FOCK SPACE RESOLUTIONS OF THE VIRASORO
HIGHEST WEIGHT MODULES WITH $c \leq 1$

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Abstract
We extend Felder’s construction of Fock space resolutions for the Virasoro minimal models to all irreducible modules with $c \leq 1$. In particular, we provide resolutions for the representations corresponding to the boundary and exterior of the Kac table.

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1. Introduction

The problem we address in this letter arises quite naturally in the so-called free field approach to conformal field theories. The mathematical question one wants to answer within this framework is whether a given irreducible module $\mathcal{L}$ of the chiral algebra $\mathcal{A}$ has a resolution in terms of Fock spaces. More precisely, one wants to construct a family of free field Fock spaces $\mathcal{F}^{(i)}$, which are $\mathcal{A}$-modules, and a set of $\mathcal{A}$-homomorphisms (intertwiners) $d^{(i)} : \mathcal{F}^{(i)} \rightarrow \mathcal{F}^{(i+1)}$, such that these spaces with the maps between them form a complex whose (co)homology is isomorphic to $\mathcal{L}$. The first example of such a construction was given by Felder [1] for the class of representations of the Virasoro algebra corresponding to minimal models [2]. Later, this was extended to other conformal field theories, in particular WZNW and their coset models (for review see [3] and references therein).

Felder’s construction relies on the complete classification of submodules of Fock spaces given by Feigin and Fuchs [4,5], and the explicit form of the intertwiners. The latter are defined, following Thorn [6], as multiple integrals of products of the screening currents. One may wonder whether the restriction to the representations in the fundamental range of the minimal series is important. The answer turns out to be negative, and already examples of similar resolutions outside this series have been discussed in [7]. In the following we will argue that a Fock space resolution can be explicitly constructed for any irreducible highest weight module of the Virasoro algebra provided one introduces additional intertwiners besides those considered in [1,7]. The existence of such intertwiners, and their properties needed in the computation of the cohomology, were demonstrated by Tsuchiya and Kanie [8]. We will discuss some of their results within the Dotsenko-Fateev formalism [9], as used in [1,3], which may be more familiar. Not to make our presentation too long we will restrict to modules with $c \leq 1$.

Our three main results, which cover the cases not analyzed previously, are summarized in Theorems 5.2, 6.1 and 7.1. In particular, the first gives the resolution for modules corresponding to the so-called boundary of the Kac table [10], whilst the last extends Felder’s construction to modules outside the fundamental range.

This letter is organized as follows: In Section 2, as well as introducing some definitions and notation, we summarize the results of Feigin and Fuchs which will be used later. Then, in Sections 3 and 4, we introduce intertwiners and recall the construction of Felder’s complex. After this review, we discuss the boundary case in detail in Sections 5 and 6, and in Section 7 describe the extension of the resolution for the irreducible modules beyond the fundamental range. We conclude in Section 8 with some remarks on possible applications of these results.
2. Feigin-Fuchs modules

The generators of the Virasoro algebra, \( \text{Vir} \), satisfy

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}n(m^2 - 1)\delta_{m+n,0}, \quad [L_n, c] = 0, \quad m, n \in \mathbb{Z}.
\] (2.1)

In this letter we will consider three classes of highest weight modules of \( \text{Vir} \): the irreducible modules \( L_{h,c} \), the Verma modules \( M_{h,c} \), and the Feigin-Fuchs modules \( F_{p,Q} \). We recall that the central charge, \( c \), and the conformal dimension, \( h \), determine \( L_{h,c} \) and \( M_{h,c} \) completely, and that the latter is generated freely by \( L_{-n}, n \geq 1 \), acting on the vacuum \( v_{h,c} \), \( L_0v_{h,c} = hv_{h,c} \).

The Feigin-Fuchs module \( F_{p,Q} \) is just the Fock space of a scalar field, \( \phi(z) \), with a background charge \( Q \) and momentum \( p \) such that

\[ c = 1 - 12Q^2, \quad h = \frac{1}{2}p(p - 2Q). \] (2.2)

We follow the convention that the two-point function of \( \phi(z) \) is

\[ \langle \phi(z)\phi(w) \rangle = -\ln(z - w), \] (2.3)

and the stress energy tensor, \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \), has the form

\[ T(z) = -\tfrac{1}{2} :\partial \phi(z)\partial \phi(z) : + iQ \partial^2 \phi(z). \] (2.4)

In terms of modes \( i\partial \phi(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} \),

\[ L_n = \tfrac{1}{2} \sum_{m \in \mathbb{Z}} :\alpha_m \alpha_{n-m} : -(n + 1)Q \alpha_n, \] (2.5)

where

\[ [\alpha_m, \alpha_n] = m\delta_{m+n,0}. \] (2.6)

The Fock space \( F_{p,Q} \) is generated from the vacuum \( v_p \), \( \alpha_0 v_p = pv_p \), by the free action of the creation operators \( \alpha_{-m}, m \geq 1 \).

