BERTINI AND HIS TWO FUNDAMENTAL THEOREMS

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Abstract. After reviewing Bertini’s life story, a fascinating drama, we make a critical examination of the old statements and proofs of Bertini’s two fundamental theorems, the theorem on variable singular points and the theorem on reducible linear systems. We explain the content of the statements in a way that is accessible to a nonspecialist, and we develop versions of the old proofs that are complete and rigorous by current standards. In particular, we prove a new extension of Bertini’s first theorem, which treats variable $r$-fold points for any $r$.

1. Preface.

Eugenio Bertini (1846–1933) studied in the 1860s with Luigi Cremona (1830–1903), the father of Italian algebraic geometry. Bertini was Cremona’s first student, and one of his best. At the time, Cremona was developing the first general theory of birational transformations of the plane and of 3-space, the transformations that now bear his name. Bertini was attracted to the subject and advanced it; thus he was led to prove, in his paper [2] dated December 1880 and published in 1882, the two theorems that now bear his name: Bertini’s theorem on variable singular points, and Bertini’s theorem on reducible linear systems.

Bertini’s two theorems soon became fundamental tools in algebraic geometry. In his comprehensive treatise of 1931 on plane curves [18], Julian Coolidge (1873–1954) wrote on p. 115 that Bertini’s first theorem “will be of utmost importance to us,” and then called it “vitally important” on the next page. In his obituary [16, p. 747] of Bertini, Guido Castelnuovo (1865–1952) wrote that both theorems “come into play at every step in all the work” of the Italian school of algebraic geometry. In his obituaries, [8, p. 621] and [9, p. 150], Luigi Berzolari (1863–1949)
wrote that the theorems are “now classical and of constant use in current research in algebraic geometry.”

Bertini’s theorems are today as fundamental as ever in algebraic geometry. Moreover, from about 1880 until 1950, their statements and proofs evolved in generality, rigor, and clarity in the hands notably of Bertini himself, of Frederigo Enriques (1871–1946), of Bartel van der Waerden (1903–96), and of Oscar Zariski (1899–1986). For these reasons alone, the theorems make an appropriate subject for a historical analysis. However, in addition, it turns out that the old statements and proofs are rather interesting from a purely mathematical point of view. For example, Bertini’s original first theorem treated the variable $r$-fold points for an arbitrary $r$, not simply the variable singular points. Furthermore, Bertini’s second theorem used to be derived from the first. Yet the old statements and proofs have been nearly forgotten, replaced by various new ones.

The bulk of the present article is devoted to a critical examination of the old statements and proofs. Section 3 discusses the first theorem for an ambient projective space over the complex numbers, the case that Bertini considered originally. Section 4 discusses the extension of the first theorem to an arbitrary ambient variety; in particular, it proves a new general version, Theorem (4.4), for an arbitrary $r$. Finally, Section 5 discusses the second theorem. Nothing is said anywhere about the newer theory, despite its importance in contemporary algebraic geometry. On the other hand, an attempt is made throughout to develop versions of the old statements and proofs that are acceptably complete, rigorous, and clear by current standards. At the same time, an attempt is made to explain the content of the statements and the spirit of the proofs in a way that is accessible to a nonspecialist.

The life stories of mathematicians are often fascinating human dramas, which show the enormous influence that social circumstances have on the development of mathematics. Thus it is with Bertini’s life story, which we’ll review in Section 2, drawing on a number of sources, including the historical articles of Bottazzini, [10] and [11], and of Vito Volterra (1860–1940) [43], the historical monographs of Aldo Brigaglia and Ciro Ciliberto [13] and of Enriques [22, pp. 281–92], and the announcement [15] of the death of Enrico Betti (1823–92) made by Francesco Brioschi (1824–97). We’ll also draw on the obituary of Bertini written by his close colleague Castelnuovo [16] (who received 185 con-
served personal letters from him, see [12, p. XXIX]), the obituaries by Bertini’s student at Pavia, Berzolari, [8] and [9], and the obituaries by his students at Pisa, Alberto Conti (1873–1940) [17], Guido Fubini (1879–1943) [25] and Gaetano Scorza (1876–1939) [37].

The obituaries carry all the authority and compassion of their authors’ firsthand knowledge of Bertini’s character, his lecturing and his writing. Further evidence of his marvelous character is found in nearly 60 of the 668 letters from Enriques to Castelnuovo, published in [12]. Each source contains its own gems, and all can be recommended. The obituaries also contain surveys of Bertini’s scientific work, and some of this material will be repeated briefly below; see also the technical monographs of Berzolari [7, p. 328], Coolidge [18, p. 481], and Jean Dieudonne (1906–92) [19, p. 111]. Two of the obituaries, Berzolari’s [9] and Fubini’s [25], contain Bertini’s scientific bibliography; of these two, the first is more carefully prepared, and it also contains the fullest discussion (19 pages) of the works themselves. However, of Bertini’s original discoveries, aside from the two fundamental theorems, only the classification of plane involutions remains significant, and it is of less general interest.

Because Bertini’s life was so intertwined with Cremona’s, the latter will also be reviewed, although more briefly, following Greitzer’s short biography [26] and following the obituaries written by his two students, Giuseppe Veronese (1857–1917) [42] and Bertini [4].

The present article is an expanded version of a talk given at the conference, *Algebra e Geometria (1860–1940): Il contributo Italiano*, which took place in Cortona, Italy, 4–8 May 1992 under the scientific direction of Brigaglia, Ciliberto and Edoardo Sernesi; see the proceedings [14]. The author’s conversations with Rick Miranda and with Sernesi in Cortona and recently with Anders Thorup in Copenhagen led to the proof of the new extension, Theorem (4.4), of Bertini’s first theorem. Beverly Kleiman provided valuable help with the copy-editing of this article. Its appearance, at last, is due to the long-standing persistent encouragement of Umberto Bottazzini, who also kindly provided copies of a fair number of hard-to-find articles. It is a pleasure for the author to express his heartfelt thanks to all of these individuals, especially to Umberto.
2. Bertini’s Life.

Eugenio Bertini was born on 8 November 1846 in Forli, Italy, about halfway between Rimini and Bologna, to Vicenzo Bertini, a typographer, and his wife Agata, née Bezzi. Bertini attended secondary school for two years at the technical institute in Forli, where he showed distinct aptitude in mathematics. The family was poor, but with the financial assistance of the Congregazione di Carità di Forli, Bertini was able to go on for a higher education. In 1863, nearly 17, he entered the University of Bologna, intending to become an engineer, but he was drawn into pure mathematics by the lectures of Luigi Cremona, who was nearly twice his age at the time.

This was the period of the unification of Italy. There were three wars of independence, of 1848, 1859 and 1866. In 1859 Lombardy was annexed by Sardinia–Piedmont, after Cavour had gotten France to help drive out Austria. In 1860 the northern states of Tuscany, Modena, Parma, and Romagna rose up against their princes, and were annexed with France’s permission in return for Savoy and Nice. Scientists, including the mathematicians Betti and Brioschi, played prominent roles in the enlightened new government, which promptly established chairs of higher mathematics. Professors and students could move more freely and easily from university to university, as the universities were now under a single ministry.

Just before, in the spring of 1858, Betti, Brioschi and several others met in Genoa to found the journal *Annali di matematica pura e applicata*, modeling it after *Crelle’s Journal für die reine und angewandte Mathematik* and *Liouville’s Journal de mathématiques pures et appliquées*; a decade later under the editorship of Brioschi and Cremona, it grew into one of the great European journals. In the fall of 1858, Betti, Brioschi and Felice Casorati (1835–90) visited the mathematical centers of Göttingen, Berlin, and Paris, and made many important mathematical contacts.

