Oscillation and non oscillation criteria for linear nonhomogeneous systems of two first-order ordinary differential equations

G. A. Grigorian

Institute of Mathematics NAS of Armenia
E-mail: mathphys2@instmath.sci.am

Abstract. The Riccati equation method is used to establish an oscillatory and a non oscillatory criteria for nonhomogeneous linear systems of two first-order ordinary differential equations. It is shown that the obtained oscillatory criterion is a generalization of J. S. W. Wong’s oscillatory criterion.

Key words: nonhomogeneous linear systems of ordinary differential equations, oscillation, non oscillation, second order linear ordinary differential equations, the Riccati equation method.

1. Introduction. Let \( p(t), q(t), r(t), s(t), f(t) \) and \( g(t) \) be real-valued continuous functions on \( [t_0, +\infty) \). Consider the linear nonhomogeneous systems of differential equations

\[
\begin{cases}
\phi' = p(t)\phi + q(t)\psi + f(t), \\
\psi' = r(t)\phi + s(t)\psi + g(t),
\end{cases}
\tag{1.1}
\]

and the corresponding homogeneous one

\[
\begin{cases}
\phi' = p(t)\phi + q(t)\psi, \\
\psi' = r(t)\phi + s(t)\psi,
\end{cases}
\tag{1.2}
\]

Definition 1.1. The system (1.1) ((1.2)) is called oscillatory if for each of its solutions \((\phi(t), \psi(t))\) the function \(\phi(t)\) has arbitrary large zeroes, otherwise it is called non oscillatory.

Definition 1.2. The system (1.1) ((1.2)) is called oscillatory on the interval \([t_1, t_2]\) \((\subset [t_0, +\infty)\)) if for each of its solutions \((\phi(t), \psi(t))\) the function \(\phi(t)\) has a zero on \([t_1, t_2]\).

Let \( a(t), b(t), c(t) \) and \( d(t) \) be real-valued continuous function on \([t_0, +\infty)\), and let \( a(t) > 0, \ t \geq t_0 \). Along with the systems (1.1) and (1.2) consider the second order linear
non homogeneous equation

\[(a(t)\phi')' + b(t)\phi' + c(t)\phi = d(t), \quad t \geq t_0. \quad (1.3)\]

The substitution

\[\psi = a(t)\phi', \quad t \geq t_0. \quad (1.4)\]

reduces Eq. (1.3) to the following linear system

\[
\begin{cases}
\phi' = \frac{1}{a(t)}\psi, \\
\psi' = -c(t)\phi - \frac{b(t)}{a(t)}\psi + d(t), \quad t \geq t_0.
\end{cases} \quad (1.5)
\]

**Definition 1.3.** Eq. (1.3) is called oscillatory if each of its solutions has arbitrary large zeroes, otherwise it is called non oscillatory.

**Definition 1.4.** Eq. (1.3) is called oscillatory on the interval \([t_1, t_2](\subset [t_0, +\infty))\) if each of its solutions has a zero on \([t_1, t_2]\).

Obviously Eq. (1.3) is oscillatory (non oscillatory, oscillatory on the interval \([t_1, t_2] (\subset [t_0, +\infty))\)) if and only if the system (1.5) is oscillatory (non oscillatory, oscillatory on the interval \([t_1, t_2](\subset [t_0, +\infty))\)).

Note that the system (1.5) is a particular case of the system (1.1) therefore the system (1.1) can be considered as a generalization of Eq. (1.3). The system (1.1) (especially Eq. (1.3)) has an important theoretical interest and practical applications. Therefore study the question of oscillation and non oscillation of the system (1.1) (in particular of Eq. (1.3)) is an important problem of qualitative theory of differential equations. To study of this question for Eq. (1.3) are devoted many works (see [1 - 10] and cited works therein). Among them notice the El-Saedy’s oscillation theorem for undamped \((b(t) \equiv 0)\) Eq. (1.3) (see [1], or [2, Theorem A]). Combining the Riccati equation method with a variational technique J. S. W. Wong obtained the following generalization of El-Saedy’s result (see [2, Theorem 1]).

