On Physical States in $c < 1$ String Theory

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Abstract: The BRST cohomology analysis of Lian and Zuckerman leads to physical states at all ghost number for $c < 1$ matter coupled to Liouville gravity. We show how these states are related to states at ghost numbers zero (pure vertex operator states – DK states) and ghost number one (ring elements) by means of descent equations. These descent equations follow from the double cohomology of the String BRST and Felder BRST operators. We briefly discuss how the ring elements allow one to determine all correlation functions on the sphere.

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1. INTRODUCTION

Non-critical string theory received a boost when the discretised model – the matrix models were exactly solved to all orders in string perturbation theory. The continuum limit of the matrix models – Liouville theory has not been understood to the same extent as matrix models. Early progress was made in calculating the torus partition function\cite{1,2} followed by the calculation of three point functions on the sphere\cite{3,4,5,6}. The BRST analysis of Lian and Zuckerman\cite{7} and subsequently by BMP\cite{8} has shown that physical states for $c < 1$ string occur at all ghost numbers. This is unlike in the critical bosonic string where it is restricted to three ghost numbers. In this talk, I shall describe how the physical states in $c < 1$ string theory can be represented by states at three ghost numbers just as in the critical bosonic string. We shall work in the free field formulation of the minimal models. The truncation of the Hilbert space of a single scalar field to that of the finite space of primaries of the minimal models is easily seen due to the existence of a BRST operator – the Felder BRST\cite{9}. On coupling to gravity, one also has to deal with the usual String BRST operator. The double cohomology of the the two BRST operators provides descent equations which relate the LZ states of arbitrary ghost number to those of ghost numbers similar to those occuring in the usual critical bosonic string. In this talk, I will be discussing work appearing in \cite{10,11}. I shall also describe new results which have enabled computation of all correlation functions on the sphere\cite{12}. These results agree with the results obtained in matrix models.

2. MINIMAL MODELS

It is well known that a single scalar field with a background charge provides a Fock space realisation of all minimal models. We shall briefly explain how this works(see Dotsenko-Fateev\cite{13} and Felder\cite{9} for more details). Consider a scalar field with the following energy-momentum tensor

\[ T_M = -\frac{1}{4} \partial X \partial X + i \alpha_0 \partial^2 X , \tag{2.1} \]

where $2\alpha_0 = \frac{p'}{\sqrt{pp'}}$. The pure vertex operators of the form $V_\alpha = e^{i\alpha X}$ provide a set of (Virasoro) primary fields of conformal dimension $\Delta_\alpha = \alpha (\alpha - 2\alpha_0)$. In this set, there are two operators of dimension 1 which correspond to the values of $\alpha$ given by $\alpha_+ \equiv \frac{p'}{\sqrt{pp'}}$ and $\alpha_- \equiv \frac{p}{\sqrt{pp'}}$. Consider the set of primary fields labelled by $\alpha_{m',m}$ where $\alpha_{m',m} \equiv$
\[ \frac{(1-m')}{2} \alpha_- + \frac{(1-m)}{2} \alpha_+ \] with \((m', m)\) restricted to the \textit{conformal grid} i.e., \(0 < m' < (p' - 1)\) and \(0 < m < (p - 1)\). It can be seen that the dimensions of these operators agree with the Kac dimension formula for the primary fields of the \((p', p)\) minimal model \(^a\). However, there is a doubling – this corresponds to the fact that given the dimension of the primary field, there are two values of \(\alpha\) which give the same dimension. This is referred to as \textit{minimal model duality}. Hence, modulo this doubling one can obtain all the states of the minimal models in this system. Dotsenko and Fateev have shown that in order to reproduce the minimal model correlation functions with these operators one has to introduce screening operators into the correlation functions to saturate the background charge. The screening operators are given by \(Q_{\pm} \sim \int V_{\alpha_{\pm}}\). The screening operators are of conformal dimension 0 and hence their introduction into correlation functions does not alter the conformal properties of the correlator. On the sphere, the charge conservation equation takes the form

\[ \sum_i \alpha_i + r \alpha_+ + r' \alpha_- = 2\alpha_0 \] (2.2)

for a correlation function involving \(V_{\alpha_i}\)'s. We also have introduced \(r Q_+\) and \(r' Q_-\) to obtain charge conservation.

