(2,2)-Formalism of General Relativity: An Exact Solution

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Abstract. We discuss the (2,2)-formalism of general relativity based on the (2,2)-fibration of a generic 4-dimensional spacetime of the Lorentzian signature. In this formalism general relativity is describable as a Yang-Mills gauge theory defined on the (1+1)-dimensional base manifold, whose local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. After presenting the Einstein’s field equations in this formalism, we solve them for spherically symmetric case to obtain the Schwarzschild solution. Then we discuss possible applications of this formalism.

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1. Introduction

There have been considerable efforts and some success to reformulate general relativity as a gauge theory[1, 2]. The major advantages of such a gauge theory description of general relativity are, first of all, that it allows us to understand general relativity in terms of familiar notions of gauge theories, and second, that the gauge constraints associated with the gauge symmetry become relatively easy to handle. However, it seems that a gauge theory formulation of general relativity as a second-order system of partial differential equations of standard gauge fields as in a Yang-Mills theory is still missing. Such a description would put the spacetime physics into a new perspective, and hopefully, it might shed light on some unsolved issues related to the problem of constraints, in particular.

Recently, we have proposed a (2,2)-formalism[3, 4, 5] of general relativity based on the (2,2)-fibration[6, 7] of a generic 4-dimensional spacetime of the Lorentzian signature. In this formalism the 4-dimensional spacetime manifold is regarded as a local product of a (1+1)-dimensional base manifold of the Lorentzian signature and a 2-dimensional spacelike fibre manifold. By introducing Yang-Mills connections adapted to this fibration, it was shown that general relativity of 4-dimensional spacetimes can be written as a Yang-Mills gauge theory defined on the (1+1)-dimensional base manifold, whose local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre space. The appearance of the diffeomorphisms of the 2-dimensional fibre space as the Yang-Mills gauge symmetry, among others, is the most distinguishable feature of this formalism, which is valid for a generic spacetime that does not have any isometries.

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In this paper, after a brief introduction to the general formalism[5] which will fix
the notations, we shall present the Einstein’s field equations written in terms of the
gauge theory variables. As an application of the (2,2)-formalism, we shall solve the
field equations for spherically symmetric case to find the Schwarzschild solution. Then
we discuss a few applications of this formalism.

