Analysis of a projection method for the Stokes problem using an $\varepsilon$-Stokes approach

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Received: 27 December 2018 / Revised: 1 June 2019 / Published online: 12 July 2019
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Abstract
We generalize pressure boundary conditions of an $\varepsilon$-Stokes problem. Our $\varepsilon$-Stokes problem connects the classical Stokes problem and the corresponding pressure-Poisson equation using one parameter $\varepsilon > 0$. For the Dirichlet boundary condition, it is proven in Matsui and Muntean (Adv Math Sci Appl, 27:181–191, 2018) that the solution for the $\varepsilon$-Stokes problem converges to the one for the Stokes problem as $\varepsilon$ tends to 0, and to the one for the pressure-Poisson problem as $\varepsilon$ tends to $\infty$. Here, we extend these results to the Neumann and mixed boundary conditions. We also establish error estimates in suitable norms between the solutions to the $\varepsilon$-Stokes problem, the pressure-Poisson problem and the Stokes problem, respectively. Several numerical examples are provided to show that several such error estimates are optimal in $\varepsilon$. Our error estimates are improved if one uses the Neumann boundary conditions. In addition, we show that the solution to the $\varepsilon$-Stokes problem has a nice asymptotic structure.

Keywords Stokes problem · Pressure-Poisson equation · Asymptotic analysis · Finite element method

Mathematics Subject Classification 76D03 · 35Q35 · 35B40 · 65N30

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n (n \geq 2, n \in \mathbb{N})$ with Lipschitz continuous boundary $\Gamma$ and let $F : \Omega \to \mathbb{R}^n$ be a given applied force field and $u_b : \Gamma \to \mathbb{R}^n$ be a given Dirichlet boundary data satisfying $\int_{\Gamma} u_b \cdot \nu = 0$, where $\nu$ is the unit outward normal vector on $\Gamma$. A strong form of the Stokes problem is given as follows. Find $u_S : \Omega \to \mathbb{R}^n$ and $p_S : \Omega \to \mathbb{R}$ such that

$$\begin{cases}
-\Delta u_S + \nabla p_S = F \quad \text{in} \quad \Omega, \\
\operatorname{div} u_S = 0 \quad \text{in} \quad \Omega, \\
u = u_b \quad \text{on} \quad \Gamma,
\end{cases}$$

(S)

where $u_s$ and $p_s$ are the velocity and the pressure of the flow governed by (S), respectively. We refer to [21] for the details on the Stokes problem (i.e., physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u_S + \nabla p_S) = -\Delta (\operatorname{div} u_S) + \Delta p_S = \Delta p_S. \quad (1.1)$$

This equation is often called the pressure-Poisson equation and is used in numerical schemes such as MAC (marker and cell), SMAC (simplified MAC) and the projection methods (see, e.g., [1,3,6,11,12,15,18,20]). Based on the above, we consider a similar problem. Find $u_{pp} : \Omega \to \mathbb{R}^n$ and $p_{pp} : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases}
-\Delta u_{pp} + \nabla p_{pp} = F \quad \text{in} \quad \Omega, \\
-\epsilon \Delta p_{pp} = -\operatorname{div} F \quad \text{in} \quad \Omega, \\
u_{pp} = u_b \quad \text{on} \quad \Gamma,
\end{cases}$$

(PP)

We call this problem the pressure-Poisson problem. The idea of using (1.1) instead of $\operatorname{div} u_S = 0$ is useful for calculating the pressure numerically in the Navier–Stokes equation. For example, this idea is used in MAC, SMAC and projection methods. The Dirichlet boundary condition for the pressure is used in an outflow boundary [2,22]. See also [4,5,16].

We introduce an “interpolation” between problems (S) and (PP). For $\epsilon > 0$, find $u_\epsilon : \Omega \to \mathbb{R}^n$ and $p_\epsilon : \Omega \to \mathbb{R}$ such that

$$\begin{cases}
-\Delta u_\epsilon + \nabla p_\epsilon = F \quad \text{in} \quad \Omega, \\
-\epsilon \Delta p_\epsilon + \operatorname{div} u_\epsilon = -\epsilon \operatorname{div} F \quad \text{in} \quad \Omega, \\
u_\epsilon = u_b \quad \text{on} \quad \Gamma,
\end{cases}$$

(ES)

This problem is called the $\epsilon$-Stokes problem (ES) in [17]. In [7,10,14], the authors treat this problem as an approximation of the Stokes problem to avoid numerical instabilities. The $\epsilon$-Stokes problem approximates the Stokes problem (S) as $\epsilon \to 0$ and the pressure-Poisson problem (PP) as $\epsilon \to \infty$ (Fig. 1). It is shown in [17] that if $p_S \in H^1(\Omega)$ then there exists a constant $c > 0$ independent of $\epsilon$ such that
where $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ is the standard trace operator [9]. From the first inequality, if we have a good prediction value for pressure on $\Gamma$, then $u_{PP}$ is a good approximation of $u_S$. Moreover, $u_\varepsilon$ is also a good approximation of $u_S$ from the second inequality.

Next we specify the boundary conditions for $p_{PP}$ and $p_\varepsilon$. We assume that the boundary $\Gamma$ is a union of two open subsets $\Gamma_D$ and $\Gamma_N$ such that

$$|\Gamma \setminus (\Gamma_D \cup \Gamma_N)| = 0, \quad |\Gamma_D| > 0, \quad \Gamma_D \cap \Gamma_N = \emptyset,$$

and number of connected components of $\Gamma_D$ and $\Gamma_N$ with respect to the relative topology of $\Gamma$ are finite. We consider a Neumann boundary condition (1.2) and a mixed boundary condition (1.3),

\[
\begin{align*}
\frac{\partial p_{PP}}{\partial v} &= g_b \quad \text{on } \Gamma, \\
\frac{\partial p_\varepsilon}{\partial v} &= g_b \quad \text{on } \Gamma, \\
\begin{cases}
\frac{\partial p_{PP}}{\partial v} &= g_b \quad \text{on } \Gamma_D, \\
\frac{\partial p_{PP}}{\partial v} &= g_b \quad \text{on } \Gamma_N, \\
p_{PP} &= p_b \quad \text{on } \Gamma_D, \\
p_\varepsilon &= p_b \quad \text{on } \Gamma_D, \\
\frac{\partial p_\varepsilon}{\partial v} &= g_b \quad \text{on } \Gamma_N,
\end{cases}
\end{align*}
\]

where $p_b : \Gamma_D \rightarrow \mathbb{R}$ and $g_b = \Gamma \rightarrow \mathbb{R}$ satisfying $\int_{\Gamma} g_b = \int_{\Gamma} \text{div} F$ are given boundary data.

In [17], the authors impose Dirichlet boundary conditions for $p_{PP}$ and $p_\varepsilon$ (i.e., (1.3) with $\Gamma_D = \Gamma$ and $\Gamma_N = \emptyset$). For such boundary conditions, they introduce a weak solution $(u_\varepsilon, p_\varepsilon)$ to the $\varepsilon$-Stokes problem (ES) and prove that $(u_\varepsilon, p_\varepsilon)$ strongly converges in $H^1(\Omega)^n \times H^1(\Omega)$ to a weak solution to the pressure-Poisson problem (PP) as $\varepsilon \rightarrow \infty$ and weakly converges in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to a weak solution $(u_S, p_S)$ to the Stokes problem (S) as $\varepsilon \rightarrow 0$. Moreover, if $p_S \in H^1(\Omega)$, then strong convergence of $(u_\varepsilon, p_\varepsilon)$ to $(u_S, p_S)$ as $\varepsilon \rightarrow 0$ holds.

In this paper, we generalize the Dirichlet boundary condition of $p_{PP}$ and $p_\varepsilon$ (i.e., (1.3) with $\Gamma_D = \Gamma$ and $\Gamma_N = \emptyset$), for which boundary conditions, they introduce a weak solution $(u_\varepsilon, p_\varepsilon)$ to the $\varepsilon$-Stokes problem (ES) and prove that $(u_\varepsilon, p_\varepsilon)$ strongly converges in $H^1(\Omega)^n \times H^1(\Omega)$ to a weak solution to the pressure-Poisson problem (PP) as $\varepsilon \rightarrow \infty$ and weakly converges in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to a weak solution $(u_S, p_S)$ to the Stokes problem (S) as $\varepsilon \rightarrow 0$. Moreover, if $p_S \in H^1(\Omega)$, then strong convergence of $(u_\varepsilon, p_\varepsilon)$ to $(u_S, p_S)$ as $\varepsilon \rightarrow 0$ holds.

The organization of this paper is as follows. In Sect. 2 we introduce the notation used in this work and the weak form of these problems. We also prove the well-posedness of the problem.
of the problems (PP) and (ES) and show some of their properties. In Sect. 3, we study
that the solution to (ES) converges to the solution to (PP) in the strong topology as
$\varepsilon \to \infty$. We also explore here the structure of the regular perturbation asymptotics.
Section 4 is devoted to proving that the solution to (ES) converges to the solution to
(S) in the weak and strong topology as $\varepsilon \to 0$. Finally, in Sect. 5, we show several
numerical examples of these problems and the numerical errors between the problems
(ES) and (PP) and between the problems (ES) and (S) using the P2/P1 finite element
method. Proofs for several theorems which are similar to ones in [17] are described
in the appendix.

2 Well-posedness

In this section, we introduce the notation and the weak form of the problems (S), (PP)
and (ES), and prove their well-posedness. We give estimates between these solutions
by using a pressure error on the boundary $\Gamma$.

