Inversion of the attenuated geodesic X-ray transform over functions and vector fields on simple surfaces

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Abstract

We derive explicit reconstruction formulas for the attenuated geodesic X-ray transform over functions and, in the case of non-vanishing attenuation, vector fields, on a class of simple Riemannian surfaces with boundary. These formulas partly rely on new explicit approaches to construct continuous right-inverses for backprojection operators (and, in turn, holomorphic integrating factors), which were previously unavailable in a systematic form. The reconstruction of functions is presented in two ways, the latter one being motivated by numerical considerations and successfully implemented at the end. Constructing the right-inverses mentioned require that certain Fredholm equations, first appearing in [22], be invertible. Whether this last condition reduces the applicability of the overall approach to a strict subset of simple surfaces remains open at present.

1 Introduction

We consider explicit inversion formulas for the two-dimensional attenuated X-ray (or, equivalently in two dimensions, Radon) transform of a function or vector field. Let \((M,g)\) a non-trapping Riemannian surface-with-boundary with unit circle bundle

\[ SM = \{ (x,v) \in TM, \ g(v,v) = 1 \}, \]

and let us denote the geodesic flow by \(\varphi_t(x,t) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\). For \(x \in \partial M\), let \(\nu_x\) the unit inner normal and define the influx/outflux boundaries \(\partial_{\pm}SM = \{ (x,v) \in \partial SM, \pm g(\nu_x,v) > 0 \}\). Fix \(a\) a smooth enough function on \(M\). Then for a function \(f \in L^2(SM)\), we define the attenuated (geodesic) X-ray transform of \(f\) by

\[ I_afa(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \exp \left( - \int_0^t a(\gamma_{x,v}(s)) \ ds \right) \ dt, \quad (x,v) \in \partial_+SM, \]

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where $\tau(x,v)$ denotes the first exit time of the geodesic $\varphi_t(x,v)$ (the non-trapping condition implies that $\tau$ is uniformly bounded above on $SM$). Cases of interest here are 
(i) the case where $f$ is a function on $M$, i.e., $f(\varphi_t(x,v)) = f(\gamma_{x,v}(t))$ with applications to X-ray Tomography in media with variable refractive index, a topic receiving much interest at the moment [17, 11], and
(ii) the case where $f$ represents a vector field on $M$, in which case the integrand above takes the form $f(\varphi_t(x,v)) = g(F(\gamma_{x,v}(t)), \dot{\gamma}_{x,v}(t))$ for $F$ some vector field on $M$, and whose applications to Doppler tomography justify its nickname of *Doppler transform*.

Such a transform over functions was studied earlier in the Euclidean setting. Inversion formulas with known attenuation were obtained independently by Arbuzov, Bukhgeim and Kasantzev using $A$-analytic function theory à la Bukhgeim in 1998 [1], and by Novikov via complexification methods in 2002 [19] deriving an explicit inverse operator, see also [18, 16] and [3] for a joint study of both approaches. The latter method was extended to the attenuated Radon transform in hyperbolic geometry and in the case of the horocyclic transform [2], finally generalized to a certain class of curves [7]. The former method was adapted to the study of the attenuated transform over functions and vector fields in *fan-beam geometry* in [9], describing both a solution via $A$-analytic functions and an approach studying fiberwise holomorphic solutions to certain transport equations related to the problem. This approach was then generalized by Salo and Uhlmann to simple Riemannian surfaces (i.e. surfaces with strictly convex boundary and no conjugate points) in [27], where a method reconstructing functions from knowledge of their geodesic X-Ray transform was developed. The Doppler transform was studied microlocally in the Riemannian case in [8], and a range characterization in the Euclidean case was recently given in [25]. Additional range characterizations of the Euclidean transform in convex domains of $\mathbb{R}^2$ were provided in [26], and a study of the attenuated Euclidean transform over second-order tensors was recently provided in [24].

The X-ray transform can also be considered over matrix-valued unknowns, complex-valued attenuations, connections and Higgs fields, for which recent results can be found in [21, 20, 5]. More general settings include the case of *weighted* X-ray transforms (of which the attenuated case is a particular example). In this setting, general results provided in [4] lead to injectivity and stability of the attenuated transform over functions when both the metric and the attenuation coefficient are real-analytic. In dimension $d \geq 3$, local injectivity of weighted X-ray transforms near convex boundary points was recently established in [30], following methods in [29] where the unattenuated case was first treated.

While laying the groundwork of non-Euclidean inversions in [27], the authors there pointed out two open questions: 
(i) how to explicitly invert the unattenuated X-ray transform over functions and solenoidal vector fields (we call them $I_0$ and $I_\perp$ here), and
(ii) how to explicitly construct *(fiber-)*holomorphic integrating factors, which are so crucial for the approach. An implementation answering (i) was presented by the author in [13], by implementing the reconstruction formulas [22, 10].

The present article addresses point (ii) by providing explicit ways of constructing holomor-
phic integrating factors, in particular by constructing preimages of the adjoint operators $I^*_0$ and $I^*_\perp$ explicitly, and by deriving reconstruction algorithms, some of which are similar in spirit to those in [9], another one similar to the approach in [22, 14]. In the first approach, the main novelty here, partly motivated by the approach in [9], consists in decomposing $v = ue^{-w}$ (with $u$ the solution of $Xu + au = -f$ and $w$ a holomorphic integrating factor) into the sum of a holomorphic function and a function that is constant along geodesics. This can be done as soon as one can construct explicit preimages of the operators $I^*_0$ and $I^*_\perp$. A second approach reconstructing functions, more similar to approaches in [22, 14], is then presented, and a numerical implementation is provided. Additionally, these reconstruction formulas are fast in that no full three-dimensional transport equation needs to be solved unlike [27]. The class of surfaces where the current approach is valid is that of simple surfaces where some Fredholm equations (see Equations (9) and (10) below), which first appeared in [22, Theorem 5.4], are invertible. It remains open at present whether this latter additional requirement applies to all or a strict subset of simple surfaces, though past numerical experiments done in [13] by the author have showed that such Fredholm equations were invertible on some family of surfaces which could become arbitrarily close to non-simple.

Recent work by the author with P. Stefanov and G. Uhlmann [15] shows that stable inversion of the attenuated ray transform should still be possible in some surfaces with conjugate points occurring in pairs. Adapting the current approach to this latter setting will be the object of future work.

**Outline.** The structure of the paper is as follows. We first introduce in Sec. 2 the basic setting (Sec. 2.1), some new notation and operators (e.g., $I^*_\perp$) which play an important role in the inversion process (Sec. 2.2), the construction of explicit, continuous right-inverses for $I^*_0$ and $I^*_\perp$ (Sec. 2.3) and how they allow the explicit construction of holomorphic integrating factors (Sec. 2.4). Section 3 presents the reconstruction formulas for functions and vector fields, following initial ideas in [9], first adapted in [27]. Section 4 proposes an alternative approach to reconstruction of functions, stating in passing $L^2 \rightarrow L^2$ continuity and bounds for a certain family of operators, the proof of which is relegated to appendix A. Section 5 presents a numerical implementation of inversions as set up in Theorem 4.2.

## 2 Notation, preliminaries and the unattenuated case

### 2.1 Geometry of the unit circle bundle

We briefly recall the geometry and notation associated with the unit circle bundle. With $(M,g)$ as above, let $X = \frac{d}{dt} \varphi_t|_{t=0}$ be the geodesic vector field on $SM$. The requirement that $(M,g)$ be non-trapping implies that it is simply connected, so that we may put a global isothermal chart on $M$ for which $g$ is scalar, of the form $g = e^{2\lambda(x,y)}(dx^2 + dy^2)$, and $(x, v) \in SM$ is parameterized by $(x, y, \theta)$ where $x = (x, y)$ and $v = e^{-\lambda(x,y)}(\cos \theta, \sin \theta)$ for $\theta \in S^1$. In these coordinates, the geodesic
flow is defined on the following domain
\[ D = \{(x, \theta, t) : (x, \theta) \in SM, \ t \in (-\tau(x, \theta + \pi), \tau(x, \theta))\}. \]

There is a circle action on \( SM \) with infinitesimal generator \( V = \frac{\partial}{\partial \theta} \), and, upon defining \( X_\perp := [X, V] \), the triple \((X, X_\perp, V)\) forms a global frame of \( T(SM) \), with structure equations
\[ [X, V] = X_\perp, \quad [X_\perp, V] = X, \quad [X, X_\perp] = -\kappa V \quad (\kappa : \text{Gaussian curvature}). \]

We use the Sasaki metric on \( SM \) for which the basis \((X, X_\perp, V)\) is orthonormal, with volume form \( d\Sigma^3 \) (in isothermal coordinates, \( d\Sigma^3 = e^{2\lambda} dx \, dy \, d\theta \)), preserved by the frame. Introducing the inner product
\[ (u, v) = \int_{SM} \bar{u} \, v \, d\Sigma^3, \quad u, v : SM \to \mathbb{C}, \]
the space \( L^2(SM, \mathbb{C}) \) decomposes orthogonally as a direct sum
\[ L^2(SM, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}} H_k, \quad \text{where} \quad H_k := \ker(V - i k \text{Id}). \]

We also denote \( \Omega_k := C^\infty(SM) \cap H_k \). A smooth function \( u : SM \to \mathbb{C} \) has a Fourier series expansion
\[ u = \sum_{k=-\infty}^{\infty} u_k(x, \theta), \quad \text{where} \quad u_k(x, \theta) = e^{ik\theta} \tilde{u}_k(x), \quad \tilde{u}_k(x) = \frac{1}{2\pi} \int_{S^1} u(x, \theta) e^{-ik\theta} \, d\theta. \]

Such functions admit an even/odd decomposition w.r.t. to the involution \( \theta \mapsto \theta + \pi \), denoted
\[ u = u_+ + u_-, \quad \text{where} \quad u_+ := \sum_{k \text{ even}} u_k \quad \text{and} \quad u_- := \sum_{k \text{ odd}} u_k. \]

An important decomposition of \( X \) and \( X_\perp \) due to Guillemin and Kazhdan (see [6]) is given by defining \( \eta_\pm := \frac{X \pm i X_\perp}{2} \), so that one has the following decomposition
\[ X = \eta_+ + \eta_- \quad \text{and} \quad X_\perp = \frac{1}{i}(\eta_+ - \eta_-), \]
with the important property that \( \eta_\pm : \Omega_k \to \Omega_{k \pm 1} \) for any \( k \in \mathbb{Z} \), so that both \( X \) and \( X_\perp \) map odd functions on \( SM \) into even ones and vice-versa.

