On the $C^1$ regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients

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Dedicated to Haim Brezis with friendship and admiration

Abstract We prove $C^1$ regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients.

This note addresses a question raised to the author by Haim Brezis, in connection with his solution of a conjecture of Serrin concerning divergence form second order elliptic equations (see [1] and [2]). If the coefficients of the equations (or systems) are Hölder continuous, then $H^1$ solutions are known to have Hölder continuous first derivatives. The question is what minimal regularity assumption of the coefficients would guarantee $C^1$ regularity of all $H^1$ solutions. See [3] for answers to some other related questions of Haim.

Consider the elliptic system for vector-valued functions $u = (u^1, \ldots, u^N)$,

$$\partial_\alpha (A^\alpha_{ij}(x) \partial_\beta u^j) = 0, \quad \text{in } B_4, \quad i = 1, 2, \ldots, N,$$

where $B_4$ is the ball in $\mathbb{R}^n$ of 4 centered at the origin. The coefficients $\{A^\alpha_{ij}\}$ satisfy, for some positive constants $\Lambda$ and $\lambda$,

$$|A^\alpha_{ij}(x)| \leq \Lambda, \quad x \in B_4, \quad \text{(1)}$$

$$\int_{B_4} A^\alpha_{ij}(x) \partial_\alpha \eta^i \partial_\beta \eta^j \geq \lambda \int_{B_4} |\nabla \eta|^2, \quad \forall \eta \in H^1_0(B_4, \mathbb{R}^N), \quad \text{(2)}$$

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and
\[ \int_0^1 r^{-1} \varphi(r) dr < \infty, \tag{3} \]
where
\[ \varphi(r) := \sup_{x \in B_3} \left( \int_{B_r(x)} |A - A(x)|^2 \right)^{\frac{1}{2}}. \tag{4} \]

**Main Theorem.** Suppose that \( \{ A_{ij}^{\alpha \beta} \} \) satisfy the above assumptions, and \( u \in H^1(B_4, \mathbb{R}^N) \) is a solution of the elliptic system. Then \( u \) is \( C^1 \) in \( B_1 \).

**Remark.** For elliptic equations with coefficients satisfying \( \alpha \)-increasing Dini conditions, a proof of the \( C^1 \) regularity of \( u \) can be found, in e.g. [5], Theorem 5.1 as pointed out in [1] and [2].

**Question.** If we replace \( \bar{\varphi} \) in (3) by
\[ \hat{\varphi}(r) := \sup_{x \in B_3} \left( \int_{B_r(x)} |A - A_B(r, x)|^2 \right)^{\frac{1}{2}}, \tag{5} \]
with \( A_B(r, x) := \int_{B_r(x)} A \), does the conclusion of the Main Theorem still hold?

Let \( B_r(x) \subset \mathbb{R}^n \) denote the ball of radius \( r \) and centered at \( x \). We often write \( B_r \) for \( B_r(0) \), and \( rB_1 \) for \( B_r \). Consider elliptic systems
\[ \partial_\alpha (A_{ij}^{\alpha \beta}(x) \partial_\beta u^j) = h_i + \partial_\beta f_i^\beta, \quad \text{in } B_4, \quad i = 1, \cdots, N, \tag{6} \]
where \( \alpha, \beta \) are summed from 1 to \( n \), while \( i, j \) are summed from 1 to \( N \). The coefficients \( \{ A_{ij}^{\alpha \beta} \} \), often denoted by \( A \), satisfy, for some positive constants \( \Lambda \) and \( \lambda \), (1), (2) and (3), with \( \bar{\varphi} \) given by (4).

