THE TIME ASYMPTOTIC EXPANSION FOR THE COMRESSIBLE EULER EQUATIONS WITH TIME-DEPENDENT DAMPING

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Abstract. In this paper, we study the compressible Euler equations with time-dependent damping
\(-\frac{1}{(1+t)^\lambda}\rho u\). We propose a time asymptotic expansion around the self-similar solution of the generalized
porous media equation (GPME) and rigorously justify this expansion as
\(\lambda \in \left(\frac{1}{7}, 1\right)\). In other word,
instead of the self-similar solution of GPME, the expansion is the best asymptotic profile of the solution
to the compressible Euler equations with time-dependent damping.

1. Introduction

In this paper, we consider the compressible Euler equations with time-dependent damping as follows:
\begin{align*}
\rho_t + m_x &= 0, \\
\rho_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x &= -\frac{1}{(1+t)^\lambda}m,
\end{align*}
(1.1)
with the initial data
\((\rho, m)(x, 0) = (\rho_0, m_0)(x) \to (\rho_\pm, m_\pm)\) as \(x \to \pm \infty\),
(1.2)
where \(\rho_\pm > 0\) and \(m_\pm\) are constants. Here \(\rho = \rho(x, t), m = m(x, t)\) and \(p = p(\rho)\) denote the density,
momentum and pressure, respectively. We assume that the pressure \(p(\rho)\) is a smooth function and satisfies
\(p'(\rho) > 0\) for \(\rho > 0\). The damping term \(-\frac{1}{(1+t)^\lambda}m\) represents the time-dependent friction effect, where
\(0 < \lambda < 1\) is constant.

When \(\lambda = 0\), the system (1.1) becomes the compressible system of Euler equations with damping
modeling the compressible flow through porous media. There has a huge literature on the investigations
of global existence and large time behaviors of smooth solutions to the compressible Euler equations
with damping. Among them, Hsiao and Liu \cite{13} firstly showed that the solution of (1.1) tends
time-asymptotically to the self-similar solution of porous media equation (PME), called by diffusion wave.
Since then, this problem has attracted considerable attentions, see \cite{3, 6, 7, 11, 13–16, 19, 21, 25–27} and
the references therein. When \(0 < \lambda < 1\), Cui-Yin-Zhang-Zhu \cite{2} showed that the asymptotic behavior of
the solution to the problem (1.1) is the so-called diffusion waves in the self-similar form of \((\bar{\rho}, \bar{m})(x, t) =
(\bar{\rho}, \bar{m})(x/\sqrt{(1+t)^{1+\lambda}})\) satisfying
\begin{align*}
\bar{\rho}_t + \bar{m}_x &= 0, \\
\bar{p}(\bar{\rho})_x &= -\frac{1}{(1+t)^\lambda}\bar{m}, \quad \text{equivalently,} \quad \frac{1}{(1+t)^\lambda}\bar{p}_t - p(\bar{\rho})_{xx} = 0 \quad \text{(GPME)}
\end{align*}
(1.3)

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with
\[
\lim_{x \to \pm \infty} (\tilde{\rho}, \tilde{m})(x, t) = (\rho_\pm, 0).
\] (1.4)

The convergence rate obtained in [2] is in the form of
\[
\| (\rho - \tilde{\rho})(t) \|_{L^\infty} \leq \begin{cases} 
C(1 + t)^{-\frac{1}{4}(1 + \lambda)}, & 0 < \lambda < \frac{1}{7}, \\
C_\varepsilon(1 + t)^{-\frac{1}{4} + \varepsilon}, & \lambda = \frac{1}{7}, \\
C(1 + t)^{(1 - \lambda)}, & \frac{1}{7} < \lambda < 1,
\end{cases}
\] (1.5)

for any small $\varepsilon > 0$. Similar results were obtained in [10] for the bipolar Euler-Poisson equation with time-dependent damping. For the other interesting works on the compressible Euler equations with time-dependent damping [13], see [13] and reference therein.

It is noted that the decay rate [13] for $\frac{1}{7} < \lambda < 1$ is different from the one for $0 < \lambda \leq \frac{1}{7}$. We guess that the solution $\tilde{\rho}$ of GPME may not be the best time-asymptotic profile of the solution $\rho$ of (1.1) for $\frac{1}{7} < \lambda < 1$. We would also like to know more information on the large time behavior of $\rho(x, t)$. Thus, we propose a time asymptotic expansion as follows:

\[
\begin{cases}
\rho = \tilde{\rho} + \sum_{i=1}^{k} (1 + t)^{-\sigma_i} \rho_i(\xi) + P_k =: \tilde{\rho}_k + P_k, \\
m = \tilde{m} + \sum_{i=1}^{k} (1 + t)^{-1/2 + \sigma_i} m_i(\xi) + Q_k =: \tilde{m}_k + Q_k,
\end{cases}
\] (1.6)

where $(\tilde{\rho}, \tilde{m})$ is the diffusion wave of GPME given in (1.3), and $(\rho_i, m_i)(\xi)$ is a solution of a linear equation given in section 2 below. Once (1.6) is justified, the optimal decay rate, the main part and even more subsequent order terms of $(\rho - \tilde{\rho})(x, t)$ are clearly known.

For convenience, we focus on the case of $m_+ = m_- = 0$. And the precise statement of our main results are as follows.

**Theorem 1.1.** For any $\lambda \in (\frac{1}{7}, 1)$, there exists a unique positive integer $k_0(\lambda) = \sup_{0 < \lambda < \frac{1}{7}} \lfloor k(\lambda) \rfloor$, where $k(\lambda) = \frac{3(1 + \lambda)}{2(1 - \lambda)}$ and $\lfloor \cdot \rfloor$ stands for the floor function. If the initial data $(y_0, y_1)(x) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$, where

\[
y_0(x) = -\int_{-\infty}^{x} (\rho_0(r) - \tilde{\rho}_{k_0(\lambda)}(r, 0))dr \quad \text{and} \quad y_1(x) = m_0(x) - \tilde{m}_{k_0(\lambda)}(x, 0),
\]

then there is a small constant $\delta_0 > 0$ such that if $\delta_1 =: \| y_0 \|_3 + \| y_1 \|_2 + |\rho_+ - \rho_-| \leq \delta_0$, there exists a unique and global solution $(\rho, m)(x, t)$ of the problem (1.1)-(1.2). Moreover, the remainders $P_{k_0(\lambda)}$ and $Q_{k_0(\lambda)}$ in (1.6) satisfy

\[
\sum_{s=0}^{2} \left[ (1 + t)^{(s+1)(1 + \lambda)} \| \partial_x^s P_{k_0(\lambda)}(t) \|^2 + (1 + t)^{2 + s(1 + \lambda)} \| \partial_x^s Q_{k_0(\lambda)}(t) \|^2 \right] 
\leq \begin{cases} 
C\delta_1^2, & \text{if} \quad k(\lambda) \notin \mathbb{N}^+, \\
C\delta_1^2 \ln^2(1 + t), & \text{if} \quad k(\lambda) \in \mathbb{N}^+.
\end{cases}
\]

Using the Sobolev inequality, we can derive the following estimates.

**Corollary 1.1.** Under the assumptions of Theorem 1.1, the remainders $P_{k_0(\lambda)}$ and $Q_{k_0(\lambda)}$ in (1.6) satisfy

\[
\| \partial_x^i (P_{k_0(\lambda)}, Q_{k_0(\lambda)})(t) \|_{L^\infty} 
\leq \begin{cases} 
C\delta_1 [(1 + t)^{-\frac{1}{4} + (1 + \lambda)}, (1 + t)^{-1 - \frac{1}{4} + (1 + \lambda)}], & i = 0, 1, \quad \text{if} \quad k(\lambda) \notin \mathbb{N}^+, \\
C\delta_1 [(1 + t)^{-\frac{1}{4} + (1 + \lambda)} \ln(1 + t), (1 + t)^{-1 - \frac{1}{4} + (1 + \lambda) \ln(1 + t)}], & i = 0, 1, \quad \text{if} \quad k(\lambda) \in \mathbb{N}^+.
\end{cases}
\]
Remark 1.1. Note that \( \frac{3(1+\lambda)}{4} = k(\lambda)\sigma \) and
\[
k_0(\lambda) = \begin{cases}
[k(\lambda)], & \text{if } k(\lambda) \notin \mathbb{N}^+,
k(\lambda) - 1, & \text{if } k(\lambda) \in \mathbb{N}^+,
\end{cases}
\]
the decay rate of the remainder \( P_{k_0(\lambda)} \) obtained in Corollary 1.1 is faster than \( (1 + t)^{-k_0(\lambda)\sigma} \), which justifies the expansion (1.6) until the order of \( k_0(\lambda) \). Nevertheless, we conjecture that the expansion (1.6) still holds for any order.

Remark 1.2. If
\[
\int_{-\infty}^{\infty} (\rho_0(r) - \bar{\rho}_{k_0(\lambda)}(r,0)) dr \neq 0, \quad \text{i.e.,} \quad y_0(x) \notin H^3(\mathbb{R}),
\]
we can still prove that the above results hold by replacing \( \xi = \frac{x}{(1+t)^{\frac{1}{2}}} \) with \( \xi = \frac{x + x_0}{(1+t)^{\frac{1}{2}}} \) for some shift \( x_0 \), where \( x_0 \) is determined by the initial data.

Remark 1.3. For the case that \( m_- \neq 0 \) or \( m_+ \neq 0 \), we claim the above results still hold by introducing a correction function \( (\check{\rho}, \check{m})(x,t) \) with exponential decay rate to delete the gap at \( x = \pm \infty \), see [2, 6] for the details.

The arrangement of the present paper is as follows. In Section 2, we propose the time asymptotic expansion \((\check{\rho}, \check{m})_k(x,t)\) of the solution to (1.3). In Section 3, we justify the expansion \((\check{\rho}_{k_0}, \check{m}_{k_0})\) and prove Theorem 1.1.

