Eigenvectors from eigenvalues: the case of one-dimensional Schrödinger operators

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Abstract
We revisit an archive submission by Denton et al. (Eigenvectors from eigenvalues: a survey of a basic identity in linear algebra. arXiv:1908.03795v3 [math.RA], 2019) on \( n \times n \) self-adjoint matrices from the point of view of self-adjoint Dirichlet Schrödinger operators on a compact interval.

Keywords  
Eigenvalues · Eigenvectors · Green’s function

Mathematics Subject Classification  
34B24 · 34B27 · 34L15 · 34L40 · 47A10

1 Introduction

To set the stage for this note we briefly summarize comments by Tao [20] on the archive submission by Denton, Parke, Tao, and Zhang [5], and recall the following facts that illustrate an identity involving eigenvectors of a self-adjoint \( n \times n \) matrix, \( n \in \mathbb{N} \), its eigenvalues, and those of a closely related \( (n-1) \times (n-1) \) matrix.

Let \( A \) be a self-adjoint \( n \times n \) matrix with eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \) and corresponding normalized eigenvectors \( v_1, \ldots, v_n \), and denote by \( v_{k,\ell} \) the \( \ell \)th component of \( v_k \). Let \( M_j \) be the \( (n-1) \times (n-1) \) matrix obtained from \( A \) by deleting the \( j \)th row and column of \( A \). Let \( \lambda_1(M_j), \ldots, \lambda_{n-1}(M_j) \) denote the eigenvalues of \( M_j \). Then the identity proven in [5] reads,
Thus, generically, that is, as long as the product on the left-hand side in (1.1) does not vanish, the component $v_{k,\ell}$ of the eigenvector $v_k$ of $A$ is expressed in terms of the eigenvalues of $A$ and $M_j$.

A complex analytic proof of (1.1) notes that if temporarily all eigenvalues of $A$ are assumed to be simple, then (1.1) holds, and taking residues in (1.4) yields (1.1). The general case (1.1) then follows from this by continuity.

The history of identity (1.1) is rather involved as pointed out in great detail in [5] and hence we refer to the extensive list of references therein. (Here we just note that, as discussed in [5], a related identity was already mentioned by Jacobi [9] in 1834 and 100 years later again by Löwner [15]).

In the next section we demonstrate how to extend this circle of ideas to a class of self-adjoint second-order differential operators on a compact interval.

2 The case of one-dimensional Dirichlet Schrödinger operators on a compact interval

Identity (1.1) for self-adjoint $n \times n$ matrices can be extended to one dimensional self-adjoint Dirichlet Schrödinger operators on compact intervals as follows.

Consider Schrödinger operators on a compact interval $[a, b]$ with potential $V$ satisfying

$$V \in L^1((a, b); dx), \quad V \text{ real-valued a.e. on } (a, b).$$

(2.1)

The pair of matrices $(A, M_j)$ in Section 1 is then replaced by a pair of Schrödinger operators $(H^D, H^D_{x_0})$ in $L^2((a, b); dx)$, $x_0 \in (a, b)$, given by
\[
(H^D f)(x) = -f''(x) + V(x)f(x), \quad x \in (a, b),
\]
\[
f \in \text{dom}(H^D) = \{ f \in L^2(a, b); dx \mid f, f' \in AC([a, b]); f(a) = 0 = f(b); \}
\[
\quad (-f'' + Vf) \in L^2(a, b); dx \},
\]
\[
(H^D f)(x) = -f''(x) + V(x)f(x), \quad x \in (a, x_0) \cup (x_0, b),
\]
\[
f \in \text{dom}(H^D) = \{ f \in L^2(a, b); dx \mid f, f' \in AC([a, b]); f(a) = 0 = f(b); \}
\[
\quad f(x_0) = 0; (-f'' + Vf) \in L^2(a, b); dx \},
\]
in particular, \(H^D\) is thus a direct sum of two Dirichlet Schrödinger operators in \(L^2((a, b); dx) \simeq L^2((a, x_0); dx) \oplus L^2((x_0, b); dx)\).

