MODELS OF AFFINE CURVES AND $G_a$-ACTIONS

KEVIN LANGLOIS

Dedicated to Mikhail Zaidenberg on
the occasion of his 70-th birthday

ABSTRACT. Using the approach of Barkatou and El Kaoui, we classify certain affine curves over discrete valuation rings having a free additive group action. Our classification generalizes results of Miyanishi in equi-characteristic 0.

1. Introduction

Let $O$ be a discrete valuation ring. Choose a uniformizer $t \in O$ such that $k = O/(t)$ is the residue field, and write $K$ for the fraction field of $O$. A faithful flat integral affine scheme of finite type over $O$ is an affine $O$-curve if it has relative dimension 1 and if $K$ is algebraically closed in its function field. Our aim is to classify the models of the affine line (i.e., affine $O$-curves whose generic fiber is isomorphic to $A^1_k$). In particular, the arithmetic surface in question inherits a non-trivial action of the additive group scheme $G_a$.

Miyanishi described any affine $O$-curve with a (free) $G_a$-action, such that the special fiber is integral, and under the condition that $O$ is equi-characteristic 0 [Miy09, Theorem 4.3]. Barkatou and El Kaoui extended this result in [BE12] for reduced special fibers over an equi-characteristic 0 principal ideal domain. Using the approach of loc. cit., we obtain the following generalization which is valid in any characteristic.

Theorem 1.1. Assume that $k$ is perfect. Let $C$ be an affine $O$-curve with a free $G_a$-action and reduced special fiber. Then there exist a natural number $n \geq 1$ and polynomials $f_i \in O[x_1, \ldots, x_i]$ for $1 \leq i \leq n$ such that

$$C \simeq \text{Spec} O[x_1, \ldots, x_{n+1}]/ (tx_2 - f_1, \ldots, tx_{n+1} - f_n).$$

Moreover, the following properties are fulfilled.

(i) The ideal $(t, f_1, \ldots, f_n)$ is 0-dimensional and radical, and the reduction modulo $t$ of the $\partial f_i/\partial x_i$’s are invertible.

(ii) Consider the subalgebra $B = O[\alpha_1(x), \ldots, \alpha_n(x)] \subseteq K[x]$, where the $\alpha_i$’s are defined as

$\alpha_1(x) = x$ and $\alpha_i(x) = t^{-1}f_{i-1}(\alpha_1(x), \ldots, \alpha_{i-1}(x))$ for $2 \leq i \leq n$. Then, under the previous isomorphism, the $G_{a,\sigma}$-actions on $C$ are in one-to-one correspondence with the $G_{a,K}$-actions $x \mapsto x + \sum_{j=1}^r c_j \lambda_j^{s_j}$ on $A^1_k = \text{Spec} K[x]$ that let stable the algebra $B$, where $s_j \in \mathbb{Z}_{\geq 0}, c_j \in K$, and $p$ is the characteristic exponent of the field $K$.

Section 2 sets the notation of the paper, while the proof of Theorem 1.1 is in Section 3.

2010 Mathematics Subject Classification 14L15, 14L30.

Key Words and Phrases: Algebraic curves, $G_a$-actions.
2. Basics

A \( \mathbb{G}_{a,\mathcal{O}} \)-action on the affine \( \mathcal{O} \)-curve \( C = \text{Spec} B \) is equivalent to a sequence

\[ \delta^{(i)} : B \to B, \quad i = 0, 1, 2, \ldots \]

of \( \mathcal{O} \)-linear maps sharing the conditions (see [Miy68]):

(a) The map \( \delta^{(0)} \) is the identity,

(b) for any \( b \in B \) there is \( i \in \mathbb{Z}_{>0} \) such that \( \delta^{(j)}(b) = 0 \) for any \( j \geq i \),

(c) we have the Leibniz rule

\[ \delta^{(i)}(b_1 \cdot b_2) = \sum_{i_1+i_2=i} \delta^{(i_1)}(b_1) \cdot \delta^{(i_2)}(b_2), \]

where \( i \in \mathbb{Z}_{\geq 0} \) and \( b_1, b_2 \in B \), and

(d) for all indices \( i, j \in \mathbb{Z}_{\geq 0} \):

\[ \delta^{(i)} \circ \delta^{(j)} = \left( \begin{array}{c} i+j \\ i \end{array} \right) \delta^{(i+j)}. \]

