EXISTENCE THEOREMS FOR GENERALIZED NONLINEAR QUADRATIC INTEGRAL EQUATIONS VIA A NEW FIXED POINT RESULT

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Dedicated to Professor Patrizia Pucci for her 65th birthday anniversary

ABSTRACT. The existence of $L^2$-nonnegative solutions for nonlinear quadratic integral equations on a bounded closed interval is investigated. Two existence results for different classes of functions are shown. As a consequence an existence theorem for the Chandrasekhar integral quadratic equation, well-known in theory of radiative transfer, is obtained. The aim is achieved by means of a new fixed point theorem for multimaps in locally convex linear topological spaces.

1. Introduction. The aim of this paper is to prove the existence of nonnegative solutions for the following nonlinear quadratic integral equation

$$x(t) = u(t, x(t)) + g(t, x(t)) \int_0^{\phi(t)} v(t, \tau, x(\tau)) d\tau, \; t \in [0, T]$$

which generalizes many quadratic integral equations in the literature (see, e.g. [2, 3, 12, 15]).

The quadratic integral equations find various applications in several branches of mathematical physics, like in the radiative transfer, in the neutron transport, in the kinetic of gases, in the traffic and in the queuing theories (see, e.g [1, 4, 5, 6, 7, 14] and references therein). The interest on these equations began around fifties when the astrophysicist Chandrasekhar proposed in his studies the following

$$x(t) = a + x(t) \int_0^T \frac{t}{t+\tau} w(\tau)x(\tau)d\tau, \; t \in [0, T]$$

currently known as the Chandrasekhar integral equation (see, e.g. [4, 5, 10, 14]).

More recently, the existence of continuous solutions for several classes of nonlinear quadratic integral equations has been studied. For example, Liu and Kang in [15] provided existence results for the following nonlinear quadratic integral equation of Volterra type

$$x(t) = a(t) + b(x(t)) \int_0^t v(t, \tau, x(\tau)) d\tau, \; t \in [0, T],$$

2010 Mathematics Subject Classification. Primary: 45G10, 47H10; Secondary: 47N20, 45B05, 45D05.

Key words and phrases. Quadratic integral equations, Chandrasekhar integral equations, fixed point theorems.
studied also in [2] by Banaś and Martinon in the case \( b(x) = x \).

Our nonlinear quadratic integral equation (1) is a generalization of (3) because of the presence of \( u(t,x) \) and \( g(t,x) \) instead of \( a(t) \) and \( b(x) \). Moreover, the insertion of the function \( \varphi \) makes it possible to examine situations that are also different from that of Volterra, in particular that of Fredholm.

In our study we do not assume continuity hypotheses on the involved functions, leading us to consider a wide class of problems. In this context the solutions cannot be continuous, hence their existence is investigated in the space \( L^2 \). Because of the absence of hypotheses of continuity, our results extend in a broad sense the ones of the above mentioned papers.

As usual in large part of the literature, we read the solutions of our equation as fixed points of a suitable operator. In our setting, the fixed point theorems that are of our knowledge can not be used. For this reason we preliminarly state and prove a new fixed point theorem for multifunctions in Hausdorff locally convex topological vector spaces.

We underline that our fixed point result has an importance himself, since it improves recent fixed point theorems given for single-value functions in less general spaces (cf. [13] and [16]). In view of further possible applications, we have also provided a second fixed point theorem that extends but does not improve the former. Finally, we remark that, unlike most of the results known in the literature, both our fixed point theorems do not require linear or topological properties on the values of the multifunctions involved. This study is collected in Section 2.

In Section 3 we present two theorems on the existence of nonnegative solutions for (1), namely Theorems 3.2 and 3.3, which deal with different classes of problems. Finally, as an application of Theorem 3.3, we prove the existence of nonnegative solutions for the following nonlinear Chandrasekhar-type integral equation

\[
x(t) = a(t) + g(t, x(t)) \int_0^{\varphi(t)} \frac{p(t, \tau, x(\tau))}{t + \tau} d\tau, \quad t \in [0, T].
\]

2. On a fixed point theorem. This Section is devoted to the study of the existence of fixed points for multimaps in Hausdorff locally convex topological vector spaces.

