ON THE GENERALIZED OF $p$-HARMONIC MAPS

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ABSTRACT

In this paper, we extend the definition of $p$-harmonic and $p$-biharmonic maps between Riemannian manifolds. We present some new properties for the generalized stable $p$-harmonic maps.

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1 Introduction

Consider a smooth map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds, and let $p$ be a smooth positive function on $M$ such that $p(x) \geq 2$ for all $x \in M$. For any compact domain $D$ of $M$ the $p$-energy functional of $\varphi$ is defined by

$$E_p(\varphi; D) = \int_D \left| d\varphi \right|^p(x) v_g,$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and $v_g$ is the volume element on $(M, g)$. A map is called $p$-harmonic if it is a critical point of the $p$-energy functional over any compact subset $D$ of $M$. $p$-harmonic maps is a natural generalization of harmonic map ([1, 5]) and $p$-harmonic map ([2, 3, 6]). We denote by

$$\tau_p(\varphi) = \text{trace}_g \nabla |d\varphi|^p(x) - 2 d\varphi,$$

the tension field of $\varphi$, where $\{e_i\}_{i=1}^m$ is an orthonormal frame on $(M, g)$, $\nabla^M$ is the Levi-Civita connection of $(M, g)$, and $\nabla \varphi$ denote the pull-back connection on $\varphi^{-1}TN$.

In this paper, we investigate some properties for $p$-harmonic maps between two Riemannian manifolds. In particular, we present the first and the second variation of the $p$-energy. We also extend the definition of $p$-biharmonic maps between two Riemannian manifolds ([8]).

2 $p$($\cdot$)-Harmonic Maps

Theorem 1 (The first variation of the $p$($\cdot$)-energy). Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds, $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ a smooth variation of $\varphi$ supported in compact domain $D$ of $M$. Then

$$\frac{d}{dt} E_{p} (\varphi_t; D) \bigg|_{t=0} = - \int_D h(v, \tau_{p} (\varphi)) v_g,$$

where $\tau_{p} (\varphi)$ denotes the $p$($\cdot$)-tension field of $\varphi$ given by

$$\tau_{p} (\varphi) = \text{trace}_g \nabla |d\varphi|^{p(x) - 2} d\varphi,$$
we have

\[ \phi(x, t) = \varphi(x), \quad \forall (x, t) \in M \times (-\epsilon, \epsilon). \]

We have \( \phi(x, 0) = \varphi(x) \) for all \( x \in M \), and the variation vector field associated to the variation \( \{ \varphi_t \}_{t \in (-\epsilon, \epsilon)} \) is given by

\[ v(x) = d(x_0)\phi \left( \frac{\partial}{\partial t} \right) \bigg|_{t=0}, \quad \forall x \in M. \]

Let \( \{ e_i \}_{i=1}^m \) be an orthonormal frame on \( (M, g) \). We compute

\[
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D) \bigg|_{t=0} = \frac{d}{dt} \int_D \frac{|d\varphi_t|^2}{p(x)} v_g \bigg|_{t=0} \\
= \int_D \frac{d}{dt} \frac{|d\varphi_t|^2}{p(x)} v_g \\
= \int_D \frac{d}{dt} \frac{|d\varphi_t|^2}{p(x)} \frac{2}{p(x)} v_g \\
= \sum_{i=1}^m \int_D \left( \frac{2}{p(x)} \lambda(d\varphi_t(e_i), d\varphi_t(e_i)) \right) v_g \\
= \sum_{i=1}^m \int_D \lambda(d\varphi_t(e_i), d\varphi_t(e_i)) v_g \\
= \sum_{i=1}^m \int_D \lambda(d\varphi_t(e_i), d\varphi_t(e_i)) v_g \\
= \sum_{i=1}^m \int_D \lambda(d\varphi_t(e_i), d\varphi_t(e_i)) v_g. \tag{5}
\]

