A central limit theorem and moderate deviations for 2-D Stochastic Navier-Stokes equations with jumps

Ran Wang ∗ Jianliang Zhai †

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences,
School of Mathematical Sciences, University of Science and Technology of China,
No. 96 Jinzhai Road, Hefei, 230026, P. R. China

Abstract

We study the small noise asymptotics for two-dimensional Navier-Stokes equations driven by Lévy noise. A central limit theorem and a moderate deviation are established under appropriate assumptions, which describes the exponential rate of convergence of the stochastic solution to the deterministic solution.

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1 Introduction

Let $D$ be a bounded open domain in $\mathbb{R}^2$ with smooth boundary $\partial D$. Denote by $u$ and $p$ the velocity and the pressure fields. The Navier-Stokes equation, an important model in fluid dynamics, is given as follows:

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = h \quad \text{in} \quad D \times [0,T],$$

with the conditions

$$\begin{cases}
\nabla \cdot u = 0 & \text{in} \quad D \times [0,T]; \\
u = 0 & \text{in} \quad \partial D \times [0,T]; \\
u_0 = x & \text{in} \quad D,
\end{cases}$$

where $\nu > 0$ is the viscosity, $h$ stands for the external force. Without loss of generality, set $T = 1$.

To formulate the Navier-Stokes equation, we introduce the following standard spaces: let

$$V = \{ v \in H_0^1(D; \mathbb{R}^2) : \nabla \cdot v = 0, \text{a.e. in } D \},$$

with the norm

$$\|v\|_V := \left( \int_D |\nabla v|^2 \, dx \right)^{\frac{1}{2}}.$$
and let $H$ be the closure of $V$ in the $L^2$-norm

$$
\|v\|_H := \left( \int_D |v|^2 \, dx \right)^{\frac{1}{2}}.
$$

Define the operator $A$ (Stokes operator) in $H$ by the formula

$$
Au := -\nu P_H \Delta u, \quad \forall u \in H^2(D; \mathbb{R}^2) \cap V,
$$

where the linear operator $P_H$ (Helmholtz-Hodge projection) is the projection operator from $L^2(D; \mathbb{R}^2)$ to $H$, and define the nonlinear operator $B$ by

$$
B(u, v) := P_H ((u \cdot \nabla) v),
$$

with the notation $B(u) := B(u, u)$ for short.

By applying the operator $P_H$ to each term of the above Navier-Stokes equation (NSE for short), we can rewrite it in the following abstract form:

$$
du_t + Au_t \, dt + B(u_t) \, dt = f_t \, dt \quad \text{in} \quad L^2([0,1]; V'),
$$

(1.1)

with the initial condition $u_0 = x \in H$.

The purpose of this paper is to study the small noise asymptotics for two-dimensional Navier-Stokes equations perturbed both by Brownian motion and by Poisson random measure, that is

$$
du^n_t + Au^n_t \, dt + B(u^n_t) \, dt = f_t \, dt + \frac{1}{\sqrt{n}} \sigma(t, u^n_t) \, d\beta_t + \frac{1}{n} \int_X G(t, u^n_{t-}, v) \tilde{N}^n (dt, dv),
$$

(1.2)

with the initial condition $u^n_0 = x \in H$. Here $\beta$ is an $H$-valued Brownian motion, and $\tilde{N}^n$ is a compensated Poisson random measure on $[0,1] \times X$ with intensity measure $n dt \vartheta(dx)$, where $\vartheta$ is a $\sigma$-finite measure on $X$. $\sigma$ and $G$ are measurable mappings specified later.

As the parameter $n$ tends to infinity, the solution $u^n$ of (1.2) will tend to the solution of the deterministic Navier-Stokes equation (1.1). In this paper, we shall investigate the asymptotic behavior of the trajectory,

$$
Y^n_t := \lambda(n) (u^n_t - u_t), \quad t \in [0,1],
$$

(1.3)

where $\lambda(n)$ is some deviation scale which strongly influences the asymptotic behavior of $Y^n$.

(1) The case $\lambda(n) = 1$ provides some large deviation estimates, which have been extensively studied in recent years. Wentzell-Freidlin type large deviation results for the 2-D stochastic Navier-Stokes equations with Gaussian noise have been established in [24], and the case of Lévy noise has been established in [32] and [33]. Large deviations for other stochastic partial differential equations also have been investigated in many papers, see [6], [22] and references therein.

(2) If $\lambda(n) = \sqrt{n}$, we are in the domain of the central limit theorem (CLT for short). We will show that, as $n$ increases to $\infty$, $\sqrt{n}(u^n - u)$ converges in distribution to the solution $V^\infty$ of stochastic equation (3.21), which is driven by Brownian motion.
Much of the problem is caused by the fact that $V^\infty \in C([0, T], H)$, but $\sqrt{n}(u^n - u) \in D([0, T], H)$. Roughly speaking, in order to solve this difficulty, we need to establish some tightness properties in $D([0, T], H)$ and apply some relationship between $D([0, T], H)$ and $C([0, T], H)$. Recently, Wang et al. [27] established a central limit theorem for 2-D stochastic Navier-Stokes equation with Gaussian noise, in their paper, they only need to focus on the space $C([0, T], H)$. Another difficulty is to deal with the highly nonlinear term $B(u, u)$, which makes the problem more complicated.

(3) To fill in the gap between the CLT scale $[\lambda(n) = \sqrt{n}]$ and the large deviations scale $[\lambda(n) = 1]$, we will study the so-called moderate deviation principle (MDP for short, cf. [8]), that is when the deviation scale satisfies

$$\lambda(n) \to +\infty, \quad \lambda(n)/\sqrt{n} \to 0 \quad \text{as} \quad n \to +\infty. \quad (1.4)$$

Throughout this paper, we assume that (1.4) is in place.

On one hand, like the large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The estimates of moderate deviations can provide us with the exponential rate of convergence and a useful method for constructing asymptotic confidence intervals, for example, see recent works [12], [15] and references therein. On the other hand, the quadratic form of the MDP’s rate function allows for the explicit minimization and in particular, it allows to obtain an asymptotic evaluation for the exit time, see [18].

In this paper, for the additive noise case, we obtain a moderate deviations of (1.2) by the generalized contraction principle together with a moderate deviation result for Lévy process. In addition to the difficulties caused by the Lévy noise, much of the problem is to deal with the highly nonlinear term $B(u, u)$. We have to prove a number of exponential estimates for the energy of the solutions as well as the exponential convergence of the approximating solutions.

At the end of this part, we mention that Budhiraja et al. [5] obtained a general moderate deviation principle for measurable functionals of a Poisson random measure by the weak convergence approach. Applying this abstract criteria, the second named author and coauthors establish a moderate deviation principle for two-dimensional stochastic Navier-Stokes equations driven by multiplicative Lévy noises in [9]. However, the moderate deviations results in [9] do not cover the results in this paper, because the assumptions in those two papers (see Condition B in [9] and Condition 1.1 in this paper) have no subordinate relationship. Wang et al. [27] also established a moderate deviation principle for 2-D stochastic Navier-Stokes equation with Gaussian noise.

There exists a great amount of literature on other properties for the stochastic Navier-Stokes equation, we only refer to [3, 4, 11, 24] for its existence and uniqueness of solutions, [10] and [14] for its ergodic properties and invariant measures. We also mention some results on MDP. Results on the MDP for processes with independent increments were obtained in De Acosta [1] and Ledoux [19]. The study of the MDP estimates for other processes has been carried out as well, e.g., Wu [29] for Markov processes, Guillin and Liptser [13] for diffusion processes, Wang and Zhang [28] for stochastic reaction-diffusion equations.

The organization of this paper is as follows. In Section 2, we shall give some preliminary results on 2-D stochastic Navier-Stokes equations. Section 3 is devoted to establishing a central limit theorem in the multiplicative noise case. In Section 4, we first put a number
of exponential estimates and several preliminary results on moderate deviations for lévy
process, then we establish a moderate deviation principle in the additive noise case.

Throughout this paper, \(c_K, c_p, \cdots\) are positive constants depending on some parameters
\(K, p, \cdots\), independent of \(n\), whose value may be different from line to line.

2 Preliminaries

Let \(V'\) be the dual of \(V\). Identifying \(H\) with its dual \(H'\), we have the dense, continuous
embedding

\[
V \hookrightarrow H \cong H' \hookrightarrow V'.
\]

In this way, we may consider \(A\) as a bounded operator from \(V\) to \(V'\). Moreover, we denote
by \((\cdot, \cdot)\) the duality between \(V\) and \(V'\) and by \(\langle \cdot, \cdot \rangle\) the inner production in \(H\). Hence, for
\(u = (u_i) \in V\), \(w = (w_i) \in V\), we have

\[
(\tilde{A}u, w) = \frac{1}{\nu}(Au, w) = \sum_{i,j=1}^{2} \int_D \partial_i u_j \partial_i w_j dx. \tag{2.5}
\]

Since \(V\) coincides with \(D(\tilde{A}^{1/2})\), we can endow \(V\) with the norm \(\|u\|_V = \|\tilde{A}^{1/2}u\|_H\). Because
the operator \(\tilde{A}\) is positive selfadjoint with compact resolvent, there is a complete orthonormal
system \(\{e_1, e_2, \cdots\}\) in \(H\) made of eigenvectors of \(\tilde{A}\), with corresponding eigenvalues \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty\) \((\tilde{A}e_i = \lambda_i e_i)\).

Define \(b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}\) by

\[
b(u, v, w) = \sum_{i,j=1}^{2} \int_D u_i \partial_i v_j w_j dx, \tag{2.6}
\]

in particular,

\[
(B(u), w) = \sum_{i,j=1}^{2} \int_D u_i \partial_i v_j w_j dx = b(u, v, w).
\]

\(B(u)\) will be used to denote \(B(u, u)\). By integration by parts,

\[
b(u, v, w) = -b(u, w, v), \tag{2.7}
\]

therefore

\[
b(u, v, v) = 0, \quad \forall u, v \in V. \tag{2.8}
\]

There are some well-known estimates for \(b\) (see [24] and [25] for example), which will be
required in the rest of this paper.

\[
|b(u, v, w)| \leq c \|u\|_V \cdot \|v\|_V \cdot \|w\|_V, \tag{2.9}
\]

\[
|b(u, v, w)| \leq 2 \|u\|_V^{1/2} \cdot \|v\|_V^{1/2} \cdot \|w\|_V^{1/2} \cdot \|v\|_H^{1/2}, \tag{2.10}
\]

\[
|b(u, u, v)| \leq \frac{1}{2} \|u\|_V^{1/2} + c \|v\|_V^{1/2} \cdot \|u\|_H^{1/2}, \tag{2.11}
\]

\[
|(B(u) - B(v), u - v)| \leq \frac{\nu}{2} \|u - v\|_V^{1/2} + \frac{32}{\nu^3} \|u - v\|_H^{1/2}, \tag{2.12}
\]


where
\[ \|v\|_{L^4}^4 \leq \|v\|_V^2 \|v\|_H^2. \] (2.13)

Let us set up the stochastic basis. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a filtration \(\{\mathcal{F}_t, t \geq 0\}\) satisfying the usual condition. Let \(\beta\) be an \(H\)-valued Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with the covariance operator \(Q\), which is positive, symmetric, trace class operator on \(H\). Let \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) be a Polish space and \(\vartheta(dx)\) a \(\sigma\)-finite measure on it. Let \(p = (p(t), t \in D_p)\) be a stationary \(\mathcal{F}_t\)-Poisson point process on \(X\) with characteristic measure \(\vartheta(dx)\), where \(D_p\) is a countable subset of \([0, \infty)\) depending on random parameter, see [16]. Denote by \(\tilde{N}(dt, dx)\) the Poisson counting measure associated with \(p\), i.e., \(\tilde{N}(t, A) = \sum_{s \in D_p, s \leq t} I_A(p(s))\). Let \(\tilde{N}(dt, dx) := N(dt, dx) - dt\vartheta(dx)\) be the compensated Poisson random measure. Denote \(\tilde{N}_n(dt, dx)\) the compensated Poisson random measure with the characteristic measure \(ndt\vartheta(dx)\).

Let \(H_0 := Q^{1/2}H\). Then \(H_0\) is a Hilbert space with the inner product
\[ \langle u, v \rangle_0 = \langle Q^{-1/2}u, Q^{-1/2}v \rangle \quad \forall u, v \in H_0. \] (2.14)

Let \(\cdot\|\cdot\|_0\) denote the norm in \(H_0\). Clearly, the embedding of \(H_0\) in \(H\) is Hilbert-Schmidt, since \(Q\) is a trace class operator. Let \(L_Q(H_0; H)\) denote the space of linear operators \(S\) such that \(SQ^{1/2}\) is a Hilbert-Schmidt operator from \(H\) to \(H\). Define the norm on the space \(L_Q(H_0; H)\) by \(\|S\|_{L_Q} = \sqrt{\text{tr}(SQS^*)}\).

