Nambu and the Ising Model

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1 Abstract

2021 was Yōichirō Nambu's birth centenary. To mark the occasion, we engaged in writing a historical/scientific description of his most incisive papers, to be published by WSP. This turned out to be a most demanding but also rewarding enterprise. Most papers we have chosen are world classics, but Nambu was the humblest genius we have known, and we expected to find some of his greatest unknown insights.

One was very early in his career; on September 1, 1949, the physics journal founded by H. Yukawa, Progress of Theoretical Physics, received a note.

“A Note on the Eigenvalue Problem in Crystal Statistics”.

Written in Osaka in the newly started group created for him, it contained a curious acknowledgment “The main part of the present work had been completed nearly two years ago. It is through the kindness of Professor Husimi and Mr. Syōzi of Osaka University that the author enjoys the opportunity of publishing this note.”

We think it was at the suggestion of Professor Sin-Itiro Tomonaga, that Nambu put aside his crystal work to calculate the Lamb shift, which he published independently but after Schwinger.

We find this paper and the techniques developed in it so remarkable and even relevant for today that we have decided to write this in greater detail than the one in the book.

2 Introduction

Yōichirō Nambu came back to Tokyo University after the war. Nambu was always a very modest person never boasting about his achievements, but it is clear that he must have been an outstanding undergraduate, graduating in 2.5 years since it was cut short by half a year. Upon graduation he was awarded a graduate fellowship. He came back to a devastated Tokyo and for three years

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\footnote{Y. Nambu, Prog. Theo. Phys. \textbf{5}, 1, (1950)}
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\footnote{News of the Lamb-Retherford experiment reached Japan in the September 29 1947 issue of \textit{Time Magazine}.}
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he lived in his office sleeping on his desk mainly living on potatoes. It is difficult for us to understand how hard life was, the years after the war.

The older theoretical physicists at Tokyo University were mostly engaged in research in statistical physics and condensed matter theory, and it was natural for Nambu to take up a study of the 2-dimensional Ising model which had been solved by Lars Onsager in 1944. He was well into a new formulation and solution of the model in the fall of 1947 when news reached Japan about the experimental discovery of the Lamb shift. When asked by Tomonaga to study this problem, he put his work on the Ising model aside. Two years later he took it up again and published the result.

The paper is Nambu’s formulation of the two-dimensional Ising model. Compared with Onsager’s formidable solution\cite{Onsager} which diagonalized a \((2^N \times 2^N)\), Nambu’s Ising model lives in a \(2^N\)-dimensional Hilbert space of \(N\) qubits. A four-page computation of the eigenvalues of the transfer matrix suffices to reproduce Onsager’s results! It is a remarkable unexplored aspect of this well-studied system. Nambu said at much, in his characteristically humble manner:

*Though as yet no substantial applications has been attempted, nor anything physically new has been derived, it may be hoped that it will do some profit for those who are interested in such problems.*

A little background on the Ising model and its place in physics: when asked after the war if anything new had happened in fundamental physics, Pauli replied “not much, except for Onsager’s solution of the Ising Model”.

In 1920, Wilhelm Lenz suggested “Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern”\cite{Lenz} that ferromagnetism could be explained in terms of interacting nearest-neighbor magnets which could flip in opposite directions (“umklapping”). He asked his student Ernst Ising to solve his model. Ising did find an analytical solution\cite{Ising} “Beitrag zur Theorie des Ferromagnetismus” but only on a linear lattice and found no ferromagnetic transition.

