Two Applications of Nilpotent Higgsing in Class-S

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Abstract

We introduce a class of Higgs-branch RG flows in theories of class-S, which flow between $d = 4$ $\mathcal{N} = 2$ SCFTs of the same ADE type. We discuss two applications of this class of RG flows: 1) determining the current-algebra levels in SCFTs where they were previously unknown — a program we carry out for the class-S theories of type $E_6$ and $E_7$ — and 2) constructing a multitude of examples of pairs of $\mathcal{N} = 2$ SCFTs whose “conventional invariants” coincide. We disprove the conjecture of [1] that the global form of the flavour symmetry group is a reliable diagnostic for determining when two such theories are isomorphic.
1. Introduction

In some sense, “most” quantum field theories are non-Lagrangian. This presents a challenge to understanding their properties. In the case of $d = 4$, $\mathcal{N} = 2$ superconformal field theories of class-S, considerable progress can be made. These theories are obtained as the compactification of a 6d (2,0) SCFT on a Riemann surface, $C_{g,n}$, with punctures labeled by nilpotent orbits in the appropriate Lie algebra. For each pants-decomposition of $C_{g,n}$, we have a presentation of the SCFT as a gauging (with vanishing (super-)Yang-Mills $\beta$-function) of a product of SCFTs associated to the compactification on 3-punctured spheres (“fixtures”). Different pants-decompositions yield different (“S-dual”) presentations of the same SCFT. The central task, then, is to understand the properties of the SCFTs associated to fixtures.

Many of those properties are calculable, in algorithmic fashion, from the data of the (2,0) theory and the choice of a triple of nilpotent orbits. But others have proven elusive. One example concerns the current-algebra levels associated to the “enhanced” flavour symmetries (enhanced relative to the manifest flavour symmetries associated to the nilpotent orbits at the punctures) of some of these SCFTs. In most cases, these levels can be determined from the current algebra levels of the manifest flavour symmetries. But the authors of [2–5] found a long list of fixtures in the class-S theories of type $E_6$, $E_7$, $E_8$ for which some of the levels could not be so-determined. In §3, we rectify this by determining all of the missing levels in type $E_6$ and all but four of them in type $E_7$. Our results are tabulated in §3.3 and the online application for the $E_7$ theory has been updated accordingly.

The key tool which allows this progress is the Higgs-branch RG flows between SCFTs in class-S. One well-known class of such RG flows is known as partial puncture closure, where one starts with the full puncture (which has a manifest $g$ flavour symmetry) and turns on a VEV for the highest root moment map of some $\mathfrak{su}(2) \subset g$. After decoupling some Nambu-Goldstone bosons, the resulting SCFT is the one associated to replacing the full puncture with the puncture associated to the nilpotent orbit determined (via Jacobson-Morozov) by that $\mathfrak{su}(2)$ embedding (see [6] for a review).

Unfortunately, that Higgsing does not yield any useful information for us. A more useful one, introduced in [7] and further explored in [8] is one where we turn on a VEV for the highest root moment map for some simple factor $f_i$ in the flavour symmetry of the theory. As with partial puncture closure, the resulting RG flow is realized in the VOA as Drinfeld-Sokolov reduction [9, 7]. This has implications for the IR SCFT: certain quantities are RG-invariant and are the same in the UV and IR SCFTs; other quantities change, but in computable ways, in flowing from the UV to the IR.

As a second application of these ideas, we revisit a question posed in [1]: how can we determine whether two $\mathcal{N} = 2$ SCFTs are isomorphic? That paper found examples of pairs of SCFTs whose easy-to-compute (“conventional”) invariants are identical, but where the SCFTs themselves are in fact not isomorphic. That paper suggested that the global form of the flavour symmetry group, $F$, (supplementing the conventional invariants) might be an effective diagnostic for determining whether two theories are isomorphic.

In §4 we examine this proposal further, by studying families of pairs of theories (whose conventional invariants coincide) which are related by a sequence of Higgs branch RG flows.
Generalizing the proposal of [10] for determining the global form of $F$ to include the $\mathbb{Z}_2$ symmetry $e^{2\pi i(R+j_1+j_2)}$ (which is the center of the $\mathcal{N} = 2$ superconformal supergroup), we find that the conjecture of [1] seems to hold in many examples. Unfortunately, the same technique also generates a family of counter-examples: pairs of theories whose conventional invariants and global form of the flavour symmetry coincide, but which are nonetheless non-isomorphic.

2. Nilpotent Higgsings

Let $\mathfrak{f} \subset \mathfrak{f}_T$ be a simple subalgebra of the flavour symmetry of some SCFT, $\mathcal{T}$, of class-S. We will restrict ourselves to the case where $\mathfrak{f}$ is a simple factor in the manifest flavour symmetry associated to a puncture, $O$. Turning on a VEV for the moment map ($\hat{B}_1$ operator) which is the highest root of $\mathfrak{f}$ induces an RG flow which, in favourable circumstances [8], yields in the infrared another class-S SCFT, $\mathcal{T}'$, with the puncture $O$ replaced by the puncture $O'$, where the nilpotent orbit $O$ lies in the closure of the orbit $O'$. The key observation of [7] is that the flow from $\mathcal{T}$ to $\mathcal{T}'$ is implemented on the level of the chiral algebras as Drinfeld-Sokolov reduction.

This has several implications [7,8]:

1. The rest of the current algebra of $\mathcal{T}$ is unmolested. In particular the flavour symmetry algebra of $\mathcal{T}'$, $\mathfrak{f}_{\mathcal{T}'}$, is some infrared enhancement of $\mathfrak{f}_T/\mathfrak{f}$.

2. The rank of the theory is either preserved ($k_\mathfrak{f}$ odd) or decreases by 1 ($k_\mathfrak{f}$ even). More specifically,

- When $k$ is even, the dimension of the Coulomb branch decreases by one. We lose a Coulomb branch parameter, $\varepsilon$, with $\Delta(\varepsilon) = k/2$.
- When $k$ is odd, the dimension of the Coulomb branch is preserved, But a Coulomb branch parameter, $\varepsilon$, with $\Delta(\varepsilon) = k - 1$, is replaced by $\varepsilon'$, where $\varepsilon = (\varepsilon')^2$ (i.e. $\Delta(\varepsilon') = (k - 1)/2$). The nilpotent orbits in question lie in the same special piece (i.e., $d(O) = d(O')$), such that the non-special orbit is Higgsed to the special one.

3. $n_v$ decreases by $k_\mathfrak{f} - 1$.

4. The (quaternionic) dimension of the Higgs branch decreases by $h^\vee(\mathfrak{f}) - 1$.

The diagram of nilpotent Higgsings for $E_6$ is

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$^1$This was the key observation in [7], though there is was expressed in terms of $\delta c$ and $\delta \dim_{\mathbb{H}}(\text{Higgs})$.  

2
This is just a decorated version of the Hasse diagram for nilpotent orbit closure, flipped on its head and decorated by the $f_k$ which get Higgsed at each stage. A nilpotent Higgsing from $O$ to $O'$ is denoted by a downward-directed solid edge from $O$ to $O'$. A dashed edge indicates that the nilpotent orbit $O$ is contained in the closure of the orbit $O'$, though there’s no corresponding nilpotent Higgsing. Some of the nilpotent orbits in the diagram are denoted in red. These have a simple factor in their flavour symmetry (e.g. $SU(3)_{24} \subset SU(3)_{24} \times SU(2)_{13}$, in the case of $3A_1$) which cannot be Higgsed in this fashion. More precisely, the Higgsing (whether or not it yields a nontrivial SCFT in the IR) does not keep us within this family of class-S theories.

The solid edges in the diagram are the ones which correspond to minimal singularities in the work of [11]; the transverse slice to the singularity is the minimal nilpotent orbit for the (complexified) Lie algebra listed. In the classical Lie algebras [12,13], the transverse slices are either minimal nilpotent orbits or ADE surface singularities $\mathbb{C}^2/\Gamma$. For nilpotent orbit
closures in the exceptional Lie algebras, the classification is more complicated [11].

While the dimension of the minimal nilpotent orbit in \( \mathfrak{f} \) is also \( h^\vee(\mathfrak{f}) - 1 \), there isn’t an a-priori connection between our Higgsings and the nilpotent orbit closures considered by Kraft-Procesi and Fu et al. Here, when we Higgs \( O \xrightarrow{\mathcal{F}_k} O' \), the dimension of the Higgs branch decreases by \( h^\vee(\mathfrak{f}) - 1 \). By contrast, the dimension of the nilpotent orbit \( \dim(O') \) is greater than \( \dim(O) \). For these particular orbits, \( \dim_{\mathfrak{g}_k}(O') = \dim_{\mathfrak{g}_k}(O) + (h^\vee(\mathfrak{f}) - 1) \) and the infinitesimal neighbourhood of a generic point in \( O \) is isomorphic to the minimal nilpotent orbit of \( \mathfrak{f} \).

In the twisted sector of \( E_6 \), punctures are labeled by nilpotent orbits in \( F_4 \). The associated diagram of nilpotent Higgsings is

\[
\begin{array}{c}
0 \\
(F_4)_{18} \\
A_1 \\
Sp(3)_{13} \\
A_1 \\
SU(4)_{12} \\
A_1 + \tilde{A}_1 \\
SU(2)_{10} \\
A_2 \\
SU(3)_{16} \\
A_2 + \tilde{A}_1 \\
SU(2)_{7} \\
C_3(a_1) \\
SU(2)_{7} \\
F_4(a_3) \\
B_3 \\
SU(2)_{24} \\
F_4(a_2) \\
SU(2)_{6} \\
F_4(a_1) \\
F_4
\end{array}
\]

The diagram of nilpotent Higgsings for \( E_7 \) is
The diagram for $E_8$ is
3. Determining Unknown Current-Algebra Levels

Fixtures are class-S theories obtained by compactifying the (2,0) theory on a sphere with three punctures. Each puncture has an associated flavour symmetry and the fixture has a flavour symmetry which is (possibly an enhancement of) the product of the flavour symmetries associated to each puncture. The latter (the “manifest” flavour symmetry) embeds as a subalgebra of the full flavour symmetry. The flavour symmetries of fixtures obtained from the exceptional (2,0) theories we will discuss were obtained in [2], [3], and [4].

