OPTIMAL CONTROL OF FRACTIONAL SEMILINEAR PDES

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Abstract. In this paper we consider the optimal control of semilinear fractional PDEs with both spectral and integral fractional diffusion operators of order $2s$ with $s \in (0, 1)$. We first prove the boundedness of solutions to both semilinear fractional PDEs under minimal regularity assumptions on domain and data. We next introduce an optimal growth condition on the nonlinearity to show the Lipschitz continuity of the solution map for semilinear elliptic equations with respect to data. This removes the usually used local Lipschitz continuity assumption on the nonlinearity. We further apply our ideas to show existence of solution to optimal control problems with semilinear fractional equations as constraints. Under the standard assumptions on the nonlinearity (twice continuously differentiable) we derive the first and second order optimality conditions. We conclude with two numerical examples.

Key words. Optimal growth condition, semilinear PDEs, integral and spectral fractional operators, semilinear optimal control problems, regularity of weak solutions.

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1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set with boundary $\partial \Omega$. In this paper we investigate the well-posedness of the semilinear fractional equation

$$(L_D)^s u + f(x, u) = z \quad \text{in } \Omega,$$

and we also consider an optimal control problem

$$\min_{z \in Z_{ad}} J(u, z) := J_1(u) + J_2(z)$$

subject to the state equation (1.1) and the control constraints

$$z \in Z_{ad} := \left\{ z \in L^\infty(\Omega) : z_a \leq z \leq z_b, \text{ a.e. in } \Omega \right\}.$$

Here $z_a, z_b \in L^\infty(\Omega)$ with $z_a(x) \leq z_b(x)$ for a.e. $x \in \Omega$. The precise conditions on $J_1$ and $J_2$ will be given in Section 4 and Remark 4.4.

In (1.1), $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and satisfies certain conditions (that we shall specify later) and $(L_D)^s (0 < s < 1)$ denotes the spectral fractional $s$ powers of the realization in $L^2(\Omega)$ of the operator $L$ formally given by

$$Lu := -\sum_{i,j=1}^N D_i \left( a_{ij}(x) D_j u \right), \quad D_i := \frac{\partial}{\partial x_i},$$

with the zero Dirichlet boundary condition $u = 0$ on $\partial \Omega$. The coefficients $a_{ij}$ are assumed to be measurable, belong to $L^\infty(\Omega)$, are symmetric, that is,

$$a_{ij}(x) = a_{ji}(x) \forall i, j = 1, \cdots, N \text{ and for a.e. } x \in \Omega,$$
and satisfy the ellipticity condition, that is, there exists a constant $\gamma > 0$ such that
\[ \sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j \geq \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega. \]

Besides Equation (1.1), we also consider the following elliptic system
\[
\begin{cases}
(-\Delta)^s u + f(x, u) = z & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where $(-\Delta)^s$ denotes the integral fractional Laplace operator (see Section 3.3 below), together with the optimal control problem (1.2) and the control constraints (1.3).

Notice that both $(L_D)^s$ and $(-\Delta)^s$ are nonlocal operators if $0 < s < 1$ and $f$ is nonlinear with respect to $u$. This makes it challenging to identify the minimum assumptions on $\Omega$, $f$ and $z$ in the study of the existence, uniqueness, regularity and the numerical analysis of (1.1) and (1.5). The main contributions of this paper are summarized as follows:

(i) We identify minimal conditions on $f$ without any regularity assumption on the domain $\Omega$ that leads to existence uniqueness and boundedness of solution to (1.1) and (1.5). Our main assumption reads that $f$ is monotone in the second variable and $f(x, t) \to \infty$ as $t \to \infty$.

(ii) We introduce an optimal growth condition on $f$ (see (3.7) below) that allows us to prove the Lipschitz continuity of the solution map: $z \mapsto S(z) := u$. Usually local Lipschitz continuity on $f$ is assumed in most of the literature. In absence of this Lipschitz continuity we also prove existence of solution to (1.2). Our growth condition is not a regularity assumption on $f$ and therefore is weaker that local Lipschitz continuity.

(iii) We study the optimality conditions for the optimization problem and under standard assumptions on $f$ we derive second order sufficient conditions.

To the best of our knowledge all these results are new not only for the spectral case but also the fractional case. We further notice that the results of (ii) can be applied to classical semilinear problems as well. When $a_{ij} = \delta_{ij}$ where the latter denotes the Kronecker delta, we developed a complete analysis, including discretization, and error estimates, for (1.1) in [7]. Such an error analysis can be directly applied to (1.1) under the usual assumptions on $\Omega$ and the coefficients $a_{ij}$. By following the approach of [7] in conjunction with the estimates for the linear problem [2] a similar error analysis can be developed for (1.5).

In order to avoid repetition we will focus on the semilinear problem (1.1) with spectral fractional operator $(L_D)^s$. However, to prove our crucial results in (i) and (ii) we rely on an equivalent integral representation of $(L_D)^s$ (cf. (2.5)). This integral representation is similar to the representation of the fractional operator $(-\Delta)^s$ (cf. Section 3.3) and all the results discussed for $(L_D)^s$ directly transfer to $(-\Delta)^s$ under minor modifications. We refer to Section 3.3 and Remarks 4.5 and 5.10 for details.

The remainder of the paper is organized as follows. In Section 2 we state some preliminary results and introduce our function spaces. Our main work starts from Section 3 where we first prove the existence of solution to (1.1) in Sobolev spaces. We next show that the inverse of solution operator is bounded and continuous under the newly introduced growth condition in (3.7), we also study compactness of such an operator. We prove $L^\infty$ bound on $u$ in Theorem 3.5. We also derive an $L^\infty$ bound on the difference of two solutions $u_1, u_2$ corresponding to given $z_1, z_2$ in Proposition 3.9.
without any additional assumptions on \( f \). In Section 3.3 we show that all our results also hold for the system (1.5) with very minor modifications in the proofs. An example of \( f \) is given in Section 3.4. We next prove the existence of our optimal control problem in Section 4 by just assuming the above mentioned growth condition on \( f \). Under additional regularity assumptions on \( f \) with respect to \( u \) we derive the first order necessary and second order sufficient conditions in Section 5. We conclude with two numerical examples in Section 6 using the spectral operator \((L_D)^s\).

2. Notation and Preliminary results. Throughout this section without any mention, \( \Omega \subset \mathbb{R}^N \) denotes an arbitrary bounded open set with boundary \( \partial \Omega \). For each result, if a regularity of \( \Omega \) is needed, then we shall specify and if no specification is given, then we mean that the result holds without any regularity assumption on the open set.

2.1. Fractional order Sobolev spaces. Let \( H^1_0(\Omega) = \overline{D(\Omega)}^{H^1(\Omega)} \) where \( D(\Omega) \) is the space of infinitely continuously differentiable functions with compact support in \( \Omega \), and

\[
H^1(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} |\nabla u|^2 \, dx < \infty \right\}.
\]

is the first order Sobolev space endowed with the norm

\[
\|u\|_{H^1(\Omega)} = \left( \int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.
\]

Next, for \( 0 < s < 1 \), we define the fractional order Sobolev space

\[
H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\},
\]

and we endow it with the norm defined by

\[
\|u\|_{H^s(\Omega)} = \left( \int_{\Omega} |u|^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

We also let

\[
H^s_0(\Omega) := \overline{D(\Omega)}^{H^s(\Omega)},
\]

and

\[
H^s_{00}(\Omega) := \left\{ u \in H^s_0(\Omega) : \int_{\Omega} \frac{u^2(x)}{\text{dist}(x, \partial \Omega)} \, dx < \infty \right\}.
\]

Note that

\[
\|u\|_{H^s_0(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}} \tag{2.1}
\]

defines a norm on \( H^s_0(\Omega) \) if \( \frac{1}{2} < s < 1 \). It is well-known (see e.g. [23, Theorem 1.4.2.4 p.25]) that if \( \Omega \) has a Lipschitz continuous boundary, then \( H^s(\Omega) = H^s_0(\Omega) \) if and only
if $0 < s \leq 1/2$. If $1/2 < s < 1$, then $H^s_0(\Omega)$ is a proper closed subspace of $H^s(\Omega)$. In particular, we also have that $H^1_{00}(\Omega) \subsetneq H^{1/2}_0(\Omega) = H^{1/2}(\Omega)$. A complete description of this fact for arbitrary bounded open sets is contained in [32]. The fractional order Sobolev spaces can be also defined by using interpolation theory. That is, for every $0 < s < 1$,

$$H^s(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1-s}$$

and for every $s \in (0, 1)$ we have that

$$H^s_0(\Omega) = [H^1_0(\Omega), L^2(\Omega)]_{1-s}$$

if $s \in (1/2, 1)$ and $H^s_{00} = [H^1_0(\Omega), L^2(\Omega)]_{1/2}$.

Here for $0 < \theta < 1$, $[,]_\theta$ denotes the complex interpolation space.

Since $\Omega$ is assumed to be bounded we have the following continuous embedding:

$$H^s_0(\Omega) \hookrightarrow \left\{ \begin{array}{ll}
L^{2-N}(-\Omega) & \text{if } N > 2s, \\
L^p(\Omega), & \text{if } N = 2s, \\
C^{0,s-\frac{N}{2}}(\Omega) & \text{if } N < 2s.
\end{array} \right. \tag{2.2}$$

We notice that if $N \geq 2$, then $N \geq 2 > 2s$ for every $0 < s < 1$, or if $N = 1$ and $0 < s < \frac{1}{2}$, then $N = 1 > 2s$, and thus the first embedding in (2.2) will be used. If $N = 1$ and $s = \frac{1}{2}$, then we will use the second embedding. Finally, if $N = 1$ and $\frac{1}{2} < s < 1$, then $N = 1 < 2s$ and hence, the last embedding will be used.

For more details on fractional order Sobolev spaces we refer the reader to [3, 20, 26, 32] and their references.

**2.2. The fractional powers of the elliptic operator.** Let $L_D$ be the realization on $L^2(\Omega)$ of $L$ given in (1.4) with the zero Dirichlet boundary condition $u = 0$ on $\partial \Omega$. That is, $L_D$ is the positive and self-adjoint operator on $L^2(\Omega)$ associated with the closed, bilinear and symmetric form

$$\mathcal{E}_D(u, v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) D_i u D_j v \, dx, \quad u, v \in H^1_0(\Omega),$$

in the sense that

$$\left\{ \begin{array}{l}
D(L_D) = \{ u \in H^1_0(\Omega) : \exists w \in L^2(\Omega), \mathcal{E}_D(u, v) = (w, v)_{L^2(\Omega)} \forall v \in H^1_0(\Omega) \}, \\
L_D u = w.
\end{array} \right.$$  

It is well-known that $L_D$ has a compact resolvent and its eigenvalues form a non-decreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ of real numbers satisfying $\lim_{n \to \infty} \lambda_n = \infty$. We denote by $\varphi_n$ the orthonormal eigenfunction associated with $\lambda_n$.

For any $\theta \geq 0$, we also introduce the following fractional order Sobolev space:

$$H^\theta(\Omega) := \left\{ u = \sum_{n=1}^{\infty} u_n \varphi_n \in L^2(\Omega) : \| u \|^2_{H^\theta(\Omega)} := \sum_{n=1}^{\infty} \lambda_n^{\theta} u_n^2 < \infty \right\},$$

where $u_n = (u, \varphi_n)_{L^2(\Omega)} = \int_{\Omega} u \varphi_n \, dx$. 

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If $0 < s < 1$, then it is well-known that

\[
\mathbb{H}^s(\Omega) = \begin{cases} 
H_0^s(\Omega) & \text{if } s \neq \frac{1}{2}, \\
H^\frac{1}{2}_{00}(\Omega) & \text{if } s = \frac{1}{2}.
\end{cases}
\tag{2.3}
\]

It follows from (2.3) that the embedding (2.2) holds with $H_0^s(\Omega)$ replaced by $\mathbb{H}^s(\Omega)$.