By using the property

\[ \nabla_X^\phi d\phi(Y) = \nabla_Y^\phi d\phi(X) + d\phi([X, Y]), \]

with \( X = \frac{\partial}{\partial x}, Y = (e_i, 0), \) and \( \left( \frac{\partial}{\partial x}, (e_i, 0) \right) = 0 \), the equation (5) becomes

\[
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D) \bigg|_{t=0} = \sum_{i=1}^m \int_D h(\nabla_{(e_i, 0)}^\phi d\phi(e_i)) |d\varphi_t|^{2/p(x)} \left| \frac{2}{p(x)} \right| v_g \\
= \sum_{i=1}^m \int_D h(\nabla_{e_i}^\phi v, |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) v_g \\
= \sum_{i=1}^m \int_D \left[ e_i h(v, |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) - h(v, |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) \right] v_g. \tag{6}
\]

Let \( \omega \in \Gamma(T^*M) \) defined by

\[ \omega(X) = h(v, |d\varphi|^{2/p(x) - 2} d\varphi(X)), \quad \forall X \in \Gamma(TM) \]

The divergence of \( \omega \) is given by

\[ \text{div}^M \omega = \sum_{i=1}^m \left[ e_i h(v, |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) - h(v, |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) \right]. \tag{7} \]

By equations (5), (7), and the divergence Theorem (11), we get

\[
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D) \bigg|_{t=0} = \sum_{i=1}^m \int_D h(v, |d\varphi|^{2/p(x) - 2} d\varphi(\nabla_{e_i}^M e_i) - \nabla_{e_i}^* |d\varphi|^{2/p(x) - 2} d\varphi(e_i)) v_g \\
= - \sum_{i=1}^m \int_D h(v, \left[ \nabla_{e_i}^* |d\varphi|^{2/p(x) - 2} d\varphi(e_i) \right] v_g. \tag{8}
\]
Corollary 2. A smooth map \( \varphi : (M, g) \rightarrow (N, h) \) between two Riemannian manifolds is \( p(\cdot) \)-harmonic if and only if
\[
\tau_{p(\cdot)}(\varphi) = |d\varphi|^p(x) - 2\tau(\varphi) + d\varphi(\text{grad}_M |d\varphi|^p(x)) = 0.
\]

Example 3. The restriction of inversion
\[
\varphi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \frac{x}{\|x\|^2},
\]
to \( M = \{ x \in \mathbb{R}^n \setminus \{0\} ; \|x\|^2 > \sqrt{n} \} \) is \( p(\cdot) \)-harmonic, where the function \( p \) is given by
\[
p(x) = n + \frac{c}{2 \ln(\|x\|^2) - \ln(n)}, \quad \forall x \in M,
\]
for some constant \( c \geq 0 \). Here, \( |d\varphi|(x) = \frac{\sqrt{n}}{|x|} \) for all \( x \in \mathbb{R}^n \setminus \{0\} \).

Example 4. Let \( F : \mathbb{R} \rightarrow [2, \infty) \) be a smooth function. The map
\[
\varphi : \mathbb{R}^n \setminus \{0\} \rightarrow S^n, \quad x \mapsto \frac{x}{\|x\|},
\]
is \( p(\cdot) \)-harmonic, where \( p(x) = F(\|x\|^2) \) for all \( x \in \mathbb{R}^n \setminus \{0\} \). The Hilbert-Schmidt norm of \( d\varphi \) is given by \( |d\varphi|(x) = \frac{\sqrt{n}}{|x|} \) for all \( x \in \mathbb{R}^n \setminus \{0\} \).

Remark 5. A smooth harmonic map, i.e., \( \tau(\varphi) = 0 \), with constant energy density \( \frac{|d\varphi|^2}{2} \) is not always \( p(\cdot) \)-harmonic. The previous examples prove the following results: There is no equivalence between the \( p(\cdot) \)-harmonicity and the harmonicity of a smooth map \( \varphi : (M, g) \rightarrow (N, h) \). There are \( p(\cdot) \)-harmonic maps which have non-constant Hilbert-Schmidt norm and they are not harmonic.

