A COPOSITIVE FORMULATION FOR THE STABILITY NUMBER OF INFINITE GRAPHS

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Abstract. In the last decade, copositive formulations have been proposed for a variety of combinatorial optimization problems, for example the stability number (independence number). In this paper, we generalize this approach to infinite graphs and show that the stability number of an infinite graph is the optimal solution of some infinite-dimensional copositive program. For this we develop a duality theory between the primal convex cone of copositive kernels and the dual convex cone of completely positive measures. We determine the extreme rays of the latter cone, and we illustrate this theory with the help of the kissing number problem.

1. Introduction

One way to deal with problems in combinatorial optimization is to (re-)formulate them as convex optimization problems. This is beneficial because it allows a geometric interpretation of the original combinatorial problem and because the convexity provides ways to find bounds or even to certify optimality of solutions.

In the last decade copositive formulations have been proposed for many NP-hard problems; see the survey of Dür [8] and references therein. In these formulations the hardness is entirely moved into the copositivity constraint. Therefore any progress in understanding this constraint immediately provides new insights for a variety of problems.

Bomze, Dür, de Klerk, Roos, Quist, and Terlaky [5] were the first to give a copositive formulation of an NP-hard combinatorial problem, namely the clique number of a graph. Similarly, de Klerk and Pasechnik [12] considered the stability number of a graph. The
The stability number of a finite, undirected, simple graph $G = (V, E)$ is

$$\alpha(G) = \max \{|S| : S \text{ is a stable set}\},$$

where $S \subseteq V$ is a stable set if for all $x, y \in S$ we have $\{x, y\} \not\in E$. Finding the stability number of a graph is a fundamental problem in combinatorial optimization and has many applications. It is one of the most difficult NP-hard problems, in the sense that even providing an approximation of any reasonable quality is NP-hard, see Håstad [11]. In [12, Theorem 2.2] de Klerk and Pasechnik gave the following copositive formulation: For $V = \{1, \ldots, n\}$,

$$\alpha(G) = \min \ t$$

$$t \in \mathbb{R}, \ K \in C_n$$

$$K(i, i) = t - 1 \quad \text{for all } i \in V$$

$$K(i, j) = -1 \quad \text{for all } \{i, j\} \not\in E,$$

where $C_n$ denotes the convex cone of copositive $n \times n$-matrices. Recall that a real symmetric matrix $K \in \mathbb{R}^{n \times n}$ is called copositive if

$$\forall a_1, \ldots, a_n \geq 0 : \sum_{i=1}^{n} \sum_{j=1}^{n} K(i, j)a_i a_j \geq 0. \quad (1)$$

The space of symmetric matrices is equipped with the usual Frobenius trace inner product

$$\langle K, L \rangle = \text{trace}(KL).$$

With this inner product the dual of the copositive cone is the cone of the completely positive matrices

$$C^*_n = \{L \in \mathbb{R}^{n \times n} : L \text{ symmetric}, \langle K, L \rangle \geq 0 \text{ for all } K \in C_n\}.\$$

In [10] Theorem 3.1 (iii) Hall and Newman determined the extreme rays of this cone. They showed that a matrix generates an extreme ray of $C^*_n$ if and only if it is of the form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j e_i e_j^T \quad \text{with } N \in \mathbb{N}, a_1, \ldots, a_N \geq 0,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$.

The concept of stable sets in a graph is also useful in infinite graphs, for instance to model geometric packing problems in metric spaces. Let $V$ be a compact metric space with Borel measure $\omega$ and distance function $d$. Finding the densest packing of balls with radius $r$ in $V$ is equivalent to finding the stability number of the graph $G = (V, E)$ where the vertices of $G$ are the elements of the compact metric space $V$ and $\{x, y\} \in E$ iff $d(x, y) \in (0, 2r)$. For
example, one can formulate the kissing number problem in this way (see, e.g., the exposition by Pfender and Ziegler \[17\]): Take $V$ to be the unit sphere $S^{n-1} = \{ x \in \mathbb{R}^n : x^T x = 1 \}$, let $\omega$ be the induced Lebesgue measure, $d$ the angular distance $d(x,y) = \arccos x^T y$, and $r = \pi/6$.

In this paper we only consider graphs $G = (V, E)$ which satisfy the following condition:

There is no open subset $U$ of $V$ such that $U \times U \subset D \cup \overline{E}$, \hspace{1cm} (2)

where $D = \{ (x,x) : x \in V \}$ and $\overline{E} = \{ (x,y) \in V \times S V : x \neq y, \{x,y\} \not\in E \}$ is the complement of the edge set. This condition, which is fulfilled for instance for the geometric graphs described above, implies that the stability number of $G$ is finite.

The aim of the present paper is to give a copositive formulation of the stability number of infinite graphs which satisfy condition \[2\], and thereby to initiate the study of copositive formulations also for other combinatorial problems in a continuous setting.

