Abstract. We study Duflo-Serganova functor for non-twisted affine Lie superalgebras and affine vertex superalgebras.

0. Introduction

Let $\mathfrak{g}$ be a Lie superalgebra and let $x \in \mathfrak{g}_1$ satisfy the condition $[x, x] = 0$. The operator $\text{ad}_x$ defines an odd square zero endomorphism of any $\mathfrak{g}$-module. This yields a functor $N \mapsto \text{DS}_x(N) := \text{Ker} N_x/\text{Im} N_x$ from the category of $\mathfrak{g}$-modules to the category of modules over $\mathfrak{g}_x := \text{DS}_x(\mathfrak{g})$.

The functor $\text{DS}_x$ was introduced in [DS] (see also [S2]) as a means to assign an analog of singular support to representations of Lie superalgebras. This functor preserves superdimension and tensor product of representations.

Recall that the defect of a finite-dimensional Lie superalgebra $\mathfrak{g}$ is the dimension of a maximal isotropic subspace in $\mathbb{Q}\Delta$; for $A(m-1, n-1), B(m,n), D(m,n)$ the defect is equal to $\min(m,n)$; for other cases of non Lie algebras it is one. It is well-known that the defect is equal to the maximal number of mutually orthogonal isotropic simple roots. A finite-dimensional simple Lie superalgebra of zero defect is either a simple Lie algebra or $\mathfrak{osp}(1|2l)$; the finite-dimensional modules over these Lie superalgebras are completely reducible (and these are the only simple Lie superalgebras with this property).

If $\mathfrak{g}$ is a finite-dimensional Lie superalgebra, then $\text{DS}_x(\mathfrak{g})$ is a finite-dimensional Lie superalgebra of a smaller defect. If $\mathfrak{g}$ is the affinization of $\mathfrak{g}$ and $x \in \mathfrak{g}_1$, then $\mathfrak{g}_x$ is the affinization of $\mathfrak{g}_x$, see [GS].

In this paper we consider the DS functors for affine Lie superalgebras $\mathfrak{g} = \mathfrak{g}^{(1)}$ and affine vertex superalgebras $V^k_\lambda(\mathfrak{g})$; we always assume that $x \in \mathfrak{g}_1$.

Let $V^k_\lambda(\mathfrak{g})$ be a vacuum $\mathfrak{g}$-module of level $k$ and $V^k_\lambda(\mathfrak{g})$ be its simple quotient. It is easy to see that $\text{DS}_x(V^k_\lambda(\mathfrak{g})) = V^k_\lambda(\text{DS}_x(\mathfrak{g}))$. We prove that for a non-negative integral $k$ one has $\text{DS}_x(V^k_\lambda(\mathfrak{g})) = V^k_\lambda(\text{DS}_x(\mathfrak{g}))$ if $\mathfrak{g}_x$ has zero defect and $\mathfrak{g}_x \neq \mathbb{C}$, see Theorem 2.2. As a result, the corresponding vertex algebras are isomorphic, see Corollary 3.4.2.

The principal admissible modules for an affine Lie algebra $\mathfrak{t}$ were classified in [KW5]. A level $k$ is called principal admissible if $V^k_\lambda(\mathfrak{t})$ is principal admissible. From Theorem

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of Arakawa \cite{A} it follows that for a principal admissible level \( k \) the \( V_k(t) \)-modules in the category \( \mathcal{O} \) are completely reducible and the irreducible modules are the principal admissible modules of level \( k \).

We introduce the principal admissible levels for an affine Lie superalgebra \( g \) using Kac-Wakimoto definition for Lie algebra case. We prove that if \( \hat{g}_x \) is a simple Lie algebra and \( \hat{g}_x \neq B(n+1|n) \), then for a principal admissible level \( k \) one has \( DS_x(Vac_k(g)) = Vac_k(g_x) \). This implies the isomorphism of the corresponding vertex algebras. The proof is based on Arakawa’s Theorem and the fact that the maximal proper submodule in \( Vac_k(g_x) \) is generated by a singular vector (if \( g_x \) is a Lie algebra, this can be easily deduced from \cite{F}). We believe that the statement holds for \( g_x = \mathfrak{osp}(1|2n) \), however both Arakawa’s and Fiebig’s results are not established in this case.

In Section 1 we recall the construction of Duflo-Serganova functor \( DS_x \) and summarize the results which we use later.

In Section 2 we study \( DS_x \) functor for integrable vacuum modules and prove Theorem 2.2. Since integrable vacuum modules have principal admissible levels, this theorem for the case, when \( \hat{g}_x \) is a simple Lie algebra and \( \hat{g}_x \neq B(n+1|n) \), is a particular case of Theorem 4.3.3. However, the proof of Theorem 2.2 is different: it does not use vertex algebras and Arakawa’s Theorem. In § 2.4 we give an example when \( DS_x(Vac_k(g)) \neq Vac_k(g_x) \) \((g = \mathfrak{sl}(1|2))\), \( k \) is critical.

In Section 3 we introduce the \( DS_x \) functor for vertex superalgebras. In particular, we prove that if \( DS_x(Vac_k(g)) = Vac_k(g_x) \), then \( DS_x \) maps the simple affine vertex superalgebra \( V_k(g) \) to the simple affine vertex superalgebra \( V_k(g_x) \). As a result, for any \( V_k(g) \)-module \( N \) the image \( DS_x(N) \) is a \( V_k(g_x) \)-module.

In Section 4 we study \( Vac_k(g) \) if \( k \) is a principal admissible level, (this notion we define in § 4.2 similarly to the Lie algebra case). In § 4.4 we prove that \( DS_x(Vac_k(g)) = Vac_k(g_x) \) if \( \hat{g}_x \) is a simple Lie algebra and \( \hat{g}_x \neq B(n+1|n) \).

Let \( \Sigma \) be a set of simple roots which contains a maximal isotropic subset \( S = \{ \beta_1, \ldots, \beta_r \} \) (\( r \) is the defect of \( \hat{g} \)). We consider \( DS_x \) for \( x = \sum_{i=1}^r x_i \), where \( x_i \) is a non-zero vector in \( g_{\beta_i} \). In this case \( \hat{g}_x = DS_x(\hat{g}) \) has zero defect. All our results are valid also for the composition \( DS_S := DS_{x_1} \circ DS_{x_2} \circ \ldots \circ DS_{x_r} \). Note that \( DS_S(g) \cong DS_x(g) \).

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1. Preliminaries

Throughout the paper \( g = \hat{g}^{(1)} \), where \( \hat{g} \neq D(2|1, a) \) is a finite-dimensional Kac-Moody superalgebra with a set of simple roots (a base) \( \Sigma \) and a Cartan subalgebra \( \mathfrak{h} \). We denote by \( \mathfrak{h} \) the Cartan subalgebra of \( g \): \( h = \hat{h} \oplus CK \oplus Cd \).

Let \( \Delta \subset h^* \) (resp., \( \hat{\Delta} \)) be the set of roots of \( g \) (resp., of \( \hat{g} \)). We denote by \( \Delta^- \) and \( \Delta^+ \) the subsets of even and odd roots. We denote by \( W \) the Weyl group of \( g \). Recall that \( \Delta^- \) is a union of a finite number of root systems of affine Lie algebras with the same minimal imaginary root \( \delta \). Throughout the paper we fix \( \hat{\Delta} \) and denote by \( \Lambda_0 \) the corresponding fundamental weight, i.e. \( (\Lambda_0, \delta) = 1 \) and \( (\Lambda_0, \hat{\Delta}) = (\Lambda_0, \Lambda_0) = 0 \).

We assume that \( \hat{\Delta} \) is indecomposable (i.e., \( g, \hat{g} \) are quasisimple in the sense of [S3]). We fix \( \Delta^+ \) and consider the subsets of positive roots \( \Delta^+ \) which contain \( \Delta^+_0 \). The choice of \( \Delta^+ \) gives a triangular decomposition of \( g \), compatible with the triangular decomposition of \( g_0 \), corresponding to \( \Delta^+_0 \). For a fixed subset of positive roots \( \Delta^+ \) we denote by \( \Sigma \) the corresponding base (i.e., the set of simple roots) and by \( \rho \) the corresponding Weyl vector.

For a fixed base \( \Sigma \) we denote by \( \alpha_0 \) the affine root, i.e. \( \Sigma = \hat{\Sigma} \cup \{\alpha_0\} \), where \( \hat{\Sigma} \) is the base of \( \hat{\Delta}^+ \).

We denote by \( O \) the BGG category of finitely generated \( g \)-modules with a diagonal action of \( h \) and a locally finite action of \( g_\alpha \) with \( \alpha \in \Delta^+_0 \). The category \( O \) is equipped by a duality functor \( N \mapsto N^\# \) and the simple modules are self-dual.

We normalize the form \( (-, -) \) on \( g \) as in [KW3] and set \( \hat{\Delta}^\# := \{\alpha \in \hat{\Delta}^-| (\alpha, \alpha) > 0\} \).

