Abstract. Although regular semisimple Hessenberg varieties are smooth and irreducible, semisimple Hessenberg varieties are not necessarily smooth in general. In this paper we determine the irreducible components of semisimple Hessenberg varieties corresponding to the standard Hessenberg space. We prove that these irreducible components are smooth and give an explicit description of their intersections, which constitute the singular locus. We conclude with an example of a semisimple Hessenberg variety corresponding to another Hessenberg space which is singular and irreducible, showing that results of this nature do not hold for all semisimple Hessenberg varieties.

1. Introduction

This paper initiates the study of the irreducible components and singular locus of semisimple Hessenberg varieties. Our main results prove that semisimple Hessenberg varieties corresponding to the standard Hessenberg space (see Equation (1.1)) have smooth irreducible components. We also give an explicit description of these irreducible components and their intersections using the associated GKM graph.

Hessenberg varieties are a collection of subvarieties of the full flag variety that generalize both Springer fibers and toric varieties associated to the Weyl chambers of the associated root system. These varieties were first defined as subvarieties of the flag variety by DeMari, Procesi, and Shayman in [DMPS92], and they appear in connection with the study of quantum cohomology of partial flag varieties [Kos96, Rec03], geometric representation theory [Spr76, Pro90, Ste92, Tym08, Tef11], numerical analysis [DMS88], and Schubert calculus [AT10, Ins15, IT16, HT11, Dre15].

Let $G$ be a linear, reductive algebraic group over $\mathbb{C}$, $B$ be a Borel subgroup, and $\mathcal{B} = G/B$ denote the corresponding flag variety. As usual, $\mathfrak{g}$ and $\mathfrak{b}$ denote the Lie algebras of $G$ and $B$ respectively, and $W$ is the Weyl group of $G$. We define a Hessenberg space $H$ to be a subspace of $\mathfrak{g}$ that contains $\mathfrak{b}$ and is closed under the Lie bracket with any element of $\mathfrak{b}$. In this paper, we focus our attention on the standard Hessenberg space given by

$$H_\Delta := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}$$

where $\Delta$ denotes the simple roots of $\mathfrak{g}$ and $\mathfrak{g}_{-\alpha}$ is the root space associated to the negative simple root $-\alpha \in -\Delta$.

Given an element $X \in \mathfrak{g}$ and a fixed Hessenberg space $H$, the Hessenberg variety $\mathcal{B}(X, H)$ is the subvariety of $\mathcal{B}$ consisting of all cosets $gB$ such that $Ad(g^{-1})(X)$ is an element of $H$. We say that $\mathcal{B}(X, H)$ is semisimple when $X$ is a semisimple element of $\mathfrak{g}$ and that $\mathcal{B}(X, H)$ is nilpotent when $X$ is a nilpotent element of $\mathfrak{g}$. Similarly, if $X \in \mathfrak{g}$ is a regular element we call the Hessenberg variety $\mathcal{B}(X, H)$ a regular Hessenberg variety. As another example, when $X \in \mathfrak{g}$ is nilpotent and $H = \mathfrak{b}$ the nilpotent Hessenberg variety $\mathcal{B}(X, \mathfrak{b})$ is the Springer fiber of $X$.

Since DeMari, Procesi, and Shayman initiated their study of these varieties [DMPS92], the geometric and topological properties of Hessenberg varieties have received considerable attention leading to fruitful and surprising discoveries. Several authors have proven Hessenberg varieties...
are paved by affines for increasingly general classes of Hessenberg varieties \cite{DMPS92, Tym06-2, Pre13}. These pavings show that some Hessenberg varieties are equivariantly formal \cite{GKM98, Tym05} and yield methods for computing their Betti numbers and other topological invariants using combinatorial formulas.

If $X \in \mathfrak{g}$ is a regular nilpotent element, the regular nilpotent Hessenberg variety $B(X, H_\Delta)$ is called the Peterson variety. Peterson and Kostant used Peterson varieties to construct the quantum cohomology of the flag variety \cite{Kos96}. More recently, several authors have studied equations defining local coordinate patches to analyze singular loci and describing some Hessenberg varieties as local complete intersections \cite{LY12, ADGH16}, including the Peterson variety. We discuss these methods in Sections 4 and 5 below. Understanding these geometric properties helps identify possible obstructions to studying (equivariant) cohomology, intersection theory, K-theory, and Newton-Okounkov bodies for Hessenberg varieties.

DeMari, Procesi, and Shayman showed that regular semisimple Hessenberg varieties are smooth and equidimensional for any Hessenberg space $H$. They also showed that the regular semisimple Hessenberg variety corresponding to the standard Hessenberg space is in fact the toric variety associated to the Weyl chambers \cite[Theorem 11]{DMPS92}. The Weyl group action on the cohomology of this variety had been studied independently by Procesi and Stembridge \cite{Pro90, Ste92}. There is an action of the Weyl group on the cohomology of regular semisimple Hessenberg varieties defined by Tymoczko \cite{Tym08} which generalizes the Weyl group action in the toric variety case. This representation has gained recent notoriety due to a conjecture posed by Shareshian and Wachs in 2011 which was proved by Brosnan and Chow in \cite{BC15} and again by Guay-Paquet \cite{GP15} using different methods.

The geometry of regular semisimple Hessenberg varieties has received a great deal of attention in the literature, due in large part to the representation discussed in the previous paragraph. However, there are only a few papers which consider non-regular Hessenberg varieties specifically (such as \cite{Pre15} and \cite{Tym06}). In this manuscript we focus primarily on non-regular semisimple Hessenberg varieties corresponding to the standard Hessenberg space. We prove that their geometry is determined by the combinatorics of the Weyl group and its cosets. Our main result is as follows.

**Theorem.** Let $S \in \mathfrak{g}$ be a non-regular semisimple element of $\mathfrak{g}$ and $W_M$ be the Weyl group of the centralizer of $S$ in $G$. The semisimple Hessenberg variety $B(S, H_\Delta)$ is a union of irreducible components

$$B(S, H_\Delta) = \bigcup_{v \in S} X_v$$

where $S \subseteq W$ is a subset of coset representatives for $W_M \backslash W$. Each of the irreducible components is smooth so the singular locus of $B(S, H_\Delta)$ consists precisely of those points in the intersection of two or more irreducible components.

The statement of this theorem is a combination of Theorems 3.16 and 4.5 below. A precise description of the irreducible components $X_v$ and the elements in $S$ are given in Theorem 3.16.

Following the proof of Theorem 4.5, we describe the GKM graphs of the irreducible components and their intersections as subgraphs of the GKM graph of $B(S, H_\Delta)$.

As Tymoczko notes in \cite{Tym06}, “It is usually difficult to identify the irreducible components of Hessenberg varieties.” This makes our results all the more surprising. In the same paper, Tymoczko poses the following questions.

- **(Question 5.2)** Let $X$ be any linear operator. If the Hessenberg space $H$ is in banded form, is the Hessenberg variety $B(X, H)$ necessarily pure-dimensional?
- **(Question 5.4)** Are all semisimple Hessenberg varieties smooth?
Our description of the irreducible components of non-regular semisimple Hessenberg varieties corresponding to the standard Hessenberg space (which is in banded Hessenberg form) shows that the answer to both of these questions is no.

The rest of this article is structured as follows. In the second section we provide a survey of notation and known results that will be used to prove our main theorems. In Section 3 we prove Theorem 3.16 which describes the irreducible components of semisimple Hessenberg varieties corresponding to the standard Hessenberg space. Section 4 contains the proof Theorem 4.5, proving that these irreducible components are smooth varieties. In Corollary 4.6 we use Theorems 3.16 and 4.5 to calculate the singular locus of these varieties. The end of Section 4 gives a description of the GKM graph of the singular locus of $B(S,H_\Delta)$ as a subgraph of the GKM graph of $B(S,H_\Delta)$ and includes many examples. While many of our arguments rely heavily on Lie-theoretic terminology, we hope that the reader who is more interested in studying the combinatorics and GKM theory of Hessenberg varieties will find this subsection visually appealing.

Finally in Section 5 we describe how to use commutative algebra and computational software to analyze the geometry of semisimple Hessenberg varieties associated to any Hessenberg space for algebraic groups in this paper are assumed to be complex and linear. Let $G$ be as in the introduction.

Let $B \subset G$ be a fixed Borel subgroup and $B_-$ denote the opposite Borel subgroup so that $T = B \cap B_-$ is a torus. Denote the Lie algebra of $B$ by $\mathfrak{b}$ and the Lie algebra of $T$ by $\mathfrak{t}$. Write $U$ for the maximal unipotent subgroup of $B$ and let $\mathfrak{u}$ denote the Lie algebra of $U$. Similarly, $U^-$ denotes the maximal unipotent subgroup of the opposite Borel.

Let $\Phi$ be the root system associated to $B$, with $\Phi^+$, $\Phi^-$, and $\Delta$ the subsets of positive, negative and simple roots in $\Phi$, respectively. Denote the negative simple roots by $\Delta^-$. Each positive root $\gamma \in \Phi^+$ can be written uniquely as $\gamma = \sum n_\alpha \alpha$ for $n_\alpha \in \mathbb{Z}_{\geq 0}$. The height of $\gamma$ is $\text{ht}(\gamma) = \sum_{\alpha \in \Delta} n_\alpha$.

Fix root vectors $E_\gamma$ in each root space $\mathfrak{g}_\gamma$ such that $\text{ad}_{\mathfrak{g}_\gamma}(E_\gamma) = [S,E_\gamma] = \gamma(S)E_\gamma$ for all $S \in \mathfrak{b}$ and

$$\text{ad}_{\mathfrak{g}_\gamma}(E_\beta) = [E_\gamma,E_\beta] = \begin{cases} m_{\gamma,\beta} E_{\gamma+\beta} & \text{if } \gamma + \beta \in \Phi^+ \\ 0 & \text{otherwise} \end{cases}$$

for all $\gamma, \beta \in \Phi^+$ where $m_{\gamma,\beta}$ is a nonzero integer. Let $U_\gamma = \exp(x_\gamma E_\gamma)$ for $x_\gamma \in \mathbb{C}$ be the 1-dimensional unipotent subgroup corresponding to $\gamma \in \Phi$.

The Weyl group of $G$ is $W = N_G(T)/T$. Throughout this paper, we fix a representative $w \in G$ such that $wU_\gamma w^{-1} = U_{w(\gamma)}$ for each $w \in W$ and use the same letter for both. Write $s_\gamma$ for the reflection corresponding to $\gamma \in \Phi$. The Weyl group of $G$ is generated by the simple reflections $s_\gamma = s_{\alpha_i}$ for each $\alpha_i \in \Delta$. Given $w \in W$, the length of $w$ is the number of simple reflections in any reduced word $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ for $w$, denoted by $\ell(w)$.

Our main example throughout this paper is the case in which $G = GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is the collection of $n \times n$ matrices, also known as the type $A$ case. In this setting, we take $B$ to be the subgroup of invertible upper triangular matrices, $B_-$ to
be the opposite subgroup of invertible lower triangular matrices, and \( T \) to be diagonal subgroup. The Weyl group of \( GL_n(\mathbb{C}) \) is the symmetric group \( S_n \).

### 2.1. Hessenberg varieties

The main focus of this paper is a collection of subvarieties of \( B \) which we will now define.

**Definition 2.2.** A subspace \( H \subseteq \mathfrak{g} \) is a *Hessenberg space* with respect to \( b \) if \( b \subseteq H \) and \([b, H] \subseteq H\).

If \( H \subseteq \mathfrak{g} \) is such a Hessenberg space then

\[
H = b \oplus \bigoplus_{\gamma \in \Phi_H^-} \mathfrak{g}_\gamma
\]

for a subset \( \Phi_H^- = \{ \gamma \in \Phi^- : \mathfrak{g}_\gamma \subseteq H \} \) of negative roots. A Hessenberg variety is a subvariety of the flag variety defined as follows.

