Multilateral basic hypergeometric summation identities and hyperoctahedral group symmetries

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Abstract

We give new proofs for certain bilateral basic hypergeometric summation formulas using the symmetries of the corresponding series. In particular, we present a proof for Bailey’s $3\psi_3$ summation formula as an application. We also prove a multiple series analogue of this identity considering hyperoctahedral group symmetries of higher ranks.

Keywords: bilateral basic hypergeometric series, hyperoctahedral group symmetries, Bailey’s $3\psi_3$ summation formula, theta functions

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1. Introduction

Let $(a; q)_\alpha$ denote the $q$–Pochhammer symbol which is formally defined by

$$(a)_\alpha = (a; q)_\alpha := \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \quad (1)$$

where the parameters $a, q, \alpha \in \mathbb{C}$, and $(a; q)_\infty$ denotes the infinite product $(a; q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i)$. Note here that when $\alpha = k$ is a positive integer, then the $q$-Pochhammer symbol reduces to $(a)_k = \prod_{i=1}^{k} (1 - aq^{i-1})$. We often use the shorthand notation $(a_1, a_2, \ldots, a_r)_\alpha$ for the product $\prod_{i=1}^{r} (a_i)_\alpha$.

The series $\sum_{k=0}^{\infty} c_k$, where the ratio $c_{k+1}/c_k$ is a rational function of $q^k$, is called a basic hypergeometric series [1]. Using the $q$–Pochhammer symbol (1), the general basic hypergeometric series with $r$ numerator parameters and $s$ denominator parameters is defined by

$$r\varphi_s \left[ \begin{array}{c} a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s \end{array} ; x, q \right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_k}{(q, b_1, b_2, \ldots, b_s)_k} x^k \left( (-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \quad (2)$$

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where we assume that none of the denominator factors vanish.

Note that if one of the numerator parameters is of the form $q^{-n}$, for some non-negative integer $n$ and $q \neq 0$, the series terminates from above since $(q^{-n})_k = 0$ when $k > n$. The denominator factor $(q)_k$ terminates the series from below, that is the factor $1/(q)_n$ causes the sum to vanish when $n < 0$.

In general, when dealing with non-terminating series it is assumed for convergence that $|q| < 1$. In that case, the series $r+1\varphi_r$ converges absolutely for $|x| < 1$.

When $r = s + 1$, the basic hypergeometric series (2) is called well-poised if the parameters satisfy the relation

$$qa_1 = a_2b_1 = a_3b_2 = \ldots = a_{s+1}b_s,$$

and very well-poised if, in addition, $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$. An $r+1\varphi_r$ series is called $k$-balanced if $b_1 \ldots b_r = q^k a_1 \ldots a_{s+1}$, and $x = q$.

There are numerous classical one-dimensional results, summation and transformation formulas for basic hypergeometric series. One of the most general summation formulas, for example, is the $(q$–Dougall or) Jackson sum

$$8\varphi_7 \left[ a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n} \left| a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1}; q, q \right. \right] = \frac{(aq, aq/bc, aq/bd, aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n}$$

where $qa^2 = bcdeq^{-n}$. An important general transformation formula is Bailey’s $10\varphi_9$ transformation

$$10\varphi_9 \left[ a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n} \left| a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f, aq^{n+1}/ef, q^{-n}/\lambda, aq^{n+1}; q, q \right. \right] = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q/ef)_\infty} \cdot 10\varphi_9 \left[ \lambda, q\lambda^{1/2}, -q\lambda^{1/2}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n} \left| \lambda^{1/2}, -\lambda^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f, aq^{n+1}/a, \lambda q^{n+1}; q, e \right. \right]$$

where $\lambda = qa^2/bcd$.

The basic hypergeometric series (2) is Heine’s generalization of the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{\{a_1\}_n \{a_2\}_n \ldots \{a_r\}_n}{n! \{b_1\}_n \{b_2\}_n \ldots \{b_s\}_n} x^n$$

where $\{a\}_n$ denotes the shifted factorial (or Pochhammer symbol) defined by

$$\{a\}_0 := 1, \quad \{a\}_n := a(a+1)\ldots(a+n-1) \text{ for } n \in \mathbb{Z}_>.$$

The basic hypergeometric series (2) reduces to (5) if we replace parameters $a_i$ and $b_i$ by $qa_i$ and $qb_i$ in (2) respectively, and let $q \to 1$. 