In the harmonic decomposition above, a diagonal operator of particular interest is the so-called fiberwise Hilbert transform \( H : C^\infty(SM) \to C^\infty(SM) \), whose action on each component is described by
\[ Hu_k := -i \text{sign}(k) u_k, \quad k \in \mathbb{Z}, \quad \text{with the convention} \quad \text{sign}(0) = 0, \]
and we denote $H_{+/-}$ the composition of $H$ with projection onto even/odd Fourier modes. We say that a function $u \in L^2(SM)$ is (fiber-)holomorphic if $(Id + iH)u = u_0$, i.e. if $u$ has only nonnegative Fourier components. An important identity first proved in [23] is the commutator between the Hilbert transform and the geodesic flow: denoting $\pi_0 : L^2(SM) \to L^2(M)$ the projection onto $H_0$ (fiberwise average)

$$[H, X] = \pi_0 X - X \pi_0, \quad [H, X_\perp] = -\pi_0 X - X \pi_0.$$ 

Note also that $H^2 = -Id + \pi_0$ and that $H \pi_0 = \pi_0 H = 0$. Using these observations and the commutators above, we write

$$[H^2, X] = H[H, X] + [H, X]H = H \pi_0 X_\perp + H X_\perp \pi_0 + X_\perp \pi_0 H + \pi_0 X_\perp H$$

On the other hand, $[H^2, X] = [-Id + \pi_0, X] = \pi_0 X - X \pi_0$. Upon splitting into odd and even parts, we arrive at the following equalities, to be used subsequently

$$\pi_0 X = \pi_0 X_\perp H, \quad \text{and} \quad X \pi_0 = -HX_\perp \pi_0. \quad (4)$$

### 2.2 An alternative notation for the unattenuated case

We now introduce some notation that emphasizes the $L^2(M)$-duality arising in the Pestov-Uhlmann reconstruction formulas [22]. The main novelty below is the introduction of $I_\perp$. The general unattenuated transform can be defined over functions $f \in L^2(SM)$ as

$$If(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \, dt, \quad (x,v) \in \partial SM. \quad (5)$$

Considering integrands of the form $f \in L^2(M)$ (i.e. $f(x,v) = f(x)$), and $X_\perp h$ for some $h \in H^1_0(M)$, let us define the unattenuated transforms

$$I_0 f := If, \quad I_\perp h := I(X_\perp h).$$

These transforms are continuous in the following settings when $(M,g)$ is non-trapping

$$I_0 : L^2(M) \to L^2_\mu(\partial_+ SM), \quad I_\perp : H^1_\mu(M) \to L^2_\mu(\partial_+ SM),$$

where $L^2_\mu$ is a weighted $L^2$ space with weight $\mu(x,v) = g(\nu_x, v)$. Recall the definitions of $A_\pm, A^*_\pm$, introduced in [22], with $\alpha$ the scattering relation:

$$A_\pm : L^2_\mu(\partial_+ SM) \to L^2_{[\mu]}(\partial SM)$$

$$A_\pm w(x, \theta) = \begin{cases} w(x, \theta) & \text{on } \partial_+ SM \\ \alpha^* w(x, \theta) & \text{on } \partial_- SM \end{cases}$$

$$A^*_\pm w = (u \pm u \circ \alpha)|_{\partial_+ SM}.$$
Note, via the fundamental theorem of calculus, that

\[ IXu = (u \circ \alpha - u)|_{\partial_+ SM} = -A^* u. \] (6)

With \( \psi : SM \to \partial_- SM \) the endpoint map \( \psi(x, \theta) = \phi_{\tau(x, \theta)}(x, \theta) \) and \( h \in L^2_\mu(\partial_+ SM) \), we denote by \( h_\psi := h \circ \alpha \circ \psi \) the function extended by free geodesic transport to \( SM \), i.e. solution of the equation

\[ Xu = 0 \quad (SM), \quad u|_{\partial_\pm SM} = h. \]

Straightforward computations using Santaló’s formula yield that for \( h \in L^2_\mu(\partial_+ SM) \),

\[ I^*_0 h = 2\pi(h_\psi)_0, \quad I^*_\perp h = -2\pi(X_\perp h_\psi)_0. \] (7)

The \( V \pm \) decomposition. Let us define the additional involution \( \alpha_1 : \partial_+ SM \to \partial_+ SM \) as \( (v \mapsto -v) \circ \alpha \), and write \( L^2_\mu(\partial_+ SM) = V_+ \oplus V_- \), where \( h \in V_+ \) (resp. \( V_- \)) iff \( h \) is even (resp. odd) with respect to the involution \( \alpha_1 \). Since a function of \( x \) only can be regarded as an even function of \( v \) on \( SM \), and a vector field can be regarded as an odd function of \( v \) on \( SM \), it is straightforward to establish that

\[ \text{Range } I_0 \subset V_+ \quad \text{and} \quad \text{Range } I_\perp \subset V_- . \]

Moreover, we have the following lemma.

**Lemma 2.1.** The direct sum \( L^2_\mu(\partial_+ SM) = V_+ \oplus V_- \) is orthogonal.

*Proof.* For \( h \in L^2_\mu(\partial_+ SM) \), define \( u = \left( \frac{h}{\sqrt{\tau}} \right)_\psi \) where \( \tau = I_0(1) \) (\( 1 \) denotes the constant function equal to 1 on \( M \)). Santaló’s formula allows to show that the map \( h \mapsto u \) is \( L^2_\mu(\partial_+ SM) \to L^2(SM) \) continuous and \( \|u\|_{L^2(SM)} = \|h\|_{L^2_\mu(\partial_+ SM)} \). Moreover, \( u \) is even/odd in \( v \) whenever \( h \in V_+/V_- \), so that, if \( h \in V_+ \) and \( g \in V_- \), and upon calling \( u = \left( \frac{h}{\sqrt{\tau}} \right)_\psi \) and \( v = \left( \frac{g}{\sqrt{\tau}} \right)_\psi \), we have

\[ (h, g)_{L^2_\mu(\partial_+ SM)} = (u, v)_{L^2(SM)} = 0 , \]

hence the proof. \( \square \)

**Inversion of \( I_0 \) and \( I_\perp \).** We now revisit the inversion of the operators \( I_0 \) and \( I_\perp \), previously established in [22], adapted here to the present notation. Recall the notation \( u^f \) (\( f \in L^2(SM) \)) for the solution of a transport problem of the form

\[ Xu = -f \quad (SM), \quad u|_{\partial_- SM} = 0, \] (8)

and for \( f \in C^\infty(M) \), define \( Wf = (X_\perp u^f)_0 \). It is established in [22] that \( W \) extends as a smoothing (hence compact) operator \( W : L^2(M) \to C^\infty(M) \) and that the \( L^2 \)-adjoint of \( W \) is given by \( W^* h = (u X_\perp)^h_0 \).
Proposition 2.2. Let $(M, g)$ a simple surface with boundary. Then we have for every $f \in L^2(M)$ and every $h \in H^1_0(M)$,

\[
  f + W^2 f = \frac{1}{2\pi} I_\perp^* w, \quad w = \frac{1}{4} A^*_+ H_- A_- I_0 f, \quad (9)
\]

\[
  h + (W^*)^2 h = -\frac{1}{2\pi} I_0^* w, \quad w = \frac{1}{4} A^*_+ H_+ A_- I_\perp h. \quad (10)
\]

Remark 2.3. Formulas (9) and (10) differ slightly from [22, Theorem 5.4] because it is stated there that the solution to the transport problem

\[
  Xu = -f, \quad u|_{\partial_+ SM} = w,
\]

is $u^f + w_\psi$, which is what the Fredholm equation there is based upon. The correct answer would be $u^f + (w - I_0 f)_\psi$, which in turn yields the modified formula (9).

