SIMPLICES WITH EQUiareAL FACES

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Abstract. We study simplices with equiareal faces in the Euclidean 3-space by means of elementary geometry. We present an unexpectedly simple proof of the fact that, if such a simplex is non-degenerate, than every two of its faces are congruent. We show also that this statement is wrong for degenerate simplices and find all degenerate simplices with equiareal faces.

1. Introduction

This paper deals with simplices with equiareal faces in the Euclidean 3-space. A simplex is called a simplex with equiareal faces if all its faces have the same area and is called degenerate if all its vertices lie on a single plane. Our primary interest is the following

Problem 1. Prove that all faces of any non-degenerate simplex with equiareal faces in the Euclidean 3-space are congruent to each other.

Problem 1 already appeared in several contexts. We mention here some of those appearances though we cannot say that our knowledge is complete.

In the 1960s Professor Hans Vogler in Vienna used Problem 1 to convince his students how powerful the synthetic geometry is. As far as we know, his descriptive-geometrical solution was never published.

Independently, in the 1970’s and 1980’s the following form of Problem 1 was used to prevent ‘undesirable applicants’ from joining the Moscow State University: ‘The faces of a triangular pyramid have the same area. Show that they are congruent.’ An elementary solution to the latter problem given in [1] shows that indeed the problem is far from being trivial.

In [2], among other results P. McMullen has proved that a non-degenerate simplex is a simplex with equiareal faces if and only if its opposite pairs of edges have the same lengths. It is easy to see that this statement is equivalent to Problem 1. McMullen’s proof is very short and natural, but it is not elementary since it rests on Minkowski’s theorem on uniqueness and existence of a closed convex polyhedron with given directions and areas of faces (see, e.g., [3]; the direction of a face is determined by the outward unit normal to the face).

In 2007 Professor Robert Connelly in Cornell University has brought our attention to the fact that it is reasonable to study degenerate simplices with equiareal faces. He argued that when we fix three vertices, say $A$, $B$, and $C$, and move arbitrarily the fourth vertex, say $D$, nothing special happens when $D$ occurs in the plane $ABC$: the degenerate simplex is obtained as a limit of non-degenerate ones;
the notions of the vertex, edge, and face are clear for it; the notion of the face area is well defined; from combinatorial point of view there is no difference between degenerate and non-degenerate simplices. Besides, the degenerate polyhedra play very important role in ‘advanced’ study of convex polytopes, see, e.g., [3].

Also Professor Robert Connelly has brought our attention to the fact that, in contrast with Problem 1, there are degenerate simplices with non-congruent equiareal faces. In fact, every parallelogram equipped with its diagonals may be treated as a degenerate simplex with equiareal faces as well as every four points on a line may be treated as a vertex-set of a degenerate simplex with equiareal faces (with all areas equal zero). The problem is whether there are some other degenerate simplices with equiareal faces.

In Section 2 we give the shortest available for us elementary solution to Problem 1. As far as we know, its main idea should be attributed to Professor Hans Vogler who is now at the University of Innsbruck. In Section 3 we use Heron’s formula to study all simplices with equiareal faces, both degenerate and non-degenerate.

2. A SHORT ELEMENTARY SOLUTION TO PROBLEM 1

Note that in order to solve Problem 1 it is sufficient to solve the following problem 2 which is of independent interest.

Problem 2. Let $ABCD$ be a non-degenerate simplex, let $a(\triangle ABC)=a(\triangle ABD)$, and let $a(\triangle ACD) = a(\triangle BCD)$, where $a(\triangle XYZ)$ stands for the area of the triangle $\triangle XYZ$. Prove that $|AC| = |BD|$ and $|BC| = |AD|$, where $|XY|$ stands for the length of the straight line segment $XY$ (see Fig. 1).

