The Logical Theory of Canonical Maps:
The Elements & Distinctions Analysis of the Morphisms,
Duality, Canonicity, and Universal Constructions in Sets

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Abstract

Category theory gives a mathematical characterization of naturality but not of canonicity. The purpose of this paper is to develop the logical theory of canonical maps based on the broader demonstration that the dual notions of elements & distinctions are the basic analytical concepts needed to unpack and analyze morphisms, duality, canonicity, and universal constructions in Sets, the category of sets and functions. The analysis extends directly to other Sets-based concrete categories (groups, rings, vector spaces, etc.). Elements and distinctions are the building blocks of the two dual logics, the Boolean logic of subsets and the logic of partitions. The partial orders (inclusion and refinement) in the lattices for the dual logics define the canonical morphisms (where ‘canonical’ is always relative to the given data, not an absolute property of a morphism). The thesis is that the maps that are canonical in Sets are the ones that are defined (given the data of the situation) by these two logical partial orders and by the compositions of those maps.

Keywords: canonical maps, category theory, duality, elements & distinctions analysis, logic of subsets, logic of partitions

AMS: 03, 18

1 Elements & Distinctions Analysis

1.1 Introduction

Category theory gives a mathematical characterization of naturality but not of canonicity, the canonical nature of certain morphisms. The purpose of this paper is to present the logical theory of canonical maps that provides such a characterization. That logical theory of canonical maps is one of the main results in the broader analysis showing that the dual notions of “elements & distinctions” (or “its & dits”) are the basic analytical concepts needed to unpack and analyze morphisms, duality, canonicity (or canonicalness), and universal constructions in Sets, the category of sets and functions. The analysis extends directly to other Sets-based concrete categories (groups, rings, vector spaces, etc.) where the objects are sets with a certain type of structure and the morphisms are set functions that preserve or reflect that structure. Then the elements & distinctions-based definitions can be abstracted in purely arrow-theoretic way for abstract category theory.

One way to approach the concepts of “elements” (or “its”) and “distinctions” (or “dits”) is to start with the category-theoretic duality between subsets and quotient sets (= partitions = equivalence relations): “The dual notion (obtained by reversing the arrows) of ‘part’ [subobject] is the notion of partition.” [13, p. 85]. That motivates the two dual forms of mathematical logic: the Boolean logic of subsets and the logic of partitions ([5]; [7]). If partitions are dual to subsets, then what is the dual concept that corresponds to the notion of elements of a subset? The notion dual to...
the elements of a subset is the notion of the *distinctions of a partition* (pairs of elements in distinct blocks of the partition).

### 1.2 Set functions transmit elements and reflect distinctions

The duality between elements ("its") of a subset and distinctions ("dits") of a partition already appears in the very notion of a function between sets in the category $Sets$. The concepts of elements and distinctions provide the natural notions to specify the binary relations, i.e., subsets $R \subseteq X \times Y$, that define functions $f : X \rightarrow Y$.

A binary relation $R \subseteq X \times Y$ *transmits elements* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.

A binary relation $R \subseteq X \times Y$ *reflects elements* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.

A binary relation $R \subseteq X \times Y$ *transmits distinctions* if for any pairs $(x, y)$ and $(x', y')$ in $R$, if $x \neq x'$, then $y \neq y'$.

A binary relation $R \subseteq X \times Y$ *reflects distinctions* if for any pairs $(x, y)$ and $(x', y')$ in $R$, if $y \neq y'$, then $x \neq x'$.

The dual role of elements and distinctions can be seen if we translate the usual characterization of the binary relations that define functions into the elements-and-distinctions language. In the usual treatment, a binary relation $R \subseteq X \times Y$ defines a function $X \rightarrow Y$ if it is defined everywhere on $X$ and is single-valued. But "being defined everywhere" is the same as transmitting (or "preserving") elements, and being single-valued is the same as reflecting distinctions so the more natural definition is:

A binary relation $R$ is a *function* if it transmits elements and reflects distinctions.

What about the other two special types of relations, i.e., those which transmit (or preserve) distinctions or reflect elements? The two important special types of functions are the injections and surjections, and they are defined by the other two notions:

- a function is *injective* if it transmits distinctions, and
- a function is *surjective* if it reflects elements.

Given a set function $f : X \rightarrow Y$ with *domain* $X$ and *codomain* $Y$, a subset of the codomain $Y$ is determined as the *image* $f(X) \subseteq Y$, and a partition on the domain $X$ is determined as the *coimage* or *inverse-image* $\{f^{-1}(y)\}_{y \in f(X)}$. It might also be noted that the empty set of ordered pairs $\emptyset \times Y$ satisfies the definition of a function $\emptyset \rightarrow Y$ whose image is the empty subset $\emptyset$ of $T$ and coimage is the empty partition $\emptyset$ on $\emptyset$.

### 1.3 The logical theory of canonical maps based on the its & dits analysis

Jean-Pierre Marquis [15] has raised the question of characterizing canonical maps in mathematics in general and category theory in particular. Category theory gives a mathematical notion of "natural-ity" but not of canonicalness or canonicity. Marquis gives the intuitive idea (maps defined "without any arbitrary decision"), a number of examples (most of which we will analyze in $Sets$), and a set of criteria stated in terms of limits (and thus dually for colimits).

We are now in a position to circumscribe more precisely what we want to include in the notion of canonical morphisms or maps.

1. Morphisms that are part of the data of a limit are canonical morphisms; for instance, the projection morphisms that are part of the notion of a product;
2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism: and
3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism. [15] p. 101

The elements & distinctions (or its & dits) analysis provides a mathematical characterization of "canonical maps" in Sets (and thus in Sets-based concrete categories) that satisfies the Marquis criteria. The characterization of canonicity is always relative to the given data; canonicity is not an 'absolute' property of a morphism. For instance, given a cone \( f : Z \rightarrow X \) and \( g : Z \rightarrow Y \), the canonical map \( Z \rightarrow X \times Y \) is only canonical relative to the data \( f \) and \( g \).

