THE WEYL REALIZATIONS OF LIE ALGEBRAS AND LEFT–RIGHT DUALITY

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Abstract. We investigate dual realizations of non–commutative spaces of Lie algebra type in terms of formal power series in the Weyl algebra. To each realization of a Lie algebra \( \mathfrak{g} \) we associate a star–product on the symmetric algebra \( S(\mathfrak{g}) \) and an ordering on the enveloping algebra \( U(\mathfrak{g}) \). Dual realizations of \( \mathfrak{g} \) are defined in terms of left–right duality of the star–products on \( S(\mathfrak{g}) \). It is shown that the dual realizations are related to an extension problem for \( \mathfrak{g} \) by shift operators whose action on \( U(\mathfrak{g}) \) describes left and right shift of the generators of \( U(\mathfrak{g}) \) in a given monomial. Using properties of the extended algebra, in the Weyl symmetric ordering we derive closed form expressions for the dual realizations of \( \mathfrak{g} \) in terms of two generating functions for the Bernoulli numbers. The theory is illustrated by considering the \( \kappa \)–deformed space.

1. Introduction

This paper deals with some aspects of realizations of finite dimensional Lie algebras with emphasis on applications to non–commutative (NC) spaces. Realizations of Lie algebras by vector fields play a major role in group analysis of differential equations, such as calculation of symmetry groups and group–invariant solutions [1, 2], group classification of PDE’s [3] and construction of difference schemes for numerical solutions of differential equations [4]. Recently, realizations of Lie algebras have been used extensively in the study of NC spaces and their deformed symmetries. The study of NC spaces is motivated by physical evidence that the classical concept of point at the Planck scale \( (l_P = \sqrt{G\hbar/c^3} \approx 1.62 \times 10^{-35} \text{ m}) \) is no longer valid due to quantum fluctuations. Einstein’s theory of gravity coupled with Heisenberg’s uncertainly principle suggests that space–time coordinates should satisfy uncertainty relations \( \Delta \hat{x}_\mu \Delta \hat{x}_\nu \geq l_P^2 \) (Refs. [5, 6]). One of the possible approaches towards description of space–time structure at the Planck scale is in the framework of NC geometry. In this approach one introduces non–commutativity in space–time coordinates via the commutation relations \( [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}(\hat{x}) \). The anti–symmetric tensor \( \theta_{\mu\nu} \) generally

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depends on $\hat{x}_\mu$ and a deformation parameter $h \in \mathbb{R}$, and satisfies the classical limit condition $\lim_{h \to 0^+} \theta_{\mu\nu} = 0$. There is also evidence from string theory suggesting that space–time coordinates are non-commutative [7]. The coordinates $\hat{x}_\mu$ can be realized as formal power series in the Weyl algebra semicompleted with respect to the degree of differential operator. Algebraic relations between space–time coordinates lead to various models of NC spaces such as the Moyal space [8, 9], $\kappa$–deformed space [10, 11] and generalized $\kappa$–deformed space [12]. A review of applications of NC spaces in physics can be found in Refs. [13, 14].

The present paper deals with realizations of NC spaces of Lie algebra type whose coordinates satisfy the Lie algebra relations $[\hat{x}_\mu, \hat{x}_\nu] = \sum_{\alpha=1}^{n} C_{\mu\nu\alpha} \hat{x}_\alpha$. An important example is the $\kappa$–deformed space defined by $[\hat{x}_\mu, \hat{x}_\nu] = i (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)$ introduced in Refs. [10, 11]. The goal of the present work is to generalize the results on realizations of the $\kappa$–deformed space to an arbitrary finite dimensional Lie algebra $\mathfrak{g}$. We also want to describe some general features of these results in a realization independent setting. This is done by introducing an associative algebra $\mathcal{H}$ which contains the enveloping algebra $U(\mathfrak{g})$ and studying an action of the generators of $\mathcal{H}$ on $U(\mathfrak{g})$.

The paper is organized as follows. In section 2 we outline the results needed in later sections. For a given Lie algebra $\mathfrak{g}$ with basis $\{X_1, X_2, \ldots, X_n\}$ we study extension of $U(\mathfrak{g})$ to an associative algebra $\mathcal{H}$ generated by $X_\mu$ and $2n^2$ generators $T_{\mu\nu}$ and $T_{\mu\nu}^{-1}$. We introduce an action of $T_{\mu\nu}$ and $T_{\mu\nu}^{-1}$ on $U(\mathfrak{g})$ describing the right and left shift of basis elements $X_\mu$ in a monomial $X_1^{i_1}X_2^{i_2} \ldots X_n^{i_n} \in U(\mathfrak{g})$. The actions of $T_{\mu\nu}$ and $T_{\mu\nu}^{-1}$ are given by the coproducts $\Delta T_{\mu\nu} = \sum_{\alpha=1}^{n} T_{\mu\alpha} \otimes T_{\alpha\nu}$ and $\Delta T_{\mu\nu}^{-1} = \sum_{\alpha=1}^{n} T_{\alpha\nu}^{-1} \otimes T_{\mu\alpha}^{-1}$.

Section 3 introduces realizations of $\mathfrak{g}$ by formal power series of differential operators in a semicompleted Weyl algebra $\hat{A}_n$. To each realization we associate a star–product on the symmetric algebra $S(\mathfrak{g})$ and an ordering prescription on $U(\mathfrak{g})$. We define left–right dual star–products on $S(\mathfrak{g})$ via $f \hat{\star} g = \tau(f \star g)$ where $\tau$ is the flip operator $\tau(f \star g) = g \star f$. We then study properties of such products in terms of the realizations of $\mathfrak{g}$. The role of the generators $T_{\mu\nu}$ is to provide transition between the left–right dual star–products and also between the associated dual realizations. In section 4 we investigate in more detail dual realizations of $\mathfrak{g}$ in the Weyl symmetric ordering. Using the operators $T_{\mu\nu}$ we find a novel proof of the dual realizations of $\mathfrak{g}$ in terms of the generating functions for the Bernoulli numbers $B_n$ (corresponding to conventions $B_1 = \pm 1/2$). In section 5 the theory is illustrated by finding the dual realizations of the $\kappa$–deformed space and the associated
star–products. We show that the star–product is in fact a deformation quantization of the Lie–Poisson bracket on the dual of the \( \kappa \)–deformed space.

## 2. Extensions of Lie algebras and left–right duality

Throughout the article \( \mathfrak{g} \) denotes a Lie algebra of dimension \( n \) over a field \( \mathbb{K} \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)). Let \( \{X_\mu \mid 1 \leq \mu \leq n\} \) be an ordered basis of \( \mathfrak{g} \) satisfying the Lie bracket

\[
[X_\mu, X_\nu] = \sum_{\alpha=1}^{n} C_{\mu\nu\alpha} X_\alpha.
\]

The structure constants obey \( C_{\mu\nu\alpha} = -C_{\nu\mu\alpha} \) and the Jacobi identity

\[
\sum_{\rho=1}^{n} \left( C_{\mu\alpha\rho} C_{\rho\beta\nu} + C_{\alpha\beta\rho} C_{\rho\mu\nu} + C_{\beta\mu\rho} C_{\rho\alpha\nu} \right) = 0.
\]

To motivate our discussion let us consider the following simple observation. In the enveloping algebra \( U(\mathfrak{g}) \), relations (1) can be written as

\[
X_\mu X_\nu = \sum_{\alpha=1}^{n} \left( \delta_{\mu\alpha} X_\nu + C_{\mu\nu\alpha} \right) X_\alpha.
\]

In general, if \( X = X_1^n X_2^m \ldots X_n^r \in U(\mathfrak{g}) \) is a monomial, then shifting \( X_\mu \) to the far right in the product \( X_\mu X \) generates polynomials \( p_{\mu\alpha}(X) \) such that

\[
X_\mu X = \sum_{\alpha=1}^{n} p_{\mu\alpha}(X) X_\alpha.
\]