Notations. Throughout this paper, the symbol \( C \) will be used to represent a generic constant which is independent of \( x \) and \( t \) and may vary from line to line. \( L^2(\mathbb{R}) \) is the space of square integrable real valued function defined on \( \mathbb{R} \) with the norm \( \| \cdot \| \), and \( H^k(\mathbb{R}) \) (\( H^k(\mathbb{R}) \) without any ambiguity) denotes the usual Sobolev space with the norm \( \| \cdot \|_k \), especially \( \| \cdot \|_0 = \| \cdot \| \). In addition, for \( r, s \in \mathbb{N} \), we adopt the convention that
\[
\sum_{i=r}^{s} a_i = 0 \quad \text{if} \quad s < r.
\]

2. The time asymptotic expansion

We first list some properties on the diffusion wave \((\check{\rho}, \check{m})(\xi)\) of GPME (1.3)-(1.4) as follows.

Lemma 2.1 ([2, 10]). For the diffusion wave \((\check{\rho}, \check{m})(\xi)\) of (1.3)-(1.4), it holds that
\[
\begin{align*}
|\check{\rho}(\xi)| - \rho_+ |\xi| < 0 + |\check{\rho}(\xi)| - \rho_- |\xi| < 0 & \leq C |\rho_+ - \rho_-| e^{-c\xi^2}, \\
|\partial_x^k \partial_t^l \check{\rho}| & \leq C |\rho_+ - \rho_-| (1 + t)^{-\frac{k(1+\lambda)+2}{2} - kl} e^{-c\xi^2}, \quad k, l \geq 1, k + l \geq 0, \\
|\partial_x^k \partial_t^l |\partial_x^i \check{\rho}|^2 & \leq C |\rho_+ - \rho_-| (1 + t)^{-\frac{k(1+\lambda)+2}{2} - 2l} e^{-c\xi^2}, \quad k, l \geq 1.
\end{align*}
\]

Then we consider the following time asymptotic expansion:
\[
\begin{align*}
\rho &= \check{\rho}(\xi) + \sum_{i=1}^{k} (1 + t)^{-i\sigma} \check{\rho}_i(\xi) + P_k =: \check{\rho}_k + P_k, \\
m &= \check{m}(\xi) + \sum_{i=1}^{k} (1 + t)^{-i\sigma} \check{m}_i(\xi) + Q_k =: \check{m}_k + Q_k,
\end{align*}
\]
where \( k \) is a positive integer determined later, and
\[
\xi = \frac{x}{(1+t)^{\frac{1}{2} \lambda}}, \quad \sigma = 1 - \lambda.
\]
Note that \( \check{\rho}_t + \check{m}_x = 0 \), we expect
\[
((1 + t)^{-i\sigma} \check{\rho}_i)_t + (1 + t)^{-i\sigma} \check{m}_i)_x = 0,
\]
which implies that
\[ m_{i\xi} = i\sigma \rho_i + \frac{1 + \lambda}{2}\xi\rho_i. \]
We hope \( \int_0^\infty \rho_i(\xi)d\xi = 0 \), which leads to
\[ m_i(\xi) = (i\sigma - (1 + \lambda))G_i + \frac{1 + \lambda}{2}(\xi G_i)\xi, \quad G_i(\xi) := \int_0^\xi \rho_i(\eta)d\eta. \tag{2.2} \]
Plugging (2.1) and (2.2) into (1.1), we derive a formal hierarchy of ODEs satisfied by the functions \( \tilde{\rho} \) and \( \rho_i, i = 1, \ldots, k \). Define the source term
\[ S(\tilde{\rho}_k) = \hat{m}_{kt} + \left( \frac{\hat{\theta}_i^2}{\tilde{\rho}_k} + p(\tilde{\rho}_k) \right)_t + \frac{1}{(1 + t)^\lambda}\hat{m}_k. \tag{2.3} \]

**Lemma 2.2.** For \( k \in \mathbb{N}^+ \), the source term \( S(\tilde{\rho}_k) \) given by (2.3) satisfies
\[ S(\tilde{\rho}_k) = \sum_{i=1}^k (1 + t)^{-\frac{1 + \lambda}{2} - i\sigma} \left[ (P(\tilde{\rho})G_i(\xi))_\xi + \frac{1 + \lambda}{2}(\xi G_i)_\xi + c_{1,i}G_i - c_{2,i}G_{i-1} + h_i\xi \right] \]
\[ + (1 + t)^{-\frac{1 + \lambda}{2} - (k+1)\sigma}R_k, \tag{2.4} \]
where
\[ c_{1,i} = i\sigma - (1 + \lambda), \quad c_{2,i} = (i\sigma - 1)((i - 1)\sigma - (1 + \lambda)), \quad i = 1, 2, \ldots, k, \]
\( G_0, h_i \) and \( R_k \) are given by (2.1), (2.13) and (2.9) below.

**Proof.** From (2.3), the Taylor expansion gives that
\[ P(\tilde{\rho}_k) = P(\tilde{\rho}) + \sum_{i=1}^k (1 + t)^{-i\sigma} \sum_{j=1}^i h_{1,i,j} + (1 + t)^{-(k+1)\sigma}R_{1,k}, \tag{2.5} \]
and
\[ \hat{m}_k^2 = (1 + t)^{-\sigma} \frac{(P(\tilde{\rho})\xi)^2}{\tilde{\rho}} + \sum_{i=2}^k (1 + t)^{-i\sigma} \sum_{j=1}^{i-1} h_{2,i,j} + \sum_{i=3}^k (1 + t)^{-i\sigma} \sum_{j=2}^{i-1} h_{3,i,j} + (1 + t)^{-(k+1)\sigma}R_{2,k}, \tag{2.6} \]
where
\[ h_{1,i,j} = \frac{P^{(j)}(\tilde{\rho})}{j!} \sum_{l_1+\cdots+l_j=i, l_1, \ldots, l_j \geq 1} \rho_{l_1} \cdots \rho_{l_j}, \quad 1 \leq j \leq i, \]
\[ h_{2,i,j} = \frac{(-1)^j(P(\tilde{\rho})\xi)_\xi^2}{(\tilde{\rho})^{j+1}} \sum_{l_1+\cdots+l_j=i-1, l_1, \ldots, l_j \geq 1} \rho_{l_1} \cdots \rho_{l_j} \]
\[ + \frac{2(-1)^jP(\tilde{\rho})}{(\tilde{\rho})^j} \sum_{l_1+\cdots+l_j=i-1, l_1, \ldots, l_j \geq 1} m_{l_1}\rho_{l_2} \cdots \rho_{l_j} , \quad 1 \leq j \leq i-1, \]
\[ h_{3,i,j} = \frac{(-1)^j}{(\tilde{\rho})^{j-1}} \sum_{l_1+\cdots+l_j=i-1, l_1, \ldots, l_j \geq 1} m_{l_1}m_{l_2}\rho_{l_3} \cdots \rho_{l_j} , \quad 2 \leq j \leq i-1, \]
and the remainder terms \( R_{1,k}, R_{2,k} \) are some functions depending on \( \tilde{\rho} \) and \( \rho_l \) with \( l \in \{1, \ldots, k\} \).

On the other hand, the direct computations give that
\[ \hat{m}_{kt} + \frac{1}{(1 + t)^\lambda}\hat{m}_k \]
\[ = -(1 + t)^{-\frac{1 + \lambda}{2}}P(\tilde{\rho})\xi + (1 + t)^{-\frac{1 + \lambda}{2}-\sigma} \left[ -\lambda P(\tilde{\rho})\xi + \frac{1 + \lambda}{2}(\xi P(\tilde{\rho})\xi + m_1) \right] + \sum_{i=2}^k (1 + t)^{-\frac{1 + \lambda}{2} - i\sigma} \left[ m_i - (i\sigma - 1)m_{i-1} + \frac{1 + \lambda}{2}(\xi m_{i-1})_\xi \right] + (1 + t)^{-\frac{1 + \lambda}{2} - (k+1)\sigma}R_{3,k}, \tag{2.7} \]
where
\[ R_{3,k} = -(k + \frac{1}{2})\sigma m_k - \frac{1 + \lambda}{2} \xi m_k. \]

Thus, we use (2.10) and (2.11) to obtain
\[
S(\tilde{\rho}_k) = (1 + t)^{-\frac{i + \lambda}{2} - \sigma} \left[ m_1 + \left( P^{(1)}(\tilde{\rho}) \rho_1 \right) \xi - \lambda \rho \xi + \frac{1 + \lambda}{2} (\xi \rho \xi + \left( \frac{P(\rho) \xi}{\rho} \right) \xi \right] \\
+ \sum_{i=2}^{k} (1 + t)^{-\frac{i + \lambda}{2} - i \sigma} \left[ m_i + \left( P^{(1)}(\tilde{\rho}) \rho_i \right) \xi + \tilde{h}_i \xi - (i \sigma - 1)m_{i-1} \right] \\
+ (1 + t)^{-\frac{i + \lambda}{2} - (k+1) \sigma} R_k,
\]
where
\[ \tilde{h}_i = \sum_{j=2}^{i} h_{1,i,j} + \sum_{j=1}^{i-1} h_{2,i,j} + \sum_{j=2}^{i-1} h_{3,i,j} + \frac{1 + \lambda}{2} \xi m_{i-1}, \quad \text{for} \ i = 2, 3, \ldots, k, \]
\[ R_k = (R_{1,k} + R_{2,k})\xi + R_{3,k}. \]

Moreover, substituting (2.10) into (2.11) yields
\[
S(\tilde{\rho}_k) = (1 + t)^{-\frac{i + \lambda}{2} - \sigma} \left[ \left( P^{(1)}(\tilde{\rho}) \rho_1 \right) \xi + \frac{1 + \lambda}{2} (\xi G_1) \xi - 2 \lambda G_1 + \lambda \rho \xi + \left( \frac{1 + \lambda}{2} \xi \rho \xi + \left( \frac{P(\rho) \xi}{\rho} \right) \xi \right) \right] \\
+ \sum_{i=2}^{k} (1 + t)^{-\frac{i + \lambda}{2} - i \sigma} \left[ \left( P^{(1)}(\tilde{\rho}) \rho_i \right) \xi + \frac{1 + \lambda}{2} (\xi G_i) \xi + c_{1,i} G_i - c_{2,i} G_{i-1} + h_{i}\xi \right] \\
+ (1 + t)^{-\frac{i + \lambda}{2} - (k+1) \sigma} R_k,
\]
where
\[ c_{1,i} = i \sigma - (1 + \lambda), \]
\[ c_{2,i} = (i \sigma - 1)((i - 1) \sigma - (1 + \lambda)), \]
\[ h_i = \tilde{h}_i - \frac{1 + \lambda}{2} (i \sigma - 1) \xi G_{i-1} \]
with \( i = 2, \ldots, k \). Note that \( c_{1,1} = -2 \lambda \), we may supply
\[ G_0 = \frac{P(\tilde{\rho}) \xi}{1 + \lambda}, \]
\[ h_1 = \frac{1 + \lambda}{2} \xi P(\tilde{\rho}) \xi + \left( \frac{P(\rho) \xi}{\rho} \right) \xi, \]
so that (2.10) holds with \( i = 1, \ldots, k \). Therefore, the proof of Lemma 2.2 is completed. \( \square \)

