On the Hausdorff dimension faithfulness connected with $Q_{\infty}$-expansion

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Received 21 February 2016, revised 17 March 2017
Accepted for publication 20 March 2017
Published 19 April 2017

Abstract

In this paper, we show that, the family of all possible unions of finitely many consecutive cylinders of the same rank of a $Q_{\infty}$-expansion is faithful for the Hausdorff dimension calculations. Applying this result, we give a necessary and sufficient condition for the family of all cylinders of the $Q_{\infty}$-expansion to be faithful for Hausdorff dimension calculation on the unit interval. This answers an open problem mentioned in Albeverio et al (2016 Math. Nachr. at press (https://doi.org/10.1002/mana.201500471)).

Keywords: $Q_{\infty}$-expansion, faithfulness, Hausdorff dimension
Mathematics Subject Classification numbers: 11K55, 28A80

1. Introduction

The notion Hausdorff dimension is well known and plays an important role in fractal geometry. The Hausdorff dimension has the advantage of being defined for any set, as it is based on measures. A major disadvantage however is that in many cases it is a rather non-trivial problem to calculate the exact Hausdorff dimension of a set. Recently, there has been a greater interest in determining the Hausdorff dimension by restricting the family of admissible coverings. In [1, 2], Albeverio et al introduced the notion of faithfulness of families for Hausdorff dimension calculation. The faithfulness of the family allows us to calculate the Hausdorff dimension considering only narrow families of admissible coverings.

We recall some definitions. Let $\Phi$ be a fine family of coverings on $[0,1)$, i.e. a family of subsets of $[0,1)$ such that for any $\epsilon > 0$, there exists, at most, a countable $\epsilon$-covering $\{E_j\}$ of $[0,1)$ with $E_j \in \Phi$. The $\alpha$-dimensional Hausdorff measure of a set $E \subset [0,1)$ with respect to a fine family of coverings $\Phi$ is defined by
\[ H^\alpha(E, \Phi) = \liminf_{\epsilon \to 0} \left\{ \sum_j |E_j|^\alpha |\{E_j\} \text{ is an } \epsilon - \text{covering of } E, E_j \in \Phi \right\} \]

and the non-negative number
\[ \dim_H(E, \Phi) = \inf \{ \alpha | H^\alpha(E, \Phi) = 0 \} \]

is called the Hausdorff dimension of the set \( E \subset [0,1) \) with respect to the family \( \Phi \). If we take \( \Phi \) to be the family of all the subsets of \([0,1), \) we denote \( \dim_H(E, \Phi) \) by \( \dim_H(E) \), which is equal to the classical Hausdorff dimension of set \( E \subset [0,1) \). For more properties of the Hausdorff dimension, one is referred to [6, 9].

A fine covering family \( \Phi \) is said to be a faithful family of coverings for Hausdorff dimension calculation if
\[ \dim_H(E, \Phi) = \dim_H(E), \text{ for any } E \in [0,1]. \]

The first result concerning the problem of faithful coverings is due to A S Besicovitch, who proved this for the family of cylinders of a binary expansion. This result was extended to the family of s-adic cylinders by Billingsley. Albeverio, Ivanenko, Lebid and Torbin have done a series of work in this direction, e.g. [1–4, 8]. Especially, in [2], Albeverio, Kondratiev, Nikiforov and Torbin [2] gave a sufficient general condition for the family of all cylinders of a \( Q_\infty \)-expansion to be faithful. Recently, we showed that the family of all possible unions of finitely many consecutive cylinders of the same rank of continued fraction expansion is also faithful [7].

In this paper, we focus on the family of the finite union of cylinders of \( Q_\infty \)-expansion and prove that the family of all possible unions of finitely many consecutive cylinders of the same rank of the \( Q_\infty \)-expansion is faithful for the Hausdorff dimension calculation. Applying this result, we give necessary and sufficient conditions for the family of all cylinders of a \( Q_\infty \)-expansion to be faithful for Hausdorff dimension calculation on the unit interval. This answers an open problem mentioned in [2].

### 2. Statement of main results

First of all, we briefly recall the definition of and some properties related to \( Q_\infty \)-expansions.

Let \( Q = (q_0,q_1,\cdots,q_n,\cdots) \) be a \( Q_\infty \)-vector, i.e. \( \{q_i\}_{i \geq 0} \) are positive reals and satisfy \( \sum_{i \geq 0} q_i = 1 \). Now we give the \( Q_\infty \)-expansion with respect to the \( Q_\infty \)-vector in the following way.