Here \( p_{\mu\alpha}(X) \) is the unique polynomial of the form \( p_{\mu\alpha}(X) = \delta_{\mu\alpha} X + \text{lower order terms} \). Roughly speaking, computation of the polynomials \( p_{\mu\alpha}(X) \) is related to the problem of extending the Lie algebra \( \mathfrak{g} \) by \( n^2 \) generators \( T_{\mu\nu} \) and defining an action of \( T_{\mu\nu} \) on \( U(\mathfrak{g}) \) such that \( T_{\mu\alpha} \triangleright X = p_{\mu\alpha}(X) \). Similarly, if \( X_\mu \) is shifted to the far left in \( XX_\mu \) so that

\[
XX_\mu = \sum_{\alpha=1}^{n} X_\alpha \tilde{p}_{\alpha\mu}(X),
\]

we want to find another set of generators, say \( S_{\mu\alpha} \), satisfying \( S_{\mu\alpha} \triangleright X = \tilde{p}_{\alpha\mu}(X) \). The generators \( T_{\mu\nu} \) and \( S_{\mu\nu} \) were introduced in construction of the Hopf algebroid structure of the Lie algebra type NC phase space (see Refs. [15, 16]). In this paper we study properties of \( T_{\mu\nu} \) and \( S_{\mu\nu} \) associated to a general Lie algebra \( \mathfrak{g} \). We construct an associative algebra \( \mathcal{H} \) by extending \( U(\mathfrak{g}) \) with \( T_{\mu\nu} \) and \( S_{\mu\nu} \), and use this to prove certain results about realizations of \( \mathfrak{g} \).
Let $g^L \supset g$ be the Lie algebra with basis \{\(X_\mu, T_{\mu\nu}\mid 1 \leq \mu, \nu \leq n\}\) satisfying relations (1) and

\[
[T_{\mu\nu}, T_{\alpha\beta}] = 0, \tag{5}
\]

\[
[T_{\mu\nu}, X_\lambda] = \sum_{\alpha=1}^{n} C_{\mu\lambda\alpha} T_{\alpha\nu}. \tag{6}
\]

Our first task is to construct an action of the enveloping algebra $U(g^L)$ on the subalgebra $U(g)$. To keep the notation simple, we identify the elements of $g^L$ with their canonical images in $U(g^L)$.

**Theorem 1.** Let $\triangleright: U(g^L) \otimes U(g) \to U(g)$ be a linear map, $a \otimes X \mapsto a \triangleright X$, defined by

\[
1 \triangleright X = X, \quad X_\mu \triangleright X = X_\mu X, \quad T_{\mu\nu} \triangleright 1 = \delta_{\mu\nu} \tag{7}
\]

and $(ab) \triangleright X = a \triangleright (b \triangleright X)$ for $X \in U(g)$ and $a, b \in U(g^L)$. Then $\triangleright$ is a left action of $U(g^L)$ on $U(g)$ satisfying

\[
T_{\mu\nu} \triangleright (XY) = \sum_{\alpha=1}^{n} (T_{\mu\alpha} \triangleright X)(T_{\alpha\nu} \triangleright Y) \tag{8}
\]

for all $X,Y \in U(g)$.

**Proof.** First we show that (8) is uniquely fixed by the relations in $U(g^L)$ and conditions (7). Relation (6) and the normalization condition $T_{\mu\nu} \triangleright 1 = \delta_{\mu\nu}$ imply

\[
T_{\mu\nu} \triangleright X_\lambda = \delta_{\mu\nu} X_\lambda + C_{\mu\lambda\nu}. \tag{9}
\]

Now, for any $Y \in U(g)$, the identity $T_{\mu\nu} \triangleright (X_\lambda Y) = [T_{\mu\nu}, X_\lambda] \triangleright Y + X_\lambda (T_{\mu\nu} \triangleright Y)$ together with Eqs. (7) and (9) yields

\[
T_{\mu\nu} \triangleright (X_\lambda Y) = \sum_{\alpha=1}^{n} C_{\mu\lambda\alpha} (T_{\alpha\nu} \triangleright Y) + X_\lambda (T_{\mu\nu} \triangleright Y) = \sum_{\alpha=1}^{n} (T_{\mu\alpha} \triangleright X_\lambda)(T_{\alpha\nu} \triangleright Y). \tag{10}
\]

This shows that decomposition (8) holds for monomials $X$ of degree one. By induction, assume that (8) holds for monomials $X$ of degree $k$. Then for $k + 1$ degree monomials we have

\[
T_{\mu\nu} \triangleright ((X_\lambda X)Y) = \sum_{\alpha=1}^{n} (T_{\mu\alpha} \triangleright X_\lambda)(T_{\alpha\nu} \triangleright (XY))
\]

\[
= \sum_{\beta=1}^{n} \left( \sum_{\alpha=1}^{n} (T_{\mu\alpha} \triangleright X_\lambda)(T_{\alpha\beta} \triangleright X) \right) (T_{\beta\nu} \triangleright Y) = \sum_{\beta=1}^{n} \left( T_{\mu\beta} \triangleright (X_\lambda X) \right)(T_{\beta\nu} \triangleright Y). \tag{11}
\]
By linearly extending the action we find that Eq. (8) holds for all $X, Y \in U(g)$. In order to prove that the action is well-defined, it suffices to show that the defining relations of $U(g)$ are in the kernel of $\triangleright$, and that the action is consistent with the relations in $U(g^L)$.

This is obvious for the action of $X_\mu$, hence we only consider $T_{\mu\nu}$. Using Eq. (8) we find

$$T_{\mu\nu} \triangleright [X_\alpha, X_\beta] = \delta_{\mu\nu}[X_\alpha, X_\beta] + \sum_{\rho=1}^n \left( C_{\mu\rho\alpha} C_{\rho\beta\nu} + C_{\beta\mu\rho} C_{\rho\alpha\nu} \right).$$

(12)

In combination with Eq. (9), this implies

$$T_{\mu\nu} \triangleright \left( [X_\alpha, X_\beta] - \sum_{\rho=1}^n C_{\alpha\beta\rho} X_\rho \right) = \sum_{\rho=1}^n \left( C_{\mu\rho\alpha} C_{\rho\beta\nu} + C_{\beta\mu\rho} C_{\rho\alpha\nu} + C_{\alpha\beta\rho} C_{\rho\mu\nu} \right) = 0$$

(13)

due to the Jacobi identity (2). To show consistency of the action with relation (5) note that $[T_{\alpha\beta}, T_{\mu\nu}] \triangleright X_\rho = 0$. By induction, assume that $[T_{\alpha\beta}, T_{\mu\nu}] \triangleright X = 0$ for all monomials of degree $k$. Then a short computation shows that

$$[T_{\alpha\beta}, T_{\mu\nu}] \triangleright (X_\rho X) = X_\rho [T_{\alpha\beta}, T_{\mu\nu}] \triangleright X + \sum_{\kappa=1}^n C_{\alpha\kappa\rho} [T_{\kappa\beta}, T_{\mu\nu}] \triangleright X + \sum_{\kappa=1}^n C_{\mu\kappa\rho} [T_{\alpha\beta}, T_{\mu\nu}] \triangleright X = 0. \quad (14)$$

Hence, $[T_{\alpha\beta}, T_{\mu\nu}] \triangleright X = 0$ for all $X \in U(g)$. Finally, consistency with relation (6) follows from Eqs. (8)–(9) and noting that for any monomial $X \in U(g)$ we have

$$[T_{\mu\nu}, X_\lambda] \triangleright X = \sum_{\alpha=1}^n \left( \delta_{\mu\alpha} X_\lambda + C_{\mu\lambda\alpha} (T_{\alpha\nu} \triangleright X) - X_\lambda (T_{\mu\nu} \triangleright X) \right) = \left( \sum_{\alpha=1}^n C_{\mu\lambda\alpha} T_{\alpha\nu} \right) \triangleright X. \quad (15)$$

This completes the proof. ■

We remark that $T_{\mu\nu} \triangleright X$ can be computed recursively from

$$T_{\mu\nu} \triangleright (X_\alpha X) = X_\alpha (T_{\mu\nu} \triangleright X) + \sum_{\rho=1}^n C_{\mu\rho\alpha} (T_{\rho\nu} \triangleright X). \quad (16)$$

The next result shows that the action of $T_{\mu\alpha}$ on $X$ generates precisely the polynomials $p_{\mu\alpha}(X)$ defined by Eq. (4).

