Quaking neutron star deriving radiative power of oscillating magneto-dipole emission from energy of Alfvén seismic vibrations

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ABSTRACT

It is shown that monotonic depletion of the magnetic field pressure in a quaking neutron star undergoing Lorentz-force-driven torsional seismic vibrations about axis of its dipole magnetic moment is accompanied by the loss of vibration energy of the star that causes its vibration period to lengthen at a rate proportional to the rate of magnetic field decay. Highlighted is the magnetic-field-decay induced conversion of the energy of differentially rotational Alfvén vibrations into the energy of oscillating magneto-dipole radiation. A set of representative examples of magnetic field decay illustrating the vibration energy powered emission with elongating periods produced by quaking neutron star are considered and discussed in the context of theory of magnetars.

1. Introduction

Most, if not all, reported up to now computations of frequency spectra of poloidal and toroidal Alfvén vibration modes in pulsars and magnetars rest on tacitly adopted assumption about constant-in-time magnetic field in which a perfectly conducting neutron star matter undergoes Lorentz-force-driven oscillations (e.g. Carroll et al. 1986, Bastrukov et al. 1997, 1999, Lee 2007, 2008, Shaisultanov, Eichler 2009, Sotani et al. 2009; see, also, references therein). A special place in the study of the above Alfvén modes of pure shear magneto-mechanical seismic vibrations (\(a\)-modes) occupies a homogeneous model of a solid star with the uniform density \(\rho\) and frozen-in poloidal static magnetic field of both homogeneous and inhomogeneous internal and dipolar external configuration. This field can be conveniently represented in the form

\[
B(\mathbf{r}) = B \mathbf{b}(\mathbf{r}), \quad \mathbf{b}(\mathbf{r}) = [b_r(\mathbf{r}) \neq 0, b_\theta(\mathbf{r}) \neq 0, b_\phi(\mathbf{r}) = 0], \quad B = \text{constant}, \quad (1)
\]

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where $B$ is the field intensity [in Gauss] and $b(r)$ stands for the dimensionless vector-function of spatial distribution of the field. In particular, the above form of $B(r)$ has been utilized in recent works (Bastrukov et al. 2009a, 2009b, 2009c) in which the discrete spectra of frequencies of node-free torsional Alfvén oscillations has been computed in analytic form on the basis of equations of linear magneto-solid mechanics (Bastrukov et al. 2009a)

\[
\rho \ddot{\mathbf{u}} = \frac{1}{c} [\delta \mathbf{j} \times \mathbf{B}], \quad \nabla \cdot \mathbf{u} = 0, \quad (2)
\]

\[
\delta \mathbf{j} = \frac{c}{4\pi} [\nabla \times \delta \mathbf{B}], \quad \delta \mathbf{B} = \nabla \times [\mathbf{u} \times \mathbf{B}], \quad \nabla \cdot \mathbf{B} = 0. \quad (3)
\]

Here $\mathbf{u} = u(r,t)$ is the field of material displacement, the fundamental dynamical variable of solid-mechanical theory of elastically deformable (non-flowing) material continua. These equations describe Lorentz-force-driven non-compressional vibrations of a perfectly conducting elastic matter of a solid star with Ampère form of fluctuating current density $\delta \mathbf{j}$. Equation for $\delta \mathbf{B}$ describing coupling between fluctuating field of material displacements $\mathbf{u}$ and background magnetic field $\mathbf{B}$ pervading stellar material is the mathematical form of Alfvén theorem about frozen-in lines of magnetic field in perfectly conducting matter$^1$. As for the general asteroseismology of compact objects is concerned, the above equations seems to be appropriate not only for neutron stars but also white dwarfs (Molodtsova et al. 2010) and quark stars (Heyvaerts et al. 2009). The superdense material of these latter yet hypothetical compact stars is too expected to be in solid state (Xu 2003, 2009), that is, in the aggregate state possessing mechanical property of shear elasticity.

$^1$Equation (2) can be represented in the following equivalent tensor form (Bastrukov et al. 2009a),

\[
\ddot{u}_i = \nabla_k \tau_{ik}, \quad \text{where stress tensor } \tau_{ik} = (1/4\pi)[B_i \delta B_k + B_k \delta B_i - B_j \delta B_j \delta_{ik}],
\]

This last equation for $u_i$ is identical in appearance to canonical equation of solid-mechanics

\[
\ddot{u}_i = \nabla_k \sigma_{ik}, \quad \text{where stress tensor } \sigma_{ik} = 2\mu u_{ik} + [\kappa - (2/3)\mu] u_{jj} \delta_{ik} \text{ is the Hookean stress tensor and } u_{ik} = (1/2)[\nabla_i u_k + \nabla_k u_i],
\]