Step 1. We decompose \([0,1)\) (from the left to the right) into the union of semi-intervals \( \Delta_{\alpha_1}, \alpha_1 \in \mathbb{N} \) (without common points) and with length \( |\Delta_{\alpha_1}| = q_{\alpha_1} \), such that
\[ [0,1) = \bigcup_{\alpha_1=0}^{\infty} \Delta_{\alpha_1}. \]

Each interval \( \Delta_{\alpha_1} \) is called a 1-rank cylinder.

Step \( n \geq 2 \). We decompose (from the left to the right) each \((n-1)\)-rank cylinder (semi-intervals) \( \Delta_{\alpha_1\cdots\alpha_{n-1}} \) into the union of \( n \)-rank cylinders \( \Delta_{\alpha_1\cdots\alpha_{n-1}\alpha_n} \) (without common points), i.e.
\[ \Delta_{\alpha_1\cdots\alpha_{n-1}} = \bigcup_{\alpha_n=0}^{\infty} \Delta_{\alpha_1\cdots\alpha_{n-1}\alpha_n}. \]
Every $n$-rank cylinder has the length
\[ |\Delta_{\alpha_1\cdots\alpha_n}| = \prod_{i=1}^{n} q_{\alpha_i}. \]

It is clear that any sequence of indices $\{\alpha_i\}$ generates the corresponding sequence of embedded cylinders
\[ \Delta_{\alpha_1} \supset \Delta_{\alpha_1\alpha_2} \supset \cdots \supset \Delta_{\alpha_1\alpha_2\cdots\alpha_n} \supset \cdots \]
and there exists a unique point $x \in [0,1)$ belonging to all of them.

Conversely, for any point $x \in [0,1)$ there is a unique sequence of embedded cylinders
\[ \Delta_{\alpha_1} \supset \Delta_{\alpha_1\alpha_2} \supset \cdots \supset \Delta_{\alpha_1\alpha_2\cdots\alpha_n} \supset \cdots \]
containing the point $x$, i.e.
\[ x = \bigcap_{n=1}^{\infty} \Delta_{\alpha_1\cdots\alpha_n} = \bigcap_{n=1}^{\infty} \Delta_{\alpha_1(x)\cdots\alpha_n(x)} := \Delta_{\alpha_1(x)\cdots\alpha_n(x)} \cdots. \]

The expression is called the $Q_\infty$-expansion of $x$. Real numbers which are the endpoints of $n$th rank cylinders are said to be $Q_\infty$-rational, and their $Q_\infty$-expansion contains only finitely many non-zero digits.

Let $\Phi$ be the family of all possible cylinders of a $Q_\infty$-partition of the interval $[0,1)$, i.e.
\[ \Phi = \{ \Delta_{\alpha_1\cdots\alpha_n} : \alpha_i \in \mathbb{N}, i = 1, 2, \cdots, n; n \in \mathbb{N} \}. \]

In a recent paper [2], Albeverio, Kondratiev, Nikiforov and Torbin give a sufficient general condition for the family $\Phi$ to be faithful. However, based on a method which was invented by Yuval Peres, they proved the following result showing that the family $\Phi$ is not necessarily faithful.

Theorem 2.1 ([2]). If constants $m_0 > 1$, $A > 0$ and $B > 0$ exist, such that
\[ \frac{A}{m_0} \leq q_i \leq \frac{B}{m_0}, \forall i \in \mathbb{N}, \]
then the family $\Phi$ is non-faithful.

In this paper, let $A_n$ be the family of all possible unions of finitely many consecutive $n$th rank cylinders, i.e.
\[ A_n = \left\{ \bigcup_{i=m}^{m+k} \Delta_{\alpha_1\cdots\alpha_i} : m, k, \alpha_j \in \mathbb{N}, 1 \leq j \leq n-1 \right\} \]
and
\[ A = A(Q) = \bigcup_{n \geq 1} A_n. \]

We mainly consider the covering family $A$ of all possible unions of finitely many cylinders, and show that $A$ is faithful for the Hausdorff dimension calculation without any additional condition.