Using the screening operator to obtain a BRST operator, Felder\(^9\) has shown how the Hilbert space of a scalar field is truncated to the finite set of primaries of the minimal models and their secondaries. The BRST operator also explained the decoupling of (Virasoro) null states in correlation functions. All the primaries of the minimal model correspond to states in the cohomology of this BRST operator (We shall henceforth refer to this operator generically as \(Q_F\)). Further, all null states are either \(Q_F\)-exact or not physical (i.e., not \(Q_F\)-closed).

Given a Virasoro secondary over a primary field, one can map it onto a state in the Fock space using equation (2.1). This is done after expanding the stress-tensor into modes. In general, this map is non-vanishing. However, for some of the Virasoro nulls, this map vanishes. We shall now illustrate this. Consider the identity operator which is a conformal field of dimension 0. Let us represent it by \(|\Delta = 0\rangle\). It has a level one null \(-L_{-1}|\Delta = 0\rangle\). In the Fock space, due to minimal model duality, the identity is represented by two different

\(^a\) Note that the operators \(V_\alpha\) and \(V_{(2\alpha_0 - \alpha)}\) have the same conformal dimension.
vertex operators: $V_{\alpha=0}$ and $V_{\alpha=2\alpha_0}$. On mapping the level one null to the Fock space, we have for 
\[ L_{-1} |\Delta = 0 \rangle \mapsto \begin{cases} \partial (e^{2i\alpha_0 X}) = 2i\alpha_0 \partial X \ e^{2i\alpha_0 X} \text{ for } \alpha = 2\alpha_0 \\ \partial 1 = 0 \text{ for } \alpha = 0 \end{cases} \tag{2.3} \]

The vanishing of the null is clearly seen for $\alpha = 0$. Since the number of states over a primary at the given level are the same in the Virasoro module as well as the Fock space (given by the number of partitions of the level), there is an oscillator state over the primary at the same level which is not mapped onto from the Virasoro module. So every vanishing null is replaced by an oscillator state which we shall refer to as $|w\rangle$ or just ‘$w$’ (= $-\partial X$ in the example just considered). The non-vanishing null state will be referred to as $|u\rangle$ or just ‘$u$’ (= $2i\alpha_0 \partial X e^{2i\alpha_0 X}$ in the example just considered). The $u$’s and $w$’s will be useful in understanding minimal models after coupling to gravity. On taking the dual in momenta, the nulls exchange their roles i.e., $u \leftrightarrow w$.

3. MINIMAL MODELS COUPLED TO GRAVITY

We shall work in the conformal gauge and treat the Liouville mode $\phi$ as a free field with (imaginary) background charge. The stress-tensor for the Liouville field is
\[ T_L = -\frac{1}{4} \partial \phi \partial \phi + i\beta_0 \partial^2 \phi \quad , \tag{3.1} \]
where $c_L = 1 - 24\beta_0^2$ and $c_M + c_L = 26$. We obtain that for the $(p', p)$ model, $2\beta_0 = \frac{i(p+p')}{\sqrt{pp'}}$. The vertex operator $e^{i\beta \phi}$ has conformal weight $\beta (\beta - 2\beta_0)$. Just as in the case of the critical bosonic string, in the conformal gauge, physical states are in the cohomology of the string BRST. It is given by
\[ Q_B = \oint : c(z)(T_M(z) + T_L(z) + \frac{1}{2} T_{gh}(z)) : \quad , \tag{3.2} \]
where $T_{gh}$ is the stress-tensor for the ghosts. The tensor product of Fock spaces $\mathcal{F}(\alpha) \otimes \mathcal{F}(\beta) \otimes \mathcal{F}(gh)$ provides the space on which $Q_B$ acts. Unlike, in the case of the critical bosonic string one has to deal with the cohomology of two BRST operators $- Q_B$ and $Q_F^b$.

\[ b \quad \text{Please refer to [14] for an introduction to } c < 1 \text{ non-critical strings.} \]
The physical states of $c < 1$ matter coupled to gravity has been studied in [7] and [8]. They have shown that there exist an infinite number of BRST invariant states in $c < 1$ theories coupled to gravity. The Liouville momenta of these states are such as to provide the gravitational dressing of the matter null states of the minimal model. Further, there is one state at ghost number $\pm n$ for every matter Virasoro representation whose Liouville momenta are $\beta > \beta_0$ for ghost number $+n$ states and $\beta < \beta_0$ for ghost number $-n$ states. These states of non-trivial ghost number will be called the LZ states.

