COMPARING COMMUTATIVE AND ASSOCIATIVE UNBOUNDED DIFFERENTIAL GRADED ALGEBRAS OVER $\mathbb{Q}$ FROM HOMOTOPICAL POINT OF VIEW

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Abstract. In this paper we establish a faithfulness result, in a homotopical sense, between a subcategory of the model category of augmented differential graded commutative algebras CDGA and a subcategory of the model category of augmented differential graded algebras DGA over the field of rational numbers $\mathbb{Q}$.

Introduction

It is well known that the forgetful functor from the category of commutative $k$-algebras to the category of associative $k$-algebras is fully faithful. We have an analogue result between the category of unbounded differential graded commutative $k$-algebras $\text{dgCAlg}_k$ and the category of unbounded differential graded associative algebras $\text{dgAlg}_k$. The question that we want explore is the following: Suppose that $k = \mathbb{Q}$, is it true that forgetful functor $U : \text{dgCAlg}_k \to \text{dgAlg}_k$ induces a fully faithful functor at the level of homotopy categories

$$RU : \text{Ho}(\text{dgCAlg}_k) \to \text{Ho}(\text{dgAlg}_k).$$

The answer is no. A nice and easy counterexample was given by Lurie. He has considered $k[x, y]$ the free commutative CDGA in two variables concentrated in degree 0. It follows obviously that

$$\text{Ho}(\text{dgCAlg}_k)(k[x, y], S) \simeq H^0(S) \oplus H^0(S),$$

while

$$\text{Ho}(\text{dgAlg}_k)(k[x, y], S) \simeq H^0(S) \oplus H^0(S) \oplus H^{-1}(S).$$

Something nice happens if we consider the category of augmented CDGA denoted by $\text{dgCAlg}^*_k$ and augmented DGA denoted by $\text{dgAlg}^*_k$.

Theorem 0.1 [3.1]. For any $R$ and $S$ in $\text{dgCAlg}^*_k$, the induced map by the forgetful functor

$$\Omega \text{Map}_{\text{dgCAlg}^*_k}(R, S) \to \Omega \text{Map}_{\text{dgAlg}^*_k}(R, S),$$

has a retract, in particular

$$\pi_i \text{Map}_{\text{dgCAlg}^*_k}(R, S) \to \pi_i \text{Map}_{\text{dgAlg}^*_k}(R, S)$$

is injective $\forall i > 0$.

2000 Mathematics Subject Classification. Primary 55, Secondary 14, 16, 18.

Key words and phrases. DGA, CDGA, Mapping Space, Noncommutative and Commutative Derived Algebraic Geometry, Rational Homotopy Theory.

Supported by the project CZ.1.07/2.3.00/20.0003 of the Operational Programme Education for Competitiveness of the Ministry of Education, Youth and Sports of the Czech Republic.
Let $S$ be a differential graded commutative algebra which is a "loop" of an other CDGA algebra $A$, i.e. $S = \text{Holim}(k \to A \leftarrow k)$, where the homotopy limit is taken in the model category $\text{dgCAlg}_k$. A direct consequence of our theorem is that the right derived functor $RU$ is a faithful functor i.e., the induced map $\text{Ho}(\text{dgCAlg}_k^*)(R, S) \to \text{Ho}(\text{dgAlg}_k^*)(R, S)$ is injective.

**Interpretation of the result in the derived algebraic geometry.** Rationally, any pointed topological $X$ space can be viewed as an augmented (connective) commutative differential graded algebra via its cochain complex $C^*(X, \mathbb{Q})$. In case where $X$ is a simply connected rational space, the cochain complex $C^*(X, \mathbb{Q})$ carries the whole homotopical information about $X$, by Sullivan Theorem [5]. Moreover, the bar construction $\text{BC}^*(X, \mathbb{Q})$ is identified (as $E_\infty$-DGA) to $C^*(\Omega X, \mathbb{Q})$ and $\Omega C^*(X, \mathbb{Q})$ is identified (as $E_\infty$-DGA) to $C^*(\Sigma X, \mathbb{Q})$ cf. [4]. This interpretation allows us to make the following definition: A generalized rational pointed space is an augmented commutative differential graded $\mathbb{Q}$-algebra (possibly unbounded). In the same spirit, we define a pointed generalized noncommutative rational space as an augmented differential graded $\mathbb{Q}$-algebra (possibly unbounded). Let $A$ be any augmented CDGA resp. DGA, we will call a CDGA resp. DGA of the form $\Omega A$ a op-suspended CDGA resp. DGA. Our theorem 3.1 can be interpreted as follows:

The homotopy category of op-suspended generalized commutative rational spaces is a subcategory of the homotopy category of op-suspended generalized noncommutative rational spaces.

