Deep Decomposition for Stochastic Normal-Abnormal Transport

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Abstract

Advection-diffusion equations describe a large family of natural transport processes, e.g., fluid flow, heat transfer, and wind transport. They are also used for optical flow and perfusion imaging computations. We develop a machine learning model, D2-SONATA, built upon a stochastic advection-diffusion equation, which predicts the velocity and diffusion fields that drive 2D/3D image time-series of transport. In particular, our proposed model incorporates a model of transport atypicality, which isolates abnormal differences between expected normal transport behavior and the observed transport. In a medical context such a normal-abnormal decomposition can be used, for example, to quantify pathologies. Specifically, our model identifies the advection and diffusion contributions from the transport time-series and simultaneously predicts an anomaly value field to provide a decomposition into normal and abnormal advection and diffusion behavior. To achieve improved estimation performance for the velocity and diffusion-tensor fields underlying the advection-diffusion process and for the estimation of the anomaly fields, we create a 2D/3D anomaly-encoded advection-diffusion simulator, which allows for supervised learning. We further apply our model on a brain perfusion dataset from ischemic stroke patients via transfer learning. Extensive comparisons demonstrate that our model successfully distinguishes stroke lesions (abnormal) from normal brain regions, while reconstructing the underlying velocity and diffusion tensor fields.

1. Introduction

Partial differential equations (PDEs) are used to describe many transport phenomena, e.g., fluid dynamics, heat conduction, and wind dynamics [12]. However, it is expensive to numerically solve PDEs especially in high spatial dimensions and across large timescales [6]. Recent deep learning approaches have made data-driven solutions to PDEs possible [8,16,33,38,46,51,55,56].

While the aforementioned deep learning approaches help to speed-up PDE solutions, we are instead interested in inverse PDE problems [34,59]. Specifically, our goal is to estimate the spatially varying velocity and diffusion tensor fields (referred to as advection-diffusion parameters throughout this manuscript) for general advection-diffusion PDEs. Limited work on estimating the parameter fields of advection-diffusion equations exists. Tartakovsky et al. [59] use a deep neural network (DNN) to estimate 2D diffusion fields from diffusion PDEs. Bézenac et al. [12] learn velocity and diffusion fields of advection-diffusion PDEs by DNNs in 2D. Koundal et al. [29] use optimal mass transport combined with a spatially constant diffusion. Optimization approaches were proposed [35,36,63] to estimate the advection-diffusion parameters of an advection-diffusion equation in 3D. Though promising, the numerical optimization approach is time-consuming, especially when dealing with large datasets. Moreover, the methods above assume isotropic diffusion (i.e., they do not estimate more general diffusion tensors), which may be insufficient to accurately model complex materials (e.g., anisotropic porous media, brain tissue) where diffusion is mostly anisotropic.

The YETI approach by Liu et al. [37] is a deep learning framework to estimate advection-diffusion parameters from transport image time-series. It addresses some key challenges regarding identifiability (i.e., whether the resulting transport should be attributed to advection or diffusion) and physical constraints (e.g., vector fields should be divergence-free for fluid flow, diffusion tensors should be positive semi-definite (PSD)). Applied on brain perfusion images for stroke lesion detection, YETI achieved clear improvements over existing approaches. However, there is still room for improvement. First, while YETI estimates the advection-diffusion parameters well, it would be useful to quantify the inherent uncertainty in the resulting solutions instead of interpreting the results deterministically. This
could be especially important when estimating advection-diffusion parameters from abnormal transport processes, e.g., brain perfusion processes for a stroke patient. Second, YETI predicts advection-diffusion parameters without knowledge of regional anomalies. Hence, additional post-processing is required for lesion detection in real perfusion analysis applications. Furthermore, YETI uses pre-training based on a simulation dataset which only considers normal transport; fine-tuning is then performed using real perfusion data including stroke lesions. This simulation and fine-tuning approach may result in pre-training biased toward normal parameterizations, as abnormal transport data is not simulated. (See the comparisons in Sec. 4.1-4.2.)

We introduce, \( D^2 - \text{SONATA} \) (Deep Decomposition for Stochastic Normal-Abnormal Transport), a new deep-learning-based stochastic model designed to predict the underlying advection-diffusion parameters from observed transport time-series with potential anomalies, in both 2D and 3D. Our main contributions are threefold:

1) A learning-based stochastic advection-diffusion model. \( D^2 - \text{SONATA} \) models advection-diffusion processes as a stochastic system. Given a transport process, this stochastic model permits not only reconstructing the transport dynamics with its underlying advection-diffusion parameters, but also captures the epistemic uncertainties with Brownian motion.

