Measurement of geometric phases by robust interferometric methods

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Abstract. We present a novel interferometric arrangement that makes it possible to measure with great versatility geometric phases produced in polarization states of classical light. Our arrangement is robust against thermal and mechanical disturbances and can be set up in a Mach-Zehnder, a Michelson or a Sagnac configuration. We present results concerning the geometric phase as an extension of previous measurements of the Pancharatnam, or total phase. The geometric phase is obtained by compensating the dynamical contribution to the total phase, so as to extract out of it a purely geometric phase. This can be achieved over trajectories on the Poincaré sphere that are not necessarily restricted to be great circles (geodesics). We thus demonstrate the feasibility of our method for dynamical extraction of the geometric contribution to the total phase, a prerequisite for building geometric quantum gates. Although our results correspond to polarization states of classical light, the same methodology could be applied in the case of polarization states of single photons.

1. Introduction

In a previous paper [1], we have demonstrated the versatility of an interferometric array that allows accurate measurements of the Pancharatnam phase [2] for two arbitrary polarization states. The array was shown to be robust to thermal and mechanical disturbances and could be set up in different configurations, like those corresponding to a Mach-Zehnder, a Sagnac, or a Michelson interferometer. Its robustness was shown to be similar to that of a polarimetric array [3]. Alternative, robust interferometric setups have been recently demonstrated [4]. Although such interferometers were used for other purposes, they could represent a promising alternative to our setup, in case one aims at scaling the latter down, so as to construct a compact device.

Besides being interesting on its own as a feature which exposes a common root underlying different phenomena in quantum and classical physics, the geometric phase could also be a useful tool for quantum computation. This is due to the fact that it is largely immune to those disturbances that usually cause decoherence. As is well known, decoherence is one of the central problems precluding the construction of a quantum computer. Indeed, computational tasks on a qubit should be performed in a time that is short compared to the decoherence time, and this is usually difficult to achieve. A promising route towards the construction of fault-tolerant quantum computers requires having at one’s disposal some devices with the help of which one can move a qubit around a parameter space. In this way one could implement quantum gates based on exploiting the capability of manipulating the phase acquired by the qubit as it completes a closed path on the chosen parameter space. This is called geometric (or holonomic) quantum computation. Its theoretical foundations were laid by Zanardi and
Among the different parametrizations of states we consider are formally spinors (whose global phase is physically irrelevant), are interested in the case where definition for the phase between two arbitrary – though non-orthogonal – polarization states. We extracting Pancharatnam’s phase:

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2. The interferometric method for measuring the geometric phase
In what follows, we use Dirac’s notation of bras and kets to stress that our results are equally valid for the classical and for the quantum case. A ket $|\psi\rangle$ will thus denote a polarization state, whose general form can be parametrized as

$$|\psi\rangle = \left( \begin{array}{c} \cos \theta \\ e^{i\phi} \sin \theta \end{array} \right).$$

Pancharatnam’s phase $\Phi_P$ between two non-orthogonal states, $|i\rangle$ and $|f\rangle$, is defined as $\Phi_P = \arg \langle i|f \rangle$. It can be exhibited through interferometry by applying a phase-shift $\phi$ to one of the states, so that the resulting intensity pattern is given by

$$I = \left| e^{i\phi} |i\rangle + |f\rangle \right|^2 = 2 + 2 |\langle i|f \rangle| \cos (\phi - \arg \langle i|f \rangle).$$

By noting that the maxima of $I$ occur for $\phi = \arg \langle i|f \rangle = \Phi_P$, Pancharatnam gave an operational definition for the phase between two arbitrary – though non-orthogonal – polarization states. We are interested in the case where $|f\rangle = U |i\rangle$, with $U$ being a unitary transformation. Because the states we consider are formally spinors (whose global phase is physically irrelevant), $U \in SU(2)$. Among the different parametrizations of $U$, the following one is particularly well suited for extracting Pancharatnam’s phase:

$$U(\beta, \gamma, \delta) = \exp \left( i \left( \frac{\delta + \gamma}{2} \right) \sigma_z \right) \exp \left( -i \beta \sigma_y \right) \exp \left( i \left( \frac{\delta - \gamma}{2} \right) \sigma_z \right),$$

with $\sigma_{i=x,y,z}$ being the Pauli matrices. Indeed, taking $|i\rangle = |+\rangle_z$, i.e., the eigenstate of $\sigma_z$ that belongs to the eigenvalue $+1$, and setting $|f\rangle = U |+\rangle_z$ we have

$$\langle i|f \rangle = _z \langle +| U(\beta, \gamma, \delta) |+\rangle_z = e^{i\delta} \cos \beta.$$  

