Machines as Programs

P ≠ NP

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Abstract

The Curry–Howard correspondence is often called the proofs-as-programs result. I offer a generalization of this result, something which may be called machines as programs. Utilizing this insight, I introduce two new Turing Machines called “Ceiling Machines.” The formal ingredients of these two machines are nearly identical. But there are crucial differences, splitting the two into a “Higher Ceiling Machine” and a “Lower Ceiling Machine.” A potential graph of state transitions of the Higher Ceiling Machine is then offered. This graph is termed the “canonically nondeterministic solution” or CNDS, whose accompanying problem is its own replication, i.e., the problem, “Replicate CNDS” (whose accompanying algorithm is cast in Martin–Löf type theory). I then show that while this graph can be replicated (solved) in polynomial time by a nondeterministic machine—of which the Higher Ceiling Machine is a canonical example—it cannot be solved in polynomial time by a deterministic machine, of which the Lower Ceiling Machine is also canonical. It is consequently proven that P ≠ NP.

Keywords: P versus NP; computational complexity; proof calculi; Curry–Howard correspondence; Martin–Löf type theory

1 Preamble to Proof

Professor of computer science Oded Goldreich has claimed that he will “refuse to check claims regarding the resolution of famous open problems such as P versus NP [...] unless the claim is augmented by a clear and convincing indication as to how this work succeeds where many others have failed.” As homage to Dr. Goldreich—and numerous others who, no doubt, share his sentiment—I will first say a few words on how this work succeeds where many others have failed.

According to an unofficial source, there have been 116 total “proofs” of P versus NP. Whatever the precise number may be, one thing is for certain: a great deal of theoretical “rubber” has been burnt, with little explanatory traction to show for it. Now, I obviously cannot go down the line of proposed proofs and pick out each one’s “fatal flaw” (perhaps to do so would require a nondeterministic Turing Machine!). However, I can show how my proof differs from the others dramatically.

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1“The P-versus-NP page” (GJ Woeginger)
Although I can say with no certainty that my approach is the only feasible one, I can emphasize that it is the most general possible solution to the problem. Obviously, this is a tremendous claim. And it must be backed up. First, I have to precisely formulate what exactly I mean by “general.” Certainly, I will not be able to provide a formal proof to the effect that this much is the case, but I can impart the basic intuition behind the assertion.

Consider Tarski’s “hierarchy” of metalanguages and their corresponding object languages. Recall that the definitions in a given object language $L$ are necessarily given in a metalanguage of $L$, $M$. Anything that can be said in $L$ can be said in $M$, but not the converse. From these simple notions (among others), Tarski was able to define truth in a formal system “from without,” or by invoking semantic notions such as “meaning” or “denoting” only within the metalanguage $M$ and not the object language $L$. Consider the following:

For all $x$, $\text{True}(x)$ if and only if $\phi(x)$, where the predicate $\text{True}$ does not occur in $\phi$.

A hierarchy of languages can then be seen as a sort of transitive ordering $\{M_1 > M_2 > ... > M_n > L\}$ in which the relation $>$ expresses that the first language can “talk about” the second language (and all subsequent languages) but not the converse.

This metamathematical notion perfectly captures the intuition behind supposed “generality” of solution. Naively, we can say that a given solution to a problem $S_i$ is “more general than” another $S_j$, just in case that $S_i$ is formulated in a language $L_i$ which can “talk about” the language in which $S_j$ is formulated in, $L_j$. So, when I claim that my solution to the problem is “more general” than other proposed solutions, what I mean is that I utilize just such a language $L_i$ which can “talk about” the language(s) in which the other solutions are formulated. Of course, now, we must be precise about which “languages” we are talking about exactly.

Superficially, a majority of the supposed 116 “proofs” of the problem are formulated mathematically. This is not exactly what I mean when I say “language.” Quite obviously, a great many mathematical techniques are required to solve such a problem (and I make use of many myself). What I have in mind with “language” can perhaps better be called “governing paradigm.” Forgive me if that sounds too high-flyingly philosophical. Although the term cuts straight to the issue, perhaps something like “canonical object language” would be more appropriate.

Although buzzwords like governing paradigm and canonical object language might sound forbidding or needlessly wordy, the basic idea behind it all is rather simple. Consider that in any branch of mathematics there are established ways of doing things, canonical methods and structures and so forth. Computational complexity theory is no different, neither is the theory of computation in general.

The new “paradigm” I offer I term machines as programs. This approach is inspired by the Curry–Howard correspondence, often termed the “proofs-as-programs” result.

Where a traditionally conceived Turing Machine consists of states, a state transition function, an input alphabet, etc., a machine adhering to the machines-as-programs maxim simply adjoins to such a machine a programmatic extension $\langle \mathcal{T}, A, \mathcal{L} \rangle$. As far as $\langle \mathcal{T}, A, \mathcal{L} \rangle$ is concerned, $\mathcal{T}$ holds theorems, $A$ holds states and input symbols and $\mathcal{L}$ holds all the requisite symbols for $\mathcal{T}$. Superficially, this 3-tuple appears as a logical structure, sans interpretation function. But this “programmatic extension” cannot be reduced to such a thing (neither can
it be reduced to mere model-theoretic construction). Why is this so? For this simple reason: \( \mathcal{T} \) and \( \mathcal{L} \) are not merely abstract mathematical objects, but they are stored data in a CM. More specifically, \( \mathcal{T} \) and \( \mathcal{L} \) are “activated” by input into \( A \). In this manner, programs of a CM are instances of \( A \)-transitions which conform to the stored data in \( \mathcal{T} \) and \( \mathcal{L} \).

Moreover, we can adjoin any generic \( n \)-tuple machine to such a programmatic extension, simply by expanding the \( \mathcal{L} \) of \( \langle \mathcal{T}, A, \mathcal{L} \rangle \) such that \( \mathcal{L}' = \mathcal{L} \cup M \) for any arbitrary definition of \( M \) (say \( M = \{Q, \Gamma, b, \Sigma, q_0, A, \delta\} \)). In this manner, we will build up to two different machines, one with only deterministic machines (DTM) in its expansion \( (\mathcal{L}' = \mathcal{L} \cup DTM_{i_1}, ..., DTM_{i_n}) \) and another with only nondeterministic machines in said expansion, \( (\mathcal{L}' = \mathcal{L} \cup NTM_{i_1}, ..., NTM_{i_n}) \). The former we will call the Lower Ceiling Machine and the latter the Higher Ceiling Machine.

This is machines as programs in action.

This conceptual innovation—allowing machines to accept “expansions” of other machines—is crucial to establishing the language \( L_i \), corresponding to the machines-as-programs approach, as the “most general” manner of attack in the Tarskian sense discussed above. This “innovation” centers on one crucial goal (among many others)—a solution to the dilemma first proposed by Baker, Gill, and Solovay in 1975.

In their paper, “Relativizations of the P \( \neq \) NP Question,” these gentlemen discovered something troubling for computational complexity theorists: so-called ordinary diagonalization methods of proof, when taken to apply equally as well to two relative classes, would imply a non-solution to all such “relativized” P \( \neq \) NP questions. Baker, Gill and Solovay went on to claim that “[our] results suggest that the study of natural, specific decision problems offers a greater chance of success in showing P \( \neq \) NP than constructions of a more general nature.” This was a massive blow to hopeful computational complexity theorists everywhere. It seemed that—whatever technical tools were utilized to finally prove the problem—a sort of “hunt and peck” method would need to be utilized. Thankfully, I contend, a general solution to the problem is alive and well. And whatever Baker, Gill and Solovay had in mind when they mentioned “constructions of a [...] general nature,” they did not bother to consider the possibility of a general solution capable of obviating the issues associated with relativization.

