An inverse problem for distributed order
time-fractional diffusion equations

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Abstract This paper deals with the distributed order time-fractional diffusion equations with non-homogeneous Dirichlet boundary condition. We first prove that the wellposedness of the solution by means of eigenfunction expansion. We next give a Harnack type inequality of the solution in the frequency domain under the Laplace transform, from which we further show a uniqueness result for an inverse problem in determining the weight function in the distributed order time derivative.

Keywords distributed order fractional diffusion, inverse problem, uniqueness, Laplace transform, Harnack’s inequality.

AMS Subject Classifications 35R11, 35R30, 26A33, 44A10.

1 Introduction and main results

In this paper, we consider the following initial-boundary value problem (IBVP)
\[
\begin{align*}
D_t^{(\mu)}u &= \Delta u \quad \text{in } Q, \\
u|_{t=0} &= 0 \quad \text{in } \Omega, \\
u &= g \quad \text{on } \Sigma,
\end{align*}
\]
(1)
where the bounded domain $\Omega$ is open and connected in $\mathbb{R}^d$, $d = 1, 2, 3$ with a smooth boundary $\partial \Omega$, which is defined e.g., by some $C^2$ functional relations, $Q := \Omega \times (0, T]$ and $\Sigma := \partial \Omega \times (0, T]$.

In (1), $D_t^{(\mu)}$ denotes a distributed order fractional derivative defined by
\[
D_t^{(\mu)} \varphi(t) = \int_0^1 \partial_t^\alpha \varphi(t) \mu(\alpha) d\alpha,
\]
where $\partial_t^\alpha$ is the Caputo derivative of order $\alpha$:
\[
\begin{align*}
\partial_t^\alpha \varphi(t) = \begin{cases}
\varphi(t), & \alpha = 0, \\
\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\varphi'(\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\
\varphi'(t), & \alpha = 1.
\end{cases}
\end{align*}
\]

The conditions on the boundary condition and the coefficient $\mu$ involved in $D_t^{(\mu)}$ will be specified later in the statement of the main theorem.
In the case when the weight function \( \mu := \sum_{j=1}^{\ell} q_j \delta(\cdot - \alpha_j) \), where \( \delta \) is the Dirac-delta function, we have the corresponding fractional diffusion model:

\[
\sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u = \Delta u \quad \text{in } Q,
\]

which has received great attention in applied disciplines due to the succeeding in modeling the anomalous diffusion processes whose mean square displacement (MSD) admits the non-Fickian growth rates, say, MSD behaves like \( \langle \Delta x^2 \rangle \sim Ct^{\min\{\alpha_j\}} \) as \( t \to \infty \) (e.g., [21]). Also see, e.g., in describing anomalous phenomenon in space such as skewness and the non-Gaussian profile: long-tailed profile (see, e.g., [1], [10] and the references therein), which are poorly characterized by the classical diffusion equations. Soon the multi-term time-fractional diffusion attracted attention of mathematicians, we refer to [11], [12], [13], [15] and the references therein. Most recently, some ultraslow diffusion processes whose MSD is of logarithmic growth were found in different application areas including polymer physics and kinetics of particles moving in the quenched random force fields (see e.g. [4], [22], [25] and the references therein). What is the governing equation for the ultraslow diffusion processes? One of the approaches for modeling of such processes is to accumulate the fractional derivative on the range e.g. \([0, 1]\), say, the diffusion model (1) with \( \mu \in C[0, 1] \), which will be studied in details in next sections.

The distributed order fractional derivative was firstly considered by Caputo [2]. After that, the mathematical researches on the analyzing the forward problem, such as the IBVPs for the diffusion equation with distributed order fractional derivatives, were growing rapidly, see, e.g. [8], [17], [18], [20] and the references therein. Namely, [8] investigated the properties of the fundamental solutions to the Cauchy problems for both the ordinary and the partial fractional differential equations with distributed order derivatives with \( \mu \in C^1[0, 1] \). The long- and short-time asymptotic behavior were discussed in detail in [17] by applying an argument similar to the derivation of the Watson lemma. [20] showed the uniqueness results for the IBVPs for the diffusion equation of distributed orders from an appropriate maximum principle. By using the Fourier method of variables separation, [18] constructed an explicit solutions of the distributed-order time-fractional diffusion equations, with Dirichlet boundary conditions. Very recently, [9] proved existence of a weak and regular solution for general uniformly elliptic operator under the assumption that the weight function is only integrable on the interval \([0, 1]\).

Other than the above mentioned aspects for the forward problems where all the coefficients such as \( \mu \) and \( g \) in the mathematical model are known, in most instances the parameters which characterize the diffusion processes cannot be measured directly or easily, for example, as is known, the weight function \( \mu \) in the model (1) should be determined by the inhomogeneity of the media, but it is not clear which physical law can relate the inhomogeneity to \( \mu \), which requires one to use inverse problems to identify these physical quantities from some additional information that can be observed or measured practically. For this, we propose the following inverse problem.

**Problem 1.1.** Let \( x_0 \in \Omega \) be arbitrarily fixed. We want to determine the weight function \( \mu \) in \([0, 1]\) by the overposed data \( u(x_0, t), t \in (0, T) \).

