Index growth of hypersurfaces with constant mean curvature

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Abstract. In this paper we give the precise index growth for the embedded hypersurfaces of revolution with constant mean curvature (cmc) 1 in $\mathbb{R}^n$ (Delaunay unduloids). When $n = 3$, using the asymptotics result of Korevaar, Kusner and Solomon, we derive an explicit asymptotic index growth rate for finite topology cmc 1 surfaces with properly embedded ends. Similar results are obtained for hypersurfaces with cmc bigger than 1 in hyperbolic space.

Résumé. Dans cet article, nous estimons de manière précise la croissance de l’indice des hypersurfaces de révolution plongées, de courbure moyenne constante (cmc) égale à 1, dans $\mathbb{R}^n$ (onduloides de Delaunay). Quand $n = 3$, utilisant le résultat de Korevaar, Kusner et Solomon, nous en déduisons une estimée de la croissance de l’indice des surfaces de topologie finie, de cmc 1 et dont les bouts sont proprement plongés. Nous obtenons des résultats similaires pour les hypersurfaces à cmc strictement plus grande que 1 dans l’espace hyperbolique.

1 Introduction

A complete cmc nonminimal surface without boundary in $\mathbb{R}^3$ has finite index if and only if it is compact [LR], [S]. If it is noncompact, the index is infinite, so it is natural to ask at what rate the index grows to infinity on an exhaustion of the surface by bounded regions. In this paper, we prove, under some natural geometric conditions, that certain complete non-compact cmc hypersurfaces have linear index growth. Let us give a typical statement:

Let $M \subset \mathbb{R}^3$ be a complete properly embedded finite-topology cmc-1 surface. The finitely many ends $E_j, j = 1 \ldots N$, of $M$ were shown by [KKS] to be asymptotic to Delaunay unduloids $D(\mu_j)$ with weight parameters $\mu_j > 0$. Let $T(\mu_j)$ denote the period of the Delaunay unduloid $D(\mu_j)$ and let $B(R)$ be the radius $R$ ball in $\mathbb{R}^3$ centered at the origin.

Theorem 1.1 With $M$ as above, the asymptotic growth of the index of $M$ is given by

$$\lim_{R \to \infty} \frac{\text{Ind}(M \cap B(R))}{R} = \sum_{j=1}^{N} \frac{2}{T(\mu_j)}.$$  

(1.1)

Using the fact that $2 \leq T(\mu_j) \leq \pi$, we can conclude from the preceding theorem that the index growth provides upper and lower bounds on the number of ends of the surface.

There are many known surfaces to which this theorem (or Theorem 5.1) applies. Complete finite-topology cmc-1 surfaces with asymptotically Delaunay ends have been constructed by N. Kapouleas [K], R. Mazzeo et al. [MP], and K. Grosse-Brauckmann et al. [GKS]. And there are other works in progress for constructing such surfaces (e.g. that of J. Dorfmeister,
The structure of such surfaces is well understood [KK], [KKS].

The rough idea of the proof is to decompose the surface into components, one which is a fixed compact part and the others which are compact pieces of ends and are close to parts of Delaunay unduloids, and then to apply Dirichlet-Neumann bracketing. We need to show that the indexes of these end pieces are close to the indexes of the actual Delaunay pieces, and then the heart of the proof becomes to carefully study the indexes of the Delaunay pieces (with both Dirichlet and Neumann boundary conditions).

**Remark.** Dirichlet–Neumann bracketing can be applied to other situations. We can for example prove quadratic or cubic index growth for certain infinite-topology cmc surfaces in [K] (Subsection 5.4).

In Section 2, we describe the framework of the paper. In Section 3 we recall the basic facts on Delaunay unduloids (in Euclidean and hyperbolic space) and we define some special domains on them. Section 4 is devoted to estimating the index of these special domains. The main results are stated in Subsection 5.3, and the other subsections of Section 5 contain technical results needed in the proofs.

## 2 Framework

We consider hypersurfaces $M^n$ with cmc $H$ in the simply connected $(n + 1)$-dimensional space form $\mathbb{M}^{n+1}$ with constant sectional curvature $c \in \{-1, 0\}$. We assume $H > |c|$. Such hypersurfaces are critical for a variational problem whose associated second order stability operator is

$$L := \Delta - nc - ||B||^2,$$

where $||B||$ is the norm of the second fundamental form of the immersion, and $\Delta$ is the (non-negative) Laplace–Beltrami operator for the induced metric on $M$. (When $\mathbb{M} = \mathbb{R}^3$, we have $L = \Delta + 2K - 4H^2$, where $K$ is the Gauss curvature.)

For $M$ compact, we define $\text{Ind}(M)$ as the index (number of negative eigenvalues) of the quadratic form $\int_M u Lu \, dv_M$ on some subspace

$$\{u \in H^1(M) \mid u|_{\Gamma} = 0\},$$

where $\Gamma \subset \partial M$ is a portion of the boundary of $M$ (this means that we consider the operator $L$ with Dirichlet boundary conditions on $\Gamma$ and with Neumann boundary conditions on $\partial M \setminus \Gamma$). The choice of $\Gamma$ will be clear from the context.

For $M$ non-compact, $\text{Ind}(M)$ is defined as the supremum of $\text{Ind}(\Omega)$ over all relatively compact subregions $\Omega \subset M$ (for a fixed choice of $\Gamma$).