Theorem 2.2. Let $Q = (q_0, q_1, \cdots, q_n, \cdots)$ be a $Q_\infty$-vector. Then the family $A = A(Q)$ is faithful for the Hausdorff dimension calculation on the unit interval.
If we take \( q_i = \frac{1}{(i+1)(i+2)} \), the \( Q_\infty \)-expansion leads to the classical Lüroth expansion [5]. As a special case of the above theorem, we have the following corollary.

**Corollary 2.3.** The family of all possible unions of finitely many consecutive cylinders of the same rank generated by Lüroth expansion is faithful.

Applying theorem 2.2 we give necessary and sufficient conditions for the family \( \Phi \) to be faithful for the Hausdorff dimension calculation. This answers an open problem mentioned in [2].

**Theorem 2.4.** Let \( Q = (q_0, q_1, \cdots, q_n, \cdots) \) be a \( Q_\infty \)-vector. Then the family \( \Phi = \Phi(Q) \) is faithful for Hausdorff dimension calculation on the unit interval if and only if for any \( \alpha \in (0, 1) \), and \( \delta \in (0, \alpha) \) there exists a positive integer \( N = N(\alpha, \delta) \) such that for any \( n > N \) and \( M \in \mathbb{N} \),

\[
\sum_{i=n}^{n+M} q_i \alpha - \delta \geq \sum_{i=n}^{n+M} q_i^\alpha.
\] (2.1)

### 3. Proofs of main results

Before we prove the main results, we give the following useful lemma, which can be proved easily.

**Lemma 3.1.** Let \( \{a_n\}_{n \geq 0} \) be a sequence of positive reals, if \( \sum_{n \geq 0} a_n < \infty \), then for any \( \alpha \in (0, 1) \) there exists a sequence \( \{n_k\}_{k \geq 1} \) of integers with

\[
\left( \sum_{i=0}^{n_k} a_i \right)^\alpha \geq \sum_{i=1}^{\infty} \left( \sum_{j=n_k+1}^{\infty} a_j \right)^\alpha.
\]

**Proof of the theorem 2.2.** We only need to show that

\[
\dim_H(E) = \dim_H(E, A)
\]

for any set \( E \subset [0, 1) \). Without loss of generality, we assume \( \dim_H(E) < 1 \).

Since the set of all \( Q_\infty \)-rational is dense in \([0,1)\), for the calculation of the Hausdorff dimension of a set \( E \subset [0, 1) \), we need only consider the coverings of intervals with \( Q_\infty \)-rational endpoints.

For a given set \( E, \alpha \in (0, 1), \delta \in (0, \alpha) \) and \( \epsilon > 0 \), let \( \{E_j\}_{j \in \mathbb{N}} \) be an \( \epsilon \)-covering of \( E \) with \( E_j = [a_j, b_j) \) and \( a_j, b_j \) both \( Q_\infty \)-rational. For each interval \( E_j, j \in \mathbb{N} \), there exists a cylinder \( \Delta_{\alpha_1, \alpha_2, \cdots, \alpha_j} \in \Phi \) such that:

1. \( E_j \subset \Delta_{\alpha_1, \alpha_2, \cdots, \alpha_j} \);
2. any interval of rank \( n_j + 1 \) does not contain \( E_j \).

Without loss of generality, we may assume that \( a_j \) has the following \( Q_\infty \)-expansion:

\[
a_j = \Delta_{\alpha_1, \cdots, \alpha_{n_j+1}, 00 \cdots}
\]

for some integer \( l_j \), where \( \alpha_k = \alpha_k(a_j) \) is the \( k \)th digit of \( a_j \). Then the point \( c_j = \Delta_{\alpha_1, \cdots, \alpha_{n_j}} (\alpha_{n_j+1} + 1)00 \cdots \) belongs to \( [a_j, b_j) \).
For simplicity of notation, take $\beta_k = \alpha_n + k(\alpha_j)$. To cover $E_j$ by a union of finitely many cylinders we consider the coverings of $[a_j, c_j)$ and $[c_j, b_j)$ separately.

- The coverings of $[c_j, b_j)$.

