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Lagrangian formulation, a general relativity analogue, and a symmetry of the Vialov equation of glaciology

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Abstract Using a suitable rescaling of the independent variable, a Lagrangian is found for the nonlinear Vialov equation ruling the longitudinal profiles of glaciers and ice caps in the shallow ice approximation. This leads to a formal analogy between the (rescaled) Vialov equation and the Friedmann equation of relativistic cosmology, which is explored. This context provides a new symmetry of the (rescaled) Vialov equation and gives, at least formally, all its solutions using a generating function, which is the Nye profile for the degenerate case of perfectly plastic ice.

1 Introduction

An important geomorphological feature of ice caps and alpine glaciers is their longitudinal profile, i.e., the ice thickness $h$ as a function of a coordinate $x$ running downstream along the glacier bed. Some knowledge of longitudinal profiles is needed in estimating the volume of a glacier or ice sheet [1], and the volume-area scaling for these systems constitutes a major tool in estimating ice content, ice loss and sea-level rise related to climate change [2].

The mathematical modeling of longitudinal glacier profiles is relatively well developed and is based on nonlinear differential equations. The rheology of glacier ice plays a crucial role in determining the shape $h(x)$ of an ice cap or valley glacier. The response of glacier ice to applied stresses is well described by Glen’s law relating the strain rate tensor $\dot{\epsilon}_{ij}$ with the stresses in the ice [3]

$$\dot{\epsilon}_{ij} = A \sigma_{\text{eff}}^{n-1} s_{ij}$$

where $A$ is a (temperature-dependent) constant [4–7], $\hat{s} = (s_{ij})$ is the deviatoric stress tensor, and

$$\sigma_{\text{eff}} = \sqrt{\frac{1}{2} \text{Tr} (\hat{s}^2)}$$

is the effective stress. Glen’s law describes the rheology of ice. The strain rate $\dot{\epsilon}_{ij}$ depends on the deviatoric stresses in all directions through $\sigma_{\text{eff}}$. Different values of the exponent $n$ corresponds to very different physical situations. At very low stresses, the value $n = 1$ corresponds to a perfectly viscous material with viscosity coefficient $\eta = A^{-1}$ and satisfying the linear relation $s_{ij} = \dot{\epsilon}_{ij} / A$. The value $n = 2$ is sometimes adopted to describe basal sliding.

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of a glacier. The value \( n = 3 \) corresponds to ice flow, and is the one commonly adopted in the modeling of ice flow in glaciers. The limit \( n \to +\infty \) corresponds to perfectly plastic ice, a poor approximation to real ice flow [4–7].

Longitudinal profiles \( h(x) \) are modeled under the assumptions of incompressible and isotropic ice, steady state, a flat bed (a plane which, in general, has non-zero slope), the shallow ice approximation, and Glen’s law to describe the ice behavior [4–7] (these are major simplifications and modern numerical models attempt to go beyond these approximations). Then, the longitudinal glacier profile \( h(x) \) satisfies the Vialov ordinary differential equation [4–8]

\[
x c(x) = \frac{2A}{n + 2} \left( \rho_{\text{ice}} g h \frac{dh}{dx} \right)^{n} h^2,
\]

where \( \rho_{\text{ice}} \) is the ice density, \( g \) is the acceleration of gravity, \( c(x) \) is the accumulation rate of ice, i.e., the flux density of ice volume in the \( z \)-direction, which is perpendicular to the \( x \)-axis and the flat bed (\( c \) has the dimensions of a velocity). The Vialov equation is essentially the mass conservation equation in the shallow ice approximation for flat glacier bed.

Analytic expressions describing longitudinal glacier profiles are needed in studying several aspects of glaciology (e.g., [1,2,9–11]). It is customary to search for solutions in the finite interval \( x \in [0, L] \), where the glacier summit is located at \( x = 0 \) and the terminus is at \( x = L \), and \( L \) is the length of the glacier or ice cap. In this case, \( dh/dx \) is negative and its absolute value must be taken in the Vialov equation. One could switch the location of summit and terminus, then \( dh/dx > 0 \) in \( (0, L) \) and the absolute value appearing in Eq. (3) takes care of both situations.