**Definition 2.1.** Let $0 < s < 1$. The spectral fractional $s$ powers of $L_D$ is defined on $\mathbb{H}^s(\Omega)$ by

\[
(L_D)^s u := \sum_{n=1}^{\infty} \lambda_n^s u_n \varphi_n \quad \text{with } u_n = \int_{\Omega} u \varphi_n \, dx.
\]

We notice that in this case we have

\[
\|u\|_{\mathbb{H}^s(\Omega)} = \| (L_D)^s u \|_{L^2(\Omega)}.
\tag{2.4}
\]

In addition $\mathcal{D}(\Omega) \hookrightarrow \mathbb{H}^s(\Omega) \hookrightarrow L^2(\Omega)$, so, the operator $(L_D)^s$ is unbounded, densely defined and with bounded inverse $(L_D)^{-s}$ in $L^2(\Omega)$. But it can also be viewed as a bounded operator from $\mathbb{H}^s(\Omega)$ into its dual $\mathbb{H}^{-s}(\Omega) := (\mathbb{H}^s(\Omega))^\ast$. The following integral representation of $(L_D)^s$ given in [15, Theorem 2.3] will be useful. For every $u, v \in \mathbb{H}^s(\Omega)$, we have that

\[
\langle (L_D)^s u, v \rangle_{\mathbb{H}^{-s}(\Omega), \mathbb{H}^s(\Omega)} = \int_{\Omega} \int_{\Omega} \left( u(x) - u(y) \right) \left( v(x) - v(y) \right) K_s(x, y) \, dx \, dy + \int_{\Omega} \kappa_s(x) u(x) v(x) \, dx,
\tag{2.5}
\]

where

\[
0 \leq K_s(x, y) := \frac{s}{\Gamma(1 - s)} \int_0^\infty \frac{W^D_{\Omega}(t, x, y)}{t^{1+s}} \, dt, \quad x, y \in \Omega,
\]

and

\[
0 \leq \kappa_s(x) = \frac{s}{\Gamma(1 - s)} \int_0^\infty \left( 1 - e^{-tL_D} 1(x) \right) \frac{dt}{t^{1+s}}, \quad x \in \Omega.
\]

Here, $\Gamma$ is the usual Gamma function, $(e^{-tL_D})_{t \geq 0}$ denotes the strongly continuous semigroup on $L^2(\Omega)$ generated by $-L_D$ and $W^D_{\Omega}$ is the associated heat kernel, that is,

\[
W^D_{\Omega}(t, x, y) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \varphi_n(x) \varphi_n(y), \quad t > 0, \ x, y \in \Omega.
\]

From the representation (2.5) we immediately see that $(L_D)^s$ is a nonlocal operator. We also notice that the case of fractional powers of elliptic operators with non-zero boundary condition has been investigated in [6].

For more details on fractional powers of more general operators we refer the reader to [1, 9, 15, 24, 30] and the references therein.
2.3. Some results on Orlicz spaces. Here we give some important properties of Orlicz type spaces that will be used throughout the paper.

Assumption 2.2. For a function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) we consider the following assumption:

\[
\begin{align*}
 f(x, \cdot) & \text{ is odd, strictly increasing for a.e. } x \in \Omega, \\
f(x, 0) & = 0 \quad \text{for a.e. } x \in \Omega, \\
f(x, \cdot) & \text{ is continuous for a.e. } x \in \Omega, \\
f(\cdot, t) & \text{ is measurable for all } t \in \mathbb{R}, \\
\lim_{t \to \infty} f(x, t) & = \infty \quad \text{for a.e. } x \in \Omega.
\end{align*}
\]

Since \( f(x, \cdot) \) is strictly increasing for a.e. \( x \in \Omega \), it has an inverse which we denote by \( \tilde{f}(x, \cdot) \). Let \( F, \tilde{F} : \Omega \times \mathbb{R} \to [0, \infty) \) be defined for a.e. \( x \in \Omega \) by

\[
F(x, t) := \int_0^{|t|} f(x, \tau) \, d\tau \quad \text{and} \quad \tilde{F}(x, t) := \int_0^{|t|} \tilde{f}(x, \tau) \, d\tau. \tag{2.6}
\]

The functions \( F \) and \( \tilde{F} \) are complementary Musielak-Orlicz functions such that \( F(x, \cdot) \) and \( \tilde{F}(x, \cdot) \) are complementary \( N \)-functions for a.e. \( x \in \Omega \) (in the sense of [3, p.229]).

Assumption 2.3. Under the setting of Assumption 2.2, and for a.e. \( x \in \Omega \), let both \( F(x, \cdot) \) and \( \tilde{F}(x, \cdot) \) satisfy the global \( (\Delta_2) \)-condition, that is, there exist two constants \( c_1, c_2 \in (0, 1] \) independent of \( x \), such that for a.e. \( x \in \Omega \) and for all \( t \in \mathbb{R} \),

\[
c_1 t f(x, t) \leq F(x, t) \leq t f(x, t) \quad \text{and} \quad c_2 t \tilde{f}(x, t) \leq \tilde{F}(x, t) \leq t \tilde{f}(x, t). \tag{2.7}
\]

Assumption 2.3 is equivalent to saying that the Musielak-Orlicz functions \( F \) and \( \tilde{F} \) satisfy the \( (\Delta_0^a) \)-condition in the sense that there exist two constants \( C_1, C_2 > 0 \) such that (see e.g. [3, p.232])

\[
F(x, 2t) \leq C_1 F(x, t) \quad \text{and} \quad \tilde{F}(x, 2t) \leq C_2 \tilde{F}(x, t), \quad \forall t \in \mathbb{R} \quad \text{and a.e. } x \in \Omega.
\]

In that case, we let

\[
L_F(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable : } F(\cdot, u(\cdot)) \in L^1(\Omega) \right\}
\]

be the Musielak-Orlicz space. The space \( L_F(\Omega) \) is defined similarly with \( F \) replaced by \( \tilde{F} \).

Remark 2.4. If Assumption 2.3 holds, then by [21, Theorems 1 and 2] (see also [3, Theorem 8.19]), \( L_F(\Omega) \) endowed with the Luxemburg norm given by

\[
\|u\|_{F,\Omega} := \inf \left\{ k > 0 : \int_{\Omega} F \left( x, \frac{u(x)}{k} \right) \, dx \leq 1 \right\},
\]

is a reflexive Banach space. The same result also holds for \( L_{\tilde{F}}(\Omega) \). Moreover, we have the following improved Hölder inequality for Musielak-Orlicz spaces (see e.g. [3, Formula (8.11) p.234]):

\[
\left| \int_{\Omega} uv \, dx \right| \leq 2\|u\|_{F,\Omega}\|v\|_{\tilde{F},\Omega}, \quad \forall u \in L_F(\Omega), \; v \in L_{\tilde{F}}(\Omega). \tag{2.8}
\]

In addition, by [12, Corollary 5.10], we have that

\[
\lim_{\|u\|_{F,\Omega} \to \infty} \frac{\int_{\Omega} F(x, u) \, dx}{\|u\|_{F,\Omega}} = \infty. \tag{2.9}
\]
We have the following result.

**Lemma 2.5.** ([7, Lemma 1.5]) Let Assumption 2.3 hold. Then \( f(\cdot, u(\cdot)) \in L_F^\infty(\Omega) \) for all \( u \in L_F^1(\Omega) \).

**Definition 2.6.** Let \( 0 < s < 1 \). Under Assumption 2.3 we can define the Banach space \( V_0 \) by

\[
V_0 := V_0(\Omega, F) := \left\{ u \in H^s(\Omega) : F(\cdot, u(\cdot)) \in L^1(\Omega) \right\}
\]

and we endow it with the norm defined by

\[
\|u\|_{V_0} := \|u\|_{H^s(\Omega)} + \|u\|_{F,\Omega}.
\]

In this case \( V_0 \) is a reflexive Banach space. In addition, it follows from (2.2) that we have the continuous embedding

\[
V_0 \hookrightarrow H^s(\Omega) \hookrightarrow L^{2^*}(\Omega),
\]

where we have set

\[
2^* = \frac{2N}{N - 2s} \quad \text{if} \quad N \geq 2 > 2s \quad \text{or if} \quad N = 1 \quad \text{and} \quad 0 < s < \frac{1}{2}.
\]

If \( N = 1 \) and \( s = \frac{1}{2} \), then \( 2^* \) is any number in the interval \([1, \infty)\). If \( N = 1 \) and \( \frac{1}{2} < s < 1 \), then we have the continuous embedding

\[
V_0 \hookrightarrow H^s(\Omega) \hookrightarrow C^{0,s-\frac{1}{2}}(\Omega).
\]

We refer to the monographs [3, 28] and their references for further properties of Orlicz type spaces.

**3. Analysis of the semilinear elliptic problem.** In this section we give some existence, uniqueness and regularity results of weak solutions to the problem (1.1). We also introduce an optimal growth condition on the nonlinearity \( f \) which leads to the Lipschitz continuity of the solution map.

**3.1. Existence of weak solutions.** Now we can introduce our notion of weak solutions to (1.1). We recall that we have set \( V_0 := H^s(\Omega) \cap L_F^1(\Omega) \). We shall denote by \( (V_0)^* = (H^s(\Omega) \cap L_F^1(\Omega))^* \) the dual of the reflexive Banach spaces \( V_0 \). Throughout the remainder of the paper, given a reflexive Banach space \( X \) and its dual \( X^* \), we shall denote by \( \langle \cdot, \cdot \rangle_{X^*,X} \) their duality map.

**Definition 3.1.** A function \( u \in V_0 \) is said to be a weak solution of (1.1) if the identity

\[
\mathcal{F}_D(u, v) := \int_\Omega (L_D)^s u(L_D)^s v \, dx + \int_\Omega f(x, u)v \, dx = \langle z, v \rangle_{(V_0)^*, V_0}, \tag{3.1}
\]

holds for every \( v \in V_0 \) and the right hand side of (3.1) makes sense.

We have the following result of existence and uniqueness of weak solution.

**Proposition 3.2 (Existence of weak solution).** Let Assumption 2.3 hold. Then for every \( z \in (V_0)^* \), (1.1) has a unique weak solution \( u \). In addition, if \( z \in H^{-s}(\Omega) \), then there exists a constant \( C > 0 \) such that

\[
\|u\|_{H^s(\Omega)} \leq C \|z\|_{H^{-s}(\Omega)}.
\]
Proof. Let \( u \in \mathcal{V}_0 \) be fixed. First it follows from Lemma 2.5 that \( f(\cdot, u(\cdot)) \in L^2(\Omega) \). Next, using the classical Hölder inequality and (2.8) we have that for all \( v \in \mathcal{V}_0 \),

\[
|\mathcal{F}_D(u, v)| \leq ((L_D)^2 u)_{\mathcal{V}_2(\Omega)}((L_D)^2 v)_{\mathcal{V}_2(\Omega)} + 2\|f(\cdot, u)\|_{F,\mathcal{V}_2(\Omega)} \|v\|_{F,\mathcal{V}_2(\Omega)}
\]

\[
\leq \left(\|(L_D)^2 u\|_{\mathcal{V}_2(\Omega)} + 2\|f(\cdot, u)\|_{F,\mathcal{V}_2(\Omega)}\right) \|v\|_{\mathcal{V}_0}.
\]

(3.3)

Since \( \mathcal{F}_D(\cdot, \cdot) \) is linear (in the second variable) we have shown that \( \mathcal{F}_D(u, \cdot) \in (\mathcal{V}_0)^* \) for every \( u \in \mathcal{V}_0 \). Since \( f(x, \cdot) \) is strictly monotone, we have that every \( u, v \in \mathcal{V}_0 \), \( u \neq v \),

\[
\mathcal{F}_D(u, u - v) - \mathcal{F}_D(v, u - v) > 0.
\]

(3.4)