2.1 Notation

We set

$\mathcal{C}^\infty_0(\Omega) := \{ \varphi \in \mathcal{C}^\infty(\Omega) \mid \text{supp}(\varphi) \subset \Omega \}$,

$L^2(\Omega)/\mathbb{R} := \left\{ \psi \in L^2(\Omega) \mid \int_\Omega \psi = 0 \right\}$,

$H^1(\Omega)/\mathbb{R} := H^1(\Omega) \cap (L^2(\Omega)/\mathbb{R})$,

$H^1_{0,D}(\Omega) := \{ \psi \in H^1(\Omega) \mid \psi|_{\Gamma_D} = 0 \}$.

For $m \in \mathbb{N}$, $H^{-1}(\Omega)^m := (H^1_0(\Omega)^m)^*$ is equipped with the dual norm

$$\| f \|_{H^{-1}(\Omega)^m} := \sup_{\varphi \in S_m} \langle f, \varphi \rangle \text{ for } f \in H^{-1}(\Omega)^m,$$

where

$$S_m := \{ \varphi \in H^1_0(\Omega)^m \mid \| \nabla \varphi \|_{L^2(\Omega)^n \times m} = 1 \}.$$

Let $Q \subset H^1(\Omega)$ be a closed subspace such that there exists a constant $c > 0$
for which $\| q \|_{L^2(\Omega)} \leq c \| \nabla q \|_{L^2(\Omega)^n}$ for all $q \in Q$. We set the norm $\| \psi \|_Q := \| \nabla \psi \|_{L^2(\Omega)^n}$. By the definition of $Q$, the norm $\| \cdot \|_Q$ is equivalent to the norm $\| \cdot \|_{H^1(\Omega)}$. The dual space $Q^*$ is equipped with the norm

$$\| f \|_{Q^*} := \sup_{\psi \in S_Q} \langle f, \psi \rangle.$$
for \( f \in Q^* \), where

\[
S_Q := \{ \psi \in Q \mid \|\psi\|_Q = 1 \} = \{ \psi \in Q \mid \|\nabla\psi\|_{L^2(\Omega)^n} = 1 \}.
\]

We define

\[
[p] := p - \frac{1}{|\Omega|} \int_{\Omega} p,
\]

\[
\|p\|_{L^2(\Omega)/\mathbb{R}} := \inf_{a \in \mathbb{R}} \|p - a\|_{L^2(\Omega)} = \|[p]\|_{L^2(\Omega)},
\]

\[
\langle \nabla p, \varphi \rangle := - \int_{\Omega} p \div \varphi
\]

\[
\nabla u : \nabla \varphi := \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_i}
\]

for all \( p \in L^2(\Omega), u, \varphi \in H^1_0(\Omega)^n \), where \(|\Omega|\) is the volume of \( \Omega \).

### 2.2 Preliminary results

Let \( \gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma)) \) be the standard trace operator. The trace operator \( \gamma_0 \) is surjective and satisfies \( \text{Ker}(\gamma_0) = H^1_0(\Omega) \) [9, Theorem 1.5]. Let \( \nu \) be the unit outward normal for \( \Gamma \). Since \( \nu \) is a unit vector, \( H^1(\Omega)^n \ni u \mapsto u \cdot \nu := (\gamma_0 u) \cdot \nu \in L^2(\Gamma) \) is a linear continuous map. For all \( u \in H^1(\Omega)^n \) and \( \omega \in H^1(\Omega) \), the following Gauss divergence formula holds:

\[
\int_{\Omega} u \cdot \nabla \omega + \int_{\Omega} (\div u) \omega = \int_{\Gamma} (u \cdot \nu) \omega.
\]

We recall the following four embedding theorems which plays an important role in the proof of the existence of pressure solutions to the Stokes problem. For the proof of Theorem 2.1, see [19, Lemma 7.1] and [8, Theorem 3.2 and Remark 3.1].

**Theorem 2.1** There exists a constant \( c > 0 \) such that

\[
\|f\|_{L^2(\Omega)} \leq c(\|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{H^{-1}(\Omega)^n})
\]

for all \( f \in L^2(\Omega) \).

The following result follows from Theorem 2.1.

**Theorem 2.2** [9, Corollary 2.1, 2°] There exists a constant \( c > 0 \) such that

\[
\|f\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla f\|_{H^{-1}(\Omega)^n}
\]

for all \( f \in L^2(\Omega) \).

The following two embedding theorems are often called the Poincaré inequality.
Theorem 2.3 [19, Theorem 7.8] There exists a constant $c > 0$ such that

$$
\| \psi \|_{L^2(\Omega)} \leq c \| \nabla \psi \|_{L^2(\Omega)^n}
$$

for all $\psi \in H^1(\Omega)/\mathbb{R}$.

Theorem 2.4 [9, Lemma 3.1] There exists a constant $c > 0$ such that

$$
\| \psi \|_{L^2(\Omega)} \leq c \| \nabla \psi \|_{L^2(\Omega)^n}
$$

for all $\psi \in H^1_{0,D}(\Omega)$.

2.3 Weak formulations of the problems (PP), (S) and (ES)

We assume the following conditions for $F$, $u_b$, $g_b$ and $p_0$:

$$
F \in L^2(\Omega)^n, \quad u_b \in H^{1/2}(\Gamma), \quad \int_{\Gamma} u_b \cdot v = 0,
$$

(2.1)

$$
g_b \in L^2(\Gamma), \quad \text{div} F \in L^2(\Omega),
$$

(2.2)

$$
p_0 \in H^1(\Omega), \quad p_0 = p_b \quad \text{in} \ H^{1/2}(\Gamma_D).
$$

(2.3)

Moreover, if we apply the Neumann boundary condition (1.2) for (PP) and (ES), then we also assume that

$$
\int_{\Gamma} g_b = \int_{\Omega} \text{div} F.
$$

(2.4)

We will not discuss the existence of $p_0 \in H^1(\Omega)$ such that $p_0 = p_b$ in $H^{1/2}(\Gamma_D)$ for a given $p_b \in H^{1/2}(\Gamma_D)$. We will simply study the problems (PP) and (ES) with the mixed boundary condition (1.3) assuming the existence of $p_0 \in H^1(\Omega)$.

We start by defining the weak solution to (S). For all $\varphi \in H^1_0(\Omega)^n$, we obtain from the first equation of (S) that

$$
\int_{\Omega} F \cdot \varphi = -\int_{\Gamma} \frac{\partial u_s}{\partial v} \cdot \varphi + \int_{\Omega} \nabla u_s : \nabla \varphi + \int_{\Omega} \nabla p_s \cdot \varphi
$$

$$
= \int_{\Omega} \nabla u_s : \nabla \varphi + \int_{\Omega} \nabla p_s \cdot \varphi.
$$

Using this expression, the weak form of the Stokes problem becomes as follows: find $u_s \in H^1(\Omega)^n$ and $p_s \in L^2(\Omega)/\mathbb{R}$ such that

$$
\begin{cases}
\int_{\Omega} \nabla u_s : \nabla \varphi + \langle \nabla p_s, \varphi \rangle = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\text{div} u_s = 0 & \text{in } L^2(\Omega), \\
u_s = u_b & \text{in } H^{1/2}(\Gamma)^n.
\end{cases}
$$

(S')
Next, we define the weak formulations of (PP) and (ES) first for the Neumann boundary condition (1.2) and then for the mixed boundary condition (1.3). After that, we define generalized weak formulations for (PP) and (ES) which cover both cases.

First, we apply the Neumann boundary condition (1.2) for (PP) and (ES). We take a test function \( \psi \in H^1(\Omega) \). From the second equation of (PP), we obtain
\[
- \int_{\Omega} (\text{div}\, F) \psi = - \int_{\Omega} (\Delta p_{\text{PP}}) \psi = - \int_{\Gamma} \frac{\partial p_{\text{PP}}}{\partial n} \psi + \int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi
\]
\[
= - \int_{\Gamma} g_b \psi + \int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi.
\]

Hence,
\[
\int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi = \int_{\Gamma} g_b \psi - \int_{\Omega} (\text{div}\, F) \psi.
\]

We note that \( \int_{\Gamma} g_b \psi - \int_{\Omega} (\text{div}\, F) \psi = \int_{\Gamma} g_b [\psi] - \int_{\Omega} (\text{div}\, F) [\psi] \) for all \( \psi \in H^1(\Omega) \) by (2.4). Therefore, the weak form of the pressure-Poisson problem with the Neumann boundary condition (1.2) becomes as follows. Find \( u_{\text{PP}} \in H^1(\Omega) \) and \( p_{\text{PP}} \in H^1(\Omega) / \mathbb{R} \) such that
\[
\begin{cases}
\int_{\Omega} \nabla u_{\text{PP}} : \nabla \varphi + \int_{\Omega} \nabla p_{\text{PP}} \cdot \varphi = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega), \\
\int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi = \langle G_1, \psi \rangle & \text{for all } \psi \in H^1(\Omega) / \mathbb{R}, \\
u_{\text{PP}} = u_b
\end{cases}
\]

where \( G_1 \in H^1(\Omega)^* \) such that \( \langle G_1, \psi \rangle = \int_{\Gamma} g_b \psi - \int_{\Omega} (\text{div} F) \psi \) (\( \psi \in H^1(\Omega) \)).

The weak form of (ES) with the Neumann boundary condition can be defined similarly to that of (PP). Find \( u_{\varepsilon} \in H^1(\Omega)^n \) and \( p_{\varepsilon} \in H^1(\Omega) / \mathbb{R} \) such that
\[
\begin{cases}
\int_{\Omega} \nabla u_{\varepsilon} : \nabla \varphi + \int_{\Omega} \nabla p_{\varepsilon} \cdot \varphi = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega), \\
\varepsilon \int_{\Omega} \nabla p_{\varepsilon} \cdot \nabla \psi + \int_{\Omega} (\text{div} u_{\varepsilon}) \psi = \varepsilon \langle G_1, \psi \rangle & \text{for all } \psi \in H^1(\Omega) / \mathbb{R}, \\
u_{\varepsilon} = u_b
\end{cases}
\]

where \( G_1 \in H^1(\Omega)^* \) such that \( \langle G_1, \psi \rangle = \int_{\Gamma} g_b \psi - \int_{\Omega} (\text{div} F) \psi \) (\( \psi \in H^1(\Omega) \)).