Our first result is a fixed point theorem we will use in Section 3 for proving the existence of solutions to equations (1) and (4).

Let \( T : S \to 2^S \), be a multimap taking nonempty values on a subset \( S \) of a Hausdorff locally convex topological vector space \( X \).

We will suppose that the multimap \( T \) satisfies the following property:

\[ (P) \quad \{x\} \cap \mathcal{T}x \subset Tx, \text{ for every } x \in S \]

where, denoted \( \mathcal{V}(0) \) the family of all the convex and balanced neighborhoods of 0, the multimap \( \mathcal{T} : S \to 2^S \) is defined by

\[
\mathcal{T}(x) = \bigcap_{V \in \mathcal{V}(0)} \text{co} T((x + V) \cap S), \quad x \in S.
\]

We wish to underline that property (P) does not mean that \( T \) has a fixed point. Indeed, if we consider the map \( \mathcal{T} : \mathbb{R} \to \mathbb{R} \) defined by \( \mathcal{T}(x) = x + 1 \), the corresponding map \( T : \mathbb{R} \to \mathbb{R} \) is \( T(x) = \mathcal{T}(x) \) for every \( x \in \mathbb{R} \), so (P) is satisfied since \( \{x\} \cap \mathcal{T}(x) = \emptyset \) for every \( x \in \mathbb{R} \), but \( T \) does not have any fixed point.
Theorem 2.1. Let $S$ be a nonempty closed convex subset of a Hausdorff locally convex topological vector space $X$ and $T : S \to 2^S$ a multimap satisfying (P) and such that the set $T(S)$ is relatively compact.

Then $T$ has at least one fixed point.

Proof. Let us note that, since $T(x) \subset Tx$ for every $x \in S$, the multimap $\mathbb{T} : S \to 2^S$ defined as in (5) assumes nonempty values, obviously closed and convex too.

Now we prove that $T$ has closed graph. To this end, we fix a net $(x_\delta)_{\delta \in \Delta}$ converging to $\bar{x}$ in $X$ and a net $(y_\delta)_{\delta \in \Delta}$, $y_\delta \in \mathbb{T}(x_\delta)$, converging to $\bar{y}$ in $X$. First we underline that

$$y_\delta \in \overline{co} T((x_\delta + V) \cap S), \quad V \in \mathcal{V}(0), \quad \delta \in \Delta.$$  \hspace{1cm} (6)

Next, put a neighborhood $V \in \mathcal{V}(0)$, we consider $W \in \mathcal{V}(0)$ such that $W + W \subset V$. \hspace{1cm} (7)

We can say that there exists $\delta(W) \in \Delta$ such that, for every $\delta \in \Delta$ with $\delta \succcurlyeq \delta(W)$, we have

$$x_\delta \in \bar{x} + W,$$ \hspace{1cm} (8)

and, by using (6), we can also say that

$$y_\delta \in \overline{co} T((\bar{x} + W + W) \cap S).$$ \hspace{1cm} (9)

Therefore, for every $\delta \succcurlyeq \delta(W)$, taking into account of (9), (8) and (7) we can write

$$y_\delta \in \overline{co} T((\bar{x} + V) \cap S) \subset \overline{co} T((\bar{x} + V) \cap S),$$

and so we have $\bar{y} \in \overline{co} T((\bar{x} + V) \cap S)$. The arbitrariness of $V \in \mathcal{V}(0)$ allows to conclude that $\bar{y} \in \mathbb{T}(\bar{x})$. Therefore, the multimap $T$ has closed graph.

