Perpendicular Diffusion Coefficient of Cosmic Rays: 
The Presence of Weak Adiabatic Focusing

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Abstract

The influence of adiabatic focusing on particle diffusion is an important topic in astrophysics and plasma physics. In the past, several authors have explored the influence of along-field adiabatic focusing on the parallel diffusion of charged energetic particles. In this paper, using the unified nonlinear transport theory developed by Shalchi and the method of He and Schlickeiser, we derive a new nonlinear perpendicular diffusion coefficient for a non-uniform background magnetic field. This formula demonstrates that the particle perpendicular diffusion coefficient is modified by along-field adiabatic focusing. For isotropic pitch-angle scattering and the weak adiabatic focusing limit, the derived perpendicular diffusion coefficient is independent of the sign of adiabatic focusing characteristic length. For the two-component model, we simplify the perpendicular diffusion coefficient up to the second order of the power series of the adiabatic focusing characteristic quantity. We find that the first-order modifying factor is equal to zero and that the sign of the second order is determined by the energy of the particles.

Key words: diffusion – magnetic fields – turbulence

1. Introduction

Energetic charged particles propagating through magnetized plasmas is an important topic in the study of the modulation of galactic cosmic rays in the heliosphere (e.g., Parker 1963; Zhao et al. 2014), solar energetic particle transport in the solar wind (e.g., Parker 1965; Roelof 1969; Ng & Reames 1994; Qin et al. 2005), energetic particle shock acceleration in interplanetary space or from supernova remnants (e.g., Zank et al. 2000; Ferrand et al. 2014), etc. According to observations, the magnetic field configuration of magnetized plasmas can be understood as the superposition of a large-scale background magnetic field \( B_0 = B_0 \hat{e}_z \) and a turbulent component \( \delta B \) in many scenarios. Both the mean magnetic field and the turbulent component can affect the properties of the particles’ propagation. Since the large-scale magnetic field breaks the symmetry of magnetized plasmas, particle transport has to be distinguished between parallel and perpendicular components relative to the mean magnetic field direction. For the purpose of simplicity, one usually assumes that the mean magnetic field is uniform (e.g., Schlickeiser 2002; Shalchi 2009). However, different observations, e.g., radio continuum surveys of interstellar space and direct in-situ measurements in the solar system, show the spatially varying large-scale magnetic fields for many scenarios, which leads to the adiabatic focusing effect of charged energetic particles and introduces a modification to the particle diffusion coefficients (Roelof 1969; Earl 1976; Shalchi 2011a; Litvinenko 2012a, 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014; Wang & Qin 2016).

For the weak adiabatic focusing limit and the distribution function with small anisotropy, along-field adiabatic focusing gives rise to convective transport along and across the guide field, as well that as in momentum space (Schlickeiser & Shalchi 2008). In addition, by considering the spatial gradient and curvature of the mean magnetic field, Schlickeiser & Jenko (2010) explored the influence of adiabatic focusing effects on particle perpendicular propagation. Since only a weak limit is considered, the influence of along-field adiabatic focusing on the particle parallel and perpendicular diffusion coefficient was not found.

By using the Unified NonLinear Transport (UNLT) theory (Shalchi 2010) for the arbitrary anisotropic distribution function, Shalchi (2011a) derived a parallel diffusion coefficient formula with along-field adiabatic focusing. Employing the stochastic differential equations, completely equivalent to the Fokker–Planck equation, another version of the parallel diffusion coefficient formula with adiabatic focusing was obtained (Litvinenko 2012a). This formula is also obtained in other papers (Beeck & Wibberenz 1986; Bieber & Burger 1990; Kóta 2000; Litvinenko 2012a; He & Schlickeiser 2014). Analytical and numerical studies demonstrated that the two versions are related to each other (Shalchi & Danos 2013).

Analogous to parallel diffusion, perpendicular diffusion is another important particle transport process. In the past, some approaches were developed to derive the perpendicular diffusion coefficient, e.g., the NonLinear Guiding Center (NLGC) theory (Matthaeus et al. 2003), the Weakly NonLinear Theory (Shalchi et al. 2004), the Modification of NLGC (Qin & Zhang 2014), the UNLT theory (Shalchi 2010), and so on. However, these theories regarding the perpendicular diffusion coefficient are all only applicable to the uniform mean magnetic field and the adiabatic focusing effect is not considered.

In summary, many studies have been done for perpendicular diffusion in the uniform mean magnetic field, and parallel diffusion with along-field adiabatic focusing is also extensively investigated so far. But it is always an open topic whether along-field adiabatic focusing has influence on perpendicular diffusion. The perpendicular diffusion coefficient formula for uniform mean magnetic field derived in Shalchi (2010; hereafter SH2010) is a well-known result and agrees well with simulations (e.g., Tautz & Shalchi 2011; Shalchi 2013; Hussein & Shalchi 2014; Shalchi & Hussein 2014). In this paper, by employing the UNLT theory and the method of...
He & Schlickeiser (2014; hereafter HS2014) we explore the perpendicular diffusion coefficient with the along-field adiabatic focusing effect.

The paper is organized as follows. In the Section 2, we introduce the definitions of the perpendicular diffusion coefficient and along-field adiabatic focusing length, and from the standard Fokker–Planck equation by employing the UNLT theory we obtain the equations of the quantities that are related to the spatial perpendicular diffusion coefficient. In Section 3, by using the method of HS2014, we derive the anisotropic distribution function and obtain the quantities related to the spatial perpendicular diffusion coefficient, which occur in the equations in Section 2. And, in Section 4, the spatial perpendicular diffusion coefficient with along-field adiabatic focusing is deduced, and for isotropic scattering and the weak focusing limit the first- and second-order modifying factors of the perpendicular diffusion coefficient are derived. In Section 5, for the two-component model, the first and second modifying factors are explored. Section 6 provides a discussion and conclusions.

2. Simplification of the Standard Fokker–Planck Equation

Similar to SH2010, our starting point is the single particle motion equation (Matthaeus et al. 2003)

\[ v_x(t) = a v_z(t) \frac{\delta B_z(x(t), t)}{B_0(z(t))}, \tag{1} \]

where \( B_0(z(t)) \) is the large-scale magnetic field, which is not necessarily uniform, \( \delta B_z \) is the x-component of random magnetic field, \( v_x(t) \) and \( v_z(t) \) are the \( x \) and \( z \) components of particle speed \( v \) respectively. For convenience, we focus on transverse fluctuations \( B_0 \cdot \delta B = 0 \) in this paper. Because the energetic charged particle feels the magnetic field at its own position, variable \( z \) in the mean magnetic field \( B_0(z) \) is a random variable with time. The parameter \( a^2 \) indicates the relation between single particle and the magnetic field lines, and the relevant details can be found in the papers of NLGC and UNLT theories and so on (Matthaeus et al. 2003; Shalchi 2015, 2016).

In this paper, the well-known Taylor–Green–Kubo (TGK) formulation (see, Taylor 1922; Green 1951; Kubo 1957) is used to compute the perpendicular diffusion coefficient

\[ \kappa_\perp = \int_0^\infty dt \langle v_x(t)v_x(0) \rangle. \tag{2} \]

2.1. Along-field Adiabatic Focusing Length

The along-field adiabatic focusing length for the spatially varying guiding field is shown as (Schlickeiser & Shalchi 2008)

\[ L^{-1}(z) = -\frac{1}{B_0(z)} \frac{dB_0(z)}{dz}. \tag{3} \]

For the limit \( L \to \infty \), the spatially varying background magnetic field \( B_0(z) \) tends to be uniform. In this paper, for mathematical tractability, we assume that the adiabatic focusing length is a constant, i.e., \( \partial L/\partial z = 0 \). Thus the mean magnetic field can be written as

\[ B_0(z) = B_0(0)e^{-z/L}. \tag{4} \]

For the convergent background magnetic field along the \( z \) positive direction, the adiabatic focusing length satisfies \( L < 0 \), and for the divergent case \( L > 0 \).

