Implicit Anyon or Single Particle Boson Mechanism of HTCS and Pseudogap Regime

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Abstract
We propose a single particle boson mechanism of High $T_c$ Superconductivity (HTCS) and pseudogap regime. Bosons appear in it due to the coupling of spins of the two-dimensional (2D) fermions with statistical magnetic field induced by anyon vector potential. The ground state of 2D gas is pure bosonic if gas is not dense. At the dense limit of gas the interaction of effective (coupled with the statistical magnetic field) spins of bosons leads to the increasing of their fluctuations, which destroy the coupling. An experimental phase diagram of the hole doped superconducting cuprates discussed in the paper of Tallon and Loram might qualitatively and quantitatively be clarified in the framework of this mechanism. The vicinity of the structural phase transition to superconducting state might strengthen the possible quadratic striction in the sample and the phase transition of bosons into Bose-Einstein condensate (BEC), which is responsible for the superconductivity (SC), is not second order, but first, close to second one. According this treatment the pseudogap regime is the region of meta stable bosons, which are out of the BEC. At the pseudogap boundary, $E_g$, the bosons finally undergo the phase transition into fermions. Non-Fermi liquid like property of quasi-particles discussed in the literature might be related to bosons with spins in the pseudogap regime.

1 Introduction
A mechanism of SC based on the BEC of bosons, irrespective of nature of particles, has been proposed by Ogg, Ref. [1], in 1946 and then by Schafroth together with collaborators, Refs. [2, 3], (see also the review of Ginzburg, Ref. [4] ) during the decade since 1950. It seems a single particle boson nature of SC is mysterious, because no reason for these particles to appear in the solid materials. Only astrophysical objects, like stars, allow [1] at the extremal physical conditions this
scenario of SC. It is believed [5] that, as metals at low temperatures, the HTCS materials are described by Cooper like pairs of fermions and the set of 2D superconducting planes of atoms CuO$_2$ plays the important role. For these 2D systems has been suggested the alternative semion (anyons with fractional parameter $\nu = 1/2$) picture of HTCS, which was latter not supported in the experiment (the references on the theoretical and experimental papers with respect to the subject see in Ref. [6]). The extended discussion of this treatment, where anyons are the quasi-particle excitations, is outlined in [7].

In general, the concept of anyons, as result of non-relativistic Chern-Simons Quantum Field theory [6], discovers the physical richness of two space dimensions. The topology of configuration space for orbital motion of particles in 2D allows for fractional exchange statistics [8], characterized by a continuous parameter $\nu$ that may attain values between 0 (for bosons) and 1 (for fermions). Particles with $0 < \nu < 1$ are generically called anyons [9].

Assuming anyons are spinless, it is believed [6] that particles in this concept can continuously transform from canonical bosons into canonical fermions [10]. This means, for instance, the symmetric wave function of boson ground state can continuously transform into antisymmetric one of fermion ground state. In the paper we follow this notion.

The interesting and not yet considered in the literature on anyons problem is the relation of fractional statistics and real spins of particles. From standard courses of Quantum Electrodynamics (see, for example, Ref. [11]) it is well known that particles with the integer number of $\hbar/2$ spins possess a Fermi statistics with Pauli exclusion principle for occupation of one quantum state by particle and antisymmetry constraint for the many-body wave function. In the same time, for 2D systems the concept of anyons (at $\nu = 1$) gives an opportunity to introduce this antisymmetry property of the wave function into Hamiltonian. In the paper we assume that anyons (like electrons or holes) have the spin $\hbar/2$.

The goal of the present paper is to outline the results of paper [12], where we have studied the simultaneous effect of spins and fractional statistics and found that the value $\hbar/2$ is crucial for 2D systems. We have introduced in the Hamiltonian of anyon gas the Zeeman term of the interaction of spins of particles with magnetic field induced by anyon vector potential, i.e. the statistical magnetic field [6, 13], and showed that the calculation of an expectation value for ground state energy exhibits the total cancellation of terms connected with statistics. As the cancellation occurs at any $\nu \neq 0$ this would mean the bosonization of anyons and, at particular case $\nu = 1$, of 2D fermions in the ground state. Expecting that this effect would be general for any 2D gas of fermions, we have applied it for the interpretation of phase diagram of HTSC and pseudogap regime suggested by Tallon and Loram in the review of experimental papers, Ref. [14].

Previously, we have derived an approximate analytic expression for the ground state energy of $N$ charged anyons confined in a 2D harmonic potential [15]. This was
achieved by using the bosonic representation of anyons and a gauge vector potential to account for the fractional statistics, which allowed working with the product ansatz for the $N$-body wave function. A variational principle has been applied by constructing this wave function from single-particle gaussians of variable shape. As in many other perturbative treatments of anyons in an oscillator potential (see references in Ref. [15]) our expression for the ground state energy had a logarithmic divergence connected with a cut-off parameter for the interparticle distance. Making use of the physical argument (see Ref. [6]) that for $\nu \neq 0$ this distance has to have some finite value, we have regularized the formula obtained for the ground state energy by an appropriate procedure that takes into account the numerical results for electrons in quantum dots in the case with Coulomb interaction.

In our treatment, Ref. [12], the normal and superconducting states were separated by gap – the difference of the ground state energies of fermions and bosons and thus, at the evaluation of phase diagram for BEC we have used the approximate analytic expressions for these energies obtained in our recent paper [16] for the 2D homogeneous Coulomb Fermi and Bose gases.

The paper is organized as follows. After brief discussion in the Section 2 of our approach and results [15] for the confined in harmonic potential anyons with Coulomb interaction, in the Section 3 we describe the harmonic potential regularization procedure [16] to get the thermodynamic limit and then obtain the approximate analytic expression for the ground state energy of the 2D homogeneous Coulomb anyon gas. The results of paper [12] about the implicit anyon or sigle particle boson mechanism of HTCS and pseudogap regime will be outlined in the Section 4 and we summarize and conclude the present paper by the Section 5.