The corresponding algebra \( \hat{g}^\# \) is a simple Lie algebra; we denote its highest root by \( \theta \). We will use bases \( \Sigma \) such that \( \theta \) is the highest root in \( \Delta^+(\Sigma) \); then \( \alpha_0 = \delta - \theta \).

Let \( \hat{\Omega} \) be the Casimir operator for \( \hat{g} \) which corresponds to the invariant bilinear form \( (-, -) \), see [K1], Ch. II. Recall that the dual Coxeter number \( h^\vee \) is half of the eigenvalue of the Casimir operator \( \hat{\Omega} \) on the adjoint representation \( \hat{g} \) and \( h^\vee = (\rho, \delta) \). We always choose the Weyl vector \( \rho \) in the form \( \rho = h^\vee \Lambda_0 + \hat{\rho} \).

We say that \( k \in \mathbb{C} \) is non-critical if \( k \neq -h^\vee \) and \( \lambda \in h^* \) is non-critical if \( K(\lambda) \neq -h^\vee \), i.e. \( (\lambda + \rho, \delta) \neq 0 \).

We use the following notations: if \( X, Y \subset h^* \) we set \( (X, Y) = \{(x, y)| x \in X, y \in Y\} \); for a vector space \( V \), \( X \subset V \) and \( R \subset \mathbb{C} \) we use the notation \( RX = \{\sum_{i=1}^n r_i x_i| r_i \in R, x_i \in X\} \) (for instance, \( \mathbb{Z}\Delta \) is the root lattice). For \( S \subset h^* \) we set \( S^\perp := \{\nu \in h^*| \forall \beta \in S \ (\beta, \nu) = 0\} \).
If $\alpha \in \Delta$ is a non-isotropic root, we say that a $\frak g$-module $N$ is $\alpha$-integrable if $\frak g_{\pm \alpha}$ act locally nilpotently on $N$. We use conventions of [GK]. We say that a $\frak g$-module $N \in \mathcal O$ is integrable if $N$ is integrable as a $\frak g_0$-module and $N$ is $\alpha$-integrable for each $\alpha \in \Delta$ satisfying $||\alpha||^2 > 0$.

For $A(m|1)_{(1)}, C(m)_{(1)}, N \in \mathcal O$ is integrable if $N$ is integrable as a $\frak g_0$-module and $h$ acts diagonally.

1.1. DS functor for affine Lie superalgebras. Take $x \in \frak g_1$ satisfying $[x,x] = 0$. Recall that Duflo-Serganova functor $DS_x$ is defined by $DS_x(N) := \text{Ker} N_x/\text{Im} N_x$; we view $DS_x(N)$ as a module over $\frak g^x$ (where $\frak g^x$ is the centralizer of $x$ in $\frak g$). Note that $[x,\frak g] \subset \frak g^x$ acts trivially on $DS_x(N)$ and that $\frak g^x := DS_x(\frak g) = \frak g^x/[x,\frak g]$ is a Lie superalgebra. Thus $DS_x(N)$ is a $\frak g_\mathfrak x$-module and $DS_x$ is a functor from the category of $\frak g$-modules to the category of $\frak g_\mathfrak x$-modules. This is a tensor functor $(DS_x(N \otimes N')) = DS_x(N) \otimes DS_x(N')$, see [DS].

An exact sequence of $\frak g$-modules

$$0 \to N_1 \to N \to N_2 \to 0$$

induces the exact sequence of $\frak g_\mathfrak x$-modules

$$0 \to E \to DS_x(N_1) \to DS_x(N) \to DS_x(N_2) \to \Pi(E) \to 0.$$  

Recall that for a $\frak g$-module $N$ with a diagonal action of $\frak h$ one has

$$\text{sch } N := \sum_{\nu \in \frak h^*} \text{sdim } N_\nu e^\nu.$$  

If $0 \to N_1 \to N \to N_2 \to 0$ is exact, then $\text{sch } DS_x(N) = \text{sch } DS_x(N_1) + \text{sch } DS_x(N_2)$.

1.2. Choice of $x$. In this paper we consider $DS_x$ for $x \in \hat{\frak g}$:

(1) 

$$x \in \hat{\frak g}_\mathfrak T, \quad [x, x] = 0.$$

1.2.1. Definition. For $a \in \frak g$ we denote by $\text{supp}(a)$ the subset of $\Delta \cup \{0\}$ such that

$$a = \sum_{\beta \in \text{supp}(a)} a_\beta,$$

where $a_\beta$ is a non-zero vector in $\frak g_\beta$.

1.2.2. Definition. We call $S \subset \hat{\Delta}_\mathfrak T$ an isotropic set if $S$ is a basis of an isotropic subspace in $\frak h^*$.

Note that if $\text{supp}(x)$ is an isotropic set, then $x$ satisfies (1).
1.2.3. Let $\hat{G}$ be the Lie group of $\hat{g}_x$. By [DS], Thm. 4.2 each $x$ satisfying (1) is $\hat{G}$-conjugate to $x'$, where $\text{supp}(x')$ is an isotropic set; this gives a one-to-one correspondence between the $\hat{G}$-orbits for $x$ satisfying (1) and $\hat{W}$-orbit of isotropic sets in $\hat{\Delta}$. In particular, for each $x$ satisfying (1) there exists a base $\hat{\Sigma}$ such that $x$ is $\hat{G}$-conjugate to $x'$ such that $\text{supp}(x')$ is an isotropic set and $\text{supp}(x') \subset \hat{\Sigma}$. We call the cardinality of $\text{supp}(x')$ the rank of $x$; 0 has the zero rank and the maximal rank is equal to the defect of $\hat{g}_x$.

Let $t$ be a Lie subalgebra of $\hat{g}_x$ and $N$ be a $g$-module which is $t$-finite (i.e., $U(t)v$ is finite-dimensional for each $v \in N$). Then the Lie group of $t$ acts on $N$. Moreover, any element $g$ in this Lie group induces an isomorphism between the algebras $\hat{g}_x$ and $\hat{g}_{Ad_g(x)}$ and the corresponding modules $\hat{D}_x(N)$ and $\hat{D}_{Ad_g(x)}(N)$.

In particular, for a $\hat{g}$-integrable $g$-module $N$, this construction gives an isomorphism between $\hat{D}_x(N)$ and $\hat{D}_{x'}(N)$ with $x'$ as above ($\text{supp}(x')$ is an isotropic subset of a certain base $\hat{\Sigma}$).

1.2.4. Assume that $S := \text{supp}(x)$ is an isotropic set.

It is shown in [DS], Lemma 6.3 that $\hat{g}_x$ a finite-dimensional Kac-Moody superalgebra with the roots

$$\hat{\Delta}_x := (S^\perp \cap \hat{\Delta}) \setminus (S \cup (-S));$$

$\hat{g}_x$ can be identified with a subalgebra of $\hat{g}$ generated by the root spaces $\hat{g}_\alpha$ with $\alpha \in \hat{\Delta}_x$ and $\hat{h}_x \subset \hat{h}^x = \{ h \in \hat{h} | S(h) = 0 \}$ such that

$$\hat{h}_x \oplus \left( \sum_{\beta \in S} \mathbb{C} h_\beta \right) = \hat{h}^x, \ [\hat{g}_\alpha, \hat{g}_\alpha] \subset \hat{h}_x \ \forall \alpha \in \hat{\Delta}_x.$$ 

Moreover, $\hat{h}_x$ is a Cartan subalgebra of $\hat{g}_x$ and $\hat{g}^x = \hat{g}_x \oplus [x, \hat{g}]$. Note that $\hat{h}_x^x$ is identified with a subspace in $\hat{h}^*$ and $S^\perp = CS \oplus \hat{h}_x^x$.

If $\hat{\Delta}_x$ is not empty, then $\hat{\Delta}_x$ is the root system of the Lie superalgebra $\hat{g}_x$. One can choose a set of simple roots $\hat{\Sigma}_x$ such that $\Delta^+(\hat{\Sigma}_x) = \Delta^+ \cap \hat{\Delta}_x$.

Let $r$ be the rank of $x$ (i.e., $|S| = r$). If $\hat{g} = A(m|n), B(m|n)$ or $D(m|n)$, then $\hat{g}_x = A(m-r|n-r), B(m-r|n-r)$ or $D(m-r|n-r)$ respectively. If $\hat{g} = C(n), G_3$ or $F_4$, then $r = 1$ and $\hat{g}_x$ is the Lie algebra of type $C_{n-2}$, $A_1$ and $A_2$ respectively. If $\hat{g} = D(2|1; a)$, then $r = 1$ and $\hat{g}_x = \mathbb{C}$. One has

$$\text{defect } \hat{g}_x = \text{defect } \hat{g} - \text{rank } x.$$ 

It is easy to show (see [GS]) that $\hat{D}_x(g) = \hat{g}_x$ is the affinization of $\hat{g}_x$; we identify this algebra with

$$\hat{g}_x = \sum_{s=-\infty}^{\infty} (\hat{g}_x t^s) \oplus \mathbb{C} K \oplus \mathbb{C} d, \ \hat{h}_x := \hat{h} \oplus \mathbb{C} K \oplus \mathbb{C} d;$$
then \( \Delta_x := \Delta(g_x) \) is the affinization of \( \tilde{\Delta}_x \). One has
\[
h^*_x = \hat{h}^* \oplus \mathbb{C} \delta \oplus \mathbb{C} \Lambda_0 \subset h^* \quad \text{and} \quad S^\perp = h^*_x \oplus \mathbb{C} S.
\]

Set \( \Delta^+_x := \Delta^+(\Sigma) \cap \Delta_x \) and consider the corresponding triangular decomposition of \( g_x \).
We will describe the base \( \Sigma \) of \( \Delta^+_x \) below.