The sequence \( \delta = (\delta^{(i)}) \) is called a locally finite iterative higher derivations (LFIHD). The kernel \( \ker(\delta) \) is the intersection of the linear subspaces \( \ker(\delta^{(i)}) \) where \( i \) runs over \( \mathbb{Z}_{>0} \). Since \( K \) is algebraically closed in the fraction field of \( B \), we have \( \ker(\delta) = \mathcal{O} \) if the action is nontrivial. The \( \mathbb{G}_{a,\mathcal{O}} \)-action on \( C \) is free if the ideal generated by \( \{ \delta^{(i)}(b) ; i \in \mathbb{Z}_{>0} \) and \( b \in B \} \) is \( B \). The exponential morphism is

\[ \exp(\delta T) : B \to B[T], \quad b \mapsto \sum_{i \in \mathbb{Z}_{\geq 0}} \delta^{(i)}(b)T^i. \]

**Lemma 2.1.** Assume that the \( \mathbb{G}_{a,\mathcal{O}} \)-action on \( C \) is free. Then there exist an LFIHD \( \delta \) on \( B \) corresponding to a free action and \( x \in B \) such that \( B \otimes_\mathcal{O} K = K[x] \) and

\[ \exp(\delta T)(x) = x + \sum_{i=1}^{m} t^{n_i}T^{e_i} \]

for some natural numbers \( n_i \) and some powers

\[ 1 \leq e_1 = p^{r_1} < \ldots < e_m = p^{r_m} = e, \]

where \( p \) is the characteristic exponent of \( \mathcal{O} \). Moreover, for any \( \kappa \gg 0 \) we may choose \( \delta \) such that \( \delta^{(e)} \) is an \( \mathcal{O} \)-derivation on the monomials of \( K[x] \) of degree less than \( \kappa \). Assume further that the special fiber of \( C \) is reduced. Then \( B = \mathcal{O}[x] \) provided that \( \min_{1 \leq i \leq m} n_i = 0 \).

**Proof.** We may assume that \( p > 1 \). Let \( \delta_1 \) be the LFIHD defined by the \( \mathbb{G}_{a,\mathcal{O}} \)-action. Choose \( x \in B \) such that \( \exp(T\delta_1)(x) \) has positive minimal degree. Then \( B \otimes_\mathcal{O} K = K[x] \) \([\text{CM05, Lemma 2.2 (c)}]\). As the extension \( \delta_K \) of \( \delta_1 \) on \( K[x] \) corresponds to a \( \mathbb{G}_{a,K} \)-action on \( \mathbb{A}^1_K \), we have \( \delta^{(j)}_1(x) \in \ker(\delta_K) \cap B = \mathcal{O} \) for any \( j \in \mathbb{Z}_{>0} \). So if \( \delta^{(j)}_1(x) \neq 0 \), then \( \delta^{(j)}_1(x) = c_j t^{m_j} \) for some \( m_j \in \mathbb{Z}_{\geq 0} \) and \( c_j \in \mathcal{O}^* \). Consequently, we modify \( \delta_1 \) by changing the \( c_j \)'s by \( 1 \). Now write

\[ \exp(\delta_1 T)(x) = x + \sum_{i=1}^{m-1} t^{n_i}T^{e_i}, \text{ where } m \geq 1. \]

We introduce a new LFIHD \( \delta \) on \( K[x] \) (trivial on \( K \)) defined by

\[ \exp(\delta T)(x) = \exp(\delta_1 T)(x) + t^nT^e. \]
where $e$ and $n$ satisfy the following conditions. Let $b_1, \ldots, b_s \in B$ such that $B = \mathcal{O}[b_1, \ldots, b_s]$ and consider a relation

$$1 = \sum_{j=1}^{\alpha} c_j \delta_1^{(\beta_j)}(d_j)$$

for $c_j, d_j \in B$ and $\beta_j \in \mathbb{Z}_{>0}$, which is guaranteed from the freeness assumption. Let $\kappa$ be a constant greater than the degrees in $x$ of the $b_i \in K[x]$ and take $e$ a power of $p$ verifying

$$e > (\kappa + 2) \max \{e_j, \beta_\ell \mid 1 \leq j < m \text{ and } 1 \leq \ell \leq \alpha \}.$$

Finally, let $n \in \mathbb{Z}_{>0}$ such that $t^n b_j \in \mathcal{O}[x]$ for $1 \leq j \leq s$. We claim that $\delta$ induces an LFIHD on $B$ with the required properties. Indeed, if $i < e$, then $\delta(i)(b) = \delta(i)(b)$ for any $b \in B$. Now assume that $i \geq e$ and let

$$b_j = \sum_u \lambda_u x^u,$$  

where $\lambda_u \in K$ and $d = \deg_x(b_j)$.