Introduce the following conditions:

**Condition 2.1** There exists a constant \(K > 0\) such that

(C.1) (Growth) For all \(t \in [0, 1]\), and \(u \in H\),
\[ \|\sigma(t, u)\|_{L_Q}^2 + \int_X \|G(t, u, v)\|_H^2 \vartheta(dv) \leq K(1 + \|u\|_H^2); \]

(C.2) (Lipschitz) For all \(t \in [0, 1]\), and \(u_1, u_2 \in H\),
\[ \|\sigma(t, u_1) - \sigma(t, u_2)\|_{L_Q}^2 + \int_X \|G(t, u_1, v) - G(t, u_2, v)\|_H^2 \vartheta(dv) \leq K\|u_1 - u_2\|_H^2. \]

(C.3) The force term \(f\) is in \(L^4([0, 1]; V')\), that is
\[ \int_0^1 \|f_s\|_{V'}^4 ds < \infty. \]

Using the similar approach in [24], one can show that Eq. (1.2) has a unique solution \(u^n\) in \(D([0, 1]; H) \cap L^2([0, 1]; V)\), where \(D([0, 1]; H)\) be the space of all the càdlàg paths from \([0, 1]\) to \(H\) endowed with the uniform convergence topology. Also refer to [4].

Denoted by \(D_s([0, 1]; H)\) be the space of all the càdlàg paths from \([0, 1]\) to \(H\) endowed with the Skorokhod topology, see [16].


# 3 Central Limit Theorem

In this section, we will establish the central limit theorem.

The following estimates can be proved by Itô’s calculus, see Theorem 1.2 in [4].

**Lemma 3.1** Under Condition 2.1, for all \( n \geq 1 \),

(i) \[
E \left( \sup_{0 \leq t \leq 1} \| u^n_t \|_H^2 + \int_0^1 \| u^n_t \|_V^2 dt \right) \leq c_{f,K}; \tag{3.15}
\]

(ii) \[
E \left( \sup_{0 \leq t \leq 1} \| u^n_t \|_H^4 + \int_0^1 \| u^n_t \|_H^2 \cdot \| u^n_t \|_V^2 dt \right) \leq c_{f,K}. \tag{3.16}
\]

Moreover,

\[
\sup_{0 \leq t \leq 1} \| u_t \|_H^2 + \int_0^1 \| u_t \|_V^2 dt \leq c_{f,K}; \tag{3.17}
\]

hence, by (2.13),

\[
\int_0^1 \| u_t \|_H^4 dt \leq c_{f,K}. \tag{3.18}
\]

The next result is concerned with the convergence of \( u^n \) as \( n \to +\infty \).

**Proposition 3.1** Under Condition 2.1, for all \( n \geq 1 \),

\[
E \left( \sup_{0 \leq t \leq 1} \| u^n_t - u_t \|_H^2 + 2\nu \int_0^1 \| u^n_t - u_t \|_V^2 dt \right) \leq \frac{1}{n} c_{f,K}. \tag{3.19}
\]

**Proof:** By Itô’s formula,

\[
\begin{align*}
&\| u^n_t - u_t \|_H^2 + 2\nu \int_0^t \| u^n_s - u_s \|_V^2 ds \\
&= -2 \int_0^t (B(u^n_s) - B(u_s), u^n_s - u_s) ds \\
&\quad + \frac{2}{\sqrt{n}} \int_0^t \sigma(s, u^n_s) d\beta_s, u^n_s - u_s \right) + \frac{1}{n} \int_0^t \| \sigma(s, u^n_s) \|_{L_Q}^2 ds \\
&\quad + \frac{2}{n} \int_0^t \int_{\mathcal{X}} \left( u^n_{s-} - u_{s-}, G(s, u^n_{s-}, v) \right) \tilde{N}(ds, dv) \\
&\quad + \frac{1}{n^2} \int_0^t \int_{\mathcal{X}} \| G(s, u^n_{s-}, v) \|_H^2 N^n(ds, dv) \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t). \tag{3.20}
\end{align*}
\]

For the first term, by (2.12),

\[
|I_1(t)| \leq \int_0^t \left( \nu \| u^n_s - u_s \|_V^2 + \frac{64}{\nu^3} \| u^n_s - u_s \|_H^2 \cdot \| u_s \|_H^4 \right) ds.
\]
For the second term, by the Burkholder-Davis-Gundy inequality and Condition 2.1,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_2(s)| \right] = \frac{2}{\sqrt{n}} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s \langle \sigma(I_t^n), d\beta_t \rangle, u_t^n - u_t \right| \right)
\leq \frac{4}{\sqrt{n}} \mathbb{E} \left( \int_0^t K(1 + \|u_s^n\|_H^2)\|u_s^n - u_s\|_H^2 \, ds \right)^{1/2}
\leq \frac{4}{\sqrt{n}} \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u_s^n - u_s\|_H \cdot \left( \int_0^t K(1 + \|u_s^n\|_H^2) \, ds \right)^{1/2} \right)
\leq \frac{1}{4} \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u_s^n - u_s\|_H^2 \right) + \frac{16}{n} \mathbb{E} \left( \int_0^t K(1 + \|u_s^n\|_H^2) \, ds \right).
\]

For the third term,
\[
|I_3(t)| \leq \frac{1}{n} \int_0^t K(1 + \|u_s^n\|_H^2) \, ds.
\]

For the martingale term \( I_4 \), by Burkholder-Davis-Gundy's inequality,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_4(s)| \right] \leq \frac{4}{n} \mathbb{E} \left( \int_0^t \int_X \|u_{s-}^n - u_{s-}\|_H^2 \|G(s, u_{s-}^n, v)\|_H^2 N^n(ds, dv) \right)^{1/2}
\leq \frac{4}{n} \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u_s^n - u_s\|_H \left( \int_0^t \int_X \|G(s, u_{s-}^n, v)\|_H^2 N^n(ds, dv) \right)^{1/2} \right)
\leq \frac{1}{4} \mathbb{E} \left( \sup_{0 \leq s \leq t} \|u_s^n - u_s\|_H^2 \right) + \frac{16}{n} \mathbb{E} \left( \int_0^t K(1 + \|u_s^n\|_H^2) \, ds \right).
\]

For the fifth term \( I_5 \), by Condition 2.1,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} I_5(s) \right] = \frac{1}{n^2} \mathbb{E} \left( \int_0^t \int_X \|G(s, u_{s-}^n, v)\|_H^2 N^n(ds, dv) \right)
= \frac{1}{n} \mathbb{E} \left( \int_0^t \int_X \|G(s, u_s^n, v)\|_H^2 \mathcal{W}(dv)ds \right)
\leq \frac{1}{n} \mathbb{E} \left( \int_0^t K(1 + \|u_s^n\|_H^2) \, ds \right).
\]

Therefore, combining the above inequalities, we get
\[
\mathbb{E} \left( \frac{1}{2} \sup_{0 \leq s \leq t} \|u_s^n - u_s\|_H^2 + \nu \int_0^t \|u_s^n - u_s\|_V^2 \, ds \right)
\leq \mathbb{E} \left( \int_0^t \frac{64}{\nu^3} \|u_s^n - u_s\|_H^2 \|\|u_s\|_L^2 \, ds \right) + \frac{34}{n} \mathbb{E} \left( \int_0^t K(1 + \|u_s^n\|_H^2) \, ds \right), \quad \forall t \in [0, 1].
\]

By Lemma 3.1 and applying the Gronwall's inequality, we have
\[
\mathbb{E} \left( \sup_{0 \leq s \leq 1} \|u_s^n - u_s\|_H^2 + 2\nu \int_0^1 \|u_s^n - u_s\|_V^2 \, ds \right)
\leq \exp \left\{ \frac{128}{\nu^3} \int_0^1 \|u_s\|_L^2 \, ds \right\} \times \frac{68}{n} \mathbb{E} \left( \int_0^1 K(1 + \|u_s^n\|_H^2) \, ds \right).
\]
\[ \frac{1}{n} c_{f,K}, \]
which is (3.19). The proof is complete.

Set \( \beta_1 \) be an \( H \)-valued Brownian motion with covariance operator \( Q \), \( \beta_2 \) be an \( H \)-valued cylindrical Brownian motion, \( \beta_1 \) and \( \beta_2 \) are independent. Let \( V^\infty \) be the solution of the following SPDE:

\[
dV^\infty_t + \left( AV^\infty_t + B(V^\infty_t, u_t) + B(u_t, V^\infty_t) \right) dt = \sigma(t, u_t) d\beta_1(t) + \tilde{\sigma}(t, u_t) d\beta_2(t), \quad (3.21)
\]
with the initial value \( V^\infty_0 = 0 \), and

\[
(\tilde{\sigma}^* \tilde{\sigma}(t, u_t))_{i,j} = \int_X \langle G(t, u_t, v), e_i \rangle \langle G(t, u_t, v), e_j \rangle \vartheta(dv). \quad (3.22)
\]

Using the classical Galerkin method, the existence and uniqueness of the solution for (3.21) can be proved similarly as for the case of 2-D stochastic Navier-Stokes equation. Furthermore, the solution has the following estimate

\[
E \left( \sup_{0 \leq t \leq 1} \| V^\infty_t \|_H^2 \right) + E \left( \int_0^1 \| V^\infty_t \|_V^2 dt \right) \leq c_{f,K}.
\]

Our first main result of this paper is the following central limit theorem.

**Theorem 3.2 (Central Limit Theorem)** Under Condition 2.1, \( \sqrt{n}(u^n - u) \) converges in distribution to \( V^\infty \) in the space \( D([0, 1]; H) \cap L^2([0, 1]; V) \).

**Proof:** Theorem 3.2 follows from the Proposition 3.3 and Proposition 3.4 below. \( \blacksquare \)

Consider the following SPDE:

\[
dV^n_t + \left( AV^n_t + B(V^n_t, u_t) + B(u_t, V^n_t) \right) dt = \sigma(t, u_t) \frac{1}{\sqrt{n}} \int_X G(t, u_t, v) \tilde{N}^n(dt, dv).
\]

**Proposition 3.3** Under Condition 2.1

\[
\lim_{n \to \infty} E \left\{ \sup_{0 \leq t \leq 1} \left\| \sqrt{n}(u^n_t - u_t) - V^n_t \right\|_H^2 \right\} + \int_0^1 \left\| \sqrt{n}(u^n_t - u_t) - V^n_t \right\|_V^2 \, dt = 0. \quad (3.24)
\]

**Proof:** The proof is similar to that of Theorem 3.2 in [27], we omit it here. \( \blacksquare \)

**Proposition 3.4** Under Condition 2.1 \( V_n \) converges in distribution to \( V^\infty \) in the space \( D([0, 1]; H) \cap L^2([0, 1]; V) \).

After giving several lemmas, we will establish Proposition 3.4 at the end of this section. Let us recall the following two lemmas (see [2] and [17]).

**Lemma 3.2** Let \( E \) be a separable Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \). For an orthonormal basis \( \{ \chi_k \}_{k \in \mathbb{N}} \) in \( E \), define the function \( r^2_L : E \to \mathbb{R}^+ \) by

\[
r^2_L(x) = \sum_{k \geq L+1} \langle x, \chi_k \rangle^2 \quad L \in \mathbb{N}.
\]

Let \( \mathcal{D} \) be a total and closed under addition subset of \( E \). Then a sequence \( \{ X^n \}_{n \in \mathbb{N}} \) of stochastic processes with trajectories in \( D_e([0, 1], E) \) is tight iff the following two conditions hold:
Lemma 3.4 with the initial value $c$ following $Y$

Proof: The proof is divided into the following two parts.

Part 1. We have the following three results (a) (b) and (c).

Let $Z^n$ be the solution of the following SPDE,
\begin{align}
\begin{cases}
    dZ^n_t + AZ^n_t dt = dY^n_t, \\
    dY^n_t = \frac{1}{\sqrt{n}} \int_X G(t, u_{t-}, v) \tilde{N}^n(dt, dv),
\end{cases}
\end{align}
(3.25)
with the initial value $Z^n_0 = 0, Y^n_0 = 0$. Then $Z^n \in D([0,1]; H) \cap L^2([0,1]; V)$ and it has the following estimate
\begin{align}
    \mathbb{E} \left( \sup_{0 \leq t \leq 1} \|Z^n_t\|^2_H \right) + \mathbb{E} \left( \int_0^1 \|Z^n_t\|^2_Y dt \right) \leq c_{f,K},
\end{align}
(3.26)
where $c_{f,K}$ is independent of $n$.

Lemma 3.3 Suppose that $\{Y^n\}_{n \in \mathbb{N}}$ satisfies Assumption (A), and either $\{Y^n(0)\}$ and $\{J(Y^n)\}_{n \in \mathbb{N}}$ are tight or $\{Y^n_t\}$ is tight on the line for each $t \in [0,1]$, then $\{Y^n\}$ is tight in $D_s([0,1], \mathbb{R})$.

