Shadow Wave Tracking Procedure and Initial Data Problem for Pressureless Gas Model

Sanja Ružičić · Marko Nedeljkov

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Abstract In this paper the new procedure for a construction of an approximated solution to initial data problem for one-dimensional pressureless gas dynamics system is introduced. The procedure is based on solving the Riemann problems and tracking singular wave interactions. For that system the new problem with initial data containing Dirac delta function is solved whenever two waves interact. Use of the shadow waves as singular solutions to such problems enables us to easily solve the interaction problems. That permits us to make a simple extension of the well known Wave Front Tracking algorithm. A non-standard part of the new algorithm is dealing with delta functions as a part of a solution. In the final part of the paper we show that the approximated solution has a subsequence converging to a signed Radon measure.

Mathematics Subject Classification 35L65 · 35L67 · 35Q35

Keywords Shadow waves · Initial data problem · Wave front tracking · Pressureless gas

1 Introduction

In the last few decades, a lot of conservation law systems with non-classical, unbounded weak solutions were analyzed. One can find a lot of examples in the references at the end of the paper. Almost all these solutions contain the Dirac delta function that is not suitable for nonlinear operations. That is a source of big problems in solving some conservation law systems. There are several methods for dealing with that, and some of them can be found in the references below. Riemann problem is almost fully understood for these systems, so a natural next step is to look for a solution to a general initial data problem. Because of that
we will use shadow waves defined in [22]. Shadow wave solutions (SDW) are represented by nets of piecewise constant functions with respect to the time variable depending on a small parameter \( \varepsilon > 0 \) tending to zero. A shadow wave approximates a significant number of different types of singular solutions that differ from classical solutions by containing the Dirac delta function supported by a shock curve. Their use permits one to easily find a solution to the interaction problem and that will be of the greatest importance for the construction of a solution here. To demonstrate these ideas, we will use the well known pressureless gas dynamics system

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) &= 0
\end{align*}
\]

(1.1)

that describes an evolution of density \( \rho \geq 0 \) and velocity \( u \) of a fluid. The equations in (1.1) express conservation of mass and linear momentum in an absence of pressure. That means that changes in internal energy manifested through temperature or specific entropy are neglected. The above system is sometimes called the sticky particle model. That name comes from the fact that colliding particles fuse into a single particle that combines their masses and moves with a velocity that conserves the total linear momentum (see [3] or [8] for example). For example, it models one-dimensional isentropic flow in the Eulerian description of a thermoelastic fluid in a duct. System (1.1) is weakly hyperbolic with the double eigenvalue \( \lambda_i(\rho, u) = u, \ i = 1, 2 \) with both fields being linearly degenerate. It allows a mass concentration that leads to singular, unbounded solutions containing the Dirac delta function. The system attracts great attention in the literature. Riemann problems for the pressureless gas dynamics system with a source are analyzed in [10, 28], two–dimensional case can be found in [29], while the system with added energy conservation law is investigated in [22]. Besides it, there are a significant number of conservation laws admitting unbounded solutions. More about their origin and history one can find in [16, 17, 26]. Unbounded solutions for weakly hyperbolic systems like (1.1) were firstly found and they are called delta shocks. Some other interesting solutions called singular shocks appearing in some strictly hyperbolic systems ([18]), or in chromatography system that changes type ([19, 30]). It is known that a Riemann problem for (1.1) with the left and right initial states \((\rho_l, u_l)\) and \((\rho_r, u_r)\) has a self-similar, classical entropy solution that consists of two contact discontinuities connected with the vacuum state if \( u_l < u_r \), or a single contact discontinuity if \( u_l = u_r \). If \( u_l > u_r \), there exists a non-classical solution containing the delta function.

The authors in [13] constructed a global weak solution to the initial data problem for (1.1) by using generalized variational method. Almost at the same time, the existence of a weak solution to the same problem was proved in [3]. Uniqueness is proved in [15] for initial data belonging to the space of Radon measures by using methods from [13]. In [2], the author proved existence of a solution to classical initial data problem for (1.1) by using viscosity approximation. The solution is understood in the sense of duality that is defined in [1]. Global existence of a measure–theoretic solution where \( \rho \) belongs to Borel measures space and \( u \) is square integrable with respect to \( \rho \) was proved in [6] by using the theory of first-order differential inclusions in the space of monotone transport maps introduced in [20]. The authors in [21] were using the usual entropy solution to a scalar conservation law to obtain a global solution, while initial data could contain a Borel measure. Methods used in all the papers cited above are specific for the pressureless gas (sticky particles) model. Our idea is to use a procedure of shadow wave tracking because it can be adapted to some other system possessing unbounded solutions. Model (1.1) should be understood as a starting
point for using this method in a general case. The logical and straightforward generalization is $3 \times 3$ pressureless gas dynamic system

$$\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0 \\
\partial_t (\rho u) + \partial_x (\rho u^2) &= 0 \\
\partial_t (\rho u^2/2 + \rho e) + \partial_x ((\rho u^2/2 + \rho e)u) &= 0
\end{align*}$$

(1.2)
described in [22]. That system has the similar structure as (1.1), and we will note small changes in the procedure.

The main idea for the approximate solution construction procedure comes from the well known Wave Front Tracking (WFT) algorithm (see [4, 5, 14, 27]). The procedure starts with an approximation of initial data by piecewise constant functions and tracking the waves and monitoring their interactions later on. The shadow waves are approximations of delta shock solution and due to their construction one can use an algorithm similar to the WFT one. One of the main difficulties in the WFT algorithm was the fact that the number of wave fronts may approach infinity within the finite time for $n \times n$ systems, with $n > 2$. Here, we are dealing with $2 \times 2$ system in which this problem does not occur. In this particular case, a number of waves decreases after each interaction as one can see below. But, the resulting wave front here is not necessarily a straight line (i.e. the wave propagates with a non–constant speed), which is not a case with WFT algorithm for BV solutions. That is a consequence of the fact that a shadow wave interaction with some wave produces a new shadow wave with non-constant speed in general. So, we have to deal with the additional problem of analyzing such wave front curves.

As we already mentioned, the procedure for finding an approximate solution to the initial data problem presented in this paper can be used for general 1D conservation law systems. It is only required that they admit a unique solution to the corresponding Riemann problem consisting of elementary and shadow waves combinations. That is the first advantage of our solution construction compared to the methods previously discussed that depend on a particular form of conservation law system. There are some peculiarities in the pressureless gas model. The absence of rarefaction waves makes the procedure simpler. But, on the other side, the appearance of vacuum in the approximate solution was the main source of difficulties in the approximate solution construction. Also, that makes a limiting process harder to follow since there are no vacuum areas in a local smooth solution to the system. The ultimate step would be to generalize the procedure for (1.1) to obtain a general algorithm for solving a wide class of conservation law systems admitting unbounded solutions. Note that there is an example of shadow wave interactions that cannot be handled in the way used here, as proved in [23] for the model of Chaplygin gas. The second advantage of the procedure is that it can be adapted for a numerical implementation. For the system (1.1) the obtained ODE system (3.1) describing a behaviour of a shadow wave after an interaction can be easily integrated in the closed form. That is not a common situation so one would need a good approximation for the shadow wave front to obtain a good approximation for the next interaction point for almost any other system containing shadow wave solutions.

The first main result in this paper is the construction of a global admissible approximate solution to the initial data problem for (1.1). The initial data are bounded piecewise $C^1$ functions with a finite number of jumps. The second one is the existence of subsequence converging in the space of signed Radon measures. Moreover, there exists a subsequence converging to a measure that consists of classical solutions connected by delta function at least for a small time interval. In that time interval, the approximate solution can be explicitly describe using a kind of well–balanced partitions.
Note that the Lax entropy condition (a convex entropy–entropy flux pair) does not suffice to single out all non-physical solutions for (1.1) as proved in [13]. One has to use overcompressibility to extract a proper solution. It means that all characteristics run into a shock front (especially, \( u_l > u_r \) for system (1.1)). The admissibility condition for the system in the context of dual solutions, see [1], [2] and [3], is so-called the OSL condition, \( \partial_x u(x,t) \leq 1/t \). The procedure for construction of the approximate solution given in this paper satisfies that condition, too, since velocity component is interpolated by a shifts of the function \( 1/t \) in the area where it is nondecreasing.

Concerning other systems admitting singular solutions, there are some interesting facts about relations between these two admissibility conditions. As it was shown in [22] for (1.2) they are equivalent for all semi–convex entropies \( \eta \). But the overcompressibility condition can be weaker as shown in [24].

When dealing with isentropic gas dynamics systems, the authors often use the energy inequality, derived from energy conservation law as an additional criterion for admissibility check (see [12] for example). The energy density for pressureless gas is \( E = \frac{1}{2} \rho u^2 \). Here we present a simple analysis of energy propagation but we did not use it for choosing a proper solution.

The paper is organized as follows. Section 2 contains a statement of the problem as well as an overview of all waves which appear as a part of a solution to the Riemann problem. Section 3 is devoted to an analysis of shadow wave interactions. We describe all interactions between two or more waves which may occur at some time in Sect. 4. After that, a detailed presentation of procedure which provides a scheme for constructing the admissible approximate solution to the initial value problem is given. The procedure is based on the approximation of initial data and tracking interactions between the waves which are obtained as solutions to the Riemann problems. A relation between each pair of consecutive states obtained by the initial data approximation contains all information needed for the construction of a solution after each interaction point. Details depend on monotonicity of the initial functions \( u(x) \) and \( \rho(x) \). Section 5 contains proofs of admissible approximated solution existence to the initial data problem when the function \( u(x) \) is monotone. That result is then extended for \( u(x) \) having a finite number of extremes. In Sect. 6 we briefly discuss entropy changes across a shadow wave and after the interactions and we prove that the total entropy decreases after the interaction between two shadow waves. The remainder of the paper is devoted to proving that solution converges in the space of measures and that a limit is unique in some sense and at least for some time.

2 Riemann Problems

In the rest of this paper we will write \( a_\varepsilon \sim b_\varepsilon \) if there exists \( A > 0 \) such that \( \lim_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} = A \). The sign “\( \sim \)” will denote the distributional limit as \( \varepsilon \to 0, \varepsilon > 0 \). Landau symbols \( O(\cdot) \) and \( o(\cdot) \) will be used under the assumption \( \varepsilon \to 0 \) which will be often omitted after their use.

Suppose that \( \rho(x) > 0 \) and \( u(x) \) are in \( C^1_b([R, \infty)) \). Let \( \rho_0, u_0 \in \mathbb{R}, \rho_0 > 0 \). Here, \( C^1_b \) denotes a space of bounded functions with a bounded derivative. The initial data for (1.1) are

\[
(\rho, u)(x, 0) = \begin{cases} 
(\rho_0, u_0), & x \leq R \\
(\rho(x), u(x)), & x > R.
\end{cases}
\]  (2.1)

Let us make a net of piecewise constant approximations \( (\rho^\varepsilon(x), u^\varepsilon(x))_\varepsilon \) of the initial data \( (\rho(x), u(x)) \). Take a fixed \( \varepsilon > 0 \) and a corresponding partition \( \{Y_1\}_{t \in \mathbb{N}_0}, R := Y_0 < Y_1 < \)
The conditions for $Y_i + 1$ will be obtained in Sect. 5. The approximation is chosen such that $\rho^\varepsilon(x) = \rho(Y_{i+1}) =: \rho_{i+1}$ and $u^\varepsilon(x) = u(Y_{i+1}) =: u_{i+1}$ for $x \in (Y_i, Y_{i+1}]$, $i \in \mathbb{N}_0$, and $(\rho^\varepsilon(x), u^\varepsilon(x)) = (\rho_0, u_0)$ for $x \leq R$. Construction of a global solution is based on tracking wave fronts and analyzing interactions between waves. We need some preparations to do it.

**Remark 2.1** With a slight abuse of notation in the rest of the paper, we will use the same notation ($u$ and $\rho$) for the initial function (which only depends on space variable $x$) and for a solution (which depends on $x$ and $t$). A missing argument means that it equals $(x, t)$.

**Definition 2.1** (Shadow waves) A shadow wave is a piecewise constant function with respect to time of the form

$$U^\varepsilon(x,t) = \begin{cases} 
(\rho_l, u_l), & x < c(t) - a_\varepsilon(t) - x_{l,\varepsilon} \\
(\rho_{l,e}(t), u_{l,e}(t)), & c(t) - a_\varepsilon(t) - x_{l,\varepsilon} < x < c(t) \\
(\rho_{r,e}(t), u_{r,e}(t)), & c(t) < x < c(t) + b_\varepsilon(t) + x_{r,\varepsilon} \\
(\rho_r, u_r), & c(t) + b_\varepsilon(t) + x_{r,\varepsilon} < x, 
\end{cases} \quad (2.2)$$

where $a_\varepsilon(t)$, $b_\varepsilon(t)$, $x_{l,\varepsilon}$, $x_{r,\varepsilon} \sim \varepsilon$ for each $t$. The states $U_{*\varepsilon}(t) = (\rho_{*,\varepsilon}(t), u_{*,\varepsilon}(t))$, $* \in \{l, r\}$ are called intermediate states. The curves $x = c(t) - a_\varepsilon(t) - x_{l,\varepsilon}$ and $x = c(t) + b_\varepsilon(t) + x_{r,\varepsilon}$ are the external, while $x = c(t)$ is the central shadow wave line. The limit $\lim_{\varepsilon \to 0} (a_\varepsilon(t) + x_{l,e})U_{l,e}(t) + (b_\varepsilon(t) + x_{r,e})U_{r,e}(t)$ is the strength of shadow wave, while its speed is given by $c'(t)$. Shadow waves with constant speed and constant intermediate values are called the simple ones. Sometimes we use the prefix “weighted” for shadow waves with variable intermediate state. We say that (2.2) solves (1.1) in the approximated sense if its substitution into the right-hand side of the system gives terms converging to zero as $\varepsilon \to 0$.

Let us note that in the case of system (1.1) one can use that $U^\varepsilon(t) = U_{l,e}(t) = U_{r,e}(t)$ without loss of generality, and we shall do it. Also, note that all necessary calculations when (2.2) is substituted into (1.1) can be done by using the classical Rankine-Hugoniot conditions. In the sequel, we shall often skip the word “approximate” and use only the word “solution”.

Approximation of the initial data using the partition $\{Y_i\}_{i \in \mathbb{N}_0}$ generates an infinite number of Riemann problem for (1.1)

$$(\rho, u)(x, 0) = \begin{cases} 
(\rho_i, u_i), & x < Y_i \\
(\rho_{i+1}, u_{i+1}), & x > Y_i, \quad i = 0, 1, 2, \ldots 
\end{cases} \quad (2.3)$$

There are three kinds of solutions to (1.1). If $u_i = u_{i+1}$, a solution is a single contact discontinuity

$$U(x, t) := (\rho, u)(x, t) = \begin{cases} 
(\rho_i, u_i), & x - Y_i < u_{i}t \\
(\rho_{i+1}, u_{i+1}), & x - Y_i > u_{i}t. 
\end{cases}$$
It will be denoted by CD\(_{i,i+1}\). If \(u_i < u_{i+1}\), solution to the Riemann problem is given by

\[
U(x, t) = \begin{cases} 
(r_i, u_i), & x - Y_i < u_i t \\
(0, u_i(x, t)), & u_i t < x - Y_i < u_{i+1} t \\
(r_{i+1}, u_{i+1}), & x - Y_i > u_{i+1} t,
\end{cases}
\]

with \(u_i(x, t)\) being an arbitrary continuous function satisfying \(u_i(Y_i + u_i t, t) = u_i, u_i(Y_i + u_{i+1} t, t) = u_{i+1}\). Such solution is denoted by CD\(_1\) + Vac\(_{i,i+1}\) + CD\(_2\). Both of the above two solutions are classical and thus admissible. If \(u_i > u_{i+1}\), the simple shadow wave

\[
U(x, t) = \begin{cases} 
(r_i, u_i), & x - Y_i < \tilde{c}(t) - \frac{\xi}{2} t \\
(r_{i,e}, u_{i,e}), & \tilde{c}(t) - \frac{\xi}{2} t < x - Y_i < \tilde{c}(t) + \frac{\xi}{2} t \\
(r_{i+1}, u_{i+1}), & x - Y_i > \tilde{c}(t) + \frac{\xi}{2} t
\end{cases}
\]

(2.4)
solves (1.1). The shock is supported by the curve \(c(t) := Y_i + \tilde{c}(t) > 0\), where \(\tilde{c}(0) = 0\). Strength of the wave is \(\lim_{\varepsilon \to 0} \varepsilon \rho_{i,e} \tilde{c'}(t)\) and \(\rho_{i,e} \sim \varepsilon^{-1}\). More precisely, (2.4) satisfies system (1.1) in the approximated sense if the terms containing the \(\delta_{|x=c(t)}\) are balanced:

\[
c'(t)(\rho_{i+1} - \rho_i) - (\rho_{i+1}u_{i+1} - \rho_i u_i) \approx \varepsilon \rho_{i,e} \\
n'(t)(\rho_{i+1}u_{i+1} - \rho_i u_i) - (\rho_{i+1}u_{i+1}^2 - \rho_i u_i^2) \approx \varepsilon \rho_{i,e} u_{i,e}.
\]

