Classical Velocity in $\kappa$–deformed Poincare Algebra and a Maximum Acceleration

S. Kalyana Rama

Institute of Mathematical Sciences, C. I. T. Campus,
Taramani, CHENNAI 600 113, India.
email: krama@imsc.ernet.in

ABSTRACT

We study the commutators of the $\kappa$–deformed Poincare Algebra ($\kappa$PA) in an arbitrary basis. It is known that the two recently studied doubly special relativity theories correspond to different choices of $\kappa$PA bases. We present another such example. We consider the classical limit of $\kappa$PA and calculate particle velocity in an arbitrary basis. It has standard properties and its expression takes a simple form in terms of the variables in the Snyder basis. We then study the particle trajectory explicitly for the case of a constant force. Assuming that the spacetime continuum, velocity, acceleration, etc. can be defined only at length scales greater than $x_{\text{min}} \neq 0$, we show that the acceleration has a finite maximum.
1. Recently, a Lorentz invariant length scale has been incorporated in the so called doubly special relativity (DSR) theories. It was originally formulated in the theory, dubbed DSR1, of [1]. A second version, dubbed DSR2, has also been proposed recently [2]. Soon after, it has been found in [3, 4] that they both can be incorporated within the framework of $\kappa$–deformed Poincare algebra ($\kappa$PA) [5], and correspond to different choices of $\kappa$PA bases. Lorentz transformation and addition properties of the energy momentum vectors have also been found [6], and incorporated within the $\kappa$PA framework [7].

As shown in [3], the commutators of the $\kappa$PA can be expressed succinctly in the Snyder basis [8]. The energy, momentum, and mass in this basis can be related to those in another basis by three arbitrary functions in general. Lorentz transformation and addition properties of the energy momentum vectors in these bases are then related. For details, see [3, 4, 6, 7].

Many DSR theories can be constructed within the $\kappa$PA framework, characterised by three functions. In the present paper, we exhibit one such example by making a particular choice of these functions, chosen to reproduce the commutators in the position momentum sector studied in [4, 10, 11] in the context of generalised uncertainty principle.

There are many possible ways to define the velocity in these theories [12]. Following [12], we study the classical limit of $\kappa$PA, where the commutators are replaced by Poisson brackets. Velocity, acceleration, etc. can then be defined naturally. We compute the velocity in any basis, characterised by the three arbitrary functions. Its expression takes a simple form in terms of the variables in the Snyder basis which also shows that the maximum speed, as also the speed of a massless particle, is given by the speed of light in vacuum. With a natural assumption regarding the Lorentz transformation of energy and momentum, the velocities can be shown to obey the standard relativistic addition law.

We also study the motion of a particle under the action of a potential obtaining, in particular, the particle trajectory explicitly for the case of a constant force. Assuming that the spacetime continuum, velocity, acceleration, etc. can be defined only at length scales greater than $x_{\text{min}} \neq 0$, we show that the acceleration has a finite maximum. Our arguments can be extended straightforwardly, leading to the same conclusion, in any theory where the above assumption is valid. An upper bound on acceleration has also been obtained previously in other theories [13, 14] and studied further in [15].

The plan of the paper is as follows. In section 2, we present the commu-
tators of the \( \kappa \)PA in a general basis, the choices of functions leading to DSR1 and DSR2, and another example of DSR. In section 3, we study the classical limit and obtain the velocity. In section 4, we present the particle motion under the action of a constant force and obtain an upper bound on the acceleration. We conclude in section 5 with a brief summary and a mention of a few issues for further studies.