Let $Z^n$ be the solution of the following SPDE,
\begin{align}
\begin{cases}
    dZ^n_t + AZ^n_t dt = dY^n_t, \\
    dY^n_t = \frac{1}{\sqrt{n}} \int_X G(t, u_{t-}, v) \tilde{N}^n(dt, dv),
\end{cases}
\end{align}
(3.27)
with the initial value $Z^n_0 = 0, Y^n_0 = 0$, where $\tilde{N}$ is the compensated Poisson random measure with intensity measure $dt \vartheta(dx)$. Let $(Z_i, \tilde{N}_i)_{i \in \mathbb{N}}$ be a sequence of independent copies of $(Z, \tilde{N})$.
on some probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \in [0,1]}, \tilde{\mathbb{P}})\). Then the compensated Poisson random measure \(\tilde{N}^n\) and \(\sum_{i=1}^n \tilde{N}_i\) have the same distribution. Consequently, \((Z^n, Y^n)\) is identically distributed with \(\frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_i, Y_i)\) in \(D_s([0,1]; H) \cap L^2([0,1]; V), (D_s([0,1]; H))\).

(a) Since
\[
\mathbb{E}\tilde{\mathbb{P}}[|Y(t)|^2] = \int_0^t \int_X \|G(s, u_s, v)\|_H^2 \vartheta(\mathrm{d}v) \mathrm{d}s
\]
is finite and continuous, by the central limit theorem for semimartingales (see [16, Chapter VIII, Theorem 3.46]), \(\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i\) converges in distribution to a Wiener process with characteristics \((0, \mathcal{Q}(t))\), where
\[
\mathcal{Q}_{ij}(t) = \int_0^t \int_X \langle G(s, u_s, v), e_i \rangle \cdot \langle G(s, u_s, v), e_j \rangle \vartheta(\mathrm{d}v) \mathrm{d}s.
\] (3.29)

(b) By the central limit theorem in Banach space, see [20, Theorem 6], \(\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i\) satisfies the central limit theorem in the space \(L^2([0,1]; V)\).

(c) Now we prove that \(Z^n\) is tight in \(D_s([0,1]; H)\). We will prove that \(Z^n\) satisfies conditions (1) and (2) in Lemma 3.2.

**Step 1.** Fix \(h \in D(A)\). By (3.25), \(\{\langle Z^n, h \rangle_H\}_{n \in \mathbb{N}}\) is tight in \(\mathbb{R}\) for each \(t \in [0,1]\).

Let \(\{\tau_n, \delta_n\}\) satisfy (a) and (b) in Assumption (A). By (3.25), we have
\[
Z^n_{\tau_n+\sigma_n} - Z^n_{\tau_n} = -\int_{\tau_n}^{\tau_n+\delta_n} A Z^n_s \mathrm{d}s + \int_{\tau_n}^{\tau_n+\delta_n} \frac{1}{\sqrt{n}} \int_X G(s, u_{s-}, v) \tilde{N}^n(ds, dv) \quad \text{in} \quad V'
\]
\[
=: J^1_n + J^2_n.
\] (3.30)

For \(J^1_n\),
\[
\mathbb{E}\|\langle J^1_n, h \rangle_H\| \leq \delta_n \|Ah\|_H \cdot \mathbb{E}\left[ \sup_{s \in [0,1]} \|Z^n_s\|_H \right].
\] (3.31)

For \(J^2_n\),
\[
\mathbb{E}\|\langle J^2_n, h \rangle_H\|^2 \leq \|h\|^2_H \cdot \mathbb{E}\|J^2_n\|^2_H
\]
\[
\leq \|h\|^2_H \cdot \mathbb{E}\left( \int_{\tau_n}^{\tau_n+\delta_n} \int_X \|G(s, u_s, v)\|_H^2 \vartheta(\mathrm{d}v) \mathrm{d}s \right)
\]
\[
\leq \|h\|^2_H \cdot (1 + \sup_{s \in [0,1]} \|u_s\|^2_H) K \delta_n.
\] (3.32)

Combining (3.30) and (3.32), we get
\[
\mathbb{E}\|\langle Z^n_{\tau_n+\sigma_n} - Z^n_{\tau_n}, h \rangle_H\| \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.
\]

By Lemma 3.3, \(\langle Z^n, h \rangle_H\) is tight in \(D_s([0,1], \mathbb{R})\).

**Step 2.** For any \(x \in H\), let
\[
x^L = \sum_{k \geq L+1} \langle x, e_k \rangle_H \cdot e_k, \quad L \in \mathbb{N}.
\]
Then $Z^{n,L}$ satisfies that
\[
\text{d}Z^{n,L}_t = A Z^{n,L}_t \text{d}t + \frac{1}{\sqrt{n}} \int_{\mathbb{R}} G^L(t, u_{-}, v) \text{d}N^n(dt, dv) \quad \text{in } V'.
\]
By Itô’s formula,
\[
\|Z^{n,L}_t\|_H^2 + 2\nu \int_0^t \|Z^{n,L}_s\|_V^2 \text{d}s = 2 \int_0^t \left( \frac{1}{\sqrt{n}} \int_{\mathbb{R}} G^L(s, u_{-}, v) \text{d}N^n(ds, dv), Z^{n,L}_s \right) + \frac{1}{n} \int_0^t \|G^L(s, u_{-}, v)\|_H^2 \text{d}N^n(ds, dv).
\]
By the Burkholder-Davis-Gundy’s inequality,
\[
\mathbb{E} \left( \sup_{0 \leq t \leq 1} \|Z^{n,L}_t\|_H^2 \right) + 2\nu \mathbb{E} \left( \int_0^1 \|Z^{n,L}_t\|_V \text{d}t \right) \leq 2 \mathbb{E} \left[ \int_0^1 \int_{\mathbb{R}} \left( \frac{1}{\sqrt{n}} \int_{\mathbb{R}} G^L(s, u_{-}, v) \text{d}N^n(ds, dv) \right)^2 \text{d}s \right] + \int_0^1 \int_{\mathbb{R}} \|G^L(s, u_{-}, v)\|_H^2 \text{d}(dv) \text{d}s
\]
\[
\leq 2 \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \|Z^{n,L}_t\|_H \cdot \left( \int_0^1 \int_{\mathbb{R}} \|G^L(s, u_{-}, v)\|_H^2 \text{d}N^n(ds, dv) \right)^{\frac{1}{2}} \right] + \int_0^1 \int_{\mathbb{R}} \|G^L(s, u_{-}, v)\|_H^2 \text{d}(dv) \text{d}s
\]
\[
\leq \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq 1} \|Z^{n,L}_t\|_H^2 \right) + 3 \int_0^1 \int_{\mathbb{R}} \|G^L(s, u_{-}, v)\|_H^2 \text{d}(dv) \text{d}s.
\]
By Fatou’s lemma,
\[
\mathbb{E} \left( \sup_{0 \leq t \leq 1} \|Z^{n,L}_t\|_H^2 \right) \leq 6 \int_0^1 \int_{\mathbb{R}} \|G^L(s, u_{-}, v)\|_H^2 \text{d}(dv) \text{d}s \to 0, \text{ as } L \to +\infty.
\]
Hence \{$Z^n\}_{n \in \mathbb{N}}$ is tight in the Skorokhod space $D_s([0, 1]; H)$ by Lemma 3.2.

**Part 2:** By (a) (b) (c), and the fact that $(Z^n, Y^n)$ is identically distributed with $\sqrt{n} \sum_{i=1}^{\infty} (Z_i, Y_i)$ in $(D_s([0, 1]; H) \cap L^2([0, 1]; V), D_s([0, 1]; H))$, $(Z^n, Y^n)$ converges in distribution in $(D_s([0, 1]; H) \cap L^2([0, 1]; V), D_s([0, 1]; H))$. Let $(\mathcal{Z}, \mathcal{Y})$ be any limit of $(Z^n, Y^n)$. We now prove that $(\mathcal{Z}, \mathcal{Y})$ has the same distribution with $(\overline{Z}, \overline{Y})$.

By the Skorokhod representation theorem, there exit a stochastic basis $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t \in [0, 1]}, \mathbb{P}_1)$, and $(D_s([0, 1]; H) \cap L^2([0, 1]; V), D_s([0, 1]; H))$-valued random variables $(Z^n_1, Y^n_1)$ and $(\mathcal{Z}_1, \mathcal{Y}_1)$ on this basis, such that $(Z^n_1, Y^n_1)$ (resp. $(\mathcal{Z}_1, \mathcal{Y}_1)$) has the same law as $(Z^n, Y^n)$ (resp. $(\mathcal{Z}, \mathcal{Y})$), and $(Z^n_1, Y^n_1) \to (\mathcal{Z}_1, \mathcal{Y}_1)$ in $(D_s([0, 1]; H) \cap L^2([0, 1]; V), D_s([0, 1]; H))$, $\mathbb{P}_1$-a.s.

From the equation satisfied by $(Z^n, Y^n)$, we see that $(Z^n_1, Y^n_1)$ satisfies the following integral equation
\[
\langle Z^n_1(t), v \rangle = \int_0^t \langle Z^n_1(s), Av \rangle \text{d}s + \langle Y^n_1(t), v \rangle, \quad \forall v \in D(A), \mathbb{P}_1\text{-a.s.} \tag{3.33}
\]

Since $\mathcal{Y}$ is a Wiener process with characteristics $(0, \overline{Q}(t))$, $\mathcal{Y}_1$ is also a Wiener process with characteristics $(0, \overline{Q}(t))$ and hence has a continuous version. Since $\lim_{n \to \infty} Y^n_1 = \mathcal{Y}_1$ in $D_s([0, 1], H)$, $\mathbb{P}_1$-a.s., by [16] Chapter VI, Proposition 1.17, we have
\[
\lim_{n \to \infty} \sup_{t \in [0, 1]} \|Y^n_1(t) - \mathcal{Y}_1(t)\|_H^2 = 0, \mathbb{P}_1\text{-a.s.}
\]
Since \( \lim_{n \to \infty} Z^n_1 = Z_1 \) \( \mathbb{P}_1 \)-a.s. in \( D_s([0, 1], H) \cap L^2([0, 1]; V) \),
\[
\lim_{n \to \infty} \int_0^1 \| Z^n_1(t) - Z_1(t) \|^2_H dt \to 0, \quad \mathbb{P}_1 \text{-a.s.,}
\]
and for any \( t_0 \in \mathcal{T} = \{ s \in [0, 1] : Z_1(s+) = Z_1(s-) \} \)
\[
\lim_{n \to \infty} \| Z^n_1(t_0) - Z_1(t_0) \|^2_H = 0.
\]
Combining above inequalities with (3.33), and taking limits of \( n \),
\[
\langle Z_1(t_0), v \rangle = \int_0^{t_0} \langle Z_1(s), Av \rangle ds + \langle \mathcal{Y}_1(t_0), v \rangle, \quad \forall v \in D(A).
\]
Since \( \mathcal{T} \) is dense in \([0, 1]\), by the uniqueness of (3.27), we have that \((Z_1, \mathcal{Y}_1)\) has the same distribution with \((\tilde{Z}, \tilde{Y})\). The proof is complete. \( \blacksquare \)

**Proof:** [Proof of Proposition 3.4] Let \((X, M)\) be the solution of the following SPDE,
\[
\begin{cases}
 dX_t + AX_t dt = dM_t, \\
 dM_t = \sigma(t, u_t) d\beta_t
\end{cases}
\]
with the initial value \( X_0 = 0, M_0 = 0 \).

By Lemma 3.4 it is easy to see that \( \{(Z^n, Y^n, X, M)\}_{n \in \mathbb{N}} \) converges in distribution to a random vector \((\tilde{Z}^0, \tilde{Y}^0, \tilde{X}^0, \tilde{M}^0)\) in \( \Pi \), where
\[
\Pi = \left( D_s([0, 1]; H) \cap L^2([0, 1]; V), D_s([0, 1]; H), C([0, 1]; H) \cap L^2([0, 1]; V), C([0, 1]; H) \right),
\]
and \((\tilde{Z}^0, \tilde{Y}^0, \tilde{X}^0, \tilde{M}^0)\) satisfies the following conditions
(1) \((\tilde{Z}^0, \tilde{Y}^0)\) has the same distribution with \((\tilde{Z}, \tilde{Y})\) in Lemma 3.4
(2) \((\tilde{X}^0, \tilde{M}^0)\) has the same distribution with \((X, M)\);
(3) \((\tilde{Z}^0, \tilde{Y}^0)\) and \((\tilde{X}^0, \tilde{M}^0)\) are independent.