is the tensor of shear deformations in an isotropic elastic continuous matter with shear modulus $\mu$ and bulk modulus $\kappa$. The adopted for Alfvén vibrations of compact solid stars terminology [which is not new of course, e.g., Moon (1984)], namely, magneto-solid mechanics and solid-magnetics, is used as a solid-mechanical counterpart of well-known terms like magneto-fluid mechanics, magnetohydrodynamics (MHD) and hydromagnetics, $\rho \ddot{\mathbf{v}} = (1/c)[\delta \mathbf{j} \times \mathbf{B}], \quad \delta \mathbf{j} = (c/4\pi)[\nabla \times \delta \mathbf{B}], \quad \delta \mathbf{B} = \nabla \times [\mathbf{v} \times \mathbf{B}]$, describing flowing magento-active plasma in terms of the velocity of fluctuating flow $\dot{\mathbf{v}} = \dot{\mathbf{u}}$ and fluctuating magnetic field $\delta \mathbf{B}$. The MHD approach is normally utilized in astrophysics of main-sequence (MS) liquid stars. The advantage and virtue of the Rayleigh energy method of computing frequencies of node-free Alfvén vibrations (Bastrukov et al. 2009a, 2009b, 2009c, Molodtsova et al. 2010) is that it can too be applied to the above MHD equations underlying the study of $a$-modes in liquid MS-stars such, for instance, as rapidly oscillating Ap (roAp) stars, chemically peculiar magnetic stars exhibiting high-frequency oscillations that have been and still remain the subject of extensive investigation (e.g., Ledoux and Walraven 1958, Rincon and Rieutord 2003).
Inserting (2) in (3) the former equation of solid-magnetics takes the form
\[
\rho \ddot{u}(r, t) = \frac{1}{4\pi} [\nabla \times [\nabla \times [u(r, t) \times B(r)]]] \times B(r).
\] (4)

The obtained equation serves as a basis of our further analysis. The studied in above works vibration regime is of some interest in that the rate of differentially rotational material displacements
\[
\dot{u}(r, t) = [\omega(r, t) \times r], \quad \omega(r, t) = [\nabla \chi(r)] \dot{\alpha}(t),
\] (5)
\[
\nabla^2 \chi(r) = 0, \quad \chi(r) = f_\ell P_\ell(\cos \theta), \quad f_\ell(r) = A_\ell r^\ell
\] (6)
has one and the same form as in torsion elastic mode of node-free vibrations under the action of Hooke’s force of mechanical shear stresses (Bastrukov et al. 1999, 2007, 2008). Hereafter \(P_\ell(\cos \theta)\) stands for Legendre polynomial of degree \(\ell\) specifying the overtone of \(a\)-mode. The time-dependent amplitude \(\alpha(t)\) describes temporal evolution of above vibrations; the governing equation for \(\alpha(t)\) is obtained form equation (4). The prime purpose of above works was to get some insight into difference between spectra of discrete frequencies of toroidal \(a\)-modes in neutron star models having one and the same mass \(M\) and radius \(R\), but different shapes of constant-in-time poloidal magnetic fields. By use of the energy method, it was found that each specific form of spatial configuration of static magnetic field about axis of which the neutron star matter undergoes nodeless torsional oscillations is uniquely reflected in the discrete frequency spectra by form of dependence of frequency upon overtone \(\ell\) of nodeless vibration.

The assumption about constant in time undisturbed magnetic field means that the internal magnetic field pressure, \(P_B\), the velocity \(v_A\) of Alfvén wave in the star bulk (e.g.

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2It may be worth noting that in standing-wave regime of elastic-force-driven differentially rotational vibrations, the toroidal field of material displacements is described one and the same equation (5), but function \(\chi = A_\ell j_\ell(kr)P_\ell(\cos \theta)\), is the solution of Helmholtz equation \(\nabla^2 \chi + k^2 \chi = 0\) where \(k^2 = \omega^2/c_t^2\) and \(c_t^2 = \mu/\rho\) is the speed of shear wave of material displacements in solid stellar matter of density \(\rho\) and whose shear modulus is \(\mu\). The discrete frequency spectrum \(\omega(\ell) = 2\pi \nu(\ell)\) of elastic torsional vibration of multipole degree \(\ell\) is computed from by boundary condition \(n_k \sigma_{ik} = 0\) resulting in dispersion equation, \(dj_\ell(z)/dz = j_\ell(z)/z\), where \(z = kR\). The characteristic feature of the long-wave regime of node-free vibrations, \(z << 1\), with the toroidal field of material displacements (5)-(6) is the absence of dipole overtone in the spectrum of discrete frequencies (Bastrukov 2007a, 2007b): \(\nu_\ell(\ell) = \nu_\ell[(2\ell + 3)(\ell - 1)]^{1/2}\), with basic frequency of shear elastic vibrations, \(\nu_c = c_t/R\), where \(R\) stands for the star radius. In the non-studied before regime of short-wave length limit of torsional elastic vibrations (the limit \(z >> 1\) taken in the exact dispersion equation for dipole overtone \(dj_1(z)/dz = j_1(z)/z\)) leads to the spectrum of standing-wave dipole vibrations of the following analytic form \(\nu_c(\ell = 1) = \nu_c(p + 1)\), where \(p = 0, 1, 2, 3...\) is the node number of short-wavelength vibration regime.
Fig. 1.— The basic frequency and period of global Alfvén oscillations, equations (8), as functions of magnetic field intensity in the neutron star models with indicated mass and radius.