Proof. Let $f \in C^\infty(M)$ and define $u^f$ as in (8) so that $u^f|_{\partial_+ SM} = I_0 f$. Applying $\pi_0 X = \pi_0 X_\perp H$ (derived in (4)) to (8), we obtain

\[
  f = \pi_0 f = -\pi_0 X u^f = -\pi_0 X_\perp H u^f = -(X_\perp H u^f)|_0, \quad (X_\perp H u^f)|_0 = -W f,
\]

so $f$ can be obtained from the last equation if we can relate $Hu^f_\perp$ to the known data $I_0 f$. In order to do so, we use the commutator $[H, X]$ to write a transport equation for $Hu^f_\perp$

\[
  X u^f_\perp = -f \quad \Rightarrow \quad X (Hu^f_\perp) = -X_\perp (u^f_\perp)|_0 - (X_\perp u^f_\perp)|_0 = -W f,
\]

so that $Hu^f_\perp$ satisfies the transport problem

\[
  X (Hu^f_\perp) = -W f, \quad Hu^f_\perp|_{\partial_+ SM} = \frac{1}{2} H_- A_- I_0 f|_{\partial_+ SM} := \eta,
\]

which means that

\[
  Hu^f_\perp = u^f_W + w_\psi, \quad w = \eta \circ \alpha.
\]

Upon applying $(X_\perp \cdot)|_0$, we obtain

\[
  (X_\perp Hu^f_\perp)|_0 = W^2 f - \frac{1}{2\pi} I_\perp^* w.
\]

Since we have established that $f = -(X_\perp Hu^f_\perp)|_0$, we conclude that

\[
  f + W^2 f = \frac{1}{2\pi} I_\perp^* w, \quad w = \left(\frac{1}{2}(H_- A_- I_0 f)|_{\partial_- SM}\right) \circ \alpha.
\]
In terms of $A^\pm$ operators, we can also write $w$ as $w = \frac{1}{2} A^+ - \frac{1}{2} A^- H_- A_- I_0 f$. Moreover, it can be seen that the function $A^+ H_- A_- I_0 f$ has $V_+$ symmetry, so it is annihilated by $I_0^+$. Thus, Equation (9) follows, extended to every $f \in L^2(M)$ by density of $C^\infty(M)$ in $L^2(M)$.

**Proof of (10).** Let $h \in C^\infty(M)$ with $h|_{\partial M} = 0$, and let $u^{X^\perp h}$ solve the transport equation

$$ Xu = -X_\perp h \quad (SM), \quad u|_{\partial_-SM} = 0. $$

Upon projecting onto odd functions of $v$, we have $Xu^{X^\perp h} = -X_\perp h$. Direct manipulations and the use of the commutator formula imply

$$ Xu^{X^\perp h} = -X_\perp h \quad \Rightarrow \quad X(Hu^{X^\perp h} - h) = -X_\perp W^* h - (X_\perp u^{X^\perp h})|_{\partial SM} = -X_\perp W^* h. $$

Moreover, the trace of $u^{X^\perp h}$ is given by $u^{X^\perp h}|_{\partial SM} = \frac{1}{2} A_- I_- h$. Since the function $Hu^{X^\perp h}$ satisfies the transport problem

$$ X(Hu^{X^\perp h} - h) = -X_\perp W^* h, \quad Hu^{X^\perp h}|_{\partial_-SM} = \frac{1}{2} H_+ A_- I_- h|_{\partial_-SM}, $$

we deduce that

$$ Hu^{X^\perp h} - h = u^{X^\perp W^* h} + w, \quad \text{where} \quad w := \left( \frac{1}{2} H_+ A_- I_- h|_{\partial_-SM} \right) \circ \alpha. $$

Upon averaging over $\theta$ and rearranging terms, we obtain

$$ h + (W^*)^2 h = -\frac{1}{2\pi} I_0^* w. $$

In terms of $A^\pm$ operators, $w$ can also be written as $w = \frac{1}{2} A^+ - \frac{1}{2} H_+ A_- I_\perp f$. We can see that $A^+ H_+ A_- I_\perp f$ has $V_-$ symmetry and as such is annihilated by $I_0^-$. Thus, Equation (10) follows, extended to every $h \in H_0^1(M)$ by density.

**Remark 2.4.** Inspection of symmetries shows that $A_- I_0 f$ in (9) is odd in $v$ and $A_- I_\perp f$ in (10) is even in $v$. This tells us that the expressions $H_- A_- I_0 f$ and $H_+ A_- I_\perp f$ are redundant, as one could just replace $H_- \pm H_+$ by $H$. This further emphasizes the similarity between formulas (9) and (10), for both of which the operator $A^+ H A_-$ acts as a first step in the postprocessing of data, though on a different subspace depending on the formula.

### 2.3 Construction of explicit continuous right-inverses for $I_0^+$ and $I_\perp^+$

The question of surjectivity of $I_0^+$ and $I_\perp^+$ have proved to be crucial for answering boundary rigidity questions (see [23] where a proof of surjectivity of $I_0^+$ appears) and constructing holomorphic integrating factors in a prior study of the attenuated transform (see [27] where Lemma 4.5 states
that $I^*_\perp$ is surjective), which are so important to the present approach. Such proofs relied on pseudodifferential arguments on an extended simple compact manifold, and did not construct explicit preimages of either operator. In order to derive and implement explicit inversions, constructing explicit preimages becomes a necessity, and we notice here that, while writing formulas (9) and (10) in a way that emphasizes duality, we also notice that the right-hand sides involve $I_0^*$ and $I_\perp^*$ directly. Under the assumption that the operators $I_\perp + W^2$ and $I_\perp + (W^*)^2$ are invertible, this allows for an explicit construction of continuous right-inverses of $I_0^*$ and $I_\perp^*$.

**Remark 2.5.** Although it would be enough to show that $I_\perp + W^2$ is injective, which is open at present for general simple surfaces, it is shown in [13] that the operator $W$ admits a bound of the form $\|W\|_{L^2 \to L^2} \leq C\|\kappa\|_{\infty}$. This implies that if curvature is close enough to constant, the operators $I_\perp + W^2$ and $I_\perp + (W^*)^2$ are invertible via Neumann series. Whether this qualitative assumption covers the case of all simple surfaces remains open at present.

**Proposition 2.6.** Suppose the operators $I_\perp + W^2$ and $I_\perp + (W^*)^2$ are $L^2(M) \to L^2(M)$ invertible. Then for every $k \in \mathbb{N}$, the operators $R_\perp : C^k(M) \to C^k(\partial_+ SM)$ and $R_0 : C^{k+1}(M) \to C^k(\partial_+ SM)$ defined by

$$R_\perp := \frac{1}{8\pi} A_+^* H_- A_- I_0 (I_\perp + W^2)^{-1}, \quad R_0 := -\frac{1}{8\pi} A_+^* H_+ A_- I_\perp (I_\perp + (W^*)^2)^{-1},$$

(11)

are continuous and satisfy $I_\perp^* R_\perp f = f$ and $I_0^* R_0 f = f$ for $f$ smooth enough.

**Proof.** Suppose that $I_\perp + W^2$ and its adjoint are invertible. Then their inverses map any $C^k(M)$ to itself. This is because the kernels of $W, W^*$ are proved in [22] to be smooth so that, e.g., if $f \in L^2(M)$ solves $f + W^2 f = f_1$, where $f_1 \in C^k(M)$ and $W^2 f$ is smooth, then $f = f_1 - W^2 f$ is $C^k$. Since the operators $W, W^*$ are also $L^2 \to C^k$ continuous, then similar arguments allow to show that $(I_\perp - W^2)^{-1}$ and its adjoint are $C^k(M) \to C^k(M)$ continuous.

The relations $I_\perp^* R_\perp = I_\perp$ and $I_0^* R_0 = I_\perp$ are straightforward to check, as a directly application of equations (9) and (10) and the invertibility of $I_\perp + W^2$ and $I_\perp + (W^*)^2$.

It remains to prove that

$$\|R_\perp f\|_{C^k(\partial_+ SM)} \leq C\|f\|_{C^k(M)}, \quad \|R_0 g\|_{C^k(\partial_+ SM)} \leq C\|g\|_{C^{k+1}(M)}.$$  

(12)

Looking at the compound expression of these operators, we see that $A_-$ and $A_+^*$ preserve $C^k$ norms since the scattering map is smooth and the function $\tau(x, \theta)$ is smooth on $\partial_+ SM$ whenever $\partial M$ is strictly convex (see [28, Lemma 4.1.1 p.115]). $H_\pm$ preserve $C^k$ norms as convolution operators, and $I_0 : C^k(M) \to C^k(\partial_+ SM)$ and $I_\perp : C^{k+1}(M) \to C^k(\partial_+ SM)$ are continuous since the geodesic flow is smooth.

