On thermal Nieh-Yan anomaly in Weyl superfluids

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We discuss the possibility of the Nieh-Yan anomaly of the type $\nabla_\mu j^\mu = \gamma T^2 T \wedge T$ in Weyl superfluids, where $T$ is temperature and $T$ is the effective torsion. As distinct from the parameter $\Lambda$ in the conventional Nieh-Yan anomaly, the parameter $\gamma$ is dimensionless. This may suggest, that such dimensionless parameter is fundamental, being determined by the geometry, topology and number of the quantum fields. It is shown here that the analog of such term does exist in the hydrodynamics of superfluid $^3$He-A, but the dimensionless prefactor $\gamma$ differs from that, which was calculated in the frame of pure relativistic theory.

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\section{I. INTRODUCTION}

Nieh and Yan\textsuperscript{3} introduced the 4-dimensional invariant in terms of torsion and curvature:

$$N = T \wedge T + e \wedge e \wedge R. $$

(1)

This invariant can be written as

$$N = dQ, \quad Q = e \wedge T, $$

(2)

which means that $N$ is the locally exact 4-from. They suggested that this invariant contributes to the anomalous production of the chiral current:

$$\partial_\mu j^\mu = \frac{\Lambda^2}{4\pi^2} N(r,t), $$

(3)

where the parameter $\Lambda$ has dimension of relativistic mass $[\Lambda] = [M]$, which is determined by the ultraviolet energy scale.

In a nonrelativistic matter with the linear spectrum of quasiparticles at low energy, the fully relativistic description of different phenomena may emerge, such as chiral anomaly in Weyl semimetals and Weyl superfluids/superconductors, see e.g.\textsuperscript{2,4}. There were several attempts to apply the Nieh-Yan anomaly in condensed matter systems with Weyl fermions, see e.g.\textsuperscript{2,5}. However, in the nonrelativistic systems the relativistic energy cut-off is not a well defined parameter, since it can be anisotropic. Moreover, the anomalous hydrodynamics of superfluid $^3$He at zero temperature suggests that the chiral anomaly is completely exhausted by the effective gauge field, leaving no space for the gravitational Nieh-Yan anomaly.

It looks that the relativistic analogs work only for such terms in the effective action, which have the dimensionless coefficients. Example is the temperature correction to the analog of the gravitational coupling (Newton constant) in the gravitational term $\int T^2 R$ in action, where $R$ is the analog of scalar curvature. Since $[T^2] R = [M]^4$, the prefactor of this term is dimensionless, and thus can be the fundamental constant. The prefactor of this term is fully determined by the number of the fermionic and bosonic species, both in relativistic and nonrelativistic systems\textsuperscript{6}. However, this is valid only for this subdominant term. The dominant term in the Einstein action, $\int G^{-1} R$, contains the dimensionful parameter $[G] = [M]^{-2}$ and as a result is not fully reproduced in condensed matter\textsuperscript{2}.

The same situation takes with the terms describing the chiral magnetic and chiral vortical effects in Weyl superfluid $^3$He-A, where the coefficients are dimensionless. These coefficients are fundamental, being determined by symmetry and topology\textsuperscript{2}. Another example is the topological Chern-Simons terms describing the intrinsic QHE in topological insulators in odd space dimension. They are expressed in terms of elasticity tetrads $E$ with dimension $[E] = [M]$, which allows to have the topological terms of the type $E \wedge A \wedge F$ with fundamental coefficients\textsuperscript{2}.

It is quite possible that the same situation may hold for the Nieh-Yan term\textsuperscript{3}. The main term in the axial current production $\Lambda^2 (T \wedge T + e \wedge e \wedge R)$ is still not confirmed, since the ultra-violet cut-off parameter $\Lambda$ is not well defined in the nonrelativistic matter, and it can be even zero. However, the term of the type $T^2 (T \wedge T + e \wedge e \wedge R)$ has the proper dimensionality $[M]^4$, and its prefactor could be fundamental, being expressed via the topological invariant – Chern number. We discuss the possibility of the Nieh-Yan anomaly in chiral superfluid $^3$He-A with

$$\partial_\mu j^\mu = \gamma T^2 N(r,t), $$

(4)

and check whether the parameter $\gamma$ can be fundamental.
II. TEMPERATURE CORRECTION TO THE NIEH-YAN TERM

We start with the result obtained by Khaidukov and Zubkov for the temperature contribution to the chiral current using two complementary methods of regularization:

$$j^k_5 = -\frac{T^2}{24}e^{0kij}T^0_{ij}.$$  \hfill (5)

