Indistinguishability of Warm Dark Matter, Modified Gravity, and Coupled Cold Dark Matter

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ABSTRACT

The current accelerated expansion of our universe could be due to an unknown energy component with negative pressure (dark energy) or a modification to general relativity (modified gravity). On the other hand, recently warm dark matter (WDM) remarkably rose as an alternative of cold dark matter (CDM). Obviously, it is of interest to distinguish these different types of models. In fact, many attempts have been made in the literature. However, in the present work, we show that WDM, modified gravity and coupled CDM form a trinity, namely, they are indistinguishable by using the cosmological observations of both cosmic expansion history and growth history. Therefore, to break this degeneracy, the other complementary probes beyond the ones of cosmic expansion history and growth history are required.

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I. INTRODUCTION

As is well known, the current accelerated expansion of our universe \([1]\) could be due to an unknown energy component with negative pressure (dark energy) \([1]\) or a modification to general relativity (modified gravity) \([1, 2]\). So, it is of interest to distinguish these two types of models. However, most cosmological observations merely probe the expansion history of our universe. On the other hand, it is easy to build models that share the same cosmic expansion history by means of reconstruction. Thus, in order to distinguish various models, some independent and complementary probes are required. In fact, it was proposed that the measurement of growth function \(\delta(z)\) might be capable (see e.g. \([3, 4, 30]\)). If the dark energy model and the modified gravity model share the same cosmic expansion history, their growth histories might be different, and hence they might be distinguished from each other.

However, up to now, in most works on this issue it is commonly assumed that there is no interaction between dark matter and dark energy. In fact, since the nature of both dark energy and dark matter are still unknown, there is no physical argument to exclude the possible interaction between them. On the contrary, some observational evidences of this possible interaction have been found. For instance, Bertolami et al. \([7]\) showed that the Abell Cluster A586 exhibits evidence of the interaction between dark energy and dark matter, and they argue that this interaction might imply a violation of the equivalence principle. On the other hand, Abdalla et al. \([8]\) found the signature of interaction between dark energy and dark matter by using optical, X-ray and weak lensing data from 33 relaxed galaxy clusters. Recently, Salvatelli et al. \([29]\) claimed that the Planck 2013 data favor a non-zero interaction between dark energy and dark matter. If the possible interaction between cold dark matter (CDM) and dark energy is allowed, it has been shown in e.g. \([2, 11]\) that the coupled CDM model can share both the same cosmic expansion history and growth history with the modified gravity model, and hence they cannot be distinguished.

On the other hand, although the well-known \(\Lambda CDM\) model is very successful in many aspects, it has been seriously challenged recently. According to the brief reviews in e.g. \([11]\), these serious challenges include, for instance, (1) \(\Lambda CDM\) predicts significantly smaller amplitude and scale of large-scale velocity flows than observations; (2) \(\Lambda CDM\) predicts fainter Type Ia supernova (SNIa) at high redshift \(z\); (3) \(\Lambda CDM\) predicts more dwarf or irregular galaxies in voids than observed; (4) \(\Lambda CDM\) predicts shallow low concentration and density profiles of cluster haloes in contrast to observations; (5) \(\Lambda CDM\) predicts galaxy halo mass profiles with cuspy cores and low outer density while observations indicate a central core of constant density and a flattish high dark mass density outer profile; (6) \(\Lambda CDM\) predicts a smaller fraction of disk galaxies due to recent mergers expected to disrupt cold rotationally supported disks. Even when one replaces the cosmological constant \(\Lambda\) with other (dynamical) dark energy candidates, these challenges still cannot be successfully addressed. In particular, the main source of the challenges on the small/galactic scale might be CDM. We refer to e.g. \([11]\) for details.

Recently, warm dark matter (WDM) remarkably rose as an alternative of CDM. The leading WDM candidates are the keV scale sterile neutrinos. In fact, the keV scale WDM is an intermediate case between the eV scale hot dark matter (HDM) and the GeV scale CDM. Unlike CDM which is challenged on the small/galactic scale (as mentioned above), it is claimed that WDM can successfully reproduce the astronomical observations over all the scales (from small/galactic to large/cosmological scales) \([12]\). The key is the connection between the mass of dark matter (DM) particles and the free-streaming length \(\ell_{fs}\) (structure smaller than \(\ell_{fs}\) will be erased). The eV scale HDM is too light, and hence all structures below the Mpc scale will be erased; the GeV scale CDM is too heavy, and hence the structures below the kpc scale cannot be erased (therefore, CDM is challenged on the small/galactic scale). In between, the keV scale WDM works well \([12]\). We refer to e.g. \([12]\) for a comprehensive review.