Interestingly enough, a hint to the dilemma lies in Baker, Gill and Solovay’s very own paper. In their introduction, the theorists stated the following:

By slightly altering the machine model, we can obtain differing answers to the relativized question. This suggests that resolving the original question requires careful analysis of the computational power of machines. (emphasis added)

In other words, the particular machine model of any given machine is likely not as important as once thought to the solution of P \( \neq \) NP. And, indeed, my proposed general solution to the problem conforms to this intuition of Baker, Gill and Solovay. As it so happens, a higher-order analogue of formal software verification factors into my solution in a prominent way.

Where any given formal software verification problem consists of formally establishing that a given implementation satisfies a specification, an interesting “angle of attack” is to designate two machines, the so-called Higher Ceiling Machine from earlier (a nondetermi-
istic machine) and the so-called Lower Ceiling Machine (a deterministic one) and consider the former to be a specification and the latter to be an implementation. Informally:

$$\text{NTM} \Rightarrow \text{specification}$$

$$\text{DTM} \Rightarrow \text{implementation}$$

In essence, the NTM acts as a “template” or an abstracted structural description of an “ideal software.” And, if the DTM is to solve a problem characteristically solvable in polynomial time by a nondeterministic machine, it must conform to this ideal software of NTM.

In the “real world,” however, it is far from obvious what such a thing could mean. How can we even talk about the “ideal software” of a nondeterministic machine, if we have no idea what such a thing could look like, in the real world? This is an understandable qualm, but there are, no doubt, rigorous means of skirting around it. We can actually abstract over the issue of what exactly a real-world piece of NTM software could “look like,” and simply consider what sort of logical rules it would obey. Consider this a sort of formal software structuralism.

When we view HCM and LCM in this light, an interesting question to ask is this: how does this specification-implementation metaphor “translate” into talk of the respective state transitions of HCM and LCM? A deeper question still is this: if LCM cannot quite “conform” to the specification that is HCM (as could be expected as a nondeterministic machine could posses fundamentally “different” software than a deterministic one), what ramifications would this have on the ability of LCM to reproduce the state transitions of HCM as input query? Although the technical details of this portion of the proof are quite involved, I can offer a good bit of intuition behind the specific maneuvers.

Here is perhaps the most important “trick”—fashion a solution whose leading algorithm replicated it. But, without context, this likely sounds inscrutably obscure, so let me parse this. First, let us consider the following:

$$\{(a^\sigma \times a^q)^i \times (a^\sigma \times a^q)^j \times \cdots (a^\sigma \times a^q)^n\} = \text{a polynomial-time computation produced by HCM}$$

$$\text{REP} = \text{an algorithm designed to replicate the above computation}$$

$$\{(a^\sigma \times a^q)^{i} \prec \cdots \prec (a^\sigma \times a^q)^{n-x}\}, \text{ with accompanying intermediary sets } x = \text{a solution produced by REP}$$

The first item of above, the computation produced by HCM, we call the canonically non-deterministic solution or CNDS. What we can do is associate with CNDS the decision problem or “query”, $CNDS^Q$, outputting an affirmative answer if the computation produced by HCM can be reproduced by LCM and a negative answer if it cannot be. Recall the statement from above, “a machine adhering to the machines-as-programs maxim simply adjoins to such a machine a programmatic extension $\langle T, A, L \rangle$. ” Each $T$, of both the HCM and LCM contains programs as stored in memory (or program templates).

The gist of our proof is quite simple. We simply prove that the LCM does not contain the requisite members of $T$ to carry out the algorithm REP, in order to answer the query $CNDS^Q$. And, as an LCM is proven equivalent to a deterministic machine and HCM equivalent to a nondeterministic one, we thus have an example of a query, $CNDS^Q$, which can be trivially answered by a nondeterministic machine but cannot be answered by a deterministic one (an LCM in this case). It is consequently proven that $P \neq NP$. 

4
2 Building a Ceiling Machine

Definition 1. The main formal device we will require is intuitionistic type theory, also known as Martin–Löf type theory. We will make use of the following canonical interpretation of logical constants as type formers:

\[
\begin{align*}
\bot &= \emptyset \\
\top &= 1 \\
A \lor B &= A + B \\
A \land B &= A \times B \\
A \supset B &= A \rightarrow B \\
\exists x : A. \ B &= \Sigma x : A. \ B \\
\forall x : A. \ B &= \Pi x : A. \ B
\end{align*}
\]

\(\Sigma x : A. \ B\) is the disjoint sum of the \(A\)-indexed family of types \(B\), while \(\Pi x : A. \ B\) is the cartesian product of the \(A\)-indexed family of types \(B\). Elements of \(\Sigma x : A. \ B\) are pairs \((a,b)\) and the elements of \(\Pi x : A. \ B\) are computable functions \(f\).

When we prove sentences or theorems in Martin–Löf type theory we build a construction or proof-object which “witnesses” the truth of the sentence or theorem in question. From here on, I will refer to such a construction simply as a proof-object.

Remark 1.1. In what follows, we will make great use of the Curry–Howard correspondence. Informally, this result casts proofs as programs. Although there have been numerous syntheses of this result, all we will require for our result is Martin–Löf type theory.

However, I do not intend to rely upon only the Curry–Howard result as it is traditionally conceived. I would also like to show another possible “rendition” of this correspondence: machines as programs. No, I do not intend to reduce all machines to mere “coding.” Neither do I intend to discount the structural properties of the wonderfully diverse computing machines.

Perhaps the best way to put it is like so: the machines-as-programs result allows us to embed any well-defined machine in a programmatic extension of said machine. By “programmatic extension” I mean that every well-defined function of said “embedded” machine can be executed through the programming of another, “higher” machine. Just what is this “higher machine”? For now, we might as well call it a Ceiling Machine or CM for short.

Simplistically, a Ceiling Machine is just a 3-tuple addition, \(\langle T, A, L \rangle\) to the usual conception of a Turing Machine. As far as \(\langle T, A, L \rangle\) is concerned, \(T\) holds theorems, \(A\) holds states and input symbols and \(L\) holds all the requisite symbols for \(T\). Superficially, this 3-tuple appears as a logical structure, sans interpretation function. But this “programmatic extension” cannot be reduced to such a thing (neither can it be reduced to mere model-theoretic construction). Why is this so? For this simple reason: \(T\) and \(L\) are not merely abstract mathematical objects, but they are stored data in a CM. More specifically, \(T\) and \(L\) are “activated” by input into \(A\). In this manner, programs of a CM are instances of \(A\)-transitions which conform to the stored data in \(T\) and \(L\). This is machine as program in action.
Definition 1.2. For now, we might as well define a Ceiling Machine as the 4-tuple \( \langle T, A, L, \delta \rangle \). We said above that machines as programs “allows us to embed” any well-defined machine in a programmatic extension of said machine. Thus, we can now expand on this. Take the 4-tuple \( \langle T, A, L, \delta \rangle \) of any generic CM and expand \( L \) of CM by the symbols of the generic 6-tuple \( \{Q, \Gamma, b, \Sigma, q_0, A\} \) of machine \( M \), such that \( L' = L \cup M \). This preserves the intuition that any generic tuple like \( \{Q, \Gamma, b, \Sigma, q_0, A\} \) is really just a language whose definitions inhabit a corresponding theory \( T \).

Note that any \( M \) of the expansion \( L' = L \cup M \) of a CM is such that the transition function or relation is omitted. The transition function/relation in the extension \( \langle T, A, L, \delta \rangle \) of a CM serves as the governing function/relation for each embedded machine \( M \) of CM. This aspect of CMs will be extremely important for our main results.

Denote \( \langle T, A, L', \delta \rangle \) as a CM which has \( L' = L \cup M \) by any arbitrary tuple corresponding to \( M \). Call \( M \) in this case an embedded machine of CM. For our results, we assume that each expansion corresponds to an appropriate expansion of theory, \( T' = T \cup M \). A theory of an arbitrary \( M \) simply constrains the potential behavior of \( M \) (possible input, state transitions, etc.). The specifics behind expansion of theory are not material to the logic of the proof.