**Theorem 1.1** For \( B_4 \subset \mathbb{R}^n, n \geq 1 \), let \( A, \Lambda, \lambda, \varphi \) be as above, \( \{ h_i \}, \{ f_i^\beta \} \in C^\alpha(B_4) \) for some \( \alpha > 0 \), and let \( u \in H^1(B_4, \mathbb{R}^N), N \geq 1 \), be a solution of (6). Then \( u \in C^1(B_1) \). Moreover, the modulus of continuity of \( \nabla u \) in \( B_1 \) can be controlled in terms of \( \varphi, n, N, \Lambda, \lambda, \alpha, \| h \|_{C^\alpha(B_4)} \) and \( [f]_{C^\alpha(B_2)} \).

**Remark 1.1** Assumption (3) is weaker than \( A \) being Dini-continuous.

**Remark 1.2** The conclusion of Theorem 1.1 still holds (the dependence on \( \alpha, \| h \|_{C^\alpha(B_2)} \) and \( [f]_{C^\alpha(B_2)} \) is changed accordingly) if \( \{ h_i \} \in L^p(B_4) \) for some \( p > n \), and \( f \) satisfies
\[ \int_0^1 r^{-1} \bar{\psi}(r) dr < \infty, \quad \text{where } \bar{\psi}(r) := \sup_{x \in B_3} \left( \int_{B_r(x)} |f - f(x)|^2 \right)^{\frac{1}{2}}. \]

**Remark 1.3** This note was written in 2008. It was intended to be published after having an answer to the question raised above.
Theorem 1.1 follows from the following two propositions.

**Proposition 1.1** For \( B_4 \subset \mathbb{R}^n \), \( n \geq 1 \), let \( \Lambda, \lambda, N \) be as above, and let \( A \) satisfy (1), (2), and

\[
(\int_{B_r} |A - A(0)|^2)^{\frac{1}{2}} \leq \varphi(r), \quad 0 < r < 1, \tag{7}
\]

for some non-negative function \( \varphi \) on \((0,1)\) satisfying, for some \( \mu > 1 \),

\[
\max_{r/2 \leq s \leq 2r} \varphi(s) \leq \mu \varphi(r), \quad \int_0^1 r^{-1} \varphi(r)dr < \infty. \tag{8}
\]

Assume that \( h, f \in C^\alpha(B_4) \) for some \( \alpha > 0 \), and \( u \in H^1(B_4, \mathbb{R}^N) \) is a solution of (6). Then there exist \( a \in \mathbb{R} \) and \( b \in \mathbb{R}^n \) such that

\[
\int_{B_r} |u(x) - [a + b \cdot x]|dx \leq r \delta(r)[||u||_{L^2(B_2)} + ||h||_{C^\alpha(B_2)} + ||f||_{C^\alpha(B_2)}], \quad \forall \ 0 < r < 1, \tag{9}
\]

where \( \delta(r) \), depending only on \( \varphi, n, \lambda, N, \mu, \alpha \), satisfies \( \lim_{r \to 0} \delta(r) = 0 \).

**Proposition 1.2** Let \( u \) be a Lebesgue integrable function on \( B_1 \subset \mathbb{R}^n, \ n \geq 1 \), and let \( \delta(r) \) be a monotonically increasing positive function defined on \((0,1)\) satisfying \( \lim_{r \to 0} \delta(r) = 0 \). Assume that for every \( \bar{x} \in B_{1/4} \), there exist \( a(\bar{x}) \in \mathbb{R}, \ b(\bar{x}) \in \mathbb{R}^n \) such that

\[
\int_{B_r(\bar{x})} |u(x) - [a(\bar{x}) + b(\bar{x}) \cdot (x - \bar{x})]|dx \leq r \delta(r), \quad \forall \ 0 < r < 1/2. \tag{10}
\]

Then \( u \), after changing its values on a zero Lebesgue measure set, belongs to \( C^1(B_{1/4}) \), with \( u \equiv a \) and \( \nabla u \equiv b \). Moreover, for some dimensional constant \( C \),

\[
|\nabla u(x) - \nabla u(y)| \leq C \delta(4|x - y|), \quad \forall \ x, y \in B_{1/4}. \tag{11}
\]