**Definition 2.3.** Fix \( X \in \mathfrak{g} \) and a Hessenberg space \( H \) with respect to \( b \). The *Hessenberg variety* associated to \( X \) and \( H \) is

\[
\mathcal{B}(X, H) = \{ gB \in \mathcal{B} : g^{-1} \cdot X \in H \}
\]

where \( g \cdot X \) denotes the adjoint action \( Ad(g)(X) \).

In later sections we will specialize to the case in which \( \Phi_H^- = \Delta^- \) and write \( H_\Delta \) for this Hessenberg space, which we refer to as the *standard Hessenberg space*.

**Example 2.4.** When \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \), the standard Hessenberg space is the subspace of matrices

\[
H = \left\{ \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1,n-1} & a_{1n} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2,n-1} & a_{2n} \\
0 & a_{32} & a_{33} & \ldots & a_{3,n-1} & a_{3n} \\
0 & 0 & a_{43} & \ldots & a_{4,n-1} & a_{4n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_{n,n-1} & a_{nn}
\end{bmatrix} : a_{ij} \in \mathbb{C}, 1 \leq i \leq n, i-1 \leq j \leq n \right\} \subseteq \mathfrak{gl}_n(\mathbb{C}).
\]

When \( S \) is a semisimple element of \( \mathfrak{g} \), we let \( M = Z_G(S) \) be the centralizer of \( S \), so \( M \) is a Levi subgroup of \( G \), i.e. it is a closed reductive subgroup of \( G \). \( M \) acts on \( \mathcal{B}(S, H) \) by translation. Indeed, if \( gB \in \mathcal{B}(S, H) \) then \( mgB \in \mathcal{B}(S, H) \) for all \( m \in M \) since

\[
(mg)^{-1} \cdot S = g^{-1}m^{-1} \cdot S = g^{-1} \cdot S \in H.
\]

If \( X \subseteq \mathcal{B}(S, H) \) is a subvariety, we denote the \( M \)-orbit of \( X \) by \( M(X) \). Since conjugate elements of \( \mathfrak{g} \) correspond to isomorphic Hessenberg varieties (see [Pre13, Remark 2.3]) we may assume without loss of generality that \( M \) is a standard Levi subgroup of \( G \).

### 2.2. Cosets in the Weyl group

Each standard Levi subgroup \( L \subseteq G \) corresponds to a unique subset \( \Delta_L \subseteq \Delta \) of simple roots. We let \( B_L = L \cap B \) denote the Borel subgroup of \( L \) determined by \( B \) and \( U_L = L \cap U \) be the maximal unipotent subgroup of \( B_L \) with Lie algebra \( u_L \). Similarly, let \( U^-_L = L \cap U^- \) be the maximal unipotent subgroup of the opposite Borel, \( L \cap B^- \). The Weyl group of \( L \) is \( W_L = \langle s_\alpha : \alpha \in \Delta_L \rangle \subseteq W \). Let \( \Phi_L \) denote the root subsystem of \( \Phi \) associated to \( B_L \subset L \) with positive roots \( \Phi_L^+ \) and negative roots \( \Phi_L^- \).

**Example 2.5.** If we take \( M = Z_G(S) \) as above, then \( \Delta_M = \{ \alpha \in \Delta : \alpha(S) = 0 \} \) so the root system of \( M = Z_G(S) \) is uniquely defined by the condition that \( \gamma \in \Phi_M \) if and only if \( \gamma(S) = 0 \).
For each $w \in W$, let
\[ N(w) = \{ \gamma \in \Phi^+ : w(\gamma) \in \Phi^- \}. \]
We call $N(w)$ the inversion set of $w$. It is a well-known fact that $|N(w)| = |N(w^{-1})| = \ell(w)$. We similarly define $N^-(w) = \{ \gamma \in \Phi^- : w(\gamma) \in \Phi^+ \}$. If $\alpha \in N(w^{-1}) \cap \Delta$, then $s_\alpha$ is called a left descent of $w$. Similarly, if $\alpha \in N(w) \cap \Delta$, then $s_\alpha$ is a right descent of $w$.

**Remark 2.6.** The elements of $N(w^{-1})$ can be characterized as follows: $\gamma \in N(w^{-1})$ if and only if $\ell(s_\gamma w) < \ell(w)$ (see [Hum90 Section 1.6]).

Consider the sets
\[ L^W = \{ v \in W : N(v^{-1}) \subseteq \Phi^+ \setminus \Phi^+_L \} \]
and
\[ W^L = \{ v \in W : N(v) \subseteq \Phi^+ \setminus \Phi^+_L \}. \]
The elements of $L^W$ and $W^L$ form a set of shortest-length coset representatives for $W_L \setminus W$ and $W/W_L$ respectively in the following sense (see [BB05 Proposition 2.4.4]).

**Lemma 2.7.** Each $w \in W$ can be written uniquely as
\begin{itemize}
  \item $w = yv$ for some $y \in W_L$ and $v \in L^W$, and
  \item $w = v'y'$ for some $y' \in W_L$ and $v' \in W^L$
\end{itemize}
such that $\ell(w) = \ell(y) + \ell(v) = \ell(v') + \ell(y')$.

We will also use the following standard fact about inversion sets; especially in the context of the previous result.

**Lemma 2.8.** Let $w \in W$ and $w = yv$ for $y, v \in W$ such that $\ell(w) = \ell(y) + \ell(v)$. Then $N(w^{-1}) = N(y^{-1}) \sqcup yN(v^{-1})$ and $N(w) = N(v) \sqcup v^{-1}N(y)$.

It’s also a well-known fact that $W_L$ must normalize $\Phi^+ \setminus \Phi^+_L$. In particular
\begin{equation}
(2.9) \quad y(\Phi^+ \setminus \Phi^+_L) \subseteq \Phi^+ \setminus \Phi^+_L \quad \text{and} \quad y(\Phi^- \setminus \Phi^-_L) \subseteq \Phi^- \setminus \Phi^-_L
\end{equation}
for all $y \in W_L$.

### 2.3. Cellular decompositions.

The Bruhat decomposition of $G$ yields a corresponding cellular decomposition of the flag variety. Namely, $B = \bigsqcup_{w \in W} C_w$ where each $C_w = BwB/B$ denotes the Schubert cell indexed by $w \in W$. Each Schubert cell has the following explicit description:
\begin{equation}
(2.10) \quad C_w = U^w wB/B \quad \text{where} \quad U^w = \{ u \in U : w^{-1}uw \in U^- \} \cong \prod_{\gamma \in N(w^{-1})} U_{\gamma}.
\end{equation}
Therefore $C_w \cong \mathbb{C}^{\ell(w)}$ and it is furthermore known that $C_w = \bigsqcup_{y \leq w} C_y$ where $\leq$ denotes the Bruhat order on $W$. We say that the affine cells $C_w$ pave $B$.

Let $S \in \mathfrak{g}$ be a semisimple element. To begin our analysis of the semisimple Hessenberg variety $B(S, H)$ consider
\[ B(S, H) = \bigsqcup_{w \in W} C_w \cap B(S, H). \]

We refer to the intersections on the right-hand side of the equation above as Hessenberg-Schubert cells. The second author proved in [Pre13 Theorem 5.4] that each Hessenberg-Schubert cell is isomorphic to affine space and computes their dimension in [Pre13 Corollary 5.8]. These results are summarized below.
Proposition 2.11. Suppose $S \in \mathfrak{h}$ is a semisimple element and $M = Z_G(S)$. Given $w \in W$, write $w = yv$ with $y \in W_M$ and $v \in MW$. Then $C_w \cap B(S, H) \cong \mathbb{C}^{d_w}$ where
\[ d_w = |N(y^{-1})| + |N(v^{-1}) \cap v(\Phi_H^\perp)|. \]

Remark 2.12. Since each Hessenberg-Schubert cell $C_w \cap B(S, H)$ is isomorphic to affine space, the closure $C_w \cap B(S, H)$ is an irreducible subvariety of $B(S, H)$.

Example 2.13. If $S \in \mathfrak{g}$ is a regular semisimple element then $M = \{e\}$ so $W_M = \{e\}$ and $MW = W$. Proposition 2.11 yields
\[ \dim(C_w \cap B(S, H)) = |N(w) \cap w(\Phi_H^\perp)| = |N^-(w) \cap \Phi_H^\perp| \]
for all $w \in W$, using the fact that $w^{-1}N(w^{-1}) = N^-(w)$. If $H$ is the standard Hessenberg space then $\dim(C_w \cap B(S, H)) = |N^-(w) \cap \Delta^-| = |N(w) \cap \Delta|$ is the number of right descents of $w$.

2.4. Regular semisimple Hessenberg varieties. In this paper we initiate the study of the singular locus of $B(S, H)$. When $S \in \mathfrak{h}$ is a regular semisimple element then $B(S, H)$ is a smooth variety [DMPS92, Theorem 6].

Proposition 2.14 (DeMari-Procesi-Shayman). Suppose $S \in \mathfrak{h}$ is a regular semisimple element and $H$ is a Hessenberg space in $\mathfrak{g}$ with respect to $\mathfrak{b}$. Then $B(S, H)$ is a smooth variety and $\dim(B(S, H)) = |\Phi_H^\perp|$.

In the same paper, DeMari-Procesi-Shayman also prove that the regular semisimple Hessenberg variety corresponding to the standard Hessenberg space is the toric variety associated with the Weyl chambers of the root system [DMPS92, Theorem 11]. In general, one obtains the following corollary to the above proposition from [AT10, Proposition A1].

Corollary 2.15. Suppose $S \in \mathfrak{h}$ is a regular semisimple element and $H$ is a Hessenberg space in $\mathfrak{g}$ with respect to $\mathfrak{b}$. If $\Delta^- \subseteq \Phi_H^\perp$ then $B(S, H)$ is an irreducible variety such that
\[ B(S, H) = C_{w_0} \cap B(S, H) \]
where $w_0 \in W$ denotes the longest element of the Weyl group.

Below we consider the case in which $S \in \mathfrak{h}$ is not necessarily regular. When the Hessenberg space is fixed as the standard one and there is no possible confusion, we write $B(S, H_\Delta) = B(S)$.

3. Irreducible components

Throughout this section and the next we assume that $H = H_\Delta$ is the standard Hessenberg space. Let $S \in \mathfrak{h}$ denote a non-regular semisimple element. In this section we identify the irreducible components of $B(S)$. Each one is the $M$-orbit of the closure of a certain Hessenberg-Schubert cell. We begin by associating each $v \in W$ to a subset of simple roots, $R(v)$. These subsets will be used later to identify which Hessenberg-Schubert cells correspond to irreducible components of $B(S)$.

Definition 3.1. For each $v \in W$, set $R(v) := N(v) \cap \Delta$. In other words, $R(v)$ is the set of simple roots such that the simple reflections $s_\alpha$ for $\alpha \in R(v)$ are right descents of $v$.

By Proposition 2.11 when $H$ is the standard Hessenberg space and $w = yv$ with $y \in W_M$ and $v \in MW$, then
\[ \dim(C_w \cap B(S)) = |N(y^{-1})| + |N(v^{-1}) \cap v(\Delta^-)| = |N(y^{-1})| + |R(v)|. \]
In particular, if $v \in MW$ then $\dim(C_v \cap B(S)) = |R(v)|$. 


We now establish some notation for use in this section and the next. For each \( v \in MW \), let \( L \subseteq G \) denote the Levi subgroup corresponding to \( R(v) \) (in other words, \( \Delta_L = R(v) \)) with Lie algebra \( L \subseteq g \) and associated flag variety \( BL = L/B_L \). If there is any ambiguity, we write \( L_v \) to indicate that \( L_v \) is the Levi subgroup associated to \( R(v) \) for \( v \in MW \).

**Remark 3.3.** Let \( w_v \in WL \) denote the longest element of \( WL \). Since \( R(w_v) = R(v) \subseteq N(v) \), it follows from Lemma \( 2.7 \) that \( v \) can be written uniquely as \( v = x_vw_v \) for some \( x_v \in WL \) and \( \ell(v) = \ell(x_v) + \ell(w_v) \).

