ESSENTIAL VARIABLES AND SEPARABLE SETS IN UNIVERSAL ALGEBRA

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Abstract. The study of essential and strongly essential variables in functions defined on finite sets is a part of $k$-valued logic. We extend the main definitions from functions to terms. This allows us to apply concepts and results of Universal Algebra. On the basis of the concept of a separable set of variables in a term we introduce a new notion of complexity of terms, algebras and varieties and give examples.

1. Introduction

Several authors considered essential variables of functions under different aspects (see e.g. [1] [2] [3] [5]). The unary function $f : A \rightarrow A$ depends essentially on its input $x$ if it takes on at least two values, i.e. if $f$ is not constant. The $n$-ary function $f : A^n \rightarrow A$ depends essentially on its $i$-the input $x_i$ if there are elements $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ such that the unary function defined by

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots a_n)$$

is not constant on $A$.

In this case it is very common to say, the variable $x_i$ is essential for $f(x_1, \ldots, x_i, \ldots, x_n)$ and to consider the set $Ess(f)$ of all variables which are essential for $f(x_1, \ldots, x_n)$. As usual, instead of the function $f$ one writes the term $f(x_1, \ldots, x_n)$. So, it is very natural to define essential variables for terms. One can do this by using the former definition and going from terms to term operations which are induced by terms on a given algebra $U$. This is a new point of view and allows us to apply the methods of Universal Algebra to the study of essential variables.

We will give the necessary definitions and prove some consequences. Moreover, on the basis of essential variables we define a new concept of complexity for terms, for polynomials, and for algebras. This concept

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depends not only on the syntax of the term but also on its meaning in a given model.

2. BASIC CONCEPTS

We will use the denotation $O^n_A$ for the set of all $n$-ary functions defined on the set $A$, i.e., $O^n_A := \{ f \mid f : A^n \to A \}$ and $O_A := \bigcup_{n=1}^{\infty} O^n_A$. An algebra of type $\tau$ is a pair $\mathcal{U} = (A; (f^i_{i \in I}))$ where $(f^i_{i \in I})$ is an indexed subset of $O_A$ and $f^i_{i \in I}$ is $n_i$-ary, $n_i \geq 1$. By $Alg(\tau)$ we denote the class of all algebras of type $\tau$.

Terms of type $\tau$ are defined in the following inductive way. Let $(f^i_{i \in I})$ be an indexed set of operation symbols where $f^i_{i \in I}$ is $n_i$-ary and let $X_n = \{ x_1, \ldots, x_n \}$ be an $n$-element set of variables. Then we define $n$-ary terms of type $\tau$ as follows

(i) $x^i \in X_n$ is an $n$-ary term for all $i \in \{1, \ldots, n\}$,
(ii) if $t_1, \ldots, t_n$ are $n$-ary terms and if $f^i_{i \in I}$ is $n_i$-ary then $f^i_{i \in I}(t_1, \ldots, t_n)$ is an $n$-ary term.

By $W_\tau(X_n)$ we denote the set of all $n$-ary terms of type $\tau$. If $X = \{ x_1, \ldots, x_n, \ldots \}$ is a countably infinite alphabet then $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type $\tau$. For $t \in W_\tau(X)$ by $\text{var}(t)$ we denote the set of all variables which occur in $t$.

Polynomials of type $\tau$ are defined in a similar way. Besides the set $X$ we need a set $\mathcal{A}$ of constant symbols with $\mathcal{A} \cap X = \emptyset$. Polynomials of type $\tau$ over $\mathcal{A}$ are defined in three steps:

(i) variables from $X$ are polynomials of type $\tau$ over $\mathcal{A}$,
(ii) elements $\overline{a}$ from $\mathcal{A}$ are polynomials of type $\tau$ over $\mathcal{A}$,
(iii) if $p_1, \ldots, p_n$ are polynomials of type $\tau$ over $\mathcal{A}$ and if $f^i_{i \in I}$ is an $n_i$-ary operation symbol then $f^i_{i \in I}(p_1, \ldots, p_n)$ is a polynomial of type $\tau$ over $\mathcal{A}$.

Let $P_\tau(X, \mathcal{A})$ be the set of all polynomials of type $\tau$ over $\mathcal{A}$. Clearly, $W_\tau(X) \subset P_\tau(X, \mathcal{A})$. Let $\mathcal{B}$ be an algebra of type $\tau$ containing a subalgebra $\mathcal{U}$ whose universe has the same cardinality as $\mathcal{A}$. Then every $n$-ary polynomial $p$ of type $\tau$ over $\mathcal{A}$ induces on the algebra $\mathcal{B}$ an $n$-ary polynomial operation $p^B$ defined by the following steps:

(i) if $x^i \in X_n$, then $x^i_B := e^n_B$, $1 \leq i \leq n$, where $e^n_B : (b_1, \ldots, b_n) \mapsto b_i$ is the $n$-ary projection onto the $i$-th coordinate,
(ii) if \( \overline{a} \in \mathcal{A} \) then \( \overline{a}^B := c^a_n \) is the \( n \)-ary constant operation on \( B \) with value \( a \in A \) and every element from \( A \subseteq B \) is uniquely induced by an element from \( \mathcal{A} \).

