The linearization of the Kodama state

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We study the question of whether the linearization of the Kodama state around classical deSitter spacetime is normalizable in the inner product of the theory of linearized gravitons on deSitter spacetime. We find the answer is no in the Lorentzian theory. However, in the Euclidean theory the corresponding linearized Kodama state is delta-functional normalizable. We discuss whether this result invalidates the conjecture that the full Kodama state is a good physical state for quantum gravity with positive cosmological constant.
I. INTRODUCTION

A central problem of loop quantum gravity is to show whether or not the theory has a good low energy limit [1]. This must reproduce as an appropriate approximation, classical general relativity and quantum field theory on fixed backgrounds. This is a hard problem for the same reason it is hard to derive the properties of liquids or solids from first principles given only the quantum theory of atoms. Even if a particular choice of hamiltonian constraint or spin foam amplitude enjoys all the properties that could be asked from a fundamental point of view, it is far from trivial to derive the low energy behavior. An infinite number of exact solutions to all the constraints are known, but little is known about the low energy description of most of them. A number of directions are under development to do address this question, based on extensions to quantum gravity of coherent states [2] or the renormalization group [3].

A possibly important piece of evidence for this question is the Kodama state [4, 5]. This is the, so far, unique, precisely known, quantum state of the gravitational field that has both an exact description at the Planck scale and a semiclassical description. Furthermore, it exists only for nonzero cosmological constant, Λ, which supports the intuition that the cosmological constant is an essential parameter of any quantum theory of gravity.

The understanding of the Kodama state as a semiclassical description is straightforward1. In the Ashtekar formulation, the configuration variable of the gravitational field is the self-dual connection, Aa i. This is a complex variable for the Lorentzian theory, so it is valued in the complexification of SU(2). This fact will be important for what follows.

To construct a semiclassical description of deSitter spacetime2 we require a Hamilton-Jacobi functional S(A) with the property that deSitter spacetime is one of its trajectories. A convenient way to describe deSitter spacetime in the language of the Ashtekar formalism is that it is the unique Lorentzian self-dual spacetime. The self-dual condition is expressed as

\[ B^a_i = \frac{1}{2} e^{abc} F^i_{bc} = \frac{1}{3} E^a_i, \]  

(1)

E^a_i is the densitized spatial triad (or, equivalently the pull back to the spatial slice of the self-dual two form) and the left-handed components of the spacetime curvature are given by

\[ F_{ab}^i = \partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk} A^i_a A^k_b \]  

(2)

E^a_i is canonically conjugate to Aa i (up to a factor of ıG). Thus, it must be true that,

\[ E^{ai} = \frac{3}{\Lambda} \epsilon^{abc} F^i_{bc} \frac{\partial S(A)}{\partial A^a_i} \]  

(3)

There is, up to a constant, a single unique solution to this equation, which is

\[ S(A) = -i \left( \frac{3}{GA} \right) \int_\Sigma Y(A)_{CS} \]  

(4)

where Y(A)CS is the Chern-Simons three form

\[ Y(A)_{CS} = \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A. \]  

(5)

Thus, the semiclassical state for deSitter must be the Kodama state

\[ \Psi_K(A) = e^{\frac{3}{2} \int_\Sigma Y(A)_{CS}} \]  

(6)

1 More details are contained in a pedagogical introduction to the Kodama state see [6].

2 We work in this paper with the case of Λ > 0, but most results extend also to negative Λ.
where \( \lambda = G \hbar \Lambda \) is the dimensionless cosmological constant. We note there is no \( \imath \) in the exponent because

\[
\hat{E}^{ai} = \hbar G \frac{\partial}{\partial A_{ai}} \tag{7}
\]

It is then straightforward to use the Born-Oppenheimer method to develop a semiclassical expansion around the Kodama state. This is described in [6, 7, 8] and applications to cosmology are described in [9]. However, the Kodama state has other properties that suggest it may be more than just a semiclassical approximation to the quantum state of the gravitational field. First, it is in fact an exact solution to the constraints of quantum gravity. This is easy to see naively, but it is also true when consideration is given to issues of regularization and operator ordering. Also, in the case of the Euclidean theory all the properties described so far hold, with all the \( \imath \)'s removed. In that case there are many self-dual solutions, so the Kodama state can be the basis for a semiclassical expansion around any of them. The Euclidean theory is simpler in several respects, one of them is that the connection \( A_1 \) is real, so it is valued in the real form of \( SU(2) \). Because of this, the Euclideanized Kodama state is a phase,

\[
\Psi_{Eucl-K}(A) = e^{\frac{3 \imath}{\hbar} \int_{\Sigma} Y(A)_{CS}} \tag{8}
\]

because there is now an \( \imath \) in (7). Moreover, the loop transform of the Euclidean Kodama state, defined by

\[
\tilde{\Psi}_{Eucl-K}(\Gamma) = \int d\mu(A) T[\Gamma, A] \Psi_{K}(A), \tag{9}
\]

where \( T[\Gamma, A] \) is the traced holonomy of the spin network \( \Gamma \) and \( d\mu(A) \) is an appropriate measure, is well understood and has a striking property: It is proportional to the Kauffman bracket for framed quantum spin networks with a unimodular quantum group parameter. In the case of the Lorentzian Kodama state the issue is more subtle, in order to define the Loop transform we have to take the integration contour along a real section of \( SL(2, \mathbb{C}) \) connections defined by the reality conditions. There is no rigorous proof on what the result should be. However if we assume that we can deform the contour of integration to be over \( SU(2) \) connection, and disregards convergence issue we expect the result to also be proportional to the Kauffman bracket with real quantum deformation parameter.