In order to go from the finite to the infinite setting we have to provide the right generalizations of copositive and completely positive matrices.

On the primal (copositive) side, the natural generalization of finite $n \times n$-matrices (where rows and columns are indexed by $\{1, \ldots, n\}$) are real-valued continuous Hilbert-Schmidt kernels on the compact metric space $V$. The set of continuous Hilbert-Schmidt kernels is defined as follows:

$$\mathcal{C}(V \times_s V) = \{ K : V \times_s V \to \mathbb{R} : K \text{ is symmetric and continuous} \}.$$  

Symmetry means here that for all $x, y \in V$ we have $K(x,y) = K(y,x)$, and we write $V \times_s V$ to denote the symmetric product.

In analogy to \[1\] we call a kernel $K$ copositive if

$$\forall f \in C(V)_{\geq 0} : \int_V \int_V K(x,y) f(x) f(y) d\omega(x)d\omega(y) \geq 0,$$ \hspace{1cm} (3)

where $C(V)_{\geq 0}$ is the convex cone of all nonnegative continuous functions on $V$. We denote the convex cone of copositive kernels by $C_V$.

The dual space of $C(V \times_s V)$ equipped with the supremum norm consists of all continuous linear functionals, and by Riesz’ representation theorem it can be identified with the space of signed symmetric Borel measures $M(V \times_s V)$ equipped with the total variation norm. Let $K$
be a continuous kernel and \( \mu \) be a Borel measure on \( V \times S V \). The duality is given by

\[
\langle K, \mu \rangle = \int_{V \times S V} K(x, y) \, d\mu(x, y).
\]  

Then the dual cone of the cone \( C_V \) of copositive kernels is a cone which we baptize the cone of completely positive measures,

\[
C_V^* = \{ \mu \in M(V \times S V) : \langle K, \mu \rangle \geq 0 \text{ for all } K \in C_V \}.
\]

Whereas in the finite setting the convex cones of copositive and completely positive matrices are mutually dual to each other, this symmetry between primal and dual breaks in the infinite setting. The dual of the cone of copositive kernels is the cone of completely positive measures, but the dual of that cone only strictly contains the copositive kernels: \( C_V \subsetneq (C_V^*)^* \).

Using these definitions we can state our main theorem which gives a copositive formulation of the stability number of infinite graphs.

**Theorem 1.1.** Let \( G = (V, E) \) be a graph whose the vertex set is a compact metric space \( V \), and which satisfies condition \((2)\). Then

\[
\alpha(G) = \inf_{t \in \mathbb{R}, K \in C_V} \quad (P)
\]

\[
t \in \mathbb{R}, \quad K(x, x) = t - 1 \text{ for all } x \in V
\]

\[
K(x, y) = -1 \quad \text{for all } \{x, y\} \not\in E.
\]

The remainder of the paper is organized as follows: In Section 2 we analyze properties of the two infinite-dimensional cones \( C_V \) and \( C_V^* \); we give a characterization of copositive kernels and we determine the extreme rays of the cone \( C_V^* \) of completely positive measures. In Section 3 we prove our main result, Theorem 1.1. There we first derive a completely positive formulation of the stability number — which we will denote by \((D)\) —, that is the dual of \((P)\). Then by proving that there is no duality gap between the primal and the dual we derive Theorem 1.1. We also give a version of Theorem 1.1 for the weighted stability number. In Section 4 we provide an interpretation of our copositive formulation for the kissing number problem. Then we end by posing a question for possible future work.

2. Copositive kernels and completely positive measures

2.1. Copositive kernels. In \((3)\) we defined a kernel to be copositive by integrating it with nonnegative continuous functions. Instead of using nonnegative continuous functions we can
also define copositivity by means of finite nonnegative delta measures. For the larger class of positive definite kernels this is a classical fact, realized for instance by Bochner [4, Lemma 1], see also Folland [9, Proposition 3.35]:

A kernel $K \in C(V \times_S V)$ is called positive (semi-)definite if

$$\forall f \in C(V) : \int_V \int_V K(x, y) f(x)f(y) \, d\omega(x) d\omega(y) \geq 0,$$

where $C(V)$ denotes the space of continuous functions. Bochner [4] proved that a kernel $K \in C(V \times_S V)$ is positive semidefinite if and only if for any choice $x_1, \ldots, x_N$ of finitely many points in $V$, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is positive semidefinite.

The following lemma shows that a similar characterization holds for copositive kernels. For the reader’s convenience we provide a proof here.

**Lemma 2.1.** A kernel $K \in C(V \times_S V)$ is copositive if and only if for any choice of finitely many points $x_1, \ldots, x_N \in V$, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is copositive.

**Proof.** Since $V \times_S V$ is compact, the continuous function $K$ is uniformly continuous and bounded on $V \times_S V$.

