ON INVERTIBILITY PRESERVING LINEAR MAPPINGS, SIMULTANEOUS TRIANGULARIZATION AND PROPERTY L

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1. Introduction.

The investigation leading to this publication was motivated by a desire to try to understand the structure of a linear unital mapping \( \varphi \) from a unital algebra \( \mathcal{A} \) of matrices contained in \( M_n(\mathbb{C}) \) into \( M_m(\mathbb{C}) \) which has the property that an invertible element in \( \mathcal{A} \) is mapped into an invertible in \( M_n(\mathbb{C}) \). The interest in this question was raised by some earlier results on a linear invertibility preserving mapping from a Banach algebra into \( M_n(\mathbb{C}) \). This lead to the article [3] which on the other hand basically is of finite dimensional nature. During the search for results of this type in the literature on linear algebra we found similar questions discussed either as operators on \( M_n(\mathbb{C}) \) [13], or in the form of results on the structure of sets of matrices having the Property L of Motzkin and Taussky [10,11,12,15,16,17,18,19]. In [13] Marcus and Purves describe the structure of an operator \( \varphi \) on \( M_n(\mathbb{C}) \) which preserves invertibility. It turns out that \( \varphi \) must have one of the forms

\[
\varphi(A) = UAV \quad \text{or} \quad \varphi(A) = UA^tV
\]

where \( U \) and \( V \) are in \( \text{GL}_n(\mathbb{C}) \). If further \( \varphi \) is unital \( (\varphi(I) = I) \) then \( V = U^{-1} \) so one gets in this case that \( \varphi \) is either an automorphism or an anti-automorphism of the algebra \( M_n(\mathbb{C}) \).

Kaplansky [9] discussed the problem whether an invertibility preserving, unital, continuous linear mapping between Banach algebras might have some algebraic properties, and he suggested mainly on the basis of results from [5, 6, 13] that such mappings might be Jordan homomorphisms \( (\varphi(a^2) = \varphi(a)^2) \). Some experiments with low dimensional matrices (Example 3.1) show that this is too much to expect. At first we thought that Kaplansky was right modulo the Jacobson radical in the algebra generated by the image of such a mapping. On the other hand a closer look at the simple case where \( \varphi \) is a mapping of \( \mathbb{C}^3 \) into \( M_n(\mathbb{C}) \) shows that this is not so. An example by Wielandt [19], which also has been used by Motzkin and Taussky to give an L-pair which is not simultaneously triangularizable yields an example which kills this modified conjecture definitely. But still we had some
results from [3] and some evidence from notably [10,11,16,17] that some reasonable extra conditions might imply that an invertibility preserving mapping with these extra properties is a Jordan homomorphism.

Regarding this question we consider in section 4 a unital algebra $A \subseteq M_h(\mathbb{C})$ and a linear mapping $\varphi : A \to M_n(\mathbb{C})$ such that $\varphi(I_A) = I_{M_n(\mathbb{C})}$ and the image of any invertible element in $A$ is invertible in $M_n(\mathbb{C})$. For $k$ in $\mathbb{N}$ we define $\varphi_k : M_k(A) = A \otimes M_k(\mathbb{C}) \to M_k(M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ by $\varphi_k = \varphi \otimes \text{id}$, i.e. $\varphi$ operates on each of the entries in $M_k(A)$. We will then say that $\varphi$ is $k$-invertibility preserving if $\varphi_k$ is invertibility preserving and our main result says that $\varphi$ is a homomorphism modulo the Jacobson radical if and only if it is $k$-invertibility preserving for some sufficiently large $k$. In particular $\varphi$ will be $m$-invertibility preserving for any $m$ larger than this $k$. The main problem is then to find an estimate for the least $k$ which has the property that $k$-invertibility implies that $\varphi$ is a homomorphism modulo the radical. The bound becomes especially nice when the image is an algebra. In other cases the bound, we have, depends on how far the image is from being an algebra. We prove that $\varphi$ is a homomorphism if it is $k$-invertibility preserving for $k = \text{dim(alg(}\varphi(A)\text{))} - \text{dim(}\varphi(A)\text{)} + 3$. This estimate is fairly crude, so we introduce a concept, which we call the semi-simple defect of a set of matrices, as a measure for how far this set is from being an algebra modulo the Jacobson radical. If $\varphi(A) = \text{alg(}\varphi(A)\text{)}$ then the semi-simple defect is 0 and $\varphi$ is a Jordan-homomorphism modulo the Jacobson radical without further conditions, if it further is 2-invertibility preserving then it is a homomorphism modulo the Jacobson radical.