To each simple flavour symmetry factor one may associate a positive integer $k$ that is the flavour central charge or “level”. The flavour central charge of a simple factor of the flavour symmetry is defined by

$$J^a_\mu(x)J^b_\nu(0) \sim \frac{3k}{4\pi^4}5^{ab}g_{\mu\nu} - \frac{2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}f_{abc}x_\mu x_\nu x \cdot J^c(0) + \ldots$$

where the normalization is such that $k = 1$ for a free half-hypermultiplet in the defining representation of $Sp(n)$. The levels of the manifest flavour symmetries are readily determined from the decomposition of the adjoint representation [14], as described in §2.4.1 of [2].

In most cases, when the flavour symmetry is enhanced, knowing the levels of the “manifest” subalgebra suffices to determine the levels of the full flavour symmetry. There are, however, two notable exceptions.

- When a manifest factor of $G_k$ (we denote the level $k$ by a subscript) is enhanced to $G_{k_1} \times G_{k_2}$, where $G$ is embedded diagonally in $G \times G$. We know that $k_1 + k_2 = k$, but — without more information — we cannot determine $k_1, k_2$ individually.

- When a manifest $U(1)$ factor is enhanced to some nonabelian $G_k$, the freedom to change the normalization of the $U(1)$ generator prevents us, in most cases, from being able to compute the level $k$.

There are a number of fixtures with unknown levels in the exceptional case. In this paper we will determine the vast majority of these levels in the twisted and untwisted $E_6$ case as well as the $E_7$ case. There are four fixtures in the $E_7$ theory that have unknown levels that could not be determined with our methods. However our results do place constraints on some of these levels as well as relate the unknown levels of different fixtures. Finally, we note that when one of the punctures is a simple puncture ($E_n(a_1)$ in the $E_n$ theory), the levels can also be determined using the methods of [15]. In those cases, our results are consistent with those.

3.1. The $E_6$ Theory

There were two fixtures in the untwisted $E_6$ theory with unknown levels
These were fixtures #63, #66 in section 3.4 of [2]. In both cases, the manifest $SU(2)_{54}$ of the $A_2 + 2A_1$ puncture was enhanced to $SU(3)_{54}$ and thence to a product of $SU(3)$s. Hence we know the sum of the $SU(3)$ levels, but not the levels themselves. These fixtures can be obtained by Higgsing

$$A_2 + A_1 \xrightarrow{SU(3)_{12}} A_2 + 2A_1$$

in fixtures #64, #67:

Higgsing the $SU(3)_{12}$ leaves the rest of the flavour symmetry unmolested. In the infrared, that flavour symmetry is enhanced. In the theory on the left, the $SU(2)_{18}^3 \times U(1)^3$ is enhanced $SU(3)_{18}^3 \times U(1)$. In the theory on the right, the $SU(2)_{36} \times SU(2)_{18} \times U(1)^2$ is enhanced to $SU(3)_{36} \times SU(3)_{18} \times U(1)$ (the $SU(2)_9$ is unaffected). So we learn that $k_1 = k_2 = k = 18$.

Moreover, we can Higgs

$$A_3 + A_1 \xrightarrow{SU(2)_{9}} D_4(a_1)$$

(either before or after the first Higgsing) to obtain the fixture on the left from the fixture on the right. The effect is to enhance the $SU(3)_{36}$ to $SU(3)_{18}^2$ in the infrared, consistent with what we found.

A more complicated example (which, to our dismay, reveals a typo\footnote{Another error along these lines was pointed out to us by Martone and Zafrir [16]. Mixed fixture #5} in one entry in the...
tables of [2] is as follows. Consider the 4-punctured sphere

\[ (a_1) \]

\[ A_3 + A_1 \]

\[ Spin(9) \]

\[ 0 \]

\[ E_6(a_1) \]

\[ D_5 \]

\[ (0, Spin(9)) \]

\[ [(E_7)_8] \times [(E_7)_{16} \times SU(2)_9] \]

This is a \( Spin(9) \) gauging of the product the rank-1 and rank-2 \( E_7 \) Minahan-Nemeschansky theories, with an additional hypermultiplet in the 9. The centralizer of \( Spin(9) \) in \( (E_7)_k \) is \( SU(2)_{2k} \times SU(2)_k \), so the flavour symmetry of this family of SCFTs is

\[ F = SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8 \]

S-dualizing, we obtain

\[ A_3 + A_1 \]

\[ A_4 + A_1 \]

\[ SU(2) \]

\[ SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8^2 \]

where the fixture on the right is \#125 from the table in section 3.4 of [2].

Higgsing \( A_4 \xrightarrow{SU(2)_8} A_4 + A_1 \), we obtain fixture \#89:

\[ A_3 + A_1 \]

\[ A_4 + A_1 \]

\[ A_3 + A_1 \]

\[ [SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8] \]

In the table, the flavour symmetry is listed as "\( \cdots \times SU(2)_8^2 \)", which is incorrect. One readily checks from the superconformal index that the enhancement of manifest flavour symmetry of the fixture is \( SU(2)^2 \times U(1)^3 \rightarrow SU(2)^6 \). Here, we’ve determined the levels to be as-stated.

is the rank-2 \( SU(10)_{10} \) SCFT, first found as a fixture in the \( A_4 \) theory (in [17], where it was called "\( S_5 \)") with an additional 9 free hypermultiplets.
As a further consistency check, we can Higgs $A_3 + A_1 \xrightarrow{SU(2)_9} D_4(a_1)$ as before. Before S-dualizing, this corresponds to turning on a VEV for the hypermultiplet in the $(9)$ of $Spin(9)$, which breaks $Spin(9) \rightarrow Spin(8)$:

\begin{align*}
\circ E_6(a_1) & \quad \circ D_4(a_1) \\
(0, Spin(8)) & \quad Spin(8) & \quad 0 \\
\circ A_3 + A_1 & \quad \circ D_5 \\
\emptyset & \quad [(E_7)_8] \times [(E_7)_{16} \times SU(2)_9]
\end{align*}

which has flavour symmetry

$$F = SU(2)^3_1 \times SU(2)_9 \times SU(2)^3_8$$

This theory has two other S-duality frames. In one,

\begin{align*}
\circ E_6(a_1) & \quad \circ D_4(a_1) \\
(0, Spin(9)) & \quad Spin(9) & \quad 0 \\
\circ A_3 + A_1 & \quad \circ D_5 \\
1(9) & \quad [(E_7)_8]^3
\end{align*}

we have Higgsed the rank-2 Minahan-Nemeschansky theory to 2 copies of the rank-1 theory. In the other,

\begin{align*}
\circ E_6(a_1) & \quad \circ D_4(a_1) \\
(A_4, SU(2)) & \quad SU(2) & \quad A_4 \\
\circ D_5 & \quad \circ A_3 + A_1 \\
\emptyset & \quad [SU(2)_8 \times SU(2)^3_1 \times SU(2)_9 \times SU(2)^3_8]
\end{align*}

the fixture on the right is #121. Higgsing $A_4 \xrightarrow{SU(2)_8} A_4 + A_1$, we obtain fixture #85:
Finally, Higgsing the other $A_3 + A_1 \xrightarrow{SU(2)_9} D_4(a_1)$, we obtain the same relationship between fixtures #120 and #84 (whose flavour symmetries are, respectively, $SU(2)^{10}_8$ and $SU(2)^9_8$).

### 3.2. Some Product SCFTs in the $E_7$ Theory

Let us start our examination of the $E_7$ theory with the SCFTs discussed in §3 of [18]. There, we have 8 SCFTs where the $(E_7)_36$ of the full puncture, 0, is enhanced to $(E_7)_{36-k} \times (E_7)_k$. They are of the form

$$[(E_7)_8] \times [(E_7)_{28} \times SU(2)_{10} \times U(1)]$$

where $O$ is one of $D_4$, $D_4 + A_1$, $D_5(a_1)$ or $D_5(a_1) + A_1$. These were all determined to be product SCFTs. When $O = D_5(a_1)$, one of the factors in the product was identified as the $(E_7)_8$ Minahan-Nemenschansky theory. Hence the level of the other $E_7$ factor in the flavour symmetry is $k = 28$.

$$[(E_7)_8] \times [(E_7)_{28} \times SU(2)_{12} \times SU(2)_{10} \times U(1)]$$

From the Higgsing diagram, we see that these are all related by the Higgsings

$$D_4 \xrightarrow{Sp(3)_{12}} D_4 + A_1 \xrightarrow{Sp(2)_{11}} D_5(a_1) \xrightarrow{SU(2)_{10}} D_5(a_1) + A_1$$

So we see that all of these are product SCFTs with $[(E_7)_8]$ as one of the factors and the level of the $E_7$ in the other factor is $k = 28$. 

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3.2.1. S-duality

Having identified the unknown levels in these theories, related by \([2]\), we can exploit S-duality to determine others. Consider

\[
\begin{array}{c}
\circ \\
O_1 \\
\circ \\
O_2 \\
\circ \\
A_6 \\
\circ \\
\end{array}
\]

where \(O_{1,2}\) are again chosen from the set \(\{D_4, D_4 + A_1, D_5(a_1), D_5(a_1) + A_1\}\). There are 10 such theories in all and, in each of them, the \(SU(2)_{36}\) symmetry of the \(A_6\) puncture is enhanced to \(SU(2)_{36-k} \times SU(2)_k\).