2 The Coulomb interacting anyons in a 2D harmonic potential

The Hamiltonian of $N$ spinless anyons of mass $M$ and charge $e$ confined to a 2D harmonic potential, interacting through Coulomb repulsions, is given by

$$
\hat{H} = \frac{1}{2M} \sum_{k=1}^{N} \left[ (\vec{p}_{k} + \vec{A}_{\nu}(\vec{r}_{k}))^{2} + M^{2} \omega_{0}^{2} |\vec{r}_{k}|^{2} \right] + \frac{1}{2} \sum_{k,j \neq k}^{N} \frac{e^{2}}{|\vec{r}_{kj}|}.
$$

(1)

Here $\vec{r}_{k}$ and $\vec{p}_{k}$ represent the position and momentum operators of the $k$th anyon in two space dimensions,

$$
\vec{A}_{\nu}(\vec{r}_{k}) = \hbar \nu \sum_{j \neq k}^{N} \frac{\vec{e}_{z} \times \vec{r}_{kj}}{|\vec{r}_{kj}|^{2}}.
$$

(2)

is the anyon gauge vector potential [17] [18], $\vec{r}_{kj} = \vec{r}_{k} - \vec{r}_{j}$, and $\vec{e}_{z}$ is the unit vector normal to the 2D plane. The factor $\nu$ determines the fractional statistics of the anyon: it varies between $\nu = 0$ (bosons) and $\nu = 1$ (fermions).
We employ a variational scheme by minimizing the expression for the total energy

\[ E = \int \frac{\Psi_T^*(\vec{R}) \hat{H} \Psi_T(\vec{R})}{\Psi_T^*(\vec{R}) \Psi_T(\vec{R})} d\vec{R} \]  

with a trial wave function \( \Psi_T(\vec{R}) \) depending on the configuration \( \vec{R} = \{\vec{r}_1, ..., \vec{r}_N\} \) of the \( N \) anyons.

It is reasonable, in the bosonic representation of anyons when the many-body wave function takes the product form

\[ \Psi_T(\vec{R}) = \prod_{k=1}^{N} \psi_T(\vec{r}_k) \]  

(4)

to adopt the single-particle trial functions \( \psi_T(\vec{r}_k) \) in the form

\[ \psi_T(\vec{r}_k) = C \exp \left( - (\alpha' + \nu) \frac{(x_k^2 + y_k^2)}{2L^2} \right) \]  

(5)

Here \( C \) is a normalization constant and \( \alpha' \) a variational parameter. We identify \( L \) with the characteristic length \( (\hbar/M\omega_0)^{1/2} \) of the harmonic oscillator.

When energies are expressed in units of \( \hbar \omega_0 \) and lengths in units of \( L \) the normalized trial wave function reads

\[ \Psi_T(\vec{R}) = \left( \frac{\alpha}{\pi} \right)^{N/2} \prod_{k=1}^{N} \exp \left( - \alpha \frac{(x_k^2 + y_k^2)}{2} \right) \]  

(6)

where \( \alpha = \alpha' + \nu \).

In evaluating the expectation value \( E \) (Eq. 3) it is convenient to consider the local energy \( E_L(\vec{R}) = \Psi_T^{-1}(\vec{R}) \hat{H} \Psi_T(\vec{R}) \) [19]. In general \( E_L(\vec{R}) \) is a complex function

\[ E_L(\vec{R}) = ReE_L(\vec{R}) + iImE_L(\vec{R}) \]  

(7)

with

\[ ImE_L(\vec{R}) = -\alpha \sum_{k=1}^{N} \left( \langle \vec{A}_\nu(\vec{r}_k) \rangle + e\vec{A}_{ext}(\vec{r}_k)/c \cdot \vec{r}_k \right) \]  

(8)

However, evaluation of the expectation value \( E = \int \Psi_T(\vec{R}) E_L(\vec{R}) \Psi_T(\vec{R}) d\vec{R} \) immediately yields

\[ \int \Psi_T(\vec{R}) ImE_L(\vec{R}) \Psi_T(\vec{R}) d\vec{R} = 0 \]  

(9)

and, therefore, the only quantity to consider in the following is \( ReE_L(\vec{R}) \). Before proceeding, we would like to emphasize that the absolute ground state of the anyon system is a non-analytic function of \( \nu \). Our calculations will simply provide a smooth interpolation.
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In the non-interacting case the local energy is

\[ \text{Re} E_L(\vec{R}) = \sum_{k=1}^{N} \left[ \alpha + \frac{x_k^2 + y_k^2}{2}(1 - \alpha^2) + \frac{\nu^2}{2}(\vec{A}_\nu(\vec{r}_k))^2 \right]. \]  

(10)

The calculation [15] of the expectation value for \( \text{Re} E_L(\vec{R}) \) gives

\[ E = \frac{N}{2} \left( N\alpha + \frac{1}{\alpha} \right) \]  

(11)

with

\[ N = 1 + \nu^2(N - 1)[\ln \left( \frac{1}{2\delta} \right) - G(N - 2)] , \]  

(12)

and \( G = 3^{1/2} \ln(4/3) \), which attains a minimum \( \left( \frac{dE}{d\alpha} = 0 \right) \) for

\[ \alpha_0 = N^{-1/2} . \]  

(13)

Thus, the resulting expression for the ground state energy is

\[ E_0 = N \sqrt{N} . \]  

(14)

The logarithmic divergence displayed in \( E_0 \) when \( \delta \rightarrow 0 \) has also been found in other approximate perturbative treatments of the problem and is widely discussed in the literature (see, for example, Refs. [20, 21, 22]). Here we have assumed as in Ref. [6] that the cut-off parameter \( \delta \) cannot be zero for \( \nu > 0 \), away from the bosonic limit, since it corresponds to the square of the nearest distance between the particles (in Ref. [15] we have supposed that \( r_0 \approx L \), where \( r_0 \) is the mean distance between particles). Thus, for anyons in the parabolic confining potential \( \delta \) is definitely smaller than 1 (in units of \( L^2 \)).

