BRACE OPERATIONS AND DELIGNE’S CONJECTURE FOR
MODULE-ALGEBRAS

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ABSTRACT. It is observed that Kaygun’s Hopf-Hochschild cochain complex for a module-algebra is a brace algebra with multiplication. As a result, (i) an analogue of Deligne’s Conjecture holds for module-algebras, and (ii) the Hopf-Hochschild cohomology of a module-algebra has a Gerstenhaber algebra structure.

1. Introduction

Let $H$ be a bialgebra and let $A$ be an associative algebra. The algebra $A$ is said to be an $H$-module-algebra if there is an $H$-module structure on $A$ such that the multiplication on $A$ becomes an $H$-module morphism. For example, if $S$ denotes the Landweber-Novikov algebra $[15, 21]$, then the complex cobordism $MU^*(X)$ of a topological space $X$ is an $S$-module-algebra. Likewise, the singular mod $p$ cohomology $H^*(X; F_p)$ of a topological space $X$ is an $A_p$-module-algebra, where $A_p$ denotes the Steenrod algebra associated to the prime $p$ $[7, 19]$. Other similar examples from algebraic topology can be found in $[4]$. Important examples of module-algebras from Lie and Hopf algebras theory can be found in, e.g., $[12, V.6]$.

In $[14]$, Kaygun defined a Hochschild-like cochain complex $CH^H_{Hopf}(A, A)$ associated to an $H$-module-algebra $A$, called the Hopf-Hochschild cochain complex, that takes into account the $H$-linearity. In particular, if $H$ is the ground field, then Kaygun’s Hopf-Hochschild cochain complex reduces to the usual Hochschild cochain complex $C^*(A, A)$ of $A$ $[11]$. Kaygun $[14]$ showed that the Hopf-Hochschild cohomology of $A$ shares many properties with the usual Hochschild cohomology. For example, it can be described in terms of derived functors, and it satisfies Morita invariance.

The usual Hochschild cochain complex $C^*(A, A)$ has a very rich structure. Namely, it is a brace algebra with multiplication $[10]$. Combined with a result of McClure and Smith $[18]$ concerning the singular chain operad associated to the little squares operad $\mathcal{E}_2$, the brace algebra with multiplication structure on $C^*(A, A)$ leads to a positive solution of Deligne’s Conjecture $[6]$. Also, passing to cohomology, the brace algebra with multiplication structure implies that the Hochschild cohomology modules $HH^*(A, A)$ form a
Gerstenhaber algebra, which is a graded version of a Poisson algebra. This fact was first observed by Gerstenhaber [8].

The purpose of this note is to observe that Kaygun’s Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ of a module-algebra $A$ also admits the structure of a brace algebra with multiplication. As in the classical case, this leads to a version of Deligne’s Conjecture for module-algebras. Also, the Hopf-Hochschild cohomology modules $\mathrm{HH}^*_{\text{Hopf}}(A, A)$ form a Gerstenhaber algebra. When the bialgebra $H$ is the ground field, these structures reduce to the ones in Hochschild cohomology.

A couple of remarks are in order. First, there is another cochain complex $\mathcal{F}^*(A)$ that can be associated to an $H$-module-algebra $A$ [23]. The cochain complex $\mathcal{F}^*(A)$ is a differential graded algebra. Moreover, it controls the deformations of $A$, in the sense of Gerstenhaber [9], with respect to the $H$-module structure, leaving the algebra structure on $A$ fixed. It is not yet known whether $\mathcal{F}^*(A)$ is a brace algebra with multiplication and whether the cohomology modules of $\mathcal{F}^*(A)$ form a Gerstenhaber algebra.

Second, the results and arguments here can be adapted to module-coalgebras, comodule-algebras, and comodule-coalgebras. To do that, one replaces the crossed product algebra $X$ (§2.3) associated to an $H$-module-algebra $A$ by a suitable crossed product (co)algebra [1, 2, 3] and replaces Kaygun’s Hopf-Hochschild cochain complex by a suitable variant.

