TENSOR FIELDS OF MIXED YOUNG SYMMETRY TYPE AND N-COMPLEXES

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Abstract
We construct $N$-complexes of non completely antisymmetric irreducible tensor fields on $\mathbb{R}^D$ which generalize the usual complex ($N = 2$) of differential forms. Although, for $N \geq 3$, the generalized cohomology of these $N$-complexes is non trivial, we prove a generalization of the Poincaré lemma. To that end we use a technique reminiscent of the Green ansatz for parastatistics. Several results which appeared in various contexts are shown to be particular cases of this generalized Poincaré lemma. We furthermore identify the nontrivial part of the generalized cohomology. Many of the results presented here were announced in [10].

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1 Introduction

Our aim in this paper is to develop differential tools for irreducible tensor fields on $\mathbb{R}^D$ which generalize the calculus of differential forms. By an irreducible tensor field on $\mathbb{R}^D$, we here mean, a smooth mapping $x \mapsto T(x)$ of $\mathbb{R}^D$ into a vector space of (covariant) tensors of given Young symmetry. We recall that this implies that the representation of $GL_D$ in the corresponding space of tensors is irreducible.

Throughout the following $(x^\mu) = (x^1, \ldots, x^D)$ denotes the canonical co-ordinates of $\mathbb{R}^D$ and $\partial_\mu$ are the corresponding partial derivatives which we identify with the corresponding covariant derivatives associated to the canonical flat torsion-free linear connection $(0)\nabla$ of $\mathbb{R}^D$. Thus, for instance, if $T$ is a covariant tensor field of degree $p$ on $\mathbb{R}^D$ with components $T_{\mu_1 \ldots \mu_p}(x)$, then $(0)\nabla T$ denotes the covariant tensor field of degree $p + 1$ with components $\partial_{\mu_{p+1}} T_{\mu_1 \ldots \mu_p}(x)$. The operator $(0)\nabla$ is a first-order differential operator which increases by one the tensorial degree.

In this context, the space $\Omega(\mathbb{R}^D)$ of differential forms on $\mathbb{R}^D$ is the graded vector space of (covariant) antisymmetric tensor fields on $\mathbb{R}^D$ with graduation induced by the tensorial degree whereas the exterior differential $d$ is up to a sign the composition of the above $(0)\nabla$ with antisymmetrisation, i.e.

$$d = (-1)^p A_{p+1} \circ (0)\nabla : \Omega^p(\mathbb{R}^D) \to \Omega^{p+1}(\mathbb{R}^D)$$

where $A_p$ denotes the antisymmetrizer on tensors of degree $p$. The sign factor $(-1)^p$ arises because $d$ acts from the left, while we defined $(0)\nabla T_{\mu_1 \ldots \mu_{p+1}} = \partial_{\mu_{p+1}} T_{\mu_1 \ldots \mu_p}$. One has $d^2 = 0$ and the Poincaré lemma asserts that the cohomology of the complex $(\Omega(\mathbb{R}^D), d)$ is trivial, i.e. that one has $H^p(\Omega(\mathbb{R}^D)) = 0$. 

2
∀p ≥ 1 and $H^0(\Omega(\mathbb{R}^D)) = \mathbb{R}$ where $H(\Omega(\mathbb{R}^D)) = \text{Ker}(d)/\text{Im}(d) = \oplus_p H^p(\Omega(\mathbb{R}^D))$ with $H^p(\Omega(\mathbb{R}^D)) = \text{Ker}(d : \Omega^p(\mathbb{R}^D) \to \Omega^{p+1}(\mathbb{R}^D))/d(\Omega^{p-1}(\mathbb{R}^D))$.

From the point of view of Young symmetry, antisymmetric tensors correspond to Young diagrams (partitions) described by one column of cells, corresponding to the partition $(1^p)$, whereas $A_p$ is the associated Young symmetrizer, (see next section for definitions and conventions).

There is a relatively easy way to generalize the pair $(\Omega(\mathbb{R}^D), d)$ which we now describe. Let $(Y) = (Y_p)_{p \in \mathbb{N}}$ be a sequence of Young diagrams such that the number of cells of $Y_p$ is $p$, $\forall p \in \mathbb{N}$ (i.e. such that $Y_p$ is a partition of the integer $p$ for any $p$). We define $\Omega^p_{(Y)}(\mathbb{R}^D)$ to be the vector space of smooth covariant tensor fields of degree $p$ on $\mathbb{R}^D$ which have the Young symmetry type $Y_p$ and we let $\Omega_{(Y)}(\mathbb{R}^D)$ be the graded vector space $\bigoplus_p \Omega^p_{(Y)}(\mathbb{R}^D)$. We then generalize the exterior differential by setting

$$d = (-1)^p Y_{p+1} \circ \nabla : \Omega^p_{(Y)}(\mathbb{R}^D) \to \Omega^{p+1}_{(Y)}(\mathbb{R}^D)$$

(2)

where $Y_p$ is now the Young symmetrizer on tensor of degree $p$ associated to the Young symmetry $Y_p$. This $d$ is again a first order differential operator which is of degree one, (i.e. it increases the tensorial degree by one), but now, $d^2 \neq 0$ in general. Instead, one has the following result.

**Lemma 1** Let $N$ be an integer with $N \geq 2$ and assume that $(Y)$ is such that the number of columns of the Young diagram $Y_p$ is strictly smaller than $N$ (i.e. $\leq N - 1$) for any $p \in \mathbb{N}$. Then one has $d^N = 0$.

In fact the indices in one column are antisymmetrized (see below) and $d^N \omega$ involves necessarily at least two partial derivatives $\partial$ in one of the columns.
since there are $N$ partial derivatives involved and at most $N - 1$ columns.

Thus if $(Y)$ satisfies the condition of Lemma 1, the pair $(\Omega_{(Y)}(\mathbb{R}^D), d)$ is a $N$-complex (of cochains) [19], [6], [12], [20], [7], i.e. here a graded vector space equipped with an endomorphism $d$ of degree 1, its $N$-differential, satisfying $d^N = 0$. Concerning $N$-complexes, we shall use here the notations and the results of [7] which will be recalled when needed.

Notice that $\Omega^p_{(Y)}(\mathbb{R}^D) = 0$ if the first column of $Y_p$ contains more than $D$ cells and that therefore, if $Y$ satisfies the condition of Lemma 1, then $\Omega^p_{(Y)}(\mathbb{R}^D) = 0$ for $p > (N - 1)D$.

One can also define a graded bilinear product on $\Omega_{(Y)}(\mathbb{R}^D)$ by setting

$$(\alpha \beta)(x) = Y_{a+b}(\alpha(x) \otimes \beta(x))$$

for $\alpha \in \Omega^a_{(Y)}(\mathbb{R}^D), \beta \in \Omega^b_{(Y)}(\mathbb{R}^D)$ and $x \in \mathbb{R}^D$. This product is by construction bilinear with respect to the $C^\infty(\mathbb{R}^D)$-module structure of $\Omega_{(Y)}(\mathbb{R}^D)$ (i.e. with respect to multiplication by smooth functions). It is worth noticing here that one always has $\Omega^0_{(Y)}(\mathbb{R}^D) = C^\infty(\mathbb{R}^D)$.

In this paper we shall not stay at this level of generality; for each $N \geq 2$ we shall choose a maximal $(Y)$, denoted by $(Y^N) = (Y^N_p)_{p \in \mathbb{N}}$, satisfying the condition of lemma 1. The Young diagram with $p$ cells $Y^N_p$ is defined in the following manner: write the division of $p$ by $N-1$, i.e. write $p = (N-1)n_p + r_p$ where $n_p$ and $r_p$ are (the unique) integers with $0 \leq n_p$ and $0 \leq r_p \leq N - 2$ ($n_p$ is the quotient whereas $r_p$ is the remainder), and let $Y^N_p$ be the Young diagram with $n_p$ rows of $N - 1$ cells and the last row with $r_p$ cells (if $r_p \neq 0$). One has $Y^N_p = ((N-1)^{n_p}, r_p)$, that is we fill the rows maximally.
We shall denote $\Omega(Y_N)(\mathbb{R}^D)$ and $\Omega^p_{(Y_N)}(\mathbb{R}^D)$ by $\Omega_N(\mathbb{R}^D)$ and $\Omega^p_N(\mathbb{R}^D)$, respectively. It is clear that $(\Omega_2(\mathbb{R}^D), d)$ is the usual complex $(\Omega(\mathbb{R}^D), d)$ of differential forms on $\mathbb{R}^D$. The $N$-complex $(\Omega_N(\mathbb{R}^D), d)$ will be simply denoted by $\Omega_N(\mathbb{R}^D)$. We recall [7] that the (generalized) cohomology of the $N$-complex $\Omega_N(\mathbb{R}^D)$ is the family of graded vector spaces $H^k(\Omega_N(\mathbb{R}^D)) = \text{Ker}(d^k)/\text{Im}(d^{N-k})$, i.e.

$$H^p_k(\Omega_N(\mathbb{R}^D)) = \bigoplus_{\rho \in \mathbb{N}} H^p_{\rho}(\Omega_N(\mathbb{R}^D))$$

with

$$H^p_{\rho}(\Omega_N(\mathbb{R}^D)) = \text{Ker}(d^k : \Omega^p_N(\mathbb{R}^D) \to \Omega^{p+k}_N(\mathbb{R}^D))/d^{N-k}(\Omega^{p+k-N}(\mathbb{R}^D)).$$

The following statement is our generalization of the Poincaré lemma.

**THEOREM 1** One has $H^{(N-1)n}_{(k)}(\Omega_N(\mathbb{R}^D)) = 0$, $\forall n \geq 1$ and $H^0_{(k)}(\Omega_N(\mathbb{R}^D))$ is the space of real polynomial functions on $\mathbb{R}^D$ of degree strictly less than $k$ (i.e. $\leq k - 1$) for $k \in \{1, \ldots, N - 1\}$.

This statement reduces to the Poincaré lemma for $N = 2$ but it is a non-trivial generalization for $N \geq 3$ in the sense that, as we shall see, the spaces $H^p_{(k)}(\Omega_N(\mathbb{R}^D))$ are nontrivial for $p \neq (N-1)n$ and, in fact, are generically
infinite dimensional for $D \geq 3$, $p \geq N$.

The connection between the complex of differential forms on $\mathbb{R}^D$ and the theory of classical gauge field of spin 1 is well known. Namely the subcomplex

$$
\Omega^0(\mathbb{R}^D) \xrightarrow{d} \Omega^1(\mathbb{R}^D) \xrightarrow{d} \Omega^2(\mathbb{R}^D) \xrightarrow{d} \Omega^3(\mathbb{R}^D)
$$

has the following interpretation in terms of spin 1 gauge field theory. The space $\Omega^0(\mathbb{R}^D)(= C^\infty(\mathbb{R}^D))$ is the space of infinitesimal gauge transformations, the space $\Omega^1(\mathbb{R}^D)$ is the space of gauge potentials (which are the appropriate description of spin 1 gauge fields to introduce local interactions). The subspace $d\Omega^0(\mathbb{R}^D)$ of $\Omega^1(\mathbb{R}^D)$ is the space of pure gauge configurations (which are physically irrelevant), $d\Omega^1(\mathbb{R}^D)$ is the space of field strengths or curvatures of gauge potentials. The identity $d^2 = 0$ ensures that the curvatures do not see the irrelevant pure gauge potentials whereas, at this level, the Poincaré lemma ensures that it is only these irrelevant configurations which are forgotten when one passes from gauge potentials to curvatures (by applying $d$). Finally $d^2 = 0$ also ensures that curvatures of gauge potentials satisfy the Bianchi identity, i.e. are in $\text{Ker}(d : \Omega^2(\mathbb{R}^D) \to \Omega^3(\mathbb{R}^D))$, whereas at this level the Poincaré lemma implies that conversely the Bianchi identity characterizes the elements of $\Omega^2(\mathbb{R}^D)$ which are curvatures of gauge potentials.

Classical spin 2 gauge field theory is the linearization of Einstein geometric theory. In this case, the analog of (4) is a complex $\mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \mathcal{E}^3 \xrightarrow{d} \mathcal{E}^4$ where $\mathcal{E}^1$ is the space of covariant vector field $(x \mapsto X_\mu(x))$ on $\mathbb{R}^D$, $\mathcal{E}^2$ is the space of covariant symmetric tensor fields of degree 2 $(x \mapsto h_{\mu\nu}(x))$ on $\mathbb{R}^D$, $\mathcal{E}^3$ is the space of covariant tensor fields of degree 4 $(x \mapsto R_{\lambda\mu,\rho\sigma}(x))$ on $\mathbb{R}^D$ having the symmetries of the Riemann curvature tensor and where $\mathcal{E}^4$ is the space of covariant tensor fields of degree 5 on $\mathbb{R}^D$ having the symmetries of
the left-hand side of the Bianchi identity. The arrows \( d_1, d_2, d_3 \) are given by

\[
(d_1 \mathbf{X})_{\mu \nu} (x) = \partial_\mu \mathbf{X}_\nu (x) + \partial_\nu \mathbf{X}_\mu (x)
\]

\[
(d_2 \mathbf{h})_{\lambda \mu, \rho \nu} (x) = \partial_\lambda \partial_\rho \mathbf{h}_{\mu \nu} (x) + \partial_\mu \partial_\nu \mathbf{h}_{\lambda \rho} (x) - \partial_\mu \partial_\rho \mathbf{h}_{\lambda \nu} (x) - \partial_\lambda \partial_\nu \mathbf{h}_{\mu \rho} (x)
\]

\[
(d_3 \mathbf{R})_{\lambda \mu, \alpha \beta} (x) = \partial_\lambda \mathbf{R}_{\mu \nu, \alpha \beta} (x) + \partial_\mu \mathbf{R}_{\nu \lambda, \alpha \beta} (x) + \partial_\nu \mathbf{R}_{\lambda \mu, \alpha \beta} (x).
\]

The symmetry of \( x \mapsto \mathbf{R}_{\lambda \mu, \rho \nu} (x) \), shows that \( \mathcal{E}^3 = \Omega_3^1 (\mathbb{R}^D) \) and that \( \mathcal{E}^4 = \Omega_3^2 (\mathbb{R}^D) \); furthermore one canonically has \( \mathcal{E}^1 = \Omega_3^1 (\mathbb{R}^D) \) and \( \mathcal{E}^2 = \Omega_3^2 (\mathbb{R}^D) \). One also sees that \( d_1 \) and \( d_3 \) are proportional to the 3-differential \( d \) of \( \Omega_3 (\mathbb{R}^D) \), i.e. \( d_1 \sim d : \Omega_3^1 (\mathbb{R}^D) \to \Omega_3^2 (\mathbb{R}^D) \) and \( d_3 \sim d : \Omega_3^1 (\mathbb{R}^D) \to \Omega_3^3 (\mathbb{R}^D) \).

