Strong coupling quantum Otto cycle

Mao Kaneyasu* and Yoshihiko Hasegawa†

Department of Information and Communication Engineering,
Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo 113-8656, Japan
(Dated: June 30, 2022)

We propose a quantum Otto cycle model without the weak coupling assumption that the interaction between the system and reservoirs is negligible. Our model replaces the thermalization process in the weak coupling model with a process comprising thermalization and decoupling. We analytically calculate the efficiency of our model and show that in the weak interaction limit, when the contribution of the interaction terms is neglected, the efficiency reduces to that of the earlier model. We derive the sufficient condition for the efficiency of our model not to surpass that of the weak coupling model and find that the condition is satisfied when the decoupling processes have a positive cost. Using a simple two-level system, we numerically examine the relation between the interaction strength and the efficiency of our model, and demonstrate that our model does not exceed the efficiency of the weak coupling model.

I. INTRODUCTION

Constructing and analyzing heat engines is one of the fundamental themes of thermodynamics. In classical thermodynamics, it is a universal principle, rigorously shown by Carnot, that no heat engine operating between two reservoirs can exceed the efficiency limit \( \eta_C = 1 - T_c/T_h \), where \( T_c \) and \( T_h \) denotes the temperatures of the cold and hot reservoirs, respectively. The Carnot limit assumes that heat engines operate at the macroscopic scale, where fluctuations and quantum effects do not come into play. Recently, the notion of thermodynamics has been applied to mesoscopic systems, such as protein motors and biochemical clocks, that are described by stochastic processes. In the mesoscopic regime, thermodynamic quantities, e.g., entropy, work, and heat, become stochastic and the second law of thermodynamics does not necessarily hold due to fluctuations [1, 2]. Moreover, heat engines have been studied in microscopic systems, where the quantum effects play fundamental roles [3–24]. Quantum extensions of the Carnot and Otto cycles, the most fundamental heat engines in thermodynamics, are summarized in Refs. [3–5]. Additionally, various heat engines which utilize quantum effects, such as measurement, coherence, and entanglement, have been proposed [6–9]. In particular, it has been theoretically proved that quantum heat engines may exceed the classical efficiency limit [4, 10–13]. For instance, the classical Carnot limit is violated in heat engines using squeezed reservoirs [10–12], although this phenomenon does not violate the second law of thermodynamics. This fact indicates that quantum resources can be used to enhance heat engines. Research into the experimental realization of quantum heat engines is currently being undertaken [12, 25–29] and quantum heat engines have already been implemented in a variety of physical platforms, such as trapped ions and nuclear magnetic resonance (see Ref. [30] for a review).

In many models of quantum heat engines, the interaction between the system and the reservoirs is assumed to be weak so that their interaction is negligible. This approximation facilitates the theoretical analysis, because the thermal equilibrium state can be described as the product state of the system and the reservoir. However, it has been pointed out that this assumption cannot be justified in systems where quantum behavior appears, because the ratio of the surface to the volume is large [31, 32]. In recent years, theories have been studied which do not assume weak coupling and take into account the contribution of the interaction [31–41]. The reaction coordinate mapping enables the analysis of quantum heat engines strongly coupled to reservoirs [31, 34, 35]. Although this method can be applied to arbitrary quantum systems, it is also the case that it must specify harmonic oscillators as reservoirs, and to the best of our knowledge, a general and straightforward model of strongly coupled quantum heat engine has not yet been proposed.

In this paper, we construct a quantum Otto cycle model without the approximation that the interaction between the system and the reservoirs is negligible. This model can be applied generally: it does not specify the details of the system and the reservoirs, except that the decoupling processes are assumed to be realized isentropically. By considering a weak coupling limit in the proposed model, we show that the efficiency reduces to that of the weak coupling model previously derived. Furthermore, we derive the sufficient condition for \( \eta_{\text{str}} \leq \eta_{\text{weak}} \), where \( \eta_{\text{str}} \) and \( \eta_{\text{weak}} \) are the efficiencies of our model (strong coupling) and the existing model (weak coupling), respectively. This inequality holds for positive costs to decouple the system from the two reservoirs. We also demonstrate numerically that this inequality holds under a simple two-level quantum Otto cycle.

---

* kaneyasu@biom.t.u-tokyo.ac.jp
† hasegawa@biom.t.u-tokyo.ac.jp
II. WEAK COUPLING MODEL

In this section, we review the widely discussed quantum Otto cycle model. In this model, we adopt an approximation that the interaction between the system and the reservoirs is negligible. We call this the “weak coupling model” to clearly distinguish it from the model we describe in the next section.

We consider a quantum system $S$ and two heat reservoirs $B_h$ and $B_c$, whose inverse temperatures are $\beta_h$ and $\beta_c$, respectively ($\beta_h < \beta_c$). The Hamiltonian of the total system is expressed as follows:

$$H_{\text{tot}} = H_S + H_B^h + H_B^c + H_{SB}^h + H_{SB}^c.$$  

(1)

$H_S$, $H_B^h$, and $H_B^c$ are the self-Hamiltonians of $S$, $B_h$, and $B_c$, respectively, and $H_{SB}^h$ corresponds to the interaction between $S$ and $B_h$. By assuming that the two reservoirs are both in Gibbs states, the states of the reservoirs can be expressed as

$$\rho_B^i = \frac{e^{-\beta_i H_B^i}}{Z_i^B} \quad (i = h, c),$$  

(2)

where $Z_i^B = \text{Tr}[e^{-\beta_i H_B^i}]$ is the partition function.

In this model, the cycle consists of the following four processes: (A) adiabatic compression, (B) hot isochoric thermalization, (C) adiabatic expansion, and (D) cold isochoric thermalization. We will describe the state change of the system in each process and calculate the transferred heat and the exerted work during these processes.

