BRST approach to Lagrangian formulation for mixed-symmetry fermionic higher-spin fields

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Abstract

We construct a Lagrangian description of irreducible half-integer higher-spin representations of the Poincare group with the corresponding Young tableaux having two rows, on a basis of the BRST approach. Starting with a description of fermionic higher-spin fields in a flat space of any dimension in terms of an auxiliary Fock space, we realize a conversion of the initial operator constraint system (constructed with respect to the relations extracting irreducible Poincare-group representations) into a first-class constraint system. For this purpose, we find auxiliary representations of the constraint subsuperalgebra containing the subsystem of second-class constraints in terms of Verma modules. We propose a universal procedure of constructing gauge-invariant Lagrangians with reducible gauge symmetries describing the dynamics of both massless and massive fermionic fields of any spin. No off-shell constraints for the fields and gauge parameters are used from the very beginning. It is shown that the space of BRST cohomologies with a vanishing ghost number is determined only by the constraints corresponding to an irreducible Poincare-group representation. To illustrate the general construction, we obtain a Lagrangian description of fermionic fields with generalized spin $(3/2,1/2)$ and $(3/2,3/2)$ on a flat background containing the complete set of auxiliary fields and gauge symmetries.

1 Introduction

The study of various aspects of higher-spin (HS) field theory has attracted a considerable attention for a long time due to the hope of discovering new possible approaches to the unification of the fundamental interactions. Higher-spin field theory is closely related to superstring theory, which operates with an infinite tower of bosonic and fermionic higher-spin fields. The problem of a covariant Lagrangian description of fields with an arbitrary spin propagating on flat \cite{11-15} and (A)dS \cite{16-27} backgrounds as well as the problem of constructing an interacting higher-spin field theory are in the permanent focus of research (for reviews and more references, see, e.g., \cite{28}). One of the attractive features of investigating higher-spin gauge theories in AdS spaces is due to a possible relation of this study to the tensionless limit of superstring theory on the $AdS_5 \times S_5$.
Ramond–Ramond background [29, 30] and the conformal $\mathcal{N} = 4$ SYM theory in the context of the AdS/CFT correspondence [31].

At present, the dynamics of totally symmetric higher-spin fields presents the most developed direction in the variety of unitary representations of the Poincare and AdS algebras [2, 3, 16, 17, 21]. To a great extent, this is caused by the fact that in a 4d space-time there is no place for mixed-symmetry irreducible representations with the exception of dual theories. In higher space-time dimensions, there appear mixed-symmetry representations determined by more than one spin-like parameters, and the problem of their field-theoretic description is not so well-developed as for totally symmetric irreps. Starting from the papers of Fierz–Pauli and Singh–Hagen [1, 2] for higher-spin field theories in the Minkowski space, it has been known that all such theories include, together with the basic fields of a given spin, also some auxiliary fields of lower spins, necessary to provide a compatibility of the Lagrangian equations of motion with the relations that determine irreducible representations of the Poincare group. Attempts to construct Lagrangian descriptions of free and interacting higher-spin field theories have resulted in consistency problems, which are not completely resolved until now.

The present work is devoted to the construction of gauge-invariant Lagrangians for both massless and massive mixed-symmetry spin-tensor fields of rank $n_1 + n_2 + \ldots + n_k$, with any integer numbers $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$ for $k = 2$ in a $d$-dimensional Minkowski space, the fields being elements of Poincare-group irreps with a Young tableaux having two rows. In the case of the Minkowski space, several approaches have been proposed to study mixed-symmetry higher-spin fields [7, 8, 11, 12]. Our approach is based on the BFV–BRST construction [33], see also the reviews [34, 35], which was initially developed for a Hamiltonian quantization of dynamical systems subject to first-class constraints. Following a tradition accepted in string theory and higher-spin field theory, we further refer to this method as the BRST method, and to the corresponding BFV charge, as the BRST operator. The application of the BRST construction to higher-spin field theory consists of three steps. First, the conditions that determine the representations with a given spin are regarded as a system of first- and second-class operator constraints in an auxiliary Fock space. Second, the system of the initial constraints is converted, with a preservation of the initial algebraic structure, into a system of first-class constraints alone in an enlarged Fock space (see [36] for the development of conversion methods), with respect to which one constructs the BRST charge. Third, the Lagrangian for a higher-spin field is constructed in terms of the BRST charge in such a way that the corresponding equations of motion reproduce the initial constraints. We emphasize that this approach automatically implies a gauge-invariant Lagrangian description reflecting the general fact of BV–BFV duality [37, 38], realized in order to reproduce a Lagrangian action or a probability amplitude by means of a Hamiltonian object.

The construction of the flat dynamics of mixed-symmetry gauge fields has been examined in [4, 5, 7, 8, 9, 11, 12], including the construction of Lagrangians in the BRST approach for massless interacting bosonic higher-spin fields with two rows of the Young tableaux [9], and recently also for interacting bosonic HS fields [39] and for those of lower spins [40] on the basis of the BV cohomological deformation theory [11]. Lagrangian descriptions of massless mixed-symmetry fermionic and bosonic higher-spin fields in the (A)dS spaces have been suggested within a “frame-like” approach in [27], whereas for massive fields of lower superspins in the flat and (A)dS spaces they have been examined in [13]. To be complete, note that for free totally symmetric higher-spin fields of integer spins the BRST approach has been used to derive Lagrangians in the flat space [8, 42] and in the (A)dS space [43]. The corresponding programme of a Lagrangian description of fermionic HS fields has been realized in the flat space [44] and in the (A)dS space [45].

In this paper, we construct a gauge-invariant Lagrangian description of fermionic HS fields

\footnote{For a detailed discussion of dual theories in various dimensions, see [11, 28, 32].}
in Minkowski space of any dimension, corresponding to a unitary irreducible Poincare-group representation with a Young tableaux having two rows of length \( n_1, n_2 \) \((n_1 \geq n_2)\).

The paper is organized as follows. In Section 2, we formulate a closed Lie superalgebra of operators, based on the constraints in an auxiliary Fock space that determines an irreducible representation of the Poincare group with a generalized spin \( s = (n_1 + 1/2, n_2 + 1/2) \). In Section 3 we construct a Verma module, being an auxiliary representation for a rank-2 subsuperalgebra of the superalgebra of the initial constraints corresponding to the subsystem of second-class constraints. This representation is then realized in terms of new (additional) creation and annihilation operators in Fock space. Note that a similar construction for bosonic HS fields in a flat space has been presented in [46]. In Section 4, we carry out a conversion of the initial system of first- and second-class constraints into a system of first-class constraints in the space being the tensor product of the initial and new Fock spaces. Next, we construct a BRST operator for the converted constraint superalgebra. The construction of an action and of a sequence of reducible gauge transformations describing the propagation of a mixed-symmetry fermionic field of an arbitrary spin is realized in Section 5. We demonstrate that the Lagrangian description of a massive half-integer mixed-symmetry HS field in a \( d \)-dimensional Minkowski space can be deduced not only by using the same algorithm, with allowance for the presence of 4 additional second-class constraints instead of the respective first-class ones with first-order derivatives, but also by using dimensional reduction for a massless HS field theory of the same type in a \((d+1)\)-dimensional flat space. In Section 6, we sketch a proof of the fact that the resulting action reproduces the correct conditions for a field that determine an irreducible representation of the Poincare group with a fixed \( s = (n_1 + 1/2, n_2 + 1/2) \) spin. We illustrate the general formalism by a construction of gauge-invariant Lagrangian actions for massless and massive spin-(1 + 1/2, 1/2) and spin-(1 + 1/2, 1 + 1/2) fields in Section 7. In Conclusion, we summarize the results of this work and outline some open problems.

In addition to the conventions of [9, 44, 46], we use the notation \( \varepsilon(A), gh(A) \) for the respective values of Grassmann parity and ghost number of a quantity \( A \), and denote by \( [A, B] \) the supercommutator of quantities \( A, B \), which in the case of definite values of Grassmann parity is given by \( [A, B] = AB - (-1)^{\varepsilon(A)\varepsilon(B)}BA \).

## 2 Half-integer HS Symmetry Algebra in Flat Space-time

In general, a massless half-integer irreducible representation of the Poincare group in a \( d \)-dimensional Minkowski space is described by a spin-tensor field \( \Phi_{\mu_1...\mu_{n_1},\nu_1...\nu_{n_2},...\nu_{n_k}}(x) \), with the Dirac index being suppressed, of rank \( n_1 + n_2 + ... + n_k \) and generalized spin \( s = (n_1 + 1/2, n_2 + 1/2, ..., n_k + 1/2) \), which corresponds to a Young tableaux with \( k \) rows of length \( n_1, n_2, ..., n_k \), respectively, and \( k \leq [d/2] \). This field is symmetric with respect to the permutations of each type of indices \( \mu_i, \nu_i \), \( i = 1, ..., k \).

In this paper, we restrict ourselves to the fields characterized by a Young tableaux with \( k = 2 \) rows:

\[
\begin{array}{cccccccc}
\mu_1 & \mu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \mu_{n_1} \\
\nu_1 & \nu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \nu_{n_2}
\end{array}
\]  

(2.1)

The field \( \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) \equiv \Phi_{\mu_1...\mu_{n_1},\nu_1...\nu_{n_2}}(x) \), as an element of a Poincare-group irrep, obeys the
mass-shell and $\gamma$-traceless conditions for each type of indices\footnote{Throughout the paper, we use the mostly minus signature $\eta_{\mu\nu} = diag(+, -, ..., -)$, $\mu, \nu = 0, 1, ..., d - 1$, and the Dirac matrices satisfy the relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.}

\begin{align}
  i\gamma^\mu &\partial_\mu \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) = 0, \quad \text{(2.2)} \\
  \gamma^{\mu_1} &\Phi_{\mu_1\mu_2...\mu_{n_1},(\nu)_{n_2}}(x) = 0, \quad \text{(2.3)} \\
  \gamma^{\mu_1} &\Phi_{(\mu)_{n_1},\nu_1\nu_2...\nu_{n_2}}(x) = 0. \quad \text{(2.4)}
\end{align}

The correspondence with a given Young tableaux implies that after the symmetrization of all the vector indices of the first row with any vector index of the second row the field $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ becomes equal to zero:

$$\Phi_{(\mu)_{n_1},\nu_1\nu_2...\nu_{n_2}}(x) \equiv \sum_{i=1}^{n_1} \Phi_{\mu_1...\mu_{i-1}\mu_i\mu_{i+1}...\mu_{n_1},\nu_1\nu_2...\nu_{n_2}}(x) + \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) = 0, \quad \text{(2.5)}$$

where in the case $i = 1$ it is implied that $\Phi_{\mu_0\nu_1\nu_2...\mu_{n_1},\mu_1\nu_2...\nu_{n_2}}(x) \equiv \Phi_{\mu_1\mu_2...\mu_{n_1},\mu_1\nu_2...\nu_{n_2}}(x)$.

In order to describe all the irreducible representations simultaneously, it is convenient to introduce an auxiliary Fock space $\mathcal{H}$ generated by creation and annihilation operators $a^i_\mu, a^j_\mu$ with additional internal indices, $i, j = 1, 2$,

$$[a^i_\mu, a^j_\nu] = -\eta_{\mu\nu}\delta^{ij}, \quad \delta^{ij} = diag(1, 1). \quad \text{(2.6)}$$

The general state (a Dirac-like spinor) of the Fock space has the form

$$|\Phi\rangle = \sum_{n_1=0}^\infty \sum_{n_2=0}^{n_1} \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) a^{+\mu_1}_1 ... a^{+\mu_{n_1}}_1 a^{+\nu_1}_2 ... a^{+\nu_{n_2}}_2 |0\rangle, \quad \text{(2.7)}$$

providing the symmetry property of $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ under the permutation of indices of the same type. We refer to the vector (2.7) as the basic vector.

Because of the property of translational invariance of the vacuum, $\partial_\mu |0\rangle = 0$, the conditions (2.2)–(2.4) can be equivalently expressed in terms of the bosonic operators

$$\tilde{t}_0 = i\gamma^\mu \partial_\mu, \quad \tilde{t}^i = \gamma^\mu a^i_\mu, \quad \text{(2.8)}$$

$$t = a^{1+}_\mu a^{2+}_\mu \quad \text{(2.9)}$$

as follows:

$$\tilde{t}_0 |\Phi\rangle = \tilde{t}^i |\Phi\rangle = t|\Phi\rangle = 0. \quad \text{(2.10)}$$

Thus, the constraints (2.10) with each component $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ of the vector (2.7) subject to (2.2)–(2.4) describe a field of spin $(n_1 + 1/2, n_2 + 1/2)$.

Because of the fermionic nature of equations (2.2)–(2.4) with respect to the standard Lorentz-like Grassmann parity, and due to the bosonic nature of the primary constraint operators $\tilde{t}_0, \tilde{t}^i$, $\varepsilon(\tilde{t}_0) = \varepsilon(\tilde{t}^i) = 0$, in order to equivalently transform these operators into fermionic ones, we now introduce a set of $d+1$ Grassmann-odd gamma-matrix-like objects $\tilde{\gamma}^\mu, \tilde{\gamma}$, subject to the conditions

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}\} = 0, \quad \tilde{\gamma}^2 = -1, \quad \text{(2.11)}$$

and related to the conventional gamma-matrices as follows\footnote{For more details, see [44]. The quantities $\tilde{\gamma}^\mu, \tilde{\gamma}$ may be viewed as intermediate objects as compared to $\gamma^\mu$. Indeed, the final Lagrangian description in terms of ghost-independent or spin-tensor forms depends only on the standard Grassmann-even matrices $\gamma^\mu$, and does not depend on $\tilde{\gamma}^\mu, \tilde{\gamma}$, because the latter enter the Lagrangian only in even degrees, and also due to the homogeneity of the reducible gauge transformations w.r.t. $\tilde{\gamma}$, as shown by the examples in Section 7.}

$$\gamma^\mu = \tilde{\gamma}^\mu \tilde{\gamma}. \quad \text{(2.12)}$$
We can now define Grassmann-odd constraints,

\[ t_0 = -i \tilde{\gamma}^\mu \partial_\mu, \quad t^i = \tilde{\gamma}^\mu a_i^\mu, \]  

(2.13)

related to the operators (2.8) as follows:

\[ (t_0, t^i) = \tilde{\gamma} (\tilde{t}_0, \tilde{t}^i). \]  

(2.14)

We next define an odd scalar product:

\[ \langle \Psi | \Phi \rangle = \int d^d x \sum_{n_1, k_1, n_2, k_2=0}^\infty \langle 0 | a_1^{\rho_1} \ldots a_1^{\rho_{k_1}} a_2^{\sigma_1} \ldots a_2^{\sigma_{k_2}} \Psi^+_{(\rho)_{k_1},(\sigma)_{k_2}}(x) \tilde{\gamma}_0 \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) \times a_1^{\mu_1} \ldots a_1^{\mu_{n_1}} a_2^{\nu_1} \ldots a_2^{\nu_{n_2}} | 0 \rangle. \]  

(2.15)

The operators \( t_0, t^i, t \) in (2.9) and (2.13), with \( t^{i+} = \tilde{\gamma}^\mu a_{i+}^\mu \) and \( t^+ = a_2^{-1+} a_1^{1\mu} \) being Hermitian conjugate, respectively, to \( t^i, t \) with reference to the scalar product (2.15), generate an operator Lie superalgebra composed of the operators

\[ t_0 = -i \tilde{\gamma}^\mu \partial_\mu, \quad t^i = \tilde{\gamma}^\mu a_i^\mu, \quad t^+ = \tilde{\gamma}^\mu a_{i+}^\mu, \]  

(2.16)

\[ t = a_2^{1+} a_1^{1\mu}, \quad t^i = a_2^{1+} a_1^{i\mu}, \quad t^i = a_2^{1+} a_1^{i\mu}, \]  

(2.17)

\[ l = -i a_2^{1+} \partial_\mu, \quad l^{i+} = -i a_2^{1+} \partial_\mu, \quad l^{i+} = -i a_2^{1+} \partial_\mu, \]  

(2.18)

\[ l^{ij} = \frac{1}{2} a_2^{1+} a_1^{i\mu}, \quad l^{ij} = \frac{1}{2} a_2^{1+} a_1^{i\mu}, \quad l^{ij} = \frac{1}{2} a_2^{1+} a_1^{i\mu}, \]  

(2.19)

\[ \Delta = \partial^\mu \partial_\mu, \quad g_0 = -a_2^{1+} a_1^{i\mu} + \frac{d}{2}, \]  

(2.20)

which is invariant under Hermitian conjugation.

The operators (2.16)–(2.21) form a superalgebra given by Table 1 with an omission of the Poincare-group Casimir operator \( l_0 \) being the central charge of this algebra, where the quantities \( A^{ik}, B^{k,ij}, C^{k,ij}, D^{ij}, E^{ij}, F^{i}, G^{ij}, H^{ij}, J^{k,ij}, K^{k,ij}, L^{kl,ij} \) are defined by the relations

\[ A^{ik} = -2 g_0^{ij} \delta^{ik} + 2 t^{i+} \delta^{k1} + 2 t^+ \delta^{i1} \delta^{k2}, \]  

(2.22)

\[ B^{k,ij} = -\frac{1}{2} t^{(i+\delta)^j k}, \]  

(2.23)

\[ C^{k,ij} = \frac{1}{4} t^{(ij)k}, \]  

(2.24)

\[ D^{ij} = t^{(ij)^1}, \]  

(2.25)

\[ F^{i} = t(\delta^{i2} - \delta^{i1}), \]  

(2.26)

\[ G^{ij} = t^{(ij)^2}, \]  

(2.27)

\[ J^{k,ij} = -\frac{1}{2} t^{(i+\delta)^k}, \]  

(2.28)

\[ L^{kl,ij} = \frac{1}{8} \delta^{ik} \delta^{lj} \left[ 2 g_0^{kl} \delta^{ij} + g_k^i + g_l^j \right] - \delta^{ik} \left[ t^{(i+\delta)^j k} + t^+(\delta^{i1} + \delta^{k1} \delta^{j2}) + t^+(\delta^{i2} + \delta^{j2} \delta^{k1}) \right] - \delta^{lj} \left[ t^{(k+\delta)^i j} + t^+(\delta^{k1} + \delta^{i1} \delta^{j2}) + t^+(\delta^{k2} + \delta^{j2} \delta^{i1}) \right]. \]  

We call this algebra the half-integer higher-spin symmetry algebra in Minkowski space with a Young tableaux having two rows.

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4For the operators \( t^{12}, t^{12+} \) in (2.20), we have used a definition slightly different from that of [46], where \((\tilde{t}^{12}, \tilde{t}^{12+}) = 2(t^{12}, t^{12+})\).
From the viewpoint of constraint system theory, the above superalgebra is a system of constraints, except for the operators $g_0^k$, being non-degenerate in $H$. These operators, as follows from Table 1, determine an invertible operator supermatrix of commutators for the subsystem of second-class constraints, \{t_0, t_0^+, l_i, l_i^+, t, t^+\}, with the other constraints, $t_0, l_0, l_i, l_i^+$, being first-class ones. A conversion of this constraint system $\{\alpha_I\}$, including the operators $g_0^k$, into a first-class constraint system $\{O_I\}$ by means of an additive composition of $\alpha_I$ with certain operators $\alpha'_I$ depending on some new creation and annihilation operators, $\alpha_I \rightarrow O_I = \alpha_I + \alpha'_I$, can be effectively realized only for the subsuperalgebra of the entire symmetry superalgebra that contains the subsystem of second class-constraints and $g_0^k$. The only requirement, as shown in [16], is that each of the Hermitian operators $g_0^k$ should have a linear dependence on an arbitrary parameter $h^k$, whose values are to be determined later.

3 Auxiliary Representation for the Superalgebra with Second-class Constraints

In this section, we describe the method of Verma module construction for the Lie superalgebra with second-class constraints alone. Having denoted $\{\alpha_a\} = \{t_k, t_k^+, l_{ij}, l_{ij}^+, t, t^+\}$, $\alpha_a \in \{\alpha_I\}$, as the basis elements of the above superalgebra, and using the requirements that $\alpha_a$, $\alpha'_a$ must supercommute, $\{\alpha_a, \alpha'_a\} = 0$, and that the converted constraints must be in involution, $\{O_a, O_b\} \sim O_c$, we find that the superalgebra of the additional parts $\alpha'_a$ is uniquely determined by the same algebraic conditions as those for the initial constraints. In this case, it is unnecessary to convert the
subsystem of the initial first-class constraints not entering \{o_\alpha\}, and therefore they remain intact.

Following [44] and the general method of Verma module construction for mixed-symmetry integer-spin HS fields [46], let us denote \(E^\alpha \equiv (t^k; i^{ij}, t) = (E^{\alpha_0}; E^{\alpha_1}), (\alpha_0 > 0, \alpha_1 > 0)\) for \(i \leq j\), and define

\[
\mathcal{H}^i = g_0^i + g_0^i, \quad g_0^i = h^i + \ldots, \quad \mathcal{E}^\alpha = E^\alpha + E^{\alpha_0}(h), \quad \alpha_0 = 1, 2, \alpha_1 = 1, 2, 3, 4. \tag{3.1}
\]

The quantities \(g_0^i, E^{\alpha_1}, E^{-\alpha_1}\) are the Cartan generators, positive and negative root vectors, except for \(\alpha_1 = 2\) (see footnote 4) of the subalgebra \(so(3, 2)\) in the superalgebra of second-class constraints, and the odd generators \(E^{\alpha_0}, E^{-\alpha_0}\) supplement the basis \(so(3, 2)\) up to that of the above superalgebra. The quantities \(g_0^i, E^{\alpha_1}, E^{-\alpha_1}\) and \(\mathcal{H}^i, \mathcal{E}^\alpha, \mathcal{E}^{-\alpha}\) have the same identification respectively for the additional and enlarged operators of the symmetry superalgebra.