The structure of \( d_2 \) looks different, it is of second order and increases by 2 the tensorial degree. However it is easy to see that it is proportional to \( d^2 : \Omega_3^2 (\mathbb{R}^D) \to \Omega_3^4 (\mathbb{R}^D) \). Thus the analog of (4) is (for spin 2 gauge field theory)

\[
\Omega_3^1 (\mathbb{R}^D) \xrightarrow{d} \Omega_3^2 (\mathbb{R}^D) \xrightarrow{d^2} \Omega_3^4 (\mathbb{R}^D) \xrightarrow{d} \Omega_3^5 (\mathbb{R}^D)
\]

and the fact that it is a complex follows from \( d^3 = 0 \) whereas our generalized Poincaré lemma (Theorem 1) implies that it is in fact an exact sequence. Exactness at \( \Omega_3^2 (\mathbb{R}^D) \) is \( H^2_2 (\Omega_3 (\mathbb{R}^D)) = 0 \) and exactness at \( \Omega_3^4 (\mathbb{R}^D) \) is \( H^4_1 (\Omega_3 (\mathbb{R}^D)) = 0 \), (the exactness at \( \Omega_3^5 (\mathbb{R}^D) \) is the main statement of [17]).

Thus what plays the role of the complex of differential forms for the spin 1 (i.e. \( \Omega_2 (\mathbb{R}^D) \)) is the 3-complex \( \Omega_3 (\mathbb{R}^D) \) for the spin 2. More generally, for the spin \( S \in \mathbb{N} \), this role is played by the \((S + 1)\)-complex \( \Omega_{S+1} (\mathbb{R}^D) \). In particular, the analog of the sequence (4) for the spin 1 is the complex

\[
\Omega_{S+1}^{S-1} (\mathbb{R}^D) \xrightarrow{d} \Omega_{S+1}^S (\mathbb{R}^D) \xrightarrow{d^2} \Omega_{S+1}^{2S} (\mathbb{R}^D) \xrightarrow{d} \Omega_{S+1}^{2S+1} (\mathbb{R}^D)
\]

for the spin \( S \). The fact that (3) is a complex was known, [4], it here follows from \( d^{S+1} = 0 \). One easily recognizes that \( d^S : \Omega_{S+1}^S (\mathbb{R}^D) \to \Omega_{S+1}^{2S} (\mathbb{R}^D) \) is
the generalized (linearized) curvature of \(4\). Our theorem 1 implies that sequence (6) is exact: exactness at \(\Omega^S_{S+1}(\mathbb{R}^D)\) is \(H^S_{(S)}(\Omega^S_{S+1}(\mathbb{R}^D)) = 0\) whereas exactness at \(\Omega^2_{S+1}(\mathbb{R}^D)\) is \(H^2_{(1)}(\Omega^2_{S+1}(\mathbb{R}^D)) = 0\), (exactness at \(\Omega^S_{S+1}(\mathbb{R}^D)\) was directly proved in [5] for the case \(S = 3\)).

Finally, there is a generalization of Poincaré duality for \(\Omega_N(\mathbb{R}^D)\), which is obtained by contractions of the columns with the Kroneker tensor \(\varepsilon^{\mu_1...\mu_D}\) of \(\mathbb{R}^D\), that we shall describe in this paper. When combined with Theorem 1, this duality leads to another kind of results. A typical result of this kind is the following one. Let \(T^{\mu\nu}\) be a symmetric contravariant tensor field of degree 2 on \(\mathbb{R}^D\) satisfying \(\partial_{\mu}T^{\mu\nu} = 0\), (like e.g. the stress energy tensor), then there is a contravariant tensor field \(R^{\lambda\mu\rho\nu}\) of degree 4 with the symmetry \(\lambda\rho\mu\nu\) (i.e. the symmetry of Riemann curvature tensor), such that

\[T^{\mu\nu} = \partial_{\lambda}\partial_{\rho}R^{\lambda\mu\rho\nu}\]  \hspace{1cm} (7)

In order to connect this result with Theorem 1, define \(\tau_{\mu_1...\mu_{D-1}\nu_1...\nu_{D-1}} = T^{\mu\nu}\varepsilon_{\mu_1...\mu_{D-1}\nu_1...\nu_{D-1}}\). Then one has \(\tau \in \Omega^2_{3}(\mathbb{R}^D)\) and conversely, any \(\tau \in \Omega^2_{3}(\mathbb{R}^D)\) can be expressed in this form in terms of a symmetric contravariant 2-tensor. It is easy to verify that \(d\tau = 0\) (in \(\Omega^3(\mathbb{R}^D)\)) is equivalent to \(\partial_{\mu}T^{\mu\nu} = 0\). On the other hand, Theorem 1 implies that \(H^2_{(1)}(\Omega^3(\mathbb{R}^D)) = 0\) and therefore \(\partial_{\mu}T^{\mu\nu} = 0\) implies that there is a \(\rho \in \Omega^2_{3}(\mathbb{R}^D)\) such that \(\tau = d^2\rho\). The latter is equivalent to (10) with \(R^{\mu_1\mu_2\nu_1\nu_2}\) proportional to \(\varepsilon^{\mu_1\mu_2...\mu_D}\varepsilon^{\nu_1\nu_2...\nu_D}\rho_{\mu_3...\mu_D\nu_3...\nu_D}\) and one verifies that, so defined, \(R\) has the correct symmetry. That symmetric tensor fields identically fulfilling \(\partial_{\mu}T^{\mu\nu} = 0\) can be rewritten as in Eq. (7) has been used in [23] and more recently in [3] in the investigation of the consistent deformations of the free spin two gauge field action.
Beside their usefulness for computations (and for unifying various results) through the generalization of Poincaré lemma (Theorem 1) and the generalization of the Poincaré duality, the $N$-complexes described in this paper give a class of nontrivial examples of $N$-complexes which are not related with simplicial modules. Indeed most nontrivial examples of $N$-complexes considered in [6], [7], [8], [19], [21], [20] are of simplicial type and it was shown in [7] that such $N$-complexes compute the ordinary (co)homologies of the simplicial modules (see also in [20] for the Hochschild case). Furthermore that kind of results have been recently extended to the cyclic context in [24] where new proofs of above results have been carried over. This does not mean that $N$-complexes associated with simplicial modules are not useful; for instance in [14] such a $N$-complex (related with a simplicial Hochschild module) was needed for the construction of a natural generalized BRS-theory [1], [18] for the zero modes of the $SU(2)$ WZNW-model, see in [3] for a general review. It is however very desirable to produce useful examples which are not of simplicial type and, apart from the universal construction of [12] (and some finite-dimensional examples [7], [12]), the examples produced here are the first ones escaping from the simplicial frame.

Many results of this paper where announced in our letter [10] so an important part of it is devoted to the proofs of these results in particular to the proof of Theorem 1 above which generalizes the Poincaré lemma. In order that the paper be self contained we recall some basic definitions and results on Young diagrams and representations of the linear group which are needed here. Throughout the paper, we work in the real setting, so all vector spaces are on the field $\mathbb{R}$ of real numbers (this obviously generalizes to any commutative field $\mathbb{K}$ of characteristic zero).
The plan of the paper is the following. After this introduction we discuss Young diagrams, Young symmetry types for tensor and we define in this context a notion of contraction. Section 3 is devoted to the construction of the basic $N$-complex of tensor fields on $\mathbb{R}^D$ considered in this paper, namely $\Omega_N(\mathbb{R}^D)$, and the description of the generalized Poincaré (Hodge) duality in this context. In Section 4 we introduce a multicomplex on $\mathbb{R}^D$ and we analyse its cohomological properties; Theorem 2 proved there, which is by itself of interest, will be the basic ingredient in the proof of our generalization of the Poincaré lemma i.e. of Theorem 1. Section 5 contains this proof of Theorem 1. In Section 6 we analyse the structure of the generalized cohomology of $\Omega_N(\mathbb{R}^D)$ in the degrees which are not exhausted by Theorem 1. The $N$-complex $\Omega_N(\mathbb{R}^D)$ is a generalization of the complex $\Omega(\mathbb{R}^D) = \Omega_2(\mathbb{R}^D)$ of differential forms on $\mathbb{R}^D$; in Section 7 we define another generalization $\Omega_{[N]}(\mathbb{R}^D)$ of the complex of differential forms which is also a $N$-complex and which is an associative graded algebra acting on the graded space $\Omega_N(\mathbb{R}^D)$. In Section 8 which plays the role of a conclusion we sketch another possible proof of Theorem 1 based on a generalization of algebraic homotopy for $N$-complexes. In this section we also define natural $N$-complexes of tensor fields on complex manifolds which generalize the usual $\bar{\partial}$-complex (of forms in $d\bar{z}$).

2 Young diagrams and tensors

For the Young diagrams etc. we use throughout the conventions of [16]. A Young diagram $Y$ is a diagram which consists of a finite number $r > 0$ of rows of identical squares (refered to as the cells) of finite decreasing lengths $m_1 \geq m_2 \geq \cdots \geq m_r > 0$ which are arranged with their left hands under one another. The lengths $\tilde{m}_1, \ldots, \tilde{m}_c$ of the columns of $Y$ are also decreasing.
\(\tilde{m}_1 \geq \cdots \geq \tilde{m}_c > 0\) and are therefore the rows of another Young diagram \(\tilde{Y}\) with \(\tilde{r} = c\) rows. The Young diagram \(\tilde{Y}\) is obtained by flipping \(Y\) over its diagonal (from upper left to lower right) and is referred to as the \textit{conjugate} of \(Y\). Notice that one has \(\tilde{m}_1 = r\) and therefore also \(m_1 = \tilde{r} = c\) and that \(m_1 + \cdots + m_r = \tilde{m}_1 + \cdots + \tilde{m}_c\) is the total number of cells of \(Y\) which will be denoted by \(|Y|\). It is convenient to add the empty Young diagram \(Y_0\) characterized by \(|Y_0| = 0\). The figure below describes a Young diagram \(Y\) and its conjugate \(\tilde{Y}\):

\[
Y = \begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array} \\
\tilde{Y} = \begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

In the following \(E\) denotes a finite-dimensional vector space of dimension \(D\) and \(E^*\) denotes its dual. The \(n\)-th tensor power \(E^\otimes n\) of \(E\) identifies canonically with the space of multilinear forms on \((E^*)^n\). Let \(Y\) be a Young diagram and let us consider that the \(|Y|\) copies of \(E^*\) in \((E^*)^{|Y|}\) are labelled by the cells of \(Y\) so that an element of \((E^*)^{|Y|}\) is given by specifying an element of \(E^*\) for each cell of \(Y\). The \textit{Schur module} \(E^Y\) is defined to be the vector space of all multilinear forms \(T\) on \((E^*)^{|Y|}\) such that:

\(i\) \(T\) is completely antisymmetric in the entries of each column of \(Y\),

11
(ii) complete antisymmetrization of $T$ in the entries of a column of $Y$ and another entry of $Y$ which is on the right-hand side of the column vanishes.

Notice that $E^Y = 0$ if the first column of $Y$ has length $\tilde{m}_1 > D$. One has $E^Y \subset E^{\otimes |Y|}$ and $E^Y$ is an invariant subspace for the action of $GL(E)$ on $E^{\otimes |Y|}$ which is irreducible. Furthermore each irreducible subspace of $E^{\otimes n}$ for the action of $GL(E)$ is isomorphic to $E^Y$ with the above action of $GL(E)$ for some Young diagram $Y$ with $|Y| = n$.

Let $Y$ be a Young diagram and let $T$ be an arbitrary multilinear form on $(E^*)^{|Y|}$, $(T \in E^{\otimes |Y|})$. Define the multilinear form $\mathcal{Y}(T)$ on $(E^*)^{|Y|}$ by

$$\mathcal{Y}(T) = \sum_{p \in R} \sum_{q \in C} (-1)^{\varepsilon(q)} T \circ p \circ q$$

where $C$ is the group of the permutations which permute the entries of each column and $R$ is the group of the permutations which permute the entries of each row of $Y$. One has $\mathcal{Y}(T) \in E^Y$ and the endomorphism $\mathcal{Y}$ of $E^{\otimes |Y|}$ satisfies $\mathcal{Y}^2 = \lambda \mathcal{Y}$ for some number $\lambda \neq 0$. Thus $Y = \lambda^{-1} \mathcal{Y}$ is a projection of $E^{\otimes |Y|}$ into itself, $Y^2 = Y$, with image $\text{Im}(Y) = E^Y$. The projection $Y$ will be referred to as the Young symmetrizer (relative to $E$) of the Young diagram $Y$. The element $e_Y = \lambda^{-1} \sum_{p \in R} \sum_{q \in C} (-1)^{\varepsilon(q)} pq$ of the group algebra of the group $S_{|Y|}$ of permutation of $\{1, \ldots, |Y|\}$ is an idempotent which will be referred to as the Young idempotent of $Y$.