Definition 1.3. We can now expand on what exactly it means to be a program of a CM. The Curry–Howard correspondence allows us to define programs as proofs in Martin–Löf type theory. We said above that \( T \) and \( L \) are not merely abstract mathematical objects, but they are stored data in a CM. We also noted that \( T \) and \( L \) are “activated” by input into \( A \), thus entailing that, programs of a CM are instances of \( A \)-transitions which conform to the stored data in \( T \) and \( L \). This makes a proof in a CM the following: a transition of the schema \( (A \to A \to \cdots A) \) such that this string corresponds to a sequence of derivable formulae in \( T \) comprising members of \( L \). In this manner, each derivation step of any given CM program must be already stored in \( T \). Thus, for any given proof-object \( p_i \) of \( T \), \( T \) must also contain \( j_1, \ldots, j_i \) of judgements such that \( j_1, \ldots, j_i, p_i \). Call this property of CM’s \( T \) derivational closure. Note that any given \( p_i \) may correspond to multiple sets of judgements such that \( j_1, \ldots, j_i, p_i \). It is immaterial to our main results exactly which judgement sets any given \( p_i \) contains. More precisely speaking then, a proof/program in any generic CM is the following:

а string \( (a_i \to a_j \to \cdots a_n) \) of the schema \( (A \to A \to \cdots A) \),

such that sets of \( (a_i, \ldots, a_{n-x}) \) of \( (a_i \to a_j \to \cdots a - n) \) correspond to combinations of \( j \) and \( p \) of some sequence of derivations \( (j_1, \ldots, j_i, p_i) \) where each \( j \) and \( p \) of \( (j_1, \ldots, j_i, p_i) \) is a formula \( \varphi \) in CM’s \( T \) \footnote{More precisely, this is a computation. But we will refer to such computations as strings, hopefully without too much confusion.}

Definition 1.4. We stated above that \( A \) holds states and input symbols. But this is rather vague. All that was meant by this cryptic-sounding remark is that, per any CM, we have no reason to distinguish between input symbols and states of the CM. Accordingly, we can offer the following transition function for a CM:

\[ a \to a' \to \cdots a_n \to \cdots \]
\[ \delta: (A^\sigma \times A^\sigma) \to (A^\sigma \times A^\sigma) \times \{L, R\}, \]

where traditionally conceived input \( \sigma \) is accompanied by traditionally conceived state \( q \).

Thus, where we talk about a string \( (a_i \to a_j \to \cdots a_n) \) of the schema \( (A \to A \to \cdots A) \), it is more precise to talk about a string \( \{(a^\sigma \times a^\sigma)^i \to (a^\sigma \times a^\sigma)^j \to \cdots (a^\sigma \times a^\sigma)^n\} \) and thus sets of \( \{(a^\sigma \times a^\sigma)^i \cdots (a^\sigma \times a^\sigma)^{n-x}\} \) as opposed to sets of \( (a^i, \ldots, a^{n-x}) \).

**Remark 1.5.** Definition 1.4 allows us to note the following: just as programs on arbitrary machines are simply bit strings, programs on Ceiling Machines are simply combinations of input (bits) and state.

**Definition 1.6.** We can now offer a more precise definition of a Ceiling Machine. Take the 4-tuple \( \langle T, A, L, \delta \rangle \). Suppose that this 4-tuple has been expanded by the 6-tuple \( \{Q, \Gamma, b, \Sigma, q_0, A\} \) of machine \( M \), such that \( L' = L \cup M \). We now write \( \langle T, A, L', \delta \rangle \) for our CM which has \( L' = L \cup M \). Adjoin to \( \langle T, A, L', \delta \rangle \) the set \( \Gamma \) of tape alphabet symbols such that \( A \subseteq \Gamma \), the set \( b \) of blank symbols such that \( b \in \Gamma \). And adjoin also to \( \langle T, A, L', \delta \rangle \) the set \( q_0 \in A \) (the initial state) and the set \( F \subseteq A \) (the set of final states). We now have:

\[ \langle T, A, L', \Gamma, \delta \rangle, \]

where,

\[ \delta: (A^\sigma \times A^\sigma) \to (A^\sigma \times A^\sigma) \times \{L, R\}. \]

**Definition 1.7.** Now we will elaborate a bit on the specific programs of a CM. We can now make use of Martin–Löf type theory. Say that we have a Ceiling Machine \( \langle T, A, L', \delta \rangle \) which has expanded its \( L \) to accommodate a generic 6-tuple machine \( M \). And say that our CM wishes to define some property that \( M \) should have, some specific limit on its behavior. Suppose we have the following proof-object:

\[ \Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i))). \]

In everyday language, “for all input symbols \( \sigma_i \) and \( \sigma_j \) of the set \( P \) of input satisfying some property \( R^x \) there exist a \( q_i \) and a \( q_j \) of the set \( Q \) of state satisfying some property \( R^y \) and behaving such that each \( q_i \) leads to \( q_j \) or the converse.”

A program leading to this proof-object in a CM would simply be a string \( \{(a^\sigma \times a^\sigma)^i \to (a^\sigma \times a^\sigma)^j \to \cdots (a^\sigma \times a^\sigma)^n\} \) such that sets of \( \{(a^\sigma \times a^\sigma)^i \cdots (a^\sigma \times a^\sigma)^{n-x}\} \) correspond to combinations of \( j \) and \( p \) of some sequence of derivations \( (j_1, \ldots, j_{i-1}, p_i) \) where each \( j \) and \( p \) of \( (j_1, \ldots, j_{i-1}, p_i) \) is a formula \( \varphi \) in CM’s \( T \). We can call the proof-object,

\[ \Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i))), \]

and its \( (j_1, \ldots, j_{i-1}, p_i) \) an initialization program or IP for machine \( M \) on CM. Thus:
\[ \text{IP}_x \equiv \Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). \]
\[ R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i))), \]

including \((j_1, \ldots, j_{i-1})\).

**Definition 1.8.** Now that we have introduced an example of a so-called initialization program, \(\text{IP}_x\), we can offer another sort of program of a \(CM\). We will introduce a *schema program* or \(\text{SP}\). A Ceiling Machine uses an \(\text{SP}\) to “structure” the behavior of its *embedded machine* \(M\) of which its \(\mathcal{L}'\) is an expansion \(\mathcal{L}' = \mathcal{L} \cup M\). We can define the proof-object of a schema program \(\text{SP}\) as the following:

\[ \Pi \_; \mathcal{S}(\_). \Sigma.I(x, \_), \]

where “\_” denotes a variable for a relevant “spot” in the \(n\)-tuple and corresponding transition function of machine \(M\), \(\mathcal{S}\) is called a schema set and encodes the relevant structure of \(M\), “\(D\)” denotes a type former of proof-objects of declarations (variable assignments in this case), and \(x\) is any given declaration of the variable \(\_\).

An English translation of this is something like, “for every variable \(\_\) of machine \(M\!’s\) \(n\)-tuple structure, there exists a declaration of the variable, \(x\).”

