TWO APPLICATIONS OF THE INTEGRAL REGULATOR

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Abstract. We review Li’s refinement of the KLM regulator map, and use it to detect torsion phenomena in higher Chow groups.

1. Introduction

The KLM formula is a morphism of complexes inducing the Bloch-Beilinson regulator map with rational coefficients, developed by the first author together with J. Lewis and S. Müller-Stach [Ke1, KLM, KL] (see §3). The second author’s refinement now enables the direct computation of the integral regulator on the level of higher Chow complexes [Li]. In this note, we shall briefly review that construction (§4) and show how it may be used to find explicit torsion generators in higher Chow groups of number fields (§5). We also apply the formula to integrally calculate a branch of the higher normal function arising from the mirror of local $\mathbb{P}^2$ (§6).

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2. Higher Chow groups

Invented by Spencer Bloch [Bl1, Bl2] in the mid-1980s to geometrize Quillen’s higher algebraic $K$-theory, these generalize the usual Chow groups of cycles modulo rational equivalence (the $n = 0$ case). In particular, for $X$ smooth quasi-projective over an infinite field $k$, they satisfy

$$CH^p(X, n) \otimes \mathbb{Q} \cong Gr^n_kK^\text{alg}_n(X) \otimes \mathbb{Q}.$$
For such $X$, Voevodsky [Vo] proved they were *integ rally* isomorphic to his motivic cohomology groups:

$$\text{CH}^p(X, n) \cong H^{2p-n}_M(X, \mathbb{Z}(p)).$$

Beyond their role in arithmetic geometry (e.g. Beilinson’s conjectures [Be]), they have recently shown up in several branches of physics (e.g. quantum field theory [BKV1] and topological string theory [7K]) and mirror symmetry [DK2, BKV2]. We focus on the cubical presentation of $\text{CH}^p(X, n)$ as the $n^{th}$ homology of a complex of higher Chow precycles [Le]

$$\cdots \to Z^p(X, n + 1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n - 1) \to \cdots$$

or its (integrally quasi-isomorphic) subcomplex of normalized precycles [Bl4]

$$\cdots \to N^p(X, n + 1) \xrightarrow{\partial} N^p(X, n) \xrightarrow{\partial} N^p(X, n - 1) \to \cdots.$$  

A *higher Chow cycle* is an element of $\ker(\partial)$. Roughly speaking, these are relative codimension-$p$ cycles on

$$(X \times \mathbb{A}^n, X \times \bigcup \mathbb{A}^{n-1})$$

where the $\mathbb{A}^{n-1}$’s are inserted into $\mathbb{A}^n$ as a “cubical” configuration of hyperplanes. More precisely, writing

$$\square^n := (\mathbb{P}^1 \setminus \{1\})^n \supset \partial \square^n := \bigcup_i \{z_i = 0 \text{ or } \infty\}$$

we set

1. Normalized precycles may be represented (in $Z^p(X, n)$) by $Z$ satisfying $Z \cdot \{z_i = 0\} = 0 \ (\forall i)$ and $Z \cdot \{z_i = \infty\} = 0 \ (i < n)$ simply by adding an element of $d^p(X, n)$.  

$$c^p(X, n) := \{\text{cycles meeting faces of } X \times \partial \square^n \text{ properly}\}$$

$$d^p(X, n) := \{\text{cycles “constant” in some } z_i\}$$

$$Z^p(X, n) := c^p(X, n)/d^p(X, n)$$

$$N^p(X, n) := \{Z \mid Z \cdot \{z_i = 0\} = Z \cdot \{z_i = \infty\} = 0 \ (\forall i < n)\}$$
and for $Z \in Z^p(X, n)$ or $N^p(X, n)$,

$$\partial Z := \sum_{i=1}^{n} (-1)^i (Z \cdot \{z_i = \infty\} - Z \cdot \{z_i = 0\}).$$

If $X = \text{Spec}(k)$, write $Z^p(k, n)$ etc. for short.

**Example 2.1.** Parametrize a cycle in $N^2(\mathbb{Q}(\zeta_\ell), 3)$ by $t \in \mathbb{P}^1$:

$$Z_\ell^2 := \left(1 - \frac{\zeta_\ell}{t}, 1 - t, t - \ell \right).$$

Intersections with facets $\{z_i = 0, \infty\}$ are given by $t = 0, 1, \zeta_\ell, \infty$. But all these intersections have some $z_j = 1$, so are trivial (as $1 \notin \Box$). We also record the cycle

$$Z_5^2 := Z_1^2 + \left(1 - \frac{\zeta_5}{t}, 1 - t, t^{-5}\right) + \left(1 - \frac{\overline{\zeta}_5}{t}, 1 - t, t^{-5}\right)$$

in $N^2(\mathbb{Q}(\sqrt{5}), 3)$ for later reference.

### 3. Abel-Jacobi maps

These simultaneously generalize two classical invariants:

1. Griffiths’s AJ map $\text{Gr}$

   $$\text{CH}^p(X, 0) \to H^{2p}_\partial(X, \mathbb{Z}(p))$$

   for $X$ smooth projective over $\mathbb{C}$; and

2. $[A \text{-lift of}]$ Borel’s regulator map $\text{Bo2 \text{ Bo}}$

   $$\text{CH}^p(k, 2p - 1) \to \mathbb{C}/\mathbb{Z}(p)$$

   for $k \subset \mathbb{C}$ a number field.