**Remark:** We will not need to take into account that there are actually two different notions of index for cmc hypersurfaces (see [BB], [BdCE], [LiRo]). Indeed, for compact subsets these indexes differ by at most one, so their asymptotic properties are the same.

## 3 Delaunay unduloids

Here we describe Delaunay unduloids with nonzero cmc in Euclidean and hyperbolic space.
3.1 Delaunay unduloids in Euclidean space, with cmc \( H > 0 \). Consider a rotation hypersurface \( M \) in \( \mathbb{R}^{n+1} \) parametrized by

\[
\mathbb{R} \times S^{n-1} \ni (x, \omega) \mapsto F(x, \omega) = (x, f(x) \omega) .
\]

We assume \( f > 0 \) and \( f \) is defined on \((-\infty, \infty)\). We choose the unit normal vector as

\[
N(x, \omega) = (1 + f'^2(x))^{-1/2} (f'(x), -\omega) .
\]

Assume that \( f' \neq 0 \) and fix the normalized mean curvature to be \( H = 1 \). The profile curves of Delaunay unduloids are given by the differential equation

\[
\mu = \frac{f^{n-1}(x)}{(1 + f'^2(x))^{1/2}} - f^n(x) .
\]

where \( \mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}) \). The extreme values correspond to a chain of spherical beads of radii 1 (when \( \mu = 0 \)), and to a cylinder with radius \( \frac{a-1}{n} \) (when \( \mu = \frac{1}{n}(\frac{n-1}{n})^{n-1} \)).

Given \( \mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}) \), let \( a_\pm(\mu) \) be the two positive roots of the equation \( X^n - X^{n-1} + \mu = 0 \) with \( a_-(\mu) \leq a_+(\mu) \).

Let \( D(\mu) \) be the Delaunay unduloid with cmc 1 and weight parameter \( \mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}) \), whose profile curve \( f \) satisfies Equation (3.5). One can show that the function \( f \) is defined over \( \mathbb{R} \), pinched between the two positive values \( a_\pm(\mu) \),

\[
a_-(\mu) \leq f(x) \leq a_+(\mu),
\]

and \( T(\mu) \)-periodic, where \( T(\mu) \) is the distance between two consecutive values of \( x \) at which \( f \) achieves its least value \( a_-(\mu) \). (For a true cylinder, \( \mu = \frac{1}{n}(\frac{n-1}{n})^{n-1} \) and \( f(x) = a_-(\mu) = a_+(\mu) = \frac{a-1}{n} \) is constant. In that case, \( T(\frac{1}{n}(\frac{n-1}{n})^{n-1}) \) is the limiting value of \( T(\mu) \) as \( \mu \) increases up to \( \frac{1}{n}(\frac{n-1}{n})^{n-1} \).)

The stability operator of the \( n \)-dimensional Euclidean Delaunay unduloid \( D(\mu) \) is given by

\[
L = \Delta - V , \quad \text{where} \quad V = \|B\|^2 = n \left(1 + (n-1)\mu^2 f^{-2n}\right) .
\]

**Lemma 3.1** For \( n \geq 2 \) and for any \( x \in \mathbb{R} \) the function \( V \) in equation (3.7) satisfies \( V(x) f^2(x) \leq n^2 \).

**Proof.** We have already seen that the weight parameter \( \mu \) of \( D(\mu) \) satisfies \( 0 < \mu \leq \frac{1}{n}(\frac{n-1}{n})^{n-1} \). Consider the polynomial \( P(t) = t^n - t^{n-1} + \mu \), whose positive roots are the numbers \( a_\pm(\mu) \). The function \( P(t) \), considered on the domain \( \mathbb{R}_+ \), achieves its non-positive minimum \( \mu = \frac{1}{n}(\frac{n-1}{n})^{n-1} \) at \( t = \frac{n-1}{n} \). Since \( P(\mu^{1/(n-1)}) \) and \( P(1) \) are both positive, it follows that

\[
\mu^{1/(n-1)} \leq a_-(\mu) \leq \frac{n-1}{n} \leq a_+(\mu) \leq 1 .
\]

Consider the function \( Q(t) = nt^2(1 + (n-1)\mu^2 t^{-2n}) \), for \( t > 0 \). When \( t \) varies from 0 to \( \infty \), \( Q \) decreases from \( \infty \) to its minimum \( Q((n-1)^\frac{n}{2}\mu^{\frac{n}{n-1}}) \geq 0 \) and then increases to \( \infty \). It follows immediately that, for all \( x \in \mathbb{R} \),

\[
(V f^2)(x) \leq \max\{Q(a_-), Q(a_+)\} \leq \max\{Q(\mu^{1/(n-1)}), Q(1)\} .
\]

Using the fact that \( \mu \leq \frac{1}{n}(\frac{n-1}{n})^{n-1} \), it follows that \( V f^2 \leq n^2 \) on \( \mathbb{R} \) as claimed. \( \blacksquare \)
3.2 Special parts of Euclidean Delaunay unduloids $D(\mu)$. Without loss of generality, we may assume that the function $f$ defining the profile curve of $D(\mu)$ satisfies $f(0) = a_-(\mu)$. It follows easily that $f(T(\mu)) = a_-(\mu)$, $f(T(\mu)/2) = a_+(\mu)$ and that $f$ is symmetric with respect to the values $kT(\mu)/2$, $k \in \mathbb{Z}$.

