From Littlewood-Richardson Sequences to Subgroup Embeddings and Back

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Abstract. Let $\alpha$, $\beta$, and $\gamma$ be partitions describing the isomorphism types of the finite abelian $p$-groups $A$, $B$, and $C$. From theorems by Green and Klein it is well-known that there is a short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups if and only if there is a Littlewood-Richardson sequence of type $(\alpha, \beta, \gamma)$. Starting from the observation that a sequence of partitions has the LR property if and only if every subsequence of length 2 does, we demonstrate how LR-sequences of length two correspond to embeddings of a $p^2$-bounded subgroup in a finite abelian $p$-group. Using the known classification of all such embeddings we derive short proofs of the theorems by Green and Klein.

1. Littlewood-Richardson sequences.

Let $\Gamma = [\gamma_0, \ldots, \gamma_r]$ be an increasing sequence of partitions defining a tableau $T$. We visualize $T$ by putting a number $h$ into each box in the skew diagram $\gamma_h - \gamma_h - 1$, for each $h = 1, \ldots, r$. As usual, $\Gamma$ is a Littlewood-Richardson sequence or LR-sequence provided (LR1) each skew diagram $\gamma_h - \gamma_h - 1$ is a horizontal strip and (LR2) the word $w(T)$ obtained by reading off the numbers in $T$ row-wise, and within each row from right to left, is a lattice permutation.

Conditions (LR1) and (LR2) can be expressed in terms of the parts of the $\gamma_h$; but two different descriptions are obtained dependent on whether those parts represent the rows (as in [4, II.3]) or the columns of the corresponding Young diagram. In this manuscript, the parts of a partition will represent the columns, and then we have the following characterization (as in [1, 2.1]):

(LR1) For each $h \geq 1$ and every $k$, we have $0 \leq \gamma_h - \gamma_h - 1 \leq 1$.
(LR2) For each $h \geq 2$ and every $k$, the following inequality holds:

$$\sum_{i \geq k} (\gamma_h - \gamma_h - 1) \leq \sum_{i \geq k} (\gamma_h - \gamma_h - 2)$$

The type of the sequence $\Gamma = [\gamma_0, \ldots, \gamma_r]$ is the triple $(\alpha, \gamma_r, \gamma_0)$ of partitions where $\alpha$ is such that its conjugate $\alpha'$ has $h$-th part $\alpha'_h = |\gamma_h - \gamma_h - 1|$, counting the number of $h$'s in the tableau.

Key Observation. The sequence $\Gamma$ is an LR-sequence if and only if for each $h \geq 2$, the sequence $[\gamma_h - 2, \gamma_h - 1, \gamma_h]$ is an LR-sequence. Here we put $\gamma_h = \gamma_r$ for $h \geq r$. Moreover, if $\Gamma$ has type $(\alpha, \gamma_r, \gamma_0)$ then each sequence $[\gamma_h - 2, \gamma_h - 1, \gamma_h]$ has type $((\alpha'_{h-1}, \alpha'_h), \gamma_h, \gamma_h - 2)$.

In this sense, arbitrary LR-sequences are generated by sequences of length 2. Such sequences are of the following form:

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LEMMA 1.1. An increasing sequence \( \Gamma = [\gamma^0, \gamma^1, \gamma^2] \) of partitions is an LR-sequence if and only if two conditions are satisfied:

1. The set of columns in the tableau \( T \) for \( \Gamma \) is a totally ordered subset of the poset \( \mathcal{L} \) with the ordering given by the horizontal position.

2. There is an injection \( \tau_{21} \) from the list \( T_2 \) of columns in \( T \) of type \( 2 \) into the list \( T_1 \) of columns of type \( 1 \) which assigns to a column of length \( \ell \) a column of length less than \( \ell \).