Lee 2007)

\[ P_B = \frac{B^2}{8\pi}, \quad v_A = \sqrt{\frac{2P_B}{\rho}} \]  

(7)

and, hence, the frequency \( \nu_A = \omega_A / 2\pi \) (where \( \omega_A = v_A/R \)) and the period \( P_A = \nu_A^{-1} \) of global Alfvén oscillations

\[ \nu_A = \frac{B}{2\pi} \sqrt{\frac{R}{3M}}, \quad P_A = \frac{2\pi}{B} \sqrt{\frac{3M}{R}} \]  

(8)

remain constant in the process of vibrations whose amplitude \( \alpha(t) \) subjects to standard equation of undamped harmonic oscillator (e.g., Bastrukov et al. 2009a). The allow for viscosity of stellar material leads to exponential damping of amplitude, but the frequency \( \nu_A \) and, hence, the period \( P_A \) preserve one and the same values as in the case of non-viscous vibrations (Bastrukov et al. 2009c). In Fig.1 these latter quantities are plotted as functions of
intensity $B$ of undisturbed poloidal magnetic field in the neutron star models with indicated mass $M$ and radius $R$. The practical usefulness of chosen logarithmic scale in this figure is that it shows absolute vales of $\nu_A$ and $P_A$ for global vibrations of typical in mass and radius neutron stars.

In this work, continuing a line of investigations reported in (Bastrukov et al. 2009a, 2009b, 2009c, Molodtsova et al. 2010), we relax the assumption about constant-in-time magnetic field and examine the impact of its decay on the vibration energy and period. In section 2, a mathematical background of quaking neutron star model is briefly outlined with emphasis on the loss of vibration energy caused by depletion of internal magnetic field pressure and resulting vibration-energy powered magneto-dipole radiation. The decreasing of magnetic field pressure in the star is presumed to be caused by coupling between vibrating star and outgoing material which is expelled by quake, but mechanisms of star-envelope interaction resulting in the decay of magnetic field, during the time of vibrational relaxation, are not considered. It is shown that physically meaningful inferences regarding radiative activity of quaking neutron star can be made even when detailed mechanisms of depletion of magnetic field pressure in the process of vibrations triggered by quakes are not exactly known. In section 3, a set of representative examples illustrating the newly disclosed effects is considered. The summary of developed theory with emphasis on its relevance to electromagnetic activity of quaking magnetars is presented in section 4.

2. Vibration-energy powered magneto-dipole emission of neutron star

In what follows we focus on dependence on time of the intensity of poloidal magnetic field, $B = B(t)$, about axis of which the star undergoes global torsional vibrations and confine our consideration to the model of uniform internal field

$$B(r, t) = B(t) \mathbf{b}(r), \quad [b_r = \cos \theta, \ b_\theta = -\sin \theta, \ b_\phi = 0]. \tag{9}$$

On account of this the equation of solid-magnetics, (4), takes the form

$$\rho \ddot{u}(r, t) = \frac{B^2(t)}{4\pi} [\nabla \times [\nabla \times [u(r, t) \times \mathbf{b}(r, t)]]] \times \mathbf{b}(r). \tag{10}$$

Inserting here the following separable form of fluctuating material displacements

$$u(r, t) = a(r) \alpha(t) \tag{11}$$

we obtain

$$\{\rho a(r)\} \ddot{\alpha}(t) = 2P_B(t)\{[\nabla \times [\nabla \times [a(r) \times \mathbf{b}(r)]]] \times \mathbf{b}(r)\} \alpha(t), \quad P_B(t) = \frac{B^2(t)}{8\pi}. \tag{12}$$
Scalar product of [12] with the time-independent field of instantaneous displacements \( \mathbf{a}(\mathbf{r}) \) followed by integration over the star volume leads to equation for amplitude \( \alpha(t) \) having the form of equation of oscillator with depending on time spring constant

\[
\mathcal{M} \ddot{\alpha}(t) + \mathcal{K}(t) \alpha(t) = 0, \quad \mathcal{M} = \rho \, m_\ell, \quad \mathcal{K}(t) = 2 \mathcal{P}_B(t) \, k_\ell, \tag{13}
\]

\[
m_\ell = \int \mathbf{a}(\mathbf{r}) \cdot \mathbf{a}(\mathbf{r}) \, dV, \quad \mathbf{a} = A_\ell \nabla \times [\mathbf{r} \, r^\ell \, P_\ell(\cos \theta)], \tag{14}
\]

\[
k_\ell = \int \mathbf{a}(\mathbf{r}) \cdot [\mathbf{b}(\mathbf{r}) \times [\nabla \times [\nabla \times [\mathbf{a}(\mathbf{r}) \times \mathbf{b}(\mathbf{r})]]]] \, dV. \tag{15}
\]

The solution of equation of non-isochronal (non-uniform in duration) and non-stationary vibrations with time-dependent frequency \([\ddot{\alpha}(t) + \omega^2(t) \alpha(t) = 0 \text{ where } \omega^2(t) = \mathcal{K}(t)/\mathcal{M}]\) is non-trivial and fairly formidable task (e.g., Vakman and Vainshtein 1977). But solution of such an equation, however, is not a prime purpose of this work. The main subject is the impact of depletion of magnetic-field-pressure on the total energy of Alfvén vibrations

\[
E_A(t) = \frac{\mathcal{M} \dot{\alpha}^2(t)}{2} + \frac{\mathcal{K}(t) \alpha^2(t)}{2} \tag{16}
\]

and the discrete spectrum of frequency of the toroidal \(a\)-mode

\[
\omega^2_\ell(t) = \omega^2_A(t) \, s^2_\ell, \quad \omega^2_A(t) = \frac{v^2_A(t)}{R^2}, \quad s^2_\ell = \frac{k_\ell}{m_\ell} \, R^2, \tag{17}
\]

\[
\omega^2_\ell(t) = B^2(t) \kappa^2_\ell, \quad \kappa^2_\ell = \frac{s^2_\ell}{4 \pi \rho \, R^2}, \quad s^2_\ell = \left[\left(\ell^2 - 1\right) \frac{2\ell + 3}{2\ell - 1}\right], \quad \ell \geq 2. \tag{18}
\]