Consider a \(CM\!’s\) transition function,

\[ \delta: (A^\sigma \times A^q) \rightarrow (A^\sigma \times A^q) \times \{L, R\}. \]

The program \(\text{SP}\) is simply an assignment and corresponding transition mapping,

\[ \delta: (A^\sigma \times A^q) \rightarrow (A^\sigma \times A^q) \times \{L, R\}. \]

of specific \(A^\sigma\) and \(A^q\) and a corresponding transition between these assigned \(A^\sigma\) and \(A^q\). The program \(\text{SP}\) is thus not a mere “simulation” of an embedded machine. Rather, the programs leading up to \(\text{SP}\) comprehensively *emulate* (produce) the behavior of the embedded machine. The specifics of how this process is implemented are immaterial to the logic of the proof.
Graphically, the program \( SP \) can be identified with the following blue transitions and states labeled with “\( \tau \),” specific programmatic types:

![Graphical representation of \( SP \)](image)

Such a schema program \( SP \) operates alongside classes of initialization programs \( IP \) and another distinguished classes of \( CM \) program: something we might as well call the schema configuration program or \( SCP \) for short. The \( SCP \) ensures that specific initialization programs for \( CM \)’s embedded machine \( M \) get properly “connected to” programs which lead to the proof-object of \( SP \):

\[
\Pi \_; \mathcal{G}(\_). \Sigma.I(x, \_).
\]

**Remark 1.9.** Where \((j_1, ..., j_{i-1}, p_i)\) of the proof object of \( SP \) is stored in a \( CM \)’s \( T \), the actualization or implementation of \( SP \) through \( A \)-transitions corresponds to a program performing a computation on the embedded machine \( M \). It is in this sense that the \( CM \) is not merely simulating its embedded machine \( M \), but it is literally executing its computations. \( CM \) is an emulator of its embedded machines. This is a fine point that will be of great importance in our main results.

**Remark 2.0.** An immediate analogy between \( IP \) and \( SP \) is that between a high-level programming language and a machine language. We might say, \( IP : SP :: \) (High-Level Programming Language) : (Machine Language). For instance, the proof object,

\[
\Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i)),
\]

is an instruction on \( CM \), delimiting the behavior of \( M \) (whose output the programs of \( CM \) ultimately produce). Such a proof object as \( IP_x \), in its current form, obviously cannot be recognized and executed by the machine \( M \). \( IP_x \) resides in a “domain” of a higher level than does \( M \). On the other hand, the proof-object of \( SP \),

\[
\Pi \_; \mathcal{G}(\_). \Sigma.I(x, \_).
\]
decares variables of \( M \) itself (members of its \( n \)-tuple definition). The precise data that \( SP \) declares has a direct impact on the behavior of \( M \).

Under this metaphor, the schema configuration program \( SCP \) serves as the compiler. \( SCP \) ensures that the proof object,

\[
\Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i)),
\]

“makes it” to the proof-object,

\[
\Pi \_; \mathcal{G}(\_). \Sigma.I(x, \_).
\]
in a format such that (1) the \( n \)-tuple structure and behavior of \( M \) is not impaired and, thus, (2) \( M \) “recognizes” this input from \( CM \).
Definition 2.1. We can now define the program $\mathcal{SCP}$. Take the proof-object,

$$\Pi_{\sigma_i, \sigma_j : P(\sigma_i, \sigma_j)} \cdot R^x(\sigma_i, \sigma_j) \cdot \Sigma q_i, q_j : Q(q_i, q_j) \cdot R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i)))$$

Each type former in this proof-object ($P, R^x, R^y$, etc.) is such that it adheres to a binary condition. For example, the type former $R^x$ could have the following conditions:

$$R^x = \begin{cases} 
  \text{true} & \text{if } \sigma_i \text{ is such that } \sigma_{i-1} q \sigma_i \Rightarrow M \sigma_{i} q \sigma_j \\
  \text{false} & \text{if otherwise}
\end{cases}$$

In this case, the type former $R^x$ would be of proof-objects of classes of input symbols $\sigma$ such that

$$\sigma_{i-1} q \sigma_i \Rightarrow_M \sigma_{i} q \sigma_j$$

In the case of the type former $Q$ of proof-objects of state symbols, the condition for true would simply look like so:

$$\text{true if } q_i \text{ and } q_j \text{ are members of } Q.$$

These sorts of conditions for type formers comprise the first component of the program $\mathcal{SCP}$. The second component of $\mathcal{SCP}$ is related to the specific assignments of symbols to each type former. For instance, if the condition for $R^x: \text{true}$ is met, then we would have something like:

$$\sigma_1 q \sigma_2 \Rightarrow_M \sigma_2 q \sigma_3$$

with each $\sigma_n$ denoting a specific symbol from the alphabet $\Sigma$.

The third component of program $\mathcal{SCP}$ simply ensures that the assignment of specific symbols to each type former accords with the usual notion of Tarskian satisfaction. That is, if we had the clause,

$$\neg \Sigma q_i, q_j : Q(q_i, q_j) \cdot R^y(q_i, q_j),$$

$\mathcal{SCP}$ would ensure that there were no $q_i, q_j$ meeting the condition of $R^y$. The rest of the nature of this component of $\mathcal{SCP}$ should be self-explanatory.

The fourth and final component of the program $\mathcal{SCP}$ simply ensures that the specific symbols as in the condition,

$$\sigma_1 q \sigma_2 \Rightarrow_M \sigma_2 q \sigma_3$$

are mapped to the proper portions of the so-called schema set of $M$. Recall the following proof-object of $\mathcal{SP}$:

$$\Pi_{\Sigma \mathcal{S}(\_)} \cdot \Sigma.I(x, \_).$$
A schema set is simply a data structure or a portion thereof containing all the relevant structure of the embedded machine $M$’s behavior (as according to its $n$-tuple definition). It matters not the precise implementation of such a schema set. All that matters is that the information from initialization programs for $M$, $\mathbf{IP}^M$, is properly translated into the proof-object of $\mathbf{SP}$.

According to all four specified components above, the program $\mathcal{SCP}$ may be defined on the following proof-object (that is, the precise definition of $\mathcal{SCP}$ is the proof-object $p_i$ below, in addition to all corresponding judgements: $j_1, ..., j_{i-1}, p_i$):

\[(\Pi \kappa, a. \kappa : K. a : A. C(\kappa, a)) \Rightarrow (\Pi d_1, ..., d_n. \mathcal{G}(d, \_))\]

where $\kappa$ are the conditions of type formers of $\mathbf{IP}$-program proof-objects, $a$ are the specific instances of symbols from $M$’s $n$-tuple definition, $C$ is a type former of proof-objects of checked conditions $\kappa$ and instances $a$; $d_1, ..., d_n$ are declarations of the variable $\_$ in the schema set $\mathcal{G}$ of the machine $M$.

**Remark 2.2.** Given the $\mathcal{CM} \langle T, A, L', \Gamma, \delta \rangle$ and the generic machine $M \{Q, \Gamma, b, \Sigma, q_0, A, \delta\}$, it might not be immediately apparent how the output of the two machines is defined. But the structure of the output of a $\mathcal{CM} \langle T, A, L', \Gamma, \delta \rangle$ is actually quite straightforward. All we must do is endow our $\mathcal{CM}$ with a program we may call $\text{Skip}$. And, if need be, a similar program could be defined for input.

Say, for example, that $M$ has just erased the tape symbol $\sigma_x$ and written the symbol $\sigma_y$. $\mathcal{CM}$ then stores this symbol $\sigma_y$ and its specific square on the tape $s$ in memory. Accordingly, when it comes time for $\mathcal{CM}$ to operate upon tape symbols, its behavior will be constrained by each corresponding tape symbol $\sigma$ and square $s$ it has stored in memory. Thus, if $\mathcal{CM}$ comes across a $(\sigma_i | s_i)$ pair from $M$, it will simply skip over to the next available square. A precise, inductive definition of such a program should be immediately obvious.

**Lemma 2.3.** We will now prove that any generic $\mathcal{CM}$ is Turing equivalent. This is extremely straightforward, nearly so much as to be trivial. But this lemma is crucial, for obvious reasons.

