Another determinantal inequality involving partial traces

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Abstract

Let $A$ be a positive semidefinite $m \times m$ block matrix with each block $n$-square, then the following determinantal inequality for partial traces holds

$$(\text{tr}A)^{mn} - \det(\text{tr}_2 A)^n \geq |\det A - \det(\text{tr}_1 A)^m|,$$

where $\text{tr}_1$ and $\text{tr}_2$ stand for the first and second partial trace, respectively. This result improves a recent result of Lin \cite{14}.

Key words: Partial traces; Block matrices; Determinantal inequality; Numerical range in a sector.

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1 Introduction

Throughout the paper, we use the following standard notation. The set of $n \times n$ complex matrices is denoted by $\mathbb{M}_n(\mathbb{C})$, or simply by $\mathbb{M}_n$, and the identity matrix of order $n$ by $I_n$, or $I$ for short. If $A = [a_{ij}]$ is of order $m \times n$ and $B$ is of order $s \times t$, the tensor product of $A$ with $B$, denoted by $A \otimes B$, is an $ms \times nt$ matrix, partitioned into $m \times n$ block matrix with the $(i, j)$-block the $s \times t$ matrix $a_{ij}B$. In this paper, we are interested in complex block matrices.

Let $\mathbb{M}_m(\mathbb{M}_n)$ be the set of complex matrices partitioned into $m \times m$ blocks with each block being a $n \times n$ matrix. The element of $\mathbb{M}_m(\mathbb{M}_n)$ is usually written as $A = [A_{i,j}]_{i,j=1}^m$, where $A_{i,j} \in \mathbb{M}_n$ for all $i, j$. By convention, if $X \in \mathbb{M}_n$ is positive semidefinite, we write $X \geq 0$.

For two Hermitian matrices $A$ and $B$ of the same size, $A \geq B$ means $A - B \geq 0$. It is easy to see that $\geq$ is a partial ordering on the set of Hermitian matrices, referred to as L"{o}wner ordering.

Now we introduce the definition of partial traces, which comes from Quantum Information Theory \cite{16} p. 12. For $A \in \mathbb{M}_m(\mathbb{M}_n)$, the first partial trace (map) $A \mapsto \text{tr}_1 A \in \mathbb{M}_n$ is

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defined as the adjoint map of the imbedding map \( X \mapsto I_m \otimes X \in M_m \otimes M_n \). Correspondingly, the second partial trace (map) \( A \mapsto \text{tr}_2 A \in M_m \) is defined as the adjoint map of the imbedding map \( Y \mapsto Y \otimes I_n \in M_m \otimes M_n \). Therefore, we have

\[
\langle I_m \otimes X, A \rangle = \langle X, \text{tr}_1 A \rangle, \quad \forall X \in M_n,
\]

and

\[
\langle Y \otimes I_n, A \rangle = \langle Y, \text{tr}_2 A \rangle, \quad \forall Y \in M_m.
\]

Assume that \( A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n) \) is positive semidefinite, it is easy to see that both \( \text{tr}_1 A \) and \( \text{tr}_2 A \) are positive semidefinite; see, e.g., [18]. To some extent, these two partial traces are closely related. For instance, Audenaert [1] proved an inequality for Schatten \( p \)-norms,

\[
\text{tr} A + \|A\|_q \geq \|\text{tr}_1 A\|_q + \|\text{tr}_2 A\|_q.
\] (1)

Inequality (1) was used to prove the subadditivity of Tsallis entropies.

Moreover, Ando (see [2]) established the following,

\[
(\text{tr} A)I_m \otimes I_n + A \geq I_m \otimes (\text{tr}_1 A) + (\text{tr}_2 A) \otimes I_n,
\] (2)

where \( \geq \) means the L"{o}wner ordering. Furthermore, Motivated by inequalities (1) and (2), Lin [13] proved an analogous result for determinant, which states that

\[
(\text{tr} A)^m + \det A \geq \det(\text{tr}_1 A)^m + \det(\text{tr}_2 A)^n.
\] (3)

In this paper, we improve Lin’s result (3) as follows.

**Theorem 1.1** Let \( A \in M_m(M_n) \) be positive semidefinite. Then

\[
(\text{tr} A)^m - \det(\text{tr}_2 A)^n \geq |\det A - \det(\text{tr}_1 A)^m|.
\]

The paper is organized as follows. We first present some auxiliary results, and then we show our proof of Theorem 1.1. Finally, we extend our result to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector (Theorem 2.7).