By composition of $Y$ as above with the canonical multilinear mapping of $E^{|Y|}$ into $E^{\otimes |Y|}$ one obtains a multilinear mapping $v \mapsto v^Y$ of $E^{|Y|}$ into $E^Y$. The Schur module $E^Y$ together with the mapping $v \mapsto v^Y$ are characterized uniquely up to an isomorphism by the following universal property: For any
multilinear mapping $\phi : E^{[Y]} \rightarrow F$ of $E^{[Y]}$ into a vector space $F$ satisfying

(i) $\phi$ is completely antisymmetric in the entries of each column of $Y$,  

(ii) complete antisymmetrization of $\phi$ in the entries of a column of $Y$ and another entry of $Y$ which is on the right-hand side of the column vanishes,

there is a unique linear mapping $\phi^Y : E^Y \rightarrow F$ such that $\phi(v) = \phi^Y(v^Y)$. By construction $v \mapsto v^Y$ satisfies the conditions (i) and (ii) above.

There is an obvious notion of inclusion for Young diagrams namely $Y'$ is included in $Y$, $Y' \subset Y$, if one has this inclusion for the corresponding subsets of the plane whenever their upper left cells coincide. This means for instance that $Y' \subset Y$ whenever the length $c = m_1$ of the first row of $Y$ is greater than the length $c' = m'_1$ of the first row of $Y'$ and that for any $1 \leq i \leq c'$ the length $\tilde{m}_i$ of the $i$-th column of $Y$ is greater than the length $\tilde{m}'_i$ of the $i$-th column of $Y'$, ($c \geq c'$ and $\tilde{m}_i \geq \tilde{m}'_i$ for $1 \leq i \leq c'$).

In the following we shall need a stronger notion. A Young diagram $Y'$ is strongly included in another one $Y$ and we write $Y' \subset \subset Y$ if the length of the first row of $Y$ is greater than the length of the first row of $Y'$ and if the length of the last column of $Y$ is greater than the length of the first column of $Y'$. Notice that this relation is not reflexive, one has $Y \subset \subset Y$ if and only if $Y$ is rectangular which means that all its columns have the same length or equivalently all its rows have the same length. It is clear that $Y' \subset \subset Y$ implies $Y' \subset Y$. 

13
Let \( Y \) and \( Y' \) be Young diagrams such that \( Y' \subset \subset Y \) and let \( \tilde{m}_1 \geq \cdots \geq \tilde{m}_c > 0 \) be the lengths of the columns of \( Y \) and \( \tilde{m}'_1 \geq \cdots \geq \tilde{m}'_{c'} > 0 \) be the lengths of the columns of \( Y' \); one has \( c \geq c' \) and \( \tilde{m}_c \geq \tilde{m}'_1 \). Define the contraction of \( Y \) by \( Y' \) to be the Young diagram \( C(Y|Y') \) obtained from \( Y \) by dropping \( \tilde{m}'_1 \) cells of the last, i.e., the \( c \)-th column of \( Y \), \( \tilde{m}'_2 \) cells of the \((c - 1)\)-th column of \( Y \), ..., \( \tilde{m}'_{c'} \) cells of the \((c - c' + 1)\)-th column of \( Y \). If \( \tilde{m}_c \) is strictly greater than \( \tilde{m}'_1 \) then \( C(Y|Y') \) has \( c \) columns as \( Y \), however if \( \tilde{m}_c = \tilde{m}'_1 \) then the number of columns of \( C(Y|Y') \) is strictly smaller than \( c \) (it is \( c - 1 \) if \( \tilde{m}_{c-1} \) is strictly greater than \( \tilde{m}'_2 \), etc.). Notice that if \( Y \) is rectangular then \( C(Y|Y') \subset \subset Y \) and \( C(Y|C(Y|Y')) = Y' \) so that \( Y' \mapsto C(Y|Y') \) is then an involution on the set of Young diagrams \( Y' \) which are strongly included in \( Y \) \( (Y' \subset \subset Y) \).

Let again \( Y \) and \( Y' \) be Young diagrams with \( Y' \subset \subset Y \). Our aim is now to define a bilinear mapping \((T, T') \mapsto C(T|T') \) of \( E^Y \times E^{*Y'} \) into \( E^{C(Y|Y')} \). This will be obtained by restriction of a bilinear mapping \((T, T') \mapsto C(T|T') \) of \( E^{|Y|} \times E^{*|Y'|} \) into \( E^{C(|Y|)} \) which will be an ordinary (complete) tensorial contraction. Any such tensorial contraction associates to a contravariant tensor \( T \) of degree \(|Y|\) (i.e. \( T \in E^{|Y|} \)) and a covariant tensor \( T' \) of degree \(|Y'|\) (i.e. \( T' \in E^{*|Y'|} \)) a contravariant tensor of degree \(|C(Y|Y')|\), \( (Y' \subset \subset Y) \). In order to specify such a contraction, one has to specify the entries of \( T \), that is of \( Y \), to which each entry of \( T' \), that is of \( Y' \), is contracted (recalling that \( T \) is a linear combination of canonical images of elements of \( E^{|Y|} \) and that \( T' \) is a linear combination of canonical images of elements of \( E^{*|Y'|} \)). In order that \( C(T|T') \) has the right antisymmetry in the entries of each column of \( C(Y|Y') \) when \( T \in E^Y \) and \( T' \in E^{*Y'} \), one has to contract the entries of \( T' \) corresponding to the \( i \)-th column of \( Y' \) with entries of \( T \) corresponding to
the \((c - i + 1)\)-th column of \(Y\). The precise choice and the order of the latter entries is irrelevant up to a sign in view of the antisymmetry in the entries of a column. Our choice is to contract the first entry of the \(i\)-th column of \(Y'\) with the last entry of the \((c - i + 1)\)-th column of \(Y\), the second entry of the \(i\)-th column of \(Y'\) with the penultimate entry of the \((c - i + 1)\)-th column of \(Y\), etc. for any \(1 \leq i \leq c'\) (with obvious conventions). This fixes the bilinear mapping \((T, T') \mapsto \mathcal{C}(T|T')\) of \(E^{|Y|} \times E^{*|Y'|}\) into \(E^{|\mathcal{C}(Y|Y')|}\). The following figure describes picturatively in a particular case the construction of \(\mathcal{C}(Y|Y')\) as well as the places where the contractions are carried over in the corresponding construction of \(\mathcal{C}(T|T')\):

\[
\begin{align*}
Y = \begin{array}{|c|c|c|}
\hline
\hline
\hline
Y' = \begin{array}{|c|c|}
\hline
\hline
\end{array}
\end{align*}
\]

\[
\begin{align*}
= \mathcal{C}(Y|Y')
\end{align*}
\]

**PROPOSITION 1** Let \(T\) be an element of \(E^Y\) and \(T'\) be an element of \(E^{*Y'}\) with \(Y' \subset Y\). Then \(\mathcal{C}(T|T')\) is an element of \(E^{\mathcal{C}(Y|Y')}\).

**Proof** As before, we identify \(\mathcal{C}(T|T') \in E^{|\mathcal{C}(Y|Y')|}\) with a multilinear form on \(E^{*|\mathcal{C}(Y|Y')|}\). To show that \(\mathcal{C}(T|T')\) is in \(E^{\mathcal{C}(Y|Y')}\) means verifying properties (i) and (ii) above. Property (i), i.e. antisymmetry in the columns entries of \(\mathcal{C}(Y|Y')\), is clear. Property (ii) has to be verified for each column of \(\mathcal{C}(Y|Y')\)
and entry on its right-hand side which can be chosen to be the first entry of a column on the right-hand side (in view of the column antisymmetry). If the column is the last one it has no entry on the right-hand side so there nothing to verify and if the column is a full column of \( Y \), i.e. has not be contracted, which is the case for the \( i \)-th column with \( i \leq c - c' \), the property (\( ii \)) follows from the same property for \( T \) (assumption \( T \in E^Y \)). Thus to achieve the proof of the proposition we only need to verify property (\( ii \)) in the case where both \( Y \) and \( Y' \) have exactly two columns of lengths say \( \tilde{m}_1 \geq \tilde{m}_2 \) for \( Y \) and \( \tilde{m}'_1 \geq \tilde{m}'_2 \) for \( Y' \) with \( \tilde{m}_2 > \tilde{m}'_1 \). In this case \( C(Y|Y') \) has also two columns of lengths \( \tilde{m}_1 - \tilde{m}'_2 \) and \( \tilde{m}_2 - \tilde{m}'_1 \) (\( \tilde{m}_1 - \tilde{m}'_2 \geq \tilde{m}_2 - \tilde{m}'_1 > 0 \)) and one has to verify that antisymmetrization of the first entry of the second column of \( C(Y|Y') \) with the entries of the first column (of length \( \tilde{m}_1 - \tilde{m}'_2 \)) of \( C(Y|Y') \) in \( C(T|T') \) gives zero. We know that antisymmetrization with all entries of the first column of \( Y \) give zero (for \( T \)); however when contracted with \( T' \) this identity implies a sum of antisymmetrizations of the entries of the first column of \( Y' \) with the successive entries of its second column for \( T' \) which gives zero (\( T' = E^*Y' \)) and reduces therefore to desired antisymmetrization with the \( \tilde{m}_1 - \tilde{m}'_2 \) first entries. □

3 Generalized complexes of tensor fields

Throughout this section \((Y)\) denotes not just one Young diagram but a sequence \((Y) = (Y_p)_{p \in \mathbb{N}}\) of Young diagrams \(Y_p\) such that the number of cells of \(Y_p\) is equal to \(p\) that is \(|Y_p| = p\), \(\forall p \in \mathbb{N}\). Notice that there is no freedom for \(Y_0\) and \(Y_1\): \(Y_0\) must be the empty Young diagram and \(Y_1\) is the Young diagram with one cell. Let us denote by \(\wedge_{(Y)} E\) the direct sum \(\oplus_{p \in \mathbb{N}} E^{Y_p}\) of the Schur modules \(E^{Y_p}\). This is a graded vector space with \(\wedge_{(Y)} E = E^{Y_0}\). The origin of this notation is that for the sequence \((Y^2) = (Y_p^2)\) of the one
column Young diagrams, i.e. $Y^2_p$ is the Young diagram with $p$ cells in one column for any $p \in \mathbb{N}$, then $\wedge_{\{Y^2\}} E$ is the exterior algebra $\wedge E$ of $E$.

In the following, we shall be interested in particular sequences $(Y^N) = (Y^N_p)_{p \in \mathbb{N}}$ of Young diagrams satisfying the assumption of Lemma 1 (as explained in the introduction). The sequence $(Y^N)$ contains Young diagrams $Y^N_p$ in which all the rows but the last one are of length $N - 1$, the last one being of length smaller than or equal to $N - 1$ in such a way that $|Y^N_p| = p$ ($\forall p \in \mathbb{N}$). Picturally one has for instance for $N = 5$

\[
Y^5_3 = \begin{array}{c}
\hline
\hline
\end{array} \quad Y^5_{22} = \begin{array}{cccc}
\hline
\hline
\end{array} \quad Y^5_{21} = \begin{array}{cc}
\hline
\hline
\end{array}
\]

and so on. In this case $\wedge_{\{Y^N\}} E$ and $\wedge_{\{Y^N\}}^p E = E_{Y^N_p}$ will be simply denoted by $\wedge_N E$ and $\wedge_N^p E$ respectively. Notice that $\wedge_N^p E = 0$ for $p > (N - 1)D$, ($D = \dim E$), so that $\wedge_N E = \bigoplus_{p=0}^{(N-1)D} \wedge_N^p E$ is finite-dimensional.

Let us assume that $E$ is equipped with a dual volume, i.e. a non-vanishing element $\varepsilon$ of $\wedge^D E (= \wedge_2^D E)$, which is therefore a basis of the 1-dimensional space $\wedge^D (E)$. It is straightforward that $\varepsilon \otimes^{(N-1)}$ is in $\wedge_N^{(N-1)D} E = E^{Y^N_{(N-1)D}}$ because (i) is obvious whereas (ii) is trivial i.e. empty. The Young diagram $Y^N_{(N-1)D}$ is rectangular so that each Young diagram which is included in $Y^N_{(N-1)D}$ is in fact strongly included in $Y^N_{(N-1)D}$; this is in particular the
case for the $Y_p^N$ for $p \leq (N - 1)D$. One then defines a linear isomorphism $\ast : \wedge_N E^* \to \wedge_N E$ generalizing the algebraic part of the Poincaré (Hodge) duality by setting

$$\ast \omega = C(\varepsilon \otimes (N-1) | \omega)$$

(8)

for $\omega \in \wedge_N E^*$. One has

$$\ast \wedge_N E^* = \wedge_N^{(N-1)D-p} E$$

(9)

for $p = 0, \ldots, (N - 1)D$.