If \( \tilde{\Delta}_x \) is empty, then \( \hat{g}_x = 0 \) or \( \hat{g}_x = gl_1 \). If \( \hat{g}_x = 0 \) (i.e., \( \hat{g} = A(n|n), A(n + 1|n), B(n|n), D(n|n), C(2), \)), then \( g_x = CK \times \mathbb{C} \). If \( \hat{g}_x = gl_1 \) (i.e., \( \hat{g} = D(n + 1|n) \) or \( D(2|1, a) \)), then \( g_x = g_{l_1} ^{(1)} \)

1.3. \textbf{Casimir operator}. Take \( x \) as in (1). The bilinear form \((-,-)\) induces an invariant bilinear form \((-,-)_x\) on \( g_x \). If \( N \) is an integrable \( g \)-module, then \( DS_x(N) \) is an integrable \( g_x \)-module.

Let \( \Omega \) be the Casimir operator for \( g \) which corresponds to the invariant bilinear form \((-,-)\), see [K1], Ch. II. Let \( \Delta_x \neq \emptyset \). By [GS], the image of \( \Omega \) is the Casimir operator for \( g_x \). This implies \( ||\rho||^2 = ||\rho_x||^2 \) and
\[
\text{(2)} \quad [DS_x(L_\rho(\lambda)) : L_{\rho_x} (\lambda')] \neq 0 \implies (\lambda + 2\rho, \lambda) = (\lambda' + 2\rho_x, \lambda')_x,
\]
where \( \lambda' \in h^*_x = S^\perp / \mathbb{C} S \).

1.4. \textbf{Duality}. The duality in \( \mathcal{O} \) is defined by an anti-automorphism \( \sigma \) of \( g \) which stabilizes the elements of \( h \). By above, \( g_x, g_{\sigma(x)} \) are identified with a subalgebra of \( g \) which is \( \sigma \)-stable (in particular, \( g_x = g_{\sigma(x)} \)). It is not hard to see that the map \( \Psi : DS_x(N^2) \rightarrow (DS_{\sigma(x)}(N))^2 \)
defined by \( \Psi(f)(v) := f(v) \) is an isomorphism of \( g_x \)-modules if \( N \in \mathcal{O} \).

2. \textbf{INTEGRABLE VACUUM MODULES}

In this section \( \hat{g} \) is a finite-dimensional Kac-Moody algebra and \( x \in \hat{g}_T \) is such that \( \text{supp}(x) \) has a maximal rank, that is \( \hat{g}_x \) has zero defect. Recall that \( g_x \) is the affinization of \( g_x \).

2.1. \textbf{Vacuum modules}. If \( p \) is a Kac-Moody superalgebra with a Cartan subalgebra \( t \), we denote by \( L_p(\lambda) \) a simple highest \( p \)-module with the highest weight \( \lambda \in \mathfrak{t}^* \). For an affine Kac-Moody superalgebra \( p \) we denote by \( Vac^k \) the vacuum module of level \( k \) and by \( |0\rangle \) the vacuum vector. For \( p = g \) we write simply \( L(\lambda), Vac^k \). Note that \( L(k\Lambda_0) = Vac^k \) is the simple quotient of \( Vac^k \) and so it does not depend on the choice of \( \Sigma \); we call \( L(k\Lambda_0) \) a simple vacuum module; if \( L(k\Lambda_0) \) is integrable, we call it an integrable vacuum module.
2.1.1. The character of an integrable vacuum module is given by the Kac-Wakimoto character formula, see [GK]. From the proof it follows that an integrable vacuum module is a unique integrable quotient of $\text{Vac}^k$ (since the proof uses only the fact that $L(k\Lambda_0)$ is a $g^\#$-integrable quotient of $\text{Vac}^k$, so all integrable quotients have the same character and thus such quotient is unique).

Recall that $\hat{g}^\# \neq D_2$. Let $\theta$ be the highest root of $\hat{\Delta}^\#$ and $e \in \hat{g}^\#$ be the corresponding root vector. From [K1], Lem. 3.4 it follows that a quotient $\text{Vac}^k/I$ is $g^\#$-integrable if and only if $k \in \mathbb{Z}_{\geq 0}$ and $I$ contains $f_k^{k+1}|0\rangle$ for $f_0 := e t^{-1}$. Therefore $L(k\Lambda_0)$ is integrable if and only if $k \in \mathbb{Z}_{\geq 0}$; in this case $L(k\Lambda_0) = \text{Vac}^k/I(k)$, where $I(k)$ is generated by $f_k^{k+1}|0\rangle$.

Note that the vector $f_k^{k+1}|0\rangle$ is singular if $\delta - \theta \in \Sigma$.

2.1.2. Remark. Recall that $L(k\Lambda_0)$ does not depend on the choice of $\hat{\Sigma}$. Combining §1.2 and §5, we see that computing $\text{DS}_x(L(k\Lambda_0))$ we can always assume that $\text{supp}(x)$ is an isotropic set which lies in $\hat{S}$ satisfying (P1), (P2), (P3) in §5.

2.2. Theorem. Let $x \in \hat{g}_T$ be such that $[x,x] = 0$ and $\text{supp}(x)$ has a maximal rank.

(i) If $\hat{g}_x = 0$ and $k \neq -h^\vee$, then $\text{DS}_x(L(k\Lambda_0))$ is one-dimensional.

(ii) Assume that $\hat{\Sigma}$ contains $S := \text{supp}(x)$ and the following inclusion holds
\[ (Q \geq 0 \Sigma \cap S^\perp) \subset (QS + Q \geq 0 \Sigma_x). \]
If $L(\lambda)$ is integrable and $(\lambda,S) = 0$, then
\[ \text{DS}_x(L(\lambda)) \cong L_{\hat{g}_x}(\lambda|_{h_x}). \]

(iii) If $\hat{g}_x \neq \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$, one has
\[ \text{DS}_x(L(k\Lambda_0)) \cong L_{\hat{g}_x}(k\Lambda_0). \]

2.3. Proof of Theorem 2.2. For (i), (iii) we set $\lambda := k\Lambda_0 \in \mathfrak{h}^*$ (so $L(\lambda) = L(k\Lambda_0)$) and $S := \text{supp}(x)$. Using Remark 2.1.2 we assume for (i), (iii) that $S, \hat{\Sigma}$ satisfies (P1), (P3) of §5 i.e. $S \subset \hat{\Sigma}$ and (3) holds except for $\hat{g} = D(n + 2|n), D(n + 1|n)$.

We introduce
\[ \lambda' := \lambda|_{h_x} \in h_x^*. \]

Since $(\lambda,S) = 0$ one has $\dim L(\lambda)_{\lambda - \nu} = \delta_{0,\nu}$ for $\nu \in \mathbb{Z}S$. Thus the singular vector in $L(\lambda)$ has a non-trivial image in $\text{DS}_x(L(\lambda))$ which is singular; moreover,
\[ [\text{DS}_x(L(\lambda)) : L_{\hat{g}_x}(\lambda')] = 1. \]
For (i) $g_x = \mathbb{C}K \times \mathbb{C}d$. By §\[1.3\] the Casimir $\Omega_x = 2(K + h^\vee)d$ acts on $DS_x(L(k\Lambda_0))$ by a scalar, so $d$ acts on $DS_x(L(k\Lambda_0))$ by a scalar. Now (i) follows from \[1\].

For (ii), (iii) assume that $[DS_x(L(\lambda)) : L_{g_x}(\lambda' - \nu')] \neq 0$ for some $\nu' \in h^*_x$ with $\nu' \neq 0$. Since $\lambda' - \nu'$ is a weight of $DS_x(L(\lambda))$, there exists $\nu \in h^*$ such that $\nu|_{g_x} = \nu'$, $(\nu, S) = 0$ and $L(\lambda)_{\lambda - \nu} \neq 0$. In particular,

$$\nu \in \mathbb{Z}_{\geq 0} \Sigma \cap S^\perp.$$  

Let us prove (ii). Combining (5) and (3), $L_{g_x}(\lambda'), L_{g_x}(\lambda' - \nu')$ are integrable modules and $||\lambda' - \nu' + \rho_x||^2 = ||\lambda' + \rho_x||^2$, that is

$$(\lambda' - \nu' + \rho_x, \nu') + (\lambda' + \rho_x, \nu') = 0.$$  

Since $g_x$ has zero defect, the integrability of $L_{g_x}(\lambda')$ and $L_{g_x}(\lambda' - \nu')$ gives $(\lambda', \nu'), (\lambda' - \nu', \nu') \geq 0$ and $(\nu', \rho_x) > 0$ (for $\nu' \neq 0$), a contradiction. This establishes (ii). Recall that for our choice of $(S, \Sigma)$, \[3\] holds except for $\hat{g} = D(n + 2|n), D(n + 1|n)$. Thus (ii) implies (iii).