## 3 Nambu’s Crystal Statistics

In 1944 came Lars Onsager’s epochal analytic solution which inspired Nambu’s paper we now present

**The Linear Single Spin Array**

Nambu first discussed the simplest one-dimensional array of \(N\) identical particles with a different two-valued spin at each site, \(n = 1, 2, \ldots, N\), with nearest-neighbor interactions,

\cite{Onsager, Lenz, Ising}

\[\text{L. Onsager, Phys. Rev. 65, (1944) 117}\]
\[\text{W. Lenz, Physik. Z. XXI, 1920, 613-615}\]
\[\text{E. Ising, Zeits. f. Physik 31, 253 (1925)}\]
\[ P = \sum_{n=1}^{N} P_{n,n+1} = \sum_{n=1}^{N} 1 + \sigma_n \sigma_{n+1}. \]

Inspired by quantum field theory, he introduced a different fermi oscillators at each site, and commute with one another at different sites,

\[ \{a_n, a_m^\dagger\} = 1, \quad [a_n, a_m] = 0, \quad n \neq m, \]

with periodic boundary conditions. In terms of these,

\[ P = \sum_{n=1}^{N} [a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n + 2a_n^\dagger a_n a_{n+1}^\dagger a_{n+1} - 2a_n^\dagger a_n + 1]. \]

This is the conventional approach. Now comes Nambu’s fundamental observation: \( P \), as a function of quadratic combinations, is the same whether the operators at different sites commute or anticommute. In an audacious leap, Nambu suggested an alternate description of \( P \) in terms of new ladder operators

\[ \{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0; \quad \{a_n, a_m^\dagger\} = \delta_{m,n}, \]

for all \( n, m \). \( P \) now lives in a much smaller \( 2N \)-dimensional Hilbert space, rather than in a \( 2^N \)-dimensional one in the conventional approach. It should lead to the same physics.

The rest of the paper is the exploitation of this generalization, first for \( P \), then for the isotropic \( X-Y \) model, and culminating in a much simpler solution of the Ising model for both square and “screw” arrays.

Nambu wrote \( P \) in a manifestly Hermitian form,

\[ P = \sum_{n,m} \left( a_n^\dagger a_m \delta(n - m + 1) - a_n^\dagger a_m \delta(n - m) \right) + a_n^\dagger a_n a_m^\dagger a_m \delta(n - m + 1) + \text{h.c.} \]

and introduced the operator Fourier transforms,

\[ \tilde{a}_k = \frac{1}{\sqrt{N}} \sum_n a_n \eta^{-k}, \quad \tilde{a}_k^\dagger = \frac{1}{\sqrt{N}} \sum_n a_n^\dagger \eta^{-k}. \]

where \( \eta \) are the \( N \) roots of unity (\( \eta = e^{2\pi i/N} \)). \( P \) emerges as,

\[ P = \sum_{k=1}^{N} \left[ \tilde{a}_k^\dagger \tilde{a}_k \eta^{-k} - \tilde{a}_k^\dagger \tilde{a}_k + \text{c.c.} \right] + \sum_{k,l=1}^{N} \tilde{a}_k^\dagger \tilde{a}_k \tilde{a}_l^\dagger \tilde{a}_l \delta(k - l \pm 1). \]

Nambu interpreted \( P \) as Hamiltonian sum of a quadratic “kinetic” term and a quartic expression describing a “short-range” (across the sites) potential.

For large \( N \), the potential becomes insignificant and \( P \) describes “free” \( N \) Bloch spin waves with energies \( \epsilon_k = 2(\cos \frac{2\pi k}{N} - 1), \ k = 1, 2, \ldots, N \). He concluded that it was “… a good approximation when the magnetization is nearly complete (low temperature)”.

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Nambu now applies his formalism to the isotropic $X - Y$ model, where he introduces new techniques before considering the Ising model.

**The Isotropic X-Y Model**

In order to simplify the notation, Nambu replaced the ladder operators by $2N$ “real coordinates” $(a_n^† + a_n)$, $i(a_n^† - a_n)$, that is Grassmann coordinate and momentum at each site. They span an orthogonal basis in a $2N$-dimensional vector space,

$$\{x_n, x_m\} = 2\delta_{rs}, \quad n, m = 1, 2, \ldots, 2N.$$  

The permutation operator of the “Isotropic X-Y” model includes two Pauli spin matrices, $\sigma_x$ and $\sigma_y$,

$$P_{X - Y} = \sum_{n=1}^{N} (\sigma_{n,x}\sigma_{n+1,x} + \sigma_{n,y}\sigma_{n+1,y}) \equiv \sum_{n=1}^{N} (A_n + B_n).$$