The Property $L$ was mentioned above. It is closely related to the invertibility preserving question for an abelian algebra $A$. In section 3 we introduce for $k \in \mathbb{N}$ a Property $kL$ which is stronger than $L$ and mimics the ideas presented above in the way that Property $kL$ is a condition on the characteristic roots for matrices of the form $a \otimes x + b \otimes y$ with $a, b$ fixed in $M_n(\mathbb{C})$ and $x, y$ general in $M_k(\mathbb{C})$. One can say it is a set of conditions on matrix pencils [5, Matrizenbüschel] but with coefficients in $M_k(\mathbb{C})$ rather than in $\mathbb{C}$. We prove that for some set $S$ contained in $M_n(\mathbb{C})$ this set can be triangularized simultaneously if and only if it has property $kL$ where $k = \text{dim(alg(S))} - \text{dim(span(S))} + 3$. As for the $k$-invertibility case it is relatively easy to see that $(k + 1)L$ implies $kL$ so the problem is to determine some estimate for the lower bound for the set of those $k$ for which property $kL$ implies simultaneous triangularization. The least usable $k$ we find is the semi-simple defect plus 3, but we find it is likely that it could be of the order of the square root of the semi-simple defect rather than of first order.

Finally section 2 contains the results which explain why we have such estimates on $k$ and why it works. The content of Section 2 is not new, but this section has its focus on traces, on the Jacobson radical and on algebras of matrices rather than on a single matrix or pairs of matrices, and we have not found this point of view presented in the literature on matrix theory. Generally speaking one can say that section 2 repeats some well known elementary algebra and tells you how much you have to pay in order to use these results if you just have a set of matrices
instead of an algebra. We estimate the size of the degree of polynomials involved in order to get from a subspace to the algebra generated by this subspace. In the sections 3 and 4 this estimate is used as the size $k$ on the matrix algebra we have to tensor with in order to get the desired properties. The estimate given in section 2 is probably not optimal, on the other hand some rather concrete computations made at the end of section 2 show that it may be hard to be more precise.

2. Basic observations.

Through this section $\mathcal{B}$ will denote an algebra of $n \times n$ matrices over $\mathbb{C}$ such that the unit $I$ of $M_n(\mathbb{C})$ is in $\mathcal{B}$. By Wedderburn's Theorem [2, p. 143] and Wedderburn-Artin's Theorem [2, p. 69], $\mathcal{B}$ decomposes as a direct sum of a semi-simple sub-algebra $\mathcal{C}$ and the Jacobson radical $J$. Further $\mathcal{C}$ is isomorphic to $\mathcal{B}/J$ and is a direct sum of full matrix algebras hence $\mathcal{C} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$ and

\begin{equation}
\mathcal{B} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus J, \quad \forall j \in J : \quad j^n = 0.
\end{equation}

It is important and will be used in the following arguments that in the sum decomposition chosen $\mathcal{B} = \mathcal{C} \oplus J$, the algebra $\mathcal{C}$ is a sub-algebra of $\mathcal{B}$. We will let $\text{tr}_n$ denote the trace on $M_n(\mathbb{C})$ and $I$ the unit in $M_n(\mathbb{C})$ so $\text{tr}_n(I) = n$. The restriction of $\text{tr}_n$ to $J$ vanishes since the elements in $J$ are all nilpotent. The restriction of $\text{tr}_n$ to the summand $M_{n_i}(\mathbb{C})$ in $\mathcal{C}$ is a trace on this algebra ($\text{tr}_n(xy) = \text{tr}_n(yx)$) and for a minimal idempotent $e$ in $M_{n_i}(\mathbb{C})$ we have $d_i = \text{tr}_n(e) \in \mathbb{N}$. Since all traces on $M_{n_i}(\mathbb{C})$ are proportional to the canonical one $- \text{tr}_{n_i}$ we get $\text{tr}_n | M_{n_i}(\mathbb{C}) = d_i \text{tr}_{n_i}$. With this notation in mind we can formulate the first observation, which is a known fact to which we do not have a suitable reference.