For ice caps and ice sheets, once a solution for the longitudinal profile of half of a glacier is found in \( [0, L] \), it is extended by continuity to the interval \( [-L, L] \) (or to \( [0, 2L] \), respectively) by reflecting it about the vertical line \( x = 0 \) (or \( x = L \), respectively) passing through the summit, where the diffusion degenerates. The surface profile \( h(x) \) of an ice cap or ice sheet is continuous but not differentiable at the summit, since the left and right derivatives of \( h \) have opposite signs there.

At the terminus, the slope \( dh/dx \) and the basal stress \( \tau_b = -\rho_{\text{ice}} g h \frac{dh}{dx} \) diverge; this well-known shortcoming of the modeling is due to the breakdown of the shallow ice approximation at the terminus [4–7] (this is not a general feature of the shallow ice approximation).

The formal solution of the Vialov equation (3) is

\[
h(x) = \left\{ \mp \frac{2(n + 1)}{n\rho_{\text{ice}} g} \left( \frac{n + 2}{2A} \right)^{1/n} \int dx \left[ x c(x) \right]^{1/n} \right\}^{n/(n+1)} \equiv A \left[ I(x) \right]^{n/(n+1)}, \quad (4)
\]

where the upper sign applies to the case in which the summit is at \( x = 0 \) (and \( dh/dx < 0 \)), and the lower sign if \( x = L \) is the summit. Here

\[
A \equiv \left[ \frac{2(n + 1)}{n\rho_{\text{ice}} g} \left( \frac{n + 2}{2A} \right)^{1/n} \right]^{n/(n+1)}, \quad (5)
\]

and the integral

\[
I(x) \equiv \int dx \left[ x c(x) \right]^{1/n}. \quad (6)
\]

is defined up to an arbitrary integration constant. A function \( c(x) \) modeling the accumulation rate of ice must be prescribed in the model, ideally based on atmospheric models for precipitation in the region. Even simple choices of \( c(x) \) make the integral (6) impossible to
compute explicitly in terms of elementary functions. A few analytic solutions of the Vialov equation are available [8,12–16].

The Chebyshev theorem of integration can be used to characterize the models in which the solution can be expressed in terms of elementary functions [17]. Other analytic profiles follow from the rather unrealistic assumption of perfectly plastic ice used in the early modeling, and only appropriate when the deformation of the ice is negligible [18–21], which is formally obtained as the limit \( n \to +\infty \) of the Vialov equation [4–7].

As shown in the next section, by suitably rescaling the independent variable \( x \), it is found that the Vialov equation admits Lagrangian and Hamiltonian formulations, thus solving an inverse variational problem. This is rather surprising because the Vialov equation was not derived originally from a variational principle or from the extremization of a physical quantity. Finding a Lagrangian and a Hamiltonian from a given equation constitutes the inverse variational problem of mathematical physics (sometimes called Helmoltz problem [22]), which can be solved in a surprisingly wide number of cases [23] and, recently, has been approached using non-standard Lagrangians for dissipative-like autonomous differential equations (e.g., [24–27]). Similar to some of the equations considered in Ref. [27], our procedure is based on a redefinition of variables.

Once a Hamiltonian is available, one quickly realizes that it is conserved and the energy conservation equation leads to a straightforward analogy between the profile \( h(x) \) solving the (rescaled) Vialov equation and the motion of a particle in one dimension under a suitable potential energy. What is more, this formulation opens the door to a formal analogy between the Vialov equation (using the rescaled bed coordinate) and the Friedmann equation of spatially homogeneous and isotropic cosmology. This analogy is explored in Sect. 4 and used in Sect. 5 to generate all the solutions of the Vialov equation, at least formally, from a generating function which is the Nye parabolic profile for perfectly plastic ice. Section 6 contains the conclusions.