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The basic hypergeometric series are further generalized in the literature in several directions. Bilateral basic hypergeometric series is a generalization where the index of summation is no longer restricted to non-negative integers, but it runs over all integers. The most general result of this type is Bailey’s $\psi_6$ summation formula which can be written as

\[
\psi_6\left[\begin{array} {c} qa^{1/2}, \ldots, -qa^{1/2}, b, \ldots, c, d, e \\ a^{1/2}, \ldots, -a^{1/2}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}; \frac{qa^2}{bcde} \end{array}\right] = \frac{(aq, \frac{aq}{be}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, q/a)_{\infty}}{(aq/b, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, q/b, q/c, q/d, q/e, \frac{qa^2}{bcde})_{\infty}}
\]

provided that $|qa^2/bcde| < 1$, where

\[
\psi_5\left[\begin{array} {c} a_1, \ldots, a_r; b_1, \ldots, b_s; x, q \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_r)_n}{(b_1)_n(b_2)_n \cdots (b_s)_n} \frac{(-1)^{(s-r)n} q^{(s-r)\frac{n}{2}}}{x^n}
\]

There are other important summation formulas such as Ramanujan’s $\psi_1$ sum, and useful transformation formulas for bilateral series as well. The former identity, for example, may be written as

\[
\psi_5\left[\begin{array} {c} a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; x, q \end{array}\right] = \frac{(a_1)_n(a_2)_n \cdots (a_r)_n}{(b_1)_n(b_2)_n \cdots (b_s)_n} \frac{(-1)^{(s-r)n} q^{(s-r)\frac{n}{2}}}{x^n}
\]

for $|qa^2/bcde| < 1$. The elliptic $q$–Pochhammer symbol is then given by

\[
(a; q, p)_n = \prod_{k=0}^{n-1} \theta(aq^k)
\]
for $n > 0$. The definition is extended to negative $n$ by the relation $(a; q, p)_n = 1/(aq^n; q, p)_n$ analogous to the standard $q$-Pochhammer symbol. When $n = 0$, we have $(a; q, p)_0 = 1$. Note also that when $p = 0$ this reduces to standard definition of the $q$-Pochhammer symbol.

The definition of a balanced, very–well–poised elliptic basic hypergeometric series now may be written [13] as

$$r+1\omega_r(a_1; a_4, \ldots, a_{r+1}; q, p) = \sum_{k=0}^{\infty} \frac{\theta(a_1 q^{2k})}{\theta(a_1)} \frac{(a_1, a_4, \ldots, a_{r+1}; q, p)_k q^k}{(q, a_1 q/a_4, \ldots, a_1 q/a_{r+1}; q, p)_k}$$

(12)

where $(a_4 \ldots a_{r+1})^2 = a_1^{-3} q^{-5}$. By defining the partition generalization of the elliptic $q$–Pochhammer symbol in the form

$$(a)_\lambda = (a; q, p, t)_\lambda := \prod_{k=0}^{n-1} (at^{1-i}; q, p)_\lambda_i$$

(13)

the definition of elliptic basic hypergeometric series is generalized to various root systems of rank $n$. The following shorthand notation will also be used.

$$(a_1, \ldots, a_k)_\lambda = (a_1, \ldots, a_k; q, p, t)_\lambda := (a_1)_\lambda \ldots (a_k)_\lambda.$$  

(14)

In [5] we proved a multiple elliptic analogue of the classical Jackson sum and other important results including a multiple analogue of Bailey’s $10\varphi 9$ transformation formula. The multiple elliptic Jackson sum may be written in the form

$$W_\lambda(z; q, p, t, at^{-2n}, bt^{-n})$$

$$= \frac{(s)_\lambda (as^{-1}t^{-n-1})_\lambda}{(qbs^{-1}t^{-1})_\lambda (qbt^n s/a)_\lambda} \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i+1})_{\lambda_i} (qbt^{-i-j+1})_{\lambda_i} (qbt^{-j-i})_{\lambda_i}}{(t^{j-i})_{\lambda_i} (qbt^{-i-j})_{\lambda_i}} \right\}$$

