Article
Symmetric Properties of Carlitz’s Type $q$-Changhee Polynomials

Yunjae Kim, Byung Moon Kim and Jin-Woo Park

1 Department of Mathematics, Dong-A University, Busan 49315, Korea; kimholzi@gmail.com
2 Department of Mechanical System Engineering, Dongguk University, Gyeongju 38066, Korea; kbm713@dongguk.ac.kr
3 Department of Mathematics Education, Daegu University, Gyeongsan 38066, Korea
* Correspondence: a0417001@knu.ac.kr; Tel.: +82-53-850-4212

Received: 23 October 2018; Accepted: 10 November 2018; Published: 13 November 2018

Abstract: Changhee polynomials were introduced by Kim, and the generalizations of these polynomials have been characterized. In our paper, we investigate various interesting symmetric identities for Carlitz’s type $q$-Changhee polynomials under the symmetry group of order $n$ arising from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$.

Keywords: fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$; $q$-Euler polynomials; $q$-Changhee polynomials; symmetry group

MSC: 33E20; 05A30; 11B65; 11S05

1. Introduction

For an odd prime number $p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completions of algebraic closure of $\mathbb{Q}_p$, respectively, throughout this paper. The $p$-adic norm is normalized as $|p|_p = \frac{1}{p}$, and let $q$ be an indeterminate in $\mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{p-1}}$. The $q$-analogue of number $x$ is defined as

$$[x]_q = \frac{1 - q^x}{1 - q}. \quad (1)$$

Note that $\lim_{q \to 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p) = \{ f : \mathbb{Z}_p \to \mathbb{R} \text{ is continuous} \}$. Then, a fermionic $p$-adic $q$-integral of $f (\in C(\mathbb{Z}_p))$ is defined by Kim as $[1-6]:$

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q \left( x + p^N \mathbb{Z}_p \right)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x$$

$$= \lim_{N \to \infty} \frac{2q}{2} \sum_{x=0}^{p^N-1} f(x) (-q)^x. \quad (2)$$
On the other hand, it is well known that the Euler polynomial \( E_n(x) \) is given by the Appell sequence with \( g(t) = \frac{1}{2} (e^t + 1) \), giving the the generating function
\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]
(see [7–17]). In particular, if \( x = 0 \), \( E_n = E_n(0) \) \( (n \in \mathbb{N}) \) is called the Euler number.

As a \( q \)-analogue of Euler polynomials, the Carlitz’s type \( q \)-Euler polynomial \( \mathcal{E}_{n,q}(x) \) is defined by
\[
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]q^{-t}} d\mu_{-q}(y),
\]
(see [2,13–17]). In particular, if \( x = 0 \), \( \mathcal{E}_{n,q} = \mathcal{E}_{n,q}(0) \) is called the \( q \)-Euler number.

By (3), the Carlitz’s type \( q \)-Euler polynomial \( \mathcal{E}_{n,q}(x) \) is obtained as
\[
\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]^n d\mu_{-q}(y), \quad (n \geq 0).
\]

From the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \), the degenerate \( q \)-Euler polynomial \( \mathcal{E}_{n,\lambda,q}(x) \) is defined as [16]:
\[
\sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[x+y]^n}{\lambda^n} d\mu_{-q}(y).
\]

By the binomial expansion of \( (1 + \lambda t) \frac{[x+y]^n}{\lambda^n} \), we get
\[
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[x+y]^n}{\lambda^n} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \left( \frac{[x+y]^n}{\lambda^n} \right) d\mu_{-q}(y) \frac{t^n}{n!},
\]
\[
\left( \begin{array}{c}
\lambda^n \\
\lambda^n \end{array} \right) = \lambda^n \sum_{l=0}^{n} S_1(n,l) \lambda^l,
\]
\[
\mathcal{E}_{n,\lambda,q}(x) = \lambda^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \left( \frac{[x+y]^n}{\lambda^n} \right)^l d\mu_{-q}(y)
\]
\[
= \lambda^n \sum_{l=0}^{n} \lambda^{-l} S_1(n,l) \mathcal{E}_{l,q}(x),
\]
where \( S_1(n,m) \) is the Stirling number of the first kind (see [2,7,8,12,17,18]).