Secondly, we apply the mixed boundary condition (1.3) for (PP) and (ES). We take a test function \( \psi \in H^1_{0, D}(\Omega) \). From the second equation of (PP), we obtain
\[
- \int_{\Omega} (\text{div} F) \psi = - \int_{\Omega} (\Delta p_{\text{PP}}) \psi = - \int_{\Gamma} \frac{\partial p_{\text{PP}}}{\partial n} \psi + \int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi
\]
\[
= - \int_{\Gamma_N} g_b \psi + \int_{\Omega} \nabla p_{\text{PP}} \cdot \nabla \psi.
\]
Hence,
\[ \int_{\Omega} \nabla p_{PP} \cdot \nabla \psi = \int_{\Gamma_N} g_b \psi - \int_{\Omega} (\text{div} F) \psi. \]

The weak form of the pressure-Poisson problem with the mixed boundary condition (1.3) becomes as follows. Find \( u_{PP} \in H^1(\Omega)^n \) and \( p_{PP} \in H^1(\Omega) \) such that

\[
\begin{align*}
\begin{cases}
\int_{\Omega} \nabla u_{PP} : \nabla \varphi + \int_{\Omega} \nabla p_{PP} \cdot \varphi = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\int_{\Omega} \nabla p_{PP} \cdot \nabla \psi = \langle G_2, \psi \rangle & \text{for all } \psi \in H^1_{0,D}(\Omega), \\
u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\
p_{PP} = p_0 & \text{in } H^{1/2}(\Gamma_D),
\end{cases}
\end{align*}
\]

where \( G_2 \in H^1_{0,D}(\Omega)^* \) such that

\[ \langle G_2, \psi \rangle = \int_{\Gamma_N} g_b \psi - \int_{\Omega} (\text{div} F) \psi \]

for \( \psi \in H^1_{0,D}(\Omega) \). The weak form of (ES) with the mixed boundary condition (1.3) can be defined similarly to that of (PP). It reads as follows. Find \( u_\varepsilon \in H^1(\Omega)^n \) and \( p_\varepsilon \in H^1(\Omega) \) such that

\[
\begin{align*}
\begin{cases}
\int_{\Omega} \nabla u_\varepsilon : \nabla \varphi + \int_{\Omega} \nabla p_\varepsilon \cdot \varphi = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla p_\varepsilon \cdot \nabla \psi + \int_{\Omega} (\text{div} u_\varepsilon) \psi = \varepsilon \langle G_2, \psi \rangle & \text{for all } \psi \in H^1_{0,D}(\Omega), \\
u_s = u_b & \text{in } H^{1/2}(\Gamma)^n, \\
p_s = p_0 & \text{in } H^{1/2}(\Gamma_D).
\end{cases}
\end{align*}
\]

Finally, we generalize (PP1) and (PP2) to an abstract pressure-Poisson problem. Let \( Q \subset H^1(\Omega) \) be a closed subspace as defined in Sect. 2.1. Find \( u_{PP} \in H^1(\Omega)^n \) and \( p_{PP} \in Q \) such that

\[
\begin{align*}
\begin{cases}
\int_{\Omega} \nabla u_{PP} : \nabla \varphi + \int_{\Omega} \nabla p_{PP} \cdot \varphi = \int_{\Omega} F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\int_{\Omega} \nabla p_{PP} \cdot \nabla \psi = \langle G, \psi \rangle & \text{for all } \psi \in Q, \\
u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\
p_{PP} - p_0 \in Q & \text{in } Q^*.
\end{cases}
\end{align*}
\]

with \( G \in Q^* \). Indeed, by Theorems 2.3 and 2.4, we obtain (PP1) from (PP') by putting \( Q := H^1(\Omega)/\mathbb{R} \) and \( G := G_1 \). Similarly, we obtain (PP2) from (PP') by putting \( Q := H^1_{0,D}(\Omega) \) and \( G := G_2 \).

We generalize (ES1) and (ES2) to an abstract \( \varepsilon \)-Stokes problem. Find \( u_\varepsilon \in H^1(\Omega)^n \) and \( p_\varepsilon \in Q \) such that
\[ \begin{align*}
\int_{\Omega} \nabla u_\varepsilon \cdot \nabla \varphi + \int_{\Omega} \nabla p_\varepsilon \cdot \varphi &= \int_{\Omega} F \cdot \varphi \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla p_\varepsilon \cdot \nabla \psi + \int_{\Omega} (\text{div} u_\varepsilon) \psi &= \varepsilon \langle G, \psi \rangle \quad \text{for all } \psi \in Q, \\
u_\varepsilon &= u_0 \\
p_\varepsilon &= p_0 \in Q.
\end{align*} \]

Indeed, by Theorems 2.3, 2.4, we obtain (ES1) from (ES') by putting \( Q := H^1_0(\Omega)/\mathbb{R} \) and \( G := G_1 \). Similarly, we also obtain (ES2) from (ES') by putting \( Q := H^1_0(\Omega) \cap D(\Omega) \) and \( G := G_2 \).

### 2.4 Well-posedness of \((S'), (PP')\) and \((ES')\)

We show the well-posedness of problems \((S'), (PP')\) and \((ES')\) in Theorems 2.5, 2.6 and 2.7.

**Theorem 2.5** Under the condition \((2.1)\), there exists a unique solution \((u_S, p_S) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})^n\) satisfying \((S')\).

See [21, Theorem 2.4 and Remark 2.5] for the proof.

**Theorem 2.6** Under the conditions \((2.1)\) and \((2.3)\), for \( G \in Q^* \), there exists a unique solution \((u_{pp}, p_{pp}) \in H^1(\Omega)^n \times Q\) satisfying \((PP')\).

**Proof** Using the Lax–Milgram theorem, since \(Q \times Q \ni (p, \psi) \mapsto \int_{\Omega} \nabla p \cdot \nabla \psi \in \mathbb{R}\) is a continuous and coercive bilinear form, \( p_{pp} \in H^1(\Omega) \) is uniquely determined from the second and fourth equations of \((PP')\). Then \( u_{pp} \in H^1(\Omega)^n \) is also uniquely determined from the first and third equations, again using the Lax–Milgram theorem.

\( \square \)

**Theorem 2.7** Under the conditions \((2.1)\) and \((2.3)\), for \( \varepsilon > 0 \) and \( G \in Q^* \), there exists a unique solution \((u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)\) satisfying \((ES')\).

This is a generalization of Theorem 2.6 in [17]. See Appendix for the proof.

From now on, let the solutions of \((S'), (PP')\) and \((ES')\) be denoted by \((u_S, p_S)\), \((u_{pp}, p_{pp})\) and \((u_\varepsilon, p_\varepsilon)\), respectively. We show their properties in connection with a pressure error on the boundary \( \Gamma \).

**Proposition 2.8** Suppose that \( p_S \in H^1(\Omega), H^1_0(\Omega) \subset Q \) and \( (G, \psi) = -\int_{\Omega} (\text{div} F) \psi \) for all \( \psi \in H^1_0(\Omega) \). Then there exists a constant \( c > 0 \) independent of \( \varepsilon \) such that

\[ \begin{align*}
\|u_S - u_{pp}\|_{H^1(\Omega)^n} &\le c \|\gamma_0 p_S - \gamma_0 p_{pp}\|_{H^{1/2}(\Gamma)}, \\
\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\le c \|\gamma_0 p_S - \gamma_0 p_{pp}\|_{H^{1/2}(\Gamma)}.
\end{align*} \]

In particular, if \( \gamma_0 p_S = \gamma_0 p_{pp} \), then \((u_S, p_S) = (u_{pp}, p_{pp}) = (u_\varepsilon, p_\varepsilon)\) holds for all \( \varepsilon > 0 \).
This is a generalization of Proposition 2.7 in [17]. See Appendix for the proof.

Since $H^1_0(\Omega) \not\subset H^1(\Omega)/\mathbb{R}$, Proposition 2.8 does not apply directly for the case of the Neumann boundary condition (1.2). However, we add natural assumptions, then it leads to (2.5).

**Proposition 2.9** Suppose that $p_s \in H^1(\Omega)$, $Q = H^1(\Omega)/\mathbb{R}$. If $G \in Q^*$ is such that $G \in H^1(\Omega)^*$ and $\langle G, \psi \rangle = \int_{\Gamma} gb\psi - \int_{\Omega}(\text{div}F)\psi$ for all $\psi \in H^1(\Omega)$, then we have (2.5).

**Proof** Since $G \in H^1(\Omega)^*$ satisfies $\langle G, \psi \rangle = \int_{\Gamma} gb\psi - \int_{\Omega}(\text{div}F)\psi$ for all $\psi \in H^1(\Omega)$, it holds that

$$\int_{\Omega} \nabla p_{PP} \cdot \nabla \psi = -\int_{\Omega} (\text{div}F)\psi$$

for all $\psi \in H^1_0(\Omega)$ from the second equation of (PP'). Hence, it leads the second equation of (A.4). Using the proof of Proposition 2.8, we obtain (2.5). □

### 3 Links between (ES) and (PP)

In this section, we show that $(u_{\varepsilon}, p_{\varepsilon})$ converges to $(u_{PP}, p_{PP})$ strongly in $H^1(\Omega)^n \times H^1(\Omega)$ as $\varepsilon \to \infty$. We also treat the case of the regular perturbation asymptotics by exploring the structure of the lower order terms and their effect on the convergence rate.

#### 3.1 Convergence as $\varepsilon \to \infty$

We use the following Lemma 3.1 for the proofs of the theorems in this section.