Defined abstractly by Bloch $\text{Bl3}$, they map higher Chow groups to Deligne cohomology

$$\text{CH}^p(X, n) \xrightarrow{\text{AJ}_{p,n}} H^{2p-n}_\partial(X, \mathbb{Z}(p)).$$

\[^2\text{or (better) to absolute Hodge cohomology} \text{KL \text{ \S}2} \text{ in the smooth quasiprojective case.}\]
Kerr, Lewis, and Müller-Stach [KLM] constructed a morphism of complexes
\[ \widetilde{AJ}_{KLM}^{p,n} : Z_{\mathbb{R}}^p(X, -\bullet) \rightarrow C_{2p+}^{2p}(X, Z(p)) := C_{2p+}^{2p}(X; Z(p)) \oplus F_p D^{2p+}(X) \oplus D^{2p-1+}(X) \]
with differential \( D(\alpha, \beta, \gamma) = (-\partial \alpha, -d \beta, d \gamma - \beta + \alpha) \) on the right. For \( Z \in Z_{\mathbb{R}}^p(X, n) \) with projections \( \pi_1 \) (to \( \square^n \)) and \( \pi_2 \) (to \( X \)), they define
\[ \widetilde{AJ}_{KLM}^{p,n}(Z) := (2\pi i)^{p-n} (T_n, \Omega_n, R_n) \]
where \( T_n := \bigcap_{i=1}^n T_{z_i} = \mathbb{R}_{<0}^n \), \( \Omega_n := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \), and
\[ R_n := \log(z_1) \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n} - (-1)^n (2\pi i) R_{n-1} \cdot \delta_{T_{z_1}}. \]
Here \( \log(z) \) has a branch cut along \( T_z = \{ z \in \mathbb{R}_{<0} \} \), and \( R_1 = \log(z) \).
(Note that \( \widetilde{AJ}_{KLM}^{p,n} \) vanishes identically on \( d^p(X, n) \).

The subcomplex \( Z_{\mathbb{R}}^p(X, -\bullet) \subset Z^p(X, -\bullet) \) consists of cycles \( Z \) for which \( Z^{an} \) properly intersects the various combinations of \( \{ T_{z_i} \} \) and \( \{ z_j = 0, \infty \} \). We call such precycles \( \mathbb{R} \)-proper. Kerr and Lewis [KL] proved the inclusion is a rational quasi-isomorphism, by appealing to Kleiman transversality in \( K \)-theory. Unfortunately, the claimed integral moving lemma in [KLM] (which would have made this quasi-isomorphism integral) was incorrect, and [KL] was only written after a prolonged effort to repair the integral version.

Now suppose we have a cycle \( Z \in \ker(\partial) \subset Z_{\mathbb{R}}^p(X, n) \) with
\[ [\widetilde{AJ}_{KLM}^{p,n}(Z)] \in H_{2p-n}^{2p}(X, Z(p)) \]
torsion of order \( M \). This implies \( [Z] \in H_n \{ Z_{\mathbb{R}}^p(X, \bullet) \} \) is at least of this order. But for \( [Z] \in H_n \{ Z^p(X, \bullet) \} = CH^p(X, n) \), it means no such thing: there could be a \( W \in Z^p(X, n+1) \setminus Z_{\mathbb{R}}^p(X, n+1) \) with \( \partial W = Z \). So the KLM map only induces a homomorphism
\[ AJ_{\mathbb{Q}}^{p,n} : CH^p(X, n) \rightarrow H_{2p-n}^{2p}(X, \mathbb{Q}(p)) \]
consistent with Bloch’s \( AJ^{p,n} \). This is frustrating, as the KLM formulas are well-adapted to detecting torsion!
For $X = \text{Spec}(k)$ and $(p, n) = (2, 3)$, consider the portion
\[
\cdots \to Z^2_R(k, 4) \to Z^p_R(k, 3) \to Z^p_R(k, 2) \to \cdots
\]
of the KLM map of complexes. We want to use the middle map to detect torsion. Denote its image on a cycle $Z$ by $R(Z) \in C/\mathbb{Z}(2)$.

**Example 3.1** (Petras [Pe]). We calculate $R(Z^2) = \frac{1}{2\pi i} \int_{Z^2} R_3 = \frac{1}{2\pi i} \int_{Z^2} \left( \log(z_1)dz_2/z_2 \wedge dz_3/z_3 + (2\pi i)\log(z_2)dz_3/z_3 \cdot \delta_{T_{z_1}} + (2\pi i)^2 \log(z_3)\delta_{T_{z_1} \cap T_{z_2}} \right) = \int_{Z_t \cap T_{z_1}} \log(z_2) \frac{dz_3}{z_3} = -\int_{T_{1 - \zeta \ell}} \log(1 - t) \frac{dt}{t} = -\int_0^\zeta \log(1 - t) \frac{dt}{t} = \text{Li}_2(\zeta\ell)$.

For $\ell = 1$, this is $\frac{\pi^2}{6} \in \mathbb{C}/\mathbb{Z}(2)$, which is 24-torsion, while (for the second cycle of Example 2.1) $R(Z_5^2) = \text{Li}_2(1) + \text{Li}_2(\zeta_5) + \text{Li}_2(\overline{\zeta_5}) = \frac{7\pi^2}{30}$ is 120-torsion. To deduce that these orders of torsion exist in $CH^2(\mathbb{Q}, 3)$ resp. $CH^2(\mathbb{Q}(\sqrt{5}), 3)$, we need an improvement in technology.

**4. The integral regulator**

A few basic strategies come to mind:

1. proving an integral moving lemma ($Z^p_R(X, \bullet) \xrightarrow{\sim} Z^p(X, \bullet)$)

2. extending KLM to a map of complexes on $Z^p(X, \bullet)$

are probably too naive;

3. extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $Z^p(X, \bullet)$

seemed promising; but what ultimately worked was

4. extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $N^p(X, \bullet)$. 
The heuristic idea of (3) was to perturb the branch cuts $T_{z_i} = \{ z_i \in \mathbb{R}_{<0} \}$ in $\log(z_i)$ to $T_{z_i}^\epsilon = \{ z_i/e^\epsilon \in \mathbb{R}_{<0} \}$ and take a limit as $\epsilon \to 0$, an approach that had been successfully applied in [Ke2, §9]. Unfortunately, there are cycles in $Z^2(\mathbb{C}, 3)$ whose intersection with $T_{z_1}^\epsilon \cap T_{z_2}^\epsilon \cap T_{z_3}^\epsilon$ is improper for every real $\epsilon$ near 0 [Li, §3]. So we need to deform the branches by distinct $\{ \epsilon_i \}$; but then we cannot expect a morphism of complexes (or “limit” thereof) on $Z_p(X, \bullet)$. This forces us into strategy (4), and working with normalized subcomplexes.