3 Stable \( p(\cdot) \)-Harmonic Maps

Theorem 6 (The second variation of the \( p(\cdot) \)-energy). Let \( \varphi \) be a smooth \( p(\cdot) \)-harmonic map between two Riemannian manifolds \( (M, g) \) and \( (N, h) \). Then we have
\[
\frac{\partial^2}{\partial t \partial s} E_{p(\cdot)}(\varphi_{t,s}; D)\bigg|_{t=s=0} = \int_D h(J_{p(\cdot)}^\varphi(v), w)v_g,
\]
where \( \{ \varphi_{t,s} \}_{t,s \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)} \) is a smooth variation supported in compact domain \( D \subset M \) of \( \varphi \),
\[
v = \left. \frac{\partial \varphi_{t,s}}{\partial t} \right|_{t=s=0}, \quad w = \left. \frac{\partial \varphi_{t,s}}{\partial s} \right|_{t=s=0},
\]
and \( J_{p(\cdot)}^\varphi \) the generalized Jacobi operator of \( \varphi \) given by
\[
J_{p(\cdot)}^\varphi(v) = -|d\varphi|^p(x) - 2\text{trace}_g R^N(v, d\varphi)d\varphi - \text{trace}_g \nabla^g |d\varphi|^p(x) - \nabla^g v
- \text{trace}_g \nabla (p(x) - 2)|d\varphi|^p(x) - 2\nabla^g v, d\varphi|d\varphi.
\]

Here \( \langle \cdot, \cdot \rangle \) denote the inner product on \( T^*M \otimes \varphi^{-1}TN \).

Proof. Let \( \phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow N \) be a smooth map defined by \( \phi(x, t, s) = \varphi_{t,s}(x) \). We have \( \phi(x, 0, 0) = \varphi(x) \), and the variation vectors fields associated to the variation \( \{ \varphi_{t,s} \}_{t,s \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)} \) are given by
\[
v(x) = d_{(x, 0, 0)} \phi \left( \frac{\partial}{\partial t} \right), \quad w(x) = d_{(x, 0, 0)} \phi \left( \frac{\partial}{\partial s} \right), \quad \forall x \in M.
\]

Let \( \{ e_i \}_{i=1}^m \) be an orthonormal frame with respect to \( g \) on \( M \) such that \( \nabla^M_M e_j = 0 \) at \( x \in M \) for all \( i, j = 1, \ldots, m \). We compute
\[
\frac{\partial^2}{\partial t \partial s} E_{p(\cdot)}(\varphi_{t,s}; D)\bigg|_{t=s=0} = \frac{\partial^2}{\partial t \partial s} \int_D |d\varphi_{t,s}|^p(x) \frac{p(x)}{p(x)} v_g\bigg|_{t=s=0}
= \int_D \frac{1}{p(x)} \frac{\partial^2}{\partial t \partial s} |d\varphi_{t,s}|^p(x) \bigg|_{t=s=0} v_g.
\]
First, note that
\[
\frac{1}{p(x)} \frac{\partial^2}{\partial t \partial s} |d\varphi_{t,s}|^{p(x)} = \frac{1}{p(x)} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial s} \left( |d\varphi_{t,s}|^2 \right)^{\frac{p(x)}{2}} \right)
\]
\[
= \frac{1}{2} \frac{\partial}{\partial t} \left( \left( |d\varphi_{t,s}|^2 \right)^{\frac{p(x)}{2} - 1} \frac{\partial}{\partial s} |d\varphi_{t,s}|^2 \right)
\]
\[
= \sum_{i=1}^{m} \frac{\partial}{\partial t} \left( \left( |d\varphi_{t,s}|^2 \right)^{\frac{p(x)}{2} - 1} h(\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), d\phi(e_i, 0, 0)) \right).
\]

Thus
\[
\frac{1}{p(x)} \frac{\partial^2}{\partial t \partial s} |d\varphi_{t,s}|^{p(x)} = \sum_{i=1}^{m} \left( p(x) - 2 \right) |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), d\phi(e_i, 0, 0))
\]
\[
+ \sum_{i=1}^{m} |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), d\phi(e_i, 0, 0))
\]
\[
+ \sum_{i=1}^{m} |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0)).
\]

So that
\[
\frac{1}{p(x)} \frac{\partial^2}{\partial t \partial s} |d\varphi_{t,s}|^{p(x)} = \sum_{i=1}^{m} \left( p(x) - 2 \right) |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), d\phi(e_i, 0, 0))
\]
\[
+ \sum_{i=1}^{m} |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), d\phi(e_i, 0, 0))
\]
\[
+ \sum_{i=1}^{m} |d\varphi_{t,s}|^{p(x) - 2} h(\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0)).
\]