Suppose that for any choice $x_1, \ldots, x_N$ of finitely many points in $V$, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is copositive. Let $\varepsilon > 0$ and $f \in C(V)_{\geq 0}$. Since $K$ is uniformly continuous, we can partition $V$ into a finite number of measurable sets $V_1, \ldots, V_N$ and find points $x_i \in V_i$ such that

$$|K(x, y) - K(x_i, x_j)| \leq \varepsilon \quad \text{for all } x \in V_i, y \in V_j. \quad (5)$$

Set $a_i = \int_{V_i} f(x) \, d\omega(x)$. Then $a_i \geq 0$ and

$$0 \leq \sum_{i=1}^N \sum_{j=1}^N K(x_i, x_j) a_i a_j - \int_V \int_V K(x, y) f(x)f(y) \, d\omega(x) d\omega(y)$$

$$= \sum_{i=1}^N \sum_{j=1}^N \int_{V_i} \int_{V_j} (K(x_i, x_j) - K(x, y)) f(x)f(y) \, d\omega(x) d\omega(y)$$

$$\leq \sum_{i=1}^N \sum_{j=1}^N \int_{V_i} \int_{V_j} |K(x_i, x_j) - K(x, y)| f(x)f(y) \, d\omega(x) d\omega(y)$$

$$\leq \varepsilon \int_V \int_V f(x)f(y) \, d\omega(x) d\omega(y).$$

Letting $\varepsilon \to 0$ and noting that the last integral is independent of $\varepsilon$ one obtains
\[
\int_V \int_V K(x, y)f(x)f(y) \, d\omega(x)d\omega(y) \geq 0
\]

and hence \( K \) is copositive.

Conversely, assume \( K \) is copositive. For \( x_1, \ldots, x_N \in V \) and \( a_1, \ldots, a_N \geq 0 \) we construct disjoint neighborhoods \( V_i \) of \( x_i \) such that (5) holds. The function

\[
f(x) = \begin{cases} 
a_i \omega(V_i) & \text{if } x \in V_i, 
0 & \text{otherwise,}
\end{cases}
\]

is nonnegative on \( V \), and \( K(x_i, x_j)a_i a_j \) can be expressed as

\[
K(x_i, x_j)a_i a_j = \int_{V_i} \int_{V_j} K(x_i, x_j)f(x)f(y) \, d\omega(x)d\omega(y).
\]

Then,

\[
\left| \sum_{i=1}^N \sum_{j=1}^N K(x_i, x_j)a_i a_j - \int_V \int_V K(x, y)f(x)f(y) \, d\omega(x)d\omega(y) \right| \leq e \sum_{i=1}^N \sum_{j=1}^N a_i a_j.
\]

By letting \( \varepsilon \to 0 \) and by approximating the step function \( f \) by a continuous function, one obtains \( \sum_{i=1}^N \sum_{j=1}^N K(x_i, x_j)a_i a_j \geq 0 \), which concludes the proof.

If the compact metric space \( V \) has infinite cardinality, then one has a “simpler” characterization of copositive kernels not using the coefficients \( a_1, \ldots, a_n \) as in (1). This was first noted by Pfender [16, Lemma 3.3] in the case when \( V \) is the unit sphere \( S^{n-1} \) and for copositive kernels which are invariant under the orthogonal group. However, his arguments hold for arbitrary compact metric spaces and arbitrary copositive kernels.

**Lemma 2.2.** Assume that \( V \) has infinite cardinality. Then a kernel \( K \in C(V \times_S V) \) is copositive if and only if for any choice of finitely many points \( x_1, \ldots, x_N \in V \), the sum \( \sum_{i=1}^N \sum_{j=1}^N K(x_i, x_j) \) is nonnegative.

**Proof.** Assume that \( K \) is copositive and take any finite set of points \( x_1, \ldots, x_N \in V \). Then choosing \( a_1 = \ldots = a_N = 1 \) in the previous lemma gives \( \sum_{i=1}^N \sum_{j=1}^N K(x_i, x_j) \geq 0 \).

To show the converse, let \( f \in C(V)_{\geq 0} \). By scaling we may assume that \( \int_V f(x) \, d\omega(x) = 1 \), i.e., \( f \) is a probability density function on \( V \). Picking points \( x_1, \ldots, x_N \in V \) from this
distribution gives
\[
0 \leq \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j)
\]
\[
= \frac{1}{N} \int_{V} K(x, x) f(x) d\omega(x) + \frac{N - 1}{N} \int_{V} \int_{V} K(x, y) f(x) f(y) d\omega(x) d\omega(y).
\]
By letting \(N \to \infty\), we see that the double integral is nonnegative, and hence \(K\) is copositive.

\[\square\]

2.2. Completely positive measures. In the finite setting, the rank-1-matrices
\[
aa^T = \left( \sum_{i=1}^{n} a_i e_i \right) \left( \sum_{i=1}^{n} a_i e_i \right)^T \quad \text{with} \quad a_1, \ldots, a_n \geq 0,
\]
determine all the extreme rays of the cone of completely positive matrices \(C_n^*\). So we have an explicit description of this cone,
\[
C_n^* = \text{cone} \left\{ aa^T : a_1, \ldots, a_n \geq 0 \right\},
\]
where the cone operator denotes taking the convex conic hull. In a sense, this fact generalizes to the infinite setting as we shall show soon in Theorem 2.5.

