ON THE STRUCTURE AND REPRESENTATIONS OF THE INSERTION-ELIMINATION LIE ALGEBRA

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Abstract. We examine the structure of the insertion-elimination Lie algebra on rooted trees introduced in [CK]. It possesses a triangular structure $g = n_+ \oplus \mathbb{C}.d \oplus n_-$, like the Heisenberg, Virasoro, and affine algebras. We show in particular that it is simple, which in turn implies that it has no finite-dimensional representations. We consider a category of lowest-weight representations, and show that irreducible representations are uniquely determined by a ”lowest weight” $\lambda \in \mathbb{C}$. We show that each irreducible representation is a quotient of a Verma-type object, which is generically irreducible.

1. Introduction

The insertion-elimination Lie algebra $g$ was introduced in [CK] as a means of encoding the combinatorics of inserting and collapsing subgraphs of Feynman graphs, and the ways the two operations interact. A more abstract and universal description of these two operations is given in terms of rooted trees, which encode the hierarchy of subdivergences within a given Feynman graph, and it is this description that we adopt in this paper. More precisely, $g$ is generated by two sets of operators $\{D^+_t\}$ and $\{D^-_t\}$, where $t$ runs over the set of all rooted trees, together with a grading operator $d$. In [CK] $g$ was defined in terms of its action on a natural representation $\mathbb{C}\{T\}$, where the latter denotes the vector space spanned by rooted trees. For $s \in \mathbb{C}\{T\}$, $D^+_t . s$ is a linear combination of the trees obtained by attaching $t$ to $s$ in all possible ways, whereas $D^-_t . s$ is a linear combination of all the trees obtained by pruning the tree $t$ from branches of $s$. $n_+ = \{D^+_t\}$ and $n_- = \{D^-_t\}$ form two isomorphic nilpotent Lie subalgebras, and $g$ has a triangular structure

$$g = n_+ \oplus \mathbb{C}.d \oplus n_-$$

as well as a natural $\mathbb{Z}$-grading by the number of vertices of the tree $t$. The Hopf algebra $U(n_{\pm})$ is dual to Kreimer’s Hopf algebra of rooted trees [K].

This note aims to establish a few basic facts regarding the structure and representation theory of $g$. We begin by showing that $g$ is simple, which together with its infinite-dimensionality implies that it has no non-trivial finite-dimensional representations, and that any non-trivial representation is necessarily faithful. We then proceed to develop a highest-weight theory for $g$ along the lines of [K1] [K2].

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In particular, we show that every irreducible highest-weight representation of $\mathfrak{g}$ is a quotient of a Verma-like module, and that these are generically irreducible.

One can define a larger, "two-parameter" version of the insertion-elimination Lie algebra $\tilde{\mathfrak{g}}$, where operators are labelled by pairs of trees $D_{t_1, t_2}$ (roughly speaking, in acting on $\mathbb{C}\{\mathcal{T}\}$, this operator replaces occurrences of $t_1$ by $t_2$). In the special case of ladder trees, $\tilde{\mathfrak{g}}$ was studied in [M, KM1, KM2]. The finite-dimensional representations of the nilpotent subalgebras $\mathfrak{n}_\pm$ as well as many other aspects of the Hopf algebra $U(\mathfrak{n}_\pm)$ were studied in [F].

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### 2. The insertion-elimination Lie algebra on rooted trees

In this section, we review the construction of the insertion-elimination Lie algebra introduced in [CK], with some of the notational conventions introduced in [M].

Let $\mathcal{T}$ denote the set of rooted trees. An element $t \in \mathcal{T}$ is a tree (finite, one-dimensional contractible simplicial complex), with a distinguished vertex $r(t)$, called the root of $t$. Let $V(t)$ and $E(t)$ denote the set of vertices and edges of $t$, and let

$$|t| = \# V(t)$$

Let $\mathbb{C}\{\mathcal{T}\}$ denote the vector space spanned by rooted trees. It is naturally graded,

$$\mathbb{C}\{\mathcal{T}\} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}\{\mathcal{T}\}_n$$

where $\mathbb{C}\{\mathcal{T}\}_n = \text{span}\{t \in \mathcal{T}| |t| = n\}$. $\mathbb{C}\{\mathcal{T}\}_0$ is spanned by the empty tree, which we denote by $\mathbf{1}$. We have

$$\mathbb{C}\{\mathcal{T}\}_0 = \langle 1 \rangle \quad \mathbb{C}\{\mathcal{T}\}_1 = \langle \bullet \rangle \quad \mathbb{C}\{\mathcal{T}\}_2 = \langle \bullet \rangle$$