If $b_j$ is the endpoint of some $(n_j + 1)$th rank cylinder, then $b_j$ has the following $Q_\infty$-expansion:

$$b_j = \Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_0)00 \cdots$$

for some integer $s_0 \geq 2$. Now $[c_j, b_j)$ is the union of finitely many cylinders of rank $n_j + 1$, i.e.

$$[c_j, b_j) = \bigcup_{i=\beta_1+1}^{\beta_1+n_j-1} \Delta_{\alpha_1 \cdots \alpha_{n_j-1} \alpha_{n_j}} \in \mathcal{A}. \quad (3.1)$$

If $b_j$ is not the endpoint of any $(n_j + 1)$th rank cylinder, then there exists an integer $s_1 \in \mathbb{N}$ such that

$$b_j \in \text{Int}(\Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)), \quad \text{where } \text{Int}(A) \text{ denotes the interior of set } A. \text{ Let } s_2 \text{ be the integer with }$$

$$b_j \in \Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)00 \cdots \underbrace{0}_{s_2}$$

and

$$\Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)00 \cdots \underbrace{0}_{s_2+1} \subset [c_j, b_j).$$

Then we have

$$|\Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)00 \cdots 0| = \frac{1}{q_0} |\Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)00 \cdots 0| \leq \frac{1}{q_0} |E_j|.$$  

Take

$$J_0 = \bigcup_{i=1}^{s_1} \Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + i) \text{ if } s_1 \geq 1 \quad (J_0 = \emptyset \text{ if } s_1 = 0)$$

and

$$J_1 = \Delta_{\alpha_1 \cdots \alpha_{n_j}}(\beta_1 + s_1 + 1)00 \cdots \underbrace{0}_{s_2}$$

$[c_j, b_j)$ can be covered by $\{J_0, J_1\} \subset \mathcal{A}$, so the corresponding $\alpha$-volume does not exceed

$$(1 + \frac{1}{q_0})|E_j|^\alpha. \quad (3.2)$$

- The coverings of $[a_j, c_j)$.
We can divide the interval \([a_j, c_j]\) into the union of pairwise disjoint cylinders of the following ranks.

\[
\text{Rank}(n_j + 2): \quad \Delta_{\alpha_1 \ldots \alpha_j \beta_{i+1}}, \quad i \geq \beta_2 + 1,
\]

\[
\text{Rank}(n_j + 3): \quad \Delta_{\alpha_1 \ldots \alpha_j \beta_{i+2}}, \quad i \geq \beta_3 + 1,
\]

\[\vdots\]

\[
\text{Rank}(n_j + l_j - 1): \quad \Delta_{\alpha_1 \ldots \alpha_j \beta_{i+l_j-2}}, \quad i \geq \beta_{l_j-1} + 1,
\]

\[
\text{Rank}(n_j + l_j): \quad \Delta_{\alpha_1 \ldots \alpha_j \beta_{i+l_j}}, \quad i \geq \beta_{l_j}.
\]

For the cylinders of rank \(n_j + k, 2 \leq k \leq l_j - 1\), by lemma 3.1, there exists a sequence \(\{t_k(0) = 0, t_k(i)\}_{i \geq 1}\) of integers such that

\[
\left| \bigcup_{i = \beta_k + t_k(1)}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}} \right|^{\alpha} \geq \sum_{m=1}^{\infty} \left| \bigcup_{i = \beta_k + t_k(m) + 1}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}} \right|^{\alpha}
\]

and

\[
\left| \bigcup_{i = \beta_k}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}} \right|^{\alpha} \geq \sum_{m=1}^{\infty} \left| \bigcup_{i = \beta_k + t_k(m) + 1}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}} \right|^{\alpha}.
\]

Take

\[
J_k(m) = \bigcup_{i = \beta_k + t_k(m)+1}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}}, \quad 2 \leq k \leq l_j - 1, \quad m \in \mathbb{N}
\]

and

\[
J_k(m) = \bigcup_{i = \beta_k + t_k(m)+1}^{\beta_k + t_k(m+1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}}, \quad m \geq 1.
\]

\[
J_k(0) = \bigcup_{i = \beta_k}^{\beta_k + t_k(1)} \Delta_{\alpha_1 \ldots \alpha_j \beta_{i-1}}.
\]

Then \([a_j, b_j]\) can be covered by family \(\{J_k(m) : 2 \leq k \leq l_j, m \in \mathbb{N}\} \subset \mathcal{A}\) and corresponding \(\alpha\)-volume,

\[
V_j(\alpha) = \sum_{k=2}^{l_j} \sum_{m=0}^{\infty} |J_k(m)|^{\alpha} \leq 2 \left( \sum_{k=2}^{l_j} |J_k(0)|^{\alpha} \right).
\]