Let $B_\epsilon$ denote the set of infinite sequences $\{ \epsilon_i \}_{i>0}$, with

$$0 < \epsilon_1 < \epsilon, \quad 0 < \epsilon_2 < e^{-1/\epsilon_1}, \quad 0 < \epsilon_3 < e^{-1/\epsilon_2}, \quad \text{etc.,}$$

so that when $\epsilon \to 0$ its projection to any $(S^1)^n$ eventually avoids any given analytic subvariety. Let $N^p_\epsilon(X, \bullet) \subset N^p(X, \bullet)$ denote the (nested) subcomplexes of cycles $Z$ with $Z^\text{an}$ properly intersecting (for each $\epsilon \in B_\epsilon$) certain combinations of $\{ T_{z_i}^\epsilon \}$ and $\{ z_j = 0, \infty \}$.

**Lemma 4.1** ([Li], Thms. 4.2 and 7.2). We have

$$\bigcup_{\epsilon>0} N^p_\epsilon(X, n) = N^p(X, n) \quad (\forall n)$$

and

$$\lim_{\epsilon \to 0} H_n(N^p_\epsilon(X, \bullet)) \cong H_n(N^p(X, \bullet)) \cong CH^p(X, n).$$

For any $\epsilon \in B_\epsilon$, replacing $T_{z_i}$ by $T_{z_i}^\epsilon$ everywhere in the KLM formula yields a morphism of complexes

$$\widetilde{AJ}^p_\epsilon(\bullet) : N^p_\epsilon(X, -\bullet) \longrightarrow C^{2p+\bullet}(X, \mathbb{Z}(p)).$$

**Lemma 4.2** ([Li], Thm. 6.1). Given $\epsilon, \epsilon' \in B_\epsilon$, $\widetilde{AJ}^p_\epsilon$ and $\widetilde{AJ}^p_{\epsilon'}$ are (Z-)homotopic.

**Sketch.** Truncating at some $N$, we may view

$$R^\epsilon_n = \{ R^\epsilon_n := ((2\pi i)^n T_{z_i}^\epsilon, \Omega_n, R_{\Omega_n}^z) \}_{n, \epsilon}^\mathcal{R}.$$
\[0 \leq n \leq N; \{\epsilon_1, \ldots, \epsilon_n\} \subset \{\epsilon_1, \ldots, \epsilon_N\}\] subsequence) as a 0-cocycle in the double complex

\[E^{a,b} = C^{2a+b}((\mathbb{P}^1)^a) \otimes (N^a) \delta_{\text{Gysin}}, D_\varphi.\]

Construct a \((-1)\)-cochain \(S_{\epsilon, \epsilon'}^{\square}\) with \(D S_{\epsilon, \epsilon'}^{\square} = R_\epsilon - R_{\epsilon'}\), and with respect to whose wavefront set the precycles in \(N^p(X, \bullet)\) remain proper. \(\square\)

We therefore have well-defined, compatible maps

\[AJ_{\epsilon, n}^p : H_n(N_{\epsilon}^p(X, \bullet)) \rightarrow H_{2p-n}^{2p-n}(X, \mathbb{Z}(p))\]

for each \(\epsilon > 0\), essentially given by \(\lim_{\epsilon \to 0} \tilde{AJ}_{\epsilon, \epsilon}^p\) (where the limit is taken so that \(\epsilon > \epsilon_1 \gg \epsilon_2 \gg \epsilon_3 \gg \cdots > 0\)), and recovering \(\tilde{AJ}_{KLM}^p\) on \(N_{\epsilon}^p(X, \bullet) := Z_{\epsilon}^p(X, \bullet) \cap N^p(X, \bullet)\).

More precisely:

**Theorem 4.3** ([Li], §7). The \(\{\tilde{AJ}_{\epsilon, \epsilon}^{p, n}\}\) induce a homomorphism

\[AJ_{\epsilon, n}^p : CH^p(X, n) \cong \lim_{\epsilon \to 0} H_n(N_{\epsilon}^p(X, \bullet)) \rightarrow H_{2p-n}^{2p-n}(X, \mathbb{Z}(p))\]

factoring \(AJ_{\epsilon, n}^p\).

This theorem has the

**Corollary 4.4.** If a class \(\xi \in CH^p(X, n)\) is represented by

\[Z \in \ker(\partial) \subset N_{\epsilon}^p(X, n) \left(\subset \bigcup_{\epsilon > 0} N_{\epsilon}^p(X, n)\right),\]

then

\[AJ_{\epsilon, n}^p(\xi) = \lim_{\epsilon \to 0} \tilde{AJ}_{\epsilon}^{p, n}(Z) = \tilde{AJ}_{KLM}^{p, n}(Z).\]

So the KLM formula holds verbatim on normalized, \(\mathbb{R}\)-proper representatives, validating the deductions at the end of Example 3.1. It is this statement that we (primarily) use in the applications that follow.

5. **Torsion generators**

Let \(\mu_{\infty} = \bigcup_{m \in \mathbb{N}} \mu_m \subset \mathbb{C}^*\) denote the roots of unity, and \(w_\tau(k) := \left|\left(\mu_{\infty}^{\mathbb{Q}}\right)^{\text{Gal}(\mathbb{Q}/k)}\right|\) for any number field \(k \subset \mathbb{C}\). By the universal coefficient
sequence for motivic cohomology

\[ H^0_M(k, \mathbb{Z}(r)) \rightarrow H^0_M(k, \mathbb{Z}/m\mathbb{Z}(r)) \rightarrow H^1_M(k, \mathbb{Z}(r)) \rightarrow H^1_M(k, \mathbb{Z}(r)) \rightarrow H^2_M(k, \mathbb{Z}(r)) \rightarrow \cdots \]

and vanishing of \( H^0_M(k, \mathbb{Z}(r)) \) (we have

\[ CH^r(k, 2r - 1)[m] \cong H^1_M(k, \mathbb{Z}(r))[m] \cong H^0_M(k, \mathbb{Z}/m\mathbb{Z}(r)). \]