1.1. Organization. The rest of this paper is organized as follows.

In the following section, we recall the construction of the Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ from Kaygun [14]. In Section 3, it is observed that $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ has the structure of an operad with multiplication (Theorem 3.4). This leads in Section 4 to the desired brace algebra with multiplication structure on $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ (Corollary 4.4). Explicit formulas for the brace operations are given.

In Section 5, it is observed that the brace algebra with multiplication structure on $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ leads to a homotopy $G$-algebra structure (Corollary 5.3). The differential from this homotopy $G$-algebra and the Hopf-Hochschild differential are then identified, up to a sign (Theorem 5.5). Passing to cohomology, this leads in Section 6 to a Gerstenhaber algebra structure on the Hopf-Hochschild cohomology modules $\mathrm{HH}^*_{\text{Hopf}}(A, A)$ (Corollary 6.6). The graded associative product and the graded Lie bracket on $\mathrm{HH}^*_{\text{Hopf}}(A, A)$ are explicitly described.
In the final section, by combining our results with a result of McClure and Smith [18], a version of Deligne’s Conjecture for module-algebras is obtained (Corollary 7.1). This section can be read immediately after Section 4 and is independent of Sections 5 and 6.

2. HOPF-HOCHSCHILD COHOMOLOGY

In this section, we fix some notations and recall from [14, Section 3] the Hopf-Hochschild cochain complex associated to a module-algebra.

2.1. Notations. Fix a ground field $K$ once and for all. Tensor product and vector space are all meant over $K$.

Let $H = (H, \mu_H, \Delta_H)$ denote a $K$-bialgebra with associative multiplication $\mu_H$ and coassociative comultiplication $\Delta_H$. It is assumed to be unital and counital, with its unit and counit denoted by $1_H$ and $\varepsilon: H \to K$, respectively.

Let $A = (A, \mu_A)$ denote an associative, unital $K$-algebra with unit $1_A$ (or simply 1).

In a coalgebra $(C, \Delta)$, we use Sweedler’s notation [22] for comultiplication: \[ \Delta(x) = \sum x(1) \otimes x(2), \Delta^2(x) = \sum x(1) \otimes x(2) \otimes x(3), \text{ etc.} \]

These notations will be used throughout the rest of this paper.

2.2. Module-algebra. Recall that the algebra $A$ is said to be an $H$-module-algebra [5, 12, 20, 22] if and only if there exists an $H$-module structure on $A$ such that $\mu_A$ is an $H$-module morphism, i.e.,

\[ b(a_1a_2) = \sum (b(1_a)1_a)(b(2_a)1_a) \] (2.2.1)

for $b \in H$ and $a_1, a_2 \in A$. It is also assumed that $b(1_A) = \varepsilon(b)1_A$ for $b \in H$.

We will assume that $A$ is an $H$-module-algebra for the rest of this paper.

2.3. Crossed product algebra. Let $X$ be the vector space $A \otimes A \otimes H$. Define a multiplication on $X$ [14 Definition 3.1] by setting

\[ (a_1 \otimes a_1' \otimes b^1)(a_2 \otimes a_2' \otimes b^2) \overset{\text{def}}{=} \sum a_1 \left( b^{1}_{(1)}a_1 \right) \otimes \left( b^{1}_{(2)}a_1' \right) \otimes b^2 \] (2.3.1)

for $a_1 \otimes a_1' \otimes b^1$ and $a_2 \otimes a_2' \otimes b^2$ in $X$. It is shown in [14 Lemma 3.2] that $X$ is an associative, unital $K$-algebra, called the crossed product algebra.

Note that if $H = K$ (= the trivial group bialgebra $K[[e]]$), then $X$ is just the enveloping algebra $A \otimes A^{\text{op}}$, where $A^{\text{op}}$ is the opposite algebra of $A$.