Hence, \( \mathcal{F}_D \) is strictly monotone. It follows from the continuity of the norm function and the continuity of \( f(x, \cdot) \) that \( \mathcal{F}_D \) is semi-continuous. It follows also from the \((\Delta_2)\)-condition and (2.9) that

\[
\lim_{\|u\|_{F,\mathcal{V}_2(\Omega)} \to \infty} \int_\Omega f(x, u) \, dx = \infty,
\]

and this implies that

\[
\lim_{\|u\|_{\mathcal{V}_0} \to \infty} \frac{\mathcal{F}_D(u, u)}{\|u\|_{\mathcal{V}_0}} = \infty.
\]

(3.5)

Hence, \( \mathcal{F}_D \) is coercive. We have shown that for every \( u \in \mathcal{V}_0 \) there exists a unique \( A_F \in (\mathcal{V}_0)^* \) such that \( \mathcal{F}_D(u, v) = (A_F(u), v)_{(\mathcal{V}_0)^*,\mathcal{V}_0} \) for every \( v \in \mathcal{V}_0 \). This defines an operator \( A_F : \mathcal{V}_0 \to (\mathcal{V}_0)^* \) which is semi-continuous, strictly monotone, coercive and bounded (the boundedness follows from (3.3)). Therefore \( A_F(\mathcal{V}_0) = (\mathcal{V}_0)^* \) and hence, by the Browder-Minty theorem, for every \( z \in (\mathcal{V}_0)^* \), there exists a unique \( u \in \mathcal{V}_0 \) such that \( A_F(u) = z \), or equivalently, \( \mathcal{F}_D(u, v) = (z, v)_{(\mathcal{V}_0)^*,\mathcal{V}_0} \) for every \( v \in \mathcal{V}_0 \). Now assume that \( z \in \tilde{H}^{-s,\varepsilon}(\Omega) \to (\mathcal{V}_0)^* \). Then taking \( v = u \) in (3.1), using the fact that \( f(x, u) \geq 0 \) and noticing that \( (z, u)_{(\mathcal{V}_0)^*,\mathcal{V}_0} = (z, u)_{\tilde{H}^{-s,\varepsilon}(\Omega),\tilde{H}^{s,\varepsilon}(\Omega)} \) we get that

\[
\|u\|_{\tilde{H}^{s,\varepsilon}(\Omega)}^2 \leq |(z, u)| \leq \|z\|_{\tilde{H}^{-s,\varepsilon}(\Omega)} \|u\|_{\tilde{H}^{s,\varepsilon}(\Omega)}.
\]

We have shown (3.2) and the proof is finished.

The following result gives further estimates for the difference of two solutions.

**Proposition 3.3.** Let Assumption 2.3 hold. Let \( z_1, z_2 \in \tilde{H}^{-s,\varepsilon}(\Omega) \) and \( u_1, u_2 \in \mathcal{V}_0 \) be the corresponding weak solution of (1.1). Then there exists a constant \( C = C(N, s, \Omega) > 0 \) such that

\[
C\|u_1 - u_2\|_{\mathcal{V}_2(\Omega)} \leq \|u_1 - u_2\|_{\tilde{H}^{s,\varepsilon}(\Omega)} \leq \|z_1 - z_2\|_{\tilde{H}^{-s,\varepsilon}(\Omega)}.
\]

(3.6)

**Proof.** Taking \( v = u_1 - u_2 \) as a test function in (3.1), we get that

\[
\int_\Omega |(L_D)^2 (u_1 - u_2)|^2 \, dx + \int_\Omega \left[f(x, u_1) - f(x, u_2)\right] (u_1 - u_2) \, dx
\]

\[
= (z_1 - z_2, u_1 - u_2)_{\tilde{H}^{-s,\varepsilon}(\Omega),\tilde{H}^{s,\varepsilon}(\Omega)}.
\]

(8)
Since \( f(x, \cdot) \) is monotone for a.e. \( x \in \Omega \), we have that \([f(x, u_1) - f(x, u_2)](u_1 - u_2) \geq 0\). It follows form the preceding identity that

\[
\|u_1 - u_2\|^2_{H^s(\Omega)} = \int_{\Omega} |(L_D)^{\frac{1}{2}} (u_1 - u_2)|^2 \, dx = \langle z_1 - z_2, u_1 - u_2 \rangle_{H^{-s}(\Omega), H^s(\Omega)} \\
\leq \|z_1 - z_2\|_{H^{-s}(\Omega)} \|u_1 - u_2\|_{H^s(\Omega)}.
\]

The above estimate together with \( H^s(\Omega) \hookrightarrow L^2(\Omega) \) imply the estimate (3.6).

Next we give further qualitative properties of the above constructed operator.

**Proposition 3.4.** Let \( A_F : V_0 \to (V_0)^* \) be the surjective, continuous and bounded operator constructed in the proof of Proposition 3.2. Then \( A_F \) is also injective, hence invertible and its inverse \( A_F^{-1} \) is bounded from \((V_0)^* \) into \( V_0 \). In addition, if \( f \) satisfies the following growth condition: there exists a constant \( c \in (0, 1] \) such that

\[
c|f(x, \xi - \eta)| \leq |f(x, \xi) - f(x, \eta)|
\]

for a.e. \( x \in \Omega \) and for all \( \xi, \eta \in \mathbb{R} \), then \( A_F^{-1} \) is also continuous from \((V_0)^* \) into \( V_0 \). Furthermore, if \( r > (2^*)' = \frac{2N}{N + 2} \), then \( A_F^{-1} : L^r(\Omega) \to V_0 \) and \( A_F^{-1} : L^{r'}(\Omega) \to L^p(\Omega) \) are compact for every \( p \in (1, 2^*) \).

**Proof.** Recall that we have shown in the proof of Proposition 3.2 that the operator \( A_F \) is strictly monotone. More precisely, (3.4) implies that

\[
\langle A_F(u) - A_F(v), u - v \rangle_{(V_0)^*} = F_D(u, u - v) - F_D(v, u - v) > 0,
\]

for all \( u, v \in V_0 \) with \( u \neq v \). This shows that \( A_F \) is injective and hence, \( A_F^{-1} \) exists. The estimate

\[
\|A_F(u)\|_{(V_0)^*} \leq \|A_F(u)\|_{(V_0)^*} \leq \|A_F(u)\|_{(V_0)^*} \|u\|_{V_0}
\]

together with the coercivity of \( F_D \) (more precisely (3.5)) imply that

\[
\lim_{\|u\|_{V_0} \to \infty} \|A_F(u)\|_{(V_0)^*} = \infty.
\]

Thus \( A_F^{-1} : (V_0)^* \to V_0 \) is bounded.

Next, assume that the nonlinearity \( f \) satisfies (3.7). Notice that it follows from (3.7) that

\[
(f(x, \xi) - f(x, \eta))(\xi - \eta) \geq cf(x, \xi - \eta)(\xi - \eta),
\]

for a.e. \( x \in \Omega \) and for all \( \xi, \eta \in \mathbb{R} \). The estimate (3.8) together with the \((\Delta_2)\)-condition (2.7) imply that for every \( u, v \in V_0 \),

\[
\int_{\Omega} (f(x, u) - f(x, v))(u - v) \, dx \geq \int_{\Omega} cf(x, u - v)(u - v) \, dx \geq \int_{\Omega} F(x, u - v) \, dx.
\]

We show that \( A_F^{-1} : (V_0)^* \to V_0 \) is continuous. Assume that \( A_F^{-1} \) is not continuous. Then there exist a sequence \( z_n \in (V_0)^* \) with \( z_n \to z \) in \((V_0)^* \) as \( n \to \infty \), and a constant \( K > 0 \) such that

\[
\|A_F^{-1}(z_n) - A_F^{-1}(z)\|_{V_0} \geq K \text{ for all } n \in \mathbb{N}.
\]
Let $u_n := A_F^{-1}(z_n)$ and $u := A_F^{-1}(z)$. Since $\{z_n\}_{n \in \mathbb{N}}$ is a bounded sequence and $A_F^{-1}$ is bounded, we have that $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{V}_0$. Since $\mathcal{V}_0$ is a reflexive Banach space, by possibly passing to a subsequence if necessary, we may assume that $u_n$ converges weakly to some $v \in \mathcal{V}_0$ as $n \to \infty$. Since $A_F(u_n) - A_F(v) \to z - A_F(v)$ in $(\mathcal{V}_0)^*$ as $n \to \infty$, and $(u - u_n) \to 0$ in $\mathcal{V}_0$ as $n \to \infty$, it follows that

$$\lim_{n \to \infty} (A_F(u_n) - A_F(v), u_n - v)_{(\mathcal{V}_0)^*, \mathcal{V}_0} = 0. \tag{3.11}$$

Using (3.9) we get that for every $n \in \mathbb{N}$,

$$\int_{\Omega} |(L_D) \hat{u}(u_n - v)|^2 \, dx + \int_{\Omega} F(x, u_n - v) \, dx \leq c(\|A_F(u_n) - A_F(v), u_n - v\|_{(\mathcal{V}_0)^*, \mathcal{V}_0}).$$

This estimate together with (3.11) imply that

$$\lim_{n \to \infty} \int_{\Omega} |(L_D) \hat{u}(u_n - v)|^2 \, dx = 0 \text{ and } \lim_{n \to \infty} \int_{\Omega} F(x, u_n - v) \, dx = 0.$$

Thus, using the $(\Delta_2)$-condition (2.7), we get that $u_n \to v$ in $\mathcal{V}_0$ as $n \to \infty$. Since $A_F$ is demi-continuous (this follows from the fact that $A_F$ is hemi-continuous, monotone and bounded by the proof of Proposition 3.2), it follows that

$$z_n = A_F(u_n) \to A_F(v) \text{ in } (\mathcal{V}_0)^* \text{ and } z_n \to z = A_F(v) \text{ in } (\mathcal{V}_0)^* \text{ as } n \to \infty.$$

The uniqueness of the weak limit implies that $A_F(u) = z = A_F(v)$ and hence, by the injectivity of $A_F$ we get that $u = v$. We have shown that

$$\lim_{n \to \infty} \|A_F^{-1}(z_n) - A_F^{-1}(z)\|_{\mathcal{V}_0} = \lim_{n \to \infty} \|u_n - u\|_{\mathcal{V}_0} = 0,$$

and this contradicts (3.10). Therefore $A_F^{-1} : (\mathcal{V}_0)^* \to \mathcal{V}_0$ is continuous.

Next let $1 < q < 2^*$. Since the embedding $\mathcal{V}_0 \hookrightarrow L^q(\Omega)$ is compact, then by duality, the embedding $L^r(\Omega) \hookrightarrow (\mathcal{V}_0)^*$ is compact for every $r > (2^*)' = \frac{2N}{N+2s}$. This, together with the fact that $A_F^{-1} : (\mathcal{V}_0)^* \to \mathcal{V}_0$ is continuous and bounded, imply that $A_F^{-1} : L^r(\Omega) \to \mathcal{V}_0$ is compact for every $r > (2^*)' = \frac{2N}{N+2s}$. It remains to show that $A_F^{-1}$ is also compact as a map into $L^p(\Omega)$ for every $p \in (1, 2^*)$. Since $A_F^{-1}$ is bounded, we have to show that the image of every bounded set $B \subset L^r(\Omega)$ is relatively compact in $L^p(\Omega)$ for every $1 < p < 2^*$. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $A_F^{-1}(B)$. Let $z_n := A_F(u_n) \in B$. Since $B$ is bounded, it follows that the sequence $\{z_n\}_{n \in \mathbb{N}}$ is bounded. Since $A_F^{-1}$ is compact as a map into $\mathcal{V}_0$, we have that there is a subsequence denoted again $\{z_n\}_{n \in \mathbb{N}}$ such that $A_F^{-1}(z_n) \to u$ in $\mathcal{V}_0$ as $n \to \infty$, and hence also in $L^2(\Omega)$. We have to show that $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. Let $p \in [2, 2^*)$. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, a standard interpolation inequality shows that there exists $\tau \in (0, 1)$ such that

$$\|u_n - u_m\|_{L^p(\Omega)} \leq \|u_n - u_m\|_{L^2(\Omega)}^{\frac{\tau}{2}} \|u_n - u_m\|_{L^{2^*/(1-\tau)}(\Omega)}^{1-\tau} \leq \|u_n - u_m\|_{L^2(\Omega)}. \tag{3.12}$$

More precisely $\tau$ is such that

$$\frac{1}{p} = \frac{\tau}{2} + \frac{1-\tau}{2^*}.$$
Since \(2 \leq p < 2^*\), a simple calculation gives that
\[
0 < \tau = \frac{2(2^* - p)}{p(2^* - 2)} < 1.
\]
Now as \(u_n\) converges in \(L^2(\Omega)\), it follows from the estimate (3.12) that \(\{u_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^p(\Omega)\) and therefore converges in \(L^p(\Omega)\). Hence, \(A_F^{-1} : L^r(\Omega) \to L^p(\Omega)\) is compact for every \(p \in [2, 2^*)\). The case \(p \in (1, 2)\) follows from the fact that \(L^2(\Omega) \hookrightarrow L^p(\Omega)\). The proof is finished.

