A UNIFIED ANALYSIS FOR SCHEDULING PROBLEMS WITH VARIABLE PROCESSING TIMES

Ji-Bo Wang*
School of Science
Shenyang Aerospace University
Shenyang 110136, China

Bo Zhang
School of Science
Shenyang Aerospace University
Shenyang 110136, China

Hongyu He
Post-doctoral Mobile Station
Dalian Commodity Exchange
Dalian, China

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Abstract. This paper considers single-machine scheduling problems with variable processing times, in which the actual processing time of a job is a function of its additional resources, starting time, and position in a sequence. Four problems arising from two criteria (a scheduling cost and a total resource consumption cost) are investigated. Under the linear and convex resource consumption functions, we provide unified approaches and consequently prove that these four problems are solvable in polynomial time.

1. Introduction. In scheduling problems and models, we often encounter settings in which the processing time of a job may be subject to change due to the phenomenon of learning effects, resource allocation and/or deterioration effects, i.e., variable processing times. Different scheduling problems involving jobs with variable processing times can be found in Shabtay and Steiner [21], Gawiejnowicz [8, 9], Biskup [5], Leyvand et al. [11], Wang and Wang [28], Yin et al. [33], Oron [17], Agnetis et al [2] (chap. 6 concerns agent scheduling with variable processing times), Wang et al. [24], Yin et al. [34], Strusevich and Rustogi [22] (chaps. 2-3 include a detailed discussion of general methods of solving scheduling problems, including those concerning objectives mentioned by the authors in Sect. 2) Azzouz et al. [3], Blazewicz et al [6] (chap. 13 concerns resource-dependent scheduling) and Wang and Liang [25].

"The phenomena of learning effects, resource allocation and deterioration effects occurring simultaneously can be found in steel production, more precisely, in the..."
process of preheating ingots by gas to prepare them for hot rolling on the blooming mill. Before the ingots can be hot rolled, they have to achieve the required temperature. However, the preheating time of the ingots depends on their starting temperature, i.e., the longer ingots wait for the start of the preheating process, the lower goes their temperature and therefore the longer lasts the preheating process. The preheating time can be shortened by the increase of the gas flow intensity, i.e., the more gas is consumed, the shorter lasts the preheating process. Thus, the ingot preheating time depends on the starting moment of the preheating process and the amount of gas consumed during it” (Bachman and Janiak [4]). “On the other hand, the learning effects reflect that the workers become more skilled to operate the machines through experience accumulation (Wright [31], Gawiejnowicz [7], Wang et al. [29]). ” Wang et al. [29] considered the single-machine scheduling in which the actual job processing time dependents on position (i.e., learning effects), starting time (i.e., deterioration effects) and allotted resource. Li et al. [12] considered a single-machine resource allocation scheduling problem with learning and deterioration effects. For the cost function including makespan, total completion time (total waiting time), absolute difference of total completion time (total absolute differences in waiting times), and total resource cost, they proposed a polynomial time algorithm to solve this problem. Sun et al. [23] considered resource allocation scheduling problems with deteriorating performance of jobs and learning under common due date assignment, slack due date assignment, and different due date assignment. They demonstrated that a cost function (including earliness, tardiness, due date allocation and total resource consumption) minimization could be solved in polynomial time. Liu et al. [15] considered the linear resource consumption and the convex resource consumption models with general position-dependent processing time. They proved that some due window assignment problems can be solved in polynomial time, respectively. For the summary of these papers, please see Section 2, literature survey.

In this paper, we consider the single-machine scheduling problems simultaneously and investigate of variable processing times. We summarize the main contributions as follows: first, this paper presents an attempt to study the scheduling and total resource consumption measures. Second, under the linear and convex resource consumption functions, we provide unified approaches and consequently solution algorithms are proposed. Finally, we prove that these problems can be solved in polynomial time.

The structure of this paper is as follows. The second section gives the problem formulation and a brief review of models studied in relevant literature. In Section 3, we discuss these four problems and give the algorithm respectively. Finally, Section 4 gives the conclusions.

2. Problems description. The notations used throughout this paper are defined in Table 1.

A set of independent jobs $J = \{J_1, J_2, ..., J_n\}$ is available for processing on a single machine, all jobs are available at time zero and with no preemption allowed, and the machine processes one job at a time. Let $\bar{p}_i$, $d_i$ and $C_i$ be the normal processing time (the processing without any learning effect, deterioration effect and resource allocation), due date and completion time of job $J_i$, respectively. In this paper, we consider two resource consumption functions.
The linear resource consumption functions:

Type 1a : \( p_i = (\bar{p}_i + bt) \max\{g(r), A\} - \theta_i u_i, \) (1)

Type 1b : \( p_i = \bar{p}_i \max\{g(r), A\} + bt - \theta_i u_i, \) (2)

where \( p_i \) is the actual processing time of job \( J_i \), \( b \) is a common deterioration rate for all jobs (i.e., deterioration effect, Gawiejnowicz, [8]), \( t \) denotes the starting process time of job \( J_i \) \((i = 1, 2, ..., n)\), \( g(r) \) is a general function of the \( r \)th position (i.e., learning effect, \( 0 < g(n) \leq g(n-1) \leq ... \leq g(1) \leq 1 \)), \( A \) is a truncation parameter with \( 0 < A < 1 \) (Wang et al. [30]), \( u_i \) \((0 \leq u_i \leq \bar{u}_i)\) is the amount of resources allocated to job \( J_i \), \( \bar{u}_i \) is the upper bound of \( u_i \) \((i = 1, 2, ..., n)\), \( \theta_i > 0 \) is compression rate corresponding to job \( J_i \).