The purpose of this paper is to address several questions that have been raised concerning the Kodama state [10]. One of them is that a similar state exists in the case of \( SU(2) \) Yang-Mills theory [11]. However, in that case it is believed to be unphysical, for the following reasons, which we now discuss.

In the \( SU(2) \) Yang-Mills theory (in Lorentzian signature) the analogue to the Kodama state is the solution to \( [E_1^a - \imath B_1^a] \Psi(A) = 0 \), where \( E_1^a = yg^2 \delta/\delta A_1^a \), where \( g \) is the Yang-mills coupling. The solution is real, and is given by

\[
\Psi(A) = e^{\frac{3 \imath}{\hbar} \int_{\Sigma} Y(A)_{CS}} \tag{10}
\]

This state is not normalizable. Because the theory is not diffeomorphism invariant, and there is no Hamiltonian constraint, the physical norm is

\[
<\Psi|\Psi>_{YM} = \int dA|\Psi(A)|^2 \tag{11}
\]

This tells us that in Yang-Mills theory

\[
<K|K>_{YM} = \int dAe^{\frac{3 \imath}{\hbar} \int_{\Sigma} Y(A)_{CS}}, \tag{12}
\]

This is not finite because there are directions in the configuration space which are unbounded.

Does this argument extend to quantum gravity? It is easy to see that it does not directly. First of all, the physical inner product is not given by (11). This is what is called the “kinematical inner product” as it defines
the kinematical Hilbert space, which is the arena for the definition of the quantum constraints. But physical states live in the subspace of states annihilated by the diffeomorphism and hamiltonian constraints. This subspace should be given an inner product structure defined by the physical reality conditions. This is that real physical observables (meaning observables that commute with all the constraints) must be represented by Hermitian operators. The physical states are then not expected to be normalizable in the kinematical inner product.

It is unfortunate that there is not known a closed form expression for the physical inner product in loop quantum gravity, in either the Euclidean or the Lorentzian theory. The physical inner product can be, however, expressed as a path integral and, in loop quantum gravity, there are explicit proposals for the physical inner product in terms of the spin foam formalism [12]. Recent convergence results on spin foam amplitudes show that the projection operator will be ultraviolet finite [13] and the fact that the representation theory for non-zero cosmological constant is q-deformed implies that the spin foam summations are also infrared finite [14]. Unfortunately, it is so far not possible to use this to test whether or not the Kodama state is normalizable with respect to the physical inner product.

We can ask a simpler question: is the Kodama state normalizable in the kinematical inner product. In the Euclidean case it is delta-functional normalizable, because the state \( S \) is a pure phase. In the Lorentzian case the situation is more complicated than in Yang-Mills theory because \( A^i_a \) is a complex variable, so the inner product must be defined by a choice of a contour. Furthermore, \( S(A) \) is a complex function, so the required integral has the form

\[
<K|K> = \int d\mu(A)e^{\frac{3}{2}\int_S 2\text{Re}Y(A)_{LCS} \Delta(A, \bar{A})}
\]

The contour is discussed in [13, 14], and \( \Delta \) is a functional implementing the projection on the kernel of the hamiltonian constraint, but the result is that it is not presently known whether this expression is convergent or divergent.

There is another question we can ask, which will be the subject of the rest of this paper. We can linearize classical general relativity on deSitter spacetime. This gives us a classical theory of tensor fields, \( a_{ab} \) propagating linearly on deSitter spacetime. We can quantize this theory. This can of course be done both in the usual ADM variables and in the Ashtekar variables. The result is the quantum theory of linearized gravitons on deSitter spacetime. This is not hard to do, following previous results [17, 18] in the linearized Ashtekar formalism at \( \Lambda = 0 \), and we will carry it out in the next section.

In the course of constructing this theory one solves the linearization versions of the constraints to find the physical Hilbert space of the linearized theory, which we may call \( \mathcal{H}_{\text{linear}} \). It has an inner product which can be found exactly by solving the linearization of the reality conditions. One can also linearize the Kodama state, arriving, for the Lorentzian case, at

\[
\Psi_{LK}(a) = e^{\frac{3}{2}\int_S Y(a)_{LCS}}
\]

where \( Y(a)_{LCS} \) is the quadratic truncation of the Chern-Simons three form. This state is annihilated by the linearized quantum constraints. So it is a functional of the same variables that linearized quantum states in \( \mathcal{H}_{\text{linear}} \) depend on. (These are not surprisingly the symmetric, trace-free, transverse components of the connection.)

One can then ask if \( \Psi_{LK}(a) \) is normalizable or even delta-functional normalisable in the inner product of the linearized theory. The answer, as we show in the next two sections is no, for the Lorentzian case. In the Euclidean case however, the linearized Euclidean Kodama state is delta-functional normalizable in the inner product of the linearized Euclidean theory. In the next sections we carry out the construction of the linearized theory and compute the norm of the linearized Kodama state. In the last section we discuss the implications of the results found.