Denoting by \(G^D(z, x, x') = (H^D - zI)^{-1}(x, x')\), \(x, x' \in [a, b]\), the Green’s function (i.e., integral kernel of the resolvent) of \(H^D\), and by \(G^D(x_0, z, x')\) that of \(H^D_{x_0}\), one verifies (the Krein-type resolvent formula, see, e.g., [7])
\[
G^D_{x_0}(z, x, x') = G^D(z, x, x') - [G^D(z, x_0, x_0)]^{-1} G^D(z, x_0) G^D(z, x_0, x'),
\]
\[
z \in \mathbb{C} \setminus (\sigma(H^D) \cup \sigma(H^D)), \quad (2.4)
\]
and hence the fact
\[
\text{tr} \left( (H^D_{x_0} - zI)^{-1} - (H^D - zI)^{-1} \right) = -\frac{d}{dz} \ln \left( G^D(z, x_0, x_0) \right),
\]
\[
z \in \mathbb{C} \setminus (\sigma(H^D) \cup \sigma(H^D)), \quad (2.5)
\]
Here we employed \((d/dz)(H^D - zI)^{-1} = (H^D - zI)^{-2}\) and the fact that the second term on the right-hand side of (2.4) represents the integral kernel of a rank-one operator. To compute \(\text{tr}((H^D_{x_0} - zI)^{-1})\) and \(\text{tr}((H^D - zI)^{-1})\) as the integral of the diagonal Green’s functions (i.e., integral kernels on the diagonal) one first assumes that \(z < 0\) is sufficiently negative to render \((H^D_{x_0} - zI)^{-1}\) and \((H^D - zI)^{-1}\) nonnegative operators in \(L^2((a, b); dx)\) and applies [19, p. 65–66], or Mercer’s theorem as in [4, Proposition 5.6.9], followed by analytic continuation with respect to \(z \in \mathbb{C} \setminus (\sigma(H^D) \cup \sigma(H^D))\).

A comparison of (2.5) with the standard trace formula
\[
\text{tr} \left((B - zI)^{-1} - (B_0 - zI)^{-1}\right) = -\frac{d}{dz} \ln(D_{B_0, B}(z)), \quad (2.6)
\]
where \(B - B_0\) is trace class and \(D_{B_0, B}(\cdot)\) denotes the (relative) Fredholm determinant
\[
D_{B_0, B}(z) = \det \left((B - zI)(B_0 - zI)^{-1}\right) = \det \left(I + (B - B_0)(B_0 - zI)^{-1}\right) \quad (2.7)
\]
for the pair of operators \((B_0, B)\) (this extends to much more general situations where only \([B - zI)^{-1} - (B_0 - zI)^{-1}\] is trace class), shows that \(G^D(z, x_0, x_0)\) is the precise
analog of the Fredholm determinant for the pair \( (H^D, H^D_{x_0}) \). In particular, \( G^D(z, x_0, x_0) \) is the Schrödinger operator analog of

\[
\det_{C^{n-1}}(M_j - zI_{n-1})/\det_{C^r}(A - zI_n)
\]

in (1.3).

Denoting

\[
\sigma(H^D) = \{ \lambda_k(H^D) \}_{k \in \mathbb{N}},
\]

(2.9)

(all eigenvalues of \( H^D \) are necessarily simple in this self-adjoint context) the analytic structure of \( G^D(z, x, x') \) is of the form

\[
G^D(z, x, x') = \sum_{k \in \mathbb{N}} (\lambda_k(H^D) - z)^{-1} e_k(x)e_k(x'), \quad z \in \mathbb{C} \setminus \sigma(H^D), \ x, x' \in [a, b],
\]

with \( \{e_k(x)\}_{k \in \mathbb{N}} \subset L^2((a, b); dx) \) the normalized eigenfunctions of \( H^D \) associated with the eigenvalues \( \lambda_k(H^D) \) of \( H^D \). Moreover, the Green’s function of any such Schrödinger operator is of the form

\[
G^D(z, x, x') = W(\psi_a(z, \cdot), \psi_a(z, \cdot))^{-1}\left\{ \psi_a(z, x)\psi_b(z, x'), \ a \leq x \leq x' \leq b, \right.
\]

\[
\left. \psi_a(z, x')\psi_b(z, x), \ a \leq x' \leq x \leq b, \right\}
\]

(2.11)

where \( \psi_a(z, x) \) (resp., \( \psi_b(z, x) \)) are distributional solutions of \( H^D\psi(z, \cdot) = z\psi(z, \cdot) \), entire in \( z \) for fixed \( x \in [a, b] \), satisfying the Dirichlet boundary condition at \( a \), respectively, at \( b \), that is,

\[
\psi_a(z, a) = 0, \quad \psi_b(z, b) = 0, \quad z \in \mathbb{C}.
\]

(2.12)

Here \( W(f, g) = f(x)g'(x) - f'(x)g(x), \ x \in [a, b] \), denotes the Wronskian of \( f \) and \( g \) (with \( f, g \) absolutely continuous on \([a, b]\)).