Since the norm residue map

\[ H^0_M(k, \mathbb{Z}/m\mathbb{Z}(r)) \rightarrow H^0_{\text{ét}}(k, \mu_m^r) \cong (\mu_m^r)^{\text{Gal}(\overline{\mathbb{Q}}/k)} \]

is an isomorphism by a celebrated theorem of Rost-Voevodsky (cf. [HW]), we conclude that \( CH^r(k, 2r - 1)[m] \cong \mathbb{Z}/(m, w_r(k))\mathbb{Z} \) hence

\[ CH^r(k, 2r - 1)_{\text{tors}} \cong \mathbb{Z}/w_r(k)\mathbb{Z}. \]

**Example 5.1.** If \( k = \mathbb{Q} \), one has \(^5\) \( w_{2n}(k) = \) denominator of \( \frac{|B_{2n}|}{4n} \) (written in lowest terms) and \( w_{2n+1} = 2 \) for \( n \geq 1 \); so

\[ CH^2(\mathbb{Q}, 3) \cong \mathbb{Z}/24\mathbb{Z}, \ CH^3(\mathbb{Q}, 5)_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}, \ CH^4(\mathbb{Q}, 7) \cong \mathbb{Z}/240\mathbb{Z}. \]

For real quadratic fields \( k = \mathbb{Q}(\sqrt{d}) \), the situation is more complicated (cf. [We, VI.2]); one computes for instance

| \( d \) | \( 2 \) | \( 3 \) | \( 5 \) | \( 7 \) |
|---|---|---|---|---|
| \( w_2(\mathbb{Q}(\sqrt{d})) \) | 48 | 24 | 120 | 24 |
| \( w_4(\mathbb{Q}(\sqrt{d})) \) | 480 | 240 | 240 | 240 |

so that only \( CH^2(\mathbb{Q}(\sqrt{2}), 3) \) and \( CH^2(\mathbb{Q}(\sqrt{5}), 3) \) [resp. \( CH^4(\mathbb{Q}(\sqrt{2}), 7) \)] are different from the \( k = \mathbb{Q} \) case. Finally, for cyclotomic \( k = \mathbb{Q}(\zeta_p) \) (\( p \geq 5 \) prime) one can show that \( w_3(\mathbb{Q}(\zeta_p)) = 2p \), while \( w_3(\mathbb{Q}(\zeta_3)) = 18 \).

For computing torsion orders of images under

\[ AJ_{\mathbb{Z}}^{2r-1} : H_{2r-1}(N^r_{\mathbb{Z}}(k, \bullet)) \rightarrow \mathbb{C}/(2\pi i)^r\mathbb{Z} \]

\[ Z \mapsto \frac{1}{(2\pi i)^{r-1}} \int_Z R_{2r-1} =: \mathcal{R}(Z) \]

we use the following basic calculation:

\(^4\)See [We, Ex. VI.4.6]: since \( H^1_M(k, \mathbb{Z}/m\mathbb{Z}(r)) \cong H^1_{\text{ét}}(k, \mu_m^r) = \{0\} \), \(-m\) is injective on \( H^0_M(k, \mathbb{Z}(r)) \) (universal coefficient sequence), which is thus torsion-free; it has rank 0 since \( H^0_M(k, \mathbb{Z}(r)) \cong \text{Gr}_{r, k}(2, k) \otimes \mathbb{Q} = \{0\} \) by Borel’s theorem [Bo1].

\(^5\)Bernoulli numbers: \( |B_{2n}| = \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \ldots \) for \( n = 1, 2, 3, \ldots \).
Proposition 5.2. Suppose that for a given \( r \in \mathbb{N} \) there exists a collection of closed precycles \( Z_{\ell,a}^r \in N_R^r(\mathbb{Q}(\zeta_\ell), 2r - 1) \) with

\[
\mathcal{R}(Z_{\ell,a}^r) = (r - 1)!^{r-1} \text{Li}_r(\zeta_\ell^a).
\]

Then for \( Z := \sum_{a=0}^{\ell-1} f(a)Z_{\ell,a}^r \) with \( f(-a) = (-1)^r f(a) \), \( AJ(Z) \) is torsion of order given by the denominator of

\[
\tau(Z) := \left| (2\pi i)^{-r} \mathcal{R}(Z) \right| = \pm \frac{\ell^{r-1} \ell - 1}{2r} \sum_{a=0}^{\ell-1} f(a)B_r(\frac{a}{\ell}).
\]

Proof. By [Ke3, Thm. 3.9],

\[
\sum_{a=0}^{\ell-1} f(a)\text{Li}_r(\zeta_\ell^a) = \frac{1}{2} \sum_{a=0}^{\ell-1} f(a) \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{\zeta_\ell^{ka}}{k^r} =: \frac{(-1)^r}{2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{f}(k) \frac{\zeta_\ell^r}{k^r} =: \frac{(-1)^r}{2} \hat{L}(\hat{f}, r)
\]

\[
= \frac{(2\pi i)^r}{2 \cdot r!} \sum_{a=0}^{\ell-1} f(a)B_r(\frac{a}{\ell}),
\]

where \( B_r(\cdot) \) are the Bernoulli polynomials. \( \Box \)

In practice, the \( \{Z_{\ell,a}^r\} \) will be obtained from a single cycle \( Z_{\ell,1}^r = Z_{\ell,1}^r \) by Galois conjugation. For \( r = 2 \), we already have this from Examples 2.1 and 3.1.

Now \( CH^r(k, 2r - 1) = CH^r(k, 2r - 1)_{\text{tors}} \iff r \) is even and \( k \) is totally real. In particular, assuming Prop. 5.2’s hypothesis, we obtain generators of \( CH^{2n}(k, 4n - 1) \) as follows:

- \( k = \mathbb{Q} \), any \( n \) : \( \tau(Z_1^{2n}) = \frac{B_{2n}}{4n} = \frac{1}{24}, \frac{1}{240}, \ldots \);
- \( k = \mathbb{Q}(\sqrt{2}) \), \( n = 1, 2 \) : \( \tau(Z_8^{2n} + Z_8^{2n}) = \frac{11}{48}, \frac{1313}{480} \);
- \( k = \mathbb{Q}(\sqrt{3}) \), \( n = 1 \) : \( \tau(Z_5^{2n} + Z_5^{2n}) = \frac{7}{120} \).