2 Lagrangian formulation of the Vialov equation

Rewrite the Vialov equation (3) as

\[
\frac{a^{n/2}}{n} h' = \left[ \frac{x c(x)}{\alpha} \right]^{1/n},
\]

(7)

where

\[
\alpha = \frac{2A}{n+2} \left( \rho_{\text{ice}} g \right)^n.
\]

(8)

Now change the independent variable, \( x \to \bar{x} \), with \( \bar{x} \) defined by

\[
d\bar{x} = \left[ x c(x) \right]^{1/n} dx,
\]

(9)

or

\[
\bar{x}(x) = \int_0^x d\bar{x'} \left[ \bar{x}' c(\bar{x}') \right]^{1/n}.
\]

(10)

This relation is invertible and one-to-one where \( c(x) > 0 \), then \( d\bar{x}/dx > 0 \); \( c(x) \) is required to be continuous and piecewise differentiable.\(^1\)

\(^1\) The exception in the literature is the Weertman-Paterson model [13,14] in which \( n = 3 \) and \( c(x) \) is a step function defined on two intervals \((0, R)\) and \((R, L)\) on which it is piecewise constant and has opposite sign. In this case, one studies the two intervals separately and reduces to the previous situation.
With the new independent variable $\bar{x}$, the Vialov equation assumes the form

$$h^{\frac{n+2}{n}} \left| \frac{dh}{d\bar{x}} \right| = \frac{1}{\alpha^{1/n}}. \quad (11)$$

Dividing both sides by $\sqrt{2} h^{\frac{n+2}{n}}$ and squaring, one obtains

$$\frac{1}{2} \left( \frac{dh}{d\bar{x}} \right)^2 = \frac{1}{2\alpha^{2/n}} h^{\frac{2(n+2)}{n}}, \quad (12)$$

which looks like an energy conservation equation for a particle of unit mass and position $h$ in one-dimensional motion in the potential energy

$$V(h) = -\frac{V_0}{h^{\frac{2(n+2)}{n}}}, \quad V_0 = \frac{1}{2\alpha^{2/n}}, \quad (13)$$

with $\bar{x}$ playing the role of time. It is then intuitive to write the Lagrangian

$$\mathcal{L} \left( h(\bar{x}), \frac{dh}{d\bar{x}}(\bar{x}) \right) = \frac{1}{2} \left( \frac{dh}{d\bar{x}} \right)^2 - \frac{V_0}{h^{\frac{2(n+2)}{n}}}. \quad (14)$$

In the limit of perfectly plastic ice $n \to +\infty$, the variables $\bar{x}$ and $x$ coincide, the accumulation function $c(x)$ disappears from the picture, and the Lagrangian (14) reduces to

$$\mathcal{L}_\infty \left( h(\bar{x}), \frac{dh}{d\bar{x}}(\bar{x}) \right) = \frac{1}{2} \left( \frac{dh}{d\bar{x}} \right)^2 - \frac{V_0}{h^2}. \quad (15)$$

In the general case with finite $n$, the Lagrangian (14) does not depend on $\bar{x}$ and the corresponding Hamiltonian is conserved, yielding the Beltrami identity

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial (dh/d\bar{x})} \frac{dh}{d\bar{x}} - \mathcal{L} = \frac{1}{2} \left( \frac{dh}{d\bar{x}} \right)^2 - \frac{1}{2\alpha^{2/n} h^{\frac{2(n+2)}{n}}} = E, \quad (16)$$

where $E$ is the constant energy of the analogous particle. The Vialov equation is reproduced for vanishing energy $E$.

When a solution $h(\bar{x})$ of this equation is found, one still has to replace $\bar{x}$ with $x$, which may not be possible to do explicitly using only elementary functions. The situations when the integral $I(x)$ can be reduced to elementary functions by means of the Chebyshev theorem are discussed in [17].