First we have to find the proper replacement of the rank-1-matrices \(aa^T\). The topological vector spaces \(C(V \times_S V)\) and \(M(V \times_S V)\) form a duality with the inner product (4). In this way it defines the weak topology on \(C(V \times_S V)\) and the weak* topology on \(M(V \times_S V)\). We will show next that delta measures of the form
\[
\sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \quad \text{with} \quad x_1, \ldots, x_N \in V, \ a_1, \ldots, a_N \geq 0 \quad (6)
\]
defined by
\[
\left\langle K, \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \right\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) \quad \text{for} \ K \in C(V \times_S V)
\]
play the role of the rank-1-matrices.

**Proposition 2.3.** The cone of completely positive measures equals
\[
C_V^* = \text{cl cone} \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, \ x_i \in V, \ a_i \geq 0 \right\},
\]
where the closure is taken with respect to the weak* topology.
Proof. One inclusion is straightforward: By definition, the cone $C_\mathcal{V}^*$ of completely positive measures is closed, and by Lemma 2.1 delta measures of the form (6) lie in $C_\mathcal{V}^*$.

For the other inclusion we use the Hahn-Banach theorem for locally convex topological vector spaces. For this note that $M(V \times_S V)$ with the weak* topology is a locally convex topological space, and all continuous linear functionals are given by $\langle K, \cdot \rangle$ for some $K \in C(V \times_S V)$. Take $\mu \in M(V \times_S V) \setminus \text{cl cone } \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, x_i \in V, a_i \geq 0 \right\}$.

By Hahn-Banach there exists a kernel $K \in C(V \times_S V)$ such that $\langle K, \mu \rangle < 0$ and $\langle K, \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \rangle \geq 0$ for all $N \in \mathbb{N}, x_i \in V$ and $a_i \geq 0$ ($i = 1, \ldots, N$). Hence, again by Lemma 2.1, the kernel $K$ is copositive and therefore $\mu \notin C_\mathcal{V}^*$.

Proposition 2.4. The set
$$B = \text{cl conv } \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, x_i \in V, a_i \geq 0, \sum_{i=1}^{N} a_i = 1 \right\}$$
forms a weak* compact base of the cone of completely positive measures, i.e.,
$$C_\mathcal{V}^* = \bigcup_{\lambda \geq 0} \lambda B.$$  

Proof. The set $B$ is closed by definition, so in order to prove the weak* compactness it suffices to show that $B$ is contained in a compact set. But this is clear since
$$B \subseteq \{ \mu \in M(V \times_S V) : \mu(V \times_S V) \leq 1, \mu \geq 0 \}$$
and the latter set is compact in the weak* topology by the Theorem of Banach-Alaoglu. The second statement of the proposition is obvious.

Theorem 2.5. A measure generates an extreme ray of the cone $C_\mathcal{V}^*$ of completely positive measures if and only if it is a product measure of the form $\mu \otimes \mu$, where $\mu \in M(V)$ is a nonnegative measure on $V$.

Proof. By the previous proposition it suffices to show that the extreme points of the set $B$ are product measures of probability measures defined on $V$:
$$\text{ex } B = \{ \mu \otimes \mu : \mu \geq 0, \mu(V) = 1 \}.$$
First we show that delta measures of the form \( \mu \) with \( \sum_{i=1}^{N} a_i = 1 \) are extreme points of \( \mathcal{B} \). Suppose that we can decompose a delta measure \( \mu \in \mathcal{B} \) as

\[
\mu = \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} = \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \quad \text{with} \quad \nu_1, \nu_2 \in \mathcal{B}.
\]

(7)

Since in the weak* topology \( \nu_1 \) and \( \nu_2 \) can be approximated by convex combinations of delta measures of the form \( \delta \) lying in \( \mathcal{B} \), there is no cancellation possible in (7). Hence, \( \nu_1 = \nu_2 = \mu \), and so \( \mu \) is extreme.

Because of this and because of Proposition 2.4, we have by Milman’s converse of the Krein-Milman theorem that

\[
\text{ex} \mathcal{B} \subseteq \overline{\left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, x_i \in V, a_i \geq 0, \sum_{i=1}^{N} a_i = 1 \right\}}
\]

holds. Since the probability measures \( \mu \in M(V) \) form the closure of the delta probability measures \( \sum_{i=1}^{N} a_i \delta_{x_i} \) this concludes the proof of the theorem.

3. Copositive formulation for the stability number of infinite graphs

In order to develop our copositive formulation of the stability number we make use of Kantorovich’s approach to linear programming over cones in the framework of locally convex topological vector spaces. This theory is thoroughly explained in Barvinok [3, Chapter IV] and we follow his notation closely.