We assume that this current can be generalized to the 4-current:

$$j^\mu_5 = -\frac{T^2}{24}e^{\mu\nu\alpha\beta}e_\nu^aT^a_{\alpha\beta}.$$  \hfill (6)

Then one obtains

$$\nabla_\mu j^\mu_5 = -\frac{T^2}{48}e^{\mu\nu\alpha\beta}T_{a\mu\nu}T^a_{\alpha\beta}.$$  \hfill (7)

When the curvature $R$ is added, this becomes the Nieh-Yan term in Eq. (4), where the uncertain cut-off $\Lambda$ is substituted by the well defined temperature $T$, and the dimensionless parameter $\gamma = 1/48$:

$$\nabla_\mu j^\mu_5 = -\frac{T^2}{12}N(r, t).$$  \hfill (8)

Let us look for the possible equation in the $^3$He-A hydrodynamics, which has the structure of the thermal Nieh-Yan term in Eq.(8).

III. FROM RELATIVISTIC PHYSICS TO CHIRAL WEYL SUPERFLUID

The chiral anomaly for the chiral current leads to the anomalous production of the linear momentum $\bar{P}_{anom}$ The reason for that is that the spectral flow of chiral quasiparticles is accompanied by the spectral flow of linear momentum $\mathbf{K}$ of the Weyl point. In $^3$He-A there are two Weyl points with opposite chirality and opposite momenta, $\mathbf{K}_\pm = \pm p_F \hat{l}$, where $\hat{l}$ is the unit vector of the orbital angular momentum of the liquid. Then the anomalous production of the linear momentum from two Weyl points sum up and one has:

$$\dot{P}_{anom} = -p_F \hat{l} (\nabla_\mu j^\mu_5).$$  \hfill (9)

Thus Eq.(9) gives the temperature correction to this anomalous momentum production:

$$\dot{P}_{anom}(T) - \dot{P}_{anom}(0) = -p_F \hat{l} \nabla_\mu j^\mu_5 = p_F \hat{l} \frac{T^2}{24} N = p_F \hat{l} \frac{T^2}{48} e^{\mu\nu\alpha\beta}T_{a\mu\nu}T^a_{\alpha\beta}.$$  \hfill (10)

Let us express Eq.(10) in terms of the hydrodynamic variables of chiral superfluid. Here we ignore the superfluid velocity. Then the only hydrodynamic variable is the unit vector of the orbital momentum $\hat{l}$. The vierbein $e^i_a$ in the vicinity of the Weyl point has the form:

$$e^i_a = \begin{pmatrix} 1 & 0 \\ 0 & c_\perp \hat{m} \\ 0 & c_\perp \hat{n} \\ 0 & c_\parallel \hat{l} \end{pmatrix}, \quad a, i = 0, 1, 2, 3.$$  \hfill (11)

Here $c_\perp (\hat{m} + i\hat{n})$ is the order parameter in the $p + ip$ chiral superfluid; $\hat{l} = \hat{m} \times \hat{n}$ is the unit vector in the direction of the orbital angular momentum of Cooper pairs; $c_\parallel$ and $c_\perp$ are effective speeds of light in Weyl equation along $\hat{l}$ and in transverse directions respectively. In the weak coupling BCS theory $c_\perp \ll c_\parallel$. For the inverse vierbein $e^a_i$, we have

$$e^a_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\perp \hat{m} & c_\perp \hat{n} & c_\parallel \hat{l} \end{pmatrix}, \quad a, i = 0, 1, 2, 3.$$  \hfill (12)
According to Ref. 5, this vierbein produces the following Nieh-Yan invariant $N$, if the effect of the superfluid velocity is ignored:

$$N = \frac{2}{c^2}(\hat{l} \cdot (\partial_t \hat{l} \times (\hat{l} \cdot \nabla)\hat{l})).$$  \hspace{1cm} (13)

Assuming that Eq. (10) is applicable to $^{3}$He-A, one obtains that the temperature correction to momentum production determined by the Nieh-Yan invariant has the following form:

$$\dot{P}_{anom}(T) = \dot{P}_{anom}(0) + p_F \hat{l} \frac{T^2}{24c^2_\perp}(\hat{l} \cdot (\partial_t \hat{l} \times (\hat{l} \cdot \nabla)\hat{l})) .$$  \hspace{1cm} (14)

