On the general elephant conjecture for Mori conic bundles

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Abstract

Let \( f : X \to S \) be an extremal contraction from a threefolds with terminal singularities onto a surface (so called Mori conic bundle). We study some particular cases of such contractions: quotients of usual conic bundles and index two contractions. Assuming Reid’s general elephants conjecture we also obtain a rough classification. We present many examples.

Introduction.

This paper continues our study of extremal contractions from threefolds to surfaces \([13], [21], [22]\). Such contractions occur naturally in birational classification theories of three-dimensional algebraic varieties. We are interested in the local situation.

(0.1) Definition. Let \((X,C)\) is a germ of three-dimensional normal complex space \(X\) along a compact reduced curve \(C\) and let \((S,o)\) be a germ of a normal two-dimensional complex space \(S\) in \(o \in S\). Assume that \(X\) has at worst terminal singularities. We say that proper morphism \(f : (X,C) \to (S,o)\) is a Mori conic bundle if

(i) \( f^{-1}(o)_{\text{red}} = C \),
(ii) \( f_* O_X = O_S \),
(iii) \(-K_X\) is \(f\)-ample.

The first example of Mori conic bundles are conic bundles in the classical sense: \(f : (X,C) \to (S,o)\) is called a (usual) conic bundle if \((S,o)\) is nonsingular and there exists an embedding \(i : (X,C) \to \mathbb{P}^2 \times (S,o)\) such that \(O_{\mathbb{P}^2 \times S}(X) = O_{\mathbb{P}^2 \times S}(2,0)\).

The general question arising very naturally lies in the classification problem of Mori conic bundles. The following two conjectures are interesting for application of the Sarkisov program to study of birational properties of threefolds with structure of conic bundle (see \([1], [8]\)).

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Conjecture I (special case of Reid’s general elephants conjecture). Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle. Then a general member of the anticanonical linear system \( | - K_X | \) has only DuVal singularities.

Conjecture II. Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle. Then \((S, o)\) is either nonsingular or a DuVal singularity of type \( A_n \).

One can expect that a Mori conic bundle with very singular base is a quotient of a usual conic bundle by a cyclic group (for example it follows from the conjecture (0.2), see (4.2)). So this particular case seems to be general. We also study index two Mori conic bundles in Section 2. Surprisingly situation here is not very complicated and there are only three cases. We present many examples. The methods are completely elementary. Almost all the results of this work were announced in [20] and [21].

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1 Preliminary results

Throughout this paper a variety is a reduced irreducible complex space. On a normal variety \( X \) by \( K_X \) we will denote its canonical (Weil) divisor. If \( X \) is a variety and \( C \subset X \) is its closed subvariety, then \((X, C)\) is a germ of \( X \) along \( C \). Sometimes we will replace \((X, C)\) by its sufficiently small representative \( X \). When we say that a variety \( X \) has terminal singularities, it means that singularities are not worse than that, so \( X \) can be nonsingular.

We need some facts about three-dimensional terminal singularities.

Let \((X, P)\) be a terminal singularity of index \( m \geq 1 \) and let \( \pi : (X^2, P^2) \to (X, P) \) be the canonical cover. Then \((X^2, P^2)\) is a terminal singularity of index 1. It is known [24] that \((X^2, P^2)\) is a hypersurface singularity, i.e. there exists an \( \mathbb{Z}_m \)-equivariant embedding \((X^2, P^2) \subset (\mathbb{C}^4, 0)\).

Theorem 5. If in notations above \((X^2, P^2)\) is smooth, then it is \( \mathbb{Z}_m \)-isomorphic to \((\mathbb{C}^3_{x_1,x_2,x_3}, 0)\) with the action of \( \mathbb{Z}_m \) by

\[
(x_1, x_2, x_3) \to (\varepsilon^a x_1, \varepsilon^{-a} x_2, \varepsilon^b x_3),
\]

where \( \varepsilon = \exp(2\pi i/m) \), and \( a, b \) are integers prime to \( m \). Conversely every such singularity is terminal.

Such singularity is denoted by \( \frac{1}{m}(a,-a,b) \) or \( \mathbb{C}^3/\mathbb{Z}_m(a,-a,b) \).
(1.1.2) **Theorem** ([17], [23]). In notations above assume that \((X^2, P^2)\) is singular, then it is \(\mathbb{Z}_m\)-isomorphic to a hypersurface \(\{\phi(x_1, x_2, x_3, x_4) = 0\}\) in \((\mathbb{C}^4_{x_1, x_2, x_3, x_4}, 0)\), where there are two cases for the action of \(\mathbb{Z}_m\):

(i) (the main series) \((x_1, x_2, x_3, x_4; \phi) \rightarrow (\varepsilon^a x_1, \varepsilon^{-a} x_2, \varepsilon^b x_3, x_4; \phi)\), where \(\varepsilon = \exp(2\pi i/m)\), and \(a, b\) are integer prime to \(m\).

(ii) (the exceptional series) \(m = 4\) and \((x_1, x_2, x_3, x_4; \phi) \rightarrow (ix_1, -ix_2, x_3, -x_4; -\phi)\), where \(a = 1\) or \(3\).

(1.1.3) **Remark.** Terminal singularities of index \(> 1\) are classified by type of general member \(\in | - K_{X,P}|\). For example there are four types of index two terminal singularities \(cA/2, cAx/2, cD/2\) and \(cE/2\) ([23] (we use notations of [13]).

(1.2) **Lemma.** Let \((X, P)\) be a germ of terminal threefold singularity and let \(F \in | - K_{X,P}|\) be an element of anticanonical linear system. If \(F\) is an irreducible nonsingular surface, then \((X, P)\) is nonsingular.

(1.2.1) **Proof.** If \((X, P)\) is of index one, then \(F\) is Cartier and \((X, P)\) is nonsingular in this case. So we assume that \((X, P)\) is of index \(m > 1\). Consider the canonical cover \(\pi : (X^2, P^2) \rightarrow (X, P)\) and let \(F^2 = \pi^{-1}(F)\). Since \(\pi\) is étale outside \(P\) and \(F - \{P\}\) is simply connected, the restriction \(\pi : F^2 \rightarrow F\) splits non-trivially, so \(F^2 = F^2_1 + \ldots + F^2_m\). But then \(F^2\) is the only point of intersection of components \(F^2_i\). On the other hand each \(F^2_i\) is \(\mathbb{Q}\)-Cartier, a contradiction. \(\square\)

It is well known that every DuVal (and, more general, log-terminal) singularity \((F, P)\) is a quotient of a nonsingular germ \((\mathbb{C}^2, 0)\) by a finite group \(G\) acting on \(\mathbb{C}^2\) free outside \(0\). The order of \(G\) is called the topological index of \((F, P)\). Similar to ([12]) one can prove the following.

(1.2.2) **Lemma.** Let \((X, P)\) be a germ of a terminal threefold singularity of index \(m > 1\) and \(F \in | - K_{X,P}|\) be an anticanonical divisor. Assume that the surface \(F\) is reduced, irreducible and the point \((F, P)\) is DuVal of topological index \(n\). Then \(n\) is divisible by \(m\). Moreover if \(n = m\), then \((X, P)\) is a cyclic quotient singularity and \((F, P)\) is of type \(A_{m-1}\).

Now we present some elementary properties of Mori conic bundles.

(1.3) **Theorem** ([4] (see also (1.7.1))). Let \(f : X \rightarrow S\) be a Mori conic bundle. Assume that \(X\) has only singularities of index \(1\). Then \(S\) is a nonsingular surface and \(f\) is a conic bundle (possibly singular).

The following statement is a consequence of the Kawamata-Viehweg vanishing theorem.

(1.4) **Proposition** (cf. [18], (1.2))). Let \(f : X \rightarrow S\) be a Mori conic bundle. Then

\[ R^i f_* O_X = 0, \quad i > 0. \]

(1.4.1) **Corollary** (cf. [18], (1.3))]. Let \(f : (X, C) \rightarrow (S, o)\) be a Mori conic bundle. Then

(i) the fiber \(C\) is a tree of rational curves, i. e. \(p_a(C_0) = 0\) for any one-dimensional subscheme \(C_0 \subset C\).

(ii) \(\text{Pic}(X) \simeq H^2(C, \mathbb{Z}) \simeq \mathbb{Z}^\rho\), where \(\rho\) is the number of components of \(C\).
(1.5) **Construction.** Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle. Assume that \((S, o)\) is a singular point. By easy remark in [4] (see also [6]), \((S, o)\) is a quotient singularity. It means that \((S, o)\) is a quotient of nonsingular point \((S', o')\) by a finite group \(G\), where the action of \(G\) on \(S' - o'\) is free. Therefore there exists a faithful representation \(G \hookrightarrow GL(T_{o', S'}) = GL_2(\mathbb{C})\). Let \( g : S' \to S \) be the quotient morphism and let \(X'\) be the normalization of \(X \times_S S'\). We have the following commutative diagram

\[
\begin{array}{ccc}
(X', C') & \xrightarrow{h} & (X, C) \\
\downarrow f' & & \downarrow f \\
(S', o') & \xrightarrow{g} & (S, o)
\end{array}
\]

where \(C' := f'^{-1}(o')\). Then \(G\) acts on \((X', C')\) and obviously \(X = X'/G, C = C'/G\). Since the action of \(G\) on \(S' - o'\) is free, so is the action on \(X' - C'\). Therefore \(X'\) has only terminal singularities, because \(\text{codim}(C') = 2\) (see e. g. [3, 6.7]). It gives us also that the action is free outside a finite number of points \(Q_1, \ldots, Q_k \in X'\) (for each \(Q_i\) its image \(P_i = h(Q_i)\) has index \(> 1\)). Since \(h : X' \to X\) has no ramification divisors, the anticanonical divisor \(-K_{X'} = h^*(-K_X)\) is ample over \(S'\). We obtain a new Mori conic bundle \(f' : (X', C') \to (S', o')\) over a nonsingular base.