1. **DGA, CDGA and $E_\infty$-DGA.**

We work in the setting of unbounded differential graded $k$-modules $\text{dgMod}_k$. This is a symmetric monoidal closed model category ($k$ is a commutative ring). We denote the category of (reduced) operads in $\text{dgMod}_k$ by $\text{Op}_k$. We follow notations and definitions of [2], we say that an operad $P$ is admissible if the category of $P-\text{dgAlg}_k$ admits a model structure where the fibrations are degree wise surjections and weak equivalence are quasi-isomorphisms. For any map of operads $\phi : P \to Q$ we have an induced adjunction of the corresponding categories of algebras:

$$P-\text{dgAlg}_k \xrightarrow{\phi^!} Q-\text{dgAlg}_k.$$ 

A $\Sigma$-cofibrant operad $P$ is an operad such that $P(n)$ is $k[\Sigma_n]$-cofibrant in $\text{dgMod}_k[\Sigma_n]$. Any cofibrant operad $P$ is a $\Sigma$-cofibrant operad [2 Proposition 4.3]. We denote the associative operad by $\text{Ass}$ and the commutative operad by $\text{Com}$. The operad $\text{Ass}$ is an admissible operad and $\Sigma$-cofibrant, while the operad $\text{Com}$ is not admissible in general. In the rational case, when $k = \mathbb{Q}$ the operad $\text{Com}$ is admissible but not $\Sigma$-cofibrant. More generally any cofibrant operad $P$ is admissible [2 Proposition 4.1, Remark 4.2]. We define a symmetric tensor product of operads by the formulae

$$(P \otimes Q)(n) = P(n) \otimes Q(n), \quad \forall n \in \mathbb{N}.$$ 

**Lemma 1.1.** Suppose that $\phi : \text{Ass} \to P$ is a cofibration of operads. The operad $P$ is admissible and the functor $\phi^* : P-\text{dgAlg}_k \to \text{dgAlg}_k$ preserves fibrations, weak equivalences and cofibrations with cofibrant domain in the iinderling category $\text{dgMod}_k$.
Proof. First of all, the operad $P$ is admissible, indeed we use the cofibrant resolution $r : E_\infty \to \text{Com}$ and consider the following pushout in $\text{Op}_k$ given by:

\[
\begin{array}{ccc}
\text{Ass}_\infty & \rightarrow & \text{E}_\infty \\
\downarrow \sim & & \downarrow \alpha \\
\text{Ass} & \rightarrow & \text{E}_\infty \\
\end{array}
\]

Where $\text{Ass}_\infty$ in the cofibrant replacement of $\text{Ass}$ in $\text{Op}_k$ and $\text{Ass}_\infty \rightarrow \text{E}_\infty$ is a cofibration. Since the category $\text{Op}_k$ is left proper in the sense of [8, Theorem 3], we have that $\alpha : \text{E}_\infty \rightarrow \text{E}_\infty'$ is an equivalence. We denote by $I$ the unit interval in the category $\text{dgMod}_k$ which is strictly coassociative. The opposite endomorphism operad $\text{End}^{op}(I)$ has a structure of $E_\infty$-algebra and $\text{Ass}_\infty$-algebra which factors through $\text{Ass}$ i.e., we have two compatible maps of operads:

\[
\begin{array}{ccc}
\text{Ass}_\infty & \rightarrow & \text{E}_\infty \\
\downarrow \sim & & \downarrow \alpha \\
\text{Ass} & \rightarrow & \text{E}_\infty \\
\end{array}
\]

