A HEAT FLOW FOR DIFFEOMORPHISMS OF FLAT TORI

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Abstract. In this paper we study the parabolic evolution equation \( \partial_t u = (|Du|^2 + 2|\det Du|)^{-1}\Delta u \), where \( u \): \( M \times [0, \infty) \to N \) is an evolving map between compact flat surfaces. We use a tensor maximum principle for the induced metric \( Du^T Du \) to establish two-sided bounds on the singular values of \( Du \), which shows that unlike harmonic map heat flow, this flow preserves diffeomorphisms. A change of variables for \( Du \) then allows us to establish a \( C^\alpha \) estimate for the coefficient of the tension field, and thus (thanks to the quasilinear structure and the Schauder estimates) we get full regularity and long-time existence. We conclude with some energy estimates to show convergence to an affine diffeomorphism.

1. Introduction

Unlike many similar problems in geometry, there is no widely applicable heat-flow technique to produce harmonic (or otherwise geometrically nice) representatives of diffeomorphisms. Unfortunately harmonic map heat flow does not work in dimensions greater than one: by choosing the right 2-jet at a point, one can arrange for the derivative to become singular a short time afterwards, so the flow does not stay inside the space of diffeomorphisms. In this paper we define and study a heat flow for diffeomorphisms of flat compact surfaces. Let \( M, N \) be oriented flat tori; i.e. \( M = \mathbb{R}^2/\Gamma_1, N = \mathbb{R}^2/\Gamma_2 \) for some integer lattices \( \Gamma_1 \) and \( \Gamma_2 \) acting by translation. Without loss of generality we can assume the initial data \( u_0 \in \text{Diff} (M, N) \) is orientation-preserving, i.e. \( \det Du > 0 \) when we choose oriented coordinates on both \( M \) and \( N \). Then we consider the Cauchy problem for a family of maps \( u \): \( M \times [0, T) \to N \) defined by

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\Delta u}{|Du|^2 + 2|\det Du|} \\
u(x, 0) = u_0
\end{cases}
\]

where \( \Delta u \) is the tension field (or simply the component-wise Laplacian in Cartesian coordinates). In this notation the denominator is somewhat opaque - see the next section for a more geometrically enlightening formula. Our main result is the following:

Theorem 1. There exists a solution \( u \in C^\infty (M \times [0, \infty), N) \) of (1) such that each \( u(\cdot, t) \) is a diffeomorphism, \( u(0, t) = u_0 \) and \( u(\cdot, t) \) converges smoothly to a harmonic map \( u_\infty \) as \( t \to \infty \).

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That is, additive subgroups of \( \mathbb{R}^2 \) isomorphic to \( \mathbb{Z}^2 \).
We will frequently (and often silently) swap between thinking of \( u(\cdot, t) \) as a map between compact manifolds \( M \to N \) and as the corresponding map \( \mathbb{R}^2 \to \mathbb{R}^2 \) between universal covers - the former is conceptually what we care about but the latter means we can choose Cartesian coordinates and work with a very concrete global PDE system.

**Notation.** From here forward \( u \) will always be a solution of (1), \( F = (|Du|^2 + 2|\det Du|)^{-1} \) the diffusion coefficient and \( P = \partial_t - F(Du) \Delta \) (acting on scalar functions) the linear parabolic operator associated with (1). We will use subscript notation \( u_{jk} = \frac{\partial^2 u}{\partial x^j \partial x^k} \) for the derivatives of \( u \) (in any Cartesian coordinate system on \( \mathbb{R}^2 \)) and the summation convention for repeated indices. Symmetrisation is denoted by \( T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) \). The derivative \( Du \) of maps/functions defined on \( M \times [0, T) \) refers to the spatial components alone - time derivatives will only appear explicitly as \( \partial_t \).