**Lemma 3.1** Let $h \in Q^*$ and $(v_{\varepsilon}, q_{\varepsilon}) \in H^1_0(\Omega)^n \times Q$ satisfy

$$\begin{cases}
\int_{\Omega} \nabla v_{\varepsilon} \cdot \nabla \varphi + \int_{\Omega} (\nabla q_{\varepsilon}) \cdot \varphi = 0 & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla q_{\varepsilon} \cdot \nabla \psi + \int_{\Omega} (\text{div}v_{\varepsilon})\psi = \langle h, \psi \rangle & \text{for all } \psi \in Q
\end{cases} \tag{3.1}$$

for an arbitrarily fixed $\varepsilon > 0$. Then there exists a constant $c > 0$ such that

$$\|v_{\varepsilon}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon} \|h\|_{Q^*}, \quad \|q_{\varepsilon}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon} \|h\|_{Q^*}.$$  

**Proof** Putting $\varphi := v_{\varepsilon}$ and $\psi := q_{\varepsilon}$ and adding two equations of (3.1), we obtain

$$\|\nabla v_{\varepsilon}\|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \|\nabla q_{\varepsilon}\|_{L^2(\Omega)^n}^2 \leq \|h\|_{Q^*} \|\nabla q_{\varepsilon}\|_{L^2(\Omega)^n}.$$  

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where we have used $\int_\Omega \nabla q_\varepsilon \cdot v_\varepsilon = - \int_\Omega (\text{div} v_\varepsilon) q_\varepsilon$. Thus

$$\|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \leq \frac{1}{\varepsilon} \|h\|_{Q^*}.$$ 

In addition, from the first equation of (3.1) by putting $\varphi := v_\varepsilon$, we have

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)^n}^2 = \int_\Omega \nabla v_\varepsilon : \nabla v_\varepsilon = - \int_\Omega (\nabla q_\varepsilon) \cdot v_\varepsilon \leq \|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \|v_\varepsilon\|_{L^2(\Omega)^n}$$

$$\leq c \|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \|\nabla v_\varepsilon\|_{L^2(\Omega)^{n \times n}},$$

and then

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)^n} \leq c \|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \leq \frac{c}{\varepsilon} \|h\|_{Q^*}.$$ 

The second inequality of this lemma also follows from the Poincaré inequality for $q_\varepsilon \in Q$. \qed

Using Lemma 3.1, we obtain Theorem 3.2.

**Theorem 3.2** There exists a constant $c > 0$ independent of $\varepsilon > 0$ such that

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon} \|\text{div} u_{PP}\|_{Q^*}, \quad \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon} \|\text{div} u_{PP}\|_{Q^*}$$

for all $\varepsilon > 0$. In particular, we have

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \to 0, \quad \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \to 0 \text{ as } \varepsilon \to \infty.$$ 

**Proof** Combining (PP’) and (ES’), we obtain

$$\begin{cases}
\int_\Omega \nabla v_\varepsilon : \nabla \varphi + \int_\Omega \nabla q_\varepsilon \cdot \varphi = 0 \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_\Omega \nabla q_\varepsilon \cdot \nabla \psi + \int_\Omega (\text{div} v_\varepsilon) \psi = - \int_\Omega (\text{div} u_{PP}) \psi \quad \text{for all } \psi \in Q,
\end{cases}$$

(3.2)

where $v_\varepsilon := u_\varepsilon - u_{PP}, q_\varepsilon := p_\varepsilon - p_{PP}$ and $h := \text{div} u_{PP}$. By Lemma 3.1, we conclude the proof. \qed

**Corollary 3.3** If $u_{PP}$ satisfies $\text{div} u_{PP} = 0$, then $u_\varepsilon = u_{PP}$ and $p_\varepsilon = p_{PP}$ hold for all $\varepsilon > 0$. Furthermore, $u_S = u_\varepsilon = u_{PP}$ and $p_S = \{p_\varepsilon\} = \{p_{PP}\}$ hold for all $\varepsilon > 0$, since $(u_{PP}, p_{PP})$ satisfies the first and third equations in (PP’) and $\text{div} u_{PP} = 0$. 
3.2 Regular Perturbation Asymptotics

By Theorem 3.2, we have that \(\|\varepsilon(u_\varepsilon - u_{PP})\|_{H^1(\Omega)} \leq c\) and \(\|\varepsilon(p_\varepsilon - p_{PP})\|_{H^1(\Omega)} \leq c\) for all \(\varepsilon > 0\). It implies that there exists a subsequence of \((u_\varepsilon - u_{PP})_n\), \((\varepsilon(p_\varepsilon - p_{PP}))_n\) which converges weakly to \((v^{(1)}, q^{(1)})\) in \(H^1_0(\Omega)^n \times Q\) if \(\varepsilon \to \infty\). The next theorem states properties of the limit functions \(v^{(1)}\) and \(q^{(1)}\).

**Theorem 3.4** Let \(v_\varepsilon^{(1)} := \varepsilon(u_\varepsilon - u_{PP}) \in H^1_0(\Omega)^n\), \(q_\varepsilon^{(1)} := \varepsilon(p_\varepsilon - p_{PP}) \in Q\) and let \((v^{(1)}, q^{(1)}) \in H^1_0(\Omega)^n \times Q\) satisfy

\[
\begin{align*}
\int_\Omega \nabla v^{(1)} \cdot \nabla \varphi + \int_\Omega (\nabla q^{(1)} - \varepsilon \varphi) &= 0, \\
\int_\Omega \nabla q^{(1)} \cdot \nabla \psi &= -\int_\Omega (\text{div} u_{PP}) \psi 
\end{align*}
\]

Then there exists a constant \(c > 0\) independent of \(\varepsilon\) such that

\[
\|v_\varepsilon^{(1)} - v^{(1)}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon}\|\text{div} v^{(1)}\|_Q, \quad \|q_\varepsilon^{(1)} - q^{(1)}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\|\text{div} v^{(1)}\|_Q.
\]

In order to prove Theorem 3.4, we define two spaces and two operators. Let \(X := H^1_0(\Omega)^n \times Q\) and \(Y := H^{-1}(\Omega)^n \times Q^*\) be equipped with norms

\[
\|(u, p)\|_X := \|u\|_{H^1(\Omega)^n}^2 + \|p\|_Q^2, \quad \|(f, g)\|_Y^2 := \|f\|_{H^{-1}(\Omega)^n}^2 + \|g\|_{Q^*}^2
\]

for \((u, p) \in X, (f, g) \in Y\), and let \(A\) and \(B\) be

\[
A : X \quad \rightarrow \quad Y \quad B : X \quad \rightarrow \quad Y
\]

\[
(u, p) \mapsto (-\Delta u + \nabla p, -\Delta p), \quad (u, p) \mapsto (0, \text{div} u),
\]

where \(\langle \Delta u, \varphi \rangle := -\int_\Omega \nabla u \cdot \nabla \varphi\) and \(\langle \Delta p, \psi \rangle := -\int_\Omega \nabla \cdot \nabla \psi\) for all \(\varphi \in H^1_0(\Omega)^n\) and \(\psi \in H^1_0(\Omega)\). Then \((u_{PP}, p_{PP})\) and \((u_\varepsilon, p_\varepsilon)\) satisfy

\[
A(u_{PP} - u_0, p_{PP} - p_0) = H, \quad \left(\frac{A}{\varepsilon} + B\right)(u_\varepsilon - u_0, p_\varepsilon - p_0) = H,
\]

where \(H := (F - \Delta u_0, G - \Delta p_0)\) and \(u_0\) is defined by (A.1). We have \(A + tB \in \text{Isom}(X, Y)\) for an arbitrary \(t \geq 0\) by the analogy of Theorem 2.6 \((t = 0)\) and Theorem 2.7 \((t = 1/\varepsilon)\).

**Proof of Theorem 3.4** The existence and the uniqueness of the pair \((v^{(1)}, q^{(1)})\) \(\in H^1_0(\Omega)^n \times Q\) as a solution to (3.3) follows from Theorem 2.6, and \((v^{(1)}, q^{(1)})\) can be written as

\[
(v^{(1)}, q^{(1)}) = A^{-1}(0, -\text{div} u_{PP}) = -A^{-1}B(u_{PP} - u_0, p_{PP} - p_0) = -A^{-1}BA^{-1}H.
\]
By the following identity
\[
\left( A + \frac{1}{\varepsilon} B \right) \left( \varepsilon \left( A + \frac{1}{\varepsilon} B \right)^{-1} - \varepsilon A^{-1} + A^{-1} BA^{-1} \right) H = \frac{1}{\varepsilon} (BA^{-1})^2 H,
\]
we have
\[
\left( A + \frac{1}{\varepsilon} B \right) (v^{(1)}_\varepsilon - v^{(1)}, q^{(1)}_\varepsilon - q^{(1)}) = -\frac{1}{\varepsilon} (0, \text{div} v^{(1)}).
\]
Hence,
\[
\begin{align*}
\int_\Omega \nabla v_\varepsilon : \nabla \varphi + \int_\Omega \nabla q_\varepsilon \cdot \varphi &= 0 \\
\varepsilon \int_\Omega \nabla q_\varepsilon \cdot \nabla \psi + \int_\Omega (\text{div} v_\varepsilon) \psi &= -\int_\Omega (\text{div} v^{(1)}) \psi
\end{align*}
\]
for all \( \varphi \in H^1_0(\Omega)^n \), \( \psi \in Q \), where \( v_\varepsilon := v^{(1)}_\varepsilon - v^{(1)} \) and \( q_\varepsilon := q^{(1)}_\varepsilon - q^{(1)} \). Therefore, by Lemma 3.1, there exists a constant \( c > 0 \) such that
\[
\| v^{(1)}_\varepsilon - v^{(1)} \|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon} \| \text{div} v^{(1)} \|_Q^* , \quad \| q^{(1)}_\varepsilon - q^{(1)} \|_{H^1(\Omega)} \leq \frac{c}{\varepsilon} \| \text{div} v^{(1)} \|_Q^*
\]
for all \( \varepsilon > 0 \). \( \square \)

