AN INEXACT ALTERNATING DIRECTION METHOD OF MULTIPLIERS FOR A KIND OF NONLINEAR COMPLEMENTARITY PROBLEMS

JIE-WEN HE, CHI-CHON LEI, CHEN-YANG SHI* AND SEAK-WENG VONG

Department of Mathematics, University of Macau
Avenida da Universidade, Taipa, Macau, China

(Communicated by Chongyang Liu)

Abstract. Many kinds of practical problems can be formulated as nonlinear complementarity problems. In this paper, an inexact alternating direction method of multipliers for the solution of a kind of nonlinear complementarity problems is proposed. The convergence analysis of the method is given. Numerical results confirm the theoretical analysis, and show that the proposed method can be more efficient and faster than the modulus-based Jacobi, Gauss-Seidel and Successive Overrelaxation method when the dimension of the problem being solved is large.

1. Introduction. Complementarity problems were first introduced by Cottle and studied by Lemke and Howson in their work [14] on computing equilibrium points of bimatrix games in 1964. These problems have wide application in fields such as scientific computing, engineering and economy [28]. For instance, they appear prominently in equilibrium problems, traffic network design, optimal control, and asset pricing [9].

A nonlinear complementarity problem (NCP) with respect to a vector-valued function $F : D \rightarrow \mathbb{R}^n$, where $D$ is a subset of $\mathbb{R}^n$ containing at least the nonnegative cone, is to find a vector $u \in \mathbb{R}^n$ that satisfies the system of equations and inequalities

$$F(u) \geq 0, \quad u \geq 0, \quad u^TF(u) = 0.$$ 

Over the past decades, several algorithms have been devised to solve nonlinear complementarity problems. These include projection-type methods [25], merit-function methods [5], smoothing Newton methods [22, 23], interior point methods [20, 21], linearization methods [8] and matrix multisplitting methods [1, 2, 3, 4]. See [13] and the references therein for more details.

Very recently, alternating direction method [6, 10, 24] was employed to solve linear complementarity problems [32] and a class of linear complementarity problems

2010 Mathematics Subject Classification. 90C33, 65F10, 65F50, 65G40.

Key words and phrases. Inexact alternating direction method, Augmented Lagrangian, Nonlinear complementarity problem, Iterative methods, Symmetric positive definite.

This research is funded by The Science and Technology Development Fund, Macau SAR (File no. 0005/2019/A) and the grant MYRG2018-00047-FST, MYRG2017-00098-FST from University of Macau.

* Corresponding author: C.-Y. Shi.
arising from free boundary problems encountered in mathematical physics were studied in [33], where the authors proposed an inexact alternating direction method of multipliers to solve these problems. This paper generalizes the result by considering an inexact alternating direction method of multipliers (ADMM) for solving a kind of nonlinear complementarity problems.

In this paper, we consider a class of nonlinear complementarity problems deriving from the following problem defined in $D \subset \mathbb{R}^2$:

$$
\begin{align*}
- Lv(x, y) + \psi(v(x, y)) + f(x, y) &\geq 0, \\
v(x, y) &\geq 0, \\
v(x, y)[-Lv(x, y) + \psi(v(x, y)) + f(x, y)] &= 0,
\end{align*}
$$

subject to $v(x, y) = g(x, y)$ on the boundary of the region $D$, where $f$, $L$ and the boundary condition $g$ are given. When this problem is solved numerically, the solution $v(x, y)$ is approximated by a vector $u$ defined on the grid points, which can usually be put into the following form:

$$
\begin{align*}
Au + \phi(u) &\geq 0, \\
u &\geq 0, \\
u^T[Au + \phi(u)] &= 0,
\end{align*}
$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix. The nonlinear term is

$$
\phi(u) = (\psi_1(u_1), \psi_2(u_2), \ldots, \psi_n(u_n))^T + b,
$$

where $b$ is a vector coming from the boundary condition and the function $f(x, y)$. Denote this problem as $NCP(\phi, A)$. Since $NCP(\phi, A)$ is derived from (1), the coefficient matrix $A$ in (2) usually has some special structure. For some practical applications, this discretization matrix can be separated into two parts as

$$
A = H + V,
$$

where $H$ and $V$ correspond to the discretization matrices of the differential operator in each spatial direction. Furthermore, they satisfy

- both the matrices $H$ and $V$ are symmetric and positive definite.
- they are large and sparse.