It remains to verify (iii) for $\hat{g} = D(n + 2|n)$. Consider the short exact sequence

$$0 \rightarrow I(k) \rightarrow V\mathfrak{ac}^k \rightarrow L(k\Lambda_0) \rightarrow 0,$$

where $I(k)$ is the maximal proper submodule of $V\mathfrak{ac}^k$. It is easy to see that $DS_x(V\mathfrak{ac}^k) = V\mathfrak{ac}^k(g_x)$. Thus the corresponding long exact sequence is

$$0 \rightarrow E \rightarrow DS_x(I(k)) \xrightarrow{\psi} V\mathfrak{ac}^k(g_x) \xrightarrow{\psi} DS_x(L(k\Lambda_0)) \rightarrow \Pi(E) \rightarrow 0.$$  

Since $DS_x(L(k\Lambda_0))$ is $g_x$-integrable, the image of $\psi$ is an integrable quotient of $V\mathfrak{ac}^k$, that is $L_{g_x}(k\Lambda_0)$. Since $L_{g_x}(k\Lambda_0 - \nu')$ is a subquotient of $DS_x(L(k\Lambda_0))$ and $\nu' \neq 0$, it is a subquotient of $\Pi(E)$. Therefore $\Pi(L_{g_x}(k\Lambda_0 - \nu'))$ is a subquotient of $DS_x(I(k))$. Take $\Sigma$ such that $||\alpha_0||^2 > 0$. By §\[2.1\] $I(k)$ is generated by a singular vector of the weight $k\Lambda_0 - (k + 1)\alpha_0$. Therefore

$$\lambda' - \nu' = k\Lambda_0 - (k + 1)\alpha_0 - \mu',$$

where $\mu' = \mu|_{g_x}$ for some $\mu \in h^*$ such that $\mu \in \mathbb{Z}_{\geq 0} \Sigma$. Then

$$\nu' = \mu|_{g_x} \geq k + 1.$$  

Take $S := \{\varepsilon_i \pm \delta_i\}_{i=1}^n$ and $\Sigma = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \ldots, \varepsilon_{n+1} - \delta_n, \delta_n \pm \varepsilon_{n+2}\}$; then $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, so $||\alpha_0||^2 = 2$. One has

$$\Sigma_x = \{\varepsilon_1 \pm \varepsilon_{n+2}; \delta - (\varepsilon_1 \pm \varepsilon_{n+2})\}, \quad \rho_x = 2\Lambda_0 + \varepsilon_1.$$  

One readily sees that $(\Sigma \cap S^\perp) \subset (\mathbb{C}S + \mathbb{C}\Sigma_x)$, so $\nu' \in \mathbb{C}\Sigma_x$, so

$$\nu' = j\delta - s_+(\varepsilon_1 + \varepsilon_{n+2}) - s_-(\varepsilon_1 - \varepsilon_{n+2}).$$
The integrability of $L_{g_x}(k\Lambda_0 - \nu')$ implies $0 \leq s_\pm \leq k/2$. In addition, \textsuperscript{(2)} gives

$$(k + 2)j - s_+ - s_- = s_+^2 + s_-^2,$$

so $j \leq k/2$. However, $j = (\nu', \Lambda_0) \geq k + 1$ by \textsuperscript{(6)}. This contradiction completes the proof.

2.4. Example: $g = \mathfrak{sl}(1|2)^{(1)}$, $k = -1$. Take $g = \mathfrak{sl}(1|2)^{(1)}$ with $\Sigma = \{\delta - \varepsilon_1, \delta_1 - \delta_2\}$ and $S = \{\varepsilon_1 - \delta_1\}$. Using the character formula (3.20) in [KW4] it is not hard to show that $\text{DS}_x((L(\Lambda_0))$ is not one-dimensional.

3. DS functor for vertex superalgebras

3.1. Vertex algebras. Recall that a vertex (super)algebra $V = V_0 \oplus V_1$ is a vector superspace endowed with a vacuum vector $|0\rangle$, an even linear endomorphism $T$ and a parity preserving linear map $Y: V \to (\text{End} V)[[z, z^{-1}]]$, $a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_M (n) z^{-n-1}$ subject to the following axioms ($a, b \in V, m, n \in \mathbb{Z}$)

- (translation covariance) $[T, Y(a, z)] = \partial_z Y(a, z)$;
- (vacuum) $T |0\rangle = 0$; $Y(\langle 0 |, z) = \text{Id}_V$; $a_{(-1)} |0\rangle = a$, $a(n) |0\rangle = 0$ for $n \geq 0$;
- and the locality axiom which we use in the Borcherds form

$$(a_M (m)b_M (n)) = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (a_M (m-i)b_M (n+i) - (-1)^{m+p(a)p(b)}b_M (m+n-i)a_M (i)).$$

For $m = 0$ this gives

$$a_M (0)b_M (n) = [a_M (0), b_M (n)].$$

(7)

Note that $T$ is “determined” by $Y$, i.e.

$$Ta = a_{(-2)} |0\rangle.$$  

(8)

3.1.1. Modules. A weak module over a vertex superalgebra $V$ in a vector superspace $M$ with a parity preserving linear map

$$Y^M: V \to (\text{End} M)[[z, z^{-1}]], a \mapsto Y_M (a, z) = \sum_{n \in \mathbb{Z}} a_M (n) z^{-n-1},$$

such that for each $v \in M$ one has $a_M (n)v = 0$ for $n >> 0$, $Y_M (\langle 0 |, z) = \text{Id}_M$ and $a_M (m), b_M (n)$ satisfy the locality axiom. As above, the locality axiom gives

$$(a_M (0)b_M (n)) = [a_M (0), b_M (n)].$$

(9)
An ideal of a vertex algebra is a subspace \( I \subset V \) such that \( a_{(m)}b, b_{(m)}a \in I \) for each \( a \in I, b \in V, m \in \mathbb{Z} \). If \( I \) is an ideal of \( V \), then the quotient \( V/I \) inherits the structure of a vertex algebra.

If \( I \) is an ideal of \( V \), the \( V/I \)-modules are the \( V \)-modules annihilated by \( I \), that is \( a_{(m)} = 0 \) for each \( a \in I, m \in \mathbb{Z} \). We say that an ideal \( I \) is generated by a set \( E \) if \( I \) is a minimal ideal containing \( E \). The locality axiom implies that in this case \( M \) is \( V/I \)-module if and only if \( a_{(m)} = 0 \) for each \( a \in E, m \in \mathbb{Z} \).

3.2. Definition of \( DS_x(V) \). Let \( V \) be a vertex superalgebra and \( x \in V \) be such that

\[
x \in V_0, \quad x_{(0)}x = 0 \quad \text{and} \quad |0\rangle \not\in \text{Im}x_{(0)}.
\]

By \( (7) \) one has \( x_{(0)}^2 = 0 \). We define the vector space \( DS_x(V) \) as follows:

\[
DS_x(V) := \text{Ker}_V x_{(0)}/\text{Im}_V x_{(0)}.
\]

By the vacuum axiom, \( |0\rangle \in \text{Ker} x_{(0)} \), so \( |0\rangle \) has a non-zero image \( |0\rangle' \in DS_x(V) \). From the translation axiom \( [T, x_{(0)}] = 0 \), so \( T \) induces an even map \( T' \in \text{End} DS_x(V) \).

Take \( b \in \text{Ker}_V x_{(0)} \). From \( (7) \) it follows that \( |b_{(n)}x_{(0)}\rangle = 0 \) for each \( n \), so \( b_{(n)} \) induces \( b'_{(n)} \in \text{End}(DS_x(V)) \) and \( b'_{(n)} = 0 \) if \( b \in \text{Im}_V x_{(0)} \). This gives a parity preserving linear map

\[
DS_x(V) \to (\text{End} DS_x(V))[z, z^{-1}] \quad b \mapsto Y'(b, z) = \sum_{n \in \mathbb{Z}} b'_{(n)} z^{-n-1}.
\]

The space \( DS_x(V) \) equipped with \( |0\rangle', T' \) and the fields \( Y'(b, z) \) form a vertex algebra (the axioms for \( V' \) follow from the corresponding axioms for the vertex algebra \( V \)). We denote this vertex algebra by \( DS_x(V) \).

3.2.1. Remark. A vertex algebra \( V \) is \( \mathbb{Z}_{\geq 0} \)-graded if \( V = \oplus_{s=0}^{\infty} V_s \) with

\[
\text{deg}(a_{(j)}b) = \text{deg}(a) + \text{deg}(b) - j - 1,
\]

where \( \text{deg} \) stands for the degree of a homogeneous vector in \( V \). We claim that the condition \( |0\rangle \not\in \text{Im}a_{(0)} \) holds for each \( a \in V \) if \( V \) is a \( \mathbb{Z}_{\geq 0} \)-graded vertex algebra with \( V_0 = \mathbb{C}|0\rangle \).