The algebra $A$ is an $X$-module via the action

\[ (a \otimes a' \otimes b)a_0 = a(ba_0)a' \]
for \(a \otimes a' \otimes b \in X\) and \(a_0 \in A\). Likewise, the vector space \(A^\otimes(n+2)\) is an \(X\)-module via the action
\[
(a \otimes a' \otimes b)(a_0 \otimes \cdots \otimes a_{n+1}) = \sum a b_{(1)} a_0 \otimes b_{(2)} a_1 \otimes \cdots \otimes b_{(n+1)} a_n \otimes b_{(n+2)} a_{n+1} a'
\]
for \(a_0 \otimes \cdots \otimes a_{n+1} \in A^\otimes(n+2)\).

2.4. Bar complex. Consider the chain complex \(CB_n(A)\) of vector spaces with \(CB_n(A) = A^\otimes(n+2)\), whose differential \(d_{CB}^n : CB_n(A) \to CB_{n-1}(A)\) is defined as the alternating sum
\[
d_{CB}^n = \sum_{j=0}^n (-1)^j \partial_j,
\]
where
\[
\partial_j(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes a_{n+1}.
\]

It is mentioned above that each vector space \(CB_n(A) = A^\otimes(n+2)\) is an \(X\)-module. Using the module-algebra condition (2.2.1), it is straightforward to see that each \(\partial_j\) is \(X\)-linear. Therefore, \(CB_\ast(A)\) can be regarded as a chain complex of \(X\)-modules.

Note that in the case \(H = K\), the chain complex \(CB_\ast(A)\) of \(A \otimes A^{op}\)-modules is the usual bar complex of \(A\).

2.5. Hopf-Hochschild cochain complex. The Hopf-Hochschild cochain complex of \(A\) with coefficients in \(A\) is the cochain complex of vector spaces:
\[
(CH_{Hopf}^\ast(A, A), d_{CH}) \overset{\text{def}}{=} \text{Hom}_X((CB_\ast(A), d_{CB}), A).
\]
(2.5.1)

Its \(n\)th cohomology module, denoted by \(HH^n_{Hopf}(A, A)\), is called the \(n\)th Hopf-Hochschild cohomology of \(A\) with coefficients in \(A\).

When \(H = K\), the cochain complex \((CH_{Hopf}^\ast(A, A), d_{CH})\) is the usual Hochschild cochain complex of \(A\) with coefficients in itself \([11]\), and \(HH^n_{Hopf}(A, A)\) is the usual Hochschild cohomology module.

In what follows, we will use the notation \(CH_{Hopf}^\ast(A, A)\) to denote (i) the Hopf-Hochschild cochain complex \((CH_{Hopf}^\ast(A, A), d_{CH})\), (ii) the sequence \([CH_{Hopf}^n(A, A)]\) of vector spaces, or (iii) the graded vector space \(\oplus_n CH_{Hopf}^n(A, A)\). It should be clear from the context what \(CH_{Hopf}^\ast(A, A)\) means.

3. Algebraic operad

The purpose of this section is to show that the vector spaces \(CH_{Hopf}^\ast(A, A)\) in the Hopf-Hochschild cochain complex of an \(H\)-module-algebra \(A\) with self coefficients has the structure of an operad with multiplication.
3.1. **Operads.** Recall from [16, 17] that an operad $\mathcal{O} = \{\mathcal{O}(n), \gamma, \text{Id}\}$ consists of a sequence of vector spaces $\mathcal{O}(n)$ ($n \geq 1$) together with structure maps

$$\gamma: \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k),$$

(3.1.1)

for $k, n_1, \ldots, n_k \geq 1$ and an identity element $\text{Id} \in \mathcal{O}(1)$, satisfying the following two axioms.

(1) The structure maps $\gamma$ are required to be associative, in the sense that

$$\gamma(\gamma(f; g_1, h_{1,N}); g_2, \ldots, h_{N-1,N,N}) = \gamma(f; \gamma(g_1; h_{1,N_1}), \ldots, \gamma(g_k; h_{N_{k-1},N_{k+1}})).$$

(3.1.2)

Here $f \in \mathcal{O}(k)$, $g_i \in \mathcal{O}(n_i)$, $N = n_1 + \cdots + n_k$, and $N_i = n_1 + \cdots + n_i$. Given elements $x_i, x_{i+1}, \ldots$, the symbol $x_{i,j}$ is the abbreviation for the sequence $x_i, x_{i+1}, \ldots, x_j$ or $x_i \otimes \cdots \otimes x_j$ whenever $i \leq j$.