These properties motivate the study of an inexact alternating direction method for solving the problem. In the next section, we present our proposed method and the convergence analysis is given in Section 3. In Section 4, numerical examples are given to demonstrate the theoretical results.

2. The proposed method. We propose an inexact ADMM for the solution of a kind of nonlinear complementarity problems in this section. It is worthwhile to point out that an assumption of $\frac{d\psi_i}{du_i} \geq 0$ is set throughout this paper. Note that $u^*$ solves (2) if and only if $u^*$ solves the following constrained programming problem:

$$
\begin{align*}
\min \ S(u) &= \frac{1}{2} u^T A u + \Phi(u), \\
\text{subject to} \ u &\geq 0,
\end{align*}
$$

where $\Phi(u) = \sum_{i=1}^{n} \int_{-\infty}^{u_i} (\psi_i(u_i) + b_i) du_i$ is convex on $u$ because $\frac{d\psi_i}{du_i}$ is nonnegative. Therefore we can solve (2) by considering (4) instead.
Let $R^n_+$ be the nonnegative cone, i.e., $R^n_+ = \{ u \in R^n : u_i \geq 0, i = 1, 2, \ldots, n \}$. Let $G(u)$ be the indicator function of $R^n_+$, i.e.,

$$G(u) = \begin{cases} 
0, & \text{for } u \in R^n_+, \\
+\infty, & \text{otherwise}. 
\end{cases}$$

By arguments similar to that of Lemma 1 in [33], we see that the problem (4) can be equivalently written as

$$\min \{ S(u) + G(w) \},$$

subject to $\Lambda(w - u) = 0$, $w \in R^n_+$, (6)

where $\Lambda$ is a given diagonal matrix with positive diagonal entries. For simplicity, here we take $\Lambda = \mu I$, $\mu > 0$.

The Augmented Lagrangian function of (6) is given as:

$$L(u, w, \lambda) = S(u) + G(w) + \lambda^T \Lambda(w - u) + \frac{\beta}{2} \| \Lambda(w - u) \|^2,$$

where $\lambda \in R^n$ is the Lagrangian multiplier and $\beta > 0$ is a penalty parameter.

Then the ADMM for the problem (6), or equivalently for the problem (4), is as follows:

$$\begin{align*}
\text{minimize} \quad & S(u) + G(w) \\
\text{subject to} \quad & \Lambda(w - u) = 0, \ w \in R^n_+.
\end{align*}$$

We now consider the minimization problems in (7). For the first minimization problem in (7), we have

$$\begin{align*}
(A + \beta \Lambda^2)u^{k+1} + \phi(u^{k+1}) &= \Lambda \lambda^k + \beta \Lambda^2 w^k. \quad (8)
\end{align*}$$

Note that $A$ has the form of (3). Then, we employ the alternating direction implicit method (ADI) to solve (8). Namely, given $u^k$, $w^k$ and $\lambda^k$, we find $u^{k+1}$ inexactly by performing the following steps:

$$\begin{align*}
(H + \beta \mu^2 I)u^{k+1} &= -Vu^k + (\mu \lambda^k + \beta \mu^2 w^k - \phi(u^k)), \\
(V + \beta \mu^2 I)w^{k+1} &= -Hu^{k+1} + (\mu \lambda^k + \beta \mu^2 w^k - \phi(u^k)).
\end{align*}$$

For the second minimization problem in (7), we have

$$u^{k+1} = \left( u^{k+1} - \frac{\mu^{-1} \lambda^k}{\beta} \right)_+.$$  

Based on (9) and (10), we get our algorithm finally.

\textbf{Algorithm 1: Inexact ADMM}

\textbf{Input} $H$, $V$ with $A = H + V$, $\phi$, $\mu$ and $\beta > 0$; give initial points $u^0$, $w^0$ and $\lambda^0$.

While not converged, Do

Generate $u^{k+1}$ according to (9);

Generate $w^{k+1}$ according to (10);

Update $\lambda^k$ according to $\lambda^{k+1} = \lambda^k + \beta \mu (w^{k+1} - u^{k+1})$.