Inverse problems in determining these unknown parameters in the model are not only important by itself, but also significant in its applications. However, the publications on the inverse problems to fractional diffusion equations are rather limited to the best of the authors’ knowledge. As mentioned above, the multi-term time-fractional diffusion equations can be formally obtained by letting \( \mu := \sum_{j=1}^{\ell} q_j \delta(\cdot - \alpha_j) \), where \( \delta \) is the Dirac-delta function. Compared with the case of \( \mu \in C[0, 1] \), for the multi-term counterpart, there exists a large and rapidly growing number of publications related to the inverse problems in determining \( \mu \). We refer to [16] in which the authors pointed out that the one interior point observation is enough in reconstructing the unknown fractional orders, and [14] where the uniqueness for reconstructing the
unknown fractional order and potential was proved with the infinite measurement: Dirichlet-to-
Neumann map. We also refer to [3] for the recovery of fractional order and diffusion coefficient
simultaneously from one endpoint observation. The uniqueness result was proved based on the
eigenfunction expansion of the weak solution to the IBVP and the Gel’fand-Levitan theory. It
reveals that analyticity of the solution to the IBVPs was well performed in determining the
fractional orders in the multi-term case. In the case of \( \mu \in C[0,1] \), we refer to [18] in which
the uniqueness for the inverse problem in determining the weight function \( \mu \) was proved after
establishing the analyticity of the solution. Very recently, in one-dimensional case, [24] studied
an inverse problem similar to that in [18] by using the value of the solution \( u \) in the time interval
\((0, \infty)\).

However, it turns out that the study on this kind of inverse problems of the recovery of the
fractional orders or the continuous counterpart \( \mu \) in the model \( \mathfrak{1} \) is far from satisfactory since
all the publications either assume the homogeneous boundary condition (\([3, 14, 16]\) and \([18]\))
or study this inverse problem by the measurement on \( t \in (0, \infty) \) (\([7]\) and \([24]\)).

In this paper, by establishing a Harnack type inequality and using strong maximum principle
of the elliptic equations, we generalize the result in \([24]\). Before giving the main result for
our inverse problem, we first give a definition which gives the class of weight function under
determination.

**Definition 1.1.** We call a function \( \mu \in C[0,1] \) as a finite oscillation function if for any \( c \in \mathbb{R} \),
the following set \( \{ \alpha ; \mu(\alpha) = c \} \) has at most finite number of isolated point.

We then introduce an admissible set of the weight function \( \mu \):

\[ U := \{ \mu \in C[0,1]; \mu \text{ is a finite oscillatory function.} \}. \]

Now we are ready to state:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^d, d \leq 3 \), be connected, open and bounded, and let \( T > 0 \) be fixed
constant. Let \( g \in C_0^\infty((0,T);H^2(\partial \Omega)) \) ( \( g \) is regarded as a \( C^\infty \) function on \( (0,T) \) with compact
support) such that \( g \geq 0, \notin 0 \) in \( \Sigma \). We further suppose that \( u, \tilde{u} \) are the solutions to the problem
\( \mathfrak{1} \) with respect to \( \mu, \tilde{\mu} \in U \). Then \( \mu = \tilde{\mu} \) in \([0,1]\) provided the overposed data
\( u(x_0, \cdot) = \tilde{u}(x_0, \cdot) \) in \((0,T)\).

The rest of this paper is organized as follows. Section 2 is devoted to the wellposedness
of the IBVP \( \mathfrak{1} \). In Section 3 preparing all necessities about the solution of \( \mathfrak{1} \), say, the
wellposedness and Harnack’s inequality, we finish the proof of Theorem 1.1. Finally, concluding
remarks are given in Section 4.

### 2 Forward problem

As is known, most of the solvability of inverse problems is very dependent on forward
problems no matter whether it is the pure theory or numerical theory of the inverse problems.
In this section, we will consider the wellposedness of the IBVP \( \mathfrak{1} \) which mainly asserts the
continuity of the solutions of \( \mathfrak{1} \) so that enables the measurement of solutions make sense at
one interior point \( \{x_0\} \times (0,T) \).

**Lemma 2.1.** Let \( g \in C_0^\infty((0,T);H^2(\partial \Omega)) \). We assume the weight function \( \mu \in C[0,1] \) is
nonnegative, and not vanish in \([0,1]\). Then the problem \( \mathfrak{1} \) admits a unique solution \( u \in C_0^\infty((0,T);H^3(\Omega)) \), satisfying

\[ \|u\|_{C_m([0,T];H^4(\Omega))} \leq C T \max\{1,T\} \|g\|_{C_{m+3}([0,T];H^2(\partial \Omega))}, \quad m = 0, 1, \ldots. \]

