The Weyl-Wigner formalism for the interpretation of parametric down-conversion experiments

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Abstract

We show that the Weyl-Wigner formalism in the Heisenberg picture may be used for the interpretation of experiments involving entangled photon pairs produced in nonlinear crystals via spontaneous parametric down-conversion. The calculations are usually no more involved than those with the Hilbert-space formalism. The WW formalism suggests a physical picture in terms of random variables and stochastic processes, but the picture is shown untenable in some instances.

1 Introduction

Entanglement is a characteristic trait of quantum mechanics[1]. It plays a crucial role in foundations of quantum physics, a role reinforced by the Bell inequalities[2] and the development of quantum information theory. The production of entangled photon pairs via spontaneous parametric down-conversion (SPDC) in nonlinear crystals has made quantum optics a suitable place for the study of entanglement. It has been investigated not only as a basic quantum feature but also for applications, including quantum cryptography[3] and quantum spectroscopy[4] among others. The correlations implied by entanglement have been of interest in connection with quantum interference effects and various classically counterintuitive experimental
results have been reported\cite{5}. In this paper we shall show that such results may be conveniently interpreted if quantum optics is formulated within the Weyl-Wigner (WW), rather than the usual Hilbert-space formalism of quantum mechanics. Most suitable is the WW formalism in the Heisenberg picture, where the evolution is studied in terms of quantum observables with states fixed.

In this paper we continue a theoretical interpretation of SPDC experiments within the WW formalism in the Heisenberg picture, that was initiated in the nineties of the past century \cite{6} - \cite{17}. In many of those early studies the approach was heuristic and one of the purposes of this paper is to provide a more formal foundation. The WW formalism suggests an intuitive picture for photon entanglement and the interpretation of SPDC experiments in terms of random variables and stochastic processes. However there are difficulties with the picture that will be discussed in section 4 of this paper.

\section{The Weyl-Wigner formalism in quantum optics}

\subsection{Definition}

The WW formalism was developed for non-relativistic quantum mechanics, where the basic observables involved are positions, $\hat{x}_j$, and momenta, $\hat{p}_j$, of the particles\cite{18},\cite{19}, \cite{20}, \cite{21}, \cite{22}, \cite{23}. It may be trivially extended to quantum optics provided we interpret $\hat{x}_j$ and $\hat{p}_j$ to be the sum and the difference of the creation, $\hat{a}_j$, and annihilation, $\hat{a}_j^\dagger$, operators of the $j$ normal mode of the radiation. That is

\begin{equation}
\hat{x}_j \equiv \frac{c}{\sqrt{2\omega_j}} (\hat{a}_j + \hat{a}_j^\dagger), \hat{p}_j \equiv \frac{i\hbar \omega_j}{\sqrt{2}c} (\hat{a}_j - \hat{a}_j^\dagger)
\end{equation}

\Rightarrow \begin{align*}
\hat{a}_j &= \frac{1}{\sqrt{2}} \left( \frac{\omega_j}{c} \hat{x}_j + \frac{ic}{\hbar \omega_j} \hat{p}_j \right), \\
\hat{a}_j^\dagger &= \frac{1}{\sqrt{2}} \left( \frac{\omega_j}{c} \hat{x}_j - \frac{ic}{\hbar \omega_j} \hat{p}_j \right).
\end{align*}

(1)

Here $\hbar$ is Planck constant, $c$ the velocity of light and $\omega_j$ the frequency of the normal mode. In the following I will use units $\hbar = c = 1$. For the sake of clarity I shall represent the operators in a Hilbert space with a ‘hat’, e. g. $\hat{a}_j, \hat{a}_j^\dagger$ and the amplitudes in the WW formalism without ‘hat’, e. g. $a_j, a_j^\ast$. 2
The connection with the Hilbert-space formalism is made via the Weyl transform as follows. For any trace class operator $\hat{M}$ of the former we define its Weyl transform to be a function of the field operators $\{\hat{a}_j, \hat{a}_j^\dagger\}$, that is

$$W_{\hat{M}} = \frac{1}{(2\pi)^n} \prod_{j=1}^n \int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{\infty} d\mu_j \exp \left[-2i\lambda_j \text{Re}a_j - 2i\mu_j \text{Im}a_j\right] \times \text{Tr} \left\{ \hat{M} \exp \left[i\lambda_j \left(\hat{a}_j + \hat{a}_j^\dagger\right) + i\mu_j \left(\hat{a}_j - \hat{a}_j^\dagger\right)\right]\right\}.$$ 

The transform is invertible, that is if $\hat{f}$ is the transform of $\hat{f}$ and $\hat{g}$ the transform of $\hat{g}$, then the transform of $\hat{f} + \hat{g}$ is $f + g$.

The use of the WW formalism in quantum optics has the following features in comparison with the Hilbert-space formalism:

1. It is just quantum optics, therefore the predictions for experiments are the same.

2. The calculations using the WW formalism are generally no more involved than corresponding ones in Hilbert space, and many times they are easier because no problem of non-commutativity arises.