2. Kinematics

Let us start by recalling that any 4-dimensional manifold of the Lorentzian signature
can be viewed as a local product of the (1+1)-dimensional base manifold $M_{1+1}$
of the Lorentzian signature, which is generated by $\partial / \partial x( = \partial _\mu; \mu = +, - )$, and a 2-
dimensional spacelike fibre manifold $N_2$, which is generated by $\partial / \partial y( = \partial _a; a = 2, 3 )$
(see Figure 1). It is convenient to introduce the horizontal lift vector fields $\hat{\partial _\mu}$
orthogonal to $\partial / \partial y$, which in general can be written as linear combinations of $\partial _\mu$ and $\partial _a$,
$$\hat{\partial _\mu} := \partial _\mu - A_\mu ^a \partial _a, \quad (1)$$
where $- A_\mu ^a$ are the coefficient functions of the linear combinations. The fields $A_\mu ^a$
play the role of gauge connections whose value lies in the Lie algebra of the diffeomorphisms
of $N_2$. Let the inverse metric of the horizontal space spanned by the horizontal
lift vector fields $\hat{\partial _\mu}$ be $\gamma _{\mu \nu}$, and the inverse metric on the fibre space
$N_2$ be $\phi _{ab}$, respectively. Then, in the horizontal lift basis that consists of \{ $\hat{\partial _\mu}, \partial _a$ \}, the inverse
metric of the 4-dimensional spacetime in this (2,2)-fibration can be written as
$$\left( \frac{\partial }{\partial s} \right)^2 = \gamma _{\mu \nu} \left( \partial _\mu - A_\mu ^a \partial _a \right) \otimes \left( \partial _\nu - A_\nu ^b \partial _b \right) + \phi _{ab} \partial _a \otimes \partial _b. \quad (2)$$
In the corresponding dual basis \{ $dx ^\mu, dy ^a + A_\mu ^a dx ^\mu$ \}, which preserves the orthogonality
of the decomposition, the metric of the spacetime becomes
$$ds^2 = \gamma _{\mu \nu} dx ^\mu dx ^\nu + \phi _{ab} \left( dy ^a + A_\mu ^a dx ^\mu \right) \left( dy ^b + A_\nu ^b dx ^\nu \right). \quad (3)$$
Using the gauge freedoms associated with the spacetime diffeomorphisms, we can always gauge fix the horizontal space metric $\gamma _{\mu \nu}$ to the following form
$$\gamma _{\mu \nu} = \begin{pmatrix} -2h & -1 \\ -1 & 0 \end{pmatrix}. \quad (4)$$
Let us denote the new coordinates \{ $x'^+, x'^-, y'^a$ \} in which $\gamma _{\mu \nu}$ assumes the above form as
$$x'^+ = u, \quad x'^- = v, \quad y'^a = y ^a, \quad (5)$$
and decompose the metric $\phi _{ab}$ on the fibre space $N_2$ as
$$\phi _{ab} = e ^\sigma \rho _{ab}, \quad (6)$$
where
$$\det \rho _{ab} = 1. \quad (7)$$
The field $e ^\sigma$ is a measure of the area element and $\rho _{ab}$ is the conformal 2-geometry of
$N_2$, respectively. Then the metric (3) becomes
$$ds^2 = -2du dv - 2hdudv + e ^\sigma \rho _{ab} \left( dy ^a + A_\mu ^a du + A_\nu ^a dv \right) \left( dy ^b + A_\mu ^b du + A_\nu ^b dv \right). \quad (8)$$
The spacetime geometry of the metric (8) can be also understood as follows. The surface \( u = \text{constant} \) defines a null hypersurface, and the surface \( v = \text{constant} \) is either a timelike, null, or spacelike hypersurface, depending on the signature of \( 2h \). The intersection of two hypersurfaces \( u, v = \text{constant} \) is the spacelike two-surface \( N_2 \). Notice that the horizontal lift vector fields \( \{ \hat{\partial}_+ , \hat{\partial}_- \} \) become

\[
\hat{\partial}_+ = \partial_+ - A_+^a \partial_a,
\]

\[
\hat{\partial}_- = \partial_- - A_-^a \partial_a,
\]

but now \( \hat{\partial}_+ \) has a norm \(-2h\), and \( \hat{\partial}_- \) is a null vector field since its norm is zero. By a suitable rescaling, the inner product of \( \hat{\partial}_+ \) and \( \hat{\partial}_- \) is normalized to \(-1\), which makes \( v \) an affine parameter. Thus two metric functions are gauged away in (8), and the remaining eight fields \( \{ h, \sigma, \rho_{ab}, A_{\pm}^a \} \) are functions of all the coordinates \( (u, v, y^a) \), since we assume no isometries. If we further introduce the condition that

\[
A_{\pm}^a = 0,
\]

then the above metric becomes identical to the Newman-Unti metric[8]. In this paper, however, we shall retain the \( A_{\pm}^a \) field, since its presence will make the gauge theory aspect of the formalism transparent.

The integral of the scalar curvature of the spacetime described by the metric (8) is given by

\[
I_4 = \int du \, dv \, d^2 y \, e^\sigma \, R_4
\]

\[
= \int du \, dv \, d^2 y \left[ -\frac{1}{2} e^{2\sigma} \rho_{ab} F_{\pm}^a F_{\pm}^b + e^\sigma (D_+ \sigma)(D_- \sigma) - \frac{1}{2} e^{\sigma} \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) + e^\sigma R_2 - 2e^\sigma (D_- h)(D_+ \sigma) - he^\sigma (D_+ \sigma)^2 + \frac{1}{2} he^\sigma \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right]
\]

+ surface terms. \hspace{1cm} (12)