As can easily be seen from (2.5), \( F_{p,Q} \) and \( F_{-p,-Q} \) are isomorphic as Virasoro modules. Choosing one solution for \( Q \) in (2.2) leaves us, for a given \( h \) and \( c \), with two Fock spaces \( F_{p,Q} \) and \( F_{2Q-p,Q} \), dual to each other, i.e. \( F_{p,Q}^* \approx F_{2Q-p,Q} \).

* See, e.g. [11] for basic definitions and [2,12] for a review of conformal field theory techniques used throughout this letter.
The detailed structure of submodules of all Verma and Feigin-Fuchs modules of the Virasoro algebra has been obtained in [4,5]. There are three main types of modules, $I$, $II$ and $III$, depending on whether the equation

$$x\alpha_+ + y\alpha_- + \sqrt{2}(p - Q) = 0,$$  

(2.7)

has respectively zero, one or infinitely many integral solutions for $x$ and $y$. Here we have introduced $\alpha_+$ and $\alpha_-$, $\alpha_+\alpha_- = -1$, which also parametrize the background charge,

$$Q = \sqrt{\frac{1}{2}(\alpha_+ + \alpha_-)}.$$  

(2.8)

In the first case (type $I$) the Verma module and the Fock space are isomorphic and irreducible [5], and obviously no resolution is needed. Thus we will assume that (2.7) has at least one integral solution, or, equivalently, that the momentum $p$ can be parametrized by a pair of integers $n$ and $n'$ (i.e. there is exactly one such pair for type $II$, and for modules of type $III$ infinitely many such),

$$p = p_{n,n'} = \sqrt{\frac{1}{2}}((1 - n)\alpha_+ + (1 - n')\alpha_-).$$  

(2.9)

For $p_{n,n'}$ as in (2.9) we denote the corresponding conformal dimension, computed from (2.2), by $h_{n,n'}$, and, for a fixed $Q$ (and $c$) write $\mathcal{F}_{n,n'}$ and $\mathcal{M}_{n,n'}$ instead of $\mathcal{F}_{p_{n,n'},Q}$ and $\mathcal{M}_{p_{n,n'},c}$. Virasoro modules with the momentum given in (2.9) arise in the generalized Dotsenko-Fateev minimal models [9,7].

In case $III$, in order to have infinitely many integral solutions to (2.7), there must exist (relatively prime) integers $p$, $p'$, $pp' \neq 0$, such that $p\alpha_+ + p'\alpha_- = 0$. We can then take $\alpha_+ = \sqrt{p'/p}$ and $\alpha_- = -\sqrt{p/p'}$. Clearly the central charge must be rational, and, depending on whether $pp' > 0$ or $pp' < 0$ we have $c \leq 1$ or $c \geq 25$, respectively. In the following we will restrict ourselves to $c \leq 1$ and take $p' \geq p \geq 1$. In particular $c = 1$ corresponds to $p' = p = 1$, and for $c < 1$ we must have $p' > p$. Using the freedom in the parametrization of the momentum, $p_{n,n'} = p_{n+jp,n'+jp'}$, $j \in \mathbb{Z}$, we can always set $0 \leq n' \leq p' - 1$ (or, equivalently, $0 \leq n \leq p - 1$). If, in addition, $p' > p > 1$ and

$$1 \leq n \leq p - 1, \quad 1 \leq n' \leq p' - 1,$$  

(2.10)

we will say that the module belongs to the fundamental range of the minimal series [2], or to the interior of the Kac table [10]. The exterior of the table is defined by letting $n \not\equiv 0 \mod p$ to be outside the range (2.10). Finally, $n = 0 \mod p$ or $n' = 0 \mod p'$ correspond to the boundary of the table.

In cases $II$ and $III$, the detailed structure of $\mathcal{F}_{n,n'}$ as a Virasoro module depends on the momentum $p_{n,n'}$ (2.9), and all possible subcases (in case $III$ with $c \leq 1$) are listed in the following theorem which summarizes the results of Feigin and Fuchs.
Theorem 2.1. ([5]) In case II there are four subcases:

- Case II\(_{\pm}\) (respectively II\(_0\)): If \(nn' < 0\) (respectively \(nn' = 0\)) then \(F_{n,n'}\) and \(M_{n,n'}\) are isomorphic and irreducible.

- Case II\(_+\)(\(-\)): If \(n, n' < 0\) then \(F_{n,n'} \simeq M_{n,n'}\) are reducible. The maximal submodule is isomorphic with \(F_{-n,n'} \simeq M_{-n,n'}\).

- Case II\(_+\)(\(+\)): This case, where \(n, n' > 0\), is dual to II\(_+\)(\(-\)), e.g. \(F_{n,n'} \simeq M_{-n,-n'}^*\).

In case III for \(c \leq 1\) there are four subcases, in all of which the Fock space is a reducible module. In the first three, III\(_{-}\) and III\(_0\)(\(\pm\)) below, \(p' > p > 1\) (i.e. \(c < 1\)), and \(m, m'\) denote labels in the fundamental range, \(1 \leq m \leq p-1, 1 \leq m' \leq p' - 1\). In III\(_0\)(\(0\)) we have \(p' \geq p \geq 1\) (i.e. \(c \leq 1\)). In all cases \(j \in \mathbb{Z}\).

