A REPRESENTATION OF TWISTED GROUP ALGEBRA OF SYMMETRIC GROUPS ON WEIGHT SUBSPACES OF FREE ASSOCIATIVE COMPLEX ALGEBRA

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Abstract

Here we consider two algebras, a free unital associative complex algebra (denoted by $B$) equipped with a multiparametric $q$-differential structure and a twisted group algebra (denoted by $A(S_n)$), with the motivation to represent the algebra $A(S_n)$ on the (generic) weight subspaces of the algebra $B$. One of the fundamental problems in $B$ is to describe the space of all constants (the elements which are annihilated by all multiparametric partial derivatives). To solve this problem, one needs some special matrices and their factorizations in terms of simpler matrices. A simpler approach is to study first certain canonical elements in the twisted group algebra $A(S_n)$. Then one can use certain natural representation of $A(S_n)$ on the weight subspaces of $B$, which are the subject of this paper.

Keywords: $q$-algebras, twisted derivations, symmetric group, polynomial ring, twisted group algebra, representation

Subject Classification: 05E10

1 Introduction

First we recall free unital associative complex algebra $B$ generated by $N$ generators $\{e_i\}_{i \in N}$ (each of degree one, $N = \text{Card} \mathcal{N}$) with a multiparametric $q$-($= \{q_{ij}\}$)-differential structure such that $q$-differential operators $\{\partial_i\}_{i \in \mathcal{N}}$ act on $B$ according to the twisted Leibniz rule:

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x),$$

for each $x \in B$ with $\partial_i(1) = 0$ and $\partial_i(e_j) = \delta_{ij}$.
(where $q_{ij}$’s are given complex numbers). The algebra $B$ is graded by total degree and more generally it could be considered as multigraded, because the operators $\partial_i$ (of degree $-1$) respect the (direct sum) decomposition of $B$ into multigraded subspaces $B_Q$.

We also recall a twisted group algebra $A(S_n)$ of the symmetric groups $S_n$ with coefficients in the polynomial ring $R_n$ in $n^2$ commuting variables $X_{ab}$, $1 \leq a, b \leq n$ (c.f [10] for details). The algebra $A(S_n)$ is defined as a semidirect product of $R_n$ and the usual group algebra $\mathbb{C}[S_n]$ of the symmetric group $S_n$. The elements of $A(S_n)$ are the linear combinations $\sum_{g_i \in S_n} p_i g_i$ with $p_i \in R_n$.

The multiplication in $A(S_n)$ is given by the formula (6) in the Section 3, where we impose that $S_n$ naturally acts on $R_n$.

In Section 4 we define a natural representation of the twisted group algebra $A(S_n)$, $n \geq 0$ on the weight subspaces $B_Q$ (Card $Q = n$) of $B$. In this representation some factorizations of certain canonical elements from $A(S_n)$ will immediately give the corresponding matrix factorizations and also determinant factorizations.

## 2 Free unital associative $\mathbb{C}$-algebra

We recall first a free (unital) associative $\mathbb{C}$-algebra (see also [9]), denoted by $B = \mathbb{C} \langle e_{i_1}, \ldots, e_{i_N} \rangle$, where $\mathcal{N} = \{i_1, \ldots, i_N\}$ is a fixed subset of the set $\mathbb{N}_0 = \{0, 1, \ldots\}$ of nonnegative integers. The algebra $B$ is generated by $N$ generators $\{e_i\}_{i \in \mathcal{N}}$ (each of degree one), so we can think of $B$ as an algebra of noncommutative polynomials in $N$ noncommuting variables $e_{i_1}, \ldots, e_{i_N}$. The algebra $B$ is naturally graded by the total degree $B = \bigoplus_{n \geq 0} B^n$, where $B^0 = \mathbb{C}$ and $B^n$ consists of all homogeneous polynomials of total degree $n$ in variables $e_{i_1}, \ldots, e_{i_N}$.

In view of the fact that every sequence $l_1, \ldots, l_n \in \mathcal{N}$, $l_1 \leq \cdots \leq l_n$ can be thought of as a multiset $Q = \{l_1 \leq \cdots \leq l_n\}$ over $\mathcal{N}$ of size $n = \text{Card } Q$ (cardinality of $Q$), we see that each weight subspace $B_Q = B_{l_1 \ldots l_n}$, corresponding to a multiset $Q$, is given by

$$B_Q = \text{span}_\mathbb{C}\left\{e_{j_1 \ldots j_n} := e_{j_1} \cdots e_{j_n} \mid j_1 \ldots j_n \in \hat{Q}\right\},$$

(1)

where $\hat{Q}$ denotes the set of all distinct permutations of the multiset $Q$. It follows that $\dim B_Q = \text{Card } \hat{Q}$. Thus, we obtain a finer decomposition of $B$ into multigraded components (= weight subspaces): $B = \bigoplus_{n \geq 0, l_1 \leq \cdots \leq l_n, l_j \in \mathcal{N}} B_{l_1 \ldots l_n}$.

Note that if $l_1 < \cdots < l_n$, then $Q$ is a set and the corresponding weight subspace $B_Q$ we call generic. Other (nongeneric) weight subspaces $B_Q$ we call degenerate.
We can interpret \( q \) as a map \( q : \mathcal{N} \times \mathcal{N} \to \mathbb{C}, \ (i, j) \mapsto q_{ij} \) for all \( i, j \in \mathcal{N} \). Then, on the algebra \( \mathcal{B} \), we introduce \( N \) linear operators \( \partial_i = \partial_i^q : \mathcal{B} \to \mathcal{B} \), \( i \in \mathcal{N} \), defined recursively, as follows:

\[
\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij},
\]

\[
\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \quad \text{for each } x \in \mathcal{B}, \ i, j \in \mathcal{N}.
\]

(\( \delta_{ij} \) is a standard Kronecker delta i.e. \( \delta_{ij} = 1 \) if \( i = j \), and 0 otherwise.) These \( q \)-differential operators \( \{ \partial_i \}_{i \in \mathcal{N}} \) act as a generalized \( i \)-th partial derivatives on the algebra \( \mathcal{B} \); they depend on additional parameters (complex numbers) \( q_{ij} \). Therefore, we can say that \( \partial_i^q \) is a multiparametrically deformed \( i \)-th partial derivative or shortly \( q \)-deformed \( i \)-th partial derivative. It is easy to see that if all \( q_{ij} \)’s are equal to one, then \( \partial_i^q \) coincides with a usual \( i \)-th partial derivative \( \partial_i \).