### 3.1. Descent Equations

Due to the existence of two BRST operators, there exist descent equations which begin at LZ states [10]. It was also shown in [10] that the descent equations relate LZ states with pure vertex operators with matter momenta outside the conformal grid. The simplest LZ state of ghost number $-1$ illustrates descent equations.

$$Q_B|LZ\rangle^{-1} = Q_F|DK\rangle^0 = |u_{m',m}\rangle_M \otimes |\beta\rangle \otimes c_1|0\rangle_{gh}$$

where $|u_{m',m}\rangle_M$ is the ‘$u$'(non-vanishing null) over the Fock primary labelled by $(m', m)$. This generalises to

$$Q_B|LZ\rangle^{-n} = Q_F|I_1\rangle^{-n+1}$$

$$Q_B|I_1\rangle^{-n+1} = -Q_F|I_2\rangle$$

$$\vdots$$

$$Q_B|I_{n-1}\rangle^{-1} = (-)^{n+1}Q_F|DK\rangle^0$$

where $|DK\rangle^0 = |\alpha_{m',m}\rangle \otimes |\beta\rangle \otimes c_1|0\rangle_{gh}$ and the ghost-numbers are given by the superscript. The matter labels $(m', m)$ take values from outside the conformal grid with suitable gravitational dressing($\beta < \beta_0$). These descent equations follow from $(|LZ\rangle + |I_1\rangle + \ldots + |DK\rangle)$ being closed under $(Q_B - (-)^GQ_F)$. Dotsenko and Kitazawa used these states to calculate three-point correlators and obtained agreement with matrix-model results.

However, not all descents end at ghost number zero. There are other descents which end at ghost number $-1$ states (or ghost number zero as operators) [11]. These descents begin with LZ states with matter momentum dual to that in (3.4). Again, let us consider

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$c$ The state $c_1|0\rangle_{gh}$ is assigned ghost number 0. However, in going from states to operators, we increase the ghost number by 1. So the state $b_{-1}c_1|0\rangle_{gh}$ has ghost number $-1$ while the corresponding operator $b_{-1}c_1$ has ghost number 0.
the simpler example given in (3.3). On taking the dual in matter momentum of the LZ state, the non-vanishing matter null is replaced by the vanishing null (as explained earlier). Hence, we obtain

\[ Q_B |\tilde{LZ}\rangle^{-1} = \text{vanishing matter null} = 0 \quad , \tag{3.5} \]

where the \( \tilde{LZ} \) refers to taking LZ state in (3.3) with its matter momentum flipped to its dual. For the general case, again a vanishing null is encountered precisely one step earlier than in (3.4). The descent is

\[
\begin{align*}
Q_B |\tilde{LZ}\rangle^{-n} &= Q_F |I'_1\rangle^{-n+1} , \\
Q_B |I'_1\rangle^{-n+1} &= -Q_F |I'_2\rangle , \\
&\vdots \\
Q_B |I'_{n-2}\rangle^{-2} &= (-)^n Q_F |R\rangle^{-1} , \\
Q_B |R\rangle^{-1} &= 0 .
\end{align*}
\tag{3.6}
\]

Hence, there are two possible end-points for descents: states at zero ghost number – DK states and states at ghost number \(-1\). The latter are precisely the ring elements for \( c < 1 \). We shall describe them in the next subsection.

So far the discussion has been restricted to the negative ghost number sector. The positive ghost number states are partners to the negative ghost number states in the sense that the norm on the sphere is obtained as

\[ +n \langle LZ | c_0 | LZ \rangle^{-n} \tag{3.7} \]

Given an LZ state of ghost number \(-n\), it is now trivial to construct a \( Q_B \) closed state of ghost number \(+n\) which has a non-zero norm with the given LZ state. This new state of ghost number \(+n\) is

\[ |LZ\rangle^{+n} = M^n |\tilde{LZ}\rangle^{-n} , \tag{3.8} \]

where \( M = \{Q_B, c_0\} \) and \( |\tilde{LZ}\rangle^{-n} \) is the LZ state with the matter and Liouville momenta flipped to their duals. The state given in (3.8) is obviously not exact and hence a good element of the cohomology. The Liouville dressing is \( \beta > \beta_0 \) as given by the analysis of Lian and Zuckerman. It can also be shown that the LZ states of positive ghost number thus obtained are equivalent to those obtained by the construction described in [10] up to exact pieces.
3.2. Rings in \( c < 1 \)

