Asymptotics behaviour in one dimensional model of interacting particles

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Introduction
We study the asymptotic behaviour of solutions to the one-dimensional initial value problem
\[ u_t = u_{xx} + (u K^t u)_x \quad \text{for } x \in \mathbb{R}, \quad t > 0, \quad (1) \]
\[ u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \quad (2) \]
where the initial datum \( u_0 \in L^1(\mathbb{R}) \) is nonnegative and \( \varepsilon \geq 0 \).

Motivations
Equation (1) arises in studies of an animal aggregation as well as in some problems in mechanics of continuous media. The unknown function \( u = u(x, t) \) represents either the population density of a species or, in the case of materials applications, a particle density. Under our assumptions on interaction kernel \( K^t = K^t(u) \), equation (1) describes a model in which particles are under some repulsive force. Notice also that the one-dimensional parabolic-elliptic system of chemotaxis
\[ u_t = u_{xx} + (u v)_x - u v - u, \quad x \in \mathbb{R}, \quad t > 0 \quad (3) \]
can be written as equation (1). Indeed, if we put \( K(x) = -2 \varepsilon \delta(x) \) into the (1), which is the fundamental solution of the operator \( \frac{1}{2} \Delta + (\varepsilon^2 - 1) Id \), one can rewrite the second equation of (3) as \( v = K^t v \). Moreover, however, we should recall that we consider repulsive phenomena.

Main assumptions
First of all, we assume that the interaction kernel has the form
\[ K^t(x) = \frac{1}{2} H(x) + V(x), \quad (4) \]
where, \( H \) is the classical Heaviside function given by the formula:
\[ H(x) := \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0. \end{cases} \]
Moreover, we assume that \( A \in (0, \infty) \) is a constant and the function \( V \) satisfy
\[ V \in W^{1,1} (\mathbb{R}), \quad ||V||_{L^1} < A \quad (5) \]
\[ ||V||_{L^\infty} < A \quad (6) \]

Remark
Notice that, under assumptions on function \( V(x) \), we have the representation
\[ V(x) = \int_0^x V(y) \, dy. \] Hence, we get immediately that \( V \in L^1(\mathbb{R}) \cap C(\mathbb{R}), \lim_{x \to \pm \infty} V(x) = 0 \) and the following estimate \( ||V||_{L^\infty} \leq ||V||_{L^1} < A \) hold true.

Recent works (existence)
Karch and Suzuki in their publication[2] showed that the initial value problem (1)-(2) have a unique and global-in-time solution for a large class of initial conditions and interaction kernels. In particular, our assumption imply that \( K^t \in L^1(\mathbb{R}) \), hence the kernel \( K^t \) is mildly singular in the sense stated in [2, Thm 2.5]. In this case, results from [2] can be summarized as follows: for every \( u_0 \in L^1(\mathbb{R}) \) such that \( u_0 \geq 0 \), there exists the unique global-in-time solution \( u_0 \) of problem (1)-(2) satisfying \( u_0 \in C([0, \infty), L^1(\mathbb{R})) \cap C([0, \infty), W^{1,1}(\mathbb{R})) \cap C([0, \infty), L^1(\mathbb{R})) \).
In addition, the condition \( u_0(x) \geq 0 \) implies \( u_0(x, t) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \). Moreover we obtain the conservation of the \( L^1(\mathbb{R}) \) norm of nonnegative solutions:
\[ ||u(t)||_{L^1} = \int_0^\infty u(x,t) \, dx = \int_0^\infty u_0(x) \, dx = ||u_0||_{L^1}. \]

Recent works (asymptotic)
Karch and Suzuki in [1] studied the large time asymptotics of solutions to (1)-(2) under the assumption that \( K^t \in L^1(\mathbb{R}) \). They showed that either the fundamental solution of heat equation or a nonlinear diffusion wave appear in the asymptotic expansion of solutions as \( t \to \infty \). We would like to emphasise that, in all those results, a diffusion phenomena play a crucial role in the large time behaviour of solutions to problem (1)-(2).