### 2. Gradient bounds and Preservation of Diffeomorphisms

Since the initial map is a diffeomorphism and any homotopy preserves degree, the only way for the map to no longer be a diffeomorphism at a later time is if it is not even a local diffeomorphism; that is, if \( Du \) becomes singular at some point. Thus we can establish that the flow preserves diffeomorphisms by preserving the invertibility of \( Du \) using the maximum principle. In particular we will preserve bounds on the singular values \( \lambda_i \) in the singular value decomposition

\[
Du(e_i) = \lambda_i v_i, \ i = 1, 2.
\]

Recall here that \( e_i, v_i \) are orthonormal bases for \( T_xM, T_u(x)N \) respectively and \( \lambda_i \geq 0 \) are the square roots of the eigenvalues of the induced metric \( h = u^*g_N = Du^T Du \). Note that \( \lambda_1 \lambda_2 = |\det Du| \) and \( \lambda_1^2 + \lambda_2^2 = |Du|^2 \), so our flow can be written more naturally as:

\[
\frac{\partial u}{\partial t} = \frac{\Delta u}{(\lambda_1 + \lambda_2)^2}.
\]

We need lower bounds on \( \lambda_i \) to preserve diffeomorphisms and upper bounds to establish the H"older estimate in the next section. Rather than studying the evolution of the singular values directly (which requires some hairy computations and gets unwieldy at singular points \( \lambda_1 = \lambda_2 \) where regularity fails), we will focus on the evolution of \( h_{ij} = u_i^*u_j^* \). Since the eigenvalues of \( h \) are \( \lambda_1^2 \) and \( \lambda_2^2 \), preserving bounds \( m \leq \lambda_i \leq M \) is equivalent to preserving the inequality \( m^2 \delta \leq h \leq M^2 \delta \) of quadratic forms.

**Proposition 2.** If the singular values of \( Du \) satisfy \( m \leq \lambda_i \leq M \) at the initial time, then this inequality persists for all later times.

**Proof.** We will show \( h \geq m^2 \delta \) is preserved using [And, Theorem 3.2], a refinement of Hamilton’s maximum principle for tensors. The argument for the upper bound \( h \leq M^2 \delta \) is very similar but easier, since there the negative-definite term in (3) below is a help rather than a hindrance. Let \( S = h - m^2 \delta \) so that we are trying to preserve the non-negative definiteness of \( S \). Then after some computation we arrive at the evolution equation

\[
\partial_t S_{ij} = F\Delta S_{ij} + N_{ij}
\]
where:

\[ N_{ij} = -2F \left( u^a_{i_k} u^a_{k_j} + \frac{2u^a_k u^a_j \Delta u^a}{\lambda_1 + \lambda_2} \right) \]

If \( S \geq 0 \) and \( S(p)(v, v) = 0 \), we can choose Cartesian coordinates for the domain and target so that \( v = \partial_1 \) and \( Du = \text{diag}(\lambda_1, \lambda_2) \) with \( \lambda_0 \geq \lambda_1 = m \). Then at \( p \) we have \( S = h - \lambda_1 \delta = \text{diag} \left( 0, \lambda_2 - \lambda_1^2 \right) \); and since \( h(\partial_1, \partial_1) \) is at a spatial minimum we have \( \partial_h h_{11} = 2\lambda_1 u^2_{11} = 0 \) and thus \( u^2_{11} = 0 \). The other derivatives \( \partial_i S = \partial_i h \) are given at \( p \) by:

\[
\begin{align*}
\partial_1 S &= \begin{pmatrix} 0 \\ \lambda_2 u^2_{11} \\ 2\lambda_2 u^2_{12} \end{pmatrix}, \\
\partial_2 S &= \begin{pmatrix} 0 \\ \lambda_1 u^2_{12} + \lambda_2 u^2_{12} \\ 2\lambda_2 u^2_{12} \end{pmatrix}
\end{align*}
\]

To preserve the inequality it suffices to show that

\[ Q = N_{11} + 2F \sup_{\Gamma \subset \mathbb{R}^{2} \times 2} \left( 2\Gamma^2_k \partial_k S_{12} - \Gamma^2_k \Gamma^2_k S_{22} \right) \geq 0 \]

at this point.

**Case 1.** If \( \lambda_1 = \lambda_2 \) then \( S_{22} = 0 \) and \( h(\partial_2, \partial_2) \) is also at a spatial minimum, so we also have \( u^2_{22} = 0 \). If \( u^2_{11} = u^2_{12} = 0 \) then the supremum and the reaction term \( N_{11} \) are both zero. Otherwise, we are taking the supremum of a non-constant linear function and thus get \( Q = \infty \geq 0 \).

