SOME BASIC RESULTS ON ACTIONS OF NON-AFFINE ALGEBRAIC GROUPS

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Abstract. We study actions of connected algebraic groups on normal algebraic varieties, and show how to reduce them to actions of affine subgroups.

0. Introduction

Algebraic group actions have been extensively studied under the assumption that the acting group is affine or, equivalently, linear; see [KSS, MPK, PV]. In contrast, little seems to be known about actions of non-affine algebraic groups. In this paper, we show that these actions may be reduced to actions of affine subgroup schemes, in the setting of normal varieties.

Our starting point is the following theorem of Nishi and Matsumura (see [Ma]). Let $G$ be a connected algebraic group of automorphisms of a nonsingular algebraic variety $X$ and denote by

$$\alpha_X : X \to A(X)$$

the Albanese morphism, that is, the universal morphism to an abelian variety (see [Se2]). Then $G$ acts on $A(X)$ by translations, compatibly with its action on $X$, and the kernel of the induced homomorphism $G \to A(X)$ is affine.

When applied to the action of $G$ on itself via left multiplication, this shows that the Albanese morphism

$$\alpha_G : G \to A(G)$$

is a surjective group homomorphism having an affine kernel. Since this kernel is easily seen to be smooth and connected, this gives back Chevalley’s structure theorem: any connected algebraic group $G$ is an extension of an abelian variety $A(G)$ by a connected affine algebraic group $G_{aff}$ (see [Co] for a modern proof).

The Nishi–Matsumura theorem may be reformulated as follows: for any faithful action of $G$ on a nonsingular variety $X$, the induced homomorphism $G \to A(X)$ factors through a homomorphism $A(G) \to A(X)$ having a finite kernel (see [Ma] again). This easily implies the existence of a $G$-equivariant morphism

$$\psi : X \to A,$$
where $A$ is an abelian variety, quotient of $A(G)$ by a finite subgroup scheme (see Section 3 for details). Equivalently, $A \cong G/H$ where $H$ is a closed subgroup scheme of $G$ such that $H \supset G_{\text{aff}}$ and the quotient $H/G_{\text{aff}}$ is finite; in particular, $H$ is affine, normalized by $G$, and uniquely determined by $A$. Then there is a $G$-equivariant isomorphism

$$X \cong G \times^H Y,$$

where the right-hand side denotes the homogeneous fiber bundle over $G/H$ associated to the scheme-theoretic fiber $Y$ of $\psi$ at the base point.

In particular, given a faithful action of an abelian variety $A$ on a nonsingular variety $X$, there exist a positive integer $n$ and a closed $A_n$-stable subscheme $Y \subset X$ such that $X \cong A \times^A Y$, where $A_n \subset A$ denotes the kernel of the multiplication by $n$. For free actions (that is, abelian torsors), this result is due to Serre, see [Se1, Prop. 17].

Next, consider a faithful action of $G$ on a possibly singular variety $X$. Then, besides the Albanese morphism, we have the Albanese map

$$\alpha_{X,r} : X \to A(X)_r,$$

i.e., the universal rational map to an abelian variety. Moreover, the regular locus $U \subset X$ is $G$-stable, and $A(U) = A(U)_r = A(X)_r$. Thus, $G$ acts on $A(X)_r$ via a homomorphism $A(G) \to A(X)_r$ such that the canonical homomorphism

$$h_X : A(X)_r \to A(X)$$

is equivariant; $h_X$ is surjective, but generally not an isomorphism, see [Se2] again. Applying the Nishi–Matsumura theorem to $U$, we see that the kernel of the $G$-action on $A(X)_r$ is affine, and there exists a $G$-equivariant rational map $\psi_r : X \to A$ for some abelian variety $A$ as above.

However, $G$ may well act trivially on $A(X)$; then there exists no morphism $\psi$ as above, and $X$ admits no equivariant embedding into a nonsingular $G$-variety. This happens for several classes of examples constructed by Raynaud, see [Ra, XII 1.2, XIII 3.2] or Examples 5.2, 5.3, 5.4; in the latter example, $X$ is normal, and $G$ is an abelian variety acting freely. However, we shall show that such an equivariant embedding (in particular, such a morphism $\psi$) exists locally for any normal $G$-variety.

To state our results in a precise way, we introduce some notation and conventions. We consider algebraic varieties and schemes over an algebraically closed field $k$; morphisms are understood to be $k$-morphisms. By a variety, we mean a separated integral scheme of finite type over $k$; a point will always mean a closed point. As a general reference for algebraic geometry, we use the book [Ha], and [DG] for algebraic groups.
We fix a connected algebraic group $G$, that is, a $k$-group variety; in particular, $G$ is smooth. A $G$-variety is a variety $X$ equipped with an algebraic $G$-action
\[ \varphi : G \times X \rightarrow X, \quad (g, x) \mapsto g \cdot x. \]
The kernel of $\varphi$ is the largest subgroup scheme of $G$ that acts trivially. We say that $\varphi$ is faithful if its kernel is trivial. The $G$-variety $X$ is homogeneous (resp. almost homogeneous) if it contains a unique orbit (resp. an open orbit). Finally, a $G$-morphism is an equivariant morphism between $G$-varieties.

We may now formulate our results, and discuss their relations to earlier work.

**Theorem 1.** Any normal $G$-variety admits an open covering by $G$-stable quasi-projective subsets.

When $G$ is affine, this fundamental result is due to Sumihiro (see \[KSS, Su\]); on the other hand, it has been obtained by Raynaud in another setting, namely, for actions of smooth, connected group schemes on smooth schemes over normal bases (in particular, for actions of connected algebraic groups on nonsingular varieties), see \[Ra, Cor. V 3.14\].

Theorem 1 implies readily the quasi-projectivity of homogeneous varieties and, more generally, of normal varieties where a connected algebraic group acts with a unique closed orbit. For the latter result, the normality assumption cannot be omitted, as shown by Example 5.2.

Next, we obtain a version of the Nishi–Matsumura theorem:

**Theorem 2.** Let $X$ be a normal, quasi-projective variety on which $G$ acts faithfully. Then there exists a $G$-morphism
\[ \psi : X \rightarrow A, \]
where $A$ is the quotient of $A(G)$ by a finite subgroup scheme. Moreover, $X$ admits a $G$-equivariant embedding into the projectivization of a $G$-homogeneous vector bundle over $A$.

The second assertion generalizes (and builds on) another result of Sumihiro: when $G$ is affine, any normal quasi-projective $G$-variety admits an equivariant embedding into the projectivization of a $G$-module (see \[KSS, Su\] again). It implies that any normal, quasi-projective $G$-variety has an equivariant embedding into a nonsingular $G$-variety, of a very special type. Namely, a vector bundle $E$ over an abelian variety $A$ is homogeneous with respect to a connected algebraic group $G$ (acting transitively on $A$) if and only if $a^*(E) \cong E$ for all $a \in A$; see \[Muk\] for a description of all homogeneous vector bundles on an abelian variety.

Clearly, a morphism $\psi$ as in Theorem 2 is not unique, as we may replace $A$ with any finite quotient. In characteristic zero, we can always
impose that the fibers of $\psi$ are normal varieties, by using the Stein factorization. Under that assumption, there may still exist several morphisms $\psi$, but no universal morphism (see Example 5.1).