Professor S. Mukai pointed out that actually \(X \times_S S'\) is normal, so we can take \(X' = X \times_S S'\).

(1.6) **Lemma (see [19], [20], [12]).** In notations above \(G\) is cyclic and has at least one fixed point on \(X'\). In particular, \((S, o)\) is a cyclic quotient singularity.

(1.7) **Proof.** Since \(p_a(C') = 0\), it is easy to prove by induction on the number of components of \(C\) that \(G\) has either a fixed point \(Q \in C'\) or an invariant component \(C'_0 \subset C', C'_0 \simeq \mathbb{P}^1\). But in the second case we have two inclusions \(G \subset PGL_2(\mathbb{C})\) and \(G \subset GL_2(\mathbb{C})\), where \(G \subset GL_2(\mathbb{C})\) contains no quasireflections. By the classification of finite subgroups in \(PGL_2(\mathbb{C})\) \(\simeq SO_3(\mathbb{C})\), \(G\) is cyclic and has two fixed points on \(C'_0\). Therefore in any case \(G\) has a fixed point \(Q \in X'\). Take a small neighborhood \(U \subset X\) of \(P := h(Q)\). We have a surjective map \(\pi_1(U - P) \to G\). But \(\pi_1(U - P)\) is cyclic because \((X, P)\) is a quotient of a hypersurface singularity by a cyclic group. Whence so is \(G\). This proves our claim.

(1.7.1) **Corollary (cf. [4]).**

(i) If \(X\) has index one, then \((S, o)\) is nonsingular.

(ii) If \(X\) has index two, then \((S, o)\) is either nonsingular or DuVal of type \(A_1\).

(iii) Let \(f : (X, C) \to (S, o)\) be a Mori conic bundle. If \((S, o)\) is a cyclic quotient singularity of type \(\frac{1}{n}(a, b)\), then \(X\) contains at least one point of index \(\geq n\).

2. **Quotients of conic bundles.**

By [1.5] and by [1.6] every Mori conic bundle \(f : (X, C) \to (S, o)\) over a singular base is a quotient of another Mori conic bundle \(f' : (X', C') \to (S', o')\) with a nonsingular base by a cyclic group. In this section we classify Mori conic bundles \(f : (X, C) \to (S, o)\) under the assumption that \(X'\) is Gorenstein (and then \(f' : (X', C') \to (S', o')\) is a conic bundle by [1.3]). First we present several examples.
(2.1) **Example (Toric example).** Let \( \mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{C}^2 \) be the standard projection. Define the action of the group \( \mathbb{Z}_n \) on \( \mathbb{C}^2_{u,v} \) and \( \mathbb{P}_x^1 \times \mathbb{C}^2_{u,v} \):

\[
(x_0, x_1; u, v) \to (x_0, \varepsilon x_1; \varepsilon^a u, \varepsilon^{-a} v),
\]

where \( \varepsilon = \exp(2\pi i/n) \), \( a \in \mathbb{N} \) and \( (n, a) = 1 \). Denote \( X = (\mathbb{P}^1 \times \mathbb{C}^2)/\mathbb{Z}_n, S = \mathbb{C}^2/\mathbb{Z}_n. \) Then the projection \( f : X \to S \) is a Mori conic bundle. The threefold \( X \) has on the fiber \( f^{-1}(0) \) exactly two terminal points \( P_1, P_2 \) which are cyclic quotients of type \( \frac{1}{n}(a, -a, \pm 1) \), the surface \( S \) has in 0 a DuVal point of type \( A_{n-1} \).

(2.2) **Example.** Consider the following hypersurface in \( \mathbb{P}_x^2_{x_0,x_1,x_2} \times \mathbb{C}^2_{u,v} \):

\[
X' := \{ x_0^2 + v x_1^2 + \psi(u, v)x_1 x_2 + u x_2^2 = 0 \},
\]

where \( \psi(0, 0) = 0 \). Let \( n \) be an odd integer, \( n = 2q + 1 \), where \( q \in \mathbb{N} \). Define an action of \( \mathbb{Z}_n \) on \( \mathbb{P}^2 \times \mathbb{C}^2 \) by

\[
(x_0, x_1, x_2, u, v) \to (x_0, \varepsilon^{-q} x_1, \varepsilon^q x_2, \varepsilon u, \varepsilon^{-1} v),
\]

where \( \varepsilon = \exp(2\pi i/n) \). If \( \psi(u, v) \) is an invariant, then \( \mathbb{Z}_n \) acts naturally on \( X' \). As in (2.1) the \( f : X'/\mathbb{Z}_n \to \mathbb{C}^2/\mathbb{Z}_n \) is a Mori conic bundle. The singular locus of \( X'/\mathbb{Z}_n \) consist of two terminal cyclic quotient points of index \( n \). The point \( (S, o) \) is DuVal of type \( A_{n-1} \).

(2.3) **Example.** Let \( X' \) be a hypersurface in \( \mathbb{P}_x^2_{x_0,x_1,x_2} \times \mathbb{C}^2_{u,v} \), defined by the equation

\[
x_0^2 + x_1^2 + x_2^2 \phi(u, v) = 0,
\]

where \( \phi(u, v) \) has no multiple factors and contains only monomials of even degree. Denote by \( f' : X' \to \mathbb{C}^2 \) the natural projection. Then \( X' \) has only one singular point \( P' = (x_0 = x_1 = u = v = 0) \) on \( f'^{-1}(0) \). Define the action of \( \mathbb{Z}_2 \) on \( X' \) and \( \mathbb{C}^2 \) by

\[
(x_0, x_1, x_2, u, v) \to (-x_0, x_1, x_2, -u, -v).
\]

Let \( X = X'/\mathbb{Z}_2, S = \mathbb{C}^2/\mathbb{Z}_2. \) The only fixed point on \( X' \) is \( P' \) it gives us a unique point \( P \in X \) of index two. The variety \( X \) has no other singular points. The surface \( S \) has a DuVal singularity of type \( A_1 \) at 0. There are two cases for \( \phi(u, v)\):

1. \( \text{mult}_{(0,0)}(\phi) = 2 \), then \( (X, P) \) is terminal of type \( cA/2 \);
2. \( \text{mult}_{(0,0)}(\phi) \geq 4 \), then \( (X, P) \) is terminal of type \( cAx/2 \).

(2.4) **Theorem.** Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle. Assume that \( f \) is a quotient of a conic bundle \( f' : (X', C') \to (C'^2, 0) \) by \( \mathbb{Z}_n \) such that the action of \( \mathbb{Z}_n \) on \( C^2 - \{0\} \) is free. Then there exists an analytic isomorphism between \( f : (X, C) \to (S, o) \) and one of examples [2.1], [2.2] or [2.3]. In particular, \( (S, o) \) is DuVal of type \( A_{n-1} \).

(2.5) **Proof.** By Cartan’s lemma [1], one can choose coordinates \( u, v \) in \( \mathbb{C}^2 \) such that the action of \( \mathbb{Z}_m \) is \( (u, v) \mapsto (e^a u, e^b v) \), where \( e := \exp(2\pi i/m) \), \( (a, m) = (b, m) = 1 \).

Let \( X_0' := f'^{-1}(0) \) be the scheme-theoretical fiber over 0. Then \( X_0' \) is isomorphic to a conic in \( \mathbb{P}^2 \) (see [1.3]). There are the following cases.
(2.6) CASE I. \( X'_0 \) is a non-degenerate conic. Then in the analytic situation \( X' \simeq \mathbb{P}^1 \times \mathbb{C}^2 \) by the Grauert-Fisher theorem and we may assume that the action of \( \mathbb{Z}_n \) in some coordinate systems \((x_0, x_1) \in \mathbb{P}^1 \) and \((u, v) \in \mathbb{C}^2 \) is

\[
(x_0, x_1; u, v) \rightarrow (x_0, \varepsilon x_1; \varepsilon^a u, \varepsilon^b v), \quad (a, m) = (b, m) = 1,
\]

where we may take \( \text{wt}(x_0) = 0 \) because \((x_0, x_1)\) are homogeneous coordinates. There are exactly two fixed points on \( X' \):

\[
Q_0 = \{x_0 = u = v = 0\}, \quad Q_1 = \{x_1 = u = v = 0\}.
\]

They give us two points of index \( n \) on \( X'/\mathbb{Z}_n \) of types \( \frac{1}{n}(-1, a, b) \) and \( \frac{1}{n}(1, a, b) \), respectively. By (1.1.1) these two points are terminal only if \( a + b = n \) (recall that \( a, b \) are defined modulo \( n \)). We obtain the example (2.1).