$$\mathbb{C}\{\mathcal{T}\}_3 = \langle \bullet, \bullet, \bullet \rangle$$

where $\langle, \rangle$ denotes span, and the root is the vertex at the top. If $e \in E(t)$, by a *cut along* $e$ we mean the operation of cutting $e$ from $t$. This divides $t$ into two components - $R_e(t)$ containing the root, and $P_e(t)$, the remaining one. $R_e(t)$ and $P_e(t)$ are naturally rooted trees, with $r(R_e(t)) = r(t)$ and $r(P_e(t)) = \text{endpoint of } e$. Note that $V(t) = V(R_e(t)) \cup V(P_e(t))$. 
Let $g$ denote the Lie algebra with generators $D^+_t, D^-_t, d, t \in T$, and relations

\begin{equation}
[D^+_{t_1}, D^+_{t_2}] = \sum_{v \in V(t_2)} D^+_{t_2 \cup v t_1} - \sum_{v \in V(t_1)} D^+_{t_1 \cup v t_2}
\end{equation}

\begin{equation}
[D^-_{t_1}, D^-_{t_2}] = \sum_{v \in V(t_1)} D^-_{t_1 \cup v t_2} - \sum_{v \in V(t_2)} D^-_{t_2 \cup v t_1}
\end{equation}

\begin{equation}
[D^-_{t_1}, D^+_{t_2}] = \sum_{t \in T} \alpha(t_1, t_2; t) D^+_t + \sum_{t \in T} \beta(t_1, t_2; t) D^-_t
\end{equation}

\begin{equation}
[D^-_t, D^+_t] = d
\end{equation}

\begin{equation}
[d, D^-_t] = -|t| D^-_t
\end{equation}

\begin{equation}
[d, D^+_t] = |t| D^+_t
\end{equation}

where for $s, t \in T$, and $v \in V(s) s \cup v t$ denotes the rooted tree obtained by joining the root of $t$ to $s$ at the vertex $v$ via a single edge, and

- $\alpha(t_1, t_2; t) = \#\{e \in E(t_2) | R_e(t_2) = t, P_e(t_2) = t_1\}$
- $\beta(t_1, t_2; t) = \#\{e \in E(t_1) | R_e(t_1) = t, P_e(t_1) = t_2\}$

Thus, for example

\begin{align*}
[D^+_t, D^+_t] &= D^+_t + 2D^+_t - D^+_t \\
[D^-_t, D^-_t] &= -D^-_t - 2D^-_t + D^-_t \\
[D^-_t, D^+_t] &= 2D^+_t
\end{align*}

$g$ acts naturally on $\mathbb{C}\{T\}$ as follows. If $s \in T$, viewed as an element of $\mathbb{C}\{T\}$, and $t \in T$, then

\begin{align*}
D^+_t(s) &= \sum_{v \in V(s)} s \cup v \ t \\
D^-_t(s) &= \sum_{e \in E(s), P_e(s) = t} R_e(s) \\
d(s) &= |s| s
\end{align*}
3. Structure of $\mathfrak{g}$

Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be the Lie subalgebras $s$ of $\mathfrak{g}$ generated by $D^+_t$ and $D^-_t$, $t \in \mathbb{T}$. We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}d \oplus \mathfrak{n}_-$$

The relations 2.5, 2.6, and 2.7 imply that for every $t \in \mathbb{T}$

$$\mathfrak{g}_t = < D^+_t, D^-_t, d >$$

forms a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. We have that $\mathfrak{g}_t \cap \mathfrak{g}_s = \mathbb{C}d$ if $s \neq t$.

Assigning degree $|t|$ to $D^+_t$, $-|t|$ to $D^-_t$, and 0 to $d$ equips $\mathfrak{g}$ with a $\mathbb{Z}$–grading.

$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$

$\mathfrak{g}$ possesses an involution $\iota$, with

$$\iota(D^+_t) = D^-_t, \quad \iota(D^-_t) = D^+_t, \quad \iota(d) = -d$$

Thus $\iota$ is a gradation-reversing Lie algebra automorphism exchanging $\mathfrak{n}_+$ and $\mathfrak{n}_-$.

