VARIATIONAL POISSON–NIJENHUIS STRUCTURES FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We explore variational Poisson–Nijenhuis structures on nonlinear PDEs and establish relations between Schouten and Nijenhuis brackets on the initial equation with the Lie bracket of symmetries on its natural extensions (coverings). This approach allows to construct a framework for the theory of nonlocal structures.

Introduction

Poisson–Nijenhuis structures [1] play an important role both in classical differential geometry (see, for example [1, 10]) and in geometry of partial differential equations, see [11, 14]. In the latter case existence of a Poisson–Nijenhuis structure virtually amounts to complete integrability of the equation under consideration.

Infinite-dimensional Poisson–Nijenhuis structures are well described for the case of jets and for evolutionary differential equations regarded as flows on the jet space. As for general differential equation, the corresponding theory was not introduced for a long time. In our relatively recent works [7, 6] we outlined an approach to the theory in application to evolution equation in geometrical setting. This approach is based on the notion of ∆-coverings and reduces construction of both recursion operators and Hamiltonian structures to solution of the linearised equation

$$\ell \mathcal{E}(\Phi) = 0$$

on special extensions of the initial equation $\mathcal{E}$. We call these extensions the $\ell$- and $\ell^*$-coverings and they play the role of tangent and cotangent bundles in the category of differential equations.

The above mentioned approach seems to work for general equations as well and we expose its generalisation below.

In Section 1 we define Poisson–Nijenhuis structures in the “absolute case”, i.e., for the manifold of infinite jets. To this end, we redefine the Schouten and Frölicher–Nijenhuis brackets (cf. with [7, 12], see also [17]). We also express the compatibility condition between a Poisson bi-vector and a Nijenhuis operator in terms of a special bracket closely related to Vinogradov’s unified bracket, see [3]. The main result of this section is Theorem 2 that states the existence of infinite families of pairwise compatible Hamiltonian structures related to the initial Poisson–Nijenhuis structure. In Section 2 Poisson–Nijenhuis structures on evolution equations are introduced. We show that an invariant (with respect to the flow determined by the equation) Nijenhuis tensors are recursion operators for the symmetries, while invariant Poisson bi-vectors amount to Hamiltonian structures. We also define the $\ell$- and $\ell^*$-coverings and reduce construction of recursion operators and Hamiltonian

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The Schouten and Frölicher–Nijenhuis brackets as well as the compatibility conditions are reformulated in terms of the Jacobi brackets of the corresponding solutions and explicit formulas for these brackets are obtained. Section 3 generalises the results to arbitrary nonlocal partial differential equation. Finally, in Section 4 we outline an approach to deal with nonlocal Poisson–Nijenhuis structures.

1. Variational Poisson–Nijenhuis structures on $J^\infty(\pi)$

1.1. Geometrical structures. Let us recall definitions and results we shall use. For details we refer to [2].

Let $\pi: E \to M$ be a vector bundle over an $n$-dimensional manifold $M$ and $\pi_\infty: J^\infty(\pi) \to M$ be the infinite jet bundle of local sections of the bundle $\pi$. If $x_1, \ldots, x_n$ are local coordinates in the base and $u^1, \ldots, u^m$ are coordinates along the fiber of $\pi$ the canonical coordinates $u^i_\sigma$ arise in $J^\infty(\pi)$ defined by

$$j_\infty(s)^*(u^i_\sigma) = \frac{\partial^{\|s\|} |_{x_\sigma}}{\partial x_\sigma},$$

where $s = (s^1, \ldots, s^m)$ is a local section of $\pi$, $j_\infty(s)$ is its infinite jet and $\sigma = i_1i_2\ldots i_{\|s\|}$, $i_\sigma = 1, \ldots, n$, is a multi-index. Denote by $\mathcal{F}(\pi)$ the algebra of smooth functions on $J^\infty(\pi)$.

The basic geometrical structure on $J^\infty(\pi)$ is the Cartan distribution $C$ that is spanned by total derivatives

$$D_\iota = \frac{\partial}{\partial x_\iota} + \sum_{j,\sigma} u^j_{\sigma1} \frac{\partial}{\partial u^j_{\sigma}}.$$

Differential operators on $J^\infty(\pi)$ in total derivatives will be called $C$-differential operators. In local coordinates, they have the form $\| \sum_{\sigma} a^s_{\sigma} D_\sigma \|$, where $a^s_{\sigma} \in \mathcal{F}(\pi)$. Let $P$ and $Q$ be $\mathcal{F}(\pi)$-modules of sections of some vector bundles over $J^\infty(\pi)$. All $C$-differential operators from a $P$ to $Q$ form an $\mathcal{F}(\pi)$-module denoted by $\text{CDiff}(P, Q)$.

The adjoint operator to a $C$-differential operator $\Delta: P \to Q$ is denoted by $\Delta^*: \hat{Q} \to \hat{P}$, where $\hat{P} = \text{Hom}_{\mathcal{F}(\pi)}(P, \Lambda^\infty(\pi))$ and $\Lambda^\infty(\pi)$ is the $\mathcal{F}(\pi)$-module of horizontal $n$-forms on $J^\infty(\pi)$, i.e., forms $\omega = a dx_1 \wedge \cdots \wedge dx_n$.

Denote by $\text{CDiff}_{sk-ad}^{sk}(P, Q)$ the module of $k$-linear skew-symmetric $C$-differential operators $P \times \cdots \times P \to Q$ and by $\text{CDiff}_{sk-ad}^{sk-ad}(P, \hat{P}) \subset \text{CDiff}_{sk-ad}^{sk}(P, \hat{P})$ the subset of operators skew-adjoint in each argument.