As an example, in the Vialov model [8] it is $c(x) = c_0 \equiv \text{const.}$ and

$$\bar{x}(x) = \frac{n c_0^{1/n}}{n+1} x^{\frac{n+1}{n}}. \quad (17)$$

Searching, as customary in the literature, for a solution in the interval $x \in [0, L]$ with the glacier summit at $x = \bar{x} = 0$ (where $h(0) = H$) and terminus at $x = L$ (where $h = 0$ and, necessarily, $dh/dx \to \infty$), one has

$$h^{\frac{n+2}{n}} \frac{dh}{d\bar{x}} = -\frac{1}{\alpha^{1/n}}, \quad (18)$$

which integrates to

$$[h(\bar{x})]^{\frac{2(n+1)}{n}} = H^{\frac{2(n+1)}{n}} \left[ 1 - \frac{2(n+1)\bar{x}}{n\alpha^{1/n} H^{\frac{2(n+1)}{n}}} \right]. \quad (19)$$
Fig. 1 The potential (13) (plotted for $2\alpha^{2/n} = 1$) for the realistic value $n = 3$. Since the total energy is $E = 0$, the particle is forced to remain on the $h$-axis.

Raising both sides to the power $\frac{n}{2(n+1)}$, one obtains the solution

$$h(\bar{x}) = H \left[ 1 - \frac{2(n+1)\bar{x}}{n\alpha^{1/n} H^{2(n+1)/n}} \right]^{\frac{n}{2(n+1)}}$$

and finally, using Eq. (17) to return to the original independent variable $x$,

$$h(x) = H \left[ 1 - \left( \frac{x}{L} \right)^{\frac{n+1}{n}} \right]^{\frac{n}{2(n+1)}}$$

where

$$L = \frac{1}{2\pi^{\frac{n}{n+1}}} H^2 \left( \frac{\alpha}{c_0} \right)^{\frac{1}{n+1}}$$

which is the well-known Vialov profile [8].

3 Mechanical analogy

Equation (16) has the form of an energy conservation equation for a particle of unit mass and total mechanical energy $E$ in the potential energy $V(h)$ given by Eq. (13). The Vialov equation is reproduced for the value $E = 0$ of this energy. This brings about an obvious analogy. Since $n > 0$ for all practical applications (and $n = 3$ for ice creep), the potential $V(h)$ has the shape illustrated in Fig. 1, to which we refer.

Only the interval $h \in [0, H]$ is relevant, where $V(h) \to -\infty$ as $h \to 0^+$ and $V(h) \leq V(H) < 0$. The motion of the particle with total mechanical energy $E = 0$ is truncated artificially at the glacier summit where $h = H$ and the solution is extended by reflecting about the vertical line through the summit. In the $(h, V)$ plane, this corresponds to replacing $V(h)$ in $h \in (H, 2H)$ with the reflection of $V$ in $(0, H)$ about the line $h = H$ (see Fig. 2). The region $h > 2H$ is irrelevant.
Fig. 2  The effective potential $V(h)$ (in black) is truncated at $h = H$ (arbitrarily set to 3 here) and reflected about the vertical line $h = H$ (blue portion of the curve). The resulting potential and the solution $h(x)$ of the Vialov equation are not differentiable at $h = H$.

Although, without this reflection, the motion of the particle would be unbounded, the truncation and the artificial confinement to $h \in (0, L)$ change the picture from what would be deduced by applying the usual Weierstrass method.

One feature that emerges is that, in order for the particle to have finite total energy $E = 0$, its kinetic energy must diverge as it falls into the origin $h = 0$ corresponding to the glacier terminus, in order to compensate for the infinitely negative potential energy at $h = 0$. This means that the derivative $dh/dx$ necessarily diverges at the terminus, a known feature of the solutions of the Vialov equation, which is associated with unphysically divergent stress at the bed $\tau_b = -\rho_{\text{ice}} gh dh/dx$. The shallow ice approximation, which is used to derive the Vialov equation, breaks down at the glacier terminus.