Similar results hold for Dirichlet problem: Let \( \Omega \subset \mathbb{R}^n, \ n \geq 1 \), be a domain with smooth boundary, let \( \Lambda \) and \( \lambda \) be positive constants, and let \( A \) satisfy, for \( N \geq 1 \),

\[
|A^{\alpha \beta}_{ij}(x)| \leq \Lambda, \ x \in \Omega,
\]

\[
\int_{\Omega} A^{\alpha \beta}_{ij}(x) \partial_\alpha \eta^i \partial_\beta \eta^j \geq \lambda \int_{\Omega} |\nabla \eta|^2, \quad \forall \ \eta \in H^1_0(\Omega, \mathbb{R}^N),
\]

\[
\int_0^1 r^{-1} \bar{\psi}(r)dr < \infty,
\]

where

\[
\bar{\psi}(r) := \sup_{x \in \Omega} \left( \int_{B_r(x) \cap \Omega} |A - A(x)|^2 \right)^{\frac{1}{2}}.
\]

Consider

\[
\partial_\alpha (A^{\alpha \beta}_{ij}(x) \partial_\beta u^i) = h_i + \partial_\beta f_i^\beta, \quad \text{in } \Omega, \quad i = 1, \ldots, N,
\]

\[
u = g, \quad \text{on } \partial \Omega.
\]
Theorem 1.2 ([4]) Assume the above, and let \( h, f \in C^\alpha(\overline{\Omega}) \) and \( g \in C^{1,\alpha}(\partial\Omega) \) for some \( \alpha > 0 \). Then an \( H^1(\Omega, \mathbb{R}^N) \) solution \( u \) to the above Dirichlet problem is in \( C^1(\overline{\Omega}) \).

Our proof of Proposition 1.1, based on the general perturbation Lemma 3.1 in [6], is similar to that of Proposition 4.1 in [6].

Proof of Proposition 1.2. For any \( \bar{x} \in B_1 \), we see from (10) that as \( r \to 0 \),

\[
\int_{B_r(\bar{x})} |u(x) - a(x)| dx \leq \int_{B_r(\bar{x})} |u(x) - [a(\bar{x}) + b(\bar{x}) \cdot (x - \bar{x})]| dx + \int_{B_r(\bar{x})} |b(\bar{x}) \cdot (x - \bar{x})| dx \to 0.
\]

Thus, by a theorem of Lebesgue, \( a = u \) a.e. in \( B_1 \). We now take \( u \equiv a \), after changing the values of \( u \) on a zero measure set. Let \( \bar{x}, \bar{y} \in B_{1/4} \) satisfy, for some positive integer \( k \),

\[
2^{-(k+1)} \leq |\bar{x} - \bar{y}| \leq 2^{-k}.
\]

By (10), we have, for some dimensional constant \( C \),

\[
|u(\bar{x}) - [u(\bar{y}) + b(\bar{y}) \cdot (\bar{x} - \bar{y})]| = \int_{B_{2^{-k}}(\bar{x})} \{[u(\bar{x}) + b(\bar{x}) \cdot (x - \bar{x})] - [u(\bar{y}) + b(\bar{y}) \cdot (x - \bar{y})]\} dx \\
\leq \int_{B_{2^{-k}}(\bar{x})} |u(x) - [u(\bar{x}) + b(\bar{x}) \cdot (x - \bar{x})]| dx + \int_{B_{2^{-k}}(\bar{x})} |u(x) - [u(\bar{y}) + b(\bar{y}) \cdot (x - \bar{y})]| dx \\
\leq \int_{B_{2^{-k}}(\bar{x})} |u(x) - [u(\bar{x}) + b(\bar{x}) \cdot (x - \bar{x})]| dx + 2^n \int_{B_{2^{-(k-1)}}(\bar{y})} |u(x) - [u(\bar{y}) + b(\bar{y}) \cdot (x - \bar{y})]| dx \\
\leq C 2^{-k} \delta(2^{-(k-1)}) \leq C |\bar{x} - \bar{y}| \delta(4|\bar{x} - \bar{y}|).
\]