2) Representation theorems for anomaly-decomposed divergence-free vector fields and symmetric PSD tensor fields. Our estimates are grounded in theorems ensuring realistic constraints on the learned advection-diffusion parameter fields by construction. The estimation automatically provides a decomposition based on an anomaly value field and the “anomaly-free” advection-diffusion parameters. This provides insights into both the anomaly patterns and what should locally be considered normal advection-diffusion parameters.

3) 2D/3D normal-abnormal advection-diffusion dataset. We develop a simulator for quasi-realistic advection-diffusion that can be used for supervised model pretraining based on the advection-diffusion parameters. Importantly, the simulator is able to generate velocity vector and diffusion tensor fields and applies artificial anomalies. We show that such simulated data boosts model performance for advection-diffusion parameter prediction, in particular for abnormal transport.

2. Related work

Neural Differential Equations and Optical Flow Significant developments in deep learning have recently led to an explosive growth of deep-learning-based solutions for PDEs. This stream of work either directly models the solution via DNNs [16, 51, 56] or learns mesh-free, infinite-dimensional operators using DNNs [8, 33, 38, 46, 55].

While the above methods focus on solving PDEs, i.e., the forward problem, we are interested in solving inverse PDE problems [34, 59]. In the context of advection equations such inverse problems have been extensively studied for optical flow and general image registration [25, 58], where the parameter to be estimated is a deformation or a velocity vector field [7, 9, 23, 24, 43]. DNN solutions for the fast prediction of these vector fields have also been studied [5, 14, 54, 62]. In contrast, our goal is the estimation of the parameters for more general advection-diffusion PDEs: specifically, their associated velocity vector and diffusion tensor fields. Further, while optical flow and registration approaches most commonly deal with image pairs, our parameter estimation for advection-diffusion PDEs is based on image time-series acquired over multiple time points.

Perfusion Imaging Perfusion imaging measures the blood flow through parenchyma by serial imaging [19]. Common perfusion measurement techniques include injecting an intravascular tracer, e.g., Dynamic Susceptibility Contrast-enhanced (DSC) and Dynamic Contrast-Enhanced (DCE) magnetic resonance (MR) perfusion [13, 19, 36], using magnetically-labeled arterial blood water protons as an endogenous tracer (Arterial spin labeling (ASL)) [50], or using positron emission tomography (PET) [21]. Derived quantitative measures help clinical diagnosis and decision-making. So far, the mainstream approach for quantitative perfusion measures is to use tracer kinetic models to estimate hemodynamic parameters and to obtain 3D perfusion parameter maps [19, 44]. However, substantial differences exist in perfusion parameter maps across institutions, mainly caused by different arterial input function (AIF) selection procedures, deconvolution techniques, and interpretations of perfusion parameters [44, 52, 53]. Moreover, these approaches are performed on individual voxels, disregarding spatial dependencies of the injected tracer’s dynamics.

Works to fit tracer transport via PDEs exist, where the observed tracer concentration time-series is assumed to reflect the blood flow within the vessels (advection) while diffusion captures the movements of freely-diffusive tracer within capillaries and the macroscopic effect of capillary transport [11, 11, 22, 36, 57]. However, these works assume that both velocity and diffusion are constant over the entire domain, which is unrealistic in real tissue. Zhou et al. [64, 65] and Liu et al. [35, 36] proposed to model the perfusion process via a transport model using numerical optimization. However, Zhou et al. [64, 65] assume the diffusion process is negligible; therefore, only the velocity field is estimated similar to optical flow [9, 25, 48]. Liu et al. [35, 36] and Zhang et al. [63] estimate both the spatial-varying velocity and diffusion fields, yet modeling the diffusion as a...
scalar field, which cannot express the diffusion anisotropy.

Liu et al. [37] propose to predict the perfusion process via an advection-diffusion model with intrinsically constructed divergence-free velocity vector fields and symmetric positive-semi-definite (PSD) diffusion tensor fields, using a deep-learning approach. This work allows for modeling diffusion anisotropy via tensors, and significantly reduces the inference time by resorting to deep learning approaches. However, regional anomalies are not explicitly modeled. Hence, additional post-processing steps, such as lesion detection, are required in perfusion analysis applications. In contrast, $D^2$-SONATA not only predicts the spatial dependency of advection-diffusion parameters in a stochastic manner, but it also decomposes transport into its normal and abnormal parts and can thereby provide additional insights into existing patterns of anomaly.