The \(\delta'\)-terms are balanced if \(c'(t) = u_s\). Put \(u_s := \lim_{\varepsilon \to 0} \varepsilon \rho_{i,e}\) and \(\xi := \lim_{\varepsilon \to 0} \varepsilon \rho_{i,e}\). The above imply that \(\tilde{c}(t) = u_s t\), i.e. the speed of shadow wave is constant, \(c'(t) = u_s\). Also,

\[
\begin{align*}
\xi &= u_s (\rho_{i+1} - \rho_i) - (\rho_{i+1}u_{i+1} - \rho_i u_i) \\
u_s \xi &= u_s (\rho_{i+1}u_{i+1} - \rho_i u_i) - (\rho_{i+1}u_{i+1}^2 - \rho_i u_i^2).
\end{align*}
\]

(2.5)

The system (2.5) reduces to

\[
u_s^2(\rho_{i+1} - \rho_i) - 2u_s(\rho_{i+1}u_{i+1} - \rho_i u_i) + (\rho_{i+1}u_{i+1}^2 - \rho_i u_i^2) = 0.
\]

If \(\rho_{i+1} = \rho_i\), the solution of the above quadratic equation is

\[
u_s = \frac{\rho_{i+1}u_{i+1} - \rho_i u_i \pm \sqrt{(\rho_{i+1}u_{i+1} - \rho_i u_i)^2 - (\rho_{i+1} - \rho_i)(\rho_{i+1}u_{i+1}^2 - \rho_i u_i^2)}}{\rho_{i+1} - \rho_i}.
\]

We say that wave (2.4) is overcompressive if \(\lambda_i(\rho_i, u_i) \geq \lambda_i(\rho_r, u_r)\), \(i = 1, 2\). That will be true if we choose the + sign above \(u_s\) is a convex combination of \(u_i\) and \(u_{i+1}\). So, if we denote \(y_{i,i+1} := u_s\), then overcompressibility condition becomes

\[
u_i \geq y_{i,i+1} \geq u_{i+1}\text{ and } y_{i,i+1} = \frac{\sqrt{\rho_{i+1}u_{i+1} + \rho_i u_i}}{\sqrt{\rho_{i+1}} + \sqrt{\rho_i}}.
\]

(2.6)

Substituting \(y_{i,i+1}\) in (2.5) one gets that the strength of the shadow wave equals \(\xi_{i,i+1}t\), where \(\xi_{i,i+1} := \xi = \sqrt{\rho_i \rho_{i+1}}(u_i - u_{i+1})\). If \(\rho_{i+1} = \rho_i\), there exists unique solution to the system (2.5) with \(y_{i,i+1} = \frac{u_i + u_i}{2}, \xi_{i,i+1} = \rho_i(u_i - u_{i+1})\). The condition (2.6) is satisfied in this case, too.
3 The Elementary Interactions

The first step in construction is the analysis of all possible interactions between waves obtained after the initial data approximation by step functions.

Suppose that two approaching waves interact. Then the right state of the left incoming wave equals the left state of the right incoming wave. That will be called the middle state in the interaction. So, the interaction problem including shadow waves can be viewed as an initial value problem containing the delta function.

Lemma 3.1 Let (1.1) with the initial data

\[
(\rho, u)(x, 0) = \left\{ \begin{array}{ll}
(\rho_l, u_l), & x < X \\
(\rho_r, u_r), & x > X \\
+\gamma \delta(x, 0), &
\end{array} \right.
\]

be given, and denote \((pu)_{t=0} = \tilde{\gamma} \delta(x,0)\), where \(u_l \geq \tilde{\gamma}/\gamma \geq u_r, \gamma > 0, \rho_l, \rho_r \geq 0\). Then there exists an overcompressive shadow wave that solves the above initial data problem. A strength \(\xi(t)\) and a speed \(u_s(t)\) are solutions to

\[
ξ'(t) = (ρ_r - ρ_l)u_s(t) - (ρ_l u_r - ρ_r u_l), \quad ξ(0) = γ \tag{3.1}
\]

The front of the resulting shadow wave is given by \(x = c(t) := \int_0^t u_s(τ) dτ + X\).

Proof Substitution of the shadow wave

\[
U^ε(x,t) = \left\{ \begin{array}{ll}
(ρ_l, u_l), & x < c(t) - \frac{ε}{2}t - x_ε \\
(ρ_ε(t), u_ε(t)), & c(t) - \frac{ε}{2}t - x_ε < x < c(t) + \frac{ε}{2}t + x_ε \\
(ρ_r, u_r), & x > c(t) + \frac{ε}{2}t + x_ε 
\end{array} \right.
\]

into system (1.1), where \(ρ_ε(t) \sim ε^{-1}, x_ε \sim ε\) and \(u_ε(t) = \lim_{ε→0} u_ε(t), ξ(t) = \lim_{ε→0} 2(\frac{ε}{2}t + x_ε) ρ_ε(t), c(0) = X\) reduces to system (3.1) with the initial data \(ξ(0) = γ, u_s(0) = \tilde{γ}/γ =: c\).

The condition \(ξ(0) = γ\) is satisfied by choosing \(x_ε\) such that \(\int_{X-x_ε}^{X+X} ρ(x,0) dx = γ\). That makes a distributional solutions being continuous in time. Then, the solution is

\[
ξ(t) = \sqrt{γ^2 + ρ_1 ρ_r [u]^2 t^2 + 2γ(ρ[ρ] - [ρu])t}
\]

\[
u_s(t) = \begin{cases}
\frac{1}{[ρ]} \left( \frac{ρu + ρ_1 ρ_r [u]^2 t + γ(ρ[ρ] - [ρu])}{ξ(t)} \right), & \text{if } ρ_l ≠ ρ_r \\
\frac{1}{ξ^2(t)} \left( c - \frac{u_l + u_r}{2} \right) + \frac{u_l + u_r}{2}, & \text{if } ρ_l = ρ_r, \tag{3.2}
\end{cases}
\]

where \([·] := ·_r - ·_l\) denotes a jump across a shock front. If \(ρ_l ≠ ρ_r\), we have

\[
u_s'(t) = -\frac{[ρ]}{ξ(t)} (u_s(t) - y_{l,r})(u_s(t) - z_{l,r}) \quad \text{or} \tag{3.3}
\]

\[
u_s'(t) = -\frac{γ^2 [ρ]}{ξ^3(t)} (c - y_{l,r})(c - z_{l,r}),
\]

where

\[
y_{l,r} := \frac{u_l \sqrt{ρ_l} + u_r \sqrt{ρ_r}}{\sqrt{ρ_l} + \sqrt{ρ_r}}, \quad z_{l,r} := \frac{u_l \sqrt{ρ_l} - u_r \sqrt{ρ_r}}{\sqrt{ρ_l} - \sqrt{ρ_r}}. \tag{3.4}
\]
Overcompressibility in the case $\rho_l \neq \rho_r$ follows from the fact that $u_l \geq u_s(0) \geq u_r$. The functions $\rho$ and $\xi(t)$ are positive, and from the second line in (3.3) we have $\text{sign}(u_s'(t)) = -\text{sign}(\rho(c - y_{l,r})(c - z_{l,r})) = -\text{sign}(c - y_{l,r})$, i.e. if $u_s(0) > y_{l,r}$, $u_s$ decreases. But it cannot go below value $y_{l,r}$ because its derivative would be positive there due to the first line in (3.3). The case $u_s(0) < y_{l,r}$ can be handled analogously. One can see that $\lim_{t \to \infty} u_s(t) = y_{l,r}$. If $u_s(0) = y_{l,r}$, $u_s$ is a constant, i.e. the shadow wave has a constant speed. In any case, $u_s(t) \in [u_r, u_l]$ and the shadow wave is overcompressive. The proof in the case $\rho_l = \rho_r$ is similar.

Remark 3.1 The above lemma corresponds to Theorem 10.1 from [22], so it can be used for (1.2), too. In that case the third component in the intermediate state $U_\epsilon(t) = (\rho_\epsilon(t), u_\epsilon(t), e_\epsilon(t))$ satisfies $e_\epsilon(t) = \lim_{\epsilon \to 0} e_{\epsilon}(t)$ and

$$c'(t)\left[\rho\left(\frac{u^2}{2} + e\right)\right] - \left[\rho u\left(\frac{u^2}{2} + e\right)\right] = \frac{d}{dt}\left(\frac{u^2(t)}{2} \xi(t) + e_s(t) \xi(t)\right).$$

(3.5)

 Remark 3.2 Note, one could not expect that (3.1) can be explicitly solvable for some other systems admitting a shadow wave solution.

Corollary 3.1 With the above notation and assumptions, we have

$$u_l \geq u_s(t) \geq u_r \text{ (overcompressibility condition)} \text{ and}$$

$$\gamma + \min\{\rho_l, \rho_r\}(u_l - u_r)t \leq \xi(t) \leq \gamma + \max\{\rho_l, \rho_r\}(u_l - u_r)t.$$  (3.6)

Proof It follows from the proof of Lemma 3.1.

Lemma 3.1 is used to solve the interaction problem. If the interaction occurs at the point $(X, T)$ the initial data is translated to the interaction point, while the initial strength of the resulting shadow wave is equal to the sum of strengths of incoming waves at interaction time $t = T$. That is,$$
\gamma = \xi(T) = \xi_l(T) + \xi_r(T),$$

(3.7)

where $\xi_l(t)$ and $\xi_r(t)$, $t < T$ are the strengths of the incoming waves. Also, denote by $u_{s_l}(t)$ and $u_{s_r}(t)$, $t < T$ the speeds of incoming waves. Due to linear momentum conservation the value $\gamma$ from Lemma 3.1 equals $\gamma = \xi(T)u_s(T) - \xi_l(T)u_{s_l}(T) + \xi_r(T)u_{s_r}(T)$. Then

$$c = u_s(T) = \frac{\xi_l(T)u_{s_l}(T) + \xi_r(T)u_{s_r}(T)}{\xi_l(T) + \xi_r(T)}.$$  (3.8)

One can neglect the fact that interaction including at least one shadow wave actually occurs a bit earlier. Let us show why. Suppose that an interaction occurs between shadow waves with the external shadow wave lines $x = c(t) = \frac{c}{2}(t - T) \pm x_e$ and contact discontinuity $x = Y_i + u_{i+1}t$ at time $t = T$. The area bounded by the external shadow wave line $x = c(t) + \frac{c}{2}(t - T) + x_e$, the contact discontinuity $x = Y_i + u_{i+1}t$, and the line $t = T$ is of the order $\epsilon^2$, and $\rho_s(t) \sim \epsilon^{-1}$. All terms of growth order less than $\epsilon$ are neglected, so one can neglect that area. The situation is quite similar in the case of a double shadow wave interaction (see Fig. 1).

The following lemma is based on the above arguments and will be used repeatedly in the rest of the paper.
Lemma 3.2 Let two approaching shadow waves with the central lines given by $x = c_l(t)$ and $x = c_r(t)$ interact at time $t = \tilde{T}$. The value of $\tilde{T}$ is obtained by solving the equation

$$c_l(t) + \frac{\epsilon}{2} (t - T_l) + x_{l,\epsilon} = c_r(t) - \frac{\epsilon}{2} (t - T_r) - x_{r,\epsilon},$$

where $x = c_l(t) + \frac{\epsilon}{2} (t - T_l) + x_{l,\epsilon}$ is the right external SDW line of the first approaching shadow wave, while $x = c_r(t) - \frac{\epsilon}{2} (t - T_r) - x_{r,\epsilon}$ is the left external SDW line of the second approaching shadow wave. Also, let $x_{l,\epsilon}, x_{r,\epsilon} \sim \epsilon$. A solution $T$ to $c_l(t) = c_r(t)$ will be called the interaction time since the area bounded by two external shadow wave lines, and the lines $t = T$ and $t = \tilde{T}$ is of order $\epsilon^2$ and all terms of order $\epsilon^\alpha, \alpha > 1$ are neglected. Note that $T = \tilde{T} + O(\epsilon)$.

The assertion stays true if one of the shadow waves is substituted by a contact discontinuity.

Proof Denote by $\tilde{U}^\epsilon$ the resulting shadow wave obtained using real interaction time $t = \tilde{T}$ and by $\hat{U}^\epsilon$ the shadow wave obtained at approximate interaction time $t = T$. The corresponding speed and strength will be denoted by $\tilde{u}_l(t), \tilde{u}_r(t)$ and $\hat{u}_l(t), \hat{u}_r(t)$, respectively. It is enough to prove that

$$\int_0^\infty \int_{-\infty}^\infty (f(\tilde{U}^\epsilon) \partial_t \varphi + g(\tilde{U}^\epsilon) \partial_x \varphi) dx dt - \int_0^\infty \int_{-\infty}^\infty (f(\bar{U}^\epsilon) \partial_t \varphi + g(\bar{U}^\epsilon) \partial_x \varphi) dx dt = O(\epsilon),$$

where $f(\rho, u) = (\rho, \rho u)^T, g(\rho, u) = (\rho u, \rho u^2)^T$, holds for each test function $\varphi$. Denote by

$$\Omega_{<\tilde{T}} = \{(x, t) | 0 < t < \tilde{T}, x \in \mathbb{R}\},$$

$$\Omega_{[\tilde{T}, T]} = \hat{\omega}_l \cup \hat{\omega}_s \cup \hat{\omega}_r = \hat{\omega}_l \cup \hat{\omega}_s \cup \hat{\omega}_r,$$

$$\hat{\omega}_l = \{(x, t) | \tilde{T} < t < T, x < \tilde{c}(t) - \frac{\epsilon}{2} (t - \tilde{T}) - \tilde{x}_\epsilon\},$$

$$\hat{\omega}_s = \{(x, t) | \tilde{T} < t < T, \tilde{c}(t) - \frac{\epsilon}{2} (t - \tilde{T}) - \tilde{x}_\epsilon < x < \tilde{c}(t) + \frac{\epsilon}{2} (t - \tilde{T}) + \tilde{x}_\epsilon\},$$

$$\hat{\omega}_r = \{(x, t) | \tilde{T} < t < T, x > \tilde{c}(t) + \frac{\epsilon}{2} (t - \tilde{T}) + \tilde{x}_\epsilon\},$$

$$\hat{\omega}_l = \{(x, t) | \tilde{T} < t < T, x < c_l(t) - \frac{\epsilon}{2} (t - T_l) - x_{l,\epsilon}\},$$

$$\hat{\omega}_s = \{(x, t) | \tilde{T} < t < T, c_l(t) - \frac{\epsilon}{2} (t - T_l) - x_{l,\epsilon} < x < c_r(t) + \frac{\epsilon}{2} (t - T_r) + x_{r,\epsilon}\},$$

$$\hat{\omega}_r = \{(x, t) | \tilde{T} < t < T, x > c_r(t) + \frac{\epsilon}{2} (t - T_r) + x_{r,\epsilon}\},$$

$$\Omega_{>T} = \{(x, t) | t > T, x \in \mathbb{R}\}.$$

This Lemma can be proved by dividing $\mathbb{R} \times [0, \infty)$ into three disjoint sets, $\Omega_{<\tilde{T}}, \Omega_{[\tilde{T}, T]}$ and $\Omega_{>T}$. We have $\tilde{U}^\epsilon \equiv \hat{U}^\epsilon$ on $\Omega_{<\tilde{T}}$, so

$$\int_0^\tilde{T} \int_{-\infty}^\infty (f(\tilde{U}^\epsilon) \partial_t \varphi + g(\tilde{U}^\epsilon) \partial_x \varphi) dx dt = \int_0^\tilde{T} \int_{-\infty}^\tilde{T} (f(\tilde{U}^\epsilon) \partial_t \varphi + g(\tilde{U}^\epsilon) \partial_x \varphi) dx dt.$$

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Further,

\[
\int_\tilde{T}^{T} \int_{-\infty}^{\infty} (f(\tilde{U}^\eps)\partial_t\varphi + g(\tilde{U}^\eps)\partial_x\varphi)dxdt = \int_\tilde{T}^{T} \int_{-\infty}^{\infty} (f(\bar{U}^\eps)\partial_t\varphi + g(\bar{U}^\eps)\partial_x\varphi)dxdt + \mathcal{O}(\eps),
\]

since \(\tilde{U}^\eps\) and \(\bar{U}^\eps\) are equal to the same constant state on \(\tilde{\Omega} \setminus (\tilde{\omega}_s \cup \hat{\omega}_s)\), while \(\tilde{\rho}^\eps(x,t) = \mathcal{O}(\eps^{-1})\) for each \(t\) (at least one component is unbounded) on the set \(\tilde{\omega}_s \cup \hat{\omega}_s\) which area is of order \(\eps^2\). Thus, the total error in \([\tilde{T}, T]\) is of order \(\eps\). The same can be proved for \(t > T\) using the above reasoning and

\[
\tilde{c}(t) = \hat{c}(t) + \mathcal{O}(\eps), \quad \tilde{\xi}(t) = \hat{\xi}(t) + \mathcal{O}(\eps), \quad \tilde{u}_s(t) = \hat{u}_s(t) + \mathcal{O}(\eps) \quad \text{for } t > T.
\]

\begin{remark}
One should have in mind that a phrase “waves interact at the same time” actually means that interactions between those waves occur in the neglected area of order \(\eps^2\) described above. That is, waves interact in a time interval of the order \(\eps\).
\end{remark}

4 The Algorithm

Let us fix some notation. A shadow wave joining \((\rho_i, u_i)\) on the left and \((\rho_j, u_j)\) on the right, \(i < j\), \(\rho_i, \rho_j > 0\) is denoted by SDW\(_{i,j}\).