3.2. Regularity of weak solutions. The following theorem is the first main result of this section.

THEOREM 3.5. Let Assumption 2.3 hold and assume that \(z \in L^p(\Omega)\) with
\[
\begin{cases}
p > \frac{N}{2s} & \text{if } N > 2s, \\
p > 1 & \text{if } N = 2s, \\
p = 1 & \text{if } N < 2s.
\end{cases}
\] (3.13)

Then every weak solution \(u\) of (1.1) belongs to \(L^\infty(\Omega)\) and there exists a constant \(C = C(N, s, p, \Omega) > 0\) such that
\[
\|u\|_{L^\infty(\Omega)} \leq C\|z\|_{L^p(\Omega)}.
\] (3.14)

REMARK 3.6. We mention that if \(N = 1\) and \(\frac{1}{2} < s < 1\), then it follows from (2.11) that the weak solution of (1.1) is globally Hölder continuous on \(\Omega\) and in this case there is nothing to prove. Thus we need to prove the theorem only in the cases \(N \geq 2\), or \(N = 1\) and \(0 < s \leq \frac{1}{2}\).

To prove the theorem we need the following lemma which is of analytic nature and will be useful in deriving some a priori estimates of weak solutions of elliptic type equations (see e.g. [25, Lemma B.1.]).

LEMMA 3.7. Let \(\Xi = \Xi(t)\) be a nonnegative, non-increasing function on a half line \(t \geq k_0 \geq 0\) such that there are positive constants \(c, \alpha\) and \(\delta (\delta > 1)\) with
\[
\Xi(h) \leq c(h - k)^{-\alpha}\Xi(k)^{\delta} \text{ for } h > k \geq k_0.
\]
Then
\[
\Xi(k_0 + d) = 0 \quad \text{with} \quad d^\alpha = c\Xi(k_0)^{\delta - 1}2^{\alpha(\delta - 1)}.
\]
The proof of Theorem 3.5 will also use heavily the following result.

LEMMA 3.8. Let \(u \in V_0\), \(k \geq 0\) and set \(u_k := (|u| - k)^+ \text{sgn}(u)\). Then \(u_k \in V_0\) and
\[
\mathcal{F}_D(u_k, u_k) \leq \mathcal{F}_D(u, u_k),
\] (3.15)
for every \(k \geq 0\).

Proof. Let \(u \in V_0\) and \(k \geq 0\). Using [32, Lemma 2.7] we get that \(u_k \in V_0\) for every \(k \geq 0\). Let \(A_k := \{x \in \Omega : |u(x)| \geq k\}, A_k^+ := \{x \in \Omega : u(x) \geq k\}\) and \(A_k^- := \{x \in \Omega : u(x) \leq -k\}\) so that \(A_k = A_k^+ \cup A_k^-\). Then
\[
u_k = \begin{cases}
  u - k & \text{in } A_k^+,
  u + k & \text{in } A_k^-,
  0 & \text{in } \Omega \setminus A_k.
\end{cases}
\] (3.16)
Since \( f(x, \cdot) \) is odd, monotone increasing and \( 0 \leq u_k = u - k \leq u \) on \( A^+_k \), we have that for a.e. \( x \in A^+_k \),
\[
f(x, u_k)u_k = f(x, u - k)u_k \leq f(x, u)u_k. \tag{3.17}
\]
Similarly, since \( u \leq u + k = u_k \leq 0 \) on \( A^-_k \), it follows that for a.e. \( x \in A^-_k \),
\[
f(x, u_k)u_k = f(x, u + k)u_k \leq f(x, u)u_k. \tag{3.18}
\]
It follows from (3.17) and (3.18) that for every \( k \geq 0 \),
\[
\int_{\Omega} f(x, u_k)u_k \, dx \leq \int_{\Omega} f(x, u)u_k \, dx. \tag{3.19}
\]
Next, we show that for every \( k \geq 0 \),
\[
\int_{\Omega} (L_D)^2 u_k(L_D)^2 u_k \, dx \leq \int_{\Omega} (L_D)^2 u(L_D)^2 u_k \, dx. \tag{3.20}
\]
We notice that it follows from the integral representation (2.5) that
\[
\int_{\Omega} (L_D)^2 u_k(L_D)^2 u_k \, dx = \|u_k\|_{H^1(\Omega)}^2
\]
\[
= \frac{1}{2} \int_{\Omega} \int_{\Omega} |u_k(x) - u_k(y)|^2 K^D_s(x, y) \, dy \, dx + \int_{\Omega} \kappa_s(x)|u_k(x)|^2 \, dx.
\]
Calculating and using (3.16) we get that for every \( k \geq 0 \),
\[
\int_{\Omega} \int_{\Omega} |u_k(x) - u_k(y)|^2 K^D_s(x, y) \, dy \, dx
\]
\[
= \int_{A^+_k} \int_{A^+_k} (u(x) - u(y))(u_k(x) - u_k(y))K^D_s(x, y) \, dx \, dy
\]
\[
+ \int_{A^+_k} \int_{A^-_k} |u(x) - u(y) - 2k|^2 K^D_s(x, y) \, dx \, dy
\]
\[
+ \int_{A^-_k} \int_{A^-_k} (u(x) - u(y))(u_k(x) - u_k(y))K^D_s(x, y) \, dx \, dy
\]
\[
+ \int_{A^-_k} \int_{A^-_k} |u(x) - u(y) + 2k|^2 K^D_s(x, y) \, dx \, dy
\]
\[
+ \int_{\Omega \setminus A_k} \int_{A^-_k} |u_k(y)|^2 K^D_s(x, y) \, dx \, dy
\]
\[
+ \int_{A_k} \int_{\Omega \setminus A_k} |u_k(x)|^2 K^D_s(x, y) \, dx \, dy.
\]
Since \( u(x) - u(y) - 2k \geq 0 \) for a.e. \((x, y) \in A^+_k \times A^-_k\), we have that for a.e. \((x, y) \in A^+_k \times A^-_k\),
\[
(u(x) - u(y) - 2k)^2 \leq (u(x) - u(y))(u(x) - u(y) - 2k)
\]
\[
= (u(x) - u(y))(u_k(x) - u_k(y)). \tag{3.22}
\]
Since \( u(x) - u(y) + 2k \leq 0 \) for a.e. \((x, y) \in A_k^- \times A_k^+\), it follows that for a.e. \((x, y) \in A_k^- \times A_k^+\),
\[
(u(x) - u(y) + 2k)^2 \leq (u(x) - u(y))(u(x) - u(y) + 2k)
= (u(x) - u(y))(u_k(x) - u_k(y)).
\]
(3.23)

For a.e. \((x, y) \in (\Omega \setminus A_k) \times A_k\), we have that (recall that \(u_k(x) = 0\)),
\[
(u(x) - u(y))(u_k(x) - u_k(y)) = -(u(x) - u(y))u_k(y) = (u(y) - u(x))u_k(y).
\]
(3.24)

Using (3.24) we get the following estimates:
- For a.e. \((x, y) \in (\Omega \setminus A_k) \times A_k^+\) we have that (as \(k - u(x) > 0\) and \(u(y) - k \geq 0\))
  \[
  (u(x) - u(y))(u_k(x) - u_k(y)) = (u(y) - k + k - u(x))(u(y) - k) = (u(y) - k)^2 + (k - u(x))(u(y) - k) \geq (u(y) - k)^2 = |u_k(y)|^2.
  \]
  (3.25)
- For a.e. \((x, y) \in (\Omega \setminus A_k) \times A_k^-\) we have that (as \(k + u(x) > 0\) and \(u(y) + k \leq 0\))
  \[
  (u(x) - u(y))(u_k(x) - u_k(y)) = (u(y) + k + k - u(x))(u(y) + k) = (u(y) + k)^2 - (k + u(x))(u(y) + k) \geq (u(y) + k)^2 = |u_k(y)|^2.
  \]
  (3.26)

Combining (3.25) and (3.26) yields for a.e. \((x, y) \in (\Omega \setminus A_k) \times A_k\)
\[
(u(x) - u(y))(u_k(x) - u_k(y)) \geq |u_k(y)|^2.
\]
(3.27)

Proceeding in the same manner, we also get that for a.e. \((x, y) \in A_k \times (\Omega \setminus A_k)\) (recall that here \(u_k(y) = 0\)),
\[
(u(x) - u(y))(u_k(x) - u_k(y)) \geq |u_k(x)|^2.
\]
(3.28)

Using (3.22), (3.23), (3.27), and (3.28) we get from (3.21) that for every \(k \geq 0\) (recall that \(K^D_{\lambda}(x, y) \geq 0\) for a.e. \(x, y \in \Omega\)),
\[
\int_\Omega \int_\Omega |u_k(x) - u_k(y)|^2 K^D_{\lambda}(x, y) \, dxdy \leq \int_\Omega \int_\Omega (u(x) - u(y))(u_k(x) - u_k(y)) K^D_{\lambda}(x, y) \, dxdy.
\]
(3.29)

As for (3.19) we have that for every \(k \geq 0\) (recall that \(\kappa_{\lambda}(x) \geq 0\) for a.e. \(x \in \Omega\)),
\[
\int_\Omega \kappa_{\lambda}(x)|u_k(x)|^2 \, dx \leq \int_\Omega \kappa_{\lambda}(x)u(x)u_k(x) \, dx.
\]
(3.30)

Now the estimate (3.20) follows from (3.29) and (3.30) since according to (2.5) there holds
\[
\int_\Omega (L_D)^{\hat{\gamma}} u(L_D)^{\hat{\gamma}} u_k \, dx = \frac{1}{2} \int_\Omega \int_\Omega (u(x) - u(y))(u_k(x) - u_k(y))K^D_{\lambda}(x, y) \, dxdy + \int_\Omega \kappa_{\lambda}(x)u(x)u_k(x) \, dx.
\]
We have shown \((3.15)\) and the proof is finished.

\[\]

**Proof of Theorem 3.5.** Invoking Assumption 2.3 and \(z \in L^p(\Omega)\) with \(p\) satisfying \((3.13)\), it follows from \((2.10)\) that \(z \in (V_0)^*\). Hence, \((1.1)\) has a unique weak solution \(u \in V_0\) (by Proposition 3.2). Let \(k \geq 0\), \(u_k\), \(A_k\), \(A_k^+\) and \(A_k^-\) be as in the proof of Lemma 3.8. Let \(p_1 \in [1, \infty]\) be such that \(\frac{1}{p} + \frac{1}{2} + \frac{1}{p_1} = 1\) where we recall that \(2^* = \frac{2N}{N-2} > 2\). Since \(p > \frac{N}{2} = \frac{2^*}{2-2}\), we have that

\[
\frac{1}{p_1} = 1 - \frac{1}{2^*} - \frac{1}{p} > 2^* - \frac{1}{2} - \frac{2^* - 2^* - 2}{2^*} = \frac{1}{2^*} \implies p_1 < 2^*. 
\]