**Case 2.** If \( \lambda_1 \neq \lambda_2 \), \( S_{22} = \lambda_2^2 - \lambda_1^2 \) is positive, so the expression being maximised is a quadratic polynomial in \( \Gamma^2_k \) and \( \Gamma^2_{kk} \) with negative leading coefficients. Thus it achieves a unique maximum at \( \Gamma^2_k = \partial_k S_{12}/S_{22} \), where it is equal to:

\[
\frac{(\partial_1 S_{12})^2 + (\partial_2 S_{12})^2}{S_{22}} = \frac{\lambda_2^2 u^2_{11} u^3_{11} + \lambda_1^2 u^2_{12} u^3_{12} + \lambda_2^2 u^2_{12} u^3_{12} + 2\lambda_1 \lambda_2 u^2_{22} u^2_{12}}{\lambda_2^2 - \lambda_1^2}
\]

Combining this with \( N_{11} \) we get some nice cancellations resulting in

\[
Q = \frac{2\lambda_1^2 F}{\lambda_2^2 - \lambda_1^2} \left( (u^2_{11})^2 + (u^2_{12} + u^2_{22})^2 \right) \geq 0
\]

as required.

\[ \square \]

**Remark.** There are other choices of \( F \) (or anisotropic flows \( \partial_t u = a^{ij} (Du) \nabla_i \nabla_j u \)) satisfying this result - rather than requiring perfect cancellation of the \( u^2_{22} u^2_{12} \) terms between \( N_{11} \) and the \( \Gamma \) contribution to form the perfect square, we could be a little more permissive and require some differential inequality on the coefficients that leads to the desired definiteness. We will not discuss these other flows, however, as the regularity estimate in the next section works only for \( \square \).

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2The supremum term here comes from noting that at \( p \), the scalar function \( x \rightarrow S(V_x, V_x) \) must be at a minimum for any vector field \( V \) extending \( v \). Choosing \( V = v + \Gamma x \) and computing the evolution equation of \( S(V, V) \) rather than \( S(v, v) = S_{11} \) gives the extra term.

3Remark: this optimal \( \Gamma \) is achieved at the singular vector field \( V = v + \Gamma x = e_1 + O(|x|^2) \); so the \( Q \) we get in this case is actually \( P(\lambda_1^2) \).
Since the domain manifold is compact and the initial data is a diffeomorphism, we know there must be some $m, M$ for which these bounds hold. To keep things simple from here on out, we will consolidate these persistent derivative bounds into a single constant:

**Corollary 3.** There is some $\Lambda > 0$ depending only on the initial data $u_0$ such that $\lambda_1, \lambda_2, |Du|, \lambda_1 + \lambda_2$ are all in $[\Lambda^{-1}, \Lambda]$ for all time.

3. *Regularity Estimate*

Since we are dealing with a quasilinear PDE system, there is very little general theory available; so to get a Hölder estimate on the coefficient $F$ we will need to exploit the particular form of our system. To get this estimate we need to define some new quantities: fix Cartesian coordinates on both the domain and target, and let $\theta \in S^1$ and $r > 0$ be defined by

\[
\begin{align*}
    r \cos \theta &= u_1 + u_2, \\
    r \sin \theta &= u_1^2 - u_2.
\end{align*}
\]

One can check that $r = \lambda_1 + \lambda_2$ and $\theta$ is the “rotational component” of $Du$; that is, the angle between the singular frames $e$ and $v$. Since $F = r^{-2}$ and Prop 2 gives us time-independent bounds above and below on $r$, to get a $C^\alpha$ estimate for $F$ it suffices to get one for $r$. This is our goal for this section. We start with the result that explains why we are interested in $\theta$ along with $r$:

**Lemma 4.** The evolution of the quantities $r, \theta$ forms a closed system of PDE given by

\[
\begin{align*}
    Pr &= -\frac{|D\theta|^2}{r} - \frac{2|Dr|^2}{r^3} + \frac{Dr \times D\theta}{r^2}, \\
    P\theta &= 0.
\end{align*}
\]

Proof. As with the previous evolution equations, differentiate (1) to get the evolution of $Du$, use this to find the evolution of $r, \theta$ and do a lot of algebraic simplification. This result relies strongly on the particular $F$ we have chosen, and is related to the fact that our choice of $F$ makes the flow preserve maps with symmetric $Du$.