The convex resource consumption functions:

Type 2a : \( p_i = \left( \frac{\bar{p}_i}{u_i} \right)^k + bt \max\{g(r), A\} \) (3)

Type 2b : \( p_i = \left( \bar{p}_i \max\{g(r), A\} \right)^k + bt \) (4)

where \( k \) is a positive number.

Let \( v_i \) denote the cost allocated to job \( J_i \), \( u_i \) be the amount of resources allocated to \( i \)th job, \( \bar{u}_i \) be the upper bounds of \( u_i \) \((i = 1, 2, ..., n)\). The scheduling measure in this paper can be expressed as \( \sum_{i=1}^{n} p[i] \varphi_i \), and the resource consumption measure
in this paper is \( \sum_{i=1}^{n} \bar{v}_i \bar{u}_i \), where \([i]\) denotes the job that is placed in \(i\)th position, \(p_{[i]}\) is the actual processing time of job \(J_i\), \(\psi_i\) is the positional weight of \(i\)th position.

The initial problems are to find the optimal sequence \(\sigma\) of all jobs, and optimal resource allocation such that the following objective function is minimized:

\[
Z = \sum_{i=1}^{n} p_{[i]} \varphi_i + \eta \sum_{i=1}^{n} v_i u_i
\]

(5)

where \(\eta\) denotes the weight of resource consumption measure. Using extensions of the notation (for details see, e.g., books by Agnetis et al [2], Strusevich and Rustogi [22] or review by Gawiejnowicz, [9]), the above problems can be described as follows:

Problem P1:

\[
1|p_i \in \{\text{type 1a, type 1b}\}| \sum_{i=1}^{n} p_{[i]} \varphi_i + \eta \sum_{i=1}^{n} v_i u_i
\]

Problem P2:

\[
1|p_i \in \{\text{type 2a, type 2b}\}| \sum_{i=1}^{n} p_{[i]} \varphi_i + \eta \sum_{i=1}^{n} v_i u_i
\]

For the convex resource consumption function, the next two problems are:

Problem P3:

\[
1|p_i \in \{\text{type 2a, type 2b}\}| \sum_{i=1}^{n} p_{[i]} \varphi_i \leq \bar{Z} \sum_{i=1}^{n} v_i u_i
\]

where \(\bar{Z}\) is a fixed value.

Problem P4:

\[
1|p_i \in \{\text{type 2a, type 2b}\}| \sum_{i=1}^{n} v_i u_i \leq \bar{V} \sum_{i=1}^{n} p_{[i]} \varphi_i
\]

where \(\bar{V}\) is a fixed value.

Similar to Leyvand et al. [11], and Rustogi and Strusevich [19], the results of the scheduling measure \(\sum_{i=1}^{n} p_{[i]} \varphi_i\) can be applied to solve some scheduling measure, such as, for the makespan \(C_{\text{max}}\), \(\varphi_i = 1\); for the total completion time \(\sum_{i=1}^{n} C_i\), \(\varphi_i = n - i + 1\); for the total waiting time \(\sum_{i=1}^{n} W_i\) (\(W_i\) is the waiting time of job \(J_i\), \(W_i = C_i - p_i\)), \(\varphi_i = n - i\); for the total absolute differences in completion times \(\sum_{i=1}^{n} \sum_{j=1}^{n} |C_i - C_j|\), \(\varphi_i = (i - 1)(n - i + 1)\); for the total absolute differences in waiting times \(\sum_{i=1}^{n} \sum_{j=1}^{n} |W_i - W_j|\), \(\varphi_i = i(n - i)\).