Indeed, assume that \( \text{deg}(a_{(0)}b) = 0 \) for homogeneous \( a, b \). Then \( \text{deg}(a) + \text{deg}(b) = 1 \), so \( a \) or \( b \) lie in \( V_0 = \mathbb{C}|0\rangle \). However, \( |0\rangle_{(0)} = 0 \) and \( a_{(0)}|0\rangle = 0 \) for each \( a \), that is \( a_{(0)}b = 0 \). Hence \( |0\rangle \not\in \text{Im}a_{(0)} \) for each \( a \in V \).

3.2.2. Modules. Let \( M \) be a \( V \)-module. The condition \( x_{(0)}x = 0 \) gives \( [x^M_{(0)}, x^M_{(0)}] = 0 \). We introduce

\[
DS_x(M) = \text{Ker}_M x^M_{(0)}/\text{Im}_M x^M_{(0)}.
\]

Using \( (9) \) it is easy to check that \( DS_x(M) \) inherits a structure of \( DS_x(V) \)-module (i.e., \( Y^M \) induces a map \( DS_x(V) \to (\text{End} DS_x(M))[z, z^{-1}] \) which satisfy the corresponding axioms).
3.3. **Affine vertex superalgebras.** Let \( \mathfrak{g} \) be a finite-dimensional Kac-Moody superalgebra and let \( \mathfrak{g} = \mathfrak{g}^{(1)} \).

By [FZ], the vacuum module \( \text{Vac}^k(\mathfrak{g}) \) has a structure of a vertex superalgebra with
\[
Y(at^{-1}|0), z) = \sum_{n \in \mathbb{Z}} (at^n)z^{-n-1} \quad \text{for } a \in \mathfrak{g}.
\]
We denote this vertex superalgebra by \( V^k(\mathfrak{g}) \).

The weak \( V^k(\mathfrak{g}) \)-modules are the restricted \( [\mathfrak{g}, \mathfrak{g}] \)-module of level \( k \) (\( M \) is ”restricted” if for each \( v \in M \) one has \( \langle \mathfrak{g} t^s | v = 0 \) for \( s >> 0 \)).

The above correspondence between \( V^k(\mathfrak{g}) \)-modules and \([\mathfrak{g}, \mathfrak{g}] \)-modules implies that the maximal proper submodule \( I(k) \) of \( \text{Vac}^k(\mathfrak{g}) \) is the maximal ideal in the vertex algebra \( V^k(\mathfrak{g}) \). Moreover, if \( I(k) \) is generated by \( E \) as a \( \mathfrak{g} \)-module, then \( I(k) \) is generated by \( E \) as an ideal in \( V^k(\mathfrak{g}) \). In particular, \( L(k\Lambda_0) \) inherits a structure of a vertex superalgebra, which is simple; it is denoted by \( V^k(\mathfrak{g}) \).

For \( \mathfrak{g} = 0 \) or \( \mathfrak{g} = \mathbb{C} \), the vacuum module \( \text{Vac}^k(\mathfrak{g}) \) is one-dimensional and \( V^k(\mathfrak{g}) = V_k(\mathfrak{g}) \) is a one-dimensional vertex algebra.

If \( \mathfrak{g} \) is a Lie algebra, then for \( k \in \mathbb{Z}_{\geq 0} \) the \( V_{\mathfrak{g}}(\mathfrak{g}) \)-modules correspond to the restricted integrable \( \mathfrak{g} \)-modules of level \( k \), see [DLM], Thm. 3.7; these modules are completely reducible and the irreducible modules are the integrable highest weight modules of level \( k \) (there are infinitely many such modules and \( V^k(\mathfrak{g}) \) is a rational vertex algebra). The following result was proven in [GS], Thm. 6.3.1 for \( \mathfrak{g}^\# \neq D_2 \).

3.3.1. **Theorem.** If \( L(k\Lambda_0) \) are integrable, then \( V_{\mathfrak{g}}^k(\mathfrak{g}) \)-modules are the restricted \( [\mathfrak{g}, \mathfrak{g}] \)-module of level \( k \) which are \([\mathfrak{g}^\#, \mathfrak{g}^\#] \)-integrable.

**Proof.** Let \( I(k) \) be the maximal proper submodule of \( \text{Vac}^k \), so \( L(k\Lambda_0) = \text{Vac}^k/I(k) \).

For \( \mathfrak{g}^\# \neq D_2 \) consider the natural embedding \( \text{Vac}^k(\mathfrak{g}^\#) \subset \text{Vac}^k \) and denote by \( I^\# \) the maximal proper submodule of \( \text{Vac}^k(\mathfrak{g}^\#) \).

If \( \mathfrak{g}^\# = D_2 = A_1 \times A_1 \) consider the natural embeddings \( \text{Vac}^k(A_1)', \text{Vac}^k(A_1)'' \subset \text{Vac}^k \) which correspond to two copies of \( A_1 \) in \( D_2 \); let \( I', I'' \) be the maximal proper submodules in \( \text{Vac}^k(A_1)', \text{Vac}^k(A_1)'' \) respectively. Set \( I^\# := I' + I'' \).

By §21 the submodule \( I(k) \) is generated by \( I^\# \). By above, a restricted \( [\mathfrak{g}, \mathfrak{g}] \)-module \( N \) of level \( k \) is \( V_k(\mathfrak{g}) \)-module if and only if it is annihilated by \( a_{(m)} \) for each \( a \in I^\# \), \( m \in \mathbb{Z} \). Since \( \mathfrak{g}^\# \) is a Lie algebra, \( N \) is annihilated by \( a_{(m)} \) for each \( a \in I^\# \), \( m \in \mathbb{Z} \) if and only if \( N \) is \( \mathfrak{g}^\# \)-integrable ([DLM], Thm. 3.7). \( \square \)

3.4. **DS\(_x\)** for affine vertex algebras. Fix \( x \in \mathfrak{g}_\tau \) satisfying \([x, x] = 0\). View
\[
x' := xt^{-1}|0)
\]
as a vector in \( V^k(\mathfrak{g}) \) and \( V_k(\mathfrak{g}) \) respectively. Note that \( x'(0) = x \).
One has \( x'(0) x' = x(x t^{-1}|0) = 0 \). The vertex algebras \( V^k(\mathfrak{g}) \), \( V_k(\mathfrak{g}) \) are \( \mathbb{Z}_{\geq 0} \)-graded (the grading is given by the action of \( -d \in \mathfrak{g} \)) and the zero component is spanned by \( |0\). Hence \( x' \) satisfies (10).

Consider the vertex algebras \( DS_x'(V^k(\mathfrak{g})) \). \( DS_x'(V^k(\mathfrak{g})) \).

It is easy to see that \( DS_x(Vac^k(\mathfrak{g})) \) is canonically isomorphic to \( V^k(\mathfrak{g})_x \) as a \( \mathfrak{g}_x \)-module. Choose a vacuum vector \( |0\) \ in \( V^k(\mathfrak{g}) \) and let the vacuum vector \( |0\)' \ in \( V^k(\mathfrak{g})_x \) be the image of \( |0\).

3.4.1. \textbf{Theorem.} \ Let \( \mathfrak{g} \) be an affine (non-twisted) Lie superalgebra and let \( x \in \hat{\mathfrak{g}}_r \) be such that \( [x, x] = 0 \); set \( x' := x t^{-1}|0\).

(i) The canonical isomorphism \( DS_x(\text{Vac}^k(\mathfrak{g})) \xrightarrow{\sim} \text{Vac}^k(\mathfrak{g}_x) \) induces a vertex algebra isomorphism \( DS_x'(V^k(\mathfrak{g})) \xrightarrow{\sim} V^k(\mathfrak{g}_x) \).

(ii) If \( DS_x(L_q(k\Lambda_0)) \cong L_{\mathfrak{g}_x}(k\Lambda_0) \), then \( \iota \) induces the vertex algebra isomorphism \( DS_x'(V^k(\mathfrak{g})) \xrightarrow{\sim} V^k(\mathfrak{g}_x) \).

\textit{Proof.} By above, \( \iota \) is an isomorphism of \( \mathfrak{g}_x \)-modules and \( \iota(|0\) \) \( \) \( |0\)' \). If \( V^k(\mathfrak{g}_x) \) is one-dimensional, this implies (i) and (ii). Assume that \( V^k(\mathfrak{g}_x) \) is not one-dimensional. Then \( \Delta(\mathfrak{g}_x) \neq \emptyset \). Since \( \iota \) is an isomorphism of \( \mathfrak{g}_x \)-modules,

\[
\iota(at^{-1}|0\) \) = \( (DS_x(a)t^{-1})|0\) \),

for each \( a \in \mathfrak{g} \) such that \( [x, a] = 0 \). By (11) we obtain

\[
Y(\iota(v), z) = \iota(Y(v, z))
\]

for each \( v = bt^{-1}|0\) \) with \( b \in \hat{\mathfrak{g}}_x \).