Denoting

\[
\sigma(H^D_{x_0}) = \{ \lambda_k(H^D_{x_0}) \}_{k \in \mathbb{N}},
\]

(2.13)

and abbreviating the multiplicity of \( \lambda_k(H^D_{x_0}) \) by \( m^D_{k}(x_0) \in \{1, 2\} \), a first analog of (1.1) then reads as follows.

**Theorem 2.1** Assume (2.1) and introduce \( H^D, H^D_{x_0}, \sigma(H^D), \sigma(H^D_{x_0}) \) as in (2.1)–(2.3), (2.9), (2.13) and suppose that \( 0 \not\in \sigma(H^D) \cap \sigma(H^D_{x_0}) \). Then,
The case of one-dimensional Schrödinger operators

The case of one-dimensional Schrödinger operators

Proof Since for fixed $x \in [a, b]$, $\psi_a(z, x), \psi_b(z, x)$ and $W(\psi_a(z, \cdot), \psi_b(z, \cdot))(x)$ are entire functions with respect to $z$ of order 1/2 (cf. [10]), $G^D(z, x_0, x_0)$ has a product representation of the form

\[
G^D(z, x_0, x_0) = \prod_{\ell \in \mathbb{N}} \left(1 - \frac{z}{\lambda_{\ell}(H^D)}\right)^{m_{\ell}(x_0)} \prod_{j \in \mathbb{N}} \left(1 - \frac{z}{\lambda_j(H)}\right)^{m_j(x_0)} \times \lambda_k(H^D),
\]

as $G^D(z, x_0, x_0)$ is meromorphic in $z$ with simple poles at $\sigma(H^D)$ and at most double zeros at $\sigma(H_{x_0})$. Thus, (2.10) and (2.15) imply

\[
e_k(x_0)^2 = \left[ \lim_{z \to -\infty} |z|^{-1/2} \prod_{\ell \in \mathbb{N}} \left(1 - \frac{z}{\lambda_{\ell}(H^D)}\right)^{m_{\ell}(x_0)} \prod_{j \in \mathbb{N}} \left(1 - \frac{z}{\lambda_j(H)}\right)^{m_j(x_0)} \prod_{m \in \mathbb{N} \setminus \{j\}} \left(1 - \frac{z}{\lambda_m(H)}\right)^{m_m(x_0)} \right]^2
\]

\[
\prod_{\ell \in \mathbb{N}} \left(1 - \frac{z}{\lambda_{\ell}(H_{x_0})}\right)^{m_{\ell}(x_0)} \prod_{j \in \mathbb{N}} \left(1 - \frac{z}{\lambda_j(H)}\right)^{m_j(x_0)} \times \lambda_k(H^D)\]

\[
G^D(z, x_0, x_0) = C(x_0) \prod_{\ell \in \mathbb{N}} \left(1 - \frac{z}{\lambda_{\ell}(H_{x_0})}\right)^{m_{\ell}(x_0)} \prod_{j \in \mathbb{N}} \left(1 - \frac{z}{\lambda_j(H)}\right)^{m_j(x_0)} \times \lambda_k(H^D),
\]

\[
G^D(0, x_0, x_0) = G^D(0, x_0, x_0),
\]
To determine the normalization constant $C(x_0)$ from the two sets of eigenvalues (2.9), (2.13) one can use the representation (2.15) and the asymptotics (see, [7, Eqs. (3.28), (A.40)] and the references cited therein), which yield