by the universality of the pushout, we have a map of operads $\text{E}_\infty' \rightarrow \text{End}^{op}(I)$. This means that the unit interval $I$ has a structure of $E_\infty$-colagebra $[2, \text{p}4]$. Moreover, we have a commutative diagram in $\text{Op}_k$ given by

\[
\begin{array}{ccc}
\text{Ass} & \xrightarrow{\Delta} & \text{Ass} \otimes \text{Ass} \\
\downarrow \phi & & \downarrow \phi \otimes f \\
P & \rightarrow & P \otimes \text{E}_\infty'
\end{array}
\]

where the diagonal map $\Delta : \text{Ass} \rightarrow \text{Ass} \otimes \text{Ass}$ is induced by the diagonals $\Sigma_n \rightarrow \Sigma_n \times \Sigma_n$. Hence, the map $P \otimes \text{E}_\infty' \rightarrow P$ admits a section. It implies by $[2, \text{Proposition 4.1}]$, that $P$ is admissible and $\Sigma$-cofibrant. Since all objects in $P - \text{dgAlg}_k$ are fibrant and $\phi^*$ is a right Quillen adjoint, it preserves fibrations and weak equivalences.

Since $P$ is an admissible operad, we have a Quillen adjunction

\[
d\text{Alg}_k \xleftarrow{\phi} P - \text{dgAlg}_k,
\]

where the functor $\phi^*$ is identified to the forgetful functor. Moreover, the model structure on $P - \text{dgAlg}_k$ is the transferred model structure from the cofibrantly generated model structure $\text{dgAlg}_k$ via the adjunction $\phi_!, \phi^*$. Suppose that $f : A \rightarrow C$ is a cofibration in $P - \text{dgAlg}_k$ such that $A$ is cofibrant in $\text{dgMod}_k$. We factor this map as a cofibration followed by a trivial fibration

\[
\begin{array}{ccc}
A & \xrightarrow{i} & P & \xrightarrow{p} & B
\end{array}
\]
in the category $\text{dgAlg}_k$. By \cite{7} Lemma 4.1.16, we have an induced map of endomorphism operads (of diagrams):

$$\text{End}_{(A \rightarrow P \rightarrow B)} \rightarrow \text{End}_{A \rightarrow B}$$

which is a trivial fibration. Moreover, we have the following commutative diagram in $\text{Op}_k$

\[
\begin{array}{ccc}
\text{Ass} & \longrightarrow & \text{End}_{(A \rightarrow P \rightarrow B)} \\
\downarrow & & \downarrow \sim \\
\text{P} & \longrightarrow & \text{End}_{(A \rightarrow B)} \\
\end{array}
\]

Since $\text{Op}_k$ is a model category, it implies that we have a lifting map of operads $\text{P} \rightarrow \text{End}_{(A \rightarrow P \rightarrow B)}$, hence $i$ and $p$ are maps of $\text{P} - \text{dgAlg}_k$. Therefore, we consider the following commutative square in the category $\text{P} - \text{dgAlg}_k$

\[
\begin{array}{ccc}
A & \xrightarrow{i} & P \\
\downarrow f & & \downarrow \sim \\
B & \xrightarrow{id} & B \\
\end{array}
\]

the lifting map $r$ exists since $\text{P} - \text{dgAlg}_k$ is a model category, we conclude that $p \circ r = id$ and $r \circ f = i$, which means that $f$ is a retract of $i$, hence $f$ is a cofibration in $\text{dgAlg}_k$. \hfill \square

**Remark 1.2.** With the same notation as in 1.1 if $A$ is a cofibrant object in $\text{P} - \text{dgAlg}_k$ then $A$ is a cofibrant object in $\text{dgMod}_k$. Indeed $k \rightarrow A$ is a cofibration in $\text{P} - \text{dgAlg}_k$, by the previous lemma $k \rightarrow A$ is a cofibration in $\text{dgAlg}_k$. Therefore, $k \rightarrow A$ is a cofibration in $\text{dgMod}_k$.

2. **Suspension in CGDA and DGA**

We denote the the operad $E'_\infty$ of the previous section by $E_\infty$, and $k = \mathbb{Q}$.