2. Let \( X^\mu \) and \( K_\mu, \mu = (0, i), \) \( i = 1, 2, \ldots, d, \) denote the position and momentum operators, and \( M_{\mu\nu}, \) explicitly \( M_0 \equiv N_i \) and \( M_{ij}, \) denote the boost and angular momentum generators in \((d + 1)\)-dimensional spacetime. Also, let \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots), \) and \( \bar{h} = c = 1. \) Then, in the Snyder basis [8] for the \( \kappa \)-deformed Poincare Algebra (\( \kappa \)PA) the commutators of the above operators can be written succinctly [8] as \([K_\mu, K_\nu] = 0\) and

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\sigma}) \\
[M_{\mu\nu}, V_\sigma] = i (\eta_{\mu\sigma}V_\nu - \eta_{\nu\sigma}V_\mu), \quad V_\sigma = X_\sigma, \quad K_\sigma \\
[X_\mu, K_\nu] = i (\eta_{\mu\nu} + \alpha K_\mu K_\nu), \quad [X_\mu, X_\nu] = i \alpha M_{\mu\nu}
\]

where \( \alpha = \lambda^2 = \kappa^{-2} \) is the deformation parameter, with dimension \((\text{length})^2\). The Casimir \( C_s \) in the Snyder basis, which commutes with \( N_i, \) is given by

\[
C_s = K_0^2 - K^2 \quad \text{where} \quad K^2 = \sum_{i=1}^{d} K_i^2.
\] (2)

When \( \alpha \neq 0, \) there is no clear physical criterion to identify the physical momentum operator, \( P_\mu. \) Various definitions of \( P_\mu(K_0, K_i) \) have been employed, which amount to choosing different bases for the \( \kappa \)PA [3, 4]. Thus, let

\[
P_0 = h(K_0, K), \quad P_i = g(K_0, K)K_i
\]

(3)

where \( h \to K_0 \) and \( g \to 1 \) in the limit \( \alpha \to 0, \) and the above relations can be inverted to obtain \( K_0(P_0, P_i) \) and \( K_i(P_0, P_i). \) It is then straightforward to obtain the new commutators. Thus, the commutators involving \( P_\mu \) become \([M_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \) \([M_{ij}, P_0] = 0, \) and

\[
[N_i, P_j] = i \left( g\delta_{ij} + \left( \frac{g_{K_0}}{K_0} + \frac{g_K}{K} \right) K_iK_j \right) K_0
\]
\[ [X_i, P_j] = i \left( g \delta_{ij} + \left( \frac{g}{K} + \alpha (g + K_0 g_{K_0} + K g_K) \right) K_i K_j \right) \]
\[ [X_0, P_i] = i \left( -\frac{g_{K_0}}{K_0} + \alpha (g + K_0 g_{K_0} + K g_K) \right) K_0 K_i \]
\[ [N_i, P_0] = i \left( \frac{h_{K_0}}{K_0} + \frac{h_K}{K} \right) K_0 K_i \]
\[ [X_i, P_0] = i \left( \frac{h_{K_0}}{K_0} + \alpha (K_0 h_{K_0} + K h_K) \right) K_i \]
\[ [X_0, P_0] = i \left( -\frac{h_{K_0}}{K_0} + \alpha (K_0 h_{K_0} + K h_K) \right) K_0 \]

(4)

where, here and in the following, a subscript denotes partial differentiation with respect to it. The commutators in (4) are obtained by setting \( g = 1 \) and \( h = K_0 \). It is also easy to verify that the Casimir \( C(P_0, P_i) \) in the new basis, which commutes with \( N_i \), is given by

\[ \alpha C = \mathcal{F}(\alpha(K_0^2 - K^2)) \quad \text{with} \quad \mathcal{F}(0) = 0 \quad \text{and} \quad \mathcal{F}'(0) = 1 \]

(5)

where \( \mathcal{F}' \) is the derivative of \( \mathcal{F} \) with respect to its argument. The restriction on \( \mathcal{F} \) and \( \mathcal{F}' \) ensures that the correct Casimir is obtained in the limit \( \alpha \to 0 \). The function \( \mathcal{F} \) is otherwise arbitrary. In equations (4) and (5), \( K_0 \) and \( K_i \) are to be expressed in terms of \( P_0 \) and \( P_i \). The mass shell condition \( C = m^2 \) gives the Hamiltonian \( H = P_0 \), equivalently energy \( E \), as a function of \( P \), \( m \), and \( \alpha \), and also that \( \alpha m^2 = \mathcal{F}(\alpha m_0^2) \) where \( C_s = m_0^2 \) in the Snyder basis. Thus, the function \( \mathcal{F} \) defines \( m \) in terms of \( m_s \).