2.2. The Formula of the Perpendicular Diffusion Coefficient

By combining Equations (1) and (2), perpendicular diffusion coefficient \( \kappa_\perp \) can be obtained

\[ \kappa_\perp = \frac{a^2v^2}{B_0^2(0)} \int d^3k P_{xx}(k) T(k) \tag{5} \]

with

\[ T(k) = \frac{1}{v^2} \int_0^\infty dt \langle v_x(t)v_x(0)e^{i\theta(t)/L}e^{ik\cdot x(0)} \rangle = \frac{1}{4} \int_0^\infty dt \int_{-1}^{+1} d\mu \int_{-1}^{+1} d\mu_0 \int d^3xe^{i\mu k\cdot x} e^{i\theta(t)/L} f_0[x(t), \mu, \mu_0, t], \tag{6} \]

here \( \mu \) is pitch-angle cosine and \( \mu_0 \) is initial pitch-angle cosine, \( P_{xx}(k) \) is the magnetic field correlation tensor. In Equation (5), the Corrsin independence hypothesis (see Corrsin 1959) is employed, and the quantity \( T(k) \) is required to be real. Note that we also use the standard assumptions of magnetic turbulence, e.g., axisymmetry, homogeneity, magnetostatic, and so on. Furthermore, \( f_0[x(t), \mu, \mu_0, t] \) in Equation (6) is the distribution function of the standard Fokker–Planck equation with the adiabatic focusing effect, which is shown as follows.

\[ \frac{\partial f_0}{\partial t} + v \mu \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f_0}{\partial \mu} \right) - \frac{v(1 - \mu^2)}{2L} \frac{\partial f_0}{\partial \mu} + D_{L} \Delta_z f_0. \tag{7} \]
The latter equation includes the Fokker–Planck coefficients of pitch-angle diffusion $D_{\mu\mu}$ and the perpendicular diffusion $D_{\perp}$, along-field adiabatic focusing length $L$, and the differential operator $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. However, the source term is neglected.

Equation (6) can also be written as

$$T(k) = \frac{1}{2} \int_{-1}^{1} d\mu \mu S(k, \mu),$$  

(8)

with the quantity $S(k, \mu)$ as

$$S(k, \mu) = \frac{1}{2} \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \int d^3x e^{ikx} e^{\xi(t)/L_f}[x(t), \mu, \mu_0, t].$$  

(9)

From Equations (5), (8), and (9) we find that in order to get the perpendicular diffusion coefficient $\kappa_\perp$, the formula $S(k, \mu)$ has to be obtained.

### 2.3. The Governing Equation of $S(k, \mu)$

Let us set $f[x(t), \mu, \mu_0, t] = e^{\xi(t)/L_f}[x(t), \mu, \mu_0, t]$ and combining the standard Fokker–Planck equation (see Equation (7)) we can obtain the following equation.

$$\frac{\partial f}{\partial t} + v_{\mu} \frac{\partial f}{\partial x} + D_{\mu\mu} \frac{\partial f}{\partial \mu} + \frac{v}{2L} (1 - \mu^2) f = \frac{v_{\mu}}{L} f + D_{\perp} \Delta_\perp f.$$  

(10)

In what follows, from the latter equation, we use the UNLT theory to derive the governing equation of $S(k, \mu)$. By employing the relation $f[x(t), \mu, \mu_0, t] = e^{\xi(t)/L_f}[x(t), \mu, \mu_0, t]$ the formula of quantity $S(k, \mu)$ (see, (9)) can be rewritten as

$$S(k, \mu) = \frac{1}{2} \int_{0}^{\infty} dt \int_{-1}^{1} d\mu_0 \mu_0 \Gamma[k(t), \mu, \mu_0, t].$$  

(11)

with

$$\Gamma[k(t), \mu, \mu_0, t] = \int d^3x e^{ikx} f[x(t), \mu, \mu_0, t].$$  

(12)

After operating Fourier transformation on Equation (10) over spatial variable $x$, we can get

$$\frac{\partial \Gamma}{\partial t} - ik_{\parallel} v_{\mu} \mu \Gamma = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} - \frac{v}{2L} (1 - \mu^2) \Gamma \right] + \frac{v_{\mu}}{L} \Gamma - k_{\parallel}^2 D_{\perp} \Delta_\parallel \Gamma,$$  

(13)

where we assume that $D_{\mu\mu}$ and $D_{\parallel} \perp$ are independent of $x$.

To multiply the time integral of Equation (13) from 0 to $\infty$ with the initial pitch-angle cosine $\mu_0$, and then integrate the results over $\mu_0$ from $-1$ to 1, we can finally obtain

$$\int_{-1}^{1} d\mu_0 \Gamma(t = \infty) = \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \Gamma = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \int_{0}^{\infty} dt \Gamma - \frac{v}{2L} (1 - \mu^2) \int_{0}^{\infty} dt \Gamma \right] + \frac{v_{\mu}}{L} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \Gamma.$$  

(14)

The initial distribution can be defined as

$$\Gamma(t = 0) = 2\delta(\mu - \mu_0) h(k_0) = 2\delta(\mu - \mu_0) \int d^3x_0 e^{ik_0x} h(x_0),$$  

(15)

which denotes that the particles have a well-defined initial pitch-angle cosine, and $h(k_0)$ is the distribution function in phase space for initial time $t = t_0$. By assuming that $\Gamma(t \rightarrow \infty)$ is independent of the initial pitch-angle cosine $\mu_0$, as in Shalchi (2010), we can obtain the following equation

$$-2\mu h(k_0) - ik_{\parallel} v_{\mu} \mu_0 \int_{0}^{\infty} dt \Gamma = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \int_{0}^{\infty} dt \Gamma - \frac{v}{2L} (1 - \mu^2) \int_{0}^{\infty} dt \Gamma \right] + \frac{v_{\mu}}{L} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \Gamma.$$  

(16)
here, since $x = x_0$ for time $t = t_0$, Fourier transformation $\int d^3x e^{i k \cdot x} h(x)$ becomes $\int d^3x_0 e^{i k \cdot x_0} h(x_0)$. And, in the latter equation, the following formula is used.

$$h(k_0) = \int d^3x_0 e^{i k \cdot x_0} h(x_0),$$

where the normalization condition $\int d^3k_0 h(k_0) = 1$ must be satisfied for the initial time $t = t_0$.

