Standard maximum likelihood drift parameter estimator in the homogeneous diffusion model is always strongly consistent✩

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Abstract

We consider the homogeneous stochastic differential equation with unknown parameter to be estimated. We prove that the standard maximum likelihood estimate is strongly consistent under very mild conditions. There are also established the conditions for strong consistency of the discretized estimator.

Keywords: Stochastic differential equation with homogeneous coefficients, drift parameter, strong consistency, discretized model

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1. Introduction

There is an extended literature devoted to standard and nonstandard approaches to the drift parameter estimation in the diffusion models, both for discrete and continuous observations. We mention only the books [1], [2], [3], [4] and references therein. Many complicated models have been studied. However, there was a curious gap even in the case of simplest homogeneous diffusion model: there were no conditions for the strong consistency of the standard maximum likelihood estimator that are close to be necessary and are sufficiently mild. We have filled the gap, applying the results of the paper [5] and have proved that, in some sense, the standard maximum likelihood estimator is always strongly consistent unless the drift coefficient is identically zero.

The paper is organized as follows. In Section 2 we prove that the denominator in the stochastic representation of the maximum likelihood estimator tends to infinity under very mild conditions and deduce from here the strong consistency. In Section 3 we establish the sufficient conditions for the strong consistency of the discretized version of the maximum likelihood estimator. Some simulation results are included.

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2. Preliminaries

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete probability space with filtration that satisfies the standard conditions. Let \(W = \{W_t, \mathcal{F}_t, t \geq 0\}\) be a standard Wiener process. Consider a homogenous diffusion process \(X = \{X_t, \mathcal{F}_t, t \geq 0\}\) that is a solution to the stochastic differential equation

\[
X_t = x_0 + \theta \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s. \quad (2.1)
\]

Here \(x_0 \in \mathbb{R}; \theta \in \mathbb{R}\) is unknown parameter to be estimated, \(a, b : \mathbb{R} \to \mathbb{R}\) are measurable functions, \(b(x) \neq 0\) for any \(x \in \mathbb{R}\), \(a\) is not zero identically.

In general, we only need the existence and uniqueness of the weak solution of equation (2.1) on the whole axis. Recall that any of the following groups of conditions on \(a\) and \(b\) supplies the existence-uniqueness for the strong solution:

Yamada conditions [6, 7]:

(i) Linear growth: there exists \(K > 0\) such that for any \(x \in \mathbb{R}\)
\[
|a(x)| + |b(x)| \leq K(1 + |x|);
\]

(ii) There exists such convex increasing function \(k : \mathbb{R}_+ \to \mathbb{R}_+\) that \(k(0) = 0\), \(\int_{0^+} k^{-1}(u)du = +\infty\) and for any \(x, y \in \mathbb{R}\)
\[
|a(x) - a(y)| \leq k(|x - y|);
\]

(iii) There exists such strictly increasing function \(\rho : \mathbb{R}_+ \to \mathbb{R}_+\) that \(\rho(0) = 0\), \(\int_{0^+} \rho^{-2}(u)du = +\infty\) and for any \(x, y \in \mathbb{R}\)
\[
|b(x) - b(y)| \leq \rho(|x - y|).
\]

Krylov–Zvonkin conditions [8]:

(i) Function \(a\) is bounded, function \(b\) is separated from 0: \(b(x) \geq \alpha > 0, x \in \mathbb{R}\);

(ii) Function \(b\) has locally bounded variation: for any \(N > 0\)
\[
var_{[-N, N]} b := \sup_{-N = x_0 < x_1 < \ldots < x_n = N} \sum |b(x_{k+1}) - b(x_k)| < \infty.
\]

Existence of the weak solution of equation (2.1) holds under continuity and linear growth of the coefficients. It was initially proved in [6]. Then the conditions of existence and uniqueness of the weak solution were generalized in [10] and the most general conditions were obtained in [11] and [12].