It is to be stated clearly from the onset that it is not our goal here to speculate about possible mechanisms of neutron star demagnetization and advocate conceivable laws of magnetic field decay. The main purpose is to get some insight into the effect of arbitrary law of magnetic field decay in quaking neutron star on period of Lorentz-force-driven seismic vibrations and radiative activity of the star brought about by such vibrations. In the reminder of the paper we remove index \( \ell \) in the expression for discrete spectrum of frequency of toroidal Alfvén mode putting \( \omega(t) = \omega_\ell(t) \). It seems worth noting that first computation of discrete spectra of frequencies of toroidal Alfvén stellar vibrations in standing wave-regime, has been reported by Chandrasekhar (1956). The extensive review of other earlier computations of discrete frequency spectra of \(a\)-modes, \( \omega_\ell = \omega_A \, s_\ell \), is given in well-known review of Ledoux and Walraven (1958). In present work, we follow the line of the energy method developed in works (Bastrukov et al 2009a, 2009b, Molodtsova et al 2010) for computing discrete frequencies of node-free Alfvén vibrations and preserve one and the same notations for frequency spectra as in this latter canonical paper on variable stars.
2.1. Magnetic-field-decay induced loss of vibration energy

The total energy stored in quake-induced Alfvén seismic vibrations of the star is given by

\[ E_A(t) = \frac{M\dot{\alpha}^2(t)}{2} + \frac{K(B(t))\alpha^2(t)}{2}, \quad K(B(t)) = \omega^2(B(t))M. \]  

(19)

Perhaps most striking consequence of the magnetic-field-pressure depletion during the post-quake vibrational relaxation of neutron star is that it leads to the loss of vibration energy at a rate proportional to the rate of magnetic field decay

\[ \frac{dE_A(t)}{dt} = \left\{ \dot{\alpha}(t)[M\dot{\alpha}(t) + K(B(t))\alpha(t)] + \frac{\alpha^2(t)dK(B(t))}{2} \right\} = \frac{\alpha^2(t)dK(B)dB(t)}{2}dt. \]  

(20)

\[ \frac{d\omega^2(B)}{dB} = \frac{M\kappa^2\alpha^2(t)B(t)}{dB}dt. \]  

(21)

It is worth emphasizing at this point that the magnetic-field-decay induced loss of vibration energy is substantially different from the vibration energy dissipation caused by shear viscosity of matter resulting in heating of stellar material (Mestel 1999). As was noted, the characteristic feature of this latter mechanism of vibration energy conversion into the heat (i.e., into the energy of non-coherent electromagnetic emission responsible for the formation of photosphere of the star) is that the frequency and, hence, period of vibrations are the same as in the case of viscous-free vibrations (Bastrukov et al. 2009c). However, it is no longer so in the case under consideration. It follows from above that depletion of magnetic field pressure resulting in the loss of total energy of Alfvén vibrations of the star causes its vibration period to lengthen at a rate proportional to the rate of magnetic field decay. The process of magnetic field decay depends on many factors of demagnetization of neutron star matter and likely primarily on the star-environment communication. Also, it seems quite likely that, contrary to viscous dissipation, the loss of vibration energy due to decay of magnetic field must be accompanied by coherent (non-thermal) electromagnetic radiation. Adhering to this supposition in the next sections we consider several representative scenarios of magnetic field decay in quaking neutron stars. In so doing, special emphasis is made on conversion of the energy of Lorentz-force-driven seismic vibrations into the energy of magneto-dipole emission whose flux oscillates with frequency of torsional Alfvén magneto-mechanical vibrations of the star.
2.2. Conversion of vibration energy into power of magneto-dipole radiation

The point of departure in the study of vibration-energy powered magneto-dipole emission of the star (whose radiation power, \( P \), is given by Larmor’s formula) is the equation

\[
\frac{dE_A(t)}{dt} = P, \quad P = \frac{2}{3c^3} \delta \mu^2(t). \tag{22}
\]

Consider a model of quaking neutron star whose torsional magneto-mechanical oscillations are accompanied by fluctuations of total magnetic moment preserving its initial (in seismically quiescent state) direction: \( \mu = \mu_n = \text{constant} \). The total magnetic dipole moment should execute oscillations with frequency \( \omega(t) \) equal to that for magneto-mechanical vibrations of stellar matter which are described by equation for \( \alpha(t) \). This means that \( \delta \mu(t) \) and \( \alpha(t) \) must be subjected to equations of similar form, namely

\[
\delta \ddot{\mu}(t) + \omega^2(t) \delta \mu(t) = 0, \tag{23}
\]

\[
\ddot{\alpha}(t) + \omega^2(t) \alpha(t) = 0, \quad \omega^2(t) = B^2(t) \kappa^2. \tag{24}
\]

It is easy to see that equations (23) and (24) can be reconciled if

\[
\delta \mu(t) = i \mu \alpha(t). \tag{25}
\]