## 2 Auxiliary results and proofs

For \( A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n) \), we define the partial transpose of \( A \) by \( A^\tau = [A_{i,j}]_{i,j=1}^m \). It is clear that \( A \geq 0 \) does not necessarily imply \( A^\tau \geq 0 \). If both \( A \) and \( A^\tau \) are positive semidefinite, then \( A \) is called to be positive partial transpose (or PPT for short). Recall that a linear map \( \Phi : M_n \to M_k \) is called positive if it maps positive matrices to positive matrices. A linear map \( \Phi : M_n \to M_k \) is said to be \( m \)-positive if for \( [A_{i,j}]_{i,j=1}^m \in M_m(M_n) \),

\[
[A_{i,j}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{i,j})]_{i,j=1}^m \geq 0.
\] (4)
It is said to be \emph{completely positive} if (4) holds for any integer \(m \geq 1\). It is well known that both the trace map and determinant map are completely positive; see, e.g., [20, p. 221, p. 237]. On the other hand, a linear map \(\Phi\) is said to be \(m\)-copositive if for \([A_{i,j}]_{i,j=1}^m \in M_m(M_n)\),
\[
[A_{i,j}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{i,j})]_{i,j=1}^m \geq 0,
\] and \(\Phi\) is said to be \emph{completely copositive} if (5) holds for any positive integer \(m \geq 1\). Furthermore, \(\Phi\) is called a \emph{completely PPT map} if it is completely positive and completely copositive. A comprehensive survey of the standard results on completely positive maps can be found in [3, Chapter 3] or [15].

We need the following lemma, which is the main result in [11]; see, e.g., [10].

\textbf{Lemma 2.1} (see [11]) \textit{The map }\(\Phi(X) = (\text{tr}X)I + X\text{ is a completely PPT map.}\)

In the proof of the next proposition, we only employ the fact that \(\Psi(X) = (\text{tr}X)I + X\) is 2-copositive. Proposition 2.2 first proved by the authors [5] recently, which is a complement of Ando’s result [2] and play a vital role in our derivation of Theorem 1.1. We here provide an alternative proof for convenience of readers. Our proof is slightly more transparent than the original proof in [5].

\textbf{Proposition 2.2} \textit{Let }\(A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n)\text{ be positive semidefinite. Then}
\[
(\text{tr}A)I_n \otimes I_n - (\text{tr}A) \otimes I_n \geq A - I_m \otimes (\text{tr}A).
\] \textit{(6)}

\textbf{Proof.} The proof is by induction on \(m\). When \(m = 1\), there is nothing to prove. We now prove the base case \(m = 2\). In this case, the required inequality is
\[
\begin{bmatrix}
(\text{tr}A)I_n & 0 \\
0 & (\text{tr}A)I_n
\end{bmatrix}
\geq
\begin{bmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2}
\end{bmatrix}
\geq
\begin{bmatrix}
A_{1,1} + A_{2,2} & 0 \\
0 & A_{1,1} + A_{2,2}
\end{bmatrix},
\]
or equivalently (note that \(\text{tr}A = \text{tr}A_{1,1} + \text{tr}A_{2,2}\)),
\[
H := \begin{bmatrix}
(\text{tr}A_{2,2})I_n + A_{2,2} & -A_{1,2} - (\text{tr}A_{1,2})I_n \\
-A_{2,1} - (\text{tr}A_{2,1})I_n & (\text{tr}A_{1,1})I_n + A_{1,1}
\end{bmatrix}
\geq 0.
\] \textit{(7)}

By Lemma 2.1 we have
\[
\begin{bmatrix}
(\text{tr}A_{1,1})I_n + A_{1,1} & (\text{tr}A_{2,1})I_n + A_{2,1} \\
(\text{tr}A_{1,2})I_n + A_{1,2} & (\text{tr}A_{2,2})I_n + A_{2,2}
\end{bmatrix}
\geq 0,
\]
and so
\[
H = \begin{bmatrix}
0 & -(I_n) \\
(I_n) & 0
\end{bmatrix}
\begin{bmatrix}
(\text{tr}A_{1,1})I_n + A_{1,1} & (\text{tr}A_{2,1})I_n + A_{2,1} \\
(\text{tr}A_{1,2})I_n + A_{1,2} & (\text{tr}A_{2,2})I_n + A_{2,2}
\end{bmatrix}
\begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix}
\geq 0,
\]
which confirms the desired (7).
After some rearrangement, we have