Next, we generalize Theorem 3.4 to the following theorem:

**Theorem 3.5** Let \( k \in \mathbb{N} \) be arbitrary \((k \geq 1)\) and let \( v^{(0)} := u_{PP} \). If functions \( v^{(1)}, v^{(2)}, \ldots, v^{(k)} \in H^1_0(\Omega)^n \) and \( q^{(1)}, q^{(2)}, \ldots, q^{(k)} \in Q \) satisfy
\[
\begin{align*}
\int_\Omega \nabla v^{(i)} : \nabla \varphi + \int_\Omega (\nabla q^{(i)}) \cdot \varphi &= 0 \quad \text{for all} \ \varphi \in H^1_0(\Omega)^n , \\
\int_\Omega \nabla q^{(i)} \cdot \nabla \psi &= -\int_\Omega (\text{div} v^{(i-1)}) \psi \quad \text{for all} \ \psi \in Q,
\end{align*}
\]
for all \( 1 \leq i \leq k \), then there exists a constant \( c > 0 \) independent of \( \varepsilon \) satisfying
\[
\begin{align*}
\left\| u_\varepsilon - \left( u_{PP} + \frac{1}{\varepsilon} v^{(1)} + \cdots + \left( \frac{1}{\varepsilon} \right)^k v^{(k)} \right) \right\|_{H^1(\Omega)^n} &\leq \frac{c}{\varepsilon^{k+1}} \| \text{div} v^{(k)} \|_Q^* , \\
\left\| p_\varepsilon - \left( p_{PP} + \frac{1}{\varepsilon} q^{(1)} + \cdots + \left( \frac{1}{\varepsilon} \right)^k q^{(k)} \right) \right\|_{H^1(\Omega)} &\leq \frac{c}{\varepsilon^{k+1}} \| \text{div} v^{(k)} \|_Q^* .
\end{align*}
\]

**Proof** Equation (3.4) states that
\[
A(v^{(i)}, q^{(i)}) = -B(v^{(i-1)}, q^{(i-1)})
\]
\( \square \) Springer
for $i = 1, \ldots, k$, i.e.

$$(v^{(i)}, q^{(i)}) = -A^{-1} B (v^{(i-1)}, q^{(i-1)}) = \cdots = (-A^{-1} B)^i (u_{pp}, p_{pp}) = A^{-1} (-BA^{-1})^i H.$$  

By the following identity

$$\left( A + \frac{1}{\varepsilon} B \right) \left\{ \left( A + \frac{1}{\varepsilon} B \right)^{-1} - A^{-1} \sum_{i=0}^{k} \left( -\frac{1}{\varepsilon} BA^{-1} \right)^i \right\} H = \left( -\frac{1}{\varepsilon} BA^{-1} \right)^{k+1} H,$$

we have

$$\left( A + \frac{1}{\varepsilon} B \right) \left( u_{\varepsilon} - u_{pp}, p_{\varepsilon} - p_{pp} - \sum_{i=1}^{k} \frac{1}{\varepsilon^i} (v^{(i)}, q^{(i)}) \right) = -\frac{1}{\varepsilon^{k+1}} (0, \text{div} v^{(k)}).$$

Therefore, by Lemma 3.1, there exists a constant $c > 0$ such that

$$\| u_{\varepsilon} - (u_{pp} + \frac{1}{\varepsilon} v^{(1)} + \cdots + \frac{1}{\varepsilon^k} v^{(k)}) \|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon^{k+1}} \| \text{div} v^{(k)} \|_{Q^*},$$

$$\| p_{\varepsilon} - (p_{pp} + \frac{1}{\varepsilon} q^{(1)} + \cdots + \frac{1}{\varepsilon^k} q^{(k)}) \|_{H^1(\Omega)} \leq \frac{c}{\varepsilon^{k+1}} \| \text{div} v^{(k)} \|_{Q^*}$$

for all $\varepsilon > 0$.  

\[\square\]

### 4 Convergence of (ES) to (S)

In this section, we show that $(u_{\varepsilon}, p_{\varepsilon})$ converges to $(u_S, p_S)$ weakly in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ as $\varepsilon \to 0$. Moreover, if $p_S \in H^1(\Omega)$, then $(u_{\varepsilon}, p_{\varepsilon})$ converges to $(u_S, p_S)$ strongly in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ as $\varepsilon \to 0$.

The outline of the proof of our convergence results (Theorems 4.2, 4.3 and 4.4) is as follows. First, we prove the boundedness of the sequence $((u_{\varepsilon}, p_{\varepsilon}))_{\varepsilon > 0}$ in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. By the reflexivity of $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$, the sequence has a subsequence converging weakly in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. In the end, we show that the limit pair of functions satisfies $(S')$.

We start this section with a useful lemma.

**Lemma 4.1** If $v \in H^1(\Omega)^n$, $q \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)^n$ satisfy

$$\int_{\Omega} \nabla v : \nabla \varphi + \langle \nabla q, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H_0^1(\Omega)^n,$$

then there exists a constant $c > 0$ such that

$$\| q \|_{L^2(\Omega)/\mathbb{R}} \leq c (\| \nabla v \|_{L^2(\Omega)^{n \times n}} + \| f \|_{H^{-1}(\Omega)^n}).$$
Proof Let $c$ be the constant from Theorem 2.2. Then we obtain
\[
\|q\|_{L^2(\Omega)/\mathbb{R}} \leq c \|\nabla q\|_{H^{-1}(\Omega)^n} = c \sup_{\varphi \in S_n} |\langle \nabla q, \varphi \rangle|
\]
\[
\leq c \sup_{\varphi \in S_n} \left( \left| \int_{\Omega} \nabla v : \nabla \varphi \right| + |\langle f, \varphi \rangle| \right)
\]
\[
\leq c \left( \|\nabla v\|_{L^2(\Omega)^{n \times n}} + \|f\|_{H^{-1}(\Omega)} \right).
\]
\[\Box\]

Theorem 4.2 There exists a constant $c > 0$ independent of $\varepsilon$ such that
\[
\|u_\varepsilon\|_{H^1(\Omega)^n} \leq c, \quad \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq c \quad \text{for all } \varepsilon > 0.
\]
Furthermore, if $C^\infty_0(\Omega) \subset Q$, then we obtain
\[
u_\varepsilon \rightharpoonup u_S \text{ weakly in } H^1(\Omega)^n, \quad [p_\varepsilon] \rightharpoonup p_S \text{ weakly in } L^2(\Omega)/\mathbb{R} \text{ as } \varepsilon \to 0.
\]
See Appendix for the proof.

If we add a regularity assumption of $p_S$, then $(u_\varepsilon, p_\varepsilon)$ converges strongly in $H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$.

Theorem 4.3 Suppose that $p_S \in H^1(\Omega)$. Then we obtain
\[
u_\varepsilon \to u_S \text{ strongly in } H^1(\Omega)^n, \quad [p_\varepsilon] \to p_S \text{ strongly in } L^2(\Omega)/\mathbb{R} \text{ as } \varepsilon \to 0.
\]
See Appendix for the proof.

Theorem 4.3 does not give the convergence rate. If $Q = H^1(\Omega)/\mathbb{R}$ (corresponding to the Neumann boundary condition (1.2)), then the convergence rate becomes $\sqrt{\varepsilon}$.

Theorem 4.4 Suppose that $Q = H^1(\Omega)/\mathbb{R}$ and $p_S \in H^1(\Omega)$. Then there exists a constant $c > 0$ independent of $\varepsilon$ such that
\[
\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \leq c \sqrt{\varepsilon}, \quad \|p_\varepsilon - p_S\|_{L^2(\Omega)} \leq c \sqrt{\varepsilon}.
\]
Proof We obtain from (ES') and (S') that
\[
\begin{cases}
\int_{\Omega} \nabla (u_\varepsilon - u_S) : \nabla \varphi + \int_{\Omega} (\nabla (p_\varepsilon - p_S)) \cdot \varphi = 0 \text{ for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla p_\varepsilon \cdot \nabla \psi + \int_{\Omega} (\text{div} u_\varepsilon) \psi = \varepsilon \langle G, \psi \rangle \quad \text{for all } \psi \in H^1(\Omega)/\mathbb{R}.
\end{cases}
\]
Putting $\varphi := u_\varepsilon - u_S \in H^1_0(\Omega)^n$ and $\psi := p_\varepsilon - p_S \in H^1(\Omega)/\mathbb{R}$, we get
\[
\|\nabla (u_\varepsilon - u_S)\|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \int_{\Omega} \nabla p_\varepsilon \cdot \nabla (p_\varepsilon - p_S)
\]
\[
\begin{aligned}
&= - \int_{\Omega} (\nabla (p_\varepsilon - p_S)) \cdot (u_\varepsilon - u_S) \\
&\quad - \int_{\Omega} (\text{div} u_\varepsilon) (p_\varepsilon - p_S) + \varepsilon \langle G, p_\varepsilon - p_S \rangle \\
&\quad = \int_{\Omega} (\text{div} u_\varepsilon - \text{div} u_S)(p_\varepsilon - p_S) \\
&\quad - \int_{\Omega} (\text{div} u_\varepsilon)(p_\varepsilon - p_S) + \varepsilon \langle G, p_\varepsilon - p_S \rangle \\
&\quad = \varepsilon \langle G, p_\varepsilon - p_S \rangle. 
\end{aligned}
\] (4.1)