Let the basic Dirichlet block for $D(\mu)$ be the compact domain

$$B(\mu) := F([0, \frac{T(\mu)}{2}] \times S^{n-1}) \quad \text{or} \quad F(\frac{T(\mu)}{2}, T(\mu)] \times S^{n-1}) ,$$

where $F$ is the parametrization (3.3), see Figure 1. We also introduce the pieces $B_\ell(\mu)$ obtained by gluing $\ell$ basic Dirichlet blocks,

$$B_\ell(\mu) := F([0, \ell \frac{T(\mu)}{2}] \times S^{n-1}) \quad \text{or} \quad F(\frac{T(\mu)}{2}, (\ell + 1) \frac{T(\mu)}{2}] \times S^{n-1}) .$$

Let $a$ be the function

$$a(x) = \langle N(x, \omega), (1, 0, ..., 0) \rangle = (1 + f'^2(x))^{-1/2} f'(x) ,$$

where $f$ is the profile curve of the Euclidean Delaunay unduloid $D(\mu)$.

**Lemma 3.2** The function $a$ satisfies $(\Delta - V)a = 0$ and vanishes precisely at the half-integer multiples of $T(\mu)$. Furthermore, $a'$ has exactly two zeroes $\zeta_1(\mu), \zeta_2(\mu)$ in the interval $[0, T(\mu)]$, with

$$0 < \zeta_1(\mu) < \frac{T(\mu)}{2} < \zeta_2(\mu) < T(\mu) .$$

(For a true cylinder, $a(x)$ is constant, so the values $\zeta_1(\frac{1}{n}(2\pi-1)n^{-1})$ and $\zeta_2(\frac{1}{n}(2\pi-1)n^{-1})$ must be determined by the limits of $\zeta_1(\mu)$ and $\zeta_2(\mu)$ as $\mu$ increases to $\frac{1}{n}(2\pi-1)n^{-1}$.)

**Proof.** The first assertion is classical: the scalar product of the unit normal vector of a cmc hypersurface with a Killing field is a solution of $Lu = 0$ (see [Ch], page 196, or the proof of Theorem 2.7 in [BGS]). The second assertion is obvious. The assertion on the zeroes of $a'$ follows from the fact that $(\Delta - V)a = 0$ reduces to a Sturm–Liouville equation, since the functions $a$ and $V$ depend on the variable $x$ only. \hfill \blacksquare

Let the basic Neumann block for $D(\mu)$ be the compact domain

$$C(\mu) := F([\zeta_1(\mu), T(\mu) + \zeta_1(\mu)] \times S^{n-1}) .$$

We also introduce the pieces $C_\ell(\mu)$ obtained by gluing $\ell$ basic Neumann blocks, see Figure 1,

$$C_\ell(\mu) := F([\zeta_1(\mu), \ell T(\mu) + \zeta_1(\mu)] \times S^{n-1}) .$$
3.3 Delaunay unduloids with cmc \( H > 1 \) in hyperbolic space. We choose the half-space model \( \{(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1} \mid y > 0\} \) for hyperbolic space \( \mathbb{H}^{n+1} \) (with the hyperbolic space metric), and we fix the geodesic \( \gamma(t) = (0, \ldots, 0, e^t) \).

The profile curve of a hyperbolic Delaunay unduloid is described, say in the vertical 2-dimensional plane \( \{(x_1, y)\} \), as a geodesic graph. The point \( m(t) \) on the profile curve is at geodesic distance \( \rho(t) \) from the point \( \gamma(t) \). Let \( \varphi(t) \) be the angle \( \langle \gamma(t), 0, m(t) \rangle \), see Figure 2. Then, \( \sinh \rho(t) = \tan \varphi(t) \).

With these notations, the profile curve is given by \( (e^t \sin \varphi(t), e^t \cos \varphi(t)) \), where \( \varphi \) satisfies the differential equation ([KKMS], Equation (6.3) page 34)

\[
\frac{\mu}{\cos \varphi} = \frac{(\tan \varphi)^{n-1}}{\cos \varphi \sqrt{1 + \varphi'^2}} - H(\tan \varphi)^n.
\]

Here, \( \mu > 0 \) is the weight parameter, and the (normalized) mean curvature \( H \) satisfies \( H > 1 \). (Note that the mean curvature is not normalized in [KKMS].) The hyperbolic Delaunay unduloids \( \mathcal{D}_H(\mu) \) are given by

\[
\mathbb{R} \times S^{n-1} \ni (t, \omega) \mapsto (e^t \sin \varphi(t) \omega, e^t \cos \varphi(t)) =: (f(t) \omega, g(t)) \in \mathbb{H}^{n+1}.
\]

As in the case of the Euclidean Delaunay unduloids, it can be shown that the function \( \varphi \) (or equivalently \( \rho \)) is pinched between two values \( 0 < \alpha_-(\mu) \leq \varphi(t) \leq \alpha_+(\mu) \) and periodic with period \( \tau(\mu) \). The Delaunay unduloids obtained in this way with \( \mu > 0 \) are embedded.