**Theorem 1.1.** Suppose that \(b(t) \equiv 0\) and for any \(T \geq t_0\) there exist \(T \leq s_1 < t_1 \leq s_2 < t_2\) such that

\[
d(t) \begin{cases}
\leq 0, & t \in [s_1, t_1], \\
\geq 0, & t \in [s_2, t_2].
\end{cases}
\]

Denote \(D(s_i, t_i) \equiv \{u \in C^1[s_i, t_i] : u(t) \neq 0, \ u(s_i) = u(t_i) = 0\}, \ i = 1, 2..\) If there exists \(u \in D(s_i, t_i)\) such that

\[
\int_{s_i}^{t_i} \left( c(\tau)u^2(\tau) - a(\tau)u'(\tau)^2 \right) d\tau \geq 0 \quad (1.6)
\]
for \( i = 1, 2 \), then Eq. (1.3) is oscillatory.

Later using the same approach of the proof of Theorem 1.1 Q. Kong and M. Pasic extended the J. S. W. Wong’s result to the damped Eq. (1.3) (see [3, Theorem 2]). Unfortunately despite the significance of these results their conditions contain free parameter-functions. This makes it hard to further application them to the concrete equations (despite the fact that Q. Kong and M. Pasic found in [3] an one parameter family of equations satisfying the conditions of Theorem 1.1).

Unlike of the approach used for proving mentioned above oscillatory criteria of works [2,3] in this paper we use only the Riccati equation method for proving an oscillatory and a non oscillatory criteria for the system (1.1). We show that the obtained oscillatory criterion is a generalization of Theorem 1.1.

2. Auxiliary propositions. Let \( f_k(t), \ g_k(t), \ h_k(t), \ k = 1, 2 \) be real-valued continuous functions on \([t_0, +\infty)\). Consider the Riccati equations

\[
y' + f_k(t)y^2 + g_k(t)y + h_k(t) = 0, \quad t \geq t_0. \tag{2.1_k}\]

\( k = 1, 2 \) and the differential inequalities

\[
\eta + f_k(t)\eta^2 + g_k(t)\eta + h_k(t) \geq 0, \quad t \geq t_0. \tag{2.2_k}\]

\( k = 1, 2 \).

**Remark 2.1.** Every solution of Eq. (2.1_2) on \([t_0, t_1]\) is also a solution of the inequality (2.2_2) on \([t_0, t_1]\).

**Remark 2.2.** If \( f_1(t) \geq 0, \ t \in [t_0, t_1], \) then every solution of the linear equation

\[
\zeta' + g_1(t)\zeta + h_1(t) = 0, \quad t \in [t_0, t_1]
\]

is also a solution of the inequality (2.2_1) on \([t_0, t_1]\).

**Theorem 2.1.** Let \( y_2(t) \) be a solution of Eq. (2.1_2) on \([t_0, \tau_0]\) \((t_0 < \tau_0 \leq +\infty)\) and let \( \eta_1(t) \) and \( \eta_2(t) \) be solutions of the inequalities (2.2_1) and (2.2_2) respectively on \([t_0, \tau_0]\) such that \( y_2(t_0) \leq \eta_k(t_0), \ k = 1, 2 \). In addition let the following conditions be satisfied:

\[
f_1(t) \geq 0, \ \gamma - y_2(t_0) + \int_{t_0}^{t} \exp\left\{ \int_{t_0}^{\tau} [f_1(s)(\eta_1(s) + \eta_2(s)) + g_1(s)]ds \right\} [(f_2(\tau) - f_1(\tau))^2y_2^2(\tau) + (g_2(\tau) - g_1(\tau))y_2(\tau) + h_2(\tau) - h_1(\tau)]d\tau \geq 0, \ \tau \in [t_0, \tau_0] \text{ for some } \gamma \in [y_2(t_0), \eta_1(t_0)].
\]

Then Eq. (2.1_1) has a solution \( y_1(t) \) on \([t_0, \tau_0]\) with \( y_1(t_0) \geq \gamma \) and \( y_1(t) \geq y_2(t), \ t \in [t_0, \tau_0]. \)

See the proof in [11].

For any \( \lambda \in \mathbb{R} \) set:

\[
\alpha_\lambda(t) \equiv \exp\left\{ \int_{t_0}^{t} p(\tau)d\tau \right\} \left[ \lambda + \int_{t_0}^{t} \exp\left\{ -\int_{t_0}^{\tau} p(s)ds \right\} f(\tau)d\tau \right], \quad t \geq t_0
\]
and in the system (1.1) substitute
\[ \phi = \phi_1 + \alpha_\lambda(t). \] (2.3)