**Lemma 1.** Let $X$ be a monomial in $U(g) \subset U(g^L)$. If $X_\mu$ is shifted to the far right in the product $X_\mu X$, then

$$X_\mu X = \sum_{\alpha=1}^n (T_{\mu\alpha} \triangleright X) X_\alpha. \quad (17)$$
Proof. For monomials of degree one, Eq. (17) follows directly from Eq. (9). By induction, assume that (17) holds for monomials \( X \) of degree \( k \). Then, in view of Eq. (8) we find

\[
X_\mu(X_\lambda X) = \sum_{\alpha=1}^{n} (T_\mu \triangleright X_\lambda) (X_\alpha X) = \sum_{\alpha=1}^{n} (T_\mu \triangleright X_\lambda) \left( \sum_{\beta=1}^{n} T_\alpha \triangleright X \right) X_\beta
\]

\[
= \sum_{\beta=1}^{n} \left[ \sum_{\alpha=1}^{n} (T_\mu \triangleright X_\lambda)(T_\alpha \triangleright X) \right] X_\beta = \sum_{\beta=1}^{n} \left( T_\mu \triangleright (X_\lambda X) \right) X_\beta.
\]

Hence, Eq. (17) holds for all monomials \( X \in U(\mathfrak{g}) \).

As noted earlier, shifting \( X_\mu \) to the far left in the product \( XX_\mu \) generates polynomials \( \tilde{p}_{\mu\alpha}(X) \) defined by Eq. (4). The following result is analogous to theorem 1. It describes an extension of the Lie algebra \( \mathfrak{g} \) by \( n^2 \) commuting generators \( S_{\mu\alpha} \) such that \( S_{\mu\alpha} \triangleright X = \tilde{p}_{\mu\alpha}(X) \). Theorem 2.

Let \( \mathfrak{g}^R \) be the Lie algebra with basis \( \{ X_\mu, T_{\mu\nu}^{-1} \mid 1 \leq \mu, \nu \leq n \} \) defined by relations (1) and

\[
[T_{\alpha\beta}^{-1}, T_{\mu\nu}^{-1}] = 0, \quad [T_{\mu\nu}^{-1}, X_\lambda] = \sum_{\alpha=1}^{n} C_{\lambda\alpha\nu} T_{\mu\alpha}^{-1}.
\]

Then there exists a left action \( \triangleright : U(\mathfrak{g}^R) \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \) satisfying

\[
1 \triangleright X = X, \quad X_\mu \triangleright X = X_\mu X,
\]

\[
T_{\mu\nu}^{-1} \triangleright 1 = \delta_{\mu\nu}, \quad T_{\mu\nu}^{-1} \triangleright (XY) = \sum_{\alpha=1}^{n} (T_{\mu\nu}^{-1} \triangleright X)(T_{\mu\nu}^{-1} \triangleright Y)
\]

for all \( X, Y \in U(\mathfrak{g}) \).

The action of \( T_{\mu\nu}^{-1} \) is computed recursively from \( T_{\mu\nu}^{-1} \triangleright X_\lambda = \delta_{\mu\nu} X_\lambda - C_{\mu\lambda\nu} \) and

\[
T_{\mu\nu}^{-1} \triangleright (X_\alpha X) = X_\alpha (T_{\mu\nu}^{-1} \triangleright X) - \sum_{\rho=1}^{n} C_{\rho\alpha\nu} (T_{\mu\rho}^{-1} \triangleright X), \quad X \in U(\mathfrak{g}).
\]

Using induction as in lemma 1 one can prove that the right multiplication by \( X_\mu \) can be written as

\[
XX_\mu = \sum_{\alpha=1}^{n} X_\alpha (T_{\mu\alpha}^{-1} \triangleright X).
\]

At this point it seems natural to extend the enveloping algebra \( U(\mathfrak{g}) \) by both sets of generators \( T_{\mu\nu} \) and \( T_{\mu\nu}^{-1} \). If such extension exists, then the normalization conditions for \( T_{\mu\nu} \) and \( T_{\mu\nu}^{-1} \) necessarily imply that \( \sum_{\alpha=1}^{n} (T_{\mu\alpha}^{-1} T_{\alpha\nu}) \triangleright 1 = \sum_{\alpha=1}^{n} (T_{\mu\alpha} T_{\alpha\nu}^{-1}) \triangleright 1 = \delta_{\mu\nu} \).
This suggests that $U(\mathfrak{g})$ should have a natural inclusion into a unital associative algebra defined as follows.

**Definition 1.** Let $\mathcal{H}$ be a unital associative algebra with generators $X_\mu, T_{\mu\nu}$ and $T_{\mu\nu}^{-1}$, $1 \leq \mu, \nu \leq n$, subject to relations (1), (5)–(6), (19) and the additional relations

$$\sum_{\alpha=1}^{n} T_{\mu\alpha}^{-1} T_{\alpha\nu} = \sum_{\alpha=1}^{n} T_{\mu\alpha} T_{\alpha\nu}^{-1} = \delta_{\mu\nu}. \quad (24)$$

It is straightforward, albeit lengthy, to verify that $\mathcal{H}$ is well defined, i.e. that the defining relations for $\mathcal{H}$ are consistent. Clearly, we have the embedding $\mathfrak{g} \hookrightarrow \mathcal{H}$ where $\mathcal{H}$ inherits the actions of $U(\mathfrak{g}^L)$ and $U(\mathfrak{g}^R)$ on the subalgebra $U(\mathfrak{g})$. In the rest of the paper important role is played by the elements of $\mathcal{H}$ defined by

$$Y_\mu = \sum_{\alpha=1}^{n} X_\alpha T_{\mu\alpha}^{-1}. \quad (25)$$

The elements $Y_\mu$ generate the left–right dual of the Lie algebra $\mathfrak{g}$ introduced in Sec. 3. We note that Eq. (23) implies $Y_\mu \triangleright X = \sum_{\alpha=1}^{n} X_\alpha (T_{\mu\alpha}^{-1} \triangleright X) = XX_\mu$. Hence, $X_\mu$ and $Y_\mu$ act as left and right multiplication operators on $U(\mathfrak{g})$ since $X_\mu \triangleright X = X X_\mu$ and $Y_\mu \triangleright X = XX_\mu$ for all $X \in U(\mathfrak{g})$. Furthermore, relations (1) and (19) imply that

$$[X_\mu, Y_\nu] = \sum_{\alpha=1}^{n} \left( [X_\mu, X_\alpha] T_{\mu\alpha}^{-1} - X_\alpha [T_{\mu\alpha}^{-1}, X_\mu] \right) = 0. \quad (26)$$

reflecting commutativity of the left and right multiplication in $U(\mathfrak{g})$.

In the following section we consider realizations of $\mathfrak{g}$ by formal power series of differential operators. To each realization of $\mathfrak{g}$ we associate a star–product on the symmetric algebra $S(\mathfrak{g})$. Using realizations of the generators $T_{\mu\nu}$ and $T_{\mu\nu}^{-1}$ we construct left–right dual realizations of $\mathfrak{g}$ that correspond to left–right dual star–products on $S(\mathfrak{g})$. In the course of our discussion we present a novel proof of the Weyl symmetric realization of $\mathfrak{g}$ found in Ref. [17]. The theory presented here is illustrated by means of the $\kappa$–deformed space in section 5.