A unit normal vector to the hypersurface \( \mathcal{D}_H(\mu) \) is given (with the above notations) by

\[
N(t, \omega) = \frac{\cos \varphi}{\sqrt{1 + \varphi'^2}} (g' \omega, -f').
\]

The stability operator \( L \) is of the form

\[
L = \Delta - V , \quad \text{where} \quad V = -n + \|B\|^2.
\]

Note that \( V \) is periodic and hence bounded on \( \mathbb{R} \), as in the Euclidean case. There is a nice expression for the function \( V \) in the hyperbolic case:

**Lemma 3.3** With the notations as in Equations (3.14) and (3.16), we have, for the \( n \)-dimensional hyperbolic Delaunay unduloid \( \mathcal{D}_H(\mu) \), that

\[
V(\Phi(t, \omega)) = n(H^2 - 1) + n(n-1)\mu^2(\tan \varphi)^{-2n}.
\]
Proof. For \( n = 2 \) a proof is in [C]. The case \( n \geq 3 \) is similar, using computations in [Hs].

In order to estimate the index of certain pieces of \( \mathcal{D}_H(\mu) \), we need the following lemma.

**Lemma 3.4** With the notations as in Equations (3.16) and (3.13), there exists a constant \( c(n, H) \) depending only on \( n \) and \( H \) so that

\[
V \tan^2 \varphi \leq c(n, H)
\]
on \( \mathcal{D}_H(\mu) \) with weight parameter \( \mu \) and \( \text{cmc} \ H > 1 \).

**Proof.** The proof is left to the reader (use Equation (3.13), Lemma 3.3, and compute).

3.4 Special parts of hyperbolic Delaunay unduloids \( \mathcal{D}_H(\mu) \). Without loss of generality, we may assume \( \varphi \) satisfies \( \varphi(0) = \alpha_-(\mu) \). Thus \( \varphi(\tau(\mu)) = \alpha_-(\mu), \varphi(\tau(\mu)/2) = \alpha_+(\mu) \) and \( \varphi \) is symmetric with respect to the values \( k \tau(\mu)/2, k \in \mathbb{Z} \).

Analogous to the Euclidean case, we define the basic Dirichlet block \( \mathcal{B}(\mu) \) for \( \mathcal{D}_H(\mu) \), the glueing of \( \ell \) basic Dirichlet blocks \( \mathcal{B}_\ell(\mu) \), and the function \( a \) as:

\[
\mathcal{B}(\mu) := \Phi([0, \tau(\mu)/2] \times S^{n-1}) \text{ or } \Phi([\tau(\mu)/2, \tau(\mu)] \times S^{n-1}),
\]

\[
\mathcal{B}_\ell(\mu) := \Phi([0, \ell \tau(\mu)/2] \times S^{n-1}) \text{ or } \Phi([\ell \tau(\mu)/2, (\ell + 1) \tau(\mu)/2] \times S^{n-1}),
\]

\[
a(x) = \langle N(t, \omega), \mathcal{Y} \rangle = \frac{\varphi'(x)}{\cos \varphi(x) \sqrt{1 + \varphi'^2(x)}},
\]

where \( \Phi \) is the parametrization (3.14), and \( \varphi \) satisfies (3.13), and \( \mathcal{Y} \) is the Killing field corresponding to hyperbolic translation along the axis of the Delaunay unduloid. The following lemma is proved in the same way as Lemma 3.2:

**Lemma 3.5** The function \( a \) satisfies \( (\Delta - V)a = 0 \) and vanishes precisely at the half-integer multiples of \( \tau(\mu) \). Furthermore, \( a' \) has exactly two zeroes \( \zeta_1(\mu), \zeta_2(\mu) \) in the interval \( [0, \tau(\mu)] \), with

\[
0 < \zeta_1(\mu) < \frac{\tau(\mu)}{2} < \zeta_2(\mu) < \tau(\mu).
\]

(Again, the values \( \zeta_j(\mu) \) and \( \tau(\mu) \) for the true hyperbolic cylinder are determined as limiting values of the \( \zeta_j(\mu) \) and \( \tau(\mu) \) for noncylindrical hyperbolic Delaunay unduloids.)

Let the basic Neumann block \( \mathcal{C}(\mu) \) for \( \mathcal{D}_H(\mu) \) and the glueing of \( \ell \) basic Neumann blocks \( \mathcal{C}_\ell(\mu) \) be

\[
\mathcal{C}(\mu) := \Phi([\zeta_1(\mu), \tau(\mu) + \zeta_1(\mu)] \times S^{n-1}),
\]

\[
\mathcal{C}_\ell(\mu) := \Phi([\zeta_1(\mu), \ell \tau(\mu) + \zeta_1(\mu)] \times S^{n-1}).
\]

4 Index estimates for pieces of Delaunay unduloids

4.1 Preliminary results. We state the following lemma for later reference.

**Lemma 4.1** Let \( A, B, V \) be smooth bounded functions on \( \mathbb{R} \). Assume that \( A, B \) are bounded from below by a positive constant. Let \( P \) be the manifold \([a, b] \times S^{n-1}\) equipped with the metric

\[
g := A(x)dx^2 + B^2(x)gs,
\]

where \( gs \) is the canonical metric on \( S^{n-1} \). We are interested in the eigenvalue problem \((\Delta - V)Y(x, \omega) = \lambda Y(x, \omega)\), with Dirichlet or Neumann conditions on \( \partial P = \{a\} \times S^{n-1} \cup \{b\} \times S^{n-1} \). Let \( \Lambda_k = k(k + n - 2), k \geq 0 \), denote the eigenvalues

\[...\]
of the Laplacian on $S^{n-1}$ and let $m(\Lambda_k)$ denote the multiplicity of $\Lambda_k$ (this is a polynomial in $k$, of degree $n-2$). Let $L := \Delta_{g} - V$ and define the operators $L_k, k \geq 0$, by

$$L_k u = -\frac{d}{dx} \left( A^{-1} B^{n-1} \frac{du}{dx} \right) + AB^{n-3} \left( \Lambda_k - B^2 V \right) u.$$  