Consider the highest-weight representation of the superalgebra of the additional parts with the highest-weight vector \(\langle 0 \rangle_V\) annihilated by the positive roots and being the proper vector of the Cartan generators:

\[
E^{\alpha_1}\langle 0 \rangle_V = 0, \alpha_0 > 0, \quad g_0^i\langle 0 \rangle_V = h^i\langle 0 \rangle_V. \tag{3.2}
\]

Following the Poincare–Birkhoff–Witt theorem, the basis space of this representation, called the Verma module in the mathematical literature [47], is given by the vectors

\[
|n_k, n_{ij}, n\rangle_V = (E^{-\alpha_1})^{n_0}(E^{\alpha_1})^{n_1}(E^{-\alpha_1})^{n_{11}}(E^{\alpha_1})^{n_{12}}(E^{-\alpha_1})^{n_{22}}(E^{\alpha_1})^{n_2}\langle 0 \rangle_V, \tag{3.3}
\]

where \(n_k^0 = (n_1^0, n_0^0), n_{ij} = (n_{11}, n_{12}, n_{22}), n_1^0, n_2^0 = 0, 1, n_{ij} \in \mathbb{N}_0\). Note that the restriction for the values of \(n_k^0\) in (3.3) is due to the identities

\[
\{E^{-\alpha_0}, E^{-\alpha_0}\} = 4(E^{\alpha_1}, E^{\alpha_1}, E^{\alpha_1}) = 4(l^{l_i}, l^{l_j}, l^{l_{ij}}), \quad i, j = 1, 2, i \leq j. \tag{3.4}
\]

Using the commutation relations of the superalgebra given by Table [41] and the formula for the product of graded operators,

\[
AB^n = \sum_{k=0}^{n}(-1)^{\varepsilon(A)\varepsilon(B)(n-k)}C(s)^{k}B^{n-k}\text{ad}_BA, \quad n \geq 0, s = \varepsilon(B), \tag{3.5}
\]

we can calculate the explicit form of the Verma module. In (3.5), we have introduced generalized coefficients for a number of graded combinations, \(C(s)^{n-k}\), that coincide with the standard ones only for the bosonic operator \(B\): \(C^{(0)}_k = C^0_k = \frac{n_k^1}{k(n-k)!}\). These coefficients are defined recursively, by the relations

\[
C^{(s)^{n+1}}_k = (-1)^{s(n+k+1)}C^{(s)^{n}}_{k-1} + C^{(s)^{n}}_k, \quad n, k \geq 0, \tag{3.6}
\]

\[
C^{(s)^{n}}_0 = C^{(s)^{n}}_n = 1, \quad C^{(s)^{n}}_k = 0, \quad n < k \tag{3.7}
\]

and possess the properties \(C^{(s)^{n}}_k = C^{(s)^{n}}_{n-k}\). The corresponding values of \(C^{(1)^{n}}_k\) are defined, for \(n \geq k\), by the formulae

\[
C^{(1)^{n}}_k = \sum_{i_k=1}^{n-k+1} \sum_{i_{k-1}=1}^{n-i_k-k+2} \cdots \sum_{i_2=1}^{n-\sum_{j=3}^{k}i_j-1} \sum_{i_1=1}^{n-\sum_{j=2}^{k}i_j} (-1)^{k(n+1)+\sum_{j=1}^{[k+1]2}[(i_2j-1)+1]}, \tag{3.8}
\]

which follow by induction, and in which \([a]\) stands for the integer part of the number \(a\). For our purposes, due to \(n_k^0 = 0, 1\) in (3.3), (3.4), it is sufficient to know that \(C^{(1)^{0}}_0 = C^{(1)^{1}}_0 = 1\) and \(C^{(1)^{n}}_1 = n_k^0\).
Then, following [48] and making use of the mapping

\[ |\vec{n}^0_k, \vec{n}_{ij}, n\rangle_V \leftrightarrow |\vec{n}^0_k, \vec{n}_{ij}, n\rangle = (f^+_1)^{n_1} (f^+_2)^{n_2} (b^+_1)^{n_{11}} (b^+_2)^{n_{12}} (b^+_3)^{n_{22}} (b^+) n |0\rangle , \]

(3.9)

where \(|\vec{n}^0_k, \vec{n}_{ij}, n\rangle\), for \(n^0_k = 0, 1\), \(n_{ij} \in \mathbb{N}_0\), are the basis vectors of a Fock space \(\mathcal{H}'\) generated by new fermionic, \(\vec{f}_k^+\), \(f_k^+\), \(k = 0, 1\), and bosonic, \(b^+_i, b_i, b^+, i, j = 1, 2, i \leq j\), creation and annihilation operators with the standard (only nonvanishing) commutation relations

\[ \{ f_k^+, f_l^+ \} = \delta_{kl}, \quad [b_{ij}, b_{ik}^+] = \delta_{ij} \delta_{jk}, \quad i \leq j, k \leq l, \quad [b, b^+] = 1, \]

(3.10)

we can represent the Verma module as polynomials in the creation operators of the Fock space \(\mathcal{H}'\).

First, we find the action of the negative root operators \(E^\alpha\) on the basis vectors. After a simple calculation, one obtains

\[ t^+ |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = \delta_{11} \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) |n^0_1 + 1 mod 2, n^0_2, n_{11} + \left[ \frac{n^0_2 + 1}{2} \right], n_{12}, n_{22}, n \rangle_V \]

(3.11)

\[ + \delta_{12} (-1)^{n_1^0} \left( 1 + \left[ \frac{n^0_2 + 1}{2} \right] \right) |n^0_1, n^0_2 + 1 mod 2, n_{11}, n_{12}, n_{22} + \left[ \frac{n^0_2 + 1}{2} \right], n \rangle_V \]

\[ - 4n_1^0 |n^0_1 - 1, n^0_2, n_{11}, n_{12} + 1, n_{22}, n \rangle_V \}, \]

\[ t^+_l |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = |\vec{n}^0_k, \vec{n}_{lm} + \delta_{il} \delta_{jm}, n\rangle_V , \]

(3.12)

\[ t^+_i |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = |\vec{n}^0_k, \vec{n}_{lm}, n + 1\rangle_V - 2n_{11} |\vec{n}^0_k, n_{11} - 1, n_{12} + 1, n_{22}, n \rangle_V \]

(3.13)

\[ - n^0_1 \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) |n^0_1 - 1, n^0_2 + 1 mod 2, n_{11}, n_{12}, n_{22} + \left[ \frac{n^0_2 + 1}{2} \right], n \rangle_V \]

\[ - n_{12} |\vec{n}^0_k, n_{11}, n_{12} - 1, n_{22} + 1, n \rangle_V \}, \]

\[ g^+_0 |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = \left( n^0_1 \delta_{ik} + \sum_{l \leq m} n_{lm} (\delta^{il} + \delta^{im}) + n (\delta^{i2} - \delta^{i1}) + h^i \right) |\vec{n}^0_k, \vec{n}_{lm}, n \rangle_V . \]

(3.14)

Second, for the positive root operators \(E^{\alpha}\) we find

\[ t^+ |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = - 2n_1^0 (2n_{11} + n_{12} - n + h^1) |n^0_1 - 1, n^0_2, \vec{n}_{lm}, n \rangle_V \]

(3.15)

\[ - (-1)^{n_1^0} \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) n_{11} |n^0_1, n^0_2 + 1 mod 2, n^0_2, n_{11} - 1 + \left[ \frac{n^0_2 + 1}{2} \right], n_{12}, n_{22}, n \rangle_V \]

\[ - (-1)^{n_1^0 + n_2^0} \frac{n_{12}}{2} \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) |n^0_1, n^0_2 + 1 mod 2, n_{11}, n_{12} - 1, n_{22} + \left[ \frac{n^0_2 + 1}{2} \right], n \rangle_V \]

\[ + 2(-1)^{n_1^0} n_2^0 \left( - n_{12} |n^0_1, n^0_2 - 1, n_{11}, n_{12} - 1, n_{22} + 1, n \rangle_V \right) \]

\[ + |n^0_1, n^0_2 - 1, \vec{n}_{lm}, n + 1 \rangle_V \}, \]

\[ t^2 |\vec{n}^0_k, \vec{n}_{lm}, n\rangle_V = (-1)^{n_1^0} \left\{ - 2n_1^0 (2n_{22} - n^0_1 + n + h^2) |n^0_1, n^0_2 - 1, \vec{n}_{lm}, n \rangle_V \right\} \]

(3.16)

\[ - \frac{n_{12}}{2} \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) |n^0_1 + 1 mod 2, n^0_2, n_{11} + \left[ \frac{n^0_1 + 1}{2} \right], n_{12} - 1, n_{22}, n \rangle_V \]

\[ - (-1)^{n_1^0} n_{22} \left( 1 + \left[ \frac{n^0_1 + 1}{2} \right] \right) |n^0_1, n^0_2 + 1 mod 2, n_{11}, n_{12}, n_{22} - 1 + \left[ \frac{n^0_1 + 1}{2} \right], n \rangle_V \]

\[ - 2n_1^0 \left( n(h^1 - h^2 - n + 1) |n^0_1 - 1, n^0_2, \vec{n}_{lm}, n - 1 \rangle_V \right) \]

\[ - 2n_{22} |n^0_1 - 1, n^0_2, n_{11} + 1, n_{22} - 1, n \rangle_V \]

\[ - n_{12} \left( n^0_1 - 1, n^0_2, n_{11} + 1, n_{12} - 1, n_{22}, n \right) \} \}, \]
Using expressions (3.11)–(3.20) and the mapping (3.9), we reconstruct the action of the operators
\[ t'_{11} |\vec{n}_k^0, \vec{n}_{lm}, n \rangle_V = n_{11}(n_{11} + n_{12} + n_0^1 - n - 1 + h^1) |\vec{n}_k^0, n_{11} - 1, n_{12}, n_{22}, n \rangle_V \]
\[ - \frac{112}{2} |\vec{n}_0^0, n_{11}, n_{12} - 1, n_{22}, n + 1 \rangle_V + \frac{112(n_{12} - 1)}{4} |\vec{n}_0^0, n_{11}, n_{12} - 2, n_{22} + 1, n \rangle_V \]
\[ - n_0^0 \left\{ 2n_2^0 \left( |n_k^0 - 1, \vec{n}_{lm}, n + 1 \rangle_V - n_{12} |n_k^0 - 1, n_{11}, n_{12} - 1, n_{22} + 1, n \rangle_V \right) \right\} \]
\[ - \frac{(n_0^0 + 1)}{2} n_{12} \left( 1 + \left[ \frac{n_0^0 + 1}{2} \right] \right) |n_0^0 - 1, n_0^2 + 1 \mod 2, n_{11}, n_{12} - 1, n_{22} + \left[ \frac{n_0^0 + 1}{2} \right], n \rangle_V \right\} \].

\[ t'_{12} |\vec{n}_k^0, \vec{n}_{lm}, n \rangle_V = \frac{1}{4} \left( 2n_{11} + n_{12} + 2n_{22} + \sum_k (n_k^0 + h^k) - 1 \right) |\vec{n}_k^0, n_{11}, n_{12} - 1, n_{22}, n \rangle_V \]
\[ + \frac{1}{2} n_{11} (h^2 - h^1 + n - 1) |\vec{n}_k^0, n_{11} - 1, n_{12}, n_{22}, n - 1 \rangle_V \]
\[ + n_{11} n_{22} |\vec{n}_k^0, n_{11} - 1, n_{12} + 1, n_{22} - 1, n \rangle_V \]
\[ - \frac{n_2^0}{2} |\vec{n}_k^0, n_{11}, n_{12}, n_{22} - 1, n + 1 \rangle_V \]
\[ + \frac{n_2^0}{2} \left( 2n_2^0 (n + 2n_{22} + h^2) |n_k^0 - 1, \vec{n}_{lm}, n \rangle_V + (-1)^n_2 \left( 1 + \left[ \frac{n_0^0 + 1}{2} \right] \right) \times \right. \]
\[ \times n_{22} |n_0^0 - 1, n_0^2 + 1 \mod 2, n_{11}, n_{12}, n_{22} - 1 + \left[ \frac{n_0^0 + 1}{2} \right], n \rangle_V \right\} \]
\[ + \frac{n_2^0}{2} n_{11} \left( 1 + \left[ \frac{n_0^0 + 1}{2} \right] \right) |n_0^0 + 1 \mod 2, n_0^0 - 1, n_{11} - 1 + \left[ \frac{n_0^0 + 1}{2} \right], n_{12}, n_{22}, n \rangle_V \].

\[ t'_{22} |\vec{n}_k^0, \vec{n}_{lm}, n \rangle_V = n_{22} (n_{12} + n + n_0^2 + n_{22} - 1 + h^2) |\vec{n}_k^0, n_{11}, n_{12} - 1, n_{22} - 1, n \rangle_V \]
\[ + \frac{n_{22}}{2} (n - 1 + h^2 - h^1) |\vec{n}_k^0, n_{11}, n_{12} - 1, n_{22}, n - 1 \rangle_V \]
\[ + \frac{n_{12} (n_{12} - 1)}{4} |\vec{n}_k^0, n_{11} + 1, n_{12} - 2, n_{22}, n \rangle_V \]
\[ + \frac{n_{22}}{2} \left( 1 + \left[ \frac{n_0^0 + 1}{2} \right] \right) |n_0^0 + 1 \mod 2, n_0^2 - 1, n_{11} + \left[ \frac{n_0^0 + 1}{2} \right], n_{12} - 1, n_{22}, n \rangle_V \].

\[ t' \| |\vec{n}_k^0, \vec{n}_{lm}, n \rangle_V = n (h^1 - h^2 - n + 1) |\vec{n}_k^0, \vec{n}_{lm}, n - 1 \rangle_V \]
\[ - n_{12} |\vec{n}_k^0, n_{11} + 1, n_{12} - 1, n_{22}, n \rangle_V - 2n_{22} |\vec{n}_k^0, n_{11}, n_{12} + 1, n_{22} - 1, n \rangle_V \]
\[ - n_0^0 \left( 1 + \left[ \frac{n_0^0 + 1}{2} \right] \right) |n_0^0 + 1 \mod 2, n_0^0 - 1, n_{11} + \left[ \frac{n_0^0 + 1}{2} \right], n_{12}, n_{22}, n \rangle_V \].

Using expressions (3.11)–(3.20) and the mapping (3.9), we reconstruct the action of the operators
\[ E^+ \alpha, E^- \alpha, g_0^\alpha \] in the Fock space \( \mathcal{H}' \), namely,
\[ t'^{i+} = f_i^+ + 2b_i^+ f_i + 4 \delta_{i} b_{12} f_1 \],
\[ t'^{+} = b^+ - 2b_{12} b_{11} - b_{22} b_{12} - f_2^+ f_1 + 2b_{22} f_1 f_2 \],
\[ g_0^\alpha = f_i^+ f_i + \sum_{\ell \leq m} b_{lm} (\delta^{\ell \ell} + \delta^{i m}) + b^+ b (\delta^{\ell 2} - \delta^{i 1}) + h^i \],
\[ t'^{i+}_{ij} = b_i^+ b_j^+ \].
\[ l'^{12} = \frac{1}{4} \left[ \sum_i \left( 2b^+_i b_{ii} + f^+_i f_i + h^i \right) + b^+_{i2} b_{12} \right] b_{12} + \frac{1}{2} (b^+ b + h^2 - h^1) b b_{11} \] (3.28)

\[ l'^{22} = \left( b^+_{22} b_{22} + b^+_{12} b_{12} + b^+ b + f^+_2 f_2 + h^2 \right) b_{22} + \frac{1}{2} (b^+ b + h^2 - h^1) b_{12} b \] (3.29)

\[ t' = (h^1 - h^2 - b^+ b) b - b^+_{12} b_{12} - 2b^+_2 b_{22} - f^+_2 f_2 - 2b^+_{12} f_2 f_1 . \] (3.30)

Note that the additional parts \( E'^\alpha, E'^{-\alpha} \) do not obey the usual properties
\[ (E'^\alpha)^+ \neq E'^{-\alpha} , \] (3.31)
if one should use the standard rules of Hermitian conjugation for the new creation and annihilation operators,
\[ (b^i)^+ = b^i, \quad (b)^+ = b^+, \quad (f_1)^+ = f_1^+ . \] (3.32)

To restore the proper Hermitian conjugation properties for the additional parts, we change the scalar product in the Fock space \( \mathcal{H}' \) as follows:
\[ \langle \tilde{\Psi}_1 | \Psi_2 \rangle_{\text{new}} = \langle \tilde{\Psi}_1 | K' | \Psi_2 \rangle , \] (3.33)
for any vectors \( |\Psi_1\rangle, |\Psi_2\rangle \) with some, yet unknown, operator \( K' \). This operator is determined by the condition that all the operators of the algebra must have the proper Hermitian properties with respect to the new scalar product:
\[ \langle \tilde{\Psi}_1 | K' E'^{-\alpha} | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' E'^{\alpha} | \Psi_1 \rangle^* , \quad \langle \tilde{\Psi}_1 | K' g^h_0 | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' g^h_0 | \Psi_1 \rangle^* . \] (3.34)

These relations permit one to determine the operator \( K' \), Hermitian with respect to the usual scalar product \( \langle | \rangle \), as follows:
\[ K' = Z^+ Z , \quad Z = \sum_{(n_{lm}, n)} \sum_{(n_{lm})} \frac{1}{n_t !} \langle n_t | b^n_{11} b^n_{12} b^n_{22} f^n_1 f^n_2 , \] (3.35)
where \( (n_{lm})! = n_{11}! n_{12}! n_{22}! \). One can show by direct calculation that the following relation holds true: \( V (\tilde{n}_{k}, \tilde{n}_{lm}, n | \tilde{n}_{k}, \tilde{n}_{lm}, n) _V \sim \delta_{n_{lm}+2n_{11}+n_{12}-n} \delta_{n_{lm}+2n_{22}+n} \delta_{n_{lm}+2n_{11}+n_{12}-n} \delta_{n_{lm}+2n_{22}+n} \). For low pairs of numbers \( (n_{1}^0 + 2n_{11} + n_{12} - n, n_{2}^0 + n_{12} + 2n_{22} + n) \), with \( n_{12} \) being the numbers of “particles” associated with \( b^+, b^+_i \) for \( i \leq j \) (where \( b^+ \)) decreases the spin number \( s_1 \) by one unit and increases the spin number \( s_2 \) by one unit simultaneously) and \( n_{10}^0 \) being the number of “particles” associated with \( f^+_k \), the operator \( K' \) reads
\[ K' = |0\rangle \langle 0| + (h^1 - h^2) b^+ |0\rangle \langle 0| b - 2h^1 f^+_i |0\rangle \langle 0| f_i + 2f^+_2 |0\rangle \langle 0| (h^1 - h^2) b f_1 \]
\[ + f^+_i b^+_1 |0\rangle \langle 0| \left( 2b f_1 (h^2 - h^1) (h^1 - 1) + 2f_2 (h^1 - h^2) \right) \]
\[ + b^+_1 b^+_2 |0\rangle \langle 0| \left( \frac{1}{4} b^+ b + f^+_1 f^+_2 + \frac{1}{2} b^+_1 b (h^2 - h^1) \right) + b^+_i b^+_i |0\rangle \langle 0| b_{ii} \]
\[ + b^+_i b^+_1 b^+_2 |0\rangle \langle 0| \left( b^+_1 b (h^1 - h^2) (h^1 - 1) + \frac{1}{2} b^+_1 b (h^2 - h^1) + 2f^+_2 f^+_1 (h^2 - h^1) \right) \]
\[ + f^+_i f^+_2 |0\rangle \langle 0| \left( 4f^+_1 f^+_2 (h^2 h^1 + h^2 - h^1) + b^+_1 b h^2 + 2b^+_1 b (h^2 - h^1) \right) + \ldots . \] (3.36)

This expression for the operator \( K' \) will be used later in constructing the examples of Section 7.

Thus, we have constructed the additional parts \( o'^\alpha, (3.21) \sim (3.30) \), for the constraints \( o^\alpha \). In the next section, we determine the algebra of the extended constraints and find the BRST operator corresponding to this algebra.
4 The Converted Superalgebra and the BFV–BRST Operator

The superalgebra of the converted operators $O_I$,

$$O_I = (O_a, O_p), \quad O_a = o_a + o_a', \quad O_p = o_p, \quad o_p \in \{t_0, l_0, l_i, l_i^+\},$$

has the same form as the superalgebra of the initial operators $o_I$, and therefore it is determined by the relations of Table [1] under the replacement $o_I \leftrightarrow O_I$. Despite the fact that the operators $\mathcal{H}^i$ do not belong to the constraint system, and in order to provide a Lorentz-covariant description of BRST cohomology spaces, we do not impose the restrictions $\mathcal{H}^i | \chi \rangle_{def} = 0$ on the vector $| \chi \rangle_{def}$, being the vector $| \Phi \rangle$ (2.7) enlarged into the tensor product of the Fock spaces $\mathcal{H}_{def} = \mathcal{H} \otimes \mathcal{H}'$,

$$| \chi \rangle_{def} = \sum_{k_a} (f_i^+)^k (h_{lm}^+)^{klm} (h^+)^k a_1^{+\mu_1} \ldots a_1^{+\mu_{k_{10}}} a_2^{+\nu_1} \ldots a_2^{+\nu_{k_{20}}} |x, \sigma_i \rangle |0\rangle,$$ (4.2)

and include $\mathcal{H}^i$ into the converted first-class constraint system, with respect to which we construct the BRST operator $Q'$. The sum in (4.2) is taken over $k_{i0}, k_{lm}, k$, running from 0 to infinity, and over $k_i$, running from 0 to 1 for $i = 0, 1, l, m = 1, 2, l \leq m$. Having constructed $Q'$, we extract from it the operators $\mathcal{H}^i$, enlarged by means of the ghost variables $\mathcal{C}, \mathcal{P}$ up to new operators $\sigma_i, \sigma^i = (\mathcal{H}^i - h^i + O(\mathcal{C}\mathcal{P}))$, which will be used to describe, by virtue of the equations $(\sigma^i + h^i)|\chi\rangle = 0$, the direct sum of the Fock subspaces $\mathcal{H}_{(n_1, n_2)}$ of a definite generalized spin $s = (n_1 + \frac{1}{2}, n_2 + \frac{1}{2})$ in the enlarged Hilbert space $\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh}$ for $|\chi\rangle \in \mathcal{H}_{tot}$. In this case, the remaining operator $Q$, independent of the ghost variables $\eta^i_H, \mathcal{P}^i_H$ associated with $\mathcal{H}^i$, in $Q' = Q + O(\eta^i_H, \mathcal{P}^i_H)$, is covariant and nilpotent in each space $\mathcal{H}_{(n_1, n_2)}$ for the converted constraint system $O_I$ without $\mathcal{H}^i$. Then, substituting instead of the parameters $-h^i$ the operators $\sigma^i$, we obtain a nilpotent BRST operator in the complete space $\mathcal{H}_{tot}$ without $\eta^i_H, \mathcal{P}^i_H$, which encodes the superalgebra of the converted constraints $\{O_I\} \setminus \{\mathcal{H}^i\}$ for fermionic HS fields with two rows of the Young tableaux.