### II. NOTATIONS AND DEFINITIONS

We denote by \( A \) the restriction of the self dual connection to a 3 dimensional spacelike slice. The phase space variables are pairs \((A^i_a, E^j_a)\) where \( A \) is a self dual connection and \( E \) is the densitized inverse frame field.
This means that $E_i^a/\sqrt{\det(E)} = e_i^a$ is the inverse frame field. The self dual connection can be expressed in terms of the usual geometrical variables as follows

$$A_i^a = \Gamma_i^a + iK_{a\overline{b}}e_{\overline{b}}^i$$

where $K_{ab}$ is the extrinsic curvature tensor, and $\Gamma$ is the spin connection satisfying

$$de^{i} + \epsilon_{ijk}\Gamma^j \wedge e^{k} = 0,$$

where $e^i = e^i_a dx^a$ is the frame field. From this relation it is clear that the quantum commutation relations are

$$[A_i^a(x), E_j^b(y)] = iP_\delta^b_{\delta^a_i} \delta(x-y),$$

where $P = \sqrt{\hbar G}$ is the Planck length and the reality conditions are given by

$$\bar{E}_i^a = E_i^a \quad \text{(18)}$$

$$A_i^a + \bar{A}_i^a = 2\Gamma_i^a(E) \quad \text{(19)}$$

There are three types of constraints. The gauge constraint

$$G^i = \nabla_a E_i^a = \partial_a E_i^a + \epsilon^{ijk} A_{aj} E_k^a = 0, \quad \text{(20)}$$

the diffeomorphism constraint

$$D_a = E_j^b F_{ab}^i = 0, \quad \text{(21)}$$

and the hamiltonian constraint

$$C(x) = \frac{1}{2\sqrt{\det(E)}\epsilon^{abc}\epsilon^{ijk} E_i^a E_j^b (B_{k\overline{c}} + \Lambda E_{k\overline{c}})} = 0, \quad \text{(22)}$$

where the magnetic field is defined by $\epsilon_{abc}B_{c\overline{k}} = F_{a\overline{k}}$, and we have introduced a constant $H$ (the Hubble constant) which is related to the cosmological constant $\Lambda$ by

$$H^2 = \frac{\Lambda}{3}. \quad \text{(23)}$$

Note that all the constraints are weight one density. One particular set of solutions are the ‘self dual’ solutions for which the electric field is proportional to the magnetic field

$$B_{k\overline{c}} + \frac{\Lambda}{3} E_{k\overline{c}} = 0. \quad \text{(24)}$$

de Sitter space is a self dual solution, this can be seen by choosing a particular gauge where $E$ is flat

$$A_i^a = i f H \delta_i^a, \quad E_i^a = f^2 \delta_i^a \quad \text{(25)}$$

$f$ labels gauge equivalent solutions, it is related to the usual time of inflationary coordinates.

$$f = \exp(+Ht), \quad \text{(26)}$$

$$ds^2 = -dt^2 + e^{2Ht} d\bar{x}^2. \quad \text{(27)}$$

Note that $\eta = (Hf)^{-1}$ is the conformal time in flat slicing

$$ds^2 = \frac{1}{H^2 \eta^2}(-d\eta^2 + d\bar{x}^2). \quad \text{(28)}$$
A. linearized gravity

We consider a perturbation of dS background

\[ A^i_a = iHF\delta^i_a + a^i_a \]  \hspace{1cm} (29)
\[ E^a_i = f^2\delta^a_i + \epsilon^a_i \]  \hspace{1cm} (30)

The constraints of gravity linearized around de Sitter background are the gauge constraint

\[ g^i = \partial_a e^{ai} + iHe^{ak}\epsilon_{ak} + \epsilon^{ij}a_{aj} = 0, \]  \hspace{1cm} (31)

the diffeomorphism constraint

\[ d_a = f^2(\partial_b a^b_a - \partial_a a^b_b + iHF\epsilon_a^{bc}a_{bc} + H^2\epsilon_a^{bc}e_{bc}) = 0, \]  \hspace{1cm} (32)

and the hamiltonian constraint

\[ c = fe^{abc}\partial_da_{bc} + 2iHF^2a^b_b + H^2f^2a^a_a = 0. \]  \hspace{1cm} (33)

The gauge fixing conditions we choose are such that \( \partial_a a^a_b = 0 \) to gauge fix the gauge constraint and \( e^{[ab]} = 0 \) in order to gauge fix the diffeomorphism constraint. Together with \( a^a_b = 0 \) for the hamiltonian constraint. This means that the zero mode part of \( a^a_b \) is a good time variable. This is clear since \( A^a_b = 3iHF + a^a_b \) so a non zero constant \( a^a_b \) can be understood as a shift in time

\[ \delta f = \frac{a^b_b}{3iH} \]  \hspace{1cm} (34)

Moreover it is easy to see that \( \partial_b f = \{\delta f, \int Nc\} = N(x)HF \). So if we choose \( N = 1 \) (resp. \( N = f \)) we recover the parametrization \( \pi \) (resp. \( \pi \)) of \( f \) in terms of inflationary time (resp. conformal time).

