A NEW FORMULA FOR THE $L^p$ NORM

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In memory of Ka-Sing Lau

Abstract. Recently, Brezis, Van Schaftingen and the second author \cite{BBM} established a new formula for the $\dot{W}^{1,p}$ norm of a function in $C_c^\infty(\mathbb{R}^N)$. The formula was obtained by replacing the $L^p(\mathbb{R}^2N)$ norm in the Gagliardo semi-norm for $W^{s,p}(\mathbb{R}^N)$ with a weak-$L^p(\mathbb{R}^2N)$ quasi-norm and setting $s = 1$. This provides a characterization of such $\dot{W}^{1,p}$ norms, which complements the celebrated Bourgain-Brezis-Mironescu (BBM) formula \cite{BBM}. In this paper, we obtain an analog for the case $s = 0$. In particular, we present a new formula for the $L^p$ norm of any function in $L^p(\mathbb{R}^N)$, which involves only the measures of suitable level sets, but no integration. This provides a characterization of the norm on $L^p(\mathbb{R}^N)$, which complements a formula by Maz’ya and Shaposhnikova \cite{MS}. As a result, by interpolation, we obtain a new embedding of the Triebel-Lizorkin space $F^s_{p,2}(\mathbb{R}^N)$ (i.e. the Bessel potential space $(I - \Delta)^{-s/2}L^p(\mathbb{R}^N)$), as well as its homogeneous counterpart $\dot{F}^s_{p,2}(\mathbb{R}^N)$, for $s \in (0, 1)$, $p \in (1, \infty)$.

1. Introduction

The purpose of this paper is to prove a new characterization of the $L^p$ norm on $\mathbb{R}^N$, by lifting to the product space $\mathbb{R}^N \times \mathbb{R}^N$ and considering a weak-$L^p$ quasi-norm over there instead. Indeed, for a measurable function $F(x, y)$ on $\mathbb{R}^N \times \mathbb{R}^N$ and $1 \leq p < \infty$, we denote the weak-$L^p$ quasi-norm of $F$ by $[F]_{L^p,\infty(\mathbb{R}^N \times \mathbb{R}^N)}$, where

$$[F]_{L^p,\infty(\mathbb{R}^N \times \mathbb{R}^N)} := \sup_{\lambda > 0} \left( \lambda^p \mathcal{L}^{2N} \{(x, y) \in \mathbb{R}^{2N} : |F(x, y)| \geq \lambda \} \right)^{1/p} \quad (1.1)$$

and $\mathcal{L}^{2N}$ denotes the Lebesgue measure on $\mathbb{R}^{2N}$ (see e.g., \cite{GP,MS}). Then our first result reads:

**Theorem 1.1.** For every $N \in \mathbb{N}$, there exist constants $c_1 = c_1(N) > 0$ and $c_2 = c_2(N) > 0$, such that for all $1 \leq p < \infty$ and all $u \in L^p(\mathbb{R}^N)$,

$$c_1^{1/p} \|u\|_{L^p(\mathbb{R}^N)} \leq \left[ \frac{u(x) - u(y)}{|x - y|^{N/p}} \right]_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)} \leq 2c_2^{1/p} \|u\|_{L^p(\mathbb{R}^N)} \quad (1.2)$$

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Moreover, for \(1 \leq p < \infty\) and \(u \in L^p(\mathbb{R}^N)\), if we write

\[
E_\lambda := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{2}} \geq \lambda \right\},
\]

then

\[
\lim_{\lambda \to 0^+} \lambda^p L^2N (E_\lambda) = 2\kappa_N \|u\|_{L^p(\mathbb{R}^N)}^p,
\]

where \(\kappa_N := \frac{\pi^{N/2}}{\Gamma(\frac{N}{2} + 1)}\) is the volume of the unit ball in \(\mathbb{R}^N\).

We remark that the power of \(|x - y|\) in the denominator of the quantity in the middle of (1.2) is the natural one dictated by dilation invariance. Furthermore, the main thrust of (1.2) is in the first inequality. In fact, the second inequality has already been observed by e.g. Dominguez and Milman in [8]. On the other hand, the first inequality of (1.2) is an easy consequence of (1.4), with \(c_1(N) := 2\kappa_N\) (because the supremum over \(\lambda > 0\) always dominates the limit as \(\lambda \to 0^+\)). In addition, we emphasize that (1.4) is not true unless we assume \(u \in L^p(\mathbb{R}^N)\) to begin with; indeed, if \(1 \leq p < \infty\) and \(u\) is identically 1, then \(L^2N (E_\lambda) = 0\) for every \(\lambda > 0\), while \(\|u\|_{L^p(\mathbb{R}^N)} = +\infty\). So the proof of (1.4) is a little delicate, which we give in detail in Section 2.

Our point of view of lifting to the product space \(\mathbb{R}^N \times \mathbb{R}^N\) and using the weak-\(L^p\) quasi-norm there is motivated by recent work of the second author with Haïm Brezis and Jean Van Schaftingen [4], which established an analog of the above theorem for the Sobolev semi-norm \(\|\nabla u\|_{L^p(\mathbb{R}^N)}\). The article [4] in turn drew important inspiration from the BBM formula for the Sobolev space \(W^{1,p}\), which first appeared in a celebrated paper [1] of Bourgain, Brezis and Mironescu. An analogue of the BBM formula for \(L^p\) in place of \(W^{1,p}\) was first obtained by Maz’ya and Shaposhnikova [12]. Our Theorem 1.1 can be thought of as a counterpart of the Maz’ya-Shaposhnikova formula for the \(L^p\) norm, in the same way that the main result in [4] relates to the BBM formula for \(W^{1,p}\).

To describe all these developments in more detail, let’s introduce some notations. Let \(\Omega\) be a domain (i.e. an open, connected set) in \(\mathbb{R}^N\). For \(1 \leq p < \infty\) and \(0 < s < 1\), the Gagliardo semi-norm of a function \(u \in L^p(\Omega)\) is defined as

\[
|u|_{W^{s,p}(\Omega)} := \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p},
\]

where \(|\cdot|\) in the denominator on the right hand side denotes the Euclidean norm on \(\mathbb{R}^N\). (The dot above \(W^{s,p}\) indicates that this semi-norm is homogeneous with respect to dilations.) This semi-norm is an important tool in the study of many partial differential equations, and has found numerous important applications (see e.g. [2, 7, 10]).

A well-known ‘defect’ of this semi-norm is that \(|u|_{W^{s,p}(\Omega)}\) does not converge to the Sobolev semi-norm \(\|\nabla u\|_{L^p(\Omega)}\) as \(s \to 1^-\). Indeed, it is easy to see (c.f. [1]) that if \(u\) is any smooth, non-constant function on a domain \(\Omega \subset \mathbb{R}^N\), then \(\|u\|_{W^{s,p}(\Omega)}^p \to \infty\) as \(s \to 1^-\) (see also
Furthermore, it was shown that for $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega) := \{ u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega) \}$, one has what is now known as the BBM formula:

$$
\lim_{s \to 1^-} (1 - s)|u|^p_{W^{s,p}(\Omega)} = \frac{1}{p} k(p, N) \| \nabla u \|^p_{L^p(\Omega)},
$$

(1.7)

where

$$
k(p, N) := \int_{S^{N-1}} |e \cdot \omega|^p d\omega = \frac{2\Gamma((p + 1)/2)\pi^{(N-1)/2}}{\Gamma((N + p)/2)}. \tag{1.8}
$$

Here $e \in S^{N-1}$ is any fixed vector, $e \cdot \omega$ is the inner product of $e$ with $\omega$, and $d\omega$ is the surface measure on $S^{N-1}$ induced from the Lebesgue measure on $\mathbb{R}^N$. See also Dávila [6] for an extension to the space of functions of bounded variation on $\Omega$.

