell,t} \) is a continuous \((\mathcal{F}_{\ell}^W)\)-martingale and
\[ [W_{\ell,t}] = \ell_t^e, \]
by Burkholder’s inequality (cf. [12, p. 279, Proposition 15.7]) and (2.4), we have for any \( p > 0 \),
\[ \mathbb{E}|\eta_{n,e} - \eta_e|^p = \mathbb{E} \left| \int_0^1 (\xi_s^n - \xi_s) dW_{\ell,t} \right|^p \leq C \mathbb{E} \left( \int_0^1 |\xi_s^n - \xi_s|^2 d\ell_t^e \right)^{\frac{p}{2}} \leq C(\ell_1^e)^{\frac{p}{2}} \mathbb{E} \left( \sup_{s \in [0,1]} |\xi_s^n - \xi_s|^p \right). \]

Hence, by the dominated convergence theorem and (2.12),
\[ \lim_{n \to \infty} \sup_{e \in (0,1)} \mathbb{E}|\eta_{n,e} - \eta_e|^p = 0. \]

On the other hand, for fixed \( n \in \mathbb{N} \), by (2.4) we have
\[ \mathbb{E}|\eta_{n,e} - \eta_n|^p = \mathbb{E} \left| \sum_{k=0}^{n-1} \xi_{\ell_k} (W_{\ell_{k+1}} - W_{\ell_k} - W_{\ell_{k+1}} + W_{\ell_k}) \right|^p \]
\[ \leq C_{n,p} K^p \sum_{k=0}^{n-1} \mathbb{E}|W_{\ell_{k+1}} - W_{\ell_k} - W_{\ell_{k+1}} + W_{\ell_k}|^p \to 0. \]

Combining the above calculations, we obtain (2.9).

(ii) We first assume that \( \xi_t \) is continuous and bounded by \( K \). Let \( \mathbb{Q} \) be the set of all rational numbers in [0, 1]. By (2.9), there exists a subsequence \( \ell_k \downarrow 0 \) such that
\[ \int_0^t \xi_s dW_{\ell_k} \to \int_0^t \xi_s dW_{\ell_t}, \quad \forall t \in \mathbb{Q}, \quad P - a.s. \]

Therefore,
\[ \mathbb{E} \left( \sup_{e \in [0,1]} \left| \int_0^t \xi_s dW_{\ell_k} \right|^p \right) = \mathbb{E} \left( \sup_{e \in \mathbb{Q}} \left| \int_0^t \xi_s dW_{\ell_k} \right|^p \right) = \mathbb{E} \left( \sup_{e \in [0,1]} \left| \int_0^t \xi_s dW_{\ell_k} \right|^p \right). \]

Since \( W_{\ell_k} \) is a continuous \((\mathcal{F}_{\ell_k}^W)\)-martingale, by Fatou’s lemma and Burkholder’s inequality again, we have
\[ \mathbb{E} \left( \sup_{e \in [0,1]} \left| \int_0^t \xi_s dW_{\ell_k} \right|^p \right) \leq \lim_{\ell_k \downarrow 0} \mathbb{E} \left( \sup_{e \in \mathbb{Q}} \left| \int_0^t \xi_s dW_{\ell_k} \right|^p \right) \]
\[ \leq C_p \lim_{\ell_k \downarrow 0} \mathbb{E} \left( \int_0^t |\xi_s|^2 d\ell_k^e \right)^{\frac{p}{2}} = C_p \mathbb{E} \left( \int_0^t |\xi_s|^2 d\ell^e_t \right)^{\frac{p}{2}}, \quad (2.13) \]

where the last limit can be proved as in (i).

Next, assume that \( \xi_t \) is left-continuous and bounded by \( K \). Define \( \xi^e : = \frac{1}{e} \int_{(t-\varepsilon)v_0}^t \xi_s ds, \varepsilon > 0. \)
Then $\xi^\varepsilon_t$ is a continuous and bounded $(\mathcal{F}^\varepsilon_{t\varepsilon})$-adapted process. By (2.13) we have

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left( \sup_{t \in [0,1]} \left| \int_0^t \xi^\varepsilon_s dW_{t\varepsilon} - \int_0^t \xi^{\varepsilon'}_s dW_{t\varepsilon} \right|^p \right) \leq C \lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left( \int_0^1 \left| \xi^\varepsilon_s - \xi^{\varepsilon'}_s \right|^2 d\ell_s \right)^{\frac{p}{2}} = 0.$$

So, (2.11) holds for left-continuous and bounded $(\mathcal{F}^\varepsilon_{t\varepsilon})$-adapted process.

