Initial boundary value problems for a fractional differential equation with hyper-Bessel operator

Fatma Al-Musalhi*, Nasser Al-Salti*, and Erkinjon Karimov*

*Department of Mathematics, Sultan Qaboos University, P.O. Box 36 Al-Khoudh, Oman

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Abstract

Direct and inverse source problems of a fractional diffusion equation with regularized Caputo-like counterpart hyper-Bessel operator are considered. Solutions to these problems are constructed based on appropriate eigenfunction expansion and results on existence and uniqueness are established. To solve the resultant equations, a solution to a non-homogeneous fractional differential equation with regularized Caputo-like counterpart hyper-Bessel operator is also presented.

1 Introduction

In this paper, we consider the following fractional differential equation involving hyper-Bessel operator with a source term $F$:

$$F^C(t^\theta \frac{\partial}{\partial t})^\alpha u(x,t) - u_{xx}(x,t) = F.$$  \hfill (1)

We study both a direct problem where $F = f(x,t)$ is a known function of space and time and an inverse source problem where $F = f(x)$ is an unknown function of space only. Here $F^C(t^\theta \frac{\partial}{\partial t})^\alpha$ stands for a regularized Caputo-like counterpart hyper-Bessel operator of order $0 < \alpha < 1$ (see formula (5)). The hyper-Bessel operator was introduced by Dimovski in [2] and it arises in various problems, such as, fractional relaxation [3] and fractional diffusion models [4]. As an example, authors in [4] used hyper-Bessel operator to describe heat diffusion for fractional...
Brownian motion. Their analysis based on converting fractional power of hyper-Bessel operator into Erdélyi-Kober (E-K) fractional integral. For more details about fractional Brownian motion, the reader is referred to [4, 7].

In fact, expressing hyper-Bessel operator in terms of Erdélyi-Kober fractional integral plays a key role in finding solution to fractional differential equations involving hyper-Bessel operator as illustrated in this paper as well. Some results related to hyper-Bessel operator are given in [3], [6].

In [1], AL-Saqabi and Kiryakova considered Volterra integral equation of second kind and a fractional differential equation, involving (E-K) integral or differential operator. They found explicit solutions to these equations using transmutation method which reduces solutions to known integral solutions of Riemann-Liouville fractional equations.

The purpose of this paper is to prove existence and uniqueness of solution to a fractional diffusion equation involving a regularized Caputo-like counterpart hyper-Bessel operator considering both direct and inverse source problems.

2 Preliminaries

In this section, we recall some definitions and results related to fractional hyper-Bessel operator which will be used later in this paper. We start by writing down the definition of Erdélyi-Kober fractional integral.

**Definition 2.1.** (see [1, 3]) Erdélyi-Kober fractional integral of a function $f(t)$ with arbitrary parameters $\delta > 0$, $\gamma \in \mathbb{R}$ and $\beta > 0$ is defined as

$$I_\gamma^{\beta \delta} f(t) = \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^{\beta} - \tau^{\beta})^{\delta-1} \tau^{\beta \gamma} f(\tau) d(\tau^{\beta}),$$

(2)

which is reduced to the well-known Riemann-Liouville fractional integral when $\gamma = 0$ and $\beta = 1$ with a power weight.

For $\delta < 0$, the interpretation is via integro-differential operator

$$I_\gamma^{\beta \delta} f(t) = (\gamma + \delta + 1) I_\gamma^{\beta, \delta+1} f(t) + \frac{1}{\beta} I_\gamma^{\beta, \delta+1} \left( t \frac{d}{dt} f(t) \right).$$

In the following theorem, we present an explicit solution to an integral equation involving E-K fractional integral.

**Theorem 2.2.** (see [1], Theorem 1) The unique solution $y(t) \in C_{\beta \mu}$, $\mu \geq \max \{0, -\gamma \} - 1$ to the following fractional integral equation of a second kind :

$$y(t) - \lambda t^{\beta \delta} I_\gamma^{\beta \delta} y(t) = f(t),$$
or equivalently,
\[ y(t) - \lambda t^{-\beta} \int_0^t \frac{(t^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta^\gamma} y(\tau) d(\tau^\beta) = f(t), \]
with \( f \in C_{\beta\mu} \), has the explicit form of a convolutional type integral:
\[ y(t) = f(t) + \lambda t^{-\beta} \int_0^t \frac{(t^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta^\gamma} y(\tau) d(\tau^\beta). \tag{3} \]

Next, we use E-K integral to define the regularized counterpart hyper-Bessel operator.