A. Process

Process A: adiabatic compression—In this process, the system does not interact with the reservoirs. The Hamiltonian of the system is initialized to $H_S^i = \sum \epsilon_i |\psi_i\rangle \langle \psi_i|$, where each $\epsilon_i$ is an energy eigenvalue of $H_S^i$, and $|\psi_i\rangle$ is the corresponding eigenvector. We assume that there is no degeneracy in the eigenvalues. We take the initial state of the system to be the Gibbs state at inverse temperature $\beta_c$:

$$\rho_S^{c, \text{eq}} = \frac{e^{-\beta_c H_S^c}}{Z_S^c} = \sum \epsilon \rho_i |\psi_i\rangle \langle \psi_i|,$$  

(3)

where $Z_S^c = \text{Tr}[e^{-\beta_c H_S^c}]$ and $\rho_i = e^{-\beta \epsilon_i}/Z_S^c$. $H_S$ is dependent on a controllable external parameter $\lambda$. The initial value of $\lambda$ is $\lambda_i$, which corresponds to $H_S^i$. During this process, we vary $\lambda$ from $\lambda_i$ to $\lambda_f$, and consequently, $H_S$ changes from $H_S^i$ to $H_S^f$. The state change of the system in this process can be expressed by a unitary operator $U_{\text{com}} = T \exp[-i \int H_S(t) dt]$, where $T$ is the time-ordering operator. If the change of $\lambda$ is sufficiently slow, the time-evolution induced by $U_{\text{com}}$ does not change the probability distribution $\{p_i\}$. The final state of the system can be expressed as

$$\rho_B^f = U_{\text{com}} \rho_S^{c, \text{eq}} U_{\text{com}}^\dagger = \sum \epsilon \rho_i |\phi_i\rangle \langle \phi_i|,$$  

(4)

where $|\phi_i\rangle = U_{\text{com}} |\psi_i\rangle$ and $|\phi_i\rangle$ is the eigenvector of $H_S^f$ corresponding to the energy eigenvalue $\epsilon'$ of $H_S^f$. We note that the eigenvalue $\epsilon$ of $H_S^c$ and the eigenvalue $\epsilon'$ of $H_S^f$ have a one-to-one correspondence and assume that there is no reversal of the magnitude relationship between eigenvalues and no degeneracy during this process.

Because the system does not interact with the reservoirs, no heat flows into the system during this process. Therefore, we regard the change in internal energy of the system as the work performed on the system, which is given by

$$W_{\text{com}} = \text{Tr}[H_B^f \rho_B^f] - \text{Tr}[H_B^i \rho_B^{c, \text{eq}}].$$  

(5)

Process B: hot isochoric thermalization—In this process, the Hamiltonian of the system is constant at $H_S^i$. There is a weak interaction between the system and the hot reservoir. We assume the reservoir is so large that its state does not change throughout this process. After a sufficiently long time, the state of the system converges to the Gibbs state at inverse temperature $\beta_h$ [6]. The final state of the system can be expressed as

$$\rho_S^{h, \text{eq}} = \frac{e^{-\beta_h H_S^h}}{Z_S^h} = \sum \epsilon' q_{\epsilon'} |\phi_{\epsilon'}\rangle \langle \phi_{\epsilon'}|,$$  

(6)

where $Z_S^h = \text{Tr}[e^{-\beta_h H_S^h}]$ and $q_{\epsilon'} = e^{-\beta_h \epsilon'}/Z_S^h$. Each $\epsilon'$ is an eigenvalue of $H_S^h$ and $|\phi_{\epsilon'}\rangle$ is the corresponding eigenvector, which is equal to that used in Eq. (4).

Here, we note that there is a crucial approximation in this model in that the interaction between the system and the reservoir is ignored. More precisely, the final state of the system should be the Gibbs state, taking into account the interaction Hamiltonian $H_{SB}^h$. However, in this model, the contribution of $H_{SB}^h$ is neglected with the
assumption that the interaction is sufficiently weak, that is
\[
e^{-eta_h (H_S^h + H_B^h + H_B^h)} Z \approx e^{-eta_h H_S^h} Z_S^h \otimes e^{-eta_h H_B^h} Z_B^h = \rho_{S}^{h,eq} \otimes \rho_{B}^{h}.
\]

Because the Hamiltonian is constant throughout this process, the work performed on the system is equal to 0. Therefore, we can regard the change in internal energy of the system as the heat transferred from the reservoir to the system, which is given by
\[
Q_{in} = \text{Tr}[H_S^h \rho_{S}^{h,eq}] - \text{Tr}[H_S^h \rho_{S}^{h,eq}].
\]

**Process C: adiabatic expansion**—Similar to Process A, in Process C, there is no interaction between the system and the reservoirs. We vary the parameter \( \lambda \) from \( \lambda_f \) to \( \lambda_i \) sufficiently slowly. As a result, the Hamiltonian of the system changes from \( H_S^h \) to \( H_S^c \). \( U_{exp} \), the time-evolution operator of this process, is equal to \( U_{exp}^{\dagger} \). Therefore, the final state can be expressed as follows:
\[
\rho_S^{c} = U_{exp} \rho_S^{h,eq} U_{exp}^{\dagger} = \sum \epsilon q_{\epsilon} |\psi_{\epsilon}\rangle \langle \psi_{\epsilon}|,
\]
where \( q_{\epsilon} \) is equal to that in Eq. (6) and \( |\psi_{\epsilon}\rangle \) is equal to that in Eq. (3).

Because the system does not interact with the reservoirs, no heat flows into the system and the work performed on the system is equal to the change in internal energy of the system, which is given by
\[
W_{exp} = \text{Tr}[H_S^c \rho_{S}^{c,eq}] - \text{Tr}[H_S^h \rho_{S}^{h,eq}].
\]

**Process D: cold isochoric thermalization**—In this process, the Hamiltonian of the system is constant at \( H_S^{c} \) and the system weakly interacts with the cold reservoir. Similar to Process B, after a sufficiently long time, the state of the system converges to the Gibbs state at inverse temperature \( \beta_c \), i.e., the final state is \( \rho_S^{c,eq} \). We note that we also neglect the contribution of the interaction Hamiltonian \( H_S^{c} \).