Let us denote by $\sigma(L)$ the set of eigenvalues of $L$, counted with multiplicities, and by $\sigma(L_k)$ the eigenvalues of the problem $L_k u = \lambda AB^{n-1} u$. Then

$$\sigma(L) = \bigcup_{k=0}^{\infty} m(\Lambda_k) \sigma(L_k),$$  

where the expression in the right-hand side means that each eigenvalue of $L_k$ appears with multiplicity $m(\Lambda_k)$ in $\sigma(L)$ (summing up multiplicities if the same number $\lambda$ appears in several $\sigma(L_k)$). In particular, the index (number of negative eigenvalues) of $L$ is given by

$$\text{Index}(L) = \sum_{k=0}^{\infty} \text{Index}(L_k),$$  

and the sum on the right-hand side involves only finitely many terms.

**Proof.** If $y(x, \omega)$ satisfies $Ly = \lambda y$, then

$$-\frac{\partial}{\partial x} \left( A^{-1} B^{n-1} \frac{\partial y}{\partial x} \right) + AB^{n-3} \left( \Delta_S y - B^2 V y \right) = \lambda AB^{n-1} y.$$  

In order to prove the lemma, it suffices to decompose the function $y(x, \omega)$ into a series of spherical harmonics. The generic term in this series will be of the form $u(x)Y(\omega)$ where $Y$ is a $k$-spherical harmonic and the preceding equation becomes

$$-\frac{d}{dx} \left( A^{-1} B^{n-1} \frac{du}{dx} \right) + AB^{n-3} \left( \Lambda_k - B^2 V \right) u = \lambda AB^{n-1} u.$$  

The first assertion of the lemma follows easily. For the final assertion we need only remark that $\Lambda_k$ tends to infinity with $k$ and hence the operators $L_k$ are positive for $k$ large enough (this is because $A, B$ are bounded from below by positive constants and $V$ is bounded).

**Remark.** The $L_k$ operate on functions of a single variable, hence their eigenvalues are all simple, and an eigenfunction associated to the $j$'th eigenvalue has exactly $j$ nodal domains. We use these properties in upcoming arguments.

Let $\mathcal{D}$ be a Delaunay unduloid in $\mathbb{R}^{n+1}$ (with $H = 1, \mu > 0$) or in $\mathbb{H}^{n+1}$ (with $H > 1, \mu > 0$), with cmc $H$ and weight parameter $\mu$. Let $\mathbb{R}^{n+1}_+$ and $\mathbb{H}^{n+1}_+$ denote one of the closed half-spaces defined by a geodesic hyperplane containing the axis of $\mathcal{D}$. Then we have:

**Proposition 4.1** The stability operator $\Delta - V$ of the Delaunay unduloid $\mathcal{D}$ is positive in any $\Omega$ contained in $\mathcal{D}_+ := \mathcal{D} \cap \mathbb{R}^{n+1}_+$ or $\mathcal{D}_+ := \mathcal{D} \cap \mathbb{H}^{n+1}_+$, with respect to Dirichlet boundary conditions. In particular, the half-Delaunay unduloids $\mathcal{D}_+$ are (strongly) stable.

**Proof.** This result is well-known for Euclidean graphs.

In the hyperbolic case, one has to be more careful, as certain kinds of graphs are not stable. To prove the proposition, it suffices to find a positive solution of $(\Delta - V) y = 0$ on $\mathcal{D}_+$. Such a solution will be given by the normal component of a well chosen Killing field. We consider the Killing field $\mathcal{Y}_0(\omega, t) = (\theta, 0)$ in $\mathbb{H}^{n+1}$, where $\theta \in S^{n-1}$ is chosen so that $\mathcal{Y}_0(\omega, t)|_{\partial \mathcal{D}_+}$ is perpendicular to the geodesic hyperplane containing $\partial \mathcal{D}_+$. The function

$$a_0(t, \omega) := \langle N, \mathcal{Y}_0 \rangle$$

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satisfies \((\Delta - V)a_\theta = 0\) and is equal to \(g'(t)(\theta, \omega)\) up to a positive factor (recall that \(g(t) = e^t \cos \varphi(t)\), see Equation (3.14)). To prove that \(a_\theta > 0\) in the interior of \(D_+\), it suffices to look at the sign of \(g'(t) = e^t (\cos \varphi(t) - \varphi'(t) \sin \varphi(t))\). Assume there is a point \(t_0\) at which \(g'\) vanishes, then
\[
\frac{1}{\varphi'(t_0)} = \tan \varphi(t_0) > 0
\]
and Equation (3.13) implies \(0 < \mu = (1 - H)/((\varphi'(t_0))^n) < 0\), a contradiction. \(\square\)

4.2 Index estimates for certain Delaunay pieces. We have the following estimates for the indexes of the Delaunay pieces \(B_\ell\) and \(C_\ell\), in both Euclidean and hyperbolic cases:

**Proposition 4.2** The index of the Delaunay piece \(B_\ell(\mu)\) with Dirichlet conditions at both boundary components is exactly \(\ell - 1\).