The construction of a nilpotent fermionic BRST operator for a Lie superalgebra is based on a principle similar to those developed in [34, 35]: see the general analysis of the BFV quantization in the reviews [33, 34, 35]. Following the prescription of [34], the BRST operator constructed on a basis of the superalgebra presented in Table [1] can be found in an exact form, with the use of the $(\mathcal{C}\mathcal{P})$-ordering of the ghost coordinate $\mathcal{C}^I$ and momenta $\mathcal{P}_I$ operators, as follows:

$$Q' = O_I \mathcal{C}^I + \frac{1}{2} \mathcal{C}^J \mathcal{C}^J f^K_{IJ} \mathcal{P}_K (-1)^{\varepsilon(O_K) + \varepsilon(O_I)}$$ (4.3)

with the constants $f^K_{IJ}$ written in a compact $x$-local representation, $\{O_I, O_J\} = f^K_{IJ} O_K$, and,
according to Table 13, $Q'$ has the form

$$Q' = g_0 T_0 + q_i^+ T_i^+ + q_i^0 T_i^0 + \eta_0 L_0 + \eta_i^+ L_i^+ + L_i^0 \eta_i^0 + \eta_{lm}^+ L_{lm}^+ + \eta^+ T + T^+ \eta + \eta_i^+ \mathcal{H}_i$$

$$+ i(\eta^+_i q^i - \eta^i q^+_i) p_0 + (\eta_0 q_i + \eta_i q^+_i) p_i^+ + (\eta_i q_i^+ + \eta_i^+ q^i) p_i$$

$$- i(q^+ i - \eta^i q^0) p_0 - i(2q^+_i q_i^0 - \eta^i q^0 i \eta^i i) p_i^+ + (\eta_0 q_i^0 + \eta_i^0 q^i p_i^+ + 2(\eta_i q_i^+ - \eta_i^+ q^i) p_i^+$$

$$- 2 \left[ \frac{1}{2} (\eta_0^+ + \eta_0^+ q^i) \eta_i^+ - \eta^+_i \eta_i - \eta^+_i \eta_i^0 + 2q_0 q_i^+ \right] p_i^+ + 2(\eta_i q_i^0 - \eta_i^0 q_i^0) p_i^+$$

$$+ 2 \left[ \frac{1}{2} (\eta_0^+ + \eta_0^+ q^i) \eta_i^0 - \eta^+_i \eta_i^0 + \eta_i^0 q_i^0 + 2q_0 q_i^+ \right] p_i - \eta_0^+ \eta_i^0 P_{1i} - \eta_i^0 \eta_0^+ P_{1i}$$

$$- \eta^+_i \eta_i^0 P_{2i} + \frac{1}{2} \eta^+_i \eta_i^0 P_{2i} + \frac{1}{2} \eta^+_i \eta_i^0 P_{2i}$$

(4.4)

Here, we imply summation over the repeated index $i$, and the raising (lowering) of the indices $i, j$ in quantities $i^{ij}$ is made by the two-dimensional Euclidian metric tensor $g^i_j (g_{ij})$, $g^i_j = \text{diag}(1, 1)$. The quantities $q_0, q_i, q^+_i$ and $\eta_0, \eta_i, \eta_i^0, \eta_0^0, \eta_{lm}$ are, respectively, bosonic and fermionic ghost “coordinates” corresponding to their canonically conjugate ghost “momenta” $p_0, p_i^+, p_i, P_0, P_i, P_{0i}, P_{il}, P_{im}, P_{il}^+, P_{il}$ for $i, l, m = 1, 2, 3 \leq m$. They form a set of Wick ghost pairs, $\{(q_i, p_i^+), (p_i^+, q_i), (\eta_i, P_i^+), (\eta_i^0, P_{0i}^+), (\eta_{lm}, P_{lm}^+), (\eta_{lm}^0, P_{0lm}^+), (\eta, P^+), (\eta^0, P^0)\}$, and a set of zero-mode pairs, $\{(q_0, p_0), (\eta_0, P_0), (\eta_0^0, P_{00}^0)\}$. Following (34), they obey the nonvanishing (anti)commutation relations

$$\{\eta, P^+\} = \{P, \eta^+\} = \{\eta_i, P_i^+\} = \{P_i^+, \eta^+_i\} = 1,$$

$$[q_i, p_i^+] = [p_i, q_i^+] = 1,$$

$$[\eta_i, P_{0i}^+\} = \{P_{0i}, \eta_i^0\} = \{P_{0i}^+\} = \{P_{0i}^0\} = \{\eta_{0i}^0, P_{0i}^0\} = \{\eta_i^{0i}, P_{0i}^+\}$$

(4.5)

they also possess the standard ghost number distribution, $gh(C^i) = -gh(P_i) = 1$, providing the property $gh(Q') = 1$, and have the Hermitian conjugation properties of zero-mode pairs

$$\left(g_0, \eta_0, \eta_0^0, p_0, P_0, P_0^0\right)^+ \Rightarrow \left(g_0, \eta_0, \eta_0^0, p_0, -P_0, -P_0^0\right)$$

(4.6)

The property of the BRST operator to be Hermitian is defined by the rule

$$Q'^+ K = KQ', \quad (4.7)$$

and is calculated with respect to the scalar product $\langle \ | \rangle$ in $\mathcal{H}_{\text{tot}}$ with the measure $d^d x$, which, in its turn, is constructed as the direct product of the scalar products in $\mathcal{H}, \mathcal{H}'$ and $\mathcal{H}_{gh}$. The operator $K$ in (4.7) is the tensor product of the operator $K'$ in $\mathcal{H}'$ and the unit operators in $\mathcal{H}, \mathcal{H}_{gh}$

$$K = \hat{1} \otimes K' \otimes \hat{1}_{gh}.$$  

(4.8)

Thus, we have constructed a Hermitian BRST operator for the entire superalgebra of $O_1$. In the next section, this operator will be used to construct a Lagrangian action for fermionic HS fields of spin $(s_1, s_2)$ in a flat space.
5 Construction of Lagrangian Actions

The construction of Lagrangians for fermionic higher-spin fields in a \(d\)-dimensional Minkowski space can be developed by partially following the algorithm of [11], which is a particular case of our construction, corresponding to \(n_2 = 0\). As a first step, we extract the dependence of the BRST operator \(Q' (4.2)\) on the ghosts \(\eta_i^+ \mathcal{P}_i\), so as to obtain the BRST operator \(Q\) only for the system of converted first-class constraints \(\{O_I\} \setminus \{\mathcal{H}'\}\):

\[
Q' = Q + \eta_i^+ (\sigma^i + h^i) + \mathcal{A}^i \mathcal{P}_i^+, \quad (5.1)
\]

where

\[
Q = q_0 T_0 + q_i^+ T^i + T_i^+ q^i + \eta_0 L_0 + \eta_i^+ L_i^+ + L_i^+ \eta^i + \eta_{lm}^+ L_{lm}^+ + L_{lm}^+ \eta_{lm} + \eta^+ T + T^+ \eta
\]

\[
\begin{align*}
&+ \eta_{ii}^+ \eta_{ii} + \eta_{i1}^+ \eta_{i1} + \eta_{i2}^+ \eta_{i2} + \eta_{i3}^+ \eta_{i3} + \eta_{i4}^+ \eta_{i4} - i(\eta_{ii}^+ \eta_{ii}^+ - \eta_{i1}^+ \eta_{i1}^+ - \eta_{i2}^+ \eta_{i2}^+ - \eta_{i3}^+ \eta_{i3}^+ - \eta_{i4}^+ \eta_{i4}^+) \mathcal{P}_0 + (\eta_{ii}^+ \eta_{ii}^+ - 2q_0 q_i^+) \mathcal{P}_i \\
&+ (\eta_{ii}^+ \eta_{ii}^+ - 2q_0 q_i^+) \mathcal{P}_i^+ - 2q_0^2 \mathcal{P}_i^+ - 2q_0^2 \mathcal{P}_i \\
&- 2 \left[ 2q_1^+ q_2^+ - \eta^+ \eta_{12} - \eta \eta_{11} \right] \mathcal{P}_{12}^+ + \eta^+ \eta_{12} \mathcal{P}_{11}^+ + \eta \eta_{12} \mathcal{P}_{22}^+ + \eta^+ \eta_{12} \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \\
&+ \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right] + \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \\
&- \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right] + \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \\
&- \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right] + \left[ \frac{1}{2} q_1^+ q_2^+ + \eta^+ \eta_{11} + \eta_{12} \right] \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right].
\]

(5.2)

Extended by the ghost Wick-pair variables, has the form

\[
\sigma^i = \mathcal{H}^i - h^i + q_i^+ q_i^+ + q_i^+ q_i^+ - \eta_i^+ \mathcal{P}_i + \eta_i^+ \mathcal{P}_i - 2 \eta_i^+ \mathcal{P}_{ii}^+ + 2 \eta_i^+ \mathcal{P}_{ii}^+ - \left[ \eta_i^+ \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right] + \left[ \eta_i^+ \mathcal{P}_{12}^+ - \eta \eta_{12} \mathcal{P}_{22}^+ \right].
\]

(5.3)

Second, we choose a representation of the Hilbert space permitting us to find the BRST cohomology spaces for the first-class constraint system,

\[
(p_0, q_i, p_i, \mathcal{P}_0, \mathcal{P}_i^+, \eta_i, \mathcal{P}_{ii}, \mathcal{P}_{lm}, \eta, \mathcal{P}) \ | 0 \rangle = 0,
\]

(5.5)

and to extract from \(\mathcal{H}_{tot}\) the Hilbert subspace that does not depend on the \(\eta_i^+\) operators (since \(\mathcal{H}'\) are not first-class constraints as the other \(O_I\)),

\[
| \chi \rangle = \sum_{k_r} (q_0)^{k_1} (q_2)^{k_2} (p_2)^{k_3} (\eta_0)^{k_4} (f^+)^{k_5} (\eta_i^+)^{k_6} (\mathcal{P}_i^+)^{k_7} (\mathcal{P}_{ii}^+)^{k_8} (\mathcal{P}_{lm}^+)^{k_9} (\eta^+)^{k_{10}} (\mathcal{P}^+)^{k_{11}} \times
\]

\[
\times \left( b_{\mu_1 \nu_1}^{a_1} a_1^{+ \mu_1} \ldots a_1^{+ \mu_{k_1}} a_2^{+ \nu_1} \ldots a_2^{+ \nu_{k_2}} \ldots a_{k_4}^{+ \nu_{k_4}} \ldots \chi(\mu)_{k_4 \nu_4}(\nu) \right) | 0 \rangle.
\]

(5.6)

The sum in (5.6) is taken over \(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}\), running from 0 to infinity, for \(i, l, m, n, o = 1, 2, l \leq m, n \leq o\), and over the other indices, running from 0 to 1. Next, we derive from the equations determining the physical vector, \(Q' | \chi \rangle = 0\), and from the reducible gauge transformations, \(\delta | \chi \rangle = Q' | \Lambda \rangle\), \(\delta | \Lambda \rangle = Q' | \Lambda^{(1)} \rangle\), \ldots, \(\delta | \Lambda^{(s-1)} \rangle = Q' | \Lambda^{(s)} \rangle\), a sequence of relations:

\[
Q | \chi \rangle = 0,
\]

(5.7)

\[
\delta | \chi \rangle = Q | \Lambda \rangle,
\]

(5.8)

\[
\delta | \Lambda \rangle = Q | \Lambda^{(1)} \rangle,
\]

(5.9)

\[
\delta | \Lambda^{(s-1)} \rangle = Q | \Lambda^{(s)} \rangle,
\]

(5.10)
The middle set of equations in (5.7)-(5.10) determines the possible values of the parameters $h^i$ and the eigenvectors of the operators $\sigma^i$. Solving these equations, we obtain a set of eigenvectors, $|\chi\rangle_{(n_1,n_2)}$, $|\Lambda\rangle_{(n_1,n_2)}$, ..., $|\Lambda^{(s)}\rangle_{(n_1,n_2)}$, $n_1 \geq n_2 \geq 0$, and a set of eigenvalues,

$$-h^i = n^i + \frac{d - 4}{2} - \delta^{2i} 2, \quad i = 1, 2, \quad n_1 \in \mathbb{Z}, n_2 \in \mathbb{N}_0,$$

with $(n_1, n_2)$ related to spin, $s = (n_1, n_2) + (1/2, 1/2)$. The values of $n_1, n_2$ are related to the spin components $s_1, s_2$ of the field, because the proper vector $|\chi\rangle_{(n_1,n_2)}$ corresponding to $(h_1, h_2)$ has the leading term $\lambda_{(\mu_1,\nu_1)} (\tau, x)$, independent of the auxiliary and ghost operators, which corresponds to the field $\Phi_{(\mu_1,\nu_1)} (\tau, x)$ with the initial value of spin $s = (s_1, s_2)$ in the decomposition (5.6),

$$|\chi\rangle_{(n_1,n_2)} = \left[ a^{+\mu_1} \ldots a^{+\mu_{n_1}} a^{'+\mu_1} \ldots a^{'+\nu_{n_2}} \chi_{(\mu_1,\nu_1)} (\tau, x) + b^{+\mu_1} \ldots a^{'+\mu_{n_1+1}} a^{'+\nu_{n_2}} \chi_{(\mu_1+1,\nu_1)} (\tau, x) + b^{+n_{n_1}} \ldots a^{'+\nu_{n_2+1}} \chi_{(\mu_1,\nu_1+1)} (\tau, x) + \ldots \right]|0\rangle,$$

where the values of $(n_1, n_2)$ can be composed of the set of coefficients $\{k_i\} \setminus \{k_1, k_4\}$ in (5.6) by the formulae

$$n_i = k_2 + k_3 + k_5 + k_6 + k_7 + 2k_{12} + 2k_{13} + 2k_{14} + 2k_{15},$$

Therefore, relations (5.7)-(5.10) guarantee both the extraction of vectors with the required value of spin and the nilpotency of $Q$ in the corresponding Hilbert subspace. If one fixes the value of spin, then the parameters $h^i$ are also fixed by (5.11). Having fixed the value of $h^i$, we should substitute it into each of the expressions (5.7)-(5.10).

Third, we should extract the zero-mode ghosts from the operator $Q$ as follows:

$$Q = q_0 T_0 + \eta_0 L_0 + i(\eta^+_i q_i - \eta q^+_i)p_0 - i(q^0_0 - \eta^+ \eta)P_0 + \Delta Q,$$

where

$$T_0 = T_0 - 2q_i^+ P_i - 2q_i P_i^+,$$

$$\Delta Q = q_i^+ T^i + T^i q^+_i + \eta^+ L^i + L^i \eta^+ + \eta_{m} L_{im} L_{im} + \eta^+ T + T^+ \eta + \eta_{j} q^+_i p^+_i + \eta q_i p_i + \eta^+ \eta P_i + \eta^+ \eta P_i^+ - 2q_i^2 P_i - 2q_i^2 P_i^+ - 2q_1^2 q_2^2 q_i^+ \eta_{11} \eta_{11} \eta_{22} P_{12} - \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22} + \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22} + \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22} + \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22} + \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22} + \eta_{12}^2 \eta_{11}^2 P_{12} + \eta_{12}^2 \eta_{22}^2 P_{12} + \eta_{12}^2 \eta_{11}^2 P_{22} + \eta_{12}^2 \eta_{22}^2 P_{22}.$$
Following the procedure described in \cite{21, 44}, we get rid of all the fields except two, \( |\chi_0^0\rangle, |\chi_1^1\rangle \).

Namely, after the extraction of zero-mode ghosts from the BRST operator \( Q \) (5.14), as well as from the state vector and the gauge parameter (5.17), (5.18), the gauge transformation for the fields \( |\chi_0^k\rangle, k \geq 2 \) has the form
\[
\delta |\chi_0^k\rangle = \Delta Q |\Lambda_0^k\rangle + \eta_i \eta^+_i |\Lambda_1^k\rangle + (k+1)(q_i \eta^+_i - \eta_i q^+_i) |\Lambda_0^{k+1}\rangle + \tilde{T}_0 |\Lambda_0^{k-1}\rangle + |\Lambda_1^{k-2}\rangle,
\]
(5.19)
implying, by induction, that we can make all the fields \( |\chi_0^k\rangle, k \geq 2 \) equal to zero by using the gauge parameters \( |\Lambda_1^k\rangle \). Then, considering the equations of motion for the fields with a fixed value of spin: \( q^0_k, k \geq 3 \) and taking into account that \( |\chi_0^k\rangle = 0, k \geq 2 \), we can see that these equations contain the subsystem
\[
|\chi_1^{k-2}\rangle = \eta_i \eta^+_i |\chi_1^k\rangle, \quad k \geq 3,
\]
(5.20)
which permits us to find, by induction, that all the fields \( |\chi_1^k\rangle, k \geq 1 \) are equal to zero. Finally, we examine the equations of motion for the power \( q^2_0 \):
\[
|\chi_1^0\rangle = -\tilde{T}_0 |\chi_0^0\rangle,
\]
(5.21)
in order to express the vector \( |\chi_0^0\rangle \) in terms of \( |\chi_1^0\rangle \). Thus, as in the totally symmetric case, there remain only two independent fields: \( |\chi_0^0\rangle, |\chi_1^1\rangle \). The first equation in (5.7), (5.14), the decomposition (5.17), and the above analysis then imply that the independent equations of motion for these vectors have the form
\[
\Delta Q |\chi_0^0\rangle + \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} |\chi_1^1\rangle = 0,
\]
(5.22)
\[
\tilde{T}_0 |\chi_0^0\rangle + \Delta Q |\chi_0^1\rangle = 0,
\]
(5.23)
where \( \{ F, G \} = FG + GF \) for any quantities \( F, G \).

Then, due to the fact that the operators \( Q, \tilde{T}_0, \eta_i^+ \eta_i \) commute with \( \sigma^i \), we obtain from (5.22), (5.23) the equations of motion for the fields with a fixed value of spin:
\[
\Delta Q |\chi_0^0\rangle^{(n_1, n_2)} + \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} |\chi_1^1\rangle^{(n_1, n_2)} = 0,
\]
(5.24)
\[
\tilde{T}_0 |\chi_0^0\rangle^{(n_1, n_2)} + \Delta Q |\chi_0^1\rangle^{(n_1, n_2)} = 0.
\]
(5.25)
where the fields \( |\chi_0^k\rangle^{(n_1, n_2)}, k = 0, 1 \) are assumed to obey the relations
\[
\sigma^i |\chi_0^k\rangle^{(n_1, n_2)} = (n_i + (d - 4)/2 - 2\delta_{i2}) |\chi_0^k\rangle^{(n_1, n_2)} , \quad k = 0, 1.
\]
(5.26)
The field equations (5.24), (5.25) are Lagrangian ones and can be deduced, in view of the invertibility of the operator \( K \), from the following Lagrangian action\(^6\)
\[
S^{(n_1, n_2)} = \langle \chi_0^0 | K^{(n_1, n_2)} \tilde{T}_0 |\chi_0^0\rangle^{(n_1, n_2)} + \frac{1}{2} \langle \chi_0^0 | K^{(n_1, n_2)} \{ \tilde{T}_0, \eta_i^+ \eta_i \} |\chi_0^1\rangle^{(n_1, n_2)} ,
\]
(5.27)
\[
+ \langle \chi_0^0 | K^{(n_1, n_2)} \Delta Q |\chi_0^1\rangle^{(n_1, n_2)} + \langle \chi_0^1 | K^{(n_1, n_2)} \Delta Q |\chi_0^0\rangle^{(n_1, n_2)} ,
\]
where the standard scalar product for the creation and annihilation operators is assumed, and \( K^{(n_1, n_2)} \) is the operator \( K \) (4.8) with the following substitution: \( h^i \rightarrow -(n_i + (d - 4)/2 - 2\delta_{i2}) \).

The equations of motion (5.24), (5.25) and the action (5.27) are invariant with respect to the gauge transformations
\[
\delta |\chi_0^0\rangle^{(n_1, n_2)} = \Delta Q |\Lambda_0^0\rangle^{(n_1, n_2)} + \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} |\Lambda_0^1\rangle^{(n_1, n_2)} ,
\]
(5.28)
\[
\delta |\chi_0^1\rangle^{(n_1, n_2)} = \tilde{T}_0 |\Lambda_0^0\rangle^{(n_1, n_2)} + \Delta Q |\Lambda_0^1\rangle^{(n_1, n_2)} ,
\]
(5.29)
\(^6\)As usual, the action is defined up to an overall factor.
which are reducible, with the gauge parameters \(|\Lambda^{(s)j}_{0} - \Lambda^{(s)j}_{0}(n_{1,n_{2}})\), \(j = 0, 1\) subject to the same conditions as those for \(|\chi^{j}_{0} - \Lambda^{(s)j}_{0}(n_{1,n_{2}})\) in (5.26),

\[
\delta|\Lambda^{(s)0}_{0}(n_{1,n_{2}}) = \Delta Q|\Lambda^{(s+1)0}_{0}(n_{1,n_{2}}) + \frac{1}{2} - T_{0}, \eta_{l}^{2} |\Lambda^{(s+1)1}_{0}(n_{1,n_{2}}),
\]

\[
\delta|\Lambda^{(s)1}_{0}(n_{1,n_{2}}) = - T_{0}|\Lambda^{(s+1)1}_{0}(n_{1,n_{2}}) + \Delta Q|\Lambda^{(s+1)0}_{0}(n_{1,n_{2}}),
\]

\[
|\Lambda^{(0)0}_{0} = |\chi^{0}_{0}, \]

\[
|\Lambda^{(0)1}_{0} = |\chi^{1}_{0},
\]

and with a finite number of reducibility stages at \(s_{\text{max}} = n_{1} + n_{2}\) for spin \(s = (n_{1} + 1/2, n_{2} + 1/2)\).

In addition to restrictions (5.13), the set of coefficients \(\{k_{s}\}\) \(\{k_{1}, k_{2}\}\) in (5.6) for fixed values of \(n_{1}\), satisfies the following equations for \(|\chi^{j}_{0}(n_{1,n_{2}})\) \(|\Lambda^{(s)j}_{0}(n_{1,n_{2}})\), respectively,

\[
|\chi^{j}_{0}(n_{1,n_{2}}) = \sum_{i} (k_{2i} - k_{3i} + k_{6i} - k_{7i}) + \sum_{i \leq m} (k_{lm}^{8} - k_{lm}^{9}) + k_{10} - k_{11} = -j,
\]

\[
|\Lambda^{(s)j}_{0}(n_{1,n_{2}}) = \sum_{i} (k_{2i} - k_{3i} + k_{6i} - k_{7i}) + \sum_{i \leq m} (k_{lm}^{8} - k_{lm}^{9}) + k_{10} - k_{11} = -(s + j + 1),
\]

due to the ghost number distribution (5.17), (5.18).

Thus, we have constructed, by using the BRST procedure, a gauge-invariant Lagrangian description of fermionic fields with a mixed symmetry of any fixed spin \(s\).