Moreover, $\overline{co} \mathbb{T}(S)$ is a compact subset of $S$ and the following inclusion hold

$$\mathbb{T}(S) = \bigcup_{x \in S} \bigcap_{V \in \mathcal{V}(0)} \overline{co} T((x + V) \cap S) \subset \overline{co} \mathbb{T}(S)$$

so the set $\mathbb{T}(S)$ is relatively compact. Therefore, the multimap $\mathbb{T}$ satisfies all the hypotheses of [9, Theorem III] and then there exists a fixed point $\hat{x} \in S$ for $\mathbb{T}$.

Finally, property (P) allows us to conclude that $\hat{x} \in \mathbb{T}(\hat{x})$. \hfill $\square$

Remark 1. We wish to note that, in the particular case when $X$ is a Banach space and $T$ is a single-valued map, our Theorem 2.1 comes down to Theorem 2.7 in [13].

Further, if $T$ is again a single-valued map and $X$ is just normed, Theorem 2.1 improves Theorem 3.1 in [16].

By means of the multimap $T$ defined in (5), we can provide another fixed point result, but in a Hausdorff locally convex topological vector space $X$ satisfying the Krein-Smulian property

: (X1) if $A$ is a compact subset of $X$, then $\overline{co} A$ is compact.

Remark 2. Notice that there exist Hausdorff locally convex topological vector space which does not satisfy this property (cf. [17, Ch.II, Exercise 27]).

We recall that a nonnegative real function $\gamma$ defined on the bounded subsets of $X$ is said to be a nonsingular measure of noncompactness on $X$ if the following properties are satisfied:

: $(\gamma_1)$ $\gamma(\Omega) = 0$ if and only if $\overline{\Omega}$ is compact;

: $(\gamma_2)$ $\gamma(\overline{co}(\Omega)) = \gamma(\Omega)$;

: $(\gamma_3)$ $\gamma(\{x\} \cup \Omega) = \gamma(\Omega)$, for every $x \in X$. 

Theorem 2.2. Let $S$ be a nonempty closed convex subset of a Haussorff locally convex linear topological space $X$ satisfying property $(X1)$, $\gamma$ a nonsingular measure of noncompactness on $X$ and $T : S \rightarrow 2^S$ a multimap satisfying $(P)$ such that

(i) $T(S)$ is bounded;
(ii) $\gamma(T(B)) < \gamma(B)$, for every $B$ bounded subset of $S$ such that $\gamma(B) > 0$;

where $T$ is defined in (5).

Then $T$ has at least one fixed point.

Proof. Using the same arguments as the proof of Theorem 2.1, we can either say that the multimap $T : S \rightarrow 2^S$ has nonempty closed convex values and that it has closed graph.

We show that $T(S)$ is bounded. Fixed $V \in \mathcal{V}(0)$, there exists $W$ such that $W \subset V$. (10)

By hypothesis (i), there exists $\rho > 0$ such that $T(S) \subset \rho W$. Hence the convexity of $W$ easily implies that $\operatorname{co}T((x + V) \cap S) \subset \rho W$, $x \in S$. (11)

Then by (11) and (10) we deduce that $T(x) \subset \rho V$, $x \in S$.

Then the boundedness of $T(S)$ is achieved.

Hence, bearing in mind also assumption (ii), we can say that the multimap $T$ satisfies all hypotheses of [8, Theorem 4.1] and so there exists a fixed point $\hat{x} \in S$ for $T$. Finally by using $(P)$ we can conclude that $\hat{x} \in T\hat{x}$. □

Remark 3. Let us note that Theorem 2.2 extends in a broad sense Theorem 2.1, but does not improve it (see Remark 2).

Moreover, unlike most of the results known in the literature, Theorems 2.1 and 2.2 do not require linear or topological properties on the values of the multifunctions involved.

3. Existence theorems. In this Section we provide our results on the existence of nonnegative solutions for equations (1) and (4).