On the other hand, the results can be applied to solve some scheduling measure with due date assignment, such as, for the common (CON) due date assignment (see Panwalkar et al. [18], and Yin et al. [35]), \(\sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma d) = \sum_{i=1}^{n} p_{[i]} \varphi_i\), where \(\alpha, \beta, \gamma\) are given constants, \(d\) is the common due date for all the jobs, \(E_i = \max\{0, d - C_i\}\) (\(T_i = \max\{0, C_i - d\}\)) is the earliness (tardiness) of job \(J_i\), \(\varphi_i = \min\{n \gamma + (i - 1) \alpha, (n + 1 - i) \beta\}\); for the slack (SLK) due date assignment (see Adamopoulos and Pappis [1], and Yin et al. [34]), \(\sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma q) = \sum_{i=1}^{n} p_{[i]} \varphi_i\), where the due date of job \(J_i\) is \(d_i = p_i + q\), \(q\) is the common float allowance for all the jobs, \(E_i = \max\{0, d_i - C_i\}\) (\(T_i = \max\{0, C_i - d_i\}\)) is the earliness (tardiness) of job \(J_i\), \(\varphi_i = \min\{n \gamma + i \alpha, (n - i) \beta\}\); for the different (DIF) due date assignment (see Seidmann et al. [20]), \(\sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i) = \sum_{i=1}^{n} p_{[i]} \varphi_i\), where, the due date \(d_i\) is a decision variable, \(\varphi_i = \min\{\gamma(n + 1 - i), (n + 1 - i) \beta\}\).
Finally, the results can be applied to solve some scheduling measure with due window assignment, such as, for the common (CON) due window assignment (see Liman et al. [13, 14]), \( \sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i + \delta D) = \sum_{i=1}^{n} p_{ij} \varphi_i, \) where \( \alpha, \beta, \gamma \) and \( \delta \) are given constants, \( d \) is the starting time of common due window, \( D \) is the size of a common due window, \( E_i = \max\{0, d - C_i\} \) \( (T_i = \max\{0, C_i - d - D\}) \) is the earliness (tardiness) of job \( J_i, \varphi_i = \min\{n\gamma + (i - 1)\alpha, n\delta, (n + 1 - i)\beta\}; \) for the slack (SLK) due window assignment (see Mosheiov and Oron [16], Wang et al. [26]), \( \sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma q_i' + \delta D) = \sum_{i=1}^{n} p_{ij} \varphi_i, \) where the due window of job \( J_i \) is \([d'_i, d''_i]\), \( d'_i = p_i + q'_i, \) \( d''_i = p_i + q''_i, \) \( D = q'' - q' \), \( E_i = \max\{0, d'_i - C_i\} \) \( (T_i = \max\{0, C_i - d''_i\}) \) is the earliness (tardiness) of job \( J_i, \varphi_i = \min\{n\gamma + i\alpha, n\delta, (n - i)\beta\}; \) for the different (DIF) due window assignment (see Wang et al. [26]), \( \sum_{i=1}^{n} (\alpha E_i + \beta T_i + \gamma d''_i + \delta D'_i) = \sum_{i=1}^{n} p_{ij} \varphi_i, \) where, the due window of job \( J_i \) is \([d'_i, d''_i]\), the starting (finishing) time of due window \( d'_i (d''_i) \) is a decision variable, the size of due window \( D_i = d''_i - d'_i, \varphi_i = \min\{\gamma(n + 1 - i), (n + 1 - i)\beta\}. \)

The literature review related to the scheduling problems with variable processing times (i.e., simultaneously consider the phenomenon of learning effects, resource allocation and deterioration effects) is given as follows (see Table 2).

3. Main results.

3.1. \( \textbf{P1:} \ 1|p_i \in \{\text{type 1a}, \text{type 1b}\}|\sum_{i=1}^{n} p_{ij} \varphi_i + \eta \sum_{i=1}^{n} v_i u_i, \) The formula of completion time and actual processing time can be proved by mathematical induction, assuming that processing times are in the form of (1), i.e.,

\[ C_{[i]} = \sum_{l=1}^{i} \left( \tilde{p}_{[l]} \max\{g(l), A\} - \theta_{[l]} u_{[l]} \right) \prod_{j=l+1}^{i} (1 + b \max\{g(j), A\}) \]

and

\[ p_{[i]} = \left( \tilde{p}_{[i]} \max\{g(i), A\} - \theta_{[i]} u_{[i]} \right) \prod_{j=1}^{i-1} \left( \tilde{p}_{[j]} \max\{g(j), A\} - \theta_{[j]} u_{[j]} \right) \prod_{j=i+1}^{n-1} (1 + b \max\{g(j), A\}). \]

Then, objective function \( Z \) can be transformed into

\[ Z_{F_1} = \sum_{i=1}^{n} p_{[i]} \psi_i + \eta \sum_{i=1}^{n} v_i u_i, \]

\[ = \left( \psi_1 + b \max\{g(2), A\} \varphi_2 + \ldots + b \max\{g(n), A\} \prod_{j=2}^{n-1} (1 + b \max\{g(j), A\}) \psi_n \right) \]

\[ \times \left( \tilde{p}_{[1]} \max\{g(1), A\} - \theta_{[1]} u_{[1]} \right) \]

\[ + \left( \varphi_2 + b \max\{g(3), A\} \varphi_3 + \ldots + b \max\{g(n), A\} \prod_{j=3}^{n-1} (1 + b \max\{g(j), A\}) \psi_n \right) \]

\[ \times \left( \tilde{p}_{[2]} \max\{g(2), A\} - \theta_{[2]} u_{[2]} \right) \]
| Ref.                          | Problem                                                                 | Relation with this article | Due date methods | Complexity       |
|----------------------------|-------------------------------------------------------------------------|-----------------------------|-----------------|-----------------|
| Wang et al. [26]            | $1|p_i = \bar{p}_i r^n + b_t - \theta u_i s_1 C_{\text{max}} + 2 \sum_{c=1}^{n} W_i = \sum_{j=1}^{n} |C_i - C_j| + \sum_{j=1}^{n} v_i u_i$ | $p_i$ is type 1b         | $O(n^3)$        |
|                            | $1|p_i = \bar{p}_i r^n + b_t - \theta u_i s_1 C_{\text{max}} + 2 \sum_{c=1}^{n} W_i = \sum_{j=1}^{n} |C_i - C_j| + \sum_{j=1}^{n} v_i u_i$ | $g(r) = r^n, A = 0$       | $O(n^3)$        |
| Li et al. [12]              | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} s_1 C_{\text{max}} + 2 \sum_{c=1}^{n} C_i = \sum_{j=1}^{n} |C_i - C_j| + \sum_{j=1}^{n} v_i u_i$ | $p_i$ is type 2a         | $O(n \log n)$   |
|                            | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} W_i + 2 \sum_{c=1}^{n} C_i = \sum_{j=1}^{n} |C_i - C_j| + \sum_{j=1}^{n} v_i u_i$ | $g(r) = r^n, A = 0$       | $O(n \log n)$   |
| Sun et al. [23]             | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} s_1 u_i \leq U |\sum_{c=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i)\right)$ | $p_i$ is type 2a         | CON/SLK/DIF     | $O(n \log n)$   |
|                            | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} s_1 u_i \leq U |\sum_{c=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i)\right)$ | $g(r) = r^n, A = 0$       | CON/SLK/DIF     | $O(n \log n)$   |
| Wang et al. [27]            | $1|p_i = \bar{p}_i \max\{r^n, A\} + b_t - \theta u_i s_1 P + 2 \sum_{c=1}^{n} v_i u_i, P \in \{C_{\text{max}}, \sum_{c=1}^{n} C_i, \sum_{j=1}^{n} \sum_{c=1}^{n} |C_i - C_j|\}$ | $p_i$ is type 1b         | $O(n \log n)$   |
|                            | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} s_1 u_i \leq U |\sum_{c=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i)\right)$ | $g(r) = r^n, A = 0$       | CON/SLK/DIF     | $O(n \log n)$   |
|                            | $1|p_i = \left( \left( \frac{b_t}{n} \right)^k + b_t \right)^{r^n} s_1 u_i \leq U |\sum_{c=1}^{n} (\alpha E_i + \beta T_i + \gamma d_i)\right)$ | $g(r) = r^n, A = 0$       | CON/SLK/DIF     | $O(n \log n)$   |
Continued