Returning to arbitrary characteristics, there does exist a universal morphism $\psi$ when $X$ is normal and almost homogeneous, namely, the Albanese morphism:

**Theorem 3.** Let $X$ be a normal, almost homogeneous $G$-variety. Then $A(X) = A(X)_G = G/H$, where $H \subset G$ is a closed subgroup scheme containing $G_{\text{aff}}$, and the quotient group scheme $H/G_{\text{aff}}$ is finite. Moreover, each fiber of the Albanese morphism is a normal variety, stable under $H$ and almost homogeneous under $G_{\text{aff}}$.

This Albanese fibration is well-known in the setting of complex Lie groups acting on compact Kähler manifolds (the Remmert–van de Ven theorem, see e.g. [Ak, Sec. 3.9]), and easily obtained for nonsingular varieties (see [Br, Sec. 2.4]).

In particular, Theorem 3 applies to any normal $G$-equivariant embedding of $G$, i.e., to a normal $G$-variety $X$ containing an open orbit isomorphic to $G$: then $A(X) = A(G) = G/G_{\text{aff}}$ and hence

$$X \cong G \times^{G_{\text{aff}}} Y,$$

where $Y$ is a normal $G_{\text{aff}}$-equivariant embedding of $G_{\text{aff}}$. (Here again, the normality assumption cannot be omitted, see Example 5.2). If $G$ is a semi-abelian variety, that is, $G_{\text{aff}}$ is a torus, then $Y$ is a toric variety, and the $G$-equivariant embedding $X$ is called a semi-abelic variety. In that case, the above isomorphism has been obtained by Alexeev under the assumption that $X$ is projective (see [Al, Sec. 5]). As another application, consider a normal algebraic monoid $X$ with unit group $G$; then $X$ may be regarded as a $(G \times G)$-equivariant embedding of $G$, and the above isomorphism yields another proof of the main result of [BR].

For a complete homogeneous variety, it is known that the Albanese fibration is trivial. In the setting of compact homogeneous Kähler manifolds, this is the classical Borel–Remmert theorem (see e.g. [Ak, Sec. 3.9]); in our algebraic setting, this is part of a structure result due to Sancho de Salas:

**Theorem 4.** ([Sa Thm. 5.2]). Let $X$ be a complete variety, homogeneous under a faithful action of $G$. Then there exists a canonical isomorphism $G \cong A(G) \times G_{\text{aff}}$, and $G_{\text{aff}}$ is semi-simple of adjoint type. Moreover, there exists a canonical isomorphism $G$-varieties $X \cong A(G) \times Y$, where $Y$ is a complete, homogeneous $G_{\text{aff}}$-variety; the Albanese morphism of $X$ is the first projection $X \to A(G)$. 

Here we present a short proof of that result by analyzing first the structure of $G$ and then the Albanese morphism of $X$, while [Sa] proceeds by constructing the second projection $X \to Y$ via the study of the $G_{\text{aff}}$-action on $X$. In characteristic zero, this projection is the Tits morphism that assigns to any point of $X$ its isotropy Lie algebra; this yields a very simple proof of Theorem 4, see [Br, Sec. 1.4]. But this approach does not extend to positive characteristics, as the $G$-action on $X$ may well be non-smooth; equivalently, the orbit maps may be non-separable. In fact, there exist many complete varieties $Y$ that are homogeneous under a non-smooth action of a semi-simple group $G_{\text{aff}}$. These varieties of unseparated flags are classified in [HL, Sa, We]. Any such variety $Y$ is projective (e.g., by Theorem 1) and rational. Indeed, $Y$ contains only finitely many fixed points of a maximal torus of $G_{\text{aff}}$; thus, $Y$ is paved by affine spaces, see [Bi].

In positive characteristics again, the homogeneity assumption of Theorem 1 cannot be replaced with the assumption that the tangent bundle is globally generated. Indeed, there exist nonsingular projective varieties which have a trivial tangent bundle, but fail to be homogeneous (see [El] or Example 5.5). Also, for $G$-varieties of unseparated flags, the subbundle of the tangent bundle generated by Lie($G$) is a proper direct summand (see Example 5.6).

The proofs of Theorems 1–4 are given in the corresponding sections. The final Section 5 presents examples illustrating the assumptions of these theorems. Three of these examples are based on constructions, due to Raynaud, of torsors under abelian schemes, that we have adapted so as to make them known to experts in algebraic groups.

In the opposite direction, it would be interesting to extend our results to group schemes. Our methods yield insight into actions of abelian schemes (e.g., the proof of Theorem 2 adapts readily to that setting), but the general case requires new ideas.

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1. Proof of Theorem 1

As in the proof of the projectivity of abelian varieties (see e.g. [Mi Thm. 7.1]), we first obtain a version of the theorem of the square. We briefly present the setting of this theorem, referring to [Ral Chap. IV] and [BLR Sec. 6.3] for further developments.

Consider an invertible sheaf $L$ on a $G$-variety $X$. For any $g \in G$, let $g^*(L)$ denote the pull-back of $L$ under the automorphism of $X$ induced by $g$. In loose words, $L$ satisfies the theorem of the square if there are compatible isomorphisms

\[(gh)^*(L) \otimes L \cong g^*(L) \otimes h^*(L)\] \hspace{1cm} (1.1)
for all \( g, h \) in \( G \). Specifically, consider the action \( \varphi : G \times X \to X \), the second projection \( p_2 : G \times X \to X \), and put

\[
(1.2) \quad \mathcal{L} := \varphi^*(L) \otimes p_2^*(L)^{-1}.
\]

This is an invertible sheaf on \( G \times X \), satisfying

\[
\mathcal{L}|_{\{g\} \times X} \cong g^*(L) \otimes L^{-1}
\]

for all \( g \in G \). Next, consider the variety \( G \times G \times X \) over \( X \), and the \( X \)-morphisms

\[
m, p_1, p_2 : G \times G \times X \to G \times X
\]

given by the multiplication and the projections \( G \times G \to G \). Then

\[
\mathcal{M} := m^*(\mathcal{L}) \otimes p_1^*(\mathcal{L})^{-1} \otimes p_2^*(\mathcal{L})^{-1}
\]

is an invertible sheaf on \( G \times G \times X \), such that

\[
\mathcal{M}|_{\{(g,h)\} \times X} \cong (gh)^*(L) \otimes L \otimes g^*(L)^{-1} \otimes h^*(L)^{-1}
\]

for all \( g, h \) in \( G \). Now \( L \) satisfies the theorem of the square, if \( \mathcal{M} \) is the pull-back of some invertible sheaf under the projection

\[
f : G \times G \times X \to G \times G.
\]

Then each \( \mathcal{M}|_{\{(g,h)\} \times X} \) is trivial, which implies (1.1).

Also, recall the classical notion of a \( G \)-linearization of the invertible sheaf \( L \), that is, a \( G \)-action of the total space of the associated line bundle which is compatible with the \( G \)-action on \( X \) and commutes with the natural \( \mathbb{G}_m \)-action (see [KSS, MFK]). The isomorphism classes of \( G \)-linearized invertible sheaves form a group denoted by \( \text{Pic}^G(X) \). We will use the following observation (see [MFK, p. 32]):

**Lemma 1.1.** Let \( \pi : X \to Y \) be a torsor under \( G \) for the fppf topology. Then the pull-back under \( \pi \) yields an isomorphism \( \text{Pic}(Y) \cong \text{Pic}^G(X) \).