Under a Lorentz transformation, let \( K_\mu \) transform in the standard way i.e. \( K_\mu \to K'_\mu = \Lambda^\mu_\nu K_\nu \). Then, the transformation \( P_\mu \to P'_\mu \) can be taken to be given by

\[ P'_0 = h(K'_0, K'_i), \quad P'_i = g(K'_0, K'_i) K'_i. \]

(6)

Similarly, let the addition law for \( K_\mu \) be the standard one, i.e. \( K'_\mu = \sum_a K'^{(a)}_\mu \). Then, the addition law for \( P_\mu \) can be taken to be given by \( P'_\mu = P'_\mu \) with \( P'_\mu \) given by (6) [3, 4, 6, 12].

In the above formulation, with \( \alpha = \lambda^2 \), the DSR theories [1, 2] correspond to different choices of the functions \( g \), \( h \), and \( \mathcal{F} \). Thus, for DSR1, \( P_\mu \), \( K_\mu \), the Casimir \( C \), and the function \( \mathcal{F} \) in (5) are given by

\[ e^{\lambda P_0} = \lambda K_0 + \sqrt{1 + \lambda^2(K_0^2 - K^2)}, \quad P_i = K_i e^{-\lambda P_0} \]
\[ \lambda K_0 = \text{Sinh} \lambda P_0 + \frac{\lambda^2 P^2}{2} e^{\lambda P_0}, \quad K_i = P_i e^{\lambda P_0} \]

\[ \lambda^2 C = 2(\sqrt{1 + \lambda^2(K_0^2 - K^2)} - 1) = (2 \text{Sinh} \frac{\lambda P_0}{2})^2 - \lambda^2 P^2 e^{\lambda P_0} \]  \hspace{1cm} (7)

where \( P^2 = \sum P_i^2 \). For DSR2, they are given by \( \frac{P_i}{P_0} = \frac{K_i}{K_0} \) and

\[ P_0 = \frac{K_0}{\lambda K_0 + \sqrt{1 + \lambda^2(K_0^2 - K^2)}} \], \quad K_0 = \frac{P_0}{\sqrt{1 - 2\lambda P_0 + \lambda^2 P^2}} \]

\[ \lambda^2 C = \frac{\lambda^2(K_0^2 - K^2)}{1 + \lambda^2(K_0^2 - K^2)} = \frac{\lambda^2(P_0^2 - P^2)}{(1 - \lambda P_0)^2}. \]  \hspace{1cm} (8)

The Lorentz transformation and the addition laws for \( P_\mu \) follow from (3).

See [1, 2, 3, 4, 6] for more details.

Consider the \([X_i, K_j]\) commutators, of the form

\[ [X_i, K_j] = i (A(K) \delta_{ij} + B(K) K_i K_j) \], \hspace{1cm} (9)

which appear in the context of generalised uncertainty principle, studied in [9, 10, 11, 16]. Let \( P_i = g(K) K_i \). One then gets

\[ [X_i, K_j] = i (\tilde{A} \delta_{ij} + \tilde{B} K_i K_j) \]  \hspace{1cm} (10)

where \( \tilde{A} \) and \( \tilde{B} \) are given by

\[ \tilde{A} = g A, \quad P^2 \tilde{B} = AKg + BK^2(g + Kg). \]  \hspace{1cm} (11)

For \( A = 1 \) and \( B = 0 \), for example, one has \( \tilde{A} = g \) and \( \tilde{B} = \frac{2gg^2}{g^2 - 2P^2g^2} \), which is the case studied in [16]. Let \( A = 1 \) and \( B = \alpha \), which corresponds to the \([X_i, K_j]\) commutators in (1) in the Snyder basis. Furthermore, let \( g \) be chosen such that \( \tilde{B} = 0 \). It then follows that

\[ \tilde{A} = g, \quad g^2 = \frac{P_0^2}{K_0^2} = \frac{a^2}{1 + \alpha K^2} = a^2 - \alpha P^2 \]  \hspace{1cm} (12)

where \( a \) is an integration constant. For \( a^2 = 1 - \alpha m^2 \) and \( \alpha = \pm \lambda^2 \), this is the case studied in [3, 4, 5].