Taking \(v = u_k\) as a test function in \((3.1)\), using the classical Hölder inequality and noticing that \(u_k = 0\) on \(\Omega \setminus A_k\), we get that there exists a constant \(C = C(N, s, p) > 0\) such that for every \(k \geq 0\),

\[
\mathcal{F}_D(u, u_k) = \int_{\Omega} z u_k \, dx = \int_{A_k} z u_k \, dx \leq \|z\|_{L^p(\Omega)} \|u_k\|_{L^{2^*}(\Omega)} \|\chi_{A_k}\|_{L^{p_1}(\Omega)},
\]

where \(\chi_{A_k}\) denotes the characteristic function of the set \(A_k\). Using \((3.15)\), \((3.32)\), \((2.10)\) and the fact that \(\int_{\Omega} f(x, u_k) u_k \, dx \geq 0\), we get that there exist two constants \(C, C_1 > 0\) such that for every \(k \geq 0\),

\[
C \|u_k\|_{L^{2^*}(\Omega)}^2 \leq \|u_k\|_{L^{2^*}(\Omega)}^2 \leq \mathcal{F}_D(u, u_k) \leq \mathcal{F}_D(u, u_k) \leq C_1 \|z\|_{L^p(\Omega)} \|u_k\|_{L^{2^*}(\Omega)} \|\chi_{A_k}\|_{L^{p_1}(\Omega)},
\]

and this implies that there exists a constant \(C > 0\) such that for every \(k \geq 0\),

\[
\|u_k\|_{L^{2^*}(\Omega)} \leq C \|z\|_{L^p(\Omega)} \|\chi_{A_k}\|_{L^{p_1}(\Omega)}. 
\]

Let \(h > k\). Then \(A_h \subset A_k\) and on \(A_h\) we have that \(|u_k| \geq h - k\). Therefore, it follows from \((3.33)\) that for every \(h \geq k > 0\),

\[
\|\chi_{A_h}\|_{L^{2^*}(\Omega)} \leq C(h - k)^{-1} \|z\|_{L^p(\Omega)} \|\chi_{A_k}\|_{L^{p_1}(\Omega)}. 
\]

Let \(\delta := \frac{2^*}{p_1} > 1\) by \((3.31)\). Then using the Hölder inequality again we get that there exists a constant \(C > 0\) such that for every \(h \geq 0\), we have

\[
\|\chi_{A_k}\|_{L^{p_1}(\Omega)} \leq C \|\chi_{A_k}\|_{L^{2^*}(\Omega)}. 
\]

It follows from \((3.34)\) and \((3.35)\) that there exists a constant \(C > 0\) such that for every \(h > k \geq 0\),

\[
\|\chi_{A_h}\|_{L^{2^*}(\Omega)} \leq C(h - k)^{-1} \|z\|_{L^p(\Omega)} \|\chi_{A_k}\|_{L^{2^*}(\Omega)}. 
\]

It follows from Lemma 3.7 with \(\Xi(k) = \|\chi_{A_k}\|_{L^{2^*}(\Omega)}\) that there exists a constant \(C_1 > 0\) such that

\[
\|\chi_{A_k}\|_{L^{2^*}(\Omega)} = 0 \quad \text{with} \quad K = CC_1 \|z\|_{L^p(\Omega)}.
\]

We have shown \((3.14)\) and the proof is finished.  \(\square\)
Next we give an $L^\infty$-estimate for the difference of two solutions which is the second main result of this section.

**Proposition 3.9.** Assume that Assumption 2.3 hold and that $f$ satisfies the growth condition (3.7). Let $z_1, z_2 \in L^p(\Omega)$ with $p$ as in (3.13) and let $u_1, u_2 \in \mathcal{V}_0 \cap L^\infty(\Omega)$ be the corresponding weak solutions. Then there exists a constant $C = C(N, p, s, \Omega) > 0$ such that

$$
\|u_1 - u_2\|_{L^\infty(\Omega)} \leq C \|z_1 - z_2\|_{L^p(\Omega)}.
$$

(3.36)

**Proof.** Let Assumption 2.3 hold and assume that $f$ satisfies the growth condition (3.7). We prove the proposition in two steps.

**Step 1.** Let $k \geq 0$. Set $u := u_1 - u_2$ and $u_k := (|u| - k)^+ \operatorname{sgn}(u)$. We claim that $u_k \in \mathcal{V}_0$ and

$$
c F_D(u_k, u_k) \leq F_D(u_1, u_k) - F_D(u_2, u_k),
$$

(3.37)

for every $k \geq 0$, where $c \in (0, 1]$ is the constant appearing in (3.7). Indeed, using [32, Lemma 2.7] we get that $u_k \in \mathcal{V}_0$. Let $A_k := \{x \in \Omega : |u(x)| \geq k\}, A_k^- := \{x \in \Omega : u(x) \geq k\}$ and $A_k^- := \{x \in \Omega : u(x) \leq -k\}$ so that $A_k = A_k^+ \cup A_k^-$. It follows from the representation (2.5) that

$$
F_D(u_1, u_k) - F_D(u_2, u_k) = \int_\Omega (L_D)^\sharp (u_1 - u_2)(L_D)^\sharp (u_k) \, dx + \int_\Omega \left(f(x, u_1) - f(x, u_2)\right) u_k \, dx
$$

$$
= \frac{1}{2} \int_\Omega \int_\Omega \left( (u_1 - u_2)(x) - (u_1 - u_2)(y) \right) \left( u_k(x) - u_k(y) \right) K^D_s(x, y) \, dxdy
$$

$$
+ \int_\Omega \kappa_s(x)(u_1 - u_2)(x)u_k(x) \, dx + \int_\Omega \left(f(x, u_1) - f(x, u_2)\right) u_k \, dx.
$$

(3.38)

Proceeding exactly as the proof of Lemma 3.8 we get that

$$
\frac{1}{2} \int_\Omega \int_\Omega \left( (u_1 - u_2)(x) - (u_1 - u_2)(y) \right) \left( u_k(x) - u_k(y) \right) K^D_s(x, y) \, dxdy
$$

$$
+ \int_\Omega \kappa_s(x)(u_1 - u_2)(x)u_k(x) \, dx
$$

$$
\geq \frac{1}{2} \int_\Omega \int_\Omega |u_k(x) - u_k(y)|^2 K^D_s(x, y) \, dxdy + \int_\Omega \kappa_s(x)|u_k(x)|^2 \, dx
$$

$$
= \int_\Omega |(L_D)^\sharp (u_k)|^2 \, dx.
$$

(3.39)

We notice that

$$
\int_\Omega \left(f(x, u_1) - f(x, u_2)\right) u_k \, dx
$$

$$
= \int_\Omega cf(x, u_k) u_k \, dx + \int_\Omega \left(f(x, u_1) - f(x, u_2) - cf(x, u_k)\right) u_k \, dx,
$$

(4.0)

where $c \in (0, 1]$ is the constant appearing in (3.7). For a.e. $x \in A_k^+$, we have that

$$
cf(x, u_k(x)) = cf(x, u_1(x) - u_2(x) - k) \leq cf(x, u_1(x) - u_2(x))
$$

$$
\leq f(x, u_1(x)) - f(x, u_2(x)).
$$
Multiplying this inequality with \( u_k(x) \geq 0 \) gives for a.e. \( x \in A_k^+ \):

\[
\left[ f(x, u_1(x)) - f(x, u_2(x)) - cf(x, u_k(x)) \right] u_k(x) \geq 0. \tag{3.41}
\]

Similarly, for a.e. \( x \in A_k^- \), we have that

\[
cf(x, u_k(x)) = cf(x, u_1(x) - u_2(x) + k) \geq cf(x, u_1(x) - u_2(x))
\]

\[
\geq f(x, u_1(x)) - f(x, u_2(x)).
\]

Hence multiplying this inequality with \( u_k(x) \leq 0 \), we get that for a.e. \( x \in A_k^- \),

\[
\left[ f(x, u_1(x)) - f(x, u_2(x)) - cf(x, u_k(x)) \right] u_k(x) \geq 0. \tag{3.42}
\]

Combining (3.40), (3.41) and (3.42) we get that for every \( k \geq 0 \),

\[
\int_{\Omega} \left( f(x, u_1) - f(x, u_2) \right) u_k \, dx \geq c \int_{\Omega} f(x, u_k) u_k \, dx. \tag{3.43}
\]

Now it follows from (3.38), (3.39) and (3.43) that

\[
F_D(u_1, u_k) - F_D(u_2, u_k) \geq \int_{\Omega} |(L_D)\varphi(u_k)|^2 \, dx + c \int_{\Omega} f(x, u_k) u_k \, dx
\]

\[
\geq c \left( \int_{\Omega} |(L_D)\varphi(u_k)|^2 \, dx + \int_{\Omega} f(x, u_k) u_k \, dx \right)
\]

\[
= cF_D(u_k, u_k),
\]

and we have shown the claim (3.37).

**Step 2.** It follows from (3.37) that there exists a constant \( C > 0 \) such that for every \( k \geq 0 \),

\[
C\|u_k\|^2_{L^2(\Omega)} \leq c\|u_k\|^2_{H^2(\Omega)} \leq cF_D(u_k, u_k) \leq F_D(u_1, u_k) - F_D(u_2, u_k)
\]

\[
= \int_{\Omega} (z_1 - z_2) u_k \, dx.
\]

Now following line by line the proof of Theorem 3.5 we get the estimate (3.36).

**Remark 3.10.** We mention that all the results given in Proposition 3.9 remain true if one replaces the growth condition (3.7) with the following local Lipschitz continuity condition: For all \( M > 0 \) there exists a constant \( L_M > 0 \) such that \( f \) satisfies

\[
|f(x, \xi) - f(x, \eta)| \leq L_M|\xi - \eta| \tag{3.44}
\]

for a.e. \( x \in \Omega \) and \( \xi, \eta \in \mathbb{R} \) with \( |\eta|, |\xi| \leq M \). A condition such as (3.44) is needed to prove the \( H^{2+\beta}(\Omega) \) regularity for \( u \) provided \( g \in H^3(\Omega) \) where \( 0 \leq \beta < 1 \) (see [7, Corollary 2.15]). This higher regularity is important for finite element error estimates as shown in [7, Section 4].
3.3. The case of the fractional Laplace operator. In this section we consider the semilinear elliptic problem (1.5). Firstly, the fractional Laplace operator \((-\Delta)^s\) is given formally for \(x \in \mathbb{R}^N\) by

\[
(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy
\]

\[
= C_{N,s} \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^N : |x-y| \geq \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,
\]

whenever the limit exists, and \(C_{N,s}\) is a normalization constant depending only on \(N\) and \(s\). We refer to [13, 14, 20] for the class of functions for which the limit exists and for further properties and applications of this operator.

Secondly, in order to give our notion of solutions to (1.5) we need to introduce the fractional order Sobolev space

\[
H^s_0(\Omega) := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.
\]

Then for every \(0 < s < 1\), we have that \(H^s_0(\Omega)\) endowed with the norm

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}
\]

is a Hilbert space, and we shall denote by \(H^{-s}(\Omega)\) its dual. It is well known (see e.g. [20]) that the embedding (2.2) holds with \(H^s_0(\Omega)\) replaced by \(H^s_0(\Omega)\). We notice that there is a priori no obvious inclusion between \(H^s_0(\Omega)\) and \(H^s_0(\Omega)\). In fact for an arbitrary bounded open set, the two spaces are different since \(D(\Omega)\) is not always dense in \(H^s_0(\Omega)\). But if \(\Omega\) has a continuous boundary, then \(H^s_0(\Omega) = H^s_0(\Omega)\) for \(\frac{1}{2} < s < 1\). For \(0 < s \leq \frac{1}{2}\), even if \(\Omega\) is smooth, the two spaces do not coincide.