Wu [17] has computed the ground state energy of \( N \) anyons in a 2D harmonic potential near the bosonic limit \( \nu \approx 0 \) and obtained

\[ E \approx [N + N(N - 1)\nu/2] . \]  

(15)

To regularize the expression for \( E_0 \) we have made use of this result by expanding \( E_0 \), Eq. (14), for \( \nu \rightarrow 0 \) and identify the leading term in \( \nu^2 \) with the term linear in \( \nu \) of Eq. (15), with the result

\[ \delta = \frac{1}{2} \exp \left[ -\frac{1 + \nu G(N - 2)}{\nu} \right] . \]  

(16)

With this value of the cut-off parameter the final analytic expression for the ground state energy is

\[ E_0 = N[1 + \nu(N - 1)]^{1/2} . \]  

(17)
This formula for large $N$ is consistent (up to a numerical factor) with the approximate expression $E \approx \nu^{1/2} N^{3/2}$ of Chitra and Sen [22] calculated perturbatively from the bosonic end for $\nu > 1/N$.

It should be noted, that due to this regularization procedure the ground state energy obtained, Eq. (17), is not an upper bound to the ground state as one would expect from a variational principle. This is a consequence of the fitting of $\delta$ to the result of Wu Ref. [17], which for $N = 2$ leads to a square root dependence in $\nu$, while the exact result for this case gives a linear dependence. On the other hand, Eq. (17) applies for the whole range of parameters of the system $N, \nu$, and $\omega_0$.

We now include the effect of the Coulomb repulsions between anyons

$$\frac{L}{2a_B} \sum_{k,j \neq k}^{N} \frac{1}{|\vec{r}_{kj}|}$$

(18)

in the expression for the real part of local energy $ReE_L(\vec{R})$, Eq. (10).

The Coulomb interaction part contributes with $N(N-1)$ integrals of the form

$$\int \frac{\Psi_T(\vec{R})}{|\vec{r}_{kj}|} \Psi_T(\vec{R}) \, d\vec{R}.$$These integrals have been evaluated in Ref. [15] and the averaged (real part of the) local energy is

$$E = \frac{N}{2} \left( N \alpha + \frac{1}{\alpha} + 2\mathcal{M} \alpha^{1/2} \right),$$

(19)

with

$$\mathcal{M} = \left( \frac{\pi}{2} \right)^{1/2} \frac{N - 1}{2} \frac{L}{a_B},$$

(20)

where $a_B$ is the Bohr radius. The extremum condition $\frac{dE}{d\alpha} = 0$ leads to the equation

$$X^4 - \mathcal{M}X - N = 0$$

(21)

for $X = 1/\alpha^{1/2}$. The minimum energy is given by the expression

$$E_0 = \frac{N}{2} \left[ \frac{N}{X_0^2} + X_0^2 + \frac{2\mathcal{M}}{X_0} \right]$$

(22)

and it is achieved at the point

$$X_0 = (A + B)^{1/2} + [-(A + B) + 2(A^2 - AB + B^2)^{1/2}]^{1/2},$$

(23)

where

$$A = \left[ \mathcal{M}^2/128 + ((N/12)^3 + (\mathcal{M}^2/128)^2)^{1/2} \right]^{1/3},$$

$$B = \left[ \mathcal{M}^2/128 - ((N/12)^3 + (\mathcal{M}^2/128)^2)^{1/2} \right]^{1/3}. $$

(24)
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Figure 1: Coulomb interaction parameter $L/a_B$ ($r_0 \approx L$, \cite{15}) dependence of the ground state energy for 7 – 10 electrons calculated by variational \cite{23} and fixed-node quantum Monte Carlo methods \cite{24} - dashed curves (results of both calculations are indistinguishable in these curves) and by formula (22) - solid curves.

Again, the ground state energy $E_0$, Eq. (22), has a logarithmic divergence in the limit $\delta \to 0$ and the quantity $N$ should be regularized.

In order to determine the cut-off parameter $\delta$, and due to the lack of analytic results, we had to fit to known numerical results for the ground state energy at special values of the parameter $L/a_B$.

In Figure 1 we have compared the ground state energies calculated for 7-10 electrons using Eq. (22), with the non-interacting $N = 1 + \nu(N - 1)$, to variational \cite{24} and fixed-node quantum Monte Carlo calculations \cite{24}.

3 Harmonic potential regularization

In the Section 2 we have outlined a variational procedure for the ground state energy of confined interacting anyons (starting from the bosonic end) and achieved, after regularization of a logarithmic expression by means of a cut-off parameter for the particle-particle interaction, approximate analytic formulas in terms of $N$, $\nu$ and $L/a_B$. For the non-interacting anyon system we found

$$E_0(N, \nu) = \hbar \omega_0 N^{1/2},$$

(25)
while for the interacting anyon system the approximate analytic ground state energy is given by

$$E_0(N, \nu) = \frac{\hbar \omega_0 N}{2} \left[ \frac{N}{X_0} + \frac{X_0^2}{X_0} + \frac{2M}{X_0} \right]$$

(26)

with the expression for $X_0$, Eq. (23). In these expressions we have used $N = 1 + \nu(N - 1)$ and Eq. (20) for $M$.

In order to obtain the corresponding expressions for the 2D homogeneous anyon gas, which have been derived in the paper [16] and will be briefly outlined in this section, we had to flatten out the parabolic confining potential while increasing the number $N$ of anyons, keeping the density $\rho = N/S = 1/\pi r_0^2$ constant, i.e. we performed the thermodynamic limit while making the confining potential disappearing. Here $\pi r_0^2$ is the area of the jellium disc carrying the positive counter charge and the mean particle distance $r_0 = a_B r_s$ can be expressed in units of the Bohr radius by the dimensionless density parameter $r_s$.