3. The formalism suggests a physical picture in terms of random variables and stochastic processes. In particular the counterparts of creation and annihilation operators look like random amplitudes. However there are difficulties for such picture that will be commented section 4.

Here we shall use the formalism in the Heisenberg picture, where the evolution appears in the observables (usually functions of the fields) whose evolution resembles classical stochastic processes. On the other hand the concept of photon does not appear in the WW formalism.

2.2 Properties

All properties of the WW transform in particle systems may be translated to quantum optics via eqs. The transform allows getting a function of
(c-number) amplitudes for any trace-class operator (e.g. any function of the creation and annihilation operators of ‘photons’). In particular we may get the (Wigner) function corresponding to any quantum state. For instance the vacuum state, represented by the density matrix $|0\rangle\langle0|$, is associated to the following Wigner function

$$W_0 = \prod_j \frac{2}{\pi} \exp \left(-2|a_j|^2\right).$$

This function may be interpreted as a (positive) probability distribution. Hence the picture that emerges is that the quantum vacuum of the electromagnetic field (also named zeropoint field, ZPF) consists of stochastic fields with a probability distribution independent for every mode, having a Gaussian distribution with mean energy $\frac{1}{2}\hbar\omega$ per mode.

Similarly there are functions associated to the observables. For instance the following Weyl transforms are obtained

$$\hat{a}_j \leftrightarrow a_j, \hat{a}_j^\dagger \leftrightarrow a_j^*, \frac{1}{2} \left(\hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger\right) \leftrightarrow a_j a_j^* = |a_j|^2,$$

$$\hat{a}_j^\dagger \hat{a}_j = \frac{1}{2} \left(\hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger\right) + \frac{1}{2} \left(\hat{a}_j^\dagger \hat{a}_j - \hat{a}_j \hat{a}_j^\dagger\right) \leftrightarrow |a_j|^2 - \frac{1}{2},$$

$$\left(\hat{a}_j + \hat{a}_j^\dagger\right)^n \leftrightarrow \left(a_j + a_j^*\right)^n, \left(\hat{a}_j^\dagger - \hat{a}_j\right)^n \leftrightarrow \left(a_j - a_j^*\right)^n, n \text{ an integer.}$$

I stress that the quantities $a_j$ and $a_j^*$ are c-numbers and therefore they commute with each other. The former eqs. (3) mean that in expressions linear in creation and/or annihilation operator the Weyl transform just implies “removing the hats”. However this is not the case in nonlinear expressions in general. In fact from the latter two eqs. (3) plus the linearity property it follows that for a product in the WW formalism the canonical counterpart is

$$a_j^k a_j^{*l} \leftrightarrow (\hat{a}_j^k \hat{a}_j^{*l})_{sym},$$

where the subindex $sym$ means writing the product with all possible orderings and dividing for the number of terms.

Other properties may be easily obtained from well known results of the standard Weyl-Wigner formalism in particle quantum mechanics. I will present the more relevant properties omitting the proofs that are practically the same than those of the formalism when applied to quantum mechanics of particles.
Expectation values may be calculated in the WW formalism as follows. In the Hilbert-space formalism they read \( \text{Tr}(\hat{\rho} \hat{M}) \), or in particular \( \langle \psi | \hat{M} | \psi \rangle \), whence the translation to the WW formalism is obtained taking into account that the trace of the product of two operators becomes

\[
\text{Tr}(\hat{\rho} \hat{M}) = \int W_\rho \{ \hat{a}_j, \hat{a}_j^\dagger \} W_\hat{M} \{ \hat{a}_j, \hat{a}_j^\dagger \} \prod_j d\text{Re}a_j d\text{Im}a_j.
\]

That integral is the WW counterpart of the trace operation in the Hilbert-space formalism. Particular instances are the following expectations that will be of interest later on

\[
\langle 0 | \hat{a}_j^\dagger \hat{a}_j | 0 \rangle = \int d\Gamma(a_j^* a_j - \frac{1}{2}) W_0 = 0,
\]

\[
\langle |a_j|^2 \rangle \equiv \int d\Gamma W_0 |a_j|^2 = \frac{1}{2},
\]

\[
\langle 0 | \hat{a}_j \hat{a}_j^\dagger | 0 \rangle = \int d\Gamma (|a_j|^2 + \frac{1}{2}) W_0 = 2 \langle |a_j|^2 \rangle = 1,
\]

where \( W_0 \) is the Wigner function of the vacuum, eq.(2). This means that in the WW formalism the field amplitude \( a_j \) (coming from the vacuum) behaves like a complex random variable with Gaussian distribution and mean square modulus \( \langle |a_j|^2 \rangle = 1/2 \). I point out that the integral for any mode not entering in the function \( M(\{a_j, a_j^*\}) \) gives unity in the integration due to the normalization of the Wigner function eq.(2).