2. Finite length modules.

Let \( R \) be a commutative principal ideal domain and \( p \) a generator of a maximal ideal. A \( p \)-module is a finite length \( R \)-module which is annihilated by some power of \( p \). There is a one-to-one correspondence between the set of partitions and the set of isomorphism classes of \( p \)-modules given by

\[
\lambda = (\lambda_1, \ldots, \lambda_s) \mapsto M(\lambda) = \bigoplus_{i=1}^{s} R/(p^{\lambda_i})
\]

(see [3, (1.4)]). The partition corresponding to the \( p \)-module is its type.

Given a \( p \)-module \( B \) and a submodule \( A \) of \( B \) of exponent \( r \) then there is an increasing sequence of partitions \([\gamma^0, \ldots, \gamma^r]\) given by the types of the factor modules \( B/p^hA \), where \( 0 \leq h \leq r \).

Embeddings \((A \subseteq B)\) where \( p^2A = 0 \) have been classified ([2, Theorem 7.5], [3, Corollary 5.4]):

**Theorem 2.1.** Let \( B \) be a \( p \)-module and \( A \) a submodule of \( B \) which is \( p^2 \)-bounded. Then the embedding \((A \subseteq B)\) has a direct sum decomposition, unique up to isomorphy and reordering, into finitely many indecomposable embeddings of type \( P^\ell_m \) and \( Q^s_2 \) defined as follows:

\[ P^\ell_m : (p^{\ell-m}) \subseteq R/(p^\ell) \quad \text{for } \ell \in \mathbb{N}, \ 0 \leq m \leq \max\{\ell, 2\}, \]

\[ Q^s_2 : (p^{\ell-s-2}, p^{s-1}) \subseteq R/(p^\ell) \oplus R/(p^s) \quad \text{for } \ell, s \in \mathbb{N}, \ s < \ell - 1. \]

3. LR-sequences of length 2 and \( p^2 \)-bounded submodules

The partition sequences of the indecomposable pairs \((A \subseteq B)\) with \( p^2A = 0 \) are as follows. They are all LR-sequences.
When dealing with the direct sum of two pairs, the partition sequence of the sum is given by the union $\Gamma \cup \Delta = \{\gamma_0 \cup \delta_0, \gamma_1 \cup \delta_1, \gamma_2 \cup \delta_2\}$ taken componentwise where $\Gamma = [\gamma_0, \gamma_1, \gamma_2]$ and $\Delta = [\delta_0, \delta_1, \delta_2]$ are the partition sequences of the two summands. Note that the list of columns in the tableau for $\Gamma \cup \Delta$ is a reordering of the columns in the tableaux for $\Gamma$ and $\Delta$; however, each (in $L$ incomparable) pair of columns arising from a sum $P_2^{\ell+1} \oplus P_0^\ell$ is replaced as follows:

Let $\Gamma = [\gamma_0, \gamma_1, \gamma_2]$ be the partition sequence of an embedding $(A \subseteq B)$ with $p^2A = 0$. We have seen that the columns of the corresponding tableau form a totally ordered subset of $L$. Columns of type $\begin{array}{c|c|c} & & \\ \hline & & \\ \end{array}$ arise for each summand of type $Q_2^s$, where $s < \ell - 1$, and for each pair of summands of type $P_2^{\ell+1} \oplus P_0^\ell$, as above. In each case there is a corresponding shorter column of type $\begin{array}{c|c|c} & & \\ \hline & & \\ \end{array}$. The map $\tau_21$ given by this correspondence is a monomorphism, and Lemma 1.1 yields that $\Gamma$ is an LR-sequence. The type of $\Gamma$ is $(\alpha, \gamma_0)$ where the partition $\alpha$ is given by $\alpha_1 = \text{len} A/pA = |\gamma_2 - \gamma_1|$, and $\alpha_2 = \text{len} pA = |\gamma_1 - \gamma_0|$.

We have shown:

**Lemma 3.1.** Let $A$, $B$, $C$ be $p$-modules of type $\alpha$, $\beta$, and $\gamma$, respectively, such that $p^2A = 0$ and such that there is a short exact sequence:

$$0 \rightarrow A \xrightarrow{\mu} B \rightarrow C \rightarrow 0$$

For $h = 0, 1, 2$, denote the type of $B/p^h\mu(A)$ by $\gamma^h$. Then the partitions $[\gamma_0, \gamma_1, \gamma_2]$ form an LR-sequence of type $(\alpha, \beta, \gamma)$. 