4 Cosmological analogy

Using the rescaled variable $\bar{x}$, the Vialov equation is formally analogous to the Friedmann equation describing spatially homogeneous and isotropic (or Friedmann-Lemaître-Robertson-Walker, hereafter FLRW) universes. Before unveiling the analogy, let us recall the basics of FLRW cosmology.

In general relativity, gravity is described by geometry, i.e., by the Lorentzian metric tensor $g_{ab}$ which satisfies the Einstein field equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi GT_{ab},$$

(23)

where $R_{ab}$ is the Ricci tensor of the metric $g_{ab}$, $R$ is its trace, and the stress-energy tensor $T_{ab}$ describes the matter content of spacetime (we adopt the notation of Refs. [28,29] using units in which the speed of light is unity, while $G$ denotes Newton’s constant). Spatially homogeneous and isotropic universes have four-dimensional spacetime geometries described by the FLRW line element.
\[
\begin{align*}
\text{ds}^2 = g_{ab} dx^a dx^b &= -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]. \\
\end{align*}
\]

in comoving polar coordinates \((t, r, \theta, \phi)\). The scale factor \(a(t)\) describes the expansion history of the universe. The constant \(K\) is associated with the constant curvature of the 3-dimensional Riemannian spatial geometries (the physical 3-dimensional spaces) obtained by setting \(dt = 0\). If \(K > 0\), the FLRW line element \((24)\) belongs to a closed universe; if \(K = 0\), the spatial 3-sections are flat (Euclidean); if \(K < 0\), 3-space is hyperbolic \([28–31]\). The requirements of spatial homogeneity and isotropy are very stringent and, as a consequence, the previous classification includes all the possible FLRW geometries.

In FLRW cosmology, the matter content of the universe is usually described by a perfect fluid with stress-energy tensor of the form

\[
T_{ab} = (P + \rho) u_a u_b + P g_{ab},
\]

where \(u_a\) is the fluid 4-velocity, \(\rho(t)\) is the energy density, and \(P(t)\) is the isotropic pressure. Comoving coordinates are adapted to observers comoving with the fluid, who see the universe spatially homogeneous and isotropic around them (that is, comoving coordinates are adapted to the symmetries of the metric). \(P\) and \(\rho\) are related by an equation of state, usually of the barotropic form \(P = P(\rho)\). Very often in the literature, this takes the form

\[
P = w\rho, \quad w = \text{const}.
\]

with constant “equation of state parameter” \(w\).

The Einstein equations \((23)\) simplify greatly by requiring spatial homogeneity and isotropy, and reduce to the Einstein–Friedmann equations satisfied by \(a(t), \rho(t), \text{and } P(t)\) \([28–31]\)

\[
H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\rho - K}{a^2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P),
\]

\[
\dot{\rho} + 3H (P + \rho) = 0.
\]

Here an overdot denotes differentiation with respect to the comoving time \(t\), while \(H(t) \equiv \dot{a}/a\) is the Hubble function \([28–31]\). Among the three Eqs. \(27\)–\(29\), only two are independent. Given any two of them, the third one is derived from them. Without loss of generality, here we choose the Friedmann equation \(27\) and the energy conservation Eq. \(29\) as our independent equations, then the acceleration Eq. \(28\) is derived from them. If the cosmic fluid obeys the equation of state \(26\) with \(w \neq -1\), the covariant conservation Eq. \(29\) is integrated, giving

\[
\rho(a) = \frac{\rho_0}{a^{3(w+1)}}.
\]

and the Friedmann equation becomes

\[
H^2 = \frac{8\pi G \rho_0}{a^{3(w+1)}} - \frac{K}{a^2}.
\]