WDM has a fairly small but non-zero equation-of-state parameter (EoS), while the EoS of CDM is zero. In the literature (e.g. \([13, 16]\)), many attempts have been made to constrain the EoS of WDM \(w_m\), and it was found that \(w_m\) is about \(O(10^{-3}) \sim O(10^{-2})\) by using the current cosmological data. Of course, \(w_m\) is not constant in general. Let us consider the energy conservation equation of WDM, namely

\[
\dot{\rho}_m + 3H\rho_m (1 + w_m) = 0, \tag{1}
\]

where a dot denotes a derivative with respect to cosmic time \(t\); \(H \equiv \dot{a}/a\) is the Hubble parameter; \(a = (1 + z)^{-1}\) is the scale factor (we have set \(a_0 = 1\); the subscript “0” indicates the present value of corresponding quantity; \(z\) is the redshift); \(\rho_m\) is the energy density of WDM (we assume that the baryon
component is negligible). One can naively rewrite Eq. (1) as

$$\dot{\rho}_m + 3H\rho_m = -\Gamma_{\text{eff}}, \quad \text{and} \quad \Gamma_{\text{eff}} = 3H\rho_m w_m \neq 0. \quad (2)$$

Obviously, Eq. (2) could be regarded as the energy conservation equation of coupled CDM, while the term \(\Gamma_{\text{eff}}\) could be regarded as the interaction between CDM and dark energy. Correspondingly, the original energy conservation equation of dark energy \(\dot{\rho}_{\text{de}} + 3H\rho_{\text{de}}(1 + w_{\text{de}}) = 0\) can be rewritten as

$$\dot{\rho}_{\text{de}} + 3H\rho_{\text{de}}(1 + w_{\text{de}}) = \Gamma_{\text{eff}}, \quad \text{and} \quad w_{\text{eff}} = w_{\text{de}} + \Gamma_{\text{eff}}/(3H\rho_{\text{de}}). \quad (3)$$

Obviously, this procedure can be reversed. So, we can naively see that WDM could be equivalent to coupled CDM in this sense. Of course, it should be checked that WDM and coupled CDM can share both the same cosmic expansion history and growth history before we can say they are really indistinguishable. This will be part of our goal of this work. As mentioned above, in e.g. [9, 10] it was shown that modified gravity and CDM coupled with dark energy are indistinguishable since they can share both the same cosmic expansion history and growth history. Thus, it is very natural to speculate that modified gravity and WDM are also indistinguishable. In fact, the main goal of this work is to check the idea mentioned here. If these three types of cosmological models are really indistinguishable, some complementary probes beyond the ones of cosmic expansion history and growth history are required.

Before we go further, it is important to clarify a pitfall in the above naive discussions. In fact, the route from Eq. (1) to Eq. (2) is not unique. One can rather rewrite Eq. (1) as

$$\dot{\rho}'_m + 3H\rho'_m = -\Gamma_{\text{eff}}, \quad \rho'_m = \rho_m + X, \quad \Gamma_{\text{eff}} = 3H\rho_m w_m - X - 3HX, \quad (4)$$

where \(X\) can be any quantity. Obviously, \(\rho'_m \neq \rho_m\) and \(\Gamma_{\text{eff}} \neq 3H\rho_m w_m\) in general. So, the energy density of coupled CDM \(\rho'_m\) is not necessarily equal to the one of WDM \(\rho_m\), and hence the fractional energy density \(\Omega'_m \neq \Omega_m\) in general.

In Sec. [II] we firstly consider modified gravity and WDM. We propose a general approach to construct a WDM model that shares both the same expansion history and growth history with modified gravity. Then, an explicit example will be shown. In Sec. [III] we turn to WDM and coupled CDM. Also, we propose a general approach to construct a coupled CDM model that shares both the same expansion history and growth history with the WDM model. Of course, we will show an explicit example, too. Finally, some brief concluding remarks are given in Sec. [IV].