(iii) if \( p = f_i(p_1, \ldots, p_{n_i}) \) and if we assume that the polynomial operations \( p_1^B, \ldots, p_{n_i}^B \) are already defined, then \( p^B = f_i^B(p_1^B, \ldots, p_{n_i}^B) \) where the right hand side is the usual composition of operations defined by

\[
 f_i^B(p_1^B, \ldots, p_{n_i}^B)(b_1, \ldots, b_{n_i}) := f_i^B(p_1^B(b_1, \ldots, b_{n_i}), \ldots, p_{n_i}^B(b_1, \ldots, b_{n_i})).
\]

The set of all operations induced by arbitrary polynomials of type \( \tau \) over \( \mathcal{A} \) on the algebra \( B \) is denoted by \( P_{\mathcal{A}}(B) \). By \( T(B) \) we denote the set of all operations induced by arbitrary terms from \( W_\tau(X, \mathcal{A}) \subseteq P_{\tau}(X, \mathcal{A}) \). The elements from \( T(B) \) are called term operations of \( B \) and \( T(B) \) is called clone of term operations of \( B \). The elements of \( P_{\mathcal{A}}(B) \) are called polynomial operations of \( B \). Polynomial operations can also be defined by mappings \( h : X \cup \mathcal{A} \to B \) which are extensions of a distinguished mapping \( h' : \mathcal{A} \to A \) with \( h'(\overline{a}) = a \in A \subseteq B \) for \( \overline{a} \in \mathcal{A} \).

Any such mapping \( h \) is called evaluation of \( X \cup \mathcal{A} \) with \( B \). It is well-known that an evaluation mapping can be uniquely extended to a mapping \( \overline{h} : P_{\tau}(X, \mathcal{A}) \to B \).

Here for operation symbols \( f_i \) occurring in a polynomial \( p \) one has to substitute the corresponding operations \( f_i^B \). For terms the extension \( \overline{h} \) maps \( W_\tau(X) \) to \( B \).

If \( s, t \) are terms of type \( \tau \) then \( s \approx t \) is called identity satisfied in the algebra \( B \) of type \( \tau \) if the induced term operations are equal, i.e. if \( s^B = t^B \). In this case we write \( B \models s \approx t \). Polynomial identities in \( B \) are pairs of polynomials \( p \approx q \) such that the polynomial operations induced by \( p \) and \( q \) on \( B \) are equal. A variety \( V \) of type \( \tau \) is a class of algebras of type \( \tau \) such that there exists a set \( \Sigma \) of equations of type \( \tau \) with the property that \( V \) consists exactly of all algebras of type \( \tau \) such that every equation from \( \Sigma \) is satisfied as identity.

As usual, by \( \mathbb{H}, \mathbb{S}, \mathbb{P} \) we denote the operators of forming homomorphic images, subalgebras, and direct products of a given algebra or a given class of algebras. A class \( V \) of algebras of the same type \( \tau \) is a variety iff \( V \) is closed under the operators \( \mathbb{H}, \mathbb{S}, \) and \( \mathbb{P} \).

3. Essential variables and separable sets of variables in terms with respect to an algebra

Definition 3.1. Let \( t \in W_\tau(X_n) \) be an \( n \)-ary term of type \( \tau \) and let \( \mathcal{U} \) be an algebra of type \( \tau \). Then the variable \( x_i \), \( 1 \leq i \leq n \), is
called essential in $t$ with respect to the algebra $U$ if the term operation $t^U : A^n \rightarrow A$ induced by $t$ on the algebra $U$ depends essentially on its $i$-th input $x_i$. By $Ess(t,U)$ we denote the set of all variables which are essential in $t$ with respect to the algebra $U$. For a polynomial $p$ we define in the same way when $x_i$ is essential in $p$ with respect to $U$.

Remarks

(1) The definition means that $x_i$ is essential in the term $t$ with respect to $U$ if there are two different evaluation mappings $h, h' : \mathbb{X}_n \rightarrow A$ with $h/\mathbb{X}_n \setminus \{x_i\} = h'/\mathbb{X}_n \setminus \{x_i\}$ such that $\overrightarrow{h}(t) \neq \overrightarrow{h'}(t)$ where $\overrightarrow{h} : W_r(\mathbb{X}_n) \rightarrow A$ is the extension of $h$.

(2) If $U$ is isomorphic to $B$, then $x_i$ is essential in $t$ with respect to $U$ iff $x_i$ is essential in $t$ with respect to $B$. Indeed, if $\varphi : U \rightarrow B$ is an isomorphism and $x_i$ is essential in $t$ with respect to $U$ then there are elements $a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n$, $b_i \neq a_i$ such that $t^U(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq t^U(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)$.

But then

$$\varphi(t^U(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)) = t^B(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{i-1}), \varphi(a_i), \varphi(a_{i+1}), \ldots, \varphi(a_n)) \neq t^B(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{i-1}), \varphi(b_i), \ldots, \varphi(a_n))$$

with $\varphi(a_i) \neq \varphi(b_i)$. Therefore $x_i$ is essential in $t$ with respect to the algebra $B$.

Examples

(1) The variable $x_i$ is essential in the term $x_i$ with respect to the algebra $U$ of type $\tau$ iff $|A| > 1$.

(2) If $t^U$ is constant then no variable is essential in $t$ with respect to $U$.