(2.7) CASE II. The fiber \( X'_0 \) is reducible. Then \( X'_0 = L_1 + L_2 \) is pair of lines intersecting each other in one point, say \( Q \), which is a fixed point. Let \( P := h(Q) \).

We fix a generator \( s \in \mathbb{Z}_n \) and for \( \mathbb{Z}_n \)-semi-invariant \( z \) define weight \( \text{wt}(z) \) as an integer defined modulo \( n \) such that

\[
\text{wt}(z) \equiv a \mod n \quad \text{iff} \quad s(z) = \varepsilon^a z,
\]

where \( \varepsilon = \exp \frac{2\pi i}{n} \).

Similar to (2.6) consider an \( \mathbb{Z}_n \)-equivariant embedding \( X' \subset \mathbb{P}^2_{x_0,x_1,x_2} \times \mathbb{C}_{u,v} \) such that \( x_0, x_1, x_2 \) are semi-invariants with

\[
\text{wt}(x_0, x_1, x_2; u, v) = (0, p, q; a, b), \quad (a, n) = (b, n) = 1,
\]

where we may take \( \text{wt}(x_0) = 0 \) because \((x_0, x_1, x_2)\) are homogeneous coordinates. There are two subcases.

(2.7.1) SUBCASE (i). Components of \( X'_0 \) are \( \mathbb{Z}_n \)-invariant. We will derive a contradiction. One can change homogeneous coordinates \( x_0, x_1, x_2 \) in \( \mathbb{P}^2 \) so that \( X'_0 \subset \mathbb{P}^2 \) is \( \{x_0 x_1 = 0\} \). Then \( X' \) is given by the equation

\[
x_0 x_1 + \varphi_0(u, v) x_0^2 + \varphi_1(u, v) x_0 x_1 + \varphi_2(u, v) x_0 x_2 + \phi(u, v) x_1^2 + \psi(u, v) x_1 x_2 + \zeta(u, v) x_2^2 = 0,
\]

where \( \varphi_0(0, 0) = \varphi_1(0, 0) = \varphi_2(0, 0) = \phi(0, 0) = \psi(0, 0) = \zeta(0, 0) = 0 \). By taking \( x'_1 = x_1 + \varphi_0 x_0 + \varphi_1 x_1 + \varphi_2 x_2 \) we obtain a new equation for \( X' \):

\[
x_0 x_1 + \phi(u, v) x_1^2 + \psi(u, v) x_1 x_2 + \zeta(u, v) x_2^2 = 0,
\]

where \( \phi(u, v), \psi(u, v), \zeta(u, v) \) are semi-invariants with suitable weights and \( \phi(0, 0) = \psi(0, 0) = \zeta(0, 0) = 0 \). Then \( Q = \{x_0 = x_1 = u = v = 0\} \) and \( y_0 := x_0/x_2, y_1 := x_1/x_2, u, v \) are local coordinates in \( \mathbb{P}^2 \times \mathbb{C}^2 \) near \( Q \). We have

\[
(X, P) = \left\{ y_0 y_1 + \phi(u, v) y_1^2 + \psi(u, v) y_1 + \zeta(u, v) = 0 \right\}/\mathbb{Z}_n(-q, p, q, a, b).
\]

The action of \( \mathbb{Z}_n \) on \( X' \) has two more fixed points \( Q_1 := \{x_0 = x_2 = u = v = 0\} \) and \( Q_2 := \{x_1 = x_2 = u = v = 0\} \). Then \( z_0 = x_0/x_1, z_2 = x_2/x_1, u, v \) are local
coordinates in $\mathbb{P}^2 \times \mathbb{C}^2$ near $Q_1$ and similarly $t_1 = x_1/x_0, t_2 = x_2/x_0, u, v$ are local coordinates in $\mathbb{P}^2 \times \mathbb{C}^2$ near $Q_2$. Let $P_i = h(Q_i), i = 1, 2$. Similar to (2.7.1.b)

(2.7.1.c) \[(X, P_1) = \{z_0 + \phi(u,v) + \psi(u,v)z_2 + \zeta(u,v)z_0z_2 = 0\}/\mathbb{Z}_n(-p, q, -p, a, b) \simeq \mathbb{C}^3_{\{x_0, y_0, v\}/\mathbb{Z}_n(q - p, a, b),}

(2.7.1.d) \[(X, P_2) = \{t_1 + \phi(u,v)t_2^2 + \psi(u,v)t_1t_2 + \zeta(u,v)t_0^2 = 0\}/\mathbb{Z}_n(p, q, a, b) \simeq \mathbb{C}^3_{\{x_2, y_2, u\}/\mathbb{Z}_n(q, a, b)}.

Since the action has a zero-dimension fixed locus,

(2.7.1.e) \[(q, n) = (p - q, n) = 1.

By (1.1.2) $(X, P)$ a cyclic quotient. It is possible only if $\zeta(u,v)$ contains either $u$ or $v$ terms. Up to permutation of $u, v$ we may assume that $\zeta = u + \ldots$. From (2.7.1.a) we have $\text{wt}(x_0 x_1) = \text{wt}(u x_0^2)$. Thus $p = a + 2q$. Then

(2.7.1.f) \[(X, P) \simeq \mathbb{C}^3_{y_0, y_1, u}/\mathbb{Z}_n(-q, p - q, b) = \mathbb{C}^3/\mathbb{Z}_n(-a + q, b).

We claim that $a + b = 0$. Indeed otherwise $n > 2$ and from (2.7.1.e) we have $b = a + q$ because $p = a + 2q$ and $(q, n) = 1$. Whence $(X, P) = \mathbb{C}^3/\mathbb{Z}_n(-b, b)$. This point cannot be terminal if $n > 2$.

The contradiction shows that $a + b = 0$. Point $(X, P)$ is terminal in this case only if $q = b$ (see (2.7.1f)). But then $p - q = a + 2q - q = 0$, a contradiction with (2.7.1.e).

(2.7.2) Subcase (ii). $\mathbb{Z}_n$ permutes components of $X'_0$. Then $n$ is even, $n = 2k$. If $k > 1$, then the quotient $X'/\mathbb{Z}_k \to \mathbb{C}^2/\mathbb{Z}_k$ is a Mori conic bundle as in (2.7.1). We have proved that this is impossible. Therefore $k = 1, n = 2$. Then it is easy to see that $\text{wt}(u) = \text{wt}(v) = 1$. We may assume also that the fiber $X'_0 \subset \mathbb{P}^2$ is $\{x_0^2 + x_1^2 = 0\}$. Since $\mathbb{Z}_2$ permutes components of $\{x_0^2 + x_1^2 = 0\}$, we may assume that $\mathbb{Z}_2$ acts on $x_0, x_1$ by $x_0 \to -x_0, x_1 \to x_1$. Then one can change the coordinate system $(x_0, x_1, x_2; u, v)$ such that in $\mathbb{P}^2 \times \mathbb{C}^2$ the variety $X'$ is given by the equation

(2.7.2.a) \[x_0^2 + x_1^2 + \phi(u, v)x_2^2 = 0,

where $\phi(u, v)$ is an invariant with $\phi(0, 0) = 0$. It means that $\phi$ contains only monomials of even degree. Then $Q = \{x_0 = x_1 = u = v = 0\}$ is the only fixed point on $C'$ and $y_0 := x_0/x_2, y_1 := x_1/x_2, u, v$ are local coordinates in $\mathbb{P}^2 \times \mathbb{C}^2$ near $Q$. Whence

(2.7.2.b) \[(X, P) = \{y_0^2 + y_1^2 + \phi(u, v) = 0\}/\mathbb{Z}_2.

If this point is terminal, then up to permutation of $y_0, y_1$ we may assume that $\text{wt}(y_0) = 1, \text{wt}(y_1) = 0$. Thus $\text{wt}(x_2) = 0$. Finally singularities of $X'$ are isolated only if $\phi = 0$ is a reduced curve. Thus we have the example (2.3).

(2.8) Case III. $X'_0$ is a double line. As above consider an $\mathbb{Z}_n$-equivariant embedding $X' \subset \mathbb{P}^2_{x_0, x_1, x_2} \times \mathbb{C}^2{u, v}$ such that the action of $\mathbb{Z}_n$ is

$\text{wt}(x_0, x_1, x_2; u, v) = (0, p, q; a, b), \quad (a, n) = (b, n) = 1$. 