Let \(i\) and \(k\) be a given pair of indices. Then \(\text{SDW}_{i,k}, i \leq k\) denotes a shadow wave joining \(\text{Vac}_{i-1,i} := (0, u_{i-1}(x,t))\) on the left to \((\rho_k, u_k)\), \(\rho_k > 0\) on the right. Note that \(\text{SDW}_{i,k} = \text{CD}_{2}^i\).

A shadow wave joining \((\rho_i, u_i)\), \(\rho_i > 0\) on its left to \(\text{Vac}_{k,k+1}(0, u_k(x,t))\) on its right will be denoted by SDW\(_{i,k}\), \(i \leq k\). Again, SDW\(_{i,k} = \text{CD}_{2}^i\).

A shadow wave joining \((0, u_{i-1}(x,t))\) on the left and \((0, u_k(x,t))\) on the right is denoted by SDW\(_{i,k}\).
Remark 4.1 A wave SDW$_{i,j}$ exists only if $u_i \geq u_j$. Waves SDW$^r_i$ and SDW$^r_r$ are special solutions to (3.2). If $\rho_l > 0$ and $\rho_r = 0$,
\begin{align*}
\xi(t) &= \sqrt{\gamma^2 + 2\rho_l \gamma (u_l - c)} t \\
u_s(t) &= u_l - \frac{\gamma (u_l - c)}{\sqrt{\gamma^2 + 2\rho_l \gamma (u_l - c)}} t \\
c(t) &= X + u_l t - \frac{1}{\rho_l} \xi(t) + \frac{\gamma}{\rho_l}.
\end{align*}
(4.1)

If $\rho_l = 0$ and $\rho_r > 0$, the solution is given by (4.1), with $\rho_l$ and $u_l$ replaced by $\rho_r$ and $u_r$. Finally, if $\rho_l = \rho_r = 0$, the resulting wave SDW$^r$ propagates with constant speed and strength.

A situation when three or more waves interact at the same time in the sense of Remark 3.3 is treated in the same way. Suppose that there are $m$ incoming waves, $W_1, \ldots, W_m$. A resulting single wave depends on a state on the left to $W_1$, a state on the right of $W_m$, wave speeds and a sum of their strengths. The middle states are lost in the interaction and there are the following possibilities.

(A1): The wave $W_1$ has a left state $(\rho_l, u_l)$, $\rho_l > 0$ and $W_m$ has a right state Vac$_{r,r+1}$. The result is a single SDW$^r_l$, $l < r$.

(A2): The wave $W_1$ has a left state $(\rho_l, u_l)$, $\rho_l > 0$ and $W_m$ has a right state $(\rho_r, u_r)$, $\rho_r > 0$. The result is a single SDW$_{l,r}$, $l < r$.

(A3): The wave $W_1$ has a left state Vac$_{l-1,l}$ and $W_m$ a right state $(\rho_r, u_r)$, $\rho_r > 0$. The result is a single SDW$^r_r$, $l < r$.

(A4): The wave $W_1$ has a left state Vac$_{l-1,l}$ and $W_m$ a right state Vac$_{r,r+1}$. The result is a single SDW$^r_r$, $l < r$.

If the incoming waves are overcompressive, the resulting wave is overcompressive, too. That follows from Corollary 3.1 and relation (3.8). We are in a position to construct an approximated solution.

Algorithm:
Suppose that given $\varepsilon$ is small enough.

**STEP 0.** Let $u_0 \in \mathbb{R}$, $\rho_0 > 0$ be constants from (2.1). The set of initial states $\{u_i\}_{i \in \mathbb{N}_0}$ and $\{\rho_i\}_{i \in \mathbb{N}_0}$ are sequences generated by the piecewise constant approximations of the functions $u(x)$ and $\rho(x)$, respectively, described in the paragraph below (2.1).

**STEP 1.** Denote by $S_0 := \{U_k : k = 0, 1, 2, \ldots\}$ the set of the initial states and by $I_0 := \{0, 1, 2, \ldots\}$ the set of corresponding indexes. A solution obtained by solving Riemann problems (1.1, 2.3) generated by states in $S_0$ is stopped at $t = T_1$ when the first interaction between two or more waves occurs. If there are no interactions, all wave fronts continue to propagate to infinity and the procedure finishes. Each interaction between two or more waves belongs to one of the four types (A1–A4) and gives a single shadow wave as a result.

The resulting wave(s) as well as all other (non-interacting) waves constitute a new set of states $S_1$ and a corresponding set of indexes $I_1 \subset I_0$ after $t > T_1$.

**STEP j TO j + 1.** Suppose that $j$-th interaction occurs at a time $t = T_j$. Then we eliminate all middle states from $S_{j-1}$ and obtain a new set $S_j$ and a corresponding $I_j = \{0, j_1, j_2, j_3, \ldots\} \subset I_{j-1}$, $1 \leq j_1 < j_2 < \cdots, k \in I_{j-1} \setminus I_j$ means that the state $U_k$ was a middle one in $S_{j-1}$. All non-interacting waves are prolonged after $t > T_j$. The procedure repeats with $j$ substituted by $j + 1$ after a new interaction at $t = T_{j+1}$. The algorithm stops when there is no $T_{j+1}$. 

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It will be proved below that the procedure presented above gives a global admissible solution to the problem (1.1, 2.1).

Remark 4.2 The above types (A1-A4) cover all possible interactions between two or more waves. So the above procedure can also be applied to the problem with initial data

\[(\rho(x,0), u(x,0)) = \begin{cases} 
(\rho(x), u(x)), & x \leq R \\
(\rho_0, u_0), & x > R, 
\end{cases}\]

or any initial data

\[\rho(x,0) = \rho(x), \quad u(x,0) = u(x), \quad x \in \mathbb{R},\]

where \(\rho(x) > 0, \quad u(x)\) having a finite number of jumps and being piecewise \(C^1_b(\mathbb{R})\).

5 Global Existence and Admissibility of a Solution

The proof that our algorithm gives an admissible solution is divided into cases depending on monotonicity of a function \(u(x)\) and relations between \(u_0\) and \(u(R)\). A function \(u(x)\) is called increasing (or decreasing) if \(u(x) \leq u(y)\) (or \(u(x) \geq u(y)\)) for each \(x < y\). The function \(u(x)\) is strictly increasing (or decreasing) if the inequality is strict.

CASE I. \(u(x)\) is increasing function for \(x > R\) and \(u_0 \leq u(R)\).

This is a simple case with no interactions. The solution is a piecewise continuous function whose jumps are located along contact discontinuity lines. That is the consequence of the fact that \(u_i \leq u_{i+1}\) for each \(i = 0, 1, \ldots\) Such waves never interact since the one in front has a larger or the same speed.

CASE II. \(u(x)\) is increasing function for \(x > R\) and \(u_0 > u(R)\).

Due to the boundedness assumption, there exists \(\bar{u}, \lim_{i \to \infty} u_i = \bar{u}\). The wave SDW_{0,1} emanating from the point \((R, 0)\) is a solution to (1.1, 2.3) for \(i = 0\). Solutions to (1.1, 2.3) are \(\text{CD}_i + \text{Vac}_{i,i+1} + \text{CD}_{i+1}\) emanating from \((Y_i, 0), i = 1, 2, \ldots\). If \(u_i = u_{i+1}\), the combination reduces to a single \(\text{CD}_{i,i+1}\). Note that all the interactions in this case are of types (A1)
or (A2). After each interaction exists only shadow wave that started at \((R, 0)\). Denote by \((X_{0,i}, T_{0,i})\) a point where it meets the first contact discontinuity in the \(i\)-th wave combination \(CD^+_i + \text{Vac}_{i,i+1} + CD^+_{2,i+1}\). \((X_{1,i}, T_{1,i})\) is the interaction point of the shadow wave and the second contact discontinuity (see Fig. 2). The overcompressibility follows from Corollary 3.1 and interactions continue to infinity if \(\bar{u} \leq u_0\) because of it. If \(\bar{u} > u_0\), the solution is same until a point where the shadow wave enters the vacuum state and interactions stop, again due to the overcompressibility.

The case of a single contact discontinuity when \(u_i = u_{i+1}\) for some \(i\) makes no real difference in the analysis.

Let \(U^\varepsilon = (\rho^\varepsilon, u^\varepsilon)\) be a function obtained by the above procedure for a fixed \(\varepsilon\). Denote by \(\hat{U}^\varepsilon\) its singular part represented by the shadow wave approximation

\[
\hat{U}^\varepsilon(x, t) = \begin{cases} 
0, & x < c(t) - a_\varepsilon(t) \\
(\rho_\varepsilon, u_\varepsilon)(t), & c(t) - a_\varepsilon(t) < x < c(t) + a_\varepsilon(t) \\
0, & x > c(t) + a_\varepsilon(t)
\end{cases}
\]

where

\[
a_\varepsilon(t) = \begin{cases} 
\frac{\varepsilon}{2} t, & t \in (0, T_{0,1}] \\
\frac{\varepsilon}{2} (t - T_{0,i}) + x^0_{\varepsilon,i}, & t \in (T_{0,i}, T_{1,i}], \quad i = 1, 2, \ldots \\
\frac{\varepsilon}{2} (t - T_{1,i}) + x^k_{\varepsilon,i}, & t \in (T_{1,i}, T_{0,i+1}]
\end{cases}
\]

Here, the values \((\rho_\varepsilon, u_\varepsilon)(t)\) and \(x^k_{\varepsilon,i}\), \(k = 0, 1\) are determined by Lemma 3.1 for each interval \((T_{0,i}, T_{1,i}], (T_{1,i}, T_{0,i+1}], i = 1, 2, \ldots\) separately. That part of a solution is called 0-SDW and it approximates a weighted delta function with variable speed.

**Theorem 5.1** Let \(u(x), \rho(x) \in C_b([R, \infty))\), \(\rho(x) > 0\). Assume that \(u(x)\) is increasing and let \(\rho_0 > 0\) and \(u_0 > u(R)\). Take a partition \(\{Y_i\}_{i \in \mathbb{N}}\) of \([R, \infty)\), \(Y_0 = R\) such that \(C\sqrt{\varepsilon} \geq Y_i - Y_{i-1} \geq \sqrt{\varepsilon}\) for every \(i = 1, 2, \ldots\) and a constant \(C \geq 1\). There exists an admissible global solution to (1.1, 2.1), i.e. there exists a function \(U^\varepsilon = (\rho^\varepsilon, u^\varepsilon)\) satisfying

\[
\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) \approx 0, \quad \partial_t (u^\varepsilon) + \partial_x (\rho^\varepsilon (u^\varepsilon)^2) \approx 0,
\]

\(\rho^\varepsilon(x, 0) \approx \rho(x, 0), u^\varepsilon(x, 0) \approx u(x, 0)\) as \(\varepsilon \rightarrow 0\) and the admissibility condition.

1. If \(\bar{u} \leq u_0\), there are infinitely many interaction points.
2. If \(\bar{u} > u_0\), the interactions will stop with the interaction point \((X_{0,k}, T_{0,k})\) where \(k \in \mathbb{N}\) is taken such that \(u_k < u_0 \leq u_{k+1}\) holds. In that case \(u_k(t) \rightarrow u_0\) as \(t \rightarrow \infty\).

**Remark 5.1** One can use any \(\mu(\varepsilon) \rightarrow 0, \varepsilon \rightarrow 0\) instead of \(C\sqrt{\varepsilon}\) above. We have used that one because of Theorem 5.3 where \(u(x)\) is not necessarily monotone. Also, any lower bound of order \(\varepsilon^\alpha\), \(0 < \alpha < 1\) can be used instead of \(\sqrt{\varepsilon}\) here.

**Proof** For a reader’s convenience we will present the complete proof here. Later on, we will skip technical details since they are similar to the ones in this proof.
(1) Let \( \tilde{u} \leq u_0 \). We have to prove that a solution \( U^\varepsilon, t \geq 0, \ x \in \mathbb{R} \) satisfies the following relations

\[
E_1 := \int_0^\infty \int_{-\infty}^\infty \left( (\rho^\varepsilon \partial_x \varphi)(x, t) + (\rho^\varepsilon u^\varepsilon \partial_v \varphi)(x, t) \right) dx \, dt + \int_{-\infty}^\infty (\rho^\varepsilon \varphi)(x, 0) dx \approx 0
\]

\[
E_2 := \int_0^\infty \int_{-\infty}^\infty \left( (\rho^\varepsilon u^\varepsilon \partial_x \varphi)(x, t) + (\rho^\varepsilon (u^\varepsilon)^2 \partial_v \varphi)(x, t) \right) dx \, dt + \int_{-\infty}^\infty (\rho^\varepsilon u^\varepsilon \varphi)(x, 0) dx \approx 0
\]

for every test function \( \varphi \in C_0^\infty(\mathbb{R} \times [0, \infty)) \). We use the Taylor expansion of the test function \( \varphi \),

\[
\begin{align*}
\varphi(c(t) - a_\varepsilon(t), t) &= \varphi(c(t), t) - \partial_x \varphi(c(t), t) a_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \\
\varphi(c(t) + a_\varepsilon(t), t) &= \varphi(c(t), t) + \partial_x \varphi(c(t), t) a_\varepsilon(t) + \mathcal{O}(\varepsilon^2)
\end{align*}
\]

(5.1)

\[
\varphi(x, t) = \varphi(c(t), t) + \mathcal{O}(\varepsilon) \text{ for } x \in (c(t) - a_\varepsilon(t), c(t) + a_\varepsilon(t)).
\]

Thus

\[
\begin{align*}
&\int_0^\infty \int_{-\infty}^\infty (\rho^\varepsilon \partial_x \varphi)(x, t) \, dx \, dt = I_0 + \int_{-\infty}^\infty ((\rho^\varepsilon \varphi)(x, T_{0,1} - 0) - (\rho^\varepsilon \varphi)(x, 0)) \, dx \\
&\quad + \sum_{i=1}^\infty I_{0,i} + \sum_{i=1}^\infty \int_{-\infty}^\infty \left( (\rho^\varepsilon \varphi)(x, T_{1,i} - 0) - (\rho^\varepsilon \varphi)(x, T_{0,i} + 0) \right) \, dx \\
&\quad + \sum_{i=1}^\infty I_{1,i} + \sum_{i=1}^\infty \int_{-\infty}^\infty \left( (\rho^\varepsilon \varphi)(x, T_{0,i+1} - 0) - (\rho^\varepsilon \varphi)(x, T_{1,i} + 0) \right) \, dx,
\end{align*}
\]

where \( I_0, I_{0,i} \) and \( I_{1,i} \) are integrals over \([0, T_{0,1}], [T_{0,i}, T_{1,i}] \) and \([T_{1,i}, T_{0,i+1}]\), respectively. All other terms cancel with the initial data and mutually because we asked for a continuity of \( U^\varepsilon \) with respect to \( t \). In the same way the flux–part can be decomposed

\[
\begin{align*}
&\int_0^\infty \int_{-\infty}^\infty \rho^\varepsilon u^\varepsilon \partial_x \varphi \, dx \, dt = J_0 + \sum_{i=1}^\infty J_{0,i} + \sum_{i=1}^\infty J_{1,i},
\end{align*}
\]

where \( J_0, J_{0,i} \) and \( J_{1,i} \) are integrals over \([0, T_{0,1}], [T_{0,i}, T_{1,i}] \) and \([T_{1,i}, T_{0,i+1}]\), respectively. Note that we have finitely many intervals due to the compactness of \( \text{supp} \varphi \). If \( u(x) \) is not strictly increasing, then some of the points \( T_{0,i} \) and \( T_{1,i} \) would coincide. That does not influence the analysis.