A DSR theory can be constructed within the framework of \( \kappa \)PA by choosing functions \( g, h, \) and \( F \) in equations (3) and (4). This was illustrated above
for DSR1 and DSR2. We now present another example. Let \( h = gK_0 \) and the function \( g \) be given by equation (12) with \( a = 1 \) and \( \alpha = \lambda^2 \). Then, one obtains \( \frac{P_0}{P_0} = \frac{K_0}{K_0} \) and

\[
P_0 = \frac{K_0}{\sqrt{1 + \lambda^2 P_0^2}}, \quad K_0 = \frac{P_0}{\sqrt{1 - \lambda^2 P_0^2}}.
\]

The Casimir depends on the choice of \( \mathcal{F} \). For example,

\[
\lambda^2 C = \lambda^2 (K_0^2 - K^2) = \frac{\lambda^2 (P_0^2 - P^2)}{1 - \lambda^2 P^2}.
\]

The Lorentz transformation and the addition laws for \( P_\mu \) follow from (6).

Note that, when \( \lambda = 0 \), the Casimir \( C \), and thus the mass shell condition \( C = m^2 \), is symmetric under \( P_\mu \leftrightarrow -P_\mu \). However, when \( \lambda \neq 0 \), this symmetry is absent for DSR1 and DSR2 (see equations (7) and (8)) but is present for the example presented above (see equations (14)).

3. The mass shell condition \( C = m^2 \) gives the Hamiltonian \( H(P) = P_0 \) as a function of \( P, m, \) and \( \alpha \). The corresponding eigenvalues then give the energy \( E(p) \). Now, there are many possible ways to define the velocity. For example, \( v_i \) can be defined as \( \frac{\partial E}{\partial p_i} \), or as the eigenvalues of the operator \( i[H, X_i] \), or, in \( \kappa \)PA, as the left— or right— covariant velocities [12]. See [17] also.

In the classical limit the commutators are replaced by the Poisson brackets. Given a Hamiltonian function \( \mathcal{H}(Y_a) \), where \( Y_a = (x_\mu, p_\mu) \), the evolution of an observable \( O \) with respect to an evolution parameter \( s \) is given by

\[
\frac{dO}{ds} = [O, Y_a]_{PB} \frac{\partial \mathcal{H}}{\partial Y_a}.
\]

Then, velocity \( v_i \) and acceleration \( a_i \) can be defined naturally to be given by

\[
v_i = \frac{dx_i/ds}{dx_0/ds}, \quad a_i = \frac{dv_i/ds}{dx_0/ds}.
\]

The interpretation of \( s \) depends on the choice of \( \mathcal{H} \). For example, if

\[
\mathcal{H} = p_0 - E(p) ,
\]
then, in the limit $\alpha \to 0$, the parameter $s$ can be identified with time \cite{12}.

It has been shown in \cite{10}, for the case of DSR1, that the velocity given by (16) is the same as the right−covariant velocity in the bicross product basis of the $\kappa$PA. Also, the maximum speed $v_{\text{max}} = 1$ and the velocities satisfy the standard relativistic addition law.

In the following, we take the velocity $v_i$ and the Hamiltonian function $\mathcal{H}$ to be given by (16) and (17), and compute $v_i$ for the general case specified by the arbitrary functions $g(k_0, k)$, $h(k_0, k)$, and $\mathcal{F}(\alpha(k_0^2 - k^2))$, defined in equations (3) and (5).

Using the Poisson brackets obtained from the commutators (4), and after some algebra, the velocity $v_i$ given by (16) can be written explicitly as

$$v_i = \frac{k_i}{k_0} \left( \frac{\mathcal{A}}{\mathcal{A} - \mathcal{B}} \right)$$

where

$$\mathcal{A} = \frac{E_p}{k} \left( g + k g_k + \alpha k^2 (g + k_0 g_{k_0} + k g_k) \right) - \left( \frac{h_k}{k} + \alpha (k_0 h_{k_0} + k h_k) \right)$$

$$\mathcal{B} = \frac{h_{k_0}}{k_0} + \frac{h_k}{k} - \frac{E_p}{k} \left( g + k g_k + \frac{k^2 g_{k_0}}{k_0} \right),$$