\[ e_k(x_0)^2 = \lim_{z \to \lambda_j(H^D)} \left[ \lambda_k(H^D) - z \right] G^D(z, x_0, x_0) \]
\[ = C(x_0) \lim_{z \to \lambda_k(H^D)} \left[ \lambda_k(H^D) - z \right] \prod_{j \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_j(H^D)} \right) \frac{1}{\prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{z}{\lambda_m(H^D)} \right) \left[ 1 - \frac{z}{\lambda_k(H^D)} \right]} \]
\[ = C(x_0) \lambda_k(H^D) \prod_{j \in \mathbb{N}} \left( 1 - \frac{\lambda_k(H^D)}{\lambda_j(H^D)} \right) \frac{1}{\prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{\lambda_k(H^D)}{\lambda_m(H^D)} \right)} \]  

(2.17)

To determine the normalization constant $C(x_0)$ from the two sets of eigenvalues (2.9), (2.13) one can use the representation (2.15) and the asymptotics (see, [7, Eqs. (3.28), (A.40)] and the references cited therein),

\[ \lim_{z \to -\infty} |z|^{1/2} G^D(z, x_0, x_0) = 1, \]  

(2.18)

which yield

\[ C(x_0) = \lim_{z \to -\infty} |z|^{-1/2} \prod_{n \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_n(H^D)} \right) \prod_{j \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_j(H^D)} \right) \]  

\[ \prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{z}{\lambda_m(H^D)} \right)^{m_p(x_0)} \]  

(2.19)

**Remark 2.1** If $0 \in \sigma(H^D) \cup \sigma(H^D_{x_0})$, one can always shift $V$ by a sufficiently small additive constant so that $0$ does not belong to $\sigma(H^D) \cup \sigma(H^D_{x_0})$. Alternatively, one can derive the analog of (2.14) directly as follows:

(i) Suppose $\lambda_{k_0}(H^D) = 0$ and $0 \notin \sigma(H^D_{x_0})$. Then the same considerations as in the proof of Theorem 2.1 yield
The case of one-dimensional Schrödinger operators

(ii) Suppose \( \lambda_{k_0} (H^D) = 0 \) and \( \lambda_{\ell_0} (H^D_{x_0}) = 0 \). Then 0 is a twice degenerate eigenvalue of \( H^D_{x_0} \) and

\[
e_{k_0}(x_0) = 0,
\]

\[
e_{k_0}^2(x_0) = \lim_{z \to -\infty} |z|^{-1/2} \prod_{n \in \mathbb{N} \setminus \{k_0\}} \left( 1 - \frac{z}{\lambda_n (H^D)} \right) \prod_{\ell \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_{\ell} (H^D_{x_0})} \right)^{m^\epsilon_{\ell_0}(x_0)},
\]

\[
(\text{2.20})
\]

(iii) Suppose \( 0 \notin \sigma(H^D) \) and \( \lambda_{\ell_0} (H^D_{x_0}) = 0 \). Then 0 is a simple eigenvalue of \( H^D_{x_0} \) and

\[
e_k(x_0)^2 = \lim_{z \to -\infty} |z|^{-3/2} \prod_{n \in \mathbb{N} \setminus \{k_0\}} \left( 1 - \frac{z}{\lambda_n (H^D)} \right) \prod_{\ell \in \mathbb{N} \setminus \{\ell_0\}} \left( 1 - \frac{z}{\lambda_{\ell} (H^D_{x_0})} \right)^{m^\epsilon_{\ell}(x_0)} \prod_{j \in \mathbb{N} \setminus \{\ell_0\}} \left( 1 - \frac{\lambda_j (H^D_{x_0})}{\lambda^D_j (H^D_{x_0})} \right)^{m^\epsilon_j(x_0)} \prod_{m \in \mathbb{N} \setminus \{k_0, k\}} \left( 1 - \frac{\lambda_k (H^D)}{\lambda_m (H^D)} \right), \quad k \in \mathbb{N} \setminus \{k_0\}.
\]

\[
(\text{2.21})
\]
\[ \begin{align*}
e_k(x_0)^2 &= - \left[ \lim_{\varepsilon \to 0} \left| \varepsilon \right|^{-3/2} \prod_{n \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_n(H^D)} \right) \prod_{\ell \in \mathbb{N}\setminus\{\ell_0\}} \left( 1 - \frac{z}{\lambda_{\ell}(H^D)} \right) \prod_{j \in \mathbb{N}\setminus\{k\}} \left( 1 - \frac{\lambda_k(H^D)}{\lambda_j(H^D)} \right) \right] \prod_{m \in \mathbb{N}\setminus\{k\}} \left( 1 - \frac{\lambda_k(H^D)}{\lambda_m(H^D)} \right), \quad k \in \mathbb{N}. \end{align*} \] (2.22)

Before continuing this discussion we consider an exactly solvable example next.