In the first interval \([0, T_{0,1}]\), we have

\[
\begin{align*}
I_0 &= -\int_0^{T_{0,1}} \rho_i u_1 \varphi(Y_1 + u_1 t, t) \, dt - \int_0^{T_{0,1}} (\rho_0 - \rho_{0,\varepsilon}) \varphi \left( R + y_{0,1} t - \frac{\varepsilon}{2}, t \right) \left( y_{0,1} - \frac{\varepsilon}{2} \right) \, dt \\
&\quad - \int_0^{T_{0,1}} (\rho_{0,\varepsilon} - \rho_1) \varphi \left( R + y_{0,1} t + \frac{\varepsilon}{2}, t \right) \left( y_{0,1} + \frac{\varepsilon}{2} \right) \, dt \\
&\quad + \sum_{i=1}^{\infty} \int_0^{T_{0,1}} \rho_i u_{i+1} \left( \varphi(Y_i + u_{i+1} t, t) - \varphi(Y_{i+1} + u_{i+1} t, t) \right) \, dt,
\end{align*}
\]

\[\therefore A_0 \]
\[ J_0 := \int_0^{T_{0,1}} \int_{-\infty}^{0} \rho^* u^* \partial_x \varphi \, dx \, dt = \int_0^{T_{0,1}} \left( \rho_0 u_0 - \rho_{0,e} u_{0,e} \right) \varphi \left( R + y_{0,1} t - \frac{\varepsilon}{2} t \right) \, dt \]
\[ \quad - \int_0^{T_{0,1}} \left( \rho_1 u_1 - \rho_{0,e} u_{0,e} \right) \varphi \left( R + y_{0,1} t + \frac{\varepsilon}{2} t \right) \, dt + \int_0^{T_{0,1}} \rho_1 u_1 \varphi \left( Y_1 + u_1 t \right) \, dt \]
\[ \quad - \sum_{i=1}^{\infty} \int_0^{T_{0,1}} \rho_{i+1} u_{i+1} \left( \varphi \left( Y_i + u_{i+1} t \right) - \varphi \left( Y_{i+1} + u_{i+1} t \right) \right) \, dt, \]
\[ \quad := B_0 \]

since \( \rho_{0,e} \) does not depend on \( t \). Using \( \xi_{0,1} = y_{0,1} (\rho_1 - \rho_0) - (\rho_1 u_1 - \rho_0 u_0) \), (5.1) and the fact that \( A_0 = B_0 \) we get

\[ I_0 + J_0 = \int_0^{T_{0,1}} (\xi_{0,1} - \varepsilon \rho_{0,e}) \varphi \left( R + y_{0,1} t, t \right) \, dt \]
\[ \quad - \int_0^{T_{0,1}} t \varepsilon \rho_{0,e} (y_{0,1} - u_{0,e}) \partial_x \varphi \left( R + y_{0,1} t, t \right) \, dt + O(\varepsilon). \]

From the fact that \( \lim_{\varepsilon \to 0} u_{0,e} = y_{0,1} \), \( \lim_{\varepsilon \to 0} \varepsilon \rho_{0,e} = \xi_{0,1} \), we have \( E_1 = O(\varepsilon) \) in the strip \([0, T_{0,1}]\). The same relations with \( \rho \) substituted by \( \rho u \) and \( \rho u \) by \( \rho u^2 \) give us \( E_2 = O(\varepsilon) \) in \([0, T_{0,1}]\).

For \( t \in [T_{0,i}, T_{1,i}] \) we have the following relations

\[ I_{0,i} = - \int_{T_{0,i}}^{T_{1,i}} \int_{\xi_{0,i} - T_{0,i}}^{\xi_{0,i} + T_{0,i}} \partial_t \rho_e (t) \varphi (x, t) \, dx \, dt \]
\[ \quad - \int_{T_{0,i}}^{T_{1,i}} \left( \rho_0 - \rho_e (t) \right) \varphi \left( c(t) - \frac{\varepsilon}{2} (t - T_{0,i}) - x_{e,0,i}, t \right) \left( c'(t) - \frac{\varepsilon}{2} \right) \, dt \]
\[ \quad - \int_{T_{0,i}}^{T_{1,i}} \rho_e (t) \varphi \left( c(t) + \frac{\varepsilon}{2} (t - T_{0,i}) + x_{e,0,i}, t \right) \left( c'(t) + \frac{\varepsilon}{2} \right) \, dt \]
\[ \quad + \sum_{k=i}^{\infty} \int_{T_{0,i}}^{T_{1,i}} \rho_{k+1} u_{k+1} \left( \varphi \left( Y_k + u_{k+1} t, t \right) - \varphi \left( Y_{k+1} + u_{k+1} t, t \right) \right) \, dt, \]
\[ \quad := A_{0,i} \]

and

\[ J_{0,i} = \int_{T_{0,i}}^{T_{1,i}} \left( \rho_0 u_0 - \rho_e (t) u_e (t) \right) \varphi \left( c(t) - \frac{\varepsilon}{2} (t - T_{0,i}) - x_{e,0,i}, t \right) \, dt \]
\[ \quad + \int_{T_{0,i}}^{T_{1,i}} \rho_e (t) u_e (t) \varphi \left( c(t) + \frac{\varepsilon}{2} (t - T_{0,i}) + x_{e,0,i}, t \right) \, dt \]
\[ \quad - \sum_{k=i}^{\infty} \int_{T_{0,i}}^{T_{1,i}} \rho_{k+1} u_{k+1} \left( \varphi \left( Y_k + u_{k+1} t, t \right) - \varphi \left( Y_{k+1} + u_{k+1} t, t \right) \right) \, dt, \]
\[ \quad := B_{0,i} = A_{0,i} \]
The proof that $I_{0,i} + J_{0,i} = O(\varepsilon)$ for $t \in [T_{0,i}, T_{1,i}]$ follows from Lemma 3.1 and the method given above. Again, we have $E_2 = O(\varepsilon)$ in the same interval by following the same arguments.

Finally, let $t \in [T_{1,i}, T_{0,i+1}]$. Then

$$I_{1,i} = - \int_{T_{1,i}}^{T_{0,i+1}} \int_{T_{1,i}}^{T_{0,i+1}} \left( \frac{\rho_0}{2} (t - T_{1,i}) - x_i^1, t \right) \left( \frac{\rho}{2} (t - T_{1,i}) - x_i^1, t \right) dt$$

$$- \int_{T_{1,i}}^{T_{0,i+1}} \left( \frac{\rho_0 - \rho_i(t)}{2} (t - T_{1,i}) - x_i^1, t \right) \left( \frac{\rho_i(t)}{2} (t - T_{1,i}) + x_i^1, t \right) c'(t) \, dt$$

$$- \int_{T_{1,i}}^{T_{0,i+1}} \left( \frac{\rho_i(t)}{2} (t - T_{1,i}) + x_i^1, t \right) \left( \frac{\rho_i(t) + \rho_{i+1}}{2} \right) \left( \frac{\rho_i(t) + \rho_{i+1}}{2} \right) \, dt$$

$$+ \sum_{k=i+1}^{\infty} \int_{T_{1,i}}^{T_{0,i+1}} \rho_{k+1} u_{k+1} \left( \varphi \left( Y_k + u_{k+1} t, t \right) - \varphi \left( Y_{k+1} + u_{k+1} t, t \right) \right) dt,$$

and

$$J_{1,i} = \int_{T_{1,i}}^{T_{0,i+1}} \left( \rho_0 u_0 - \rho_i(t) u_i(t) \right) \varphi \left( c(t) - \frac{\varepsilon}{2} (t - T_{1,i}) - x_i^1, t \right) \, dt$$

$$+ \int_{T_{1,i}}^{T_{0,i+1}} \left( \rho_i(t) u_i(t) - \rho_{i+1} u_{i+1} \right) \varphi \left( c(t) + \frac{\varepsilon}{2} (t - T_{1,i}) + x_i^1, t \right) \, dt$$

$$+ \int_{T_{1,i}}^{T_{0,i+1}} \rho_{i+1} u_{i+1} \varphi \left( Y_{i+1} + u_{i+1} t, t \right) dt$$

$$- \sum_{k=i+1}^{\infty} \int_{T_{1,i}}^{T_{0,i+1}} \rho_{k+1} u_{k+1} \left( \varphi \left( Y_k + u_{k+1} t, t \right) - \varphi \left( Y_{k+1} + u_{k+1} t, t \right) \right) dt.$$

The same arguments as above and Lemma 3.1 imply $E_1 = O(\varepsilon)$. Proof for $E_2$ is the same.

Note that the proof holds even if $\rho_{i+1} = \rho_0$ for some $i$, since (3.2) implies $\xi'(t) = -\rho_0(u_{i+1} - u_0)$ and the expression for $c'(t)$ does not have an influence in the proof.

Due to the fact that the test function $\varphi$ has a compact support and from $Y_i - Y_{i-1} \geq \sqrt{\varepsilon}$, one can see that there are at most $\frac{\text{const}(\varphi)}{\sqrt{\varepsilon}}$ interactions. Thus, $E_1$ and $E_2$ are of order $O(1) O(\varepsilon) = O(\sqrt{\varepsilon})$, $\varepsilon \to 0$. That proves the existence in the case $\bar{u} \leq u_0$. The admissibility of the obtained solution follows from the uniqueness of the classical solutions and piecewise overcompressibility of the shadow wave in each segment. If $\bar{u} = u_0$, then $y_{0,i+1} \to u_0$ as $i \to \infty$. That is, the overcompressibility implies that the speed of the shadow wave is close to $u_0$ for $i$ large enough.

(2) If $\bar{u} > u_0$, then there exists $k \in \mathbb{N}$ such that $u_{k+1} \geq u_0$ and $u_k < u_0$. Consequently, a curve $x = c(t)$ will stay in vacuum area between two contact discontinuities emanating from $Y_k$ and the interactions will stop after the interaction point $(X_{0,k}, T_{0,k})$. 

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CASE III. \( u(x) \) is decreasing function for \( x > R \) and \( u_0 \geq u(R) \)

The solution formed at the initial time is a piecewise constant function with constant states connected by simple shadow waves. Each SDW\(_{i,i+1}\) emanates from a point \( Y_i \) and joints (\( \rho_i, u_i \)) and (\( \rho_{i+1}, u_{i+1} \)). Thus, all possible cases of interactions are covered by type (A2). With notation from (3.4) we have

\[
y_{i,k} - y_{k,j} = \frac{\sqrt{\rho_i}}{\sqrt{\rho_i} + \sqrt{\rho_k}} \frac{\sqrt{\rho_k}}{\sqrt{\rho_k} + \sqrt{\rho_j}} (u_i - u_k) + \frac{\sqrt{\rho_i}}{\sqrt{\rho_i} + \sqrt{\rho_k}} \frac{\sqrt{\rho_j}}{\sqrt{\rho_j} + \sqrt{\rho_j}} (u_j - u_i)
\]

\[
+ \frac{\sqrt{\rho_k}}{\sqrt{\rho_i} + \sqrt{\rho_k}} \frac{\sqrt{\rho_j}}{\sqrt{\rho_j} + \sqrt{\rho_j}} (u_k - u_j) \geq 0 \quad \text{for} \quad i < k < j,
\]

since \( u(x) \) decreases. Due to overcompressibility each pair of shadow waves is approaching.

The interaction point \((\hat{X}_i, \hat{T}_i)\) between SDW\(_{i-1,i}\) and SDW\(_{i,i+1}\), \( i = 1, 2, \ldots \) is determined by

\[
\hat{X}_i = Y_{i-1} + y_{i-1,i}, \quad \hat{T}_i = Y_i + y_{i,i+1}, \quad \hat{T}_i = Y_i - Y_{i-1} - y_{i-1,i}, y_{i,i+1}.
\]

(Note that \( y_{i,i+1} \) is a speed of SDW\(_{i,i+1}\).) Then \( \Delta u_i := u_i - u_{i-1} < \text{const}_u(\varepsilon) \), \( \sup_{x > R} |u'(x)| \leq \text{const}_u \), since \( \Delta Y_i := Y_i - Y_{i-1} \sim \mu(\varepsilon) =: \mu, i = 1, 2, 3, \ldots \). Relation (5.2) implies

\[
y_{i-1,i} - y_{i,i+1} < (u_{i-1} - u_i) + (u_{i-1} - u_{i+1}) + (u_i - u_{i+1}) = 2(u_{i-1} - u_{i+1}) < \text{const}_u \mu,
\]

i.e. \( \hat{T}_i = O(1) \). Note that it is not possible to determine which interaction takes place first and if more than two shadow waves interact at the same time, since the relationship between \( y_{i-1,i} - y_{i,i+1} \) and \( y_{i,i+1} - y_{i+1,i+2} \) depends on \( \rho(x) \), too.

If \( u_0 > u(R) \), the time of interaction between SDW\(_{0,1}\) and SDW\(_{1,2}\) is of order \( O(\mu) \).

So, the first interaction that occurs is one between SDW\(_{0,1}\) and SDW\(_{1,2}\). Resulting SDW\(_{0,2}\) propagates until the next interaction. Due to Lemma 3.1 the solution is overcompressive for \( t \geq 0 \).

As in the Case II, 0-SDW is defined to be the shadow wave connecting all piecewise defined SDW\(_{i,i}\), \( i = 1, 2, \ldots \) (see \( \hat{U}^\varepsilon \) above). Eventually, 0-SDW will overtake each SDW\(_{i,i+1}\) and any other shadow wave obtained by their mutual interactions. All waves except the 0-SDW are called “small” shadow waves.

CASE IV. \( u(x) \) is decreasing function for \( x > R \) and \( u_0 < u(R) \)

This case is similar to the previous one. The solution shortly after the initial time consists of a wave combination CD\(_1^0\) + Va\(_{0,1}\) + CD\(_1^1\) emanating from \( x = R \) followed by a sequence SDW\(_{1,2}\), SDW\(_{2,3}\), \ldots . Possible types of interactions are (A2) and (A3), and the order of interactions cannot be determined in advance. Note that CD\(_1^0\) does not interact with other waves, while CD\(_1^1\) interacts with some SDW\(_{1,k}\). The resulting \( ^1SDW_k \) continues to propagate and collide with shadow waves approaching from the right since it has a larger speed than all waves from its right side. That conclusion follows from the overcompressibility of the solution in each time interval.

**Theorem 5.2** (Case III and IV) Let \( u(x), \rho(x) \in C_b([R, \infty)), \rho(x) > 0 \). Assume that \( u(x) \) is decreasing and let \( \rho_0 > 0 \) and \( u_0 \in \mathbb{R} \). Take a partition \( \{Y_i\}_{i \in \mathbb{N}_0} \) such that \( C\sqrt{\varepsilon} \geq Y_i - Y_{i-1} \geq \sqrt{\varepsilon}, i = 1, 2, \ldots, C \geq 1 \) and \( Y_0 = R \). For \( \varepsilon > 0 \) small enough there exists admissible global solution \( \hat{U}^\varepsilon \) to the problem (1.1, 2.1).
Proof Denote by

\[ S^{III}_0 := \{ (\rho_i, u_i) : i = 0, 1, 2, \ldots \} \]

the set of initial states for the case \( u_0 \geq u(R) \) and by

\[ S^{IV}_0 := \{ (\rho_0, u_0), \text{Vac}_0 \} \cup \{ (\rho_i, u_i) : i = 1, 2, \ldots \} \]

the set of initial states for the case \( u_0 < u(R) \). Denote by \( I^*_0 = \{ 0, 1, 2, \ldots \} \) the initial set of indexes corresponding to \( S^*_0, * \in \{ III, IV \} \) as above. The analysis below is the same for both cases.

Suppose that an interaction occurs at \( t = T_k \) for waves corresponding to states in \( S^*_k \). A new set of states \( S^*_k \) is constructed by eliminating all the middle ones in interactions. The new set of indexes is now denoted by \( I^*_k = \{ 0, k^*_1, k^*_2, k^*_3, \ldots \} \) where \( 1 \leq k^*_1 < k^*_2 < k^*_3 < \ldots \).

Let us prove that

\[
\int_0^\infty \int_{-\infty}^\infty \left( \rho^e \partial_t \varphi(x,t) + (\rho^e u^e \partial_x \varphi)(x,t) \right) dx dt + \int_{-\infty}^\infty (\rho^e \varphi)(x,0) dx \approx 0
\]

for any \( \varphi \in C^\infty_0(\mathbb{R} \times [0, \infty)) \).

Again, values \( x_\varepsilon \sim \varepsilon \) are chosen such that the sum of strengths of incoming waves is equal to the initial strength of outgoing shadow wave (we use Lemma 3.1). We proceed in the same way as in the proof of Theorem 5.1.