Since \( \alpha_\lambda(t) \) is a solution of the equation
\[ \alpha' = p(t)\alpha + f(t), \quad t \geq t_0 \]
we obtain
\[
\begin{align*}
\phi_1' &= p(t)\phi_1 + q(t)y(t), \\
\psi' &= r(t)\phi_1 + s(t)\psi + g_\lambda(t), \quad t \geq t_0,
\end{align*}
\] (2.4)
where \( g_\lambda(t) \equiv r(t)\alpha_\lambda(t) + g(t), \quad t \geq t_0 \). In the obtained system substitute
\[ \psi = y\phi_1. \]

We get
\[
\begin{align*}
\phi_1' &= [p(t) + q(t)y]y, \\
y'\phi_1 + [q(t)y^2 + E(t)y - r(t)]\phi_1 = g_\lambda(t), \quad t \geq t_0,
\end{align*}
\]
where \( E(t) \equiv p(t) - s(t), \quad t \geq t_0 \). It follows from here that if \((\phi_1(t), \psi(t))\) is a solution of the system (2.4) such that \( \phi_1(t) \neq 0, \quad t \in [t_1, t_2) \) \((t_0 \leq t_1 < t_2 \leq +\infty)\) then
\[
\phi_1(t) = \phi_1(t_1) \exp \left\{ \int_{t_0}^{t} \left[ p(\tau) + q(\tau)y(\tau) \right] d\tau \right\}, \quad \psi(t) = y(t)\phi_1(t), \quad t \in [t_1, t_2),
\] (2.5)
where \( y(t) \) is the solution of the Riccati equation
\[ y' + q(t)y^2 + E(t)y - h_{\phi_1}(t) = 0, \quad t \in [t_1, t_2), \]
with \( y(t_1) = \frac{\psi(t_1)}{\phi_1(t_1)} \); where \( h_{\phi_1}(t) \equiv r(t) + \frac{g_\lambda(t)}{\phi_1(t)} \), \( t \in [t_1, t_2) \).

3. Main results. In this section we prove a non oscillatory and an oscillatory criteria for the system (1.1). We show that the obtained oscillatory criterion is a generalization of Theorem 1.1.

**Theorem 3.1.** Let the following conditions be satisfied:

1) \( q(t) \geq 0, \quad t \geq t_0; \)

2) there exists \( \lambda \geq 0 \) such that

2.1) \( \alpha_\lambda(t) \geq 0, \quad t \geq t_0; \)
2) \( r(t)\alpha(t) + g(t) \geq 0, \quad t \geq t_0. \)

Then if the system (1.2) is non oscillatory the system (1.1) is also non oscillatory.

Proof. Assume the system (1.2) is non oscillatory. Then there exists its a solution \((\phi_0(t), \psi_0(t))\) such that \(\phi_0(t) \neq 0, \quad t \geq t_1\) for some \(t_1 \geq t_0\). It follows from here that 
\[
y_0(t) \equiv \frac{\psi_0(t)}{\phi_0(t)}, \quad t \geq t_0.
\]

Then if the system (1.2) is non oscillatory the system (1.1) is also non oscillatory.

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y_0(t) \equiv \frac{\psi_0(t)}{\phi_0(t)}, \quad t \geq t_0.
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\[
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\]

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\[
y_0(t) \equiv \frac{\psi_0(t)}{\phi_0(t)}, \quad t \geq t_0.
\]
which contradicts (3.5). The obtained contradiction proves (3.4). It follows from the condition 2.1) that
\[ \phi(t) = \phi_1(t) + \alpha_\lambda(t) \geq \phi_1(t), \quad t \geq t_1. \]
This together with (3.4) implies that \( \phi(t) > 0, \quad t \geq t_1. \) Therefore the system (1.1) is non oscillatory. The theorem is proved.

**Remark 3.1.** Some explicit non oscillatory criteria for the system (1.2) are obtained in [12 - 14].

**Theorem 3.2.** Let the following conditions be satisfied:

1) \( q(t) \geq 0, \quad t \geq t_0; \)

2) for every \( T > t_0 \) there exist \( T \leq s_1 < t_1 \leq s_2 < t_2 \) and \( \lambda \in \mathbb{R} \) such that

3.1) \( (-1)^k \alpha_\lambda(t) \geq 0, \quad t \in [s_k, t_k], \quad k = 1, 2; \)

3.2) \( (-1)^k [r(t) \alpha_\lambda(t) + g(t)] \geq 0, \quad t \in [s_k, t_k], \quad k = 1, 2; \)

3.3) the system (1.1) is oscillatory on the intervals \([s_k, t_k], \quad k = 1, 2.\)

Then the system (1.1) is oscillatory.