End Do
3. Convergence analysis. In this section, we prove that \((u^k, w^k)\) generated by Algorithm 1 converges to the solution \((u^*, w^*)\) of the problem (6), where the solution \(u^*\) of (6) is also the solution of (2) according to the discussion in Section 2. We note that, following arguments in [33], \((u^*, w^*, \lambda^*)\) satisfies

\[
\begin{aligned}
&\left\{
(Au^* + \phi(u^*)) - \Lambda \lambda^* = 0,
\right.

&\left.\quad (w - w^*)^T \Lambda \lambda^* \geq 0, \forall w \in R^n_+,
\right.

&\left.\Lambda (u^* - w^*) = 0.
\right.
\end{aligned}
\] (11)

The main result in this paper is

**Theorem 3.1.** Suppose that \(H, V\) are symmetric positive definite matrices and \(\lambda_{\text{min}}(A) > d^*_0\), where \(d^*_0 \equiv \max_{i=1, \ldots, n} \left\{ \max_{x \in R} \phi'_i(x) \right\}\) and \(\phi'_i(x) \geq 0, \forall x \in R, i = 1, 2, \ldots, n\). If \(\beta \mu^2\) is a number such that \(S \equiv \beta \mu^2 V + \beta \mu^2 H - H^2 - V^2 - \beta \mu^2 d_0^* I\) is a symmetric positive definite matrix. Then the sequence \((u^k, w^k)\) generated by Algorithm 1 converges to the solution \((u^*, w^*)\) of the problem (6).

**Proof.** Denote \(e^k_u = u^k - u^*, e^k_w = w^k - w^*\) and \(e^k_\lambda = \lambda^k - \lambda^*\). Combining (11) with (9), we have

\[
\begin{aligned}
&\left\{
(H + \beta \mu^2 I) e^k_u + (\mu e^k_u + \beta \mu^2 e^k_w - \phi(u^k) + \phi(u^*))
\right.

&\left.\quad + (V + \beta \mu^2 I) e^k_w = -2 \beta \mu e^k_w (e^k_u + e^k_\lambda),
\right.

&\left.\quad - \beta \mu^2 (e^k_u + e^k_\lambda)
\right.

&\left.\quad - 2 \beta \mu e^k_w (e^k_u + e^k_\lambda) - \beta \mu^2 (e^k_u + e^k_\lambda),
\right.

&\left.\quad - 2 \beta \mu^2 (e^k_u + e^k_\lambda).
\right.
\end{aligned}
\] (12)

We can now combine them to get

\[
2[H + \beta \mu^2 I][V + \beta \mu^2 I] e^k_u = 2HV e^k_u + 2\beta \mu^2 (\mu e^k_u + \beta \mu^2 e^k_w - \phi(u^k) + \phi(u^*))
\] (13)

From \(\lambda^k+1 = \lambda^k + \beta \mu (w^{k+1} - u^{k+1})\) in Algorithm 1, we can get that

\[
\|e^k_u\|^2 - \|e^k_\lambda\|^2 = -2 \beta \mu (e^k_u, (e^k_w - e^k_u)) - \beta \mu^2 (e^k_u, e^k_u).
\]

Following [33], we have

\[
2\beta^2 \mu^2 \|e^k_u + e^k_w\|^2 - 2\beta^2 \mu^2 (e^k_u, e^k_w) + 2\mu \beta (e^k_u, e^k_w) \leq 0,
\]

which implies that

\[
\|e^k_u\|^2 - \|e^k_\lambda\|^2 \geq 2 \beta \mu (e^k_u, e^k_u) + \beta \mu^2 (e^k_u, e^k_u) - 2 \beta^2 \mu^2 (e^k_u, e^k_u),
\]

or equivalently,

\[
\mu^2 \|e^k_w\|^2 - \mu^2 \|e^k_u\|^2 \geq 2 \beta \mu^3 \|e^k_u\|^2 - \beta \mu^2 \|e^k_u\|^2 - 2 \beta^2 \mu^4 \|e^k_u\|^2.
\] (13)