(2) The identity element $\text{Id} \in \mathcal{O}(1)$ is required to satisfy the condition that the linear map

$$\gamma(-; \text{Id}, \ldots, \text{Id}): \mathcal{O}(k) \to \mathcal{O}(k)$$

(3.1.3)

is equal to the identity map on $\mathcal{O}(k)$ for each $k \geq 1$.

What is defined above is usually called a non-$\Sigma$ operad in the literature.

3.2. **Operad with multiplication.** Let $\mathcal{O}$ be an operad. A multiplication on $\mathcal{O}$ [10, Section 1.2] is an element $m \in \mathcal{O}(2)$ that satisfies

$$\gamma(m; m, \text{Id}) = \gamma(m; \text{Id}, m).$$

(3.2.1)

In this case, $(\mathcal{O}, m)$ is called an operad with multiplication.

3.3. **Operad with multiplication structure on $\text{CH}_n^{\text{Hopf}}(A, A)$.** In what follows, in order to simplify the typography, we will sometimes write $\mathcal{C}(n)$ for the vector space $\text{CH}_n^{\text{Hopf}}(A, A)$. To show that the vector spaces $\text{CH}_n^{\text{Hopf}}(A, A)$ form an operad with multiplication, we first define the structure maps, the identity element, and the multiplication.

**Structure maps:** For $k, n_1, \ldots, n_k \geq 1$, define a map

$$\gamma: \mathcal{C}(k) \otimes \mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \to \mathcal{C}(N)$$

(3.3.1)

by setting

$$\gamma(f; g_1, \ldots, g_k)(a_0, a_{N+1}) = \text{def} f(a_0 \otimes g_1(1 \otimes a_1, N_1 \otimes 1) \otimes \cdots \otimes g_k(1 \otimes a_{N_{k-1}+1}, N_1 \otimes 1) \otimes \cdots \otimes a_{N+1}).$$

(3.3.2)

Here the notations are as in the definition of an operad above, and each $a_i \in A$. 
Identity element: Let \( \text{Id} \in \mathcal{C}(1) \) be the element such that
\[
\text{Id}(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 a_2.
\] (3.3.3)

This is indeed an element of \( \mathcal{C}(1) \), since the identity map on \( A \) is \( H \)-linear.

Multiplication: Let \( \pi \in \mathcal{C}(2) \) be the element such that
\[
\pi(a_0 \otimes a_1 \otimes a_2 \otimes a_3) = a_0 a_1 a_2 a_3.
\] (3.3.4)

This is indeed an element of \( \mathcal{C}(2) \), since the multiplication map \( A^\otimes 2 \to A \) on \( A \) is \( H \)-linear.

Theorem 3.4. The data \( \mathcal{C} = \{ \mathcal{C}(n), \gamma, \text{Id} \} \) forms an operad. Moreover, \( \pi \in \mathcal{C}(2) \) is a multiplication on the operad \( \mathcal{C} \).

Proof. It is immediate from (3.3.2) and (3.3.3) that \( \gamma(-; \text{Id}^\otimes k) \) is the identity map on \( \mathcal{C}(k) \) for each \( k \geq 1 \).

To prove associativity of \( \gamma \), we use the notations in the definition of an operad and compute as follows:
\[
\gamma(\gamma(f; g_1, k); h_{1,N})(a_0 \otimes \cdots \otimes a_{M+1})
\]
\[
= \gamma(f; g_1, k)(a_0 \otimes \cdots \otimes h_j(1 \otimes a_{M_j+1,M_j} \otimes 1) \otimes \cdots \otimes a_{M+1})
\]
\[
= f(a_0 \otimes \cdots \otimes g_i(1 \otimes z_i \otimes 1) \otimes \cdots \otimes a_{M+1})
\]
\[
= \gamma(f; \ldots, \gamma(g_i; h_{N_i+1,N_i}) \ldots)(a_0 \otimes \cdots \otimes a_{M+1}).
\] (3.4.1)