Let \( V \) be a vertex algebra and \( E \) be a subspace of \( V \). Denote by \( \langle E \rangle \) the smallest subspace \( V' \) of \( V \) which contains \( E \) and such that \( b_{ij} v \in \langle E \rangle \) for each \( b, v \in V' \) and \( j \in \mathbb{Z} \). The locality axiom and (8) imply that if \( V \) admits two vertex algebra structures \( (|0\), \( T, Y \) \) and \( (|0\), \( T', Y' \) \) such that \( Y(v, z) = Y'(v, z) \) for each \( v \in E \), then these structures coincide on \( \langle E \rangle \) (i.e., \( TV = T'v \) and \( Y(v, z) = Y'(v, z) \) for each \( v \in \langle E \rangle \)).

Now let \( E \subset V^k(\mathfrak{g}_x) \) (resp., \( E \subset V^k(\mathfrak{g}_x) \)) be the span of \( |0\) \) and \( bt^{-1}|0\) \) with \( b \in \hat{\mathfrak{g}}_x \). Since \( V^k(\mathfrak{g}_x) \) and \( V_k(\mathfrak{g}_x) \) are generated by \( |0\) \) as a \( [\mathfrak{g}_x, \mathfrak{g}_x] \)-modules, \( V^k(\mathfrak{g}_x) = \langle E \rangle \) (resp., \( V^k(\mathfrak{g}_x) = \langle E \rangle \)). Thus \( \iota \) is an isomorphism of the vertex algebras. \hfill \Box

Using Theorem 2.22 we obtain the

3.4.2. \textbf{Corollary.} \ If \( k \) is a non-negative integer, \( x \) has a maximal rank and \( \hat{\mathfrak{g}} \) differ from \( D(n + 1|n), D(2|1, a) \), then \( DS_x'(V^k(\mathfrak{g})) \) and \( V_k(\mathfrak{g}_x) \) are isomorphic as vertex algebras.
3.4.3. Take $x$ as above. Let $M$ be a weak $V^k(g_x)$-module which we view as a restricted $g_x$-modules of level $k$. One readily sees that, as a $g_x$-module, the $DS_x'(V^k(g))$-module $DS_x'(M)$ is $DS_x(M)$, so $DS_x'$ for $V^k(g)$-modules correspond to $DS_x$ for $[g, g]$-modules. We will denote the functor $DS_x'$ by $DS_x$.

3.4.4. Corollary. Let $L(k\Lambda_0)$ be integrable. For any $V^k(g)$-module $M$ the $V^k(g_x)$-module $DS_x(M)$ is a $V^k(g_x)$-module.

Proof. By Theorem 3.3.1 $M$ is $[g^\#, g^\#]$-integrable. Note that $g_x^\#$ is the image of $g^\# \cap \ker g^x$ in $g_x = \ker g^x / \text{Im} g$. Therefore $DS_x(M)$ is $[g^\#, g_x^\#]$-integrable, so $DS_x(M)$ is a $V^k(g_x)$-module by Theorem 3.3.1.

4. Principal admissible vacuum modules

In this section we define admissible weights for affine Lie superalgebras and prove Theorem 4.4.2.

4.1. Affine Lie algebra case. Let $\mathfrak{i}$ be a finite-dimensional simple Lie algebra; let $\mathfrak{t} = \mathfrak{i}^{(1)}$ be the corresponding affine Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We denote by $\Delta_{re}$ the set of real roots of $\mathfrak{t}$.

4.1.1. For a non-critical weight $\lambda \in \mathfrak{h}^*$ the set of $\lambda$-integral real roots is defined as

$$\Delta_{re}(\lambda) = \{\alpha \in \Delta_{re} \mid \frac{2(\lambda + \rho, \alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\}.$$ 

For our purposes we consider only $\lambda$s where $\mathbb{C}\Delta_{re}(\lambda) = \mathbb{C}\Delta_{re}$. In this case $\Delta_{re}(\lambda)$ is the set of real roots of an affine Lie algebra algebra $\mathfrak{i}$ with the same Cartan algebra $\mathfrak{h}$ and the triangular decomposition induced by the triangular decomposition of $\mathfrak{t}$, i.e.

$$\Delta_{re}(\lambda)^+ := \Delta_{re}(\lambda) \cap \Delta^+.$$ 

We denote by $\rho, \overline{\rho}$ the Weyl vectors of $\mathfrak{t}, \mathfrak{i}$ respectively. The character of $L_\rho(\lambda)$ and the character of the highest weight $\tau$-module $L_\tau(\lambda + \rho - \overline{\rho})$ are related by the following formula:

$$Re^\rho \text{ch} L_\rho(\lambda) = \overline{Re}^\tau \text{ch} L_\tau(\lambda + \rho - \overline{\rho}),$$

where $R, \overline{R}$ stand for the respective Weyl denominators (see [KT1], [KT2] and references there).
4.1.2. Admissible weights. A non-critical weight $\lambda \in \mathfrak{h}^*$ is called admissible if $C_{\Delta_{re}}(\lambda) = C_{\Delta_{re}}$ and $L_{\mathfrak{g}}(\lambda + \rho - \overline{\rho})$ is an integrable $\mathfrak{g}$-module.

If $\lambda$ is admissible, then $\text{ch} L_{\mathfrak{g}}(\lambda + \rho - \overline{\rho})$ is given by the Weyl-Kac character formula and $\text{ch} L(\lambda)$, suitably normalized, is a ratio of theta functions, which is a modular function, see [KW1], [KW2]. The admissible weights were classified in [KW2]. An admissible weight $\lambda$ (and a module $L_{\mathfrak{g}}(\lambda)$) is called principal admissible if $\Delta_{re}(\lambda) \sim \Delta_{re}$, that is $\mathfrak{t} \cong \mathfrak{t}$; the principal admissible weights were classified in [KW5].

4.1.3. Principal admissible levels. A level $k$ is called principal admissible if $k\Lambda_0$ is principal admissible.

It is easy to see that $k$ is principal admissible if and only if

$$k + h^\vee = \frac{p + h^\vee}{u},$$

where $p, u \in \mathbb{Z}_{\geq 0}, u > 0 \ (p + h^\vee, u) = (u, r^\vee) = 1$,

where $r^\vee$ is the lacity of $\hat{\mathfrak{t}}$ (see the definition below in 4.2.1).

4.1.4. The following Adamović-Milas conjecture [AM] was proven by T. Arakawa in [A].

Theorem, Arakawa, 2014.

Let $k$ be an admissible level for an affine Lie algebra $\mathfrak{t}$. The $V_k(\mathfrak{t})$-modules in the category $\mathcal{O}$ are completely reducible and the irreducible modules are $L_{\mathfrak{g}}(\lambda)$, where $\lambda$ are the principal admissible weights of level $k$.

4.2. Admissibility for affine Lie superalgebras. Let $\mathfrak{g}$ be a (non-twisted) affine Lie superalgebra.

4.2.1. Lacity. Let $\hat{\mathfrak{g}} \neq D(2|1, a)$. We call $\alpha \in \Delta$ (resp., $\alpha \in \hat{\Delta}$) a short root if $|\langle \alpha, \alpha \rangle|$ takes the smallest non-zero value. We define the lacity for $\mathfrak{g}$ and for $\hat{\mathfrak{g}}$ as

$$r^\vee = \frac{2}{|\langle \alpha, \alpha \rangle|},$$

where $\alpha$ is a short root. Observe that the lacities for $\Delta$ and for $\hat{\Delta}$ are equal. Moreover, this lacity is equal to the lacity of $\hat{\mathfrak{g}}^\#$ if $\hat{\mathfrak{g}} \neq B(0|n)$; for $\hat{\mathfrak{g}} = B(0|n)$ one has $r^\vee = 4$.

The set $\Delta_{re}(\lambda)$ was introduced in [GK]. As for Lie algebra case, we define the admissible weights as follows.

4.2.2. Definitions. A non-critical weight $\lambda \in \mathfrak{h}^*$ is admissible if $C_{\Delta_{re}}(\lambda) = C_{\Delta_{re}}$ and $L_{\mathfrak{g}}(\lambda + \rho - \overline{\rho})$ is an integrable $\mathfrak{g}$-module.

An admissible weight $\lambda$ is called principal admissible if $\Delta_{re}(\lambda) \cong \Delta_{re}$.

We say that $k$ is an admissible (resp., principal admissible) level if $k\Lambda_0$ is admissible (resp., principal admissible).
By [GK], Thm. 11.2.3, \( \text{ch} L(k\Lambda_0) \) is given by (13) if \( k \) is admissible.

4.3. **Principal admissible levels.** It is not hard to show that for \( \hat{\mathfrak{g}} \neq D(2|1, a) \) the level \( k \) is principal admissible if and only if

\[
k + h^\vee = \frac{p + h^\vee}{u}, \quad \text{where } p, u \in \mathbb{Z}_{\geq 0}, u > 0 \quad (r^\vee(p + h^\vee), u) = 1,
\]

where \( r^\vee \) is the lacity of \( \hat{\mathfrak{g}} \). Note that \( r^\vee(p + h^\vee) \) is integral: \( h^\vee \) is integral for \( \hat{\mathfrak{g}} \neq B(m|n), m \leq n \), and \( h^\vee = n - m + \frac{1}{2} \) for \( \hat{\mathfrak{g}} = B(m|n), m \leq n \).

