Duality Invariance and Higher Derivatives

Camille Eloy\textsuperscript{1}, Olaf Hohm\textsuperscript{2} and Henning Samtleben\textsuperscript{1}

\textsuperscript{1}Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France
\textsuperscript{2}Institute for Physics, Humboldt University Berlin, Zum Gro\ssen Windkanal 6, D-12489 Berlin, Germany

Abstract

We dimensionally reduce the spacetime action of bosonic string theory, and that of the bosonic sector of heterotic string theory after truncating the Yang-Mills gauge fields, on a $d$-dimensional torus including all higher-derivative corrections to first order in $\alpha'$. A systematic procedure is developed that brings this action into a minimal form in which all fields except the metric carry only first order derivatives. This action is shown to be invariant under $O(d, d, \mathbb{R})$ transformations that acquire $\alpha'$-corrections through a Green-Schwarz type mechanism. We prove that, up to a global pre-factor, the first order $\alpha'$-corrections are uniquely determined by $O(d, d, \mathbb{R})$ invariance.
1 Introduction

String theory features the T-duality property according to which there is a non-linear group action of $O(d, d, \mathbb{Z})$ on $d$-dimensional toroidal backgrounds such that all backgrounds in one orbit are physically equivalent. When restricting to the massless fields for compactifications on tori, i.e., when performing dimensional reduction, this duality implies invariance under the continuous symmetry group $O(d, d, \mathbb{R})$.

For the two-derivative effective action this symmetry was first shown explicitly for the (cosmological) reduction to one dimension by Veneziano and Meissner in Refs. [1,2] and later generalized to arbitrary $d$ by Maharana and Schwarz [3].

It was proven by Sen, using closed string field theory, that the $O(d, d, \mathbb{R})$ symmetry of dimensionally reduced theories is present to all order in $\alpha'$ [4], but it remains as a highly non-trivial problem to
actually display this symmetry when higher-derivative \( \alpha' \)-corrections are included. First significant progress was due to Meissner who investigated the dimensional reduction to one dimension including the four-derivative terms that appear in bosonic string theory to first order in \( \alpha' \) \[5\]. He uncovered the expected \( O(d, d, \mathbb{R}) \) symmetry, but this required a series of elaborate field redefinitions (that in particular cannot all originate from covariant field redefinitions before reduction). Subsequent work considered the reduction on a single circle \[6\] and reductions on a general torus but truncating out all ‘off-diagonal’ field components \[7\]. In all these truncations there is a choice of field variables for which the \( O(d, d, \mathbb{R}) \) transformations are undeformed, as is also suggested by string field theory \[8\]. In particular, this fact was used to classify all higher-derivative corrections in cosmology that, somewhat surprisingly, only require (higher powers of) first-order time derivatives \[9, 10\].

Recently, the higher-derivative \( \alpha' \)-corrections of string theory have been the focus of attention in the framework of double field theory. Double field theory is a formulation featuring a manifest \( O(d, d, \mathbb{R}) \) invariance before dimensional reduction by virtue of a generalized spacetime with doubled coordinates transforming covariantly under \( O(d, d, \mathbb{R}) \) \[11–14\]. While the two-derivative double field theory can be written naturally in terms of a ‘generalized metric’ that encodes metric and \( B \)-field \((c.f. \text{ Eq. (2.9) below})\), there are obstacles when including higher derivatives that require a deformation of the framework, see Refs. \[15–22\]. It was proven in Refs. \[23, 24\] that the general \( \alpha' \)-corrections of bosonic and heterotic string theory cannot be written in terms of the generalized metric, so that in particular the \( O(d, d, \mathbb{R}) \) transformations of double field theory get \( \alpha' \)-deformed. Alternatively, one may set up a generalized frame formalism for which \( O(d, d, \mathbb{R}) \) remains undeformed while the local frame transformations receive \( \alpha' \)-corrections \[18, 23, 24\].

In this paper we complete the existing literature by giving the complete dimensionally reduced action for bosonic string theory to first order in \( \alpha' \), i.e., including all four-derivative terms, and prove its \( O(d, d, \mathbb{R}) \) invariance, presenting results that have recently been announced in Ref. \[25\]. In particular, we prove that the first order \( \alpha' \)-corrections are uniquely determined by \( O(d, d, \mathbb{R}) \) invariance, up to an overall constant whose value depends on the string theory under consideration. While this \( O(d, d, \mathbb{R}) \) invariance is also implied by the existence of \( \alpha' \)-deformed double field theory, whose dimensional reduction has already been explored in Ref. \[21\], until now it has not been systematically investigated whether some of the unexpected new features arising in double field theory also show up in the dimensional reduction of conventional (non-extended) theories, nor has the dimensionally reduced action been displayed in a sufficiently simplified form that allows for applications (and comparison with some of the earlier results cited above). To our surprise we find that there is no choice of field variables so that the full dimensionally reduced action can be written in terms of familiar \( O(d, d, \mathbb{R}) \) covariant variables (like the generalized metric); rather, a generalized Green-Schwarz mechanism is required under which the (external) singlet \( B \)-field acquires non-trivial transformations under \( O(d, d, \mathbb{R}) \), hence implying that the \( O(d, d, \mathbb{R}) \) action gets \( \alpha' \)-deformed. This effect has been invisible in all truncations investigated so far, but it does mimic the situation in double field theory before reduction. Intriguingly, the \( \alpha' \)-deformations needed in double field theory can thus not be blamed entirely on its novel geometric structure, but such deformations also emerge in completely conventional dimensional reductions.

On a technical level, the present investigation requires full control over all possible field redefinitions, both redefinitions that are covariant in the usual sense (i.e. \( \text{GL}(d) \) covariant) and covariant with respect to \( O(d, d, \mathbb{R}) \). As one of the main technical results of this paper we present a fully systematic procedure to test \( O(d, d, \mathbb{R}) \) invariance, generalizing that of Refs. \[9, 10\] to higher dimensions. One
first dimensionally reduces the action as usual and then uses GL($d$) covariant field redefinitions to bring the action into a form in which all fields apart from the metric appear only with first-order derivatives. Next, one employs O($d,d,\mathbb{R}$) covariant redefinitions in order to find the minimal set of O($d,d,\mathbb{R}$) invariant four-derivative terms, which then are decomposed under GL($d$) with the aim to match with the dimensionally reduced terms. Our analysis applies to bosonic string theory but also to the bosonic sector of heterotic string theory after truncating out the Yang-Mills gauge fields, which still features a gravitational Chern-Simons form (due to the original Green-Schwarz mechanism).

The rest of the paper is organized as follows. In Sec. 2, we review the dimensional reduction of the leading two-derivative action of the bosonic string, and its manifestly O($d,d,\mathbb{R}$) symmetric formulation revealed in Ref. [3]. In order to set up a systematic analysis of its higher-order corrections, we outline how to organize and fix the ambiguities related to partial integration and higher-order field redefinitions. In Sec. 3, we present a general counting of independent higher-derivative terms upon modding out these ambiguities. At order $\alpha'$, we construct an explicit 61-dimensional basis of independent O($d,d,\mathbb{R}$) invariant four-derivative terms, which is algebraic in first order derivatives and the Riemann tensor. Sec. 4 presents the explicit torus reduction of the four-derivative action of the bosonic string. In particular, we show how all second-order derivatives in the reduced action can be eliminated by suitable field redefinitions. Comparing the result to our explicit basis, we show in Sec. 5 that apart from a single term the entire reduced action can be rewritten in terms of manifestly O($d,d,\mathbb{R}$) invariant terms. Restoring O($d,d,\mathbb{R}$) invariance of the full action then requires a Green-Schwarz type mechanism inducing a non-trivial O($d,d,\mathbb{R}$) transformation of order $\alpha'$ of the two-form $B_{\mu\nu}$ . In Sec. 6, we embed this structure into a frame formalism in which the O($d,d,\mathbb{R}$) symmetry remains undeformed, while the local frame transformations acquire $\alpha'$ deformations. Finally, in Sec. 7, we extend the analysis to the bosonic sector of heterotic supergravity and present its dimensionally reduced action in manifestly O($d,d,\mathbb{R}$) invariant form. The appendices collect a number of explicit technical results.

2 Two-derivative action and systematics of field redefinitions

A main goal of this paper is to compute the dimensional reduction of the bosonic string on a $d$-dimensional torus including the first order in $\alpha'$ and to make the resulting O($d,d,\mathbb{R}$) symmetry manifest. In this section, we review the reduction of the two-derivative action and its manifestly O($d,d,\mathbb{R}$) symmetric formulation first exhibited in Ref. [3]. We then discuss its field equations and the systematics of non-linear field redefinitions as a starting point for the subsequent systematic analysis of the higher order corrections.

2.1 Reduction and O($d,d,\mathbb{R}$) symmetry

Let us start from the two-derivative effective action for the bosonic string in $D + d$ dimensions, with metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, antisymmetric Kalb-Ramond field $\hat{B}_{\hat{\mu}\hat{\nu}}$ and dilaton $\hat{\phi}$:

$$\hat{I}_0 = \int d^{D+d}X \sqrt{-\hat{g}} e^{-\hat{\phi}} \left( \hat{R} + \partial_{\hat{\mu}}\hat{\phi} \partial^{\hat{\mu}}\hat{\phi} - \frac{1}{12} \hat{H}^2 \right),$$

(2.1)

where indices $\hat{\mu}$ run over the $(D + d)$ dimensional space, and $\hat{H}^2 = \hat{H}^{\hat{\mu}\hat{\nu}\hat{\phi}} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\phi}}$ with the field strength $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\phi}} = 3 \partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\phi}]}$. To compactify on the spatial torus $T^d$, we use the index split $X^{\hat{\mu}} = (x^\mu, y^m)$, with
\( \mu \in [1, D], \ m \in [1, d] \), for curved indices and \( \{ \hat{\alpha} \} = \{ \alpha, a \} \), with \( \alpha \in [1, D], \ a \in [1, d] \) for flat indices, and drop the dependence of all fields on the internal coordinates \( y^m \). For the metric \( \hat{g}_{\hat{\mu}\hat{\nu}} \), we use the vielbein formalism and consider the standard Kaluza-Klein ansatz

\[
\hat{e}_{\hat{\mu}}{}^\hat{\alpha} = \begin{pmatrix} e_{\mu}{}^\alpha & A_{\mu}^{(1)n} F_{n}{}^a \\ 0 & E_{m}{}^a \end{pmatrix},
\]

(2.2)
in terms of the \( D \)-dimensional vielbein \( e_{\mu}{}^\alpha \), Kaluza-Klein vector fields \( A_{\mu}^{(1)m} \), and the internal vielbein \( E_{m}{}^a \). The metric \( \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^\alpha \eta_{\hat{\alpha}\hat{\beta}} \hat{e}_{\hat{\nu}}{}^{\hat{\beta}} \) then takes the form

\[
\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A_{\mu}^{(1)p} G_{pq} A_{\nu}^{(1)q} & A_{\mu}^{(1)p} G_{pm} \\ G_{mp} A_{\nu}^{(1)p} & G_{mn} \end{pmatrix},
\]

(2.3)

where \( g_{\mu\nu} = e_{\mu}{}^\alpha \eta_{\alpha\beta} e_{\nu}{}^{\beta} \) and \( G_{mn} = E_{m}{}^a \delta_{ab} E_{n}{}^b \) denote the \( D \)-dimensional metric and the internal metric, respectively.

Similarly, the 2-form \( \hat{B}_{\hat{\mu}\hat{\nu}} \), is parametrized as \([3]\)

\[
\hat{B}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B_{\mu\nu} - A_{\mu}^{(1)m} A_{\nu}^{(2)m} + A_{\mu}^{(1)m} B_{mn} A_{\nu}^{(1)n} - A_{\nu}^{(2)m} + B_{mp} A_{\nu}^{(1)p} \\ -A_{\nu}^{(2)m} + B_{mp} A_{\nu}^{(1)p} B_{mn} \end{pmatrix},
\]

(2.4)
in terms of \( D \)-dimensional scalars \( B_{mn} = -B_{nm} \), vector fields \( A_{\mu}^{(2)m} \), and a 2-form \( B_{\mu\nu} \). The lower-dimensional components of \( \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \) are defined using the standard Kaluza-Klein procedure \([3]\): first converting \( \hat{H} \) to flat indices, block decomposing, and finally converting back to curved indices using the lower-dimensional blocks \( e_{\mu}{}^\alpha \) and \( E_{m}{}^a \). This amounts to converting a curved index \( \hat{\mu} \) to a curved index \( \mu \) using contraction with \( e_{\mu}{}^\alpha \varepsilon_{\alpha}{}^{\hat{\mu}} \) and to \( m \) contracting with \( E_{m}{}^a \varepsilon_{a}{}^{\hat{\mu}} \), such that the resulting fields transform covariantly under internal diffeomorphisms\(^1\). With Eq. \((2.3)\), this leads to

\[
\begin{align*}
H_{\mu\nu\rho} &= 3 \partial_{[\mu} B_{\nu\rho]} - \frac{3}{2} \left( A_{[\mu}^{(1)m} F_{\nu\rho]}^{(2)} - A_{[\mu}^{(1)m} F_{\nu\rho]}^{(1)n} A_{\nu]}^{(2)n} - A_{\nu]}^{(2)m} + B_{\nu\rho} A_{\nu]}^{(1)p} - A_{\nu]}^{(2)m} + B_{\nu\rho} A_{\nu]}^{(1)p} B_{mn} \right), \\
H_{\mu mn} &= F_{\mu m} - B_{mn} F_{\mu n}, \\
H_{mn\rho} &= \nabla_{\mu} B_{mn}, \\
H_{mnp} &= 0,
\end{align*}
\]

(2.5)

where we have defined the abelian field strengths

\[
\begin{align*}
F_{\mu\nu}^{(1)m} &= \partial_{\mu} A_{\nu}^{(1)m} - \partial_{\nu} A_{\mu}^{(1)m}, \\
F_{\mu\nu}^{(2)m} &= \partial_{\mu} A_{\nu}^{(2)m} - \partial_{\nu} A_{\mu}^{(2)m}.
\end{align*}
\]

(2.6)

In terms of these objects, after dimensional reduction, the action \((2.1)\) then takes the form \([3]\)

\[
I_0 = \int d^D x \sqrt{-g} e^{-\Phi} \left( R + \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \text{Tr} \left( \partial_{\mu} G \partial^{\mu} G^{-1} \right) \\
+ \frac{1}{4} \text{Tr} \left( G^{-1} \partial_{\mu} B G^{-1} \partial^{\mu} B \right) - \frac{1}{4} F_{\mu\nu}^{(1)m} G_{mn} F_{n}^{(1)\mu\nu} - \frac{1}{4} H_{\mu\nu m} G_{mn} H_{\mu\nu n} \right),
\]

(2.7)

\(^1\) Note that it is not the procedure that is used on \( \dot{B} \), as pointed out in Ref. \([26]\).
with the rescaled dilaton $\Phi = \hat{\phi} - \frac{1}{2} \log(\det(G_{mn}))$. In this form, the action features an explicit $GL(d)$ symmetry, as guaranteed by toroidal reduction. The symmetry enhancement to $O(d,d,\mathbb{R})$ can be made manifest upon regrouping the vector fields $A_{\mu}^{(1)m}$ and $A_{\mu}^{(2)m}$ into a single $O(d,d,\mathbb{R})$ vector

$$A_{\mu}^M = \begin{pmatrix} A_{\mu}^{(1)m} \\ A_{\mu}^{(2)m} \end{pmatrix},$$

(2.8)

and the scalar fields $G_{mn}$, $B_{mn}$ into an $O(d,d,\mathbb{R})$ matrix $\mathcal{H}_{MN}$ as

$$\mathcal{H}_{MN} = \begin{pmatrix} G_{mn} - B_{mp}G_{pq}B_{qn} & B_{mp}G_{mn} \\ -G_{mp}B_{pn} & G_{mn} \end{pmatrix}.$$

(2.9)

Throughout, the fundamental $O(d,d,\mathbb{R})$ indices are raised and lowered using the constant $O(d,d,\mathbb{R})$-invariant matrix

$$\eta^{MN} = \begin{pmatrix} 0 & \delta^m_n \\ \delta_m^n & 0 \end{pmatrix},$$

(2.10)

so that $\mathcal{H}^{-1}$ is defined as $\mathcal{H}^{MN} = \eta^{MP} \mathcal{H}_{PQ} \eta^{QN}$. In terms of the fields (2.8), (2.9), the reduced action (2.7) may be cast into the manifestly $O(d,d,\mathbb{R})$ invariant form [3]

$$I_0 = \int d^{D}x \sqrt{-g} e^{-\Phi} \left( R + \partial_{\mu} \Phi \partial^{\mu} \Phi + \frac{1}{8} \partial_{\mu} \mathcal{H}_{MN} \partial^{\mu} \mathcal{H}^{MN} - \frac{1}{4} \mathcal{F}_{\mu\nu}^M \mathcal{H}_{MN} \mathcal{F}^{\mu\nu N} - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \right),$$

(2.11)

where $\mathcal{F}_{\mu\nu}^M = 2\partial_{[\mu} A_{\nu]}^M$ is the abelian field-strength associated to the vectors (2.8). In terms of the covariant objects (2.8) and (2.9), the infinitesimal $O(d,d,\mathbb{R})$ variations of the fields are given by

$$\begin{cases}
\delta g_{\mu\nu} = 0, \\
\delta B_{\mu\nu} = 0,
\end{cases}
\begin{cases}
\delta \mathcal{H}_{MN} = \Gamma_M^P \mathcal{H}_{PN} + \Gamma_N^P \mathcal{H}_{MP},
\delta \mathcal{F}_{\mu\nu}^M = -\mathcal{F}_{\mu\nu}^N \Gamma_N^M,
\end{cases}$$

(2.12)

for $\Gamma_N^M \in \mathfrak{o}(d,d,\mathbb{R})$. The action (2.11) is manifestly invariant under these transformations. For later convenience, we also rewrite the action in terms of the matrix $S_{MN} = \mathcal{H}_{MP} \eta^{PN}$

$$I_0 = \int d^{D}x \sqrt{-g} e^{-\Phi} \left( R + \partial_{\mu} \Phi \partial^{\mu} \Phi + \frac{1}{8} \text{Tr} (\partial_{\mu} S \partial^{\mu} S) - \frac{1}{4} \mathcal{F}_{\mu\nu}^M S_{MN} \mathcal{F}^{\mu\nu N} - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} \right).$$

(2.13)

Note that $SS = 1$, so that $S$ is a constrained field.

### 2.2 GL($d$) fields redefinitions

Our aim is an extension of the previous construction to higher orders in $\alpha'$. As usual, the study of higher-derivative terms requires to carefully handle the ambiguities due to the possible non-linear field redefinitions. In particular, the symmetry enhancement to $O(d,d,\mathbb{R})$ will only be possible after identification of the proper field redefinitions. In this section, we describe the systematics of higher-order field redefinitions based on the two-derivative action (2.11), inspired by Refs. [10,9].