Solution to Problem 2. Let $P$ be the plane which is parallel to the line $AB$ and contains the line $CD$ (see Fig. 2). We are going to study the orthogonal projection of the simplex $ABCD$ into the plane $P$. Denote by $X^\perp$ the image of a point $X$ under that projection and denote by $X^*$ the foot of the perpendicular to the line $AB$ emanated from the point $X$.
Since \( AB \) is parallel to \( P \) we have \(|D^*D^+| = |C^*C^+|\). Since the triangles \( \triangle ABC \) and \( \triangle ABD \) have equal areas and the common side \( AB \) we conclude that \(|DD^*| = |CC^*|\). Applying Pythagoras theorem to the right-angled triangles \( \triangle DD^*D^+ \) and \( \triangle CC^*C^+ \) we get \(|DD^*|^2 = |DD^+|^2 - |D^*D^+|^2 = |CC^*|^2 - |C^*C^+|^2 = |CC^+|^2\). In terms of the quadrilateral \( A^\perp CB^\perp D \) this means that the vertices \( D \) and \( C \) lie at the same distance from the diagonal \( A^\perp B^\perp \). Note also that the triangles \( \triangle_A \) and \( \triangle_B \) and \( \triangle_C \) triangles \( \triangle_D \) and \( \triangle_E \) have the same area and the points \( D \) and \( C \) lie on the different sides of the line passing through the points \( A^\perp \) and \( B^\perp \). In fact, if they lie on the same side, the line through the points \( A^\perp \) and \( B^\perp \) must be parallel to the line through the points \( D \) and \( C \) and, thus, the points \( A, B, C, \) and \( D \) should be coplanar. A contradiction.

Similar arguments applied to the triangles \( \triangle AC \) and \( \triangle BD \) show that the vertices \( A^\perp \) and \( B^\perp \) of the quadrilateral \( A^\perp CB^\perp D \) lie at the same distance from the line through the points \( D \) and \( C \) and, moreover, lie on the different sides of that line. Hence, the quadrilateral \( A^\perp CB^\perp D \) is convex.

Recall that if a convex planar quadrilateral is such that every two opposite vertices lie at the same distance from the diagonal that joins the rest two vertices then the quadrilateral is a parallelogram. This implies that the quadrilateral \( A^\perp CB^\perp D \) is a parallelogram and, thus, that \(|A^\perp C| = |B^\perp D|\) and \(|B^\perp C| = |A^\perp D|\).

Applying Pythagoras theorem to the right-angled triangles \( \triangle AA^\perp C \) and \( \triangle BB^\perp D \) and taking into account that \(|AA^\perp| = |BB^\perp|\) we get \(|AC|^2 = |AA^\perp|^2 + |A^\perp C|^2 = |BB^\perp|^2 + |B^\perp D|^2 = |BD|^2\).

Similar arguments applied to the triangles \( \triangle AA^\perp D \) and \( \triangle BB^\perp C \) show that \(|BC| = |AD|\).

Q.E.D.

3. A STUDY OF DEGENERATE AND NON-DEGENERATE SIMPLEXES WITH EQUIAREAL FACES

From Section 1 we know the following three types of equiareal non-degenerate and degenerate simplices in the Euclidean 3-space:

Type 1: non-degenerate simplex with all faces congruent;
Type 2: parallelogram equipped with its diagonals; and
Type 3: four points on a line treated as a vertex-set of a degenerate simplex with equiareal faces of zero area.

In this Section we use Heron’s formula to study the following

**Problem 3.** Prove that there are no simplices with equiareal faces which do not belong to Types 1–3.

**Solution** to Problem 3. Let \( T \) be a simplex with equiareal faces. Let the first face (I) of \( T \) has edges of the lengths \( a, b, \) and \( c \); the second face (II) has edges of the lengths \( a, y, \) and \( z \); the third face (III) — \( b, x, \) and \( z \); and the forth face (IV) — \( c, x, \) and \( y \). (Equivalently, we can say that side \( x \) is opposite to \( a \); side \( y \) — to \( b \); and \( z \) — to \( c \)). Let \( S \) be common area of the faces of \( T \).

Heron’s formula for the face (I) yields

\[
(4S)^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 = -(a^2 - b^2 + c^2)^2 + 4a^2c^2.
\]
Now let’s use Heron’s formula to express the fact that the faces (I) and (II) have the same areas

\[(a + y + z)(-a + y + z)(a - y + z)(a + y - z) - (4S)^2\]

(2) = \(2a^2y^2 + 2a^2z^2 + 2x^2y^2 - a^4 - y^4 - z^4 - (4S)^2\) = \(-(a^2 - y^2)^2 - (4S)^2\) + \(4y^2a^2 = 0\).