**Definition 2.3.** (see [3]) The hyper-Bessel operator of order \( 0 < \alpha < 1 \) is defined in terms of E-K integral as follows
\[
\left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = \begin{cases} 
(1 - \theta)^\alpha t^{-(1-\theta)\alpha} I^{0,\alpha}_{1-\theta} f(t), & \text{if } \theta < 1, \\
(\theta - 1)^\alpha I^{1,\alpha}_{1-\theta} t(1-\theta)^\alpha f(t), & \text{if } \theta > 1.
\end{cases} \tag{4}
\]

Note that when \( \theta = 0 \), the hyper-Bessel operator coincides with Riemann-Liouville fractional derivative.

Now, recall that Caputo and Riemann-Liouville fractional derivatives of order \( 0 < \alpha < 1 \) are defined as (see [5]):
\[
C_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t^\alpha - \tau^\alpha)^{\alpha-1} f'(\tau) d\tau,
\]
\[
D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t^\alpha - \tau^\alpha)^{\alpha-1} f(\tau) d\tau,
\]
respectively, and they are related by ([5]):
\[
C_{0+}^\alpha f(t) = D_{0+}^\alpha (f(t) - f(0^+)).
\]

Using the above relation, we can express the regularized Caputo-like counterpart hyper-Bessel operator as:
\[
C \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) - \frac{f(0) t^{-\alpha(1-\theta)}}{(1-\theta)^{-\alpha}\Gamma(1-\alpha)}, \tag{5}
\]
and in terms of E-K fractional integral:
\[
C \left( t^\theta \frac{d}{dt} \right)^\alpha f(t) = (1 - \theta)^\alpha t^{-\alpha(1-\theta)} I^{0,\alpha}_{1-\theta} (f(t) - f(0)). \tag{6}
\]

Also, we need to recall the Mittag-Leffler function of one parameter:
\[
E_{\alpha}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \text{Re}(\alpha) > 0, \ z \in C,
\]
and the Mittag-Leffler of two parameters
\[
E_{\alpha,\beta^*}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta^*)}, \quad \text{Re}(\alpha) > 0, \ \text{Re}(\beta^*) > 0, \ z \in C.
\]

Now, we need the following results related to the Mittag-Leffler function
Theorem 2.4. (see\[11\]) If \( \text{Re}(\mu) > 0, \text{Re}(\beta^*) > 0 \), \( \lambda \) is a complex number and \( f(t) \) is an integrable function, then
\[
\int_a^x (x-u)^{\beta^*-1} E_{\alpha,\beta^*}(\lambda(x-u)^{\alpha}) \, du \int_a^u \frac{(u-t)^{\mu-1}}{\Gamma(\mu)} f(t) \, dt = \int_a^x (x-t)^{\beta^*+\mu-1} E_{\alpha,\beta^*+\mu}(\lambda(x-t)^{\alpha}) f(t) \, dt.
\]

Theorem 2.5. (see\[10\]) Let \( \alpha < 2, \beta^* \in \mathbb{R} \) and \( \frac{\pi \alpha}{2} < \mu < \min\{\pi, \pi \alpha\} \). Then we have the following estimate
\[
|E_{\alpha,\beta^*}(z)| \leq \frac{M}{1+|z|}, \quad \mu \leq |\text{arg}z| \leq \pi, \quad |z| \geq 0.
\]

(7)

Here and in the rest of the paper, \( M \) denotes a positive constant.

In the following theorem, we present a homogeneous fractional equation with regularized Caputo-like counterpart hyper-Bessel operator and its explicit solution as proved in [3].

Theorem 2.6. (see [3], Theorem 2.1) The function
\[
u(t) = E_{\alpha} \left( -\frac{\lambda t^{\alpha(1-\theta)}}{(1-\theta)^{\alpha}} \right),
\]
solves the fractional Cauchy problem
\[
\left\{
\begin{array}{l}
C \left( t^\theta \frac{d}{dt} \right)^\alpha u(t) = -\lambda u(t), \quad \alpha \in (0, 1), \quad \theta < 1, \quad t \geq 0, \\
u(0) = 1.
\end{array}
\right.
\]

In this paper, we consider a more general case, namely, a non-homogeneous fractional differential equation with a regularized Caputo-like counterpart hyper-Bessel operator presented in the following lemma:

Lemma 2.7. Consider the following non-homogeneous fractional differential equation
\[
C \left( t^\theta \frac{d}{dt} \right)^\alpha u(t) = -\lambda u(t) + f(t), \quad \alpha \in (0, 1), \quad \theta < 1, \quad t \geq 0,
\]
with \( u(0) = u_0 \), where \( u_0 \) is a constant. Then, its solution is given in the integral form
\[
u(t) = u_0 E_{\alpha} (\lambda^* t^{\rho \alpha}) + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - x^\rho)^{\alpha-1} f(x) \, d(x^\rho)
+ \frac{\lambda^*}{\rho^\alpha} \int_0^t (t^\rho - x^\rho)^{2\alpha-1} E_{\alpha,2\alpha}(\lambda^* (t^\rho - x^\rho)^{\alpha}) f(x) \, d(x^\rho),
\]
where, \( \rho = 1 - \theta \) and \( \lambda^* = -\frac{\lambda}{\rho^\alpha} \). Moreover, if \( f = f_0 \) is constant, then the solution reduces to
\[
u(t) = C^* E_{\alpha} (\lambda^* t^{\rho \alpha}) + \frac{f_0}{\lambda},
\]
where \( C^* = \left( u_0 - \frac{f_0}{\lambda} \right) \). In particular, when \( f = 0 \), we have
\[
u(t) = u_0 E_{\alpha} (\lambda^* t^{\rho \alpha}).
\]
Proof. First, using relation $[3]$, equation $[3]$ can be written as

$$
\rho^\alpha t^{-\alpha} I^0_{\rho} (u(t) - u_0) = -\lambda u(t) + f(t),
$$

which, on dividing by $\rho^\alpha t^{-\alpha}$, becomes

$$
I^0_{\rho} (u(t) - u_0) = \lambda^* t^\alpha u(t) + \frac{I^\alpha}{\rho^\alpha} f(t),
$$

where $\lambda^* = \frac{\lambda}{\rho^\alpha}$. Using the following property of the inverse of E-K integral (see $[3]$, Theorem 2.7):

$$
(I_n^{\eta,\alpha})^{-1} = I_n^{\eta+\alpha,-\alpha},
$$

the above equation can be written as an integral equation, namely,

$$
u(t) - \lambda^* I^0_{\rho} (t^\alpha u(t)) = u_0 + \frac{1}{\rho^\alpha} I_{\rho}^{-\alpha,\alpha} (t^\alpha f(t)),
$$

or equivalently,

$$
u(t) - \frac{\lambda^*}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} u(\tau) d(\tau^\rho) = u_0 + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} f(\tau) d(\tau^\rho).
$$

Whereupon using Theorem [2.2] we have

$$
u(t) = f^*(t) + \lambda^* \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} E_{\alpha,\alpha}(\lambda^*(t^\rho - \tau^\rho)^\alpha) f^*(\tau) d(\tau^\rho)
\quad + \quad u_0 \left( 1 + \lambda^* \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} E_{\alpha,\alpha}(\lambda^*(t^\rho - \tau^\rho)^\alpha) d(\tau^\rho) \right),
$$

where, $f^*(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - x^\rho)^{\alpha-1} f(x) d(x^\rho)$.

The first integral in (10) can be simplified using Theorem 2.2 to the following:

$$
\frac{\lambda^*}{\rho^\alpha} \int_0^t (t^\rho - x^\rho)^{2\alpha-1} E_{\alpha,2\alpha}(\lambda^*(t^\rho - x^\rho)^\alpha) f(x) d(x^\rho),
$$

and the second integral in (10) can be also simplified as follows:

$$
u_0 \left( 1 + \lambda^* \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} E_{\alpha,\alpha}(\lambda^*(t^\rho - \tau^\rho)^\alpha) d(\tau^\rho) \right)
\quad = \quad u_0 \left\{ 1 + \sum_{k=0}^{\infty} \frac{\lambda^*^{k+1}}{\Gamma(\alpha k + \alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha k + \alpha - 1} d(\tau^\rho) \right\}
\quad = \quad u_0 \left\{ 1 + \sum_{k=0}^{\infty} \frac{\lambda^*^{k+1}}{\Gamma(\alpha k + 1)} \frac{\tau^\rho}{\rho^\alpha \Gamma(\alpha k + 1)} \right\}
\quad = \quad u_0 \left\{ 1 + \sum_{m=1}^{\infty} \frac{\lambda^*^m}{\Gamma(\alpha m + 1)} \frac{\tau^\rho \rho^m}{\rho^\alpha \Gamma(\alpha m + 1)} \right\}
\quad = \quad u_0 E_{\alpha}(\lambda^* t^\alpha).
$$
Substituting the two simplified forms (11) and (12) into (10), we get the integral solution (9). Now, if \( f(t) = f_0 \) is constant, then evaluating the first integral in the expression (9) gives

\[
\frac{f_0}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - x^\rho)^{\alpha-1} \, d(x^\rho) = \frac{f_0 t^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)}.
\]