A study of symmetries with respect to the involution $\alpha_1$ shows that $p = R_\perp f$ and $q = R_0 g$ thus constructed satisfy $p \in \mathcal{V}_-$ and $q \in \mathcal{V}_+$. This is compliant with the continuity statements (12), as any component of $p$ in $\mathcal{V}_+$ would be annihilated by $I_\perp^*$ and any component of $q$ in $\mathcal{V}_-$ would be annihilated by $I_0^*$.
2.4 Holomorphic solutions to certain transport equations

A crucial tool in the inversion of attenuated ray transforms is the construction of holomorphic integrating factors, whose existence relies on the surjectivity of $I_0^*$ and $I_\perp^*$. In the simple Riemannian setting, it is proved in [20, Theorem 4.1] that the transport equation $Xu = -F$ (for $F \in C^\infty(SM)$) admits holomorphic solutions if and only if $F$ is of the form $F = f_1 + X_\perp f_2$ for some functions $f_1, f_2 \in C^\infty(M)$. Although uniqueness of such solutions may not hold (e.g. adding a constant to such a solution makes another solution), a constructive approach, inspired in part by [20, Theorem 4.1], is to look for an ansatz, holomorphic by construction, of the form

$$u = (Id + iH)p_\psi + (Id + iH)q_\psi,$$

where $p$ is a smooth element in $\mathcal{V}_-$ and $q$ is a smooth element in $\mathcal{V}_+$, so that $p_\psi$ is odd and $q_\psi$ is even. Plugging this ansatz into $Xu = -f_1 - X_\perp f_2$, we obtain

$$Xu = X(Id + iH)p_\psi + X(Id + iH)q_\psi = -i[H, Xu]p_\psi - i[H, Xu]q_\psi = (Xp_\psi = Xq_\psi = 0)$$

$$= -i(X_\perp p_\psi)_0 - iX_\perp(p_\psi)_0 = -iX_\perp(p_\psi)_0 - iX_\perp(q_\psi)_0$$

$$= \frac{i}{2\pi}I_\perp^*p - \frac{i}{2\pi}X_\perp I_0^*q \quad \text{(using (7)).}$$

Therefore, a sufficient condition for $u$ to solve $Xu = -F$ is if $p, q$ satisfy

$$\frac{-i}{2\pi}I_\perp^*p = f_1, \quad \text{and} \quad \frac{i}{2\pi}I_0^*q = f_2,$$

which we may solve explicitly using the previous section. Using Proposition 2.6, we summarize this construction in the following result, whose proof is straightforward and omitted.

**Proposition 2.7.** Under the hypotheses of Proposition 2.6, and given $k \in \mathbb{N}$, $f_1 \in C^k(M)$ and $f_2 \in C^{k+1}(M)$, the function

$$u = 2\pi i[(Id + iH)(R_\perp f_1)_\psi - (Id + iH)(R_0 f_2)_\psi],$$

is a holomorphic solution of $Xu = -f_1 - X_\perp f_2$ on $SM$, satisfying the estimate

$$\|u\|_{C^k(SM)} \leq C(\|f_1\|_{C^k(M)} + \|f_2\|_{C^{k+1}(M)}),$$

where $R_0$ and $R_\perp$ are defined in (11).

3 Inversion of the attenuated ray transform over functions and vector fields à la Kasantzev-Bukhgeim

In [9], the authors provide reconstruction formulas for functions and vector fields from knowledge of their ray transforms in the case where the metric is Euclidean and the domain is the unit disk. The present section generalizes these ideas to the case of simple Riemannian surfaces.
3.1 Reconstruction of a function

In this first approach, we follow the idea in [9] that, if we can find a solution \( u^* \), holomorphic with \( u^*_0 = 0 \) of \( X u^* + a u^* = -f \), then projecting this equation onto \( \Omega_0 \) gives

\[
    f = -\eta_+ u^* - \eta_- u^*_1 - au^*_0 = -\eta_- u^*_1,
\]

after which one must explain how to express \( u^*_1 \) in terms of known data. Here and below, assuming that the operators \( Id + W^2 \) and \( Id + (W^*)^2 \) are invertible, we denote \( I_{0,\perp} \) the ray transform restricted to integrands of the form \( f_1 + X_\perp f_2 \), where \( f_1, f_2 \in C^\infty(M) \). If \( D := I_0 f_1 + I_\perp f_2 = I(f_1 + X_\perp f_2) \), we can reconstruct \( f_1, f_2 \) (and in turn, \( f_1 + X_\perp f_2 \)) from \( D \) by carrying out the following steps:

1. Compute the \( \mathcal{V}_+/\mathcal{V}_- \) decomposition of \( D \). This decomposition will be given by \( \frac{1}{2}(D \pm \alpha_1 D) \in \mathcal{V}_\pm \), where \( \alpha_1 \) is the involution associated with this decomposition.

2. Reconstruct \( f_1 \) from the projection of \( D \) onto \( \mathcal{V}_+ \) by inverting (9).

3. Reconstruct \( f_2 \) from the projection of \( D \) onto \( \mathcal{V}_- \) by inverting (10).

The above procedure we summarize below by writing \( I_{0,\perp}^{-1}D = f_1 + X_\perp f_2 \).

**Theorem 3.1.** Let \((M,g)\) a simple Riemannian surface with boundary and \( f, a \in C^\infty(M, \mathbb{R}) \). Then \( f \) is uniquely determined by its attenuated geodesic transform via the reconstruction formula

\[
    f = 2i\eta_- ((\text{Im}(e^w h'_\psi)))_1, \quad h' := \frac{1}{2}(D - w')|_{\partial a SM}, \quad (13)
\]

where \( w \) is an odd holomorphic solution of \( Xw = -a \), \( D = ((Id - iH)(e^{-w}Iaf)_- \), and \( w' \) is a holomorphic solution of \( Xw' = -I_{0,\perp}^{-1}(A^* D) \), given by Proposition 2.7.

**Proof.**

**Step 1:** find \( u^* \) a holomorphic solution of \( Xu^* + au^* = -f \). Call \( u \) the solution to \( Xu + au = -f \), \( u|_{\partial SM} = 0 \) so that \( u|_{\partial a SM} = Iaf \). Let \( w \) a holomorphic, odd solution of \( Xw = -a \). Then \( v = e^{-w}u \) solves the transport problem

\[
    Xv = e^{-w}(Xu - (Xu)u) = e^{-w}(Xu + au) = -e^{-w}f, \quad v|_{\partial a SM} = e^{-w}Iaf, \quad v|_{\partial SM} = 0, \quad (14)
\]

The right-hand side \( b := e^{-w}f \) is holomorphic. This is because, since the product of holomorphic functions is holomorphic, so are (convergent) powers series of holomorphic functions. Moreover, plugging the expansion \( w = w_1 + w_3 + \ldots \) \((w_j \in \Omega_j)\) into the exponential, we see that the Fourier series in \( \theta \) of \( b(x, \theta) \) looks like

\[
    b = f(1 + w + \frac{w_2}{2} + \ldots) = \sum_{b_0} f_0 + \sum_{b_1} f_1 w_1 + \sum_{b_2} \frac{f_2}{2} w_1^2 + \ldots,
\]

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so $b_0 = f$. We now want to decompose $v$ into $v = v^* + h^*_\psi$, where $v^*$ is holomorphic and $h^*_\psi$ is constant along geodesics, then we will have that $v^*$ solves $Xv^* = -b$. We proceed as follows. First decompose $v = \frac{1}{2}(v^+ + v^-)$, where $v^\pm = (Id \pm iH)v$. The function $v^-$ solves the transport equation

\[
Xv^- = X(v - iHv) = (Id - iH)Xv + i[H, X]v = -(Id - iH)b + i(X_\perp v)_0 + iX_\perp v_0 \\
= -(f - i(X_\perp v)_0) + iX_\perp v_0.
\]

Integrating along geodesics, we deduce that

\[
I_0(f - i(X_\perp v)_0) - iX_\perp v_0 = A^*_-v^-(\partial_{\partial SM}),
\]

where the right-hand-side $A^*_-v^-(\partial_{\partial SM}) = A^*_-D$, where $D$ has the expression in the statement of the theorem and is known from data $I_0f$. Therefore the right-hand side $(f - i(X_\perp v)_0) - iX_\perp v_0$ can be reconstructed upon inverting $I_0$ and $I_\perp$, a relation which we denote

\[
(f - i(X_\perp v)_0) - iX_\perp v_0 = I^{-1}_{0, \perp}A^*_-D.
\]

Let $w'$ a second holomorphic function such that $Xw' = -(f - i(X_\perp v)_0) + iX_\perp v_0$, constructed following Proposition 2.7, that is, $w' = 2\pi i(Id + iH)(p_\psi + q_\psi)$ with $p, q$ solving

\[
I^*_p = (f - i(X_\perp v)_0) \quad \text{and} \quad I^*_q = iv_0.
\]

With $w'$ thus constructed, we have $X(v^- - w') = 0$, i.e. $(v^- - w')$ is constant along geodesics. Upon rewriting $v$ as $v = \frac{1}{2}(v^+ + w') + \frac{1}{2}(v^- - w')$, we see that the first term is holomorphic and the second is constant along geodesics. In other words, $v$ is of the form $v = v^* + h^*_\psi$, where $v^* = \frac{1}{2}(v^+ + w')$ and $h^*_\psi = \frac{1}{2}(v^- - w')$. Additionally, with this choice of $w'$, we have

\[
v_0^* = \frac{1}{2}(v_0^+ + w_0') = \frac{1}{2}(v_0 + 2\pi i(q_\psi)_0) = \frac{1}{2}(v_0 + iI_0^*q) = 0.
\]

Finally, defining $u^* = v^*e^w$, we see that $u^*$ is holomorphic as the product of holomorphic functions, and, using the last equation, we see that $u_0^* = (v^*e^w)_0 = v_0^* = 0$, and that it solves

\[
Xu^* + au^* = -be^w = -f.
\]

Projecting the equation above into $\Omega_0$, we obtain

\[
-f = \eta_+u^*_{1-} + \eta_-u^*_1 + au^*_0 \quad \Rightarrow \quad f = -\eta_-u^*_1,
\]

so it remains to show how to compute $u^*_1$ in terms of known data.
Step 2: express $u_1^*$ in terms of known data. We now write

$$u_1^* = u_1^* - \overline{u_{-1}^*} = u_1^* - (\overline{u^*})_1 = 2i(\mathbb{I}m(u^*))_1 = 2i(\mathbb{I}m(u^* - u))_1,$$

where the first equality comes from the fact that $u_{-1}^* = 0$ and the last comes from the fact that $u$ is real valued. We now write, by definition,

$$u^* - u = (v^* - v)e^w = -e^w h'_\psi,$$

so that $\mathbb{I}m(u^* - u) = -\mathbb{I}m(e^w h'_\psi)$, and we arrive at the expression

$$u_1^* = -2i(\mathbb{I}m(e^w h'_\psi))_1.$$

Now looking to compute $h'$, since we proved that $h'_\psi = \frac{1}{2}(v(\cdot) - w')$, then

$$h' = \frac{1}{2}(v(\cdot) - w')|_{\partial_{+} SM}$$

$$= \frac{1}{2}v(\cdot)|_{\partial_{+} SM} - \frac{1}{2}(2\pi i)[(Id + iH)(p_\psi + q_\psi)]|_{\partial_{+} SM},$$

where $I_{+}^w p = g, I_{0}^w q = iv_0, I_0 g - iI_1 v_0 = A_\perp^w (v(\cdot)|_{\partial_{SM}})$.  