- Case III\(_{-}\): The submodules of \(F_{m+jp,m'}\) are generated by the vectors \(u_i, v_i,\) and \(w_i\) given by the following diagram

\[
\begin{array}{cccccccc}
  v_0 & \leftrightarrow & w_0 & \rightarrow & v_1 & \leftrightarrow & w_1 & \rightarrow & v_2 & \leftrightarrow & w_2 & \rightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  u_1 & \leftrightarrow & v_{-1} & \rightarrow & u_2 & \leftrightarrow & v_{-2} & \rightarrow & u_3 & \leftrightarrow & \cdots \\
\end{array}
\]

whose conformal weights are (see [1])

\[
\begin{align*}
  h(v_0) &= h_{m+jp,m'}, \\
  h(u_i) &= h_{m+((j+2i)p)m'}, \quad i \geq 1, \\
  h(w_i) &= h_{m-((j+2i)p)m'}, \quad i \geq 0, \\
  h(v_{-i}) &= h_{m-((j+2i)p)m'}, \quad i \geq 1, \\
  h(v_i) &= h_{m+(j+2i)p,m'}, \quad i \geq 1.
\end{align*}
\]

- Case III\(_0\)(\(-\))\(_0\): The submodules of i) \(F_{m+jp,0}\), \(j \geq 0\), and ii) \(F_{0,m'+jp,0}\), \(j \geq 0\), are generated by the vectors \(u_i\) and \(v_i\),

\[
\begin{array}{cccccccc}
  v_0 & \rightarrow & u_1 & \leftrightarrow & v_1 & \rightarrow & u_2 & \leftrightarrow & v_2 & \rightarrow & u_3 & \leftrightarrow & \cdots \\
\end{array}
\]

where

\[
\begin{align*}
  i) \quad h(v_i) &= h_{m+(j+2i)p,0}, \quad i \geq 0, \\
  &\quad h(u_i) = h_{m-(j+2i)p,0}, \quad i \geq 1; \\
  ii) \quad h(v_i) &= h_{0,m'+(j+2i)p,0}, \quad i \geq 0, \\
  &\quad h(u_i) = h_{0,m'-(j+2i)p}, \quad i \geq 1.
\end{align*}
\]

- Case III\(_0\)(\(+\))\(_0\): The submodules of i) \(F_{m+jp,0}\), \(j < 0\), and ii) \(F_{0,m'+jp,0}\), \(j < 0\), are generated by the vectors \(u_i\) and \(v_i\),

\[
\begin{array}{cccccccc}
  u_1 & \leftrightarrow & v_1 & \rightarrow & u_2 & \leftrightarrow & v_2 & \rightarrow & u_3 & \leftrightarrow & v_3 & \rightarrow & \cdots \\
\end{array}
\]

where

\[
\begin{align*}
  i) \quad h(v_i) &= h_{m-(j-2(i-1))p,0}, \quad i \geq 1, \\
  &\quad h(u_i) = h_{m+(j-2(i-1))p,0}, \quad i \geq 1; \\
  ii) \quad h(v_i) &= h_{0,m'-(j-2(i-1))p}, \quad i \geq 1, \\
  &\quad h(u_i) = h_{0,m'+(j-2(i-1))p}, \quad i \geq 1.
\end{align*}
\]
Case III\textsuperscript{00}: The Fock space $\mathcal{F}_{jp,0} \simeq \mathcal{F}_{-jp,0}$ is the direct sum of irreducible modules, 
$\mathcal{F}_{jp,0} = \bigoplus_{k=0}^{\infty} \mathcal{L}((|j|+2k)p,0)$.

In the diagrams (2.11), (2.13), and (2.15) vectors $u_i$ correspond to singular vectors in the Fock space $\mathcal{F}_{*,*}$ and generate the submodule, $\mathcal{F}'_{*,*}$, which is a direct sum of irreducible highest weight modules. In the quotient $\mathcal{F}_{*,*}/\mathcal{F}'_{*,*}$ vectors $v_i$ become singular, and generate $\mathcal{F}''_{*,*}$ which is a direct sum of irreducible highest weight modules. The arrows, $1 \rightarrow 2$, indicate that the second vector is in the submodule generated by the first one.

There are also similar diagrams for the composition series of singular vectors of the Verma modules, in which the singular vectors occur at precisely the same conformal weights as above. Moreover, some arrows must be reversed, so that all of them describe embeddings of the Verma submodules.