2.1 Lemma. An element $b$ in $\mathcal{B}$ belongs to $J$ if and only if for each $x \in \mathcal{B}$ 
\[\text{tr}_n(xb) = 0.\]

Proof. If $b$ is in $J$ then so is $xb$ and since $J$ is an ideal, $\text{tr}_n$ vanishes on $xb$.

If $b$ does not belong to $J$ then according to the decomposition of $\mathcal{B}$ in (1)

\[b = b_1 + \cdots + b_i + \cdots + b_s + j\]

and we may assume that $b_i \neq 0$. Let

\[x = 0 + \cdots + 0 + b_i^* + 0 + \cdots + 0 + 0,\]

where $b_i^*$ is the adjoint matrix in $M_{n_i}(\mathbb{C})$ to $b_i$. Then $x$ is in $\mathcal{B}$ and

\[\text{tr}_n(xb) = d_i \text{tr}_{n_i}(b_i^*b_i) > 0.\]
2.2 Definition. For natural numbers \( i, j \) let \( \text{PNC}(i, j) \) denote the space of polynomials of degree less than or equal to \( j \) in \( i \) non-commuting variable.

It should be noted that \( I \) is assumed to be in \( \text{PNC}(i, j) \) and that not all \( i \) variables need to be represented in every element of \( \text{PNC}(i, j) \).

In the rest of this section we will consider a fixed linear subset \( \mathcal{L} \) of \( B \) such that \( I \in \mathcal{L} \) and \( \mathcal{L} \) generates \( B \) algebraically. We will let \( d \) denote the linear dimension of \( \mathcal{L} \) and consider a fixed basis \( \{ \ell_1, \ell_2, \ldots, \ell_d \} \) for \( \mathcal{L} \). Our aim is to get an estimate of the costs involved in order to use Lemma 2.1. For \( k \in \mathbb{N} \) we will let \( \mathcal{L}_k = \text{span}\{\ell_{i_1} \cdots \ell_{i_k} | i_1, \ldots, i_k \in \{1, \ldots, d\}\} \). Since \( I \in \mathcal{L} \) we have \( \mathcal{L}_{k+1} \supseteq \mathcal{L}_k \) so either \( \mathcal{L}_k = \mathcal{L}_{k+1} \) or \( \dim(\mathcal{L}_{k+1}) \geq \dim(\mathcal{L}_k) + 1 \). If \( \mathcal{L}_k = \mathcal{L}_{k+1} \) then \( \mathcal{L}_k = B \). Hence we get \( B = \mathcal{L}_k \) at least for \( k = \dim(B) - d + 1 \). On the other hand this is a very rough estimate relating the linear dimension of \( \mathcal{L} \) to that of \( B \). In fact in order to apply Lemma 2.1 we do only need to get to the least \( k \) such that \( (\mathcal{L}_k + J) = B \), hence we define:

2.3 Definition. The semi-simple defect \( \text{sd}(\mathcal{L}) \) of \( \mathcal{L} \) is the smallest natural number \( t \) such that

\[
(\mathcal{L}_t + J) = B
\]

The reason why we think it is worth introducing this term is based on an example where the matrices in \( \mathcal{L} \) generate the upper triangular matrices. In this case \( \text{sd}(\mathcal{L}) \leq (d - 1)n \) which for a small \( d \) and a large \( n \) is far from \( \dim(B) - d = n(n+1)/2 - d \). Using these definitions and the remarks made we have the following immediate application of Lemma 2.1.

2.4 Proposition. Let \( b \in B \) then \( b \in J \) if and only if for any \( p \in \text{PNC}(d - 1, \text{sd}(\mathcal{L}) + 1) \):

\[
\text{tr}_n (bp(\ell_2, \cdots, \ell_d)) = 0.
\]

The semi-simple defect satisfies \( \text{sd}(\mathcal{L}) \leq (\dim(B) - \dim(\mathcal{L})) \).

The real content of this proposition is mostly that it calls the attention to the interplay between traces and algebraic properties. As we shall see below a well known result of McCoy follows easily from Proposition 2.4, and in order to provide basic results for the coming sections we formulate and prove some results based on this type of arguments.