Liu et al. [15]

\[ p_i = (p_i - b_i) g(r) - \sum_{j \leq i} [i - u_i + \sum (\alpha E_i + \beta T_i + \gamma d_i^{|i|} + D_i) + \sigma C_{max} + \gamma \sum u_i] \]

1. \( \left( \frac{r}{u_i} \right)^{b_i} - b_i \) \( g(r) \sum (\alpha E_i + \beta T_i + \gamma d_i^{|i|} + D_i) + \sigma C_{max} + \gamma \sum u_i \)

1. \( \left( \frac{r}{u_i} \right)^{b_i} - b_i \) \( g(r) \sum (\alpha E_i + \beta T_i + \gamma d_i^{|i|} + D_i) + \sigma C_{max} \leq V \sum u_i \)

1. \( \left( \frac{r}{u_i} \right)^{b_i} - b_i \) \( g(r), \sum u_i \leq U \| \sum (\alpha E_i + \beta T_i + \gamma d_i^{|i|} + D_i) + \sigma C_{max} \)

This paper

| P1: \( 1 \cdot p_i \in \{ \text{type 1a, type 1b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 1a/1b | O(n^2^) |
|---|---|---|
| P1: \( 1 \cdot p_i \in \{ \text{type 1a, type 1b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 1a/1b | O(n^2^) |
| P2: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |
| P2: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |
| P3: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |
| P3: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |
| P4: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |
| P4: \( 1 \cdot p_i \in \{ \text{type 2a, type 2b} \} \| P_j | p_i + \gamma \sum_{j \leq i} u_i \) | p_i is type 2a/2b | O(n log n) |

\( W_i, T_i, E_i \) is the waiting time, earliness, tardiness of job \( j_i \), respectively; \( C_{max} \) is the makespan of all jobs; \( d_1, d_2, \ldots, d_n, \alpha, \beta, \gamma, \delta, \sigma, \beta_1, \beta_2, U \) and \( V \) are given constants.
\[
+ \ldots + (\varphi_{n-1} + b \max\{g(n), A\}) \psi_n \left(\bar{p}_{n-1} \max\{g(n-1), A\} - \theta_{i[n]} u_{i[n-1]}\right)
\] 
\[
+ \psi_n \left(\bar{p}_{i[n]} \max\{g(n), A\} - \theta_{i[n]} u_{i[n]}\right) + \eta \sum_{i=1}^{n} v_{i[n]} u_{i[n]}
\] 
\[
= \sum_{i=1}^{n} \Omega_i \bar{p}_{i[n]} \max\{g(i), A\} + \sum_{i=1}^{n} (\eta v_{i[n]} - \Omega_i \theta_{i[n]} u_{i[n]})
\] 

(6)

where

\[
\Omega_1 = \varphi_1 + b \max\{g(2), A\} \varphi_2 + \ldots + b \max\{g(n), A\} \prod_{j=2}^{n-1} (1 + b \max\{g(j), A\}) \varphi_n
\]

\[
\Omega_2 = \varphi_2 + b \max\{g(3), A\} \varphi_3 + \ldots + b \max\{g(n), A\} \prod_{j=3}^{n-1} (1 + b \max\{g(j), A\}) \varphi_n
\]

\[
\ldots
\]

\[
\Omega_{n-1} = \varphi_{n-1} + b \max\{g(n), A\} \varphi_n
\]

\[
\Omega_n = \varphi_n
\]

From (2), the actual processing time for type 1b can be obtained as follows:

\[
p_{i[n]} = (\bar{p}_{i[n]} \max\{g(i), A\} - \theta_{i[n]} u_{i[n]} + b \sum_{j=1}^{i-1} (\bar{p}_{j[n]} \max\{g(j), A\} - \theta_{j[n]} u_{j[n]}) (1 + b)^{i-j-1}
\]

Then

\[
Z^{P_1} = \sum_{i=1}^{n} p_{i[n]} \varphi_i + \eta \sum_{i=1}^{n} v_i u_i = \sum_{i=1}^{n} \Omega'_{i[n]} \bar{p}_{i[n]} \max\{g(i), A\} + \sum_{i=1}^{n} (\eta v_{i[n]} - \Omega'_{i[n]} \theta_{i[n]} u_{i[n]})
\] 