A $\pi_\infty$-vertical vector field on $J^\infty(\pi)$ is called evolutionary if it preserves the Cartan distribution. There is a one-to-one correspondence between evolutionary vector fields and sections of the bundle $\pi_{\infty\infty}$. Denote by $\mathcal{X}(\pi)$ the corresponding module of sections. In local coordinates, the evolutionary vector field that corresponds to a section (the generating section, or function) $\varphi = (\varphi^1, \ldots, \varphi^m)$ is of the form

$$\mathcal{D}_\varphi = \sum_{j, \sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_{\sigma}}.$$

The commutator of evolutionary vector fields induces in $\mathcal{X}(\pi)$ a Lie algebra structure that is given by the higher Jacobi bracket defined as a unique section $\{\varphi, \psi\}$ satisfying $[\mathcal{D}_\varphi, \mathcal{D}_\psi] = \mathcal{D}_{\{\varphi, \psi\}}$. The bracket $\{\varphi, \psi\}$ is expressed by

$$\{\varphi, \psi\} = \mathcal{D}_\varphi(\psi) - \mathcal{D}_\psi(\varphi).$$

Following [4, 8], we shall call elements of $\mathcal{X}(\pi)$ variational vectors, elements of the module $\text{CDiff}_{sk-ad}^{sk-ad}(\mathcal{X}, \mathcal{X})$ will be called variational $k$-vectors, while elements of $\mathcal{X}$ will be called variational 1-forms and elements of the module $\text{CDiff}_{sk-ad}^{sk-ad}(\mathcal{X}, \mathcal{X})$
variational $k$-forms, respectively. The Lie derivative on variational vectors $L_\varphi: \mathcal{X} \to \mathcal{X}$ takes the form

$$L_\varphi = \mathcal{D}_\varphi - \ell_\varphi,$$

(2)

where the linearization operator $\ell_\varphi$ is defined by the equality $\ell_\varphi(\alpha) = \mathcal{D}_\alpha(\varphi)$, $\alpha \in \mathcal{X}$. In local coordinates, it has the form

$$\ell_\varphi(\alpha) = \sum_{j,\sigma} \frac{\partial \varphi}{\partial u^\sigma_j} D_\sigma(\alpha^j), \quad \alpha = (\alpha^1, \ldots, \alpha^m).$$

The Lie derivative on variational forms $L_\varphi: \mathcal{X} \to \mathcal{X}$ is of the form

$$L_\varphi = \mathcal{D}_\varphi + \ell_\varphi^*.$$

(3)

1.2. Variational Poisson–Nijenhuis structures. Recall (see [17]) that the variational Schouten bracket of two operators $A, B \in \text{CDiff}^{sk-ad}(\mathcal{X}, \mathcal{X})$ is defined by

$$[A, B](\psi_1, \psi_2) = -\ell_A(\varphi_1(B\psi_2) + \ell_A(\varphi_2(B\psi_1) - A(\ell^*_B, \psi_1(\psi_2)))$$

$$- \ell_B(\varphi_1(A\psi_2) + \ell_B(\varphi_2(A\psi_1) - B(\ell^*_A, \psi_1(\psi_2))), \quad \psi_1, \psi_2 \in \mathcal{X}. \quad (4)$$

An operator $A$ is called Hamiltonian if the $[A, A] = 0$ and two Hamiltonian operators $A$ and $B$ are compatible if their Schouten bracket vanishes.

**Remark 1.** Here and below the notation $\ell_{\Delta,p_1,\ldots,p_n}(\varphi)$, $\varphi \in \mathcal{X}$, for a $C$-differential operator $\Delta: P \times \cdots \times P \to Q$ means

$$\ell_{\Delta,p_1,\ldots,p_n}(\varphi) = \mathcal{D}_\varphi(\Delta)(p_1, \ldots, p_n), \quad p_1, \ldots, p_n \in P.$$

For two operators $R, S \in \text{CDiff}(\mathcal{X}, \mathcal{X})$ their Frölicher–Nijenhuis bracket (cf. with [12], see also [17]) is defined by

$$[R, S]_{\mathcal{F}\mathcal{N}}(\varphi_1, \varphi_2) = \{R\varphi_1, S\varphi_2\} + \{S\varphi_1, R\varphi_2\} - R(\{S\varphi_1, \varphi_2\} + \{\varphi_1, S\varphi_2\}$$

$$- S(\{R\varphi_1, \varphi_2\} + \{\varphi_1, R\varphi_2\} - R(\varphi_1, \varphi_2)), \quad \varphi_1, \varphi_2 \in \mathcal{X}. \quad (5)$$

If $[R, R]_{\mathcal{F}\mathcal{N}} = 0$ we shall refer to $R$ as a Nijenhuis operator. For particular computations it is convenient to use the equality

$$[R, S]_{\mathcal{F}\mathcal{N}}(\varphi_1, \varphi_2) = -\ell_R(\varphi_1(S\varphi_2) - \ell_S(\varphi_1(R\varphi_2) + \ell_R(\varphi_2(S\varphi_1) + \ell_S(\varphi_2(R\varphi_1)$$

$$+ R(\ell_S(\varphi_1(\varphi_2) - \ell_S(\varphi_2(\varphi_1)) + S(\ell_R(\varphi_1(\varphi_2) - \ell_R(\varphi_2(\varphi_1))). \quad (6)$$

**Definition 1** (cf. with [11]). A Hamiltonian operator $A \in \text{CDiff}^{sk-ad}(\mathcal{X}, \mathcal{X})$ and a Nijenhuis operator $R \in \text{CDiff}(\mathcal{X}, \mathcal{X})$ constitute a variational Poisson–Nijenhuis structure $(A, R)$ on $J^\infty(\pi)$ if the following compatibility conditions hold

(i) $R \circ A = A \circ R^*$,

(ii) $C(A, R)(\psi_1, \psi_2) = L_{A\psi_1}(R^*\psi_2) - L_{A\psi_2}(R^*\psi_1) + R^*L_{A\psi_2}(\psi_1) - R^*L_{A\psi_1}(\psi_2) + E(\psi_1, AR\psi_2) - R^*E(\psi_1, A\psi_2) = 0,$

where $E: \dot{H}^n(\pi) \to \mathcal{X}$ is the Euler operator and $\dot{H}^n(\pi)$ is the $n$th horizontal de Rham cohomology group, while $(\cdot, \cdot): \mathcal{X} \times \mathcal{X} \to \dot{H}^n(\pi)$ is the natural pairing.

