Solutions of the Yang-Baxter equation associated with a left brace

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Abstract
Given a left brace $G$, a method to construct all the involutive, non-degenerate set-theoretic solutions $(Y, s)$ of the YBE, such that $G(Y, s) \cong G$ is given. This method depends entirely on the brace structure of $G$.

1 Introduction
The quantum Yang-Baxter equation is an important equation coming from theoretical physics, first appearing in the works of Yang [26] and Baxter [3]. Recall that a solution of the Yang-Baxter equation is a linear map $R : V \otimes V \rightarrow V \otimes V$, where $V$ is a vector space, such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij}$ denotes the map $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ acting as $R$ on the $(i, j)$ tensor factor and as the identity on the remaining factor. A central open problem is to construct new families of solutions of this equation. It is this problem which initially motivated the definition of quantum groups, and one of the reasons of the recent interest in Hopf algebras, see [20].

Note that if $X$ is a basis of the vector space $V$, then a map $\mathcal{R} : X \times X \rightarrow X \times X$, such that

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

where $\mathcal{R}_{ij}$ denotes the map $X \times X \times X \rightarrow X \times X \times X$ acting as $\mathcal{R}$ on the $(i, j)$ components and as the identity on the remaining component, induces a solution of the Yang-Baxter equation. In this case, one says that $(X, \mathcal{R})$ (or $\mathcal{R}$) is a set-theoretic solution of the quantum Yang-Baxter equation. Drinfeld, in [9], posed the question of finding these set-theoretic solutions.

A subclass of this type of solutions, the non-degenerate involutive ones, has received a lot of attention in the last years [6, 7, 8, 11, 13, 14, 15, 19, 18, 19, 21, 22, 23]. This class of solutions is not only studied for the applications of

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the Yang-Baxter equation in physics, but also for its connection with other topics in mathematics of recent interest: semigroups of I-type and Bieberbach groups \[16\], bijective 1-cocycles \[11\], radical rings \[23\], triply factorized groups \[25\], construction of semisimple minimal triangular Hopf algebras \[10\], regular subgroups of the holomorf and Hopf-Galois extensions \[5, 12\], and groups of central type \[4\].

Gateva-Ivanova and Van den Bergh \[16\], and Etingof, Schedler and Soloviev \[11\] introduced this subclass of solutions and associated with each solution \((X, r)\) of this type two groups which are fundamental for its study: the structure group \(G(X, r)\), and the permutation group \(\mathcal{G}(X, r)\). In order to study this class of solutions, Rump in \[23\] introduced a new algebraic structure called brace. Recall that a left brace is a set \(B\) with two operations, \(+\) and \(\cdot\), such that 

\[ x \cdot (y + z) + x = x \cdot y + x \cdot z, \]

for all \(x, y, z \in B\). This structure is connected with the solutions of the Yang-Baxter equation because, besides the group structure of \(G(X, r)\) and \(\mathcal{G}(X, r)\), any solution \((X, r)\) induces a sum over these two groups that defines a structure of left brace.

In spite of all this progress, it is still an open problem to construct all the non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation. Inspired by \[8\], and using the new techniques available when we introduce the brace structure, we separate this problem in two parts:

1. Classify all the left braces.
2. For each left brace \(G\), construct all the non-degenerate involutive set-theoretic solutions \((Y, s)\) with \(\mathcal{G}(Y, s) \cong G\) as left braces, and classify them up to isomorphism.

In this article, we focus on the second problem. We give a translation of the problem completely in terms of the structure of left brace of \(G\). Specifically, we show that the construction of solutions with \(\mathcal{G}(Y, s) \cong G\) is equivalent to find some subgroups of \((G, \cdot)\) and some orbits of \(G\) with respect to an action \(\lambda\) that is defined in any left brace. Note that, when \(Y\) is finite, the associated permutation group \(\mathcal{G}(Y, s) \leq \text{Sym}_X\) is finite, so the set of subgroups and of orbits of \(\mathcal{G}(Y, s)\) is finite, and our result reduces the question of finding all the solutions of the Yang-Baxter equation with the same permutation group (which are infinite) to a problem of finding a finite set of objects.

For previous results on the second problem, Cedó, Jespers and del Río initiated the study of the problem presenting a particular construction of this type in \[3\] Section 5: for a given solution \((X, r)\), they give a method to construct a solution \((X^2, r^{(2)})\) over \(X^2 = X \times X\) such that \(\mathcal{G}(X^2, r^{(2)}) \cong \mathcal{G}(X, r)\). This result was then generalized in \[2\], giving a method to construct solutions \(r^{(n)}\) over \(X^n\) for each \(n\) such that \(\mathcal{G}(X^n, r^{(n)}) \cong \mathcal{G}(X, r)\). It was trying to generalize \[2\] that we found the results of the present article. We also have to mention
that [4, Corollary D] provides a homological solution to the second problem, but that solution is not constructive. Another initial motivation for this paper was to give an explicit and constructive solution.

The content of the paper is as follows. In Section 2, we recall some definitions and known results about left braces and the Yang-Baxter equation that we need in the next sections. Then, in Section 3 we present our method of construction of solutions with respect to some subgroups and some orbits of $G$, and proof that any solution can be obtained using this construction. It is possible that two solutions constructed with this method are isomorphic, so in Section 4 we present a method to detect isomorphism between solutions, which essentially says that isomorphisms between solutions are induced by automorphisms of the left brace $G$. So we also manage to translate the isomorphism problem completely in terms of the brace structure of $G$. Finally, in Section 5, we suggest a possible way to classify all the non-degenerate set-theoretic solutions of the Yang-Baxter equation with fixed permutation group through the concept of basic solution, and we apply the two theorems of Sections 3 and 4 to find all the basic solutions with permutation group equal to a brace with multiplicative group isomorphic to $\mathbb{Z}/(p^n)$ for some prime $p$.

2 Preliminary results

**Definition 2.1** A left brace is a set with two binary operations, an addition $+$ and a multiplication $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group, and

$$x \cdot (y + z) + x = x \cdot y + x \cdot z,$$

for all $x, y, z \in X$.

A right brace is defined similarly, but changing the last property by $(y + z) \cdot x + x = y \cdot x + z \cdot x$. When $B$ is both a left and a right brace, we say that $B$ is a (two-sided) brace. A morphism between two left braces $B_1$ and $B_2$ is a map $f : B_1 \rightarrow B_2$ such that $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$.

Now we define some important concepts in the study of left braces. For $x \in B$, we define a map $\lambda_x : B \rightarrow B$ by $\lambda_x(y) = xy - x$ for all $y \in B$. It is known that $\lambda_x$ is an automorphism of the additive group of $B$ and the map $\lambda : B \rightarrow \text{Aut}(B, +)$, defined by $x \mapsto \lambda_x$, is a morphism of groups from the multiplicative group of $B$ to $\text{Aut}(B, +)$. The kernel of this morphism is called the socle

$$\text{Soc}(B) := \{ g \in B \mid \lambda_g = \text{id} \}.$$

We will use in a fundamental way the next result about extensions of braces with respect to the socle.