Now, we apply these polynomials to Changhee polynomials, introduced by Kim et al. [19]. The Changhee polynomial of the first kind \( \text{Ch}_n(x) \) is defined by the generating function to be
\[
\sum_{n=0}^{\infty} \text{Ch}_n(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + t)^x y d\mu_{-1}(y)
\]
\[
= \frac{2}{2 + t} (1 + t)^x,
\]
(see [20,21]).
In view point of (3) and (9), Carlitz’s type $q$-Changhee polynomial $Ch_{n,q}(x)$ is defined by

$$\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + t)^{|x+y|}_q d\mu_{-q}(y),$$  \hfill (10)

(see [18,22]).

By the binomial expansion of $(1 + t)^{|x+y|}_q$,

$$\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + t)^{|x+y|}_q d\mu_{-q}(y)
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (|x+y|)_q^n d\mu_{-q}(y) \frac{t^n}{n!},$$  \hfill (11)

and so the equation (10) and (11) yield the following:

$$Ch_{n,q}(x) = \int_{\mathbb{Z}_p} (|x+y|)_q^n d\mu_{-q}(y),$$  \hfill (12)

(see [20,21]).

In the past decade, many different generalizations of Changhee polynomials have been studied (see [19,20,22–32]), and the relationship between important combinatorial polynomials and those polynomials was found.

Symmetric identities of special polynomials are important and interesting in number theory, pure and applied mathematics. Symmetric identities of many different polynomials were investigated in [5,10,14,16,32–39]. In particular, C. Cesarano [40] presented some techniques regarding the generating functions used, and these identities can be applicable to the theory of porous materials [41].

In current paper, we construct symmetric identities for the Carlitz’s type $q$-Changhee polynomials under the symmetry group of order $n$ arising from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and the proof methods which was used in the Kim’s previous researches are also used as good tools in this paper (see [5,10,14,16,32–39]).

2. Symmetric Identities for the Carlitz’s Type $q$-Changhee Polynomials

Let $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{\nu-1}}$, and let $S_n$ be the symmetry group of degree $n$. For positive integers $w_1, w_2, \ldots, w_n$ with $w_i \equiv 1 \pmod{2}$ for each $i = 1, 2, \ldots n$, we consider the following integral equation for the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$;

$$\int_{\mathbb{Z}_p} (1 + t)^\left[\left(\prod_{i=1}^{n-1} w_i \right) y + \left(\prod_{i=1}^{n-1} w_i \right) x + w_0 \sum_{i=1}^{n-1} \left(\prod_{j=1, j \neq i}^{n-1} w_j \right) \right]_q d\mu_{-q^{w_1 \cdots w_{n-1}}}(y)
= \frac{2q^{w_1 \cdots w_{n-1}}}{2} \lim_{N \to \infty} \sum_{m=0}^{w_0-1} \sum_{y=0}^{p^{N-1}-1} (1 + t)^{\left(\prod_{i=1}^{n-1} w_i \right)(m+w_0 y) + \left(\prod_{i=1}^{n-1} w_i \right) x + w_0 \sum_{i=1}^{n-1} \left(\sum_{j=1, j \neq i}^{n-1} w_j \right) \right]_q
\times (-1)^{m+w_0 y} q^{w_1 w_2 \cdots w_{n-1}(m+w_0 y)}. \hfill (13)$$
From (13), we get

\[
\frac{2}{[2]^q} \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} (-1)^{\sum_{m=1}^{n-1} k_m} q^\alpha \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j
\]

\[
\times \int_{Z_p} (1 + t) \left[ (\prod_{m=1}^{n-1} w_i)^y + (\prod_{m=1}^{n-1} w_i)^x + w_n \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j \right] \mu_{q^a w_2^m w_{n-1}} (y)
\]

\[
= \lim_{N \to \infty} \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} \sum_{y=0}^y \sum_{l=0}^{y} (1 + t) \left[ (\prod_{m=1}^{n-1} w_i)^y + (\prod_{m=1}^{n-1} w_i)^x + w_n \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j \right]
\]

\[
\times (-1)^{\sum_{m=1}^{n-1} k_m + y q} (\prod_{m=1}^{n-1} w_i)^y + (\prod_{m=1}^{n-1} w_i)^x + w_n \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j
\]