(3) If the variable $x_i$ does not occur in the term $t$ then $x_i$ is not essential in $t$ with respect to any algebra $U$ of type $\tau$ since $Ess(t,U) \subseteq \text{var}(t)$.

(4) Let $SC$ be a two-element semilattice. Then $x_i$ is essential in a term $t$ with respect to $SC$ iff $x_i \in \text{var}(t)$ (indeed, terms have the form $t(x_1, \ldots, x_n) = x_{i_1} \cdots x_{i_n}$, $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, n\}$ if the binary operation symbol is written as $\cdot$).

(5) Let $BU$ be a two-element Boolean algebra with conjunction, disjunction, and negation ($\land_{BU}, \lor_{BU}, \neg_{BU}$) as fundamental operations. Then the variable $x_1$ is essential in a term $t$ with respect to $BU$ for the following terms: $x_1, x_1 \lor x_2, x_2 \land (x_3 \lor x_1), (x_1 \land
Variables which are essential in a term \( t \in W_{\tau}(X_n) \) with respect to an algebra \( U \in \text{Alg}(\tau) \) can also be characterized by evaluation mappings which we have already introduced in Section 2. We will use the following more general definition.

**Definition 3.2.** Let \( U = (A; (f_i^U)_{i \in I}) \) be an algebra of type \( \tau \) and let \( A \) be a set with \(|A| = |A|\) and \( A \cap X = \emptyset \). A mapping \( h : X_n \cup A \to P_\tau(X_n, A) \) is called evaluation of the set \( M = \{x_i, \ldots, x_m\} \subseteq X_n \) with the sequence \( C = (\overline{t}_{i_1}, \ldots, \overline{t}_{i_m}) \in A^m \) if

\[
h(x_j) = \begin{cases} x_j & \text{for } x_j \notin M \\ \overline{t}_{i_j} & \text{for } x_j \in M \end{cases} \quad \text{and } h(\overline{a}) = \overline{a} \quad \text{for every } \overline{a} \in A
\]

Clearly, every evaluation of the set \( M \) with \( C \in A^m \) is uniquely determined by \( M \) and \( C \) and can be extended to a uniquely determined mapping (endomorphism) \( \overline{h} : P_\tau(X_n, A) \to P_\tau(X_n, A) \). For the extension there holds \( \overline{h}(\overline{h}(p)) = \overline{h}(p) \) for every polynomial \( p \) over \( A \), i.e. \( \overline{h} \) is idempotent. Moreover we have \( \text{var}(\overline{h}(p)) = \text{var}(p) \setminus M \) (see e.g. [4, 7, 9]).

If \( M = X_n \), \( C \in A^n \) and if we substitute the constants \( \overline{a} \in A \) by its corresponding elements from \( A \) we obtain the evaluation mapping of \( X \) with \( A \) introduced in Section 2. Note that the result of an evaluation mapping defined by Definition 3.2 is a polynomial over \( A \).

Now we have

**Proposition 3.3.** A variable \( x_i \in X_n \) is essential in the term \( t \) (of type \( \tau \)) if and only if there exists an evaluation \( h \) of the set \( M = X_n \setminus \{x_i\} \) with some sequence of \( C \in A^{n-1} \) (\(|A| = A\)) such that the unary polynomial operation \( \overline{h}(t)^{\overline{h}} \) takes on at least two values, i.e. is not constant.

This can also be expressed in the following form: A variable \( x_i \in X_n \) is essential in the term \( t \) (of type \( \tau \)) with respect to the algebra \( U \) (of type \( \tau \)) iff there exists at least one evaluation \( h \) of \( X_n \setminus \{x_i\} \) with \( C \in A^{n-1} \) (\(|A| = |A|\)) such that \( x_i \in \text{Ess}(\overline{h}(t), U) \).

Another easy consequence of the definition is

**Proposition 3.4.** Let \( M \subseteq X_n \) be a nonempty subset of \( X_n \) and \( x_j \notin M \). If for every evaluation \( h \) of \( M \) with \( C \in A^{|M|} \), \(|A| = |A|\) there holds \( x_j \notin \text{Ess}(\overline{h}(t), U) \), where \( U \) is an algebra of type \( \tau \), then \( x_j \notin \text{Ess}(t, U) \).
where $h : X_n \to W_r(X_{n+1})$ is a mapping defined by $h(x_i) = x_{n+1}$ and $h(x_j) = x_j$ for all $j \neq i$, $j \in \{1, \ldots, n\}$ and where $\overline{h}$ is the extension of $h$, i.e., $\overline{h} : W_r(X_n) \to W_r(X_{n+1})$.

**Proof.** The variable $x_i$ does not occur in the term $\overline{h}(t)$. Therefore, $x_i$ is not essential for $\overline{h}(t)$ with respect to $U$. By Definition 3.1 $\overline{h}(t)^U$ does not depend essentially on its $i$-th input $x_i$.

