LOGARITHM LAWS FOR UNIPOTENT FLOWS
ON HYPERBOLIC MANIFOLDS

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ABSTRACT. We prove logarithm laws for unipotent flows on non-compact finite-volume hyperbolic manifolds. Our method depends on the estimate of norms of certain incomplete Eisenstein series.

1. INTRODUCTION

Let $G$ denote a connected real semisimple Lie group with no compact factors and $\Gamma \subset G$ be a non-uniform irreducible lattice, that is, $\Gamma$ is discrete, the homogeneous space $\Gamma \backslash G$ is non-compact and has finite co-volume with respect to the Haar measure of $G$. Let $\sigma$ denote the right $G$-invariant probability measure on $\Gamma \backslash G$. Any unbounded one-parameter subgroup $\{g_t\}_{t \in \mathbb{R}} \subset G$ acts on $\Gamma \backslash G$ by right multiplication. By Moore’s Ergodicity Theorem this action is ergodic with respect to $\sigma$, hence for $\sigma$-a.e. $x \in \Gamma \backslash G$ the orbit $\{xg_t\}$ is dense. In particular, these orbits will make excursions into the cusp(s) of $\Gamma \backslash G$. A natural question to ask is at what rate these cusp excursions occur.

We first fix some notations throughout this paper. We write $A \sim B$ if there is some constant $c > 1$ such that $\frac{1}{c} A \leq B \leq c A$. And we write $A \lesssim B$ or $A = O(B)$ to indicate that $A \leq cB$ for some positive constant $c$. We will use subscripts to indicate the dependence of the constant on some parameters.

The above question can be restated as a shrinking target problem. Let $K$ be a maximal compact subgroup of $G$, $\Gamma \backslash G$ has a naturally defined distance function, $\text{dist}$, induced from a left $G$-invariant and bi-$K$-invariant Riemannian metric on $G$. For a fixed $o \in \Gamma \backslash G$ we define the cusp neighborhoods by

$$B_r := \{ x \in \Gamma \backslash G \mid \text{dist}(o,x) > r \}$$

for any $r > 0$. By [13] there exists a constant $\varkappa > 0$ such that $\sigma(B_r) \approx e^{-\varkappa r}$. For $\{r_\ell\}$ a sequence of positive real numbers with $r_\ell \to \infty$, consider the family of shrinking cusp neighborhoods $\{B_{r_\ell}\}$, we define a corresponding sequence of random variables on $\Gamma \backslash G$ by

$$X_\ell (x) := \begin{cases} 1 & \text{if } xg_\ell \in B_{r_\ell} \\ 0 & \text{otherwise.} \end{cases}$$
Note that $X_\ell(x) = 1$ if and only if the $\ell$th orbit of $x$ makes excursion into the $\ell$th cusp neighborhood $B_{r_\ell}$. In this setting, one can vary the sequence $\{r_\ell\}$ to enlarge or shrink the family of cusp neighborhoods $\{B_{r_\ell}\}$, and then ask whether the events $X_\ell(x) = 1$ happen finitely or infinitely many times for a generic point $x$. We note that the first half of Borel-Cantelli lemma implies that if $\sum_{\ell=1}^\infty \rho(B_{r_\ell}) < \infty$, then for $\sigma$-a.e. $x \in \Gamma \backslash G$, the events $X_\ell(x) = 1$ happen for finitely many $\ell$. Thus $\limsup_{\ell \to \infty} \frac{\text{dist}(o, xg_\ell)}{r_\ell} \leq 1$ for $\sigma$-a.e. $x \in \Gamma \backslash G$. In particular, for any small positive number $\epsilon$, choosing $r_\ell = \frac{(1+\epsilon) \log(\ell)}{x}$, a standard continuity argument implies that $\limsup_{t \to \infty} \frac{\text{dist}(o, xg_\ell)}{\log(t)} \leq \frac{1}{x}$ for $\sigma$-a.e. $x \in \Gamma \backslash G$. Letting $\epsilon \to 0$, we get $\limsup_{t \to \infty} \frac{\text{dist}(o, xg_\ell)}{\log(t)} = \frac{1}{x}$ for $\sigma$-a.e. $x \in \Gamma \backslash G$. If the bound is sharp, for $\sigma$-a.e. $x \in \Gamma \backslash G$, we say that the flow $\{g_\ell\}_{\ell \in \mathbb{R}}$ satisfies the logarithm law. Following [2], we say a sequence of cusp neighborhoods $\{B_{r_\ell}\}$ is Borel-Cantelli for $\{g_\ell\}$ if $\sum_{\ell=1}^\infty \rho(B_{r_\ell}) = \infty$ and for $\sigma$-a.e. $x \in \Gamma \backslash G$, $X_\ell(x) = 1$ for infinitely many $\ell$. Note that $\{g_\ell\}$ satisfying logarithm law is equivalent to the statement that for any $\epsilon > 0$, any sequence of cusp neighborhoods $\{B_{r_{1,\ell}}\}$ with $\rho(B_{r_{1,\ell}}) \approx \frac{1}{\ell^{2-\epsilon}}$ is Borel-Cantelli for $\{g_\ell\}$.

The problem of logarithm laws in the context of homogeneous space was first studied by Sullivan [18] where he proved logarithm laws for geodesic flows on non-compact finite-volume hyperbolic manifolds. The general case of one-parameter diagonalizable flows on non-compact finite-volume homogeneous spaces was proved by Kleinbock and Margulis [13]. The main ingredient of their proof is the exponential decay of matrix coefficients of diagonalizable flows, from which they deduced the quasi-independence of the above events $X_\ell$. Then the logarithm law follows from a quantitative Borel-Cantelli lemma.

The problem of logarithm laws for unipotent flows is more subtle since the matrix coefficients of unipotent flows only decay polynomially. Nevertheless, using a random analogy of Minkowski’s theorem Athreya and Margulis [4] proved logarithm laws for one-parameter unipotent subgroups on the space of lattices $X_d := SL_d(\mathbb{Z}) \backslash SL_d(\mathbb{R})$. Later Kelmer and Mohammadi [12] proved the case when $G$ is a product of copies of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ and $\Gamma$ is any irreducible non-uniform lattice. We note that in the above two cases, their methods are closely related and both rely on the estimate of $L^2$-norms of certain transform functions.

In [3], Athreya studied the cusp excursion of the full horospherical group with respect to some one-parameter diagonalizable subgroup on $X_d$. Surprisingly, he was able to relate the cusp excursion rates for diagonalizable and horospherical actions and certain Diophantine properties for every $x \in X_d$. In particular, his result implies logarithm laws for unipotent flows on $X_d$. The most general result known for unipotent flows was obtained by Athreya and Margulis [5]. More precisely, for $G$ a semisimple Lie group without compact factors, $\Gamma \subset G$
an irreducible non-uniform lattice in $G$ and $\{g_t\}_{t \in \mathbb{R}}$ a one-parameter unipotent subgroup in $G$, they proved that for any $o \in \Gamma \backslash G$ and $\sigma$-a.e. $x \in \Gamma \backslash G$, there exists $0 < \beta \leq 1$ such that $\limsup_{t \to \infty} \frac{\text{dist}(o, x g_t)}{\log t} = \frac{\beta}{2}$. Moreover, they asked whether such $\beta$ can always attain 1, which is the upper bound coming from the first half of Borel-Cantelli lemma.

In this paper, we generalize the approach in [4] and [12] to give a positive answer to this question when $\Gamma \backslash G$ is the frame bundle of hyperbolic manifolds. Before stating our main result, we first fix some notations. Let $\mathbb{H}^{n+1}$ be the $(n+1)$-dimensional real hyperbolic space with $n \geq 2$ and $\text{Iso}^+(\mathbb{H}^{n+1})$ denote the orientation preserving isometry group of $\mathbb{H}^{n+1}$. Fix a maximal compact subgroup $K$ and identify $G/K$ with $\mathbb{H}^{n+1}$.

**Theorem 1.1.** Let $G = \text{Iso}^+(\mathbb{H}^{n+1})$ with $n \geq 2$, $\Gamma \subset G$ a non-uniform lattice and $\{g_t\}_{t \in \mathbb{R}}$ a one-parameter unipotent subgroup of $G$. Let $\text{dist}(\cdot, \cdot)$ denote the distance function obtained from hyperbolic metric on the hyperbolic manifold $\Gamma \backslash \mathbb{H}^{n+1}$. Then for any fixed $o \in \Gamma \backslash G$,

\[
\limsup_{t \to \infty} \frac{\text{dist}(o, x g_t)}{\log t} = \frac{1}{n},
\]

for $\sigma$-a.e. $x \in \Gamma \backslash G$.  

We give a brief outline of our proof here. We first note that if (1.1) holds for $\Gamma$, then it also holds for any $\Gamma'$ conjugate to $\Gamma$ (see Section 5). Hence after suitable conjugation we can assume $\Gamma$ has a cusp at $\infty$ (see Section 2.3 for the definition of cusps).

The upper bound, as mentioned above, follows from the first half of the Borel-Cantelli lemma. For the lower bound, we first note that it suffices to show that the set

\[\mathcal{A}_\epsilon := \left\{ x \in \Gamma \backslash G \mid \limsup_{t \to \infty} \frac{\text{dist}(o, x g_t)}{\log t} \geq \frac{1 - \epsilon}{n} \right\}\]

has positive measure for any $\epsilon > 0$. This is because $\mathcal{A}_\epsilon$ is invariant under the action of $\{g_t\}_{t \in \mathbb{R}}$, hence by ergodicity, if $\mathcal{A}$ is of positive measure it must have full measure. Then the theorem follows by letting $\epsilon$ approach zero.

Next, in order to show that $\mathcal{A}_\epsilon$ has positive measure, we construct a subset $\mathcal{B}_\epsilon \subset \mathcal{A}_\epsilon$ which we show has positive measure. To describe our construction, we need some additional notations. Fix an Iwasawa decomposition $G = NAK$ with the maximal unipotent subgroup $N$ fixing $\infty$. Let $M \subset K$ be the centralizer of $A$ in $K$, $P = NAM$ the stabilizer of $\infty$ in $G$ and $\Gamma_\infty = \Gamma \cap P$ the stabilizer of $\infty$ in $\Gamma$. Let $Q = NM$ be the maximal subgroup of $P$ containing $\Gamma_\infty$ such that $\Gamma_\infty \backslash Q$ is relatively compact. See Section 2.1 for explicit descriptions of these groups. For any $\mathcal{D} \subset Q \backslash G$ we let

\[Y_\mathcal{D} = \{ \Gamma g \in \Gamma \backslash G \mid Q \gamma g \in \mathcal{D} \text{ for some } \gamma \in \Gamma \} .\]

Here by abuse of notation, for $o, x g_t \in \Gamma \backslash G$, we write $\text{dist}(o, x g_t)$ for the distance between their projections to $\Gamma \backslash \mathbb{H}^{n+1}$.
In Section 5.1, for any $\epsilon > 0$ we construct a sequence of sets $\mathcal{D}_m \subset Q \setminus G$ explicitly by taking unions of certain translations of cusp neighborhoods and we show that $\{\sigma (Y_{\mathcal{D}_m})\}_{m \in \mathbb{N}}$ is uniformly bounded from below and each $Y_{\mathcal{D}_m}$ satisfies

$$\forall x \in Y_{\mathcal{D}_m} \exists \ell \geq m \text{ such that } \frac{\text{dist}(x, xg_{\ell})}{\log \ell} \geq \frac{1-\epsilon}{n}. \tag{1.2}$$

By (1.2) it is clear that the limit superior set $\mathcal{B}_\epsilon := \cap_{\ell=1}^{\infty} \cup_{m=1}^{\infty} Y_{\mathcal{D}_m}$ is contained in $\mathcal{A}_\epsilon$. Moreover, since $\{\sigma (Y_{\mathcal{D}_m})\}_{m \in \mathbb{N}}$ is uniformly bounded from below, $\mathcal{B}_\epsilon$ has positive measure. Hence $\mathcal{A}_\epsilon$ is of positive measure.

To show that $\{\sigma (Y_{\mathcal{D}_m})\}$ has a uniform lower bound, we find nice subsets $\mathcal{D}'_m \subset \mathcal{D}_m$ with $|\mathcal{D}'_m| = |\mathcal{D}_m|$ (here $|\cdot|$ denotes a right $G$-invariant measure on $Q \setminus G$) and we show $\{\sigma (Y_{\mathcal{D}'_m})\}$ is uniformly bounded from below. One standard way to handle $\sigma (Y_{\mathcal{D}'_m})$ to $\mathcal{D}'_m$. More precisely, for any compactly supported function $f$ on $Q \setminus G$ the corresponding incomplete Eisenstein series $\Theta_f \in L^2(G \setminus G)$ attached to $f$ is defined by

$$\Theta_f (g) = \sum_{\gamma \in \Gamma \gamma \Gamma} f(\gamma g).$$

Note that if $f$ is supported on $\mathcal{D}$, then $\Theta_f$ is supported on $Y_{\mathcal{D}}$. To show that $\{\sigma (Y_{\mathcal{D}'_m})\}$ is bounded from below, it is enough to show that the $L^2$-norm (with respect to the measure $\sigma$) of the incomplete Eisenstein series $\Theta_{1_{\mathcal{D}'_m}}$ is not too large compared to the measure of $\mathcal{D}'_m$, where $1_{\mathcal{D}'_m}$ is the characteristic function of $\mathcal{D}'_m$. To show this, we bound $||\Theta_{1_{\mathcal{D}'_m}}||_2$ in terms of $|\mathcal{D}'_m|$. In fact, for any parameter $\lambda > 0$ we define a family of functions $\mathcal{A}_\lambda \subset L^2(Q \setminus G)$ (see description of $\mathcal{A}_\lambda$ in Section 4.2) and we prove the following bound for functions in $\mathcal{A}_\lambda$.