II. MODIFIED GRAVITY AND WDM

A. General formalism

Throughout this work, we consider a flat Friedmann-Robertson-Walker (FRW) universe. We firstly consider modified gravity and WDM. In this section, for the WDM model, we assume that the universe contains only WDM and dark energy (note that in general relativity, it is required that dark energy coexists with WDM to accelerate the cosmic expansion). The Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{\text{de}}), \quad (5)$$

where \(\rho_m\) and \(\rho_{\text{de}}\) are the energy densities of WDM and dark energy, respectively. In the WDM model under consideration, we assume that there is no interaction between WDM and dark energy, and hence their energy conservation equations are given by

$$\dot{\rho}_m + 3H\rho_m (1 + w_m) = 0, \quad (6)$$

$$\dot{\rho}_{\text{de}} + 3H\rho_{\text{de}} (1 + w_{\text{de}}) = 0, \quad (7)$$

where \(w_m\) and \(w_{\text{de}}\) are the EoS of WDM and dark energy, respectively. From Eqs. (6) and (7), we have

$$\Omega'_m = -\Omega_m \left[3(1 + w_m) + \frac{H'}{H}\right], \quad (8)$$

$$\Omega'_{\text{de}} = -\Omega_{\text{de}} \left[3(1 + w_{\text{de}}) + \frac{H'}{H}\right], \quad (9)$$
where a prime denotes a derivative with respect to $\ln a$, and $\Omega_i \equiv 8\pi G \rho_i/(3H^2)$ are the fractional energy densities for WDM and dark energy. On the side of growth history, in general relativity, the perturbation equation for WDM in the sub-horizon regime is given by (see e.g. [17–19])

$$\ddot{\delta} + \left[2 - 3\left(2w_m - c_s^2\right)\right]H\dot{\delta} = 4\pi G \rho_m \delta \left(1 - 6c_s^2 + 8w_m - 3w_m^2\right),$$

(10)

where $\delta \equiv \delta \rho_m/\rho_m$ is the linear matter density contrast, and $c_s^2 \equiv p_m/\rho_m$ is the sound speed squared of WDM. Using $p_m = w_m \rho_m$ and Eq. (6), we find that

$$c_s^2 = w_m - w_m' \frac{3(1 + w_m)}{3(1 + w_m)}.$$  

(11)

Obviously, if $w_m = \text{const.}$, we have $c_s^2 = w_m = \text{const.}$. Therefore, for CDM (namely $w_m = 0$), it is easy to see that Eq. (10) reduces to the well-known form in general relativity [3–6, 9, 10, 30]

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G \rho_m \delta.$$  

(12)

For convenience, we recast Eq. (10) as

$$\ddot{\delta} + \left[2 - 3\left(2w_m - c_s^2\right)\right]H\dot{\delta} = 3\frac{2}{2} \left(1 - 6c_s^2 + 8w_m - 3w_m^2\right)$$

(13)

On the other hand, in modified gravity, the perturbation equation reads [4, 20, 21]

$$\ddot{\tilde{\delta}} + 2\tilde{H}\dot{\tilde{\delta}} = 4\pi G_{\text{eff}} \tilde{\rho}_m \tilde{\delta},$$

(14)

where the quantities in modified gravity are labeled by a tilde “$\sim$”; $G_{\text{eff}}$ is the effective local gravitational “constant” measured by Cavendish-type experiment, which is time-dependent. Note that in the modified gravity model under consideration, the role of dark matter is played by CDM (its EoS is zero). In general, $G_{\text{eff}}$ can be written as [4, 20, 21]

$$G_{\text{eff}} = G \left(1 + \frac{1}{3\beta}\right),$$

(15)

where $\beta$ is a conventional quantity introduced just for convenience (which is equivalent to $G_{\text{eff}}$ in fact). $\beta$ will be determined once we specify the modified gravity model (see below). Eq. (14) can be recast as

$$\ddot{\tilde{\delta}} + \left(2 + \frac{\tilde{H}'}{\tilde{H}}\right)\dot{\tilde{\delta}} = \frac{3}{2} \left(1 + \frac{1}{3\beta}\right) \tilde{\Omega}_m \tilde{\delta}.$$  