Let $(e_\mu)_{\mu \in \{1, \ldots, D\}}$ be a basis of $E$ and let $(\theta^\mu)$ be the dual basis of $E^*$. Our aim is to be able to compute in terms of the components of tensors for the various concepts connected with Young diagrams. For this, one has to decide the linear order in which one writes the components of a tensor $T \in E^{\otimes |Y|}$ or, which is the same, of a multilinear form $T$ on $E^{*|Y|}$ for any given Young diagram $Y$. Since we have labelled the arguments (entries) of such a $T$ by the cells of $Y$ and since the components are obtained by taking the arguments among the $\theta^\mu$, this means that one has to choose an order for the cells of $Y$ (i.e a way to “read the diagram” $Y$). One natural choice is to read the rows of $Y$ from left to right and then from up to down (like a book); another natural choice is to read the columns of $Y$ from up to down and then from left to right. Although the first choice is very natural with respect to the sequences $(Y^N)$ of Young diagrams introduced above and will be used later, we shall choose the second way of ordering in the following. The reason is that when $T$ belongs to the Schur module $E^Y$, then it is (property (i)) antisymmetric in the entries of each columns. Thus if $Y$ has columns of lengths $\tilde{m}_1 \geq \cdots \geq \tilde{m}_c$ ($> 0$ for $|Y| \neq 0$) our choice is induced by the canonical identification

$$E^Y \subset \wedge^{\tilde{m}_1} E \otimes \cdots \otimes \wedge^{\tilde{m}_c} E$$

(10)
of the Schur module $E_Y$ as a subspace of $\wedge^{\tilde{m}_1} E \otimes \cdots \otimes \wedge^{\tilde{m}_c} E$ where $\wedge^p E = \wedge^p E$ is the $p$-th exterior power of $E$. With the above choice, the components (relative to the basis $(e_\mu)$ of $E$) of $T \in E^{\otimes |Y|}$ read $T_{\mu_1 \ldots \mu_{\tilde{m}_1} \ldots \mu_{\tilde{m}_c}}$ and $T \in E_Y$ if and only if these components are completely antisymmetric in the $\mu_1^1, \ldots, \mu_{\tilde{m}_r}$ for each $r \in \{1, \ldots, c\}$ and such that complete antisymmetrization in the $\mu_1^1, \ldots, \mu_{\tilde{m}_c}$ and $\mu_1^1$ gives zero for any $1 \leq r < s \leq c$.

We have defined for a sequence $(Y) = (Y_p)$ of Young diagrams with $|Y_p| = p$ ($\forall p \in \mathbb{N}$) the graded vector space $\wedge(Y) E$ which can be considered as a generalization of the exterior algebra $\wedge E$ as explained above. We now wish to define the corresponding generalization of differential forms. Let $M$ be a $D$-dimensional smooth manifold. For any Young diagram $Y$ one has the smooth vector bundle $T^*Y(M)$ over $M$ of the Schur modules $(T^*_x(M))^Y$, $x \in M$. Correspondingly, for $(Y)$ as above, one has the smooth bundle $\wedge(Y) T^*_x(M)$ over $M$ of graded vector spaces $\wedge(Y) T^*_x(M)$. The graded $C^\infty(M)$-module $\Omega(Y)(M)$ of smooth sections of $\wedge(Y) T^*_x(M)$ is the generalization of differential forms corresponding to $(Y)$. In order to generalize the exterior differential one has to choose a connection $\nabla$ on the vector bundle $T^*_x(M)$ that is a linear connection $\nabla$ on $M$. Such a connection extends canonically as linear mappings

$$\nabla : \Omega^p(Y)(M) \rightarrow \Omega^p(Y)(M) \otimes_{C^\infty(M)} \Omega^1(M)$$

where $\Omega^1(M) = \Omega^1(Y)(M)$ is the $C^\infty(M)$-module of smooth sections of $T^*_x(M)$ (i.e. of differential 1-forms) satisfying

$$\nabla(\alpha f) = \nabla(\alpha) f + \alpha \otimes df$$

for any $\alpha \in \Omega^p(Y)(M)$ and $f \in C^\infty(M)$ and where $d$ is the ordinary differential of $C^\infty(M)$ into $\Omega^1(M)$. Notice that for any sequence $(Y)$ of Young diagrams
as above, one has $\Omega_{(Y)}^0(M) = C^\infty(M)$ and $\Omega_{(Y)}^1(M) = \Omega^1(M)$ since one has no choice for $Y_0$ and $Y_1$. Let us define the generalization of the covariant exterior differential $d\nabla : \Omega_{(Y)}(M) \to \Omega_{(Y)}(M)$ by

$$d\nabla = (-1)^p Y_{p+1} \circ \nabla : \Omega_{(Y)}^p(M) \to \Omega_{(Y)}^{p+1}(M) \quad (11)$$

for any $p \in \mathbb{N}$. Notice that $d\nabla = d$ on $C^\infty(M) = \Omega_{(Y)}^0(M)$ and that $d\nabla$ is a first order differential operator. Lemma 1 in the introduction admits the following generalization.

**LEMMA 2** Let $N$ be an integer with $N \geq 2$ and assume that $(Y)$ is such that the number of columns of the Young diagram $Y_p$ is strictly smaller than $N$ for any $p \in \mathbb{N}$. Then $(d\nabla)^N$ is a differential operator of order strictly smaller than $N$. If $\nabla$ is torsion-free, then $d\nabla^N$ is order strictly smaller than $N - 1$. If furthermore $\nabla$ has vanishing torsion and curvature then one has $(d\nabla)^N = 0$.

The proof is straightforward. In the case $N = 2$, if $\nabla$ is torsion free, $(d\nabla)^2$ is not only an operator of order zero but $(d\nabla)^2 = 0$ follows from the first Bianchi identity; however in this case, for $(Y^2)$, $d\nabla$ coincides with the ordinary exterior differential. For the sequences $(Y^N) = (Y^N_p)$ we denote $\Omega_{(Y^N)}(M)$ and $\Omega_{(Y^N)}^p(M)$ simply by $\Omega_N(M)$ and $\Omega_N^p(M)$. As already mentioned $\Omega_2(M)$ is the graded algebra $\Omega(M)$ of differential forms on $M$.

Not every $M$ admits a flat torsion-free linear connection. In the following we shall concentrate on $\Omega_N(\mathbb{R}^D)$ equipped with $d = d_{(\nabla)}^{(0)}$ where $\nabla$ is the canonical flat torsion-free connection of $\mathbb{R}^D$. So equipped, $\Omega_N(\mathbb{R}^D)$ is a $N$-complex. One has of course $\Omega_N(\mathbb{R}^D) = \wedge_N \mathbb{R}^D^* \otimes C^\infty(\mathbb{R}^D)$. Let us equip $\mathbb{R}^D$ with the dual volume $\epsilon \in \wedge^D \mathbb{R}^D$ which is the completely antisymmetric contravariant tensor of maximal degree with component $\epsilon^{1...D} = 1$ in the
canonical basis of $\mathbb{R}^D$. Then the corresponding isomorphism $\ast : \bigwedge_N \mathbb{R}^D \to \bigwedge_N \mathbb{R}^D$ extends by $C^\infty(\mathbb{R}^D)$-linearity as an isomorphism of $C^\infty(\mathbb{R}^D)$-modules, again denoted by $\ast$, of $\Omega_N(\mathbb{R}^D)$ into the space (of contravariant tensor fields on $\mathbb{R}^D$) $\bigwedge_N \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$ with

$$\ast \Omega^p_N(\mathbb{R}^D) = \bigwedge_{N-1}^{(N-1)p} \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$$

for any $0 \leq p \leq (N-1)D$. Let us define the first-order differential operator $\delta$ of degree $-1$ on $\bigwedge_N \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$

$$\delta : \bigwedge_{N-1}^{(N-1)p+r} \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D) \to \bigwedge_{N-1}^{(N-1)p+r-1} \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$$

by setting

$$\delta T = Y^N_{(N-1)p+r-1} \circ \tilde{\delta} T$$

(12)

for $T \in \bigwedge_{N-1}^{(N-1)p+r} \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$ with $0 \leq p < D$ and $1 \leq r \leq N-1$, $\tilde{\delta}$ being defined by

$$(\tilde{\delta} T)_{\mu_1^{\prime} \ldots \mu_{p+r-1}^{\prime} = \partial_{\mu_1} T_{\mu_1^{\prime} \ldots \mu_{p+r-1}^{\prime}}}$$

where we have used the canonical identification (11) and the conventions explained below (11). It is worth noticing here that in view (essentially) of Proposition 1, one has $\delta T = \tilde{\delta} T$ for $r = N - 1$, i.e. in this case (well-filled case) the projection is not necessary in formula (12). So defined $(\delta T)(x)$ is by construction in $\bigwedge_{N-1}^{(N-1)p+r-1} \mathbb{R}^D$ and the operator $\delta$ is in each degree proportional to the operator $\ast d \ast^{-1}$, i.e. that one has

$$\delta = c_n \ast d \ast^{-1} : \bigwedge_N \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D) \to \bigwedge_N \mathbb{R}^D \otimes C^\infty(\mathbb{R}^D)$$

(13)

for some $c_n \in \mathbb{R}$, $1 \leq n \leq (N-1)D$ ($\delta = 0$ in degree zero).
4 Digression on a related multicomples

In this section, we introduce a multicomples which will be related to our
$N$-complex $\Omega_N(\mathbb{R}^D)$ in the next section. We also derive some useful coho-
mological results in this multicomples, which will be the key for proving our
generalization of the Poincaré lemma that is Theorem [1].

Let $\mathfrak{A}$ be the graded tensor product of $N - 1$ copies of the exterior algebra
$\wedge \mathbb{R}^{D*}$ of the dual space $\mathbb{R}^{D*}$ of $\mathbb{R}^D$ with $C^\infty(\mathbb{R}^D)$,

$$
\mathfrak{A} = (\otimes^{N-1} \wedge \mathbb{R}^{D*}) \otimes C^\infty(\mathbb{R}^D) = \otimes^{N-1} C^\infty(\mathbb{R}^D) \Omega(\mathbb{R}^D).
$$

An element of $\mathfrak{A}$ is as a sum of products of the $(N - 1)D$ generators $d_i x^\mu$
($i = 1, \ldots, N - 1, \mu = 1, \ldots, D$) with smooth functions on $\mathbb{R}^D$. Elements
of $\mathfrak{A}$ will be refered to as multiforms. The space $\mathfrak{A}$ is a graded-commutative
algebra for the total degree, in particular one has

$$
d_i x^\mu d_j x^\nu = -d_j x^\nu d_i x^\mu, \quad x^\mu d_i x^\nu = d_i x^\nu x^\mu.
$$

One defines $N - 1$ antiderivations $d_i$ on $\mathfrak{A}$ by setting

$$
d_i f = \partial_\mu f d_i x^\mu \quad (f \in C^\infty(\mathbb{R}^D)), \quad d_i(d_j x^\mu) = 0. \quad (14)
$$

These antiderivations anticommute,

$$
d_i d_j + d_j d_i = 0 \quad (15)
$$
in particular each $d_i$ is a differential. The graded algebra $\mathfrak{A}$ has a natural
multidegree ($d_1, d_2, \ldots, d_{N-1}$) for which $d_i(d_j x^\mu) = \delta_{ij}$.

It is useful to consider the subspaces $\mathfrak{A}^{(k)}$ of multiforms that vanish at
the origin, together with all their successive derivatives up to order $k - 1$
included ($k \geq 1$). If $\omega \in \mathfrak{A}^{(k)}$, one says that $\omega$ is of order $k$. The terminology
comes from the fact that a smooth function belongs to $\mathfrak{A}^{(k)}$ if and only if its limited Taylor expansion starts with terms of order $\geq k$. If $l \geq k$, $\mathfrak{A}^{(l)} \subset \mathfrak{A}^{(k)}$. The subspaces $\mathfrak{A}^{(k)}$ are not stable under $d_i$ but one has $d_i \mathfrak{A}^{(k)} \subset \mathfrak{A}^{(k-1)}$ for $k \geq 1$ (with $\mathfrak{A}^{(0)} \equiv \mathfrak{A}$). The vector space $H^{(k)}(d_i, \mathfrak{A})$ is defined as

$$H^{(k)}(d_i, \mathfrak{A}) \equiv \frac{Z^{(k)}(d_i, \mathfrak{A})}{d_i \mathfrak{A}^{(k+1)}}$$

where $Z^{(k)}(d_i, \mathfrak{A})$ is the set of $d_i$-cocycles $\in \mathfrak{A}^{(k)}$. Note that any multiform $\omega \in \mathfrak{A}$ can be written as $\omega = p^{(k)} + \beta$ where $p^{(k)}$ is a polynomial multiform of polynomial degree $k$ and $\beta \in \mathfrak{A}^{(k+1)}$. This decomposition is unique which implies in particular that $\mathfrak{A}^{(k)} \cap d_i \mathfrak{A} = d_i \mathfrak{A}^{(k+1)}$.

It follows from the standard Poincaré lemma that

$$H^{(1)}(d_i, \mathfrak{A}) = 0.$$ (16)

Indeed, the cohomology of $d_i$ in $\mathfrak{A}$ is isomorphic to the space of constant multiforms not involving $d_i x^\mu$. The condition that the cocycles belong to $\mathfrak{A}^{(1)}$, i.e., vanish at the origin, eliminates precisely the constants. One has also $H^{(m)}(d_i, \mathfrak{A}) = 0 \forall m \geq 1$ since $\mathfrak{A}^{(m)} \subset \mathfrak{A}^{(1)}$ for $m \geq 1$ and $\mathfrak{A}^{(m)} \cap d_i \mathfrak{A} = d_i \mathfrak{A}^{(m+1)}$.