From now on, we will put $I = [0, T]$; further, by $L^p(I)$ we will denote the space of all $\mathbb{R}$-valued functions on $I \subset \mathbb{R}$ such that their $p$-power is Lebesgue integrable with norm $\|v\|_p = \left[\int_I \|v(z)\|^p dz\right]^{\frac{1}{p}}$, $p = 1, 2$; moreover, we set $L^1_+(I) = \{f \in L^1(I) : f(t) \geq 0, \text{ for a.a. } t \in I\}$.

Finally, we will say that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is $L^2$-supmeasurable if $f(\cdot, x(\cdot))$ is measurable, for every $x \in L^2(I)$.

As a preliminary, we provide the following result.

Lemma 3.1. Let $u : I \times \mathbb{R} \rightarrow \mathbb{R}^+$, $g : I \times \mathbb{R} \rightarrow \mathbb{R}^+$, $\varphi : I \rightarrow I$, $v : I \times I \times \mathbb{R} \rightarrow \mathbb{R}^+$ be functions such that

(i) $u$ is $L^2$-supmeasurable;
(ii) there exists $a \in L^2_+(I) : u(t, x) \leq a(t), x \in \mathbb{R}, \text{ a.e. } t \in I$;
(g1) $g$ is $L^2$-supmeasurable;
(g2) there exists $m \in L^2_+(I) : g(t, x) \leq m(t), x \in \mathbb{R}, \text{ a.e. } t \in I$;

or

(g2)* there exists $\alpha > 0 : g(t, x) \leq \alpha(1 + |x|), x \in \mathbb{R}, \text{ a.e. } t \in I$;
(φ1) \( \varphi \) is nondecreasing;
(v1) for every \( t \in I \), the function \( v(t, \cdot , \cdot ) \) is \( L^2 \)-supmeasurable;
(v2) for every \( (\tau , x) \in I \times \mathbb{R} \), the function \( v(\cdot , \tau , x) \) is nondecreasing;
(v3) for every \( t \in I \), there exists \( h_t \in L^2_v(I) : v(t, \tau , x) \leq h_t(\tau) \), a.e. \( \tau \in I \), all \( x \in \mathbb{R} \).

Then, the map \( V : L^2(I) \to L^2(I) \)

\[
V(x)(t) = u(t, x(t)) + g(t, x(t)) \int_0^{\varphi(t)} v(t, \tau, x(\tau))d\tau, \quad x \in L^2(I), t \in I \quad (12)
\]

is well-defined.

Proof. First, by virtue of (v1) and (v3) we can say that, for every \( t \in I \) and \( x \in L^2(I) \), \( v(t, \cdot , x(\cdot )) \) is Lebesgue integrable. Moreover by using (v2) and (v3) we have

\[
\int_0^{\varphi(t)} v(t, \tau, x(\tau))d\tau \leq \int_0^T v(t, \tau, x(\tau))d\tau \leq \int_0^T h_T(\tau)d\tau \quad (13)
\]

hence by (g2) it follows that

\[
0 \leq g(t, x(t)) \int_0^{\varphi(t)} v(t, \tau, x(\tau))d\tau \leq m(t) \int_0^T h_T(\tau)d\tau, \text{ a.e. } t \in I. \quad (14)
\]

Now, taking into account also of (u1), (u2) and (φ1), we can say that the map \( V(x) \in L^2(I) \).

If we assume \((g2)^*\) instead of \((g2)\), fixed \( x \in L^2(I) \), by (13) we obtain the following estimate

\[
0 \leq g(t, x(t)) \int_0^{\varphi(t)} v(t, \tau, x(\tau))d\tau \leq \alpha(1 + |x(t)|) \int_0^T h_T(\tau)d\tau, \text{ a.e. } t \in I \quad (15)
\]

so also in this case (u1), (u2) and (φ1) imply that \( V(x) \in L^2(I) \).