$$\cdot \sum_{\mu \subseteq \lambda} \frac{(bs^{-1}t^{-n})_\mu (qbt^n/a)_\mu}{(qt^n-1)_\mu (as^{-1}t^{-n-1})_\mu} \cdot \prod_{i=1}^{n} \left\{ \frac{(1 - bs^{-1}t^{-1-i}q^{2\mu_i})_\mu}{(1 - bs^{-1}t^{-2})} (qt^{2i-2})_{\mu_i} \right\}$$

$$\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\mu_i} (q^{t-i-j})_{\mu_i} (q^{t-j-i})_{\mu_i}}{(qt^{j-i-1})_{\mu_i} (t^{j-i-1})_{\mu_i - \mu_j} (bs^{-1}q^{t-j-i})_{\mu_i + \mu_j}} \cdot W_\mu(q^{\lambda t^{\delta(n)}}; q, t, bt^{-2n}, bs^{-1}t^{-n}) \cdot W_\mu(zs; q, t, as^{-2}t^{-2n}, bs^{-1}t^{-n})$$

(15)

where $z \in \mathbb{C}^n$ and $W_\lambda$ denotes the symmetric Macdonald function [5] that is defined as follows.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions of at most $n$ parts for a positive integer $n$ such that the skew partition $\lambda/\mu$ is a horizontal strip;
i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \lambda_n \geq \mu_n \geq \lambda_{n+1} = \mu_{n+1} = 0$. Setting

$$H_{\lambda/\mu}(q, p, t, b)$$

$$= \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{\mu_i - \mu_j - 1}t^{-i})_{\mu_j - 1 - \lambda_j}(q^{\lambda_i + \lambda_j}t^{j-i}b)_{\mu_j - 1 - \lambda_j}}{(q^{\mu_i - \mu_j - 1}t^{-i})_{\mu_j - 1 - \lambda_j}(q^{\lambda_i + \lambda_j}t^{j-i}b)_{\mu_j - 1 - \lambda_j}} \right\} \prod_{1 \leq i < (j-1) \leq n} \frac{(q^{\mu_i + \lambda_j + 1}t^{j-i}b)_{\mu_j - 1 - \lambda_j}}{(q^{\mu_i + \lambda_j}t^{j-i}b)_{\mu_j - 1 - \lambda_j}} \right\} \prod_{1 \leq i < (j-1) \leq n} \frac{(q^{\mu_i + \lambda_j + 1}t^{j-i}b)_{\mu_j - 1 - \lambda_j}}{(q^{\mu_i + \lambda_j}t^{j-i}b)_{\mu_j - 1 - \lambda_j}}$$

we define

$$W_{\lambda/\mu}(x; q, p, t, a, b) := H_{\lambda/\mu}(q, p, t, b) \left( \frac{1}{x-a} \right)_\lambda (qbt/xt, qbt/(axt))_{\mu} \left( \frac{1}{x-a} \right)_\mu (qbt/xt, qbt/(axt))_{\lambda}$$

where $q, p, t, x, a, b \in \mathbb{C}$. Note that $W_{\lambda/\mu}(x; q, p, t, a, b)$ vanishes unless $\lambda/\mu$ is a horizontal strip. The symmetric function $W_{\lambda/\mu}(y, z_1, \ldots, z_\ell; q, p, t, a, b)$ is extended to $\ell+1$ variables $y, z_1, \ldots, z_\ell \in \mathbb{C}$ through the following recursion formula

$$W_{\lambda/\mu}(y, z_1, \ldots, z_\ell; q, p, t, a, b) = \sum_{\nu \prec \lambda} W_{\lambda/\nu}(y; q, p, t, a, b) W_{\nu/\mu}(z_1, \ldots, z_\ell; q, p, t, a, b).$$

These functions generalize Macdonald polynomials and interpolation Macdonald polynomials [7, 9] and are closely related to $BC_n$ abelian functions [10].