In the general case, we truncate $\xi_t$ as follows: for $K > 0$, define

$$\xi^K_t := (-K) \vee \xi_t \wedge K.$$

By the dominated convergence theorem, we have

$$\lim_{K,K' \to \infty} \mathbb{E} \left( \sup_{t \in [0,1]} \left| \int_0^t \xi^K_s dW_{t\varepsilon} - \int_0^t \xi^{K'}_s dW_{t\varepsilon} \right|^p \right) \leq C_p \lim_{K,K' \to \infty} \mathbb{E} \left( \int_0^1 \left| \xi^K_s - \xi^{K'}_s \right|^2 d\ell_s \right)^{\frac{p}{2}} = 0.$$

The proof is complete. □

**Remark 2.4.** For $p \geq 1$, by Burkholder’s inequality (cf. [12] p.443, Theorem 23.12), we have

$$\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t \xi_s dW_{t\varepsilon} \right|^p \right) \leq \mathbb{E} \left( \int_0^T |\xi_s|^2 d[W_t]_s \right)^{\frac{p}{2}},$$

where $\approx$ means that both sides are comparable by multiplying a constant, and $[W_t]_t$ denotes the quadratic variation of $(W_{t\varepsilon})_{t \geq 0}$ given by

$$[W_t]_t = \ell_t - \sum_{0 < \ell \leq t} \Delta \ell_s + \sum_{0 < \Delta W_{t\varepsilon} < 2} |\Delta W_{t\varepsilon}|^2.$$

Except for the case of $p = 2$, it is not known whether the right hand sides of (2.11) and (2.14) are comparable.

**Lemma 2.5.** For all $t \geq 0$, we have

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left( \int_0^t \sigma^{-1} \cdot \nabla h X^\varepsilon_s (x) dW_{t\varepsilon} \right) = \mathbb{E} \left( \int_0^t \sigma^{-1} \cdot \nabla h X^\varepsilon_s (x) dW_{t\varepsilon} \right).$$

**Proof.** For simplicity of notation, we drop the variable “$x$” below. Since $\ell_s \geq 0$, by definition, $X^\varepsilon_s$ and $X^\varepsilon_s$ are $(\mathcal{F}^\varepsilon_{t\varepsilon})$-adapted. Thus, for proving (2.15), it suffices to prove the following two limits:

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left| \int_0^t (\sigma^{-1} \cdot \nabla h X^\varepsilon_s - \sigma^{-1} \cdot \nabla h X^{\varepsilon'}_s) dW_{t\varepsilon} \right|^2 = 0$$

and

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left| \int_0^t \sigma^{-1} \cdot \nabla h X^\varepsilon_s dW_{t\varepsilon} - \int_0^t \sigma^{-1} \cdot \nabla h X^{\varepsilon'}_s dW_{t\varepsilon} \right|^2 = 0.$$

For (2.16), by the isometry property of stochastic integrals and (2.7), we have

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left| \int_0^t (\sigma^{-1} \cdot (\nabla h X^\varepsilon_s - \nabla h X^{\varepsilon'}_s) dW_{t\varepsilon} \right|^2 = \lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left( \int_0^t \left| \sigma^{-1} \cdot (\nabla h X^\varepsilon_s - \nabla h X^{\varepsilon'}_s) \right|^2 d\ell_s \right) \leq \|\sigma^{-1}\|^2 \lim_{\varepsilon,\varepsilon' \downarrow 0} \mathbb{E} \left( \sup_{s \in [0,t]} |\nabla h X^\varepsilon_s - \nabla h X^{\varepsilon'}_s|^2 \right) \ell_t = 0.$$
Proof of Theorem 2.1 By (2.4), (2.6) and (2.15), the right hand sides of (2.5) converges to the one of (2.7). On the other hand, by (2.6) and (2.7), we have
\[ \nabla_h \mathbb{E}(X_t^h(x)) = \mathbb{E}(\nabla f(X_t^h(x)) \cdot \nabla_h X_t^h(x)) \rightarrow \mathbb{E}(\nabla f(X_t^h(x)) \cdot \nabla_h f(X_t^h(x))). \]

3. Proof of Theorem 1.1

The following lemma follows by the monotone class theorem.