Multiplying (12) with \(e^k_u\) and combining with (13), we have

\[
mu^2 \|e^k_w\|^2 - \mu^2 \|e^k_u\|^2 \geq 2 \beta^2 \mu^4 (\|e^k_u\|^2 - 2(e^k_u, e^k_w))
\]

\[
- (e^{k+1}_u, 2HV e^k_u + 2\beta \mu^2 [-\phi(u^k) + \phi(u^*)])
\]

\[
+ (e^{k+1}_u, 2[H + \beta \mu^2 I][V + \beta \mu^2 I] e^k_u) - \beta^2 \mu^4 \|e^k_u\|^2.
\]

This gives

\[
\mu^2 (\|e^k_w\|^2 - \|e^k_u\|^2) + \beta \mu^4 (\|e^k_u\|^2 - \|e^k_w\|^2)
\]

\[
\geq 2 \beta^2 \mu^4 \|e^k_w - e^{k+1}_u\|^2 - (e^k_u, 2HV e^k_u + 2\beta \mu^2 [-D^k_\phi] e^k_u)
\]

\[
+ (e^{k+1}_u, 2[H + \beta \mu^2 I][V + \beta \mu^2 H] e^k_u)
\]

\[
\geq 2 \beta^2 \mu^4 \|e^k_w - e^{k+1}_u\|^2 - \|H e^k_u\|^2 - \|V e^k_u\|^2 - \beta \mu^2 d_0^* \|e^k_u\|^2
\]

\[
- \beta \mu^2 d_0^* \|e^k_u\|^2 - \|H e^k_u\|^2 - \|V e^{k+1}_u\|^2 + 2(e^k_u, [\beta \mu^2 V + \beta \mu^2 H] e^k_u),
\]
where
\[ D^k_\varphi = \text{diag}\left( \phi_1(u^k_1) - \varphi_1(u^*_1), \ldots, \phi_n(u^k_n) - \varphi_n(u^*_n) \right) = \text{diag}(\phi'_1(\xi^k_1), \ldots, \phi'_n(\xi^k_n)) \]
with \( \xi^k_i \) lying between \( u^k_i \) and \( u^*_i \) for \( i = 1, \ldots, n \).

Finally we get
\[
\begin{align*}
\mu^2(\|e^k_A\|^2 - \|e^{k+1}_A\|^2) + \beta^2\mu^4(\|e^k_w\|^2 - \|e^{k+1}_w\|^2) + & \|Ve^k_u\|^2 - \|Ve^{k+1}_u\|^2 \\
+ \beta^2\mu^2d_\varphi(\|e^k_w\|^2 - \|e^{k+1}_w\|^2) & \geq \beta^2\mu^4|e^k_w - e^{k+1}_w|^2 + 2(e^{k+1}_w, [\beta\mu^2V + \beta\mu^2H - H^2 - V^2 - \beta\mu^2d_\varphi I]e^{k+1}_w),
\end{align*}
\]
(14)
where we suppose that \( \beta\mu^2 \) is a number such that \( S \equiv \beta\mu^2V + \beta\mu^2H - H^2 - V^2 - \beta\mu^2d_\varphi I \) is a symmetric positive definite matrix. Since we suppose that \( \lambda_{\min}(A) > d_\varphi \), the assumption of \( S \) symmetric positive definite is reasonable as long as \( \beta\mu^2 \) is large enough. Thus, we have shown that the proposed method is convergent based on (14).

4. Numerical Experiments. Matrix splitting methods were found to be efficient methods to solve NCPs (See [15, 18, 26, 27, 29, 30, 31]). Here we give a brief review of the modulus-based Jacobi method (MJ), modulus-based Gauss-Seidel method (MGS) and modulus-based successive overrelaxation method (MSOR) [15, 27] for solving NCPs, which will then be used to compare with the proposed method in numerical tests. Specifically, these iteration methods are based on a splitting \( A = F - G = D - L - U \), where \( D \) is a diagonal matrix and \( L, U \) are strictly lower-triangular and upper-triangular matrices respectively. With this formulation, the iteration is given as follows:

\[
(\Omega + F) x^{k+1} = Gx^k + (\Omega - A)|x^k| - \gamma [b + \psi\left(\frac{1}{\gamma}(|x^k| + x^k)\right)]
\]
(15)
with \( u^{k+1} = \frac{1}{\gamma}(|x^{k+1}| + x^{k+1}) \).