On the other hand, for $s \in (0, 1)$ and $1 \leq p < \infty$, let $W^{s,p}_0(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under the Gagliardo semi-norm $| \cdot |_{W^{s,p}(\mathbb{R}^N)}$. Parallel to the BBM formula (1.7), Maz’ya and Shaposhnikova [12] showed that for any $u \in \bigcup_{0 < s < 1} W^{s,p}_0(\mathbb{R}^N)$, we have

$$
\lim_{s \to 0^+} s \| u \|^p_{W^{s,p}(\mathbb{R}^N)} = \frac{2N}{p} \kappa_N \| u \|^p_{L^p(\mathbb{R}^N)}, \tag{1.9}
$$

where $\kappa_N$ is the volume of the unit ball in $\mathbb{R}^N$.

Recently, Brezis, Van Schaftingen and the second author [3] considered what happened when one replaces the $L^p$ norm on $\mathbb{R}^N \times \mathbb{R}^N$ in the Gagliardo semi-norm $| \cdot |_{W^{s,p}(\mathbb{R}^N)}$ by the weak-$L^p$ quasi-norm, and evaluates it at $s = 1$. This leads to the following characterization of $\| \nabla u \|_{L^p(\mathbb{R}^N)}$ in [3, Theorem 1.1]: they proved the existence of two positive constants $c = c(N)$ and $C = C(N)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$ and $1 \leq p < \infty$,

$$
c^p \| \nabla u \|^p_{L^p(\mathbb{R}^N)} \leq \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right]^p \leq C \| \nabla u \|^p_{L^p(\mathbb{R}^N)}. \tag{1.10}
$$

Furthermore, it was shown that for $u \in C_0^\infty(\mathbb{R}^N)$ and $1 \leq p < \infty$, if

$$
\tilde{E}_\lambda := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} \geq \lambda \right\}, \tag{1.11}
$$

then

$$
\lim_{\lambda \to \infty} \lambda^p L^2(\tilde{E}_\lambda) = \frac{1}{3} k(p, N) \| \nabla u \|^p_{L^p(\mathbb{R}^N)}. \tag{1.12}
$$
Thus, the first inequality in (1.10), with \( c(N) := \inf_{p \in [1, \infty)} (k(p, N)/N)^{1/p} > 0 \), is a direct consequence of (1.12) and (1.1). See also Poliakovsky [11, Lemma 3.1] for an extension of the second inequality in (1.10) to functions \( u \in W^{1,p}(\mathbb{R}^N) := \{ u \in L^p(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N) \} \).

In light of the formula (1.9) of Maz’ya and Shaposhnikova mentioned above, which establishes an analog of the BBM formula (1.7) when \( \lambda \rightarrow 0^+ \), a natural question is whether one has an analog of (1.10) and (1.12) for \( L^p \) instead of \( W^{1,p} \). Our Theorem 1.1 can be thought of as an affirmative answer to this question. Our proof is technically simpler than the corresponding one for (1.10) and (1.12) in [4], in that our proof relies only on Fubini’s theorem, but not on any covering lemma nor any Taylor expansion. On the other hand, it came as a mild surprise that while (1.12) involves a limit as \( \lambda \rightarrow +\infty \), its cousin (1.1) involves instead a limit where \( \lambda \rightarrow 0^+ \): the former is natural since large values of \( \lambda \) captures what happens to \( |u(x) − u(y)| \) when \( x \) and \( y \) are close to each other, which in turn relates to the size of \( |\nabla u(x)| \), but we do not have a good explanation of the latter.

We next turn to two results obtained by interpolating the upper bound in (1.2), with the upper bound in (1.10). The first result can be formulated using the Bessel potential spaces \((I − \Delta)^{-s/2}L^p(\mathbb{R}^N)\):

**Theorem 1.2.** For every \( N \in \mathbb{N} \) and \( p \in (1, \infty) \), there exists a constant \( C' = C'(p, N) \) such that for all \( s \in (0, 1) \) and all \( u \in (I − \Delta)^{-s/2}L^p(\mathbb{R}^N) \), we have

\[
\left[ \frac{|u(x) − u(y)|}{|x − y|^s} \right]_{L^p,\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq C' \|(I − \Delta)^{s/2}u\|_{L^p(\mathbb{R}^N)}. \tag{1.13}
\]

This theorem follows from complex interpolation by considering the following holomorphic family of linear operators

\[
u(x) \mapsto T_z u(x, y) := \frac{u(x) − u(y)}{|x − y|^s} \tag{1.14}
\]

where \( z \in \mathbb{C} \) takes value in the strip \( \{ 0 \leq \text{Re } z \leq 1 \} \). Indeed, the second inequality in (1.2) shows that when \( \text{Re } z = 0 \), \( T_z \) maps \( L^p(\mathbb{R}^N) \) to \( L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N) \). On the other hand, as observed by Poliakovsky [11, Lemma 3.1], the second inequality in (1.10) continues to hold for all \( u \in W^{1,p}(\mathbb{R}^N) = (I − \Delta)^{-1/2}L^p(\mathbb{R}^N) \). Thus when \( \text{Re } z = 1 \), \( T_z \) maps \( (I − \Delta)^{-1/2}L^p(\mathbb{R}^N) \) to \( L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N) \). Theorem 1.2 now follows by complex interpolation.

One drawback of Theorem 1.2 is that the left-hand side concerns a homogeneous norm, while the right-hand side contains an inhomogeneous norm. But it is only slightly harder to prove a variant of Theorem 1.2 concerning a homogeneous Triebel-Lizorkin space instead.

First, let’s recast Theorem 1.2 in terms of (inhomogeneous) fractional Triebel-Lizorkin spaces \( F^{s}_{p,q} \) on \( \mathbb{R}^N \), which we define as follows. Let \( \mathcal{S}(\mathbb{R}^N) \) be the Fréchet space of Schwartz functions on \( \mathbb{R}^N \), and \( \mathcal{S}'(\mathbb{R}^N) \) the space of all tempered distributions on \( \mathbb{R}^N \). Let \( \mathcal{F}^{-1} \) be the
inverse Fourier transform on $\mathbb{R}^N$ given by

$$F^{-1}\phi(x) = \int_{\mathbb{R}^N} \phi(\xi)e^{2\pi i x \cdot \xi}d\xi,$$

(1.15)

for $\phi \in S(\mathbb{R}^N)$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a fixed function supported on $\{|\xi| \leq 2\}$ such that $\varphi(\xi) = 1$ whenever $|\xi| \leq 1$. Write

$$\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$$

(1.16)

so that $\psi \in C_c^\infty(\mathbb{R}^N)$ is supported on $\{1/2 \leq |\xi| \leq 2\}$ with

$$\varphi(\xi) + \sum_{j \in \mathbb{N}} \psi(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^N.$$  

(1.17)