Without Coulomb interaction and in the case of fermions ($\nu = 1$) the ground state energy of the 2D homogeneous electron system of density $\rho$ is determined by the Pauli exclusion principle and is given by

$$E_0(\rho) = \pi \hbar^2 \rho N/M ,$$

(27)

while from (25) we have

$$E_0(N, \nu = 1) = \hbar \omega_0 N^{3/2} .$$

(28)

In the thermodynamic limit both expressions have to become identical and we obtain the relation

$$\omega_0(N) = \pi \hbar \rho/(MN^{1/2}) ,$$

(29)

which means that, in fact, the thermodynamic limit ($N \to \infty$) is obtained for vanishing parabolic confinement potential. We can extend this consideration for the fermionic limit, $\nu = 1$, to the general anyon case, $\nu \neq 1$, by assuming instead of (27) the relation

$$E_0(\rho, \nu) = \pi \hbar^2 \rho N \phi(\nu)/M ,$$

(30)

where the function $\phi(\nu)$ is still to be determined under the constraint $\phi(\nu = 1) = 1$. This form is in the accordance with the fact that close to the bosonic limit ($\nu \approx 0$) the ground state energy of the infinite anyon gas depends linearly on $\nu$ [25, 26, 27]. Consequently, we have for this case

$$\omega_0(N, \nu) = \pi \hbar \rho f(\nu)/(MN^{1/2})$$

(31)

with another unknown function $f(\nu)$ and the constraint $f(\nu = 1) = 1$. It turns out $\phi(\nu)$ is determined by $f(\nu)$.

In the thermodynamic limit and including the Coulomb interaction, the parabolic confinement has to be replaced by the jellium contribution, which for
a disc of radius $R_0$ (containing $N$ counter charges) gives a potential energy contribution \[ V(\vec{r}_k) = -\rho \int_S \frac{e^2}{|\vec{r}_k - \vec{r}|} \, d^2r. \] (32)

Here $S = \pi R_0^2$ is the area of the jellium disc for $N$ charges and we have $R_0 = N^{1/2} r_0$. For $\nu = 1$, there is a relation between characteristic length $L$ of the oscillator with the mean particle distance $L = N^{1/4} r_0$ and we have found $r_0 \ll L \ll R_0$ for $N \gg 1$.

In the general case of $\nu \neq 1$ and the Coulomb interaction, the approximate analytic expression for the ground state (30) can be written in the form

\[ E_0(\nu, r_s) = \pi \hbar^2 \rho N \phi(\nu, r_s)/M, \] (33)

thus, in a similar way to generalize $f(\nu)$ by $f(\nu, r_s)$.

Before deriving the approximate analytic expression for the ground state energy of Coulomb interacting anyon gas, in Ref. [16] we have obtained the formula for one of the 2D Coulomb Bose gas at high densities. Performing the calculation for 2D case, which is the analog of Foldy’s one, Ref. [29], for the 3D case, we have found

\[ E = -c_{BG} r_s^{-2/3}, \] (34)

where $c_{BG} = 1.29355$. Thus, we had an exact analytic expression for the ground state energy per particle for the 2D Coulomb Bose gas valid at high densities. It has next used together with the known expression for the 2D Coulomb gas in the low density limit (the 2D Wigner crystal) to derive a form for the unknown function $f(\nu, r_s)$ introduced above in this section.

In the paper Ref. [16] we have also obtained the spectrum of collective excitations of the 2D Coulomb Bose gas. It has the form

\[ E_k = [\hbar^2 2\pi e^2 \rho k/M + (\hbar^2 k^2/(2M))^2]^{1/2}, \] (35)

which for small $k$ is the 2D plasmon dispersion and approaches for large $k$ the free particle dispersion.

For the derivation of an approximate analytic expression for the ground state energy of the Coulomb anyon gas, in paper [16] we have at beginning calculated the contribution of the jellium term, Eq. (32), into expectation value for energy. We have obtained

\[ \rho \sum_{k=1}^{N} \int_S \int_{-\infty}^{\infty} \Psi_T(\vec{r}_k) \frac{e^2}{|\vec{r}_k - \vec{r}|} \Psi_T(\vec{r}_k) \, d^2r = \frac{e^2}{L} N^{1/2} \alpha^{1/2}. \] (36)

It was achieved at the condition $\alpha R_0^2 \ll 1$. A validity of this condition had been shown in [16] for almost all numerical values of $\nu$ and $r_s$. 
To bring the contribution of the jellium term in our approximation together with the Coulomb interaction evaluated in [15] and outlined in the previous section we might write the ground state energy as in Eq. (26) but with \( \bar{\omega}_0 = \hbar^2/(ML^2) \) and
\[
\mathcal{M} = \frac{\pi^{1/2}NL}{a_B} \left( \frac{1}{2^{3/2}} - 1 \right). \tag{37}
\]
However, this expression for \( \mathcal{M} \) we have replaced (the reason for that see [16]) by \( \mathcal{M} = -c_{WC} N^{3/4}r_s \) where the constant \( c_{WC} \) has been fixed by the ground state energy of the Wigner crystal (limit \( r_s \gg 1 \)).