**Proposition 4.3** There is a constant \(c_1(n, H)\), which depends only on the dimension \(n\) and the mean curvature \(H\), such that the index of the Delaunay piece \(C_\ell(\mu)\) with Neumann conditions at both boundary components satisfies
\[
2\ell \leq \text{Neumann Index}(C_\ell(\mu)) \leq 2\ell + c_1(n, H).
\]

**Proofs.** The proofs of these two propositions are quite similar.

**Step 1.** The induced metric on the pieces \(B_\ell(\mu)\) or \(C_\ell(\mu)\) is of the type described in Lemma 4.1, with \(A = \sqrt{1 + (f')^2}\), \(B = f\) in the Euclidean case, and \(A = \sqrt{1 + (\varphi')^2}(1 + \tan^2 \varphi)\), \(B = \tan \varphi\) in the hyperbolic case. Hence, to estimate the index we only need to look at the indexes of the corresponding operators \(A^{-1}B^{1-n}L_k\), and we already know that for \(k\) large enough the operator \(A^{-1}B^{1-n}L_k\) is positive, implying that its index is zero. In fact, looking at the bounds we have for \(B^2V\) in Lemmas 3.1 and 3.4, we see that there exists a constant \(c(n, H)\) such that \(A^{-1}B^{1-n}L_k\) is positive whenever \(k \geq c(n, H)\).

**Proof of Proposition 4.2, Step 2.** The following proof applies when the Delaunay unduloid is not a cylinder. (For cylinders the index estimates are trivial, and we do not include the arguments here.) By Lemmas 3.2 and 3.5, the function \(a\) in (3.10) and (3.19) satisfies \((\Delta - V)a = 0\) and \(a|_{B_\ell} = 0\). This function \(a\) has precisely \(\ell\) nodal domains in \(B_\ell\). It follows that 0 is the \(\ell\)-th eigenvalue of the operator \(A^{-1}B^{1-n}L_0\). So the Dirichlet index of \(B_\ell\) is at least \(\ell - 1\) and is bigger than \(\ell - 1\) if and only if some of the operators \(A^{-1}B^{1-n}L_k, k \geq 1\), have negative eigenvalues. Assume this is the case and that some \(u(x)\) with Dirichlet boundary conditions satisfies \(A^{-1}B^{1-n}L_k u = \lambda u\) for some \(\lambda < 0\). This implies that
\[
(\Delta - V)uY = \lambda uY
\]
for any spherical harmonic \(Y\) of degree \(k\). Choosing, for example, a radial spherical harmonic \(Y\), we can always find a domain \(D_k \subset D_+\) such that the function \(uY\) is positive in \(D_k\) and satisfies
\[
\begin{cases}
(\Delta - V)uY = \lambda uY & \text{in } D_k, \\
uY = 0 & \text{on } \partial D_k.
\end{cases}
\]
This contradicts Proposition 4.1.

**Proof of Proposition 4.3, Step 2.** We use the same function \(a\) as before, so \((\Delta - V)a = 0\). The domain \(C_\ell\) was designed so that \(a' = \frac{\partial a}{\partial n} = 0\) on \(\partial C_\ell\) and so that \(a\) has exactly \((2\ell + 1)\) nodal domains in \(C_\ell\). It follows, as in the preceding argument, that 0 is the \((2\ell + 1)\)-st eigenvalue of \(L_0\) and hence that the Neumann index of \(C_\ell\) is at least \(2\ell\). In order to obtain the upper bound, we remark that the Neumann index of \(A^{-1}B^{1-n}L_k, k \geq 1\), is at most 2. Indeed, assume it is at least 3. Then there is an eigenfunction \(u\) of \(A^{-1}B^{1-n}L_k, k \geq 1\),
5 Index growth results

5.1 Eigenvalue estimates for almost Delaunay pieces. Fix a Delaunay unduloid $\mathcal{D}$ and a piece $\mathcal{E} \subset \mathcal{D}$ which is bounded by two “parallel spheres” in geodesic hyperplanes orthogonal to the axis of revolution. We call $\mathcal{E}$ an almost Delaunay piece if it is a cylindrical graph over $\mathcal{E}$.

**Lemma 5.1** There exists a constant $c_2(n, H)$, depending only on the dimension $n$ and mean curvature $H$, such that if $\mathcal{E}$ is close enough to $\mathcal{E}$ in the $C^2$-sense, then

$$\text{Ind}(\mathcal{E}) \leq \text{Ind}(\mathcal{E}) \leq \text{Ind}(\mathcal{E}) + c_2(n, H),$$

where $\text{Ind}$ denotes the index for either Dirichlet or Neumann conditions on the corresponding boundary components of $\partial \mathcal{E}, \partial \mathcal{E}$.

**Proof.** Indeed, once the piece $\mathcal{E}$ is fixed, we can write the eigenvalues of the operator $L$ on $\mathcal{E}$ (with respect to some Dirichlet or Neumann conditions on the boundary components) as

$$\lambda_1(\mathcal{E}) < \lambda_2(\mathcal{E}) \leq \ldots \lambda_k(\mathcal{E}) < 0 \leq \lambda_{k+1}(\mathcal{E}) \leq \ldots$$

where $k = \text{Ind}(\mathcal{E})$. If $\mathcal{E}$ is close enough to $\mathcal{E}$ in the $C^2$-sense, the negative eigenvalues of the operator $L$ corresponding to $L$ are close to the corresponding eigenvalues of $L$. It follows that $\text{Ind}(\mathcal{E}) \leq \text{Ind}(\mathcal{E})$ because $\lambda_k(\mathcal{E}) < 0$, with $\text{Ind}(\mathcal{E}) = \text{Ind}(\mathcal{E})$ unless $\lambda_{k+1}(\mathcal{E}) = 0$, in which case we may have $\lambda_{k+1}(\mathcal{E}) < 0$ and the constant $c_2(n, H)$ takes the possible multiplicity of $\lambda_{k+1}(\mathcal{E})$ into account. This multiplicity can be bounded as indicated in the proof of Proposition 4.3. 