A corresponding family of Littlewood-Paley projections is given by

$$P_0 u(x) := u \ast F^{-1}\varphi(x)$$

(1.18)

and

$$\Delta_j u(x) := u \ast F^{-1}\psi_j(x),$$

(1.19)

where $\psi_j(\xi) := \psi(2^{-j}\xi)$. For $s \in \mathbb{R}$, $p \in (1, \infty)$ and $q \in (1, \infty)$, we define the \textit{(inhomogeneous) Triebel-Lizorkin space} $F^s_{p,q}(\mathbb{R}^N)$ to be the space of all $u \in S'(\mathbb{R}^N)$ for which

$$\|u\|_{F^s_{p,q}(\mathbb{R}^N)} := \left\| \left( |P_0 u|^q + \sum_{j \in \mathbb{N}} |2^{js}\Delta_j u|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} < \infty.$$  

(1.20)

Standard Littlewood-Paley theory shows that for $1 < p < \infty$, $s \in \mathbb{R}$, we have

$$F^s_{p,2}(\mathbb{R}^N) = (I - \Delta)^{-s/2}L^p(\mathbb{R}^N)$$

(1.21)

with comparable norms: for all $u \in S'(\mathbb{R}^N)$, we have

$$\|u\|_{F^s_{p,2}(\mathbb{R}^N)} \simeq_{p,N} \|(I - \Delta)^{s/2}u\|_{L^p(\mathbb{R}^N)}.$$  

(1.22)

Thus we could have replaced the Bessel potential spaces $(I - \Delta)^{-s/2}L^p(\mathbb{R}^N)$ in Theorem 1.2 by the inhomogeneous $F^s_{p,2}(\mathbb{R}^N)$.

This motivates us to consider a variant of Theorem 1.2 for homogeneous Triebel-Lizorkin spaces instead. To introduce these spaces, we denote by $Z(\mathbb{R}^N)$ the subspace of all $u \in S(\mathbb{R}^N)$ for which $\int_{\mathbb{R}^N} u(x)p(x)dx = 0$ for every polynomial $p(x) \in \mathbb{R}[x]$, and denote by $Z'(\mathbb{R}^N)$ the space of all continuous linear functionals on $Z(\mathbb{R}^N)$, which we identify with the quotient $S'(\mathbb{R}^N)/\{\text{polynomials on } \mathbb{R}^N\}$. If $\psi \in S(\mathbb{R}^N)$ is as in (1.16) and $\psi_j(\xi) := \psi(2^{-j}\xi)$ for $j \in \mathbb{Z}$, then

$$\sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$  

(1.23)

We denote by $\{\Delta_j\}_{j \in \mathbb{Z}}$ the family of Littlewood-Paley projections given by

$$\Delta_j u(x) := u \ast F^{-1}\psi_j(x),$$

(1.24)
which is well-defined for all \( u \in \mathcal{Z}'(\mathbb{R}^N) \) (because \( \int_{\mathbb{R}^N} \mathcal{F}^{-1} \psi_j(x)p(x)dx = 0 \) for all polynomials \( p(x) \in \mathbb{R}[x] \)). The homogeneous Triebel-Lizorkin space \( \dot{F}_{p,q}^s(\mathbb{R}^N) \) is then defined to be the space of all \( u \in \mathcal{Z}'(\mathbb{R}^N) \) for which

\[
\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^N)} := \left\| \left( \sum_{j \in \mathbb{Z}} |2^j \Delta_j u|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} < \infty. \tag{1.25}
\]

It was known (c.f. proof of Theorem in [15, Chapter 5.1.5]) that \( \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \), the space of (Schwartz) functions on \( \mathbb{R}^N \) given by inverse Fourier transforms of \( C_c^\infty \), compactly supported functions on \( \mathbb{R}^N \setminus \{0\} \), is a dense subset of \( \dot{F}_{p,q}^s(\mathbb{R}^N) \) for \( s \in \mathbb{R}, p \in (1, \infty) \) and \( q \in (1, \infty) \) (see Appendix below for a sketch of proof). Also, for \( 1 < p < \infty \), we have

\[
\|u\|_{\dot{F}_{p,2}^s(\mathbb{R}^N)} \simeq_{p,N} \begin{cases} \|u\|_{L^p(\mathbb{R}^N)} & \text{if } s = 0, \\ \|\nabla u\|_{L^p(\mathbb{R}^N)} & \text{if } s = 1, \end{cases} \tag{1.26}
\]

at least if \( u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \) (indeed this holds as long as \( u \in \mathcal{S}'(\mathbb{R}^N) \) for which the right hand side of the above display equation is finite). This allows us to prove the next result, concerning the homogeneous space \( \dot{F}_{p,2}^s(\mathbb{R}^N) \):

**Theorem 1.3.** For every \( N \in \mathbb{N} \) and \( p \in (1, \infty) \), there exists a constant \( C' = C'(p, N) \) such that for all \( s \in (0, 1) \) and all \( u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \),

\[
\left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \right]_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C' \|u\|_{\dot{F}_{p,2}^s(\mathbb{R}^N)}. \tag{1.27}
\]

As a result, for \( s \in (0, 1) \) and \( p \in (1, \infty) \), one may define the left-hand side of (1.27) for all \( u \in \dot{F}_{p,2}^s(\mathbb{R}^N) \) by density, and the inequality (1.27) continues to hold.

We note that Dominguez and Milman [8, Theorem 4.1] had actually proved a stronger embedding, namely that the \( \dot{F}_{p,2}^s \) norm above can be replaced by \( \dot{F}_{p,\infty}^s \), but their constant might blow up as \( s \to 0^+ \), whereas ours remain bounded uniformly for all \( 0 < s < 1 \).

Theorem 1.3 is the most powerful in the case \( 1 < p < 2 \), as one can see by comparing (1.27) with the following known inequality for \( \dot{F}_{p,p}^s(\mathbb{R}^N) \) (so \( q = p \) as opposed to \( q = 2 \) in Theorem 1.3):

**Proposition 1.4.** For every \( N \in \mathbb{N} \) and every \( p \in (1, \infty) \), there exists a constant \( C'' = C''(N, p) \) so that for all \( u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \), \( s \in (0, 1) \), one has

\[
\left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \right]_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C'' \left[ \frac{1}{s(1 - s)} \right] \max_{\frac{1}{2}, \frac{1}{p}} \|u\|_{\dot{F}_{p,p}^s(\mathbb{R}^N)}. \tag{1.28}
\]

The left hand side of (1.27) is smaller than the left hand side of (1.28) by Chebyshev’s inequality, but the norm on the right hand side of (1.27) is also smaller than the norm on
the right hand side of \((1.28)\) if \(1 < p < 2\) (because \(\|u\|_{\dot{F}_{p,2}^s} \leq \|u\|_{\dot{F}_{p,p}^s}\) if \(p < 2\)). In addition, the constant \(C'\) in \((1.27)\) does not blow up if we fix \(p\) and let \(s \to 0^+\) or \(1^-\).