Substituting back into (9) and proceeding in a similar way as in (12), the expression of \( u(t) \) can be reduced to

\[
u(t) = u_0 E_\alpha(\lambda^* t^{\rho\alpha}) - \frac{f_0}{\lambda} \sum_{k=1}^\infty \frac{\lambda^k t^{\rho k}}{\Gamma(\alpha k + 1)},
\]

which can be rewritten as

\[
u(t) = \frac{f_0}{\lambda} + C^* E_\alpha(\lambda^* t^{\rho\alpha}),
\]

where \( C^* = \left( u_0 - \frac{f_0}{\lambda} \right) \). Finally, if \( f(t) = f_0 = 0 \), the the expression of \( u(x, t) \) can be further reduced to

\[
u(t) = u_0 E_\alpha(\lambda^* t^{\rho\alpha}),
\]

which is consistent with Theorem 2.6.

The rest of this paper is devoted for the main results. In the remaining two sections, we present existence and uniqueness results of solutions to direct and inverse source problems involving a regularized Caputo-like counterpart hyper-Bessel operator.

3 A Direct Problem

3.1 Statement of Problem and Main Result

Find a function \( u(x, t) \) in a domain \( \Omega = \{0 < x < 1, 0 < t < T\} \) satisfying

\[
C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t) = f(x, t), \quad (x, t) \in \Omega,
\]

the boundary conditions

\[
u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T,
\]

and the initial condition

\[
u(x, 0) = \psi(x), \quad 0 \leq x \leq 1,
\]

where \( f(x, t) \) is a given function, \( \theta < 1, 0 < \alpha < 1 \) and \( C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha \) is the regularized Caputo-like counterpart hyper-Bessel operator defined in (5). Our aim is to prove the existence and uniqueness of solution to the problem (13) - (15) as stated in the following theorem:
Theorem 3.1. Assume that the following conditions hold

- \( \psi(x) \in C[0, 1] \) such that \( \psi(0) = \psi(1) = 0 \) and \( \psi'(x) \in L^2(0, 1) \),
- \( f(\cdot, t) \in C^3[0, 1] \) such that \( f(0, t) = f(1, t) = f_{xx}(0, t) = f_{xx}(1, t) = 0 \), and \( \frac{\partial^4}{\partial x^4} f(\cdot, x) \in L(0, 1) \),

then, the problem (13) – (15) has a unique solution given by

\[
u(x, t) = \sum_{k=1}^{\infty} \left[ \psi_k E_\alpha \left( \frac{-k^2 \pi^2}{(1 - \theta)^\alpha} (1 - \theta) \alpha \right) + F_k(t) \right] \sin(k\pi x),
\]

where,

\[
F_k(t) = \frac{1}{(1 - \theta)^\alpha \Gamma(\alpha)} \int_0^t \left( \frac{(t(1 - \theta) - \tau(1 - \theta))^{\alpha-1}}{\tau(1 - \theta) \alpha} \right) f_k(\tau) d\tau,
\]

\[
\psi_k = 2 \int_0^1 \psi(x) \sin(k\pi x) dx, \quad k = 1, 2, 3, \ldots
\]

\[
f_k(t) = 2 \int_0^1 f(x, t) \sin(k\pi x) dx, \quad k = 1, 2, 3, \ldots
\]

3.2 Proof of Result

3.2.1 Existence of Solution

Using separation of variables method for solving the homogeneous equation corresponding to (13) along with the homogeneous boundary conditions (14) yields the following spectral problem:

\[
\begin{cases}
X'' + \lambda X = 0, \\
X(0) = 0, \quad X(1) = 0.
\end{cases}
\]

It is known that the above problem is self adjoint and has the following eigenvalues

\[
\lambda_k = (k\pi)^2, \quad k = 1, 2, 3, \ldots
\]

and the corresponding eigenfunctions are

\[
X_k = \sin(k\pi x) \quad k = 1, 2, 3, \ldots.
\]