Motivated by Lemma 2.2, we define the hierarchy of ODEs as
\[
(P'(\tilde{\rho}) G_i) \xi + \frac{1 + \lambda}{2} (\xi G_i) \xi + c_{1,i} G_i = c_{2,i} G_{i-1} - h_{i}\xi, \quad \text{for} \ i = 1, 2, \ldots, k, \]
so that \( S(\tilde{\rho}_k) = O(1)(1 + t)^{-\frac{i + \lambda}{2} - (k+1) \sigma} \). We will seek for the solution \( G_i \in \chi^l(\mathbb{R}) \) to (2.14), where
\[ \chi^l(\mathbb{R}) = \{ f : \xi^s \partial^r_f \in L^2(\mathbb{R}), \ \forall r, s \in \{0, 1, \ldots, l\} \} \]
equipped with the norm
\[ \| f \|_{\chi^l(\mathbb{R})} = \left( \sum_{0 \leq s, r \leq l} \int_{\mathbb{R}} (\xi^s \partial^r_f)^2 d\xi \right)^{\frac{1}{2}}. \]

Then integrating (2.14) with respect to \( \xi \) over \( \mathbb{R} \) gives that
\[ \int_{\mathbb{R}} G_i d\xi = \frac{c_{2,i}}{c_{1,i}} \int_{\mathbb{R}} G_{i-1} d\xi \quad \text{for} \ i = 1, 2, \ldots, k. \]
We obtain the existence of the smooth solution \( G_i \) of (2.14) and (2.16) as follows.

**Lemma 2.3.** Let \( \delta = |\rho_+ - \rho_-| \ll 1 \), there exists a solution \( G_i \in \chi^{m_i}(\mathbb{R}) \) to (2.14) and (2.16) for large integer \( m_i > 0 \). Furthermore, it holds that
\[
\|G_i\|_{\chi^{m_i}(\mathbb{R})} \leq C\delta
\]
and
\[
\|\xi^{s_0} \partial_{\xi_{m}} G_i\|_{L^\infty} \leq C\delta,
\]
where \( 0 \leq s_0 \leq m_i \) and \( 0 \leq r_0 \leq m_i - 1 \).

The detailed proof is left in the Appendix.

Thanks to Lemmas 2.2-2.3 we get the estimates of source term \( S(\tilde{\rho}_k) \) in (2.3).

**Lemma 2.4.** It holds that
\[
S(\tilde{\rho}_k) = O(1)\delta(1 + t)^{-\frac{1 + \lambda}{2}-(k+1)\sigma}
\]  
(2.17)
and
\[
\|\partial_t^{2j} \partial_{x}^{l} S(\tilde{\rho}_k)\|_{L^{2}(\mathbb{R})}^{2} \leq C\delta^{2}(1 + t)^{-2j-2(k+1)\sigma -(l+\frac{3}{2})(1+\lambda)}, \quad j, l \geq 0.
\]  
(2.18)

**Proof.** It follows from (2.4) and Lemma 2.3 that
\[
S(\tilde{\rho}_k) = (1 + t)^{-\frac{1 + \lambda}{2}-(k+1)\sigma} R_k,
\]
where \( R_k \) is given by (2.8). Then, (2.17) and (2.18) can be obtained by direct computations. Thus, the proof of Lemma 2.4 is completed. \(\square\)

### 3. The Estimates of the Remainder Terms

This section is devoted to Theorem 1.1 by the classical energy method with the continuation argument based on the local existence and the a priori estimates. For any \( \lambda \in (\frac{1}{2}, 1) \), let
\[
k_0 =: k_0(\lambda) = \sup_{0 < \varepsilon < \frac{1}{2}} \left[ k(\lambda) - \varepsilon \right] = \sup_{0 < \varepsilon < \frac{1}{2}} \left[ \frac{3(1 + \lambda)}{4(1 - \lambda)} - \varepsilon \right]
\]
and the time asymptotic expansion is
\[
\rho = \tilde{\rho} + \sum_{i=1}^{k_0} (1 + t)^{-i\sigma} \rho_i(\xi) + P_{k_0} =: \tilde{\rho}_{k_0} + P_{k_0},
\]
\[
m = \tilde{m} + \sum_{i=1}^{k_0} (1 + t)^{-(i+\frac{3}{2})\sigma} m_i(\xi) + Q_{k_0} =: \tilde{m}_{k_0} + Q_{k_0}.
\]  
(3.1)

We shall show that the remainder \( P_{k_0} \) decays faster than \((1 + t)^{-k_0}\sigma\). Denote
\[
y = -\int_{-\infty}^{x} P_{k_0}(r, t)dr,
\]
then
\[
y_x = -P_{k_0}, \quad y_t = Q_{k_0}.
\]
Thus the system (1.1) can be rewritten as a quasilinear wave equation for \( y \):
\[
\begin{cases}
y_{tt} - (P(\tilde{\rho}_{k_0})y_x)_x + \frac{y_t}{(1 + t)^{\frac{3}{2}}} = g_1 + g_2 + S(\tilde{\rho}_{k_0}), \\
y(x, y)|_{(x, 0)} = (y_0, y_1)(x),
\end{cases}
\]  
(3.2)
where
\[
g_1 = -(P(\rho) - P(\tilde{\rho}_{k_0}) + P'(\tilde{\rho}_{k_0})y_x)_x, \quad g_2 = -\left( \frac{m^2}{\rho} - \frac{\tilde{m}^2}{\tilde{\rho}_{k_0}} \right)_x.
\]

Motivated by the work of [15], we seek for the solution of (3.2) in the following solution space
\[
X_T =: \{ y \in C([0, T); H^3(\mathbb{R})); y_t \in C([0, T); H^2(\mathbb{R})) \}. \]
Since the local existence of the solution of (3.2) can be proved by the standard iteration method, see [12], the main effort in this section is to establish the a priori estimates for the solution.

For any $T \in (0, +\infty)$, define

$$N(T)^2 = \sup_{0 \leq t \leq T} \{ \| y \|^2 + \sum_{i=0}^{2} (1 + t)^{(i+1)(1+\lambda)} \| (\partial_x^i y_t) \|^2 + (\partial_x^i y_x) \|^2 \}.$$ 

We assume

$$N(T) \leq \begin{cases} 
\epsilon, & \text{if } k(\lambda) \notin \mathbb{N}^+, \\
\epsilon \ln(1 + T), & \text{if } k(\lambda) \in \mathbb{N}^+. 
\end{cases} \quad (3.3)$$

where $\epsilon$ is sufficiently small and will be determined later. Then it follows from Sobolev inequality $\| \partial_x^i f \|_{L^\infty} \leq C \| \partial_x^i f \|^{1/2} \| \partial_x^{i+1} f \|^{1/2}$ for $i = 0, 1$ that

$$\| \partial_x^i y_t \|_{L^\infty} + \| \partial_x^i y_x \|_{L^\infty} \leq \begin{cases} 
C \epsilon (1 + t)^{-\frac{4}{2(2+3)(1+\lambda)}}, & \text{if } k(\lambda) \notin \mathbb{N}^+, \\
C \epsilon (1 + t)^{-\frac{4}{2(2+3)(1+\lambda)} \ln(1 + t)}, & \text{if } k(\lambda) \in \mathbb{N}^+. 
\end{cases} \quad (3.4)$$

We first establish the following basic energy estimate. For abbreviation, let $(\hat{\rho}, \hat{m})$ stand for $(\hat{\rho}_{k_0}, \hat{m}_{k_0})$ in what follows.

**Lemma 3.1.** For any $T > 0$, assume that $y(x, t) \in \mathcal{X}_T$ is the solution of (3.2). If $\epsilon$ and $\delta$ are small, then it holds that

$$\int_{\mathbb{R}} \left[ (1 + t)^{\beta+1}(y_t^2 + y_x^2) + (1 + t)^{\beta-\lambda}y^2 \right] dx + \int_0^t \int_{\mathbb{R}} (1 + \tau)^{\beta+1-\lambda}y_t^2 dx d\tau$$

$$+ \int_0^t \int_{\mathbb{R}} (1 + \tau)^{\beta}y_x^2 dx d\tau \leq C N(0)^2 + \delta^2 + \delta \epsilon, \quad (3.5)$$

where $\beta < \lambda$.