7
We may assume also that the fiber $X'_0 \subset \mathbb{P}^2$ is \( \{ x_0^2 = 0 \} \).

Then in some semi-invariant coordinate system \((x_0, x_1, x_2; u, v)\) in \( \mathbb{P}^2 \times \mathbb{C}^2 \) the variety \( X \) is given by the equation

\[
x_0^2 + \phi(u, v)x_1^2 + \psi(u, v)x_1x_2 + \zeta(u, v)x_2^2 = 0,
\]

where \( \phi(u, v), \psi(u, v), \zeta(u, v) \) are semi-invariants with \( \phi(0, 0) = \psi(0, 0) = \zeta(0, 0) = 0 \). As above we can take \( \text{wt}(x_0) = 0 \). Denote \( p = \text{wt}(x_1) \), \( q = \text{wt}(x_2) \). The local coordinate along the fiber \( X'_0 \) is \( x_1/x_2 \) and \( \text{wt}(x_1/x_2) = p - q \). Since the action of \( \mathbb{Z}_n \) on \( X' \) is free in codimension two, \( (p - q, n) = 1 \). Changing a generator of \( \mathbb{Z}_n \) we can get \( p - q = 1 \). Fixed points are \( Q_1 = \{ u = v = 0, x_0 = x_1 = 0 \} \) and \( Q_2 = \{ u = v = 0, x_0 = x_2 = 0 \} \). We can take the local coordinates near \( Q_1 \) and \( Q_2 \) in \( \mathbb{P}^2 \times \mathbb{C}^2 \) as and \((y_0 = x_0/x_2, y_1 = x_1/x_2, u, v)\) and \((z_0 = x_0/x_1, z_2 = x_2/x_1, u, v)\), respectively. The point \( Q_1 \in X' \) gives the singular point \( P_1 \in X \) of type

\[
(2.8.1.a) \quad \{ y_0^2 + \phi(u, v)y_1^2 + \psi(u, v)y_1 + \zeta(u, v) = 0 \} / \mathbb{Z}_n(-q, 1, a, b).
\]

Similarly, the point \( Q_2 \in X' \) gives the singular point \( P_2 \in X \) of type

\[
(2.8.1.b) \quad \{ z_0^2 + \phi(u, v) + \psi(u, v)z_2 + \zeta(u, v)z_2^2 = 0 \} / \mathbb{Z}_n(-p, -1, a, b).
\]

(2.8.2) We claim that \((X, P_1)\) and \((X, P_2)\) are from the main series. Indeed assume for example that \((X, P_1)\) is from the special series. Then \( n = 4, q = 2, p = 3 \). For \( P_2 \) all the weights are prime to \( n = 4 \). By \( (1.1.2) \) \( P_2 \) is a cyclic quotient singularity and \( X' \) is nonsingular at \( Q_2 \). It is possible only if \( \phi(u, v) \) contains a linear term. Then \( \text{wt}(\phi) = \text{wt}(z_0^2) = 2 \). It gives us \( a = \text{wt}(u) = 2 \) or \( b = \text{wt}(v) = 2 \), a contradiction.

(2.8.3) Now we claim that \( X' \) is nonsingular at \( Q_1 \) and \( Q_2 \). As above if \( Q_1 \in X' \) is singular, then by \( (1.1.2) \) we have \( q = 0, p = 1 \). Again by \( (1.1.2) \), \( P_2 \) is a cyclic quotient singularity and \( X' \) is nonsingular at \( Q_2 \). It is possible only if \( \phi(u, v) \) contains a linear term. Up to permutation \( u, v \) we may assume that \( \phi = u + \cdots \).

Since \( \text{wt}(\phi) = \text{wt}(z_0^2) = -2 \), it gives us \( a = \text{wt}(u) = -2 \). Therefore \( n \) is odd and

\[
(2.8.3.a) \quad (X, P_1) = \{ y_0^2 + \phi(u, v)y_1^2 + \psi(u, v)y_1 + \zeta(u, v) = 0 \} / \mathbb{Z}_n(0, 1, -2, b).
\]

\[
(2.8.3.b) \quad (X, P_2) = \{ z_0^2 + (u + \cdots) + \psi(u, v)z_2 + \zeta(u, v)z_2^2 = 0 \} / \mathbb{Z}_n(-1, -1, -2, b)
\]

\[
\simeq \mathbb{C}_{z_0, z_2, v}/\mathbb{Z}_n(-1, -1, b).
\]

Since \( P_2 \) is a terminal point and \( n \) is odd, from \( (2.8.3.b) \) we have \( b = 1 \). But on the other hand from \( (2.8.3.a) \) \( 1 + b = 0 \) or \( -2 + b = 0 \), a contradiction.

(2.8.4) Therefore \( X' \) is nonsingular at \( Q_1 \) and \( Q_2 \). It follows from \( (2.8.1.a) \) and \( (2.8.1.b) \) that both \( \phi(u, v) \) and \( \zeta(u, v) \) contain linear terms. Up to permutation of \( u, v \) we may assume that \( \zeta = u + \cdots \). Whence \( a = -2q \). Moreover \( (X, P_1) = \mathbb{C}(-q, 1, b) \). If \( \phi = u + \cdots \), then from \( (2.8.1.b) \) \( -2p = a = -2 + a \), so \( n = 2, a = 0 \), a contradiction. Therefore \( \phi = v + \cdots \). It gives us \( -2p = b = a - 2 \) and
\((X, P_2) = \frac{1}{n}(-p, -1, a)\). Since \(b\) is even, \(n\) is odd. Further \(a = -2q, p = q + 1, b = -2q - 2\). Thus

\((2.8.4.a) \quad (X, P_1) = \frac{1}{n}(-q, 1, -2q - 2), \quad (X, P_2) = \frac{1}{n}(-q - 1, -1, -2q)\)

\((2.8.5) \quad \) Now we show that \(n = 2q + 1\). Indeed assume the opposite. Then since \((X, P_2)\) is terminal, from \([2.8.4.a]\) we have either \(q + 2 = 0\) or \(3q + 1 = 0\). But in the first case \((X, P_1) = \frac{1}{n}(2, 1, 2)\). This point is terminal only if \(n = 3, q = 1\). In the second case \((X, P_1) = \frac{1}{n}(-q, 1, -2q - 2)\) cannot be terminal. Thus our claim is proved. In particular, we have \(a = -2q = 1, b = -2q - 2 = -1\). Finally in this case \(|\psi(x_0, x_1, x_2, u, v)| = (0, q, q, 1, -1)\) and by changing coordinates in \(\mathbb{C}^2\) by \(u' = \zeta(u, v), v' = \phi(u, v)\) we obtain

\[X \simeq \{x_0^2 + vx_1^2 + \psi(u, v)x_1x_2 + ux_2^2 = 0\}/\mathbb{Z}_{2q+1}(0, -q, q, 1, -1)\]

Therefore \(X\) is as in \([2.2]\). This proves our theorem. \(\Box\)

3 Index two Mori conic bundles.

In this section index two Mori conic bundles will be investigated. First we consider the case when the base of a Mori conic bundle \(f : (X, C) \to (S, o)\) is nonsingular, i.e. \((S, o) \simeq (\mathbb{C}^2, 0)\). We need some elementary results about extremal neighborhoods \([18]\), \([18]\). Note that if \(f : (X, C) \to (S, o)\) is a Mori conic bundle with reducible central fiber \(C\), then the Mori cone \(\overline{\text{NE}}(X/S)\) is generated by extremal rays, because \(-K_X\) is ample. Since we consider a germ \((X, C)\), every extremal ray is generated by a component of \(C\). On the other hand the dimension of \(\overline{\text{NE}}(X/S)\) is equal to the number of components of \(C\) by \([1.4.1]\). Therefore \(\overline{\text{NE}}(X/S)\) is simplicial and generated by classes of components of \(C\). So every irreducible component \(C_i \subseteq C\) gives us (not necessary isolated) extremal neighborhood in the sense of \([18]\).

\((3.1)\) Proposition. Let \((X, C)\) be an extremal neighborhood (not necessary isolated). Assume that \(X\) has index two and let \(P \in X\) be an index two point. Then

\((3.1.1)\) \([18], (4.6)\] \(P\) is the only point of index two. \(C\) has at most three components they all pass through \(P\) and they do not intersect elsewhere.

\((3.1.2)\) \([18], (2.3.2)\] For every component \(C_i \subseteq C\) one has \((-K_X \cdot C_i) = 1/2\).

\((3.1.3)\) \([18], (7.3)\] A general member \((F, P) \in |-K_{(X, P)}|\) satisfies \(F \in |-K_X|, F \cap C = \{P\}\) and \((-K_{(X, C_i)} \cdot C_i) = 1/2\) for every component \(C_i \subseteq C\).

The first main result of this section is the following.

\((3.2)\) Theorem. Let \(f : (X, C) \to (S, o)\) be a Mori conic bundle over a nonsingular base surface \((S, o) \simeq (\mathbb{C}^2, 0)\). Assume that \(X\) is of index two. Then

\((3.2.1)\) \(X\) contains exactly one index two point \(P\), the central fiber \(C\) has at most four components they all pass through \(P\) and they do not intersect elsewhere.

\((3.2.2)\) There exists a flat elliptic fibration \(g : (Y, L) \to (S, o)\) where \((Y, L)\) is a germ of threefold with only isolated Gorenstein terminal singularities along a
reducible curve $L$ such that $L = (g^{-1}(o))_{\text{red}}$ and a general fiber $Y_s := g^{-1}(s)$, $s \in S$ is a nonsingular elliptic curve. Further $K_Y$ is trivial along $L$ and $g$ factores as

$$g : (Y, L) \xrightarrow{h} (X, C) \xrightarrow{f} (S, o),$$

where $h$ is a quotient morphism by $\mathbb{Z}_2$.