and $k_0$, $k$, $\mathcal{A}$, $\mathcal{B}$, etc. are all functions of $p_0$ and $p$. Now, given the function $\mathcal{F}$, the Casimir given by (3) can be written as

$$F^2(p_0, p) - G^2(p_0, p) p^2 \equiv \mathcal{F}(\alpha(k_0^2 - k^2)),$$

the precise form of $F$ and $G$ depending on the choice of functions $g$, $h$, and $\mathcal{F}$. Using $p_0 = h(k_0, k)$ and performing the operation $\left( \frac{1}{k_0} \frac{\partial}{\partial k_0} + \frac{1}{k} \frac{\partial}{\partial k} \right)$ on both sides of equation (20) gives

$$(F^2 - G^2 p^2)_h \left( \frac{h_{k_0}}{k_0} + \frac{h_k}{k} \right) + (F^2 - G^2 p^2)_p \left( \frac{p_{k_0}}{k_0} + \frac{p_k}{k} \right) = 0.$$

The energy $E(p) = p_0 = h$ is obtained from the mass shell condition $\mathcal{C} = m^2$.

Thus, $F^2(E, p) - G^2(E, p) p^2 = \alpha m^2$, differentiating which gives

$$(F^2 - G^2 p^2)_h E_p + (F^2 - G^2 p^2)_p = 0.$$
Furthermore, using \( p_i = g(k_0, k)k_i \) one obtains \( p = kg(k_0, k) \) and, hence,

\[
\frac{p_{k_0}}{k_0} + \frac{p_k}{k} = \frac{1}{k} \left( g + kg_k + \frac{k^2g_{k_0}}{k_0} \right).
\]

(23)

Equation (21) can now be written, using equations (22) and (23), as

\[
(F^2 - G^2p^2)_B = 0
\]

(24)

where \( B \) is given in (19). Since \( F \) and \( G \) are arbitrary functions, it follows that \( B = 0 \). Then, equation (18) becomes simply

\[
v_i = \frac{k_i(p_0, p)}{k_0(p_0, p)}
\]

(25)

which is valid for the general case specified by the arbitrary functions \( g(k_0, k) \), \( h(k_0, k) \), and \( F(\alpha(k_0^2 - k^2)) \). Note that \( k_0 \) and \( k_i \) are the momentum variables in the Snyder basis, whereas \( p_0 \) and \( p_i \) are those in a particular basis under consideration. Thus, using equations (7), (8), and (13), \( v_i \) for the case of DSR1 [12, 17], DSR2 [2], and the present example are given by

\[
v_i|_{DSR1} = \frac{2\lambda p_i}{1 - e^{-2\lambda p_0} + \lambda^2 p^2}, \quad v_i|_{DSR2} = v_i|_{Example} = \frac{p_i}{p_0}.
\]

(26)

Equation (25) implies that the maximum speed \( v_{max} = 1 \), as is the speed of a massless particle.\(^1\) Also, using equation (6) and following [12], it is straightforward to prove that the velocities defined above satisfy the standard relativistic addition law for any choice of functions \( g, h, \) and \( F \).

4. We now study the motion of a particle under the action of a potential in the classical limit of the \( \kappa \)PA, with the Hamiltonian function \( \mathcal{H}(x, p) \) given by

\[
\mathcal{H} = p_0 - E(p) - V(x)
\]

(27)

where the potential \( V(x) \) is a function of \( x^i \) only. For this purpose, it is convenient to work in the bicross product basis. With \( \alpha = \lambda^2 \), the relevant

\(^1\)However, if \( v_i \) is given, for example, by \( \frac{\partial E}{\partial p_i} \) or the eigenvalues of the operator \( i[H, X_i] \) then \( v_{max} \neq 1 \) in general. In fact, the functions \( g \) and \( h \) can be chosen such that \( v_{max} > 1 \).
Poisson brackets in this basis are given by

\[ [p_i, x_j]_{PB} = \delta_{ij}, \quad [p_0, x_i]_{PB} = 0, \quad [x_0, x_i]_{PB} = \lambda x_i \]
\[ [p_i, x_0]_{PB} = \lambda p_i, \quad [p_0, x_0]_{PB} = -1, \quad [p_0, p_\nu]_{PB} = 0. \] (28)