**Theorem 3.1.** $\mathfrak{g}$ is a simple Lie algebra

**Proof.** Suppose that $\mathcal{I} \subset \mathfrak{g}$ is a proper Lie ideal. If $x \in \mathcal{I}$, let $x = \sum_i x_i, \ x_i \in \mathfrak{g}_i$ be its decomposition into homogenous components. We have

$$[d, x] = \sum_n nx_n$$

which implies that $x_n \in I$ for every $n$ (because the Vandermonde determinant is invertible) i.e. $\mathcal{I} = \bigoplus_{n \in \mathbb{Z}}(\mathcal{I} \cap \mathfrak{g}_n)$. Suppose now that $x_n \in \mathfrak{g}_n, \ n > 0$. We can write $x_n$ as a linear combination of $n$–vertex rooted trees

$$x_n = \sum_{t \in T_n} \alpha_t \cdot t$$

We proceed to show that $D^+_\bullet \in \mathcal{I}$, where $\bullet$ is the rooted tree with one vertex. Let $S(x_n) \subset \mathbb{T}_n$ be the subset of n-vertex trees occurring with a non-zero $\alpha_t$ in $\mathfrak{g}_n$. Given a rooted tree $t$, let $St(t)$ denote the set of rooted trees obtained by removing all the edges emanating from the root. Let

$$St(x_n) = \bigcup_{s \in S(x_n)} St(s)$$

and let $\xi \in St(x_n)$ be of maximal degree. It is easy to see that $[D^-_\bullet, x_n]$ is a non-zero element of $\mathfrak{g}_{n-|\xi|}$. Starting with $x_n \in \mathfrak{n}_+, \ x_n \neq 0$, and repeating this process if necessary, we eventually obtain a non-zero element of $\mathfrak{g}_1 = < D^+_\bullet >$. Now, $[D^-_\bullet, D^+_\bullet] = d$, and since $[d, \mathfrak{g}] = \mathfrak{g}$, this implies $\mathcal{I} = \mathfrak{g}$. We have thus shown that if $\mathcal{I}$ is proper, then

$$\mathcal{I} \cap \mathfrak{n}_+ = 0$$
Applying $\iota$ shows that $\mathcal{I} \cap n_0$ as well, and it is clear that $\mathcal{I} \cap \mathbb{C}d = 0$.

We can now use this result to deduce a couple of facts about the representation theory of $\mathfrak{g}$.

**Corollary 3.1.** If $V$ is a non-trivial representation of $\mathfrak{g}$, then $V$ is faithful.

**Corollary 3.2.** $\mathfrak{g}$ has no non-trivial finite-dimensional representations.

The latter can also be easily deduced by analyzing the action of the $\mathfrak{sl}_2$ subalgebras $\mathfrak{g}^t$ as follows. Suppose that $V$ is a finite-dimensional representation of $\mathfrak{g}$. To show that $V$ is trivial, it suffices to show that it restricts to a trivial representation of $\mathfrak{g}^t$ for every $t \in \mathbb{T}$. This in turn, will follow if we can show that for a *single* tree $t \in \mathbb{T}$, $\mathfrak{g}^t$ acts trivially, because this implies that $d$ acts trivially, and $\mathbb{C}d \subset \mathfrak{g}^t$ plays the role of the Cartan subalgebra. Let

$$V = \bigoplus_{\delta} V_{\delta}$$

be a decomposition of $V$ into $d$-eigenspaces - i.e. if $v \in V_{\delta_i}$, then $d.v_i = \delta_i v$. Since $V$ is finite-dimensional, the set $\{\delta_i\}$ is bounded, and so lies in a disc of radius $R$ in $\mathbb{C}$. If $v \in V_{\delta_i}$ then $[d, D_i^+] = |t|D_i^+$ implies that $D_i^+.v \in V_{\delta_i+|t|}$. Choosing a $t \in \mathbb{T}$ such that $|t| > 2R$ shows that $D_i^+.v = 0$ for every $v \in V$.

3.1. **Lowest-weight representations of $\mathfrak{g}$**. We begin by examining the ”defining” representation $\mathbb{C}\{\mathbb{T}\}$ of $\mathfrak{g}$ introduced in section 2. Its decomposition into $d$-eigenspaces is given by $\mathbb{C}\{\mathbb{T}\}$. Given a representation $V$ of $\mathfrak{g}$ on which $d$ is diagonalizable, with finite-dimensional eigenspaces, and writing

$$V = \bigoplus_{\delta} V_{\delta}$$

for this decomposition, we define the *character* of $V$, $\text{char}(V, q)$ to be the formal series

$$\text{char}(V, q) = \sum_{\delta} \text{dim}(V_{\delta})q^\delta$$

The case $V = \mathbb{C}\{\mathbb{T}\}$, where $\text{dim}(V_n)$ is the number of rooted trees on $n$ vertices, suggests that representations of $\mathfrak{g}$ may contain interesting combinatorial information. The triangular structure of $\mathfrak{g}$ suggests that a theory of highest– or lowest–weight representations may be appropriate.