Lemma 3.1. Let \( t \geq 0 \) and \( A \in \mathcal{F}_t \). For any \( \ell \in \mathbb{S} \), we have
\[ \{ w \in \mathcal{W} : w_\ell \in A \} \in \mathcal{F}_t^W. \]

Below we recall a result due to Giné and Marcus [10].

Theorem 3.2. Let \( \xi \) be a left continuous \((\mathcal{F}_t)\)-adapted \( \mathbb{R}^d \)-valued process and satisfies
\[ \int_0^T \mathbb{E}|\xi|^{\alpha} ds < +\infty, \ \forall T > 0. \]

Then there exists a constant \( C = C(\alpha) > 0 \) such that for all \( \lambda > 0 \) and \( T > 0 \),
\[ P\left\{ \sup_{\tau \in [0,T]} \left| \int_0^\tau \xi_s dW_{S_s} \right| > \lambda \right\} \leq C \lambda^{-\alpha} \int_0^T \mathbb{E}|\xi|^{\alpha} ds. \]

In particular, for any \( p \in (0, \alpha) \) and some \( C = C(\alpha, p) > 0 \),
\[ \mathbb{E} \left( \sup_{\tau \in [0,T]} \left| \int_0^\tau \xi_s dW_{S_s} \right|^p \right) \leq C \left( \int_0^T \mathbb{E}|\xi|^{\alpha} ds \right)^{\frac{p}{\alpha}}. \quad (3.1) \]

Proof. Define
\[ \zeta := \sup_{\tau \in [0,T]} \left| \int_0^\tau \xi_s dW_{S_s} \right|. \]

Then for any \( \eta \geq 0 \),
\[ \mathbb{E}\zeta^p = p \int_0^\infty \lambda^{p-1} P(\zeta > \lambda) d\lambda = p \left( \int_0^\eta + \int_\eta^\infty \right) \lambda^{p-1} P(\zeta > \lambda) d\lambda \]
\[ \leq p \int_0^\eta \lambda^{p-1} d\lambda + pC \int_0^T \mathbb{E}|\xi|^{\alpha} ds \int_\eta^\infty \lambda^{-\alpha - 1} d\lambda \]
\[ \leq \eta^p + \frac{Cp}{p - \alpha} \int_0^T \mathbb{E}|\xi|^{\alpha} ds. \]

Taking \( \eta = \left( \int_0^T \mathbb{E}|\xi|^{\alpha} ds \right)^{\frac{1}{\alpha}} \), we obtain (3.1). \( \square \)

Below we prove a substitution formula about stochastic integrals with respect to the subordinated Brownian motion \( W_{S_s} \).

Proposition 3.3. Assume that \( \xi(W_S) \) is a bounded and left continuous \((\mathcal{F}_t)\)-adapted \( \mathbb{R}^d \)-valued process. Then for any \( T \geq 0 \), we have
\[ \int_0^T \xi_s(W_S) dW_{S_s} = \int_0^T \xi_s(W_\ell) dW_{\ell} \bigg|_{\ell = S} \quad P - a.s. \quad (3.2) \]