Consider an arbitrary $\mathcal{CM} \langle T, A, L', \Gamma, \delta \rangle$, such that $L' = L \cup M$ for some arbitrary $M \{Q, \Gamma, b, \Sigma, q_0, A\}$. Any program that can be run on $M$ (any graph of configurations) can also be run on $\mathcal{CM}$, as $\langle T, A, L', \Gamma, \delta \rangle$ is simply a Turing Machine with a programmatic extension $\langle T, A, L \rangle$. Recall that a $\mathcal{CM}$ has the following state transition function:

$\delta : (A^\sigma \times A^\eta) \rightarrow (A^\sigma \times A^\eta) \times \{L, R\}$

Thus, any state transition graph $q_i \rightarrow q_j \cdots \rightarrow q_n$ of a generic deterministic machine $M$ can be reproduced by a $\mathcal{CM}$ (a precise definition of a nondeterministic $\mathcal{CM}$ will be offered shortly). 

\[\blacksquare\]
Definition 2.4. We will now demarcate two sorts of Ceiling Machines: a Higher Ceiling Machine or HCM and a Lower Ceiling Machine or LCM. The former is defined as the following:

\[ \langle T, A, L', \Gamma, \delta \rangle, \]

such that \((L' = L \cup M_i, ..., M_n)\) where each \(M\) is an \(n\)-tuple definition corresponding to a nondeterministic machine (the relevant changes to this CM’s transition \(\delta\) will be defined shortly).

And the latter is defined as the 5-tuple:

\[ \langle T, A, L', \Gamma, \delta \rangle, \]

such that \((L' = L \cup M_i, ..., M_n)\) where each \(M\) is an \(n\)-tuple definition corresponding to a deterministic machine.

Remark 2.5. We now arrive at the motivation behind such phrases as “the canonically nondeterministic machine” and the “canonically deterministic machine.” Any HCM can be called a canonically nondeterministic machine, as each one of its M-expansions of \((L' = L \cup M_i, ..., M_n)\) is from a nondeterministic machine. Likewise, any LCM can be called a canonically deterministic machine, as each one of its M-expansions is from a deterministic machine.

Where HCM is a nondeterministic machine, it might not be immediately apparent that we are permitted to endow it with programs at all. It might be tempting to object to the use of the Curry–Howard correspondence in the context of an NTM equivalent. However, all we must consider is this: where we are concerned only with the sets \(\{ (a^\sigma \times a^\varphi)^i \cdots (a^i \times a^\varphi)^n \} \) of a CM—recall Definition 1.4—which correspond to formulae \(\varphi_1, ..., \varphi_n\) in its \(T\), nondeterminism does not pose a threat to our maxim of machines as programs. In the opening section of the paper, we mentioned that we need not know what hypothetical programs on a NTM “look like.” Rather, we can apply a sort of formal software structuralism and consider solely the logical rules (proof structure) which the programs conform to. In the case of a NTM, all we must do to have a program corresponding to one on a DTM is have some instance along the graph of transition relation of computation sets \(\{ (a^\sigma \times a^\varphi)^i \cdots (a^i \times a^\varphi)^n \} \) corresponding to the same (concordantly programmed) formulae \(\varphi_1, ..., \varphi_n\) in the DTM’s \(T\). The only difference with an NTM is that it can run multiple programs at the same time.

Proposition 2.6. Take any generic HCM \(\langle T, A, L', \Gamma, \delta \rangle\) with arbitrary expansion \((L' = L \cup M_i, ..., M_n)\). Single out any specific \(M\) of the set \(\{M_i, ..., M_n\}\) of HCM’s expansion. Recall the following:

We can call the proof-object,

\[ \Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^x(\sigma_i, \sigma_j) \sum q_i, q_j : Q(q_i, q_j). R^y(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i)), \]

and its \((j_1, ..., j_{i-1}, p_i)\) an initialization program or IP for machine \(M\) on CM. Thus:
\[ \text{IP}_x \equiv \Pi \sigma_i, \sigma_j : P(\sigma_i, \sigma_j). R^\sigma(\sigma_i, \sigma_j) \Sigma q_i, q_j : Q(q_i, q_j). \]
\[ \quad R^\eta(q_i, q_j) \times ((\delta(q_i, q_j) + \delta(q_j, q_i)), \]
\[ \quad \text{including } (j_1, ..., j_{i-1}). \]

Fashion a class IP consisting of arbitrary initialization programs like \( \text{IP}_x \) above. Or, to simplify things, we may simply consider that \( \text{IP}_x \) is the only initialization program (assuming that the conditions associated with each type former hold). We might say that, outside the specific initialization programs fed to \( M \) by \( \text{HCM} \), that \( M \) can be left to “make its own choice” through its finite control, beyond adhering to the members of \( \text{IP} \). In this case, it would be easy enough to adjoin to \( \text{HCM} \) the relevant program specifying as much. For now, however, consider an arbitrary class \( \text{IP} \).

Recall that the \( \text{SCP} \) program is defined like so:
\[ (\Pi \kappa, a, \kappa : K.a : A. C(\kappa, a)) \Rightarrow (\Pi d_i, ..., d_n, \mathcal{G}(d, _)), \]
\[ \text{including } (j_1, ..., j_{i-1}). \]

Also recall that the proof-object of \( \text{SP} \) is defined as the following:
\[ \Pi _\_ \mathcal{G}(_). \Sigma I(x, _). \]

Now associate with a specific \( M \) chosen from the set \( \{M_1, ..., M_n\} \) of \( \text{HCM} \)'s expansion a class \( \text{IP} \). Then, consider that the program \( \mathcal{SCP} \) ensures that the data from \( \text{IP} \) delivers to the proof object of \( \text{SP} \) declared variables in \( \mathcal{G} \).

Insert a strict partial order into the class \( \text{IP} \), such that \( \text{IP} = \{\text{IP}_1 < \text{IP}_2 < ... < \text{IP}_n\} \). Then fashion a string \( \{(a^\sigma \times a^g)^i \times (a^\sigma \times a^g)^j \times \cdots (a^\sigma \times a^g)^{n-x}\} \) of \( \text{HCM} \) such that for each \( \text{IP} \) of the set \( \text{IP} \) there is a corresponding set from \( \{(a^\sigma \times a^g)^i \cdots (a^\sigma \times a^g)^{n-x}\} \) which corresponds to a sequence of judgements \( (j_1, ..., j_{i-1}, p_i) \) where each \( j \) and \( p \) of \( (j_1, ..., j_{i-1}, p_i) \) is a formula \( \varphi \) in \( \text{HCM} \)'s \( \mathcal{T} \). Also allow room for any intermediary programs of the set \( \text{IP} = \{\text{IP}_1 < \text{IP}_2 < ... < \text{IP}_n\} \). In other words, whenever there are \( x \) such that \( (x < \text{IP}) \) or \( (\text{IP} < x) \), ensure that \( \text{HCM} \)'s string \( \{(a^\sigma \times a^g)^i \times (a^\sigma \times a^g)^j \times \cdots (a^\sigma \times a^g)^n\} \) accommodates for such \( x \).

Call this string corresponding to the set \( \text{IP} \), with any intermediary programs or computations \( x \), \( \text{String IP} \). Then, we may adjoin \( \text{String IP} \) to the program \( \mathcal{SCP} \) taking all \( \text{IP} \) to \( \Pi : \mathcal{G}(_). \Sigma I(x, _) \).

To do so, we can start by fashioning a set \( \{(a^\sigma \times a^g)^i \cdots (a^\sigma \times a^g)^{n-x}\} \) of \( \text{HCM} \) corresponding to:
\[ (\Pi \kappa, a, \kappa : K.a : A. C(\kappa, a)) \Rightarrow (\Pi d_i, ..., d_n, \mathcal{G}(d, _)), \]
\[ \text{including } (j_1, ..., j_{i-1}), \]
\[ \text{String IP}, \text{which need not be expressed in } (j_1, ..., j_{i-1}, p_i) \text{ of } \mathcal{SCP}. \]

Call this string \( \text{String SCP} \).

In addition, we can also fashion the set \( \{(a^\sigma \times a^g)^i \cdots (a^\sigma \times a^g)^{n-x}\} \) of \( \text{HCM} \) corresponding to,
\[ \Pi \Sigma(x, \_), \ \Sigma.I(x, \_), \]

including \((j_1, \ldots, j_{i-1})\),

where \((j_1, \ldots, j_{i-1})\) excludes any \((j_1, \ldots, j_{i-1}, p_i)\) already expressed in \(HCM\)'s \text{String IP} or \text{String SCp} which need not be expressed in \((j_1, \ldots, j_{i-1}, p_i)\) of \text{SP}.