(16)

To be indistinguishable, we require that the WDM model shares both the same expansion history and growth history with the modified gravity model, namely, we identify

$$H = \tilde{H} \quad \text{and} \quad \delta = \tilde{\delta}.$$  

(17)

Comparing Eq. (13) with Eq. (16), it is easy to find that

$$2 \left(2w_m - c_s^2\right) \ddot{\delta} = \tilde{\Omega}_m \left(1 + \frac{1}{3\beta}\right) - \tilde{\Omega}_m (1 - 6c_s^2 + 8w_m - 3w_m^2).$$

(18)

Note that $\Omega_m \neq \tilde{\Omega}_m$ in general.

Let us describe the prescription to construct the WDM model that shares both the same expansion history and growth history with a given modified gravity model. For a given modified gravity model, all its $\tilde{\Omega}_m$, $\beta$, and $\tilde{H}$ are known. Then, we can solve the differential equation (16) and obtain $\tilde{\delta}$ as a function of $\ln a$. Note that $H = \tilde{H}$ and $\delta = \tilde{\delta}$. Substituting Eq. (11) into Eq. (18), we find that Eqs. (18) and (8) become two coupled first-order differential equations for $w_m$ and $\Omega_m$ with respect to $\ln a$. Obviously, we
can numerically solve them and get \( w_m \) and \( \Omega_m \) as functions of \( \ln a \). Then, \( \Omega_{de} = 1 - \Omega_m \) and \( c_s^2 \) in Eq. (17) are on hand. From Eq. (19), we obtain the EoS of dark energy, namely

\[
  w_{de} = -1 - \frac{1}{3} \left( \frac{\Omega_{de}}{\Omega_{rc}} + 2 \frac{H'}{H} \right),
\]

in which we note that \( H = \dot{H} \) from Eq. (17). Of course, if one needs \( \rho_i = 3H^2\Omega_i/(8\pi G) \) for WDM and dark energy, noting \( H = \dot{H} \), they are also ready. So, all the physical quantities of the corresponding WDM model are known. The resulting WDM model is really indistinguishable from the given modified gravity model by using the observations of both expansion history and growth history.

### B. Explicit example

Here, we would like to give an explicit example following the prescription proposed in Sec. IIA. As an example, we consider the simplest modified gravity model, namely, the flat Dvali-Gabadadze-Porrati (DGP) braneworld model [22] (see also e.g. [4, 20, 21, 23]). Note that here we only consider the self-accelerating branch of DGP model, for which the reduced Hubble parameter is given by [4, 22, 23]

\[
  \tilde{E} \equiv \frac{\dot{H}}{H_0} = \sqrt{\tilde{\Omega}_{m0}(1 + z)^3 + \tilde{\Omega}_{rc}} + \sqrt{\tilde{\Omega}_{rc} = \sqrt{\tilde{\Omega}_{m0} e^{-3\ln a} + \tilde{\Omega}_{rc} + \tilde{\Omega}_{rc}.}}
\]

where \( \tilde{\Omega}_{rc} \) is a constant. Requiring \( \tilde{E}(z = 0) = 1 \) by definition, it is easy to see that

\[
  \tilde{\Omega}_{m0} = 1 - 2 \sqrt{\tilde{\Omega}_{rc}}.
\]

Thus, the DGP model has only an independent model parameter, namely \( \tilde{\Omega}_{rc} \). Note that \( 0 \leq \tilde{\Omega}_{rc} \leq 1/4 \) is required by \( 0 \leq \tilde{\Omega}_{m0} \leq 1 \). The fractional energy density of dark matter in the DGP model is given by [4, 22, 23]

\[
  \tilde{\Omega}_m = \frac{\tilde{\Omega}_{m0}(1 + z)^3}{E^2(z)} = \frac{\tilde{\Omega}_{m0} e^{-3\ln a}}{E^2(\ln a)}.
\]

On the other hand, the \( \beta \) in Eq. (15) for the flat DGP model is given by [4, 20, 21]

\[
  \beta = \frac{1 + \tilde{\Omega}_m^2}{1 - \tilde{\Omega}_m^2}.
\]