3. Method

3.1. Problem Setup

Let $C = C(x,t)$ denote the mass concentration at location $x$ in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), at time $t \in [0, T]$. The local mass concentration changes of an advection-diffusion process can be modeled as:

$$\frac{\partial C}{\partial t} = -V \cdot \nabla C + \nabla \cdot (D \nabla C) + \sigma \partial W_t$$

(1)

with specified boundary conditions (B.C.). The advection term captures the transport related to the flow of the fluid and the diffusion is driven by the gradient of mass concentration. The spatially-varying velocity field $V (V(x) \in \mathbb{R}^d)$ and diffusion tensor field $D (D(x) \in \mathbb{R}^{d \times d})$ describe the advection and diffusion. Eq. (1) contains the common assumption that fluids show negligible density variation in practice [26, 37], which means $V$ has zero divergence (i.e., is divergence-free).

As in [37], we model diffusion tensors $D(x)$ as symmetric PSD, to capture the predominant diffusion direction and diffusion anisotropy [47].

In contrast to existing approaches we explicitly model abnormalities via an anomaly value field $A (A(x) \in \mathbb{R}_{[0,1]}$, higher $A(x)$ means closer to normal), which modulates both the velocity and diffusion fields. Specifically, we assume that the velocity field can be decomposed as $V = A \circ \tilde{V}$ and the diffusion tensor field as $D = A \circ \tilde{D}$, where $\tilde{V}$ and $\tilde{D}$ denote the “anomaly-free” velocity and diffusion tensor fields respectively. Here, $\circ$ and $\odot$ denote the chosen interactions between the “anomaly-free” velocity and diffusion tensor fields and the anomaly value field respectively (see the definition in Eq. (4)). This interaction can be chosen in different ways. See Sec. 3.2 (Theorems 1 & 2 and Definition 1) for details on the anomaly-decomposed representations (and how $\circ$ and $\odot$ are operationalized) which assure that the decompositions can express divergence-free velocity fields and PSD diffusion tensor fields.

In addition, we model the advection-diffusion process via a stochastic PDE (SPDE), where $\sigma \ (\sigma(x) \in \mathbb{R})$ denotes the variance of the Brownian motion $W_t (W(x,t) \in \mathbb{R})$ [4] and represents the epistemic uncertainty for the dynamical system. With this additional stochastic term, the existence and uniqueness of the solution to Eq. (1) still holds [4, 49, 60].

3.2. Anomaly-decomposed Constraint-free Representation

As discussed in Sec. 3.1, incompressibility and symmetric PSD-ness are commonly used assumptions for fluid flow and diffusion. In YETI [37], divergence-free vectors and PSD tensors are obtained by construction via a suitable parameterization. Hence, when integrated into a deep-learning model no extra losses need to be imposed during training to obtain these properties. However, YETI does not provide an explicit abnormality model for the predicted $V$ and $D$ fields. $D^2$-SONATA, therefore introduces two representation theorems which allow constructing divergence-free velocity fields and symmetric PSD diffusion tensor fields in such a way that a decomposition into normal and abnormal value fields is obtained.

Anomaly-decomposed Divergence-free Velocity Vectors

Bézenac et al. [12] penalize deviations of predicted velocity fields from zero divergence during training, with the assumption that test time predictions will be approximately divergence free. Kim et al. [26] parametrize velocity vectors via the curl of vector fields, yet do not account for the boundary conditions that need to be imposed in bounded domain scenarios [2, 3, 15, 40]. Liu et al. [37] propose to predict a parameterization of velocity fields $V$ which is by construction divergence-free; however, abnormal regions are not explicitly modeled for $V$ and extra post-processing steps are required for applications targeted at anomaly detection. Therefore, our goal is a representation strategy for $V$ on a domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary such that (1) $V$ is divergence-free by construction; (2) Any divergence-free $V$ can be represented; (3) $V$ can be decomposed into an anomaly value field $A$ which characterizes the abnormal fields, and $\tilde{V}$ which refers to the corresponding normal value field without anomaly.

Theorem 1 (Anomaly-decomposed Divergence-free Vector Representation). For any vector field $V \in L^p(\Omega)^d$ and scalar field $A$ in $\mathbb{R}_{[0,1]}(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial \Omega$. If $V$ satisfies $\nabla \cdot V = 0$, there
exists a potential \( \Psi \) in \( L^p(\Omega)^{\alpha} \) (\( \alpha = 1(3) \) when \( d = 2(3) \)):

\[
V = \nabla \times (A \Psi), \quad (A \Psi) \cdot n |_{\partial \Omega} = 0. \quad (2)
\]

Conversely, for any \( A \in \mathbb{R}^{(0,1]}(\Omega) \), \( \Psi \in L^p(\Omega)^{\alpha} \), \( \nabla \cdot V = 0 \) implies \( \nabla \times (A \Psi) = 0 \). (See complete proof in Supp. A.)