Put

\[ Q_k := \int_{T_k}^{T_{k+1}} \int_{-\infty}^\infty \rho^e \partial_t \varphi(x,t) dx dt + \int_{T_k}^{T_{k+1}} \int_{-\infty}^\infty \rho^e u^e \partial_x \varphi(x,t) dx dt. \]

Then,

\[
\int_0^\infty \int_{-\infty}^\infty \rho^e \partial_t \varphi dx dt + \int_0^\infty \int_{-\infty}^\infty \rho^e u^e \partial_x \varphi dx dt = \sum_{k=0}^{\infty} Q_k.
\]

It is enough to prove \( \varepsilon \)-bounds for \( Q_k \) due to Lemma 3.1. The sum is finite because \( \text{supp} \varphi \) is compact. Take two successive \( i, j \) from \( I^*_k \). There exists a shadow wave or a contact discontinuity with the states corresponding to indexes \( i, j \). If it is a shadow one, denote its speed and strength by \( u_s(t) \) and \( \xi(t) \), respectively. The intermediate state is denoted by \( (\rho^e(t), u^e(t)) \). Then, \( Q_k \) is a sum of terms

\[
C_{i,j}^k := \int_{T_k}^{T_{k+1}} \left( A_{i,j}(t) - B_{i,j}(t) \right) dt + \int_{T_k}^{T_{k+1}} \int_{c(t)-\frac{\xi(t-T)}{\varepsilon}+x_\varepsilon}^{c(t)+\frac{\xi(t-T)}{\varepsilon}+x_\varepsilon} \partial_t \rho^e(t) \varphi(x,t) dx dt,
\]
where

\[ A_{i,j}(t) := (\rho_i - \rho_\epsilon(t)) \varphi \left( c(t) - \frac{\varepsilon}{2} (t - T) - x_\epsilon, t \right) \left( c'(t) - \frac{\varepsilon}{2} \right) + (\rho_\epsilon(t) - \rho_j) \varphi \left( c(t) + \frac{\varepsilon}{2} (t - T) + x_\epsilon, t \right) \left( c'(t) + \frac{\varepsilon}{2} \right) \]

\[ B_{i,j}(t) := (\rho_i u_i - \rho_\epsilon(t) u_\epsilon(t)) \varphi \left( c(t) - \frac{\varepsilon}{2} (t - T) - x_\epsilon, t \right) + (\rho_\epsilon(t) u_\epsilon(t) - \rho_j u_j) \varphi \left( c(t) + \frac{\varepsilon}{2} (t - T) + x_\epsilon, t \right), \]

for each pair \( i, j \) of sequential indexes. After some calculations analogous to the ones performed in the proof of Theorem 5.1,

\[ C_{i,j}^k = \int_{T_k}^{T_{k+1}} \left( \partial_t \rho_\epsilon(t) \left( \frac{\varepsilon}{2} (t - T) + x_\epsilon \right) + \frac{\varepsilon}{2} \rho_\epsilon(t) \right) \varphi(c(t), t) dt \]

\[ + \int_{T_k}^{T_{k+1}} \left( (\rho_i - \rho_j) c'(t) - (\rho_i u_i - \rho_j u_j) \right) \varphi(c(t), t) dt \]

\[ + \int_{T_k}^{T_{k+1}} 2 \rho_\epsilon(t) \left( c'(t) - u_\epsilon(t) \right) \left( \frac{\varepsilon}{2} (t - T) - x_\epsilon \right) \partial_x \varphi(c(t), t) dt + O(\varepsilon) = O(\varepsilon). \]

If \( u_i, u_j \) are connected by a contact discontinuity, then \( C_{i,j}^k = 0 \). The condition \( \Delta Y_i \geq \sqrt{\varepsilon} \) ensures that at most \( \frac{\text{const}(\varphi)}{\sqrt{\varepsilon^2}} \) interactions occur, since a test function \( \varphi \) has a compact support.

For the same reason, solution in the interval \([T_k, T_{k+1}]\) consists of at most \( \frac{\text{const}(\varphi)}{\sqrt{\varepsilon^2}} \) wave fronts. Then

\[ Q_k = \frac{\text{const}(\varphi)}{\sqrt{\varepsilon^2}} O(\varepsilon) = O(\sqrt{\varepsilon^2}) \text{ as } \varepsilon \to 0. \]

The proof for the second equation goes analogously (\( \rho^\varepsilon \) is replaced by \( \rho^\varepsilon u^\varepsilon \) and \( \rho^\varepsilon u^\varepsilon \) by \( \rho^\varepsilon (u^\varepsilon)^2 \)). Admissibility of a solution follows from the overcompressibility in each time interval \([T_k, T_{k+1}]\). That concludes the proof.

\[ \square \]

5.1 The General Case

Suppose that a function \( u(x) \) has a finite number of local extremes. We will give a short analysis of the cases when a function has only one local extremum. Cases when function changes monotonically more than one time can be treated in the same way.

Suppose that a local maximum of a piecewise constant approximation is reached at a point \( Y_{m_1} \in \{ Y_i \}_{i \in \mathbb{N}} \). If \( u_0 \leq u(R) \), the solution before the first interaction consists of the combinations CD+Vac+CD which do not interact with each other. The last of them emanates from the point \( (Y_{m_1-1}, 0) \). Starting from the point \( (Y_{m_1}, 0) \), the solution is like in Case III with \( R = Y_{m_1} \) and \( u_0 = u(Y_{m_1}) \). Those waves continue to propagate until the first interaction. If \( u_0 > u(R) \), the solution before it is a combination of waves obtained in Cases II and III. Unlike the case \( u_0 \leq u(R) \), the wave front propagating from \( (R, 0) \) is SDW_{0,1}. Figure 3 illustrates that case.

Similarly, if \( u^\varepsilon(x) \) has a local minimum at \( Y_{m_1} \) and \( u_0 \geq u(R) \), the solution before the first interaction is a combination of shadow waves and contact discontinuities (Cases II and III):
A sequence \( \{SDW_{i,i+1}\}_{i=0}^{m-1} \) is followed by a sequence of wave combinations CD+Vac+CD. If \( u_0 < u(R) \), a CD\(^0\) + Vac\(_1\) + CD\(^2\) emanates from \((R, 0)\) instead of SDW\(_{0,1}\) (Cases II and IV).

The proof of the following theorem will be omitted since technical details are combined in proofs of Theorems 5.1 and 5.2.

**Theorem 5.3** *(Global existence)* Suppose that \( u(x), \rho(x) \in C_b([R, \infty)) \). Moreover, suppose that \( u(x) \) has a finite number of local extremes and that \( \rho(x) > 0 \). Let \( \rho_0 > 0, u_0 \in \mathbb{R} \) and consider a partition \( \{Y_i\}_{i \in \mathbb{N}_0}, Y_0 = R \) such that \( C \sqrt{\varepsilon} \geq Y_i - Y_{i-1} \geq \sqrt{\varepsilon} \), \( i = 1, 2, \ldots \), \( C \geq 1 \). For \( \varepsilon > 0 \) small enough there exists an admissible global solution \( U^\varepsilon \) to \((1.1, 2.1)\) (in the approximated sense).

**Remark 5.2** One could easily check that the above statement also holds true for \((1.2)\) when \( e(x) \in C_b([R, \infty)) \) is positive. The energy variable does not have an influence on the approximated solution behaviour.

**Remark 5.3** The approximate solution constructed above can be related to the duality solution from [1] where the authors considered the system of \( N \) particles. The exact solution is obtained letting \( N \to \infty \) which is analogous to initial data approximation using parameter \( \mu \to 0 \). We will return to this connection in the last section. The initial data in [1] are more general and they belong to space of measures. Here we deal with piecewise continuous initial data. The delta measure initial data for a class of conservation law systems for gasses with non-positive pressure are analysed in [25].

### 6 Entropy Dynamics and Dissipation of Energy

In the case of system \((1.2)\), the semi-convex entropy pair is given by

\[
\begin{align*}
\eta(\rho, u, e) &= \rho(R(u) + S(e)), \\
Q(\rho, u, e) &= \rho(R(u) + S(e)),
\end{align*}
\]

where \( R''(x) \geq 0 \), \( S'(x) \leq 0 \) and \( S''(x) \geq 0 \) for each \( x \) (see [22]). The constructed solution should satisfy the entropy inequality \( \partial_t \eta + \partial_x Q \leq 0 \). Physically, it means that the mathematical entropy cannot increase. This condition is necessary and sufficient condition for uniqueness of \( 3 \times 3 \) pressureless gas dynamics system.
For the pressureless gas dynamics system (1.1) it is known that using semi-convex entropy pairs \((\eta, Q)\) is not sufficient to extract a proper solution (we have to use overcompressibility). We will examine a dynamics of the physical energy and its flux,

\[
\eta(\rho, u) = \frac{1}{2} \rho u^2, \quad Q(\rho, u) = \frac{1}{2} \rho u^3. \tag{6.2}
\]

If shadow wave connecting states \(U_l\) and \(U_r\) emanating at the time \(t = T_1\) satisfies the entropy condition, then

\[
D_{l,r}(t) := -u_s(t)[\eta] + [Q] + \lim_{\varepsilon \to 0} \frac{d}{dt} \left( (\varepsilon(t - T_1) + 2x_s) \eta(U_\varepsilon(t)) \right) \leq 0,
\]
as proved in [22]. There is the second entropy condition given in the same paper, but it is always satisfied here due to the fact that \(u_s(t) \approx u_s(t)\). The overcompressibility implies \(D_{l,r}(t) \leq 0\).

Here \(D_{l,r}(t)\) is consistent with entropy production measure defined in [9] for general conservation law systems possessing bounded variation solutions. We will call it the entropy production across the SDW\(_{l,r}\) at time \(t \geq T_1\). Denote by

\[
E(t) = \int_{-M}^{M} \eta(U(x, t)) dx
\]
the total entropy at time \(t\) of a solution \(U(x, t)\). Here \(M > 0\) is taken to be large enough to avoid the total entropy being infinite in finite time.

**Theorem 6.1** Consider the system (1.1) (or (1.2)). The total entropy decreases after the interaction between two shadow waves.

**Proof** Suppose that two shadow wave interact at time \(t = T\). One from the left SDW\(_{l,m}\) propagates with speed \(u_{s_l}(t)\) and strength \(\xi_l(t)\), while the right one SDW\(_{m,r}\) propagates with speed \(u_{s_r}(t)\) and strength \(\xi_r(t)\). (The corresponding specific internal energies are denoted by \(e_{s_l}(t)\) and \(e_{s_r}(t)\).) The speed and the strength of the resulting shadow wave SDW\(_{l,r}\) are denoted by \(u_s(t)\) and \(\xi(t)\) (and the internal energy is \(e_s(t)\)). The total entropy at time \(t < T\) is given by \(E^-(t)\), while the total entropy at time \(t > T\) (across SDW\(_{l,r}\)) is \(E^+(t)\).

Consider first the system (1.1) and entropy pair (6.2). Then

\[
E^+(T + 0) - E^-(T - 0) = \frac{1}{2} \left( \frac{\xi(T) u_s^2(T) - \xi_l(T) u_{s_l}^2(T) - \xi_r(T) u_{s_r}^2(T)}{\xi(T)} \right)
\]

\[
= -\frac{1}{2} \frac{\xi_l(T) \xi_r(T)}{\xi(T)} (u_{s_l}(T) - u_{s_r}(T))^2 \leq 0.
\]

The above inequality follows from relations (3.7) and (3.8).

In the case of system (1.2) and entropy pair (6.1) we have

\[
E^+(T + 0) - E^-(T - 0) = \xi(T) \left( R(u_s(T)) - \frac{\xi_l(T)}{\xi(T)} R(u_{s_l}(T)) - \frac{\xi_r(T)}{\xi(T)} R(u_{s_r}(T)) \right)
+ S(u_s(T)) - \frac{\xi_l(T)}{\xi(T)} S(u_{s_l}(T)) - \frac{\xi_r(T)}{\xi(T)} S(u_{s_r}(T)) \right).
\]
Using the relation
\[ e_s(T) = \frac{\xi_l(T)}{\xi(T)} e_{s_l}(T) + \frac{\xi_r(T)}{\xi(T)} e_{s_r}(T) + \frac{1}{2} \frac{\xi_l(T)\xi_r(T)}{\xi^2(T)} (u_{s_l}(T) - u_{s_r}(T))^2 \]
which follows from the continuity of energy across the interaction time, and conditions imposed on functions \( R \) and \( S \), we get
\[
R(u_s(T)) \leq \frac{\xi_l(T)}{\xi(T)} R(u_{s_l}(T)) + \frac{\xi_r(T)}{\xi(T)} R(u_{s_r}(T)) \\
S(e_s(T)) \leq S\left(\frac{\xi_l(T)}{\xi(T)} e_{s_l}(T) + \frac{\xi_r(T)}{\xi(T)} e_{s_r}(T)\right) \\
\leq \frac{\xi_l(T)}{\xi(T)} S(u_{s_l}(T)) + \frac{\xi_r(T)}{\xi(T)} S(u_{s_r}(T)).
\]
Since \( \xi(T) > 0 \) we have \( E^+(T + 0) - E^-(T - 0) \leq 0. \)

\[ \square \]

**Remark 6.1** The interaction between shadow wave and contact discontinuity can be treated as a special case of Theorem 6.1. It is enough to take the strength of the wave corresponding to contact discontinuity equal to zero. Then the entropy is constant across the interaction time.

If the total entropy across SDW\(_{l,r}\) at time \( t \geq T_1 \) is denoted by \( E_{l,r}(t) \), the entropy rate is given by \( \frac{d}{dt} E_{l,r}(t) \) and the following relation holds
\[
\frac{d}{dt} E_{l,r}(t) = D_{l,r}(t) + Q(U_l) - Q(U_r).
\]  
(6.3)

For (1.1) and energy-entropy pair (6.2) we can explicitly calculate the energy production,
\[
D_{l,r}(t) = -\frac{1}{2} (\rho_l(u_l - u_s(t))^3 + \rho_r(u_s(t) - u_r))^3 =: E(t).
\]  
(6.4)

The condition \( D_{l,r}(t) \leq 0 \) means that the energy is dissipative. The value \( \frac{d}{dt} E_{l,r}(t) \) is called the energy dissipation rate across SDW\(_{l,r}\) at time \( t \geq T_1 \). Let
\[
A(t) := [\rho] u_s^2(t) - 2[\rho u] u_s(t) + [\rho u^2] = \rho_r(u_s(t) - u_r)^2 - \rho_l(u_l - u_s(t))^2.
\]

If \( \rho_l \neq \rho_r \), then \( u_s'(t) = -A(T_1) \frac{\gamma^2}{\xi^3(t)} \) and
\[
D_{l,r}'(t) = -\frac{3}{2} u_s'(t) A(t) - \frac{3}{2} \frac{\gamma^2}{\xi^3(t)} A(T_1) A(T_1).
\]

Using \( \xi(t) > 0 \) and the fact that \( u_s(T_1) \geq y_{l,r} (u_s(T_1) < y_{l,r}) \) implies \( u_s(t) \geq y_{l,r} (u_s(t) < y_{l,r}, \text{ respectively}) \) as proved in Lemma 3.1, we get \( D_{l,r}'(t) \geq 0, t > T_1 \). If \( \rho_l = \rho_r \neq 0 \), then
\[
u_s'(t) = -2 \frac{\gamma^2}{\xi^3(t)} \left( c - \frac{u_l + u_r}{2} \right) \rho_l(u_l - u_r),
\]
and \( D_{l,r}'(t) \geq 0 \). If \( \rho_l = \rho_r = 0 \), then \( D_{l,r}(t) = D_{l,r}'(t) = 0 \) for \( t > T_1 \). So, \( D_{l,r}(t) \) is non-positive and increasing function of time. For a contact discontinuity \( D_{l,r} \) is equal to 0 (energy is conserved). Also, it is constant for a simple shadow wave since \( u_s(t) \) does not depend on \( t \).
If $Y_r - Y_l < C \sqrt{\varepsilon}$ as needed in Theorem 5.3, then $D_{t,r}(t) = O(\varepsilon)$. That means that dissipation across a small shadow wave is negligible.