For demonstration, we choose the single independent model parameter as \( \tilde{\Omega}_{rc} = 0.125 \), which is well consistent with the current observational data (see e.g. [24, 25]). Substituting this given \( \tilde{\Omega}_{rc} \) into Eq. (21) and then Eqs. (20), (22) and (23), the corresponding \( \tilde{E} \), \( \tilde{\Omega}_m \), and \( \beta \) as functions of \( \ln a \) are known. Noting that \( H'/H = H'/\tilde{H} = \tilde{E}'/\dot{\tilde{E}} \), and following the prescription proposed in Sec. IIA we can easily construct the desired WDM model which shares both the same expansion history and growth history with this given DGP model. Firstly, we obtain \( \delta = \delta \) by solving Eq. (16). As is well known, \( \delta' = \delta = a \) at \( z \gg 1 \) (see e.g. [3, 4]). Therefore, we use the initial condition \( \delta' = \delta = a_{ini} = 1000 \) for the differential equation (16). The resulting \( \delta' = \delta \) and \( \ln E = \ln \tilde{E} \) as functions of \( \ln a \) are shown in Fig. [1]. Secondly, substituting Eq. (11) into Eq. (15), noting \( H'/H = H'/\tilde{H} = \tilde{E}'/\dot{\tilde{E}} \), Eqs. (15) and (5) become two coupled first-order differential equations for \( w_m \) and \( \Omega_m \) with respect to \( \ln a \). We can numerically solve them with the demonstrative initial conditions \( \Omega_m(z = z_{ini}) = 0.995, w_m(z = z_{ini}) = 0.005 \) (which is well consistent with the observational data [16]), and then obtain \( w_m, \Omega_m \) as functions of \( \ln a \). They are also shown in Fig. [1]. Finally, \( \Omega_{de} = 1 - \Omega_m \), as well as \( c_s^2 \) in Eq. (11) and \( w_{de} \) in Eq. (19) are available, and they can also be found in Fig. [1]. So far, we have successfully constructed the WDM model that shares both the same expansion history and growth history with a given modified gravity model, namely, the flat DGP model. These two models are indistinguishable in this sense. Therefore, to distinguish the modified gravity model and WDM model, it is required to seek some complementary probes beyond the ones of cosmic expansion history and growth history (for instance, the observations on the small/galactic scale).
FIG. 1: $\ln E = \ln \tilde{E}$, $\delta = \tilde{\delta}$, $\Omega_m$, $\Omega_{de}$, $w_{de}$, $w_m$ and $c_s^2$ as functions of $\ln a$. See the text for details.
III. WDM AND COUPLED CDM

A. General formalism

In this section, we turn to WDM and coupled CDM. Since we have shown that WDM and modified gravity are indistinguishable in Sec. II and we have already shown that modified gravity and coupled CDM are indistinguishable in \[9\] (see also e.g. \[10\]), it is reasonable to expect that WDM and coupled CDM are also indistinguishable. Here, we try to construct a coupled CDM model that shares both the same expansion history and growth history with the WDM model.

Following \[9\], we consider the case of CDM coupled with quintessence (which is a main candidate of dark energy). It is well known that the pressure and energy density for the homogeneous quintessence are given by

\[
\hat{p}_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad \hat{\rho}_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi),
\]

(24)

where \(V\) is the potential of the scalar field \(\phi\), and the quantities in coupled CDM model are labeled by a hat “\(^\wedge\)”. The corresponding Friedmann equation reads

\[
3\dot{H}^2 = \kappa^2 (\hat{\rho}_m + \hat{\rho}_\phi),
\]

(25)

where \(\kappa^2 \equiv 8\pi G\). We assume that CDM and quintessence interact through \[26–28\]

\[
\dot{\rho}_m + 3H \dot{\rho}_m = -\kappa Q \dot{\rho}_m \dot{\phi}, \quad \dot{\phi} + 3H (\dot{\phi} + \dot{\rho}_\phi) = \kappa Q \dot{\rho}_m \dot{\phi},
\]