**Proof.** Multiplying (3.2) by $(\alpha + t)^{\beta} y$ and integrating the result over $\mathbb{R}$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} [(\alpha + t)^{\beta} y_t y + \frac{(\alpha + t)^{\beta}}{2(1 + t)^{\lambda}} y^2] dx + (\alpha + t)^{\beta} \int_{\mathbb{R}} P'(\hat{\rho}) y_t^2 dx + \frac{(\alpha + t)^{\beta}}{(1 + t)^{1+\lambda}} \lambda \alpha - \beta + (\lambda - \beta) t \int_{\mathbb{R}} \frac{1}{2} y^2 dx$$

$$= (\alpha + t)^{\beta} \int_{\mathbb{R}} y_t^2 dx + \beta (\alpha + t)^{\beta-1} \int_{\mathbb{R}} y_t y dx + (\alpha + t)^{\beta} \int_{\mathbb{R}} g_1 y dx$$

$$+ (\alpha + t)^{\beta} \int_{\mathbb{R}} g_2 y dx + (\alpha + t)^{\beta} \int_{\mathbb{R}} S(\hat{\rho}) y dx, \quad (3.6)$$

where $\alpha$ is a positive constant to be determined later. From Lemmas 2.1, 2.3 the a priori assumption (3.3) and the expansion (3.1), we have

$$(\alpha + t)^{\beta} \int_{\mathbb{R}} g_1 y dx = (\alpha + t)^{\beta} \int_{\mathbb{R}} (P(\rho) - P(\hat{\rho}) + P'(\hat{\rho}) y_x) y dx \leq C \epsilon (\alpha + t)^{\beta} \int_{\mathbb{R}} y_x^2 dx \quad (3.7)$$

and

$$(\alpha + t)^{\beta} \int_{\mathbb{R}} g_2 y dx = -(\alpha + t)^{\beta} \int_{\mathbb{R}} \left( \frac{m^2}{\rho} - \frac{m^2}{\rho} \right) y dx$$

$$\leq C(\delta + \epsilon) \frac{(\alpha + t)^{\beta}}{(1 + t)^{1+\lambda}} \int_{\mathbb{R}} y^2 dx + C(\delta + \epsilon)(1 + t)^{\beta+\lambda-1} \int_{\mathbb{R}} (y_x^2 + y_y^2) dx. \quad (3.8)$$

In addition, it is easy to check that

$$\beta (\alpha + t)^{\beta-1} \int_{\mathbb{R}} y_t y dx \leq \nu_1 \frac{(\alpha + t)^{\beta}}{(1 + t)^{1+\lambda}} \int_{\mathbb{R}} y^2 dx + C(\nu_1)(\alpha + t)^{\beta+\lambda-1} \int_{\mathbb{R}} y_t^2 dx. \quad (3.9)$$

Moreover, from the a priori assumption (3.3) and the estimates (2.18), for the case of $k(\lambda) \notin \mathbb{N}^+$ we have

$$(\alpha + t)^{\beta} \int_{\mathbb{R}} S(\hat{\rho}) y dx \leq (\alpha + t)^{\beta} \| y \| \| S(\hat{\rho}) \| \leq C \delta \epsilon (1 + t)^{-\frac{1-\sigma(k_0+1)-k(\lambda)}{-(\lambda-\beta)}} \quad (3.10)$$
and for the case of \( k(\lambda) \in \mathbb{N}^+ \) we have

\[
(\alpha + t)^\beta \int_{\mathbb{R}} S(\tilde{\rho}) y dx \leq (\alpha + t)^\beta \| y \| S(\tilde{\rho}) \leq C\delta \epsilon (1 + t)^{-1 - \sigma[(k_0+1)-k(\lambda)]} - (\lambda - \beta) \ln(1 + t).
\]  

(3.11)

Substituting (3.7)-(3.11) into (3.6) and choosing \( \nu \) and \( \beta \) small enough give that

\[
\frac{d}{dt} \int_{\mathbb{R}} [(\alpha + t)^\beta y y + (\alpha + t)^\beta \frac{2}{(1 + t)^\lambda} y^2] dx \geq \frac{(\alpha + t)^\beta}{(1 + t)^{\lambda+1}} \int_{\mathbb{R}} \frac{\lambda - \beta}{4} y^2 dx
\]

\[
\leq C(\alpha + t)^\beta \int_{\mathbb{R}} y^2 dx + C\delta \epsilon (1 + t)^{-1 - \sigma[(k_0+1)-k(\lambda)]} - (\lambda - \beta)(1 + \ln(1 + t)),
\]  

(3.12)

where we choose \( \beta < \lambda \) such that

\[
\frac{\lambda - \beta}{\alpha + t} = \lambda - \beta + \frac{\beta(\alpha - 1)}{\alpha + t} \geq \lambda - \beta > 0.
\]

We multiply by \((\alpha + t)^{\beta+1}y_t\) and integrate the result over \( \mathbb{R} \) to obtain

\[
\frac{d}{dt} \int_{\mathbb{R}} [(\alpha + t)^{\beta+1} \frac{1}{2} y_t^2 + (\alpha + t)^{\beta+1} P'(\tilde{\rho}) \frac{1}{2} y_t^2] dx + \frac{(\alpha + t)^{\beta+1}}{(1 + t)^\lambda} \left[ 1 - (\frac{1}{2(\alpha + t)})^\lambda \right] \int_{\mathbb{R}} y_t^2 dx \leq C(\alpha + t)^\beta \int_{\mathbb{R}} y_t^2 dx.
\]  

(3.13)

A direct computation yields that

\[
(\alpha + t)^{\beta+1} \int_{\mathbb{R}} g_1 y_t dx = \frac{d}{dt} \left\{ (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \left[ - \int_{\tilde{\rho}} \bar{P}(s) ds - P(\tilde{\rho}) y_x + \frac{1}{2} P'(\tilde{\rho}) y_t^2 \right] dx \right\}
\]

\[
- (1 + \beta)(\alpha + t)^\beta \int_{\mathbb{R}} \left[ - \int_{\tilde{\rho}} \bar{P}(s) ds - P(\tilde{\rho}) y_x + \frac{1}{2} P'(\tilde{\rho}) y_t^2 \right] dx
\]

\[
+ (\alpha + t)^{\beta+1} \int_{\mathbb{R}} (P(\tilde{\rho} - y_x) - P(\tilde{\rho}) - P'(\tilde{\rho}) y_x - \frac{1}{2} P''(\tilde{\rho}) y_x^2) \bar{\rho}_t dx
\]

\[
\leq \frac{d}{dt} \left\{ (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \left[ - \int_{\tilde{\rho}} \bar{P}(s) ds - P(\tilde{\rho}) y_x + \frac{1}{2} P'(\tilde{\rho}) y_t^2 \right] dx \right\}
\]

\[
+ C(\epsilon + \delta)(\alpha + t)^\beta \int_{\mathbb{R}} y_t^2 dx.
\]  

(3.14)

Next, we estimate the term \((\alpha + t)^{\beta+1} \int_{\mathbb{R}} g_2 y_t dx\). Note that

\[
g_2 = -\left( \frac{m^2}{\rho} - \frac{\tilde{m}^2}{\tilde{\rho}} \right) x = -\left( \frac{m^2}{\rho^2} y_{xx} - \frac{2m}{\rho} y_{xt} + \frac{m^2}{\rho^2} \tilde{\rho}_x - \frac{2m}{\rho} \tilde{\rho}_t \right) \tilde{m}_x,
\]  

(3.15)

and

\[
\left\{ \begin{array}{l}
|m| \leq |\tilde{m}| + |y_t| \leq C(\delta + \epsilon)(1 + t)^{-\frac{\lambda-1}{\lambda}}; \\
|\rho_t| + |m_x| \leq |\tilde{\rho}_t| + |y_{xt}| + |\tilde{m}_x| \leq C(\delta + \epsilon)(1 + t)^{-1}; \\
|\rho_x| + |m_t| \leq |\tilde{\rho}_x| + |y_{xx}| + |\tilde{m}_t| + |y_t| \leq C(\delta + \epsilon)(1 + t)^{-\frac{\lambda+1}{\lambda}};
\end{array} \right.
\]  

(3.16)

we have

\[
(\alpha + t)^{\beta+1} \int_{\mathbb{R}} \frac{m^2}{\rho^2} y_{xx} y_t dx
\]

\[
= -\frac{1}{2} \frac{d}{dt} \left\{ (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \frac{m^2}{\rho^2} y_t^2 dx \right\} + \frac{\beta + 1}{2} (\alpha + t)^\beta \int_{\mathbb{R}} \frac{m^2}{\rho^2} y_t^2 dx + \frac{1}{2} (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \left( \frac{m^2}{\rho^2} \right) y_t^2 dx
\]

\[
- (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \left( \frac{m^2}{\rho^2} \right) y_{xt} y_t dx
\]

\[
\geq -\frac{1}{2} \frac{d}{dt} \left\{ (\alpha + t)^{\beta+1} \int_{\mathbb{R}} \frac{m^2}{\rho^2} y_t^2 dx \right\} - C(\delta + \epsilon)(\alpha + t)^\beta \int_{\mathbb{R}} y_t^2 dx - C(\delta + \epsilon)(\alpha + t)^\beta \int_{\mathbb{R}} y_t^2 dx
\]  

(3.17)
Thus, it follows from (3.15) and (3.17)-(3.18) that
\[
\xi(t) = \left(\int_0^t \left(\frac{m^2}{\rho^2} - \frac{m\tilde{m}}{\rho}ight) \tilde{m}_x y_t dx + \frac{2m}{\rho} - \frac{2\tilde{m}}{\rho}\right) y_t dx
\]

Thus, the proof of Lemma 3.1 is completed.

In addition, it is straightforward to check from (2.18) that
\[
(\alpha + t)^{3+1} \int_\mathbb{R} S(\bar{\rho}) y_t dx \leq \nu_1 \frac{(\alpha + t)^{3+1}}{(1 + t)\lambda} \int_\mathbb{R} y_t^2 dx + C(\nu_1) \int_\mathbb{R} (\alpha + t)^{3+1} (1 + t)^\lambda S^2(\bar{\rho}) dx
\]

Substituting (3.14) and (3.19) into (3.18) and choosing \(\nu_1\) small enough, together with the fact that \(|\xi'| \leq C\delta\), give that
\[
\frac{d}{dt}\left[\frac{1}{2} (\alpha + t)^{3+1} \int_\mathbb{R} \left( y_t^2 + P(\bar{\rho})y_t^2 + \frac{m^2}{\rho^2} y_t^2 \right) dx \right] + \frac{1}{2} \frac{(\alpha + t)^{3+1}}{(1 + t)\lambda} \int_\mathbb{R} y_t^2 dx
\]

Integrating \(C_1 \times (3.12) + (3.21)\) in \((0, t)\) for large constant \(C_1\) and choosing \(\alpha\) large enough, we obtain
\[
(1 + t)^{3+1} \int_\mathbb{R} (y_t^2 + y_x^2) dx + \frac{\alpha + t}{(1 + t)\lambda} \int_\mathbb{R} y_t^2 dx + \int_0^t \int_\mathbb{R} \frac{(\alpha + \tau)^\beta}{(1 + \tau)^{\lambda - 1}} y_t^2 dx d\tau + \int_0^t \int_\mathbb{R} \frac{(1 + \tau)^{3+1}}{(1 + \tau)^{1+\lambda}} y_t^2 dx d\tau
\]

where we have used the facts that \(\beta < \lambda\) and that
\[
\begin{cases}
(k_0 + 1) - k(\lambda) > 0 & \text{if } k(\lambda) \notin \mathbb{N}^+,
(k_0 + 1) - k(\lambda) = 0 & \text{if } k(\lambda) \in \mathbb{N}^+,
\end{cases}
\]

from \(k_0 = \sup_{0<\varepsilon<\frac{1}{2}} [k(\lambda) - \varepsilon] \) with \(k(\lambda) = \frac{3(1+\lambda)}{4(1-\lambda)}\). Thus, the proof of Lemma 3.1 is completed. \(\square\)