**Anomaly-decomposed Symmetric PSD Diffusion Tensors**

We seek a representation for \( D \) (for a \( D(x) \)) such that (1) \( D \) is a symmetric PSD tensor by construction; (2) any symmetric PSD \( D \) can be represented; (3) \( D \) can be decomposed into an anomaly value field \( A \) indicating the abnormal patterns, and \( \tilde{D} \) which is the corresponding normal value field. We resort to the spectral decomposition theorem and the surjective Lie exponential mapping on \( SO(n) \) (exp : \( so(n) \rightarrow SO(n) \), \( so(n) \) is the group of skew-symmetric matrices, \( SO(n) \) is the real orthogonal group) [31, 32, 37].

**Theorem 2 (Anomaly-decomposed Symmetric PSD Tensor Representation).** For any \( n \times n \) symmetric PSD tensor \( D \) and \( A \in \mathbb{R}^{(0,1]}(\Omega) \), there exist an upper triangular matrix with zero diagonal entries, \( B \in \mathbb{R}^{n(n-1)/2} \), and a non-negative diagonal matrix, \( A \in SD(n) \), satisfying:

\[
D = U(AA)U^T, \quad U = \exp(B-B^T) \in SO(n). \quad (3)
\]

Conversely, for \( \forall A \in \mathbb{R}^{(0,1]}(\Omega), \forall B \in \mathbb{R}^{n(n-1)/2}, \) and \( \forall A \in SD(n) \), Eq. (3) results in a symmetric PSD tensor, \( D \). (See complete proof in Supp. B.)

**Definition 1 (“Anomaly-free” Fields).** According to Eq. (2-3), when \( A \) equals 1 over the entire domain \( (\Omega) \), we denote the corresponding parameters as “anomaly-free”. For the sake of convenience, we write the relation:

\[
V = A \odot \hat{V}, \quad D = A \odot \tilde{D}, \quad (4)
\]

where the overline is used to denote “anomaly-free” fields. Obviously, when \( A = 1 \), \( V, D \) are closer to normal. (See explicit expressions of the \( \odot, \odot \) operations in Supp. A & B.)

In this way, we not only obtain the underlying advection-diffusion parameters (\( V, D \)) satisfying the divergence-free and PSD constraints from the observed transport processes by construction, but we also gain knowledge as to where the anomalies exist and what the normal patterns of the advection-diffusion parameters are. Note that in our proposed approach we will directly predict \( A, V, D \). I.e., we will directly work in the decomposed domain rather than predicting \( V \) and \( D \) followed by a decomposition.

**3.3. \( D^2 \)-SONATA: Deep Decomposition for StOchastic Normal-Abnormal TrAnsport**

Sec. 3.2 described the representation theorems of \( D^2 \)-SONATA for anomaly-encoded divergence-free velocity fields and symmetric PSD diffusion fields. This enables expressing the advection-diffusion parameters with realistic constraints along with the corresponding anomaly patterns by construction. In particular, using these representations within a deep network will assure these constraints during training and testing. This section introduces \( D^2 \)-SONATA’s two-stage learning framework, to predict the velocity vector and diffusion tensor fields driving a potentially anomalous observed advection-diffusion process:

1. **Physics-informed learning**: reconstructs the anomaly value field (\( A \)) and “anomaly-free” advection-diffusion parameters (\( V, D \)), from observed transport time-series. The anomaly-encoded parameters (\( V, D \)) can then be obtained (Eq. (4)). In this stage, the model is trained on a simulated dataset (Sec. 4.1-4.2) under the supervision of the ground truth parameters and a corresponding anomaly value field;

2. **Transport-informed learning**: reconstructs the transport concentration time-series, where the ground truth advection-diffusion parameters can be unknown. During this stage, the supervision is imposed by the observed mass transport dynamics, via the integration of the stochastic advection-diffusion PDE in Eq. (1).

**Patch-based Input Time-series**

Given a time-series \( C = \{C^i \in \mathbb{R}(\Omega) \mid i = 1, 2, ..., N_T \} \), we first extract \( 32^2 \) (32^2 for 2D domains) patches randomly (uniform over the space) over the same spatial domain (\( \Omega_p \subset \Omega \)) at \( \text{N}_n \) sequential time points, starting from a random \( t_i \) (\( i = \{1, 2, ..., N_T - \text{N}_n + 1 \} \)). Each training sample \( (C_p) \) is denoted as \( C_p = \{C^i_p \in \mathbb{R}(\Omega_p) \mid j = i, ..., i + \text{N}_n - 1 \} \) (Fig. 1 (top left)).