In terms of linearizations operators condition (ii) has the form

$$C(A, R)(\psi_1, \psi_2) = -\ell_{R^*, \psi_1}(A\psi_2) + \ell_{R^*, \psi_2}(A\psi_1) + \ell^*_{A, \psi_1}(R^*\psi_2)$$

$$+ \ell^*_{R^*, \psi_1}(A\psi_2) - R^*(\ell^*_R, \psi_1(\psi_2)) = 0.$$
Proof. By straightforward computations one can prove that
\[
\begin{multline}
[RA,RA](\psi_1,\psi_2) - 2R[A,RA](\psi_1,\psi_2) \\
+ R^2[A,A](\psi_1,\psi_2) - [R,R]_{FN}(A\psi_1,A\psi_2) = 0
\end{multline}
\] (9)
and
\[
\begin{multline}
2[A,RA](\psi_1,\psi_2) - [A,A](R^*\psi_1,\psi_2) \\
- [A,A](\psi_1,R^*\psi_2) - 2A(C(A,R)(\psi_1,\psi_2)) = 0,
\end{multline}
\] (10)
from where the statement follows immediately. \qed

Theorem 2. Let a Hamiltonian operator \( A \in CDiff^{sk-ad}(\mathbb{R},\mathbb{R}) \) and a Nijenhuis operator \( R \in CDiff(\mathbb{R},\mathbb{R}) \) define Poisson–Nijenhuis structure on \( J^\infty(\pi) \). Then on \( J^\infty(\pi) \) there is a hierarchy of iterated Hamiltonian operators, that is a sequence of Hamiltonian operators \( R^iA, i \geq 0 \), which are pair-wise compatible, i.e., \([R^iA,R^jA] = 0, i, j \geq 0\).

Proof. The proof is by induction on \( n = \max(i, j) \). For \( n = 1 \) the statement follows from the proposition above. Assume now that \([R^iA,R^jA] = 0, i, j = 0, \ldots, n, \) and \( A(C(R^iA,R^j)) = 0, i + j \leq n, \) and let us prove that \([R^iA,R^{i+1}A] = 0, i = 0, \ldots, n + 1, \) and \( A(C(R^iA,R^{i+1})) = 0, i + j \leq n + 1. \)

First, note that by direct computations one can prove the following formulas
\[
\begin{align*}
[R,A,B](\psi_1,\psi_2) &= R[R,A,B](\psi_1,\psi_2) - [A,RA,B](\psi_1,\psi_2) + R^2[A,B](\psi_1,\psi_2) \\
&- [R,R]_{FN}(A\psi_1,B\psi_2) - [R,R]_{FN}(B\psi_1,A\psi_2) = 0, \quad (11) \\
\end{align*}
\]
\[
\begin{align*}
[R,A,B](\psi_1,\psi_2) &= [A,RB](\psi_1,\psi_2) - [A,B](R^*\psi_1,\psi_2) + [A,B](\psi_1,R^*\psi_2) \\
&- A(C(B,R)(\psi_1,\psi_2)) - B(C(A,R)(\psi_1,\psi_2)) = 0, \quad (12) \\
\end{align*}
\]
\[
\begin{align*}
C(RA,R)(\psi_1,\psi_2) &= C(A,R^2)(\psi_1,\psi_2) - C(A,R)(R^*\psi_1,\psi_2) \\
&- C(A,R)(\psi_1,R^*\psi_2) - R^n(C(A,R)(\psi_1,\psi_2)) = 0. \quad (13)
\end{align*}
\]
Let us substitute \( R^nA \) and \( R^{n-1}A, l = 1, \ldots, n, \) for \( A \) and \( B \) in (12), respectively. Then we get
\[
\begin{multline}
[R^{n+1}A,R^{n-1}A] - R^{n-1}A(C(R^nA,R)) = 0. \quad (14)
\end{multline}
\]
Now let us take \( R^nA, R^{n-1}A, l = 1, \ldots, n, \) and \( R^2 \) for \( A, B \) and \( R \) in (12), respectively. Then we have
\[
\begin{multline}
[R^{n+1}A,R^{n-1}A] - R^{n-1}A(C(R^nA,R^2)) = 0. \quad (15)
\end{multline}
\]
If we have in (13) \( R^{n-1}A \) for \( A \) we get
\[
C(R^nA,R) + C(R^{n-1}A,R^2) = 0. \quad (16)
\]
Therefore, taking the sum of (14) and (15) we obtain that \([R^{n+1}A,R^{n-1}A] = 0 \) for \( l = 1, \ldots, n. \)

Let us substitute now \( R^{n-1}A \) and \( R^nA \) for \( A \) and \( B \) in (11), respectively, and then \( R^nA \) for \( A \) in (9). Thus we get \([R^{n+1}A,R^nA] = 0 \) and \([R^{n+1}A,R^{n+1}A] = 0. \) In order to prove that \( A(C(R^2A,R^2)) = 0 \) for \( i + j \leq n + 1, \) one has to put \( B = R^{n-1}A, l = 0, \ldots, n \) and take \( R^{l+1} \) for \( R \) in (12). \qed

For subsequent constructions we need the operator
\[
C^*(A,R)(\psi,\varphi) = -\ell_{A,\psi}(R\varphi) + \ell_{R,\psi}(A\psi) + R(\ell_{A,\psi}(\varphi)) \\
+ A(\ell_{R,\psi}(\psi) - \ell_{R^*,\psi}(\varphi))
\]
defined by
\[
\langle C(A,R)(\psi_1,\psi_2),\varphi \rangle = \langle \psi_2, C^*(A,R)(\psi_1,\varphi) \rangle. \quad (17)
\]
2. Variational Poisson–Nijenhuis structures on evolution equations

2.1. Symmetries and cosymmetries. Consider a system of evolution equations

\[ \mathcal{E} = \{ F = u_t - f(x, t, u, u_1, \ldots, u_k) = 0 \} , \]  

where both \( u = (u^1, \ldots, u^n) \) and \( f = (f^1, \ldots, f^m) \) are vectors and \( u_t = \partial u / \partial t, \ u_k = \partial^k u / \partial x_k \). For simplicity, we consider the case of one space variable \( x \), though everything works in general situation as well. Denote by \( \mathcal{F}(\mathcal{E}) \) the algebra of smooth functions on \( \mathcal{E} \).