For \( CH^3(k, 5)_{\text{tors}} \) one computes for example

- \( k = \mathbb{Q}(\zeta_3) \) : \( \tau(Z_{3,1}^3 - Z_{3,2}^3) = \frac{1}{5} \) and
- \( k = \mathbb{Q}(\zeta_5) \) : \( \tau(Z_{5,1}^3 - Z_{5,4}^3) = \frac{2}{5} \),

which miss only the 2-torsion element from \( CH^3(\mathbb{Q}, 5) \leftrightarrow CH^3(k, 5) \).

(So far we have no \( N_k^3(\mathbb{Q}, 5) \) representative for this element.)

\[ B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \text{ etc.} \]
It remains to construct the cycles of the Proposition for \( r = 3, 4 \).
From \([KY \ S4.2]\), for \( r = 3 \) we have\(^7\)

\[
Z^3_t := -2 \left( \frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{t_3}{t_3 - 1}, 1 - \zeta t_1 t_2 t_3, t^t_1, t^t_2, t^t_3 \right)
\]

\[
- \left( \frac{t}{t - 1}, \frac{1}{1 - \zeta t}, \frac{(u - t^t)(u - t^t)}{(u - 1)^2}, t^t u, \frac{u}{t^t} \right)
\]

which is normalized since all “boundaries” occur in the third coordinate. We have \( R(Z^3_t) = 2Li_3(\zeta t) \) by \([op. cit., Tm. 3.6]\), with only the first term contributing. (This gives in particular \( R(Z^3_1) = 2Li_3(1) = 2\zeta(3) \).

For \( r = 4 \), the first construction in \([KY \ S4.3]\) would be in \( N^4_R(Q(\zeta), 7) \), but there is an error in the computation of the boundary of the last component \( \mathcal{H}_2 \); in fact, it is degenerate\(^8\) and the cycle \( \tilde{Z} \) is therefore not closed. A correct application of the strategy in \([op. cit., S3.1]\) yields \( Z^4_t := \)

\[
6 \left( \frac{t_1}{t_1 - 1} - 1, \frac{t_2}{t_2 - 1} - 1, \frac{t_3}{t_3 - 1} - 1, 1 - \zeta t_1 t_2 t_3, t^t_1, t^t_2, t^t_3 \right)
\]

\[
+ \left( \frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta t_1 t_2}, \frac{(u - t^t)(u - t^t)(u - t^t)}{(u - 1)^3}, t^t_1, t^t_2, \frac{1}{u}, \frac{1}{u}, \frac{1}{ut^t_1 t^t_2} \right)
\]

\[
+ \left( \frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta t_1 t_2}, \frac{(u - t^t)(u - t^t)(u - t^t)}{(u - 1)^3}, t^t_1, t^t_2, \frac{1}{u}, \frac{1}{u}, \frac{1}{ut^t_1 t^t_2} \right)
\]

\[
+ \left( \frac{(v - u)(v - u)}{(v - 1)^2}, \frac{t}{i - 1}, \frac{1}{1 - \zeta t}, \frac{(u - t^t)(u - t^t)}{(u - 1)^2}, \frac{1}{u}, \frac{1}{u}, \frac{1}{ut^t} \right)
\]

\[
+ \left( \frac{t}{t - 1}, \frac{(v - u)(v - u)}{(v - 1)^2}, \frac{1}{1 - \zeta t}, \frac{(u - t^t)(u - t^t)}{(u - 1)^2}, \frac{1}{u}, \frac{1}{u}, \frac{1}{ut^t} \right)
\]

\[
+ \left( \frac{t}{t - 1}, \frac{1}{1 - \zeta t}, \frac{(v - u)(v - u)}{(v - 1)^2}, \frac{(u - t^t)(u - t^t)}{(u - 1)^2}, \frac{1}{u}, \frac{1}{u}, \frac{1}{ut^t}, v \right)
\]

\(^7\)The components are parametrized by \((t_1, t_2)\) and \((t, u)\) respectively.

\(^8\)i.e. belongs to \( d^4(Q(\zeta), 7) \), as can be seen by substituting \( v = uw \).
which belongs to \( \ker(\partial) \cap N^4_K(\mathbb{Q}(\zeta_8), 7) \). Only the first term contributes to \( \mathcal{R}(Z) = -6\epsilon^3 f_{[0,1]} \log(1 - \zeta t_1 t_2 t_3) \frac{dt_1}{t_1} \land \frac{dt_2}{t_2} \land \frac{dt_3}{t_3} = 6\epsilon^3 \text{Li}_4(\zeta_8) \), see [op. cit., §3.2].

**Remark 5.3.** An example of a cycle for which the log-branch perturbations are required for the integral regulator computation is \( Z := Z_- - Z_+ := \left( \frac{(z - i)}{z + i}, \frac{(z - 1)}{z + 1} \right) \) (parametrized by \( z \in \mathbb{P}^1 \)) in \( N^2(\mathbb{Q}(i), 3) \). Indeed, \( T_{z^2} [\text{resp. } T_{(\frac{z+1}{z+1})^2}] \) has support on \( i\mathbb{R} [\text{resp. the unit circle } S^1] \), so that the triple intersection (essentially \( i\mathbb{R} \cap S^1 \cap S^1 \)) is nonempty. Though this cycle is non-torsion, we briefly describe the computation. After making the deformation, \( T_{(\frac{z+1}{z+1})^2} \) and \( T_{(\frac{z-1}{z+1})^2} \) intersect twice with opposite orientations, at points near \( i \) and \(-i\) with phase just greater than \( \frac{\pi}{2} \) resp. \( \frac{3\pi}{2} \). Since \( \epsilon_3 \to 0 \) much faster than \( \epsilon_1 \) and \( \epsilon_2 \), in the limit the