Here the constant \( C > 0 \) only depends on \( m, \mu, d, \Omega \).
Before giving the proof of the above lemma, we first introduce the eigensystem \( \{ \lambda_n, \varphi_n \} \) of the operator \(-\Delta\), that is, \( \{ \varphi_n \}_{n=1}^{\infty} \) satisfy
\[
-\Delta \varphi_n = \lambda_n \varphi_n, \quad \varphi_n \in H^1_0(\Omega) \cap H^2(\Omega).
\]
For short we denote \( w(s) := \int_0^1 \mu(\alpha)s^{-1}d\alpha \) and we define an operator \( I^{(\mu)} \) as
\[
I^{(\mu)} \phi(t) := \int_0^t \kappa(t-\tau)\phi(\tau)d\tau, \quad \text{where} \quad \kappa(t) := \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{sw(s)}e^{st}ds.
\]
We now turn to considering the following IBVP
\[
\begin{aligned}
\left\{
\begin{array}{ll}
\mathbb{D}_t^{(\mu)} v - \Delta v = F & \text{in } Q, \\
v|_{t=0} = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \Sigma,
\end{array}
\right.
\end{aligned}
\tag{2}
\]
where \( F \in C^0([0,T); H^2(\Omega)) \). With reference to Corollary 3.1 in [24] and the above notations, the solution \( v \) to (2) can be represented in the form
\[
v(x,t) = \sum_{n=1}^{\infty} \int_0^t (\partial_t F(\cdot, \tau), \varphi_n) I^{(\mu)}v_n(t-\tau)\varphi_n(x), \quad (x,t) \in Q,
\]
where \((\cdot, \cdot)\) denotes the inner product in \( L^2(\Omega) \), and \( v_n(t) \) is the unique solution of the distributed ordinary differential equation
\[
\begin{aligned}
\left\{
\begin{array}{ll}
\mathbb{D}_t^{(\mu)} v_n(t) = -\lambda_n v_n(t), & t > 0, \\
v_n(0) = 1.
\end{array}
\right.
\end{aligned}
\]
Armed with the above argument, in what following, we will give the formal representation of the solution to the IBVP (2) and give the proof of the conclusions stated in Lemma 2.1. For this, we introduce the operator \( \Lambda : L^2(\partial\Omega) \to H^2(\Omega) \) by
\[
\Lambda \phi(x) := -\sum_{n=1}^{\infty} \frac{1}{\lambda_n} (\phi, \partial_{\nu} \varphi_n)_{L^2(\partial\Omega)} \varphi_n(x), \quad \phi \in L^2(\partial\Omega).
\]
Here we set \( \partial_{\nu}u := \sum_{i=1}^d \nu_i \partial_i u \) where \((\nu_1, \cdots, \nu_d)\) denotes the unit outwards normal vector to the boundary \( \partial\Omega \). Meanwhile, it is not very difficult to see that \( \Lambda g \) solves the following boundary value problem for the elliptic equation
\[
\begin{aligned}
\left\{
\begin{array}{ll}
-\Delta(\Lambda g) = 0 & \text{in } \Omega, \\
\Lambda g = g & \text{on } \partial\Omega.
\end{array}
\right.
\end{aligned}
\tag{4}
\]
in view of Lemma 2.1 in [6], or one can prove directly by integration by parts.

As a byproduct of the regularity estimate for above boundary value problem (4) (see, e.g., [19]) and the assumption \( g \in C^\infty((0,T); H^2(\partial\Omega)) \), we find that \( \Lambda g \in C^\infty((0,T); H^4(\Omega)) \) and there exists a positive constant \( C \) which is independent of \( t, T \) and \( g \) such that the following estimate
\[
\|\partial_t^i \Lambda g(t)\|_{H^4(\Omega)} \leq C \|\partial_t^i g(t)\|_{H^2(\partial\Omega)}, \quad t \in (0,T), \ i = 0, 1, \cdots
\]
holds true.

Now letting \( w(x,t) := u(x,t) - (\Lambda g)(x,t) \), we see that \( w \) reads
\[
\begin{aligned}
\begin{cases}
\mathbb{D}_t^{(\mu)} w - \Delta w = -\mathbb{D}_t^{(\mu)} (\Lambda g) & \text{in } Q, \\
w|_{t=0} = u|_{t=0} - \Lambda g|_{t=0} = 0 & \text{on } \Omega, \\
w = u - \Lambda g = 0 & \text{on } \Sigma.
\end{cases}
\end{aligned}
\]
Consequently, with reference to (3), we obtain
\[ w(x, t) = -\sum_{n=1}^{\infty} \int_0^t (\partial_t D_t^{(\mu)}(\cdot, \tau) \varphi_n) v_n(t - \tau) d\tau \varphi_n, \quad t \in (0, T), \]
and hence
\[ u(x, t) = (\Lambda g)(x, t) - \sum_{n=1}^{\infty} \int_0^t (\partial_t D_t^{(\mu)}(\cdot, \tau) \varphi_n) v_n(t - \tau) d\tau \varphi_n, \quad t \in (0, T). \] (6)

We are now ready to give the proof of Lemma 2.1

**Proof of Lemma 2.1** We shall treat each of each term on the right-hand side of (6) separately. Firstly, by arguing as in the derivation of Corollary 3.1 in [24], we find that \( w \in C_0^\infty((0, \infty); H^4(\Omega)) \) and the following inequality
\[ \| w \|_{C^m([0,T];H^4(\Omega))} \leq C \sum_{i=1}^{m+2} \| \partial_t^i D_t^{(\mu)} \Lambda g \|_{L^2([0,T];H^4(\Omega))}, \quad m = 0, 1, \ldots \]
is valid, which gives an evaluation for \( u \):
\[ \| u \|_{C^m([0,T];H^4(\Omega))} \leq C \sum_{i=0}^m \| \partial_t^i \Lambda g \|_{L^2([0,T];H^4(\Omega))} + C \sum_{i=1}^{m+2} \| \partial_t^i D_t^{(\mu)} \Lambda g \|_{L^2([0,T];H^4(\Omega))} \]
\[ \leq C \sum_{i=0}^m \| \partial_t^i g \|_{L^2([0,T];H^4(\partial\Omega))} + C \sum_{i=1}^{m+2} \| \partial_t^i D_t^{(\mu)} \Lambda g \|_{L^2([0,T];H^4(\Omega))} \]
upon applying the above estimate for \( \Lambda g \).