By the definition of the curvature tensor of \((N, h)\) and the properties
\[
\nabla_\varphi ^{\phi} d\phi(e_i, 0, 0) = \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0), \quad \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0) = \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0),
\]
with \([\frac{\partial}{\partial s}, (e_i, 0, 0)] = 0\), we obtain the following equation
\[
\frac{1}{p(x)} \frac{\partial^2}{\partial t \partial s} |d\varphi_{t,s}|^{p(x)} \bigg|_{t=s=0} = \sum_{i=1}^{m} \left( p(x) - 2 \right) |d\varphi_{t,s}|^{p(x) - 4} \langle \nabla_\varphi ^{\phi} v, d\varphi \rangle d\varphi(e_i)
\]
\[
- |d\varphi|^{p(x) - 2} \sum_{i=1}^{m} h(R^N(v, d\varphi(e_i))d\varphi(e_i), w)
\]
\[
+ \sum_{i=1}^{m} h \left( \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0) \left|_{t=s=0} \right. \right) \left| \frac{\partial}{\partial s} \right|_{t=s=0} |d\varphi|^{p(x) - 2} d\varphi(e_i)
\]
\[
+ \sum_{i=1}^{m} h \left( \nabla_\varphi ^{\phi} w, |d\varphi|^{p(x) - 2} \nabla_\varphi ^{\phi} v \right).
\]

Let \(\omega_1, \omega_2, \omega_3 \in \Gamma(T^* M)\) defined by
\[
\omega_1(X) = h \left( w, (p(x) - 2) |d\varphi|^{p(x) - 4} \langle \nabla_\varphi ^{\phi} v, d\varphi \rangle d\varphi(X) \right);
\]
\[
\omega_2(X) = h \left( \nabla_\varphi ^{\phi} d\phi(e_i, 0, 0) \left|_{t=s=0} \right. \right) \left| \frac{\partial}{\partial s} \right|_{t=s=0} |d\varphi|^{p(x) - 2} d\varphi(X)\);\]
\[
\omega_3(X) = h \left( w, |d\varphi|^{p(x) - 2} \nabla_\varphi ^{\phi} v \right), \quad \forall X \in \Gamma(TM).
\]
The divergence of \( \omega_1, \omega_2, \) and \( \omega_3 \) are given by

\[
\text{div}^M \omega_1 = \sum_{i=1}^{m} e_i h \left( w, (p(x) - 2)|d\phi|^{p(x)-4}(\nabla^p v, d\phi) d\phi(e_i) \right);
\]

\[
\text{div}^M \omega_2 = \sum_{i=1}^{m} e_i h \left( \nabla^\phi \frac{\partial}{\partial s} \right)_{t=s=0} |d\phi|^{p(x)-2} d\phi(e_i) ;
\]

\[
\text{div}^M \omega_3 = \sum_{i=1}^{m} e_i h \left( w, |d\phi|^{p(x)-2} \nabla^\phi v \right), \quad \forall X \in \Gamma(TM).
\]

By equations \([13], [14]\), the \( p(\cdot) \)-harmonicity condition of \( \phi \), and the divergence Theorem, we obtain

\[
\frac{\partial^2}{\partial t \partial s} E_{p(x)}(\phi, s; D) \bigg|_{t=s=0} = -\int_M h \left( w, \nabla^\phi (p(x) - 2)|d\phi|^{p(x)-4}(\nabla^\phi v, d\phi) d\phi(e_i) \right) v_g
\]

\[
- \int_D |d\phi|^{p(x)-2} \sum_{i=1}^{m} h(w, R^N v, d\phi(e_i)) v_g,
\]

\[
- \int_D \sum_{i=1}^{m} h \left( w, \nabla^\phi_i |d\phi|^{p(x)-2} \nabla^\phi v \right) v_g.
\]

(15)

The proof is completed. □

If \((M, g)\) is a compact Riemannian manifold, \( \varphi \) be a \( p(\cdot) \)-harmonic map from \((M, g)\) to Riemannian manifold \((N, h)\), and for any vector field \( v \) along \( \varphi \),

\[
I^\varphi_{p(x)}(v, v) = \int_M h(v, I^\varphi_{p(x)}(v)) v_g \geq 0,
\]

(16)

then \( \varphi \) is called a stable \( p(\cdot) \)-harmonic map. Note that, the definition of stable \( p(\cdot) \)-harmonic maps is a generalization of stable harmonic maps \([10]\), is also a generalization of stable \( p \)-harmonic maps \([4, 9]\). By using the Green Theorem \([11]\) it is easy to prove that

\[
I^\varphi_{p(x)}(v, v) = -\int_M |d\phi|^{p(x)-2} \sum_{i=1}^{m} h(v, R^N v, d\phi(e_i)) v_g
\]

\[
+ \int_M |d\phi|^{p(x)-2} |\nabla^\phi v|^2 v_g + \int_M (p(x) - 2)|d\phi|^{p(x)-4}(\nabla^\phi v, d\phi)^2 v_g.
\]

(17)

From equation \((17)\), we deduce the following result.