The treatment of canonicity is part of a broader elements & distinctions analysis of the morphisms, duality, and universal constructions (limits and colimits) in the basic 'ur-category' Sets of sets and functions and thus in Sets-based categories—which is abstracted in category theory as a whole. At that point, the elements & distinctions analysis connects to the broader philosophical literature on structuralism since category theory is essentially the natural codification of that philosophy of mathematics ([10]; [12]; [11]).

The logical theory of canonicity is that the canonical maps and the unique canonical factor morphisms in the universal mapping properties (UMPs) in Sets are always constructed in the two ways that maps are constructed from the partial orders in the two basic logics, the logic of subsets and the logic of partitions. In the powerset Boolean algebra of subsets \( \varnothing (U) \) of \( U \), the partial order is the inclusion relation \( S \subseteq T \) for \( S, T \subseteq U \), which induces the canonical injection \( S \rightarrow T \). That is the way canonical injective maps are defined from the partial order of inclusion on subsets.

In the dual algebra of partitions \( \Pi (U) \) on \( U \) (i.e., the lattice of partitions on \( U \) enriched with the implication operation on partitions), the partial order is the refinement relation between partitions and it induces a canonical map using refinement. A partition \( \pi = \{ B, B', ... \} \) on a set \( U \) is a set of non-empty subsets of \( U \) (called blocks, \( B, B', ... \)) that are mutually exclusive (i.e., disjoint) and jointly exhaustive (i.e., whose union is \( U \)). It might be noticed that the empty set \( \emptyset \), which has no nonempty subsets, is the empty partition on \( U = \emptyset \). One could also define a partition on \( U \) as the set of inverse-images \( f^{-1} (y) \subseteq U \) for \( y \in f (U) \) for any function \( f : U \rightarrow Y \) with domain \( U \). The empty partition on \( U = \emptyset \) is then the inverse-image partition of the empty function \( \emptyset \rightarrow Y \).

Given another partition \( \sigma = \{ C, C', ... \} \) on \( U \), a partition \( \pi \) is said to refine \( \sigma \) (or \( \sigma \) is refined by \( \pi \)), written \( \sigma \preceq \pi \), if for every block \( B \in \pi \), there is a block \( C \in \sigma \) (necessarily unique) such that \( B \subseteq C \). If we denote the set of distinctions or dits of a partition (ordered pairs of elements in different blocks) by \( \text{dit}(\pi) \), the ditset of \( \pi \), then just as the partial order in \( \varnothing (U) \) is the inclusion of elements, so the refinement partial order on \( \Pi (U) \) is the inclusion of distinctions, i.e., \( \sigma \preceq \pi \) iff (if and only if) \( \text{dit}(\sigma) \subseteq \text{dit}(\pi) \). And just as the inclusion ordering on subsets induces a canonical injection between subsets, so the refinement ordering \( \sigma \preceq \pi \) on partitions induces a canonical surjection between partitions, namely \( \pi \rightarrow \sigma \) where \( B \in \pi \) is taken to the unique \( C \in \sigma \) where \( B \subseteq C \). If the blocks of \( \pi = \{ B_x \}_{x \in X} \) are indexed by a set \( X \) and the blocks of \( \sigma = \{ C_y \}_{y \in Y} \) are indexed by a set \( Y \), then the refinement \( \sigma \preceq \pi \) induces a canonical surjection \( X \rightarrow Y \). That is the way canonical surjective maps are defined from the partial order of refinement on partitions.

These canonical injections and surjections are built into the partial orders of the lattice (or algebraic) structure of the two dual logics of subsets and partitions; they logically define the 'atomic' canonical maps in Sets, and other canonical maps in Sets arise out of their compositions. Note that the canonical injections are in the 'upward' direction of the partial order (more elements) in the lattice of subsets, while the the canonical surjections are in the opposite downward direction to the 'upward' (more dits) direction of the partial order in the lattice of partitions.

1 In [10], the partition algebra was defined as the partition lattice enriched with the implication and nand operations on partitions. But for purposes of comparisons with Boolean or Heyting algebras, it suffices to consider only the implication in addition to the join and meet. In any case, this does not affect the analysis here where the lattice structure suffices.

2 Thanks to Paul Blain Levy and Alex Simpson for emphasizing to me the role of empty partition for the consistent development of the whole theory, e.g., as the inverse-image partition on the domain of the empty function \( \emptyset \rightarrow Y \).
This logical theory of canonical maps is that all "canonical" maps in $\text{Sets}$ arise from the given data in these two ways or by compositions of them—which then extends to $\text{Sets}$-based concrete categories. "Canonical" always means relative to the given data. The given data only plays the role of defining the sets with the inclusion relations between them or the partitions with the refinement relation between them. That is the role of the given data. Then the canonical injections and canonical surjections are defined by those inclusions or refinements. Marquis’ informal definition of canonical maps as maps defined “without any arbitrary decision” then means that once the initial data is encoded as inclusion or refinement relations in the respective logical lattices, then the logical structure suffices to define the canonical injective or surjective maps. And the thesis is that all canonical maps in $\text{Sets}$ are those canonical injections, canonical surjections, or their compositions.

The thesis cannot be proven since “canonical” is an intuitive notion. But we will show that all the canonical maps and unique factor maps in the universal constructions (limits, colimits, and the exponential or Currying adjunction) in $\text{Sets}$ arise in this way from the partial orders of the dual lattices (or algebras) of subsets and partitions—which thus satisfies the Marquis criteria. This logical basis for this theory of canonical maps accounts for the name "logical."

1.4 Initial & terminal objects in $\text{Sets}$

The top of the powerset Boolean algebra $\wp(U)$ is $U$, where each subset $S \subseteq U$ induces the canonical injection $S \to U$. The bottom of the Boolean algebra, the null set $\emptyset$, is included in any set, e.g., $\emptyset \subseteq U$, so the induced morphism $\emptyset \to U$ is the canonical map that makes $\emptyset$ the initial object in $\text{Sets}$ (taking $U$ as any set).