Let \( k \) be a principal admissible level. Then \( \Delta_{re}(\hat{\mathfrak{g}}) = \hat{\Sigma} + Zu\delta \), where \( u \) is as above and the formula (13) takes the form

\[
(14) \quad Re^\rho \text{ch} L_{\hat{\mathfrak{g}}}(k\Lambda_0) = \overline{Re^\rho \text{ch} L_{\mathfrak{g}}(p\hat{\Lambda}_0)}.
\]

Note that \( \Delta_{re}(\hat{\mathfrak{g}}) \cap \Delta^+ \) has the base \( \hat{\Sigma} \cup \{\alpha'_0\} \), where

\[
\alpha'_0 = (u - 1)\delta + \alpha_0,
\]

where \( \Sigma = \hat{\Sigma} \cup \{\alpha_0\} \).

Recall that for \( x \in \hat{\mathfrak{g}} \) such that \( \hat{\Delta}_x \) is non-empty, \( \hat{\mathfrak{g}} \) and \( \text{DS}_x(\hat{\mathfrak{g}}) \) have the same dual Coxeter numbers. If, in addition, \( \hat{\Delta}_x \) has rank more than one, then \( \hat{\mathfrak{g}} \) and \( \text{DS}_x(\hat{\mathfrak{g}}) \) have the same lacity \( r^\vee \), so the principal admissible levels for \( \mathfrak{g} \) and \( \text{DS}_x(\hat{\mathfrak{g}}) \) coincide. If \( \hat{\Delta}_x \) has rank one, then the lacity of \( \hat{\mathfrak{g}} \) is 1 for \( A(n \pm 1|n) \) and 2 for other cases, whereas the lacity of \( \text{DS}_x(\hat{\mathfrak{g}}) \) is 1; hence each principal admissible levels for \( \mathfrak{g} \) is principle admissible for \( \text{DS}_x(\hat{\mathfrak{g}}) \).

4.4. **Vacuum modules for principal admissible levels.** Retain notation of § 4.3

4.4.1. Take \( x \in \hat{\mathfrak{g}} \) satisfying (1) such that \( x \) has a maximal rank, i.e.

\[
i := \text{DS}_x(\hat{\mathfrak{g}})
\]

has zero defect. We denote by \( I_i(k) \) the maximal proper submodule of \( \text{Vac}_i^k \). Let \( k \) be an admissible level for \( \mathfrak{t} \). The vacuum module \( \text{Vac}_i^k \) has a singular vector of weight \( r'_0, k\Lambda_0 \), where

\[
r'_0 := r_{\alpha'_0} \in W.
\]

From [F] it follows that in the case when \( \hat{i} \) is a simple Lie algebra, this singular vector generates \( I_i(k) \).
4.4.2. **Theorem.** Let $\mathfrak{g} \neq B(n+1|n)$ be a finite-dimensional Kac-Moody algebra and let $k$ be a principal admissible level. Let $x \in \mathfrak{t}$ be of the maximal rank.

Assume that $t := g_x$ satisfies the following: $t$ is simple,

(A1) $I_k(k)$ is generated by a singular vector of weight $r_0^\prime k\Lambda_0$;

(A2) any irreducible $V_k(t)$-module in the category $O$ is principal admissible.

Then

(i) $DS_x(L(k\Lambda_0)) \cong L_{g_x}(k\Lambda_0)$ as $g_x$-modules;

(ii) $DS_x(V_k(g)) \cong V_k(g_x)$ as vertex algebras;

(iii) for any $V_k(g)$-module $N$, $DS_x(N)$ is a $V_k(g_x)$-module;

(iv) if $N$ is a $V_k(g)$-module in $O$, then $DS_x(N)$ is either zero or the direct sum of principal admissible modules of level $k$.

4.5. **Proof of Theorem 4.4.2.** Note that $\mathfrak{g} \neq A(m|n), B(m|n), D(m|n)$ with $m = n, n+1$ and $D(n+2|n)$. Using Remark 2.1.2 we assume for that $S, \hat{\Sigma}$ satisfies (P1), (P2), (P3) of §5, i.e.

$$S \subset \hat{\Sigma}, \quad (S, \theta) = 0, \quad ||\theta||^2 = 2,$$

where $\theta$ is the maximal root in $\Delta^+(\hat{\Sigma})$, and (20) holds.

In particular, $\alpha_0 := \delta - \theta$ is the affine root for $\mathfrak{g}$ and for $t$.

Since $L(k\Lambda_0)$ is $\mathfrak{g}$-integrable, we can (and will) assume that $\text{supp}(x) = S$.

We fix the $\mathbb{Z}_2$-grading on $Vac^k$ and all its subquotients by letting the highest weight vector to be even. For a $\mathbb{Z}_2$-graded space $E$ we write $\dim E = (a|b)$ if $\dim E_0^\tau = a, \dim E_1^\tau = b$. Retain notation of §4.3.

4.5.1. Denote by $I(k)$ the maximal submodule of $Vac^k$ and by $I_\mathfrak{g}(p)$ the maximal submodule of the vacuum $\mathfrak{g}$-module $Vac^0_\mathfrak{g}$. One has

$$R e^{-k\Lambda_0}ch I(k) = R e^{-k\Lambda_0}(chVac^k - chL(k\Lambda_0)) = \hat{R} - R e^{-k\Lambda_0}chL(k\Lambda_0),$$

where $R, \hat{R}$ are the Weyl denominators for $\Delta^+, \hat{\Delta}^+, \hat{\Delta}^+$ respectively; recall that $\hat{\Delta} \subset \Delta$, so $\hat{R} = \hat{R} = R$.

From (14) we have $Re^{-k\Lambda_0}ch L(k\Lambda_0) = \mathfrak{T} e^{-p\Lambda_0}ch L(p\Lambda_0)$. This gives

$$Re^{-k\Lambda_0}ch I(k) = \hat{R} - \mathfrak{T} e^{-p\Lambda_0}ch L(p\Lambda_0) = R e^{-p\Lambda_0}ch I_\mathfrak{g}(p).$$

By §2.1 $I_\mathfrak{g}(p)$ is generated by a singular vector $v'$ of the weight

$$\mu := p\Lambda_0 - (p + 1)\alpha'_0.$$
Now the formula (15) can be rewritten as

\[ Re^{-r_0',k\Lambda_0}chI(k) = \overline{Re^{-\mu}chI_\Phi(p)} \]

since 

\[ k\Lambda_0 + \mu - p\Lambda_0 = k\Lambda_0 - (p + 1)\alpha'_0 = r_0'(k\Lambda_0). \]

Recall that \( v' = f_0^{p+1}|0\), where \( f_0 \in g - \alpha'_0 \). For any \( \beta \in S \) we have \( (\alpha'_0, \beta) = 0 \), so 

\[ [g - \alpha'_0, g_{\pm}] = 0. \]

Therefore \( g_{\pm}v' = 0 \) and so 

\[ \dim I_\Phi(p\Lambda_0)_\mu = (1|0), \quad I_\Phi(p\Lambda_0)_{\mu - \gamma} = 0 \quad \text{for} \ \gamma \in \mathbb{Z}S \setminus \{0\}. \]

4.5.2. Set 

\[ \mathcal{R} := \{ \sum_{\nu \in \mathbb{Z}_{\geq 0}\Sigma} a_\nu e^{-\nu} | a_\nu \in \mathbb{C} \}, \quad P_S(\sum_{\nu \in \mathbb{Z}_{\geq 0}\Sigma} a_\nu e^{-\nu}) := \sum_{\nu \in \mathbb{Z}_{\geq 0}S} a_\nu e^{-\nu}. \]

Clearly, \( \mathcal{R} \) has a ring structure; this ring does not have zero divisors. Note that \( P_S \) is a ring homomorphism (since \( S \subset \Sigma \)) and \( P_S^2 = P_S \).

The ring \( \mathcal{R} \) contains \( R, \hat{R}, \overline{R}, R^{-1}, \overline{R}^{-1} \) and 

\[ P_S(R) = P_S(\hat{R}) = P_S(\overline{R}). \]

Since \( I_\Phi(p) \) is generated by a singular vector of weight \( \mu \), one has \( e^{-\mu}chI_\Phi(p) \in \mathcal{R} \). By (17), \( P_S(e^{-\mu}chI_\Phi(p)) = 1 \). Using (16) we get 

\[ e^{-r_0',k\Lambda_0}chI(k) \in \mathcal{R}, \quad P_S(e^{-r_0',k\Lambda_0}chI(k)) = 1, \]

By (16), \( r_0',k\Lambda_0 \) is the highest weight of \( I(k) \) and \( \dim I(k)r_0',k\Lambda_0 = 1 \). Thus \( I(k)r_0',k\Lambda_0 \) is spanned by an even singular vector. By (15), this vector has non-zero image in \( DS_x(I(k)) \)

\( DS_x(I(k))r_0',k\Lambda_0 \) is spanned by this image.