We consider the $\alpha'$ extension of Eq. (2.11) as a perturbation series

$$I = I_0 + I_1 + \mathcal{O}(\alpha'^2),$$

(2.14)
with the first order term $I_1 \sim \mathcal{O}(\alpha')$. In order to organize the possible ambiguities in $I_1$, we consider field redefinitions of the form

$$\varphi \rightarrow \varphi + \alpha' \delta \varphi,$$

where $\varphi$ denotes a generic field. Under such redefinitions of its fields, the variation of $I$ to order $\alpha'$ arises exclusively from the variation of $I_0$ and takes the form

$$\delta I_0 = \alpha' \int d^D x \sqrt{-g} e^{-\Phi} \left[ \mathcal{E}_\Phi \delta \Phi + (\mathcal{E}_g)_{\mu\nu} \delta g^{\mu\nu} + (\mathcal{E}_B)_{\mu\nu} \delta B^{\mu\nu} + (\mathcal{E}_G)_{mn} \delta G^{mn} ight.\]

$$\left. + (\mathcal{E}_B)^{mn} \delta B_{mn} + (\mathcal{E}_{A(1)})^\mu_m \delta A_{(1)}^m + (\mathcal{E}_{A(2)})^m_m \delta A_{(2)}^m \right],$$

proportional to the field equations associated with the two-derivative action $I_0$

$$\mathcal{E}_\Phi = -2 \Box \Phi - R + \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{12} H^2 - \frac{1}{8} \text{Tr} (\nabla_\mu S \nabla^\mu S) + \frac{1}{4} F_{\mu\nu}^M S_M^N F^{\mu\nu}_N,$$

$$(\mathcal{E}_g)_{\mu\nu} = R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H^2 + \frac{1}{8} \text{Tr} (\nabla_\mu S \nabla_\nu S) - \frac{1}{2} F_{\mu\nu}^M S_M^N F^{\nu\rho}_N + \frac{1}{2} g_{\mu\nu} \mathcal{E}_\Phi,$$

$$(\mathcal{E}_B)_{\mu\nu} = \frac{1}{2} (\nabla^\rho H_{\rho\mu\nu} - \nabla^\rho \Phi H_{\rho\mu\nu}),$$

$$(\mathcal{E}_G)_{mn} = \frac{1}{2} \left[ - \Box G_{mn} + \nabla_\mu \Phi \nabla^\mu G_{mn} - (\nabla_\mu G \nabla^\mu G^{-1} G)_{mn} + (\nabla_\mu B \nabla^\mu B)_{mn} \right.\]

$$\left. + \frac{1}{2} G_{mp} F^{(1)\mu}_{\mu\rho} p F^{(1)\rho\nu} q G_{qn} - \frac{1}{2} H_{\mu m n} H^{\mu\nu n} \right],$$

$$(\mathcal{E}_B)^{mn} = \frac{1}{2} \left[ (G^{-1} \Box B G^{-1})^{mn} - \nabla_\mu \Phi (G^{-1} \nabla^\mu B G^{-1})^{mn} + (G^{-1} \nabla_\mu B \nabla^\mu G^{-1})^{mn} \right.\]

$$\left. + (\nabla_\mu G^{-1} \nabla^\mu B G^{-1})^{mn} + \frac{1}{2} G_{mp} H_{\mu \rho p} F^{(1)\mu\nu n} - \frac{1}{2} F^{(1)\mu\nu m} G^{\nu \rho} H_{\mu \rho p} \right],$$

$$(\mathcal{E}_{A(1)})^\nu_n = \nabla_\mu F^{(1)\mu\nu m} G_{mn} - \nabla_\mu \Phi F^{(1)\mu\nu m} G_{mn} - \frac{1}{2} H^{\mu\rho p} H_{\mu \rho p} - (\mathcal{E}_{A(2)})^\nu_m B_{mn} \]

$$+ F^{(1)\mu\nu m} \nabla_\mu G_{mn} - H^{\mu \rho m} (G^{-1} \nabla_\rho B)^m_n + (\mathcal{E}_B)^\mu_n \left( A_{(2)}^\mu_{m n} - B_{mn} A_{(1)}^m \right),$$

$$(\mathcal{E}_{A(2)})^\nu_m = \nabla_\mu H^{\mu\nu n} G_{mn} - \nabla_\mu \Phi H^{\mu\nu n} G_{mn} + H^{\mu\nu n} \nabla_\mu G_{mn} + \frac{1}{2} H^{\mu\nu p} F^{(1)\mu p}_p \]

$$+ (\mathcal{E}_B)^\mu_p A_{(1)}^{\mu p} m .$$

Here, $H^2_{\mu\nu} = H_{\mu\rho\sigma} H^{\rho\sigma}$, $\nabla_\mu$ denotes the covariant derivative with respect to $g_{\mu\nu}$ and accordingly $\Box = \nabla_\mu \nabla^\mu$. At order $\alpha'$, the action thus is unique up to contributions proportional to the lowest order field equations. In the next section, we will show that by field redefinitions (2.15), the transformation (2.16) together with partial integrations allows to map all terms at order $\alpha'$ to a basis which carries only first derivatives of all fields (except for the two-derivative terms within the Riemann tensor).

As an example, let us show how a term carrying the factor $\Box \Phi$ can be replaced by terms carrying only products of first derivatives. Consider a generic term of $I_1$ of the form

$$Z = \alpha' \int d^D x \sqrt{-g} e^{-\Phi} X \Box \Phi,$$

where $X$ is a function of $\Phi$, $R_{\mu\nu\rho\sigma}$, $H_{\mu\nu\rho}$, $G_{mn}$, $B_{mn}$, $F_{\mu\nu}^{(1)\mu}$ and $H_{\mu\nu\rho}$ (and their derivatives) which carries exactly two derivatives. Redefining the dilaton and the metric as Eq. (2.15) with $\delta g_{\mu\nu} = \lambda g_{\mu\nu}$,
Eq. (2.16) yields the transformation

\[ \delta I_0 = \alpha' \int d^D x \sqrt{-g} e^{-\Phi} \left[ \Box \Phi \left( -2 \delta \Phi + \lambda (D + 1) \right) + \frac{1}{2} R \left( -2 \delta \Phi + \lambda (D + 2) \right) \right. \]
\[ + \nabla_\mu \Phi \nabla^\mu \Phi \left( \delta \Phi - \frac{D}{2} \lambda \right) + \frac{1}{24} H^2 \left( 2 \delta \Phi - \lambda (D + 6) \right) \]
\[ \left. - \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla^\mu S) \left( -2 \delta \Phi + \lambda (D + 2) \right) \right] \]
\[ + \frac{1}{8} F_{\mu \nu} M S_{\mu \nu} N F^{\mu \nu} N \left( 2 \delta \Phi - \lambda (D + 4) \right) \]  
(2.19)

With the particular choice

\[ \left\{ \begin{array}{l}
\delta \Phi = \frac{1}{2} (D + 2) X, \\
\lambda = X,
\end{array} \right. \]

(2.20)

the new terms (2.19) cancel the term \( Z \) and replace it by

\[ Z' = \alpha' \int d^D x \sqrt{-g} e^{-\Phi} X \left( \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{6} H^2 - \frac{1}{4} F_{\mu \nu} M S_{\mu \nu} N F^{\mu \nu} N \right), \]  
(2.21)

which carries only products of first order derivatives. In the same fashion, all the four-derivative terms carrying the leading two-derivative contributions from the field equations (2.17) can be transformed into terms carrying only products of first order derivatives. We may summarize the resulting replacement rules as

\[ \Box \Phi \rightarrow Q_\Phi = \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{6} H^2 - \frac{1}{4} F_{\mu \nu} M S_{\mu \nu} N F^{\mu \nu} N, \]

\[ R_{\mu \nu} \rightarrow Q_{g \mu \nu} = - \nabla_{(\mu} \nabla_{\nu)} \Phi + \frac{1}{4} H_{\mu \nu}^2 - \frac{1}{8} \text{Tr} (\nabla_\mu S \nabla_\nu S) + \frac{1}{2} F_{\mu \rho} M S_{\mu \nu} N F_{\nu \rho} N \]
\[ - \frac{1}{12} g_{\mu \nu} \left( \nabla_\rho \Phi \nabla^\rho \Phi - \frac{1}{6} H^2 - \frac{1}{4} F_{\rho \sigma} M S_{\rho \sigma} N F^{\rho \sigma} N \right), \]

\[ \nabla^\rho H_{\mu \rho \sigma} \rightarrow Q_{B_{\rho \sigma}} = \nabla^\rho \Phi H_{\mu \rho \sigma}, \]

\[ \Box G_{mn} \rightarrow Q_{G_{mn}} = \nabla_\mu \Phi \nabla^\mu G_{mn} - \nabla_\mu G_{mp} \nabla^\mu G_{qn} - \frac{1}{2} H_{\mu \nu mn} H_{\mu \nu n} \]
\[ + \nabla_\mu B_{mp} G^{pq} \nabla^\mu B_{qn} + \frac{1}{2} F_{\mu \nu}^{(1)} G_{pm} F_{(1) \mu \nu q} G_{qn}, \]  
(2.22)

\[ \Box B_{mn} \rightarrow Q_{B_{mn}} = \nabla_\mu \Phi \nabla^\mu B_{mn} - \nabla_\mu B_{mp} \nabla^\mu G^{pq} G_{qn} - G_{mp} \nabla^\mu G^{pq} \nabla_\mu B_{qn} \]
\[ - \frac{1}{2} H_{\mu \nu mn} F_{(1) \mu \nu p} G_{pm} + \frac{1}{2} G_{mp} F_{(1) \mu \nu p} H_{\mu \nu mn}, \]

\[ \nabla_\mu F_{(1) \mu \nu} m \rightarrow Q_{A_{(1) \mu \nu} m} = \nabla_\mu \Phi \ F_{(1) \mu \nu} m + H_{\mu \nu n} G^{pq} \nabla_\mu B_{pq} G_{mn} \]
\[ + \frac{1}{2} H_{\mu \nu p} H_{\mu \rho n} G_{mn} F_{(1) \mu \nu p} G_{pn} \]
\[ \nabla_\mu H_{\mu \nu} m \rightarrow Q_{A_{(2) \mu \nu} m} = \nabla_\mu \Phi \ F_{\mu \nu} m - H_{\mu \nu n} \nabla_\mu G^{np} G_{pm} + \frac{1}{2} H_{\mu \rho n} F_{(1) \mu \rho} G_{mn}. \]

Double parenthesis ((...)) in the second line refer to traceless symmetrization. The associated field redefinitions are collected in Tab. 1. As we will show in Sec. 4, all other four-derivative terms can be mapped into the terms listed in Tab. 1 upon using partial integration and Bianchi identities.
In this section, we present the construction of an explicit $O(d,d,\mathbb{R})$-invariant basis for the four-derivative terms in $D$ dimensions. We discuss the general counting of independent terms for building an action upon modding out field redefinitions and partial integrations. At order $\alpha'$ we find that the number of independent terms is 61 and coincides with the number of terms that can be built from products of first order derivatives (and the Riemann tensor). We confirm the number by an explicit construction of a 61-dimensional basis which we use subsequently in order to organize the result of the explicit torus reduction.

### 3.1 Counting independent terms

Following the general discussion of field redefinition ambiguities of the last section, we first count the number of independent terms modulo the two-derivative field equations (2.22) and Bianchi identities. At this stage we do not yet restrict to Lorentz scalars, i.e. we keep all $D$-dimensional space-time indices uncontracted. In a second step we will restrict to Lorentz scalars and account for the freedom of partial integration. We start by defining the alphabet whose letters are the $O(d,d,\mathbb{R})$-invariant building blocks in the various matter sectors (dilaton, scalars, vectors, 2-forms, metric) before identifying all possible symmetric words in these letters. We only count manifestly $O(d,d,\mathbb{R})$ and gauge invariant terms, i.e.
neglect possible Chern-Simons and topological terms which we will have to treat separately.

**Dilaton** The independent building blocks carrying the dilaton are given by powers of derivatives

\[ \mathcal{B}_{\text{dil}} = \left\{ \nabla_{(\mu_1 \ldots \nabla_{\mu_n})} \Phi \mid n \in \mathbb{N}^*, \{\mu_1, \ldots, \mu_n\} \right\}, \tag{3.1} \]

with the double parentheses \((\ldots)\) indicating traceless symmetrization in order to divide out field equations. We may encode the set of letters (3.1) into a partition function

\[ Z_{\text{dil}} = u \left( \frac{1 - q^2}{(1 - q)^{v_D} - 1} \right), \tag{3.2} \]

such that upon expanding Eq. (3.2) into a series in \(q\) every term represents a letter with exponents counting the number of derivatives. We have also added a factor \(u\) to keep track of the dilaton power when combining Eq. (3.2) with the other building blocks of the theory. We use the notation

\[ (1 - q)^{v_D} = 1 - q v_D + q^2 (v_D \otimes v_D)_{\text{alt}} - \ldots, \]

\[ \frac{1}{(1 - q)^{v_D}} = 1 + q v_D + q^2 (v_D \otimes v_D)_{\text{sym}} + \ldots, \tag{3.3} \]

with the SO(\(D\)) vectorial representation \(v_D\), in order to describe the tower of traceless symmetrized vectors. \((v_D \otimes v_D)_{\text{alt}}\) is the antisymmetric tensor product of two SO(\(D\)) vectors,

**Coset scalars** The scalar fields parametrize the \(SO(d,d)/(SO(d) \times SO(d))\) matrix \(\mathcal{H}_{MN}\). In order to directly implement all constraints deriving from the coset structure, it is convenient to turn to the vielbeins

\[ \mathcal{H}_{MN} = E_M^A \delta_{AB} E_N^B \implies \partial_{\mu} \mathcal{H}^{-1} = 2 E P_{\mu} E^{-1}, \tag{3.4} \]

with the coset currents defined by

\[ E^{-1} \partial_{\mu} E = Q_{\mu} + P_{\mu} \in \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{so}(d,d), \tag{3.5} \]

where \(\mathfrak{t} = \mathfrak{so}(d) \oplus \mathfrak{so}(d)\) and \(\mathfrak{p}\) is its (non-compact) orthogonal complement. In terms of the currents \(Q_\mu\) and \(P_\mu\), global SO(\(d,d\)) invariance is ensured, and the counting problem reduces to identifying combinations that are invariant under local SO(\(d\) \(\times\) SO(\(d\)) transformations, i.e. built from \(P_\mu\)'s and covariant derivatives \(D_\mu = \partial_\mu + \text{ad}_{Q_\mu}\). Moreover, we have integrability conditions

\[ [D_\mu, D_\nu] = Q_{\mu\nu} \propto [P_\mu, P_\nu], \quad D_{[\mu} P_{\nu]} = 0, \tag{3.6} \]

and field equations with leading second order term \(D^\mu P_\mu\) which implies that a basis of on-shell independent combinations is given by

\[ \mathcal{B}_P = \left\{ \nabla_{(\mu_1 \ldots \nabla_{\mu_n} P_{\mu_{n+1}})} \mid n \in \mathbb{N}, \{\mu_1, \ldots, \mu_{n+1}\} \right\}, \tag{3.7} \]

counted by the partition function

\[ Z_P = p \left( \frac{1 - q^2}{(1 - q)^{v_D} - 1} \right), \tag{3.8} \]
with the charge \( p \) introduced to count the power of \( P_\mu \)'s. It remains to count the independent \( \text{SO}(d) \times \text{SO}(d) \) invariant single-trace combinations in the letters (3.7). With \( P_\mu \) transforming in the \((d, d)\) representation of \( \text{SO}(d) \times \text{SO}(d) \), this amounts to counting ordered monomials and dividing out transpositions and cyclic shifts of even length\(^2\). The result then follows from Polya’s counting theorem \([27]\) as

\[
Z_{\text{sing.trace}} = -\frac{1}{2} \sum_n \frac{\varphi(n)}{n} \log (1 - Z_{P,n}^2) + \frac{Z_{P,2}}{2(1 - Z_{P,2})},
\]

with Euler’s totient function \( \varphi(n) \) and \( Z_{P,n} = Z_P(p^n, q^n) \).

**Vectors** The (manifestly) gauge invariant building blocks in terms of the vector field are obtained by derivatives of its field strength subtracting Bianchi identities and contractions proportional to the field equations

\[
B_F = \{\nabla_{(\mu_1} \cdots \nabla_{\mu_n)} F_{\nu_1 \nu_2}^M - \text{traces \& Bianchi} \mid n \in \mathbb{N}\},
\]

counted by the partition function (see e.g. Ref. \([28]\))

\[
Z_F = \sum_{n=0}^{\infty} \left( \prod_{d=1}^{n} \prod_{p=1}^{\infty} \phi_p - \text{traces} \right) = f \left( \frac{1}{q} - \frac{1 - \nu_D q (1 - q^2) - q^4}{q (1 - q)^{\nu_D}} \right),
\]

where \( f \) is a charge for the powers of \( F_{\mu\nu}^M \). However, the letters (3.10) are not \( \text{O}(d, d, \mathbb{R}) \) singlets but rather carry a fundamental vector index. \( \text{O}(d, d, \mathbb{R}) \) invariant combinations are built from bilinears of Eq. (3.10) with the two \( \text{O}(d, d, \mathbb{R}) \) vector indices contracted by products of the \( \text{O}(d, d, \mathbb{R}) \) invariant \( \eta_{MN} \), the scalar matrix \( H_{MN} \), and its derivatives. This is most conveniently counted by using the vielbeins (3.4) to convert the \( \text{O}(d, d, \mathbb{R}) \) indices of Eq. (3.10) into \( \text{SO}(d) \times \text{SO}(d) \) indices, such that the flattened field strength \( F_{\mu\nu}^M E_M^A \) decomposes into \((d, 1) \oplus (1, d)\) contributions which we denote by \( F_L \) and \( F_R \), respectively. The flattened letters (3.10) are then contracted out by arbitrary chains of letters from Eq. (3.7). This gives rise to three different types of terms

\[
(\nabla \cdots \nabla F_L) \text{ (even chain of } \nabla \cdots \nabla P) (\nabla \cdots \nabla F_L), \quad (\nabla \cdots \nabla F_R) \text{ (even chain of } \nabla \cdots \nabla P) (\nabla \cdots \nabla F_R), \quad (\nabla \cdots \nabla F_L) \text{ (odd chain of } \nabla \cdots \nabla P) (\nabla \cdots \nabla F_R).
\]

Upon taking into account the reflection symmetries of the first two chains, the counting of \( \text{O}(d, d, \mathbb{R}) \) invariant building blocks in the vector sector yields

\[
Z_{FF} = \frac{1}{2} \left( \frac{Z_{F,2}}{1 - Z_{P,2}} + \frac{Z_{F,2}^2}{1 - Z_{P,2}^2} \right) + \frac{1}{2} \left( \frac{Z_{F,2}}{1 - Z_{P,2}} + \frac{Z_{F,2}^2}{1 - Z_{P,2}^2} \right) + Z_F \frac{Z_P}{1 - Z_P} Z_F \]

\[
= \frac{Z_{F,2}^2}{1 - Z_F} + \frac{Z_{F,2}}{1 - Z_{P,2}} Z_F.
\]

\(^2\) In this counting, we neglect all the identities induced by the finite size \((2d)\) of the \( \text{SO}(d, d) \) matrices, i.e. formally we count for \( d = \infty \).
**Two-form** Similarly, the independent (manifestly gauge-invariant) building blocks carrying the 2-form $B_{\mu\nu}$ are counted by powers of derivatives on the field strength $H_{\mu\nu\rho}$ upon subtracting Bianchi identities and contractions proportional to the field equations

$$B_H = \{ (\nabla_{(\mu_1} \cdots \nabla_{\mu_n)}) H^{\nu_1 \nu_2 \nu_3}_\alpha - \text{traces & Bianchi} \mid n \in \mathbb{N} \} ,$$

(3.14)

which gives rise to a partition function

$$Z_H = \sum_{n=0}^{\infty} \left( b_{\mu_1 \nu_1} \cdots b_{\nu_3} - \text{traces} \right)$$

$$= h \left( \frac{1 - q^6 - q(1 - q^4) v_D + q^2(1 - q^2) (v_D \otimes v_D)_{\text{alt}}}{q^2(1 - q)^{\nu_D}} - \frac{1}{q^2} \right),$$

(3.15)

where $h$ is a charge for the powers of $H_{\mu\nu\rho}$.