**Proposition 2.2** ([1, Theorem 2.1]) Let $H$ be an abelian group and $B$ be a left brace. Let $\sigma : (B, \cdot) \rightarrow \text{Aut}(H, +)$ be an injective morphism, and $h : (H, +) \rightarrow \mathbb{Z}/(p^n)$ for some prime $p$. Then there is a unique left brace $B$ such that $B$ is isomorphic to $\langle H, \sigma, h \rangle$. The brace $B$ is constructed as follows:

1. Form the set $B := H \times \mathbb{Z}/(p^n)$.
2. Define the operation $\cdot$ by $(h, m) \cdot (h', m') = (h + h', m + m')$ for all $(h, m), (h', m') \in B$.
3. Define the operation $+$ by $(h, m) + (h', m') = (h + h', m + m')$ for all $(h, m), (h', m') \in B$.
4. Define the map $\lambda_x : B \rightarrow B$ by $\lambda_x((h, m)) = (h + x, m + x)$ for all $(h, m) \in B$.

The brace $B$ is the extension of $H$ by $\sigma$ and $h$.
\((B,+)\) be a surjective morphism. Suppose that they satisfy \(h(\sigma(g)(m)) = \lambda_g(h(m))\) for all \(g \in B\) and \(m \in H\). Then, the multiplication over \(H\) given by
\[
x \cdot y := x + \sigma(h(x))(y) \quad \forall x, y \in H,
\]
defines a structure of left brace on \(H\) such that \(h\) is a morphism of left braces, \(\text{Soc}(H) = \text{Ker}(h)\) and \(H/\text{Soc}(H) \cong B\) as left braces.

Two of these structures, determined by \(\sigma, h\) and \(\sigma', h'\) respectively, are isomorphic if and only if there exists an \(F \in \text{Aut}(H,+)\) such that
\[
\sigma'(h'(m)) = F^{-1} \circ \sigma(h(F(m))) \circ F,
\]
for all \(m \in H\).

Conversely, suppose that \(G\) is a left brace. Then, the map \(\sigma: (G/\text{Soc}(G), \cdot) \rightarrow \text{Aut}(G,+)\) induced by the map \(\lambda: (G, \cdot) \rightarrow \text{Aut}(G,+),\) and the natural map \(h: G \rightarrow G/\text{Soc}(G)\) satisfy the above properties.

The importance of this algebraic structure is its relation with a class of set-theoretic solutions of the Yang-Baxter equation, the non-degenerate involutive ones. Given a set \(X\), recall that a map \(r: X \times X \rightarrow X \times X\) is a set-theoretic solution of the Yang-Baxter equation if
\[
r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}, \tag{2}
\]
where \(r_{ij}\) denotes the map \(X \times X \times X \rightarrow X \times X \times X\) acting as \(r\) in the \((i,j)\) component and as the identity on the remaining component. If we write \(r\) as
\[
r: (x, y) \mapsto (\sigma_x(y), \gamma_y(x)),
\]
\(r\) is said to be involutive if \(r^2 = \text{id}_{X^2}\). Moreover, it is said to be non-degenerate if each map \(\sigma_x\) and \(\gamma_x\) is bijective. In what follows, we will only consider non-degenerate involutive set-theoretic solutions.

**Convention:** By a solution of the YBE we mean a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

Two groups are very important to study this type of solutions. The first one, the structure group \(G(X, r)\), is defined by the presentation
\[
G(X, r) = \langle x \in X \mid xy = \sigma_x(y)\gamma_y(x), \text{ for all } x, y \in X, \rangle,
\]
where \(r(x,y) = (\sigma_x(y), \gamma_y(x))\). In [11] it is proved that the map \(X \rightarrow \mathbb{Z}^{(X)} \rtimes \text{Sym}_X\) defined by \(x \mapsto (x, \sigma_x),\) for all \(x \in X,\) extends to a homomorphism
\[
g \mapsto (\pi(g), \phi(g))
\]

\footnote{To be precise, (2) is the so-called braid equation. But, \(r\) is a solution of the braid equation if and only if \(\tau \circ r\) is a set-theoretic solution of the Yang-Baxter equation (defined in the introduction), where \(\tau(x,y) = (y,x)\).}
such that the map \( \pi : G(X, r) \to Z(X) \) is a bijective 1-cocycle. \( Z(X) \) denotes the free abelian group with basis \( X \).

It is easy to see that the group \( G(X, r) \) with the addition defined by

\[
g + h := \pi^{-1}(\pi(g) + \pi(h)),
\]

for all \( g, h \in G(X, r) \), is a left brace. Note that the additive group of \( G(X, r) \) is a free abelian group with basis \( X \). Furthermore, for \( g \in G(X, r) \) and \( x \in X \), we have that \( \lambda_g(x) = gx - g = \phi(g)(x) \in X \).

The second important group, the permutation group of \((X, r)\), is defined by

\[
\mathcal{G}(X, r) := \{ \phi(g) \mid g \in G(X, r) \} \leq \text{Sym}_X,
\]
or, equivalently,

\[
\mathcal{G}(X, r) := \langle \sigma_x \mid x \in X \rangle.
\]

Note that \( \mathcal{G}(X, r) \) with the addition defined by \( \phi(g) + \phi(h) = \phi(g + h) \), for all \( g, h \in G(X, r) \), is a left brace and the natural projection \( G(X, r) \to \mathcal{G}(X, r) \) is a homomorphism of left braces with kernel equal to the socle of \( G(X, r) \).

The next result gives a first way to construct solutions of the YBE. For references of this result, we can find a homological version of it in [4, Corollary D], and a first version without a formal statement in [11] pages 182–183.

**Proposition 2.3** Let \( G \) be a left brace, and let \( Y \) be a set. Suppose that \( h : Z(Y) \to (G, +) \) is a surjective morphism, and that \( \sigma : (G, \cdot) \to \text{Aut}(Z(Y)) \) is an injective morphism such that \( \sigma(g) \mid_Y \) is a bijection of \( Y \) for all \( g \in G \), and \( h(\sigma(g)(m)) = \lambda_g(h(m)) \) for all \( g \in G \) and \( m \in Z(Y) \). Let \( s \) be the map

\[
s : Y \times Y \to Y \times Y
\]

\[
(x, y) \mapsto (f_x(y), f_{f_x(y)}^{-1}(x)),
\]

where \( f_x(y) := \sigma(h(x))(y) \). Then \((Y, s)\) is a solution of the YBE and \( \mathcal{G}(Y, s) \cong G \) as left braces. Moreover, any solution \((Z, t)\) of the YBE with \( \mathcal{G}(Z, t) \cong G \) is of this form.

**Proof.** By Proposition 2.2, the abelian group \( Z(Y) \) with the multiplication defined by

\[
x \cdot y := x + \sigma(h(x))(y) \quad \forall x, y \in Z(Y),
\]
is a left brace and \( h \) becomes a homomorphism of left braces. Note that for \( x, y \in Y \subseteq Z(Y) \), we have that

\[
\lambda_x(y) = xy - x = x + \sigma(h(x))(y) - x = \sigma(h(x))(y) = f_x(y).
\]

Therefore \((Y, s)\) is a solution of the YBE because it is the restriction of the solution of the YBE associated with the left brace \( Z(Y) \) (cf. [7, Lemma 2]). In fact the left brace \( Z(Y) \) is equal to the left brace \( G(Y, s) \). Recall that the addition of the left brace

\[
\mathcal{G}(Y, s) = \{ \sigma(h(m)) \mid_Y \mid m \in Z(Y) \}
\]
is defined by \( \sigma(h(m_1))Y + \sigma(h(m_2))Y = \sigma(h(m_1 + m_2))Y \), for all \( m_1, m_2 \in \mathbb{Z}(Y) \). Therefore the map \( G \rightarrow \mathcal{G}(Y, s) \) defined by \( g \mapsto \sigma(g)Y \) is an isomorphism of left braces.