If $U \models t \approx \overline{h}(t)$ then there exists an evaluation $h_1$, of $X_{n+1}$ with $C_1 \in \mathcal{A}^{n+1}$, $|A| = |A|$ such that $\overline{h}_1(t) \neq \overline{h}_1(\overline{h}(t))$. We can assume that $h_1(x_i) \neq h_1(x_{n+1})$ since $t$ does not depend on $x_{n+1}$ and $\overline{h}(t)$ does not depend on $x_i$. Now we choose a second evaluation $h_2$ of $X_{n+1}$ with $C_2 \in \mathcal{A}^{n+1}$ satisfying $h_2(x_i) = h_2(x_{n+1}) = h_1(x_{n+1}), h_2(x_j) = h_1(x_j)$ if $j \neq i, n+1$. Clearly, $\overline{h}_2(\overline{h}(t)) = \overline{h}_2(t)$ and $\overline{h}_2(\overline{h}(t)) = \overline{h}_1(\overline{h}(t))$, therefore $\overline{h}_1(t) \neq \overline{h}_2(t)$. This means, $x_i$ is essential in $t$ with respect to $U$. \qed

Denote by $IdU$ the set of all identities satisfied in $U$ and by $V(U)$ the variety generated by $U$. It is well-known that $IdU$ is a fully invariant congruence relation on the absolutely free algebra $\mathcal{W}_r(X_{n+1})$ if the identities in $IdU$ contain at most $n+1$ variables. The quotient algebra $\mathcal{W}_r(X_{n+1})/IdU$ is isomorphic to the $V(U)$-free algebra with $n+1$ generators, i.e., to $\mathcal{F}_{V(U)}(X_{n+1})$. Then $U \models t \approx \overline{h}(t)$ is equivalent to the equality $[t]_{IdU} = [\overline{h}(t)]_{IdU}$, i.e. the corresponding elements of $\mathcal{F}_{V(U)}(Y)$ where $Y$ is an arbitrary set of free generators with at least $n+1$ elements agree. But this means:

**Corollary 3.6.** The variable $x_i$ is essential in the $n$-ary term $t$ with respect to the algebra $U$ iff $x_i$ is essential in $t$ with respect to any $V(U)$-free algebra with at least $n+1$ free generators.

If the algebra $U$ is a homomorphic image or a subalgebra of $B$, then for the sets of identities satisfied in $U$ and in $B$, respectively, there holds $IdU \supseteq IdB$. If $U$ is a direct power of $B$ then $IdU = IdB$. Therefore $U \models t \approx \overline{h}(t)$ implies $B \models t \approx \overline{h}(t)$ for the mapping $h$ used in Lemma 3.5. It follows:
Corollary 3.7. If the variable $x_i, 1 \leq i \leq n$, is essential in the term $t \in W_\tau(X_n)$ with respect to the algebra $\mathcal{U}$ of type $\tau$ then $x_i$ is essential in $t$ with respect to any algebra $\mathcal{B}$ (of type $\tau$) with $\mathcal{U} \in \mathbb{H}(\mathcal{B})$, $\mathcal{U} \in \mathbb{S}(\mathcal{B})$, or $\mathcal{U} \in \mathbb{P}(\mathcal{B})$ where $\mathbb{P}$ is the operator of forming direct powers. Further $x_i$ is essential for $t$ with respect to any $\mathcal{B} \in \mathbb{P}(\mathcal{U})$. □

From Corollary 3.6 we obtain:

Corollary 3.8. Let $s,t \in W_\tau(X_n), \ n \geq 1$, and assume that $\mathcal{U} \in \text{Alg}(\tau)$. If $\mathcal{U} \models s \approx t$ then $\text{Ess}(t,\mathcal{U}) = \text{Ess}(s,\mathcal{U})$. □

Corollary 3.8 suggests the definition of a variable being essential in a term with respect to a variety.

Definition 3.9. Let $\mathcal{V}$ be a variety of type $\tau$ and let $t \in W_\tau(X_n)$. Then a variable $x_i \in X_n$ is called essential in $t$ with respect to the variety $\mathcal{V}$ if it is essential in $t$ with respect to the free algebra $\mathcal{F}_\mathcal{V}(X)$ with $X = \{x_1, \ldots, x_n, \ldots\}$ as set of free generators. The set of all variables in $t$ which are essential with respect to the variety $\mathcal{V}$ is denoted by $\text{Ess}(t,\mathcal{V})$.

Example The class $\text{Alg}(\tau)$ is the biggest variety of algebras of type $\tau$. The free algebra $W_\tau(X)$ with respect to $\text{Alg}(\tau)$ is the algebra of all terms of type $\tau$ where the operations $f_i^{W_\tau(X)}$ are defined by $(t_1, \ldots, t_n) \mapsto f_i^{W_\tau(X)}(t_1, \ldots, t_n) := f_i(t_1, \ldots, t_n)$ (see e.g. [8]).

A variable $x_i \in X_n$ is essential in the term $t$ with respect to the variety $\text{Alg}(\tau)$ iff $x_i \in \text{var}(t)$. Indeed, if $x_i \in \text{Ess}(t,\text{Alg}(\tau))$ then $x_i \in \text{var}(t)$. Conversely, if $x_i \in \text{var}(t)$ then $t \not\approx \mathcal{U}(t)$ for the mapping $h : X_n \rightarrow W_\tau(X_{n+1})$ defined by $h(x_i) = x_{n+1}$ and $h(x_j) = x_j$ for all $j \neq i$. Therefore, $\text{Alg}(\tau) \not\models t \approx \mathcal{U}(t)$ and $x_i \in \text{Ess}(t,\text{Alg}(\tau))$.