**Proposition 7.** Every \( p(\cdot) \)-harmonic map from compact Riemannian manifold \((M, g)\) to Riemannian manifold \((N, h)\) has \( \text{Sect}^N \leq 0 \) is stable.

In the case where the codomain of the stable \( p(\cdot) \)-harmonic map is the standard sphere \( \mathbb{S}^n \), we have the following result.

**Theorem 8.** Let \((M, g)\) be a compact Riemannian manifold. When \( n > 2 \), any stable \( p(\cdot) \)-harmonic map \( \varphi : (M, g) \to \mathbb{S}^n \) must be constant, where \( p \) is a smooth positive function on \( M \) such that \( 2 \leq p(x) < n \) for all \( x \in M \).

**Proof.** Choose a normal orthonormal frame \( \{ e_i \}_{i=1}^{m} \) at point \( x \) in \((M, g)\). We set \( \lambda(y) = (\alpha, y)_{\mathbb{R}^{n+1}} \), for all \( y \in \mathbb{S}^n \), where \( \alpha \in \mathbb{R}^{n+1} \). Let \( v = \text{grad}^\mathbb{S}^n \lambda \). We have \( \nabla^\mathbb{S}^n v = -\lambda X \) for all \( X \in \Gamma(T\mathbb{S}^N) \), where \( \nabla^\mathbb{S}^n \) is the Levi-Civita connection on \( \mathbb{S}^n \) with respect to the standard metric of the sphere (see \([10]\)). We compute

\[
\sum_{i=1}^{m} \nabla^\phi_i |d\phi|^{p(x)-2} \nabla^\phi_{e_i} (v \circ \varphi) = \nabla^\text{grad}^\mathbb{S}^n |d\phi|^{p(x)-2} (v \circ \varphi)
\]

\[
+ \sum_{i=1}^{m} |d\phi|^{p(x)-2} \nabla^\phi_{e_i} \nabla^\phi_{e_i} (v \circ \varphi).
\]

(18)
By using the property $\nabla_X^p v = -\lambda X$, the first term of (18) is given by

$\nabla^p_{\text{grad}^M |d\varphi|^p(x)-2}(v \circ \varphi) = -(\lambda \circ \varphi)d\varphi(\text{grad}^M |d\varphi|^p(x)-2)$,  \hspace{1cm} (19)

and the second term of (18) is given by

$$\sum_{i=1}^{m} |d\varphi|^p(x)-2 \nabla^p_{e_i} \nabla^p_{e_i} (v \circ \varphi) = -\sum_{i=1}^{m} |d\varphi|^p(x)-2 \nabla^p_{e_i} (\lambda \circ \varphi)d\varphi(e_i)$$

$$= -\sum_{i=1}^{m} |d\varphi|^p(x)-2 < d\varphi(e_i), v \circ \varphi > d\varphi(e_i)$$

$$= -(\lambda \circ \varphi)|d\varphi|^p(x)-2 \tau(\varphi).$$  \hspace{1cm} (20)

Substituting the formulas (19) and (20) in (18) gives

$$\sum_{i=1}^{m} \nabla^p_{e_i} |d\varphi|^p(x)-2 \nabla^p_{e_i} (v \circ \varphi) = -(\lambda \circ \varphi)d\varphi(\text{grad}^M |d\varphi|^p(x)-2)$$

$$-\sum_{i=1}^{m} |d\varphi|^p(x)-2 < d\varphi(e_i), v \circ \varphi > d\varphi(e_i)$$

$$= -(\lambda \circ \varphi)|d\varphi|^p(x)-2 \tau(\varphi).$$  \hspace{1cm} (21)