Let’s pick one of these theories, say

\[
\begin{array}{c}
\circ \\
D_4 \\
\circ \\
D_4 \\
\circ \\
A_6 \\
\circ \\
\end{array}
\]

and, instead of Higgsing the \(Sp(3)\) associated to the \(D_4\) puncture, let’s gauge (an \(Sp(2)\) subgroup of) it instead. In other words, consider the 4-punctured sphere

\[
\begin{array}{c}
\circ \\
D_4 \\
\circ \\
D_4 \\
\circ \\
A_6 \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
E_7(a_2) \\
\circ \\
(D_4, Sp(2)) \\
\circ \\
E_6 \\
\circ \\
\end{array}
\]

\[
[Sp(3)_{12}^2 \times SU(2)_{36-k} \times SU(2)_k] \quad \emptyset
\]

Now we S-dualize and obtain

\[
\begin{array}{c}
\circ \\
E_7(a_2) \\
\circ \\
D_4 \\
\circ \\
0 \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
E_6 \\
\circ \\
A_6 \\
\circ \\
\end{array}
\]

\[
[(E_7)_8] \times [(E_7)_{28} \times Sp(3)_{12}] \quad \frac{1}{2}(2, 12)
\]
and

\[
\begin{align*}
& [(E_7)_8] \times [(E_7)_{28} \times Sp(3)_{12} \times SU(2)_{12}] & \emptyset
\end{align*}
\]

In both cases, the SCFT on the left is one of the product SCFTs we obtained in the previous subsection. The centralizer of Spin(12) (or Spin(11)) in \((E_7)_8 \times (E_7)_{28}\) is \(SU(2)_8 \times SU(2)_{28}\). Hence we have determined the \(SU(2)\) levels in

\[
[Sp(3)^2_{12} \times SU(2)_{28} \times SU(2)_8]
\]

Finally, we apply the Higgsing \(\mathfrak{2}\) to obtain the (same) \(SU(2)\) levels in the other 9 theories.

### 3.2.2. Matching to known SCFTs

Another technique we can employ is to match the theory to known SCFTs. Consider

\[
\begin{align*}
& \circ E_7(a_2) \\
& A_4 + A_1 \\
& 4A_1
\end{align*}
\]

This is a mixed fixture whose manifest flavour symmetry is \(Sp(3)_{19} \times U(1)^2\). The four free hypermultiplets transform as \(\frac{1}{2}(6) + 1(1)\). In particular, they contribute \(k = 1\) to the level of the \(Sp(3)\). Subtracting their contribution, the remaining SCFT has enhanced flavour symmetry \(SU(8)_{18} \times SU(2)_k \times U(1)\). It is a rank-4 SCFT with \(n_4 = n_5 = n_8 = n_9 = 1\) and \((n_h, n_v) = (92, 48)\). These data agree with an interacting fixture in the \(E_6\) theory (fixture \#10 in section 3.4 of \[2\]).
We can check this identification by computing the unrefined Schur index of both theories. After subtracting the contribution of the free hypers, we obtain

\[ I_{\text{Schur}} = 1 + 67\tau^2 + 188\tau^3 + 2764\tau^4 + 13496\tau^5 + 102726\tau^6 + 569632\tau^7 + 3443569\tau^8 + \ldots \]

Another example is afforded by the pair of fixtures

These are mixed fixtures. The fixture on the left has manifest global symmetry $SU(2)_{56} \times SU(2)_{36} \times SU(2)_9$ with hypermultiplets in the $1/2(3, 2, 1)$. Removing the contribution of the free hypers, the underlying interacting SCFT has enhanced flavour symmetry $SU(2)_{48-k} \times SU(2)_k \times SU(2)_9 \times SU(2)_{33-k'_{1-k'_{2}}} \times SU(2)_{k'_{1}} \times SU(2)_{k'_{2}}$. It is a rank-5 theory, with graded Coulomb branch dimensions $n_4 = 2$, $n_6 = 1$, $n_8 = 2$ and $(n_h, n_v) = (82, 55)$.

The fixture on the right has manifest global symmetry $SU(2)_{56} \times SU(2)_{36}$ with hypermultiplets in the $1/2(3, 2)$. Removing the contribution of the free hypers, the underlying interacting SCFT has enhanced flavour symmetry $SU(2)_{48-k_1-k_2} \times SU(2)_{k_1} \times SU(2)_{k_2} \times SU(2)_{33-k_3-k_4-k_5} \times SU(2)_{k_3} \times SU(2)_{k_4} \times SU(2)_{k_5}$. This theory is also rank-5, with $n_4 = 3$, $n_6 = n_8 = 1$ and $(n_h, n_v) = (73, 47)$.

We can pass from the former to the latter by Higgsing $D_6(a_2) \xrightarrow{SU(2)_9} E_7(a_5)$ (which preserves the rank). Comparing the invariants (and Schur indices), we recognize these as fixtures #89 and #85 of the $E_6$ theory, discussed in §3.1. So $k = k'_1 = k_1 = k_2 = 16$, $k'_2 = k_3 = k_4 = k_5 = 8$.

As a more complicated example, consider the quartet of fixtures
which are related by the nilpotent Higgsings \( D_6(a_2) \xrightarrow{SU(2)_9} E_7(a_5) \) and/or by \( A_3 + A_2 \xrightarrow{SU(3)_{12}} A_3 + A_2 + A_1 \). In the top row, the two hypermultiplets are singlets under the nonabelian part of the manifest flavour symmetry; in the bottom row, they transform as a triplet of the manifest \( SU(2)_{224} \). After subtracting the contributions of the free hypermultiplets, we recognize the theories on the bottom row as fixtures \#66 and \#63 from the \( E_6 \) theory that we discussed in §3.1. Since we determined the current algebra levels for those theories, we can fill in the other two SCFTs: \( k_1 = 36 \) and \( k_2 = k = k'_1 = k'_2 = 18 \).

### 3.2.3. Examples where we fail

Unfortunately, there remain a handful of fixtures in the \( E_7 \) theory, which are not amenable to the above techniques. Consider

\[
\begin{array}{c}
\circ O_1 \\
D_5(a_1) + A_1 \phantom{O_2} \\
\circ O_2
\end{array}
\]

where \( O_{1,2} \) are both chosen from Higgsing diagram

\[
\begin{array}{c}
SU(2)_{20} \xrightarrow{D_6(a_2)} SU(2)_9 \\
(A_5)' \xrightarrow{E_6(a_3)} E_7(a_5) \\
SU(2)_9 \xrightarrow{E_6(a_3)} SU(2)_{20}
\end{array}
\]
There are 10 fixtures of this form. If both $O_1, 2$ are chosen from the subset $\{D_6(a_2), E_7(a_5)\}$, then the $SU(2)_{56}$ of the $D_5(a_1)+A_1$ puncture is enhanced to $SU(2)_{56-k} \times SU(2)_k$. Otherwise, it is unenhanced. Unfortunately, none of these 10 fixtures can be gauged, so we cannot use S-duality to determine $k$ in the 3 fixtures with enhanced flavour symmetry.

Another case, where we are able to constrain but not completely fix the levels, consists of the fixtures

\[ O \]

\[ A_6 \]

\[ A_3 + A_2 + A_1 \]

where $O$ is taken from the set $\{D_5 + A_1, D_6(a_1), E_7(a_4), A_6\}$. The Higgsing Diagram for the this set is

\[ \begin{align*}
    &SU(2)_8 & D_5 + A_1 & SU(2)_{12} & E_7(a_4) & SU(2)_{36} \\
    &SU(2)_{12} & D_6(a_1) & SU(2)_8 & \\
\end{align*} \]

The case of $O = E_7(a_4)$ has manifest symmetry $SU(2)_{224} \times SU(2)_{36}$ and free hypermultiplets in the representation $1/2(3,2)$ The remaining SCFT then has a $SU(2)_{216} \times SU(2)_{33}$ symmetry which is enhanced to $SU(2)_{128} \times SU(2)_{88} \times SU(2)_{33}$ which is then further enhanced to $SU(2)_{128-k} \times SU(2)_k \times Sp(3)_{11}$.

One can set $O = A_6$, this fixture has the enhanced symmetry $SU(2)_{152} \times Spin(7)_{36}$. This tells us the manifest $SU(2)_{224}$ was enhanced to a $SU(2)_{152} \times SU(2)_{72}$. Higgsing an $SU(2)_{36}$ gives the original fixture with unknown levels. After subtracting the contribution from the hypermultiplets we have flavour symmetry $SU(2)_{152-(8-l)} \times SU(2)_{72-l} \times SU(2)_{33}$ where $0 \leq l \leq 8$. This is then embedded into $SU(2)_{128-k} \times SU(2)_k \times Sp(3)_{11}$. For this to happen we need a further enhancement of $SU(2)_{152-(8-l)} \times SU(2)_{72-l}$ to something containing a factor of $SU(2)_{88}$. This must come from the $SU(2)_{152-(8-l)}$ and so we are left with a flavour symmetry of the form $SU(2)_{64-(8-l)} \times SU(2)_{72-l} \times Sp(3)_{11}$. No other Higgsings tell us any information so this is the best we can do with these methods.