A Lagrangian description for a half-integer mixed-symmetry HS field of mass \(m\) in a \(d\)-dimensional Minkowski space can be deduced in two ways. First, one should use a modified procedure starting from the Dirac equation

\[
(i \gamma^{\mu} \partial_{\mu} - m) \Phi_{(\mu)_{n_{1},(\nu)_{n_{2}}}} = 0 \iff (i \gamma^{\mu} \partial_{\mu} - \tilde{\gamma} m) \Phi_{(\mu)_{n_{1},(\nu)_{n_{2}}}} = 0,
\]

with the relations (2.3), (2.4) being unaltered. The equation (5.34) contains a massive term in both even and odd space-time dimensions (see footnote 3), so that an equivalent description in terms of the Clifford algebra elements \(\tilde{\gamma}^{\mu}, \tilde{\gamma}\) is possible for both \(d = 2N\) and \(d = 2N + 1\):

\[
\tilde{\gamma} = \kappa_{d} \Pi \gamma^{d} \text{ for } \gamma^{d} = \frac{1}{d!} \left( \prod_{i=1}^{d} \gamma^{\mu_{i}} \right) \epsilon^{\mu_{1}...\mu_{d}} \text{ and } \kappa_{d} = \begin{cases} 1, & d = 4M, \\ \frac{1}{4}, & d = 4M + 2, \end{cases}
\]

with an odd non-degenerate supermatrix \(\Pi\), such that \(\Pi^{2} = 1\), and with the Levi-Civita tensor \(\epsilon^{\mu_{1}...\mu_{d}}\) normalized as \(\epsilon^{01...d-1} = 1\). For \(d = 2N - 1\), we enlarge all the spin-tensors \(\Phi_{A(\mu)_{n_{1},(\nu)_{n_{2}}}}(x),...\) to those with doubled components \(\hat{\Phi}_{2A(\mu)_{n_{1},(\nu)_{n_{2}}}} = (\hat{\Phi}_{A(\mu)_{n_{1},(\nu)_{n_{2}}}}, \hat{\Phi}_{A(\mu)_{n_{1},(\nu)_{n_{2}}}^{T}},...\), with the Dirac index \(A = 1,...,2^{N-1}\), while also enlarging \(\gamma^{\mu}_{AB}\) to \(\gamma^{\mu}_{(2A)(2B)}\text{=antidiag}(\gamma^{\mu}_{AB}, \gamma^{\mu}_{AB})\) (see (5.43)), which are now elements of a \(d' = d + 1\)-dimensional space-time (although with \(\gamma^{d+1}_{(2A)(2B)}\) being absent), where \(\gamma^{d+1} = \gamma^{2N}\). The Lagrangian formulation is determined by the same relations as those in the massless case, with some modifications: first, for the initial operators \(t_{0}, l_{0}\),

\[
(t_{0}, l_{0}) \rightarrow (\tilde{t}_{0}, \tilde{l}_{0}) = (t_{0} + \tilde{\gamma} m, l_{0} + m^{2}),
\]

which, along with the remaining elements of \(\sigma_{t}\), obey the same HS symmetry superalgebra, except for the additional commutators

\[
[t_{i}^{+}, l_{j}^{+}] = \delta_{ij}(\tilde{l}_{0} - \tilde{\gamma} m), \quad [t_{i}, l_{j}^{+}] = -\delta_{ij}(\tilde{l}_{0} - \tilde{\gamma} m), \quad [l_{i}, l_{j}^{+}] = \delta_{ij}(\tilde{l}_{0} - m^{2}).
\]

\(^{7}\text{In the case of a spin-tensor field } \Phi_{(\mu)_{n_{1},(\nu)_{n_{2}}},...,(\rho)_{n_{k}}} (x) \text{ with the Young tableaux having } k \text{ rows, one can show that the stage of reducibility for the corresponding Lagrangian formulation must be equal to } s_{\text{max}} = \sum_{i=1}^{k} n_{i} + k(k - 1)/2 - 1, \text{ so that for a totally symmetric field } \Phi_{(\mu)_{n_{1}}} (x) \text{ one has } s_{\text{max}} = n_{1} - 1, \text{ in accordance with } 11\).
The additional parts $\sigma'_l$ coincide with those of the massless case, whereas the converted set of constraints has the form $O^m_l = \hat{o}_l + \sigma'_l$, no longer having a central charge $\tilde{\gamma}_m$ for massive HS fields with $\hat{o}_l = o_l + \hat{o}_l(b_i, b_i^\dagger)$ (additional bosonic 4-oscillators $[b_i, b_i^\dagger] = \delta_{ij}$ acting in the Fock space $\mathcal{H}_m$), determined by adding the terms induced by dimensional reduction

\[
\begin{pmatrix}
\hat{t}_0, \hat{t}_0, \hat{t}_i, \hat{t}_{ij}, \hat{g}_0^i \\
\end{pmatrix} = \left(0, 0, mb_i, mb_i^\dagger, b_i^\dagger b_i + \frac{1}{2}\right),
\]

\[
\begin{pmatrix}
\hat{t}_i, \hat{t}_i^+, \hat{t}_{ij}, \hat{t}_{ij}^+, \hat{t}^+ \\
\end{pmatrix} = -\left(\tilde{\gamma}_b, \tilde{\gamma}_b^+; \frac{1}{2} b_i b_j, \frac{1}{2} b_i^\dagger b_j^\dagger, b_i^\dagger b_2, b_1 b_2^\dagger, b_{ij}^\dagger b_{ij}^+ \right).
\]

The set of $O^m_l$ satisfies the same HS symmetry superalgebra as the one for massless half-integer irreducible representations of the Poincare group). The generalized spin and BRST operators, $\sigma^i_m = \mathcal{H}_m + \ldots, Q_m = Q|_{O \to O^m}$, with an arbitrary vector $|\chi_m\rangle \in \mathcal{H}^m_{\text{tot}} = \mathcal{H}_\text{tot} \otimes \mathcal{H}_m$, are formally identical with those of the massless case (5.4), (5.2), after the replacement $\mathcal{H}^i \to \mathcal{H}^i_m$, whereas $|\chi_m\rangle$ has the vector $|\chi\rangle$ (5.6) as the massless limit for $b_i^\dagger = 0$:

\[
|\chi_m\rangle = \sum_{n'_l \geq (0)_l} \prod_{l=1}^2 (b_i^\dagger)^{n'_l} |\chi'_l(a^+, f^+, f^+, p^+, q^+, \eta^+, \mathcal{P}^+)) \rangle \text{ for } |\chi'_l\rangle \in \mathcal{H}_\text{tot}.
\]

Note that the $b_i^\dagger$-independent vectors $|\chi'_l\rangle$, for $l = 0, 1, 2$, have a decomposition in oscillator powers, given by (5.6). At the same time, one may follow the dimensional reduction [52 53] of a massless HS field theory of the same type in a $(d + 1)$-dimensional flat space $\mathbb{R}^{1,d}$. To this end, we make a projection $\mathbb{R}^{1,d} \to \mathbb{R}^{1,d-1}$ over the sphere $S^1$ with a simple decomposition,

\[
\begin{align*}
\partial^M &= (\partial^\mu, -um), & a^M_i &= (a^\mu_i, b_i), & a^M_i &= (a^\mu_i, b_i^\dagger), \\
M &= 0, 1, \ldots, d, & \mu &= 0, 1, \ldots, d-1, & \eta^{MN} &= \text{diag}(1, -1, \ldots, -1, -1),
\end{align*}
\]

\[
\tilde{\gamma}^M_{2^{d+1}} = \frac{\tilde{\gamma}^M_{2N}}{2^{d/2}} = \begin{cases} 
\tilde{\gamma}^M_{2[d]} & d = 2N, \\
0 & d = 2N - 1,
\end{cases}
\]

\[
\Phi_{\Lambda(M)_{n_1}(N)_{n_2}}(x, x^d) = \begin{cases} 
\exp\{imx^d\} \Phi_{A(\mu)_{n_1-r_1}(\nu)_{n_2-r_2}}(x), r_i = 0, \ldots, n_i, & d = 2N \\
\exp\{imx^d\} \left( \Phi_{L(\mu)_{n_1-r_1}(\nu)_{n_2-r_2}} \right)(x), & d = 2N - 1,
\end{cases}
\]

for $A = (a, b)$ with $d = 2N$ and $a, b = 1, \ldots, 2^{N-1}$. For odd $d = 2N - 1$, it is only the $2^{N-1}$-dimensional $\gamma$-matrices and spinors in the left-hand side of (5.43), (5.44) that correspond to an irreducible representation. The reduction from $\mathbb{R}^{1,2N}$ to $\mathbb{R}^{1,2N-1}$ leads precisely to the Dirac equation (5.43) for $(M, N) = (\mu, \nu)$: $i\tilde{\gamma}^M \partial_M \Phi_{A(\mu)_{n_1}(\nu)_{n_2}}(x, x^d) = \exp\{imx^d\} (i\tilde{\gamma}^\mu \partial_\mu + i\tilde{\gamma}^\nu) \Phi_{A(\mu)_{n_1}(\nu)_{n_2}}(x) = 0$. To make a projection from $\mathbb{R}^{1,2N-1}$ to $\mathbb{R}^{1,2N-2}$, one may use the following representation of $\gamma^M = \tilde{\gamma}^M \gamma$-matrices, which can be realized inductively, according to (5.43), as an outer product of the unity $1_2$ and Pauli $\sigma_i$-matrices, $i = 1, 2, 3$, for any even $d$:

\[
d = 2 : \quad (\gamma^0_2, \gamma^1_2, \gamma^2_2) = (\sigma_1, \sigma_2, \sigma_3 (\gamma^0_2 \gamma^1_2)) \quad \text{for} \quad (\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
d = 2N : \quad \gamma^\mu_{2N} = \text{antidiag}(\gamma^\mu_{2N-1}; \mu = 0, \ldots, 2N - 2, \gamma^\mu_{2N} = \begin{pmatrix} 0 & 1_{2N-1} \\ -1_{2N-1} & 0 \end{pmatrix},
\]

\[
\gamma^d_{2N} = (\pm 1)^{d/2+1} \prod_{i=0}^{d-1} \gamma^i_{2N} = \pm \begin{pmatrix} 0 & 1_{2N-1} \\ -1_{2N-1} & 0 \end{pmatrix}.
\]

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where the gamma-matrices of a $d = 2N$-dimensional space are used for $d = 2N + 1$, and the matrix $\gamma^\mu_{2N}$ is introduced (5.47). By the Pauli theorem, any set of $\gamma^\mu_{2N}$-matrices for a fixed $d$ can be obtained from the given set by using a similarity transformation. In this case, a respective massless spinor possesses $2N$ components and contains two Weyl spinors in a transposed form $(\Phi_L|a(M)_{n_1},(N)_{n_2}, \Phi_R|b(M)_{n_1},(N)_{n_2})^T(x, x^d)$, being an element of an $ISO(1, 2N - 1)$ reducible massless representation for a half-integer spin, and having the property of an eigenvector for $\gamma^\mu_{2N}$, $\Phi_R,L = (1/2)(1 \mp \gamma^\mu)\Phi$. Any element of an irreducible massless representation has to be chiral, e.g., right-handed, $(0, \Phi_R|b(M)_{n_1},(N)_{n_2})^T(x, x^d)$, which transforms by the projection (5.44) into the set $\Phi_R|b(M)_{1_{r_1}(\nu)_{r_2}}(x)$. According to (5.45)–(5.47), $\gamma^M_{2N}, M = 0, \ldots, d$, become $\gamma^M_{N-1}, \mu = 0, \ldots, d-1$, and $\gamma^d_{2N-1} = 1_{2N-1}$. Once again, for the Dirac equation projected from $\mathbb{R}^{1,2N-1}$ to $\mathbb{R}^{1,2N-2}$ with $(M, N) = (\mu, \nu)$, we have

$$\gamma^\mu_{M}\partial_M(0, \Phi_R|b(\mu)_{n_1},(\nu)_{n_2})^T(x, x^d) = \exp\{imx^d\}(\gamma^\mu_{M}\partial_M - m)\Phi_R|b(\mu)_{n_1},(\nu)_{n_2}(x) = 0. \quad (5.48)$$

Left-handed massless spin-tensors can be subject to a similar treatment and thereby represent after projection independent massive fermions.

For the doubled massive components $\Phi_A(\mu)_{n_1},(\nu)_{n_2} = (\Phi_R|a(\mu)_{n_1},(\nu)_{n_2}, \Phi_R|a(\mu)_{n_1},(\nu)_{n_2})^T$, the representation (5.31) with $2N$-dimensional $\gamma^\mu$-matrices is also valid.

Therefore, the quantities $\gamma T_0\gamma$, identical with $T_0$ for massless HS fields, transform for massive fields as $\gamma T_0\gamma = T_0^*$, $T_0^* = -\gamma^\mu\partial_\mu - \gamma m$. Hence, for the coefficient functions entering (along with the matrix $\gamma$) into the decomposition of any vector composed of $|\chi^j_{0}(n_1,n_2), |\Lambda^{(s)j}_{0}(n_1,n_2)$, the part of a vector $|\Xi\rangle\in\{\Lambda^{(s)j}_{0}(n_1,n_2)$, the part of a vector $|\Xi\rangle\in\{\chi^j_{0}, |\Lambda^{(s)j}_{0}\}$ homogeneous with respect to $\gamma$ is subject to

$$\gamma^M_M\partial_M|\Xi\rangle = (\gamma^\mu_M\partial_\mu - (-1)^{\deg_\gamma|\Xi\rangle}m)|\Xi\rangle, \quad \deg_\gamma|\Xi\rangle = 0, 1, \quad |\Xi\rangle \in \{\chi^j_{0}, |\Lambda^{(s)j}_{0}\}. \quad (5.49)$$

Relations (5.31) and (5.32) indicate the presence of four additional second-class constraints, $l_i, l_i^+$, with the corresponding oscillator operators $b_i, b_i^+$, $[b_i, b_j^+] = \delta_{ij}$, in comparison with the massless case.\footnote{Based on the above reasons, one can state that the procedure of dimensional reduction given by (5.31), (5.42) can be applied to massless mixed-symmetry bosonic fields \cite{11} in order to obtain a Lagrangian description of massive mixed-symmetry bosonic fields, whereas for fermionic HS fields it is necessary to make allowance for the matrix structure, according to (5.33), (5.44), (5.48), (5.50); cf. \cite{11}.}

A simultaneous construction of Lagrangian actions describing the propagation of all massless (massive) fermionic fields with two rows of the Young tableau in Minkowski space is similar to the case of totally symmetric spin-tensors in flat spaces \cite{14}, and we only note that a necessary condition for solving this problem is to replace in $Q$, $Q$, $K$ the parameters $-h^i$ by the operators $\sigma^i$ in an appropriate way, and to discard the condition (5.21) for the fields and gauge parameters. Amongst other things, this completes both the conversion procedure for the initial constraint system $\{a_I\} \setminus \{g^i_0\}$ and the construction of a nilpotent BRST operator in the entire Hilbert space for the set of converted constraints $\{O_I\} \setminus \{H^i\}$.

In the section to follow, we outline a proof of the fact that the action, in fact, reproduces the correct equations of motion (2.2)–(2.5).

### 6 Reduction to the Initial Irreducible Relations

Let us briefly show the fact that it is only the solutions of the equations of motion (2.2)–(2.5) that determine the space of BRST cohomologies of the operator $Q$ (5.2) with a vanishing ghost number.
in the Fock space $\mathcal{H}$ for the basic fermionic field with spin $s = (n_1 + 1/2, n_2 + 1/2)$. To solve this problem, we can follow two ways: the first one is realized, for instance, in [20] for massless totally-symmetric bosonic fields in a flat space-time, and the second one, for totally-symmetric fermionic fields [14] [15]. We will use the technics of [14] [15], taking into account the fact that the spectrum of component fields for an arbitrary vector $|\chi\rangle_{(n_1,n_2)}$ in (5.6) for $k_1 = k_4 = 0$ is essentially larger than the spectrum for a totally-symmetric fermionic vector $|\chi\rangle_{(n_1+n_2,0)}$, for whose description one should not use the operators $q_0^+, p_0^+, f_0^+, \eta_2^+, \eta_4^+, \eta_6^+, \eta_8^+, \eta_9^+, \eta_+^+, b^+, b_1^+, b_2^+, b_3^+, a_1^+, a_2^+$ and the corresponding conjugations.\footnote{The total number of independent “creation” operators which are necessary to compose the vector $|\chi\rangle_{(n_1,n_2)}$ is more than twice as large as the number required for $|\chi\rangle_{(n_1+n_2,0)}$: $(2d + 22)/(d + 8)$.} As a consequence, the character of proof is more involved even in comparison with the case of the AdS space [45].

In the standard manner, the proof consists of two steps. First, in order to simplify the spectrum of the gauge parameters $|\Lambda^{(s)}j\rangle_0$ and the fields $|\chi^0\rangle_j$, $j = 0, 1$, we apply to them a gauge-fixing based on the structure of gauge transformations (5.28)–(5.31) and extract the physical field $|\Phi\rangle_{(n_1,n_2)}$ alone, by using (2.7) for $s = (n_1 + 1/2, n_2 + 1/2)$:

$$
|\chi^0\rangle_{(n_1,n_2)} = |\Phi\rangle_{(n_1,n_2)} + |\Phi_A\rangle_{(n_1,n_2)}, 
|\Phi_A\rangle_{(n_1,n_2)}\big|_{c=b_{ij}^+=b^+=f_i^+=0} = 0. 
(6.1)
$$

Second, we use a part of the Lagrangian equations of motion (5.24), (5.25) in order to select from them only the equations of motion for $|\Phi\rangle_{(n_1,n_2)}$, and to remove all of the remaining auxiliary fields of lower spins.

Let us now describe the basic sequence of gauge-fixing. Our strategy consists in a successive elimination of the terms with $\mathcal{P}^+_{11}$ from the fields $|\chi^0\rangle_j$ and gauge parameters $|\Lambda^{(s)}j\rangle_0$, starting from the top of the tower of gauge transformations (5.28)–(5.31). For this purpose, it should be noted that we have a reducible gauge theory of $(n_1 + n_2)$-th stage of reducibility. Because of the restrictions for the spin (5.13) and ghost number (5.32), (5.33), the independent parameters of the lowest stage have the form

$$
|\Lambda^{(n_1+n_2)0}\rangle_{(n_1,n_2)} = (p_1^+)^{n_1} A(p_1^+,\mathcal{P}_i^+,\mathcal{P}^+)_{(n_2)}, 
|\Lambda^{(n_1+n_2)1}\rangle_{(n_1,n_2)} \equiv 0, 
(6.2)
$$

where the vector $A(p_1^+,\mathcal{P}_i^+,\mathcal{P}^+))_{(n_2-m)} \equiv A(l,n_2-m)$, $l < n_1$, $m < n_2$, has the structure

$$
|A\rangle_{(l,n_2-m)} = \mathcal{P}^+ \left\{ \sum_{k=m+1}^{n_2} (p_1^+)^k (p_2^+)^{n_2-k} |\omega^1_k\rangle_{(l+k+1,k+1-l-m)} + \mathcal{P}_1^+ \sum_{k=m}^{n_2-1} (p_1^+)^k (p_2^+)^{n_2-k-1} |\omega^2_k\rangle_{(l+k,k-l-m)} + \mathcal{P}_2^+ \sum_{k=m+1}^{n_2-1} (p_1^+)^k (p_2^+)^{n_2-k-1} |\omega^3_k\rangle_{(l+k+1,k+1-l-m)} + \mathcal{P}_1^+ \mathcal{P}_2^+ \sum_{k=m}^{n_2-2} (p_1^+)^k (p_2^+)^{n_2-k-2} |\omega^4_k\rangle_{(l+k,k-l-m)} \right\}. 
(6.3)
$$

It can be verified directly that one can eliminate the dependence on the ghost $\mathcal{P}_{11}^+$ from the gauge function $|\Lambda^{(n_1+n_2-1)0}\rangle_{(n_1+n_2-1)}$ of $(n_1 + n_2 - 1)$-th stage of reducibility, whereas the vector
\( |\Lambda^{(n_1+n_2-1)_{(0)}} \rangle \) has the same structure as \( |\Lambda^{(n_1+n_2)_{(0)}} \rangle \) in (6.2). Indeed, for \( |\Lambda^{(n_1+n_2-1)_{(0)}} \rangle \) we have the following expansion in the powers of \( \mathcal{P}_{11}^+, \mathcal{P}_{12}^+, \mathcal{P}_{22}^+ \):

\[
|\Lambda^{(n_1+n_2-1)_{(0)}} \rangle_{(n_1, n_2)} = |\Lambda^{(n_1+n_2-1)_{(0)}} \rangle_{(n_1, n_2)} + \mathcal{P}_{11}^+ (p_1^+)_{n_2-2} |\tilde{A} \rangle_{(0, n_2)} + \mathcal{P}_{12}^+ (p_1^+)_{n_2-1} |\tilde{A} \rangle_{(0, n_2-1)} + \mathcal{P}_{22}^+ (p_1^+)_{n_2} |\tilde{A} \rangle_{(0, n_2-2)},
\]

(6.4)

with \( |\tilde{A} \rangle_{(0, n_2-k)} \) defined according to (6.3), so that the gauge transformation (5.30) at \( \mathcal{P}_{11}^+ \) implies

\[
\delta |\tilde{A} \rangle_{(0, n_2)} = -2q_1^2 (p_1^+)^2 |A \rangle_{(0, n_2)}.
\]

(6.5)

After the vector \( |\tilde{A} \rangle_{(0, n_2)} \) has been removed, the theory is transformed to a theory of \( (n_1 + n_2 - 1) \)-th stage of reducibility. Then, it is possible to verify that one can remove the dependence of \( |\Lambda^{(n_1+n_2-2)_{(0)}} \rangle \) on \( \mathcal{P}_{11}^+ \) with the help of the remaining gauge parameters \( |\Lambda^{(n_1+n_2-1)_{(0)}} \rangle \), which do not depend on \( \mathcal{P}_{11}^+ \).

It then becomes possible to prove by induction that after removing the dependence on \( \mathcal{P}_{11}^+ \) from the gauge parameters up to the \( (s + 1) \)-th stage, \( |\Lambda^{(l)_{(0)}} \rangle, k = 0, 1, l \geq s + 1 \) (i.e., we have \( \eta_{11} |\Lambda^{(l)_{(k)}} \rangle = 0 \)), and applying the restricted vector \( |\Lambda^{(s+1)_{(0)}} \rangle \), one can eliminate the dependence on \( \mathcal{P}_{11}^+ \) from the gauge functions \( |\Lambda^{(s)_{(k)}} \rangle \). To this end, we introduce the following notation for the gauge parameters related to their expansion in the ghosts \( \mathcal{P}_{ij}^+ \) :

\[
|\Lambda^{(l)_{(0)}} \rangle = |\Lambda^{(l)_{(0)}} \rangle_{0} + \sum_{i \leq j} \mathcal{P}_{ij}^+ |\Lambda^{(l)_{ij}} \rangle + \mathcal{P}_{11}^+(\mathcal{P}_{12}^+|\Lambda^{(l)_{01}} \rangle

+ \mathcal{P}_{22}^+|\Lambda^{(l)_{02}} \rangle + \mathcal{P}_{12}^+ \mathcal{P}_{22}^+|\Lambda^{(l)_{03}} \rangle) + \mathcal{P}_{12} \mathcal{P}_{22}^+|\Lambda^{(l)_{04}} \rangle.
\]

(6.6)

Here and elsewhere, we omit the vector subscripts associated with the eigenvalues of the operators \( \sigma^i \) (5.26). From (5.30), (5.31), we obtain the gauge transformations for \( |\Lambda^{(s)_{(0)}} \rangle, |\Lambda^{(s)_{(0)}} \rangle, p = 1, 2, 3, \) being the coefficients at \( \mathcal{P}_{ij}^+ \), namely,

\[
\delta |\Lambda^{(s)_{01}} \rangle = -2q_1^2 |\Lambda^{(s+1)_{00}} \rangle + \eta^+ |\Lambda^{(s+1)_{02}} \rangle,
\]

(6.7)

\[
\delta |\Lambda^{(s)_{02}} \rangle = -2q_1^2 |\Lambda^{(s+1)_{01}} \rangle - \eta^+ |\Lambda^{(s+1)_{02}} \rangle,
\]

(6.8)

\[
\delta |\Lambda^{(s)_{03}} \rangle = -2q_1^2 |\Lambda^{(s+1)_{01}} \rangle - \eta^+ |\Lambda^{(s+1)_{03}} \rangle.
\]

(6.9)

(6.10)

Then, a certain choice for \( |\Lambda^{(s+1)_{02}} \rangle, |\Lambda^{(s+1)_{03}} \rangle \) removes \( |\Lambda^{(s)_{03}} \rangle, |\Lambda^{(s)_{01}} \rangle \), respectively, whereas a certain choice for \( |\Lambda^{(s+1)_{02}} \rangle, |\Lambda^{(s+1)_{03}} \rangle \) eliminates \( |\Lambda^{(s)_{02}} \rangle, |\Lambda^{(s)_{01}} \rangle \) by means of the remaining gauge transformations. Thus, we have shown that the dependence on \( \mathcal{P}_{11}^+ \) can be eliminated from \( |\Lambda^{(l)_{(0)}} \rangle \). As a consequence of the above procedure, the theory becomes a gauge theory of \( l \)-th stage of reducibility.

This algorithm is valid down to the vector \( |\Lambda^{(n_2+1)_{(0)}} \rangle \), when there arise terms linear in \( p_1^+ \). When these terms are present, one deals with gauge parameters that have remained unused after eliminating the dependence on \( \mathcal{P}_{11}^+ \). Therefore, in view of the \( \eta^+ \)-dependent terms in (6.7), (6.9), a gauge transformation with such parameters may cause some \( \mathcal{P}_{11}^+ \)-dependent terms to appear in the transformed vector \( |\Lambda^{(n_2)_{(0)}} \rangle \). Consequently, it is necessary to make a gauge transformation with parameters linear in \( p_1^+ \), or independent of it, before removing the \( \mathcal{P}_{11}^+ \)-dependence. Let us examine a gauge transformation with the gauge function \( |\Lambda^{(n_2)_{(0)}} \rangle \) more carefully.