Definition 3.11. A function \(u \in H^s_0(\Omega)\) is said to be a weak solution of (1.5) if the identity

\[
\frac{CN_s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy + \int_{\Omega} f(x,u) \, dx = (z,v)_{H^{-s}(\Omega),H^s_0(\Omega)}
\]

holds for every \(v \in H^s_0(\Omega)\) and the right hand side makes sense.

Thirdly, we notice the following.

Remark 3.12. Letting \(V_0 := H^s(\Omega) \cap L^p(\Omega)\), then all the results in Sections 3.1 and 3.2 hold for the system (1.5) if one replaces in the statements and the proofs \(V_0\), \(H^s(\Omega)\), and \(H^{-s}(\Omega)\) by \(V_0\), \(H^s_0(\Omega)\), and \(H^{-s}(\Omega)\), respectively and the form \(\mathcal{F}_D\) by

\[
\mathcal{F}_D(u,v) = \frac{CN_s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy + \int_{\Omega} f(x,u)v \, dx.
\]

All the proofs follow similarly with very minor modifications.

Next, let \((-\Delta)^s_D\) be the selfadjoint operator on \(L^2(\Omega)\) associated with the form

\[
\mathcal{E}_D(u,v) := \frac{CN_s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy, \quad u,v \in H^s_0(\Omega),
\]

\[
\text{for some } s \in (0,1),
\]
in the sense that
\[
\begin{cases}
    D((-\Delta)^D) = \{ u \in H_0^3(\Omega), \exists w \in L^2(\Omega), E_D(u, v) = (w, v)_{L^2(\Omega)} \forall v \in H_0^3(\Omega) \} \\
    (-\Delta)^D u = w.
\end{cases}
\]

With this setting, the system (1.5) can be rewritten as
\[
(-\Delta)^D u + f(x, u) = z \quad \text{in} \quad \Omega.
\]

We also mention that even if taking \(a_{ij} = \delta_{ij}\), that is \(L = -\Delta\), the operator \(L_D^s\) and \((-\Delta)^D\) are different. More precisely their eigenvalues and eigenfunctions are different. We refer to [11, 29] for more details on this topic.

3.4. An Example. We conclude this section with the following example.

**Example 3.13.** Let \(q \in [1, \infty)\) and let \(b : \Omega \to (0, \infty)\) be a function in \(L^\infty(\Omega)\), that is, \(b(x) > 0\) for a.e. \(x \in \Omega\). Define the function \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) by \(f(x, t) = b(x)|t|^q - t\). It is clear that \(f\) satisfies Assumption 2.2 and the associated function \(F : \Omega \times \mathbb{R} \to [0, \infty)\) is given by \(F(x, t) = \frac{1}{q+1}b(x)|t|^{q+1}\). For a.e. \(x \in \Omega\), the inverse \(\tilde{f}(x, \cdot)\) of \(f(x, \cdot)\) is given by \(\tilde{f}(x, t) = (b(x))^{-\frac{1}{q}}|t|^{-\frac{1}{q-1}}.\) Therefore, the complementary function \(\tilde{F}\) of \(F\) is given by \(\tilde{F}(x, t) = q \cdot (b(x))^{-\frac{1}{q}}|t|^{\frac{q+1}{q-1}}\). Hence,
\[
tf(x, t) = (q + 1)F(x, t) \quad \text{and} \quad t\tilde{f}(x, t) = \frac{q + 1}{q}\tilde{F}(x, t),
\]
and we have shown that Assumption 2.3 is also satisfied.

Next, let us show that \(f\) satisfies the growth condition (3.7). If \(\xi = 0\) or \(\eta = 0\) then the assertion (3.7) is obvious since \(f(x, 0) = 0\). Hence, we assume that \(\xi \neq 0\) and \(\eta \neq 0\). Moreover, since \(|f(x, -\gamma)| = |f(x, \gamma)|\) we may assume without loss of generality that \(|\eta| \geq |\xi|\). Hence, there exists \(\alpha \in \mathbb{R}\) with \(|\alpha| \geq 1\) such that \(\eta = \alpha\xi\). It follows that
\[
|f(x, \xi - \eta)| = |b(x)\cdot|\xi - \alpha\xi|^q = |b(x)|\cdot|1 - \alpha|^q|\xi|^q
\]
and
\[
|f(x, \xi) - f(x, \eta)| = |b(x)|\cdot||\xi|^q\sgn(\xi) - |\alpha|^q|\xi|^q\sgn(\alpha)\sgn(\xi)|
= |b(x)|\cdot|\xi|^q\cdot|1 - |\alpha|^q\sgn(\alpha)|.
\]
The proof is done if \(c(1 - |\alpha|^q) \leq |1 - |\alpha|^q\sgn(\alpha)|\) for all \(\alpha \in \mathbb{R}\ \backslash \ (-1, 1)\). For \(\alpha = 1\) this inequality is obvious. For \(\alpha > 1\) this inequality is equivalent to
\[
c(\alpha - 1)^q \leq \alpha^q - 1 \iff c \leq \frac{\alpha^q - 1}{(\alpha - 1)^q} = g(\alpha)
\]
where \(g : [1, \infty) \to \mathbb{R}\) is given by \(g(x) := (x^q - 1)/(x - 1)^q\). Differentiating shows that \(g'(x) \leq 0\), hence,
\[
\inf_{x>1}g(x) = \lim_{x \to \infty}g(x) = 1 \geq c.
\]
To finish, we prove the case \(\alpha \leq -1\). In this case, we have to show that
\[
c(1 + |\alpha|)^q \leq 1 + |\alpha|^q \iff c \leq \frac{1 + |\alpha|^q}{(1 + |\alpha|)^q} = h(|\alpha|)
\]
where \( h : [1, \infty) \to \mathbb{R} \) is given by \( h(x) = (1+x^q)/(1+x)^q \). Differentiating shows that \( h'(x) \geq 0 \), hence,

\[
\inf_{x \geq 1} h(x) = h(1) = c
\]

and this completes the proof of (3.7). In particular, we have that \( f \) also satisfies the condition (3.44).

4. Optimal control problem: existence. Now as the state equation (1.1) has a unique solution, it follows that the control-to-state map (solution map)

\[
S : L^\infty(\Omega) \to \mathcal{V}_0 \cap L^\infty(\Omega), \quad z \mapsto S(z) = u
\]
is well defined. We notice that \( S \) is also well defined as a map from \( L^p(\Omega) \) into \( \mathcal{V}_0 \cap L^\infty(\Omega) \) where \( p \) is as in (3.13). We begin by showing that under the growth assumption (3.7), the mapping \( S \) is Lipschitz continuous.

**Lemma 4.1 (S is Lipschitz continuous).** Let Assumption 2.3 hold and \( z_1, z_2 \in H^{-s}(\Omega) \) and \( u_1, u_2 \in \mathcal{V}_0 \) be the corresponding weak solution of (1.1). Then there exists a constant \( C = C(N, s, \Omega) > 0 \) such that

\[
\|u_1 - u_2\|_{L^2(\Omega)} \leq C\|u_1 - u_2\|_{H^{-s}(\Omega)} \leq \|z_1 - z_2\|_{H^{-s}(\Omega)}. \tag{4.1}
\]

In addition, let \( p \) be as in (3.13). Let Assumption 2.3 hold and assume that \( f \) satisfies the growth condition (3.7). Let \( u_1 \in \mathcal{V}_0 \cap L^\infty(\Omega) \) and \( u_2 \in \mathcal{V}_0 \cap L^\infty(\Omega) \) be the weak solutions to (1.1) with right hand sides \( z_1 \) and \( z_2 \) in \( L^p(\Omega) \), respectively. Then there exists a constant \( C = C(N, p, s, \Omega) > 0 \) such that

\[
\|u_1 - u_2\|_{L^\infty(\Omega)} + \|u_1 - u_2\|_{H^s(\Omega)} \leq C\|z_1 - z_2\|_{L^p(\Omega)}. \tag{4.2}
\]

**Proof.** Firstly, the estimate (4.1) is due to Proposition 3.3. Notice that in this case one does not need the growth assumption (3.7) on \( f \).

Next, it follows from (3.6) that

\[
\|u_1 - u_2\|_{H^{-s}(\Omega)} \leq C\|u_1 - u_2\|_{H^{-s}(\Omega)} \leq C\|z_1 - z_2\|_{L^p(\Omega)}.
\]

Secondly, since by assumption \( f \) satisfies (3.7), it follows from (3.36) in Proposition 3.9 that

\[
\|u_1 - u_2\|_{L^\infty(\Omega)} \leq C\|z_1 - z_2\|_{L^p(\Omega)}.
\]

Now (4.2) follows from the above two estimates. The estimate (4.2) shows that \( S \) is Lipschitz continuous as a map from \( L^p(\Omega) \) into \( H^s(\Omega) \cap L^\infty(\Omega) \) and hence, as a map from \( L^\infty(\Omega) \) into \( H^s(\Omega) \cap L^\infty(\Omega) \).

Now we recall the cost functional from (1.2), i.e. \( J(u, z) := J_1(u) + J_2(z) \). We let \( J_1 : \mathcal{V}_0 \to \mathbb{R} \) and \( J_2 : L^2(\Omega) \to \mathbb{R} \), and as a result we can write the reduced minimization problem

\[
\min_{z \in Z_{ad}} J(z) := J_1(S(z)) + J_2(z). \tag{4.3}
\]

Next we state the existence of solution to (4.3) and equivalently to the system (1.2)–(1.3).
Theorem 4.2 (Existence of optimal control). Let the assumptions of Theorem 3.5 hold. Assume in addition that $f$ satisfies the growth condition (3.7) and that

$$f(\cdot, w(\cdot)) \in L^2(\Omega) \quad \text{for every } w \in L^\infty(\Omega).$$

Then under the following assumptions on $J_1$ and $J_2$,

(i) $J_1 : V_0(\Omega) \to [0, +\infty)$ is weakly lower semicontinuous;

(ii) $J_2 : L^p(\Omega) \to (-\infty, +\infty]$ is proper, convex, lower-semicontinuous and bounded from below;

the minimization problem (4.3) admits a solution.

Proof. We begin by noticing that $J$ is bounded from below. Therefore, the infimum

$$j := \inf_{z \in Z_{ad}} J(z)$$

exists. Let $\{(u_n, z_n)\}_{n \in \mathbb{N}}$ be a minimizing sequence, that is, $z_n \in Z_{ad}$ and $u_n = S(z_n)$, for $n \in \mathbb{N}$ are such that $J(z_n) \to j$ as $n \to \infty$.

Notice that $Z_{ad} \subset L^\infty(\Omega) \subset L^p(\Omega)$ for every $1 < p < \infty$. Since $z_n \in Z_{ad}$, we have that $\|z_n\|_{L^p(\Omega)} \leq \|z_n\|_{L^\infty(\Omega)} \leq \max\{\|z_n\|_{L^\infty(\Omega)}, \|z_0\|_{L^\infty(\Omega)}\}$. Since $L^p(\Omega)$ is a reflexive Banach space, we have that by taking a subsequence if necessary, we may assume that $\{z_n\}_{n \in \mathbb{N}}$ converges weakly in $L^p(\Omega)$ to some $\bar{z} \in Z_{ad}$, i.e., $z_n \rightharpoonup \bar{z}$ as $n \to \infty$. This $\bar{z}$ is the candidate for our optimal control.

Next we shall show that the state $\{u_n\}_{n \in \mathbb{N}}$ converges to some $\bar{u}$ in a suitable sense and $(\bar{u}, \bar{z})$ satisfies the state equation. Towards this end we introduce a sequence

$$b_n(\cdot) = f(\cdot, u_n(\cdot)), \quad n \in \mathbb{N}.$$  

Since

$$\|u_n\|_{L^\infty(\Omega)} \leq C \|z_n\|_{L^p(\Omega)} \leq C \max\{\|z_n\|_{L^\infty(\Omega)}, \|z_0\|_{L^\infty(\Omega)}\},$$

it follows from the fat that $f(x, \cdot) = 0$ for a.e. $x \in \Omega$ and the assumption (4.4) that $\{b_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega)$. As a consequence, taking a subsequence if necessary, we may assume that $b_n \to b$ in $L^2(\Omega)$ as $n \to \infty$.