Many Italian mathematicians did pure research of international caliber, yet also made an effort to write good texts and to train skilled engineers, who were needed to build the industry and the infrastructure of the new nation. Italy felt a particular kinship with Germany, which was also, at the time, forging a nation out of a maze of states. So Italian mathematicians studied the works of Gauss, Jacobi and Riemann, and
of Plücker, Clebsch, and Noether. In 1866 Italy joined Germany (Prussia) in a war against Austria, which gave the Venetia to France, which in turn gave it to Italy. Bertini volunteered as an infantryman under Garabaldi in this brief third war for Italian independence.

Eighteen years earlier, in April 1848, Cremona had interrupted his studies to fight as a volunteer in the first war against Austrian rule, attaining the rank of sergeant. He took part in the heroic defense of Venice, which capitulated on 24 August 1849. Because of the discipline and bravery of the defenders, they were allowed by the victors to leave as a unit to serve as a model of military and civil virtue. Cremona returned to his native Pavia to find that his mother had died a few months earlier. Soon he became gravely ill with typhus, which he had picked up in the war. Strong willed, he reentered the university the same year. From his teacher Brioschi, he learned to love science and to pass on this love in his own lectures and texts, which were excellent. On 9 May 1853, he was awarded his laurea degree in civil engineering and architecture (sic!).

Cremona could not enter the official educational system right away because of his record of military service against Austria. So he became a private tutor in Pavia for several years (a common profession at the time). On 22 November 1855 he was appointed provisionally as a teacher at Pavia’s ginnasio liceale, and on 17 December 1856 he was promoted to the rank of associate. A month later he was appointed to full rank at the ginnasio in the city of Cremona. He remained there for three years, until the new government of Lombardy appointed him to the Liceo S. Alessandro in Milan and then on 10 June 1860 to the first chair of higher geometry in Bologna. There he carried out his most important original research, which concerned birational transformations and their applications. For part of this work, he shared the 1864 Steiner prize in geometry awarded by the Berlin Academy (the other winner was R. Sturm). In 1863, Brioschi founded an engineering school in Milan, and on his recommendation, Cremona was transferred there in October 1866.

After the third war, Bertini was advised by Cremona to resume his studies under Betti and Ulisse Dini (1845–1918) in Pisa. Bertini earned his laurea in 1867 at the Ateneo pisano and his teaching certificate in 1868 at the Scuola Normale Superiore. Nominated immediately afterwards to the chair of mathematics at the Liceo Parini in Milan, Bertini had the opportunity, the next academic year (1868–69), to attend a three-part course given by Brioschi, Casorati, and Cremona, on Abelian
integrals from the three different points of view: Jacobi’s analysis, Rie-
mann’s topology, and Clebsch’s algebraic geometry. This course had a
profound and lasting affect on Bertini. In particular, he was led to give,
in 1869, one of the first and simplest geometric proofs of the invariance
of the genus of a curve under birational correspondence. This proof was
included in three standard texts (Clebsch–Lindemann of 1876, Salmon–
Chemin of 1884, and Enriques–Chisini of 1918 [24, vol. 2, pp. 131-35]),
making Bertini’s name famous.

In the fall of 1871, Bertini took the chair of mathematics at the
Liceo Visconti in Rome, but in addition he taught descriptive geometry as an adjunct at the university. In the fall of 1873, he abandoned
secondary-school teaching, and began teaching projective geometry too
at the university. He remained in Rome one more year only. In 1875,
he won, with Betti’s support, a competition for the chair in advanced
gometry in Pisa. He served first as an adjunct, but after three years,
was promoted to a regular position, and he remained in Pisa two more
years, until 1880.

On 9 October 1873, Cremona was appointed director of the newly
established engineering school in Rome, and he soon became so burdened with administration that he had little time left over for creative
research. When Bertini went to Pisa in 1875, Cremona objected vocif-
erously, and only forgave him two years later when Bertini, generous as
always, offered to give up his chair in favor of his teacher, who appeared
for a moment to want to leave Rome. However, in November 1877,
Cremona was appointed to a chair at the University of Rome. On 16
March 1879 he was appointed a senator, and then his research activities
stopped completely. On 10 June 1903, he left his sickbed to act on some
legislation, had a heart attack, and died. His Opere matematiche in
three volumes (Hoepli, Milano 1914, 1915, and 1917) were edited under
Bertini’s direction.

In 1876, in Pisa, Bertini classified the plane Cremona transforma-
tions that are equal to their own inverses, the plane involutions of order
two. He proved that they decompose into products of irreducible in-
volutions, each of which is equivalent under a Cremona transformation
to one of only four types, a surprisingly simple result. The first two
types were already well known: reflections in lines, and de Jonquières
involutions. However, Bertini showed that each de Jonquières involution
is irreducible, and is given by a curve of degree $n$, for some $n$, with a
multiple point of order \( n - 2 \); a given point, in general position in the plane, corresponds to its harmonic conjugate with respect to two other points, namely, the two points of the curve on the line determined by the given point and the multiple point. An involution of the third type is given by a net of cubics through seven points; the cubics through an eighth point in general position obviously pass as well through a ninth, and the ninth corresponds to the eighth. This type had been found ten years earlier by Geiser, but described differently.

An involution of the fourth type is given by a 3-dimensional system of sextics passing doubly through eight points; the sextics through a ninth point in general position can be proved to pass as well through a tenth, and the tenth corresponds to the ninth. These involutions were new, and have become known as Bertini involutions. Later, in 1889, Bertini gave a simple geometric proof that an involution of each of the four types is rational; that is, the pairs of corresponding points form the fibers of a rational 2-to-1 map onto the plane. (This result, in essence, had already been stated by Noether in 1876, and was also proved by Lüroth in 1889, but algebraically. In 1893, Castelnuovo proved the more general result that any rational transform of the plane is birational to the plane.)

Bertini’s classification represented a philosophical break with Cremona, who saw his transformations only as a tool for reducing the complexity of given geometric figures, and not as objects of study in their own right. Cremona received Bertini’s work coldly, unjustifyably so. Bertini explained it to him in person before the appearance of his main paper [1]. However, Cremona said only that he had already discovered the fourth type of involution, and had communicated its existence in a letter to his former student, Ettore Caporali (1855–86). Bertini respectfully added Cremona’s existence proof to the page proofs; see [1, p. 273]. Moreover, Bertini needed to make a technical restriction on the fundamental locus, and did so openly in the first paragraph of his paper [1]. The restriction was eventually removed by Castelnuovo and Enriques.

Meanwhile, two years later, Caporali took up the classification from a different point of view, and suggested that there might be, in addition to Bertini’s four types, infinitely many others. The significance of this entire incident is suggested by the length and strength of the discussions of it by Castelnuovo [16, pp. 746–7] and by Scorza [37, pp. 105–7] nearly fifty years after the fact. In particular, Castelnuovo called Caporali’s
point of view “much less interesting” than Bertini’s, which “opened new horizons to algebraic geometry.” And Scorza concluded with this bit of wisdom: it takes longer to appreciate a conceptual advance when it is more original and more profound.

Bertini’s research reflects, by and large, Cremona’s influence in its subject and its style. Both geometers employed synthetic methods and analytic methods with equal facility. Both wrote succinct, precise, elegant treatments, which reflect their powerful, penetrating intellects. Many of their investigations concerned special properties of given figures; these works have lost much of their interest, but had their importance in pointing the way to the general theory.

Much of Bertini’s research grew out of his lecture preparations. For him, research and teaching were two aspects of the same activity. He kept current by reworking the latest advances, often putting them in a new and perspicacious form, which he included in his lectures and papers. Here are some examples.