\[
\Gamma := (\text{tr}A)I_k \otimes I_n + I_k \otimes (\text{tr}_1 A) - A - (\text{tr}_2 A) \otimes I_n
\]

\[
= \left( \text{tr} \sum_{i=1}^{k} A_{i,i} \right) I_k \otimes I_n + I_k \otimes \left( \sum_{j=1}^{k} A_{j,j} \right) - A - (\text{tr}A_{i,j})_{i,j=1}^k \otimes I_n
\]

\[
= \begin{bmatrix}
\sum_{i=1}^{k-1} (\text{tr}A_{i,i})I_n \\
\vdots \\
\sum_{i=1}^{k-1} (\text{tr}A_{i,i})I_n \\
0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
(\text{tr}A_{k,k})I_n \\
\vdots \\
(\text{tr}A_{k,k})I_n \\
\sum_{i=1}^{k} (\text{tr}A_{i,i})I_n
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\sum_{i=1}^{k-1} A_{i,i} \\
\vdots \\
\sum_{i=1}^{k-1} A_{i,i} \\
0
\end{bmatrix}
+ \begin{bmatrix}
A_{k,k} \\
\vdots \\
A_{k,k} \\
\sum_{i=1}^{k} A_{i,i}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{1,1} & \cdots & A_{1,k-1} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
A_{k-1,1} & \cdots & A_{k-1,k-1} & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 & A_{1,k} \\
\vdots & \vdots & \vdots & \vdots \\
A_{k,1} & \cdots & A_{k,k-1} & A_{k,k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
(\text{tr}A_{1,1})I_n & \cdots & (\text{tr}A_{1,k-1})I_n & 0 \\
\vdots & \vdots & \vdots & \vdots \\
(\text{tr}A_{k-1,1})I_n & \cdots & (\text{tr}A_{k-1,k-1})I_n & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 & (\text{tr}A_{1,k})I_n \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & (\text{tr}A_{k-1,k})I_n \\
(\text{tr}A_{k,1})I_n & \cdots & (\text{tr}A_{k,k-1})I_n & (\text{tr}A_{k,k})I_n
\end{bmatrix}
\]

After some rearrangement, we have

\[
\Gamma = \Gamma_1 + \Gamma_2,
\]

where

\[
\Gamma_1 := \begin{bmatrix}
\sum_{i=1}^{k-1} (\text{tr}A_{i,i})I_n \\
\vdots \\
\sum_{i=1}^{k-1} (\text{tr}A_{i,i})I_n \\
0
\end{bmatrix}
\begin{bmatrix}
\sum_{i=1}^{k-1} A_{i,i} \\
\vdots \\
\sum_{i=1}^{k-1} A_{i,i} \\
0
\end{bmatrix}
\]

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is that \( \Gamma \) (see [14]) Let Lemma 2.3

Now by induction hypothesis, we get that \( \Gamma \) and \( X \geq 0 \). In view of (3), it suffices to show

\[ \text{Proof of Theorem 1.1} \]

\[ A, B \]

Before showing our proof of Theorem 1.1, we need one more lemma for our purpose.

We are now in a position to present the proof of Theorem 1.1.

**Proof of Theorem 1.1** In view of (3), it suffices to show

\[ (\text{tr}A)^m + \det(\text{tr}_1 A)^m \geq \det A + \det(\text{tr}_2 A)^n. \] (8)
Let $X = (\text{tr}A)I_m \otimes I_n, Y = I_m \otimes (\text{tr}_1 A), W = A, Z = (\text{tr}_2 A) \otimes I_n$, respectively. It is easy to see that 

$$(\text{tr}A)I_m = \sum_{i=1}^{m}(\text{tr}A_{i,i})I_m = (\text{tr}(\text{tr}_2 A))I_m \geq \text{tr}_2 A,$$

which implies that $X \geq Z \geq 0$, and clearly $X \geq W \geq 0$. Moreover, by Proposition 2.2, $X + Y \geq W + Z$. That is, all conditions in Lemma 2.3 are satisfied. Therefore,

$$(\text{tr}A)^m + \det((I_m \otimes (\text{tr}_1 A))) \geq \det A + \det((\text{tr}_2 A) \otimes I_n).$$

Since $\det(X \otimes Y) = (\det X)^n(\det Y)^m$ for every $X \in \mathbb{M}_m$ and $Y \in \mathbb{M}_n$, this completes the proof.