**Example 3.4.** Let \( g = gl_4(\mathbb{C}) \) and \( S = \text{diag}(1, 1, -1, -1) \). In this case, \( M = GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \) and \( W_M = \langle s_1, s_3 \rangle \) is a Young subgroup of \( S_n \). The table below displays each element of \( MW \), the subset of simple roots \( R(v) \), and corresponding decomposition \( v = x_vw_v \).

| \( v \in MW \) | \( R(v) \) | \( x_v \) | \( w_v \) |
|----------------|---------|--------|--------|
| \( s_2s_3s_1s_2 \) | \( \{\alpha_2\} \) | \( s_2s_3s_1 \) | \( s_2 \) |
| \( s_2s_3s_1 \) | \( \{\alpha_1, \alpha_3\} \) | \( s_2 \) | \( s_3s_1 \) |
| \( s_2s_1 \) | \( \{\alpha_1\} \) | \( s_2 \) | \( s_1 \) |
| \( s_2s_3 \) | \( \{\alpha_3\} \) | \( s_2 \) | \( s_3 \) |
| \( s_2 \) | \( \{\alpha_2\} \) | \( e \) | \( s_2 \) |
| \( e \) | \( \emptyset \) | \( e \) | \( e \) |

We will see below that this information can be used to characterize the closure relations among the Hessenberg-Schubert cells \( C_v \cap B(S) \) for \( v \in MW \).

Suppose \( v \in MW \) and \( v = x_vw_v \) is the decomposition of \( v \) given in Remark \( 3.3 \). The next two lemmas establish some basic properties of this decomposition. Since \( w_v \) is the longest element of \( WL \), \( N(w_v) = \Phi_L^+ \) and by Lemma \( 2.8 \), \( N(v) = N(x_v) \cup x_v\Phi_L^+ \).

**Lemma 3.5.** If \( \gamma \in N(x_v^{-1}) \), then \( v^{-1}(\gamma) \in \Phi^- \setminus \Delta^- \).

**Proof.** Since \( \gamma \in N(x_v^{-1}) \subseteq N(v) \) we certainly have that \( v^{-1}(\gamma) \in \Phi^- \). Using Equation \( 2.9 \) and the fact that \( x_v \in WL \) we get

\[
v^{-1}(\gamma) \in w_v^{-1}x_v^{-1}N(x_v^{-1}) = w_v^{-1}N(x_v^{-1}) \subseteq w_v^{-1}(\Phi^- - \Phi_L^-) \subseteq \Phi^- - \Phi_L^-.
\]

If \( v^{-1}(\gamma) \in \Delta^- \), then there exists \( \alpha \in \Delta \) such that \( v^{-1}(\gamma) = -\alpha \) implying that \( v^{-1}(\gamma) = -\alpha \in -R(v) \subseteq \Phi_L^- \), contradicting the previous sentence. Therefore \( v^{-1}(\gamma) \in \Phi^- \setminus \Delta^- \).

**Lemma 3.6.** \( \tau = x_vz \) is an element of \( MW \) for all \( z \in WL \).

**Proof.** This is a direct implication of Lemmas \( 2.7 \) and \( 2.8 \). Since \( x_v \in WL \) it follows that \( \ell(x_vz) = \ell(x_v) + \ell(z) \) for all \( z \in WL \) by Lemma \( 2.7 \). As \( w_v \) is the longest element of \( WL \) we know \( N(z^{-1}) \subseteq \Phi_L^+ = N(w_v^{-1}) \). Now by Lemma \( 2.8 \)

\[
N(\tau^{-1}) = N(x_v^{-1}) \cup x_vN(z^{-1}) \subseteq N(x_v^{-1}) \cup x_v\Phi_L^+ = N(v^{-1}) \subseteq \Phi^+ - \Phi_M^+
\]

since \( v \in MW \).

We define \( u^v \) to be the Lie algebra of the unipotent subgroup \( U^v \) defined in Equation \( 2.10 \), so

\[
u^v := \bigoplus_{\gamma \in N(v)} g_{\gamma} = \bigoplus_{\gamma \in N(x_v^{-1})} g_{\gamma} \oplus \bigoplus_{\gamma \in x_v\Phi_L^+} g_{\gamma}.
\]

Our next result is a technical lemma which will be used to prove the proposition following it.
Lemma 3.7. Let $X \in u^v$ and write $X = \sum_{\gamma \in N(v^{-1})} c_{\gamma} E_{\gamma}$ for some $c_{\gamma} \in \mathbb{C}$. Suppose $\beta \in N(v^{-1})$ such that $\text{ht}(\beta) \geq k$ or $\beta \in x_v \Phi^+_L$. If $c_{\gamma} = 0$ for all $\gamma \in N(v^{-1})$ such that $\text{ht}(\gamma) < k$, then

$$\text{ad}^i_X(E_{\beta}) \in \bigoplus_{\delta \in N(v^{-1})} g_{\delta} \oplus \bigoplus_{\delta \in x_v \Phi^+_L} g_{\delta}$$

for all $i \geq 1$.

Proof. Fix $i \geq 1$ and let $\text{ad}^i_X(E_{\beta}) = \sum_{\delta \in \Phi^+} d_{\delta} E_{\delta}$ for some $d_{\delta} \in \mathbb{C}$. Suppose $\delta \in \Phi^+$ such that $d_{\delta} \neq 0$, i.e., $E_{\delta}$ occurs as a summand of $\text{ad}^i_X(E_{\beta})$. Since

$$X = \sum_{\gamma \in N(v^{-1})} c_{\gamma} E_{\gamma} + \sum_{\gamma \in x_v \Phi^+_L} c_{\gamma} E_{\gamma},$$

it follows from the definition of the adjoint action in (2.1) that

$$\delta = \beta + \sum_{\gamma \in N(v^{-1})} n_{\gamma} \gamma + \sum_{\gamma \in x_v \Phi^+_L} n_{\gamma} \gamma$$

for some $n_{\gamma} \in \mathbb{Z}_{\geq 0}$ such that $n_{\gamma} \neq 0$ for at least one $\gamma \in N(v^{-1})$ appearing in the index sets above because $i \geq 1$. Note that $\delta \in N(v^{-1})$ since all roots in the equation above are elements of $N(v^{-1})$, and this set is closed under addition.

If $\beta \in N(v^{-1})$ such that $\text{ht}(\beta) \geq k$, then $\text{ht}(\delta) > \text{ht}(\beta) \geq k$. If $\beta \in x_v \Phi^+_L$, we consider two possible cases. Either $n_{\gamma} = 0$ for all $\gamma \in N(v^{-1})$ or there exists at least one $\gamma \in N(v^{-1})$ such that $\text{ht}(\gamma) \geq k$ and $n_{\gamma} \neq 0$. In the latter case, $\text{ht}(\delta) > \text{ht}(\gamma) \geq k$. In the former case, $\delta \in x_v \Phi^+_L$ since $x_v \Phi^+_L$ is closed under addition (because $\Phi^+_L$ is). We conclude that in every possible case, either $\text{ht}(\delta) > k$ or $\delta \in x_v \Phi^+_L$. This proves the desired result since $N(v^{-1}) \setminus x_v \Phi^+_L = N(v^{-1})$.

The next proposition is a key step in the proof of Theorem 3.10 below.

Proposition 3.8. If $uvB \in C_v \cap B(S)$, then $u \in x_v U_L x_v^{-1}$.

Proof. Using the description of $C_v$ given in Equation (2.10), we begin by noting that if $uvB \in C_v$, then we may assume $u \in U^v = \prod_{\gamma \in N(v^{-1})} U_\gamma$. Since $U^v$ is unipotent, the exponential map $\exp : u^v \to U^v$ is a diffeomorphism. Therefore

$$u = \exp(X)$$

for some $X = \sum_{\gamma \in N(v^{-1})} c_{\gamma} E_{\gamma}$ with $c_{\gamma} \in \mathbb{C}$. Recall that $N(v^{-1}) = N(v^{-1}) \cup x_v \Phi^+_L$. To prove the proposition, it suffices to show that $c_{\gamma} = 0$ for all $\gamma \in N(v^{-1})$. Given this fact, $X = \sum_{\gamma \in x_v \Phi^+_L} c_{\gamma} E_{\gamma}$ so

$$x_v^{-1} \cdot X = \sum_{\gamma \in x_v \Phi^+_L} c_{\gamma} E_{x_v^{-1}(\gamma)} \in u_L \Rightarrow \exp(x_v^{-1} \cdot X) \in U_L \Rightarrow x_v^{-1} \exp(X) x_v \in U_L$$

which implies $u \in u_v U_L x_v$ as desired.

We will prove $c_{\gamma} = 0$ for $\gamma \in N(v^{-1})$ using induction on $\text{ht}(\gamma)$. First we outline some additional notation and recall some facts about the adjoint action. Since $u^{-1} \cdot S = \text{Ad}(\exp(-X))(S) = \text{Ad}(u^{-1}X)$,

$$x_v^{-1} \cdot X = \sum_{\gamma \in x_v \Phi^+_L} c_{\gamma} E_{x_v^{-1}(\gamma)} \in u_L \Rightarrow \exp(x_v^{-1} \cdot X) \in U_L \Rightarrow x_v^{-1} \exp(X) x_v \in U_L$$

which implies $u \in u_v U_L x_v$ as desired.
\[\exp(\text{ad}^{-X})(S)\text{ and } [S, E_v] = \gamma(S)E_v \text{ for all } \gamma \in \Phi^+, \text{ we have}\]

\[
u^{-1} \cdot S = S + \sum_{i=1}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S) = S + \text{ad}^{-X}(S) + \sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S)
\]

\[= S + \sum_{\gamma \in N(v^{-1})} c_{\gamma} E_{\gamma} + \sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S)\]

Note that although the index set is infinite, the sum above is finite since \(X \in \mathfrak{u}\) is a nilpotent element of \(\mathfrak{g}\). Consider

\[
\sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S) = \sum_{i=2}^{\infty} \text{ad}^{-X}_{i-1}(\text{ad}^{-X}(S))
\]

(3.9)

\[
= \sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_{i-1} \left( \sum_{\beta \in N(v^{-1})} c_{\beta} (S)E_{\beta} \right)
\]

\[= \sum_{i=2}^{\infty} \frac{1}{i!} \sum_{\beta \in N(v^{-1})} c_{\beta} (S) \text{ad}^{-X}_{i-1}(E_{\beta}).\]

Since \(u^v\) is closed under the adjoint action, \(\text{ad}^{-X}_{i-1}(E_{\beta}) \in u^v\) for all \(i \geq 2\) and \(\beta \in N(v^{-1})\). Thus we may write

\[
\sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S) = \sum_{\gamma \in N(v^{-1})} d_{\gamma} E_{\gamma}
\]

for some \(d_{\gamma} \in \mathbb{C}\), so \(u^{-1} \cdot S = S + \sum_{\gamma \in N(v^{-1})} (\gamma(S)c_{\gamma} + d_{\gamma}) E_{\gamma}\). Since \(uvB \in \mathcal{B}(S)\) it must be the case that

\[v^{-1} \cdot u^{-1} \cdot S = v^{-1} \cdot S + \sum_{\gamma \in N(v^{-1})} (\gamma(S)c_{\gamma} + d_{\gamma}) E_{v^{-1}(\gamma)} \in H_{\Delta} = b \oplus \bigoplus_{\alpha \in \Delta} g_{-\alpha}.\]

In particular, if \(v^{-1}(\gamma) \in \Phi^- \setminus \Delta^-\) then the above equation implies \(\gamma(S)c_{\gamma} + d_{\gamma} = 0\). By Lemma 3.5 we conclude that \(\gamma(S)c_{\gamma} + d_{\gamma} = 0\) for all \(\gamma \in N(x_v^{-1})\). To prove \(c_{\gamma} = 0\), we have only to show that \(d_{\gamma} = 0\) since \(\gamma(S) \neq 0\) because \(\gamma \in N(v^{-1}) \subseteq \Phi^+ - \Phi^+_M\).