The first thing we notice is that the evolution of $\theta$ decouples entirely - it satisfies a uniformly parabolic PDE in general form, so we can apply the standard Krylov-Safanov estimate [Lie96, Corollary 7.41]:

**Corollary 5.** $\theta$ belongs to some parabolic Hölder space, with exponent and norm depending only on $\Lambda$.

Our estimate for $r$ is inspired by the general Hölder gradient estimate for quasilinear PDE (see Lieberman [Lie96, Theorem 12.3], originally due to Ladyzhenskaya and Ural'tseva [LU64]) which perturbs $\partial_i u$ by $|du|^2$ in order to obtain super- and subsolutions of a divergence-form equation, allowing the application of a weak Harnack inequality. We will instead perturb $r$ by $\theta$, with the Hölder continuity of $\theta$ being key to getting the same regularity for $r$.

\[4Dr \times D\theta\] denotes the 2D cross product $\partial_1 r \partial_2 \theta - \partial_2 r \partial_1 \theta.$
Notation. For \( X_0 = (x_0, t_0) \in T^2 \times [0, T) \) we define the parabolic neighbourhood

\[
Q(X_0, R) = \left\{ (x, t) \in M \times [0, T) : d(x, x_0) < R, |t - t_0| < R^2, t < t_0 \right\}.
\]

For \( \alpha \in (0, 1] \), the parabolic \( \alpha \)-Hölder space is then defined by the norm

\[
\| f \|_{0; \alpha} = \sup_{X \in M \times [0, T), R > 0} \left\{ \frac{\text{osc}_{Q(X, R)} f}{R^\alpha} |X| \right\}.
\]

We will denote averages by

\[
\int_E f = \frac{1}{\mu(E)} \int_E f d\mu
\]

where \( d\mu = dA dt \) is the standard measure on \( M \times [0, T) \), and often use the shorthand \( \sup_R = \sup_{Q(X_0, R)} \) (similarly \( \inf_R, \text{osc}_R \)) since almost all the neighbourhoods we are working with in this estimate have a common centre. The relation \( A \lesssim B \) means \( A \leq CB \) for some positive constant \( C \) depending only on \( \Lambda \).

**Proposition 6.** The diffusion coefficient \( F(x, t) = (\lambda_1(x, t) + \lambda_2(x, t))^{-2} \) is bounded in the parabolic Hölder space over \( M \times [0, T) \), with norm depending only on \( \Lambda \).

**Proof.** This is a local estimate, so fix a point \( X_0 = (x_0, t_0) \) now and let \( \phi = \theta - \theta(X_0) \) (so \( D\theta = D\phi \)). Since we know \( r \in [\Lambda^{-1}, \Lambda] \), oscillations of \( r^q \) are comparable to those of \( r \) for any \( q > 0 \): to be precise we have

\[
|s^q - r^q| = \left| s - r \right| \in \left[ q\Lambda^{-1}, q\Lambda^q \right]
\]

for all \( r, s \in [\Lambda^{-1}, \Lambda] \). Thus we can take powers of \( r \) to get more favourable evolution equations: we will work with the two functions \( r^2 \) and \( w = r^p - c\phi^2 \), where \( p \) and \( c \) will be chosen later to make \( w \) a supersolution. Letting

\[
L_d: f \mapsto \partial_t f - \text{div} \left( r^{-2} Df \right)
\]

be our associated linear divergence-form operator, we compute

\[
L_dr^2 = 2r^{-1} Dr \times D\theta - 2 |D\theta|^2 - 2r^{-2} |Dr|^2
\]

\[
L_dw = pr^{p-2} \left( r^{-1} Dr \times D\theta - |D\theta|^2 + (1 - p) r^{-2} |Dr|^2 \right) + 2cr^{-2} |D\theta|^2 - 4c\phi r^{-3} \langle Dr, D\theta \rangle.
\]