Switching the roles of \( \bar{x} \) and \( \bar{y} \) leads to

\[
|u(\bar{y}) - [u(\bar{x}) + b(\bar{x}) \cdot (\bar{y} - \bar{x})]| \leq C |\bar{y} - \bar{x}| \delta(4|\bar{y} - \bar{x}|).
\]

Thus, by the above two inequalities and the triangle inequality,

\[
|b(\bar{x}) - b(\bar{y})| \leq 2C \delta(4|\bar{x} - \bar{y}|).
\]

The conclusion of Proposition 1.2 follows from (12) and (13).

Proof of Proposition 1.1. For simplicity, we prove it for \( h = 0, f = 0 \) — the general case only requires minor changes. We may assume without loss of generality that \( \varphi(1) \leq \epsilon_0, \int_0^1 r^{-1} \varphi(r) dr \leq \epsilon_0 \), for some small universal constant \( \epsilon_0 > 0 \). This can be achieved by working with \( u(\delta_0 x) \) for some \( \delta_0 \) satisfying \( \varphi(\delta_0) \leq \epsilon_0 \) and \( \int_0^{\delta_0} r^{-1} \varphi(r) dr < \epsilon_0 \). The
smallness of $\epsilon_0$ will be either obvious or specified in the proof. In the proof, a universal constant means that it depends only on $\varphi, n, \lambda, N, \mu$. We assume that $u$ is normalized to satisfy $\|u\|_{L^2(B_2)} = \varphi(4^{-1})$. We often write $\partial_\alpha(A_{ij}^\beta \partial_\beta u^j)$ as $\partial(A\partial u)$. For $k \geq 0$, let

$$A_{k+1}(x) = A(4^{-(k+1)}x), \quad \overline{A} = A(0).$$

We will find $w_k \in H^1(\frac{3}{4k+1}B_1, \mathbb{R}^N)$, such that for all $k \geq 0$,

$$\partial(\overline{A}\partial w_k) = 0, \quad \frac{3}{4k+1}B_1, \quad (14)$$

$$\|w_k\|_{L^2(\frac{2}{4k+1}B_1)} \leq C'4^{-\frac{(k+2)}{2}}\varphi(4^{-k}), \quad \|\nabla w_k\|_{L^\infty(\frac{1}{4k+1}B_1)} \leq C'\varphi(4^{-k}), \quad (15)$$

$$\|\nabla^2 w_k\|_{L^\infty(\frac{1}{4k+1}B_1)} \leq C'4^k\varphi(4^{-k}), \quad (16)$$

$$\|u - \sum_{j=0}^k w_j\|_{L^2((\frac{1}{4})^{k+1}B_1)} \leq 4^{-(k+1)(n+2)}\varphi(4^{-(k+1)}). \quad (17)$$

An easy consequence of (15) is

$$\|w_k\|_{L^\infty(4^{-(k+1)}B_1)} \leq C'4^{-k}\varphi(4^{-k}). \quad (18)$$

Here and in the following $C, C'$ and $\epsilon_0$ denote various universal constants. In particular, they are independent of $k$. $C$ will be chosen first and will be large, then $C'$ (much larger than $C$), and finally $\epsilon_0 \in (0,1)$ (much smaller than $1/C'$).