(26) \hspace{1cm} (27)

which preserve the total energy conservation equation \(\dot{\rho}_{tot} + 3\dot{H} (\dot{\rho}_{tot} + \dot{p}_{tot}) = 0\). The dimensionless coupling coefficient \(Q = Q(\phi)\) is an arbitrary function of \(\phi\). In fact, Eq. (27) is equivalent to

\[
\ddot{\phi} + 3\dot{H} \dot{\phi} + \frac{dV}{d\phi} = \kappa Q \dot{\rho}_m .
\]

(28)

Using Eqs. (25), (26) and (27), one can obtain the Raychaudhuri equation, namely

\[
\dot{H} = \frac{\kappa^2}{2} (\dot{\rho}_m + \dot{\rho}_\phi) = -\frac{\kappa^2}{2} \left( \dot{\rho}_m + \dot{\phi}^2 \right).
\]

(29)

It is worth noting that \(\dot{\rho}_m\) does not scale as \(a^{-3}\) due to the non-vanishing interaction. On the side of growth history, the perturbation equation in the sub-horizon regime is given by \[27\]

\[
\ddot{\delta}'' + \left( 2 + \frac{\dot{H}}{H} - \kappa Q \dot{\phi} \right) \dot{\delta}' = \frac{3}{2} \left( 1 + 2Q^2 \right) \Omega_m \delta .
\]

(30)

Obviously, if \(Q = 0\), Eq. (30) reduces to the well-known form (n.b. Eq. (12)). In fact, Eq. (30) from \[27\] is valid for any \(Q = Q(\phi)\) and generalizes the one of \[28\] which is only valid for constant \(Q\).

On the other hand, the perturbation equation in the WDM model has already been given in Eq. (13). To be indistinguishable, we require that the coupled CDM model shares both the same expansion history and growth history with the WDM model, namely, we identify

\[
\dot{H} = \dot{H} \quad \text{and} \quad \delta = \hat{\delta} .
\]

(31)

Comparing Eq. (13) with Eq. (30), it is easy to find that

\[
\left[ 3 \left( 2w_m - c_s^2 \right) - \kappa Q \dot{\phi} \right] \dot{\delta}' = \frac{3}{2} \delta \left[ \Omega_m (1 + 2Q^2) - \Omega_m (1 - 6c_s^2 + 8w_m - 3w_m^2) \right].
\]

(32)
Note that \( \dot{\Omega}_m \neq \Omega_m \) in general, as mentioned in Sec. I. From Eq. (29), we have

\[
(\kappa \phi')^2 = -3\dot{\Omega}_m - 2\frac{\dot{H}'}{\dot{H}}. \tag{33}
\]

From Eq. (29), it is easy to obtain

\[
\dot{\Omega}_m' = - \left( 3 + 2\frac{\dot{H}'}{\dot{H}} + \kappa Q \phi' \right) \dot{\Omega}_m. \tag{34}
\]

It turns out that

\[
\kappa Q \phi' = -3 - 2\frac{\dot{H}'}{\dot{H}} - \frac{\dot{\Omega}_m'}{\dot{\Omega}_m}. \tag{35}
\]

From Eqs. (33) and (35), we have

\[
Q^2 = \left( \frac{\kappa Q \phi'}{(\kappa \phi')} \right)^2 = \left( 3 + 2\frac{\dot{H}'}{\dot{H}} + \frac{\dot{\Omega}_m}{\dot{\Omega}_m} \right)^2 \left( -3\dot{\Omega}_m - 2\frac{\dot{H}'}{\dot{H}} \right)^{-1}. \tag{36}
\]

From Eqs. (24), (25) and (33), we obtain the dimensionless potential of quintessence, namely

\[
U \equiv \frac{\kappa^2 V}{H_0^2} = 3\hat{E}^2 \left( 1 - \frac{\dot{\Omega}_m}{2} + \frac{1}{3} \frac{\dot{H}'}{H} \right), \tag{37}
\]

where \( \hat{E} \equiv \dot{H}/H_0 \). Obviously, \( U \) is equivalent to \( V \) in fact. From Eqs. (24), (33) and (37), we find the EoS of quintessence, namely

\[
w_\phi = \frac{p_\phi}{\rho_\phi} = \left( -1 - 2\frac{\dot{H}'}{3H} \right) \left( 1 - \dot{\Omega}_m \right)^{-1}. \tag{38}
\]