### 2.3 Evolution

In the Heisenberg picture of the Hilbert-space formalism the density matrix is fixed and any observable, say \( \hat{M} \), evolves according to

\[
\frac{d}{dt} \hat{M} = i \left( \hat{H} \hat{M} - \hat{M} \hat{H} \right), \quad \hat{M} = \hat{M}(t).
\]

Translated to the WW formalism this leads to the Moyal equation (with the sign changed from the usual Moyal equation, that applies to the evolution of Wigner functions). We have

\[
\frac{\partial W_\hat{M}}{\partial t} = 2 \left\{ \sin \left[ \frac{1}{4} \left( \frac{\partial}{\partial \text{Re}a_j} \frac{\partial}{\partial \text{Im}a_j} - \frac{\partial}{\partial \text{Im}a_j} \frac{\partial}{\partial \text{Re}a_j} \right) \right] W_\hat{M} \{ a'_j, a''_j, t \} H_\hat{M} \{ a''_j, a''_j \} _{a_j} \right\} ,
\]

(6)
where \( \{ \} \) means making \( a'_j = a''_j = a_j \) and \( a'''_j = a''''_j = a^*_j \) after performing the derivatives.

A simple example is the free evolution of the field amplitude of a single mode. The Hamiltonian in the WW formalism may be trivially got translating the Hamiltonian of the Hilbert-space formalism, that is

\[
\hat{H}_{\text{free}} = \omega_j \hat{a}^j_\dagger \hat{a}_j \rightarrow H_{\text{free}} = \omega_j (|a_j|^2 - \frac{1}{2}) = \omega_j \left[ (\text{Re} a_j)^2 + (\text{Im} a_j)^2 - \frac{1}{2} \right].
\]

This leads to

\[
\frac{d}{dt} a_j = \frac{1}{2} \omega_j [2(\text{Im} a_j) - 2 (\text{Re} a_j)i] = -i\omega_j a_j \Rightarrow a_j(t) = a_j(0) \exp(-i\omega_j t)
\]

(7)

Another example is the down-conversion process in a single crystal. Avoiding a detailed study of the physics inside the crystal\,[24],\,[12] we shall study a single mode problem with the model Hamiltonian\,[25],[26]

\[
\hat{H}_I = C \hat{a}^s_\dagger \hat{a}^i_\dagger \exp(-i\omega_P t) + C^* \hat{a}_s \hat{a}_i \exp(i\omega_P t),
\]

(8)

whence taking eqs.(6) and (7) into account we have

\[
\frac{d}{dt} a_s = -i\omega_s a_s - iCa^*_i \exp(-i\omega_P t), \quad (9)
\]

\[
\frac{d}{dt} a_i = -i\omega_i a_i - iCa^*_s \exp(-i\omega_P t).
\]

We shall assume that the vacuum field \( a_s \) evolves as in eq.(7) before entering the crystal and according to eqs.(9) inside the crystal, during the time \( T \) needed to cross it. In order to get the radiation intensity to second order in \( CT \) (see below section 2.4) we must solve these two coupled equations also to second order. This leads to

\[
a_s(t) = \left( 1 - |D|^2 \right) a_s(0) \exp(-i\omega_s t) - iDa^*_i(0) \exp[i(\omega_i - \omega_P)t]
\]

\[
= \left( 1 - |D|^2 \right) a_s(0) - iDa^*_i(0) \exp(-i\omega_s t),
\]

(10)
where $|D| \equiv |C| T << 1$ and the latter equality takes the ‘energy conservation’ into account (that in the WW formalism looks like a condition of frequency matching, $\omega_F = \omega_s + \omega_i$, with no reference to photon energies).

Eq. (10) gives the time dependence of the relevant mode of signal after the crystal, but we shall take account of the field dependence on position including a factor $\exp (i \mathbf{k}_s \cdot \mathbf{r})$, that is phase depending on the path length. Therefore the correct form of eq. (10) would be, modulo a global phase,

$$a_s(\mathbf{r}, t) = \left[ (1 - |D|^2) a_s(0) - iDa_i^*(0) \right] \exp (i \mathbf{k}_s \cdot \mathbf{r} - i \omega_st). \quad (11)$$

A similar result is obtained for $a_i(t)$, that is

$$a_i(\mathbf{r}, t) = \left[ (1 - |D|^2) a_i(0) - iDa_s^*(0) \right] \exp (i \mathbf{k}_i \cdot \mathbf{r} - i \omega_it). \quad (12)$$