(3.2.3) The action of $\mathbb{Z}_2$ on $L$ does not interchange components. The locus of $\mathbb{Z}_2$-fixed points on $Y$ consists of an isolated point $Q$ such that $h(Q) = P$ and a nonsingular divisor $D \neq Q$ such that $D \cap \text{Sing}(X) = \emptyset$. All the components of $L$ are isomorphic to $\mathbb{P}^1$. They all pass through $Q$ and they do not intersect elsewhere.

(3.2.4) In notations above we have the following cases for the scheme-theoretical fiber $X_o = f^{-1}(o)$. In each case $C_1, \ldots, C_r \simeq \mathbb{P}^1$ are irreducible components of $C$.

- (3.2.4.a) $X_0 \equiv C_1 + C_2 + C_3 + C_4$,
- (3.2.4.b) $X_0 \equiv C_1 + C_2 + 2C_3$,
- (3.2.4.c) $X_0 \equiv C_1 + 3C_2$,
- (3.2.4.d) $X_0 \equiv 2C_1 + 2C_2$,
- (3.2.4.e) $X_0 \equiv 4C_1$.

(3.3) Remark. Conversely, let $g : (Y, L) \rightarrow (S, o) \simeq (\mathbb{C}^2, 0)$ be an elliptic fibration with an action of $\mathbb{Z}_2$ such as in (3.2.2) (3.2.3). If the point $(Y, Q)/\mathbb{Z}_2$ is terminal, then $f : (X, C) := (Y, L)/\mathbb{Z}_2 \rightarrow (S, o)$ is a Mori conic bundle of index two.

We prove our theorem in several steps.

(3.4) Lemma. Let $f : X \rightarrow S$ be a morphism from a normal threefold with only terminal singularities onto a surface. Assume that all the fibers of $f$ are connected and one-dimensional. Then $f : X \rightarrow S$ is flat.

(3.5) Proof. Terminal singularities are rational [10, 1-3-6] and therefore Cohen-Macaulay [11]. Then $f$ is flat by [11, 23.1]. □

(3.6) Thus $f : X \rightarrow S$ is flat. Let $X_s := f^{-1}(s)$ be a scheme-theoretical fiber over $s \in S$. Then $(X_o)_{\text{red}} = C$ and $X_0 \equiv \sum n_i C_i$ for some $n_i \in \mathbb{N}$. Since $X_s \simeq \mathbb{P}^1$ for general $s \in S$, $(-K_X \cdot X_s) = 2$. Thus we have

$$2 = (-K_X \cdot X_s) = (-K_X \cdot X_o) = \sum n_i (-K_X \cdot C_i).$$

It gives us $\sum n_i \leq 4$ because $(-K_X \cdot C_i) \in \frac{1}{2} \mathbb{Z}$. In particular, $C$ has at most four components. If $C$ is reducible then by (3.1.2) $(-K_X \cdot C_i) = 1/2$, so $\sum n_i = 4$ and for $X_0$ we have only possibilities as in (3.2.4.a) (3.2.4.d).

(3.7) Lemma. Let $P \in X$ be a point of index 2. Then every component $C_i \subset C$ contains $P$.

(3.8) Proof. Assume $C_j \neq P$ for some $j$. Then in particular, $C$ is reducible. By (3.6.1) we have $\sum_{C_i \supseteq P} n_i < \sum n_i \leq 4$. Let $F \in | -K_{(X, P)}|$ be a general member. Proposition (3.1) gives us $(F \cdot C_i) = 1/2$ for all components $C_i$ passing through $P$. On the other hand

$$\frac{1}{2} \sum_{C_i \supseteq P} n_i = \sum_{C_i \supseteq P} n_i (F \cdot C_i) = (F \cdot X_0) = (F \cdot X_s) \in \mathbb{N}.$$
Thus $1/2 \sum_{C_i \geq D} n_i = 1$. Therefore $(F \cdot X_0) = (F \cdot X_0) = 1$, i.e. the map $f|_F : F \to S$ is bimeromorphic and finite. Since $S$ is nonsingular $f|_F$ is an isomorphism. So $F$ is also nonsingular. We derive the contradiction with (1.2). □

(3.9) Lemma. The Weil divisor class group $\text{Cl}(X)$ has no torsion.

(3.10) Proof. If $\xi \in \text{Cl}(X)$ is a torsion, then $\xi$ defines a cyclic étale in codimension 1 cover $X' \to X$ (see e.g. [13, (1.11)]). The threefold $X'$ is normal and has terminal singularities of indices $\leq 2$ only. It follows also that $X' \to X$ is étale in codimension 2. Take the Stein factorization

$$
X' \xrightarrow{\pi} X \xrightarrow{f} S' \xrightarrow{\pi} S
$$

We obtain a new Mori conic bundle $X' \to S'$ and étale in codimension one cover $S' \to S$. But $S$ is nonsingular. The contradiction shows that $\text{Cl}(X)$ is torsion-free. □

(3.11) Lemma. $X$ contains a unique point $P$ of index two.

(3.12) Proof. If $C$ is reducible, then $P$ is a unique point of index two on $X$ by (3.11) and because $p_a(C) = 0$. So assume that $C \simeq \mathbb{P}^1$ and let $P_1, \ldots, P_r \in (X, C)$ be all the points of index 2.

(3.12.1) Definition. Let $X$ be a normal variety and $\text{Cl}(X)$ be its Weil divisor class group. The subgroup of $\text{Cl}(X)$ consisting of Weil divisor classes which are Cartier is called by the semi-Cartier divisor class group. We denote it by $\text{Cl}^{sc}(X)$.

(3.12.2) Theorem [24],[3]. Let $(X, P)$ be a germ of 3-dimensional terminal singularity. Then $\text{Cl}^{sc}(X, P) \simeq \mathbb{Z}_m$ and it is generated by the class of $K_{(X, P)}$.

We have the following natural exact sequence (see [18, (1.8.1)])

$$
0 \to \text{Pic}(X) \to \text{Cl}^{sc}(X) \xrightarrow{\text{res}} \bigoplus_{i=1}^r \text{Cl}^{sc}(X, P_i) \to 0,
$$

(3.12.3)

where $\text{Pic}(X) \simeq \mathbb{Z}$ by (1.4.1) and $\text{Cl}^{sc}(X, P_i) \simeq \mathbb{Z}_2$ by (3.12.2). From (3.9) we get $\text{Cl}^{sc}(X) \simeq \mathbb{Z}$. Hence $r = 1$. Thus our lemma is proved. □

Since $p_a(C) = 0$, components of $C$ do not intersect outside $P$. This proves (3.2.1).

(3.12.4) Corollary. If $C$ is irreducible, then $(-K_X \cdot C) = 1/2$.

(3.12.5) Proof. Suppose that $(-K_X \cdot C) > 1/2$. From lemma (3.9) we have $\text{Cl}^{sc}(X) = \mathbb{Z}$. Moreover by (3.12.3) $\text{Pic}(X) \subset \text{Cl}^{sc}(X)$ is a subgroup of index 2. Let $R$ be the ample generator of $\text{Cl}^{sc}(X)$. Then $(R \cdot C) = 1/2$ and $-K_X = kR$, $k \in \mathbb{Z}$. On the other hand from (3.6.1) we have $(-K_X \cdot C) = 1$, $X_0 \equiv 2C$ or $(-K_X \cdot C) = 2$, $X_0 \equiv C$. Whence $-K_X = 2R$ or $-K_X = 4R$. But then $-K_X$ is Cartier and $X$ has index one, a contradiction. □

Therefore $X_0 \equiv 4C$, if $C$ is irreducible. This proves (3.2.4).

(3.13) Construction. Let $B_i$ be a disc that intersects $C_i$ transversally in a general point and let $B = \sum B_i$. Since $(-K_X \cdot C_i) = 1/2$ and $\text{Pic}(X) = \mathbb{Z}^r$,
$B \in | - 2K_X|$. Take a double cover $h: Y \to X$ with ramification divisor $B$. Set $L := (h^{-1}(C))_{\text{red}}$ and $D := (h^{-1}(B))_{\text{red}}$. We have

$$g: (Y, L) \to (X, C) \to (S, o) \simeq (\mathbb{C}^2, 0).$$

By our construction $Y$ is normal, hence $g: Y \to \mathbb{C}^2$ has only connected fibers. In a neighborhood of each singular points on $X$ $h$ is étale in codimension 1, therefore $Y$ has only terminal singular points (see e. g. [24, (3.1)] or [3, (6.7)]). We have the following equalities for Weil divisors on $Y$

$$K_Y = h^*(K_X) + D, \quad D = h^*(-K_X).$$

It gives us $K_Y = 0$ and $Y$ has only terminal Gorenstein singularities. The morphism $g$ is flat by lemma [3.4]. Therefore $g$ is an elliptic fibration. We have proved [3.2.2]. It is clear that the locus of $\mathbb{Z}_2$-fixed points on $Y$ consists of $D$ and $h^{-1}(P)$. This proves [3.2.3]. Our theorem is proved. \hfill \square

Using [18, (6.2)], [13, (4.7)] and [24] one can improve results of [3.2.4].