### 3.3. Results

Here we present our results for the previously-unknown levels in the $E_6$ and $E_7$ theories. The enhanced flavour symmetry factors, whose levels were undetermined, are highlighted in red, as are the manifest flavour symmetry factors that underwent the enhancement. The [online application] for the $E_7$ theory has been updated accordingly.
Table 1: Untwisted $E_6$

| Fixture | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|---------|----------------------------|----------------------------|
| $D_5(a_1)$ | $A_2 + 2A_1$ | $SU(2)_{54} \times U(1)^3 \times U(1)$ | $SU(3)_{18}^3 \times U(1)$ |
| $D_4(a_1)$ | $A_2 + 2A_1$ | $SU(2)_{54} \times U(1)^2 \times SU(2)_9 \times U(1)$ | $SU(3)_{30} \times SU(3)_{18} \times SU(2)_9 \times U(1)$ |

Table 2: Twisted $E_6$

| Fixture | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|---------|----------------------------|----------------------------|
| $F_4$ | $A_1 + \tilde{A}_1$ | $SU(6)_{18} \times SU(2)_{64} \times SU(2)_{10}$ | $SU(6)_{18} \times SU(2)_{48} \times SU(2)_{16} \times SU(2)_{10}$ |
| $F_4(a_2)$ | $A_2 + 2A_1$ | $SU(2)_{54} \times U(1)$ | $SU(2)_{36} \times SU(2)_{18} \times U(1)$ |
| $F_4(a_2)$ | $A_1 + \tilde{A}_1$ | $SU(2)_{64} \times SU(2)_{10}$ | $SU(2)_{32} \times SU(2)_{16} \times SU(2)_{16} \times SU(2)_{10}$ |
| $F_4(a_2)$ | $A_1 + \tilde{A}_1$ | $SU(2)_{64} \times SU(2)_{10} \times U(1)$ | $SU(2)_{18} \times SU(2)_{16} \times SU(2)_{10} \times U(1)$ |
| $F_4(a_2)$ | $A_1 + \tilde{A}_1$ | $SU(3)_{12} \times SU(2)_{64} \times SU(2)_{10}$ | $SU(3)_{12} \times SU(2)_{48} \times SU(2)_{16} \times SU(2)_{10}$ |
| $C_3$ | $A_1 + \tilde{A}_1$ | $SU(2)_{64} \times SU(2)_{10}$ | $SU(2)_{48} \times SU(2)_{16} \times SU(2)_{10}$ |
| $C_3$ | $A_1 + \tilde{A}_1$ | $SU(2)_{64} \times SU(2)_{10} \times U(1)$ | $SU(2)_{18} \times SU(2)_{16} \times SU(2)_{10} \times U(1)$ |
| $B_3$ | $A_2 + \tilde{A}_1$ | $SU(2)_{30} \times SU(2)_{24}$ | $SU(2)_{26} \times SU(2)_{13} \times SU(2)_{24}$ |
| $\tilde{A}_2$ | $A_1 + \tilde{A}_1$ | $(G_2)_{10} \times SU(2)_{64} \times SU(2)_{10} \times U(1)$ | $(G_2)_{10} \times SU(2)_{48} \times SU(2)_{16} \times SU(2)_{10} \times U(1)$ |
| $F_4(a_2)$ | $2A_2 + A_1$ | $SU(2)_{26}$ | $SU(2)_{16} \times SU(2)_{9} \times U(1) + \frac{1}{2}(2)$ |
| $F_4(a_2)$ | $A_2 + \tilde{A}_1$ | $SU(2)_{30} \times U(1)$ | $SU(2)_{20} \times SU(2)_{18} \times U(1) + \frac{1}{2}(2)$ |
Table 3: $E_7$