Proposition 3.10. If $x_i \in X_n$ is essential in the $n$-ary term $t$ of type $\tau$ with respect to the variety $\mathcal{V}$ of type $\tau$ and if $\mathcal{W} \supseteq \mathcal{V}$, i.e. if $\mathcal{V}$ is a subvariety of $\mathcal{W}$ then $x_i$ is essential in $t$ with respect to $\mathcal{W}$.

Proof. $\mathcal{W} \supseteq \mathcal{V}$ implies $\text{Id}\mathcal{W} \subseteq \text{Id}\mathcal{V}$. If $x_i \in \text{Ess}(t,\mathcal{V})$ then $\mathcal{V} \not\models t \approx \mathcal{U}(t)$ for the mapping defined in Lemma 3.5. But then $\mathcal{W} \not\models t \approx \mathcal{U}(t)$ and $x_i \in \text{Ess}(t,\mathcal{W})$. □

Example We consider the lattice of all varieties of semigroups. An equation $s \approx t$ is called regular if $\text{var}(s) = \text{var}(t)$. A variety is regular if it is the model class of a set of regular equations. It is well-known that a variety $\mathcal{V}$ of semigroups is regular iff it contains the variety $\text{SL}$ of semilattices and that the variety of semilattices is an atom in the
lattice of all varieties of semigroups ([6]). By Example 4 after Definition 3.1 we have \( x_i \in \text{Ess}(t, SL) \) iff \( x_i \in \text{var}(t) \). Now Proposition 3.10 shows that for an arbitrary regular variety \( V \) of semigroups we have \( x_i \in \text{Ess}(t, V) \) iff \( x_i \in \text{var}(t) \).

A term \( t \in W_\tau(X) \) is called a subterm of a term \( s \in W_\tau(X) \) with respect to the algebra \( U \) of type \( \tau \) and we write \( t \prec s \) if there exists an evaluation \( h \) of some subset \( M \subset \text{var}(s) \) with a sequence \( C \in A|M|, (|A| = |A|) \) such that \( U \models t \approx h(s) \). By \( \text{Sub}(t, U) \) we denote the set of all subterms of a term \( t \) of type \( \tau \) with respect to the algebra \( U \).

Clearly, if \( t \prec s \) then \( \text{Ess}(t, U) \subseteq \text{Ess}(s, U) \).

The concept of a separable set of variables in a function defined e.g. in [2] can also be extended to terms.

**Definition 3.11.** Let \( U \) be an algebra of type \( \tau \) and let \( A \) be a set of constant symbols (for elements of \( A \)) with \( |A| = |A| \). A set \( M \) of essential variables in the term \( t \in W_\tau(X) \) with respect to the algebra \( U \) is called separable in \( t \) with respect to \( U \) if there exists at least one evaluation \( h : X_n \setminus M \to U \) such that \( M = \text{Ess}(h(t), U) \).

By \( \text{Sep}(t, U) \) we denote the set of all separable sets in \( t \) with respect to \( U \).

**Example** Consider the term \( t_1 = x_1x_2 + x_3 \) with respect to the two-element Boolean ring \( \mathcal{BR} = (\{0, 1\}; +, \cdot) \). Then \( M = \{x_1, x_2\} \) is separable in \( t \) with respect to \( \mathcal{BR} \) since \( h : \{x_3\} \to \mathcal{BR} \) gives \( h(t_1) = x_1x_2 \) and \( \text{Ess}(h(t_1), \mathcal{BR}) = \{x_1, x_2\} \).

For separable sets of variables with respect to an algebra \( U \) we obtain results which are similar to some results on essential variables. We add some facts about separable sets. The proofs are straightforward and left to the reader.

**Theorem 3.12.** Let \( s, t \) be terms of type \( \tau \) and let \( U \) be an algebra of type \( \tau \).

(i) If \( s \) and \( t \) are two terms with \( U \models s \approx t \) then \( \text{Sep}(s, U) = \text{Sep}(t, U) \).

(ii) If \( s \prec t \) then \( \text{Sep}(s, U) \subseteq \text{Sep}(t, U) \)

(iii) A set \( M = \{x_{i_1}, \ldots, x_{i_m}\} \subseteq X_n \) of essential variables in the term \( t \) with respect to the algebra \( U \) is separable in \( t \) with respect to \( U \) iff \( M \) is separable in \( t \) with respect to any free algebra in \( V(U) \) with at least \( n + 1 \) free generators. \( \square \)

The last proposition motivates the following definition:
**Definition 3.13.** Let $t$ be an $n$-ary term of type $\tau$ and let $V$ be a variety of type $\tau$. A set $M \subseteq X_n$ is called separable in $t$ with respect to $V$ if $M$ is separable in $t$ with respect to the free algebra $F_V(X_n)$. Let $\text{Sep}(t, V)$ be the set of all separable sets with respect to $V$.

4. **Complexity of terms, polynomials, and algebras**

Terms are useful tools in Theoretical Computer Science. They can be used as models for different structures in logic programming. Every term can be described by a graph with operation symbols or variables as vertices and with subterms as edges. Such graphs are also called *semantic trees* of the corresponding terms.

**Example:** Figure 1 shows the semantic trees of the terms $t_1 = x_1x_2 + x_3$ and $t_2 = x_1x_3 + x_2x_3$ of a language containing two binary operation symbols $\cdot$ and $+$ and a unary symbol $\bar{}$.