If we put

\[
F(w_1, w_2, \ldots, w_n) = \frac{2}{[2]^q} \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} (-1)^{\sum_{m=1}^{n-1} k_m} q^\alpha \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j
\]

\[
\times \int_{Z_p} (1 + t) \left[ (\prod_{m=1}^{n-1} w_i)^y + (\prod_{m=1}^{n-1} w_i)^x + w_n \sum_{j=1}^{n-1} \left( \prod_{l \neq j}^{n-1} w_l \right) k_j \right] \mu_{q^a w_2^m w_{n-1}} (y),
\]

then, by (14), we know that \(F(w_1, w_2, \ldots, w_n)\) is invariant for any permutation \(\sigma \in S_n\). Hence, by (14) and (15), we obtain the following theorem.

**Theorem 1.** Let \(w_1, w_2, \ldots, w_n\) be positive odd integers. For any \(\sigma \in S_n\), \(F(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n)})\) have the same value.

By (1), we know that

\[
\left[ \prod_{i=1}^{n-1} w_i \right] q^y + w_n x + w_n \sum_{i=1}^{n-1} k_i q^y w_2^{w_{n-1}} = \left[ \prod_{i=1}^{n-1} w_i \right] y + \left( \prod_{i=1}^{n-1} w_i \right) x + w_n \sum_{i=1}^{n-1} \left( \prod_{l \neq i} w_l \right) k_i \right] q.
\]
From (5) and (16), we derive the following identities.

$$
\int_{Z_p} (1 + t) \left[ (\prod_{j=1}^{n-1} w_j) y + (\prod_{j=1}^{n-1} w_j) x + w_n \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \right] \, d\mu_{-q^{n_1}w_2 - w_n - 1}(y) \\
= (1 + t) \left[ \prod_{j=1}^{n-1} w_j \right] \int_{Z_p} (1 + t) \left[ y + w_n x + w_n \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \right] \, d\mu_{-q^{n_1}w_2 - w_n - 1}(y) \\
= \left( \sum_{i=0}^{\infty} \left( \prod_{j=1}^{n-1} w_j \right) q^i \right) \left( \sum_{m=0}^{\infty} \left( \prod_{i=1}^{m-r} \left( \prod_{j=1}^{n-1} w_i \right) \right) \right) \left( \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \right) \left( \sum_{j=1}^{n-1} k_j \right) \, \frac{t^m}{m!},
$$

for each positive integer $n$. Thus, by Theorem 1 and (17), we obtain the following corollary.

**Corollary 1.** Let $w_1, w_2, \ldots, w_n$ be positive integers with $w_i \equiv 1 \pmod{2}$ for each $i = 1, 2, \ldots, n$, and let $m$ be a nonnegative integer. Then, for any permutation $\tau \in S_n$,

$$
\left[ \prod_{i=1}^{n-1} w_\tau(i) \right] q^r \left( \sum_{i=1}^{n-1} \left( \prod_{j=1}^{n-1} w_j \right) \right) \left( \sum_{j=1}^{n-1} k_j \right) \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \left( w_\tau(n) \right)^{n-1} \left( \sum_{i=1}^{n-1} w_\tau(i) \right) k_i \\
\times \left( \sum_{i=1}^{n-1} \left( \prod_{j=1}^{n-1} w_j \right) \right) q^r \left( \sum_{i=1}^{n-1} \left( \prod_{j=1}^{n-1} w_j \right) \right) \left( \sum_{i=1}^{n-1} k_i \right) \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \left( w_\tau(n) \right)^{n-1} \left( \sum_{i=1}^{n-1} w_\tau(i) \right)
$$

have the same expressions.