Eq. (11) may be interpreted saying that the vacuum signal amplitude is modified by the addition of an amplification of the vacuum idler, but it travels in the same direction of the incoming vacuum signal, and therefore it has sense adding the initial vacuum signal with the amplification of the idler. And similarly for $a_i$ eq. (12). Still eqs. (11) and (12), although good enough for calculations are bad representations of the physics. In fact a physical beam corresponds to a superposition of the amplitudes, $a^*_k$, of many modes with frequencies and wavevectors close to $\omega_s$ and $\mathbf{k}_s$, respectively. For instance we may represent the positive frequency part of the idler beam created in the crystal, at first order in $D$, as follows

$$E^{(+)}_i(\mathbf{r}, t) = -iD \int f_i(\mathbf{k}) d^3k a^*_k \exp \left[ i (\mathbf{k} - \mathbf{k}_s) \cdot \mathbf{r} - i (\omega - \omega_s) t \right] + E^{(+)}_{ZPF}, \quad (13)$$

where $\omega = \omega(\mathbf{k})$ and $f_i(\mathbf{k})$ is an appropriate function, with domain in some region of $\mathbf{k}$ around $\mathbf{k}_s$. The field $E^{(+)}_{ZPF}$ is the sum of amplitudes of all vacuum modes, including the one represented by $a_s$ in eq. (11). (We have neglected a term of order $|D|^2$ so that $E^{(+)}_i$ is correct to order $|D|$. These vacuum modes have fluctuating amplitudes with a probability distribution given by the vacuum Wigner function eq. (2). It may appear that the amplitude $a_s$ is lost ‘as a needle in the haystack’ within the background of many radiation modes, but it is relevant in photon correlation experiments. In fact the vacuum amplitude $a_s$ in eqs. (10) or (11) is fluctuating and the same fluctuations appear also in the signal amplitude $a_s^*$ of eq. (12). Therefore coincidence counts will be favoured when large positive fluctuations of the fields eqs. (10)
and (12) arrive simultaneously to Alice and Bob detectors. In the Hilbert-space formalism this fact is named ‘entanglement between a signal and the vacuum’ [27], [30]. In the WW formalism of quantum optics entanglement appears as a correlation between fields in distant places.

Up to here we have exhibited the most relevant properties of the WW formalism needed for the interpretation of experiments involving pure radiation field or the field interacting with macroscopic bodies, the latter defined by their bulk electric properties like the refraction index or the nonlinear electrical susceptibility. Within the WW formalism the interaction of the fields with macroscopic bodies may be treated as in classical electrodynamics. This is for instance the case for the action of a laser on a crystal with nonlinear susceptibility, studied elsewhere [24], [12].

2.4 Counts in photodetectors

As said in the previous section the propagation of electromagnetic radiation (restricted to optical frequencies) either in vacuum or in bulk matter may be interpreted in terms of (classical) stochastic processes. However specific quantum features appear in the interaction of radiation with microscopic systems like atoms or electrons in solids. In particular such interactions are essential for the behaviour of detectors like photon counters. Extending the WW formalism to atoms or electron in solids would not be trivial and the relative simplicity of the formalism would be lost. Therefore we shall evade the problem postulating directly rules for photon counting via translating the Hilbert-space rules to the WW formalism, with the condition that the results should be the same in both.

I shall study only the case where the field operator representing a beam is linear in the creation and annihilation operators. Then the positive frequency field operator in the Hilbert-space formalism and the corresponding field amplitude in the WW one may be written

\[ \hat{E}_A^{(+)} = \sum_j c_j \hat{a}_j + \sum_l d_l \hat{a}_l^\dagger, \quad E_A^{(+)} = \sum_j c_j a_j + \sum_l d_l a_l^*, \quad (14) \]

where the numerical (not operators) quantities \( c_j \) and \( d_l \) are space-time functions. Similar relations exist for \( \hat{E}_A^{(-)} \) from \( E_A^{(-)} \). The subindex \( A \) (for Alice) is introduced for later convenience. The field components in the WW formalism are easily got from the Hilbert-space formalism taking into account the rule ‘in linear expressions just remove the hats’, eq. (3). In eq. (14) the
subindices $j$ and $l$ may correspond to either the field superpositions needed to get physical representations of beams, as in eq.(13), or to different beams involved. Still we might include in the sums all modes, although only some of them will be needed in the interpretation of experiments and should appear in eq.(14). When the theory is applied to SPDC the annihilation operators $\hat{a}_j$ would correspond to the vacuum beams entering the crystals and the creation operators $\hat{a}_l^\dagger$ to the signal and idler fields created by the pumping beam inside the crystal, or to functions of them, as shown below.

From the former eq.(14) we may obtain the field intensity operator, defined as the product of field operators with the positive frequency part to the right, that is

$$\hat{I}_A = \hat{E}_A^{(-)} \hat{E}_A^{(+)} = \sum_j \left[ \hat{a}_j^\dagger \hat{a}_j |c_j|^2 + \hat{a}_j \hat{a}_j^\dagger |d_j|^2 + \hat{a}_j^2 c_j d_j^* + \hat{a}_j^\dagger c_j^* d_j \right]. \quad (15)$$

In the Hilbert-space formalism the detection counting rate, $R_A$, is (proportional to) the vacuum expectation of the field intensity operator, whence we get

$$R_A = \langle 0 | \hat{I}_A | 0 \rangle = \sum_l |d_l|^2, \quad (16)$$

modulo a proportionality constant that we will ignore everywhere.