**Theorem 1.2.** Let $G = \text{Iso}^+ (\mathbb{H}^{n+1})$ and $\Gamma \subset G$ a non-uniform lattice with a cusp at $\infty$. For any parameter $\lambda > 0$ there exists some constant $C$ (depending on $\Gamma$ and $\lambda$) such that

$$||\Theta_f||_2^2 \leq C (||f||_1^2 + ||f||_2^2) \tag{1.3}$$

for any $f \in \mathcal{A}_\lambda$, where the norms on the right are with respect to the right $G$-invariant measure on $Q \setminus G$.

Our construction of $\mathcal{D}'_m$ yields that we can take functions in $\mathcal{A}_{\lambda}$ (for some $\lambda$) to approximate $1_{\mathcal{D}'_m}$, then we can use Theorem 1.2 to bound $||\Theta_{1_{\mathcal{D}'_m}}||_2^2$ in terms of $|\mathcal{D}'_m|$. We note that our strategy of proving (1.3) is similar to the one used in [13]. To prove Theorem 1.2, we work out an explicit constant term formula for certain non-spherical Eisenstein series (for arbitrary $n \geq 2$). With this constant term formula, a formal computation ensures that we can bound the $L^2$-norm of any incomplete Eisenstein series by the right-hand side of (1.3), together with a third term expressed in terms of the exceptional poles of Eisenstein series. Thus (1.3) follows if we can bound this third term by the right-hand side of (1.3). However, to prove this bound, we need to assume the functions are from $\mathcal{A}_{\lambda}$. 
**Remark 1.** An interesting question is whether (1.3) holds uniformly for any \( \mathcal{A}_f \). In particular, for our purpose if one can prove (1.3) uniformly for linear combinations of nonnegative functions in \( \mathcal{A}_f \), then the same method implies a stronger Borel-Cantelli law: every sequence of nested cusp neighborhoods \( \{ B_r \} \) with \( \sum_{r=1}^{\infty} \sigma(B_r) = \infty \) is Borel-Cantelli for unipotent flows. Such a result was obtained in [12, Remark 8] by proving (1.3) for all nonnegative functions in \( C_c^\infty(Q) \) when \( G \) is a product of copies of \( SL_2(\mathbb{R}) \left( \cong Iso^+(\mathbb{H}^2) \right) \) and \( SL_2(\mathbb{C}) \left( \cong Iso^+(\mathbb{H}^3) \right) \) and \( \Gamma \) is any arithmetic irreducible lattice. Their proof of (1.3) is indirect and depends on the existence of a family of lattices for which the Eisenstein series have no exceptional poles. We note that in [11] Gritsenko gave an example of such a lattice in \( Iso^+(\mathbb{H}^4) \). Hence using the general constant term formula we get, one can prove the above Borel-Cantelli law (for unipotent flows) for this specific lattice (and its commensurable lattices) in \( Iso^+(\mathbb{H}^4) \).

**Remark 2.** We end the introduction by remarking that the sets \( \mathcal{D}_m \) are constructed by taking unions of translations of the neighborhoods at the cusp \( \infty \) (along the unipotent flow), hence our method implies a slightly stronger result: logarithm laws for excursions of unipotent flows into any individual cusp (for other cusps, the result can be obtained by conjugating this cusp to \( \infty \)).

### 2. Preliminaries and notations

#### 2.1. Vahlen group.** Let \( \mathbb{H}^{n+1} \) denote the \((n+1)\)-dimensional real hyperbolic space and \( G = Iso^+(\mathbb{H}^{n+1}) \) be its orientation preserving isometry group. There are various hyperbolic models of \( \mathbb{H}^{n+1} \) and each model gives an explicit description of \( G \). In this paper, we choose the upper half space model and realize \( G \) via the Vahlen group (see [1] [7] and [8] for more details about Vahlen group).

We first briefly recall some facts about Clifford algebra. The Clifford algebra \( C_n \) is an associative algebra over \( \mathbb{R} \) with \( n \) generators \( e_1, \ldots, e_n \) satisfying relations \( e_i^2 = -1, e_i e_j = -e_j e_i \). Let \( \mathcal{P}_n \) be the set of subsets of \( \{1, \ldots, n\} \). For \( I \in \mathcal{P}_n \), \( I = \{i_1, \ldots, i_r\} \) with \( i_1 < \cdots < i_r \), we define \( e_I := e_{i_1} \cdots e_{i_r} \) and \( e_{\emptyset} = 1 \). These \( 2^n \) elements \( e_I \) \((I \in \mathcal{P}_n)\) form a basis of \( C_n \). The Clifford algebra \( C_n \) has a main anti-involution \( \ast \) and a main involution \( \cdot \). Explicitly, their actions on the basis elements are given by \( (e_{i_1} \cdots e_{i_r})^* = e_{i_r} \cdots e_{i_1} \) and \( (e_{i_1} \cdots e_{i_r})' = (-1)^r e_{i_r} \cdots e_{i_1} \). Their composition \( \bar{e}_I := (e_I^*)^* \) gives the conjugation map on \( C_n \).

For any \( 1 \leq i \leq n \), let \( \mathbb{V}^i \) denote the real vector space spanned by \( 1, e_1, \ldots, e_i \). Note that \( \dim \mathbb{V}^i = i + 1 \). The Clifford group \( T_i \) is defined to be the collection of all finite products of non-zero elements from \( \mathbb{V}^i \) with group operation given by multiplication. There is a well-defined norm on \( \mathbb{V}^n \) given by \( |v| = \sqrt{v \bar{v}} \) and it extends multiplicatively to a norm on \( T_i \).

In this setting, the \((n+1)\)-dimensional hyperbolic space model is the upper half space

\begin{equation}
\mathbb{H}^{n+1} := \{ x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{V}^n \mid x_i \in \mathbb{R}, x_n > 0 \}
\end{equation}
endowed with the Riemannian metric

\begin{equation}
 ds^2 = \frac{dx_0^2 + \cdots + dx_n^2}{x_n^2}.
\end{equation}

Let $M_2(C_n)$ be the set of $2 \times 2$ matrices over $C_n$. The Vahlen group $SL(2,T_{n-1})$ is defined by

\[
 SL(2,T_{n-1}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(C_n) \mid a,b,c,d \in T_{n-1} \cup \{0\}, \begin{array}{l} ab^* + cd^* \in V^{n-1}, \\ \text{ad}^* - bc^* = 1 \end{array} \right\}.
\]

An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,T_{n-1})$ acts on $\mathbb{H}^{n+1}$ as an isometry via the Möbius transformation

\begin{equation}
 g \cdot v = (av + b)(cv + d)^{-1}.
\end{equation}

This gives a surjective homomorphism from $SL(2,T_{n-1})$ to $\text{Iso}^+ (\mathbb{H}^{n+1})$ with kernel $\pm I_2$. Hence $G$ is realized as $PSL(2,T_{n-1}) := SL(2,T_{n-1}) / \{ \pm I_2 \}$. Here $I_2$ is the $2 \times 2$ identity matrix.

We fix an Iwasawa decomposition

\[ PSL(2,T_{n-1}) = NAK, \]

with

\[ N = \left\{ u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} \in V^{n-1} \right\}, \]

\[ A = \left\{ a_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}, \]

and

\[ K = \{ g \in SL(2,T_{n-1}) \mid g \cdot e_n = e_n \} / \{ \pm I_2 \} \]

is the stabilizer of $e_n$. From this we can identify $G/K$ with $\mathbb{H}^{n+1}$ by sending $gK$ to $g \cdot e_n$.

An element in $K$ is of the form $\begin{pmatrix} q_2^* & -q_1^* \\ q_1 & q_2 \end{pmatrix}$ with $|q_1|^2 + |q_2|^2 = 1$ and $q_1 q_2^* \in V^{n-1}$. Let $M$ be the centralizer of $A$ in $K$, that is, $M$ is the subgroup of $K$ consisting of diagonal matrices. For later use we note that $K$ is isomorphic to $SO(n+1)$, $M$ is isomorphic to $SO(n)$ and $M \backslash K$ can be identified with the $n$-sphere $S^n$ via the map

\[ M \backslash K \longrightarrow S^n := \{ x_0 + x_1 e_1 + \cdots + x_n e_n \mid x_j \in \mathbb{R}, x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \} \]

\[ \begin{pmatrix} q_2^* & -q_1^* \\ q_1 & q_2 \end{pmatrix} \longrightarrow 2q_1^* q_2 + (|q_2|^2 - |q_1|^2) e_n. \]

Here this map is well-defined since $q_1 q_2^* \in V^{n-1}$ if and only if $q_1^* q_2 \in V^{n-1}$ ([16, Corollary 7.15]).
2.2. **Coordinates and normalization.** Let $G = NAK$ be the fixed Iwasawa decomposition as above and let $Q = NM$. Under the coordinates $g = u_xa_tk$, the Haar measure of $G$ is given by
\[ dg = e^{-nt}dxdt\,dk, \]
where $dx$ is the usual Lebesgue measure on $N$ (identified with $\mathbb{R}^n$), $dk$ is the probability Haar measure on $K$. Hence the probability Haar measure $\sigma = \sigma_\Gamma$ on $\Gamma \backslash G$ is given by
\[ d\sigma(g) = \frac{1}{\nu_\Gamma}e^{-nt}dxdt\,dk, \]
with $\nu_\Gamma = \int_{\mathcal{F}_\Gamma} dg$, where $\mathcal{F}_\Gamma$ is a fundamental domain for $\Gamma \backslash G$.

We normalize the measures on various spaces as following. First for $\phi \in L^2(M\backslash K)$, we view $\phi$ as a left $M$-invariant function on $K$ and normalize the Haar measure on $M\backslash K$ (also denoted by $dk$) such that
\[ \int_{M\backslash K} \phi(k)\,dk = \int_K \phi(k)\,dk. \]
Next we identify $Q\backslash G = A \times M\backslash K$ and we normalize the Haar measure on $Q\backslash G$ so that for any $f \in C_\infty^c(Q\backslash G)$ we have
\[ \int_{Q\backslash G} f(g)\,dg = \int_{\mathbb{R}}\int_{M\backslash K} f(a_tk)\,e^{-nt}dx\,dt\,dk. \]
We then normalize the Haar measure on $Q$ so that for any $f \in C_\infty^c(G)$ we have
\[ \int_G f(g)\,dg = \int_{Q\backslash G}\int_Q f(qg)\,dq\,dg. \]

2.3. **Cusps and reduction theory.** Fix the notations as above. Let $\partial \mathbb{H}^{n+1} := \{x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1} \in \mathbb{H}^{n-1} \mid x_i \in \mathbb{R}\} \cup \{\infty\}$ denote the boundary of $\mathbb{H}^{n+1}$. The action (2.3) extends naturally to $\partial \mathbb{H}^{n+1}$ by the same formula. Let $P = NAM$ be the subgroup of upper triangular matrices in $G$. We note that $P$ is the stabilizer of $\infty$. Let $\Gamma \subseteq G$ be a non-uniform lattice in $G$. Define
\[ \Gamma_\infty = \Gamma \cap P \]
and
\[ \Gamma'_\infty = \Gamma \cap N. \]
Note that $\Gamma_\infty$ is the stabilizer of $\infty$ in $\Gamma$, and $\Gamma'_\infty$ consists the identity and unipotent elements in $\Gamma_\infty$. We say that $\Gamma$ has a cusp at $\infty$ if $\Gamma'_\infty$ is nontrivial. Note that if $\Gamma$ has a cusp at $\infty$, then $\Gamma \backslash G$ having finite co-volume implies that $\Gamma'_\infty$ is a lattice (free abelian and of full rank) in $N$ (see [9, Definition 0.5 and Theorem 0.7]). Moreover, we note that by discreteness, $\Gamma_\infty \cap A = \{I_2\}$. Thus the conjugation action of $\Gamma_\infty$ on $N(= \mathbb{H}^{n-1})$ and $\Gamma'_\infty$ induces an injection
\[ \Gamma_\infty / \Gamma'_\infty \hookrightarrow SO(\mathbb{H}^{n-1}) \cap GL(\Gamma'_\infty). \]
Hence $\Gamma'_\infty$ is a finite index subgroup of $\Gamma_\infty$. Denote by $[\Gamma_\infty : \Gamma'_\infty]$ this index.
satisfies the following properties:

\[ G \text{ is left invariant and satisfies } \text{dist}\text{ to be their lifts to } \Gamma = \mathbb{H}^{n+1}. \]

The distance function.

For any \( \xi \in \partial \mathbb{H}^{n+1} \), there exists some \( g \in G \) such that \( g \cdot \xi = \infty \). We say \( \Gamma \) has a cusp at \( \xi \) if \( g \Gamma g^{-1} \) has a cusp at \( \infty \). And we say two cusps \( \xi, \xi' \) are \( \Gamma \)-equivalent if there exists some \( \gamma \in \Gamma \) such that \( \gamma \cdot \xi = \xi' \). Assume that \( \Gamma \) has a cusp at \( \infty \), define the lattice

\[ \mathcal{O}_\Gamma := \{ x \in \mathbb{H}^{n+1} \mid u_x \in \Gamma' \} \]

in \( \mathbb{H}^{n+1} \). Let \( \mathcal{F}_\mathcal{O}_\Gamma \subset \mathbb{H}^{n+1} \) be a fundamental domain for \( \mathcal{O}_\Gamma \). One easily sees that the set

\[ (2.7) \quad \mathcal{F}^\prime_\infty = \{ u_x a_t k \mid x \in \mathcal{F}_\mathcal{O}_\Gamma, t \in \mathbb{R}, k \in K \} \]

is a fundamental domain for \( \Gamma' \setminus G \). It contains \( [\Gamma_\infty : \Gamma^\prime_\infty] \) copies of \( \Gamma_\infty \setminus G \). For later use, we note that since \( \Gamma_\infty \) is a lattice in \( N \), \( \Gamma_\infty \setminus Q \) is relatively compact, hence

\[ \omega_\Gamma := \int_{\Gamma_\infty \setminus Q} dq \]

is finite.