In light of the above inequalities, it is enough to evaluate \( \| \partial_t^i D_t^{(\mu)} \Lambda g \|_{L^2([0,T];H^4(\Omega))} \) for \( i = 1, 2, \ldots, m + 2 \). We start with \( \partial_t D_t^{(\mu)} \Lambda g \). For this, as a preamble, from the definition of the distributed order time fractional derivative \( D_t^{(\mu)} \), it follows that
\[ \partial_t D_t^{(\mu)} \Lambda g = \int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)} \left[ \partial_t \int_0^t \frac{\partial_t \Lambda g(\cdot, \tau)}{(t-\tau)^\alpha} d\tau \right] d\alpha = \int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)} \left[ \int_0^t \frac{\partial_t^2 \Lambda g(\cdot, \tau)}{(t-\tau)^\alpha} d\tau \right] d\alpha, \]
where in the last equality we used the assumption that \( g \in C_0^\infty((0, T); H^2(\partial\Omega)) \), and hence
\[ \| \partial_t D_t^{(\mu)} \Lambda g \|_{H^4(\Omega)} \leq \| \mu \|_{C[0,1]} \| \partial_t^2 \Lambda g \|_{C([0,T];H^4(\Omega))} \int_0^1 \frac{1}{\Gamma(1-\alpha)} \left[ \int_0^t \tau^{-\alpha} d\tau \right] d\alpha \]
\[ \leq \| \mu \|_{C[0,1]} \| \partial_t^2 g \|_{C([0,T];H^4(\partial\Omega))} \int_0^1 \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} d\alpha, \quad t \in (0, T). \]

Here in the last equality, we used the estimate for the Gamma function \( \Gamma(1 + \beta) = \beta \Gamma(\beta), \beta > 0 \). Now from the continuity of the Gamma function on the interval \([1, 2]\), we finally derive that
\[ \| \partial_t D_t^{(\mu)} \Lambda g \|_{H^4(\Omega)} \leq \| 1/\Gamma \|_{L^\infty([1,2])} \| \mu \|_{C[0,1]} \| g \|_{C^2([0,T];H^4(\partial\Omega))} \max\{T, 1\}, \quad t \in (0, T). \]
Similarly, for \( k = 2, \ldots, m + 2 \), we have
\[ \| \partial_t^k D_t^{(\mu)} \Lambda g \|_{H^4(\Omega)} \leq \| 1/\Gamma \|_{L^\infty([1,2])} \| \mu \|_{C[0,1]} \| g \|_{C^{k+1}([0,T];H^4(\partial\Omega))} \max\{T, 1\}, \quad t \in (0, T). \]

Collecting all the above estimates leads to
\[ \| u \|_{C^m([0,T];H^4(\Omega))} \leq CT \max\{1, T\} \| g \|_{C^{m+\chi([0,T];H^4(\partial\Omega))}}, \]
where the constant \( C \) is independent of \( g \) and \( T \) but may depend on \( m, \mu, d, \Omega \).
3 Proof of Theorem 1.1

In this section, we will give the proof for Theorems 1.1. To this end, we first fix some general settings and notations. We introduce the Riemann-Liouville fractional integral operator $J^\alpha$:

$$J^\alpha \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(r)}{(t-r)^{1-\alpha}} dr, \quad t > 0,$$

where $\alpha > 0$. We see that $J^\alpha$ admits the semigroup property

$$J^\alpha J^\beta = J^{\alpha+\beta}, \quad \alpha > 0, \beta > 0,$$

(see, e.g., [23]). Furthermore, the following useful lemma holds:

**Lemma 3.1** (Harnack’s inequality). Assume $u \geq 0$ is a $C^2$ solution of

$$-\Delta u + \lambda u = 0 \text{ in } \Omega,$$

where $\lambda > 0$ is a constant. Consider a connected domain $U \subset \subset \Omega$. Then

$$\sup_U u \leq e^{C(1+\lambda)} \inf_U u \quad (7)$$

for some positive constant $C$ that depends on $d, U, \Omega$.

The proof is followed from the classical idea in deriving the Harnack inequality for parabolic equation from [3].

**Proof.** Without loss of generality, we assume $u > 0$ (else consider $u + \varepsilon$, and send $\varepsilon \to 0$). Moreover, we should point out here that if we let $v = \log u$,

then the Harnack inequality follows once we show that $\|\nabla v\|_{L^\infty(U)} \leq C\lambda$. Indeed, in this case for $x_1, x_2 \in U$ we have

$$\frac{u(x_2)}{u(x_1)} = e^{v(x_2) - v(x_1)} \leq e^{\|x_1 - x_2\|\|\nabla v\|_{L^\infty(U)}}$$

and so (7) holds true from boundedness of the domain $U$.

We conclude from the equation

$$-\Delta u + \lambda u = 0$$

that

$$|\nabla v|^2 = -\Delta v + \lambda =: \zeta. \quad (8)$$

Now we point out that a sufficient condition for (7) to hold is that $\|\zeta\|_{L^\infty(U)} \leq C(\lambda + 1)$. We shall prove the latter by looking at the elliptic equation obeyed by $\zeta$. To get an elliptic equation for $\zeta$, by direct calculation, we find

$$\partial_k \zeta = 2 \sum_{i=1}^d (\partial_i v) \partial_{ki} v,$$

$$\partial_{kk} \zeta = 2 \sum_{i=1}^d (\partial_i v) \partial_{kki} v + 2 \sum_{i=1}^d (\partial_{ki} v)^2.$$
Therefore,
\[-\Delta \zeta = -2 \sum_{i=1}^{d} \partial_i v \left( \sum_{k=1}^{d} \partial_{k,i} v \right) - 2 \sum_{i,k=1}^{d} \partial_{k,i} v^2 = -2 \sum_{i=1}^{d} (\partial_i v)(\partial_i (\Delta v)) - 2 \sum_{i,k=1}^{d} \partial_{k,i} v^2. \tag{9}\]