The Cauchy–Schwarz and Peter–Paul inequalities allow us to estimate

\[
|Dr \times D\theta|, |\langle Dr, D\theta \rangle| \leq \frac{1}{2r} |Dr|^2 + \frac{r}{2} |D\theta|^2
\]

and thus we immediately get

\[
L_dr^2 \leq -r^{-2} |Dr|^2 - |D\theta|^2 \leq 0.
\]
Getting a supersolution is more difficult: using the same estimates for the cross terms we have
\[ L_d \left( r^p - c\phi^2 \right) \geq \left( -\frac{1}{2} pr^{p-4} + p(1-p)r^{p-4} - 2c\phi r^{-4} \right)|Dr|^2 \]
\[ + \left( -\frac{1}{2} pr^{p-2} - pr^{p-2} + 2cr^{-2} - 2c\phi r^{-2} \right)|D\theta|^2. \]

For this to be non-negative we need the two inequalities
\[ p \left( \frac{1}{2} - p \right) r^p - 2c\phi \geq 0 \]
and
\[ 2c(1-p) - \frac{3}{2} pr^p \geq 0. \]
In particular these are both satisfied if \(|\phi| < \delta < 1\) and
\[ \frac{3}{4} p\Lambda \frac{p}{1 - \delta} \leq c \leq p \left( \frac{1}{2} - p \right) \Lambda^{-p}, \]
and we can choose such a \(c\) if and only if
\[ \frac{3\delta}{1 - \delta} \Lambda^{2p} \leq 1 - 2p. \]
Thus taking \(\delta < 1/4\) we can satisfy this with some \(p \in (0, \frac{1}{2})\). By the uniform Hölder continuity of \(\theta\) we can choose \(R_0\) depending only on \(\Lambda\) so that \(|\phi| < \delta\) on \(Q(X_0, 4R_0)\), and thus we have \(L_d r^2 \leq 0\) and \(L_d w \geq 0\) on this set. Now let \(R \in (0, R_0)\) be arbitrary and apply the Weak Harnack estimate [Lie96, Theorem 6.18] to the non-negative supersolutions \(\sup_{4R} r^2 - r^2\) and \(w - \inf_{4R} w\) on \(Q(4R)\), which produces estimates
\[ (5) \quad \int_{\Theta(R)} \sup_{4R} r^2 - r^2 \leq C \left( \sup_{4R} r^2 - \sup_R r^2 \right) \]
\[ (6) \quad \int_{\Theta(R)} \left[ (r^p - c\phi^2) - \inf_{4R} (r^p - c\phi^2) \right] \leq C \left( \inf_{4R} (r^p - c\phi^2) - \inf_{4R} (r^p - c\phi^2) \right), \]
where \(\Theta(R) = Q\left((x_0, t_0 - 4R^2), R\right) \subset Q(X_0, 4R)\) covers the same spatial region as \(Q(R)\) but is disjoint in time and \(C > 0\) depends only on \(\Lambda\). Using (5) we get:
\[ \int_{\Theta(R)} \left[ \sup_{4R} r^2 - r^2 \right] \geq 2\Lambda^{-1} \int_{\Theta(R)} \left[ \sup_{4R} r - r \right] \]
\[ \int_{\Theta(R)} \left[ (r^p - c\phi^2) - \inf_{4R} (r^p - c\phi^2) \right] \geq p\Lambda^{-p} \int_{\Theta(R)} \left[ r - \inf_{4R} r \right] \]
Similarly on the other side we have:
\[ \sup_{4R} r^2 - \sup_{R} r^2 \leq 2\Lambda \left( \sup_{4R} r - \sup_{R} r \right) \]
\[ \inf_{R} (r^p - c\phi^2) - \inf_{4R} (r^p - c\phi^2) \leq p\Lambda^{1-p} \left( \inf_{R} r - \inf_{4R} r \right) + \sup_{4R} c\phi^2 \]
Using these estimates in (6) and (9) and multiplying by appropriate factors yields

\[ \int_{\Theta(R)} \sup_{4R} r - r \leq C\Lambda^2 \left( \sup_{4R} r - \sup_{4R} r \right) \]

and

\[ \int_{\Theta(R)} \left[ r - \inf_{4R} r \right] \leq C\Lambda^{-2p} \left( \inf_{4R} r - \inf_{4R} r \right) + \frac{1}{p} \Lambda^{1-p} \sup \phi^2. \]