By the Skorokhod representation theorem, there exist a stochastic basis \((\Omega_1, \mathcal{F}_1, \{\mathcal{F}_{1,t}\}_{t \in [0,1]}, \mathbb{P}^1)\), \( \Pi \)-valued random variables \((\tilde{Z}^1, \tilde{Y}^1, \tilde{X}^1, \tilde{M}^1)\) on this basis, and \( \{(Z^n, Y^n, X^n, M^n)\}_{n \in \mathbb{N}} \), such that \((Z^n_1, Y^n_1, X^n_1, M^n_1)\) (resp. \((\tilde{Z}^1, \tilde{Y}^1, \tilde{X}^1, \tilde{M}^1)\)) has the same law as \((Z^n, Y^n, X, M)\) (resp. \((\tilde{Z}^0, \tilde{Y}^0, \tilde{X}^0, \tilde{M}^0)\)), and \((Z^n_1, Y^n_1, X^n_1, M^n_1) \to (\tilde{Z}^1, \tilde{Y}^1, \tilde{X}^1, \tilde{M}^1)\) in \( \Pi, \mathbb{P}^1 \)-a.s.. Since \( \tilde{Z}^1 \) is continuous, by [16] Chapter VI, Proposition1.17, we have
\[
\sup_{t \in [0,1]} \| Z^n_1(t) - \tilde{Z}^1(t) \|^2_H + \int_0^1 \| Z^n_1(t) - \tilde{Z}^1(t) \|^2_V dt \to 0, \quad \mathbb{P}^1 \text{-a.s.}
\]
We also have
\[
\sup_{t \in [0,1]} \| X^n_1(t) - \tilde{X}^1(t) \|^2_H + \int_0^1 \| X^n_1(t) - \tilde{X}^1(t) \|^2_V dt \to 0 \quad \mathbb{P}^1 \text{-a.s.}
\]
Let \( L_1^n \) be the solution of the following equation
\[
\begin{cases}
    dL_1^n(s) + AL_1^n(s)ds + B\left(L_1^n(s) + Z_1^n(s) + X_1^n(s), u_s\right)ds + B\left(u_s, L_1^n(s) + Z_1^n(s) + X_1^n(s)\right)ds = 0; \\
    L_1^n(0) = 0.
\end{cases}
\] (3.38)

With the help of (3.36) and (3.37), it is not difficult to prove that
\[
\lim_{n \to +\infty} \left[ \sup_{0 \leq t \leq 1} \|L_1^n(t) - \tilde{L}^1(t)\|^2_H + \int_0^1 \|L_1^n(s) - \tilde{L}^1(s)\|^2_{V_1^n} ds \right] = 0, \quad \mathbb{P}^1\text{-a.s.,} \quad (3.39)
\]
where \( \tilde{L}^1 \) is the solution of the following equation
\[
\begin{cases}
    d\tilde{L}^1(s) + AL\tilde{L}^1(s)ds + B\left(\tilde{L}^1(s) + \tilde{Z}^1(s) + \tilde{X}^1(s), u_s\right)ds + B\left(u_s, \tilde{L}^1(s) + \tilde{Z}^1(s) + \tilde{X}^1(s)\right)ds = 0; \\
    \tilde{L}^1(0) = 0.
\end{cases}
\]

Set \( \tilde{V}^n = L_1^n + Z_1^n + X_1^n \) and \( \tilde{V}^\infty = \tilde{L}^1 + \tilde{Z}^1 + \tilde{X}^1 \). By (3.36), (3.37) and (3.39), we obtain
\[
\lim_{n \to +\infty} \left[ \sup_{0 \leq t \leq 1} \|\tilde{V}^n(t) - \tilde{V}^\infty(t)\|^2_H + \int_0^1 \|\tilde{V}^n(s) - \tilde{V}^\infty(s)\|^2_{V_1^n} ds \right] = 0, \quad \mathbb{P}^1\text{-a.s.} \quad (3.40)
\]
Since \( V^n \) has the same distribution with \( \tilde{V}^n \) in \( D([0, 1]; H) \cap L^2([0, 1]; V) \), and \( V^\infty \) has the same distribution with \( \tilde{V}^\infty \) in \( C([0, 1]; H) \cap L^2([0, 1]; V) \), the proof is complete. \( \square \)

4 Moderate deviations

In this section, we shall study the moderate deviations for the following two-dimensional Stochastic Navier-Stokes equations driven by additive noises.
\[
du^n_i + Au^n_idt + B(u^n_i)dt = f_idt + \frac{1}{\sqrt{n}}dB_t + \frac{1}{n} \int_X G(x)\tilde{N}(dt, dx). \quad (4.41)
\]
Recall that \( \{e_i\}_{i \in \mathbb{N}} \) is a complete orthonormal system of \( H \), and \( \tilde{A}e_i = \lambda_i e_i \) (see (2.5) for \( \tilde{A} \)). Here we assume that the covariance operator of \( \beta \), denoted by \( Q \), satisfies \( Qe_i = q_ie_i \). Denote by \( TrQ \) the trace of the operator \( Q \). Since \( Q \) is a trace class operator, \( TrQ = \sum_{i=1}^{\infty} (Qe_i, e_i) = \sum_{i=1}^{\infty} q_i < \infty \).

Assume that \( G \) satisfies the following condition:

\textbf{Condition 4.1 (Exponential Integrability)} There exists \( \theta_0 > 0 \) such that
\[
C_0 := \int_X \|G(x)\|^2_H \exp(\theta_0 \|G(x)\|_H) \vartheta(dx) < +\infty. \quad (4.42)
\]

\textbf{Remark 1} Under Condition 4.1, for any \( \theta \in (0, \theta_0] \),
\[
\mathbb{E} \left[ \exp \left\{ \theta \sup_{0 \leq t \leq 1} \left\| \int_0^t \int_X G(x)\tilde{N}(ds, dx) \right\|_H \right\} \right] < +\infty,
\]
where \( \tilde{N} \) is a compensated Poisson random measure on \([0, 1] \times X\) with intensity measure \( dt\vartheta(dx) \). See the proof of Lemma 4.7 in Appendix or [23, Corollary 4.3].

We will prove that \( \lambda(u^n - u) \) satisfies a large deviation principle (LDP for short) on \( D([0, 1]; H) \cap L^2([0, 1]; V) \) where \( \lambda(u^n) \) satisfies (1.4). This special type of LDP is usually called the moderate deviation principle of \( u^n \) (cf. [3]).
4.1 Some exponential estimates

For any integer \( m \geq 1 \), let \( P_m : H \to H \) be the projection operator

\[
P_m x = \sum_{i=1}^{m} \langle x, e_i \rangle e_i,
\]

(4.43)

and denote \( H_m := P_m H \). Set

\[
Z_{t}^{n,m} := -\int_{0}^{t} AZ_{s}^{n,m} \, ds + \frac{\lambda(n)}{n} \int_{0}^{t} \int_{X} P_m G(x) \tilde{N}^n(ds, dx),
\]

(4.44)

\[
Z_{t}^{n} := -\int_{0}^{t} AZ_{s}^{n} \, ds + \frac{\lambda(n)}{n} \int_{0}^{t} \int_{X} G(x) \tilde{N}^n(ds, dx),
\]

(4.45)

and define

\[
X_{t}^{m,n} := \frac{\lambda(n)}{n} \int_{0}^{t} \int_{X} P_m G(x) \tilde{N}^n(ds, dx),
\]

\[
X_{t}^{n} := \frac{\lambda(n)}{n} \int_{0}^{t} \int_{X} G(x) \tilde{N}^n(ds, dx).
\]

Lemma 4.1 For any \( \delta > 0 \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \| Z_{t}^{n,m} - Z_{t}^{n} \|_H > \delta \right) = -\infty,
\]

(4.46)

and

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \| X_{t}^{m,n} - X_{t}^{n} \|_H > \delta \right) = -\infty.
\]

(4.47)

Proof: These two equations can be proved by the same method. Here we only prove the first one. Put \( \tilde{Z}_{t}^{n,m} = \frac{n}{\lambda(n)} (Z_{t}^{n,m} - Z_{t}^{n}) \). Then \( \tilde{Z}_{t}^{n,m} \) is the solution of the equation

\[
\tilde{Z}_{t}^{n,m} = -\int_{0}^{t} A\tilde{Z}_{s}^{n,m} \, ds + \int_{0}^{t} \int_{X} (P_m G(x) - G(x)) \tilde{N}^n(ds, dx).
\]

For any \( n, k \geq 1 \), let \( g_{n,k}(y) := \left( 1 + k \lambda^{-2}(n) \| y \|_H^2 \right)^{-\frac{1}{2}} \). Then

\[
g'_{n,k}(y) := k \lambda^{-2}(n) \left( 1 + k \lambda^{-2}(n) \| y \|_H^2 \right)^{-\frac{1}{2}} y,
\]

(4.48)

and

\[
g''_{n,k}(y) := -k^2 \lambda^{-4}(n) \left( 1 + k \lambda^{-2}(n) \| y \|_H^2 \right)^{-\frac{3}{2}} y \otimes y + k \lambda^{-2}(n) \left( 1 + k \lambda^{-2}(n) \| y \|_H^2 \right)^{-\frac{1}{2}} I_H,
\]

where \( I_H \) stands for the identity operator. Applying Itô’s formula to \( \exp(g_{n,k}(\tilde{Z}_{t}^{n,m})) \), we know that

\[
M_{t}^{n,m} := \exp \left( g_{n,k}(\tilde{Z}_{t}^{n,m}) - 1 - \int_{0}^{t} h_{n,k}(\tilde{Z}_{s}^{n,m}) \, ds \right)
\]

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is a local martingale, where
\[
h_{n,k}(y) = -\langle Ay, g'_{n,k}(y) \rangle + n \int_X \left\{ \exp \left[ g_{n,k}(y + P_m G(x) - G(x)) - g_{n,k}(y) \right] \right. \\
- 1 - \left. \langle g'_{n,k}(y), P_m G(x) - G(x) \rangle \right\} \vartheta(dx).
\] (4.49)

Using similar arguments as the proof of (4.6) in [23], and choosing \( n \) large enough such that \( k^2 \lambda^{-1}(n) < \theta_0 \), we get
\[
\begin{align*}
& \left| \int_X \left\{ \exp \left[ g_{n,k}(y + P_m G(x) - G(x)) - g_{n,k}(y) \right] - 1 - \langle g'_{n,k}(y), P_m G(x) - G(x) \rangle \right\} \vartheta(dx) \right| \\
\leq & \frac{k}{\lambda^2(n)} \int_X \|P_m G(x) - G(x)\|_H^2 \exp \left( \sqrt{k} \lambda^{-1}(n) \|P_m G(x) - G(x)\|_H \right) \vartheta(dx) \\
\leq & \frac{k}{\lambda^2(n)} \int_X \|P_m G(x) - G(x)\|_H^2 \exp \left( \theta_0 \|P_m G(x) - G(x)\|_H \right) \vartheta(dx) \\
= & \frac{k}{\lambda^2(n)} c_m.
\end{align*}
\] (4.50)

Note that by the dominated convergence theorem, \( \lim_{m \to \infty} c_m = 0 \) and
\[
\langle -Ay, g'_{n,k}(y) \rangle = k \lambda^{-2}(n) \left( 1 + k \lambda^{-2}(n) \|y\|_H^2 \right)^{-\frac{1}{2}} \langle -Ay, y \rangle \leq 0.
\]
Thus, for any \( t \in [0,1] \),
\[
h_{n,k}(\tilde{Z}_{t}^{n,m}) \leq \frac{nk^2 c_m}{\lambda^2(n)} \to 0, \text{ as } m \to \infty.
\] (4.51)

Observe that
\[
\begin{align*}
& \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Z_t^{n,m} - Z_t^{n}\|_H > \delta \right) \\
= & \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|\tilde{Z}_t^{n,m}\|_H > \frac{n \delta}{\lambda(n)} \right) \\
= & \mathbb{P} \left( \sup_{0 \leq t \leq 1} g_{n,k}(\tilde{Z}_t^{n,m}) > \left( 1 + \frac{nk^2 \delta^2}{\lambda^4(n)} \right)^{\frac{1}{2}} \right) \\
= & \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left[ g_{n,k}(\tilde{Z}_t^{n,m}) - 1 - \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds + 1 + \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds \right] > \left( 1 + \frac{nk^2 \delta^2}{\lambda^4(n)} \right)^{\frac{1}{2}} \right) \\
\leq & \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left[ g_{n,k}(\tilde{Z}_t^{n,m}) - 1 - \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds \right] > \left( 1 + \frac{nk^2 \delta^2}{\lambda^4(n)} \right)^{\frac{1}{2}} - 1 - \sup_{0 \leq t \leq 1} \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds \right).
\] (4.52)

Due to (4.51) and Doob’s inequality,
\[
\begin{align*}
& \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left[ g_{n,k}(\tilde{Z}_t^{n,m}) - 1 - \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds \right] > \left( 1 + \frac{nk^2 \delta^2}{\lambda^4(n)} \right)^{\frac{1}{2}} - 1 - \sup_{0 \leq t \leq 1} \int_0^t h_{n,k}(\tilde{Z}_s^{n,m}) ds \right)
\end{align*}
\]
\[ \leq P \left( \sup_{0 \leq t \leq 1} \left[ g_{n,k}(\tilde{Z}_{t}^{n,m}) - 1 - \int_{0}^{t} h_{n,k}(\tilde{Z}_{s}^{n,m}) ds \right] > \left( 1 + \frac{kn^{2}\delta^{2}}{\lambda^{4}(n)} \right)^{\frac{1}{2}} - 1 - \frac{nk_{c_{m}}}{\lambda^{2}(n)} \right) \]

\[ \leq \sup_{0 \leq t \leq 1} E \left[ \exp \left( g_{n,k}(\tilde{Z}_{t}^{n,m}) - 1 - \int_{0}^{t} h_{n,k}(\tilde{Z}_{s}^{n,m}) ds \right) \right] \times \exp \left[ - \left( 1 + \frac{kn^{2}\delta^{2}}{\lambda^{4}(n)} \right)^{\frac{1}{2}} + 1 + \frac{nk_{c_{m}}}{\lambda^{2}(n)} \right] \]

\[ \leq \exp \left[ - \left( 1 + \frac{kn^{2}\delta^{2}}{\lambda^{4}(n)} \right)^{\frac{1}{2}} + 1 + \frac{nk_{c_{m}}}{\lambda^{2}(n)} \right] \], \hspace{1cm} (4.53)

where in the last inequality, we used the fact that

\[ \sup_{0 \leq t \leq 1} E \left[ \exp \left( g_{n,k}(\tilde{Z}_{t}^{n,m}) - 1 - \int_{0}^{t} h_{n,k}(\tilde{Z}_{s}^{n,m}) ds \right) \right] \leq 1, \]

since \( \exp \left( g_{n,k}(\tilde{Z}_{t}^{n,m}) - 1 - \int_{0}^{t} h_{n,k}(\tilde{Z}_{s}^{n,m}) ds \right) \) is a nonnegative local martingale with the initial value 1. Putting (4.52) and (4.53) together, we have

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \lambda^{2}(n) \log P \left( \sup_{0 \leq t \leq 1} \|Z_{t}^{n,m} - Z_{t}^{n}\|_{H} > \delta \right) \]

\[ \leq \lim_{m \to \infty} \left( -k^{\frac{1}{2}} \delta + k_{c_{m}} \right) \]

\[ = -k^{\frac{1}{2}} \delta. \hspace{1cm} (4.54) \]

Since \( k \) is arbitrary, taking \( k \to +\infty \), we get (4.46). The proof is complete.  \[\blacksquare\]

By a similar but simple calculation, one can obtain the following lemma. The proof is omitted here.