Here \( R_2 \) is the scalar curvature of \( N_2 \), and the notations are summarized below:

\[
F_{\pm}^a = \partial_\pm A^a - \partial_\pm A_{\pm}^a - [A_\pm, A_\pm]_L^a
\]

\[
D_\pm \sigma = \partial_\pm \sigma - [A_\pm, \sigma]_L
\]

\[
D_\pm h = \partial_\pm h - [A_\pm, h]_L
\]

\[
D_\pm \rho_{ab} = \partial_\pm \rho_{ab} - [A_\pm, \rho]_{Lab}
\]

where \([A_\pm, \ast]\) is the Lie derivative of \( \ast \) along the vector field \( A_\pm := A_\pm^a \partial_a \). Each term in \( I_4 \) strongly suggests that the integral should be interpreted as an action integral of a Yang-Mills type gauge theory defined on the \((1+1)\)-dimensional base manifold \( M_{1+1} \), interacting with \( h, \sigma \), and \( \rho_{ab} \). The associated local gauge symmetry is the \( \text{diff}N_2 \) symmetry, the group of the diffeomorphisms of the fibre space \( N_2 \), which is built-in into the theory via the Lie derivatives. It must be mentioned here that each term in (12) is manifestly \( \text{diff}N_2 \)-invariant, and that the \( y^a \)-dependence is completely hidden in the Lie derivatives. In this sense we may regard the fibre space \( N_2 \) as a kind of an internal space as in Yang-Mills theories[9].
3. The Einstein’s equations

By varying $I_4$ with respect to the eight fields $h$, $A_{\pm \sigma}^a$, and $\rho_{ab}$ (subject to the condition $\det \rho_{ab} = 1$), one can obtain the eight field equations (eqs. (18), \ldots, (22)) in the gauge (4). In fact one can obtain the complete set of the Einstein’s field equations at once by varying the Einstein-Hilbert action before one introduces any gauge condition, and then imposing the gauge condition (4) in the resulting field equations. It turns out that the eight field equations one obtains from $I_4$ by variations are identical to the eight Einstein’s equations that follow from the general Einstein-Hilbert action by the corresponding variations. The underlying reason for this equality is because the gauge fixing condition (4) puts no restrictions on the remaining eight fields, so that the variations of the integral $I_4$ with respect to the eight fields are completely arbitrary variations subject to no restrictions at all. Hence the integral $I_4$ may be regarded an action integral, modulo two supplementary equations that should be appended to it by the Lagrange multipliers, since it reproduces the ten Einstein’s equations in the gauge (4) when implemented by the two equations. The complete set of the Einstein’s equations are found to be