**Metric** For the external metric $g_{\mu\nu}$, we count derivatives of its Weyl tensor $C_{\nu_1 \nu_2 \nu_3 \nu_4}$, subtracting traces and Bianchi identities, giving rise to the letters

$$B_C = \{ (\nabla_{(\mu_1} \cdots \nabla_{\mu_n)}) C_{\nu_1 \nu_2 \nu_3 \nu_4} - \text{traces & Bianchi} \mid n \in \mathbb{N} \} ,$$

(3.16)

which are counted as

$$Z_C = \sum_{n=0}^{\infty} \left( a_{\mu_1} a_{\nu_1} \cdots a_{\nu_4} - \text{traces} \right)$$

$$= c \left( q (1 - q^4) (v_D \otimes v_D)_{\text{sym}} - (1 - q^4) v_D \right. \left. + (v_D \otimes v_D)_{\text{alt}} + \frac{1}{q} v_D \right) ,$$

(3.17)

where $c$ is a charge for the powers of the Weyl tensor (or equivalently, the Riemann tensor).

### 3.2 Space-time singlets and partial integration

Putting everything together, we have identified the manifestly $O(d, d, \mathbb{R})$ and gauge invariant building blocks in the various sectors,

$$Z_0 = Z_{\text{dil}} + Z_H + Z_C + Z_{\text{sing, trace}} + Z_{\mathcal{F}F} ,$$

(3.18)

with the different terms defined in Eqs. (3.2), (3.15), (3.17), (3.9), and (3.13), respectively. From these objects, we can construct the most general $O(d, d, \mathbb{R})$ and gauge invariant terms as arbitrary polynomials in the letters of Eq. (3.18), counted as

$$Z_{\text{inv}} = \exp \left[ \sum_k \frac{1}{k} Z_{0,k} \right] .$$

(3.19)

So far, we have been counting combinations in all possible $SO(D)$ representations, without restricting to $SO(D)$ Lorentz scalars. In order to count the independent space-time actions, we first project $Z_{\text{inv}}$ to Lorentz scalars. Next, in order to subtract the ambiguities from partial integrations, we extract from $Z_{\text{inv}}$ all possible $SO(D)$ vectors $J_\mu$ each of which gives rise to an ambiguity $d * J$ of the space-time Lagrangian. On the other hand, currents with (off-shell) vanishing divergence $d * J = 0$
do not define ambiguities, these are of the form $J = *d *J_2$ for a 2-form $J_2$. Unless $*J_2$ is of vanishing divergence thus defined by a 3-form $J_3$, etc. To summarize, a basis of independent space-time Lagrangians, after dividing out the freedom of partial integrations, is given by

$$Z_{\text{Lag}} = Z_{\text{inv}} (1 - uq)^{\nu_D} \bigg|_{\text{SO(D) singlets}}, \quad (3.20)$$

in the notation of Eq. (3.3)

### 3.3 Some examples

**Evaluation in $D = 10$** As a first test of the counting formula (3.20), we may evaluate it to order $\alpha'$ in $D = 10$ dimensions, i.e. for $d = 0$, upon truncating out the vector and scalar sector which do not exist at $d = 0$. Then, in Eq. (3.18) only the contributions from metric, two-form and dilaton are taken into account. Evaluating Eq. (3.20) gives rise to the following types of terms at the four-derivative order

$$\left\{ R^2 \ [1], \ \nabla^2 H^2 \ [1], \ RH^2 \ [1], \ H^4 \ [3], \ H^2 \nabla^2 \Phi \ [1], \ \nabla^2 \Phi \nabla^2 \Phi \ [1] \right\}, \quad (3.21)$$

where the multiplicities $[n]$ indicate the number of independent terms of the same type. This precisely reproduces the counting from Ref. [29] (c.f. their Eq. (2.36)).

**Evaluation in $D = 1$** Upon reduction to only one dimension, we can evaluate the counting formulas to all orders in closed form. In particular $Z_H = Z_F = 0$, while

$$Z_{\text{dil}} = uq, \quad Z_P = pq, \quad Z_{\text{sing,trace}} = \frac{p^2 q^2}{1 - p^2 q^2}, \quad (3.22)$$

and

$$Z_C = -c q^2 \longrightarrow -p^2 q^2, \quad (3.23)$$

reflecting the fact that in $D = 1$ the Einstein equations pose a constraint on the energy-momentum tensor. For Eqs. (3.18), (3.19), we thus find

$$Z_0 = \frac{p^4 q^4}{1 - p^2 q^2} + uq \quad \Rightarrow \quad Z_{\text{inv}} = \prod_{n>1} \frac{1}{1 - p^{2n} q^{2n}} \times \frac{1}{1 - uq}, \quad (3.24)$$

upon removing total derivatives (3.20) thus

$$Z_{\text{Lag}} = (1 - qu) Z_{\text{inv}} = \prod_{n>1} \frac{1}{1 - p^{2n} q^{2n}}, \quad (3.25)$$

which precisely reproduces the counting from Ref. [10].

---

3 Here, we have inserted a dilaton charge $u$, since all terms carry a global dilaton power $e^{-\Phi}$ such that partial integration brings in an extra dilaton derivative.
3.4 Basis at order $\alpha'$

Evaluating the counting formula (3.20) in generic dimension $D$ we infer that at order $\alpha'$ there are 61 independent manifestly $O(d,d,\mathbb{R})$ invariant four-derivative terms. While the general counting only determines the number of independent terms without selecting a particular basis, it turns out that at order $\alpha'$ there is a distinguished explicit basis which is built from polynomials in terms carrying only first order time derivatives (and the Riemann tensor). Indeed, truncating the partition functions (3.2), (3.15), (3.17), (3.8), (3.11) to first order in derivatives, we may count from Eq. (3.19) the number of independent terms that carry first derivatives only, and find precisely 61 terms at order $\alpha'$.

The basis at order $\alpha'$ can thus be given in terms of polynomials in $R_{\mu\nu\rho}^\sigma$, $H_{\mu\nu\rho}$, $\mathcal{F}_{\mu\nu}^M$, $\nabla\mu S_M^N$, and $\nabla\mu\Phi$. Schematically, its elements take the form

$$\left\{ R^2 [1], H^4 [3], (\nabla\Phi)^4 [1], (\nabla S)^4 [5], \mathcal{F}^4 [12], RH^2 [1], R\mathcal{F}^2 [2], H^2 (\nabla\Phi)^2 [2], H^2 (\nabla S)^2 [2], H^2 \mathcal{F}^2 [8], (\nabla\Phi)^2 (\nabla S)^2 [2], (\nabla\Phi)^2 \mathcal{F}^2 [4], \right.$$

$$\left. (\nabla S)^2 \mathcal{F}^2 [10], H \nabla\Phi \mathcal{F}^2 [2], H \nabla S \mathcal{F}^2 [3], \nabla\Phi \nabla S \mathcal{F}^2 [3] \right\}. \quad (3.26)$$

We give the explicit expressions for all the basis elements in App. A. In the following we will exhibit $O(d,d,\mathbb{R})$ invariance of the dimensionally reduced action by expanding the reduced action in the basis (3.26).

4 Compactification of the four-derivative action

The first order $\alpha'$ extension of the action of the bosonic string (2.1) has been known for some time [29] and is given up to field redefinitions by

$$\tilde{I}_1 = \frac{1}{4} \alpha' \int d^{D+d} X \sqrt{-g} e^{-\phi} \left( \tilde{R}_{\mu\nu\rho\sigma} R_{\mu\nu}^{\rho\sigma} - \frac{1}{2} \tilde{H}_{\mu}^{\nu} \tilde{H}_{\nu}^{\rho} \tilde{H}_{\rho}^{\sigma} - \frac{1}{8} \tilde{H}_{\mu}^{2 \nu} \tilde{H}_{\nu}^{2 \rho} \tilde{H}_{\rho}^{2 \sigma} + \frac{1}{24} \tilde{H}_{\mu}^{\overline{\nu} \overline{\rho} \overline{\sigma}} \tilde{H}_{\overline{\nu} \overline{\sigma}} \tilde{H}_{\overline{\rho} \overline{\sigma}} \right). \quad (4.1)$$

In this section, we compactify separately all of its terms on a $d$-torus, using the ansätze (2.3) and (2.4). We fix the freedom of partial integration and possible field redefinitions, by converting all terms into polynomials of first order derivatives (and the Riemann tensor). To do so, we systematically use partial integration and Bianchi identities to bring all terms carrying second order derivatives into a form corresponding to the first column of Tab. 1, which can then be converted to the desired form by means of field redefinitions as discussed in Sec. 2.2. In the next section, we then compare the result to the $O(d,d,\mathbb{R})$ basis of Sec. 3.4.

The reduction of the three-form field strength $\tilde{H}_{\mu\nu\rho}$ is given in Eq. (2.5). For the reduction of the Riemann tensor, we follow the results of Ref. [30], and give the lower-dimensional components in flat

\footnote{At order $\alpha'^2$ this pattern breaks down. The general counting (3.20) reveals 1817 independent terms at order $\alpha'^2$ whereas there are only 1212 independent polynomials that can be constructed in terms of first order derivatives. This general case differs from the situation encountered in the reduction to $D=1$ dimensions where one can always find a basis carrying no more than first-order time derivatives [9].}
indices as

\[ \hat{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2} \left[ -G_{mn}F_{\alpha[\gamma}^{(1)m}F_{\delta]\beta}^{(1)n} + G_{mn}F_{\alpha\beta}^{(1)m}F_{\gamma\delta}^{(1)n} \right], \]

\[ \hat{R}_{\alpha\beta\gamma\delta} = \left[ \nabla_{[\alpha}F_{\beta\gamma\delta]}^{(1)p} - \frac{1}{2} \left( G_{mn}\nabla_{[\alpha}G^{np}F_{\beta\gamma\delta]}^{(1)m} - F_{\alpha\beta}^{(1)m}G_{mn}\nabla_{\gamma}G^{np} \right) \right] E_{p\,d}, \]

\[ \hat{R}_{\alpha\beta\gamma\delta} = \frac{1}{4} \left[ 2\nabla_{\alpha}\nabla_{\gamma}G^{mq} - 2\nabla_{\alpha}G^{mn}G_{np}\nabla_{\gamma}G^{pq} - \nabla_{\gamma}G^{mn}G_{np}\nabla_{\alpha}G^{pq} + F_{\gamma\delta}^{(1)m}F_{\alpha\beta}^{(1)n} \right] E_{m\,d}E_{d\,q}, \]

\[ \hat{R}_{\alpha\beta\gamma\delta} = -\frac{1}{2}\nabla_{\epsilon}G^{mn}\nabla_{\gamma}G^{pq}E_{m\,a}E_{\epsilon|\nu}E_{\nu\,d} - \frac{1}{2}\nabla_{\epsilon}G^{mn}\nabla_{\gamma}G^{pq}E_{m\,a}E_{\epsilon|\nu}E_{\nu\,d} - \frac{1}{2}\nabla_{\epsilon}G^{mn}\nabla_{\gamma}G^{pq}E_{m\,a}E_{\epsilon|\nu}E_{\nu\,d} \quad (4.2) \]

4.1 Reduction of the various terms

We reduce the action (4.1) term by term.

**Reduction of \( \hat{H}_{\mu\nu}^2 \hat{H}^2 \hat{\mu}\hat{\nu} \)** Upon compactification, we obtain

\[ \hat{H}_{\mu\nu}^2 \hat{H}^2 \hat{\mu}\hat{\nu} = H_{\mu\nu}^2H^{2\mu\nu} + 4H_{\mu\nu}^2H^\rho_\mu H^\nu_\rho - 2H^{2\mu\nu}H_{\mu\nu m}H_\nu^m + 4H_{\mu\nu m}H^{\mu mn}H^{\nu\sigma n}H_{\rho n mn} - 4H_{\mu\nu m}H^{\mu mn}H_{\rho n}H^\nu_\rho \]

\[ + 8H_{\mu\nu m}H_{\mu\nu m}H^{mn}H_{\rho n} - 8H_{\mu\nu m}H^{mn}H_{\rho n}H^\nu_\nu H_{\rho n} + 4H_{\mu\nu m}H_{\rho n}H^\nu_\rho H^{mn}H_{\rho n} \]

\[ - 4H_{\mu\nu m}H_{\rho n}H^{mn}H_{\rho n}H^\nu_\nu 4H_{\mu\nu m}H_{\rho n}H^\nu_\rho H^{mn}H_{\rho n} \quad (4.3) \]

Using Eq. (2.5), this takes the form

\[ \hat{H}_{\mu\nu}^2 \hat{H}^2 \hat{\mu}\hat{\nu} = H_{\mu\nu}^2H^{2\mu\nu} + 4\Tr\left( \nabla_\mu B G^{-1}\nabla_\nu B G^{-1}\nabla_\nu B G^{-1}\nabla_\nu B G^{-1} \right) \]

\[ + \Tr\left( \nabla_\nu B G^{-1}\nabla_\nu B G^{-1} \right) \Tr\left( \nabla_\mu B G^{-1}\nabla_\nu B G^{-1} \right) + H_{\nu\rho m}G^{mn}H_{\rho n}H^{\mu\nu m}G^{\nu\sigma n}H_{\rho n} \]

\[ + 4H_{\nu\rho m}G^{mn}H^{\mu\nu m}H_{\rho n}H^{\nu\sigma n} - 2H_{\mu\nu}^2\Tr\left( \nabla_\mu B G^{-1}\nabla_\nu B G^{-1} \right) \]

\[ + 4H^2_H\mu\nu m \rho n H_{\rho n} \quad (4.4) \]

where all terms carry first order derivatives only, i.e. are already of the desired form.

**Reduction of \( \hat{H}_{\mu\nu\rho\sigma} \hat{H}_{\lambda\delta}^\mu \hat{H}_{\lambda\delta}^\nu \hat{\mu}\hat{\nu} \hat{\rho}\hat{\sigma} \** Upon compactification, we obtain

\[ \hat{H}_{\mu\nu\rho\sigma} \hat{H}_{\lambda\delta}^\mu \hat{H}_{\lambda\delta}^\nu \hat{\mu}\hat{\nu} \hat{\rho}\hat{\sigma} = H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma}H_{\lambda\delta}^\mu H_{\lambda\delta}^\nu \hat{\mu}\hat{\nu} \hat{\rho}\hat{\sigma} + 6H_{\mu\nu\rho\sigma}H_{\lambda\delta}^\mu \hat{H}_{\mu\nu\rho\sigma}H_{\lambda\delta}^\nu \]

\[ - 12H_{\mu\nu\rho\sigma}H_{\mu\nu m}H_{\rho n}H_{\delta}^m + 4H_{\mu\nu\rho\sigma}H_{\mu\nu m}H_{\rho n}H_{\delta}^m + 3H_{\mu\nu m}H_{\rho n}H_{\delta}^mH_{\nu\rho\sigma n} \]

\[ - 12H_{\mu\nu m}H_{\rho n}H_{\nu\rho\sigma}H_{\delta}^m + 3H_{\mu\nu m}H_{\nu\rho m}H_{\rho p}H_{\nu\rho p} + 4H_{\nu\rho m}H_{\mu\nu m}H_{\rho p}H_{\nu\rho p} \]

\[ + 12H_{\mu\nu m}H_{\rho n}H_{\nu\rho p} \hat{\mu}\hat{\nu} + 6H_{\nu\rho m}H_{\nu\rho m}H_{\mu\nu m} + H_{\nu\rho m}H_{\nu\rho p} - 4H_{\nu\rho m}H_{\nu\rho p} \quad (4.5) \]
Using Eq. (2.5), this takes the form

\[
\begin{align*}
\hat{H}_{\hat{\mu} \hat{\nu}} \hat{H}^{\hat{\rho} \hat{\sigma}} & \hat{\lambda} \hat{\tau} \hat{\rho} \hat{\delta} = H_{\mu \rho} H^{\mu} \hat{\lambda} H^{\nu} \hat{\tau} H^{\rho} \hat{\sigma} + 3 H_{\mu \nu \rho} G^{mn} H_{\rho \sigma \eta} H^{\mu} \rho G^{pq} H^{\sigma} \eta \\
+ 6 H^{\mu \nu \lambda \rho \sigma} H_{\mu \rho \sigma} & G^{mn} H_{\rho \sigma \eta} + 3 \text{Tr} (\nabla_{\mu} BG^{-1} \nabla_{\nu} BG^{-1} \nabla_{\mu} BG^{-1} \nabla_{\nu} BG^{-1}) \\
- 12 H^{\mu \nu \rho} H_{\mu \nu m} (G^{-1} \nabla_{\mu} BG^{-1})^{mn} & H_{\rho} \sigma n - 12 H_{\mu \nu m} (G^{-1} \nabla_{\rho} BG^{-1} \nabla_{\nu} BG^{-1})^{mn} H^{\mu \rho n} \\
+ 4 H^{\mu \nu \rho} & \text{Tr} (\nabla_{\mu} BG^{-1} \nabla_{\nu} BG^{-1} \nabla_{\rho} BG^{-1}),
\end{align*}
\]

(4.6)

where again all terms carry first order derivatives only, i.e. are already of the desired form.

**Reduction of \( \hat{R}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{R}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \)**

Splitting the \( D + d \) indices \( \hat{\mu} \rightarrow \{ \mu, m \} \), we obtain

\[
\hat{R}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{R}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} = \hat{R}_{\mu \nu \rho \sigma} \hat{R}^{\mu \nu \rho \sigma} + 4 \hat{R}_{\mu \nu \rho m} \hat{R}^{\mu \nu \rho m} + 2 \hat{R}_{\mu \nu m n} \hat{R}^{\mu \nu m n} \\
+ 4 \hat{R}_{\mu m \nu n} \hat{R}^{\mu m \nu n} + 4 \hat{R}_{m \mu n \nu} \hat{R}^{m \mu n \nu} + \hat{R}_{m n p q} \hat{R}^{m n p q}.
\]

(4.7)

Upon using Eq. (4.2), the reduction of the first term of the action (4.1) then yields

\[
\alpha' \int d^{D+d}x \sqrt{-g} e^{-\Phi} \hat{R}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{R}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \rightarrow
\]

\[
\alpha' \int d^{D}x \sqrt{-g} e^{-\Phi} \left[ R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - \frac{3}{2} R^{\mu \nu \rho} F^{(1) m}_{\mu \nu} G_{m n} F^{(1) n}_{\rho \sigma} + \frac{3}{2} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\rho} G^{-1} \nabla_{\sigma} G) \right]
\]

\[
+ \frac{5}{8} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\sigma} G^{-1} \nabla_{\rho} G) + \frac{1}{8} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} G) \text{Tr} (\nabla_{\rho} G^{-1} \nabla_{\sigma} G)
\]

\[
+ \frac{3}{8} F^{(1) m}_{\mu \nu} G_{m n} F^{(1) n}_{\rho \sigma} + \frac{1}{8} F^{(1) m}_{\rho \mu} G_{m n} F^{(1) n}_{\nu \sigma} \frac{1}{2} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} G) F^{(1) m}_{\rho \sigma} G_{m n} F^{(1) n}_{\mu \nu}
\]

\[
- \frac{3}{2} F^{(1) m}_{\mu \nu} \left( \nabla_{\rho} G \nabla_{\sigma} G^{-1} \right)_{m n} F^{(1) n}_{\mu \nu}
\]

\[
+ \text{Tr} (\nabla_{\mu} G^{-1} G \nabla_{\nu} G^{-1} G) + 3 \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\rho} G^{-1} \nabla_{\sigma} G) - 6 \nabla_{\rho} F^{(1) m}_{\mu \nu} \nabla_{\sigma} G_{m n} F^{(1) n}_{\rho \mu \nu}
\]

\[
+ F^{(1) m}_{\mu \nu} \left( G \nabla_{\nu} G^{-1} \nabla_{\mu} G^{-1} \right)_{m n} F^{(1) n}_{\rho \mu \nu} - 2 \nabla_{\rho} \left[ F^{(1) m}_{\mu \nu} G_{m n} \nabla_{\sigma} G \nabla_{\tau} G \right].
\]

(4.8)