The top of the partition algebra $\Pi(U)$ is the discrete partition $1_U = \{\{u\}\}_{u \in U}$ of all singletons. Since every partition $\pi$ is refined by $1_U$, i.e., $\pi \preceq 1_U$, there is the canonical surjection $1_U \cong U \to \pi$ that takes the singleton $\{u\}$ or just $u$ (since blocks of $1_U$ are in one-to-one correspondence with the elements of $U$) to the unique block $B$ such that $u \in B$. The bottom of the partition algebra (or lattice) is the indiscrete partition (nicknamed the "Blob") $0_U = \{\}$ with only one block $U$ that identifies all the points in $U$ so $0_U$ is isomorphic to the one-element set 1. And $0_U$ is refined by all partitions, e.g., $0_U \preceq 1_U$. That refinement relation induces the unique map from the blocks of $1_U$ (i.e., the elements of $U$) to the blocks or rather “the block” of $0_U \cong 1$, i.e., induces the canonical map $U \to 1$ that makes the one-element set 1 into the terminal object in $\text{Sets}$ (taking $U$ as any set).

Thus the maps induced by the bottom of the inclusion/refinement relations in the two logical partial orders give the canonical maps for the initial and terminal objects in $\text{Sets}$.

| Dualities       | Subset logic | Partition logic |
|-----------------|-------------|----------------|
| 'Elements'      | Elements $u$ of $S$ | Dits $(u, u')$ of $\pi$ |
| Partial order   | Inclusion $S \subseteq T$ | $\sigma \preceq \pi$: dit $(\sigma) \subseteq \text{dit}(\pi)$ |
| Canonical map   | $S \rightarrow T$ | $\pi \rightarrow \sigma$ |
| Top of partial order | $U$ all elements | $1_U$, dit$(1_U) = U^2 - \Delta$, all dits |
| Bottom of partial orders | $\emptyset$ no elements | $0_U$, dit$(0_U) = \emptyset$, no dits |
| Extremal objects $\text{Sets}$ | $\emptyset \subseteq U$, $\emptyset \rightarrow U$ | $1 \cong 0_U \preceq 1_U$, $U \rightarrow 1$ |

Table 1: Elements and distinctions in the dual logics

There are different ways to characterize objects with universal mapping properties (like the initial and terminal objects) using ‘higher order’ machinery in category theory, e.g., using units or counits of adjunctions or using representable functors. For instance, consider the covariant functor $I : C \to \text{Sets}$ that takes any object in a category $C$ to a singleton set in $\text{Sets}$. If there is a natural isomorphism $\text{Hom}_C(0, -) \cong I(-)$ then the object 0 in $\text{C}$ represents $I(-)$ and 0 is an initial object in $C$. If there is another object $0'$ such that $\text{Hom}_C(0', -) \cong I(-)$, then $\text{Hom}_C(0, -) \cong \text{Hom}_C(0', -)$ and, by the Yoneda Lemma, there is a canonical isomorphism $0 \cong 0'$ (21). But this important characterization of the initial object 0 in $\text{Sets}$ does not offer an explanation of why 0 has that
universal property in the first place. Our claim is that the underlying fact that $\emptyset \subseteq U$ in $\wp(U)$ for any set $U$ accounts for it being the initial object in $\text{Sets}$, and dually, $1 \cong 0_U \preceq 1_U$ in $\Pi(U)$ for any set $U$ accounts for $1$ being the terminal object in $\text{Sets}$. And $\emptyset$ being the no-elements subset and $0_U$ being the no-distinctions partition accounts for them being the bottoms of the dual logical lattices $\wp(U)$ and $\Pi(U)$ where the respective partial orders are inclusions of elements and inclusions of distinctions.

1.5 The epi-mono factorization in $\text{Sets}$

Another simple application of the elements & distinctions analysis is the construction of the canonical surjection and canonical injection in the epi-mono factorization of any set function: $f : X \to Y$. The data in the function provide the coimage (or inverse-image) partition $f^{-1} = \{ f^{-1}(y) : y \in f(X) \}$ on $X$ [where the blocks of $f^{-1}$ are indexed by the $y \in f(X)$] and the image subset $f(X)$. Since $f^{-1}$ is refined by the discrete partition on $X$, $f^{-1} \preceq 1_X$, the induced surjection is the canonical map $X \twoheadrightarrow f(X)$. Note that in this case, the initial data $f$ only defined one partition $f^{-1}$ on $X$. The refinement relation used is that the discrete partition $1_X$ on $X$ refines all partitions on $X$. Similarly the initial data $f$ only defines one subset $f(X)$ of $Y$ but all such subsets are included in $Y$ so that inclusion $f(X) \subseteq Y$ induces the injection $f(X) \hookrightarrow Y$ and the epi-mono factorization of $f$ is:

$$f : X \to Y = X \twoheadrightarrow f(X) \hookrightarrow Y.$$ 

1.6 Abstracting to arrow-theoretic definitions

One of our themes is that the concepts of elements and distinctions unpack and analyze the category theoretic concepts in the basic ‘ur-category’ $\text{Sets}$, and they are abstracted into purely arrow-theoretic definitions in abstract category theory. For instance, the elements & distinctions definitions of injections and surjections yield ”arrow-theoretic” characterizations which can then be applied in any category to provide the usual category-theoretic dual definitions of monomorphisms (injections for set functions) and epimorphisms (surjections for set functions).

Two set functions $f, g : X \rightrightarrows Y$ are different, i.e., $f \neq g$, if there is an element $x$ of $X$ such that their values $f(x)$ and $g(x)$ are a distinction of $Y$, i.e., $f(x) \neq g(x)$. Hence if $f$ and $g$ are followed by a function $h : Y \to Z$, then the compositions $hf, hg : X \to Y \to Z$ must be different if $h$ preserves distinctions (so that the distinction $f(x) \neq g(x)$ is preserved as $hf(x) \neq hg(x)$), i.e., if $h$ is injective. Thus in the category of sets, $h$ being injective is characterized by: for any $f, g : X \rightrightarrows Y$, ”$f \neq g$ implies $hf \neq hg$” or equivalently, ”$hf = hg$ implies $f = g$” which is the general category-theoretic definition of a monomorphism or mono.