\[
2\pi i \int_{Z_+} \log^{\epsilon_3}(z^2) \delta_{T^{\epsilon_1}_{(\frac{z+1}{z+1})^2}} \wedge T^{\epsilon_2}_{(\frac{z-1}{z+1})^2}
\]

term of \( \frac{1}{2\pi i} \int_{Z_+} R_3 \) contributes \( 2\pi i (\pi i - \pi i) = 0 \). The remaining term yields

\[
2 \int_{Z_+} \log^{\epsilon_2} \left( \frac{1 - z}{1 + z} \right) \frac{dz}{z} \delta_{T^{\epsilon_1}_{(\frac{z+1}{z+1})^2}} ,
\]

where \( T^{\epsilon_1}_{(\frac{z+1}{z+1})^2} \) consists of two paths from \(-i\) to \( i \), along which one checks that (in the limit) \( \log^{\epsilon_2} \left( \frac{1 - z}{1 + z} \right) = 2 \log^{\epsilon_2}(1 - z) - 2 \log^{\epsilon_2}(1 + z) \); and so (5.3) becomes \( 8 \int_{-1}^{1} \log(1 - z) \frac{dz}{z} - 8 \int_{-1}^{1} \log(1 + z) \frac{dz}{z} \). Combining this with the portion from \( Z_- \), we obtain

\[
\mathcal{R}(Z) = 32 \text{Li}_2(i) - 32 \text{Li}_2(-i) = 64iL(\chi_4, 2) \in \mathbb{C}/\mathbb{Z}(2).
\]

**6. Local \( \mathbb{P}^2 \) revisited**

For a reflexive polytope \( \Delta \subset \mathbb{R}^2 \) with polar polytope \( \Delta^\circ \), the mirror of \( K_{\mathbb{P}_\Delta^\circ} \) (“local \( \mathbb{P}_{\Delta^\circ} \)” can be identified with a family of \( CH^2(\cdot, 2) \)-elements on a family of anticanonical (elliptic) curves in \( \mathbb{P}_{\Delta} \) [DKT §5].
As a second application of the integral regulator, we show how to apply it to compute the correct “torsion term” in the higher normal function associated to one of these families. That is, if $\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1$ is smooth away from $\Sigma = \{0\} \cup \Sigma^*$, with fibers $E_t = \pi^{-1}(t)$, and $\Xi \in CH^2(\mathcal{E} \setminus E_0, 2)$ has fiberwise restrictions $\xi_t \in CH^2(E_t, 2)$ ($t \not\in \Sigma$), we shall compute

$$R_t := AJ^2, 2(\xi_t) \in H^1(E_t, \mathbb{C}/\mathbb{Z}(2)) \cong \text{Hom} \left( H_1(E_t, \mathbb{Z}), \mathbb{C}/(2\pi i)^2\mathbb{Z} \right) \in \text{H}_2(E_t, \mathbb{C}/\mathbb{Z}(2)) \cong \text{Hom} \left( \text{H}_1(E_t, \mathbb{Z}), \mathbb{C}/(2\pi i)^2\mathbb{Z} \right)$$

in a neighborhood of $t = 0$. Writing $\{\omega_t\}$ for a section of $\omega_{\mathcal{E}/\mathbb{P}^1}$ vanishing at $\infty$, the constant term in (a branch of) the resulting truncated higher normal function

$$\nu(t) := \frac{1}{2\pi i} (\omega_t, R_t)$$

will play a role in forthcoming work of the first author with C. Doran on quantum curves.

To begin in a somewhat more general scenario, let $\Delta \subset \mathbb{R}^2$ be any convex polytope with integer vertices $\{p_i = (a_i, b_i)\}_{i=1}^N$ and interior integer points $\{(v_j, w_j)\}_{j=1}^g$. Define a multiparameter family

$$\rho : C \rightarrow C^g, \quad C_\Delta := \rho^{-1}(\Delta)$$

of (where smooth) genus $g$ curves by taking the Zariski closure of $C^* := \{(x, y; \lambda_1, \ldots, \lambda_g) \mid 0 = \Phi_\Delta(x, y) := \phi(x, y) - \sum_{j=1}^g \lambda_j x^{v_j} y^{w_j} \} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^g$

in $\mathbb{P}_\Delta \times \mathbb{C}^g$, where $\phi(x, y) := \sum_{(a, b) \in \partial \Delta \setminus \mathbb{Z}^2} m_{a, b} x^a y^b$ is uniquely determined by requiring its edge polynomials to be powers of $(t + 1)$. (If the edges of $\Delta$ have no interior points, then $\phi(x, y) = \sum_{i=1}^N x^{a_i} y^{b_i}$.) The symbol $\{-x, -y\}$ represents a closed precycle $\Xi^* \in Z^2(C^*, 2)$ parametrized (in $C^* \times \square^2$) by $(x, y; \lambda, -x, -y)_{(x, y, \lambda) \in C^*}$.

**Lemma 6.1.** The class of $\Xi^*$ in $CH^2(C^*, 2)$ is the restriction of a class $\Xi \in CH^2(C, 2)$.

**Proof.** We need only check that the Tame symbol of $\{-x, -y\}|_{C_\Delta} \in K^M_2(C(C_\Delta))$ is zero for general $\Delta$. The symbol $\{-x, -y\}$ is invariant
under unimodular change of toric coordinates, so we may assume that
(after shifting \(\Delta\) by \((-a_m, -b_m)\) for some \(m\)) we have a picture

\[
\Delta
\]

where the bottom edge corresponds to the toric divisor at whose inter-
section with \(C_\Delta\) we wish to compute \(\text{Tame}(\{-x, -y\} |_{C_\Delta^*}) \in \mathbb{C}^*.\) Since
the edge polynomial is \((1 + x)^c\), this intersection occurs at \((-1, 0)\), so
the Tame symbol is \(1\). □