3. Strong consistency of the drift parameter maximum-likelihood estimator constructed for continuous observations

Denote the functions \(c(x) = \frac{a(x)}{b(x)}, d(x) = \frac{a^2(x)}{b^2(x)}\). In what follows we suppose that the following condition holds:

\((A)\) functions \(\frac{1}{b^2}, d\) and \(\frac{d}{b^2}\) are locally integrable.
Furthermore, denote \( L_t(x) = b^2(x) \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1\{|X_s - x| < \varepsilon\} ds \) the local time of the process \( X \) at the point \( x \) on the interval \([0, t]\), \( t \geq 0\). Then, according, e.g., to \[13\], for any locally integrable function \( f \) the following equality holds:

\[
\int_0^t f(X_s) ds = \int_\mathbb{R} \frac{f(x)}{b^2(x)} L_t(x) dx, \quad t \geq 0.
\]

Therefore, under the condition of local integrability, \( \int_0^t d(X_s) ds < \infty \) for any \( t > 0 \). As it is well-known, a likelihood function for equation (2.1) has a form

\[
dP_\theta(t) = \exp \left\{ \theta \int_0^t \frac{a(X_s)}{b(X_s)} dW_s + \frac{\theta^2}{2} \int_0^t d(X_s) ds \right\} \exp \left\{ \int_0^t c(X_s) dX_s \right\},
\]

and the maximum likelihood estimator of parameter \( \theta \) constructed by the observations of \( X \) on the interval \([0, t]\), has a form

\[
\hat{\theta}_t = \frac{\int_0^t c(X_s) dX_s}{\int_0^t d(X_s) ds} = \theta + \frac{\int_0^t \frac{a(X_s)}{b(X_s)} dW_s}{\int_0^t d(X_s) ds}.
\] (3.1)

In order to establish the criteria of the strong consistency of \( \hat{\theta}_t \) in terms of the coefficients \( a \) and \( b \), denote \( \varphi(x) = \exp \left\{ -2\theta \int_0^x c(y) dy \right\} \), \( \Phi(x) = \int_0^x \varphi(y) dy \).

Concerning the asymptotic behavior of the integral \( \int_0^t d(X_s) ds \) under the fixed value of parameter \( \theta \neq 0 \), two cases can be considered.

Let for some \( \theta \in \mathbb{R} \) \( \Phi(+\infty) = -\Phi(-\infty) = +\infty \). In this case the diffusion process \( X \) is recurrent and its trajectories have the property: \( \lim_{t \to \infty} X_t = +\infty \) a.s. and \( \lim_{t \to \infty} X_t = -\infty \) a.s.

Furthermore, \( \int_0^t f(X_s) ds = \int_\mathbb{R} \frac{f(x)}{b^2(x)} L_\infty(x) dx \). However, as it was mentioned in \[13\] and \[14\], \( L_\infty(x) = \infty \) \( P \)-a.s. for any \( x \in \mathbb{R} \) and recurrent process \( X \). It means that \( \int_0^t f(X_s) ds = \infty \) a.s. for any \( f \) that is not identically 0, and in this case

\[
\int_0^\infty d(X_s) ds = \infty \quad P - \text{a.s.}
\] (3.2)
Now, let at least one of the integrals \( \Phi(\infty) \) or \( \Phi(-\infty) \) be finite. In this case the process \( X \) is transient. We shall apply the following result that is the reformulation of Theorem 2.12 from \([1]\). Denote \( \psi_+(x) = \frac{\int_0^\infty \varphi(y)dy}{\varphi(x)} \), \( \psi_-(x) = \frac{\int_0^\infty \varphi(y)dy}{\varphi(x)} \), \( J_\infty(f) = \int_0 f(X_s)ds \).

**Theorem 3.1.** ([1])

- Let \( \psi_+(0) < \infty, \psi_-(0) = \infty \).
  
  If \( I_1(f) := \int_0^\infty \frac{|f(x)|}{a^2(x)} \psi_+(x)dx < \infty \), then \( J_\infty(f) \in \mathbb{R} \) \( P \)-a.s.
  
  If \( I_1(f) = \infty \) then \( J_\infty(f) = \infty \) \( P \)-a.s.

- Let \( \psi_+(x) = \infty, \psi_-(x) < \infty \).
  