Call this string \text{String SP}.

We can then form the string \(|\{\text{String IP}\} \cup \{\text{String SCp}\} \cup \{\text{String SP}\}\}||

Remark 2.7. Proposition 2.6 allows us to now introduce an important phrase from the abstract—\text{CNDS} or the \textit{canonically nondeterministic solution}. In Remark 2.5 we mentioned that any arbitrary \(HCM\) can be called a canonically nondeterministic machine, as its expansion \((L' = L \cup M_i, \ldots, M_n)\) is such that each \(M\) is nondeterministic. Accordingly, we may term the string \(|\{\text{String IP}\} \cup \{\text{String SCp}\} \cup \{\text{String SP}\}\}|| the canonically nondeterministic solution or \text{CNDS} (whose accompanying \textit{problem} will be defined shortly).

Lemma 2.8. Expanding on Remark 2.7, we will now prove that \text{CNDS} is produced by a nondeterministic Turing Machine. Lemma 2.3 shows us that any arbitrary \(CM\) (\(LCM\) or \(HCM\)) is already Turing equivalent, as any \(CM\) is just a \(TM\) with the programmatic extension \(\langle T, A, L \rangle\). Remember that any \(CM\) has a transition function identical to that of a \(TM\). However, we only previously offered a transition function for an LCM or a canonically deterministic machine:

\[ \delta: (A^\sigma \times A^q) \rightarrow (A^\sigma \times A^q) \times \{L, R\}. \]

It is easy enough to fashion a transition relation for an arbitrary HCM:

\[ \delta: (A^\sigma \times A^q) \times (A^\sigma \times A^q) \times \{L, R\}. \]

Accordingly, this makes the nondeterministic version of the schema program \text{SP} simply look like so:

\[ \delta: (A^\sigma \times A^q) \times (A^\sigma \times A^q) \times \{L, R\}. \]

an assignment of specific instances, a movement of the head and a corresponding transition relation between the assigned instances.
Graphically, nondeterministic instances of SP can be identified with the following blue transitions and states labeled with “τ,” specific programmatic types:

![Diagram of transitions and states labeled with τ]

Thus, it becomes immediately clear that a HCM is just a nondeterministic TM with the extension \( \langle T, A, L \rangle \), which is solely built from the transition relation \( (A^σ \times A^q) \times (A^σ \times A^q) \times \{L, R\} \). CNDS is thus produced by a NTM equivalent.

**Remark 2.9.** We are now nearly ready for the main results. I will now briefly outline the coming steps. First, we must define a suitable problem whose answer is the solution CNDS. We must find a language \( L \) which, when fed into a CM, would allow this CM to follow the state transition path corresponding to CNDS. The obvious answer: fashion a problem which is an injunction to replicate CNDS. In this manner, if a CM accepts the language \( L \), the state transitions of CNDS would be found somewhere in the behavior of CM.

It matters not whether the CM would be able to replicate the transition path verbatim. Rather, it would suffice if each set \( \{(a^σ \times a^q)^i \cdots (a^σ \times a^q)^{n-x}\} \) were reproduced in CM, excluding any intermediary transitions. However, we will require not only a reproduction of each set \( \{(a^σ \times a^q)^i \cdots (a^σ \times a^q)^{n-x}\} \), but that each reproduced set also corresponds to the same members of \( T \) and \( L \) in CM’s memory. In what follows, we will assume this criterion.

**Definition 3.0** We will now fashion a specific problem corresponding to the solution that is CNDS. Call this problem the canonically nondeterministic problem or CNDP. We will define
CNDP on the sets \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) of CNDS. The gist of CNDP is that it necessitates an algorithm searching through LCM’s memory for instances of \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) corresponding to those from CNDS of HCM. CNDP searches LCM’s memory for such instances, and if for each set comprising \( (a^\sigma \times a^q) \) of CNDS there exists a corresponding set comprising \( (a^\sigma \times a^q) \) in HCM, CNDP’s corresponding algorithm tells the LCM to execute or run each corresponding \( (a^\sigma \times a^q) \) of the set \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) corresponding to CNDS. If the Lower Ceiling Machine does indeed have each corresponding set in its memory, the behavior of LCM is then such that \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) of CNDS are produced, alongside arbitrary intermediary sets, where \(((a^\sigma \times a^q) < x) \) or \((x < (a^\sigma \times a^q)) \), where “<” denotes a partial order corresponding to time of execution. A solution to CNDP for an LCM would thus be the following:

\[
\{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\},
\]

with the accompanying intermediary sets \( x \).

Let us now define the algorithm corresponding to CNDP. Define said algorithm as the following proof-object in conjunction with the relevant judgements \((j_1, \ldots, j_{i-1})\):

\[
(\Pi(a^\sigma \times a^q)^< : \text{CNDS}.S(a^\sigma \times a^q)^< \cdot \Sigma(a^\sigma \times a^q) : \text{LCM}.I((a^\sigma \times a^q), (a^\sigma \times a^q)) \Rightarrow (\Pi(a^\sigma \times a^q)^< : \text{LCM}.E(a^\sigma \times a^q)^<),
\]

where \( S \) is a type former of proof-objects of search statements, \( I \) is a type former of proof-objects of identity statements (equality between stored data), \( E \) is a type former of proof-objects of execute or run statements, and the superscript \( < \) is the partial order on sets \( (a^\sigma \times a^q) \); we somewhat sloppily use “\( (a^\sigma \times a^q) \)” to denote arbitrary sets of \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \).

**Proof 3.1.** We can now prove that \( P \neq NP \). Take any arbitrary \( HCM \langle \mathcal{T}, A, \mathcal{L}', \Gamma, \delta \rangle_{HCM} \) and any arbitrary \( LCM \langle \mathcal{T}, A, \mathcal{L}', \Gamma, \delta \rangle_{LCM} \). Then take the string \( \{\{\text{String IP} < \text{String GP}\} < \text{String SP}\}\} \) produced by \( HCM \). We call this string the CNDS. Lemma 2.3. shows that this string, CNDS, has been produced by a nondeterministic Turing machine. Lemma 2.3. also shows that \( LCM \) is equivalent to a deterministic Turing machine.

Thus, if we can give \( LCM \) a language \( L \) that it cannot replicate (or a program which it cannot execute!), that \( HCM \) can replicate, we can offer an instance of a problem solvable by a nondeterministic machine, yet unsolvable by a deterministic one. Such a problem would ensure that \( P \neq NP \).

Take the algorithm corresponding to CNDP, defined above as the following proof-object with relevant judgements \((j_1, \ldots, j_{i-1})\):

\[
(\Pi(a^\sigma \times a^q)^< : \text{CNDS}.S(a^\sigma \times a^q)^< \cdot \Sigma(a^\sigma \times a^q) : \text{LCM}.I((a^\sigma \times a^q), (a^\sigma \times a^q)) \Rightarrow (\Pi(a^\sigma \times a^q)^< : \text{LCM}.E(a^\sigma \times a^q)^<),
\]

where for each set of computations \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) corresponding to a program of CNDS (which is a proof, \( (j_1, \ldots, j_{i-1}, p_i) \)), LCM is such that the same computation sets \( \{(a^\sigma \times a^q)^i \cdots (a^\sigma \times a^q)^{n-x}\} \) lead to the same \((j_1, \ldots, j_{i-1}, p_i)\) in LCM’s \( \mathcal{T} \) as they do in HCM’s \( \mathcal{T} \) (and thus CNDS).
We will call this algorithm REP. We have spoken rather loosely about CNDS and its corresponding algorithm REP of CNDP above, and there is one small addendum. I have neglected something quite obvious: in order for our LCM to run the REP algorithm, it must have already been fed the input corresponding to the string CNDS. Thus, we must feed the string \( \{(a^a \times a^a)^i \times (a^a \times a^a)^{n-x}\} \) (where each symbol is fashioned into a tape symbol \( \sigma \) for input) corresponding to CNDS into our LCM before it can run the algorithm REP.