2. Multilateral Basic Series Identities

I would like to present our bilateralization argument first in one dimensional case to help make the multiple multilateral analogues easier to read. The classical Jackson sum, that is $s\psi_T$ summation formula, for example, may be written in the form

$$\sum_{k=0}^{n} \frac{(1 - bq^{2k})}{(1 - b)} \frac{(b, q^{-n})_k}{(q, bq^{1+n})_k} \frac{(\sigma, \rho, \gamma, b^2q^{1+n}/\sigma \rho \gamma)_k}{(qb/\sigma, qb/\rho, qb/\gamma, \sigma \rho \gamma q^{-n}/b)_k} q^k$$

$$= \frac{(qb, qb/\sigma \rho, qb/\sigma \gamma, qb/\rho \gamma)_n}{(qb/\sigma, qb/\rho, qb/\gamma, qb/\sigma \rho \gamma)_n}$$

By using the definition (1) of $q$–Pochhammer symbol and the identity

$$(a)_k = \frac{(-a)^k q^{\binom{k}{2}}}{(q/a)^k}$$

$$5$$
we may flip factors and write the summand in the left hand side as

\[
\frac{q^{-z^2}}{(q^{-2z})_{\infty}} = \frac{(\sigma, \rho, \sigma q^{-2z}, \rho q^{-2z})_{\infty}}{(q^{-2z})_{\infty} (q^1-2z, q^{1+n}, q, q^{1+n+2z})_{\infty}} \cdot (\gamma, q^{1+z+1+n}/\sigma \rho \gamma, \gamma q^{-2z}, q^{1+n+2z}/\sigma \rho \gamma)_{\infty}
\]

\[
\cdot (q^{1-z-(z+k)}, q^{1+n+z-(z+k)}, q^{1-z+(z+k)}, q^{1+n+z+(z+k)})_{\infty}
\]

\[
\cdot (\sigma q^{z+(k+z)}, \rho q^{z+(k+z)}, \sigma q^{z-(k+z)}, \rho q^{z-(k+z)})_{\infty}
\]

\[
\cdot \left(\gamma q^{z+(k+z)}, q^{1+z+1+n+(z+k)}/\sigma \rho \gamma, \gamma q^{z-(k+z)}, q^{1+z+1+n-(k+z)}/\sigma \rho \gamma\right)_{\infty}
\]

(21)

where we also set \( b = q^{2z} \) for some \( z \in \mathbb{C} \). It is clear that the summand is invariant under the maps \( (z + k) \leftrightarrow w(z + k) \) for all \( w \) in the hyperoctahedral group of rank 1, namely \( \mathbb{Z}_2 \). It is clear that these maps generate full weight lattice \( \mathbb{Z} \) for the root system \( C_1 \) if \( z = m/2 \) as illustrated in [2] for Rogers–Selberg identity. However, we will only consider the case when \( m = \delta \in \{0, 1\} \), that is \( b = q^\delta \).

Recall also that the Macdonald polynomial identity [8] for the root system \( C_1 \) of rank 1 may be written as

\[
1 = \frac{1}{1 - x^2} + \frac{1}{1 - x^{-2}}
\]

(22)

By setting \( x = q^{(z+k)} \) and multiplying the sum on the left hand side by the polynomial identity and simplifying, we get

\[
\sum_{k=0}^{n} \frac{(1 - q^{\delta+2k})}{(1 - b)} \frac{(q^\delta, q^{-n})_k}{(q^{\delta+1+n})_k} \left(\frac{(\sigma, \rho, \gamma, q^{2\delta+1+n}/\sigma \rho \gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, \sigma \rho \gamma q^{-\delta-n})_k} q^k\right)
\]

\[
= \sum_{k=-n-\delta}^{n} f(\delta) \frac{(q^{-n})_k}{(q^{1+\delta})_k} \left(\frac{(\sigma, \rho, \gamma, q^{2\delta+1+n}/\sigma \rho \gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, \sigma \rho \gamma q^{-\delta-n})_k} q^k\right)
\]

(23)

where

\[
f(\delta) = \begin{cases} 
1 & \text{when } \delta = 0 \\
1/(1 - q^\delta) & \text{when } \delta = 1
\end{cases}
\]

(24)