Here the element \( z_i \) (\( 1 \leq i \leq k \)) is given by
\[
z_i = \bigotimes_{l=N_{i-1}+1}^{N_i} h_l(1 \otimes a_{M_{l-1},l+1,M_l} \otimes 1)
\] (3.4.2)
\[
= h_{N_{i-1}+1}(1 \otimes a_{M_{N_{i-1}+1},N_{i-1}+1} \otimes 1) \otimes \cdots \otimes h_{N_i}(1 \otimes a_{M_{N_i},N_{i-1}+1,M_N} \otimes 1).
\]
This shows that \( \gamma \) is associative and that \( \mathcal{C} = \{ \mathcal{C}(n), \gamma, \text{Id} \} \) is an operad.

To see that \( \pi \in \mathcal{C}(2) \) is a multiplication on \( \mathcal{C} \), one observes that both \( \gamma(\pi; \pi, \text{Id})(a_0 \otimes \cdots \otimes a_4) \) and \( \gamma(\pi; \text{Id}, \pi)(a_0 \otimes \cdots \otimes a_4) \) are equal to the product \( a_0 a_1 a_2 a_3 a_4 \).

This finishes the proof of Theorem 3.4. \( \square \)

4. Brace algebra

The purpose of this section is to show that the graded vector space \( \text{CH}^*_\text{Hopf}(A, A) \) admits the structure of a brace algebra with multiplication.
4.1. **Brace algebra.** For a graded vector space $V = \oplus_{n=1}^{\infty} V^n$ and an element $x \in V^n$, set $\deg x = n$ and $|x| = n - 1$. Elements in $V^n$ are said to have degree $n$.

Recall from [10, Definition 1] that a brace algebra is a graded vector space $V = \oplus V^n$ together with a collection of brace operations $x(x_1, \ldots, x_n)$ of degree $-n$, satisfying the associativity axiom:

$$x(x_1, \ldots, x_n) = \sum_{0 \leq i_1 \leq \ldots \leq i_n \leq m} (-1)^{\varepsilon} x(y_{i_1}, x_1 ; y_{i_1+1}, \ldots, y_{i_n}, x_m ; y_{i_m+1}, \ldots, y_{i_n^m}).$$

Here the sign is given by $\varepsilon = \sum_{p=1}^{m} (|x_p| \sum_{q=1}^{i_p} |y_q|)$.

4.2. **Brace algebra with multiplication.** Let $V = \oplus V^n$ be a brace algebra. A multiplication on $V$ ([10, Section 1.2] is an element $m \in V^2$ such that

$$m(m) = 0. \quad (4.2.1)$$

In this case, we call $V = (V, m)$ a brace algebra with multiplication.

4.3. **Brace algebra from operad.** Suppose that $O = \{O(n), \gamma, \text{Id}\}$ is an operad. Define the following operations on the graded vector space $O = \oplus O(n)$:

$$x(x_1, \ldots, x_n) \overset{\text{def}}{=} \sum (-1)^{\varepsilon} \gamma(x; \text{Id}, \ldots, \text{Id}, x_1, \ldots, \text{Id}, x_n, \text{Id}, \ldots, \text{Id}). \quad (4.3.1)$$

Here the sum runs over all possible substitutions of $x_1, \ldots, x_n$ into $\gamma(x; \ldots)$ in the given order. The sign is determined by $\varepsilon = \sum_{p=1}^{n} |x_p| \sum_{q=1}^{i_p} |y_q|$, where $i_p$ is the total number of inputs in front of $x_p$. Note that

$$\deg x(x_1, \ldots, x_n) = \deg x - n + \sum_{p=1}^{n} \deg x_p,$$

so the operation (4.3.1) is of degree $-n$.