The proof of Theorem 1.3 will be given in Section 3. For the convenience of the reader, we will also give a proof of Proposition 1.4, which we adapt from [13, Chapter V.5]. We also remark in passing that it is also known that if \(p \in [2, \infty)\), then the right hand side of \((1.28)\) can also be replaced by \(C''(N, p)[s(1-s)]^{-1/p}\|u\|_{\dot{F}_{p,2}^s(\mathbb{R}^N)}\). In other words, if we control \(\|u\|_{\dot{F}_{p,2}^s(\mathbb{R}^N)}\) (which is typically bigger than the norm \(\|u\|_{\dot{F}_{p,p}^s(\mathbb{R}^N)}\) that appears in \((1.28)\)), then to control the left hand side of \((1.28)\), we only need to pay a price of a smaller constant \(C''(N, p)[s(1-s)]^{-1/p}\) (as opposed to \(C''(N, p)[s(1-s)]^{-1/2}\)). This follows from an adaptation of the arguments given for Proposition 1.4, which we will not give in detail.

An interesting related question is whether the inequality in \((1.27)\) can be reversed. Since \(\mathcal{F}_{p,2}(\mathbb{R}^N) = [L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N)]_s\), this question could be reformulated as follows: If \(N \in \mathbb{N}, p \in (1, \infty), s \in (0, 1)\) and \(u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})]\), is there a holomorphic family of functions \(\{u_z(x) : z \in \mathbb{C}, 0 \leq \Re z \leq 1\}\) so that \(u_z(x) = u(x)\), and so that

\[
\max\left\{\sup_{\Re z = 0} \|u_z\|_{L^p(\mathbb{R}^N)}, \sup_{\Re z = 1} \|\nabla u_z\|_{L^p(\mathbb{R}^N)}\right\} \lesssim \left[\frac{u(x) - u(y)}{|x - y|^{\frac{s}{p} + s}}\right]_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)}? \quad (1.29)
\]

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2. Proof of Theorem 1.1

Proof. As remarked above, the second inequality in \((1.2)\) is essentially known. It was stated without proof in [8]. But for completeness, and also because we need to use it to derive the first inequality in \((1.2)\), we give its simple proof below. Indeed, we show that for \(1 \leq p < \infty\) and all measurable functions \(u\) on \(\mathbb{R}^N\),

\[
\left[\frac{u(x) - u(y)}{|x - y|^{\frac{s}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)} \leq 2^{p+1} \kappa_N \|u\|_{L^p(\mathbb{R}^N)}^p \quad (2.1)
\]

so that the second inequality of \((1.2)\) holds with \(c_2(N) := 2\kappa_N\), where \(\kappa_N\) is the volume of the unit ball in \(\mathbb{R}^N\).

To prove \((2.1)\), given \(1 \leq p < \infty\), a measurable \(u\) on \(\mathbb{R}^N\), and \(\lambda > 0\), let \(E_\lambda\) be as in \((1.3)\). Then by the triangle inequality,

\[
E_\lambda \subset \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, |u(x)| \geq \frac{1}{2} \lambda |x - y|^{N/p}\right\} \cup \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, |u(y)| \geq \frac{1}{2} \lambda |x - y|^{N/p}\right\}
\]

\[(2.2)\]
\[
\mathcal{L}^{2N}(E_\lambda) \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1\{ (x,y) : |y-x| \leq (2|u(x)|\lambda^{-1})^{p/N} \} \, dy \, dx \\
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1\{ (x,y) : |y-x| \leq (2|u(y)|\lambda^{-1})^{p/N} \} \, dx \, dy
\]
\[
= \kappa_N \int_{\mathbb{R}^N} (2\lambda^{-1})^p |u(x)|^p \, dx + \kappa_N \int_{\mathbb{R}^N} (2\lambda^{-1})^p |u(y)|^p \, dy
\]
\[
= 2^{p+1}\kappa_N \lambda^{-p} \|u\|^p_{L^p(\mathbb{R}^N)}.
\]

(2.11) now follows by multiplying by \( \lambda^p \) on both sides and taking supremum over all \( \lambda > 0 \).

It remains to establish (1.4) for all \( u \in L^p(\mathbb{R}^N), 1 \leq p < \infty \), which would then imply the first inequality in (1.2). We first consider the case under the additional assumption that \( u \) is compactly supported on \( \mathbb{R}^N \). This extra assumption about \( u \) will then be removed by using suitable truncations of \( u \), together with (2.1) which handles the error that arises.

**Case 1.** \( u \) is compactly supported. For \( \lambda > 0 \), let \( E_\lambda \) be as in (1.3). Then

\[
\mathcal{L}^{2N}(E_\lambda) = 2\mathcal{L}^{2N}(H_\lambda)
\]

(2.4) where

\[
H_\lambda := E_\lambda \cap \{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |y| > |x| \}.
\]

(2.5)

This is because \( E_\lambda \) is the union of its three subsets, one where \( |y| > |x| \), one where \( |y| < |x| \), and one where \( |y| = |x| \). The last set has \( \mathcal{L}^{2N} \) measure zero, and the first two sets have the same \( \mathcal{L}^{2N} \) measure by symmetry of the set \( E_\lambda \). Hence we only need to estimate \( \mathcal{L}^{2N}(H_\lambda) \).

Since \( u \) is compactly supported, we may assume

\[
\text{supp } u \subseteq B_R := \{ x \in \mathbb{R}^N : |x| < R \}
\]

(2.6)

for some \( R > 0 \). Now if \( (x,y) \in H_\lambda \), then we must have \( x \in B_R \). This is because otherwise both \( x,y \) are outside \( B_R \), which by our assumption about the support of \( u \) implies that \( u(x) = u(y) = 0 \), and hence \( (x,y) \notin E_\lambda \), contradicting that \( (x,y) \in H_\lambda \). Moreover, for \( x \in B_R \), let

\[
H_{\lambda,x} := \left\{ y \in \mathbb{R}^N : |y| > |x|, \frac{|u(y) - u(x)|}{|y-x|^{N/p}} \geq \lambda \right\}
\]

(2.7)

and

\[
H_{\lambda,x,R} := \left\{ y \in \mathbb{R}^N : |y| \geq R, |y-x| \leq \left( \frac{|u(x)|}{\lambda} \right)^{p/N} \right\}.
\]

(2.8)

Then Fubini’s theorem gives

\[
\mathcal{L}^{2N}(H_\lambda) = \int_{B_R} \mathcal{L}^N(H_{\lambda,x}) \, dx,
\]

(2.9)

while

\[
H_{\lambda,x,R} = H_{\lambda,x} \setminus B_R
\]

(2.10)
because for \(|y| \geq R\), we have \(u(y) = 0\) and hence \(|u(x)| = |u(y) - u(x)|\). It follows that

\[
H_{\lambda,x,R} \subseteq H_{\lambda,x} \subseteq H_{\lambda,x,R} \cup B_R.
\]  

(2.11)

Writing \(\kappa_N = \mathcal{L}^N(B_1)\), from the first inclusion in (2.11), we have

\[
\mathcal{L}^N(H_{\lambda,x}) \geq \mathcal{L}^N(H_{\lambda,x,R}) \geq \kappa_N \frac{|u(x)|^p}{\lambda^p} - \kappa_N R^N.
\]  

(2.12)

On the other hand, from the second inclusion in (2.11), we have

\[
\mathcal{L}^N(H_{\lambda,x}) \leq \kappa_N \frac{|u(x)|^p}{\lambda^p} + \kappa_N R^N.
\]  

(2.13)

Integrating (2.12) and (2.13) over \(x \in B_R\), and using (2.9), we obtain

\[
\frac{\kappa_N}{\lambda^p} \|u\|_{L^p(\mathbb{R}^N)}^p - \kappa_N^2 R^{2N} \leq \mathcal{L}^2N(H_{\lambda}) \leq \frac{\kappa_N}{\lambda^p} \|u\|_{L^p(\mathbb{R}^N)}^p + \kappa_N^2 R^{2N}.
\]  

(2.14)

Multiplying both sides by \(\lambda^p\) and letting \(\lambda \to 0^+\), we have

\[
\lim_{\lambda \to 0^+} \lambda^p \mathcal{L}^2N(H_{\lambda}) = \kappa_N \|u\|_{L^p(\mathbb{R}^N)}^p,
\]  

(2.15)

as desired.