From $\Phi_P = \arg \langle i|f \rangle$ we obtain $\Phi_P = \delta + \arg(\cos \beta)$. Because $\cos \beta$ can take on positive and negative real values, $\arg(\cos \beta)$ equals 0 or $\pi$, and $\Phi_P$ is thus obtained modulo $\pi$. In principle,
then, we could obtain $\Phi_P$ (modulo $\pi$) with the help of an interferometric array by comparing two interferograms, one of them obtained for $\Phi_P = 0$, i.e., $U = I$, and serving us as a reference, and the second interferogram corresponding to the applied $U$. The relative shift of the latter with respect to the reference interferogram gives us $\Phi_P$. Now, the unitary transformations that we can implement using common optical devices like quarter-wave plates ($Q$) and half-wave plates ($H$) are of the form

$$U(\xi, \eta, \zeta) = \exp\left(-\frac{i\xi}{2} \sigma_y\right)\exp\left(i\frac{\eta}{2} \sigma_z\right)\exp\left(-\frac{i\zeta}{2} \sigma_y\right).$$ (5)

They can be realized with the following gadget:

$$U(\xi, \eta, \zeta) = Q\left(-\frac{3\pi}{4} + 2\xi\right) H\left(\frac{\xi - \eta - \zeta - \pi}{4}\right) Q\left(\frac{\pi - 2\zeta}{4}\right).$$ (6)

**Figure 1.** Interferometric setup for measuring the geometric phase ($P$: polarizer, $BS$: beamsplitter, $M$: mirror, $E$: beam expander, $Q$: $\lambda/4$ waveplate, $H$: $\lambda/2$ waveplate). The gadget $QHQ$ on the left arm produces the desired unitary transformations, while that on the right arm nullifies the dynamic contribution to the total phase. Two interferograms are simultaneously produced and recorded. As the inset shows, one interferogram corresponds to the vertically polarized part, while the other corresponds to the horizontally polarized part of the expanded beam. From the relative shift between these interferograms one can obtain the geometric phase.
The corresponding interferogram has an intensity pattern given by

\[
I_V = \left| \frac{1}{\sqrt{2}} \left( e^{i\phi} |+\rangle_z + U(\xi, \eta, \zeta) |+\rangle_z \right) \right|^2 = \\
= \frac{1}{2} \left[ 1 - \cos \left( \frac{\eta}{2} \right) \cos \left( \frac{\xi + \zeta}{2} \right) \cos \phi - \sin \left( \frac{\eta}{2} \right) \cos \left( \frac{\xi - \zeta}{2} \right) \sin \phi \right].
\]

(7)

\(I_V\) refers to an initial state \(|+\rangle_z\) that is vertically polarized. From the relationship connecting the two parametrizations, \(U(\xi, \eta, \zeta)\) and \(U(\beta, \gamma, \delta)\), of the same \(U \in SU(2)\), one can show that \(I_V\) can also be written as

\[
I_V = \frac{1}{2} \left[ 1 - \cos \beta \cos (\phi - \delta) \right].
\]

(8)

### Figure 2.

The geometric phase can be extracted from the relative fringe-shift between the upper and lower parts of the interferogram. The left panels show column averages of the fringes obtained after applying a low-pass filter to get rid of noise features. The column average is performed after selecting an evaluation area \(R_0\), as illustrated on the right panel.

Pancharatnam’s phase \(\Phi_P = \delta\) is thus given by the shift of the interferogram whose intensity pattern is \(I_V\), with respecto to a reference interferogram whose intensity pattern is \(I = [1 - \cos \beta \cos \phi] / 2\). By recording one interferogram after the other one could measure their relative shift. However, thermal and mechanical disturbances make it difficult to record stable reference patterns, thereby precluding accurate measurements of \(\Phi_P\). A way out of this situation follows from observing that the intensity pattern that corresponds to an initial, horizontally polarized state \(|-\rangle_z\) is given by

\[
I_H = \frac{1}{2} \left[ 1 - \cos (\beta) \cos (\phi + \delta) \right].
\]

(9)
Hence, the relative shift between $I_V$ and $I_H$ is just twice Pancharatnam’s phase. This suggests dividing the laser beam into a vertically polarized part and a horizontally polarized part, something that can be achieved with the help of a beam-displacer prism. By so doing, we have the two parts of the laser beam being subjected to the same disturbances and we can record two interferograms in a single shot. The relative shift is thus easily measurable, being robust to thermal and mechanical disturbances. With such an array we were able to measure Pancharatnam’s phase for different unitary transformations. Our results were reported in [1].

Our aim here is to measure a geometric phase $\Phi_g$. Given a curve $C_0$ in parameter space, $\Phi_g$ relates to $\Phi_P$ by

$$\Phi_g(C_0) = \Phi_P(C_0) - \Phi_{dyn}(C_0),$$

with

$$\Phi_P(C_0) = \arg\langle \psi(s_1)|\psi(s_2)\rangle,$$

$$\Phi_{dyn}(C_0) = \int_{s_1}^{s_2} \operatorname{Im}\langle \psi(s)|\dot{\psi}(s)\rangle ds$$