**Algorithm 2:** MJ, MGS and MSOR

| Input | F, G with \( A = F - G = D - L - U, \theta > 0, \Omega = \theta D \) and \( \gamma > 0 \); give initial points \( x^0 \) and get \( u^0 \) from (15); While not converged, Do Generate \( u^{k+1} \) according to (15); End Do |

(a) When \( F = D, G = L + U \), it is called the modulus-based Jacobi method.
(b) When \( F = D - L, G = U \), it is called the modulus-based Gauss-Seidel method.
(c) When \( F = \frac{1}{\alpha}D - L, G = (\frac{1}{\alpha} - 1)D + U \), it is called the modulus-based successive overrelaxation method.

We next compare our proposed method with the MJ, MGS and MSOR methods. Let \( RES(z^k) := \min(Az^k + \phi(z^k), z^k) \), where \( z^k \) is the k-th approximate solution to the NCP and the minimum is taken componentwise. For MJ and MGS methods, \( \theta \) is obtained experimentally by minimizing the corresponding iteration steps. For the MSOR method, both \( \alpha \) and \( \theta \) are obtained by minimizing the iteration steps. For our proposed method, we set \( \mu = 4 \), and \( \beta \) is taken similarly to minimize the number of iterations. Moreover, we choose all initial vectors \( x^0 = u^0 = w^0 = \)
$\lambda^0 = (0, 0, \ldots, 0)^T \in \mathbb{R}^n$ and $\gamma = 2$. Let IT denotes the number of iterations and CPU denotes the elapsed CPU time. Finally, we set the stopping criterion to be $\text{RES} \leq 10^{-6}$ or IT > 10000 for all methods.

**Example 1.** We consider the problem arisen from the discretization of free boundary problems [7, 11, 12, 19, 23, 30]. Let $D = (0, 1) \times (0, 1)$ and function $g$ satisfy $g(0, y) = y(1 - y)$, $g(x, y) = 0$ on $y = 0$, $y = 1$ or $x = 1$. Consider the following problem:

Find $u$ such that

\[
\begin{align*}
-\nabla^2 u + f(u, x, y) - 8(y - 0.5) &\geq 0 \quad \text{in } D, \\
u &\geq 0 \quad \text{in } D, \\
u(-\nabla^2 u + f(u, x, y) - 8(y - 0.5)) &\equiv 0 \quad \text{in } D, \\
u &\equiv g \quad \text{on } \partial D,
\end{align*}
\]

where $f(u, x, y)$ is continuously differentiable and $\frac{\partial f}{\partial u} \geq 0$ on $\overline{D} \times \{u : u \geq 0\}$.

Here, we set $f(u, x, y) = \log(1 + e^u)$ and apply the five-point difference discretization with mesh-step size $h = \frac{1}{m+1}$. By eliminating known values of $u$ on boundaries, we obtain a NCP of the form (2), where the matrix $A$ is of the form

\[
A = H + V = \frac{I \otimes V}{h^2} + \frac{V \otimes I}{h^2}.
\]

$I \in \mathbb{R}^{m \times m}$ is identity matrix. $V \in \mathbb{R}^{m \times m}$ is tri-diagonal matrix with diagonal elements 2 and subdiagonal elements -1.

Table 1 shows the numerical results of Example 1.

| m | Algorithm 1 | MJ | MGS | MSOR |
|---|-------------|----|-----|------|
| $2^1$ | IT | 57 | 170 | 83 | 81 |
| | CPU | 1.98E-03 | 1.35E-03 | **6.49E-04** | 6.82E-04 |
| | RES | 9.72E-07 | 9.86E-07 | 9.58E-07 | 8.21E-07 |
| $2^2$ | IT | 106 | 644 | 272 | 270 |
| | CPU | 1.16E-02 | 1.62E-02 | 7.74E-03 | **7.14E-03** |
| | RES | 9.78E-07 | 9.98E-07 | 9.25E-07 | 9.97E-07 |
| $2^3$ | IT | 210 | 2531 | 1131 | 865 |
| | CPU | 7.13E-02 | 1.36E-01 | 6.66E-02 | **5.05E-02** |
| | RES | 9.84E-07 | 9.98E-07 | 9.88E-07 | 9.89E-07 |
| $2^4$ | IT | 435 | – | 4839 | 4729 |
| | CPU | **0.613** | – | 0.981 | 1.033 |
| | RES | 9.80E-07 | – | 9.99E-07 | 9.97E-07 |
| $2^5$ | IT | 900 | – | – | – |
| | CPU | **5.57** | – | – | – |
| | RES | 9.96E-07 | – | – | – |
| $2^6$ | IT | 1875 | – | – | – |
| | CPU | **75.5** | – | – | – |
| | RES | 9.99E-07 | – | – | – |
Example 2. Let $M$ be a given positive integer, $m = 2^M - 1$, $n = m^2$, $h = \frac{1}{m+1}$, and $A = H + V \in \mathbb{R}^{n \times n}$, where $H = I \otimes V_1$, $V = V_1 \otimes I$,