On the other hand, from (8), we have
\[\partial_k \zeta = -\partial_k (\Delta v). \tag{10}\]

Combining (9) and (10) yields
\[-\Delta \zeta + \sum_{i=1}^{d} b_i \partial_i \zeta = -2 \sum_{i,k=1}^{d} \partial_{k,i} v^2 = -2|\nabla^2 v|^2, \tag{11}\]

where we have denoted
\[b_i := -2\partial_i v. \]

Let \(\chi\) be a smooth cutoff function adapted to \((U, \Omega)\), say, \(\chi \in C_0^\infty(\Omega)\) such that \(0 \leq \chi \leq 1\) and \(\chi(x) = 1\) if \(x \in U\), and define
\[z = \chi^4 \zeta. \]

The function \(z\) is continuous (recall \(u\) is \(C^2\)) and has compact support in \(U\), so it attains its maximum at some point \(x_0 \in \Omega\). At this point we have \(\nabla z(x_0) = 0\), and therefore
\[\chi(x_0) \partial_k \chi(x_0) = -4 \partial_k \chi(x_0) \chi(x_0). \tag{12}\]

Moreover, since \(x_0\) is the maximum point \(x_0 \in \Omega\) of \(z\), we also have at \(x_0\) that
\[0 \leq -\Delta z + \sum_{i=1}^{d} b_i \partial_i z = \chi^4 \left( -\Delta \zeta + \sum_{i=1}^{d} b_i \partial_i \zeta \right) + R, \tag{13}\]

where
\[|R| \leq C(\chi^2 |\zeta| + \chi^3 |\nabla \zeta|) + \chi^3 |b||\zeta| \leq C\chi^2 |\zeta| + \chi^3 |\nabla v||\zeta|. \]

In the bound for \(R\) we have used (12) and the definition of \(b_i\).

Now combining (9), (11) and (13) with the fact \(|\Delta v| \leq |\nabla^2 v|\) and obtain
\[\chi^4 \zeta^2 \leq 2\chi^4 |\Delta v|^2 + 2\lambda^2 \leq 2\chi^4 |\nabla^2 v|^2 + 2\lambda^2 \]
\[= -\chi^4 \left( -\Delta z + \sum_{i=1}^{d} b_i \partial_i \zeta \right) + 2\lambda^2 \leq C\chi^2 |\zeta| + \chi^3 |\nabla v||\zeta| + 2\lambda^2, \]

and noting that \(\zeta := |\nabla v|^2\), we further see that
\[\chi^4 \zeta^2 \leq C\chi^2 |\zeta| + \chi^3 |\zeta|^3 + C\lambda^2 \tag{14}\]

for some constant \(C\) that depends on \(U, \Omega\). But (14) shows that \(\chi^2 \zeta\) is bounded by \(C(\lambda + 1)\) on \(\Omega\), and since \(\chi \equiv 1\) on \(U\), it also gives a bound on \(\|\zeta\|_{L^\infty(U)} \leq C(\lambda + 1)\), where \(C\) only depends on \(U, \Omega\), thereby concluding the proof.

On the basis of the Harnack inequality, we can get the following corollary.

**Corollary 3.1.** Let the weight function \(\mu \in C[0,1]\) be nonnegative, and not vanish in \([0,1]\). Suppose the non-negative function \(u \in C_0^\infty((0, T_0); H^4(\Omega))\) solves the following problem
\[D_t^{(\mu)} u - \Delta u \leq 0 \text{ in } \Omega \times (0, T),\]

where \(0 < T < T_0\). Suppose also \(\Omega\) is connected, open and bounded. Then for any \(x \in \Omega\), there exists \(t_x \in (0, T)\) such that \(u(x, t_x) > 0\).
Proof. We prove this corollary by contradiction argument. For this, we assume there exists \( x_1 \in \Omega \) such that \( v(x_1, \cdot) = 0 \) in \((0, T)\). Since \( u \neq 0 \) in \( \Omega \times (0, T) \), we can choose a connected domain \( U \subset \Omega \) contains \( x_1 \) and such that \( u \neq 0 \) on \( \partial U \times (0, T) \). Moreover, from our assumption, it follows that \( u|_{\partial \Omega} \in C^\infty_C((0, \infty); H^4(\partial U)) \). Keeping this in mind, we introduce an auxiliary function \( v \) satisfies the following equation

\[
\begin{cases}
D_t^{(\mu)} v - \Delta v = 0 & \text{in } U \times (0, \infty), \\
v|_{t=0} = u|_{t=0} = 0 & \text{in } U, \\
v = u & \text{on } \partial U \times (0, \infty).
\end{cases}
\]  

(15)

By Lemma 2.4, we see that \((15)\) admits a unique solution \( v \in C^\infty_C((0, \infty); H^4(U)) \) which does not vanish in \( U \times (0, T) \). Moreover, we have

\[
\|v\|_{C([0, \infty); H^4(U))} \leq C\|u\|_{C^2([0, T]; H^2(\partial U))} < \infty,
\]

and hence the Sobolev embedding theorem implies that \( v \in C^\infty_C((0, \infty); C^2(U)) \), for \( d \leq 3 \). Moreover, from the Maximum principle (see, e.g., [20]), it follows that

\[
0 \leq v \leq u \text{ in } U \times (0, T).
\]