2.1. $E_\infty$-DGA. We have a map of operads $\text{Ass} \rightarrow \text{Com}$, which we factor as cofibration followed by a trivial fibration.

\[
\begin{array}{ccc}
\text{Ass} & \xleftarrow{\sim} & E_\infty \\
\downarrow & & \downarrow \sim \\
& \xrightarrow{\sim} & \text{Com} \\
\end{array}
\]

As a consequence, we have the following Quillen adjunctions

$$
\begin{array}{ccc}
\text{dgAlg}_k & \xleftarrow{Ab_\infty} & E_\infty \text{dgAlg}_k \\
\downarrow U & \xrightarrow{\sim} & \downarrow U' \\
\text{dgCAlg}_k & \xrightleftharpoons{str} & \text{dgCAlg}_k \\
\end{array}
$$

These adjunctions have the following properties:

- The functors $U'$ and $U \circ U'$ and are the forgetful functors, they are fully faithful cf \cite{2.3} and \cite{2.9}.
- The functors $\text{str}$, $U'$ form a Quillen equivalence since $k = \mathbb{Q}$ cf \cite{9} Corollary 1.5. The functor $\text{str}$ is the strictification functor.
- The functors $Ab_\infty$, $U$ form a Quillen pair.
- The composition $\text{str} \circ Ab_\infty$ is the abelianization functor $Ab : \text{dgAlg}_k \rightarrow \text{dgCAlg}_k$.
- The functors $\text{str}$ and $Ab$ are idempotent functors. cf \cite{2.3} and \cite{2.2}.
The model categories $\text{dgCAlg}^*_k$ and $\text{dgAlg}^*_k$ and $E_{\infty}\text{dgAlg}^*_k$ are pointed model categories. It is natural to introduce the suspension functors in these categories.

**Definition 2.1.** Let $C$ be any pointed model category, we denote the point by $1$, and let $x \in C$, a suspension $\Sigma x$ is defined as $\text{hocolim}(1 \leftarrow x \rightarrow 1)$.

**Proposition 2.2.** Any map $f : A \rightarrow S$ in $E_{\infty}\text{dgAlg}_k$, where $S$ is in $\text{dgCAlg}_k$ factors in a unique way as $A \rightarrow \text{str}(A) \rightarrow S$ and the forgetful functor $U' : \text{dgCAlg}_k \rightarrow E_{\infty}\text{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu : A \rightarrow \text{str}(A)$ is a fibration.

**Proof.** Suppose that we have a map $h : R \rightarrow S$ in $E_{\infty}\text{dgAlg}_k$ such that $R$ and $S$ are objects in $\text{dgCAlg}_k$. By definition of the operad $E_{\infty}$ the map $h$ has to be associative, therefore $h$ is a morphism in $\text{dgCAlg}_k$, since $R$ and $S$ are commutative differential graded algebras. The forgetful functor $U' : \text{dgCAlg}_k \rightarrow E_{\infty}\text{dgAlg}_k$ is fully faithful, this implies that $\text{str}(S) = S$ for any $S \in \text{dgCAlg}_k$. We have a commutative diagram induced by the unit $\nu$ of the adjunction $(U', \text{str})$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & S \\
\downarrow{\nu_A} & & \downarrow{\nu_S = \text{id}} \\
\text{str}(A) & \xrightarrow{\text{str}(f)} & \text{str}(S) = S.
\end{array}
$$

We conclude that $f = \text{str}(f) \circ \nu_A$. The surjectivity of the $\nu_A$ follows from the universal property of $\text{str}(A)$. Hence, $\nu_A$ is a fibration in $E_{\infty}\text{dgAlg}_k$. \qed

**Proposition 2.3.** Any map $f : A \rightarrow S$ in $\text{dgAlg}_k$, where $S$ is in $\text{dgCAlg}_k$ factors in a unique way as $A \rightarrow \text{Ab}(A) \rightarrow S$ and the forgetful functor $U \circ U' : \text{dgCAlg}_k \rightarrow \text{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu_A : A \rightarrow \text{Ab}(A)$ is a fibration.

