Omega-limit sets and bounded solutions

Dang Vu Giang
Hanoi Institute of Mathematics
Vietnam Academy of Science and Technology
18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
e-mail: (dangvugiang@yahoo.com)

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Abstract. We prove among other things that the omega-limit set of a bounded solution of a Hamilton system

$$\begin{cases} 
\dot{p} = \frac{\partial H}{\partial q} \\
\dot{q} = -\frac{\partial H}{\partial p}
\end{cases}$$

is containing a full-time solution so there are the limits of $\frac{1}{t} \int_0^t p(s)ds$ and $\frac{1}{t} \int_0^t q(s)ds$ as $t \to \infty$ for any bounded solution $(p, q)$ of the Hamilton system. These limits are stationary points of the Hamilton system so if a Hamilton system has no stationary point then every solution of this system is unbounded.

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1. INTRODUCTION

In this paper \((\mathbb{X}, \|\cdot\|_\mathbb{X})\) denotes a complex Banach space. Let \(A : \mathbb{X} \to \mathbb{X}\) be a bounded linear operator with compact spectrum \(\sigma(A)\) and positive spectral radius \(r(A)\). In [1] we proved that if \(\sigma(A) \cap i\mathbb{R} = \{i\xi_1, i\xi_2, \cdots, i\xi_n\}\) then every bounded full-time solution of differential equation \(\dot{x}(t) = Ax(t)\) has the form

\[ u(t) = \sum_{k=1}^{n} e^{i\xi_k t} v_k, \]

where \(v_1, v_2, \cdots, v_n\) are fixed vectors of \(\mathbb{X}\). Recall that full-time solution is the solution satisfying the differential equation for all \(t \in \mathbb{R}\). For example, periodic solutions (if exist) are full-time and bounded solutions. We used Beurling spectrum [1] and Fourier coefficients of a bounded function (on the real line) in the proof. More exactly, we proved that the Beurling spectrum of any bounded full-time solution is a subset of \(\{\xi_1, \xi_2, \cdots, \xi_n\}\). For the delay equation \(\dot{u}(t) = -u(t - \tau)\) we proved that every almost periodic solution is periodic, so if there exists an almost periodic solution then the delay \(\tau\) must be \(\pi/2\). Generally, the spectrum of any bounded full-time solution of the delay equation \(\dot{x}(t) = Ax(t - \tau)\) is a compact subset of the interval \([-r(A), r(A)]\). Now consider a bounded solution \(x\) of

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) \text{ for } t > 0 \\
x(0) &\text{ given in } \mathbb{X}.
\end{aligned}
\]

Assume that the orbit \(\{x(t) : t \geq 0\}\) is relatively compact. Then the omega-limit set \(\omega\) of \(x\) is a compact connected subset of \(\mathbb{X}\) [4]. Moreover, \(\omega\) is invariant under the group \(T(t) = e^{At}\). Let \(v\) be a point in this omega-limit set and \(u(t) = T(t)v\). Then \(u\) is a bounded full-time solution of the differential equation \(\dot{x} = Ax\). On the other hand, \(\Omega = \omega \cup \{x(t) : t \geq 0\}\) is a compact subset of \(\mathbb{X}\). Therefore, the semi-group \(\{T(t)\}_{t \geq 0}\) acts injectively on \(\Omega\). By an ergodic theorem [6] we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s)ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(s)ds.
\]

This limit is lying in the kernel of \(A\). Specially, if \(\sigma(A) \cap i\mathbb{R} = \emptyset\) then 0 is the only bounded full time solution. Thus, every bounded solution tends to 0 as \(t \to \infty\). Now let \((p, q)\) be a bounded solution of the Hamilton system

\[
\begin{aligned}
\dot{p} &= \frac{\partial H}{\partial q} \\
\dot{q} &= -\frac{\partial H}{\partial p}.
\end{aligned}
\]
Then there is an injective continuous semi-flow $T(t): \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $(p(t), q(t)) = T(t)(p(0), q(0))$. Then the omega-limit set $\omega$ of $(p, q)$ is a compact connected subset of $\mathbb{R}^{2n}$ [4]. Moreover, $\omega$ is invariant under the group $T(t)$. The dynamical system $\langle \omega, \{T(t)\}_{t \in \mathbb{R}} \rangle$ is uniquely ergodic, since the only invariant (continuous) function on $\langle \omega, \{T(t)\}_{t \in \mathbb{R}} \rangle$ is the constant function. Let $v$ be a point in this omega-limit set and $u(t) = T(t)v$. Then $u$ is a bounded full-time solution of the differential equation

$$
\begin{align*}
\dot{p} &= \frac{\partial H}{\partial q} \\
\dot{q} &= -\frac{\partial H}{\partial p}.
\end{align*}
$$

By an ergodic theorem [6] there are the limits of $\frac{1}{t} \int_0^t p(s)ds$ and $\frac{1}{t} \int_0^t q(s)ds$ as $t \to \infty$ for any bounded solution $(p, q)$ of the Hamilton system. These limits are stationary points of the Hamilton system. Therefore, we have

**Theorem A.** If the gradient $\nabla H$ of a smooth hamiltonian $H$ is nowhere 0 then every solution of the Hamilton system

$$
\begin{align*}
\dot{p} &= \frac{\partial H}{\partial q} \\
\dot{q} &= -\frac{\partial H}{\partial p}
\end{align*}
$$

is unbounded.