(8)

where

\[
\Omega'_{1} = \varphi_1 + b \varphi_2 + b(1 + b) \varphi_3 + \ldots + b(1 + b)^{n-2} \varphi_n
\]

\[
\Omega'_{2} = \varphi_2 + b \varphi_3 + \ldots + b(1 + b)^{n-3} \varphi_n
\]

\[
\ldots
\]

\[
\Omega'_{n-1} = \varphi_{n-1} + b \varphi_n
\]

\[
\Omega_n = \varphi_n
\]

From (6), for any given schedule, the optimal resource should be \(u^*_i = 0\) if \(\eta v_{i[n]} - \Omega_i \theta_{i[n]} > 0\); \(u^*_i = u_i\) of \(\eta v_{i[n]} - \Omega_i \theta_{i[n]} < 0\); \(u^*_i \in [0, \bar{u}_i]\) if \(\eta v_{i[n]} - \Omega_i \theta_{i[n]} = 0\), i.e.,

\[
u_{i[n]} = \begin{cases} 
0, & \text{if } \eta v_{i[n]} - \Omega_i \theta_{i[n]} > 0 \\
[0, \bar{u}_i], & \text{if } \eta v_{i[n]} - \Omega_i \theta_{i[n]} = 0 \\
\bar{u}_i, & \text{if } \eta v_{i[n]} - \Omega_i \theta_{i[n]} < 0 
\end{cases}
\] 

(10)

Let

\[
\lambda_{ir} = \begin{cases} 
\Omega_r \bar{p}_{i[n]} \max\{g(r), A\}, & \text{if } \eta v_{i[n]} - \Omega_r \theta_{i[n]} \geq 0 \\
\Omega_r \bar{p}_{i[n]} \max\{g(r), A\} + (\eta v_{i[n]} - \Omega_r \theta_{i[n]}) \bar{u}_i, & \text{if } \eta v_{i[n]} - \Omega_r \theta_{i[n]} < 0 
\end{cases}
\] 

(11)

\[
x_{ir} = \begin{cases} 
1, & \text{if job } J_i \text{ is scheduled in position } r \\
0, & \text{otherwise}
\end{cases}
\]
P1 of type 1a can be solved by the following 0-1 assignment problem:

\[ \min \sum_{i=1}^{n} \sum_{r=1}^{n} \lambda_{ir} x_{ir} \]  \hspace{1cm} (12) 

s.t. \[ \sum_{i=1}^{n} x_{ir} = 1, \quad r = 1, 2, ..., n \]  \hspace{1cm} (13) 

\[ \sum_{r=1}^{n} x_{ir} = 1, \quad i = 1, 2, ..., n \]  \hspace{1cm} (14) 

\[ x_{ir} = 0 \text{ or } 1 \]  \hspace{1cm} (15) 

From (11), for type 1b, P1 can be solved similarly, except that \( \Omega_i \) is replaced by \( \Omega'_i \).

Based on the above analysis, we can solve P1 via the following algorithm:

**Algorithm 1.**

Step 1.1: Calculate the value of \( \lambda_{ir} \) by (11), where for type 1a, \( \Omega_i \) is given by (7); for type 1b, \( \Omega'_i \) is given by (9), \( i, r = 1, 2, ..., n \);

Step 1.2: Solve the 0-1 assignment problem (12-15);

Step 1.3: By using (10), calculate the resource allocation.

**Theorem 1.** Algorithm 1 solves the problem \( 1 | p_i \in \{ \text{type 1a}, \text{type 1b} \} | \sum_{i=1}^{n} p_i \varphi_i + \eta \sum_{i=1}^{n} v_i u_i \) in \( O(n^3) \) time. 

**Proof.** Step 1.1 needs \( O(n^2) \) time, Step 1.3 needs linear time, and step 1.2 requires \( O(n^3) \) time. Thus the computational complexity of Algorithm 1 is \( O(n^3) \).  \( \square \)

3.2. \( P2: 1 | p_i \in \{ \text{type 2a, type 2b} \} | \sum_{i=1}^{n} p_i \varphi_i + \eta \sum_{i=1}^{n} v_i u_i \). Similar to P1, from (3), the actual processing time of \( j \)th job under convex resource consumption for type 2a can be arranged as follows:

\[ p_{i}[j] = \left( \frac{\bar{p}_{i}[j]}{u_{i}[j]} \right)^k + b \sum_{l=1}^{i-1} \max\{ g(l), A \} \left( \frac{\bar{p}_{i}[j]}{u_{i}[j]} \right)^k \prod_{j=l+1}^{i-1} (1 + b \max\{ g(j), A \}) \times \max\{ g(j), A \} \]

Then, objective function \( Z \) can be transformed into

\[ Z_{P2} = \sum_{i=1}^{n} p_{i}[j] \varphi_i + \eta \sum_{i=1}^{n} v_i u_i = \sum_{i=1}^{n} \Omega_i \max\{ g(i), A \} \left( \frac{\bar{p}_{i}[j]}{u_{i}[j]} \right)^k + \eta \sum_{i=1}^{n} v_i u_i \]  \hspace{1cm} (16)

where \( \Omega_i \) is given by (7).