**Proof.** By assumption, \( \pi \) is faithfully flat and fits into a cartesian square

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\varphi} & X \\
p_2 \downarrow & & \downarrow \pi \\
X & \xrightarrow{\pi} & Y.
\end{array}
\]

Moreover, a \( G \)-linearization of an invertible sheaf \( L \) on \( X \) is exactly a descent datum for \( L \) under \( \pi \), see [MFK, Sec. 1.3]. So the assertion follows from faithfully flat descent, see e.g. [BLR, Sec. 6.1]. \( \square \)

We now come to our version of the theorem of the square:

**Lemma 1.2.** Let \( L \) be a \( G_{\text{aff}} \)-linearizable invertible sheaf on a \( G \)-variety \( X \). Then \( L \) satisfies the theorem of the square.
Proof. Consider the action of $G$ on $G \times X$ via left multiplication on the first factor. Then $\varphi$ is equivariant, and $p_2$ is invariant. Hence the choice of a $G_{\text{aff}}$-linearization of $L$ yields a $G_{\text{aff}}$-linearization of the invertible sheaf $\mathcal{L}$ on $G \times X$ defined by (1.2). Note that the map

$$\alpha_G \times \text{id}_X : G \times X \longrightarrow A(G) \times X$$

is a $G_{\text{aff}}$-torsor. By Lemma 1.1, there exists a unique invertible sheaf $\mathcal{L}'$ on $A(G) \times X$ such that

$$\mathcal{L} = (\alpha_G \times \text{id}_X)^*(\mathcal{L}')$$

is $G_{\text{aff}}$-linearized sheaves. Then $\mathcal{M} = (\alpha_G \times \alpha_G \times \text{id}_X)^*(\mathcal{M}')$, where

$$\mathcal{M}' := m'^* (\mathcal{L}') \otimes p'_1*(\mathcal{L}')^{-1} \otimes p'_2*(\mathcal{L}')^{-1}$$

and $m', p'_1, p'_2 : A(G) \times A(G) \times X \rightarrow A(G) \times X$ are defined analogously to $m, p_1, p_2$. Thus, it suffices to show that $\mathcal{M}'$ is the pull-back of an invertible sheaf under the projection

$$f' : A(G) \times A(G) \times X \longrightarrow A(G) \times A(G).$$

Choose $x \in X$ and put

$$\mathcal{M}'_x := \mathcal{M}'|_{A(G) \times A(G) \times \{x\}}.$$  

We consider $\mathcal{M}'_x$ as an invertible sheaf on $A(G) \times A(G)$, and show that the invertible sheaf $\mathcal{M}' \otimes f'^*(\mathcal{M}'_x)^{-1}$ is trivial. By a classical rigidity result (see [Mi Thm. 6.1]), it suffices to check the triviality of the restrictions of this sheaf to $\{0\} \times A(G) \times X$, $A(G) \times \{0\} \times X$, and $A(G) \times A(G) \times \{x\}$. In view of the definition of $\mathcal{M}'_x$, it suffices in turn to show that

$$\mathcal{M}'|_{\{0\} \times A(G) \times X} \cong O_{A(G) \times X}.$$  

For this, note that $\mathcal{L}|_{G_{\text{aff}} \times X} \cong O_{G_{\text{aff}} \times X}$ since $L$ is $G_{\text{aff}}$-linearized. By Lemma 1.1, it follows that

$$\mathcal{L}'|_{\{0\} \times X} \cong O_{\{0\} \times X}.$$  

Thus, $p'_1*(\mathcal{L}')$ is trivial; on the other hand, $m' = p'_2$ on $\{0\} \times A(G) \times X$. These facts imply (1.4).□

Next, recall that for any invertible sheaf $L$ on a normal $G$-variety $X$, some positive power $L^n$ admits a $G_{\text{aff}}$-linearization; specifically, the Picard group of $G_{\text{aff}}$ is finite, and we may take for $n$ the order of that group (see [KSS, p. 67]). Together with Lemma 1.2 this yields:

**Lemma 1.3.** Let $L$ be an invertible sheaf on a normal $G$-variety. Then some positive power $L^n$ satisfies the theorem of the square; we may take for $n$ the order of $\text{Pic}(G_{\text{aff}})$.

From this, we deduce an ampleness criterion on normal $G$-varieties, analogous to a result of Raynaud about actions of group schemes on smooth schemes over a normal base (see [Ra Thm. V 3.10]):
Lemma 1.4. Let $X$ be a normal $G$-variety, and $D$ an effective Weil divisor on $X$. If \text{Supp}(D) contains no $G$-orbit, then some positive multiple of $D$ is a Cartier divisor generated by its global sections. If, in addition, the open subset $X \setminus \text{Supp}(D) \subset X$ is affine, then some positive multiple of $D$ is ample.

Proof. Consider the regular locus $X_0 \subset X$ and the restricted divisor $D_0 := D \cap X_0$. Then $X_0$ is $G$-stable, and the sheaf $\mathcal{O}_{X_0}(D_0)$ is invertible; moreover, $g^*(\mathcal{O}_{X_0}(D_0)) = \mathcal{O}_{X_0}(g \cdot D_0)$ for all $g \in G$. By Lemma 1.3 there exists a positive integer $n$ such that $\mathcal{O}_{X_0}(nD_0)$ satisfies the theorem of the square. Replacing $D$ with $nD$, we obtain isomorphisms $\mathcal{O}_{X_0}(2D_0) \cong \mathcal{O}_{X_0}(g \cdot D_0 + g^{-1} \cdot D_0)$ for all $g \in G$. Since $X$ is normal, it follows that

\begin{equation}
\mathcal{O}_X(2D) \cong \mathcal{O}_X(g \cdot D + g^{-1} \cdot D)
\end{equation}

for all $g \in G$.

Let $U := X \setminus \text{Supp}(D)$. By (1.6), the Weil divisor $2D$ restricts to a Cartier divisor on every open subset $V_g := X \setminus \text{Supp}(g \cdot D + g^{-1} \cdot D) = g \cdot U \cap g^{-1} \cdot U$, $g \in G$.

Moreover, these subsets form a covering of $X$ (indeed, given $x \in X$, the subset

$W_x := \{ g \in G \mid g \cdot x \in U \} \subset G$

is open, and non-empty since $U$ contains no $G$-orbit. Thus, $W_x$ meets its image under the inverse map of $G$, that is, there exists $g \in G$ such that $U$ contains both points $g \cdot x$ and $g^{-1} \cdot x$). It follows that $2D$ is a Cartier divisor on $X$. Likewise, the global sections of $\mathcal{O}_X(2D)$ generate this divisor on each $V_g$, and hence everywhere.

If $U$ is affine, then each $V_g$ is affine as well. Hence $X$ is covered by affine open subsets $X_s$, where $s$ is a global section of $\mathcal{O}_X(2D)$. Thus, $2D$ is ample. □

We may now prove Theorem 1. Let $x \in X$ and choose an affine open subset $U$ containing $x$. Then $G \cdot U$ is a $G$-stable open subset of $X$; moreover, the complement $(G \cdot U) \setminus U$ is of pure codimension 1 and contains no $G$-orbit. Thus, $G \cdot U$ is quasi-projective by Lemma 1.4.

2. Proof of Theorem 2

First, we gather preliminary results about algebraic groups and their actions.

Lemma 2.1. (i) Let $\pi : X \rightarrow Y$ be a torsor under a group scheme $H$. Then $\pi$ is affine if and only if $H$ is affine.