Using the fact that the system of eigenfunctions (17) forms an orthogonal basis in \( L^2(0, 1) \), we can write the solution \( u(x, t) \) in the form of a series expansion as follows:

\[
u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x),
\]

\[
u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x),
\]
and
\[ f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x), \quad (19) \]
where, \( u_k(t) \) is the unknown to be determined and \( f_k(t) \) is known and given by
\[ f_k(t) = 2 \int_{0}^{1} f(x,t) \sin(k\pi x) \, dx. \]

Substituting (18) and (19) into (13) and (15), we get the linear fractional differential equation
\[ C \left( t^\theta \frac{d}{dt} \right)^\alpha u_k(t) + k^2\pi^2 u_k(t) = f_k(t), \quad (20) \]
with the initial condition
\[ u_k(0) = \psi_k, \]
where, \( \psi_k \) is the coefficient of the series expansion of \( \psi(x) \) in terms of the orthogonal basis (17), i.e.,
\[ \psi_k = 2 \int_{0}^{1} \psi(x) \sin(k\pi x) \, dx. \]

Whereupon using Lemma 2.7, the solution of equation (20) is given by
\[ u_k(t) = \psi_k E_\alpha \left( \frac{-k^2\pi^2}{\rho^\alpha} t^{\rho\alpha} \right) + F_k(t), \]
where, \( \rho = 1 - \theta \) and
\[ F_k(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{0}^{t} (t^\rho - \tau^\rho)^{\alpha-1} f_k(\tau) \, d(\tau^\rho) \]
\[ - \frac{1}{k^2\pi^2 \rho^{2\alpha}} \int_{0}^{t} (t^\rho - y^\rho)^{2\alpha-1} E_{\alpha,2\alpha} \left( \frac{-\lambda}{\rho^\alpha} (t^\rho - y^\rho)\right) f_k'(y) \, d(y^\rho). \]

Consequently, the expression of \( u(x,t) \) can be written as
\[ u(x,t) = \sum_{k=1}^{\infty} \left( \psi_k E_\alpha \left( \frac{-k^2\pi^2}{\rho^\alpha} t^{\rho\alpha} \right) + F_k(t) \right) \sin(k\pi x). \quad (21) \]

To complete the proof of existence, we need to prove the uniform convergence of the series representations of
\[ u(x,t), \quad C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x,t), \quad u_x(x,t), \quad u_{xx}(x,t). \]

We start with the series representation of \( u(x,t) \), rewriting \( F_k(t) \) as follows
\[ F_k(t) = \frac{1}{k^2\pi^2 \rho^\alpha \Gamma(\alpha)} \int_{0}^{t} (t^\rho - \tau^\rho)^{\alpha-1} f_k''(\tau) \, d(\tau^\rho) \]
\[ - \frac{1}{\rho^{2\alpha}} \int_{0}^{t} (t^\rho - y^\rho)^{2\alpha-1} E_{\alpha,2\alpha} \left( \frac{-k^2\pi^2}{\rho^\alpha} (t^\rho - y^\rho)\right) f_k''(y) \, d(y^\rho), \]
where,
\[ f_k''(t) = 2 \int_{0}^{1} f_{xx}(x,t) \sin(k\pi x) \, dx. \]
Now, we estimate the Mittag-Leffler function using inequality (7):

\[
\left| E_{\alpha,2\alpha} \left( -\frac{k^2\pi^2}{\rho^\alpha} (t^\rho - y^\rho) \right) \right| \leq \frac{M \rho^\alpha}{\rho^\alpha + k^2\pi^2 |t^\rho - y^\rho|^\alpha},
\]

which implies the following estimate for \( u(x, t) \):

\[
|u(x, t)| \leq M \sum_{k=1}^\infty \left( \frac{|\psi_k|}{\rho^\alpha + k^2\pi^2 |t^\rho - y^\rho|^\alpha} + \frac{1}{k^2\pi^2} \int_0^t |t^\rho - \tau^\rho|^{\alpha-1} |f_k''(\tau)| d(\tau^\rho) \right.
\]
\[
+ \left. \int_0^t \frac{|t^\rho - y^\rho|^{2\alpha-1}}{\rho^\alpha + k^2\pi^2 (t^\rho - y^\rho)^\alpha} |f_k''(y)| d(y^\rho) \right).
\]

Since \( \psi(x) \in C[0, 1] \) and \( f(\cdot, t) \in C^3[0, 1] \), then the above series converges and hence, by Weierstrass M-test the series representation of \( u(x, t) \) is uniformly convergent in \( \Omega \).