**Lemma 3.2.** Assume that \(y(x, t) \in X_T\) is the solution of (3.2). If \(\epsilon\) and \(\delta\) are small, it holds that
\[
\int_\mathbb{R} (1 + t)^{3+1} (y_t^2 + y_x^2) dx + \int_0^t \int_\mathbb{R} (1 + \tau)^{\lambda} y_t^2 dx d\tau + \int_0^t \int_\mathbb{R} (1 + \tau) y_t^2 dx d\tau
\]

\[
\leq \begin{cases}
C(\lambda)(N(0)^2 + \delta^2 + \delta\epsilon), & \text{if } k(\lambda) \notin \mathbb{N}^+,
C(\lambda)(N(0)^2 + \delta^2 + \delta\epsilon) \ln^2(1 + t), & \text{if } k(\lambda) \in \mathbb{N}^+.
\end{cases}
\]
Proof. Multiplying (3.2) by \((\alpha + t)^\lambda y\) and integrating the result over \(\mathbb{R}\), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} \left[ (\alpha + t)^\lambda y_t + \frac{(\alpha + t)^\lambda}{2(1 + t)} y^2 \right] dx + (\alpha + t)^\lambda \int_{\mathbb{R}} P'(\tilde{\rho}) y_x^2 dx + \lambda(\alpha - 1) (\alpha + t)^{\lambda - 1} \int_{\mathbb{R}} \frac{1}{2} y_t^2 dx \\
= (\alpha + t)^\lambda \int_{\mathbb{R}} g_1^2 dx + \lambda(\alpha + t)^{\lambda - 1} \int_{\mathbb{R}} y_1 y dx + (\alpha + t)^\lambda \int_{\mathbb{R}} g_2 y dx \\
+ (\alpha + t)^\lambda \int_{\mathbb{R}} S(\tilde{\rho}) y dx.
\]
Different from (3.6), it is difficult to use the term \(\frac{(\alpha + t)^{\lambda - 1}}{(1 + t)\beta} \int_{\mathbb{R}} y_t^2 dx\) to control \((\alpha + t)^{-1} \int_{\mathbb{R}} y_1 y dx\) and \((\alpha + t)^\lambda \int_{\mathbb{R}} g_2 y dx\). To overcome this, we choose \(2\lambda - 1 < \beta < \lambda\) to get
\[
\lambda(\alpha + t)^{\lambda - 1} \int_{\mathbb{R}} y_t y dx \leq \frac{\lambda}{2} \frac{(\alpha + t)^\beta}{(1 + t)^{1 + \lambda}} \int_{\mathbb{R}} y_t^2 dx + \frac{\lambda}{2} (\alpha + t)^{2(\lambda - 1) - \beta + \lambda} \int_{\mathbb{R}} y_t^2 dx \\
\leq \frac{\lambda}{2} \frac{(\alpha + t)^\beta}{(1 + t)^{1 + \lambda}} \int_{\mathbb{R}} y_t^2 dx + \frac{\lambda}{2} (\alpha + t)^{\beta + 1 - \lambda} \int_{\mathbb{R}} y_t^2 dx
\]
and
\[
(\alpha + t)^\lambda \int_{\mathbb{R}} g_2 y dx = - (\alpha + t)^\lambda \int_{\mathbb{R}} \left( \frac{\tilde{m}^2}{\rho} - \frac{m^2}{\rho} \right) y dx \\
\leq C(\delta + \epsilon) \frac{(\alpha + t)^\beta}{(1 + t)^{1 + \lambda}} \int_{\mathbb{R}} y_t^2 dx + C(\delta + \epsilon) (\alpha + t)^{2\lambda - \beta} (1 + t)^{\lambda - 1} \int_{\mathbb{R}} (y_x^2 + y_t^2) dx \\
\leq C(\delta + \epsilon) \frac{(\alpha + t)^\beta}{(1 + t)^{1 + \lambda}} \int_{\mathbb{R}} y_t^2 dx + C(\delta + \epsilon) (\alpha + t)^\lambda \int_{\mathbb{R}} (y_x^2 + y_t^2) dx.
\]
In addition, we use the similar method applied in (3.7) and (3.10), (3.11) and Lemma 3.1 to get
\[
\frac{d}{dt} \int_{\mathbb{R}} \left[ (\alpha + t)^\lambda y_t + \frac{(\alpha + t)^\lambda}{2(1 + t)} y^2 \right] dx + \frac{(\alpha + t)^\lambda}{2} \int_{\mathbb{R}} P'(\tilde{\rho}) y_x^2 dx \\
\leq \frac{\lambda}{2} \frac{(\alpha + t)^\beta}{(1 + t)^{1 + \lambda}} \int_{\mathbb{R}} y_t^2 dx + C(\delta + \epsilon) (1 + t)^{-1 - \sigma_{\infty}}(k_{0} + 1) \int_{\mathbb{R}} y_t^2 dx \\
+ \begin{cases} 
C(\delta + \epsilon)(1 + t)^{-1} \ln(1 + t), & \text{if } k(\lambda) \notin \mathbb{N}^+, \\
C(\delta + \epsilon)(1 + t)^{-1}, & \text{if } k(\lambda) \in \mathbb{N}^+,
\end{cases}
\]
where we have used the fact (3.22).

On the other hand, multiplying (3.2) by \((\alpha + t)^{1 + \lambda} y_t\) and integrating the result over \(\mathbb{R}\), we use the same argument in (3.22) to obtain that
\[
\frac{1}{2} \frac{d}{dt} \left[ (\alpha + t)^{1 + \lambda} \int_{\mathbb{R}} \left( y_t^2 + P'(\tilde{\rho}) y_x^2 + \frac{m^2}{\rho^2} y_x^2 \right) dx \right] + \frac{1}{2} (\alpha + t)^{\lambda + 1} \int_{\mathbb{R}} y_t^2 dx \\
\leq \frac{d}{dt} \left[ (\alpha + t)^{1 + \lambda} \int_{\mathbb{R}} \left( \int_{\tilde{\rho}^\gamma y_x} \right) P(s) ds - P(\tilde{\rho}) y_x^2 + \frac{1}{2} P'(\tilde{\rho}) y_x^2 dx \right] \\
+ C(\alpha + t)^\lambda \int_{\mathbb{R}} y_t^2 dx + C\tilde{\delta}^2 (1 + t)^{-1 - 2\sigma_{\infty}}(k_{0} + 1) - k(\lambda)].
\]
Thanks to (3.22) and the improved estimates (3.5), integrating \(C_2 \times (3.24) + (3.25)\) in \((0, t)\) for large constant \(C_2\) and choosing \(\alpha\) large enough imply (3.22) directly. Thus the proof of Lemma 3.2 is completed. \(\square\)
Lemma 3.3. Assume that \( y(x,t) \in \mathcal{X}_T \) is the solution of (3.2). If \( \epsilon \) and \( \delta \) are small, it holds that for \( s = 1, 2 \),

\[
(\alpha + t)^{(s+1)(1+\lambda)} \int_{\mathbb{R}} [(\partial_x^{s+1})^2 + (\partial_x^s y)^2] \, dx + \int_{0}^{t} \left( \int_{\mathbb{R}} (\alpha + \tau)^{(1+s)(1+\lambda)} (\partial_x^s y)^2 \, d\tau \right) \, dx \leq \begin{cases} 
C(N(0)^2 + \delta^2 + \delta \epsilon), & \text{if } k(\lambda) \in \mathbb{N}^+; \\
C(N(0)^2 + \delta^2 + \delta \epsilon) \ln^2(1 + t), & \text{if } k(\lambda) \notin \mathbb{N}^+. 
\end{cases} \tag{3.26}
\]

**Proof.** Differentiating (3.2) with respect to \( x \), we obtain

\[
(\partial_x y)_{xt} - (\partial_x (P(\rho)y)_x)_x + \frac{\partial_x y_t}{(1 + t)^{\lambda}} = \partial_x g_1 + \partial_x g_2 + \partial_x S(\rho). \tag{3.27}
\]

Multiplying (3.27) by \((\alpha + t)^{(2+1)(1+\lambda)}\partial_x y_t\) and integrating the result with respect to \( x \) over \( \mathbb{R} \) give that

\[
\frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \right] \int_{\mathbb{R}} \frac{1}{2} (P'(\rho)(\partial_x^2 y)^2 + (\partial_x^s y)^2) \, dx + \int_{\mathbb{R}} \frac{(\alpha + t)^{2(1+\lambda)}}{(1 + t)^{\lambda}} (\partial_x y_t)^2 \, dx \\
\leq 2(1 + \lambda)(\alpha + t)^{1+2\lambda} \int_{\mathbb{R}} \frac{1}{2} (\partial_x^2 y)^2 \, dx + (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} \partial_x^2 y(P'(\rho))_x + y_x \partial_x^2 (P'(\rho)) \, dx.
\]

A direct computation yields that

\[
|I_1| \leq C \left( 1 + \frac{\delta}{1 + t} \right) (\alpha + t)^{1+2\lambda} \int_{\mathbb{R}} (\partial_x^2 y)^2 \, dx + C\delta \frac{(\alpha + t)^{2(1+\lambda)}}{(1 + t)^{\lambda}} (\partial_x y_t)^2 \, dx + C\delta \frac{(\alpha + t)^{2(1+\lambda)}}{(1 + t)^{2+\lambda}} \int_{\mathbb{R}} y_x^2 \, dx. \tag{3.29}
\]