It follows from Equations (2.1) and (3.9) that

\[
\sum_{\gamma \in N(v^{-1})} d_{\gamma} E_{\gamma} = \sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}^{-X}_i(S) \subseteq \bigoplus_{\delta \in \Phi^+_{\nu} \setminus \{0\}} g_{-\delta}.
\]

This implies \(d_{\gamma} = 0\) for all \(\gamma\) such that \(\text{ht}(\gamma) = 1\), implying \(c_{\gamma} = 0\) for all \(\gamma\) such that \(\text{ht}(\gamma) = 1\), and proving the base case.

Now assume that \(c_{\gamma} = 0\) for all \(\gamma \in N(x_v^{-1})\) such that \(\text{ht}(\gamma) < k\). Equation (3.9) now becomes

\[
\sum_{\gamma \in N(v^{-1})} d_{\gamma} E_{\gamma} = \sum_{i=2}^{\infty} \frac{1}{i!} \left( \sum_{\beta \in N(x_v^{-1})} c_{\beta} (S) \text{ad}^{-X}_{i-1}(E_{\beta}) + \sum_{\beta \in x_v \Phi^+_M} c_{\beta} (S) \text{ad}^{-X}_{i-1}(E_{\beta}) \right).\]
Lemma 3.7 implies that
\[ \sum_{\gamma \in N(x^{-1})} d_{\gamma} E_{\gamma} \in \bigoplus_{\delta \in N(x^{-1})} g_{\gamma} \oplus \bigoplus_{\delta \in N(x^{-1})} g_{\delta}. \]
Therefore \( d_{\gamma} = 0 \) for all \( \gamma \in N(x^{-1}) \) such that \( \text{ht}(\gamma) = k \). This implies \( c_{\gamma} = 0 \) for all \( \gamma \in N(x^{-1}) \) such that \( \text{ht}(\gamma) = k \) and proves the inductive step. We conclude \( c_{\gamma} = 0 \) for all \( \gamma \in N(x^{-1}) \) as desired.

Recall that if \( L \subseteq G \) is a Levi subgroup then \( B_L = L/B_L \) denotes the corresponding flag variety. Let \( \iota_L : B_L \hookrightarrow B \) denote the inclusion of \( B_L \) into the flag variety \( B \) induced from the inclusion \( L \subseteq G \). In particular, if \( uzB_L \in B_L \) for some \( u \in U^z \) and \( z \in W_L \), then \( \iota_L(uzB_L) = uzB \). Any subvariety of \( B_L \) can be viewed as a subvariety of \( B \) by considering its image under \( \iota_L \). Our next result shows that the closures of certain Hessenberg-Schubert cells are isomorphic to regular semisimple Hessenberg varieties in the flag variety of Levi subgroup.

**Theorem 3.10.** For \( v \in MW \) let \( L = L_v \) be the Levi subgroup corresponding to \( R(v) \subseteq \Delta \) and \( v = x_vw_v \) be the decomposition of \( v \) given in Remark 3.6. Set \( S_v := x_v^{-1}S \in \mathfrak{h} \). Then \( C_v \cap B(S) \cong B_L(S_v) \) where \( B_L(S_v) \) is the regular semisimple Hessenberg variety in \( B_L \) corresponding to the standard Hessenberg space \( H_{R(v)} := H_\Delta \cap I \subset I \).

**Proof.** We can view any element of \( \mathfrak{h} \) as a semisimple element of \( I \subseteq \mathfrak{g} \) by restriction. First we show that \( S_v \) is a regular semisimple element of \( I \), or equivalently that \( \gamma(S_v) \neq 0 \) for all \( \gamma \in \Phi_L \). If \( \gamma \in \Phi_L \) then \( x_v(\gamma) \in x_v\Phi_L \subseteq N(v^{-1}) \subseteq \Phi^+ \setminus \Phi^- \) because \( v \in MW \). This implies \( \gamma(S_v) = \gamma(x_v^{-1}S) = x_v(\gamma)(S) \neq 0 \) so \( S_v \) is indeed a regular element of \( I \).

By Corollary 2.15 \( B_L(S_v) = C_{L,w_v} \cap B(S_v) \) where \( C_{L,w_v} = B_L(w_vB/L) \) is a Schubert cell in \( B_L \). Let \( U_{L,w_v} := \{ u \in U_L : uw_vB_L \in B_L(S_v) \} \) so \( C_{L,w_u} \cap B_L(S_v) = U_{L,w_v}w_vB_L/B_L \) using the description of Schubert cells given in Equation (2.10). Furthermore, \( C_{L,w_u} \cap B_L(S_v) \) can be viewed as an open subset in \( B \) by identifying it with its image \( \iota_L(C_{L,w_u} \cap B_L(S_v)) = U_{L,w_v}w_vB/B \). We will prove that \( C_v \cap B(S) \) is the \( x_v \)-translate of this image, namely \( C_v \cap B(S) = x_vU_{L,w_v}w_vB/B \). Our result then follows from the fact that translation respects closure relations in the flag variety so
\[ C_v \cap B(S) = x_vU_{L,w_v}w_vB/B \cong C_{L,w_v} \cap B(S_v) = B_L(S_v). \]

Each element of \( C_{L,w_v} \cap B_L(S_v) \) is of the form \( uw_vB \) for some \( u \in U_{L,w_v} \). Consider \( \iota_L(uw_vB) = uw_vB \). Since \( uw_vB \in C_{L,w_v} \cap B_L(S_v) \) we get
\[ w_v^{-1}u^{-1} \cdot S_v \in H_{R(v)} \Rightarrow w_v^{-1}u^{-1}x_v^{-1} \cdot S \in H_\Delta \cap I \subset H_\Delta \]
so \( x_vuw_vB \in C_v \cap B(S) \), implying \( x_vU_{L,w_v}w_vB/B \subseteq C_v \cap B(S) \).

If \( uw \in C_v \cap B(S) \), Proposition 3.8 implies that \( u \in x_vU_{L,x_v^{-1}} \). Let \( u' = x_v^{-1}uw \in U_{L,x_v} \). Our assumption that \( uwB \in C_v \cap B(S) \) implies
\[ w_v^{-1}(u')^{-1}x_v^{-1} \cdot S = v^{-1}u^{-1} \cdot S \in H_\Delta \Rightarrow w_v^{-1}(u')^{-1} \cdot S \in H_\Delta \cap I \subset H_{R(v)} \]
since \( u'w_v \in L \). We conclude that \( u' \in U_{L,w_v} \) and \( uwB = x_v\iota_L(u'w_vB_L) \). Thus \( C_v \cap B(S) \subseteq x_vU_{L,w_vw_vB/B} \) and \( C_v \cap B(S) = x_vU_{L,w_v}w_vB/B \). \( \square \)

This theorem yields a cellular decomposition of \( C_v \cap B(S) \) for each \( v \in MW \) using the Hessenberg-Schubert decomposition
\[ B_L(S_v) = \bigcup_{z \in W_L} C_{L,z} \cap B_L(S_v) \]
where $C_{L,z}$ denotes the Schubert cell $B_L z B_L / B_L$ in $B_L$. Let $U_{L,z} := \{ u \in U^z : uzB_L \in B_L(S_v) \}$. It follows from the proof of Theorem 3.10 that

$$C_v \cap B(S) = \bigcup_{z \in W_L} x_v U_{L,z} z B / B \subseteq \bigcup_{z \in W_L} C_{x_v z}.$$  

(3.11)

since $C_{L,z} \cap B_L(S_v) = U_{L,z} z B_L / B_L$ and $\iota_L(U_{L,z} z B_L / B_L) = U_{L,z} z B / B$. In particular, Equation (3.11) and Lemma 3.6 imply

$$C_v \cap B(S) \subseteq \bigcup_{\tau \in M^W} C_\tau.$$  

(3.12)

The next lemma partially characterizes the closure relations among Hessenberg-Schubert cells $C_v \cap B(S)$ for $v \in M^W$. Results of this nature are rare. For example, a full understanding of the closure relations between the cells paving Springer fibers is unknown. For each $v \in M^W$ define

$$O(v) := \{ \tau \in M^W : \tau \neq v, \tau = x_v z \text{ for some } z \in W_{L_v}, \text{ and } R(\tau) = R(z) \}.$$  

Lemma 3.13. If $v$ and $\tau$ are distinct elements of $M^W$ then

$$C_\tau \cap B(S) \subset \overline{C_v \cap B(S)}$$

if and only if $\tau \in O(v)$.

Proof. If $v$ and $\tau$ are distinct elements of $M^W$ such that $C_\tau \cap B(S) \subseteq \overline{C_v \cap B(S)}$ then the description of $\overline{C_v \cap B(S)}$ in (3.11) implies $C_\tau \cap B(S) = x_v U_{L,v} z B / B$ for some $z \in W_L$, so $\tau = x_v z$ and $R(z) \subseteq R(\tau)$. Furthermore, by Equation (3.2) we have

$$|R(\tau)| = \dim(C_\tau \cap B(S)) = \dim(x_v U_{L,v} z B / B) = \dim(C_{L,v} \cap B_L(S_v)) = |R(z)|,$$

so $R(z) = R(\tau)$ and $\tau \in O(v)$.

Now we assume $\tau \in O(v)$, so $\tau = x_v z$ for some $z \in W_{L_v}$ and $R(\tau) = R(v)$. Given these assumptions, we have $x_v U_{L,v} z B / B \subseteq C_\tau \cap B(S)$. Our goal is to prove that this is an equality. Consider the decomposition of $\tau$ defined in Remark 3.3, namely $\tau = x_\tau w_\tau$. Since $\Delta_{L,\tau} = R(\tau) = R(z)$ we know $\Delta_{L,\tau} \subseteq N(z)$ and therefore $\Phi^+_{L,\tau} \subseteq N(z)$. This together with the fact that $z \in W_{L_v}$ implies $z(\Phi^+_{L,\tau}) \subseteq \Phi^+_{L,v}$ so $z U_{L,v} z^{-1} \subseteq U^z$. Let $u' = x_v^{-1} u x_v \in U^z$. Our assumption that $u_\tau B \in C_\tau \cap B(S)$ implies

$$z^{-1}(u')^{-1} x^{-1} \cdot S = z^{-1} u^{-1} \cdot S \in H_\Delta \Rightarrow z^{-1}(u')^{-1} \cdot S \in H_\Delta \cap 1 = H_{R(v)},$$

since $u' z \in L_v$. Therefore $u \tau B = x_v u' w_\tau B \in x_v U_{L,v} z B / B$ and we conclude $C_\tau \cap B(S) = x_v U_{L,v} z B / B \subseteq \overline{C_v \cap B(S)}$. \hfill $\Box$

Example 3.14. Building on Example 3.4 suppose that $g = \mathfrak{gl}_4(C)$ and $S = \text{diag}(1, 1, -1, -1)$. From the table in Example 3.4 we see that if $v = s_2 s_1 s_3$ then $O(v) = \{ s_2 s_1, s_2 s_3 \}$ and Lemma 3.13 implies

$$C_{s_2 s_1} \cap B(S), \ C_{s_2 s_3} \cap B(S) \subset \overline{C_{s_2 s_3 s_1} \cap B(S)}.$$  

The decomposition in Equation (3.11) becomes

$$\overline{C_{s_2 s_3 s_1} \cap B(S)} = (C_{s_2 s_3 s_1} \cap B(S)) \cup (C_{s_2 s_1} \cap B(S)) \cup (C_{s_2 s_3} \cap B(S)) \cup s_2 B / B$$

since $U_{L,e} = \{ e \}$. Note that $O(v)$ predicts which cells $C_\tau \cap B(S)$ are completely contained in $\overline{C_v \cap B(S)}$, but there may also be portions of other Hessenberg-Schubert cells contained in
The set $O(v)$ will be used in Theorem 3.16 to describe the irreducible components of $\mathcal{B}(S)$. Similar calculations show

$$C_{s_2s_3s_2} \cap \mathcal{B}(S) = (C_{s_2s_3s_2} \cap \mathcal{B}(S)) \sqcup s_2s_3s_1 B$$

and

$$C_{s_2} \cap \mathcal{B}(S) = (C_{s_2} \cap \mathcal{B}(S)) \sqcup eB.$$ 

In order to identify the irreducible components of $\mathcal{B}(S)$ we will need the following lemma.