In 1881 Bertini enumerated the 5-secant conics to the quintic space curve. In 1884 he wrote his extensive and elaborate memoir on the cubic surface, with its 27 lines and 45 tritangent planes. In 1888 he gave a simple and ingenious geometric proof of the following one of Noether’s theorems: any plane curve can be transformed into another one that has only ordinary multiple points, via Cremona transformations of the ambient plane. As a consequence, he extended Plücker’s formulas to plane curves with arbitrary singularities. Later, in 1891 he transformed a given curve, via arbitrary birational transformations, into a plane curve with ordinary double points; this result had already been in use for a long time, but not yet proved rigorously. In 1890 and in 1908, he advanced the theory of linear series on an abstract curve. In 1896 he studied the 21 quadruple tangents of the Cayleyian of a plane quartic. In 1898 he studied pencils of quadrics and the linear spaces on a quadric of even dimension.

In 1880 Bertini left Pisa for Pavia, exchanging positions with Riccardo De Paolis (1854–92), another one of Cremona’s former students. De Paolis wanted to be in Pisa, and Bertini wanted to be with his two close colleagues, Eugenio Beltrami (1835–1899) and Casorati. Beltrami had taught algebra and analytic geometry at Bologna when Bertini was a student there. In 1890 Casorati died, and Bertini took over the responsibility for teaching analysis. Two years later, De Paolis died pre-
maturely, and Bertini reclaimed his old position in Pisa on the encouragement of Luigi Bianchi (1856–1928) and of Dini. Bertini remained in Pisa for the rest of his life. Among his students during this period were Carlo Rosati (1876–1929), Ruggiero Torelli (1884–1915), and Giacomo Albanese (1890–1947).

Bertini was a scrupulous and zealous teacher, who viewed teaching as a ministry. Each year he covered a different subject. He prepared his lectures with great care, both in plan and in detail, obtaining a precise, efficient, clean development. His delivery was clear and dry, yet lively. He often asked questions and proposed problems to engage his audience, and he knew how to maintain decorum. He demanded hard work of his students, but they loved him, and called him “Papá Bertini.”

Bertini was a tall heavyset man. He had a fresh open mind, and a noble upright character. He was modest in every way, and found a friend in every one. He was kind and affectionate. He was frank, optimistic, and generous. He was full of advice, encouragement, and help. He had a strong sense of duty, and an inflexible sense of justice. He was looked up to as an inspiring example of high morality.

Bertini’s lectures on projective algebraic geometry at the Ateneo pisano were written up and lithographed during the academic year 1889–99 by his assistant and former student, Scorza. They were then revised, expanded with an appendix, and published in 1907 as the book [5]. It was republished in 1923, and translated into German in 1924. The book contains all the essential results about projective varieties that had been obtained in the preceding decades. The appendix treats algebraic curves and their singularities; it was based in part on the extensive summaries prepared by Corrado Segre (1863–1924) for his own courses. Bertini’s exposition is systematic, extensive, and lucid. It was the crowning achievement of his scientific work, and is still studied and cited today.

On 1 August 1922 Bertini had to retire because of age (75). He was succeeded by Rosati, who had been his assistant. However, for the next ten years, Bertini continued to teach for free, as professor emeritus, because teaching was so important to him. He also turned these lectures into a text book [6], which he wrote himself at the age of 82. The book was aimed at the second and third year students; it discusses assorted topics in the geometry of projective space, such as quadratic transformations of the plane, the formulas of Plücker and Cayley relat-
ing numerical characters of plane and space curves, various types of line complexes, the cubic surface, and the Steiner surface. The book nobly closed Bertini’s scientific production. He gave his scientific library to the faculty of science, and funded an annual prize for the best laurea dissertation in pure mathematics.

On 24 February 1933, Eugenio Bertini died in Pisa after a brief illness, survived only by his daughter, Eugenia. He also had a son, Giulio, with his wife Giulia, née Boschi. She died in Pisa on 23 January 1915 after a long and difficult illness, and Giulio died tragically, 27 years old, on 22 September 1922 in a traffic accident. Bertini was buried, as he wished, without the pomp and ceremony of a traditional large outdoor funeral, in the cemetery of Forli, along side his wife. Nevertheless, a year later, on 24 February 1934, a remarkable number of people, including immediate family, former students, close colleagues, academic administrators, and political dignitaries, met at the R. Istituto Tecnico di Forli to celebrate Bertini’s life and to unveil a memorial plaque; the event is lovingly described by Conti in his chronicle [17]. Throughout his life, Bertini dedicated himself humbly and altruistically to family, school, and science.

3. The original first theorem.

In this section and the next, we’ll discuss Bertini’s first fundamental theorem on linear systems, which concerns the multiple points on the general member. Originally, in 1882, Bertini [2] worked only with systems on the full \( n \)-dimensional projective space \( \mathbb{P}^n \) over the complex numbers, and this is the only case that we’ll consider case here. In the next section, we’ll go on to discuss the extension of the theorem to an arbitrary ambient variety in arbitrary characteristic. In this section, we’ll begin by reviewing some basic notions in the form in which they were considered in Bertini’s time. Then we’ll consider the content of the theorem. Finally, we’ll go through Bertini’s proof, which is simple, rigorous, and interesting.

On the projective space \( \mathbb{P}^n \), a **linear system** is simply the family of all hypersurfaces \( U_t \) of the form,

\[
U_t : t_0 u^{(0)} + \cdots + t_s u^{(s)} = 0,
\]

where \( \mathbf{t} := (t_0, \ldots, t_s) \) is a point of a projective space \( \mathbb{P}^s \), the **parameter space**, and where \( u^{(0)}, \ldots, u^{(s)} \) are linearly independent homogeneous
polynomials of the same degree in \( n + 1 \) variables \( x_0, \ldots, x_n \). The system is called a (linear) pencil if \( s = 1 \) and a net if \( s = 2 \).

If all the members \( U_t \) contain a common hypersurface \( U : u = 0 \), then \( U \) is said to be fixed, and the system defined by the quotients \( u^{(0)}/u, \ldots, u^{(s)}/u \) is called the residual system. There is a smallest residual system: it is the one where \( u \) is the greatest common divisor of the \( u^{(i)} \). If all the members of this smallest residual system contain a common point or a common variety, then it is called a base point or base variety of the original system. So a base variety has dimension at most \( n - 2 \). (Nowadays, it is more common to consider a fixed component to be a base variety as well.)

Let \( U : u = 0 \) be a hypersurface. A point of \( U \) at which all the partial derivatives of order \( r - 1 \),

\[
u_{i_1 \ldots i_{r-1}} := \frac{\partial^{r-1} u}{\partial x_{i_1} \ldots \partial x_{i_{r-1}}} \quad \text{where} \quad 0 \leq i_j \leq n,
\]

vanish is called a point of multiplicity \( r \), or an \( r \)-fold point. It may also be an \((r + 1)\)-fold point (although some authors, including Bertini, sometimes exclude this possibility as part of the definition). It is called a multiple point, or a singular point, if \( r \geq 2 \), but the exact value of \( r \) is unimportant. If a point is not multiple, then it is called simple. Given \( r \), the various \( r \)-fold points of the various members \( U_t \) of the linear system form a closed subset \( M_r \) of \( \mathbb{P}^n \times \mathbb{P}^s \), and a given \( r \)-fold point is said to be variable if it lies in an irreducible closed subset of \( M_r \) that covers \( \mathbb{P}^s \).