Recall that equation \(18\) \( \mathcal{E} \) can be understood as the space \( J^\infty(\pi) \times \mathbb{R} \) with the Cartan distribution generated by the fields \( D_x \) and \( D_t = \partial / \partial t + \partial f / \partial t \). Here \( t \) the coordinate along \( \mathbb{R} \). In local coordinates, these fields are of the form

\[ D_x = \partial / \partial x + \sum_{j=1}^{m} u_j \partial / \partial u_j , \quad D_t = \partial / \partial t + \sum_{j=1}^{m} D^k_j(f_j) \partial / \partial u_j. \]

A symmetry of the equation \( \mathcal{E} \) is a \( \pi_\infty \)-vertical vector field on \( \mathcal{E} \) that preserves the Cartan distribution. The set of all symmetries forms a Lie algebra over \( \mathbb{R} \) denoted by \( \text{sym}(\mathcal{E}) \) and there is a one-to-one correspondence between \( \text{sym}(\mathcal{E}) \) and smooth sections \( \varphi \in \Gamma(\pi_\infty^*(\pi)) = \mathcal{K}(\mathcal{E}) \) satisfying the equation

\[ \ell_\mathcal{E}(\varphi) = 0, \]

where \( \ell_\mathcal{E} = D_t - \ell_f \) is the linearization operator of \( \mathcal{E} \).

A conservation law for the equation \(18\) is a horizontal 1-form \( \eta = Xdx + Tdt \) closed with respect to horizontal de Rham differential \( \check{d} : \Lambda^1(\mathcal{E}) \to \Lambda^2(\mathcal{E}) \), i.e., such that

\[ D_t(X) = D_x(T), \]

where \( X, T \in \mathcal{F}(\mathcal{E}) \). A conservation law \( \eta \) is trivial if it is of the form \( \eta = \check{d} h, \ h \in \mathcal{F}(\mathcal{E}) \). The space of equivalence classes of conservation laws coincides with the first horizontal de Rham cohomology group and is denoted by \( \check{H}^1(\mathcal{E}) \). To any conservation law \( \eta = Xdx + Tdt \) there correspond its generating function \( \psi_\eta = E(\eta) \) that satisfies the equation

\[ \ell_E^* (\psi_\eta) = 0. \]

Solutions of the last equation are called cosymmetries of equation \( \mathcal{E} \) and the space of cosymmetries of \( \mathcal{E} \) will be denoted by \( \text{sym}^*(\mathcal{E}) \).

2.2. Invariant Poisson–Nijenhuis structures. Consider the modules \( \mathcal{K} \) and \( \hat{\mathcal{K}} \) on the space \( J^\infty(\pi) \times \mathbb{R} \) of extended jets, i.e., we admit explicit dependence of their elements on \( t \). Since vector fields on \( \mathcal{E} \) act on \( \mathcal{K} \) and \( \hat{\mathcal{K}} \) by Lie derivatives, we can give the following

**Definition 2.** An operator \( O \) acting from \( \mathcal{K} \) to \( \mathcal{K} \) (or from \( \mathcal{K} \) to \( \hat{\mathcal{K}} \), etc.) is called invariant if \( L_{D_t} \circ O = O \circ L_{D_t} \).

**Proposition 3.** An operator \( A \): \( \hat{\mathcal{K}} \to \mathcal{K} \) is invariant iff

\[ \ell_F \circ A + A \circ \ell_F^* = 0. \]  

An operator \( R \): \( \mathcal{K} \to \mathcal{K} \) is invariant iff

\[ \ell_F \circ R - R \circ \ell_F = 0. \]  

**Proof.** Indeed, from \(19\) one has for the action on \( \mathcal{K} \)

\[ L_{D_t} = L_{D_t^\mathcal{K} + \partial f / \partial t} = \frac{\partial}{\partial t} + \partial f / \partial t - \ell_f = \ell_F, \]

while from \(19\) for the action on \( \hat{\mathcal{K}} \) we obtain

\[ L_{D_t} = L_{D_t^\hat{\mathcal{K}} + \partial f / \partial t} = \frac{\partial}{\partial t} + \partial f / \partial t + \ell_f^* = -\ell_F^*. \]
and this proves both (21) and (22).

\[ \square \]

**Remark 2.** From (22) we see that an invariant C-differential operator takes symmetries of equation \( E \) to symmetries, i.e., is a recursion operator for symmetries of \( E \). On the other hand, a C-differential operator \( A \) that enjoys (21) takes cosymmetries to symmetries. As it was shown in [7], if \( A \) is skew-adjoint and satisfies \( [[A, A]] = 0 \) it is a Hamiltonian structure for \( E \) and all Hamiltonian structures can be found in such a way.

**Definition 3.** Let \((A, R)\) be a Poisson–Nijenhuis structure on \( \mathcal{J}^\infty(\pi) \times \mathbb{R} \). We say that it is a Poisson–Nijenhuis structure on \( E \) if both \( A \) and \( R \) are invariant operators.

Thus, a Poisson–Nijenhuis on \( E \) consists of a Hamiltonian structure \( A \) and a recursion operator which is a Nijenhuis operator compatible with \( A \).

**2.3. The \( \ell^* \) - and \( \ell \)-coverings.** We shall now look at Poisson–Nijenhuis structures from a different point of view. To this end, consider the following extension of the equation \( E \). Let us add to \( E \) new odd variable \( p = (p_1, \ldots, p_m) \) that satisfies

\[ p_t = -\ell^*_f(p). \tag{23} \]

The system consisting of the initial equation \( E \) and equation (23) is called the \( \ell^* \)-covering of \( E \) and is denoted by \( \mathcal{L}^*E \). The extended total derivatives on \( \mathcal{L}^*E \) are

\[ \tilde{D}_x = D_x + \sum_{j=1}^m \sum_{k \geq 0} p_{k+1} \frac{\partial}{\partial p_k}, \quad \tilde{D}_t = D_t - \sum_{j=1}^m \sum_{k \geq 0} \tilde{D}_k(\ell^*_f(p')) \frac{\partial}{\partial p_k}. \]

Note that any C-differential operator \( O \) on \( E \) can be lifted to an operator \( \tilde{O} \) on \( \mathcal{L}^*E \) by superscribing tildes over corresponding total derivatives.