To proceed, by using the formula $S(k, \mu) = (1/2) \int_0^\infty dt f_1^1 d\mu_0 \mu_0^\Gamma[k(t), \mu, \mu_0, t]$ (see Equation (11)) the governing equation of $S(k, \mu)$ can be derived from Equation (16)

$$-\mu h(k_0) - ik \gamma \mu S = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial S}{\partial \mu} - \frac{\nu}{2L} (1 - \mu^2) S \right] + \frac{\nu \mu S}{L} - k_+^2 D_+ S.$$

(18)

Splitting $S(\mu, k)$ into an even function $S_+(\mu, k)$ and an odd function $S_-(\mu, k)$, i.e., $S(\mu, k) = S_+(\mu, k) + S_-(\mu, k)$, and then inserting this formula into Equation (18) yields

$$-\mu h(k_0) - ik \gamma \mu S_+ - \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial S_+}{\partial \mu} - \frac{\nu}{2L} (1 - \mu^2) \frac{\partial S_+}{\partial \mu} + k_+^2 D_+ S_+ \right] - \frac{\nu}{2L} (1 - \mu^2) \frac{\partial S_+}{\partial \mu} - k_+^2 D_+ S_+.$$

(19)

To proceed, by integrating Equation (19) over $\mu$ from $-1$ to $1$ and from $0$ to $1$ respectively, there after using the standard assumption $D_{\mu\mu}(\mu = \pm 1) = 0$, we can solve the following integral equations.

$$\left( ik \gamma + \frac{\nu}{L} \right) \int_{-1}^1 d\mu \mu S_+ = k_+^2 \int_{-1}^1 d\mu D_+ S_+$$

(20)

and

$$-\frac{1}{2} h(k_0) - \left( ik \gamma + \frac{\nu}{L} \right) \int_0^1 d\mu \mu S_+ + \left[ D_{\mu\mu} \frac{\partial S_+}{\partial \mu} \right](\mu = 0) - \frac{\nu}{2L} S_+(\mu = 0) + k_+^2 \int_0^1 d\mu D_+ S_+ = - \left[ D_{\mu\mu} \frac{\partial S_+}{\partial \mu} \right](\mu = 0) + \frac{\nu}{2L} S_+(\mu = 0).$$

(21)

If employing the pitch-angle isotropic scattering model $D_{\mu\mu} = D(1 - \mu^2)$ with a constant $D$ (e.g., Shalchi et al. 2009; Shalchi 2010) and the perpendicular diffusion coefficient model $D_\perp = \frac{2}{k_\perp} \mu_0 |\mu|$ (e.g., Qin & Shalchi 2014; Shalchi 2010, 2017), Equation (21) can be simplified to

$$-\frac{1}{2} h(k_0) - \left( ik \gamma + \frac{\nu}{L} \right) \int_0^1 d\mu \mu S_+ + \left[ D_{\mu\mu} \frac{\partial S_+}{\partial \mu} \right](\mu = 0) - \frac{\nu}{2L} S_+(\mu = 0) + 2k_+^2 \int_0^1 d\mu S_+ = - \left[ D_{\mu\mu} \frac{\partial S_+}{\partial \mu} \right](\mu = 0) + \frac{\nu}{2L} S_+(\mu = 0).$$

(22)

Since a uniform mean magnetic field was considered in SH2010, only the two terms $\partial S_+ / \partial \mu(\mu = 0)$ and $\partial S_+ / \partial \mu(\mu = 0)$ occurred in the governing equations (see Equations (15) and (16) in SH2010). Since the adiabatic focusing effect is considered in this paper, we have to determine the four terms $\partial S_+ / \partial \mu(\mu = 0)$, $\partial S_+ / \partial \mu(\mu = 0)$, $S_+(\mu = 0)$, and $S_-(\mu = 0)$ in Equations (20) and (22). If the approximation formula $\partial S_+ / \partial \mu \approx \partial S_- / \partial \mu(\mu = 0)$ in SH2010 and the approximation method in Shalchi (2011b) are not used in Equations (20) and (22), the specific expression of $S(k, \mu)$ has to be obtained.

3. The Formula of $S(k, \mu)$

The distribution function $f(x, \mu, \mu_0, t)$ of Equation (10) can be split into the isotropic distribution part $F(x, t)$ and anisotropic distribution part $g(x, \mu, \mu_0, t)$

$$f(x, \mu, t) = F(x, t) + g(x, \mu, \mu_0, t).$$

(23)

By inserting Equation (23) into the formula of $S(k, \mu)$ (see Equation (11)), since $F(x, t)$ does not contain $\mu_0$, we can get

$$S(\mu, k) = \frac{1}{2} \int_{-1}^1 d\mu_0 \mu_0 \int_0^\infty dt \int d^3x e^{i k \cdot x} g(x, \mu, \mu_0, t).$$

(24)

If the anisotropic distribution function $g(x, \mu, \mu_0, t)$ is obtained, the specific expressions of $S(\mu, k)$, $S_+(\mu, k)$, and $S_-(\mu, k)$ can be obtained from Equation (24). In the next subsection, we will employ the method of HS2014 to obtain $g(x, \mu, \mu_0, t)$. Consequently, the terms $\partial S_+ / \partial \mu(\mu = 0)$, $\partial S_- / \partial \mu(\mu = 0)$, $S_+(\mu = 0)$, and $S_-(\mu = 0)$ contained in Equations (20) and (22) will also be obtained.

3.1. The Anisotropic Distribution Function $g(x, \mu, \mu_0, t)$

By using the method of HS2014, from Equation (10), the exact anisotropic distribution function can be written as

$$g(\mu) = \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}} \right] L \left[ \frac{\partial E}{\partial x} - 2E \right] + e^{M(\mu)} \left[ \int_{-1}^1 d\nu R(\nu) - \frac{\int_{-1}^1 d\mu_0 e^{M(\mu_0)}}{\int_{-1}^1 d\mu e^{M(\mu)}} \right].$$

(25)
where \( R(\mu) \) can be written as

\[
R(\mu) = R_1(\mu) + R_2(\mu) + R_3(\mu),
\]

with

\[
R_1(\mu) = \frac{e^{-M(\mu)}}{D_{\mu\mu}} \left( \mu \frac{\partial F}{\partial t} + \int_{-1}^{\mu} d\mu' \frac{\partial g}{\partial t} \right)
\]

\[
R_2(\mu) = \frac{e^{-M(\mu)}}{D_{\mu\mu}} \left( \mu \frac{\partial F}{\partial t} - \frac{\nu}{L} \int_{-1}^{\mu} d\mu' g + \frac{1}{2} \int_{-1}^{\mu} d\mu' g \right)
\]

\[
R_3(\mu) = \frac{e^{-M(\mu)}}{2D_{\mu\mu}} \left( \int_{-1}^{\mu} d\mu D_\perp \Delta_\perp F + \Delta_\perp \int_{-1}^{\mu} d\mu D_\perp g \right) - \frac{e^{-M(\mu)}}{D_{\mu\mu}} \left( \Delta_\perp F \int_{-1}^{\mu} d\mu D_\perp + \Delta_\perp \int_{-1}^{\mu} d\mu D_\perp g \right)
\]

and the quantity \( M(\mu) \) is introduced as in HS2014

\[
M(\mu) = \frac{\nu}{2L} \int_{-1}^{\mu} d\mu' \frac{1 - \nu^2}{D_{\mu\mu}(\nu)}.
\]

For pitch-angle isotropic scattering \( D_{\mu\mu} = D(1 - \mu^2) \) the quantity \( M(\mu) \) can be simplified as

\[
M(\mu) = \xi(1 + \mu),
\]

with \( \xi = \nu/(2DL) \). From Equations (25)–(29), we can see that the anisotropic distribution function \( g(\mu) \) is an iteration function by itself. In other words, \( R(\mu) \) is an iteration function by itself.

### 3.2. The Formula of \( S(k, \mu) \)

In Section 3.1, the anisotropic distribution function \( g(\mu) \) is obtained by using the method of HS2014. In what follows, we derive the specific formula of \( S(k, \mu) \), and then obtain the expressions of \( S_+(k, \mu) \) and \( S_-(k, \mu) \), which are the keys to expressing the perpendicular diffusion coefficient.