**Definition 3.1.** We say that a representation $V$ of $\mathfrak{g}$ is *lowest–weight* if the following properties hold

1. $V = \bigoplus V_{\delta}$ is a direct sum of finite-dimensional eigenspaces for $d$.
2. The eigenvalues $\delta$ are bounded in the sense that there exists $L \in \mathbb{R}$ such that $\text{Re}(\delta) \geq L$. 
We call the $\delta$ the weights of the representation, and category of such representations $\mathcal{O}$. If $V \in \mathcal{O}$, we say $v \in V_\delta$ is a lowest-weight vector if $n_- v = 0$. Since $D_t^-$ decreases the weight of a vector by $|t|$, and the weights all lie in a half-plane, it is clear that every $V \in \mathcal{O}$ contains a lowest-weight vector.

Recall that a representation $V$ of $g$ is indecomposable if it cannot be written as $V = V_1 \oplus V_2$ for two non-zero representations. Let $U(\mathfrak{h})$ denote the universal enveloping algebra of a Lie algebra $\mathfrak{h}$.

**Lemma 3.1.** If $v \in V_\lambda$ is a lowest-weight vector, then $U(n_+) v$ is an indecomposable representation of $g$.

**Proof.** $U(g) v$ is clearly the smallest sub-representation of $V$ containing $v$. The decomposition together with the PBW theorem implies that

$$U(g) = U(n_+) \otimes \mathbb{C}[d] \otimes U(n_-)$$

Because $v$ is a lowest-weight vector, $\mathbb{C}[d] \otimes U(n_-) v = \mathbb{C} v$. It follows that $U(g) v = U(n_+) v$. That the latter is indecomposable follows from the fact that in $U(n_+) v$, the weight space corresponding to $\lambda$ is one-dimensional, and so if $U(n_+) v = V_1 \oplus V_2$, then $v \in V_1$ or $v \in V_2$. □

Observe that

$$U(n_+) v = \oplus (U(n_+) v)_{\lambda+k}, \quad k \in \mathbb{Z}_{\geq 0}$$

where $(U(n_+) v)_{\lambda+k}$ is spanned by monomials of the form

$$D_{t_1}^+ D_{t_2}^+ \cdots D_{t_i}^+ v$$

with $|t_1| + \cdots + |t_i| = k$.

The category $\mathcal{O}$ contains Verma-like modules. For $\lambda \in \mathbb{C}$, let $\mathbb{C}_\lambda$ denote the one-dimensional representation of $\mathbb{C} d \oplus n_-$ on which $n_-$ acts trivially, and $d$ acts by multiplication by $\lambda$.

**Definition 3.2.** The $g$–module

$$W(\lambda) = U(g) \otimes_{\mathbb{C}[d] \otimes U(n_-)} \mathbb{C}_\lambda$$

will be called the Verma module of lowest weight $\lambda$.

Choosing an ordering on trees yields a PBW basis for $n_+$, and thus also a basis of the form (3.3) for $W(\lambda)$.

Given a representation $V \in \mathcal{O}$, and a lowest weight vector $v \in V_\lambda$, we obtain a map of representations

(3.4) $W(\lambda) \rightarrow V$

$1 \mapsto v$

**Lemma 3.2.** If $V \in \mathcal{O}$ is an irreducible representation, then $V$ is the quotient of a Verma module.
Proof. Since \( V \in \mathcal{O} \), \( V \) possesses a lowest-weight vector \( v \in V_\lambda \) for some \( \lambda \in \mathbb{C} \). Since \( V \) is irreducible, \( V = U(\mathfrak{g}).v = U(\mathfrak{n}_+).v \). The latter is a quotient of \( W(\lambda) \). \[ \square \]

We have

\[
\text{Char}(W(\lambda)) = q^\lambda \sum_{n \in \mathbb{Z}_\geq 0} \dim(\mathbb{C}\{T\}_n)q^n
\]
\[
= q^\lambda \prod_{n \in \mathbb{Z}_\geq 0} \frac{1}{(1 - q^n)^{P(n)}},
\]

where \( P(n) \) is the number of primitive elements of degree \( n \) in \( \mathcal{H}_K \).

3.2. Irreducibility of \( W(\lambda) \). It is a natural question whether \( W(\lambda) \) is irreducible. In this section we prove the following result:

**Theorem 3.2.** For \( \lambda \) outside a countable subset of \( \mathbb{C} \) containing 0, \( W(\lambda) \) is irreducible.