**Corollary.** In notations and conditions of [3.2] we have

**(3.13.1.a)** If $X_0 \equiv C_1 + C_2 + C_3 + C_4$, then $P$ is the only singular point of $X$, and $(X, P)$ is of type $cA/2$.

**(3.13.1.b)** If $X_0 \equiv C_1 + C_2 + 2C_3$, then $X$ may have one more singular point of index one on $C_3$, and $(X, P)$ is of type $cA/2$.

**(3.13.1.c)** If $X_0 \equiv C_1 + 3C_2$, then $X$ may have one more singular point of index one on $C_3$.

**(3.13.1.d)** If $X_0 \equiv 2C_1 + 2C_2$, then $X$ may have one more singular point of index one on each component $C_i$, $i = 1, 2$.

**(3.13.1.e)** If $X_0 \equiv 4C_1$, then $X$ may have at most two more singular points of one index 1.

Now we use the construction [3.13] to get examples index two Mori conic bundles.

**Examples.** In the following examples a Mori conic bundle $f: (X, C) \to (\mathbb{C}^2, 0)$ is constructed as a quotient $X = Y/\mathbb{Z}_2 \to \mathbb{C}^2$, where $Y \subset \mathbb{P}^3 \times \mathbb{C}^2$ is an intersection of two quadrics, $\mathbb{Z}_2$ acts on $Y \subset \mathbb{P}^3_{x_0, x_1, x_2, x_3} \times \mathbb{C}^2_{u, v}$ by

$$(x_0, x_1, x_2, x_3; u, v) \mapsto (-x_0, -x_1, -x_2, x_3; u, v)$$

and $f$ is induced by the projection of $g: Y \subset \mathbb{P}^3 \times \mathbb{C}^2 \to \mathbb{C}^2$ on the second factor. We will use the notations $cA/2$, $cAx/2$, $cD/2$, $cE/2$ to distinguish index two terminal singularities [13].

**Example.** Let $Y \subset \mathbb{P}^3_{x_0, x_1, x_2, x_3} \times \mathbb{C}^2_{u, v}$ is given by the equations

$$\left\{ \begin{array}{l}
x_0x_1 = (au + bu^2 + cuv)x_3^2 \\
(x_0 + x_1 + x_2)x_2 = vx_3^2,
\end{array} \right.$$

where $a, b, c \in \mathbb{C}$ are constants. It is easy to check that $Y$ is nonsingular near the central fiber $g^{-1}(0)$ if $a \neq 0$ and has an isolated hypersurface singularity in
The central fiber $X$ has the following possibilities: nonsingular. As above the quotient $X = Y/\mathbb{Z}_2 \to \mathbb{C}^2$ is a Mori conic bundle with only one singular point of type $eA/2$ and the central fiber $X_o \equiv C_1 + C_2 + C_3 + C_4$. If $a \neq 0$, then the singular point is a cyclic quotient of type $\frac{1}{2}(1,1,1)$.

(3.16) Example. Let $Y \subset \mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}$ is given by the equations

$$\begin{align*}
  x_0x_1 &= u(x_0^2 + x_1^2 + x_2^2 - x_3^2) + avx_3^2 \\
  (x_0 + x_1)x_2 &= vx_3^2 + bux_0^2 \\
  a, b &\in \mathbb{C}
\end{align*}$$

Then the quotient $X := Y/\mathbb{Z}_2 \to \mathbb{C}^2$ is a Mori conic bundle with the central fiber $X_o \equiv C_1 + C_2 + 2C_3$. $X$ contains exactly one non-Gorenstein point that is of type $\frac{1}{2}(1,1,1)$. If $a = b = 0$, then $X$ has also an ordinary double point on $C_3$.

(3.17) Example. Let $Y \subset \mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}$ is given by the equations

$$\begin{align*}
  x_0x_1 - x_2^2 &= ux_3^2 \\
  x_0x_2 &= ux_1^2 + v(x_2^2 + x_3^2)
\end{align*}$$

Then the quotient $X := Y/\mathbb{Z}_2 \to \mathbb{C}^2$ is a Mori conic bundle with only one singular point of type $\frac{1}{2}(1,1,1)$ and the central fiber $X_o \equiv C_1 + 3C_2$.

The following example shows that all the types of terminal singularities of index two can appear on Mori conic bundles such as in [3.2.4.d].

(3.18) Example. Let $Y \subset \mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}$ is given by the equations

$$\begin{align*}
  x_0x_1 &= ux_3^2 \\
  x_2^2 &= u(x_0^2 + x_1^2) + (av + bv^2 + cv^3 + duv + eu^2v)x_3^2
\end{align*}$$

where $a, b, c, d, e \in \mathbb{C}$ are constants. If at least one of $a, b, c$ is non-zero, then $Y$ has near the central fiber $Y_0$ an isolated singularity at $\{x_0 = x_1 = x_2 = u = v = 0\}$ (or nonsingular). As above the quotient $X := Y/\mathbb{Z}_2 \to \mathbb{C}^2$ is a Mori conic bundle with the central fiber $X_o \equiv 2C_1 + 2C_2$ and only one singular point $P := C_1 \cap C_2$. We have the following possibilities:

- $a \neq 0$, then $Y$ is nonsingular, so $P \subset X$ is of type $\frac{1}{2}(1,1,1)$,
- $a = 0, b \neq 0$, then $P \subset X$ is of type $cAx/2$,
- $a = b = e = 0, c \neq 0, d \neq 0$, then $P \subset X$ is of type $cD/2$,
- $a = b = d = 0, c \neq 0, e \neq 0$, then $P \subset X$ is of type $cE/2$.

(3.19) Example. Let $Y \subset \mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}$ is given by the equations

$$\begin{align*}
  x_0^2 &= ux_2^2 + vx_3^2 \\
  x_1^2 &= ux_3^2 + vx_2^2
\end{align*}$$

Then the quotient $X := Y/\mathbb{Z}_2 \to \mathbb{C}^2$ is a Mori conic bundle with irreducible central fiber that contains one singular point of type $\frac{1}{2}(1,1,1)$ and two ordinary double points.

Now we investigate index two Mori conic bundles with singular base.

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\( \text{(3.20) Theorem.} \) Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle of index 2 over a surface. Assume that the point \((S, o)\) is singular. Then \((S, o)\) is DuVal of type \( A_1 \) and \( f \) is either as in example \( (2.3) \) or as in example \( (2.1) \) with \( n = 2 \).

\( \text{(3.21) Proof.} \) We can use construction \( (1.5) \). Since \( X \) contains only points of indices \( \leq 2 \) by \( (1.7.1), \) \((S, o)\) is a singularity of type \( \frac{1}{2}(1, 1) = A_1 \). We claim that \( f' : (X', C') \to (\mathbb{C}^2, 0) \) is a conic bundle. It is sufficient to show that \( X' \) is Gorenstein \( (1.3) \). Assume the opposite. We remark that \( X' \) contains only points of indices \( \leq 2 \) because \( X' \to X \) is étale in codimension 1. By theorem \( (3.2) \) \( X' \) contains a unique point, denote it by \( Q \), of index 2. Then \( Q \) is a \( \mathbb{Z}_2 \)-fixed point. But then the point \( P := h(Q) \) has index 4, a contradiction. Therefore \( f' \) is a conic bundle and by theorem \( (2.4) \) \( f(X, C) \to (X, C) \) is such as in example \( (2.1) \) or example \( (2.3) \). This proves our theorem. \( \Box \)

4 The general elephant conjecture.

In this section we study Mori conic bundles under the assumption the existence of a good member in \( | - K_X | \).

\( \text{(4.1) Example.} \) Let \( f : (X, C) \to (S, o) \) be as in example \( (2.2) \). Consider the open subset \( U_2 = \{ x_2 \neq 0 \} \subset X \). The local coordinates in \( U_2 \) \( (t_0 = x_0/x_2, t_1 = x_1/x_2, v) \). Consider also the rational differential \( \sigma = (1/t_0)(dt_0 \wedge dt_1 \wedge dv) \) on \( U_2 \). Then it is easy to see that \( \sigma \) can be extended on \( X' \) near \( C' \). Since \( \sigma \) is \( \mathbb{Z}_n \)-invariant, \( \sigma^{-1} \) defines an element \( F \in | - K_X | \), the image of \( \{ x_0 = 0 \} \). It is easy to check that \( F \) contains the central fiber \( C = (f^{-1}(o))_{\text{red}} \) and has two singular points of type \( A_{n-1} \).

Similarly one can check that in example \( (2.1) \) a general member \( F \in | - K_X | \) does not contain \( C \) and has two connected components. It contains two singular points of type \( A_{n-1} \) on each component.

The following theorem improves results of \( [K] \).

\( \text{(4.2) Theorem.} \) Let \( f : (X, C) \to (S, o) \) be a Mori conic bundle. Assume that conjecture \( (0.2) \) holds. Then we have one of the following:

\( \text{(4.2.1) } (S, o) \text{ is nonsingular,} \)

\( \text{(4.2.2) } (S, o) \text{ is DuVal of type } A_1, \)

\( \text{(4.2.3) } (S, o) \text{ is DuVal of type } A_3, \text{ in this case } C \text{ is irreducible, } (X, C) \text{ has a cyclic quotient singularity } P \text{ of index } 8 \text{ and has no another points of index } \geq 1 \) (see \( [23] \) for more detailed study of this case).