Let $e_\delta$ reaching the maximum of the $e_j$'s. By a direct induction on $u \leq d$, $t^n$ divides $\delta^{(\beta)}(x^u)$ if $\beta > ue_\delta$ (note that $\delta^{(\beta)}(x^u) = ux^{u-1}t^n$ if $\beta = e$). Thus $\delta(i)(b_j) \in \mathcal{O}[x]$ for any $j$ and $\delta$ induces a free $G_{a,K}$-action on $C$. This yields the first claim.

Let us show the second one. The assumption $\min_{1 \leq i \leq m} n_i = 0$ implies that $\delta^{(\gamma)}(x) = 1$ for some $\gamma$ and that the residue class $\bar{x}$ of $x$ modulo $t$ is not algebraic over $k$. Indeed, if $\bar{x}$ would admit an algebraic dependence relation, then, applying the exponential map (from the $G_{a,K}$-action on $\text{Spec } B/tB$) to this relation, we would get a contradiction. Let $b \in B \setminus tB$. Since $B \subseteq K[x]$, there is a primitive polynomial $s(T) \in \mathcal{O}[T]$ such that $s(x) = t^n b$ for some $r \in \mathbb{Z}_{>0}$. Observe that $s(x) \equiv 0 \mod tB$ if $r > 0$. So the previous step implies that $r = 0$ and $b \in \mathcal{O}[x]$. Now let $c \in tB$. Write $c = t^\ell a$ for some $\ell \in \mathbb{Z}_{>0}$ and $a \in B \setminus tB$. As $a \in \mathcal{O}[x]$, we have $c \in \mathcal{O}[x]$. Thus $B = \mathcal{O}[x]$, as required.

### 3. Proof of the main result

Let $C = \text{Spec } B$ be an affine $\mathcal{O}$-curve with a free $G_{a,K}$-action. Assume that the special fiber is reduced and that $k$ is perfect. Let $\delta$ be an LFIHD on $B$ as in the proof of Lemma 2.1. Set $e := e_m$ and $n := n_m$. Consider $B_1 := \mathcal{O}[x], I_1 := (tB) \cap B_1$ and inductively define the other rings and ideals by

$$B_{i+1} = B_i[t^{-1}I_i]$$

for $i = 1, \ldots, n$.

The inclusions $B_i \subseteq B_{i+1}$ yield a sequence of $G_{a,K}$-equivariant affine modifications (cf. [KZ99])

$$\text{Spec } B_{n+1} \to \text{Spec } B_n \to \ldots \to \text{Spec } B_2 \to A^1_{\mathcal{O}} = \text{Spec } B_1,$$

where $\text{Spec } B_{i+1}$ is the complement of the hypersurface $\mathbb{V}(t)$ in the blow-up of $\text{Spec } B_i$ with center $I_i$. The next lemma is analogous to [BE12, Lemma 4.4]. We use the adapted result [BE12, Lemma 4.3] for the ring $\mathcal{O}$ where $k$ needs to be perfect. Note that in the argument of the proof we will vary $\delta$ and $e$, while the number will be constant in the entire paper.

**Lemma 3.1.** There exist $x_1, \ldots, x_{n+1} \in B$ and $f_i \in \mathcal{O}[T_1, \ldots, T_i]$ for $1 \leq i \leq n$ such that the following hold.

1. $x_1 = x, t$ divides $f_i(x_1, \ldots, x_i)$, and if $x_{i+1} = t^{-1}f_i(x_1, \ldots, x_i)$,

then $t^{n-i+1}$ divides $\delta^{(e)}(x_i)$ for an appropriate choice of $\delta$. 


(ii) \( B_i = \mathcal{O}[x_1, \ldots, x_i], \quad I_i = (t, f_1(x_1), \ldots, f_i(x_1, \ldots, x_i)) \subseteq B_i, \) and the reduction modulo \( t \) of \( \partial f_i / \partial T_i \) is invertible.

(iii) We have \( C = \text{Spec} \ B_{n+1} \).

Proof. We show the existence of the \( f_i \)'s and (iii) by induction on \( i \). We treat the case \( i = 1 \). Let \( \delta_k \) be the LFIHD corresponding to the \( \mathbb{G}_{a,k} \)-action on \( \text{Spec} \ B/tB \) and set \( R := \ker(\delta_k) \). By our assumption and [CM05, Lemmata 2.1, 2.2], the scheme \( \text{Spec} \ R \) is 0-dimensional and reduced. Therefore \( B_1/I_1 \subseteq R \). By [BE12, Lemma 4.3], there is \( f_1 \in \mathcal{O}[T_1] \) such that \( I_1 = (t, f_1(x)) \subseteq B_1 \) and the reduction modulo \( t \) of \( \partial f_1 / \partial T_1 \) is a unit.