Next, we show the uniform convergence of series representation of \( u_{xx}(x, t) \), which is given by

\[
u_{xx}(x, t) = -\sum_{k=1}^\infty k^2\pi^2 \left( \psi_k E_{\alpha,1} \left( \frac{-k^2\pi^2}{\rho^\alpha} \mu^\alpha \right) + F_k(t) \right) \sin(k\pi x).
\]

To prove this assertion, we have the following estimate

\[
u_{xx}(x, t) \leq M \sum_{k=1}^\infty \left( \frac{k^2\pi^2 |\psi_k|}{\rho^\alpha + k^2\pi^2 |t^\rho - y^\rho|^\alpha} + \frac{1}{k^2\pi^2} \int_0^t |t^\rho - \tau^\rho|^{\alpha-1} |f_k''(\tau)| d(\tau^\rho) \right.
\]
\[
+ \left. \int_0^t \frac{|t^\rho - y^\rho|^{2\alpha-1}}{\rho^\alpha + k^2\pi^2 (t^\rho - y^\rho)^\alpha} |f_k''(y)| d(y^\rho) \right),
\]

where

\[
f_k''(t) = 2 \int_0^1 \frac{\partial^4}{\partial x^4} f(x, t) \sin(k\pi x) \, dx.
\]

Since \( \psi(0) = \psi(1) = 0 \) and \( \partial^4 f / \partial x^4 (\cdot, t) \in L(0, 1) \), then using integration by parts, we arrive at the following estimate

\[
u_{xx}(x, t) \leq M \sum_{k=1}^\infty \left( \frac{1}{k\pi} |\psi_k^{(1)}| + \frac{1}{k^2\pi^2} \right)
\]
\[
\leq M \left( \sum_{k=1}^\infty \frac{1}{(k\pi)^2} + \sum_{k=1}^\infty |\psi_k^{(1)}|^2 \right),
\]

where, we have used the inequality \( 2ab \leq a^2 + b^2 \) and

\[
\psi_k^{(1)} = 2 \int_0^1 \psi'(x) \cos(k\pi x) \, dx.
\]

Then, Bessel’s inequality for trigonometric functions

\[
\sum_{k=0}^\infty g_k^2 \leq \|g\|_{L^2(0,1)}^2,
\]

implies

\[
u_{xx}(x, t) \leq M \sum_{k=1}^\infty \frac{1}{(k\pi)^2} + \|\psi'(x)\|_{L^2(0,1)}^2.
\]
Thus, the series in expression of $u_{xx}(x, t)$ is bounded by a convergent series which implies that its uniformly convergent by Weierstrass M-test. Finally, series representation of $C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t)$ is given by

$$C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) = -\sum_{k=1}^{\infty} k^2 \pi^2 \left( \psi_k E_\alpha \left( \frac{-k^2 \pi^2}{\rho^{\omega}} + t^\omega \right) + F_k(t) \right) \sin(k\pi x) + f(x, t),$$

and convergence of the above series follows directly from the uniform convergence of $u_{xx}(x, t)$, which also ensures the uniform convergence of $u_x(x, t)$.

### 3.2.2 Uniqueness of Solution:

Suppose that $u_1(x, t)$ and $u_2(x, t)$ are two solutions of the problem (13) - (15), then $\hat{u}(x, t) = u_1(x, t) - u_2(x, t)$ satisfies the following boundary value problem:

$$C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha \hat{u} - \frac{\partial^2 \hat{u}}{\partial x^2} = 0, \quad (x, t) \in \Omega, \quad (22)$$

$$\hat{u}(0, t) = 0, \quad \hat{u}(1, t) = 0, \quad 0 \leq t \leq T, \quad (23)$$

$$\hat{u}(x, 0) = 0, \quad 0 \leq x \leq 1. \quad (24)$$

Define the following function:

$$u_k(t) = 2 \int_0^1 \hat{u}(x, t) \sin(k\pi x) dx. \quad (25)$$

Then, the initial condition (24) implies

$$u_k(0) = 0. \quad (26)$$

Applying regularized Caputo-like counterpart hyper-Bessel operator to (25), we get

$$C \left( t^\theta \frac{d}{dt} \right)^\alpha u_k(t) = 2 \int_0^1 C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha \hat{u}(x, t) \sin(k\pi x) dx, \quad (27)$$

$$= 2 \int_0^1 \hat{u}_{xx}(x, t) \sin(k\pi x) dx. \quad (28)$$

Then, integrating by parts twice and using boundary conditions (23), we obtain the following fractional differential equation

$$C \left( t^\theta \frac{d}{dt} \right)^\alpha u_k(t) - (k\pi)^2 u_k(t) = 0. \quad (29)$$