In Section 3.1 we cast the copositive problem (P) into the general framework of conic problems in Hilbert spaces as studied by Barvinok, and using this general theory, we derive the dual of (P) which will turn out to be an infinite-dimensional completely positive problem. Then we prove our main theorem, Theorem 1.1 in two steps. In the first step, we show in Section 3.2 that the stability number of \( G \) equals the optimal value of the dual problem. In particular we show that the optimum is attained. In the second step, Section 3.3 we establish the fact that there is no duality gap between primal and dual. In Section 3.4 we extend these results and give a copositive formulation for the weighted stability number.

3.1. Primal-dual pair. Let \( G = (V, E) \) be a graph whose vertex set is a compact metric space \( V \). For this graph, the copositive problem (P) can be seen as a general conic problem of the form

\[
\inf_{x \in \mathcal{K}, Ax = b} \langle x, c \rangle_1
\]

(8)
with the following notations:

\[ x = (t, K) \in \mathbb{R} \times C(V \times_S V) \]

\[ c = (1, 0) \in \mathbb{R} \times M(V \times_S V) \]

\[ \langle \cdot, \cdot \rangle_1 : (\mathbb{R} \times C(V \times_S V)) \times (\mathbb{R} \times M(V \times_S V)) \to \mathbb{R} \]

\[ K = \mathbb{R}_{\geq 0} \times C_V \]

\[ A : \mathbb{R} \times C(V \times_S V) \to C(V) \times C(\overline{E}) \]

\[ A(t, K) = (x \mapsto K(x, x) - t, (x, y) \mapsto K(x, y)) \]

\[ b = (-1, -1) \in C(V) \times C(\overline{E}). \]

In this, we recall that \( \overline{E} = \{(x, y) \in V \times_S V : x \neq y, \{x, y\} \notin E\} \) is the complement of the edge set. Note that we can replace the constraint \( t \in \mathbb{R} \) in (P) by \( t \in \mathbb{R}_{\geq 0} \), since \( t \geq 1 \) holds automatically because diagonal elements of copositive kernels are nonnegative.

The dual problem of (8) is

\[ \sup_{c - A^* y \in K^*} \langle b, y \rangle_2 \]  \( (9) \)

Applying this to our setting, it is not difficult to see that we need

\[ \langle \cdot, \cdot \rangle_2 : (C(V) \times C(\overline{E})) \times (M(V) \times M(\overline{E})) \to \mathbb{R} \]

\[ y = (\mu_0, \mu_1) \in M(V) \times M(\overline{E}) \]

\[ A^* : M(V) \times M(\overline{E}) \to \mathbb{R} \times M(V \times_S V) \]

\[ A^*(\mu_0, \mu_1) = (-\mu_0(V), \mu_0 + \mu_1) \]

\[ K^* = \mathbb{R}_{\geq 0} \times C^*_V. \]

The map \( A^* \) is the adjoint of \( A \) because

\[ \langle A(t, K), (\mu_0, \mu_1) \rangle_2 = \int_V K(x, x) - t \, d\mu_0(x) + \int_{\overline{E}} K(x, y) \, d\mu_1(x, y) \]

\[ = -t\mu_0(V) + \int_{V \times_S V} K(x, y) \, d(\mu_0 + \mu_1)(x, y) \]

\[ = \langle (t, K), A^*(\mu_0, \mu_1) \rangle_1. \]
Above, when we add the measures \( \mu_0 \) and \( \mu_1 \), we consider them as measures defined on the product space \( V \times S \), where we see the measure \( \mu_0 \) as a measure defined on the diagonal \( D = \{(x,x) : x \in V\} \).

With this, the dual of (P) is the completely positive program

\[
\sup -\mu_0(V) - \mu_1(\bar{E})
\]
\[
\mu_0 \in M(V), \ \mu_1 \in M(\bar{E})
\]
\[
1 + \mu_0(V) \geq 0
\]
\[
-\mu_0 - \mu_1 \in C^*_V.
\]

In order to simplify this, we define the support of a Borel measure \( \mu \) as follows:

\[
\text{supp } \mu = \{(x,y) \in V \times S : \mu(U) \neq 0 \text{ for all open Borel neighborhoods } U \text{ of } (x,y)\}.
\]

Then the dual, completely positive program equals

\[
\sup \mu(V \times S)
\]
\[
\mu \in C^*_V
\]
\[
\mu(D) \leq 1
\]
\[
\text{supp } \mu \subseteq D \cup \bar{E}.
\]

One can that argue by scaling the inequality constraint \( \mu(D) \leq 1 \) can be replaced by the equality constraint \( \mu(D) = 1 \) in (D).