**Physics-informed Learning**

We use a U-Net [1, 10] as the backbone for the parameter prediction network (\( \mathcal{F}_p \)) in this stage (Fig. 1 (middle)). A transport encoder is first used to encode the input \( C_p \) into latent features. The number of the input time points determines the input channels. Two decoders then learn the mappings to the representations of the “anomaly-free” parameters (\( V, D \)). Specifically, the \( V \)-decoder predicts the potential \( \hat{\Psi} \) to represent a divergence-free velocity field \( \hat{V} \) via Theorem 1, and the \( \tilde{D} \)-decoder predicts \( \tilde{D} \) along with \( \Lambda \) to represent a symmetric PSD diffusion tensor field \( \tilde{D} \) via Theorem 2. Another decoder outputs the predicted anomaly value field (\( A \)). The resulting \( \hat{A}, \hat{V}, \tilde{D} \) then allow computing the final prediction of the estimated velocity (\( \hat{V} \)) and diffusion (\( D \)) fields (Eq. (4)).

The reconstruction loss in this stage is three-fold. It is a supervised loss based on the actual and “anomaly-free” advection-diffusion parameters and the anomaly value field, which are all known from simulation:

\[
\mathcal{L}_{VD} = \frac{1}{|\Omega_p|} \int_{\Omega_p} \left\{ ||\nabla - \hat{\nabla}||_2^2 + ||V - \hat{V}||_2^2 + ||\tilde{D} - \hat{\tilde{D}}||_F + ||D - \hat{D}||_F + ||A - \hat{A}||_F \right\} dx,
\]

where the hatted symbols denote the predictions, \( || \cdot ||_2 \) and \( || \cdot ||_F \) the vector 2-norm and tensor Frobenius norm respectively. Similar to YETI [37], we additionally impose supervision on the eigenvectors (\( \hat{U} \)) and eigenvalues (\( \hat{A} \)) of \( \hat{D} \) to improve the network’s ability to capture the anisotropic struc-
The model is trained under the supervision of the ground truth anomaly value field (A), and the “anomaly-free” advection-diffusion parameters (V, D). In the transport-informed stage, the model is trained based on the observed concentration time-series, which allows recovering the underlying V, D that best fit the input time-series. The transport-informed stage is therefore applicable to real data where only the concentration time-series are observed.

The loss for the physics-informed learning stage is

\[
L_{\text{phy}} = L_{\text{vd}} + w_{\text{ua}}L_{\text{ua}}, \quad w_{\text{ua}} > 0. \tag{7}
\]

**Transport-informed Learning** In this stage (Fig. 1), the model is trained under the supervision of the transport dynamics. It contains two parts: (1) an uncertainty prediction network (\(\mathcal{F}_p\)); (2) a stochastic advection-diffusion PDE integrator. Specifically, \(\mathcal{F}_p\) is used to represent the model’s epistemic uncertainty, namely \(\sigma\) in Eq. (1). Intuitively, the model’s epistemic uncertainty should capture the following: (1) For values within the normal range, the variance of the Brownian motion should be small (low uncertainty). In this case the system state is dominated by the advection-diffusion term. (2) For anomaly regions with parameter values outside the normal range, the variance of the Brownian motion should be large and the system should exhibit higher uncertainty [28]. Based on this desired property, we design our training loss for model uncertainty as

\[
L_\sigma = \frac{1}{|\Omega_p|} \int_{\Omega_p} (1 - A - \tilde{\sigma})^2 \, dx, \tag{8}
\]

with the prior assumption that uncertainties should be small when the ground truth \(A\) is close to the normal value, 1.

Next, we implement a stochastic advection-diffusion PDE solver to integrate the initial state \(C_p^0\) forward in time to \(t + N_{\text{test}} - 1\) via Eq. (1) (Fig. 1 (right)), and to train the model by minimizing the differences between the predicted \((\tilde{C}_p)\) and the input \((C_p)\) time-series. We follow the patch-based Cauchy-B.C. and the spatial discretization scheme described in Liu et al. [30, 37, 59]. We also use RK45 for time-integration (\(\delta t\)) for the concentration prediction \(\tilde{C}_p\).

4. Experiments

Secs. 4.1-4.2 present results on simulated time-series in 2D/3D. We analyze the individual contribution of \(D^2\)-SONATA’s components, and compare the improvements achieved to the state of the art approaches. In Sec. 4.3, we further transfer our model pre-trained on simulated data to real time-series of magnetic resonance (MR) perfusion images from ischemic stroke patients. We demonstrate \(D^2\)-SONATA’s ability to distinguish stroke lesions.
Figure 2. Anisotropic moving Gaussians (Sec. 4.1), examples for normal (1st row) and abnormal (2nd row) transport time-series.