Consider an interaction between SDW$_{l,r}^i$ and CD$_i^1$ at $t = T_1$. The total energy production before the interaction equals

$$D_1(T_1 - 0) = -\frac{1}{2} \left( \rho_l (u_l - u_s(T_1 - 0))^3 + \rho_r (u_s(T_1 - 0) - u_r)^3 \right),$$

and

$$D_1(T_1 + 0) = \frac{1}{2} \rho_l (u_s(T_1 + 0) - u_l)^3$$

after it. The speed continuity, $u_s(T_1 - 0) = u_s(T_1 + 0)$ implies

$$\Delta D_1(T_1) := D_1(T_1 + 0) - D_1(T_1 - 0) = \frac{1}{2} \rho_r (u_s(T_1) - u_r)^3 > 0.$$ 

The resulting SDW$_{r}^r$ further interacts with CD$_{r+1}^i$ at time $t = T_2$. Then

$$D_2(T_2 - 0) = \frac{1}{2} \rho_l (u_s(T_2 - 0) - u_l)^3$$

and

$$D_2(T_2 + 0) = -\frac{1}{2} \left( \rho_l (u_l - u_s(T_2 + 0))^3 + \rho_{r+1} (u_s(T_2 + 0) - u_{r+1})^3 \right).$$

Thus,

$$\Delta D_2(T_2) = -\frac{1}{2} \rho_{r+1} (u_s(T_2) - u_{r+1})^3 < 0.$$ 

Let us consider an interaction between SDW$_{l,m}$ and SDW$_{m,r}$ now. The SDW$_{l,m}$ propagates with a speed $u_{s1}(t)$ and a strength $\xi_{s1}(t)$, while SDW$_{m,r}$ propagates with a speed $u_{s2}(t)$ and a strength $\xi_{s2}(t)$. The initial speed of the resulting SDW$_{l,r}$ equals $u_s := u_s(T) = \alpha u_{s1} + (1 - \alpha) u_{s2}$, where $\alpha := \frac{\xi_{s1}(T-0)}{\xi_{s1}(T-0) + \xi_{s2}(T-0)}$ and $u_{s1} := u_{s1}(T - 0), i = 1, 2$. Then

$$D(T - 0) = -\frac{1}{2} \left( \rho_m (u_{s1} - u_m)^3 + \rho_l (u_l - u_{s1})^3 + \rho_r (u_{s2} - u_r) + \rho_m (u_m - u_{s2})^3 \right)$$

$$D(T + 0) = -\frac{1}{2} \left( \rho_r (u_s - u_r)^3 + \rho_l (u_l - u_s)^3 \right)$$

$$\Delta D(T) = -\frac{1}{2} (u_{s1} - u_{s2}) \left( \alpha \rho_r ((u_s - u_r)^2 + (u_s - u_r) (u_{s2} - u_r) + (u_{s2} - u_r)^2) 

- \rho_m ((u_{s1} - u_m)^2 - (u_{s1} - u_m) (u_m - u_{s2}) + (u_m - u_{s2})^2) 

+ (1 - \alpha) \rho_l ((u_l - u_s)^2 + (u_l - u_s) (u_l - u_{s1}) + (u_l - u_{s1})^2) \right).$$

Note that the sign of $\Delta D(t)$ depends on $\rho(x)$ and $u(x)$.

**Example 6.1** Suppose that $u(x)$ is a decreasing function, $u_0 > u(R)$ and $\rho(x) = \rho_0$ for each $x > R$. A simple SDW$_{i,i+1}$ emanating at the $x$-axis propagates with speed $y_{i,i+1} = \frac{u_l + u_{i+1}}{2}$. 
for every \( i \). The result of an interaction at \( t = T \) between SDW\(_{i,i+1}\) and SDW\(_{i+1,i+2}\) is a new SDW\(_{i,i+2}\) with the constant speed and strength given by

\[
y_{i,i+2} = \frac{u_i + u_{i+2}}{2}, \quad \xi_{i,i+2} = \rho_0 (u_i - u_{i+2}) t, \quad t \geq T.
\]

It can be proved by an induction that a solution in this case is piecewise constant function, with the constant states connected by simple shadow waves, i.e. all jumps are located along straight lines. The energy production across SDW\(_{l,m}\) and SDW\(_{m,r}\) before and after their interaction at \( t = T \) is given by

\[
D(T - 0) = -\frac{\rho_l}{8} \left( (u_l - u_m)^3 + (u_m - u_r)^3 \right), \quad D(T + 0) = -\frac{\rho_0}{8} (u_l - u_r)^3.
\]

Thus, in this case the energy dissipation rate decreases after the interaction (that follows from (6.3)),

\[
\Delta D(T) = -\frac{3}{8} \rho_l (u_l - u_r)(u_l - u_m)(u_m - u_r) < 0.
\]

That is, the solution dissipates more energy after the interaction.

When the pressure vanishes the entropy relation for gases \( kT dS = dU + pdV \), where \( k \) is constant, \( T \) is a temperature, \( S \) is an entropy, \( U \) is an internal energy and \( V \) is a volume, reduces to \( S(e) = \text{const} \) for a fixed temperature. Thus, let us put \( \eta(\rho, u, e) = -\rho e \), \( Q(\rho, u, e) = -\rho u e \) for the system (1.2). Then the entropy production is given by

\[
D_{l,r}(t) = -u_s(t)(-\rho_r u_r + \rho_l u_l) + (-\rho_r u_r e_r + \rho_l u_l e_l) - \frac{d}{dt}(\xi(t)e_s(t))
\]

\[
= \rho_r e_r (u_s(t) - u_r) + \rho_l e_l (u_l - u_s(t)) - \frac{d}{dt}(\xi(t)e_s(t)).
\]

Combining (3.1) and (3.5) it can be easily proved that

\[
\frac{d}{dt}(\xi(t)e_s(t)) = -E(t) + \rho_r e_r (u_s(t) - u_r) + \rho_l e_l (u_l - u_s(t)),
\]

where \( E(t) \) is defined in (6.4). So, we have \( D_{l,r}(t) = E(t) \), as for the \( 2 \times 2 \) system and the previous analysis also holds for (1.2).

### 7 Existence of a Measure Valued Limit

A natural choice for a function space corresponding to our solution is the space of signed Radon measures due to the presence of delta function. Radon measures are Borel regular and locally finite measures, and can be understood as distributions of zero order.

We shall use the fact that for every signed measure \( M \) there exist unique nonnegative mutually singular measures \( M^+ \) and \( M^- \) such that \( M = M^+ - M^- \). Measures \( M^+ \) and \( M^- \) are called positive and negative variations of \( M \) and \( M = M^+ + M^- \) is Jordan decomposition of \( M \) (see [7] for details). The nonnegative measure \( |M| = M^+ + M^- \) is called variation of \( M \). The Riesz’s representation theorem gives the following characterization of the space of signed Radon measures whose positive and negative variations are Radon measures.
Definition 7.1 A space of signed Radon measures $\mathcal{M}(\Omega)$ consists of linear forms $M$ defined on $C_0(\Omega)$ such that for every compact set $K \subset \Omega$ there exists a constant $C_K$ such that

$$|\langle M, \varphi \rangle| \leq C_K \|\varphi\|_{L^\infty} \text{ for all } \varphi \in C_0(\Omega), \ \text{supp} (\varphi) \subset K.$$ 

Denote by $\mathcal{M}_f (\Omega)$ the space of signed Radon measures with a finite mass, i.e. $M \in \mathcal{M}_f (\Omega)$ if there exist a constant $C$ such that

$$|\langle M, \varphi \rangle| \leq C \|\varphi\|_{L^\infty} \text{ for all } \varphi \in C_0(\Omega), \ \text{supp} \varphi \subset K.$$ 

Proposition 7.1 (Proposition 2.5. from [11]) Let $\{M_\nu\}_{\nu \in \mathbb{N}_0}$ be a sequence of nonnegative uniformly locally bounded measures. Then there exists its subsequence still denoted by $\{M_\nu\}_{\nu \in \mathbb{N}_0}$ and a Radon measure $M$ such that $M_\nu \rightharpoonup M$.

The following existence theorem is a continuation of the above construction. We believe that can be extended to other systems having SDW solutions. One will see below that it is not optimal for system (1.1, 2.1) due to a lack of uniqueness.

Theorem 7.1 (Existence of a weak limit) Suppose that $u(x), \rho(x) \in C_b([R, \infty), \rho(x) > 0, \rho_0 > 0$ and $u(x)$ having a finite number of local extremes. Take any sequence $\{\varepsilon_\nu\}_{\nu \in \mathbb{N}_0}$, $\varepsilon_\nu \to 0+$ satisfying $\sqrt[3]{\varepsilon_\nu} \leq Y_i - Y_{i-1} \leq C \sqrt[3]{\varepsilon_\nu}, C \geq 1$ for partition $\{Y_\nu^i\}_{i \in \mathbb{N}_0}$ corresponding to $\varepsilon_\nu$. Denote $\{U^\nu\}_{\nu \in \mathbb{N}_0}$ a corresponding sequence of solutions to problem (1.1, 2.1) constructed as in Theorem 5.3. There exists a subsequence still denoted by $\{U^\nu\}_{\nu \in \mathbb{N}_0}$ and a signed Radon measure $U^*$ such that $U^\nu$ converges weakly to $U^*$ as $\nu \to \infty$.

To prove the existence of a limit $U^*$ we have to show that the components of $|U^\nu| := (\rho^\nu, |u^\nu|)$ are uniformly locally bounded measures for each $\nu \in \mathbb{N}_0$. Note that $|\rho^\nu| = \rho^\nu$ since $\rho^\nu$ is nonnegative. The proof will rely on three lemmas given in the sequel.

Remark 7.1 Note that we will not emphasize that $U^\nu$, as well as $U^*$ are vector-valued measures since one can easily distinguish vector from scalar valued measures.

Lemma 7.1 (Finite propagation speed) Suppose that $\rho(x)$ and $u(x)$ are continuous and bounded functions and $u(x)$ has a finite number of local extremes. Then a speed of any wave which is part of an admissible solution to problem (1.1, 2.1) is bounded.

The proof of the above Lemma is straightforward. Each shadow wave is overcompressive, while a speed of each contact discontinuity is constant that equals to a value of $u(x)$ at some point $x > R$. Thus, the propagation speed is between $\min \{u_0, \inf_{x \geq R} u(x)\}$ and $\max \{u_0, \sup_{x \geq R} u(x)\}$.

Lemma 7.2 Let $U^\nu$ be the admissible solution to (1.1, 2.1), with $u(x)$ and $\rho(x)$ satisfying the assumptions from the previous lemma. Then

$$\inf_{i \in \mathbb{N}_0} \xi_i t_{|t=0} < \xi(t) < \sup_{i \in \mathbb{N}_0} \xi_i t_{|t=0} + \bar{\rho} \left( \max_{x \geq R} \left\{ u_0, \sup_{x \geq R} u(x) \right\} - \min_{x \geq R} \left\{ u_0, \inf_{x \geq R} u(x) \right\} \right) t,$$

for some $\bar{\rho}$, where $\xi_i t$ is the strength of $i$-th wave emerging at the initial time, $i \in \mathbb{N}_0$. 

\[ \text{Springer} \]
Proof Define \( \tilde{\rho} := \max \left\{ \rho_0, \sup_{x > R} \rho(x) \right\} \). Let \( c_l \) (or \( c_r \)) and \( \sigma_l \) (or \( \sigma_l \)) be a speed and a strength of an incoming wave from left (or right). Let \( t = T \) be a time of the interaction. Then, the initial speed and the strength of the resulting wave are
\[
c := u_r(T + 0) = \frac{\sigma_l c_l + \sigma_r c_r}{\sigma_l + \sigma_r}, \quad \sigma := \xi(T + 0) = \sigma_l + \sigma_r.
\]
The global bounds for a strength of any wave propagating at time \( t \) follow from estimate (3.6)_2.

\[\square\]

**Lemma 7.3** Suppose that all assumptions of Theorem 7.1 hold. Denote by \( \{U^v\}_{v \in \mathbb{N}_0} \) the sequence defined in that theorem. Then \( \rho^v \) and \( |u^v| \) are (nonnegative) uniformly locally bounded measures for each \( v \in \mathbb{N}_0 \).

**Proof** Due to construction of the solution, boundedness of \( u(x) \) and Lemma 7.1 we have that \( u^v \) is uniformly globally bounded function for \( v \in \mathbb{N}_0 \). In order to prove that \( \rho^v \) is uniformly \( L^1 \)-bounded for \( v \in \mathbb{N}_0 \), we will use the conservation of mass principle, boundedness of \( \rho(x) \) and the finite propagation speed property. For each \( E \subset \mathbb{R} \) there exists a \( C_E > 0 \) such that
\[
0 \leq \int_{E \times (t_0, T)} \rho^v(x, t) \, dx \, dt \leq (T - t_0) \cdot C_E \sup_{x \in \mathbb{R}} \rho(x, 0) < \infty.
\]
Thus, \( |u^v| \) and \( \rho^v \) are bounded in \( L^1(K) \) for every compact set \( K \subset \mathbb{R}^2_+ \), i.e. \( |U^v| \) is uniformly locally bounded measure.

\[\square\]

**Proof of Theorem 7.1** Due to Lemma 7.3 we know that \( \rho^v \) and \( |u^v| \) are (nonnegative) uniformly locally bounded measures. Thus, there exist uniformly locally bounded measures \( U^v_+ \) and \( U^v_- \) such that \( U^v = U^v_+ - U^v_- \) and \( |U^v| = U^v_+ + U^v_- \). From Proposition 7.1 it follows that there exist subsequences \( \{U^v_+\}_{v \in \mathbb{N}_0}, \{U^v_-\}_{v \in \mathbb{N}_0} \) and locally finite measures \( U^*_+, U^*_- \) such that \( U^v_+ \rightharpoonup_* U^*_+ \) and \( U^v_- \rightharpoonup_* U^*_- \). Thus, \( U^v \) converges weakly to \( U^* := U^*_+ - U^*_- \). Note that one can also use Proposition 7.1 directly to obtain the subsequence \( \{|U^v|\}_{v \in \mathbb{N}_0} \) that converges weakly to \( |U^*| \).

In the case of system (1.1) there is a unique solution in the space of Borel measures for a wide class of measure initial data. The initial data (2.1) satisfy the conditions of Theorem 2.8 from [1], \( \partial_t u \leq 1/t \) by the construction, so the following theorem holds.

**Theorem 7.2** (Special case of Theorem 2.8 in [1]) There is a unique solution to (1.1, 2.1) in the space of Borel measures.

The proof of Theorem 2.8 is based on the sticky particles procedure from [3]. For the proof of the Theorem we will find a correspondence between the procedure in that paper and the shadow wave tracking one, even though the analysis of mass distribution shows differences between them. The following lemma proves that approximate solution with finite mass constructed in this paper satisfies the scalar conservation law used in [3] to recover a solution to pressureless gas dynamics.

**Lemma 7.4** Consider the initial data problem (1.1, 4.2) with \( u(x) \) and \( \rho(x) \) being bounded and piecewise continuous on \( \mathbb{R} \), and suppose \( \rho(x) = 0 \) for \( x \notin [-L, L], L > 0 \). Denote by
$U^\mu = (\rho^\mu, u^\mu)$ the approximate admissible solution to that problem obtained in Theorem 5.3. Take a partition $\{Y_i\}_{i \in \mathbb{N}}$ such that $Y_i - Y_{i-1} < \mu$ for each $i$, with each shadow wave defined by a parameter $\varepsilon \ll \mu$ and let $\varepsilon \to 0$ in $U^\mu$. Define

$$M^\mu(x,t) = \int_{-\infty}^{x} \rho^\mu(y,t) \, dy, \quad M_0^\mu(x) = \int_{-\infty}^{x} \rho(y,0) \, dy,$$

$$A^\mu(z) = \int_{0}^{z} g(y,t) \, dy,$$

where $g(M^\mu(x,t),t) := u^\mu(x,t)$. The function $M^\mu$ is the unique entropy solution of the scalar conservation law

$$\partial_t (M^\mu(x,t)) + \partial_x \left( A^\mu(M^\mu(x,t)) \right) = 0.$$

**Proof** Fix $t > 0$. In order to prove the assertion it is enough to prove that Rankine-Hugoniot condition is satisfied across any shadow wave that forms $U^\mu$. Take arbitrary shadow wave $U^\varepsilon$ existing at time $t$ with the distributional limit $U_\varepsilon(x,t) := \lim_{\varepsilon \to 0} U^\varepsilon(t,x) = \begin{cases} U_l, & x < c(t) \\ U_r, & x > c(t) \end{cases} + (\xi(t),0)\delta(x - c(t)), \ u_\varepsilon(t) = c'(t)$.

By definition of shadow wave, jump of mass across $U^\varepsilon$ equals

$$M^\mu(c(t)+,t) - M^\mu(c(t)-,t) = \xi(t),$$

while

$$A^\mu(M^\mu(c(t)+),t) - A^\mu(M^\mu(c(t)-),t) = (M^\mu(c(t)+,t) - M^\mu(c(t)-,t))u_\varepsilon(t),$$

since $g(M^\mu(c(t),t),t) = u_\varepsilon(t)$. □

The above lemma shows that the shadow wave tracking solution $U^\varepsilon$ from Theorem 5.3 tends to the unique solution given in Theorem 7.2 by letting $\varepsilon \to 0$ and $\mu \to 0$ afterwards.

### 7.1 The Explicit Limit

In certain cases, it is possible to find an explicit form of a measure–valued limit $U^*$ at least for some small time interval as one can see in the following theorem.

**Theorem 7.3** Suppose that all the assumptions of Theorem 7.1 hold, as well as the notation. Let $u_0 > u(R)$. There exists $T_{\text{max}} > 0$ such that $U^*$ is the weighted $\delta$ measure supported by a curve $\Gamma: \ x = c(t)$ that connects $U_0$ from the left and a classical solution $U(x,t)$ to (1.1) to the right in the strip $t < T_{\text{max}}$. The life-span $T_{\text{max}}$ is a positive infimum of $-\frac{1}{u'(x)}$, $x > R$ such that $D_\chi := \left( x - \frac{u(x)}{u'(x)}, -\frac{1}{u'(x)} \right)$ lies above the curve $\Gamma$.