Let \( A : \hat{\kappa} \to \kappa \) be a C-differential operator of the form \( \| \sum_{i \geq 0} a_{ij} D^i_x \| \). Consider a \( p \)-linear vector function \( \mathcal{H}_A = (\mathcal{H}_A^1, \ldots, \mathcal{H}_A^m) \),

\[ \mathcal{H}_A^l = \sum_{i,j} a_{ij}^l p_i, \quad a_{ij}^l \in \mathcal{F}(E). \tag{24} \]

**Theorem 4 (see [7]).** A skew-adjoint operator \( A \) satisfies (21) iff

\[ \ell_E(\mathcal{H}_A) = 0. \tag{25} \]

In the terminology of the covering theory [13], a vector function satisfying (25) is nothing but a shadow of symmetry of \( E \) in the \( \ell^* \)-covering or, more precisely, \( \mathcal{H}_A \) is generating section of the shadow

\[ \tilde{\mathcal{H}}_A = \sum_{k \geq 0} \partial_k (\mathcal{H}_A^j) \frac{\partial}{\partial u_k}. \]

In coordinate-free terms, a shadow is a derivation of \( \mathcal{F}(E) \) with values in the function algebra on \( \mathcal{L}^*E \) that preserves the Cartan distribution.

**Lemma 5.** Let \( A \in \text{CDiff}^{sk-ad} (\mathfrak{z}, \mathfrak{z}) \) and \( \tilde{\mathcal{H}}_A \) be the corresponding shadow. Then there exists a symmetry of the equation \( \mathcal{L}^*E \) such that its restriction to \( \mathcal{F}(E) \) coincides with \( \tilde{\mathcal{H}}_A \).

**Proof.** Consider the vector function \( \alpha = (\alpha^1, \ldots, \alpha^m) \) defined by

\[ \alpha = -\frac{1}{2} \ell^*_A(p) \]

and set

\[ \tilde{\mathcal{H}}_{A, \bar{\mathcal{H}}_A} = \tilde{\mathcal{H}}_A + \sum_{k,j} \tilde{D}_k(\alpha^j) \frac{\partial}{\partial p_k}. \tag{26} \]
It is easily checked that the field $\tilde{\mathcal{H}}_{A,\tilde{A}}$ is the desired symmetry. \hfill \Box

Consider now two operators $A, B \in C\text{Diff}^{sk-ad}(\tilde{x}, \kappa)$ and using Lemma 5 define the Jacobi bracket of $\mathcal{H}_A$ and $\mathcal{H}_B$ by

$$\{\mathcal{H}_A, \mathcal{H}_B\} = \tilde{\mathcal{H}}_{A,\tilde{A}}(\mathcal{H}_B) + \tilde{\mathcal{H}}_{B,\tilde{B}}(\mathcal{H}_A)$$  \hspace{1cm} (27)

**Remark 3.** The plus sign in the right-hand side of equation (27) is due to the fact that both the fields $\tilde{\mathcal{H}}_{A,\tilde{A}}, \tilde{\mathcal{H}}_{B,\tilde{B}}$ and the functions $\mathcal{H}_B, \mathcal{H}_A$ are odd.

**Remark 4.** In more explicit terms the Jacobi bracket can be rewritten in the form

$$\{\mathcal{H}_A, \mathcal{H}_B\} = -\dot{\mathcal{H}}_A(\mathcal{H}_B) - \dot{\mathcal{H}}_B(\mathcal{H}_A) - \frac{A(\dot{\mathcal{H}}_B(p)) + B(\dot{\mathcal{H}}_A(p))}{2}.$$  \hspace{1cm} (28)

Note that there exists a one-to-one correspondence between $C$-differential operators from the module $C\text{Diff}^{sk}_{(k)}(\tilde{x}, \kappa)$ and functions on $L^*\mathcal{E}$ that are $(k-1)$-linear with respect to the odd variables. For $\Delta \in C\text{Diff}^{sk}_{(k)}(\tilde{x}, \kappa)$ denote by $\mathcal{H}_\Delta$ the corresponding function. Then by direct computations one can prove the following

**Theorem 6.** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be shadows of symmetries in the $E^*$-covering to which there correspond operators $A, B \in C\text{Diff}^{sk-ad}(\tilde{x}, \kappa)$. Then

$$\{\mathcal{H}_A, \mathcal{H}_B\} = \mathcal{H}_{[A,B]}.$$  \hspace{1cm} (29)

If $\psi_1$ and $\psi_2$ are sections of the $E^*$-covering such that $(d\psi_i)(C) \subset \tilde{C}$, $i = 1, 2$, where $C$ and $\tilde{C}$ are the Cartan distributions on $\mathcal{E}$ and $L^*\mathcal{E}$, respectively, then equation (29) can be rewritten as follows

$$\psi_1^* \psi_2^* \{\mathcal{H}_A, \mathcal{H}_B\} = [A,B](\psi_1, \psi_2).$$

We now introduce the object dual to the $E^*$-covering. Namely, consider the extension of $\mathcal{E}$ by new odd variables $q = (q^1, \ldots, q^m)$ that satisfy the equation

$$q_t = \ell_f(q).$$  \hspace{1cm} (30)

The system consisting of the initial equation and equation (30) will be called the $E^*$-covering of $\mathcal{E}$ and denoted by $E^*\mathcal{E}$. The extended total derivatives on $E^*\mathcal{E}$ are

$$\tilde{D}_x = D_x + \sum_{l=1}^m \sum_{k \geq 0} a^l_{k+1} \frac{\partial}{\partial q^k}, \quad \tilde{D}_t = D_t + \sum_{l=1}^m \sum_{k \geq 0} \tilde{D}^k_{\ell_f}(q^l) \frac{\partial}{\partial q^k}.$$  