For sufficiently large \( N \), we used \( M = -c_{WC} N^{3/4}r_s \) and \( N = \nu N \) to write the expression for the ground state energy, Eq. (26), with \( X_0 = N^{1/4}X \), where
\[
K_X = (K_A + K_B)^{1/2} + \left( -(K_A + K_B) + 2(K_A^2 - K_A K_B + K_B^2)^{1/2} \right)^{1/2}, \tag{38}
\]
and
\[
K_A = \left[ K^2/128 + \left( (\nu/12)^3 + (K^2/128)^2 \right)^{1/2} \right]^{1/3}, \tag{39}
K_B = \left[ K^2/128 - \left( (\nu/12)^3 + (K^2/128)^2 \right)^{1/2} \right]^{1/3},
\]
with \( K = c_{WC}r_s \). It took the form (in Ry units)
\[
E_0(\nu,r_s) = \frac{2}{r_s^2} \left[ \frac{\nu}{2K_X^2} + \frac{K_X^2}{2} - \frac{K}{K_X} \right]. \tag{40}
\]
If we remember that we have introduced a function \( f(\nu,r_s) \) in \( L \), therefore, now \( K = c_{WC}r_s/f^{1/2}(\nu,r_s) \) and the final expression for the ground state energy per particle is
\[
E_0(\nu,r_s) = \frac{2f(\nu,r_s)}{r_s^2} \left[ \frac{\nu}{2K_X^2} + \frac{K_X^2}{2} - \frac{K}{K_X} \right]. \tag{41}
\]
We have determined the function \( f(\nu,r_s) \) by fitting Eq. (41) to the known asymptotic limits for the ground state energy of spin polarized electrons and Coulomb 2D Bose gases at very small and very big values of \( r_s \). At very big \( r_s \) the ground state energy does not depend on statistics and equals the energy of the classical 2D Wigner crystal [30], \( E_{WC} = -2.2122/r_s \). For the Bose gas at small \( r_s \) we used Eq. (34).

For the bosons at \( \nu = 0 \) we had from Eq. (41)
\[
E_0(0,r_s) = -\frac{2^{3/2}c_{WC}s^{2/3}(0,r_s)}{r_s^{4/3}} \tag{42}
\]
with
\[
f(0,r_s) \approx \frac{c_{BG}s/r_s}{c_{WC}} \left( \frac{c_{BG}s/r_s}{c_{WC}} \right)^{3/2}, \tag{43}
\]
where $c_{WC}^{2/3} = 2.2122$ or $c_{WC} = 3.2903$.

The asymptotics of the ground state energy, Eq. (41), for $\nu \neq 0$ and at big $r_s$ has the form

$$\mathcal{E}_0(\nu, r_s \to \infty) = \frac{\frac{2}{3} c_{WC}^2 f^{2/3} (\nu, r_s)}{r_s^{4/3}} \left( -1 + \frac{7\nu f^{2/3} (\nu, r_s)}{3 c_{WC}^{4/3} r_s^{4/3}} \right).$$

(44)

The function $f(\nu, r_s)$ has to fulfil the constraints: $f(1, r_s = 0) = 1$ for the dense ideal Fermi gas; $f(\nu, r_s = 0) = \nu^{1/2}$ for the ideal anyon gas close to the bosonic limit (see [25, 26, 27]), and $f(0, r_s)$ given by Eq. (43) for the 2D Coulomb Bose gas. The form

$$f(\nu, r_s) \approx \nu^{1/2} (1 - r_s) e^{-r_s} \frac{c_{BG}^{3/2} r_s^{1/2}}{1 + c_{BG}^{3/2} r_s^{1/2}} ,$$

(45)

is consistent with these requirements.

Using this form of $f(\nu, r_s)$ in Eq. (41), we have calculated the ground state energy per particle in the boson ($\nu = 0$) and fermion ($\nu = 1$) limits and compared the results for the latter case with the data obtained by Tanatar and Ceperley [31] for the spin polarized electron system.

If we look at the results for large $r_s$ in Figure 2 with ground state energies calculated from Eqs. (41) and (44) (the results coincide on the plotted scale, solid line) and from Eq. (42) (dash-dotted line) in comparison with the data of [31], we see the results from our approximate analytic formula are very close to the "exact" data of [31] for the spin-polarized 2D electron system obtained numerically.

4 Implicit anyon or single particle boson mechanism of HTCS and pseudogap regime

We outline in this section results of paper [12], where it has been shown that inside of the ground state the interaction of spins of anyons with statistical magnetic field can induce the bosonization of these particles, i.e. the transformation of anyons (as also 2D fermions) into bosons.

In the paper [12] we have considered the gas of $N$ spinless anyons, which was described by the Hamiltonian, Eq. (1), with positive background jellium term, Eq. (32), included inside of the first sum.

We have introduced in the Hamiltonian the term

$$\frac{\hbar}{M} \sum_{k=1}^{N} \vec{s} \cdot \vec{b}_k$$

(46)

with statistical magnetic field [6, 13]

$$\vec{b}_k = -2\pi \hbar \nu \vec{e}_z \sum_{j(k \neq j)} \delta^{(2)}(\vec{r}_k - \vec{r}_j) ,$$

(47)
which can be derived if calculates \( \vec{b}_k = \vec{\nabla} \times \vec{A}_\nu(r_k) \) by using Eq. (2). The sign in Eq. (16) is taken for electrons with charge \( e = -|e| \). It is chosen to minimize the energy [32]. For holes with charge \( e = |e| \), we need to change the sign for \( \nu \) in Eqs. (2) and (17), then Eq. (16) and the expectation value for energy (see below Eq. (50)) retain the sign.