In a similar manner, if we have functions $f, g : X \rightrightarrows Y$ where $f \neq g$, i.e., where there is an element $x$ of $X$ such that their values $f(x)$ and $g(x)$ are a distinction of $Y$, then suppose the functions are preceded by a function $h : W \to X$. Then the compositions $fh, gh : W \to X \to Y$ must be different if $h$ reflects elements (so that the element $x$ where $f$ and $g$ differ is sure to be in the image of $h$), i.e., if $h$ is surjective. Thus in the category of sets, $h$ being surjective is characterized by: for any $f, g : X \rightrightarrows Y$, ”$f \neq g$ implies $fh \neq fg$” or ”$fh = gh$ implies $f = g$” which is the general category-theoretic definition of an epimorphism or epi.

Hence the dual interplay of the notions of elements & distinctions can be seen as yielding the arrow-theoretic characterizations of injections and surjections which are lifted into the general categorical dual definitions of monomorphisms and epimorphisms.

1.7 Duality interchanges elements & distinctions

In the duality of plane projective geometry, every proof of a theorem involving points and lines yields another proof of the theorem with points and lines interchanges. Similarly, any arrow-theoretic proof of a result in category theory yields a proof of a result in the opposite category with the arrows
reversed. In $\text{Sets}$, what is interchanged (like points and lines) to reverse the arrows? The reverse-the-arrows duality of category theory is the abstraction from the reversing of the roles of elements & distinctions (or its & dits) in dualizing $\text{Sets}$ to $\text{Sets}^{\text{op}}$. That is, a concrete morphism in $\text{Sets}^{\text{op}}$ is a binary relation, which might be called a *cofunction*, that preserves distinctions and reflects elements—instead of preserving elements and reflecting distinctions.

**Function:** A binary relation that preserves Its and reflects Dits.

**Cofunction:** A binary relation that preserves Dits and reflects Its.

Figure 1: Interchange Its & Dits dualizes between functions and cofunctions

Equivalently, when we reverse the direction of a binary relation defining a function, we just interchanged "reflects" and "preserves" (or "transmits"). Thus with every binary relation $f \subseteq X \times Y$ that is a function $f : X \to Y$, there is a binary relation $f^{\text{op}} \subseteq Y \times X$ that is a cofunction $f^{\text{op}} : Y \to X$ in the opposite direction.

**Function:** A binary relation that preserves Its and reflects Dits.

**Cofunction:** A binary relation that reflects Its and preserves Dits.

Figure 2: Interchange preserves and reflects dualizes between functions and cofunctions

For the universal constructions in $\text{Sets}$, the interchange in the roles of elements and distinctions interchanges each construction and its dual: products and coproducts, equalizers and coequalizers, and in general limits and colimits. That is then abstracted to make the reverse-the-arrows duality in abstract category theory.

This begins to illustrate our theme that the language of elements & distinctions is the conceptual language in which the category of sets and functions is written, and abstract category theory gives the abstract-arrows version of those definitions. Hence we turn to universal constructions for further analysis.

## Chapter 2 The Elements & Distinctions Analysis of Products and Coproducts

### 2.1 The coproduct in $\text{Sets}$

Given two sets $X$ and $Y$ in $\text{Sets}$, the idea of the *coproduct* is to create the set with the maximum number of elements starting with $X$ and $Y$. Since $X$ and $Y$ may overlap, we must make two copies of the elements in the intersection. Hence the relevant operation is not the union of sets $X \cup Y$ but the disjoint union $X \sqcup Y$. To take the disjoint union of a set $X$ with itself, a copy $X^* = \{x^* : x \in X\}$ of $X$ is made so that $X \sqcup X$ can be constructed as $X \sqcup X^*$. In a similar manner, if $X$ and $Y$ overlap, then $X \sqcup Y = X \sqcup Y^*$.

Then the inclusions $X, Y \subseteq X \sqcup Y$, give the canonical injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$.

The universal mapping property for the coproduct in $\text{Sets}$ is that given any other ‘cocone’ of maps $f : X \to Z$ and $g : Y \to Z$, there is a unique map $f \sqcup g : X \sqcup Y \to Z$ such that the triangles commute in the following diagram.
Then each element \( x \in X \) for the equalizer and coequalizer, the data is not just two sets but two parallel maps \( f, g : Y \to Z \). The map \( f : X \to Z \) defines the image \( f(X) \subseteq Z \) and \( g : Y \to Z \) defines the image \( g(Y) \subseteq Z \) and the subset lattice join in \( \mathcal{V}(Z) \) gives \( f(X) \cup g(Y) \subseteq Z \) so there is a canonical injection \( f(X) \cup g(Y) \to Z \). Since each \( x \in X \cup Y \) is associated with \( f(x) \in f(X) \cup g(Y) \) and each \( y \in X \cup Y \) is associated with \( g(y) \in f(X) \cup g(Y) \), that epi completes the factor map \( f \sqcup g : X \cup Y \to f(X) \cup g(Y) \to Z \).

2.2 The product in \( \textbf{Sets} \)

Given two sets \( X \) and \( Y \) in \( \textbf{Sets} \), the idea of the product is to create the set with the maximum number of distinctions starting with \( X \) and \( Y \). The product in \( \textbf{Sets} \) is usually constructed as the set of ordered pairs in the Cartesian product \( X \times Y \). But to emphasize the point about distinctions, we might employ the same trick of ‘marking’ the elements of \( Y \), particularly when \( Y = X \), with an asterisk. Then an alternative construction of the product in \( \textbf{Sets} \) is the set of unordered pairs \( X \bowtie Y = \{ \{ x, y \* \} : x \in X; y \* \in Y \* \} \) which in the case of \( Y = X \) would be \( X \bowtie X = \{ \{ x, x \* \} : x \in X; x \* \in X \* \} \). This alternative construction of the product (isomorphic to the Cartesian product) emphasizes the distinctions formed from \( X \) and \( Y \) so the ordering in the ordered pairs of the usual construction \( X \times Y \) is only a way to make the same distinctions.