Going back now to the Vialov equation, Eq. \((16)\) can be rewritten as

\[
\left( \frac{1}{h} \frac{dh}{dx} \right)^{2} = \frac{1}{2\alpha^{2/n}} \frac{4(\alpha+1)}{\alpha} + \frac{E}{h^{2}}.
\]
This is formally a Friedmann equation for a universe with scale factor $a(t)$ analogous to $h(\bar{x})$, curvature index $K = -E$, filled with a perfect fluid with constant equation of state parameter

$$w = \frac{n + 4}{3n}$$  \hspace{1cm} (33)

and energy density $\rho(a) = \rho_0/a^{3(w+1)} = \rho_0/a^{4(n+1)/n}$, with

$$\rho_0 = \frac{3}{16\pi G \alpha^{2/n}}.$$  \hspace{1cm} (34)

It is meaningful that the analogous energy density comes out positive: this property is not to be taken for granted when building such analogies and its failure would take much value away from the analogy.

Ice creep, corresponding to the value $n = 3$, gives $w = 7/9 \approx 0.778$. The equation of state parameter (33) is a positive and always decreasing function of $n$; it diverges at $n = 0$ and has a horizontal asymptote $w = 1/3$ corresponding to the radiation era of the spatially flat universe and, in the analogy, to the limit of perfectly plastic ice $n \rightarrow +\infty$.

An inspection of the Friedmann equation shows that the Hubble function $H$ diverges when $a \rightarrow 0$, which signals a Big Bang (if the universe is expanding, $\dot{a} > 0$) or a Big Crunch (if the universe is contracting, $\dot{a} < 0$) spacetime singularity. According to the Hawking-Penrose singularity theorems [28], this singularity is generic and unavoidable if the cosmic fluid satisfies the weak energy condition $\rho > 0$ and $\rho + 3P > 0$ (corresponding to $w > -1/3$ for a barotropic perfect fluid), which is always satisfied in our case.

In the Vialov equation, the singularity in $h'$ at the terminus $x = L$ is due to the breakdown of the shallow ice approximation. In the cosmic analogue, the shallow ice approximation would mean small $\dot{a}$ or $a(t)/t \ll 1$. Since $a(t) = a_0 t^{2/3(w+1)}$, we have that $a(t)/t = a_0/t^{3w+1} \rightarrow +\infty$ at the Big Bang or Big Crunch. While some cosmologists tend to think that quantum mechanics will change the behavior of matter and circumvent the singularity theorems, thus avoiding the singularity, the analogy with the Vialov equation would suggest instead that the Friedmann equation only holds when the scale factor $a(t)$ is much smaller than the age of the universe $t$ at that time, $a(t) \ll t$, therefore away from the beginning. This intuition matches the idea that there is a fundamental length, the Planck length $\ell_{Pl}$ below which a continuous spacetime manifold does not exist.

Recent work has established that all the solutions of the Friedmann equation are roulettes [32]. A roulette is the curve described by a point attached to a closed convex curve as that curve rolls without slipping along a second, given, curve. This result, translated to the glaciology side of the analogy, states that all solutions of the rescaled Vialov equation are roulettes, a property hitherto undiscovered for this equation.

The Einstein–Friedmann equations for a spatially flat ($K = 0$) universe and a perfect fluid with constant barotropic equation of state $P = w\rho$ enjoy the symmetry [33]

$$a \rightarrow \tilde{a} = a^s,$$  \hspace{1cm} (35)

$$dt \rightarrow \tilde{t} = sa^{3(w+1)(s-1)/2} dt,$$  \hspace{1cm} (36)

$$\rho \rightarrow \tilde{\rho} = a^{-3(w+1)(s-1)} \rho,$$  \hspace{1cm} (37)

where $s \neq 0$ is a real number. These symmetry transformations form a one-parameter Abelian group as $s$ varies in $\mathbb{R} \setminus \{0\}$ [33]. Each symmetry transformation translates into the symmetry
of the Vialov equation (18) written using the variable \( \bar{x} \)

\[
\begin{align*}
  h &\to \tilde{h} = h^s, \\
  d\bar{x} &\to d\tilde{x} = s h^{2(n+1)(s-1)} d\bar{x}.
\end{align*}
\]

(38) (39)

It is straightforward to check that the left-hand side of Eq. (18) is invariant under the transformations (38) and (39), while the right-hand side is invariant because it is constant. Again, the transformations form a one-parameter Abelian group as \( s \neq 0 \) varies.