**Lemma 4.2** For any \( m \geq 1 \),

\[ \lim_{M \to \infty} \limsup_{n \to \infty} \lambda^{2}(n) \log P \left( \sup_{0 \leq t \leq 1} \|Z_{t}^{n,m} - Z_{t}^{n}\|_{H} > M \right) = -\infty, \]

and

\[ \lim_{M \to \infty} \limsup_{n \to \infty} \lambda^{2}(n) \log P \left( \sup_{0 \leq t \leq 1} \|Z_{t}^{n}\|_{H} > M \right) = -\infty. \]

**Lemma 4.3** For any \( \delta > 0 \),

\[ \lim_{M \to \infty} \limsup_{n \to \infty} \lambda^{2}(n) \log P \left( \left( \int_{0}^{1} \|Z_{t}^{n,m} - Z_{t}^{n}\|_{V}^{2} dt \right)^{\frac{1}{2}} > \delta \right) = -\infty. \]

**Proof:** We keep use the notions in the proof of Lemma 4.1. As for any \( m, n, k \geq 1 \),

\[ \left( \int_{0}^{1} \|Z_{t}^{n,m} - Z_{t}^{n}\|_{V}^{2} dt \right)^{\frac{1}{2}} \]

\[ = \frac{\lambda(n)}{n} \left( \int_{0}^{1} \|\tilde{Z}_{t}^{n,m}\|_{V}^{2} dt \right)^{\frac{1}{2}} \]

\[ = \frac{\lambda(n)}{n} \left[ \int_{0}^{1} \left( 1 + \frac{k}{\lambda^{2}(n)} \|\tilde{Z}_{t}^{n,m}\|_{H}^{2} \right)^{\frac{1}{2}} \cdot \|\tilde{Z}_{t}^{n,m}\|_{V} \cdot \left( 1 + \frac{k}{\lambda^{2}(n)} \|\tilde{Z}_{t}^{n,m}\|_{H}^{2} \right)^{-\frac{1}{2}} dt \right]^{\frac{1}{2}} \]
Recall (4.48), it suffices to show that

\[ \lambda(n) \left[ \sup_{0 \leq t \leq 1} \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{\frac{1}{2}} \cdot \left( \int_0^1 \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{-\frac{1}{2}} \cdot \| \tilde{Z}_{t}^{n,m} \|^2_V dt \right)^{\frac{1}{2}} \right] \]

\[ \leq \frac{\lambda^2(n)}{2n} \sup_{0 \leq t \leq 1} \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{\frac{1}{2}} + \frac{1}{2n} \int_0^1 \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{-\frac{1}{2}} \cdot \| \tilde{Z}_{t}^{n,m} \|^2_V dt, \]

it suffices to show that

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{\frac{1}{2}} > \frac{n}{\lambda^2(n)} \delta \right) = -\infty, \quad (4.55) \]

and

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \int_0^1 \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{-\frac{1}{2}} \cdot \| \tilde{Z}_{t}^{n,m} \|^2_V dt > n\delta \right) = -\infty. \quad (4.56) \]

Since

\[ \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{\frac{1}{2}} > \frac{n}{\lambda^2(n)} \delta \right) = \mathbb{P} \left( \sup_{0 \leq t \leq 1} \| \tilde{Z}_{t}^{n,m} - Z_t^n \|^2_H > \delta^2 - \frac{\lambda^4(n)}{n^2k} \right), \]

(4.55) follows from Lemma 4.1. Next we prove (4.55). Set

\[ R_{n,m,k} := \int_0^1 \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{-\frac{1}{2}} \cdot \| \tilde{Z}_{t}^{n,m} \|^2_V dt. \]

Recall (4.48),

\[ \frac{k\nu}{\lambda^2(n)} \left( 1 + \frac{k}{\lambda^2(n)} \| \tilde{Z}_{t}^{n,m} \|^2_H \right)^{-\frac{1}{2}} \cdot \| \tilde{Z}_{t}^{n,m} \|^2_V = (A \tilde{Z}_{t}^{n,m}, g_{n,k}(\tilde{Z}_{t}^{n,m})), \]

Combining this with (4.49), (4.50) and (4.51), we get

\[ \int_0^1 h(\tilde{Z}_{t}^{n,m}) ds + \frac{k\nu}{\lambda^2(n)} R_{n,m,k} \leq \frac{nkc_m}{\lambda^2(n)}, \]

where \( c_m \) is defined in (4.50), which converges to 0 as \( m \to \infty \).

Therefore,

\[ \mathbb{P} \left( R_{n,m,k} > n\delta \right) \]

\[ = \mathbb{P} \left( \frac{\nu k}{\lambda^2(n)} R_{n,m,k} > \frac{\nu nk\delta}{\lambda^2(n)} \right) \]

\[ \leq \mathbb{P} \left( g_{n,k}(\tilde{Z}_{1}^{n,m}) + \frac{\nu k}{\lambda^2(n)} R_{n,m,k} > \frac{\nu nk\delta}{\lambda^2(n)} \right) \]

\[ = \mathbb{P} \left( g_{n,k}(\tilde{Z}_{1}^{n,m}) - 1 - \int_0^1 h(\tilde{Z}_{t}^{n,m}) dt + 1 + \int_0^1 h(\tilde{Z}_{t}^{n,m}) dt + \frac{\nu k}{\lambda^2(n)} R_{n,m,k} > \frac{\nu nk\delta}{\lambda^2(n)} \right) \]

\[ \leq \mathbb{P} \left( g_{n,k}(\tilde{Z}_{1}^{n,m}) - 1 - \int_0^1 h(\tilde{Z}_{t}^{n,m}) dt > \frac{\nu nk\delta}{\lambda^2(n)} - 1 - \frac{nkc_m}{\lambda^2(n)} \right) \]
\[
\exp \left( -\frac{nk\delta}{\lambda^2(n)} + 1 + \frac{nkc_m}{\lambda^2(n)} \right),
\]
which yields
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} (R_{n, mk} > n\delta) = -k\delta \nu.
\]
Since \(k\) is arbitrary, taking \(k \to +\infty\), we get (4.56). The proof is complete. \(\blacksquare\)

By a similar but simple calculation, one can obtain the following lemma. The proof is omitted here.

**Lemma 4.4** For any \(m \geq 1\),
\[
\lim_{M \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \int_0^1 \|Z_{t,m}^n\|_V^2 dt > M \right) = -\infty,
\]
and
\[
\lim_{M \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \int_0^1 \|Z_t^n\|_V^2 dt > M \right) = -\infty.
\]

Recall (1.3), then
\[
dY_t^n = -AY_t^n dt - B(u_t^n, Y_t^n) dt - B(Y_t^n, u_t) dt + \frac{\lambda(n)}{\sqrt{n}} d\beta_t + \frac{\lambda(n)}{n} \int_{\mathbb{X}} G(x) \tilde{N}^n(dt, dx).
\]

Denote by \(Y_{n,m}^n\) the solution of the following equation
\[
dY_{t,m}^n = -AY_{t,m}^n dt - B(u_t^n, Y_{t,m}^n) dt - B(Y_{t,m}^n, u_t) dt + \frac{\lambda(n)}{\sqrt{n}} d\beta_t^m + \frac{\lambda(n)}{n} \int_{\mathbb{X}} P_m G(x) \tilde{N}^n(dt, dx).
\]

**Lemma 4.5**
(1) For any \(\delta > 0\),
\[
\lim_{Y \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Y_{t,m}^n - Y_t^n\|_H^2 + \nu \int_0^1 \|Y_{t,m}^n - Y_t^n\|_V^2 dt > \delta \right) = -\infty.
\]

(2) For any \(m \geq 1\),
\[
\lim_{Y \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Y_t^n\|_H^2 + \nu \int_0^1 \|Y_t^n\|_V^2 dt > Y \right) = -\infty,
\]
and
\[
\lim_{Y \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Y_t^n\|_V^2 dt > Y \right) = -\infty.
\]
Proof: Set
\[ \hat{Y}_t^{n,m} = Y_t^{n,m} - Z_t^{n,m}, \quad \bar{Y}_t^n = Y_t^n - Z_t^n. \] (4.62)

Then \( \hat{Y}_t^{n,m} \) and \( \bar{Y}_t^n \) solve the following equations, respectively,
\[ d\hat{Y}_t^{n,m} = - A\hat{Y}_t^{n,m} dt - B(u_t, \hat{Y}_t^{n,m} + Z_t^{n,m}) dt - B(\bar{Y}_t^n + Z_t^n, u_t) dt + \frac{\lambda(n)}{\sqrt{n}} d\beta_t^m, \]
and
\[ d\bar{Y}_t^n = - A\bar{Y}_t^n dt - B\left(u_t + \frac{\bar{Y}_t^n + Z_t^n}{\lambda(n)}, \bar{Y}_t^n + Z_t^n\right) dt - B(\bar{Y}_t^n + Z_t^n, u_t) dt + \frac{\lambda(n)}{\sqrt{n}} d\beta_t. \]

Note that
\[ \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| Y_t^{n,m} - Y_t^n \|_H^2 + \nu \int_0^1 \| Y_t^{n,m} - Y_t^n \|_V^2 dt > \delta \right) \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| Z_t^{n,m} - Z_t^n \|_H^2 + \nu \int_0^1 \| Z_t^{n,m} - Z_t^n \|_V^2 dt > \frac{\delta}{2} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| \hat{Y}_t^{n,m} - \bar{Y}_t^n \|_H^2 + \nu \int_0^1 \| \hat{Y}_t^{n,m} - \bar{Y}_t^n \|_V^2 dt > \frac{\delta}{2} \right). \]

By Lemma 4.1 and Lemma 4.3, for any \( \delta > 0 \),
\[ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| Z_t^{n,m} - Z_t^n \|_H^2 + \nu \int_0^1 \| Z_t^{n,m} - Z_t^n \|_V^2 dt > \delta \right) = -\infty. \] (4.63)

To prove (4.59), it remains to prove that for any \( \delta > 0 \),
\[ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| \hat{Y}_t^{n,m} - \bar{Y}_t^n \|_H^2 + \nu \int_0^1 \| \hat{Y}_t^{n,m} - \bar{Y}_t^n \|_V^2 dt > \delta \right) = -\infty. \] (4.64)

For any \( M > 0, \delta_0 > 0 \), define stopping times
\[ \tau_{\delta_0}^{n,m} := \inf \left\{ t \geq 0, \| Z_t^{n,m} - Z_t^n \|_H > \delta_0 \text{ or } \int_0^t \| Z_s^{n,m} - Z_s^n \|_V^2 ds > \delta_0 \right\}, \]
\[ \tau_{Y,1,M}^{n} := \inf \{ t \geq 0, \| Y_t^n \|_H > M \}, \]
and
\[ \tau_{Y,2,M}^{n} := \inf \{ t \geq 0, \int_0^t Y_s^n \|_V^2 ds > M \}. \]

We also define similar stopping times for \( Y^{n,m}, Z^n, Z^{n,m} \), denoting by \( \tau_{Y,1,M}^{n,m}, \tau_{Y,2,M}^{n,m}, \tau_{Z,1,M}^{n,m}, \tau_{Z,2,M}^{n,m} \), respectively. Let
\[ \tau_M^{n,m} := \tau_{Y,1,M}^{n,m} \wedge \tau_{Y,2,M}^{n,m} \wedge \tau_{Z,1,M}^{n,m} \wedge \tau_{Z,2,M}^{n,m}. \]