\( \text{(4.2.4) } f : (X, C) \to (S, o) \text{ is quotient of a nonsingular conic bundle } f' : (X', C') \to (S', o') \text{ with irreducible } C' \text{ by the group } \mathbb{Z}_n, \text{ where } n \geq 3 \text{ and the action } \mathbb{Z}_n \text{ on } (S' , o') \simeq (\mathbb{C}^2, 0) \text{ is free in codimension } 1 \text{ (i.e. } f : (X, C) \to (S, o) \text{ is such as in } (2.1) \text{ or } (2.2) \text{). In particular, } (S, o) \text{ has type } A_{n-1} \text{ in this case.} \)

\( \text{(4.3) Proof.} \) Assume that \((S, o)\) is singular and it is not of type \( A_1 \). We will use the notations of construction \( (1.5) \). If \( X' \) is of index one, then by \( (1.3) \) \((S, o)\) is nonsingular. So we assume that \( X' \) is of index \( > 1 \) and show in this case that \( f : (X, C) \to (S, o) \) is such as in \( (4.2.3) \). Let \( F \in | - K_X | \) be a general member and \( F' := h^{-1}(F) \). By our conditions \( F \) has only DuVal singularities and since
$F' \to F$ is étale in codimension 1, so has $F'$ (In particular, $F'$ is irreducible). Since $(-K_X \cdot X_s) = 2$, where $X_s$ is a general fiber of $f$, the restriction $f|_{F'} : F' \to S$ is generically finite of degree 2. There are the following cases.

(4.4) **Case I.** $F' \cap C'$ is disconnected. As above since $F' \in |-K_X|$, we see that $f|_{F'} : F' \to S'$ is generically finite of degree 2 and $F'$ has two connected components $F'_1$ and $F'_2$. Let

$$f' : F' \xrightarrow{f'_1} D' \xrightarrow{f'_2} S'$$

be the Stein factorization. Then $f'_1 : F' \to D'$ is bimeromorphic and $f'_2 : D' \to S'$ is finite of degree 2. In our case $D'$ has exactly two irreducible components $D'_1$ and $D'_2$. Therefore we have $D'_1 \simeq D'_2 \simeq S'$ and the divisor $D'$ is nonsingular, because so is $S'$. On the other hand by the adjunction formula, $K_{F'} = 0$. Whence the morphism $f'_1$ is crepant (i.e. $K_{F'} = f_1^* K_{D'}$). It means that $F'$ is nonsingular and $f'_1 = \text{id}$. By \([1.2]\), $X'$ has no points of indices $> 1$, a contradiction with our assumption in \((4.3)\).

(4.5) **Case II.** $F' \cap C'$ is one point $P$. Then $F \cap C = \{ h(Q) \}$ is also one point, say $P$. Therefore the morphism $f|_{F'} : F' \to S$ is finite of degree 2 and $P$ is a unique point of index $> 1$ and $(F, P) \to (S, o)$ is double cover of isolated singularities. Thus $(S, o)$ is a quotient of DuVal singularity $(F, P)$ by an involution $\tau$.

Actions of involutions on DuVal singularities were classified by Catanese \([2]\). Recall that $(S, o)$ is of topological index $n \geq 3$ (otherwise we have cases \((4.2.1)\) or \((4.2.2)\) of our theorem). Taking into account that $(S, o)$ is a cyclic quotient from the list in \([2]\) we obtain the following

(4.6) **Lemma** \([2]\). Let $(F, P)$ be a germ of DuVal singularity and let $\tau$ be an analytic involution acting on $(F, P)$. Assume that the quotient $(S, o) := (F, P)/\tau$ is a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ with $n \geq 3$ (and $(a, n) = (b, n) = 1$). Then there are the following possibilities for $(F, P) \to (S, o)$:

$$(F, P) \to (S, o) \quad \quad \quad n$$

\[
\begin{align*}
(4.6.1) & \quad E_6 \xrightarrow{2:1} A_2, \quad n = 3, \\
(4.6.2) & \quad A_{2n-1} \xrightarrow{2:1} A_{n-1}, \quad n \geq 1, \\
(4.6.3) & \quad A_{2k} \xrightarrow{2:1} \frac{1}{2k+1}(k, 2k-1), \quad n = 2k + 1, \\
(4.6.4) & \quad A_k \xrightarrow{2:1} A_{2k+1}, \quad n = 2k + 1, \\
(4.6.5) & \quad A_{2k+1} \xrightarrow{2:1} \frac{1}{4k+4}(2k+1, 2k+1), \quad n = 4k + 4.
\end{align*}
\]

The restriction $(F', Q) \to (F, P)$ is étale outside $P$ and of degree $n$. Therefore $(F', Q)$ is DuVal. It is easy to see (see e.g. \([23\ 4.10]\)), that $(F, P)$ has no such covers in cases \((4.6.4), (4.6.5)\). The singularity $(F, P)$ from \((4.6.3)\) admits only cover by nonsingular $(F', P')$ of degree $n = 2k + 1$. But then $(X', Q)$ is a nonsingular point by \((1.2)\), a contradiction with our assumption in \((4.3)\).

Let $m$ be the index of $(X, P)$, $\pi : (X^2, P^2) \to (X, P)$ be the canonical cover and $F^2 := \pi^{*-1} F$. As above we have étale in codimension one $\mathbb{Z}_m$-cover $\pi : (F^2, P^2) \to (F, P)$ of DuVal singularities. Since $\pi$ factores through $(X', Q)$ we have $m \geq n$.
In case \([4.6.1]\) \((F, P) = E_6\) by \([23, 4.10]\), admits only cyclic cover \(D_4 \overset{3:1}{\rightarrow} E_6\).

Then \(n = m = 3\). Therefore \((X^2, P^2) \simeq (X', P')\) has index 1, a contradiction.

Finally, consider case \([4.6.2]\). Since \((X', P')\) has index \(> 1\), we have \(m > n \geq 3\) and by \([23, 4.10]\) \((F^2, P^2) \overset{m/n}{\rightarrow} (F, P)\) is of type (nonsingular) \(\overset{2ni}{\rightarrow}\) \(A_{2k+1}\), \(m = 2n\).

Then the index of \((X', P')\) is equal to \(m/n = 2\), \((X^2, P^2)\) is nonsingular, hence \((X', P')\) is a cyclic quotient singularity of type \(\frac{1}{n} (1, 1, 1)\). In particular, \(f\) is as in theorem \([3.2]\). Let \(X'_f = f^{n-1}(o')\) be the scheme-theoretical fiber of \(f'\) over \(o'\). Further \(P'\) is the only fixed point on \(C'\) under the action of \(Z_n\). Whence \(Z_n\) permutes components \(\{C_i'\}\) of \(C'\). In particular, the number of components \(\geq n \geq 3\) and multiplicities of components in \(X'_f\) are the same. Therefore for \(f' : (X', C') \rightarrow (S', o')\) we have the only possibility \([3.2.4.a]\). Hence we have exactly four components of \(X'_0, n = 4\) and the fiber \(X'_0\) is reduced. Then \(X'\) is nonsingular outside \(P'\). We obtain case \([4.2.3]\) of our theorem.

\((4.7)\) Case III. \(F' \cap C'\) is one-dimensional and connected. Then so is \(F \cap C\). Let \(L := F' \cap C'\) (with reduced structure). On the surface \(F' = h^{-1}(F)\) we have by the adjunction formula that \(L\) is Gorenstein and \(\omega_L = O_L(L)\), because \(\omega_{F'} = O_{F'}\).

Since \(L\) is contracted by \(f'\), the dualizing sheaf \(\omega_L\) is anti-ample. Whence \(L\) is a reduced conic in \(\mathbb{P}^2\) (see e. g. \([23]\)).

Let

\[f_F : F \xrightarrow{f_1} D \xrightarrow{f_2} S\]

be the Stein factorization. Then \(f_1 : F \rightarrow D\) is bimeromorphic and \(f_2 : D \rightarrow S\) is finite of degree 2. By the adjunction formula, \(K_F = 0\). Therefore the morphism \(f_1\) is crepant (i. e. \(K_F = f_1^*K_D\)) and \(D\) has only DuVal singularities.

There exists the common minimal resolution \(\sigma : \tilde{F} \rightarrow F \rightarrow D\). In our case \(f_2^{-1}(o)\) consist of only one point \(R\). Let \(\Gamma = \Gamma(\tilde{F}/D)\) be a dual graph for \(\sigma\). Denote vertices corresponding to components of \(h(L)\) (resp. \(f_2\)-exceptional divisors) by \(\bullet\) (resp. \(o\)). Then white vertices form connected subgraphs corresponding singular points of \((F, C)\) and black vertices correspond components of \(C\) that contained in \(F\). For \(f_2 : (D, R) \rightarrow (S, o)\) we have the same possibilities as for \((F, P) \rightarrow (S, o)\) in \([4.6]\). Let us consider these cases.