Hence, in both cases we can conclude that \( V \) is well-posed. \( \square \)

RemarK 4. In the setting of Lemma 3.1, for every \( r > 0 \) the multimap \( \mathbb{V} : L^2(I) \to 2L^2(I) \)

\[
\mathbb{V}(x) = \bigcap_{U \in \mathbb{L}(0)} \mathbb{c}o \mathbb{V} \left( (x + U) \cap \overline{B}^+_L(0, r) \right), \quad x \in L^2(I) \quad (16)
\]

is well-defined, where \( \overline{B}^+_L(0, r) = \{ x \in L^2(I) : \| x \|_{L^2} \leq r, \ x(t) \geq 0 \text{ a.e. } t \in I \} \).

We can state and prove the following existence result.

Theorem 3.2. Let \( u : I \times \mathbb{R} \to \mathbb{R}^+, g : I \times \mathbb{R} \to \mathbb{R}^+, \varphi : I \to I, v : I \times I \times \mathbb{R} \to \mathbb{R}^+ \) be functions satisfying (u1), (u2), (g1), (φ1), (v1), (v2), (v3).

Assume: \((g2)\) and

: (H) there exists \( r_0 > 0 : \| a \|_{L^2} + A \| m \|_{L^2} \leq r_0; \)

or in alternative: \((g2)^*\) and

: \((H)^*\) there exists \( r_0 > 0 : \frac{\| a \|_{L^2} + A \sqrt{T}}{1 - A \sqrt{T}} \leq r_0, \)

where (cf. (v3)) \( A = \int_0^T h_T(\tau)d\tau \).

Further, we suppose that functions \( V \) and \( \mathbb{V} \), respectively by (12) and (16), have the following property

: \((V)\) \( \{ x \} \cap \mathbb{V}(x) \subset V(x), \text{ for all } x \in \overline{B}^+_L(0, r_0). \)
Then there exists at least one a nonnegative solution \( x \in L^2(I) \) for the generalized quadratic integral equation (1).

Proof. We notice that, thanks to Lemma 3.1 and Remark 4, the maps \( V \) and \( \mathcal{V} \) are well-defined.

Let us consider the restriction of \( V \) to the set \( B^+_{L^2}(0, r_0) \), where \( r_0 \) is from \( (H) \) or, in alternative, from \( (H)^* \). For the sake of simplicity, we will denote this restriction again as \( V \).

Fixed \( x \in B^+_{L^2}(0, r_0) \), we show that \( V(x) \in B^+_{L^2}(0, r_0) \). First of all, it is easily seen that \( V(x) \) is a nonnegative function. Moreover, the following inequality holds (see (12) and (u2)):

\[
\|V(x)\|_{L^2}^2 = \int_0^T \left[ u(t, x(t)) + g(t, x(t)) \int_0^t v(t, \tau, x(\tau)) d\tau \right]^2 dt \\
\leq \int_0^T a^2(t) dt + \int_0^T |g(t, x(t))|^2 \left[ \int_0^t v(t, \tau, x(\tau)) d\tau \right]^2 dt \\
+ 2 \int_0^T a(t) g(t, x(t)) \left[ \int_0^t v(t, \tau, x(\tau)) d\tau \right] dt.
\]

Hence, if we assume \((g2)\) and \( (H) \), by (14) and the Hölder inequality, we have

\[
\|V(x)\|_{L^2}^2 \leq \|a\|_{L^2}^2 + \int_0^T |g(t, x(t))|^2 A^2 dt + 2 \int_0^T a(t) g(t, x(t)) A dt \\
\leq \|a\|_{L^2}^2 + A^2 \|m\|_{L^2}^2 + 2A \|a\|_{L^2} \|m\|_{L^2} = (\|a\|_{L^2} + A \|m\|_{L^2})^2,
\]

then \( \|V(x)\|_{L^2} \leq r_0 \).

With analogous arguments, if we assume \((g2)^*\) and \((H)^*\), we obtain (cf. [11, Proposition 2.24])

\[
\|V(x)\|_{L^2}^2 \leq \|a\|_{L^2}^2 + A^2 \|m\|_{L^2}^2 + 2A \|a\|_{L^2} \|m\|_{L^2} = (\|a\|_{L^2} + A \|m\|_{L^2})^2
\]

then \( \|V(x)\|_{L^2} \leq r_0 \).