Accordingly, we define the input into our LCM—before the execution of REP—as the following: \( \{\{\text{String IP}\} \prec \{\text{String SCP}\} \prec \{\text{String SP}\}\} \) in hopes that it will produce the transitions (via the algorithm REP), \( \{(a^a \times a^a)^i \times (a^a \times a^a)^{n-x}\} \), with accompanying intermediary sets \( x \), such that the complement of all said intermediary sets is equivalent to the string that is CNDS.

More simply, suppose that we give an arbitrary LCM the string \( \{\{\text{String IP}\} \prec \{\text{String SCP}\} \prec \{\text{String SP}\}\} \) plus the query “Can this string be replicated on this machine?” or “Does the algorithm REP terminate in the accepting state?” as input. We then implement the algorithm REP on LCM, in hopes that it can reproduce the input string of CNDS as state transition (computation) and give an affirmative answer to the query. In other words, we hope that the algorithm REP can do its job.

Suppose that LCM has only one machine \( M \) in its expansion, such that \( (L' = L \cup M) \).

Recall that leading up to the proof object of SP, \( \Pi \preceq \mathcal{Q} \). \( \Sigma.I(x, \_ \_ \_) \), we have the following program:

\[
(\Pi_k.a.K : C(k, a)) \Rightarrow (\Pi d_1, ..., d_n. S(d, \_ \_)),
\]

including \( (j_1, ..., j_{i-1}) \).

We have referred to this program as SCP.

Also recall that leading up to the program SCP we have various initialization programs, comprising the class IP = \( \{\text{IP}_1 \prec \text{IP}_2 \prec ... \prec \text{IP}_n\} \). In order to more straightforwardly deduce the \( P \neq \text{NP} \) result, we can endow our HCM with a particular member of IP. We will call this member the canonical initialization program of its sole embedded nondeterministic \( M \), symbolized as IP. We will define IP like so:

\[
\Sigma q_i.q_j: Q(q_i, q_j). \Sigma \delta : S(\delta) : (\delta(q_i, q_j) \times \delta(q_x, q_j)),
\]

including \( (j_1, ..., j_{i-1}) \), where \( Q \) is the type former of proof-objects of state statements, \( S \) is the type former of proof-objects of transition relation statements, and \( q_x \) is an arbitrary state also transitioning to \( q_j \) (in addition to \( q_i \)).

---

3 More precisely, we can fashion an input CNDS\(^Q\), a decision problem which asks whether or not CNDS can be replicated. All this does is eliminate the final execution (replication) clause from the algorithm REP. In other words, if all corresponding sets could be found in memory, the query CNDS\(^Q\) would be answered. We will use this version of CNDS for our final results.

4 Of course, if we dropped the requirement that each subprogram (or set \( (j_1, ..., j_{i-1}, p_i) \)) of formulae in a CM’s T) of CNDS have the same corresponding computation sets \( \{(a^a \times a^a)^i \times \cdots \times (a^a \times a^a)^{n-x}\} \), all we would need to do is ask our LCM to replicate the set \( \{(j_1, ..., j_{i-1}, p_i)^i \times \cdots \times (j_1, ..., j_{i-1}, p_i)^{n-x}\} \) of proofs/programs encoded in HCM’s memory. It would be vacuous, in such a case, to ask an LCM to replicate only the corresponding computation sets.
We account for \( \text{IP}_C \) in the string \( \{\text{String IP}\} \cup \{\text{String SP}\} \), such that \( \text{IP}_C \in \{\text{String IP}\} \). In the context of \( \text{IP}_C \in \{\text{String IP}\} \), the program \( \mathcal{GCP} \) ensures that all members of \( \text{IP} \) “make it” to the proof object of \( \text{SP} \), \( \Pi \vdash \mathcal{G}(\_). \Sigma.I(x, \_). \). Recall that \( \text{CNDS} \) is a computation or string produced by a non-deterministic machine equivalent.

Suppose that an \( \text{LCM} \) were fed the input corresponding to the decision problem \( \text{CNDS}^Q \), i.e., the question “can the language corresponding to \( \text{CNDS} \) be replicated as a computation on \( \text{LCM} \)?” To answer this query, \( \text{LCM} \) runs the algorithm \( \text{REP} \) (the version of \( \text{REP} \) corresponding to the query \( \text{CNDS}^Q \)),

\[
(\Pi(a^n \times a^n)^{\prec}: \text{CNDS}.S(a^n \times a^n)^{\prec}. \Sigma(a^n \times a^n): \text{LCM}.I((a^n \times a^n), (a^n \times a^n)),
\]

where for each set of computations \( \{(a^n \times a^n)^i \cdots (a^n \times a^n)^{n-x}\} \) corresponding to a program of \( \text{CNDS} \) (which is a proof, \( (j_1, \ldots, j_{i-1}, p_i) \)), \( \text{LCM} \) is such that the same computation sets \( \{(a^n \times a^n)^i \cdots (a^n \times a^n)^{n-x}\} \) lead to the same \( (j_1, \ldots, j_{i-1}, p_i) \) in \( \text{LCM} \)’s \( T \) as they do in \( \text{HCM} \)’s \( T \) (and thus \( \text{CNDS} \)),

which searches the memory of \( \text{LCM} \) for programs corresponding to formulae (so-called program templates) \( (\varphi_i, \ldots, \varphi_n) \) in its theory \( T \).

However, \( \text{LCM} \) cannot possibly run the algorithm \( \text{REP} \) to the solution above. To see why this is the case we need only consider the proof object \( \Pi \vdash \mathcal{G}(\_). \Sigma.I(x, \_). \text{LCM} \). Since our \( \text{LCM} \)’s expansion consists of only one deterministic machine, its corresponding species set \( \mathcal{G} \) is tailored to the data structure of \( \text{LCM} \)’s version of \( \text{LCM} \). Accordingly, it is impossible that the proof-object,

\[
\Sigma_{q_i:q_j}.Q(q_i, q_j). \Sigma:\delta.S(\delta): (\delta(q_i, q_j) \times \delta(q_x, q_j)),
\]

exists as an executable program along the computation path to the proof-object, \( \Pi \vdash \mathcal{G}(\_). \Sigma.I(x, \_). \text{LCM} \). There are no proof-objects of transition relation statements “\( S \)” for \( \text{LCM} \) because of the definition of \( \text{LCM} \):

\[
\langle T, A, L', \Gamma, \delta, \rangle,
\]

such that \( (L' = L \cup M_1, \ldots, M_n) \) where each \( M \) is an \( n \)-tuple definition corresponding to a deterministic machine and such that \( CM \)’s \( \delta \) is a function.

The language \( L' \) of \( \text{LCM} \) is such that there is no transition relation symbol (there is only the transition function of the \( \text{LCM} \) itself!). Accordingly, there is no group of formulae \( \varphi_1, \ldots, \varphi_n \) in its theory \( T \) corresponding to a program of its implementation (and, if there were, it would be vacuous, unable to execute the program \( \text{SP} \)). This is a sort of programmatic diagonalization which clearly, does not relativize.