Note that

\[
(\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} \partial_x g_1 \partial_x y_t \, dx \\
= (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} [(P'(\rho) - P'(\rho))y_{xx} + (P'(\rho) - P'(\rho) + P''(\rho)y_x)\partial_x^2 y \, dx \\
= \frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} (P'(\rho) - P'(\rho))\frac{1}{2} (\partial_x^2 y)^2 \right] + I_{2,1} + I_{2,2},
\]

where

\[
I_{2,1} = -2(1 + \lambda)(\alpha + t)^{1+2\lambda} \int_{\mathbb{R}} (P'(\rho) - P'(\rho))\frac{1}{2} (\partial_x^2 y)^2 \, dx - (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} (P'(\rho) - P'(\rho))\frac{1}{2} (\partial_x^2 y)^2 \, dx,
\]

\[
I_{2,2} = (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} [(P'(\rho) - P'(\rho) + P''(\rho)y_x)\partial_x^2 y + (P'(\rho) - P'(\rho) + P''(\rho)y_x)\partial_x^2 y] \, dx,
\]

then it follows from the a priori assumption (3.4) and the estimates (3.16) that

\[
(\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} \partial_x g_1 \partial_x y_t \, dx \\
\leq \frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \int_{\mathbb{R}} (P'(\rho) - P'(\rho))\frac{1}{2} (\partial_x^2 y)^2 \right] + C(\delta + \epsilon + \delta \frac{\alpha + t}{1 + t})(\alpha + t)^{1+2\lambda} \int_{\mathbb{R}} (\partial_x y_t)^2 \, dx \\
+ C\delta \frac{(\alpha + t)^{2(1+\lambda)}}{(1 + t)^{\lambda}} (\partial_x y_t)^2 \, dx + C\delta (1 + t)^{\lambda} \int_{\mathbb{R}} y_x^2 \, dx. \tag{3.30}
\]

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In addition, it holds that
\[
(\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \partial_x g_2 \partial_x y_t dx
\]
\[
= - (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \partial_x \left[ \frac{m^2}{\rho^2} y_{xx} + \frac{2m}{\rho} y_{xt} - \left( \frac{m^2}{\rho^2} - \frac{m^2}{\rho^2} \right) \partial_x \tilde{\rho} + \left( \frac{2m}{\rho} - \frac{2m}{\rho} \right) \partial_x \tilde{m} \right] dx
\]
\[
= \frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \frac{m^2}{\rho^2} \left( \partial_x^2 y \right)^2 dx \right] + I_{3,1} + I_{3,2} + I_{3,3},
\]
where
\[
I_{3,1} = -2(1+\lambda)(\alpha + t)^{1+2\lambda} \int_\mathbb{R} \frac{m^2}{\rho^2} \left( \partial_x^2 y \right)^2 dx - (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \left( \frac{m^2}{\rho^2} \right) \frac{1}{2} \left( \partial_x^2 y \right)^2 dx;
\]
\[
I_{3,2} = - (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \frac{2m}{\rho} \frac{1}{2} \left( \partial_x y_t \right)^2 dx,
\]
\[
I_{3,3} = (\alpha + t)^{2(1+\lambda)} \sum_{l+r=1} \int_\mathbb{R} C_1 \left( \frac{m^2}{\rho^2} - \frac{m^2}{\rho^2} \right) \left( \partial_x \tilde{\rho} \right)^{(l)} \left( \partial_x \tilde{m} \right)^{(r)} \partial_x y_t dx.
\]
Similarly, a tedious computation shows that
\[
(\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \partial_x g_2 \partial_x y_t dx
\]
\[
\leq \frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \frac{m^2}{\rho^2} \left( \partial_x^2 y \right)^2 dx \right] + C(\delta + \epsilon)(\alpha + t)^{1+2\lambda} \int_\mathbb{R} \left( \partial_x^2 y \right)^2 dx
\]
\[
+ C(\delta + \epsilon)(\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \left( \partial_x y_t \right)^2 dx + C(1 + t)^{\lambda} \int_\mathbb{R} \left( y^2 + y_t^2 \right) dx.
\] (3.31)

Moreover, Lemma 2.4 yields that
\[
(\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \partial_x S(\tilde{\rho}) \partial_x y_t dx
\]
\[
\leq \nu_1 (\alpha + t)^{2(1+\lambda)\alpha} \int_\mathbb{R} \left( \partial_x y_t \right)^2 dx + C(\nu_1)\delta^2 (1 + t)^{-1-2\sigma[(k_0+1)-k(\lambda)]},
\] (3.32)

where \( \nu_1 \) is a small constant. Substituting \( \nu_1 \) into (3.31), and choosing \( \nu_1 \) small and \( \alpha \) large enough give that
\[
\frac{d}{dt} \left[ (\alpha + t)^{2(1+\lambda)} \int_\mathbb{R} \frac{1}{2} \left[ \left( P'(\rho) - \frac{m^2}{\rho^2} \right) (\partial_x^2 y)^2 + (\partial_x y_t)^2 \right] dx \right] + \int_\mathbb{R} \frac{2m}{\rho} \frac{1}{2} \left( \partial_x^2 y \right)^2 dx
\]
\[
\leq C(1 + \delta \frac{\alpha + t}{1 + t}) (\alpha + t)^{1+2\lambda} \int_\mathbb{R} \left( \partial_x y_t \right)^2 dx + C(\delta + \epsilon)(1 + t)^{\lambda} \int_\mathbb{R} \left( y^2 + y_t^2 \right) dx
\]
\[
+ C(\nu_1)\delta^2 (1 + t)^{-1-2\sigma[(k_0+1)-k(\lambda)]}.
\] (3.33)

It remains to estimate the term \( (\alpha + t)^{1+2\lambda} \int_\mathbb{R} \left( \partial_x^2 y \right)^2 dx \). We multiply \( (\alpha + t)^{1+2\lambda} \partial_x y_t \) and integrate the result over \( \mathbb{R} \) to obtain
\[
\frac{d}{dt} \left[ \int_\mathbb{R} \frac{(\alpha + t)^{1+2\lambda}}{2(1 + t)^{\lambda}} \left( \partial_x y_t \right)^2 dx \right] + (\alpha + t)^{1+2\lambda} \int_\mathbb{R} P'(\tilde{\rho}) \left( \partial_x^2 y \right)^2 dx
\]
\[
\leq (\alpha + t)^{1+2\lambda} \int_\mathbb{R} \left( \partial_x g_1 + \partial_x g_2 + \partial_x S(\tilde{\rho}) \right) \partial_x y_t dx + \sum_{i=1}^{3} I_{4,i},
\] (3.34)
Proof. Differentiating (3.2) with respect to \(t\), we have

\[
I_{4,1} = (1 + 2\lambda)(\alpha + t)^{2\lambda} \int_x \partial_x y \partial_x y_t dx + (\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx,
\]

\[
I_{4,2} = (\alpha + t)^{1+2\lambda} \int_x (P'(\rho))_x y \partial_x^2 y dx,
\]

\[
I_{4,3} = \frac{1}{2} [(1 + 2\lambda)\alpha + t^{2\lambda} - \lambda (\alpha + t)^{1+2\lambda}] \int_x y_t^2 dx.
\]

Integrating (3.33) + (3.34) and choosing \(\nu\) where we have used the fact (3.22).

It is easy to check that

\[
\frac{3}{\nu_1} \leq (\alpha + t)^{1+2\lambda} \int_x y_t^2 dx + C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(1 + t)^{1+2\lambda} \int_x y_t^2 dx,
\]

(3.35)

where \(\nu_1\) is a small constant. In addition, a direct computation shows that

\[
(\alpha + t)^{1+2\lambda} \int_x \partial_x S(\rho) \partial_x y dx \leq (\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\nu_1) \delta^2 (1 + t)^{1-2\sigma[k(0)-k(\lambda)]}.
\]

(3.36)

and

\[
(\alpha + t)^{1+2\lambda} \int_x \partial_x (\rho) \partial_x y dx \leq C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\alpha + t)^{1+2\lambda} (1 + t)^{1-2\sigma[k(0)-k(\lambda)]} \int_x (y_t^2 + y_t^2) dx + C(\nu_1) \delta^2 (1 + t)^{1-2\sigma[k(0)-k(\lambda)]}.
\]

(3.37)

Substituting (3.35) - (3.37) into (3.34) and choosing \(\nu_1\) small enough to verify that

\[
\frac{d}{dt} \int_x (\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + (\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + (\alpha + t)^{1+2\lambda} \int_x P'(\rho) \partial_x^2 y dx + C(\alpha + t)^{1+2\lambda} (1 + t)^{1-2\sigma[k(0)-k(\lambda)]} \int_x (y_t^2 + y_t^2) dx + C(\nu_1) \delta^2 (1 + t)^{1-2\sigma[k(0)-k(\lambda)]}.
\]

(3.38)

Integrating (3.33) + (3.35) - (3.37) in \((0, t)\) for large constants \(C_3\) and \(\alpha\) and using Lemma 3.2, we obtain

\[
(\alpha + t)^{2(1+\lambda)} \int_x [(\partial_x y_t)^2 + (\partial_x y_t)^2] dx + (\alpha + t)^{1+2\lambda} \int_x (\partial_x^2 y_t)^2 dx + (\alpha + t)^{1+2\lambda} \int_x (\partial_x^2 y_t)^2 dx + C(\alpha + t)^{1+2\lambda} (1 + t)^{1-2\sigma[k(0)-k(\lambda)]} \int_x (y_t^2 + y_t^2) dx + C(\nu_1) \delta^2 (1 + t)^{1-2\sigma[k(0)-k(\lambda)]}.
\]

(3.39)

where we have used the fact (3.22).