**Lemma 3.15.** [TY05 Lemma 39.2.1] Let $M$ be an algebraic group, $K$ a parabolic subgroup of $M$, $X$ a variety with an $M$-action, and $Y$ a $K$-stable closed subset of $X$. Then the $M$ orbit $M(Y)$ is a closed subset of $X$.

Recall that $M$ acts on $\mathcal{B}(S)$, and since $M$ is a connected algebraic group, the irreducible components of $\mathcal{B}(S)$ are $M$-invariant [Hum75, §8.2, Proposition (d)]. In fact, we have the following characterization of the irreducible components of $\mathcal{B}(S)$.

**Theorem 3.16.** The irreducible components of $\mathcal{B}(S)$ are of the form

$$\mathcal{X}_v := M(C_v \cap \mathcal{B}(S))$$

for $v \in S := MW \setminus (\cup_{v \in MW} O(v))$. In particular, $\mathcal{X}_v \subset \mathcal{X}_w$ if and only if $\tau \in O(v)$.

*Proof.* Let $L = L_v$ be the standard Levi subgroup associated to $R(v)$. First, $C_v \cap \mathcal{B}(S)$ with $v \in MW$ is a closed subvariety of $\mathcal{B}(S)$ which is clearly $B_M$-invariant since both $C_v$ and $\mathcal{B}(S)$ are $B_M$-invariant. By Lemma 3.15, the $M$-orbit $\mathcal{X}_v := M(C_v \cap \mathcal{B}(S))$ is a closed subvariety of $\mathcal{B}(S)$. It must also be irreducible since $C_v \cap \mathcal{B}(S)$ is irreducible.

Next we have that

$$M(C_v \cap \mathcal{B}(S)) = M(C_v) \cap \mathcal{B}(S) = \bigcup_{y \in W_M} C_{yv} \cap \mathcal{B}(S).$$

On the other hand, using the Bruhat decomposition for $M$, Equation 3.11 implies

$$M(C_v \cap \mathcal{B}(S)) = \bigcup_{y \in W_M} U^y yx_v U_{L, w_v} w_v B / B$$

so $C_{yv} \cap \mathcal{B}(S) = U^y yx_v U_{L, w_v} w_v B / B$ for all $y \in W_M$. The previous two sentences imply

$$\mathcal{B}(S) = \bigcup_{v \in MW} \bigcup_{y \in W_M} C_{yv} \cap \mathcal{B}(S) \subseteq \bigcup_{v \in MW} \mathcal{X}_v$$

so $\mathcal{B}(S) = \bigcup_{v \in MW} \mathcal{X}_v$ is a decomposition of $\mathcal{B}(S)$ into irreducible components. This decomposition will be unique after we remove all $\mathcal{X}_v$ with $\tau \in MW$ such that $\mathcal{X}_v \subseteq \mathcal{X}_w$.

We now prove that $\mathcal{X}_v \subset \mathcal{X}_w$ if and only if $C\mathcal{X}_v \cap \mathcal{B}(S) \subset C\mathcal{X}_w \cap \mathcal{B}(S)$. It is clear that if $C\mathcal{X}_v \cap \mathcal{B}(S) \subset C\mathcal{X}_w \cap \mathcal{B}(S)$ then $\mathcal{X}_v \subset \mathcal{X}_w$. For the opposite direction, suppose $\mathcal{X}_v \subset \mathcal{X}_w$ and consider

$$gb \in C\mathcal{X}_v \cap \mathcal{B}(S) \subseteq \mathcal{X}_v \cap \mathcal{X}_w = M(C_v \cap \mathcal{B}(S)).$$

By assumption, there exists $m \in M$ so that $mgb \in C_v \cap \mathcal{B}(S)$. The description of $C_v \cap \mathcal{B}(S)$ given in Equation 3.12 implies $m \in B_M$. If not, then $m = by_1b_2$ for some $b_1, b_2 \in B_M$ and $e \neq y \in W_M$ so

$$mgb = by_1b_2 (C\mathcal{X}_v \cap \mathcal{B}(S)) \subseteq C_y.$$ 

On the other hand, $ygb \cap (C_v \cap \mathcal{B}(S)) = \emptyset$ by Equation 3.12 since $y \neq MW$, so we obtain a contradiction. Since $m \in B_M$ and $C_v \cap \mathcal{B}(S)$ is $B_M$-invariant, we conclude $gb \in C_v \cap \mathcal{B}(S)$. The description of the set $S$ and final assertion of the theorem now follows from Lemma 3.13. □
As in the statement of Theorem 3.16, we adopt the notation \( \mathcal{X}_v := M(C_v \cap \mathcal{B}(S)) \).

**Example 3.17.** Let \( g = \mathfrak{gl}_4(\mathbb{C}) \) and \( S = \text{diag}(2, 2, -1, -3) \). In this case, \( M = \langle s_1 \rangle \) and \( \mathcal{M} \) contains 12 elements. Fix \( v = s_2 s_3 s_1 s_2 s_1 \) so \( R(v) = \{ s_1, s_2 \} \), \( x_v = s_2 s_3 \), and \( w_v = s_1 s_2 s_1 = (s_1, s_2) \). In the table below, we consider the set of all \( \tau = x_v z \) for \( z \in W_L \). We compute the simple roots \( R(\tau) \) and display the corresponding element \( z \in W_L \), and \( R(z) \).

| \( \tau \in \mathcal{M} \) | \( x_v z \in W_L \) | \( R(\tau) \) | \( R(z) \) |
|---|---|---|---|
| \( s_2 s_3 s_1 s_2 s_1 \) | \( s_1 s_2 s_1 \) | \{ \( \alpha_1, \alpha_2 \) \} | \{ \( \alpha_1, \alpha_2 \) \} |
| \( s_2 s_3 s_1 s_2 \) | \( s_1 s_2 \) | \{ \( \alpha_2 \) \} | \{ \( \alpha_2 \) \} |
| \( s_2 s_3 s_2 s_1 \) | \( s_2 s_1 \) | \{ \( \alpha_1, \alpha_3 \) \} | \{ \( \alpha_1 \) \} |
| \( s_2 s_3 s_1 \) | \( s_1 \) | \{ \( \alpha_1, \alpha_3 \) \} | \{ \( \alpha_1 \) \} |
| \( s_2 s_3 s_2 \) | \( s_2 \) | \{ \( \alpha_2, \alpha_3 \) \} | \{ \( \alpha_2 \) \} |
| \( s_2 s_3 \) | \( e \) | \{ \( \alpha_3 \) \} | \{ \( \alpha_3 \) \} |

From the table, we see that \( O(v) = \{ s_2 s_3 s_1 s_2 \} \), \( \mathcal{X}_{s_2 s_3 s_1 s_2} \subseteq \mathcal{X}_v \), but this is the only \( \mathcal{X}_\tau \) such that \( \mathcal{X}_\tau \subset \mathcal{X}_v \). Doing similar computations for each \( v \in \mathcal{M} \) shows that

\[
\mathcal{S} = \{ s_2 s_3 s_1 s_2 s_1, s_2 s_3 s_2 s_1, s_2 s_3 s_1, s_2 s_3 s_2 \}
\]

so \( \mathcal{B}(S) \) has four irreducible components.

Our next two results give an explicit description of the cellular decomposition of each irreducible component \( \mathcal{X}_v \) and describe the dimensions of these components and \( \mathcal{B}(S) \) combinatorially.

**Corollary 3.18.** Suppose \( v \in \mathcal{S} \) and let \( v = x_v w_v \) be the decomposition of \( v \) defined in Remark 3.3. Then

\[
\mathcal{X}_v = \bigcup_{y \in \mathcal{M}} \bigcup_{z \in W_L} U^y y x_v U_{L, z} z B / B.
\]

Furthermore, \( \mathcal{X}_v = C_{y_0 v} \cap \mathcal{B}(S) \) where \( y_0 \in \mathcal{M} \) is the longest element.

**Proof.** Applying the Bruhat decomposition for \( M \) and using the fact that each \( \mathcal{X}_v \) is \( \mathcal{B}(S) \)-stable yields the decomposition above from the description of \( C_v \cap \mathcal{B}(S) \) in (3.11). From the proof of Theorem 3.16, we know \( C_{y_0 v} \cap \mathcal{B}(S) = U^{y_0} y_0 x_v U_{L, w_v} w_v B / B \). The cellular decomposition given above shows that \( C_{y_0 v} \cap \mathcal{B}(S) \subset \mathcal{X}_v \) for all \( v \in \mathcal{M} \), so \( C_{y_0 r} \cap \mathcal{B}(S) \subset \mathcal{X}_v \) for all \( r \in O(v) \) and \( C_{y_0 v} \cap \mathcal{B}(S) \subseteq \mathcal{X}_v \). Equality follows from Theorem 3.16 since \( \mathcal{B}(S) = \bigcup_{r \in \mathcal{M}} C_{y_0 r} \cap \mathcal{B}(S) \) is another decomposition of \( \mathcal{B}(S) \) into irreducible components.

**Corollary 3.19.** For each \( v \in \mathcal{S} \), \( \dim(\mathcal{X}_v) = \ell(y_0) + |R(v)| \) where \( y_0 \) denotes the longest element of \( W_M \). In particular,

\[
\dim(\mathcal{B}(S)) = \ell(y_0) + \max_{v \in \mathcal{S}} |R(v)|.
\]

**Proof.** The dimension of \( \mathcal{B}(S) \) will be the maximum dimension of its irreducible components. Combining Corollary 3.18 and Equation 3.2, the dimension of each irreducible component is \( \dim(\mathcal{X}_v) = \dim(C_{y_0 v} \cap \mathcal{B}(S)) = \ell(y_0) + |R(v)| \).

**Example 3.20.** Suppose \( g = \mathfrak{gl}_4(\mathbb{C}) \) and \( S = \text{diag}(1, 1, -1, -1) \). The table in Example 3.4 implies that the corresponding semisimple Hessenberg variety has three irreducible components

\[
\mathcal{B}(S) = \mathcal{X}_{s_2 s_1 s_3 s_2} \sqcup \mathcal{X}_{s_2 s_1 s_3} \sqcup \mathcal{X}_{s_2}
\]

since \( \mathcal{S} = \{ s_2 s_1 s_3 s_2, s_2 s_1 s_3, s_2 \} \). By Corollary 3.19

\[
\dim(\mathcal{B}(S)) = \dim(\mathcal{X}_{s_2 s_1 s_3}) = |\Phi_M^+| + |R(s_2 s_1 s_3)| = 2 + 2 = 4.
\]
Notice that \( \dim(B(S)) > \dim(C_{w_0} \cap B(S)) = \dim(\mathcal{X}_{s_2 s_1 s_2}) = 3 \). This shows that one cannot use the intersection of the Hessenberg variety \( B(S) \) and the big open cell \( C_{w_0} \) to compute the dimension of \( B(S) \), which is always true in the regular case.

4. The singular locus

We now prove that each of the irreducible components of \( B(S) \) described in the previous section is smooth. Our main tool for investigating these components are patches. In the first portion of this section, we use similar methods as Insko and Yong in [IY12] to analyze the local properties of each irreducible component. In the second, we describe the GKM graphs of each irreducible component and the singular locus of \( B(S) \) as subgraphs of the GKM graph of \( B(S) \). Before going further, we make the following simplifying remark.