**Remark 7.2** Theorem 7.3 holds for $u_0 \leq u(R)$ and increasing $u(x)$ too. That is a trivial case since a solution converges to a smooth solution obtained by the method of characteristics.
Proof Let $T > 0$ be arbitrary but fixed. First, we will show that $\hat{U}^v$ has a subsequence that converges. It is bounded in $L^1_{\text{loc}}(\mathbb{R}^2_+)$ uniformly for $v \in \mathbb{N}_0$ by Lemmas 7.1 and 7.2. Therefore, it has a subsequence that converges to some $\hat{U}^* \in \mathcal{M}(\mathbb{R}^2_+)$. From the construction, it is obvious that its support is the curve $\Gamma$.

On the other hand, a part of $U^v$ lying to the right of $\hat{U}^v$ converges to a classical solution $U$ obtained by method of characteristics as long the classical solution exists. Let us show that.

Suppose that $u(x)$ is increasing. The procedure from Sect. 4 gives the admissible solution $U^v = (\rho^v, u^v)$ to (1.1) consisting of a sequence of contact discontinuities connected by a vacuum state. The classical initial value problem $(\rho, u)|_{t=0} = (\rho(x), u(x))$ can be solved by method of characteristics. For smooth solutions and away from vacuum state one gets the Burgers equation $\partial_t u + u \partial_x u = 0$. Its characteristics are integral curves of ordinary differential equation $\frac{dx}{dt} = u(x(t), t)$ and a solution is given by

$$u(x, t) = u(\psi(x, t)),$$

where a function $\psi = \psi(x, t)$ satisfies $x = u(\psi)t + \psi$. The existence of function $\psi$ for each $t > 0$ and in the region where $u(x)$ is strictly increasing follows from the Implicit Function Theorem. From the first equation in (1.1), one can see that $\rho$ satisfies the equation $\partial_t \rho + u \partial_x \rho = -\rho \partial_u u$. That is,

$$\rho(x, t) = \rho(\psi(x, t)) \exp\left(-\int_0^t \frac{u'(\psi(x, s))}{u'(\psi(x, s))} ds + 1\right) \in C^1.$$

The solution $(\rho, u)$ corresponding to the region where $u(x)$ is constant is also constant.

For each interval $[X_-, X_+]$ and time $T > 0$, let us show that

$$I_v := \int_{X_-}^{X_+} \rho^v(x, T) \, dx \rightarrow \int_{X_-}^{X_+} \rho(x, T) \, dx \quad \text{as} \quad v \rightarrow \infty,$$

$$\int_{X_-}^{X_+} u^v(x, T) \, dx \rightarrow \int_{X_-}^{X_+} u(x, T) \, dx \quad \text{as} \quad v \rightarrow \infty.$$ (7.1)

For any $v \in \mathbb{N}_0$, let $U^v$ be the solution constructed by using the partition $\{Y_i\}_{i \in \mathbb{Z}}$ such that $Y_i - Y_{i-1} < C(\sqrt{\varepsilon_v})$, $C \geq 1$, $i \in \mathbb{Z}$. (To simplify notation we will drop superscript $v$ in $\{Y_i\}_{i \in \mathbb{Z}}$.) There exist $Y_-, Y_+$ such that $X_- = Y_- + u(Y_-)T$ and $X_+ = Y_+ + u(Y_+)T$. Suppose that $Y_\in \in (Y_{l-1}, Y_l)$, $Y_+ \in [Y_m, Y_{m+1})$ for some $l, m \in \mathbb{Z}$. Denote by $X_{0,i} := Y_l + u(Y_l)T$, $X_{1,i} := Y_l + u(Y_{l+1})T$, $i \in \mathbb{Z}$. The function $u^v(x, t)$ is a good approximation of $u(x, t)$ since it is uniquely determined in non-vacuum part and its value in vacuum part is continuously interpolated. We will use the conservation of mass to prove (7.1)$_l$. Note that $\rho^v(x, T) = 0$, $x \in (X_{0,i}, X_{1,i})$ and

$$I_v = \int_{X_-}^{X_+} \rho^v(x, T) \, dx = \int_{X_{0,i}}^{X_{1,m}} \rho^v(x, T) \, dx + \int_{X_{0,i}}^{X_{1,m}} \rho^v(x, T) \, dx + \int_{X_{0,i}}^{X_{1,m}} \rho^v(x, T) \, dx$$

$$= \sum_{i=1}^{m-1} \rho_{i+1}(X_{0,i+1} - X_{1,i}) + O(\sqrt{\varepsilon_v}) = \sum_{i=1}^{m-1} \rho(Y_{i+1}) (Y_{i+1} - Y_i) + O(\sqrt{\varepsilon_v})$$

$$\approx \int_{Y_l}^{Y_m} \rho(x) \, dx \rightarrow \int_{Y_1}^{Y_+} \rho(x) \, dx \quad \text{as} \quad v \rightarrow \infty.$$
We have used that $\rho(x)$ and $u(x)$ together with their first derivatives are bounded in order to get that $I' = O(1/\varepsilon)$. Due to the mass conservation and the fact that flow maps $[Y_-, Y_+]$ to $[X_-, X_+]$ we have

$$\mathcal{M}([Y_-, Y_+]) := \int_{Y_-}^{Y_+} \rho(x, 0) \, dx = \int_{X_-}^{X_+} \rho(x, T) \, dx =: \mathcal{M}([X_-, X_+]),$$

and (7.1) is proved. A value of $T_{\text{max}}$ is arbitrary here.

Next, suppose that $u(x)$ is decreasing. The solution $U^\nu$ consists of shadow waves separating constant states in the beginning. The first interaction occurs in a non-negligible time (see (5.2) and the analysis there), since $Y_{t-1} - Y_t < C \sqrt{\varepsilon}$ for each $i$. A classical solution to (1.1, 4.2) with decreasing $u(x)$ exists only until some time $T_{\text{max}}$ when the first pair of characteristics intersect. That is, shadow waves intersect approximately at the same time as nearby characteristics.

Let $T < T_{\text{max}}$. Take an interval $[X_-, X_+]$, where $X_+ = Y_+ + u(Y_+) T, * \in \{+, -\}$. It is clear that $[Y_-, Y_+]$ maps to $[X_-, X_+]$ when $t = T$, so (7.1) follows. Suppose that $Y_- \in (Y_{t-1}, Y_t)$, $Y_+ \in (Y_m, Y_{m+1})$ for some $l, m \in \mathbb{Z}$ as in the previous case. Denote $X_i := Y_i + y_i,i+1 T$. Then

$$S_k := \int_{X_k}^{X_{k+1}} \rho^\nu(x, T) \, dx = \frac{1}{2} \xi_{k-1/k} T + \int_{X_k + \xi_{k/k} T}^{X_{k+1} - \xi_{k/k} T} \rho_{k+1} \, dx + \frac{1}{2} \xi_{k,k+1} T$$

$$= \frac{1}{2} (\xi_{k-1/k} + \xi_{k,k+1}) T + \rho_{k+1} (X_{k+1} - X_k) - \varepsilon T \rho_{k+1}$$

$$= \frac{1}{2} (\xi_{k-1/k} + \xi_{k,k+1}) T + \rho_{k+1} (Y_{k+1} - Y_k) + \rho_{k+1} (y_{k,k+1} - y_{k-1,k}) T - \varepsilon T \rho_{k+1},$$

since

$$\xi_{k,k+1} T = \sqrt{\rho_k \rho_{k+1} (u_k - u_{k+1}) T} = \lim_{\nu \to \infty} \int_{X_k + \xi_{k/k} T}^{X_{k+1} - \xi_{k/k} T} \rho^\nu(x, T) \, dx.$$

Using (5.2) with $i = k - 1, j = k + 1$ and $\rho_{k+1} = \rho_k + O(1/\varepsilon)$,

$$y_{k,k+1} - y_{k,k-1} = -\frac{1}{2} (u_{k-1} - u_{k+1}) + O(\sqrt{\varepsilon}^2). \tag{7.2}$$

Boundedness of $\rho(x)$ implies $\sqrt{\rho_k \rho_{k+1}} = \rho_{k+1} + \frac{1}{2} (\rho_k - \rho_{k+1}) + O(\sqrt{\varepsilon}^2)$, and $\sqrt{\rho_{k-1} \rho_k} = \rho_{k+1} + \frac{1}{2} (\rho_{k-1} + \rho_k - 2 \rho_{k+1}) + O(\sqrt{\varepsilon}^2)$. Together with (7.2), it implies

$$\beta_k := \frac{1}{2} (\xi_{k-1,k} + \xi_{k,k+1}) + \rho_{k+1} (y_{k,k+1} - y_{k-1,k})$$

$$= \frac{1}{2} \rho_{k+1} (u_{k-1} - u_k) + \frac{1}{2} \rho_{k+1} (u_k - u_{k+1}) - \frac{1}{2} \rho_{k+1} (u_{k-1} - u_{k+1}) + O(\sqrt{\varepsilon}^2)$$

$$+ \frac{1}{4} ((\rho_{k-1} + \rho_k - 2 \rho_{k+1})(u_{k-1} - u_k) + (\rho_k - \rho_{k+1})(u_k - u_{k+1})) = O(\sqrt{\varepsilon}^2). \quad \sim \frac{3}{\sqrt{\varepsilon}^2}$$
Thus,

\[ I_v := \int_{X^-}^{X^+} \rho^v(x, T) \, dx \]

\[ = \sum_{k=l}^{m-1} S_k + \rho_l(X_l - X_-) + \frac{1}{2} (\xi_{l,l+1} + \xi_{m,m+1}) T + \rho_{m+1}(X_+ - X_m) \]

\[ = \sum_{k=l}^{m-1} \rho_{k+1}(Y_{k+1} - Y_k) + \rho_l(Y_l - Y_-) + \rho_{m+1}(Y_+ - Y_m) + T \sum_{k=l}^{m-1} \beta_k \]

\[ + \left( \frac{1}{2} (\xi_{l,l+1} + \xi_{m,m+1}) - \rho_l(u(Y_-) - y_{l,l+1}) - \rho_{m+1}(y_{m,m+1} - u(Y_+)) \right) T \]

\[ - \varepsilon_v T \sum_{k=l}^{m-1} \rho_{k+1}, \ v \to \infty. \]

The above sum \( \sum_{k=l}^{m-1} \rho_{k+1} \) has \( O(1/\sqrt{\varepsilon_v}) \) globally bounded elements due to the assumption \( \sqrt{\varepsilon_v} \leq Y_l - Y_{l-1} \leq C \sqrt{\varepsilon_v} \) from Theorem 7.1. Thus, it is bounded from above by \( \text{const} \cdot \sqrt{\varepsilon_v} \).

Then

\[ \varepsilon_v T \sum_{k=l}^{r-1} \rho_{k+1} \leq \sqrt{\varepsilon_v} T \sum_{k=l}^{r-1} \rho_{k+1}(Y_{k+1} - Y_k) \to 0 \text{ as } v \to \infty, \]

since \( \rho(x) \) is bounded. Therefore,

\[ I_v \to \int_{Y_-}^{Y_+} \rho(x) \, dx = M([Y_-, Y_+]) = M([X_-, X_+]) \text{ as } v \to \infty. \]

The limit \( U^* \) for \( t < T_{\text{max}} \) is the weighted delta measure \( \hat{U}_a \) connecting \( (\rho_0, u_0) \) and the classical solutions obtained by the above procedure. The life-span \( T_{\text{max}} \) is determined by the fact that we can use the above arguments as long as the classical solution exists below \( \Gamma \). That is, as long as characteristics intersect above it. For a neighborhood of a point \( x > R \) their intersection is at the point around \( D_x \) and the assertion follows.

The case when \( u(x) \) changes monotonicity finitely many times reduces to combining these two cases. \( \square \)

**Remark 7.3** The life–span \( T_{\text{max}} \) equals infinity if \( u(x) \) is increasing or if \( u'(x) \leq 0 \) with small enough absolute value. For a finite \( T_{\text{max}} \) we do not know what is distributional limit of solution for \( t \geq T_{\text{max}} \), but a solution becomes a single delta shock connecting \( (\rho(R), u(R)) \) and \( (\rho(\infty), u(\infty)) \) for \( t \gg 1 \).

**Remark 7.4** Again, the above result is easily extended to system (1.2) with the additional energy variable, so all the assertions in this section hold for that system, too. Smooth energy component solves the equation \( \partial_t e + u \partial_x e = 0 \).
7.2 Partitions of Equidistant Type

Proofs of Theorems 7.1 and 7.3 are based on the compactness argument without any information about a uniqueness of the limit. We shall now prove that the limit $U^*$ given in Theorem 7.3 is unique at least for $t < T_{\text{max}}$ if partitions of the interval $[R, \infty)$ satisfy the equidistant property: Take $\epsilon$ small enough and define a family of partitions $\{P^\nu\}_{\nu \in \mathbb{N}_0}$ in the following way. If $P^\nu = \{Y^\nu_i\}_{i \in \mathbb{N}_0}$, then $P^{\nu+1} = P^\nu \cup \{Y^\nu_{i+1}\}_{i \in \mathbb{N}_0}$, where $\sqrt{\epsilon} \leq Y^{0}_{k+1} - Y^{0}_{k} \leq C \sqrt{\epsilon}$ for each $k$ and some constant $C \geq 1$. If $\frac{3\gamma}{2\nu} \leq Y^{0}_{k+1} - Y^{0}_{k} \leq C \frac{3\gamma}{2\nu} =: \mu_\nu$ for every $k \in \mathbb{N}_0$ and $\nu \in \mathbb{N}_0$, the family is said to have the equidistant property. For each partition $P^\nu$ a corresponding $U^\nu$ is defined in Theorem 7.1 for $\epsilon_\nu = \epsilon / 3^\nu$. Denote by $\Gamma^\nu : x = c^\nu(t)$ the 0-SDW curve in $U^\nu$.

**Assumption 7.1** Suppose that $u(x)$ and $\rho(x) > 0$ are continuous and bounded together with their first derivatives, and $u(x)$ has a finite number of local extremes. The values $u_0 > \sup_{x \geq R} u(x)$ and $\rho_0 > 0$ are chosen such that the minimum distance between a slope of the curve $\Gamma^\nu$ and $u(c^\nu(t), t)$ is uniformly greater than zero.

We want to show the uniqueness of the limit $U^*$ for sequences $\{U^\nu\}_{\nu \in \mathbb{N}_0}$ defined by partitions of equidistant type. It suffices to show that the curve $\Gamma$ from Theorem 7.3 is unique since it connects $U_0$ and the unique classical solution $U(x, t)$.

**Theorem 7.4** If Assumption 7.1 holds, then a sequence $\{U^\nu\}_{\nu \in \mathbb{N}_0}$ defined by partitions $\{P^\nu\}_{\nu \in \mathbb{N}_0}$ of equidistant type converges to the unique bounded measure $U^*$ in $\mathbb{R} \times (0, T_{\text{max}})$ as $\nu \to \infty$.

**Proof** Let $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$, then there exists $\tau_0, 0 < \tau_0 < T_{\text{max}}$, independent of $\nu \in \mathbb{N}_0$ such that $\tau(s) \geq \tau_0$. Our aim is to prove that $\Gamma^\nu \to \Gamma$ as $\nu \to \infty$ in the strip $0 < t < T_{\text{max}}$. Suppose that $c^\nu(t_0) = c(t_0)$ and $\gamma_0 \leq \xi^\nu(t_0)$ independently of a partition. Without loss of generality, assume that $t = t_0$ is interaction time between 0-SDW and a contact discontinuity (or a shadow wave). That is, for each $\nu$ there exists some $Y_i^\nu \in P^\nu$ such that $Y_i^\nu + u(Y_i^\nu) t_0 = c^\nu(t_0)$ (or $Y_{i+1}^\nu + y_{i+1,0} t = c^\nu(t_0)$), where $y_{i+1,0}$ from (3.4) corresponds to the states $u_i^\nu, u_{i+1}^\nu$ in $P^\nu$. This may not be true in general, but a difference would be negligible. The compactness of a test function support permits us to take a points $Y_i^0 > R$ and $T > 0$ as a boundary of the limit analysis.