Using Lemma (27) the above equation with the initial condition (26) has the trivial solution $u_k(t) \equiv 0$, and hence we have

$$\int_0^1 \hat{u}(x, t) \sin(k\pi x) dx = 0. \quad (30)$$

Therefore, using the completeness property of system (17), we deduce that $\hat{u}(x, t) = 0$ in $\Omega$, which implies the uniqueness of solution to the problem (13) - (15).
4 Inverse source problem

Here, we consider an inverse source problem of finding a pair of functions \( \{u(x, t), f(x)\} \) in a rectangular domain \( \Omega = \{(x, t) : 0 < x < 1, \ 0 < t < T\} \), which satisfies the following initial-boundary value problem:

\[
C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) - u_{xx}(x, t) = f(x), \quad (x, t) \in \Omega \tag{27}
\]

\[
u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \tag{28}
\]

\[
u(x, 0) = \psi(x), \quad u(x, T) = \phi(x), \quad 0 \leq x \leq 1, \tag{29}
\]

where \( \phi \) and \( \psi \) are given functions, such that

\[
\psi(0) = \psi(1) = 0, \quad \phi(0) = \phi(1) = 0,
\]

which follows directly from (28) and (29). As in the previous section, we seek solution to problem (27) - (29) in a form of series expansions using the orthogonal system \( (17) \) as follows:

\[
u(x, t) = \sum_{k=1}^{\infty} u_k(t) \sin(k\pi x),
\]

\[
f(x) = \sum_{k=1}^{\infty} f_k \sin(k\pi x).
\]

where \( f_k, u_k \) are the unknowns to be determined. Substituting the above expressions for \( u(x, t) \) and \( f(x) \) into (27) and (29) gives the following fractional differential equation:

\[
C \left( t^\theta \frac{d}{dt} \right)^\alpha u_k(t) + k^2\pi^2 u_k(t) = f_k,
\]

with the following conditions

\[
u_k(0) = \psi_k, \quad u_k(T) = \phi_k,
\]

where \( \psi_k, \phi_k \) are called the Fourier sine coefficients and defined as

\[
\psi_k = 2 \int_0^1 \psi(x) \sin(k\pi x) \, dx, \quad \phi_k = 2 \int_0^1 \phi(x) \sin(k\pi x) \, dx.
\]

Solving the above equation, using Lemma 2.7, we obtain

\[
u_k(t) = C_k E_\alpha \left( -\frac{k^2\pi^2}{(1 - \theta)^\alpha} t^{(1-\theta)\alpha} \right) + \frac{f_k}{k^2\pi^2},
\]

and using the given initial conditions, we have

\[
C_k = \frac{\psi_k - \phi_k}{1 - E_\alpha \left( -\frac{k^2\pi^2}{(1 - \theta)^\alpha} T^{(1-\theta)\alpha} \right)}, \quad f_k = k^2\pi^2(\psi_k - C_k).
\]
Hence, the expressions for \( u(x, t) \) and \( f(x) \) can be written as,

\[
\begin{align*}
    u(x, t) &= \sum_{k=1}^{\infty} C_k E_\alpha \left( - \frac{k^2 \pi^2}{(1 - \theta)^\alpha} t^{(1 - \theta)\alpha} \right) \sin(k\pi x) + (\psi_k - C_k) \sin(k\pi x) \\
    &= \psi(x) - \sum_{k=1}^{\infty} \frac{1 - E_\alpha}{1 - E_\alpha} \left( - \frac{k^2 \pi^2}{(1 - \theta)^\alpha} T^{(1 - \theta)\alpha} \right) (\psi_k - \phi_k) \sin(k\pi x),
\end{align*}
\]

and

\[
\begin{align*}
    f(x) &= \sum_{k=1}^{\infty} k^2 \pi^2 (\psi_k - C_k) \sin(k\pi x) \\
    &= \psi''(x) - \sum_{k=1}^{\infty} \frac{k^2 \pi^2 (\psi_k - \phi_k)}{1 - E_\alpha} \left( - \frac{k^2 \pi^2}{(1 - \theta)^\alpha} T^{(1 - \theta)\alpha} \right) \sin(k\pi x).
\end{align*}
\]