Then, from (23), it follows \( \delta \ddot{\mu} = -i \omega^2 \mu \alpha \). Given this and equating

\[
\frac{dE_A(t)}{dt} = \mathcal{M} \kappa^2 \alpha^2(t) B(t) \frac{dB(t)}{dt}, \tag{26}
\]

with

\[
P = -\frac{2}{3c^3} \mu^2 \kappa^2 B^4(t) \alpha^2(t) \tag{27}
\]

we arrive at the equation of time evolution of magnetic field

\[
\frac{dB(t)}{dt} = -\gamma B^3(t), \quad \gamma = \frac{2 \mu^2 \kappa^2}{3 \mathcal{M} c^3} = \text{constant} \tag{28}
\]

which yields the following law of its decay

\[
B(t) = \frac{B(0)}{\sqrt{1 + t/\tau}}, \quad \tau^{-1} = 2 \gamma B^2(0). \tag{29}
\]

The lifetime of magnetic field \( \tau \) is regarded as a parameter whose value is established from above given relations between the period \( P \) and its time derivative \( \dot{P} \) which are taken from observations. And knowing from observations \( P(t) \) and \( \dot{P}(t) \) and estimating \( \tau \) one can get information about the strength of magnetic field in the star and the magnitude of its total magnetic moment in undisturbed state.\(^3\)

\(^3\)Understandably that in the model of vibration-energy powered magneto-dipole emission under consideration, the equation of magnetic field evolution is obtained in similar fashion as equation for the angular
2.3. Lengthening of vibration period

The immediate consequence of above line of argument is the magnetic-field-decay induced lengthening of vibration period

\[ P(t) = \frac{2\pi}{\omega(t)} = \frac{C}{B(t)}, \quad C = \frac{2\pi}{\kappa}. \]  

(30)

The rate of period elongation is given by

\[ \frac{dP(t)}{dt} = -\frac{C}{B^2(t)} \frac{dB(t)}{dt}, \quad \frac{dB(t)}{dt} < 0. \]  

(31)

Combining these equations we obtain the following general relations

\[ P(t) B(t) = \text{constant}, \]  

(32)

\[ \frac{\dot{P}(t)}{P(t)} = -\frac{\dot{B}(t)}{B(t)}. \]  

(33)

velocity \( \Omega \) does in the standard model of rotation-energy powered emission of neutron star. However, in the model of quaking neutron star vibrating in toroidal \( a \)-mode, the elongation of period is attributed to magnetic field decay, whereas in canonical Pacini-Gold model of radio-pulsar the period lengthening is attributed to the spin-down of rotating neutron star.
which are generic to solely Alfvén vibration mode and independent of specific form of the magnetic field decay law. The difference between periods evaluated at successive moments of time $t_1 = 0$ and $t_2 = t$ is given by

$$\Delta P(t) = P(t) - P(0) = -P(0) \left[ 1 - \frac{B(0)}{B(t)} \right] > 0, \quad B(t) < B(0). \quad (34)$$

The practical usefulness of these general relations is that they can be used as a guide in search for fingerprints of Alfvén seismic vibrations in data on oscillating emission from quaking neutron star.

In the model under consideration

$$P(t) = \frac{C}{B(t)}, \quad B(t) = \frac{B(0)}{\sqrt{1 + t/\tau}} \quad (35)$$

$$P(0) = \frac{C}{B(0)}, \quad B(0) = B(t = 0). \quad (36)$$

and the time evolution of vibrational period and its derivative is described by

$$P(t) = P(0) \left[ 1 + (t/\tau) \right]^{1/2}, \quad \Delta P(t) = P(0) \left[ 1 - \sqrt{1 + t/\tau} \right], \quad (37)$$

$$\dot{P}(t) = \frac{1}{2\tau} \frac{P(0)}{\left[ 1 + (t/\tau) \right]^{3/2}}, \quad \rightarrow \quad \dot{P}(0) = \frac{P(0)}{2\tau} \quad (38)$$

$$\Delta \dot{P}(t) = \dot{P}(t) - \dot{P}(0) = -\frac{P(0)}{2\tau} \left[ 1 - \frac{1}{\sqrt{1 + t/\tau}} \right]. \quad (39)$$

It follows that lifetime $\tau$ is determined by

$$\dot{P}(t) \cdot P(t) = \frac{P^2(0)}{2\tau} = \text{constant} \quad (40)$$

and the ratio $\dot{P}$ to $P$ is given by

$$\frac{\dot{P}(t)}{P(t)} = \frac{1}{2\tau} \left[ 1 + (t/\tau) \right]^{-1}. \quad (41)$$

For the sake of illustration of general trends in $P$ and $\dot{P}$, in Fig.2 these quantities are plotted as functions of the fraction time $x = t/\tau$ which is ranged in the interval $0 < x < 1$. In Fig. 3, we plot $\dot{P}(t)/P(0)$ versus $P(t)/P(0)$.
Fig. 3.— The interrelation between time derivative of period $\dot{P}(x)$ versus period $P(x)$, both normalized to period at initial moment of time $P(0)$, computed as functions of $x = t/\tau$.