3. **Realizations of Lie algebras and left–right duality of associated star–products**

Let $\mathfrak{g}_h$ be a Lie algebra over $\mathbb{K}$ defined by

$$[X_\mu, X_\nu] = \sum_{\alpha=1}^{n} C_{\mu\nu\alpha}(h) X_\alpha, \quad 1 \leq \mu, \nu \leq n. \quad (27)$$
We assume that the structure constants $C_{\mu\nu\alpha}(h)$ depend on a deformation parameter $h \in \mathbb{R}$ such that $\lim_{h \to 0} C_{\mu\nu\alpha}(h) = 0$. Any Lie algebra $\mathfrak{g}$ can be deformed in this way by simply rescaling the structure constants of $\mathfrak{g}$, $C_{\mu\nu\alpha} \mapsto hC_{\mu\nu\alpha}$. To simplify the notation, we omit explicit dependence of $C_{\mu\nu\alpha}$ on $h$. The enveloping algebra $U(\mathfrak{g}_h)$ is the coordinate algebra of the NC space defined by relations (27). In this section we study realizations of $X_{\mu}$ by formal power series of differential operators in a semi-completed Weyl algebra. Recall that the $n$–th Weyl algebra $A_n$ is a unital associative algebra over $K$ generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ satisfying the commutation relations $[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0$, $[\partial_\mu, x_\nu] = \delta_{\mu\nu}$. (In the physics literature this is usually called the Heisenberg algebra). The algebra $A_n$ has a faithful representation on the vector space of polynomials $K[x_1, \ldots, x_n]$ where $x_\mu$ stands for multiplication operator by $x_\mu$ and $\partial_\mu$ is the partial derivative $\partial/\partial x_\mu$. We define $\hat{A}_n$ to be the semicompletion of $A_n$ by the order of differential operators. Thus, $\hat{A}_n$ contains formal power series in $\partial_\mu$ but only polynomial expressions in $x_\mu$.

**Definition 2.** A realization of the Lie algebra $\mathfrak{g}_h$ is a Lie algebra monomorphism $\varphi: \mathfrak{g}_h \to \hat{A}_n$ defined on the basis of $\mathfrak{g}$ by

$$\varphi(X_\mu) = \sum_{\alpha=1}^{n} x_\alpha \varphi_{\alpha\mu}(\partial),$$

where $\varphi_{\alpha\mu}(\partial)$ is a formal power series in $\partial_1, \ldots, \partial_n$ depending on $h$ such that $\lim_{h \to 0} \varphi_{\alpha\mu}(\partial) = \delta_{\alpha\mu}$.

The map $\varphi$ extends to a unique homomorphism of associative algebras $\varphi: U(\mathfrak{g}_h) \to \hat{A}_n$. The coordinates $\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \varphi_{\alpha\mu}(\partial) \in \hat{A}_n$ are interpreted as deformations of ordinary coordinates $x_\mu$ since $\lim_{h \to 0} \hat{x}_\mu = x_\mu$. Let $\hat{\mathcal{X}}$ and $\mathcal{X}$ denote the subalgebras of $\hat{A}_n$ generated by $\hat{x}_1, \ldots, \hat{x}_n$ and $x_1, \ldots, x_n$, respectively. Since $\varphi$ is injective and $[\hat{x}_\mu, \hat{x}_\nu] = \sum_{\alpha=1}^{n} C_{\mu\nu\alpha} \hat{x}_\alpha$, the algebra $\hat{\mathcal{X}}$ is isomorphic with $U(\mathfrak{g}_h)$ (and $\mathcal{X}$ is trivially isomorphic with $S(\mathfrak{g}_h)$). The commutation relations for $\hat{x}_\mu$ hold if and only if the functions $\varphi_{\mu\nu}$ satisfy a system of formal PDE’s:

$$\sum_{\alpha=1}^{n} \left( \frac{\partial \varphi_{\lambda\mu}}{\partial \varphi_{\alpha\nu}} - \frac{\partial \varphi_{\lambda\nu}}{\partial \varphi_{\alpha\mu}} \right) = \sum_{\alpha=1}^{n} C_{\mu\nu\alpha} \varphi_{\lambda\alpha}, \quad 1 \leq \mu, \nu \leq n.$$ (29)

This is generally an under–determined system admitting infinitely many solutions parameterized by arbitrary real–analytic functions. The order in which $x_\alpha$ and $\varphi_{\alpha\mu}(\partial)$ appear
in the realization is immaterial since any linear combination
\[ \hat{x}_\mu = c \sum_{\alpha=1}^{n} x_\alpha \varphi_{\alpha \mu}(\partial) + (1 - c) \sum_{\alpha=1}^{n} \varphi_{\alpha \mu}(\partial)x_\alpha \] (30)
is also a realization of $\mathfrak{g}_\hbar$. Hermitian realizations are obtained for $c = 1/2$ (see Ref. [18]).

In the rest of the paper we set $c = 1$. Examples of different realizations of NC spaces such as the $\kappa$–deformed space, generalized $\kappa$–deformed space and $su(2)$–type NC space were found in Refs. [12, 17, 19, 20, 21, 22].

Realizations of Lie algebras are related to two important concepts: ordering on the enveloping algebra $U(\mathfrak{g}_\hbar)$ and star–product on the symmetric algebra $S(\mathfrak{g}_\hbar) \simeq X$.

To establish the connection we introduce a left action $\triangleright: \mathcal{A}_n \otimes X \to X$, $a \otimes f \mapsto a \triangleright f$, defined by
\[ x_\mu \triangleright f = x_\mu f, \quad \partial_\mu \triangleright f = \frac{\partial f}{\partial x_\mu}, \quad (ab) \triangleright f = a \triangleright (b \triangleright f), \quad f \in X. \] (31)
The action extends to formal power series in $\hat{A}_n$ in the obvious way. For a given realization $\varphi: \mathfrak{g}_\hbar \to \hat{A}_n$ we define the vector space isomorphism $\Omega_{\varphi}: \hat{X} \to X$ by $\Omega_{\varphi}(\hat{f}) = \hat{f} \triangleright 1$.

Note that for $\hat{f} = \hat{x}_{\mu_1} \hat{x}_{\mu_2} \ldots \hat{x}_{\mu_k}$ we have $\Omega_{\varphi}(\hat{f}) = x_{\mu_1} x_{\mu_2} \ldots x_{\mu_k} + p_{k-1}$ where $p_{k-1}$ is a polynomial of degree $k-1$ in the variables $x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_k}$. The isomorphism $\Omega_{\varphi}$ induces a star–product on the algebra $X$ as follows.

**Definition 3.** The star–product $\star: X \otimes X \to X$ associated to realization $\varphi: \mathfrak{g}_\hbar \to \hat{A}_n$ is defined by
\[ f \star g = \Omega_{\varphi}(\Omega_{\varphi}^{-1}(f) \Omega_{\varphi}^{-1}(g)), \quad f, g \in X. \] (32)
The algebra $X^\star = (X, +, \star)$ is a unital associative algebra which is isomorphic to $\hat{X}$ since $\Omega_{\varphi}(\hat{f} \hat{g}) = \Omega_{\varphi}(\hat{f}) \star \Omega_{\varphi}(\hat{g})$ for all $\hat{f}, \hat{g} \in \hat{X}$. Furthermore, the generators of $X^\star$ satisfy the commutation relations $x_{\mu} \star x_{\nu} - x_{\nu} \star x_{\mu} = \sum_{\alpha=1}^{n} C_{\mu \nu \alpha} x_{\alpha}$. A few remarks about the star–product are in order. The star–product is a deformation of the commutative product in $X$ since $f \star g = fg + O(h)$. In deformation quantization, for a given Poisson manifold $(M, \{ , \})$ one looks for a formal star–product such that $f \star g - g \star f = ih \{ f, g \} (\mod h^2)$ for $f, g \in C^\infty(M)[[h]]$. The proof of existence of star–products for a general Poisson manifold was given in Ref. [23] by Kontsevich’s formality theorem. In our approach the starting point is not a Poisson manifold but the non–commutative algebra $\hat{X}$. The vector space isomorphism $\Omega_{\varphi}: \hat{X} \to X$ is then used to transfer the non–commutative multiplication in $\hat{X}$ to the star–product in $X$. Here the star–product (32) is treated formally since it may
fail to converge when extended to power series in the variables $x_1, \ldots, x_n$. For technical issues about convergence see Ref. \[21\].

Next we introduce left–right duality of the star–product (32). Let $g_h$ be the Lie algebra with basis \{ $Y_1, Y_2, \ldots, Y_n$ \} closing the bracket relations

$$[Y_\mu, Y_\nu] = -\sum_{\alpha=1}^{n} C_{\mu\nu\alpha} Y_\alpha, \quad 1 \leq \mu, \nu \leq n.$$  \tag{33}

We say that $\tilde{g}_h$ is the “left–right dual” of the Lie algebra (27). Although $g_h$ and $\tilde{g}_h$ are trivially isomorphic via $X_\mu \mapsto -Y_\mu$, the relation between their realizations is generally non–trivial.