The new operators $A_n$ and $B_n$ commute, except at adjacent sites where they anticommute,

$$\{A_n, B_{n\pm 1}\} = 0,$$

and obey the constraints $A_n^2 = B_n^2 = 1$. These algebraic requirements are solved by expressing $A_n$ and $B_n$ as quadratic combinations of “Nambu’s basis” coordinates $\{x_n\}$,

$$A_n = ix_{2n}x_{2n+1}, \quad B_n = ix_{2n-1}x_{2n+2}, \quad n = 1, 2, \ldots, N,$$

so that $A_n$ links adjacent sites and $B_n$ hops over three sites. For $N$ even, and periodicity, the constraints collapse into one $1 = \prod A_n = \prod B_n \equiv x$.

As in the one-spin linear case, Nambu introduced Fourier transforms

$$\tilde{x}_k = \frac{1}{\sqrt{2N}} \sum_{n=1}^{N} x_{2n}\eta^{nk}, \quad \tilde{y}_k = \frac{1}{\sqrt{2N}} \sum_{n=1}^{N} x_{2n+1}\eta^{nk},$$

for even or odd sites. They describe for each $k$ two fermion oscillators since,

$$\{\tilde{x}_k, \tilde{x}_{-l}\} = \{\tilde{y}_k, \tilde{y}_{-l}\} = \delta_{kl}, \quad \{\tilde{x}_k, \tilde{y}_{-l}\} = 0,$$

where $k, l$ run from $-N$ to $N$ in integer steps. Then

$$P_{X - Y} = -2\sum_{k=1}^{N} (\tilde{x}_k\tilde{y}_{-k} + \tilde{y}_k\tilde{x}_{-k}\eta^{2k}).$$

After some algebra,

$$P_{X - Y} = -4\sum_{k=1}^{N/2} z_k \sin \frac{2\pi k}{N},$$

where
is a sum of quadratic forms in $\tilde{x}_k$ and $\tilde{y}_l$, which is readily be diagonalized.

For each $k$, Nambu found the $(4 \times 4)$ matrix representation,

$$\tilde{x}_k = \sigma_0 \otimes \sigma_3,$$  
$$\tilde{y}_k = \sigma_0 \otimes \sigma_-, $$

that is,

$$z_k = \begin{pmatrix} 0 & 0 & \frac{e^{-2\pi ik}}{2N} & 0 \\ 0 & 0 & \frac{e^{2\pi ik}}{2N} & 0 \\ \frac{e^{-2\pi ik}}{2N} & \frac{e^{2\pi ik}}{2N} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues follow,

$$\epsilon_k = 0, 0, 1, -1, \quad \longrightarrow \quad P_{X-Y} = 4 \sum_{k=1}^{N/2} \frac{1}{N} \epsilon_k \sin \frac{2\pi k}{N},$$

but restricted by the one boundary condition $x = 1$, ($N$ even). $x$ can be expressed in terms of rotations,

$$R(\theta) \equiv e^{2\theta \sum_n x_{2n} x_{2n+1}} = \prod_{n=1}^{N/2} e^{2\theta (\tilde{x}_k \tilde{y}_k + \tilde{x}_k \tilde{y}_k)}.$$

It can be expressed as,

$$R(\theta) \equiv \prod_{n=1}^{N/2} e^{2\theta (\tilde{x}_k \tilde{y}_k + \tilde{x}_k \tilde{y}_k)} = \prod_{n=1}^{N/2} \left[ 1 + (\cos(2\theta) - 1)R_k^2 + i \sin(2\theta)R_k \right],$$

where

$$R_k \equiv i(\tilde{x}_k \tilde{y}_k + \tilde{x}_k \tilde{y}_k),$$

commutes with $z_k$ and satisfies $R_k = R_k^3$. Comparing these two expressions at $\theta = \pi/2,$

$$R(\pi/2) = x_2x_4\ldots x_{2N}x_1 = -x = \prod_{k=1}^{N/2} (1 - 2R_k^2),$$

$$R(\pi/2) = x_2x_4\ldots x_{2N}x_1 = -x.$$

The number of non-zero eigenvalues is restricted to
\[ x = 1 \rightarrow \prod_{n=1}^{N/2} (1 - 2x_k^2) = (-1)^{N/2}. \]

which completes the solution of the isotropic \( X - Y \) model. The next sections will truly highlight the power of his method applied to the two-dimensional Ising model.