The proof is complete. \[\square\]

Remark 3.2. This proof generalizes the one completed in [9] in the Euclidean case when the domain is a disk. In order to complete the argument there, it is required to relate the fiberwise Hilbert transform with the Hilbert transform of the domain for the so-called divergent beam transform (see [9, Lemma 4.1]), which in turn uses the singular value decomposition (SVD) of that operator, established in earlier references (see [9] for detail and references there).

While this SVD, specific to the choice of metric and domain, does not seem straightforward to generalize systematically, the present construction of holomorphic solutions allows here to simplify the proof by bypassing these steps altogether.

In isothermal coordinates, Equation (13) takes the following form:

$$f = -\eta_+ u_1^* = -e^{-2\lambda} \bar{\partial} \bigg( \overline{u_1^*} e^\lambda \bigg), \quad \overline{u_1^*}(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \mathbb{I}m(e^w(x,\theta) h'_\psi(x,\theta)) \, d\theta. \quad (16)$$

Constructing both holomorphic solutions $w, w'$ using Proposition 2.7, we arrive at the following reconstruction procedure:

1. Construct $w = 2\pi i (I + iH)(R_\perp a)_\psi$, odd holomorphic solution of $Xw = -a$.
2. Compute $v|_{\partial_{+} SM} = e^w I_w f$ and $v|_{\partial_{-} SM} = 0$, then $v(\cdot)|_{\partial_{SM}} = \frac{1}{2}(Id - iH)(v|_{\partial_{SM}})$. 

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3. Reconstruct \( g \) and \( v_0 \) from \( I_0 g - i I_\perp v_0 = A^* (v^-|_{\partial SM}) \). (inversion of \( I_0 \) and \( I_\perp \))

4. Construct \( p \) and \( q \) from \( i I_\perp^* p = g \) and \( I_0^* q = iv_0 \). (inversion of \( I_0^* \) and \( I_\perp^* \), see Sec. 2.4)

5. Construct \( h' \) from (15).

6. Reconstruct \( \widetilde{u}_{1}^* \) then \( f \) according to (16).

### 3.2 Reconstruction of vector fields

As in [9], the method of proof above can easily be generalized to the case of reconstruction of vector fields, which we now present. Such a problem has applications to Doppler tomography in media with variable index of refraction, previously studied in [8, 9]. Integrands of the form \( f(x, \theta) = f_0(x) + f_1(x) \cos(\theta - \alpha) \) are also considered in [2] in the Euclidean setting. For convenience, we write the unknown vector field in holomorphic coordinates, in which \( \ker(\partial_\theta - ikId) \) is spanned by \( e^{ik\theta} \) for any \( k \in \mathbb{Z} \).

**Theorem 3.3.** Let \((M, g)\) a simple Riemannian surface with boundary and \( f_1, f_2, a \in C^\infty(M, \mathbb{R}) \).

Then at every \( x \in M \) where \( a(x) \neq 0 \), the vector field \( V(x, \theta) = f_1(x) \cos \theta + f_2(x) \sin \theta \) can be uniquely reconstructed from data \( I_a V \) via the formula

\[
(f_1(x) - if_2(x))e^{i\theta} = -2i\eta \left( \frac{1}{a} \eta - \text{Im}(e^{i\theta} h'_{\psi}) \right), \quad h' := \frac{1}{2}(D - w')|_{\partial_{\psi}, SM},
\]

where \( w \) is an odd holomorphic solution of \( Xw = -a \), \( D = ((1d - iH)(e^{-w} I_a V))_{\partial SM} \), and \( w' \) is a holomorphic solution of \( Xw' = -I_{0, \perp}^{-1}(A^* D) \).

**Proof.** Start from the equation

\[
X u + au = -f_1(x) \cos \theta - f_2(x) \sin \theta = -(F(x) e^{i\theta} + \overline{F}(x) e^{-i\theta}), \quad u|_{\partial, SM} = I_a V,
\]

where we have defined \( F(x) = f_1(x) + if_2(x) \).

**Step 1:** find \( u^* \) a holomorphic solution of \( Xu^* + au^* = -f \). Let \( w = w_1 + w_2 + \ldots \) be an odd, holomorphic integrating factor satisfying \( Xw = -a \) and define \( v = e^{-w} u \). Then \( v \) satisfies the transport problem

\[
Xv = -e^{-w}(F(x) e^{i\theta} + \overline{F}(x) e^{-i\theta}) = -G(x, \theta) = -(G_{-1} + G_0 + G_1 + G_2 \ldots),
\]

where \( G_{-1}(x, \theta) = \overline{F}(x) e^{-i\theta} \) and \( G_0(x) = \overline{w}_1(x) \overline{F}(x) \). Split \( v \) into an holomorphic and anti-holomorphic part, i.e. look at \( v = \frac{1}{2}(v^+ + v^-) \), where \( v^\pm = ((Id \pm H) v) \). \( v^- \) satisfies the transport equation

\[
Xv^(-) = X(v - iHv) = (Id - iH)Xv + i[H, X]v
= -(Id - iH)G + I(x_\perp v)_0 + iX_\perp v_0
= -(G_0 - i(X_\perp v)_0 + iX_\perp v_0 - G_{-1}.
\]
The data \( D \) defined is the statement is \( D = v(-)|_{\partial SM} \). Using the Hodge decomposition, we now write \( G_{-1} = Xg + X_{\perp}h \) for \( g, h \) two functions on \( M \), where \( g \) fulfills the additional prescription \( g|_{\partial M} = 0 \). Then the previous transport equation becomes

\[
X(v^{(-)} + g) = -(G_0 - i(X_{\perp}v)_0) + X_{\perp}(iv_0 - h).
\]

Note that \((v^{(-)} + g)|_{\partial_{+}SM} = v^{(-)}|_{\partial_{+}SM} = (Id - iH)(e^{-w}I_a f)|_{\partial_{+}SM} \) is known, so that upon integrating the transport equation along each geodesic we can see that the data gives us

\[
I_0(G_0 - i(X_{\perp}v)_0) - I_\perp(i v_0 - h) = A^* D,
\]

with \( D \) defined as in the statement of the theorem. Now construct \( w' \) a holomorphic solution of

\[
Xw' = -(G_0 - i(X_{\perp}v)_0) + X_{\perp}(iw_0 - h) = -I_{0,1}[A^* D],
\]

so that \( X(v^{(-)} + g - w') = 0 \), i.e. there exists \( h \) defined on \( \partial_{+}SM \) such that \( v^{(-)} + g - w' = h_\psi \).

We now decompose \( v = \frac{1}{2}(v^{(+) + w'} - g) + \frac{1}{2}(v^{(-) - w'} + g) = v^* + h'_\psi \), where the first term is holomorphic and the second is constant along geodesics. In particular, we get that \( Xv^* = -G(x, \theta) \) where, now, \( v^* \) is holomorphic. Then defining \( u^* = e^w v^* \), we find that \( u^* \) is holomorphic and satisfies

\[
Xu^* + au^* = -(F e^{i\theta} + \overline{F} e^{-i\theta}).
\]

Looking at the projections on \( \Omega_{-1} \) and \( \Omega_0 \) yields the equations

\[
\eta_{-} u_0^* = -\overline{F}(x) e^{-i\theta}, \quad \eta_{-} u_1^* + au_0^* = 0,
\]

which implies the reconstruction formula, at each point where \( a \) does not vanish:

\[
\overline{F}(x) e^{-i\theta} = \eta_{-} \left( \frac{1}{a} \eta_{-} u_1^* \right).
\]

**Step 2: obtain \( u_1^* \) from the measurements.** This part is, again, similar to the proof of Theorem 3.1. We write

\[
u_1^* = u_1^* - \overline{u_1^*} = u_1^* - \overline{(u^*)}_1 = 2i(\text{Im}(u^*))_1 = 2i(\text{Im}(u^* - u))_1,
\]

where the first equality comes from the fact that \( u_1^* = 0 \) and the last comes from the fact that \( u \) is real valued. Next we have the relation

\[
u^* - u = (v^* - v) e^w = -e^w h'_\psi,
\]

so it remains to compute \( h'_\psi \), which after unrolling definitions,

\[
h' = h'_\psi|_{\partial_{+}SM} = \frac{1}{2}(v^{(-)} - w' + g)|_{\partial_{+}SM} = \frac{1}{2}(v^{(-)} - w')|_{\partial_{+}SM},
\]

where both terms are, again, computable from data: \( v^{(-)}|_{\partial_{+}SM} = D|_{\partial_{+}SM} \) and \( w' \) is a holomorphic solution of \( Xw' = -I_{0,1}[A^* D] \), a solution of which can be explicitly constructed following Proposition 2.7. This ends the proof. \( \square \)
4 A second approximate formula for functions, conditionally invertible via Neumann series

While Theorem 3.1 reconstructs functions exactly and, in some sense, in a “one-shot” fashion, it presents a couple of weaknesses: (i) it involves all values of \( w(x, \theta) \) throughout \( SM \), which in turn involves storing three dimensions of data when everything should be dealt with using two-dimensional structures, and (ii) the effects of curvature (which need iterative corrections as in [13]) are not being corrected in the right place.