Note that, by the definition of $[x]_q$, 

$$
\left[ y + w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right] q^{n_1 w_2 - w_n - 1} \\
= \left[ \frac{[w_n]}{q^{n-1} w_1} \right] \left[ \prod_{i=1}^{n-1} \left( \prod_{j=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \right) \right] k_i \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \left( w_\tau(n) \right)^{n-1} \left( \sum_{i=1}^{n-1} w_\tau(i) \right) \left( y + w_n x \right) q^{n_1 w_2 - w_n - 1}.
$$

By (12), we get 

$$
\left[ y + w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right] q^{n_1 w_2 - w_n - 1} \\
= \int_{Z_p} \left[ y + w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right] q^{n_1 w_2 - w_n - 1} \, d\mu_{-q^{n_1}w_2 - w_n - 1},
$$

for each positive integer $n$.
and by (8) and (18),

\[
\left( y + w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} q^{w_i w_{i+1} \cdots w_{n-1}} \right)_m
\]

\[
= \left( \frac{[w_n]_q}{[\prod_{j=1}^{n-1} w_j]_q} \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) k_i \right)_m + q \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) k_i
\]

\[
= \sum_{i=0}^{m} S_1(m, l) \left( \frac{[w_n]_q}{[\prod_{j=1}^{n-1} w_j]_q} \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) k_i \right)
\]

\[
\times q \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \left( y + w_n x_1 q^{w_1 w_2 \cdots w_{n-1}} \right)^{l-i}
\]

From (4), (19) and (20), we have

\[
Ch_{m,q}^{w_n w_{n-1}} \left( w_n x + w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right)
\]

\[
= \sum_{i=0}^{m} S_1(m, l) \left( \frac{[w_n]_q}{[\prod_{j=1}^{n-1} w_j]_q} \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) k_i \right)
\]

\[
\times q \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) \int_{Z_p} \left( y + w_n x_1 q^{w_1 w_2 \cdots w_{n-1}} \right)^{l-i} d\mu_{-q^{w_1 w_2 \cdots w_{n-1}}} (y)
\]

\[
= \sum_{i=0}^{m} S_1(m, l) \left( \frac{[w_n]_q}{[\prod_{j=1}^{n-1} w_j]_q} \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) k_i \right)
\]

\[
\times q \sum_{i=1}^{n-1} \left( \prod_{j \neq i}^{n-1} w_j \right) Ch_{l,q}^{q^{w_1 w_2 \cdots w_{n-1}}} (w_n x).
\]
From (21), we have

\[
\begin{aligned}
&\frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}}
\sum_{r=0}^{m-1} \prod_{l=1}^{m} \sum_{k_1=0}^{n-1} (-1)^{\sum_{j=1}^{n-1} k_j} q^{w_n \sum_{j=1}^{n-1} \left(\sum_{j=1}^{n-1} w_j\right)} k_i \\
&\times \left(\prod_{i=1}^{l} w_i\right)_{q, m-r} \frac{m}{r} \mathrm{Ch}_{r,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_n x + w_n \sum_{j=1}^{n-1} \frac{k_i}{w_i} \right) \\
&= \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}}
\sum_{r=0}^{m-1} \prod_{l=1}^{m} \sum_{k_1=0}^{n-1} (-1)^{\sum_{j=1}^{n-1} k_j} q^{w_n \sum_{j=1}^{n-1} \left(\sum_{j=1}^{n-1} w_j\right)} k_i \\
&\times \left(\prod_{i=1}^{l} w_i\right)_{q, m-r} \frac{m}{r} \mathrm{Ch}_{r,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_n x \right) \\
&= \sum_{r=0}^{m} \sum_{l=1}^{1} S_1(r, l) \left(\frac{m}{r}\right) \left(\frac{[w_n]_q}{\prod_{i=1}^{l} w_i}\right)^{p-i} \left(\prod_{i=1}^{n-1} w_i\right)_{q, m-r} \mathrm{Ch}_{i,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_n x \right) \\
&\times \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}}
\prod_{l=1}^{m} \sum_{k_1=0}^{n-1} (-1)^{\sum_{j=1}^{n-1} k_j} q^{w_n \sum_{j=1}^{n-1} \left(\sum_{j=1}^{n-1} w_j\right)} k_i \\
&\times \left(\prod_{i=1}^{l} w_i\right)_{q, m-r} \frac{m}{r} \mathrm{Ch}_{i,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_n x \right) F_{n,q^{w_1 \cdots w_{n-1}}} \left( w_1, \ldots, w_{n-1} | i + 1 \right),
\end{aligned}
\]