(ii) Let $X$ be a variety on which $G$ acts faithfully. Then the orbit map

$\varphi_x : G \rightarrow G \cdot x$, $g \mapsto g \cdot x$

is affine, for any $x \in X$. 
Let $C(G)$ denote the center of $G$, and $C(G)^0$ its reduced neutral component. Then $G = C(G)^0 G_{\text{aff}}$.

Let $M$ be an invertible sheaf on $A(G)$, and $L := \alpha_G^*(M)$ the corresponding $G_{\text{aff}}$-linearized invertible sheaf on $G$. Then $L$ is ample if and only if $M$ is ample.

Proof. (i) follows by faithfully flat descent, like Lemma 1.1. (ii) Since $\varphi_x$ is a torsor under the isotropy subgroup scheme $G_x \subset G$, it suffices to check that $G_x$ is affine or, equivalently, admits an injective representation in a finite-dimensional $k$-vector space. Such a representation is afforded by the natural action of $G_x$ on the quotient $\mathcal{O}_{X,x} / m_x^n$ for $n \gg 0$, where $\mathcal{O}_{X,x}$ denotes the local ring of $X$ at $x$, with maximal ideal $m_x$; see [Ma, Lem. p. 154] for details.

(iii) By [Se1, Lem. 2], $\alpha_G$ restricts to a surjective morphism $C \to A(G)$. Hence the restriction $C^0 \to A(G) = G/G_{\text{aff}}$ is surjective as well.

(iv) Since the morphism $\alpha_G$ is affine, the ampleness of $M$ implies that of $L$. The converse holds by [Ra, Lem. XI 1.11.1]. □

Next, we obtain our main technical result:

**Lemma 2.2.** Let $X$ be a variety on which $G$ acts faithfully. Then the following conditions are equivalent:

(i) There exists a $G$-morphism $\psi : X \to A$, where $A$ is the quotient of $A(G)$ by a finite subgroup scheme.

(ii) There exists a $G_{\text{aff}}$-linearized invertible sheaf $L$ on $X$ such that $\varphi_x^*(L)$ is ample for any $x \in X$.

(iii) There exists a $G_{\text{aff}}$-linearized invertible sheaf $L$ on $X$ such that $\varphi_{x_0}^*(L)$ is ample for some $x_0 \in X$.

Proof. (i) $\Rightarrow$ (ii) Choose an ample invertible sheaf $M$ on the abelian variety $A$. We check that $L := \psi^*(M)$ satisfies the assertion of (ii). Indeed, by the universal property of the Albanese morphism $\alpha_G$, there exists a unique $G$-morphism $\alpha_x : A(G) \to A$ such that the square

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi_x} & X \\
\downarrow{\alpha_G} & & \downarrow{\psi} \\
A(G) & \xrightarrow{\alpha_x} & A
\end{array}
$$

is commutative. By rigidity (see e.g. [Mi, Cor. 3.6]), $\alpha_x$ is the composite of a homomorphism and a translation. On the other hand, $\alpha_x$ is $G$-equivariant; thus, it is the composite of the quotient map $A(G) \to A$ and a translation. So $\alpha_x$ is finite, and hence $\alpha_x^*(M)$ is ample. By Lemma 2.1 (iv), it follows that $\varphi_x^*(L) = \alpha_x^*(\alpha_x^*(M))$ is ample.

(ii) $\Rightarrow$ (iii) is left to the reader.

(iii) $\Rightarrow$ (i) Consider the invertible sheaves $\mathcal{L}$ on $G \times X$, and $\mathcal{L}'$ on $A(G) \times X$, defined by (1.2) and (1.3). Recall from (1.5) that $\mathcal{L}'$ is
equipped with a rigidification along \( \{0\} \times X \). By [BLR, Sec. 8.1], it follows that \( \mathcal{L}' \) defines a morphism

\[
\psi : X \rightarrow \text{Pic}(A(G)), \quad x \mapsto \mathcal{L}'|_{A(G) \times \{x\}},
\]

where \( \text{Pic}(A(G)) \) is equipped with its scheme structure (reduced, locally of finite type) for which the connected components are exactly the cosets of the dual abelian variety, \( A(G)^\vee \). Since \( X \) is connected, its image under \( \psi \) is contained in a unique coset.

By construction, \( \psi \) maps every point \( x \in X \) to the isomorphism class of the unique invertible sheaf \( M_x \) on \( A(G) \) such that \( \alpha^*_G(M_x) = \varphi_x^*(L) \) as \( G_{\text{aff}} \)-linearized sheaves on \( G \). When \( x = x_0 \), the invertible sheaf \( \varphi_x^*(L) \) is ample, and hence \( M_x \) is ample by Lemma 2.1 (iv). It follows that the points of the image of \( \psi \) are exactly the ample classes \( a^*(M_{x_0}) \), where \( a \in A(G) \).

Moreover, \( \varphi_{gx} = \varphi_x \circ \rho(g) \) for all \( g \in G \) and \( x \in X \), where \( \rho \) denotes right multiplication in \( G \). Thus, \( \alpha^*_G(M_{gx}) = \rho(g)^*(\alpha^*_G(M_x)) \), that is, \( \psi(g \cdot x) = \alpha_g^*(\varphi_x^*(L)) \). In other words, \( \psi \) is \( G \)-equivariant, where \( G \) acts on \( \text{Pic}(A(G)) \) via the homomorphism \( \alpha^*_G : G \rightarrow A(G) \) and the \( A(G) \)-action on \( \text{Pic}(A(G)) \) via pull-back.

Hence the image of \( \psi \) is the \( G \)-orbit of \( \psi(x_0) \), that is, the quotient \( A(G)/F \) where \( F \) denotes the scheme-theoretic kernel of the polarization homomorphism

\[
A(G) \rightarrow A(G)^\vee, \quad a \mapsto a^*(M_{x_0}) \otimes M_{x_0}^{-1}.
\]

\[\square\]

We are now in a position to prove Theorem 2. Under the assumptions of that theorem, we may choose an ample invertible sheaf \( L \) on \( X \). Then \( \varphi_x^*(L) \) is ample for any \( x \in X \), as follows from Lemma 2.1 (ii). Moreover, replacing \( L \) with some positive power, we may assume that \( L \) is \( G_{\text{aff}} \)-linearizable. Now Lemma 2.2 yields a \( G \)-morphism

\[
\psi : X \rightarrow A,
\]

where \( A \cong G/H \) for an affine subgroup scheme \( H \subset G \). So \( X \cong G \times^HY \), where \( Y \subset X \) is a closed \( H \)-stable subscheme. To complete the proof, it suffices to show that \( X \) (and hence \( Y \)) admits an \( H \)-equivariant embedding into the projectivization of an \( H \)-module. Equivalently, \( X \) admits an ample, \( H \)-linearized invertible sheaf. We already know this when \( H \) is smooth and connected; the general case may be reduced to that one, as follows.

We may consider \( H \) as a closed subgroup scheme of some \( \text{GL}_n \). Then \( G_{\text{aff}} \) embeds into \( \text{GL}_n \) as the reduced neutral component of \( H \). The homogeneous fiber bundle \( \text{GL}_n \times^{G_{\text{aff}}}X \) is a normal \( \text{GL}_n \)-variety, see [Sc1, Prop. 4]; it is quasi-projective by [MFK, Prop. 7.1]. The finite
group scheme $H/G_{\text{aff}}$ acts on that variety, and this action commutes with the $GL_n$-action. Hence the quotient

$$(GL_n \times^{G_{\text{aff}}} X)/(H/G_{\text{aff}}) = GL_n \times^H X$$

is a normal, quasi-projective variety as well (see e.g. [Mum, § 12]). So we may choose an ample, $GL_n$-linearized invertible sheaf $L$ on that quotient. The pull-back of $L$ to $X \subset GL_n \times^H X$ is the desired ample, $H$-linearized invertible sheaf.