Using the same idea in previous proof and combining [5] Proposition 2.3, one could also get the following Proposition 2.4. We leave the details for the interested reader.

**Proposition 2.4** Let $A \in \mathbb{M}_m(\mathbb{M}_n)$ be PPT. Then

$$(\text{tr}A)^m + \det(\text{tr}_2 A)^n \geq \det A + \det(\text{tr}_1 A)^n.$$}

At the end of the paper, we extend the determinantal inequality [5] to a larger class of matrices whose numerical ranges are contained in a sector. The same extension of [5] can be found in [17]. Before showing our extension, we first introduce some standard notations.

For $A \in \mathbb{M}_n$, the Cartesian (Toeplitz) decomposition $A = RA + iZA$, where $RA = \frac{1}{2}(A + A^*)$ and $ZA = \frac{1}{2i}(A - A^*)$. Let $|A|$ denote the positive square root of $A^*A$, i.e., $|A| = (A^*A)^{1/2}$. We denote the $i$-th largest singular value of $A$ by $s_i(A)$, then $s_i(A) = \lambda_i(|A|)$, the $i$-th largest eigenvalue of $|A|$. Recall that the numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$}

For $\alpha \in [0, \frac{\pi}{2})$, let $S_\alpha$ be the sector on the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \alpha\} = \{re^{i\theta} : r > 0, |\theta| \leq \alpha\}.$$}

Obviously, if $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$, then $\Re(A)$ is positive definite and if $W(A) \subseteq S_0$, then $A$ is positive definite. Such class of matrices whose numerical ranges are contained in a sector is called the **sector matrices class**. Clearly, the concept of sector matrices is an extension of that of positive definite matrices. Over the past years, various studies on sector matrices have been obtained in the literature; see, e.g., [4, 7, 8, 13, 17, 19].

First, we list two lemmas which are useful to establish our extension (Theorem 2.7).

**Lemma 2.5** (see [13]) Let $0 \leq \alpha < \frac{\pi}{2}$ and $A \in \mathbb{M}_n$ with $W(A) \subseteq S_\alpha$. Then

$$|\det A| \leq (\sec \alpha)^n \det(\Re A).$$}

**Lemma 2.6** (see [6, p. 510]) Let $A$ be an $n$-square complex matrix. Then

$$\lambda_i(\Re A) \leq s_i(A), \quad i = 1, 2, \ldots, n.$$}

Moreover, if $X$ has positive definite real part, then

$$\det \Re A + |\det \Im A| \leq |\det A|.$$}
Now, we provide the extension of (8).

**Theorem 2.7** Let $H \in M_m(M_n)$ be such that $W(H) \subseteq S_\alpha$. Then

$$(\text{tr}|A|)^{mn} + |\det(\text{tr}_1 A)|^m \geq (\cos \alpha)^{mn} \det |A| + (\cos \alpha)^{mn} |\det(\text{tr}_2 A)|^n.$$ 

**Proof.** By Lemma 2.6 we have

$$\text{tr}|A| = \sum_{i=1}^{mn} s_i(A) \geq \sum_{i=1}^{mn} \lambda_i(\Re A) = \text{tr}(\Re A) \geq 0. \tag{9}$$

Since $W(A) \subseteq S_\alpha$, it is noteworthy that $W(\text{tr}_1 A) \subseteq S_\alpha$ and $W(\text{tr}_2 A) \subseteq S_\alpha$; see, e.g., [8]. Observe that $\Re(\text{tr}_1 A) = \text{tr}_1(\Re A)$ and $\Re(\text{tr}_2 A) = \text{tr}_2(\Re A)$. Therefore,

$$(\text{tr}|A|)^{mn} + |\det(\text{tr}_1 A)|^m \geq (\text{tr}(\Re A)^{mn} + \det(\Re(\text{tr}_1 A))^m$$

$$= (\text{tr}(\Re A)^{mn} + \det(\text{tr}_1(\Re A))^m$$

$$\geq \det(\Re A) + \det(\text{tr}_2(\Re A))^n$$

$$= \det(\Re A) + \det(\Re(\text{tr}_2 A))^n$$

$$\geq (\cos \alpha)^{mn} |\det A| + (\cos \alpha)^{mn} |\det(\text{tr}_2 A)|^n,$$

where the first inequality follows from (9) and Lemma 2.6, the second one follows by applying (8) to $\Re A$, the last one is by Lemma 2.5.

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