Suppose that the dependence on the ghost \( \mathcal{P}_{11}^+ \) in \( |\Lambda^{(n_2+1)_{(0)}} \rangle \) has been removed by a gauge
transformation, and hence the functions \( |\Lambda^{(n_2+1)0}_{\ell}\rangle \), \( |\Lambda^{(n_2)0}_{\ell}\rangle \) admit the following representation:

\[
|\Lambda^{(n_2+1)0}_{\ell}\rangle = p_1^+ \left( |\Lambda^{(n_2+1)0}_{\ell}\rangle + p_{12}^+ |\Lambda^{(n_2)0}_{\ell12}\rangle \right),
\]

\[
|\Lambda^{(n_2+1)1}_{\ell}\rangle = (p_1^+)^2 \left( |\Lambda^{(n_2+1)1}_{\ell1}\rangle + p_{12}^+ |\Lambda^{(n_2)0}_{\ell12}\rangle \right),
\]

\[
|\Lambda^{(n_2)0}_{\ell}\rangle = |\Lambda^{(n_2)0}_{\ell0}\rangle + \sum_{i,j} \mathcal{P}_{ij} |\Lambda^{(n_2)0}_{ij0}\rangle ,
\]

\[
|\Lambda^{(n_2)1}_{\ell}\rangle = p_1^+ \left( |\Lambda^{(n_2)1}_{\ell0}\rangle + p_{12}^+ |\Lambda^{(n_2)1}_{\ell12}\rangle + p_{22}^+ |\Lambda^{(n_2)1}_{\ell22}\rangle \right),
\]

where the vectors \( |\Lambda^{(i)0}_{\ell0}\rangle \), \( |\Lambda^{(i)k}_{\ell0}\rangle \), \( l = n_2, n_2 + 1 \), \( i, j = 1, 2, i \leq j \), possess terms having no dependence on \( p_1^+ \), except for \( |\Lambda^{(n_2)0}_{\ell0}\rangle \), and the vectors in \( (6.14) \) have a structure analogous to the corresponding structure in \( (6.11) \). Then, one has to make a transformation with parameters linear in \( p_1^+ \). We will use \( |\Lambda^{(n_2+1)0}_{\ell0}\rangle \), \( |\Lambda^{(n_2)0}_{\ell12}\rangle \) to make such gauge transformations. Since

\[
\delta |\Lambda^{(n_2)0}_{\ell0}\rangle = T_1^+ |\Lambda^{(n_2)0}_{\ell0}\rangle , \quad |\hat{\Lambda}^{(n_2+1)0}_{\ell0}\rangle = q_1 |\Lambda^{(n_2+1)0}_{\ell0}\rangle ,
\]

\[
\delta |\Lambda^{(n_2)0}_{\ell12}\rangle = T_1^+ |\Lambda^{(n_2)0}_{\ell12}\rangle , \quad |\hat{\Lambda}^{(n_2+1)0}_{\ell12}\rangle = q_1 |\Lambda^{(n_2+1)0}_{\ell12}\rangle ,
\]

one can use the vectors \( |\Lambda^{(n_2+1)0}_{\ell0}\rangle \), \( |\Lambda^{(n_2)0}_{\ell12}\rangle \) to eliminate the dependence on \( b_{11}^+ \) and \( f_1^+ \) from \( |\Lambda^{(n_2)0}_{\ell0}\rangle \) and \( |\Lambda^{(n_2)0}_{\ell12}\rangle \), respectively, due to the fact that the \( b_{11}^+ \) - and \( f_1^+ \)-linear components of the latter vectors are identical to the corresponding components of the previous vectors. As a result, we obtain the gauge-fixing

\[
b_{11} |\Lambda^{(n_2)0}_{\ell0}\rangle = f_1 |\Lambda^{(n_2)0}_{\ell0}\rangle = b_{11} |\Lambda^{(n_2+1)0}_{\ell0}\rangle = f_1 |\Lambda^{(n_2)0}_{\ell12}\rangle = 0 ,
\]

and then remove the \( \mathcal{P}_{11}^+ \)-dependence from \( |\Lambda^{(n_2)j}_{\ell0}\rangle \), as has been described in the case of the system \( (6.7) - (6.10) \).

Proceeding by induction, we may use the algorithm which has been applied to the treatment of the vectors \( (6.13) \), \( (6.14) \) in order to eliminate the dependence on \( \mathcal{P}_{11}^+ \) related to all the vectors down to \( |\Lambda^{(0)0}_{\ell0}\rangle \), whereas the \( \mathcal{P}_{22}^+ \)-independent terms in \( |\Lambda^{(i)0}_{\ell0}\rangle \) for \( l \geq n_2 \) are restricted by relations of the form \( (6.17) \).

Let us now turn to the gauge-fixing of the fields. We expand the fields in the powers of the ghosts \( \mathcal{P}_{ij}^+ \) by analogy with the gauge parameters:

\[
|\chi_{\ell0}^k\rangle = |\chi_{\ell0}^{k0}\rangle + \sum_{i \leq j} \mathcal{P}_{ij}^+ |\chi_{\ell0ij}^k\rangle + \mathcal{P}_{11}^+ \left( \mathcal{P}_{12}^+ |\chi_{\ell01}^k\rangle + \mathcal{P}_{22}^+ |\chi_{\ell02}^k\rangle \right) + \mathcal{P}_{12}^+ \mathcal{P}_{22}^+ |\chi_{\ell04}^k\rangle .
\]

Further, we need to restrict the vectors by the gauge conditions \( (6.17) \), which follow from the gauge transformations, and then we eliminate the terms coupled to \( \mathcal{P}_{11}^+ \).

Having completed the above procedure, we briefly mention that the remaining gauge ambiguity is sufficient to eliminate the auxiliary oscillators \( b_{ij}^+, b^+, f_i^+ \) from the field \( |\chi_{\ell0000}^{0000}\rangle \),

\[
|\chi_{\ell00}^{k0}\rangle = \sum_{l_i \geq 0} \sum_{m_j = 0} 1 \left( p_i^+ \right)^{l_i} \left( p_j^+ \right)^{m_j} |\chi_{\ell0}^{klm0}\rangle ,
\]

and therefore, in view of \( gh(|\chi_{\ell0}^{0000}\rangle) = 0 \), this field has no dependence on the ghost “coordinates”, so that, after the gauge-fixing, we conclude

\[
|\chi_{\ell0}^{0000}\rangle = |\Phi\rangle .
\]
The second step of establishing an equivalence of equations (2.2)–(2.5) with the Lagrangian equations (5.24), (5.25) is more involved and is based on a detailed expansion of equations (5.24), (5.25) in the powers of $p_i^+; q_i^+; \eta_i^+; \eta_{ij}^+; \eta^+$ and then in the powers of $b_i^+; f_i^+$. We only state the result that after gauge-fixing $|\chi_{00}^k\rangle$ and $|\chi_{012}\rangle, |\chi_{022}\rangle, |\chi_{044}\rangle$, expanded by analogy with (6.19), the only independent equations among (5.24), (5.25) have the form
\[
t_0|\Phi\rangle = t_1|\Phi\rangle = t|\Phi\rangle = 0,
\]
and all of the auxiliary fields can be made equal to zero.

In what follows, we consider some examples of the Lagrangian formulation procedure.

## 7 Examples

Here, we shall realize the general prescriptions of our Lagrangian formulation in the case of fermionic fields of lowest spins.

### 7.1 Spin-(3/2,1/2) Field

In the case of a field of spin (3/2,1/2), we have $(n_1, n_2) = (1, 0)$, $(h^1, h^2) = (1 - d/2, 4 - d/2)$. Since $s_{\text{max}} = 0$, the corresponding Lagrangian formulation is an irreducible gauge theory and describes a totally symmetric fermionic field of spin $s = 3/2$. The nonvanishing fields $|\chi_{0}^0\rangle, |\chi_{0}^1\rangle, |\chi_{0}^2\rangle$ and gauge parameters $|A_{0}^0\rangle, |A_{0}^1\rangle, |A_{0}^2\rangle$, (for $|A_{0}^0\rangle, |A_{0}^1\rangle, |A_{0}^2\rangle$, due to $gh(|A_{0}^1\rangle, |A_{0}^2\rangle) = -2$), have the following Grassmann grading and ghost number distribution:
\[
(\varepsilon, gh) (|\chi_{0}^0\rangle, |\chi_{0}^1\rangle) = (1, -i), \quad (\varepsilon, gh) (|A_{0}^0\rangle, |A_{0}^1\rangle) = (0, -1).
\]

These conditions determine the dependence of the fields and gauge parameters on the oscillator variables in a unique form, with the help of the operators corresponding only to the first row of the Young tableaux,
\[
|\chi_{0}^0\rangle = |\chi_{0}^1\rangle = \left[ -i a^+ \mu \psi_{\mu}(x) + f_1^+ \tilde{\gamma} \psi(x) \right] |0\rangle, \quad |\chi_{0}^2\rangle = \left[ \mathcal{P}_{1}^+ \tilde{\gamma} \chi(x) + p_{i}^+ \chi_{1}(x) \right] |0\rangle, \quad |A_{0}^0\rangle = \left[ \mathcal{P}_{1}^+ \xi_{1}(x) + p_{i}^+ \xi_{2}(x) \right] |0\rangle,
\]
\[
|\chi_{0}^1\rangle = \left[ \tilde{\psi}_{\mu}^+ a_{\mu} + \psi^+(x) \tilde{\gamma} f_1 \right] \tilde{\gamma}^0, \quad |\chi_{0}^2\rangle = \left[ \tilde{\chi}^0 \right].
\]

Substituting (7.2), (7.3) into (5.27), we find the action (up to an overall factor) for a free massless field of spin (3/2, 1/2) on a flat background:
\[
\mathcal{S}_{(1,0)} = \int d^d x \left[ \tilde{\psi}_{\mu} \left\{ i \gamma^\mu \partial_\mu \psi_{\mu} - \partial_\mu \chi - i \gamma_{\mu} \chi_{1} \right\} + (d - 2) \tilde{\psi} \left\{ i \gamma^\mu \partial_\mu \psi_{\mu} + \chi_{1} \right\} \right] + \tilde{\chi} \left\{ i \gamma^{\mu} \partial_\mu \chi_{1} - \chi_{1} + \partial^\mu \psi_{\mu} \right\} + \chi_{1} \left\{ i \gamma^\mu \psi_{\mu} + (d - 2) \psi - \chi_{1} \right\}.
\]

In deriving the action (7.5), we have used the expressions (3.36), (4.10) for the operators $K_{(1,0)}^{10}$. A substitution of (7.2), (7.4) into (5.28), (5.29) permits one to find the gauge transformations (5.28), (5.29) in the form
\[
\delta \psi_{\mu} = \partial_\mu \xi_{1} + i \gamma_{\mu} \xi_{2}, \quad \delta \psi = \xi_{2}, \quad \delta \chi = i \gamma_{\mu} \partial_\mu \xi_{1} - 2 \xi_{2}, \quad \delta \chi_{1} = -i \gamma_{\mu} \partial_\mu \xi_{2}.
\]

\(\text{For } n_2 = 0, \text{ we have the case of totally symmetric spin-tensors in a } d\text{-dimensional flat space} \[44\], \text{ so that the total Hilbert space } \mathcal{H}_{\text{tot}} \text{ and all of the operators acting on it can be factorized from } q_{ij}^+, \eta_{ij}^+, \eta_i^+, q_2, q_{12}, q_{22}, q_2, f_{ij}^+, b_{ij}^+, b^+ \text{ and their canonically conjugate operators. In the expressions for the action (5.27) and the sequence of gauge transformations (5.28), (5.29), we must set } n_2 = 0 \text{ and use the above restrictions for } \mathcal{H}_{\text{tot}}. \text{ In particular, the operator } K_{(n_1,0)}^0 \text{ has an exact form} \[44\]. K_{(n_1,0)}^0 = \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left( |n_111\rangle \langle n_111| C(n_11, h_{n_1}) - 2 f_{1}^{+} |n_{11}\rangle \langle n_{11}| f_1 C(n_{11} + 1, h_{n_1}) \right), \text{ for } C(n, h) = h(h+1) \cdots (h+n-1), C(0, h) = 1.\]
Let us present the action in terms of the physical field $\psi_\mu$ alone. To this end, we get rid of the field $\psi$ by using its gauge transformation and the gauge parameter $\xi_2$. Having expressed the field $\chi$ by using the equation of motion $\chi = i\gamma^\mu \psi_\mu$, we can see that the terms with the Lagrangian multiplier $\chi_1$ turn to zero. As a result, we obtain the action

\[
S_{(1,0)} = \int d^d x \left\{ i\bar{\psi}^\mu \gamma^\nu \partial_\nu \psi_\mu - i\bar{\psi}^\mu (\gamma_\nu \partial_\nu + \gamma_\mu \partial_\nu) \psi^\nu + i\bar{\psi}^\nu \gamma_\sigma \partial_\sigma \gamma^\mu \psi_\mu \right\},
\]

which is invariant with respect to the residual gauge transformation $\delta \psi_\mu = \partial_\mu \xi_1$.

To obtain a Lagrangian description for a massive fermionic field, we may take two ways, either following the dimensional reduction procedure (5.41)–(5.48), starting directly from the action (7.7) presented for a $(d+1)$-dimensional Minkowski space, or following the prescription (5.36)–(5.40), starting from the expansion (7.2), (7.3), where it is only $|\chi_{0m}^0\rangle$ that changes to $|\chi_{0m}^0\rangle$.

\[
|\chi_{0m}^0\rangle_{(1,0)} = |\chi_{0}^0\rangle_{(1,0)} + b_+ \varphi(x)|0\rangle,
\]

\[
|\chi_{0m}^0\rangle_{(1,0)} = |\chi_{0}^0\rangle_{(1,0)} + \langle 0|\varphi^+(x)b_1\tilde{\gamma}^0. \tag{7.8}
\]

In the latter case, the Lagrangian action and the gauge transformations for $h^1_m = (1-d)/2$ read as follows:

\[
S_{(1,0)}^{\gamma} = \int d^d x \left\{ \bar{\psi}^\mu \left\{ [i\gamma^\nu \partial_\nu - m] \psi_\mu \partial_\mu \chi - i\gamma_\mu \chi_1 \right\} + (d-1) \bar{\psi} \left\{ [i\gamma^\nu \partial_\nu + m] \psi + \chi_1 \right\}
+ \bar{\chi} \left\{ i\gamma^\nu \partial_\nu + m \right\} \chi_1 \right\} \right\}
- \bar{\varphi} \left\{ i\gamma^\nu \partial_\nu - m \right\} \varphi - m \chi_1 - \chi_1 \right\}, \tag{7.9}
\]

\[
\delta \psi_\mu = \partial_\mu \xi_1 + i\gamma_\mu \xi_2, \quad \delta \chi = \chi_2, \quad \delta \varphi = m \xi_1 + \xi_2, \quad \delta \chi_1 = -\left[ i\gamma^\mu \partial_\mu + m \right] p \xi_2. \tag{7.10}
\]

Then one gets rid of the fields $\psi, \varphi$ by using the respective gauge transformations and gauge parameters $\xi_2, \xi_1$. Once again, using the corresponding equation of motion, we express the field $\chi$ as $\chi = i\gamma^\mu \psi_\mu$ and obtain the Rarita–Schwinger Lagrangian in a $d$-dimensional flat space [4],

\[
\mathcal{L}_{RS} = \bar{\psi}^\mu (i\gamma^\nu \partial_\nu - m) \psi_\mu - i\bar{\psi}^\mu (\gamma^\nu \partial_\nu + \gamma_\mu \partial^\nu) \psi_\nu + \bar{\psi}^\nu \gamma^\mu (i\gamma^\sigma \partial_\sigma + m) \gamma^\nu \psi_\nu. \tag{7.11}
\]

The same result follows from dimensional reduction applied to a massless theory in $R^{1,d}$ (e.g., with an odd $d+1$), for a spin-3/2 massive fermionic field $\psi^M = (\psi, \varphi)$, which contains the Stueckelberg field $\varphi$, with $i\gamma^M \partial_M \psi^N = (i\gamma^M \partial_\mu - m) \psi^N$, $i\gamma^M \partial_M \gamma_N \psi^N = (i\gamma^M \partial_\mu + m) \gamma_N \psi^N$, in view of (5.40) and due to the relation $\chi = i\gamma_N \psi^N$ before Eq. (7.7), with allowance for the structure of $|\chi_{0}^0\rangle_{(1,0)}$ in (7.2) and the resolution of the respective gauge transformations $\delta (\psi, \varphi) = (\partial_\mu, m) \xi$.

### 7.2 Rank-2 antisymmetric spin-tensor field

In the case of a spin-(3/2, 3/2) field, we have $n_i = 1, (h^1, h^2) = (1-d/2, 3-d/2)$. Since $s_{\text{max}} = 2$, the corresponding Lagrangian description is a reducible gauge theory of second-stage reducibility. The nonvanishing fields $|\chi_{0}^0\rangle_{(1,1)}, |\chi_{1}^1\rangle_{(1,1)}$, gauge parameters $|\Lambda_{0}^{(1)}\rangle_{(1,1)}$, first-stage gauge parameters $|\Lambda_{0}^{(1)k}\rangle_{(1,1)}$, and second-stage gauge parameters (for $|\Lambda_{0}^{(2)}\rangle_{(1,1)} \equiv 0$, due to $gh(|\Lambda_{0}^{(2)}\rangle_{(1,1)}) = -4$), have the following Grassmann grading and ghost number distribution:

\[
(\varepsilon, gh) (|\chi_{0}^0\rangle_{(1,1)}) = (1, -i), \quad (\varepsilon, gh) (|\Lambda_{0}^{(1)}\rangle_{(1,1)}) = (0, -1 - k), \tag{7.12}
\]

\[
(\varepsilon, gh) (|\Lambda_{0}^{(1)k}\rangle_{(1,1)}) = (1, -2 - k), \quad (\varepsilon, gh) (|\Lambda_{0}^{(2)}\rangle_{(1,1)}) = (0, -3). \tag{7.13}
\]
These conditions allow one, first, to extract the dependence on the ghost variables from the fields and gauge parameters:

\[
|\chi^0_0(1,1)\rangle = |\Psi_1(1,1) + \eta_1^\dagger P^+_2|\Psi_1(0,0) + P^+_1\eta_2^\dagger|\Psi_2(0,0) + q_1^+P^+_2|\Psi_3(0,0) + p_1^+q_2^+|\Psi_4(0,0)
+ p_1^+q_2^+|\Psi_4(0,0) + \eta_1^\dagger P^+_2|\Psi_5(0,0) + q_1^+P^+_2|\Psi_6(0,0) + P^+_1q_2^+|\Psi_7(0,0)
+ q_1^+P^+_2|\Psi_8(0,0) + P^+_1\eta^+|\Psi_9(0,0) + \eta_1^\dagger P^+_1|\Psi_{10}(0,0)
+ q_1^+P^+_2|\varphi_1(1,0) + \eta_1^\dagger P^+_2|\varphi_2(1,0) + P^+_1\eta^+|\varphi_3(1,0) + p_1^+\eta^+|\varphi_4(1,0)
+ q_1^+P^+_2|\rho_1(-1,1) + q_1^+P^+_2|\rho_2(-1,1) + \eta_1^\dagger P^+_2|\rho_3(-1,1) + \eta_1^\dagger P^+_1|\rho_4(-1,1),
\]

\[
|\chi^1_0(1,1)\rangle = p_1^+|\chi_1(0,1) + p_2^+|\chi_2(1,0) + P^+_1\eta^+|\chi_3(0,1) + P^+_2\eta^+|\chi_4(1,0)
+ P^+_1\eta^+|\chi_5(0,0) + q_1^+P^+_1\eta^+|\chi_6(0,0) + \eta_1^\dagger P^+_1|\chi_7(0,0)
+ q_1^+P^+_2|\varphi_1(0,0) + p_1^+\eta^+P^+|\varphi_2(0,0) + p_1^+\eta^+|\varphi_3(0,0)
+ p_1^+\eta^+|\varphi_4(0,0) + P^+_1\eta^+|\chi_8(1,0) + P^+_2\eta^+|\chi_9(1,0)
+ P^+_1\eta^+|\chi_{10}(1,0) + P^+_2\eta^+|\chi_{12}(0,0) + P^+_1\eta^+|\chi_{13}(-1,1),
\]

\[
|\Lambda^0_0(1,1)\rangle = p_1^+\varphi_1(0,1) + p_2^+\varphi_2(1,0) + P^+_1\varphi_3(0,1) + P^+_2\varphi_4(1,0)
+ P^+_1\varphi_5(0,0) + q_1^+P^+_1\varphi_6(0,0) + \eta_1^\dagger P^+_2|\varphi_7(0,0)
+ q_1^+P^+_2|\varphi_8(0,0) + p_1^+\eta^+P^+|\varphi_9(0,0) + p_1^+P^+_1|\varphi_{10}(0,0)
+ P^+_1\eta^+|\varphi_{11}(0,0) + P^+_2\eta^+|\varphi_{12}(0,0) + P^+_1\eta^+|\varphi_{13}(-1,1),
\]

\[
|\Lambda^0_1(1,1)\rangle = p_1^+\varphi_1(0,1) + p_1^+\varphi_2(0,0) + P^+_1\varphi_3(0,0) + P^+_2\varphi_4(0,0)
+ P^+_1\varphi_5(0,0) + q_1^+P^+_1\varphi_6(0,0) + \eta_1^\dagger P^+_2|\varphi_7(0,0)
+ q_1^+P^+_2|\varphi_8(0,0) + p_1^+\eta^+P^+|\varphi_9(0,0) + p_1^+P^+_1|\varphi_{10}(0,0)
+ P^+_1\eta^+|\varphi_{11}(0,0) + P^+_2\eta^+|\varphi_{12}(0,0) + P^+_1\eta^+|\varphi_{13}(-1,1),
\]

\[
|\Lambda^0_0(1,0)\rangle = p_1^+\varphi_1(0,1) + p_1^+\varphi_2(0,0) + P^+_1\varphi_3(0,0) + P^+_2\varphi_4(0,0)
+ P^+_1\varphi_5(0,0) + q_1^+P^+_1\varphi_6(0,0) + \eta_1^\dagger P^+_2|\varphi_7(0,0)
+ q_1^+P^+_2|\varphi_8(0,0) + p_1^+\eta^+P^+|\varphi_9(0,0) + p_1^+P^+_1|\varphi_{10}(0,0)
+ P^+_1\eta^+|\varphi_{11}(0,0) + P^+_2\eta^+|\varphi_{12}(0,0) + P^+_1\eta^+|\varphi_{13}(-1,1),
\]

\[
|\Lambda^0_1(1,0)\rangle = (p_1^+)^2P^+\varphi_1(1,0) + p_1^+P^+_1|\varphi_2(1,0),
\]

\[
|\Lambda^0_0(2,0)\rangle = (p_1^+)^2P^+\varphi_1(2,0) + p_1^+P^+_1|\varphi_2(2,0),
\]

where the coefficient fermionic fields and gauge parameters in the right-hand side of equations \((7.14) - (7.20)\) are independent of ghost operators. The bra-vectors \((1,1)\langle \bar{\chi}_0^0 |\) corresponding to expansion \((7.14), (7.15)\) have the form

\[
(1,1)\langle \bar{\chi}_0^0 | = (1,0)\langle \bar{\Psi}| + (0,0)\langle \bar{\Psi}_1|P^+_2\eta_1 + (0,0)\langle \bar{\Psi}_2|\eta_2P_1 + (0,0)\langle \bar{\Psi}_3|q_1p_2
+ (0,0)\langle \bar{\Psi}_4|q_2p_1 + (0,0)\langle \bar{\Psi}_5|\gamma p_2\eta_1 + (0,0)\langle \bar{\Psi}_6|\gamma p_2\eta_1 + (0,0)\langle \bar{\Psi}_7|\gamma q_2p_1
+ (0,0)\langle \bar{\Psi}_8|\gamma \eta_2p_1 + (0,0)\langle \bar{\Psi}_9|\eta P_{11} + (0,0)\langle \bar{\Psi}_{10}|P\eta_{11}
+ (1,0)\langle \varphi_1|\gamma P_1 + (1,0)\langle \varphi_2|\gamma P_1 + (1,0)\langle \varphi_3|\gamma P_1 + (1,0)\langle \varphi_4|\gamma \eta P_1
+ (-1,1)\langle \rho_1|P_1q_1 + (-1,1)\langle \rho_2|\gamma P_1q_1 + (-1,1)\langle \rho_3|\gamma \eta P_1q_1 + (-1,1)\langle \rho_4|P\eta_{11},
\]

\[
(1,1)\langle \bar{\chi}_0^1 | = (0,1)\langle \bar{\chi}_1|P_1 + (1,0)\langle \bar{\chi}_2|P_2 + (0,1)\langle \bar{\chi}_3|\gamma P_1 + (1,0)\langle \bar{\chi}_4|\gamma p_2
+ (0,0)\langle \bar{\chi}_5|\gamma P_1 + (0,0)\langle \bar{\chi}_6|\gamma P_1q_1 + (0,0)\langle \bar{\chi}_7|\gamma p_1\eta P_{11}
+ (0,0)\langle \bar{\chi}_8|P^+_1q_1 + (0,0)\langle \bar{\chi}_9|P^+_1q_1 + (0,0)\langle \bar{\chi}_{10}|\eta P_{11}p_1
+ (0,0)\langle \bar{\chi}_{11}|\gamma \eta P^+_{11} + (2,0)\langle \bar{\chi}_{12}|\gamma P^+ + (-1,1)\langle \bar{\chi}_{13}|\gamma P^+_{11}.
\]