We now propose an algorithm based on a more direct interplay of the Hilbert transform with transport equations as in [14, 22], which leads to a Neumann-series based inversion, faster in implementation, and valid when curvature is close enough to constant and the attenuation is small enough in \( C^2 \) norm. Such an algorithm is then implemented in Section 5.

We first state a result about a certain family of operators generalizing the operator \( Wf = (X_u f)_0 \) first defined in [22]. These operators appear as error operators of the next reconstruction formula. We relegate the proof and some remarks about these operators to the appendix.

**Proposition 4.1.** Let \( h \in L^\infty(SM) \) such that \( Vh \in C^1(SM) \). Then the operator \( K_h : L^2(M) \to L^2(M) \) defined by \( K_h f := (X_u f h)_0 \) is well-defined and continuous, and there exists a constant \( C \) such that

\[
\|K_h\|_{L^2 \to L^2} \leq C(\|\nabla \kappa\|_\infty \|h\|_\infty + \|Vh\|_{C^1}). \tag{18}
\]

We now state the main result of the section.

**Theorem 4.2 (Iterative inversion).** If \( f \in L^2(M) \) and \( I_a f \) denotes its attenuated GXRT with given attenuation \( a \in C^2(M) \), then the function \( f \) satisfies the following equation

\[
f + Kf = \frac{1}{2\pi} I_0^a \eta, \quad \eta = \frac{1}{4} A^a_+ H(e^{-w} I_a f)_-, \tag{19}
\]

where \((e^{-w} I_a f)_-\) denotes extension from \( \partial_+ SM \) to \( \partial SM \) by oddness w.r.t. \( \theta \mapsto \theta + \pi \), \( w \) is an odd, holomorphic function solving \( Xw = -a \) and \( K : L^2(M) \to L^2(M) \) is a bounded operator satisfying the following estimate

\[
\|K\|_{L^2 \to L^2} \leq C \left( \|\nabla \kappa\|_\infty^2 + e\|a\|_\infty(\|a\|_\infty \|\nabla \kappa\|_\infty + \|a\|_{C^2}) \right). \tag{20}
\]

**Proof.** **Proof of (19).** Let \( w \) a holomorphic, odd solution to \( Xw = -a \). If \( u \) is the solution to \( Xu + au = -f \) with boundary condition \( v|_{\partial_+ SM} = 0 \), then the function \( v = e^{-w} u \) solves the transport problem \( Xv = -b := -f e^{-w} \) with boundary condition \( v|_{\partial_+ SM} = 0 \), so that \( v \) is no other than \( u e^{-w} \). Applying \( \pi_0 X = \pi_0 X_H \) (derived in (4)) to the equation \( Xv = -b \), we obtain

\[-f = -\pi_0 b = \pi_0 Xv = (X_u H v)_0 = (X_u H v_-)_0. \]

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The task now is to make \( Hv_- \) more explicit. Hitting the transport equation satisfied by \( v \) with the Hilbert transform \( H \), we write a transport equation for \( Hv \)

\[
X(Hv) = -Hb - X_\perp v_0 - (X_\perp v)_0.
\]

As \( b \) is holomorphic, we have \( Hb = fH(e^{-w}) = -if(e^{-w} - 1) \) and upon taking the even part of the last equation w.r.t. \( \theta \mapsto \theta + \pi \), the function \( Hv_- \) solves the transport equation

\[
X(Hv_-) = if(cosh w - 1) - (X_\perp v)_0,
\]

where we have used that \( w(x, \theta + \pi) = -w(x, \theta) \). This tells us that

\[
Hv_- = u^{-if(cosh w-1)} + u(X_\perp v)_0 + \eta_{\psi},
\]

where \( \eta_{\psi} \) is constant along geodesics. We can deduce \( \eta \) from the fact that the first two terms in the last r.h.s. vanish on \( \partial_\perp SM \), so that \( \eta = (Hv_-|_{\partial_\perp SM}) \circ \alpha \), and since \( v_- \) is known from data on \( \partial SM \), so is \( \eta \). More precisely, we have, for any \( (x, \theta) \in \partial_\perp SM \)

\[
v_-(x, \theta) = \frac{1}{2}(e^{-w}I_a f)_-(x, \theta) := \begin{cases} \frac{1}{2}e^{-w(x, \theta)}I_a f(x, \theta) & \text{if } (x, \theta) \in \partial_\perp SM, \\ \frac{1}{2}e^{-w(x, \theta+\pi)}I_a f(x, \theta + \pi) & \text{if } (x, \theta) \in \partial_\perp SM. \end{cases} \tag{21}
\]

Upon applying the operator \( \pi_0 X_\perp \), we obtain

\[
f = -(X_\perp Hv_-)_0 \\
= i(X_\perp uf^{(cosh w-1)})_0 - W(X_\perp v)_0 - (X_\perp \eta_{\psi})_0 \\
= i(X_\perp uf^{(cosh w-1)})_0 - W(X_\perp uf^{e^{-w}})_0 - (X_\perp \eta_{\psi})_0
\]

i.e. this equation takes the form

\[
f + Kf = -(X_\perp \eta_{\psi})_0, \quad \eta = (Hv_-)|_{\partial_\perp SM} \circ \alpha, \quad \text{where} \quad Kf := W(X_\perp uf^{e^{-w}})_0 - i(X_\perp uf^{(cosh w-1)})_0. \tag{22}
\]

Note that upon rewriting \( uf^{e^{-w}} = u^f + uf^{(e^{-w}-1)} \), the operator \( K \) can be rewritten as

\[
Kf = W(X_\perp uf^f)_0 + W(X_\perp uf^{(e^{-w}-1)})_0 - i(X_\perp uf^{(cosh w-1)})_0 \\
= W^2 f + W(X_\perp uf^{(e^{-w}-1)})_0 - i(X_\perp uf^{(cosh w-1)})_0.
\]

Equation (22) now corresponds to (19) upon noticing that \( I^*_\perp \eta = -2\pi (X_\perp \eta)_0 \), and that

\[
I_{\perp}^* \eta = I_\perp^* \left( \frac{1}{2}(H(e^{-w}I_a f)_-)|_{\partial_\perp SM} \circ \alpha \right) = I_\perp^* \left( \frac{1}{2} A^*_\perp - \frac{A^*_\perp}{2} (H(e^{-w}I_a f)_-) \right) = \frac{1}{4} I^*_{\perp} A^*_\perp H(e^{-w}I_a f)_-,
\]

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where the last equality comes from the fact that $A^*_\nu$ applied to an odd function on $\partial SM$ makes a function with $\nu_+$ symmetry, which in turn is annihilated by $I^*_\perp$.

**Proof of (20).** The theorem will be proved once we show that the operator $K$ satisfies the bound (20). From the last equation, we see that $K = W^2 + WK_1 - iK_2$, where $h_1 = e^{-w} - 1$ and $h_2 = \cosh w - 1$ and the notation $K_{h_j}$ refers to Proposition 4.1. It is proved in [10] that $\|W\|_{L^2 \to L^2} \leq C\|\nabla \kappa\|_{\infty}$ for some constant $C$. By virtue of Proposition 4.1, we deduce that for $j = 1, 2$

$$\|K_{h_j}\|_{L^2 \to L^2} \leq C(\|\nabla \kappa\|_{\infty} \|h_j\|_{\infty} + \|Vh_j\|_{C^1}).$$

Now we bound $\|h_j\|_{\infty} \leq \|w\|_{\infty} e^{\|w\|_{\infty}}$ and $\|Vh_j\|_{C^1} \leq C\|Vw\|_{C^1} e^{\|w\|_{\infty}} \leq C\|w\|_{C^2} e^{\|w\|_{\infty}}$ for $j = 1, 2$, and together with the fact that $w$ is constructed following Prop. 2.7, it satisfies estimates of the form

$$\|w\|_{C^k} \leq C\|a\|_{C^k}, \quad k = 0, 1, 2, \ldots$$

Combining all these estimates together, we obtain estimate (20).

**Remark 4.3.** In the absence of attenuation $a \equiv 0$, equation (19) is exactly the Fredholm equation (9), in which case $w(x, \theta) \equiv 0$ so that $Kf = W^2 f$ and the right-hand side of (22) is the post-processing of unattenuated data, since in the latter case, $I_0 f$ has $\nu_+$ symmetry so that $(I_0 f)^- \ $ coincides with $A^- I_0 f$.