For any \( \tau \in \mathbb{R} \), let us denote \( A(\tau) = \{ a_t \mid t \geq \tau \} \). Recall that a Siegel set is a subset of \( G \) of the form \( \Omega_{\tau, U} = U A(\tau) K \) where \( U \) is an open, relatively compact subset of \( N \). Since \( G \) is of real rank one, we can apply the reduction theory of Garland and Raghunathan ([9] Theorem 0.6). That is, there exists \( \tau_0 \in \mathbb{R} \), an open, relatively compact subset \( U_0 \subset N \), a finite set \( \Xi = \{ \xi_1, \cdots, \xi_h \} \subset G \) (corresponding to a complete set of \( \Gamma \)-inequivalent cusps) and an open, relatively compact subset \( \mathcal{C} \) of \( G \) such that the Siegel fundamental domain

\[ (2.8) \quad \mathcal{F}_{\tau, \tau_0, U_0} = \mathcal{C} \bigcup \left( \bigcup_{\xi_j \in \Xi} \xi_j \Omega_{\tau_0, U_0} \right) \]

satisfies the following properties:

1. \( \Gamma \mathcal{F}_{\tau, \tau_0, U_0} = G \);
2. the set \( \{ \gamma \in \Gamma \mid \gamma \mathcal{F}_{\tau, \tau_0, U_0} \cap \mathcal{F}_{\tau, \tau_0, U_0} \neq \emptyset \} \) is finite;
3. \( \gamma \xi_j \Omega_{\tau_0, U_0} \cap \xi_j \Omega_{\tau_0, U_0} = \emptyset \) for all \( \gamma \in \Gamma \) whenever \( \xi_i \neq \xi_j \in \Xi \).

In other words, the restriction to \( \mathcal{F}_{\tau, \tau_0, U_0} \) of the natural projection of \( G \) onto \( \Gamma \setminus G \) is surjective, at most finite-to-one, and the cusp neighborhood of each cusp of \( \Gamma \setminus G \) can be taken to be disjoint. We will fix this Siegel fundamental domain \( \mathcal{F}_{\tau, \tau_0, U_0} \) throughout the paper. For further use, we note that \( U_0 \) contains a fundamental domain of \( \Gamma_\infty \setminus N \).

2.4. The distance function. Fix a non-uniform lattice \( \Gamma \) in \( G \). Let \( \text{dist}_G \) and \( \text{dist} = \text{dist}_\Gamma \) denote the hyperbolic distance functions on \( G/K = \mathbb{H}^{n+1} \) and \( \Gamma \setminus G/K = \Gamma \setminus \mathbb{H}^{n+1} \) respectively. By slight abuse of notation, we also denote \( \text{dist}_G \) and \( \text{dist} \) to be their lifts to \( G \) and \( \Gamma \setminus G \) respectively. In particular, \( \text{dist}_G \) is left \( G \)-invariant and satisfies \( \text{dist}_G (I_2, a_t k) = t \) for any \( t \geq 0 \) and \( k \in K \), where \( I_2 \) is the identity matrix in \( G \). Moreover, for any \( g, h \in G \), \( \text{dist}_G \) and \( \text{dist} \) satisfy the relation

\[ \text{dist}(\Gamma g, \Gamma h) = \inf_{\gamma \in \Gamma} \text{dist}_G (g, \gamma h) \].
Clearly, $\dist(\Gamma g, \Gamma h) \leq \dist_G(g, h)$. Conversely, if $g, h$ are from the Siegel set $\Omega_{\tau_0, U_0}$, then there exists a constant $D$ such that $\dist_G(\xi_i g, \gamma \xi_j h) \geq \dist_G(g, h) - D$ for any $\xi_i, \xi_j \in \Xi$ and any $\gamma \in \Gamma$ (see [6, Theorem C]). In particular, this implies
\[
\dist(\Gamma \xi_j g, \Gamma \xi_j h) \geq \dist_G(g, h) - D
\]
for any $\xi_j \in \Xi$ and any $g, h \in \Omega_{\tau_0, U_0}$. We then have

**Lemma 2.1.** For $o \in \mathcal{F}_{\Gamma, \tau_0, U_0}$ fixed, there exists a constant $D'$ such that
\[
(2.9) \quad \dist_G(o, \xi_j g) - D' \leq \dist(o, \xi_j g) \leq \dist_G(o, \xi_j g)
\]
for any $\xi_j \in \Xi$ and any $g \in \Omega_{\tau_0, U_0}$.

**Remark 3.** We view $o, \xi_j g$ as elements in $\Gamma \backslash G$ when we write $\dist(o, \xi_j g)$, and as elements in $G$ when we write $\dist_G(o, \xi_j g)$.

**Proof.** Only the first inequality needs a proof. Fix an arbitrary $h \in \Omega_{\tau_0, U_0}$, we have
\[
\dist(o, \xi_j g) \geq \dist(\xi_j h, \xi_j g) - \dist(o, \xi_j h) \\
\geq \dist_G(h, g) - D - \dist_G(o, \xi_j h) \\
= \dist_G(\xi_j h, \xi_j g) - D - \dist_G(o, \xi_j h) \\
\geq \dist_G(o, \xi_j g) - 2\dist_G(o, \xi_j h) - D.
\]
Then $D' = 2\sup_{\xi_j \in \Xi} \dist_G(o, \xi_j h) + D$ satisfies (2.9). \qed

Note that any $g \in \Omega_{\tau_0, U_0}$ can be written as $g = ua_t k$ with $u \in U_0, t \geq \tau_0, k \in K$. Since $U_0$ is relatively compact, $\Xi$ is finite and dist is right $K$-invariant, in view of Lemma 2.1 we have
\[
(2.10) \quad \dist(o, \xi_j g) = \dist_G(o, a_t) + O(1) = t + O(1).
\]
In particular, if $\Gamma$ has a cusp at $\infty$, $\Xi$ can be taken such that it contains the identity element. Hence in this case,
\[
(2.11) \quad \dist(o, g) = t + O(1)
\]
for any $g = ua_t k \in \Omega_{\tau_0, U_0}$. Finally, we note that when $r$ is sufficiently large, $B_r$ is a collection of neighborhoods at all cusps. In view of the above reduction theory and the Haar measure (2.4) we have
\[
(2.12) \quad \sigma(B_r) \asymp e^{-nr}
\]
for any $r > 0$.

3. **Incomplete Eisenstein series**

Let $G$ be as before, $\Gamma \subset G$ be a non-uniform lattice in $G$ with a cusp at $\infty$. Given a compactly supported function $f \in L^2(Q \backslash G)$, we define the incomplete Eisenstein series attached to $f$ by
\[
\Theta_f(g) = \sum_{\gamma \in \Gamma \cap \Gamma g} f(\gamma g).
\]
Note that \( \Theta_f \) is left \( \Gamma \)-invariant since \( f \) is left \( \Gamma_\infty \)-invariant \( (\Gamma_\infty \subset Q) \). Moreover, since \( f \) is compactly supported, the summation is a finite sum. Hence it is a well-defined function on \( \Gamma \backslash G \). We first give a simple but useful identity related to \( \Theta_f \) given by the standard unfolding trick.

**Lemma 3.1.** For \( \Theta_f \) as above and any \( F \in L^2(\Gamma \backslash G) \)
\[
\int_{\Gamma \backslash G} \Theta_f(g)F(g)\,d\sigma(g) = \int_{\Gamma_\infty \backslash G} f(g)\,d\sigma(g).
\]

**Proof.** Let \( \mathcal{F}_\Gamma \) be a fundamental domain for \( \Gamma \backslash G \). Note that \( \mathcal{F}_\infty = \cup_{\gamma \in \Gamma_\infty} \gamma \mathcal{F}_\Gamma \) form a fundamental domain for \( \Gamma_\infty \backslash G \), hence
\[
\int_{\mathcal{F}_\Gamma} \Theta_f(g)F(g)\,d\sigma(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}_\Gamma} f(\gamma g)F(g)\,d\sigma(g)
\]
\[
= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}_\Gamma} f(g)\,d\sigma(g)
\]
\[
= \int_{\cup_{\gamma \in \Gamma_\infty \backslash \Gamma} \gamma \mathcal{F}_\Gamma} f(g)\,d\sigma(g)
\]
\[
= \int_{\Gamma_\infty \backslash G} f(g)\,d\sigma(g). 
\]

In particular, taking \( F = \overline{\Theta_f} = \Theta_T \) we get
\[
||\Theta_f||^2_2 = \int_{\Gamma_\infty \backslash G} \overline{f(g)}\Theta_f(g)\,d\sigma(g).
\]

Moreover, by (2.4) and (2.7) we have
\[
||\Theta_f||^2_2 = \frac{1}{[\Gamma_\infty : \Gamma_\infty \cap \Gamma]_K} \int_K \int_{\mathcal{F}_{\mathcal{V}_T}} f(a_t k)e^{-nt} \int_{\mathcal{F}_{\mathcal{V}_T}} \Theta_f(u_x a_t k)\,dx\,dt\,dk.
\]

Next we will compute \( ||\Theta_f||^2_2 \) by expressing the term \( \int_{\mathcal{F}_{\mathcal{V}_T}} \Theta_f(u_x a_t k)\,dx \) as an integral of certain non-spherical Eisenstein series (see Lemma 4.2). Before we can do that, we need to recall some facts of spherical Eisenstein series of real rank one groups (see [20] for the statements and [15] for the general theory).

### 3.1. Spherical Eisenstein series
Denote by \( C^\infty(Q \backslash G / K) \) the space of smooth left \( Q \)-invariant and right \( K \)-invariant functions on \( G \). For any \( s \in \mathbb{C} \), define the function \( \varphi_s \in C^\infty(Q \backslash G / K) \) by
\[
\varphi_s(ua_t k) = e^{st}.
\]

Given a lattice \( \Gamma \subset G \) with a cusp at \( \infty \), the spherical Eisenstein series (corresponding to the cusp at \( \infty \)) is defined by
\[
E(s, g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_s(\gamma g).
\]

This series converges for \( \text{Re}(s) > n \), it is right \( K \)-invariant and satisfies the following differential equation
\[
(\Delta + s(n - s)) E(s, g) = 0,
\]
where \( \Delta \) is the Laplacian on \( G \backslash \mathbb{R}^n \).
where $\Delta = e^{2t} \left( \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} \right) + \left( \frac{\partial^2}{\partial t^2} - n \frac{\partial}{\partial t} \right)$ is the Laplace-Beltrami operator on the upper half space

$$\mathbb{H}^{n+1} = \{ x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} + e^t e_n \mid x_i, t \in \mathbb{R} \}.$$  

The constant term of the Eisenstein series (corresponding to the cusp at $\infty$) is defined as

$$E^0 (s, g) = \frac{1}{|\mathbb{F}_\mathbb{C}|} \int_{x \in \mathbb{F}_\mathbb{C}} E(s, u_x g) \, dx.$$  

It has the form

$$E^0 (s, g) = \phi_s (g) + \mathcal{C}_\Gamma (s) \phi_{n-s} (g),$$

where the function $\mathcal{C} (s) = \mathcal{C}_\Gamma (s)$ can be extended to a meromorphic function on the half plane $\text{Re}(s) \geq \frac{n}{2}$ with a simple pole at $s = n$ and only possibly finitely many simple poles (called exceptional poles) on the interval $(\frac{n}{2}, n)$. Finally we note that using the functional equation satisfied by the Eisenstein series and the same argument in [13, p. 10], $|\mathcal{C} (s)| \leq 1$ for $\text{Re}(s) = \frac{n}{2}$.

### 3.2. The raising operator.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of $G$ and $K$ respectively. Let $\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}$ and $\mathfrak{k} = \mathfrak{k} \otimes \mathbb{C}$ be their complexifications. As a real vector space, $\mathfrak{k}$ is spanned by the matrices

$$- \frac{1}{2} \begin{pmatrix} e_i e_j & 0 \\ e_i e_j & 0 \end{pmatrix} (1 \leq i < j \leq n-1), \quad - \frac{1}{2} \begin{pmatrix} e_l & 0 \\ 0 & -e_l \end{pmatrix} (1 \leq l \leq n-1),$$

$$\frac{1}{2} \begin{pmatrix} 0 & e_m \\ e_m & 0 \end{pmatrix} (1 \leq m \leq n-1), \quad \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $e_1, \ldots, e_{n-1}$ are elements in the Clifford algebra $C_n$ as before. The Lie algebra $\mathfrak{g}$ is spanned by the matrices as above and

$$\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} (1 \leq i \leq n-1), \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

### 3.2.1. Root-space decomposition of $\mathfrak{k}_C$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}_C$. Since $\mathfrak{k}_C$ is a complex semisimple Lie algebra, it has a root-space decomposition with respect to $\mathfrak{h}$:

$$\mathfrak{k}_C = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi (\mathfrak{k}_C, \mathfrak{h})} \mathfrak{k}_\alpha,$$

where $\Phi = \Phi (\mathfrak{k}_C, \mathfrak{h})$ is the corresponding set of roots, and for each $\alpha \in \Phi$ the root-space $\mathfrak{k}_\alpha$ is given by

$$\mathfrak{k}_\alpha := \{ X \in \mathfrak{k}_C \mid [H, X] = \alpha (H) \, X \text{ for any } H \in \mathfrak{h} \}.$$  

Each root-space is one-dimensional and satisfies $[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha + \beta}$ for any $\alpha, \beta \in \Phi$. Fix a set of simple roots $\Delta$ and let $\Phi^+$ denote the corresponding set of positive roots. Then $\Phi = \Phi^+ \cup (-\Phi^+)$. For backgrounds on complex semisimple Lie algebra, see [14, Chapter II]. In this subsection, we first give an explicit isomorphism between $K$ and $SO (n + 1)$, then use this isomorphism and the classical root-space decomposition of $so (n + 1, \mathbb{C})$ to get an explicit root-space decomposition of $\mathfrak{k}_C$.  