Proposition 1 in [10] establishes that the operations (4.3.1) make the graded vector space $\oplus O(n)$ into a brace algebra. Moreover, a multiplication on the operad $O$ in the sense of §3.2 is equivalent to a multiplication on the brace algebra $\oplus O(n)$. In fact, for an element $m \in O(2)$, one has that

$$m(m) = \gamma(m ; m, \text{Id}) - \gamma(m ; \text{Id}, m). \quad (4.3.2)$$

It follows that the condition (3.2.1) is equivalent to (4.2.1). In other words, an operad with multiplication $(O, m)$ gives rise to a brace algebra with multiplication $(\oplus O(n), m)$. Combining this discussion with Theorem 3.4, we obtain the following result.

**Corollary 4.4.** The graded vector space $CH_{\text{Hopf}}(A, A)$ is a brace algebra with brace operations as in (4.3.1) and multiplication $\pi$ (3.3.4).
The brace operations on $\mathrm{CH}^*_\mathrm{Hopf}(A, A)$ can be described more explicitly as follows. For $f \in \mathbb{C}(k)$ and $g_i \in \mathbb{C}(m_i)$ $(1 \leq i \leq n)$, we have

$$f\{g_1, \ldots, g_n\} = \sum (-1)^{\varepsilon} f \{g_1, \ldots, g_{i-1}, g_i, \ldots, g_n\}, \quad \varepsilon = \sum_{p=1}^n (m_p - 1) i_p. \quad (4.4.1)$$

where $\text{Id}' = \text{Id}^{br}$. Here the $r_j$ are given by

$$r_j = \begin{cases} i_1 & \text{if } j = 1, \\ i_j - i_{j-1} - 1 & \text{if } 2 \leq j \leq n, \\ k - i_n - 1 & \text{if } j = n + 1, \end{cases} \quad (4.4.2)$$

and

$$\varepsilon = \sum_{p=1}^n (m_p - 1) i_p. \quad (4.4.3)$$

Write $M = \sum_{i=1}^n m_i$ and $M_j = \sum_{i=1}^j m_i$. Then for an element $a_{0, k+M-n+1} \in A^{\otimes (k+M-n)}$, we have

$$f\{g_1, n\}(a_{0, k+M-n+1}) = \sum (-1)^{\varepsilon} f(a_{0, i_1} \otimes \cdots \otimes a_{i_{j-1} + M_{j-1} - (j-1)+1, i_{j-1} + M_{j-1} - (j-1)+1}) \otimes \cdots \otimes a_{i_n + M_{n+2}, n+1} \otimes (1 \otimes a_{i_{n+1} + M_{n+1} - (n+2)} \otimes \cdots \otimes a_{i_n + M_{n+1}}). \quad (4.4.4)$$

5. Homotopy Gerstenhaber algebra

The purpose of this section is to observe that the brace algebra with multiplication structure on $\mathrm{CH}^*_\mathrm{Hopf}(A, A)$ induces a homotopy Gerstenhaber algebra structure.

5.1. Homotopy $G$-algebra. Recall from [10, Definition 2] that a homotopy $G$-algebra $(V, d, \cup)$ consists of a brace algebra $V = \oplus V^n$, a degree $+1$ differential $d$, and a degree $0$ associative $\cup$-product that make $V$ into a differential graded algebra, satisfying the following two conditions.

1. The $\cup$-product is required to satisfy the condition

$$(x_1 \cup x_2)\{y_1, n\} = \sum_{k=0}^n (-1)^{\varepsilon} x_1\{y_1, k\} \cup x_2\{y_{k+1}, n\},$$

where $\varepsilon = |x_2| \sum_{p=1}^k |y_p|$, for $x_i, y_j \in V$. 


Theorem 5.5. The cohomology modules defined by these two differentials are the same.

Corollary 5.3. For an H-module-algebra A, \( CH^*_H(A, A) \) is a homotopy G-algebra.

5.4. Comparing differentials. At this moment, there are two differentials on the graded vector space \( CH^*_H(A, A) \), namely, the differential \( d^n \) (5.2.1) induced by the multiplication \( \pi \) and the Hopf-Hochschild differential \( d^n_{CH} \). The following result ensures that the cohomology modules defined by these two differentials are the same.