Moreover, for any nonnegative random variable \( g \) on \( \mathbb{S} \) and \( p > 0 \), we have
\[ \mathbb{E} \left( g(S) \sup_{\tau \in [0,T]} \left| \int_0^\tau \xi_s(W_S) dW_{S_s} \right|^p \right) \leq C_p \int_\mathbb{S} g(\ell) \mathbb{E}^{\mu_\mathbb{S}} \left( \int_0^T |\xi_s(W_\ell)|^2 d\ell_s \right)^{\frac{p}{2}} \mu_\mathbb{S}(d\ell). \quad (3.3) \]
Thus, by (3.3) we have
\[ \xi^n_s(W) := \xi^n_0(W)1_{[0]}(s) + \sum_{i=0}^{n-1} \xi^n_s(W_{t_{i+1}})1_{(t_i, t_{i+1}]}(s). \]
Then, by definition we have
\[ \int_0^1 \xi^n_s(W) \, dW_s = \sum_{i=0}^{n-1} \xi^n_s(W_{t_{i+1}}) - W_{t_i} \]
\[ = \sum_{i=0}^{n-1} \xi^n_s(W_{t_{i+1}}) - W_{t_i} \bigg|_{t_{i+1} = S} = \int_0^1 \xi^n_s(W_t) \, dW_t \bigg|_{t = S}, \]
and by the left continuity of \( s \mapsto \xi_s(W) \),
\[ \lim_{n \to \infty} |\xi^n_s(W) - \xi_s(W)| = 0, \quad s \geq 0. \tag{3.4} \]
For \( p \in (0, \alpha) \), by (3.1) we have
\[ \mathbb{E} \left| \int_0^1 (\xi^n_s(W) - \xi_s(W)) \, dW_s \right|^p \leq C \int_0^1 \mathbb{E} \left| \xi^n_s(W) - \xi_s(W) \right|^p \, ds \to 0. \]
On the other hand, by Lemma [3.1] for each \( \ell \in \mathbb{S} \) and \( s \geq 0 \), \( \xi_s(W_\ell) \) is \( (\mathcal{F}_\ell) \)-adapted. Thus, by (2.11) we have
\[ \mathbb{E} \left| \int_0^1 (\xi^n_s(W_\ell) - \xi_s(W_\ell)) \, dW_\ell \right|_{t = S}^p \leq C \int_0^1 \mathbb{E} \left| \xi^n_s(W_\ell) - \xi_s(W_\ell) \right|^p \, \mu_{2}(d\ell) \]
\[ = C \mathbb{E} \left( \int_0^1 |\xi^n_s(W) - \xi_s(W)|^2 \, dS \right)^{\frac{p}{2}}. \]
which converges to zero by (3.4) and the dominated convergence theorem. Combining the above calculations, we obtain (3.2). Lastly, (3.3) is an easy consequence of (3.2) and (2.11).

We are now in a position to give

**Proof of Theorem 1.1** Formula (1.6) follows by (2.1) and (3.2). We now prove gradient estimate (1.7). By Hölder’s inequality, we have
\[ |\nabla_h \mathbb{E}(X_t(x))| \leq \mathbb{E} \left( \frac{1}{S_t} \left| f(X_t(x)) \int_0^t \sigma^{-1} \cdot \nabla_h X_s(x) \, dW_s \right| \right) \]
\[ \leq \left( \mathbb{E}|f(X_t(x))|^p \right)^{\frac{1}{p}} \left( \mathbb{E} \left( \frac{1}{S_t} \int_0^t \sigma^{-1} \cdot \nabla_h X_s(x) \, dW_s \right)^q \right)^{\frac{1}{q}}, \]
where \( \frac{1}{p} + \frac{1}{q} = 1 \). By (1.4), it is easy to see that
\[ |\nabla_h X_t(x)| \leq |h| e^{\|b\|_{\infty} \cdot t}. \]
Thus, by (3.3) we have
\[ \mathbb{E} \left( \frac{1}{S_t} \int_0^t \sigma^{-1} \cdot \nabla_h X_s(x) \, dW_s \right)^q \leq C_q \int_{\mathbb{P}_t} \mathbb{E} \mu^{x}( \int_0^t |\sigma^{-1} \cdot \nabla_h X_s(x)|^2 \, d\ell_s ) \frac{q}{2} \mu_{2}(d\ell). \]
Theorem 4.1. Our aim of this section is to prove that

\[ S_{1} \text{an orthogonal basis} \]

Recalling that the distributional density of \( \alpha \)-stable subordinator satisfies (cf. [5, (14)])

\[
P \circ S_{t}^{-1}(ds) \leq C t s^{-\frac{\alpha}{2}} e^{-t s^{-\frac{\alpha}{2}}} ds,
\]

we have

\[
\int_{S} \frac{\mu_{S}(df)}{t^{q/2}} = \mathbb{E} \left( \frac{1}{S_{t}^{q/2}} \right) \leq C \int_{0}^{\infty} t s^{-1-\frac{\alpha}{2}} e^{-t s^{-\frac{\alpha}{2}}} ds = C t \int_{0}^{\infty} u^{\frac{\alpha}{2}} e^{-u} du,
\]

where the last equality is due to the change of variable \( u = ts^{-\frac{\alpha}{2}} \), and \( C \) only depends on \( \alpha, q \). Combining the above calculations, we obtain (1.7).