Similarly, we can obtain the desired estimates (3.30) for the case of \(s = 2\). Thus, the proof is completed.

\[
\square
\]

**Lemma 3.4.** Assume that \(y(x, t) \in X_T\) is the solution of (3.2). If \(\epsilon\) and \(\delta\) are small, it holds that for \(s = 0, 1,\)

\[
(1 + t)^{3+\lambda+s(1+\lambda)} \int_x [(\partial_x y_t)^2 + (\partial_x y_t)^2] dx + (\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx + C(\alpha + t)^{1+2\lambda} \int_x (\partial_x y_t)^2 dx.
\]

(3.40)

**Proof.** Differentiating (3.2) with respect to \(t\) gives

\[
y_{tt} - (P'(\rho)y_x)_{xt} + \frac{y_t}{(1 + t)^{1+\lambda}} = g_{tt} + g_{xt} + S(\rho)_t.
\]

(3.40)
Then we multiply (3.40) by \((\alpha + t)^{3+\lambda} y_{tt}\) and integrate the result over \(\mathbb{R}\) to obtain that
\[
\frac{d}{dt} \int_{\mathbb{R}} \frac{(\alpha + t)^{3+\lambda}}{2} \left[ y_{tt}^2 + P'(\rho) y_{xx}^2 \right] dx + \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx \\
\leq \frac{3 + \lambda}{2} (\alpha + t)^{2+\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + (\alpha + t)^{3+\lambda} \int_{\mathbb{R}} g_{tt} y_{tt} dx + (\alpha + t)^{3+\lambda} \int_{\mathbb{R}} g_{2tt} y_{tt} dx \\
+ (\alpha + t)^{3+\lambda} \int_{\mathbb{R}} S(\tilde{\rho}) y_{tt} dx + I_5,
\]
where
\[
I_5 = \frac{3 + \lambda}{2} (\alpha + t)^{2+\lambda} \int_{\mathbb{R}} P'(\rho) y_{tt}^2 dx + (\alpha + t)^{3+\lambda} \left[ \int_{\mathbb{R}} P''(\rho) \tilde{\rho}_{tt} \frac{1}{2} y_{tt}^2 dx + \int_{\mathbb{R}} P''(\rho) \tilde{\rho}_{tt} y_{xx} y_{tt} dx \\
+ \int_{\mathbb{R}} P''(\rho) \tilde{\rho}_{x2} y_{tt} dx + \int_{\mathbb{R}} P''(\rho) \tilde{\rho}_{y2} y_{tt} dx \right] + \lambda (\alpha + t)^{3+\lambda} \int_{\mathbb{R}} y_{tt}^2 dx.
\]
A direct computation shows that
\[
|I_5| \leq \nu_1 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + C(\alpha + t)^{3+\lambda} \left[ \frac{1}{\alpha + t} + \frac{\delta}{1 + t} \right] \int_{\mathbb{R}} y_{tt}^2 dx + C\delta^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^2} \int_{\mathbb{R}} y_{tt}^2 dx,
\]
\[
(\alpha + t)^{3+\lambda} \int_{\mathbb{R}} g_{tt} y_{tt} dx \leq \frac{d}{dt} \left[ (\alpha + t)^{3+\lambda} \int_{\mathbb{R}} (P'(\rho) - P'(\rho)) \frac{1}{2} y_{tt}^2 dx \right] + \nu_1 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx \\
+ C(\nu_1)^2 \left( \frac{\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{xx}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^2} \int_{\mathbb{R}} y_{tt}^2 dx \\
+ C(\nu_1)^2 \frac{(\alpha + t)^{2+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^2} \int_{\mathbb{R}} y_{tt}^2 dx \right) \int_{\mathbb{R}} y_{tt}^2 dx
\]
and
\[
(\alpha + t)^{3+\lambda} \int_{\mathbb{R}} S(\tilde{\rho}) y_{tt} dx \leq \nu_1 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx
\]
In addition, it follows from Lemma 2.24 that
\[
(\alpha + t)^{3+\lambda} \int_{\mathbb{R}} S(\tilde{\rho}) y_{tt} dx \leq \nu_1 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx
\]
where \(\nu_1\) is a small constant. Next, substituting (3.41) into (3.44) and choosing \(\alpha\) large and \(\nu_1\) small enough yield that
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}} \frac{(\alpha + t)^{3+\lambda}}{2} \left[ y_{tt} + \left( P'(\rho) - \frac{m^2}{\rho^2} \right) y_{xx} \right] dx \right] + \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^\lambda} \int_{\mathbb{R}} y_{tt}^2 dx \\
\leq C(\alpha + t)^{3+\lambda} \left[ \frac{1}{\alpha + t} + \frac{\delta}{1 + t} \right] \int_{\mathbb{R}} y_{xx}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^2} \int_{\mathbb{R}} y_{tt}^2 dx \\
+ C(\alpha + t)^{3+\lambda} \int_{\mathbb{R}} y_{tt}^2 dx + C(\nu_1)^2 \frac{(\alpha + t)^{3+\lambda}}{(1 + t)^2} \int_{\mathbb{R}} y_{tt}^2 dx
\]
From Lemmas 3.2-3.3 and the fact (3.51), we integrate (3.40) over \([0, t]\) to obtain that
\[
(\alpha + t)^{3+\lambda} \int_{\mathbb{R}} (y_{it}^2 + y_{tt}^2)dx + \int_0^t \int_{\mathbb{R}} (\alpha + \tau)^3 y_t^2 dx d\tau \leq \begin{cases} C(N(0)^2 + \delta^2 + \delta\epsilon), & \text{if } k(\lambda) \notin \mathbb{N}^+, \\ C(N(0)^2 + \delta^2 + \delta\epsilon) \ln^2(1 + t), & \text{if } k(\lambda) \in \mathbb{N}^+. \end{cases}
\]

Similarly, we can verify that (3.39) holds for the case of \(s = 1\). Thus the proof is completed. \(\square\)

**Lemma 3.5.** Assume that \(y(x, t) \in X_T\) is the solution of (3.2). If \(\epsilon\) and \(\delta\) are small, it holds that
\[
(1 + t)^2 \int_{\mathbb{R}} y_t^2 dx \leq \begin{cases} C(N(0)^2 + \delta^2 + \delta\epsilon), & \text{if } k(\lambda) \notin \mathbb{N}^+, \\ C(N(0)^2 + \delta^2 + \delta\epsilon) \ln^2(1 + t), & \text{if } k(\lambda) \in \mathbb{N}^+. \end{cases}
\]  

(3.47)

**Proof.** Multiplying (3.40) by \((1 + t)^{2+\lambda} y_t\) and integrating the result over \(\mathbb{R}\) lead to
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}} (1 + t)^{2+\lambda} y_t y_{tt} dx + \int_{\mathbb{R}} \frac{1}{2} (1 + t)^2 y_t^2 dx \right] 
\leq C(1 + t)^{2+\lambda} \int_{\mathbb{R}} (y_{xt}^2 + y_{tt}^2) dx + C(1 + t)^3 \int_{\mathbb{R}} y_t^2 dx + C(1 + t) \int_{\mathbb{R}} y_t^2 dx + (1 + t)^{2+\lambda} \int_{\mathbb{R}} (g_{1t} + g_{2t} + S(\tilde{\rho}_t)) y_t dx.
\]

(3.48)

It follows from (3.16) that
\[
(1 + t)^{2+\lambda} \int_{\mathbb{R}} g_{1t} y_t dx + (1 + t)^{2+\lambda} \int_{\mathbb{R}} g_{2t} y_t dx
\leq C(1 + t)^{2+\lambda} \int_{\mathbb{R}} (y_{xt}^2 + y_{tt}^2) dx + C(1 + t)^3 \int_{\mathbb{R}} y_t^2 dx + C(\nu_1) \sigma^2 (1 + t)^{1-2\sigma} \int_{\mathbb{R}} \|y_t\|^2 dx.
\]

(3.49)

Moreover, it follows from Lemma 2.4 that
\[
(1 + t)^{2+\lambda} \int_{\mathbb{R}} S(\tilde{\rho}_t) y_t dx \leq (1 + t) \int_{\mathbb{R}} y_t^2 dx + C(\nu_1) \sigma^2 (1 + t)^{1-2\sigma} \int_{\mathbb{R}} \|y_t\|^2 dx + C(\nu_1) \sigma^2 (1 + t)^{1-2\sigma} \int_{\mathbb{R}} \|y_t\|^2 dx.
\]

(3.50)

Substituting 2.39 - 2.40 into 2.38 and integrating the result over \([0, t]\), we deduce 2.47 from Lemmas 3.2-3.3. Thus the proof is completed. \(\square\)

**Proof of Theorem 1.1** Lemmas 3.2-3.3 show that there exists some positive constant \(C_0\) such that
\[
\sum_{s=0}^{2} \left[ (1 + t)^{(s+1)(1+\lambda)} \|\partial_s^s y_s(t)\|^2 + (1 + t)^{2+s(1+\lambda)} \|\partial_s^s y_s(t)\|^2 \right]
\leq \begin{cases} C_0(N(0)^2 + \delta^2 + \delta\epsilon), & \text{if } k(\lambda) \notin \mathbb{N}^+, \\ C_0(N(0)^2 + \delta^2 + \delta\epsilon) \ln^2(1 + t), & \text{if } k(\lambda) \in \mathbb{N}^+. \end{cases}
\]

provided that \(\epsilon \ll 1\). Choose \(\epsilon = 4C_0(N(0) + \delta)\) and suppose that \(N(0) + \delta \ll 1\), then we can obtain from 2.31 that
\[
N(T) \leq \begin{cases} \frac{\phi^2}{4}, & \text{if } k(\lambda) \notin \mathbb{N}^+, \\ \frac{\phi^2}{4} \ln(1 + T), & \text{if } k(\lambda) \in \mathbb{N}^+. \end{cases}
\]

which closes the a priori assumption 3.3. Therefore the proof is completed. \(\square\)

**Appendix A. Proof of Lemma 2.3**

For the case \(i = 1\), we consider the following ODE
\[
(P'(\tilde{\rho})G_1)_{\xi} + \frac{1 + \lambda}{2} (\xi G_1)_{\xi} - 2\lambda G_1 = \lambda P(\tilde{\rho})_{\xi} - h_{1\xi}
\]

(A.1)

with the condition
\[
\int_{\mathbb{R}} G_1 dx = \frac{P(\rho_+) - P(\rho_-)}{2}.
\]

(A.2)
Let \( \tilde{G}_1 = G_1 + \frac{1}{2} P(\tilde{\rho}) \xi \), then (A.1)-(A.2) can be rewritten as
\[
\begin{align*}
\int_R \tilde{G}_1(\xi)d\xi &= 0,
\end{align*}
\]
where
\[
\tilde{h}_1 = \frac{1}{2} P'(\tilde{\rho}) P(\tilde{\rho}) \xi - \frac{1}{4} \xi P(\tilde{\rho}) - h_1.
\]