Now we send \( n \to \infty \) applying the dominated convergence theorem for infinite series [3] to get

\[
\sum_{k=-\infty}^{\infty} f(\delta) \left(\frac{q^{(\delta+1)}}{\sigma \rho \gamma}\right)^k \left(\frac{(\sigma, \rho, \gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma)}\right)
\]

\[
= \left(\frac{q^{1+\delta}}{\sigma \rho \gamma}, q^{1+\delta}/\sigma \rho, q^{1+\delta}/\sigma \gamma, q^{1+\delta}/\rho \gamma\right)_{\infty}
\]

(25)
Here we also used the limit rule
\[
\lim_{a \to 0} a^k(x/a)_k = (-1)^k x^k q(\frac{k}{2})
\] (26)

Now, setting \(\delta = 0\) gives Bailey’s \(3\psi_3\) bilateral summation formula. The \(\delta = 1\) case appears to be a new bilateral sum.

We now give a multilateral analogue of Bailey’s \(3\psi_3\) bilateral summation formula. Recall [5] that when \(z = xt^3\) for some \(x \in \mathbb{C}\) and \(t^3 = (t^{n-1}, t^{n-2}, \ldots, t, 1)\), the multiple Jackson sum (15) may be written as

\[
\frac{(sx^{-1}, asx)_\lambda}{(qbx, qb/ax)_\lambda} = \sum_{\mu \leq \lambda} q^{\mu \cdot 2n(\mu)} \frac{(s, as)_\lambda}{(qb, qb/a)_\lambda} \frac{(bt^{1-n}, qb/as)_\mu}{(qt^{n-1}, as)_\mu}
\]

\[
\cdot \prod_{i=1}^{n} \left\{ \frac{(1 - bt^{2-i}q^{2\mu_i})}{(1 - bt^{2-2i})} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j} (bt^{3-i-j})_{\mu_i + \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (bt^{2-i-j})_{\mu_i + \mu_j}} \right\}
\]

\[
\cdot W_{\mu}(q^{\lambda} t^{\delta(n)}; q, t, bst^{2-2n}, bt^{1-n}) \frac{(x^{-1}, ax)_\mu}{(qbx, qb/ax)_\mu} \tag{27}
\]

It was also shown [4] that the summand which includes the \(W_{\mu}\) function is invariant under the hyperoctahedral group action of permutations and sign changes for arbitrary partitions \(\lambda\). More precisely, it was shown that under the specialization \(t = q^k\) and \(b = q^{2z_i + 2k(i-1)}\) where \(z_i \in \mathbb{C}\) and \(k \geq 0\) is a non-negative integer, the summand is invariant under the action \((\mu_i + z_i) \leftrightarrow w(\mu_i + z_i)\) for all elements \(w \in W\) of the hyperoctahedral group or rank \(n\). It was further verified that this action generates the full weight lattice \(\mathbb{Z}^n\) only if \(z_i = m/2 + k(n - i)\) for some non-negative integers \(m, k \geq 0\).

The proof of the invariance follows from the duality formula and the flip identity [2] for \(W_{\lambda}\) functions. The symmetry under permutations are given by the duality formula

\[
W_{\lambda}(k^{-1} q^\nu t^\delta; q, t, k^2 a, kb) \cdot \frac{(qb^{n-1})_\lambda(qb/a)_\lambda}{(nak^{n-1})_\lambda}
\]

\[
\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\lambda_i - \lambda_j} (qa't^{2n-i-j-1})_{\lambda_i + \lambda_j}}{(t^{j-i-1})_{\lambda_i - \lambda_j} (qa't^{2n-i-j})_{\lambda_i + \lambda_j}} \right\}
\]

\[
= W_{\nu}(h^{-1} q^{\lambda} t^\delta; q, t, h^2 a', hb') \cdot \frac{(qbt^{n-1})_\nu(qb/a')_\nu}{(h)_\nu(ha't^{n-1})_\nu}
\]

\[
\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\nu_i - \nu_j} (qa't^{2n-i-j-1})_{\nu_i + \nu_j}}{(t^{j-i-1})_{\nu_i - \nu_j} (qa't^{2n-i-j})_{\nu_i + \nu_j}} \right\} \tag{28}
\]

where \(k = a't^{n-1}/b\) and \(h = at^{n-1}/b\). The invariance under sign changes follows from the flip identity