Now set \(R_{i\lambda} := AJ_{2,2}(\Xi |_{C_\Delta^*}) \in \text{Hom}(H_1(C_\Delta, \mathbb{Z}), \mathbb{C}/\mathbb{Z}(2))\). Picking any
vertex \(p_m\) of \(\Delta\), we can (via unimodular coordinate change) put it in
the position (6.1). In the new coordinates (still denoted \((x, y)\), \(C_\Delta\) is
cut out by an equation of the form

\[
0 = \tilde{\Phi}_\lambda := x^{-a_m} y^{-b_m} \Phi(x, y)
\]

\[
= (1 + x)^{\kappa_m} + y\{\Psi_m(x, y) - \sum_{j=1}^g \lambda^j x^{v_j - a_m} y^{w_j - b_m - 1}\},
\]

and acquires a node at \((0, 0)\) as \(\lambda_\ell \to \infty\). (Note that \(\ell\) is determined
by \(m\).) The corresponding vanishing cycle \(\alpha_m\) has image \(|x| = |y| = \epsilon\)
under \(H_1(C_\Delta, \mathbb{Z}) \xrightarrow{\text{Tube}} H_2(\mathbb{P}_\Delta \setminus C_\Delta)\) for large \(|\lambda_\ell|\).

**Proposition 6.2.** For \(i\lambda_\ell \in \Delta\) and \(|\lambda_\ell| \gg 0\), and \(\lambda_{j \neq \ell}\) sufficiently
small\(^{10}\) we have

\[
\mathcal{R}_\lambda(\alpha_m) = 2\pi i (-\log(\lambda_\ell) + \sum_{k \geq 1} \frac{1}{k} [\Psi_{k,\lambda,\ell}]_0) \in \mathbb{C}/\mathbb{Z}(2),
\]

where \(\Psi_{k,\lambda,\ell} := \frac{1}{\lambda_\ell} (x^{-v_\ell} y^{-w_\ell} \Phi(x, y) + \lambda_\ell)\) and \([\cdot]_0\) takes the constant term in
a Laurent polynomial.

**Proof.** We use the notation \(R\{f_1, f_2\} = \log(f_1) \frac{df_1}{f_1} - 2\pi i \log(f_2)\delta_{T_1}\) and
\(R\{f_1, f_2, f_3\} = \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} + 2\pi i \log(f_2) \frac{df_3}{f_3}\delta_{T_1} + (2\pi i)^2 \log(f_3) \delta_{T_1 \cap T_2}\)

\(^9\)that is, replacing \(x, y\) by \(x^a y^b, x^c y^d\) with \(ad - bc = 1\); the \(a_i, b_i, v_i, w_i\) are changed accordingly.

\(^{10}\)e.g. if \(c := |\Delta \cap \mathbb{Z}^2|\), then \(|\lambda_\ell| > \frac{c}{c^2}\) and \(|\lambda_{j \neq \ell}| < \frac{1}{c^2}\) will do.
for $R_2$ and $R_3$ with $f_i$ replacing $z_i$. Writing $D$ for the bottom-edge divisor in (6.1), we have

$$T_{\text{ame}}\{\tilde{\Phi}_{\lambda}, -x, -y\} = \{\tilde{\Phi}_{\lambda}(x, 0), -x\} = \{(1 + x)^\epsilon, -x\}(= 1).$$

So writing $\Gamma = \{|x| = \epsilon \geq |y|\} \implies \alpha_m = \Gamma \cap C_{\lambda}$ gives $R_t(\alpha_m) = \int_{\alpha_m} R\{-x, -y\} = \int_\Gamma R\{-x, -y\} \cdot \delta_{C_{\lambda}}$

$$= \frac{-1}{2\pi i} \int_\Gamma d[R\{\tilde{\Phi}_{\lambda}, -x, -y\}] - \int_\Gamma R\{(1 + x)^\epsilon, -x\} \cdot \delta_D$$

$$= \frac{-1}{2\pi i} \int_{\partial \Gamma} R\{\tilde{\Phi}_{\lambda}, -x, -y\} - \int_{|x| = \epsilon} R\{(1 + x)^\epsilon, -x\}$$

$$= \frac{-1}{2\pi i} \int_{|x| = |y| = \epsilon} R\{x^{-\epsilon}y^{-\epsilon}\tilde{\Phi}_{\lambda}, -x, -y\}$$

$$= \frac{-1}{2\pi i} \int_{|x| = |y| = \epsilon} \{\log(\lambda) + \log(1 - \Psi_{\lambda, \epsilon})\} \frac{dx}{x} \wedge \frac{dy}{y}$$

$$= -2\pi i \log(\lambda) + 2\pi i \sum_{k \geq 1} \int_{|x| = |y| = \epsilon} \Psi_{\lambda, \epsilon} \frac{k}{x} \wedge \frac{dy}{y}$$

modulo $\mathbb{Z}(2)$. Here only the first term of $R_3$ enters since $T_{\lambda(1 - \Psi)} \cap |x| = |y| = \epsilon$ is empty under the given assumptions.

Returning to the more specific scenario at the beginning of this section, if $g = 1$ and $\lambda_1 =: \lambda =: \frac{1}{t}$, then $\Phi_{\lambda} = \phi(x, y) - \lambda$ and $\Psi_{\lambda, 1} = t\phi(x, y)$, so that (writing $R_t$ instead of $R_{\lambda}$), Prop. 6.2 yields:

**Corollary 6.3.** If $\Delta$ is reflexive, then the $\alpha_m$ are all homologous (=: $\alpha$), and $R_t(\alpha) \equiv \frac{2\pi}{\mathbb{Z}(2)} \left(\log(t) + \sum_{k \geq 1} \frac{[\phi_k, 0]}{k}\right)$ for $t$ small in the right-half-plane.

