Nullification of multi-Higgs threshold amplitudes in the Standard Model

E.N. Argyres
Institute of Nuclear Physics, NRCPS ‘Δημόκριτος’, Greece

Ronald H.P. Kleiss,
NIKEHF-H, Amsterdam, the Netherlands

Costas G. Papadopoulos
TH Division, CERN, Geneva, Switzerland

Abstract

We show that nullification of all tree-order threshold amplitudes involving Higgs particles in the Standard Model occurs, provided that certain equations relating the masses of all existing elementary particles to the mass of the Higgs scalar are satisfied. The possible role of these relations in restoring the high-multiplicity unitarity and their phenomenological relevance are briefly discussed.

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The high-multiplicity limit of processes involving scalar particles has been studied recently \[1, 2\]. At tree order the amplitudes $A(H^* \rightarrow nH)$ as well as the cross section $\sigma(ff \rightarrow nH)$, to leading order in the Yukawa coupling \[3, 4\], grow as $n!$, where $n$ is the number of produced scalar particles, and violate the unitarity bounds for sufficiently high energies. The reason for this behaviour is rather simple and relies on the coherence properties of scalar amplitudes: all amplitudes at tree order add coherently, so the $n!$ simply counts the number of Feynman diagrams.

The study of the high-multiplicity limit of amplitudes, within the framework of perturbative quantum field theory, might have a profound theoretical and phenomenological interest. The situation resembles that of the high-energy limit of amplitudes involving longitudinal bosons. As is well known \[5\], the latter amplitudes violate the unitarity bound, unless specific relations hold among the different couplings. The existence of such relations causes very delicate cancellations at tree order, which are responsible for the restoration of high-energy unitarity. The persistence of these cancellations at higher orders is naturally understood in the framework of the $SU(2)_L \times U(1)_Y$ gauge symmetry, which guarantees a consistent behaviour of the amplitudes in the high-energy limit, by discarding from all physical processes the ‘unphysical’ Goldstone bosons. Nevertheless, since in the high-energy limit ($\sqrt{s} \rightarrow \infty, n$ fixed) one can neglect any mass dependence, no direct predictions can be made concerning the masses \[4\]. The situation is opposite in the case of the high-multiplicity limit ($\sqrt{s} \rightarrow \infty, \sqrt{s}/nm$ fixed), since the unitarity violation is now related to the threshold behaviour of the amplitudes, so that the mass dependence can no longer be neglected. As we will show, it is the advantage of the high-$n$ limit of multi-Higgs amplitudes at threshold that confronts the mass parameters with the unitarity limits, and that it provides, in principle, a way to directly extract information on the masses in the framework of the Standard Model.

In this letter we investigate the possibility that, given certain relations among the masses of fermions, Higgs particle and gauge bosons, all physical multiboson amplitudes respect unitarity at tree order in the high-$n$ limit. The idea is rather simple: we seek a mechanism that may discard the bad high-$n$ behaviour of the amplitude $A(H^* \rightarrow nH)$ from all physical processes.

First of all it is easy to see that the physical amplitudes $A(HH \rightarrow nH)$ \[6, 7, 8\] not only do not exhibit factorial growth, but are actually zero (!) for $n \geq 3$. As was shown in ref.\[8\] this nullification phenomenon is independent of the values of the mass and the self-coupling of the scalar particle, and depends only on the specific form of the interaction. It is worth while to recall that the only interaction exhibiting the nullification phenomenon and incorporating $\phi^3$ as well as $\phi^4$ terms, is the spontaneously broken $\phi^4$ theory.

The next step is to consider the couplings with the fermions. The recursion relation for the amplitudes $d(n) \equiv A(f(p)f'(p') \rightarrow nH)$ is given by (see Fig.1):

\[1\] Except for the ratio $m_W/m_Z$. 

Figure 1: Diagrammatic representation of the recursion relation for the process $f \bar{f} \rightarrow nH$. The blob connected with a dashed line corresponds to the amplitude $H^* \rightarrow nH$.