Conversely, every LR-sequence of length 2 gives rise to a short exact sequence of $p$-modules:

**Lemma 3.2.** Given an LR-sequence $\Gamma = [\gamma_0, \gamma_1, \gamma_2]$ of length 2 and type $(\alpha, \beta, \gamma)$, a $p$-module $B$ of type $\beta = \gamma_2$ and a semisimple submodule $U$ of $B$ such that $B/U$ has type $\gamma_1$. Then there is a submodule $A$ of $B$ containing $U$ and satisfying the following conditions:

- The type of $A$ is $\alpha$,
- $U = pA$ and
- the type of $B/A$ is $\gamma_0 = \gamma$.

**Remark.** A submodule $U$ as in the lemma can be obtained as follows. Let $\lambda = \gamma_2$ and define $\kappa_i$ by letting $\kappa_i = \gamma_2^i - \gamma_1^i$. Then the direct sum $\bigoplus_i P_{\kappa_i}$ defines an embedding $(U' \subseteq B')$ with $B \cong \varphi B'$. Since $[\gamma_1, \gamma_2]$ is an LR-sequence, $0 \leq \kappa_i \leq 1$ follows, and hence $U'$ is semisimple. Put $U = \varphi^{-1}(U')$. 

\[
\begin{array}{c|c|c|c|c}
P_0^\ell & P_1^\ell & P_2^\ell & Q_2^s \\
\hline
& & & \\
\end{array}
\]
Proof. By Theorem 2.1 the embedding \((U \subseteq B)\) is isomorphic to a direct sum of embeddings of type \(P^0_i\) and \(P^1_i\) as above. We may assume that \(\varphi\) is the identity map on \(B\) and write:

\[
(*) \quad (U \subseteq B) = \bigoplus_i P^{\lambda_i}_i \quad \text{where } \kappa \text{ is as above and } \lambda = \gamma^2.
\]

Since \(\Gamma\) is an LR-sequence, say defining the tableau \(T\), Lemma 1.1 gives us an injective map \(\tau_{21} : T_2 \to T_1\) between the two lists of columns containing only a 2 and containing only a 1, respectively. If the \(i\)-th column of the tableau (which is defined by the numbers \((\gamma^0_i, \gamma^1_i, \gamma^2_i)\)) does not occur among the columns in \(T_2 \cup \im \tau_{21}\), let \((A_i \subseteq B_i)\) denote the \(i\)-th summand \(P^{\lambda_i}_i\) in \((*)\). Otherwise \(\tau_{21}\) defines a pair \((i,j) \in T_2 \cup T_1\) where \(j = \tau_{21}(i)\) such that the \(i\)-th column has length \(\ell\) and type \(\begin{array}{c} 2 \end{array}\) and the \(j\)-th column has length \(s < \ell\) and type \(\begin{array}{c} 1 \end{array}\). Let \(B_i\) be the sum of the \(i\)-th and the \(j\)-th summand in \((*)\), then choose \(A_i \subseteq B_i\) such that

\[
(A_i \subseteq B_i) \cong \begin{cases} P^s_{j} \oplus P^s_{0} & \text{if } s = \ell - 1 \\ P^s_{2} \oplus Q^s_{1} & \text{if } s < \ell - 1 \end{cases}.
\]

Putting \(A = \bigoplus_{i \in \{1, \ldots, n\} \setminus \im \tau_{21}} A_i\) yields a submodule of \(B\) of type \(\alpha\) such that \(U = pA\). The type of \(B/A\) is computed from the types of the summands, as above Lemma 3.1.

\[
\Box
\]

4. The Theorems by Green and Klein

Following \(\Pi\), we can now establish the correspondence between LR-sequences and short exact sequences of \(p\)-modules.