4. Strong Feller property of SPDEs driven by subordinated cylindrical Brownian motions

Let \( \mathbb{H} \) be a real separable Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_{\mathbb{H}} \). The norm in \( \mathbb{H} \) is denoted by \( \| \cdot \|_{\mathbb{H}} \). Let \( A \) be a negative self-adjoint operator in \( \mathbb{H} \) with discrete spectrals, i.e., there exists an orthogonal basis \( \{e_{k}\}_{k \in \mathbb{N}} \) and a sequence of real numbers \( 0 < \lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{k} \to \infty \) such that

\[ Ae_{k} = -\lambda_{k} e_{k}. \]

Let \( \{W_{k}^{t}, t \geq 0\}_{k \in \mathbb{N}} \) be a sequence of independent standard 1-dimensional Brownian motion. Let \( \{L_{t}\}_{t \geq 0} \) be the subordinated cylindrical Brownian motion in \( \mathbb{H} \) defined by

\[ L_{t} := \sum_{k=1}^{\infty} \beta_{k} W_{S_{t}}^{k} e_{k}, \quad \beta_{k} \in \mathbb{R}, \]

where \( S_{t} \) is an independent \( \alpha/2 \)-stable subordinator.

Consider the following SPDE in Hilbert space \( \mathbb{H} \):

\[
dx_{t}(x) = [Ax_{t}(x) + F(X_{t}(x))]dt + dL_{t}, \quad X_{0} = x \in \mathbb{H}. \tag{4.1}
\]

Our aim of this section is to prove that

**Theorem 4.1.** Assume that for some \( \delta > 0 \),

\[ \beta_{k} \geq \delta, \quad \forall k \in \mathbb{N} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\beta_{k}^{2}}{\lambda_{k}} < +\infty, \tag{4.2} \]

and one of the following conditions holds:

(i) \( \alpha \in (1, 2) \) and \( F : \mathbb{H} \to \mathbb{H} \) is Lipschitz continuous;

(ii) \( \alpha \in (0, 2) \) and \( F : \mathbb{H} \to \mathbb{H} \) is bounded and Lipschitz continuous.

Then there exists a constant \( C = C(\alpha) > 0 \) such that for any bounded Borel measurable function \( f : \mathbb{H} \to \mathbb{R}, x, y \in \mathbb{H} \) and \( t > 0 \),

\[
|\mathbb{E} f(X_{t}(x)) - \mathbb{E} f(X_{t}(y))| \leq C \delta e^{\|F\|_{\text{Lip}} t^{-\frac{1}{2}}} \|f\|_{\infty} \|x - y\|_{\mathbb{H}}, \tag{4.3}
\]

where \( \|F\|_{\text{Lip}} := \sup_{x \neq y} \frac{\|F(x) - F(y)\|_{\mathbb{H}}}{\|x - y\|_{\mathbb{H}}} \).

Let us first prove a result about the following stochastic convolution (cf. [15, Theorem 4.5]):

\[
Z^{k}_{t} := \int_{0}^{t} e^{A(t-s)} dL_{s} = \sum_{k=1}^{\infty} \beta_{k} \int_{0}^{t} e^{-\lambda_{k}(t-s)} dW_{S_{t}}^{k} e_{k}.
\]
Proposition 4.2. If \( \sum_{k=1}^{\infty} \beta_k^2 / \lambda_k < +\infty \), then for any \( p \in (0, \alpha) \) and \( t \geq 0 \), we have

\[
\mathbb{E}[|Z_t^A|^p] \leq C_{\alpha, p} \left( \sum_{k=1}^{\infty} \frac{\beta_k^2}{\lambda_k} \right)^{\frac{p}{2}} t^{1-\frac{p}{2}}.
\]  