2.5 Proposition. Let \( x_1, \ldots, x_s \) be linearly independent matrices in \( M_n(\mathbb{C}) \) and let \( t = \text{sd}(\text{span}(I, x_1, \ldots, x_s)) \), then \( t \leq n^2 - s \) and the set \( \{x_1, \ldots, x_s\} \) can be triangularized simultaneously if and only if for any pair \( i, j \in \{1, \ldots, s\} \) and any \( p \in \text{PNC}(s, t + 1) \)

\[
(\ast) \quad \text{tr}_n ((x_i x_j - x_j x_i)p(x_1, \cdots, x_s)) = 0.
\]

Proof. The condition is clearly satisfied for sets of upper triangular matrices.
Next suppose (∗) is valid and let \( \mathcal{B} \) denote the algebra generated by \( I = I_{M_n(\mathbb{C})} \) and the set \( \{x_1, \cdots, x_s\} \), and let the decomposition from (1) be

\[
\mathcal{B} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus \mathcal{J}.
\]

By Proposition 2.4 we find that \( x_i x_j - x_j x_i \in \mathcal{J} \) so the semi-simple part \( \mathcal{C} \) of \( \mathcal{B} \) is commutative and \( n_1 = n_2 = \cdots = n_s = 1 \). This means that for some suitable basis for \( \mathbb{C}^n \); \( \mathcal{B} \) can be represented as a sub-algebra of the upper triangular matrices. The reason being that \( \mathcal{B} \), as a Lie algebra under the usual commutator product, must be solvable since \( [\mathcal{B}, \mathcal{B}] \subseteq \mathcal{J} \) and \( \mathcal{J}^n = 0 \). An application of Lie’s Theorem [7, Cor. A p. 17] tells that \( \mathcal{B} \) is triangularizable. □

The proposition has an immediate corollary

2.6 Corollary. Let \( t = \text{sd}(I, x_1, \cdots, x_s) \) then \( \{x_1, \cdots, x_s\} \) are simultaneously triangularizable if for any monomial \( x_{i_1} x_{i_2} \cdots x_{i_m} \) with \( m \leq t + 3 \) and any permutation \( \sigma \in \sum_m \)

\[
\text{tr}_n(x_{i_1} x_{i_2} \cdots x_{i_m}) = \text{tr}_n(x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \cdots x_{i_{\sigma(m)}}).
\]

Another immediate corollary is the following

2.7 Corollary [14]. Let \( x, y \) be in \( M_n(\mathbb{C}) \) then \( x \) and \( y \) are simultaneously triangularizable if and only if for any polynomial \( p \) in 2 non-commuting variables \( p(x,y)(xy - yx) \) is nilpotent.

Proof. The trace of a nilpotent element vanishes. □

In the rest of this section we will do some computations for 2 matrices in \( M_2(\mathbb{C}) \) and \( M_3(\mathbb{C}) \) respectively. We get some estimates on how large the semi-simple defect can be in these low dimensional cases.

2.8 Corollary. Let \( x, y \in M_2(\mathbb{C}) \) then \( x \) and \( y \) are simultaneously triangularizable if and only if

\[
\text{tr}_2(x^2 y^2) = \text{tr}_2((xy)^2).
\]

Proof. If \( \dim(\text{span}(I, x, y)) \leq 2 \) then \( x \) and \( y \) commute and hence they are simultaneously triangularizable. If \( \dim(\text{span}(I, x, y)) = 3 \) then \( \text{alg}(I, x, y) = \{p(x, y) \mid p \in \text{PNC}(2, 2)\} \) so the condition in Proposition 2.5 becomes

\[
0 = \text{tr}_2((xy - yx)p(x, y)), p \in \text{PNC}(2, 2).
\]

The only relations which do not vanish automatically are those where \( p \) contains \( xy \) or \( yx \). In both cases we then get the condition \( \text{tr}_2((xy)^2) = \text{tr}_2(x^2 y^2) \).