Substituting \((7.14), (7.15), (7.21), (7.22)\) into \((5.27)\), we find the action (up to an overall factor) for a spin\((3/2,3/2)\) free massless field on a flat background in the form of a scalar product for
vectors defined only in $\mathcal{H} \otimes \mathcal{H}'$,

$$
\mathcal{S}_{(1,1)} = \left[ \langle \bar{\Psi} | K_{(1,1)} \{ \frac{1}{2} T_0 | \Psi \rangle + T^+_1 | \chi_1 \rangle + T^+_2 | \chi_2 \rangle + \tilde{\gamma} L^+_1 | \chi_3 \rangle \\
+ \tilde{\gamma} L^+_2 | \chi_4 \rangle + \tilde{\gamma} L^+_{12} | \chi_5 \rangle + \tilde{\gamma} T^+ | \chi_{12} \rangle + \tilde{\gamma} L^+_{11} | \chi_{13} \rangle \} \\
+ \langle \bar{\Psi}_1 | K_{(1,1)} \{ T_0 | \Psi_2 \rangle - 2 \tilde{\gamma} | \Psi_8 \rangle - \tilde{\gamma} L_2 | \chi_3 \rangle + \frac{1}{2} \tilde{\gamma} | \chi_5 \rangle + \tilde{\gamma} | \chi_7 \rangle \} \\
+ \langle \bar{\Psi}_2 | K_{(1,1)} \{ \frac{1}{2} \tilde{\gamma} | \Psi_5 \rangle + \gamma L^+_1 | \chi_4 \rangle - \frac{1}{2} \gamma | \chi_5 \rangle - \gamma | \chi_7 \rangle \} \\
+ \langle \bar{\Psi}_3 | K_{(1,1)} \{ T_0 | \Psi_4 \rangle - 2 \tilde{\gamma} | \Psi_8 \rangle + T_2 | \chi_1 \rangle + \frac{1}{2} \tilde{\gamma} | \chi_5 \rangle - \tilde{\gamma} | \chi_6 \rangle - 4 \tilde{\gamma} | \chi_{11} \rangle \} \\
+ \langle \bar{\Psi}_4 | K_{(1,1)} \{ -2 \tilde{\gamma} | \Psi_5 \rangle + T_1 | \chi_2 \rangle + \frac{1}{2} \tilde{\gamma} | \chi_5 \rangle - \tilde{\gamma} | \chi_6 \rangle \} \\
+ \langle \bar{\Psi}_5 | K_{(1,1)} \{ -T_0 | \Psi_7 \rangle - T_2 | \chi_3 \rangle + \tilde{\gamma} | \chi_8 \rangle + 2 \tilde{\gamma} | \chi_{10} \rangle \} \\
+ \langle \bar{\Psi}_6 | K_{(1,1)} \{ -T_0 | \Psi_8 \rangle + \gamma L_2 | \chi_1 \rangle + \gamma | \chi_9 \rangle \} \\
+ \langle \bar{\Psi}_7 | K_{(1,1)} \{ \frac{1}{2} \gamma L^+_1 | \chi_2 \rangle + \gamma | \chi_9 \rangle \} + \langle \bar{\Psi}_8 | K_{(1,1)} \{ -T_1 | \chi_4 \rangle + \gamma | \chi_8 \rangle \} \\
+ \langle \bar{\Psi}_9 | K_{(1,1)} \{ T_0 | \Psi_{10} \rangle - \frac{1}{2} \tilde{\gamma} | \chi_5 \rangle + \tilde{\gamma} | \chi_7 \rangle + \tilde{\gamma} L^+_{11} | \chi_{12} \rangle \} \\
+ \langle \bar{\Psi}_{10} | K_{(1,1)} \{ \frac{1}{2} \tilde{\gamma} | \chi_5 \rangle - 4 \tilde{\gamma} | \chi_{11} \rangle - \tilde{\gamma} T | \chi_{13} \rangle \}
\right] + c.c.,
$$

(7.23)

where we have omitted the lower spin subscripts of the component fields. In deriving the action (7.23), we have used the expressions for the operators $K_{(1,1)}$ (3.30), (4.3), and then, substituting (7.22)–(7.24) into (7.23), (5.29), we find the gauge transformations for the vectors $| \Psi \rangle, | \Psi_k \rangle$,

$$
\delta | \Psi \rangle = -\tilde{\gamma} (T^+_1 | \xi_1 \rangle + T^+_2 | \xi_2 \rangle) + L^+_1 | \xi_3 \rangle + L^+_2 | \xi_4 \rangle + L^+_{12} | \xi_5 \rangle + L^+_{11} | \xi_{13} \rangle + T^+ | \xi_{12} \rangle,
$$

(7.24)

$$
\delta | \Psi_1 \rangle = L_1 | \xi_4 \rangle - \frac{1}{2} | \xi_5 \rangle - | \xi_7 \rangle - | \lambda_2 \rangle + 2 | \lambda_3 \rangle - \tilde{\gamma} T_0 | \lambda_4 \rangle,
$$

(7.25)

$$
\delta | \Psi_2 \rangle = -L_2 | \xi_3 \rangle + \frac{1}{2} | \xi_5 \rangle + | \xi_7 \rangle - 2 | \lambda_2 \rangle + | \lambda_3 \rangle - \tilde{\gamma} T_0 | \lambda_4 \rangle,
$$

(7.26)

$$
\delta | \Psi_3 \rangle = -\tilde{\gamma} T_1 | \xi_2 \rangle + \frac{1}{2} | \xi_5 \rangle - | \xi_6 \rangle - | \lambda_3 \rangle,
$$

(7.27)

$$
\delta | \Psi_4 \rangle = -\tilde{\gamma} T_2 | \xi_1 \rangle + \frac{1}{2} | \xi_5 \rangle - | \xi_6 \rangle - 4 | \xi_{11} \rangle - | \lambda_2 \rangle,
$$

(7.28)

$$
\delta | \Psi_5 \rangle = L_1 | \xi_2 \rangle + | \xi_9 \rangle - | \lambda_1 \rangle + \tilde{\gamma} T_0 | \lambda_3 \rangle,
$$

(7.29)

$$
\delta | \Psi_6 \rangle = \tilde{\gamma} T_1 | \xi_4 \rangle + | \xi_8 \rangle - | \lambda_4 \rangle,
$$

(7.30)

$$
\delta | \Psi_7 \rangle = \tilde{\gamma} T_2 | \xi_3 \rangle + | \xi_8 \rangle + 2 | \xi_{10} \rangle + | \lambda_4 \rangle,
$$

(7.31)

$$
\delta | \Psi_8 \rangle = L_2 | \xi_1 \rangle + | \xi_9 \rangle + \tilde{\gamma} T_0 | \lambda_2 \rangle - | \lambda_1 \rangle,
$$

(7.32)

$$
\delta | \Psi_9 \rangle = | \xi_5 \rangle - 4 | \xi_{11} \rangle - T | \xi_{13} \rangle,
$$

(7.33)

$$
\delta | \Psi_{10} \rangle = -\frac{1}{2} | \xi_5 \rangle + | \xi_6 \rangle + | \xi_7 \rangle + L_{11} | \xi_{12} \rangle,
$$

(7.34)
for the vectors $|\varphi_i\rangle, |\rho_i\rangle$,

$$
\delta|\varphi_1\rangle = -2|\xi_2\rangle + \gamma T^+_1|\xi_6\rangle + L^+_1|\xi_8\rangle + \gamma T_1|\xi_{12}\rangle - |\lambda_6\rangle ,
$$  \hspace{1cm} (7.35) \\

$$
\delta|\varphi_2\rangle = -L^+_1|\xi_7\rangle - \gamma T^+_1|\xi_9\rangle + L_1|\xi_{12}\rangle - \gamma T_0|\lambda_6\rangle - |\lambda_7\rangle ,
$$  \hspace{1cm} (7.36) \\

$$
\delta|\varphi_3\rangle = -T|\xi_3\rangle + |\xi_4\rangle - \gamma T^+_1|\xi_{10}\rangle ,
$$  \hspace{1cm} (7.37) \\

$$
\delta|\varphi_4\rangle = T|\xi_1\rangle - |\xi_2\rangle + L^+_1|\xi_{10}\rangle + 2\gamma T^+_1|\xi_{11}\rangle ,
$$  \hspace{1cm} (7.38) \\

$$
\delta|\rho_1\rangle = -\gamma T^+_1|\xi_1\rangle + T^+|\xi_6\rangle + |\xi_{13}\rangle - |\lambda_9\rangle ,
$$  \hspace{1cm} (7.39) \\

$$
\delta|\rho_2\rangle = \gamma T^+_1|\xi_3\rangle - T^+|\xi_8\rangle ,
$$  \hspace{1cm} (7.40) \\

$$
\delta|\rho_3\rangle = L_1|\xi_1\rangle - T^+|\xi_9\rangle - 2|\lambda_8\rangle + \gamma T_0|\lambda_9\rangle ,
$$  \hspace{1cm} (7.41) \\

$$
\delta|\rho_4\rangle = L_1|\xi_3\rangle + T^+|\xi_7\rangle - |\xi_{13}\rangle + |\lambda_9\rangle ,
$$  \hspace{1cm} (7.42)

and for the vectors $|\chi_m\rangle$,

$$
\delta|\chi_1\rangle = -\gamma T_0|\xi_1\rangle - \gamma T^+_2|\xi_1\rangle + L^+_2|\lambda_2\rangle + T^+|\lambda_7\rangle - 2\gamma T^+_1|\lambda_8\rangle + L^+_1|\lambda_9\rangle ,
$$  \hspace{1cm} (7.43) \\

$$
\delta|\chi_2\rangle = -\gamma T_0|\xi_2\rangle - \gamma T^+_1|\lambda_1\rangle + L^+_1|\lambda_3\rangle - |\lambda_7\rangle ,
$$  \hspace{1cm} (7.44) \\

$$
\delta|\chi_3\rangle = -2|\xi_1\rangle + \gamma T_0|\xi_4\rangle + \gamma T^+_2|\lambda_4\rangle - L^+_2|\lambda_4\rangle + T^+|\lambda_9\rangle + \gamma T^+_1|\lambda_9\rangle ,
$$  \hspace{1cm} (7.45) \\

$$
\delta|\chi_4\rangle = -2|\xi_2\rangle + \gamma T_0|\xi_4\rangle + \gamma T^+_1|\lambda_2\rangle + L^+_1|\lambda_4\rangle + |\lambda_6\rangle ,
$$  \hspace{1cm} (7.46) \\

$$
\delta|\chi_5\rangle = \gamma T_0|\xi_5\rangle - 4|\lambda_1\rangle + 2|\lambda_5\rangle ,
$$  \hspace{1cm} (7.47) \\

$$
\delta|\chi_6\rangle = \gamma T_0|\xi_6\rangle - 2|\lambda_9\rangle - 2|\lambda_1\rangle + |\lambda_5\rangle + \gamma T_1|\lambda_7\rangle ,
$$  \hspace{1cm} (7.48) \\

$$
\delta|\chi_7\rangle = \gamma T_0|\xi_7\rangle + 2|\lambda_9\rangle - |\lambda_5\rangle + L_1|\lambda_6\rangle ,
$$  \hspace{1cm} (7.49) \\

$$
\delta|\chi_8\rangle = -\gamma T_0|\xi_8\rangle - 2|\xi_6\rangle - 2|\xi_7\rangle + 2|\lambda_3\rangle - \gamma T_1|\lambda_6\rangle ,
$$  \hspace{1cm} (7.50) \\

$$
\delta|\chi_9\rangle = -\gamma T_0|\xi_9\rangle + L_1|\lambda_7\rangle ,
$$  \hspace{1cm} (7.51) \\

$$
\delta|\chi_{10}\rangle = -\gamma T_0|\xi_{10}\rangle - 4|\xi_{11}\rangle + |\lambda_2\rangle + |\lambda_3\rangle - T|\lambda_9\rangle ,
$$  \hspace{1cm} (7.52) \\

$$
\delta|\chi_{11}\rangle = \gamma T_0|\xi_{11}\rangle - |\lambda_3\rangle + T|\lambda_8\rangle ,
$$  \hspace{1cm} (7.53) \\

$$
\delta|\chi_{12}\rangle = \gamma T_0|\xi_{12}\rangle + L^+_1|\lambda_5\rangle + L^+_1|\lambda_6\rangle + \gamma T^+_1|\lambda_7\rangle ,
$$  \hspace{1cm} (7.54) \\

$$
\delta|\chi_{13}\rangle = \gamma T_0|\xi_{13}\rangle - T^+|\lambda_5\rangle - 4|\lambda_8\rangle .
$$  \hspace{1cm} (7.55)

Then, substituting expressions (7.16)–(7.19) into (5.30), (5.31), we obtain the gauge transformations for the zero-stage gauge vectors $|\xi_m\rangle, |\lambda_n\rangle$,

$$
\delta|\xi_1\rangle = -\gamma T^+_2|\xi^{(1)}_1\rangle + L^+_2|\xi^{(1)}_2\rangle + T^+|\xi^{(1)}_7\rangle - 2\gamma T^+_1|\xi^{(1)}_8\rangle + L^+_1|\xi^{(1)}_9\rangle ,
$$  \hspace{1cm} (7.56) \\

$$
\delta|\xi_2\rangle = -\gamma T^+_1|\xi^{(1)}_1\rangle + L^+_1|\xi^{(1)}_3\rangle - |\xi^{(1)}_7\rangle ,
$$  \hspace{1cm} (7.57) \\

$$
\delta|\xi_3\rangle = \gamma T^+_2|\xi^{(1)}_3\rangle - L^+_2|\xi^{(1)}_4\rangle - T^+|\xi^{(1)}_6\rangle + \gamma T^+_1|\xi^{(1)}_9\rangle ,
$$  \hspace{1cm} (7.58) \\

$$
\delta|\xi_4\rangle = \gamma T^+_1|\xi^{(1)}_4\rangle + L^+_1|\xi^{(1)}_4\rangle + |\xi^{(1)}_6\rangle ,
$$  \hspace{1cm} (7.59) \\

$$
\delta|\xi_5\rangle = -4|\xi^{(1)}_1\rangle + 2|\xi^{(1)}_5\rangle ,
$$  \hspace{1cm} (7.60) \\

$$
\delta|\xi_6\rangle = -2|\xi^{(1)}_2\rangle + |\xi^{(1)}_5\rangle + \gamma T_1|\xi^{(1)}_7\rangle - |\xi^{(1)}_2\rangle ,
$$  \hspace{1cm} (7.61) \\

$$
\delta|\xi_7\rangle = -|\xi^{(1)}_5\rangle + L_1|\xi^{(1)}_6\rangle + |\xi^{(1)}_2\rangle ,
$$  \hspace{1cm} (7.62) \\

$$
\delta|\xi_8\rangle = 2|\xi^{(1)}_3\rangle - \gamma T_1|\xi^{(1)}_6\rangle ,
$$  \hspace{1cm} (7.63) \\

$$
\delta|\xi_9\rangle = L_1|\xi^{(1)}_7\rangle - 2|\lambda^{(1)}_1\rangle - \gamma T_0|\lambda^{(1)}_2\rangle ,
$$  \hspace{1cm} (7.64) \\

$$
\delta|\xi_{10}\rangle = |\xi^{(1)}_2\rangle + |\xi^{(1)}_3\rangle - T|\xi^{(1)}_9\rangle ,
$$  \hspace{1cm} (7.65) \\

$$
\delta|\xi_{11}\rangle = -|\xi^{(1)}_1\rangle + T|\xi^{(1)}_8\rangle ,
$$  \hspace{1cm} (7.66) \\

$$
\delta|\xi_{12}\rangle = L^+_1|\xi^{(1)}_5\rangle + L^+_1|\xi^{(1)}_6\rangle + \gamma T^+_1|\xi^{(1)}_7\rangle ,
$$  \hspace{1cm} (7.67) \\

$$
\delta|\xi_{13}\rangle = -T^+|\xi^{(1)}_5\rangle - 4|\xi^{(1)}_8\rangle ,
$$  \hspace{1cm} (7.68)
Finally, using the equations (5.30), (5.31) for the vectors (7.19)–(7.20), we find the gauge transformations for the first-stage gauge vectors $|\xi_n^{(1)}\rangle$, $|\lambda_n^{(1)}\rangle$,

$$
\delta|\xi_1^{(1)}\rangle = -2|\xi_2^{(2)}\rangle, \quad \delta|\xi_2^{(1)}\rangle = |\xi_2^{(2)}\rangle, \quad \delta|\xi_3^{(1)}\rangle = |\xi_2^{(2)}\rangle, \quad \delta|\xi_4^{(1)}\rangle = 0, \quad \delta|\xi_5^{(1)}\rangle = -4|\xi_1^{(2)}\rangle, \quad \delta|\xi_6^{(1)}\rangle = -2|\xi_1^{(2)}\rangle - 2|\xi_2^{(2)}\rangle - \gamma T_1^+|\lambda_2^{(1)}\rangle, \quad \delta|\xi_7^{(1)}\rangle = 2\gamma T_1^+|\xi_2^{(2)}\rangle + L_1^+|\lambda_2^{(1)}\rangle, \quad \delta|\xi_8^{(1)}\rangle = -\gamma T_0|\xi_1^{(2)}\rangle + T^+|\lambda_1^{(1)}\rangle, \quad \delta|\lambda_1^{(1)}\rangle = \gamma T_0|\xi_1^{(2)}\rangle, \quad \delta|\lambda_2^{(1)}\rangle = -\gamma T_0|\xi_2^{(2)}\rangle - 4|\xi_1^{(2)}\rangle. \quad (7.78)
$$

In order to derive the action $S_{(1,1)}$ (7.23) only in terms of the component fields, we, first of all, present the vectors $|\Psi\rangle$, $|\Psi_k\rangle$, $|\chi_1\rangle$, $|\rho_1\rangle$, $|\chi_m\rangle$ and the corresponding bra-vectors as expansions with respect to the initial and auxiliary creation operators:

$$
|\Psi\rangle_{(1,1)} = (a_1^{+\mu}a_2^{+\nu}\psi_{\mu,\nu}(x) + f_2^{+\mu}\gamma\psi_1^{+(1)}(x) + f_1^{+\mu}\gamma\psi_2^{+(1)}(x) + b_{12}^{+}\psi(x) + f_1^{+}\gamma\psi_1^{+(1)}(x) + f_2^{+}\gamma\psi_2^{+(1)}(x) + b_{12}^{+}\psi(x))|0\rangle, \quad (7.84)
$$

$$
|\bar{\Psi}\rangle_{(1,1)} = (0)(\psi_1^{+(1)}(x)a_2^{+\mu}\alpha_1^{\mu}(x) + \psi_2^{+(1)}(x)a_1^{+\mu}\alpha_2^{\mu}(x) + b_{12}^{+}\psi(x) + f_1^{+}\gamma\psi_1^{+(1)}(x) + f_2^{+}\gamma\psi_2^{+(1)}(x) + b_{12}^{+}\psi(x))|0\rangle, \quad (7.85)
$$

$$
|\chi_m\rangle_{(0,1)} = (a_1^{+\mu}b_1^{+\mu}(x) + a_2^{+\mu}b_1^{+\mu}(x) + f_1^{+}\gamma b_1^{+}\chi_m^{(1)}(x) + f_2^{+}\gamma b_1^{+}\chi_m^{(1)}(x))|0\rangle, \quad m = 1, 3, \quad (7.86)
$$

$$
|\bar{\chi}_m\rangle_{(0,1)} = (0)(\chi_m^{(1)}(x)b_1^{+\mu}(x) + \chi_m^{(1)}(x)b_1^{+\mu}(x) + \chi_m^{(1)}(x)b_1^{+}\gamma f_1 + \chi_m^{(1)}(x)b_1^{+}\gamma f_1)\bar{\gamma}_0, \quad m = 1, 3, \quad (7.87)
$$

$$
|\chi_{12}\rangle_{(2,0)} = (0)(\frac{1}{2}a_1^{+\mu}a_1^{+\nu}\chi_{12}^{(1)}(x) + a_1^{+\mu}f_1^{+}\gamma\chi_{12}^{(1)}(x) + b_{11}^{+}\chi_{12}(x))|0\rangle, \quad (7.88)
$$

$$
|\bar{\chi}_{12}\rangle_{(2,0)} = (0)(\frac{1}{2}a_1^{+\mu}a_1^{+\nu}\chi_{12}^{(1)}(x) + a_1^{+\mu}f_1^{+}\gamma\chi_{12}^{(1)}(x) + b_{11}^{+}\chi_{12}(x))\bar{\gamma}_0, \quad (7.89)
$$

$$
|\chi_{m}\rangle_{(1,0)} = (0)(\chi_m^{(1)}(x) + f_1^{+}\gamma\bar{\chi}_m(x))|0\rangle, \quad m = 2, 4, \quad (7.90)
$$

$$
|\bar{\chi}_m\rangle_{(1,0)} = (0)(\chi_m^{(1)}(x) + f_1^{+}\gamma\bar{\chi}_m(x))\bar{\gamma}_0, \quad (7.91)
$$

27
Second, let us fix preliminarily a part of the gauge ambiguity starting from the first-stage gauge parameters. To this end, we can use the choice of the second-stage independent vectors $|\xi_{k}^{(2)}\rangle, |\xi_{1}^{(2)}\rangle$, entering relations (7.78)–(7.83) as shift parameters, in order to get rid, for instance, of the vectors $|\xi_{k}^{(1)}\rangle, k = 1, 3$, so that the description of the model is transformed to a first-stage reducible theory with independent first-stage gauge parameters, $|\lambda_{l}^{(1)}\rangle, l = 1, 2 |\xi_{n}^{(1)}\rangle, n = 2, 4, ..., 9$, and without restrictions (7.78)–(7.83).