\[ d(n) = -i g_Y \sum_{n_1 \geq 0, n_2 \geq 1} \frac{i a(n_1)}{n_1! P(n_1)} \frac{i a(n_2)}{n_2! (n_2^2 - 1)} \tag{1} \]

where $P(n) = n \hat{q} - \hat{p} - m$, $m$ is the mass of the fermion and $g_Y$ the Yukawa coupling. Throughout this paper we assume $m_H = 1$, and we restore the $m_H$ dependence only when this is necessary. Defining

\[ d(n) = -in! \hat{d}(n) P(n) \quad \text{and} \quad f = \sum_{n \geq 0} \hat{d}(n) x^n \tag{2} \]

and using that

\[ a(n) = -in!(n^2 - 1)b(n), \quad f_0(x) = \sum b(n)x^n = \frac{x}{1 - \sqrt{\frac{\lambda}{12}} x} \tag{3} \]

we have

\[ x f'(x)\hat{q} - f(x)(\hat{p} + m) - 2m \frac{z}{1 - z} f(x) = 0 \tag{4} \]

where $z = \sqrt{\frac{\lambda}{12}} x$, $d(1) = -i g_Y \bar{u}(p)$ and $f(0) = \bar{u}(p)$. Taking now $y = -z/(1 - z)$ and $(f(x) = h(y))$, we find

\[ y(1 - y)h'(y)\hat{q} - h(y)(\hat{p} + m) + 2myh(y) = 0 \tag{5} \]

Writing $h(y) = \alpha(y)\bar{u}(p) + \beta(y)\bar{u}(p)\hat{q}$ we obtain

\[ y(1 - y)\beta'(y) - 2p \cdot q\beta(y) + 2my\alpha(y) = 0 \]

\[ -2m\beta(y) + y\alpha'(y) = 0 \tag{6} \]
which gives rise to
\[ y(1 - y)\alpha''(y) + \alpha'(y)(c - y) + 4m^2\alpha(y) = 0 \] (7)
with \( c = -2p \cdot q + 1 \), which is similar to the equations obtained for the amplitudes \( \mathcal{A}(HH \rightarrow nH) \) \cite{7, 8}. The results are
\[
\begin{align*}
\alpha(y) &= F(\nu, -\nu; c; y) \\
\beta(y) &= -\frac{2m}{c}yF(1 + \nu, 1 - \nu; 1 + c; y)
\end{align*}
\] (8)
with \( \nu = 2m \) and \( F(a, b; c; z) \) is the hypergeometric function. It is easy to see that the amplitude \( \mathcal{A}(f \bar{f} \rightarrow nH) \) vanishes for any \( n \geq N \) provided that
\[
\frac{2m_f}{m_H} = N
\] (9)
where \( N \) is a non-negative integer \cite{9}.

Figure 2: Diagrammatic representation of the recursion formula for the amplitude \( VV \rightarrow nH \) at threshold. The blobs connected with a dashed line corresponds to the amplitude \( H^* \rightarrow nH \).

We now turn to the amplitudes \( VV \rightarrow nH \), where \( V \) stands for either a \( W^\pm \) or a \( Z^0 \). The recursion relation for
\[
\mathcal{A}(V(k_1)V(k_2) \rightarrow nH(q)) = a_{\mu\nu}(n)e^\mu(k_1; \lambda_1)e^\nu(k_2; \lambda_2)
\] (10)
is given by (see Fig.2):
\[
\begin{align*}
\frac{a_{\mu\nu}(n)}{n!} &= igM \sum_{n_1 \geq 0, n_2 \geq 1 \atop n_1 + n_2 = n} \mathcal{P}_\nu^\rho(n_1) \frac{ia_{\rho\mu}(n_1)}{n_1!} \frac{ia(n_2)}{n_2!(n_2^2 - 1)} \\
&\quad + \frac{ig^2}{4} \sum_{n_1 \geq 0, n_2, n_3 \geq 1 \atop n_1 + n_2 + n_3 = n} \mathcal{P}_\nu^\rho(n_1) \frac{ia_{\rho\mu}(n_1)}{n_1!} \frac{ia(n_2)}{n_2!(n_2^2 - 1)} \frac{ia(n_3)}{n_3!(n_3^2 - 1)}
\end{align*}
\] (11)
where

\[ P_{\mu\nu}(n) = \frac{1}{Q^2 - M^2} \left( -g_{\mu\nu} + \frac{Q_\nu Q_\mu}{M^2} \right) \]  

(12)

\[ Q^\mu = k_1^\mu - nq^\mu, \quad k_1 = (E; k_1), \quad g \text{ is the gauge coupling constant and } M = m_W. \]

Since the only momenta available are \( k_1 \) and \( q \), the general form of \( a_{\mu\nu}(n) \), taking into account that \( k_1 \cdot e(k_1; \lambda_1) = 0 \), is given by:

\[ a_{\mu\nu}(n) = a_1(n)g_{\mu\nu} + a_2(n)k_{1\nu}q_\mu + a_3(n)q_\nu q_\mu. \]  

(13)