□

The content of Corollary 2.8 is closely related to the description given by Friedland in [4]. Here it is proven that \( x, y \) in \( M_2(\mathbb{C}) \) are simultaneously triangularizable if and only if

\[
(2 \text{tr}_2(x^2) - \text{tr}_2(x)^2)(2 \text{tr}_2(y^2) - \text{tr}_2(y)^2) = (2 \text{tr}_2(xy) - \text{tr}_2(x) \text{tr}_2(y))^2.
\]
If one applies Cayley-Hamiltons Theorem to \( x^2, y^2 \) and \((xy)^2\) in the relation presented in Corollary 2.8, one can obtain Friedlands relation quite easily.

We will now investigate the case where \( x, y \) are in \( M_3(\mathbb{C}) \). An immediate application of Proposition 2.5 in order to determine whether \( x, y \) are simultaneously triangularizable or not would involve traces of polynomials in \( x, y \) of total degree 9, \((9 = (3^2 - 3) + 3)\). It should however be remarked that the problems we are facing when we have to compute traces of polynomials are reduced considerably by the fact that the trace is invariant under cyclic permutations of monomials. We will take a closer look into this problem for \( n = 3 \) in order to see that degree 6 suffices whereas 5 does not. We start with an example demonstrating that polynomials of degree 5 are not sufficient to settle the question for \( x, y \) in \( M_3(\mathbb{C}) \).

2.9 Example.

Let \( x, y \) in \( M_3(\mathbb{C}) \) be given by

\[
x = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1 + i\sqrt{3}
\end{pmatrix}, \quad y = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Clearly \( \text{alg}(x, y) = M_3(\mathbb{C}) \) so the matrices can not be triangularized simultaneously. On the other hand we have

\[
\forall p \in \text{PNC}(2, 3) : \text{tr}_3((xy - yx)p(x, y)) = 0.
\]

This is easily seen once it is observed that for any monomial \( m = x^{i_1}y^{j_1} \cdots x^{i_k}y^{j_k} \) we have \( \text{tr}_3(m) = 0 \) unless \( j_1 + \cdots + j_k \in 3\mathbb{Z} \).

Hence we only have to evaluate

\[
\text{tr}_3((xy - yx)y^2), \quad \text{tr}_3((xy - yx)xy^2), \quad \text{tr}_3((xy - yx)yxy), \quad \text{tr}_3((xy - yx)y^2x).
\]

The first of these terms vanishes because \( \text{tr}_3(ab) = \text{tr}_3(ba) \). Using this property over and over again, the other expressions are seen to vanish because they can be reduced to

\[
\text{tr}_3(y^2xy - y^3x^2), \quad 0, \quad \text{tr}_3(y^3x^2 - y^2xy),
\]

and a simple computation shows that \( \text{tr}_3(y^3x^2) = \text{tr}_3(y^2xy) \). \( \square \)

2.10 Proposition. Let \( x, y \) be in \( M_3(\mathbb{C}) \) then \( x \) and \( y \) can be triangularized simultaneously if and only if for any monomial \( m = x^{i_1}y^{j_1}x^{i_2}y^{j_2}x^{i_3}y^{j_3} \) of degree 6 or less \( \text{tr}_3(m) = \text{tr}_3(x^{(i_1+i_2+i_3)}y^{(j_1+j_2+j_3)}) \).

Proof. Suppose the condition is satisfied then by Corollary 2.6 it suffices to show that the algebra, say \( \mathcal{B} \), generated by \( \{I, x, y\} \) is spanned by polynomials \( p(x, y) \) of degree 4 or less. Let \( \mathcal{B} = \mathcal{C} \oplus \mathcal{J} \) be the algebraic Wedderburn decomposition (1) where \( \mathcal{C} \) denotes the semi-simple summand. Then \( \mathcal{C} \) can be of one of the forms \( \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C} \), \( \mathcal{C} \oplus M_2(\mathbb{C}) \) or \( M_3(\mathbb{C}) \), so we will have to exclude the latter 2 under
the assumptions made on $x$ and $y$. Let us first suppose $\mathcal{C} = \mathbb{C} \oplus M_2(\mathbb{C})$. Then for suitable elements $\lambda_x, \lambda_y \in \mathbb{C}$, $m_x, m_y \in M_2(\mathbb{C})$ and $j_x, j_y$ in $\mathcal{J}$ we have
\[ x = \lambda_x + m_x + j_x \quad y = \lambda_y + m_y + j_y. \]

Clearly $m_x, m_y$ are not simultaneously triangularizable since they must generate $M_2(\mathbb{C})$. By Corollary 2.8 we have $\text{tr}_3(m_x^2m_y^2) \neq \text{tr}_3((m_xm_y)^2)$ and then
\[ \text{tr}_3(x^2y^2) - \text{tr}_3(xyxy) = \text{tr}_3(m_x^2m_y^2 - (m_xm_y)^2) \neq 0, \]
which contradicts the assumptions made on $x$ and $y$.