**The Square Ising Model**

Nambu’s starting point is Onsager’s operator (neglecting the prefactor) which describes the square Ising model with different interaction strengths for vertical and horizontal nearest neighbors \( J \) and \( J' \),

\[ H = \exp \left( H' \sum_{n=1}^{N} s_n s_{n+1} \right) \exp \left( H^* \sum_{n} c_n \right) , \]

where \( H' = J'/kT \), and \( H^* \) is the the Kramers-Wannier\(^6\) dual of \( H = J/kT \). The spins satisfy,

\[ s_n^2 = c_n^2 = 1, \quad \{ s_n, c_n \} = 0, \]

and commute with one another at different sites.

Onsager’s *tour de force* was to determine the eigenvalues of this operator, and prove the existence of a ferromagnetic transition in the thermodynamic limit.

As he did for the \( X - Y \) model, Nambu introduced new variables,

\[ S_n = s_n s_{n+1}, \quad C_n = c_n, \]

which commute with one another except at adjacent sites,

\[ \{ S_n, C_{n\pm 1} \} = 0. \]

and boundary conditions, \( S \equiv S_1 S_2 \cdots S_N = 1, \ C \equiv C_1 C_2 \cdots C_N = \pm 1 \).

\( S_n \) and \( C_n \) are now expressed in the “Nambu basis” \( \{ x \} \),

\[ S_n = ix_{2n} x_{2n+1}, \quad C_n = ix_{2n-1} x_{2n}, \quad (1) \]

for even \( N \) with boundary conditions,

\[ C = i^N x_1 x_2 x_3 \cdots x_{2N-1} x_{2N} \equiv X, \quad S = i^N x_2 x_4 x_5 \cdots x_{2N} x_1 = -X. \]

The stage is set for Nambu’s computation of the eigenvalues and eigenfunctions of the transfer matrix,

\[^6\text{H.A. Kramers and G. H. Wannier, Phys. Rev. 60 (1941) 252}\]
\[ H = \exp \left[ iH' \sum x_{2n}x_{2n+1} \right] \exp \left[ iH^* \sum x_{2n-1}x_{2n} \right] \equiv H_2 H_1. \] 

(2)

in the \( \{x\} \) basis.

It is a product of operators,

\[ U = e^{\theta x_n x_m}, \quad n \neq m, \quad e^{\theta/2 x_n x_m} = \cos \theta + \sin \theta x_n x_m, \]

which describe a rotation by \( \theta \) in the \( x_n - x_m \) plane. \( H \) is just a rotation by an angle \( iH^* \) followed by another rotation by \( iH' \). In some basis \( \{x'\} \), \( H \) will be expressed in Jordan’s canonical form,

\[ H = \exp \left[ i \sum x'_{2n} x'_{2n+1} \gamma_n \right] \]

with eigenfunctions that satisfy \( H \Psi = \epsilon \sum \gamma_n \Psi \), \( \epsilon_n = \pm 1 \), the largest eigenfunction is simply \( H_{\text{max}} = \epsilon \sum \gamma_n \).

Using periodicity, \( x_1 = x_{2n+1} \), we obtain,

\[
\begin{align*}
H_1 &= \exp \left[ iH^* (x_1x_2 + x_3x_4 + \cdots x_{2N-1}x_{2N}) \right], \\
H_2 &= \exp \left[ iH' (x_2x_3 + x_4x_5 + \cdots x_{2N}x_1) \right],
\end{align*}
\]

so that \( H_1 \) rotates the “odd-even” pairs \( (x_{2n-1}, x_{2n}) \),

\[
H_1: \left( \begin{array}{c} x_{2n-1} \\ x_{2n} \end{array} \right) \rightarrow \left( \begin{array}{c} y_{2n-1} \\ y_{2n} \end{array} \right) = R(2iH^*) \left( \begin{array}{c} x_{2n-1} \\ x_{2n} \end{array} \right),
\]