We conclude that \( DS_x(I(k))r_0',k\Lambda_0 \) is spanned by an even singular vector, which we denote by \( v_0 \).

4.5.3. Recall that \( DS_x(Vac^k) = Vac^k \). Consider the short exact sequence 

\[ 0 \rightarrow I(k) \rightarrow Vack \rightarrow Vac_k \rightarrow 0 \]

and the corresponding long exact sequence 

\[ 0 \rightarrow E \rightarrow DS_x(I(k)) \xrightarrow{\phi} Vac^k \xrightarrow{\psi} DS_x(Vac_k) \rightarrow \Pi(E) \rightarrow 0. \]

By (A1), \( I_t(k) \) is generated by a singular vector \( v'_0 \) of weight \( r_0',k\Lambda_0 \). Since \( v_0, v'_0 \) are singular, \( \phi(v_0) \) is proportional to \( v'_0 \). There are two possibilities: either \( \phi(v_0) = v'_0 \) (up to a non-zero scalar) or \( \phi(v_0) = 0 \).
Assume that $\phi(v_0) = 0$. Since $v_0$ spans $\mathbb{D}_L(I(k))_{r_0,k \Lambda_0}$ one has $v'_0 \not\in \text{Im} \phi = \text{Ker} \psi$. Since $v_0 \in \text{Ker} \phi$ one has

$$\dim \mathbb{D}_L(I(k))_{r_0,k \Lambda_0} = \dim \mathbb{D}_L(I(k))_{r'_0,k \Lambda_0} = (1|0).$$

Since $v'_0 \not\in \text{Ker} \psi$, the $t$-module $\mathbb{D}_L(Vac_k)$ has an even indecomposable subquotient of length two with the socle $L_1(r'_0,k \Lambda_0)$ and the cosocle $L_1(k \Lambda_0)$. Since $Vac_k \cong L(k \Lambda_0)$ is self-dual, $\mathbb{D}_L(Vac_k)$ is also self-dual (see § 3.1.3); thus $\mathbb{D}_L(Vac_k)$ has an even indecomposable subquotient of length two with the cosocle $L_1(r'_0,k \Lambda_0)$ and the socle $L_1(k \Lambda_0)$. Since $I_1(k)$ is generated by a singular vector of weight $r'_0,k \Lambda_0$, one has $[Vac_k^k : L_1(r'_0,k \Lambda_0)] = 1$, so $\text{Im} \psi$ does not have such subquotient. Then $\Pi(E)$ has an even subquotient $L_1(r'_0,k \Lambda_0)$, which contradicts to (19).

We conclude that $\phi(v_0) = v'_0$ up to a non-zero scalar. Denote by $a$ a preimage of $v_0$ in $I(k) \subset Vac^k$. Let $N$ be a $V_k(g)$-module and $\mathbb{D}_L(N) \neq 0$. Since $Vac_k = Vac^k/I(k)$, § 3.1.3 gives $Y(a,z)N = 0$, so $Y(v_0,z)$ $\mathbb{D}_L(N) = 0$. Since $Vac_{k,k} = Vac_k^k/I(k)$ with $I(k)$ generated by $v'_0$, § 3.1.3 implies that $\mathbb{D}_L(N)$ is a $V_k(t)$-module. This establishes (iii).

Let us prove that $\mathbb{D}_L(Vac_k) = L_1(k \Lambda_0)$. Clearly, $[\mathbb{D}_L(Vac_k) : L_1(k \Lambda_0)] = 1$. Let $L_1(\lambda'')$ be a subquotient of $\mathbb{D}_L(Vac_k)$ and $\lambda'' \neq k \Lambda_0$. By (iii) and (A2), $\lambda''$ is a $t$-admissible weight. One has $\lambda'' = k \Lambda_0 - (\nu|_{\Sigma_S})$ for some $\nu \in (\mathbb{Z}_{\geq 0} \Sigma \cap S^\perp)$. Recall that $(S, \Sigma)$ satisfies (20), so $\nu|_{\Sigma_S} \in Z \Sigma_S$, which contradicts to Lemma 3.1.4. This gives (i); (ii) follows from Theorem 3.1.1 (ii). \hfill \Box

4.6. **Corollary.** Let $\hat{g}$ is one of the following algebras: $A(m|n), C(n); B(m|n), m \geq n + 2; D(m|n), m \neq n + 1, n + 2, B(n|m), F(4)$ or $G(2)$. Take $x \in \hat{g}_x$ such that $\text{supp}(x)$ is maximal. Let $k$ be an admissible level. Then

(i) $\mathbb{D}_L(k \Lambda_0) \cong L_{\hat{g}_x}(k \Lambda_0)$ as $\hat{g}_x$-modules;

(ii) $\mathbb{D}_L(V_k(g)) \cong V_k(\hat{g}_x)$ as vertex algebras;

(iii) for any $V_k(g)$-module $N$, $\mathbb{D}_L(N)$ is a $V_k(\hat{g}_x)$-module;

(iv) if $N$ is a $V_k(g)$-module in $\mathcal{O}$, then $\mathbb{D}_L(N)$ is either zero or the direct sum of principal modules of level $k$.

**Proof.** For $\hat{g}_x \neq 0$, the assumption (A1) of Theorem 4.4.2 follows from $[E]$ and the assumption (A2) follows from Theorem 4.1.3. This gives (i)–(iii) for $\hat{g}_x \neq 0$; (iv) follows from (iii) and Theorem 3.1.4.

If $\hat{g}_x = 0$, then (i) is a particular case of Theorem 2.2 (i); moreover, (ii)-(iv) follow from (i). \hfill \Box

4.7. The following lemma was used in the proof.

**Lemma.** Let $\hat{t}$ has zero defect and let $k$ be a principal admissible level. If $\lambda$ is an admissible weight such that $k \Lambda_0 - \lambda \in \mathbb{Z} \Sigma$, then $\lambda = k \Lambda_0$. 

Proof. Since $k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$ one has $\Delta_{re}(\lambda) = \Delta_{re}(k\Lambda_0)$. Set

$$\lambda' := \lambda + (p + h')(1 - \frac{1}{\alpha})\Lambda_0.$$ 

One readily sees that $\lambda'$ is a dominant weight of level $p$. One has $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$. Since $\lambda'$ is dominant,

$$0 \leq (\Lambda_0, p\Lambda_0 - \lambda') = p(\Lambda_0, \lambda') - (\lambda', \lambda').$$

Therefore $(\lambda', \lambda') = (\Lambda_0, \lambda) = 0$. Since $p\Lambda_0 - \lambda' = k\Lambda_0 - \lambda \in \mathbb{Z}\Sigma$, we obtain $k\Lambda_0 - \lambda = 0$ as required.

\section{Appendix}

We fix the standard triangular decomposition in $\hat{\Delta}_\Sigma$ and consider the bases $\hat{\Sigma}$ which are compatible with this decomposition. For each base $\hat{\Sigma}$ denote by $\theta_{\hat{\Sigma}}$ the maximal root of $\Delta + (\hat{\Sigma})$. Let $\mathcal{S}$ be the set of maximal isotropic subsets of $\Delta_{\mathfrak{g}}$. Consider the action of the Weyl group $\hat{\mathfrak{W}}$ on $\mathcal{S}$. For each orbit it is not hard to give an example of a pair $(S, \hat{\Sigma})$ such that

- (P1) $S \subset \hat{\Sigma}$;
- (P2) $\theta_{\hat{\Sigma}} \in \hat{\Delta}^\#$ and $(\theta_{\hat{\Sigma}}, S) = 0$ for $\hat{\mathfrak{g}} \neq A(m|n), B(m|n), D(m|n)$ with $m = n, n + 1$;
- (P3) if $\hat{\mathfrak{g}} \neq D(n + 1, n), D(n + 2|n)$, then the following inclusion holds

\begin{equation}
(Q_{\geq 0} \Sigma \cap S^\perp) \subset (Q S + Q_{\geq 0} \Sigma S),
\end{equation}

where $\Sigma = \{\delta - \theta_{\hat{\Sigma}}\} \cup \hat{\Sigma}$ is a base for $\Delta = \hat{\Delta}^{(1)}$.

For instance, for $B(m, n), D(m, n), n > m$ one has $\hat{\Delta}^\# = C_n$. We take $S := \{\epsilon_i - \delta_i + 1\}_{i=1}^m$ and

$$\hat{\Sigma} := \{\delta_1 - \epsilon_1, \epsilon_1 - \delta_2, \ldots, \epsilon_m - \delta_{m+1}, \delta_m + 1 - \delta_{m+2}, \ldots, \delta_n - 1 - \delta_n, a\delta_n\},$$

where $a = 1$ for $B(m|n)$ and $a = 2$ for $D(m|n)$. One has $\theta = 2\delta_1$, so (P1), (P2) are satisfied. One has

$$\hat{\Sigma}_S = \{\delta_1 - \delta_{m+2}, \ldots, \delta_n - \delta_n, a\delta_n\},$$

of type $B(0|n - m)$ for $B(m|n)$ and $C_{n-m}$ for $D(m|n)$; it is easy to see that [20] holds.

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