Let $K$ be an arbitrary subset of $\{1, 2, \ldots, N-1\}$. We define $\mathfrak{A}_K$ as the quotient space

$$\mathfrak{A}_K = \frac{\mathfrak{A}}{\sum_{j \in K} d_j \mathfrak{A}}$$

(for $K = \emptyset$, $\mathfrak{A}_K = \mathfrak{A}$). The differential $d_i$ induces, for each $i$, a well-defined differential in $\mathfrak{A}_K$ which we still denote by $d_i$. Of course, the induced $d_i$ is equal to zero if $i \in K$.

**Lemma 3** For every proper subset $K$ of $\{1, 2, \ldots, N-1\}$ and for every $i \notin K$, one has

$$H^{(k+1)}(d_i, \mathfrak{A}_K) = 0 \quad (k = \#K)$$

23
The proof proceeds by induction on the number \( k \) of elements of \( K \). The lemma clearly holds for \( k = 0 \) (\( K = \emptyset \)) since then \( \mathfrak{A}_K = \mathfrak{A} \) and the lemma reduces to Eq. \((16)\). Let us now assume that the lemma holds for all subsets \( K \) (not containing \( i \)) with \( k \leq \ell \) elements. Let \( K' \) be a subset not containing \( i \) with \( \ell + 1 \) elements. Let \( j \in K' \) and \( K'' = K' \setminus \{j\} \). The induction hypothesis implies \( H^{(\ell+1)}(d_i, \mathfrak{A}_{K''}) = H^{(\ell+1)}(d_j, \mathfrak{A}_{K''}) = 0 \). By standard “descent equation” arguments (see below), this leads to

\[
H^{p,q,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}) \simeq H^{p+1,q-1,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}).
\]

In \( H^{p,q,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}) \), the first superscript \( p \) stands for the \( \mathfrak{A}_i \)-degree, the second superscript \( q \) stands for the \( \mathfrak{A}_j \)-degree while \( (\ell + 2) \) is the polynomial order. Repeated application of this isomorphism yields

\[
H^{p,q,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}) \simeq H^{p+q,0,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}).
\]

But \( H^{p+q,0,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}) \equiv H^{p+q,0,(\ell+2)}(d_i, \mathfrak{A}_{K''}) = 0 \). Hence, the cohomological spaces \( H^{p,q,(\ell+2)}(d_i|d_j, \mathfrak{A}_{K''}) \) vanish for all \( p, q \), which is precisely the statement \( H^{(\ell+2)}(d_i, \mathfrak{A}_{K''}) = 0 \). \( \square \)

The precise descent equation argument involved in this proof runs as follows: let \( \alpha^{p,q,(\ell+2)} \) be a \( d_i \)-cocycle modulo \( d_j \) in \( \mathfrak{A}_{K''} \), i.e., a solution of \( d_i \alpha^{p,q,(\ell+2)} + d_j \alpha^{p+1,q-1,(\ell+2)} \approx 0 \) for some \( \alpha^{p+1,q-1,(\ell+2)} \), where the notation \( \approx \) means “modulo terms in \( \sum_{j \in K''} d_j \mathfrak{A} \). Applying \( d_i \) to this equation yields \( d_j d_i \alpha^{p+1,q-1,(\ell+2)} \approx 0 \) and hence, since \( d_i \alpha^{p+1,q-1,(\ell+2)} \) is of order \( \ell + 1 \) and \( H^{(\ell+1)}(d_j, \mathfrak{A}_{K''}) = 0 \), \( d_i \alpha^{p+1,q-1,(\ell+2)} + d_j \alpha^{p+2,q-2,(\ell+2)} \approx 0 \) for some \( \alpha^{p+2,q-2,(\ell+2)} \). Hence, \( \alpha^{p+1,q-1,(\ell+2)} \) is also a \( d_i \)-cocycle modulo \( d_j \) in \( \mathfrak{A}_{K''} \). Consider the map \( \alpha^{p,q,(\ell+2)} \mapsto \alpha^{p+1,q-1,(\ell+2)} \) of \( d_i \)-cocycles modulo \( d_j \). There is an arbitrariness in the choice of \( \alpha^{p+1,q-1,(\ell+2)} \) given \( \alpha^{p,q,(\ell+2)} \) so this map is ambiguous, however \( H^{(\ell+1)}(d_j, \mathfrak{A}_{K''}) = 0 \) implies that it induces a well-defined
linear mapping $H^{p,q,\ell+2}(d_i| d_j, \mathcal{A}_{K''}) \rightarrow H^{p+1,q-1,\ell+2}(d_j| d_j, \mathcal{A}_{K''})$ in cohomology. This map is injective and surjective since $H^{\ell+1}(d_i, \mathcal{A}_{K''}) = 0$ and thus one has the isomorphism $H^{p,q,\ell+2}(d_i| d_j, \mathcal{A}_{K''}) \simeq H^{p+1,q-1,\ell+2}(d_j| d_i, \mathcal{A}_{K''})$ (see [11] for additional information).

A direct application of this lemma is the following

**PROPOSITION 2** Let $J$ be any non-empty subset of $\{1, 2, \ldots, N - 1\}$. Then

$$\left( \prod_{j \in J} d_j \right) \alpha = 0 \text{ and } \alpha \in \mathcal{A}(\# J) \Rightarrow \alpha = \sum_{j \in J} d_j \beta_j$$

for some $\beta_j$’s.

**Proof** The property is clearly true for $\# J = 1$ (see Eq. (16)). Assume then that the property is true for all proper subsets with $k \leq \ell < N - 1$ elements. Let $J$ be a proper subset with exactly $\ell$ elements and $i \notin J$. Let $\alpha$ be a multiform in $\mathcal{A}(\ell+1)$ such that $d_i(\prod_{j \in J} d_j)\alpha = 0$. This is equivalent to $(\prod_{j \in J} d_j)d_i \alpha = 0$. Application of the recursive assumption to $d_i \alpha$, which belongs to $\mathcal{A}(\ell)$, implies then $d_i \alpha = \sum_{j \in J} d_j \beta_j$, from which one derives, using the previous lemma, that $\alpha = d_i \rho + \sum_{j \in J} \rho_j$ for some $\rho, \rho_j$. Therefore, the property passes on to all subsets with $\ell + 1$ elements, which establishes the theorem. $\square$

We are now in position to state and prove the main result of this section.

**THEOREM 2** Let $K$ be an arbitrary non-empty subset of $\{1, 2, \ldots, N - 1\}$. If the multiform $\omega$ is such that

$$\left( \prod_{i \in I} d_i \right) \omega = 0 \quad \forall I \subset K \mid \# I = m$$

(17)
(with \( m \leq \#K \) a fixed integer), then

\[
\omega = \sum_{J \subset K} \left( \prod_{j \in J} d_j \right) \alpha_J + \omega_0
\]

(18)

where \( \omega_0 \) is a polynomial multiform of degree \( \leq m - 1 \).

**Proof** The polynomial multiform \( \omega_0 \) is clearly a solution of the problem, so we only need to check that if \( \omega \in A^{(m)} \) in addition to (17), then (18) is replaced by

\[
\omega = \sum_{J \subset K} \left( \prod_{j \in J} d_j \right) \alpha_J
\]

(19)

The \( \alpha_J \)'s can be assumed to be of order \( \#K + 1 \) since one differentiates them \( \#K - m + 1 \) times to get \( \omega \). To prove (19), we proceed recursively, keeping \( m \) fixed and increasing the size of \( K \) step by step from \( \#K = m \) to \( \#K = N - 1 \). If \( \#K = m \), there is nothing to be proven since \( I = K \) and the theorem reduces to the previous theorem. So, let us assume that the theorem has been proven for \( \#K = k \geq m \) and let us show that it extends to any set \( U = K \cup \{ \ell \} \), \( \ell \notin K \) with \( \#U = k + 1 \) elements. If (17) holds for any subset \( I \subset U \) of \( U \) (with \( \#I = m \)), it also holds for any subset \( I \subset K \) of \( K \subset U \) (with \( \#I = m \)), so the recursive hypothesis implies

\[
\omega = \sum_{J \subset K} \left( \prod_{j \in J} d_j \right) \alpha_J
\]

(20)

Let now \( A \) be an arbitrary subset of \( U \) with \( \#A = m \), which contains the added element \( \ell \). Among all the subsets \( J \) occurring in the sum (20), there
is only one, namely $J' = U \setminus A$ such that $J' \cap A = \emptyset$. The condition \((17)\) with $I = A$ implies, when applied to the expression \((20)\) of $\omega$,

\[
\left( \prod_{j \in U} d_j \right) \alpha_{J'} = 0
\]

(if $J \neq J'$, the product $(\prod_{i \in A} d_i)(\prod_{j \in J} d_j)$ identically vanishes because at least one differential $d_f$ is repeated). But since $\alpha_{J'}$ is of order $k + 1 = \#U$, the previous proposition implies that

\[
\alpha_{J'} = \sum_{j \in U} d_j \beta_{J'}^j.
\]

When injected into \((21)\), this yields in turn

\[
\omega = \sum_{L \subset U} \left( \prod_{j \in L} d_j \right) \alpha'_L.
\]

\[\text{for some } \alpha'_L, \text{ and shows that the required property is also valid for sets with cardinal equal to } k + 1, \text{ completing the proof of the theorem.}\]

5 The generalization of the Poincaré lemma

With the result of last section, Theorem 2, we can now proceed to the proof of Theorem 1 that is to the proof of the generalization of the Poincaré lemma announced in the introduction.

Let us first show that $\Omega_N(\mathbb{R}^D)$ identifies canonically as graded $C^\infty(\mathbb{R}^D)$-module with the image of a $C^\infty(\mathbb{R}^D)$-linear homogeneous projection $\pi$ of $\mathfrak{A}$ into itself: $\Omega_N(\mathbb{R}^D) = \pi(\mathfrak{A}) \subset \mathfrak{A}$. Indeed by using the canonical identification \((10)\) of Section 3, one has the identification

\[
\bigwedge_N^{(N-1)n+i} E \subset \bigwedge_{i}^{n+1} E \otimes \cdots \otimes \bigwedge_{i}^{n+1} E \otimes \bigwedge_{N-1-i}^n E \otimes \cdots \otimes \bigwedge_{N-1-i}^n E
\]

(22)
of the Schur module $E^N_{(N-1)n+i} = \Lambda^N_{N}^{(n+1)n+i} E$ as vector subspace of the right-hand side. However by decomposing the right-hand side of (22) into irreducible subspaces for the action of $GL(E)$ one sees that there is only one irreducible factor isomorphic to $E^N_{(N-1)n+i}$ which is therefore the image of a $GL(E)$-invariant projection. It follows that $E^N_{(N-1)n+i} \subset \otimes^{N-1} E$ is the image of a $GL(E)$-invariant projection $P$ of $\otimes^{N-1} E$ into itself which is homogeneous for the total degree. The result for $\Omega^N_{R}$ follows by choosing $E$ to be the dual space $R^*_D$ of $R^D$ and by setting $\pi = P \otimes I_{C^\infty(R^D)}$ in view of $\Omega^N_{R} = \Lambda^N_{N}^{(n+1)n+i}$ with obvious notations.

We now relate the $N$-differential $d$ of $\Omega^N_{R}$ to the differentials $d_i$ of $\mathfrak{A}$. Let $\omega$ be an element of $\Omega^{n}_{R}$ with $n = (N - 1)n + i$, $0 \leq i < N - 1$. One has

$$d\omega = c_\omega \pi(d_{i+1}\omega)$$

(23)

where $c_\omega$ is a non-vanishing number that depends on the degrees of $\omega$. In general, the projection is non-trivial, in the sense that $d_{i+1}\omega$ has components not only along the irreducible Schur module $E^N_{(N-1)n+i}$ ($E = R^*_D$), but also along other Schur modules not occurring in $\Omega^N_{R}$. For instance, with $N = 3$, the covariant vector with components $v_\alpha$ defines the element $v = v_\alpha d_1 x^\alpha$ of $\mathfrak{A}$. One has $d_2 v = -\partial_\beta v_\alpha d_1 x^\alpha d_2 x^\beta$. This expression contains both a symmetric ($dv$) and an antisymmetric part, so $d_2 v = dv - v_{[\alpha,\beta]} d_1 x^\alpha d_2 x^\beta$. The projection removes $v_{[\alpha,\beta]} d_1 x^\alpha d_2 x^\beta$, which does not vanish in general. Because the projection is non-trivial, the conditions $d\omega = 0$ and $d_{i+1}\omega = 0$ are inequivalent for generic $i$. However, if $\omega$ is a well-filled tensor that is if
\( i = 0, \) then
\[
d\omega = d_1\omega \quad (i = 0)
\] (24)
Indeed, \( d_1\omega \) has automatically the correct Young symmetry. Thus the conditions \( d_1\omega = 0 \) and \( d\omega = 0 \) are equivalent. Furthermore, because of the symmetry between the columns, if \( d_1\omega = 0 \), then, one has also \( d_2\omega = d_3\omega = \cdots = 0 \). For instance, again for \( N = 3 \), the derivative of the symmetric tensor \( g = g_{\alpha\beta}d_1x^\alpha d_2x^\beta \) (\( g_{\alpha\beta} = g_{\beta\alpha} \)) is given by
\[
dg = d_1g = \frac{1}{2}(g_{\alpha\beta,\rho} - g_{\rho\beta,\alpha})d_1x^\rho d_1x^\alpha d_2x^\beta.
\]
The completely symmetric component \( g_{(\alpha\beta,\rho)} \) is absent because \( d_1x^\rho d_1x^\alpha = -d_1x^\alpha d_1x^\rho \). Also, it is clear that if \( d_1g = 0 \), then, \( d_2g = \frac{1}{2}(g_{\alpha\beta,\rho} - g_{\alpha\rho,\beta})d_1x^\alpha d_2x^\beta d_2x^\rho = 0 \). This generalizes to the following lemma:

**Lemma 4** Let \( \omega \in \Omega^{(N-1)n}(\mathbb{R}^D) \) (well-filled, or rectangular, tensor). Then
\[
d^k\omega = 0 \iff \left( \prod_{j \in J, \#J = k} d_j \right)\omega = 0.
\] (25)

**Proof** One has \( d^k\omega = (-1)^md_1d_2\cdots d_k\omega \). Indeed, it is clear that the multiform \( d_1d_2\cdots d_k\omega \in \mathfrak{A}^{n+1,n+1,\cdots,n+1,n,\cdots,n} \) belongs to \( \Omega_N(\mathbb{R}^D) \) because it cannot have components along the invariant subspaces corresponding to Young diagrams with first column having \( i > r + 1 \) boxes, since one cannot put two derivatives \( \partial_{\mu}, \partial_{\nu} \) in the same column. Hence, \( d^k\omega = 0 \) is equivalent to \( d_1d_2\cdots d_k\omega = 0 \). One completes the proof by observing that for well-filled tensors, the condition \( d_1d_2\cdots d_k\omega = 0 \) is equivalent to the conditions \( d_{i_1}d_{i_2}\cdots d_{i_k}\omega = 0 \ \forall (i_1, \cdots, i_k) \) because of symmetry in the columns. □

**Lemma 5** Let \( \omega \in \Omega^{(N-1)n}(\mathbb{R}^D) \) with \( n \geq 1 \). Then
\[
\omega = \sum_{J, \#J = N-k} \left( \prod_{j \in J} d_j \right) \alpha_J \Rightarrow \omega = d^{N-k}\alpha
\] (26)
\[29\]
for some \( \alpha \in \Omega^{(N-1)n-N+k}_N(\mathbb{R}^D) \), \( k \in \{1, \ldots, N-1\} \).

**Proof** First, we note that the \( \alpha_J \) occurring in (26) can be chosen to have \( d_i \)-degrees equal to \( n-1 \) or \( n \) according to whether \( d_i \) acts or does not act on \( \alpha_J \), since \( \omega \) has multidegree \( (n, n, \cdots, n) \). Second, one can project the right-hand side of (26) on \( \Omega^{(N-1)n}_N(\mathbb{R}^D) \) without changing the left-hand side, since \( \omega \in \Omega^{(N-1)n}_N(\mathbb{R}^D) \). It is easy to see that \( \pi[(\prod_{j \in J} d_j) \alpha_J] \sim d^{N-k} \alpha'_J \), with \( \alpha'_J = \pi(\tilde{\alpha}_J) \), where \( \tilde{\alpha}_J \) is the element in \( \mathfrak{A}^{n-1,n-1,n-1,n-1,\cdots,n-1} \) obtained by reordering the “columns” of \( \alpha_J \) so that they have non-increasing length. In fact, when differentiated, the other irreducible components of \( \tilde{\alpha}_J \) do not contribute to \( \omega \) because their first column is too long to start with or because two partial derivatives find themselves in the same column, yielding zero. Injecting the above expression for \( \pi[(\prod_{j \in J} d_j) \alpha_J] \) into (26) yields the desired result. \( \square \)

**Lemma 6** Let \( \omega \in \Omega^{(N-1)n}_N(\mathbb{R}^D) \) with \( n \geq 1 \) be a polynomial multiform of degree \( k - 1 \). Then,

\[
\omega = d^{N-k} \alpha
\]

for some polynomial multiform \( \alpha \in \Omega^{(N-1)n-N+k}_N(\mathbb{R}^D) \) of degree \( N-1 \), with \( k \in \{1, \ldots, N-1\} \).

**Proof** The proof amounts to play with Young diagrams. The coefficients of \( \omega \) transform in the tensor product of the representation associated with \( Y^{N}_{(N-1)n} \) (symmetry of \( \omega \)) and the completely symmetric representation with \( k-1 \) boxes (symmetric polynomials in the \( x^{\mu} \)'s of degree \( k-1 \)). Let \( T \) be this representation and \( V_T \) be the carrier vector space. Similarly, the multiform \( \alpha \) transforms (if it exists) in the tensor product of the representation associated
with $Y_{(N-1)n-N+k}^N$ (symmetry of $\alpha$) and the completely symmetric representation with $N-1$ boxes (symmetric polynomials in the $x^\mu$'s of degree $N-1$).

Let $S$ be this representation and $W_S$ be the carrier vector space. Now, the linear operator $d^{N-k}: W_S \to V_T$ is an intertwiner for the representations $S$ and $T$. To analyse how it acts, it is convenient to decompose both $S$ and $T$ into irreducible representations.

The crucial fact is that all irreducible representations occurring in $T$ also occur in $S$. That is, if

$$T = \bigoplus_i T_i, \quad V_T = \bigoplus_i V_i$$

(where each irreducible representation $T_i$ has multiplicity one), then

$$S = \bigoplus_i T_i \oplus (\bigoplus \alpha T_\alpha), \quad W_S = \bigoplus_i W_i \oplus (\bigoplus \alpha W_\alpha)$$

where $T_\alpha$ are some other representations, irrelevant for our purposes. Because $T_i$ is irreducible, the operator $d^{N-k}$ maps the invariant subspace $W_i$ on the invariant subspace $V_i$, and furthermore, $d^{N-k}|_{W_i}$ is either zero or bijective. It is easy to verify by taking simple examples that $d^{N-k}|_{W_i}$ is not zero. Hence, $d^{N-k}|_{W_i}$ is injective, which implies that $d^{N-k}: W_S \to V_T$ is surjective, so that $\omega$ can indeed be written as $d^{N-k}\alpha$ for some $\alpha$. □

**Proof of Theorem** The theorem is a direct consequence of the above two lemmas. (i) Let $\omega \in \Omega_{N}^{(N-1)n}(\mathbb{R}^D)$ (with $n \geq 1$) be annihilated by $d^k$, $d^k\omega = 0$. We write $\omega = \omega' + \omega_0$, where $\omega'$ is of order $k$ and where $\omega_0$ is a polynomial multiform of polynomial degree $< k$. Both $\omega'$ and $\omega_0$ have the symmetry of $\omega$. Also, since $\omega_0$ is trivially annihilated by $d^k$, one has separately $d^k\omega' = 0$ and $d^k\omega_0 = 0$. We consider first $\omega'$. The first lemma implies $(\prod_{j \in I, \#j=k} d_j)\omega' = 0$, from which it follows, using the theorem of
the previous section, that

\[ \omega' = \sum_{J, \#J = N-k} \left( \prod_{j \in J} d_j \right) \alpha_J \]

(see (19)). By the second lemma above, this term can be written as \( d^{N-k} \alpha \).

As we have also seen, the same property holds for \( \omega_0 \). This proves the theorem for \( n \geq 1 \). (ii) The case of \( H_{(k)}^0(\Omega_N(\mathbb{R}^D)) \) is even easier to discuss: for a function, the condition \( d^k f = 0 \) is equivalent to \( \partial_{\mu_1 \cdots \mu_k} f = 0 \) and thus, \( f \) must be of degree strictly less than \( k \). Moreover, it can never be the \( d^{N-k} \) of something, since there is nothing in negative degree. \( \Box \)

It is worth noticing here that, as explained in the introduction, Theorem \( \[ \] \) has a dual counterpart for the \( \delta \)-operator introduced at the end of Section 3 which allows to integrate lots of generalized currents conservation equations. In the last section of this paper we shall sketch another approach for proving Theorem \( \[ \] \) which is based on the appropriate generalization of homotopy for \( N \)-complexes.

6 Structure of \( H_{(k)}^m(\Omega_N(\mathbb{R}^D)) \) for generic \( m \)

In the previous section we have shown that \( H_{(k)}^m(\Omega_N(\mathbb{R}^D)) \) vanishes whenever \( m = (N-1)n \) with \( n \geq 1 \). In the case \( N = 2 \) this is the usual Poincaré lemma which means that the cohomology vanishes in positive degrees. For \( N \geq 3 \) there are degrees \( m \) which do not belong to the set \( \{ (N-1)(n+1) | n \in \mathbb{N} \} \) and it turns out that for such a (generic) degree \( m \), the spaces \( H_{(k)}^m(\Omega_N(\mathbb{R}^D)) \) are non trivial \( (k \in \{1, \ldots, N-1\}) \). More precisely for \( m \in \{0, \ldots, N-2\} \) these spaces are finite-dimensional of strictly positive dimensions whereas for \( m \geq N \) with \( m \neq (N-1)n \) these spaces are infinite-dimensional. In the following we shall explicitly display the case \( N = 3 \) and indicate how to
proceed for the general case \( N \geq 3 \).

In order to simplify the notations let us denote the spaces \( H^m_{(k)}(\Omega_N(\mathbb{R}^D)) \) by \( H^m_{(k)} \) and the graded spaces \( H_{(k)}(\Omega_N(\mathbb{R}^D)) \) by \( H_{(k)}(= \oplus_m H^m_{(k)}) \).

For \( N = 3 \), one has only \( H^1_{(1)} \) and \( H^2_{(2)} \) and Theorem 1 states that \( H^2_{n(1)} = H^2_{n(2)} = 0 \) for \( n \geq 1 \) and that \( H^0_{(1)} \simeq \mathbb{R} \) is the space of constant functions on \( \mathbb{R}^D \) whereas \( H^0_{(2)} \) is the space of polynomial functions of degree less or equal to one on \( \mathbb{R}^D \), i.e. \( H^0_{(1)} \simeq \mathbb{R} \oplus \mathbb{R}^D \). On the other hand, the elements of \( H^1_{(1)} \) identify with the covariant vector fields (or one-forms) \( x \mapsto X(x) \) on \( \mathbb{R}^D \) satisfying

\[
\partial_\mu X_\nu + \partial_\nu X_\mu = 0
\] (28)

which is the equation characterizing the Killing vector fields (i.e. infinitesimal isometries) of the standard euclidean metric \( \sum_{\mu=0}^{D}(dx^\mu)^2 \) of \( \mathbb{R}^D \). The general solution of (28) is \( X_\mu = v_\mu + a_{\mu\nu}x^\nu \) with \( v \in \mathbb{R}^D \) (infinitesimal translations) and \( a \in \wedge^2 \mathbb{R}^D \) i.e. \( a_{\mu\nu} = -a_{\nu\mu} = \text{Cte} \) (infinitesimal rotations). Thus one has \( H^1_{(1)} \simeq \mathbb{R}^D \oplus \wedge^2 \mathbb{R}^D \). Notice that with this terminology we have implicitly identified covariant vector fields with contravariant ones by using the standard metric of \( \mathbb{R}^D \). Notice also that as far as \( H^0_{(1)}, H^0_{(2)} \) and \( H^1_{(1)} \) are concerned nothing change if \( N \geq 3 \). For \( N = 3 \), \( H^1_{(2)} \) identifies with the space of covariant vector fields \( x \mapsto X(x) \) on \( \mathbb{R}^D \) satisfying

\[
\partial_\lambda(\partial_\mu X_\nu - \partial_\nu X_\mu) = 0
\] (29)