We may therefore suppose that $\mathcal{B} = M_3(\mathbb{C})$ and we prove below that under this assumption one has
\[ (**): \quad M_3(\mathbb{C}) = \text{alg}(I, x, y) = \{ p(x, y) \mid p \in \text{PNC}(2, 4) \}. \]

By Corollary 2.6 $x$ and $y$ are then simultaneously triangularizable and the assumption $\mathcal{B} = M_3(\mathbb{C})$ can not be true.

The set $\{I, x, y, x^2, y^2, xy, yx\}$ has 7 elements. If it is linearly independent then since $\mathcal{B} = M_3(\mathbb{C})$ the linear dimension of $\{ p(x, y) \mid p \in \text{PNC}(2, 4) \}$ must be at least 9 and hence 9 so we have (**). On the other hand if the set is not linearly independent we will prove that there exists a linear relation of the type
\[ \lambda_1 + \lambda_2x + \lambda_3y + \lambda_4x^2 + \lambda_5y^2 + \lambda_6xy + \lambda_7yx = 0, \]
where either $\lambda_6 \neq 0$ or $\lambda_7 \neq 0$. Let us postpone the proof of this and see how this can be used to prove the proposition. Suppose for instance $\lambda_7 \neq 0$ and say $\lambda_7 = 1$ then
\[ yx = -\lambda_1 - \lambda_2x - \lambda_3y - \lambda_4x^2 - \lambda_5y^2 - \lambda_6xy, \]
so any polynomial $p(x, y)$ can be written as a sum of products of polynomials in just one variable
\[ p(x, y) = q_0(x) + r_0(y) + \sum_{i=1}^{t} q_i(x)r_i(y). \]

By Cayley-Hamiltons Theorem we then get $p(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2} \alpha_{ij}x^iy^j$ and it turns out that $\mathcal{B} = \{ p(x, y) \mid p \in \text{PNC}(2, 4) \}$, and the argument is completed.

Let us then assume that the set $\{1, x, y, x^2, y^2, xy, yx\}$ is linearly dependent and satisfies a non trivial linear relation as above with $\lambda_6 = \lambda_7 = 0$. If $\lambda_4 = \lambda_5 = 0$ also then $x$ and $y$ commutes so we may and will assume that $\lambda_5 = 1$. let $z = y - x$ then $y = x + z$ and the linear relation becomes - when expressed in $x$ and $z$:
\[ \lambda_1 + (\lambda_2 + \lambda_3)x + \lambda_3z + (1 + \lambda_4)x^2 + z^2 + xz + zx = 0, \]
and we are back in a situation already covered. \[ \square \]
3. Property L and tensor products.

Let \( x, y \) be 2 matrices in \( M_n(\mathbb{C}) \), then according to [12, 15] \( x \) and \( y \) are said to have Property L if there exist sets \((s_1, \cdots, s_n) \) \((t_1, \cdots, t_n)\) of complex numbers such that for \( \lambda, \mu \in \mathbb{C} \) the characteristic roots of \((\lambda x + \mu y)\) is the set \(\{\lambda s_i + \mu t_i \mid 1 \leq i \leq n\} \). Notably O. Taussky [16, 17] and T. Laffey [10,11] have obtained results on such pairs of matrices. Especially sufficient conditions which together with Property L implies that \( x \) and \( y \) are simultaneously triangularizable have been searched for. The following example comes from Wielandt [19] but was used in [15] to show that there exist 2 nilpotent matrices \( x \) and \( y \) in \( M_3(\mathbb{C}) \) such that every element \( \lambda x + \mu y \) in the matrix pencil is nilpotent and \( \text{alg}(x, y) = M_3(\mathbb{C}) \). The matrices \( x \) and \( y \) then do have Property L, but they are not simultaneously triangularizable.