Because the Hamiltonian does not change during this process, the work performed on the system is 0 and we consider the decrease in internal energy of the system as the heat transferred into the cold reservoir, which is given by
\[
Q_{out} = \text{Tr}[H_S^c \rho_{S}^{c,eq}] - \text{Tr}[H_S^h \rho_{S}^{h,eq}].
\]

**B. Efficiency**

With \( Q_{in} \) [Eq. (8)] and \( W_{out} = -(W_{com} + W_{exp}) \), the heat absorbed by the system from the hot reservoir and the net work performed by the system during one cycle, the efficiency of the weak coupling model is defined as
\[
\eta_{weak} = \frac{W_{out}}{Q_{in}}.
\]

Because \( W_{com} + Q_{in} + W_{exp} - Q_{out} = 0 \), we can rewrite \( \eta_{weak} \) as
\[
\eta_{weak} = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}}.
\]

Using the von Neumann entropy \( S(\rho) := -\text{Tr}[\rho \ln \rho] \) and the quantum relative entropy \( D(\rho||\sigma) := \text{Tr}[\rho \ln \rho - \rho \ln \sigma] \), the heat transferred between the system and the reservoirs can be expressed as follows (see Appendix A):
\[
\beta_h Q_{in} = \Delta S - D(\rho_{S}^{h}||\rho_{S}^{h,eq}),
\]
\[
\beta_c Q_{out} = \Delta S + D(\rho_{S}^{c}||\rho_{S}^{c,eq}),
\]
where \( \Delta S = S(\rho_{S}^{h,eq}) - S(\rho_{S}^{h}) = S(\rho_{S}^{c}) - S(\rho_{S}^{c,eq}) \). Consequently, we can express \( \eta_{weak} \) as
\[
\eta_{weak} = 1 - \frac{\beta_h \Delta S + D(\rho_{S}^{h}||\rho_{S}^{h,eq})}{\beta_c \Delta S - D(\rho_{S}^{c}||\rho_{S}^{c,eq})}.
\]

Because the quantum relative entropy is non-negative, the following inequality holds:
\[
\eta_{weak} \leq 1 - \frac{\beta_h}{\beta_c} = \eta_C.
\]

Hence, the efficiency of the weak coupling model does not exceed \( \eta_C \), the efficiency limit of classical heat engines.

**III. STRONG COUPLING MODEL**

In the weak coupling model, we assume that the interaction Hamiltonians are negligible. However, in quantum systems, this assumption is often unrealistic because the surface area of such systems is not insignificant compared with their volume [31, 32]. In this section, we develop a quantum Otto cycle model without applying the weak coupling assumption. We call this the “strong coupling model” to distinguish it from the weak coupling model.

The strong coupling model consists of the following six processes: (A) adiabatic compression, (B-1) hot isochoric thermalization, (B-2) decoupling from the hot reservoir, (C) adiabatic expansion, (D-1) cold isochoric thermalization, and (D-2) decoupling from the cold reservoir. Although there are some differences in details, this division is similar to the model proposed in Ref. [34].

**A. Process**

**Process A: adiabatic compression**—This process is equivalent to Process A in the weak coupling model. There is no interaction between the system and the reservoirs, so no heat flows into the system. The state of the system changes from \( \rho_{S}^{h,eq} \) to \( \rho_{S}^{h} \), and the work...
With the von Neumann entropy and the quantum relative entropy, we can rewrite $Q_{th}^h$ as follows: (see Appendix A)

$$\beta_h Q_{th}^h = S(\rho_S^h) - S(\tilde{\rho}_S^h) - \{D(\rho_{SB}^h|\rho_S^h \otimes \rho_B^h) + D(\rho_{SB}^h||\rho_{SB}^h)\}. \quad (21)$$

This expression is useful for the calculation of the efficiency.

**Process B-2: decoupling from hot reservoir**—We next conduct an operation to detach the system from the hot reservoir. As in Ref. [38], we consider the situation where the decoupling and the thermalization proceed simultaneously, i.e., the decoupling operation is sufficiently slow and the state of the compound system $S + B_h$ is the Gibbs state throughout this process. The initial state is $\rho_{SB}^h$ and the final state is $\rho_S^{h,eq} \otimes \rho_B^h$. Here, we impose a restriction that this process can be realized under the Schrödinger equation. From this restriction, the time evolution of the compound system is unitary and the following equality holds:

$$S(\rho_S^{h,eq} \otimes \rho_B^h) = S(\rho_{SB}^h). \quad (22)$$

Hereafter, we refer to this restriction as “isentropic restriction.”

We introduce the definition of the heat proposed in Ref. [38]. Using this definition, the heat transferred into the system can be calculated as follows (see Appendix B):

$$\beta_h Q_d^h = S(\rho_S^{h,eq}) - S(\tilde{\rho}_S^h) + D(\rho_{SB}^h|\tilde{\rho}_S^h \otimes \rho_B^h), \quad (23)$$

where $\tilde{\rho}_B^h = Tr[S|\rho_{SB}^h]$. We define the work performed on the system as the difference between the change in internal energy of the system and $Q_d^h$, which is given by

$$W_d^h = Tr[H_S^h(\rho_S^{h,eq} - \tilde{\rho}_S^h)] - Tr[H_{SB}^h(\rho_{SB}^h) - Q_d^h]. \quad (24)$$

**Process C: adiabatic expansion**—This process is completely equivalent to Process C of the weak coupling model. The state of the system changes from $\rho_S^{h,eq}$ to $\rho_S^h$ without interaction with the reservoirs. No heat flows into the system and the work performed on the system is

$$W_{exp} = Tr[H_S^h(\rho_S^{h,eq}) - Tr[H_{SB}^h(\rho_{SB}^h)]]$$

**Process D-1: cold isochoric thermalization**—In this process, the Hamiltonian of the system is constant at $H_S^c$. First, the system is coupled to the cold reservoir, and we assume $Tr[H_{SB}^c(\rho_S^c \otimes \rho_B^c)|0$ as in Process B-1. $H_{SB}^c$ corresponds to the interaction between the system and the cold reservoir, and it is also constant in this process. We stress that $H_{SB}^c$ is not necessarily negligible. After a sufficiently long time, the state of the compound system $S + B_c$ transitions to the Gibbs state at inverse temperature $\beta_c$. The final state is

$$\rho_{SB}^c = \frac{e^{-\beta_c(H_S^c + H_B^c + H_{SB}^c)}}{Z_{SB}^c}, \quad (25)$$
where \(Z_{SB}^c = \text{Tr}[e^{-\beta_c(\tilde{H}_S^c + \tilde{H}_B^c + \tilde{H}_{SB}^c)}]\).