**Case 2.** \(u \in L^p(\mathbb{R}^N), \text{ not necessarily compactly supported.}\) Let \(u_R = u \cdot 1_{B_R}\) be the truncation of \(u\) with \(|x| \leq R\) for some \(R > 0\). Let \(v_R = u - u_R\). Later we will crucially use that \(\|v_R\|_{L^p(\mathbb{R}^N)} \to 0\) as \(R \to \infty\), which holds only because \(u \in L^p(\mathbb{R}^N)\) and \(1 \leq p < \infty\).

Now since \(u = u_R + v_R\), for any \(\sigma \in (0, 1)\), we have

\[
E_{\lambda} = \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{N/p}} \geq \lambda \right\} \subseteq A_1 \cup A_2
\]  

(2.16)

where

\[
A_1 := \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_R(x) - u_R(y)|}{|x - y|^{N/p}} \geq \lambda(1 - \sigma) \right\}
\]  

(2.17)

and

\[
A_2 := \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v_R(x) - v_R(y)|}{|x - y|^{N/p}} \geq \lambda \sigma \right\}.
\]  

(2.18)

Hence

\[
\mathcal{L}^2N(E_{\lambda}) \leq \mathcal{L}^2N(A_1) + \mathcal{L}^2N(A_2).
\]  

(2.19)

Since \(u_R\) is compactly supported in \(B_R\), by (2.13) with \(\lambda\) replaced by \(\lambda(1 - \sigma)\), we obtain

\[
\mathcal{L}^2N(A_1) \leq \frac{2\kappa_N}{\lambda^p(1 - \sigma)^p} \|u_R\|_{L^p(\mathbb{R}^N)}^p + 2(\kappa_N R^N)^2.
\]  

(2.20)

For \(A_2\), by using (2.1) for \(v_R\), we obtain

\[
\mathcal{L}^2N(A_2) \leq \frac{2^{p+1}\kappa_N}{(\lambda \sigma)^p} \|v_R\|_{L^p(\mathbb{R}^N)}^p.
\]  

(2.21)
Combining (2.19), (2.20) and (2.21), and multiplying by $\lambda^p$, we obtain
\[
\lambda^pL^2N(E_\lambda) \leq \frac{2\kappa N}{(1-\sigma)^p}u_R^p_{L^p(\mathbb{R}^N)} + 2\lambda^p(\kappa_N R^N)^2 + \frac{2^{p+1}\kappa N}{\sigma_p}v_R^p_{L^p(\mathbb{R}^N)}.
\] (2.22)

We now first let $\lambda \to 0^+$, then let $R \to \infty$ and finally let $\sigma \to 0^+$. Since
\[
\lim_{R \to \infty} \|u_R\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} \quad \text{and} \quad \lim_{R \to \infty} \|v_R\|_{L^p(\mathbb{R}^N)} = 0,
\] (2.23)
we obtain
\[
\limsup_{\lambda \to 0^+} \lambda^pL^2N(E_\lambda) \leq 2\kappa N\|u\|_{L^p(\mathbb{R}^N)}^p.
\] (2.24)

Similarly, for any $\sigma > 0$, we have
\[
E_\lambda = \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{N/p}} \geq \lambda \right\} \supseteq A_3 \setminus A_2
\] (2.25)
where
\[
A_3 := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_R(x) - u_R(y)|}{|x - y|^{N/p}} \geq \lambda(1 + \sigma) \right\}
\] (2.26)
and $A_2$ is as in (2.18). Hence
\[
L^2N(E_\lambda) \geq L^2N(A_3) - L^2N(A_2).
\] (2.27)

Since $u_R$ is compactly supported in $B_R$, by (2.12) with $\lambda$ replaced by $\lambda(1 + \sigma)$, we have
\[
L^2N(A_3) \geq \frac{2\kappa_N}{\lambda^p(1 + \sigma)^p}\|u_R\|^p_{L^p(\mathbb{R}^N)} - 2(\kappa_N R^N)^2.
\] (2.28)

Combining (2.27), (2.28) and (2.21), and multiplying by $\lambda^p$, we obtain
\[
\lambda^pL^2N(E_\lambda) \geq \frac{2\kappa_N}{(1 + \sigma)^p}\|u_R\|^p_{L^p(\mathbb{R}^N)} - 2\lambda^p(\kappa_N R^N)^2 - \frac{2^{p+1}\kappa N}{\sigma_p}v_R^p_{L^p(\mathbb{R}^N)}.
\] (2.29)

We now first let $\lambda \to 0^+$, then let $R \to \infty$ and finally let $\sigma \to 0^+$. We obtain
\[
\liminf_{\lambda \to 0^+} \lambda^pL^2N(E_\lambda) \geq 2\kappa N\|u\|_{L^p(\mathbb{R}^N)}^p.
\] (2.30)

(1.4) then follows from (2.24) and (2.30). □

3. Embeddings of homogeneous fractional Triebel-Lizorkin spaces

In this section, we first prove Theorem 1.3. Its proof is similar to that of Theorem 1.2, in that it also proceeds via complex interpolation, for the holomorphic family of linear operators $\{T_z\}$ defined in (1.14). On the other hand, it is not clear whether $T_1$ maps $\dot{F}^1_{p,2}(\mathbb{R}^N)$ to $L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)$. Thus we provide a more careful proof below, explaining why interpolation works.
First, we recall the subspace $\mathcal{F}^{-1}[C^\infty_c(\mathbb{R}^N \setminus \{0\})]$ which consists of Schwartz functions on $\mathbb{R}^N$ and is dense in $\dot{F}^s_{p,2}(\mathbb{R}^N)$ when $s \in (0,1), p \in (1,\infty)$. For $u \in \mathcal{F}^{-1}[C^\infty_c(\mathbb{R}^N \setminus \{0\})]$, say $u = \mathcal{F}^{-1}\tilde{u}$ and $\tilde{u} \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$, we may define complex powers of Laplacian:
\[
(-\Delta)^z u(x) := \int_{\mathbb{R}^N} (2\pi|\xi|)^{2z}\tilde{u}(\xi)e^{2\pi i x \cdot \xi}d\xi, \quad z \in \mathbb{C}.
\] (3.1)

For every fixed $x \in \mathbb{R}^N$, this defines an entire function of $z \in \mathbb{C}$. Furthermore, for every fixed $z \in \mathbb{C}$, this defines a function in $\mathcal{F}^{-1}[C^\infty_c(\mathbb{R}^N \setminus \{0\})] \subset \mathcal{S}(\mathbb{R}^N)$.

**Lemma 3.1.** For every $N \in \mathbb{N}$ and $1 < p < \infty$, there exists a constant $A = A(p, N)$ such that for any $u \in \mathcal{F}^{-1}[C^\infty_c(\mathbb{R}^N \setminus \{0\})]$ and any $s \in (0,1)$, the following estimates hold.