Theorem (Decays of \( L^1 \) norm)
Assume that \( u = u(x,t) \) is a nonnegative solution to problem (1)-(2) where the interaction kernel satisfy assumptions (4)-(6). Suppose also that \( u_0 \in L^1(\mathbb{R}) \) is nonnegative and \( \varepsilon \geq 0 \). Then for every \( p \in [1, \infty) \) the following inequality hold true:
\[ ||u(t)||_{L^p} \leq (A - ||V||_{L^1})^\frac{1}{p} ||u_0||_{L^1}^\frac{1}{p} + \frac{1}{p} \quad (7) \]
for all \( t > 0 \).

Primitive of solution
From now on, without loss of generality, we assume that
\[ \int_0^\infty u_0(x) \, dx = \int_0^\infty u_0(x) \, dx = 1. \] Indeed, it suffices to replace \( u \) in equation (1) by \( \frac{u}{\int_0^\infty u_0(x) \, dx} \) and \( K^t \) by \( \frac{K^t}{\int_0^\infty u_0(x) \, dx} \). Now, let us put
\[ u(x,t) = \int_0^\infty u(x,t) \, dy - \frac{1}{p} \quad (8) \]
where \( u(x,t) \) is the solution of (1)-(2). Then, we show that the large time behaviour of \( U \) is described by a self-similar profile, given by a rarefaction wave, namely, the unique entropy solution of the Riemann problem for the scalar conservation law
\[ W_t^\infty + A V W_t^\infty = 0 \]
\[ W_t^\infty (x,0) = \frac{1}{p} \quad (9) \]
\[ W_t^\infty (x) = 1 \]
It is well-known that this rarefaction wave is given by explicit formula
\[ W_t^\infty (x,t) = \begin{cases} \frac{1}{2} & \text{for } x < \frac{At}{2}, \\ \frac{x}{At} & \text{for } \frac{At}{2} < x < \frac{At}{2}, \\ \frac{1}{2} & \text{for } x > \frac{At}{2} \end{cases} \quad (11) \]

Theorem (Convergence towards rarefaction waves)
Let the assumptions of above Theorem hold true and \( ||u||_{L^1} = 1 \). Assume, moreover, that
\[ \int_0^\infty u_0(y) \, dy \leq L^1(\mathbb{R}), \quad \text{and} \int_0^\infty u_0(y) \, dy = 1 \in L^0(\mathbb{R}). \]
\[ \text{Then, there exist a constant } C > 0 \text{ such that } \forall \varepsilon > 0 \text{ and each } p \in [1, \infty] \text{ the following estimate hold true:} \]
\[ ||u(t) - W_t^\infty (\cdot, t)||_p \leq C_t (\log \big(2 + ||V||_{L^1} \big) )^{1-p} \]
where \( U = U(x,t) \) is the primitive of solution of problem (1)-(2) given by (8) and \( W_t^\infty \) is the rarefaction wave given by (11).

Corollary
Let the assumptions of the second Theorem hold true. For the solution \( u = u(x,t) \) of problem (1)-(2) we define its rescaled version \( u_c (x,t) = u(x,t) \lambda^t \) for \( \lambda > 0, x \in \mathbb{R} \) and \( t > 0 \). Then, for every test function \( \varphi \in C_c ^0(\mathbb{R}) \) and each \( \lambda > 0 \)
\[ \int_0^\infty u_c (x, t) \varphi(x) \, dx \]
\[ \lim_{\lambda \to \infty} \int_0^\infty u_c (x, t) \varphi(x) \, dx = \int_0^\infty u_c (x, t) \varphi(x) \, dx. \]

Final result
In other words, for each \( \lambda > 0 \), the family of functions \( u_c (\cdot, t) \) converges weakly as \( \lambda \to \infty \) to \( (W_t^\infty)_{\lambda^2} \).