**Example 2.1** Let \( V = 0 \) a.e. on \((a, b)\), and introduce in \( L^2((a, b); dx) \),

\( (H^D_0f)(x) = -f''(x), \quad x \in (a, b), \)

\( f \in \text{dom}(H^D_0) = \{ f \in L^2((a, b); dx) | f, f' \in AC([a, b]); f(a) = 0 = f(b); f'' \in L^2((a, b); dx) \} \)

\[ = H^2((a, b)) \cap H^1_0((a, b)), \] (2.23)

\( (H^D_{0,x_0}f)(x) = -f''(x), \quad x \in (a, x_0) \cup (x_0, b), \)

\( f \in \text{dom}(H^D_{0,x_0}) = \{ f \in L^2((a, b); dx) | f, f' \in AC([a, b]); f(a) = 0 = f(b); \}

\( \quad f(x_0) = 0; f'' \in L^2((a, b); dx) \} \).

Then one obtains,

\[ G^D_0(z; x, x') = \frac{1}{z^{1/2} \sin \left( z^{1/2}(b - a) \right)} \times \left\{ \begin{array}{ll}
\sin \left( z^{1/2}(x - a) \right) \sin \left( z^{1/2}(b - x') \right), & a \leq x \leq x' \leq b, \\
\sin \left( z^{1/2}(x' - a) \right) \sin \left( z^{1/2}(b - x) \right), & a \leq x' \leq x \leq b,
\end{array} \right\} \] (2.25)

\[ C_0(x) = G^D_0(0, x, x) = (x - a)(b - x)/(b - a), \] (2.26)

\[ \sigma(H^D_0) = \{ [k\pi/(b - a)]^2 \}_{k \in \mathbb{N}^+}, \] (2.27)

\[ e_{0,k}(x) = [2/(b - a)]^{1/2} \sin(k\pi(x - a)/(b - a)), \quad a \leq x \leq b, \] (2.28)
\[ \sigma(H_{0,x_0}^D) = \{ [j\pi/(x_0 - a)]^2 \}_{j \in \mathbb{N}} \cup \{ [\ell\pi/(b - x_0)]^2 \}_{\ell \in \mathbb{N}}, \quad x_0 \in (a, b). \] (2.29)

Applying (2.17) one computes
\[ e_{0,k}(x_0)^2 = \frac{(x_0 - a)(b - x_0)}{(b - a)} \frac{(k\pi)^2}{(b - a)^2} \prod_{j \in \mathbb{N}} \left( 1 - \frac{k^2(x_0 - a)^2}{j^2(b - a)^2} \right) \times \prod_{\ell \in \mathbb{N}} \left( 1 - \frac{k^2(b - x_0)^2}{\ell^2(b - a)^2} \right) / \prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{k^2}{m^2} \right) \]
\[ = \frac{(x_0 - a)(b - x_0)}{(b - a)} \frac{(k\pi)^2}{(b - a)^2} \times \frac{\sin(k\pi(x_0 - a)/(b - a))}{k\pi(x_0 - a)/(b - a)} \frac{\sin(k\pi(b - x_0)/(b - a))}{k\pi(b - x_0)/(b - a)} \times \left[ \prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{k^2}{m^2} \right) \right]^{-1} \]
\[ = \left[ 2/(b - a) \right] \left[ \sin(k\pi(x_0 - a)/(b - a)) \right]^2, \quad x_0 \in [a, b], \]
confirming (2.28).