Overall, this means that both \( a \) and \( e \) can be taken to be symmetric, transverse and traceless tensors. Such fields carry two degree of freedom per spacetime points, the positive and negative helicity. In order to describe these degree of freedom it is convenient to work in momentum space. In this space we introduce for each momentum \( k \) a basis \( m^a(k), \overline{m}^a(k), k^a (m, \overline{m} \text{ being complex conjugate}) \) satisfying

\[ k^a m_a(k) = 0, m_a(k)m^a(k) = 0, \overline{m}_a(k)m^a(k) = 1 \]  \hspace{1cm} (35)

and

\[ \epsilon_{abc}ik^a m^b(k) = |k|m^c(k). \]  \hspace{1cm} (36)

these definitions imply that \( m^a(-k) = \overline{m}^a(k) \). The symmetric traceless transverse fields \( a_{ab}(x), e_{ab}(x) \) can be expressed in terms of positive and negative helicity fields \( a^\pm(k), e^\pm(k) \)

\[ a_{ab}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} (a^+(k)m_a(k)m_b(k) + a^-(k)m_a(k)m_b(k), \]  \hspace{1cm} (37)

\[ e_{ab}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} (e^+(k)m_a(k)m_b(k) + e^-(k)m_a(k)m_b(k). \]  \hspace{1cm} (38)

B. linearized reality conditions and commutation relations

The commutation relations in terms of the helicity fields are given by

\[ [a^\pm(k), e^{\mp}(p)] = l^2_p\delta^4(k + p); [a^\pm(k), e^{\mp}(p)] = 0. \]  \hspace{1cm} (39)
The linearized spin connection associated with the linearized E field $\gamma_{a}^i$ is given by

$$\gamma_{a}^i = -f^{-2} \epsilon^{ijk}(\partial_j e_{(ka)} + \cdots),$$  

so when $e$ is symmetric and traceless

$$\gamma_{a}^i = -f^{-2} \epsilon^{ijk} \partial_j e_{(ka)}.$$  

The linearized reality conditions are therefore given by

$$\hat{e}^\pm(k) = e^\pm(-k),$$  
$$a^+(k) + \bar{a}^+(-k) = -2f^{-2}|k|e^+(k),$$  
$$a^-(k) + \bar{a}^-(k) = +2f^{-2}|k|e^-(k).$$  

The reality condition on $e$ is a consequence of the other ones when $k \neq 0$. Also, it is important to realize that there is a sign difference between the two helicities. The commutators between the self dual connection and its complex conjugate are given by

$$[\bar{a}^+, a^+(p)] = +2l_p^2 f^{-2}|k|\delta^i(k - p),$$  
$$[\bar{a}^-, a^-(p)] = -2l_p^2 f^{-2}|k|\delta^i(k - p),$$  
$$[\bar{a}^-, a^+(p)] = 0.$$  

This clearly shows that $a^+$ is a creation operator (raising energy) whereas $a^-$ is an annihilation operator (lowering energy).

### III. PHYSICAL STATES

At the quantum level we decide to work in the polarization where the self dual connection $A$ is diagonal. The physical states are solutions of the linear constraints and we have seen that we can choose the gauge fixing such that the wave function depends only on the symmetric transverse traceless part of the perturbation so in this polarization the wave function $\psi$ is a functional of $(z^+(k), z^-(k))$ and

$$a^\pm(k)\psi(z^+, z^-) = z^\pm(k)\psi(z^+, z^-).$$  

The representation we work with is the one for which the frame fields acts as a derivative operator

$$e^\pm(-k)\psi(z^+, z^-) = -l_p^2 \frac{\partial\psi(z^+, z^-)}{\partial z^\pm(k)}.$$  

Given this representation of the commutator algebra the scalar product is now uniquely determined by the reality conditions. The scalar product is expressible as an infinite dimensional integral

$$<\phi|\psi> = \int D^2 z^+ D^2 z^- \bar{\phi}(\bar{z}^+, \bar{z}^-)e^{F(z^+, \bar{z}^+)}\psi(z^+, z^-),$$  

where $D^2 z = \prod_k dz(k) d\bar{z}(k)$ is the usual path integral measure and the functional $F$ satisfies the following reality conditions

$$\frac{\partial F}{\partial z^+(k)} = \frac{\partial F}{\partial z^+(-k)} = -\frac{f^2}{2|k|l_p^2} (z^+(k) + \bar{z}^+(k)),$$  
$$\frac{\partial F}{\partial z^-(k)} = \frac{\partial F}{\partial z^-(k)} = +\frac{f^2}{2|k|l_p^2} (z^-(k) + \bar{z}^-(k)).$$
So $F$ can be written as $F = F^+(z^+, \bar{z}^+) + F^-(z^-, \bar{z}^-)$ where

$$F^\pm (z^\pm, \bar{z}^\pm) = \mp f^2 \int \frac{d^3k}{4|k|^2} (z^\pm(k) + \bar{z}^\pm(-k))(z^\pm(-k) + \bar{z}^\pm(+k)).$$

(53)

The representation we just described is equivalent for the positive helicity to the usual Bargmann-Fock coherent state representation. If we define

$$\phi_B(z^+) \equiv e^{-f^2 \int \frac{d^3k}{4|k|^2} z^+(k)z^+(-k)} \phi(z^+)$$

(54)

The scalar product in term of these functionals is

$$<\psi_B|\phi_B> = \int D^2z^+ \bar{\psi}_B(z^+) e^{-f^2 \int \frac{d^3k}{4|k|^2} z^+(k)z^+(-k)} \phi_B(z^+).$$

(55)

The Fock vaccua is annihilated by $\bar{a}^+$ and given by the functional $\psi_B^{(0)}(z^+) = 1$. This state is normalizable in the sense that each mode $k$ is normalizable, the normalization being

$$\int d^2z(k)e^{\frac{f^2|\bar{z}(k)|^2}{2|k|^2}} = \frac{2\pi|k|^2}{f^2}.$$  

(56)