Again Lemma 2.4 entails that the Laplace transform of the solution \( v \) exists. Then taking the Laplace transforms on both sides of \((15)\) derives

\[
\begin{cases}
- \Delta \hat{v} + sw(s)v = 0 & \text{in } U, \\
\hat{v} = \hat{u} & \text{in } \partial U.
\end{cases}
\]

Since the function \( v \) does not vanish in \( U \times (0, T) \), we choose \((x_2, t_2) \in U \times (0, T)\) such that \( v(x_2, t_2) > 0 \). From the Harnack inequality proved in Lemma 3.4, for the connected domain \( V \) such that \( V \subset U \) and \( x_1, x_2 \in V \), we have

\[
\sup_V \hat{v}(x; s) \leq e^{C(1+|sw(s)|)} \inf_V \hat{v}(x; s), \quad s > 0.
\]

From the choice of \( x_1, x_2 \) and \( V \), we see that

\[
\hat{v}(x_2; s) \leq \sup_V \hat{v}(x; s) \leq e^{C(1+|sw(s)|)} \inf_V \hat{v}(x; s) \leq e^{C(1+|sw(s)|)} \hat{v}(x_1; s), \quad s > 0.
\]

Now since \( u(x_1, t) = 0 \) for \( t \in (0, T) \), and \( v \leq u \), we note that \( v(x_1, t) = 0 \) if \( t \in (0, T) \), which implies that

\[
\hat{v}(x_1; s) = \int_T^\infty v(x_1, t)e^{-st}dt \leq Cs^{-1}e^{-Ts}, \quad s > 0.
\]

On the other hand, we have

\[
\hat{v}(x_2; s) \geq \int_{t_2-\delta}^{t_2+\delta} v(x_2, t)e^{-st}dt \geq c_1 s^{-1}e^{-(t_2-\delta)s}, \quad s > 0.
\]

Here \( c_1 := \inf_{t \in (t_2-\delta, t_2+\delta)} v(x_2, t) > 0 \). Combining all the estimates, we find

\[
e^{-(t_2-\delta)s} \leq e^{C(1+|sw(s)|)}Ce^{-Ts}, \quad s > 0.
\]

Here \( t_2 \in (0, T) \). Moreover, from the notation of \( sw(s) := \int_0^1 s^\alpha \mu(\alpha) d\alpha \), we have

\[
|sw(s)| \leq \|\mu\|_{C[0, 1]} \int_0^1 e^{a \log s} d\alpha \leq \|\mu\|_{C[0, 1]} s - \frac{1}{\log s}, \quad s > 0.
\]
which implies
\[ e^{-(t_2-\delta)s} \leq e^{C(1+\frac{1}{\log s})s}, \quad s > 1. \]
Letting \( s \to \infty \), we get a contradiction in view of \( t_2-\delta < T \) and \( 1/\log s << T \) if \( s \) is sufficiently large. We must have that for any \( x \in \Omega \) there exists \( t_x \in (0, T) \) such that \( u(x, t_x) > 0 \). This completes the proof of the corollary.

**Proof of Theorem 1.1.** We first extend the function \( g \) to the interval \([0, \infty)\) by letting \( g = 0 \) outside of \((0, T)\) and we denote the extension by \( G \). We notice that \( g \in C^\infty_0((0, T); H^1(\partial\Omega)) \) and hence \( G \in C^\infty_0((0, \infty); H^1(\partial\Omega)) \), and when no confusion can arise, we still denote \( u \) as the solution to the following IBVP

\[
\begin{aligned}
\mathcal{D}_t^\alpha u &= \Delta u \quad \text{in } \Omega \times (0, \infty), \\
|u|_{t=0} &= 0 \quad \text{in } \Omega, \\
|u| &= G \quad \text{on } \partial\Omega \times (0, \infty).
\end{aligned}
\]  

(16)

In view of the fact that \( d \leq 3 \), we conclude from Lemma 2.1 that \( u \in C^\infty_0((0, \infty); C^2(\Omega)) \). Moreover, the non-negativity of the function \( G \) combined with the maximum principle (see, e.g., [20]) yields that \( u(t) \geq 0 \) in \( \Omega \) for any \( t > 0 \). The same property holds true for the solution \( \tilde{u} \) to the IBVP (10) with weight function \( \mu \).

Taking the operator \( J^2 \) on both sides of the equation (16), and noting that \( u(0) = 0 \) implies \( J^2\partial_t^\alpha u = \partial_t^\alpha J^2u = J^{2-\alpha}u \), we find

\[
\begin{aligned}
\mathcal{D}_t^\alpha (J^2u) &= \Delta (J^2u) \quad \text{in } \Omega \times (0, \infty), \\
J^2u|_{t=0} &= 0 \quad \text{in } \Omega, \\
J^2u &= J^2G \quad \text{on } \partial\Omega \times (0, \infty).
\end{aligned}
\]

Now by taking the difference of the above systems of \( J^2u \) and \( J^2\tilde{u} \), it turns out that the system for \( v := J^2u - J^2\tilde{u} \) reads

\[
\begin{aligned}
\mathcal{D}_t^\alpha v - \Delta v &= \int_0^1 (\bar{\mu}(\alpha) - \mu(\alpha)) J^{2-\alpha} \tilde{u} \, d\alpha \quad \text{in } \Omega \times (0, \infty), \\
v|_{t=0} &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial\Omega \times (0, \infty).
\end{aligned}
\]