This example shows that the graph of $t_2$ is more complex than the graph of $t_1$ since the number of edges and vertices in the graph of $t_2$ is greater than in $t_1$. There are different kinds of concepts of *complexity* of terms. Roughly spoken, there are two classes of term complexity, i.e. of functions from $W_\tau(X) \to \mathbb{N}$. The first one is based on a linguistic point of view and counts the number of variables or of operation symbols occurring in the term. Let us give two examples. If $\ell_i$ denotes the number of occurrences of the variable $x_i$ in the $n$-ary term $t$ then the first complexity measure is defined as

$$Cp^{(1)}(t) = \sum_{i=1}^{n} \ell_i.$$
The second one counts the operation symbols occurring in the term and is defined inductively by

$$C_{p}^{(2)}(x_{i}) := 0, \quad C_{p}^{(2)}(f_{i}(t_{1}, \ldots, t_{n_{i}})) := \sum_{j=1}^{n_{i}} C_{p}^{(2)}(t_{j}) + 1.$$  

For our terms $t_{1}, t_{2}$ we obtain $C_{p}^{(1)}(t_{1}) = 3, \quad C_{p}^{(1)}(t_{2}) = 4$ and $C_{p}^{(2)}(t_{1}) = 2, \quad C_{p}^{(2)}(t_{2}) = 4$.

The second class is based on the consideration of a term as a word in a given language together with its “meaning” in an algebra. In the theory of switching circuits one uses so-called binary decision diagrams (bdd) to measure the complexity of terms. In this case the underlying algebra is the two-element Boolean algebra $\mathcal{BU} = (\{0, 1\}; +, \cdot, -)$ of type $(2,2,1)$ (or another two-element algebra). One starts to give the variable with the greatest index the value 0 or 1, respectively, and continues with the arising subterms in the same way. In our examples we obtain the diagrams given on the Figure 2.

The bdd of $t_{1}$ looks more complex. This reflects the fact that in the first case there are more evaluation mappings. If one starts with another variable instead of $x_{3}$ one gets different bdd’s. Figure 3 gives the bdd’s of $t_{1}$ and $t_{2}$ if we replace the variables following the order $x_{2}, x_{3}, x_{1}$ by 0 and 1. The reason is that $\{x_{1}, x_{2}\}$ is not separable in $t_{2}$.

Now we want to count all evaluations of $X_{n} \setminus M$ for arbitrary (not only one-element) subsets $M$ of $X_{n}$ with elements of an arbitrary algebra $\mathcal{U}$ of type $\tau$. While $C_{p}^{(1)}$ and $C_{p}^{(2)}$ are complexities of terms, $C_{p}^{(3)}$
as the complexity of a function, is connected with the computational complexity.

**Definition 4.1.** Let \( t \in W_\tau(X_n) \) be a term and let \( \emptyset \neq M \subseteq X_n \) be a subset. The number \( C_p^{(3)}(t, M, \mathcal{U}) \) of all those evaluations of \( X_n \setminus M \) with \( C \in \mathcal{A}^{n-|M|}, |A| = |A|, \) for which \( M = \text{Ess}(\mathcal{H}(t), \mathcal{U}) \) is called complexity of \( t \) with respect to \( M \) in \( \mathcal{U} \). Further we define the complexity of \( t \) in \( \mathcal{U} \) by

\[
C_p^{(3)}(t, \mathcal{U}) := \sum_{\emptyset \subset M \subseteq X_n} C_p^{(3)}(t, M, \mathcal{U}).
\]

**Example:** We consider again the terms \( t_1, t_2 \) in the two-element Boolean algebra \( \mathcal{B}U = \{\{0, 1\}; +, \cdot, -\} \) and obtain

\[
C_p^{(3)}(t_2, \{x_3\}, \mathcal{B}U) = 2, C_p^{(3)}(t_2, \{x_2\}, \mathcal{B}U) = 2, C_p^{(3)}(t_2, \{x_1\}, \mathcal{B}U) = 2, C_p^{(3)}(t_2, \{x_1, x_2\}, \mathcal{B}U) = 2, C_p^{(3)}(t_2, \{x_1, x_3\}, \mathcal{B}U) = 2, C_p^{(3)}(t_2, \{x_2, x_3\}, \mathcal{B}U) = 0, C_p^{(3)}(t_2, \{x_1, x_2, x_3\}, \mathcal{B}U) = 1
\]

and therefore \( C_p^{(3)}(t_2, \mathcal{B}U) = 11. \)

In a similar way we calculate \( C_p^{(3)}(t_1, \mathcal{B}U) = 13. \)

Since we defined \( C_p^{(3)}(t, \mathcal{U}) \) using evaluation mappings we can expect that for terms \( t, t' \) which form an identity in \( \mathcal{U} \) we get the same complexity.