**Definition 4.** The star–products $*$ and $\tilde{*}$ associated with realizations $\varphi: g_h \to \hat{A}_n$ and $\tilde{\varphi}: \tilde{g}_h \to \hat{A}_n$, respectively, are left–right dual if

$$f \star g = g \tilde{*} f, \quad f, g \in \mathcal{X}. \tag{34}$$

We remark that the notion of duality introduced here refers to the flip operator $\tau$ since $f \tilde{*} g = \tau(f \star g)$, and is not related to standard duality between coordinates and momenta in the Weyl algebra. For future reference we refer to $\varphi$ and $\tilde{\varphi}$ as dual realizations. It is not difficult to show that the following result characterizes such realizations.

**Lemma 2.** Let $\varphi$ and $\tilde{\varphi}$ be realizations of Lie algebras (27) and (33) given by

$$\varphi(X_\mu) \equiv \hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \varphi_{\alpha\mu}(\partial), \quad \tilde{\varphi}(Y_\mu) \equiv \hat{y}_\mu = \sum_{\alpha=1}^{n} x_\alpha \tilde{\varphi}_{\alpha\mu}(\partial). \tag{35}$$

Then the star–products $*$ and $\tilde{*}$ are left–right dual if and only if $[\hat{x}_\mu, \hat{y}_\nu] = 0$ for all $\mu, \nu = 1, \ldots, n$.

The vector space isomorphism $\Omega_{\varphi}^{-1}: \mathcal{X} \to \hat{\mathcal{X}}$ associates to a realization $\varphi$ an ordering on the algebra $\hat{\mathcal{X}} \simeq U(\mathfrak{g}_h)$ by mapping the standard basis of $\mathcal{X}$ to a basis of $\hat{\mathcal{X}}$. Of particular interest is the Weyl symmetric realization which makes $\Omega_{\varphi}^{-1}$ the symmetrization map. This property is characterized by

$$\left(\sum_{\mu=1}^{n} k_\mu \hat{x}_\mu\right)^m \succ 1 = \left(\sum_{\mu=1}^{n} k_\mu x_\mu\right)^m, \quad \forall k_\mu \in \mathbb{K}, \ m \geq 1. \tag{36}$$

In the following section we derive an explicit form of the Weyl symmetric realization and compute its left–right dual. The approach followed here differs from that in Ref. \[17\] since the key role is played by the algebra $\mathcal{H}$ introduced in definition 1.
4. The Weyl symmetric realization of the algebra $\mathcal{H}$

This section deals with the realization of the algebra $\mathcal{H}$ that corresponds to the Weyl symmetric ordering on the subalgebra $U(g_h) \subset \mathcal{H}$. The realization uses the generating function for the Bernoulli numbers $B_k$, 

$$\psi(t) \equiv \frac{t}{1 - e^{-t}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} B_k t^k$$  \hfill (37)

(with convention $B_1 = -1/2$). Let $C$ be the $n \times n$ operator–valued matrix with elements $C_{\mu\nu} = \sum_{\alpha=1}^{n} C_{\mu\alpha\nu} \partial_{\alpha}$ where $C_{\mu\nu}$ are the structure constants of the Lie algebra \cite{27}. Let $e^C$ denote the formal matrix exponential $e^C = \sum_{k=0}^{\infty} C_k / k!$.

**Theorem 3.** The algebra $\mathcal{H}$ in definition \[4\] admits the following realization:

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \psi_{\mu\alpha}(C), \quad \hat{T}_{\mu\nu} = (e^C)_{\mu\nu}, \quad \hat{T}^{-1}_{\mu\nu} = (e^{-C})_{\mu\nu},$$ \hfill (38)

where $\psi_{\mu\nu}(C)$ denotes the $(\mu, \nu)$ element of the matrix $\psi(C)$.

**Proof.** Define the realization of $T_{\mu\nu}$ by $\hat{T}_{\mu\nu} = (e^C)_{\mu\nu}$. Then clearly $[\hat{T}_{\mu\nu}, \hat{T}_{\alpha\beta}] = 0$. Next we seek a realization of $X_\mu \in U(g_h) \subset \mathcal{H}$ defined by $\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \varphi_{\alpha\mu}(\partial)$ such that $\hat{T}_{\mu\nu}$ and $\hat{x}_\lambda$ close relations \[39\]. It is shown in lemma \[3\] (see Appendix) that the formal derivative of $\hat{T}_{\mu\nu}$ is given by

$$\frac{\partial}{\partial \lambda} \hat{T}_{\mu\nu} = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left( \frac{1 - e^{-C}}{C} \right)_{\lambda\alpha} \hat{T}_{\beta\nu},$$ \hfill (39)

hence

$$[\hat{T}_{\mu\nu}, \hat{x}_\lambda] = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left[ \sum_{\kappa=1}^{n} \left( \frac{1 - e^{-C}}{C} \right)_{\kappa\alpha} \varphi_{\kappa\lambda}(\partial) \right] \hat{T}_{\beta\nu}. \quad (40)$$

If we choose $\varphi_{\kappa\lambda}(\partial) = \psi(C)_{\lambda\kappa}$, then

$$[\hat{T}_{\mu\nu}, \hat{x}_\lambda] = \sum_{\beta=1}^{n} C_{\mu\lambda\beta} \hat{T}_{\beta\nu}. \quad (41)$$

For this choice of the realization $\varphi_{\mu\nu}(\partial)$, the generators $\hat{x}_\mu$ are given by the power series

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \psi_{\mu\alpha}(C) = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \frac{(-1)^k}{k!} B_k x_\alpha (C^k)_{\mu\alpha}. \quad (42)$$

Next we show that $[\hat{x}_\mu, \hat{x}_\nu] = \sum_{\alpha=1}^{n} C_{\mu\alpha\nu} \hat{x}_\alpha$. Since the algebra $\hat{A}_n$ is associative, the operators $\hat{x}_\mu, \hat{T}_{\alpha\beta} \in \hat{A}_n$ satisfy the Jacobi identity $[[\hat{x}_\mu, \hat{x}_\nu], \hat{T}_{\alpha\beta}] + [[\hat{T}_{\alpha\beta}, \hat{x}_\nu], \hat{x}_\mu] + [[\hat{T}_{\alpha\beta}, \hat{x}_\mu], \hat{x}_\nu] = 0$. 

$$[[\hat{x}_\mu, \hat{x}_\nu], \hat{T}_{\alpha\beta}] + [[\hat{T}_{\alpha\beta}, \hat{x}_\nu], \hat{x}_\mu] + [[\hat{T}_{\alpha\beta}, \hat{x}_\mu], \hat{x}_\nu] = 0. \quad (43)$$
Substituting (41) into this identity and using Eq. (2) we find
\[
[[\hat{x}_\mu, \hat{x}_\nu], \hat{T}_{\alpha\beta}] = \sum_{\lambda, \rho=1}^n C_{\mu\nu\rho} C_{\rho\alpha\lambda} \hat{T}_{\lambda\beta}.
\] (43)

The matrix \( \psi(C) \) is invertible, hence \( x_\mu = \sum_{\alpha=1}^n \hat{x}_\alpha (\psi(C)^{-1})_{\mu\alpha} \). This implies that \( [\hat{x}_\mu, \hat{x}_\nu] = \sum_{\rho=1}^n \hat{x}_\rho \theta_{\mu\rho}(\partial) \) for some formal power series \( \theta_{\mu\rho}(\partial) \in \hat{A}_n \). Consequently, Eq. (41) yields
\[
[[\hat{x}_\mu, \hat{x}_\nu], \hat{T}_{\alpha\beta}] = \sum_{\rho=1}^n [\hat{x}_\rho, \hat{T}_{\alpha\beta}] \theta_{\mu\rho}(\partial) = \sum_{\lambda, \rho=1}^n \theta_{\mu\rho}(\partial) C_{\rho\alpha\lambda} \hat{T}_{\lambda\beta}.
\] (44)