Next, we will use a contradiction argument to finish the proof. For this, we assume that \( \mu \neq \bar{\mu} \). More precisely, since \( \mu \) is a finite oscillatory function, without loss of generality, we can assume that there exists \( \alpha_0 \in (0, 1) \) such that \( \bar{\mu}(\alpha) < \mu(\alpha) \) for \( \alpha \in [\alpha_0 - 4\varepsilon, \alpha_0] \) and \( \bar{\mu}(\alpha) = \mu(\alpha) \) if \( \alpha \in (\alpha_0, 1] \). Then we can assert that the following inequality

\[ RHS := \int_0^1 (\bar{\mu}(\alpha) - \mu(\alpha)) J^{2-\alpha} \tilde{u} \, d\alpha \leq 0 \text{ in } \Omega \]

is valid for any \( 0 < t < \delta \) with a sufficiently small constant \( \delta > 0 \). Indeed, notice that \( \mu, \bar{\mu} \) is continuous on \([0, 1]\), since \( \bar{\mu}(\alpha) < \mu(\alpha) \) for \( \alpha \in [\alpha_0 - 4\varepsilon, \alpha_0] \), one can choose a constant \( c_0 \) and sufficiently small constant \( \varepsilon > 0 \) such that

\[ \mu(\alpha) - \bar{\mu}(\alpha) > c_0 > 0, \quad \alpha \in (\alpha_0 - 3\varepsilon, \alpha_0 - \varepsilon), \]

which combined with the fact \( \tilde{u} \geq 0 \) implies that

\[
RHS \leq c_1 \int_0^{\alpha_0 - 3\varepsilon} J^{2-\alpha} \tilde{u} \, d\alpha - c_0 \int_{\alpha_0 - 2\varepsilon}^{\alpha_0 - \varepsilon} J^{2-\alpha} \tilde{u} \, d\alpha,
\]

where \( c_1 \) is a constant.
where \( c_1 := \|\mu\|_{C[0,1]} + \|\overline{\mu}\|_{C[0,1]} \). We change the variable \( \alpha \rightarrow (\alpha_0 - 2\varepsilon)(1 - \frac{\alpha}{\alpha_0 - 3\varepsilon}) + (\alpha_0 - \varepsilon) \), then the above formula can be rephrased as follows

\[
RHS \leq \int_{\alpha_0 - 2\varepsilon}^{\alpha_0 - \varepsilon} \left( \frac{c_1(\alpha_0 - 3\varepsilon)}{\varepsilon} J^{2-\beta} \overline{u} - c_0 J^{2-\alpha} \overline{u} \right) d\alpha.
\]

where \( \beta := \frac{\alpha(\alpha - 3\varepsilon)(\alpha - 3\varepsilon + 2\varepsilon)}{\varepsilon} \). Moreover, again from the semigroup property of the Riemann-Liouville fractional operator \( J^\alpha \), it follows that

\[
RHS \leq \int_{\alpha_0 - 2\varepsilon}^{\alpha_0 - \varepsilon} \left( \frac{c_1(\alpha_0 - 3\varepsilon)}{\varepsilon} J^1 J^{\alpha - \beta} J^{1-\alpha} \overline{u} - c_0 J^{2-\alpha} \overline{u} \right) d\alpha.
\]

Again by the non-negativity of \( \overline{u} \) and noticing the definition of \( J^\alpha \), \( \alpha > 0 \), we have

\[
J^1 J^{\alpha - \beta} J^{1-\alpha} \overline{u}(t) = \|J^{\alpha - \beta} J^{1-\alpha} \overline{u}\|_{L^1(0,t)} = \frac{1}{\Gamma(\alpha - \beta)} \left\| \int_0^t (t - \tau)^{\alpha - \beta - 1} J^{1-\alpha} \overline{u}(\tau) d\tau \right\|_{L^1(0,t)}
\]

which entails

\[
J^1 J^{\alpha - \beta} J^{1-\alpha} \overline{u}(t) \leq \frac{t^{\alpha - \beta}}{\Gamma(1 + \alpha - \beta)} \int_0^t J^{1-\alpha} \overline{u}(\tau) d\tau = \frac{t^{\alpha - \beta}}{\Gamma(1 + \alpha - \beta)} J^{2-\alpha} \overline{u}(t)
\]

upon applying the Young inequality and \( \Gamma(1 + \gamma) = \gamma \Gamma(\gamma), \gamma > 0 \). Here in the last equality we again used the definition of the Riemann-Liouville fractional integral. Moreover, since \( \alpha \in [\alpha_0 - 2\varepsilon, \alpha_0 - \varepsilon] \), we further see that

\[
0 < 2\varepsilon \leq \alpha - \beta \leq \alpha_0 - 2\varepsilon < 1,
\]

which implies that \( t^{\alpha - \beta} \leq t^{2\varepsilon} \) if \( t \in (0,1) \), and from the continuity of the Gamma function on the interval \([2\varepsilon,1]\), it then follows that

\[
J^1 J^{\alpha - \beta} J^{1-\alpha} \overline{u}(t) \leq c_2 t^{2\varepsilon} J^{2-\alpha} \overline{u}(t), \quad 0 < t < 1,
\]

where \( c_2 := \|1/\Gamma(\cdot)\|_{C[2\varepsilon,1]} \), and combining all the above estimates, we find

\[
RHS \leq \int_{\alpha_0 - 2\varepsilon}^{\alpha_0 - \varepsilon} \left( \frac{c_1 c_2 (\alpha_0 - 3\varepsilon)}{\varepsilon} t^{2\varepsilon} - c_0 \right) J^{2-\alpha} \overline{u} d\alpha.
\]