Inserting the anisotropic distribution function (25) into Equation (24), we can get

\[
S(k, \mu) = \frac{1}{2} e^{M(\mu)} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v) - \frac{1}{2} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v),
\]

from which we can see that in order to get the expression for \( S(k, \mu) \) we have to obtain the results of \( \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v) \). By assuming the pitch-angle isotropic scattering \( D_{\mu\mu} = D(1 - \mu^2) \) and using Equations (26)–(29) the following formula can be obtained

\[
\int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v) = \frac{1}{D} \int_{-1}^{1} d\mu e^{-M(\mu)} \int_{-1}^{1} d\mu e^{-M(\mu)} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v) + \Phi(k, \mu_0, \mu),
\]

with

\[
\Phi(k, \mu_0, \mu) = \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \int d^3x e^{i k x} \int_{-1}^{\mu} dv R(v) \left[ \frac{1}{2} \left( -i k x - \frac{v}{L} \right) \int_{-1}^{1} d\mu g(\mu) - \int_{-1}^{1} d\mu D_\perp g(\mu) \right]
\]

\[
+ \frac{k^2}{2} \left( \int_{-1}^{1} d\mu D_\perp g(\mu) - \int_{-1}^{1} d\mu D_\perp g(\mu) \right),
\]

where the formula \( g(k) = \int d^3x e^{i k x} g(x) \) and the original condition (15) are used. Combining Equations (32) and (33) gives

\[
S(k, \mu) = \frac{1}{2D} e^{M(\mu)} \left[ \int_{-1}^{1} d\mu e^{-M(\mu)} - \int_{-1}^{1} d\mu e^{-M(\mu)} \int_{-1}^{1} d\mu e^{-M(\mu)} \right] h(k_0) + \Lambda(k, k_0, \mu).
\]
with

\[ \Lambda(k, k_0, \mu) = \frac{1}{2} e^{M(\mu)} \left[ \phi(k, k_0, \mu) - \int_{-1}^{1} d\mu e^{M(\mu)} \phi(k, k_0, \mu) \right] \]

where \( \phi(k, k_0, \mu) \) is Equation (34), and from Equations (35) and (36) we can find that \( S(k, \mu) \) contains variable \( k_0 \), i.e., \( S(k, \mu) = S(k, k_0, \mu) \). From Equations (25), (34)–(36) we can see that in order to get the formula of \( S(k, \mu) \) we have to explore the expression of \( R(\mu) \). Since \( R(\mu) \) is an infinite iteration expression by itself, so \( \phi(k, k_0, \mu) \) is the infinite iteration function of \( R(\mu) \). Therefore, \( \Lambda(k, k_0, \mu) \) as well as \( S(k, k_0, \mu) \) is an infinite iteration function of \( R(\mu) \). Using the assumption \( D_{\mu\mu} = D(1 - \mu^2) \) Equation (35) can be simplified as

\[ S(k, k_0, \mu) = \frac{1}{2D} \left[ \frac{\xi \cosh(\mu \xi) + \xi \sinh(\mu \xi) - \sinh \xi h(k_0) + \Lambda(k, k_0, \mu)}{\xi \sinh \xi} \right]. \]  

### 3.3. The Formulas of \( S_+(k, k_0, \mu) \) and \( S_-(k, k_0, \mu) \)

The quantity \( S(k, \mu) \) can be split as the sum of the even function \( S_+(k, \mu) \) and the odd function \( S_-(k, \mu) \) of \( \mu \), and \( S_+(k, \mu) \) and \( S_-(k, \mu) \) are shown as follows

\[ S_+(k, k_0, \mu) = \frac{\xi \cosh(\mu \xi) - \sinh \xi}{2D \xi \sinh \xi} h(k_0) + \Lambda_+(k, k_0, \mu), \]  

\[ S_-(k, k_0, \mu) = \frac{\sinh(\mu \xi)}{2D \sinh \xi} h(k_0) + \Lambda_-(k, k_0, \mu), \]

where we use the even function \( \Lambda_+(k, k_0, \mu) \) and the odd function \( \Lambda_-(k, k_0, \mu) \) of \( \mu \), which satisfy the formula \( \Lambda(k, k_0, \mu) = \Lambda_+(k, k_0, \mu) + \Lambda_-(k, k_0, \mu) \). Obviously, \( S_+(k, k_0, \mu) \) and \( S_-(k, k_0, \mu) \) are iteration functions of \( R(\mu) \).

By using formulas (38) and (39), we can get the following quantities

\[ S_+(k, k_0, \mu = 0) = \frac{\xi - \sinh \xi}{2D \xi \sinh \xi} h(k_0) + \Lambda_+(k, k_0, \mu = 0), \]  

\[ S_-(k, k_0, \mu = 0) = 0, \]  

\[ \frac{\partial S_-(k, k_0, \mu = 0)}{\partial \mu} = 0, \]  

\[ \frac{\partial S_-(k, k_0, \mu = 0)}{\partial \mu} = \frac{\xi}{2D \sinh \xi} h(k_0) + \frac{\partial \Lambda_-(k, k_0, \mu)}{\partial \mu}(\mu = 0). \]

From Equation (39), the following quantity can be obtained

\[ \frac{\partial S_-(k, k_0, \mu)}{\partial \mu} = \frac{\xi \cosh(\mu \xi)}{2D \sinh \xi} h(k_0) + \frac{\partial \Lambda_-(k, k_0, \mu)}{\partial \mu}. \]

The formula \( \Lambda_-(k, k_0, \mu) \) is very complicated, and if \( \partial \Lambda_-(k, k_0, \mu)/\partial \mu \) can be neglected, the following approximation can be obtained

\[ \frac{\partial S_-(k, k_0, \mu)}{\partial \mu} \approx \frac{\partial S_-(k, k_0, \mu)}{\partial \mu}(\mu = 0), \]

where the weak adiabatic focusing limit \( \cosh(\mu \xi) \approx 1 \) is used. Note that the Equation (45) is precisely the approximation used in SH2010.

### 4. The Perpendicular Diffusion Coefficient with Along-field Adiabatic Focusing

Next, Equation (8) can be rewritten as

\[ T(k, k_0) = \frac{1}{2} \int_{-1}^{1} d\mu S_-(k, k_0, \mu). \]

By considering Equation (39), \( T(k, k_0) \) becomes

\[ T(k, k_0) = \frac{\xi \cosh(\mu \xi) - \sinh \xi}{2D \xi ^2 \sinh \xi} h(k_0) + \frac{1}{2} \int_{-1}^{1} d\mu \Lambda_-(k, k_0, \mu). \]

In fact, using Equations (5) and (47), the perpendicular diffusion coefficient can also be obtained, but we want to leave this for future work.
Equation (47) can be rewritten as

\[
T(\mathbf{k}, k_0) = \left[ \frac{\xi \cosh(\mu\zeta) - \sinh(\mu\zeta)}{2\xi^2 \sinh(\mu\zeta)} + \frac{1}{h(k_0)} \int_0^\infty d\mu \Lambda_-(\mathbf{k}, k_0, \mu) \right] h(k_0).
\]  
(48)

Inserting Equations (40)–(43) and (48) into Equations (20) and (22), we can get

\[
T(\mathbf{k}, k_0) = \frac{1}{3} (4/3) k_+^2 (k_0^2 - (k_0^2 + 2D\xi^2 / (3\kappa^2 + \Xi(\mathbf{k}, k_0, \xi))) - h(k_0)^{-1}
\]
(49)

with

\[
\Xi(\mathbf{k}, k_0, \xi) = \frac{2D\xi}{3} + \frac{[2D/h(k_0)](\partial\Lambda_-(\mathbf{k}, k_0, \mu)/\partial\mu)(\mu = 0) - 2D\xi^2 / (3\kappa^2 + \Xi(\mathbf{k}, k_0, \xi))}{(\xi \cosh(\mu\zeta) - \sinh(\mu\zeta)) / (\xi^2 \sinh(\mu\zeta)) + 2D \int_0^\infty d\mu \Lambda_-(\mathbf{k}, \mu) / h(k_0)}
\]
(50)

where the perpendicular diffusion coefficient model \(D_p = 2\kappa_0^2 / vL\) and \(2D\xi = vL\) are employed.