By the $p(\cdot)$-harmonicity condition of $\varphi$

$$\tau_{p(\cdot)}(\varphi) = |d\varphi|^p(x)-2 \tau(\varphi) + d\varphi(\text{grad}^M |d\varphi|^p(x)-2) = 0,$$

and equation (21), we get

$$\sum_{i=1}^{m} \langle \nabla^p_{e_i} |d\varphi|^p(x)-2 \nabla^p_{e_i} (v \circ \varphi), v \circ \varphi \rangle = -\sum_{i=1}^{m} |d\varphi|^p(x)-2 \langle d\varphi(e_i), v \circ \varphi \rangle^2.$$  \hspace{1cm} (22)

Since the sphere $S^n$ has constant curvature, we have

$$\sum_{i=1}^{m} \langle |d\varphi|^p(x)-2 R^{S^n}(v \circ \varphi, d\varphi(e_i))d\varphi(e_i), v \circ \varphi \rangle = |d\varphi|^p(x)(v \circ \varphi, v \circ \varphi)$$

$$-\sum_{i=1}^{m} |d\varphi|^p(x)-2 \langle d\varphi(e_i), v \circ \varphi \rangle^2.$$  \hspace{1cm} (23)

By the definition of generalized Jacobi operator, and (22), (23), we obtain

$$\langle J^p_{\varphi}(v \circ \varphi), v \circ \varphi \rangle = 2|d\varphi|^p(x)-2 \sum_{i=1}^{m} \langle d\varphi(e_i), v \circ \varphi \rangle^2$$

$$-|d\varphi|^p(x) \langle v \circ \varphi, v \circ \varphi \rangle$$

$$-\sum_{i=1}^{m} \langle \nabla^p_{e_i} (p(x) - 2)|d\varphi|^p(x)-4 \langle \nabla^p_{e_i} (v \circ \varphi, d\varphi) d\varphi(e_i), v \circ \varphi \rangle,$$  \hspace{1cm} (24)

Using $\langle \nabla^p_{e_i} (v \circ \varphi, d\varphi) \rangle = -(\lambda \circ \varphi)|d\varphi|^2$, and equation (24), we find that

$$\text{trace}_{\alpha}(J^p_{\varphi}(v \circ \varphi), v \circ \varphi) = (p(x) - n)|d\varphi|^p(x).$$  \hspace{1cm} (25)

Hence Theorem 8 follows from (25), and the stable $p(\cdot)$-harmonicity condition of $\varphi$, with $2 \leq p(x) < n$ for all $x \in M$. 

\qed
4 \( p(\cdot) \)-Biharmonic Maps

Let \( \varphi : (M, g) \rightarrow (N, h) \) be a smooth map between two Riemannian manifolds, the \( p(\cdot) \)-bienergy of \( \varphi \) is defined by

\[
E_{2,p(\cdot)}(\varphi; D) = \frac{1}{2} \int_D |\tau_{p(\cdot)}(\varphi)|^2 v_g,
\]

where \( p \geq 2 \) is a smooth function on \( M \), and \( D \) a compact subset of \( M \). A smooth map \( \varphi \) is called \( p(\cdot) \)-biharmonic if it is a critical point of the \( p(\cdot) \)-bienergy functional for any compact domain \( D \).

**Theorem 9** (The first variation of the \( p(\cdot) \)-bienergy). Let \( \varphi \) be a smooth map between two Riemannian manifolds \((M, g)\) and \((N, h)\). Then we have

\[
\frac{d}{dt} E_{2,p(\cdot)}(\varphi_t; D) \bigg|_{t=0} = -\int_D h(v, \tau_{2,p(\cdot)}(\varphi)) v_g,
\]

where \{\( \varphi_t \)\} \(_{t \in (-\epsilon, \epsilon)} \) is a smooth variation of \( \varphi \) supported in \( D \), \( v = \frac{d\varphi_t}{dt} \big|_{t=0} \) denotes the variation vector field, and \( \tau_{2,p(\cdot)}(\varphi) \) the \( p(\cdot) \)-bienergy field of \( \varphi \) given by

\[
\tau_{2,p(\cdot)}(\varphi) = -|d\varphi|^2 R^N(\tau_{p(\cdot)}(\varphi), d\varphi) - \text{trace}_g \nabla \nabla^2 d\varphi \cdot d\varphi - \text{trace}_g \left( (p(x) - 2) |d\varphi|^2 \nabla^2 (\tau_{p(\cdot)}(\varphi), d\varphi) d\varphi \right).
\]