**Proof.** The proof is the same as in 2.2. \qed

**Proposition 2.4.** Suppose that we have a trivial cofibration $k \rightarrow k$ in $E_{\infty}\text{dgAlg}_k$. Then the universal map $\pi : \text{Ab}(k) \rightarrow \text{str}(k)$ is a trivial fibration and admits a section in the category $\text{dgCAlg}_k$.

**Proof.** We consider the following commutative diagram in $E_{\infty}\text{dgAlg}_k$:

$$
\begin{array}{ccc}
k & \xrightarrow{\sim} & \text{str}(k) \\
\downarrow{id} & & \downarrow{\text{id}} \\
k = \text{str}(k) & \xrightarrow{\sim} & \text{str}(k).
\end{array}
$$

The map $k \rightarrow \text{str}(k)$ is an equivalence since $\text{str}$ is left Quillen functor, the same thing holds for the abelization functor i.e., $k \rightarrow \text{Ab}(k)$ is a trivial fibration, since $k \rightarrow k$ is a trivial cofibration in $\text{dgAlg}_k$ and $\text{Ab}$ is a left Quillen functor. On another hand the map $k \rightarrow \text{str}(k)$, which is a trivial fibration in $E_{\infty}\text{dgAlg}_k$ and hence in $\text{dgAlg}_k$, can be factored (cf. 2.3) as $k \rightarrow \text{Ab}(k) \rightarrow \text{str}(k)$, where $\text{Ab}(k) \rightarrow \text{str}(k)$ is a trivial fibration between cofibrant object in $\text{dgCAlg}_k$. It follows that we have a retract $l : \text{str}(k) \rightarrow \text{Ab}(k)$. \qed

**Definition 2.5.** The suspension functor in the pointed model categories $\text{dgCAlg}_k$, $\text{dgAlg}_k$ and $E_{\infty}\text{dgAlg}_k$ are denoted by $B$, $\Sigma$ and $B_{\infty}$. 

Lemma 2.6. Suppose that $A$ is a cofibrant object in $E_\infty \text{dgAlg}_k^*$, and $i : A \to k$ a cofibration, then $str(B_\infty A)$ is a retract of $Ab(\Sigma A)$ in the category $\text{dgCAlg}_k$.

Proof. First of all if a map $f$ is associative, commutative resp. $E_\infty$-map we put an index $f_a$, $f_c$ resp. $f_\infty$, notice that by definition of the operad $E_\infty$ any $E_\infty$-map is a strictly associative map. Suppose that $A$ is a cofibrant object in $E_\infty \text{dgAlg}_k$.

Consider the following commutative square:

\[
\begin{array}{c}
\Sigma A \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
Ab(\Sigma A) & \xrightarrow{x_c} & str[B_\infty A] = B[str(A)].
\end{array}
\]

where $\Sigma A$ is the (homotopy) pushout in $\text{dgAlg}_k$ and $B_\infty A$ is the (homotopy) pushout in $E_\infty \text{dgAlg}_k$. By proposition 2.2 and proposition 2.3 we have a following commutative square in $\text{dgAlg}_k$:

\[
\begin{array}{c}
\Sigma A \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
Ab(\Sigma A) & \xrightarrow{x_c} & str[B_\infty A] = B[str(A)].
\end{array}
\]

By 2.4 we have an inclusion of commutative differential graded algebras $l_c : str(k) \to Ab(k)$ and after strictification we obtain on another (homotopy) pushout square in $\text{dgCAlg}_k$ given by

\[
\begin{array}{c}
\text{str}(A) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{str}(k) & \xrightarrow{f_c} & B[str(A)]
\end{array}
\]

\[
\begin{array}{c}
\text{str}(k) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
Ab(k) & \xrightarrow{h_c} & Ab(\Sigma A).
\end{array}
\]

In order to prove that $B[str(A)]$ is a retract of $Ab(\Sigma A)$ it is sufficient to prove that

\[ x_c \circ h_c \circ l_c = f_c. \]
By proposition 2.2 and proposition 2.3, the composition of $E_\infty$-maps

$$
\begin{array}{c}
\mathbb{k} \\
\xrightarrow{f_c} \\
\mathbb{B}_\infty A \\
\mathbb{str} \mathbb{B}_\infty A
\end{array}
$$

can be factored in a unique way as

$$
\begin{array}{c}
\mathbb{k} \\
\xrightarrow{h_c} \\
\mathbb{B}_\infty A \\
\mathbb{x_c} \\
\mathbb{str} \mathbb{B}_\infty A
\end{array}
$$