  If \( I_2(f) := \int_\infty^0 \frac{|f(x)|}{a^2(x)} \psi_-(x)dx < \infty \), then \( J_\infty(f) \in \mathbb{R} \) \( P \)-a.s.
  
  If \( I_2(f) = \infty \) then \( J_\infty(f) = \infty \) \( P \)-a.s.

- Let \( \psi_+(x) < \infty, \psi_-(x) < \infty \).
  
  If \( I_1(f) < \infty \), then \( J_\infty(f) \in \mathbb{R} \) \( P \)-a.s. on the set \( X^*_f \rightarrow +\infty \).
  
  If \( I_1(f) = \infty \) then \( J_\infty(f) = \infty \) \( P \)-a.s. on the set \( X^*_f \rightarrow +\infty \).
  
  If \( I_2(f) < \infty \), then \( J_\infty(f) \in \mathbb{R} \) \( P \)-a.s. on the set \( X^*_f \rightarrow -\infty \).
  
  If \( I_2(f) = \infty \) then \( J_\infty(f) = \infty \) \( P \)-a.s. on the set \( X^*_f \rightarrow -\infty \).

**Theorem 3.2.** (1) Let for some \( \theta \neq 0 \) \( \Phi(\infty) < +\infty \). Then

\[
I_1(d) = \int_0^\infty \frac{d(x)}{b^2(x)} \psi_+(x)dx = +\infty.
\]

(2) Let for some \( \theta \neq 0 \) \( \Phi(-\infty) < \infty \). Then \( I_2(d) := \int_\infty^0 \frac{d(x)}{b^2(x)} \psi_-(x)dx = \infty \).

**Proof.** We prove only the first statement since the second one can be proved similarly. Note that \( \frac{\varphi(y)}{\varphi(x)} = \exp \left\{ -2\theta \int_x^y c(u)du \right\} \) and \( \frac{d(x)}{b^2(x)} = \theta^2 c^2(x) \). It means that without loss of generality, we can put \( \theta = 1 \). Therefore, applying Fubini
theorem for nonnegative integrands and Schwartz inequality, we get

\[
I_1(d) = \int_0^\infty c^2(x) \int_0^\infty \exp \left\{ -2 \int_x^y c(u) du \right\} dy dx
\]

\[
= \int_0^\infty \int_0^y c^2(x) \exp \left\{ -2 \int_x^y c(u) du \right\} dx dy
\]

\[
\geq \int_0^\infty \left( \int_0^y c(x) \exp \left\{ -\int_x^y c(u) du \right\} dx \right)^2 \frac{dy}{y}
\]

\[
\geq \int_1^\infty \left( \int_1^y c(x) \exp \left\{ -\int_1^y c(u) du \right\} dx \right)^2 \frac{dy}{y}
\]

\[
= \int_1^\infty \left( 1 - \exp \left\{ -\int_1^y c(u) du \right\} \right)^2 \frac{dy}{y}.
\]

(3.3)

It is sufficient to prove that the last integral in (3.3) diverges. However, it consists of three terms, one of which, \( \int_1^\infty \frac{dy}{y} \) diverges, and two other converge:

\[
\int_1^\infty \exp \left\{ -\int_1^y c(u) du \right\} \frac{dy}{y} \leq (\Phi(\infty))^{\frac{1}{2}} \left( \int_0^\infty \frac{dy}{y^2} \right)^{\frac{1}{2}} < \infty
\]

and

\[
\int_1^\infty \exp \left\{ -2 \int_1^y c(u) du \right\} \frac{dy}{y} \leq \Phi(\infty) < \infty.
\]

\( \Box \)

Corollary 3.1. As an immediate consequence of Theorems 3.1 and 3.2 and formula (3.2), we get the following statement: let equation (2.1) have the weak solution, the coefficients \( a \) and \( b \) satisfy the condition (A) and \( a \) be not identically zero. Then \( \int_0^\infty d(X_s)ds = +\infty \) P-a.s.

The next theorem is the main result in this section.

Theorem 3.3. Let equation (2.1) has the weak solution, coefficients \( a \) and \( b \) satisfy condition (A) and \( a \) is not identically zero. Then maximum likelihood estimator \( \hat{\theta}_t \) is strongly consistent as \( t \to \infty \).