Let us describe the prescription to construct the coupled CDM model which shares both the same expansion history and growth history with a given WDM model. For a given WDM model, all its \( w_m, c_s^2, \Omega_m, \) and \( H \) are known. Then, we can solve the differential equation (13) and obtain \( \delta \) as a function of \( \ln a \).
Note that $\dot{H} = H$ and $\dot{\delta} = \delta$. Substituting $\delta$, Eqs. (35) and (36) into Eq. (32), noting $\dot{H} = H$, Eq. (32) becomes a first-order differential equation for $\Omega_m$ with respect to $\ln a$. Obviously, we can numerically solve this differential equation and get $\Omega_m$ as a function of $\ln a$. So, $\Omega_m = 1 - \Omega_m$ is available. Substituting $\Omega_m$ into Eqs. (35), (35), (37) and (38), noting $\dot{H} = H$, we find $\kappa \phi'$, $\kappa Q \phi'$, $U$ and $w_\phi$ as functions of $\ln a$. Then, $Q = (\kappa Q \phi') / (\kappa \phi')$ is on hand. By integrating $\kappa \phi'$, we get $\kappa \phi$ as a function of $\ln a$. Therefore, we can finally obtain $Q$ and $U$ as functions of $\kappa \phi$. So, all the physical quantities of the corresponding coupled CDM model are known. The resulting coupled CDM model is really indistinguishable from the given WDM model by using the observations of both expansion history and growth history.

B. Explicit example

Here, we give an explicit example following the prescription proposed in Sec. III A. As an example, we consider the simplest WDM model, namely the so-called $\Lambda$WDM model, in which the role of dark energy is played by a cosmological constant $\Lambda$, while the EoS of WDM $w_\phi$ is a constant. In this case, from Eq. (11), we have $c_s^2 = w_\phi = \text{const}$. As is well known, the corresponding reduced Hubble parameter of the $\Lambda$WDM model reads (see e.g. [16])

$$E \equiv \frac{H}{H_0} = \left[ \Omega_{m0}(1 + z \delta) + (1 - \Omega_{m0}) \right]^{1/2} = \left[ \Omega_{m0} e^{-3(1+w_\phi)\ln a} + (1 - \Omega_{m0}) \right]^{1/2}. \quad (39)$$

There are two independent model parameters. For demonstration, we choose the model parameters of $\Lambda$WDM as $\Omega_{m0} = 0.28$ and $w_\phi = 0.003$, which is well consistent with the observational data (see e.g. [16]). On the other hand, the fractional energy density of WDM is given by

$$\Omega_m \equiv \frac{8\pi G \rho_m}{3H^2} = \frac{\Omega_{m0} e^{-3(1+w_\phi)\ln a}}{E^2(\ln a)}. \quad (40)$$

Following the prescription proposed in Sec. III A we can easily construct the desired coupled CDM model that shares both the same expansion history and growth history with this given WDM model. Firstly, substituting Eqs. (39) and (40) into Eq. (13), noting $\dot{H}' / H' = E' / E$, we can numerically solve this differential equation (13) and get $\delta$ as a function of $\ln a$. As is well known, $\delta' = \delta = a$ at $z \gg 1$ (see e.g. [3]). Therefore, we use the initial condition $\delta' = \delta = a_{\text{ini}}$ at $z_{\text{ini}} = 1000$ for the differential equation (13). The resulting $\delta = \delta$ and $\ln E = \ln E$ as functions of $\ln a$ are shown in Fig. 2. Secondly, substituting Eqs. (35), (36) and (40) into Eq. (32), noting $\dot{H}' / H' = H' / H = E' / E$, it is easy to see that Eq. (32) becomes a first-order differential equation for $\Omega_m$ with respect to $\ln a$. Obviously, we can numerically solve this differential equation with the demonstrative initial condition $\Omega_m(z = z_{\text{ini}}) = 0.995$, and get $\Omega_m$ as a function of $\ln a$. We show the resulting $\Omega_m$ and $\Omega_\phi = 1 - \Omega_m$ in Fig. 3. Substituting $\Omega_m$ into Eqs. (33), (35), (37) and (38), noting $\dot{H}' / H' = H' / H = E' / E$, we find $\kappa \phi'$, $\kappa Q \phi'$, $U$ and $w_\phi$ as functions of $\ln a$. Then, $Q = (\kappa Q \phi') / (\kappa \phi')$ is on hand. By integrating $\kappa \phi'$, we get $\kappa \phi$ as a function of $\ln a$. Note that for demonstration, we choose the negative branch for $\kappa \phi'$, and choose $\phi_0 = 0$ when we get $\kappa \phi$. In Fig. 3 we also show the resulting $w_\phi$, $\kappa \phi'$, $Q$, $U$ and $\kappa \phi$ as functions of $\ln a$. Once we obtain $Q$, $U$ and $\kappa \phi$ as functions of $\ln a$, it is easy to find $Q$ and $U$ as functions of $\kappa \phi$. The results are shown in Fig. 4. So far, we have successfully constructed the coupled CDM model that shares both the same expansion history and growth history with a given WDM model, namely, the $\Lambda$WDM model. These two models are indistinguishable in this sense. Therefore, to distinguish the coupled CDM model and WDM model, it is required to seek some complementary probes beyond the ones of cosmic expansion history and growth history (for instance, the observations on the small/galactic scale).