$$V_1 = \frac{1}{h^2} \times \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{m \times m}.$$  

and $I \in \mathbb{R}^{m \times m}$ is a unit matrix.

The nonlinear term is $\phi(u) = u - \sin u + b$, in which

$$b = -[0, h_1, 2h_1, \ldots, 10, 0, h_1, 2h_1, \ldots, 10, \ldots, 0, h_1, 2h_1, \ldots, 10] \in \mathbb{R}^n;$$

and $h_1 = \frac{10}{m-1}$.

Test the problem for different mesh-step sizes $h$. Table 2 shows the numerical results of Example 2.

| M | Algorithm 1 | MJ | MGS | MSOR |
|---|-------------|----|-----|------|
| 4 | IT 82 | 925 | 365 | 308 |
|   | CPU 5.80E-03 | 7.80E-03 | 3.40E-03 | 2.90E-03 |
|   | RES 8.99E-07 | 9.97E-07 | 9.76E-07 | 9.56E-07 |
| 5 | IT 157 | 3862 | 1578 | 1203 |
|   | CPU 4.95E-02 | 1.15E-01 | 5.34E-02 | 4.14E-02 |
|   | RES 9.95E-07 | 9.99E-07 | 9.93E-07 | 9.94E-07 |
| 6 | IT 310 | – | 7536 | 7402 |
|   | CPU 0.370 | – | 0.976 | 0.972 |
|   | RES 9.99E-07 | – | 1.00E-06 | 9.97E-07 |
| 7 | IT 622 | – | – | – |
|   | CPU 3.52 | – | – | – |
|   | RES 9.99E-07 | – | – | – |
| 8 | IT 1255 | – | – | – |
|   | CPU 43.2 | – | – | – |
|   | RES 1.00E-06 | – | – | – |
| 9 | IT 2551 | – | – | – |
|   | CPU 584 | – | – | – |
|   | RES 9.98E-07 | – | – | – |

Example 3. We consider the Porous Flow Through a Dam Problem in [33] under the effect of a nonlinear function $\arctan v$. Referring to Fig. 1, it can be formulated as:

Find $v$ on the rectangle $R = ABCF$ such that

$$\begin{cases} -\nabla^2 v + 1 + \arctan v \geq 0 & \text{in } R, \\ v \geq 0 & \text{in } R, \\ v(-\nabla^2 v + 1 + \arctan v) = 0 & \text{in } R, \end{cases}$$

(17)
Here, we approximate \( R \) by a uniform grid \( G_d \) with grid size \( h \), and then apply the classical five-point difference discretization to the Laplace operator in (17). By eliminating known values of \( v \) on boundaries, we obtain a NCP of the form (2), where the matrix \( A \) is of the form

\[
A = H + V = \frac{I_1 \otimes V_1}{h^2} + \frac{V_2 \otimes I_2}{h^2}.
\]

\( I_1 \in \mathbb{R}^{n \times n} \) and \( I_2 \in \mathbb{R}^{m \times m} \) are the identity matrices. \( V_1 \in \mathbb{R}^{m \times m} \) and \( V_2 \in \mathbb{R}^{n \times n} \) are tri-diagonal matrices with diagonal elements 2 and subdiagonal elements -1, where \( h = \frac{l}{2^M}, \) \( m = 2^M + 1 \) and \( n = 2^M - 1 \).