modulo the ones of the form \( X_\mu = \partial_\mu \varphi \) for some \( \varphi \in C^\infty(\mathbb{R}^D) \). The general solution of (29) is \( X_\mu = a_{\mu\nu}x^\nu + \partial_\mu \varphi \) with \( a \in \wedge^2 \mathbb{R}^D \) and \( \varphi \in C^\infty(\mathbb{R}^D) \) so that one has \( H^1_{(2)} \simeq \wedge^2 \mathbb{R}^D \). Let us now show that \( H^3_{(1)} \) is infinite-dimensional.
for \( N = 3 \). For this, consider an arbitrary 2-form \( \omega \) i.e. an arbitrary covariant antisymmetric tensor field of degree 2 on \( \mathbb{R}^D \) and consider the element 
\[
t = Y_3^3 \circ \nabla (0) \omega \in \Omega^3(\mathbb{R}^D).
\]
Up to an irrelevant normalization constant, the components of \( t \) are given by
\[
t_{\mu\lambda\nu} = 2 \partial_\lambda \omega_{\mu\nu} + \partial_\mu \omega_{\lambda\nu} - \partial_\nu \omega_{\lambda\mu}
\]
and one verifies that one has \( dt = 0 \) in \( \Omega_3(\mathbb{R}^D) \). On the other hand one has \( t = dh \in \Omega_3(\mathbb{R}^D) \) that is
\[
2 \partial_\lambda \omega_{\mu\nu} + \partial_\mu \omega_{\lambda\nu} - \partial_\nu \omega_{\lambda\mu} = \partial_\nu h_{\mu\lambda} - \partial_\mu h_{\nu\lambda}
\]
for some symmetric covariant tensor field \( h \in \Omega^2_3(\mathbb{R}^D) \) if and only if \( \omega \) is of the form
\[
\omega_{\mu\nu} = a_{\rho\mu\nu} x^\rho + \partial_\mu X_\nu - \partial_\nu X_\mu
\]
for \( a \in \Lambda^3 \mathbb{R}^D \) and some covariant vector field \( X \in \Omega^1_3(\mathbb{R}^D) \) and then \( t \) is proportional to \( d^2(X) \) in \( \Omega_3(\mathbb{R}^D) \) i.e. \( t \) is trivial in \( H^3_1 \). This argument shows firstly that \( H^3_1 \) contains the quotient of the space of 2-forms by the ones of the form given by (32) which is infinite-dimensional and secondly that the same space identifies with a subspace of \( H^3_2 \) which is therefore also infinite-dimensional. In fact as will be shown below one has an isomorphism
\[
H^3_1 \simeq H^3_2 \text{ which is induced by the inclusion } \text{Ker}(d) \subset \text{Ker}(d^2).
\]
By replacing the 2-form \( \omega \) by an irreducible covariant tensor field \( \omega_n \) of degree \( 2n + 2 \) on \( \mathbb{R}^D \) with Young symmetry type given by the Young diagram with \( n \) lines of length two and two lines of length one, it can be shown similarly that \( H^{2(n+1)+1}_1 \) and \( H^{2(n+1)+1}_2 \) are infinite-dimensional spaces (we shall see that they are in fact isomorphic).
The last argument for \( N = 3 \) admits the following generalization for \( N \geq 3 \). Let \( Y^N_m \) be a Young diagram of the sequence \( (Y^N) \) and let \( Y'_{m-p} \) be a Young diagram obtained by deleting \( p \) boxes of \( Y^N_m \) with \( 0 < p < N - 1 \) such that it does not belong to \( (Y^N) \) (i.e. \( Y'_{m-p} \neq Y^N_{m-p} \)) and such that by applying \( p \) derivatives (i.e. \( (0) \nabla^p \)) to a generic tensor field with Young symmetry \( Y'_{m-p} \) one obtains a tensor which has a nontrivial component \( t \) with Young symmetry \( Y^N_m \). Then generically the latter \( t \in \Omega^m_N(\mathbb{R}^D) \) is a nontrivial generalized cocycle and one obtains by this procedure an infinite dimensional subspace of the corresponding generalized cohomology, i.e. of \( H^m_{(k)} \) for the appropriate \( k \). Notice that this is only possible for \( m \geq N \) with \( m \neq (N - 1)n \). We conjecture that the whole nontrivial part of the generalized cohomology of \( \Omega_N(\mathbb{R}^D) \) in degree \( m \geq N \) is obtained by the above construction \( (N \geq 3) \).

In order to complete the discussion for \( N \geq 3 \) in degree \( m \leq N - 2 \) as well as to show the isomorphisms \( H^{2n+1}_{(1)} \simeq H^{2n+1}_{(2)} \) for \( N = 3 \), \( n \geq 1 \) and their generalizations for \( N \geq 3 \), we now recall a basic lemma of the general theory of \( N \)-complexes [7], [12]. This lemma was formulated in [7] in the more general framework of \( N \)-differential modules (Lemma 1 of [7]) that is of \( \mathbf{k} \)-modules equipped with an endomorphism \( d \) such that \( d^N = 0 \) where \( \mathbf{k} \) is a unital commutative ring. In this paper we only discuss \( N \)-complexes of (real) vector spaces. Let \( E \) be a \( N \)-complex of cochain [7] like \( \Omega_N(\mathbb{R}^D) \), that is here \( E = \oplus_{m \in \mathbb{N}} E^m \) is a graded vector space equipped with an endomorphism \( d \) of degree one such that \( d^N = 0 \) (\( N \geq 2 \)). The inclusions \( \text{Ker}(d^k) \subset \text{Ker}(d^{k+1}) \) and \( \text{Im}(d^{N-k}) \subset \text{Im}(d^{N-k-1}) \) induce linear mappings \( [i] : H_{(k)} \to H_{(k+1)} \) in generalized cohomology for \( k \) such that \( 1 \leq k \leq N - 2 \). Similarly the linear mappings \( d : \text{Ker}(d^{k+1}) \to \text{Ker}(d^k) \) and \( d : \text{Im}(d^{N-k-1}) \to \text{Im}(d^{N-k}) \) obtained by restriction of the \( N \)-differential \( d \).
induce linear mappings $[d] : H_{(k+1)} \to H_{(k)}$. One has the following lemma (for a proof we refer to [12] or [7]).

**LEMMA 7** Let the integers $k$ and $\ell$ be such that $1 \leq k$, $1 \leq \ell$, $k + \ell \leq N - 1$. Then the hexagon of linear mappings

$$
\begin{array}{cccccc}
H_{(\ell+k)}(E) & \xrightarrow{[d]^k} & H_{(\ell)}(E) & \xrightarrow{[i]^{\ell}} & H_{(k)}(E) \\
& \downarrow{[i]^{\ell}} & & & \downarrow{[d]^{N-(\ell+k)}} \\
H_{(N-k)}(E) & & H_{(N-\ell)}(E) & \xleftarrow{[i]^{k}} & [d]^{\ell} \\
& [d]^{N-(\ell+k)} & & & \\
\end{array}
$$

is exact.

Since $[i]$ is of degree zero while $[d]$ is of degree one, these hexagons give long exact sequences.

Let us apply the above result to the $N$-complex $\Omega_N(\mathbb{R}^D)$. For $N = 3$, there is only one hexagon as above ($k = \ell = 1$) and, by using $H_{(k)}^{2n} = 0$ for $n \geq 1$, $k = 1, 2$ it reduces to the exact sequences

$$
0 \xrightarrow{[d]} H_{(1)}^{0} \xrightarrow{[i]} H_{(2)}^{0} \xrightarrow{[d]} H_{(1)}^{1} \xrightarrow{[i]} H_{(2)}^{1} \xrightarrow{d} 0 \quad (33)
$$

and

$$
0 \xrightarrow{d} H_{(1)}^{2n+1} \xrightarrow{[i]} H_{(2)}^{2n+1} \xrightarrow{d} 0 \quad (34)
$$

for $n \geq 1$. The sequences (34) give the announced isomorphisms $H_{(1)}^{2n+1} \simeq H_{(2)}^{2n+1}$ while the 4-terms sequence (33) allows to compute the finite dimension of $H_{(2)}^{1}$ knowing the one of $H_{(1)}^{0}$, $H_{(2)}^{0}$ and $H_{(1)}^{1}$. For $N \geq 3$ one has several
hexagons and by using $H^{(N-1)n}_k = 0$ for $n \geq 0$, the sequence (34) generalizes as the following \[(N-2)(N-1)\] four-terms exact sequences

\[
0 \xrightarrow{d^k} H^{k-1}_\ell \xrightarrow{[i]^{N-k-\ell}} H^{k-1}_{N-k} \xrightarrow{d^\ell} H^{k+\ell-1}_{N-k-\ell} \xrightarrow{[i]^k} H^{k+\ell-1}_{N-\ell} \xrightarrow{d^{N-k-\ell}} 0 \quad (35)
\]

for $1 \leq k, \ell$ and $k + \ell \leq N - 1$. There are also two-terms exact sequences generalizing (34) giving similar isomorphisms but, for $N > 3$, there are other longer exact sequences (which are of finite lengths in view of $H^{(N-1)n}_k = 0$ for $n \geq 1$). Suppose that the spaces $H^m_k$ are finite-dimensional for $k + m \leq N - 1$ and that we know their dimensions. Then the exact sequences (35) imply that all the $H^m_k$ for $m \leq N - 2$ are finite-dimensional and allows to compute their dimensions in terms of the dimensions of the $H^m_k$ for $k + m \leq N - 1$. To complete the discussion it thus remains to show the finite-dimensionality of the $H^m_k$ for $k + m \leq N - 1$. For $k + m \leq N - 1$, the space $H^m_k$ identifies with the space of (covariant) symmetric tensor fields $S$ of degree $m$ on $\mathbb{R}^D$ such that

\[
\sum_{\pi \in S_{k+m}} \partial_{\mu_{\pi(1)}} \cdots \partial_{\mu_{\pi(k)}} S_{\mu_{\pi(k+1)}} \cdots \mu_{\pi(k+m)} = 0 \quad (36)
\]

for $\mu_i \in \{1, \ldots, D\}$ where $S_{k+m}$ is the group of permutation of $\{1, \ldots, k+m\}$. In particular, for $k = 1$ the equation (36) means that $S$ is a Killing tensor field of degree $m$ for the canonical metric of $\mathbb{R}^D$ and it is well known and easy to show that the components of such a Killing tensor field of degree $m$ are polynomial functions on $\mathbb{R}^D$ of degree less or equal to $m$. In fact the Killing tensor fields on $\mathbb{R}^D$ form an algebra for the symmetric product over each point of $\mathbb{R}^D$ which is generated by the Killing vector fields (which are polynomial of degree $\leq 1$). Thus $H^m_{(1)}$ is finite-dimensional for $1 + m \leq N - 1$. By using this together with Theorem 1, one shows by induction on $k$ that $H^m_{(k)}$ is finite-dimensional for $k + m \leq N - 1$, more precisely, that the solutions
of (36) are polynomial functions on $\mathbb{R}^D$ of degree less than $k + m$.

The results of this section concerning the generic degrees show that our generalization of the Poincaré lemma, i.e. Theorem 1, is far from being a straightforward result and that it is optimal.

7 Algebras

Let $E \cong \mathbb{R}^D$ be a $D$-dimensional vector space, $(Y)$ be a sequence $(Y) = (Y_p)_{p \in \mathbb{N}}$ of Young diagrams such that $|Y_p| = p$ ($\forall p \in \mathbb{N}$) and let us use the notations and conventions of Section 3. As we have seen, the graded space $\wedge(Y)E$ is a generalization of the exterior algebra of $E$ in the sense that as graded vector space it reduces to the latter when $(Y)$ coincides with the sequence $(Y^2) = (Y^2_p)_{p \in \mathbb{N}}$ of the one-column Young diagrams. It is also a generalization of the symmetric algebra of $E$ since it reduces to it when $(Y)$ coincides with the sequences $(\tilde{Y}^2) = (\tilde{Y}^2_p)_{p \in \mathbb{N}}$ of the one-line Young diagrams (which are the conjugates of the $Y^2_p$). In fact, for general $(Y)$ the graded vector space $\wedge(Y)E$ is also a graded algebra if one defines the product by setting

$$TT' = Y_{p+p'}(T \otimes T')$$

for $T \in E^{Y_p}$ and $T' \in E^{Y_{p'}}$ where $Y_n$ is the Young symmetrizer defined in Section 2. However, although it generalizes the exterior product, this product is generically a nonassociative one. Thus $\wedge(Y)E$ is a generalization of the exterior algebra $\wedge E$ in which each homogeneous subspace is irreducible for the action of $GL(E) \cong GL_D$ but in which one loses the associativity of the product. There is another closely related generalization of the exterior algebra connected with the sequence $(Y)$ in which what is retained is the
associativity of the graded product but in which one generically loses the $GL(E)$-irreducibility of the homogeneous components. This generalization, denoted by $\wedge_{[\{Y\}]} E$, is such that $\wedge_{\{Y\}} E$ is a graded (right) $\wedge_{[\{Y\}]} E$-module. We now describe its construction.

Let $T(E)$ be the tensor algebra of $E$, we use the product defined by (37) to equip $\wedge_{\{Y\}} E$ with a right $T(E)$-module structure by setting

$$T \lambda_{\{Y\}}(X_1 \otimes \cdots \otimes X_n) = (\cdots (TX_1) \cdots) X_n$$

for any $X_i \in E$ and $T \in \wedge_{\{Y\}} E$. By definition the kernel $\text{Ker}(\lambda_{\{Y\}})$ of $\lambda_{\{Y\}}$ is a two-sided ideal of $T(E)$ so that the right action of $T(E)$ on $\wedge_{\{Y\}} E$ is in fact an action of the quotient algebra $\wedge_{[\{Y\}]} E = T(E) / \text{Ker}(\lambda_{\{Y\}})$. So $\wedge_{\{Y\}} E$ is a graded right $\wedge_{[\{Y\}]} E$-module.

**Lemma 8** Let $N$ be an integer with $N \geq 2$ and assume that $(Y)$ is such that the number of columns of the Young diagram $Y_p$ is strictly smaller than $N$ for any $p \in \mathbb{N}$. Then $\text{Ker}(\lambda_{\{Y\}})$ contains the two-sided ideal of $T(E)$ which consists of the tensors which are symmetric with respect to at least $N$ of their entries; in particular $(\lambda_{\{Y\}}(X))^N = 0$, $\forall X \in E$.

Stated differently, under the assumption of the lemma for $(Y)$, a monomial $X_1 \cdots X_n \in \wedge_{[\{Y\}]} E$ with $X_i \in E$ vanishes whenever it contains $N$ times the same argument, that is if there are $N$ distinct elements $i_1, \ldots, i_N$ of $\{1, \ldots, n\}$ such that $X_{i_1} = \cdots = X_{i_N}$.