Apart from the Riemann tensor, only the five last terms contain second order derivatives. Using partial integration and Bianchi identities, it is possible to transform those terms so that all second order derivatives appear as the leading two-derivative contribution from the field Eqs. (2.17), i.e. appear within the first column of Tab. 1. Details are given in App. B. Specifically, the remaining second order derivative terms combine into

\[
\alpha' \int d^{D}x \sqrt{-g} e^{-\Phi} \left[ \text{Tr} (\Box G^{-1} G \Box G^{-1} G) - 2 \nabla_{\mu} G \text{Tr} (\Box G^{-1} G \nabla_{\mu} G^{-1}) \right]
\]

\[
+ 2 \text{Tr} (\Box G^{-1} G \nabla_{\mu} G^{-1} \nabla_{\nu} G) + \frac{1}{2} \text{Tr} (\Box G^{-1} G \nabla_{\mu} G \nabla_{\nu} G^{-1}) - \frac{5}{4} F^{(1) m}_{\mu \nu} \Box G_{m n} F^{(1) n}_{\mu \nu}
\]

\[
+ (R_{\mu \nu} + \nabla_{\mu} \nabla_{\nu} \Phi) \left( \text{Tr} (\nabla_{\nu} G^{-1} \nabla_{\mu} G) - 2 F^{(1) m}_{\mu \nu} G_{m n} F^{(1) n}_{\rho \nu} \right)
\]

\[
+ 2 \nabla_{\rho} F^{(1) m}_{\mu \nu} G_{m n} \left( \nabla_{\rho} F^{(1) n}_{\mu \nu} - \nabla_{\rho} \Phi F^{(1) n}_{\mu \nu} \right)
\]

\[
+ \left( -2 \nabla_{\rho} \Phi F^{(1) m}_{\mu \nu} G_{m n} + 3 F^{(1) m}_{\mu \nu} \nabla_{\rho} G_{m n} \right) \nabla_{\rho} F^{(1) n}_{\mu \nu} \right].
\]

(4.9)
and can be eliminated by field redefinitions according to the rules defined in Tab. 1. The explicit induced field redefinitions are collected in Eq. (B.4). The final result of the reduction (4.8) then takes the form

\[ \frac{\alpha'}{4} \int d^{D+d} X \sqrt{-g} e^{-\Phi} \tilde{R}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \tilde{R}^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \rightarrow \]

\[ \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{2} R_{\mu\nu\rho\sigma} F^{\mu\nu}_{\mu_1} G^{m_{\mu_1}}_{\rho_1 n_{\rho_1}} + \frac{1}{2} \text{Tr} \left( \nabla_{\mu} G \nabla^\mu G^{-1} \nabla_{\nu} G^{-1} \nabla^\nu B G^{-1} \right) + \frac{1}{8} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\rho} B G^{-1} \nabla_{\sigma} B G^{-1} \right) + \frac{1}{8} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla^\mu G^{-1} \nabla^\nu G \right) \right] \]

\[ \frac{1}{8} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \right) \left( \nabla^\mu G^{-1} \nabla_{\nu} B G^{-1} \right) + \frac{1}{4} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\rho} B G^{-1} \nabla_{\sigma} B G^{-1} \right) + \frac{1}{4} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla^\mu G^{-1} \nabla^\nu G \right) \]

\[ + \frac{1}{4} H_{\mu\nu\rho\sigma} G^{mn} H_{\rho\sigma\tau\sigma} H_{\mu\nu\tau\rho} \left( \frac{1}{16} F^{m}_{\mu\nu} G^{n}_{\rho\sigma} F^{(1) n}_{\rho\sigma} + \frac{1}{8} F^{(1) m}_{\mu\nu} G^{n}_{\rho\sigma} F^{(1) n}_{\rho\sigma} \right) + \frac{1}{8} H_{\mu\nu\rho\sigma} G^{mn} H_{\rho\sigma\tau\sigma} H_{\mu\nu\tau\rho} \left( \frac{1}{16} F^{m}_{\mu\nu} G^{n}_{\rho\sigma} F^{(1) n}_{\rho\sigma} + \frac{1}{8} F^{(1) m}_{\mu\nu} G^{n}_{\rho\sigma} F^{(1) n}_{\rho\sigma} \right) \]

\[ + \frac{1}{2} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \right) \left( \nabla^\mu G^{-1} \nabla_{\nu} B G^{-1} \right) + \frac{1}{4} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla_{\rho} B G^{-1} \nabla_{\sigma} B G^{-1} \right) + \frac{1}{4} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla^\mu G^{-1} \nabla^\nu G \right) \]

\[ - 2 H^{\mu\nu\rho} H_{\mu\nu\rho} \left( G^{-1} \nabla_{\nu} B G^{-1} \right)_{mn} H_{\rho\sigma\tau\sigma} - \frac{1}{2} H^{\mu\nu\rho} H_{\mu\nu\rho} \left( G^{-1} \nabla_{\nu} G \right)_{mn} F^{(1) n} \]

\[ - \frac{1}{4} F^{(1) m}_{\mu\nu} \left( \nabla_{\rho} B G^{-1} \nabla_{\sigma} B G^{-1} \right)_{mn} \left( \nabla_{\nu} B G^{-1} \nabla_{\rho} G \nabla_{\sigma} G \right)_{mn} H_{\mu\nu}^{\rho\sigma} \]

\[ - H_{\mu\nu\rho} \left( G^{-1} \nabla_{\rho} B G^{-1} \nabla_{\nu} B G^{-1} \right)_{mn} H_{\mu\nu}^{\rho\sigma} - \frac{1}{2} F^{(1) m}_{\mu\nu} \left( \nabla_{\rho} G \nabla_{\nu} G \right)_{mn} F^{(1) n} \]

\[ - 2 H_{\mu\nu\rho} \left( G^{-1} \nabla_{\nu} G \nabla_{\rho} B G^{-1} \right)_{mn} H_{\mu\nu}^{\rho\sigma} - H_{\mu\nu\rho} \left( G^{-1} \nabla_{\nu} B G^{-1} \nabla_{\rho} G \right)_{mn} F^{(1) n} \]

\[ \right] . \quad (4.10) \]

**Reduction of \( \tilde{R}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \tilde{H}^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \)**

Let us finally consider the reduction of the term \( \text{RHH} \). The index split gives

\[ \tilde{R}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \tilde{H}^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} = \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} + \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} - 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} - 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} \]

\[ - 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} + 2 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} + 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} + 2 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} \]

\[ + 2 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} + 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} + 4 \tilde{R}_{\mu\nu\rho\sigma} \tilde{H}^{\mu\nu\rho\sigma} \tilde{H}^{\rho\sigma} \]

Then, using Eqs. (2.5) and (4.2), the corresponding term of the action (4.1) gives

\[- \frac{1}{8} \alpha' \int d^{D+d} X \sqrt{-g} e^{-\Phi} \tilde{R}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \tilde{H}^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} \rightarrow \]

\[ \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ - \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} + \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} G^{mn} H^{\rho\sigma} \right] \]

\[ \left. - \frac{1}{4} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} B \nabla_{\rho} G^{-1} \nabla_{\sigma} B \right) - \text{Tr} \left( \nabla_{\mu} B \nabla_{\rho} G^{-1} \nabla_{\nu} B \nabla_{\rho} G^{-1} \nabla_{\sigma} B \right) \right] \]
Apart from the Riemann tensor, the four last terms contain second order derivatives. Just as for the second order derivatives appear as the leading two-derivative contribution from the field Eqs. (4.8). Specifically, the remaining second order derivative terms combine into

\[
\frac{1}{2} \int d^D x \sqrt{-g} e^{-\Phi} \alpha' \left[ \frac{1}{2} \text{Tr} (\Box G^{-1} \nabla_{\rho} B \nabla_{\nu} B G^{-1} \nabla_{\rho} B) - \text{Tr} (G^{-1} \nabla_{\nu} B G^{-1} \nabla_{\rho} B) \right]
\]

(4.13)

and can be eliminated by field redefinitions according to the rules defined in Tab. 1. The explicit induced field redefinitions are collected in Eq. (B.9). The final result of the reduction (4.12) then takes the form

\[
- \frac{1}{8} \alpha' \int d^{D+d} X \sqrt{-g} e^{-\Phi} \hat{R}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{R}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{H}^{\hat{\rho} \hat{\sigma}} \rightarrow \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ -\frac{1}{2} R_{\mu \nu \rho \sigma} H^{\mu \nu} H^{\rho \sigma} - \frac{1}{2} R_{\mu \nu \rho \sigma} H^{\mu \nu} mG^{mn} H^{\rho \sigma} n + \frac{1}{4} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} B \nabla_{\rho} G^{-1} \nabla_{\sigma} B G^{-1} \nabla_{\rho} B) - \frac{1}{2} \text{Tr} (\nabla_{\mu} G^{-1} \nabla_{\nu} B G^{-1} \nabla_{\rho} B G^{-1} \nabla_{\sigma} B G^{-1} \nabla_{\rho} B) \right]
\]

(4.12)
By partial integration and suitable field redefinitions, we have thus cast the reduced action at order $4.2$ Field redefinitions

d, $d,$

In the next section, we will match the result of the explicit reduction against the basis \((3.26)\) in order to establish \(O(d, d, \mathbb{R})\) invariance of the reduced action.

### 4.2 Field redefinitions

By partial integration and suitable field redefinitions, we have thus cast the reduced action at order $\alpha'$ into a form which is polynomial in first order derivatives and the Riemann tensor. As an illustration and for potential applications requiring the dictionary between the lower-dimensional fields and the fields featuring in the original action \((2.1)\), let us list the full set of induced field redefinitions, put together from Eqs. \((B.4)\) and \((B.9)\):

\[
\begin{align*}
\delta \Phi &= \frac{1}{4} \left[ - F_{\mu \nu}^{(1) m} G_{mn} F^{(1) \mu \nu n} + \frac{1}{2} \text{Tr} (\nabla_{\mu} G - 1 \nabla_{\mu} G) \right] , \\
\delta g_{\mu \nu} &= \frac{1}{4} \left[ 2 F_{\mu \nu}^{(1) m} G_{mn} F_{\nu \rho}^{(1) n} - \text{Tr} (\nabla_{\mu} G - 1 \nabla_{\nu} G) \right] , \\
\delta B_{\mu \nu} &= \frac{1}{8} \left[ - 2 \nabla^{\rho} F_{\mu \nu}^{(1) m} + 2 \nabla^{\rho} \Phi F_{\mu \nu}^{(1) m} + \frac{1}{2} H_{\mu \rho \sigma} H^{\rho \sigma \rho} G_{\rho \mu \nu} \\
&\quad + F_{\mu \nu}^{(1) p} (\nabla_{\rho} G_{G} - 1)_{\rho}^{m} + H_{\mu \rho \sigma} (G^{(1) \rho} B G - 1)_{\rho}^{m} \left( A_{\nu}^{(2)} - B_{mn} A_{\nu}^{(1) n} \right) \\
&\quad - A_{\mu}^{(1) m} (G_{\nabla G} - 1)_{m}^{n} H_{\nu \rho n} - A_{\mu}^{(1) m} \nabla^{\rho} B_{mn} F_{\nu \rho}^{(1) n} + 2 F_{\mu \nu}^{(1) m} H_{\nu \rho \nu} \\
&\quad + \frac{1}{2} A_{\mu}^{(1) m} H_{\nu \rho \sigma} G_{mn} F^{(1) \sigma \rho n} - (\mu \leftrightarrow \nu) , \\
\delta G_{\mu \nu} &= \frac{1}{4} \left[ - 2 G_{\mu \nu}^{(1) m} + 2 \nabla_{\mu} \Phi \nabla_{\nu} G_{\mu \nu} - \frac{1}{2} G_{\mu \rho \sigma} H_{\mu \rho \nu} G_{\mu \nu} - 3 \frac{3}{2} F_{\mu \nu}^{(1) m} F_{\nu \rho}^{(1) n} \\
&\quad - (G^{(1) \rho} B G - 1)_{\rho}^{m} + (G^{(1) \rho} B G - 1)_{\rho}^{m} \right] , \\
\delta B_{mn} &= \frac{1}{4} \left[ (G_{\nabla G} - 1)_{mn} + (G_{\nabla G - 1} \nabla_{\mu} B)_{mn} \\
&\quad - \frac{1}{2} H_{\mu \rho \sigma} F^{(1) \mu \nu p} G_{\rho \mu \nu} + \frac{1}{2} G_{\mu \nu \rho} F^{(1) \mu \nu p} H_{\mu \nu \rho} \right] , \\
\delta A_{\mu}^{(1) m} &= \frac{1}{4} \left[ - 2 \nabla^{\nu} F_{\nu \mu}^{(1) m} + 2 \nabla^{\nu} \Phi F_{\nu \mu}^{(1) m} + \frac{1}{2} H_{\mu \rho \nu} H_{\nu \rho \nu} G_{\nu \rho \nu} \\
&\quad + F_{\nu \mu}^{(1) n} (\nabla_{\nu} B G - 1)_{\nu}^{m} + H_{\nu \rho \nu} (G^{(1) \rho} B G - 1)_{\rho}^{m} \right] , \\
\delta A_{\mu}^{(2) m} &= \frac{1}{4} \left[ 2 \nabla^{\nu} F_{\nu \mu}^{(1) n} B_{mn} - 2 \nabla^{\nu} \Phi F_{\nu \mu}^{(1) n} B_{mn} - \frac{1}{2} H_{\mu \rho \nu} H_{\rho \nu \nu} (G^{(1)} B)_{mn} \\
&\quad - F_{\nu \mu}^{(1) n} (\nabla_{\nu} G B G - 1)_{\nu}^{m} - H_{\nu \rho \nu} (G^{(1)} \nabla_{\rho} B G - 1)_{\rho}^{m} + H_{\rho \nu \nu} (\nabla_{\rho} G^{(1)} G)_{\rho}^{m} \right] ,
\end{align*}
\]
where we used the convention of Eq. (2.15).

5 $O(d,d,\mathbb{R})$ invariance and a Green-Schwarz type mechanism

We have now set up all the elements allowing to systematically exhibit the $O(d,d,\mathbb{R})$ invariance of the dimensionally reduced theory at order $\alpha'$. Having brought the reduced action into a form that is polynomial in first derivatives (and the Riemann tensor), we have fully fixed the ambiguities due to field redefinitions and partial integration. We can then compare the result to the distinguished manifestly $O(d,d,\mathbb{R})$ invariant basis constructed in Sec. 3.4, after breaking up the latter under $GL(d)$\textsuperscript{5}. Different terms of the $O(d,d,\mathbb{R})$ basis (3.26) do not share common terms in the decomposition under $GL(d)$, i.e. every $GL(d)$ invariant term we have obtained in the reduction in the previous section has a unique ancestor within the $O(d,d,\mathbb{R})$ basis (3.26). It becomes thus a straightforward – albeit lengthy – task to recombine (if possible) any collection of $GL(d)$ terms into $O(d,d,\mathbb{R})$ invariant expressions.

The dimensionally reduced action is given by the sum of Eqs. (4.4), (4.6), (4.10), and (4.14). Upon combining these terms into the $O(d,d,\mathbb{R})$ invariant expressions of the basis (3.26), we can bring it into the form

$$I_1 = I_1 + O_1,$$

where $I_1$ is the part of $I_1$ that can be organized into a linear combination of manifestly $O(d,d,\mathbb{R})$ invariant basis elements as

$$I_1 = \frac{1}{4} \alpha' \int d^D x \sqrt{-g} e^{-\Phi} \left[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\lambda} H^{\rho\sigma\lambda} - \frac{1}{8} H^2 \nu^2 + \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\lambda} H^{\rho\tau\sigma} + \frac{1}{2} R_{\mu\nu\rho\sigma} F^{\mu\nu} S_{M}^{N} F_{\rho\sigma}^{N} + \frac{1}{16} \text{Tr} (\nabla_{\mu} S \nabla_{\nu} S \nabla^{\mu} S \nabla^{\nu} S) - \frac{1}{32} \text{Tr} (\nabla_{\mu} \nabla_{\nu} S) \text{Tr} (\nabla^{\mu} S \nabla^{\nu} S) \right. \\
+ \frac{1}{8} F^{\mu\nu} M S_{M}^{N} F_{\rho\sigma}^{N} - \frac{1}{2} \mathcal{F}_{\mu\nu}^{\rho\sigma} \mathcal{F}_{\rho\sigma}^{\mu\nu} P_{\rho} \mathcal{F}_{\sigma}^{Q} Q_{\rho} Q + \frac{1}{2} \mathcal{F}_{\mu\nu}^{\rho\sigma} M S_{M}^{N} F_{\rho\sigma}^{\mu\nu} P_{\rho} \mathcal{F}_{\sigma}^{Q} Q + \frac{1}{2} \mathcal{F}_{\mu\nu}^{\rho\sigma} M S_{M}^{N} F_{\rho\sigma}^{\mu\nu} P_{\rho} \mathcal{F}_{\sigma}^{Q} Q \\
+ \frac{1}{4} H^{\mu\nu\rho} F_{\mu\nu}^{\rho} \mathcal{F}_{\rho\sigma}^{M} (S \nabla_{\rho} S)_{M}^{N} F_{\rho}^{\sigma} N - \frac{1}{2} \mathcal{F}_{\mu\nu}^{\rho\sigma} M (S \nabla_{\rho} S)_{M}^{N} F_{\rho}^{\sigma} N \right],$$

whereas the remaining part of the action $O_1$ is not manifestly $O(d,d,\mathbb{R})$ invariant, but given by

$$O_1 = \frac{1}{8} \alpha' \int d^D x \sqrt{-g} e^{-\Phi} H^{\mu\nu\rho} \text{Tr} \left[ \nabla_{\mu} G^{-1} G \nabla_{\nu} G^{-1} \nabla_{\rho} G^{-1} \nabla_{\mu} B G^{-1} \nabla_{\nu} B G^{-1} \right].$$

This suggests the definition

$$\Omega_{\mu\nu\rho} = \frac{3}{4} \text{Tr} (\partial_{\mu} G^{-1} G \partial_{\nu} G^{-1} \partial_{\rho} B) + \frac{1}{4} \text{Tr} (\partial_{\mu} B G^{-1} \partial_{\nu} B G^{-1} \partial_{\rho} B G^{-1}),$$

\textsuperscript{5} See App. C for the $GL(d)$ expressions of the relevant $O(d,d,\mathbb{R})$ terms.
such that \( O_1 \) takes the form

\[
O_1 = \frac{1}{6} \alpha' \int d^D x \sqrt{-g} e^{-\phi} H_{\mu\nu\rho} \Omega^{\mu\nu\rho} .
\]  

(5.5)

The 3-form (5.4) descends from the non-vanishing cohomology \( H^4 \) of \( O(d, d, \mathbb{R})/(O(d) \times O(d)) \) \([31,32]\), although it is not \( O(d, d, \mathbb{R}) \) invariant, its exterior derivative is\(^6\)

\[
4 \partial_{[\mu} \Omega_{\nu\rho]} = \frac{3}{8} \mathrm{Tr} (S \partial_{[\mu} S \partial_{\nu} S \partial_{\rho]} S) .
\]  

(5.6)

For \( \Gamma \in \mathfrak{o}(d, d, \mathbb{R}) \) this implies that \( d\delta \Gamma \Omega = \delta \Gamma d\Omega = 0 \), i.e. the \( O(d, d, \mathbb{R}) \) variation of \( \Omega_{\mu\nu\rho} \) is closed and can locally be integrated to a 2-form \( X_{\mu\nu} \) such that

\[
\delta \Gamma \Omega_{\mu\nu\rho} = 3 \partial_{[\mu} X_{\nu\rho]} .
\]  

(5.7)

This observation together with the particular form of (5.5) suggests a Green-Schwarz type mechanism in order to restore \( O(d, d, \mathbb{R}) \) invariance of the \( D \)-dimensional action. Specifically, the term (5.5) can be absorbed into a deformation of the two-derivative action (2.11) upon redefining

\[
\tilde{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - \alpha' \Omega_{\mu\nu\rho} ,
\]  