2.4. Time evolution of the vibration amplitude

It is interesting to note that considered model permits exact analytic solution of equation for vibration amplitude $\alpha(t)$ which is convenient to represented as

$$\ddot{\alpha}(t) + \omega^2(t)\alpha(t) = 0, \quad \omega^2(t) = \frac{\omega^2(0)}{1+t/\tau}, \quad \omega(0) = \omega_A k = \text{constant.} \quad (42)$$

The procedure is as follows. Let us introduce new variable

$$s = 1 + t/\tau, \quad t = (s - 1)\tau \quad (43)$$

$$\frac{\partial s}{\partial t} = \frac{1}{\tau}, \quad \dot{\alpha}(t) = \frac{\partial \alpha(t)}{\partial t}, \quad \alpha'(s) = \frac{\partial \alpha(s)}{\partial s} \quad (44)$$

so that

$$\dot{\alpha}(t) = \frac{\partial \alpha(t)}{\partial t} = \frac{\partial \alpha(s)}{\partial s} \frac{\partial s}{\partial t} = \frac{\alpha'(s)}{\tau}, \quad \ddot{\alpha}(t) = \frac{\alpha''(s)}{\tau^2}. \quad (45)$$

In terms of $\alpha(s)$, equation (42) takes the form

$$s\alpha''(s) + \beta^2\alpha(s) = 0, \quad \beta^2 = \omega^2(0)\tau^2 = \text{constant.} \quad (46)$$

This equation permits exact analytic solution (e.g. Polyanin and Zaitsev, 2004)

$$\alpha(s) = s^{1/2}\{C_1 J_1(2\beta s^{1/2}) + C_2 Y_1(2\beta s^{1/2})\} \quad (47)$$

where $J_1(2\beta s^{1/2})$ and $Y_1(2\beta s^{1/2})$ are Bessel functions (Abramowitz and Stegun 1972)

$$J_1(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - \theta) d\theta, \quad Y_1(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \theta) d\theta, \quad z = 2\beta s^{1/2}. \quad (48)$$
The arbitrary constants $C_1$ and $C_2$ can be eliminated from two conditions
\[ \alpha(t = 0) = \alpha_0, \quad \alpha(t = \tau) = 0. \] (49)
where zero-point amplitude
\[ \alpha_0^2 = \frac{2 \bar{E}_A(0)}{M \omega^2(0)} = \frac{2 \bar{E}_A(0)}{K(0)}, \quad \omega^2(0) = \frac{K(0)}{M} \] (50)
is related to the average energy $\bar{E}_A(0)$ stored in torsional Alfvén vibrations at initial (before magnetic field decay) moment of time $t = 0$. So, knowing from observations the amplitude of vibrations $\alpha_0$, the average energy $\bar{E}_A(0)$ is established from last equation.4

Fig. 4.— Vibration amplitude $\alpha(t)$ computed at fixed $C$ and $\beta$ and different values of $\eta$. This shows that variation of this parameter is manifested in change of magnitude of $|\alpha|$, but period elongation is not changed.

As a result, the general solution of (42) can be represented in the form
\[ \alpha(t) = [1 + (t/\tau)^{1/2}] C \{J_1(2\beta [1 + (t/\tau)]^{1/2}) - \eta Y_1(2\beta [1 + (t/\tau)^{1/2}])\}, \] (51)
\[ \eta = \frac{J_1(z(\tau))}{Y_1(z(\tau))}, \quad C = \alpha_0[J_1(z(0)) - \eta Y_1(z(0))]^{-1}. \] (52)

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4Before magnetic field decay, the star oscillates in the harmonic in time regime, that is, with amplitude $\alpha = \alpha_0 \cos(\omega(0)t)$, so that $\tilde{a}^2 = (1/2)\alpha_0^2$ and $\tilde{a}^2 = (1/2)\omega^2(0)\alpha_0^2$. The average energy stored in such oscillations is given by $\bar{E}_A(0) = (1/2)M \tilde{a}^2 + (1/2)K(0) \tilde{a}^2 = (1/2)M \omega^2(0)\alpha_0^2 = (1/2)K(0)\alpha_0^2$ from which equation (49) is obtained.
The vibration period lengthening in the process of vibrations is illustrated in Fig.4 and Fig.5, where we plot $\alpha(t)$, equation (51), at different values of parameters $\beta$ and $\eta$ pointed out in the figures. Fig.4 clearly shows that $\eta$ is the parameter regulating magnitude of vibration amplitude, the larger $\eta$, the higher is amplitude. However this parameter does not affect the rate of period lengthening. As it is clearly seen from Fig.5 both the elongation rate of vibration period and magnitude of vibration amplitude are highly sensitive to parameter $\beta$.

![Fig. 5.— Vibration amplitude $\alpha(t)$ computed at fixed $C$ and $\eta$ and different values of $\beta$.](image-url)

3. Representative examples

The considered law of magnetic field decay cannot be, of course, regarded as universal because it reflects a quite concrete line of argument regarding the fluctuations of magnetic moment of the star. The interrelation between quake-induced oscillations of total magnetic moment of the star $\delta \mu(t)$ and the amplitude $\alpha(t)$ of its seismic magneto-mechanical oscillations can be consistently interpreted with the aid of the function of dipole demagnetization $f(B(t))$ which is defined by the following condition of self-consistency in $\alpha$ of right and left
hand sides of equation (22), namely
\[
\dot{\delta} \mu(t) = i f(B(t)) \alpha(t)
\]  
(53)

The vector-function of dipole demagnetization \(f(B(t))\) depending on decaying magnetic field reflects temporal changes of electromagnetic properties of neutron star matter as well as evolution of magnetic-field-promoted coupling between neutron star and its environment. This means that specific form of this phenomenological function should be motivated by heuristic arguments taking into account these factors. With this form of \(\dot{\delta} \mu(t)\), equation (22) is transformed to magnetic field decay of the form
\[
\frac{dB(t)}{dt} = -\eta f^2(B(t)) \frac{B(t)}{B(t)}
\]
\(\eta = \frac{2}{3 \mathcal{M} \kappa^2 c^3}\) = constant.  
(54)

