A complete hybrid quantization in inhomogeneous cosmology

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A complete quantization of a homogeneous and isotropic spacetime with closed spatial sections coupled to a massive scalar field is provided, within the framework of Loop Quantum Cosmology. We identify solutions with their initial data on the minimum volume section, and from this we construct the physical Hilbert space. Moreover, a perturbative study allows us to introduce small inhomogeneities. After gauge fixing, the inhomogeneous part of the system is reduced to a linear field theory. We then adopt a standard Fock representation to quantize these degrees of freedom. For the considered case of compact spatial topology, the requirements of: i) invariance under the spatial isometries, and ii) unitary implementation of the quantum dynamics, pick up a unique Fock representation and a particular set of canonical fields (up to unitary equivalence).

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1. INTRODUCTION

Loop Quantum Cosmology (LQC) is one of the most promising approaches to the quantization of cosmological spacetimes. In particular, its application to homogeneous and isotropic spacetimes (or even to anisotropic ones) has shown that the classical singularity is always replaced with a quantum bounce.

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In inflationary scenarios \([3]\), for instance in the presence of a large scalar field, a careful analysis assuming some associated effective dynamics proves the overwhelming probability that the inflaton produces the necessary amount of e-foldings \([4]\), solving the fine tuning problem inherent to General Relativity. But such an effective dynamics does not arise from any genuine quantization within LQC; it is rather adopted as a reasonable hypothesis. On the other hand, if one introduces small inhomogeneities –coming from the vacuum fluctuations of the inflaton–, a valid description for them should be provided by a standard Fock representation, at least in certain regimes. Nevertheless, the infinite number of possible inequivalent Fock representations leads to an uncontrollable ambiguity in the physical predictions, unless one can find a privileged Fock quantization.

In this work, we consider a closed Friedmann-Robertson-Walker (FRW) spacetime coupled to a massive scalar field. We provide a complete polymeric quantization of this model, where each solution to the Hamiltonian constraint is characterized by its data on the minimum volume section. The completion of the space of these data with respect to a suitable inner product provides the physical Hilbert space. Moreover, the small inhomogeneities around homogeneous solutions, after a suitable gauge fixing, are described by a (linear) scalar field theory. Finally, we determine a unique \(SO(4)\)-invariant Fock quantization for them by requiring a unitary implementation of the dynamics.

2. HOMOGENEOUS SYSTEM

The homogeneous and isotropic system has two global (canonical pairs of) degrees of freedom. One is the pair \((\phi, p_{\phi})\) for the matter content. Besides, the geometry is described by a densitized triad, which essentially reduces to \(p\) (the square of the scale factor), and an Ashtekar-Barbero \(su(2)\)-connection \(c\). The definition of these variables takes into account the \(S^3\) topology, and uses a fiducial volume \(l_0^3 = 2\pi^2\) (see \([5]\)). The Poisson bracket is \(\{c, p\} = 8\pi G \gamma /3\), with \(G\) being the Newton constant and \(\gamma\) the Immirzi parameter.

We carry out a polymeric quantization of the geometry degrees of freedom. The basic variables in LQC are fluxes of densitized triads through surfaces enclosed by four geodesic
edges, which are determined essentially by $p$, and holonomies of the connection along integral curves of the fiducial triads, with fiducial length $\mu_0$. Specifically, we adopt the improved dynamics scheme, in which $\mu = \sqrt{\Delta/p}$, with $\Delta$ equal to the minimum nonzero eigenvalue allowed for the area in Loop Quantum Gravity [6]. Then, the gravitational part of the kinematical Hilbert space is $H_{\text{grav}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, where $\mathbb{R}_{\text{Bohr}}$ is the Bohr compactification of the real line, and $d\mu_{\text{Bohr}}$ is the natural Haar measure associated with it. In this Hilbert space one can find a basis of normalizable states in which the action of the matrix elements of the holonomies is given by $\hat{N} \mu |v\rangle = |v + 1\rangle$, whereas, for the triad, $\hat{p} |v\rangle = \text{sgn}(v)(2\pi\gamma G\hbar\sqrt{\Delta}|v|^{2/3}|v\rangle)$. Here, $\hbar$ is the Planck constant. For the scalar field, on the other hand, we employ a standard Schrödinger representation. Adopting a suitable factor ordering, we obtain the following operator representation for the Hamiltonian constraint:

$$\hat{C}_0 = \left[\frac{1}{\hat{V}}\right]^{1/2} \left[8\pi G (\hat{p}_\phi^2 + m^2\hat{V}^2\phi^2) - \frac{6}{\gamma^2} \left(\hat{\Omega}^2 + (1 + \gamma^2)\hat{\Delta}^2\hat{V}^{4/3} - \frac{\hat{V}^2}{\Delta} \sin^2 \mu \right)\right] \left[\frac{1}{\hat{V}}\right]^{1/2},$$

where $[1/\hat{V}]$ is the regularized inverse of the volume operator $\hat{V}$ (see [5, 6]), and

$$\hat{\Omega} = \frac{1}{4i\sqrt{\Delta}} \hat{V}^{1/2} \left[\text{sgn}(v) \left(e^{-i\hat{\Delta}n} \hat{N}_{2\mu} e^{-i\hat{\Delta}n} - e^{i\hat{\Delta}n} \hat{N}_{-2\mu} e^{i\hat{\Delta}n}\right) + \left(e^{-i\hat{\Delta}n} \hat{N}_{2\mu} e^{-i\hat{\Delta}n} - e^{i\hat{\Delta}n} \hat{N}_{-2\mu} e^{i\hat{\Delta}n}\right) \text{sgn}(v)\right] \hat{V}^{1/2}.$$

The constraint is a second order difference operator that only relates states with support on semilattices of the type $\mathcal{L}_\varepsilon^\pm = \{v = \pm(\varepsilon + 4n), n \in \mathbb{N}\}$, where $\varepsilon \in (0, 4]$ is a continuous parameter labeling each subspace, which can be interpreted as a superselection sector. Moreover, on any solution $\Psi$ to the constraint, the value at all the volumes $v = \varepsilon + 4n$ with $n > 0$ is totally determined by the value on the minimum volume section, namely $\Psi(\phi, v = \varepsilon)$. The physical Hilbert space can thus be identified with the completion of the functional space of initial data at $v = \varepsilon$ with respect to an inner product which is uniquely determined by the requirement of self-adjointness on a complete set of observables. Hence, the physical Hilbert space of this homogeneous system can be taken as

$$H_{\text{phy}} = L^2(\mathbb{R}, d\phi).$$
3. INHOMOGENEOUS MODEL AND GAUGE FIXING

Small inhomogeneities may arise from vacuum fluctuations of the inflaton. With this motivation, we introduce inhomogeneities in our model and adopt a perturbative treatment for them. We will consider scalar perturbations only (this is possible because vector and tensor modes are dynamically decoupled from the scalar ones at first perturbative order). Around the homogeneous solutions, we have

\[
N = \sigma (N_0 + \delta N), \quad N_a = \sigma^2 \delta N_a, \\
h_{ab} = \sigma^2 e^{2\alpha} (\Omega_{ab} + \Delta_{ab}) \quad \text{and} \quad \Phi = \sigma^{-1} (l_{0}^{-3/2} \varphi + \delta \varphi),
\]

for the lapse, shift, spatial metric and scalar field, respectively. Here \( \sigma^2 = 4\pi G / 3l_0^3 \), and \( e^\alpha \) is the scale factor \([7]\). Besides, the spatial sections are isomorphic to \( S^3 \), for which we have a natural basis of spherical harmonics. In terms of it, we get the expansions:

\[
\delta N = l_0^{3/2} N_0 \sum_n g_n Q_n, \quad \delta N_a = l_0^{3/2} e^\alpha \sum_n j_n P_n^a, \\
\Delta_{ab} = 2l_0^{3/2} \sum_n a_n Q_n \Omega_{ab} + 3b_n P_{ab}^n, \quad \delta \varphi = \sum_n f_n Q_n.
\]

We will consider perturbations to second order in the Hamiltonian, which takes then the form

\[
H = N_0 \left[ C_0 + \sum_n (C_2^n + g_n C_1^n) \right] + \sum_n j_n D_1^n,
\]

where \( C_0 \) is the scalar constraint of the homogeneous system \( (\alpha, \pi_\alpha, \varphi, \pi_\varphi) \) [with \( p \) replaced with \( \alpha \) and \( \pi_\alpha \) being its momentum], each \( C_2^n \) is quadratic in the scalar modes \( (a_n, b_n, f_n) \) and their canonically conjugate momenta \( (\pi_{a_n}, \pi_{b_n}, \pi_{f_n}) \), and \( C_1^n \) and \( D_1^n \) are the linear contributions of the perturbations to the scalar and the diffeomorphism constraints (spanned in Fourier modes), respectively.