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**References**

[13, p. 10]  
[14, Chapter II]
Recall the identification
\[ M \setminus K \rightarrow S^n := \{ x_0 + x_1 e_1 + \cdots + x_n e_n \mid x_i \in \mathbb{R}, x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \} \]

Embedding \( S^n \) in \( \mathbb{V}^n \) and fix an inner product on \( \mathbb{V}^n \) such that \( \{1, e_1, \cdots, e_n\} \) form an orthonormal basis of \( \mathbb{V}^n \). Then the right regular action of \( K \) on \( M \setminus K = S^n \) induces an isomorphism from \( K \) to \( SO(n + 1) \). In particular, it induces an isomorphism between \( \mathfrak{t}_C \) and \( \mathfrak{s} \mathfrak{o} (n + 1, \mathbb{C}) \). Explicitly, for any \( 0 \leq i < j \leq n \) define \( L_{i,j} \in \mathfrak{t}_C \) as following:

\[ L_{i,j} = \begin{cases} 
\frac{1}{2} \begin{pmatrix} e_i e_j & 0 \\
0 & e_i e_j \end{pmatrix} & \text{if } 1 \leq i < j \leq n - 1 \\
\frac{1}{2} \begin{pmatrix} 0 & -e_i \\
e_i & 0 \end{pmatrix} & \text{if } i = 0, 1 \leq j \leq n - 1 \\
\frac{1}{2} \begin{pmatrix} e_i & 0 \\
0 & e_i \end{pmatrix} & \text{if } 1 \leq i \leq n - 1, j = n \\
\frac{1}{2} \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix} & \text{if } i = 0, j = n.
\end{cases} \tag{3.4} \]

By direct computation, the induced isomorphism from \( \mathfrak{t}_C \) to \( \mathfrak{s} \mathfrak{o} (n + 1, \mathbb{C}) \) is given by sending \( L_{i,j} \) to \( E_{i,j} \) for any \( 0 \leq i < j \leq n \), where \( E_{i,j} \) is the antisymmetric \((n + 1) \times (n + 1)\) matrix with \( (i,j) \)th entry equals one, \( (j,i) \)th entry equals negative one and zero elsewhere. Using the classical commutator relations of \( E_{i,j} \) we get the commutator relations of \( L_{i,j} \). To ease the notation, we let \( L_{i,i} = 0 \) for \( 0 \leq i \leq n \) and \( L_{i,j} = -L_{j,i} \) for \( 0 \leq j < i \leq n \). Explicitly, \( L_{i,j} \) satisfy the following commutator relations

\[ [L_{i,j}, L_{i,m}] = \delta_{jl} L_{i,m} - \delta_{il} L_{j,m} - \delta_{jm} L_{i,l} + \delta_{im} L_{j,l} \tag{3.5} \]

for any \( 0 \leq i, j, l, m \leq n \), where \( \delta_{ij} \) is the Kronecker symbol. Moreover, using the root-space decomposition of \( \mathfrak{s} \mathfrak{o} (n + 1, \mathbb{C}) \) (see [14, p. 127-129]) we get the following root-space decomposition of \( \mathfrak{t}_C \) depending on the parity of \( n + 1 \).

**Case 1:** \( n + 1 = 2k + 1 \) is odd. For each \( 0 \leq i \leq k - 1 \), let

\[ H_i = \sqrt{-1} L_{2i,2i+1} \]

with \( L_{2i,2i+1} \) defined in (3.4). Let \( \mathfrak{h} \) be the complex vector space spanned by the set \( \{ H_i \mid 0 \leq i \leq k - 1 \} \). For each \( 0 \leq i \leq k - 1 \), let \( \epsilon_i : \mathfrak{h} \to \mathbb{C} \) be the linear functional on \( \mathfrak{h} \) characterized by \( \epsilon_i (H_j) = \delta_{ij} \). Using the above isomorphism between \( \mathfrak{t}_C \) and the root-space decomposition of \( \mathfrak{s} \mathfrak{o} (2k + 1, \mathbb{C}) \), we know \( \mathfrak{h} \) is a Cartan subalgebra and we can choose the set of simple roots to be

\[ \Delta = \{ \epsilon_0 - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{k-2} - \epsilon_{k-1}, \epsilon_{k-1} \}. \]

The corresponding positive roots are given by

\[ \Phi^+ = \{ \epsilon_i \pm \epsilon_j \mid 0 \leq i < j \leq k - 1 \} \cup \{ \epsilon_l \mid 0 \leq l \leq k - 1 \}. \]
Moreover, for any $0 \leq i < j \leq k - 1$ and $0 \leq l \leq k - 1$, the positive root-spaces $\mathfrak{t}_{\varepsilon_i \pm \varepsilon_j}$ and $\mathfrak{t}_{\varepsilon_i}$ are given as following:

\[
\mathfrak{t}_{\varepsilon_i \pm \varepsilon_j} = \mathbb{C} \left\langle \left( L_{2i,2j} - \sqrt{-1} L_{2i+1,2j} \right) \pm \left( L_{2i+1,2j+1} + \sqrt{-1} L_{2i,2j+1} \right) \right\rangle,
\]

and

\[
\mathfrak{t}_{\varepsilon_i} = \mathbb{C} \left\langle L_{2i,2k} - \sqrt{-1} L_{2i+1,2k} \right\rangle.
\]

**Case II:** $n + 1 = 2k$ is even. Similar to the odd case, for each $0 \leq i < k - 1$, let

\[
H_i = \sqrt{-1} L_{2i,2i+1}
\]

and let $\mathfrak{h}$ be the complex vector space spanned by $\{H_i \mid 0 \leq i \leq k - 1\}$. For each $0 \leq i \leq k - 1$, denote $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ the linear functional on $\mathfrak{h}$ characterized by $\varepsilon_i(H_j) = \delta_{ij}$. The set of simple roots can be chosen to be

\[
\Delta = \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{k-2} - \varepsilon_{k-1}, \varepsilon_{k-2} + \varepsilon_{k-1}\}.
\]

The corresponding positive roots are given by

\[
\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid 0 \leq i < j < k - 1\},
\]

with $\mathfrak{t}_{\varepsilon_i \pm \varepsilon_j}$ also given by (3.6).

**Remark 4.** The commutator relations and root-space decomposition above can both be checked directly using the relations $e_i e_j + e_j e_i = -2\delta_{i,j}$ for any $1 \leq i, j \leq n$.

3.2.2. **Spherical principal series representation.** Let $G = NAK$ be the fixed Iwahori decomposition and $M$ be the centralizer of $A$ in $K$ as before. For any $s \in \mathbb{C}$, recall the function $\varphi_s$ on $NA$ defined by

\[
\varphi_s(ua_i) = e^{st}
\]

for any $u \in N$ and $a_i \in A$. Consider the corresponding spherical principal series representation $I^s = \text{Ind}_{NAM}^G(\varphi_s \otimes 1_M)$, where $1_M$ is the trivial representation of $M$. Elements in $I^s$ are measurable functions $f : G \to \mathbb{C}$ satisfying

\[
\int (uawg) = \varphi_s(a)f(g) \quad \text{for} \quad \sigma \text{-a.e.} \quad g \in G,
\]

with $u \in N, a \in A$ and $w \in M$.

$G$ acts on $I^s$ by right regular action. We note that due to condition (3.8), $f \in I^s$ is a function on $A \times M \setminus K$, thus by the identification between $M \setminus K$ and $S^n$, $f$ is a functions in coordinates $(t, x_0, x_1, \ldots, x_n)$ with the restriction $x_0^2 + x_1^2 \cdots + x_n^2 = 1$.

Let $I_\infty^s$ be the space of smooth functions in $I^s$. We note that $I_\infty^s$ is a dense subspace of $I^s$ and the right regular action of $G$ on $I^s$ induces a $g$-module structure on $I_\infty^s$: For any $X \in g$ and any $f \in I_\infty^s$, define the Lie derivative, $\pi(X)$, by

\[
\left( \pi(X)f \right)(g) = \frac{d}{dy} f \left( g \exp(yX) \right) \bigg|_{y=0}.
\]

We note that the Lie derivative respects the Lie bracket, that is, $[\pi(X), \pi(Y)] = \pi([X, Y])$ for any $X, Y \in g$, where the first Lie bracket is the Lie bracket of endomorphisms. Since functions in $I_\infty^s$ are complex-valued, we can complexify the Lie derivative by defining $\pi(X + \sqrt{-1} Y) := \pi(X) + \sqrt{-1} \pi(Y)$ for any $X, Y \in g$. 


Thus $I^s_\infty$ becomes a $\mathfrak{g}_C$-module. In particular, $I^s_\infty$ is also a $\mathfrak{t}_C$-module. Let $\mathfrak{h}$ and $\Phi^+$ be as before. Let $\mathfrak{h}^*$ denote the complex dual of $\mathfrak{h}$. Given a $\mathfrak{t}_C$-module $V$ and $\rho \in \mathfrak{h}^*$, we say $v \in V$ is of $K$-weight $\rho$ if $H \cdot v = \rho(H) v$ for any $H \in \mathfrak{h}$. We say $v \in V$ is a highest weight vector if $v$ is of $K$-weight $\rho$ for some $\rho \in \mathfrak{h}^*$ and $X \cdot v = 0$ for any $\alpha \in \Phi^+$ and any $X \in \mathfrak{g}_\alpha$. We note that every irreducible representation of $K$ is a finite-dimensional irreducible $\mathfrak{t}_C$-module by differentiating the group action at the identity, and every finite-dimensional irreducible $\mathfrak{t}_C$-module admits a unique (up to scalars) highest weight vector (see [14, Theorem 5.5 (b)]).

Due to condition (3.8), $I^s$ is isomorphic to $L^2(M \backslash K)$ as $K$-representations by sending $f$ to $f|_K$. Identify $M \backslash K$ with $S^n$ as above, we have the following decomposition of $L^2(M \backslash K)$ as $K$-representations:

$$L^2(M \backslash K) = \bigoplus_{m \geq 0} L^2(M \backslash K, m),$$

where $L^2(M \backslash K, m)$ is the space of degree $m$ harmonic polynomials in $n + 1$ variables restricted to $S^n$ (see [10, Corollary 5.0.3]) and $\bigoplus$ denote the Hilbert direct sum. Moreover, let $\mathcal{H}^m$ be the space of degree $m$ harmonic polynomials in coordinates $(x_0, x_1, \cdots, x_n) \in \mathbb{V}^n$. Then $\mathcal{H}^m$ is an irreducible $K$-representation and is isomorphic to $L^2(M \backslash K, m)$ via the map $\phi \mapsto \phi|_{S^n}$ ([19, Theorem 0.3 and 0.4]). Finally, we note that $(x_0 - \sqrt{-1} x_1)^m \in \mathcal{H}^m$ is of $K$-weight $m \varepsilon_0$ ([14, p. 277-278]) and $\mathcal{H}^m$ is of highest weight $m \varepsilon_0$ ([14, p. 339 Prob. 9.2]). Hence $(x_0 - \sqrt{-1} x_1)^m$ is the unique (up to scalars) highest weight vector in $\mathcal{H}^m$.

Correspondingly, let $I^s_\infty(K, m) := \{ f \in I^s_\infty \mid f|_K \in L^2(M \backslash K, m) \}$. Then we have a decomposition of $I^s_\infty$

$$I^s_\infty = \bigoplus_{m=0}^{\infty} I^s_\infty(K, m).$$

Moreover, $I^s_\infty(K, m)$ is an irreducible $\mathfrak{t}_C$-module of highest weight $m \varepsilon_0$, and the highest weight vector is given by

$$\varphi_{s, m}(t, x_0, x_1, \cdots, x_n) := e^{st} (x_0 - \sqrt{-1} x_1)^m.$$ 

Now we define the raising operator $R^+ \in \mathfrak{g}_C$ by

$$R^+ = \frac{1}{2} \pi \left( \begin{array}{cc} 0 & -1 + \sqrt{-1} e_1 \\ -1 - \sqrt{-1} e_1 & 0 \end{array} \right) = -\frac{1}{2} \pi \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) + \frac{\sqrt{-1}}{2} \pi \left( \begin{array}{cc} 0 & e_1 \\ -e_1 & 0 \end{array} \right).$$

To compute $R^+$ explicitly, we use the spherical coordinates on $S^n$: Let $(x_0, x_1, \cdots, x_n)$ be the coordinates on $S^n$ as above, define $(\theta_0, \theta_1, \ldots, \theta_{n-1}) \in [0, 2\pi)^{n-1} \times$
[0, \pi) such that
\[
\begin{align*}
    x_0 &= \cos \theta_0, \\
    x_1 &= \sin \theta_0 \cos \theta_1, \\
    &\vdots \\
    x_{n-1} &= \sin \theta_0 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\
    x_n &= \sin \theta_0 \cdots \sin \theta_{n-2} \sin \theta_{n-1}.
\end{align*}
\]
Hence under the coordinates \((t, \theta_i)\), \(\varphi_{s,m}\) is given by
\[
\varphi_{s,m}(t, \theta_i) = e^{st\left(\cos \theta_0 - \sqrt{-1} \sin \theta_0 \cos \theta_1\right)}^m.
\]
Moreover, in these coordinates, for any \(X \in g_C\), the Lie derivative \(\pi(X)\) is a first order differential operator of the form
\[
\pi(X) = F \frac{\partial}{\partial t} + \sum_{i=0}^{n-1} F_i \frac{\partial}{\partial \theta_i},
\]
where \(F, F_i\) are functions in \((t, \theta_i)\). For our purpose, we define
\[
\pi(X) := F \frac{\partial}{\partial t} + F_0 \frac{\partial}{\partial \theta_0} + F_1 \frac{\partial}{\partial \theta_1}.
\]
Since \(\varphi_{s,m}\) only depends on the variables \((t, \theta_0, \theta_1)\), \(\pi(X) \varphi_{s,m} = \pi(X) \varphi_{s,m}\) for any \(X \in g_C\).