To any $C$-differential operator $R: \kappa \to \kappa$ of the form $\sum_{i \geq 0} a_{ij}^l D_x^l$ let us put into correspondence a $q$-linear vector function $\mathcal{N}_R = (\mathcal{N}_R^1, \ldots, \mathcal{N}_R^m)$, where

$$\mathcal{N}_R^l = \sum_{i,j} a_{ij}^l q^i, \quad a_{ij}^l \in \mathcal{F}(\mathcal{E}).$$

Similar to Theorem 6 we have the following

**Theorem 7** (see [7, 6]). An operator $R$ satisfies relation (22), i.e., is a recursion operator for the equation $\mathcal{E}$ iff

$$\tilde{\ell}_E(\mathcal{N}_R) = 0,$$  \hspace{1cm} (31)

i.e., $\mathcal{N}_R$ is a shadow of a symmetry in the $E^*$-covering.

The corresponding derivation is of the form

$$\tilde{\mathcal{E}}_{\mathcal{N}_R} = \sum_{k \geq 0} \tilde{D}^k_x(\mathcal{N}_R^j) \frac{\partial}{\partial q^k}.$$  

In parallel to Lemma 5 we have the following auxiliary result.
Lemma 8. Let $R \in \text{CDiff}(\mathcal{X})$ and $\bar{\partial}_{\mathcal{N}_R}$ be the corresponding shadow. Then there exists a symmetry of the equation $\mathcal{L}\mathcal{E}$ such that its restriction to $\mathcal{F}(\mathcal{E})$ coincides with $\bar{\partial}_{\mathcal{N}_R}$.

Proof. Consider the vector function $\beta = (\beta^1, \ldots, \beta^m)$ with

$$\beta^j = \sum_{k,j} \frac{\partial N_{R_j}^{\bar{\partial}^j_{R_k}}}{\partial \tau_{k}} q_k^j,$$

and set

$$\bar{\partial}_{\mathcal{N}_R} = \bar{\partial}_{\mathcal{N}_R} + \sum_{k,j} \tilde{D}_k^j(\beta^j) \frac{\partial}{\partial q_k^j}. \quad (32)$$

This is the symmetry we are looking for. \qed

Using Lemma 8 we define the Jacobi bracket of vector functions $\mathcal{N}_R$ and $\mathcal{N}_S$ by

$$\{\mathcal{N}_R, \mathcal{N}_S\} = \bar{\partial}_{\mathcal{N}_R, \mathcal{N}_S}(\mathcal{N}_S) + \bar{\partial}_{\mathcal{N}_S, \mathcal{N}_R}(\mathcal{N}_R). \quad (33)$$

In explicit terms the Jacobi bracket has the form

$$\{\mathcal{N}_R, \mathcal{N}_S\} = -\tilde{\epsilon}_{\mathcal{N}_R}(\mathcal{N}_S) - \tilde{\epsilon}_{\mathcal{N}_S}(\mathcal{N}_R) + R(\tilde{\epsilon}_{\mathcal{N}_S}(q)) + S(\tilde{\epsilon}_{\mathcal{N}_R}(q)). \quad (34)$$

Denote by $\mathcal{N}(R, S)_{FN}$ the function on $\mathcal{L}\mathcal{E}$ bilinear with respect to the variables $q$ and corresponding to the operator $[R, S]_{FN} \in \text{CDiff}^{\text{skew}}(\mathcal{X}, \mathcal{X})$. Then by the direct computations we obtain

Theorem 9. Let $\mathcal{N}_R$ and $\mathcal{N}_S$ be shadows of symmetries in the $\ell$-covering to which there correspond operators $R$ and $S \in \text{CDiff}(\mathcal{X}, \mathcal{X})$. Then

$$\{\mathcal{N}_R, \mathcal{N}_S\} = \mathcal{N}(R, S)_{FN}. \quad (35)$$

If $\varphi_1, \varphi_2$ are sections of the $\ell$-covering preserving the Cartan distributions then

$$\varphi_1^\ast \varphi_2^\ast \{\mathcal{N}_R, \mathcal{N}_S\} = [R, S]_{FN}(\varphi_1, \varphi_2). \quad (36)$$

2.4. Compatibility condition. We shall now express the compatibility condition of a Hamiltonian structure $A$ and a recursion operator $R$ on equation (15) in similar geometric terms. To this end, recall that both $\mathcal{L}\mathcal{E}$ and $\mathcal{L}^\ast \mathcal{E}$ are fibered over the equation $\mathcal{E}$ and denote the corresponding fiber bundles by $\tau: \mathcal{L}\mathcal{E} \rightarrow \mathcal{E}$ and $\tau^*: \mathcal{L}^\ast \mathcal{E} \rightarrow \mathcal{E}$, respectively. Consider the Whitney product $\tau \otimes \tau^*: \mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E} \rightarrow \mathcal{E}$ of these bundles. One can extend the total derivatives to the space $\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E}$ by setting

$$\tilde{D}_x = D_x + \sum_{l=1}^{m} \sum_{k \geq 0} \left( q_{k+1} \frac{\partial}{\partial q_k} + p_{k+1} \frac{\partial}{\partial p_k} \right),$$

$$\tilde{D}_t = D_t + \sum_{l=1}^{m} \sum_{k \geq 0} \left( \tilde{D}_x^k(\hat{\epsilon}_f(q^l)) \frac{\partial}{\partial q_k} - \tilde{D}_x(\tilde{\epsilon}_f(p^l)) \frac{\partial}{\partial p_k} \right).$$

Thus the equation $\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E}$ amounts to $\mathcal{E}$ extended both by (23) and (30).

Due to the natural projections $\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E} \rightarrow \mathcal{L}^\ast \mathcal{E}$ and $\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E} \rightarrow \mathcal{L}\mathcal{E}$, the vector fields $\bar{\partial}_{\mathcal{N}_A, \mathcal{N}_A}$ and $\bar{\partial}_{\mathcal{N}_R, \mathcal{N}_R}$ (see equalities (20) and (32)) may be considered as derivations from $\mathcal{F}(\mathcal{L}^\ast \mathcal{E})$ and $\mathcal{F}(\mathcal{L}\mathcal{E})$ to $\mathcal{F}(\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E})$, respectively.