**Proof** This is straightforward, as for the proof of Lemma 1, since one has more than $N$ symmetrized entries which are distributed among less than $N - 1$ columns in which the entries are antisymmetrized. □

The right action $\lambda_{\{Y\}}(N)$ of $T(E)$ on $\wedge_{N} E$ will also be simply denoted by $\lambda_N$. In the case $N = 2$, $\wedge_{2} E$ is the usual exterior algebra $\wedge E$ of $E$ and
the right action $\lambda_2$ of $T(E)$ factorizes through the right action of $\wedge E$ on itself, in particular $\ker(\lambda_2)$ is the two-sided ideal of $T(E)$ generated by the $X \otimes X$ for $X \in E$. Thus the graded algebra $\wedge_{[Y]}E = T(E)/\ker(\lambda_Y)$ is also a generalization of the exterior algebra of $E$. For $(Y) = (Y^N)$, $\wedge_{[Y \cdot N]}E$ will be simply denoted by $\wedge_{[N]}E$. One clearly has $\wedge_{[2]}E = \wedge_2E = \wedge E$ for $N = 2$. In the case $N = 3$, it can be shown that $\ker(\lambda_3)$ is the two-sided ideal of $T(E)$ generated by the

$$X \otimes Y \otimes Z + Z \otimes X \otimes Y + Y \otimes Z \otimes X$$

and the

$$X \otimes Y \otimes X \otimes X$$

for $X, Y, Z \in E$. This implies that one has

$$\lambda_3(X)\lambda_3(Y)\lambda_3(Z) + \lambda_3(Z)\lambda_3(X)\lambda_3(Y) + \lambda_3(Y)\lambda_3(Z)\lambda_3(X) = 0$$

and

$$\lambda_3(X)\lambda_3(Y)(\lambda_3(X))^2 = 0$$

for any $X, Y, Z \in E$ and that these are the only independent relations in the associative algebra $\text{Im}(\lambda_3) = \wedge_{[3]}E$. This means that $\wedge_{[3]}E$ is the associative unital algebra generated by the subspace $E$ with relations $XYZ + ZXY + YZX = 0$ and $XYX^2 = 0$ for $X, Y, Z \in E$. The graduation is induced by giving the degree one to the elements of $E$ which is consistent since the relations are homogeneous. It is clear on this example that the homogeneous subspaces $\wedge_{[N]}^pE$ of $\wedge_{[N]}E$ are generally not irreducible for the (obvious) action of $GL(E)$. It is not hard to see that one has

$$\omega_0 \wedge_{[N]}E = \wedge_N E$$

40
where $\omega_0$ is a generator ($\simeq 1$) of $\wedge_N E \simeq \mathbb{R}$, that is $\omega_0$ is a cyclic vector for the action of $\wedge_N E$ on $\wedge_N E$.

Corresponding to the generalization $\wedge_{[(Y)]} E$ of the exterior algebra there is a generalization $\Omega_{[(Y)]}(M)$ of differential forms on a smooth manifold $M$ which is defined in a similar way as $\Omega_{(Y)}(M)$ was defined in Section 3. This $\Omega_{[(Y)]}(M)$ is then a graded associative algebra and $\Omega_{(Y)}(M)$ is a right graded $\Omega_{[(Y)]}(M)$-module (etc.). In the case $(Y) = (Y^N)$ one writes $\Omega_{[N]}(M)$ for this generalization. For $M = \mathbb{R}^D$ one has

$$\Omega_{[N]}(\mathbb{R}^D) = \wedge_{[N]} \mathbb{R}^{D*} \otimes C^\infty(\mathbb{R}^D)$$

and, by identifying $\Omega_{[N]}(\mathbb{R}^D)$ as a graded-subspace of $T(\mathbb{R}^{D*}) \otimes C^\infty(\mathbb{R}^D)$ and by using the canonical flat torsion-free linear connection of $\mathbb{R}^D$ one can define a $N$-differential $d$ on $\Omega_{[N]}(\mathbb{R}^D)$ by appropriate projection. One can proceed similarly for $\Omega_{[(Y)]}(\mathbb{R}^D)$ when $(Y)$ satisfies the assumption of Lemma 1 (or Lemma 2, Lemma 7). More precisely, the $N$-complexes constructed so far are particular cases of the following general construction.

Let $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A^n$ be an associative unital graded algebra generated by $D$ elements of degree one $\theta^\mu$ for $\mu \in \{1, \ldots, D\}$ such that

$$\sum_{\rho \in S_N} \theta_{\mu(1)}^\rho \ldots \theta_{\mu(N)}^\rho = 0 \quad (39)$$

for any $\mu_1, \ldots, \mu_N \in \{1, \ldots, D\}$. Then the algebra $\mathcal{A}(\mathbb{R}^D)$ defined by $\mathcal{A}(\mathbb{R}^D) = \mathcal{A} \otimes C^\infty(\mathbb{R}^D)$ is a graded algebra and one defines a $N$-differential $d$ on $\mathcal{A}(\mathbb{R}^D)$, i.e. a linear mapping $d$ of degree one satisfying $d^N = 0$, by setting

$$d(a \otimes f) = (-1)^n a \theta^\mu \otimes \partial_\mu f \quad (40)$$
for \(a \in \mathcal{A}^n\) and \(f \in C^\infty(\mathbb{R}^D)\). Let \(\mathcal{M} = \bigoplus_n \mathcal{M}^n\) be a graded right \(\mathcal{A}\)-module, then \(\mathcal{M}(\mathbb{R}^D) = \mathcal{M} \otimes C^\infty(\mathbb{R}^D)\) is a graded space which is a graded right \(\mathcal{A}(\mathbb{R}^D)\)-module and one defines a \(N\)-differential \(d\) on \(\mathcal{M}(\mathbb{R}^D)\) by setting
\[
d(m \otimes f) = (-1)^n m \mathcal{A} \otimes \partial f
\]
for \(m \in \mathcal{M}^n\) and \(f \in C^\infty(\mathbb{R}^D)\). The (irrelevant) sign \((-1)^n\) in formulas (40) and (41) is here in order to recover the usual exterior differential in the case where \(\mathcal{A} = \wedge \mathbb{R}^{D*} = \mathcal{M}\).

It is clear that \(\Omega_{[N]}(\mathbb{R}^D) = \mathcal{A}(\mathbb{R}^D)\) for \(\mathcal{A} = \wedge_{[N]} \mathbb{R}^{D*}\) and that \(\Omega_N(\mathbb{R}^D) = \mathcal{M}(\mathbb{R}^D)\) for \(\mathcal{M} = \wedge_N \mathbb{R}^{D*}\). If \((Y)\) satisfies the assumption of Lemma 1 one can take (in view of Lemma 7) \(\mathcal{A} = \wedge_{([Y])} \mathbb{R}^{D*}\) and \(\mathcal{M} = \wedge_{([Y])} \mathbb{R}^{D*}\) and then \(\Omega_{([Y])}(\mathbb{R}^D) = \mathcal{A}(\mathbb{R}^D)\) and \(\Omega_{([Y])}(\mathbb{R}^D) = \mathcal{M}(\mathbb{R}^D)\).

8 Further remarks

Our original unpublished project for proving Theorem 1 was based on the construction of generalized algebraic homotopy in appropriate degrees. Let us explain what it means, why it is rather cumbersome and why the proof given here, based on the introduction of the multigraded differential algebra \(\mathfrak{A}\), is much instructive and general and is related to the ansatz of Green for the fermionic parastatistics of order \(N - 1\) (in the case \(d^N = 0\)).

Let \(\Omega = \bigoplus_n \Omega^n\) be a \(N\)-complex (of cochains) with \(N\)-differential \(d\). An algebraic homotopy for the degree \(n\) will be a family of \(N\) linear mappings
\[
h_k : \Omega^{n+k} \rightarrow \Omega^{n+k-N+1}
\]
for \(k = 0, \ldots, N - 1\) such that \(\sum_{k=0}^{N-1} d^{N-1-k}h_kd^k\) is the identity mapping \(I_n\) of \(\Omega^n\) onto itself. If such a homotopy exists for the degree \(n\), then one has
\( H^k_{(k)} = 0 \) for \( k \in \{0, \ldots, N-1\} \). Indeed let \( \omega \in \Omega^n \) be such that \( d^k \omega = 0 \) then one has \( \omega = d^{N-k} \left( \sum_{p=0}^{k-1} d^{k-1-p} h_p \partial^p \omega \right) \).

Our original strategy for proving Theorem 1 was to show that one can construct inductively such homotopies for the degrees \((N-1)p\) with \( p \geq 1 \) in the case of the \( N \)-complex \( \Omega_N(\mathbb{R}^D) \) and our idea was to exhibit explicit formulas. Unfortunately this latter task seems very difficult in general. We only succeeded in producing formulas in a closed form in the case \( N = 3 \) and we refrain to give them here because this would imply explanations of our normalization conventions which have no character of naturality. The difficulty is indeed a problem of normalization. For the classical case \( N = 2 \), one obtains a homotopy formula by using the inner derivation \( i_x \) with respect to the vector field \( x \) with components \( x^\mu \). In this case one uses the fact that both \( d \) and \( i_x \) are antiderivations and that the Lie derivative \( L_x = di_x + i_x d \) is the sum of the form-degree and of the degree of homogeneity in \( x \). This gives homotopy formulas for forms which are homogeneous polynomials in \( x \) and one gets rid of the above degree by appropriately weighted radial integration and obtains thereby the usual homotopy formula for positive form-degree. In this case the normalizations are fixed by the (anti)derivation properties. In the case \( N \geq 3 \), \( d \) has no derivation property and one has to generalize \( i_x \) which is possible with \( i_x^N = 0 \) but there is no natural normalization since \( i_x \) cannot possess derivation property. As a consequence the appropriate generalization of the Lie derivative involves a linear combination of products of powers of \( d \) and \( i_x \) with coefficients which have to be fixed at each step. That this is possible constitutes a cumbersome proof of Theorem 1 but does not allow easily to write closed formulas.
The interest of the proof of Theorem 1 presented here lies in the fact that it follows from the more general Theorem 2 which can be applied to other situations in particular to investigate the generalized cohomology of \( \Omega_{[N]}(\mathbb{R}^D) \). Moreover, the realization of \( \Omega_N(\mathbb{R}^D) \) embedded in \( \mathfrak{A} \) is related to the Green ansatz for the parafermionic creation operators of order \( N - 1 \). Indeed if instead of equipping \( \mathfrak{A} \) with the graded commutative product one replaces in the definition of \( \mathfrak{A} \) the graded tensor products of graded algebras by the ordinary tensor products of algebras (applying the appropriate Klein transformation) then the \( d_i x^\mu \) and the \( d_j x^\nu \) commute for \( i \neq j \) and the \( d_i \) defined by the same formulas (14) commute, i.e. satisfy \( d_i d_j = d_j d_i \) instead of (15), from which it follows that \( \sum_i d_i \) is only a \( N \)-differential. This latter \( N \)-differential induces the \( N \)-differential of \( \Omega_N(\mathbb{R}^D) \subset \mathfrak{A} \) and the relation with the Green ansatz becomes obvious after Fourier transformation.

The basic \( N \)-complexes considered in this paper are \( N \)-complexes of smooth tensor fields on \( \mathbb{R}^D \) and we have seen the difficulty to extend the formalism on an arbitrary \( D \)-dimensional manifold \( M \). In the case of a complex (holomorphic) manifold \( M \) of complex dimension \( D \), there is an extension of the previous formalism at the \( \bar{\partial} \)-level which we now describe shortly.

Let \( M \) be a complex manifold of complex dimension \( D \) and let \( T \) be a smooth covariant tensor field of type \((0, p)\) (i.e. of \( d\bar{z} \)-degree \( p \)) with local components \( T_{\bar{\mu}_1...\bar{\mu}_p} \) in local holomorphic coordinates \( z^1, \ldots, z^D \). Then \( \partial_{\bar{\mu}_{p+1}} T_{\bar{\mu}_1...\bar{\mu}_p} \) are the components of a well-defined smooth covariant tensor field \( \nabla T \) of type \((0, p + 1)\) since the transition functions are holomorphic, where \( \partial_{\bar{\mu}} \) denotes the partial derivative \( \partial/\partial \bar{z}^\mu \) of smooth functions. Let \( (Y) \) be a sequence \( (Y_p)_{p \in \mathbb{N}} \) of Young diagrams such that \( |Y_p| = p \) (\( \forall p \in \mathbb{N} \)
and denote by $\Omega^{0,p}_{(Y)}(M)$ the space of smooth covariant tensor fields of type $(0,p)$ with Young symmetry type $Y_p$ (with obvious notation). Let us set $\Omega^{0,\ast}_{(Y)}(M) = \oplus_p \Omega^{0,p}_{(Y)}(M)$ and generalize the $\bar{\partial}$-operator by setting

$$\bar{\partial} = (-1)^{p}Y_{p+1} \circ \bar{\nabla} : \Omega^{0,p}_{(Y)}(M) \to \Omega^{0,p+1}_{(Y)}(M)$$

with obvious conventions. It is clear that if $(Y)$ is such that for any $p \in \mathbb{N}$ the number of columns of $Y_p$ is strictly less than $N$, then one has $\bar{\partial}^N = 0$ so $\Omega^{0,\ast}_{(Y)}(M)$ is a $N$-complex (for $\bar{\partial}$). In particular one has the $N$-complex $\Omega^{0,\ast}_{N}(M)$ for $\bar{\partial}$ by taking $(Y) = (Y^N)$. One has an obvious extension of Theorem 1 ensuring that the generalized $\bar{\partial}$-cohomology of $\Omega^{0,\ast}_{(Y)}(\mathbb{C}^D)$ vanishes in degree $(N-1)p$ (i.e. bidegree or type $(0,(N-1)p)$) for $p \geq 1$. It is thus natural to seek for an interpretation of this generalized cohomology for $\Omega^{0,\ast}_{N}(M)$ in degrees $(N-1)p$ with $p \geq 1$ for an arbitrary complex manifold $M$ and one may wonder whether it can be computed in terms of the ordinary $\bar{\partial}$-cohomology of $M$.

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