**Lemma 3.** The optimal resource allocation for \( P2 \) (type 2a) is given by

\[ u_{i}[j]^{*} = \frac{(k \Omega_i)^{1/k} \left( \max\{ g(i), A \} (\bar{p}_{i}[j])^k \right)^{1/k}}{\eta v_{i}[j]^{1/k}}, \quad i = 1, 2, ..., n \]  \hspace{1cm} (17)

**Proof.** Take the derivative of (16) with respect to \( u_{i}[j] \), we have

\[ -k \max\{ g(i), A \} \Omega_i (\bar{p}_{i}[j])^k (u_{i}[j])^{-(k+1)} + \eta v_{i}[j] = 0, \]

then the optimal resource allocation (17) can be obtained.  \( \square \)
Lemma 5. The optimal resource allocation for P3 (type 2a) can be obtained by

\[ Z_{P3} = \sum_{i=1}^{n} \Omega_i \max\{g(i), A\} \left(\frac{\bar{p}[i]}{u[i]}\right)^k + \eta \sum_{i=1}^{n} v_i u_i \]

\[ = (k^{-\frac{k}{k+1}} + k^{\frac{k}{k+1}}) \cdot \eta \sum_{i=1}^{n} (v_i[p][i])^{\frac{k}{k+1}} (\Omega_i \max\{g(i), A\})^{\frac{k}{k+1}} \]  

(18)

From (4), the actual processing time for type 2b can be obtained as follows:

\[ p[i] = \left(\frac{\bar{p}[i] \max\{g(i), A\}}{u[i]}\right)^k + b \sum_{j=1}^{i-1} \left(\frac{\bar{p}[j] \max\{g(j), A\}}{u[j]}\right)^k (1 + b)^{i-j-1} \]

Substituting \( u, \lambda \) into \( Z_{P2} \) (see (16)) yields

\[ Z_{P2} = \sum_{i=1}^{n} \Omega_i \left(\frac{\bar{p}[i] \max\{g(i), A\}}{u[i]}\right)^k + \eta \sum_{i=1}^{n} v_i u_i \]

(19)

where \( \Omega_i \) is given by (9).

Lemma 4. The optimal resource allocation for \( P2 \) (type 2b) is given by

\[ u[i] = \frac{(k\Omega_i)^{\frac{1}{k+1}} (\bar{p}[i] \max\{g(i), A\})^{\frac{1}{k+1}}}{(\eta v[i])^{\frac{1}{k+1}}}, i = 1, 2, ..., n \]  

(20)

By substituting \( u[i] \) into \( Z_{P2} \) (see (19)) yields

\[ Z_{P2} = \sum_{i=1}^{n} \Omega_i \left(\frac{\bar{p}[i] \max\{g(i), A\}}{u[i]}\right)^k + \eta \sum_{i=1}^{n} v_i u_i \]

\[ = \eta^{\frac{1}{k+1}} \cdot (k^{\frac{k}{k+1}} + k^{\frac{k}{k+1}}) \sum_{i=1}^{n} \Omega_i^{\frac{1}{k+1}} (v_i[p][i] \max\{g(i), A\})^{\frac{k}{k+1}} \]  

(21)

where \( \Omega_i \) is given by (9).

3.3. The double-standard problem: P3: 1\( |p| \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^{n} p[i] \times \varphi_i \leq Z | \sum_{i=1}^{n} v_i u_i \). Similar to Subsection 3.2, we have

Lemma 5. The optimal resource allocation for P3 (type 2a) can be obtained by

\[ u^*_i = \frac{(\Omega_i \max\{g(i), A\})^{\frac{1}{k+1}} (\sum_{i=1}^{n} (v_i[p][i])^{\frac{k}{k+1}} (\Omega_i \max\{g(i), A\})^{\frac{k}{k+1}})^{\frac{k}{k+1}}}{(v_i[p][i])^{\frac{k}{k+1}}} \cdot Z^{-\frac{1}{k}}, i = 1, 2, ..., n. \]  

(22)

where \( \Omega_i \) is given by (7).

Proof. For any given job sequence, by using the Lagrange multiplier method:

\[ L(u, \lambda) = \sum_{i=1}^{n} v_i u_i + \lambda \left( \sum_{i=1}^{n} \omega_i |L[i]| + \omega_0 d_{opt} - Z \right) \]

\[ = \sum_{i=1}^{n} v_i u_i + \lambda \left( \sum_{i=1}^{n} \Omega_i \max\{g(i), A\} \left(\frac{\bar{p}[i]}{u[i]}\right)^k - Z \right) \]  

(23)

where \( \lambda \) is the Lagrange multiplier, \( \lambda \geq 0 \).
Take the derivative of (23) with respect to \( u_{[i]} \) and \( \lambda \) respectively, then make them equal to 0:

\[
\frac{\partial L(u, \lambda)}{\partial u_{[i]}} = v_{[i]} - \frac{k\Omega_i \max\{g(i), A\}(\bar{p}_{[i]})^k}{(u_{[i]})^{k+1}} = 0 \tag{24}
\]

\[
\frac{\partial L(u, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} \Omega_i \max\{g(i), A\} \left( \frac{\bar{p}_{[i]}}{u_{[i]}} \right)^k - \bar{Z} = 0 \tag{25}
\]

From (24), we can obtain

\[
u_{[i]} = \frac{(k\Omega_i \max\{g(i), A\}(\bar{p}_{[i]})^k)}{(v_{[i]})^{\frac{1}{k+1}}} \tag{26}
\]

Substituting (26) into (25), we can obtain

\[
(k\lambda)^{\frac{1}{k}} = \frac{\sum_{i=1}^{n} (v_{[i]}\bar{p}_{[i]})^{\frac{1}{k+1}}(\Omega_i \max\{g(i), A\})^{\frac{1}{k+1}}}{\bar{Z}} \tag{27}
\]

Substituting (27) into (26), we can obtain the optimal resource allocation \( u^*_{[i]} \).