Proof. According to representation (3.1), \( \hat{\theta}_t \) is strongly consistent if for locally square-integrable martingale \( M_t = \int_0^t \frac{a(X_s)}{b(X_s)} dW_s \) we have that \( \frac{M_t}{\langle M \rangle_t} \to 0 \) P-a.s.

However, according to the strong law of large numbers for martingales (Theorem 10, §6, Chapter 2, [15]), under condition \( \langle M \rangle_t \to \infty, t \to \infty \) P-a.s., we have that \( \frac{M_t}{\langle M \rangle_t} \to 0 \) P-a.s. The proof immediately follows now from Corollary 3.1. \( \Box \)

4. Discretization and strong consistency

In this section we suppose that the coefficients \( a, b \) and \( c \) are bounded and Lipschitz, more precisely, satisfy condition: for some \( a_0 > 0 \) and \( K > 0 \) and for any \( x, y \in \mathbb{R} \)
Let $0 < \alpha < \frac{1}{2}$. Suppose that we observe the process $X$ that is the solution of equation (2.1), only at discrete moments of time $t^n_k = \frac{k}{n}, 0 \leq k \leq n^{1+\alpha}$.

Consider a discretized version of the estimate $\hat{\theta}_t^d$:

$$\hat{\theta}_t^d = \frac{\sum_{k=0}^{n^{1+\alpha}} c(X_{\frac{k}{n}}) \triangle X_{\frac{k}{n}}}{\frac{1}{n} \sum_{k=0}^{n^{1+\alpha}} d(X_{\frac{k}{n}})},$$

where $\triangle X_{\frac{k}{n}} = X_{\frac{k+1}{n}} - X_{\frac{k}{n}}$. Then

$$\hat{\theta}_t^d = \left(\frac{1}{n} \sum_{k=0}^{n^{1+\alpha}} d(X_{\frac{k}{n}})\right)^{-1} \left(\sum_{k=0}^{n^{1+\alpha}} d(X_{\frac{k}{n}}) \left(\theta \int_{\frac{k}{n}}^{\frac{k+1}{n}} a(X_s)ds + \int_{\frac{k}{n}}^{\frac{k+1}{n}} b(X_s)dW_s\right)\right) =$$

$$= \theta + \left(\frac{1}{n} \sum_{k=0}^{n^{1+\alpha}} d(X_{\frac{k}{n}})\right)^{-1} \left(\sum_{k=0}^{n^{1+\alpha}} c(X_{\frac{k}{n}}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (a(X_s) - a(X_{\frac{k}{n}}))ds + \sum_{k=0}^{n^{1+\alpha}} \frac{a(X_{\frac{k}{n}})}{b(X_{\frac{k}{n}})} \triangle W_k^n + \sum_{k=0}^{n^{1+\alpha}} c(X_{\frac{k}{n}}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (b(X_s) - b(X_{\frac{k}{n}}))dW_s\right) =: \theta + I^n_1 + I^n_2 + I^n_3.$$

**Theorem 4.1.** Let equation (2.1) has the weak solution, coefficients $a$ and $b$ satisfy condition (B) and $a$ is not identically zero. Then $\hat{\theta}_t^d$ is strongly consistent as $n \to \infty$.

**Proof.** It is sufficient to prove that $I^n_r \to 0, r = 1, 2, 3$ a.s. as $n \to \infty$. Evidently, the denominator $\left(\frac{1}{n} \sum_{k=0}^{n^{1+\alpha}} d(X_{\frac{k}{n}})\right)^{-1}$ tends to infinity a.s. as $n \to \infty$. Consider the numerator, say $J^n_1$, for $I^n_1$:

$$|J^n_1| = \left|\sum_{k=0}^{n^{1+\alpha}} c(X_{\frac{k}{n}}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (a(X_s) - a(X_{\frac{k}{n}}))ds\right| \leq |\theta| a_0 n^\alpha \sup_{0 \leq k \leq n^{1+\alpha}, \frac{k}{n} \leq s \leq \frac{k+1}{n}} |a(X_s) - a(X_{\frac{k}{n}})| \leq K |\theta| a_0 n^\alpha \sup_{0 \leq k \leq n^{1+\alpha}, \frac{k}{n} \leq s \leq \frac{k+1}{n}} |X_s - X_{\frac{k}{n}}|. $
In turn, \( |X_s - X_{\frac{k}{n}}| \leq \frac{\mu n}{\alpha} + \frac{\int b(X_u) dW_u}{\frac{k}{n}} \). Therefore,

\[
J_N^1 \leq K|\theta|a_0^2n^{\alpha - 1} + K|\theta|a_0 \xi_n,
\]

where \( \xi_n = n^{\alpha} \sup_{0 \leq k \leq n^{1+\alpha}} \frac{k}{n} \leq s \leq \frac{k+1}{n} \left| \int b(X_u) dW_u \right| \). For any \( \varepsilon > 0 \) denote \( A_n = \{ \omega : \xi_n \geq \varepsilon \} \). Then it follows from Burkholder-Gundy inequality that for any \( p > 1 \)

\[
P(A_n) \leq C \varepsilon^{-p} n^{\alpha p} \sum_{k=1}^{\frac{k+1}{n}} E(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |b(X_u)|^2 du)^{\frac{p}{2}} \leq C \varepsilon^{-p} n^{\alpha p - \frac{p}{2} + 1},
\]

and \( \sum_{n=1}^{\infty} P(A_n) < +\infty \) if we choose \( p > \frac{4}{1 - 2\alpha} \). It means that for any \( \omega \in \Omega \) there exists \( n(\omega) \) such that for \( n > n(\omega) \)

\[
n^{\alpha} \sup_{0 \leq k \leq n^{1+\alpha}, \frac{k}{n} \leq s \leq \frac{k+1}{n}} \left| \int b(X_n) dW_u \right| \leq \varepsilon.
\]

Therefore, \( I_n^1 \to 0, \) \( n \to \infty \) P-a.s. Consider the term \( I_n^2 \). Denote martingale \( N_n : = \sum_{k=0}^{n^{1+\alpha}} a(X_{\frac{k}{n}}) \triangle W^\nu_k \). Then \( I_n^2 = \frac{N_n}{\langle N \rangle_n} \to 0, t \to \infty \) a.s. since \( \langle N \rangle_t \to \infty \) a.s.

Consider the numerator for \( I_n^3 \). It is the square-integrable martingale with respect to the discretized filtration \( \{ \mathcal{F}_{\frac{k}{n}} = \sigma \{ X_{\frac{i}{n}}, 0 \leq i \leq k \}, 0 \leq k \leq n^{1+\alpha} \} \). Denote it as \( P_n \). Its quadratic characteristic equals

\[
\langle P \rangle_n = \sum_{k=0}^{n^{1+\alpha}} c^2(X_{\frac{k}{n}}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} E((b(X_s) - b(X_{\frac{k}{n}}))^2 / \mathcal{F}_{\frac{k}{n}}) ds,
\]

and

\[
\langle P \rangle_n \leq a_0^2 n^{\alpha} \sup_{0 \leq k \leq n^{1+\alpha}, \frac{k}{n} \leq s \leq \frac{k+1}{n}} E((b(X_s) - b(X_{\frac{k}{n}}))^2 / \mathcal{F}_{\frac{k}{n}})
\]

\[
\leq 2K^2 a_0^2 n^{\alpha - 2} + 2K^2 a_0^2 n^{\alpha - 1} \leq C n^{\alpha - 1}
\]

with some constant \( C > 0 \). Now we use the fact from [13] that for any locally square integrable martingale \( Y \) and for any constant \( a > 0 \) \( \frac{Y}{a + \langle Y \rangle_t} \) converges a.s. to some finite random variable as \( t \to \infty \). Therefore, we can take some \( a > 0 \) and conclude that

\[
\frac{P_n}{a + \langle P \rangle_n} \to \xi \text{ a.s.,}
\]
where $\xi$ is some random variable and consequently

$$\frac{P_n}{\frac{1}{n} \sum_{k=1}^{n+\alpha} d(X_k \frac{1}{n})} = \frac{P_n}{a + \langle P \rangle_n} \cdot \frac{a + \langle P \rangle_n}{\frac{1}{n} \sum_{k=1}^{n+\alpha} d(X_k \frac{1}{n})} \to 0$$

a.s. as $n \to \infty$.