The evolution of an observable \( O(s) \) is given by equation (15) where \( s \) is an evolution parameter which, in the limit \( \lambda \to 0 \), can be identified with time. Equation (15) determines, in particular, that the evolution of \( p_i(s), x_i(s), \) and \( x_0(s) \) is given by the differential equations

\[ \frac{dp_i}{ds} = f_i(x), \quad \frac{dx_i}{ds} = u_i(p), \quad \frac{dx_0}{ds} = D(x, p) \] (29)

where we have defined

\[ f_i = -\frac{\partial V(x)}{\partial x_i}, \quad u_i = \frac{\partial E(p)}{\partial p_i}, \quad D = 1 + \lambda (p_j u_j + x_j f_j). \]

Note that \( f_i \) is the \( i \)th component of the force due to the potential \( V(x) \) and, in the limit \( \lambda \to 0 \), \( u_i \) is the \( i \)th component of the velocity. Also, it follows that

\[ \frac{d}{ds} (E(p) + V(x)) = 0 \iff E(p) + V(x) = \text{constant}. \] (30)

The velocity \( v_i \) and acceleration \( a_i \), defined in (16), are now given by

\[ v_i = \frac{u_i}{D}, \quad a_i = \frac{f_i}{D^2} \left( \frac{\partial u_i}{\partial p_j} \right) - \frac{\lambda u_i f_j}{D^3} \left( 2u_j + p_k \frac{\partial u_k}{\partial p_j} \right) - \frac{\lambda u_i u_j x_k}{D^3} \left( \frac{\partial f_k}{\partial x_j} \right). \] (32)

Solutions to equations (29) will describe the motion of a particle under the action of a potential \( V(x) \) in the classical limit of the \( \kappa \)PA. Note that, generically, there are two arbitrary functions, namely \( E(p) \) and \( V(x) \), and solutions to equations (29) are difficult to obtain in the general case.

Let \( V(x) = -fx_1 \) with \( f = \text{constant} \), which corresponds to a constant force, chosen to be along \( x_1 \) with no loss of generality. Let \( x_0 = x_i = p_i = 0 \) at \( s = 0 \). Then, it follows that \( x_i = \delta_{i1} x \) and \( p_i = \delta_{i1} p \). Equations (29) can be solved easily, with the resulting solution given simply by

\[ p = sf, \quad x = \frac{E(p) - E(0)}{f}, \quad x_0 = s (1 + xf), \] (33)
which is valid for any function $E(p)$. Note that equation (30) is also satisfied.

To obtain a complete solution, $E(p)$ must be specified. As an example, consider the case of DSR1 with the Casimir $C$, equivalently the function $F$, given in (7). The energy $E(p)$, obtained by setting $C = m^2$ with $p_0 = E(p)$, is given by

$$e^{-\lambda E} = b - \sqrt{b^2 - (1 - \lambda^2 p^2)}$$

where $b = 1 + \frac{\lambda^2 m^2}{2}$. Equations (33) now determine $p(s)$, $x(s)$, and $x_0(s)$ completely.

As follows from equations (1) and (28), the spacetime is noncommutative at length scales smaller than $O(1) \lambda$. Therefore, spacetime continuum, velocity, acceleration etc. can be defined only at length scales greater than $x_{\min} = O(1) \lambda$. Hence, solutions (33) are also applicable only for $x \geq x_{\min}$.

Consider a particle with mass $m$ such that $\lambda m \ll 1$. Its motion under a constant force is given by equations (33), taken to be valid for $x \geq x_{\min}$ only; equivalently, in the limit $\lambda m \ll 1$, for $s \geq s_{\min} = \left(\frac{2m}{f} x_{\min}\right)^\frac{1}{2}$ only. The acceleration at $s_{\min}$ is given by

$$a_{\text{cont}}(s_{\min}) = \frac{m^2 f}{(m^2 + 2mf x_{\min})^\frac{3}{2}} (1 + \cdots), \quad (34)$$

where $\cdots$ represents terms which are negligible in the limit $\lambda m \ll 1$, and $a_{\text{cont}}(s)$, $s \geq s_{\min}$, is the acceleration of the particle under a force $f$, which can be defined and measured in the conventional way using the concept of spacetime continuum.