Now we describe the strategy to compute the Lie derivatives. We first show how to extract the coordinates \((t, \theta_i)\) from a given element \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\). Write
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t \frac{i}{2}} & 0 \\ 0 & e^{-t \frac{i}{2}} \end{pmatrix} \begin{pmatrix} q_2 & -q_1' \\ q_1' & q_2' \end{pmatrix}
\]
by Iwasawa decomposition. Comparing the second row of the matrices on both sides, we get
\[
e^{-t} = |c|^2 + |d|^2
\]
and
\[
x_0 + x_1 e_1 + \cdots + x_n e_n = \frac{2\bar{c}d + (|d|^2 - |c|^2) e_n}{|c|^2 + |d|^2},
\]
where \(x_i\) are expressed by \(\theta_i\) as above. Fix an element \(g \in G\). For any \(X \in g\), the coordinates \((t, \theta_i)\) of \(g \exp \{yX\}\) can be viewed as functions in \(y\) as \(y\) varies. Denote \(\{t(y), \theta_i(y)\}\) to indicate this dependence on \(y\). Then the Lie derivative \(\pi(X)\) is exactly given by
\[
\pi(X) = t'(0) \frac{\partial}{\partial t} + \theta_0'(0) \frac{\partial}{\partial \theta_0} + \theta_1'(0) \frac{\partial}{\partial \theta_1},
\]
**Lemma 3.2.** Let \( B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( B_2 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix} \). Then

\[
\pi(B_1) = -2 \cos \theta_0 \frac{\partial}{\partial t} - 2 \sin \theta_0 \frac{\partial}{\partial \theta_0},
\]

and

\[
\pi(B_2) = -2 \sin \theta_0 \cos \theta_1 \frac{\partial}{\partial t} + 2 \cos \theta_0 \cos \theta_1 \frac{\partial}{\partial \theta_0} - 2 \sin \theta_1 \frac{\partial}{\sin \theta_0 \partial \theta_1}.
\]

**Proof.** Using the formula \( \exp(yB_1) = \sum_{i=0}^{\infty} \frac{(yB_1)^i}{i!} \) we get

\[
\exp(yB_1) = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}.
\]

Thus

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix} = \begin{pmatrix} \ast & \ast \\ c \cosh y + d \sinh y & c \sinh y + d \cosh y \end{pmatrix}.
\]

Using (3.10) we get

\[
e^{-t(y)} = | \cosh y + d \sinh y |^2 + |c \sinh y + d \cosh y |^2 = e^{-t} \left( \cosh(2y) + \sinh(2y) \cos \theta_0 \right).
\]

Taking derivatives with respect to \( y \) and evaluating at 0 on both sides we get

\[
t'(0) = -2 \cos \theta_0.
\]

Similarly, using (3.11) and comparing the constant term and coefficient of \( e_1 \), we get

\[
\cos(\theta_0(y)) = \frac{\sinh(2y) + \cosh(2y) \cos \theta_0}{\cosh(2y) + \sinh(2y) \cos \theta_0}
\]

and

\[
\sin(\theta_0(y)) \cos(\theta_1(y)) = \frac{\sin \theta_0 \cos \theta_1}{\cosh(2y) + \sinh(2y) \cos \theta_0}.
\]

Taking derivatives with respect to \( y \) and evaluating at 0 we get

\[
\theta_0'(0) = -2 \sin \theta_0 \quad \text{and} \quad \theta_1'(0) = 0.
\]

Thus \( \pi(B_1) = -2 \cos \theta_0 \frac{\partial}{\partial t} - 2 \sin \theta_0 \frac{\partial}{\partial \theta_0} \).

Similarly, for \( B_2 \) we have \( \exp(yB_2) = \begin{pmatrix} \cosh y & \sinh y e_1 \\ -\sinh y e_1 & \cosh y \end{pmatrix} \) and

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cosh y & \sinh y e_1 \\ -\sinh y e_1 & \cosh y \end{pmatrix} = \begin{pmatrix} \ast & \ast \\ c \cosh y - d e_1 \sinh y & c e_1 \sinh y + d \cosh y \end{pmatrix}.
\]

Using (3.10) and (3.11), after some tedious but straightforward computations we get

\[
e^{-t(y)} = e^{-t} \left( \cosh(2y) + \sinh(2y) \sin \theta_0 \cos \theta_1 \right),
\]

\[
\cos(\theta_0(y)) = \frac{\cos \theta_0}{\cosh(2y) + \sinh(2y) \sin \theta_0 \cos \theta_1},
\]

\[
\cos(\theta_0(y)) = \frac{\cos \theta_0}{\cosh(2y) + \sinh(2y) \sin \theta_0 \cos \theta_1}.
\]
and 
\[
\sin(\theta(y)) \cos(\theta_1(y)) = \frac{\sinh(2y) + \cosh(2y) \sin \theta_0 \cos \theta_1}{\cosh(2y) + \sinh(2y) \sin \theta_0 \cos \theta_1}.
\]

Hence by taking derivatives with respect to \( y \) and evaluating at 0 we get
\[
t'(0) = -2 \sin \theta_0 \cos \theta_1, \quad \theta_0'(0) = 2 \cos \theta_0 \cos \theta_1 \quad \text{and} \quad \theta_1'(0) = -2 \frac{\sin \theta_1}{\sin \theta_0}.
\]

Hence \( \pi(B_2) = -2 \sin \theta_0 \cos \theta_1 \frac{\partial}{\partial t} + 2 \cos \theta_0 \cos \theta_1 \frac{\partial}{\partial \theta_0} - 2 \frac{\sin \theta_1}{\sin \theta_0} \frac{\partial}{\partial \theta_1} \).

In view of Lemma 3.2 we get
\[
\tilde{R}^+ = \left( \cos \theta_0 - \sqrt{-1} \sin \theta_0 \cos \theta_1 \right) \frac{\partial}{\partial t} + \left( \sin \theta_0 + \sqrt{-1} \cos \theta_0 \cos \theta_1 \right) \frac{\partial}{\partial \theta_0} - \sqrt{-1} \frac{\sin \theta_1}{\sin \theta_0} \frac{\partial}{\partial \theta_1}.
\]

Since \( R^+ \varphi_{s,m} = \tilde{R}^+ \varphi_{s,m} \), applying \( \tilde{R}^+ \) to \( \varphi_{s,m} \) we get
\[
(3.12) \quad R^+ \varphi_{s,m} = (s + m) \varphi_{s,m+1}.
\]

**Remark 5.** We note that using the explicit root-space decomposition described as above, one can directly check that the raising operator \( R^+ \) (more explicitly, the matrix representing \( R^+ \)) satisfies
\[
(3.13) \quad [H, R^+] = \epsilon_0(H) R^+ \quad \text{and} \quad [R^+, \xi_a] = 0
\]

for any \( H \in \mathfrak{h} \) and any \( \alpha \in \Phi^+ \). The first part of (3.13) implies that \( R^+ \) sends a vector of \( K \)-weight \( \rho \) to a vector of \( K \)-weight \( \rho + \epsilon_0 \) and the second part of (3.13) implies that \( R^+ \) sends a highest weight vector to either zero or another highest weight vector. Since \( \varphi_{s,m} \) is a highest weight vector of \( K \)-weight \( me_0 \), \( R^+ \varphi_{s,m} \) is either zero or a highest weight vector of \( K \)-weight \( (m + 1)e_0 \). But since \( I^s_m = \sum_{m=0}^{\infty} I^s_m(K, m) \) and each \( I^s_m(K, m) \) has a unique (up to scalars) highest weight vector \( \varphi_{s,m} \), the set of highest weight vectors in \( I^s_m \) is exactly \( \{ \varphi_{s,m} \mid m \geq 0 \} \). Thus \( R^+ \varphi_{s,m} \) is a multiple of \( \varphi_{s,m+1} \). In fact, (3.13) is the characterization we used to find \( R^+ \). However, once we have found \( R^+ \), (3.13) is no longer essential for our proof, since we get (3.12) (which trivially implies that \( R^+ \varphi_{s,m} \) is a multiple of \( \varphi_{s,m+1} \)) by explicit computation.

### 3.3. Non-spherical Eisenstein series.

Given \( \phi \in L^2(M \backslash K, m) \), we view \( \phi \) as a function on \( G \) by setting \( \phi(\tau u a k) = \phi(k) \) for any \( u \in N \) and any \( a \in A \). We define the non-spherical Eisenstein series by
\[
E(\phi, s, g) = \sum_{g \in \Gamma \backslash \Gamma} \varphi_s(\gamma g) \phi(\gamma g).
\]

Note that \( E(\phi, s, g) \) is no longer right \( K \)-invariant. Its constant term (corresponding to the cusp at \( \infty \)) is defined by
\[
E^0(\phi, s, g) = \frac{1}{|\mathcal{F}_0|} \int_{x \in \mathcal{F}_0} E(\phi, s, u_x g) \, dx.
\]

Now using (3.12) we can get an explicit formula for \( E^0(\phi, s, g) \).
PROPOSITION 3.3. For any $\phi \in L^2(M \setminus K, m)$,

$$E^0(h_m, s, g) = (\varphi_s(g) + P_m(s) \mathcal{C}(s) \varphi_{n-s}(g)) \phi(g),$$

(3.14)

where $P_0(s) = 1$ and $P_m(s) = \prod_{k=0}^{m-1} \frac{n-s+k}{s+k}$ for $m \geq 1$.

Proof. For any $m \geq 0$, let $h_m$ be the highest weight vector in $L^2(M \setminus K, m)$. We first prove (3.14) for $h_m$. We prove by induction. If $m = 0$, then (3.14) follows from the constant term formula for spherical Eisenstein series. Assume that it holds for some integer $m \geq 0$, we want to show it also holds for $m + 1$. We apply the raising operator $R^+$ to the constant term $E^0(h_m, s, g)$. On the one hand, by induction,

$$E^0(h_m, s, g) = \varphi_s, m(g) + P_m(s) \mathcal{C}(s) \varphi_{n-s, m}(g).$$

Hence by (3.12) we get

$$R^+ E^0(h_m, s, g) = (s + m) \varphi_{s, m+1}(g) + (n - s + m) P_m(s) \mathcal{C}(s) \varphi_{n-s, m+1}(g).$$

On the other hand, since $R^+$ commutes with the left regular action, we have

$$R^+ E^0(h_m, s, g) = \frac{1}{|F_{C_t}|} \int_{x \in F_{C_t}} \sum_{\gamma \in \Gamma_{\infty} \Gamma} R^+ \varphi_{s, m}(\gamma u x g) \, dx$$

$$= \frac{s + m}{|F_{C_t}|} \int_{x \in F_{C_t}} \sum_{\gamma \in \Gamma_{\infty} \Gamma} \varphi_{s, m+1}(\gamma u x g) \, dx$$

$$= (s + m) E^0(h_{m+1}, s, g).$$

Hence

$$E^0(h_{m+1}, s, g) = \frac{1}{s + m} ((s + m) \varphi_{s, m+1}(g) + (n - s + m) P_m(s) \mathcal{C}(s) \varphi_{n-s, m+1}(g))$$

$$= \varphi_{s, m+1}(g) + \frac{n - s + m}{s + m} P_m(s) \mathcal{C}(s) \varphi_{n-s, m+1}(g)$$

$$= \varphi_{s, m+1}(g) + P_{m+1}(s) \mathcal{C}(s) \varphi_{n-s, m+1}(g).$$

Now for general $\phi \in L^2(M \setminus K, m)$. Since $L^2(M \setminus K, m)$ is an irreducible $\xi_C$-module, $\phi$ can be written as $\phi = \mathcal{D} h_m$ with $\mathcal{D}$ some differential operator on $L^2(M \setminus K, m)$ generated by $\pi(\xi_C)$. Since $\pi(\xi_C)$ acts trivially on the character $\varphi_s$, we have $\mathcal{D} \varphi_s, m = \varphi_s, \mathcal{D} h_m = \varphi_s, \phi$. Hence on the one hand,

$$\mathcal{D} E^0(h_m, s, g) = \frac{1}{|F_{C_t}|} \int_{x \in F_{C_t}} \sum_{\gamma \in \Gamma_{\infty} \Gamma} \mathcal{D} \varphi_{s, m}(\gamma u x g) \, dx$$

$$= \frac{1}{|F_{C_t}|} \int_{x \in F_{C_t}} \sum_{\gamma \in \Gamma_{\infty} \Gamma} \varphi_s(\gamma u x g) \phi(\gamma u x g) \, dx$$

$$= E^0(\phi, s, g).$$

On the other hand, using (3.14) for $h_m$ we get

$$\mathcal{D} E^0(h_m, s, g) = \mathcal{D} ((\varphi_s(g) + P_m(s) \mathcal{C}(s) \varphi_{n-s}(g)) h_m(g))$$

$$= (\varphi_s(g) + P_m(s) \mathcal{C}(s) \varphi_{n-s}(g)) \phi(g).$$

This completes the proof.
4. Bounds for Incomplete Eisenstein Series

4.1. Explicit formula. For each $L^2 (M\backslash K, m)$ we fix an orthonormal basis

$$\{ \psi_{m,l} \mid 1 \leq l \leq \dim L^2 (M\backslash K, m) \}.$$  

For any $f \in C_c^\infty (Q\backslash G)$ let

$$\hat{f}_{m,l} (a) = \int_K f (ak) \overline{\psi_{m,l}(k)} dk,$$

and we define the following function

$$M_f (s) = \sum_{m,l} P_m (s) \left| \int_{\mathbb{R}} \hat{f}_{m,l} (a_t) e^{-st} dt \right|^2,$$

with $P_m (s)$ as in Proposition 3.3. We then have

**Proposition 4.1.** (cf. [12, Proposition 2.3]) Let $\frac{n}{2} < s_p < \cdots < s_1 < s_0 = n$ denote the poles of $c^s$ (s) and let $c_j = \text{Res}_{s=s_j} c^s$ be the residue of $c^s$ (s) at $s_j$ for $0 \leq j \leq p$. Then for any $f \in C_c^\infty (Q\backslash G)$ we have

$$\| \Theta f \|_2^2 \leq \frac{|\mathcal{F}_{\psi}|}{|\Gamma_\infty : \Gamma'_\infty|} \left( 2 \| f \|_2^2 + \sum_{j=0}^p c_j M_f (s_j) \right).$$