The preceding concepts are illustrated in the two simple examples shown in Fig. 1. (All the figures were drawn using the \TeX\-PostScript macro package \texttt{PSpictures} written by Thorup.) In both examples, the ambient space is the plane \( \mathbb{P}^2 \) with inhomogeneous coordinates \( x, y \), and the parameter space is the line \( \mathbb{P}^1 \) with inhomogeneous coordinate \( t \). In the first example (the one on the left), each curve \( U_t \) consists of two components: one is the vertical line through the point \( x(t, 0) \), and the other is the \( x \)-axis with multiplicity two. The latter is a fixed component; \( x \) is a variable 3-fold point, or triple point; and the point at infinity on each vertical line is a base point, and the only one. In the second example, the vertical lines of the first are replaced by the pencil of lines through the origin. So it is a base point, and the only one; moreover, it is a variable triple point, although \( x \) is fixed in the plane. Once again, the \( x \)-axis is a fixed component with multiplicity two.
In general, when \( n \geq 2 \), each \( U_t \) will have an \( r \)-fold point where \( r \geq 2 \) if there is a hypersurface \( U : u = 0 \) such that \((r-1)U : u^{r-1} = 0\) is fixed. Indeed, then \( U_t \) is defined by an equation of the form \( u^{r-1}v_t = 0 \), and all the partial derivatives of order \( r-1 \) of the product \( u^{r-1}v_t \) vanish at every point of the intersection \( W_t \) of \( U \) and \( V_t \), where \( V_t : v_t = 0 \) is the residual hypersurface. So the points of \( W_t \) are \( r \)-fold points of \( U_t \), and each component \( W'_t \) of \( W_t \) has dimension at least \( n-2 \). Suppose that some \( W'_t \) is not contained in the base locus, and let \( U'_t \) be any component of \( U \) that contains \( W'_t \), but not \( V_t \). Then each point \( x \) of \( U'_t \) must lie on some member \( V'_t \) of the residual system (otherwise \( W'_t \times P^s \) would be a component of the intersection of \( U'_t \times P^s \) with the total space of the residual system); so \( x \) is an \( r \)-fold point of \( U'_t \). Hence \( U'_t \) consists entirely of variable \( r \)-fold points.

This is not the only way that there can be a variable \( r \)-fold point off the base locus, although Bertini asserted that it was so in his original formulation of his first theorem \([2, p. 26]\). For instance, if there is a fixed hypersurface with an \( r \)-fold point, off the base locus or not, then this point will be a variable \( r \)-fold. In his book \([5, p. 227]\), Bertini was more careful, and put the theorem essentially as follows.

**Theorem (3.1)** (On variable \( r \)-fold points). On \( P^n \), a variable \( r \)-fold point of a linear system is an \((r-1)\)-fold point of every member.

The most important case of Theorem (3.1) occurs when \( r \) is 2 and there are no fixed components. Then the result is always put in contrapositive form; Bertini himself \([2, p. 26]\) did so essentially as follows.

**Theorem (3.2)** (On variable singular points). On \( P^n \), if a linear system has no fixed components, then a general member has no singular points outside the base locus.

By a general member of a linear system is meant one that is represented by a point in the parameter space \( P^s \) that satisfies no special algebraic-geometric conditions; in other words, the point lies in no subvariety that depends on the issue in question. Curiously, Bertini changed
terminology over the years, doubtless reflecting the prevailing usage. Thus, in his 1877 paper [1, p. 246], he wrote “curva generale”; in his 1882 paper [2, p. 26], he wrote “spazio arbitrario”; and in his book [5, p. 227], he wrote “ipersuperficie generica.”

Without the hypothesis of linearity of the system, the theorem may fail. A simple counterexample is given in Fig. 2. Here the curves $U_t$ are the horizontal translates of an irreducible plane cubic with a cusp at the origin. Thus each curve has a point of multiplicity two, or double point, but the various curves have no point in common, not even at infinity.

![Fig. 2. A nonlinear system with a variable double point](image)

Theorem (3.2) had already been proved in 1871, as Bertini [2, p. 26] noted, by Rosanes [36, p. 100] in a particular case, that of a homoloidal net of plane curves. This is the case where $s = 2$, and the $u^{(i)}$ have no common factor. Moreover, they must define a Cremona transformation of the plane; that is, the rational map,

$$\mathbf{x} \mapsto (u^{(0)}(\mathbf{x}), u^{(1)}(\mathbf{x}), u^{(2)}(\mathbf{x})),$$

is a birational transformation of the plane onto itself. Rosanes’ computational proof is special to this case, and gives no hint of a more general result.

In 1877 Bertini stated Theorem (3.2) for an arbitrary linear system of plane curves on p. 246 of his main paper [1] on plane involutions. It may be true, as Castelnuovo [16, p. 747] put it, that Bertini established this result “by means of intuitive considerations.” However, these considerations do belong to the standard theory of envelopes, which can be justified using calculus, as Bertini did in his book [5, Fn. 1, p. 225]. Nevertheless, in the book, Bertini gave a somewhat different proof of the theorem: he repeated the proof in his paper [2], see below. On the other hand, such complete analytic local proofs of the theorem in the plane are given in the texts of Picard and Simart [34, p. 51], of Francesco Severi (1879–1961) [39, p. 27] and of Enriques and Chisini [24, p. 181]. In fact, Severi proved the full Theorem (3.1) itself in the plane, although he attributed only the special case of Theorem (3.2) to Bertini. A more general version of this local proof is given at the end of the next section.
Theorem (3.1) applies equally well to a variable $r$-fold base point with $r \geq 3$. A simple example with $r = 3$ is shown in Fig. 3. Here $U_t$ is the cone in $\mathbf{P}^3$ with vertex at $x_t := (0, 0, t)$ over a fixed irreducible curve $C$ in the $(x, y)$-plane with a double point at the origin. The vertex $x_t$ is a triple point of $U_t$, but a double point of $U_{t'}$ for $t' \neq t$. As $t$ varies, $x_t$ sweeps out the $z$-axis, which is the full base locus.

Bertini proved his theorem rigorously, more or less as follows, although he did not pay as much attention to the details in the first part. Let $x \in U_{t_0}$ be the variable $r$-fold point; by definition, $(x, t_0)$ lies in a component $M$ of $M_r$ such that the projection $\mu: M \to \mathbf{P}^s$ is surjective. Hence $\mu$ is smooth on some dense open subset $M^0$ because the characteristic is zero. Now, we have to prove that $x$ lies in the closed subset $N$ of $\mathbf{P}^n$ where all the partial derivatives of order $r - 2$ of $u^{(k)}$ vanish for $0 \leq k \leq s$. Since $M^0$ is dense in $M$, it is enough to prove that the image of $M^0$ is contained in $N$. Thus we may assume that $\mu$ is smooth at $x$.

We may replace $u^{(0)}, \ldots, u^{(s)}$ by any invertible linear combination of themselves; so we may assume that $U_{t_0} : u^{(0)} = 0$. Given $k$, set
\[
v(t) := u^{(0)} + tu^{(k)}.
\]
Since $\mu$ is smooth at $x$, there are power series $x_0(t), \ldots, x_n(t)$ such that
\[
x(t) := (x_0(t), \ldots, x_n(t)) \in M, \ x = x(0), \ 
\mu(x(t)) = (1, 0, \ldots, 0, t, 0, \ldots, 0)
\]
where the last $t$ is placed in the $k$th coordinate. Then all the partial derivatives of order $r - 1$, with respect to the $x_i$, of $v(t)$ vanish at $x(t)$;
in other words,
\[ u^{(0)}_{i_1...i_{r-1}}(x(t)) + tu^{(k)}_{i_1...i_{r-1}}(x(t)) = 0 \text{ where } 0 \leq i_j \leq n. \quad (*) \]

Equation (*) holds identically in \( t \). So differentiating it yields
\[
\sum_l u^{(0)}_{i_1...i_{r-1}l}(x(t)) \frac{dx_l(t)}{dt} + t \sum_l u^{(k)}_{i_1...i_{r-1}l}(x(t)) \frac{dx_l(t)}{dt} + u^{(k)}_{i_1...i_{r-1}}(x(t)) = 0.
\]

Multiply by \( x_{i_1}(t) \) for \( i_1 = 0, \ldots, n \), and take the sum. Let \( m \) be the common degree of \( u^{(0)} \) and \( u^{(k)} \). Then Euler’s identity yields
\[
(m - r + 1) \sum_l \left( u^{(0)}_{i_2...i_{r-1}l}(x(t)) + tu^{(k)}_{i_2...i_{r-1}l}(x(t)) \right) \frac{dx_l(t)}{dt} + (m - r + 2) u^{(k)}_{i_2...i_{r-1}}(x(t)) = 0.
\]

However, the expression between the large parentheses vanishes by (*). Hence \( u^{(k)}_{i_2...i_{r-1}}(x(t)) = 0 \). Setting \( t = 0 \) yields \( u^{(k)}_{i_2...i_{r-1}}(x) = 0 \), which was to be proved, and the proof of Theorem (3.1) is complete.