In the above considered case this function is given by This implies that the function of dipole demagnetization \(f(B(t))\) has the form
\[
f(B(t)) = \beta B(t) B(t), \quad \beta = \kappa^2 \mu = \text{constant.}
\]  
(55)

As is shown below, the practical usefulness of function of dipole demagnetization, \(f(B(t))\), consists in that it provides economic way of studying a vast variety of heuristically motivated laws of magnetic field decay, \(B = B(t)\), whose inferences can ultimately be tested by observations. In the next section, with no discussion of any specific physical mechanism which could be responsible for magnetic field decay, we consider a set of representative examples of demagnetization function \(f(B(t))\) some of which lead to the magnetic field decay laws that have been regarded before, though in a somewhat different context (e.g. Sang, Chanmugam 1987, Srinivasan et al. 1990, Goldreich, Reisenegger 1992, Urpin et al. 1994, Wang 1997, Livio et al. 1998, Geppert et. al 2001, Bisnovatyi-Kogan, 2002). In this connection it worth stressing that the model under consideration deals with magnetic field decay in the course of vibrations triggered by starquake, not with long-term secular decay which has been the subject of these latter investigations (see also references therein). In the case in question, a fairly rapid decay of magnetic field during the time of post-quake vibrational relaxation of the star is though of as caused, to a large extent, by coupling of the vibrating star with material expelled by quake. In other words, escaping material removing a part of magnetic flux density from the star is considered to be a most plausible reason of depletion of internal magnetic field pressure in the star. While physical processes responsible for magnetic field decay remain uncertain, practical significance of above, admittedly heuristic, line of argument is that it leads to meaningful conclusion (regarding elongation of periods of oscillating magneto-dipole emission) even when detailed mechanisms of depletion of magnetic field pressure in the course of vibrations are not exactly known.
Fig. 6.— The upper panel – the period $P(x)$ and difference between periods $\Delta P(x)$ of toroidal $a$-mode as functions of $x = t/\tau$. The middle panel shows that in this model of magnetic field decay the time derivatives of period $\dot{P}(x)$ and difference between periods $\Delta \dot{P}(x)$ are decreasing functions of $x$. The lower panel, the caused by field decay rate of the $a$-modes vibration period lengthening, normalized to period of initial moment of time, caused by magnetic field decay with time evolution of considered in the text representative examples $i = 1$ and $2$.

1. As a first representative example, consider a model of quaking neutron star in which function of dipole demagnetization is given by

$$f(B(t)) = \beta B(t).$$

This leads to exponential decay of magnetic filed

$$\frac{dB(t)}{dt} = -\frac{B(t)}{\tau}, \quad \rightarrow \quad B(t) = B(0) e^{-t/\tau}, \quad \tau^{-1} = \eta \beta^2$$

where $B(0)$ is the intensity of magnetic field before quake. The period and its derivative are given by

$$P(t) = P(0) e^{t/\tau}, \quad \dot{P}(t) = \frac{P(0)}{\tau} e^{t/\tau}.$$
Fig. 7.— The rate of period versus period of toroidal $a$-mode computed in 1-st and 2-nd representative models.

The difference between periods, evaluated at the moment of time $t_2 = t$ and $t_1 = 0$ is given by

$$
\Delta P(t) = P(t) - P(0) = P(0) e^{t/\tau_1} (1 - e^{-t/\tau_1}).
$$

(59)

In the initial stage of decay, when $(t/\tau) \ll 1$, the last equation is reduced to $\Delta P(t) \approx P(0) (t/\tau_1) [1 + (t/\tau_1)]$. The parameter of magnetic field lifetime is given by

$$
\tau^{-1} = \frac{\dot{P}(t)}{P(t)}.
$$

(60)

In Fig.6 the above period and its derivative are plotted as functions of fraction time $x = t/\tau$.

2. Consider a neutron star model whose function of demagnetization is given by

$$
f(B(t)) = \beta \sqrt{B(t) B(t)}.
$$

(61)

The resultant equation of magnetic field evolution reads

$$
\frac{dB(t)}{dt} = -\gamma B^2(t), \quad \gamma = \eta \beta^2
$$

(62)

and thus implying that rate of magnetic field decay is proportional to the density of magnetic field energy stored in the star. In this case one has

$$
B(t) = B(0) \left(1 + \frac{t}{\tau}\right)^{-1}.
$$

(63)

The elongation of vibration period and a rate of its lengthening are given by

$$
P(t) = P(0) \left(1 + \frac{t}{\tau}\right), \quad \dot{P}(t) = \frac{P(0)}{\tau} = \text{constant}.
$$

(64)
The difference between periods is given by \( \Delta P(t) = P(0)(t/\tau) \). The decay time is eliminated from the ratio

\[
\frac{\dot{P}(t)}{P(t)} = \tau^{-1} \left( 1 + \frac{t}{\tau} \right)^{-1} = \frac{\dot{P}(0)}{P(0)}.
\]

The inferences of this model regarding period and its derivative as functions of time are presented in Fig. 7.

![Graph](image)

Fig. 8.— The rate of frequency \( \dot{\nu}(x) \) normalized to \( \nu(0) \) as a function of \( x = t/\tau \) for logarithmic law of magnetic field decay.