The set \( X \) defines a partition \( \pi_X \) on \( X \times Y \) whose blocks are \( B_x = \{ \{ x, y \} : y \in Y \} = \{ x \} \times Y \) for each \( x \in X \), and \( Y \) defines a partition \( \pi_Y \) whose blocks are \( B_y = \{ \{ x, y \} : x \in X \} = X \times \{ y \} \) for each \( y \in Y \). Since \( \pi_X, \pi_Y \preceq 1_{X \times Y} \), the induced maps (surjections if \( X \) and \( Y \) are non-empty) are the canonical projections \( p_X : X \times Y \to X \) and \( p_Y : X \times Y \to Y \).

The universal mapping property for the product in \( \textbf{Sets} \) is that given any other ‘cone’ of maps \( f : Z \to X \) and \( g : Z \to Y \), there is a unique map \( [f,g] : Z \to X \times Y \) such that the triangles commute in the following diagram.

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y \\
\downarrow & \searrow & \downarrow \\
X & \xrightarrow{p_X} & X \times Y & \xleftarrow{p_Y} & Y \\
\end{array}
\]

Product diagram

From the data \( f : Z \to X \) and \( g : Z \to Y \), we need to canonically construct the unique factor map \( [f,g] : Z \to X \times Y \). The map \( f \) contributes the coimage \( f^{-1} \) partition on \( Z \) and \( g \) contributes the coimage \( g^{-1} \) partition on \( Z \) so we have the partition lattice join \( f^{-1} \vee g^{-1} \) in \( \Pi(Z) \) whose blocks have the form \( f^{-1}(x) \cap g^{-1}(y) \). To define the unique factor map \( [f,g] : Z \to X \times Y \), the discrete partition \( 1_Z \) refines \( f^{-1} \vee g^{-1} \) so there is a canonical surjection \( 1_Z \cong Z \to f^{-1} \vee g^{-1} \) which takes each \( z \in Z \) to the unique block of the form \( f^{-1}(x) \cap g^{-1}(y) \) containing \( z \). Each such block of the form \( f^{-1}(x) \cap g^{-1}(y) \) defines a mono \( f^{-1} \vee g^{-1} \to X \times Y \) so the factor map is: \( [f,g] : Z \to f^{-1} \vee g^{-1} \to X \times Y \).

3 The Elements & Distinctions Analysis of Equalizers and Coequalizers

3.1 The coequalizer in \( \textbf{Sets} \)

For the equalizer and coequalizer, the data is not just two sets but two parallel maps \( f, g : X \rightrightarrows Y \). Then each element \( x \in X \), gives us a pair \( f(x) \) and \( g(x) \) so we take the equivalence relation ~
defined on \( Y \) that is generated by \( f(x) \sim g(x) \) for any \( x \in X \). Then the coequalizer is the quotient set \( C = Y/\sim \). When \( \sim \) is represented as a partition on \( Y \), then it is refined by the discrete partition \( 1_Y \) on \( Y \), and that refinement defines the canonical surjection \( Y \cong 1_Y \to Y/\sim \).

For the UMP, let \( h : Y \to Z \) be such that \( hf = hg \). Then we need to show there is a unique refinement-defined map \( h^* : Y/\sim \to Z \) such that the triangle commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow h \downarrow \nearrow & \sim \\
& & Z
\end{array}
\]

Coequalizer diagram

We already have one partition \( \sim \) on \( Y \) which was generated by \( f(x) \sim g(x) \). Since \( hf = hg \), we are given that \( hf(x) = hg(x) \) so the coimage \( h^{-1} \) has to at least identify \( f(x) \) and \( g(x) \) (and perhaps identify other elements) which means that \( h^{-1} \cong Y/\sim \) in the partition lattice on \( Y \). Hence the induced surjection map \( Y/\sim \to h^{-1} \) and the mono \( h^{-1} \cong h(Y) \to Z \) (taking \( h^{-1}(z) \) to \( z \)) completes the factor map \( h^* : Y/\sim \to h^{-1} \to Z \).

3.2 The equalizer in \( \text{Sets} \)

For the same data \( f, g : X \to Y \), the equalizer is the \( \text{Eq} = \{ x \in X : f(x) = g(x) \} \subseteq X \) so the map induced by that inclusion is the canonical map \( \text{Eq} \to X \).

The UMP is that for any other map \( h : Z \to X \) such that \( fh = gh \), then \( \exists h_* : Z \to \text{Eq} \) such that the triangle commutes.

\[
\begin{array}{ccc}
\text{Eq} & \xrightarrow{\sim} & X \\
& \nearrow f \searrow & g \\
& \downarrow h_* & \nearrow h \\
& \sim & Z
\end{array}
\]

Equalizer diagram

The image of \( h(Z) \subseteq X \) must satisfy \( fh(z) = gh(z) \) for all \( z \in Z \), so \( f \) and \( g \) agree on \( h(z) \in X \) so \( h(Z) \subseteq \text{Eq} \), which gives the canonical injection \( h(Z) \to \text{Eq} \) and the epi \( Z \to h(Z) \) completes the factor map \( h_* : Z \to h(Z) \Rightarrow \text{Eq} \).

4 The Elements & Distinctions Analysis of Pushouts and Pullbacks

4.1 The pushout or co-Cartesian square in \( \text{Sets} \)

It is a standard theorem of category theory that if a category has products and equalizers, then it has all limits, and if it has coproducts and coequalizers, then it has all colimits. Since we have presented the elements & distinctions analysis of the canonical maps for products and coproducts, and for equalizers and coequalizers, the analysis extends to all limits and colimits. Hence we have shown that the logical characterization of canonical maps in \( \text{Sets} \) satisfies Marquis's criteria:

1. Morphisms that are part of the data of a limit are canonical morphisms; for instance, the projection morphisms that are part of the notion of a product;
2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism; and
3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism. [15, p. 101]
However, the theme would be better illustrated by considering some more complicated limits and colimits such as Cartesian and co-Cartesian squares, i.e., pullbacks and pushouts.