If we take \( s_z = \hbar/2 \) and take into account that length unit is \( L \), so \( \delta^{(2)}(\vec{r}) \) should be replaced by \( \delta^{(2)}(\vec{r})/L^2 \), then

\[
\frac{\hbar}{M} \sum_{k=1}^{N} \hat{s}_z \cdot \vec{b}_k = -\pi \nu \frac{\hbar^2}{ML^2} \sum_{k,j(k\neq j)} \delta^{(2)}(\vec{r}_k - \vec{r}_j). \tag{48}
\]

The calculation of the expectation value, Eq. (3), with the Hamiltonian, Eq. (48), and wave function \( \Psi_T(\vec{R}) \), Eq. (6), gives (in \( \hbar^2/(ML^2) \) units)

\[
-\pi \nu \sum_{k,j (k\neq j)} \int \Psi_T(\vec{R}) \delta^{(2)}(\vec{r}_k - \vec{r}_j) \Psi_T(\vec{R}) d\vec{R} = -\nu \alpha N(N - 1) / 2. \tag{49}
\]

The total expectation value for the ground state energy including all terms of the Hamiltonian is

\[
E = \frac{NN\alpha}{2} + \frac{N}{2\alpha} + \frac{N\alpha^{1/2}}{N} - \frac{\nu \alpha N(N - 1)}{2}, \tag{50}
\]
where the quantity $\mathcal{M}$ contains the Coulomb interaction and jellium background term (see previous section).

As in previous sections we have used the expression $\mathcal{N} = 1 + \nu(N - 1)$ for $\mathcal{N}$, which is result of the cut-off parameter regularization [15] of the logarithmic divergence, revealing in the bosonic representation, when two particles come up to each other. Substituting this expression for $\mathcal{N}$ into expectation value for energy $E$, Eq. (50), we see the exact cancellation of terms containing the $\nu$ factor. This can be also achieved putting in the formula for ground state energy $\nu = 0$, i.e. the case of bosons. As the energy of bosons is lower than one for fermions or anyons with $\nu \neq 0$, there appears a coupling of spin with statistical magnetic field for every particle or bosonization of 2D fermions and anyons. Of course, this effect occurs when the particles have the spin $\hbar/2$.

As next step of this coupling scenario might be the fluctuations of spins of bosons around of statistical magnetic field. Hence, the bosons might reveal an effective spin and look like as Fermi particles. However, Fermi gases with different spins are independent [33]. Thus, we have presumed that in the zero order approximation the spins of bosons interact only with each other and do not interact with spins of another fermions if they exist in the system. Interaction of spins might take place inside of some length- spin correlation radius, which for temperature $T = 0$ we denote as $\xi_0$. The final feature of this model might be a destruction of spin and magnetic field coupling at the increasing of spin fluctuations when bosons become the fermions. This would occur when the gain in the energy of bosons due to fluctuations is equal to energy difference between the Fermi and Bose ground states.

The interaction term of boson spins in the Hamiltonian we brought in the form

$$e^{-r_0/\xi_0} \sum_{k=1}^{N} \hat{s}_{k+\delta} \cdot \hat{s}_k.$$  \hspace{1cm} (51)

Here $r_0$ is the mean distance between particles. We introduced in this expression a factor $e^{-r_0/\xi_0}$, which takes into account the exchange character of interaction of spins [34]. However, if the typical scale of $\xi_0$ is the nearest interatomic distance, for our screened by magnetic field spins, $\xi_0$ is to be assumed phenomenological and taken from experiment.

For the $T = 0$ case, a following argument allowed us to establish the explicit form of Eq. (51). The boson ground state energy was obtained when the term, Eq. (16), was included in the Hamiltonian. To get again the fermion ground state energy, we need to cancel it. Therefore, for dense ($r_0 < \xi_0$) boson gas, there should be $\hat{s}_{k+\delta} = -\hbar \hat{b}_k/M$. Comparing with Eq. (47) we see that the summation index $\delta$ includes all $N - 1$ numbers of index $j$.

The expression for the Hamiltonian of bosonized infinite Coulomb anyon gas
with interaction of spins is

\[
\hat{H} = \frac{1}{2M} \sum_{k=1}^{N} \left[ \left( \vec{p}_k + \vec{A}_\nu(\vec{r}_k) \right)^2 + MV(\vec{r}_k) \right] + \frac{1}{2} \sum_{k,j \neq k}^{N} \frac{e^2}{|\vec{r}_{kj}|} + \frac{\hbar(1 - e^{-r_s/\xi_0})}{M} \sum_{k=1}^{N} \vec{\hat{s}} \cdot \vec{b}_k .
\]

(52)

We can bring here the results of Ref. [16] and previous section. The ground state energy of the Coulomb anyon gas at \( r_s > 2 \) with the high accuracy (see Figure 2) is described by formula, Eq. (44), with the expression for function \( f(\nu, r_s) \), Eq. (45).

For the HTCS, actual for us is the region \( r_s > 9 \) (see below Figure 4), where one can put \( f(\nu, r_s) \approx f(\nu = 0, r_s) \). The ground state energy for the 2D Bose and Fermi gases is obtained putting in Eq. (44) \( \nu = 0 \) and \( \nu = 1 \), respectively.

The analogical to Ref. [16] and Section 3 calculation, with the Hamiltonian, Eq. (52), gave for \( r_s > 2 \) the same expression, Eq. (44), for ground state energy of bosonized anyons, but in it one needs to replace \( \nu \) by \( \nu e^{-r_s/\xi_0} \) (now \( \xi_0 \) is expressed in \( a_B \) units). Considering below the bosonization of 2D fermions we have putted now \( \nu = 1 \).

To become the pure fermions bosons had to overcome the energy difference

\[
\Delta^B_0 = \frac{7(1 - e^{-r_s/\xi_0})f^{4/3}(0, r_s)}{3c^{2/3}W_c r_s^{8/3}} .
\]

(53)

One needs a special remark here. Our previous calculations have been related to ground state of spinless or fully spin polarized fermions (electrons). It would be preferable to deal with one of normal, i.e. with no spin polarized, electron liquid state. However, the difference of their ground state energy is essentially lower [31] than accuracy of our calculations.