For transverse \( V \)'s it is easy to see that only \( a_1 \) survives after the contraction with the polarization vectors. The equation for \( a_1 \) is:

\[ \frac{a_1(n)}{n!} = -igM \sum_{\begin{subarray}{c} n_1 \geq 0, n_2 \geq 1 \\ n_1 + n_2 = n \end{subarray}} \frac{ia_1(n_1)}{n_1!(n_1^2 - 1)} \frac{ia_1(n_2)}{n_2!(n_2^2 - 1)} - \frac{g^2}{4} \sum_{\begin{subarray}{c} n_1 \geq 0, n_2, n_3 \geq 1 \\ n_1 + n_2 + n_3 = n \end{subarray}} \frac{ia_1(n_1)}{n_1!(n_1^2 - 1)} \frac{ia_1(n_2)}{n_2!(n_2^2 - 1)} \frac{ia_1(n_3)}{n_3!(n_3^2 - 1)} \]  

(14)

The above equation has been studied extensively in refs. [7, 8] and it leads to the nullification of \( a_1(n) \) for \( n > N \) provided that

\[ \frac{4m_W^2}{m_H^2} = N(N + 1). \]  

(15)

For longitudinal \( V \)'s, things are slightly more complicated. Defining

\[ b_\mu(n) = a_{\mu\nu}(n)e^\nu(k_1; \lambda_1 = 0) \]  

(16)

we have

\[ \frac{b_\mu(n)}{n!} = igM \sum_{\begin{subarray}{c} n_1 \geq 0, n_2 \geq 1 \\ n_1 + n_2 = n \end{subarray}} P_{\mu\nu}^\nu(n_1) \frac{ib_\mu(n_1)}{n_1!} \frac{ia_1(n_2)}{n_2!(n_2^2 - 1)} + \frac{g^2}{4} \sum_{\begin{subarray}{c} n_1 \geq 0, n_2, n_3 \geq 1 \\ n_1 + n_2 + n_3 = n \end{subarray}} P_{\mu\nu}^\nu(n_1) \frac{ib_\mu(n_1)}{n_1!} \frac{ia_1(n_2)}{n_2!(n_2^2 - 1)} \frac{ia_1(n_3)}{n_3!(n_3^2 - 1)}. \]  

(17)

Using the obvious ansatz:

\[ b_\mu(n) = in! \left( (Q^2 - M^2)g_\mu - Q_\mu Q^\nu \right) c_\nu(n) \]  

(18)

and writing \( c_\mu(n) = c_1(n)k_{1\mu} + c_2(n)q_\mu \) we obtain the following system of recursion relations:

\[ (n^2 - En - M^2)c_1(n) + (n - E)c_2(n) = 4M^2 \sum_{k=0}^{n-1} c_1(k)(n-k) \left( \frac{\lambda}{12} \right)^{(n-k)/2} \]

\[ n(M^2 - nE)c_1(n) - nEc_2(n) = 4M^2 \sum_{k=0}^{n-1} c_2(k)(n-k) \left( \frac{\lambda}{12} \right)^{(n-k)/2} \]
Defining as usual
\[ f_1(x) = \sum_{n \geq 0} c_1(n)x^n \quad \text{and} \quad f_2(x) = \sum_{n \geq 0} c_2(n)x^n \]  
we arrive at the following system of second-order differential equations:
\[ (D^2 - ED - M^2)f_1 + (D - E)f_2 = \frac{4M^2z}{(1 - z)^2}f_1 \]  
\[ (-ED^2 + M^2D)f_1 - EDf_2 = \frac{4M^2z}{(1 - z)^2}f_2 \]
where \( D \) is the differential operator:
\[ D \equiv x \frac{d}{dx}. \]  
Defining \( x = -\sqrt{12/\lambda} e^{2\tau} \) and
\[ G = e^{-2E\tau} \left( \frac{1 - z}{1 + z} \right) (Df_1 + f_2) \]
we arrive after some straightforward manipulations, at the equation
\[ \left( \frac{d^2}{d\tau^2} - 4E^2 + \frac{4M^2}{\cosh^2 \tau} - \frac{2}{\sinh^2 \tau} \right) G = 0. \]  
This is nothing but the Schrödinger equation with a Pöschl-Teller potential. The explicit (but ugly) form of the \( G \) is given in ref. [10], from which the explicit form of \( f_1, f_2 \) can be derived (see also Eq.\((23))\), taking into account that
\[ f_1 = \frac{1}{M^2} \left( \frac{1 - z}{1 + z} \right)^2 D \left( \frac{1 + z}{1 - z} \right) G. \]  
As usual [8], the poles of the function \( G \) in \( E \) are the only non-zero \( \mathcal{A}(VV \rightarrow nH) \) amplitudes. It is easy to see that \( \mathcal{A}(VV \rightarrow nH) \) vanishes for \( n \geq N + 2 \) provided that
\[ \frac{4m_W^2}{m_H^2} = N(N + 1). \]  
Note that the nullification for longitudinal polarized bosons starts from \( n = N + 2 \) whereas in the case of transversely polarized bosons this happens from \( n = N + 1 \).