Therefore, choosing \( \delta > 0 \) sufficiently small such that \( \frac{c_1 c_2 (\alpha_0 - 3\varepsilon)}{\varepsilon} t^{2\varepsilon} - c_0 \), and then for \( 0 < t < \delta \), we can assert that

\[
RHS \leq, \neq 0 \text{ in } \Omega.
\]

We finally obtain

\[
\begin{cases}
\mathbb{D}_t^\mu v - \Delta v \leq, \neq 0 & \text{in } \Omega \times (0,\delta), \\
v|_{t=0} = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega \times (0,\delta).
\end{cases}
\]

From Corollary 3.1 we assert that for any \( x \in \Omega \), there exists \( t_x \in (0,\delta) \) such that \( v(x, t_x) > 0 \), that is

\[
J^2 u(x, t_x) - J^2 \overline{u}(x, t_x) > 0, \quad x \in \Omega
\]

in view of the notation of \( v \). This is a contradiction since the overposed data \( u(x_0, \cdot) = \overline{u}(x_0, \cdot) \) in \((0,T)\) implies that \( J^2 u(x_0, t) = J^2 \overline{u}(x_0, t) \) for any \( 0 < t < T \). By contradiction, we must have

\[
\mu(\alpha) = \overline{\mu}(\alpha), \quad \alpha \in [0,1].
\]

This completes the proof of the theorem. \( \square \)
Remark 3.1. Our method cannot work in the case when \( \mu \) is not a finite oscillatory function. However, if we use the measurement data similar to the one in [24], say, \( u(x, \cdot) \) in \((0, \infty)\), it is expected to obtain the uniqueness of the inverse problem in determining the weight function.

Indeed, since we now consider all the problem in the infinite time interval \((0, \infty)\), we can employ the Laplace transform \( \hat{\cdot} \) on both sides of the equation (1) with respect to \( \mu, \tilde{\mu} \in C[0, 1] \), we find

\[
\begin{aligned}
sw(s)\hat{u}(s) - \Delta \hat{u}(s) &= 0 \quad \text{in } \Omega, \\
\hat{u}(s) &= \hat{g}(s), \quad \text{on } \partial\Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
s\hat{w}(s)\hat{u}(s) - \Delta \hat{u}(s) &= 0 \quad \text{in } \Omega, \\
\hat{u}(s) &= \hat{g}(s) \quad \text{on } \partial\Omega,
\end{aligned}
\]

for any \( s > 0 \), where

\[
w(s) := \int_0^1 s^\alpha \mu(\alpha) d\alpha, \quad \hat{w}(s) := \int_0^1 s^\alpha \tilde{\mu}(\alpha) d\alpha.
\]

From the strong maximum principle for the elliptic equations, we see that \( \hat{u}(s) \) and \( \hat{\tilde{u}}(s) \) are strictly positive in the domain \( \Omega \) for any \( s > 0 \).

Now by taking the difference of the above systems, it turns out that the system for \( v := \hat{u} - \hat{\tilde{u}} \) reads

\[
\begin{aligned}
sw(s)v(s) - \Delta v(s) &= -s\hat{\tilde{u}}(s)(w(s) - \hat{w}(s)) \quad \text{in } \Omega, \\
\hat{v}(s) &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]

We prove by contradiction. Let us assume that \( \mu \neq \tilde{\mu} \) in \([0, 1]\) and we claim that there exists \( s_0 > 0 \) such that

\[
\int_0^1 s_0^\alpha (\mu(\alpha) - \tilde{\mu}(\alpha)) d\alpha \neq 0.
\]

This can be done by an argument similar to the proof of Theorem 2.2 in [13]. Without loss of generality, we assume that

\[
\int_0^1 s_0^\alpha (\mu(\alpha) - \tilde{\mu}(\alpha)) d\alpha > 0.
\]

Then we conclude from the strong maximum principle for the elliptic equations that \( v(x; s_0) > 0 \) in \( \Omega \) because of \( \hat{\tilde{u}}(x; s_0) \geq 0 \), hence that

\[
\hat{u}(x; s_0) > \hat{\tilde{u}}(x; s_0), \quad x \in \Omega.
\]

This is a contradiction since \( u(x_0, \cdot) = \tilde{u}(x_0, \cdot) \) in \((0, \infty)\) implies that \( \hat{u}(x_0; s) = \hat{\tilde{u}}(x_0; s) \) for any \( s > 0 \). By contradiction, we must have \( \mu = \tilde{\mu} \) in \([0, 1]\).

4 Concluding remarks

In this paper, the initial-boundary value problem for the diffusion equation with distributed order derivatives was investigated. On the basis of eigenfunction expansion, we first gave a representation formula of the solution via Fourier series and showed the convergence as well as several estimates for the solution. In Theorem 2.1, we can relax the regularity of \( g \) via the argument used in [17] but we do not discuss here.

For the inverse problem, on the basis of Laplace transform, we first transferred the time-fractional diffusion equation to the corresponding elliptic equation with the Laplacian parameter and established a Harnack type inequality for this elliptic equation, which were further used...
to imply the uniqueness of the inverse problem in determining the weight function $\mu$ from one interior point observation provided the unknown weight function $\mu$ lies in the admissible set $U$. The inverse problem in determining the weight function $\mu$ in general case, say, $\mu \notin U$, by the overposed data $u(x_0, \cdot)$ in $(0,T)$ remains open. It should be mentioned here that the proof of the above uniqueness result heavily relies on the setting of Dirichlet boundary condition. It would be interesting to investigate what happens about this inverse problem in the framework of Neumann boundary condition.

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