(5.8)

such that the kinetic term now produces

\[
- \frac{1}{12} \tilde{H}^{\mu\nu\rho} \tilde{H}_{\mu\nu\rho} = - \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} + \frac{\alpha'}{6} H^{\mu\nu\rho} \Omega_{\mu\nu\rho} + O(\alpha'^2) .
\]  

(5.9)

In view of Eq. (5.7), the deformed field strength (5.8) remains \( O(d, d, \mathbb{R}) \) invariant, if we impose on \( B_{\mu\nu} \) a non-trivial \( O(d, d, \mathbb{R}) \) transformation for \( \Gamma \in \mathfrak{o}(d, d, \mathbb{R}) \) as

\[
\delta \Gamma B_{\mu\nu} = \alpha' X_{\mu\nu} \quad \implies \quad \delta \Gamma \tilde{H}_{\mu\nu\rho} = 0 .
\]  

(5.10)

The resulting theory is then fully \( O(d, d, \mathbb{R}) \)-invariant to first order in \( \alpha' \). In order to compute an explicit expression for \( X_{\mu\nu} \), we start from a general \( \mathfrak{o}(d, d, \mathbb{R}) \) matrix parametrized as

\[
\Gamma_M^N = \begin{pmatrix} a_m^n & b_{mn} \\ c_{mn} & -a_n^m \end{pmatrix} ,
\]  

(5.11)

with \( c^{mn} \) and \( b_{mn} \) antisymmetric. Further defining the \( \mathfrak{o}(d, d, \mathbb{R}) \) matrices

\[
\mathfrak{A}(a)_M^N = \begin{pmatrix} a_m^n & 0 \\ 0 & -a_n^m \end{pmatrix} , \quad \mathfrak{B}(b)_M^N = \begin{pmatrix} 0 & b_{mn} \\ 0 & 0 \end{pmatrix} , \quad \mathfrak{C}(c)_M^N = \begin{pmatrix} 0 & 0 \\ c^{mn} & 0 \end{pmatrix} ,
\]  

(5.12)

the \( \mathfrak{o}(d, d, \mathbb{R}) \) algebra takes the form

\[
\begin{align*}
[\mathfrak{A}(a_1), \mathfrak{A}(a_2)] &= \mathfrak{A}([a_1, a_2]) , \\
[\mathfrak{A}(a), \mathfrak{B}(b)] &= \mathfrak{B}(ab + ba^t) , \\
[\mathfrak{A}(a), \mathfrak{C}(c)] &= -\mathfrak{C}(ca + a^tc) , \\
[\mathfrak{B}(b_1), \mathfrak{B}(b_2)] &= 0 , \\
[\mathfrak{B}(b), \mathfrak{C}(c)] &= \mathfrak{A}(bc) , \\
[\mathfrak{C}(c_1), \mathfrak{C}(c_2)] &= 0 .
\end{align*}
\]  

(5.13)

\(^6\)See App. C for the \( \mathrm{GL}(d) \) expression.
The action of these generators on $G_{mn}$ and $B_{mn}$ is obtained from Eq. (2.9) as

$$\begin{align*}
\delta\Gamma_G &= a G + G a^t - G c B - B c G , \\
\delta\Gamma_B &= a B + B a^t - B c B - G c G + b ,
\end{align*}$$

which, together with Eq. (5.4), yields the general $O(d, d, \mathbb{R})$ variation of $\Omega_{\mu\nu\rho}$

$$\delta\Omega_{\mu\nu\rho} = -\frac{3}{2} \left[ \text{Tr} \left( c \partial_{[\mu} G \partial_{\nu]} G^{-1} \partial_{\rho]} G \right) + \text{Tr} \left( c \partial_{[\mu} B \partial_{\nu]} G^{-1} \partial_{\rho]} B \right) \right].$$

Pulling out one derivative, we extract the explicit form of $X_{\mu\nu}$ from Eq. (5.7):

$$X_{\mu\nu} = \frac{1}{2} \text{Tr} \left( c \partial_{[\mu} (G + B) G^{-1} \partial_{\nu]} (G + B) \right).$$

According to Eq. (5.10), the 2-form thus acquires new transformations only along the nilpotent $\sigma(d, d, \mathbb{R})$ generators $c^{mn}$. This is consistent with the fact that all the other $\sigma(d, d, \mathbb{R})$ generators have a geometric origin and by construction represent manifest symmetries of the dimensionally reduced action. Moreover, with the expression (5.16), one can verify that the algebra of $\sigma(d, d, \mathbb{R})$ transformations (5.13) closes on $B_{\mu\nu}$. Crucially, the deformed $\sigma(d, d, \mathbb{R})$ action (5.10) cannot be absorbed into a redefinition of the fields but represents a genuine deformation of the $O(d, d, \mathbb{R})$ transformation rules.

We may also consider the behavior of Eq. (5.5) under the $\mathbb{Z}_2$ invariance of bosonic string theory that sends $\hat{B} \rightarrow -\hat{B}$. On the $O(d, d, \mathbb{R})$ matrix (2.9) this symmetry acts as \[ Z \rightarrow Z^T Z , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]

The matrix $Z$ is not $O(d, d, \mathbb{R})$-valued since the metric (2.10) transforms as

$$\eta \rightarrow Z\eta Z^T = -\eta \quad \Longrightarrow \quad S \rightarrow -ZSZ .$$

Thus, the $O(d, d, \mathbb{R})$ invariant defined by the r.h.s. of Eq. (5.6) is $\mathbb{Z}_2$ odd. This ensures $\mathbb{Z}_2$ invariance of the action (5.5) since $B_{\mu\nu}$ and its field strength $H_{\mu\nu\rho}$ are also $\mathbb{Z}_2$ odd.

Let us summarize the previous discussion. The bosonic string effective action, including its first order $\alpha'$-corrections, upon compactification on a $d$-torus exhibits a global $O(d, d, \mathbb{R})$ symmetry, provided the $O(d, d, \mathbb{R})$ transformations of the two-derivative action acquire $\alpha'$-corrections according to Eq. (5.10). The full $\alpha'$-corrected transformations are given by

$$\begin{align*}
\delta\Gamma g_{\mu\nu} &= 0 , \\
\delta\Gamma B_{\mu\nu} &= \frac{\alpha'}{2} \text{Tr} \left( c \partial_{[\mu} (G + B) G^{-1} \partial_{\nu]} (G + B) \right) , \\
\delta\Gamma H_{MN} &= \Gamma_{M}^{P} H_{PN} + \Gamma_{N}^{P} H_{MP} , \\
\delta\Gamma F_{\mu\nu}^{M} &= -F_{\mu\nu}^{N} \Gamma_{N}^{M} ,
\end{align*}$$

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for \( \Gamma_{MN} \in \mathfrak{o}(d,d,\mathbb{R}) \) parametrized as Eq. \((5.11)\). To order \( \alpha' \), the \( \mathbb{O}(d,d,\mathbb{R}) \) invariant action is given by

\[
I = \int \! d^Dx \sqrt{-g} \, e^{-\Phi} \left[ R + \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{12} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} + \frac{1}{8} \text{Tr} (\partial_{\mu} S \partial^{\mu} S) - \frac{1}{4} F_{\mu\nu}^M S_M^N F_{N}^{\mu\nu} \\
+ \frac{1}{4} \alpha' \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\lambda} H^{\rho\sigma}_\lambda + \frac{1}{24} H_{\mu\nu\rho} H^\mu_{\rho \lambda} H^\nu_{\lambda \tau} H^\rho_{\tau \sigma} \right) \\
- \frac{1}{8} H_{\mu\nu}^2 H^{2\mu\nu} + \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla_\nu S^\mu \nabla^\nu S) - \frac{1}{32} \text{Tr} (\nabla_\mu S \nabla_\nu S) \text{Tr} (\nabla^\mu S \nabla^\nu S) \\
+ \frac{1}{8} F_{\mu\nu}^M S_M^N F_{\rho\sigma}^N F_{\rho\sigma}^M S^P_Q F_{P}^{\mu\nu} Q - \frac{1}{2} F_{\mu\nu}^M S_M^N F_{\rho\sigma}^N F_{\rho\sigma}^P S^P_Q F_{Q}^{\mu\nu} \\
+ \frac{1}{8} F_{\mu\nu}^M F_{\rho\sigma}^P T_{\mu\rho}^N F_{\nu\sigma}^N - \frac{1}{2} R_{\mu\nu\rho\sigma} F_{\mu\nu}^M S_M^N F_{\rho\sigma}^N \\
+ \frac{1}{8} F_{\mu\nu}^M (S_{\rho\sigma} \nabla^\mu S) - \frac{1}{2} H_{\mu\nu}^2 H^{2\mu\nu} \text{Tr} (\nabla_\rho S \nabla^\mu S) + \frac{1}{4} H^{\mu\nu\lambda} H^{\rho\sigma}_\lambda F_{\mu\rho}^M S_M^N F_{\nu\sigma}^N \\
- \frac{1}{2} F_{\mu\nu}^M (S_{\rho\sigma} \nabla^\mu S) + \frac{1}{4} F_{\mu\nu}^M S_M^N F_{\nu\rho}^N \text{Tr} (\nabla_\mu S \nabla^\rho S) \\
- \frac{1}{2} H^{\mu\nu\rho} F_{\rho\sigma}^M (S_{\nu\sigma} S)^N_{\mu\rho} F_{\rho\sigma}^N \right] + \mathcal{O}(\alpha'^2), \tag{5.20}
\]

with the deformed field-strength \( \tilde{H}_{\mu\nu\rho} \) defined in Eq. \((5.8)\). This constitutes the main result of this paper.

Let us comment on the relation to Ref. \([7]\), where a similar analysis of the first order \( \alpha' \)-corrections is performed, however restricted to the scalar sector, i.e. setting \( A_{\mu}^{(1)m} = A_{\mu}^{(2)m} = B_{\mu\nu} = 0, \eta_{\mu\nu} = g_{\mu\nu} \). Their result is given in their Eq. \((74)\):

\[
I_1 = \frac{1}{8} \alpha' \int \! d^Dx \sqrt{-g} \, e^{-\Phi} \left[ - \text{Tr} (\nabla_\mu \nabla_\nu S \nabla^\mu S \nabla^\nu S) + \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla_\nu S) \text{Tr} (\nabla^\mu S \nabla^\nu S) \\
+ \text{Tr} (\nabla_\mu S \nabla^\mu S \nabla_\nu S \nabla^\nu S) + \frac{1}{8} \text{Tr} (\nabla_\mu S \nabla_\nu S \nabla^\mu S \nabla^\nu S) \right]. \tag{5.21}
\]

Upon partial integration, this can be rewritten as

\[
I_1 = \frac{1}{8} \alpha' \int \! d^Dx \sqrt{-g} \, e^{-\Phi} \left[ - \text{Tr} \left( (\Box S - \nabla_\mu \Phi \nabla^\mu S) (\Box S - \nabla_\nu \Phi \nabla^\nu S) \right) \\
+ (R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi) \text{Tr} (\nabla^\mu S \nabla^\nu S) + \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla_\nu S) \text{Tr} (\nabla^\mu S \nabla^\nu S) \\
+ \text{Tr} (\nabla_\mu S \nabla^\mu S \nabla_\nu S \nabla^\nu S) + \frac{1}{8} \text{Tr} (\nabla_\mu S \nabla_\nu S \nabla^\mu S \nabla^\nu S) \right]. \tag{5.22}
\]

As discussed in Sec. \( 2.2 \), we can then remove the second order derivative terms by performing the \( \mathbb{O}(d,d,\mathbb{R}) \) covariant field redefinitions

\[
\begin{aligned}
\delta \Phi &= \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla^\mu S), \\
\delta g_{\mu\nu} &= - \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla_\nu S), \\
\delta S &= - \frac{1}{2} (\Box S - \nabla_\mu \Phi \nabla^\mu S) + \frac{1}{2} S \nabla_\mu S \nabla^\mu S, \tag{5.23}
\end{aligned}
\]
in the convention of Eq. \((2.15)\), to bring the result into the equivalent form

\[
I_1 = \frac{1}{4} \alpha' \int d^D x \sqrt{-g} e^{-\Phi} \left[ \frac{1}{16} \text{Tr} (\nabla_\mu S \nabla_\nu S \nabla_\mu S \nabla_\nu S) - \frac{1}{32} \text{Tr} (\nabla_\mu S \nabla_\nu S) \text{Tr} (\nabla_\mu \nabla_\nu S) \right].
\]

(5.24)

This precisely coincides with the truncation of Eq. \((5.2)\) to the scalar fields. Our result reproduces also the first order \(\alpha'\) expressions of Refs. \([5, 9]\) for the reduction to \(D = 1\) dimensions.

Let us finally point out that considering the most generic four-derivative action \([29]\)

\[
I_1 = \alpha' \int d^{D+d} x \sqrt{-g} e^{-\phi} \left( \gamma_1 \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} + \gamma_2 \hat{H}^{\mu\nu\rho\sigma} \hat{H}^{\mu\nu\rho\sigma} + \hat{R}_{\mu\nu\rho\sigma} + \gamma_3 \hat{H}^{\mu\rho\sigma} \hat{H}^{\mu\rho\sigma} \hat{H}^{\nu\rho\sigma} \hat{H}^{\nu\rho\sigma} \right)
\]

\[
+ \gamma_4 \hat{R}_{\mu\rho}^{2} \hat{R}_{\nu\sigma}^{2} + \gamma_5 (\hat{H}^{2})^2 + \gamma_6 \hat{H}_{\mu\nu}^{2} \partial^{\rho} \phi \partial^{\sigma} \phi + \gamma_7 \hat{H}^{2} \partial_{\mu} \phi \partial^{\mu} \phi + \gamma_8 \partial_{\mu} \phi \partial^{\mu} \phi \partial_{\nu} \phi \partial^{\nu} \phi \right),
\]

(5.25)

the only choice of coefficients that give rise to an \(O(d,d,\mathbb{R})\) invariant action after reduction on a generic \(d\)-dimensional torus is

\[
\gamma_2 = -\frac{\gamma_1}{2}, \quad \gamma_3 = \frac{\gamma_1}{24}, \quad \gamma_4 = -\frac{\gamma_1}{8}, \quad \gamma_5 = 0, \quad \gamma_6 = 0, \quad \gamma_7 = 0, \quad \gamma_8 = 0,
\]

(5.26)

corresponding to the action \((4.1)\). Indeed, as the definition of \(\Phi\) imposes

\[
\partial_\mu \phi = \partial_\mu \Phi + \frac{1}{2} \text{Tr} \left( G^{-1} \partial_\mu G \right),
\]

(5.27)

the terms proportional to \(\gamma_6, \gamma_7\) and \(\gamma_8\) respectively in Eq. \((5.25)\) produce terms carrying a factor \(\text{Tr} \left( G^{-1} \partial_\mu G \right)\). However, there is no \(O(d,d,\mathbb{R})\)-invariant term in the basis \((3.26)\) that contains such a factor, as shown in App. \(C\). Moreover, these terms cannot cancel each other, as they come with different contraction structures. This imposes \(\gamma_6 = \gamma_7 = \gamma_8 = 0\). The computations detailed in Secs. \(2\) and \(4\) finally implies the remaining coefficients of Eq. \((5.26)\). Only with this choice do the \(\text{GL}(d)\) terms combine into the \(O(d,d,\mathbb{R})\) invariant terms of the basis \((3.26)\). Up to field redefinition, the action \((4.1)\) thus is the unique four-derivative correction exhibiting \(O(d,d,\mathbb{R})\) invariance upon dimensional reduction.

6 Frame formulation

In the previous section we have shown that invariance under rigid \(O(d,d,\mathbb{R})\) transformations requires an \(\alpha'\)-deformation of the transformation rules that resembles a Green-Schwarz mechanism. We will now make this analogy more precise by introducing a frame formalism for which the \(O(d,d,\mathbb{R})\) symmetry remains undeformed, while the local frame transformations acquire \(\alpha'\)-deformations. This formulation uses the standard Green-Schwarz mechanism, albeit with composite gauge fields.

We introduce a frame field \(E \equiv (E_M^A)\) with inverse \(E^{-1} \equiv (E_A^M)\) from which the scalar matrix \((2.9)\) encoding \(G\) and \(B\) can be reconstructed via

\[
\mathcal{H}_{MN} = E_M^A E_N^B \kappa_{AB},
\]

(6.1)

where flat indices are split as \(A = (a, \bar{a})\), and \(\kappa_{AB}\) is a block-diagonal matrix with components \(\kappa_{ab}\) and \(\kappa_{\bar{a}\bar{b}}\). Furthermore, we constrain the frame field by demanding that the ‘flattened’ \(O(d,d,\mathbb{R})\) metric is
also block-diagonal according to
\[
\eta_{AB} \equiv E_A^M E_B^N \eta_{MN} = \begin{pmatrix} \kappa_{ab} & 0 \\ 0 & -\kappa_{ab} \end{pmatrix}, \quad (6.2)
\]
with a relative sign in the space of barred indices reflecting the signature of the \(O(d,d,\mathbb{R})\) metric. In this formalism \(\kappa_{ab}\) and \(\kappa_{\bar{a}\bar{b}}\) need not be Kronecker deltas, and in particular can be spacetime dependent, and so there is a local \(\text{GL}(d) \times \text{GL}(d)\) frame invariance, with transformation rules
\[
\delta \Lambda E_A^M = \Lambda A^B E_B^M, \quad \Lambda A^B = \begin{pmatrix} \Lambda_{ab} & 0 \\ 0 & \bar{\Lambda}_{\bar{a}\bar{b}} \end{pmatrix}. \quad (6.3)
\]

We could partially gauge fix \(\kappa_{AB} = \delta_{AB}\), which reduces the frame transformations to \(\text{SO}(d) \times \text{SO}(d)\), but in the following another gauge fixing is convenient: we identify the components of \(\kappa\) with the metric \(G\) according to
\[
\kappa = \begin{pmatrix} 2G & 0 \\ 0 & 2G \end{pmatrix}, \quad (6.4)
\]
where we used matrix notation. A frame field satisfying the constraint (6.2) and leading to the familiar form of \(\mathcal{H}_{MN}\) is then given by
\[
E \equiv (E^A_M) \equiv \frac{1}{2} \begin{pmatrix} 1 + BG^{-1} & 1 - BG^{-1} \\ G^{-1} & -G^{-1} \end{pmatrix}. \quad (6.5)
\]

In order to derive composite connections from the frame field we define the Maurer-Cartan forms
\[
(E^{-1} \partial_\mu E)_{AB} \equiv \begin{pmatrix} Q_{\mu a}^b & P_{\mu a}^{\bar{b}} \\ P_{\mu \bar{a}}^b & \bar{Q}_{\mu \bar{b}} \end{pmatrix}. \quad (6.6)
\]
From this definition one finds that under \(\text{GL}(d) \times \text{GL}(d)\) transformations (6.3) the \(P_\mu\) transform as tensors, and the \(Q_\mu\) transform as connections:
\[
\delta \Lambda Q_{\mu a}^b = -D_\mu \Lambda_{a}^b, \quad \delta \Lambda \bar{Q}_{\mu \bar{a}}^{\bar{b}} = -D_\mu \bar{\Lambda}_{\bar{a}\bar{b}}, \quad (6.7)
\]
with \(D_\mu \Lambda_{a}^b = \partial_\mu \Lambda_{a}^b + [Q_\mu, \Lambda]_a^b\) and a similar formula for the barred expression. We can evaluate these connections for the gauge choice (6.5),
\[
\begin{cases} 
Q_\mu = -\frac{1}{2} \partial_\mu (G - B)G^{-1}, \\
\bar{Q}_\mu = -\frac{1}{2} \partial_\mu (G + B)G^{-1}, 
\end{cases} \quad (6.8)
\]
using again matrix notation.