Recall briefly that the non-deterministic (\( \text{HCM} \)’s) version of the program \( \text{SP} \) looks like so:

\[
\delta: (A^n \times A^n) \times (A^n \times A^n) \times \{L, R\}.
\]

Also recall that the deterministic (\( \text{LCM} \)’s) version of \( \text{SP} \) looks like:

\[
\delta: (A^n \times A^n) \rightarrow (A^n \times A^n) \times \{L, R\}.
\]

From this vantage it is exceedingly clear that any attempt by an \( \text{LCM} \) traverse \( \text{CNDS} \) and thus execute the program \( \text{SP} \) begets a type error.
For more clarity, consider the following graphics:

Figure 1: This is a potential tree of state relations of an HCM. The blue arrows represent executions of the program SP. The states and accompanying transition relation at the given executions are each labeled with a “τ,” denoting that each has a corresponding type τᵢ in the programs of HCM. Where the program SP is defined on both the states and the transition itself, an LCM wishing to replicate CNDS would have to not only replicate the states, qᵢ, qⱼ, themselves but programmatically replicate their corresponding types, τᵢ, τⱼ, and the type of the transition relation itself, τⱼ₋₁, all in an executable program. The term “embedded machine” takes on a pictorial character above, where each instance of SP is identified with a transition of an embedded machine M. Whatever the particular ingredients are, for any given M, it is embedded in the transitions of its CM. Additionally, since a CM is a general-purpose programmatic extension of each of its M’s, any specifics of each M can be incorporated into its programming.
Figure 2: This is a potential graph of state transition of an \textit{LCM}. As above, each blue transition and its accompanying states labeled with “\(\tau\)” represent an execution of the program \textit{SP}. Regardless of the specific embedded machine \(M\) that an \textit{LCM} emulates, its computational path is \textit{deterministic}. This entails that each transition type \(\tau\) (blue path) corresponds to a deterministic, \textit{function} type.

We have offered an example of a language \textit{CNDS} which \textit{can} be trivially solved by a nondeterministic machine (as \textit{HCM} is the one which produced \textit{CNDS} and can thus trivially reproduce it) but cannot be solved by a deterministic machine (as \textit{LCM} cannot possess the requisite structure in its extension \(\langle T, A, L \rangle\) to enable it to execute the replication algorithm \textit{REP}). Moreover, the emulation of an arbitrary \textit{NTM} by an \textit{HCM}—as found in \textit{CNDS}—is clearly a polynomial-time algorithm. \textit{HCM} is endowed with a pre-specified template of \(M\)’s \(n\)-tuple structure, and it must simply match the relevant variables with the relevant places in this structure. As far as the program \textit{SCP} is concerned, all our \textit{HCM} must do is ensure that these assignments adhere to the usual notions of satisfaction. This amounts to an elementary constraint satisfaction problem. And, since we are permitted to define \textit{CNDS} on only a handful of \textit{IP} proof-objects, such a procedure is ensured to be in \(P\).

Although it \textit{is} the case that a \textit{DTM} can \textit{simulate} an \textit{NTM}, it is \textit{not} the case that a \textit{DTM} can \textit{execute} the computation structure (transition of state via \textit{relation}) of an \textit{NTM}. And, as we have defined the algorithm \textit{REP} on the eventual \textit{programmatic execution} or \textit{emulation} of a given embedded machine (as the proof object of \textit{SP})—as opposed to a mere \textit{simulation} of it—an \textit{LCM} cannot affirmatively answer the query \textit{CNDS}\(^Q\).

Even more succinctly: no comprehensive (applicable to all possible \textit{DTMs}) deterministic machine can follow the programmatic path of a nondeterministic machine emulator, as it computes a state transition for one of its emulated machines.

This much suffices to prove that \(P \neq \text{NP}\). \hfill \blacksquare

\textbf{Remark 3.2.} Now we will address a few possible objections to our result. When we say that our \textit{LCM} is a so-called canonical example of a deterministic machine, this entails that \textit{no possible embedded deterministic machine can be fashioned to affirmatively answer \textit{CNDS}\(^Q\)}. Remember that each embedded machine \(M\) of any arbitrary \textit{LCM} has \textit{LCM}’s transition function \textit{as its own} (see Definition 1.2). And as far as deterministic machines are \textit{classically} conceived—without an accompanying \textit{CM}—it is obvious that they are not equipped with the programmatic structure to solve \textit{CNDS}. But there is another, more potent possible sticking point: one may wonder if we could \textit{break the definition of a Lower Ceiling Machine} and endow one of its embedded machines \(M\) with a transition \textit{relation} (in hopes to enable the \textit{LCM} to affirmatively answer the query \textit{CNDS}\(^Q\)). We will address this objection below.
Remark 3.3. Consider that we had some embedded nondeterministic machine $M$ of a generic LCM (going against the definitions we have utilized). Say that we endowed the embedded machine $M$ with its own transition relation $\delta$, such that $M = \{Q, \Gamma, b, \Sigma, q_0, A, \delta\}$ (again going against previous definitions). Thus, we would try to prove that a deterministic machine (the LCM) could run along the computation path of CNDS, to the transition relation of $M$. However, this would be a problem. Recall what was said in Definition 1.2—“The transition function/relation in the extension $\langle T, A, L, \delta \rangle$ of a CM serves as the governing function/relation for each embedded machine $M$ of CM.” As a result of this, the way the program SP is defined—the program which carries out the computations of the embedded $M$—is such that even if we had some LCM able to execute the program IP$_C$ (as would trivially be the case with an embedded NTM of an arbitrary LCM),

$$\Sigma q_i, q_j: Q(q_i, q_j). \Sigma \delta: S(\delta): (\delta(q_i, q_j) \times \delta(q_x, q_j)),$$

including $(j_1, \ldots, j_{i-1})$,

such an LCM would not be able to reach the proof object,

$$\Pi \models \mathcal{S}(\_). \Sigma I(x, \_).$$

Such would entail that the transition function of LCM would execute the computation of a nondeterministic machine (no, not “simulate” an NTM, but compute along a nondeterministic path or emulate an NTM). This much is clearly a contradiction.

Recall Remark 1.9. A CM does not merely “simulate” its embedded machines. Obviously, if this were the case, it would make little difference whether the CM possessed a transition function or relation, given its corresponding embedded machines (more precisely speaking, it would no longer have embedded machines). On the contrary, the embedded machines of a CM have computations which accord with the transition function or relation of their “host machine,” CM. When a CM executes the program SP, its transition function/relation “acts as” that of its embedded machine. Although it is possible for a DTM to simulate an NTM, it is not possible for a DTM to perform the operation of state transition relation, as opposed to function.

Note also that bizarre “combined Ceiling Machines” with both deterministic and nondeterministic machines in their extension ($L' = L \cup M_1, \ldots, M_n$) are indeed permitted. In such a case, one would simply need to make the relevant distinctions between the specific means of computation at use at any given time by the Ceiling Machine (transition function or transition relation). At any rate, such oddities do not impair our main result.

Remark 3.4. As one final remark, it may be objected (sloppily!) that Rice’s theorem impairs our results. This is not the case, however. When our LCM reads the input corresponding to CNDS (the graph of computations of an HCM) it does not decide on any properties of CNDS itself. Rather, our LCM is given the query as to whether, through its own programming, it can reproduce the computations of CNDS. Our LCM simply searches its memory for programs corresponding to formulae (proofs) $\varphi_1, \ldots, \varphi_n$ of its $T$. No “semantic properties” are involved.
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Bibliography

[1] T. Baker, J. Gill, and R. Solovay. “Relativizations of the P =? Question.” In: SIAM Journal on Computing 4(4) (1975), pp. 431–442. DOI: https://doi.org/10.1137/0204037

[2] P Martin-Löf. “An intuitionistic theory of types.” In: Twenty Five Years of Constructive Type Theory. (1998). DOI: https://doi.org/10.1093/oso/9780198501275.003.0010

[3] B. Mills. Practical formal software engineering: Wanting the software you get. Cambridge University Press, 2019.

[4] A. Tarski. “The semantic conception of truth: And the foundations of semantics.” In: Philosophy and Phenomenological Research 4(3) (1944), pp. 341–376. DOI: https://doi.org/10.2307/2102968

[5] GJ Woeginger. P-versus-NP page. URL: https://www.win.tue.nl/~gwoegi/P-versus-NP.htm (Retrieved September 13, 2021).