Theorem 5.5. The equality \( d^n_{CH} = (-1)^{n+1}d^n \) holds for each \( n \).

Proof. Pick \( f \in CH^*_H(A, A) \). Then we have

\[
d^n f = \pi(f) + (-1)^n f(\pi)
\]

\[
= (-1)^{n-1} \gamma(\pi; \mathrm{Id}, f) + \gamma(\pi; f, \mathrm{Id})
\]

\[
+ (-1)^n \sum_{i=1}^n (-1)^{i-1} \gamma(f; \mathrm{Id}_{i-1}, \pi, \mathrm{Id}_{n-i}).
\]

(5.5.1)

It follows that

\[
(-1)^{n+1}d^n f = \gamma(\pi; \mathrm{Id}, f) + (-1)^{n+1} \gamma(\pi; f, \mathrm{Id}) + \sum_{i=1}^n (-1)^i \gamma(f; \mathrm{Id}_{i-1}, \pi, \mathrm{Id}_{n-i}).
\]

(5.5.2)

Observe that [14] page 8

\[
g(a_{0,n+1}) = a_0 g(1 \otimes a_{1,n} \otimes 1) a_{n+1}
\]

(5.5.3)
for \( g \in \text{CH}_{\text{Hopf}}^\ast(A, A) \). Using (5.5.3) and applying the various terms in (5.5.2) to an element 
\( a_{0,n+2} \in \text{CB}_{n+1}^\ast(A) = A^{\otimes(n+3)} \), we obtain
\[
\gamma(\pi; \text{Id}, f)(a_{0,n+2}) = f(a_0 a_1 \otimes a_{2,n+2}),
\]
\[
\gamma(\pi; f, \text{Id})(a_{0,n+2}) = f(a_0 \otimes a_{n+1}a_{n+2}),
\]
\[
\gamma(f; \text{Id}^{-1}, \pi, \text{Id}^{-1})(a_{0,n+2}) = f(a_{0,j-1} \otimes a_{i+1} \otimes a_{i+2,n+2}).
\]
\[(5.5.4)\]

The Theorem now follows immediately from (5.5.2) and (5.5.4). □

**Corollary 5.6.** There is an isomorphism of cochain complexes
\[(\text{CH}^\ast_{\text{Hopf}}(A, A), d_{\text{CH}}) \cong (\text{CH}^\ast_{\text{Hopf}}(A, A), d).\]

Moreover, the cohomology modules on \( \text{CH}^\ast_{\text{Hopf}}(A, A) \) defined by the differentials \( d_{\text{CH}} \) and \( d \) are equal.

### 6. Gerstenhaber algebra

The purpose of this section is to observe that the homotopy \( G \)-algebra structure on \( \text{CH}^\ast_{\text{Hopf}}(A, A) \) gives rise to a \( G \)-algebra structure on the Hopf-Hochschild cohomology modules \( H^\ast_{\text{Hopf}}(A, A) \).

#### 6.1. Gerstenhaber algebra

Recall from [10, Section 2.2] that a \( G \)-algebra \((V, \cup, [-, -])\) consists of a graded vector space \( V = \oplus V^n \), a degree 0 associative \( \cup \)-product, and a degree \(-1\) graded Lie bracket
\[[-, -]: V^m \otimes V^n \to V^{m+n-1},\]
satisfying the following two conditions:
\[(6.1.1)\]
\[x \cup y = (-1)^{\deg x \deg y} y \cup x,\]
\[[x, y \cup z] = [x, y] \cup z + (-1)^{\deg y} y \cup [x, z].\]

In other words, the \( \cup \)-product is graded commutative, and the Lie bracket is a graded derivation for the \( \cup \)-product. In particular, a \( G \)-algebra is a graded version of a Poisson algebra. This algebraic structure was first studied by Gerstenhaber [3].