Similar to Process B-1, the work performed on the system is 0, so the change in internal energy of the system can be regarded as \(Q_{\text{th}}^c\), the heat transferred into the system during this process. \(Q_{\text{th}}^c\) is given by

\[
Q_{\text{th}}^c = \text{Tr}[(\tilde{H}_S^c \otimes I_B)\tilde{\rho}_{SB}^c] - \text{Tr}[\tilde{H}_B^c\tilde{\rho}_S^c] + \text{Tr}[\tilde{H}_{SB}^c\rho_{SB}^c],
\]

where \(\tilde{\rho}_S^c = \text{Tr}_B[\rho_{SB}^c]\). We note that \(Q_{\text{th}}^c\) is calculated with the flow into the system in a positive direction, although the positive energy actually transfers from the system to the cold reservoir. Using \(Q_{\text{th}}^c\), we can write the work performed on the system as

\[
W_{\text{th}}^c = \text{Tr}[\tilde{H}_S^c(\tilde{\rho}_S^c - \tilde{\rho}_S^c)] + \text{Tr}[\tilde{H}_{SB}^c\rho_{SB}^c] - Q_{\text{th}}^c = 0. \tag{27}
\]

Similar to \(Q_{\text{th}}^h\), we get another expression for \(Q_{\text{th}}^c\) as follows:

\[
\beta_c Q_{\text{th}}^c = S(\tilde{\rho}_S^c) - S(\tilde{\rho}_S^c) - \{D(\tilde{\rho}_{SB}^c||\tilde{\rho}_S^c \otimes \rho_B^c) + D(\tilde{\rho}_S^c \otimes \rho_B^c||\tilde{\rho}_{SB}^c)\}. \tag{28}
\]

**Process D-2: decoupling from cold reservoir**—Similar to Process B-2, in this process, we decouple the system from the cold reservoir sufficiently slowly, and the thermalization proceeds simultaneously. The compound system \(S + B_c\) is the Gibbs state at inverse temperature \(\beta_c\) throughout this process. The final state is \(\tilde{\rho}_{SB}^{c,\text{eq}} \otimes \rho_B^c\). Here, we impose the isentropic restriction as in Process B-2, i.e., this process can be realized under the Schrödinger equation and the following equality holds:

\[
S(\rho_{SB}^{c,\text{eq}} \otimes \rho_B^c) = S(\tilde{\rho}_{SB}^c). \tag{29}
\]

By adopting the same definition of heat as in Process B-2, the heat the system absorbs from the cold reservoir can be calculated as follows:

\[
\beta_c Q_{\text{d}}^c = S(\tilde{\rho}_S^c) - S(\tilde{\rho}_S^c) + D(\tilde{\rho}_{SB}^c||\tilde{\rho}_S^c \otimes \tilde{\rho}_B^c), \tag{30}
\]

where \(\tilde{\rho}_B^c = \text{Tr}_S[\rho_{SB}^c]\). The work performed on the system is defined as the difference between the change in internal energy of the system and \(Q_{\text{d}}^c\):

\[
W_{\text{d}}^c = \text{Tr}[\tilde{H}_S^c(\tilde{\rho}_S^c - \tilde{\rho}_S^c)] - \text{Tr}[\tilde{H}_{SB}^c\rho_{SB}^c] - Q_{\text{d}}^c. \tag{31}
\]

### B. Efficiency

Similar to the weak coupling model, we define the efficiency of the strong coupling model as

\[
\eta_{\text{str}} = \frac{W'_{\text{out}}}{Q_{\text{in}}'}, \tag{32}
\]

where \(Q_{\text{in}}'\) is the sum of the heat the system absorbs from the hot reservoir in Process B-1 and Process B-2, and \(W_{\text{out}}'\) is the net work the system performs during one cycle. Hence, \(W_{\text{out}}' = -(W_{\text{com}} + W_{\text{th}}^h + W_{\text{exp}} + W_{\text{th}}^c + W_{\text{d}})\) and \(Q_{\text{in}}' = Q_{\text{th}}^h + Q_{\text{th}}^c + Q_{\text{d}}^c\). Additionally, \(Q_{\text{out}}' = -(Q_{\text{th}}^c + Q_{\text{d}}^c)\) denotes the heat transferred from the system to the cold reservoir. We can derive

\[
W_{\text{out}}' = Q_{\text{th}}^h + Q_{\text{th}}^c + Q_{\text{d}}^c = Q_{\text{in}}' - Q_{\text{out}}'. \tag{33}
\]

and we can rewrite \(\eta_{\text{str}}\) as

\[
\eta_{\text{str}} = \frac{Q_{\text{in}}' - Q_{\text{out}}'}{Q_{\text{in}}'} = 1 - \frac{Q_{\text{out}}'}{Q_{\text{in}}'}. \tag{34}
\]

Moreover, from Eqs. (21) and (23), we can calculate \(Q_{\text{in}}'\) as follows:

\[
\beta_h Q_{\text{in}}^h = S(\rho_{SB}^h) - S(\rho_{SB}^h) - \{D(\rho_{SB}^h||\rho_{SB}^h - D(\rho_{SB}^h||\rho_{SB}^h - D(\rho_{SB}^h||\rho_{SB}^h)\}
\]

\[
= S(\rho_{SB}^h) - S(\rho_{SB}^h) - \{D(\rho_{SB}^h||\rho_{SB}^h)\}. \tag{35}
\]

Here, we used the following relation:

\[
D(\rho_{SB}^h||\rho_{SB}^h) = D(\rho_{SB}^h||\rho_{SB}^h) = \{-S(\rho_{SB}^h) + S(\rho_{SB}^h) - \text{Tr}[\rho_{SB}^h \ln \rho_{SB}^h]) - \{-S(\rho_{SB}^h) + S(\rho_{SB}^h) - \text{Tr}[\rho_{SB}^h \ln \rho_{SB}^h]\}
\]

\[
= \text{Tr}[\rho_{SB}^h \ln \rho_{SB}^h] - \text{Tr}[\rho_{SB}^h \ln \rho_{SB}^h]\]

\[
= D(\rho_{SB}^h||\rho_{SB}^h). \tag{36}
\]

Similarly, from Eqs. (28) and (30), we get

\[
\beta_c Q_{\text{out}}^c = S(\tilde{\rho}_S^c) - S(\tilde{\rho}_S^c) + \{D(\tilde{\rho}_B^c||\tilde{\rho}_B^c) + D(\tilde{\rho}_S^c \otimes \rho_B^c||\tilde{\rho}_{SB}^c)\}. \tag{37}
\]

Hence, using \(\Delta S = S(\tilde{\rho}_S^{h,\text{eq}}) - S(\tilde{\rho}_S^c) - S(\tilde{\rho}_S^{c,\text{eq}})\), \(\eta_{\text{str}}\) can be expressed as

\[
\eta_{\text{str}} = 1 - \frac{\beta_h \Delta S + \{D(\tilde{\rho}_B^c||\tilde{\rho}_B^c) + D(\tilde{\rho}_S^c \otimes \rho_B^c||\tilde{\rho}_{SB}^c)\}}{\beta_c \Delta S - \{D(\tilde{\rho}_B^c||\tilde{\rho}_B^c) + D(\tilde{\rho}_S^c \otimes \rho_B^c||\tilde{\rho}_{SB}^c)\}}. \tag{38}
\]
This expression resembles $\eta_{\text{weak}}$ in Eq. (16). Equation (38) is the main result of this paper. From the non-negativity of the quantum relative entropy, we get

$$\eta_{\text{str}} \leq 1 - \frac{\beta_h}{\beta_c} = \eta_c,$$  

(39)
i.e., similar to the weak coupling model, the efficiency of our strong coupling model does not exceed the classical limit $\eta_c$.