| Fixture | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|---------|----------------------------|---------------------------|
| $A_5 + A_1$ | $D_5(a_1) + A_1$ | $SU(2)_{26} \times SU(2)_{56} \times SU(2)_9$ | $SU(2)_{26} \times SU(2)_{28} \times SU(2)_{28} \times SU(2)_9$ |
| $D_5(a_2)$ | | | |
| $A_5 + A_1$ | $D_5(a_1) + A_1$ | $SU(2)_{26} \times SU(2)_{56}$ | $SU(2)_{26} \times SU(2)_{28} \times SU(2)_{28}$ |
| $E_7(a_3)$ | | | |
| $D_4$ | $A_6$ | $Sp(3)^2_{12} \times SU(2)_{36}$ | $Sp(3)^2_{12} \times SU(2)_{28} \times SU(2)_8$ |
| $D_4$ | $A_6$ | $Sp(3)_{12} \times Sp(2)_{111} \times SU(2)_{36}$ | $Sp(3)_{12} \times Sp(2)_{111} \times SU(2)_{28} \times SU(2)_8$ |
| $D_4 + A_1$ | $A_6$ | $Sp(2)_{11} \times SU(2)_{36}$ | $Sp(2)_{11} \times SU(2)_{28} \times SU(2)_8$ |
| $D_4 + A_1$ | $A_6$ | $Sp(2)_{11} \times SU(2)_{10} \times U(1) \times SU(2)_{36}$ | $Sp(2)_{11} \times SU(2)_{10} \times U(1) \times SU(2)_{28} \times SU(2)_8$ |
| $D_5(a_1)$ | $A_6$ | $SU(2)_{10} \times U(1)^2 \times SU(2)_{36}$ | $SU(2)_{10} \times U(1)^2 \times SU(2)_{28} \times SU(2)_8$ |
| $D_5(a_1)$ | $A_6$ | $Sp(3)_{12} \times SU(2)_{56} \times SU(2)_{36}$ | $Sp(3)_{12} \times SU(2)_{56} \times SU(2)_{28} \times SU(2)_8$ |
| $D_4$ | $A_6$ | $Sp(3)_{111} \times SU(2)_{56} \times SU(2)_{36}$ | $Sp(3)_{111} \times SU(2)_{56} \times SU(2)_{28} \times SU(2)_8$ |
| $D_5(a_1) + A_1$ | $A_6$ | $SU(2)_{10} \times U(1) \times SU(2)_{36} \times SU(2)_{20}$ | $SU(2)_{10} \times SU(4)_{20} \times SU(2)_{28} \times SU(2)_8$ |
| $D_5(a_1) + A_1$ | $A_6$ | $SU(2)_{56} \times SU(2)_{36} \times SU(2)_{20}$ | $Spin(7)_{20} \times SU(2)_{16} \times SU(2)_{28}$ |
| $D_5(a_1) + A_1$ | $A_6$ | $SU(2)_{10} \times U(1) \times SU(2)_{56} \times SU(2)_{36}$ | $SU(2)_{10} \times SU(2)_{96} \times SU(2)_{12} \times SU(2)_{12}$ |
| $D_5(a_1) + A_1$ | $A_6$ | $SU(2)_{20} \times SU(2)_{9} \times SU(2)_{108} \times SU(2)_{12}$ | $SU(2)_{20} \times SU(2)_{9} \times SU(2)_{96} \times SU(2)_{12} \times SU(2)_{12}$ |
| $D_5(a_1) + A_1$ | $A_6$ | $SU(2)_{56} \times SU(2)_{108} \times SU(2)_{12} \times SU(2)_{12}$ | $SU(2)_{56} \times SU(2)_{96} \times SU(2)_{12} \times SU(2)_{12}$ |
| Fixture | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|---------|---------------------------|---------------------------|
| $A_4 + A_2\newline D_6(a_2)$ | $D_5 + A_1\newline SU(2)_9 \times SU(2)_{108} \times SU(2)_{12}$ | $SU(2)_9 \times SU(2)_{48} \times SU(2)_{24} \times (G_2)_{12}$ |
| $A_4 + A_2\newline E_6(a_3)$ | $D_5 + A_1\newline SU(2)_{20} \times SU(2)_{12} \times SU(2)_{108}$ | $SU(2)_{20} \times SU(2)_{12} \times SU(2)_{24} \times SU(2)_{60}$ |
| $A_4 + A_2\newline E_7(a_5)$ | $D_5 + A_1\newline SU(2)_{108} \times SU(2)_{12}$ | $(G_2)_{12} \times SU(2)_{12}^3$ |
| $(A_3 + A_1)'\newline A_6$ | $D_5 + A_1\newline SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}^2 \times SU(2)_{36}$ | $SU(2)_{13} \times SU(2)_{24}^2 \times SU(2)_{24} \times SU(2)_{24} \times SU(2)_{12}$ |
| $D_4(a_1)\newline A_6$ | $D_5 + A_1\newline SU(2)_{12}^4 \times SU(2)_{36}$ | $SU(2)_{12}^4 \times SU(2)_{12}^3$ |
| $D_4(a_1) + A_1\newline A_6$ | $D_5 + A_1\newline SU(2)_{12}^2 \times SU(2)_{36} \times SU(2)_{12}$ | $SU(2)_{12}^2 \times SU(2)_{12}^2 \times Sp(2)_{12}$ |
| $A_2 + 2 A_1\newline D_5 + A_1$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{12}^2 \times SU(2)_{34}$ | $SU(2)_{16} \times SU(2)_{28}^2 \times SU(2)_{28} \times SU(2)_{28} \times SU(2)_{56}$ |
| $D_5(a_1)\newline E_7(a_4)$ | $A_4 + A_2\newline SU(2)_{10} \times U(1) \times SU(2)_{108}$ | $SU(2)_{10} \times U(1) \times SU(2)_{96} \times SU(2)_{12}$ |
| $A_3 + A_2 + A_1\newline A_5 + A_1$ | $E_7(a_4)\newline SU(2)_{26} \times SU(2)_{224}$ | $SU(2)_{26} \times SU(2)_{112} \times SU(2)_{112}$ |
| $D_5(a_1) + A_1\newline A_4 + A_2$ | $E_7(a_4)\newline SU(2)_{56} \times SU(2)_{108}$ | $SU(2)_{56} \times SU(2)_{96} \times SU(2)_{12}$ |
| $A_3 + A_2 + A_1\newline D_6(a_2)$ | $E_7(a_4)\newline SU(2)_{9} \times SU(2)_{224}$ | $SU(2)_{9} \times SU(2)_{112} \times SU(2)_{112}$ |
| $A_3 + A_2 + A_1\newline E_6(a_3)$ | $E_7(a_4)\newline SU(2)_{20} \times SU(2)_{108}$ | $SU(2)_{24}^2 \times (G_2)_{20}$ |
| $A_3 + A_2 + A_1\newline E_7(a_4)$ | $SU(2)_{224}$ | $SU(2)_{112} \times SU(2)_{112}$ |
| $A_2 + 2 A_1\newline D_5 + A_1$ | $E_7(a_4)\newline SU(2)_{16} \times SU(2)_{28} \times SU(2)_{12}^2 \times SU(2)_{34}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{28} \times SU(2)_{28} \times SU(2)_{56}$ |
| $A_2 + A_1\newline E_7(a_4)$ | $SU(4)_{18} \times U(1)$ | $SU(4)_{18} \times U(1) \times SU(2)_{56}$ |
| $D_4\newline A_3 + A_2$ | $E_6(a_1)\newline Sp(3)_{12} \times SU(2)_{12} \times U(1) \times U(1)$ | $Sp(3)_{12} \times SU(2)_{12} \times U(1) \times U(1) \times SU(2)_{56}$ |
| $D_4\newline A_3 + A_2$ | $E_6(a_1)\newline Sp(2)_{11} \times SU(2)_{12} \times U(1) \times U(1)$ | $Sp(2)_{11} \times SU(2)_{12} \times U(1) \times U(1) \times SU(2)_{56}$ |
| $D_5(a_1)\newline A_3 + A_2$ | $E_6(a_1)\newline SU(2)_{10} \times SU(2)_{12} \times U(1)^2 \times U(1)$ | $SU(2)_{10} \times SU(2)_{12} \times U(1)^2 \times U(1) \times SU(2)_{56}$ |
| Fixture            | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|--------------------|---------------------------|---------------------------|
| $A_4$              | $E_6(a_1)$                | $SU(3)_{12} \times U(1)^2 \times SU(2)_{108}$ | $SU(3)_{12} \times U(1)^2 \times SU(2)_{54} \times SU(2)_{54}$ |
| $A_4 + A_2$        | $E_6(a_1)$                | $U(1)^3 \times SU(2)_{108}$ | $U(1)^3 \times SU(2)_{54} \times SU(2)_{54}$ |
| $D_3(a_1) + A_1$   | $E_6(a_1)$                | $SU(2)_{56} \times SU(2)_{12} \times U(1) \times U(1)$ | $SU(2)_{56} \times SU(2)_{12} \times U(1) \times U(1) \times SU(2)_{56}$ |
| $A_3 + A_2$        | $E_6(a_1)$                | $SU(2)_{12} \times U(1)^2 \times SU(2)_{20}$ | $SU(2)_{12} \times U(1)^2 \times SU(4)_{20} \times SU(2)_{36}$ |
| $A_2$              | $E_6(a_1)$                | $SU(6)_{20} \times U(1) \times SU(2)_{36}$ | $SU(6)_{20} \times U(1) \times SU(2)_{18} \times SU(2)_{18}$ |
| $A_2 + A_1$        | $E_6(a_1)$                | $SU(4)_{18} \times U(1) \times SU(2)_{36} \times U(1)$ | $SU(4)_{18} \times U(1) \times SU(3)_{18} \times SU(3)_{18}$ |
| $A_2 + 2A_1$       | $E_6$                     | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $D_4(a_1)$         | $E_6$                     | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $A_2 + 2A_1$       | $E_6$                     | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $A_3 + 2A_1$       | $E_6$                     | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $0$                | $E_6$                     | $Sp(3)_{12} \times SU(2)_{12} \times (E_7)_{36}$ | $Sp(3)_{12} \times SU(2)_{12} \times (E_7)_{28} \times (E_7)_{8}$ |
| $A_2 + 2A_1$       | $E_6$                     | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{3}^{12} \times SU(2)_{36}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{3}^{12} \times SU(2)_{28}$ |
| $A_2$              | $E_6$                     | $SU(6)_{20} \times SU(2)_{36}^{12} \times U(1)$ | $SU(6)_{20} \times SU(2)_{36}^{12} \times U(1) \times SU(2)_{36}$ |
| $A_2 + A_1$        | $E_6$                     | $SU(4)_{18} \times SU(2)_{36}^{12} \times U(1) \times U(1)$ | $SU(4)_{18} \times SU(2)_{12} \times U(1) \times U(1) \times SU(2)_{56}$ |
| $0$                | $E_6$                     | $Sp(2)_{11} \times SU(2)_{12} \times (E_7)_{36}$ | $Sp(2)_{11} \times SU(2)_{12} \times (E_7)_{28} \times (E_7)_{8}$ |
| $0$                | $E_6$                     | $SU(2)_{56} \times SU(2)_{12} \times (E_7)_{36}$ | $SU(2)_{56} \times SU(2)_{12} \times (E_7)_{28} \times (E_7)_{8}$ |
| $A_3 + 2A_1$       | $E_7(a_1)$                | $SU(2)_{13} \times SU(2)_{24} \times SU(2)_{224}$ | $SU(2)_{13} \times SU(2)_{24} \times SU(2)_{112}$ |
| $A_3 + A_2 + A_1$  | $E_7(a_1)$                | $SU(2)_{12}^{12} \times SU(2)_{224}$ | $SU(2)_{12}^{12} \times SU(2)_{112}$ |
| $A_2 + A_1$        | $E_7(a_1)$                | $SU(4)_{18} \times SU(2)_{10} \times U(1) \times U(1)$ | $SU(4)_{18} \times SU(2)_{10} \times U(1) \times SU(2)_{28}$ |
| Fixture       | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|--------------|---------------------------|---------------------------|
| $4A_1$       | $E_7(a_3)$                | $Sp(3)_{19} \times SU(2)_{56}$ | $Sp(3)_{19} \times SU(2)_{28}$ |
| $D_5(a_1) + A_1$ |                           |                           |
| $A_2 + 2A_1$ | $E_7(a_2)$                | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $(A_1 + A_1)'$ |                           | $\times SU(2)_{24} \times SU(2)_{12} \times SU(2)_{84}$ | $\times SU(2)_{24} \times SU(2)_{12} \times SU(2)_{56} \times SU(2)_{28}$ |
| $A_2 + 2A_1$ | $E_7(a_2)$                | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ | $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{13}$ |
| $D_4(a_1)$   |                           | $\times SU(2)_{12} \times SU(2)_{84}$ | $\times SU(2)_{12} \times SU(2)_{56}$ |
| $A_2 + A_1$  | $E_7(a_2)$                | $SU(4)_{18} \times SU(2)_{13} \times SU(2)_{24} \times U(1)$ | $SU(4)_{18} \times SU(2)_{13} \times SU(2)_{24} \times U(1) \times SU(2)_{56}$ |
| $A_3 + 2A_1$ |                           |                           |
| $A_2 + A_1$  | $E_7(a_2)$                | $SU(6)_{20} \times SU(2)_{12} \times U(1)$ | $SU(6)_{20} \times SU(2)_{12} \times U(1) \times SU(2)_{56}$ |
| $A_3 + A_2$  |                           |                           |
| $A_3 + A_2$  | $E_7(a_2)$                | $SU(4)_{18} \times SU(2)_{12} \times U(1) \times U(1)$ | $SU(4)_{18} \times SU(2)_{12} \times U(1) \times SU(3)_{56}$ |
| $D_4 + A_1$  |                           |                           |
| $A_5 + A_1$  | $E_7(a_2)$                | $Sp(3)_{19} \times SU(2)_{224}$ | $Sp(3)_{19} \times SU(2)_{112}$ |
| $D_5(a_1) + A_1$ |                           |                           |
| $D_7(a_2)$   | $E_7(a_2)$                | $SU(2)_{26} \times SU(2)_{56} \times SU(2)_{36}$ | $SU(2)_{26} \times SU(2)_{48} \times SU(2)_{24} \times SU(2)_{9} + \frac{1}{2}(1,3,2)$ |
| $D_5(a_1)$   |                           |                           |
| $D_7(a_2)$   | $E_7(a_2)$                | $SU(2)_{9} \times U(1) \times SU(2)_{10} \times SU(2)_{36}$ | $SU(2)_{9} \times U(1) \times SU(2)_{16} \times SU(2)_{8} \times Sp(2)_{10} + \frac{1}{2}(1,1,2)$ |
| $D_5(a_1) + A_1$ |                           |                           |
| $E_7(a_5)$   | $E_7(a_2)$                | $SU(2)_{9} \times SU(2)_{12} \times SU(2)_{36}$ | $SU(2)_{9} \times SU(2)_{16} \times SU(2)_{32} \times SU(2)_{8} \times Sp(2)_{10} + \frac{1}{2}(1,3,2)$ |
| $D_5(a_1)$   |                           |                           |
| $E_7(a_5)$   | $E_7(a_2)$                | $U(1) \times SU(2)_{10} \times SU(2)_{36}$ | $U(1) \times U(1)^2 \times SU(2)_{8} \times Sp(2)_{10} + \frac{1}{2}(1,2)$ |
| $D_5(a_1) + A_1$ |                           |                           |
| $2A_1 + A_1$ | $E_7(a_2)$                | $SU(2)_{56} \times SU(2)_{36}$ | $SU(2)_{12} \times SU(2)_{24} \times SU(2)_{16} \times SU(2)_{13} + \frac{1}{2}(1,1,2,1)$ |
| $D_5 + A_1$  |                           |                           |