In what follows we will consider \( \mathcal{B} \) with this ‘\( q \)-differential structure’.

Let us denote by \( \mathcal{B}_Q = \{ e_{\tilde{j}} \mid \tilde{j} \in \tilde{Q} \} \) the monomial basis of \( \mathcal{B}_Q \), where \( \tilde{j} := j_1 \ldots j_n \). Then the action of \( \partial_i = \partial_i^q \) on a typical monomial \( e_{\tilde{j}} \in \mathcal{B}_Q \) is given explicitly by the formula:

\[
\partial_i(e_{\tilde{j}}) = \sum_{1 \leq k \leq n, j_k = i} q_{ij_1} \cdots q_{ij_{k-1}} e_{\tilde{j}_1 \cdots \hat{j}_k \cdots \tilde{j}_n}, \tag{2}
\]

where \( \hat{j}_k \) denotes the omission of the corresponding index \( j_k \).

The number of terms in this sum is equal to the number of appearances (multiplicity) of the generator \( e_i \) in the monomial \( e_{\tilde{j}} \).

In the generic case, when \( Q \) is a set, the formula (2) is reduced to:

\[
\partial_i(e_{\tilde{j}}) = q_{ij_1} \cdots q_{ij_{k-1}} e_{\tilde{j}_1 \cdots \hat{j}_k \cdots \tilde{j}_n}. \tag{3}
\]

If \( j_k = i \) for all \( 1 \leq k \leq n \), then from (2) we get the following important special case

\[
\partial_i(e_i^n) = (1 + q_{ii} + q_{ii}^2 + \ldots + q_{ii}^{n-1}) e_i^{n-1} = [n]_{q_{ii}} e_i^{n-1}, \tag{4}
\]

where \( [n]_q = 1 + q + \cdots + q^{n-1} \) is a \( q \)-analogue of a natural number \( n \). For \( q_{ii} = 1 \) the formula (4) can be read as the classical formula \( \partial_i(e_i^n) = n \cdot e_i^{n-1} \).

For \( x \in \mathcal{B}_{l_1 \ldots l_n} \) and \( y \in \mathcal{B} \) we have a more general formula:

\[
\partial_i(xy) = \partial_i(x) y + q_{id_1} \cdots q_{id_m} x \partial_i(y) \quad \text{for each } i \in \mathcal{N}.
\]

On the other hand, with the motivation of treating better the matrices of \( \partial_i|_{\mathcal{B}_Q} \), we introduce a multidegree operator \( \partial : \mathcal{B} \to \mathcal{B} \) with \( \partial = \sum_{i \in \mathcal{N}} e_i \partial_i \), where
$e_i : B \rightarrow B$ are considered as (multiplication by $e_i$) operators on $B$. The operator $\partial$ preserves the direct sum decomposition of the algebra $B$, i.e. each subspace $B_Q$ is an invariant subspace of $\partial$. Moreover, we denote by $\partial^Q : B_Q \rightarrow B_Q$ the restriction of $\partial : B \rightarrow B$ to the subspace $B_Q$ (i.e. $\partial^Q x = \partial x$ for every $x \in B_Q$). Hence for each $j_1 \ldots j_n \in \hat{Q}$ we get

$$
\partial^Q (e_{j_1 \ldots j_n}) = \sum_{i \in \mathcal{N}} e_i \partial_i (e_{j_1 \ldots j_n}) = \sum_{i \in \mathcal{N}} e_i \sum_{1 \leq k \leq n, j_k = i} q_{ij_i} \cdots q_{ij_{k-1}} e_{j_1 \ldots \hat{j}_k \ldots j_n} = \sum_{1 \leq k \leq n} q_{jk,j_1} \cdots q_{jk,j_{k-1}} e_{j_k j_1 \ldots \hat{j}_k \ldots j_n}.
$$

If $B_Q$ denotes the matrix of $\partial^Q$ w.r.t basis $\mathcal{B}_Q$ (totally ordered by the Johnson-Trotter ordering on permutations c.f [11]) of $B_Q$, then we can write

$$
B_Q e_{j_1 \ldots j_n} = \sum_{1 \leq k \leq n} q_{jk,j_1} \cdots q_{jk,j_{k-1}} e_{j_k j_1 \ldots \hat{j}_k \ldots j_n}.
$$

Note that for any multiset $Q = \{k_1^{n_1}, \ldots, k_p^{n_p}\}$ ($k_i$ distinct) of cardinality $n$ (where $n = n_1 + \cdots + n_p$) the size of the matrix $B_Q$ is equal to the following multinomial coefficient

$$
\frac{n!}{n_1! \cdots n_p!} = \binom{n}{n_1, \ldots, n_p} (= \dim B_Q).
$$

The entries of $B_Q$ are polynomials in $q_{ij}$'s, hence its determinant is also a polynomial in $q_{ij}$'s. Clearly, in the generic case (i.e. $Q$ is a set) the entries of $B_Q$ are monomials in $q_{ij}$'s and the size of $B_Q$ is equal to $n!$.

In the algebra $B$ of particular interest are the elements called constants. They are by definition annihilated by all multiparametric partial derivatives. In other words, an element $C \in B$ is called a constant in $B$ if $\partial_i(C) = 0$ for every $i \in \mathcal{N}$. It is obvious that $\partial C = \sum_{i \in \mathcal{N}} e_i \partial_i C = 0$ iff $\partial_i C = 0$ for all $i \in \mathcal{N}$.

We denote by $\mathcal{C}$ the space of all constants in $B$ and similarly by $\mathcal{C}_Q$ the space of all constants belonging to $B_Q$. Then $\mathcal{C} = \ker \partial$ (where $\ker \partial$ denotes the kernel of the multidegree operator $\partial$). By using the fact that the operator $\partial$ preserves the direct sum decomposition of the algebra $B$, we have that the space $\mathcal{C}$ inherits the direct sum decomposition into multigraded subspaces $\mathcal{C}_Q$. Hence the fundamental problem of determining the space $\mathcal{C}$ can be reduced to determining all finite dimensional spaces $\mathcal{C}_Q$ (= $\ker \partial^Q$) for all multisets $Q$ over $\mathcal{N}$. Thus of particular interest is the study of $\det B_Q$. The special role play the actual values of the parameters $q_{ij}$'s (called singular values or singular parameters) for which $\det B_Q$ vanishes. In Section 4 a formula for the factorization of the matrix $B_Q$ and its determinant will be given.
3 A twisted group algebra of the symmetric group

Here, we firstly recall some basic factorizations in the twisted group algebra \(A(S_n)\) of the symmetric group \(S_n\) with coefficients in a polynomial algebra in commuting variables \(X_{ab}\) \((1 \leq a, b \leq n)\).