It remains to compute $R_t(\beta)$ for a cycle $\beta$ complementary to $\alpha$ (so that $\mathbb{Z}\langle \alpha, \beta \rangle = H_1(E_t, \mathbb{Z})$, which we shall do for the local $\mathbb{P}^2$ setting only: $\Delta$ the convex hull of $\{(1, 0), (0, 1), (-1, -1)\}$, and $\phi = x + y + x^{-1}y^{-1}$. Taking $t > 0$ small, write $(0 <) x_0(t) < x_-(t) < x_+(t) < \infty$ for the branch points of

$$E_t(=\lambda^{-1}) : y^2 + (x - \lambda)y + x^{-1} = 0$$
over $\mathbb{P}^1_x$, and $y^\pm(x) = \frac{1}{2} \{(\lambda - x) \pm \sqrt{(x - \lambda)^2 - 4x^{-1}}\}$. Then $\beta$ [resp. $\alpha$] is given by the difference of paths (on the two branches) between $x_0(t)$ and $x_-(t)$ [resp. $x_-(t)$ and $x_+(t)$].

Now $T_x^- = \mathbb{R}_{>0} \subset \mathbb{P}^1_x$, so taking the $y^+$- [resp. $y^-$] branch of $\beta$ to run from $x_0$ to $x_-$ [resp. $x_-$ to $x_0$] in $\mathcal{S}$ [resp. $\mathcal{S}$], we have $\beta \cap T_x^- = (x_0, y_0) \cup (x_-, y_-)$; moreover, $\log(-x) = \log(x) \mp i\pi$ on the $y^\pm$-branch of $\beta$. The upshot is that

$$
\int_{\beta} R\{-x, -y\}|_{E_t} = \int_{\beta} \log(-x) \frac{dy}{y} - 2\pi i \sum_{\beta \cap T_x^-} \log(y)
$$

$$
= -\int_{x_0(t)}^{x_-(t)} \log(x) \frac{dy}{y} \frac{y^+(x)}{y^-(x)}
$$

$$
= \int_{x_0(t)}^{x_-(t)} \log \left( \frac{y^+(x)}{y^-(x)} \right) \frac{dx}{x}
$$

$$
= \int_{x_0(t)}^{x_-(t)} \log \left( \frac{1 + \sqrt{1 - \frac{4t^2}{x(1 - xt)^2}}}{1 - \sqrt{1 - \frac{4t^2}{x(1 - xt)^2}}} \right) \frac{dx}{x}
$$

where $\xi = \frac{4t^2}{x(1 - xt)^2}$. Writing for $\xi \in (0, 1)$

$$
\log \left( \frac{1 + \sqrt{1 - \frac{4t^2}{x(1 - xt)^2}}}{1 - \sqrt{1 - \frac{4t^2}{x(1 - xt)^2}}} \right) =: -\sum_{m \geq 1} \alpha_m \xi^m,
$$

the above integral decomposes into

$$
-2 \log(t) \int_{x_0}^{x_-} \frac{dx}{x} + \int_{x_0}^{x_-} \log(x) \frac{dx}{x} + 2 \int_{x_0}^{x_-} \log(1 - xt) \frac{dx}{x} - \sum_{m \geq 1} \alpha_m \int_{x_0}^{x_-} \xi^m \frac{dx}{x}.
$$

Using the approximations $x_0 \simeq 4t^2(1 + 8t^3)$ and $x_- \simeq t^{-1}(1 - 2t^2 - 2t^3)$, a lengthy direct computation gives that

$$
\mathcal{R}(\beta) = \frac{9}{2} \log^2(t) - \frac{a^2}{2} + \mathcal{O}(t \log(t)).
$$

Let $\delta_t := t \frac{d}{dt}$. By a general result of [DK1], one knows that $\nabla_{\delta_t} \mathcal{R} = \{[\omega_t]\}$, where

$$
\omega_t := \text{Res}_{E_t} \left( \frac{dx \wedge dy}{1 - t\phi(x, y)} \right)
$$

has its periods $\omega_t(\gamma) = \int_{\gamma} \omega_t$ annihilated by the Picard-Fuchs operator

$$
\mathcal{L} = \delta_t^2 - 27t^3(\delta_t + 1)(\delta_t + 2).
$$
The regulator periods $R_t(\gamma)$ are therefore killed by $L \circ \delta_t$. Since $L(\cdot) = 0$ is known to have basis of solutions

$$\pi_1 = \sum_{n \geq 0} a_n t^{3n}$$

$$\pi_2 = 3 \log(t) \pi_1 + \sum_{n \geq 1} a_n b_n t^{3n}$$

with $a_n = \frac{(3n)!}{(m!)^3}$ and $b_n = \sum_{k=0}^{n-1} \left( \frac{3}{3k+1} + \frac{3}{3k+2} - \frac{2}{k+1} \right)$, it now follows that (writing $B_n = b_n - \frac{1}{n}$)

$$R_t(\alpha) \equiv \frac{\pi i}{2} \left( \log(t) + \sum_{n \geq 1} \frac{a_n}{3n} t^{3n} \right)$$

$$R_t(\beta) \equiv \frac{9}{2} \log^2(t) + 3 \log(t) \sum_{n \geq 1} \frac{a_n}{n} t^{3n} + \sum_{n \geq 1} a_n B_n t^{3n} - \frac{\pi^2}{2}$$

$$\omega_t(\alpha) = 2\pi i \sum_{n \geq 0} a_n t^{3n}$$

$$\omega_t(\beta) = 9 \log(t) \sum_{n \geq 0} a_n t^{3n} + 3 \sum_{n \geq 1} a_n b_n t^{3n}$$

for $0 < |t| < \frac{1}{3}$. For the truncated normal function, this yields (modulo $\mathbb{Z}(2) \otimes \{\omega_t\text{-periods}\}$)

$$\nu(t) = \langle \frac{\omega}{2\pi i}, R_t \rangle = \frac{1}{2\pi i} \left( R_t(\alpha) \omega_t(\beta) - R_t(\beta) \omega_t(\alpha) \right)$$

$$= \frac{9}{2} \log^2(t)(1 + 6t^3) + 3 \log(t)(9t^3) + \frac{\pi^2}{2} + (3\pi^2 - 9)t^3 + O(t^6 \log^2 t).$$

**Remark 6.4.** This is closely related to computations in [Ho] and [MOY]; the main difference – and the salient result here – is the identification of $\frac{\pi^2}{2}$ as the correct torsion offset for our motivically defined $\nu$.

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