(3)

while \( H_2 \) rotates the “even-odd” pairs \( (y_{2n}, y_{2n+1}) \),

\[
H_2: \left( \begin{array}{c} y_{2n} \\ y_{2n+1} \end{array} \right) \rightarrow \left( \begin{array}{c} z_{2n} \\ z_{2n+1} \end{array} \right) = R(2iH') \left( \begin{array}{c} y_{2n} \\ y_{2n+1} \end{array} \right)
\]

(4)

where

\[
R(2it) = \begin{pmatrix} \cos(2it) & \sin(2it) \\ -\sin(2it) & \cos(2it) \end{pmatrix} = \begin{pmatrix} \cosh(2t) & i \sinh(2t) \\ -i \sinh(2t) & \cosh(2t) \end{pmatrix}
\]

The combined action of \( H = H_2 H_1 \) amounts to a linear transformation on the original pair,

\[
\left( \begin{array}{c} x_{2n-1} \\ x_{2n} \end{array} \right) \rightarrow \left( \begin{array}{c} z_{2n-1} \\ z_{2n} \end{array} \right) \equiv \lambda \left( \begin{array}{c} x_{2n-1} \\ x_{2n} \end{array} \right),
\]

(5)

where \( \lambda \) is the eigenvalue. Define the coefficients

\[
\begin{align*}
a &= i \cosh(2H^*) \sinh(2H'), \\
b &= i \sinh(2H^*) \cosh(2H'), \\
c &= -\sinh(2H^*) \sinh(2H'), \\
d &= \cosh(2H^*) \cosh(2H').
\end{align*}
\]

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with \[ ab - cd = 0, \quad a^2 + b^2 + c^2 + d^2 = 1. \]

Nambu’s clever choice of the pairs on which \( \mathcal{H} \) acts, has reduced the characteristic equation to two equations for each \( n \),
\[
\begin{align*}
-bx_{2n-1} + (d - \lambda)x_{2n} + ax_{2n+1} + cx_{2n+2} &= 0, \\
-ax_{2n} + (d - \lambda)x_{2n+1} + bx_{2n+2} + cx_{2n-1} &= 0.
\end{align*}
\]

Nambu’s elegant solution of these equations is to introduce two matrices and an eigenfunction,
\[
A = \begin{pmatrix} a & c \\ \lambda - d & -b \end{pmatrix}, \quad B = \begin{pmatrix} b & \lambda - d \\ c & -a \end{pmatrix}, \quad \psi_n = \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix},
\]
so that the characteristic equations become one matrix equation,
\[ A\psi_{n+1} = B\psi_n, \]
resulting in a recursion relation (\( A \) is not singular),
\[ \psi_{n+1} = A^{-1}B\psi_n \equiv D\psi_n, \]
that is readily solved,
\[ \psi_{n+1} = D^n\psi_1. \]
Finally, the periodicity constraint \( \psi_{N+1} = \psi_1 \) leads to the characteristic equation,
\[ \det(1 - D^N) = 0. \tag{6} \]
It is solved by means of the “well-known” identity,
\[ 1 - D^N = \prod_{k=1}^{N}(\eta^k - D), \quad \eta = e^{2\pi i/N}, \tag{7} \]
which reduces to \( N \) equations,
\[ \det(\eta^k - D) = 0 \quad \rightarrow \quad |A\eta^k - B| = 0. \quad k = 1, 2, \ldots, N. \]
Explicitly,
\[ \begin{vmatrix} \eta^k a - b & \eta^k c - (\lambda - d) \\ \eta^k(\lambda - d) - c & -\eta^k b + a \end{vmatrix} = 0. \]
This simple quadratic equation,
\[ \lambda^2 - 2\lambda\left[d + c\cos \varphi_k\right] + 1 = 0, \quad \varphi_k = \frac{2\pi k}{N}, \]
has two solutions for each \( k \),
\[ \lambda_{k \pm} = \cosh(2\gamma_k) \pm \sinh(2\gamma_k), \quad (8) \]