3. Proof of Theorem

We begin with some observations about the Albanese morphism of a $G$-variety $X$, based on the results of [Se2]. Since the Albanese morphism of $G \times X$ is the product morphism $\alpha_G \times \alpha_X$, there exists a unique morphism of varieties $\beta: A(G) \times A(X) \to A(X)$ such that the following square is commutative:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\varphi} & X \\
\downarrow \alpha_G \times \alpha_X & & \downarrow \alpha_X \\
A(G) \times A(X) & \xrightarrow{\beta} & A(X).
\end{array}
\]

Then $\beta$ is the composite of a homomorphism and a translation. Moreover, $\beta(0, z) = z$ for all $z$ in the image of $\alpha_X$. Since this image is not contained in a translate of a smaller abelian variety, it follows that $\beta(0, z) = z$ for all $z \in A(X)$. Thus, $\beta(y, z) = \alpha_{\varphi}(y) + z$, where

$$\alpha_{\varphi}: A(G) \to A(X)$$

is a homomorphism. In other words, the $G$-action on $X$ induces an action of $A(G)$ on $A(X)$ by translations via $\alpha_{\varphi}$.

Given an abelian variety $A$, the datum of a morphism $X \to A$ up to a translation in $A$ is equivalent to that of a homomorphism $A(X) \to A$. Thus, the datum of a $G$-equivariant morphism

$$\psi : X \to A$$

up to a translation in $A$, is equivalent to that of a homomorphism

$$\alpha_{\psi} : A(X) \to A = A(G)/F$$

such that the composite $\alpha_{\psi} \circ \alpha_{\varphi}$ equals the quotient morphism

$$q : A(G) \to A(G)/F$$

up to a translation. Then the kernel of $\alpha_{\varphi}$ is contained in $F$, and hence is finite.

Conversely, if $\alpha_{\varphi}$ has a finite kernel, then there exist a finite subgroup scheme $F \subset A(G)$ and a homomorphism $\alpha_{\psi}$ such that $\alpha_{\psi} \circ \alpha_{\varphi} = q$. Indeed, this follows from Lemma 2.2 applied to the image of $\alpha_{\varphi}$; alternatively, this is a consequence of the Poincaré complete reducibility theorem (see e.g. [Mi, Prop. 12.1]). We also see that for a fixed
subgroup scheme \( F \subset A(G) \) (or, equivalently, \( H \subset G \)), the set of homomorphisms \( \alpha_\varphi \) is a torsor under \( \text{Hom}(A(X)/\alpha_\varphi(A(G)), A(G)/F) \), a free abelian group of finite rank.

So we have shown that the existence of a \( G \)-morphism \( \psi \) as in Theorem 2 is equivalent to the kernel of the \( G \)-action on \( A(X) \) being affine, and also to the kernel of \( \alpha_\varphi \) being finite.

Next, we assume that \( X \) is normal and quasi-homogeneous, and prove Theorem 3. Choose a point \( x \) in the open orbit \( X_0 \) and denote by \( G_x \subset G \) its isotropy subgroup scheme, so that \( X_0 \) is isomorphic to the homogeneous space \( G/G_x \). Then \( G_x \) is affine by Lemma 2.1, therefore, the product \( H_0 := G_x G_{\text{aff}} \) is a closed affine subgroup scheme of \( G \), and the quotient \( F := H_0/G_{\text{aff}} \) is finite. As a consequence, the homogeneous space

\[
G/H_0 \cong (G/G_{\text{aff}})/(H_0/G_{\text{aff}}) = A(G)/F
\]

is an abelian variety, isogenous to \( A(G) \).

We claim that the Albanese morphism of \( X_0 \) is the quotient map \( G/G_x \to G/H_0 \). Indeed, let \( f : X_0 \to A \) be a morphism to an abelian variety. Then the composite

\[
G \xrightarrow{\varphi_x} X_0 \xrightarrow{f} A
\]

factors through a unique morphism \( A(G) = G/G_{\text{aff}} \to A \), which must be invariant under \( G_x \). Thus, \( f \) factors through a unique morphism \( G/H_0 \to A \).

Next, we claim that the Albanese morphism of \( X_0 \) extends to \( X \). Of course, such an extension is unique if it exists; hence we may assume that \( X \) is quasi-projective, by Theorem 1. Let \( \psi : X \to A \) be a morphism as in Theorem 2. Then \( \psi \) factors through a \( G \)-morphism \( \alpha_\psi : A(X) \to A \), and the composite \( \alpha_\psi \circ \alpha_\varphi : A(G) \to A \) is an isogeny. Thus, \( \alpha_\varphi \) is an isogeny, and hence the canonical morphism \( h_X : A(X)_r \to A(X) \) is an isogeny as well (since \( A(X)_r = A(X_0) = A(G)/F \)). Together with Zariski’s main theorem, it follows that the rational map \( \alpha_{X,r} : X \dashrightarrow A(X)_r \) is a morphism, i.e., \( h_X \) is an isomorphism.

4. Proof of Theorem 4

Choose \( x \in X \) so that \( X \cong G/G_x \). The radical \( R(G_{\text{aff}}) \) fixes some point of \( X \), and is a normal subgroup of \( G \); thus, \( R(G_{\text{aff}}) \subset G_x \). By the faithfulness assumption, it follows that \( R(G_{\text{aff}}) \) is trivial, that is, \( G_{\text{aff}} \) is semi-simple. Moreover, the reduced connected center \( C(G)^0 \) satisfies \( C(G)^0_{\text{aff}} \subset R(G_{\text{aff}}) \). Thus, \( C(G)^0 \) is an abelian variety, and hence \( C(G)^0 \cap G_{\text{aff}} \) is a finite central subgroup scheme of \( G \).
We claim that $C(G)^0 \cap G_{\text{aff}}$ is trivial. Consider indeed the reduced neutral component $P$ of $G_x$. Then $P \subset G_{\text{aff}}$ by Lemma 2.1 (ii). Moreover, the natural map
\[ \pi : G/P \longrightarrow G/G_x \cong X \]
is finite, and hence $G/P$ is complete. It follows that $G_{\text{aff}}/P$ is complete as well. Thus, $P$ is a parabolic subgroup of $G_{\text{aff}}$, and hence contains $C(G)^0 \cap G_{\text{aff}}$. In particular, $C(G)^0 \cap G_{\text{aff}} \subset G_x$; arguing as above, this implies our claim.

By that claim and the equality $G = C(G)^0 G_{\text{aff}}$ (Lemma 2.1 (iii)), we obtain that $C(G)^0 = A(G)$ and $G \cong A(G) \times G_{\text{aff}}$, where $G_{\text{aff}}$ is semi-simple and adjoint. Moreover,
\[ G/P \cong A(G) \times (G_{\text{aff}}/P). \]
Next, recall from Section 3 that the Albanese morphism $\alpha_X$ is the natural map $G/G_x \rightarrow G/H$, where $H := G_x G_{\text{aff}}$ is affine. So we may write $H = F \times G_{\text{aff}}$, where $F \subset A(G)$ is a finite subgroup-scheme, and hence a central subgroup-scheme of $G$. Then
\[ X \cong A(G) \times^F Y \]
and $A(X) \cong A(G)/F$, where $Y := H/G_x \cong G_{\text{aff}}/(G_{\text{aff}} \cap G_x)$. We now show that $F$ is trivial; as above, it suffices to prove that $F \subset G_x$.