where

\[
F_{n,q}(w_1, \ldots, w_{n-1} | i) = \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}}
\prod_{l=1}^{m} \sum_{k_1=0}^{n-1} (-1)^{\sum_{j=1}^{n-1} k_j} q^{w_n \sum_{j=1}^{n-1} \left(\sum_{j=1}^{n-1} w_j\right)} k_i \\
\times \left(\frac{[w_n]_q}{\prod_{i=1}^{l} w_i}\right)^{p-i} \left(\prod_{i=1}^{n-1} w_i\right)_{q, m-r} \mathrm{Ch}_{i,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_n x \right) F_{n,q^{w_1 \cdots w_{n-1}}} \left( w_1, \ldots, w_{n-1} | i + 1 \right).
\]

**Theorem 2.** For each nonnegative odd integers \( w_1, w_2, \ldots, w_n \) and for any permutation \( \sigma \) in the symmetry group of degree \( n \), the expressions

\[
\sum_{r=0}^{m} \sum_{l=1}^{1} S_1(r, l) \left(\frac{m}{r}\right) \left(\frac{[w_{\sigma(r)}]_q}{\prod_{i=1}^{l} w_{\sigma(i)}}\right)^{p-i} \left(\prod_{i=1}^{n-1} w_{\sigma(i)}\right)_{q, m-r} \mathrm{Ch}_{i,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_{\sigma(n)} x \right) F_{n,q^{w_1 \cdots w_{n-1}}} \left( w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} | i + 1 \right)
\]

\[
\times \frac{2}{[2]_{q^{w_1 w_2 \cdots w_n}}}
\prod_{l=1}^{m} \sum_{k_1=0}^{n-1} (-1)^{\sum_{j=1}^{n-1} k_j} q^{w_n \sum_{j=1}^{n-1} \left(\sum_{j=1}^{n-1} w_j\right)} k_i \\
\times \left(\frac{[w_{\sigma(n)}]_q}{\prod_{i=1}^{l} w_{\sigma(i)}}\right)^{p-i} \left(\prod_{i=1}^{n-1} w_{\sigma(i)}\right)_{q, m-r} \mathrm{Ch}_{i,q^{w_1 w_2 \cdots w_{n-1}}} \left( w_{\sigma(n)} x \right) F_{n,q^{w_1 \cdots w_{n-1}}} \left( w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} | i + 1 \right)
\]
have the same.

3. Conclusion

The Changhee numbers are closely related with the Euler numbers, the Stirling numbers of the first kind and second kind and the harmonic numbers, and so on. Throughout this paper, we investigate that the function $F(w_{v(1)}, w_{v(2)}, \ldots, w_{v(n)})$ for the Carlitz’s type $q$-Changhee polynomials is invariant under the symmetry group $\sigma \in S_n$. From the invariance of $F(w_{v(1)}, w_{v(2)}, \ldots, w_{v(n)})$, $\sigma \in S_n$, we construct symmetric identities of the Carlitz’s type $q$-Changhee polynomials from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$. As Bernoulli and Euler polynomials, our properties on the Carlitz’s type $q$-Changhee polynomials play an crucial role in finding identities for numbers in algebraic number theory.

Author Contributions: All authors contributed equally to this work; All authors read and approved the final manuscript.