C. Weak coupling limit

Here, we demonstrate that our strong coupling model agrees with the weak coupling model in the limit of weak interaction. When we add a condition that the interaction Hamiltonians are negligible in the strong coupling model, the final state of Process B-1 is approximated as follows:

$$\rho_{SB}^h = \frac{e^{-\beta_h(H_S^h + H_B^h)}Z_{SB}^h}{Z_S^h Z_B^h} = \frac{e^{-\beta_h H_S^h}}{Z_S^h} \otimes \frac{e^{-\beta_h H_B^h}}{Z_B^h} = \rho_{S}^{\text{eq}} \otimes \rho_B^{\text{eq}}.$$  

(40)

Consequently, $\rho_S^h = \text{Tr}_B[\rho_{SB}^h] = \rho_{S}^{\text{eq}}$ and $\rho_B^h = \text{Tr}_S[\rho_{SB}^h] = \rho_B^{\text{eq}}$ hold. Therefore, we can rewrite the heat exchanged between the system and the hot reservoir as follows:

$$Q_{in}^h = \text{Tr}[H_B^h(\rho_B^h - \rho_{SB}^h)] + \text{Tr}[H_{SB}^h \rho_{SB}^h] \simeq \text{Tr}[H_B^h(\rho_{S}^{\text{eq}} - \rho_{SB}^h)],$$  

(41)

$$\beta_h Q_{in}^h = S(\rho_{S}^{\text{eq}}) - S(\rho_{SB}^h) + D(\rho_{SB}^h || \rho_{SB}^h)$$

$$\simeq S(\rho_{S}^{\text{eq}}) - S(\rho_{SB}^{\text{eq}}) + D(\rho_{SB}^{\text{eq}} || \rho_B^{\text{eq}}).$$  

(42)

From Eqs. (41) and (42), we get $Q_{in}^h = Q_{in}^h + Q_{out}^h \simeq \text{Tr}[H_B^h(\rho_{S}^{\text{eq}} - \rho_{SB}^h)] = Q_{in}^h$. By some similar calculations, we can show $Q_{out}^h \simeq Q_{out}$ and $W_{in}^h \simeq W_{out}$. These relations mean the cycle of the strong coupling model agrees with that of the weak coupling model in the limit of weak interaction. Furthermore, the following calculation shows that $\eta_{\text{str}}$ in Eq. (38) agrees with $\eta_{\text{weak}}$ in the same limit:

$$\eta_{\text{str}} = 1 - \frac{\beta_h}{\beta_c} \Delta S + \frac{D(\rho_{SB}^h || \rho_{SB}^h)}{\beta_c} = 1 - \frac{\beta_h}{\beta_c} \Delta S + \frac{D(\rho_{SB}^{\text{eq}} || \rho_B^{\text{eq}})}{\beta_c} = \eta_{\text{weak}}.$$  

(43)

These discussions support the consistency of our strong coupling model with the weak coupling model. We can consider our model to be a valid extension of the existing weak coupling model.

D. Comparison of efficiencies

We now compare the efficiency of the two models discussed previously. From Eqs. (16) and (38), we derive the sufficient condition for $\eta_{\text{str}} \leq \eta_{\text{weak}}$. First, from the non-negativity of the quantum relative entropy, we get $D(\rho_{SB}^h || \rho_B^h) + D(\rho_{SB}^{\text{eq}} || \rho_B^{\text{eq}}) \geq D(\rho_{SB}^{\text{eq}} || \rho_{SB}^h)$. By the following calculation, we examine the magnitude relation between $D(\rho_{SB}^{\text{eq}} || \rho_B^h)$ and $D(\rho_{SB}^h || \rho_B^{\text{eq}})$:

$$D(\rho_{SB}^h || \rho_B^h) - D(\rho_{SB}^{\text{eq}} || \rho_B^{\text{eq}})$$

$$= -S(\rho_{SB}^h) - S(\rho_{SB}^{\text{eq}}) + \text{Tr}[\rho_{SB}^h \ln Z_{SB}^h]$$

$$= \text{Tr}[\rho_{SB}^h \ln e^{-\beta_h(H_S^h + H_B^h + H_{SB}^h)}] + \text{Tr}[\rho_{SB}^h \ln e^{-\beta_h H_{SB}^h}]$$

$$= \beta_h \left\{ F(\rho_{SB}^{\text{eq}} \otimes \rho_B^h) - F(\rho_{SB}^h) \right\}.$$  

(44)

Here, we introduced the free energy $F(\rho) = -\frac{1}{\beta} \ln Z$, defined for an arbitrary Gibbs state. $\beta$ is the inverse temperature and $Z$ is the partition function. Hence, if $F(\rho_{SB}^{\text{eq}} \otimes \rho_B^h) \geq F(\rho_{SB}^h)$ holds, we get $D(\rho_{SB}^h || \rho_B^h) + D(\rho_{SB}^{\text{eq}} || \rho_B^{\text{eq}}) \geq D(\rho_{SB}^{\text{eq}} || \rho_{SB}^h)$. Similarly, we also get $D(\rho_B^h || \rho_B^{\text{eq}}) + D(\rho_{SB}^{\text{eq}} || \rho_{SB}^h) \geq D(\rho_{SB}^{\text{eq}} || \rho_{SB}^h)$ under the condition that $F(\rho_{SB}^{\text{eq}} \otimes \rho_B^h) \geq F(\rho_{SB}^h)$. From the two inequalities, we can derive the following relation:

$$\eta_{\text{str}} = 1 - \frac{\beta_h}{\beta_c} \Delta S + \frac{D(\rho_{SB}^{\text{eq}} || \rho_B^h)}{\beta_c} = \eta_{\text{weak}}.$$  