\begin{align}
(a) &\quad e^\sigma D_+ D_- \sigma + e^\sigma D_- D_+ \sigma + 2e^\sigma (D_+ \sigma)(D_- \sigma) - 2e^\sigma (D_- h)(D_- \sigma) \\
&\quad - \frac{1}{2} e^{2\sigma} \rho_{ab} F_+^a F_+^b - e^\sigma R_2 - he^\sigma \left\{ (D_- \sigma)^2 - \frac{1}{2} \sigma^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\} = 0, \quad (16) \\
(b) &\quad - e^\sigma D_+^2 \sigma - \frac{1}{2} e^\sigma (D_- \sigma)^2 - e^\sigma (D_- h)(D_- \sigma) + e^\sigma (D_+ h)(D_- \sigma) \\
&\quad + 2he^\sigma (D_- h)(D_- \sigma) + e^\sigma F_+^a \partial_a h - \frac{1}{4} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_+ \rho_{bd}) + \partial_a \left( \rho^{ab} \partial_b h \right) \\
&\quad + h \left\{ - e^\sigma (D_+ \sigma)(D_- \sigma) + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) + \frac{1}{2} e^{2\sigma} \rho_{ab} F_+^a F_+^b + e^\sigma R_2 \right\} \\
&\quad + h^2 e^\sigma \left\{ (D_- \sigma)^2 - \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\} = 0, \quad (17) \\
(c) &\quad 2e^\sigma (D_-^2 \sigma) + e^\sigma (D_- \sigma)^2 \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) = 0, \quad (18) \\
(d) &\quad D_+ \left( e^{2\sigma} \rho_{ab} F_+^b \right) - e^\sigma \partial_a (D_- \sigma) - \frac{1}{2} e^\sigma \rho^{bc} \rho^{de} (D_- \rho_{bd})(\partial_a \rho_{ce}) + \partial_b \left( e^\sigma \rho^{bc} D_- \rho_{ac} \right) \\
&\quad = 0, \quad (19) \\
(e) &\quad - D_+ \left( e^{2\sigma} \rho_{ab} F_+^b \right) - e^\sigma \partial_a (D_- \sigma) - \frac{1}{2} e^\sigma \rho^{bc} \rho^{de} (D_+ \rho_{bd})(\partial_a \rho_{ce}) \\
&\quad + \partial_b \left( e^\sigma \rho^{bc} D_+ \rho_{ac} \right) + 2he^\sigma \partial_a (D_- \sigma) + he^\sigma \rho^{bc} \rho^{de} (D_- \rho_{bd})(\partial_a \rho_{ce}) + 2e^\sigma \partial_a (D_- h) \\
&\quad - 2\partial_b \left( he^\sigma \rho^{bc} D_- \rho_{ac} \right) = 0, \quad (20) \\
(f) &\quad - 2e^\sigma D_-^2 h - 2e^\sigma (D_- h)(D_- \sigma) + e^\sigma D_+ D_- \sigma + e^\sigma D_- D_+ \sigma + e^\sigma (D_+ \sigma)(D_- \sigma) \\
&\quad + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) + e^{2\sigma} \rho_{ab} F_+^a F_+^b - 2he^\sigma \left\{ D_+^2 \sigma + \frac{1}{2} (D_- \sigma)^2 \right\} \\
&\quad + \frac{1}{4} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) = 0, \quad (21) \\
(g) &\quad h \left\{ e^\sigma D_+^2 \rho_{ab} - e^\sigma \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) + e^\sigma (D_- \rho_{ab})(D_- \sigma) \right\}
\end{align}
\[-\frac{1}{2}e^{\sigma}(D_{+}D_{-}\rho_{ab} + D_{-}D_{+}\rho_{ab}) + \frac{1}{2}e^{\sigma}\rho_{ab}d\left((D_{-}\rho_{ac})(D_{+}\rho_{bd}) + (D_{-}\rho_{bc})(D_{+}\rho_{ad})\right)\]
\[-\frac{1}{2}e^{\sigma}\left((D_{-}\rho_{ab})(D_{+}\sigma) + (D_{+}\rho_{ab})(D_{-}\sigma)\right)\]
\[+ e^{\sigma}(D_{-}\rho_{ab})(D_{-}h) + \frac{1}{2}e^{2\sigma}\rho_{ac}\rho_{bd}F_{+}^{-}F_{+}^{d} - \frac{1}{4}e^{2\sigma}\rho_{ab}\rho_{cd}F_{+}^{-}F_{+}^{d} = 0. \tag{22}\]

4. A spherically symmetric solution

The equations (16), \ldots, (22) are the Einstein’s field equations in the gauge theory variables in the (2,2)-fibration of a generic spacetime. It will be instructive to find some solution of the above equations, since it will give us an idea how to use this formalism. There are several classes of spacetimes\[10, 11\] to which this (2,2)-formalism is directly applicable, but in this paper, we are interested in solving the above equations for spherically symmetric vacuum case.