Subtracting \( \varepsilon \int_{\Omega} \nabla p_S \cdot \nabla (p_\varepsilon - p_S) \) from both sides of (4.1), we obtain
\[
\| \nabla (u_\varepsilon - u_S) \|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \| \nabla (p_\varepsilon - p_S) \|_{L^2(\Omega)^n}^2 \\
= - \varepsilon \int_{\Omega} \nabla p_S \cdot \nabla (p_\varepsilon - p_S) + \varepsilon \langle G, p_\varepsilon - p_S \rangle \\
\leq \varepsilon (\| \nabla p_S \|_{L^2(\Omega)^n} + \| G \|_{(H^1(\Omega)/\mathbb{R})^*}) \| \nabla (p_\varepsilon - p_S) \|_{L^2(\Omega)^n}. 
\] (4.2)

To clarify the following estimates, we set
\[
\alpha := \| \nabla (u_\varepsilon - u_S) \|_{L^2(\Omega)^{n \times n}}, \beta := \| \nabla (p_\varepsilon - p_S) \|_{L^2(\Omega)^n}, a := \| \nabla p_S \|_{L^2(\Omega)^n} + \| G \|_{(H^1(\Omega)/\mathbb{R})^*}. \]

The estimate (4.2) reads as
\[
\alpha^2 + \varepsilon \beta^2 \leq \varepsilon a \beta, \quad \left( \frac{\alpha}{\sqrt{\varepsilon}} \right)^2 + (\beta - \frac{a}{2})^2 \leq \left( \frac{a}{2} \right)^2.
\]

Hence, \( \alpha \leq a \sqrt{\varepsilon}/2 \), i.e., \( \| \nabla (u_\varepsilon - u_S) \|_{L^2(\Omega)^{n \times n}} \leq (\sqrt{\varepsilon}/2)(\| \nabla p_S \|_{L^2(\Omega)^n} + \| G \|_{(H^1(\Omega)/\mathbb{R})^*}). \) By Lemma 4.1, we obtain
\[
\| p_\varepsilon - p_S \|_{L^2(\Omega)} \leq c \| \nabla (u_{\varepsilon k} - u_S) \|_{L^2(\Omega)^{n \times n}} = c\alpha \leq c \frac{a \sqrt{\varepsilon}}{2} = c \frac{\sqrt{\varepsilon}}{2}(\| \nabla p_S \|_{L^2(\Omega)^n} + \| G \|_{(H^1(\Omega)/\mathbb{R})^*}).
\]

\( \square \)

5 Numerical examples

For our simulations, we consider \( \Omega = (0, 1) \times (0, 1) \). We put \( F = 0 \) and take the following boundary conditions:
\[
\begin{aligned}
u_b &= (x(x - 1), y(y - 1))^T, \\
g_b &= (2, 2)^T \cdot \nu
\end{aligned}
\]
on \( \Gamma \). The exact solutions for (PP1) are \( u_{PP} = (x(x - 1), y(y - 1))^T \) and \( p_{PP} = 2x + 2y - 2 \). We solve the problems (PP1), (ES1) and (S’) numerically by using the finite element method with P2/P1 elements by the software FreeFem++ [13].
numerical solutions \((u_{PP}, p_{PP}), (u_\varepsilon, p_\varepsilon)\) \((\varepsilon = 1, 10^{-2} \text{ or } 10^{-4})\) and \((u_S, p_S)\) to the problems \((PP_1), (ES_1)\) and \((S')\), respectively, are illustrated in Figs. 2, 3 and 4. From these pictures we observe that \((u_\varepsilon, p_\varepsilon)\) seems to converge to \((u_{PP}, p_{PP})\) as \(\varepsilon \to \infty\) and to \((u_S, p_S)\) as \(\varepsilon \to 0\) (as expected from Theorems 3.2 and 4.3.)

Next we compute the error estimate between the numerical solutions of \((ES_1)\) and \((PP_1)\). The numerical errors \(\|u_\varepsilon - u_{PP}\|_{L^2(\Omega)^n}, \|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^{n\times n}}, \|p_\varepsilon - p_{PP}\|_{L^2(\Omega)}\) and \(\|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n}\) are shown in Figs. 5 and 6. Based on these values, we have fitted a constant \(c\) such that \(\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \sim \frac{c}{\varepsilon}\) and \(\|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \sim \frac{c}{\varepsilon}\) for \(\varepsilon\) large. Figures 5 and 6 indicate that there exists a constant \(c\) such that \(\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon}\) and \(\|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\), as expected from Theorem 3.2.

We also compute the error estimate between the problems \((ES_1)\) and \((S')\) by numerical calculation. The numerical error estimate \(\|u_\varepsilon - u_S\|_{L^2(\Omega)^n}, \|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}}, \|p_\varepsilon - p_S\|_{L^2(\Omega)}\) and \(\|\nabla(p_\varepsilon - p_S)\|_{L^2(\Omega)^n}\) are shown in Figures 7 and 8. Based on these values, we have fitted a constant \(c\) such that \(\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \sim \frac{c}{\varepsilon}\) and \(\|p_\varepsilon - p_S\|_{L^2(\Omega)} \sim \frac{c}{\varepsilon}\) for \(\varepsilon\) small. Figures 7 and 8 indicate that there exists a constant \(\tilde{c}\) such that \(\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \leq \tilde{c} \sqrt{\varepsilon}\) and \(\|p_\varepsilon - p_S\|_{L^2(\Omega)} \leq \tilde{c} \sqrt{\varepsilon}\), as expected from Theorem 4.4.

**Appendix**

Theorems 2.7, 4.2, 4.3, and Proposition 2.8 are generalizations of several theorems stated in [17]. In this appendix, however, we give their proofs for the readers’ convenience. We define a continuous coercive bilinear form depending on \(\varepsilon\) and prove Theorem 2.7 by the Lax–Milgram Theorem.

**Proof of Theorem 2.7** We take arbitrary \(u_1 \in H^1(\Omega)^n\) with \(\gamma_0 u_1 = u_b\). Since \(\text{div} : H^1_0(\Omega)^n \to L^2(\Omega)/\mathbb{R}\) is surjective [9, Corollary 2.4, 2°], there exists \(u_2 \in H^1_0(\Omega)^n\) such that \(\text{div} u_2 = \text{div} u_1\). We put

\[
 u_0 := u_1 - u_2, \tag{A.1}
\]
and note that $\gamma_0 u_0 = u_b$ and $\text{div} u_0 = 0$. To simplify the notation, we set $u := u_\varepsilon - u_0 \in H_0^1(\Omega)^n$, $p := p_\varepsilon - p_0 \in Q$, and define $f \in H^{-1}(\Omega)$ and $g \in Q^*$ by

$$\langle f, v \rangle := \int_\Omega F v - \int_\Omega \nabla u_0 : \nabla v - \int_\Omega (\nabla p_0) \cdot v \quad \text{for all } v \in H_0^1(\Omega)^n,$$

$$\langle g, q \rangle := \langle G, q \rangle - \int_\Omega \nabla p_0 \cdot \nabla q \quad \text{for all } q \in Q. \quad \text{(A.2)}$$
Fig. 4 $p_S$ (left) and $u_S$ (right). The color scale indicates the length of $|u_S(\xi)|$ at each node $\xi$.

Fig. 5 $\|u_\varepsilon - u_{PP}\|_{L^2(\Omega)}^n$ (left, solid line) and $\|\nabla (u_\varepsilon - u_{PP})\|_{L^2(\Omega)}^n$ (right, solid line) as functions of $\varepsilon$.

Fig. 6 $\|p_\varepsilon - p_{PP}\|_{L^2(\Omega)}^n$ (left, solid line) and $\|\nabla (p_\varepsilon - p_{PP})\|_{L^2(\Omega)}^n$ (right, solid line) as functions of $\varepsilon$.

Then, $(u_\varepsilon, p_\varepsilon)$ satisfies (ES') if and only if $(u, p)$ satisfies

\[
\begin{aligned}
\int_{\Omega} \nabla u : \nabla \varphi + \int_{\Omega} (\nabla p) \cdot \varphi &= \langle f, \varphi \rangle \quad \text{for all } \varphi \in H_0^1(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla p \cdot \nabla \psi + \int_{\Omega} (\text{div} u) \psi &= \varepsilon \langle g, \psi \rangle \quad \text{for all } \psi \in Q.
\end{aligned}
\]  

(A.3)
Adding the equations in (A.3), we get

\[
\left(\begin{array}{c}
u \\
p
\end{array}\right), \left(\begin{array}{c}
\varphi \\
\psi
\end{array}\right)\right)_\epsilon := \int_\Omega \nabla u : \nabla \varphi + \epsilon \int_\Omega \nabla p : \nabla \psi + \int_\Omega (\nabla p) \cdot \varphi + \int_\Omega (\text{div} u) \psi \\
= \langle f, \varphi \rangle + \epsilon \langle g, \psi \rangle.
\]

We check that \((\cdot, \cdot)_\epsilon\) is a continuous coercive bilinear form on \(H^1_0(\Omega)^n \times Q\). The bilinearity and continuity of \((\cdot, \cdot)_\epsilon\) are obvious. The coercivity of \((\cdot, \cdot)_\epsilon\) is obtained in the following way. Take \((v, q)^T \in H^1_0(\Omega)^n \times Q\). We have the following sequence of inequalities:

\[
\left(\begin{array}{c}
v \\
q
\end{array}\right), \left(\begin{array}{c}
v \\
q
\end{array}\right)\right)_\epsilon := \int_\Omega \nabla v : \nabla v + \epsilon \int_\Omega \nabla v : \nabla q + \int_\Omega v : \nabla q + \int_\Omega (\text{div} v) q \\
= \|\nabla v\|_{L^2(\Omega)}^2 + \epsilon \|\nabla q\|_{L^2(\Omega)}^2 \\
\geq \min\{1, \epsilon\} \left(\|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla q\|_{L^2(\Omega)}^2\right) \\
\geq c \min\{1, \epsilon\} \left(\|v\|_{H^1(\Omega)^n}^2 + \|q\|_{H^1(\Omega)}^2\right).
\]
Summarizing, \((\cdot, \cdot)_{\epsilon}\) is a continuous coercive bilinear form and \(H^1_0(\Omega)^n \times Q\) is a Hilbert space. Therefore, the conclusion of Theorem 2.7 follows from the Lax–Milgram Theorem.