(a) For $z \in \mathbb{C}$ with $\text{Re} \, z = 0$, we have
\[
\|(\Delta)^{(s-z)/2} u\|_{L^p(\mathbb{R}^N)} \leq A(1 + |\text{Im} \, z|)^{N+1} \|u\|_{\dot{F}^s_{p,2}(\mathbb{R}^N)}.
\] (3.2)

(b) For $z \in \mathbb{C}$ with $\text{Re} \, z = 1$, we have
\[
\|\nabla (\Delta)^{(s-z)/2} u\|_{L^p(\mathbb{R}^N)} \leq A(1 + |\text{Im} \, z|)^{N+1} \|u\|_{\dot{F}^s_{p,2}(\mathbb{R}^N)}.
\] (3.3)

**Proof.** The proof is a standard application of the theory of singular integrals. We verify the $z$-dependence of the constants in (3.2) and (3.3) by providing the necessary details below.

We first prove (a). Let $z \in \mathbb{C}$ with $\text{Re} \, z = 0$. Write $z = it$ for some $t \in \mathbb{R}$. Let $\{\tilde{\Delta}_j\}_{j \in \mathbb{Z}}$ be another family of Littlewood-Paley projections, given by
\[
\tilde{\Delta}_j u(x) := u \ast \mathcal{F}^{-1}\tilde{\psi}_j(x),
\] (3.4)

where $\tilde{\psi}_j(\xi) := \tilde{\psi}(2^j\xi)$ for some $C^\infty$ function supported on $\{1/4 \leq |\xi| \leq 4\}$, so that $\tilde{\psi}(\xi) = 1$ on the support of $\psi$; this gives $\tilde{\Delta}_j \Delta_j = \Delta_j$ for all $j \in \mathbb{Z}$. As a result, for $u \in \mathcal{F}^{-1}[C^\infty_c(\mathbb{R}^N \setminus \{0\})], j \in \mathbb{Z}$ and $s \in (0,1)$, we have
\[
\Delta_j(-\Delta)^{(s-z)/2} u = (2^{js} \Delta_j u) \ast K_j
\] (3.5)

where $K_j := \mathcal{F}^{-1}[2^{-js}(2\pi|\xi|)^{s-it}\tilde{\psi}_j](\xi)$ satisfies
\[
\sup_{j \in \mathbb{Z}} |\nabla K_j(x)| \lesssim_N (1 + |t|)^{N+1} |x|^{-(N+1)}.
\] (3.6)

(This is because for any multiindices $\alpha$, one has
\[
|\partial_\xi^\alpha [2^{-js}(2\pi|\xi|)^{s-it}\tilde{\psi}_j(\xi)]| \lesssim (1 + |t|)^{|\alpha|}2^{-j|\alpha|} \chi_{|\xi| \geq 2^j}
\] (3.7)

with implicit constant independent of $j \in \mathbb{Z}, s \in (0,1)$ and $t \in \mathbb{R}$; we may apply this with $|\alpha| = N$ and $N+1$ to bound $|x|^{N+1} |\nabla K_j(x)|$ in $L^\infty(\mathbb{R}^N)$.) We may now apply a vector-valued singular integral theorem to the operator
\[
(f_j(x))_{j \in \mathbb{Z}} \mapsto (f_j \ast K_j(x))_{j \in \mathbb{Z}},
\] (3.8)

which is clearly bounded on $L^2(\ell^2)$ with norm $\lesssim 1$; by [13, Chapter II, Theorem 5], or [14 Chapter I.6.4], this operator is also bounded on $L^p(\ell^2)$ for all $1 < p < \infty$, with operator...
norm \lesssim_{p,N} (1 + |t|)^{N+1}. Combined with the Littlewood-Paley inequality (which holds because
\((-\Delta)^{(s-z)/2} u \in L^p(\mathbb{R}^N)\)), we now have

\|
(-\Delta)^{(s-z)/2} u \|_{L^p(\mathbb{R}^N)} \lesssim_{p,N} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j (-\Delta)^{(s-z)/2} u|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^N)}
\lesssim_{p,N} (1 + |t|)^{N+1} \left\| \left( \sum_{j \in \mathbb{Z}} |2^j \Delta_j u|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^N)}
\leq A(p, N)(1 + |t|)^{N+1} \| u \|_{F^s_p,2(\mathbb{R}^N)}
(3.9)

the middle inequality following from (3.5) and the boundedness of the operator in (3.8) on
\(L^p(l^2)\). This completes the proof of (a).

To deduce (b), one can either appeal to the boundedness of the Riesz transform \(\nabla (-\Delta)^{-1/2}\)
on \(L^p(\mathbb{R}^N)\), or repeat the argument above. We omit the details. \(\square\)

Furthermore, we will need to consider, for \(1 < p < \infty\), the Lorentz space \(L^{p,1}(\mathbb{R}^N \times \mathbb{R}^N)\),
which is defined to be the set of all measurable functions \(g(x,y)\) on \(\mathbb{R}^N \times \mathbb{R}^N\) for which

\[ [g]_{L^{p,1}(\mathbb{R}^N \times \mathbb{R}^N)} := p \int_0^\infty \mathcal{L}^{2N}(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |g(x,y)| \geq \lambda\})^{1/p} d\lambda < \infty. \tag{3.10} \]

Just like \([\cdot]_{L^{p,\infty}}\), the quantity \([\cdot]_{L^{p,1}}\) is not a norm because it does not satisfy the triangle
inequality; it is only a quasi-norm. Nevertheless, for \(1 < p < \infty\), both \(L^{p,1}\) and \(L^{p,\infty}\) admit
a comparable norm, which make them Banach spaces, and \(L^{p,\infty}\) is the dual space of \(L^{p',1}\)
whenever \(1/p + 1/p' = 1\): in fact, the easiest way to norm \(L^{p',1}\) is to define

\[ \|g\|_{L^{p',1}(\mathbb{R}^N \times \mathbb{R}^N)} := \sup \left\{ \left| \int_{\mathbb{R}^N \times \mathbb{R}^N} g(x,y) G(x,y) dxdy \right| : [G]_{L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)} = 1 \right\}. \tag{3.11} \]

If \(p \in (1, \infty)\), every \(g \in L^{p,1}(\mathbb{R}^N \times \mathbb{R}^N)\) can be approximated in the \(L^{p',1}\) norm by functions
in \(L^{p',1}(\mathbb{R}^N \times \mathbb{R}^N)\) that are compactly supported in the open set \(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\}\)
(because such approximation is possible in the comparable \(L^{p',1}(\mathbb{R}^N \times \mathbb{R}^N)\) quasi-norm by
the dominated convergence theorem). We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** We fix \(s \in (0,1), p \in (1, \infty), u \in \mathcal{F}^{-1}[C^\infty(\mathbb{R}^N \setminus \{0\})]\), and \(g \in L^{p',1}(\mathbb{R}^N \times \mathbb{R}^N)\) with compact support in \(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\}\). Consider the function

\[ H(z) = \int_{\mathbb{R}^N \times \mathbb{R}^N} g(x,y) \frac{(-\Delta)^{(s-z)/2} u(x) - (-\Delta)^{(s-z)/2} u(y)}{|x - y|^{n+z}} dxdy. \tag{3.12} \]