We have applied the model for possible clarification of summarized experimental phase diagram of hole doped High-\( T_c \) superconductors, which was recently proposed by Tallon and Loram [14]. As it was said above it is believed that the main contribution into HTCS is provided by \( ab \) planes of set of \( CuO_2 \) atoms and therefore, 2D is responsible for it. The SC state occurs when holes transit into BEC. We note that for the ideal 2D systems BEC can exist only at \( T = 0 \) [35]. At \( T \neq 0 \) the evolution of fluctuations for the order parameter destroys BEC. However, if 2D gas is situated inside of third dimension this evolution is suppressed (with respect to SC see book of Abrikosov [36]). Furthermore, we used in the Hamiltonian, Eq. (52), a three dimensional Coulomb potential. Therefore, for HTCS the 2D BEC of bosons can also exist at not zero temperature.

We used \( \Delta^B_0 \) as gap for SC and studied its asymptotics as function of density of holes \( n_s = 1/(\pi r_s^2) \). Below we show that \( n_s \sim p \), where \( p \), called in the experiment as concentration of holes, is the fractional part of hole per atom \( Cu \). At big values of \( r_s \) or small \( n_s \) one can neglect the exponential factor in Eq. (53) and \( \Delta^B_0 \sim n_s \sim p \).

At small \( r_s \) or big \( n_s \), without this factor, \( \Delta^B_0 \) would have \( \Delta^B_0 \sim n_s^{2/3} \sim p^{2/3} \), but \( e^{-r_s/\xi_0} \) suppresses this dependence to zero and the law of it depends from function
\(\xi_0(p)\). At this limit of \(r_s\) we presumed that the function \(\Delta_0^B(p)\) coincides with experimental dependence \(E_g(p)\). Extrapolating this asymptotics of \(\Delta_0^B(p)\) to small values of \(p\) and equating it to \(E_g(p)\) one finds the empirical dependence \(\xi_0(p)\). For it \(\xi_0(p) \sim 1/E_g(p)\).

The experimental values for energy are expressed in the Kelvin temperature, \(K\), units. To be sure in the correctness of our 2\(D\) density of holes, we expressed it in \(cm^{-2}\) units and compared with experiment. For elementary structural cell of almost all cuprate \(ab\) planes \(a = 3.81\AA\) and \(b = 3.89\AA\). Assuming that it has one atom of \(Cu\) with \(p\) fraction of hole, the density is \(n_{ab} = N_{ab} \cdot p \ cm^{-2}\), where \(N_{ab}\) is number of elementary cells per \(1 cm^2\) square. We have \(n_{ab} = 6.7472 \cdot p \cdot 10^{14} \ cm^{-2}\), from where \(n_s \sim p\). At the optimal doping of holes, \(p \approx 0.16\), we compared \(n_{ab}\) with experimental one of [87]. In it for Y-123 compound the square of three dimensional plasma frequency, being expressed in square of wave number units, for SC carriers has a value \(1.1 \cdot 10^8 \ cm^{-2}\). We find the density of holes per \(1 cm^3\) and then, assuming that for Y-123 compound there are two \(CuO_2\) planes [5] per elementary cell, responsible for SC, and in the \(c\) axis the spacing is \(c = 11.7\AA\), transform it into \(2D\) \(n_{ab}^{exp} = 0.913710^{14} \ cm^{-2}\). Our optimal doping value for \(n_{ab}\) is \(1.0795 \cdot 10^{14} \ cm^{-2}\). Therefore, typical density for SC carriers for the optimal doping state is \(\sim 10^{14} \ cm^{-2}\). From the experiment [87] also leads important information about approximate equality of SC carriers density to whole one. Thus, we can bring \(n_{ab}\) as SC density. A comparison of this value for \(n_{ab}\) with \(n_{FQHE} \sim 10^{11} \cdot 10^{12} \ cm^{-2}\) for Fractional Quantum Hall Effect (FQHE) experiment [38] shows importance of the pure, not screened, Coulomb potential interaction, which is essence of FQHE [28], for HTCS. The screened Coulomb potential is supposed to be as justification [39] for HTCS.

On the Figure 3 we presented the summarized curve for pseudogap boundary energy \(E_g\) (Fig. 11 from paper [14]), as result of interpolation of sets of experimental data, SC gap energy \(\Delta_0 = 4K_B T_c\), which was evaluated by empirical formula \(T_c = T_{c, max}[1 - 82.6(p - 0.16)^2]\) with \(T_{c, max} = 95 K\) for \(Bi\ 2212\) compound [40], and calculated by formula Eq. (53) SC gap energy for bosons \(\Delta_0^B\) as function of concentration of holes \(p\). Despite a \(\Delta_0^B(p)\) dependence has the qualitatively different form than experimental one, it is in accordance with the conclusion of Tallon and Loram that \(E_g\), being going to zero at the critical doping concentrate \(p_c \approx 0.19\), separates BEC into two parts, where in first one the density of SC providing pairs is small (even though that conclusion is not expressed explicitly by Tallon and Loram, however, Fig. 10 of their paper more than obviously displays it). As we see from the figure the magnitudes of this and experimental diagrams have the same order. We would like to note that, apparently, it is the first estimate for BEC energy of canonical bosons, which can be appropriate for experiments of SC.

Figure 4 demonstrates the \(p\) dependencies of \(r_s\) and \(\xi_0\). The spin correlation radius \(\xi_0\) becomes to be sharply increased when \(p\) approaches \(p_c\), which might mean the vicinity of phase transition, where many and many spins of bosons are involved.
in correlation process before the transforming of them into fermions. We established a correspondence with second important conclusion of Tallon and Loram that "the pseudogap is intimately connected with (though not equivalent to) short-range Anti Ferromagnetic (AF) correlations, which disappear at the same critical doping state" \(p_c\), and results shown in Figure 4. The paper \[14\] shows that experimental short-range AF correlations scale like \(E_g(p)\) dependence and vanish at \(p_c\) (see Fig. 6 of this paper). It was also suggested in Ref. \[14\] that two these quantities are the temperature independent. Our treatment is not for the temperature \(T \neq 0\).