Note that $f \in C_c^\infty (Q\backslash G)$ can be written as $f = \sum_{m,l} f_{m,l}$ with $f_{m,l} (ak) = \hat{f}_{m,l} (a) \psi_{m,l}(k)$. We first prove a preliminary estimate for each $f_{m,l}$ and then deduce the proposition from this estimate.

**Lemma 4.2.** Let $f \in C_c^\infty (Q\backslash G)$ be of the form $f (a_t) = v(t) \phi(k)$ where $v(t) \in C_c^\infty (\mathbb{R})$ and $\phi \in L^2 (M\backslash K, m)$ for some $m$. Let $s_j$ and $c_j$ be as above, we then have

$$\| \Theta f \|_2^2 \leq \frac{|\mathcal{F}_{\psi}|}{|\Gamma_\infty : \Gamma'_\infty|} \left( 2 \| f \|_2^2 + \sum_{j=0}^p c_j M_f (s_j) \| \phi \|_2^2 \left| \int_{\mathbb{R}} v(t) e^{-s_j t} dt \right|^2 \right).$$

**Proof.** Let $\hat{v} (r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(t) e^{-irt} dt$ denote the Fourier transform of $v$. Note that $\hat{v} (r)$ extends to an entire function in $r$ since $v$ is smooth and compactly supported. Recall the Fourier inversion formula

$$v (t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{v} (r) e^{irt} dr.$$  

Making the substitution $s = ir$ we get

$$v (t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{v} (-is) e^{ist} ds.$$  

For any $\sigma > n$ shifting the contour of integration to the line $\text{Re} (s) = \sigma$ we get

$$v (t) = \frac{1}{\sqrt{2\pi}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{v} (-is) e^{ist} ds = \frac{1}{\sqrt{2\pi}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{v} (-is) \phi_s (a_t) ds.$$  

Consequently we can write

$$f (g) = \frac{1}{\sqrt{2\pi}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{v} (-is) \phi_s (g) \phi (g) ds.$$
and summing over $\Gamma_\infty \setminus \Gamma$ we get

$$\Theta_f (g) = \frac{1}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \varphi_s (\gamma g) \phi (\gamma g) \, ds$$

$$= \frac{1}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) E (\phi, s, g) \, ds.$$ Integrating this over $\mathcal{F}_{\Theta_f}$ gives

$$\int_{\mathcal{F}_{\Theta_f}} \Theta_f (u_x \alpha k) \, dx = \frac{|\mathcal{F}_{\Theta_f}|}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \varphi_s (a) \, ds$$

$$= \frac{|\mathcal{F}_{\Theta_f}|}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) E^0 (\phi, s, a \alpha k) \, ds.$$ Using the formula (3.14) for $E^0 (\phi, s, g)$ we get

$$\int_{\mathcal{F}_{\Theta_f}} \Theta_f (u_x \alpha k) \, dx = \frac{|\mathcal{F}_{\Theta_f}|}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \varphi_s (a) \, ds$$

$$+ \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \mathcal{C} (s) P_m (s) \varphi_{n-s} (a) \, ds$$

Now shift the contour of integration to the line $\Re (s) = \frac{n}{2}$ (picking up possible poles) to get

$$\int_{\mathcal{F}_{\Theta_f}} \Theta_f (u_x \alpha k) \, dx = \frac{|\mathcal{F}_{\Theta_f}|}{\sqrt{2\pi i}} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \varphi_s (a) \, ds$$

$$+ \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \mathcal{C} (s) P_m (s) \varphi_{n-s} (a) \, ds$$

$$+ 2\pi i \sum_{j=0}^{p} c_j P_m (s_j) \hat{\nu}(-is_j) \varphi_{n-s_j} (a).$$ Where $c_j = \text{Res}_{s=s_j} \mathcal{C} (s)$ is the residue of $\mathcal{C} (s)$ at the pole $s_j$. Now applying formula (3.2) we get

$$||\Theta_f||_2^2 = \frac{1}{|\Gamma_\infty : \Gamma'_\infty| v^T} \int_{K} \int_{\Gamma} \int_{\Gamma} e^{-nt} \int_{\mathcal{F}_{\Theta_f}} \Theta_f (u_x \alpha k) \, dx \, dk$$

$$= \frac{|\mathcal{F}_{\Theta_f}|}{|\Gamma_\infty : \Gamma'_\infty| v^T} \int_{K} \int_{\Gamma} |\phi(k)|^2 \, dk \frac{1}{\sqrt{2\pi i}} \int_{\mathcal{F}_{\Theta_f}} \hat{\nu}(-is) \int_{\Gamma} e^{-nt} \varphi_s (a_t) \, dt \, ds$$

$$+ \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{\nu}(-is) \mathcal{C} (s) P_m (s) \int_{\Gamma} e^{-nt} \varphi_{n-s} (a_t) \, dt \, ds$$

$$+ 2\pi i \sum_{j=0}^{p} c_j P_m (s_j) \hat{\nu}(-is_j) \int_{\Gamma} e^{-nt} \varphi_{n-s_j} (a_t) \, dt.$$
Note that for \( s \in \mathbb{C} \) we have

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(t)e^{-nt} \varphi_s(a_t) \, dt = \vartheta(i(s - m)).
\]

Hence (use the substitution \( s = \frac{n}{2} + ir \))

\[
\|\Theta_f\|_2^2 = \left( \int_{\mathbb{R}} |\vartheta(r - \frac{n}{2}i)|^2 \, dr + \int_{\mathbb{R}} |\vartheta(r - \frac{n}{2}i)\vartheta(-r - \frac{n}{2}i)\mathcal{E}(\frac{n}{2} + ir)P_m(\frac{n}{2} + ir)\, dr + 2\pi \sum_{j=0}^{P} c_j P_m(s_j) |\vartheta(-is_j)|^2 \right) \frac{\|\mathcal{F}\varphi_f\|_{L^1}}{|\Gamma_{\infty} : \Gamma_{\infty}'|} \|\varphi\|_2^2.
\]

Now for the first term, applying Plancherel’s theorem for \( v(t) e^{-\frac{n}{2}t} \) we get

\[
\|\varphi\|_2^2 \int_{\mathbb{R}} |\vartheta(r - \frac{n}{2}i)|^2 \, dr = \|\varphi\|_2^2 \int_{\mathbb{R}} |v(t)|^2 e^{-nt} \, dt = \|f\|_2^2.
\]

For the second term, using the fact that \(|\mathcal{E}(\frac{n}{2} + ir)| \leq 1, |P_m(\frac{n}{2} + ir)| = 1 \) and Cauchy-Schwarz, we see that its absolute value is bounded by the first term. Finally, for the last term we have for each pole \( 2\pi |\vartheta(-is_j)|^2 = | \int_{\mathbb{R}} v(t) e^{-s_j t} \, dt |^2 \).

This finishes the proof. \( \square \)

We can now give the proof of Proposition 4.1. Note that for \( f = \sum_{m,l} f_{m,l} \) as above, by orthogonality we have

\[
\|\Theta_f\|_2^2 = \sum_{m,l} \|\Theta_{f_{m,l}}\|_2^2.
\]

We can use Lemma 4.2 to estimate each of the terms \( \|\Theta_{f_{m,l}}\|_2^2 \) separately and sum all the contributions.

First, for the \( L^2 \)-norms we have \( \sum_{m,l} \| f_{m,l} \|_2^2 = \|f\|_2^2 \). Next, the contribution of the exceptional poles \( \frac{n}{2} < s_j < n \) is \( \sum_{j=1}^{P} c_j M_f(s_j) \). Finally, for the pole \( s_0 = n \), note that \( P_m(n) = 0 \) except \( m = 0 \). Thus its contribution is

\[
c_0 \left| \int_{\mathbb{R}} \int_{\mathbb{R}} f_{0,0}(a_t) e^{-nt} \, dt \right|^2 = c_0 \left| \int_{Q\setminus G} f(g) \, dg \right|^2 \leq c_0 \|f\|_1^2.
\]

**Remark 6.** We note that if \( f \) is of the form \( f(a_t,k) = v(t) \phi(k) \), with \( v \) smooth, compactly supported and \( \phi \) any function in \( L^2(M\setminus K) \), then Proposition 4.1 still holds by exactly the same computation.

4.2. **Proof of Theorem 1.2.** Fix a parameter \( \lambda > 0 \), define \( \mathcal{A}_\lambda \subset L^2(Q\setminus G) \) to be the set of functions of the form

\[
f(a_t,k) = v(t) \phi(k)
\]

with \( v \) smooth, nonnegative, compactly supported and satisfying

\[
\int_{\mathbb{R}} v(t)e^{-st} \, dt \leq \lambda \left( \frac{\int_{\mathbb{R}} v(t)e^{-nt} \, dt}{\left( \frac{\int_{\mathbb{R}} v(t)e^{-nt} \, dt} {v^2(t)e^{-nt} \, dt} \right)^{\frac{1}{n}}} \right)^{1-\frac{1}{n}}.
\]
for any \( s \in \left( \frac{n}{2}, n \right) \), and \( \phi \) any function in \( L^2(M \setminus K) \). In this section we will show that for any \( s \in \left( \frac{n}{2}, n \right) \),

\[
M_f(s) \lesssim_s \lambda \|f\|_1^2 + \|f\|_2^2
\]

for all \( f \in M_\lambda \). In particular, this bound holds at the finitely many exceptional poles \( s_j \) determined by \( \Gamma \). Hence in view of Proposition 4.1 it implies Theorem 1.2.

To show (4.2), we first give two preliminary lemmas.

**Lemma 4.3.** If \( f \in L^2(Q \setminus G) \) is of the form \( f(a_t,k) = v(t) \phi(k) \), then \( M_f(s) \) can be written as

\[
M_f(s) = \left( \sum_{m=0}^\infty P_m(s) \|\phi_m\|_2^2 \right) \left| \int_R v(t) e^{-st} dt \right|^2,
\]

where \( \phi_m \) denotes the projection of \( \phi \) into \( L^2(M \setminus K, m) \).

**Proof.** For each \( m \geq 0 \), let

\[
\{\psi_{m,l} \mid 1 \leq l \leq \dim L^2(M \setminus K, m)\}
\]

be the fixed orthonormal basis of \( L^2(M \setminus K, m) \) as before. Then we have

\[
\phi_m(k) = \sum_l c_{m,l} \psi_{m,l}(k)
\]

with \( c_{m,l} = \int_K \phi(k) \overline{\psi_{m,l}(k)} \, dk \). Moreover, we have

\[
\|\phi_m\|_2^2 = \sum_l |c_{m,l}|^2
\]

and

\[
\hat{f}_{m,l}(a_t) = v(t) \int_K \phi(k) \overline{\psi_{m,l}(k)} \, dk = c_{m,l} v(t).
\]

Hence

\[
M_f(s) = \sum_{m,l} P_m(s) \left| \int_R \hat{f}_{m,l}(a_t) e^{-st} dt \right|^2
\]

\[
= \sum_{m=0}^\infty P_m(s) \sum_l |c_{m,l}|^2 \left| \int_R v(t) e^{-st} dt \right|^2
\]

\[
= \left( \sum_{m=0}^\infty P_m(s) \|\phi_m\|_2^2 \right) \left| \int_R v(t) e^{-st} dt \right|^2.
\]

**Lemma 4.4.** For any \( s \in \left( \frac{n}{2}, n \right) \), \( P_m(s) \approx_s (m+1)^{(n-2s)} \).

**Proof.** Since \( P_m(s) = \prod_{k=0}^{m-1} \frac{n-s+k}{s+k} \) we have

\[
\log(P_m(s)) = \log \left( \frac{n}{s} \right) + \sum_{k=1}^{m-1} \left( \log \left( 1 + \frac{n-s}{k} \right) - \log \left( 1 + \frac{s}{k} \right) \right)
\]

\[
= (n-2s) \sum_{k=1}^{m-1} \frac{1}{k} + O_s(1) = (n-2s) \log(m+1) + O_s(1). \]

\[\]
Thus for $f(a_t k) = v(t) \phi(k) \in \mathcal{A}_\lambda$ we define

$$\tilde{M}_f(s) := \left( \int_{\mathbb{R}} v(t) e^{-st} dt \right)^2 \sum_{m=0}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{(2s-n)}},$$

where $\phi_m$ is the projection of $\phi$ into $L^2(M \setminus K, m)$. In view of these two lemmas it suffices to prove $\tilde{M}_f(s) \leq ||f||_1^2 + ||f||_2^2$ for all $f \in \mathcal{A}_\lambda$.