Funding: This research was supported by the Daegu University Research Grant, 2018.

Acknowledgments: The authors would like to thank the referees for their valuable and detailed comments which have significantly improved the presentation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kim, T. $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals. *J. Nonlinear Math. Phys.* 2007, 14, 15–27. [CrossRef]
2. Kim, T. Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$. *Russ. J. Math. Phys.* 2009, 16, 484–491. [CrossRef]
3. Kim, T. On $q$-analogue of the $p$-adic log gamma functions and related integral. *J. Number Theory* 1999, 76, 320–329. [CrossRef]
4. Kim, T. $q$-Volkenborn integration. *Russ. J. Math. Phys.* 2002, 9, 288–299.
5. Kim, T. Symmetry of power sum polynomials and multivariate fermionic $p$-adic invariant integral on $\mathbb{Z}_p$. *Russ. J. Math. Phys.* 2009, 16, 93–96. [CrossRef]
6. Kim, D.S.; Kim, T. Some $p$-adic integrals on $\mathbb{Z}_p$ associated with trigonometric Functions. *Russ. J. Math. Phys.* 2018, 25, 300–308. [CrossRef]
7. Kim, T. A study on the $q$-Euler numbers and the fermionic $q$-integrals of the product of several type $q$-Bernstein polynomials on $\mathbb{Z}_p$. *Adv. Stud. Contemp. Math.* 2013, 23, 5–11.
8. Bayad, A.; Kim, T. Identities involving values of Bernstein, $q$-Bernoulli, and $q$-Euler polynomials. *Russ. J. Math. Phys.* 2011, 18, 133–143. [CrossRef]
9. Gaboury, S.; Tremblay, R.; Fugere, B.-J. Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials. *Proc. Jangjeon Math. Soc.* 2014, 17, 115–123.
10. Kim, D.S.; Kim, T. Some symmetric identities for the higher-order $q$-Euler polynomials related to symmetry group $S_3$ arising from $p$-adic $q$-fermionic integral on $\mathbb{Z}_p$. *Filomat* 2016, 30, 1717–1721. [CrossRef]
11. Sharma, A. $q$-Bernoulli and Euler numbers of higher order. *Duke Math. J.* 1958, 25, 343–353. [CrossRef]
12. Srivastava, H. Some generalizations and basic (or $q$-)extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inf. Sci.* 2011, 5, 390–444.
13. Zhang, Z.; Yang, H. Some closed formulas for generalized Bernoulli-Euler numbers and polynomials. *Proc. Jangjeon Math. Soc.* 2008, 11, 191–198.
14. Kim, T.; Kim, D.S. Identities of symmetry for degenerate Euler polynomials and alternating generalized falling factorial sums. *Iran. J. Sci. Technol. Trans. A Sci.* 2017, 41, 939–949. [CrossRef]
15. Carlitz, L. $q$-Bernoulli and Eulerian numbers. *Trans. Amer. Math. Soc.* 1954, 76, 332–350.
16. Kim, D.S.; Kim, T. Symmetric identities of higher-order degenerate $q$-Euler polynomials. *J. Nonlinear Sci. Appl.* 2016, 9, 443–451. [CrossRef]
17. Comtet, L. *Advanced Combinatorics*; Reidel: Dordrecht, The Netherlands, 1974.
18. Dolgy, D.V.; Jang, G.W.; Kwon, H.I.; Kim, T. A note on Carlitzs type $q$-Changhee numbers and polynomials. *Adv. Stud. Contemp. Math.* 2017, 27, 451–459.
19. Kim, D.S.; Kim, T.; Seo, J.J. A note on Changhee polynomials and numbers. *Adv. Stud. Theor. Phys.* **2013**, *7*, 993–1003. [CrossRef]
20. Kim, T.; Kim, D.S. A note on nonlinear Changhee differential equations. *Russ. J. Math. Phys.* **2016**, *23*, 88–92. [CrossRef]