\((4.7.1)\) Subcase \([4.6.1]\). (i. e. \((D, R) = E_6, (S, o) = A_2, n = 3\)). The group \(Z_3\) naturally acts on the curve \(L = F' \cap C'\). Since \(L\) is a conic, \(Z_3\) cannot permute its components. Therefore \(Z_3\) has two or three fixed points \(Q_i\) on \(L\). Then there exists at least two points \(P_1, P_2 \in X\) of indices \(\geq 3\). These points on \(F\) have by \([1.2.2]\) topological indices \(\geq 3\). On the other hand, the graph of the minimal resolution of \((F, P_i)\) must be a white subgraph of \(\Gamma (\simeq E_6)\). Whence each \((F, P_i)\) is of type \(A_2\) and there are only two fixed points on \(L\). It is possible only \(L\) is irreducible and so is \(h(L)\). Thus we have only one possibility for \(\Gamma\).

\[\begin{array}{cccccccc}
\bullet \\

| \\

\circ -- \circ -- \bullet -- \circ -- \circ \circ
\end{array}\]

Whence \(F\) contains exactly two singular points \(P_1, P_2\) and they have type \(A_2\).

The variety \(X\) has at these points index 3. Since \(h^{-1}(P_i) = Q_i\) by \([1.2.2]\) \(X'\) is Gorenstein. We obtain a contradiction with assumptions in \([4.3]\).
(4.7.2) **Subcase** \([4.6.2]\) i.e. \((D,R) = A_{2n-1}, (S,o) = A_{n-1}\). If \(L\) is irreducible or \(\mathbb{Z}_n\) doesn’t permute components of \(L\), then as above \(F\) contains at least two points of topological indices \(\geq n\). Then \(\Gamma\) is

\[(4.7.2.a)\]

\[\begin{array}{c}
\circ \circ \cdots \circ \circ \bullet \circ \circ \cdots \circ \\
n_{-1} & n_{-1}
\end{array}\]

Whence \(F\) contains exactly two singular points and they have type \(A_{n-1}\). By \([1.2.2]\) \(X'\) is Gorenstein, a contradiction.

Therefore \(L\) is reducible and \(\mathbb{Z}_n\) interchanges its components. Then \(h(L) = F \cap C\) is irreducible, so \(\Gamma\) has only one black vertex. Further a \(\mathbb{Z}_n\)-fixed point \(Q_1\) on \(X'\) gives us the point \(P_1 \in F\) of topological index \(nk, k \in \mathbb{N}\). This point corresponds to a white subgraph in \(\Gamma\) of type \(A_{nk-1}\). Thus \(\Gamma\) again has the form \([4.7.2.a]\). So \((F,P_1)\) is of type \(A_{n-1}\). In particular, by \([1.2]\) \(Q_1\) is nonsingular. The second white subgraph in \(\Gamma\) also corresponds to the singular point \(P_2 \in F\) of type \(A_{n-1}\). Then the index of \((X,P_2)\) divides \(n\). Since \(\mathbb{Z}_n\) interchanges two components of \(L\), \(n = 2k\) is even and there are three \(\mathbb{Z}_k\)-fixed points on \(L\): \(Q_1, Q_2\) and \(Q_3\), where \(h(Q_2) = h(Q_3) = P_2\). Whence the index of \(P_2\) is divisible by \(k\). By our assumption \((X',C')\) is not Gorenstein, so \((X',Q_2)\), \((X',Q_3)\) have (the same) index \(m_0 \geq 1\) and \((X',C')\) contains no another points of index \(> 1\). But then index of \((X,R)\) is \(m_0n/2 \leq n\). Hence \(m_0 = 2\), it means that \(f' : (X',C') \to (S',o')\) is an index two Mori conic bundle that contains two non-Gorenstein points \(Q_2, Q_3\), a contradiction with \([3.2]\).

Finally subcases \([4.6.3]\), \([4.6.4]\) and \([4.6.5]\) are impossible as above. \(\square\)

5 **An example of Mori conic bundle of index three.**

In this section we construct an example of index three Mori conic bundle over a nonsingular base (that is not such as in examples \([2.1]\) or \([2.2]\)).

(5.1) Let \(V \subseteq \mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}\) is given by two equations

\[
\begin{align*}
& a_1x_1^3 + a_2x_2^3 + a_3x_1^2x_2 + a_4x_2^2x_1 + a_5x_3^2x_1 + a_6x_3^2x_2 = uw_0^3 \\
& b_1x_1^3 + b_2x_2^3 + b_3x_1^2x_2 + b_4x_2^2x_1 + b_5x_3^2x_1 + b_6x_3^2x_2 = vx_0^3
\end{align*}
\]

where \(a_i, b_i \in \mathbb{C}\) are general constants. Then \(V\) is nonsingular and the projection on the second factor \(q : V \to \mathbb{C}^2\) gives us a fibration of curves of degree 9 in \(\mathbb{P}^3_{x_0,x_1,x_2,x_3} \times \mathbb{C}^2_{u,v}\). The central fiber \(q^{-1}(0)\) has exactly nine irreducible components \(\Gamma_0, \ldots, \Gamma_8\), where \(\Gamma_0 := \{u = v = x_1 = x_2 = 0\}\). Define an action of \(\mathbb{Z}_9\) on \(V\) by

\[(x_0, x_1, x_2, x_3; u, v) \mapsto (\varepsilon x_0, x_1, x_2, -x_3; u, v), \quad \varepsilon = \exp(2\pi i/6).\]

First we consider the quotient \(p : V \to Y := V/\mathbb{Z}_3\) and the standard projection \(g : Y \to \mathbb{C}^2\). The set of \(\mathbb{Z}_3\)-fixed points is a divisor \(R := \{x_0 = 0\} \cap V\) and an isolated point \(O := \{x_1 = x_2 = x_3 = u = v = 0\}\). The local coordinates on \(V\) near \(O\) are \((x_1/x_0, x_2/x_0, x_3/x_0)\) with \(\mathbb{Z}_3\)-weights \(\text{wt}(x_1/x_0) = \text{wt}(x_2/x_0) = \text{wt}(x_3/x_0)\).
Therefore the singular locus of $Y$ consists of one point of type $\frac{1}{3}(1,1,1)$ (denote it by $Q$) that is canonical and Gorenstein. Further by the Hurwitz formula

$$K_Y = p^*K_Y + 2R, \quad p^*K_Y = K_Y - 2R = 0.$$  

Whence the canonical divisor of $Y$ is trivial and $g: Y \to \mathbb{C}^2$ is an elliptic fibration. Since $\mathbb{Z}_3$ does not permute $\Gamma_0, \ldots, \Gamma_8$, the central fiber $g^{-1}(0)$ consists of nine components $L_0, L_1, \ldots, L_8$ which are proper transforms of corresponding components $\Gamma_0, \ldots, \Gamma_8$ of $g^{-1}(0)$. Now consider the quotient $h: Y \to X := Y/\mathbb{Z}_2$ and the natural morphism $f: X \to \mathbb{C}^2$. The component $L_0$ is $\mathbb{Z}_2$-invariant and $\mathbb{Z}_2$ permutes $L_i$, $i = 1, \ldots, 8$ non-trivially. Hence $f^{-1}(0) = g^{-1}(0)/\mathbb{Z}_2$ has exactly five components. Denote them by $C_0, \ldots, C_4$. To prove that $f: X \to \mathbb{C}^2$ is a Mori conic bundle, we have to investigate the singular locus of $X$. The set of $\mathbb{Z}_2$-fixed points on $V$ is an irreducible divisor $D := \{x_3 = 0\}$. Therefore the set of $\mathbb{Z}_2$-fixed points on $Y$ is an irreducible Weil divisor $F := p(D)$ such that $3F$ is Cartier. Moreover $D$ intersects components $\Gamma_0, \ldots, \Gamma_8$ transversally.

The following is an easy exercise.

(5.2) Lemma. Let $(Y, Q)$ be a cyclic quotient singularity of type $\frac{1}{3}(1,1,1)$ with action of $\mathbb{Z}_2$. Assume that the locus of fixed points of this action is a Weil divisor $F$ such that $3F = 0$. Then $(Y, Q)/\mathbb{Z}_2$ is terminal of type $\frac{1}{3}(1,1,-1)$. \(\square\)

(5.3) By the Hurwitz formula $0 = K_Y = h^*(K_X) + F$. Since $(L_i \cdot F) > 0$, we have $(h^*(-K_X) \cdot L_i) = (F \cdot L_i) > 0, i = 0, \ldots, 8$. Hence $-K_X \cdot C_i > 0, i = 0, \ldots, 4$ i.e. $-K_X$ is relatively ample (at least near $f^{-1}(0)$).

(5.4) Conclusion. We get a Mori conic bundle $f: (X, C) \to (\mathbb{P}^2, 0)$ such that its central fiber $C$ has exactly five components and the only singular point $P$ of $X$ is a cyclic quotient singularity of type $\frac{1}{3}(1,1,-1)$. All the components of $C$ pass through $P$ and they do not intersect elsewhere. Note that $h: Y \to X$ is nothing else but Kawamata’s double cover trick [3].

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