For the negative helicity we can get an analogous description by defining

$$\phi_B(z^-) \equiv e^{+f^2 \int \frac{d^3k}{4|k|^2} z^-(k)z^-(+k)} \phi(z^-)$$

(57)

The scalar product being

$$<\psi_B|\phi_B> = \int D^2z^+ \bar{\psi}_B(z^+) e^{+f^2 \int \frac{d^3k}{4|k|^2} z^+(k)z^+(k)} \phi_B(z^+).$$

(58)

The state annihilated by $\bar{a}^-$ is also given by the functional $\psi_B^{(0)}(z^-) = 1$. However due to the wrong sign in the exponent this state is not normalizable. This is understandable since we have seen in eq (55) that $\bar{a}^-$ is in fact a creation operator, so the state annihilated by it is in fact a maximal energy state instead of a minimal energy state.

It is possible to represent the Fock vaccua in term of holomorphic wave function if one allow the wave function to be distributional in that case the Fock vaccua can be written as a product for each mode of $\delta(z^-)(k) \exp(z^+(k)z^+(k)/4|k|^2)$.  

**IV. LINEARIZED KODAMA STATE**

The Kodama state $\psi_K(A) = \exp(S_{CS}(A)/H^2i^2)$, where

$$S_{CS}(A) = \int \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A$$

(59)

is the unique solution of the quantum self-duality equation

$$B_k^c \psi_K(A) - H^2i^2 \frac{\delta \psi_K}{\delta A^c_k} = 0.$$  

(60)

We want to expand the Chern-Simons functional around de Sitter background, for symmetric transverse traceless perturbation one obtains $S_{CS} = S_0 + S(a) + I(a)$ where

$$S_0 = -iH^3 \int f^3,$$  

(61)
the quadratic fluctuation is given, up to boundary terms, by

\[ S(a) = \frac{1}{2} \int \epsilon^{abc} \partial_a a^b a^c - iHf a^b a^b, \]  

and

\[ I(a) = \frac{1}{6} \int d^3x \epsilon_{ijk} \epsilon^{abc} a^i a^j a^k. \]

It is convenient to express the quadratic fluctuation in terms of fourier modes

\[ S^\pm(a^\pm) = \frac{1}{2} \int d^3k (|\pm k| - iHf) a^\pm(k) a^{\pm(-k)}. \]

We are interested into small fluctuation around the de Sitter background. The linearized Kodama state is given by

\[ \psi_{LK} = e^{S_0/H^2 \bar{i}_2} e^{S^+(z^+) / H^2 \bar{i}_2 + S^-(z^-) / H^2 \bar{i}_2}. \]

\[ S_0 \] is infinite since it is the integral of a constant \( f \) which contains an infinite volume factor. One way to deal with that is by cutting off the flat slice at a fix volume \( V \) and then let \( V \) goes to infinity. The linearized Kodama state is, as a function of \( a \) and up to a constant, the unique solution of the linearized self-duality equation

\[ (|\pm k| - iHf) a^\pm(k) \psi_{LK}(z^+, z^-) - H^2 \bar{i}_2 \partial \psi(z^+, z^-) \]  

and it is explicitly time dependent. This state is a good approximation of the full Kodama state if we can neglect the cubic term \( I(a) \), this is the case when the fluctuations satisfy \( | \int d^3q a^\pm(q - k) a^{\pm(-q)} | \ll |ka^\pm(k)|. \)

### A. scalar product

The linearized Kodama state is given by a product of an holomorphic function for positive helicity with an holomorphic function for negative helicity. The measure of integration of the scalar product has the same property, so the norm of \( \psi_{LK} \) factorizes \( ||\psi_{LK}||^2 = ||\psi_{LK}||_+^2 ||\psi_{LK}||^-_2 \), with

\[ ||\psi_{LK}||^2_{\pm} = \int D^2 z^\pm e^{Q^\pm(z^\pm, \bar{z}^\pm).} \]  

\( Q^\pm \) are quadratic forms given by

\[ Q^\pm(z^\pm, \bar{z}^\pm) = \int d^3k \left( \begin{array}{c} z^\pm(k) \\ \bar{z}^\pm(-k) \end{array} \right)^t Q^\pm(k) \left( \begin{array}{c} z^\pm(k) \\ \bar{z}^\pm(-k) \end{array} \right), \]

\[ Q^\pm(k) = \frac{\mp 1}{k^2} \left( \begin{array}{cc} f^2 & \frac{|k|}{2\pi} + i \frac{f}{2\pi} \\ \frac{|k|}{2\pi} - i \frac{f}{2\pi} & f^2 \end{array} \right). \]

In order to know the eigenvalues of \( Q \) lets compute

\[ \text{tr} Q^\pm(k) = \mp \frac{f^2}{2|k|^2}, \]
\[ \det Q^\pm(k) = -\left( \frac{|k|}{2H^2f_2} \right)^2. \]
Since the determinant is always negative, we see that for both helicities at least one of the eigenvalue is positive so the corresponding mode is not normalizable. This lead to the drastic conclusion that for both helicities there is always a non normalizable mode. In order to better understand the nature of this non normalisability of the Kodama state at the quadratic level we can decompose $a$ into real and imaginary part $z^\pm(k) = x^\pm(k) + iy^\pm(k)$, and $x, y$ are real in the sense that $\bar{x}(k) = x(-k)$, $\bar{y}(k) = y(-k)$. With this variables we can write the quadratic form as

$$Q^\pm(k) = \pm \frac{|k|}{H^2 P^2} \left\{ x^\pm(k)x^\pm(k) - [y^\pm(k) \mp \frac{Hf}{|k|} x^\pm(k)] [\bar{y}^\pm(k) \mp \frac{Hf}{|k|} \bar{x}^\pm(k)] \right\}$$