By Lemma 3.1 in [6], we can find $w_0 \in H^1(\frac{3}{7}B_1, \mathbb{R}^N)$ such that

$$\partial(\overline{A}\partial w_0) = 0, \quad \text{in } \frac{3}{4}B_1,$$

$$\|u - w_0\|_{L^2(\frac{1}{2}B_1)} \leq C_0\|u\|_{L^2(B_1)} \leq 4^{-\frac{n+2}{2}}\varphi(4^{-1}).$$

So

$$\|w_0\|_{L^2(\frac{1}{4}B_1)}, \quad \|\nabla w_0\|_{L^\infty(\frac{1}{4}B_1)}, \quad \|\nabla^2 w_0\|_{L^\infty(\frac{1}{4}B_1)} \leq C\varphi(1) \leq C'\varphi(1).$$

We have verified (14)-(17) for $k = 0$. Suppose that (14)-(17) hold up to $k \ (k \geq 0)$; we will prove them for $k+1$. Let

$$W(x) = [u - \sum_{j=0}^k w_j](4^{-(k+1)}x),$$

$$g_{k+1}(x) = 4^{-(k+1)}\{[\overline{A} - A_{k+1}](x) \sum_{j=0}^k (\partial w_j)(4^{-(k+1)}x)\}.$$
Then $W$ satisfies
\[ \partial(A_{k+1}\partial W) = \partial(g_{k+1}), \quad B_1. \]

A simple calculation yields, using (8),
\[ \|A_{k+1} - \overline{A}\|_{L^2(B_1)} = \sqrt{\|B_1\|\varphi(4^{-(k+1)})} \leq C(n, \mu)\varphi(4^{-(k+2)}). \]

By the induction hypothesis (see (15)-(17)),
\[ \sum_{j=0}^{k} \left| (\nabla w_j)(4^{-(k+1)}x) \right| \leq C' \sum_{j=0}^{k} \varphi(4^{-j}) \leq C(n)C' \int_0^1 r^{-1}\varphi(r)dr \leq C(n)C'\epsilon_0, \quad x \in B_1, \]

\[ \sum_{j=0}^{k} \left| (\nabla^2 w_j)(4^{-(k+1)}x) \right| \leq C' \sum_{j=0}^{k} 4^j\varphi(4^{-j}), \quad x \in B_1, \]

\[ \|W\|_{L^2(B_1)} \leq 4^{-(k+1)}\varphi(4^{-(k+1)}) \leq C(\mu)4^{-(k+2)}\varphi(4^{-(k+2)}), \]

\[ \|g_{k+1}\|_{L^2(B_1)} \leq C(n, \mu)C'\epsilon_0 4^{-k}4^{-(k+2)}\varphi(4^{-(k+2)}). \]

By Lemma 3.1 in [6], there exists $v_{k+1} \in H^1(\frac{3}{4}B_1, \mathbb{R}^N)$ such that
\[ \partial(A\partial v_{k+1}) = 0, \quad \text{in} \quad \frac{3}{4}B_1, \]

and, for some universal constant $\gamma > 0$,
\[ \|W - v_{k+1}\|_{L^2(\frac{1}{4}B_1)} \leq C(\|g_{k+1}\|_{L^2(B_1)} + \epsilon_0^2\|W\|_{L^2(B_1)}) \leq C(C'\epsilon_0 + \epsilon_0^2)4^{-(k+2)}\varphi(4^{-(k+2)}). \tag{19} \]

Let
\[ w_{k+1}(x) = v_{k+1}(4^{k+1}x), \quad x \in \frac{3}{4k+2}B_1. \]

A change of variables in (19) and in the equation of $v_{k+1}$ yields (14) and (17) for $k + 1$. It follows from the above that
\[ \|\nabla^2 v_{k+1}\|_{L^2(\frac{1}{4}B_1)} + \|\nabla^2 v_{k+1}\|_{L^\infty(\frac{1}{4}B_1)} \leq C\|v_{k+1}\|_{L^2(\frac{1}{4}B_1)} \leq C4^{-(k+1)}\varphi(4^{-(k+1)}). \]

Estimates (15) for $k + 1$ follow from the above estimates for $v_{k+1}$. We have, thus, established (14)-(17) for all $k$.