For any \( M \geq \sup_{0 \leq t \leq 1} \| u_t \|_H \vee \int_0^1 \| u_t \|_V^2 dt \), set
\[ A_M^{n,m} := \left\{ \sup_{0 \leq t \leq 1} \| Y_t^{n,m} \|_H \leq M \right\} \cap \left\{ \sup_{0 \leq t \leq 1} \| Y_t^n \|_H \leq M \right\} \cap \left\{ \sup_{0 \leq t \leq 1} \| Z_t^{n,m} \|_H \leq M \right\}. \]
Thus, \( \sup_{0 \leq t \leq 1} \| Z_t^n \|_H \leq M \)

\[
B_{M}^{n,m} := \left\{ \int_0^1 \| Y_t^n \|_V^2 \, dt \leq M \right\} \cap \left\{ \int_0^1 \| Y_t^n \|_V^2 \, dt \leq M \right\} \cap \left\{ \int_0^1 \| Z_t^n \|_V^2 \, dt \leq M \right\},
\]

and

\[
C_{\delta_0}^{n,m} := \left\{ \sup_{0 \leq t \leq 1} \| Z_t^{n,m} - Z_t^n \|_H \leq \delta_0, \sup_{0 \leq t \leq 1} \| Z_t^{n,m} - Z_t^n \|_V^2 \, dt \leq \delta_0 \right\}.
\]

Then

\[
\mathbb{P}\left( \left\{ \sup_{0 \leq t \leq 1} \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_H \geq \nu \int_0^1 \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_V^2 \, dt \geq \delta \right\} \cap A_{M}^{n,m} \cap B_{M}^{n,m} \cap C_{\delta_0}^{n,m} \right) = \mathbb{P}\left( \left\{ \sup_{0 \leq t \leq 1} \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_H \geq \nu \int_0^1 \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_V^2 \, dt \geq \delta \right\}, 1 \leq \tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m} \right) \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_H^2 = \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_V^2 \, dt \geq \delta \right).
\]

Applying Itô’s formula to \( \| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_H^2 \), we have

\[
\| \tilde{Y}_t^{n,m} - \tilde{Y}_t^n \|_H^2 = \frac{2}{\lambda(n)} \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \langle B \left( \tilde{Y}_s^n + Z^n_s, \tilde{Y}_s^n + Z^n_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \, ds
\]

\[
- 2 \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \langle B \left( u_s, \tilde{Y}_s^{n,m} - \tilde{Y}_s^n + Z^n_s, Z^n_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \, ds
\]

\[
- 2 \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \langle B \left( \tilde{Y}_s^{n,m} - \tilde{Y}_s^n + Z^n_s, Z^n_s, u_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \, ds
\]

\[
+ \frac{2\lambda(n)}{\sqrt{n}} \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \langle Y_{s}^{n,m} - \tilde{Y}_{s}^{n,m}, d(\beta_{s}^m - \beta_s) \rangle + \frac{\lambda^2(n)}{n} \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \sum_{i=m+1}^\infty q_i \, ds.
\]

Thus,

\[
\sup_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \| \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \|_H^2 = \| \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \|_V^2 \, ds
\]

\[
\leq \frac{2}{\lambda(n)} \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \| \langle B \left( \tilde{Y}_s^n + Z^n_s, \tilde{Y}_s^n + Z^n_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \| \, ds
\]

\[
+ 2 \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \| \langle B \left( u_s, \tilde{Y}_s^{n,m} - \tilde{Y}_s^n + Z^n_s, Z^n_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \| \, ds
\]

\[
+ 2 \int_0^{\tau_{M}^{n,m} \wedge \tau_{\delta_0}^{n,m}} \| \langle B \left( \tilde{Y}_s^{n,m} - \tilde{Y}_s^n + Z^n_s, Z^n_s, u_s \right), \tilde{Y}_s^{n,m} - \tilde{Y}_s^n \rangle \| \, ds
\]
\[ + \frac{2\lambda(n)}{\sqrt{n}} \sup_{0 \leq s \leq t} \left| \int_0^s (\bar{Y}^n_m - \bar{Y}^n_s, d(\beta^m - \beta)) \right| + \frac{\lambda^2(n)}{n} \sum_{i=m+1}^{\infty} q_i \] 

\[ =: II_1 + II_2 + II_3 + \frac{2\lambda(n)}{\sqrt{n}} \sup_{0 \leq s \leq t} \left( \int_0^s (\bar{Y}^n_m - \bar{Y}^n_s, d(\beta^m - \beta)) \right) + \frac{\lambda^2(n)}{n} \sum_{i=m+1}^{\infty} q_i. \] 

(4.65)

By the virtue of the properties to \( b(\cdot, \cdot, \cdot) \), for the first term,

\[ II_1 \leq \frac{4}{\lambda(n)} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_s + Z^n_s \|_V \cdot \| \bar{Y}^n_s + Z^n_s \|_H \cdot \| \bar{Y}^n_m - \bar{Y}^n_s \|_V ds \]

\[ \leq \frac{\nu}{4} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_m - \bar{Y}^n_s \|_V^2 ds \]

\[ + \frac{16}{\nu \lambda^2(n)} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_s + Z^n_s \|_V^2 \cdot \| \bar{Y}^n_s + Z^n_s \|_H^2 ds \]

\[ \leq \frac{\nu}{4} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_m - \bar{Y}^n_s \|_V^2 ds \]

\[ + \frac{16}{\nu \lambda^2(n)} \sup_{0 \leq s \leq t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_s + Z^n_s \|_H^2 \left( \int_{0}^{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_s + Z^n_s \|_V^2 ds \right) \]

\[ \leq \frac{\nu}{4} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_m - \bar{Y}^n_s \|_V^2 ds + \frac{16}{\nu \lambda^2(n)} M^3. \]

(4.66)

For the second term,

\[ II_2 = 2 \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \left| \langle B(u_s, Z^n_s - Z^n), \bar{Y}^n_m - \bar{Y}^n_s \rangle \right| ds \]

\[ \leq 4 \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| u_s \|_H \cdot \| u_s \|_V \cdot \| Z^n_s - Z^n \|_H \cdot \| Z^n_m - Z^n \|_V \cdot \| \bar{Y}^n_m - \bar{Y}^n_s \|_V ds \]

\[ \leq \frac{\nu}{4} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_m - \bar{Y}^n_s \|_V^2 ds \]

\[ + \left( \frac{16}{\nu} \int_{0}^{t^{\lambda\tau^m_m} \wedge \tau_0} \| u_s \|_V^2 ds \right) \left( \int_{0}^{t^{\lambda\tau^m_m} \wedge \tau_0} \| Z^n_m - Z^n_s \|_V^2 ds \right) \]

\[ \leq \frac{\nu}{4} \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \| \bar{Y}^n_m - \bar{Y}^n_s \|_V^2 ds + \frac{16}{\nu} (M\delta_0)^{\frac{3}{2}}. \]

(4.67)

For the third term, we have

\[ II_3 \leq 2 \int_{t^{\lambda\tau^m_m} \wedge \tau_0} \left| \langle B(\bar{Y}^n_m - \bar{Y}^n_s, u_s), \bar{Y}^n_m - \bar{Y}^n_s \rangle \right| ds \]
\[ + 2 \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \langle B(Z_{s}^{n,m} - Z_{s}^{n}, u_{s}), \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \rangle \, ds \]

\[ \leq 4 \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{H} \cdot \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V} \cdot \| u_{s} \|_{V} \, ds \]

\[ + 4 \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| Z_{s}^{n,m} - Z_{s}^{n} \|_{H}^{2} \cdot \| Z_{s}^{n,m} - Z_{s}^{n} \|_{V}^{2} \cdot \| u_{s} \|_{H} \cdot \| u_{s} \|_{V} \cdot \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V} \, ds \]

\[ \leq \nu \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \, ds + \frac{16}{\nu} \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \| u_{s} \|_{V} \, ds \]

\[ \times \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| u_{s} \|_{H} \cdot \left( \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| Z_{s}^{n,m} - Z_{s}^{n} \|_{V}^{2} \, ds \right)^{\frac{1}{2}} \left( \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| u_{s} \|_{V}^{2} \, ds \right)^{\frac{1}{2}} \]

\[ \leq \nu \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \, ds + \frac{16}{\nu} \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \| u_{s} \|_{V} \, ds + \frac{16}{\nu} (M \delta_{0})^{\frac{2}{3}}. \] (4.68)

Set \( M_{t} = \frac{2 \lambda(n)}{\lambda^{2}(n) \nu} \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V} \, ds + d(\beta_{n} - \beta_{s}) \). Putting (4.65)–(4.68) together, one obtains that for any \( 0 \leq t \leq 1 \),

\[ \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{H} + \nu \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \, ds \]

\[ \leq \left( \frac{16 M^{3}}{\lambda^{2}(n) \nu} + \frac{32 (M \delta_{0})^{\frac{2}{3}}}{\nu} + \lambda^{2}(n) \sum_{i=m+1}^{\infty} q_{i} + \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} | M_{s} \right) \]

\[ + \frac{16}{\nu} \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{H} \cdot \| u_{s} \|_{V} \, ds. \]

Set \( C_{n,M,\delta_{0}} := \frac{16 M^{3}}{\lambda^{2}(n) \nu} + \frac{32 (M \delta_{0})^{\frac{2}{3}}}{\nu} + \frac{\lambda^{2}(n)}{n} \sum_{i=m+1}^{\infty} q_{i} \). Applying Gronwall’s inequality to previous inequality, we get

\[ \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{H} + \nu \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \, ds \]

\[ \leq \left( C_{n,M,\delta_{0}} + \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} | M_{s} \right) \times \exp \left( \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \frac{16}{\nu} \| u_{s} \|_{V}^{2} \, ds \right) \]

\[ \leq \left( C_{n,M,\delta_{0}} + \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} | M_{s} \right) \cdot \exp \left( \frac{16 M}{\nu} \right). \]

Set \( C_{M} := \exp \left( \frac{16 M}{\nu} \right) \). Applying the martingale inequality in [7] to \( M_{t} \), it follows that

\[ \left[ \mathbb{E} \left( \sup \limits_{0 \leq s \leq t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{H} + \nu \int_{0}^{t \wedge \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \| \bar{Y}_{s}^{n,m} - \bar{Y}_{s}^{n} \|_{V}^{2} \, ds \right) \right]^{\frac{1}{2}} \]
Lemma 4.2 and 4.4, it suffices to show that

\[ \leq 2C_{n,M}^2 \delta_0 C_M^2 + 2 \left( \mathbb{E} \left( \sup_{0 \leq s \leq \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} |M_s|^p \right) \right)^{\frac{2}{p}} C_M^2 \]

\[ \leq 2C_{n,M}^2 \delta_0 C_M^2 + \frac{\lambda^2(n) C_M^2}{n} \mathbb{E} \left( \left( \int_0^{\tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} \sum_{i=m+1}^{\infty} q_i \|Y_s^{n,m} - Y_s^{n}\|_H^2 \, ds \right)^{\frac{2}{p}} \right) \]

\[ \leq 2C_{n,M}^2 \delta_0 C_M^2 + \frac{\lambda^2(n) C_M^2}{n} \left( \sum_{i=m+1}^{\infty} q_i \right)^2 \cdot \exp \left( \frac{\lambda^2(n) C_M^2}{n} \right). \quad (4.69) \]

where we have used the Minkowski’s inequality [21, page 47] in the last inequality.

Applying Gronwall’s inequality to (4.69), we have

\[ \left[ \mathbb{E} \left( \sup_{0 \leq s \leq \tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} ||Y_s^{n,m} - Y_s^{n}||_H^2 + \nu \int_0^{\tau_{M}^{n,m} \wedge \tau_{0}^{n,m}} ||Y_s^{n,m} - Y_s^{n}||_V^2 \, ds \right) \right]^{\frac{2}{p}} \]

\[ \leq \left[ 2C_{n,M}^2 \delta_0 C_M^2 + \frac{\lambda^2(n) C_M^2}{n} \left( \sum_{i=m+1}^{\infty} q_i \right)^2 \right] \times \exp \left( \frac{\lambda^2(n) C_M^2}{n} \right). \quad (4.70) \]

Taking \( p = \frac{2n}{\lambda^2(n)} \), by Chebychev’s inequality, we have

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Y_t^{n,m} - Y_t^{n}\|_H^2 + \nu \int_0^{1} \|Y_t^{n,m} - Y_t^{n}\|_V^2 \, dt > \delta \right) \]

\[ \leq \lim_{m \to \infty} \limsup_{n \to \infty} \log \left[ \mathbb{E} \left( \sup_{0 \leq t \leq 1} \|Y_t^{n,m} - Y_t^{n}\|_H^2 + \nu \int_0^{1} \|Y_t^{n,m} - Y_t^{n}\|_V^2 \, dt \right) \right]^{\frac{2}{p}} - \log \delta^2 \]

\[ \leq \log \left( \frac{64}{\nu} (M \delta_0)^2 C_M^2 \right) + 2C_M^2 - 2 \log \delta. \quad (4.71) \]

To finish the proof of (4.59), we now turn to proving (4.60) and (4.61). The proofs of (4.60) and (4.61) are similar, we shall only prove the second one. Recall (4.62) and Lemma 4.2 and 4.3 it suffices to show that

\[ \lim_{Y \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|Y_t^n\|_H^2 + \nu \int_0^1 \|Y_t^n\|_V^2 \, dt > Y \right) = -\infty. \]

For any \( \theta > \sup_{0 \leq t \leq 1} \|u_t^n\|_H \vee \left( \int_0^1 \|u_s^n\|_V^2 \, ds \right) \), define the stopping time

\[ \tau_{Z,\theta}^n := \inf \{ t \geq 0, \|Z_t^n\|_H > \theta \text{ or } \int_0^t \|Z_s^n\|_V^2 \, ds > \theta \}, \]

and set

\[ A^n_{\theta} := \left\{ \sup_{0 \leq t \leq 1} \|Z_t^n\|_H \leq \theta \right\} \cap \left\{ \int_0^1 \|Z_t^n\|_V^2 \, ds \leq \theta \right\}. \]
Using the same method as in the proof of (4.71), one obtains
\[
\lim_{\Upsilon \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq \Upsilon} \| \hat{Y}_t^n \|^2_H + \nu \int_0^1 \| \hat{Y}_t^n \|^2_V ds \right) = -\infty.
\]

Because of Lemma 4.2 and 4.3, for any \( R > 0 \), there exists some \( \theta \) large enough such that
\[
\limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(A^n_\theta) \leq -R.
\]

Thus for the above choice of \( \theta \), we have
\[
\lim_{\Upsilon \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq \Upsilon} \| \hat{Y}_t^n \|^2_H + \nu \int_0^1 \| \hat{Y}_t^n \|^2_V ds \right) \leq -R.
\]

Due to the arbitrariness of \( R \), (4.61) follows. Similarly, we have (4.60).