Noting \( \Lambda^{-2p} \leq 1 \) and adding these together we find

\[ \text{osc}_{4R} r \leq C_1 \left( \text{osc}_{4R} r - \text{osc}_{4R} r + \sup \phi^2 \right) \]

where the new constant \( C_1 = \max \left( \Lambda^2, \frac{p}{p} \Lambda^1, 1 \right) \) depends only on \( \Lambda \). Writing this as

\[ \text{osc}_{4R} r \leq \left( 1 - C_1^{-1} \right) \text{osc}_{4R} r + \sup \phi^2 \]

and applying Lemma 8.23 of Gilbarg–Trudinger [GT83] with \( \sigma(R) = \sup_R \phi^2 \) yields the estimate

\[ \text{osc}_{4R} r \leq C_2 \left( \left( \frac{R}{R_0} \right)^{\alpha} \text{osc}_{R_0} r + \sup_{R^\mu R_0^{1-\mu}} \phi^2 \right) \]

where \( C_2 > 0 \) and \( \alpha \in (0, 1) \) depend only on \( \Lambda \) and a freely chosen \( \mu \in (0, 1) \). The Hölder bound on \( \theta \) implies \( \sup_{R} \phi^2 \lesssim R^\beta \) for some \( \beta \in (0, 1) \), and thus we have (for \( R < R_0 \))

\[ \text{osc}_{R} r \lesssim R^\alpha + R^\mu \lesssim R^{\min(\alpha, \mu, \beta)}, \]

i.e. a Hölder bound on \( r \) with constant and exponent depending only on \( \Lambda \) (since \( \beta \) and \( R_0 \) depend only on \( \Lambda \)). Since \( F = r^{-2} \) and we already know \( r \) is restricted to \( [\Lambda^{-1}, \Lambda] \), we get the same result for \( X \mapsto F(Du(X)) \). \( \square \)

A standard bootstrap argument using the parabolic Schauder estimate now gives \( C^\alpha \) regularity of all derivatives; so we get the estimate we need for long-time existence:

**Corollary 7.** All derivatives of \( u \) are bounded on \( M \times [0, T) \).

**Proof.** Repeatedly differentiating the evolution equation \( \partial_t u = F \Delta u \) gives

\[ P \left( D^k u \right) = \partial_t D^k u - F \Delta D^k u = D^k F * D^2 u + \cdots + D^k F * D^{k+1} u \]

where \( A * B \) denotes an arbitrary contraction of \( A \otimes B \). By our control on \( Du \) we know that \( \|F\|_{k, \alpha} \lesssim \|u\|_{k+1, \alpha} \), so \( \|P \left( D^k u \right)\|_\alpha \lesssim \|u\|_{k+2, \alpha} \). Combining this with the Schauder estimate we see that \( \|u\|_{k+1, \alpha} < \infty \) implies \( \|u\|_{k+2, \alpha} < \infty \) for all \( k \geq 1 \); so once we have \( C^{2, \alpha} \) control we get bounds on all derivatives. Applying the Schauder estimate to the components of \( u \) we see that our \( C^\alpha \) control of the coefficients implies this \( C^{2, \alpha} \) control on \( u \). \( \square \)
4. Existence and Convergence

We come now to the proof of Theorem 4. The first technicality to address is short-time existence, which is (as is usually the case) fairly straightforward for our flow.

**Lemma 8.** For any \( t_0 \in \mathbb{R} \) and \( u_0 \in \text{Diff}(M, N) \), there exists some time \( \epsilon > 0 \) and a smooth solution \( u: M \times [t_0, t_0 + \epsilon) \to N \) of \( \partial_t u = F \Delta u \) such that \( u(\cdot, t_0) = u_0 \).

**Proof.** We refer the reader to Baker [Bak11, Main Theorem 1] for a local existence theorem for solutions of nonlinear parabolic systems. The conditions are easy to check - since the initial data is a diffeomorphism of a compact manifold we have \( \Lambda^2 \leq F|_{t=0} \leq \Lambda^2 \), so we immediately satisfy the Legendre-Hadamard condition and can modify \( F \) on \( B(0, \Lambda^{-1}/2) \) (i.e. away from the image of \( Du \)) to make it continuously differentiable without changing the dynamics. \( \square \)

We now have all the ingredients we need for the standard long-time existence argument:

**Proposition 9.** Given any \( u_0 \in \text{Diff}(M, N) \), there is a smooth solution \( u: M \times [0, \infty) \to N \) of \( F \) such that \( u(\cdot, 0) = u_0 \) and each \( u(\cdot, t) \) is a diffeomorphism.