For example, consider the system $\ddot{x} = -\sin x$ with $x(0) = 0$. If $\dot{x}(0) > 2$ then $x(t)$ is unbounded. If $\dot{x}(0) = 2$ then

$$
x(t) = 2\arcsin \left( \frac{e^{2t} - 1}{e^{2t} + 1} \right)
$$

which is increasingly tending to $\pi$ as $t \to \infty$. If $\dot{x}(0) \in (0, 2)$ then $x(t)$ is periodic and bounded by $\pi$ in the time and both $\frac{1}{t} \int_0^t x(s)ds$ and $\frac{1}{t} \int_0^t \dot{x}(s)ds$ tend to 0 as $t \to \infty$. Moreover, the period of this solution is

$$
2 \int_0^A \frac{dx}{\sqrt{2 \cos x - 2 + \dot{x}(0)^2}},
$$

3
where $A = \arccos \left( 1 - \frac{\dot{x}(0)^2}{2} \right)$ is the maximal value of $x(t)$.

2. MAIN RESULTS

Let $T(t) : X \to X$ for $t \geq 0$ denote a semi-group with (unbounded and close) generator $A$. Let $x(t) = T(t)x(0)$ denote a bounded solution of the differential equation $\dot{x} = Ax$. Assume that the orbit $\{x(t) : t \geq 0\}$ is relatively compact. Then the omega-limit set $\omega$ of $x$ is a compact connected subset of $X$. Moreover, $\omega$ is invariant under the semi-group $\{T(t)\}_{t \geq 0}$. Clearly, $T(t) : \omega \to \omega$ is bijective. It is easy to prove that the dynamical system $\langle \omega, \{T(t)\}_{t \in \mathbb{R}} \rangle$ is uniquely ergodic [6]. In fact, the only invariant (continuous) function on $\langle \omega, \{T(t)\}_{t \in \mathbb{R}} \rangle$ is the constant function. Hence, there is a unique Borel probability measure $\mu$ on $\omega$ such that

$$\lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} \varphi(u(s)) \, ds = \int_{\omega} \varphi(v) \, d\mu(v).$$

Here, $\varphi$ denotes a continuous function on $\omega$ and $u(s) = T(s)v$ for some $v \in \omega$. Therefore, there is the limit of $\frac{1}{t} \int_{0}^{t} x(s) \, ds$ as $t \to \infty$. Similarly, the limit of $\frac{1}{t} \int_{0}^{t} x(s) \, ds$ exists as $t \to \infty$.

**Theorem B.** Let $A$ denote the generator of a linear semigroup $T(t) : X \to X$ for $t \geq 0$. Let $x(t) = T(t)x(0)$ denote a bounded solution of the differential equation $\dot{x} = Ax$. Assume that the orbit $\{x(t) : t \geq 0\}$ is pre-compact. Then the limit of $\frac{1}{t} \int_{0}^{t} x(s) \, ds$ exists as $t \to \infty$. This limit is a vector in the kernel of the operator $A$. If $\sigma(A) \cap i \mathbb{R} = \{i\xi_1, i\xi_2, \ldots, i\xi_n\}$ then every bounded full-time solution of differential equation $\dot{x}(t) = Ax(t)$ has the form $u(t) = \sum_{k=1}^{n} e^{i\xi_k t}v_k$, where $v_1, v_2, \ldots, v_n$ are fixed vectors of $X$. Specially, if $\sigma(A) \cap i \mathbb{R} \subseteq \{0\}$ then every bounded solution of pre-compact orbit tends to a vector in the kernel of $A$ as $t \to \infty$.

**Proof:** As we have mentioned before, the dynamics on the omega limit set of $x$ is uniquely ergodic. Moreover, this limit set contains a full time bounded solution. Let $u$ denote a bounded full-time solution of $\dot{x}(t) = Ax(t)$. Then $(\lambda - D)^{-1}u(t) = (\lambda - A)^{-1}u(t)$ for any $t \in \mathbb{R}$ and $\lambda \notin i \mathbb{R} \cup \sigma(A)$.
Here $D$ denotes the differential operator with spectrum $i\mathbb{R}$. Therefore, for any point $\xi$ in the Beurling spectrum of $u$ we have $i\xi \in \sigma(A)$. Hence, if $\sigma(A) \cap i\mathbb{R} = \{i\xi_1, i\xi_2, \ldots, i\xi_n\}$ then the Beurling spectrum of any bounded full-time solution is a subset of $\{\xi_1, \xi_2, \ldots, \xi_n\}$. Thus, $u(t) = \sum_{k=1}^{n} e^{i\xi_k t}v_k$, where $v_1, v_2, \ldots, v_n$ are fixed vectors of $X$ [1], [5]. Now consider a bounded solution $x$ of pre-compact orbit. Then the omega-limit set of $x$ should contain a bounded full time solution $u(t) = \sum_{k=1}^{n} e^{i\xi_k t}v_k$. Specially, if $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ then the omega limit set of any bounded solution with pre-compact orbit has only one element. This element is a vector of the kernel of $A$. The proof is now complete.

Remark. The last statement in our Theorem makes a significant extension of results in [2], [3]. Indeed, the authors have proved the existence of the limit $\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} x(s)ds$ only.

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