Theorem 4.1 (Green). Suppose \(A, B,\) and \(C\) are \(p\)-modules of type \(\alpha, \beta,\) and \(\gamma\) such that there is a short exact sequence

\[
0 \longrightarrow A \xrightarrow{\mu} B \longrightarrow C \longrightarrow 0.
\]

Then the sequence of partitions \(\Gamma = [\gamma^0, \ldots, \gamma^r]\) where \(r = \alpha_1\) is the exponent of \(A\) and \(\gamma^h\) is the type of \(B/p^h\mu(A)\) for \(0 \leq h \leq r\), is an LR-sequence of type \((\alpha, \beta, \gamma)\).

Proof. We may assume that \(\mu\) is an inclusion, then \(A\) is a submodule of \(B\). For each \(2 \leq h \leq r\), we have the short exact sequence

\[
0 \longrightarrow p^{h-2}A/p^hA \xrightarrow{\mu} B/p^hA \longrightarrow B/p^{h-2}A \longrightarrow 0.
\]

The modules \(B/p^hA, \ (B/p^hA)/(p^{h-1}A/p^hA) \cong B/p^{h-1}A,\) and \(B/p^{h-2}A\) have type \(\gamma^h, \gamma^{h-1},\) and \(\gamma^{h-2}\), respectively, and hence, by Lemma 3.1, the sequence \([\gamma^{h-2}, \gamma^{h-1}, \gamma^h]\) is an LR-sequence of type \((\alpha'_{h-1}, \alpha'_h, \gamma^h, \gamma^{h-2})\) where \(\alpha'_h = |\gamma^h - \gamma^{h-1}|\) and \(\alpha'_{h-1} = |\gamma^{h-1} - \gamma^{h-2}|\). It follows from the Key Observation in Section \(\Pi\) that \(\Gamma\) is an LR-sequence of type \((\alpha, \gamma^r, \gamma^0)\).

\[
\Box
\]

Theorem 4.2 (Klein). If \(\Gamma = [\gamma^0, \ldots, \gamma^r]\) is an LR-sequence of type \((\alpha, \beta, \gamma)\), and if \(A, B,\) and \(C\) are \(p\)-modules of type \(\alpha, \beta,\) and \(\gamma,\) respectively, then there is a short exact sequence

\[
0 \longrightarrow A \xrightarrow{\mu} B \longrightarrow C \longrightarrow 0
\]

such that \(\gamma^h\) is the type of \(B/p^h\mu(A)\) for each \(0 \leq h \leq r\).
Proof. We may assume that $r \geq 2$. First consider the LR-sequence $[\gamma^{r-2}, \gamma^{r-1}, \gamma^{r}]$. By Lemma 3.2 and the following remark there is a $p$-module $B$ and a submodule $U_{r-2} \subseteq B$ such that $B$, $B/pU_{r-2}$ and $B/U_{r-2}$ have type $\gamma^{r}$, $\gamma^{r-1}$, and $\gamma^{r-2}$, respectively. We put $U_{r} = 0$, $U_{r-1} = pU_{r-2}$ and define for $h = r - 3, r - 4, \ldots, 0$ successively submodules $U_{h}$ of $B$ such that the conditions

$$p^{s}U_{h} = U_{h+s} \quad \text{and} \quad B/U_{h} \text{ has type } \gamma^{h}$$

hold for all $0 \leq h \leq h + s \leq r$. Suppose $U_{h+1}$ has been constructed. Consider the module $B' = B/pU_{h+1}$ and the semisimple submodule $U' = U_{h+1}/pU_{h+1}$. By Lemma 3.2 the LR-sequence $[\gamma^{h}, \gamma^{h+1}, \gamma^{h+2}]$ yields a submodule $A'$ of $B'$ such that $pA' = U'$ and $B'$, $B/U'$, $B/A'$ have type $\gamma^{h+2}$, $\gamma^{h+1}$, and $\gamma^{h}$, respectively. Let $U_{h}$ be the inverse image of $A'$ under the canonical map $B \to B'$.

This process yields a submodule $A = U_{r}$ of $B$ with the property that the type of $B/p^{h}A = B/U_{h}$ is $\gamma^{h}$ for each $0 \leq h \leq r$. \hfill \square

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