In order to solve the field equations in this case, we have to construct a coordinate system adapted to the metric (8)\[12\]. Recall that the spherical symmetry with respect to a given observer means that the metric is independent of the orientation at each point on the worldline \(C\) of that observer (see Figure 2). Let \(\vartheta\) and \(\varphi\) be the angular coordinates that define the orientation at that point. Then, by the spherical symmetry, it suffices to consider the (1+1)-dimensional subspace defined by \(\vartheta, \varphi = \text{constant}\). Let us define the coordinates \((u, v)\) of an arbitrary event \(E\) in this subspace as follows: (a) Given an event \(E\), draw a past-directed null geodesic from \(E\) cutting the worldline \(C\) at \(P\). The coordinate \(v\) is defined as the affine distance of the event \(E\) from \(P\) along the null geodesic. (b) The coordinate \(u\) measures the location of the event \(P\) from a certain reference point \(O\) along the worldline \(C\). The affine parameter \(v\) has the coordinate freedom
\[v \rightarrow v' = A(u)v + B(u), \tag{23}\]
on each null hypersurface defined by \(u = \text{constant}\). Also there is a reparametrization invariance
\[u \rightarrow u' = f(u), \tag{24}\]
where \(f(u)\) is an arbitrary function of \(u\). Notice that the equation \(du = 0\) defines a null geodesic in the \((u, v)\)-subspace. If we further choose \(A = 1\) and \(B = 0\) in (23), then we can write the metric on this subspace as a product of \(-2du\) and \(dv + h(u, v)du\), where \(h\) is an arbitrary function of \((u, v)\). Thus the metric of the spherically symmetric spacetime is given by
\[ds^2 = -2du dv - 2h(u, v)du^2 + H(u, v)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \tag{25}\]
Here the fibre space \(N_2\) is a two-sphere \(S_2\) of radius \(H^{1/2}(H > 0)\), whose scalar curvature is
\[R_2 = -\frac{2}{H}. \tag{26}\]
If we compare (8) with (25), we find that
\[A^{\vartheta}_{\pm} = A^{\varphi}_{\pm} = 0, \]
\[\rho_{\vartheta\vartheta} = \frac{1}{\sin \vartheta}, \quad \rho_{\varphi\varphi} = \sin \vartheta, \quad \rho_{\vartheta\varphi} = 0, \]
\[e^{\sigma} = H \sin \vartheta. \tag{27}\]
Notice that the diffN2-covariant derivatives $D_\pm$ reduce to $\partial_\pm$ since $A^0_\pm = A^\varphi_\pm = 0$. Therefore the equations (16), · · · , (21) become

\[ a' \quad \partial_+ \sigma + 2(\partial_+ \varphi) - 2(\partial_- \varphi)(\partial_- h) + \frac{2}{H} - h(\partial_+ \sigma)^2 = 0, \]

\[ b' \quad \partial_\varphi^2 \sigma + \frac{1}{2}(\partial_+ \varphi)^2 + (\partial_+ \varphi)(\partial_- h) - (\partial_\varphi \varphi)(\partial_- h) - 2h(\partial_- \varphi)(\partial_- h) + h\left\{(\partial_+ \varphi)(\partial_- \varphi) + \frac{2}{H}\right\} - h^2(\partial_- \varphi)^2 = 0, \]

\[ c' \quad 2\partial_\varphi^2 - (\partial_- \varphi)^2 = 0, \]

\[ d' \quad \partial_\varphi \partial_- \varphi = 0, \]

\[ e' \quad \partial_+ \varphi - 2h \partial_\varphi \partial_- \varphi - 2\partial_\varphi \partial_- h = 0, \]

\[ f' \quad \partial_+ \partial_- \varphi + \partial_- \partial_\varphi = (\partial_\varphi \varphi)(\partial_- \varphi) - 2\partial_\varphi^2 h - 2(\partial_- h)(\partial_- \varphi) = 0, \]

respectively, and eq. (22) turns out to be trivial. Let us integrate eq. (30) first. It can be written as

\[ 2\partial_- X + X^2 = 0, \]

\[ X := \partial_- \varphi. \]

Solving eq. (34), we find that

\[ X = \frac{2}{v + 2F}, \]

where $F$ is an arbitrary function of $(u, \vartheta, \varphi)$. Therefore $\varphi$ becomes

\[ \varphi = 2\ln (v + 2F) + G = \ln H + \ln \sin \vartheta, \]

where $G$ is another arbitrary function of $(u, \vartheta, \varphi)$. Since the spacetime must be asymptotically flat, we have to choose the integral constants $F = 0$ and $G = \ln \sin \vartheta$. Then $\varphi$ and $H$ becomes

\[ \varphi = 2\ln v + \ln \sin \vartheta, \]

\[ H = v^2, \]

from which it follows that

\[ \partial_- \varphi = \frac{2}{v}, \quad \partial_+ \varphi = 0. \]

Therefore we find that eqs. (31) and (32) are trivially satisfied, and the remaining equations become

\[ a', (b') \quad 2\partial_- h + \frac{2}{v} h - \frac{1}{v} = 0, \]

\[ f' \quad \partial^2 h + \frac{2}{v} (\partial_- h) = 0. \]

Notice that eqs. (28) and (29) reduce to eq. (40) identically, and that eq. (41) is also trivial since it follows from eq. (40) by taking a derivative with respect to $v$. Therefore we need to solve eq. (40) only. Assuming the asymptotic flatness as $v \to \infty$, we find that $h$ becomes

\[ 2h = 1 - \frac{2m}{v}, \]
where \( m \) is a constant. Plugging (38) and (42) into (25), we find that the spherically symmetric solution of the vacuum Einstein’s equations is given by

\[
 ds^2 = -2dudv - (1 - \frac{2m}{v})dv^2 + v^2(du^2 + \sin^2 \vartheta d\varphi^2),
\]

(43)

which is just the Schwarzschild solution. Notice that the metric (43) is independent of \( u \), so \( u \) is the Killing time, as implied by the Birkhoff’s theorem.