On the other hand, observe that, if \((Z, t)\) is a solution of the YBE and \(\eta: G \rightarrow \mathcal{G}(Z, t)\) is an isomorphism of left braces, then the unique homomorphism \( h: \mathbb{Z}(Z) \rightarrow (G, +) \) such that \( h(z) = \eta^{-1}(\pi(z)) \), for all \( z \in Z \), where \( \pi: G(Z, t) \rightarrow \mathcal{G}(Z, t) \) is the natural projection, is surjective, and the map \( \sigma : (G, \cdot) \rightarrow \text{Aut}(\mathbb{Z}(Z)) \), where \( \sigma(g) \) is the unique automorphism of \( \mathbb{Z}(Y) \) such that \( \sigma(g)(z) = \eta(g)(z) \), for all \( z \in Z \), is an injective homomorphism. Furthermore, for \( g \in G \), there exists \( m \in \mathbb{Z}(Z) \) such that \( g = h(m) \) and

\[
  h(\sigma(g)(z)) = h(\eta(g)(z)) = \eta^{-1}(\pi(\eta(h(m))(z)))
  = \eta^{-1}(\pi(\pi(m)(z))) = \eta^{-1}(\pi(m)\pi(z) - \pi(m))
  = h(m)h(z) - h(m) = gh(z) - g
  = \lambda_g(h(z)),
\]

for all \( z \in Z \). Therefore \( h(\sigma(g)(n)) = \lambda_g(h(n)) \), for all \( g \in G \) and all \( n \in \mathbb{Z}(Z) \). Now we have that

\[
  \sigma(h(x))(y) = \eta(h(x))(y) = \pi(x)(y),
\]

for all \( x, y \in Z \). Hence \((Z, t)\) is exactly the same solution of the YBE that the solution obtained by the given construction using the maps \( h \) and \( \sigma \).

Note that this reduces the problem of finding all the solutions of the YBE to the problem of finding the two maps \( h \) and \( \sigma \) with the required properties. This has the disadvantage that we do not know in principle how to construct these two maps. Our aim in the next sections is trying to solve this difficulty.

### 3 Construction of solutions

Given a left brace \( G \), we will try to construct all the involutive non-degenerate set-theoretic solutions of the Yang-Baxter equation \((Y, s)\) such that \( \mathcal{G}(Y, s) \cong G \) as left braces. As we proved in the previous section, this is equivalent to construct two maps with some properties.

The next result gives a translation of the problem that only depends on the brace structure of \( G \). Recall that, for a subgroup \( K \) of a group \( H \), we define the core of \( K \) in \( H \) as \( \text{core}(K) := \bigcap_{g \in H} gKg^{-1} \). It is the maximal normal subgroup of \( H \) contained in \( K \).

**Theorem 3.1** Let \( G \) be a left brace. Let \( X \) be a subset of \( G \), invariant by the action \( \lambda \), such that \( X \) generates \( G \) additively. Let \( X = \bigcup_{i \in I} X_i \) be the decomposition of \( X \) as disjoint union of orbits, and choose an element \( x_i \) of \( X_i \) for any \( i \in I \). Let \( H_i \) be the stabilizer \( \text{St}(x_i) \) of \( x_i \) in \( G \). For each \( i \in I \), let \( J_i \) be a nonempty set, and let \( (K_{i,j})_{j \in J_i} \) be a family of subgroups of \( H_i \) such that
\[ \bigcap_{i \in I} \bigcap_{j \in J_i} \text{core}(K_{i,j}) = \{1\} \]. Let \( Y := \bigcup_{i \in I} \bigcup_{j \in J_i} G/K_{i,j} \) be the disjoint union of the sets of left cosets \( G/K_{i,j} \). Then, \((Y, s)\), where \( s \) is the map

\[
\begin{align*}
  s : \quad Y \times Y & \to Y \times Y \\
  (g_1 K_{i_1, j_1}, g_2 K_{i_2, j_2}) & \to (f_{g_1 K_{i_1, j_1}}(g_2 K_{i_2, j_2}), f_{g_1 K_{i_1, j_1}}^{-1}(g_2 K_{i_2, j_2})(g_1 K_{i_1, j_1})),
\end{align*}
\]

with \( f_{g_1 K_{i_1, j_1}}(g_2 K_{i_2, j_2}) = \lambda_{g_1}(x_{i_1})g_2 K_{i_2, j_2} \), is a solution of the YBE such that \( \mathcal{G}(Y, s) \cong G \) as left braces.

Moreover, any solution \((Z, t)\), with \( \mathcal{G}(Z, t) \cong G \) as left braces, is isomorphic to one of this form.

**Proof.** First we shall prove that \((Y, s)\) is a solution of the YBE such that \( \mathcal{G}(Y, s) \cong G \) as left braces. We define \( h : Y \to X \) by \( h(g K_{i,j}) = \lambda_g(x_i) \), and define \( \sigma : G \to \text{Sym}_Y \) to be the natural action of \( G \) on \( Y \) given by left multiplications on the cosets in \( G/K_{i,j} \); i.e. \( \sigma(g)(x K_{i,j}) := gx K_{i,j} \). Note that \( h \) extends to a unique epimorphism \( h : \mathbb{Z}^Y \to (G, +) \) and, for each \( g \in G \), \( \sigma(g) \) extends to a unique automorphism of \( \mathbb{Z}^Y \). Since

\[
h(\sigma(g)(x K_{i,j})) = h(g x K_{i,j}) = \lambda_g(x_i)
\]

we have that \( h(\sigma(g)(m)) = \lambda_g(h(m)) \), for all \( g \in G \) and \( m \in \mathbb{Z}^Y \). Let \( g \in G \) be an element such that \( \sigma(g) = \text{id} \). Hence \( gx K_{i,j} = x K_{i,j} \), for all \( x \in G \), \( i \in I \) and \( j \in J_i \). Thus \( g \in \text{core}(K_{i,j}) \), for all \( i \in I \) and \( j \in J_i \). Since \( \bigcap_{i \in I} \bigcap_{j \in J_i} \text{core}(K_{i,j}) = \{1\} \), we have \( g = 1 \). Therefore \( \sigma \) is injective. Hence, by Proposition 2.3, \((Y, s)\) is a solution of the YBE and \( \mathcal{G}(Y, s) \cong G \) as left braces.