Integrating Equation (49) over \(k_0\) for all values and then inserting it into Equation (5), the following equation can be obtained

\[
\kappa_+ = \frac{a^2 v^2}{3\kappa^2(0)} \int d^3k \int d^3k_0 \overline{P_{\alpha\beta}(k)} h(k_0)
\]
(51)

The latter equation can be rewritten as

\[
\kappa_+ = \frac{a^2 v^2}{3\kappa^2(0)} \int d^3k \int d^3k_0 \overline{P_{\alpha\beta}(k)} h(k_0)
\]
(52)

The real part is

\[
\kappa_+ = \frac{a^2 v^2}{3\kappa^2(0)} \int d^3k \int d^3k_0 \overline{P_{\alpha\beta}(k)} h(k_0)
\]
(53)

So far, Equations (50)–(53) are accurate except for some common assumptions.

From Equations (38), (39), (50), and (53), we can find that \(\Xi(\mathbf{k}, k_0, \xi)\) in Equation (53) is a complicated function of \(\Lambda(\mathbf{k}, k_0, \mu)\), which is the function of \(R(\mu)\). However, \(R(\mu)\) is an iteration function by itself. Therefore, in order to obtain a specific expression of \(\kappa_+\), we have to truncate \(R(\mu)\).

4.1. The Conditions of Truncating \(R(\mu)\)

From the above discussion, we can see that \(R(\mu)\) has to be truncated. Comparing \(R_0(\mu)\) with \(R_1(\mu)\) and \(R_2(\mu)\) can be ignored under some conditions that can be satisfied by some special cases. In what follows, we explore these conditions. Furthermore, in this paper, we only explore the special cases that satisfy these conditions.

First, we compare the relative importance between \(R_1(\mu)\) and \(R_2(\mu)\). Let us set \(Z^*_g\) and \(T^*_p\) as the along-field spatial characteristic scale and temporal characteristic scale of the anisotropic distribution function \(g\) corresponding to the significant magnitude variation. Here, for the purpose of simplification, we set the order of significant magnitude variation as \(O(1)\), i.e., \(\Delta g = g' \sim O(1)\). And similarly, \(T^*_g\) is set as the temporal characteristic scales of isotropic distribution function \(F\) corresponding to the significant magnitude variation. We also approximate the order of significant magnitude variation as \(O(1)\), i.e., \(\Delta F = F' \sim O(1)\). Thus, \(R_0(\mu)\) and \(R_2(\mu)\) can be written as

\[
R_0(\mu) = \frac{e^{-M(\mu)}}{Dp} \left[ \frac{\mu}{T_g} \partial F' / \partial t' + \frac{1}{T_p} \int_0^\mu \frac{d\mu}{\partial g'/\partial \zeta'} \right],
\]
(54)

\[
R_2(\mu) = \frac{e^{-M(\mu)}}{Dp} \left[ \frac{\nu}{Z^*_g} \left( \int_1^\mu \frac{d\mu}{\partial g'/\partial \zeta'} - \frac{1}{2} \int_1^\mu \frac{d\mu}{\partial \zeta'} \right) - \frac{\nu}{L} \left( \int_1^\mu \frac{d\mu}{\partial g'} - \frac{1}{2} \int_1^\mu \frac{d\mu}{\partial \zeta} \right) \right],
\]
(55)

where the dimensionless time \(t'\) and dimensionless along-field distance \(\zeta'\) are used. Then, according to the above discussion, derivative terms in Equation (54) can be approximated as \(\partial F' / \partial t' \sim \partial g'/\partial \zeta' \sim O(1)\) (e.g., Fay 1994). Since pitch-angle cosine \(\mu\) belongs to the interval \([-1, 1]\), we can approximate the integral terms in Equation (55) as \(\int_1^\mu d\mu / (\partial g'/\partial \zeta') - (1/2) \int_1^\mu d\mu (\partial g'/\partial \zeta') \sim O(1)\) and \(\int_1^\mu d\mu / (\partial g') - (1/2) \int_1^\mu d\mu / \partial \zeta' \sim O(1)\). Let us set \(T^*\) equal to the maximum value of \(T^*_p\) and \(T^*_g\). By comparing Equation (54) with Equation (55), we can obtain the relation \(vT^* \ll Z^*_g\) for the case \(L \gg Z^*_g\), and the relation \(vT^* \ll L\) for the case \(Z^*_g \gg L\). If the case \(L \sim Z^*_g\) occurs, the relations \(vT^* \ll Z^*_g\) and \(vT^* \ll L\) have to be satisfied simultaneously. Therefore, if the conditions \(vT^* \ll Z^*_g\) and \(vT^* \ll L\) are satisfied at the same time, we find that \(R_2(\mu)\) can be neglected compared to \(R_0(\mu)\) for any case.
In general, if particle motion is completely passive in turbulence, stronger turbulence leads to a faster change of the distribution function, i.e., stronger turbulence leads to a smaller characteristic temporal scale $T^*$. It can also be assumed that a smaller gradient of the anisotropic distribution function leads to larger spatial characteristic length $L^*$. Therefore, for relatively strong turbulence, a relatively small along-field gradient of anisotropic distribution function, and not too high energy of particles, the condition $vT^* \ll Z^*$ might be satisfied. The condition $vT^* \ll L$ denotes that the adiabatic focusing characteristic length $L$ is far greater than the characteristic distance $vT^*$. Next, we only explore the cases that satisfy the requirements $vT^* \ll Z^*$ and $vT^* \ll L$. In addition, we assume that perpendicular effects are not much larger than parallel effects, so we can ignore both $R_L(\mu)$ and $R_R(\mu)$ compared to $R_{\parallel}(\mu)$. Then Equation (26) can be simplified as

$$R(\mu) \approx R_{\parallel}(\mu) = \frac{e^{-M(\mu)}}{D_{\mu\parallel}} \left( \frac{\partial F}{\partial \mu} + \int_{-1}^{\mu} d\nu \frac{\partial g}{\partial \mu} \right).$$

(56)

Combining Equations (25) and (56) gives

$$g(\mu) = \left[ 1 - \frac{2e_{M(\mu)}}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \right] \left( \frac{\partial F}{\partial \mu} - \frac{2F}{L} \right) + e_{M(\mu)} \left[ \int_{-1}^{\mu} d\nu \frac{e_{M(\nu)}}{D_{\mu\nu}} \left( \frac{\partial F}{\partial \nu} + \int_{-1}^{\nu} d\rho \frac{\partial g}{\partial \nu} \right) ight. - \left. \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \int_{-1}^{\mu} d\mu e_{M(\mu)} \int_{-1}^{\mu} d\nu e_{M(\nu)} \left( \frac{\partial F}{\partial \nu} + \int_{-1}^{\nu} d\rho \frac{\partial g}{\partial \nu} \right) \right].$$

(57)

Next, we operate Fourier transformation on Equation (57) and then insert the result into Equation (34) to yield

$$\int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} d\tau \int d^3x e^{i k z} \int_{-1}^{1} d\mu R(\mu) = \frac{1}{D} \int_{-1}^{1} d\nu e^{-M(\nu)} \int d^3x_0 e^{i k_0 x_0} h(x_0) + \Phi(k, \mu) h(k_0).$$