By the same argument, we can make the zero-stage gauge vectors $|\xi_{m}\rangle, m = 2, 4, 5, 6, 10, 13, |\lambda_{l}\rangle, l = 1, 3$ to vanish by using a choice for the parameters $|\xi_{m}^{(1)}\rangle, m = 7, 6, 5, 2, 9, 8, |\lambda_{l}^{(1)}\rangle, l = 1, 2$, respectively, in the gauge transformations (7.56)–(7.71). As a result, the remaining gauge transformations with the independent first-stage gauge parameters $|\xi_{4}^{(1)}\rangle$ for the remaining zero-stage vectors have the form

$$
\delta|\xi_{k}\rangle = 0, \; k = 1, 9, 11, \quad \delta|\xi_{3}\rangle = -(L_{2}^{+} + T^{+}L_{1}^{+})|\xi_{4}^{(1)}\rangle, \quad (7.97)
$$

$$
\delta|\xi_{7}\rangle = -L_{0}|\xi_{4}^{(1)}\rangle, \quad \delta|\xi_{8}\rangle = -\tilde{\gamma}T_{0}|\xi_{4}^{(1)}\rangle, \quad (7.98)
$$

$$
\delta|\xi_{12}\rangle = -(L_{1}^{+})^{2}|\xi_{4}^{(1)}\rangle, \quad \delta|\lambda_{4}\rangle = -\tilde{\gamma}0|\xi_{4}^{(1)}\rangle, \quad (7.99)
$$

$$
\delta|\lambda_{6}\rangle = \tilde{\gamma}T_{0}L_{+}|\xi_{4}^{(1)}\rangle, \quad \delta|\lambda_{l}\rangle = 0, \; l = 2, 5, 7, 8, 9. \quad (7.100)
$$

Finally, in the same manner, we can get rid of the fields $|\Psi_{k}\rangle, k = 4, ..., 7, 9, 10, |\varphi_{1}\rangle, |\rho_{1}\rangle, |\chi_{l}\rangle, l = 2, 5, 11$, with the help of a corresponding choice for the independent (except $|\xi_{7}\rangle, |\xi_{8}\rangle, |\lambda_{4}\rangle, |\lambda_{l}\rangle$ which may be used in pairs in order to take account of its reducibility) gauge parameters $|\lambda_{2}\rangle, |\xi_{9}\rangle, |\lambda_{4}\rangle, |\xi_{11}\rangle, |\xi_{7}\rangle, |\lambda_{l}\rangle, l = 6, 9, 7, 5, 8$ respectively, in the gauge transformations (7.72)–(7.55). The resultant gauge transformations for the remaining fields $|\Psi\rangle, |\Psi_{k}\rangle, k = 1, 2, 3, 8, |\varphi_{1}\rangle, l = 2, 3, 4, |\rho_{m}\rangle, m = 2, 3, 4, |\chi_{k}\rangle, k = 1, 3, 4, 6, 7, 8, 9, 10, 12, 13$, with the zero-stage gauge vectors $|\xi_{k}\rangle, k = 1, 3, 12$, that have not been used previously, are reduced to

$$
\delta|\Psi\rangle = -\tilde{\gamma}T_{1}^{+}|\xi_{1}\rangle + L_{1}^{+}|\xi_{3}\rangle + T^{+}|\xi_{12}\rangle, \quad (7.101)
$$

$$
\delta|\Psi_{1}\rangle = L_{11}|\xi_{12}\rangle + \tilde{\gamma}T_{2}|\xi_{1}\rangle + \frac{1}{2}T_{0}T_{2}|\xi_{3}\rangle, \quad \delta|\Psi_{3}\rangle = 0, \quad (7.102)
$$

$$
\delta|\Psi_{2}\rangle = 2\tilde{\gamma}T_{2}|\xi_{1}\rangle - L_{11}|\xi_{12}\rangle - \frac{1}{2}T_{2}T_{0}|\xi_{3}\rangle, \quad \delta|\Psi_{8}\rangle = (L_{2} - T_{0}T_{2})|\xi_{1}\rangle, \quad (7.103)
$$

$$
\delta|\varphi_{2}\rangle = (L_{11}L_{1}^{+} + T_{1}T_{0})|\xi_{12}\rangle + \frac{1}{2}T_{0}L_{1}^{+}T_{2}|\xi_{3}\rangle, \quad \delta(|\varphi_{3}\rangle, |\varphi_{4}\rangle) = T(|\xi_{3}\rangle, |\xi_{1}\rangle), \quad (7.104)
$$

$$
\delta|\rho_{2}\rangle = \tilde{\gamma}(T_{1} + \frac{1}{2}T_{0}T_{2})|\xi_{3}\rangle, \quad \delta|\rho_{3}\rangle = (L_{1} - T_{0}T_{1})|\xi_{1}\rangle, \quad (7.105)
$$

$$
\delta|\rho_{4}\rangle = L_{1}|\xi_{3}\rangle - T^{+}L_{11}|\xi_{12}\rangle - \tilde{\gamma}T_{1}|\xi_{1}\rangle, \quad \delta|\xi_{k}\rangle = 0, \; k = 6, 9, 13, \quad (7.106)
$$

$$
\delta|\chi_{1}\rangle = -\tilde{\gamma}(T_{0} + \frac{1}{2}L_{1}^{+}T^{+}T_{2})|\xi_{3}\rangle - \tilde{\gamma}T^{+}T_{1}|\xi_{12}\rangle, \quad (7.107)
$$

$$
\delta|\chi_{3}\rangle = -(2 + T_{1}^{+}T_{1})|\xi_{1}\rangle + \tilde{\gamma}(T_{0} + \frac{1}{2}L_{1}^{+}T^{+}T_{2})|\xi_{3}\rangle - \tilde{\gamma}T^{+}T_{1}|\xi_{12}\rangle, \quad (7.108)
$$
\(\delta |\chi_4\rangle = -T_1^+ T_2 |\xi_1\rangle - \tilde{\gamma} L_1^+ T_2 |\xi_3\rangle + \tilde{\gamma} T_1 |\xi_{12}\rangle, \quad \delta |\chi_8\rangle = \frac{1}{2} L_1^+ T_1 T_2 |\xi_3\rangle, \quad (7.109)\)

\(\delta |\chi_7\rangle = \tilde{\gamma} (L_1 T_1 - T_0 L_{11}) |\xi_{12}\rangle - \frac{\tilde{\gamma}}{2} L_1^+ T_2 |\xi_3\rangle, \quad \delta |\chi_{10}\rangle = -\tilde{\gamma} T_1 T |\xi_1\rangle, \quad (7.110)\)

\(\delta |\chi_{12}\rangle = \tilde{\gamma} (T_0 + L_1^+ T_1) |\xi_{12}\rangle - \frac{\tilde{\gamma}}{2} (L_1^+)^2 T_2 |\xi_3\rangle. \quad (7.111)\)

One can easily prove that the gauge transformations (7.101)–(7.111) are invariant with respect to their gauge transformations (7.97)–(7.100) for the gauge parameters \(|\xi_m\rangle, m = 1, 3, 12\). We then take into account the internal structure of the above gauge parameters \(|\xi_m\rangle\) (having the same respective form as that for the fields \(|\chi_m\rangle, m = 1, 3, 12\) in (7.88), (7.89)) in order to gauge away the fields \(|\varphi_l\rangle, l = 3, 4\), and to simplify the structure of the basic field \(|\Psi\rangle\). As a result, the gauge transformations have the form

\(\delta |\Psi\rangle = -\tilde{\gamma} T_1^+ |\xi_1\rangle + L_1^+ |\xi_3\rangle + T^+ |\xi_{12}\rangle, \quad (7.112)\)

\(\delta |\Psi_1\rangle = L_{11} |\xi_1\rangle + \tilde{\gamma} T_2 |\xi_1\rangle + \frac{1}{2} T_0 T_2 |\xi_3\rangle, \quad \delta |\Psi_3\rangle = 0, \quad (7.113)\)

\(\delta |\Psi_2\rangle = 2 \tilde{\gamma} T_2 |\xi_1\rangle - L_{11} |\xi_{12}\rangle - \frac{1}{2} T_2 T_0 |\xi_3\rangle, \quad \delta |\Psi_8\rangle = (L_2 - T_0 T_2) |\xi_1\rangle, \quad (7.114)\)

\(\delta |\varphi_2\rangle = (L_{11} L_1^+ + T_1 T_0) |\xi_{12}\rangle + \frac{1}{2} T_0 L_1^+ T_2 |\xi_3\rangle, \quad \delta |\rho_2\rangle = \tilde{\gamma} (T_1 + \frac{1}{2} T^+ T_2) |\xi_3\rangle \equiv 0, \quad (7.115)\)

\(\delta |\rho_4\rangle = L_1 |\xi_3\rangle - T^+ L_{11} |\xi_{12}\rangle - \tilde{\gamma} T_1 |\xi_1\rangle, \quad \delta |\rho_3\rangle = (L_1 - T_0 T_1) |\xi_1\rangle, \quad (7.116)\)

\(\delta |\chi_k\rangle = 0, \quad k = 6, 9, 10, 13, \quad \delta |\chi_1\rangle = -\tilde{\gamma} (T_0 + L_2^+ T_2 + L_1^+ T_1) |\xi_1\rangle, \quad (7.117)\)

\(\delta |\chi_3\rangle = -(2 + T_1^+ T_1) |\xi_1\rangle + \tilde{\gamma} (T_0 + \frac{1}{2} L_1^+ T^+ T_2) |\xi_3\rangle - \tilde{\gamma} T^+ T_1 |\xi_{12}\rangle, \quad (7.118)\)

\(\delta |\chi_4\rangle = -T_1^+ T_2 |\xi_1\rangle - \tilde{\gamma} L_1^+ T_2 |\xi_3\rangle + \tilde{\gamma} T_1 |\xi_{12}\rangle, \quad \delta |\chi_8\rangle = \frac{1}{2} L_1^+ T_1 T_2 |\xi_3\rangle = 0, \quad (7.119)\)

\(\delta (|\chi_7\rangle; |\chi_{12}\rangle) = \tilde{\gamma} \left( (L_1 T_1 - T_0 L_{11}) , (T_0 + L_1^+ T_1) \right) |\xi_{12}\rangle - \frac{\tilde{\gamma}}{2} (L_0 T_2, (L_1^+)^2 T_2) |\xi_3\rangle, \quad (7.120)\)

where the vectors \(|\xi_m^r\rangle, m = 1, 3\), are solutions of the equations \(T |\xi_m\rangle = 0,\)

\(|\xi_m^r\rangle_{(0,1)} = (a_1^{\mu} b^+ - 2 a_2^{\mu}) \xi_{1m}^l (x) - \tilde{\gamma} (f_1^{\mu} b^+ - 2 f_2^{\mu}) \xi_{3m}^l (x) |0\rangle, m = 1, 3, \quad (7.121)\)

and \(|\xi_{12}^r\rangle_{(2,0)}\) has the form

\(|\xi_{12}^r\rangle_{(2,0)} = (\frac{1}{2} a_1^{\mu} a_1^{\nu} \xi_{12}^l (x) + a_1^{\mu} f_1^+ \tilde{\gamma} \xi_{12}^l (x) + b_1^{\nu} \xi_{12}^l (x)) |0\rangle. \quad (7.122)\)

As a consequence, the action (7.23) is simplified as follows:

\(S_{(1,1)} = \left[ \langle \tilde{\Psi} | K_{(1,1)} \{ \frac{1}{2} T_0 |\Psi\rangle + T_1^+ |\chi_1\rangle + \tilde{\gamma} L_1^+ |\chi_3\rangle + \tilde{\gamma} L_2^+ |\chi_4\rangle + \tilde{\gamma} T^+ |\chi_{12}\rangle + \tilde{\gamma} L_1^+ |\chi_{13}\rangle \right] \right. \)

\(\left. + \langle \tilde{\Psi}_1 | K_{(1,1)} \{ T_0 |\Psi_2\rangle - 2 \tilde{\gamma} |\Psi_8\rangle - \tilde{\gamma} L_2 |\chi_3\rangle + \tilde{\gamma} |\chi_7\rangle \right) \)

\(\left. + \langle \tilde{\Psi}_2 | K_{(1,1)} \{ \tilde{\gamma} L_1 |\chi_4\rangle - \tilde{\gamma} |\chi_{12}\rangle \right) + \langle \tilde{\Psi}_3 | K_{(1,1)} \{ -T_1 |\chi_4\rangle + \tilde{\gamma} |\chi_8\rangle \right) \)

\(\left. + \langle \tilde{\varphi}_2 | K_{(1,1)} \{ -2 \tilde{\gamma} |\Psi_8\rangle + T_2 |\chi_1\rangle - \tilde{\gamma} |\chi_6\rangle \right) \)

\(+ \langle \tilde{\varphi}_3 | K_{(1,1)} \{ -\tilde{\gamma} T |\chi_3\rangle + \tilde{\gamma} |\chi_4\rangle + T_1^+ |\chi_{10}\rangle \right) \)
\[ + \langle \hat{\rho}_2 | K_{(1,1)} \{-T_0 | \rho_3 \} + \hat{\gamma} L_1 | \chi_1 \} - \hat{\gamma} T^+ | \chi_9 \} \]
\[ + \langle \hat{\rho}_2 | K_{(1,1)} \{-2 \hat{\gamma} | \rho_4 \} - T_1 | \chi_3 \} - \hat{\gamma} T^+ | \chi_8 \} \]
\[ + \langle \hat{\rho}_4 | K_{(1,1)} \{-\frac{1}{2} T_0 | \rho_4 \} - \hat{\gamma} L_1 | \chi_3 \} - \hat{\gamma} T^+ | \chi_7 \} + \hat{\gamma} | \chi_{13} \} \]
\[ + \langle \hat{\chi}_1 | K_{(1,1)} \{-\hat{\gamma} | \chi_3 \} - \frac{1}{2} \langle \hat{\chi}_3 | K_{(1,1)} T_0 | \chi_3 \} - \frac{1}{2} \langle \hat{\chi}_4 | K_{(1,1)} T_0 | \chi_4 \} \]
\[ + \langle \hat{\chi}_0 | K_{(1,1)} \{-\hat{\gamma} | \chi_{10} \} \} + \langle \hat{\chi}_7 | K_{(1,1)} \{-\hat{\gamma} | \chi_{10} \} \} + \langle \hat{\chi}_8 | K_{(1,1)} T_0 | \chi_{10} \} \} + \text{c.c.} \text{ (7.123)} \]

Then, we gauge away the symmetric part \( \psi_{(\mu \nu)} = \frac{1}{2} (\psi_{\mu \nu} + \psi_{\nu \mu}) \) from the basic spin-tensor \( \psi_{\mu \nu} \) as well as the fields \( \psi^k, k = 2, 3, \psi_l, l = 1, 2, \) from the basic field \( | \Psi \rangle (1,1) \) using all the components of the gauge parameters \( | \xi_{12} \rangle (2,0) \) and \( | \xi_1 \rangle (0,1) \) in the gauge transformations \( (7.112) \) \( (7.120) \), so that only the following nontrivial relations for the used gauge parameters,

\[ \xi_{12}^1 (x) = 2 \partial_\mu \xi_{12}^1 (x), \quad \xi_{12}^2 (x) = 2 \partial_\mu \xi_{12}^2 (x), \quad \xi_{12}^3 (x) = - i \partial_\mu \xi_{12}^3 (x), \quad (7.124) \]

hold true. The resulting spectrum of the fields, \( \psi_{\mu \nu}, \psi_{\mu}, \psi_k, k = 1, 2, 3, 8, \varphi_2, \varphi_2, \rho_l = 2, 3, 4, \chi^l_1, \chi^l_2, \chi^l_3, \chi^l_4, \mu = 1, 3, \chi^l, \chi_k, k = 4, 6, ..., 10, 12, 13, \chi^l_1, \chi^l_2, \) and their nontrivial gauge transformations have the component form, for \( \gamma_{\rho \sigma} = \frac{1}{2} (\gamma_{\rho} \gamma_{\sigma} - \gamma_{\sigma} \gamma_{\rho}) \),

\[ \delta \psi_{\mu \nu} (x) = 2 i \partial_\mu \xi_{12}^1 (x) + 2 i \gamma_{\mu \nu} \partial_\nu \xi_{12}^1 (x), \quad (7.125) \]
\[ \delta \psi_1 (x) = i \gamma_{\mu \nu} \partial_\nu \psi_{\mu 1} (x), \quad (7.126) \]
\[ \delta \psi_2 (x) = i \gamma_{\mu \nu} \partial_\nu \psi_{\mu 2} (x), \quad (7.127) \]
\[ \delta \chi^4_1 (x) = \frac{2}{3} \left( \gamma_{\mu \nu} \partial_\mu \xi_{12}^1 (x) + \gamma_{\mu \nu} \partial_\nu \xi_{12}^1 (x) \right), \quad (7.128) \]
\[ \delta \chi^{23} (x) = -i \left( 2 \gamma_{\mu \nu} \partial_\mu \xi_{12}^1 (x) + \gamma_{\mu \nu} \partial_\nu \xi_{12}^1 (x) \right) \}

whereas the nontrivial gauge transformation for the gauge parameter is written as follows:

\[ \delta \xi_{12}^1 (x) = -i \partial_\mu \xi_{12}^1 (x). \quad (7.130) \]

Then let us remove the additional fields from the spectrum of the above fields by means of their equations of motion. Thus, for the extremals \( \frac{\delta S_{(1,1)}}{\delta \chi_k} = \frac{\delta S_{(1,1)}}{\delta \rho_3} = 0, \) with the vectors \( | \chi_k \rangle, \)

\( k = 1, 6, 7, 9, 12, 13, | \rho_3 \rangle \) considered as Lagrangian multipliers, we have the solutions

\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{12}} = \hat{\gamma} T | \Psi \rangle = 0 \quad \Rightarrow \psi_{\mu \nu} = \psi_{\mu} = \psi = 0, \quad (7.131) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{13}} = \hat{\gamma} L_{11} | \Psi \rangle + \hat{\gamma} | \rho_4 \rangle = 0 \quad \Rightarrow \rho_4 = 0, \quad (7.132) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{14}} = -\hat{\gamma} T | \rho_2 \rangle = 0 \quad \Rightarrow \rho_2 = 0, \quad (7.133) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{15}} = -\hat{\gamma} (| \Psi_3 \rangle + | \chi_{10} \rangle) = 0 \quad \Rightarrow \chi_{10} = -\psi_3, \quad (7.134) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{16}} = \hat{\gamma} (| \Psi_1 \rangle - | \Psi_2 \rangle - | \chi_{10} \rangle) = 0 \quad \Rightarrow \chi_{10} = \psi_1 - \psi_2, \quad (7.135) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{17}} = T_1 | \Psi \rangle + T_2^+ | \Psi_3 \rangle - \hat{\gamma} | \chi_3 \rangle = 0 \quad \Rightarrow \gamma^{\mu} \psi_{\mu \nu} = \gamma_{\nu} \chi^2_3 - \chi^2_3, \quad (7.136) \]
\[ K_{(1,1)}^{\chi_{12}} \frac{\delta S_{(1,1)}}{\delta \chi_{18}} = -T_1 | \chi_3 \rangle - \hat{\gamma} T^+ | \chi_8 \rangle = 0 \quad \Rightarrow \chi_8 = -2 \chi^2_3, \quad \chi^2_3 = \chi^2_3 = 0, \quad (7.137) \]
(with \( K_{(1,1)}^{-1} \) being the inverse of \( K_{(1,1)} \) and thus the basic vector \( |\Psi| \) contains only the antisymmetric spin-tensor \( \psi_{\mu\nu}(x) \). After a substitution of expressions (7.84)–(7.96) into (7.123), we find the action for a field of spin \((3/2, 3/2)\) in a manifest form, with the remaining auxiliary fields:

\[
S_{(1,1)} = \int d^4x \left[ \bar{\psi}_{\mu\nu} \left\{ -\frac{i}{2} \gamma^\rho \partial_\rho \psi_{\mu\nu} - i\partial^{\left[\mu\nu\right]} \left[ \gamma^\rho (\psi_2 - \psi_1) + \gamma^\mu \psi_{[\nu|\rho]} \right] - i\partial^{\left[\mu\nu\right]} \chi^4 \right\} \\
+ \bar{\psi}_1 \left\{ -i\gamma^\rho \partial_\rho \psi_2 - 2\psi_8 - i\partial^{\left[\mu\nu\right]} \left[ \gamma^\mu (\psi_2 - \psi_1) + \gamma^\nu \psi_{[\rho]} \right] \right\} \\
+ i\bar{\psi}_2 \partial^{\left[\mu\nu\right]} \chi^4 - 2(\bar{\psi}_2 - \bar{\psi}_1)\psi_6 + \bar{\psi}_8 \left\{ \gamma^\mu \chi^4_4 + (2 - d)\chi_4 - 2(\psi_2 - \psi_1) \right\} \\
- \partial^\mu \left( \gamma^\nu \psi_{[\mu]} + \chi^4 \right) - (2 - d)\psi^2 \chi^4 \\
- \frac{i}{2} \left[ (\bar{\psi}_2 - \bar{\psi}_1)\gamma^\nu \bar{\psi}_{\nu} + \bar{\psi}_{\bar{\psi}} \gamma^\nu \right] \gamma^\sigma \partial_\sigma \left[ \gamma^\mu (\psi_2 - \psi_1) + \gamma^\nu \psi_{[\rho]} \right] \\
+ \frac{i}{2} (6 - d)(\bar{\psi}_2 - \bar{\psi}_1)\gamma^\nu \partial_\nu (\psi_2 - \psi_1) - \frac{i}{2} \chi^4 \gamma^\nu \partial_\nu \chi^4 \\
+ \frac{i}{2} (6 - d)(\bar{\psi}_2 - \bar{\psi}_1)\gamma^\nu \partial_\nu (\psi_2 - \psi_1) \right] + c.c. \right]\right.
\]

(7.138)

Then, from the extremals for the fields \( \bar{\psi}^2, \bar{\psi}_8, \bar{\psi}_8 \), we have their respective solutions

\[
\chi_4 = 0, \quad \chi^4 = \gamma^\mu \psi_{[\mu]} \quad \psi_1 = 2\psi_2 + \frac{1}{2} \gamma^\mu \psi_{[\mu]},
\]

so that the action (7.138) is transformed as follows:

\[
S_{(1,1)} = \int d^4x \left[ \bar{\psi}_{\mu\nu} \left\{ -\frac{i}{2} \gamma^\rho \partial_\rho \psi_{\mu\nu} - i\partial^{\left[\mu\nu\right]} \left[ \gamma^\rho (\psi_2 - \psi_1) + \gamma^\mu \psi_{[\nu]} \right] \right\} \\
- i\partial^{\left[\mu\nu\right]} \gamma^\rho \psi_{[\mu]} \right\} \right\} \\
+ \left\{ -i\bar{\psi}_2 + \bar{\psi}_8 \right\} \left\{ -i\partial^{\left[\mu\nu\right]} \left[ \gamma^\rho \psi_{[\mu]} + \gamma^\mu \psi_{[\nu]} \right] \right\} \\
- \frac{i}{2} \left\{ -\bar{\psi}_2 + \bar{\psi}_8 \right\} \left[ \gamma^\rho \psi_{[\mu]} + \gamma^\mu \psi_{[\nu]} \right] \right\} \\
- \frac{i}{2} \left\{ -\bar{\psi}_2 + \bar{\psi}_8 \right\} \gamma^\nu \partial_\nu \psi_{[\mu]} \right\} \\
- \frac{i}{2} \left\{ -\bar{\psi}_2 + \bar{\psi}_8 \right\} \gamma^\nu \partial_\nu \psi_{[\mu]} \right\} \\
+ \left\{ -\bar{\psi}_2 + \bar{\psi}_8 \right\} \gamma^\nu \partial_\nu \psi_{[\mu]} \right\} + c.c. \right]\right]
\]