**4. The extended first theorem.**

In this section, we’ll discuss the extension of Bertini’s first theorem to an arbitrary ambient variety. We’ll trace the evolution of its statement and of its proof in the hands of Bertini and others. In particular, we’ll discuss some rather sketchy and intuitive proofs given by Enriques and by Severi. These proofs will only be described briefly to give their flavor. Doubtless, with some effort, they could be completed and made rigorous, but doing so would provide involved proofs that offer no new insight. Until explicit mention is made to the contrary about half way through the section, the ground field will be the complex numbers.

To date, Theorem (3.1) has been extended only in the special case of Theorem (3.2), where the minimal order of multiplicity \( r \) is 2. However, at the end of this section, by refining, developing and completing some of the old arguments, we’ll prove a full extension, Theorem (4.2), of Theorem (3.1) itself.

In the extensions of Theorem (3.1) and Theorem (3.2), the original statements must be refined to make allowance for the possible presence of singularities in the ambient variety, which is arbitrary now. Thus Theorem (3.2) acquires the following familiar statement.
Theorem (4.1) (On variable singular points, extended). On an arbitrary ambient variety, if a linear system has no fixed components, then the general member has no singular points outside of the base locus of the system and of the singular locus of the ambient variety.

Here the ambient variety is abstract, and need not be embeddable in any projective space, although only embeddable varieties were considered before about 1950. However, this matter is of no real importance since the question is local.

The ambient variety was a surface when Theorem (4.1) was stated for the first time and by Enriques. He did so in 1893 in his first major work [20, p. 42], which initiated a general theory of linear systems (notably including adjoint systems) on an abstract surface. For Enriques, a linear system was obtained by taking a suitable projective model of the surface, cutting it by the members of a linear system of hypersurfaces, and then stripping away some, or all, of the fixed components, which are the components common to all the members of the induced system. Thus the members of a linear system are divisorial cycles, or linear combinations of subvarieties of codimension 1. These cycles need not be defined by a single equation locally about any given singular point of the ambient surface. However, this matter too is of no real importance since the theorem asserts nothing about these singular points.

When Enriques stated the theorem, he did not simply assume that the system has no fixed components; curiously, he also assumed that a general member is irreducible. Enriques did not dwell on the details of his proof, but in brief he argued by contradiction as follows (and repeated the argument, in slightly more detail, in his 1896 paper [21, pp. 230–31]): if the theorem were false, then the given system would contain a pencil with a variable singular point, and the surface would have a birationally equivalent model in $\mathbb{P}^3$ so that the equivalent pencil consists of the sections by the planes through a line; however, each of these planes would be tangent to the model, and this situation is absurd.

The theorem was extended further, to an ambient variety $X$ of arbitrary dimension, by Severi in his 1906 paper [38, p. 169]. On p. 168, he credited Bertini’s 1882 paper [2] for the original statement and Enriques 1896 paper [21, pp. 230–31] for the extension to an arbitrary surface. At the same time, Severi acknowledged the receipt of a letter from Bertini, in which Enriques’ extension was used. Severi then sketched a proof,
which is in some ways similar to and in some ways different from Enriques’ proof and also from Bertini’s proof in his book [5, §18, p. 239], which had already appeared in lithograph in 1899, but not yet in print. Generously, Severi [38, Oss., p. 170] described his proof as nothing but a geometric form of Bertini’s 1882 proof.

Severi’s sketch looks much like this. Let \( x \) be a variable singular point at which \( X \) is smooth. We may assume that \( X \) is embedded in a \( \mathbb{P}^n \) as a hypersurface and that the linear system on \( X \) is cut out, off a fixed component not containing \( x \), by a linear system of hypersurfaces \( U_t \). If this system has \( x \) as a variable singular point too, then it is a base point by Bertini’s Theorem (3.2) for the ambient \( \mathbb{P}^n \). Otherwise, a general \( U_t \) is smooth at \( x_t \); replace it by its tangent hyperplane, use the dual variety to conclude that this hyperplane contains \( x \), and finally deduce that every \( U_t \) too contains \( x \); thus again \( x \) is a base point.

As noted above, Bertini himself had already proved the extended theorem in arbitrary dimension in his book [5]. Curiously, Bertini did not cite Enriques or Severi, although he did cite many other relevant works at appropriate places throughout the book. Bertini’s proof and Severi’s are basically alike. However, Bertini did not require \( X \) to be a hypersurface in the ambient \( \mathbb{P}^n \). On the other hand, he too used his Theorem (3.2) for \( \mathbb{P}^n \) to reduce to the case that the general \( U_t \) is smooth at \( x_t \). However, he did not then replace this \( U_t \) by its tangent hyperplane; rather, he gave a direct analytic argument that every \( U_t \) must contain \( x \). In fact, it is unnecessary to make this application of his original theorem for either of the following two reasons: first, if we re-embed \( X \) using a Veronese embedding, then we may assume that all the \( U_t \) are hyperplanes, so smooth; second, Bertini’s analytic argument works even when the general \( U_t \) is not smooth, as we’ll see now.

Bertini’s analytic argument is rigorous here again, and so it provides a rigorous proof of Theorem (4.1). The argument is much like that in his proof of Theorem (3.1), but there is a twist. The argument runs as follows. The setup here is the same as that in the proof of Theorem (3.1), except that now the ambient variety \( X \) is a subvariety of \( \mathbb{P}^n \) and the parameterized variable point \( x(t) \) lies outside the smooth locus of \( X \) and inside the singular locus of \( X \cap U_t \), where \( U_t : v(t) = 0 \) is a parameterized hypersurface in \( \mathbb{P}^n \). These conditions imply that the vector,

\[
\left( \frac{dx_0}{dt}(0), \ldots, \frac{dx_r}{dt}(0) \right),
\]
lies in the tangent space to $X$ at $x = x(0)$; they also imply that this
tangent space lies in the tangent hyperplane to $U_{t_0}$ at $x$ if $U_{t_0}$ is smooth
there. So, in any event,

$$\sum_i v_i(0)(x) \frac{dx_i}{dt}(0) = 0$$

where $v_i := \frac{\partial v}{\partial x_i}$.

On the other hand, $v(t)(x(t)) = 0$. Differentiating this equation yields

$$\sum_i v_i(0)(x) \frac{dx_i}{dt}(0) + \frac{\partial v}{\partial t}(x) = 0.$$

Hence $\frac{\partial v}{\partial t}(x) = 0$. Now $v(t) := u^{(0)} + tu^{(k)}$. So $u^{(k)}(x) = 0$, as was
required. Thus the proof of Theorem (4.1) is complete.

