A Vector Supersymmetry Killing IR Divergences in Non-Commutative Gauge Theories

Talk presented by Daniel N. Blaschke

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[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}
VSUSY Killing
IR
Divergences in NCGFT

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Introduction

Slavnov Term Idea
Symmetries & Consequences

Generalization

Conclusion and Outlook

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Weyl-Moyal correspondence

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \]

- definition of the Weyl-Moyal \(*\)-product:

\[
A_\rho(x) \ast A_\sigma(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu} A_\rho(x) A_\sigma(y) \bigg|_{x=y} \\
\neq A_\sigma(x) \ast A_\rho(x)
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**Weyl-Moyal correspondence**

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- invariance under cyclic permutations of the integral

\[
\int d^4 x A_\mu(x) \ast A_\rho(x) \ast A_\sigma(x) = \int d^4 x A_\sigma(x) \ast A_\mu(x) \ast A_\rho(x) \\
\implies \int d^4 x A_\mu(x) \ast A_\rho(x) = \int d^4 x A_\mu(x) A_\rho(x)
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- interaction vertices gain phases, whereas propagators remain unchanged
QFT on $\theta$-deformed space-time

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\int d^4k \frac{e^{ik\tilde{p}}}{k^2 + i\epsilon} \propto \frac{1}{\tilde{p}^2} \quad \text{with} \quad \tilde{p}^\mu = \theta^{\mu\nu} p_\nu
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- phases act as UV-regulators,

\[\Rightarrow\] origin of the \textit{UV/IR mixing} problem
An action such as

\[ S = -\frac{1}{4} \int d^4x F_{\mu\nu} \star F^{\mu\nu} + \ldots \]

with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \)

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\[ \Pi_{\text{IR}}^{\mu\nu}(k) = \frac{2g^2}{\pi^2} \frac{\tilde{k}^\mu \tilde{k}^\nu}{(\tilde{k}^2)^2} \quad \text{with} \quad \tilde{k}^\mu = \theta^{\mu\nu} k_\nu \quad (1) \]

⇒ Graphs with this insertion are IR divergent!
Slavnov’s extension

Slavnov has proposed a modification of Yang-Mills theories, adding to the action a term

\[
\frac{1}{2} \int d^4 x \, \lambda \star \theta^{\mu \nu} F_{\mu \nu}
\]  

⇒ makes gauge field propagator transversal with respect to \( \tilde{k}^\mu \)
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**Figure:** this graph has now become *IR finite*
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- a \( \lambda \)-propagator, a mixed \( \lambda A \)-propagator and a \( \lambda AA \)-vertex
- Slavnov trick does not work for certain diagrams, i.e.

**Figure:** this graph is *IR divergent*
To avoid unitarity problems we choose the non-commutativity tensor spacelike, i.e.

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Gauge fixing chosen axial in the plane of the non-commutative coordinates:

\[ n^I = 0, \quad I = 0, 3 \]
Slavnov Term and BF Model

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- Gauge fixing chosen axial in the plane of the non-commutative coordinates:

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- The Slavnov term, together with the gauge fixing terms, now have the form of a 2-dimensional topological BF model.
Slavnov Term and BF model

Action:

\[ S_{\text{inv}} = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\theta}{2} \lambda \star \epsilon^{ij} F_{ij} \right) \]

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu \star, A_\nu] \]
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and

\[ S_{\text{gf}} = \int d^4x \left( B \star n^i A_i - \bar{c} \star n^i D_i c \right) \]

with

\[ D_\mu c = \partial_\mu c - ig \left[ A_\mu \star, c \right] \]
Symmetries of the Action

$S$ is invariant under

**BRS:**  
$$sA_\mu = D_\mu c, \quad s\bar{c} = B, \quad s\lambda = -ig [\lambda, c], \quad sB = 0, \quad sc = \frac{ig}{2} [c, c], \quad s^2 = 0.$$
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**VSUSY:**

$$\delta_i A_\mu = 0 , \quad \delta_i c = A_i ,$$
$$\delta_i \bar{c} = 0 , \quad \delta_i B = \partial_i \bar{c} ,$$
$$\delta_i \lambda = \frac{\epsilon_{ij}}{\theta} n^j \bar{c} , \quad \delta^2 = 0 .$$
Note: Only the interplay of appropriate choices for $\theta^{\mu\nu}$ and $n^\mu$ lead to the existence of the VSUSY.
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In contrast to the pure topological theories, we have an additional vectorial symmetry:

$$\hat{d}_i A_J = -F_{iJ} , \quad \hat{d}_i \lambda = -\frac{\epsilon_{ij}}{\theta} D_K F^{Kj} ,$$

$$\hat{d}_i \Phi = 0 \quad \text{for all other fields} .$$
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$\Rightarrow$ The algebra involving $s$, $\delta_i$, $\hat{d}_i$ and the $(1,2)$-plane translation generator $\partial_i$ closes on-shell.
Ward id. for Green functions