For the pushout or co-Cartesian square, the data are two maps \( f : Z \to X \) and \( g : Z \to Y \) so we have the two parallel maps \( Z \xrightarrow{f} X \xleftarrow{g} Y \) and \( Z \xrightarrow{g} Y \xleftarrow{f} X \) and then we can take their coequalizer \( C \) formed by the equivalence relation \( \sim \) on the common codomain \( X \sqcup Y \) which is the equivalence relation generated by \( x \sim y \) if there is a \( z \in Z \) such that \( f(z) = x \) and \( g(z) = y \). The canonical maps \( X \to X \sqcup Y/\sim \) and \( Y \to X \sqcup Y/\sim \) are just the canonical injections into the disjoint union followed by the canonical map of the coequalizer construction analyzed above. As the composition of a canonical injection with a canonical surjection, those canonical maps need not be either injective or surjective.

4.2 The pullback or Cartesian square in \( \mathbf{Sets} \)

For the Cartesian square or pullback, the data are two maps \( f : X \to Z \) and \( g : Y \to Z \). Then \( h^{-1} \) is a partition on \( X \) and \( h'^{-1} \) is a partition on \( Y \) so let \( h^{-1} \sqcup h'^{-1} \) be the disjoint union partition on \( X \sqcup Y \). The condition that for any \( z \in Z \), \( h f(z) = h'g(z) = u \) for some \( u \in U \) means that \( h^{-1} \sqcup h'^{-1} \) must make at least the identifications of the coequalizer (and perhaps more) so that \( h^{-1} \sqcup h'^{-1} \) is refined by \( \sim \) as partitions on \( X \sqcup Y \). Since \( h^{-1} \sqcup h'^{-1} \not\sim \sim \) so each block \( b \) in \( \sim \) is contained in a block of the form \( h^{-1}(u) \) for some \( u \) or a block of the form \( h'^{-1}(u) \) for some \( u \). Hence that block \( b \) of \( \sim \) is mapped by \( h^* \) to the appropriate \( u \) depending on the case which defines the surjection from \( X \sqcup Y/\sim \) to \( h(X) \sqcup h'(Y) \subseteq U \) and the inclusion defines the injection to complete the definition of the canonical factor map \( h^* : C = X \sqcup Y/\sim \to U \).

For the universal mapping property, consider any \( h : X \to U \) and \( h' : Y \to U \) such that \( h f = h' g \). Then \( h^{-1} \) is a partition on \( X \) and \( h'^{-1} \) is a partition on \( Y \) so let \( h^{-1} \sqcup h'^{-1} \) be the disjoint union partition on \( X \sqcup Y \). The condition that for any \( z \in Z \), \( h f(z) = h'g(z) = u \) for some \( u \in U \) means that \( h^{-1} \sqcup h'^{-1} \) must make at least the identifications of the coequalizer (and perhaps more) so that \( h^{-1} \sqcup h'^{-1} \) is refined by \( \sim \) as partitions on \( X \sqcup Y \). Since \( h^{-1} \sqcup h'^{-1} \not\sim \sim \) so each block \( b \) in \( \sim \) is contained in a block of the form \( h^{-1}(u) \) for some \( u \) or a block of the form \( h'^{-1}(u) \) for some \( u \). Hence that block \( b \) of \( \sim \) is mapped by \( h^* \) to the appropriate \( u \) depending on the case which defines the surjection from \( X \sqcup Y/\sim \) to \( h(X) \sqcup h'(Y) \subseteq U \) and the inclusion defines the injection to complete the definition of the canonical factor map \( h^* : C = X \sqcup Y/\sim \to U \).

For the universality property, consider any other maps \( h : U \to X \) and \( h' : U \to Y \) such that \( fh = gh' \). Hence \( h'(u) \) and \( h(u) \) are elements such that \( f(h(u)) = g(h'(u)) \) so \( (h(u), h'(u)) \in E \) and thus for the images, there is the inclusion \( h(U) \times h'(U) \subseteq E \). Now \( h \) contributes the coimage partition \( h^{-1} \) on \( U \) and \( h' \) contributes the coimage partition \( h'^{-1} \) on \( U \) and the join \( h^{-1} \sqcup h'^{-1} \) is refined by the discrete partition on \( U \). Hence each \( u \in U \) is contained in a unique block \( h^{-1}(x) \cap h'^{-1}(y) \) of the join so the refinement-induced canonical map \( U \to h(U) \times h'(U) \subseteq E \) is defined by \( u \mapsto (x, y) \) and the inclusion-defined injection \( h(U) \times h'(U) \to E \) completes the definition of the canonical factor map \( h_* : U \to E \).
5 The Elements & Distinctions Analysis of the Exponential Adjunction

The adjunction $\text{Hom}_{\text{Sets}}(X \times Y, Z) \cong \text{Hom}_{\text{Sets}}(X, \text{Hom}(Y, Z))$ is entirely in $\text{Sets}$. In the following diagram:

$$
\begin{array}{ccc}
X \xrightarrow{\text{can}} \text{Hom}(Y, X \times Y) & \xrightarrow{g} & \text{Hom}(Y, Z) \\
\downarrow f & & \downarrow f \\
X \times Y & \xrightarrow{\exists ! f} & Z
\end{array}
$$

the two canonical maps that need to be analyzed are counit $X \xrightarrow{\text{can}} \text{Hom}(Y, X \times Y)$ and the unique factor map $f : X \times Y \to Z$ where the given data for the factor map is the map $g : X \to \text{Hom}(Y, Z)$. That given data $g$ defines the partition $B_z = \{(x, y) \in X \times Y : g(x)(y) = z\}$ on $X \times Y$ indexed by $z \in Z$ and then the factor map $f : X \times Y \to Z$ is induced by the refinement $\{B_z\}_{z \in Z} \preceq 1_{X \times Y}$.

To analyze $X \xrightarrow{\text{can}} \text{Hom}(Y, X \times Y)$, for each $x \in X$, there is a (discrete) partition on $Y$ with the block $\{y\}$ indexed by the ordered pair $(x, y)$. Since the discrete partition on $Y$ refines itself, there is the induced map $h_x : Y \to X \times Y$ defined by $y \mapsto (x, y)$. And those functions $h_x \in \text{Hom}(Y, X \times Y)$ can also index the blocks $\{x\}$ of the discrete partition on $X$ which induces the injection $X \to \text{Hom}(Y, X \times Y)$ defined by $x \mapsto h_x$.