Here we employed \( \sin(\zeta) = \zeta \prod_{j \in \mathbb{N}} \left[ 1 - \zeta^2 j^{-2} \pi^{-2} \right] \), \( \zeta \in \mathbb{C} \), which also yields the following identity, required to arrive at (2.30),
\[ \prod_{m \in \mathbb{N} \setminus \{k\}} \left( 1 - \frac{k^2}{m^2} \right) = (-1)^{k+1}/2, \quad k \in \mathbb{N}. \] (2.31)

Next, we employ Example 2.1 to elaborate on the computation of \( C(x_0) \) in (2.19). In this manner the squared eigenvectors \( e_k(x_0)^2 \) can be expressed in terms of the two sets of eigenvalues (2.9), (2.13) somewhat more explicitly.

**Theorem 2.2** Assume (2.1) and introduce the linear operators \( H^D, H_{x_0}^D, H_{0}^D, H_{0,x_0}^D \), and the spectra \( \sigma(H^D) \), \( \sigma(H_{x_0}^D) \), \( \sigma(H_{0}^D) \), \( \sigma(H_{0,x_0}^D) \) as in (2.1)–(2.3), (2.9), (2.13), (2.23), (2.24), (2.27), and (2.29). Then,
\[ e_k(x)^2 = \frac{(x_0 - a)(b - x_0)}{(b - a)} \frac{\lambda_k(H_{0,x_0}^D)}{\lambda_k(H_{0}^D)} \frac{\prod_{j \in \mathbb{N}} \left( \lambda_j(H_{0,x_0}^D) - \lambda_k(H_{0,x_0}^D) \right) \prod_{m \in \mathbb{N} \setminus \{k\}} \left( \lambda_m(H_{0,x_0}^D) - \lambda_k(H_{0,x_0}^D) \right)}{\prod_{j \in \mathbb{N}} \left( \lambda_j(H_{0}^D) - \lambda_k(H_{0}^D) \right) \prod_{m \in \mathbb{N} \setminus \{k\}} \left( \lambda_m(H_{0}^D) - \lambda_k(H_{0}^D) \right)}. \] (2.32)

**Proof** Recalling (2.27) and (2.29), \( 0 \notin \sigma(H_{0,x_0}^D) \cap \sigma(H_{0,x_0}^D) \), and \( \lambda_k(H_{0,x_0}^D) = [k\pi/(b - a)]^2, \quad k \in \mathbb{N} \). Thus, employing [13, Eq. (1.6.6)], one has
\[ \lambda_k(H_{0,x_0}^D)^{1/2} = \frac{k\pi}{b - a} + O\left( \frac{1}{k} \right) \] (2.33)
and hence
\[ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{0,0})} \xrightarrow{k \to \infty} 1 + O\left(\frac{1}{k^2}\right). \]  
(2.34)

Similarly, labeling the eigenvalues of \( H^D_{0,0} \) and \( H^D_{0,x_0} \) in a convenient manner so that the even ones correspond to the problem on \((a, x_0)\) and the odd ones to the problem on \((x_0, b)\) (e.g., \( \lambda_{2k}(H^D_{0,0}) = [k\pi/(x_0 - a)]^2 \), \( \lambda_{2k-1}(H^D_{0,0}) = [k\pi/(b - x_0)]^2 \), \( k \in \mathbb{N} \)), one gets
\[ \frac{\lambda_k(H^D_{0,x_0})}{\lambda_k(H^D_{0,0})} \xrightarrow{k \to \infty} 1 + O\left(\frac{1}{k^2}\right). \]  
(2.35)

Then it follows that the infinite products
\[ \prod_{k \in \mathbb{N}} \left[ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{0,0})} \right] \quad \text{and} \quad \prod_{k \in \mathbb{N}} \left[ \frac{\lambda_k(H^D_{0,x_0})}{\lambda_k(H^D_{0,0})} \right] \]  
(2.36)
converge to finite nonzero values and moreover one has
\[ \lim_{n \to \infty} \prod_{k=n}^{\infty} \min \left\{ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{0,0})}, 1 \right\} = 1, \quad \lim_{n \to \infty} \prod_{k=n}^{\infty} \max \left\{ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{0,0})}, 1 \right\} = 1, \]  
\[ \lim_{n \to \infty} \prod_{k=n}^{\infty} \min \left\{ \frac{\lambda_k(H^D_{0,x_0})}{\lambda_k(H^D_{0,0})}, 1 \right\} = 1, \quad \lim_{n \to \infty} \prod_{k=n}^{\infty} \max \left\{ \frac{\lambda_k(H^D_{0,x_0})}{\lambda_k(H^D_{0,0})}, 1 \right\} = 1. \]  
(2.37)