Secondly, (in the next section) we will consider natural representation of the algebra \(A(S_n)\) on the weight subspaces \(B_Q\) of \(B\).

In [10] we have introduced a twisted group algebra \(A(S_n)\) of the symmetric group \(S_n\), which we recall now.

The elements of the twisted group algebra \(A(S_n) = R_n \rtimes \mathbb{C}[S_n]\) are the linear combinations \(\sum_{g_i \in S_n} p_i g_i\), where the coefficients are polynomials \(p_i = p_i(\ldots, X_{ab}, \ldots)\) in commuting variables \(X_{ab}\), \(1 \leq a, b \leq n\). The multiplication in \(A(S_n)\) is given by the formula

\[
(p_1 g_1) \cdot (p_2 g_2) = (p_1 \cdot (g_1 \cdot p_2)) g_1 g_2, 
\]

where \(g_1 \cdot p_2 = g_1 \cdot p_2(\ldots, X_{ab}, \ldots) = p_2(\ldots, X_{g_1(a)g_1(b)}, \ldots) g_1\).

The algebra \(A(S_n)\) is associative but not commutative (i.e. \(g \cdot p \neq p \cdot g\)).

In the algebra \(A(S_n)\) we have introduced more specific elements, denoted by \(g^*\) and defined by

\[
g^* := \left( \prod_{(a,b) \in I(g^{-1})} X_{ab} \right) g
\]

for every \(g \in S_n\), where \(I(g) = \{(a,b) \mid 1 \leq a < b \leq n, g(a) > g(b)\}\) denotes the set of inversions of the permutation \(g\).

Of particular interest are the elements \(t_{b,a}^* \in A(S_n)\), \(1 \leq a \leq b \leq n\), where for \(a < b\), \(t_{b,a} \in S_n\) denotes the inverse of the cyclic permutation \(t_{a,b} \in S_n\) i.e

\[
t_{b,a} = \begin{pmatrix}
1 & \cdots & a - 1 & a & a + 1 & \cdots & b - 1 & b & b + 1 & \cdots & n \\
1 & \cdots & a - 1 & a + 1 & a + 2 & \cdots & b & a & b + 1 & \cdots & n
\end{pmatrix}
\]

(see also [3]). Every permutation \(g \in S_n\) can be decomposed into cycles (from the left) as follows:

\[
g = t_{k_n,n} \cdot t_{k_{n-1,n-1}} \cdots t_{k_{j,j}} \cdots t_{k_2,2} \cdot t_{k_1,1} \left( = \prod_{1 \leq j \leq n} t_{k_{j,j}} \right).
\]

By applying (6) we obtain the following formula

\[
g_1^* \cdot g_2^* = X(g_1, g_2) (g_1 g_2)^*,
\]
where the multiplication factor $X(g_1, g_2)$ takes care of the reduced number of inversions in the group product of $g_1, g_2 \in S_n$ and it is given by

$$X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) \setminus I((g_1 g_2)^{-1})} X_{\{a,b\}} = \prod_{(a,b) \in I(g_1) \cap I(g_2^{-1})} X_{\{g_1(a), g_1(b)\}}.$$ 

By studying in details the elements $g^* \in A(S_n)$ and more specific properties arising from (9) and (8) (see also [3] and [4]) we came to the following conclusions.

Let $\alpha_n^* := \sum_{g \in S_n} g^*$, $n \geq 1$.

1. If we define simpler elements $\beta_k^* \in A(S_n)$ ($1 \leq k \leq n$) as follows

$$\beta_1^* = t_{n,1}^* + t_{n-1,1}^* + \cdots + t_{2,1}^* + t_{1,1}^*,$$

$$\beta_{n-1}^* = t_{n,2}^* + t_{n-1,2}^* + \cdots + t_{3,2}^* + t_{2,2}^*,$$

$$\vdots$$

$$\beta_{n-k+1}^* = t_{n,k}^* + t_{n-1,k}^* + \cdots + t_{k+1,k}^* + t_{k,k}^*,$$

$$\vdots$$

$$\beta_2^* = t_{n,n-1}^* + t_{n-1,n-1}^*,$$

$$\beta_1^* = t_{n,n}^* (= id)$$

then the element $\alpha_n^* \in A(S_n)$ can be decomposed into the product of elements $\beta_k^* \ (1 \leq k \leq n)$ i.e we have

$$\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^* = \prod_{1 \leq k \leq n} \beta_k^*. \tag{10}$$

2. Now we define yet simpler elements in $A(S_n)$ for all $1 \leq k \leq n-1$

$$\gamma_{n-k+1}^* = (id - t_{n,k}^*) \cdot (id - t_{n-1,k}^*) \cdots (id - t_{k+1,k}^*),$$

$$\delta_{n-k+1}^* = (id - (t_k^*)^2 t_{n,k+1}^*) \cdot (id - (t_k^*)^2 t_{n-1,k+1}^*) \cdots (id - (t_k^*)^2 t_{k+1,k+1}^*),$$

where $t_{k+1,k+1}^* = id$ and

$$(t_k^*)^2 = X_{\{k, k+1\}} \ id \tag{11}$$

because

$$(t_k^*)^2 = (X_{k+1} t_k) \cdot (X_{k+1}^{-1} t_k) = X_{k+1} \cdot X_{k+1} (t_k)^2 = X_{\{k, k+1\}} \ id$$

\footnote{For more details c.f the Ph.D. thesis \cite{7}.}
with $t_k = t_{k+1,k} (= t_{k,k+1})$.

Then each element $\beta_k^*, 2 \leq k \leq n$ can be further factored as

$$\beta_k^* = \delta_k^* \cdot (\gamma_k^*)^{-1}. \quad (12)$$

4 Representation of the twisted group algebra $\mathcal{A}(S_n)$ on the subspaces $B_Q$ of $\mathcal{B}$

Recall that $B_Q = \text{span}_\mathbb{C}\left\{e_{j_1...j_n} = e_{j_1} \cdots e_{j_n} \mid j_1 \cdots j_n \in \hat{Q}\right\}$ denotes the weight subspace of the free associative complex algebra $\mathcal{B}$, where $\hat{Q}$ is the set of all distinct permutations of the multiset $Q$ (see Section 2).