Remark 4.1. Any two points in the same \( M \)-orbit of \( X_v \) have isomorphic tangent spaces. In addition, if the \( T \)-fixed points of \( X_v \) are smooth points then \( X_v \) is smooth [DMPS92, Lemma 4]. Combining these statements, our strategy is to prove that \( \tau B \in X_v \) is a smooth point for all \( \tau \in M \) such that \( \tau B \in X_v \).

4.1. Patches. The opposite big cell \( B_-B/B \cong \mathbb{C}^{|w_0|} \) provides an affine open neighborhood of \( eB \in B \) and we obtain an affine open neighborhood \( N_g := gB_-B/B \) of each \( gB \in B \) by translation.

Definition 4.2. Let \( X \) be a subvariety of \( B \). The affine open neighborhood, \( N_{g,X} = N_g \cap X \) of \( gB \in X \) is called a patch of \( X \) at the point \( gB \).

Given \( g \in G \) such that \( gB \in X \subseteq G/B \), we now give explicit coordinates for \( N_{g,X} \) as in as in [IY12, §3]. Consider the projection \( \pi : G \to G/B \), and let \( U^- \) denote the maximal unipotent subgroup of \( B_- \). Since \( \pi \) is a trivial fibration over \( B_-B/B \) with fiber \( B \), it admits a local section

\[
\sigma : B_-B/B \to G
\]

such that \( \sigma(B_-B/B) = U^- \subseteq G \). Similarly, \( \pi \) admits a local section

\[
\sigma_g : gB_-B/B \to G
\]

defined by \( \sigma_g = g \sigma g^{-1} \) so \( \sigma_g(N_g) = gU^- \). This provides a scheme-theoretic isomorphism \( N_g \cong gU^- \). The section \( \sigma_g \) identifies explicit coordinates for \( N_{g,X} \) by restricting \( \pi \) and \( \sigma_g \) to \( X \).

\[
(4.3) \quad N_{g,X} = N_g \cap X \cong \sigma_g(N_g \cap X) \cong \pi^{-1}(X) \cap gU^-.
\]

In particular, we view \( N_{g,X} \cong \pi^{-1}(X) \cap gU^- \) as a subscheme of \( gU^- \). The patch of \( X \) at \( gB \) can be used to investigate the local structure of \( X \). For example, the variety \( X \) is singular at \( gB \) if and only if \( N_{g,X} \) is singular at \( g \).

For the case in which \( X = X_v \) and \( g = \tau \in M \), \( N_{\tau,X_v} \) is the subscheme of \( \tau U^- \) defined by the condition that \( \tau u \in N_{\tau,X_v} \) for \( u \in U^- \) and only if \( \tau u B \in X_v \). We make use of the fact that the unipotent subgroup \( U^- \subseteq G \) can be factored as a product of root subgroups,

\[
(4.4) \quad U^- \cong U_{\gamma_1}^r U_{\gamma_2} \cdots U_{\gamma_r}
\]

where \( \Phi^- = \{ \gamma_1, \gamma_2, \ldots, \gamma_r \} \) is any ordering of the negative roots [Hum75 §28.1] in our proof below.

Theorem 4.5. The subvariety \( X_v \subseteq B \) is smooth.

Proof. Let \( L = L_v \) be the Levi subgroup associated to the subset of simple roots \( R(v) \) and write \( v = x_v w_v \) for \( w_v \in W_L \) the longest element and \( x_v \in W^L \) as in the previous section. By Corollary 3.18
τ ∈ ℳW satisfies τB ∈ ℧v if and only if τ = x_vz for some z ∈ W_L. As in the proof of Theorem 3.10 let S_v := x_v⁻¹⋅S ∈ h, which is a regular semisimple element of ℓ. We will prove
\[ \mathcal{N}_τ X_v \cong U_M⁻ × N_{z,z_B(L(S_v))} \]
where N_{z,z_B(L(S_v))} is the patch of B_L(S_v) at zB_L. Each element of N_{z,z_B(L(S_v))} is of the form zu for some u ∈ U_L such that zuB_L ∈ B_L(S_v).

First we note that τ⁻¹U_M⁻τ ⊆ U⁻ and τ⁻¹U_M⁻τ ∩ U_L⁻ = {e}. The first statement is an obvious implication of the fact that τ ∈ ℳW. To prove the second, it suffices to show that τ⁻¹(Φ_M⁻) ∩ Φ_L⁻ = ∅. If not, then there exists γ ∈ Φ_M and β ∈ Φ_L such that τ⁻¹(γ) = β. Now,
\[ w_v z_v⁻¹(γ) = w_v τ⁻¹(γ) = w_v(β) ∈ Φ⁺ \]
since w_v(Φ_L) ⊆ Φ_L⁺ by definition of w_v as the longest element of W_L. On the other hand, zw_v ∈ W_L so x_v(zw_v) ∈ ℳW by Lemma 3.6 implying w_v z_v⁻¹(x_v⁻¹(γ)) ∈ w_v⁻¹(x_v⁻¹(φ_M⁻)) ⊆ Φ⁻, a contradiction.

Given u_1 ∈ U_M and z u_2 ∈ N_{z,z_B(L(S_v))}, consider the product u_1τ u_2. We claim that u_1τ u_2 ∈ N_τ X_v. Since τ⁻¹U_M⁻τ ⊆ U⁻ we have u_1τ u_2 ∈ τ⁻¹U⁻. Furthermore, since zu_2B_L ∈ B_L(S_v), we get that τ u_2 B = x_vu_1(zu_2B_L) ∈ C_v × B(S) by the proof of Theorem 3.10. Therefore u_1τ u_2 B ∈ ℧v, and our claim follows.

Define the map
\[ \phi : U⁻ M⁻ × N_{z,z_B(L(S_v))} → N_τ X_v \]
by \( \phi(u_1, z u_2) = u_1 x_v z u_2 = u_1 τ u_2 \) for all \( u_1 ∈ U_M⁻ \) and \( z u_2 ∈ N_{z,z_B(L(S_v))} \). The paragraph above shows that \( \phi \) is well defined. If \( \phi(u_1, z u_2) = \phi(u_3, z u_4) \) for some \( u_1, u_3 ∈ U_M⁻ \) and \( z u_2, z u_4 ∈ N_{z,z_B(L(S_v))} \), then
\[ u_1 τ u_2 = u_3 τ u_4 \Rightarrow τ⁻¹ u_3⁻¹ u_1 τ = u_4 u_2⁻¹ ∈ τ⁻¹U_M⁻τ ∩ U_L⁻. \]
Since τ⁻¹U_M⁻τ ∩ U_L⁻ = {e}, this forces \( u_1 = u_3 \) and \( u_2 = u_4 \) so \( \phi \) is injective.

Now suppose τ u ∈ N_τ X_v, i.e. \( u ∈ U⁻ \) such that \( τ u B ∈ ℧v \). Using Equation 4.4,
\[ U⁻ ≜ τ⁻¹U_M⁻τ × \prod_{γ ∈ Φ⁻ \setminus τ(γ)} U_γ \]
so we may write \( u = u_1 u_2 \) where \( u_1 ∈ τ⁻¹U_M⁻τ \) and \( u_2 ∈ \prod_{γ ∈ Φ⁻ \setminus τ(γ)} U_γ \). Since \( u_1' = τ u_1⁻¹ \in U_M⁻ \) and \( τ u_2 ∈ U_L⁻ \), it follows that \( τ u B ∈ ℧v = M(C_v ∩ B(S)) \) if and only if \( τ u_2 B ∈ C_v ∩ B(S) \). Using the proof of Theorem 3.10,
\[ τ u_2 B ∈ C_v ∩ B(S) ⇔ z u_2 B ∈ U_L⁻, w_v B \Rightarrow z u_2 B ∈ B_L(S_v). \]
The above equation makes sense only if \( u_2 ∈ U_L⁻ \) and \( z u_2 ∈ N_{z,B_L(S_v)} \). Thus \( u_1', z u_2 ∈ U_M⁻ \) and \( N_{x,v}(z u_2) \) such that \( \phi(u_1', z u_2) = τ u \) and we conclude that \( \phi \) is surjective.

Finally, since \( N_{x,v}(z u_2) \) is smooth at \( z \) by Proposition 2.14 and \( U_M⁻ ≜ C^{ℓ(m)} \) it follows that \( N_{x,v} \) is smooth at \( (e, z) = τ \). \[ \square \]

We obtain the following corollary from Theorems 3.10 and 4.5.

**Corollary 4.6.** In the semisimple Hessenberg variety \( B(S) \), a nonempty intersection of any two irreducible components is singular, and the singular locus of \( B(S) \) is the union of all such intersections. Furthermore,
\[ ℧v ∩ ℧τ = M(C_v ∩ B(S) ∩ C_τ ∩ B(S)). \]
for all \( v, τ ∈ S \).
Proof. The first part of the corollary is obvious from Theorems 3.16 and 4.5. The second assertion follows from the equality
\[ M(C_v \cap B(S)) \cap M(C_\tau \cap B(S)) = M(C_v \cap B(S) \cap C_\tau \cap B(S)). \]
The right-hand side of the above equation is clearly a subset of the left. We have only to show that the left-hand side is also a subset of the right. Let \( gB \in M(C_v \cap B(S)) \cap M(C_\tau \cap B(S)) \), so there exists \( m_1, m_2 \in M \) such that \( gB = m_1g_1B = m_2g_2B \) where \( g_1B \in C_v \cap B(S) \) and \( g_2B \in C_\tau \cap B(S) \). Now,
\[ m_1^{-1}m_2g_2B = g_1B \in C_v \cap B(S). \]
It follows that \( m_1^{-1}m_2 \in B_M \) from Equation (3.12) using a similar argument as in the proof of Theorem 3.16. Therefore, \( g_1B \in m_1^{-1}m_2C_\tau \cap B(S) = C_\tau \cap B(S) \) and \( gB = m_1g_1B \in M(C_v \cap B(S)) \cap C_\tau \cap B(S) \), proving our claim. \( \square \)

The example below illustrates that regular semisimple Hessenberg varieties corresponding to the standard Hessenberg space can be singular and not pure-dimensional. This answers Tymoczko's Questions 5.2 and 5.4 from [Tym06-2] in the negative, and one can use Corollary 4.6 to identify many examples of singular and non-pure-dimensional Hessenberg varieties. More examples are included in Subsection 4.2.

Example 4.7. Consider \( B(S) \subset GL_4(\mathbb{C})/B \) for \( S = \text{diag}(1,1,-1,-1) \). In Example 3.20 we saw that \( S = \{s_2s_1s_3s_2, s_2s_1s_3, s_2\} \) and a description of the closure \( \overline{C_v \cap B(S)} \) for each \( v \in S \) was given in Example 3.14. From these, we conclude that the singular locus of \( B(S) \) is
\[ \mathcal{X}_{s_2s_1s_3s_2} \cap \mathcal{X}_{s_2s_1s_3} = M(s_2s_1s_3B) \quad \text{and} \quad \mathcal{X}_{s_2s_1s_3} \cap \mathcal{X}_{s_2} = M(s_2B) \]
by Corollary 4.6. We saw in Example 3.20 that \( B(S) \) is not pure dimensional.

4.2. GKM graphs. Every semisimple Hessenberg variety \( B(S,H) \) has an action of the maximal torus \( T \). In fact, \( B(S,H) \) is a GKM space and there are combinatorial methods available for computing the \( T \)-equivariant cohomology of these varieties (see [GKM98]; [Tym05] provides an overview of GKM theory and examples of such computations). Although our proofs above do not rely on GKM theory, it is frequently convenient to use the GKM graph (or moment graph) to help visualize examples.

We now give a description for the GKM graph of \( B(S) \) and identify the subgraph associated to each irreducible component and the singular locus. By definition, the GKM graph of a GKM space \( \mathcal{X} \) has a vertex set of \( T \)-fixed points of \( \mathcal{X} \) and edges connecting two fixed points if there exists a one-dimensional \( T \)-orbit whose closure contains these points. The following definition summarizes this information for \( \mathcal{X} = B(S) \).