Let \((u_S, p_S)\), \((u_{\text{PP}}, p_{\text{PP}})\) and \((u_\epsilon, p_\epsilon)\) be the solutions of \((S')\), \((\text{PP}')\) and \((\text{ES}')\), respectively, as guaranteed by Theorems 2.5, 2.6 and 2.7. We show that the subtract \(p_S - p_{\text{PP}}\) satisfies

\[\Delta(p_S - p_{\text{PP}}) = 0\]

in distributions sense. The weak harmonicity is the key ingredient to proving Proposition 2.8.

**Proof of Proposition 2.8** First, we prove that there exists a constant \(c > 0\) independent of \(\epsilon\) such that

\[\|u_S - u_{\text{PP}}\|_{H^1(\Omega)^n} \leq c\|\gamma_0 p_S - \gamma_0 p_{\text{PP}}\|_{H^{1/2}(\Gamma)}\]

and if \(\gamma_0 (p_S - p_{\text{PP}}) = 0\), then \(p_{\text{PP}} = p_S\). Taking the divergence of the first equation of \((S')\), we obtain

\[\text{div} F = \text{div}(-\Delta u_S + \nabla p_S) = -\Delta(\text{div} u_S) + \Delta p_S = \Delta p_S.\]

in distributions sense. Since \(p_S \in H^1(\Omega)\) and \(C^\infty_0(\Omega)\) is dense in \(H^1_0(\Omega)\), it follows that

\[\int_\Omega \nabla p_S \cdot \nabla \psi = -\int_\Omega (\text{div} F) \psi\]

for all \(\psi \in H^1_0(\Omega)\). Together with \((S')\), \((\text{PP}')\) and \(H^1_0(\Omega) \subset Q\), we obtain

\[
\begin{cases}
\int_\Omega \nabla (u_S - u_{\text{PP}}) : \nabla \varphi = -\int_\Omega (\nabla (p_S - p_{\text{PP}})) \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\int_\Omega \nabla (p_S - p_{\text{PP}}) \cdot \nabla \psi = 0 & \text{for all } \psi \in H^1_0(\Omega)
\end{cases}
\]

from the assumption \(\langle G, \psi \rangle = \int_\Omega \nabla F \cdot \psi\). Putting \(\varphi := u_S - u_{\text{PP}} \in H^1_0(\Omega)^n\) in \((A.4)\), we get

\[
\|\nabla (u_S - u_{\text{PP}})\|_{L^2(\Omega)^{n \times n}}^2 = -\int_\Omega (\nabla (p_S - p_{\text{PP}})) \cdot (u_S - u_{\text{PP}}) \\
\leq \|\nabla (p_S - p_{\text{PP}})\|_{L^2(\Omega)^n} \|u_S - u_{\text{PP}}\|_{L^2(\Omega)^n}.
\]

Hence,

\[\|u_S - u_{\text{PP}}\|_{H^1(\Omega)^n} \leq c_1 \|\nabla (p_S - p_{\text{PP}})\|_{L^2(\Omega)^n} \quad (A.5)\]

holds. From the second equation of \((A.4)\), we obtain

\[\|\nabla (p_S - p_{\text{PP}} - \psi)\|_{L^2(\Omega)^n}^2 \geq \int_\Omega (\nabla (p_S - p_{\text{PP}})) : \nabla (p_S - p_{\text{PP}} - \psi)
\]

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\[\begin{align*}
= \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega)}^2 - 2\int_\Omega \nabla(p_S - p_{PP}) \cdot \nabla\psi \\
= \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega)}^2 \\
\geq \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)}^2
\end{align*}\]

for all \(\psi \in H^1_0(\Omega)\). Thus we find
\[\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)}^2 \leq \inf_{\psi \in H^1_0(\Omega)} (\|\nabla(p_S - p_{PP} - \psi)\|_{L^2(\Omega)}^2).
\]

Since \(\gamma_0\) is surjective and the space \(\text{Ker}(\gamma_0) = H^1_0(\Omega)\), \(H^1(\Omega)/H^1_0(\Omega)\) and \(H^{1/2}(\Gamma)\) are isomorphic, there exists a constant \(c_2 > 0\) such that \(\|q\|_{H^1(\Omega)/H^1_0(\Omega)} \leq c_2\|\gamma_0 q\|_{H^{1/2}(\Gamma)}\) for all \(q \in H^1(\Omega)\). Hence, we obtain
\[\begin{align*}
\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)}^2 &\leq \inf_{\psi \in H^1_0(\Omega)} \|\nabla(p_S - p_{PP} - \psi)\|_{L^2(\Omega)}^2 \\
&\leq \inf_{\psi \in H^1_0(\Omega)} \|p_S - p_{PP} - \psi\|_{H^1(\Omega)} \\
&= \|p_S - p_{PP}\|_{H^1(\Omega)/H^1_0(\Omega)} \\
&\leq c_2\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}.
\end{align*}\]

Together with (A.5), we obtain \(\|u_S - u_{PP}\|_{H^1(\Omega)}^n \leq c_1 c_2\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}\).

Moreover, if \(\gamma_0(p_S - p_{PP}) = 0\), then \(p_{PP} = p_S\).

Next, we prove that there exists a constant \(c > 0\) independent of \(\varepsilon\) such that
\[\|u_S - u_\varepsilon\|_{H^1(\Omega)}^n \leq c\|\gamma_0 p_S - \gamma_0 p_\varepsilon\|_{H^{1/2}(\Gamma)}\]
and if \(\gamma_0(p_S - p_{PP}) = 0\), then \(p_{PP} = p_\varepsilon\).

Let \(w_\varepsilon := u_S - u_\varepsilon \in H^1_0(\Omega)^n\) and \(r_\varepsilon := p_{PP} - p_\varepsilon \in Q\). By (S'), (PP') and (ES'), we obtain
\[\begin{align*}
\begin{cases}
\int_\Omega \nabla w_\varepsilon : \nabla \varphi + \int_\Omega (\nabla r_\varepsilon) \cdot \varphi = -\int_\Omega (\nabla(p_S - p_{PP})) \cdot \varphi &\text{for all } \varphi \in H^1_0(\Omega)^n, \\
\int_\Omega \nabla r_\varepsilon \cdot \nabla \psi + \int_\Omega (\text{div} w_\varepsilon) \psi = 0 &\text{for all } \psi \in Q.
\end{cases}
\end{align*}\]

Putting \(\varphi := w_\varepsilon\) and \(\psi := r_\varepsilon\) and adding the two equations of (A.7), we get
\[\|\nabla w_\varepsilon\|_{L^2(\Omega)^{n \times n}}^2 + \|\nabla r_\varepsilon\|_{L^2(\Omega)^n}^2 \leq \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} \|w_\varepsilon\|_{L^2(\Omega)^n} \tag{A.8}\]
from \(\int_\Omega (\nabla r_\varepsilon) \cdot w_\varepsilon = -\int_\Omega (\text{div} w_\varepsilon) r_\varepsilon\). Thus we find
\[\|w_\varepsilon\|_{H^1(\Omega)^n} \leq c_3 \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}.
\]

Together with (A.6), we obtain
\[\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} = \|w_\varepsilon\|_{H^1(\Omega)^n} \leq c_2 c_3 \|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}.
\]
Moreover, by (A.8), we obtain
\[ \varepsilon \|p_{PP} - p_\varepsilon\|_{L^2(\Omega)}^2 = \varepsilon \|r_\varepsilon\|_{L^2(\Omega)}^2 \leq c_4 \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} \|u_\varepsilon\|_{L^2(\Omega)^n}. \]

Hence, if \( \gamma_0(p_S - p_{PP}) = 0 \), then \( p_{PP} = p_\varepsilon \).

We show that the sequence \(((u_\varepsilon, p_\varepsilon))_{\varepsilon > 0}\) is bounded in \( H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R}) \). By the reflexivity of \( H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R}) \), the sequence \(((u_\varepsilon, p_\varepsilon))_{\varepsilon > 0}\) has a subsequence converging weakly to somewhere in \( H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R}) \). It is sufficient to check that the limit satisfies (S'). Since the solution of (S') is unique, the sequence \(((u_\varepsilon, p_\varepsilon))_{\varepsilon > 0}\) converges weakly.

**Proof of Theorem 4.2** We take \( u_\varepsilon \in H^1(\Omega)^n, f \in H^{-1}(\Omega)^n \) and \( g \in Q^\star \) as (A.1) and (A.2) in the proof of Theorem 2.7. We put \( \tilde{u}_\varepsilon := u_\varepsilon - p_\varepsilon \in H^1_0(\Omega)^n \), \( \tilde{p}_\varepsilon := p_\varepsilon - p_0 \in Q \). Then we obtain

\[
\begin{align*}
\int_\Omega \nabla \tilde{u}_\varepsilon : \nabla \varphi + \int_\Omega (\nabla \tilde{p}_\varepsilon) \cdot \varphi &= (f, \varphi) \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_\Omega \nabla \tilde{p}_\varepsilon \cdot \nabla \psi + \int_\Omega (\text{div} \tilde{u}_\varepsilon) \psi &= \varepsilon (g, \psi) \quad \text{for all } \psi \in Q. 
\end{align*}
\]

Putting \( \varphi := \tilde{u}_\varepsilon \), \( \psi := \tilde{p}_\varepsilon \) and adding the two equations of (A.9), we get

\[
\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \|\nabla \tilde{p}_\varepsilon\|_{L^2(\Omega)^n}^2 \leq \|f\|_{H^{-1}(\Omega)^n} \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^{n \times n}} + \varepsilon \|g\|_{Q^\star} \|\nabla \tilde{p}_\varepsilon\|_{L^2(\Omega)^n}
\]

since \( \int_\Omega (\nabla \tilde{p}_\varepsilon) \cdot \tilde{u}_\varepsilon = -\int_\Omega (\text{div} \tilde{u}_\varepsilon) \tilde{p}_\varepsilon \). Hence,

\[(\|\tilde{u}_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon > 0} \text{ and } (\|\sqrt{\varepsilon} \tilde{p}_\varepsilon\|_{H^1(\Omega)})_{\varepsilon > 0} \text{ are bounded.} \quad (A.10)\]

Moreover, by Lemma 4.1, we obtain

\[ \|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq c(\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^{n \times n}} + \|f\|_{H^{-1}(\Omega)^n}), \]

i.e., \((\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon > 0} \) is bounded. By Theorem 3.2, \((\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon \geq 1}\) and \((\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon \geq 1}\) are bounded, and thus \((\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon > 0}\) and \((\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon > 0}\) are bounded.