This completely positive program using measures is a generalization of the finite-dimensional completely positive program for finite graphs \( G = (V,E) \), with \( V = \{1, \ldots, n\} \), of de Klerk, Pasechnik [12]:

\[
\max \sum_{i=1}^{n} \sum_{j=1}^{n} X(i,j)
\]
\[
X \in C^*_n
\]
\[
\sum_{i=1}^{n} X(i,i) = 1
\]
\[
X(i,j) = 0 \quad \text{for all } \{i,j\} \in E.
\]

3.2. Completely positive formulation. We next show that the optimal value of problem (D) equals the stability number.

Theorem 3.1. Let \( G = (V,E) \) be a graph whose vertex set is a compact metric space \( V \), and which satisfies property (2). Then the optimal value of the completely positive program (D) is attained and equals \( \alpha(G) \).
Proof. Let \( \lambda \) be the optimal value of (D). For the ease of notation we write \( \alpha \) for \( \alpha(G) \) in this proof.

Let \( x_1, \ldots, x_\alpha \in V \) be a stable set in \( G \) of maximal cardinality. Then the measure

\[
\frac{1}{\alpha} \left( \sum_{i=1}^{\alpha} \delta_{x_i} \right) \otimes \left( \sum_{i=1}^{\alpha} \delta_{x_i} \right)
\]

is a feasible solution of (D) with objective value \( \alpha \). Hence, \( \lambda \geq \alpha \).

In order to prove the reverse inequality we first show that set \( F_D \) of feasible solutions of (D) is weak* compact. We will show in fact that \( F_D = \{ t(\mu_0 + \mu_1) : (\mu_0, \mu_1) \in S_1, t \in [1, \alpha] \} \), where

\[
S_1 = \{ (\mu_0, \mu_1) \in M(V) \times M(E) : \mu_0 + \mu_1 \in C^*_V, \mu_0(V) + \mu_1(\overline{E}) \leq 1 \}.
\]

By Theorem of Banach-Alaoglu, the set \( S_1 \) is weak* compact, so \( F \) is weak* compact as well.

It is clear that \( F \subseteq F_D \). For the reverse inclusion, consider the convex cone

\[
\mathcal{M}_G = \{ \mu \in C^*_V : \text{supp} \mu \subseteq V \cup \overline{E} \}.
\]

It follows from Theorem 2.5 that the extreme rays of \( \mathcal{M}_G \) are product measures. Furthermore, since the graph \( G \) satisfies condition (2), the extreme rays of \( \mathcal{M}_G \) have to be of the form

\[
\left( \sum_{i=1}^{N} a_i \delta_{x_i} \right) \otimes \left( \sum_{i=1}^{N} a_i \delta_{x_i} \right) \quad \text{with} \quad a_i \geq 0, \ x_1, \ldots, x_N \text{ a stable set of } G. \tag{10}
\]

Now let

\[
\mu = \left( \sum_{i=1}^{N} a_i \delta_{x_i} \right) \otimes \left( \sum_{i=1}^{N} a_i \delta_{x_i} \right) \in F_D
\]

be a feasible solution of (D) which lies in an extreme ray of \( \mathcal{M}_G \). We have

\[
\mu(V \times S V) = \left( \sum_{i=1}^{N} a_i \right)^2 \quad \text{and} \quad \mu(D) = \sum_{i=1}^{N} a_i^2 \leq 1.
\]

Write \( \mu = \nu_0 + \nu_1 \) with \( \nu_0 \in M(V) \) and \( \nu_1 \in M(\overline{E}) \). Let \( s \) be a real number such that \( s(\nu_0 + \nu_1)(V \times S V) = 1 \). Then setting \( \mu_0 = s\nu_0, \mu_1 = s\nu_1, t = \frac{1}{s} \), shows that \( s \in \left[ \frac{1}{\alpha}, 1 \right] \) because of the Cauchy-Schwartz inequality

\[
1 \geq s = \frac{1}{\left( \sum_{i=1}^{N} a_i \right)^2} \geq \frac{1}{N} \sum_{i=1}^{N} a_i^2 \geq \frac{1}{N} \geq \frac{1}{\alpha}.
\]

Hence \( \mu \in F \), and consequently \( F_D = F \), which proves that \( F_D \) is weak* compact.
Because of this compactness, the supremum of \( D \) is attained at an extreme point of \( F_D \). Suppose \((\sum_{i=1}^{N} a_i \delta_{x_i}) \otimes (\sum_{i=1}^{N} a_i \delta_{x_i})\) is a maximizer of \( D \). Then again by Cauchy-Schwartz we get that
\[
\lambda = \left( \sum_{i=1}^{N} a_i \right)^2 \leq N \sum_{i=1}^{N} a_i^2 \leq N \leq \alpha,
\]
and the claim of the theorem follows. \( \square \)

3.3. **Copositive formulation.** In this section, we prove our main result, Theorem 1.1, by showing that we have strong duality between \( P \) and \( D \).

**Theorem 3.2.** There is no duality gap between the primal copositive program \( P \) and the dual completely positive program \( D \). In particular, the optimal value of both programs equals \( \alpha(G) \).