**Lemma 4.2.** If \( \mathcal{U} \models t \approx t' \) then for each \( \emptyset \subset M \subseteq X_n \) we have

\[
C_p^{(3)}(t, M, \mathcal{U}) = C_p^{(3)}(t', M, \mathcal{U})
\]

and thus also

\[
C_p^{(3)}(t, \mathcal{U}) = C_p^{(3)}(t', \mathcal{U}).
\]
Proof. If $\mathcal{U} \models t \approx t'$ then $\text{Ess}(t, \mathcal{U}) = \text{Ess}(t', \mathcal{U})$ by Corollary 3.8 of Lemma 3.5. Let $h$ be an evaluation of $X_n \setminus M$ with $C \in \mathcal{A}^{n-|M|} (|\mathcal{A}| = |A|)$ satisfying $M = \text{Ess}(\overline{h}(t), \mathcal{U})$. From $\mathcal{U} \models t \approx t'$ for every evaluation of $X_n \setminus M$ with elements from $\mathcal{A}$ we obtain polynomials $\overline{h}(t), \overline{h}(t')$ which give equal polynomial operations $\overline{h}(t)^{\mathcal{U}} = \overline{h}(t')^{\mathcal{U}}$, i.e. which form a polynomial identity. But this means $\text{Ess}(\overline{h}(t), \mathcal{U}) = \text{Ess}(\overline{h}(t'), \mathcal{U})$. Therefore, $h$ is an evaluation of $X_n \setminus M$ with $C \in \mathcal{A}^{n-|M|}$ satisfying $M = \text{Ess}(h(t), \mathcal{U})$ iff $h$ is an evaluation of $X_n \setminus M$ with $\mathcal{A}$ satisfying $M = \text{Ess}(h(t'), \mathcal{U})$. □

This motivates to consider only the elements of the free algebra $F_{V(\mathcal{U})}(X_n)$, freely generated by $X_n$. Let $t^*$ be the equivalence class with respect to the equivalence relation $\text{Id}\mathcal{U}$ containing the term $t$. Then we define

**Definition 4.3.** The sum
$$\sum_{t^* \in F_{V(\mathcal{U})}(X_n)} Cp^{(3)}(t^*, \mathcal{U})$$

is called $n$-complexity of $\mathcal{U}$.

**Example** A finite algebra $\mathcal{U} = (A; (f_i^{\mathcal{U}})_{i \in I})$ is called primal if $T(\mathcal{U}) = O_A$, i.e. if every operation defined on $A$ is a term operation of $\mathcal{U}$. We consider the variety $V(\mathcal{U})$ generated by a primal algebra with $|A| = k \geq 2$. Let $F_{V(\mathcal{U})}(X_n)$ be the free algebra in $V(\mathcal{U})$ generated by an $n$-element alphabet $X_n$.

Then there are exactly $k^{kn}$ different equivalence classes with respect to the equivalence relation $\text{Id}\mathcal{U}$, i.e. $|F_{V(\mathcal{U})}(X_n)| = k^{kn}$. The algorithm for calculating the complexity of $\mathcal{U}$ can be described in the following way: Let us set $p = k^{kn} - 1$ and $s = 2^n$.

begin
1 Coding all term operations from $\mathcal{U}$ by the
2 integers from the set $\{0, 1, \ldots, p\}$;
3 CPA:=0;
4 for $i := 0$ to $p$ do
5 begin
6 $CP := 0$;
7 Coding all subsets of $X_n$ by the integers
8 from the set $\{1, 2, \ldots, s\}$;
9 for $j := 1$ to $s$ do
begin
    \[ r := k^{\|\text{Decode}(j)\|} - 1; \]
    Coding all evaluations of the set \( X_n \setminus \text{Decode}(j) \) by the integers from \( \{0, \ldots, r\} \);
    for \( m := 0 \) to \( r \) do
        if \( \text{Decode}(j) = \text{Ess}(\text{Decode}(i, m, U)) \) then
            \( CP := CP + 1; \)
            Print \( CP \);
        end;
        \( CPA := CPA + CP; \)
    end;
    Print \( CPA \) as complexity of \( U \);
end.

Here \( \text{Decode}(j) \) is the subset of \( X_n \) which is decoded by \( j \), and \( \text{Decode}(i, m) \) is the image of the polynomial coded by \( i \) under the evaluation coded by \( m \).

This algorithm is constructed by full exhaustion of all cases and therefore it works not so quickly, but for small values of \( k \) and \( n \) it can be used. In the case \( k = 2 \) and \( n = 3 \) we obtained the following results. All \( n \)-ary functions (unequivalent classes of terms over \( V(U) \)) are classified with respect to the complexity. There exist 2 terms with complexity 19, 16 with complexity 16, 40 with complexity 13, 72 with complexity 12, 24 with complexity 11, 6 with complexity 10, 48 with complexity 9, 24 with complexity 8, 16 with complexity 7, 6 with complexity 4 and 2 terms with complexity 0.

Consequently, the 3-complexity of the two-element Boolean algebra \( BU \) is:

\[
Cp^{(3)}(U) = 2 \cdot 19 + 16 \cdot 16 + 40 \cdot 13 + 72 \cdot 12 + 24 \cdot 11 + 6 \cdot 10 + 48 \cdot 9 + 24 \cdot 8 + 16 \cdot 7 + 6 \cdot 4 + 2 \cdot 0 = \\
= 38 + 256 + 520 + 864 + 264 + 60 + 432 + 192 + 112 + 24 + 0 = 2762.
\]

We can also define the \( n \)-complexity of a variety.

**Definition 4.4.** Let \( V \) be a variety of type \( \tau \) and let \( \mathcal{F}_V(X) \) be the \( V \)-free algebra generated by the \( n \)-element alphabet \( X_n \). Then

\[
\sum_{t^* \in \mathcal{F}_V(X_n)} Cp^{(3)}(t^*, \mathcal{F}_V(X))
\]

is called \( n \)-complexity of \( V \).
Lemma 4.2 gave an example which showed that different terms can have the same complexity. The following results give examples for equal complexities if we have different M’s or different algebras.