In the other UMP diagram for the adjunction:

$$
\begin{array}{ccc}
Z \xleftarrow{\text{eval}} \text{Hom}(Y, Z) \times Y & \xleftarrow{\exists ! g} & \text{Hom}(Y, Z) \\
\uparrow g \times 1_Y & & \uparrow g \\
X \times Y & \xrightarrow{g} & X
\end{array}
$$

the two canonical maps that need to be analyzed are counit $Z \xleftarrow{\text{eval}} \text{Hom}(Y, Z) \times Y$ and the unique factor map $g : X \to \text{Hom}(Y, Z)$ where the given data is the map $f : X \times Y \to Z$. The given data $f$ defines for each $x \in X$, a $Y$-partition $B_z = \{y : f(x, y) = z\}$ indexed by $z \in f(X \times Y)$ which is refined by the discrete partition on $Y$. Thus each $x \in X$ determines a function $g_x : Y \to Z$, so the factor map is $g : X \to \text{Hom}(Y, Z)$ where $g(x) = g_x \in \text{Hom}(Y, Z)$.

To analyze the canonical evaluation map $\text{Hom}(Y, Z) \times Y \to Z$, each $z \in Z$ determines a partition on the domain by the blocks $B_z = \{(h, y) : h(y) = z\}$, and that partition is refined by the discrete partition on $\text{Hom}(Y, Z) \times Y$ and the induced map is the evaluation map.

6 Example: A more complex canonical map

Marquis [15] gives the standard examples of canonical maps that arise from limits and colimits but also mentions a more complex example that will be analyzed. Let $\mathcal{C}$ be a category with finite products, finite coproducts, and a null object (an object that is both initial and terminal). Then a canonical morphism can be constructed from the coproduct of two (or any finite number of) objects to the products of the objects: $X \sqcup Y \to X \times Y$. In such a category abstractly specified, the map could be constructed from the ‘atomic’ canonical morphisms that are already given by the arrow-theoretic definitions of products, coproducts, and the null object. But the its & dits analysis shows how all these ‘atomic’ canonical morphisms and their ‘molecular’ compositions are not just assumed but are constructed in $\text{Sets}$ or $\text{Sets}$-based categories according to the logical theory of canonical maps.

There is a simple $\text{Sets}$-based category that has finite products, finite products, and a null object, namely the category $\text{Sets}_*$ of pointed sets where the objects are sets with a designated element (or basepoint), e.g., $(X, x_0)$ with $x_0 \in X$, and the morphisms are set functions that preserve the basepoints. The designation of the basepoint can be given by a set map $1 \xrightarrow{x_0} X$ in $\text{Set}$ which is taken as part of the structure and is thus assumed canonical in $\text{Set}_*$. A basepoint preserving map $(X, x_0) \to (Y, y_0)$ is a set map $X \to Y$ in $\text{Sets}$ so that the following diagram commutes:
Hence $\text{Sets}_*$ can also be seen as the slice category $1/\text{Sets}$ of $\text{Sets}$ under 1.

The null object is the one-point set $1$ and instead of assuming the canonical morphisms that make it initial and terminal, we need to construct them using the its & dits analysis. We have already seen that the refinement relation $\mathbf{0}_X \not\leq 1_X$ induces the unique map $X \rightarrow 1$ that makes $1$ the terminal object in $\text{Sets}$. And since $1 \xrightarrow{x_0} X \rightarrow 1 = 1 \xrightarrow{id} 1$, it is also the terminal object in $\text{Sets}_*$. Moreover, the basepoint in $(Y, y_0)$ is given by the structurally canonical map $1 \xrightarrow{y_0} Y$ and since $1 \xrightarrow{\mathbf{id}} 1 \xrightarrow{y_0} Y = 1 \xrightarrow{y_0} Y$, that map $1 \xrightarrow{y_0} Y$ is the unique map that makes $1$ also the initial object in $\text{Sets}_*$. Hence in $\text{Sets}_*$, there is always a canonical map formed by the composition: $X \rightarrow 1 \rightarrow Y = X \rightarrow Y$ (called the zero arrow).

To build up the its & dits analysis of the canonical morphism $X \sqcup_{\ast} Y \rightarrow X \times_{\ast} Y$ from the coproduct to the product in $\text{Sets}_*$, we begin with the construction of the coproduct $X \sqcup_{\ast} Y$ which is just the pushout in $\text{Sets}$ of the two canonical basepoint maps:

$$
\begin{array}{ccc}
1 & \xrightarrow{x_0} & X \\
\downarrow{y_0} & \searrow{\text{can.}} & \\
Y & \xrightarrow{\text{can.}} & X \sqcup_{\ast} Y = X \sqcup Y/ \sim \\
\parallel & \downarrow{\exists !_Y} & \searrow{h_*} \\
Y & \xrightarrow{h'_Y} & U
\end{array}
$$

Since the only points in $X$ and $Y$ that are the image of an elements in $1$ are the basepoints, the equivalence relation $\sim$ only identifies the basepoints $x_0$ and $y_0$. Hence $X \sqcup_{\ast} Y$ is like $X \sqcup Y$ except that the two basepoints are identified in the quotient $X \sqcup_{\ast} Y = X \sqcup Y/ \sim$ and that block identifying the basepoints is the basepoint of $X \sqcup_{\ast} Y$. Then for any two set maps $h : X \rightarrow U$ and $h' : Y \rightarrow U$ that are also $\text{Sets}_*$ morphisms (i.e., preserve basepoints), there is a unique canonical factor map $h_* : X \sqcup_{\ast} Y \rightarrow U$ by the UMP for the pushout in $\text{Sets}$ to make the triangles commute (and thus preserve basepoints). Hence $X \sqcup_{\ast} Y$ is the coproduct in $\text{Sets}_*$.