Comparing equations (43) and (44), and taking into account that the matrix \( \hat{T} = e^C \) is regular, we find \( \sum_{\rho=1}^n \theta_{\mu\rho}(\partial) C_{\rho\alpha\lambda} = \sum_{\rho=1}^n C_{\mu\rho\alpha} C_{\rho\alpha\lambda} \). Thus, the power series \( \theta_{\mu\rho}(\partial) \) has only the zero–order term \( \theta^0_{\mu\rho} \in \mathbb{K} \). We claim that \( \theta^0_{\mu\rho} = C_{\mu\rho\alpha} \). Note that the action \( \hat{T}_{\mu\nu} = \theta^0_{\mu\nu} \) yields \( [\hat{x}_\mu, \hat{x}_\nu] \triangleright 1 = \sum_{\rho=1}^n \theta^0_{\mu\rho} x_\rho \). A short computation using expansion (42) shows that
\[
\hat{x}_\mu \triangleright x_\nu = x_\mu x_\nu + \frac{1}{2} \sum_{\rho=1}^n C_{\mu\rho\alpha} x_\rho.
\] (45)

Therefore, \( [\hat{x}_\mu, \hat{x}_\nu] \triangleright 1 = \hat{x}_\mu \triangleright x_\nu - \hat{x}_\nu \triangleright x_\mu = \sum_{\rho=1}^n C_{\mu\rho\alpha} x_\rho \) which proves that \( \theta^0_{\mu\rho} = C_{\mu\rho\alpha} \). Thus the generators \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \) close the Lie algebra \( \mathfrak{g}_h \).

In a similar fashion one can show that the algebra \( \mathbb{U}(\mathfrak{g}_h^R) \subset \mathcal{H} \) admits the realization given by Eq. (42) and \( \hat{T}^{-1}_{\mu\nu} = (e^{-C})_{\mu\nu} \). Obviously, \( \sum_{\alpha=1}^n \hat{T}_{\mu\alpha} \hat{T}^{-1}_{\alpha\nu} = \sum_{\alpha=1}^n \hat{T}^{-1}_{\mu\alpha} \hat{T}_{\alpha\nu} = \delta_{\mu\nu} \), which proves that (38) defines a realization of \( \mathcal{H} \).

We note that the realizations of the generators of \( \mathcal{H} \) are defined in terms of the structure constants \( C_{\mu\nu\lambda} \) which describe the adjoint representation of the Lie algebra \( \mathfrak{g}_h \). The realization (42) was found in Ref. [17] where several different proofs were given using direct computation, formal geometry and a coalgebra structure. The proof given here is based on the realization of the extended algebra \( \mathcal{H} \) which is then utilized to construct the left–right dual realizations of the algebra \( \mathfrak{g}_h \). We remark that the star–product associated to realization (42) appears implicitly in Ref. [24] as the Gutt star–product.

**Theorem 4.** The realization of the Lie algebra \( \mathfrak{g}_h \) given by (42) satisfy the symmetrization property (36).

**Proof.** We prove relation (36) by induction on \( m \). The claim is obvious for \( m = 1 \) since \( \hat{x}_\mu \triangleright 1 = x_\mu \). Assume that Eq. (36) holds for some \( m > 1 \). Then by the induction
assumption
\[
\left( \sum_{\mu=1}^{n} k_{\mu} \hat{x}_{\mu} \right)^{m+1} \triangleright 1 = \left( \sum_{\mu=1}^{n} k_{\mu} \hat{x}_{\mu} \right) \triangleright \left( \sum_{\mu=1}^{n} k_{\mu} x_{\mu} \right)^{m}.
\]
(46)

Define homogeneous polynomials \( P_{m}(x) = \left( \sum_{\mu=1}^{n} k_{\mu} x_{\mu} \right)^{m} \). We write the realization of \( \hat{x}_{\mu} \) as \( \hat{x}_{\mu} = x_{\mu} + \sum_{k=1}^{\infty} \sum_{\alpha=1}^{n} a_{k} x_{\alpha} (C^{k})_{\mu \alpha} \) where \( a_{k} = (-1)^{k} \frac{B_{k}}{k!} \). Substituting this into Eq. (46) we find
\[
\left( \sum_{\mu=1}^{n} k_{\mu} \hat{x}_{\mu} \right)^{m+1} \triangleright 1 = P_{m+1}(x) + \sum_{k=1}^{\infty} \sum_{\alpha=1}^{n} a_{k} x_{\alpha} \left( \sum_{\mu=1}^{n} k_{\mu} (C^{k})_{\mu \alpha} \triangleright P_{m}(x) \right). \]
(47)

Define coefficients \( K_{\mu \alpha} = \sum_{\rho=1}^{n} C_{\mu \rho \alpha} k_{\rho} \). Then the action of the operator \( (C^{k})_{\mu \alpha} \) on \( P_{m}(x) \) is found to be
\[
(C^{k})_{\mu \alpha} \triangleright P_{m}(x) = \frac{m!}{(m-k)!} P_{m-k}(x) K_{\mu \alpha}^{k}, \quad k \geq 1,
\]
where
\[
K_{\mu \alpha}^{1} = K_{\mu \alpha}, \quad K_{\mu \alpha}^{k} = \sum_{\beta_{1}, \beta_{2}, \ldots, \beta_{k-1}=1}^{n} K_{\mu \beta_{k-1} \beta_{k} \ldots \beta_{k-2}} K_{\beta_{k-1} \beta_{k-2} \ldots \beta_{1} \alpha}, \quad k \geq 2.
\]
(49)

Note that \( \sum_{\mu=1}^{n} k_{\mu} K_{\mu \alpha} = 0 \) since \( C_{\mu \nu \lambda} = -C_{\nu \mu \lambda} \). This yields
\[
\sum_{\mu=1}^{n} k_{\mu} K_{\mu \alpha}^{k} = 0 \quad \text{for all} \quad k \geq 1.
\]
(50)

Now, substituting Eq. (48) into (47) and using Eq. (50) we find
\[
\left( \sum_{\mu=1}^{n} k_{\mu} \hat{x}_{\mu} \right)^{m+1} \triangleright 1 = P_{m+1}(x) = \left( \sum_{\mu=1}^{n} k_{\mu} x_{\mu} \right)^{m+1}.
\]
(51)

This completes the proof. ■

4.1. Left–right duality in the Weyl–symmetric realization. Recall that in the associative algebra \( H \) the elements \( Y_{\mu} = \sum_{\alpha=1}^{n} X_{\alpha} T_{\mu \alpha}^{-1} \) act as right multiplication operators by \( X_{\mu}, Y_{\mu} \triangleright X = XX_{\mu} \) for \( X \in U(g) \). In view of theorem 3, the realization of \( Y_{\mu} \) is given by
\[
\hat{y}_{\mu} = \sum_{\alpha=1}^{n} \hat{x}_{\alpha} T_{\mu \alpha}^{-1} = \sum_{\alpha=1}^{n} x_{\alpha} \left( e^{-C \psi(C)} \right)_{\mu \alpha}.
\]
(52)

Interestingly,
\[
\tilde{\psi}(t) \equiv e^{-t} \psi(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \tilde{B}_{k} t^{k}
\]
(53)
is also a generating function for the Bernoulli numbers \( \tilde{B}_{k} \) with convention \( \tilde{B}_{1} = 1/2 \). The following result shows that \( \hat{x}_{\mu} = \sum_{\alpha=1}^{n} x_{\alpha} \psi_{\mu \alpha}(C) \) and \( \hat{y}_{\mu} = \sum_{\alpha=1}^{n} x_{\alpha} \tilde{\psi}_{\mu \alpha}(C) \) actually define dual realizations of the Lie algebra \( g_{h} \).
Theorem 5. Let \( \mathfrak{g}_h \) and \( \tilde{\mathfrak{g}}_h \) denote the left–right dual Lie algebras \((27)\) and \((33)\), respectively. Define linear maps \( \psi : \mathfrak{g}_h \to \mathfrak{A}_n \) and \( \tilde{\psi} : \tilde{\mathfrak{g}}_h \to \tilde{\mathfrak{A}}_n \) by \( X_\mu \mapsto \hat{x}_\mu = \sum_{\alpha=1}^n x_\alpha Y_{\mu\alpha}(C) \) and \( Y_\mu \mapsto \hat{y}_\mu = \sum_{\alpha=1}^n x_\alpha \tilde{\psi}_{\mu\alpha}(C) \) where \( \psi(t) = t/(1 - e^{-t}) \) and \( \tilde{\psi}(t) = t/(e^t - 1) \). Then the associated star–products
\[
f \star g = \Omega_{\psi}(\Omega_{\psi}^{-1}(f) \Omega_{\psi}^{-1}(g)) \quad \text{and} \quad f \star g = \Omega_{\tilde{\psi}}(\Omega_{\tilde{\psi}}^{-1}(f) \Omega_{\tilde{\psi}}^{-1}(g))
\] (54)
are left–right dual, i.e. \( f \star g = g \star f \) for all \( f, g \in \mathcal{X} \).