Let \((Z, t)\) be a solution of the YBE such that \( \mathcal{G}(Z, t) \cong G \) as left braces. Let \( \eta : \mathcal{G}(Z, t) \to G \) be an isomorphism of left braces. Let \( h = \eta \circ \phi \), where \( \phi : G(Z, t) \to \mathcal{G}(Z, t) \) is the natural projection. Then \( h(Z) \) is a subset of \( G \), invariant by \( \lambda \) which generates \( G \) additively. Let \( X = h(Z) \).

We also have an injective morphism \( \sigma : (G, \cdot) \to \text{Aut}(\mathbb{Z}^Z) \), such that \( \sigma(g)(z) = \eta^{-1}(g)(z) \), for all \( g \in G \) and \( z \in Z \). Therefore \( Z \) is a left \( G \)-set with the action induced by \( \sigma \). Let \( a \in G \) and \( z \in Z \). Let \( g \in G(Z, t) \) such that \( \phi(g) = \eta^{-1}(a) \). We have

\[
\begin{align*}
  h(\sigma(a)(z)) &= \eta(\phi(\eta^{-1}(a)(z))) = \eta(\phi(g)(z)) \\
  &= \eta(\phi(gz - g)) = \eta(\phi(g))\eta(\phi(z)) - \eta(\phi(g)) \\
  &= ah(z) - a = \lambda_a(h(z)).
\end{align*}
\]

Therefore the restriction \( h|_Z : Z \to X \) of \( h \) is a \( G \)-map.

Let \( X = \bigcup_{i \in I} X_i \) be the decomposition of \( X \) as disjoint union of orbits under the action \( \lambda \). Since \( h|_Z \) is a surjective \( G \)-map, for all \( i \in I \), the action \( \sigma \) splits \((h|_Z)^{-1}(X_i)\) into orbits: \((h|_Z)^{-1}(X_i) = \bigcup_{j \in J_i} Z_{i,j} \) and \( h(Z_{i,j}) = X_i \). So we have \( Z = \bigcup_{i \in I} \bigcup_{j \in J_i} Z_{i,j} \), where \( h(Z_{i,j}) = X_i \) for all \( i, j \).

For each \( i \in I \), we choose an element \( x_i \in X_i \), and for each \( j \in J_i \), we choose \( z_{i,j} \in Z_{i,j} \) such that \( h(z_{i,j}) = x_i \). Note that \( \text{St}(z_{i,j}) \leq \text{St}(x_i) \leq G \),
Therefore \( f \) is an isomorphism of solutions of the YBE.

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Summarizing, given a left brace \( G \), to construct all the solutions \((Y,s)\) of the YBE such that \( G(Y,s) \cong G \) as left braces one can proceed as follows:

1. Find the decomposition of \( G \) as disjoint union of orbits, \( G = \bigcup_{i \in K} G_i \), by the action \( \lambda : (G,\cdot) \to \text{Aut}(G,+) \). Then choose one element \( x_i \) in each orbit \( G_i \) for all \( i \in K \).

2. Find all the subsets \( I \) of \( K \) such that the subset \( X = \bigcup_{i \in I} G_i \) generates the additive group of \( G \).

3. Given such an \( X \), find for each \( i \in I \) a non-empty family \((K_{i,j})_{j \in I_i}\) of subgroups of \( \text{St}(x_i) \) such that \( \bigcap_{i,j} \text{core}(K_{i,j}) = \{1\} \). Note that the \( K_{i,j} \) could be equal for different \((i,j)\).
4. Construct a solution as in the statement of Theorem 3.1 using the families $(K_{i,j})_{j \in J_i}$ for $i \in I$.

Note that by Theorem 3.1, any solution $(Y, s)$ of the YBE such that $G(Y, s) \cong G$ as left braces is isomorphic to one constructed in this way. It could happen that different solutions of the YBE constructed in this way from a left brace $Y$ are in fact isomorphic. In the next section we characterize when two of these solutions are isomorphic.

4 Isomorphism of solutions

We begin with a left brace $G$ with orbits $\{G_i\}_{i \in I}$ under the action $\lambda: G \to \text{Aut}(G, +)$ defined as usual $\lambda(g) = \lambda_g$. Choose an element $x_i \in G_i$ in each orbit $G_i$. Let $I_1, I_2$ be subsets of $I$, such that $X_1 = \cup_{i \in I_1} G_i$ and $X_2 = \cup_{j \in I_2} G_j$ satisfy $G = (X_1)_+ = (X_2)_+$. For each $i \in I_1$, let $\{K_{i,k}\}_{k \in A_i}$ be a non-empty family of subgroups of $\text{St}_{G_i}(x_i)$ such that

$$\bigcap_{i \in I_1} \bigcap_{k \in A_i} \text{core}(K_{i,k}) = \{1\}.$$ 

For each $j \in I_2$, let $\{L_{j,l}\}_{l \in B_j}$ be a non-empty family of subgroups of $\text{St}_{G_j}(x_j)$ such that

$$\bigcap_{j \in I_2} \bigcap_{l \in B_j} \text{core}(L_{j,l}) = \{1\}.$$ 

Let $Y_1$ be the disjoint union of the family of left $G$-sets $G/K_{1,k}$, for $i \in I_1$ and $k \in A_i$. Let $Y_2$ be the disjoint union of the family of left $G$-sets $G/L_{j,l}$, for $j \in I_2$ and $l \in B_j$. Let $s_1: Y_1^2 \to Y_1^2$ and $s_2: Y_2^2 \to Y_2^2$ be maps defined by

$$s_1(g_1 K_{i_1,k_1}, g_2 K_{i_2,k_2}) = (\lambda_{g_1} (x_{i_1}) g_2 K_{i_2,k_2}, \lambda_{g_1} (x_{i_1}) g_2 (x_{i_2})^{-1} g_1 K_{i_1,k_1}),$$

$$s_2(g_1 L_{j_1,l_1}, g_2 L_{j_2,l_2}) = (\lambda_{g_1} (x_{j_1}) g_2 L_{j_2,l_2}, \lambda_{g_1} (x_{j_1}) g_2 (x_{j_2})^{-1} g_1 L_{j_1,l_1}).$$

We know that $(Y_1, s_1)$ and $(Y_2, s_2)$ are solutions of the YBE (and any solution of the YBE is isomorphic to one constructed in this way). We shall characterize when $(Y_1, s_2)$ and $(Y_2, s_2)$ are isomorphic in the following result.

**Theorem 4.1** The solutions $(Y_1, s_1)$ and $(Y_2, s_2)$ are isomorphic if and only if there exist an automorphism $\psi$ of the left brace $G$, a bijective map $\alpha: I_1 \to I_2$, a bijective map $\beta: A_i \to B_{\alpha(i)}$ and $z_{i,k} \in G$, for each $i \in I_1$ and $k \in A_i$, such that

$$\psi(x_i) = \lambda z_{i,k}(x_{\alpha(i)}) \quad \text{and} \quad \psi(K_{i,k}) = z_{i,k} L_{\alpha(i),\beta_i(k)} z_{i,k}^{-1},$$

for all $i \in I_1$ and $k \in A_i$.