**Proof.** Let \( T \in [0, \infty] \) denote the largest time such that there is a smooth solution on \([0, T)\). By Lemma 8 we know that \( T > 0 \). If \( 0 < T < \infty \), Corollary 7 tells us that all derivatives of \( u \) are equicontinuous, so by the Arzela-Ascoli theorem we can smoothly extend the solution to \([0, T)\). But then \( u(\cdot, T) \) is also a diffeomorphism (by continuity of the derivative and the strong lower bound \( \lambda_i > \Lambda^{-1} \)), so by Lemma 8 again we can extend our solution to some \([0, T + \epsilon)\), which contradicts the assumption that \( T \) is maximal. Thus \( T = \infty \). \( \square \)

**Proposition 10.** For any solution \( u: M \times [0, \infty) \to N \) of \( F \) there is a sequence of times \( t_k \to \infty \) such that \( u(t_k) \) converges to a harmonic diffeomorphism; that is, a map of the form \( u(x) = Ax + y \) for some \( y \in \mathbb{R}^2 \) and \( A \in GL(2, \mathbb{R}) \) sending \( \Gamma_1 \) onto \( \Gamma_2 \).

**Proof.** First let’s establish the characterisation of harmonic diffeomorphisms \( M \to N \). If \( u \) is harmonic then integration by parts over a fundamental domain of \( M \) yields

\[
\int |D^2 u|^2 = \int (\Delta u)^2 = 0;
\]

so \( u \) is an affine function \( u(x) = Ax + y \) when considered as a map \( \mathbb{R}^2 \to \mathbb{R}^2 \). In order for this to descend to a the quotients \( M \to N \) it must satisfy some periodicity \( u(x + z) = u(x) + Bz \) for all \( x \in \mathbb{R}^2, z \in \mathbb{Z}^2 \), where \( B: \Gamma_1 \to \Gamma_2 \) is a homomorphism of abelian groups; and putting these together we see that in fact \( y = u(0) \) and \( A = B \). Since \( u \) is a diffeomorphism, \( u^{-1} \) must be of the same form; i.e. \( A \) must be an isomorphism.

We can get convergence on a sequence of times by studying the Dirichlet energy \( E(u) = \frac{1}{2} \int |Du|^2 \). Integration by parts shows that \( E(t) = E(u(\cdot, t)) \) is a positive non-increasing function, and thus it converges to some limit \( E_\infty \) as \( t \to \infty \). Since \( \int_0^\infty E'(t) dt = E_\infty - E(0) \) is finite and \( E'(t) \) is nonpositive, we can extract a sequence of times \( t_k \to \infty \) such that \( E'(t_k) \to 0 \). From the evolution equation we then see that

\[
\frac{dE}{dt} = - \int F(\Delta u)^2 \to 0;
\]
so the lower bound on $F$ implies that $\Delta u(t_k) \to 0$ in $L^2$. Since $u$ is bounded in every $C^k$ norm, all of its derivatives are equicontinuous and thus we can use Arzela-Ascoli to pass to a diagonal subsequence of times (replacing $t_k$ from now on) on which $u(t_k)$ converges smoothly to a limit $u_\infty$. In particular we have $\Delta u(t_k) \to \Delta u_\infty$ uniformly and thus in $L^2$ as well, so $\Delta u_\infty = 0$. \qed

To improve this convergence on a sequence to convergence for all times, we estimate the second-order energy $q(t) = \|D^2 u(t)\|_{L^2}^2 = \int |\Delta u|^2$.