3. Finally, consider a model with quite sophisticated function of dipole demagnetization

\[
f(B(t)) = \sqrt{\frac{B(0)}{\tau + t}} \frac{B(t)}{\sqrt{\eta B(t)}}
\]

which is interesting in that it leads to equation

\[
\frac{dB(t)}{dt} = -\frac{B(0)}{\tau} \left( 1 + \frac{t}{\tau} \right)^{-1} = -\frac{B(0)}{\tau + t}
\]

\[\text{Eq. 67}\]

\[\text{Eq. 66}\]

Similar analysis can be performed for the demagnetization function of the form

\[
f(B(t)) = \beta \sqrt{B^{m-1}(t)} B(t), \quad m = 3, 4, 5...
\]

which leads to

\[
\frac{dB(t)}{dt} = -\gamma B^m(t), \quad \gamma = \eta \beta^2,
\]

\[\text{Eq. 66}\]

\[\text{Eq. 67}\]

\[\text{Eq. 66}\]

\[\text{Eq. 67}\]
yielding a fairly non-trivial logarithmic law of magnetic field decay

\[ B(t) = B(0) \left[ 1 - \ln \left( 1 + \frac{t}{\tau} \right) \right]. \quad (68) \]

The corresponding frequency of toroidal \( a \)-mode as a function of time is plotted in Fig. 8 and it may be interesting to note that similar shape of \( \dot{\nu}(t)/\nu(0) \) exhibit data on post-glitch emission of PSR J1846-0258 (Livingstone et al. 2010).

In this model

\[ P(t) = P(0) \left[ 1 - \ln \left( 1 + \frac{t}{\tau} \right) \right]^{-1}, \quad (69) \]
\[ \dot{P}(t) = \frac{dP(t)}{dt} = \frac{P(0)}{\tau + t} \left[ 1 - \ln \left( 1 + \frac{t}{\tau} \right) \right]^{-2}, \quad (70) \]
\[ \tau^{-1} = \frac{\dot{P}(0)}{P(0)}. \quad (71) \]

These examples show that while the period elongation is the common effect of depletion of magnetic field pressure, the interrelations between periods and its derivatives substantially depend on specific form of the magnetic field decay law.

4. Summary

The main purpose of this work was to examine inferences of asteroseismic model of neutron star undergoing quake-induced torsional Alfvén vibrations about axis of its dipole magnetic moment which are accompanied by monotonic depletion of internal magnetic field pressure. It is shown this latter process results in the loss of vibration energy and changes the character of temporal evolution of vibrations from harmonic in time vibrations (with constant frequency) in static, time-independent, magnetic field to non-isochronal vibrations with monotonically decreasing frequency in decaying magnetic field. The presented analysis suggests that the loss of vibration energy caused by depletion of magnetic field pressure should be accompanied by coherent (non-thermal) oscillating magneto-dipole radiation (with frequency of luminosity oscillations equal to the frequency of Alfvén seismic vibrations of the star). In this context it is appropriate to remind a seminal work of Hoyle, Narlikar and Wheeler (1964) in which it has been pointed out for the first time that vibrating neutron star should operate like Hertzian magnetic dipole deriving radiative power of magneto-dipole emission from the energy of magneto-mechanical vibrations (see, also, Pacini 2008). What is newly disclosed here is that conversion of vibration energy into the energy of magneto-dipole
electromagnetic radiation can be realized when, and only when, torsional Alfvén vibrations are accompanied by decay of magnetic field.

All above suggests that considered for the first time theory of vibration-energy powered emission of neutron star is relevant to electromagnetic activity of magnetars - neutron stars endowed with magnetic field of extremely high intensity the radiative activity of which is ultimately related to the magnetic field decay (Duncan & Thompson 1992, Thompson & Duncan 1995). This subclass of highly magnetized compact objects is commonly associated with soft gamma repeaters and anomalous X-ray pulsars (e.g., Harding 1999, Kouveliotou 1999, Woods and Thompson 2006, Mereghetti 2008) – young isolated and seismically active neutron stars (Chen et al. 1996, Franco et al. 2000). The magnetar quakes are exhibited by short-duration thermonuclear gamma-ray flash followed by rapidly oscillating X-ray flare of several-hundred-seconds duration. During this latter stage of quake-induced radiation of magnetar, a long-periodic (5-12 sec) modulation of brightness was observed lasting for about 3-4 minutes (e.g., Harding 1999, Kouveliotou 1999). Taking into account such long-periodic and rapidly dying variability of magnetar pulsating radiation is the feature which is non-typical for young neutron stars powered by energy of rotation, it has been suggested (Bastrukov et al. 2002) that detected long-periodic pulsating emission is powered by the energy of torsional magneto-elastic vibrations of magnetar triggered by quake. In view of key role of ultra-strong magnetic field it is quite likely that quasi-periodic oscillations (QPOs) of outburst flux from SGR 1806-20 and SGR 1900+14 discovered in observations reported in (Israel et al. 2005, Watts & Strohmayer 2006, Terasawa et al. 2006) are produced by torsional seismic vibrations predominately sustained by Lorentz force (Bastrukov et al 2009b, 2009c) which are accompanied, as was argued above, by monotonic decay of background magnetic field. If so, the predicted elongation of QPOs period of oscillating outburst emission from quaking magnetars should be traced in existing and future observations.

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