\[
a^{[\lambda]} b^{-[\lambda]} q^{-[\lambda]} t^{-n([\lambda]+(n-1)[\lambda])} W_{\lambda}(x_1^{-1}, \ldots, x_n^{-1}, q^{-1}, p, t^{-1}, a^{-1}, b^{-1})
\]

\[
= a^{-[\lambda]} b^{[\lambda]} q^{[\lambda]} t^{n([\lambda]-(n-1)[\lambda])} W_{\lambda}(x_1, \ldots, x_n; q, p, t, a, b) \tag{29}
\]
The invariance of other factors follows immediately form the definition of the $q$–Pochhammer symbol.

We will give the multiple $3\psi_3$ summation using the specialization $m = \delta \in \{0, 1\}$ as in the classical one dimensional case, and for $k = 1$ or $t = q$. In other words, we let $t \to q$ and $b \to q^{\delta+2(n-1)}$ and write the identity above in the form

$$
\begin{aligned}
\frac{(sx^{-1}, asx)_\lambda}{(s, as)_\lambda} &= \sum_{\mu \in \mathbb{Z}^n} f(\delta) q^{\mu|+2n(\mu)} \frac{q^{\delta+2n-1}/as\mu}{(as)_\mu} \\
&\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{i-j+1})_{\mu_i-\mu_j}(q^{\delta+2n+1-i-j})_{\mu_i+\mu_j}}{(q^{j-i})_{\mu_i-\mu_j}(q^{\delta+2n-i-j})_{\mu_i+\mu_j}} \right\} \\
&\cdot W_\mu(q^{\lambda+\delta(n)}; q, q, sq^\delta, q^{\delta+(n-1)}) (x^{-1}, ax)_\mu 
\end{aligned}
$$

where

$$
f(\delta) := \frac{1}{2^n} \prod_{i=1}^{n-1} \frac{1}{(1+q^{n-i})}, \quad \text{if } \delta = 0
$$

and

$$
f(\delta) := \frac{1}{2^n} \prod_{i=1}^{n} \frac{1}{(1-q^{1+2n-2i})}, \quad \text{if } \delta = 1
$$

Note also that although the series is written over $\mathbb{Z}^n$, it actually terminates from above by $\lambda$ and from below by $(-\lambda_i - 2n - 2i + \delta)$.

The analogue of Weyl degree formula [4] for $W_\mu$ functions implies that

$$
W_\mu(q^{N+\delta(n)}; q, q, sq^\delta, q^{\delta+(n-1)}) = \frac{(q^{-N}, sq^{N+N+n+1})_\mu}{(q^{N+\delta+2n-1}, q^{-N+n}/s)_\mu} \prod_{1 \leq i < j \leq n} \frac{(q^{i-j+1})_{\mu_i-\mu_j}(q^{\delta+2n+1-i-j})_{\mu_i+\mu_j}}{(q^{j-i})_{\mu_i-\mu_j}(q^{\delta+2n-i-j})_{\mu_i+\mu_j}}
$$

Therefore, by setting $\lambda = N^n = (N, N, \ldots, N)$ and sending $N \to \infty$ we get

$$
\begin{aligned}
\frac{(sx^{-1}, asx)_{\infty}}{(s, as)_{\infty}} &= \sum_{\mu \in \mathbb{Z}^n} q^{(1-n)|\mu|+2n(\mu)} \frac{(q^{\delta+2n-1}/as\mu)}{(as)_\mu} \frac{1}{(q^{\delta+2n-1}/ax)_\mu} 2^n f(\delta) \\
&\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(1-q^{i-j+1})_{\mu_i-\mu_j}(1-q^{\delta+2n+1-i-j})_{\mu_i+\mu_j}}{(1-q^{j-i})^2_{\mu_i-\mu_j}(1-q^{\delta+2n-i-j})^2_{\mu_i+\mu_j}} \right\}
\end{aligned}
$$

This is the multilateral analogue of Bailey’s bilateral $3\psi_3$ summation formula as desired.

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3. Conclusion

The multilateralization technique applied here can be used to prove other multilateral series when the invariance property is satisfied. We will explore similar identities in other publications.

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