Consider $a_{\text{cont}}$ as a function of the applied force $f$. If $\lambda = 0$ then $x_{\min} = 0$ and the maximum value of the acceleration $a_{\text{cont}}$ is infinite, achieved when $f \to \infty$. If $\lambda \neq 0$ then $a_{\text{cont}}$ has a finite maximum, $a_{\text{max}}$, achieved when $f = f_s = \frac{m}{x_{\min}}$. Upto numerical coefficients of $O(1)$, $a_{\text{max}}$ is given by

$$a_{\text{cont}} \leq a_{\text{max}} \simeq \frac{1}{x_{\min}} \simeq \frac{1}{\lambda}, \quad (35)$$

where we have used $x_{\min} = O(1) \lambda$. The above expression remains valid also when $\lambda m$ is not negligible, but with different numerical coefficients.

Note that for $f = f_s$, $p \simeq E \simeq O(1) m$ at $s = s_{\min}$. This suggests that when $f > f_s$, the excess force is likely to create more particle–antiparticle
pairs, and not increase the particle acceleration beyond $a_{\text{max}}$ (cf. footnote 2 below).

Consider a composite of $N$ particles, of average mass $m$ with $\lambda m \ll 1$. Then $m_{\text{total}} = Nm$. For such a composite, it has been proposed that $\lambda$ must be replaced by $\lambda_{\text{eff}} = \frac{\lambda}{N}$ [2]. We assume this to be the case. Since one must still have $x > x_{\text{min}} = O(1)\lambda$ in (33), and since $\lambda_{\text{eff}}m_{\text{total}} = \lambda m \ll 1$, it follows for a composite object also that the acceleration $a_{\text{cont}}$ must obey the bound given in (35).

Our arguments can be extended straightforwardly to show that an upper bound on the acceleration exists, and is of the form given in (35), in any theory where there is a non zero length scale only above which one can define the spacetime continuum, velocity, acceleration, etc.

Note that an upper bound on acceleration has also been obtained previously in other theories [13] and studied further in [13]. It also follows, through Unruh effect, in theories with a limiting temperature such as string theory with Hagedorn temperature $T$ [14]. It will be interesting to study whether conversely an upper bound on acceleration, such as that in the present work, implies a limiting temperature.

5. We now summarise briefly the present work and mention a few issues for further studies. We consider arbitrary bases for the $\kappa\text{PA}$, characterised by three functions which redefine energy, momentum, and mass. DSR1 and DSR2 theories are obtained by particular choices of these functions, and another example of DSR is presented.

We study the Hamiltonian evolution in the classical limit, defining velocity $v_i$, acceleration $a_i$, etc. following [12]. We find that $v_i$ thus defined is given simply by equation (25). Also, with a natural assumption regarding the Lorentz transformation of momentum and energy (6), the velocities obey the standard relativistic addition law.

We also study the motion of a particle under the action of a potential obtaining, in particular, the particle trajectory explicitly for the case of a constant force. Assuming that the spacetime continuum, velocity, acceleration, etc. can be defined only at length scales greater than $x_{\text{min}} \neq 0$, we then show that the acceleration has a finite maximum given in (35).

\footnote{In string theory, excess energy pumped in to increase the temperature goes, instead, into increasing the length of one ‘long’ string.}
We mention a few issues for further studies. For $\alpha = \lambda^2$, the theory can be viewed as that with a de Sitter momentum space [7]. It may be of interest to study the case $\alpha = -\lambda^2$ which is likely to be a theory with anti de Sitter momentum space (see [11]).

An important issue is to understand the physical significance of different bases, equivalently whether any particular basis is preferred physically. It is also equally important to understand how to describe a composite of a collection of particles within the framework of $\kappa$PA. See [18] for a recent proposal for such a description.

Acknowledgement: We thank G. Date for a discussion and G. Lambiase for bringing [15] to our attention.