(58)

with

$$\Phi(k, \mu) = \frac{\nu}{2D} k z \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\mu\nu}} \left\{ 2 \int_{-1}^{\mu} d\rho e_{M(\rho)} \left[ \int_{-1}^{\mu} d\sigma e^{-M(\sigma)} - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \right] \right\} \left[ \int_{-1}^{\mu} d\nu e^{-M(\nu)} - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \right] \left[ \int_{-1}^{\mu} d\rho D_{\nu\rho} e_{M(\rho)} \right] \left\{ 2 \int_{-1}^{\mu} d\rho D_{\nu\rho} e_{M(\rho)} \right\} \int_{-1}^{\mu} d\mu e_{M(\mu)} \left[ \int_{-1}^{\mu} d\nu e^{-M(\nu)} - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \right] \right\},$$

(59)

where we assume that $M(\mu)$ and $D_{\mu\nu}(\mu)$ are all independent of the initial pitch-angle cosine $\mu_0$. Substituting Equation (58) into Equation (32) yields

$$S(k, k_0, \mu) = \frac{e_{M(\mu)}}{2D} \left[ \int_{-1}^{\mu} d\nu e^{-M(\nu)} - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \right] h(k_0) + \Lambda(k, \mu) h(k_0),$$

(60)

with

$$\Lambda(k, \mu) = \frac{1}{2} e_{M(\mu)} \left[ \Phi(k, \mu) - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \int_{-1}^{\mu} d\nu e_{M(\mu)} \right].$$

(61)

Equation (60) is the result of a first-order iteration of $R(\mu)$. For more iterations of $R(\mu)$, we can get more complicated $S(k, k_0, \mu)$ corresponding to more complicated $\Lambda(k, \mu)$. Therefore, if Equation (56) is used directly in Equation (32), the formula $S(k, k_0, \mu) = S^{(0)}(k, k_0, \mu)$ can be obtained corresponding to $\Lambda(k, \mu) = \Lambda^{(0)}(k, \mu) = 0$. Note that we call $S^{(0)}(k, k_0, \mu)$ the zeroth-order $S(k, k_0, \mu)$. After inserting Equation (25) into Equation (26) and using Equation (56), the first-order $S(k, k_0, \mu)$, or $S^{(1)}(k, k_0, \mu)$ can be found corresponding to the first-order $\Lambda^{(1)}(k, \mu) = 0$, which is just Equation (61). If we perform n iterations of $R(\mu)$ and use Equation (56), the nth-order $S(k, k_0, \mu)$, i.e., $S^{(n)}(k, k_0, \mu)$, can be written as

$$S^{(n)}(k, k_0, \mu) = \Theta(\mu) h(k_0) + \Lambda^{(n-1)}(k, \mu) h(k_0),$$

(62)

with

$$\Theta(\mu) = \frac{e_{M(\mu)}}{2D} \left[ \int_{-1}^{\mu} d\nu e^{-M(\nu)} - \frac{1}{\int_{-1}^{\mu} d\mu e_{M(\mu)}} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \right].$$

(63)
Here $N_{n-1}(k, \mu)$ is a more complicated formula. Note that Equation (35) denotes the result for infinite iterations of $R(\mu)$.

4.2. Truncating $\Xi(k, k_0, \xi)$

According to the truncation condition in the above subsection, if the approximations

$$[\partial \Lambda_-(k, k_0, \mu)/\partial \mu](\mu = 0) \approx h(k_0)[\partial \Lambda_{\alpha}^n(k, k_0, \mu)/\partial \mu](\mu = 0),$$

$$\Lambda_+(k, k_0, \mu = 0) \approx h(k_0)\Lambda_{\alpha}^n(k, \mu = 0),$$

$$\Lambda_-(k, k_0, \mu) \approx h(k_0)\Lambda_{\alpha}^n(k, \mu)$$

are used, Equations (51) and (50) become

$$\kappa_\alpha = \frac{a^2v^2}{3B_0^2(0)} \int d^3k P_\alpha(k) \frac{(4/3)\kappa_\alpha k_\alpha^2 + (k_\parallel v)^2/(3\kappa_\alpha k_\alpha^2) - (2D)^2\xi^2/(3\kappa_\alpha k_\alpha^2) + \Xi(k, \xi)}{[(4/3)\kappa_\alpha k_\alpha^2 + (k_\parallel v)^2/(3\kappa_\alpha k_\alpha^2) - (2D)^2\xi^2/(3\kappa_\alpha k_\alpha^2) + \Xi(k, \xi)]^2 + (4vDk_\parallel)^2\xi^2/(3\kappa_\alpha k_\alpha^2)^2}.$$  \hspace{1cm} (67)

with

$$\Xi(k, \xi) = \frac{2D}{3} + \frac{2D(\partial \Lambda_{\alpha}^n(k, \mu)/\partial \mu)(\mu = 0) + 2D\xi\Lambda_{\alpha}^n(k, \mu = 0)}{(\xi \cosh \xi - \sinh \xi)/(\xi^2 \sinh \xi) + 2D\int_0^1 d\mu \Lambda_{\alpha}^n(k, \mu)},$$  \hspace{1cm} (68)

where the normalization condition $\int d^3k_0 h(k_0) = 1$ is used. Equation (67) with formula (68) is the nth-order result of $\kappa_\alpha(\xi)$.

The expression $\Xi(k, \xi)$ is a very complicated function. For mathematical tractability in this paper, we only consider the zeroth-order $\kappa_\alpha(\xi)$ corresponding to $N^0(k, \mu = 0)$ and $\partial \Lambda_{\alpha}^{n0}(k, \mu)/\partial \mu)(\mu = 0) = 0$ in Equation (68). Then Equation (67) can be simplified as

$$\kappa_\alpha = \frac{a^2v^2}{3B_0^2(0)} \int d^3k P_\alpha(k) \frac{(4/3)\kappa_\alpha k_\alpha^2 + (k_\parallel v)^2/(3\kappa_\alpha k_\alpha^2) - (2D)^2\xi^2/(3\kappa_\alpha k_\alpha^2) + \Xi(\xi)}{[(4/3)\kappa_\alpha k_\alpha^2 + (k_\parallel v)^2/(3\kappa_\alpha k_\alpha^2) - (2D)^2\xi^2/(3\kappa_\alpha k_\alpha^2) + \Xi(\xi)]^2 + (4vDk_\parallel)^2\xi^2/(3\kappa_\alpha k_\alpha^2)^2}.$$  \hspace{1cm} (69)

with

$$\Xi(\xi) = \frac{2D}{3} \frac{\xi^2 \sinh \xi}{\xi \cosh \xi - \sinh \xi}.$$  \hspace{1cm} (70)

From the latter equation, we can see that $\kappa_\alpha(\xi)$ is a complicated function of $\xi$. And for a weak limit $|\xi| \ll 1$, Equation (69) can be written as

$$\kappa_\alpha \approx \frac{a^2v^2}{3B_0^2(0)} \int d^3k P_\alpha(k) \frac{1}{\left[2D + \frac{4}{3}\kappa_\alpha k_\alpha^2 + \frac{(k_\parallel v)^2}{3\kappa_\alpha k_\alpha^2} \right] + \left[\frac{2D}{15} - \frac{(2D)^2}{3\kappa_\alpha k_\alpha^2} + \frac{(4vDk_\parallel)^2}{3\kappa_\alpha k_\alpha^2} \right] \xi^2}.$$  \hspace{1cm} (71)

$$\approx \frac{a^2v^2}{3B_0^2(0)} \int d^3k P_\alpha(k) \frac{1}{\left[2D + \frac{4}{3}\kappa_\alpha k_\alpha^2 + \frac{(k_\parallel v)^2}{3\kappa_\alpha k_\alpha^2} \right] + \left[\frac{2D}{15} - \frac{(2D)^2}{3\kappa_\alpha k_\alpha^2} + \frac{(4vDk_\parallel)^2}{3\kappa_\alpha k_\alpha^2} \right] \xi^2}.$$  \hspace{1cm} (71)