(4.4)

Proof. Recall the following Gaussian formula (cf. [13, Appendix A.1]): Let \( \{\xi_k\}_{k \in \mathbb{N}} \) be a sequence of independent random variables defined on some probability space \((\Omega', \mathcal{F}', P')\) with normal distribution \( N(0, 1) \), and \( \{c_k\}_{k \in \mathbb{N}} \) a sequence of real numbers. Then

\[
E' \left[ \sum_{k=1}^{\infty} c_k \xi_k \right]^p = A_p \left( \sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{p}{2}}, \quad A_p := \int_{\mathbb{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.
\]

Using this formula and by Fubini’s theorem and (3.1), we have

\[
A_p \mathbb{E}[|Z_t^A|^p] = A_p \mathbb{E} \left( \sum_{k=1}^{\infty} \beta_k \int_0^t e^{-\lambda_k (t-s)} dW^k_S \right)^2 \leq \mathbb{E} \left( \sum_{k=1}^{\infty} \xi_k \beta_k \int_0^t e^{-\lambda_k (t-s)} dW^k_S \right)^p \leq CE' \int_0^t \left( \sum_{k=1}^{\infty} |\xi_k \beta_k|^2 e^{-2\lambda_k (t-s)} \right)^{\frac{p}{2}} ds \leq C \left( \sum_{k=1}^{\infty} \frac{\beta_k^2}{\lambda_k} \right)^{\frac{p}{2}} t^{1-\frac{p}{2}}.
\]

The proof is finished. \( \Box \)

We now establish the following existence and uniqueness of mild solutions to equation (4.1).

Proposition 4.3. Assume that \( F : \mathbb{H} \rightarrow \mathbb{H} \) is Lipschitz continuous and \( \sum_{k=1}^{\infty} \beta_k^2 / \lambda_k < +\infty \). Then for each \( x \in \mathbb{H} \), there exists a unique \( X_t(x) \in \mathbb{H} \) satisfying

\[
X_t = e^{At} x + \int_0^t e^{A(t-s)} F(X_s) ds + Z_t^A.
\]  

(4.5)

Proof. Consider the following deterministic equation:

\[
Y_t = e^{At} x + \int_0^t e^{A(t-s)} F(Y_s + Z_t^A) ds.
\]

Using the standard Picard’s iteration, it is easy to see that there exists a unique \( Y \in C([0, \infty); \mathbb{H}) \) satisfying the above equation. Thus, \( X_t = Y_t + Z_t^A \) satisfies (4.5). \( \Box \)

Let \( \mathbb{H}_n := \{ x = \sum_{k=1}^n c_k e_k, c_k \in \mathbb{R} \} \) and \( \Pi_n \) the projection operator from \( \mathbb{H} \) to \( \mathbb{H}_n \) defined by

\[
\Pi_n x := \sum_{k=0}^{n} \langle x, e_k \rangle_{\mathbb{H}} e_k.
\]

Let \( \rho_n \) be a sequence of nonnegative smooth functions with

\[
\text{supp}(\rho_n) \subset \{ z \in \mathbb{H}_n : |z| \leq 1/n \}, \quad \int_{\mathbb{H}_n} \rho_n(z) dz = 1.
\]

Define

\[
F_n(x) := \int_{\mathbb{H}_n} \rho_n(\Pi_n x - z) \Pi_n F(z) dz
\]

and

\[
L^n_t := \Pi_n L_t = \sum_{k=1}^{n} \beta_k W^k_{S, e_k}.
\]
Consider the following finite dimensional approximation of equation (4.1):
\[ dX^n_t = [\Pi_n A X^n_t + F_n(X^n_t)]dt + dL^n_t, \quad X^n_0 = \Pi_n x. \]

**Lemma 4.4.** Under the assumptions of Theorem 4.1, for any fixed \( t > 0 \) and \( x \in \mathbb{H} \), we have
\[ \lim_{n \to \infty} ||X^n_t(\Pi_n x) - X_t(x)||_{\mathbb{H}} = 0, \quad P - a.s. \quad (4.6) \]