Choose a maximal torus $T \subset P$ so that $x$ lies in the fixed point subscheme $X^T$. The latter is nonsingular and stable under $A(G)$. Moreover, the restriction
\[ \pi : (G/P)^T = A(G) \times (G_{\text{aff}}/P)^T \longrightarrow X^T \]
is surjective, and hence $A(G)$ acts transitively on each component of $X^T$; the Weyl group of $(G_{\text{aff}}, T)$ permutes transitively these components. Let $Z$ be the component containing $x$. To complete the proof, it suffices to show that the morphism $\alpha_X : X \rightarrow A(G)/F$ restricts to an isomorphism $Z \rightarrow A(G)/F$.

By the Bialynicki-Birula decomposition (see [Bi]), $Z$ admits a $T$-stable open neighborhood $U \subset X$ together with a $T$-equivariant retraction $\rho : U \rightarrow Z$, a locally trivial fibration in affine spaces. It follows that the Albanese morphisms of $U$ and $Z$ satisfy $\alpha_U = \alpha_Z \circ \rho$. Since $\alpha_U$ is the restriction of $\alpha_X$, we obtain that $\alpha_Z = \alpha_X|_Z$. On the other hand, $\alpha_Z$ is the identity, since $Z$ is homogeneous under $A(G)$; this completes the argument.

5. Examples

**Example 5.1.** Let $A$ be an abelian variety. Choose distinct finite subgroups $F_1, \ldots, F_n \subset A$ such that $F_1 \cap \cdots \cap F_n = \{0\}$ (as schemes), and let
\[ X := A/F_1 \times \cdots \times A/F_n. \]
Then $A$ acts faithfully on $X$ via simultaneous translation on all factors, and each projection $p_i : X \to A/F_i$ satisfies the assertion of Theorem 2. Since the fibers of $p_i$ are varieties, this morphism admit no non-trivial factorization through a $A$-morphism $\psi : X \to A/F$, where $F$ is a finite subgroup scheme of $A$.

Assume that the only endomorphisms of $A$ are the multiplications by integers, and the orders of $F_1, \ldots, F_n$ have a non-trivial common divisor. Then there exists no $A$-morphism $\psi : X \to A$ (otherwise, we would obtain endomorphisms $u_1, \ldots, u_n$ of $A$ such that each $u_i$ is zero on $F_i$ and $u_1 + \cdots + u_n = \text{id}_A$, which contradicts our assumptions). Equivalently, $A$ is not a direct summand of $X$ for its embedding via the diagonal map $a \mapsto (a + F_1, \ldots, a + F_n)$. Thus, there exists no universal morphism $\psi$ satisfying the assertions of Theorem 2.

Example 5.2. Following [Ra, XIII 3.1], we construct a complete equivariant embedding $X$ of a commutative non-affine group $G$, for which the assertions of Theorems 1–3 do not hold. (Of course, $X$ will be non-normal).

Consider an abelian variety $A$, and a point $a \in A$. Let $X_a$ be the scheme obtained from $A \times \mathbb{P}^1$ by identifying every point $(x,0)$ with $(x+a, \infty)$. (That $X_a$ is indeed a scheme follows e.g. from the main result of [Fe].) Then $X_a$ is a complete variety, and the canonical map

$$f_a : A \times \mathbb{P}^1 \longrightarrow X_a$$

is the normalization. In particular, $X_a$ is weakly normal, i.e., every finite birational bijective morphism from a variety to $X_a$ is an isomorphism.

The connected algebraic group

$$G := A \times \mathbb{G}_m$$

acts faithfully on $A \times \mathbb{P}^1$ via $(x,t) \cdot (y,u) = (x+y, tu)$, and this induces a faithful $G$-action on $X_a$ such that $f_a$ is equivariant. Moreover, $X_a$ consists of two $G$-orbits: the open orbit,

$$f_a(A \times (\mathbb{P}^1 \setminus \{0, \infty\})) \cong G,$$

and the closed orbit,

$$f_a(A \times \{0\}) = f_a(A \times \{\infty\}) \cong A \cong G/\mathbb{G}_m,$$

which is the singular locus of $X_a$.

The projection $A \times \mathbb{P}^1 \to A$ induces a morphism

$$\alpha_a : X_a \longrightarrow A/a,$$

where $A/a$ denotes the quotient of $A$ by the closure of the subgroup generated by $a$. We claim that $\alpha_a$ is the Albanese morphism: indeed, any morphism $\beta : X_a \to B$, where $B$ is an abelian variety, yields a
morphism $\beta \circ f_a : A \times \mathbb{P}^1 \to B$. By rigidity (see e.g. [Mi, Cor. 2.5, Cor. 3.9]), there exists a morphism $\gamma : A \to B$ such that

$$(\beta \circ f_a)(y, z) = \gamma(y)$$

for all $y \in A$ and $z \in \mathbb{P}^1$. Conversely, a morphism $\gamma : A \to B$ yields a morphism $\beta : X_a \to B$ if and only if $\gamma(y + a) = \gamma(y)$ for all $y \in A$, which proves our claim.

Also, note that $\alpha_G : G \to A(G)$ is just the projection $A \times \mathbb{G}_m \to A$. Thus, the kernel of the homomorphism $A(G) \to A(X_a)$ is the closed subgroup generated by $a$.

In particular, if the order of $a$ is infinite, then $X_a$ does not admit any $G$-morphism to a finite quotient of $A$, so that the assertions of Theorems 2 and 3 are not satisfied. Moreover, $X_a$ is not projective (as follows e.g. from Lemma 2.2 applied to the action of $A$), so that the assertion of Theorem 1 does not hold as well.

On the other hand, if $a$ has finite order $n$, then the fibers of $\alpha_a$ are unions of $n$ copies of $\mathbb{P}^1$ on which the origins are identified. In that case, $X_a$ is projective, but does not satisfy the assertions of Theorem 3 when $n \geq 2$.

We may also consider $X_a$ as an $A$-variety via the inclusion $A \subset G$. The projection $A \times \mathbb{P}^1 \to \mathbb{P}^1$ induces an $A$-invariant morphism

$$\pi : X_a \to C,$$

where $C$ denotes the nodal curve obtained from $\mathbb{P}^1$ by identifying 0 with $\infty$; one checks that $\pi$ is an $A$-torsor. If $a$ has infinite order, then $X_a$ is not covered by $A$-stable quasi-projective open subsets: otherwise, by Lemma 2.2 again, any such subset $U$ would admit an $A$-morphism to a finite quotient of $A$, i.e., the kernel of the $A$-action on $A(U)$ would be finite. However, when $U$ meets the singular locus of $X_a$, one may show as above that the Albanese morphism of $U$ is the restriction of $\alpha_a$, which yields a contradiction.

**Example 5.3.** Given any abelian variety $A$, we construct (after [Ra, XII 1.2]) an $A$-torsor $\pi : X \to Y$ such that $A$ acts trivially on $A(X)$.