**Proof.** Let \( v \neq 0 \) be a basis for \( W(\lambda)_\lambda \). \( W(\lambda) \) contains a proper sub-representation if and only if contains a lowest-weight vector \( w \) such that \( w \notin \mathbb{C}.v \). In \( W(0) \), \( D^+.v \in W(0)_1 \) is a lowest-weight vector, since

\[
D^- D^+.v = D^+ D^- .v + d.v = 0
\]

and \( D^- .v = 0 \) for all \( t \in \mathbb{T} \) with \( |t| \geq 2 \) by degree considerations. It follows that \( W(0) \) is not irreducible.

If \( I = (t_1, \cdots, t_k) \) is a \( k \)-tuple of trees such that

\[
t_1 \preceq t_2 \preceq \cdots \preceq t_k
\]

in the chosen order, let \( D^+_I.v \) denote the vector

\[
D^+_{t_k} \cdots D^+_1 .w \in W(\lambda)
\]

\( w \in W(\lambda)_{\lambda+n} \) is a lowest-weight vector if and only if

\[
D^- .w = 0
\]

for all \( t \) such that \( |t| \leq n \). Writing \( w \) in the basis \ref{3.5}

\[
w = \sum_{|I|=n} \alpha_I D^+_I.v
\]

the conditions \ref{3.6} translate into a system of equations for the coefficients \( \alpha_I \). For example, if \( w \in W(\lambda)_{\lambda+2} \), then

\[
w = \alpha_1 D^+_1.v + \alpha_2 D^+_2 D^+_1.v
\]
and conditions $D^-v = 0$, $D^-.w = 0$ translate into

$$\begin{align*}
\lambda\alpha_1 + \lambda\alpha_2 &= 0 \\
\alpha_1 + (2\lambda + 1)\alpha_2 &= 0
\end{align*}$$

The determinant of the corresponding matrix is $2\lambda^2$, and so for $\lambda \neq 0$, there is no lowest-weight vector $w \in W(\lambda)_{\lambda+2}$. For a general $n$, the system can be written in the form

$$(A + \lambda B)[\alpha_1] = 0$$

where $A$ and $B$ are matrices whose entries are non-negative integers. Let

$$f_n(\lambda) = \dim(Ker(A + \lambda B))$$

Then for every $r \in \mathbb{N}$

$$S_{n,r} = \{\lambda \in \mathbb{C}|f_n(\lambda) \geq r\}$$

if proper, is a finite subset of $\mathbb{C}$, since the condition is equivalent to the vanishing a finite collection of sub-determinants, each of which is a polynomial in $\lambda$. The set of $\lambda \in \mathbb{C}$ for which $W(\lambda)$ is irreducible is therefore

$$\bigcup_{n \in \mathbb{N}}\{\mathbb{C}\setminus S_{n,1}\}$$

The theorem will follow if $S_{n,1}$ is proper for each $n \in \mathbb{N}$. This follows from the following Lemma.

\[ \square \]

**Lemma 3.3.** $Z(1)$ is irreducible.

**Proof.** We begin by examining the representation $\mathbb{C}\{T\}$. The degree 0 subspace $\mathbb{C}.1$ is a trivial representation of $\mathfrak{g}$. Let $M$ denote the quotient $\mathbb{C}\{T\}/\mathbb{C}.1$. It is easily seen that the exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}\{T\} \rightarrow M \rightarrow 0$$

is non-split. $M$ has highest weight 1, and the subspace $M_1$ can be identified with the span of the tree on one vertex $\bullet$. By the universal property of Verma modules, $3.4$ we have a map

$$W(1) \rightarrow M$$

(3.7)

sending the lowest-weight vector of $W(1)$ to $\bullet$. Now, $W(1)_n$ is spanned by all vectors $3.5$ such that $|t_1| + \cdots |t_k| = n - 1$, and so can be identified with the set of forests on $n - 1$ vertices, while $M_n$ can be identified with $\mathbb{C}\{T\}_n$. The operation of adding a root to a forest on $n - 1$ vertices to produce a rooted tree with $n$ vertices yields an isomorphism $W(1)_n \cong M_n$. Thus, if the map $3.7$ is a surjection, it is an isomorphism. This in turn, follows from the fact that $M$ is irreducible.
It suffices to show that $M_n$ contains no lowest-weight vectors for $n > 1$. This follows from an argument similar to the one used to prove 3.1. Let $w \in M_n$, and write

$$w = \alpha_1 t_1 + \cdots + \alpha_k t_k$$

where $|t_i| = n$ and we may assume that $\alpha_i \neq 0$. In the notation of 3.1, let $\xi \in St(w)$ be of maximal degree. Then

$$D^- \xi . w \neq 0$$

Thus, $M$ is irreducible, and hence isomorphic to $W(1)$ by the map 3.7. $\square$

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