**Lemma 11.** There are positive constants $C_0, C_1$ depending only on $\Lambda$ such that $q(t)$ satisfies

$$\frac{dq}{dt} \leq -C_0 (1 - C_1 q(t)) \|D^3 u\|_{L^2}^2.$$

**Proof.** From the evolution equation we can compute

$$\frac{dq}{dt} = -2 \int F(Du) |D^3 u|^2 - 2 \int \hat{F}(Du) * D^2 u * D^3 u,$$

where $\hat{F}$ is the matrix of derivatives $\partial F/\partial u_j$. Estimating the second term with Peter-Paul as

$$\left| \int \hat{F}(Du) * D^2 u * D^3 u \right| \leq \|\hat{F}(Du)\|_{L^\infty} \left( \epsilon \|D^2 u\|_{L^2}^2 + \frac{1}{\epsilon} \|D^3 u\|_{L^2}^2 \right)$$

with $\epsilon = \Lambda^2 \|\hat{F}(Du)\|_{L^\infty}$, we get

$$\frac{dq}{dt} \leq -\Lambda^{-2} \|D^3 u\|_{L^2}^2 + \Lambda^2 \|\hat{F}(Du)\|_{L^\infty}^2 \|D^2 u\|_{L^4}^4.$$

Applying the Gagliardo-Nirenberg interpolation inequality $\|f\|_{L^4}^2 \leq C \|f\|_{L^2} \|Df\|_{L^2}$ to $\|D^2 u\|_{L^4}$, we arrive at

$$\frac{dq}{dt} \leq -\Lambda^{-2} \|D^3 u\|_{L^2}^2 \left( 1 - C A^4 \|\hat{F}(Du)\|_{L^\infty}^2 q \right)$$

as desired. (Since $F$ is continuously differentiable on the compact image of $Du$, $\|\hat{F}(Du)\|_{L^\infty}$ is a finite constant depending on $\Lambda$.) \qed

By the uniform subconvergence $D^2 u(t_k) \to 0$, there is some time $t'$ at which $C_1 q(t') < \frac{1}{\epsilon}$, and we see $q'(t) < 0$ whenever this is true; so this inequality is preserved for all time. Thus we have

**Corollary 12.** For $t > t'$, $q$ satisfies

$$q(t) \leq \frac{C_0}{2} \|D^3 u\|_{L^2}.$$ 

**Proposition 13.** The quantity $q$ converges exponentially to zero: that is, $q(t) \leq Ae^{-\omega t}$ for some $A, \omega > 0$. 


Proof. Applying the Poincaré inequality to \( u_{jk}^{i} \) and summing, we see \( \| D^2 u \|_{L^2}^2 \lesssim \| D^3 u \|_{L^2}^2 \), and thus for \( t > t' \) we have
\[
\frac{dq}{dt} < -\omega q
\]
for a positive constant \( \omega \). Comparison with the corresponding ODE proves that \( q(t) \leq q(t') e^{-\omega(t-t')} \).

As a consequence we obtain the full convergence, completing the proof of Theorem \footnote{}.

**Proposition 14.** The flow \( u(t) \) converges smoothly to \( u_\infty \) as \( t \to \infty \).

**Proof.** Using \( \| \Delta u \|_{L^2} = \| D^2 u \|_{L^2} \) and the Gagliardo-Nirenberg interpolation inequality \( \| f \|_{L^\infty}^2 \lesssim \| D f \|_{L^\infty} \| f \|_{L^2} \), we see
\[
\| \partial_t u \|_{L^\infty} \leq \Lambda^2 \| D^2 u \|_{L^\infty} \leq M e^{-\omega t/2}
\]
for some \( M > 0 \). Thus for \( b > a > 0 \) we have
\[
|u(x,b) - u(x,a)| \leq \int_a^b \partial_t u(x,t) \, dt \leq \int_a^b M e^{-\omega t/2} = \frac{2M}{\omega} \left( e^{-\omega a/2} - e^{-\omega b/2} \right)
\]
which converges to zero as \( a, b \to \infty \); i.e. \( u(t) \) is uniformly Cauchy as \( t \to \infty \). Since we already know it converges to \( u_\infty \) on a subsequence, we obtain the full uniform convergence \( u(t) \to u_\infty \). Once again interpolating
\[
\| D^j (u - u_\infty) \|_{L^\infty}^2 \lesssim \| u - u_\infty \|_{L^\infty} \| D^j (u - u_\infty) \|_{L^\infty}
\]
and applying the uniform \( C^{2j} \) bound, we get \( C^j \) convergence for all \( j \). \qed

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