5. Some simulation results

We simulated the model and set the discretization interval $\Delta t = 0.01$; number of the simulated trajectories is 1000; the value of the parameter to be estimated equals 1. Let us consider three cases:

(i) Let $a(x) = 1 + x$, $b(x) = x^{-1/3}$. Then for different $t$ we have the values of $\hat{\theta}_t$ as presented in the Table 1.

| $t$  | 1    | 10   | 20   | 30   |
|------|------|------|------|------|
| $\hat{\theta}_t$ | 0.870 | 0.999 | $1 - 5 \times 10^{-8}$ | $1 + 10^{-11}$ |

(ii) Let $a(x) = 1 + x$, $b(x) = 2 + \sin x$. Then for different $t$ we have the values of $\hat{\theta}_t$ as presented in the Table 2.
(iii) Let $a(x) = |x|1_{\{|x|\leq 1\}}$, $b(x) = 1$. Then for different $t$ we have the values of $\hat{\theta}_t$ as presented in the Table 3.

| $t$  | 5   | 10  | 20  | 30  |
|------|-----|-----|-----|-----|
| $\theta_t$ | 0.908 | 0.997 | $1 + 2 \times 10^{-7}$ | $1 + 6 \times 10^{-11}$ |

We see that in the last case, when the process is recurrent, the convergence is slow. It can be explained in such a way: the drift coefficient “often” equals zero. When it is zero, we can not estimate the value of parameter. So, we must wait until sufficient quantity of information comes.

References

[1] R. Liptser, A. Shiryaev, Statistics of Random Processes. II, Applications, Springer-Verlag, Berlin Heidelberg 2001.

[2] B. L. S. Prakasa Rao, Asymptotic Theory of Statistical Inference Wiley, New York 1987.

[3] C. Heyde, Quasi-Likelihood and Its Application: a General Approach to Optimal Parameter Estimation, Springer-Verlag, New York 1997.

[4] M. Kessler, A. Lindner, M. Sørensen (editors), Statistical Methods for Stochastic Differential Equations, CRC Press, Taylor and Francis group, Boca Raton 2012.

[5] A. Mijatovic, M. Urusov, Convergence of integral functionals of one-dimensional diffusions. Electronic Communications in Probability, 17(61) (2012) 1–13.

[6] T. Yamada, S. Watanabe On the uniqueness of solutions of stochastic differential equations J. Math. Kyoto Univ. 11(1) (1971) 155–167.

[7] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd ed. North-Holland Publishing Company 1989.

[8] N.V. Krylov, A.K. Zvonkin, On strong solutions of stochastic differential equations. Sel. Math. Sov. I, (1981) 19–61.

[9] A. V. Skorokhod, Studies in the theory of random processes Addison-Wesley, Reading 1965
[10] N.V. Krylov, Controlled Diffusion Processes, Springer-Verlag, Berlin Heidelberg 2009 Strong Markov Continuous Local Martingales and Solutions of One-Dimensional Stochastic Differential Equations (Part III)

[11] H. J. Engelbert, W. Schmidt, On the behaviour of certain functionals of the Wiener process and applications to stochastic differential equations, In: Stochastic differential systems, Lecture notes in Control and Information Sciences, 36 (1981), 47–55.

[12] H. J. Engelbert, W. Schmidt, Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations, I, II, III. Math. Nachr., 143(1) (1989) 167184, 144(1)(1989) 241281, 151(1)(1991) 149197.

[13] J. Pitman, M. Yor Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches, Bernoulli, 9(1) (2003) 1-24.

[14] K. Itô, H. P. McKean, Diffusion Processes and Their Sample Paths, Springer-Verlag, 1965.

[15] R. Liptser, A. Shiryaev, Theory of Martingales, Kluwer Academic Publishers, Dordrecht Boston London 1989