Having constructed composite gauge fields from the frame field we can consider the familiar Chern-Simons three-forms built from them:
\[
\text{CS}_{\mu \nu \rho}(Q) \equiv \text{Tr} \left( Q_{[\mu} \partial_{\nu} Q_{\rho]} + \frac{2}{3} Q_{[\mu} Q_{\nu} Q_{\rho]} \right). \quad (6.9)
\]
These Chern-Simons forms transform under Eq. (6.7) as
\[
\delta_\lambda \text{CS}_{\mu\nu\rho}(Q) = \partial_\mu \text{Tr}(\partial_\nu \lambda Q_\rho),
\] (6.10)
with the barred formulas being analogous. Evaluating the Chern-Simons-form with Eq. (6.8) one recovers precisely the expression (5.4) encountered in the previous section, up to a global factor 3. Therefore, we can define a 3-form curvature with Chern-Simons modification:
\[
\hat{H}_{\mu\nu\rho} \equiv H_{\mu\nu\rho} - \frac{3}{2} \alpha' (\text{CS}_{\mu\nu\rho}(Q) - \text{CS}_{\mu\nu\rho}(\bar{Q})) ,
\] (6.11)
which then reproduces the term proportional to $\Omega H$ encountered in the $O(\alpha')$ action.

We have thus succeeded to find a formulation for which the $O(d,d,\mathbb{R})$ invariance is manifestly realized without deformation. Rather, the $GL(d) \times GL(d)$ gauge symmetry is deformed by having a 2-form transforming according to the Green-Schwarz mechanism,
\[
\delta B_{\mu\nu} = \frac{1}{2} \alpha' \text{Tr}(\partial_{[\mu} \lambda Q_{\nu]}) - \frac{1}{2} \alpha' \text{Tr}(\partial_{[\mu} \bar{\lambda} \bar{Q}_{\nu]}).\] (6.12)
Performing a partial gauge fixing to $SO(d) \times SO(d)$, together with appropriate field redefinitions, this Green-Schwarz mechanism relates to the reduction of $\alpha'$-deformed double field theory [21]. This formulation is related to the one of the previous section as follows: if one fully gauge fixes $GL(d) \times GL(d)$ the $O(d,d,\mathbb{R})$ transformations acquire deformations through compensating frame transformations and hence the singlet $B_{\mu\nu}$ starts transforming non-trivially under $O(d,d,\mathbb{R})$.

Let us close this section by discussing how the $\mathbb{Z}_2$ invariance (5.17) of bosonic string theory is realized in this frame formulation. The $\mathbb{Z}_2$ acts on the frame field as
\[
E \to Z^T E \hat{Z}, \quad \hat{Z} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\] (6.13)
The matrix $\hat{Z}$ exchanges the two $GL(d)$ factors and hence exchanges the role of unbarred and barred indices. Indeed, under the transformation (6.13) the Maurer-Cartan forms (6.6) transform as $P_{\mu} \leftrightarrow \bar{P}_{\mu}$ and $Q_{\mu} \leftrightarrow \bar{Q}_{\mu}$, as one may verify by a quick computation and as is suggested by the explicit form (6.8). Thus, the relative sign in Eq. (6.11) implies that the total Chern-Simons form is $\mathbb{Z}_2$ odd, which together with $B_{\mu\nu} \to -B_{\mu\nu}$ implies $\mathbb{Z}_2$ invariance of the action.

## 7 Gravitational Chern-Simons form of the heterotic supergravity

In this section, we repeat the above analysis of the first order $\alpha'$-corrections for the case of the heterotic string. In absence of the Yang-Mills field in ten dimensions, the bosonic part of the four-derivative effective action of the heterotic string takes the form [29]
\[
\hat{I}_1 = \frac{1}{4} \alpha' \int d^{D+d}X \sqrt{-\hat{g}} \, e^{-\hat{\phi}} \left[ -\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\phi}^{(\hat{\omega})}_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{1}{2} \left( \hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} - \frac{1}{2} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \right) - \frac{1}{8} \hat{H}^{2\hat{\mu}\hat{\nu}} \hat{H}^{2\hat{\mu}\hat{\nu}} + \frac{1}{24} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \right],
\] (7.1)
where $D + d = 10$, and $\hat{\phi}^{(\hat{\omega})}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ is the gravitational Chern-Simons form, defined as
\[
\hat{\phi}^{(\hat{\omega})}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \text{Tr}(\hat{\phi}_{\hat{\mu}\hat{\nu}} \partial_{\hat{\rho}} \hat{\phi}) + \frac{2}{3} \text{Tr}(\hat{\phi}_{\hat{\mu}\hat{\nu}} \hat{\phi}_{\hat{\rho}}),
\] (7.2)
in terms of the spin connection
\[ \hat{\omega}_{\mu \dot{\alpha}} = \nabla_{\mu} \hat{\epsilon}_{\dot{\alpha}} \hat{\epsilon}^{\dot{\beta}}. \] (7.3)

Apart from the Chern-Simons contribution, the action (7.1) coincides (up to a factor 1/2) with the bosonic string (4.1) whose reduction we have already cast into \( O(d, d, \mathbb{R}) \) invariant form in Secs. 4 and 5 above. For the analysis of Eq. (7.1) it thus remains to reduce the first term \( \hat{H}^{\hat{\mu} \hat{\nu} \hat{\rho}} \hat{\Omega}^{(\hat{\omega})}_{\hat{\mu} \hat{\nu} \hat{\rho}} \). We follow the same systematics outlined in Secs. 4 and 5.

In the flat basis, after dimensional reduction, the non-vanishing components of the spin connection are given by

\[ \hat{\omega}_{\alpha, \beta} = \hat{\omega}_{\alpha, \beta}, \]
\[ \hat{\omega}_{\alpha, \beta} = \frac{1}{2} \epsilon_{\alpha}^{\mu} \epsilon_{\beta}^{\nu} \eta_{\alpha b} E_{\mu}^{\nu b} F_{\mu \nu}^{(1) m}, \]
\[ \hat{\omega}_{\alpha, ab} = \epsilon_{\alpha}^{\mu} Q_{\mu a} \epsilon_{\eta_{b c b}}, \]
\[ \hat{\omega}_{\alpha, \beta} = -\frac{1}{2} \epsilon_{\alpha}^{\mu} \epsilon_{\beta}^{\nu} \eta_{\alpha b} E_{\mu}^{\nu b} F_{\mu \nu}^{(1) m}, \]
\[ \hat{\omega}_{a, \beta} = \epsilon_{\alpha}^{\mu} \tilde{P}_{\mu a} \epsilon_{\eta_{b c b}}. \] (7.4)

Here, \( \tilde{P}_{\mu a} \) and \( \tilde{Q}_{\mu a} \) are, respectively, the symmetric and antisymmetric parts of the GL(\( d \)) Maurer-Cartan form \( \tilde{J}_{\mu a} = E_{a}^{m} \partial_{\mu} E_{m} = \tilde{P}_{\mu a} \) + \( \tilde{Q}_{\mu a} \) and verify the integrability relations
\[ \partial_{[\mu} \tilde{J}_{\nu]} a = - (\tilde{J}_{[\mu} \tilde{J}_{\nu]} a)^{b} \Leftrightarrow \begin{cases} \partial_{[\mu} \tilde{P}_{\nu]} a^{b} = - (\tilde{J}_{[\mu} \tilde{J}_{\nu]} a)^{b} - (\tilde{Q}_{[\mu} \tilde{P}_{\nu]} a)^{b}, \\ \partial_{[\mu} \tilde{Q}_{\nu]} a^{b} = - (\tilde{J}_{[\mu} \tilde{J}_{\nu]} a)^{b} - (\tilde{P}_{[\mu} \tilde{P}_{\nu]} a)^{b}. \end{cases} \] (7.5)

Defining the low-dimensional components of \( \hat{\Omega}^{(\hat{\omega})} \) in the same way as we did for \( \hat{H} \) in Eq. (2.5), we obtain

\[ \hat{\Omega}_{\mu \rho}^{(\hat{\omega})} = \hat{\Omega}_{\mu \rho}^{(\hat{\omega})} - \frac{1}{3} \text{Tr} \left( \tilde{J}_{[\mu} \tilde{J}_{\nu]} \tilde{J}_{\rho]} \right) - \frac{1}{2} F_{[\mu}^{(1) m} G_{m n} \nabla_{[\nu} F_{\rho]}^{(1) n} \sigma + \frac{1}{4} F_{[\mu}^{(1) m} \nabla^{\sigma} G_{m n} F_{\rho]}^{(1) n} \sigma \] 
\[- \frac{1}{4} \epsilon_{\alpha}^{\sigma} \nabla_{[\nu] \epsilon_{\alpha}^{\rho} F_{[\mu}^{(1) m} G_{m n} F_{\rho]}^{(1) n} \sigma}, \]
\[ \hat{\Omega}_{\mu n m}^{(\hat{\omega})} = \frac{1}{6} R_{\mu \rho \sigma} F_{[\mu}^{(1) \rho} G_{n m} - \frac{1}{12} F_{[\mu}^{(1) m} \left( \nabla_{\rho} G G^{-1} \nabla_{\rho} G \right)_{n m} - \frac{1}{6} F_{[\mu}^{(1) \rho} \left( \nabla_{\rho} G G^{-1} \nabla_{\rho} G \right)_{n m} \] 
\[- \frac{1}{24} G_{m n} F_{[\mu}^{(1) \rho} G_{[p q]} F_{\rho]}^{(1) q} \sigma - \frac{1}{12} G_{m n} F_{[\mu}^{(1) \rho} G_{[p q]} F_{\rho]}^{(1) q} \sigma - \frac{1}{12} G_{m n} F_{[\mu}^{(1) \rho} G_{[p q]} F_{\rho]}^{(1) q} \sigma - \frac{1}{6} \nabla_{\mu} \nabla_{\nu} G_{m n} F_{[\nu]}^{(1) n} \rho, \]
\[ \hat{\Omega}_{\mu n m}^{(\hat{\omega})} = \frac{1}{6} \left( \nabla_{\nu} G G^{-1} \nabla_{\nu} \nabla_{\mu} \nabla_{(n} G_{m]} \right). \] (7.6)

We can now focus on the reduction of the action. Splitting the ten-dimensional indices \( \hat{\mu} \) into \((\mu, m)\), we obtain
\[ \hat{H}^{\hat{\mu} \hat{\nu} \hat{\rho}} \hat{\Omega}^{(\hat{\omega})}_{\hat{\mu} \hat{\nu} \hat{\rho}} = H^{\mu \rho} \hat{\Omega}^{(\hat{\omega})}_{\mu \rho} + 3 H^{\mu m} \hat{\Omega}^{(\hat{\omega})}_{\mu m} + 3 H^{mn} \hat{\Omega}^{(\hat{\omega})}_{mn} + H^{mnp} \hat{\Omega}^{(\hat{\omega})}_{mnp}. \] (7.7)
Using the explicit expressions of Eqs. (2.5) and (7.6), the reduced Chern-Simons form then takes the form
\[ -\frac{1}{4} \alpha' \int d^{D+d}X \sqrt{-g} e^{-\phi} \hat{H}^{\mu\rho} \hat{\mathcal{G}}^{(\omega)}_{\mu\rho} \rightarrow \]
\[ \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\phi} \left[ -H^{\mu\rho} \mathcal{G}^{(\omega)}_{\mu\rho} + \frac{1}{3} H^{\mu\rho} \text{Tr} \left( \mathcal{J}_{\mu} \mathcal{J}_{\rho} \right) - \frac{1}{2} R^{\mu\rho\sigma\rho} F^{(1)}_{\mu \rho m} H_{\rho \sigma m} \right. \]
\[ \left. - \frac{1}{4} H^{\mu\rho} F^{(1)}_{\mu \rho} \nabla^\sigma G_{mn} F^{(1)}_{\nu \rho} + \frac{1}{8} F^{(1)}_{\mu \rho} G_{mn} F^{(1)}_{\nu \rho} F^{(1)}_{\rho \sigma \mu} \right] \]
\[ + \frac{1}{4} F^{(1)}_{\mu \rho} m \nabla^\rho G_{mn} F^{(1)}_{\nu \sigma\mu} - \frac{1}{4} H^{\mu \rho\sigma\rho} F^{(1)}_{\mu \rho m} \nabla^\rho G^{(1)} G_{mn} F^{(1)}_{\nu \sigma\mu} \]  
\[ - \frac{1}{4} F^{(1)}_{\mu \rho} m \nabla^\rho G^{(1)} G_{mn} F^{(1)}_{\nu \sigma\mu} + \frac{1}{2} H^{\mu \rho\sigma\rho} m \nabla^\rho G^{(1)} G_{mn} F^{(1)}_{\nu \sigma\mu} \]
\[ + \frac{1}{2} H^{\mu \rho\sigma\rho} m \nabla^\rho G^{(1)} m n F^{(1)}_{\nu \sigma\mu} - \frac{1}{4} F^{(1)}_{\mu \rho} m \nabla^\rho B_{mn} F^{(1)}_{\nu \sigma\mu} \]  
\[ (7.8) \]

Only the six last terms carry second order derivatives. Following the systematic of Sec. 4, these terms can be transformed by means of partial integration and Bianchi identities such that all second order derivatives appear as the leading two-derivative contribution from the field Eqs. (2.17), i.e. appear within the first column of Tab. 1. Details are given in App. B.3. Specifically, the remaining second order derivative terms combine into
\[ \frac{\alpha'}{8} \int d^D x \sqrt{-g} e^{-\phi} \left[ \nabla^\mu H^{\mu\rho} m F^{(1)}_{\sigma\rho} m m \nabla^\nu e_\sigma e_\alpha e_\alpha - \text{Tr} \left( D B^{\mu\rho} G^{\nu\sigma} G^{\gamma\delta} - D G^{\nu\sigma} G^{\gamma\delta} B \right) \right] \]
\[ - \nabla^\mu F^{(1)}_{\mu\nu\sigma\rho} m m \nabla^\nu G^{(1)} m n F^{(1)}_{\nu\sigma\rho} \]
\[ - \frac{1}{2} H^{\mu\rho\sigma\rho} m \nabla^\rho \nabla^\nu \nabla^\sigma \nabla^\rho B_{mn} m m F^{(1)}_{\nu\sigma\rho} \]  
\[ (7.9) \]

and can be eliminated by field redefinitions according to the rules defined in Tab. 1. These take the explicit form (in the convention of Eq. (2.15))
\[ \delta B_{\mu\nu} = \frac{1}{8} A^{(1)}_{[\mu,\nu]} e_\sigma e_\alpha \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m + \frac{1}{8} A^{(2)}_{[\mu,\nu]} \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m - \frac{1}{8} B_{mn} A^{(1)}_{[\mu,\nu]} \nabla^\rho G^{mp} H^{(1)}_{\sigma\rho} m \]
\[ - \frac{1}{16} A^{(2)}_{[\mu,\nu]} \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m + \frac{1}{16} B_{mn} A^{(1)}_{[\mu,\nu]} \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m - \frac{1}{8} A^{(2)}_{[\mu,\nu]} \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m \]
\[ + \frac{1}{8} B_{mn} A^{(1)}_{[\mu,\nu]} \nabla^\rho G^{mn} F^{(1)}_{\sigma\rho} m m \]
\[ \delta G^{mn} = \frac{1}{4} \left( \nabla^\mu G^{\nu\sigma} G^{\rho\delta} \right)^{mn} - \frac{1}{8} F^{(1)}_{\mu\nu\sigma\rho} m m H^{(1)}_{\rho\sigma} \]
\[ \delta B_{mn} = \frac{1}{4} \left( \nabla^\mu G^{\nu\sigma} G^{\rho\delta} \right)^{mn} \]
\[ \delta A^{(1)}_{\mu} = \frac{1}{8} \nabla^\mu G^{mn} H_{\mu\nu\sigma\rho} m m F^{(1)}_{\sigma\rho} m m + \frac{1}{16} H^{\mu\rho\sigma\rho} F^{(1)}_{\mu\rho m} m m F^{(1)}_{\sigma\rho} m m \]
\[ + \frac{1}{8} \left( G^{\nu\sigma} G^{\rho\delta} \right)^{mn} F^{(1)}_{\rho\sigma} m m \]
\[ \delta A^{(2)}_{\mu} = \frac{1}{8} \nabla^\mu G^{mn} H_{\mu\nu\sigma\rho} m m F^{(1)}_{\sigma\rho} m m + \frac{1}{16} H^{\mu\rho\sigma\rho} F^{(1)}_{\mu\rho m} m m F^{(1)}_{\sigma\rho} m m \]
This form is closed by virtue of the integrability relations (7.5). It can thus locally be integrated into a 2-form

\[ \Omega^{(j)}_{\mu \nu \rho} = 3 \partial_{[\mu} \theta_{\nu \rho]}^{WZW} . \]  

(7.14)
such that the last term in Eq. (7.12) can be absorbed into a field redefinition

$$\delta B_{\mu\nu} = \frac{1}{2} \theta_\mu^{\nu wz}.$$  (7.15)

As $\Omega^{(\tilde{j})}_{\mu\nu\rho}$ is $O(d,d,\mathbb{R})$-invariant, this does not affect the behaviour of $B_{\mu\nu}$ under $O(d,d,\mathbb{R})$ transformations. Putting everything together, the reduced action for the bosonic part of heterotic supergravity (in absence of the ten-dimensional Yang-Mills field) is obtained by combining Eqs. (5.2) and (7.12) into

$$I = \int d^Dx \sqrt{-g} e^{-\Phi} \left[ R + \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} \text{Tr} (\partial_\mu S \partial^\mu S) - \frac{1}{4} \mathcal{F}_\mu^\nu M S_M^N F_N^\rho \right]$$

$$- \frac{1}{4} \alpha' \left( H^{\mu\nu\rho} \Omega^{(\omega)}_{\mu\nu\rho} - \frac{1}{16} \text{Tr} (S \nabla_\mu S \nabla^\mu S \nabla^\nu S) - \frac{1}{16} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma M \mathcal{F}_\sigma^\rho P S_P^Q \mathcal{F}_Q^\rho \right)$$

$$+ \frac{1}{4} \mathcal{F}_\mu^\nu M \nabla^\rho \mathcal{S}_M M_N F_N^\nu + \frac{1}{8} H^{\mu\nu\rho} \mathcal{F}_\mu^\nu M \nabla^\sigma \mathcal{S}_M^N F_N^\rho \right)$$

$$+ \frac{1}{8} \alpha' \left( R_{\mu\rho\nu\sigma} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma \right)$$

$$\left[ 1 + \frac{1}{4} \text{Tr} (\nabla_\mu S \nabla_\nu S \nabla^\mu S \nabla^\nu S) - \frac{1}{32} \text{Tr} (\nabla_\mu S \nabla_\nu S) \text{Tr} (\nabla^\mu S \nabla^\nu S)$$

$$+ \frac{1}{8} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma M \mathcal{F}_\sigma^\rho \mathcal{F}_\mu^\nu P S_P^Q \mathcal{F}_Q^\sigma \right. \right]$$

$$- \frac{1}{2} R_{\mu\rho\nu\sigma} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma \right]$$

$$+ \frac{1}{8} \text{Tr} (\nabla^\mu S \nabla^\nu S) - \frac{1}{2} H^{\mu\nu\rho} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma \mathcal{S}_M M_N F_N^\sigma$$

$$- \frac{1}{2} \frac{1}{4} \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma \mathcal{F}_\mu^\nu M \mathcal{F}_\rho^\sigma \right]$$

$$- \frac{1}{2} H^{\mu\nu\rho} \mathcal{F}_\mu^\nu M \mathcal{S}_M \mathcal{F}_\rho^\sigma \mathcal{F}_\rho^\sigma \mathcal{N}_M^N \mathcal{F}_\rho^\sigma \mathcal{F}_\rho^\sigma \right]$$