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| Fixture          | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|------------------|---------------------------|---------------------------|
| $A_4 + A_1$      | $E_7(a_4)$                | $SU(2)_{26} \times U(1) \times U(1)$ | $SU(2)_{26} \times U(1) \times U(1) \times SU(2)_{54} + (1)$ |
| $A_5 + A_1$      | $E_7(a_4)$                | $SU(2)_{26} \times SU(2)_{108}$ | $SU(2)_{26} \times SU(2)_{76} \times SU(2)_{26} + \frac{1}{2}(1, 4)$ |
| $A_4 + A_1$      | $E_7(a_4)$                | $SU(2)_{9} \times U(1) \times U(1)$ | $SU(2)_{9} \times U(1) \times SU(2)_{54} \times U(1)^2 + (1)$ |
| $D_6(a_2)$       |                           |                           |                           |
| $A_4 + A_2$      | $E_7(a_4)$                | $SU(2)_{9} \times SU(2)_{108}$ | $SU(2)_{9} \times SU(2)_{48} \times SU(2)_{26} \times SU(2)_{24} + \frac{1}{2}(1, 4)$ |
| $E_7(a_4)$       |                           | $U(1) \times U(1)$        | $U(1) \times SU(2)_{54} \times U(1)^3 + (1)$ |
| $A_4 + A_2$      | $E_7(a_4)$                | $SU(2)_{108}$             | $SU(2)_{26} \times SU(2)_{33} + \frac{1}{2}(4)$ |
| $E_7(a_4)$       |                           | $Spin(7)_{16} \times SU(2)_{36}$ | $Spin(7)_{16} \times SU(2)_{24} \times SU(2)_{11} + \frac{1}{2}(1, 2)$ |
| $A_6$            | $E_7(a_4)$                | $S U(2)_{36} \times SU(2)_{38} \times SU(2)_{12}$ | $Sp(2)_{12} \times SU(2)_{12} \times SU(2)_{13} + \frac{1}{2}(1, 2, 1)$ |
| $2A_2 + A_1$     | $E_7(a_4)$                | $(G_2)_{16} \times SU(2)_{36}$ | $Spin(7)_{16} \times SU(2)_{24} \times SU(2)_{11} + \frac{1}{2}(1, 2)$ |
| $D_7 + A_1$      |                           | $SU(2)_{36} \times SU(2)_{38}$ | $Sp(2)_{24} \times SU(2)_{13} \times SU(2)_{11} + \frac{1}{2}(1, 2) + \frac{1}{2}(2, 1)$ |
| $2A_2$           | $E_7(a_4)$                | $SU(2)_{9} \times SU(2)_{12} \times U(1)^2$ | $SU(2)_{9} \times SU(3)_{12} \times SU(2)_{36} \times SU(2)_{18} + 2(1, 1)$ |
| $E_7(a_4)$       |                           | $SU(2)_{9} \times U(1) \times SU(2)_{224}$ | $SU(2)_{9} \times U(1) \times SU(3)_{36} \times SU(3)_{18} + (1, 3)$ |
| $A_3 + A_2$      | $E_6(a_1)$                | $U(1) \times SU(2)_{12} \times U(1)$ | $U(1) \times SU(3)_{12} \times SU(2)_{18} + 2(1)$ |
| $D_6(a_2)$       |                           |                           |                           |
| $A_3 + A_2 + A_1$| $E_6(a_1)$                | $U(1) \times SU(2)_{224}$ | $U(1) \times SU(3)_{18} + (3)$ |
| $E_7(a_3)$       |                           | $SU(2)_{24} \times SU(2)_{12} \times SU(2)_{36} \times SU(2)_{13}$ | $SU(2)_{24} \times SU(2)_{12} \times Sp(2)_{13} \times SU(2)_{24} \times SU(2)_{13} + \frac{1}{2}(1, 1, 1, 1, 2, 1)$ |
| $2A_2 + A_1$     | $E_6$                     | $SU(2)_{12} \times SU(2)_{36} \times SU(2)_{38}$ | $SU(2)_{12} \times SU(2)_{12} \times SU(2)_{12} + \frac{1}{2}(1, 1, 1, 1, 2)$ |
| $(A_3 + A_1)'$   |                           | $SU(2)_{12} \times SU(2)_{36} \times SU(2)_{38}$ | $SU(2)_{12} \times SU(2)_{12} \times SU(2)_{12} + \frac{1}{2}(1, 1, 1, 1, 2)$ |
| $2A_2 + A_1$     | $E_6$                     | $SU(2)_{12} \times SU(2)_{36} \times SU(2)_{38}$ | $SU(2)_{12} \times SU(2)_{12} \times SU(2)_{12} + \frac{1}{2}(1, 1, 1, 1, 2)$ |
| $D_4(a_1) + A_1$ | $E_6$                     |                           |                           |
| $A_3 + 2A_1$     | $E_7(a_3)$                | $SU(2)_{13} \times U(1) \times SU(2)_{24} \times U(1)$ | $SU(2)_{13} \times U(1) \times SU(3)_{24} \times SU(2)_{54} + (1, 1)$ |
| $A_4 + A_1$      |                           |                           |                           |
4. Theories with the Same “Conventional Invariants”

4.1. Global Form of the Flavour Symmetry Group

Each puncture \( p_i \), in a theory of class-S, contributes a factor of \( \text{f}_{p_i} \) to the flavour symmetry algebra of the theory. The “manifest” flavour symmetry algebra, \( \text{f}_{\text{manifest}} = \bigoplus_i \text{f}_{p_i} \). In general, the full symmetry algebra is some enhancement \( \text{f} \supset \text{f}_{\text{manifest}} \). Let \( F \) be the corresponding compact Lie group, whose semi-simple part is simply-connected. In addition, every \( \mathcal{N} = 2 \) SCFT has a \( \mathbb{Z}_2 \) global symmetry, \( S \), generated by \( \gamma = e^{2\pi i (R_1 + j_1 + j_2)} \). In general, \( F \times S \) does not act effectively. Some subgroup \( \Gamma \subset \mathbb{Z}(F) \times S \) acts trivially on all of the local operators in the theory. We call \( \tilde{F} = (F \times S)/\Gamma \) the global form of the flavour symmetry group. Determining \( \Gamma \), and hence \( \tilde{F} \), is the subject of this subsection. Without \( S \), a method for determining \( \Gamma \) was proposed in [10]. Here we extend that prescription to include \( S \).

To motivate the proposal of [10], consider the Schur Index of an untwisted class S theory,

\[ S \text{ is the center of the } \mathcal{N} = 2 \text{ superconformal supergroup, } SU(2,2|2). \] So the symmetries that act effectively are \( (SU(2,2|2) \times F)/\Gamma \). For brevity, we’ll omit the \( SU(2,2|2) \) factor from our formulæ below.

---

Table 4: Undetermined levels in the \( E_7 \) theory

| Fixture | Manifest Flavour Symmetry | Enhanced Flavour Symmetry |
|---------|---------------------------|---------------------------|
| \( D_6(a_2) + A_1 \) | \( SU(2)_{56} \times SU(2)^2_{36} \) | \( SU(2)_{56-k} \times SU(2)_{k} \times SU(2)^2_{j} \) |
| \( E_7(a_5) + A_1 \) | \( SU(2)_{56} \times SU(2)_{9} \) | \( SU(2)_{56-k} \times SU(2)_{k} \times SU(2)_{9} \) |
| \( A_3 + A_2 + A_1 \) | \( SU(2)_{224} \times SU(2)_{36} \) | \( SU(2)_{128-k} \times SU(2)_{k} \times Sp(3)_{11} + \frac{1}{2}(3,2) \) |
of type $j \in \text{ADE}$, with $n$ punctures $p_i$ determined in [19]

$$I_{\text{Schur}}(\tau) = \sum_{\Lambda} \left(\psi^\rho_\Lambda(\tau)\right)^{2-2g-n} \prod_{i=1}^{n} \psi^{p_i}_{\Lambda}(a_i, \tau)$$

(3)

The sum is over highest weight representations $\Lambda$ of $j$. Here $\rho$ corresponds to the trivial puncture (the regular embedding $\mathfrak{su}(2) \hookrightarrow j$) and the wave functions $\psi^{p_i}_{\Lambda}(a_i, \tau)$ are given by

$$\psi^{p_i}_{\Lambda}(a_i, \tau) = K^{p_i}_{\Lambda}(a_i, \tau) \chi^{p_i}_{\Lambda}(a_i, \tau)$$

The characters, $\chi^{p_i}_{\Lambda}(a_i, \tau)$, are those corresponding to the decomposition of $\Lambda$ under the embedding of $\mathfrak{su}(2) \times F_{p_i} \hookrightarrow j$. The $K$-factors are given by the plethystic exponential

$$\text{P.E.} \left[ \frac{1}{1 - \tau^2} \chi^{p_i}_{\text{adj}}(a_i, \tau) \right]$$

For $j \neq D_{2n}$, there’s a highest weight irreducible representation $R$ of $j$ such that all of the other highest weight representations are contained in the tensor powers of $R$. Let $R_{p_i}$ denotes the decomposition of $R$ under $\mathfrak{su}(2) \times F_{p_i}$. Then $\Gamma$ is the subgroup that leaves $R^{p_i}_\rho \otimes \left( \bigotimes_i R_{p_i} \right)$ invariant (when we take $\gamma$ to act as the center of $SU(2)$, i.e. as $\tau \rightarrow -\tau$ on the level of the superconformal index). This can be seen from the expression (3) for the superconformal index. If the above condition is satisfied then, assuming the $K$-factors are invariant as will be shown below, $\Gamma$ will leave the contribution of $\Lambda = R$ to the sum in (3) invariant. Since all other reps can be generated by tensor powers of $R$, it will leave those terms in the sum invariant as well.