References

[1] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35, gr-qc/0012051; Phys. Lett. B 510 (2001) 255, hep-th/0012238; Nature 418 (2002) 34, gr-qc/0207049. See also N. R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, Phys. Lett. B 522 (2001) 133, hep-th/0107039; G. Amelino-Camelia, D. Benedetti, and F. D’Andrea, hep-th/0201245.

[2] J. Magueijo and L. Smolin Phys. Rev. Lett. 88 (2002) 190403, hep-th/0112090.

[3] J. Kowalski-Glikman and S. Nowak, Phys. Lett. B 539 (2002) 126, hep-th/0203040; hep-th/0204245.

[4] J. Lukierski and A. Nowicki, to appear in Int. J. Mod. Phys. A, hep-th/0203063.

[5] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, Phys. Lett. B 264 (1991) 331; S. Majid and H. Ruegg, Phys. Lett. B 334 (1994) 348, hep-th/9405107. P. Kosinski, J. Lukierski, P. Maślanka, and J.Sobczyk, J. Phys. A 28 (1995) 2255, hep-th/9411115. P. Kosinski, J. Lukierski, and P. Maślanka, Phys. Rev. D 62 (2000) 025004, hep-th/9902037. J. Lukierski, H. Ruegg, and V. N. Tolstoy, in the Proceedings of XXX
Karpacz School, 1994, *Quantum Groups: Formalism and Applications*, Eds. J. Lukierski, Z. Popowicz, and J. Sobczyk, Polish Scientific Publishers PWN (1995); J. Lukierski, H. Ruegg, and W. J. Zakrzewski, Ann. Phys. 243 (1995) 90, hep-th/9312153; J. Lukierski and A. Nowicki, Proceedings of Quantum Group Symposium July 1996, Goslar, Eds. H. -D. Doebner and V. K. Dobrev, Heron Press, Sofia (1997).

[6] S. Judes and M. Visser, gr-qc/0205067
[7] J. Kowalski-Glikman, hep-th/0207279
[8] H. S. Snyder, Phys. Rev. 71 (1947) 38.
[9] M. Maggiore, Phys. Lett. B 304 (1993) 65, hep-th/9301067; Phys. Rev. D 49 (1994) 5182, hep-th/9305163; Phys. Lett. B 319 (1993) 83, hep-th/9309034.
[10] S. Kalyana Rama, Phys. Lett. B 519 (2001) 103, hep-th/0107255; Phys. Lett. B 539 (2002) 289, hep-th/0204215.
[11] M. Maggiore, hep-th/0205014.
[12] J. Lukierski and A. Nowicki, to appear in special issue of Acta Physica Polonica B. hep-th/0207022.
[13] E. Caianeillo, Lett. Nuovo Cimento 32 (1981) 65; H. E. Brandt, Found. Phys. Lett. 2 (1989) 39, 405, *ibid.* 6 (1991) 523; C. Castro, hep-th/0208138 and references therein.
[14] H. E. Brandt, Lett. Nuovo Cimento 38 (1983) 522, *ibid.* 39 (1984) 192 (Erratum); R. Parentani and R. Potting, Phys. Rev. Lett. 63 (1989) 945; M. J. Bowick and S. B. Giddings, Nucl. Phys. B 325 (1989) 631.
[15] A. Feoli *et al*, Phys. Rev. D 60 (1999) 065001, hep-th/9812130; S. Capozziello, G. Lambiase, and G. Scarpetta, Int. J. Theor. Phys. 39 (2000) 15, gr-qc/9910017; V. Bozza *et al*, Phys. Lett. A 279 (2001) 163, hep-ph/0012270; V. Bozza *et al*, Phys. Lett. A 283 (2001) 53, gr-qc/0104058; V. Bozza *et al*, Int. J. Theor. Phys. 40 (2001) 849, hep-ph/0106234; G. Papini *et al*, Phys. Lett. A 300 (2002) 603, gr-qc/0208013. See also the references therein for an extensive work on the consequences of maximum acceleration.
[16] A. Kempf, J. Phys. A 30 (1997) 2093.

[17] T. Tamaki, T. Harada, and U. Miyamoto, gr-qc/0208002.

[18] J. Magueijo and L. Smolin gr-qc/0207085.