5. Discussions

There are several potential applications of this formalism, but here we list only a few of them. First, this formalism provides a natural (1+1)-dimensional framework for a conventional gauge theory description of general relativity of 4-dimensional Lorentzian spacetimes, where the local gauge symmetry is \( \text{diff}N_2 \), the infinite dimensional group of the diffeomorphisms of the 2-dimensional fibre space \( N_2 \). This enables us to explore certain aspects of the theory, such as constructing physical observables for instance, using the \( \text{diff}N_2 \)-invariant quantities. Finding physical observables that are spacetime diffeomorphism invariant is an outstanding problem in classical general relativity, but so far none are known. It is natural to try to construct such observables using this formalism.

Second, the self-dual Einstein’s equations have been identified as some types of 2-dimensional field theories that have the area-preserving diffeomorphisms as the internal gauge symmetry. In our formalism, since a spacetime is viewed as a 4-dimensional fibre bundle whose base manifold is (1+1)-dimensional, a (1+1)-dimensional field theory description of a generic spacetime is an automatic consequence of this formalism. More specifically, 4-dimensional general relativity can be regarded as a (1+1)-dimensional gauge theory of an infinite dimensional Yang-Mills gauge symmetry, as we stressed previously. This observation leads to the expectation that (1+1)-dimensional field theoretic methods should be also applicable to the studies of 4-dimensional spacetime physics without the self-dual restriction, by treating the \( \text{diff}N_2 \) gauge symmetry as a kind of “internal” symmetry as in \( w_\infty \)-gravity theories.

Third, this formalism fits most naturally to the studies of gravitational waves, where one of the coordinates is usually adapted to the congruence of the out-going null vector field. In our formalism, one of the reasons that the integral \( I_4 \) of a generic spacetime is written in such a simple and suggestive form as in (12) is in fact due to our choice of the out-going null vector field

\[
 \frac{\partial}{\partial v} - A_{-}^{a} \frac{\partial}{\partial y^{a}}
\]

(44)
as one of the basis vector fields. Moreover, it is well-known that the physical degrees of freedom of gravitational waves reside in the conformal 2-geometry, which is precisely the non-linear sigma field \( \rho_{ab} \). Thus the study of gravitational waves is a natural problem in the (2,2)-formalism.

Finally, let us also mention that to examine exact solutions of the Einstein’s equations in the light of the gauge theory variables in this framework and interpret them from the (1+1)-dimensional gauge theory perspective is an interesting problem by itself. For instance, the Schwarzschild solution in this paper may be interpreted as the “vacuum” configuration, in the sense that the gauge fields \( A_{\pm}^{a} \) are identically zero. Those solutions that do not admit physical interpretations in a straightforward
way from the spacetime point of view might admit sensible interpretations from this (1+1)-dimensional gauge theory point of view.

Acknowledgments

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Figure 1. The geometry of a 4-dimensional spacetime in the (2,2)-fibration: the (1+1)-dimensional base manifold $M_{1+1}$ is generated by $\{\partial/\partial x^+, \partial/\partial x^-\}$, and the 2-dimensional fibre space $N_2$ is generated by $\partial/\partial y^\alpha$. The horizontal vector fields $\hat{\partial}_\pm$ are general linear combinations of $\partial/\partial x^\pm$ and $\partial/\partial y^\alpha$ with the coefficient functions $-A_\pm^\alpha$. 

$\partial_x = \partial/\partial y^\alpha (\alpha = 2, 3)$
Figure 2. The construction of the coordinates \((u, v)\) assuming the spherical symmetry about an observer. Here the angular coordinates \((\theta, \varphi)\) are suppressed.