**Proof.** Suppose that there exist an automorphism $\psi$ of the left brace $G$, a bijective map $\alpha: I_1 \to I_2$, a bijective map $\beta: A_i \to B_{\alpha(i)}$ and $z_{i,k} \in G$, for $i \in I_1$, and for $k \in A_i$, such that

$$\psi(x_i) = \lambda z_{i,k}(x_{\alpha(i)}) \quad \text{and} \quad \psi(K_{i,k}) = z_{i,k} L_{\alpha(i),\beta_i(k)} z_{i,k}^{-1}.$$
Observe that we also have \( \psi(\lambda_g(x_i)) = \lambda_{\psi(\phi)}(\alpha(x_i)) \) for every \( g \), because \( \psi \) is a morphism of braces. We define \( F: Y_1 \to Y_2 \) by \( F(gK_{i,k}) = \psi(g)(z_{i,k}L_{\alpha(\beta_{i,k})}) \), for all \( i \in I_1, k \in A_i \), and \( g \in G \). Since \( \psi(K_{i,k}) = z_{i,k}L_{\alpha(\beta_{i,k})}z_{i,k}^{-1} \), \( F \) is well defined. It is easy to check that \( F \) is an isomorphism of the solutions \((Y_1,s_1)\) and \((Y_2,s_2)\).

Conversely, suppose that there exists an isomorphism \( F: Y_1 \to Y_2 \) of the solutions \((Y_1,s_1)\) and \((Y_2,s_2)\). We can write \( F(gK_{i,k}) = \varphi(gK_{i,k})L_{\alpha(\beta_{i,k})} \), for some maps \( \varphi: Y_1 \to G, \alpha: Y_1 \to I_2 \) and \( \beta: Y_1 \to \bigcup_{i \in I_2} B_j \). We shall prove that \( \alpha(g,i,k) = \alpha(1,i,k') \) and \( \beta(g,i,k) = \beta(1,i,k) \), for all \( g \in G, i \in I_1 \) and \( k, k' \in A_i \). Since \( F \) is a morphism of solutions of the YBE, we have

\[
F(\lambda_{g_1}(x_{i_1})g_2K_{i_2,k_2}) = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)})F(g_2K_{i_2,k_2}),
\]

for all \( g_1, g_2 \in G, i_1, i_2 \in I_1, k_1 \in A_i, \) and \( k_2 \in A_{i_2} \). Hence

\[
\varphi(\lambda_{g_1}(x_{i_1})g_2K_{i_2,k_2})L_{\alpha(\lambda_{g_1}(x_{i_1})g_2,i_2,k_2),\beta(\lambda_{g_1}(x_{i_1})g_2,i_2,k_2)} = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)})\varphi(g_2K_{i_2,k_2})L_{\alpha(g_2,i_2,k_2),\beta(g_2,i_2,k_2)},
\]

for all \( g_1, g_2 \in G, i_1, i_2 \in I_1, k_1 \in A_i, \) and \( k_2 \in A_{i_2} \). Thus \( \alpha(g_1,i_1,k_2) = \alpha(g_2,i_2,k_2) \) and \( \beta(g_1,i_1,k_2) = \beta(g_2,i_2,k_2) \). Since \( G = \langle X_1 \rangle \) and \( X_1 \) is \( G \)-invariant (by the action \( \lambda \)), we know that \( X_1 \) also generates the multiplicative group of \( G \). Therefore \( \alpha(g_2,i_2,k_2) = \alpha(1,i_2,k_2) \) and \( \beta(g_2,i_2,k_2) = \beta(1,i_2,k_2) \).

Note also that

\[
\lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)})F(g_2K_{i_2,k_2}) = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)})F(g_2K_{i_2,k_2}),
\]

for all \( g_1, g_2 \in G, i_1, i_2 \in I_1, k_1 \in A_i, \) and \( k_2 \in A_{i_2} \). Since \( \bigcap_{i \in I_2} \bigcap_{j \in B_j} \text{core}(\Lambda_{j,i}) = \{1\} \), we have that

\[
\lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)}) = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(g_1,i_1,k_1)}),
\]

for all \( g_1 \in G, i_1 \in I_1 \) and \( k_1 \in A_i \). Therefore \( x_{\alpha(g_1,i_1,k_1)} \) is \( G(1,i_1,k_1) \)-invariant for all \( g_1 \in G, i_1 \in I_1 \) and \( k_1 \in A_i \), and thus \( \alpha(g,i,k) = \alpha(1,i,k') \), for all \( g \in G, i \in I_1 \) and \( k, k' \in A_i \). For each \( i \in I_1 \) we choose an element \( k_i \in A_i \). Since \( F \) is bijective, the map \( I_1 \to I_2 \) defined by \( i \mapsto \alpha(1,i,k_i) \) is bijective and for each \( i \in I_1 \) the map \( A_i \to B_{\alpha(1,i,k_i)} \) defined by \( k \mapsto \beta(1,i,k) \) is bijective. We shall see that there exists an automorphism \( \psi \) of the left brace \( G \) such that

\[
\psi(\lambda_g(x_i)) = \lambda_{\psi(\varphi)}(\alpha(1,i,k_i)) \quad \text{and} \quad \psi(K_{i,k}) = \varphi(K_{i,k})L_{\alpha(\beta_{i,k}),\beta(1,i,k')}^{-1},
\]

for all \( g \in G, i \in I_1 \) and \( k \in A_i \). Let \( 1 = \lambda_{g_1}(x_{i_1})^{\varepsilon_1} \cdots \lambda_{g_m}(x_{i_m})^{\varepsilon_m} \), for some \( g_1, \ldots, g_m \in G, i_1, \ldots, i_m \in I_1 \) and \( \varepsilon_1, \ldots, \varepsilon_m \in \{1,-1\} \). By (3), we have

\[
F(gK_{i,k}) = F(\lambda_{g_1}(x_{i_1})^{\varepsilon_1} \cdots \lambda_{g_m}(x_{i_m})^{\varepsilon_m} gK_{i,k}) = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(1,i_1,k_1)})^{\varepsilon_1} F(\lambda_{g_2}(x_{i_2})^{\varepsilon_2} \cdots \lambda_{g_m}(x_{i_m})^{\varepsilon_m} gK_{i,k}) = \lambda_{\varphi(g_1K_{i_1,k_1})}(x_{\alpha(1,i_1,k_1)})^{\varepsilon_1} \cdots \lambda_{\varphi(g_mK_{i_m,k_m})}(x_{\alpha(1,i_m,k_m)})^{\varepsilon_m} F(gK_{i,k}),
\]