Let $\sigma$ be a permutation of $X_n$ and let $f_\sigma : W_\tau(X_n) \rightarrow W_\tau(X_n)$ be the extension of $\sigma$ to terms. Then $f_\sigma$ has the following interesting property. For each subset $\emptyset \subset M \subseteq X_n$, for each term $t \in W_\tau(X_n)$ and for each algebra $U$ of type $\tau$ we have

$$Cp^{(3)}(t, M, U) = Cp^{(3)}(f_\sigma(t), \sigma(M), U).$$

For the proof we have to show that for every evaluation $h : X_n \setminus M \rightarrow A, |A| = |A|$, satisfying $M = Ess(h(t), U)$ we get an evaluation $h' : X_n \setminus \sigma(M) \rightarrow A$ satisfying $\sigma(M) = Ess(h'(f_\sigma(t)), \sigma(M), U)$ and conversely. Clearly, there is a one-to-one correspondence between evaluations $h$ and $h'$ on terms. Then $f_\sigma$ has the desired property.

Let $p$ be a polynomial of type $\tau$ over $A$ and let $U$ be an algebra of type $\tau$ with $|A| = |A|$. By $V^U_p$ we denote the set of all values of the induced polynomial operation $p^U$, i.e.

$$V^U_p := \{ a \in a \ | \ \exists h : X \cup A \rightarrow A (h(p) = a) \}.$$ 

Let $g : A \rightarrow A$ be a mapping on the set of constant symbols. This mapping can be uniquely extended to a mapping $f_g : P_\tau(X_n, A) \rightarrow P_\tau(X_n, A)$ with $f_g(x_i) = x_i$ for all $i \in \{x_1, \ldots, x_n\}$. Then we have

**Proposition 4.5.** Let $g : A \rightarrow A$ be a mapping and let $p \in P_\tau(X_n, A)$. If $|V^U_p| = |g(V^U_p)|$ then for every $\emptyset \subset M \subseteq X_n$ we have

$$Cp^{(3)}(p, M, U) = Cp^{(3)}(f_g(p), M, U)$$

and therefore also

$$Cp^{(3)}(p, U) = Cp^{(3)}(f_g(p), U).$$

**Proof.** For the proof we have to show that for every evaluation $h : X_n \setminus M \rightarrow A$, satisfying $M = Ess(h(p), M, U)$ we have an evaluation $h' : X_n \setminus M \rightarrow A$, satisfying $M = Ess(h'(f_g(p)), M, U)$. Because of $|V^U_p| = |g(V^U_p)|$ and the finiteness, the restriction of $g$ to $V^U_p$ is a one-to-one mapping. Therefore, $f_{g^{-1}}(p)$ exists and satisfies $f_{g^{-1}}(f_g(p)) = p$. If we define $h' := h \circ f_{g^{-1}}$ then we have

$$h'(f_g(p)) = (h \circ f_{g^{-1}})(f_g(p)) = h(p).$$

Therefore,

$$Ess(h'(f_g(p)), M, U) = Ess(h(p), M, U) = M.$$ 

Clearly, there is a one-to-one correspondence between evaluations $h$ and $h'$. This proves Proposition 4.5. $\blacksquare$
Theorem 4.6. Let $U$ and $B$ be isomorphic algebras of type $\tau$. If $t \in W_\tau(X)$ and $\emptyset \subset M \subseteq X_n$ then

$$Cp^{(3)}(t, M, U) = Cp^{(3)}(t, M, B).$$

Proof. Let $\varphi : U \to B$ be an isomorphism and let $h : X_n \setminus M \to \mathcal{A}$ be an evaluation satisfying $M = Ess(\overline{h}(t), U)$. Consider a set $\overline{B}$ of constant symbols for the elements of $B$. Then $\overline{\varphi} : \mathcal{A} \to \overline{B}$ is a one-to-one mapping. Consider the evaluation $h'$ of $X_n \setminus M$ with $\overline{B}$ defined by $h' = \overline{\varphi} \circ h$. Since $\overline{\varphi}$ is one-to-one we have

$$M = Ess(\overline{h}(t), U) = Ess(\overline{\varphi} \circ \overline{h}(t), B)$$

where $\overline{\varphi} \circ \overline{h}$ is the extension of $\overline{\varphi} \circ h$. To every $h$ we obtain exactly one $\overline{\varphi} \circ h$. If conversely $h^* : X_n \setminus M \to \overline{B}$ is an evaluation then by $(h^*)' := \overline{\varphi}^{-1} \circ h$ we obtain an evaluation $(h^*)' : X_n \setminus M \to \mathcal{A}$ which is uniquely determined and satisfies

$$M = Ess(\overline{h^*}(t), B) = Ess((h^*)'(t), U).$$

Therefore, the cardinalities of the sets of all evaluations of $X_n \setminus M$ with $\mathcal{A}$ and $\overline{B}$ with the property that the set of essential variables is $M$, are equal and this means,

$$Cp^{(3)}(t, M, U) = Cp^{(3)}(t, M, B).$$

\[\square\]

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