3. The intertwiners

Let us first introduce a class of intertwiners between Fock spaces. They are constructed as products of the screening currents

$$s_+(z) = \exp(i\sqrt{2}\alpha_+\phi)(z) \quad \text{and} \quad s_-(z) = \exp(i\sqrt{2}\alpha_-\phi)(z)$$

integrated over suitable multiple contours [6,5,8,1]. For positive integers $r$ and $r'$ let us consider operators

$$Q_+^r[\Omega] = \left[ (s_+)^r \right]_{\Omega_+}, \quad Q_-^{r'}[\Omega] = \left[ (s_-)^{r'} \right]_{\Omega_-},$$

where $\left[ \cdots \right]_{\Omega_k}$ is defined by

$$\left[ s_{i_1} \ldots s_{i_k} \right]_{\Omega_k} = \int_{\Omega_k} dz_1 \ldots dz_k s_{i_1}(z_1) \ldots s_{i_k}(z_k), \quad i_1, \ldots, i_k \in \{+, -\}.$$  \hspace{1cm} (3.3)

and $\Omega_k$ is a set of contours for $z_1, \ldots, z_k$. When acting on $\mathcal{F}_{n,n'}$, the integrand in (3.3) is analytic in $M_k = \mathcal{Q}^k \setminus \{ z_i = 0, z_i = z_j; i, j = 1, \ldots, k \}$ and has nontrivial monodromies due to the presence of the factor

$$\prod_{i<j}(z_i - z_j)^{2\alpha^2} \prod_k z_k^{\sqrt{2}\alpha_{\pm}p_{n,n'}}.$$  \hspace{1cm} (3.4)

In order that (3.3) be a well-defined operator we must require that $\Omega_k$ is a closed contour [8], i.e. it is an element of $H(M_k, \mathcal{S}_{\alpha^2})$, the homology of $M_k$ with coefficients in the local system $\mathcal{S}_{\alpha^2}$ corresponding to (3.4) (this system depends only on $\alpha^2_{\pm}$).
In explicit computations one needs to have convenient representatives of these homology classes. We will construct them using the following two classes of multicontours (singular at a point) that have been used in \([1,13,3]\): In the first class, which we denote \(\Gamma_k\), the integration variables \(z_1, \ldots, z_k\) are taken counterclockwise from 1 to 1 around 0, and nested according to \(|z_1| > \ldots > |z_k|\). In the second, \(\hat{\Gamma}_k\), the \(z_1\) integration is along a contour surrounding 0, while \(z_2, \ldots, z_k\) are integrated counterclockwise from the base point \(z_1\) to \(z_1\) around 0, and the nesting is the same as in \(\Gamma_k\). The ambiguity in the phase of the integrand (3.3) is fixed by analytic continuation from the positive real half-line (see \([1,13]\)). Let us denote the resulting operators (3.2) by \(Q_k^{(+)}, Q_k^{(-)}\) and \(\hat{Q}_k^{(+)}, \hat{Q}_k^{(-)}\), respectively.

We will also consider multiple contours obtained by putting several of those together. For example \(\Gamma_k \cup \Gamma_{k'}\) will denote a multiple contour \(z_1, \ldots z_k, \ldots z_{k+k'}\) in which the variables of \(\Gamma_{k'}\) are nested inside those of \(\Gamma_k\).

We may now state the main result about the intertwiners between Feigin-Fuchs modules of type III due to Tsuchiya and Kanie \([8]\).

**Theorem 3.1.** 1) Consider \(F_{n,n'}\), where \(n = m + jp\), \(1 \leq m \leq p\), \(j \in \mathbb{Z}\), and \(n'\) is arbitrary. Then for any non-negative integer \(k\) the operator

\[
Q_{m+kp}^{(+)\star}[\Omega] : F_{n,n'} \rightarrow F_{n-2(m+kp),n'},
\]

is a well defined intertwiner provided \(\Omega_{m+kp} \in H(M_{m+kp}, S_{\alpha_z^+})\).

2) These intertwiners are nontrivial in the sense that such \(\Omega_{m+kp}\) exists for any \(k \geq 0\) and

i) if \(n' + (k-j)p' < 0\) then \(Q_{m+kp}^{(+)\star}[\Omega]v_{n,n'} \neq 0\);

ii) if \(n' + (k-j)p' > 0\) then \(Q_{m+kp}^{(+)\star}[\Omega]\chi = v_{n-2(m+kp),n'}\) for some \(\chi \in F_{n,n'}\).

Here \(v_{n,n'}\) and \(v_{n-2(m+kp),n'}\) denote vacua of \(F_{n,n'}\) and \(F_{n-2(m+kp),n'}\), respectively.

The analogous result holds for the operators \(Q_\gamma(-\star)[\chi]\).

4. The Fock space resolution for type III and III modules

We begin by constructing explicitly intertwiners using contours \(\Gamma_r, \Gamma_{r'}, \hat{\Gamma}_r\) and \(\hat{\Gamma}_{r'}\), \(1 \leq r \leq p\) and \(1 \leq r' \leq p'\), introduced in the previous section.

**Lemma 4.1.** Operators \(Q_r^{(+)\star}, \hat{Q}_r^{(+)\star} : F_{n,n'} \rightarrow F_{n-2r,n'}\) and \(Q_{r'}^{(-\star)}, \hat{Q}_{r'}^{(-\star)} : F_{n,n'} \rightarrow F_{n,n'-2r'}\) are well defined intertwiners between Virasoro modules if \(r = n \mod p\) and \(r' = n' \mod p'\), respectively.