**Proof.** By Duhamel’s formula, one can write
\[ X^n_t = e^{At} \Pi_n x + \int_0^t e^{A(t-s)} F_n(X^n_s)ds + \Pi_n Z^n_t. \]

Set
\[ Y^n_t := X^n_t - \Pi_n Z^n_t, \quad Y_t := X_t - Z^A_t. \]

Then
\[ Y^n_t - Y_t = e^{At}(\Pi_n x - x) + \int_0^t e^{A(t-s)}(F_n(Y^n_s + \Pi_n Z^n_s) - F(Y_s + Z^A_s))ds. \]

Hence,
\[ ||Y^n_t - Y_t||_{\mathbb{H}} \leq ||\Pi_n x - x||_{\mathbb{H}} + \int_0^t ||F_n(Y^n_s + \Pi_n Z^n_s) - F(Y_s + Z^A_s)||_{\mathbb{H}} ds. \]

Notice that
\[ ||F_n(Y^n_s + \Pi_n Z^n_s) - F(Y_s + Z^A_s)||_{\mathbb{H}} \leq ||F||_{\text{Lip}}(||Y^n_s - Y_s||_{\mathbb{H}} + ||(\Pi_n - I)Z^A_s||_{\mathbb{H}}) \]
\[ + ||(\Pi_n - I)F(Y_s + Z^A_s)||_{\mathbb{H}} \]
and
\[ \lim_{n \to \infty} ||(\Pi_n - I)Z^A_s||_{\mathbb{H}} = 0, \quad \lim_{n \to \infty} ||(\Pi_n - I)F(Y_s + Z^A_s)||_{\mathbb{H}} = 0. \]

(i) If \( \alpha \in (1, 2) \) and \( F \) is Lipschitz continuous, by (4.4) and the dominated convergence theorem, we have
\[ \lim_{n \to \infty} ||Y^n_t - Y_t||_{\mathbb{H}} = 0. \quad (4.7) \]
which then gives
\[ \lim_{n \to \infty} ||Y^n_t - Y_t||_{\mathbb{H}} = 0. \quad (4.8) \]
(ii) If \( \alpha \in (0, 2) \) and \( F \) is bounded, by Fatou’s lemma, we also have (4.7) and (4.8). \( \square \)

**Proof of Theorem 4.1.** For any function \( f \in C^1_b(\mathbb{H}) \), by (1.7) with \( p = \infty \), there exists a constant \( C = C(\alpha) > 0 \) such that for all \( x, y \in \mathbb{H} \) and \( t > 0 \),
\[ ||\mathbb{E} f(X^n_t(\Pi_n x)) - \mathbb{E} f(X^n_t(\Pi_n y))|| \leq C\delta ||f||_{\infty} e^{\|\Pi_n x - Y^n_t\|_{\mathbb{H}}} \]
\[ \leq C\delta ||f||_{\infty} e^{\|\Pi_n y - Y^n_t\|_{\mathbb{H}}} \]
\[ < \infty. \]
By taking limits for (4.9), we get (4.3) for any \( f \in C^1_b(\mathbb{H}) \). For general bounded measurable \( f \), it follows by a standard approximation (see [7] p.125, Lemma 7.1.5)).

**Example:** Consider the following nonlinear stochastic heat equation in \([0, 1]\) with Dirichlet boundary conditions:
\[ \partial_t u = [\Delta u + b(u)]dt + dL_t, \quad u_t(0) = u_t(1) = 0, \quad u_0 = \varphi, \]
where \( b : \mathbb{R} \to \mathbb{R} \) is a Lipschitz function, and \( \varphi \in L^2([0, 1]) =: \mathbb{H} \). It is well-known that \( A = \Delta \) is a negative self-adjoint operator on \( \mathbb{H} \) with eigenvalues
\[ \lambda_k = \pi^2 k^2, \quad k \in \mathbb{N}, \]
\[ \lambda_k \]
and eigenvectors
\[ e_k(\zeta) = \sqrt{2} \sin(\pi k\zeta), \quad \zeta \in [0, 1]. \]
In particular,
\[ \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k} < +\infty. \]
Thus, if one takes \( \beta_k = 1 \), then (4.2) holds.

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