Let $V$ be a vector space over a field $F$. Then $\text{End}(V)$ denotes the algebra of all endomorphisms of $V$. If we denote by $A$ an associative algebra then by definition a representation of $A$ on $V$ is any algebra homomorphism $\varphi: A \to \text{End}(V)$.

Our next task is to define a representation (see the formula (14) below) $\varphi: \mathcal{A}(S_n) \to \text{End}(B_Q)$, where $B_Q$ is defined by (11).

We first recall that $R_n = \mathbb{C}[X_{ab} \mid 1 \leq a, b \leq n]$ denotes the polynomial ring with unit element $1 \in R_n$ and $\mathbb{C}[S_n] = \left\{\sum_{\sigma \in S_n} c_\sigma \sigma \mid c_\sigma \in \mathbb{C}\right\}$ denotes the usual group algebra. In $\mathbb{C}[S_n]$ the multiplication is given by

$$\left(\sum_{\sigma \in S_n} c_\sigma \sigma\right) \cdot \left(\sum_{\tau \in S_n} d_\tau \tau\right) = \sum_{\sigma, \tau \in S_n} (c_\sigma d_\tau) \sigma \tau.$$

Recall that $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$, so we first consider the representations $\varrho_1$ of $R_n$ and $\varrho_2$ of $\mathbb{C}[S_n]$:

$$\varrho_1: R_n \rightarrow \text{End}(B_Q),$$

$$\varrho_2: \mathbb{C}[S_n] \rightarrow \text{End}(B_Q)$$

given by Definition 4.1 and Definition 4.2 respectively.

Let $Q_{ab}, 1 \leq a, b \leq n$ denote a diagonal operator on $B_Q$ (see (11)) defined by

$$Q_{ab} e_{j_1...j_n} := q_{ja} q_{j_ab} e_{j_1...j_n}. \quad (13)$$

Note that

$$Q_{ab} \cdot Q_{cd} = Q_{cd} \cdot Q_{ab}.$$
Definition 4.1. We define a representation \( \varrho_1 : R_n \rightarrow \text{End}(\mathcal{B}_Q) \) on the generators \( X_{ab} \) by the formula
\[
\varrho_1(X_{ab}) := Q_{ab} \quad 1 \leq a, b \leq n.
\]
In other words, considering (13) we get
\[
\varrho_1(X_{ab}) e_{j_1 \ldots j_n} = q_{ab} e_{j_1 \ldots j_n}.
\]

Definition 4.2. We define a linear operator \( \varrho_2 : \mathbb{C}[S_n] \rightarrow \text{End}(\mathcal{B}_Q) \) by
\[
\varrho_2(g) e_{j_1 \ldots j_n} := e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}}
\]
for every \( g \in S_n \).

Proposition 4.3. A map \( \varrho_2 : \mathbb{C}[S_n] \rightarrow \text{End}(\mathcal{B}_Q) \) is a representation.

Proof. In fact \( \varrho_2 \) is a (right) regular representation.

Let \( \varrho : \mathcal{A}(S_n) \rightarrow \text{End}(\mathcal{B}_Q) \) be a map defined on decomposable elements by
\[
\varrho(pg) := \varrho_1(p) \cdot \varrho_2(g)
\]
for every \( p \in R_n \) and \( g \in S_n \) and extended by additivity. In the trivial cases we have
\[
(i) \quad \varrho(1 \cdot g) e_{j_1 \ldots j_n} = \varrho_1(1) \cdot \varrho_2(g) e_{j_1 \ldots j_n} = 1 \cdot e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} = e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}},
\]
\[
(ii) \quad \varrho(X_{ab} e) e_{j_1 \ldots j_n} = \varrho_1(X_{ab}) \cdot \varrho_2(e) e_{j_1 \ldots j_n} = Q_{ab} e_{j_1 \ldots j_n} = q_{j_{1} \ldots j_{n}} e_{j_1 \ldots j_n}.
\]

Remark 4.4. By applying the formula (14) on the general elements of the twisted group algebra we get:
\[
\varrho \left( \sum_{g_i \in S_n} p_i g_i \right) e_{j_1 \ldots j_n} = \sum_{g_i \in S_n} \varrho(p_i g_i) e_{j_1 \ldots j_n}
\]
\[
= \sum_{g_i \in S_n} \varrho_1(p_i (\ldots, X_{ab}, \ldots)) \cdot (\varrho_2(g_i) e_{j_1 \ldots j_n})
\]
\[
= \sum_{g_i \in S_n} p_i \left( \ldots, q_{g_i^{-1}(a)} g_{g_i^{-1}(b)} \ldots \right) e_{j_{g_i^{-1}(1)} \ldots j_{g_i^{-1}(n)}}.
\]

Note that the basic instance of the multiplication (6) in \( \mathcal{A}(S_n) \) is given by the following formula
\[
(X_{ab} g_1) \cdot (X_{cd} g_2) = (X_{ab} \cdot X_{g_1(c) g_1(a)}) g_1 g_2
\]
which are the consequences of the following two types of basic relations:
\[
X_{ab} \cdot X_{cd} = X_{cd} \cdot X_{ab},
\]
\[
g \cdot X_{ab} = X_{g(a) g(b)} g.
\]
Proposition 4.5. A map \( \varrho : \mathcal{A}(S_n) \rightarrow \text{End}(\mathcal{B}_Q) \) is a representation.

**Proof.** It is enough to check that \( \varrho \) preserves the basic relations (13) and (17), where we will apply the formula (14) and Definitions 4.1 and 4.2.