We set \( y_1 = 24, y_2 = 4, \) and \( l = 16. \) Then \( G_d^0 \) is a \((2 + 1) \times (3 + 1)\) grid with \( h_0 = 8. \) The finest grid is \( G_d^9 \) with \((512 + 1) \times (768 + 1) = 394497\) grid points.

This example does not meet the assumption “\( \lambda_{\min}(A) > d_{\phi}^* \)” in Theorem 3.1. However, we can still obtain the numerical results of Example 3 without satisfying all the assumptions, as shown in Table 3.

From these examples, it can be concluded that the proposed method has advantages comparing with the MJ, MGS and MSOR method, when the sizes of problem under consideration are large.

**Acknowledgments.** The authors would like to thank the editor and referees for their helpful comments and suggestions. These help to improve the paper significantly.
### Table 3: Numerical results of Example 3

| M | Algorithm 1 | MJ | MGS | MSOR |
|---|-------------|----|-----|------|
| 3 | IT          | 337| 730 | 311  | 307  |
|   | CPU         | 3.90E-02 | 1.09E-02 | 5.10E-03 | 4.90E-03 |
|   | RES         | 9.56E-07 | 9.91E-07 | 9.91E-07 | 9.46E-07 |
| 4 | IT          | 345| 3039| 1201 | 1031 |
|   | CPU         | 1.58E-01 | 1.37E-01 | 6.55E-02 | 5.25E-02 |
|   | RES         | 9.95E-07 | 9.96E-07 | 9.87E-07 | 9.97E-07 |
| 5 | IT          | 481| –   | 5610 | 4569 |
|   | CPU         | 0.861| –   | 1.13 | 1.02 |
|   | RES         | 9.89E-07| –   | 9.99E-07 | 9.96E-07 |
| 6 | IT          | 993| –   | –   | –   |
|   | CPU         | 8.80| –   | –   | –   |
|   | RES         | 1.00E-06| –   | –   | –   |
| 7 | IT          | 2069| –   | –   | –   |
|   | CPU         | 117 | –   | –   | –   |
|   | RES         | 9.97E-07| –   | –   | –   |
| 8 | IT          | 4340| –   | –   | –   |
|   | CPU         | 1519| –   | –   | –   |
|   | RES         | 9.95E-07| –   | –   | –   |

### REFERENCES

[1] Z. Z. Bai, New comparison theorem for the nonlinear multisplitting relaxation method for the nonlinear complementarity problems, *Comput. Math. Appl.*, 32 (1996), 41–48.

[2] Z. Z. Bai, A class of asynchronous parallel nonlinear accelerated over relaxation methods for the nonlinear complementarity problem, *J. Comput. Appl. Math.*, 93 (1998), 35–44.

[3] Z. Z. Bai, Asynchronous parallel nonlinear multisplitting relaxation methods for the large sparse nonlinear complementarity problems, *Appl. Math. Comput.*, 92 (1999), 35–100.

[4] Z. Z. Bai, V. Migallón, J. Penadés and D. B. Szyld, Block and asynchronous two-stage methods for mildly nonlinear systems, *Numer. Math.*, 82 (1999), 1–20.

[5] S. Q. Du and Y. Gao, Merit functions for nonsmooth complementarity problems and related descent algorithm, *Applied Mathematics - A Journal of Chinese Universities*, 25 (2010), 78–84.

[6] W. Deng and W. Yin, On the global and linear convergence of the generalized alternating direction method of multipliers, *J. Sci. Comput.*, 66 (2016), 889–916.

[7] C. M. Elliott and J. R. Ockendon, Weak and variational methods for moving boundary problems, in *Research Notes in Mathematics*, 59, Pitman, Boston, London, 1982.

[8] M. C. Ferris and C. Kanzow, Complementarity and related problems: A survey, in *Handbook of Applied Optimization* (eds. P. M. Pardalos, and M. G. C. Resende), Oxford University Press, New York, (2002), 514–530.

[9] M. C. Ferris, O. Mangasarian and J. Pang, *Complementarity: Applications, Algorithms and Extensions*, Springer, New York, 2011.

[10] R. Glowinski and A. Marrocco, Sur l’approximation par éléments finis d’ordre un et la résolution par pénalisation-dualité d’une classe de problèmes de Dirichlet non linéaires, Laboria Report 115, 1975.