One clearly sees that the non normalisability to quadratic order of the Kodama state is due to non normalizable fluctuation for each mode $k$ and for each helicity. This is very different in nature to the non normalisability of the Chern-simons state in electromagnetism. In the latter case we can show that the positive helicity modes are normalizable whereas the negative helicity modes are non normalizable. Also the positive helicity modes have a positive energy while the negative helicity modes have a negative energy. The Kodama state do not show such drastic birefringence properties since none of the helicity are normalizable to quadratic order.

Finally, It is not clear if the instability (non-normalisability) of the linearized Kodama state implies some instability for the full Kodama state. What we have proven so far is the non normalisability of the Kodama state to quadratic order we have neglected the influence of higher order correction. The full Kodama state is cubic, so we do not expect the higher order terms of the Chern-Simons term to improve the convergence properties. On the other hand the measure of integration implementing the reality conditions will introduce contribution to all order. This is very different from the QCD case where there are non contribution coming from the measure. So there is still the logical possibility that the Kodama state while non normalizable to quadratic order is normalizable when we take into account the contribution from the measure to all order.

V. THE LINEARIZED EUCLIDEAN THEORY

It is easy to see that the linearized Kodama state is delta-functional normalizable in the Euclidean theory. The reduction to the two helicity states goes the same way as the Lorentzian theory. Hence the theory is again reduced to linearized physical degrees of freedom, $(a^\pm(k), e^\pm(k))$. However these are separately real, because the Euclideanized reality conditions are simply that $(a^\pm(k), e^\pm(k))$ are real. There is no $\bar{n}$ in the classical Poisson brackets, so the canonical commutation relations are now,

$$[a^\pm(k), e^\mp(p)] = i\frac{\delta^3(k+p)}{P}; \quad [a^\pm(k), e^\mp(p)] = 0.$$  \hspace{1cm} (72)

The states in the linearized Hilbert space are again functionals $\psi(z^+, z^-)$. The representation is defined by

$$a^\pm(k)\psi(z^+, z^-) = z^\pm(k)\psi(z^+, z^-).$$  \hspace{1cm} (73)

The representation we work with is the one for which the frame fields acts as a derivative operator

$$e^\pm(-k)\psi(z^+, z^-) = -i\frac{\partial \psi(z^+, z^-)}{\partial z^\pm(k)}.$$  \hspace{1cm} (74)

The inner product that realizes the reality conditions is now simply

$$<\phi|\psi> = \int D^2 z^+ D^2 z^- \bar{\psi}(\bar{z}^+, \bar{z}^-)\psi(z^+, z^-),$$  \hspace{1cm} (75)

where the integration is over a real section $z(k) = \bar{z}(-k)$. The linearized self-duality condition on states is now

$$(\pm |k| - Hf)a^\pm(k)\psi_{L,k}(z^+, z^-) - i\frac{\partial \psi(z^+, z^-)}{\partial z^\pm(-k)} = 0,$$  \hspace{1cm} (76)
The $i$ in the last term is now there because there is an $i$ in the canonical commutation relation. An $i$ that is in the first term in the Lorentzian theory is absent because the connection $A_a$ corresponding to Euclidean deSitter is purely real rather than purely imaginary.

The unique solution to the linearized self-dual equation is again a linearized Kodama state. It is now

$$\psi_{ELK}(f, z^+, z^-) = e^{iS_0/H^2}\frac{e^{iS^+(z^+)/H^2} + iS^-(z^-)/H^2}{e^{iS^+} + e^{iS^-}}.$$ (77)

where $S_0$ and the $S^\pm$ are now real functionals of $z^\pm$.

The result is that the norm of the linearized Euclidean Kodama state is

$$<ELK|ELK> = \int D^2z^+ D^2z^- 1$$ (78)

This is delta functional normalizable.

VI. ALTERNATIVES

Before closing we want to make some comment on the implications of the results we described here.

We first may note that the issue or normalizability would not generally come up for a semiclassical state

$$\Psi(x) \approx e^{iS(x)_{HJ}}$$ (79)

where $S(x)_{HJ}$ is a solution to the Hamiltonian-Jacobi functions. Such states are only delta-function normalizable. As energy is one of the parameters of the Hamilton-Jacobi equation, such solutions correspond to a definite choice of energy. Normalizable states are wavepackets constructed from superposition of energy eigenstates. This is not generally a problem for normal systems. The problem for us is that on compact regions, and in the absence of matter, the Kodama state is unique. There is no parameter to vary, it depends only on $\lambda = G\hbar\Lambda$ which is a parameter of the theory.