**Proof:** Operators \(\hat{Q}_r^{(+)\star}\) and \(\hat{Q}_{r'}^{(-\star)}\) are well defined and commute with the action of the Virasoro algebra iff \(\hat{\Gamma}\) is a closed cycle. This amounts to the \(z_1\)-contour being closed. For the values of \(r\) and \(r'\) as in the lemma this is easily verified using standard methods \([1]\). For \(Q_r^{(+)\star}\) and \(Q_{r'}^{(-\star)}\) we use the following lemma which can be proven by standard manipulations with the contours \([1,13]\). \(\Box\)
Lemma 4.2. For $1 \leq r \leq p$ and $1 \leq r' \leq p'$,

$$Q_r^{(+)Q_{r}^{(-)} = \frac{1}{r} - \frac{2r}{1 - q_r^{2}}Q_{r}^{(-)}$$

$$Q_{r'}^{(+)Q_{r'}^{(-)} = \frac{1}{r'} - \frac{2r'}{q_{r'}^{2}}Q_{r'}^{(-)}, \quad (4.1)$$

where $q_{\pm} = \exp(i\pi\alpha_{\pm}^2)$.

Note that for positive $r_1, r_2, r_1 + r_2 \leq p$ we have $Q_r^{(+)}Q_{r_2}^{(+)} = Q_r^{(+)}Q_{r_1 + r_2}^{(+)}$, provided $Q_{r_1}^{(+)}$ and $Q_{r_2}^{(+)}$ act on spaces on which they are well-defined. In particular – after further investigation of the integral in $Q_{r}^{(+)}$ – (4.1) implies

$$Q_{m}^{(+)}Q_{p-m}^{(+)} = Q_{p-m}^{(+)}Q_{m}^{(+)} = Q_{p}^{(+)} = 0, \quad (4.2)$$

The same also holds for $Q_{m}^{(-)}$.

Lemma 4.1 together with identity (4.2) is the basis for the construction of the complex of Fock spaces when $m, m'$ lie in the fundamental range (2.10). This complex, $(\mathcal{F}, d) \equiv \{ (\mathcal{F}_{m,m'}, d^{(i)}), i \in \mathbb{Z} \}$, is defined as follows

$$\mathcal{F}_{m,m'}^{(2j)} = \mathcal{F}_{m-2jp,m'}, \quad d^{(2j)} = Q_{m}^{(+)}$$

$$\mathcal{F}_{m,m'}^{(2j+1)} = \mathcal{F}_{m-2jp,m'}, \quad d^{(2j+1)} = Q_{p-m}^{(+)}, \quad j \in \mathbb{Z}. \quad (4.3)$$

Using the submodule structure of Fock spaces $\mathcal{F}_{m,m'}^{(i)}$ as summarized in Theorem 2.1, Felder [1] was able to compute the kernels and the images of all the intertwiners $d^{(i)}$, and prove the following important result

**Theorem 4.3.** Let $1 \leq m \leq p - 1, 1 \leq m' \leq p' - 1$. Then the complex $(\mathcal{F}, d)$ defined in (4.3) is a (two-sided) resolution of the irreducible module $\mathcal{L}_{m,m'}$, i.e.

$$H^{(i)}(\mathcal{F}, d) \simeq \delta_{i,0}\mathcal{L}_{m,m'}. \quad (4.4)$$

In fact there are three more resolutions, one in terms of modules $\mathcal{F}_{p-m,p'-m'}^{(i)}$ (recall that $\mathcal{F}_{m,m'}^{(+)} \simeq \mathcal{F}_{p-m,p'-m'}$ and $\mathcal{L}_{p-m,p'-m'} \simeq \mathcal{L}_{m,m'}$), and two others if we use operators $Q_{m}^{(-)}$ instead of $Q_{m}^{(+)}$.

It may be worth noting that Felder’s proof also shows that the differential in the complex is nilpotent without resorting to the computation in (4.2).

A similar construction for the modules of type $II_+(\pm)$ was carried out in [7].

**Theorem 4.4.** In the notation of Theorem 2.1, let $n, n' < 0$. Then the complex

$$0 \rightarrow \mathcal{F}_{-n,n'}^{(+)Q_{n}^{(+)}} \rightarrow \mathcal{F}_{n,n'}^{(+)Q_{n}^{(+)}} \rightarrow \mathcal{F}_{n,n'}^{(+)Q_{n}^{(+)}} \rightarrow 0, \quad (4.5)$$

is a resolution for $\mathcal{L}_{n,n'}$. For the case $II_+(\pm)$ the resolution can be obtained by dualization of (4.5).
5. Resolutions for the modules of type III_\(i\) 

Using the results reviewed in the previous sections we will now construct resolutions for irreducible modules \(L_{h,c}\) of type III_\(i\), for which \(h = h_{m+jp,0}, 1 \leq m \leq p - 1\) or \(h = h_{0,m'+jp'}, 1 \leq m' \leq p' - 1, j \in \mathbb{Z}\). Since \(h_{n,n'} = h_{-n,-n'}\), we may take \(j \geq 0\).

Let us consider the Fock space \(F_{0,m'+jp'}\). According to Theorem 3.1 we know it exists. To construct it explicitly we observe that, in view of Lemma 5.1.