#### 6.2. \( G \)-algebra from homotopy \( G \)-algebra

If \((V, d, \cup)\) is a homotopy \( G \)-algebra, one can define a degree \(-1\) operation on \( V 
\[x \cup y \overset{\text{def}}{=} x[y] - (-1)^{\deg[y]} y[x].\]

Passing to cohomology, \((H^\ast(V, d), \cup, [-, -])\) becomes a \( G \)-algebra ([10, Corollary 5 and its proof]).

Combining the previous paragraph with Corollary 5.3 and Corollary 5.6, we obtain the following result.
Corollary 6.3. The Hopf-Hochschild cohomology modules $HH^*_{\text{Hopf}}(A, A)$ of an $H$-module-algebra $A$ admits the structure of a $G$-algebra.

This $G$-algebra can be described on the cochain level more explicitly as follows. Pick $\varphi \in CH^m_{\text{Hopf}}(A, A)$ and $\psi \in CH^n_{\text{Hopf}}(A, A)$. Then

$$(\psi \cup \varphi)(a_0, m+n+1) = (-1)^{m+n-1}\psi(a_0, m) \otimes 1)\varphi(1 \otimes a_{m+1, m+n+1}),$$

where, writing $a = a_0, m+n$,

$$\psi[\varphi](a) = \sum_{i=1}^{m} (-1)^{(i-1)(m-1)}\psi(a_0, i-1) \otimes \varphi(1 \otimes a_{i, i+n-1} \otimes 1) \otimes a_{i+n, m+n},$$

$$\varphi[\psi](a) = \sum_{j=1}^{n} (-1)^{(j-1)(n-1)}\varphi(a_0, j-1) \otimes \psi(1 \otimes a_{j, j+m-1} \otimes 1) \otimes a_{j+m, m+n}.$$  \hfill (6.3.1)

In particular, if $m = n = 1$, then the bracket operation

$$[\psi, \varphi](a_{0,2}) = \psi(a_0 \otimes \varphi(1 \otimes a_1 \otimes 1) \otimes a_2) - \varphi(a_0 \otimes \psi(1 \otimes a_1 \otimes 1) \otimes a_2)$$  \hfill (6.3.3)

gives $HH^1_{\text{Hopf}}(A, A)$ a Lie algebra structure. There is another description of this Lie algebra in terms of (inner) derivations in \[14\] Proposition 3.9.

7. Deligne’s Conjecture for module-algebras

The purpose of this section is to observe that a version of Deligne’s Conjecture holds for the Hopf-Hochschild cochain complex of a module-algebra. The original Deligne’s Conjecture for Hochschild cohomology is as follows.

**Deligne’s Conjecture (6).** The Hochschild cochain complex $C^*(R, R)$ of an associative algebra $R$ is an algebra over a suitable chain model of May’s little squares operad $\mathcal{C}_2$ \[16\].

A positive answer to Deligne’s conjecture was given by, among others, McClure and Smith \[18\] Theorem 1.1] and Kaufmann \[13] Theorem 4.2.2]. There is an operad $\mathcal{H}$ whose algebras are the brace algebras with multiplication (§4.2). For an associative algebra $R$, the Hochschild cochain complex $C^*(R, R)$ is a brace algebra with multiplication and hence an $\mathcal{H}$-algebra. McClure and Smith showed that $\mathcal{H}$ is quasi-isomorphic to the chain operad $S$ obtained from the little squares operad $\mathcal{C}_2$ by applying the singular chain functor, thereby proving Deligne’s Conjecture.

It has been observed that the Hopf-Hochschild cochain complex $CH^*_{\text{Hopf}}(A, A)$ is a brace algebra with multiplication (Corollary 4.4). Therefore, we can use the result of McClure and Smith \[18\] Theorem 1.1] to obtain the following version of Deligne’s Conjecture for module-algebras.
Corollary 7.1 (Deligne’s Conjecture for module-algebras). The Hopf-Hochschild cochain complex $\text{CH}^*_\text{Hopf}(A,A)$ of an $H$-module-algebra $A$ is an algebra over the McClure-Smith operad $\mathcal{K}$ that is a chain model for the little squares operad $\mathcal{C}_2$.

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