So in any case we have that \( V(x) \in B^+_{L^2}(0, r_0) \).

Now, we consider the restriction of \( V \) to \( B^+_{L^2}(0, r_0) \), that we denote \( \mathcal{V} \) as well.

Let us note that the multimap \( \mathcal{V} \) is defined on \( B^+_{L^2}(0, r_0) \), which is a weakly compact convex subset of the reflexive separable Banach space \( L^2(I) \). Moreover, the following inclusion holds

\[
\overline{\mathcal{V}(B^+_{L^2}(0, r_0))} \subset \overline{B^+_{L^2}(0, r_0)},
\]

hence the set \( \overline{\mathcal{V}(B^+_{L^2}(0, r_0))} \) is compact in the Hausdorff locally convex topological vector space \( (L^2(I), T_\omega) \). Since \( V \) satisfies (V1), we are in a position to apply our fixed point Theorem 2.1. Therefore there exists at least fixed point \( \bar{x} \) for the map
$V$, so $\bar{x}$ is the nonnegative solution for the generalized quadratic integral equation (1).

**Remark 5.** We underline that in the setting of Theorem 3.2, case of hypotheses $(g2)^*$ and $(H)^*$, we can consider the particular case when $u(t, x) = a(t)$, $g(t, x) = x$ and $\varphi(t) = t$, so that equation (1) becomes the following quadratic integral equation

$$x(t) = a(t) + x(t) \int_0^t v(t, \tau, x(\tau)) d\tau, \quad t \in I$$

studied in the literature under continuity hypotheses on the involved functions (see, e.g. [2]).

Note that the aforementioned function $g$ does not satisfy hypothesis $(g2)$, therefore equation (18) cannot be considered in the context of Theorem 3.2 with hypotheses $(g2)$ and $(H)$.

In the rest of this Section, our aim is to prove the existence of solutions for the Chandrasekhar equation. The first step is to provide a different version of Theorem 3.2, where we do not assume the monotonicity of $v$ with respect of the first variable (see $(v2)$) and strengthen hypothesis $(v3)$ by

$(v3)'$ there exists $q \in L^1_+(I)$ : $v(t, \tau, x) \leq q(\tau)$, all $t \in I$, a.e. $\tau \in I$, all $x \in \mathbb{R}$.

**Remark 6.** We observe that in this slightly modified setting the thesis of Lemma 3.1 is still valid.

Indeed, by virtue of $(v1)$ and $(v3)'$ we can say that, for every $t \in I$ and $x \in L^2(I)$, the map $v(t, \cdot, x(\cdot))$ is Lebesgue integrable. Moreover by using and $(v3)'$ we have

$$\int_0^T v(t, \tau, x(\tau)) d\tau \leq \int_0^T q(\tau) d\tau.$$

Now, by $(g2)$ or alternatively $(g2)^*$, the analogous inequalities of (14) and (15) hold. Therefore, we have that $V(x) \in L^2(I)$ and hence $V$ (see (12)) is well-defined.

Note that, as a consequence, $V$ (see (16)) is well-defined too.

As a consequence of Remark 6, we can state the following theorem. We omit the proof, as it traces that of Theorem 3.2.

**Theorem 3.3.** Let $u : I \times \mathbb{R} \to \mathbb{R}^+$, $g : I \times \mathbb{R} \to \mathbb{R}^+$, $\varphi : I \to I$, $v : I \times I \times \mathbb{R} \to \mathbb{R}^+$ be functions satisfying $(u1)$, $(u2)$, $(g1)$, $(\varphi1)$, $(v1)$ and $(v3)'$.

Assume: $(g2)$ and

$(H)$ there exists $r_0 > 0 : \|a\|_{L^2} + Qm\|_{L^2} \leq r_0$;

or in alternative: $(g2)^*$ and

$(H)^*$ there exists $r_0 > 0 : \|a\|_{L^2} + Q\alpha \sqrt{T} \leq r_0$,

where $Q = \int_0^T q(\tau) d\tau$.