While $\Phi_P(C_0)$ and $\Phi_{dyn}(C_0)$ depend on the curve $C_0$ described by $|\psi(s)\rangle$ in parameter space, which in our case is the Poincaré sphere spanned by $(\theta, \varphi)$ in Eq.(1), $\Phi_g(C_0)$ turns out to depend only on the curve $C_0$ that is described by $|\psi(s)\rangle\langle \psi(s)|$. This object and hence also the curve $C_0$ are “gauge-invariant”, i.e., invariant under parameter-dependent changes of the phase: $|\psi(s)\rangle \rightarrow \exp(i\alpha(s))|\psi(s)\rangle$. This is what makes $\Phi_g(C_0)$ a geometrical object. Exploiting such a gauge freedom we can choose an appropriate phase factor $\exp(i\alpha(s))$, so as to make $\Phi_{dyn}(C_0) = 0$ along a given curve $C_0 : |\psi(s)\rangle, s \in [s_1, s_2]$ which is traced out by our polarization states $|\psi(s)\rangle$ as a result of applying to an initial state $|\psi(0)\rangle$ some unitary transformation $U(s) : |\psi(s)\rangle = U(s)|\psi(0)\rangle$. Any $U(s)$ can be realized by making one or more of the parameters appearing in $U(\xi, \eta, \zeta)$ (see Eq.(6)) functions of $s$ while keeping the other ones fixed. Hence, any desired curve on the Poincaré sphere can be realized in this way. Setting the corresponding $QHQ$-gadget on one arm of the interferometer we make the polarization state $|\psi(s)\rangle$ follow the prescribed curve. A second $QHQ$-gadget can be set on the other arm of the interferometer in order to produce with its help the factor $\exp(i\alpha(s))$ that is needed to nullify $\Phi_{dyn}(C_0)$. This is achieved by observing that under the gauge transformation $|\psi(s)\rangle \rightarrow |\psi'(s)\rangle = \exp(i\alpha(s))|\psi(s)\rangle$ the integrand entering the definition of $\Phi_{dyn}(C_0)$, Eq.(12), changes according to $\operatorname{Im}\langle \psi(s)|\dot{\psi}(s)\rangle \rightarrow \operatorname{Im}\langle \psi'(s)|\dot{\psi'}(s)\rangle = \operatorname{Im}\langle \psi(s)|\dot{\psi}(s)\rangle + \alpha(s)$. Solving $\operatorname{Im}\langle \psi(s)|\dot{\psi}(s)\rangle + \alpha(s) = 0$ for $\alpha(s)$ we can fix the $QHQ$-gadget that makes $\Phi_{dyn}(C_0) = 0$.

Our interferometric setup is shown in Fig.(1). It is of the Mach-Zehnder type; but a Sagnac-like and a Michelson-like interferometer could be used as well. With the help of this array we could measure geometric phases stemming from non-geodesic trajectories on the Poincaré sphere. Our results will be discussed in the next section.

3. Experimental results

We carried out our measurements employing a 30 mW, cw He-Ne laser (632.8 nm) to feed the interferometric array shown in Fig.(1). The interferograms were recorded with the help of a CCD camera (1/4" Sony, video format of $640 \times 480$ pixels, frame rate adjusted to 30 fps) and digitized with a computer. The upper and lower halves of the interferograms showed a small relative shift stemming from surface irregularities and tiny misalignments. We used a first interferogram taken without phase-shifting ($U = I$) to gauge all the successive ones that correspond to transformations $U(\xi, \eta, \zeta) \neq I$. They were evaluated using an algorithm that
works as follows. First, by optical inspection one selects (defining pixel numbers) a common region $R_0$ of the images that the algorithm should work with (see Fig. (1)). The algorithm performs a column average of each half of the interferogram and the output is then submitted to a low-pass filter to get rid of noisy features. For each pair of curves the algorithm searches for relative minima and compares their locations. It gives as a first output the relative shifts between the minima. After averaging these relative shifts the algorithm produces a final output for each pair of curves. We repeated this procedure for a series of regions: $R_0 \ldots R_3$, that were defined by their pixel numbers, in order to estimate the accuracy of our results. We applied this procedure to a whole set of interferograms corresponding to different choices of $U(\xi, \eta, \zeta)$. Our results are shown in Figs. (3) to (8). As can be seen, our experimental results are in very good agreement with theoretical predictions. As expected (retarders and polarizers could be oriented to within $1^\circ$), the experimental values were within $6\%$ in accordance with the theoretical predictions.
4. Conclusions and outlook
We have implemented a robust interferometric array to measure geometric phases. Our setup allows producing and measuring geometric phases with great versatility, without the restriction of having to move polarization states along paths that are composed of geodesic segments. Given a path, we could submit the polarization state to a transformation that nullifies the dynamical contribution to its total (Pancharatnam) phase, so as to obtain a purely geometric phase. Our results represent a proof-of-principle that could be applied to different arrays. These arrays ought to be capable of implementing unitary transformations on two qubits that are carried along by a single beam, as it occurs in our case, where a laser beam was divided into two halves. One half was vertically, and the other half horizontally polarized. In this way one can get rid of the commonly encountered instabilities of interferometric arrays. The robustness of our setup is similar to the one achieved using a polarimetric approach. The results obtained by using the latter will be reported elsewhere.

As a next step, we plan to submit our polarization states to random disturbances in order to test the robustness of the geometric phase against decoherence. Thereafter, the single-photon version of our experiments should be implemented, in order to prove that the same array can be used to construct geometric quantum gates.

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