Figure 3. Mean relative absolute error (RAE) of YETI, Anom.-YETI, w/o-σ, w/o-A and D²-SONATA for 2D (ab)-normal anisotropic moving Gaussian (Sec. 4.1). X-axis: training iterations; Y-axis: the average RAE (log scale) of 100 test samples.

from normal brain regions via its predicted anomaly value field \( \hat{A} \) and reconstructed advection-diffusion parameters \((\hat{V}, \hat{D})\).

4.1 2D (Ab-)Normal Anisotropic Moving Gaussian Dataset We simulate 2D advection-diffusion with random anomalies on-the-fly (Fig. 2). Each sample is a 2D image time-series \((N_T = 40, \Delta t = 0.01 \text{s})\) on a 64² domain with 1 mm spacing. Every time-series is an advection-diffusion process initialized by a Gaussian \( \mathcal{N}(0, 2) \) at a uniformly sampled location. The “anomaly-free” advection-diffusion parameters \((\mathbf{V}, \mathbf{D})\) are obtained from randomly generated potentials \( \Psi, \mathbf{B}, \Lambda \) with uniformly sampled dominant diffusion directions. All components of \( \Lambda \) are randomly sampled from \([0, 1] \), and \( \Psi \) is randomly sampled from \([\mathcal{N}(0, 2)) \). Anomaly value fields are applied on the original “anomaly-free” parameters with a probability of 50%, where the value \((\in [0, 1])\) and area of anomaly fields are determined by a multivariate Gaussian with its center and standard deviation uniformly sampled in the domain. The initial Gaussian is then transported by \( \mathbf{V}, \mathbf{D} \). (See Supp. C for details.)

Comparisons To examine the contributions of different components in D²-SONATA, we compare five variants (Tab. 1): (1) YETI: the original setting of YETI by Liu et al. [37], where the model is trained with anisotropic Gaussians with no anomaly value field applied, and the advection-diffusion parameters are determined by anisotropically predicted without predicting an anomaly field or uncertainty; (2) Anom.-YETI: here the YETI model remains unchanged, but is trained with anisotropic Gaussians of which 50% are affected by the anomaly fields introduced in Sec. 4.1; (3) w/o-σ: D²-SONATA without the stochastic term and the model uncertainty network \( \mathcal{F}_U \); (4) w/o-A: D²-SONATA without the decoder branch for the anomaly value field \( A \) in \( \mathcal{F}_P \). In this case, the \( \mathbf{V}, \mathbf{D} \)-decoders directly output the overall value for \( V, D \) instead of predicting the anomaly value field and corresponding anomaly-free parameters separately; (5) The full D²-SONATA model.

We follow the training schedule in [37], and re-train YETI with its original settings. The time-series sample length for all models is \( N_{in} = 10 \). We set \( N_{out} = 10, w_{UA} = 0.5, w_{SS} = 0.1, w_{\sigma} = 0.5 \). We test on 100 samples after every 500 training epochs and use the mean relative absolute error (RAE) [37] for the predicted \( \hat{V}, \hat{D}, \hat{U}, \hat{A} \):

\[
Err(F) = \frac{1}{\Omega} \int_{\Omega} \| F - \hat{F} \| d\mathbf{x},
\]

where \( F(\hat{F}) \) denotes the ground truth (prediction), \( \| \cdot \| \) is the absolute, 2-norm and Frobenius norm for scalars, vectors and tensors. The time-series error \( Err(C) \) computes the RAE averaged over all predicted collocation time points.

Fig. 3 shows the testing reconstruction errors of the five models throughout training. Being exposed to abnormal data (Anom.-YETI) during training significantly boosts YETI’s reconstruction performance. The anomaly value field \( A \) prediction helps \( \mathbf{V} \)'s reconstruction, and modeling as a stochastic system \( \sigma \) further improves the time-series predictions. Overall, with anomaly-prediction and stochastic modeling, D²-SONATA outperforms all other variants with respect to both (ab)-normal time-series prediction and advection-diffusion parameter reconstruction.