**Proof.** Define \( \phi : M \times (-\epsilon, \epsilon) \rightarrow N \) by \( \phi(x, t) = \varphi_t(x) \). First, note that

\[
\frac{d}{dt} E_{2,p(\cdot)}(\varphi_t; D) = \int_D h(\nabla_{\partial/\partial t} \varphi_t, \nabla_{\partial/\partial t} \varphi_t) v_g.
\]

Calculating in a normal frame at \( x \in M \), we have

\[
\nabla_{\partial/\partial t} \varphi_t(\varphi) = \sum_{i=1}^m \nabla_{\partial/\partial t} \nabla_{(e_i, 0)} |d\varphi_t|^2 d\phi(e_i, 0).
\]

From the definition of the curvature tensor of \((N, h)\), we obtain

\[
\sum_{i=1}^m \nabla_{\partial/\partial t} \nabla_{(e_i, 0)} |d\varphi_t|^2 d\phi(e_i, 0)
\]

\[
= |d\varphi_t|^2 \sum_{i=1}^m R^N(d\phi_j \frac{\partial}{\partial t}, d\phi(e_i, 0)) d\phi(e_i, 0) + \sum_{i=1}^m \nabla_{(e_i, 0)} \nabla_{\partial/\partial t} |d\varphi_t|^2 d\phi(e_i, 0).
\]

By using the compatibility of \( \nabla^\phi \) with \( h \), we find that

\[
\sum_{i=1}^m h(\nabla_{(e_i, 0)} \nabla_{\partial/\partial t} |d\varphi_t|^2 d\phi(e_i, 0), \tau_{p(\cdot)}(\varphi_t))
\]

\[
= \sum_{i=1}^m (e_i, 0) \left[ h(\nabla_{\partial/\partial t} |d\varphi_t|^2 d\phi(e_i, 0), \tau_{p(\cdot)}(\varphi_t)) \right] - \sum_{i=1}^m h(\nabla_{\partial/\partial t} |d\varphi_t|^2 d\phi(e_i, 0), \nabla_{(e_i, 0)} \tau_{p(\cdot)}(\varphi_t)).
\]

From the property \( \nabla^\phi d\phi(Y) = \nabla^\phi d\phi(X) + d\phi([X, Y]) \), with \( X = \frac{\partial}{\partial t} \) and \( Y = |d\varphi_t|^2 d\phi(e_i, 0) \), we get
\[ \frac{\nabla \phi}{\partial t} \left|_{t=0} \right. = |d\phi|^{p(x)-2}\nabla_{e_i} v \]
\[ + \sum_{j=1}^{m} (p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{e_j} \phi, d\phi \rangle d\phi(e_j), \]
for all \( i = 1, \ldots, m \). Substituting (33) in (32), we have
\[ \sum_{i=1}^{m} h(\nabla_{e_i} \phi) \frac{\nabla \phi}{\partial t} \left|_{t=0} \right. = \sum_{i=1}^{m} e_i h(|d\phi|^{p(x)-2}\nabla_{e_i} v, \tau_{p(\cdot)}(\phi)) \]
\[ + \sum_{i=1}^{m} e_i h((p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{e_i} \phi, d\phi \rangle d\phi(e_i), \tau_{p(\cdot)}(\phi)) \]
\[ - \sum_{i=1}^{m} e_i h(v, |d\phi|^{p(x)-2}\nabla_{e_i} \tau_{p(\cdot)}(\phi)) \]
\[ + \sum_{i=1}^{m} h(v, \nabla_{e_i} |d\phi|^{p(x)-2}\nabla_{e_i} \tau_{p(\cdot)}(\phi)) \]
\[ - \sum_{j=1}^{m} e_j h(v, (p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{e_j} \tau_{p(\cdot)}(\phi), d\phi \rangle d\phi(e_j)) \]
\[ + \sum_{j=1}^{m} h(v, \nabla_{e_j} (p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{e_j} \tau_{p(\cdot)}(\phi), d\phi \rangle d\phi(e_j)). \]