For $x \in 4^{-(k+1)}B_1$, using (15), (16), (18), (8) and Taylor expansion,
\[ \left| \sum_{j=0}^{k} w_j(x) - \sum_{j=0}^\infty w_j(0) - \sum_{j=0}^\infty \nabla w_j(0) \cdot x \right| \]
Proposition 1.1 follows from the above with a proof of Theorem 1.1.

The second inequality follows. For the first inequality, we only need to show that $\mu > \delta_0$, depending only on $\varphi$, $n, \lambda, N, \mu$,

$$\int_{A_n} \sum_{j=0}^k (w_j(0) - \nabla w_j(0) \cdot x) \, dx \leq \sum_{j=0}^k \left| w_j(x) - \sum_{j=0}^k w_j(0) \cdot x \right| L^1_{\varphi} + \left| u - \sum_{j=0}^k w_j(x) \right| L^1_{\varphi} = 4^{-(k+1)(n+1)} \delta(4^{-(k+1)}).$$

Proposition 1.1 follows from the above with $a = \sum_{j=0}^k w_j(0)$ and $b = \sum_{j=0}^k \nabla w_j(0) \cdot x$.

**Proof of Theorem 1.1.** Fix a $\rho \in C^\infty_c(B_1)$, $\rho \equiv 1$ on $B_3$, and let $\varphi(r) := \sup_{x \in B_3} (\int_{B_r(x)} |(\rho A) - (\rho A)(x)|^2)^{\frac{1}{2}}.$

It is easy to see that for some $\mu > 1$, $\varphi$ satisfies (8). Indeed, since it is easily seen that $\varphi(r) \leq C(\varphi(r) + r)$,

the second inequality follows. For the first inequality, we only need to show that $\varphi(2r) \leq C(n)\varphi(r)$, since the rest is obvious. For any $\bar{x}$, let $x_1 = \bar{x}, x_2, \cdots, x_m$, $m = m(n)$, satisfy $B_{2r}(\bar{x}) \subseteq \bigcup_{i=1}^m B_{r/9}(x_i)$, and $|x_i - x_{i+1}| \leq r/9$. Then

$$\int_{B_{2r}(\bar{x})} |(\rho A) - (\rho A)(\bar{x})|^2 \leq C(n) \sum_{i=1}^m \left( \int_{B_{r/9}(x_i)} |(\rho A) - (\rho A)(\bar{x})|^2 \right)^{\frac{1}{2}}$$

$$\leq C(n) \sum_{i=1}^m \left( \int_{B_{r/9}(x_i)} |(\rho A) - (\rho A)(x_i)|^2 \right)^{\frac{1}{2}} + \sum_{i=1}^m \left( |(\rho A)(\bar{x}) - (\rho A)(x_i)| \right)^{\frac{1}{2}}$$

$$\leq C(n) \varphi(r) + C(n) \sum_{i=1}^{m-1} |(\rho A)(x_i) - (\rho A)(x_{i+1})|. $$
Since
\[ |(\rho A)(x_i) - (\rho A)(x_{i+1})| \]
\[ = \left| \int_{B_{r/9}(x_i)} [(\rho A) - (\rho A)(x_i)] - \int_{B_{r/9}(x_i)} [(\rho A) - (\rho A)(x_{i+1})] \right| \]
\[ \leq C(n) \left( \int_{B_{r}(\bar{x})} |(\rho A) - (\rho A)(\bar{x})| \right)^{1/2} \leq C(n) \varphi(r), \]
we have
\[ \left( \int_{B_{2r}(\bar{x})} |(\rho A) - (\rho A)(\bar{x})|^2 \right)^{1/2} \leq C(n) \varphi(r). \]
Thus \( \varphi(2r) \leq C(n) \varphi(r) \).

For any \( \bar{x} \in B_2 \),
\[ \left( \int_{B_r(\bar{x})} |A - A(\bar{x})|^2 \right)^{1/2} \leq \varphi(r), \quad 0 < r < 1/4. \]
Thus Theorem 1.1 follows from Proposition 1.1-1.2.

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