Solving this equation with respect to \(z^2\) yields

(3) \(z^2 = y^2 + a^2 \pm \sqrt{4a^2y^2 - (4S)^2}\).

Similarly, we use Heron’s formula to express the fact that the faces (I) and (III) have the same areas

\[(b + z + x)(-b + z + x)(b - z + x)(b + z - x) - (4S)^2\]

= \(2b^2z^2 + 2b^2x^2 + 2z^2x^2 - b^4 - z^4 - x^4 - (4S)^2\) = \(-(z^2 - x^2 - b^2)^2 - (4S)^2\) + \(4b^2x^2 = 0\).

Solving this equation with respect to \(z^2\) yields

(4) \(z^2 = x^2 + b^2 \pm \sqrt{4b^2x^2 - (4S)^2}\).

At last, we use Heron’s formula to express the fact that the faces (I) and (IV) have the same areas

\[(c + x + y)(-c + x + y)(c - x + y)(c + x - y) - (4S)^2\]

= \(2c^2x^2 + 2c^2y^2 + 2x^2y^2 - c^4 - x^4 - y^4 - (4S)^2\) = \(-(y^2 - x^2 - c^2)^2 - (4S)^2\) + \(4c^2x^2 = 0\).

Solving this equation with respect to \(y^2\)

(5) \(y^2 = x^2 + c^2 \pm \sqrt{4c^2x^2 - (4S)^2}\).

Eliminate \(z^2\) from (3) and (4)

\(y^2 + a^2 \pm \sqrt{4a^2y^2 - (4S)^2} = x^2 + b^2 \pm \sqrt{4b^2x^2 - (4S)^2}\),

then twice square this equation in order to eliminate square roots and use the formula \((a^2 - b^2 + c^2)^2 = 4a^2c^2 - (4S)^2\) (which is a consequence of (1)) to obtain

\[4a^4y^4 + 4b^4x^4 + (x^2 + b^2 - y^2 - a^2)^4 - 8a^2b^2x^2y^2 - 2a^2y^2(x^2 + b^2 - y^2 - a^2)^2\]

(6) \(-2b^2x^2(x^2 - y^2 - a^2 + b^2)^2 = -4(x^2 - y^2 - a^2 + b^2)^2\).

So, we arrive at the most computationally difficult, but still straightforward point of the solution: substitute \(y^2\) in (6) by the right-hand side of (5). After simplifications and multiple usage of the formula \((a^2 - b^2 + c^2)^2 = 4a^2c^2 - (4S)^2\) we get

(7) \[4(c^2x^2 - 4S^2)(x^2 - a^2)^2S^4 = [x^4 - x^2(a^2 + b^2 - c^2) - a^2(b^2 - c^2)^2]S^4\].

We see that \(S = 0\) is a root of (7) which corresponds to simplices of Type 3. In order to find the other roots, cancel \(S^4\) in (7), rearrange terms and use the formula \((a^2 - b^2 + c^2)^2 = 4a^2c^2 - (4S)^2\) again to arrive at

\[(x^2 - a^2)^2[x^4 - 2x^2(b^2 + c^2) + a^2(2b^2 + 2c^2 - a^2)] = 0\].

Note that the bi-quadratic expression in the brackets has two roots: \(x^2 = a^2\) and \(x^2 = 2b^2 + 2c^2 - a^2\). Note also that we can obtain similar equations for \(y\) and \(z\) just
by permuting three pairs of symbols \((a, x), (b, y),\) and \((z, c)\). As a result we find 8 solutions for \(x, y,\) and \(z\) that are accumulated as rows in the following table:

| Solutions | \(x\) | \(y\) | \(z\) |
|-----------|-------|-------|-------|
| Solution 1 | \(a\) | \(b\) | \(c\) |
| Solution 2 | \(a\) | \(b\) | \(\sqrt{2a^2 + 2b^2 - c^2}\) |
| Solution 3 | \(a\) | \(\sqrt{2a^2 - b^2 + 2c^2}\) | \(c\) |
| Solution 4 | \(a\) | \(\sqrt{2a^2 - b^2 + 2c^2}\) | \(\sqrt{2a^2 + 2b^2 - c^2}\) |
| Solution 5 | \(\sqrt{2b^2 + 2c^2 - a^2}\) | \(b\) | \(c\) |
| Solution 6 | \(\sqrt{2b^2 + 2c^2 - a^2}\) | \(b\) | \(\sqrt{2a^2 + 2b^2 - c^2}\) |
| Solution 7 | \(\sqrt{2b^2 + 2c^2 - a^2}\) | \(\sqrt{2a^2 - b^2 + 2c^2}\) | \(c\) |
| Solution 8 | \(\sqrt{2b^2 + 2c^2 - a^2}\) | \(\sqrt{2a^2 - b^2 + 2c^2}\) | \(\sqrt{2a^2 + 2b^2 - c^2}\) |