Proof.

It was already shown in theorem 3 that \( \psi \) is a realization of \( \mathfrak{g}_h \). In order to show that \( \hat{\psi} \) is a realization of \( \tilde{\mathfrak{g}}_h \), i.e. \( [\hat{y}_\mu, \hat{y}_\nu] = -\sum_{\alpha=1}^n C_{\mu\alpha\nu} \hat{y}_\alpha \), we make use of the relations in the algebra \( \mathcal{H} \). Since \( \hat{y}_\mu \) is of the form \( \hat{y}_\mu = \sum_{\alpha=1}^n \hat{x}_\alpha \tilde{T}^{-1}_{\mu\alpha} \), we have \( [\hat{y}_\mu, \hat{y}_\nu] = 0 \) in view of Eq. (40). This implies that \( [\hat{y}_\mu, \hat{y}_\nu] = \sum_{\alpha=1}^n \hat{x}_\alpha \tilde{T}^{-1}_{\mu\alpha} \). Furthermore, it follows from Eq. (19) that \( [\tilde{T}^{-1}_{\mu\alpha}, \hat{y}_\nu] = \sum_{\alpha=1}^n \hat{x}_\alpha \tilde{T}^{-1}_{\mu\alpha} \tilde{T}^{-1}_{\nu\beta} \), hence \( [\hat{y}_\mu, \hat{y}_\nu] = \sum_{\alpha, \beta, \rho=1}^n C_{\beta\rho\alpha} \hat{x}_\alpha \tilde{T}^{-1}_{\mu\rho} \tilde{T}^{-1}_{\nu\beta} \).

Expressing \( \hat{x}_\alpha \) as \( \hat{x}_\alpha = \sum_{\kappa=1}^n \hat{y}_\kappa \tilde{T}_{\alpha\kappa} \), the last relation can be written as
\[
[\hat{y}_\mu, \hat{y}_\nu] = \sum_{\kappa=1}^n \hat{y}_\kappa \left( \sum_{\alpha, \beta, \rho=1}^n C_{\beta\rho\alpha} \tilde{T}_{\alpha\kappa} \tilde{T}^{-1}_{\mu\rho} \tilde{T}^{-1}_{\nu\beta} \right).
\] (55)
According to proposition 3 (see Appendix) that the operators \( \tilde{T}_{\mu\nu} = (eC)_{\mu\nu} \) satisfy the identity
\[
\sum_{\alpha, \beta, \rho=1}^n C_{\beta\rho\alpha} \tilde{T}_{\alpha\kappa} \tilde{T}^{-1}_{\mu\rho} \tilde{T}^{-1}_{\nu\beta} = -C_{\mu\nu\kappa},
\] (56)
hence Eq. (55) yields \( [\hat{y}_\mu, \hat{y}_\nu] = -\sum_{\kappa=1}^n C_{\mu\nu\kappa} \hat{y}_\kappa \). Now, since \( [\hat{x}_\mu, \hat{y}_\nu] = 0 \), lemma 2 implies that the induced star–products (54) are left–right dual. ■

It is important to note that only in the case of the Weyl symmetric ordering, the dual realization is obtained by the simple transformation \( \tilde{\psi}_{\alpha\beta}(C) = \psi_{\alpha\beta}(-C) \). In arbitrary orderings there is no simple relation between a realization \( \varphi_{\mu\alpha}(\partial) \) and its dual \( \tilde{\varphi}_{\mu\alpha}(\partial) \).

5. DUAL REALIZATIONS OF THE \( \kappa \)–DEFORMED SPACE

The \( \kappa \)–deformed space is the enveloping algebra of the Lie algebra
\[
[X_\mu, X_\nu] = i(a_\mu X_\nu - a_\nu X_\mu), \quad 1 \leq \mu, \nu \leq n.
\] (57)
This algebra appears in the mathematical framework of deformed (doubly) special relativity theories \((25, 26)\) and it has applications in quantum gravity \((27)\) and quantum field theory \((28, 29)\). The deformation parameter \( \kappa = 1/|a| \) is usually associated with the
Planck mass or quantum gravity scale. The Lie algebra (57) represents deformations of the Euclidean or Minkowski space, depending on the metric imposed on the underlying commutative space (obtained in the classical limit $\kappa \to \infty$). Realizations of the Lie algebra (57) in different orderings have been investigated in Refs. [19, 20, 22]. Recently, realizations of Lie superalgebras have been used to construct graded differential algebras on the $\kappa$–Minkowski space in Refs. [30, 31, 32, 33, 34]. Left–right dual realizations of the $\kappa$–Minkowski space were found in Ref. [35]. Here we restrict our attention to the $\kappa$–Euclidean space, although the analysis is easily extended to the $\kappa$–Minkowski space.

For future reference, let $g_\kappa$ denote the Lie algebra (57). The structure constants of $g_\kappa$ are given by

$$C_{\mu\nu\lambda} = i(a_\mu\delta_{\nu\lambda} - a_\nu\delta_{\mu\lambda}).$$

Let us define the row vectors $a = (a_1, a_2, \ldots, a_n)$ and $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$. Then the operator–valued matrix $C_{\mu\nu} = \sum_{\alpha=1}^{n} C_{\mu\alpha\nu} \partial_\alpha$ can be written as $C = ia \otimes \partial - (ia \cdot \partial) I$ where $I$ is the $n \times n$ identity matrix and $a \cdot \partial = \sum_{\alpha=1}^{n} a_\alpha \partial_\alpha$.

The powers of $C$ are given by

$$C^k = (-1)^{k-1}A^{k-1}(ia \otimes \partial) + (-1)^{k}A^{k}I, \quad k \geq 1,$$

where $A$ denotes the differential operator $A = ia \cdot \partial$. Using the above expression the matrix $\psi(C)$, where $\psi(t)$ is the generating function (37), can be found in closed form:

$$\psi(C) = \frac{A}{e^A - 1} I - \frac{1}{A}(\frac{A}{e^A - 1} - 1)(ia \otimes \partial).$$

Hence, the Weyl symmetric realization of the $\kappa$–Euclidean space is found to be

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \psi_{\mu\alpha}(C) = x_\mu \frac{A}{e^A - 1} + ia_\mu(x \cdot \partial)(\frac{1}{A} - \frac{1}{e^A - 1}).$$