Substituting \( u^*_{[i]} \) into \( Z_{P3} \) yields

\[
Z_{P3} = \sum_{i=1}^{n} v_{i}u_{i} = \left( \frac{\sum_{i=1}^{n} (v_{[i]}\bar{p}_{[i]})^{\frac{1}{k+1}}(\Omega_i \max\{g(i), A\})^{\frac{1}{k+1}}}{(v_{[i]})^{\frac{1}{k+1}}} \right)^{1+\frac{1}{k}} \cdot \bar{Z}^{-\frac{k}{k+1}} \tag{28}
\]

**Lemma 6.** The optimal resource allocation for P3 (type 2b) can be obtained by

\[
u^*_{[i]} = \frac{(\Omega_i \max\{g(i), A\}(\bar{p}_{[i]})^k)}{(v_{[i]})^{\frac{1}{k+1}}} \cdot \bar{Z}^{-\frac{k}{k+1}} \tag{29}
\]

where \( \Omega^*_i \) is given by (9).

Substituting \( u^*_{[i]} \) into \( Z_{P3} \) yields

\[
Z_{P3} = \sum_{i=1}^{n} v_{i}u_{i} = \left( \frac{\sum_{i=1}^{n} (v_{[i]}\bar{p}_{[i]})^{\frac{1}{k+1}}(\Omega_i \max\{g(i), A\})^{\frac{1}{k+1}}}{(v_{[i]})^{\frac{1}{k+1}}} \right)^{1+\frac{1}{k}} \cdot \bar{Z}^{-\frac{k}{k+1}} \tag{30}
\]

### 3.4. The double-standard problem: P4: 1\(|p_i| \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^{n} \bar{v}_{i}u_{i} \leq \bar{V} \sum_{i=1}^{n} p_{[i]}\bar{\varphi}_{i} \). Similar to Subsection 3.2, we have

**Lemma 7.** The optimal resource allocation for P4 (type 2a) is given by

\[
u^*_{[i]} = \frac{(\Omega_i \max\{g(i), A\}(\bar{p}_{[i]})^k)}{(v_{[i]})^{\frac{1}{k+1}}} \cdot \bar{V} \tag{31}
\]

where \( \Omega_i \) is given by (7).

**Proof.** For any given job sequence, by using the Lagrange multiplier method:

\[
L(u, \lambda) = \sum_{i=1}^{n} \omega_{i}|L_{[i]}| + \omega_0d_{opt} + \lambda \left( \sum_{i=1}^{n} v_{i}u_{i} - \bar{V} \right)
\]

\[
= \sum_{i=1}^{n} \Omega_i \max\{g(i), A\} \left( \frac{\bar{p}_{[i]}}{u_{[i]}} \right)^k + \lambda \left( \sum_{i=1}^{n} v_{i}u_{i} - \bar{V} \right) \tag{32}
\]

where \( \lambda \) is the Lagrange multiplier, \( \lambda \geq 0 \).
Take the derivative of (32) with respect to $u_{[i]}$ and $\lambda$ respectively, then make them equal to 0:

$$\frac{\partial L(u, \lambda)}{\partial u_{[i]}} = \lambda v_{[i]} - k\Omega_{i} \max\{g(i), A\} \frac{(\bar{p}_{[i]})^{k}}{(u_{[i]})^{k+1} + 1} = 0 \quad (33)$$

$$\frac{\partial L(u, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} v_{[i]} u_{[i]} - \bar{V} = 0 \quad (34)$$

From (33), we can obtain

$$u_{[i]} = \left(\frac{k\Omega_{i} \max\{g(i), A\} \bar{p}_{[i]}^{k}}{\lambda v_{[i]}}\right)^{\frac{1}{k+1}} \quad (35)$$

Substituting (35) into (34), we can obtain

$$\left(\frac{k}{\lambda}\right)^{\frac{1}{k+1}} = \frac{\bar{V}}{\sum_{i=1}^{n} (v_{[i]} \bar{p}_{[i]})^{\frac{k}{k+1}} \max\{g(i), A\}^{\frac{1}{k+1}}} \quad (36)$$

Substituting (36) into (35), we can obtain the optimal resource allocation $u_{[i]}^{*}$. □

Substituting $u_{[i]}^{*}$ into $Z^{P_{4}}$, we can obtain

$$Z^{P_{4}} = \sum_{i=1}^{n} p_{[i]} \varphi_{i}$$

$$= \sum_{i=1}^{n} \Omega_{i} \max\{g(i), A\} \left(\frac{\bar{p}_{[i]}}{u_{[i]}}\right)^{k}$$

$$= \left(\sum_{i=1}^{n} (v_{[i]} \bar{p}_{[i]} \max\{g(i), A\})^{\frac{1}{k+1}}\right)^{k+1} \bar{V}^{k} \quad (37)$$

**Lemma 8.** The optimal resource allocation for $P_{4}$ (type 2b) is given by

$$u_{[i]}^{*} = \frac{\Omega_{i}^{\frac{1}{k+1}} \max\{g(i), A\}^{\frac{1}{k+1}}}{(v_{[i]}^{\frac{1}{k+1}} \sum_{i=1}^{n} \Omega_{i}^{\frac{1}{k+1}} (v_{[i]} \bar{p}_{[i]} \max\{g(i), A\})^{\frac{1}{k+1}})} \bar{V}, \; i = 1, 2, ..., n \quad (38)$$

where $\Omega_{i}'$ is given by (9).