It is easy to see that:

(i) \( \varrho(X_{ab} \cdot X_{cd}) = Q_{ab} \cdot Q_{cd} = Q_{cd} \cdot Q_{ab} = \varrho(X_{cd} \cdot X_{ab}). \)

(ii) Now we will show that \( \varrho(g \cdot X_{ab}) e_{j_1 \ldots j_n} = \varrho(X_{g(a)g(b)}) e_{j_1 \ldots j_n}. \)

\[ L \equiv \varrho(g \cdot X_{ab}) e_{j_1 \ldots j_n} = \varrho_2(g) \varrho_1(X_{ab}) e_{j_1 \ldots j_n} = \varrho_2(g) q_{ja} q_{jb} e_{j_1 \ldots j_n} = q_{ja} q_{jb} e_{j_1 \ldots j_n}; \]

\[ D \equiv \varrho(g \cdot X_{ab}) g e_{j_1 \ldots j_n} = \varrho_1(X_{g(a)g(b)}) \varrho_2(g) e_{j_1 \ldots j_n} = Q_{g(a)g(b)} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} = q_{ja} q_{jb} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}}. \]

In the case \( Q \) is a set we call the representation \( \varrho \) in Proposition 4.5 a twisted regular representation.

**Lemma 4.6.** The representation \( \varrho \) applied to element \( g^* = \left( \prod_{(a,b) \in I(g^{-1})} X_{ab} \right) g \)

is given by

\[ \varrho(g^*) e_{j_1 \ldots j_n} = \prod_{(a,b) \in I(g)} q_{ja} q_{jb} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}}. \]  

**Proof.** Note that by applying the formula (13) on \( g^* = \left( \prod_{(a',b') \in I(g^{-1})} X_{a' b'} \right) g \)

we obtain

\[ \varrho(g^*) e_{j_1 \ldots j_n} = \prod_{(a',b') \in I(g^{-1})} \varrho_1(X_{a' b'}) \varrho_2(g) e_{j_1 \ldots j_n} = \prod_{(a',b') \in I(g^{-1})} q_{ja} q_{jb} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} = \prod_{(b,a) \in I(g)} q_{ja} q_{jb} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} \]

with \( a = g^{-1}(a'), \ b = g^{-1}(b'). \)

Now it is easy to check that \( (a',b') \in I(g^{-1}) \) implies \( a' < b', \ g^{-1}(a') > g^{-1}(b') \)
i.e. \( g(a) < g(b), \ a > b. \) Thus we get \( (b,a) \in I(g). \)

\[ \square \]
Remark 4.7. A direct consequence of the Lemma 4.6: the element \( \varrho(t_{b,a}^*) \in \text{End}(B_Q) \) is given by

\[
\varrho(t_{b,a}^*) e_{j_1 \ldots j_a j_{a+1} \ldots j_b \ldots j_n} = \prod_{a \leq i \leq b-1} q_{j_i j_i} e_{j_1 \ldots j_i j_a \ldots j_b \ldots j_n}
\]

and in special case \( \varrho(t_a^*) e_{j_1 \ldots j_a j_{a+1} \ldots j_n} = q_{j_a j_{a+1}} e_{j_1 \ldots j_a j_{a+1} \ldots j_n} \) (recall \( t_a^* = t_{a+1,a}^* \)).

From the identity (11) we obtain:

\[
\varrho((t_a^*)^2) e_{j_1 \ldots j_n} = \sigma_{j_a j_{a+1}} e_{j_1 \ldots j_n},
\]

where we denoted \( \sigma_{j_a j_{a+1}} := q_{j_a j_{a+1}} q_{j_{a+1} j_a} \).

Note that the matrices of the operators in the twisted regular representation of \( A(S_n) \) on the generic (resp. degenerate) weight subspaces \( B_Q \subset B \) are square matrices of size \( n! \) (resp. \( \text{Card} \hat{Q} \)) whose entries are monomials (resp. polynomials) in \( q_{ij} \)'s.

Recall that the element \( \alpha_n^* \in A(S_n) \) is given by \( \alpha_n^* = \sum_{g \in S_n} g^*, n \geq 1 \).

Proposition 4.8. Let \( \varrho: A(S_n) \to \text{End}(B_Q) \) be the twisted regular representation on the generic weight space \( B_Q \). Then the \( (k, j) \)-entry of the matrix \( A_Q \) of the \( \varrho(\alpha_n^*) \) is given by

\[
(A_Q)_{k,j} = \prod_{(a,b) \in I(g)} q_{j_a j_a},
\]

where \( g \) satisfies \( k = g.j \) (\( j = j_1 \ldots j_n \in \hat{Q}, k = k_1 \ldots k_n \in \hat{Q} \)).

Proof. By applying Lemma 4.6 we see that \( \varrho(\alpha_n^*) \in \text{End}(B_Q) \) acts as

\[
\varrho(\alpha_n^*) e_{j_1 \ldots j_n} = \sum_{g \in S_n} \left( \prod_{(a,b) \in I(g)} q_{j_a j_a} e_{j_{g^{-1}(1)} \ldots j_{g^{-1}(n)}} \right)
\]

i.e

\[
\varrho(\alpha_n^*) e_j = \sum_{g \in S_n} \left( \prod_{(a,b) \in I(g)} q_{j_a j_a} e_k \right).
\]

Therefore we have that the \( (k, j) \)-entry of \( A_Q \) is equal to \( \prod_{(a,b) \in I(g)} q_{j_a j_a} \), so the identity (11) follows. \( \square \)
In particular, if \( \text{Card } Q = 1 \), then \( A_Q = 1 \). Thus, we suppose that \( \text{Card } Q = n \geq 2 \).

**Remark 4.9.** If \( \varrho : \mathcal{A}(S_n) \to \text{End}(\mathcal{B}_Q) \) is the representation (on the degenerate weight subspace \( \mathcal{B}_Q \)), then the \( (k, j) \)-entry of \( A_Q \) is given by

\[
(A_Q)_{k,j} = \sum_{g \in g(k,j)} \left( \prod_{(a,b) \in I(g)} q_{ja_j} \right),
\]

where \( g(k,j) := \{ g \in S_n \mid k_p = j_{g^{-1}(p)} \text{ for all } 1 \leq p \leq n \} \).