The translation of eqs.(15) and (16) to the WW formalism is straightforward taking eqs.(5) into account, that is

$$R_A = \int d\Gamma \left[ \sum_j |c_j|^2 \langle |a_j|^2 \rangle - \frac{1}{2} + \sum_l |d_l|^2 \langle |a_l|^2 \rangle + \frac{1}{2} \right] W_0$$

$$= 2 \sum_l |d_l|^2 \langle |a_l|^2 \rangle = \sum_l |d_l|^2. \quad (17)$$

The result agrees eq.(16) as it should.

This WW rule for the counting rate does not look as simple as the Hilbert-space rule. In particular the rate is not proportional to the average intensity, $I_A$, if it is defined in terms of the field eq.(14) in the obvious way, that is

$$I_A = |E_A^{(-)} E_A^{(+)}|^2 = |E_A^{(+)}|^2$$

$$\Rightarrow \langle I_A \rangle = \sum_j (|c_j|^2 + |d_j|^2) \langle |a_j|^2 \rangle = \frac{1}{2} \sum_j (|c_j|^2 + |d_j|^2). \quad (18)$$
In order to write the correct WW rule, eq.(17), in terms of intensities we must introduce at least another intensity in addition to eq.(18). I shall do that in terms of a new field, that is

\[ I_{A0} = |E_{A0}^{(+)}|^2, \quad E_{A0}^{(+)} = \sum_j c_j a_j \Rightarrow \langle I_{A0} \rangle = \sum_j |c_j|^2 \langle |a_j|^2 \rangle = \frac{1}{2} \sum_j |c_j|^2. \]

(19)

After that we may write the WW rule eq.(17) as follows

\[ R_A = 2 \langle I_A - \langle I_{A0} \rangle \rangle = \sum_j |d_j|^2. \]

(20)

One of the possible virtues of the WW formalism is to provide an intuitive picture of some quantum optical phenomena. Actually a faithful picture in terms of random variables and stochastic processes is not possible as we will discuss below. Therefore for some people that intuitive picture might be misleading rather than interesting. Nevertheless, as classical theories are an approximation of quantum theories (when Planck constant \( \hbar \) may be taken as small), the ‘classical intuitive picture’ could be of some interest and therefore I will try to develop it a little further.

To begin with we remember the well known fact that the normal ordering rule of quantum optics is equivalent to the subtraction of the ‘vacuum energy’ of a mode. In fact we have

\[ 0 = \langle 0 | \hat{\mathcal{H}}_{\omega_l} \hat{\alpha}_l^+ \hat{\alpha}_l | 0 \rangle = \hat{\mathcal{H}}_{\omega_l} \int d\Gamma W_0 |a_l|^2 - \frac{1}{2} \hat{\mathcal{H}}_{\omega_l}, \]

where we have taking eqs.(5) into account. The latter term is the energy contribution of the mode \( l \) to the zeropoint field energy. On the other hand \( I_{A0} \) eq.(19) corresponds to the intensity of the vacuum fields alone. Therefore we may interpret eq.(20) saying that only the radiation energy above the zeropoint contributes to detection, that would explain why the former term of eq.(17) does not contribute.

In summary, the detection rules in the WW formalism suggests the following intuitive picture: The quantities \( a_i, a_i^* \) may be seen as random variables representing field amplitudes. Averages of terms like \( a_i a_i^* \), \( a_i a_i \) or \( a_i^* a_i^* \) are nil because we assume that the field amplitudes have random phases. Only the field intensity above the zeropoint level may contribute to detection, whence we must subtract from \( I_A \) the part, \( I_{A0} \), coming from the zeropoint (that is the vacuum fields entering the crystal in case of SPDC). The zeropoint part is the intensity that would arrive at the detector if there was no
signal, that is if $d_j = 0$ what would happen if there was no laser pumping field. Actually the picture fails due to the fact that $I_A - \langle I_{A0} \rangle$ is not positive definite in general. Indeed if we treat this quantity as a random variable we are led to assume that its value for a particular sample of the amplitudes $\{a_j\}$ gives the probability of detection for that sample, but not being positive definite the picture is untenable. This difficulty will be discussed in more detail in section 4.

The detection counting rate by Bob may be obtained in a similar form. If Bob’s field is

$$
\hat{E}_B^{(+)} = \sum_k f_k \hat{a}_k + \sum_r g_r \hat{a}_r^\dagger,
$$

then the single detection rate $R_B$ is obtained as follows

$$
\langle I_B \rangle = \frac{1}{2} \left( \sum_k |f_k|^2 + \sum_r |g_r|^2 \right),
\langle I_{B0} \rangle = \frac{1}{2} \sum_k |f_k|^2,
R_B = 2 \langle I_B - \langle I_{B0} \rangle \rangle = \sum_r |g_r|^2.
\tag{22}
$$