For simplicity, we shall suppose first that all partitions $P^\nu$ are equidistant, i.e. $Y_{k+1}^\nu - Y_k^\nu = \mu_\nu$ for each $k \in \mathbb{N}_0$ and $\nu \in \mathbb{N}_0$. The proof for partitions with equidistant property differs only in technical details. Denote

$$M = \max\{\rho_0, \sup_{x \geq R} \rho(x)\}, \quad A = \max\{u_0, \sup_{x \geq R} u(x)\} - \min\{u_0, \inf_{x \geq R} u(x)\}.$$

The following estimates will be used below. The Taylor expansion formula implies

$$\xi(t) = \begin{cases} 
\gamma + (c[\rho] - [\rho u])t + \frac{\rho_1 \rho [a] (c[\rho] - [\rho u])}{2\gamma} t^2 + O(t^3), & \rho_1 \neq \rho_r \\
\gamma + \rho_1 (u_l - u_r)t, & \rho_1 = \rho_r = 0
\end{cases}$$

and

$$u_x(t) = \begin{cases} 
c + \frac{\rho_1 \rho [a] (c[\rho] - [\rho u])}{\gamma [\rho]} t + O(t^2), & \rho_1 \neq \rho_r \\
c - \frac{2}{\gamma} \rho_1 (u_l - u_r) (c - \frac{u_r + u_l}{2}) t + O(t^2), & \rho_1 = \rho_r \neq 0
\end{cases}$$
Thus, solution to the system of equations $x = c(t)$, $c(0) = X$ is approximated by
\[
c(t) = \begin{cases} 
X + ct + \frac{\rho \rho_r [u^2 - (c[\rho] - [\rho u])]}{2 \gamma |\rho|} t^2 + \mathcal{O}(t^3), & \rho_l \neq \rho_r \\
X + ct - \frac{1}{\gamma} \rho_l (u_l - u_r) (c - \frac{u_r + u_l}{2}) t^2 + \mathcal{O}(t^3), & \rho_l = \rho_r \neq 0.
\end{cases}
\]

Also,
\[
\rho \rho_r [u^2 - (c[\rho] - [\rho u])^2] = -[\rho]^2 (c - y_{l,r})(c - z_{l,r}),
\]
where $y_{l,r}$ and $z_{l,r}$ are defined in (3.4). The overcompressibility consequences are the following estimates
\[
0 < [\rho] (c - z_{l,i}) \leq 2 \max \{\rho_l, \rho_r\} (u_l - u_r) \leq 2MA, \quad |c - y_{l,r}| < (u_l - u_r) \leq A,
\]
with constants $A$ and $M$ independent of a partition. Finally, we have the global estimates
\[
|\xi(t) - \gamma - (c[\rho] - [\rho u])t| \leq MC_\gamma t^2, \quad |c(t) - X - ct| \leq C_\gamma t^2,
\]
and $|u_i(t) - c| \leq 2C_\gamma t$, where $C_\gamma := \frac{MA^2}{\gamma}$.

Now, let $u(x)$ be an increasing function. Take a partition $\mathcal{P}^0 = \{Y_k\}_{k \in \mathbb{N}_0}$ and its subpartition $\mathcal{P}^1 = \mathcal{P}^0 \cup \{Y_{k+\frac{1}{2}}\}_{k \in \mathbb{N}_0}$, where $Y_{k+\frac{1}{2}} = Y_k + \frac{1}{2} t$, $k \in \mathbb{N}_0$. Denote by $(X_{0,j}, T_{0,j})$ the point where 0-SDW supported by $\Gamma^1$ meets the contact discontinuity line $x = Y_j + u_{j,t}$. Denote by $(X_{1,j}, T_{1,j})$ the intersection point between $\Gamma^0$ and the second contact discontinuity line $x = X_j + u_{j,t}$ from $(Y_j, 0)$. The intersection points between $\Gamma^1$ and the first and the second contact discontinuity that originate from $(Y_j, 0)$ are denoted by $(X_{1,j}, T_{0,j})$ and $(X_{1,j}, T_{1,j})$, respectively. Note that in that case we also have contact discontinuities originating from the points $(Y_{k+\frac{1}{2}}, 0)$. That produces the new interaction points $(X_{m,k+\frac{1}{2}}, T_{m,k+\frac{1}{2}})$, $m = 0, 1$.

Using the above assumptions, we have $X_{0,i} = X_{0,i,v}$, $T_{0,i} = T_{0,i,v} = \tau_0$ for each $v \in \mathbb{N}_0$. Define
\[
\gamma_{k,j} = \xi(T_{k,j}), \quad c_{k,j} = u_k(T_{k,j}), \quad X_{k,j} = \gamma(T_{k,j}), \quad k = 0, 1, \quad j = i, i + 1,
\]
\[
\gamma_{1,j} = \xi(T_{1,j}), \quad c_{1,j} = u_k(T_{1,j}), \quad X_{1,j} = \gamma(T_{1,j}), \quad k = 0, 1, \quad j = i, i + 1, i + \frac{1}{2}, i + 1.
\]

From $X_{1,i} = c(T_{1,i}) = Y_i + u_{i+1} T_{1,i}$, $c(T_{0,i}) = X_{0,i} = Y_i + u_i T_{0,i}$, one easily finds $T_{1,i} = T_{0,i} + \tau_1 + \mathcal{O}(\tau_1^2)$, and $\tau_1 := \frac{u_{i+1} - u_i}{c_{0,i} - u_i} T_{0,i}$. Assumption 7.1 implies that there exists an $\alpha > 0$ such that $c_{0,i} - u_{i+1} > \alpha$, $i \in \mathbb{N}_0$. Then, $\mathcal{O}(\tau_1^2) = \mathcal{O}(\mu_0^2)$ since $u_{i+1} - u_i = \mathcal{O}(\mu_0)$. Note that $\mu_0 = \sqrt{\varepsilon}$. The estimate
\[
|X_{1,i} - (X_{0,i} + c_{0,i} \tau_i)| \leq C_{\gamma_0} \tau_1^2 < C_{\gamma_0} \frac{B_3^2 \tilde{T}^2 \mu_0^2}{\alpha^2},
\]
with $B_3 := \sup_{x \in \mathbb{R}} |u'(x)|$ follows from (7.3). The new interaction point $(X_{0,i+1}, T_{0,i+1})$ is a solution to the system of equations
\[
X_{0,i+1} = c(T_{0,i+1}) = Y_{i+1} + u_{i+1} T_{0,i+1}, \quad c(T_{1,i}) = X_{1,i}.
\]

Thus, $T_{0,i+1} = T_{1,i} + \tau_2 + \mathcal{O}(\mu_0^2)$, where $\tau_2 := \frac{Y_{i+1} - Y_i}{c_{1,i} - u_i} = \frac{\mu_0}{c_{1,i} - u_i}$. Note that $c_{1,i} - u_{i+1} > c_{0,i} - u_i + 1 > \alpha$ due to the fact that the speed of shadow wave is increasing in vacuum area,
and we have
\[
|X_{0,i+1} - (X_{1,i} - c_{1,i} \tau_2)| \leq C_0 \tau_2^2 < \frac{C_{\gamma_0} \mu_0^2}{\alpha^2}.
\]

Let us now consider the partition \( P^1 \). Denote \( \tau_1^1 := T_{0,i} \frac{u_{i+1} - u_i}{c_{0,i} - u_{i+1}^2}, \tau_2^1 := \frac{Y_i - Y_{i+1}}{c_{1,i} - u_{i+1}^2}, \tau_3^1 := T_{0,i+1} \frac{u_i - u_{i+1}}{c_{1,i} - u_{i+1}^2} \). In the same way as for \( P^0 \) we have
\[
T^1_{0,i+1} = T_{0,i} + \tau_1^1 + \tau_2^1 + \tau_3^1 + \tau_4^1 + O\left(\frac{\mu_0^2}{2}\right),
\]
\[
X^1_{0,i+1} = X_{0,i} + c_{0,i} \tau_1^1 + c^1_{1,i} \tau_2^1 + c^1_{0,i+1} \tau_3^1 + c^1_{1,i+1} \tau_4^1 + O\left(\frac{\mu_0^2}{2}\right),
\]
where
\[
|X^1_{1,i} - (X_{0,i} + c_{0,i} \tau_1^1)|, |X^1_{1,i+1/2} - (X^1_{0,i+1} + c_{0,i+1} \tau_3^1)| < \frac{C_{\gamma_0} B^2 \tilde{T}^2 \mu_0^2}{\alpha^2},
\]
\[
|X^1_{0,i+1/2} - (X^1_{1,i} + c_{1,i} \tau_2^1)|, |X^1_{0,i+1} - (X^1_{1,i+1/2} + c_{1,i+1} \tau_4^1)| < \frac{C_{\gamma_0} \mu_0^2}{\alpha^2}.
\]

There exist positive constants \( C_0 \) and \( C_1 \) such that
\[
|\tau_1 - (\tau_1^1 + \tau_3^1)| \leq C_0 \frac{\mu_0^2}{2}, |\tau_2 - (\tau_2^1 + \tau_4^1)| \leq C_1 \frac{\mu_0^2}{2}. \tag{7.4}
\]

That follows from the estimates
\[
\left| \frac{1}{c_{0,i} - u_{i+1}} - \frac{1}{c_{0,i} - u_{i+1}^2} \right| < B_u \frac{\mu_0}{\alpha^2},
\]
\[
\left| \frac{1}{c_{0,i} - u_{i+1}} - \frac{1}{c^1_{0,i+1} - u_{i+1}} \right| < C_{\gamma_0} \frac{\mu_0^2}{\alpha^2} (B_u \tilde{T} + 1).
\]

Thus,
\[
|T^1_{0,i+1} - T_{0,i+1}| \leq (C_0 + C_1) \frac{\mu_0}{2}.
\]

We have
\[
|X^1_{0,i+1} - (X_{0,i} + c_{0,i} \tau_1 + c_{1,i} \tau_2)| < \frac{C_{\gamma_0} \mu_0^2}{\alpha^2} (B_u^2 \tilde{T}^2 + 1),
\]
\[
|X^1_{0,i+1} - \tilde{X}^1_{0,i+1}| < 2 \frac{C_{\gamma_0} \mu_0^2}{\alpha^2} (B_u^2 \tilde{T}^2 + 1) \frac{\mu_0^2}{2},
\]
\[
|\tilde{X}^1_{0,i+1} - (X_{0,i} + c_{0,i} \tau_1 + c_{1,i} \tau_2)| < C_2 \frac{\mu_0^2}{2}.
\]
from (7.4) and the estimates
\[
|c_{1,i} - c_{0,i}| < \frac{2C\gamma_0 B_{u}\mu_0}{\alpha}, \quad |c_{1,i} - c_{0,j}| < \frac{2C\gamma_0 B_{u}\mu_0}{\alpha}, \quad |c_{0,i+1} - c_{1,i}| < \frac{2C\gamma_0}{\alpha^2}.
\]

That proves the existence of the constant \(\tilde{C} > 0\) such that
\[
|X_{0,i+1} - X_{0,i}| < \tilde{C}\mu_0^2.
\]

By repeating the process with each partition \(\mathcal{P}^v\) and its subpartition \(\mathcal{P}^{v+1}\), \(v = 1, 2, \ldots\), we obtain the same estimates with \(T_{0,i+1}, T_{0,i+1}^{v+1}\) and \(\mu_0\), respectively. Let \(T_{0,v}^\tau \leq \tilde{T}\) be the time of interaction of \(\Gamma^v\) and the contact discontinuity line \(x = Y_j + u(Y_j)\tau\). For each \(v\) and the partition \(\mathcal{P}^v\) there are at most \(2(Y_j - Y_i)/\mu_v\) interactions on the compact set. So, we have
\[
|T_{0,v}^\tau - T_{0,v+1}^\tau| \leq 2(C_0 + C_1)(Y_j - Y_i)\frac{\mu_v}{2} =: C_T \frac{\mu_v}{2},
\]
\[
|X_{0,v}^\tau - X_{0,\tau+1}^\tau| \leq 2\tilde{C}(Y_j - Y_i)\frac{\mu_v}{2} =: C_X \frac{\mu_v}{2}.
\]

Finally, since \(C_X, C_T\) do not depend on partition we conclude that a distance between the curves \(\Gamma^0\) and \(\Gamma^{m+p}\) on \((\mathbb{R} \times \mathbb{R}_+) \cap \text{supp} \varphi\) can be estimated by
\[
|X_{0,v}^{m+p} - X_{0,v}^p| \leq C_X \sum_{i=p+1}^{m+p} \frac{\mu_0}{2^i} \leq C_X \frac{\mu_0}{2^p} = C_X \mu_{p}, \quad |T_{0,v}^{m+p} - T_{0,v}^p| \leq C_T \frac{\mu_0}{2^p}.
\]

Thus, \(\{\Gamma^v\}_{v \in \mathbb{N}_0}\) forms a Cauchy sequence, and it converges for each \(t > \tau_0\). To prove the assertion for \(t > \tau_0\) it is enough to take \(\tau_0\) small enough. One can prove the assertion in the same way when the function \(u(x)\) is decreasing and \(u_0 > u(R)\) for \(t < T_{\text{max}}\), i.e. as long as characteristics do not intersect below the curve \(x = c(t)\). Take the partition \(\mathcal{P}^0\) with \(Y_k - Y_{k-1} = \mu_0\). Suppose that \(\Gamma^0\) meets a shadow wave with a front \(x = Y_k + y_{k+1}t\) at a point \((X_k, T_k)\). Assume \(X_k = X_j, \quad T_k = T_j^\nu = \tau_0\) for each \(v\). The next interaction point \((X_{i+1}, T_{i+1})\) is determined by
\[
c(t) = Y_{i+1} + y_{i+1,i+2}t, \quad c(T_i) = X_{i+1}, \quad u_s(T_i) = c_{i}.
\]

There exists an \(\alpha > 0\) such that \(c_i - y_{i+1,i+2} > \alpha\) due to Assumption 7.1. Thus, 0-SDW and SDW\(_{i+1,i+2}\) interact at \(t = T_{i+1}\),
\[
T_{i+1} = T_i + \tau_i + O\left(\frac{\mu_0^2}{2}\right), \quad \tau_i := \frac{1 + u(Y_{i+1})T_i}{c_i - y_{i+1,i+2}} \mu_0.
\]

That follows from the estimates \(y_{i+1,i+2} - y_{i+1,i} = \frac{1}{2}(u_{i+2} - u_{i}) + O(\mu_0^2)\) and \(u_{i+2} - u_{i} = 2u(Y_{i+1})\mu_0 + O(\mu_0^2)\). Then
\[
|X_{i+1} - (X_j + c_{i}\tau_i)| < C_\gamma T_i^2 < C_\gamma \frac{(1 + B_u T_{\text{max}})^2 \mu_0^2}{\alpha^2},
\]
\[
|c_{i+1} - c_i| < 2C_\gamma \frac{1 + B_u T_{\text{max}}}{\alpha} \mu_0.
\]
Take now the subpartition $P^i$ with $Y_{i+1} - Y_{i+1} = Y_{i+1} - Y_i = \frac{\mu_0}{2} = \mu_1$ for each $i \in \mathbb{N}_0$. The interaction points are denoted by $(X^1_j, T^1_j)$, $j = i, i + \frac{1}{2}, i + 1$. Similarly, as in the case of increasing $u(x)$, there exist constants $D_0, D_1 > 0$ such that

$$|T^1_{i+1} - T_i| \leq D_0 \frac{\mu^2_0}{2}, \quad |X^1_{i+1} - X_i| \leq D_1 \frac{\mu^2_0}{2}.$$

Analogous relations with $\mu_0$ replaced by $\mu_\nu$ hold for the partitions $P^\nu$ and $P^{\nu+1}$. Let $T^\nu_j < T^\nu_{\max}$ denotes the last intersection time between $\Gamma^\nu$ and shadow wave in the domain $\text{supp} \varphi$. The error accumulates with each interaction and gives

$$|X^\nu_j - X^\nu_j| \leq (Y^\nu_j - Y^\nu_i) D_1 \frac{\mu^\nu}{2}, \quad |T^\nu_j - T^\nu_{j+1}| \leq (Y^\nu_j - Y^\nu_i) D_0 \frac{\mu^\nu}{2}.$$

Hence, one concludes that $\Gamma^\nu \rightarrow \Gamma$ as $\nu \rightarrow \infty$ in the strip $t < T^\nu_{\max}$. As $\tau_0$ decreases the first point of curve $\Gamma$ tends to $(R, 0)$.

In the general case of partition with the equidistant property it is easy to prove that the maximum number of interactions between $\Gamma^0$ and shadow waves equals $\frac{2(Y^\nu_j - Y^\nu_i)}{\sqrt{\varepsilon}} : = \frac{E}{\sqrt{\varepsilon}}$ following the above procedure for equidistant case. Since the sequence of partitions $\{P^\nu\}_{\nu \in \mathbb{N}_0}$ is formed in such a way that each subinterval $[Y^\nu_k, Y^\nu_{k+1}]$ is divided into two, not necessary equal parts such that

$$\min \left\{ \frac{Y^\nu_k + Y^\nu_{k+1}}{2} \right\} \geq \frac{\mu^\nu}{C}, \quad \max \left\{ \frac{Y^\nu_k + Y^\nu_{k+1}}{2} \right\} \leq \frac{\mu^\nu}{2},$$

one concludes that the number of collisions between $\Gamma^\nu$ and shadow waves is at most $\frac{EC}{\mu^\nu}$. Thus, the above proof holds for general case, with $Y_j - Y_i$ replaced by $C(Y_j - Y_i)$.

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