Zariski gave an intrinsic proof of Theorem (4.1) in his paper [46] of
1944. (Zariski was in Italy from 1921 to 1927, and studied with Castel-
nuovo, Enriques and Severi; for a lovely description of Zariski’s stay, see
[33].) Ten years earlier, Enriques had given an intrinsic proof for sur-
faces in his book with Campedelli [23, p. 31] (again assuming a general
member is irreducible); Enriques simply asserted that the theorem can
be proved in essentially the same way as it was for the plane in his book
with Chisini [24, p. 181] (where a general member is not assumed to be
irreducible), because this proof is local in nature. Enriques’ assertion
was repeated by Zariski in his book [44, p. 26] of 1935.

Zariski’s 1944 proof is conceptually more advanced, and runs as
follows. Replacing the ambient space $X$ by an open subset, we may
assume that $X$ is smooth and that the system has no fixed components
and no base points. Then the total space of the system is smooth too,
because it is a locally trivial bundle of projective spaces; indeed, the fiber
over a point of $X$ is just the set of members that contain this point. A
general member of the system is, therefore, smooth everywhere by what
is now usually called “Sard’s theorem,” which holds in characteristic 0.

Zariski proved Sard’s theorem in the case at hand; in fact, the proof
works in general. Namely, Zariski noted that the “generic” member of
the system is regular (an intrinsic property, defined by means of uni-
formizing parameters), because, as a scheme, it is the fiber over the
“generic” point of the parameter space $\mathbb{P}^*$; so its local rings are simply
localizations of the local rings of the total space. Now, in characteristic
zero, a regular scheme is smooth [48, Thm. 7], and smoothness is an
open condition, as it is defined “by means of nonvanishing Jacobians” [46, p. 139].

It was Zariski’s work on Bertini’s two theorems that helped motivate him to study the relationship between regularity and smoothness, according to Mumford [30, p. 5]. In turn, it is clear from what Zariski himself wrote, in his paper [47, p. 474] on the reduction of singularities of 3-folds, that this work on singularities had motivated him to work on Bertini’s first theorem.

Zariski [46, p. 140] also provided (essentially) the following two examples, which show that Theorem (3.2), as stated, is false in positive characteristic $p$. (Earlier, van der Waerden [40, §3., p. 133] had noted that his version of Enriques’ 1896 proof required an appropriate hypothesis of separability.) In both examples, the ambient space is the plane $\mathbb{P}^2$ with inhomogeneous coordinates $x, y$, and the parameter space is the line $\mathbb{P}^1$ with inhomogeneous coordinate $t$. In the first example, the system consists of the curves,

$$U_t : x^p - ty^p = (x - t^{1/p}y)^p = 0,$$

which consist entirely of $p$-fold points. In the second example, $p \neq 2$, and the system consists of the (absolutely) irreducible curves,

$$U_t : y^2 - x^p + t = y^2 - (x - t^{1/p})^p = 0,$$

which have a variable double point at $(0, t^{1/p})$.

Finally, here is the new extension of Bertini’s Theorem (3.1), which treats variable $r$-fold points for an arbitrary $r$. This extension, Theorem (4.4) below, is valid for an arbitrary ambient variety $X$ in arbitrary characteristic $p$. Validity for $p > 0$ is made possible by a suitable new notion, “separable variable $r$-fold point,” which is defined below. Of course, the separability requirement is automatically satisfied when $p = 0$.

The various $r$-fold points of the members of a linear system on $X$, parameterized by $\mathbb{P}^s$, form a subset, $M_r$, say, of $X \times \mathbb{P}^s$. A given $r$-fold point will be called variable, resp., separable variable, if it lies in a irreducible subset $M$ of $M_r$ that is closed in $X \times \mathbb{P}^s$ and that projects onto $\mathbb{P}^s$, resp., and that projects separably onto $\mathbb{P}^s$. In fact, the proof of Theorem (4.2) shows that $M_r$ is closed in $X \times \mathbb{P}^s$; so $M$ is closed in $M_r$. Notice that the multiple points in Zariski’s two examples are variable, but not separable.
Theorem (4.2) (On variable $r$-fold points, extended). Let $X$ be a variety over an algebraically closed field of arbitrary characteristic, and $x$ a point of $X$ outside of its singular locus. If $x$ is a separable variable $r$-fold point of a linear system, then $x$ is an $(r-1)$-fold point of every member.

Since the matter is local on $X$, we may replace $X$ by an affine neighborhood of $x$ on which there exist "uniformizing coordinates"; these are functions $\xi_1, \ldots, \xi_n$, where $n := \dim X$, whose differentials $d\xi_i$ form a free basis of the 1-forms. Identify $\xi_i$ with its pullback to $X \times X$ via the second projection, denote by $\eta_i$ its pullback via the first projection, and set $\delta_i := \eta_i - \xi_i$. Given any (regular) function $u$ on $X$, denote by $u(\eta)$ its pullback to $X \times X$ via the first projection, and by $u(\xi)$ that via the second. Then $u(\eta)$ has a unique expansion,

$$u(\eta) = \sum_{0 \leq i_1 \leq \ldots \leq i_q \leq r} u_{i_1 \ldots i_q}(\xi) \delta_{i_1} \cdots \delta_{i_q} + w,$$

where the $u_{i_1 \ldots i_q}$ are suitable functions on $X$ and where $w$ is a suitable function on $X \times X$ that lies in the $r$th power of the ideal $I_{\Delta}$ of the diagonal. In characteristic 0, the $u_{i_1 \ldots i_q}$ are simply scalar multiples of the partial derivatives of $u$, and this expansion of $u$ is essentially the Taylor expansion. In arbitrary characteristic, the expansion can be obtained by plugging in $\delta_i + \xi_i$ for $\eta_i$ in $u(\eta)$ and collecting the terms in the products $\delta_{i_1} \cdots \delta_{i_q}$; the expansion is unique because these products provide a free basis of $I_{\Delta}^{q}/I_{\Delta}^{q+1}$ as a module over the ring of functions on $X$ because $I_{\Delta}/I_{\Delta}^2$ is equal to the module of 1-forms.

A point $x$ of $X$ is an $r$-fold point of the divisor $U : u = 0$ if and only if all the functions $u_{i_1 \ldots i_q}$ vanish at $x$. In characteristic 0, this statement is essentially the definition in Section 3 for the case $X = \mathbb{P}^n$. In general, we must see that this vanishing holds if and only if $u \in m_x^r$ where $m_x$ is the maximal ideal of $x$. To do so, identify $X$ with $X \times \{x\}$ inside $X \times X$. Then $u(\eta)$ restricts to $u$, and $u_{i_1 \ldots i_q}(\xi)$ restricts to $u_{i_1 \ldots i_q}(x)$; moreover, $I_{\Delta}$ restricts to $m_x$. Hence, if the $u_{i_1 \ldots i_q}$ vanish at $x$, then $u \in m_x^r$. The converse holds because the $\delta_i$ restrict to the functions $\pi_i := \xi_i - \xi_i(x)$, which form a system of regular parameters at $x$, and so the various products $\pi_{i_1} \cdots \pi_{i_q}$ form a vector-space basis of $m_x^r/m_x^{r+1}$.

Say that the members $U_t$ of the linear system are of the form,

$$U_t : t_0 u^{(0)} + \cdots + t_s u^{(s)} = 0,$$
where $t := (t_0, \ldots, t_s) \in \mathbf{P}^s$ and where $u^{(0)}, \ldots, u^{(s)}$ are now (regular) functions on $X$. Expand each $u^{(i)}$ as above, and let $u^{(i)}_{i_1 \ldots i_q}(\xi)$ be the coefficients. Then the set $M_r$ of various $r$-fold points is defined in $X \times \mathbf{P}^s$ by the vanishing of all the functions,

$$t_0 u^{(0)}_{i_1 \ldots i_q}(\xi) + \cdots + t_s u^{(s)}_{i_1 \ldots i_q}(\xi),$$

where now $u(\xi)$ means the pullback of the function $u$ on $X$. Thus $M_r$ is closed.