Notice that for every $n \in \mathbb{N}$, $u_n$ satisfies

$$\int_\Omega (L_D)\bar{z} u_n(L_D)\bar{z} v \, dx = \int_\Omega B_n v \, dx \quad \forall v \in V_0,$$

where $B_n := -f(x, u_n) + z_n$ converges weakly in $L^2(\Omega)$ to $-b + \bar{z}$ as $n \to \infty$. Taking $v = u_n$ in (4.6), we get that there exists a constant $C > 0$ (independent of $n$) such that

$$\|u_n\|_{H^1(\Omega)} \leq C \|B_n\|_{L^2(\Omega)}.$$

Since $\{B_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega)$, it follows from the preceding estimate that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega)$, hence as before, we may assume that $u_n \rightharpoonup \bar{u}$ in $H^1(\Omega)$ as $n \to \infty$ and hence, strongly in $L^2(\Omega)$ since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. In addition we have that $\bar{u} \in L^\infty(\Omega)$ and thus using (4.4), we get that

$$\|u_n\|_{L^\infty(\Omega)} \leq C \|B_n\|_{L^2(\Omega)}.$$

Since $\{B_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega)$, it follows from the preceding estimate that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^s(\Omega)$, hence as before, we may assume that $u_n \rightharpoonup \bar{u}$ in $H^s(\Omega)$ as $n \to \infty$ and hence, strongly in $L^2(\Omega)$ since the embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact. In addition we have that $\bar{u} \in L^\infty(\Omega)$ and thus using (4.4), we get that

$$\|u_n\|_{L^\infty(\Omega)} \leq C \|B_n\|_{L^2(\Omega)}.$$

Secondly, as $u_n, \bar{u} \in L^\infty(\Omega)$ for every
n ∈ N, it follows from (4.4) that \( f(\cdot, u_n(\cdot)), f(\cdot, \bar{u}(\cdot)) \in L^2(\Omega) \) for every \( n \in \mathbb{N} \). Thirdly, it follows from (4.5) that there exists a constant \( M > 0 \) (independent of \( n \)) such that 
\[ |u_n(x)| \leq M \quad \text{for a.e. } x ∈ Ω. \]
Since \( f(x, \cdot) \) is odd and strictly increasing for a.e. \( x ∈ Ω \) and \( f(x, 0) = 0 \), we have that 
\[ |f(x, u_n(x))| \leq f(x, M) \] and \( f(\cdot, M) \in L^2(\Omega) \) by the assumption (4.4). Therefore using the Lebesgue Dominated Convergence Theorem, we get that \( f(\cdot, u_n(\cdot)) \rightarrow f(\cdot, \bar{u}(\cdot)) \) in \( L^2(\Omega) \) as \( n \rightarrow \infty \). Thus taking the limit as \( n \rightarrow \infty \) in both sides of the following identity
\[
\int_Ω (L_D)^2 u_n(L_D)^2 v \, dx + \int_Ω f(x, u_n)v \, dx = \int_Ω z_n v \, dx \quad \forall v ∈ V_0,
\]
and using all the above convergences, we can conclude that \( \bar{u} \) is the weak solution of (1.1) corresponding to the right hand side \( \bar{z} \). It remains to show that \( \bar{z} \) is the optimal control. This is due to the weak lower semicontinuity of \( J_1 \), i.e., since \( S(z_n) → S(\bar{z}) \) in \( V_0 \) then we have that \( \liminf_{n→∞} J_1(S(z_n)) ≥ J_1(S(\bar{z})) \). Since \( J_2 \) is convex, proper and lower semicontinuous (cf. (ii)), it follows that it is weakly lower semicontinuous (see e.g. [10, Theorem 3.3.3]), i.e., \( \liminf_{n→∞} J_2(z_n) ≥ J_2(\bar{z}) \). It then follows that
\[
\min_{z ∈ Z_{ad}} J(z) = \liminf_{n→∞} J(u_n, z_n) ≥ J(S(\bar{z}), \bar{z}) ≥ \min_{z ∈ Z_{ad}} J(z).
\]
This completes the proof. \( \Box \)

**Remark 4.3.** We mention that the nonlinearity \( f \) given in Example 3.13 above also satisfies the assumption (4.4).

**Remark 4.4 (Nonsmooth cost functionals).** Notice that the condition (i) in Theorem 4.2 already allows nonsmooth \( J_1 \) such as \( J_1(u) := \|u - u_d\|_{L^1(Ω)} \) which is non-smooth but Lipschitz continuous, where \( u_d ∈ L^1(Ω) \) is given. The proof in Theorem 4.2 extends in a straightforward manner to other nonsmooth control regularizations such as BV-regularization, i.e., when \( J_2(z) := \int_Ω \| \nabla z \| \) or when \( J(u, z) := J_1(u) + J_2(z) + \int_Ω |z| \) (with \( J_1 \) and \( J_2 \) as in Theorem 4.2) by following [8, Theorem 3.4].

**Remark 4.5.** We also notice that all the results proved in this section also hold for the map \( \bar{S} : L^∞(Ω) → V_0 \cap L^∞(Ω) \), that is, the solution operator to (1.5).

### 5. Optimal control problem: first and second order optimality conditions

Before we investigate more regularity of the control-to-state map, we make the following further regularity assumptions on the nonlinearity \( f \).

**Assumption 5.1.** We assume the following.

(i) The function \( f(x, \cdot) \) is \( k \)-times, with \( k = 1, 2 \), continuously differentiable for a.e. \( x ∈ Ω \).

(ii) For all \( M > 0 \) there exist a constant \( L_M > 0 \) such that \( f \) satisfies Assumption 3.44 and
\[
\left| \frac{∂^k f}{∂u^k}(x, ξ) - \frac{∂^k f}{∂u^k}(x, η) \right| ≤ L_M |ξ - η|, \quad k = 1, 2,
\]
for a.e. \( x ∈ Ω \) and \( ξ, η ∈ \mathbb{R} \) with \( |ξ|, |η| ≤ M \).

(iii) \( \frac{∂f}{∂u}(\cdot, 0) ∈ L^∞(Ω) \).

**Remark 5.2.** For notational convenience, we will write \( f_u \) and \( f_{uu} \) instead of \( \frac{∂f}{∂u} \) and \( \frac{∂^2 f}{∂u^2} \), respectively.

Next we show that \( S \) is twice continuously Fréchet differentiable.
LEMMA 5.3 (Twice Fréchet differentiability of $S$). Let Assumptions 2.3 and 5.1(i) hold for $k = 1, 2$. Then the mapping $S : L^\infty(\Omega) \to \mathbb{H}^*(\Omega) \cap L^\infty(\Omega)$ is twice continuously Fréchet differentiable. Moreover, for all $z, \zeta \in L^\infty(\Omega)$, $S'(z)\zeta = u_\zeta \in \mathbb{H}^*(\Omega)$ is defined as the unique weak solution of

$$(L_D)^s u_\zeta + f_u(\cdot, u)u_\zeta = \zeta \quad \text{in } \Omega$$

(5.1)

where $u = S(z)$. Furthermore, for every $z, \zeta_1, \zeta_2 \in L^\infty(\Omega)$,

$$S''(z)[\zeta_1, \zeta_2] := (S''(z)\zeta_1)\zeta_2 = u_{\zeta_1, \zeta_2} \in \mathcal{V}_0$$

is the unique solution of

$$(L_D)^s u_{\zeta_1, \zeta_2} + f_u(\cdot, u)u_{\zeta_1, \zeta_2} = -f_{uu}(\cdot, u)u_{\zeta_1} u_{\zeta_2} \quad \text{in } \Omega$$

(5.2)

where $u = S(z)$ and $u_{\zeta_i} = S'(z)\zeta_i$, $i = 1, 2$.

Proof. The proof is based on the implicit function theorem. Let $p$ be as in (3.13). We introduce the space

$$\mathcal{W} = \left\{ u \in \mathbb{H}^*(\Omega) \cap L^\infty(\Omega), \quad (L_D)^s u \in L^p(\Omega) \right\}$$

with norm

$$||u||_{\mathcal{W}} = ||u||_{\mathbb{H}^*(\Omega) \cap L^\infty(\Omega)} + ||(L_D)^s u||_{L^p(\Omega)}.$$ 

We next introduce the function

$$\mathcal{F} : \mathcal{W} \times L^\infty(\Omega) \to L^p(\Omega), \quad \mathcal{F}(u, z) = (L_D)^s u + f(\cdot, u) - z.$$

Under Assumption 5.1(i), $\mathcal{F}$ is $C^2$. Moreover, Assumptions 2.3 and 5.1(i) imply that $f_u \geq 0$. Then using Theorem 3.5 with $f \equiv 0$ we deduce that

$$\mathcal{F}_u(u, z) = (L_D)^s u + f_u(\cdot, u)$$

is an isomorphism from $\mathcal{W}$ to $L^p(\Omega)$. Since $\mathcal{F}(u, z) = 0$ if and only if $u = S(z)$, we can apply the implicit function theorem to deduce that $S$ is of class $C^2$ and fulfills $\mathcal{F}(S(z), z) = 0$. Therefore (5.1) and (5.2) follow easily. The proof is finished.

Throughout the remainder of the paper we restrict ourselves to the case where

$$J_1(u) = \frac{1}{2} ||u - u_d||_{L^2(\Omega)}^2, \quad J_2(z) = \frac{\mu}{2} ||z||_{L^2(\Omega)}^2.$$ 

(5.3)

The given function $u_d \in L^2(\Omega)$ and $\mu > 0$ is the cost of the control. We further remark that these results can be directly extended to a more general setting as described in the monograph [31].

Next, we introduce the adjoint state $\phi \in \mathbb{H}^*(\Omega)$ as the unique weak solution of the adjoint equation

$$(L_D)^s \phi + f_u(\cdot, u)\phi = u - u_d \quad \text{in } \Omega,$$

(5.4)

where $u \in L^\infty(\Omega)$ is given. Using Assumptions 2.3 and 5.1(i) we have $f_u(x, u(x)) \geq 0$ for a.e. $x \in \Omega$. Moreover,

$$||f_u(\cdot, u)||_{L^\infty(\Omega)} \leq ||f_u(\cdot, u) - f_u(\cdot, 0)||_{L^\infty(\Omega)} + ||f_u(\cdot, 0)||_{L^\infty(\Omega)} \leq C ||u||_{L^\infty(\Omega)} + ||f_u(\cdot, 0)||_{L^\infty(\Omega)} < \infty$$

(5.5)

where in the last step we have used Assumption 5.1(iii).
Lemma 5.4 (Existence of solution to the adjoint equation). Let Assumptions 2.3 and 5.1(i)-(iii) hold for $k = 1$. Let $u \in L^\infty(\Omega)$ be given. Then there exists a unique $\phi \in H^s(\Omega)$ weak solution to (5.4). In addition, $\phi \in H^{2s}(\Omega)$.