Next we continue to prove (4.59). Because of Lemma 4.2, Lemma 4.4, (4.60) and (4.61), for any \( R > 0 \), there exists some \( M > 0 \) such that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(A^{n,m}_M) \leq -R,
\]
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(B^{n,m}_M) \leq -R. \quad (4.72)
\]

By Lemma 4.1 and Lemma 4.3 for any \( \delta_0 > 0 \), we have
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(C^{n,m}_{\delta_0}) = -\infty. \quad (4.73)
\]

Thus for the above choice of \( M \), by (4.71)-(4.73), we have
\[
\limsup_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{0 \leq t \leq 1} \| \hat{Y}_{t,m}^{n,m} - \hat{Y}_t^n \|^2_H + \nu \int_0^1 \| \hat{Y}_{t,m}^{n,m} - \hat{Y}_t^n \|^2_V dt \right) \leq \\
\limsup_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(A^{n,m}_M) \cup \limsup_{m \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(B^{n,m}_M) \cup \\
\limsup_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P}(C^{n,m}_{\delta_0}) \cup (-R).
\]

Letting \( \delta_0 \) go to 0, by the arbitrariness of \( R \), (4.64) follows. Consequently, we have (4.59). The proof is complete. 

\[\blacksquare\]
4.2 Moderate Deviations for Lévy process

Let $\mathcal{X}$ be a regular Hausdorff topological space equipped with some $\sigma$-algebra $\mathcal{A}$ containing an open basis and a close basis for any $x \in \mathcal{X}$, and $\{\mu_n\}_{n \in \mathbb{N}}$ a family of probability measures on $\mathcal{X}$. Recall the definition of large deviation principle from [8].

**Definition 4.1** $\{\mu_n\}_{n \geq 1}$ satisfies a large deviation principle if there exists a family of positive numbers $\{h(n)\}_{n \geq 1}$ which tends to $+\infty$ as $n \to +\infty$, and a function $I(x)$ which maps $\mathcal{X}$ into $[0, +\infty]$ satisfying the following conditions:

(i) for each $l < +\infty$, the level set $\{x : I(x) \leq l\}$ is compact in $\mathcal{X}$;

(ii) for each closed subset $F$ of $\mathcal{X}$,

$$\limsup_{n \to \infty} \frac{1}{h(n)} \log \mu_n(F) \leq -\inf_{x \in F} I(x);$$

(iii) for each open subset $G$ of $\mathcal{X}$,

$$\liminf_{n \to \infty} \frac{1}{h(n)} \log \mu_n(G) \geq -\inf_{x \in G} I(x),$$

Here $h(n)$ is called the speed function and $I(x)$ the rate function. In that case, we simply write $\{\mu_n\}_{n \geq 1} \in \text{LDP}(\mathcal{X}, h(n), I)$.

Let $(\mathcal{Y}, \rho)$ be another complete metric space equipped with some $\sigma$-field containing all balls. In large deviation theory, when $\{\mu_n\}_{n \geq 1}$ satisfies the $\text{LDP}(\mathcal{X}, h(n), I)$ and $f : \mathcal{X} \to \mathcal{Y}$ is continuous, $\{\mu_n \circ f^{-1}\}_{n \geq 1} \in (\mathcal{Y}, h(n), I_f)$, where

$$I_f(y) := \inf_{f(x)=y} I(x), \quad y \in \mathcal{Y}.$$ 

This is the so-called contraction principle. The following generalization contraction principle is taken from [31, Theorem 2.2].

**Theorem 4.1 (Generalized Contraction Principle)** Assume $\{\mu_n\}_{n \geq 1} \in \text{LDP}(\mathcal{X}, h(n), I)$ and $f_n : \mathcal{X} \to \mathcal{Y}$ be a measurable mapping up to $\mu_n$-equivalence for each $n$. Suppose that there exists a sequence of continuous measurable mappings $f^N : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{N \to +\infty} \limsup_{n \to \infty} \frac{1}{h(n)} \log \mu_n^{\text{out}}(\rho(f^N, f_n) > \delta) = -\infty, \quad \forall \delta > 0. \quad (4.74)$$

Then there exists a mapping $\tilde{f} : [I < +\infty] \to \mathcal{Y}$ such that

$$\sup_{x \in [I \leq L]} \rho(f^N(x), \tilde{f}(x)) \to 0, \quad \text{as } N \to +\infty, \quad \forall L > 0; \quad (4.75)$$

and $\{\mu_n \circ f_n^{-1}\}_{n \geq 1} \in \text{LDP}(\mathcal{Y}, h(n), I_f)$, where

$$I_f(y) := \inf \{I(x) | I(x) < +\infty, \tilde{f}(x) = y, \forall x \in \mathcal{X}\}.$$ 

**Remark 2** When $f_n = f$ is independent of $n$, the result above extends [8, Theorem 4.2.23]. The main difference is: (4.75) being an assumption with $\tilde{f} = f$ in [8, Theorem 4.2.23], becomes now a consequence of the large deviation in this new version.
Let $E$ be a Polish space with metric $d$. Consider the càdlàg space $D([0, 1]; E)$ equipped with the uniform metric $\rho(x_1, x_2) = \sup_{t \in [0, 1]} d(x_1(t), x_2(t))$. It is a complete but not separable. Thus the known MDP results for the sums of i.i.d. random vectors with values in a separable Banach space (see De Acosta [1] and references therein) do not hold. In this part, we shall establish the MDP for the Lévy process by Dawson-Gärtner’s projective limits approach, see [8].

Consider the production topological space
\[
E^{[0, 1]} := \{ x = (x_t)_{t \in [0, 1]} | x_t \in E, \forall t \in [0, 1] \}.
\]

Let $\mathcal{A} := \{ \{ t_1, t_2, \cdots, t_n \} \subset [0, 1]; n \geq 1 \}$ and $\Phi : D([0, 1]; E) \to E^{[0, 1]}$. For any $\alpha \in \mathcal{A}$, let $p_\alpha$ be the canonical projection of $E^{[0, 1]}$ to $E^\alpha$. Given a family of probability measures $\{ \mu_n; n \geq 1 \}$ on $D([0, 1]; E)$, let $\mu^\alpha_n := \mu_n \circ \Phi^{-1} \circ p_\alpha^{-1}$ for any $\alpha$ in $\mathcal{A}$.

Here we quote the criteria of the exponential tightness for càdlàg stochastic process from [30, Proposition 5.6, page 264].

**Theorem 4.2** Assume that for every finite $\alpha \subset [0, 1], \{ \mu^\alpha_n; n \geq 1 \}$ satisfies the LDP on $E^\alpha$ with speed $h(n)$ and with rate function $I_\alpha$. If for any $\eta > 0$,
\[
\lim_{\delta \to 0^+} \sup_{t \in [0, 1]} \limsup_{n \to \infty} \frac{1}{h(n)} \log \mu_n \left( \sup_{t \leq s \leq (t+\delta)\wedge 1} d(x(s), x(t)) > \eta \right) = -\infty, \quad (4.76)
\]
then $\{ \mu_n; n \geq 1 \}$ satisfies the LDP on $D([0, 1]; E, \rho)$ with the speed $h(n)$ and with rate function
\[
I(x) = \sup_{\alpha \in \mathcal{A}} I_\alpha(p_\alpha(\Phi(x))), \quad x \in D([0, 1]; E).
\]

Let us give a moderate deviation principle for Lévy process.

**Theorem 4.3** Under Condition 4.4 $\{ \frac{\lambda(n)}{n} \int_X G(x) \hat{N}^n(ds, dx); n \geq 1 \}$ satisfies a large deviation principle on $D([0, 1]; H)$ with the speed $n/\lambda^2(n)$ and with the rate function
\[
I_1(y) = \begin{cases}
\frac{1}{2} \int_0^1 \langle y(s), \Pi^{-1} y(s) \rangle ds, & \text{if } y(t) = \int_0^t y'(s) ds; \\
+\infty, & \text{otherwise},
\end{cases} \quad (4.77)
\]
where $\Pi := (\pi_{ij})_{i,j \geq 1} := (\int_X \langle G(x), e_i \rangle \langle G(x), e_j \rangle \vartheta(dx))_{i,j \geq 1}$.

**Proof:** Let $\hat{N}_k$ be a sequence of i.i.d. compensated Poisson random measures on $[0, 1] \times X$ with intensity measure $dt \vartheta(dx)$. Then
\[
\int_0^t \int_X G(x) \hat{N}^n(ds, dx) \text{ has the same distribution with } \sum_{k=1}^n \int_0^t \int_X G(x) \hat{N}_k(ds, dx).
\]
Denote $\xi_k(\cdot) := \int_0^t \int_X G(x) \hat{N}_k(ds, dx), k \geq 1$, which are i.i.d. random variables taking values in $D([0, 1]; H)$. Hence, it is equivalent to prove the theorem for $\sum_{k=1}^n \xi_k$. We divide the proof into two steps.
Step 1. LDP for finite dimensional distributions. Recall the projection operator $P_m$ in [4,3]. By Theorem 3.7.1 in [8], Remark 4.1 and the independence of increments of $\xi_k$, one obtains that for any finite subset $\alpha = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$,

$$\frac{\lambda(n)}{n} \left( \sum_{k=1}^{n} P_m \xi_k(t_1), \cdots, \sum_{k=1}^{n} P_m [\xi_k(t_N) - \xi_k(t_{N-1})] \right)$$

satisfies an LDP on $(H_m)^\alpha$ with the speed $n/\lambda^2(n)$ and with the rate function given by

$$(y_1, \cdots, y_N) \to \sum_{l=1}^{N} \frac{\langle y_l, (\Pi^2_{\alpha})^{-1} y_l \rangle }{2(t_l - t_{l-1})}, \quad y \in (H_m)^\alpha,$$

where $\Pi^\alpha_N := (\pi^m_{i,j})_{1 \leq i,j \leq N}, \pi^m_{i,j} := \int_X \langle P_m G(x), e_i \rangle \langle P_m G(x), e_j \rangle \vartheta(dx)$. Thus by the contraction principle [8, Theorem 4.2.1],

$$\frac{\lambda(n)}{n} \left( \sum_{k=1}^{n} P_m \xi_k(t_1), \cdots, \sum_{k=1}^{n} P_m \xi_k(t_N) \right)$$

satisfies an LDP on $(H_m)^\alpha$ with the speed $n/\lambda^2(n)$ and with the rate function

$$I^\alpha_m(y_\alpha) = \sum_{l=1}^{N} \frac{\langle y_l(t_l) - y(t_{l-1}), (\Pi^2_{\alpha})^{-1} (y_l(t_l) - y(t_{l-1})) \rangle }{2(t_l - t_{l-1})}, \forall y \in (H_m)^\alpha, \ y(0) = 0$$

and $I^\alpha_m(y_\alpha) = +\infty$ if $y(0) \neq 0$.