Solution 1, obviously, corresponds to the simplices of Type 1.

Solutions 2, 3, and 5 correspond to simplices of Type 2. For example, a simplex \(T\), corresponding to Solution 2, is degenerated into a parallelogram with the side lengths \(a\) and \(b\) and the diagonals \(c\) and \(\sqrt{2a^2 + 2b^2 - c^2}\) (recall that in any parallelogram the sum of the squared lengths of all sides equals the sum of the squared lengths of the both diagonals).

Solutions 4, 6, and 7 correspond to simplices of Type 2 again, but the face (I), with edge lengths \(a, b,\) and \(c\), must be a right-angled triangle this time. For example, consider Solution 4. We have

\[
(8) \quad x^2 = a^2, \quad y^2 = 2a^2 - b^2 + 2c^2, \quad \text{and} \quad z^2 = 2a^2 + 2b^2 - c^2.
\]

Using the formula (2) we get \(-(z^2 - y^2 - a^2)^2 - (4S)^2 + 4a^2y^2 = 0.\) Now we use the formula (1) and, after some simplifications, we get \(a^4 - (b^2 - c^2)^2 = 0.\) Without loss of generality, we may assume that \(b \geq c.\) This yields \(a^2 + c^2 = b^2\) and, thus, the face (I) is a right-angled triangle. Moreover, now the formula (8) implies that \(y^2 = b^2\) and \(z^2 = 4a^2 + c^2.\) Hence, the simplex \(T,\) corresponding to Solution 4, is degenerated to a parallelogram with the side lengths \(a, b, x = a,\) and \(y = b\) and the diagonals of the lengths \(c\) and \(z = \sqrt{4a^2 + c^2} = \sqrt{2a^2 + 2b^2 - c^2}.\) Hence, the simplex \(T\) is of Type 2. Solutions 6 and 7 are treated similarly.

Solution 8 does not correspond to any simplex in the Euclidean 3-space (neither degenerated nor non-degenerated). In fact, we have

\[
(9) \quad x^2 = 2b^2 + 2c^2 - a^2, \quad y^2 = 2a^2 - b^2 + 2c^2, \quad \text{and} \quad z^2 = 2a^2 + 2b^2 - c^2.
\]

Using the formula (2) we get \(-(z^2 - y^2 - a^2)^2 - (4S)^2 + 4a^2y^2 = 0.\) Now we use formula (1) and, after some simplifications, we get \(a^4 - (b^2 - c^2)^2 = 0\) or

\[
(10) \quad (a^2 - b^2 + c^2)(a^2 + b^2 - c^2) = 0.
\]

The geometric meaning of the formula (10) is that either \(b\) or \(c\) is the hypotenuse of the right-angled triangle (I) with the sides \(a, b,\) and \(c.\) Similarly we can substitute
(9) into formulas (3) and (4). Proceeding as above we get

\begin{align}
(11) & \ (a^2 + b^2 - c^2)(-a^2 + b^2 + c^2) = 0, \\
(12) & \ (a^2 - b^2 + c^2)(-a^2 + b^2 + c^2) = 0.
\end{align}

The geometric meaning of the formula (11) is that either \( c \) or \( a \) is the hypotenuse of the right-angled triangle (I) with the sides \( a, b, \) and \( c \). Similarly, the formula (12) implies that either \( b \) or \( a \) is the hypotenuse of the right-angled triangle (I) with the sides \( a, b, \) and \( c \). But the triangle (I) has only one hypotenuse! Hence the equations (10)–(12) can not hold true simultaneously. This means that Solution 8 does not correspond to any simplex.

Now we can conclude that there is no simplices with equiareal faces which do not belong to Types 1–3. Q.E.D.

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