Since the given point $x$ is a separable variable point, it lies in a closed subset $M$ of $M_r$ such that the projection $\mu: M \to \mathbf{P}^s$ is smooth on a dense open set. Arguing as above in our version of Bertini’s original proof, using the closedness of $M_r$, we may assume that $\mu$ is smooth at $x \in U_{t_0}: u^{(0)} = 0$.

We have to prove that $u^{(k)} \in m_x^{r-1}$ for each $k$. As before, fix $k$ and use power series to parameterize a variable point $x(t) \in M$ such that

$$x(0) = x \text{ and } \mu(x(t)) = (1, 0, \ldots, 0, t, 0, \ldots, 0).$$

Since $x(t) \in M_r$, the power series,

$$u^{(0)}_{i_1 \ldots i_q}(x(t)) + t u^{(k)}_{i_1 \ldots i_q}(x(t)),$$

vanishes identically. Now $I_\Delta$ is generated over the ring of functions on $X$ by the differences $g(\eta) - g(\xi)$ as $g$ runs through the functions on $X$. Hence, thanks to the expansion of $u^{(0)}(\eta) + t u^{(k)}(\eta)$, we can find functions $g_i$ and $g_{ij}$ on $X$ such that

$$u^{(0)} + t u^{(k)} = \sum_i g_i (g_{i_11} - g_{i_11}(x(t))) \cdots (g_{ir} - g_{ir}(x(t))).$$

Now, $g_{ij}(x(t)) = g_{ij}(x) + t g'_{ij}(t)$ for some power series $g'_{ij}(t)$. Plugging in and collecting the coefficient of $t$, we conclude that $u^{(k)} \in m_x^{r-1}$ because $g_{ij} - g_{ij}(x) \in m_x$. Thus Theorem (4.2) is proved.
5. The second theorem.

In this section, we’ll discuss Bertini’s second fundamental theorem. It characterizes those linear systems whose members are all reducible (or equivalently, whose general members are reducible), the so-called reducible linear systems. The theorem asserts, over the complex numbers, that, if a reducible system has no fixed components, then its members are sums of members of a pencil. The pencil is linear if the ambient space is the projective $n$-space $\mathbb{P}^n$, the only case that Bertini considered; see [2] and [5, p. 231]. However, in other cases, the pencil can be a nonlinear 1-parameter algebraic system.

Bertini’s theorem was extended to an arbitrary ambient surface by Enriques in this pioneering 1893 paper [20, p. 41]. He gave a sketchy argument; it is somewhat similar to the one that he gave at the same time for Theorem (4.1), which is described briefly in Section 4. In a footnote, Enriques suggests that his work be compared to Noether’s work, [31, p. 171] and [32, p. 524], which appeared in 1873 and 1875, about ten years before Bertini’s original paper [2]. However, Noether did not, in fact, clearly state or prove any particular case of the theorem. Moreover, Bertini did not refer to Noether’s work in either his paper [2] or in his book [5]. At any rate, the theorem was called the theorem of Bertini–Enriques by van der Waerden in his 1937 paper [41, §3], and by Zariski in his 1941 paper [45]. However, seventeen years later, when Zariski [49] published his extension of Bertini’s theorem to positive characteristic, he did not cite Enriques’ paper, nor even mention his name.

The papers of van der Waerden and of Zariski are devoted to providing rigorous proofs of Bertini’s second theorem for an arbitrary ambient variety of any dimension. Bertini, Enriques (in his later treatment [23, pp. 31–33]), and van der Waerden approached the theorem in roughly the same way, involving Bertini’s first theorem. Zariski changed this approach at several places; he introduced some serious commutative algebra, which in particular eliminated the need for Bertini’s first theorem. These approaches will now be examined and developed, but first we’ll discuss the content of the second theorem itself.

Let $X$ be an arbitrary (irreducible) variety of dimension $n$ over the complex numbers. A trivial way to construct a reducible linear system on $X$ is to add a fixed component to every member of a given system. A more sophisticated way is to begin with a pencil without
fixed components. This is an algebraic system of divisorial cycles such that precisely one member passes through a given general point \( x \) of \( X \); also, the total space \( T \) is assumed to be reduced and irreducible, and the parameter space \( C \), to be complete. If this \( x \) lies outside the singular locus of \( X \), then \( x \) lies inside the open set \( V \) over which the projection \( T \to X \) is an isomorphism (by Zariski’s “main theorem”). Thus \( V \) is nonempty. Hence the total space \( T \) is of dimension \( n \), and the parameter space \( C \) is a curve. So if a point \( y \) of \( X \) lies outside the base locus of the pencil (the intersection of all its members), then \( y \) lies in at most finitely many members. Hence, if \( y \) also lies outside the singular locus of \( X \), then \( y \) lies in \( V \).

There is a natural map \( f: V \to C \); its graph is the preimage of \( V \) in the total space \( T \). The pencil consists of the closures of the map’s fibers, aside possibly from finitely many members that have, as components, parts of the singular locus of \( X \). Now, a linear system (or linear involution) on \( C \) gives rise, via pullback, to one on \( X \); its members are composed of \( d \) members of the pencil if the system on \( C \) is of degree \( d \). Such a linear system on \( X \) is said to be composite with a pencil (or an involution in a pencil).

Thus we have constructed two sorts of reducible linear systems. Remarkably, these are the only possibilities, according to Bertini’s second fundamental theorem. Bertini himself treated only the case where \( X = \mathbb{P}^n \). In this case, the pencil in question is necessarily a linear system, and his theorem may be stated as follows.

**Theorem (5.1)** (On reducible linear systems). On \( \mathbb{P}^n \), a reducible linear system, without fixed components, is necessarily composite with a linear pencil.

Bertini also restated the theorem algebraically essentially as follows.

**Theorem (5.2)** (Algebraic restatement). Let \( u^{(0)}, \ldots, u^{(s)} \) be forms of the same degree in variables \( x_0, \ldots, x_n \), and let \( t_0, \ldots, t_n \) be indeterminate parameters. If the form in the \( x_i \),

\[
F := t_0 u^{(0)} + \cdots + t_s u^{(s)},
\]

is reducible, then either the \( u^{(i)} \) have a common factor, or they are equal to forms \( v^{(i)} \) of the same degree in two other forms \( w_0 \) and \( w_1 \) of the same degree in the \( x_i \),

\[
u^{(i)}(x_0, \ldots, x_n) = v^{(i)}(w_0(x_0, \ldots, x_n), w_1(x_0, \ldots, x_n)).
\]
Note that $F$ may be reducible in the $x_i$, although it is irreducible in the $t_i$ and the $x_i$ together. For example, $F$ might be

$$t_0x_0^2 + t_1x_1^2 = (\sqrt{t_0}x_0 + \sqrt{-t_1}x_1)(\sqrt{t_0}x_0 - \sqrt{-t_1}x_1);$$

here, $v^{(i)}(x_0, x_1) = x_i^2$ and $w^{(i)}(x_0, x_1) = x_i$.

In his book [5, p. 231], Bertini cited two of Lüroth’s works, [27] and [28], published in 1893 and 1894; in them, Theorem (5.2) is credited to Bertini and reproved via a lengthier, more elementary, and more algebraic proof. Just before, in 1891 Poincaré [35, p. 183] published a short and more algebraic proof of the theorem for the plane, but he doesn’t credit the result. Both Poincaré and Lüroth assumed that $F$ is reducible whenever the $t_i$ are given complex values, and this hypothesis amounts precisely to the corresponding hypothesis in Theorem (5.1). Bertini apparently did not see the need to prove the equivalence of reducibility for indeterminate values and of reducibility for all complex values; more about this gap will be said below.