Since \( H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R}) \) is reflexive and \(((\tilde{u}_\varepsilon, [\tilde{p}_\varepsilon])_{\varepsilon > 0}) \) is bounded in \( H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R}) \), there exist \((u, p) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})\) and a subsequence of pairs \((\tilde{u}_{\varepsilon_k}, \tilde{p}_{\varepsilon_k})_{k \in \mathbb{N}} \subset H^1_0(\Omega)^n \times Q\) such that

\[ \tilde{u}_{\varepsilon_k} \rightharpoonup u \text{ weakly in } H^1(\Omega)^n, \quad [\tilde{p}_{\varepsilon_k}] \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } k \to \infty. \]

Hence, from (A.9) with \( \varepsilon := \varepsilon_k \), taking \( k \to \infty \), we obtain

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\[
\begin{aligned}
\int_\Omega \nabla u : \nabla \varphi + \langle \nabla p, \varphi \rangle &= \langle f, \varphi \rangle \quad \text{for all } \varphi \in H^1_0(\Omega)^n \\
\int_\Omega (\text{div}u) \psi &= 0 \quad \text{for all } \psi \in Q,
\end{aligned}
\]
(A.11)

where we have used that

\[
|\varepsilon_k| \int_\Omega \nabla \tilde{\rho}_{\varepsilon_k} \cdot \nabla \psi \leq \sqrt{\varepsilon_k} \|\sqrt{\varepsilon_k} \tilde{\rho}_{\varepsilon_k}\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)} \to 0,
\]

\[
\int_\Omega \nabla \tilde{\rho}_{\varepsilon_k} \cdot \varphi = -\int_\Omega [\tilde{\rho}_{\varepsilon_k}] \text{div} \varphi \to -\int_\Omega p \text{div} \varphi = \langle \nabla p, \varphi \rangle
\]
as \(k \to \infty\). By (A.2), the first equation of (A.11) implies that

\[
\int_\Omega \nabla(u + u_b) : \nabla \varphi + \langle \nabla (p + p_0), \varphi \rangle = \int_\Omega F \cdot \varphi
\]

for all \(\varphi \in H^1_0(\Omega)^n\). From the second equation of (A.11) and \(C_0^\infty(\Omega) \subset Q\), \(\text{div}(u + u_b) = 0\) follows. Hence, we obtain that \((u + u_b, p + [p_0])\) satisfies (S'), i.e., \(u_s = u + u_b\) and \(p_s = p + [p_0]\). Then we have

\[
\begin{aligned}
u_{\varepsilon_k} - u_s &= u_{\varepsilon_k} - u - u_b = \tilde{u}_{\varepsilon_k} - u_s \to 0 \text{ weakly in } H^1(\Omega)^n, \\
[p_{\varepsilon_k}] - p_s &= [p_{\varepsilon_k} - p - p_0] = [\tilde{p}_{\varepsilon_k}] - p \to 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}
\end{aligned}
\]
as \(k \to \infty\). Since any arbitrarily chosen subsequence of \((u_{\varepsilon}, [p_{\varepsilon}])\) has a subsequence which converges to \((u_s, p_s)\), we can conclude the proof. \(\square\)

Using Theorem 4.2 and the Rellich–Kondrachov Theorem, it is easy to prove Theorem 4.3.

**Proof of Theorem 4.3** We have from (ES') and (S') that

\[
\begin{cases}
\int_\Omega \nabla(u_{\varepsilon} - u_s) : \nabla \varphi + \int_\Omega (\nabla(p_{\varepsilon} - p_s)) \cdot \varphi = 0 \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_\Omega \nabla p_{\varepsilon} \cdot \nabla \psi + \int_\Omega (\text{div}u_{\varepsilon}) \psi = \varepsilon \langle G, \psi \rangle \quad \text{for all } \psi \in Q.
\end{cases}
\]
Putting $\varphi := u_\varepsilon - u_S \in H^1_0(\Omega)^n$ and $\tilde{p}_S := p_S - p_0 \in H^1(\Omega)$, we get

$$
\|\nabla (u_\varepsilon - u_S)\|_{L^2(\Omega)^{n \times n}}^2 = - \int_\Omega (\nabla (p_\varepsilon - p_S)) \cdot (u_\varepsilon - u_S)
$$

$$
= - \int_\Omega (\nabla (p_\varepsilon - p_0)) \cdot (u_\varepsilon - u_S)
+ \int_\Omega (\nabla (p_S - p_0)) \cdot (u_\varepsilon - u_S)
= \int_\Omega (p_\varepsilon - p_0) \text{div}(u_\varepsilon - u_S) + \int_\Omega (\nabla \tilde{p}_S) \cdot (u_\varepsilon - u_S)
= \int_\Omega (p_\varepsilon - p_0) \text{div}u_\varepsilon + \int_\Omega (\nabla \tilde{p}_S) \cdot (u_\varepsilon - u_S),
$$

since $- \int_\Omega (\nabla (p_\varepsilon - p_0)) \cdot (u_\varepsilon - u_S) = \int_\Omega (p_\varepsilon - p_0) \text{div}(u_\varepsilon - u_S)$ and $\text{div}u_S = 0$. Thus,

$$
\|\nabla (u_\varepsilon - u_S)\|_{L^2(\Omega)^{n \times n}}^2 = \int_\Omega (p_\varepsilon - p_0) \text{div}u_\varepsilon + \int_\Omega (\nabla \tilde{p}_S) \cdot (u_\varepsilon - u_S). \quad (A.12)
$$

Putting $\psi := p_\varepsilon - p_0 \in Q$, we have

$$
\varepsilon \int_\Omega \nabla p_\varepsilon \cdot \nabla (p_\varepsilon - p_0) + \int_\Omega (\text{div}u_\varepsilon)(p_\varepsilon - p_0) = \varepsilon \langle G, p_\varepsilon - p_0 \rangle.
$$

Hence,

$$
\varepsilon \|p_\varepsilon - p_0\|_{L^2(\Omega)^n}^2 = -\varepsilon \int_\Omega \nabla (p_\varepsilon - p_0) \cdot \nabla p_0
- \int_\Omega (p_\varepsilon - p_0) \text{div}u_\varepsilon + \varepsilon \langle G, p_\varepsilon - p_0 \rangle. \quad (A.13)
$$

Together with (A.12) and (A.13), we obtain

$$
\|\nabla (u_\varepsilon - u_S)\|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \|\nabla (p_\varepsilon - p_0)\|_{L^2(\Omega)^n}^2
$$

$$
= \int_\Omega \nabla \tilde{p}_S \cdot (u_\varepsilon - u_S) - \varepsilon \int_\Omega \nabla (p_\varepsilon - p_0) \cdot \nabla p_0 + \varepsilon \langle G, p_\varepsilon - p_0 \rangle
\leq \|\nabla \tilde{p}_S\|_{L^2(\Omega)^n} \|u_\varepsilon - u_S\|_{L^2(\Omega)^n}
+ \varepsilon (\|\nabla p_0\|_{L^2(\Omega)^n} + \|G\|_{Q^*}) \|\nabla (p_\varepsilon - p_0)\|_{L^2(\Omega)^n}.
$$

By Theorem 4.2 and the Rellich–Kondrachov Theorem, there exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ such that

$$
u_{\varepsilon_k} \rightarrow u_S \text{ strongly in } L^2(\Omega)^n \quad \text{as } k \rightarrow \infty.
$$
Therefore, by (A.10),
\[
\| \nabla (u_{\varepsilon_k} - u_S) \|^2_{L^2(\Omega)^{n \times n}} \leq \| \nabla \tilde{p}_S \|^2_{L^2(\Omega)^n} \| u_{\varepsilon_k} - u_S \|^2_{L^2(\Omega)^n} \\
+ \varepsilon_k (\| \nabla \tilde{p}_0 \|^2_{L^2(\Omega)^n} + \| G \|^2_{Q^*}) \| \nabla (p_{\varepsilon_k} - p_0) \|^2_{L^2(\Omega)^n} \\
\rightarrow 0
\]
as \( k \rightarrow \infty \). This implies that
\[
\| [p_{\varepsilon_k}] - p_S \|^2_{L^2(\Omega)/\mathbb{R}} = \| p_{\varepsilon_k} - p_S \|^2_{L^2(\Omega)/\mathbb{R}} \\
\leq c \| \nabla (u_{\varepsilon_k} - u_S) \|^2_{L^2(\Omega)^{n \times n}} \rightarrow 0 \text{ as } k \rightarrow \infty
\]
by Lemma 4.1. Since any arbitrarily chosen subsequence of \(((u_{\varepsilon}, [p_{\varepsilon}]))_{0<\varepsilon<1}\) has a subsequence which converges to \((u_S, p_S)\), we can conclude the proof. \( \square \)

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