### 4.1 Factorization of the matrix \( A_Q \)

Let us denote \( T_{b,a} := \varrho(t^*_{b,a}) \), \( T_a := \varrho(t^*_a) \). If \( b = a \) then \( T_{b,a} = \text{I} \). The \( (k, j) \)-entry of the corresponding matrices \( T_{b,a}, 1 \leq a < b \leq n \) and \( T_a, 1 \leq a \leq n - 1 \) are given by

\[
(T_{b,a})_{k,j} = \begin{cases} 
\prod_{a \leq i \leq b-1} q_{ja_i} & \text{if } k = t_{b,a} \cdot j \\
0 & \text{otherwise}
\end{cases}
\]

(21)

with \( t_{b,a} \cdot j = j_1 \ldots ja \ldots jb-1 \ldots j_n \) and

\[
(T_a)_{k,j} = \begin{cases} 
q_{ja_{j+1}a} & \text{if } k = t_{a} \cdot j \\
0 & \text{otherwise}
\end{cases}
\]

(22)

with \( t_{a} \cdot j = j_1 \ldots ja_{j+1}a \ldots j_n \) (see Remark 4.7). Now it is easy to see that

\[
(T_a)^2 e_{\cdot j} = \sigma_{ja_{j+1}} e_{\cdot j}
\]

(23)

is the diagonal matrix with \( \sigma_{ja_{j+1}} \) as \( j \)-th diagonal entry.

Now we consider the elements \( \beta_{k}^* \in \mathcal{A}(S_n), 1 \leq k \leq n \) defined before (10) in Section 3. The corresponding elements \( \varrho(\beta_{n-k+1}^*) \in \text{End}(\mathcal{B}_Q) \) are given by

\[
\varrho(\beta_{n-k+1}^*) e_{\cdot j} = \varrho \left( \sum_{k+1 \leq m \leq n} t_{m,k}^* \right) e_{\cdot j} = \left( \sum_{k \leq m \leq n} \varrho(t_{m,k}^*) \right) e_{\cdot j}
\]

\[
= \left( \sum_{k+1 \leq m \leq n} \varrho(t_{m,k}^*) + \varrho(id) \right) e_{\cdot j}
\]

i.e

\[
\varrho(\beta_{n-k+1}^*) e_{\cdot j} = \sum_{k+1 \leq m \leq n} \varrho(t_{m,k}^*) e_{\cdot j} + e_{\cdot j}
\]

(24)
Let 
\[ B_{Q,n-k+1} := \varrho(\beta_{n-k+1}^*), \quad 1 \leq k \leq n - 1, \]
where \( B_{Q,1} = \varrho(\beta_1^*) = \varrho(id) = I \). Then in the matrix notation \([24]\) can be written
\[ B_{Q,n-k+1} = \sum_{k+1 \leq m \leq n} T_{m,k} + I \]
or shorter \( B_{Q,n-k+1} = \sum_{k \leq m \leq n} T_{m,k} \).

Note that the \((k,j)\)-entry of \( B_{Q,n-k+1} \) (1 \( \leq k \leq n - 1 \)) is equal to \( \prod_{k \leq i \leq m-1} q_{j_{m,i}} \) if \( k = t_{m,k,j} = j_1 \ldots j_m j_k \ldots j_{m-1} \ldots j_1 \), otherwise it is equal to zero.

Thus we can write
\[
(B_{Q,n-k+1})_{k,j} = \begin{cases} 
\prod_{k \leq i < m} q_{j_{m,i}} & \text{if } k = t_{m,k,j} \leq m \leq n \\
0 & \text{otherwise}
\end{cases}
\]  
(25)

for each \( 1 \leq k \leq n - 1 \).

In the special case for \( m = k \) all \( j \)-th diagonal entries of \( B_{Q,n-k+1} \) are equal to one.

**Remark 4.10.** For \( k = 1 \) we have
\[ B_{Q,n} = \sum_{1 \leq m \leq n} T_{m,1} = T_{n,1} + T_{n-1,1} + \cdots + T_{3,1} + T_{2,1} + I \]
(where \( T_{1,1} = I \)) so the \((k,j)\)-entry of \( B_{Q,n} \) is given by
\[
(B_{Q,n})_{k,j} = \begin{cases} 
q_{j_{m,j_1}} \cdots q_{j_{m,j_{m-1}}} & \text{if } k = t_{m,1,j} \leq m \leq n \\
0 & \text{otherwise}
\end{cases}
\]
(with \( k = j_m j_1 \ldots j_{m-1} j_{m+1} \ldots j_n \)). In other words we get the following identity
\[ B_{Q,n} e_j = \sum_{1 \leq m \leq n} q_{j_{m,j_1}} \cdots q_{j_{m,j_{m-1}}} e_{j_{m,j_1} \ldots j_{m-1} j_{m+1} \ldots j_n} \]
(compare with \([5]\)). Now it is easy to see that the matrix \( B_{Q,n} \) is equal to the matrix \( B_{Q} \) (i.e. the matrix of \( \partial Q \) w.r.t monomial basis of \( B_Q \subset B \); see first section). It turns out that the factorization of the matrix \( B_{Q,n} \) is equivalent to factorization of \( B_{Q} \), so the problem of computing \( \det B_{Q} \) can be reduced to the problem of computing \( \det B_{Q,n} \). With this motivation we are going to find a formula for the factorization of \( B_{Q,n} \) and also its determinant.
In what follows we will consider the additional elements $\gamma^*_k$, $\delta^*_k$, $1 \leq k \leq n$ in the algebra $\mathcal{A}(S_n)$ defined after (11) in Section 3. The corresponding elements $\varphi(\gamma^*_{n-k+1}), \varphi(\delta^*_{n-k+1}) \in \text{End}(\mathcal{B}_Q)$, $1 \leq k \leq n - 1$ are given by

\[
\varphi(\gamma^*_{n-k+1}) e_j = (id - \varphi(t^*_{n,k})) \cdot (id - \varphi(t^*_{n-1,k})) \cdots (id - \varphi(t^*_{k+1,k})) e_j
\]

\[
\varphi(\delta^*_{n-k+1}) e_j = (id - \varphi((t^*_k)^2) \varphi(t^*_{n,k+1})) \cdot (id - \varphi((t^*_k)^2) \varphi(t^*_{n-1,k+1})) \cdots (id - \varphi((t^*_k)^2) \varphi(t^*_{k+2,k+1})) \cdot (id - \varphi((t^*_k)^2)) e_j
\]

which in matrix notation corresponds to the following expressions

\[
C_{Q,n-k+1} = (I - T_{n,k}) \cdot (I - T_{n-1,k}) \cdots (I - T_{k+1,k})
\]

\[
D_{Q,n-k+1} = (I - (T_k)^2 T_{n,k+1}) \cdot (I - (T_k)^2 T_{n-1,k+1}) \cdots (I - (T_k)^2)
\]

Here we have introduced notations

\[
C_{Q,n-k+1} := \varphi(\gamma^*_{n-k+1}), \quad D_{Q,n-k+1} := \varphi(\delta^*_{n-k+1}),
\]