Definition 4.8. The GKM graph of \( B(S) \) has vertex set \( W \) and directed edges
\[ w \xrightarrow{\gamma} v \]
for all \( v, w \in W \) such that
\begin{enumerate}
  \item \( w = s_\gamma v \) and \( \ell(v) < \ell(w) \), and
  \item \( \text{either } \gamma \in \Phi_M^+ \text{ or } w^{-1}(\gamma) \in \Delta^- \).
\end{enumerate}
We say that \( w \) is the source of the edge \( w \xrightarrow{\gamma} v \) and \( v \) is the target.

In the definition above, we identify each \( T \)-fixed point \( wB \in B \) with \( w \in W \). Each edge \( w \xrightarrow{\gamma} v \) for \( w, v \in W \) satisfying (1) corresponds to the one-dimensional \( T \)-orbit \( U_\gamma wB \subset B \) whose closure contains \( wB \) and \( vB \). Every one-dimensional \( T \)-orbit in \( B \) is of this form [CK03]
Lemma 2.2]. Therefore the GKM graph of the full flag variety is the graph with vertex set $W$ and edges satisfying property (1).

Recall that $\mathcal{B}(S)^T = B^T$ [DMPS92 Proposition 3]. Thus the vertex set of the GKM-graph for $\mathcal{B}(S)$ is also $W$. Given a 1-dimensional $T$-orbit $U_w w B$, we have

$$u^{-1} \cdot S = S + \gamma(S)x_\gamma E_\gamma$$

for all $u = \exp(x_\gamma E_\gamma) \in U_\gamma$ where $x_\gamma \in \mathbb{C}$. Therefore $u^{-1} \cdot S \in w \cdot H_\Delta$ if and only if $w^{-1}(\gamma) \in \Delta^-$ or $\gamma(S) = 0$. It follows that conditions (1) and (2) from Definition 4.8 give precisely those one-dimensional $T$-orbits in $\mathcal{B}(S)$. This confirms that the information given in Definition [4.8 is correct.

Let $w = yv$ with $y \in W_M$ and $v \in M W$ be the decomposition given in Lemma 2.7. By Lemma 2.8, $N(w^{-1}) = N(y^{-1}) \cap y N(v^{-1})$. Any edge with source vertex $w$ has label $\gamma \in N(w^{-1})$ by condition (1) and Remark 2.6. Condition (2) now implies that those labels are exactly the roots in the set

$$N(y^{-1}) \cup (y N(v^{-1}) \cap w(\Delta^-)) = N(y^{-1}) \cup (N(v^{-1}) \cap v(\Delta^-)),$$

so the number of edges with source vertex $w$ is dim($C_w \cap \mathcal{B}(S)$) by Proposition 2.11.

Example 4.9. Consider the regular semisimple Hessenberg variety $\mathcal{B}(S)$ where $S \in \mathfrak{h}$ is a regular semisimple element. In this case, $M = \{e\}$ so $\Phi_M^+ = \emptyset$. Our graph includes all edges with source vertex $w$, labeled by $\gamma \in N(w^{-1})$ such that $w^{-1}(\gamma) \in \Delta^-$. Figure 1 below shows the GKM graphs of the variety $\mathcal{B}(S)$ in $GL_3(\mathbb{C})/B$ and $SP_4(\mathbb{C})/B$.

![Figure 1. The GKM graphs of the regular semisimple variety corresponding to the standard Hessenberg space in $GL_3(\mathbb{C})/B$ (left) and in $SP_4(\mathbb{C})/B$ (right).](image)

**Proposition 4.10.** Given $v \in S$ let $L = L_v$ be the standard Levi subgroup corresponding to $R(v) \subseteq \Delta$ and $v = x_v w_v$ be the decomposition of $v$ such that $w_v \in W_L$ is the longest element and $x_v \in W^L$. The GKM graph of the irreducible component $\mathcal{X}_v$ is the induced subgraph of the GKM graph of $\mathcal{B}(S)$ corresponding to the vertices

$$V(\mathcal{X}_v) := \{y x_v z : y \in W_M, z \in W_L\}.$$

**Proof.** Our proof relies on the description of $\mathcal{X}_v$ given in Corollary 3.18. By this corollary, $w B \in \mathcal{X}_v$ if and only if $w = y x_v z$ for some $y \in W_M$ and $z \in W_L$. Thus $V(\mathcal{X}_v)$ as defined above is indeed the vertex set for the GKM graph of $\mathcal{X}_v$. It remains to show that any edge in the GKM graph of $\mathcal{B}(S)$ between two vertices in $V(\mathcal{X}_v)$ corresponds to a one-dimensional $T$-orbit in $\mathcal{X}_v$. 

Suppose $\gamma$ labels an edge between two vertices in $V(\mathcal{X}_v)$, with source vertex $w = y\tau$ for some $y \in W_M$ and $\tau = x_vz \in MW$ with $z \in WL$. Applying Lemma 2.8 twice we get

$$N(w^{-1}) = N(y^{-1}) \cup yN(\tau^{-1}) = N(y^{-1}) \cup yN(x_v^{-1}) \cup yx_vN(z^{-1}).$$

Given $\gamma \in N(w^{-1})$, $\gamma$ must be an element of either $N(y^{-1})$, $yN(x_v^{-1})$, or $yx_vN(z^{-1})$. We consider each of these three cases below.

If $\gamma \in N(y^{-1}) \subseteq \Phi^+_M$, then $U_\gamma \subset U_M$ so $U_\gamma wB \subset \mathcal{X}_v$. Now suppose that $\gamma \in yx_vN(z^{-1})$ and let $\beta = x_v^{-1}y^{-1}(\gamma)$ so $\beta \in N(z^{-1}) \subseteq \Phi^+_L$. By condition (2) in Definition 4.8 we have $z^{-1}(\beta) \in \Delta^-$ and since $z \in WL$, $z^{-1}(\beta) \in \Delta^- \cap \Phi_L = \Delta^-$. Thus $U_\beta zB_L \subseteq B_L(S_v)$ so $U_\beta \subseteq U_{L,z}$. It follows that $U_\gamma wB = yx_vU_\beta zB \subset \mathcal{X}_v$ by Corollary 3.18.

Finally, consider the case in which $\gamma \in yN(x_v^{-1})$. We have that $s_vw = ys_{y^{-1}(\gamma)}x_vz$. Since $y^{-1}(\gamma) \in N(x_v^{-1})$, $\ell(s_{y^{-1}(\gamma)}x_v) < \ell(x_v)$ by Remark 2.6. In particular, $s_vw$ cannot be written in the form $y'x_vz'$ for some $y' \in W_M$ and $z' \in WL$. Thus $s_vw \notin V(\mathcal{X}_v)$ in this case, violating our assumption that the edge labeled by $\gamma$ has a target vertex in $V(\mathcal{X}_v)$. We conclude that this case will never occur. 

We can now describe the GKM graph of the singular locus of $B(S)$.

**Corollary 4.11.** For all distinct elements $v, \tau \in S$, define

$$V(v, \tau) = V(\mathcal{X}_v) \cap V(\mathcal{X}_\tau)$$

and let $\Gamma_{v,\tau}$ be the induced subgraph of the GKM graph of $B(S)$ associated to the vertex set $V(v, \tau)$. The GKM graph of the singular locus of $B(S)$ is the union of subgraphs $\Gamma_{v,\tau}$ for all distinct $v, \tau \in S$.

**Example 4.12.** Figure 2 shows the GKM graph of $B(S) \subset GL_4(\mathbb{C})/B$ for $S = \text{diag}(1,1,-1,-1)$. The GKM graph for each of the distinct irreducible components $\mathcal{X}_v$ for $v \in S = \{s_2s_1s_3s_2, s_2s_1s_3, s_2\}$ is a different color; $\mathcal{X}_{s_2s_1s_3s_2}$ is red, $\mathcal{X}_{s_2s_1s_3}$ is blue, and $\mathcal{X}_{s_2}$ is yellow. The GKM graphs of the intersections between these irreducible components are highlighted correspondingly: $\mathcal{X}_{s_2s_1s_3s_2} \cap \mathcal{X}_{s_2s_1s_3} = M(s_2s_1s_3B)$ is violet and $\mathcal{X}_{s_2s_1s_3} \cap \mathcal{X}_{s_2} = M(s_2B)$ is green. Together, these form the GKM graph of the singular locus of $B(S)$ which was calculated in Example 4.7. To make the graph easier to read, we have suppressed the specific label of each of the vertices and the edge labels, except for the vertices in $MW = \{s_2s_1s_3s_2, s_2s_1s_3, s_2s_1, s_2s_3, s_2, e\}$.

We close this section with two more examples.

**Example 4.13.** Continuing Example 3.17, we consider the semisimple Hessenberg variety $B(S) \subseteq GL_4(\mathbb{C})/B$ corresponding to $S = \text{diag}(2,2,-1,-3)$. Instead of the full GKM graph of $B(S)$, Figure 3 displays the the GKM graph for $\bigcup_{v \in MW} C_v \cap B(S)$ (the induced subgraph corresponding to $MW$). From Example 3.17 we know

$$S = \{s_2s_3s_1s_2s_1, s_3s_2s_3s_1, s_2s_3s_2, s_2s_3s_1\}.$$

Now Proposition 4.10 and Corollary 4.11 compute the GKM graphs of $\mathcal{X}_v$ for each $v \in S$ and their intersections, respectively. The GKM graph of the intersections between the closures of the cells $C_v \cap B(S)$ for $v \in S$ are highlighted in red in Figure 3. This is the induced subgraph corresponding to the vertices $\bigcup_{v,\tau \in S} (V(v, \tau) \cap MW)$. The singular locus of $B(S)$ consists of the $M$-orbit of these intersections by Corollary 4.6.

For our last example, we consider a (non-regular) semisimple Hessenberg variety in $SP_4(\mathbb{C})/B$.

**Example 14.** Let $S = \text{diag}(1,1,-1,-1) \in h \subset sp_4(\mathbb{C})$, so $M = \{s_1\}$ and

$$MW = \{e, s_2, s_2s_1, s_2s_1s_2\}.$$
In this case, $\mathcal{S} = M^W - \{e\}$ since $C_v \cap \mathcal{B}(S) \cong \mathbb{P}^1$ for all $v \in M^W$ such that $v \neq e$. The point $eB = C_e \cap \mathcal{B}(S)$ is contained in $C_{s_3} \cap \mathcal{B}(S)$. Figure 4 shows the GKM graph of each irreducible component of $\mathcal{B}(S)$ highlighted a different color; $\mathcal{X}_{s_2s_1}$ is red, $\mathcal{X}_{s_2s_3}$ is blue, and $\mathcal{X}_{s_2}$ is yellow. The GKM graphs of the intersections of these irreducible components are highlighted correspondingly: $\mathcal{X}_{s_2s_1} \cap \mathcal{X}_{s_2}$ is violet and $\mathcal{X}_{s_2s_1} \cap \mathcal{X}_{s_2}$ is green.
5. Examples and Applications

Many of the results proved in this paper began as conjectures formed using CoCalc (Sage) to analyze patch ideals in Type A (when $G = GL_n(\mathbb{C})$) following methods pioneered by Woo and Yong for Schubert varieties and Insko and Yong and Abe, Dedieu, Galetto, and Harada for regular nilpotent Hessenberg varieties [ADGH16, IY12, WY08, WY12]. In this section we will:

- provide examples of such computations,
- describe how to apply these computational techniques to study geometric properties of other semisimple Hessenberg varieties, and
- provide an example which shows that the results of the previous two sections may fail for semisimple Hessenberg varieties that do not correspond to the standard Hessenberg space.