**Proof of Theorem 1.2.** We first prove for $f = v \phi \in \mathcal{A}_\lambda$ with $||\phi||_2 \leq ||\phi||_1$. We first recall a simple inequality that for any $y_1, y_2 > 0$ and $0 < \eta < 1$, $y_1^{1-\eta} y_2 \leq \max\{y_1, y_2\} \leq y_1 + y_2$. Hence in view of (4.1), for any $s \in \left(\frac{n}{2}, n\right]$, since $(\frac{2n}{n} - 1) + (2 - \frac{2s}{n}) = 1$, we have

$$\left( \int_{\mathbb{R}} v(t) e^{-st} dt \right)^2 \leq \lambda^2 \left( \int_{\mathbb{R}} v(t) e^{-nt} dt \right)^2 + \int_{\mathbb{R}} v^2(t) e^{-nt} dt.$$

Thus

$$\tilde{M}_f(s) = \left( \int_{\mathbb{R}} v(t) e^{-st} dt \right)^2 \sum_{m=0}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{(2s-n)}}$$

$$\leq \lambda^2 \left( \int_{\mathbb{R}} v(t) e^{-nt} dt \right)^2 + \int_{\mathbb{R}} v^2(t) e^{-nt} dt \frac{||\phi||_1^2}{2}$$

$$\leq \lambda^2 \left( \int_{\mathbb{R}} v(t) e^{-nt} dt \right)^2 ||\phi||_1^2 + \int_{\mathbb{R}} v^2(t) e^{-nt} dt ||\phi||_2^2$$

$$= \lambda^2 (||f||_1^2 + ||f||_2^2).$$

Now we prove the case $||\phi||_2 > ||\phi||_1$. Let $\iota := \frac{||\phi||_2}{||\phi||_1} > 1$. We separate the summation into two parts:

$$\sum_{m=0}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{(2s-n)}} = \left( \sum_{m=0}^{\left\lfloor \frac{\iota}{\iota + 1}\right\rfloor} + \sum_{m=\left\lfloor \frac{\iota}{\iota + 1}\right\rfloor + 1}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{(2s-n)}} \right).$$

For the first part, we invoke an estimate from spherical harmonic analysis ([17, inequality (4.4)]). Namely, for any $\phi \in L^2(M \setminus K)$ and $m \geq 0$,

$$||\phi_m||_2^2 \lesssim (m+1)^{n-1} ||\phi||_1^2$$

with the implicit constant only depends on the dimension of $M \setminus K$. Thus we have

$$\sum_{m=0}^{\left\lfloor \frac{\iota}{\iota + 1}\right\rfloor} \frac{||\phi_m||_2^2}{(m+1)^{(2s-n)}} \lesssim \sum_{m=0}^{\left\lfloor \frac{1}{\iota + 1}\right\rfloor} \frac{||\phi||_1^2}{(m+1)^{(2(n-s))}}$$

$$\lesssim (\frac{1}{\iota + 1})^{2(n-s)} ||\phi||_1^2$$

(since $1 - 2(n-s) < 1$)

$$= ||\phi||_1^2 \left( \frac{n}{n-1} \right)^{4(1-\frac{s}{n})}.$$

\[\text{The exact form of inequality (4.4) in [17] is } ||\phi_m||_2 \lesssim m^{\frac{n-1}{2}} ||\phi||_1. \text{ Here we square both sides and replace } m \text{ by } m+1 \text{ to cover the case } m = 0.\]
For the second part, we have
\[
\sum_{m=1}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{2s-n}} \leq \frac{1}{(t^n)^{2s-n}} \sum_{m=1}^{\infty} ||\phi_m||_2^2 \leq t^{-2\left(\frac{2}{n}-1\right)} ||\phi||_2^2 = ||\phi||_1^2 \left(\frac{1}{2} - \frac{1}{n}\right).
\]

Hence
\[
\sum_{m=0}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{2s-n}} \lesssim_s ||\phi||_1^2 ||\phi||_2^2 \left(\frac{1}{2} - \frac{1}{n}\right).
\]

Thus by (4.1) and (4.3) we get
\[
\tilde{M}_f(s) = \left(\int_{\mathbb{R}} v(t) e^{-st} dt\right) \sum_{m=0}^{\infty} \frac{||\phi_m||_2^2}{(m+1)^{2s-n}} \lesssim_{s,A} \left(\int_{\mathbb{R}} v(t) e^{-nt} dt\right)^{2\left(\frac{2}{n}-1\right)} \left(\int_{\mathbb{R}} v^2(t) e^{-nt} dt\right)^{2-s-n} \left(||\phi||_1^2 ||\phi||_2^2 \left(\frac{1}{2} - \frac{1}{n}\right)\right) = ||f||_1^2 ||f||_2^2 
\]
\[
\leq ||f||_1^2 + ||f||_2^2.
\]

This finishes the proof. \(\square\)

5. Logarithm laws

Fix \( o \in \Gamma \backslash G \) and let \( \{g_t\} \subset G \) be a one-parameter unipotent subgroup in \( G \). Let \( \text{dist}_G \) and \( \text{dist}_\Gamma \) be the hyperbolic distance functions on \( G/K \) and \( \Gamma \backslash G/K \) respectively. In this section we will prove logarithm laws for the unipotent flow \( \{g_t\} \), that is
\[
\limsup_{t \to \infty} \frac{\text{dist}_\Gamma(o,xg_t)}{\log t} = \frac{1}{n}
\]
for \( \sigma \)-a.e. \( x \in \Gamma \backslash G \). First we note that if (5.1) holds for \( \Gamma \), then it also holds for any \( \Gamma' = g^{-1}\Gamma g \). This follows from the following identity
\[
\text{dist}_\Gamma(\Gamma h, \Gamma h') = \text{dist}_{\Gamma'}(\Gamma' g^{-1} h, \Gamma' g^{-1} h'),
\]
where \( h, h' \) are any two elements in \( G \) and \( \text{dist}_{\Gamma'} \) is the hyperbolic distance function on \( \Gamma' \backslash G/K \). Hence we can assume that \( \Gamma \) has a cusp at \( \infty \). Fix this \( \Gamma' \) and we denote dist = dist\(_{\Gamma'}\) without ambiguity. Next note that \( \{g_t\} \) can be replaced by a new flow \( \{\tilde{g}_t\} \) with \( \tilde{g}_t = k^{-1} g_{nt} k \) for some \( k \in K \) and \( \eta > 0 \). This is because
\[
\limsup_{t \to \infty} \frac{\text{dist}(o, x\tilde{g}_t)}{\log t} = \limsup_{t \to \infty} \frac{\text{dist}(o, x'g_t)}{\log t}
\]
with \( x' = xk^{-1} \). For any \( x \in \mathbb{V}^{n-1} \), denote \( u_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \) to be the corresponding lower triangular unipotent matrix.
Lemma 5.1. Every unipotent element in $G$ is $K$-conjugate to $u_x$ for some $x > 0$.

Proof. Let $g$ be an unipotent element in $G$. We first note that it suffices to show $g$ is $K$-conjugate to $u_x$ for some $x > 0$. This is because $u_x$ is conjugate to $u_{-x}$ via $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in K$. Next we note that since $g$ is unipotent, $g$ is conjugate to some element in $N$. By Iwasawa decomposition and the fact that $NA$ normalizes $N$, $g$ is $K$-conjugate to some element in $N$. Hence we can assume $g$ is contained in $N$. Finally, we note that any element in $N$ is conjugate to some $u_x$ with $x > 0$ via the group $M$ since the conjugation action of $M$ on $N$ realizes $M$ as the rotation group of $N$. 

Since we can conjugate $\{g_t\}$ by some element in $K$ and rescale it by a positive number, in view of Lemma 5.1 we can assume the unipotent flow is given by $\{g_t = u_t\}_{t \in \mathbb{R}}$.

5.1. Technical lemmas. For any $\mathcal{D} \subset Q \setminus G$, we denote $|\mathcal{D}|$ to be its measure with respect to the right $G$-invariant measure on $Q \setminus G$ as fixed in (2.5). We define the set $Y_\mathcal{D} \subset \Gamma \setminus G$ corresponding to $\mathcal{D}$ by

$$Y_\mathcal{D} := \{ \gamma g \in \Gamma \setminus G \mid Q \gamma g \in \mathcal{D} \text{ for some } \gamma \in \Gamma \}.$$ 

Let $\{r_\ell\}$ be any sequence of positive numbers such that $r_\ell \to \infty$ and $\sum_{\ell = 1}^{\infty} e^{-n r_\ell} = \infty$. For any integer $m \geq 1$, let $p(m) > m$ be an integer such that $\sum_{\ell = m}^{p(m)} e^{-n r_\ell} \geq 1$ for all $m \geq 1$. Let $N^-$ be the subgroup of lower triangular unipotent matrices and

$$B^- = \{ u_x^- \mid x \in \mathbb{V}^{n-1}, |x| < \frac{1}{2} \}$$

be the open ball with radius $\frac{1}{2}$, centered at the identity in $N^-$. We define the set

$$\mathcal{D}_m := Q \setminus \bigcup_{\ell = m}^{p(m)} QA(r_\ell) B^- g_{-\ell} \subset Q \setminus G,$$

where $A(\tau) = \{ a_t \mid t \geq \tau \}$.

Lemma 5.2. $||\mathcal{D}_m||_{m \geq 1}$ is uniformly bounded from below.

Proof. Note that every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $d \neq 0$ can be written uniquely as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d' & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} |d|^{-1} & 0 \\ 0 & |d| \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Hence $MAN^- = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid d \neq 0 \right\}$ is a Zariski open dense subset of $G$. Thus there is a Zariski open dense subset in $Q \setminus G$ which can be expressed by the coordinates $Qg = QA_t u_x^-$. We note that in these coordinates, the right $G$-invariant measure on $Q \setminus G$ (up to scalars) is given by $e^{-nt} dtdx$ since this is the right Haar measure for the group $AN^-$. Moreover, $\mathcal{D}_m$ is a disjoint union of the sets...
$Q \backslash QA(r_\ell)B^{-g_{-\ell}}$ since $B^{-g_{-\ell_1}} \cap B^{-g_{-\ell_2}} = \emptyset$ whenever $\ell_1 \neq \ell_2$. Hence one can compute

$$|\Omega_m| = \sum_{\ell = m}^{p(m)} \int_{r_\ell}^\infty e^{-n_\ell t} dt \int_{B^{-g_{-\ell}}} 1 dx \approx \sum_{\ell = m}^{p(m)} e^{-n_\ell t} \geq 1. \quad \square$$

**Lemma 5.3.** There is some sufficiently large integer $L$ such that for any $m \geq L$ and any $x \in Y_{\Omega_m}$ there exists $m \leq \ell \leq p(m)$ such that

$$\text{dist}(o, xg_\ell) \geq r_\ell + O(1).$$

*Proof.* First recall the Siegel fundamental domain $\mathcal{F}_{\Gamma, r_0, U_0}$ we fixed in (2.8), take $L$ such that $r_\ell - \log 2 \geq \tau_0$ for all $\ell \geq L$. Next using (3.10) we have that for any $\tau \in \mathbb{R}$

$$QA(\tau) B^{-\ell} \subset QA(\tau - \log 2) \subset NA(\tau - \log 2) K.$$ (5.2)

Hence for $m \geq L$, $x \in Y_{\Omega_m}$ can be written as $x = \Gamma g_{xg_\ell}$ for some $m \leq \ell \leq p(m)$ with $g \in QA(\tau_{\ell}) B^{-\ell} \subset NA(\tau_{\ell} - \log 2) K$. After left multiplying by some $\gamma \in \Gamma' \setminus \Lambda$ we can assume that $g = u_{\ell} a_{\ell} k$ is contained in the Siegel set $U_0 A[\tau_\ell - \log 2] K$ (we can do this since $U_0$ contains a fundamental domain of $\Gamma' \setminus \Lambda$). Since $\ell \geq m \geq L$, $r_\ell - \log 2 \geq \tau_0$. By (2.11) we have

$$\text{dist}(o, xg_\ell) = \text{dist}(o, u_{\ell} a_{\ell} k) = t + O(1) \geq r_\ell - \log 2 + O(1) = r_\ell + O(1). \quad \square$$

The next lemma shows that there exists a nice set sitting inside $\Omega_m$. We first identify $Q \backslash G$ with $A \times M \backslash K$. Let $Pr : Q \backslash G \to M \backslash K$ be the natural projection map and

$$K(\ell) := \text{Pr}(Q \backslash QA(\tau_{\ell}) B^{-g_{-\ell}})$$

be the $K$-part of $Q \backslash QA(\tau_{\ell}) B^{-g_{-\ell}}$. We note that $K(\ell)$ is independent of $r_{\ell}$, that is, $K(\ell) = \text{Pr}(Q \backslash QA(\tau) B^{-g_{-\ell}})$ for any $\tau \in \mathbb{R}$.