$1 \leq k \leq n - 1$. Clearly, $(T_k)^2 = (T_{k+1,k})^2$ is the diagonal matrix given by (23). By using the identity (12) we obtain

\[
B_{Q,n-k+1} = D_{Q,n-k+1} \cdot (C_{Q,n-k+1})^{-1} \quad \text{for all} \quad 1 \leq k \leq n - 1.
\]

and more precisely

\[
B_{Q,n-k+1} = (I - (T_k)^2 T_{n,k+1}) \cdot (I - (T_k)^2 T_{n-1,k+1}) \cdots (I - (T_k)^2 T_{k+2,k+1}) \cdot (I - (T_k)^2) \cdot (I - T_{k+1,k})^{-1} \cdots (I - T_{n-1,k})^{-1} \cdot (I - T_{n,k})^{-1}
\]

or in shorter form

\[
B_{Q,n-k+1} = \prod_{k+1 \leq m \leq n} (I - (T_k)^2 T_{m,k+1}) \cdot \prod_{k+1 \leq m \leq n} (I - T_{m,k})^{-1}. \quad (26)
\]

On the other hand, from the identity (10) in $\mathcal{A}(S_n)$ we get

\[
A_Q = \prod_{1 \leq k \leq n-1} (D_{Q,n-k+1} \cdot (C_{Q,n-k+1})^{-1})
\]

i.e

\[
A_Q = \prod_{1 \leq k \leq n-1} \left( \prod_{k+1 \leq m \leq n} (I - (T_k)^2 T_{m,k+1}) \cdot \prod_{k+1 \leq m \leq n} (I - T_{m,k})^{-1} \right). \quad (27)
\]

Now it is easy to see that for computing $\det B_{Q,n-k+1}$ and $\det A_Q$ it is enough to compute $\det (I - T_{b,a})$ and $\det (I - (T_{a-1})^2 T_{b,a})$ ($1 \leq a < b \leq n$).
Let us denote
\[
\binom{Q}{m} = \{ T \subseteq Q \mid \text{Card } T = m \},
\]
\[
\sigma_T = \prod_{\{i \neq j\} \subseteq T} \sigma_{ij} = \prod_{i \neq j \in T} q_{ij}.
\]

**Lemma 4.11.** Let \( \varphi: \mathcal{A}(S_n) \to \text{End}(\mathcal{B}_Q) \) be the twisted regular representation of twisted group algebra \( \mathcal{A}(S_n) \) on any generic subspace \( \mathcal{B}_Q \) of the algebra \( \mathcal{B} \). Then
\[
(i) \quad \det(I - T_{b,a}) = \prod_{T \in \binom{Q_{b-a+1}}{b-a}} (1 - \sigma_T)^{(b-a)! \cdot (n-b+a-1)!} \quad (1 \leq a < b \leq n)
\]
\[
(ii) \quad \det(I - (T_{a-1})^2 T_{b,a}) = \prod_{T \in \binom{Q_{b-a+2}}{b-a+2}} (1 - \sigma_T)^{(b-a)! \cdot (b-a+2) \cdot (n-b+a-2)!}
\end{align*}
(1 < a \leq b \leq n).

Note that this Lemma is the twisted group algebra analogue of the Lemma 1.9.1 in the paper of Svrtan and Meljanac (see [3]). Therefore the proof will be similar to the proof of Lemma 1.9.1. Here we use the factorizations in different direction.

**Proof.**

(i) Let \( H := \langle t_{b,a} \rangle \subset S_n \) be the cyclic subgroup of \( S_n \) generated by the cycle \( t_{b,a} \) (whose length is equal to \( b - a + 1 \)). Then every \( H \)-orbit on generic subspace \( \mathcal{B}_Q \) is given by
\[
\mathcal{B}_Q^{[j]^b_a} = \text{span}_\mathbb{C} \left\{ e_{i_{b,a}^k} \mid 0 \leq k \leq b - a \right\},
\]
which corresponds to a cyclic \( t_{b,a} \)-equivalence class
\[
[j]^b_a = j_1 j_2 \cdots (j_a j_{a+1} \cdots j_b) \cdots j_n \quad \text{of the sequence } \underline{j} = j_1 \cdots j_n \in \check{Q}.
\]
We get
\[
T_{b,a} \left( e_{i_{b,a}^k} \right) = c_k e_{i_{b,a}^{k+1}}, \quad 0 \leq k \leq b - a,
\]
where
\[
c_0 = q_{j_b j_a} q_{j_{b+1} j_a} \cdots q_{j_b j_{b-1}},
\]
\[
c_1 = q_{j_{b-1} j_b} q_{j_{b-1} j_a} q_{j_{b-1} j_{a+1}} \cdots q_{j_{b-1} j_{b-2}},
\]
\[
c_2 = q_{j_{b-2} j_{b-1}} q_{j_{b-2} j_b} q_{j_{b-2} j_a} \cdots q_{j_{b-2} j_{b-3}},
\]
\[\vdots\]
\[
c_{b-a-1} = q_{j_{a+1} j_a} q_{j_a+1 j_a+3} q_{j_a+1 j_a+4} \cdots q_{j_{a+1} j_a},
\]
\[
c_{b-a} = q_{j_a j_a+2} q_{j_a j_a+3} q_{j_a j_{a+3}} \cdots q_{j_a j_b}.
\]

By using (21) and by applying the identities given above, we obtain the following

\[(T_{b,a})^k e_{\frac{k}{2}} = \prod_{0 \leq i \leq k-1} c_i e_{i,k_{b,a}^*} \quad 1 \leq k \leq b - a + 1. \quad (28)\]

By considering that \( T_{b,a} | B_{Q}^{[2]\mathbb{R}} \) is a cyclic operator (which corresponds to cyclic matrix \( T_{b,a} \)) we can write

\[
\det \left( (I - T_{b,a}) | B_{Q}^{[2]\mathbb{R}} \right) = 1 - \prod_{0 \leq i \leq b-a} c_i
\]

\[
= 1 - \prod_{i \neq j \in T} q_{ij} = 1 - \prod_{\{i,j\} \subset T} \sigma_{ij}
\]

i.e

\[
\det \left( (I - T_{b,a}) | B_{Q}^{[2]\mathbb{R}} \right) = 1 - \sigma_T \quad (29)
\]

where \( \sigma_{ij} = q_{ij}q_{ji} \), \( T = \{j_a, \ldots, j_b\} \) \((1 \leq a < b \leq n)\) and \( \text{Card} \ T = b - a + 1 \).