Let $X$ be the scheme obtained from $A \times A \times \mathbb{P}^1$ by identifying every point $(x, y, 0)$ with $(x + y, y, \infty)$. Then again, $X$ is a complete variety, and the canonical map

$$f : A \times A \times \mathbb{P}^1 \to X$$

is the normalization; $X$ is weakly normal but not normal.

Let $A$ act on $A \times A \times \mathbb{P}^1$ via translation on the first factor; this induces an $A$-action on $X$ such that $f$ is equivariant. Moreover, the projection $p_{23} : A \times A \times \mathbb{P}^1 \to A \times \mathbb{P}^1$ induces an $A$-invariant morphism

$$\pi : X \to A \times C,$$

where $C$ denotes the rational curve with one node, as in the above example. One checks that $\pi$ is an $A$-torsor, and the Albanese morphism
\(\alpha_X\) is the composite of \(\pi\) with the projection \(A \times C \rightarrow A\). Thus, \(\alpha_X\) is \(A\)-invariant.

So Theorem 2 does not hold for the \(A\)-variety \(X\). Given any \(A\)-stable open subset \(U \subset X\) which meets the singular locus, one may check as above that \(\alpha_U\) is \(A\)-invariant; as a consequence, Theorem 1 does not hold as well.

**Example 5.4.** Given an elliptic curve \(E\), we construct an example of an \(E\)-torsor \(\pi: X \rightarrow Y\), where \(Y\) is a normal affine surface, and the Albanese variety of \(X\) is trivial. (In particular, \(X\) is a normal \(E\)-variety which does not satisfy the assertions of Theorem 2). For this, we adapt a construction in [Ra, XIII 3.2].

Let \(\hat{E} := E \setminus \{0\}\) and \(\hat{A}^1 := A^1 \setminus \{0\}\). We claim that there exist a normal affine surface \(Y\) having exactly two singular points \(y_1, y_2\), and a morphism

\[
f: \hat{E} \times \hat{A}^1 \rightarrow Y
\]

such that \(f\) contracts \(\hat{E} \times \{1\}\) to \(y_1\), \(\hat{E} \times \{-1\}\) to \(y_2\), and restricts to an isomorphism on the open subset \(\hat{E} \times (\hat{A}^1 \setminus \{1, -1\})\).

Indeed, we may embed \(E\) in \(\mathbb{P}^2\) as a cubic curve with homogeneous equation \(F(x, y, z) = 0\), such that the line \((z = 0)\) meets \(C\) with order 3 at the origin. Then \(\hat{E}\) is identified with the curve in \(\mathbb{A}^2\) with equation \(F(x, y, 1) = 0\). Now one readily checks that the claim holds for the surface

\[
Y \subset \mathbb{A}^2 \times \hat{A}^1
\]

with equation \(F(x, y, z^2 - 1) = 0\), and the morphism

\[
f: (x, y, z) \mapsto (x(z^2 - 1), y(z^2 - 1), z).
\]

The singular points of \(Y\) are \(y_1 = (0, 0, 1)\) and \(y_2 = (0, 0, -1)\).

Next, let \(U_1 = Y \setminus \{y_2\}\), \(U_2 := Y \setminus \{y_1\}\), and \(U_{12} := U_1 \cap U_2\). Then \(U_{12}\) is nonsingular and contains an open subset isomorphic to \(\hat{E} \times (\hat{A}^1 \setminus \{1, -1\})\). Thus, the projection \(\hat{E} \times (\hat{A}^1 \setminus \{1, -1\}) \rightarrow \hat{E}\) extends to a morphism

\[
p: U_{12} \rightarrow E.
\]

We may glue \(E \times U_1\) and \(E \times U_2\) along \(E \times U_{12}\) via the automorphism

\[
(x, y) \mapsto (x + p(y), y)
\]

to obtain a torsor \(\pi: X \rightarrow Y\) under \(E\). Arguing as in Example 5.2, one checks that the Albanese variety of \(X\) is trivial.

**Example 5.5.** Following Igusa (see [Ig]), we construct nonsingular projective varieties for which the tangent bundle is trivial, and the Albanese morphism is a non-trivial fibration.

We assume that the ground field \(k\) has characteristic 2, and consider two abelian varieties \(A, B\) such that \(A\) has a point \(a\) of order 2. Let \(X\) be the quotient of \(A \times B\) by the involution

\[
\sigma: (x, y) \mapsto (x + a, -y)
\]
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and denote by \( \pi : A \times B \to X \) the quotient morphism. Since \( \sigma \) has no fixed point, \( X \) is a nonsingular projective variety, and the natural map between tangent sheaves \( \mathcal{T}_{A \times B} \to \pi^*(\mathcal{T}_X) \) is an isomorphism. This identifies \( \Gamma(X, \mathcal{T}_X) \) with the invariant subspace

\[
\Gamma(A \times B, \mathcal{T}_{A \times B})^\sigma = \text{Lie}(A \times B)^\sigma.
\]

Moreover, the action of \( \sigma \) on \( \text{Lie}(A \times B) \) is trivial, by the characteristic 2 assumption. This yields

\[
\Gamma(X, \mathcal{T}_X) \cong \text{Lie}(A \times B).
\]

As a consequence, the natural map \( \mathcal{O}_X \otimes \Gamma(X, \mathcal{T}_X) \to \mathcal{T}_X \) is an isomorphism, since this holds after pull-back via \( \pi \). In other words, the tangent sheaf of \( X \) is trivial.

The action of \( A \) on \( A \times B \) via translation on the first factor induces an \( A \)-action on \( X \), which is easily seen to be faithful. Moreover, the first projection \( A \times B \to A \) induces an \( A \)-morphism

\[
\psi : X \to A/a,
\]

a homogeneous fibration with fiber \( B \). One checks as in Example 5.2 that \( \psi \) is the Albanese morphism of \( X \).

Let \( G \) denote the reduced neutral component of the automorphism group scheme \( \text{Aut}(X) \). We claim that the natural map \( A \to G \) is an isomorphism. Indeed, \( G \) is a connected algebraic group, equipped with a morphism \( G \to A/a \) having an affine kernel (by the Nishi–Matsumura theorem) and such that the composite \( A \to G \to A/a \) is the quotient map. Thus, \( G = AG_{\text{aff}} \), and \( G_{\text{aff}} \) acts faithfully on the fibers of \( \psi \), i.e., on \( B \). This implies our claim.

In particular, \( X \) is not homogeneous, and the fibration \( \psi \) is non-trivial (otherwise, \( X \) would be an abelian variety). In fact, the sections of \( \psi \) correspond to the 2-torsion points of \( B \), and coincide with the local sections; in particular, \( \psi \) is not locally trivial.