One possibility is to couple gravity to matter with a potential, such that the value of the cosmological constant becomes a parameter of a solution. This case one can consider wavepackets constructed by superposing different values of $\lambda$. Such a procedure has been recently proven successful [9] in the case of mini-superspace quantization of gravity coupled to a scalar field which involves only the zero modes of the Kodama state. Since the Euclidean linearized Kodama state is only delta functional normalizable this procedure is expected to be successful, but only in the Euclidean case. It is not known whether or not it works in the full theory.

There is another simple consideration that shed some light on the meaning of the truncation of the Kodama state. We earlier wrote the Kodama state in the form

$$\Psi_K(A) = e^{S_0 + S^2}$$ (80)

Truncation corresponded to dropping the $S^3$ term. In the ordering in which the full Kodama state is a solution to the constraints we may write the Hamiltonian constraint schematically as

$$H = EEJ$$ (81)

where $J = F + H^2E$ is the self-duality operator. Each has an expansion around deSitter spacetime.

$$J = J^1 + J^2, \quad E = E^0 + e$$ (82)

We can then write

$$H = H^1 + H^2 + H^3 + H^4$$ (83)

As we indicated the truncated state $\Psi_{LK} = e^{S_0 + S^2}$ is a physical state of the linearized theory

$$H^1\Psi_{LK} = 0$$ (84)
Moreover, the linearized Kodama is explicitly time dependent, the quadratic Hamiltonian is also time dependent its dynamics is governed by the quadratic Hamiltonian. One can check that

$$H^2 \Psi_{LK} = \frac{\partial}{\partial \eta} \Psi_{LK} \neq 0.$$  \hspace{1cm} (85)

The linear Kodama state thus possesses all the properties we would expect for a vacuum except that this is not a Fock vacuum.

The main question is then whether this means that the full Kodama state cannot be physical if its linearization is not a normalizable state in the inner product of the linearized theory. Since one should expect that if one has a state which is proportional to the ground state of the full theory, its truncation to a linearized state should be the ground state of the linearized theory. And, certainly, as we have seen the linearized Kodama state is not the ground state of the linearized theory.

Let us try to flesh out this argument to see how definitive it is. To state it more precisely requires that for at least some classical solutions, $A^0, E_0$ there exists a map $\mathcal{M}$ from the full to linearized physical Hilbert spaces,

$$\mathcal{M} : \mathcal{H}^{\text{physical}} \rightarrow \mathcal{H}^{\text{linearized}}$$  \hspace{1cm} (86)

satisfying some natural list of properties. What should these be,

1. $\mathcal{M}$ is defined for solutions $A^0, E_0$ of maximum symmetry

2. The image of the ground state is the ground state

3. There is a subspace $\mathcal{H}^{\text{fullgravitons}} \subset \mathcal{H}^{\text{physical}}$ which is mapped into $\mathcal{H}^{\text{linearized}}$. This corresponds to gravitons propagating on the background, fully dressed in the fully interacting theory.

4. $\mathcal{M}$ takes states that are fully diffeomorphism invariant to states that are invariant under linearized diffeomorphisms around $A^0$.

5. The orthogonal subspace to $\mathcal{H}^{\text{fullgravitons}}$ is mapped to the null vector in $\mathcal{H}^{\text{linearized}}$. Hence there is a large kernel. These correspond to states in $\mathcal{H}^{\text{physical}}$ that cannot be decomposed in the basis of graviton states on the given background.

6. If we accept the results in loop quantum gravity indicating there is an ultraviolet cutoff, such as the discreteness of area and volume, then the map $\mathcal{M}$ cannot be onto, because there will be no states in its image with wavelength or frequency shorter than $l_{\text{Planck}}$.

7. We note that the choice of a maximally symmetric Lorentzian spacetime $A^0$ does not determine a unique Hilbert space of linearized fields. Additional information is required corresponding to the choice of a timelike killing field on all or part of $A^0$. The Hilbert spaces corresponding to different choices are generally unitarily inequivalent. Examples are the Minkowski versus the Rindler states in Minkowski spacetime or the Hilbert spaces corresponding to different observers in deSitter spacetime. Thus, $\mathcal{M}$ must depend on additional information beyond the specification of $A^0$.

We may note that if we insist on properties 4 to 7 the map may not be just a simple truncation of the functional form. Hence, it may be not necessary that $\mathcal{M} \cdot \Psi_K \rightarrow \Psi_{LK}$.

Another problem with such a map $\mathcal{M}$ is contained in property 2. The problem is that there is no definition of the Hamiltonian for the full theory, in the absence of a boundary. In classical or quantum gravity, it is a simple and direct consequence of the equivalence principle that energy is only defined quasi-locally, on a

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\(^3\) We note that this implies that the symmetry group of $A^0$ is either broken or deformed as in doubly special relativity \(^1\). We note that it has recently been established that the latter is the case in \(2+1\) gravity \(^2\).
timelike or null 3-surface which may be taken to be the boundary of a region of spacetime. The boundary may be at infinity, as in the ADM energy or it may be of finite area. But without a boundary there can be no definition of energy.

Finally, It is well known that the usual choice of Fock vacua is well defined only after one has given a choice of synchronized observers, for instance by specifying a timelike Killing vector field on the background. The Kodama state is a full covariant state, in that it does not depend on the choice of a timelike killing field on a background. It cannot, for it is defined on any point in the configuration space. A point in the configuration space corresponds to a connection on a three slice, hence it is dual to a 3-geometry, not a solution. Further, only a set of measure zero correspond to spatial slices of metrics that have killing fields.