In a sense the problem is similar to the one in case II_\(\pm\), except that the structure of submodules of Fock spaces is somewhat more complicated. Since \(F_{0,m'+jp'} \simeq F_{-jp,m'}\) and \(F_{0,m'-(j+2)p'} \simeq F_{(j+2)p,m'}\), the intertwiner should involve \((j+1)p\) currents \(s_+(z)\). By Theorem 3.1 we know it exists. To construct it explicitly we observe that, in view of Lemma 4.1 and (4.2), an obvious building block for such operators is \(\hat{Q}_p^{(+)\cdot}\). Its properties are summarized in the following technical lemma whose proof will be outlined at the end of the section.

**Lemma 5.1.** Consider \(\hat{Q}_p^{(+)\cdot} : F_{0,m'+kp'} \longrightarrow F_{0,m'-(k+2)p'}, \ k \in \mathbb{Z}\). Depending on \(k\) there are four cases described by the diagrams i)-iv) below. Operator \(\hat{Q}_p^{(+)\cdot}\) maps special vectors \(u_i'\) and \(v_i'\) from the first Fock space onto \(u_i\) and \(v_i\) in the second space, as indicated by the downward arrows, i.e. \(\hat{Q}_p^{(+)\cdot}\) is nonzero along these arrows.

\[
\begin{align*}
i) \quad & k \leq -3 : \\
& \downarrow \downarrow \downarrow \\
& u_1 \leftarrow v_1 \rightarrow u_2 \leftarrow v_2 \rightarrow u_3 \leftarrow \cdots \\
\text{ii}) \quad & k = -2 : \\
& \downarrow \downarrow \downarrow \\
& v_0 \rightarrow u_1 \leftarrow v_1 \rightarrow u_2 \leftarrow v_2 \rightarrow \cdots \\
\text{iii}) \quad & k = -1 : \\
& \downarrow \downarrow \downarrow \\
& v_0 \rightarrow u_1 \leftarrow v_1 \rightarrow u_2 \leftarrow \cdots \\
\end{align*}
\]
\[ iv \] \[ k \geq 0 : \quad v'_0 \rightarrow u'_1 \leftarrow v'_1 \rightarrow u'_2 \leftarrow v'_2 \rightarrow \cdots \]
\[ \downarrow \quad \downarrow \quad \downarrow \vdots \]
\[ v_0 \rightarrow u_1 \leftarrow v_1 \rightarrow \cdots \]

The main result of this section is

**Theorem 5.2.** The intertwiner \( d = (\hat{Q}_p^{(+)})^{j+1} \) is an embedding of \( F_{0,m'-(j+2)p'} \) into \( F_{0,m'+jp'} \), i.e. the complex

\[
0 \rightarrow F_{0,m'-(j+2)p'} \xrightarrow{d} F_{0,m'+jp'} \rightarrow 0 ,
\]

is a Fock space resolution of \( L_{0,m'+jp'} \). Another resolution is given by the dual complex

\[
0 \rightarrow F_{0,-m'-jp'} \xrightarrow{d^*} F_{0,-m'-(j+2)p'} \rightarrow 0 ,
\]

where \( d^* = (\hat{Q}_p^{(+)})^{j+1} \).

**Proof:** Using Lemma 5.1 we can compute the image of \( F_{0,m'-(j+2)p'} \) when acting with subsequent \( \hat{Q}_p^{(+)} \). The result is \( d(F_{0,m'-(j+2)p'}) = F_{0,m'+jp'} \). The theorem follows from \( L_{0,m'+jp'}^* \simeq L_{0,m'+jp'} \) and \( \hat{Q}_p^{(+)\ast} = \hat{Q}_p^{(+)} \).

Clearly, for modules \( L_{m+jp,0} \) there are analogous resolutions in which the differential is \( \hat{Q}_p^{(-)} \).

**Proof of Lemma 5.1:** The general idea of the proof is the same as that of parts of Theorem 4.3 (see [1]). Let us begin with case i). By Theorem 3.1. 2.i) \( \hat{Q}_p^{(+)} u_1' \neq 0 \). To verify this we choose a covector \( \chi \in F_{0,m'+(k+2)p'} \) (see [6] and Appendix in [1]) such that

\[
\langle \chi, s_+(z_1) \cdots s_+(z_p) u_1' \rangle = \prod_{\ell=1}^{p} z_{\ell}^{-(k+1)p'-m'} \prod_{\ell \neq \ell'}(z_{\ell} - z_{\ell'})^{2\alpha_+^2} \prod_{\ell=1}^{p} z_{\ell}^{2\alpha_+^2 - p - 1} F_{0,p_0,m'+kp'}. \tag{5.4}
\]