**Example 5.6.** We assume that \( k \) has characteristic \( p > 0 \). Let \( X \) be the hypersurface in \( \mathbb{P}^n \times \mathbb{P}^n \) with bi-homogeneous equation

\[
f(x_0, \ldots, x_n, y_0, \ldots, y_n) := \sum_{i=0}^{n} x_i^p y_i = 0,
\]

where \( n \geq 2 \). The simple algebraic group \( G := \text{PGL}(n+1) \) acts on \( \mathbb{P}^n \times \mathbb{P}^n \) via

\[
[A] \cdot ([x], [y]) = ([Ax], [F(A^{-1})^T y]),
\]

where \( A \in \text{GL}(n+1) \), and \( F \) denotes the Frobenius endomorphism of \( \text{GL}(n+1) \) obtained by raising matrix coefficients to their \( p \)-th power. This induces a \( G \)-action on \( X \) which is faithful and transitive, but not smooth; the isotropy subgroup scheme of any point \( \xi = ([x], [y]) \) is the intersection of the parabolic subgroup \( G_{[x]} \) with the non-reduced parabolic subgroup scheme \( G_{[y]} \). So \( X \) is a variety of unseparated flags.
Denote by \( \pi_1, \pi_2 : X \longrightarrow \mathbb{P}^n \)
the two projections. Then \( X \) is identified via \( \pi_1 \) to the projective bundle associated to \( F^* (\mathcal{T}_{\mathbb{P}^n}) \) (the pull-back of the tangent sheaf of \( \mathbb{P}^n \) under the Frobenius morphism). In particular, \( \pi_1 \) is smooth. Also, \( \pi_2 \) is a homogeneous fibration, and \( (\pi_2)_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \), but \( \pi_2 \) is not smooth.

Let \( \mathcal{T}_{\pi_1}, \mathcal{T}_{\pi_2} \) denote the relative tangent sheaves (i.e., \( \mathcal{T}_{\pi_i} \) is the kernel of the differential \( d\pi_i : T_X \longrightarrow \pi_i^* (\mathcal{T}_{\mathbb{P}^n}) \)). We claim that

\[
T_X = T_{\pi_1} \oplus T_{\pi_2};
\]

in particular, \( T_{\pi_2} \) has rank \( n - 1 \). Moreover, \( T_{\pi_2} \) is the subsheaf of \( T_X \) generated by the global sections.

For this, consider the natural map

\[
op_X : \text{Lie}(G) \longrightarrow \Gamma(X, \mathcal{T}_X)
\]

and the induced map of sheaves

\[
op_X : \mathcal{O}_X \otimes \text{Lie}(G) \longrightarrow \mathcal{T}_X.
\]

For any \( \xi = ([x], [y]) \in X \), the kernel of the map \( \nop_{X, \xi} : \text{Lie}(G) \rightarrow \mathcal{T}_{X, \xi} \)
is the isotropy Lie algebra \( \text{Lie}(G)_\xi \), that is, \( \text{Lie}(G)_{[x]} \) (since \( \text{Lie}(G) \) acts trivially on the second copy of \( \mathbb{P}^n \)). Thus, \( (d\pi_1)_\xi : \mathcal{T}_{X, \xi} \longrightarrow \mathcal{T}_{\mathbb{P}^n, [x]} \)
restricts to an isomorphism \( \text{Im}(\nop_{X, \xi}) \cong \mathcal{T}_{\mathbb{P}^n, [x]} \). In other words, \( d\pi_1 \)
restricts to an isomorphism \( \text{Im}(\nop_X) \cong \pi_1^*(\mathcal{T}_{\mathbb{P}^n}) \). So \( \pi_1 \) is the Tits morphism of \( X \). Moreover, we have a decomposition

\[
\mathcal{T}_X = \mathcal{T}_{\pi_1} \oplus \text{Im}(\nop_X),
\]

and \( \text{Im}(\nop_X) \subset \mathcal{T}_{\pi_2} \). Since \( \mathcal{T}_{\pi_1} \cap \mathcal{T}_{\pi_2} = 0 \), this implies \( (5.1) \) and the equality \( \text{Im}(\nop_X) = \mathcal{T}_{\pi_2} \).

To complete the proof of the claim, we show that \( \nop_X \) is an isomorphism. Consider indeed the bi-homogeneous coordinate ring

\[
k[x_0, \ldots, x_n, y_0, \ldots, y_n]/(x_0^p y_0 + \cdots + x_n^p y_n)
\]

of \( X \). Its homogeneous derivations of bi-degree \((0, 0)\) are those given by \( x \mapsto Ax, y \mapsto ty \), where \( A \) is an \( n + 1 \times n + 1 \) matrix, and \( t \) a scalar; this is equivalent to our assertion.

Also, \( X \) admits no non-trivial decomposition into a direct product, by [Sa, Cor. 6.3]. This may be seen directly: if \( X \cong X_1 \times X_2 \), then the \( G \)-action on \( X \) induces actions on \( X_1 \) and \( X_2 \) such that both projections are equivariant (as follows e.g. by linearizing the pull-backs to \( X \) of ample invertible sheaves on the nonsingular projective varieties \( X_1, X_2 \)). So \( X_1 \cong G/H_1 \) and \( X_2 \cong G/H_2 \), where \( H_1, H_2 \) are parabolic subgroup schemes of \( G \). Since the diagonal \( G \)-action on \( G/H_1 \times G/H_2 \) is transitive, it follows that \( G = P_1 P_2 \), where \( P_i \) denotes the reduced scheme associated to \( H_i \) (so that each \( P_i \) is a proper parabolic subgroup of \( G \)). But the simple group \( G \) cannot be the product of two proper parabolic subgroups, a contradiction.
Since $X$ is rational and hence simply-connected, this shows that a conjecture of Beauville (relating decompositions of the tangent bundle of a compact Kähler manifold to decompositions of its universal cover, see [Be]) does not extend to nonsingular projective varieties in positive characteristics.

More generally, let $G$ be a simple algebraic group in characteristic $p \geq 5$, and $X \cong G/G_x$ a complete variety which is homogeneous under a faithful $G$-action. By [We], there exists a unique decomposition

$$G_x = \bigcap_{i=1}^r P_i G_{n_i},$$

where $P_1, \ldots, P_r$ are pairwise distinct maximal parabolic subgroups of $G$, and $(n_1, \ldots, n_r)$ is an increasing sequence of non-negative integers. Here $G_n$ denotes the $n$-th Frobenius kernel of $G$. Since $G$ acts faithfully on $X$, we must have $n_1 = 0$. Let

$$Q_1 := \bigcap_{i,n_i=0} P_i, \quad Q_2 := \bigcap_{i,n_i \geq 1} P_i G_{n_i}.$$

Then $Q_1$ is a parabolic subgroup of $G$, $Q_2$ is a parabolic subgroup scheme, $G_x = Q_1 \cap Q_2$, and $\text{Lie}(G_x) = \text{Lie}(Q_1)$. Thus, the Tits morphism of $X$ is the canonical map

$$\pi_1 : G/G_x \to G/Q_1,$$

and $d\pi_1$ restricts to an isomorphism $\text{Im}(op_X) \cong \pi_1^*(\mathcal{T}_{G/Q_1})$. This implies the decomposition $\mathcal{T}_X = \text{Im}(op_X) \oplus \mathcal{T}_{\pi_1}$, which is non-trivial unless $G_x = Q_1$ (i.e., $G_x$ is reduced). Moreover, $\text{Im}(op_X) \subset \mathcal{T}_{\pi_2}$ (where $\pi_2 : G/G_x \to G/Q_2$ denotes the canonical map), since $G_1$ acts trivially on $G/Q_2$. As $\pi_1 \times \pi_2$ is a closed immersion, it follows again that

$$\mathcal{T}_X = \mathcal{T}_{\pi_1} \oplus \mathcal{T}_{\pi_2} \quad \text{and} \quad \text{Im}(op_X) = \mathcal{T}_{\pi_2}.$$ 

On the other hand, the variety $X$ is indecomposable, since $G$ is simple (see [Sa] Cor. 6.3 again, or argue directly as above). This yields many counter-examples to the analogue of the Beauville conjecture.

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