The latter equation can be rewritten as a series expansion

$$\kappa_\alpha = \kappa_\alpha^{[0]} + \frac{\kappa_\alpha^{[1]} \xi^2}{\kappa_\alpha^{[0]}} + \frac{\kappa_\alpha^{[2]} \xi^4}{\kappa_\alpha^{[0]}} + \frac{\kappa_\alpha^{[3]} \xi^6}{\kappa_\alpha^{[0]}} + \cdots + (-1)^n \frac{\kappa_\alpha^{[n]} \xi^{2n}}{\kappa_\alpha^{[0]}} + \cdots.$$  \hspace{1cm} (72)
with

$$\kappa_0 = \frac{a^2\nu^2}{3B_0(0)} \int d^3k \frac{P_{xx}(k)}{2D + \frac{4}{3}\kappa_0 k_1^2 + \frac{(k_y\nu)^2}{3\kappa_0 k_1^2}}$$

(73)

$$\kappa_1 = \frac{a^2\nu^2}{3B_0(0)} \int d^3k \frac{P_{xx}(k)}{2D + \frac{4}{3}\kappa_1 k_1^2 + \frac{(k_y\nu)^2}{3\kappa_1 k_1^2}}$$

$$\kappa_2 = \frac{a^2\nu^2}{3B_0(0)} \int d^3k \frac{P_{xx}(k)}{2D + \frac{4}{3}\kappa_2 k_1^2 + \frac{(k_y\nu)^2}{3\kappa_2 k_1^2}}$$

(74)

(75)

(76)

$$\kappa_3 = \frac{a^2\nu^2}{3B_0(0)} \int d^3k \frac{P_{xx}(k)}{2D + \frac{4}{3}\kappa_3 k_1^2 + \frac{(k_y\nu)^2}{3\kappa_3 k_1^2}}$$

(77)

......

$$\kappa_n = \frac{a^2\nu^2}{3B_0(0)} \int d^3k \frac{P_{xx}(k)}{2D + \frac{4}{3}\kappa_n k_1^2 + \frac{(k_y\nu)^2}{3\kappa_n k_1^2}}$$

......

Obviously, when parameter $\xi$ tends to zero, Equation (71) tends to the result of Shalchi (2010). The perpendicular and parallel diffusion coefficients $\kappa_0$ and $\kappa_0$ for the uniform background magnetic field have been deeply explored (e.g., Shalchi 2009). Thus it is convenient to explore the influence of adiabatic focusing on perpendicular diffusion if the modifying factors to perpendicular diffusion coefficient $\kappa_1 (\xi)$, which is the function of $\kappa_0$ and $\kappa_0$, can be obtained.

4.3. Power Function Expanding Formula of $\kappa_1 (\xi)$

The perpendicular diffusion coefficient $\kappa_1 (\xi)$ can be expanded to a power function of $\xi$ as in the following

$$\kappa_1 (\xi) = \kappa_1^{(0)} (1 + a_1 \xi + a_2 \xi^2 + \cdots),$$

(78)

where $\kappa_1^{(0)}$ is the perpendicular diffusion coefficient for the uniform background field, and $a_1$, $a_2$, ... are the modifying factors to perpendicular diffusion coefficient introduced by adiabatic focusing. If $\xi \to 0$, the background magnetic field tends to be uniform. For $\xi = 0$, only $\kappa_1^{(0)}$ is retained in formula (78). On the other hand, the following parallel diffusion coefficient with along-field adiabatic focusing was already derived in previous papers (see, e.g., Litvinenko 2012a; Shalchi & Danos 2013; He & Schlickeiser 2014)

$$\kappa_1 (\xi) = \kappa_1^{(0)} (1 + b_2 \xi^2 + \cdots),$$

(79)

where $\kappa_1^{(0)}$ is the parallel diffusion coefficient for uniform background field, and $b_2 = -1/15$.  

10
Inserting Equations (78) and (79) into Equation (71) yields

$$a_1 = 0,$$

$$a_2 = \frac{b_2}{9B_0^2(0)\kappa_{\perp}^{(0)}(k_{\perp}^{(0)})^2} \int d^3k \frac{P_{s\parallel}(k)}{\Omega^2} + \frac{a^2v^4}{3B_0^2(0)} \int d^3k \frac{P_{s\parallel}(k)}{\Omega^2} \left[ \frac{v^2}{27\kappa_{\perp}^{(0)}(k_{\perp}^{(0)})^2} - \frac{v^2}{45\kappa_{\perp}^{(0)}(k_{\perp}^{(0)})^2} - \frac{k_{\perp}^{2}v^6}{90\kappa_{\perp}^{(0)}(k_{\perp}^{(0)})^2} \right],$$

with

$$k_{\perp}^{(0)} = \frac{a^2v^2}{3B_0^2(0)} \int d^3k \frac{P_{s\parallel}(k)}{\Omega^2},$$

and

$$\Omega = \frac{v^2}{3\kappa_{\perp}^{(0)}} + \frac{(k_{\parallel}v)^2}{3k_{\perp}^{2}\kappa_{\perp}^{(0)}} + \frac{4k_{\perp}^2\kappa_{\perp}^{(0)}}{3},$$

It is noteworthy that Equation (79) is only applicable to the cases in which perpendicular diffusion is not considered and $D_{\parallel\mu}$ is independent of adiabatic focusing. So it is more appropriate to derive the formulas of $D_{\parallel\mu}$ and $\kappa_\perp$ with adiabatic focusing and perpendicular diffusion effects, and then to re-deduce Equation (81). we would reserve this as our tasks in the future.

5. Modifying Factor $a_2$

For mathematical tractability, we only explore the modifying factor $a_2$ for the two-component model (slab+2D, Matthaeus et al. 1990). For the slab component contribution, it is more convenient to start from Equation (71). We can easily find that the slab transport is subdiffusive, i.e., $\kappa_{\perp,\text{slab}} = 0$. So we only consider the 2D-component contribution. For the pure two-dimensional model, Equation (81) can be simplified as

$$a_2 = \frac{a^2v^4}{27B_0^2(0)\kappa_{\perp}^{(0)}} W,$$

with

$$W = \int d^3k \frac{P_{s\parallel}^{2D}(k)}{\Omega^2} \left[ \frac{v^2}{27\kappa_{\perp}^{(0)}(k_{\perp}^{(0)})^2} - \frac{2}{5} \right],$$

$$\Omega = \frac{v^2}{3\kappa_{\perp}^{(0)}} + \frac{4k_{\perp}^2\kappa_{\perp}^{(0)}}{3}.$$

In this paper, the following tensor of the two-dimensional (2D) magnetic turbulence is used

$$P_{lm}^{2D}(k) = g^{2D}(k) \frac{\delta(k)}{k_\perp} \left[ \delta_{lm} - \frac{k_\perp k_m}{k^2} \right], \quad l, m = x, y,$$

here we use the spectra of the two-dimensional modes $g^{2D}(k)$,

$$g^{2D}(k) = \frac{D(s, q)}{2\pi} B_{2D}^2 \left\{ \begin{array}{ll} 0, & k_\perp < L_{2D}^{-1} \\ (k_\perp L_{2D})^q, & L_{2D}^{-1} < k_\perp < L_{2D}^{-1} \\ (k_\perp L_{2D})^s, & L_{2D}^{-1} < k_\perp < k_{\text{cut}}, \end{array} \right.$$ (88)

where $q$ is the energy range index, $s$ is the inertial range index, $L_{2D}$ is the box size, $k_{\text{cut}}$ is the random range cut-off wave number, and $D(s, q)$ is the normalized factor. The spectrum is correctly normalized for $q > -1$ and $s > 1$. A more detailed explanation can be found in Shalchi (2011c).