Then a straightforward computation yields

\[
\langle \chi, \hat{Q}_p^{(+) u_1'} \rangle = \int dz_1 \cdots dz_p \prod_{\ell \neq \ell'}(z_{\ell} - z_{\ell'})^{2\alpha_+^2} \prod_{\ell=1}^{p} z_{\ell}^{2\alpha_+^2 - p - 1} \\
= 2\pi i (-1)^{p-1} \prod_{\ell=1}^{p-1} \frac{(1 - q_+^{2\ell})^2}{1 - q_+^2} J_{0,p-1}(2\alpha_+^2, \alpha_+^2 - p' - 1, \alpha_+^2) \tag{5.5}
\]

\[
= (2\pi i)^p (-1)^{pp'+p-1} \frac{p'!}{(p-1)! \Gamma(1 + \alpha_+^2)^p} \prod_{\ell=1}^{p-1} \frac{\sin \pi \ell \alpha_+^2}{\sin \pi \alpha_+^2} \\
\neq 0 .
\]
We used here an explicit result for the Dotsenko-Fateev integral \[9\]

\[
J_{0,n}(\alpha, \beta; \rho) = \frac{1}{n!} \int_0^1 dt_1 \ldots \int_0^1 dt_n \prod_{\ell=1}^n (1-t_\ell)^{\alpha} t_\ell^\beta \prod_{\ell, l'=1}^n |t_\ell - t_{l'}|^{2\rho}
= \prod_{\ell=1}^n \frac{\Gamma(\ell \rho)}{\Gamma(\rho)} \frac{\Gamma(1+\alpha + (\ell - 1)\rho)\Gamma(1+\beta + (\ell - 1)\rho)}{\Gamma(2+\alpha + \beta + (n+\ell - 2)\rho)}.
\]

(5.6)

The rest of the lemma in this case follows entirely from the embedding patterns of submodules in both Fock spaces and \(\hat{Q}_p^{(+)}\) being an intertwiner. Let us outline the first few steps.

Normalize \(u_2\) such that \(u_2 = \hat{Q}_p^{(+)}u_1\). Since \(u'_1\) is in the submodule generated by \(v'_1\) we must have \(\hat{Q}_p^{(+)}v'_1 \neq 0\). Note that the latter remains nonzero after we divide \(F_{0,m'+(k+2)p'}\) by the submodule generated by \(u_1\) and \(u_2\). Indeed, suppose the opposite, \(i.e.\) that \(\hat{Q}_p^{(+)}v'_1 = P_1u_1 + P_2u_2\), where \(P_{1,2}\) are some polynomials in \(L_{-n}\), \(n \geq 1\). Since the submodule generated by \(u_1\) and \(u_2\) is a direct sum of two irreducible ones, we also have \(u_1 = P'_1P_1u_1\) where \(P'_1\) is some element in the enveloping algebra of \(Vir\). But if \(P_1u_1 \neq 0\) then \(u_1 = \hat{Q}_p^{(+)}P'_1(v'_1 - P_2u'_1)\), and a simple examination of the weights in both Fock spaces shows the r.h.s. must vanish, which is a contradiction. Thus we can at most have \(\hat{Q}_p^{(+)}v'_1 = P_2u_2\). But then \(\hat{Q}_p^{(+)}(v'_1 - P_2u'_1) = 0\), \(i.e.\) \(v'_1 - P_2u'_1 \neq 0\) is in the kernel of \(\hat{Q}_p^{(+)}\). Since the submodule \(ker \hat{Q}_p^{(+)}\) contains neither \(u'_1\) nor \(v'_1\), there can be no such vectors at this level, \(i.e.\) we must have \(\hat{Q}_p^{(+)}v'_1 \neq P_2u_1\). That \(\hat{Q}_p^{(+)}v'_1\) is singular after we divide out the submodule generated by \(u_1\) follows from the corresponding property for \(v'_1\). Thus, finally, we may set \(\hat{Q}_p^{(+)}v'_1 = v_2\). In the next step a similar reasoning shows that \(\hat{Q}_p^{(+)}u'_2 \neq 0\), so, up to normalization it yields \(\hat{Q}_p^{(+)}u'_2 = u_3\). And so on.

Case ii) is proven by exactly the same method. Cases iii) and iv) are dual to ii) and i), respectively. Since \(\hat{Q}_p^{(+)*} = \hat{Q}_p^{(+)}\), we deduce from \(coker \hat{Q}_p^{(+)*} \simeq ker \hat{Q}_p^{(+)}\) that \(\hat{Q}_p^{(+)}\) must be onto. Because \(\hat{Q}_p^{(+)}\) commutes with \(Vir\), \(u'_i\) are mapped onto \(u_i\), and the maps indicated by the arrows are clearly nonzero. After we divide both sides by submodules generated by \(u'_i\) and \(u_i\), respectively, \(\hat{Q}_p^{(+)}\) becomes a homomorphism from a direct sum of irreducible highest weight modules onto a direct sum of a subset of these modules, and the arrows between \(v'_i\) and \(v_i\) follow.

\[\square\]