In the case where \( V \) stands for \( Z^0 \) the analysis is exactly the same and we arrive at the following condition:
\[ \frac{4m_Z^2}{m_H^2} = N(N + 1) \]  
where \( N \) is again a positive integer.

The nullification of threshold amplitudes is the result of very delicate cancellations between \( s- \) and \( t- \) channel graphs. In general, if such cancellations do not exist, one
can take the leading term in the Yukawa coupling which is the coherent sum of the s-channel graphs. This leads to violation of unitarity\[2\. The existence of specific relations among the masses and/or the couplings could in principle restore the high-n unitarity. In order to see what the effect of cancellations beyond threshold is, we calculate the amplitudes for the processes $H(p_1) + H(p_2) \rightarrow H(p_3) + nH(k)$ in a $\phi^m$ scalar theory. The recursion formula is given by (see Fig.3 for $\phi^4$)

![Figure 3: Diagrammatic representation of the recursion relation for the process $H(p_1) + H(p_2) \rightarrow H(p_3) + nH$. The blob connected with one line corresponds to the amplitude $H^* \rightarrow nH$ and that with two lines to the process $H + H \rightarrow nH$.](#)

\[
\frac{a_3(n)}{n!} = -\frac{i\lambda}{q!} \sum P_3(n_1)n_1!(n_2^2 - 1)n_2! \cdots (n_r^2 - 1)n_r! 
\]

\[
+ \frac{-i\lambda}{(q-1)!} \sum P_2(n_1;p_1)n_1! P_2(n_2; -p_3)n_2! \cdots P_2(n_r; k)n_r! 
\]

(28)

where $a_2(n; p)$ is the amplitude for the process $H(p) + H(p') \rightarrow nH(k)$, $q = m - 2$, $r = m - 1$,

\[
P_2(n; p) = (p - nk)^2 - 1 , \quad P_3(n) = (p_1 - p_3 - nk)^2 - 1
\]

(29)

and $k^2 = p_i^2 = 1 , \ i = 1, 2, 3$.

Defining as usual

\[
f_3(x) = \sum_{n=N+q-1,N\geq0} b_3(n)x^n
\]

(30)

where $a_3(n) = -in!P_3(n)b_3(n)$ and

\[
f_2(x; \omega) = \sum_{n=N+q,n\geq0} b_2(n; \omega)x^n
\]

(31)
where \(a_2(n; p) = -in!P_2(n; p)b_3(n; \omega)\), with \(\omega = p \cdot k\), we get
\[
\left(D^2 - 2\omega_1 D + \omega_1^2 - \Omega^2\right)f_3 = \frac{\lambda}{q!}f_3f_0^q + \frac{\lambda}{(q-1)!}f_2(\omega_1)f_2(-\omega_2)f_0^{q-1}
\]
with
\[
\omega_1 = p_1 \cdot k, \quad \omega_2 = p_3 \cdot k, \quad \omega_3 = p_1 \cdot p_3, \quad \omega_{12} = \omega_1 - \omega_2, \quad \Omega = \sqrt{\omega_{12}^2 - 1 + 2\omega_3}
\]
and
\[
f_2(\omega) = (1 + u^2)^{-s}F\left(-s, -s - \frac{2\omega}{q}; 1 - \frac{2\omega}{q}; -u^2\right)
\]
with \(u^2 = -(\lambda/2m!)x^q\) and \(u = e^\tau\). The solution of the Eq.(32) is given by:
\[
f_3 = -\frac{u^{2\omega_{12}/q}}{W}\left(\psi_1 \int_{-\infty}^\infty \psi_1 F + \psi_1 \int_{-\infty}^{\tau} \psi_1 F + C_1\psi_1 + C_2\psi_1\right)
\]
where the C’s are defined so as to make \(f_3\) have an expansion in integer powers of \(x\), \(F\) is the inhomogeneous part of Eq.(32) multiplied by \(4/q^2\), and \(\psi_{1,11}\) are the two independent solutions of
\[
\left(\frac{d^2}{d\tau^2} - \omega^2 + \frac{s(s + 1)}{\cosh^2(\tau)}\right)\psi = 0
\]
with \(s(s + 1) = 2m!/q^2q!\), \(\omega = 2\Omega/q\),
\[
\psi_1 = u^\omega(1 + u^2)^{-s}F\left(-s, -s + \omega; 1 + \omega; -u^2\right)
\]
\[
\psi_{11} = u^{-\omega}(1 + u^2)^{-s}F\left(-s, -s - \omega; 1 - \omega; -u^2\right)
\]
and their Wronskian is \(W = 2\omega\).