To see that the $K$-factors are invariant, note an element of the center of the flavour symmetry must act as a multiple of the identity on each irreducible representation in the decomposition of $R$, hence it will act as a diagonal matrix on the decomposition $R_{p_i}$. If the tensor product of diagonal matrices is plus or minus the identity, then each diagonal matrix must be a multiple of the identity. Therefore if an element of the center of $F \times S$ leaves the $\bigotimes_i R_{p_i}$ invariant or anti-invariant, then the element of the center must act as a multiple of the identity on each $R_{p_i}$. Since the adjoint can be generated by tensor powers of $R$, it must act as a multiple of the identity on the decomposition of the adjoint of $j$. Since this is an embedding, the decomposition of the adjoint will contain $(3; 1)$ which is invariant under the center of $S \times F$ and hence the entire decomposition is invariant. Thus the $K$-factors are invariant.

Note that in the $E_8$ case the above implies that $\Gamma$ should leave each $R_{p_i}$ invariant. Thus one does not need to worry about any tensor products. In general, the action of elements of $\Gamma$ on $R_{p_i}$ will be multiplication by an $N$th root of unity where $N$ is the smallest power such that $R^N$ contains the adjoint representation.

To generalize to $j = D_{2n}$, one must repeat the procedure for $R = S, S'$, the two chiral spinor representations. For twisted theories, the Schur index is a sum over highest weight representations of the Langlands dual, $\mathfrak{g}$, of the subalgebra $\mathfrak{g}^\vee \subset j$ invariant under the outer automorphism [20]. In this case there is a map $\sigma$ from the set of representations of $\mathfrak{g}$ to the representations of $j$. One modifies the above procedure by finding a representation $R$ of $\mathfrak{g}$
that generates all other representations via tensor products with itself, decomposing $R$ for the twisted punctures, and decomposing the representation $\sigma(R)$ for untwisted punctures.

Note that this method only works for the manifest flavour symmetry $F_{\text{manifest}}$ and those cases where the global form of $F_{\text{manifest}}$ determines the global form of $F$. It also assumes that if all the Schur operators are invariant under $\Gamma$, then same is true of all local operators in the SCFT.

4.2. Theories with the Same Conventional Invariants

In [1] it was proposed that the global form of the flavour symmetry might distinguish theories with the same “conventional invariants”: flavour symmetry algebras (and the associated current-algebra levels), $a, c$ central charges, and graded Coulomb branch dimensions.

Using the nilpotent Higgsings introduced above, we find a way to generate many theories with the same conventional invariants. We find many families of pairs where each fixture in each pair is related to a fixture in another pair via nilpotent Higgsings. Among these families we find examples of pairs whose conventional invariants coincide but with different global form of their flavour symmetry groups, pairs whose invariants coincide and are actually the same theory, as well as pairs with the same invariants and flavour symmetry groups but which are nonetheless distinct theories, disproving the conjecture of [1].

4.2.1. $D_{2n}$

Using very even punctures in the $D_{2n}$ series many theories were found with the same invariants in [1]. For example the pairs of $D_{2n}$ theories

have the same conventional invariants, but are nonetheless distinct SCFTs. The flavour symmetry is $\mathfrak{sp}(n)_{4n} \oplus \mathfrak{sp}(n)_{4n} \oplus \mathfrak{so}(4n)_{(2n-1)}$. Let $\gamma_1$ generate the center of the $Sp(n)$ of the top puncture and $\gamma_2$ generate the center of the $Sp(n)$ of the bottom puncture. Let $\gamma_3, \gamma_4$ generate the center of $Spin(4n)$ with $\gamma_3$ acting as $-1$ on the left handed spinor and vector representation, while $\gamma_4$ acts as $-1$ on the right handed spinor and vector representation. For $n$ even, $\Gamma$ for the theories on the left is given by $\langle \gamma, \gamma_1 \gamma_2, \gamma_1 \gamma_3 \rangle$ while for the theories on the right it is $\langle \gamma \gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_2 \gamma_4 \rangle$. For $n$ odd, $\Gamma$ for the theories on the left is $\langle \gamma \gamma_3 \gamma_4, \gamma_1 \gamma_2, \gamma_1 \gamma_4 \rangle$, and for the theories on the right it is $\langle \gamma, \gamma_1 \gamma_3, \gamma_2 \gamma_4 \rangle$. We see that they do indeed have different flavour symmetry groups.

4.2.2. Generating examples from nilpotent Higgsing

Using the technology of nilpotent Higgsings it is rather straightforward to generate additional examples of pairs of theories in class-S, with the same conventional invariants. Consider a
fixture of the form

\[
\begin{array}{c}
O_1 \\
O_2 \\
O_3 \\
O_4
\end{array}
\]

in some theory of class-S, where there are nilpotent Higgsings

\[
O_1 \xrightarrow{f_k} O_3, \quad O_2 \xrightarrow{f_k} O_4
\]
such that the manifest flavour symmetries of the punctures satisfy

\[
f_{O_1} = f_k \oplus f_{O_3}, \quad f_{O_2} = f_k \oplus f_{O_4}.
\]  \hspace{1cm} (4)

for the same \( f_k \). Then (provided that we choose the third puncture \( O \) sufficiently high up on the Hasse diagram), the theories

\[
\begin{array}{c}
O_1 \\
O_2 \\
O_3 \\
O_4
\end{array}
\]

have exactly the same conventional invariants. (If we choose \( O \) too low on the Hasse diagram, then the Higgsing might yield a bad pair of theories, or one where the IR flavour symmetries are enhanced in different ways.)

As an example consider the fixture

\[
\begin{array}{c}
E_6 \\
(A_3 + A_1)'
\end{array}
\]

\[F_{\text{manifest}} = SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12} \times SU(2)_{12} \times F_O\]
in the \( E_7 \) theory. Higgsing one or the other of the \( SU(2)_{12}s \) yields the pair of theories

\[
\begin{array}{c}
E_6 \\
E_7(a_2)
\end{array}
\]
With $O$ chosen from the $E_7$ Higgsing diagram

$$
\begin{array}{c}
\text{0 (E7)$_{36}$} & \text{A$_1$} & \text{Spin(12)$_{28}$} & \text{SU(2)$_{20}$} & (3A$_1$)'' & (F$_4$)$_{24}$ & 4A$_1$ \\
\text{Spin(9)$_{24}$} & \text{Spin(3)$_{20}$} & \text{Sp(3)$_{19}$} & \text{SU(4)$_{18}$} & \text{A$_2$ + A$_1$} & \text{SU(2)$_{16}$} & \text{A$_2$ + 3A$_1$} \\
& & & & \text{SU(2)$_{28}$} & \text{2A$_2$} & \text{2A$_2$ + A$_1$} \\
& & & & \text{SU(6)$_{20}$} & \text{A$_2$ + 2A$_1$} & \text{SU(2)$_{28}$} & \text{SU(2)$_{19}$} & \text{(G$_2$)$_{16}$} & \text{SU(2)$_{28}$} & \text{(G$_2$)$_{28}$} \\
\end{array}
$$

(5)

the manifest flavour symmetry algebras are the same for each pair in this family (except for 2A$_2$, as we will explain presently). The other conventional invariants also coincide. But the theories in black have different $\Gamma$s and are non-isomorphic SCFTs, as verified by computing the respective Schur indices.

This pattern continues until we Higgs 2A$_1 \xrightarrow{\text{SU(2)$_{20}$}} (3A$_1$)''. For that pair, the $\Gamma$s are the same and a computation of the Schur index to $O(\tau^{12})$

$$
I_{\text{Schur}}(O = (3A$_1$)'') = 1 + 61\tau^2 + 10\tau^3 + 2017\tau^4 + 7987\tau^5 + 47969\tau^6 + 32690\tau^7 + 920783\tau^8 + 927216\tau^9 + 15202440\tau^{10} + 20597848\tau^{11} + 224805960\tau^{12} + O(\tau^{13})
$$

leads us to believe that the SCFTs are also isomorphic. The same holds for subsequent Higgsings, and all of the pairs marked in green appear to be isomorphic SCFTs.

A new phenomenon, however, occurs once we reach $A_2 + A_1$. For the remaining 5 punctures, the flavour symmetry is enhanced over the manifest one, and our methods do not determine the full $\Gamma$, but only the subgroup $\Gamma_{\text{manifest}}$.

Computing the Schur indices, however, leads us to believe that the pairs of theories marked in green are isomorphic, which leads to a prediction for $\Gamma$ for 3 of those 4 cases.

- For $O = A_2 + A_1$, a manifest $U(1)$ is enhanced to $U(1)^2$. Computing the manifest flavour symmetries one finds for the theory on the left that the manifest $\Gamma$ is generated by $\langle \gamma_1\delta_L\gamma_4 \rangle$ while on the right it is generated by $\langle \gamma_1\delta_R\gamma_2\gamma_3 \rangle$. Here $\delta_L$ is the generator of the $Z_4$ subgroup of the manifest $U(1)$ for the theory on the left and similarly for $\delta_R$. Assuming the two theories are the same due to their matching Schur indices this implies the manifest $U(1)$ symmetries are different. Thus we may write the $U(1)^2$ of both theories as $U(1)_{L} \times U(1)_{R}$ where $U(1)_{L}$ is the manifest $U(1)$ of the theory on the left and $U(1)_{R}$ is the manifest $U(1)$ on the theory on the right. Requiring the global forms of the flavour symmetry groups to be the same determines $\Gamma$ to be $\langle \gamma_1\delta_L\gamma_4, \gamma_1\delta_R\gamma_2\gamma_3 \rangle$.