6. Resolutions for the modules of type \(III_{00}\)

The problem of obtaining a resolution for a module \(L_{jp,0}\), \(j \geq 0\), is similar to the one discussed above. Given the Fock space \(F_{jp,0}\) we must construct an embedding from \(F_{(j+2)p,0}\) into \(F_{jp,0}\). The result is
Then \( F \) vectors in \( \hat{\mathcal{F}} \) are obtained by removing irreducible modules. Let us now consider the collection of Fock spaces obtained by removing terms of Schur polynomials) for singular vectors in the Fock space and \( \hat{\mathcal{F}} \) from \((\mathcal{F},d)\) corresponding to the resolution of \( \mathcal{L}_{m,m'} \) for \( j \) even, and \( \mathcal{L}_{p-m,m'} \) for \( j \) odd, constructed as in Section 4. An important feature of \((\mathcal{F},d)\) is that the vacuum of the 0-th Fock space has the highest weight with respect to all other states in the complex. In particular, this guarantees that the cohomology contains at least the irreducible module. Let us now consider the collection of Fock spaces obtained by removing from \((\mathcal{F},d)\) all Fock spaces whose weights are higher than \( h_{m+p,m'} \); i.e. we delete the spaces \( \mathcal{F}_{-m+2jp,m'}, \ldots , \mathcal{F}_{-m+2jp,m'} \) for \( j > 0 \), and the spaces \( \mathcal{F}_{m-jp,m'}, \ldots , \mathcal{F}_{-m+(j+2)p,m'} \) for \( j < 0 \). On both sides of the deleted segment the differential is defined as before in terms of Schur polynomials) for singular vectors in the Fock space and \( \hat{\mathcal{F}} \) is easy to check that, up to normalization, they coincide with the vectors \((\hat{\mathcal{Q}}^{(+)}_{1})(v_{kp+2\ell,0})\), \( \ell \geq 0 \), where \( v_{kp+2\ell,0} \) denotes the vacuum of \( \mathcal{F}_{kp+2\ell,0} \). The theorem is then obvious. The general case is similar.

\[ Q^{(+)}_{m+kp} = \left[ (s_{+})^{m+kp} \right]_{\mathcal{O}_{m+kp}} \quad \text{or} \quad Q^{(+)}_{p-m+k} = \left[ (s_{+})^{(p-m)+k} \right]_{\mathcal{O}_{(p-m)+k}} , \quad k \geq 1 , \]

(7.1)

where \( \mathcal{O}_{m+kp} = \Gamma_{m} \cup \hat{\Gamma}_{p} \cup \ldots \cup \hat{\Gamma}_{p} \) and \( \mathcal{O}_{(p-m)+k} = \Gamma_{(m-p)} \cup \hat{\Gamma}_{p} \cup \ldots \cup \hat{\Gamma}_{p} \). The resulting extension of the Fock space resolution to modules outside the fundamental range can be summarized by the following theorem.

**Theorem 7.1.** Let \( 1 \leq m \leq p-1, 1 \leq m' \leq p'-1 \), and \( j \geq 0 \). Then the complex of Fock spaces

\[ \ldots \longrightarrow Q^{(+)}_{m} \mathcal{F}_{m+(j+2)p,m'} \longrightarrow Q^{(+)}_{p-m} \mathcal{F}_{m+(j+2)p,m'} \longrightarrow \mathcal{F}_{m-jp,m'} \longrightarrow Q^{(+)}_{p-m} \mathcal{F}_{m-jp,m'} \longrightarrow \ldots , \]

(7.2)

is a resolution of the irreducible module \( \mathcal{L}_{m-jp,m'} \), while

\[ \ldots \longrightarrow Q^{(+)}_{m} \mathcal{F}_{m+(j+2)p,m'} \longrightarrow Q^{(+)}_{p-m} \mathcal{F}_{m+(j+2)p,m'} \longrightarrow Q^{(+)}_{m+jp} \mathcal{F}_{m-jp,m'} \longrightarrow Q^{(+)}_{p-m} \mathcal{F}_{m-jp,m'} \longrightarrow \ldots , \]

(7.3)
is a resolution of $L_{m+jp,m'}$.

**Proof:** Contours $\Omega_{(p-m)+kp}$ and $\Omega_{m+kp}$ are nontrivial cycles in local homology. This can be verified by a computation similar to the one in the proof of Lemma 5.1. Thus the intertwiners $Q_{(p-m)+jp}^{(+)}$ and $Q_{m+jp}^{(+)}$ are precisely those given in Theorem 3.1. In particular they satisfy i) and ii), respectively, which is precisely what one needs to extend the proof of Theorem 4.3 to the present case. 

\[ \Box \]

It is clear that the three other resolutions discussed in Section 4 – the dual, and the two others constructed with $Q_{(-)}$ instead of $Q_{(+)}$ – may be extended in the same way to modules outside the fundamental range.

### 8. Concluding remarks

Formally, the new intertwiners in Sections 5, 6 and 7 are proportional to operators of the form

$$
\frac{[[s^{\pm}]]_k}{[k]_{q^{\pm}} !},
$$

where $[n]_{q^{\pm}}$ denotes the usual $q^{\pm}$-number. As has been extensively discussed for free field realizations during the past two years, the screening currents inside $[[\cdot]]_k$ satisfy the defining relations of generators of a quantum group $U_{q^{\pm}}(n^{\pm})$ (see e.g. [3] and references therein). It is worth noting that in the discussion of the general class of resolutions in this letter one is automatically led to the “rescaled quantum group” generators of Lusztig [16].

An interesting application of these resolutions is to extend the computation of the BRST cohomology of minimal models coupled to 2D quantum gravity [17] to generalized Dotsenko-Fateev models using methods discussed in [18]. This is elaborated in more detail in [19].

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