Consider now the following choice of $\hat{l}(r, t)$ texture:

$$\hat{l} = \hat{z} + f(z, t),$$  \hspace{1cm} (15)

where $f \perp \hat{z}$ and $|f| \ll 1$. Then leaving only the terms quadratic in $f$, and neglecting the full space derivative, which corresponds to the surface term, one obtains

$$\hat{l}(\hat{l} \cdot (\partial_t \hat{l} \times (\hat{l} \cdot \nabla)\hat{l})) = \frac{1}{2} \frac{T}{c^2} \partial_t (\hat{l} \cdot (\nabla \times \hat{l})).$$  \hspace{1cm} (16)

This suggests the following temperature correction to the anomalous momentum in $^{3}$He-A from the thermal Nieh-Yan anomaly:

$$\dot{P}_{anom}(T) = \dot{P}_{anom}(0) + T^2 \frac{p_F}{48c^2_\perp} \hat{l}(\hat{l} \cdot (\nabla \times \hat{l})),$$  \hspace{1cm} (17)

**IV. NIEH-YAN TERM FROM HYDRODYNAMICS OF CHIRAL SUPERFLUID**

On the other hand, it is known that the hydrodynamics of $^{3}$He-A experiences anomalies related to the spectral flow through the Weyl points. In particular, the hydrodynamic anomaly provides the anomalous term in the mass current, which has the following general form:

$$\dot{P}_{anom}(T) = -\frac{1}{2} C_0(T) \hat{l}(\nabla \times \hat{l}).$$  \hspace{1cm} (18)

This term exists only on the weak coupling side of the topological Lifshitz transition, where the pair of the Weyl points appear in the quasiparticle spectrum. At zero temperature, such term with the parameter $C_0(T = 0) = p^3_F/3\pi^2$ is fully determined by the axial gauge anomaly, which takes place in the Weyl materials. We are interested in the temperature correction to the anomaly, which may come from the Nieh-Yan term. According to Cross, the anomalous parameter $C_0(T)$ has the following temperature dependence at low $T \ll T_c$:

$$C_0(T) = C_0(0) - T^2 \frac{p_F}{6c^2_\perp} \left(1 + \frac{m^*}{m}\right),$$  \hspace{1cm} (19)

where $m^*$ is the effective mass of quasiparticles in the normal Fermi liquid, which differs from the bare mass $m$ of the $^{3}$He atom due to the Fermi liquid corrections. This gives the temperature correction to the anomalous current:

$$\dot{P}_{anom}(T) = \dot{P}_{anom}(0) + T^2 \left(1 + \frac{m^*}{m}\right) \frac{p_F}{12c^2_\perp} \hat{l}(\nabla \times \hat{l}).$$  \hspace{1cm} (20)

The equation (20) has the form of Eq. (17), where the temperature correction comes from the Nieh-Yan term. This suggests the existence of the $T^2$ Nieh-Yan term in superfluid $^{3}$He in the form of Eq. (4):

$$\nabla_{\mu}j_{5}^{\mu} = -\frac{T^2}{12} \left(1 + \frac{m^*}{m}\right) N(r, t).$$  \hspace{1cm} (21)

However, the prefactor $\gamma$ in this equation differs from the prefactor $\gamma = 1/12$ in Eq. (8). The factors coincide in the limit $m^*/m \rightarrow 0$, which corresponds to the inert vacuum, when the superfluid velocity can be ignored (see Sec. 10.5 in Ref. 2).
V. CONCLUSION

We discussed the thermal Nieh-Yan anomaly where the role of the ultraviolet cut-off is played by temperature, and the prefactor $\gamma$ is dimensionless. Consideration of this anomaly in the nonrelativistic chiral superfluid demonstrates that in the hydrodynamics of this liquid there is the proper term, which can be ascribed to the Nieh-Yan anomaly. However, the prefactor $\gamma$ in the anomaly equation Eq. (21) by the factor $(1 + m^*/m)$ differs from $\gamma = 1/12$ in Eq. (8), which follows from the relativistic regularization in Ref. 8. The disagreement disappears in the limit of the so-called inert vacuum, where the bare mass $m \rightarrow \infty$. The dependence of the prefactor $\gamma$ on $m^*/m$ may suggest that the prefactor is not fundamental, i.e. it may depend on the ultraviolet physics and thus on the ratio of the effective and bare masses. The further consideration of the Nieh-Yan anomaly in Weyl superfluids and semimetals is required.

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