**Corollary 4.4.** Under the hypotheses of Theorem 4.2, if $\|\nabla \kappa\|_{\infty}$ and $\|a\|_{C^2}$ are small enough so that estimate (20) implies $\|K\|_{L^2 \to L^2} < 1$, then $f$ can be reconstructed from equation (19) via the Neumann series

$$f = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-K)^n I^*_\perp \eta, \quad \eta = \frac{1}{4} A^*_\perp H(e^{-w} I_0 f)^-.$$

**Remark 4.5.** As in the unattenuated case, this restriction on $\|\nabla \kappa\|_{\infty}$ and $\|a\|_{C^2}$ is of rather qualitative nature and does not tell us whether all simple cases will work, and whether this approach would work for attenuations less than $C^2$. This is to be contrasted with successful numerical reconstructions below, which work for both discontinuous attenuations and cases of metrics arbitrarily close to non-simple.

## 5 Numerical implementation

We now present a brief implementation of an inversion of (19) via a Neumann series approach. We use the code developed by the author in [13] for the unattenuated case, augmented with
attenuation. The domain $M$ is the unit disk \( \{ x = (x, y), x^2 + y^2 \leq 1 \} \) endowed with the metric
\[
g(x) = e^{2\lambda(x)}Id,
\]
where
\[
\lambda(x) = 0.2 \left( \exp \left( \frac{(x - x_0)^2}{2\sigma^2} \right) - \exp \left( \frac{(x + x_0)^2}{2\sigma^2} \right) \right), \quad x_0 = (0.3, 0.3), \quad \sigma = 0.25,
\]
describing a region of “low sound speed” near $x_0$ and “high sound speed” near $-x_0$. The effect on geodesic curves can be seen Fig. 1. For such a domain, it can be computed that the boundary is strictly convex and there are no conjugate points.

The influx boundary $\partial_+ SM = S^1 \times (-\pi / 2, \pi / 2)$ is parameterized by $x(\beta) = \left( \frac{\cos \beta}{\sin \beta} \right)$ and ingoing speed direction $\theta(\beta, \alpha) = \beta + \pi + \alpha$ (in this case, $\beta + \pi$ is the direction of the unit inner normal). $M$ is represented into the unit square $[-1, 1]^2$, discretized in an equispaced cartesian fashion with $N = 300$ points along each dimension.

![Figure 1: Geodesics cast from the boundary point $x(\beta) = \left( \frac{\cos \beta}{\sin \beta} \right)$ with, from left to right, $\beta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. The colorbar on the right indicates the values of the background sound speed $c(x) = e^{-\lambda(x)}$.](image)

The function $f$ and the attenuation $a$ are displayed on Fig. 2. Note that both quantities contain jump singularities.

**Strategy for inversion.** Equation (19) is of the form
\[
f + Kf = L_a I_a f,
\]
where $I_a f$ represents forward data, $L_a D = \frac{1}{\pi^2} \int_\mathbb{R} A_\perp^* A_\perp H(e^{-\nu D}) -$ is an approximate inversion and we assume that $K$ is a contraction. Once $I_a$ and $L_a$ are discretized (call their discretized versions with the same name for simplicity), discretizing $K$ separately and computing a finite sum of $\sum_{k=0}^{\infty}(-K)^k L_a I_a f$ to reconstruct $f$ may introduce additional numerical errors due to the fact that (23) may not be satisfied at the numerical level. A better approach is to set directly $K = Id - L_a I_a$ and to implement a finite sum of the series
\[
f = \sum_{k=0}^{\infty} (Id - L_a I_a)^k L_a I_a f.
\]
Figure 2: From left to right: Unknown $f$, attenuation $a$, function $n$ defined on $\partial_+ SM$ such that $I^*_a n = a$ (so that $w = 2\pi i(\Id + i H)n_\psi$ is a holomorphic solution of $Xw = -a$). Axes for $n$ are $(\beta, \alpha) \in [0, 2\pi] \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

We now briefly explain how the operators $I_a$ and $L_a$ are implemented.

**Forward operator $I_a$.** The computation of the forward data is done by discretizing the influx boundary $\partial_+ SM$ into an equispaced family $(\beta_i, \alpha_j)_{1 \leq i \leq 2N, 1 \leq j \leq N}$ and for each data point, we compute $I_a f(\beta_i, \alpha_j)$ by discretizing the system of ODEs over $t \in [0, \tau_{ij}]$ where

\[
\dot{x} = e^{-\lambda(x)} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \dot{\theta} = e^{-\lambda(x)} (\partial_x \lambda(x) \sin \theta - \partial_y \lambda(x) \cos \theta), \quad \dot{u} = -a(x) u + f(x),
\]

with initial conditions $x(0) = x(\beta_i)$, $\theta(0) = \theta(\beta_i, \alpha_j)$ and $u(0) = 0$, and setting $I_a f(\beta_i, \alpha_j) = u(\tau_{ij})$.

**Computation of a holomorphic solution $w$ of $Xw = -a$.** Following Proposition 2.7, we look for $w$ in the form $w = 2\pi i(\Id + i H)n_\psi$, where $n$ solves $I^*_a n = a$. This requires implementing $n = R_+ a$, with $R_+ = \frac{1}{8\pi} A^* H_- A_- I_0(\Id + W^2)^{-1}$. Following [13], $(\Id + W^2)^{-1}$ is computed via a few iteration of a rapidly convergent Neumann series, $I_0$ is the unattenuated X-ray transform. In the present case, $n$ is represented on the right-hand plot of Figure 2. Once $n$ is computed, as the expression of $L_a$ only involves values of $w$ on $\partial SM$, then we can compute

\[
w |_{\partial SM} = (2\pi i(\Id + i H)n_\psi) |_{\partial SM} = 2\pi i(\Id + i H)(n_\psi) |_{\partial SM} = 2\pi i(\Id + i H)A_+ n,
\]

where the Hilbert transform is processed via Fast Fourier Transform on the columns of $A_+ n$. As $n$ has $V_-$ symmetry, $A_+ n$ amounts to the same thing as extending $n$ to $\partial_- SM$ by oddness w.r.t. $\theta \mapsto \theta + \pi$ (the latter is much more straightforward numerically).
Approximate inverse $L_a$. For $D$ a data function defined on a discretization of $\partial_+ SM$, we wish to compute $L_a D = \frac{1}{8\pi} I^*_\perp A^*_\perp H(e^{-w}D)_-$. The computation of $w$ and $H$ is explained in the previous paragraph. $A^*_\perp h(\beta, \alpha)$ is computed by combining values of $h(\beta, \alpha)$ and $h$ at the endpoint of the geodesic starting from coordinate $(\beta, \alpha)$. The main technical step if the computation of $I^*_\perp$, which in isothermal coordinates can be greatly simplified as follows (see [13, Sec. 3.1.2]):

$$I^*_\perp h(x) = e^{-2\lambda(x)} \nabla_x \cdot \left( e^{\lambda(\cdot)} \int_{S^1} \left( \frac{-\sin \theta}{\cos \theta} \right) h_{\psi}(\cdot, \theta) \, d\theta \right),$$

where $\nabla_x \cdot$ is just divergence on a cartesian grid. Integrals in $S^1$ can then be discretized using finite sums and the divergence is implemented using finite differences.

Both computations of $A^*_\perp$ and $I^*_\perp$ require computing several geodesic endpoints, which is the main bottleneck of the code. An alternative option, trading memory for much shorter CPU time, is to compute and store all endpoints required at first, and reusing them in further Neumann iterations.

---

Figure 3: Left: attenuated data $I_a f$ (data for Experiment 1) with $f, a$ from Fig. 2. Right: unattenuated data $I_0 f$ for comparison (note that the attenuation takes both positive and negative values so we do not necessarily have $|I_a f(\beta, \alpha)| \leq |I_0 f(\beta, \alpha)|$).

Experiments. We present two experiments, in which $a$ and $f$ refer to the functions displayed on Figure 2. $5a$ refers to the function $M \ni x \mapsto 5a(x)$.

- **Experiment 1.** (low attenuation) Neumann series based reconstruction of $f$ from $I_a f$.
- **Experiment 2.** (high attenuation) Neumann series based reconstruction of $f$ from $I_{5a} f$.

Experiment 1 successfully and stably reconstructs $f$ within 3 Neumann iterations, as shown in Fig. 4 (up to numerical inaccuracies, and given the fact that jumps can never be fully captured exactly). As this example works even if $a$ is discontinuous, this is to be contrasted with the regularity requirements on $a$ from Theorem 4.2.
Figure 4: From left to right: reconstruction $f_{rc}$ from $I_a f$ and pointwise error $|f_{rc} - f|$ after one iteration, $f_{rc}$ from $I_a f$ and pointwise error $|f_{rc} - f|$ after three iterations.

Experiment 2 displays a divergent Neumann series, due to the fact that attenuation $5a$ is too high. This is in agreement with the smallness requirements of Corollary 4.4 on the attenuation coefficient, as the operator norm of the error operator in (20) potentially grows like $e^{|a|\infty}$.

Figure 5: From left to right: $I_{5a} f$ (data for Experiment 2), pointwise error $|f_{rc} - f|$ after one iteration, pointwise error $|f_{rc} - f|$ after nine iterations. Divergence of the algorithm ensues due to the appearing unstable modes.