Before, we may choose \( \delta \) and the reductions modulo \( \mathcal{O} \) of \( a \) and \( \partial g / \partial T_i \)'s are invertible. From Property (ii) in loc. cit. we may choose \( g_1, \ldots, g_i \) such that \( I_i = (t, g_1(x_1), \ldots, g_i(x_1, \ldots, x_i)) \). Therefore, we take \( g_j = f_j \) for any \( 1 \leq j \leq i \) (from our induction hypothesis) and let \( f_{i+1} = g_{i+1} \), as required.

Choose an LFIHD \( \delta \) with \( e \gg 0 \) as in Lemma 2.1 such that \( \delta(e) \) acts as an \( \mathcal{O} \)-derivation on \( x_1, \ldots, x_{n+1} \) (seen as polynomials in \( x \)). We show (i) by induction on \( i \). For \( i = 1 \), we have \( \delta(e)(x_1) = \delta(e)(x) = t^a \), and for \( i = 2 \), we get \( \delta(e)(f_1(x)) = t^a \partial f_1(x) / \partial x \). As \( x_2 := t^{-1} f_1(x) \), it follows that \( t^{n-1} \delta(e)(x_2) \). Assume that Statement (ii) holds for \( i < n \). By induction hypothesis, \( t^{n-1} \delta(e)(x_j) \) for any \( j \leq i \). This implies

\[
\delta(e)(x_{i+1}) = t^{-1} \left( \sum_{j=1}^{i} \partial f_j / \partial x_j \right) \delta(e)(x_j)
\]

and so \( t^{n-i} \delta(e)(x_{i+1}) \), proving (i).

(iii) If \( n = 0 \), then \( B = \mathcal{O}[x] = B_1 \) (see Lemma 2.1). Thus, we assume \( n > 0 \). Let \( b \in B \) such that \( t b \in B_{n+1} \) and write \( t b = \sum_{j=1}^{r} a_j x_{n+1}^j \) for some \( a_j \in \mathcal{O}[x_1, \ldots, x_n] \). By the reasoning we did before, we may choose \( \delta \) as in Lemma 2.1 such that \( \delta(e) \) acts as an \( \mathcal{O} \)-derivation on the polynomials \( a_j x_{n+1}^j \). We show by induction on \( r \) that \( a_j \in I_n \) for any \( j \). The case \( r = 0 \) being obvious, assume that the statement holds true for \( r - 1 \). Then

\[
t \delta(e)(b) = \sum_{j=1}^{r} \delta(e)(a_j) x_{n+1}^j + \delta(e)(x_{n+1}) \cdot \left( \sum_{j=1}^{r} j a_j x_{n+1}^{j-1} \right) .
\]

By Statement (i) and the fact that \( n > 0 \), \( t \) divides the \( \delta(e)(a_j) \)'s. From a direct computation,

\[
\delta(e)(x_{n+1}) = t \alpha + \prod_{i=1}^{n} \partial f_i / \partial x_i \text{ for some } \alpha \in B.
\]

Therefore \( t \) divides \( \sum_{j=1}^{r} j a_j x_{n+1}^{j-1} \). Let \( p \) be the characteristic exponent of \( \mathcal{O} \) and assume that \( p > 1 \). By induction assumption, \( a_j \in I_n \) for any nonzero \( j \notin p \mathbb{Z} \). Now, there is \( b' \in B \) such that

\[
t b' = \sum_{1 \leq j \leq r, j \notin p \mathbb{Z}} a_j x_{n+1}^j .
\]
Letting $b_1 = b - b'$ we may write
\[ tb_1 = \sum_{u=1}^{s} a_{up^\ell}(x_{n+1}^{p^\ell})^u \text{ for some } s, \ell \geq 1. \]

Here $\ell$ is taken so that $a_{up^\ell} \neq 0$ for some $1 \leq u \leq s$ with $u \not\in p\mathbb{Z}$. Now applying $\delta^{(ep^\ell)}$ we obtain
\[ t\delta^{(ep^\ell)}(b_1) = \sum_{u=1}^{s} \delta^{(ep^\ell)}(a_{up^\ell})(x_{n+1}^{p^\ell})^u + (\delta^{(e)}(x_{n+1}))^{p^\ell} \cdot \left( \sum_{u=1}^{s} ua_{up^\ell}(x_{n+1}^{p^\ell})^{u-1} \right). \]

By the Leibniz rule and the fact that $n > 0$, $t$ divides the $\delta^{(ep^\ell)}(a_{up^\ell})$’s. Thus $t$ divides $\sum_{u=1}^{s} ua_{up^\ell}(x_{n+1}^{p^\ell})^{u-1}$.