By Lemma 4.11 we know that for any $\delta > 0, t \in [0, 1]$

$$\lim_{m \to \infty} \limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \left\| \frac{\lambda(n)}{n} \left\| \sum_{k=1}^{n} [P_m \xi_k(t) - \xi_k(t)] \right\|_H \right\| > \delta \right) = -\infty.$$  

Using the approximation theorem [8, Theorem 4.2.16], one obtains that

$$\frac{\lambda(n)}{n} \left( \sum_{k=1}^{n} \xi_k(t_1), \sum_{k=1}^{n} \xi_k(t_2), \cdots, \sum_{k=1}^{n} \xi_k(t_N) \right)$$

satisfies an LDP on $H^\alpha$ with the speed $n/\lambda^2(n)$ and with the rate function

$$I^\alpha(y_\alpha) = \sum_{l=1}^{N} \frac{\langle y_l(t_l), \Pi^{-1}_{\alpha} (y_l(t_l) - y(t_{l-1})) \rangle }{2(t_l - t_{l-1})}, \forall y \in H^\alpha, \ y(0) = 0$$

and $I^\alpha(y_\alpha) = +\infty$ if $y(0) \neq 0$, where $\Pi_{\alpha} := (\pi_{i,j})_{1 \leq i,j \leq N}, \pi_{i,j} := \int_X \langle G(x), e_i \rangle \langle G(x), e_j \rangle \vartheta(dx)$.

Step 2. Exponential tightness [4.76]. Using similar arguments as proof of (4.54) in Lemma 4.11 one obtains that for any $\delta > 0, \eta > 0, \theta > 0, t \in [0, 1]$,

$$\limsup_{n \to \infty} \frac{\lambda^2(n)}{n} \log \mathbb{P} \left( \sup_{t \leq s \leq (t+\delta)\land t} \frac{\lambda(n)}{n} \left\| \sum_{k=1}^{n} (\xi_k(s) - \xi_k(t)) \right\|_H > \eta \right) \leq -\theta^2 \eta + \theta C_0 \delta,$$
By the classical argument for identification of the rate function (see [8, Chapter 5]), we have

\[ \frac{\lambda(n)}{n} \log \mu_n \left( \sup_{t \leq s \leq (t+\delta) \wedge 1} \frac{\lambda(n)}{n} \sum_{k=1}^{n} (\xi_k(t) - \xi_k(s)) \right) > \eta \right) = -\infty. \]

By Theorem 4.2, \( \frac{\lambda(n)}{n} \sum_{k=1}^{n} \int_{0}^{t} G(x) \tilde{N}_k(ds, dx) \) satisfies the LDP on \( D([0, 1]; H) \) with the speed \( n/\lambda^2(n) \) and with the rate function given by

\[ I_1(y) = \sup_{a \in A} I_a(p_a(\Phi(y))) \quad \text{for any } y \in D([0, 1]; H). \]

By the classical argument for identification of the rate function (see [8, Chapter 5]), we have \( I_1(y) \) satisfies (4.77).

The proof is complete. \( \blacksquare \)

By the scaling property of Brownian motion, \( \{ \frac{\lambda(n)}{\sqrt{n}} \beta; n \geq 1 \} \) satisfies a large deviation principle on \( D([0, 1]; H) \) with the speed \( n/\lambda^2(n) \) and with the rate function \( I_2 \) given by

\[ I_2(y) = \begin{cases} \frac{1}{2} \int_{0}^{1} \langle y'(s), y'(s) \rangle ds, & \text{if } y(t) = \int_{0}^{1} y'(s) ds, y' \in L^2([0, 1]; H_0); \\ +\infty, & \text{otherwise.} \end{cases} \]

where \( \langle \cdot, \cdot \rangle_0 \) is defined in (2.11). By the large deviations theory on production space, one can easily obtain the following result. The proof is omitted here.

**Corollary 4.4** Under Condition [4.4] \( \{ \frac{\lambda(n)}{\sqrt{n}} \beta + \frac{\lambda(n)}{n} \int_{0}^{\lambda} G(x) \tilde{N}_k(ds, dx); n \geq 1 \} \) satisfies a large deviation principle on \( D([0, 1]; H) \) with the speed \( n/\lambda^2(n) \) and with the rate function \( I \) given by

\[ I(y) = \inf \{ I_1(y_1) + I_2(y_2); y = y_1 + y_2 \}. \] (4.79)

### 4.3 Moderate Deviations for Stochastic Navier-Stokes equations

For any \( g \in D([0, 1]; H) \), denote by \( \phi(g) \in D([0, 1]; H) \cap L^2([0, 1]; V) \) the solution of the following equation

\[ \phi_t(g) = -\int_{0}^{t} A\phi_s(g) ds - \int_{0}^{t} B(u_s, \phi_s(g)) ds - \int_{0}^{t} B(\phi_s(g), u_s) ds + g(t). \] (4.80)

For any \( h \in D([0, 1]; H) \), set

\[ \tilde{I}(h) := \inf \{ I(g) : h = \phi(g), g \in D([0, 1]; H) \}, \] (4.81)

with the convention \( \inf \{ \emptyset \} = +\infty \), where \( I \) given by (4.79).

Our second main result is the following moderate deviation principle of \( u^n \).

**Theorem 4.5** Under Conditions [2.2] and [2.1] \( Y^n = \lambda(n)(u^n - u) \) obeys an LDP on \( D([0, 1]; H) \cap L^2([0, 1]; V) \) with speed function \( n/\lambda^2(n) \) and with rate function \( \tilde{I} \) given by (4.81).

**Proof:** By Corollary 4.4, we know that the law of

\[ \left\{ \zeta_n := \frac{\lambda(n)}{\sqrt{n}} \beta + \frac{\lambda(n)}{n} \int_{0}^{\lambda} G(x) \tilde{N}_k(dt, dx); n \geq 1 \right\} \]
satisfies a large deviation principle on $D([0, 1]; H)$ with the speed function $n/\lambda^2(n)$ and with the rate function $I$ given by (4.79).

For $g \in D([0, 1]; H)$, denote by $\phi_n(g) \in D([0, 1]; H) \cap L^2([0, 1]; V)$ the solution of the following equation

$$
\phi_n(g)(t) = -\int_0^t A\phi_n(g)(s)ds - \int_0^t B(u_s, \phi_n(g)(s))ds - \int_0^t B(\phi_n(g)(s), u_s)ds + g(t). \tag{4.82}
$$

Then $Y^n = \phi_n(\zeta_n)$.

Recall (4.83). We introduce a mapping $\phi^m(\cdot)$ from $D([0, 1]; H)$ into $D([0, 1]; H) \cap L^2([0, 1]; V)$ as follows: for $g \in D([0, 1]; H)$, define $\phi^m(g) := \phi(P_m g)$ where $\phi$ is defined in (4.80). It is easy to see that $Y^{n,m} := \phi^m(\zeta_n)$ is the solution of the following equation:

$$
dY_t^{n,m} = - AY_t^{n,m}dt - B(u_t, Y_t^{n,m})dt - B(Y_t^{n,m}, u_t)dt + \frac{\lambda(n)}{\sqrt{n}}dB_t^m + \frac{\lambda(n)}{n} \int_{\mathbb{X}} P_m G(x)\tilde{N}^n(dt, dx). \tag{4.83}
$$

By Lemma 4.6 below, the mapping $\phi^m$ is continuous from $D([0, 1]; H)$ into $D([0, 1]; H) \cap L^2([0, 1]; V)$. Then by the contraction principle in large deviations theory, $\{Y^{n,m}; n \geq 1\}$ satisfies an LDP on $D([0, 1]; H) \cap L^2([0, 1]; V)$ with speed function $n/\lambda^2(n)$ and with rate function $I^m$ as follows: for any $h \in D([0, 1]; H)$, set

$$
I^m(h) := \inf\{ I(g) : h = \phi^m(g), g \in D([0, 1]; H) \}, \tag{4.84}
$$

with the convention $\inf\{\emptyset\} = +\infty$, where $I$ is given by (4.79).

Finally, by the generalized contraction principle (see Theorem 4.1), to prove Theorem 4.5 we need to prove that $\{Y^{n,m}; n \geq 1\}$ is exponentially equivalent to $\{Y^n; n \geq 1\}$, which has been done by Lemma 4.5 The proof is complete once if Lemma 4.6 below is proved. □

**Lemma 4.6** The mapping $\phi^m(g) = \phi(P_m g)$ is continuous from $D([0, 1]; H)$ into $D([0, 1]; H) \cap L^2([0, 1]; V)$ in the topology of uniform convergence.

**Proof:** Let $v_t(g) = \phi^m_t(g) - P_m g(t)$. Then $v_t(g)$ satisfies the following equation

$$
v_t(g) = -\int_0^t A(v_s(g) + P_m g(s))ds - \int_0^t B(v_s(g) + P_m g(s), u_s)ds - \int_0^t B(u_s, v_s(g) + P_m g(s))ds.
$$

To prove this lemma, it suffices to show that, for any $\{g_n\}_{n=1}^\infty \subset D([0, 1]; H)$ such that $\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \|g_n(t) - g(t)\|_H = 0$,

$$
\lim_{n \to \infty} \left( \sup_{0 \leq t \leq 1} \|v_t(g_n) - v_t(g)\|_H^2 + \nu \int_0^1 \|v_t(g_n) - v_t(g)\|_V^2 ds \right) = 0.
$$

Now by the chain rule,

$$
\|v_t(g_n) - v_t(g)\|_H^2 + 2\nu \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds
$$

$$
= -2 \int_0^t \left( B(v_s(g_n) - v_s(g) + P_m g_n(s) - P_m g(s), u_s), v_s(g_n) - v_s(g) \right) ds
$$

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\[-2 \int_0^t \left\langle B\left(u_s, v_s(g_n) - v_s(g) + P_m g_n(s) - P_m g(s)\right), v_s(g_n) - v_s(g) \right\rangle ds \]
\[-2 \int_0^t \left\langle A\left(P_m g_n(s) - P_m g(s)\right), v_s(g_n) - v_s(g) \right\rangle ds \]
\[=: I_1 + I_2 + I_3. \quad (4.86)\]

By the virtue of the properties to \(b(\cdot, \cdot, \cdot)\), we have

\[|I_1| \leq 2 \int_0^t \left| \left\langle B\left(v_s(g_n) - v_s(g), u_s\right), v_s(g_n) - v_s(g) \right\rangle \right| ds\]
\[+ 2 \int_0^t \left| \left\langle B\left(P_m g_n(s) - P_m g(s), u_s\right), v_s(g_n) - v_s(g) \right\rangle \right| ds\]
\[\leq 4 \int_0^t \|v_s(g_n) - v_s(g)\|_V \cdot \|v_s(g_n) - v_s(g)\|_{\mu} \|u_s\|_V ds\]
\[+ 2c \int_0^t \|P_m g_n(s) - P_m g(s)\|_V \cdot \|u_s\|_V \cdot \|v_s(g_n) - v_s(g)\|_V ds\]
\[\leq \frac{\nu}{4} \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + \frac{16}{\nu} \int_0^t \|v_s(g_n) - v_s(g)\|_{\mu}^2 \|u_s\|_V^2 ds\]
\[+ \frac{\nu}{4} \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + \frac{4c^2}{\nu} \int_0^t \|P_m g_n(s) - P_m g(s)\|_V^2 \cdot \|u_s\|_V^2 ds. \quad (4.87)\]

For the second term,

\[|I_2| \leq 2 \int_0^t \left| \left\langle B\left(u_s, P_m g_n(s) - P_m g(s)\right), v_s(g_n) - v_s(g) \right\rangle \right| ds\]
\[\leq 2c \int_0^t \|u_s\|_V \cdot \|P_m g_n(s) - P_m g(s)\|_V \cdot \|v_s(g_n) - v_s(g)\|_V ds\]
\[\leq \frac{\nu}{4} \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + \frac{4c^2}{\nu} \int_0^t \|P_m g_n(s) - P_m g(s)\|_V^2 \cdot \|u_s\|_V^2 ds. \quad (4.88)\]

For \(I_3\),

\[|I_3| \leq 2 \int_0^t \|P_m g_n(s) - P_m g(s)\|_V \|v_s(g_n) - v_s(g)\|_V ds\]
\[\leq \frac{\nu}{4} \int_0^t \|v_s(g_n) - v_s(g)\|_V^2 ds + 4 \int_0^t \|P_m g_n(s) - P_m g(s)\|_V^2 ds. \quad (4.89)\]

Putting (4.86)-(4.89) together and applying Gronwall’s inequality,

\[\sup_{0 \leq s \leq 1} \|v_s(g_n) - v_s(g)\|_V^2 + \nu \int_0^1 \|v_s(g_n) - v_s(g)\|_V^2 ds \]
\[\leq \left( \int_0^1 \|P_m g_n(s) - P_m g(s)\|_V^2 \cdot \left( \frac{8c^2}{\nu} \|u_s\|_V^2 + 4\nu \right) ds \right) \cdot \exp \left( \frac{16}{\nu} \left( \|u_s\|_V^2 + 4\nu \right) ds \right) \]
\[\leq \left( \sup_{0 \leq s \leq 1} \|P_m g_n(s) - P_m g(s)\|_V^2 \cdot \int_0^1 \left( \frac{8c^2}{\nu} \|u_s\|_V^2 + 4\nu \right) ds \right) \cdot \exp \left( \frac{16}{\nu} \left( \|u_s\|_V^2 + 4\nu \right) ds \right) \]
\[\rightarrow 0, \quad \text{as } n \rightarrow +\infty, \]
where we used the fact that $\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \| g_n(t) - g(t) \|_H = 0$ implies

$$\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \| P_m g_n(t) - P_m g(t) \|_V = 0, \ \forall m \geq 1.$$ 

The proof is complete.

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