For the proof of this theorem we make use of a variant of the zero duality gap theorem of a primal-dual pair of conic linear programs, see Barvinok [3, Chapter IV.7.2]: By dualizing the statement of [3, Problem 3 in Chapter IV.7.2] we see that if the cone
\[
\{(d - A^*y, \langle b, y \rangle_2) : y \in M(V) \times M(\overline{E}), d \in \mathbb{R}_{\geq 0} \times \mathcal{C}_V^* \}
\]
is closed in \( \mathbb{R} \times M(V \times_\mathcal{S} V) \times \mathbb{R} \), then there is no duality gap.

In order to show this closedness condition we need slight modifications of [3, Lemmas III.2.10 and IV.7.3].

**Lemma 3.3.** Let \( V \) be a topological vector space and let \( C \subseteq V \) be a compact set such that \( 0 \notin C \). Then the set \( K = \{\lambda x : x \in C, \lambda \geq 0\} \) is closed.

**Proof.** Since \( 0 \notin C \), there is a neighborhood \( W \) of 0 that does not intersect \( C \). Let \( u \notin K \). Let \( U_1 \) be a neighborhood of \( u \), and \( \delta > 0 \) such that \( \alpha U_1 \subset W \) for all \( |\alpha| < \delta \) (from the continuity of \( (\alpha, x) \to \alpha x \) at \( (0, u) \)). Then \( U_1 \cap \lambda C = \emptyset \) for all \( \lambda > 1/\delta \). The image of the compact set \([0, 1/\delta] \times C\) by the continuous map \( (\alpha, x) \to \alpha x \) is compact and is contained in \( K \). Hence there is a neighborhood \( U_2 \) of \( u \) that does not intersect the image. Then the intersection \( U_1 \cap U_2 \) is a neighborhood of \( u \) that does not intersect \( \{\lambda x : x \in C, \lambda \geq 0\} = K \) which proves that \( K \) is closed. \( \square \)
Lemma 3.4. Let $V$ and $W$ be topological vector spaces, let $K \subseteq V$ be a cone such that there is a compact base $B \subseteq V$ with $0 \notin B$ and $K = \bigcup_{\lambda \geq 0} \lambda B$. Let $T: V \to W$ be a continuous linear transformation such that $\ker T \cap K = \{0\}$. Then $T(K) \subseteq W$ is a closed convex cone.

Proof. Obviously $T(K)$ is a convex cone. The set $C = T(B)$ is compact, $0 \notin C$, and $T(K) = \bigcup_{\lambda \geq 0} \lambda C$. Applying Lemma 3.3 gives that $T(K)$ is closed. \hfill \Box

Now we are ready for the proof of the theorem:

Proof of Theorem 3.2. Consider the continuous linear transformation

$$T(d, y) = (d - A^*y, \langle b, y \rangle_2).$$

We have already seen that the cone has a compact base. Suppose $(d, y)$ lie in the kernel of $T$. Then the condition $\langle b, y \rangle_2 = 0$ forces $y$ to be zero. This forces $d = 0$ and we can apply Lemma 3.4 to complete the proof of the theorem. \hfill \Box

3.4. Copositive formulation for the weighted stability number. In some situations one wishes to consider packing problems with different types of objects, having different sizes; for instance the problem of packing spherical caps having different radii as considered by de Laat, Oliveira, and Vallentin [14]. In these cases it is helpful to use a weighted version of the copositive problem formulation which is presented in the next theorem. We omit its proof here since it is completely analogous to the one of Theorems 3.2 and 3.1. The only difference is that we are now given a continuous weight function $w: V \to \mathbb{R}_{\geq 0}$ for the vertex set, and in our optimization problems we replace the objective function

$$\mu(V \times S V) = \int_V \int_V d\mu(x, y) \quad \text{by} \quad \int_V \int_V \sqrt{w(x)w(y)} d\mu(x, y).$$

Theorem 3.5. Let $G = (V, E)$ be a graph which satisfies condition (2) and let $w: V \to \mathbb{R}_{\geq 0}$ be a continuous weight function for the vertex set. Then the weighted stability number $\alpha_w(G)$ defined by

$$\alpha_w(G) = \max \left\{ \sum_{x \in S} w(x) : S \text{ stable set of } G \right\}$$

has the following copositive formulation

$$\alpha_w(G) = \inf t \quad t \in \mathbb{R}, \ K \in C_V$$

$$K(x, x) = t - w(x) \quad \text{for all } x \in V$$

$$K(x, y) = -\sqrt{w(x)w(y)} \quad \text{for all } \{x, y\} \notin E. \quad (11)$$
For the finite case, Bomze [6] showed that the maximum weight clique problem can be formulated as a standard quadratic problem. With the techniques from [5] this in turn can be written as a copositive problem of which (11) is the infinite counterpart.