According to our induction hypothesis, $a_{up^\ell} \in I_n$ for any $u \not\in p\mathbb{Z}$. Continuing this process, we arrive at $a_j \in I_n$ for any $j$. Let us show that $B = B_{n+1}$. By the previous step, we have
\[ a_j = a_{0,j}t + \sum_{i=1}^{n} a_{i,j}f_i \text{ for } a_{i,j} \in B_n \text{ and } j = 1, \ldots, r. \]

Setting $c_i = \sum_{j=1}^{r} a_{i,j}x_{n+1}^{j}$, we have
\[ b = t^{-1}\left( \sum_{j=1}^{r} a_jx_{n+1}^{j} \right) = \sum_{i=1}^{n} c_i x_i \in B_{n+1}, \text{ where } x_0 = 1. \]

Hence $b \in B_{n+1}$ provided that $t^\varepsilon b \in B_{n+1}$ for some $\varepsilon \in \mathbb{Z}_{\geq 0}$. From the equality $B(t) = K[x] = (B_{n+1}(t))$ we conclude that $B = B_{n+1}$. This completes the proof of the lemma.

**Proof of Theorem 1.1.** We follow the proof of [BE12, Theorem 3.1]. Consider the surjective morphism $\psi : E := \mathcal{O}[T_1, \ldots, T_{n+1}] \to B_{n+1} = B, T_i \mapsto x_i$, the ideal $I = (t(T_2 - f_1, \ldots, t_{n+1} - f_n)$ and let $J = \bigcup_{i}(I : t^{\varepsilon}E)$. We show that $J = \ker \psi$. Note that $J \subseteq \ker \psi$ is clear. Let $b \in \ker \psi$. By performing several Euclidean divisions we get an equality
\[ t^\ell b = \sum_{j=1}^{r} c_j(t_{j+1} - f_j) + \beta, \text{ where } \beta \in \mathcal{O}[T_1], \ell \in \mathbb{Z}_{\geq 0}, \text{ and } \gamma_j \in E. \]

Since $\psi(t^\ell b) = 0$ and $x_1 = x$ is transcendental over $K$, we have $\beta = 0$. Thus $\ker \psi = J$. It remains to prove that $I = J$. From [BE12, Lemma 4.5] (which is valid in our context) we only need to have $J \subseteq tE + I = (t, f_1, \ldots, f_n)$. Let $b \in J$. Then $b = \sum_{j=0} a_j T_{n+1}^j$ for some $a_j \in \mathcal{O}[T_1, \ldots, T_n]$. Since $\psi(b) = 0$, the argument of the proof of Lemma 3.1 (iii) implies that $\psi(a_j) \in I_n$ for any $j$. Thus $b \in tE + I$, establishing the theorem.

**Acknowledgments.** The author was supported by the Heinrich Heine University of Düsseldorf. This research is supported by ERC Consolidator Grant 615655 - NMST and also by the Basque Government through the BERC 2014-2017 program and by Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013-0323.
[BE12] Moulay A. Barkatou and M’hammed El Kahoui. Locally nilpotent derivations with a PID ring of constants. Proc. Amer. Math. Soc. 140 (2012), no. 1, 119–128.
[CM05] Anthony J. Crachiola and Leonid G. Makar-Limanov. On the rigidity of small domains. J. Algebra 284 (2005), no.1 1–12.
[KZ99] Shulim Kaliman and Mikhail Zaidenberg. Affine modifications and affine hypersurfaces with a very transitive automorphism group, Transform. Groups 4 (1999), no.1, 53–95.
[Miy68] Masayoshi Miyanishi. A remark on an iterative infinite higher derivation. J. Math. Kyoto Univ. 8 1968 411–415.
[Miy09] Masayoshi Miyanishi. Additive group scheme actions on integral schemes defined over discrete valuation rings. J. Algebra 322 (2009), no. 9, 3331–3344.

Mathematisches Institut, Heinrich Heine Universität, 40225 Düsseldorf, Germany.
E-mail address: langlois.kevin18@gmail.com