**Note.** On $\mathcal{E}$, the eigenvalue problem $L_p u = \lambda AB^{n-1} u$ introduced earlier cannot have eigenvalue 0 when $p \geq 1$ in case of Dirichlet boundary condition (by arguing like in the proof of Proposition 4.2).

5.2 Asymptotically Delaunay hypersurfaces. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface such that

1. $M$ can be decomposed as

$$M = M_0 \bigcup_1^N E_j,$$

where $M_0$ is compact with boundary and where each $E_j$ is an end of $M$ (Figure 3).

2. Each end $E_j$ is a cylindrical graph over half a Delaunay unduloid $\mathcal{D}_j(\mu_j)$ with weight $\mu_j > 0$ and semi-axis $a_j + \mathbb{R}_+$ $d_j$ for some $a_j, d_j \in \mathbb{R}^{n+1}$. The boundary of $E_j$ lies in the hyperplane through $a_j$ orthogonal to $d_j$, and $\partial M_0 = \bigcup_j \partial E_j$.

3. The graph $E_j$ above $\mathcal{D}(\mu_j)$ is given by a parametrization of the form

$$\mathbb{R} \times S^{n-1} \ni (x, \omega) \to F_j(x, \omega) = (x, f(x) + w_j(x, \omega)) \omega$$

with some function $w_j(x, \omega)$, for $(x, \omega) \in \mathbb{R}_+ \times S^{n-1}$, where $f(x)$ satisfies equation (3.5). We assume that $w_j$ tends to zero in $C^2$-norm on $[r, \infty) \times S^{n-1}$ as $r \to \infty$. 

Definition 5.1 We will say that a hypersurface which satisfies the preceding three conditions is an asymptotically Delaunay hypersurface.

This definition extends mutatis mutandis to the case of hypersurfaces in $\mathbb{H}^{n+1}$ (in this case, the axis is a geodesic ray parametrized by arc-length). Note that [KKS] and [KKMS] give sufficient conditions to insure that a cmc hypersurface is asymptotically Delaunay.

With the above notations, we also introduce the following subsets of $M$:

$$M^R = M_0 \bigcup \bigcup_{j=1}^N E_j^R$$

for $R > 0$,

where $E_j^R$ is the part of $E_j$ which lies above $a_j + [0, R]d_j$ (see Figure 3), and

$$M^{S,R} = M^R \setminus M^S = \bigcup_{j=1}^N E_j^{S,R}$$

for $R > S > 0$,

where $E_j^{S,R}$ is the part of $E_j$ which lies above $a_j + [S, R]d_j$. We can use similar notations for hypersurfaces $M$ in $\mathbb{H}^{n+1}$.

5.3 Main results. We have the following results:

Theorem 5.1 Let $M \subset \mathbb{R}^{n+1}$ be a complete asymptotically Delaunay cmc hypersurface. Let $E_j, j = 1 \ldots N$, be the ends of $M$ and let $D(\mu_j)$ be the Delaunay unduloid to which $E_j$ is asymptotic (with weight parameter $\mu_j > 0$). Denote by $T(\mu_j)$ the period of $D(\mu_j)$. Then

$$\lim_{R \to \infty} \frac{\text{Ind}(M \cap B(R))}{R} = 2 \sum_{j=1}^N \frac{1}{T(\mu_j)},$$

(5.23)

where $B(R)$ is the Euclidean ball of radius $R$ in $\mathbb{R}^{n+1}$. 
Remark. It follows from [KKKS] that the preceding theorem applies when $M$ is a properly embedded cmc 1 surface with finite topology in $\mathbb{R}^3$.

Let $M \subset H^{n+1}$ be a complete properly embedded hypersurface, with cmc $H > 1$ and finite topology. Such an $M$ is asymptotically Delaunay if $n = 3$ ([KKMS], Theorem 1.2) or if $n \geq 4$ and each end of $M$ is within a bounded distance of a geodesic ray ([KKMS], Theorem 1.3).

Theorem 5.2 Let $M \subset H^{n+1}(-1)$ be a complete properly embedded hypersurface, with cmc $H > 1$ and finite topology. Assume furthermore that each end of $M$ is within a bounded distance of some geodesic ray when $n \geq 4$. Let $E_j, j = 1, \ldots, N$, be the ends of $M$. Let $D_H(\mu_j)$ be the Delaunay unduloid (with weight parameter $\mu_j > 0$) to which $E_j$ is asymptotic. Denote by $\tau(\mu_j)$ the period of $D_H(\mu_j)$. Then

$$\lim_{R \to \infty} \frac{\text{Ind}(M \cap B(R))}{R} = 2 \sum_{j=1}^N \frac{1}{\tau(\mu_j)},$$

where $B(R)$ is the hyperbolic ball of hyperbolic radius $R$ in $H^{n+1}$.