By unicity, $\alpha_c = f_c$. On another hand, using the first pushout in $E_\infty \text{dgAlg}_k$, the previous composition $\mathbb{k} \to \mathbb{str} \mathbb{B}_\infty A$ is factored as

$$
\begin{array}{c}
\mathbb{k} \\
\xrightarrow{pr} \\
\mathbb{Ab}(k) \\
\xrightarrow{\pi} \\
\mathbb{str}(k) \\
\xrightarrow{f_c} \\
\mathbb{str} \mathbb{B}_\infty A
\end{array}
$$

We summarize the previous remarks in the following commutative diagram:

$$
\begin{array}{c}
\mathbb{k} \\
\xrightarrow{pr} \\
\mathbb{Ab}(k) \\
\xrightarrow{\pi} \\
\mathbb{str}(k) \\
\xrightarrow{f_c} \\
\mathbb{str} \mathbb{B}_\infty A
\end{array}
$$

by definition of $h_c$, the dotted map $h_c$ makes the left square commutative. Since the whole square is commutative and the map $pr$ is surjective we conclude that $x_c \circ h_c = f_c \circ \pi$. Since the map $l_c : Str(k) \to Ab(k)$ is a retract of $\pi$ (Cf. 2.4) i.e., $\pi \circ l_c = id$, we conclude that $x_c \circ h_c \circ l_c = f_c$. Finally, by unicity of the pushout, we deduce that the following composition

$$
\mathbb{B}[str(A)] \xrightarrow{x_c} \mathbb{B}[\Sigma A] \xrightarrow{f_c} \mathbb{B}(\mathbb{str}(A))
$$

is identity. □

3. MAIN RESULT AND APPLICATIONS

**Theorem 3.1.** For any $R$ and $S$ in $\text{dgCAlg}_k^*$, the induced map by the forgetful functor

$$
\Omega \text{Map}_{\text{dgCAlg}}^*(R, S) \to \Omega \text{Map}_{\text{dgAlg}}^*(R, S),
$$

has a retract, in particular

$$
\pi_i \text{Map}_{\text{dgCAlg}}^*(R, S) \to \pi_i \text{Map}_{\text{dgAlg}}^*(R, S)
$$

is injective $\forall i > 0$.

**Proof.** Suppose that $R$ is (cofibrant) object in $E_\infty \text{dgAlg}_k$ and $S$ any object in $\text{dgCAlg}_k$. By adjunction, we have that

$$
\Omega \text{Map}_{\text{dgCAlg}}^*(str(R), S) \sim \text{Map}_{\text{dgCAlg}}^*(\mathbb{B}[str(R)], S) \quad (3.1)
$$

$$
\sim \text{Map}_{\text{dgCAlg}}^*(\mathbb{str} \mathbb{B}_\infty R, S) \quad (3.2)
$$

$$
\sim \text{Map}_{E_\infty \text{dgAlg}}^*(\mathbb{B}_\infty R, S) \quad (3.3)
$$

$$
\sim \Omega \text{Map}_{E_\infty \text{dgAlg}}^*(R, S). \quad (3.4)
$$

By Lemma 2.6 we have a retract

$$
\text{Map}_{\text{dgCAlg}}^*(\mathbb{B}[str(R)], S) \to \text{Map}_{\text{dgCAlg}}^*(\mathbb{Ab}(\Sigma R), S) \to \text{Map}_{\text{dgCAlg}}^*(\mathbb{B}[str(R)], S).
$$