**Lemma 5.4.** For any $\ell \geq 1$, let $\tau_{\ell} = r_{\ell} - 2\log(\ell) + \log 2$. Then

$$A(\tau_{\ell}) \times K(\ell) \subset Q \backslash QA(\tau_{\ell}) B^{-g_{-\ell}} \quad \text{and} \quad |A(\tau_{\ell}) \times K(\ell)| = |Q \backslash QA(\tau_{\ell}) B^{-g_{-\ell}}|$$

with the implicit constant independent of $\ell$.

*Proof.* For each $k \in K(\ell)$, define

$$I(k) := \{ t \in \mathbb{R} \mid Qa_{\ell} k \in Q \backslash QA(\tau_{\ell}) B^{-g_{-\ell}} \}$$

and

$$t(k) := \inf I(k).$$

We first note that if $t \in I(k)$ (that is, $a_{\ell} k = qa_{t_0} u_{x_{t_0}}$ for some $q \in Q$, $t_0 \geq r_{\ell}$ and $|x_{t_0}| < \frac{1}{2}$), then $(t, \infty) \subset I(k)$. This is because for any $t' > t$ we have

$$a_{\ell} k = a_{t' - t} a_{\ell} k = a_{t' - t} qa_{t_0} u_{x_{t_0}} = q' a_{t' - t + t_0} u_{x_{t_0}},$$

where $q' = qa_{t_0} u_{x_{t_0}}$. Therefore

$$Q \backslash QA(r_{t'}) B^{-g_{-t'}} \cap \cup_{k \in K(\ell)} \{ qa : q \in Q \} = \emptyset.$$
and \( t' - t + t_0 > t_0 \geq r_\ell \). Here \( q' = a_{r'-t} q a_{t'-t} \in Q \). This implies that

\[
Q \setminus QA(r_\ell) B^{-} g_{-\ell} = \bigcup_{k \in K(\ell)} A(t(k)) \times \{k\}.
\]

Moreover, by (3.10), the relation \( a_t k = q a_u u_{x_{-\ell}}^{-} \) implies

\[
t = t_0 - \log(1 + |x - \ell|^2).
\]

In particular, the minimality of \( t(k) \) implies that

\[
t(k) = r_\ell - \log(1 + |x - \ell|^2)
\]

for some \( x \) (determined by \( k \)). As \( k \) ranges over \( K(\ell) \) (that is, \( x \) ranges over \( B^{-} \)),

\( t(k) \) attains the maximal value when \( x = \frac{1}{2} \) and the minimal value when \( x = -\frac{1}{2} \).

Let \( t_{\ell, \pm \frac{1}{2}} = r_\ell - \log(1 + |\ell + \frac{1}{2}|^2) \), then in view of (5.3) we have

\[
A(t_{\ell, \frac{1}{2}}) \times K(\ell) \subset Q \setminus QA(r_\ell) B^{-} g_{-\ell} \subset A(t_{\ell, -\frac{1}{2}}) \times K(\ell).
\]

Next, note that \( e^{-n\ell_{\ell, \frac{1}{2}}} \approx e^{-n\ell_{\ell, -\frac{1}{2}}} \approx e^{-n\ell} \epsilon^{2n} \), hence

\[
|A(t_{\ell, \frac{1}{2}}) \times K(\ell)| = |Q \setminus QA(r_\ell) B^{-} g_{-\ell}| \approx |A(t_{\ell, -\frac{1}{2}}) \times K(\ell)|.
\]

Finally, note that \( t_{\ell, \frac{1}{2}} \leq \tau_\ell \) and \( e^{-n\ell_{\ell, \frac{1}{2}}} \approx e^{-n\ell} \epsilon^{2n} \), hence

\[
A(\tau_\ell) \times K(\ell) \subset Q \setminus QA(r_\ell) B^{-} g_{-\ell} \quad \text{and} \quad |A(\tau_\ell) \times K(\ell)| = |Q \setminus QA(r_\ell) B^{-} g_{-\ell}|. \quad \square
\]

**Remark 7.** Later we will take \( r_\ell = \frac{1 - \epsilon}{n} \log(\ell) \) with \( \epsilon \) some fixed small positive number. We note that in this case we can take \( p(m) = 2m \). Moreover, since \( \tau_m \geq \tau_\ell \) and \( e^{-n\tau_m} \approx e^{-n\tau_\ell} \approx \epsilon^{2n(1 + \epsilon)} \) for all \( m \leq \ell \leq 2m \), in view of Lemma 5.5 we have

\[
A(\tau_m) \times K_m \subset \mathcal{D}_m \quad \text{and} \quad |A(\tau_m) \times K_m| \approx |\mathcal{D}_m|,
\]

where \( K_m := \bigcup_{\ell = m}^{2m} K(\ell) \).

### 5.2. Proof of Theorem 1.1

Now we can give the proof of logarithm laws.

**Upper bound.** Fix \( \epsilon > 0 \) and let \( r_\ell = \frac{1 + \epsilon}{n} \log(\ell) \). By (2.12) the sets

\[
\{ x \in \Gamma \setminus G | x g_\ell \in B_{r_\ell} \} = B_{r_\ell} g_{-\ell}
\]

satisfy

\[
\sum_{\ell = 1}^{\infty} \sigma(B_{r_\ell} g_{-\ell}) = \sum_{\ell = 1}^{\infty} \sigma(B_{r_\ell}) = \sum_{\ell = 1}^{\infty} \frac{1}{\ell^{1 + \epsilon}} < \infty.
\]

Hence by Borel-Cantelli lemma the set

\[
\{ x \in \Gamma \setminus G | x g_\ell \in B_{r_\ell} \text{ for finitely many } \ell \}
\]

has full measure. This implies that

\[
\limsup_{\ell \to \infty} \frac{\text{dist}(o, x g_\ell)}{\log(\ell)} \leq \frac{1 + \epsilon}{n}
\]

for \( \sigma \)-a.e. \( x \in \Gamma \setminus G \). Moreover, for all \( t \in \mathbb{R} \) let \( \ell = \lfloor t \rfloor \), we have

\[
|\text{dist}(o, x g_\ell) - \text{dist}(o, x g_\ell)| \leq \text{dist}(x g_\ell, x g_\ell) \leq \text{dist}_G(e, g_{\ell - t}) = O(1),
\]
hence we can replace the discrete limit over \( \ell \in \mathbb{N} \) with a continuous limit over \( t \in \mathbb{R} \). Finally, letting \( \epsilon \to 0 \) we get

\[
\limsup_{t \to \infty} \frac{\text{dist}(o, xg_t)}{\log t} \leq 1
\]

for \( \sigma \)-a.e. \( x \in \Gamma \setminus G \).

**Lower bound.** Fix \( \epsilon > 0 \) and let \( r_\ell = \frac{1-\epsilon}{n} \log \ell \). Let \( \mathcal{D}_m \) and \( Y_{\mathcal{D}_m} \) be as above. Note that in this case, for the definition of \( \mathcal{D}_m \) we can take \( p(m) = 2m \). We first prove the following

**Lemma 5.5.** There is a constant \( \kappa_G > 0 \) depending only on \( \Gamma \) such that \( \sigma(Y_{\mathcal{D}_m}) \geq \kappa_G \) for all \( m \geq 1 \).

**Proof.** Let \( \tau_m = r_m - 2\log(m) + \log 2 \) be as above. Let \( T \) be a sufficiently large integer such that

\[
(5.5) \quad \frac{1}{2n} e^{-rt_m} \leq \int_{T_m} e^{-nt} dt \leq \frac{2}{n} e^{-nt_m} \quad \text{and} \quad \frac{1}{2n} e^{-st_m} \leq \int_{T_m} e^{-st} dt \leq \frac{4}{n} e^{-st_m}
\]

for any \( s \in \left( \frac{n}{2}, n \right) \) and \( m \geq 1 \). We identify the subgroup \( A \) with \( \mathbb{R} \) by sending \( a_t \) to \( t \). For every \( m \geq 1 \) define the set

\[
\mathcal{D}'_m := [\tau_m, T] \times K_m,
\]

where \( K_m = \cup_{\ell=m} K(\ell) \). By Remark 7 we have \( \mathcal{D}'_m \subset \mathcal{D}_m \) and \( |\mathcal{D}'_m| = |\mathcal{D}_m| \). In particular, by Lemma 5.2, \( |\mathcal{D}'_m| \geq 1 \) are uniformly bounded from below for all \( m \geq 1 \). Using the same unfolding trick as we did in Lemma 3.1, one has the following identity

\[
\int_{\Gamma \setminus G} \Theta_{1_{\mathcal{D}'_m}}(g) d\sigma (g) = \frac{\omega}{\nu} \int_{Q \setminus G} 1_{\mathcal{D}'_m}(g) dg
\]

where \( \omega = \int_{\Gamma \setminus G} dq \) and \( \nu = \int_{\Gamma} dg \). By Cauchy-Schwartz and the fact that \( \Theta_{1_{\mathcal{D}'_m}} \) is supported on \( Y_{\mathcal{D}'_m} \), we have

\[
\left( \frac{\omega}{\nu} \right)^2 |\mathcal{D}'_m|^2 = \left( \int Y_{\mathcal{D}'_m} \Theta_{1_{\mathcal{D}'_m}}(g) d\sigma (g) \right)^2 \leq \sigma(Y_{\mathcal{D}'_m}) ||\Theta_{1_{\mathcal{D}'_m}}||^2_2.
\]

Now in view of (5.5) we can take \( f_m = \nu_m 1_{K_m} \) with \( 1_{K_m} \) the characteristic function of \( K_m \) and \( \nu_m \) approximating \( 1_{[\tau_{2m}, T]} \) from above sufficiently well such that

1. \( \nu_m \) is smooth, compactly supported and takes values in \([0, 1]\); 
2. \( \frac{1}{3n} \leq \frac{\int \nu_m(t) e^{-st} dt}{e^{-st_m}} \leq 3 \) and \( \frac{1}{3n} \leq \frac{\int \nu_m(t) e^{-st} dt}{e^{-st_m}} \leq 5 \) for any \( s \in \left( \frac{n}{2}, n \right) \); 
3. \( ||f_m||_1 \leq 2 |\mathcal{D}'_m| \).

In particular, for any \( s \in \left( \frac{n}{2}, n \right) \)

\[
\frac{\int \nu_m(t) e^{-st} dt}{(\int \nu_m(t) e^{-nt} dt)^{\frac{2}{n} - 1}} \leq \frac{5}{n} e^{-st_m}
\]

\[
(\int \nu_m(t) e^{-nt} dt)^{\frac{2}{n} - 1} \leq \frac{5}{n} e^{-st_m}
\]

\[
(\int \nu_m(t) e^{-nt} dt)^{\frac{2}{n} - 1 + \frac{1}{n}} < 15.
\]
Hence \( \{f_m\} \subset \mathcal{A}_{15} \) and by Theorem 1.2 we can bound
\[
||\Theta_{f_m}||_2^2 \leq ||\Theta_{f_m}||_2^2 \lesssim \Gamma ||f_m||_1^2 + ||f_m||_2^2.
\]
Next, since \( ||f_m||_2^2 \leq ||f_m||_1, ||f_m||_1 \leq 2|\mathcal{D}_m'| \) and \( |\mathcal{D}_m'| \gtrsim 1 \), we can bound
\[
||f_m||_2^2 + ||f_m||_1^2 \leq ||f_m||_2^2 + ||f_m||_1 \leq 4|\mathcal{D}_m'|^2 + 2|\mathcal{D}_m'| \lesssim |\mathcal{D}_m'|^2.
\]
Finally, since \( Y_{\mathcal{D}_m'} \subset Y_{\mathcal{D}_m} \), we conclude that there is a constant \( \kappa_\Gamma > 0 \) (independent of \( m \)) such that \( \sigma(Y_{\mathcal{D}_m}) \geq \sigma(Y_{\mathcal{D}_m'}) \gtrsim \kappa_\Gamma \) for any \( m \geq 1 \).

Now consider the set \( \mathcal{B}_\varepsilon := \cap_{t=1}^\infty \cup_{m=\ell}^{\infty} Y_{\mathcal{D}_m}, \) where \( L \) is as in Lemma 5.3. Then \( \sigma(\mathcal{B}_\varepsilon) \gtrsim \kappa_\Gamma > 0 \) by Lemma 5.5. Moreover, by Lemma 5.3, for any \( m \geq L, x \in Y_{\mathcal{D}_m} \) there is some \( \ell \geq m \) such that \( \text{dist}(o, xg_{\ell}) \geq r_\ell + O(1) \). Hence for any \( x \in \mathcal{B}_\varepsilon \) there is a sequence \( \ell_m \rightarrow \infty \) such that \( \text{dist}(o, xg_{\ell_m}) \gtrsim r_{\ell_m} + O(1) \). Consequently, we have
\[
\mathcal{B}_\varepsilon \subseteq \left\{ x \in \Gamma \backslash G \mid \limsup_{t \to \infty} \frac{\text{dist}(o, xg_t)}{\log t} \geq \frac{1}{n} \right\}.
\]
Since the latter set is invariant under the action of \( \{g_t\}_{t \in \mathbb{R}} \), by ergodicity it has full measure. Letting \( \varepsilon \to 0 \) we get
\[
\limsup_{t \to \infty} \frac{\text{dist}(o, xg_t)}{\log t} \geq \frac{1}{n}
\]
for \( \sigma \)-a.e. \( x \in \Gamma \backslash G \).

**Remark 8.** The same argument works for \( r_\ell = \frac{1}{n} \log \ell + O(1) \) taking \( p(m) = 2m \). Hence we can show that the sequence of nested cusp neighborhoods \( \{B_{r_\ell}\} \) with \( \sigma(B_{r_\ell}) = \frac{1}{m} \) is Borel-Cantelli for unipotent flows in this setting.

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