(45)
Eventually, the fact that both $F(\rho_{SB}^{h,eq} \otimes \rho_B^h) \geq F(\rho_{SB}^h)$ and $F(\rho_{SB}^{c,eq} \otimes \rho_B^c) \geq F(\rho_{SB}^c)$ are satisfied is the sufficient condition for $\eta_{str} \leq \eta_{weak}$. Furthermore, from the restriction that the entropy of the compound system is constant throughout the decoupling processes, the change in free energy is equal to the change in internal energy, i.e.,

\begin{equation}
F(\rho_{SB}^{h,eq} \otimes \rho_B^h) - F(\rho_{SB}^h) = \text{Tr}[H_{SB}^h(\rho_{SB}^{h,eq} - \rho_{SB}^h)] + \text{Tr}[H_B^h(\rho_B^h - \rho_B^h)] - \text{Tr}[H_{SB}^h\rho_{SB}^h], \tag{46}
\end{equation}

\begin{equation}
F(\rho_{SB}^{c,eq} \otimes \rho_B^c) - F(\rho_{SB}^c) = \text{Tr}[H_{SB}^c(\rho_{SB}^{c,eq} - \rho_{SB}^c)] + \text{Tr}[H_B^c(\rho_B^c - \rho_B^c)] - \text{Tr}[H_{SB}^c\rho_{SB}^c]. \tag{47}
\end{equation}

Therefore, using $\Delta E_d^h$ and $\Delta E_d^c$ to denote the right sides of Eqs. (46) and (47), the fact that both $\Delta E_d^h \geq 0$ and $\Delta E_d^c \geq 0$ are satisfied also expresses the sufficient condition for $\eta_{str} \leq \eta_{weak}$. We can interpret this condition that if there are positive costs on the two decoupling processes, the strong coupling model has a lower efficiency than the weak coupling model.

### IV. EXAMPLE

In this section, we apply the strong coupling model to a simple two-level system and compute its efficiency numerically. We represent the state of the reservoirs by the general density matrices and do not specify the details of the reservoirs. We introduce a parameter $\theta$ to control the strength of the interaction and examine the relationship between $\theta$ and $\eta_{str}$. In addition, we analytically calculate the efficiency of the weak coupling model and compare it with $\eta_{str}$.

We consider a two-level system $S$ using the computational basis $|0\rangle = (1,0)^T$, $|1\rangle = (0,1)^T$ and the orthonormal basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle)^T$, we set $H_S$ and $H_B^h$ as

\begin{equation}
H_S = E_g^c|0\rangle\langle 0| + E_g^h|1\rangle\langle 1|, \quad H_B^h = E_g^h|+\rangle\langle +| + E_g^h|-\rangle\langle -|. \tag{48}
\end{equation}

$\{E_g^c, E_g^h\}$ and $\{E_g^h, E_g^h\}$ are the energy eigenvalues of $H_S$ and $H_B^h$, respectively. Similarly, the eigenvalue decompositions of $H_B^h$ and $H_B^h$ can be written as follows:

\begin{equation}
H_B^h = \sum_{i=1}^{d_c} \epsilon_i^c|\psi_i\rangle\langle \psi_i|, \quad H_B^c = \sum_{i=1}^{d_h} \epsilon_i^h|\phi_i\rangle\langle \phi_i|. \tag{49}
\end{equation}

We choose the computational basis as the eigenvectors and set $d_c = d_h = 4$. Here, because $\rho_{SB}^h$ is diagonalized by $\{|\phi_i\rangle\}_i$, the following relation holds for arbitrary $\alpha_{ij} \in \mathbb{R}$:

\begin{equation}
\text{Tr}\left[\rho_{SB}^h \left( \sum_{i,j \neq j} \alpha_{ij}|\phi_i\rangle\langle \phi_j| \right) \right] = 0. \tag{50}
\end{equation}

Thus, by generating $\{\alpha_{ij}\}_{i,j}$ randomly, we construct $H_{SB}^h$ as follows to satisfy the restriction $\text{Tr}[H_{SB}^h(\rho_{SB}^{h,eq} \otimes \rho_B^h)] = 0$:

\begin{equation}
H_{SB}^h = I_S \otimes \left( \sum_{i,j \neq j} \alpha_{ij}|\phi_i\rangle\langle \phi_j| \right) \theta, \tag{51}
\end{equation}

where $\alpha_{ij} = \alpha_{ji}$ and the parameter $\theta \in \mathbb{R}$ controls the strength of the interaction. Then, we examined the relationship between $\theta$ and $\eta_{str}$.

We construct $H_{SB}^c$ in the same way.

Using Eq. (38), we can calculate $\eta_{str}$ numerically, while the efficiency of the weak coupling model can be calculated analytically (see Appendix C). Figure 3 shows $\eta_{str}$ as a function of $\theta$, together with $\eta_{weak}$. The blue dashed line represents $\eta_{weak} = 0.5$, the efficiency of the weak coupling model. The orange line shows the change of the efficiency of the strong coupling model with a change in $\theta$. For small $\theta$, the efficiencies of both models are close. As $\theta$ increases, $\eta_{str}$ decreases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Relationship between interaction strength $\theta$ and efficiency for the two quantum Otto cycle models. The parameters are set as follows: $\beta_c = 2.0, \beta_h = 0.5, E_g^c = 0.5, E_g^h = 1.5, E_f^c = 2E_g^c, E_f^h = 2E_g^h, \{\epsilon_i^c\}_i \in \{0.5, 1.5, ... , 15.5\}, \{\epsilon_i^h\}_i \in \{2.0, 3.0, ... , 17.0\}$. $\theta$ is varied from 0.01 to 0.1. With $\theta = 0.1$, we generated 5000 interaction Hamiltonians and used the one that gave the smallest change in entropy during the decoupling processes. The blue dashed line represents $\eta_{weak} = 0.5$, the efficiency of the weak coupling model. The orange line shows the change of efficiency of the strong coupling model with a change in $\theta$. For small $\theta$, the efficiencies of both models are close. As $\theta$ increases, $\eta_{str}$ decreases.
\end{figure}
Finally, we perform an additional numerical experiment that does not require the isentropic restriction made in the analytical calculation. In the preceding numerical experiment, we generated a set of $H_{SB}^h$ and $H_{SB}^c$ repeatedly and selected the set that gives the value of $\Delta S'$ described in Eq. (52) closest to 0. This operation is required because we make the isentropic restriction that the decoupling processes can be realized under the Schrödinger equation and isentropically. This restriction may not be satisfied in general; however, even though the isentropic restriction does not hold, we here demonstrate that the efficiency of the strong coupling model calculated by Eq. (38) behaves similarly to that with the isentropic restriction. In this numerical experiment, we randomly generate $H_{SB}^h$ and $H_{SB}^c$ and calculate the efficiency by using them. In contrast to the previous numerical experiment, the generated interaction Hamiltonians do not necessarily satisfy the isentropic restriction, since we do not pick up the one that is close to 0 from many randomly generated Hamiltonians. We conduct this operation repeatedly and we compare each of the efficiencies with that obtained under the isentropic restriction. Figure 4 shows the efficiencies as functions of $\theta$, where the red dashed line and solid lines represent the efficiencies with and without the isentropic restriction, respectively. Each of the solid lines corresponds to the efficiency calculated by a different random realization of $H_{SB}^h$ and $H_{SB}^c$. Although all the solid lines are below the red dashed line for all values of $\theta$, the general tendency of efficiencies agree regardless of whether the isentropic restriction is satisfied: they are very close to $\eta_{\text{weak}} = 0.5$ when $\theta$ is small, and as $\theta$ increases, they decrease. From this result, even under the situation that the isentropic restriction does not hold, it is considered that our expression of the efficiency in Eq. (38) retains a certain validity.