Now we shall get the joint detection probability in two detectors (Alice and Bob). In the Hilbert-space formalism it is given by the vacuum expectation value of four fields, that is

$$
R_{AB} = \frac{1}{2} R_{AB}^{(1)} + \frac{1}{2} R_{AB}^{(2)},
R_{AB}^{(1)} = \left\langle 0 \left| \hat{E}_{B}^{(-)} \hat{E}_{B}^{(-)} \hat{E}_{B}^{(+)} \hat{E}_{B}^{(+)} \right| 0 \right\rangle,
R_{AB}^{(2)} = \left\langle 0 \left| \hat{E}_{B}^{(-)} \hat{E}_{B}^{(-)} \hat{E}_{B}^{(+)} \hat{E}_{B}^{(+)} \right| 0 \right\rangle.
\tag{23}
$$

We must use the symmetrized form that would reduce to twice one of the terms if $[\hat{E}_A^{(+)}, \hat{E}_B^{(+)}] = 0$. In the following I will calculate $R_{AB}^{(1)}$ in some detail and then write the result $R_{AB}^{(2)}$ by analogy.

It is straightforward to get the fields in terms of the creation and annihilation operators taking eqs. (14), (21) and (23) into account, that is

$$
R_{AB}^{(1)} = \left\langle 0 \left| \sum_{jklr} \left( c_j^* \hat{a}_j^\dagger + d_j^* \hat{a}_j^\dagger \right) \left( f_k^* \hat{a}_k^\dagger + g_k^* \hat{a}_k^\dagger \right) \left( f_l^* \hat{a}_l^\dagger + g_l^* \hat{a}_l^\dagger \right) \left( c_r \hat{a}_r + d_r \hat{a}_r^\dagger \right) \right| 0 \right\rangle.
\tag{24}
$$

At this moment I assume that every sum, $\sum_j, \sum_k, \sum_l, \sum_r$, may be performed for all possible modes of the field, appropriate constraints appearing in every
case where the coefficients $c_j, d_j, \ldots$ are zero. Removing the operators that do not contribute in eq. (24) we get

$$R = \text{negligible. The same is true for the term} \quad \text{case where the coefficients} \quad c_j, d_j, \ldots \text{are zero. Removing the operators that}$$

$$\quad \text{do not contribute in eq. (24) we get} \quad R \quad \text{thus leading to} \quad R_A R_B = \sum_k d_k f_k^2 + \sum_{jl} |d_j|^2 |g_l|^2 + \sum_j d_j^* g_j = R_A R_B + \sum_k |d_k f_k|^2 + \sum_j |d_j g_j|^2. \quad (25)$$

The constraint $j \neq l$ may be removed taking the latter term into account, thus leading to

$$R_{AB}^1 = \sum_k d_k f_k^2 + \sum_{jl} |d_j|^2 |g_l|^2 + \sum_j d_j^* g_j = R_A R_B + \sum_k |d_k f_k|^2 + \sum_j |d_j g_j|^2. \quad (26)$$

The term $R_{AB}^2$ may be obtained from $R_{AB}^1$ via the changes

$$c, d, f, g \rightarrow f, g, c, d,$$

whence, taking eq. (23) into account, we get

$$R_{AB} = R_A R_B + 1 \sum_k d_k f_k^2 + 1 \sum_k c_k g_k^2 + 1 \sum_j d_j g_j^2 \quad \text{in SPDC the term} \quad |\sum_j d_j g_j|^2 \text{is fourth order in the (small) parameter} \quad |D| \quad \text{and therefore negligible. The same is true for the term} \quad R_A R_B = \sum_{jl} |d_j|^2 |g_l|^2.$$

The rule for counting rate in the WW formalism may easily be obtained via a translation of eqs. (25) to (26). Defining the following field amplitudes

$$E_{A0}^{(+)} = \sum_j c_j a_j, E_{A1}^{(+)} \equiv E_{A}^{(+)} - E_{A0}^{(+)} = \sum_l d_l a_l^*, E_{B0}^{(+)} = \sum_k f_k a_k, E_{B1}^{(+)} = \sum_r g_r a_r^*, \quad (27)$$
we get

\[ R_{AB} = R_A R_B + 2 \left| \langle E_A^{(+)} E_{B1}^{(+)} \rangle \right|^2 + 2 \left| \langle E_{A1}^{(+)} E_{B1}^{(+)} \rangle \right|^2 + 4 \left| \langle E_{A1}^{(+)} E_{B1}^{(-)} \rangle \right|^2, \]  

(28)

that leads to eq.(26) as may be checked. In SPDC both the latter term
and the product \( R_A R_B \) are fourth order in \( |D| \) and might be neglected, see eq. (26). Eq.(27) may be written in terms of the intensities rather than field
amplitudes, that is

\[ R_{AB} = R_A R_B + 2 \left[ \langle I_A I_{B1} \rangle - \langle I_A \rangle \langle I_B \rangle - \langle I_{A1} \rangle \langle I_B \rangle \right] \]

\[ + \langle I_{A1} \rangle \langle I_{B1} \rangle - \langle I_{A1} \rangle \langle I_B \rangle \]

(29)

where the intensities are the square moduli of the corresponding field
amplitudes. It is easy to prove that eq.(29) is equivalent to eq.(28), taking
into account the well known property of the average for a product of four
Gaussian random variables, \( A, B, C, D \), namely

\[ \langle ABCD \rangle = \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle. \]

The random fields \( E^{(+)} \) and \( E^{(-)} \) are Gaussian because they are linear combi-
nations of elementary field amplitudes, \( \{ a_j, a_j^* \} \) that are Gaussian, see eq.(2).