Legendre transformation

$$S_{\text{tot}}[\phi, \phi^*, \ldots] \rightarrow Z^c[j_\phi, \phi^*, \ldots]$$

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⇒ WI for VSUSY at vanishing antifields transforms into

\[ \mathcal{W}_i Z^c \bigg|_{\{A^*, \lambda^*, c^*\} \rightarrow 0} = 0 \]

\[ \Rightarrow \int d^4x \left\{ j_B \frac{\delta Z^c}{\delta j\bar{c}} - j_c \frac{\delta Z^c}{\delta j_A} + \frac{\epsilon_{ij}}{\theta} n^j j_\lambda \frac{\delta Z^c}{\delta j\bar{c}} \right\} = 0 \]
Differentiating this with respect to $j_c$ and $j_A^\mu$ yields for the gauge field propagator:

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It is impossible to construct a closed loop including a $\lambda A A$-vertex without having such a combination somewhere.

\[ \Rightarrow \text{All loop graphs involving the } \lambda A A\text{-vertex vanish!} \]
Absence of IR singularities

In particular, dangerous vacuum polarization insertions as in the following figure vanish:

\[ \Pi_{\text{IR}}^{\mu\nu}(k) = \frac{2g^2}{\pi^2} \frac{\tilde{k}^\mu \tilde{k}^\nu}{(\tilde{k}^2)^2} \quad \text{with} \quad \tilde{k}^\mu = \theta^{\mu\nu} k_\nu \]
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\Rightarrow \text{model is free of the most dangerous infrared singularities!}
Can we show cancellation of IR singular Feynman graphs for a more general choice of $\theta^{\mu\nu}$ and $n^{\mu}$?
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More general choice of $\theta^{\mu\nu}$ and $n^\mu$

- Can we show cancellation of IR singular Feynman graphs for a more general choice of $\theta^{\mu\nu}$ and $n^\mu$?

  ⇒ The answer is yes, but we need to impose stronger Slavnov constraints

- initial Slavnov constraint was

  $\theta^{12} F_{12} + \theta^{13} F_{13} + \theta^{23} F_{23} = 0$

- Upon introducing stronger constraints we may write

  $$S_{\text{inv}} = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon^{ijk} F_{ij \lambda} \lambda_k \right]$$

  with $i, j, k \in \{1, 2, 3\}$

- Looks like 3 dim. BF model coupled to Maxwell theory
The action has a second gauge symmetry

\[ \delta g_2 \lambda_k = D_k \Lambda', \quad \delta g_2 A_\mu = 0 \]
Properties of this new action

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Similar to the previous model, we have an additional bosonic vector symmetry:

\[ \hat{d}_i A_0 = -F_{i0}, \quad \hat{d}_i \lambda_j = \epsilon_{ijk} D_0 F^{0k}, \]

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**Difference:** This symmetry is broken when fixing the second
gauge symmetry!
The gauge fixed action, with space-like axial gauge

$$S_{gf} = \int d^4x \left[ Bn^i A_i + d'n^i \lambda_i - \bar{c}n^i D_i c - \bar{\phi}n^i D_i \phi \right],$$

is invariant under the linear VSUSY

$$\delta_i c = A_i, \quad \delta_i \lambda_j = -\epsilon_{ijk} n^k \bar{c},$$
$$\delta_i B = \partial_i \bar{c},$$
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\end{align*}
\]

for all other fields.

\((d' = d - ig [\bar{\phi}, c] \text{ is the rescaled multiplier field fixing the second gauge freedom.})\)
Ward identity describing the linear vector supersymmetry in terms of $Z^c$ is given by

$$\mathcal{W}_i Z^c = \int d^4 x \left[ j_B \partial_i \frac{\delta Z^c}{\delta j^\bar{c}} - j_c \frac{\delta Z^c}{\delta j^i_A} + \epsilon_{ijk} n^j j^k \frac{\delta Z^c}{\delta j^\bar{c}} \right] = 0.$$
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- model exhibits numerous further symmetries
The vector supersymmetry

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- same arguments as before show absence of IR singular graphs
- model exhibits numerous further symmetries
- generalization to higher dimensional models is possible, i.e. if $\lambda$ had $n$ indices the VSUSY would become

$$\delta_i c = A_i \quad \delta_i \lambda_{j_1 \ldots j_n} = \epsilon_{ikj_1 \ldots j_n} n^k \bar{c} \quad \delta_i B = \partial_i \bar{c}$$
Slavnov-extended Yang Mills theory can be shown to be free of worst infrared singularities, if Slavnov term is of BF-type.
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What is the role of VSUSY with respect to UV/IR mixing in topological NCGFT in general?
Slavnov-extended Yang Mills theory can be shown to be free of worst infrared singularities, if Slavnov term is of BF-type.

SUSY, in the form of VSUSY, seems to play a decisive role in theories which are not Poincaré supersymmetric.

What is the role of VSUSY with respect to UV/IR mixing in topological NCGFT in general?

What are the consequences of the additional symmetries?
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*Thank you for your attention!*