4. COPOSITIVE FORMULATION OF THE KISSING NUMBER

In this section we give a copositive formulation of the kissing number problem. We show that in this case the copositive program can be equivalently transformed into a semi-infinite linear program. We start with the original copositive formulation:

\begin{equation}
\inf t \\
\quad t \in \mathbb{R}, \ K \in \mathcal{C}_{S^{n-1}} \\
\quad K(x, x) = t - 1 \quad \text{for all } x \in S^{n-1} \\
\quad K(x, y) = -1 \quad \text{for all } x, y \in S^{n-1} \text{ with } x^T y \in [-1, 1/2].
\end{equation}

Since the packing graph is invariant under the orthogonal group, also the copositive formulation is invariant under this group. By convexity we can restrict the copositive formulation above to copositive kernels which are invariant under the orthogonal group. So \( K(x, y) \) only depends on the inner product \( x^T y \).

By Stone-Weierstrass we know that polynomials lie dense in \( C(S^{n-1} \times S^{n-1}) \), so we approximate \( K(x, y) \) by \( \sum_{k=1}^{d} c_k (x^T y)^k \). Then by Lemma 2.2 the copositivity condition \( K \in \mathcal{C}_{S^{n-1}} \) translates to

\begin{equation}
\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=0}^{d} c_k (x_i \cdot x_j)^k \geq 0 \quad \text{for all } N \in \mathbb{N} \text{ and } x_1, \ldots, x_N \in S^{n-1}.
\end{equation}

The other constraints of the above copositive problem translate likewise, and observing that \( x^T x = 1 \) for \( x \in S^{n-1} \), we get the following semi-infinite linear program whose optimal value converges to the kissing number if the degree \( d \) tends to infinity:

\begin{equation}
\inf 1 + \sum_{k=0}^{d} c_k \\
\quad c_0, \ldots, c_d \in \mathbb{R}, \\
\quad \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=0}^{d} c_k (x_i \cdot x_j)^k \geq 0 \quad \text{for all } N \in \mathbb{N} \text{ and } x_1, \ldots, x_N \in S^{n-1} \\
\quad \sum_{k=0}^{d} c_k s^k \leq -1 \quad \text{for all } s \in [-1, 1/2].
\end{equation}
We impose the condition $\sum_{k=0}^{d} c_k s^k \leq -1$ instead of $\sum_{k=0}^{d} c_k s^k = -1$ to make the problem feasible for finite degree $d$. It is easy to see that this relaxation is not effecting the optimal value when $d$ tends to infinity.

Note here that all the difficulty of the problem lies in the copositivity constraint (12). In contrast to this, the other constraint $\sum_{k=0}^{d} c_k s^k \leq -1$ for all $s \in [-1, 1/2]$ is computationally relatively easy. Although it gives infinitely many linear conditions on the coefficients $c_k$, it can be modeled equivalently as a semidefinite constraint using the sums of squares techniques for polynomial optimization; see for instance Parrilo [15] and Lasserre [13].

If, instead of requiring copositivity of the invariant kernel

$$(x, y) \mapsto \sum_{k=0}^{d} c_k (x^T y)^k$$

we impose the weaker constraint that this kernel should be positive semidefinite, then things become considerably simpler. Using harmonic analysis on the unit sphere, by Schoenberg’s theorem [19], one can identify this class of kernels explicitly, namely these are the kernels which can be written as

$$(x, y) \mapsto \sum_{k=0}^{d} g_k P_k^{((n-3)/2,(n-3)/2)}(x^T y) \text{ with } g_0, \ldots, g_d \geq 0,$$

where $P_k^{((n-3)/2,(n-3)/2)}$ is the Jacobi polynomial of degree $k$ with parameters $((n-3)/2,(n-3)/2)$. Thus requiring this weaker constraint instead of the copositivity constraints yields the linear programming bound for the kissing number due to Delsarte, Goethals, and Seidel [7].

This bound is known to be tight in a few cases only, namely for $n = 1, 2, 8, 24$. The kissing number is also known in dimensions $n = 3, 4$, where the last case $n = 4$ was settled by Musin [18]. As shown by Pfender [16, Lemma 5.2], Musin’s determination of the kissing number in dimension 4 relies on the identification of kernels lying in the copositive cone but not in the semidefinite cone.

5. Future work

In this paper we gave a copositive formulation of the stability number of infinite geometric packing graphs which satisfy condition (2). This condition guarantees in particular that all stable sets are finite. Sometimes one is also interested in stable sets which are infinite, measurable sets. For instance, what is the measurable stability number of the graph on the unit sphere where two vertices are adjacent whenever they are orthogonal? Semidefinite
relaxations for problems of this kind have been proposed by Bachoc, Nebe, Oliveira, and Vallentin [1]. However, the bound which one can obtain by this method is very weak for the orthogonality graph on the unit sphere. For this reason we think that it would be interesting to derive a copositive formulation for this problem to be able to derive stronger bounds.

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