Let \(q_1 = \max_i(q_i)\) and \(d_j \in \mathbb{N}\) such that \(2^{d_j-1} < l_j \leq 2^{d_j}\), then for any \(2 \leq k \leq l_j - 1\),
Nonlinearity 30 (2017) 2268
J Liu and Z Zhang

\[ |J_k(0)|^\alpha \leq \frac{\bigcup_{j=0}^{b_k+h_k} \Delta_{\alpha_1 \cdots \alpha_k j}}{|E_j|^{\alpha-\delta}} \]

\[ \leq \frac{\bigcup_{j=0}^{b_k+h_k} \Delta_{\alpha_1 \cdots \alpha_k j}}{|E_j|^{\alpha-\delta}} \]

\[ \leq |\Delta_{\alpha_1 \cdots \alpha_k j}|^{\delta} \leq q^{(k-1)\delta} \]

and similarly,

\[ \frac{|J_k(0)|^\alpha}{|E_j|^{\alpha-\delta}} \leq q^{(k-1)\delta}, \]

thus we have the following estimation of \( V_j(\alpha) \).

\[ V_j(\alpha) \leq 2 \left[ \sum_{i=1}^{d-1} \sum_{2^{-i} \leq k \leq 2^{-i-1}} |J_k(0)|^\alpha + \sum_{2^{-i-1} \leq k \leq 2^{-i}} |J_k(0)|^\alpha \right] \]

\[ \leq \frac{2}{q^s} |E_j|^{\alpha-\delta} \sum_{k=0}^{d-1} 2^k (q^s)^{2k} \leq \frac{2}{q^s} |E_j|^{\alpha-\delta} \sum_{j=1}^{\infty} s(q^s)^i (q^s)^j \]

\[ \leq \frac{2}{q^s} |E_j|^{\alpha-\delta} W(\delta) \sum_{j=1}^{\infty} (q^s)^j = \frac{2 W(\delta)}{(1-q^s)q^s} |E_j|^{\alpha-\delta} \]

where \( W(\delta) \) is a constant such that \( s(q^s)^j \leq W(\delta) \) for any \( s \in \mathbb{N} \).

So, for a given \( E_j = [a_j, b_j] \) there exists a countable subfamily of \( \mathcal{A} \) that covers \( E_j \) and the corresponding \( \alpha \)-volume does not exceed

\[ K(\alpha, \delta)|E_j|^{\alpha-\delta} \]

where \( K(\alpha, \delta) = 1 + \frac{1}{q^s} + \frac{2 W(\delta)}{(1-q^s)q^s} \).

Therefore, for any \( E \subset [0, 1) , \alpha \in (0, 1) , \delta \in (0, \alpha) \) we have

\[ H^\alpha(E) \leq H^\alpha(E, \mathcal{A}) \leq K(\alpha, \delta)H^{\alpha-\delta}(E), \]

which gives \( \dim_H(E, \mathcal{A}) \leq \dim_H(E) + \delta \) for any \( \delta \in (0, \alpha) \). Thus we show that

\[ \dim_H(E) = \dim_H(E, \mathcal{A}), \]

which completes the proof.

**Proof of the theorem 2.4.** Sufficiency. Suppose inequality (2.1) holds. Due to theorem 2.2, it is enough to consider the coverings coming from \( \mathcal{A} \) for the calculation of the Hausdorff dimension of a set \( E \subset [0, 1) \). Let \( \epsilon > 0 \), and \( \{E_j\} \subset \mathcal{A} \) be an \( \epsilon \)-covering of \( E \). Without loss of generality, we assume that \( E_j \) is a union of finitely many consecutive cylinders of rank \( n_j \), i.e.
\[ E_j \equiv \bigcup_{i=m}^{m+k} \Delta_{\alpha_1 \cdots \alpha_{\eta-1}} \]

for some \( k, m, \alpha_1, \ldots, \alpha_{\eta-1} \in \mathbb{N} \).