Now we give that there are \((b-a)! \cdot (n-(b-a+1))! = (b-a)! \cdot (n-b+a-1)! H\)-orbits for which determinant (29) gets the value of \( 1 - \sigma_T \), therefore we obtain \( \det(I - T_{b,a}) = \prod_{T \in (\mathcal{Q})}(1 - \sigma_T)^{(b-a)! \cdot (n-b+a-1)!} \).

(ii) Will be proven in a manner similar to (i), where we have

\[(T_{a-1})^2 T_{b,a} \left( e_{i,k_{b,a}^*} \right) = d_k e_{i,k_{b,a}^*} \quad 0 \leq k \leq b - a,
\]

with

\[
d_0 = \sigma_{j_{a-1}j_0} c_0 = \sigma_{j_{a-1}j_b}q_{j_bj_a}q_{j_bj_{a+1}}q_{j_bj_{a+2}} \cdots q_{j_bj_{a-1}}
\]

\[
d_1 = \sigma_{j_{a-1}j_{b-1}} c_1 = \sigma_{j_{a-1}j_{b-1}}q_{j_{b-1}j_b}q_{j_{b-1}j_{a+1}}q_{j_{b-1}j_{a+2}} \cdots q_{j_{b-1}j_{b-2}}
\]

\[
d_2 = \sigma_{j_{a-1}j_{b-2}} c_2 = \sigma_{j_{a-1}j_{b-2}}q_{j_{b-2}j_{b-1}}q_{j_{b-2}j_{a+1}}q_{j_{b-2}j_{a+2}} \cdots q_{j_{b-2}j_{b-3}}
\]

\[
\vdots
\]

\[
d_{b-a-1} = \sigma_{j_{a-1}j_{a+1}} c_{b-a+1} = \sigma_{j_{a-1}j_{a+1}}q_{j_{a+1}j_a}q_{j_{a+1}j_{a+2}}q_{j_{a+1}j_{a+3}}q_{j_{a+1}j_{a+4}} \cdots q_{j_{a+1}j_a}
\]

\[
d_{b-a} = \sigma_{j_{a-1}j_a} c_{b-a} = \sigma_{j_{a-1}j_a}q_{j_{a}j_{a+1}}q_{j_{a}j_{a+2}}q_{j_{a}j_{a+3}}q_{j_{a}j_{a+4}} \cdots q_{j_{a}j_a}
\]

Here we have

\[
((T_{a-1})^2 T_{b,a})^k e_{\frac{k}{2}} = \prod_{0 \leq i \leq k-1} d_i e_{i,k_{b,a}^*} \quad 1 \leq k \leq b - a + 1. \quad (30)
\]
and
\[
\det \left( (I - (T_{a-1})^2 T_{b,a})|B^{ij}_{Q} \right) = 1 - \prod_{0 \leq i \leq b-a} d_i \\
= 1 - \prod_{i \neq j \in T} q_{ij} = 1 - \prod_{\{i,j\} \subset T} \sigma_{ij}
\]
i.e
\[
\det \left( (I - (T_{a-1})^2 T_{b,a})|B^{ij}_{Q} \right) = 1 - \sigma_{T} \quad (31)
\]
where \( T = \{j_{a-1}, j_a, \ldots, j_b\} \) \((1 < a \leq b \leq n)\) and \( \text{Card } T = b - a + 2 \). There are \((b - a)! \cdot (b - a + 2) \cdot (n - b + a - 2)! \) \(H\)-orbits for which determinant (31) gets the value of \(1 - \sigma_{T}\), therefore we have proved (ii).

**Theorem 4.12.** Let \( \varphi : A(S_n) \rightarrow \text{End}(B_Q) \) be a twisted regular representation (where \( B_Q \) is generic subspace of \( B \)). Then we have

(i) \( \det A_Q = \prod_{2 \leq m \leq n} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(m-2)! \cdot (n-m+1)!} \),

(ii) \( \det B_{Q,n-k+1} = \prod_{2 \leq m \leq n-k+1} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(m-2)! \cdot (n-m)!} \), \( 1 < k \leq n-1 \).

This Theorem is similar to Theorem 1.9.2 in [3].

**Proof.** By using Lemma 4.11 we get the following:

\[
\det C_{Q,n-k+1} = \prod_{k+1 \leq p \leq n} \det(I - T_{p,k})
\]
\[
= \prod_{k+1 \leq p \leq n} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(p-k)! \cdot (n-p+k-1)!}
\]
\[
= \prod_{2 \leq m \leq n-k+1} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(m-1)! \cdot (n-m)!}
\]

\[
\det D_{Q,n-k+1} = \prod_{k+1 \leq p \leq n} \det(I - (T_k)^2 T_{p,k+1})
\]
\[
= \prod_{k+1 \leq p \leq n} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(p-k-1)! \cdot (p-k+1) \cdot (n-p+k-1)!}
\]
\[
= \prod_{2 \leq m \leq n-k+1} \prod_{T \in (Q^m)} (1 - \sigma_{T})^{(m-2)! \cdot m \cdot (n-m)!}
\]

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Therefore by applying the formula  \( \det B_{Q,n-k+1} = \frac{\det D_{Q,n-k+1}}{\det C_{Q,n-k+1}} \) (see also (26)) we get the formula given in (ii), i.e

\[
\det B_{Q,n-k+1} = \prod_{2 \leq m \leq n-k+1} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)!\cdot(n-m)!}.
\]

On the other hand by applying  \( \det A_Q = \prod_{1 \leq k \leq n-1} \det B_{Q,n-k+1} \) (see also (27)) we get

\[
\det A_Q = \prod_{1 \leq k \leq n-1} \prod_{2 \leq m \leq n-k+1} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)!\cdot(n-m)!}
\]

\[
= \prod_{2 \leq m \leq n} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)!\cdot(n-m)!\cdot(n-m+1)}
\]

i.e

\[
\det A_Q = \prod_{2 \leq m \leq n} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)!\cdot(n-m+1)!}.
\]

More results of this type will be given in a subsequent paper on computation of constants in multiparametric quon algebras by using a twisted group algebra approach (c.f [8]), which is more efficient than the approach of the paper [9].

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