A choice of a time like killing field corresponds in some sense to a choice of an observer in spacetime. We may then try to interpret the fact that the Kodama state does not map to a linearized vacua as saying that the linearized vacua depends on a choice of an observer which is not made in the specification of the Kodama state. It either means that the Kodama state is not physical because its covariance prevent its linearization from being a Fock vacua or that the map $\mathcal{M}$ should also contain in some way a choice of synchronized observer.

Thus, the conclusion is that while the lack of normalizability of the linearization of the Lorentzian Kodama state is worrying, there is not yet a definitive argument that the full state is unphysical in the Lorentzian case. We also do not yet understand the significance of the fact that the Euclidean version of the Kodama state appears to be better off, in this respect. More work is clearly needed. Among the issues left open are the question of how to evaluate the action of the spin foam projection operator onto physical states on the Kodama state. Also, to be studied in future work, is a further analysis of the hypothesis that the physics in the presence of the Kodama state reproduces in an appropriate limit the quantization of field in a de Sitter background.

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[1] C. Rovelli, Living Rev. Rel. 1 (1998) 1, gr-qc/9710008; A. Perez, Spin Foam Models for Quantum Gravity. Topical Review in Class.Quant.Grav. 20 (2003) R43, gr-qc/0301113; L. Smolin, How far are we from the quantum theory of gravity?, hep-th/0303185.
[2] H. Sahlmann, T. Thiemann, O. Winkler, Coherent States for Canonical Quantum General Relativity and the Infinite Tensor Product Extension, Nucl.Phys. B606 (2001) 401-440, gr-qc/0102038.
[3] Fotini Markopoulou, Coarse graining in spin foam models, gr-qc/0203036; An algebraic approach to coarse graining, hep-th/0006199.
[4] H. Kodama, Holomorphic Wave Function Of The Universe, Phys. Rev. D 42, 2548 (1990).
[5] H. Kodama, “Specialization Of Ashtekar’s Formalism To Bianchi Cosmology,” Prog. Theor. Phys. 80, 1024 (1988).
[6] L. Smolin, “Quantum gravity with a positive cosmological constant,” arXiv:hep-th/0209079.
[7] L. Smolin and C. Soo, “The Chern-Simons invariant as the natural time variable for classical and quantum cosmology,” Nucl. Phys. B 449, 289 (1995) arXiv:gr-qc/9405015.
[8] C. Soo and L. N. Chang, “Superspace dynamics and perturbations around ‘emptiness’,” Int. J. Mod. Phys. D 3, 529 (1994) arXiv:gr-qc/9307018.
[9] S. Alexander, J. Malecki and L. Smolin, “Quantum gravity and inflation,” arXiv:hep-th/0309045.
[10] E. Witten, “A note on the Chern-Simons and Kodama wavefunctions,” arXiv:gr-qc/0306083.
[11] R. Jackiw, Topological Investigations In Quantized Gauge Theories, p. 258, exercise 3.7, in S. B. Treiman et. al. Current Algebra And Anomalies (World Scientific, 1985).
[12] M. P. Reisenberger and C. Rovelli. “Sum-over-surface form of loop quantum gravity”, gr-qc/9612035, Phys. Rev. D 56 (1997) 3490; J. W. Barrett and L. Crane, “Relativistic spin networks and quantum gravity,” J. Math. Phys. 39, 3296 (1998) [arXiv:gr-qc/9709028]. L. Freidel and K. Krasnov, “Spin foam models and the classical action principle,” Adv. Theor. Math. Phys. 2, 1183 (1999) [arXiv:hep-th/9807092]. J. W. Barrett and L. Crane, “A Lorentzian signature model for quantum general relativity,” Class. Quant. Grav. 17, 3101 (2000) [arXiv:gr-qc/9904025]. R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, “Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space,” Nucl. Phys. B 574, 785 (2000) [arXiv:hep-th/9907154]. M. P. Reisenberger and C. Rovelli. “Spacetime as a Feynman diagram: the connection formulation”, Class. Quant. Grav., 18:121140, 2001;

[13] Louis Crane, Alejandro Perez, Carlo Rovelli, “A finiteness proof for the Lorentzian state sum spin foam model for quantum general relativity”, gr-qc/0104057

[14] , Karim Noui, Philippe Roche, “Cosmological Deformation of Lorentzian Spin Foam Models”, gr-qc/0211109

[15] C. p. Soo, “Wave function of the universe and Chern-Simons perturbation theory,” Class. Quant. Grav. 19, 1051 (2002) [arXiv:gr-qc/0109046].

[16] R. Paternoga and R. Graham, “Triad representation of the Chern-Simons state in quantum gravity,” Phys. Rev. D 62, 084005 (2000) [arXiv:gr-qc/0003111].

[17] A. Ashtekar, C. Rovelli and L. Smolin, “Gravitons And Loops,” Phys. Rev. D 44, 1740 (1991) [arXiv:hep-th/9202054].

[18] A. Ashtekar, C. Rovelli and L. Smolin, “Selfduality And Quantization,” J. Geom. Phys. 8, 7 (1992) [arXiv:hep-th/9202079].

[19] G. Amelino-Camelia, “Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale,” Int. J. Mod. Phys. D 11, 35 (2002) [arXiv:gr-qc/0012051].

[20] L. Freidel, J. Kowalski-Glikman and L. Smolin, “2+1 gravity and doubly special relativity,” arXiv:hep-th/0307085