However, last remark allow us to do the connection between short-range AF correlations and correlation radius \(\xi_0\). The presumption about correlations of spins of bosons would mean that there would exist the competing of these correlations with AF correlations, where nature of last ones is the spin interaction of closely situated fermions. Increasing of correlation radius \(\xi_0\) would lead to revealing the Fermi like spin correlations of bosons in the extended region of sample and thus, assuming that in the first order approximation these spins now interact with ones of fermions, to suppressing of short-range AF correlations inside of this region. From this a disappearing of AF correlations at \(p_c\), where \(\xi_0\) goes to infinity, would be natural and might be considered as indirect indication of destruction of BEC from canonical bosons. Mathematically the short-range AF correlations will be proportional to \(1/\xi_0(p)\), because \(E_g(p) \sim 1/\xi_0(p)\). One might suppose that the interaction of spins of fermions with statistical magnetic field (as well as the statistical magnetic field itself) of anyons would have a hidden character and might not be revealed obviously in the experiment. In this case, the described experimental behaviour of short-range AF correlations might be also the implicit indication of existing of the statistical magnetic field.

Next conclusion of Tallon and Loram paper \[14\] is an independence of pseudogap boundary energy \(E_g\) and phase diagram for BEC. As we said above, in our treatment, without \(E_g\), \(\Delta^B_0\) would have a proportionality \(\Delta^B_0 \sim p^{2/3}\). The phenomenological including of \(E_g\) determines the maximum \(T_c\) value for BEC phase and critical concentration \(p_c\). Close to \(p_c\) \(E_g(p)\) merges with \(\Delta^B_0(p)\) determining the asymptotics for latter one. In our model \(E_g\) characterizes the cuprate material. As for almost all High-\(T_c\) superconducting samples the numerical values for cell spacing constants \(a, b\) are the same \[5\], according to our approach we should been obtain the similar \(T_{c,max}\) for all of them. The value for \(T_{c,max}\) depends also from number of \(CuO_2\) planes in the elementary cell \[5\]. We have considered the sample with one plane in the hoping that \(E_g(p)\) takes effectively into account this factor.

Here we would like to say some words about the possible scenario for the phase diagram of HTCS, which might be derived from this model. Due to vicinity of the structural phase transition to superconducting state in the cuprate materials at the concentration of holes below optimal \[5\], we might do a hypothesis that the induced by it the mechanical strain would strengthen a quadratic striction and therefore, the phase transition into BEC would be not a second order, as it should be, but
first, close to second one \cite{41}. For this case a pseudogap regime would find natural
interpretation. It would correspond to meta stable phase of bosons. At pseudogap
boundary, $E_g$, bosons would finally undergo a phase transition into fermions. The
critical concentration, $p_c$, at which the pseudogap energy, $E_g$, is zero, would be
considered as critical point of first order phase transition. The first order phase
transition scenario is possibly consistent with recent experimental observations of
coeexistence of pseudogap and superconducting phases described in the review of
Pines \cite{42}, because typically the coexistence of equilibrium phases exists in this
order of phase transitions \cite{43}. In our case, these phases would be the small islands
of meta stable phase of bosons and BEC situated in close area to BEC boundary
in the phase diagram. For electron doped materials one may suppose that there
is no structural phase transition. Therefore, the phase transition into BEC would
have a pure second order, which is possibly seen in the experiments. At last, the
hypothesis might clarify the possible independence of pseduo gap energy, $E_g$, and
BEC suggested in the paper \cite{14}. As it was said above the boundary, $E_g$, would be
connected with mechanical strain property of cuprate materials, while condensation
into BEC with coupling of spins of fermions with statistical magnetic field. However,
the spin correlations of bosons, which are to be governed by $E_g$ and suppress the
BEC, might implicitly be relate two these quantities.

Now about the spectrum of collective excitations, Eq. (35). This spectrum
has no gap at small wave vectors $k$ and therefore, can not be responsible for SC.
SC is provided by particles in the BEC. On the other hand, one might expect that the
bosons with effective spins inside of correlation radius $\xi_0(p)$ and obeying the
relation, Eq. (35), at big wave vectors $k$ are not the Fermi liquid quasiparticles
like.

We considered the effect of bosonization of 2D fermions for the instance of HTCS.
One might presume that it would be important for any 2D fermion gas. FQHE, gas
of isotopes $He$ 3 and many other objects might be affected by this effect.

5 Summary

We have introduced in the Hamiltonian of anyon gas the Zeeman term of the inter-
action of spins $s_z = \hbar/2$ of particles with magnetic field induced by anyon vector
potential, i.e. statistical magnetic field. A calculation of the expectation value for
ground state energy in the framework of variational approach with cut-off para-
meter regularization has exhibited the cancellation of terms connected with fractional
statistics, which might be mean the bosonization of anyons due to a coupling of
their spins with statistical magnetic field. For fermions, as the particular case of
anyons, we have applied it for the possible clarification of summarized phase dia-
gram of HTCS suggested by Tallon and Loram for the region below pseudogap $E_g$.
Additionally to Zeeman term we have phenomenologically introduced in Hamilto-
nian the term, which was responsible for the correlations of effective spins of bosons,
Figure 3: The experimental pseudogap energy $E_g$, SC gap energy $\Delta_0 = 4K_B T_c$ (experiment for $Bi\ 2212$ compound), and calculated by formula Eq. (53) SC gap energy for bosons $\Delta_0^B$ in Kelvin temperature (K) units as function of concentration of holes $p$. 
Figure 4: The mean distance between holes $r_s$ and spin correlation radius $\xi_0$ in Bohr radius $a_B$ units as function of concentration of holes $p$. 
and connected them with $E_g$. The obtained phase diagram has been quantitatively close to experimental one, while model qualitatively described conclusions made in paper of Tallon and Loram.

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