If \( m + k \leq N \), then

\[ |E_j|^{\alpha} \geq \frac{1}{N} \sum_{i=m}^{m+k} |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha} . \]

If \( m + k > N \), then

\[ 2|E_j|^{\alpha-\delta} = 2 \left( \bigcup_{i=m}^{m+k} \Delta_{\alpha_1 \cdots \alpha_{\eta-1}} \right)^{\alpha-\delta} \]
\[ \geq \left( \bigcup_{i=m}^{N} \Delta_{\alpha_1 \cdots \alpha_{\eta-1}} \right)^{\alpha-\delta} + \left( \bigcup_{i=N+1}^{m+k} \Delta_{\alpha_1 \cdots \alpha_{\eta-1}} \right)^{\alpha-\delta} \]
\[ \geq \frac{1}{N} \sum_{i=m}^{N} |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha-\delta} + |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha-\delta} \left( \sum_{i=N+1}^{m+k} q_i \right)^{\alpha-\delta} \]
\[ \geq \frac{1}{N} \sum_{i=m}^{N} |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha} + |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha} \sum_{i=N+1}^{m+k} q_i^{\alpha} \]
\[ \geq \frac{1}{N} \sum_{i=m}^{m+k} |\Delta_{\alpha_1 \cdots \alpha_{\eta-1}}|^{\alpha} . \]

Therefore, for any \( \alpha \in (0, 1) \), \( \delta \in (0, \alpha) \) and \( E \subset [0, 1] \), there exists a constant \( C = 2N(\alpha, \delta) \) such that the following inequality holds:

\[ H^\alpha(E, A) \leq H^\alpha(E, \Phi) \leq CH^{\alpha-\delta}(E, A) . \]

Hence, \( \dim_H(E, \Phi) \leq \dim_H(E, A) + \delta, \forall \delta \in (0, \alpha) \), which proves that

\[ \dim_H(E) = \dim_H(E, A) = \dim_H(E, \Phi), \quad \forall E \subset [0, 1] . \]

**Necessity.** Suppose inequality \( (2.1) \) does not hold, i.e. we can find \( \alpha \in (0, 1) \) and \( \delta \in (0, \alpha) \) such that for any \( N \in \mathbb{N} \) there exists \( n > N, M \in \mathbb{N} \), we have

\[ \left( \sum_{i=n}^{n+M} q_i \right)^{\alpha-\delta} < \sum_{i=n}^{n+M} q_i^\alpha . \tag{3.3} \]

Now we will construct a Cantor set \( K \) such that

\[ \dim_H(K) < \dim_H(K, \Phi) . \]

In fact, we show that \( \dim_H(K) \leq \frac{\delta}{2} \) and \( \dim_H(K, \Phi) \geq \delta \).