4.2 3D (Ab-)Normal Brain Advection-Diffusion Dataset We develop a normal-abnormal brain advection-diffusion simulator based on the IXI brain dataset⁴. Following the simulation procedure in [37], we use 200 pa-

Table 1. Model properties comparison.

| Models        | Training Data | V, D | Y, D | A | σ | Type                  |
|---------------|---------------|------|------|---|---|-----------------------|
| PIANO [35]    | N/A           | ✓    | ✓    | ✓ | ✓ | Optimization-based    |
| YETI [37]     | Normal        | ✓    | ✓    | ✓ | ✓ | Learning-based        |
| Anom.-YETI    | (Ab)-Normal   | ✓    | ✓    | ✓ | ✓ | Learning-based        |
| w/o-σ         | (Ab)-Normal   | ✓    | ✓    | ✓ |   | Learning-based        |
| w/o-A         | (Ab)-Normal   | ✓    | ✓    | ✓ |   | Learning-based        |
| D²-SONATA     | (Ab)-Normal   | ✓    | ✓    | ✓ | ✓ | Learning-based        |

Table 2. Reconstruction comparisons on mean RAE (%) (Eq. (10)) for 3D (ab)-normal brain advection-diffusion dataset (Sec. 4.2).

| Models        | \( \hat{C} \) | \( \hat{V} \) | \( \hat{D} \) | \( \hat{U} \) | \( \hat{A} \) | \( \hat{A} \) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| YETI [37]     | 8.53          | 18.21         | 6.48          | 4.43          | 4.71          | N/A           |
| Anom.-YETI    | 2.52          | 12.23         | 5.15          | 4.12          | 3.54          | N/A           |
| w/o-σ        | 2.41          | 10.91         | 4.67          | 4.07          | 3.08          | 0.51          |
| w/o-A        | 2.29          | 12.10         | 4.61          | 4.11          | 3.17          | N/A           |
| D²-SONATA     | 2.12          | 10.68         | 4.21          | 3.08          | 2.94          | 0.47          |

http://brain-development.org/ixi-dataset/
patients that have T1-/T2-weighted images, a magnetic resonance angiography (MRA) image, and diffusion-weighted images (DWI) with 15 directions, to simulate the normal 3D velocity and diffusion tensor fields. For each case, we simulate both normal samples and anomaly-encoded samples, i.e., with simulated anomaly value fields applied to both $\mathbf{V}$ and $\mathbf{D}$, where the value (within $[0,1]$) and area of the anomaly fields are calculated by a multivariate Gaussian with its center and standard deviation uniformly sampled over the spatial domain. Each resulting image has isotropic spacing (1mm) and has been rigidly registered intra-subject. In summary, we simulated 2,400 brain advection-diffusion time-series, where 50% contain simulated anomalies. We randomly select 40 time-series for validation and testing respectively. (See Supp. D for more simulation details.)

**Comparisons** We compare the same five models as in Sec. 4.1 (Tab. 1). We follow the training strategy in YETI [37] and re-train YETI with its original settings. Predicted parameters ($\hat{\mathbf{V}}, \hat{\mathbf{D}}, \hat{\mathbf{U}}, \hat{\Lambda}, \hat{A}$) on the entire domain are obtained by splicing the output patches together. Integrating the stochastic advection-diffusion PDE forward with $\hat{\mathbf{V}}, \hat{\mathbf{D}}$, we obtain the predicted time-series $\hat{\mathbf{C}}$ on the original domain. For better visualization, we use the 2-norm ($\|\mathbf{V}\|_2$) map for the velocity fields, and the commonly used tensor feature maps [37,45] for the diffusion fields: (1) Trace ($tr_\mathbf{D}$), the sum of tensor eigenvalues ($\Lambda$): the overall diffusion strength; and (2) Fractional anisotropy (FA): the amount of anisotropy across diffusing directions.

Tab. 2 compares the reconstruction errors of the predicted $\hat{\mathbf{V}}, \hat{\mathbf{D}}, \hat{\mathbf{U}}, \hat{\Lambda}, \hat{A}$ from the five models, where $D^2$-SONATA consistently outperforms all other models. Specifically, training with abnormal samples and the stochastic term ($\sigma$) to model uncertainties helps achieve better overall performance, particularly in predicting the transport time-series. Predicting with the proposed decomposition based on the anomaly value field ($A$) further boosts the reconstruction of the advection-diffusion parameters. The advantages of $D^2$-SONATA can be observed more clearly in Fig. 4, where it successfully captures the anomaly value field ($A$), and accurately reconstructs the overall magnitudes of the advection-diffusion parameters ($\mathbf{V}, \mathbf{D}$). Without seeing abnormal samples during training, YETI struggles to identify the anomalies and to differentiate between advection ($\mathbf{V}$) and diffusion ($\mathbf{D}$) effectively. (2nd column), where the resulting $tr_\mathbf{D}$ reflects insufficient differences between normal and abnormal regions and the reconstructed $\|\mathbf{V}\|_2$ is noisiest among all models. Training with abnormal samples generally helps the models to locate anomalies. Interestingly, decomposing into an anomaly value field and the “anomaly-free” parameters improves the reconstruction of the diffusion’s fractional anisotropy (FA), especially within the abnormal regions.