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for all \( g \in G, i \in I_1 \) and \( k \in A_i \). Since \( \bigcap_{j \in I_2} \bigcap_{l \in B_j} \text{core}(L_{j,l}) = \{1\} \), we have that 
\[
\lambda_{\varphi(gK_{i,k})}((x_{\alpha(i),k_1}))^{g_1} \cdots \lambda_{\varphi(gK_{i,k})}((x_{\alpha(i),k_m}))^{g_m} = 1.
\]
Therefore there exists a unique morphism \( \psi: G \to G \) of multiplicative groups such that 
\[
\psi(\lambda_g(x_i)) = \lambda_{\varphi(gK_{i,k})}(x_{\alpha(i),k}).
\]
Since \( X_1 \) generates the multiplicative group of \( G \), by [3] one can see that 
\[
\varphi(gK_{i,k})L_{\alpha(1,i,k_i),\beta(1,i,k)} = F(gK_{i,k}) = \psi(g)\varphi(K_{i,k})L_{\alpha(1,i,k_i),\beta(1,i,k)}.
\]
Therefore, since \( L_{\alpha(1,i,k_i),\beta(1,i,k)} \subseteq \text{St}(x_{\alpha(1,i,k_i)}) \), we have 
\[
\lambda_{\varphi(gK_{i,k})}(x_{\alpha(1,i,k_i)}) = \lambda_{\psi(g)\varphi(K_{i,k})}(x_{\alpha(1,i,k_i)}).
\]
Hence \( \psi(\lambda_g(x_i)) = \lambda_{\psi(g)\varphi(K_{i,k})}(x_{\alpha(1,i,k_i)}) \). Now we have that 
\[
\psi(g + x_i) = \psi(g\lambda_{g^{-1}}(x_i)) = \psi(g)\psi(\lambda_{g^{-1}}(x_i)) = \psi(g)\lambda_{\psi(g^{-1}\varphi(K_{i,k})}(x_{\alpha(1,i,k_i)})) = \psi(g) + \lambda_{\varphi(K_{i,k})}(x_{\alpha(1,i,k_i)}) = \psi(g) + \psi(\lambda_1(x_i)) = \psi(g) + \psi(x_i).
\]
Now it is easy to see that \( \psi \) is a morphism of left braces. Since \( F \) is bijective and \( F(gg'K_{i,k}) = \psi(g)F(g'K_{i,k}) \), it follows that \( \psi \) is bijective. Furthermore \( g \in K_{i,k} \) if and only if 
\[
\varphi(K_{i,k})L_{\alpha(1,i,k_i),\beta(1,i,k)} = F(K_{i,k}) = F(gK_{i,k}) = \psi(g)F(K_{i,k}) = \psi(g)\varphi(K_{i,k})L_{\alpha(1,i,k_i),\beta(1,i,k)}.
\]
Therefore the result follows. }

Summarizing, the last theorem says that two solutions constructed as in Theorem 3.3 are isomorphic if we can find an automorphism of the left brace \( G \) that brings each \( K_{i,k} \) to one \( L_{j,l} \), taking into account that maybe the \( L_{j,l} \)'s are permuted (that is the reason for the \( \alpha \) and \( \beta \) maps), and that maybe we have chosen another element of the orbit in the process (that is the reason why the image \( x_i \) is \( \lambda_{z_{i,k}}(x_{\alpha(i)}) \) and not just \( x_{\alpha(i)} \), and it is also the reason why the \( L_{\alpha(i)},\beta(i,k) \) is conjugated by \( z_{i,k} \)).

The following is an example of how to use Theorems 3.3 and 4.1 to compute all the finite solutions associated to a given finite left brace up to isomorphism. We use the easiest examples of braces: trivial braces of order \( p \), where \( p \) is a prime.

**Example 4.2** Consider the trivial brace over \( G = \mathbb{Z}/(p) \). Then, the orbits are \( \{\alpha\} \) for every \( \alpha \in \mathbb{Z}/(p) \). Since any orbit has one element, then \( \text{St}(\alpha) = G \), and the possible \( K_{i,j} \)'s in this case are 0 and \( G \).

Let \( X \) be a subset of \( \mathbb{Z}/(p) \) with at least a nonzero element. Let \( K_{\alpha,j} = G \) for \( \alpha \in X \) and \( j \in \{1,\ldots,k_\alpha\} \), and let \( K'_{\alpha,k} = 0 \) for \( \alpha \in X \) and \( k \in \{1,\ldots,k_\alpha\} \).
\[ \{1, \ldots, m_\alpha\}, \text{ where } k_\alpha \text{ and } m_\alpha \text{ are non-negative integers such that } k_\alpha + m_\alpha > 0. \]

Write \( G/K_{\alpha,k} = \{y_{\alpha,j}\} \), and \( G/K'_{\alpha,k} = \{y^1_{\alpha,k}, \ldots, y^p_{\alpha,k}\} \), where \( y^l_{\alpha,k} = l + K'_{\alpha,k} \).

Assume that at least one \( m_\alpha \) is positive. Then the corresponding solution of the YBE is \((Y, r)\), where

\[
Y = \bigcup_{\alpha \in X} \left( \bigcup_{1 \leq j \leq k_\alpha} \{y_{\alpha,j}\} \right) \cup \left( \bigcup_{1 \leq k \leq m_\alpha} \{y^1_{\alpha,k}, \ldots, y^p_{\alpha,k}\} \right)
\]

and \( r(x, y) = (\sigma_x(y), \sigma^{-1}_{\sigma(y)}(x)) \), with the sigma maps given by

\[
\sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha, \text{ for all } \alpha \in X, \text{ for all } j, k \text{ and for all } l \in \{1, \ldots, p\},
\]

where \( \tau \in \text{Sym}_V \) is the product of all the cycles of length \( p \) \( \{y^1_{\alpha,k}, y^2_{\alpha,k}, \ldots, y^p_{\alpha,k}\} \) for any \( \alpha \in X \) and \( k \in \{1, \ldots, m_\alpha\} \).

Finally observe that, in this case, \( \text{Aut}(G, +, \cdot) = \text{Aut}(G, +) \cong (\mathbb{Z}/(p))^\ast \), and the effect of an automorphism of \( G \) over a solution is to change \( \sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha \) to the isomorphic solution \( \sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^A \), where \( A \in (\mathbb{Z}/(p))^\ast \).

### 5 Basic solutions and examples

It seems difficult to apply Theorems 3.1 and 4.1 to construct and classify all the solutions \((Y, s)\) of the YBE with isomorphic left brace \( G(Y, s) \) because there are a lot of different solutions (for instance, there is a lot of freedom choosing the subgroups \( K_{i,j} \)). Maybe an easier problem is to classify a smaller class of solutions from which we can recover all the other solutions. For example, consider the following ways to obtain new solutions from a given one:

1. Adding a new \( K \) (and the corresponding \( X_i \), if necessary) to \( \{K_{i,j}\} \): this gives a solution because the intersection of the cores remains trivial when we add another subgroup, and \( X \) still generates \( G \) if we add a new \( X_i \).

2. Changing a \( K_{i,j} \) by a \( K \leq K_{i,j} \): this gives a solution because the intersection of the cores remains trivial when we change one of the \( K_{i,j} \) by a smaller subgroup, and we are not changing \( X \).

Thus maybe a good definition for basic solution is the converse of constructions 1 and 2: a solution in which it is impossible to take out a \( K_{i,j} \) or to change a \( K_{i,j} \) by a \( K_{i,j} \leq K'_{i,j} \leq H_i \) without losing the property of being a solution. From these solutions, and using construction 1 and 2, we can recover all the other solutions. Note that these class of solutions can be also described as the solutions \((X, r)\) such that every surjective morphism of solutions \((X, r) \rightarrow (Y, s)\), to another solution \((Y, s)\) with \( G(X, r) \cong G(Y, s) \), is an isomorphism.