- For $O = A_2 + 2A_1$, the manifest flavour symmetry is $SU(2)_{16} \times SU(2)_{28} \times SU(2)_{84} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$. Let $\Gamma_i$ be the generator of the center of the $i$th factor in the manifest flavour symmetry group above. Then $\Gamma_{L,\text{manifest}} = \langle \gamma_1\gamma_2\gamma_6, \gamma_3\gamma_4\gamma_5 \rangle$ and $\Gamma_{R,\text{manifest}} = \langle \gamma_1\gamma_2\gamma_5\gamma_6, \gamma_3\gamma_4 \rangle$. While the manifest flavour symmetry groups have different global form, the Schur indices nonetheless agree up to $O(\tau^{10})$, which leads us to believe that these are isomorphic theories. In fact, the $SU(2)_{84}$ is enhanced to an
involves a theory. The element of $\Gamma$ of the UV
($A$ Flavour Symmetry Group $= O_{\Gamma}$ Manifest Flavour Symmetry $= O_{\Gamma}$)
and its center has an additional generator. Demanding that $\Gamma_L = \Gamma_R \equiv \Gamma$ then
determines $\Gamma = \langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6, \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \rangle$, where $\gamma_i$ is the generator of the center of the $i$th factor in $F$.

- Since there are two $SU(2)_{28}$s, the Higgsing to $2A_2$ involves a different $SU(2)_{28}$ in the
theory on the left versus the theory on the right. And, indeed, the two theories with $2A_2$
have different flavour symmetry algebras: $F_L = Spin(7)_{16} \times SU(2)_{13} \times Sp(2)_{12}$, while $F_R = Spin(7)_{16} \times Sp(2)_{24} \times SU(2)_{13} \times Sp(2)_{12}$.

- For $O = A_2 + 3A_1$, the flavour symmetry enhancement is from $(G_2)_{28}$ to $Spin(7)_{28}$
adding element to the center. However we can determine the possible additional generator of $\Gamma$ by Higgsing from the $O = A_2 + 2A_1$ theory. The element of $\Gamma$ of the UV theory $\gamma_2 \gamma_3 \gamma_6$ embeds into the IR theory as $\gamma_1 \gamma_3$. Thus we are able to completely determine $\Gamma$.

- For $O = 2A_2 + A_1$, there could be additional elements of $\Gamma$ due to the enhancements,
however we cannot determine them with our methods.

We assemble our results for this family of pairs of SCFTs in the table below. They seem to
support the conjecture of [1] that the distinct SCFTs with the same “conventional” invariants
are distinguished by the global form of the flavour symmetry group and, conversely, when
the conventional invariants and the global forms of the flavour symmetry group coincide, the
theories are isomorphic.

| $O$ | Manifest Flavour Symmetry | Flavour Symmetry Group | $\Gamma_L$ | $\Gamma_R$ |
|-----|---------------------------|------------------------|------------|------------|
| 0   | $(E_7)_{36} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $(E_7)_{36} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_2, \gamma_3 \gamma_5 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3, \gamma_2 \gamma_4 \rangle$ |
| $A_1$ | $Spin(12)_{28} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $Spin(12)_{28} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_2, \gamma_3 \gamma_4 \gamma_5 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4, \gamma_2 \gamma_5 \rangle$ |
| $2A_1$ | $Spin(9)_{24} \times SU(2)_{20} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $Spin(9)_{24} \times SU(2)_{20} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_3 \gamma_5 \gamma_6, \gamma_2 \gamma_5 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4, \gamma_2 \gamma_5 \rangle$ |
| $(3A_1)'$ | $Sp(3)_{20} \times SU(2)_{19} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $Sp(3)_{20} \times SU(2)_{19} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_2 \gamma_5, \gamma_2 \gamma_3 \gamma_4 \gamma_6 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4, \gamma_2 \gamma_5 \rangle$ |
| $A_2$ | $SU(6)_{20} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12} \times U(1)$ | $SU(6)_{20} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12} \times U(1)$ | $\langle \gamma_1 \gamma_2, \gamma_3 \gamma_4 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3, \gamma_2 \gamma_4 \rangle$ |
| $(3A_1)''$ | $F_4 \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $F_4 \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \rangle$ |
| $4A_1$ | $Sp(3)_{19} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $Sp(3)_{19} \times SU(2)_{13} \times SU(2)_{24} \times SU(2)_{12}$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \rangle$ | $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \rangle$ |
4.2.3. Another family

The previous family of examples seemed in perfect accord with the conjecture of [1]. When the global form of the flavour symmetry groups differ, the SCFTs were distinct; when the global forms were the same, the theories were isomorphic. The next family of examples will not be so obliging.

Again, take the $E_7$ theory and the fixture

$$F_{\text{manifest}} = \text{Spin}(7)_{16} \times SU(2)_{12} \times SU(2)_{12} \times SU(2)_8 \times F_0$$

We can Higgs either $SU(2)_{12}$ to obtain a pair of theories

Every pair has the same conventional invariants. Moreover, for

$$O \in \{(3A_1)'', 4A_1, A_2 + 3A_1, (A_3 + A_1)'', 2A_2 + A_1, A_3 + 2A_1, D_4(a_1) + A_1, A_4, A_3 + A_2 + A_1, (A_5)'', D_4 + A_1, A_5 + A_1, D_5(a_1) + A_1, D_6(a_2), E_7(a_5)\}$$

they even have the same global form of the flavour symmetry group. For instance, for $O = 4A_1$ the flavour symmetry for both theories is

$$\hat{F} = (\text{Spin}(7)_{16} \times \text{Sp}(3)_{19} \times SU(2)_8 \times SU(2)_{12} \times S) / \Gamma$$
where \( \Gamma = \langle \gamma_2 \gamma_3 \gamma_4 \rangle \).

Nevertheless, for 11 of those 15 punctures, one quickly discovers that the theories are not isomorphic. The Schur indices differ at \( O(\tau^4) \) because the theory on the right has an additional \( B_2 \) operator, in the 8 of the \( Spin(7)_{16} \) associated to the \((A_3 + A_1)''\) puncture, that is absent in the theory on the left.

At least in some case, perhaps we should have expected this to be the case. While it’s true that the two theories with \( O = 4A_1 \) have the same global form of the flavour symmetry group, we can Higgs \( 4A_1 \xrightarrow{Sp(3)_{19}} A_2 + A_1 \). The theories with \( O = A_2 + A_1 \) have distinct global forms of the flavour symmetry group, which ought to tell us that the parent theories with \( O = 4A_1 \) must also be distinct\(^4\). And once we’ve determined that the two theories with \( O = 4A_1 \) are distinct, then the Higgsing \((3A_1)'' \xrightarrow{(F_4)_{24}} 4A_1\) implies that the theories with \( O = (3A_1)'' \) must also be distinct.

In similar fashion, this “explains” why the pairs associated to \( O = (A_3 + A_1)'' \), \( A_4 \) and \( D_4 + A_1 \) are distinct SCFTs. Alas, there’s no similar explanation for remaining six. More surprising, the remaining four choices for \( O \), which are related by the nilpotent Higgsings

\[
(A_5)'' \xrightarrow{(G_2)_{12}} (A_5 + A_1), \quad D_6(a_2) \xrightarrow{SU(2)_{36}} E_7(a_5),
\]

do appear to lead to isomorphic pairs of SCFTs. We have checked their Schur indices up to \( O(\tau^{12}) \), and they agree.

4.2.4. Sporadic examples

In sporadic cases, the condition \(^4\) can be relaxed, with an infrared enhancement of the flavour symmetry compensating for the lack of agreement of the manifest symmetries. As an example, start with the \( E_7 \) fixture

\[
F_{\text{manifest}} = (G_2)_{16} \times SU(2)_{36} \times SU(2)_{36} \times F_O
\]

For general \( O \), Higgsing one or the other of the \( SU(2)_{36} \) yields IR theories with different flavour symmetries as Higgsing \( 2A_2 \xrightarrow{SU(2)_{36}} (A_3 + A_1)'' \) enhances the \((G_2)_{16}\) symmetry of the \( 2A_2 \) puncture to the \( Spin(7)_{16} \) symmetry of the \((A_3 + A_1)''\) puncture.

\(^4\)Note that this is unlike the family of examples in \(^5\), where there were two distinct \( A_2 + 2A_1 \xrightarrow{SU(2)_{28}} 2A_2 \) Higgsings, which led to distinct child SCFTs, even though the parents were isomorphic.
However, if we choose $O$ from the Higgsing diagram

\[
\begin{array}{c}
S U(2)_{12} \\
\downarrow \\
D_5 \\
\downarrow \\
S U(2)_8 \quad E_7(a_4) \\
\downarrow \\
D_5 + A_1 \quad S U(2)_{12}
\end{array}
\]

then the $(G_2)_{16}$ is enhanced to $S p i n(7)_{16}$ for both fixtures in the pair

\[
\begin{array}{c}
\circ A_6 \\
O \circ \\
(A_3 + A_1)''
\end{array} \quad \text{and} \quad \begin{array}{c}
\circ 2A_2 \\
O \circ \\
E_7(a_4)
\end{array}
\]

While the manifest symmetries are different (the theory on the right has a $(G_2)_{16}$ where the theory on the left has a $S p i n(7)_{16}$ factor), the enhanced flavour symmetry algebras are the same. The Schur indices agree up to $O(\tau^9)$. Due to enhancements we are unable to compute the global form of the full flavour symmetry, however the elements we are able to compute are the same. Thus we believe that each pair represents isomorphic SCFTs.

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