Let \( \{ \epsilon_k \}_{k \geq 1} \) be a decreasing sequence of positive reals with limit zero.
Step 1. We take $k_1 > N$ as large enough so that
\[
\left( \sum_{i=k_1}^{\infty} q_i \right)^{\frac{1}{2}} \leq 1 \quad \text{and} \quad \sum_{i=k_1}^{\infty} q_i \leq \epsilon_1.
\]
By inequality (3.3), there exists an integer $M_1 \in \mathbb{N}$ with
\[
\left( \sum_{i=k_1}^{k_1+M_1} q_i \right)^{\alpha - \delta} \leq \sum_{i=k_1}^{k_1+M_1} q_i^\alpha.
\]
Then we choose the following finitely many unions of cylinders of rank 1,
\[
F_1 := \left\{ k_1 + \sum_{\alpha=1}^{M_1} \Delta_\alpha \right\}.
\]
Step $n \geq 1$. Suppose $k_1, k_2, \cdots, k_n$ and $M_1, M_2, \cdots, M_n$ have been defined. We choose $k_{n+1}$ sufficiently large so that
\[
\sum_{i=k_n}^{k_n+M_n} \sum_{\alpha=1}^{\infty} \Delta_{\alpha_1=\alpha_{n+1}=k_{n+1}} \leq 1 \quad \text{and} \quad \sum_{i=k_{n+1}}^{\infty} q_i \leq \epsilon_{n+1}.
\]
Then there exists an $M_{n+1} \in \mathbb{N}$ with
\[
\left( \sum_{i=k_{n+1}}^{k_{n+1}+M_{n+1}} q_i \right)^{\alpha - \delta} \leq \sum_{i=k_{n+1}}^{k_{n+1}+M_{n+1}} q_i^\alpha.
\]
Repeating the operation, we get sequences $\{k_n\}, \{M_n\}$ of integers.
Define
\[
F_n := \left\{ \bigcup_{i=k_n}^{k_n+M_n} \Delta_{\alpha_1=\cdots=\alpha_{n+1}=k_{n+1}} : k_j \leq \alpha_j \leq k_j + M_j, 1 \leq j \leq n - 1 \right\},
\]
and let
\[
K := \bigcap_{n=1}^{\infty} \bigcup_{\Delta \in F_n} \Delta.
\]
We now show that
\[
\dim_H(K) \leq \frac{\delta}{2} \quad \text{and} \quad \dim_H(K, \Phi) \geq \delta.
\]
Note $F_n$ is an $\epsilon_n$–cover of $K$, and the $\frac{\delta}{2}$-volume of the coverings $F_n$ does not exceed 1, i.e.
\[
\sum_{\Delta \in F_n} |\Delta|^{\frac{\delta}{2}} \leq \sum_{i=1}^{n-1} \sum_{\alpha_i=k_i}^{k_i+M_i} \bigcup_{\alpha_{n+1}=k_{n+1}} \Delta_{\alpha_1=\cdots=\alpha_{n+1}=\alpha_n} \leq 1.
\]
Therefore, $H^{\frac{\delta}{2}}(K) \leq 1$, thus $\dim_H(K) \leq \frac{\delta}{2}$.
In order to prove $\dim_H(K, \Phi) \geq \delta$, we will define a probability measure on $K$.
Firstly, we give some notations for the sake of briefness. Let
\[ D_n = \{ \Delta_{\alpha_1 \cdots \alpha_n} : k_i \leq \alpha_i \leq k_i + M_i, 1 \leq i \leq n \} \]
and
\[ D^* = \cup_{n \geq 1} D_n. \]
Denote \( \gamma_n \) as the normalised constants determined by
\[ \frac{1}{\gamma_n} \sum_{i=k_n}^{k_n+M_n} q_i^{\alpha_i} = 1. \]
Then by inequality (3.3), we have
\[ \gamma_n = \frac{\sum_{i=k_n}^{k_n+M_n} q_i^{\alpha_i}}{\sum_{i=1}^{k_n} q_i^{\alpha_i}} > \left( \frac{\sum_{i=k_n}^{k_n+M_n} q_i^{\alpha_i}}{\sum_{i=1}^{k_n} q_i^{\alpha_i}} \right)^{\alpha - \delta}. \]
Now we define a set function
\[ \mu : D^* \to \mathbb{R}^+ \]
given as follows:
\[ \mu(\Delta_{\alpha_1 \cdots \alpha_n}) = \prod_{i=1}^{n} \frac{1}{\gamma_i^{\alpha_i}}, \text{ for } k_i \leq \alpha_i \leq k_i + M_i, 1 \leq i \leq n. \]
By Kolmogorov extension theorem, the set function \( \mu \) can be extended into a probability measure supported on \( K \), which is still denoted by \( \mu \).
Recalling that
\[ |\Delta_{\alpha_1 \cdots \alpha_n}| = \prod_{i=1}^{n} q_{\alpha_i}, \]
for any \( x \in K \) and \( t \in (0, \delta) \) we have
\[ \frac{\mu(\Delta_{\alpha_1(x) \cdots \alpha_n(x)})}{|\Delta_{\alpha_1(x) \cdots \alpha_n(x)}|^t} = \prod_{i=1}^{n} \frac{1}{\gamma_i^{\alpha_i-t}} \leq \prod_{i=1}^{n} \frac{1}{(\sum_{j=k_i}^{k_i+M_i} q_j)^{\alpha_i-t}} q_i^{\delta-t} \leq \prod_{i=1}^{n} q_i^{\delta-t} \leq \epsilon \delta^{-t}. \]
So
\[ \lim_{n \to \infty} \frac{\mu(\Delta_{\alpha_1(x) \cdots \alpha_n(x)})}{|\Delta_{\alpha_1(x) \cdots \alpha_n(x)}|^t} = 0 \]
for any \( t \in (0, \delta) \) and \( x \in K \). By using the mass distribution principle [6], this gives \( \dim_H(K, \Phi) \geq \delta \), which proves the theorem.

Acknowledgments
The authors would like to thank the editor and the anonymous referees for their helpful comments and suggestions. This work was supported by National Natural Science Foundation of China (Grant Nos.11501168,11626030).
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