In our formalism, the perturbations \( g_n \) and \( j_n \) are nondynamical variables. They just play the role of Lagrange multipliers, associated with first-class constraints. In order to identify the true physical degrees of freedom and reduce the model, we now introduce gauge fixing conditions. One possibility is, e.g., the gauge \( a_n = 0 = b_n \). The derived conditions \( \dot{a}_n = 0 \) and \( \dot{b}_n = 0 \), necessary for consistency with the dynamics, fix the functions \( j_n \) and \( g_n \). In addition, the constraints \( D_1^n = C_1^n = 0 \) allow us to determine \( \pi_{a_n} \) and \( \pi_{b_n} \) in terms of \( f_n \) and \( \pi_{f_n} \). Besides, in order to adapt the system to a form in which one can apply the uniqueness results of \([8, 9]\) (concerning its Fock quantization), we will make the change of variables \( \bar{f}_n = e^\alpha f_n \) and \( \bar{\pi}_{f_n} = e^{-\alpha} [\pi_{f_n} - (3\pi_\varphi^2 / \pi_\alpha + \pi_\alpha) f_n] \), which can
be easily extended into a canonical transformation (up to the perturbative order under consideration).

### 4. HYBRID QUANTIZATION

Assuming that the most important quantum geometry effects are those affecting the homogeneous part of the geometry, we now perform a complete quantization of the system in which we adopt the polymeric quantization explained above for that homogeneous part, while we adhere to a standard quantization not only for the homogeneous component of the scalar field, but for all the inhomogeneities. Remarkably, we can choose a privileged Fock quantization for the inhomogeneities: there is only one Fock representation invariant under the spatial isometries that implements the dynamics unitarily \[8\]. Actually, this criterion fixes not only the representation of the canonical commutation relations, but also the choice of a canonical pair for our field \[9\] (at least if one expects to reach regimes where deparametrization is possible).

In our gauge fixed system, those uniqueness results select the Fock representation which is naturally associated with the massless case for the field described by the modes $\vec{f}_n$ and $\vec{\pi}_n$ \[10\]. The resulting kinematical Hilbert space is the Fock space $\mathcal{F}$ constructed from the corresponding one-particle Hilbert space. In total, the kinematical Hilbert space is $\mathcal{H}^{\text{kin}} = \mathcal{H}^{\text{kin}}_{\text{grav}} \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}$. With this space as starting point, we can construct a quantum Hamiltonian constraint and look for its solutions. We expect this constraint to impose a second order difference equation with solutions totally determined by their values on the minimum volume section (at least in the studied perturbative order). Identifying then solutions with initial data, the latter can be completed with a suitable inner product to obtain the physical Hilbert space (see e.g. \[11\] and the previous section).

### 5. CONCLUSIONS

We have first analyzed a homogeneous, massive scalar field propagating in a closed FRW spacetime and quantized the system to completion, combining both Schrödinger
and polymeric representations. The physical Hilbert space has been constructed out of the functional space defined on the minimum volume section, which provides the initial data for the solutions of the scalar constraint. Along similar lines, we can also obtain the physical Hilbert space when small inhomogeneities are included. These are introduced in the system as perturbations. In the process followed to arrive to this space of physical states, we have carried out a partial gauge fixing, removing two of the three fieldlike degrees of freedom, and performed a canonical transformation (consisting in a scaling of the field) to write the reduced system in a way in which one can appeal to the uniqueness theorems of \[8,9\]. Finally, we have adopted a hybrid quantization for this reduced system, combining loop techniques for the homogeneous sector of the geometry with a standard Fock quantization of the inhomogeneities \[10\].

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