Let \( \eta_1, \eta_2, \eta_3, \eta_4 \in \Gamma(T^*M) \) defined by
\[ \eta_1(X) = h(|d\phi|^{p(x)-2}\nabla_{X} v, \tau_{p(\cdot)}(\phi)); \]
\[ \eta_2(X) = h((p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{X} \phi, d\phi \rangle d\phi(X), \tau_{p(\cdot)}(\phi)); \]
\[ \eta_3(X) = h(v, |d\phi|^{p(x)-2}\nabla_{X} \tau_{p(\cdot)}(\phi)); \]
\[ \eta_4(X) = h(v, (p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla_{X} \tau_{p(\cdot)}(\phi), d\phi \rangle d\phi(X)). \]

Finally, we calculate the divergence of \( \eta_i \) (\( i = 1, \ldots, 4 \)) and substituting in (34). The proof of Theorem 5 follows by (29), (31), (34), and the divergence Theorem. \( \square \)

From Theorem 9 we deduce:

**Corollary 10.** Let \( \varphi : (M, g) \to (N, h) \) be a smooth map between Riemannian manifolds. Then, \( \varphi \) is \( p(\cdot) \)-biharmonic if and only if
\[ \tau_{2,p(\cdot)}(\varphi) = -|d\phi|^{p(x)-2} \text{trace}_g R^N(\tau_{p(\cdot)}(\varphi), d\phi) d\phi \]
\[ - \text{trace}_g \nabla^\varphi |d\phi|^{p(x)-2} \nabla^\varphi \tau_{p(\cdot)}(\varphi) \]
\[ - \text{trace}_g \nabla (p(x) - 2)|d\phi|^{p(x)-4} \langle \nabla^\varphi \tau_{p(\cdot)}(\varphi), d\phi \rangle d\phi = 0. \]

**Remark 11.** For any smooth map \( \varphi : (M, g) \to (N, h) \) between two Riemannian manifolds, we have
\[ \tau_{2,p(\cdot)}(\varphi) = J_{p(\cdot)}^\varphi(\tau_{p(\cdot)}(\varphi)). \]
We can extract several examples of \( p(\cdot) \)-biharmonic non \( p(\cdot) \)-harmonic maps \( \varphi : (M, g) \to \mathbb{R}^n \) where the \( p(\cdot) \)-tension field is parallel along \( \varphi \), i.e., the components of \( \tau_{p(\cdot)}(\varphi) \) are constants.
Example 12. Let $M = \{ (x, y, z) \in \mathbb{R}^3, \sqrt{x^2 + y^2} > 2 \}$. The smooth map $\varphi : M \rightarrow \mathbb{R}^2$ defined by

$$\varphi(x, y, z) = (\sqrt{x^2 + y^2}, z), \quad \forall (x, y, z) \in M,$$

is $p(\cdot)$-biharmonic non $p(\cdot)$-harmonic, where

$$p(x, y, z) = \frac{\ln(x^2 + y^2)}{\ln(2)},$$

for all $(x, y, z) \in M$. Here, $\tau_{p(\cdot)}(\varphi) = (1, 0)$.

References

[1] P. Baird and J. C. Wood, Harmonic morphisms between Riemannian manifolds, Clarendon Press, Oxford (2003).
[2] P. Baird, S. Gudmundsson, $p$-Harmonic maps and minimal submanifolds, Math. Ann. 294 (1992), 611-624.
[3] B. Bojarski, and T. Iwaniec, $p$-Harmonic equation and quasiregular mappings, Partial differential equations (Warsaw, 1984), 25-38, Banach Center Publ., vol. 19, PWN, Warsaw, 1987.
[4] L-F. Cheung and P-F. Leung, Some results on stable $p$-harmonic maps, Glasgow Math. J. 36 (1994) 77-80.
[5] J. Eells and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109-160.
[6] A. Fardoun, On equivariant $p$-harmonic maps, Ann.Inst. Henri. Poincare, 15 (1998), 25-72.
[7] G. Y. Jiang, 2-Harmonic maps between Riemannian manifolds, Annals of Math., China, 7A(4) (1986), 389-402.
[8] A. Mohammed Cherif, On the $p$-harmonic and $p$-biharmonic maps, J. Geom. (2018) 109:41
[9] T. Nagano and M. Sumi, Stability of $p$-harmonic maps, Tokyo J. Math. Vol. 15, No. 2, 1992.
[10] Y. Xin, Geometry of harmonic maps, Fudan University, 1996.