V. CONCLUSION

In this paper, we developed a quantum Otto cycle model, taking into account the interaction between the system and the reservoirs, which is neglected in the existing weak coupling model. We confirmed the fact that our strong coupling model agrees with the weak coupling model in the limit of weak interaction. We expressed the efficiencies of both models in closed form and we derived the sufficient condition for $\eta_{\text{str}} \leq \eta_{\text{weak}}$. This condition is satisfied when the decoupling processes, which we introduced to the strong coupling model, have positive costs. Using numeric calculations, we demonstrated that $\eta_{\text{str}} \simeq \eta_{\text{weak}}$ in the range of weak interaction, and that when the interaction is not negligible, $\eta_{\text{str}}$ is lower than $\eta_{\text{weak}}$ in the prepared system. These results show that our strong coupling model can be regarded as a reasonable extension of the existing weak coupling model. We emphasize that our model does not specify the details of the system and the reservoirs, so it can be applied generally with the assumption that the decoupling processes can be realized under the Schrödinger equation. We hope that the present work will contribute to the development of a strong-coupling thermodynamics theory. Additionally, our study is expected to enhance the understanding of quantum correlation because incorporating the interaction terms is synonymous with considering the quantum correlation between the system and the reservoirs. Thus, we anticipate that this study contributes to several quantum thermodynamics fields, not just quantum heat engines.

ACKNOWLEDGMENTS

This work was supported by KAKENHI Grant Numbers JP19K12153 and JP22H03659.

Appendix A: Heat transferred in thermalization process

The heat transferred in the thermalization processes can be expressed by the von Neumann entropy and the quantum relative entropy. Here we derive the expressions in both the weak coupling model and the strong coupling model.
In the weak coupling model, the quantum relative entropy $D(\rho_S^h||\rho_S^{eq})$ can be calculated as follows:

$$D(\rho_S^h||\rho_S^{eq}) = \text{Tr}[\rho_S^h \ln \rho_S^h] - \text{Tr}[\rho_S^h \ln \rho_S^{eq}] = S(\rho_S^{eq}) - S(\rho_S^h) = -S(H_\rho^h - \ln Z_S^h)$$

This equality shows that the change in the von Neumann entropy of the system in the thermalization process can be calculated as follows:

$$\Delta S = S(H_\rho^h - \ln Z_S^h)$$

From these two equalities, we get Eq. (14). Furthermore, the following relation holds:

$$\Delta S = \beta_h Q_{in} + D(\rho_S^h||\rho_S^{eq}).$$

Next, we describe the heat transferred in the thermalization processes of the strong coupling model. $D(\rho_S^h||\rho_S^h)\otimes\rho_B^h$ and $D(\rho_S^h||\rho_S^h)\otimes\rho_B^h$ can be calculated as follows:

$$D(\rho_S^h||\rho_S^h)\otimes\rho_B^h = \text{Tr}[\rho_S^h \ln \rho_S^h] - \text{Tr}[\rho_S^h \ln \rho_S^h] = S(\rho_S^h) - S(\rho_S^h) = -S(H_\rho^h - \ln Z_S^h)$$

From these two equalities, we get Eq. (21) holds. By the same calculations for the low temperature side, we can derive Eq. (28).

**Appendix B: Heat transferred in decoupling process**

Here, we introduce the definition of heat proposed in Ref. [38] and calculate the heat transferred in the decoupling processes of the strong coupling model. We discuss Process B-2 and calculate only $Q_{in}$ here, but $Q_{out}$ can be calculated in the same way.

$\rho$ represents the state of the compound system $S + B_h$ in Process B-2. $\rho_S$ and $\rho_B$ are the reduced states: $\rho_S = \text{Tr}_B[\rho]$ and $\rho_B = \text{Tr}_S[\rho]$, respectively. We define the heat transferred from the reservoir to the system during an infinitesimal time $dt$ as $dQ = -i\text{Tr}[[H_S^\text{eff}, H_{\text{tot}}]C] dt$. Here, $H_S^\text{eff}$ is the effective Hamiltonian of the system, defined as $H_S^\text{eff} = H_{\text{tot}} - H_B^\text{eff}$, where $H_B^\text{eff} = -\frac{1}{\beta_B} \ln \rho_B$. $H_{\text{tot}}$ is the Hamiltonian of the compound system, including the interaction. $C = \rho - \rho_S \otimes \rho_B$ corresponds to the quantum coherence of $\rho$. Using these definitions, we can rewrite $dQ$ as follows:

$$dQ = -i\text{Tr}[[H_S^\text{eff}, H_{\text{tot}}]C] dt = -i\text{Tr}[[-H_B^\text{eff}, H_{\text{tot}}]C] dt = -\frac{1}{\beta_h} \text{Tr}[\{I_S \otimes \ln \rho_B, H_{\text{tot}}\}C] dt.$$
On the other hand, the infinitesimal change in von Neumann entropy of $\rho_S$ can be calculated as follows:

\[
dS_S = -d\text{Tr}[\rho_S \ln \rho_S] = -d\text{Tr}[\rho \ln (\rho_S \otimes I_B)] = -\text{Tr}[(\ln \rho_S \otimes I_B)\, d\rho] = i\text{Tr}[[H_{\text{tot}}, \rho] \ln (\rho_S \otimes I_B)] dt = i\text{Tr}[(\ln \rho_S \otimes I_B, H_{\text{tot}})\, \rho] dt = i\text{Tr}[(\ln \rho_S \otimes I_B, H_{\text{tot}})\, C] dt.
\]