**Example 5.1** In the case of Example 4.2, the basic solutions are, up to automorphism of \( G \):

\[
\begin{align*}
    \{1, \ldots, m_\alpha\}, & \text{ where } k_\alpha \text{ and } m_\alpha \text{ are non-negative integers such that } k_\alpha + m_\alpha > 0. \\
    \text{Write } G/K_{\alpha,j} = \{y_{\alpha,j}\}, & \text{ and } G/K'_{\alpha,k} = \{y^1_{\alpha,k}, \ldots, y^p_{\alpha,k}\}, \text{ where } y^l_{\alpha,k} = l + K'_{\alpha,k}. \\
    \text{Assume that at least one } m_\alpha & \text{ is positive. Then the corresponding solution of the YBE is } (Y, r), \text{ where }
\end{align*}
\]

\[
Y = \bigcup_{\alpha \in X} \left( \bigcup_{1 \leq j \leq k_\alpha} \{y_{\alpha,j}\} \right) \cup \left( \bigcup_{1 \leq k \leq m_\alpha} \{y^1_{\alpha,k}, \ldots, y^p_{\alpha,k}\} \right)
\]

and \( r(x, y) = (\sigma_x(y), \sigma^{-1}_{\sigma(y)}(x)) \), with the sigma maps given by

\[
\sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha, \text{ for all } \alpha \in X, \text{ for all } j, k \text{ and for all } l \in \{1, \ldots, p\},
\]

where \( \tau \in \text{Sym}_V \) is the product of all the cycles of length \( p \) \( \{y^1_{\alpha,k}, y^2_{\alpha,k}, \ldots, y^p_{\alpha,k}\} \) for any \( \alpha \in X \) and \( k \in \{1, \ldots, m_\alpha\} \).

Finally observe that, in this case, \( \text{Aut}(G, +, \cdot) = \text{Aut}(G, +) \cong (\mathbb{Z}/(p))^\ast \), and the effect of an automorphism of \( G \) over a solution is to change \( \sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha \) to the isomorphic solution \( \sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^A \), where \( A \in (\mathbb{Z}/(p))^\ast \).
1. \(X_1 = \{1\}, K_{1, 1} = 0\). It corresponds to the solution over the set \(\{1, \ldots, p\}\) given by \(\sigma_1 = \cdots = \sigma_p = (1, 2, \ldots, p)\).

2. \(\{X_1 = \{0\}, X_2 = \{1\}\}, \{K_{1, 1} = 0, K_{2, 1} = G\}\). It corresponds to the solution over the set \(\{1, \ldots, p + 1\}\) given by \(\sigma_1 = \cdots = \sigma_p = \text{id}, \sigma_{p+1} = (1, 2, \ldots, p)\).

So, when we only consider the class of basic solutions, the classification for braces of order \(p\) turns out to be much easier. This example brings some hope that maybe it is possible to classify the basic solutions associated to any finite left brace \(G\).

We are not able to solve this problem in general because our knowledge of the brace structure is still limited. Nevertheless, we will classify now the basic solutions for some concrete examples of finite left braces. Note that good candidates are multiplicative groups such that the intersection of all their non-trivial subgroups is non-trivial, because in this case we always need some \(K_{i,j}\), equal to \(\{1\}\), and this condition might restrict the possible basic solutions. Groups with this property must be \(p\)-groups, and they have been classified in \(\text{[17]}\) Theorem 5.4.10 (ii): they are the cyclic \(p\)-groups and the generalized quaternion groups \(Q_{2^m}\) of order a power of 2.

We shall study first the cyclic case. We need a complete knowledge of all the possible brace structures with cyclic multiplicative group. First of all, the next proposition describes the additive group in this case.

**Proposition 5.2** (converse result to Rump’s classification) Let \(G\) be a left brace with \((G, \cdot) \cong \mathbb{Z}/(p^n)\) and \(n \geq 3\). Then, \((G, +) \cong \mathbb{Z}/(p^n)\).

**Proof.** We prove it by induction over \(n\). The case \(n = 3\) is true by the classification of braces of order \(p^3\) in \([1]\). For \(n > 3\), assume that it is true for \(n - 1\). Then, since \((G, \cdot)\) is abelian, \(G\) is a two-sided brace, so \(\text{Soc}(G) \neq 0\) by \([7]\) Proposition 3. Thus there exists an element \(x \neq 0\) in \(\text{Soc}(G)\) of multiplicative order \(p\). Note that in this case \(\text{Soc}(G) = \text{Fix}(\lambda_\xi)\), where \(\xi\) is a multiplicative generator of \(G\). This implies that \(\langle x \rangle = \langle x \rangle_+\) is an ideal of \(G\). Then \(G/\langle x \rangle\) is a brace of order \(p^{n-1}\) with \((G/\langle x \rangle, \cdot) \cong \mathbb{Z}/(p^{n-1})\). By induction hypothesis, \((G/\langle x \rangle, +) \cong \mathbb{Z}/(p^{n-1})\). This restricts the possible additive groups of \(G\) to only two cases: \(\mathbb{Z}/(p^n)\) or \(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1})\). If \((G, +) \cong \mathbb{Z}/(p^n)\), we are done, so assume \((G, +) \cong \mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1})\) to arrive to a contradiction.

The element \(x\) in \(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1})\) must be of the form \((\alpha, p^{n-2} \beta)\) because it has additive order equal to \(p\). Moreover, it must be a fix element of the automorphism \(\lambda_\xi\). An automorphism of \(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1})\) can be written in the matrix form \(\begin{pmatrix} a & b \\ p^{n-2}c & d \end{pmatrix}\) with \(ad \neq 0\) (mod \(p\)), where the entries in the first row are elements of \(\mathbb{Z}/(p)\), and the entries in the second one are elements of \(\mathbb{Z}/(p^{n-1})\). It is a compact way to express that any endomorphism \(f\) of \(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1})\) satisfies \(f(x, y) = xf(1, 0) + yf(0, 1)\), so it is determined by \(f(1, 0) = (a, p^{n-2}c)\) (which has to be an element of order \(p\)), and \(f(0, 1) = (b, d)\). The condition \(ad \neq 0\) (mod \(p\)) ensures that \(f\) is bijective.
Thus the order of $\text{Aut}(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1}))$ is $(p - 1) \cdot p \cdot (p^n - p - 1) = p^{n+1}(p - 1)^2$. Then, a $p$-Sylow subgroup of $\text{Aut}(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1}))$ is

$$\left\{ \begin{pmatrix} 1 & b \\
   p^{n-2}c & 1 + pd 
\end{pmatrix} : b \in \mathbb{Z}/(p), \ c, d \in \mathbb{Z}/(p^{n-1}) \right\}. $$

Since $\lambda_\xi$ has prime power order, after a suitable conjugation, it can be written as $\begin{pmatrix} 1 & b \\
   p^{n-2}c & 1 + pd 
\end{pmatrix}$. So a fixed element $x = (\alpha, p^{n-2}\beta)$ must satisfy

$$\begin{pmatrix} \alpha \\
   p^{n-2}\beta 
\end{pmatrix} \begin{pmatrix} 1 & b \\
   p^{n-2}c & 1 + pd 
\end{pmatrix} \begin{pmatrix} \alpha \\
   p^{n-2}\beta 
\end{pmatrix} = \begin{pmatrix} \alpha - 2c\alpha + p^{n-2}\beta \\
   p^{n-2}\beta 
\end{pmatrix}.$$