3 Experiment of two-photon interference using two crystals

In the following we use the WW formalism for the interpretation of an ex-
periment consisting of two crystals pumped by two coherent beams obtained
dividing the laser beam by means of a balanced beam splitter[28],[29],[30].
The signal beams from both crystals are sent to the two incoming channels of
a beam splitter, BS1, and in front of one of the outgoing channels a detector,
say Alice, is placed. Similarly the idler beams of both crystals are mixed via
another beam splitter, BS2, and the field from one of the outgoing channels
goes to another detector, say Bob. The measured quantity is the coincidence
counting rate between A and B as a function of the path length of one of
beams, that may be controlled at will.

The amplitude of the light beam arriving at the detector will consists of
an appropriate superposition of modes but we consider just one of the modes
of the superposition. The positive frequency part of the field of a typical mode may be written (compare with eq.(10))

$$E_A^{(+)} = \{ [a_{s10} - iDa_{s10}^*]r \exp(i\phi_1) + [a_{s20} - iDa_{s20}^*]t \} \exp(-i\omega_1 t). \quad (30)$$

$t(r)$ is the transmission (reflection) coefficient of BS1 in front of the detector, $\phi_1$ takes into account the phase difference between the two beams mixed at that BS1, due to the path length difference that may be changed at will. Actually there is a global factor in front of the right side of eq.(30) that is irrelevant for our purposes and we ignore. In this factor we absorb the term in $|D|^2$ of eq.(10), needed to get the intensity to that order. That is the parameter $D$ of eq.(30) actually corresponds to $D/(1 - |D|^2)$ of eq.(11). Similarly the idler fields of both crystals are sent to another beam splitter BS2 and then to a detector, say Bob. The positive frequency part of the field arriving at Bob may be written, assuming that $D,r$ and $t$ are the same for Alice and Bob beams,

$$E_B^{(+)} = \{ [a_{i10} - iDa_{i10}^*]r + [a_{i20} + iDa_{i20}^*]t \} \exp(-i\omega_2 t). \quad (31)$$

The notation in eqs.(30) and (31) follows Ref.[30].

The interesting quantity to be compared with experiments is the coincidence counting rate, $R_{AB}$, of Alice and Bob. In the WW formalism we should calculate it from eq.(28) after evaluating the different averages. Taking eqs.(30) and (31) into account we have, ignoring the time-dependent factor,

$$E_{A0}^{(+)} = ra_{s10} \exp(i\phi_1) + ta_{s20}, \quad E_{A1}^{(+)} = -iDra_{s10}^* \exp(i\phi_1) - iDta_{s20}^*, \quad (32)$$

$$E_{B0}^{(+)} = ta_{i10} + a_{i20}r \exp(i\phi_2), \quad E_{B1}^{(+)} = -iDta_{i10}^* - iDra_{i20}^* \exp(i\phi_2). \quad (33)$$

Hence eq.(28) gives

$$R_{AB} = R_A R_B + 2 |D|^2 |rt|^2 \left[ |\exp(i\phi_1) + \exp(i\phi_2)|^2 \right]$$

$$\simeq 4 |D|^2 |rt|^2 \left[ 1 + \cos (\phi_2 - \phi_1) \right], \quad (34)$$

where we have neglected $R_A R_B$ that is of order $|D|^4$. This agrees with the result got with the Hilbert-space formalism[30] as it should.
4 Does the WW formalism provide a picture in terms of random variables and stochastic processes?

In the previous two sections we have seen that the WW formalism applied to experiments with photon pairs produced in nonlinear crystals via SPDC suggests a picture resting upon (classical) random variables and stochastic processes. Actually the picture is not possible due to the positivity problem mentioned above (see after eq.(20)). Now I shall illustrate the problem in the particular case of the two-photon interference experiment. To do that I will look more closely at the expression for the coincidence detection probability, latter eq.(29). There we have two similar terms contributing to the joint detection, each one been the average of the product of two random variables. I will study the former as it appears in the analysis of the experiment, eqs.(32) and (33), putting $|r|^2 = |t|^2 = 1/2$ for simplicity. The two random variables, say $x$ and $y$, are as follows

\[
x (\phi_1) \equiv I_A - \langle I_A \rangle = 1/2 |a_{s10} \exp(i\phi_1) + a_{s20} - iD a_{s10}^* \exp(i\phi_1) - iD a_{s20}^*|^2 - 1/2 \left| a_{s10} \right|^2 \left| a_{s10}^* \right|^2 + |D|^2 \left| a_{s10} \right|^2 + |D|^2 \left| a_{s10}^* \right|^2
\]

\[
= 1/2 |a_{s10} \exp(i\phi_1) + a_{s20} - iD a_{s10}^* \exp(i\phi_1) - iD a_{s20}^*|^2 - 1 - |D|^2,
\]

\[
y (\phi_2) \equiv I_{B1} = 1/2 |D|^2 \left| a_{s10} + a_{s20} \exp(i\phi_2) \right|^2.
\]