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A A certain family of operators - proof of Proposition 4.1

On simple surfaces, it is proved in [22] that the operator \( Wf = (X_\perp u^f)_0 \) is smoothing on simple surfaces, so that the equation reconstructing a function from its unattenuated ray transform is of Fredholm type.

It has been observed that if \( f \) is now a function on \( SM \), the corresponding operator no longer has such properties. However, the operators \( K_h \) introduced in Proposition 4.1 appear naturally as error operators in Theorem 4.2, and they generalize \( W \) since \( W = K_h \) with \( h \equiv 1 \). In general, \( K_h \) may no longer be smoothing, but we can still obtain \( L^2(M) \to L^2(M) \) continuity with estimates on the norm in terms of \( h \) and the ambient curvature. We recall the definition of the main non-trivial Jacobi fields, denoted by \( X_\perp \dot{\gamma} = a_{x,\theta}(t) \dot{\gamma}^\perp \) and \( \partial_{\theta} \dot{\gamma} = b_{x,\theta}(t) \dot{\gamma}^\perp \), where the scalar functions \( a \equiv a_{x,\theta}(t) \) and \( b \equiv b_{x,\theta}(t) \) are defined on \( D \) and solve the scalar Jacobi equations

\[
\ddot{a} + \kappa(\gamma_{x,\theta}(t))a = \ddot{b} + \kappa(\gamma_{x,\theta}(t))b = 0, \quad \begin{bmatrix} a & b \\ \dot{a} & \dot{b} \end{bmatrix}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

**Proof of Proposition 4.1.** For a smooth function \( \phi(x, \theta) \) on \( SM \) vanishing on \( \partial SM \), we first compute formally

\[
\int_{S^1} X_\perp \int_0^{\tau(x, \theta)} \phi(\gamma_{x,\theta}(t), \alpha_{x,\theta}(t)) \, dt \, d\theta = \int_{S^1} \int_0^{\tau(x, \theta)} (X_\perp \gamma \cdot \nabla_x \phi + (X_\perp \alpha)V\phi) \, dt \, d\theta
\]

\[= \int_{S^1} \int_0^{\tau(x, \theta)} \left( \frac{a}{b} \partial_{\theta} \gamma \cdot \nabla_x \phi + (X_\perp \alpha)V\phi \right) \, dt \, d\theta
\]

\[= \int_{S^1} \int_0^{\tau(x, \theta)} \left( \frac{a}{b} \partial_{\theta} \phi + (X_\perp \alpha - \frac{a}{b} \partial_{\theta} \alpha)V\phi \right) \, dt \, d\theta
\]

\[= \int_{S^1} \int_0^{\tau(x, \theta)} \left( -\partial_{\theta} \left( \frac{a}{b} \right) \phi + \frac{1}{b} V\phi \right) \, dt \, d\theta,
\]

where we used at the third step that \( X_\perp \alpha - \frac{a}{b} \partial_{\theta} \alpha = \frac{1}{b}, \) which follows from [14, Lemma 5.5] and the fact that the Wronskian \( \dot{a}b - \dot{b}a \equiv 1 \). Now replacing \( \phi(x, \theta) \) by \( f(x)h(x, \theta) \) and using the fact that \( V(fh) = fV(h) \), we arrive at

\[K_h(f)(x) = \frac{1}{2\pi} \int_{S^1} \int_0^{\tau(x, \theta)} \left( -\partial_{\theta} \left( \frac{a}{b} \right) h + \frac{1}{b} Vh \right) f \, dt \, d\theta = -K_{h,1}f + K_{h,2}f,
\]

upon expanding the sum. The operator \( K_{h,1} \) is just as well-behaved as the operator \( W \) and for the same reason: upon defining \( q(x, \theta, t) := \frac{1}{b_{x,\theta}(t)} \partial_{\theta} a_{x,\theta}(t) \), it is shown in [10] that \( |q(x, \theta, t)| \leq \ldots \)
\( C \| \nabla \kappa \|_\infty \) for every \((x, \theta, t) \in \mathcal{D}\). So we can rewrite

\[
K_{h,1}f(x) = \frac{1}{2\pi} \int_{S^1} \int_0^{\tau(x, \theta)} q(x, \theta, t) h(\gamma_{x, \theta}(t), \alpha_{x, \theta}(t)) f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{M} q(x, \theta(y), t(y)) h(y, \alpha_{x, \theta(y)}(t(y))) f(y) \, dM_y,
\]

so that the kernel \(k_{h,1}(x, y) = q(x, \theta(y), t(y)) h(y, \alpha_{x, \theta(y)}(t(y)))\) of \(K_{h,1}\) is bounded (hence in \(L^2(M \times M)\)), i.e. the operator \(K_{h,1} : L^2(M) \to L^2(M)\) is continuous (in fact, compact) with an operator norm less than \(C \| \nabla \kappa \|_\infty \| h \|_\infty\).

On to the study of the second term

\[
K_{h,2}f(x) = \frac{1}{2\pi} \int_{S^1} \int_0^{\tau(x, \theta)} \frac{q_2(x, \theta, t)}{b_{x, \theta}(t)} f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta, \quad q_2(x, \theta, t) := Vh(\gamma_{x, \theta}(t), \alpha_{x, \theta}(t)).
\]

The function \(q_2\) satisfies \(\int_{S^1} q_2(x, \theta, 0) \, d\theta = \int_{S^1} Vh(x, \theta) \, d\theta = 0\), so that the integral is expected to make sense as a principal value integral. Note that near \(t = 0\), we have \(b_{x, \theta}(t) = t(1 + tc_{x, \theta}(t))\) where \(c\) is smooth on \(\mathcal{D}\) and \((1 + tc_{x, \theta}(t))\) does not vanish on \(\mathcal{D}\) since \(b\) does not vanish outside \(\{t = 0\}\) by simplicity of the surface. More precisely, we write

\[
2\pi K_{h,2}f(x) = \int_{S^1} \int_0^{\tau(x, \theta)} \frac{q_2(x, \theta, t)}{b_{x, \theta}(t)} f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta = Af(x) + Bf(x) + Cf(x)
\]

where, upon writing \(b_{x, \theta}(t) = (1 + tc_{x, \theta}(t))\), we define

\[
Af(x) = \int_{S^1} \int_0^{\tau(x, \theta)} \frac{q_2(x, \theta, 0)}{t^2} f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta
\]

\[
Bf(x) = \int_{S^1} \int_0^{\tau(x, \theta)} \frac{q_2(x, \theta, t) - q_2(x, \theta, 0)}{t^2} f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta
\]

\[
Cf(x) = \int_{S^1} \int_0^{\tau(x, \theta)} \frac{q_2(x, \theta, t) (1 + 2tc_{x, \theta}(t))}{b_{x, \theta}(t)} f(\gamma_{x, \theta}(t)) \, b \, dt \, d\theta.
\]

Upon changing variable \((\theta, t) \mapsto \gamma(\theta, t) = \gamma_{x, \theta}(t)\) (with Jacobian \(dM_y = b \, dt \, d\theta\)), the \(C\) term becomes an operator with integrable kernel, i.e. \(L^2 \to L^2\) bounded, with operator norm controlled by \(\|q_2\|_\infty\), i.e. \(\|Vh\|_\infty\). On to the \(B\) term, we may write \(|q_2(x, \theta, t) - q_2(x, \theta, 0)| \leq Ct \|Vh\|_{C^1}\) uniformly on \(\mathcal{D}\), so that the kernel of \(B\) also has an integrable singularity and the \(B\) term also becomes an operator with integrable kernel, i.e. \(L^2 \to L^2\) bounded, with operator norm controlled by \(\|Vh\|_{C^1}\). On to the \(A\) term, we change variable \((\theta, t) \mapsto \gamma(\theta, t) = \gamma_{x, \theta}(t)\) to make appear

\[
Af(x) = \int_M \frac{q_2(x, \theta(y), 0)}{(d_g(x, y))^2} f(y) \, dM_y = \int_M \frac{q_2(x, \theta(y), 0)}{(d_g(x, y))^2} e^{2\lambda(y)} f(y) \, dy,
\]

24
where \( dy \) now represents the Lebesgue measure on \( \mathbb{R}^2 \). Expansions near \( x \) give that

\[
d_g(x, y) = e^{\lambda(y)}|x - y| + |x - y|^2 d_1(x, y),
\]
\[
\dot{\theta}(y) = \frac{x - y}{|x - y|} + |x - y|\dot{\theta}_1(x, y),
\]

this allows to rewrite \( A \) as a Calderón-Zygmund operator of the form

\[
Af(x) = \int_M \frac{q_2(x, \theta, 0)}{|x - y|^2} f(y) \, dy + \int_M \frac{q_3(x, y)}{|x - y|} f(y) \, dy,
\]

where \( q_3 \) is uniformly bounded by \( C\|Vh\|_{C^1} \). By virtue of [12, Theorem XI.3.1], the first term together with the zero mean value condition \( \int_{S^1} q_2(x, \theta, 0) \, d\theta = 0 \) is an operator \( L^2 \to L^2 \) continuous, with an operator norm bounded by \( C\sup_{x \in M} \|q_2(x, \cdot)\|_{L^2(S^1)} \), in turn bounded by \( C\|Vh\|_{\infty} \). The second term of (24) is another weakly singular operator whose operator norm can be bounded by \( C\|Vh\|_{C^1} \) as well.

\[\square\]

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