After some algebra (where we are interested only in the terms in Eq.(35) that have poles in \(n - \Omega - \omega_{12}\)) we get
\[
a_3(n) = -in!\frac{2m!}{(q-1)!}\left(\frac{\lambda}{2m!}\right)^N \sum_{k=0}^N G_k \frac{(a + N - k - 1)!}{(N - k)!(a - 1)!}
\]
where
\[
G_k = \frac{1}{k!}\left(\frac{d^k}{dx^k}\left(g(\Omega; x)g(\omega_1; x)g(-\omega_2; x)\right)\right)_{x=0}
\]
\[
g(\omega; x) = F\left(-s, -s - \frac{2\omega}{q}; 1 - \frac{2\omega}{q}; x\right)
\]
\(a = 3s + 2(q - 1)/q\) and \(n = Nq + q - 1\). For s integer the sum in Eq.(39) terminates, since \(G_k\) vanishes for \(k \geq 3s + 1\). This termination is the remnant of nullification beyond threshold. It shows that the cancellations caused by the mass relations survive beyond threshold. Of course an estimate of the real amplitude (not only collinear configurations) is needed in order to have a proof of unitarity restoration. Unfortunately this is a rather difficult problem, since the definition of a lower bound in
analogy with the \( H^* \rightarrow nH \) amplitude, is no longer possible because of the existence of destructive interference between \( s-\) and \( t-\) channel graphs. Nevertheless the persistence of the cancellations beyond threshold is a strong hint that unitarity might be restored when nullification at threshold is present. In any case the understanding of this phenomenon relies on the recovery of the ‘symmetry’ which is responsible for it, which will guarantee the consistency of the theory.

Although a complete understanding of the nullification phenomenon is not available, it is worth while to investigate the consequences of Eqs.\((9,15,27)\) for the masses of all elementary particles. The predictions are summarized as follows:

\[
\begin{align*}
\text{Fermions} & \quad m_f = \frac{N_f}{2} m_H \\
\text{Bosons} & \quad m_V = \sqrt{\frac{N_V(N_V+1)}{2}} m_H
\end{align*}
\]

where \( N_f \) and \( N_V \) are integers. If we try to fit the whole spectrum using these relations we need a super-light Higgs, \( m_H \leq 2m_e \), where \( m_e \) is the electron mass. Besides the experimental exclusion of this possibility, such a solution is not theoretically attractive. For instance one cannot understand why only a few integers between 1 and \( 10^5 \) \((N_f, N_W, N_Z)\) are realized in the spectrum. On the other hand a solution of the form

\[
N_W = N_Z \quad , \quad N_f = 0 \quad \text{and} \quad N_{\text{top}} \geq 1
\]

may be seen as a first-order approximation to the spectrum of the existing elementary particles\(^2\). In order to make the whole picture phenomenologically relevant, one has to perform a more detailed analysis \([11]\). Recall that we have relations between ‘bare’ couplings that will be changed by renormalization group equations, hopefully to a more realistic form. At any rate the existence of such relations could, in principle, answer some important open problems of the Standard Model. For instance, large mass splittings in the fermionic sector (as in the case of the top quark), can be explained in the context of Eq.\((3)\), since this latter suggests a kind of ‘quantization’ of the fermion masses in terms of the Higgs mass. Of course, much more effort has to be spent in order to understand these new relations. Nevertheless, we can safely conclude that the high-\( n \) limit of multi-Higgs amplitudes is deeply related to the Higgs mechanism itself and could provide us with a new tool to understand the mass spectrum of the elementary particles within the Standard Model.

\(^2\)For instance, we find, for \( m_Z \sim m_W = 80 \text{ GeV} \), that \( m_H = 67 \text{ GeV} \) \((N_{W,Z} = 2)\) and \( m_{\text{top}} = 134 \text{ GeV} \) \((N_{\text{top}} = 4)\).
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