Proof. Suppose the system (1.1) is not oscillatory. Then there exists a solution \((\phi(t), \psi(t))\) such that \( \phi(t) \neq 0, \quad t \geq T, \) for some \( T \geq t_0. \) We set: \( k = 1 \) if \( \phi(t) > 0, \quad t \geq T \)

and \( k = 2 \) if \( \phi(t) < 0, \quad t \geq T. \) Let \( \phi_1(t) \equiv \phi(t) - \alpha_\lambda(t), \quad t \geq t_0. \) Then by (2.3) \((\phi_1(t), \psi(t))\)

is a solution of the system (2.4) and by 3.1) we have \( \phi_1(t) \neq 0, \quad t \in [s_k, t_k + \varepsilon] \) for some \( \varepsilon > 0, \) so by virtue of (2.5)

\[ \phi_1(t) = \phi_1(s_1) \exp \left\{ \int_{s_k}^{t} [p(\tau) + q(\tau)y(\tau)] d\tau \right\}, \quad t \in [s_k, t_k], \]

where \( y(t) \) is a solution of the Riccati equation

\[ y' + q(t)y^2 + E(t)y - r(t) - \frac{r(t)\alpha_\lambda(t) + g(t)}{\phi_1(t)} = 0, \quad t \in [s_k, t_k + \varepsilon) \quad (3.8) \]

with \( y(s_k) = \frac{\psi(s_k)}{\phi_1(s_k)} \). Consider the Riccati equation

\[ y' + q(t)y^2 + E(t)y - r(t) = 0, \quad t \in [s_k, t_k + \varepsilon) \quad (3.9) \]

From 3.2) it follows that

\[ \frac{r(t)\alpha_\lambda(t) + g(t)}{\phi_1(t)} \leq 0, \quad t \in [s_k, t_k]. \]
Then applying Theorem 2.1 to the pair of equations (3.8) and (3.9) we conclude that Eq. (3.9) has a solution \( y_1(t) \) on \([s_k, t_k]\) with \( y_1(s_1) \geq y(s_1) \) and \( y_1(t) \geq y(t), \ t \in [s_k, t_k] \). Hence according to (2.5) the pair of functions
\[
\phi_2(t) \equiv \exp\left\{ \int_{t_0}^{t} [p(\tau) + q(\tau)y(\tau)]d\tau \right\}, \quad \psi_2(t) \equiv y_1(t)\phi_2(t), \ t \in [s_k, t_k]
\]
form a solution \((\phi_2(t), \psi_2(t))\) of the system (1.2) on \([s_k, t_k]\) and is continuable on \([s_k, t_k]\) as a solution of the system (1.2), for which \( \phi_2(t) \neq 0, \ t \in [s_k, t_k] \). Therefore the system (1.2) is not oscillatory on \([s_k, t_k]\), which contradicts the condition 3.3). The obtained contradiction completes the proof of the theorem.

**Remark 3.2.** An explicit interval oscillatory criterion for the system (1.2) is obtained in [15] (see also [16]). An interval oscillatory criteria for unforced Eq. (1.3) is obtained in [17].

One can easily show (using the same way of proof of Theorem 1.1) that under the conditions of Theorem 1.1 the equation
\[
(a(t)\phi')' + c(t)\phi = 0, \quad t \geq t_0 \tag{3.10}
\]
is oscillatory on the intervals \([s_k, t_k], \ k = 1, 2\) for \( s_k, t_k, \ k = 1, 2 \) indicated in the conditions of Theorem 1.1. on the other hand in virtue of the connection (1.4) between Eq. (1.3) and the system (1.5) the conditions of Theorem 3.2 for the system (1.5) in the particular case \( \lambda = 0 \) can be reduced (equivalently translated) for undamped Eq. (1.3) to the following ones.

For every \( T \geq t_0 \) there exist \( T \leq s_1 < t_1 \leq s_2 < t_2 \) such that
\[
d(t) \begin{cases} 
\leq 0, & t \in [s_1, t_1], \\
\geq 0, & t \in [s_2, t_2]; 
\end{cases}
\]
the equation (3.10) is oscillatory on the intervals \([s_k, t_k], \ k = 1, 2\).

Therefore Theorem 3.2 is a generalization of Theorem 1.1, moreover the condition 3.3) of Theorem 3.2 is preferable to the condition (1.6), since each explicit oscillatory criterion for the system (1.2) generates according to Theorem 3.2 an explicit oscillatory criterion, whereas (1.6) does not have this property (since no explicit conditions (or an explicit condition) on the coefficients of Eq. (3.10) that ensure (1.6) have been specified yet).
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