Substituting $u_{[i]}^{*}$ into $Z^{P_{4}}$, we can obtain

$$Z^{P_{4}} = \sum_{i=1}^{n} p_{[i]} \varphi_{i}$$

$$= \sum_{i=1}^{n} \Omega_{i}' \left(\frac{\bar{p}_{[i]} \max\{g(i), A\}}{u_{[i]}}\right)^{k}$$

$$= \left(\sum_{i=1}^{n} \Omega_{i}'^{\frac{1}{k+1}} (v_{[i]} \bar{p}_{[i]} \max\{g(i), A\})^{\frac{1}{k+1}}\right)^{k+1} \bar{V}^{k} \quad (39)$$

From (18), (21), (28), (30), (37), and (39), in order to find the optimal sequence of the problems $P_{2}$, $P_{3}$, and $P_{4}$ (type 2a, type 2b), we have to optimally match the job-dependent costs $(v_{[i]} \bar{p}_{[i]})^{\frac{1}{k+1}}$ with the positional penalties $(\Omega_{i} \max\{g(i), A\})^{\frac{1}{k+1}}$ (type 2b is $\Omega_{i}'^{\frac{1}{k+1}} (\max\{g(i), A\})^{\frac{1}{k+1}}$). The optimal matching
can be solved by sorting and matching the \( \text{ith} \) smallest \((v_i,p_i)\) to the \( \text{ith} \) largest \((\Omega, \max\{g(i), A\})\) \(\text{see Hardy et al. }[10]\).

From above analysis, the problems \( \text{P2} \): \(1|p_i \in \{\text{type 2a}, \text{type 2b}\}|\sum_{i=1}^n p_i|\sum_{i=1}^n v_i u_i\), \( \text{P3} \): \(1|p_i \in \{\text{type 2a}, \text{type 2b}\}, \sum_{i=1}^n p_i|\max_\psi \leq \bar{Z} \sum_{i=1}^n v_i u_i\), and \( \text{P4} \): \(1|p_i \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^n v_i u_i \leq \bar{V} \sum_{i=1}^n \omega_i L_i|+\omega_0 d_{\text{opt}}\leq \bar{Z} \sum_{i=1}^n v_i u_i\), and \( \text{P4} \): \(1|p_i \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^n v_i u_i \leq \bar{V} \sum_{i=1}^n \omega_i L_i|+\omega_0 d_{\text{opt}}\) can be solved by the following algorithm.

**Algorithm 2**

Step 2.1. Calculate the values of \((v_i,p_i)\) \(\text{see Hardy et al. }[10]\).

Step 2.2. The optimal sequence can be obtained by sorting and matching the \( \text{ith} \) smallest \((v_i,p_i)\) to the \( \text{ith} \) largest \((\Omega, \max\{g(i), A\})\) \(\text{see Hardy et al. }[10]\).

Step 2.3. Calculate the optimal resource allocation, \( \text{P2}, \text{P3}, \text{P4} \) (type 2a) are obtained by \((17), (22)\) and \((31)\); \( \text{P2}, \text{P3}, \text{P4} \) (type 2b) are obtained by \((20), (29)\) and \((38)\), respectively.

Step 2.4. Calculate the objective function values for \( \text{P2}, \text{P3}, \text{P4} \) (type 2a) are obtained by \((18), (28)\) and \((37)\); \( \text{P2}, \text{P3}, \text{P4} \) (type 2b) are obtained by \((21), (30)\) and \((39)\), respectively.

**Theorem 2.** Algorithm 2 solves the problems \( \text{P2} \): \(1|p_i \in \{\text{type 2a, type 2b}\}|\sum_{i=1}^n \omega_i L_i|+\omega_0 d_{\text{opt}}|+\sum_{i=1}^n v_i u_i\), \( \text{P3} \): \(1|p_i \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^n \omega_i L_i|+\omega_0 d_{\text{opt}}\leq \bar{Z} \sum_{i=1}^n v_i u_i\), and \( \text{P4} \): \(1|p_i \in \{\text{type 2a, type 2b}\}, \sum_{i=1}^n v_i u_i \leq \bar{V} \sum_{i=1}^n \omega_i L_i|+\omega_0 d_{\text{opt}}\) in \( O(n \log n) \) time, respectively.

**Proof.** Steps 2.1, 2.3, 2.4 need linear time and step 2.2 requires \( O(n \log n) \) (Hardy et al. [10]). Thus, the computational complexity of Algorithm 2 is \( O(n \log n) \).

4. Conclusions. In this paper, we studied the single-machine problems with variable processing times, in which the resource consumption functions are linear and convex resource consumption functions. Four problems arising from two criteria were investigated: 1) the scheduling cost consisting of the weighted sum of earliness, tardiness, and common due date, and 2) the total resource consumption cost. Bu using the unified approaches, we showed that four problems (say P1-P4) are polynomially solvable. An interesting direction of future research is to consider the four problems (say P1-P4) in multi-machine environments (e.g., flow shop and open shop settings). It would also be interesting to consider parallel-batching and serial-batching scheduling problems, and to propose variable neighborhood searches to solve the problems (P1-P4).

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