From Equation (84), we can find that the sign of $a_2$ is determined by formula $W$. In what follows, we explore the sign of formula $W$. From formulas (87) and (88), we can obtain

$$W = \frac{81L_{2D}^3}{32v^2(A_{\perp}^{(0)})^2} D(s, q) B_{2D}^2 \left[ \int_0^1 dxx^{q-2} \left( \frac{5\varepsilon/2 - x^2}{(3\varepsilon/4 + x^2)^2} \right)^2 + \int_1^{k_{\text{cut}}} dxx^{q-2} \left( \frac{5\varepsilon/2 - x^2}{(3\varepsilon/4 + x^2)^2} \right)^2 \right],$$

here formulas $\varepsilon = v^2L_{2D}^2/(3\kappa_{\perp}^{(0)})$, $x = k_\perp L_{2D}$, $\eta = L_{2D}/L_{2D}$, and $\Omega = v^2/(3\kappa_{\perp}^{(0)}) + 4k_{\perp}^2\kappa_{\perp}^{(0)}/3$ are used.
5.1. The Case $\varepsilon = 3l^2_{2D}/(\lambda^{(0)}_\parallel \lambda^{(0)}_\perp) \to 0$

The condition $\varepsilon \to 0$, i.e., $l^2_{2D} \ll \lambda^{(0)}_\parallel \lambda^{(0)}_\perp$, which denotes the product of parallel and perpendicular mean-free paths $\lambda^{(0)}_\parallel$ and $\lambda^{(0)}_\perp$ is far greater than the turnover scale $l^2_{2D}$, and high energy charged particles satisfy this condition. For this case, $W$ becomes

$$W \approx -\frac{81l^4_{2D}}{32v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\left[1 - \frac{\eta^{-3}}{q - 3} + \frac{1 - k^{-3}_{\text{max}}}{s + 3}\right].$$

(90)

since $k^{-3}_{\text{max}} \ll 1$, the latter equation can be simplified as

$$W \approx -\frac{81l^4_{2D}}{32v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\left[1 - \frac{\eta^{-3}}{q - 3} + \frac{1}{s + 3}\right].$$

(91)

For $q > 3$, we can find $\eta^{-3} \ll 1$, then we can get

$$W \approx -\frac{81l^4_{2D}}{32v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\frac{s + q}{(q - 3)(s + 3)} < 0.$$

(92)

For $-1 < q < 3$, Equation (91) can be simplified as

$$W \approx -\frac{81l^4_{2D}}{32v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\frac{\eta^{-3}(s + 3) + (3 - q)(s + 3)}{(3 - q)(s + 3)} < 0.$$

(93)

In both special case (92) and (93), we find that $W < 0$, then $a_2 < 0$. Thus, we can see that for high energy particles the modified factor $a_2$ makes the perpendicular diffusion coefficient decrease.

5.2. The Case $\varepsilon = 3l^2_{2D}/(\lambda^{(0)}_\parallel \lambda^{(0)}_\perp) \to \infty$

In this case, the condition $\lambda^{(0)}_\parallel \lambda^{(0)}_\perp \ll l^2_{2D}$ corresponding to low energy particles should be satisfied, and we can derive the following equation from Equation (89).

$$W \approx \frac{15l^2_{2D}}{4v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\left[1 - \frac{\eta^{-1}}{q - 1} + \frac{1 - k^{-1}_{\text{max}}}{s + 1}\right].$$

(94)

Since $k^{-1}_{\text{max}} \ll 1$, we can get from the latter equation

$$W \approx \frac{15l^2_{2D}}{4v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\left[1 - \frac{\eta^{-1}}{q - 1} + \frac{1}{s + 1}\right].$$

(95)

For $q > 1$, we get $\eta^{-1} \ll 1$ and the latter formula can be simplified as

$$W \approx \frac{15l^2_{2D}}{4v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\frac{s + q}{(q - 1)(s + 1)} > 0.$$

(96)

For $-1 < q < 1$, we can obtain from formula (95)

$$W \approx \frac{15l^2_{2D}}{4v^2(\lambda^{(0)}_\parallel)^2} D(s, q) \delta B^2_{2D}\left[\frac{\eta^{-1}}{1 - q} + \frac{1}{s + 1}\right] > 0.$$

(97)

Clearly, we find that for low energy particles the modified factor $a_2$ makes the perpendicular diffusion coefficient increase.

In summary, the specific modifying factor $a_2$ to perpendicular diffusion coefficient appears in different forms for different particle energy. However, it is independent of the energy range index $q$ and the sign of adiabatic focusing length. Considering the discussion in Section 4.1, it is might only the results obtained in Section 5.2 that can satisfy the application scope of this paper.

6. Discussion and Conclusions

In this paper, by employing the UNLT theory in SH2010 and the method of HS2014, we explore the perpendicular diffusion of energetic charged particles in a spatially varying background magnetic field, and the perpendicular diffusion coefficient with along-field adiabatic focusing is obtained. From this formula, we find that along-field adiabatic focusing can introduce a significant modification to the particle transport across the mean magnetic field.

Previous articles demonstrated that the parallel diffusion is decreased by along-field adiabatic focusing, and the parallel diffusion coefficient is independent of the sign of adiabatic focusing (Beeck & Wibberenz 1986; Bieber & Burger 1990; Kóta 2000; Shalchi 2011a; Litvinenko 2012a, 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014). In this paper, we find that for isotropic scattering
and the weak adiabatic limit the perpendicular diffusion coefficient for the non-uniform background field is independent of the sign of adiabatic focusing length, which is similar to the parallel diffusion coefficient. The perpendicular diffusion coefficient can be indicated as a power series of adiabatic focusing characteristic quantity $\xi$ and all of the modifying factors are shown as the functions of perpendicular and parallel diffusion coefficients for a uniform background magnetic field. For the two-component model, we find that the first-order modifying factor is equal to zero and the sign of the second one is determined by the energy of particles. For low energy particles, the second-order modifying factor is positive, and for high energy particles the modifying factor is negative.

However, it must be emphasized that the perpendicular diffusion coefficient derived in this paper has three requirements. The first one is that the turbulence is relatively strong, the along-field gradient of the anisotropic distribution function is small, and the energy of particles is not too high, so that $vT^{*} \ll Z_{\xi}^{2}$ can be satisfied. The second one is that the perpendicular diffusion effect is not much higher than the parallel one. The third one is that the adiabatic focusing length is very large, i.e., $vT^{*} \ll L$, so the formulas derived in this paper might only be applicable to the special case of Section 5.2 corresponding to low energy particles. It is also noted that the specific formulas shown in this paper are only for the lowest order iteration results. By combining Equations (47) and (5), the perpendicular diffusion coefficient $k_{\perp}(\xi)$ with along-field adiabatic focusing can also be obtained. This is our future task. In fact, because we only explore the weak adiabatic focusing limit $L \rightarrow \infty$ in this paper, formula $B_{0}(z) = B_{0}(0)e^{-z/L} \approx B_{0}(0)$ is established approximately. So the results derived by employing the approximation $B_{0}(z) \approx B_{0}(0)$ are also approximately valid.

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