[11] K. H. Hoffmann and J. Zou, Parallel solution of variational inequality problems with nonlinear source terms, *IMA J. Numer. Anal.*, 16 (1996), 31–45.

[12] N. Huang and C. F. Ma, The modulus-based matrix splitting algorithms for a class of weakly nonlinear complementarity problems, *Numer. Linear Algebra Appl.*, 23 (2016), 558–569.

[13] P. T. Harker and J. S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Program.*, 48 (1990), 161–220.

[14] C. E. Lemke and J. T. Howson, Equilibrium points of bimatrix games, *SIAM J. Appl. Math.*, 12 (1964), 413–423.
[15] R. Li and J. F. Yin, Accelerated modulus-based matrix splitting iteration methods for a restricted class of nonlinear complementarity problems, Numer. Algor., 75 (2017), 339–358.
[16] W. Li and W. W. Sun, Modified Gauss-Seidel type methods and Jacobi type methods for Z matrices, Linear Algebra Appl., 317 (2000), 227–240.
[17] W. Li, A general modulus-based matrix splitting method for linear complementarity problems of H-matrices, Appl. Math. Lett., 26 (2013), 1159–1164.
[18] C. F. Ma and N. Huang, Modified modulus-based matrix splitting algorithms for a class of weakly nondifferentiable nonlinear complementarity problems, Appl. Numer. Math., 108 (2016), 116–124.
[19] G. H. Meyer, Free boundary problems with nonlinear source terms, Numer. Math., 43 (1984), 463–482.
[20] M. A. Noor, Fixed point approach for complementarity problems, J. Comput. Appl. Math., 133 (1988), 437–448.
[21] F. A. Potra and S. J. Wright, Interior-point methods, J. Comput. Appl. Math., 124 (2000), 255–281.
[22] L. Qi, D. Sun and G. Zhou, A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequalities, Math. Program., 87 (2000), 1–35.
[23] Z. Sun and J. P. Zeng, A monotone semismooth Newton type method for a class of complementarity problems, J. Comput. Appl. Math., 235 (2011), 1261–1274.
[24] M. Tao and X. Yuan, On Glowinski’s open question on the alternating direction method of multipliers, Journal of Optimization Theory and Applications, 179 (2018), 163–196.
[25] N. H. Xiu and J. Zhang, Some recent advances in projection-type methods for variational inequalities, J. Comput. Appl. Math., 152 (2003), 559–585.
[26] S. L. Xie, H. R. Xu and J. P. Zeng, Two-step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems, Linear Algebra Appl., 494 (2016), 1–10.
[27] Z. Xia and C. Li, Modulus-based matrix splitting iteration methods for a class of nonlinear complementarity problem, Appl. Math. Comput., 271 (2015), 34–42.
[28] L. Yong, Nonlinear complementarity problem and solution methods, in Proceedings of the 2010 International Conference on Artificial Intelligence and Computational Intelligence, Part I, Springer-Verlag, (2010), 461–469.
[29] H. Zheng, S. Vong and L. Liu, The relaxation modulus-based matrix splitting iteration method for solving a class of nonlinear complementarity problems, Int. J. Comput. Math., 96 (2018), 1648–1667.
[30] H. Zheng and S. Vong, The modulus-based nonsmooth Newtons method for solving a class of nonlinear complementarity problems of P-matrices, Calcolo, 55 (2018), 37.
[31] H. Zheng, S. Vong and L. Liu, A direct preconditioned modulus-based iteration method for solving nonlinear complementarity problems of H-matrices, Appl. Math. Comput., 353 (2019), 396–405.
[32] J. J. Zhang, MSSOR-based alternating direction method for symmetric positive-definite linear complementarity problems, Numer. Algor., 68 (2015), 631–644.
[33] J. J. Zhang, J. L. Zhang and W. Z. Ye, An inexact alternating direction method of multipliers for the solution of linear complementarity problems arising from free boundary problems, Numer. Algor., 78 (2018), 895–910.

Received October 2019; 1st revision February 2020; Final revision March 2020.

E-mail address: jwkeng56@gmail.com
E-mail address: ccleijohn@gmail.com
E-mail address: shichenyang15@gmail.com
E-mail address: swvong@umac.mo