Next, consider the Green’s function \( G_0^D(z, x, x') \) for the operator \( H^D_0 \). Multiplying and dividing by \( G_0^D(z, x_0, x_0) \) under the limit in (2.19) and utilizing (2.15) and (2.18) for \( G_0^D(z, x_0, x_0) \) one obtains (cf. (2.26))
\[ C(x_0) = \lim_{z \downarrow -\infty} G_0^D(z, x_0, x_0) \prod_{k \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_k(H^D)} \right) \prod_{\ell \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_{\ell}(H^D_{0,0})} \right) \]  
\[ = C_0(x_0) \lim_{z \downarrow -\infty} \prod_{k \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_k(H^D_{0,0})} \right) \prod_{\ell \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_{\ell}(H^D_{0,0})} \right) \]  
\[ = C_0(x_0) \lim_{z \downarrow -\infty} \prod_{\ell \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_{\ell}(H^D_{0,0})} \right) \prod_{k \in \mathbb{N}} \left( 1 - \frac{z}{\lambda_k(H^D_{0,0})} \right). \]  
(2.38)
Without loss of generality we assumed implicitly in arriving at (2.38) that \(0 \not\in \sigma(H^D) \cap \sigma(H^D_{x_0})\). (Again, one could temporarily shift \(V\) by a sufficiently small additive constant and note that (2.32) only depends on differences of eigenvalues of \(H^D\), resp., \(H^D_{x_0}\).)

Next, we note that for any \(n \in \mathbb{N}\),

\[
\lim_{z \to \infty} \prod_{\ell = 1}^{n} \frac{1 - \left(z/\lambda_\ell(H^D_{x_0})\right)}{1 - \left(z/\lambda_\ell(H^D)\right)} = \prod_{k = 1}^{n} \frac{\lambda_k(H^D_{x_0})}{\lambda_k(H^D)}. \tag{2.39}
\]

For all sufficiently large \(\ell, k\) the eigenvalues \(\lambda_\ell(H^D_{x_0}), \lambda_k(H^D), \lambda_\ell(H^D_{x_0}), \lambda_k(H^D)\) are positive and hence for all \(z < 0\) we have the estimates

\[
\min \left\{ \frac{\lambda_\ell(H^D_{x_0})}{\lambda_\ell(H^D)}, 1 \right\} \leq \frac{1 - \left(z/\lambda_\ell(H^D_{x_0})\right)}{1 - \left(z/\lambda_\ell(H^D)\right)} \leq \max \left\{ \frac{\lambda_\ell(H^D_{x_0})}{\lambda_\ell(H^D)}, 1 \right\}, \tag{2.40}
\]

\[
\min \left\{ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{x_0})}, 1 \right\} \leq \frac{1 - \left(z/\lambda_k(H^D_{x_0})\right)}{1 - \left(z/\lambda_k(H^D)\right)} \leq \max \left\{ \frac{\lambda_k(H^D)}{\lambda_k(H^D_{x_0})}, 1 \right\}. \tag{2.41}
\]

Combining these estimates with (2.37) and (2.39) then shows that the limit in (2.38) can be evaluated term by term. In addition, noting that \(C_0(x) = G^D_0(0, x, x) = (x-a)(b-x)/(b-a)\) by (2.26), one obtains

\[
C(x_0) = \frac{(x_0-a)(b-x_0)}{(b-a)} \prod_{\ell \in \mathbb{N}} \frac{\lambda_\ell(H^D_{x_0})}{\lambda_\ell(H^D)} / \prod_{k \in \mathbb{N}} \frac{\lambda_k(H^D_{x_0})}{\lambda_k(H^D)}. \tag{2.42}
\]

Finally, combining (2.42) with (2.17) yields (2.32). \(\square\)

**Remark 2.2** (i) This is just the tip of the iceberg as more general situations can be discussed along similar lines: Three-coefficient Sturm–Liouville operators in \(L^2((a, b); r(x)dx)\) generated by differential expressions of the type

\[
r(x)^{-1}[(d/dx)p(x)(d/dx) + q(x)]; \tag{2.43}
\]

non-self-adjoint operators as long as the analog of \(\lambda_k(H^D)\) has algebraic multiplicity equal to one (i.e., the associated Riesz projection is one-dimensional); other one-dimensional systems such as Jacobi operators, CMV operators, etc.; other boundary conditions (Neumann, Robin, etc.); in principle, this circle of ideas extends to some higher-order one-dimensional systems.