The general construction which argues for the presence of ghost number \(-1\) states in the Coulomb gas method allows us to develop a ring structure in analogy with the work of Witten for \( c = 1 \). Following the suggestion of Kutasov, Martinec, and Seiberg[15], we define the two operators that generate the (chiral) ring structure,

\[
x = \mathcal{R}_{1,2} = (b_{-2}c_1 + t(L_L^{1} − L_M^{1}))e^{i\alpha_{1,2}X}e^{i\beta_{1,2}\phi} \\
y = \mathcal{R}_{2,1} = (b_{-2}c_1 + \frac{1}{t}(L_L^{1} − L_M^{1}))e^{i\alpha_{2,1}X}e^{i\beta_{2,1}\phi}
\]

(3.9)
i.e., \( x \) and \( y \) are the \( LZ^{-1} \) states with matter momenta labelled by \( \alpha_{1,2} \) and \( \alpha_{2,1} \) respectively and \( t = \frac{p}{p+1} \). We would like to point out that a target space boost of the \((X, \phi)\) system which transforms the \( c = 1 \) theory to the appropriate \( c < 1 \) theory, would in fact transform the generators \( x \) and \( y \) of Witten[16] to precisely the ones we have written above. The full set of ring elements as well as DK states obtained depends on the choice of screening operator to form the Felder BRST operator. We refer to this as a choice of resolution. For the case of the unitary \((p+1, p)\) minimal models, the ring elements in the \( Q_- \) resolution are[11]

\[
(a_-)^n, \ a_+(a_-)^n, \ldots, (a_-)^{p-1}(a_-)^n
\]

(3.10)
where \( a_\pm \) are the non-chiral ring elements obtained by composing holomorphic and anti-holomorphic ring elements \( −a_+ = −|x|^2 \) and \( a_- = −|y|^2 \). The DK states in the \( Q_- \) resolution are

\[
V_n^\gamma = e^{xp\left[p(n-2) + \gamma\right]X + [pn + \gamma + 2]iX \over 2\sqrt{p(p+1)}},
\]

(3.11)
where \( \gamma = 0, \ldots, (p-2) \) and \( n = 0, 1, \ldots \). The value \( \gamma = (p-1) \) has been excluded since these Liouville exponents are not seen in the \((p+1)\)th critical point of \( p \)-matrix model[1]. They correspond to matter momenta of the edge of the conformal grid with labels \((m', jp)\). However, their decoupling from correlation functions is not obvious.

In the \( Q_+ \) resolution, the ring elements are given by

\[
(a_+)^n, \ a_-(a_+)^n, \ldots, (a_-)^{p}(a_+)^n
\]

(3.12)
How are the ring elements in (3.10) and (3.12) related? An equivalence relation \( a_+^{p+1} \sim a_+^{p} \) imposed on the ring of monomials of the form \( \{a_+^m a_-^n\} \) makes the two rings isomorphic to each other. The DK states in the \( Q_+ \) resolution are

\[
V_n^\gamma = e^{xp\left[(p+1)(n-2) + \gamma + 2\right]X + [-\gamma - (p+1)n - \gamma]iX \over 2\sqrt{p(p+1)}},
\]

(3.13)
where \( \gamma = 0, \ldots, (p-1) \) and \( n = 0, 1, \ldots \). Here we have excluded edges of the type \((j(p+1), m)\) which correspond to \( \gamma = p \). The Liouville exponents belong to those seen in the \( p \)th critical point of the \((p+1)\) matrix model[1].
3.3. Correlation Functions

Given a resolution, one would like to calculate correlation functions involving arbitrary numbers of DK states. This has been accomplished recently\[12\]. The ring elements in (3.10) and (3.12) were utilised to obtain recursion relations amongst the DK states. These recursion relations are useful in converting integrals not in the form given by Dotsenko and Fateev into those which are of that form. A complete match with matrix model results has been obtained for all correlation functions on the sphere albeit with analytic continuation in the numbers of the cosmological constant operators and matter screening operators. See ref.\[12\] for more details. This completes the identification of matrix model observables as DK states in continuum Liouville theory.

However, one would like to see how the matrix model observables can be represented not as pure vertex operator states but rather as states built over the primaries of the minimal models. A suggestion has been presented in ref.\[11\]. The states which have been presented as candidates are given by

\[M^n |LZ\rangle^{-n} \otimes |\overline{LZ}\rangle^{-n}, \tag{3.14}\]

where \(M \equiv \{Q_B + \bar{Q}_B, (c_0 - \bar{c}_0)\}\). Not only do these states have total ghost number zero but also have the required scaling dimensions. One can now easily construct ghost number conserving correlation functions with these states. However, calculations involving these states will need techniques along the line used in the context of the string theory dilaton\[17\]. However, one can hope to relate this to the pure vertex operator states(DK states) using the descent equation in correlation functions as described in ref.\[10\].

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