This is satisfied if $c = 0$ or if $\alpha = 0$. If $\alpha = 0$, then $x = (0, p^{n-2}\beta)$. But

$$(\mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-1}))/((0, p^{n-2}\beta)) \cong \mathbb{Z}/(p) \times \mathbb{Z}/(p^{n-2}),$$

a contradiction. On the other hand, if $c = 0$, then

$$\lambda_\xi^k = \begin{pmatrix} 1 \\
   0 
\end{pmatrix} \begin{pmatrix} kb \\
   (1 + pd)^k 
\end{pmatrix}.$$

Thus, for $j = p^{n-1}$, we get

$$\text{Id} + \lambda_\xi + \lambda_\xi^2 + \cdots + \lambda_\xi^{p^{n-1}-1} = 0,$$

because $(p^{n-1}) \equiv 0 \pmod{p}$, and

$$\sum_{i=0}^{p^{n-1}-1} (1 + pd)^i = \frac{(1 + pd)^{p^{n-1}} - 1}{pd} \equiv 0 \pmod{p^{n-1}}$$

using [24, Lemma 4]. But this implies $\xi^{p^{n-1}} = (\text{Id} + \lambda_\xi + \lambda_\xi^2 + \cdots + \lambda_\xi^{p^{n-1}-1})(\xi) = 0$, a contradiction with the fact that $\xi$ has multiplicative order equal to $p^n$. 

Fortunately, the complete classification of all the left brace structures with cyclic additive group of order $p^n$ is known [24]. That allows us to determine all the basic solutions in the next example.

**Example 5.3** Now that we know that the additive group is always cyclic when the multiplicative group is $\mathbb{Z}/(p^n)$ with $n \geq 3$, we can apply Rump’s classification [24]. If the multiplicative group is also cyclic, there is a brace structure $G_i$ for every $i \in \{1, \ldots, n\}$, given as a product over $\mathbb{Z}/(p^n)$ by $x \cdot y := x + y + p^i xy$, except for $p = 2$, where $i \in \{2, \ldots, n\}$ (see [24, Theorem 1]). Note that, through all this example, we use a product with a dot to represent the multiplication in a brace, and a product without dot to represent the usual ring product over $\mathbb{Z}/(p^n)$. 

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Then, it is easy to determine the socle of these braces $G_i$:

$$\text{Soc}(G_i) = \langle p^{n-i} \rangle \cong \mathbb{Z}/(p^i).$$

We also need to determine the brace automorphism group:

$$\text{Aut}(G_i, +, \cdot) \cong \{1 + p^{n-i}k : k \in \mathbb{Z}/(p^n)\} \leq (\mathbb{Z}/(p^n))^*,$$

except when $i = n$, which is

$$\text{Aut}(G_n, +, \cdot) = \text{Aut}(G_n, +) \cong (\mathbb{Z}/(p^n))^*.$$

To compute the orbits, observe that any element $a \in \mathbb{Z}/(p^n)$ can be written as

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_{n-1} p^{n-1},$$

with $a_i \in \{0, 1, \ldots, p^i - 1\}$, and the action of the lambda maps over one of these elements is

$$\lambda_x(a) = (1 + p^i x)(a_0 + a_1 p + a_2 p^2 + \cdots + a_{n-1} p^{n-1})$$

$$= (a_0 + a_1 p + \cdots + a_{n-1} p^{n-1}) + (a_i + a_0 x)p^i + \cdots + (a_{n-1} + a_{n-1} x)p^{n-1},$$

for each $x \in \mathbb{Z}/(p^n)$. Note that if $a_0 = \cdots = a_{n-i-1} = 0$, then $\lambda_x(a) = a$ for every $x$, so the orbit has only one element. Observe also that, if $k$ is the first integer between 0 and $n-i-1$ such that $a_k = 0$, then, using different $x$'s, any element of the form $a + mp^{i+k}$, $m \in \mathbb{Z}/(p^n)$, belongs to the orbit of $a$. Thus each orbit of $G_i$ with respect to $\lambda$ belongs to one of the following classes of orbits:

(a) $X_{\alpha}^{(1)} := \{\alpha + p^i x : x \in \mathbb{Z}/(p^i)\}$, for each $\alpha \in \{1, \ldots, p^i\}$, $\alpha \neq 0 \pmod{p}$;

(b) $X_{\beta}^{(2)} := \{\beta p^k + p^{i+k} x : x \in \mathbb{Z}/(p^i)\}$ for any $\beta \in \{1, \ldots, p^i\}$, $\beta \neq 0 \pmod{p}$, and for any $k \in \{1, \ldots, n-i-1\}$;

(c) $X_{p^{n-i}}^{(3)} := \{p^{n-i} \gamma\}$, for each $\gamma \in \{0, 1, \ldots, p^i - 1\}$.

For each class of orbits, the stabilizer of any element (which corresponds to the possible $H_i$'s) is equal to:

(a) Since the cardinality of the orbit is $|\{\alpha + p^i x : x \in \mathbb{Z}/(p^i)\}| = p^{n-i}$, the stabilizer is the unique subgroup of $\mathbb{Z}/(p^i)$ of order $p^i$;

(b) Since the cardinality of the orbit is $|\{\beta p^k + p^{i+k} x : x \in \mathbb{Z}/(p^i)\}| = p^{n-i-k}$, the stabilizer is the unique subgroup of $\mathbb{Z}/(p^i)$ of order $p^{i+k}$;

(c) $G$.

To have trivial intersection, we need at least one $K_{3,1}$ equal to 0. Moreover, we need an additive generator of $\mathbb{Z}/(p^n)$ in the subset $X$, and all those elements belong to some $X_{\alpha}^{(1)}$. Thus the basic solutions are
1. \( X_1 = X_\alpha^{(1)}, K_{1,1} = 0 \).

2. \( X_1 = X_{p^{-1}n}^{(3)}, X_2 = X_\alpha^{(1)}, K_{1,1} = 0, K_{2,1} = \langle p^{n-i} \rangle \).

3. \( X_1 = X_{p^k}^{(2)}, X_2 = X_\alpha^{(1)}, K_{1,1} = 0, K_{2,1} = \langle p^{n-i} \rangle \).

Finally, to know if there is some isomorphism between these basic solutions, we need to apply brace automorphisms of \( G \) over them. Note that, since the multiplicative group is abelian, the subgroups \( K_{j,l} \) remain unchanged by the isomorphism of solutions, so we should only care about the effect over the subset \( X \), which is equal to \( X_1 \) in case 1, and equal to \( X_1 \cup X_2 \) in cases 2 and 3. Thus two of these solutions are isomorphic if and only if they belong to the same class 1, 2 or 3, and the their two invariant subsets \( X \) and \( X' \) satisfies \( X' = (1 + p^{n-1}k)X \), for some \( k \in \mathbb{Z}/(p^n) \), where here the product is the ring product of \( \mathbb{Z}/(p^n) \).

On the other hand, the case of \( Q_{2^m} \) is more difficult because we do not have a complete classification of all the possible brace structures with multiplicative group isomorphic to \( Q_{2^m} \).

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