4.3. ISLES2017: Brain Perfusion Dataset from Ischemic Stroke Patients

**Dataset** We test on the Ischemic Stroke Lesion Segmentation (ISLES) 2017 dataset [39], which contains perfusion data from 75 (43 training, 32 testing) ischemic stroke patients. Each patient has a dynamic susceptibility contrast (DSC) MR perfusion image (4D, with 40 to 80 available time points, time interval $\approx 1s$), along with the corresponding gold-standard lesion segmentation maps [17]. All images are resampled to isotropic spacing (1mm) and rigidly registered intra-subject via ITK [42] following [37]. Perfusion images are converted to the tracer concentration time-series, $\{C^i \in \mathbb{R}(\Omega)|i = 1, 2, \ldots, N_T\}$ ($t_1$ is the time-to-peak for total concentration over the entire brain, at which the injected tracer is assumed to have been fully transported into the brain [35,37]), via the relation between MR signal and tracer intensity [19]. 10 patients with lesion segmentation maps are randomly chosen for testing, while the remaining 65 concentration time-series are left-right flipped (i.e., the brain hemispheres are flipped) for data augmentation. In total, we obtain 130 time-series training samples, from which 10 samples are randomly picked for validation.

**Results** We transfer the pre-trained model from our simulated 3D normal-abnormal brain advection-diffusion dataset (Sec. 4.2) to the ISLES tracer concentration time-series dataset, using the Adam optimizer and learning rate $10^{-4}$.

We set $N_{in} = N_{out} = 5$, $w_\sigma = 0.5$, $w_{SS} = 0.1$.

Following the feature maps for comparison proposed.
Dice score, averaged from 6 randomly-selected test patients.

We compare the performance of feature maps across 10 test subjects from ISLES2017 dataset (Sec. 4.3), using Mean (Me.), Median (Med.), Standard Deviation (STD) of relative mean μ', absolute (|t|), and area under the curve (AUC).

Table 3. Quantitative comparison between D²-SONATA, YETI, PIANO and ISLES maps across 10 test subjects from ISLES2017 dataset (Sec. 4.3), using Mean (Me.), Median (Med.), Standard Deviation (STD) of relative mean μ', absolute (|t|), and area under the curve (AUC).

| Metrics                  | D²-SONATA | YETI [37] | PIANO [35] | ISLES [39] |
|--------------------------|-----------|-----------|------------|------------|
| µ'                       | V[2]      | V[2]      | V[2]       | V[2]       |
| Med.                     | 0.67      | 0.67      | 0.67       | 0.67       |
| (STD) (0.08)             | (0.10)    | (0.10)    | (0.10)     | (0.10)     |
| µ''                      | trp       | trp       | trp        | trp        |
| Med.                     | 0.49      | 0.49      | 0.49       | 0.49       |
| (STD) (0.30)             | (0.30)    | (0.30)    | (0.30)     | (0.30)     |
| AUC                      | Me.       | Me.       | Me.        | Me.        |
| V[2]                     | 0.70      | 0.70      | 0.70       | 0.70       |
| (STD) (0.01)             | (0.01)    | (0.01)    | (0.01)     | (0.01)     |
| Seg(A)                   | D         | D         | D          | D          |
| A                         | 0.76      | 0.76      | 0.76       | 0.76       |
| (STD) (0.04)             | (0.04)    | (0.04)    | (0.04)     | (0.04)     |

The chosen threshold is the optimal threshold for the segmentation Dice score, averaged from 6 randomly-selected test patients.

Comparisons We compare the performance of feature maps from D²-SONATA ([|V|[2], D, A) with (1) YETI [37]: ||V||2, D; (2) PIANO [35]: ||V||2, D; and (2) ISLES [39]

We presented D²-SONATA, a two-stage learning scheme to estimate the velocity vector fields and diffusion tensor fields that drive real-world-observed advection-diffusion processes in the presence of anomalies. The model is built upon a stochastic advection-diffusion PDE and predicts divergence-free velocity fields and symmetric PSD diffusion tensor fields by construction. Further, our model directly predicts an anomaly field and provides measures of uncertainty. Extensive experiments on 2D and 3D simulated data illustrate that our model can successfully reconstruct the advection-diffusion fields (with or without anomalies) from the time-series data of observed transport processes. When applied to real brain perfusion data from ischemic stroke patients, our model outperforms the state of the art across all metrics. By predicting the underlying advection-diffusion parameters along with an anomaly value field our model provides additional insights into real stroke data.
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