This is an entire function of \(z\), and we claim that it is a bounded function on the strip
\(\{z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1\}\). Indeed, for \(u \in \mathcal{F}^{-1}[C^\infty(\mathbb{R}^N \setminus \{0\})]\), (3.1) gives

\[ (-\Delta)^{(s-z)/2} u(x) = \int_{\mathbb{R}^N} (2\pi |\xi|)^{s-z} \hat{u}(\xi) e^{2\pi i x \cdot \xi} d\xi \quad \text{for all } z \in \mathbb{C}, \tag{3.13} \]
so

\[
|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)| \leq 2\|(-\Delta)^{(s-z)/2}u\|_{L^\infty(\mathbb{R}^N)}
\]

\[
\leq 2 \int_{\xi \in \text{supp} \hat{u}} (2\pi|\xi|)^{s-\text{Re} z} |\hat{u}(\xi)|d\xi \lesssim \exp(a_1|\text{Re} z|)
\]

(3.14)

if \(a_1 > 0\) is large enough so that \(\max\{2\pi|\xi|, \frac{1}{2\pi|\xi|}\} \leq \exp(a_1)\) for all \(\xi \in \text{supp} \hat{u}\). Also, on the support of \(g(x, y)\), we have

\[
\frac{1}{|x-y|^{\frac{N}{p}+\epsilon}} \lesssim \exp(a_2|\text{Re} z|)
\]

(3.15)

if \(a_2 > 0\) is large enough so that \(\max\{|x-y|, |x-y|^{-1}: (x, y) \in \text{supp} g\} \leq \exp(a_2)\). Finally, since \(g \in L^{p',1}(\mathbb{R}^N \times \mathbb{R}^N)\) has compact support, it is in \(L^1(\mathbb{R}^N \times \mathbb{R}^N)\) as well. So

\[
|H(z)| \lesssim \|g\|_{L^1} \exp(a|\text{Re} z|) \quad \text{for all } z \in \mathbb{C}
\]

(3.16)

where \(a = a_1 + a_2\). In particular, \(H(z)\) is bounded on the strip \(\{z \in \mathbb{C}: 0 \leq \text{Re} z \leq 1\}\), as claimed.

Furthermore, for \(\text{Re} z = 0\), the upper bound in (1.12), together with (3.2), show that

\[
|H(z)| \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \left|g(x, y)\right| \left|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)\right|dx dy
\]

\[
\leq \|g\|_{L^{p',1}(\mathbb{R}^N)} \left[\frac{|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)|}{|x-y|^{\frac{N}{p}}} \right]_{L^{p,\infty}(\mathbb{R}^N)}
\]

\[
\leq 2c_2^{1/p}\|g\|_{L^{p',1}(\mathbb{R}^N)} \left[\frac{|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)|}{|x-y|^{\frac{N}{p}}} \right]_{L^p(\mathbb{R}^N)}
\]

\[
\leq 2c_2^{1/p}A(1 + |\text{Im} z|)^{N+1}\|g\|_{L^{p',1}(\mathbb{R}^N)}\|u\|_{\mathcal{F}^{s,2}_{p,2}(\mathbb{R}^N)}.
\]

On the other hand, for \(\text{Re} z = 1\), the upper bound in (1.10), together with (3.3), show that

\[
|H(z)| \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \left|g(x, y)\right| \left|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)\right|dx dy
\]

\[
\leq \|g\|_{L^{p',1}(\mathbb{R}^N)} \left[\frac{|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)|}{|x-y|^{\frac{N}{p}+1}} \right]_{L^{p,\infty}(\mathbb{R}^N)}
\]

\[
\leq C_1^{1/p}\|g\|_{L^{p',1}(\mathbb{R}^N)} \left[\frac{|(-\Delta)^{(s-z)/2}u(x) - (-\Delta)^{(s-z)/2}u(y)|}{|x-y|^{\frac{N}{p}+1}} \right]_{L^p(\mathbb{R}^N)}
\]

\[
\leq C_1^{1/p}A(1 + |\text{Im} z|)^{N+1}\|g\|_{L^{p',1}(\mathbb{R}^N)}\|u\|_{\mathcal{F}^{s,2}_{p,2}(\mathbb{R}^N)}.
\]

(The upper bound in (1.10) applies because \((-\Delta)^{(s-z)/2}u \in \mathcal{S}(\mathbb{R}^N) \subset W^{1,p}(\mathbb{R}^N)\) when \(u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})]\), allowing us to invoke III Lemma 3.1.) This allows us to use the three lines lemma from complex analysis to the bounded holomorphic function \(H(z)/(z+1)^{N+1}\)
on the strip \( \{ z \in \mathbb{C} : 0 \leq \text{Re} \, z \leq 1 \} \), and conclude that
\[
|H(s)| \lesssim_{p,N} \|g\|_{L^{p',1}(\mathbb{R}^{2N})} \|u\|_{F_{p,2}^s(\mathbb{R}^N)}.
\]
Taking supremum over \( g \), we get
\[
\left[ \frac{u(x) - u(y)}{|x - y|^N}\right]_{L_p,\infty(\mathbb{R}^{2N})} \leq C' \|u\|_{F_{p,2}^s(\mathbb{R}^N)}
\]
where \( C' = C'(p, N) \), and this inequality holds for all \( u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \). This shows that the left-hand side may be defined by density for all \( u \in \hat{F}_{p,2}^s(\mathbb{R}^N) \), and that the inequality continues to hold after such extension for all \( u \in \hat{F}_{p,2}^s(\mathbb{R}^N) \). \( \square \)

**Proof of Proposition 1.4.** We just note that for \( u \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \), we have
\[
\left\| \frac{u(x) - u(y)}{|x - y|^N} \right\|_{L_p(\mathbb{R}^{2N})} = \left\| \frac{u(x + z) - u(x)}{|z|^N} \right\|_{L_p(\mathbb{R}^{2N})}
\]
\[
\lesssim_N \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \sup_{|z| \simeq 2^{-k}} \|u(x + z) - u(x)\|_{L_p(dx)}^p \right)^{1/p}
\]
\[
\lesssim_{N,p} \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \sup_{|z| \simeq 2^{-k}} \left( \sum_{j \in \mathbb{Z}} \| \Delta_{j+k} u(x + z) - \Delta_{j+k} u(x) \|^2 \right)^{p/2} dx \right)^{1/p},
\]
the last inequality following from Littlewood-Paley (note that the sum in \( j \) has only finitely many non-zero terms). We consider two cases.