We used the von Neumann equation $d\rho = -i[H_{\text{tot}}, \rho] dt$ for the fourth line above and $[\ln \rho_S \otimes I_B, \rho_S \otimes \rho_B] = 0$ for the last line. Therefore, we can derive the following relation:

\[
dS_S - \beta_h dQ = i\text{Tr}[(\ln (\rho_S \otimes \rho_B), H_{\text{tot}})\, C] dt
\]

By integrating both sides of Eq. (B3) from the initial state $\rho_{SB}^h$ to the final state $\rho_{SB}^{h,\text{eq}} \otimes \rho_B^h$, we get the following relation:

\[
S(\rho_{SB}^{h,\text{eq}}) - S(\rho_{SB}^h) - \beta_h Q_d^h = -\text{Tr}[(\rho_{SB}^{h,\text{eq}} \otimes \rho_B^h) \ln (\rho_{SB}^{h,\text{eq}} \otimes \rho_B^h)] + \text{Tr}[\rho_{SB}^h \ln (\rho_{SB}^{h,\text{eq}} \otimes \rho_B^h)]
= -\text{Tr}[\rho_{SB}^h \ln (\rho_{SB}^{h,\text{eq}})] + \text{Tr}[\rho_{SB}^h \ln (\rho_{SB}^{h,\text{eq}})]
= -D(\rho_{SB}^h || \rho_{SB}^{h,\text{eq}})
\beta_h Q_d^h = S(\rho_{SB}^{h,\text{eq}}) - S(\rho_{SB}^h) + D(\rho_{SB}^h || \rho_{SB}^{h,\text{eq}}).
\]

This is equal to Eq. (23). Note that we used the restriction $S(\rho_{SB}^{h,\text{eq}} \otimes \rho_B^h) = S(\rho_{SB}^h)$ for the second line in the relation above. By the same calculations, we can derive Eq. (30).

Appendix C: Analytical calculation of efficiency

Here, we detail the calculation of the efficiency of the weak coupling model prepared for the simulation. We signify the probability distributions of $\rho_{S,\text{eq}}^c$ and $\rho_{S,\text{eq}}^h$ as follows:

\[
p_g^c = \frac{e^{-\beta_c E_g^c}}{Z_S^c}, \quad p_e^c = \frac{e^{-\beta_c E_e^c}}{Z_S^c}, \quad p_g^h = \frac{e^{-\beta_h E_g^h}}{Z_S^h}, \quad p_e^h = \frac{e^{-\beta_h E_e^h}}{Z_S^h}.
\]

Using these values, we can calculate the von Neumann entropy and the quantum relative entropy as

\[
\Delta S - D(\rho_{S}^h || \rho_{S,\text{eq}}^h) = S(\rho_{S}^{h,\text{eq}}) - S(\rho_{S}^h) + \text{Tr}[\rho_{S}^h \ln (\rho_{S,\text{eq}}^h)]
= (p_g^h - p_g^c) \ln (p_g^h) + (p_e^c - p_e^h) \ln (p_e^h)
= (p_g^h - p_g^c)(-\beta_h E_g^h - \ln Z_S^h) + (p_e^c - p_e^h)(-\beta_h E_e^h - \ln Z_S^h)
= \beta_h \{ (p_g^h - p_g^c) E_g^h + (p_e^h - p_e^c) E_e^h \},
\]

\[
\Delta S + D(\rho_{S}^c || \rho_{S,\text{eq}}^c) = S(\rho_{S}^{c,\text{eq}}) - S(\rho_{S}^c) - \text{Tr}[\rho_{S}^c \ln (\rho_{S,\text{eq}}^c)]
= (p_g^c - p_g^h) \ln (p_g^c) + (p_e^c - p_e^h) \ln (p_e^c)
= (p_g^c - p_g^h)(-\beta_c E_g^c - \ln Z_S^c) + (p_e^c - p_e^h)(-\beta_c E_e^c - \ln Z_S^c)
= \beta_c \{ (p_g^h - p_g^c) E_g^c + (p_e^h - p_e^c) E_e^c \}.
\]

We used $p_g^c + p_e^c = p_g^h + p_e^h = 1$ for the last line of both calculations. Thus, from Eq. (16), we can rewrite $\eta_{\text{weak}}$ as

\[
\eta_{\text{weak}} = 1 - \frac{(p_g^h - p_g^c) E_g^c + (p_e^h - p_e^c) E_e^c}{(p_g^h - p_g^c) E_g^c + (p_e^h - p_e^c) E_e^c}.
\]
When we set $E_g^h = 2E_g^e$ and $E_c^h = 2E_c^e$, we get

$$\eta_{\text{weak}} = 1 - \frac{1}{2} = \frac{1}{2}. \quad (C5)$$
Rev. E 95, 032139 (2017).

[35] D. Newman, F. Mintert, and A. Nazir, Quantum limit to nonequilibrium heat-engine performance imposed by strong system-reservoir coupling, Phys. Rev. E 101, 052129 (2020).

[36] D. Gelbwaser-Klimovsky and A. Aspuru-Guzik, Strongly coupled quantum heat machines, J. Phys. Chem. Lett. 6, 3477 (2015).

[37] R. Gallego, A. Riera, and J. Eisert, Thermal machines beyond the weak coupling regime, New J. Phys. 16, 125009 (2014).

[38] Y. Xu, B. Chen, and J. Liu, Achieving the classical Carnot efficiency in a strongly coupled quantum heat engine, Phys. Rev. E 97, 022130 (2018).

[39] G. Katz and R. Kosloff, Quantum thermodynamics in strong coupling: Heat transport and refrigeration, Entropy 18, 186 (2016).

[40] U. Seifert, First and second law of thermodynamics at strong coupling, Phys. Rev. Lett. 116, 020601 (2016).

[41] M. Carrega, P. Solinas, M. Sassetti, and U. Weiss, Energy exchange in driven open quantum systems at strong coupling, Phys. Rev. Lett. 116, 240403 (2016).

[42] M. Esposito, K. Lindenberg, and C. Van den Broeck, Entropy production as correlation between system and reservoir, New J. Phys. 12, 013013 (2010).