Bertini’s proof runs, more or less, as follows. By Theorem (2.2), the system has no variable singular points outside of the base locus. In particular, a general member $U$ is reduced; that is, its components $U_1, \ldots, U_d$ are distinct. Among such $U$, fix one with $d$ minimal. By hypothesis, $d \geq 2$. Let $\mathcal{P}$ be any subsystem with the following three properties: (1) $\mathcal{P}$ contains $U$; (2) $\mathcal{P}$ has no fixed components; and (3) $\mathcal{P}$ is parameterized by a line $D$ in the parameter projective space $P^s$ of the given linear system. Then a general member of $\mathcal{P}$ has $d$ components because of the minimality of $d$. Furthermore, the $i$th component varies in a 1-parameter algebraic system $\mathcal{P}_i$, which may be constructed as follows.

Over a suitable finite algebraic extension field of the field of rational functions on $D$, factor the form defining the total space of $\mathcal{P}$. Say that the $i$th factor has degree $r_i$, and set $s_i := \binom{r_i + n}{n} - 1$. Then the factor’s coefficients are the coordinates of a “generic” point of a curve $D_i$ in $P^{s_i}$, which parameterizes the system $\mathcal{P}_i$ of hypersurfaces of degree $r_i$ in $P^n$. In fact, $\mathcal{P}_i$ is a linear pencil, because $\deg D_i$ is equal to the number of hypersurfaces belonging to $\mathcal{P}_i$ and passing through a general point $x$ of $P^n$, and this number is 1 as only one member of $\mathcal{P}$ passes through $x$.

The $d$ systems $\mathcal{P}_i$ are, in fact, equal to each other. Indeed, given a general member $U'_1$ of $\mathcal{P}_1$, say $U'_1$ is a component of $U' \in \mathcal{P}$, and choose a general point $x$ in $U'_1$. For each $i$, necessarily $x$ lies in some
member $U'_i \in \mathcal{P}_i$. Then $U'_i$ must also be a component of $U'$ because $\mathcal{P}$ is a pencil. Hence $U'_i$ must be equal to $U'_1$. Thus $U'_1 \in \mathcal{P}_i$. Hence $\mathcal{P}_1$ and $\mathcal{P}_i$ coincide for all $i$, as asserted. Therefore, all the components of $U$ and of $U'$ belong to $\mathcal{P}_1$. In particular, $U_1$ and $U_2$ do; so they determine this linear pencil. Hence, when $\mathcal{P}$ is varied, $\mathcal{P}_1$ remains fixed. Therefore, the given system is composite with $\mathcal{P}_1$. Thus Theorem (5.1) is proved.

Bertini gave this proof, more or less, in both [2] and [5]; in the latter, he gave more details. The only significant gap is the lack of justification for the assertion that a general member of the pencil $\mathcal{P}_i$ is irreducible, or as Bertini put it, that each component of a general member varies in an algebraic system. This gap was filled by van der Waerden [41] using the theory of algebraic cycles that he and his thesis student Wei-Liang Chow (1911–95) had just developed (and which has become known as the theory of “Chow coordinates”).

In fact, van der Waerden filled the corresponding gap in Enriques’ extension of the Bertini’s theorem. Van der Waerden cited and followed Enriques’ treatment in [23, pp. 31–33], which is rather clean. Whereas Bertini worked with an ambient projective space of arbitrary dimension, Enriques worked only with a surface; however, van der Waerden observed that the proof works equally well on an arbitrary ambient variety.

Here is the statement of the general extension of Bertini’s second theorem.

**Theorem (5.3)** (On reducible linear systems, extended). On an arbitrary variety, a reducible linear system, without fixed components, is necessarily composite with a pencil (which need not be linear).

Enriques’ proof [23, p.33], as completed by van der Waerden [41, p.709], runs, more or less, as follows. Let $B$ be the union the singular locus of the ambient variety $X$ and of the base locus of the given system $\mathcal{U}$. Let $U$ be a general member, and $U_1, \ldots, U_d$ its components. Then the $U_i$ are distinct; in fact, they have no common point outside $B$, because any such point would be a singular point of $U$, and contradict Bertini’s first theorem, Theorem (4.1).

As in the proof of Theorem (5.1), let $\mathcal{P}$ be any subsystem with the three properties stated there. Then a general member of $\mathcal{P}$ has $d$ components, and each varies in a certain pencil $\mathcal{P}_1$, which is independent of the component; this time, $\mathcal{P}_1$ needn’t be linear, and can be defined using Chow coordinates. Let $b$ be a base point of $\mathcal{P}$. Since $\mathcal{P}$ has no
fixed components, $b$ must lie in infinitely many members of $P_1$, so in all of them. So $b$ must be common to all the components $U_i$ of $U$. Hence, by the conclusion of the preceding paragraph, $b \in B$.

Consider the map $f: (X - B) \to \mathbb{P}^s$ defined by $U$. Let $x \in (X - B)$. Then

$$f^{-1}f x = \bigcap \{U \in \mathcal{U} | x \in U\}.$$ 

Suppose $x$ is general. Then $f^{-1}f x$ is equidimensional. Its codimension must be 1; otherwise, since the $U \in \mathcal{U}$ that contain $x$ form a linear system, $\mathcal{U}$ would contain a linear pencil having $x$ as a base point, but no fixed component, contradicting the preceding argument (applied to this pencil and a general member $U$). It follows that, for any choice of $\mathcal{P}$, necessarily $P_1$ is the natural pencil $N$ parameterized by the normalization of $Y$ in the function field of $X$. Therefore, $\mathcal{U}$ is composite with this pencil, and the theorem is proved.

Enriques and van der Waerden carried out the proof above in two steps, obtaining the following intermediate results about a linear system without fixed components:

1. If the system is reducible, then any two general members have only base points in common.
2. If any two general members have only base points in common, then the system is composite with a pencil (and conversely).

On a surface, a linear system has an important invariant, its degree; by definition, this is the number of points, other than the base points, that two general members have in common. In terms of this notion, Enriques [23, p. 32] restated (2) as follows:

(2') On a surface, a linear system, without fixed components, that has degree zero is composite with a pencil (and conversely).

For the plane, this result already had been proved by Bertini, [3] and [5, p. 234]. In fact, Bertini defined the degree of a linear system on $\mathbb{P}^r$ for $r \geq 3$ as well, and he extended the characterization of those systems of degree zero, but this extension is no longer equivalent to (2).

Zariski did not use the theory of Chow coordinates, but proved the necessary irreducibility results directly. In particular, he gave a direct proof of the irreducibility of a general member of the pencil $N$. In fact, he observed that the proof works for any pencil, linear or not, yielding the following result [45, p. 61]:
If a pencil, without fixed components, is not composite, then all but finitely many members are irreducible.

Zariski also eliminated the use of Bertini’s first theorem by giving a direct algebraic proof that the pencils $\mathcal{P}_1$ and $\mathcal{N}$ coincide. In the middle of this proof, he explicitly used the hypothesis of characteristic 0. Later, in his 1958 monograph [49, §I.6], he treated the case of an algebraically closed ground field of arbitrary characteristic $p$, proving this result:

If a reducible linear system is free of fixed components and is not composite, then there is a power $p^e$ such that its members are all of the form $p^eU$ where $U$ moves in an irreducible linear system.

Another proof had already been published in 1951 by Matsusaka [29]. However, the author was a student of Zariski’s in the early 1960s, and can distinctly recall several conversations with him in which he shook his head and said: “Poor Matsusaka, he didn’t know that I already had a proof.”

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