**Case 1:** \( 1 < p \leq 2 \)

In this case, we bound (3.23) using the inequality
\[
\left| \sum_j F_j \right|^{p/2} \leq \sum_j |F_j|^{p/2}
\]
with \( F_j := |\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)|^2 \). We get
\[
\lesssim^{(3.23)} \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \sup_{|z| \simeq 2^{-k}} \int_{\mathbb{R}^N} \sum_{j \in \mathbb{Z}} \| \Delta_{j+k} u(x + z) - \Delta_{j+k} u(x) \|^p dx \right)^{1/p}
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \sum_{j \in \mathbb{Z}} \sup_{|z| \simeq 2^{-k}} \| \Delta_{j+k} u(x + z) - \Delta_{j+k} u(x) \|^p_{L_p(dx)} \right)^{1/p}.
\]
Now if \( |z| \simeq 2^{-k} \), we write
\[
\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x) = \int_0^1 \frac{d}{dt}\Delta_{j+k} u(x + tz) dt = \int_0^1 z \cdot \nabla \Delta_{j+k} u(x + tz) dt,
\]
so its \( L^p \) norm with respect to \( x \) is bounded by
\[
|z| \| \nabla \Delta_{j+k} u \|_{L_p(\mathbb{R}^N)} \lesssim 2^{-k} \| \nabla \Delta_{j+k} u \|_{L_p(\mathbb{R}^N)} \lesssim_N 2^j \| \Delta_{j+k} u \|_{L_p(\mathbb{R}^N)}.
\]
This shows
\[
\sup_{|z| \geq 2^{-k}} \norm{\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)}_{L^p(dx)} \lesssim_N 2^j \norm{\Delta_j u}_{L^p(\mathbb{R}^N)}. \tag{3.29}
\]
We also have the trivial bound
\[
\sup_{|z| \geq 2^{-k}} \norm{\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)}_{L^p(dx)} \leq 2 \norm{\Delta_j u}_{L^p(\mathbb{R}^N)}. \tag{3.30}
\]
Then combining these two estimates, we write
\[
\sup_{|z| \geq 2^{-k}} \norm{\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)}_{L^p(dx)} \lesssim_N \begin{cases} 
\norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)} & \text{if } j > 0, \\
2^j \norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)} & \text{if } j \leq 0.
\end{cases} \tag{3.31}
\]
Then by substituting (3.31) into (3.26), we obtain
\[
\left| \frac{u(x) - u(y)}{|x - y|^{\frac{p}{p-s}}} \right|_{L^p(\mathbb{R}^N)} \lesssim_N \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \left( \sum_{j > 0} \norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)}^{p} + \sum_{j \leq 0} 2^{jp} \norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)}^{p} \right) \right)^{1/p} \tag{3.32}
\]
\[
= \left[ \sum_{j > 0} 2^{-jsp} \sum_{k \in \mathbb{Z}} 2^{(j+k)sp} \norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)}^{p} + \sum_{j \leq 0} 2^{j(1-s)p} \sum_{k \in \mathbb{Z}} 2^{(j+k)sp} \norm{\Delta_{j+k} u}_{L^p(\mathbb{R}^N)}^{p} \right]^{1/p} \tag{3.33}
\]
\[
\lesssim \left[ \sum_{j > 0} 2^{-js} + \sum_{j \leq 0} 2^{j(1-s)} \right]^{1/p} \norm{u}_{\dot{F}_{p,p}^{s}(\mathbb{R}^N)}. \tag{3.34}
\]
\[
\lesssim \left[ \frac{1}{s(1-s)} \right]^{1/p} \norm{u}_{\dot{F}_{p,p}^{s}(\mathbb{R}^N)}. \tag{3.35}
\]
**Case 2:** \(2 \leq p < \infty\)

We apply Minkowski inequality for \(L^{p/2}(\mathbb{R}^N)\) and obtain
\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \sup_{|z| \geq 2^{-k}} \left( \sum_{j \in \mathbb{Z}} \norm{\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)}_{L^p(dx)}^2 \right)^{p/2} \right)^{1/p} \tag{3.37}
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \left( \sum_{j \in \mathbb{Z}} \sup_{|z| \geq 2^{-k}} \norm{\Delta_{j+k} u(x + z) - \Delta_{j+k} u(x)}_{L^p(dx)}^2 \right)^{p/2} \right)^{1/p}. \tag{3.38}
\]
But by (3.31), we have
\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{ksp} \left( \sum_{j > 0} \norm{\Delta_{j+k} u}_{L^p(dx)}^2 + \sum_{j \leq 0} 2^{2j} \norm{\Delta_{j+k} u}_{L^p(dx)}^2 \right)^{p/2} \right)^{1/p} \tag{3.39}
\]
\[
= \left\| \sum_{j > 0} 2^{-2js} 2^{2(j+k)s} \norm{\Delta_{j+k} u}_{L^p(dx)}^2 + \sum_{j \leq 0} 2^{2j(1-s)} 2^{2(j+k)s} \norm{\Delta_{j+k} u}_{L^p(dx)}^2 \right\|_{L^{p/2}}^{1/2}. \tag{3.40}
\]
Applying the Minkowski inequality for \( \ell^p/2 \), we bound this by

\[
\left( \sum_{j \geq 0} 2^{-2js} + \sum_{j \leq 0} 2^{2j(1-s)} \right)^{1/2} \| u \|_{F_{p,q}(\mathbb{R}^N)} \lesssim \left[ \frac{1}{s(1-s)} \right]^{1/2} \| u \|_{\dot{F}_{p,q}(\mathbb{R}^N)}.
\]  (3.41)

\[
\square
\]

4. Appendix: Density in Triebel-Lizorkin spaces

In the proof of Theorem 1.3 we appealed to the case \( s \in (0, 1) \) and \( q = 2 \) of the following proposition. Thus we include a sketch of its proof.

**Proposition 4.1.** \( \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \) is dense in \( \dot{F}^{s \infty}_{p,q}(\mathbb{R}^N) \) for \( s \in \mathbb{R}, \ p \in (1, \infty) \) and \( q \in (1, \infty) \).

**Proof.** Fix \( s \in \mathbb{R}, \ p \in (1, \infty) \) and \( q \in (1, \infty) \). First, for \( u \in \dot{F}^{s \infty}_{p,q}(\mathbb{R}^N) \), since

\[
\lim_{J \to +\infty} \left\| \left( \sum_{|j| > J} 2^{js} \Delta_j u \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} = 0,
\]  (4.1)

we see that \( u_J := \sum_{|j| \leq J} \Delta_j u \) converges in \( \dot{F}^{s \infty}_{p,q}(\mathbb{R}^N) \) as \( J \to +\infty \).

Next, let \( \phi \in \mathcal{S}(\mathbb{R}^N) \) with \( \phi(0) = 1 \) whose Fourier transform \( \hat{\phi} \) is compactly supported on the unit ball. For every fixed \( J \in \mathbb{N} \), we let \( u_{J,\delta}(x) := \phi(\delta x)u_J(x) \). Then for \( \delta \ll 2^{-J} \), we have \( u_{J,\delta} \in \mathcal{F}^{-1}[C_c^\infty(\mathbb{R}^N \setminus \{0\})] \). Thus it remains to show that \( u_{J,\delta} \to u_J \) in \( \dot{F}^{s \infty}_{p,q} \) as \( \delta \to 0 \). To see this, note that \( u_{J,\delta} \) is \( C^\infty \) on \( \mathbb{R}^N \), so \( u_{J,\delta} \) converges pointwisely to \( u_J \) as \( \delta \to 0 \). Furthermore, \( u_{J,\delta} \) is dominated by a multiple of \( u_J \), which is in \( L^p(\mathbb{R}^N) \), so by the dominate convergence theorem,

\[
\lim_{\delta \to 0} \| u_{J,\delta} - u_J \|_{L^p(\mathbb{R}^N)} = 0.
\]  (4.2)

As a result,

\[
\lim_{\delta \to 0} \| \Delta_j(u_{J,\delta} - u_J) \|_{L^p(\mathbb{R}^N)} = 0
\]  (4.3)

for every \( j \in \mathbb{Z} \), which implies the desired convergence of \( u_{J,\delta} \) to \( u_J \) in \( \dot{F}^{s \infty}_{p,q} \) as \( \delta \to 0 \). \( \square \)

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