In the limit of perfectly plastic ice \( n \to +\infty \), the Vialov equation (written using the original variable \( x \)) can be written as

\[
\begin{align*}
  h|\dot{h}|h^{2/n} &= \frac{(n + 2)^{1/n} (xc(x))^{1/n}}{(2A)^{1/n} \rho_{\text{ice}} g} \to \frac{1}{\rho_{\text{ice}} g} \\
  h|\dot{h}| &= \frac{\tau_b}{\rho_{\text{ice}} g},
\end{align*}
\]

(40) (41)

where the constant \( \tau_b \) is the stress at the bottom of the ice and can be obtained by using Eq. (16) with the Lagrangian \( \mathcal{L}_\infty \). The limit \( n \to +\infty \) of the symmetry produces

\[
\begin{align*}
  h &\to \tilde{h} = h^s, \\
  d\bar{x} &\to d\tilde{x} = s h^{2(s-1)} dx,
\end{align*}
\]

(42) (43)

and it is straightforward to check, again, that it leaves invariant the left-hand side of Eq. (41), while the right-hand side is invariant because it is constant. The solution of Eq. (41) is the parabolic Nye profile [18]

\[
\begin{align*}
  h(x) &= H \sqrt{1 - \frac{x}{L}}, \\
  H &= \sqrt{\frac{2\tau_b L}{\rho_{\text{ice}} g}}.
\end{align*}
\]

(44) (45)

It is well known that the parabola is a roulette described by a point fixed on the involute of the circle rolling along a straight line.

5 Reducing the Vialov equation to the Nye equation

The symmetry of the Friedmann equation for \( K = 0 \) with a perfect fluid with constant barotropic equation of state essentially says that one can obtain the solution for a fluid from that for another fluid. On the other side of the analogy, this property is useful to approach the problem of solving the Vialov equation for general \( n \). Noting that the equation for perfectly plastic ice \( (n \to +\infty) \) is simpler than the Vialov equation for finite \( n \), we can try to reduce the problem of solving the latter to that of solving the former. Indeed, the Vialov equation can be reduced to the Nye equation (41) for perfectly plastic ice by making one more change of variable. Keeping \( \bar{x} \) given by Eq. (9) as the independent variable, let us change also the dependent variable according to \( h \to \tilde{h} = h^p \). Then the Vialov equation (11) becomes

\[
\begin{align*}
  \tilde{h}^{2(n+1)(n-1)} \frac{d\tilde{h}}{d\bar{x}} &= \frac{p}{\alpha^{1/n}}.
\end{align*}
\]