Lemma 10. There exist symmetries $\bar{\partial}_{\mathcal{N}_A, \mathcal{N}_A, \alpha}$ and $\bar{\partial}_{\mathcal{N}_R, \mathcal{N}_R, \rho}$ of equation $\mathcal{L}\mathcal{E} \times \mathcal{L}^\ast \mathcal{E}$ such that their restrictions to the function algebras $\mathcal{F}(\mathcal{L}\mathcal{E})$ and $\mathcal{F}(\mathcal{L}^\ast \mathcal{E})$ coincide with $\bar{\partial}_{\mathcal{N}_A, \mathcal{N}_A}$ and $\bar{\partial}_{\mathcal{N}_R, \mathcal{N}_R}$, respectively.
Proof. Let us set
\[ \tilde{\partial}_{H_A,\tilde{\partial}_{H_A,\alpha}} = \tilde{\partial}_{H_A,\tilde{\partial}_{H_A,\alpha}} + \sum_{k,j} \tilde{D}_k(\alpha^j) \frac{\partial}{\partial q_k^j}. \] (35)
where the vector function \( \alpha = (\alpha^1, \ldots, \alpha^m) \) is defined by \( \alpha = \tilde{\ell}_H(q) \). Let also consider the field
\[ \tilde{\partial}_{N_R,\tilde{\partial}_{N_R,\rho}} = \tilde{\partial}_{N_R,\tilde{\partial}_{N_R,\rho}} + \sum_{k,j} \tilde{D}_k(\rho^j) \frac{\partial}{\partial p_k^j}. \] (36)
where \( \rho = (\rho^1, \ldots, \rho^m) \) is defined by
\[ \rho = -\tilde{\ell}_{N_R}(p) - \tilde{\partial}_{q}(R^*)(p). \]
By direct computations one can check that the fields (35) and (36) possess the needed properties.

Using Lemma 10 let us define the bracket
\[ \{H_A, N_R\} = \tilde{\partial}_{H_A,\tilde{\partial}_{H_A,\alpha}}(N_R) + \tilde{\partial}_{N_R,\tilde{\partial}_{N_R,\rho}}(H_A), \]
or, in explicit terms,
\[ \{H_A, N_R\} = -\tilde{\ell}_{N_R}(H_A) - \tilde{\ell}_H(N_R) - A \left( \tilde{\ell}_{N_R}(p) + \tilde{\partial}_q(R^*)(p) \right) + R(\tilde{\ell}_H(q)). \] (37)
Denote by \( C^*_{A,R} \) the bilinear with respect to the variables \( p \) and \( q \) function on \( \mathcal{L}E \times \mathcal{L}E \) corresponding to the \( C \)-differential operator \( C^*(A, R): \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) (see equality (17)). Then the following result holds:

**Theorem 11.** Let \( H_A \) be a shadow in the \( \ell^* \)-covering to which there corresponds an operator \( A \in \text{CDiff}^{\text{sk-ad}}(\mathcal{A}, \mathcal{A}) \) and \( N_R \) be a shadow in the \( \ell \)-covering to which there corresponds an operator \( R \in \text{CDiff}(\mathcal{A}, \mathcal{A}) \). Then
\[ \{H_A, N_R\} = C^*_{A,R}. \]

From the above said we get the following

**Theorem 12.** Let a Hamiltonian operator \( A \in \text{CDiff}^{\text{sk-ad}}(\mathcal{A}, \mathcal{A}) \) and a recursion operator \( R \in \text{CDiff}(\mathcal{A}, \mathcal{A}) \) define Poisson–Nijenhuis structure on evolution equation \( \mathcal{E} \), while \( H_A \) and \( N_R \) be the corresponding shadows in the \( \ell^* \)- and \( \ell \)-covering over \( \mathcal{E} \), respectively. Then
\[ \{H_A, H_A\} = 0, \] (38)
\[ \{N_R, N_R\} = 0, \] (39)
\[ \{H_A, N_R\} = 0. \] (40)

3. Variational Poisson–Nijenhuis structures in general case

Consider now the infinite prolongation \( \mathcal{E} \subset J^\infty(\pi) \) of a general differential equation as a submanifold in the space \( J^\infty(\pi) \) of infinite jets of some locally trivial bundle \( \pi: E \to M \). In local coordinates \( \mathcal{E} \) is given by the system
\[ F^j(x^1, \ldots, x^n, u^1, \ldots, u^m, \ldots, u^j, \ldots) = 0, \quad j = 1, \ldots, r. \]
We assume that \( F = (F^1, \ldots, F^r) \in P \), where \( P \) is the module of sections of some vector bundle over \( J^\infty(\pi) \). Consider the linearization operator \( \ell_\mathcal{E}: \mathcal{A} \to P \) and its adjoint \( \ell_\mathcal{E}^*: \tilde{P} \to \tilde{\mathcal{A}} \). Similar to the evolutionary case, we can construct \( \ell \)- and \( \ell^* \)-coverings by extending \( \mathcal{E} \) with \( \ell_\mathcal{E}(q) = 0 \) and \( \ell_\mathcal{E}^*(p) = 0 \).
Following the scheme exposed in Section 2, we are looking for \( C \)-differential operators such that the diagrams

\[
\begin{align*}
\text{(i)} & \quad \kappa \ell_E \rightarrow P, \\
\text{(ii)} & \quad \hat{P} \rightarrow \hat{\kappa}
\end{align*}
\]

are commutative. Literally copying the construction of Section 2, we can put into correspondence to an operator \( R \) from the first diagram in (41) a \( q \)-linear function \( N_R = (N_{1R}, \ldots, N_{mR}) \) on \( LE \), while for \( A \) from the second diagram we construct a \( p \)-linear function \( H_A = (H_{1A}, \ldots, H_{mA}) \) on \( L \ast E \). We again use the general result proved in [7]:

**Theorem 13.** An operator \( R \) fits diagram (i) in (41) iff

\[
\tilde{\ell}_E (N_R) = 0,
\]

while \( A \) fits diagram (ii) iff

\[
\tilde{\ell}_E (H_A) = 0.
\]

Among the operators \( A \) let us distinguish those ones that enjoy the property similar to skew-adjointness. Namely, we shall consider operators such that

\[
A^* = -\bar{A}.
\]

**Remark 5.** This property means that the operator \( (A, \bar{A}) : \hat{P} \oplus \hat{\kappa} \rightarrow \kappa \oplus P \) is skew-adjoint.