IV. CONCLUDING REMARKS

In summary, we have shown that WDM, modified gravity and coupled CDM form a trinity, namely, they are indistinguishable by using the cosmological observations of both cosmic expansion history and
FIG. 3: $\Omega_m, \Omega_\phi, w_\phi, \kappa \phi', Q, U$ and $\kappa \phi$ as functions of $\ln a$. See the text for details.
growth history. In fact, they are three fairly different types of models: in the modified gravity models general relativity has been modified, while in both the WDM models and the coupled CDM models general relativity still holds. On the other hand, the interaction between CDM and dark energy is allowed in the coupled CDM models, while WDM and dark energy do not interact in the WDM models. However, we show that they are really indistinguishable by using the observations of both cosmic expansion history and growth history. This is not good news for the extensive attempts made in the literature to distinguish them (see e.g. [3–6, 30]). In particular, the indistinguishability of modified gravity and coupled CDM was shown in the previous works [9, 10], while the indistinguishability of WDM and modified gravity, as well as the indistinguishability of coupled CDM and WDM, are shown in the present work. Therefore, to distinguish them, it is required to seek some complementary probes beyond the ones of cosmic expansion history and growth history (for instance, the observations on the small/galactic scale).

Some remarks are in order. Firstly, the models considered in this work are simple in fact. There are more complicated cases. For example, in the coupled CDM model, the role of dark energy can be played by other dynamical candidates rather than quintessence. On the other hand, the interaction form considered in the coupled CDM model is the type $\propto Q\rho_m \dot{\phi}$, which in fact can be more complicated in the literature. Secondly, the situation can be more complicated by extending these models. For instance, we can allow that there is also interaction between WDM and dark energy in the WDM model, or even allow CDM/WDM non-minimally coupled with gravity in all three of these types of models. Of course, one can also replace CDM with WDM in the modified gravity models. In these more complicated cases, it is more difficult to distinguish them by using the observations of cosmic expansion history and growth history. Thirdly, to distinguish these models, the other complementary probes beyond the ones of cosmic expansion history and growth history are desirable. For example, we can consider the observations on the small/galactic scale in which the structure formation is non-linear. The local tests of gravity on Earth or in the solar system are also useful, since general relativity can be tightly tested here. Of course, the high energy experiments are helpful too. For example, one might find the evidence of extra dimensions (required by some kinds of modified gravity models) in the very high energy colliders (e.g. the LHC at CERN). Finally, from the statistical point of view (e.g. Bayesian analysis), model selection is a rather subtle subject, in which priors and the number of parameters play an essential role (we thank the anonymous referee for pointing out this issue). The simplest model with the fewest free parameters and the minimal assumptions is favored. However, such a selection is based on the available data under consideration. When the other new data are added, the conclusion might be changed correspondingly. So, to distinguish the three types of cosmological models considered here, seeking the other complementary probes is still the best way. Recently, many significant progresses have been made in the observations on the small/galactic scale, the local tests of gravity, and the very high energy colliders (e.g. the LHC at CERN). It is hopeful to distinguish these models in the near future.
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