(46)
The choice
\[ p = \frac{n + 1}{n} \tag{47} \]
transforms the Vialov equation into the Nye equation
\[ \frac{\dot{\tilde{h}}}{\tilde{h}} \left| \frac{d\tilde{h}}{dx} \right| = \frac{1}{\tilde{\alpha}^{1/n}} \tag{48} \]
for \( \tilde{h}(\tilde{x}) \), where \( \tilde{\alpha} = (\frac{n}{n+1})^n \alpha \). This property is in principle useful to generate solutions of the Vialov equation. For example, in the Vialov model with \( c(x) = c_0 = \text{const.} \), we have already seen that \( \tilde{x} = \frac{n^{1/n} c_0}{n+1} x \frac{n+1}{n} \). Defining now \( \tilde{h} = h \frac{n+1}{n} \), the Vialov equation reduces to the Nye equation, which has as solution the Nye profile
\[ \tilde{h}(\tilde{x}) = \tilde{H} \sqrt{1 - \frac{\tilde{x}}{\tilde{L}}} \tag{49} \]
This solution can be now be written in terms of \( h \) and \( x \) as
\[ h^{\frac{n+1}{n}} = \tilde{H} \sqrt{1 - \frac{n c_0^{1/n}}{(n+1) L} x \frac{n+1}{n}} \tag{50} \]
Raising both sides to the power \( \frac{n}{n+1} \) and introducing
\[ H = \tilde{H}^{\frac{n+1}{n}} \tag{51} \]
\[ L = \left[ \frac{(n+1) \tilde{L}}{n c_0^{1/n}} \right]^{\frac{n}{n+1}} \tag{52} \]
the Nye solution (49) becomes the Vialov solution
\[ h(x) = H \left[ 1 - \left( \frac{x}{L} \right) \frac{n+1}{n} \right]^{\frac{n}{2(n+1)}} \tag{53} \]
In the general case in which \( c(x) \neq \text{const.} \), one cannot express explicitly \( \tilde{x} \) in terms of elementary functions of \( x \). If this happens, the reduction in the Vialov equation to the Nye equation is purely formal; then also the generation of the solution of the Vialov equation from the generating function (49) is purely formal. Forms of the function \( c(x) \) for which the integral \( I(x) \) can be calculated explicitly in terms of elementary functions have been identified in [17].

6 Conclusions

The understanding of longitudinal profiles of ice sheets, ice caps and glaciers is necessary to estimate the volume of ice in these three-dimensional systems and the volume-area scaling relation [1], which is the basis to estimate the ice content of a region, the ice loss due to climate change, and the sea-level rise due to the melting of polar ice [2]. Realistic ice follows Glen’s law [3], while its degenerate limit of perfectly plastic ice was used in the early days of glaciology [4,18,19]. The longitudinal profile of an ice cap or glacier following Glen’s law is described by the Vialov equation [4–8], in which the accumulation rate of ice due to precipitation on the glacier is described by the accumulation function \( c(x) \). This function must be prescribed in a given model, ideally based on atmospheric models for the region.
The solution of the Vialov equation is reduced to the quadrature of an integral $I(x)$ involving the accumulation function $c(x)$.

It is not always possible (or easy) to express the integral $I(x)$ in terms of elementary functions (cf. Ref. [17]); hence, nothing is gained, in terms of solving analytically the Vialov equation, by introducing the new variable $\bar{x}(x)$ and rewriting the Vialov equation in terms of it, as we have done in Sect. 2. Nevertheless, what is gained is a Lagrangian formulation for this equation, thus solving the inverse variational problem. In turn, conservation of the corresponding Hamiltonian leads to an analogy with a particle in one-dimensional motion and to an unexpected and intriguing analogy with the Friedmann equation of general relativistic cosmology. We have explored this analogy and found that, for the physically meaningful values $n > 0$ of the Glen law exponent, the analogous universe is filled with a perfect fluid with ultrarelativistic equation of state $P > \rho/3$. The limit $n \to +\infty$ of perfectly plastic ice corresponds to a radiation-dominated universe with $P = \rho/3$. In addition, the cosmological analogy provides two new developments. First, it establishes that all solutions of the (rescaled) Vialov equation are roulettes. Second, it provides a new symmetry of the Vialov equation written in terms of the new variable $\bar{x}$. By using this idea, the solution of the Vialov equation for finite $n$ can be reduced, at least formally, to solving the Nye equation (which is trivial) and then expressing $\bar{x}$ in terms of $x$ (which is not). Then the Nye profile obtained for perfectly plastic ice [18,19] becomes a generating function for all solutions of the Vialov equation. In general (i.e., for general forms of the function $c(x)$), this solution-generating technique remains purely formal because of the difficulty in expressing the rescaled variable $\bar{x}$ in terms of the physical variable $x$ along the glacier bed (but see Ref. [17]). At least the Vialov profile (corresponding to $c(x) = \text{const.}$) for Glen law ice can be obtained in this way from the Nye profile for perfectly plastic ice.

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