Note now that explicit expressions (28), (34) and (37) for Jacobi brackets do not rely on the fact that they were given for evolutionary equations. Using this observation we give the following

**Definition 4.** Let \( \mathcal{E} \subseteq J^\infty (\pi) \) be a differential equation.

1. A \( C \)-differential operator \( A : \hat{P} \rightarrow \kappa \) is called a Hamiltonian structure on \( \mathcal{E} \) if it fits the left diagram (41), equation (44) holds and \( \{H_A, H_A\} = 0 \), where the bracket is given by (28). Two Hamiltonian structures \( A \) and \( B \) are said to be compatible if \( \{H_A, H_B\} = 0 \).
2. A \( C \)-differential operator \( R : \kappa \rightarrow \kappa \) is called a Nijenhuis operator for the equation \( \mathcal{E} \) if it fits the right diagram (41) and \( \{N_R, N_R\} = 0 \), where the bracket is given by (34).
3. A pair of \( C \)-differential operators \( (A, R) \) is called a Poisson–Nijenhuis structure on \( \mathcal{E} \) if \( R \) is a Nijenhuis operator, \( A \) is a Hamiltonian structure such that \( A^* \circ R^* = \bar{R} \circ A^* \) and \( \{N_R, H_A\} = 0 \), where the bracket is given by (37).

**Remark 6.** If \( (A, R) \) is a Poisson–Nijenhuis structure on \( \mathcal{E} \) then \( R \) is a recursion operator for symmetries of \( \mathcal{E} \). Moreover, a Hamiltonian structure \( A \), similar to the evolutionary case, determines a Poisson bracket on the group of conservation laws of \( \mathcal{E} \). Namely, if \( \omega_1, \omega_2 \) are conservation laws then we set

\[
\{\omega_1, \omega_2\}_A = L_{d^{0,n-1}_1} \omega_1 \omega_2,
\]

where \( d^{0,n-1}_1 : E^{0,n-1}_1 \rightarrow E^{1,n-1}_1 \) is the differential in Vinogradov’s \( C \)-spectral sequence, see [17]. For evolutionary equations this differential coincides with the Euler operator.

The following result generalises Theorem 2:

**Theorem 14.** If \( (A, R) \) is a Poisson–Nijenhuis structure on \( \mathcal{E} \) then \( R^i \circ A, i = 0,1,2,\ldots \), is a family of pair-wise compatible Hamiltonian structures on \( \mathcal{E} \).
4. Concluding remarks: nonlocal Poisson–Nijenhuis structures

Strictly speaking all constructions exposed above are valid for local Poisson–Nijenhuis structures. In reality, the operators $A$ or $R$ or both are nonlocal, i.e., contain terms like $D_x^{-1}$. For example, recall the recursion operator for the KdV equation. It seems that our approach can be extended to structures of this type. The general scheme is as follows.

Let $\mathcal{E}$ be a differential equation and $\tau: \tilde{\mathcal{L}} \mathcal{E} \to \mathcal{L} \mathcal{E}$ be a general covering over $\mathcal{L} \mathcal{E}$ in the sense of [13]. Then solutions of the equation

$$\tilde{\ell}_\mathcal{E} (\mathcal{N}) = 0$$

linear with respect to odd variables give rise to nonlocal $\mathcal{L}$-differential recursion operators $R_\mathcal{N}$ with nonlocalities corresponding to nonlocal variables defined by $\tau$. These solutions are shadows of symmetries in this covering. The hardest problem lies in definition of the Jacobi bracket for such shadows. Nontriviality of this problem is illustrated by observation given in [15]. A way to commute shadows can be derived from the results of [9] but the constructions given there are ambiguous.

In [5] we described a canonical construction for the Jacobi bracket $\{ \ldots \}$ of shadows of a general nature. Given this construction and taking into account the above exposed results, we can define nonlocal Nijenhuis operators $R_\mathcal{N}$ as the ones satisfying

$$\{ \mathcal{N}, \mathcal{N} \} = 0.$$

In a similar manner, we can consider coverings over $\tau^*: \tilde{\mathcal{L}}^* \mathcal{E} \to \mathcal{L}^* \mathcal{E}$ and, solving the equation

$$\tilde{\ell}_\mathcal{E} (\mathcal{H}) = 0,$$

look for nonlocal Hamiltonian operators $A_\mathcal{H}$ corresponding to shadows $\mathcal{H}$. The Hamiltonianity condition is given by

$$\{ \mathcal{H}, \mathcal{H} \} = 0.$$

Finally, the compatibility condition for $R_\mathcal{N}$ and $A_\mathcal{H}$ are expressed by

$$\{ \mathcal{H}, \mathcal{N} \} = 0,$$

where the bracket is considered in the Whitney product of $\tau$ and $\tau^*$.

A detailed theory of nonlocal Poisson–Nijenhuis structures will be given elsewhere.

Remark 7. As it was demonstrated in [7] and [6], a very efficient way to construct nonlocal recursion operators and Hamiltonian structures for evolution equations is the use of nonlocal vectors (for the $\ell^*$-covering) and covectors (for the $\ell$-covering). In particular, this method, by its nature, leads to the so-called weakly nonlocal operators.

When this paper was almost finished, Maria Clara Nucci indicated to us the work [4] where the construction of nonlocal vectors was reinvented. The author of [4] exploits the Lagrangian structure of the $\ell^*$-covering, though the reason for existence of nonlocal vectors is more general (it suffices to compare the construction with the one for nonlocal covectors on the $\ell$-covering).

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