In the former we may neglect terms containing the parameter $|D| << 1$ so that we shall study the random variables

\[
x (\phi_1) = 1/2 |a_{s10} \exp(i\phi_1) + a_{s20}|^2 - 1, x \in (-1, \infty),
\]

\[
y (\phi_2) = 1/2 |D|^2 \left| a_{s10} + a_{s20} \exp(-i\phi_2) \right|^2, y \in (0, \infty).
\]

Their single probability distributions, $P(x), P(y)$, may be easily obtained. In fact the amplitudes $a_j$ are treated as complex random variables with zero mean, Gaussian distributions and such that $\langle |a_j|^2 \rangle = 1/2$ (see eq.(2)). Hence the intensities should have exponential distributions and averages that are easy to get, leading to

\[
P(x) = \exp(1 - x), P(y) = |D|^{-2} \exp (-y/|D|^2), x \in (-1, \infty), y \in (0, \infty).
\]
It is obvious that both random variables are strongly correlated although getting the joint probability distribution is involved. The most interesting particular case is the one giving a maximum coincidence counting rate that corresponds to $\phi_1 = \phi_2$ and we may put $\phi_1 = 0$ without loss of generality. Then $y$ is a function of $x$ and the joint probability distribution is

$$y = |D|^2 (x + 1) \Rightarrow P_{12}(x, y) = \exp (1 - x) \delta \left( y - |D|^2 (x + 1) \right)$$

where $\delta (\cdot)$ is Dirac’s delta. This strong correlation corresponds to entanglement between a signal (the variable $y$) and the vacuum (the variable $x$)\cite{25},\cite{30}. The coincidence counting rate eq.(34) may be written using these variables, the former term being

$$R_{AB}^{(1)} = \int_{-1}^{\infty} dx \int_{0}^{\infty} dy x y P_{12}(x, y) = |D|^2 \int_{-1}^{\infty} dx x (x + 1) \exp (1 - x)$$

$$= |D|^2 \int_{-1}^{0} dx x (x + 1) \exp (1 - x) + |D|^2 \int_{0}^{\infty} dx x (x + 1) \exp (1 - x)$$

$$= |D|^2 \left(1 - 3e^{-1}\right) + |D|^2 3e^{-1} \approx -0.104 |D|^2 + 1.104 |D|^2 = |D|^2 (36)$$

a result in agreement with eq.(34) (for $\phi_1 = \phi_2 = 0, r = t = 1/\sqrt{2}$). Then eq.(36) suggests that the joint detection probability is positive for some samples of the field and negative for others, in such a way that the average probability is positive and agrees with the quantum prediction. Obviously that picture is untenable.

A more physical study of the positivity problem requires a better treatment of the beams incoming to the detectors, every beam involving many modes. Also we should assume that detection is not instantaneous but involves a time interval of order hundred times the inverse of the light frequency\cite{27} so that we should include an integral over time. That kind of time average would produce an effect similar to an ensemble average, represented by $\langle \rangle$, that is a reduction of the negative part in eq.(36). However a refined treatment would not solve the problem in all cases, as is shown by the loophole-free violation of Bell’s inequalities. Indeed several of these experiments have been performed using ‘entangled photon pairs’ produced via parametric down-conversion\cite{31},\cite{32} and the violation shows that no classical-like (local realistic) model is compatible with the empirical results that, in contrast, do agree with quantum predictions.
5 Conclusions

We have shown that the Weyl-Wigner formalism in the Heisenberg picture may be used for the interpretation of experiments involving entangled photon pairs produced in nonlinear crystals via spontaneous parametric down-conversion. The calculations are usually no more involved than those with the Hilbert-space formalism. The WW formalism suggests a physical picture in terms of random variables and stochastic processes. In particular the concept of ‘photon’, seen as a more or less localized particle, does not appear. The well known difficulty that the Wigner functions are not positive definite in general is avoided in the Heisenberg picture because only the Wigner function of the vacuum, eq. (2), is interpreted as a probability distribution. Also the WW formalism provides an intuitive picture of entanglement as a strong correlation between the fluctuating fields of the vacuum in distant places.

Nevertheless there is a difficulty that appears unsurmountable for an interpretation of quantum optics in terms of stochastic fields, even in the restricted domain involving experiments with nonlinear crystals, namely the reported violation of the Bell inequality, as commented on the previous section. In the WW formalism the reason for the impossibility of pictures seems to derive from the fact that the behaviour of photon counters cannot be interpreted within such models. Indeed a crucial assumption in the picture is that counters are sensitive only to radiation above the level of the ZPF, but that level is not well defined because the fields are fluctuating. Therefore a problem of negative probabilities may appear, as commented on the previous section.

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