Remark. The general idea of the proofs of these theorems is to apply Dirichlet–Neumann bracketing (see [RS] for example) to $M^R$ decomposed as $M^R = M^S \sqcup M^{S,R}$, and to use the fact that each component of $M^{S,R}$ (as $R \to \infty$) is asymptotic to a Delaunay piece for which we can estimate the index.

Proofs, main argument. Here we give the argument only for the Euclidean case. (The argument in the hyperbolic case is identical, except for some minor changes of notation.) For $M$ as in Theorem 5.1, we want to estimate the limits

$$\liminf_{R \to \infty} \frac{\text{Ind}(M \cap B(R))}{R} = \liminf_{R \to \infty} \frac{\text{Ind}(M^R)}{R}, \quad \limsup_{R \to \infty} \frac{\text{Ind}(M \cap B(R))}{R} = \limsup_{R \to \infty} \frac{\text{Ind}(M^R)}{R}.$$

Step 1, Estimating the index from below. Let $a_j$ be the scalar product of the normal to the hypersurface with the Killing field corresponding to translation along the axis of the Delaunay unduloid $D(\mu_j)$. Since the end $E_j$ is asymptotic to $D(\mu_j)$, the nodal domains of the function $a_j$ look very much like the nodal domains of the corresponding function for $D(\mu_j)$. Then, applying Dirichlet–Neumann bracketing to the decomposition $M^R = M^S \sqcup M^{S,R}$ and letting $R \to \infty$, Proposition 4.2 implies $\liminf_{R \to \infty} \text{Ind}(M^R)/R$ is greater than or equal to the value in the right hand side of (5.23).

Step 2, Estimating the index from above. For $R > S$, we decompose $M^R$ into pieces $M^R = M^S \bigcup \bigcup_{j=1}^N E_{j}^{S,R}$, chosen in such a way that the components of $\partial M^S$ lie above boundaries of $C(\mu_j)$ basic Neumann blocks of the corresponding Delaunay unduloids (this can be done with the correct choices of $M_0$ and $S$).

Fix some $\ell \in \mathbb{N}$. Each piece $E_{j}^{S,R}$ can again be decomposed into almost Delaunay pieces above $C(\mu_j)$ pieces of the Delaunay unduloids $D(\mu_j)$, plus a remainder part. We write such a decomposition

$$E_{j}^{S,R} = \bigsqcup_{p=1}^{m_j} C_{\ell,p}(\mu_j) \bigcup \mathcal{R}_j,$$

where $\mathcal{R}_j \subset C_{\ell,m_j+1}(\mu_j)$. Dirichlet–Neumann bracketing implies

$$\text{Ind}_D(M^R) \leq \text{Ind}_N(M^S) + \sum_{j=1}^N \left( \sum_{p=1}^{m_j} \text{Ind}_N(C_{\ell,p}(\mu_j)) + \text{Ind}_N(D(\mathcal{R}_j)) \right),$$

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where $\text{Ind}_D$ (resp. $\text{Ind}_N$, $\text{Ind}_{ND}$) stands for the index with Dirichlet (resp. Neumann, mixed Neumann–Dirichlet) boundary condition.

The number $\ell$ being fixed, we can choose $S$ (and $R > S$) so large that each piece $\tilde{C}_{\ell,p}(\mu_j)$ is close enough to a $C_{\ell}(\mu_j)$-piece so that $\text{Ind}_N(\tilde{C}_{\ell,p}(\mu_j)) \leq 2\ell + c(n,H)$, by Lemma 5.1 and Proposition 4.3.

We can now look at the extrinsic length $R$ and write, for each end $E_j$,

$$S + m_j \ell T(\mu_j) \leq R \leq S + (m_j + 1) \ell T(\mu_j).$$

It follows that

$$\text{Ind}_D(M^R) \leq \text{Ind}_N(M^S) + \sum_{j=1}^N \left\{ (1 + \frac{c(n,H)}{2\ell}) \frac{2(R - S)}{T(\mu_j)} + \text{Ind}_{ND}(\tilde{C}_\ell(\mu_j)) \right\}$$

$$\leq \text{Ind}_N(M^S) + \sum_{j=1}^N \left\{ (1 + \frac{c(n,H)}{2\ell}) \frac{2(R - S)}{T(\mu_j)} + \text{Ind}_{ND}(C_\ell(\mu_j)) + c(n,H) \right\}.$$ 

Dividing the preceding inequality by $R$ and letting $R$ tend to infinity, we find that

$$\limsup_{R \to \infty} \frac{\text{Ind}(M^R)}{R} \leq 2 \sum_{j=1}^N \left( 1 + \frac{c(n,H)}{2\ell} \right) \frac{1}{T(\mu_j)}.$$ 

Since $\ell$ is an arbitrary positive integer, we have that $\limsup_{R \to \infty} \text{Ind}(M^R)/R$ is less than or equal to the value in the right hand side of (5.23).

5.4 Other growth results. Kapouleas [K] has constructed examples of complete constant mean curvature surfaces in $\mathbb{R}^3$ which are periodic with respect to some 2 (resp. 3) dimensional lattice. It is not difficult to establish that, for each of the doubly (resp. triply) periodic surfaces $M$ in [K], there exist finite positive constants $c_1$ and $c_2$ such that $c_1 R^2 \leq \text{Ind}(M \cap B(R)) \leq c_2 R^2$ (resp. $c_1 R^3 \leq \text{Ind}(M \cap B(R)) \leq c_2 R^3$) for large $R$.

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