21. Kim, T.; Kim, D.S. Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind. *Sci. China Math.* **2018**. [CrossRef]
22. Kim, B.M.; Jang, L.C.; Kim, W.; Kwon, H.I. On Carlitz’s Type Modified Degenerate Changhee Polynomials and Numbers. *Discrete Dyn. Nat. Soc.* **2018**, *2018*, 9520269. [CrossRef]
23. Kim, T.; Kim, D.S. Differential equations associated with degenerate Changhee numbers of the second kind. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM* **2018**. [CrossRef]
24. Moon, E.J.; Park, J.W. A note on the generalized $q$-Changhee numbers of higher order. *J. Comput. Anal. Appl.* **2016**, *20*, 470–479.
25. Kwon, J.; Park, J.W. On modified degenerate Changhee polynomials and numbers. *J. Nonlinear Sci. Appl.* **2015**, *18*, 295–305. [CrossRef]
26. Kim, T.; Kwon, H.I.; Seo, J.J. Degenerate $q$-Changhee polynomials. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2389–2393. [CrossRef]
27. Kwon, H.I.; Kim, T.; Seo, J.J. A note on degenerate Changhee numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2015**, *18*, 295–305.
28. Arici, S.; A˘ gyüz, E.; Acikgoz, M. On a $q$-analogue of some numbers and polynomials. *J. Ineqal. Appl.* **2015**, *2015*, 19. [CrossRef]
29. Pak, H.K.; Jeong, J.; Kang, D.J.; Rim, S.H. Changhee-Genocchi numbers and their applications. *ARS Combin.* **2018**, *136*, 153–159.
30. Kim, B.M.; Jeong, J.; Rim, S.H. Some explicit identities on Changhee-Genocchi polynomials and numbers. *Adv. Differ. Equ.* **2016**, *2016*, 202. [CrossRef]
31. Rim, S.H.; Park, J.W.; Pyo, S.S.; Kwon, J. The $n$-th twisted Changhee polynomials and numbers. *Bull. Korean Math. Soc.* **2015**, *52*, 741–749. [CrossRef]
32. Simsek, Y. Identities on the Changhee numbers and Apostol-type Daehoe polynomials. *Adv. Stud. Contemp. Math.* **2017**, *27*, 199–212.
33. Kim, T.; Kim, D.S. Degenerate Bernstein polynomials. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM* **2018**. [CrossRef]
34. Liu, C.; Bao, W. Application of Probabilistic Method on Daehoe Sequences. *Eur. J. Pure Appl. Math.* **2018**, *11*, 69–78. [CrossRef]
35. Kim, D.S.; Lee, N.; Na, J.; Park, K.H. Abundant symmetry for higher-order Bernoulli polynomials (I). *Adv. Stud. Contemp. Math.* **2013**, *23*, 461–482.
36. Kim, D.S.; Lee, N.; Na, J.; Pak, K.H. Identities of symmetry for higher-order Euler polynomials in three variables (I). *Adv. Stud. Contemp. Math.* **2012**, *22*, 51–74. [CrossRef]
37. Kim, D.S.; Lee, N.; Na, J.; Pak, K.H. Identities of symmetry for higher-order Bernoulli polynomials in three variables (II). *Proc. Jangjeon Math. Soc.* **2013**, *16*, 359–378.
38. Kim, D.S. Symmetry identities for generalized twisted Euler polynomials twisted by unramified roots of unity. *Proc. Jangjeon Math. Soc.* **2012**, *15*, 303–316.
39. Kim, T.; Kim, D.S. An identity of symmetry for the degenerate Frobenius-Euler polynomials. *Math. Slovaca* **2008**, *58*, 239–243. [CrossRef]
40. Cesarano, C. Operational methods and new identities for Hermit polynomials. *Math. Model. Nat. Phenom.* **2017**, *12*, 44–50. [CrossRef]
41. Marin, M. Weak solutions in elasticity of dipolar porous materials. *Math. Probl. Eng.* **2008**, *2008*, 158908. [CrossRef]

© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).