In this section we fix the algebraic group $G = GL_n(\mathbb{C})$. Let $H \subseteq \mathfrak{gl}_n(\mathbb{C})$ denote a Hessenberg space, and $S$ be a diagonal matrix in $\mathfrak{gl}_n(\mathbb{C})$. In this case, there exists a unique weakly increasing function $h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ with $j \leq h(j)$ for all $1 \leq j \leq n$ such that

$$H = \{A = [a_{ij}] \in \mathfrak{gl}_n(\mathbb{C}) : a_{ij} = 0 \text{ for all } i > h(j)\}.$$ 

The equation above defines a bijective correspondence between Hessenberg spaces and all weakly increasing functions $h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that $j \leq h(j)$ for all $1 \leq j \leq n$. We call any such function $h$ a Hessenberg function and denote it by $(h(1), h(2), ..., h(n))$. We will use this notation whenever it is convenient.

**Example 5.1.** When $H = H_\Delta$ is the standard Hessenberg space as in Example 2.4, the corresponding Hessenberg function is $h(i) = i + 1$ for all $1 \leq i \leq n - 1$ and $h(n) = n$, or $(2, 3, ..., n-1, n, n)$.

Using the description given in Equation (4.3), the patch of $B(S, H)$ at $wB$ is

$$\mathcal{N}_{w,B(S,H)} \cong \{wu \in wU^- : A = u^{-1}w^{-1} \cdot S \in H\}.$$ 

Let $a$ denote a generic invertible lower-triangular unipotent matrix in $U^-$, and take the coordinates of $U^-$ to be $\{x_{ij} : i > j\}$. Then $A = u^{-1}w^{-1} \cdot S \in H$ if and only if $a_{ij} = 0$ for $i > h(j)$, where each $a_{ij}$ is a polynomial function in the $x_{ij}$-variables. We define the ideal $I_{w,B(S,H)} \subset \mathbb{C}[U^-]$ to be

$$I_{w,B(S,H)} := \langle a_{ij} : i > h(j) \rangle.$$ 

If this ideal is radical, then $I_{w,B(S,H)} = I(\mathcal{N}_{w,B(S,H)})$ is called the patch ideal for $B(S, H)$ at $wB$.

**Example 5.2.** The standard Hessenberg space in $\mathfrak{gl}_4(\mathbb{C})$ corresponds to the Hessenberg function $h = (2, 3, 4, 4)$. Let $S = \text{diag}(1, 1, -1, -1)$ and $w = s_2$. The reader will recognize $B(S)$ as the
semisimple Hessenberg variety appearing in many of the examples from Section 3. To obtain the patch ideal $I_{s_2, B(S)}$, we compute,

$$A = u^{-1}w^{-1} · S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & x_{43} & x_{42} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & x_{43} & x_{42} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2x_{21} & -1 & 0 & 0 \\ 2x_{21}x_{32} & 2x_{32} & 1 & 0 \\ -2x_{21}x_{32}x_{43} + 2x_{21}x_{42} - 2x_{41} & -2x_{32}x_{43} & -2x_{43} & -1 \end{bmatrix}$$

and require that

- $a_{31} = 2x_{21}x_{32} = 0$,
- $a_{42} = -2x_{32}x_{43} = 0$, and
- $a_{41} = -2x_{21}x_{32}x_{43} + 2x_{21}x_{42} - 2x_{41} = 0$.

These vanishing conditions define the ideal

$$I_{w, B(S)} = \langle g_{41}, g_{31}, g_{42} \rangle = \langle x_{21}x_{42} - x_{41}, x_{21}x_{32}, x_{23}x_{43} \rangle.$$ 

In the equation above we have simplified the polynomial $a_{41}$ to the generator $g_{41}$ by subtracting a multiple of $a_{31}$ from $a_{41}$. One can verify that $I_{s_2, B(S)}$ is radical as these three generators form a square-free Gröbner basis for $I_{s_2, B(S)}$ [Eis95, Exercise 18.9]. Hence $I(\mathcal{N}_{s_2, B(S)}) = I_{s_2, B(S)}$ is the patch ideal for $B(S)$ at $s_2 B$.

Using the Jacobian criterion, we note that $s_2 B$ is a singularity in the patch $\mathcal{N}_{s_2, B(S)}$ because the Jacobian matrix of partial derivatives of $I_{s_2, B(S)}$ has rank 1 when evaluated at the origin, but it has rank 3 when evaluated at a generic point in $\mathcal{N}_{s_2, B(S)}$.

Computing a primary decomposition of $I_{s_2, B(S)}$, we also see that the point $s_2 B$ is contained in two irreducible components corresponding to the primary ideals $I_1 = \langle x_{32}, x_{21}x_{42} - x_{41} \rangle$ and $I_2 = \langle x_{43}, x_{31}, x_{21} \rangle$. The component $\mathcal{V}(I_1)$ corresponding to $I_1$ has dimension 4 (codimension 2), and the component $\mathcal{V}(I_2)$ corresponding to $I_2$ has dimension 3 (codimension 3). This confirms our calculations in Example 3.14 which showed

$$s_2 B \in (C_{s_2 s_3 s_1} \cap B(S)) \cap (C_{s_2} \cap B(S)) \subset \mathcal{X}_{s_2 s_3 s_1} \cap \mathcal{X}_{s_2}$$

where $\dim(\mathcal{X}_{s_2 s_3 s_1}) = 4$ and $\dim(\mathcal{X}_{s_2}) = 3$ by Corollary 3.19.

Completing similar calculations at all elements of $MW$, we see that this Hessenberg variety has singular locus: $M(s_2 B) \cup M(s_2 s_3 B)$. We therefore recover the results of Example 4.7 using explicit calculations involving the patch ideal.

When applying computational techniques to study patch ideals, one needs to show that the ideals $I = I_{w, B(S,H)}$ are radical to conclude that the schemes defined by those ideals are reduced (namely, $\mathcal{N}_{w, B(S,H)}$). We have encountered two ways of doing this in the literature:

1. If the scheme Spec $\mathbb{C}[U^-]/I$ is Gorenstein and generically reduced, then it is reduced. Thus the ideal $I$ is radical [ADGH16, Y12].
2. If $I$ has a Gröbner basis with square-free lead terms, then $I$ is radical [Eis95, Exercise 18.9].

When $H$ is the standard Hessenberg space, one can prove that (2) always holds.
Lemma 5.3. For a semisimple Hessenberg variety corresponding to the standard Hessenberg space $H_{\Delta}$ in Lie type $A$, the ideals $I_{w,B(S)}$ for each $w \in W$ all have a square-free Gröbner basis and are therefore radical. In particular, $I_{w,B(S)}$ is the patch ideal of $B(S)$ at $wB$.

Outline of the Proof. When $H$ is the standard Hessenberg space, the ideal $I_{w,B(S)}$ is generated by polynomials $a_{ij}$ for $i > h(j) = j + 1$ determined by the entries of the matrix $A = u^{-1} w^{-1} \cdot S$ for $u \in U$ a generic element.

One then uses these polynomials to obtain simplified generators $g_{ij}$ for $i > j + 1$, as in Example 5.2. Each generator $g_{ij}$ has the form $g_{ij} = c_{ij} x_{ij} \pm c_k x_{kj} x_{ik}$ for $j < k < i$ where $c_{ij}$ and $c_k$ could be zero. We order the variables $x_{ij}$ of the ring $C[U]$ by first giving preference to those furthest from diagonal and then breaking ties lexicographically with respect to the first index so $x_{k\ell} < x_{ij}$ if $i < k$. Fix the lexicographic monomial ordering determined by this total order on the variables in $C[U]$.

We now apply Buchberger’s algorithm to the generators $g_{ij}$ in order to construct a Gröbner basis of $I_{w,B(S)}$ (see [CLO15, §7, Theorem 2]). For each pair of generators $g_{ij}, g_{k\ell}$ of the patch ideal $I_{w,B(S)}$, the $S$-polynomial $S(g_{ij}, g_{k\ell})$ has remainder zero when divided by $\{g_{ij}, g_{k\ell}\}$. Thus the generators $\{g_{ij} : i > j + 1\}$ form a Gröbner basis for $I_{w,B(S)}$ that is square-free. □

While we have verified that the ideals $I_{w,B(S,H)}$ for $w \in W$ are radical for any semisimple Hessenberg variety in $GL_n(\mathbb{C})/B$ with $n \leq 5$, we do not know an argument to prove this fact more generally.

Conjecture 5.4. The ideal $I_{w,B(S,H)}$ defined above is radical for any semisimple Hessenberg variety $B(S, H)$, and is therefore the patch ideal of $B(S, H)$ at $wB$.

Once we know that $I_{w,B(S,H)}$ is the patch ideal of $B(S, H)$ at $wB$, this ideal can be used investigate the local structure of $B(S, H)$ at $wB$; such as whether or not $wB$ is a singular point. Figure 5 lists some geometric properties of semisimple Hessenberg varieties in $GL_n(\mathbb{C})/B$ for $3 \leq n \leq 4$.

| Hess. fun. | Jordan Blocks of $S$ | Singular | Irreduc. | Equidimensional |
|-----------|----------------------|----------|----------|-----------------|
| (2,3,3)   | (2,1)                | Yes      | No       | Yes             |
| (2,3,4,4) | (3,1)                | Yes      | No       | Yes             |
| (2,4,4,4) | (3,1)                | Yes      | No       | No              |
| (3,3,4,4) | (3,1)                | Yes      | No       | No              |
| (2,3,4,4) | (2,2)                | Yes      | No       | No              |
| (2,4,4,4) | (2,2)                | Yes      | No       | No              |
| (3,3,4,4) | (2,2)                | Yes      | Yes      | Yes             |
| (2,3,4,4) | (2,1,1)              | Yes      | No       | No              |
| (2,4,4,4) | (2,1,1)              | Yes      | No       | Yes             |
| (3,4,4,4) | (2,1,1)              | Yes      | Yes      | Yes             |

Figure 5. Some geometric properties of semisimple Hessenberg varieties in $GL_n(\mathbb{C})/B$ for $3 \leq n \leq 4$.

This table also shows that some semisimple Hessenberg varieties are irreducible and singular (unlike those Hessenberg varieties associated to the standard Hessenberg space), as the following computation illustrates.
Example 5.5. Let $S = \text{diag}(1, 1, -1, -1)$ and $w = s_2s_1$. The Hessenberg space corresponding to $h = (3, 4, 4, 4)$ is $H = g - g_{-\theta}$ where $\theta$ denotes the root of maximum height in $\Phi$. To obtain the polynomial generators defining the ideal $I_{wB(S, H)}$, we compute $A = (w^{-1}w^{-1}) \cdot S$ and require that
\[a_{41} = 2x_{21}x_{32}x_{43} - 2x_{21}x_{42} - 2x_{31}x_{43} = 0\]

This vanishing condition defines the ideal
\[I_{wB(S, H)} = \langle 2x_{21}x_{32}x_{43} - 2x_{21}x_{42} - 2x_{31}x_{43} \rangle\]

which is radical since it is generated by a single irreducible polynomial (as can be verified with CoCalc). The Jacobian criterion now implies that $wB = s_2s_1B$ is a singularity in the affine patch $N_{wB(S, H)}$ because the Jacobian matrix of partial derivatives has rank 0 at the origin whereas the codimension of $N_{w, H}$ in $N_{w, B}$ is 1. As $I_{wB(S, H)}$ is a primary ideal, the variety is irreducible at this singularity. Completing similar computations at each $T$-fixed point of $B(S, H)$ shows that the variety has only one irreducible component, and eight singular $T$-fixed points corresponding the elements of the set $\{s_1s_2s_3s_2, s_3s_1s_2s_1, s_1s_2s_3, s_1s_2s_1, s_2s_3s_2, s_3s_2s_1, s_2s_1, s_2s_3\}$.

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**DEPT. OF MATHEMATICS, FLORIDA GULF COAST UNIVERSITY, FORT MYERS, FL 33965**

*E-mail address: einsko@fgcu.edu*

**DEPT. OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208**

*E-mail address: martha.precup@northwestern.edu*