Proof. Since $f_u(\cdot, u) \in L^\infty(\Omega)$ and $f_u(\cdot, u) \geq 0$, the existence and uniqueness follows by using Assumption 5.1(iii) and Theorem 3.5. Since,

$$\phi_k = \lambda_k^{-s} \int_\Omega (u - u_d - f_u(\cdot, u)\phi) \varphi_k \, dx,$$

then using the definition of the $H^{2s}$-norm (see (2.4)) we deduce that

$$\|\phi\|_{H^{2s}(\Omega)} = \|u - u_d - f_u(\cdot, u)\phi\|_{L^2(\Omega)}$$

and the proof is complete. \qed

Lemma 5.5 ($J$ is twice Fréchet differentiable). Let Assumption 5.1(i) and assumptions of Lemma 5.4 hold. Then the functional $J : L^\infty(\Omega) \to \mathbb{R}$ is twice continuously Fréchet differentiable. Moreover for every $z, \zeta, \zeta_1, \zeta_2 \in L^\infty(\Omega)$ there holds

$$J'(z)\zeta = \int_\Omega (\phi + \mu z)\zeta \, dx,$$

and

$$J''(z)[\zeta_1, \zeta_2] = \int_\Omega S'(z)\zeta_1 S'(z)\zeta_2 - \phi f_{uu}(\cdot, S(z)) S'(z) S'(z) \zeta_1 \zeta_2 \, dx + \mu \int_\Omega \zeta_1 \zeta_2 \, dx.$$

Proof. The proof is based on the chain rule and the results from Lemma 5.3 together with

$$\int_\Omega (S(z) - u_d) S'(z) v \, dx = \int_\Omega \phi v \, dx$$

and

$$\int_\Omega (S(z) - u_d) S''(z) [\zeta_1, \zeta_2] \, dx = -\int_\Omega \phi f_{uu}(\cdot, S(z)) S'(z) S'(z) \zeta_1 \zeta_2 \, dx$$

which can be deduced from the weak formulations of (5.1), (5.2) and (5.4). \qed

Since $J$ is non-convex, in general due to the semilinear state equation, we cannot expect a unique solution to the optimal control problem. We introduce locally optimal solutions: $\bar{z} \in Z_{ad}$ is locally optimal or local solution to (4.3) if there exists an $\varepsilon > 0$ such that

$$J'(z) \leq J'(\bar{z}) \quad \forall z \in Z_{ad} \cap B_\varepsilon(\bar{z})$$

where the $L^\infty$-ball $B_\varepsilon(\bar{z})$ centered at $\bar{z}$ with radius $\varepsilon$ is defined by

$$B_\varepsilon(\bar{z}) := \left\{ z \in L^\infty(\Omega), \| z - \bar{z}\|_{L^\infty(\Omega)} \leq \varepsilon \right\}.$$

Theorem 5.6 (First order necessary conditions). For every local solution $\bar{z}$ of the problem (4.3) there exists a unique optimal state $\bar{u} = S(\bar{z})$ and an optimal adjoint state $\phi$ such that

$$\int_\Omega (\phi + \mu \bar{z}) (z - \bar{z}) \, dx \geq 0 \quad \forall z \in Z_{ad} \quad (5.6)$$
which is equivalent to
\[
\bar{z}(x) = \Pi_{[z_a(x), z_b(x)]} \left( -\frac{1}{\mu} \bar{\phi}(x) \right) \quad \text{for a.e. } x \in \Omega.
\] (5.7)

Here \( \Pi_{[z_a(x), z_b(x)]}(w(x)) = \min \left\{ z_b(x), \max \{ z_a(x), w(x) \} \right\} \).

Proof. The proof of (5.6) is standard, see [31, Lemma 4.18]. Moreover, the equivalence between (5.6) and (5.7) is well-known [31, Pg. 217]. \( \square \)

Remark 5.7 (Nonsmooth cost functionals). We recall Remark 4.4. We let \( J_1 \) be as in (5.3). The first order optimality conditions when \( J_2(\bar{z}) := \int_{\Omega} |\nabla z| \) are technical and are part of our future project (cf. [17] for standard Laplacian case). On the other hand in case \( J(u, z) = J_1(u) + J_2(z) + \nu \| z \|_{L^1(\Omega)} \) with \( J_1, J_2 \) as in (5.3) and constant \( z_a, z_b \) fulfilling \( z_a < 0 < z_b \), the first order optimality conditions are a modification of (5.7) by using the characterization of subdifferential of the \( L^1(\Omega) \)-norm (cf. [16, Corollary 3.2] and [19] for details). In particular, we obtain that

(a) \( \bar{z}(x) = \Pi_{[z_a, z_b]} \left( -\frac{1}{\mu} (\bar{\phi}(x) + \nu \bar{\zeta}(x)) \right) \);  
(b) \( \bar{z}(x) = 0 \) if and only if \( |\bar{\phi}(x)| \leq \nu \);  
(c) \( \bar{\zeta}(x) = \Pi_{[-1, 1]} \left( -\frac{1}{\nu} \bar{\phi}(x) \right) \).

To state the second order sufficient conditions, we introduce the \( \tau \)-critical cone associated to a control \( \bar{z} \),
\[
C_\tau(\bar{z}) := \left\{ v \in L^\infty(\Omega), \ v \text{ fulfills (5.9)} \right\},
\] (5.8)

where
\[
v(x) \begin{cases} 
\geq 0, & \text{if } \bar{z}(x) = z_a, \\
\leq 0, & \text{if } \bar{z}(x) = z_b, \\
= 0, & \text{if } |\bar{\phi}(x) + \mu \bar{z}(x)| > \tau > 0.
\end{cases}
\] (5.9)

The notion of \( \tau \)-critical cone goes back to [22].

Theorem 5.8 (Quadratic growth condition). Let \( \bar{z} \in Z_{ad} \) be a control satisfying the first order optimality conditions (5.6). Assume that there are two constants \( \tau > 0 \) and \( \delta > 0 \) such that
\[
J''(\bar{z})[z, \bar{z}] \geq \delta \| z \|_{L^2(\Omega)}^2 \quad \forall v \in C_\tau(\bar{z}).
\] (5.10)

Then there exist two constants \( \beta > 0 \) and \( \varphi > 0 \) such that
\[
J(z) \geq J(\bar{z}) + \beta \| z - \bar{z} \|_{L^2(\Omega)}^2 \quad \forall z \in Z_{ad} \cap B_\varphi(\bar{z}).
\] (5.11)

Notice that in certain cases it is possible to prove (5.10), see for instance [4]. Before proving Theorem 5.8 we need the following auxiliary result.

Lemma 5.9 (\( L^2 - L^\infty \) norm discrepancy). Let Assumption 5.1 hold and \( J : L^\infty(\Omega) \to \mathbb{R} \). Then for each \( M > 0 \) there exists a constant \( L(M) > 0 \) such that
\[
|J''(z + h)[z_1, z_2] - J''(z)[z_1, z_2]| \leq L(M) \| h \|_{L^\infty(\Omega)} \| z_1 \|_{L^2(\Omega)} \| z_2 \|_{L^2(\Omega)}
\] for all \( z, h, z_1, z_2 \in L^\infty(\Omega) \) satisfying \( \max \{ \| z \|_{L^\infty(\Omega)}, \| h \|_{L^\infty(\Omega)} \} \leq M \).

Proof. We begin by setting \( u = S(z), u_h = S(z + h) \) with the corresponding adjoint state \( \phi \) and \( \phi_h \). Moreover, let \( u_i = S'(z)z_i \) and \( u_{i,h} = S'(z + h)z_i \) for \( i = 1, 2 \).
Using Lemma 5.5 we have

\[ J''(z + h)[z_1, z_2] - J''(z)[z_1, z_2] \]

\[
= \int_{\Omega} \left( u_{1,h}u_{2,h} - u_{1}u_{2} \right) dx - \int_{\Omega} \phi_h f_{uu}(x,u_{h})u_{1,h}u_{2,h} dx + \int_{\Omega} \phi f_{uu}(x,u)u_{1}u_{2} dx \\
= \int_{\Omega} \left( u_{1,h}u_{2,h} - u_{1}u_{2} \right) dx - \int_{\Omega} \phi_h (f_{uu}(x,u_{h})u_{1,h}u_{2,h} - f_{uu}(x,u)u_{1}u_{2}) dx \\
+ \int_{\Omega} (\phi - \phi_h)f_{uu}(x,u)u_{1}u_{2} dx.
\]

Therefore

\[ |J''(z + h)[z_1, z_2] - J''(z)[z_1, z_2]| \]

\[
\leq \|u_{1,h}u_{2,h} - u_{1}u_{2}\|_{L^1(\Omega)} + \|\phi - \phi_h\|_{L^\infty(\Omega)} f_{uu}(x,u)\|_{L^1(\Omega)} \\
+ \|\phi_h\|_{L^\infty(\Omega)} f_{uu}(x,u_{h})u_{1,h}u_{2,h} - f_{uu}(x,u)u_{1}u_{2}\|_{L^1(\Omega)}.
\]

The result then follows by estimating the terms on the right hand side. See for instance [31, Lemma 4.26].

**Proof of Theorem 5.8.** The second order sufficient condition (5.10) in conjunction with Lemma 5.9 imply (5.11), see [31, Theorem 4.29] for more details.

Remark 5.10. We mention that all the results in this section, except the \(H^{2s}\)-elliptic regularity result in Lemma 5.4, hold for the map \(\tilde{S}\) by replacing in all statements and proofs, \((LD)^s\) by the operator \((-\Delta)^s\). For \((-\Delta)^s\) only local maximal elliptic regularity can be achieved. More precisely, for the integral fractional Laplacian, the solution \(\phi\) of the corresponding adjoint equation belongs to \(H^{2s\alpha}_{loc}(\Omega)\) (see e.g. [11] for more details).

6. **Numerical examples.** For the optimal control problem we use the finite element discretization as in [5], see also [27] for the discretization of the linear state equation and [7] for the semilinear state equation. The implementation was carried out in Matlab under FEM library [18]. Let the discrete state and control be \(U\) and \(Z\) respectively.

**Example 6.1 (Rates of convergence).** In order to validate our implementation we first consider the following example. We let \(f \equiv u^3\) i.e., a smooth nonlinearity. We let \(n = 2, \mu = 1, \Omega = (0,1)^2\), and \(A(x) \equiv 1\). Then the eigenvalues and eigenfunctions of \(LD\) are: \(\lambda_{k,l} = \pi^2(k^2 + l^2), \varphi_{k,l}(x_1,x_2) = \sin(k\pi x_1)\sin(l\pi x_2)\), with \(k,l \in \mathbb{N}\). We first construct an exact solution as follows. Given \(\bar{u} = \lambda_{2,2}^s \varphi_{2,2}\) solving \(L^s \bar{u} = f + \bar{u}\) in \(\Omega\), \(\bar{u} = 0\) on \(\partial \Omega\), we set \(f = \varphi_{2,2} + \bar{u}^3 - \bar{u}\). Letting \(\bar{\phi} = \varphi_{2,2}\), we obtain that \(u_d = \bar{u} - (\lambda_{2,2}^s + 3\bar{u}^2)\bar{\phi}\). In view of the projection formula we notice that \(\bar{z} = \min \left\{ z_b, \max \left\{ z_a, -\bar{\phi}/\mu \right\} \right\}\). We let \(z_a = 0\) and \(z_b = 0.5\). We then compute the rate of convergence which are shown in Figure 1. These rates of convergence are in accordance with the case when \(f \equiv 0\) investigated in [5].

**Example 6.2 (Matching a nonsmooth function).** We finish with a more challenging example. Here we are interested in matching a non-smooth function \(u_d\) defined as below. Let

\[
v(z) = \begin{cases} 
  z & z < 1/2 \\
  1 - z & z \geq 1/2.
\end{cases}
\]

Then we set \(u_d(x_1,x_2) = v(x_1)v(x_2)\) as shown in Figure 2. We let \(s = 0.2, \lambda = 1 - e - 3, A(x) \equiv 1, \Omega = (0,1)^2\) and \(z_a = -2\) and \(z_b = 2\).
Let \((u, z)\) denote an explicit solution to the control problem. Let \((U, Z)\) denote the discrete solution. The left panel shows the rate of convergence for the control. The middle and right panels show the \(H^s\) and \(L^2\) error in the state. All the results are in accordance with the linear case in [5].

The computed optimal state is shown in Figure 2 (middle) with optimal control (right). We remark that in the case of the standard Laplacian we will not be able to match \(u_d\) with this precisely as the standard Laplacian will enforce higher regularity (at least \(H^2\) in this case).

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