Again by adjunction:

$$
\text{Map}_{\text{dgCAlg}}^*(\mathbb{Ab}(\Sigma R), S) \sim \text{Map}_{\text{dgAlg}}^*(\Sigma R, S) \sim \Omega \text{Map}_{\text{dgAlg}}^*(R, S).
$$
We conclude that

$$\Omega \text{Map}_{E_\infty \text{dgAlg}_k} (R, S) \xrightarrow{U} \Omega \text{Map}_{dg\text{Alg}_k} (R, S) \xrightarrow{U} \Omega \text{Map}_{E_\infty \text{dgAlg}_k} (R, S)$$

is a retract. Hence, the forgetful functor $U$ induces an injective map on homotopy groups i.e.,

$$\pi_i \text{Map}_{dg\text{CAlg}_k} (\text{str}(R), S) \simeq \pi_i \text{Map}_{E_\infty \text{dgAlg}_k} (R, S) \rightarrow \pi_i \text{Map}_{dg\text{Alg}_k} (R, S)$$

is injective $\forall \ i > 0$. □

3.1. Rational homotopy theory. We give an application of our theorem 3.1 in the context of rational homotopy theory. Let $X$ be a simply connected rational space such that $\pi_i X$ is finite dimensional $\mathbb{Q}$-vector space for each $i > 0$. Let $C^* (X)$ be the differential graded $\mathbb{Q}$-algebra cochain associated to $X$ which is a connective $E_\infty \text{dgAlg}_k$. By Sullivan theorem $\pi_i X \simeq \pi_i \text{Map}_{dg\text{CAlg}_k} (C^* (X), \mathbb{Q})$. By 3.1 we have that $\pi_i X$ is a sub $\mathbb{Q}$-vector space of $\pi_i \text{Map}_{E_\infty \text{dgAlg}_k} (R, S)$. On another hand [1], since $C^* (X)$ is connective, we have that for any $i > 1$

$$\pi_i \text{Map}_{dg\text{Alg}_k} (C^* (X), \mathbb{Q}) \simeq \text{HH}^{-1+i}(C^* (X), \mathbb{Q}),$$

where $\text{HH}^*$ is the Hochschild cohomology. Since we have assumed finiteness condition on $X$, we have that

$$\text{HH}^{-1+i}(C^* (X), \mathbb{Q}) \simeq \text{HH}_{i-1}(C^* (X), \mathbb{Q}).$$

The functor $C^*(\cdot, \mathbb{Q}) : \text{Top}^{op} \rightarrow E_\infty \text{dgAlg}_k$ commutes with finite homotopy limits, where $\text{Top}$ is the category of simply connected spaces. Hence,

$$\text{HH}_{i+1+i}(C^* (X), \mathbb{Q}) = H^{i-1} [C^* (X) \otimes_{C^* (X \times X)} \mathbb{Q}] \simeq H^{i-1}(\Omega X, \mathbb{Q}).$$

We conclude that $\pi_i X$ is a sub $\mathbb{Q}$-vector space of $H^{i-1}(\Omega X, \mathbb{Q})$. More generally by Block-Lazarev result [3] on rational homotopy theory and [1], we have an injective map of $\mathbb{Q}$-vector spaces

$$\text{AQ}^{-i}(C^* (X), C^* (Y)) \rightarrow \text{HH}^{-i+1}(C^* (X), C^* (Y)),$$

where the $C^*(\cdot)$-(bi)modules structure on $C^* (Y)$ is given by $C^* (X) \rightarrow \mathbb{Q} \rightarrow C^* (Y)$, and $\text{AQ}^*$ is the André-Quillen cohomology. We also assume that $X$ and $Y$ are simply connected and $i > 1$.

More generally,

$$\pi_i \text{Map}_{E_\infty \text{dgAlg}_k} (R, S) = \text{AQ}^{-i}(R, S) \rightarrow \text{HH}^{-i+1}(R, S) = \pi_i \text{Map}_{dg\text{Alg}_k} (R, S)$$

is an injective map of $\mathbb{Q}$-vector spaces for all $i > 1$ and any augmented $E_\infty$-differential graded connective $\mathbb{Q}$-algebras $R$ and $S$, where the action of $S$ on $R$ is given by $S \rightarrow \mathbb{Q} \rightarrow R$.

Acknowledgement: I'm grateful to Benoît Fresse for his nice explanation of Lemma [1.1] the key point of the proof is due to him.
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