with

\[ \cosh 2\gamma_k = d + c \cos \varphi_k = \cosh(2H^*) \cosh(2H') - \sinh(2H^*) \sinh(2H') \cos \varphi_k, \quad (9) \]

the same formula as Onsager’s Eq(95) of his paper:

\[ \frac{1}{2} \sum_{r=1}^{n} \gamma_{2r-1} = \frac{1}{2} \sum_{r=1}^{n} \cosh^{-1} \left[ \cosh 2H' \cosh 2H^* - \sinh 2H' \sinh 2H^* \cos((2r-1)\pi/2n)) \right], \]

with largest eigenvalue,

\[ H_{\text{max}} = \exp \left[ \sum_{k} |\gamma_k| \right]. \]

By a simple series of steps, Nambu duplicated Onsager’s result! It is a conceptual result, the Ising model realized from a 2\(N\)-dimensional Hilbert space.

Nambu also pointed out that this method applies mutatis mutandis (when necessary changes made) to certain variants of Onsager model such as the honeycomb lattice of Kodi Husimi and Itiro Syōzi\footnote{K. Husimi and I. Syōzi, Prog. Theo. Phys. V, (1950) 177}.

When he tried to apply his method to the three-dimensional case in his basis, he found that not all operators are exponentials of quadratics (i.e., rotations), some are exponentials of quartics, such as \(e^{ax_1x_2x_3x_4}\). In view of Nambu’s many prescient comments, it might be interesting to follow his path, although no analytic solution has ever been found.

**The Helical Ising Model**

In their attempt to find an analytic solution for Ising’s model, Kramers and Wannier argued in 1941 that it was simpler to describe the lattice in terms of one string of spins, lying on the the wires of an infinite solenoid, which they call the “screw lattice”. Nambu noted that “This model seems more convenient for general purposes than that used by Onsager.”

To transform Onsager’s expression into the Kramers-Wannier helical string model, Nambu rearranged the interaction as

\[
\mathcal{H} = \mathcal{H}_1 \mathcal{H}_2 = e^{iH^* x_1 x_2} e^{iH^* x_3 x_4} \ldots e^{iH^* x_2 x_3} e^{iH^* x_4 x_5} \ldots = e^{iH^* x_1 x_2} (e^{iH^* x_3 x_4} e^{iH^* x_5 x_6} e^{iH^* x_4 x_5}) \ldots, \\
= e^{iH^* x_1 x_2} \prod_{n=1}^{N} \mathcal{H}_n e^{iH^* x_{2N+1}}, \quad \mathcal{H}_n = e^{iH^* x_{2n+1} x_{2n+2}} e^{iH^* x_{2n} x_{2n+1}}.
\]

Neglecting the two boundary terms, he started from,
First step is to express a displacement operator $P$ as a product of rotations,
\[ x_n \rightarrow e^{-x_n x_{n+1} \pi/4} x_n e^{-x_n x_{n+1} \pi/4} = x_{n+1}, \]
from which
\[ \mathcal{H}_{n+1} = P \mathcal{H}_n P^{-1} = P^{n+1} \mathcal{H}_0 P^{-n-1}, \quad \mathcal{H}_0 = e^{iH' x_1 x_2} e^{iH' x_2 x_1} = \mathcal{H}_N, \]
by periodicity. The wavefunctions
\[ \Psi_n = \mathcal{H}_n \mathcal{H}_{n-1} \cdots \mathcal{H}_1 \Psi_0, \]
obey the recursion relation, “a Schrödinger equation for a discrete time variable”,
\[ \Psi_{n+1} = \mathcal{H}_{n+1} \Psi_n. \]

The modified eigenfunction
\[ \Psi'_n = P^{-n} \Psi_n. \]
also satisfies a recursion relation,
\[ \Psi'_{n+1} = \mathcal{H}_0 P^{-1} \Psi'_n \equiv A \Psi'_n, \]
but the shift operator, $\mathcal{H}_0 P^{-1}$ does not depend on $n$. Nambu calls it $A$, but we call it $F$ so as not to confuse with the matrix of the previous section. Then,
\[ \Psi'_N = F^N \Psi'_0, \quad F^N = 1, \]
since $F$ does not depend on $n$: the eigenvalues are roots of unity, $\lambda^N = 1$.