(ii) The standard results on “two spectra determine the potential uniquely” (see \([1, 2, 10]\)) and the corresponding reconstruction results of the potential (cf. \([11, 12, \text{Ch. 3}], [13, 14, \text{Sects. 6.9, 6.11}], [16, 17, \text{Sect. 3.4}]\)) differ from the results discussed in this note as the “two-spectra results” vary the boundary condition on one end, but keep the one at the opposite endpoint fixed. However, the “three spectra results” in
are related as they include as a special case the situation of Dirichlet boundary conditions on the intervals \((a, b), (a, x_0)\) and \((x_0, b)\) associated with the Dirichlet Schrödinger operators \(H^D\) and \(H^D_{(a, x_0)}, H^D_{(x_0, b)}\), where \(H^D_{x_0} = H^D_{(a, x_0)} \oplus H^D_{(x_0, b)}\) and \(H^D_{(a, x_0)}\) and \(H^D_{(x_0, b)}\) represent the Dirichlet Schrödinger operators associated with the intervals \((a, x_0)\) and \((x_0, b)\), respectively. The result in \([3, 8, 18]\) most relevant to our situation then reads as follows:

Suppose that the three sets \(\sigma(H^D), \sigma(H^D_{(a, x_0)}), \text{ and } \sigma(H^D_{(x_0, b)})\) are mutually disjoint. Then \(\sigma(H^D), \sigma(H^D_{(a, x_0)}), \text{ and } \sigma(H^D_{(x_0, b)})\) determine \(V(.)\) uniquely a.e. on \([a, b]\). (It can be shown that there are counterexamples to this statement if the disjointness condition of the three spectra is violated, see \([8]\)).

(iii) The idea of using spectra associated with the pair of operators \((H^D_{x_0}, H^D)\) is closely related to the concept of Krein’s spectral shift function \(\xi(\cdot; H^D, H^D_{x_0})\) for this pair. The latter function is explicitly given by

\[
\xi(\lambda;H^D, H^D_{x_0}) = \pi^{-1} \arg \left( \lim_{\varepsilon \downarrow 0} (G^D(\lambda + i\varepsilon, x, x)) \right), \quad x \in (a, b), \text{ for a.e. } \lambda \in \mathbb{R},
\]

such that (2.5) can be complemented by

\[
\text{tr}\left( (H^D_{x_0} - zI)^{-1} - (H^D - zI)^{-1} \right) = -\frac{d}{dz} \ln \left( G^D(z, x_0, x_0) \right),
\]

\[
= -\int_{E_0}^{\infty} \frac{\xi(\cdot; H^D, H^D_{x_0}) d\lambda}{(\lambda - z)^2},
\]

\[
E_0 = \inf \left( \sigma(H^D) \right), \quad z \in \mathbb{C}\setminus(\sigma(H^D) \cup \sigma(H)).
\]

(2.45)

In the compact interval case at hand, \(\xi(\cdot; H^D, H^D_{x_0})\) is a piecewise constant function, normalized to be 0 for \(\lambda < E_0\), jumping by 1 (resp., -1) at each eigenvalue of \(H^D\) (resp., \(H^D_{x_0}\)). One can now use the exponential Herglotz–Nevanlinna representation of \(\ln \left( G^D(z, x_0, x_0) \right)\) in terms of the measure \(\xi(\cdot; H^D, H^D_{x_0}) d\lambda\) to obtain an alternative derivation of \(G^D(z, x_0, x_0)\) as in (2.15), (2.19). In the related situation where \((a, b) = \mathbb{R}\) and \(V\) is continuous and bounded from below, this circle of ideas has been used in \([6]\) to derive a trace formula for the potential \(V(x)\) in terms of \(\xi(\cdot; H^D, H^D_{x})\), \(x \in \mathbb{R}\); moreover, a variety of inverse spectral questions associated with half-line and real line problems involving Krein’s spectral shift function were discussed in \([7]\). ☜

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