(7.140)

One can show that the terms in (7.140) with the auxiliary spinor \( \bar{\psi}_8\) vanish identically, so that we have the final form of the action and reducible gauge transformations for the spin-tensor \( \psi_{[\mu\nu]}\):

\[
S_{(1,1)} = \int d^4x \left[ \bar{\psi}_{\mu\nu} \left\{ -i\gamma^\rho \partial_\rho \psi_{[\mu\nu]} + i\partial^{\left[\mu\nu\right]} \gamma^\rho \psi_{[\mu\nu]} + 2i\partial^{\left[\mu\nu\right]} \gamma^\rho \psi_{[\mu\nu]} + 2i\gamma^\nu \partial_\nu \psi_{[\mu\nu]} \right\} \\
- \frac{i}{2} \left( \gamma^\mu \gamma^\nu \gamma^\sigma \psi_{[\mu\nu]} \right) + \gamma^\mu \partial_\mu \psi_{[\mu\nu]} - 2i\gamma^\nu \partial_\nu \psi_{[\mu\nu]} \right\} \\
\delta \psi_{[\mu\nu]} = 2i\partial_\mu \xi_{\nu} + 2i\gamma^\nu \partial_\nu \eta \right\} \right\} \right\} \right\} \equiv (\xi_{\mu}^{(1)}, \eta, \xi_{\mu}^{(1)}).
\]

(7.142)

To obtain a Lagrangian description of the massive rank-2 antisymmetric spin-tensor \( \psi_{[\mu\nu]}\), having the Young tableaux (2.21) and subject to the conditions (2.23), (2.21) and the requirement \( \gamma^\nu \partial_\nu m(x) = 0 \), instead of (2.22), we may follow the example of Section (7.17) with \( (h_m^2, h_m^2) = ([1 - d]/2, [5 - d]/2) \) and apply the prescription (5.30)–(5.40), i.e., starting from the expansion.
\[(7.14) - (7.22), (7.31) - (7.36),\] where it is only the vectors in \(|\chi_0^i\rangle, |\Lambda_0^i\rangle, |\Lambda_0^{(1)}\rangle\) that change to \(|\chi_{0m}^i\rangle, |\Lambda_{0m}^i\rangle, |\Lambda_{0m}^{(1)}\rangle\), respectively,

\[
|\Psi_m\rangle_{(1,1)} = |\Psi\rangle_{(1,1)} + \left( a_{1}^{+\mu} (b_2^+ \psi_{1m\mu}(x) + b_1^+ b_3^+ \psi_{3m\mu}(x)) + b_1^+ \psi_{1m\mu}(x) \right) + b_1^+ \psi_{1m\mu}(x)
\]

\[
(1,1)\langle \bar{\Psi}_m \rangle = (1,1)\langle \bar{\Psi} \rangle - | \langle x_{1m}^\pm | b b_1 \rangle |^2 f_1 + \psi_5^m(x) b_1 f_2 \gamma_0 ,
\]

\[
|\chi^m_{k,(0)}, f_{i}\rangle = |\chi^m_{k,(0)}\rangle + b_{i}^{+} \chi^m_{k}(x) , (1,0) \langle x_{1m}^\pm | b_1 \gamma_0 , k = 2, 4, (7.149)
\]

\[
|\varphi_{l,m}\rangle = |\varphi_{l,m}\rangle + b_{l}^{+} \varphi_{l,m}(x) , (1,0) \langle x_{1m}^\pm | b_1 \gamma_0 , l = 1, ..., 4 , (7.150)
\]

Then all the terms in (7.23), except for the action \(S_{m(1,1)}\), have the same form, with a change of the massless \(O_{l}\) to the massive \(O_{m}^{l}\), except for the vectors \(|\Psi_k\rangle, \varphi_{l}, |\varphi_{l}\rangle, |\rho_{2}\rangle, |\rho_{3}\rangle, |\chi_{l}\rangle, l = 3, 4, 6, 11, 12,\) with \(\gamma_{0}\) and \(\gamma_{1}\), \(\gamma_{2}\) are 

\[
\tilde{\gamma} T_{0}^{m} \tilde{\gamma} = T_{0}^{m} - t_{0} - \gamma_{0} \neq T_{0}^{m} \tilde{\gamma} T_{0}^{m} \gamma_{0} = T_{0}^{m} \tilde{\gamma} T_{0}^{m} \tilde{\gamma} = T_{0}^{m} \tilde{\gamma} T_{0}^{m} \gamma_{0} = T_{0}^{m} \tilde{\gamma} T_{0}^{m}
\]

In the lowest gauge transformations (7.78) - (7.83), it is only the relations (7.80), (7.83) for \(\delta|\xi_{0}^{(1)}\rangle, \delta|\xi_{2}^{(1)}\rangle\) that are modified, respectively, \(\tilde{\gamma} T_{0}^{m} \tilde{\gamma} = T_{0}^{m} - t_{0} - \gamma_{0} \neq T_{0}^{m} \tilde{\gamma} T_{0}^{m} \gamma_{0} = T_{0}^{m} \tilde{\gamma} T_{0}^{m} \gamma_{0} = T_{0}^{m} \tilde{\gamma} T_{0}^{m} \gamma_{0} = T_{0}^{m} \tilde{\gamma} T_{0}^{m}
\]

The gauge-fixing (7.97) - (7.111) remains valid so as one works only with the ghost-independent structure of the modified vectors \(|\chi_{m}^i\rangle\), with a change of the massless \(O_{l}\) to the massive \(O_{m}^{l}\), featuring the mentioned properties.

From Eqs. (7.97) - (7.100), we gauge away, using \(|\xi_{1m}^{(1)}\rangle = |\xi_{1m}^{(1)}\rangle\), the \((b_{i}^{+})\)-dependent vector of the gauge parameter \(|\xi_{1m}^{(2)}\rangle\) in (7.99), which has the same form as \(|\chi_{1m}^{(2)}\rangle\) in (7.147). The theory becomes an irreducible one, with independent \(|\xi_{1m}^{(2)}\rangle\), \(|\xi_{k,m}^{(2)}\rangle\), \(k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19\), and also in the gauge transformations for the field \(|\Psi, |\chi_{l}\rangle, l = 3, 4, 6, 8, 10, 12, 14, 16, 18, 20\), with the respective parameters \(|\lambda_{o}\rangle, o = 1, 4, 6, 8, |\xi_{e}\rangle\) in (7.24) - (7.28), (7.30) - (7.37), (7.39), (7.41), (7.50) - (7.52). The gauge-fixing (7.97) - (7.111) remains valid so as one works only with the ghost-independent structure of the modified vectors \(|\chi_{m}^i\rangle\), with a change of the massless \(O_{l}\) to the massive \(O_{m}^{l}\), featuring the mentioned properties.
\[
\delta|\chi_{3|m}\rangle = -(2 + T_1^{1+T_1^*})|\xi^r_{3|m}\rangle + \bar{\gamma}(T_1^* + \frac{1}{2}L^+T^+T_2)|\xi^r_{3|m}\rangle - \bar{\gamma}T^+T_1|\xi^r_{2|2|m}\rangle,
\]

and
\[
\delta|\chi_{4|m}\rangle = -T_1^{1+T_2^*}|\xi^r_{3|m}\rangle - \bar{\gamma}L_1^+T_2|\xi^r_{3|m}\rangle + \bar{\gamma}T_1|\xi^r_{1|2|m}\rangle,
\]

are valid for the vectors \(|\xi^r_{k|m}\rangle, k = 1, 3,\) being solutions of the equations \(T^m|\xi|_{k|m}\rangle = 0,\)

\[
|\xi^r_{k|m}\rangle_{(0,1)} = |\xi^r_{k}\rangle_{(0,1)} + (b_1^+b^- + 2b_2^+)\xi^s_{k|m}(x)|0\rangle.
\]

The simplified representation \((7.123)\) for \(S^m_{(1,1)}\) in terms of the massive objects holds true, with the following changes:

\[
\langle \tilde{\Psi}_8|K_{(1,1)}T_1|\chi_4\rangle + \langle \tilde{\rho}_2|K_{(1,1)}T_0|\rho_3\rangle + \langle \tilde{\rho}_3|K_{(1,1)}T_1|\chi_3\rangle + \frac{1}{2}\sum_{i=3}^4 \langle \tilde{\chi}_i|K_{(1,1)}T_0|\chi_i\rangle + c.c. (7.159)
\]

\[
\Rightarrow \langle \tilde{\Psi}_8|m|K_{(1,1)}T_1^*|\chi_4|m\rangle + \langle \tilde{\rho}_2|m|K_{(1,1)}T_0^*|\rho_3|m\rangle + \langle \tilde{\rho}_3|m|K_{(1,1)}T_1^*|\chi_3|m\rangle
\]

\[
+ \frac{1}{2}\sum_{i=3}^4 \langle \tilde{\chi}_i|m|K_{(1,1)}T_0^*|\chi_i|m\rangle + c.c.
\]

In addition to removing the symmetric part \(\psi_{(\mu,\nu)}\) from the basic spin-tensor \(\psi_{\mu\nu}\), and the fields \(\psi^k_\mu, k = 2, 3, \psi^l_\mu, l = 1, 2,\) from \(|\Psi_{(1,1)}\rangle\) \((7.143)\), we also gauge away the new fields \(\psi^k_m(x), k = 2, 4, 5,\) and \(\psi^l_{m\mu}(x), l = 2, 3,\) except for \(\psi^1_{m}(x), \psi^2_{m}(x), \psi^3_{m\mu}(x),\) by using the respective gauge parameters \(\xi^r_{1|m}, \xi^l_{1|m}, \xi^l_{2|m}\) and the “massless” parameters \(\xi^r_{13}(x), \xi^l_{3}(x)\), from the gauge parameters \(|\xi^r_{12|m}\rangle_{(2,0)}\) and \(|\xi^r_{k|m}\rangle_{(0,1)}\), \(k = 1,3,\) in the gauge transformations \((7.153) - (7.157)\).

The following restrictions hold true:

\[
(\xi^r_{1|m}, \xi^l_{1|m}, \xi^l_{2|m}) = (-1, m, -2m)\xi^r_3,
\]

\[
(\xi^r_{2|m}, \xi^l_{2|m}, \xi^l_{13}) = (2[|\partial^\mu - m\gamma_\mu\rangle, \gamma^\mu] \xi^r_3 \Rightarrow \delta\psi_{[\mu|\nu]} = 2|\partial^\mu\psi^r_3 + 2|\partial^\mu\psi^l_{3}(\xi^r_{13} - m\xi_3)|.
\]

From the gauge transformations for the spinors \(\Psi_{1|m}, \Psi_{2|m} \quad (7.113), (7.114)\),

\[
\delta(\Psi_{1|m}, \Psi_{2|m}) = 2(r^\mu\partial_\mu - 2m, r^\mu\partial_\mu - 4m)\xi^r_3,
\]

it follows that the difference \((\Psi_{1|m} - |\Psi_{2|m}\rangle\) obeys a Stueckelberg-type gauge symmetry, so that in the new basis of the fields \(\hat{\Psi}_{1|m}, \hat{\Psi}_{2|m}\) we have

\[
\langle \hat{\Psi}_{1|m}, \hat{\Psi}_{2|m}\rangle = \frac{1}{2}(\Psi_{1|m} \pm \Psi_{2|m}) : \delta(\hat{\Psi}_{1|m}, \hat{\Psi}_{2|m}) = 2(r^\mu\partial_\mu - 3m, m)\xi^r_3.
\]

The field \(\hat{\Psi}_{2|m}\) may be gauged away by using the last gauge parameter \(\xi^r_3\), resulting in a non-gauge theory. Once again, the resolution of the “massive” analog of equations of motion \((7.132) - (7.137)\) leads to the same solutions, augmented by \(\psi^m_{(\mu,\nu)} = \psi^m_\nu(x) = 0,\) for \(k = 1,3,\) in the first equation, so that it is only the initial massive field \(\psi_{(\mu,\nu)}\) that survives in \(|\Psi_{m}\rangle\), and also augmented by the solutions \(\chi^r_{13}(x) = 0, \chi^r_{23}(x) = -\chi^r_3(x) = -\psi_3(x)\) in the last two (adapted) equations \((7.136), (7.137)\), with allowance for \((7.159)\).

In addition, \(\hat{\Psi}_{2|m} = 0\) implies that \(\chi_8 = \chi_1 = \psi_3 = \chi_3 = \chi_2 = \psi_3 = \chi_2 = 0,\) Substituting the expressions \((7.143) - (7.150)\) into \((7.123)\), albeit for \(S^m_{(1,1)}\), we find an explicit action for a massive field of spin \((3/2, 3/2)\), with the remaining auxiliary fields:

\[
S^m_{(1,1)} = \int d^4x \left[ \tilde{\psi}_{[\mu\nu]} \left\{ -\frac{1}{2}(r^\mu\partial_\mu - m)\psi_{[\mu\nu]} - i\partial^\mu \left[ \gamma_\rho \psi^r_{[\mu\nu]} - \chi_4^s \right] \right\} + \tilde{\psi}_1 \left\{ - (r^\mu\partial_\mu - m)\psi_1 - 2\psi_8 - i\partial^\mu \left[ \gamma^\mu \psi_{[\mu\rho]} - \chi^4_\mu \right] + m\chi^4_\mu \right\} + \tilde{\psi}_8 \left\{ \varphi^\mu_\mu \chi^4_\mu + (1 - d)\chi^4_\mu - m\chi^4_\mu \right\} - \varphi^2_\mu \left[ \gamma_\rho \psi_{[\mu\rho]} + \chi^4_\mu \right] - (1 - d)\varphi^2_\mu \chi^4_\mu - \frac{1}{2} \tilde{\psi}_{[\mu\nu]} \gamma_\sigma (r^\sigma \partial_\sigma + m) \gamma^\rho \psi_{[\mu\rho]} - \frac{1}{2} \chi^4_\mu (r^\nu \partial_\nu + m) \chi^4_\mu + \frac{1}{2} \chi^4_\mu (r^\nu \partial_\nu + m) \chi^4_\mu \right] + c.c.
\]
From the extremals for the fields \( \varphi_\mu, \psi, \varphi_2, \varphi_3 \), we find the respective solutions \( \chi^{A\mu} = \gamma_\mu \psi_{[\mu\nu]} \), \( \hat{\psi}_1 = \frac{1}{2} \gamma^{\mu\rho} \psi_{[\mu\rho]} \), \( \chi^A = \gamma^A = 0 \), which allow one to present the Lagrangian for a massive antisymmetric spin-tensor (3/2, 3/2) field in a flat \( d \)-dimensional space-time:

\[
\mathcal{L}_{\hat{\psi}_{[\mu\nu]}}^m = \bar{\psi}_{[\mu\nu]} \left\{ -(\gamma^\mu \partial_\mu - m)\psi_{[\mu\nu]} + i \partial_\mu \gamma^\nu \gamma^{\rho\sigma} \psi_{[\rho\sigma]} + 2 i \partial^\nu \gamma_\rho \psi_{[\rho\mu]} + 2 \gamma^\nu \partial_\rho \psi_{[\rho\mu]} \right. \\
\left. - \frac{1}{2} \gamma^\mu (\gamma^\sigma \partial_\sigma - m) \gamma_{\rho\sigma} \psi_{[\rho\sigma]} + i \gamma^\nu \partial_\rho \gamma_\tau \psi_{[\rho\tau]} - 2 \gamma^\mu (\gamma^\rho \partial_\rho + m) \gamma_\sigma \psi_{[\sigma\nu]} \right\} . \tag{7.165}
\]

Note that one can obtain \( \mathcal{L}_{\hat{\psi}_{[\mu\nu]}}^m \) by using the dimensional reduction procedure (5.41)–(5.48), starting directly from the action (7.141) presented for a (3/2, 3/2) spin-tensor (3 for the fields: \( \psi \), \( \chi \), \( \varphi_2 \), \( \varphi_3 \)) with respect to the gauge transformations (7.141) by using the dimensional projection \( \mathcal{R} \) which, in turn, are reducible:

\[
\delta \psi_{[\mu\nu]} = 2 i \partial_\mu \xi_\nu + 2 \gamma_\mu \partial_\nu \eta, \quad \delta \varphi_\mu = -i \partial_\mu \xi - m \xi_\mu + m \gamma_\mu \eta + i \partial_\mu \eta, \tag{7.166}
\]

which, in turn, are reducible:

\[
\delta \xi_\mu (x) = i \partial_\mu \xi^{(1)}(x), \quad \delta \xi (x) = -m \xi^{(1)}(x) . \tag{7.167}
\]

Third, due to (5.49), we use the identity \( \gamma^{RS} \psi_{[RS]} = \gamma^{\rho\sigma} \psi_{[\rho\sigma]} \), with the following identification implied by (7.151), (7.152):

\[
i \gamma^M \partial_M \psi_{[NK]} = (i \gamma^M \partial_M - m) \psi_{[NK]}, \quad i \gamma^M \partial_M \gamma_N \psi_{[NK]} = (i \gamma^M \partial_M + m) \gamma_N \psi_{[NK]}, \tag{7.168}
\]

which is valid if one replaces the quantities \( \psi_{[NK]} [\gamma_N \psi_{[NK]}] \) by \( (\gamma_L)^{2k} \psi_{[NK]} [(\gamma_L)^{2k+1} \psi_{[NK]}] \), for \( k \in \mathbb{N}_0 \). After removing the gauge parameter \( \xi (x) \) by using the shift transformation, and then (in the same manner) removing the field \( \varphi_\mu \), by using the already independent gauge transformation with the parameter \( \xi_\mu (x) \), we finally obtain the Lagrangian (7.165).

8 Conclusion

In the present work, we have constructed a gauge-invariant Lagrangian description of free half-integer HS fields belonging to an irreducible representation of the Poincare group \( ISO(1, d - 1) \) corresponding Young tableaux having two rows in the “metric-like” formulation. The results of this study are the most general ones and apply to both massive and massless fermionic HS fields with a mixed symmetry in a Minkowski space of any dimension.

In the standard manner, starting from an embedding of fermionic HS fields into vectors of an auxiliary Fock space, we treat the fields as coordinates of Fock-space vectors and reformulate the theory in such terms. We realize the conditions that determine an irreducible Poincare-group representation with a given mass and generalized spin in terms of differential operator constraints imposed on the Fock space vectors. These constraints generate a closed Lie superalgebra of HS symmetry, which contains, with the exception of two basis generators of its Cartan subalgebra, a system of first- and second-class constraints.

We demonstrate that the construction of a correct Lagrangian description requires a deformation of the initial symmetry superalgebra, in order to obtain from the system of mixed-class
constraints a converted system with the same number of first-class constraints alone, whose structure provides the appearance of the necessary number of auxiliary spin-tensor fields with lower generalized spins. We have shown that this purpose can be achieved with the help of an additional Fock space, by constructing an additive extension of a symmetry subsuperalgebra which consists of the subsystem of second-class constraints alone and of the generators of the Cartan subalgebra, which form an invertible even operator supermatrix composed of supercommutators of the second-class constraints.

We have realized the Verma module construction \[47\] in order to obtain an auxiliary representation in Fock space for the above superalgebra with second-class constraints. As a consequence, the converted Lie superalgebra of HS symmetry has the same algebraic relations as the initial superalgebra; however, these relations are realized in an enlarged Fock space. The generators of the converted Cartan subalgebra contain linearly two auxiliary independent number parameters, whose choice provides the vanishing of these generators in the corresponding subspaces of the total Hilbert space extended by the ghost operators in accordance with the minimal BFV–BRST construction for the converted HS symmetry superalgebra. Therefore, the above generators, enlarged by the ghost contributions up to the “particle number” operators in the total Hilbert space, covariantly determine Hilbert subspaces in each of which the converted symmetry superalgebra consists of the first-class constraints alone, labeled by the values of the above parameters, and constructed from the initial irreducible Poincare-group relations.

It is shown that the Lagrangian description corresponding to the BRST operator, which encodes the converted HS symmetry superalgebra, yields a consistent Lagrangian dynamics for fermionic fields of any generalized spin. The resulting Lagrangian description, realized concisely in terms of the total Fock space, presents a set of generating relations for the action and the sequence of gauge transformations for given fermionic HS fields with a sufficient set of auxiliary fields, and proves to be a reducible gauge theory with a finite number of reducibility stages, increasing with the value of generalized spin. We elaborate a dimensional reduction procedure used to obtain a gauge-invariant Lagrangian description for a massive fermionic HS field in a \(d\)-dimensional Minkowski space \(R^{1,d}\) starting from a Lagrangian description for a massless fermionic HS field of the same generalized spin, albeit in \(R^{1,d-1}\).

We have outlined a proof of the fact that the solutions of the Lagrangian equations of motion \(5.22\), \(5.25\), after a partial gauge-fixing, correspond to the BRST cohomology space with a vanishing ghost number, which is determined only by the relations that extract the fields of an irreducible Poincare-group representation with a given value of generalized spin.

As examples demonstrating the applicability of the general scheme, we have derived gauge-invariant Lagrangian formulations for the field of spin \(3/2, 1/2\) and for the rank-2 antisymmetric spin-tensor in a manifest form in both massless and massive cases\[11\]. In principle, the suggested algorithm permits one to derive manifest actions for any other half-integer spin fields characterized by two rows of the corresponding Young tableaux.

The basic results of the present work are given by relations \(5.27\), where the action for a field with an arbitrary generalized half-integer spin is constructed, as well as by relations \(5.28\)–\(5.31\), where the gauge transformations for the fields are presented, along with the sequence of reducible gauge transformations and gauge parameters.

Concluding, we would like to discuss a number of additional points. First, the gauge-invariant description of massless and massive HS field theories with a mixed symmetry is an interesting starting point for a systematic construction of a Lagrangian formulation for HS interacting vertices

\[11\] Lagrangian formulations for the case of antisymmetric spin-tensors of arbitrary rank \(n\) in \(R^{1,d-1}\) and AdS_d-spaces, for \(n \leq \left[ \frac{d}{2} \right] \), was considered recently within BRST and algebraic approach respectively, in Refs. \[50, 51\], so that the Lagrangian formulations both for the massive and massless fields of spin \((3/2, 3/2)\) coincide with ones in \[50, 51\].
with mixed-symmetry fermionic HS fields, including the case of the AdS space, in order to provide a description of the high-energy limit for open superstrings; see the arguments in favor of this suggestion in [39]. Second, the role of fermionic HS fields in the above limit of superstring theory in connection with the AdS/CFT correspondence signals the importance of extending the obtained results to the case of fermionic HS fields with a mixed symmetry in the AdS space. Thus, the present Lagrangian description takes a first step towards an interacting theory with mixed-symmetry fermionic HS fields, including the case of curved backgrounds, and then towards a covariant construction (following, e.g., the BV formalism) of the generating functionals of Green’s functions, including the quantum effective action; examples of such calculations can be found, e.g., in [49]. Third, we estimate an extension of the obtained results to the case of arbitrary fermionic HS fields with any number of rows in the corresponding Young tableaux as a challenging technical problem. One of the possible approaches to this problem may rely on creating a computer algorithm which would permit one to obtain the HS symmetry superalgebra and calculate the action with the sequence of gauge transformations in an analytic component form for fermionic fields of a given generalized spin. Finally, the example of a field of spin $(3/2, 3/2)$ in Section 7.2 has demonstrated a possibility of extracting a large number of auxiliary fields until the point when the component form of the action and gauge transformations can be derived in a manifest form. In our forthcoming work [54], we plan to realize this possibility, which should permit one to significantly reduce the spectrum of fields and gauge parameters in order to simplify the component structure of the basic results of the present work, however, with a possible appearance of additional off-shell constraints for the fields and gauge parameters.

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