The eigenvalues are determined from the “eigenoperator” equation that Nambu had previously used (see last section),
\[ F X F^{-1} = \lambda X, \]
Its solution is expanded as a linear combination of $x_n$,
\[ X = \sum_{n=1}^{2N} \alpha_n x_n. \]
After inserting this expansion in the eigenoperator equation, the expansion is written in terms of three coefficients, $a, b, c$,
\[ X = \sum_{n=1}^{N-1} [a x_{2n-1} + b x_{2n}] \lambda^{n-1} + c x_{2N} + a x_{2N-1} \lambda^{N-1}, \]
reducing the eigenvalue operator equation to three coupled algebraic equations,
\[
\begin{align*}
\lambda^N x &= \cosh 2H^* \cosh 2H' x - i \sinh 2H^* \cosh 2H' y + i \sinh 2H' z, \\
\lambda z &= i \sinh 2H^* x + \cosh 2H' y, \\
\lambda^{N-1} y &= -i \sinh 2H' \cosh 2H^* x - \sinh 2H' \cosh 2H' y + \cosh 2H' z.
\end{align*}
\]
By eliminating the real variables \(x, y, z\) Nambu arrived at the consistency equation
\[
\lambda^{2N} + \sinh 2H^* \sinh 2H'(\lambda + \lambda^{-1}) - 2 \cosh 2H^* \cosh 2H' + \lambda^{-N} = 0,
\]
whose solution yields the eigenvalues. Setting \(\lambda = e^{2\gamma}\), it reduces to
\[
\cosh 2N\gamma = \sinh 2H^* \sinh 2H'(\lambda + \lambda^{-1}) - 2 \cosh 2H^* \cosh 2H' \cosh \gamma,
\]
to be solved for \(\gamma\). He assumes \(2\gamma = 2\gamma_0 + i\omega\), with \(\omega = k\pi/N, \ k = 1, 2, \ldots, 2N\). Comparing the real and imaginary parts yields
\[
\pm \cosh 2\Gamma = \cosh 2H^* \cosh 2H' \cosh \gamma,
\]
Since the rhs is positive, it follows that
\[
\cosh 2\Gamma = \cosh 2H^* \cosh 2H' - \sinh 2H^* \sinh 2H' \cos \omega, \quad \omega = \frac{k\pi}{N}, \ k = 1, 2, \ldots, N,
\]
which is Onsager's formula, for large \(N, N\gamma_0 \to \Gamma\).

**Additional Remarks**

In the solution for the screw lattice, Nambu emphasized a new mathematical method to solve eigenvalue problems which he had earlier used in his papers on "Third Quantization". He defined an "eigenoperator" \(X\) whose commutator with the operator of interest is proportional to itself,
\[
[H, X] = \lambda X.
\]
Stated without proof are its properties:
- \(\lambda\) is the difference of two eigenvalues, \(\lambda = E_n - E_m\).
- \(X\) transforms an eigenvector \(\Psi_m\) of \(H\) into another \(\Psi_n\) with eigenvalue \(\lambda_n = E_m + \lambda\).
- The product of the two eigenoperators \(X_2X_1\) is again an eigen operator with eigenvalue \(\lambda = \lambda_1 + \lambda_2\), transforming an eigenvector to another one.
- When \(H\) has a simple structure, a general eigenoperator \(X\) will be factorized into a product of eigenoperators
\[
X = X_1X_2 \cdots X_k, \quad \text{with eigenvalues} \ e^\lambda = e^{\lambda_1 + \lambda_2 + \cdots + \lambda_k}.
\]
4 Conclusions

Although Nambu’s computation of the Ising model seems to be a clever trick, the tremendous simplification suggests that there must be conceptual advantages as well, possibly in the symmetries at the critical point, and perhaps connections to quantum codes.

5 Acknowledgements

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