Further, we suppose that functions $V$ and $\mathbb{V}$, respectively by (12) and (16), have the property $(V)$.

Then there exists at least one nonnegative solution $x \in L^2(I)$ for the generalized quadratic integral equation (1).

We are now able to provide the result on the existence of nonnegative solutions for the nonlinear Chandrasekhar-type integral equation (4).
Corollary 1. Let \( u : I \times \mathbb{R} \to \mathbb{R}^+ \), \( g : I \times \mathbb{R} \to \mathbb{R}^+ \), \( \varphi : I \to I \), \( p : I \times I \times \mathbb{R} \to \mathbb{R}^+ \) be functions satisfying (u1), (u2), (g1), (\( \varphi 1 \)) and

- (p1) for every \( t \in I \), the function \( p(t, \cdot, \cdot) \) is \( L^2 \)-supmeasurable,
- (p2) there exists \( k \in L^1(I) : p(t, \tau, x) \leq k(\tau)(t + \tau) \), all \( t \in I \), a.e \( \tau \in I \), all \( x \in \mathbb{R} \).

Assume: (g2) and

- (H) there exists \( r_0 > 0 : \|a\|_{L^2} + K\|m\|_{L^2} \leq r_0 \); or in alternative: (g2)* and

- (H)* there exists \( r_0 > 0 : \frac{\|a\|_{L^2} + K\alpha \sqrt{T}}{1 - K\alpha \sqrt{T}} \leq r_0 \),

where \( K = \int_0^T k(\tau) \, d\tau \).

Further, we suppose that functions \( V \) and \( V' \), respectively by (12) and (16), have the property \( (V) \).

Then there exists at least one a nonnegative solution \( x \in L^2(I) \) for the nonlinear Chandrasekhar-type integral equation (4).

Proof. The thesis follows from Theorem 3.3, where as function \( v : I \times I \times \mathbb{R} \to \mathbb{R}^+ \) we take the one defined by

\[
v(t, \tau, x) = \begin{cases} 
p(t, \tau, x) \frac{t + \tau}{t + \tau}, & t \in [0, T], \ \tau \in [0, T], \ x \in \mathbb{R} \\
p(0, \tau, x) \frac{t}{\tau}, & t = 0, \ \tau \in [0, T], \ x \in \mathbb{R} \\
0, & t = 0, \ \tau = 0, \ x \in \mathbb{R}.
\end{cases}
\]

Indeed, by (p1) and (p2) respectively, \( v \) satisfies both assumptions \( (v1) \) and \( (v3)' \) of Theorem 3.3. \( \square \)

Remark 7. From a formal point of view the classical Chandrasekhar equation (2) is a particular case of (4), just by taking \( a(t) = a \), \( g(t, x) = x \), \( \varphi(t) = T \) and \( p(t, \tau, x) = tu(\tau)x \).

Unfortunately, our Corollary 1 does not allow us to state the existence of nonnegative \( L^2 \)-solutions for (2). In fact, while the particular functions \( a, g \) and \( \varphi \) above defined satisfy the assumptions of Corollary 1, as regards the function \( p \), it does not satisfy hypothesis (p2). Therefore in our setting the existence of solutions for equation (2) remains an open problem.

Acknowledgments. The research is carried out within the national group GNAMPA of INdAM.

This work was supported by the project Fondi Ricerca di Base 2015 “Metodi topologici nello studio delle inclusioni differenziali”, Department of Mathematics and Computer Science, University of Perugia; the first author has been also supported by the INdAM-GNAMPA Project 2018 Equazioni differenziali, integrali ed integro-differenziali ordinarie nello studio dei fenomeni reali, whereas the second author by FFABR (Fondo per il Finanziamento delle Attività Base di Ricerca): Ricercatori - anno 2017.
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Received September 2018; revised October 2018.

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