Let us finally note that one could have started equivalently from the ten-dimensional action formulated in terms of the gravitational Chern-Simons form built from the Christoffel connection

$$\hat{\Omega}^{(\hat{\phi})}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{\Gamma}^\tau_{\hat{\mu}\hat{\nu}} + \hat{\Gamma}^\sigma_{\hat{\mu}\hat{\rho}} + \frac{2}{3} \hat{\Gamma}^\tau_{\hat{\mu}\hat{\sigma}} \hat{\Gamma}^\sigma_{\hat{\nu}\hat{\rho}} \hat{\Gamma}^\rho_{\hat{\mu}\hat{\sigma}}.$$  (7.16)

This form is invariant under Lorentz transformations and related to Eq. (7.2) by [16]

$$\hat{\Omega}^{(\hat{\phi})}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{\Omega}^{(\hat{\phi})}_{\hat{\mu}\hat{\nu}\hat{\rho}} + \partial_{\hat{\nu}} \left( \hat{\partial}_{\hat{\rho}} \hat{\epsilon}^\hat{\sigma} \hat{\epsilon}^\hat{\delta} \hat{\omega}[\hat{\mu}\hat{\nu}\hat{\rho}] \right) + \frac{1}{3} \text{Tr} \left( \partial_{\hat{\mu}} \hat{\epsilon} \hat{\epsilon}^{-1} \partial_{\hat{\nu}} \hat{\epsilon} \hat{\epsilon}^{-1} \partial_{\hat{\rho}} \hat{\epsilon} \hat{\epsilon}^{-1} \right),$$  (7.17)

with the difference given by two closed terms that can be absorbed by a ten-dimensional field redefinition. Dimensional reduction of the resulting ten-dimensional action then induces a lower-dimensional action in which the $\text{Tr}(\hat{J}^3)$ term from Eq. (7.12) is no longer present. The field redefinition required in order to absorb the closed terms of Eq. (7.18) precisely corresponds to the lower-dimensional field redefinition we have encountered in Eq. (7.15).
8 Conclusions

In this paper we have set up a systematic procedure for analyzing the higher-derivative corrections of the bosonic and the heterotic string upon toroidal compactification. In particular, we have discussed how to control the ambiguities that arise due to non-linear field redefinitions and partial integration. This establishes the basis for analyzing the realization of $O(d, d, \mathbb{R})$ invariance of the dimensionally reduced action. At first order in $\alpha'$, we have presented the explicit reduction of the bosonic string and cast the result into a manifestly $O(d, d, \mathbb{R})$ invariant form upon identification of the necessary field redefinitions. In particular, the analysis confirms that at order $\alpha'$, the $O(d, d, \mathbb{R})$ invariance of the dimensionally reduced action fixes all the couplings in higher dimensions (up to an overall factor). The analysis has revealed the need for a Green-Schwarz type mechanism by which the lower-dimensional two-form (which is originally singlet under $O(d, d, \mathbb{R})$) acquires a non-trivial transformation of order $\alpha'$. This is a genuine deformation which cannot be eliminated by further field redefinitions.

We have also extended the analysis to the bosonic sector of the heterotic string (in absence of the ten-dimensional vector fields). In particular, we have given the complete set of non-linear field redefinitions (4.15), (7.10) which translate between the original ten-dimensional fields and the $O(d, d, \mathbb{R})$-covariant lower-dimensional fields. This dictionary allows to exploit the $O(d, d, \mathbb{R})$ symmetry as a solution generating method for the heterotic string [33, 34] to first order in $\alpha'$. It would be very interesting to extend the analysis to also include the ten-dimensional vector fields [35], resulting in an $O(d, d + K, \mathbb{R})$ extension of the present results with the larger group broken down by the non-abelian gauge couplings [26].

In principle, the method we have outlined is fully systematic and could be applied to higher-order $\alpha'$-corrections. In practice, the number of terms quickly explodes and calls for complementary techniques to be combined with the present approach. As noted above, already at order $\alpha'^2$ the number of manifestly $O(d, d, \mathbb{R})$ invariant terms in lower dimensions amounts to 1817. Nevertheless, it would be interesting to compare the resulting structures to related work in Ref. [36, 37]. It would also be interesting to investigate the effect of $\alpha'$-corrections on the more general Yang-Baxter type deformations recently explored in Ref. [38].

Finally, it will be interesting to further study the simplifications arising in the resulting actions upon reduction to particularly low dimensions $D$. For $D = 1$, all terms other than the scalar couplings disappear from Eqs. (5.20) and (7.16), and we recover the lowest-order result of Refs. [5, 9, 10]. At $D = 2$, the two-form couplings disappear and the vector fields may be integrated out. Particularly interesting is the three-dimensional case. At $D = 3$, the two-form may be integrated out. With a field equation of the type

$$\nabla^\mu (e^{-\Phi} H_{\mu\nu\rho}) = O(\alpha'),$$

this introduces an integration constant which in particular turns the coupling (5.4) into a three-dimensional analogue of the WZW model, c.f. Ref. [31]. Furthermore, in $D = 3$, the (abelian) vector fields may be dualized into scalars. While this dualization is still possible in the presence of $\alpha'$-corrections, the symmetry enhancement to $O(d + 1, d + 1, \mathbb{R})$ encountered for the two-derivative action breaks down at order $\alpha'$ and is replaced by the appearance of the relevant automorphic forms [39, 40].
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Appendix

A Basis at order $\alpha'$

In this appendix, we explicitly spell out the $O(d, d, \mathbb{R})$ invariant basis schematically given in Eq. (3.26), whose existence we have deduced in Sec. 3.4 and which we have used in order to bring the reduced action into manifestly $O(d, d, \mathbb{R})$ invariant form. The basis is built from 61 terms which we list according to their different structures.

$R^2$:

\[ \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\} \]  (A.1)

$H^4$:

\[ \left\{ (H^2)^2, H^{2\mu\nu} H_{\mu\nu}^2, H_{\mu\nu\alpha} H^\alpha_{\mu} H^\nu_{\beta} H^\rho_{\gamma} \right\} \]  (A.2)

$(\nabla \Phi)^4$:

\[ \left\{ \nabla_{\mu} \Phi \nabla^{\mu} \Phi \nabla_{\nu} \Phi \nabla^{\nu} \Phi \right\} \]  (A.3)

$(\nabla S)^4$:

\[ \left\{ \text{Tr} (\nabla_{\mu} S \nabla^{\mu} S \nabla_{\nu} S \nabla^{\nu} S), \text{Tr} (\nabla_{\mu} S \nabla^{\mu} S \nabla_{\nu} S \nabla^{\nu} S), \text{Tr} (S \nabla_{\mu} S \nabla^{\mu} S \nabla_{\nu} S \nabla^{\nu} S), \text{Tr} (\nabla_{\mu} S \nabla^{\mu} S \nabla_{\nu} S \nabla^{\nu} S) \right\} \]  (A.4)

$\mathcal{F}^4$:

\[ \left\{ \mathcal{F}^{\mu\nu}_{\alpha\beta}, \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta}, \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta}, \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \right\} \]  (A.5)

$R H^2$:

\[ \left\{ R_{\mu\nu\rho\sigma} H^{\mu\nu\lambda} H^{\rho\sigma}_{\lambda} \right\} \]  (A.6)

$R \mathcal{F}^2$:

\[ \left\{ R_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta}, R_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \mathcal{F}^{\mu\nu}_{\alpha\beta} \mathcal{F}^{\rho\sigma}_{\alpha\beta} \right\} \]  (A.7)
\( H^2 (\nabla \Phi)^2 : \)
\[
\left\{ H^2_{\mu\nu} \nabla^\mu \Phi \nabla^\nu \Phi, H^2 \nabla \mu \Phi \nabla^\mu \Phi \right\}
\]  \hspace{1cm} (A.8)  

\( H^2 (\nabla S)^2 : \)
\[
\left\{ H^2_{\mu\nu} \text{Tr}(\nabla^\mu S \nabla^\nu S), H^2 \text{Tr}(\nabla \mu S \nabla^\mu S) \right\}
\]  \hspace{1cm} (A.9)  

\( H^2 F^2 : \)
\[
\left\{ H^2 F_{\mu\nu}^M F^\mu_{\mu\nu}^M, H^2 F_{\mu\nu}^M S_M^N F^\mu_{\mu\nu}^N, H^2_{\mu\nu} F_{\mu\rho}^M F^\rho_{\nu M}, H^2_{\mu\nu} F^\mu_{\mu\rho} M S_M^N F^\nu_{\rho N},
H_{\mu\nu}^\lambda H_{\lambda\rho\sigma} F^\mu_{\nu M} F^\rho_{\sigma M}, H_{\mu\nu}^\lambda H_{\lambda\rho\sigma} F^\mu_{\nu M} S_M^N F^\rho_{\sigma N},
H_{\mu\nu}^\lambda H_{\lambda\rho\sigma} F^\mu_{\nu M} F^\rho_{\sigma N}, H_{\mu\nu}^\lambda H_{\lambda\rho\sigma} F^\mu_{\nu M} F^\rho_{\sigma M}, \right\}
\]  \hspace{1cm} (A.10)  

\( (\nabla \Phi)^2 (\nabla S)^2 : \)
\[
\left\{ \nabla \mu \Phi \nabla^\mu \Phi \text{Tr}(\nabla^\nu S \nabla^\nu S), \nabla \mu \Phi \nabla^\mu \Phi \text{Tr}(\nabla \nu S \nabla^\nu S) \right\}
\]  \hspace{1cm} (A.11)  

\( (\nabla \Phi)^2 F^2 : \)
\[
\left\{ \nabla \rho \Phi \nabla^\rho \Phi F^\mu_{\mu\nu} M F^\mu_{\mu\nu} M, \nabla \rho \Phi \nabla^\rho \Phi F^\mu_{\mu\nu} M S_M^N F^\mu_{\mu\nu} N,
\nabla^\rho \Phi \nabla^\rho \Phi F^\mu_{\mu\nu} M F^\rho_{\mu M}, \nabla^\rho \Phi \nabla^\rho \Phi F^\rho_{\mu M} S_M^N F^\rho_{\mu N} \right\}
\]  \hspace{1cm} (A.12)  

\( (\nabla S)^2 F^2 : \)
\[
\left\{ \text{Tr}(\nabla^\rho S \nabla^\rho S) F^\rho_{\mu\nu} M F^\rho_{\mu\nu} M, \text{Tr}(\nabla^\rho S \nabla^\rho S) F^\rho_{\mu\nu} M S_M^N F^\rho_{\mu\nu} N,
\text{Tr}(\nabla^\mu S \nabla^\nu S) F^\mu_{\rho \sigma} M, \text{Tr}(\nabla^\mu S \nabla^\nu S) F^\mu_{\rho \sigma} M S_M^N F^\rho_{\sigma N},
F_{\mu\nu}^M \nabla \rho S_M^N \nabla^\rho S_M^P F^\mu_{\mu\nu} P, F_{\mu\nu}^M \nabla \rho S_M^N \nabla^\rho S_M^P S_P^Q F^\mu_{\mu\nu} Q,
F_{\mu\nu}^M S_M^N \nabla \rho S_M^P F^\rho_{\mu \nu} P, F_{\mu\nu}^M S_M^N \nabla \rho S_M^P S_P^Q F^\rho_{\mu \nu} Q,
F_{\mu\nu}^M S_M^N \nabla^\nu S_M^P F^\rho_{\mu \nu} P, F_{\mu\nu}^M \nabla \rho S_M^N \nabla^\nu S_M^P S_P^Q F^\rho_{\mu \nu} Q \right\}
\]  \hspace{1cm} (A.13)  

\( H \nabla \Phi F^2 : \)
\[
\left\{ H^{\mu\nu} \nabla^\sigma \Phi F^\mu_{\mu\nu}^M F^\rho_{\nu M}, H^{\mu\nu} \nabla^\sigma \Phi F^\mu_{\nu \nu}^M S_M^N F^\rho_{\nu N} \right\}
\]  \hspace{1cm} (A.14)  

\( H \nabla S F^2 : \)
\[
\left\{ H^{\mu\nu} F_{\mu\sigma} M \nabla \sigma S_M^N S_N^P F^\mu_{\sigma P}, H^{\mu\nu} F_{\mu\sigma} M \nabla^\sigma S_M^N F^\mu_{\nu \nu} N, H^{\mu\nu} F_{\mu\nu}^M \nabla^\sigma S_M^N S_N^P F^\rho_{\sigma P} \right\}
\]  \hspace{1cm} (A.15)  

\( \nabla \Phi \nabla S F^2 : \)
\[
\left\{ \nabla^\rho \Phi F_{\mu\nu}^M \nabla \rho S_M^N F^\mu_{\mu\nu} N, \nabla^\mu \Phi F_{\mu\nu}^M \nabla \nu S_M^N F^\nu_{\mu \nu} N, \nabla^\mu \Phi F_{\mu\nu}^M \nabla \nu S_M^N S_N^P F^\nu_{\nu \nu} P \right\}
\]  \hspace{1cm} (A.16)
Partial integration and explicit field redefinitions

In this appendix, we give some details about the computations of the dimensionally reduced actions presented in Sec. 4.1, and Sec. 7 respectively. We show explicitly how to eliminate all second order derivatives by partial integration up to terms appearing in the first column of Tab. 1, amenable to subsequent elimination by field redefinitions.

\section*{B.1 $\hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma}$}

Let us begin with the terms appearing in the reduction of $\hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma}$, as presented in Sec. 4.1. We give the explicit expression of the five last terms in Eq. (4.8) after integration by parts (and use of Bianchi identities). Up to boundary terms (which we ignore), the first two terms can be rewritten as

$$\frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \nabla_{\mu} \nabla_{\nu} G^{-1} G^{\mu} G^{\nu} G^{-1} G \right)$$

$$= \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ \text{Tr} \left( (\Box G^{-1} - \nabla_{\mu} \Phi \nabla^{\mu} G^{-1}) G (\Box G^{-1} - \nabla_{\nu} \Phi \nabla^{\nu} G^{-1}) G \right) \right.$$ 

$$+ 2 \text{Tr} \left( (\Box G^{-1} - \nabla_{\mu} \Phi \nabla^{\mu} G^{-1}) G \nabla_{\nu} G^{-1} \nabla^{\nu} G \right) - \text{Tr} \left( (\Box G^{-1} - \nabla_{\mu} \Phi \nabla^{\mu} G^{-1}) G^{-1} \nabla_{\nu} G \nabla^{\nu} G \right)$$

$$+ \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla^{\mu} G^{-1} \nabla^{\nu} G \right) + (R_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} \Phi) \text{Tr} \left( \nabla^{\mu} G^{-1} \nabla^{\nu} G \right) \right], \quad (B.1)$$

and

$$\frac{3\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \nabla_{\mu} \nabla_{\nu} G^{-1} \nabla^{\mu} G \nabla^{\nu} G^{-1} G \right)$$

$$= \int d^D x \sqrt{-g} e^{-\Phi} \frac{\alpha'}{4} \left[ \frac{3}{2} \text{Tr} \left( (\Box G^{-1} - \nabla_{\mu} \Phi \nabla^{\mu} G) G^{-1} \nabla_{\nu} G \nabla^{\nu} G^{-1} \right) \right.$$ 

$$- \frac{3}{2} \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \nabla^{\mu} G^{-1} \nabla^{\nu} G \right) \right], \quad (B.2)$$

respectively. The last three terms can be manipulated similarly and their sum takes the following form

$$\frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ F_{(1)m}^{(1)} \left( G \nabla^{\mu} \nabla_{\rho} G^{-1} G \right)_{mn} F_{(1)^{\rho \mu n}}^{(1)} - 2 \nabla_{\rho} F_{(1)^{\mu n}}^{(1)} G_{mn} \nabla^{\mu} F_{(1)^{\rho \mu n}}^{(1)} \right.$$ 

$$- 6 \nabla_{\rho} F_{(1)^{m}}^{(1)} m \nabla^{\mu} G_{mn} F_{(1)^{\mu \rho n}}^{(1)} \right]$$

$$= \int d^D x \sqrt{-g} e^{-\Phi} \frac{\alpha'}{4} \left[ 2(\nabla_{\mu} F_{(1)m}^{(1)}) \nabla_{\rho} \Phi F_{(1)^{\mu n}}^{(1)} G_{mn} \left( \nabla_{\rho} F_{(1)^{\mu \rho n}}^{(1)} - \nabla_{\rho} \Phi F_{(1)^{\mu \rho n}}^{(1)} \right) \right.$$ 

$$- 2 (R_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} \Phi) F_{(1)^{m}}^{(1)} G_{mn} F_{(1)^{\rho n}}^{(1)} + \frac{5}{4} F_{(1)^{m}}^{(1)} G_{mn} \left( G \nabla_{\rho} G^{-1} G \right)_{mn} F_{(1)^{\rho \mu n}}^{(1)}$$

$$+ F_{(1)^{m}}^{(1)} \nabla_{\rho} G \nabla^{\mu} G^{-1} G \left( F_{(1)^{\rho \mu n}}^{(1)} - \nabla_{\rho} \Phi F_{(1)^{\rho \mu n}}^{(1)} \right) + R_{\mu\nu\rho\sigma} F_{(1)^{m}}^{(1)} G_{mn} F_{(1)^{n}}^{(1)} \right), \quad (B.3)$$

again up to boundary contributions. In the form (B.1)–(B.3), all the remaining second order derivatives are of the form appearing in the first column of Tab. 1. They can thus be reabsorbed into field
redefinitions as discussed in Sec. 2.2. Explicitly, this induces the order $\alpha'$ field redefinitions

$$
\delta \Phi = \frac{1}{8} \left[ -2 F^{(1)}_{\mu \nu} m G_{mn} F^{(1) \mu \nu n} + \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \right) \right],
$$
$$
\delta g_{\mu \nu} = \frac{1}{4} \left[ 2 F^{(1)}_{\mu \rho} m G_{mn} F^{(1) \rho \nu n} - \text{Tr} \left( \nabla_{\mu} G^{-1} \nabla_{\nu} G \right) \right],
$$
$$
\delta B_{\mu \nu} = \frac{1}{8} \left[ -2 \nabla^\rho F^{(1)}_{\mu \rho} m + 2 \nabla^\rho \Phi F^{(1)}_{\nu \mu} m + H_{\mu \rho \sigma} H^{\sigma \rho \mu} G_{\nu \mu} + F^{(1)}_{\mu \rho} \left( \nabla^\rho G^{-1} \right)_{\nu \mu} m
+ 2 H_{\mu \nu \rho} \left( G^{-1} \nabla^\rho B G^{-1} \right)_{\mu \nu} m \right] \left( A^{(2)}_{\nu m} - B_{mn} A^{(1) n} \right) - \left( \mu \leftrightarrow \nu \right),
$$
$$
\delta G^{mn} = \frac{1}{4} \left[ -2 \Box G^{mn} + 2 \nabla_{\mu} \Phi \nabla^{\mu} G^{mn} - G^{mp} H_{\mu \nu p} G^{n q} H^{m q} - \frac{3}{2} F^{(1) m}_{\mu \nu} F^{(1) \mu \nu n}
- \left( G^{-1} \nabla_{\mu} G^{-1} \nabla_{\nu} G^{-1} \right)^m_{mn}
+ 2 \left( G^{-1} \nabla_{\mu} B G^{-1} \nabla_{\nu} B G^{-1} \right)^m_{mn} \right],
$$
$$
\delta A^{(1) m}_{\mu} = \frac{1}{4} \left[ -2 \nabla^\nu F^{(1)}_{\mu \nu} m + 2 \nabla^\nu \Phi F^{(1)}_{\nu \mu} m + H_{\mu \nu \rho} H^{\nu \rho \mu} G^{mn}
+ F^{(1)}_{\mu \nu \rho} \left( \nabla^\rho G^{-1} \right)_{n m} + 2 H_{\mu \nu n} \left( G^{-1} \nabla^\nu B G^{-1} \right)^m_{mn} \right],
$$
$$
\delta A^{(2) n}_{\mu} = \frac{1}{4} \left[ 2 \nabla^\nu F^{(1)}_{\nu \mu} m B_{mn} - 2 \nabla^\nu \Phi F^{(1)}_{\nu \mu} m B_{nm} - H_{\mu \nu \rho} H^{\nu \rho \mu} (G^{-1} B)^n_m
- F^{(1)}_{\mu \nu \rho} \left( \nabla^\rho G^{-1} B \right)_{nm} - 2 H_{\mu \nu n} \left( G^{-1} \nabla^\nu B G^{-1} B \right)^n_m \right].
$$
(B.4)