In a similar manner, one shows that the product $X \times_{\ast} Y$ in $\text{Sets}_*$ is just the product $X \times Y$ in $\text{Sets}$ with $(x_0, y_0)$ as the basepoint. Using the UMP of the product $X \times_{\ast} Y$ in $\text{Sets}_*$, the two $\text{Sets}_*$ maps $1_X : X \rightarrow X$ and the canonical $X \rightarrow 1 \rightarrow Y$, we have the unique canonical factor map $X \rightarrow X \times_{\ast} Y$ in $\text{Sets}_*$ and similarly for $Y \rightarrow X \times_{\ast} Y$ in $\text{Sets}_*$.

Then we put all the canonical maps together and use the UMP for the coproduct in $\text{Sets}_*$ to construct the desired canonical map: $X \sqcup_{\ast} Y \rightarrow X \times_{\ast} Y$ in $\text{Sets}_*$.

$$
\begin{array}{ccc}
X & \xrightarrow{\text{can.}} & X \sqcup_{\ast} Y \\
\searrow{\text{can.}} & \Downarrow{\exists !_X} & \searrow{\text{can.}!

\text{Coproduct diagram in } \text{Sets}_*$

This example shows how in a $\text{Sets}$-based category like $\text{Sets}_*$, the given canonical maps for the structured sets (i.e., the basepoint maps $1 \xrightarrow{x_0} X$) are combined with the canonical maps defined by the its & dits analysis in $\text{Sets}$ to give the canonical morphisms in the $\text{Sets}$-based category. In abstract category theory, as in the case of a category $\mathcal{C}$ which is assumed to have finite products, finite coproducts, and a null object, the 'atomic' canonical morphisms are all given as part of the assumed UMPs for products, coproducts, and the null object which are then composed to define other ‘molecular’ canonical morphisms. That suggests that the only categories where a theory of canonical morphisms is needed is $\text{Sets}$ and $\text{Sets}$-based categories, and that is the theory presented here.
7 Concluding philosophical reflections

The "logical" in the logical theory of canonicity refers to the two dual mathematical logics: the Boolean logic of subsets and the logic of partitions. Note that from the mathematical viewpoint, the Boolean logic of subsets and the logic of partitions have equal intertwining roles in the whole analysis. Normally, we might say that "subsets" and "partitions" are category-theoretic duals, but we have tried to show a more fundamental analysis based on "elements & distinctions" or "its & dits" that are the building blocks of subsets and partitions and that underlie the duality in \( \text{Sets} \).

Thus instead of saying that duality explains elements and distinctions, we tried to show that the intricate and precise interplay of elements and distinctions explains morphisms, duality, canonicity, and universal constructions in \( \text{Sets} \), which generalizes to other \( \text{Sets} \)-based concrete categories and which is abstracted in abstract category theory.

Our focus here is the E&d treatment of canonicity.

- Each construction starts with certain data.
- When that data is sufficient to define inclusions in an associated subset lattice or refinements in an associated partition lattice, then the induced "logical" maps (and their compositions) are canonical.

This suggests that the dual notions of elements & distinctions (its & dits) have some broader significance. One possibility is they are respectively mathematical building blocks of the old metaphysical concepts of matter (or substance) and form (as in in-form-ation). The matter versus form idea \( \Pi \) can be illustrated by comparing the two lattices of subsets and partitions on a set—the two lattices that we saw defined the canonical morphisms and canonical factor maps in \( \text{Sets} \)-based categories.

For \( U = \{a, b, c\} \), start at the bottom and move towards the top of each lattice.

\[ \begin{align*}
\text{Elements} & \quad \text{increase.} \\
\text{Distinctions} & \quad \text{constant.} \\
\{a\} & \quad \{a\} \\
\{a,b\} & \quad \{a\} \quad \{b\} \\
\{a,b,c\} & \quad \text{Substance increases,} \\
& \quad \text{always fully} \\
& \quad \text{formed.} \\
\emptyset & \quad \text{Start with zero} \\
& \quad \text{substance.}
\end{align*} \]

\[ \begin{align*}
\text{Substance} & \quad \text{increasingly} \\
& \quad \text{in-formed} \\
& \quad \text{by making} \\
& \quad \text{distinctions.} \\
\{a\} & \quad \{a\} \\
\{b\} & \quad \{b\} \\
\{c\} & \quad \{c\} \\
\{a,b,c\} & \quad \text{Start with zero} \\
& \quad \text{form.}
\end{align*} \]

Figure 3: Moving up the subset and partition lattices.

At the bottom of the Boolean subset lattice is the empty set \( \emptyset \) which represents no substance (no elements or 'its'). As one moves up the lattice, new elements of substance, new elements, are created that are always fully distinguished or formed until finally one reaches the top, the universe \( U \). Thus new substance is created in moving up the lattice but each element is fully formed and thus distinguished from the other elements.

At the bottom of the partition lattice is the indiscrete partition or "blob" \( 0_U = \{U\} \) (where the universe set \( U \) makes one block) which represents all the substance or matter but with no distinctions to in-form the substance (no distinctions or 'dits'). As one moves up the lattice, no new substance is created but distinctions are created that in-form the indistinct elements as they become more and more distinct. Finally one reaches the top, the discrete partition \( 1_U \), where all the elements of \( U \) have been fully in-formed or distinguished. A partition combines indefiniteness (within blocks) and definiteness (between blocks). At the top of the partition lattice, the discrete partition
\(1_u = \{\{u\} : \{u\} \subseteq U\}\) is the result making all the distinctions to eliminate any indefiniteness. Thus one ends up at essentially the same place (universe \(U\) of fully formed entities) either way, but by two totally different but dual ‘creation stories’:

- Subset Creation Story: creating elements as in creating fully-formed and distinguished matter out of nothing, versus
- Partition Creation Story: creating distinctions by starting with a totally undifferentiated matter and then, in a ‘big bang,’ start making distinctions, e.g., breaking symmetries, to give form to the matter.

Moreover, we have seen that:

- the quantitative increase in substance (normalized number of elements) moving up in the subset lattice is measured by logical or Laplacian probability, and
- the quantitative increase in form (normalized number of distinctions) moving up in the partition lattice is measured by logical entropy ([4]; [8]; [13]).

Declarations:
The author has no competing or conflicting interests to declare that are relevant to the content of this article.
No funds, grants, or other support was received.
Data availability: N/A

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