Appropriate conditions on the given functions \( \psi(x) \) and \( \phi(x) \), see the Theorem 4.1 below, are assumed for establishing the uniform convergence of the series expansions of \( u(x, t) \), \( \sum \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) \), \( u_x(x, t) \), \( u_{xx}(x, t) \) and \( f(x) \). This can be done in a similar approach as presented earlier. For example, for \( f(x) \) we have the following estimate:

\[
|f(x)| \leq |\psi''(x)| + \sum_{k=1}^{\infty} \frac{k^2 \pi^2 (1 - \theta)^\alpha + k^2 \pi^2 T^{(1 - \theta)\alpha}}{(1 - M)(1 - \theta)^\alpha + k^2 \pi^2 T^{(1 - \theta)\alpha}} \left( |\psi_k| + |\phi_k| \right)
\]

\[
\leq |\psi''(x)| + M \sum_{k=1}^{\infty} \frac{1}{k\pi} \left( |\psi_k^{(3)}| + |\phi_k^{(3)}| \right)
\]

\[
\leq |\psi''(x)| + M \left( \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \sum_{k=1}^{\infty} |\psi_k^{(3)}|^2 + \sum_{k=1}^{\infty} |\phi_k^{(3)}|^2 \right)
\]

\[
\leq |\psi''(x)| + M \left( \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \|\psi''(x)\|_{L^2(0,1)}^2 + \|\phi''(x)\|_{L^2(0,1)}^2 \right),
\]

where

\[
\psi_k^{(3)} = 2 \int_0^1 \psi'''(x) \cos(k\pi x) \, dx,
\]

and

\[
\phi_k^{(3)} = 2 \int_0^1 \phi'''(x) \cos(k\pi x) \, dx.
\]

Assuming that \( \psi(x) \in C^2[0, 1] \) and \( \psi'''(x), \phi'''(x) \in L^2(0,1) \), then by Weierstrass M-test the series representation of \( f(x) \) is uniformly convergent. Also, the series representation of \( \sum \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) \), which is given by

\[
\sum \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) = - \sum_{k=1}^{\infty} \frac{E_\alpha}{1 - E_\alpha} \left( - \frac{k^2 \pi^2}{(1 - \theta)^\alpha} T^{(1 - \theta)\alpha} \right) k^2 \pi^2 (\psi_k - \phi_k) \sin(k\pi x),
\]
can be estimated as follows:

\[ C \left( t^\theta \frac{\partial}{\partial t} \right)^\alpha u(x, t) \leq M \sum_{k=1}^{\infty} \frac{k^2 \pi^2}{(1 - \theta)^\alpha} + k^2 \pi^2 t(1 - \theta)^\alpha (|\psi_k| + |\phi_k|) \]

\[ \leq M \sum_{k=1}^{\infty} \frac{1}{k \pi} \left( |\psi_k^{(1)}| + |\phi_k^{(1)}| \right) \]

\[ \leq M \left( \sum_{k=1}^{\infty} \frac{1}{(k \pi)^2} + \|\psi'(x)\|_{L^2(0,1)}^2 + \|\phi'(x)\|_{L^2(0,1)}^2 \right). \]

It is clear that the above series is uniformly convergent. The main result for this section can be summarized in the following theorem:

**Theorem 4.1.** Assume \( \psi(x), \phi(x) \in C^2[0, 1] \) such that \( \psi^{(i)}(0) = \psi^{(i)}(1) = \phi^{(i)}(0) = \phi^{(i)}(1) = 0 \), \( (i = 0, 2) \) and \( \psi''(x), \phi''(x) \in L^2(0, 1) \), then the inverse source problem (27) - (29) has a unique pair of solutions \( \{u(x, t), f(x)\} \) given by

\[ u(x, t) = \psi(x) - \sum_{k=1}^{\infty} \frac{1 - E_\alpha \left( -\frac{k^2 \pi^2}{T(1 - \theta)^\alpha} \right)}{1} (\psi_k - \phi_k) \sin(k\pi x), \]

\[ f(x) = \psi''(x) - \sum_{k=1}^{\infty} \frac{k^2 \pi^2 (\psi_k - \phi_k)}{1} \sin(k\pi x), \]

where,

\[ \psi_k = 2 \int_0^1 \psi(x) \sin(k\pi x) dx, \quad \phi_k = 2 \int_0^1 \phi(x) \sin(k\pi x) dx. \]

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