### B.2 $\hat{R}_{\mu \nu \rho \delta} \hat{H}^{\mu \nu \lambda} \hat{H}^{\rho \delta} \hat{\lambda}$

Here, we consider the four last terms in the reduction (4.12) of $RHH$. After partial integration, they can be brought into the form

$$
\frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \nabla_{\mu} \nabla_{\nu} G^{-1} \nabla_{\mu} \nabla_{\nu} G^{-1} \nabla_{\mu} \nabla_{\nu} B \right)
$$
$$
= \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ - \text{Tr} \left( \nabla_{\mu} B \nabla^{\mu} G^{-1} \nabla_{\nu} B \nabla^{\nu} G^{-1} \right) + \frac{1}{2} \text{Tr} \left( \nabla_{\mu} B \nabla_{\nu} G^{-1} \nabla_{\mu} B \nabla^{\nu} G^{-1} \right)
- \text{Tr} \left( \left( \Box B - \nabla_{\mu} \Phi \nabla^{\mu} B \right) G^{-1} \nabla_{\nu} B \nabla^{\nu} G^{-1} \right) + \frac{1}{2} \text{Tr} \left( \left( \Box G^{-1} - \nabla_{\mu} \Phi \nabla^{\mu} G^{-1} \right) \nabla_{\nu} B \nabla^{\nu} G^{-1} \right) \right],
$$
(B.5)

$$
- \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} H^{\mu \nu \rho \mu} \nabla_{\mu} \nabla_{\nu} G^{mn} H^{\rho \mu n}
$$
$$
= \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ - \left( \nabla_{\mu} H^{\mu \nu \rho \mu} - \nabla_{\nu} \Phi H^{\mu \nu \rho \mu} \right) \nabla^{\rho} G^{mn} H^{\nu \rho \mu n}
- \frac{1}{2} F^{(1) m}_{\mu \nu \rho \mu} \nabla^{\rho} G^{-1} m H^{\mu \nu \rho \mu n}
- \frac{1}{4} H^{\mu \mu \rho \mu \nu \rho} \left( \Box G^{-1} - \nabla_{\rho} \Phi \nabla^{\rho} G^{-1} \right)^{mn} H^{\mu \nu \rho \mu n}
- F^{(1) m}_{\mu \nu \rho \mu} \nabla^{\rho} G^{-1} m H^{\mu \nu \rho \mu n} \right],
$$
(B.6)

$$
- \frac{\alpha'}{2} \int d^D x \sqrt{-g} e^{-\Phi} \nabla_{\mu} F^{(1) m}_{\nu \rho \mu} \nabla^{\rho} B G^{-1} m H^{\mu \nu \rho \mu n}
$$
$$
= \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ - \left( \nabla_{\mu} H^{\mu \nu \rho \mu} - \nabla_{\nu} \Phi H^{\mu \nu \rho \mu} \right) \nabla^{\rho} G^{-1} m H^{\mu \nu \rho \mu n}
- \frac{1}{2} F^{(1) m}_{\mu \nu \rho \mu} \nabla^{\rho} G^{-1} m H^{\mu \nu \rho \mu n} \right]
$$

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\[ -\frac{1}{2} F_{\mu\nu}^{(1)} (\nabla_\rho B \nabla^\rho G^{-1})_m^n H^{\mu\nu} - \frac{1}{2} F_{\mu\nu}^{(1)} (\Box B - \nabla_\rho \Phi \nabla^\rho B) G^{-1})_m^n H^{\mu\nu} \\
+ F_{\nu\rho}^{(1)} (\nabla^\rho B \nabla_\rho G^{-1})_m^n H^{\mu\nu} - F_{\nu\rho}^{(1)} (\nabla_\rho B \nabla^\rho G^{-1})_m^n H^{\mu\nu} \\
+ F_{\mu\nu}^{(1)} (\nabla_\rho B G^{-1} \nabla^\rho B)_{mn} F^{(1)} F^{(1)} m n \] (B.7)

\[
\frac{\alpha'}{4} \int d^Dx \sqrt{-g} e^{-\Phi} 2 H^{\mu\nu\lambda} \nabla_\mu F^{(1)}_{\nu\rho} H^\rho\lambda_m
\]

\[
= \frac{\alpha'}{4} \int d^Dx \sqrt{-g} e^{-\Phi} \left[ \frac{1}{2} (\nabla_\mu H^{\mu\nu}_m - \nabla_\nu \Phi H^{\mu\nu}_m) H_{\nu\sigma\rho} F^{(1)} F^{(1)} m n - H^{\mu\nu\rho} F^{(1)} F^{(1)} m n \right] \\
- (\nabla_\mu H^{\mu\nu\rho} - \nabla_\nu \Phi H^{\mu\nu\rho}) F^{(1)} F^{(1)} m n H^{\sigma\rho\mu} + \frac{1}{2} (\nabla_\mu F^{(1)} F^{(1)} m n - \nabla_\nu \Phi F^{(1)} F^{(1)} m n) H_{\nu\sigma\rho\mu} \\
+ \frac{1}{2} H^{\mu\nu\rho} F^{(1)} F^{(1)} m n H^{\sigma\rho\mu} m F_{\mu\nu}^{(1)} n + F_{\mu\nu}^{(1)} H_{\rho\sigma\mu} F^{(1)} F^{(1)} m n H^{\sigma\rho\mu} n - \frac{1}{4} F_{\mu\nu}^{(1)} m H_{\rho\sigma\mu} F^{(1)} F^{(1)} m n H^{\sigma\rho\mu} n \\
- \frac{1}{4} F_{\mu}^{(1)} m H_{\rho\sigma\mu} F^{(1)} F^{(1)} m n H^{\sigma\rho\mu} n \right] , 
\] (B.8)

respectively. Again, all left-over terms carrying second-order derivatives can be converted to products of first order derivatives by means of the rules of Tab. 1. This induces the explicit field redefinitions

\[
\delta B_{\mu\nu} = \frac{1}{8} \left[ - A_{\mu}^{(1)m} (G\nabla_\rho G^{-1})_m^n H_{\nu\rho m} - A_{\mu}^{(1)m} \nabla_\rho B_{\nu\rho m} F_{\nu\rho}^{(1)n} + 2 F_{\mu\nu}^{(1)} H^\rho \nu m \\
+ \frac{1}{2} A_{\mu}^{(1)m} H_{\nu\rho m} G_{mn} F^{(1)} m n - \frac{1}{2} H_{\nu\sigma\rho m} G_{mn} \left( A_{\nu}^{(2)} - B_{\nu} A_{\nu}^{(1)p} \right) \\
- H_{\mu\nu m} (G^{-1} \nabla^\rho B G^{-1})_{mn} \left( A_{\nu}^{(2)} - B_{\nu} A_{\nu}^{(1)p} \right) - (\mu \leftrightarrow \nu) \right] , 
\]

\[
\delta G^{mn} = \frac{1}{4} \left[ \frac{1}{2} G^{mp} H_{\mu\rho q} G^{aq} H^{\mu\nu}_q - (G^{-1} \nabla_\mu B G^{-1} \nabla^\mu B G^{-1})^{mn} \right] , 
\]

\[
\delta B_{mn} = \frac{1}{4} \left[ \nabla_\mu B \nabla_\nu G^{-1} G \right]_{mn} + (G \nabla_\nu G^{-1} \nabla^\mu B)_{mn} \\
- \frac{1}{2} H_{\mu\nu m} F^{(1)} m n H^{\mu\nu p} \right] , 
\]

\[
\delta A_{\mu}^{(1)m} = \frac{1}{4} \left[ - H_{\mu\nu m} (G^{-1} \nabla^\nu B G^{-1})^{mn} - \frac{1}{2} H_{\mu\nu p} H^{\rho \nu p} G_{mn} \right] , 
\]

\[
\delta A_{\mu}^{(2)m} = \frac{1}{4} \left[ H_{\mu\nu m} (G^{-1} \nabla^\nu B G^{-1})^{mn} - F_{\mu}^{(1)n} \nabla_\nu B_{mn} + H_{\mu\nu m} (G^{-1} \nabla^\nu B G^{-1}) \right]_{mn} \\
- \frac{1}{2} H_{\mu\nu p} F^{(1)} m n G_{mn} + \frac{1}{2} H_{\mu\nu p} H^{\rho \nu p} G_{mn} \right] , 
\] (B.9)

B.3 $H^{\mu\nu\rho} \Omega_{\mu\nu\rho}^{(\omega)}$

Finally, we give the result for the six last terms in Eq. (7.8). After partial integration they are rewritten as

\[
\frac{\alpha'}{4} \int d^Dx \sqrt{-g} e^{-\Phi} \left[ -\frac{1}{2} H_{\mu\nu m} G^{mn} \nabla_\mu \left( \nabla_\rho \epsilon_{\sigma} \epsilon_{\alpha} \epsilon_{\beta} F_{\rho}^{(1)} \sigma p G_{\nu p} \right) \right] \\
= \frac{\alpha'}{4} \int d^Dx \sqrt{-g} e^{-\Phi} \left[ \frac{1}{2} (\nabla_\mu H^{\mu\nu} \mu - \nabla_\mu \Phi H^{\mu\nu} \mu + H^{\mu\nu m} \nabla_\mu G_{mn} G_{np} \nabla_\mu \epsilon_{\sigma} \epsilon_{\alpha} \epsilon_{\beta} F_{\rho}^{(1)} \sigma p \right] , 
\] (B.10)
\[
\frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ \frac{1}{2} \text{Tr} \left( \nabla_\mu \nabla_\nu G \nabla^\mu G^{-1} \nabla^\nu B G^{-1} \right) \right]
= \frac{\alpha'}{4} \int d^D x \sqrt{-g} e^{-\Phi} \left[ - \frac{1}{2} \text{Tr} \left( (\Box B - \nabla_\mu \Phi \nabla^\mu B) G^{-1} \nabla_\nu G \nabla^\nu G^{-1} \right) \right.
+ \frac{1}{2} \text{Tr} \left( (\Box G - \nabla_\mu \Phi \nabla^\mu G) \nabla_\nu G^{-1} \nabla^\nu B G^{-1} \right) - \text{Tr} \left( \nabla_\mu B \nabla^\mu G^{-1} \nabla_\nu G \nabla^\nu G^{-1} \right) \right],
\]

Again, the remaining terms carrying second-order derivatives can be eliminated by field redefinitions as discussed in Sec. 2.2. The explicit form of the induced field redefinitions has been given in Eq. (7.10) in the main text.

## C GL(d) expressions of some O(d, d, \mathbb{R}) terms

In this appendix, we present the GL(d) decomposition of some of the O(d, d, \mathbb{R}) invariant terms, that are relevant for the identifications made in Secs. 5 and 7.

\[
\text{Tr} \left( \nabla_\mu S \nabla_\nu S \right) = 2 \text{Tr} \left( \nabla_\mu (G \nabla_\nu G)^{-1} \right) + 2 \text{Tr} \left( \nabla_\mu B G^{-1} \nabla_\nu B G^{-1} \right).
\]

\[
\text{Tr} \left( \nabla_\mu S \nabla_\nu S \nabla^\mu S \nabla^\nu S \right) = 2 \text{Tr} \left( \nabla_\mu G^{-1} \nabla_\nu G \nabla^\mu G^{-1} \nabla^\nu G \right) + 4 \text{Tr} \left( \nabla_\mu B \nabla_\nu G^{-1} \nabla^\mu B \nabla^\nu G^{-1} \right)
+ 8 \text{Tr} \left( G^{-1} \nabla_\mu B G^{-1} \nabla_\nu G \nabla^\mu G^{-1} \nabla^\nu B \right)
+ 2 \text{Tr} \left( G^{-1} \nabla_\mu B G^{-1} \nabla^\mu B G^{-1} \nabla_\nu B G^{-1} \nabla^\nu B \right),
\]

\[
\text{Tr} \left( \nabla_\mu \Phi \nabla_\nu \Phi \nabla^\mu \Phi \nabla^\nu \Phi \right) = 2 \text{Tr} \left( \nabla_\mu G^{-1} \nabla_\nu G \nabla^\mu G^{-1} \nabla^\nu G \right) + 4 \text{Tr} \left( G^{-1} \nabla_\mu \Phi \nabla^\mu \Phi \nabla_\nu B G^{-1} \nabla^\nu B \right)
+ 4 \text{Tr} \left( \nabla_\nu G^{-1} \nabla_\mu B \nabla^\mu G^{-1} \nabla^\nu B \right) + 4 \text{Tr} \left( \nabla_\mu \Phi \nabla^\mu G^{-1} \nabla_\nu G \nabla^\nu G^{-1} \nabla_\nu B G^{-1} \nabla^\nu B \right)
+ 2 \text{Tr} \left( G^{-1} \nabla_\mu \Phi \nabla^\mu G^{-1} \nabla_\nu B G^{-1} \nabla^\nu B \right),
\]
\[
\text{Tr}(S\nabla_{\mu}S\nabla_{\nu}S_{\rho}S_{\sigma}S) = 2\left[ \text{Tr}\left( \nabla_{\mu}G^{-1}\nabla_{\nu}G\nabla_{\rho}G^{-1}\nabla_{\sigma}B \right) - \text{Tr}\left( \nabla_{\sigma}G^{-1}\nabla_{\mu}G\nabla_{\nu}G^{-1}\nabla_{\rho}B \right) \right]
\]
\[
+ \text{Tr}\left( \nabla_{\nu}G^{-1}\nabla_{\sigma}G\nabla_{\rho}B \right) - \text{Tr}\left( \nabla_{\rho}G^{-1}\nabla_{\nu}G\nabla_{\sigma}G^{-1}\nabla_{\mu}B \right)
\]
\[
- \text{Tr}\left( \nabla_{\mu}BG^{-1}\nabla_{\nu}BG^{-1}\nabla_{\rho}B \right) - \text{Tr}\left( \nabla_{\sigma}BG^{-1}\nabla_{\mu}BG^{-1}\nabla_{\nu}B \nabla_{\rho}G^{-1} \right)
\]
\[
- \text{Tr}\left( \nabla_{\rho}BG^{-1}\nabla_{\sigma}BG^{-1}\nabla_{\mu}B \right) - \text{Tr}\left( \nabla_{\nu}BG^{-1}\nabla_{\rho}BG^{-1}\nabla_{\sigma}B \nabla_{\mu}G^{-1} \right) \right], \quad (C.4)
\]

\[
\mathcal{F}_{\mu\nu}^{M} S_{M}^{N} \mathcal{F}_{\rho\sigma}^{P} = F_{\mu\nu}^{(1)m} G_{mn} F_{\rho\sigma}^{(1)n} + H_{\mu\nu} m G^{mn} H_{\rho\sigma n}, \quad (C.5)
\]

\[
\mathcal{F}_{\mu\nu}^{M} S_{M}^{N} \mathcal{F}_{\rho\sigma}^{P} = F_{\mu\nu}^{(1)m} F_{\rho\sigma}^{(2)n} + F_{\mu\nu}^{(2)m} F_{\rho\sigma}^{(2)n}, \quad (C.6)
\]

\[
\mathcal{F}_{\mu\nu}^{M} S_{M}^{N} \nabla_{\rho} S_{N}^{P} \mathcal{F}_{\sigma\lambda}^{Q} = F_{\mu\nu}^{(1)m} (G\nabla_{\rho}G^{-1})_{m}^{n} H_{\rho\lambda n} - F_{\mu\nu}^{(1)m} \nabla_{\rho} B_{mn} F_{\sigma\lambda}^{(1)n}
\]
\[
+ H_{\mu\nu m} (G^{-1}\nabla_{\rho} G^{-1})^{mn} H_{\sigma\lambda n} + H_{\mu\nu m} (G^{-1}\nabla_{\rho} G)^{m} F_{\sigma\lambda}^{(1)n}, \quad (C.7)
\]

\[
\mathcal{F}_{\mu\nu}^{M} \nabla_{\rho} S_{M}^{N} \mathcal{F}_{\sigma\lambda}^{Q} = F_{\mu\nu}^{(1)m} \nabla_{\rho} G_{mn} F_{\sigma\lambda}^{(1)n} + F_{\mu\nu}^{(1)m} (\nabla_{\rho} B G^{-1})_{m}^{n} H_{\rho\lambda n}
\]
\[
- H_{\mu\nu m} (G^{-1}\nabla_{\rho} B)^{m} F_{\sigma\lambda}^{(1)n} + H_{\mu\nu m} \nabla_{\rho} G^{mn} H_{\sigma\lambda n}, \quad (C.8)
\]

\[
\mathcal{F}_{\mu\nu}^{M} S_{M}^{N} \nabla_{\rho} S_{N}^{P} \nabla_{\sigma} S_{P}^{Q} \mathcal{F}_{\lambda\tau}^{Q} = \]
\[
F_{\mu\nu}^{(1)m} (G\nabla_{\rho}G^{-1}G_{m}^{n} F_{\lambda\tau}^{(1)n} + F_{\mu\nu}^{(1)m} (G\nabla_{\rho}G^{-1}\nabla_{\sigma}B)^{m} F_{\lambda\tau}^{(1)n}
\]
\[
- F_{\mu\nu}^{(1)m} (\nabla_{\rho} B V_{m}^{n} H_{\lambda\tau n} + H_{\mu\nu m} (G^{-1}\nabla_{\omega} B)^{m} F_{\lambda\tau}^{(1)n}
\]
\[
+ H_{\mu\nu m} (G^{-1}\nabla_{\rho} B G^{-1})^{mn} H_{\lambda\tau n} + H_{\mu\nu m} (G^{-1}\nabla_{\rho} B G^{-1})^{mn} H_{\lambda\tau n} . \quad (C.9)
\]

\[
\mathcal{F}_{\mu\nu}^{M} \nabla_{\rho} S_{M}^{N} \nabla_{\sigma} S_{N}^{P} \mathcal{F}_{\lambda\tau}^{P} = \]
\[
F_{\mu\nu}^{(1)m} (\nabla_{\rho} B G^{-1}G_{m}^{n} F_{\lambda\tau}^{(1)n} - F_{\mu\nu}^{(1)m} (\nabla_{\rho} G G^{-1})_{mn} F_{\lambda\tau}^{(1)n}
\]
\[
+ F_{\mu\nu}^{(1)m} (\nabla_{\rho} G^{-1}G_{m}^{n} H_{\lambda\tau n} + F_{\mu\nu}^{(1)m} (\nabla_{\rho} G G^{-1})_{mn} H_{\lambda\tau n}
\]
\[
+ H_{\mu\nu m} (G^{-1}\nabla_{\rho} G G^{-1})^{mn} F_{\lambda\tau}^{(1)n} + H_{\mu\nu m} (G^{-1}\nabla_{\rho} G^{-1})^{mn} F_{\lambda\tau}^{(1)n}
\]
\[
+ H_{\mu\nu m} (G^{-1}\nabla_{\rho} B G^{-1})^{mn} H_{\lambda\tau n} - H_{\mu\nu m} (G^{-1}\nabla_{\rho} B G^{-1})^{mn} H_{\lambda\tau n} . \quad (C.10)
\]

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