Taking the Fourier transformation of \( \tilde{G}_1(\xi) \) gives that
\[
\begin{align*}
\mathcal{F}_1(\eta) &= \mathcal{F}[\tilde{G}_1(\xi)], \\
\mathcal{F}_1(0) &= \mathcal{F}[\tilde{h}_1],
\end{align*}
\]
where \( \mathcal{F}_1(\eta) = \mathcal{F}[\tilde{G}_1(\xi)] \). We construct the following iterative sequences \( \{\mathcal{F}_n\} \):
\[
\begin{align*}
\mathcal{F}_{n+1}(\eta) &= -\frac{2}{1 + \lambda} \eta \int_0^\eta e^{-i \int_{\eta_1}^{\eta} P'(\rho_+) \xi^2} \int_0^{\eta_2} \frac{d\eta_2}{\eta_2} e^{-i \int_{\eta_1}^{\eta} P'(\rho_+) \xi^2} \mathcal{F}[((P'(\rho_+) - P'(\tilde{\rho})) \tilde{G}_1^\eta \xi)] + \mathcal{F}[\tilde{h}_1],
\end{align*}
\]
where \( \tilde{G}_1^\eta(\xi) = \mathcal{F}^{-1}[\mathcal{F}_1(\eta)] \) with \( n \in \mathbb{N}^+ \) and \( \tilde{G}_1^0(\xi) = 0 \). It follows from (A.4) that
\[
\mathcal{F}_{n+1}(\eta) = -\frac{2}{1 + \lambda} \eta \mathcal{F}_n(\eta) + \mathcal{F}[\tilde{h}_1],
\]
where \( \mathcal{F}_1(\eta) = \mathcal{F}[\tilde{G}_1(\xi)] \) is the Schwartz space. Indeed, since \( \tilde{G}_1(\xi) = 0 \) and \( \mathcal{F}[\tilde{h}_1] \in \mathcal{F}(\mathbb{R}) \), it is straightforward to imply that for any nonnegative integers \( \alpha \) and \( \beta \),
\[
|\eta^\alpha \partial_\eta^\beta \mathcal{F}_1 | \to 0, \quad \text{as} \quad |\eta| \to \infty.
\]
Thus \( \mathcal{F}_1 \in \mathcal{F}(\mathbb{R}) \) holds. In the same way, we can verify \( \{\mathcal{F}_n\} \subset \mathcal{F}(\mathbb{R}) \).

It follows from the above claim that \( \{\mathcal{F}_n\} \subset \chi^{m_1}(\mathbb{R}) \) for any given integer \( m_1 > 0 \), where \( \chi^{m_1}(\mathbb{R}) \) is given in (2.15). We will use the contraction principle to show that \( \mathcal{F}_n \) has a unique limit in \( \chi^{m_1}(\mathbb{R}) \). To this end, let \( \Delta_1^{n+1} = \mathcal{F}_{n+1} - \mathcal{F}_n \) and we hope that
\[
\|\Delta_1^{n+1}\|_{\chi^{m_1}(\mathbb{R})} \leq C \delta \|\Delta_1^{n}\|_{\chi^{m_1}(\mathbb{R})} \leq \frac{1}{2} \|\Delta_1^{n}\|_{\chi^{m_1}(\mathbb{R})}.
\]

It remains to prove (A.3). Note that \( \Delta_1^{n+1} \) satisfies that
\[
\begin{align*}
\frac{1 + \lambda}{2} \eta \Delta_1^{n+1} + (2 \lambda + P'(\rho_+)) \Delta_1^{n+1} &= -\mathcal{F}[((P'(\rho_+) - P'(\tilde{\rho})) (\tilde{G}_1^\eta \xi - \tilde{G}_1^{n-1}))],
\end{align*}
\]
we take the procedure as (A.6) \( \times \Delta_1^{n+1} + (A.6) \times \Delta_1^{n+1} \) and integrate the result over \( \mathbb{R} \) to obtain
\[
\begin{align*}
\int_R P'(\rho_+) \eta^2 |\Delta_1^{n+1}|^2 d\eta &= \left( \frac{1 + \lambda}{4} - 2\lambda \right) \int_R |\Delta_1^{n+1}|^2 d\eta - \frac{1}{2} \int_R \mathcal{F}[((P'(\rho_+) - P'(\tilde{\rho})) (\tilde{G}_1^\eta \xi - \tilde{G}_1^{n-1}))] \Delta_1^{n+1} d\eta \\
&\quad - \frac{1}{2} \int_R \mathcal{F}[((P'(\rho_+) - P'(\tilde{\rho})) (\tilde{G}_1^\eta \xi - \tilde{G}_1^{n-1}))] \Delta_1^{n+1} d\eta \\
&\leq \tilde{C} \int_R |\Delta_1^{n+1}|^2 d\eta + \nu_1 \int_R \eta^2 |\Delta_1^{n+1}|^2 d\eta + C(\nu_1) \int_R |F[(P'(\rho_+) - P'(\tilde{\rho})) (\tilde{G}_1^\eta \xi - \tilde{G}_1^{n-1})]|^2 d\eta \\
&\leq \tilde{C} \int_R |\Delta_1^{n+1}|^2 d\eta + \nu_1 \int_R \eta^2 |\Delta_1^{n+1}|^2 d\eta + C(\nu_1) \int_R |(P'(\rho_+) - P'(\tilde{\rho})) (\tilde{G}_1^\eta \xi - \tilde{G}_1^{n-1})|^2 d\eta \\
&\leq \tilde{C} \int_{|\eta| \geq M} |\Delta_1^{n+1}|^2 d\eta + \tilde{C} \int_{|\eta| \leq M} |\Delta_1^{n+1}|^2 d\eta + \nu_1 \int_R \eta^2 |\Delta_1^{n+1}|^2 d\eta + C(\nu_1) \delta^2 \int_R \eta^2 |\Delta_1^{n+1}|^2 d\eta,
\end{align*}
\]
where \( \bar{f} \) represents the conjugate complex of \( f \) and we have used Plancherel’s Theorem in the last two inequalities. By choosing \( \nu \) small and \( M \) large enough so that \( P'(\rho_+)M^2 > 2\bar{C} \), we deduce from (A.7) that
\[
\int_{\mathbb{R}} \eta^2 |\Delta_1^{n+1}|^2 \, d\eta \leq C \int_{\mathbb{R}} |\Delta_1^{n+1}|^2 \, d\eta + C(\nu_1) \delta^2 \int_{\mathbb{R}} \eta^2 |\Delta_1^n|^2 \, d\eta.
\]
Moreover, the explicit expression \( \Delta_1^{n+1} \) from (A.6) shows that
\[
\int_{\mathbb{R}} |\Delta_1^{n+1}|^2 \, d\eta = \int_{\mathbb{R}} \left| \eta \frac{\Delta_1}{1 + \lambda} e^{-\frac{\rho_+\eta^2}{1 + \lambda}} \int_0^\eta \frac{\Delta_1}{1 + \lambda} e^{\frac{\rho_+\eta^2}{1 + \lambda}} \{ \mathcal{F}[(P'(\rho_+) - P'(\bar{\rho}))\tilde{G}_1^\eta - \hat{G}_1^{n-1}] \} \, d\eta \right|^2 \, d\eta 
\leq C \int_{\mathbb{R}} \left| \mathcal{F}[(P'(\rho_+) - P'(\bar{\rho}))(\tilde{G}_1^\eta - \hat{G}_1^{n-1})] \right|^2 \, d\eta \, d\eta 
\leq C_M \delta^2 \int_{\mathbb{R}} \eta^2 |\Delta_1^n|^2 \, d\eta,
\]
which, together with (A.8), yields that
\[
\int_{\mathbb{R}} |\Delta_1^{n+1}|^2 \, d\eta + \int_{\mathbb{R}} \eta^2 |\Delta_1^{n+1}|^2 \, d\eta \leq C(\nu_1, M) \delta^2 \int_{\mathbb{R}} \eta^2 |\Delta_1^n|^2 \, d\eta \leq \frac{1}{2} \int_{\mathbb{R}} \eta^2 |\Delta_1^n|^2 \, d\eta.
\]
In the same way, we can verify further that (A.3) holds.

Thus, it follows from the contraction mapping principle that (A.3) admits a unique solution \( \mathcal{F}_1 \in \chi^{m_1}(\mathbb{R}) \). Moreover, applying the similar argument in (A.3) we get
\[
\| \mathcal{F}_1 \|_{\chi^{m_1}(\mathbb{R})} \leq C\delta, \quad \text{or equivalently}, \quad \| \tilde{G}_1 \|_{\chi^{m_1}(\mathbb{R})} \leq C\delta.
\]
Note that \( \tilde{G}_1 = G_1 + \frac{1}{2} P(\rho) \xi \), it holds that \( \| G_1 \|_{\chi^{m_1}(\mathbb{R})} \leq C\delta \).

Similarly, for the general case \( i \geq 2 \), let \( \tilde{G}_i = G_i - \frac{c_{i-1}}{c_i} G_{i-1} \) and we can rewrite (2.14) and (2.16) as
\[
\left\{ (P'((\rho^+ - P'(\bar{\rho})))\tilde{G}_i\xi + \frac{1 + \lambda}{c_i} \xi \tilde{G}_i\xi + c_{i-1} \tilde{G}_i = ((P'((\rho^+ - P'(\bar{\rho}))))\tilde{G}_i\xi + \tilde{h}_i, \xi, \right. \quad \text{A.9}
\]
\[
\int_{\mathbb{R}} G_i(\xi) \, d\xi = 0,
\]
where \( \tilde{h}_i = -\frac{c_{i-1}}{c_i} \left( P'(\bar{\rho})G_{i-1}\xi + \frac{1 + \lambda}{2} \xi G_{i-1} \right) - h_i \) and \( c_1, c_2, h_1 \) are given in (2.10)–(2.11). Note that \( G_1 \in \chi^{m_1}(\mathbb{R}) \) with any given integer \( m_1 > 0 \), which implies that \( \mathcal{F}[\tilde{h}_i] \in \mathcal{S}(\mathbb{R}) \). Thus in the same way we can see that (A.9) admits a unique solution \( G_i \in \chi^{m_i}(\mathbb{R}) \) and \( \| G_i \|_{\chi^{m_i}(\mathbb{R})} \leq C\delta \) for any given integer \( m_i > 0 \). Thus, the proof of Proposition 2.3 is completed.

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