This realization appears as a special case of an infinite family of covariant realizations found in Ref. [20] (see also Ref. [35]). Note that in the classical limit we find $\lim_{\kappa \to \infty} \hat{x}_\mu = x_\mu$, as required. Similarly, one finds

$$\hat{T}_{\mu\nu} = (e^{-A})_{\mu\nu} = e^{-A}\delta_{\mu\nu} - ia_\mu \partial_\nu \frac{e^{-A} - 1}{A}, \quad \hat{T}^{-1}_{\mu\nu} = (e^{-A})_{\mu\nu} = e^{A}\delta_{\mu\nu} - ia_\mu \partial_\nu \frac{e^A - 1}{A}$$

which provides a realization of the associative algebra $H$ containing the $\kappa$–deformed space (57). According to theorem 5 the dual realization in the symmetric ordering is simply given by

$$\hat{y}_\mu = \sum_{\alpha=1}^{n} x_\alpha \tilde{\psi}_{\mu\alpha}(C) = x_\mu \frac{A}{1 - e^{-A}} + ia_\mu(x \cdot \partial)(\frac{1}{A} - \frac{1}{1 - e^{-A}}).$$
It is shown in Ref. [20] that the star–product associated with the Weyl symmetric realization (60) can be written in terms of bi–differential operators as

\[ f \ast g = \exp \left( \sum_{\alpha=1}^{n} x_\alpha (\Delta \partial_\alpha - \Delta_0 \partial_\alpha) \right) (f, g) \] (63)

where

\[ \Delta_0 \partial_\alpha = \overset{\leftarrow}{\partial}_\alpha + \overset{\rightarrow}{\partial}_\alpha \quad \text{and} \quad \Delta \partial_\alpha = \overset{\leftarrow}{\partial}_\alpha \frac{\psi(\overset{\leftarrow}{A} + \overset{\rightarrow}{A})}{\psi(\overset{\leftarrow}{A})} + \overset{\rightarrow}{\partial}_\alpha \frac{\psi(\overset{\leftarrow}{A} + \overset{\rightarrow}{A})}{\psi(\overset{\rightarrow}{A})}. \] (64)

Here, the operators \( \overset{\leftarrow}{\partial}_\alpha \) and \( \overset{\rightarrow}{\partial}_\alpha \) act on \( f \) and \( g \), respectively, while \( \overset{\leftarrow}{A} \) and \( \overset{\rightarrow}{A} \) are defined by \( \overset{\leftarrow}{A} = i \sum_{\alpha=1}^{n} a_\alpha \overset{\leftarrow}{\partial}_\alpha \) and \( \overset{\rightarrow}{A} = i \sum_{\alpha=1}^{n} a_\alpha \overset{\rightarrow}{\partial}_\alpha \). As usual, the identity operator stands for pointwise multiplication. The star–product associated with the dual realization (62) is found to be given by interchanging \( \psi \) and \( \tilde{\psi} \) in Eq. (64). We then have \( f \ast g = g \ast f \), in agreement with theorem [5]. Expanding the star–product to first order in the deformation parameter \( 1/\kappa \) we obtain

\[ f \ast g = fg + i \frac{1}{2\kappa} \sum_{\alpha,\beta=1}^{n} (a^0_\alpha x_\beta - a^0_\beta x_\alpha)(\partial_\alpha f)(\partial_\beta g) + O\left(\frac{1}{\kappa^2}\right) \] (65)

where \( a^0 = \kappa a \in \mathbb{R}^n \) is a unit vector. Note that the last equation can be written as \( f \ast g = fg + (i/2\kappa)\{f, g\} + O(1/\kappa) \) where

\[ \{f, g\} = \sum_{\alpha,\beta=1}^{n} (a^0_\alpha x_\beta - a^0_\beta x_\alpha)(\partial_\alpha f)(\partial_\beta g) \] (66)

is the Lie–Poisson bracket on the dual of the Lie algebra \( g^0_\kappa \) having the structure constants \( C^0_{\mu\nu\lambda} = a^0_\mu \delta_{\nu\lambda} - a^0_\nu \delta_{\mu\lambda} \). Thus, the star–product (63) corresponding to the Weyl symmetric realization of the Lie algebra (57) is a deformation quantization of the Poisson manifold \( (g^0_\kappa)^*, \{ , \} \). More information on quantizing the dual of a Lie algebra is found in Refs. [36, 24].

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Appendix A. Identities for the Weyl symmetric realization

In this appendix we prove some identities used in the proofs of statements in section 4.
Proposition 1. The matrix elements $C_{\mu\nu} = \sum_{\alpha=1}^{n} C_{\mu\alpha\nu} \partial_{\alpha}$ satisfy the following identities

$$\sum_{\alpha=1}^{n} (C^{m})_{\mu\alpha} C_{\alpha\nu} = \sum_{\alpha,\beta=1}^{n} \left[ \sum_{k=0}^{m} \binom{m}{k} (-1)^{k} (C^{k})_{\lambda\alpha}(C^{m-k})_{\beta\nu} \right] C_{\mu\alpha\beta}, \ m \geq 1. \quad (67)$$

Proposition 2.

$$\frac{\partial}{\partial \partial}(C^{m})_{\mu\nu} = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left[ \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} (C^{k-1})_{\lambda\alpha}(C^{m-k})_{\beta\nu} \right], \ m \geq 1. \quad (68)$$

The above propositions are easily proved by induction on $m$.

Lemma 3.

$$\frac{\partial}{\partial \lambda} (e^{C})_{\mu\nu} = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left( \frac{1-e^{-C}}{C} \right)_{\lambda\alpha} (e^{C})_{\beta\nu}. \quad (69)$$

Proof. Using proposition 2 we find

$$\frac{\partial}{\partial \lambda} (e^{C})_{\mu\nu} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial}{\partial \lambda} (e^{C})_{\mu\nu} = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left[ \sum_{m=0}^{\infty} \frac{1}{m!} \binom{m}{k} (-1)^{k-1} (C^{k-1})_{\lambda\alpha}(C^{m-k})_{\beta\nu} \right]. \quad (70)$$

The formal power series in Eq. (70) can be written in closed form using the Cauchy product

$$\sum_{m=0}^{\infty} \sum_{k=1}^{m} A_{k-1} B_{m-k} (C^{k-1})_{\lambda\alpha}(C^{m-k})_{\beta\nu} = \left( \sum_{m=0}^{\infty} A_{m} (C^{m})_{\lambda\alpha} \right) \left( \sum_{m=0}^{\infty} B_{m} (C^{m})_{\beta\nu} \right) \quad (71)$$

with $A_{k} = (-1)^{k}/(k+1)!$ and $B_{k} = 1/k!$. Then

$$\sum_{m=0}^{\infty} \sum_{k=1}^{m} \frac{1}{m!} \binom{m}{k} (-1)^{k-1} (C^{k-1})_{\lambda\alpha}(C^{m-k})_{\beta\nu} =$$

$$\left( \sum_{m=0}^{\infty} \frac{1}{(m+1)!} (C^{m})_{\lambda\alpha} \right) \left( \sum_{m=0}^{\infty} \frac{1}{m!} (C^{m})_{\beta\nu} \right) = \left( 1 - \frac{e^{-C}}{C} \right)_{\lambda\alpha} (e^{C})_{\beta\nu}, \quad (72)$$

hence

$$\frac{\partial}{\partial \partial \lambda} (e^{C})_{\mu\nu} = \sum_{\alpha,\beta=1}^{n} C_{\mu\alpha\beta} \left( \frac{1-e^{-C}}{C} \right)_{\lambda\alpha} (e^{C})_{\beta\nu}. \quad (73)$$

Proposition 3.

$$\sum_{\alpha,\beta,\rho=1}^{n} C_{\beta\rho\alpha} (e^{C})_{\alpha\kappa}(e^{C})_{\mu\rho}(e^{C})_{\nu\beta} = -C_{\mu\kappa}. \quad (74)$$
Proof. In view of proposition 1 we have
\[
\sum_{\alpha=1}^{n} C_{\alpha\mu\kappa} (e^C)^{\beta\alpha} = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{\alpha=1}^{n} (C^m)^{\mu\alpha} C_{\alpha\mu\kappa} \right)
\]
\[
= \sum_{\alpha, \rho=1}^{n} \left[ \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (C^k)^{\mu\rho} (C^{m-k})^{\alpha\kappa} \right] C_{\beta\rho\alpha}.
\]
(75)
It is easily verified that the sum in the brackets is the Cauchy product
\[
(e^{-C})_{\mu\rho} (e^C)^{\alpha\kappa} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (C^k)^{\mu\rho} (C^{m-k})^{\alpha\kappa},
\]
(76)
hence
\[
\sum_{\alpha=1}^{n} C_{\alpha\mu\kappa} (e^C)^{\beta\alpha} = \sum_{\alpha, \rho=1}^{n} C_{\beta\rho\alpha} (e^{-C})_{\mu\rho} (e^C)^{\alpha\kappa}.
\]
(77)
Multiplying Eq. (77) by \((e^{-C})_{\nu\beta}\) and summing over \(\beta\) we obtain Eq. (74). ■

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