3.1 Example, [15, 19].

Let 
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\] and 
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}
\] then \( \lambda x + \mu y \) is nilpotent for \( \lambda, \mu \in \mathbb{C} \), and \( \text{alg}(x, y) = M_3(\mathbb{C}) \).

3.2 Definition. Let \( S \subseteq M_n(\mathbb{C}) \) be a set of matrices such that for each \( a \in S \) the characteristic roots of \( a \) are equipped with a given numbering \((\lambda_1^a, \cdots, \lambda_n^a)\) and let \( k \in \mathbb{N} \). Then \( S \) is said to have the property \( kL \) if for any set \((a_1, \cdots, a_j)\) from \( S \) and any set \((x_1, \cdots, x_j)\) from \( M_k(\mathbb{C}) \) and \( t \in \mathbb{C} \)
\[
\det(tI_k \otimes I_n) - (x_1 \otimes a_1 + \cdots + x_j \otimes a_j)
\]

\[
= \prod_{i=1}^{n} \det \left( tI_k - (\lambda_i^{a_1} x_1 + \cdots + \lambda_i^{a_j} x_j) \right)
\]

It is clear from the definition that the characteristic roots of the sum \( x_1 \otimes a_1 + \cdots + x_j \otimes a_j \) are grouped as the disjoint union of the sets

characteristic roots of \((\lambda_i^{a_1} x_1 + \cdots + \lambda_i^{a_j} x_j), 1 \leq i \leq n \)

when \( S \) has Property \( kL \). We have chosen \( "kL" \) rather than \( "Lk" \) because \( "Property Lk" \) has been used in [18] to mean the generalization of \( "Property L" \) to sets of \( k \) matrices.

The shift from \( "Property L" \) to \( "Property kL" \) corresponds to a shift in coefficients from scalars to matrices in matrix pencils
\[
\lambda a + \mu b \rightarrow x \otimes a + y \otimes b.
\]

This method has been very fruitful in \( K\)-theory and in many so-called non-commutative theories. Further we think it fits well with some of the early works in matrix theory by Kronecker [5,12]. After all the tensor product was known and used as the Kronecker product in matrix theory long before its general algebraic nature was described. The following lemma needs no written proof, but the observation has to be made.
3.3 Lemma. If a set $S$ in $M_n(\mathbb{C})$ has Property $kL$ then it has Property $(k-1)L$.

The following proposition shows how the logarithm can give an alternative characterization of Property $kL$ via the trace rather than the determinant.

3.4 Proposition. Let $S \subseteq M_n(\mathbb{C})$ be such that for each $a \in S$ there is a given numbering $(\lambda^a_1, \ldots, \lambda^a_n)$ of its characteristic roots. Then $S$ has property $kL$ if and only if for any $j$ in $\mathbb{N}$ and any sets $(a_1, \ldots, a_j)$ from $S$, $(x_1, \ldots, x_j)$ from $M_k(\mathbb{C})$ and $m \in \mathbb{N}$, $1 \leq m \leq nk$:

\[ (*) \quad \text{tr}_{nk}((x_1 \otimes a_1 + \cdots + x_j \otimes a_j)^m) = \sum_{i=1}^{n} \text{tr}_k((\lambda^a_i x_1 + \cdots + \lambda^a_j x_j)^m) \]

If $S$ has property $kL$ then $(*)$ is true for all $m \in \mathbb{N}$.

Proof. Suppose $S$ has property $kL$, then from Definition 3.2 we get for $z \in \mathbb{C}$

\[ \det (I_k \otimes I_n - z(x_1 \otimes a_1 + \cdots + x_j \otimes a_j)) = \prod_{i=1}^{n} \det (I_k - z(\lambda^a_i x_1 + \cdots + \lambda^a_j x_j)) \quad (**) \]

For $z$ near 0, the main branch of the logarithm can be applied on both sides of $(**)$ and we get since $\text{Log} \det(y) = \text{tr}_r \text{Log}(y)$ when $y$ is near $I$ in any $M_r(\mathbb{C})$

\[ \text{tr}_{kn} \left( \text{Log} (I_k \otimes I_n - z(x_1 \otimes a_1 + \cdots + x_j \otimes a_j)) \right) = \sum_{i=1}^{n} \text{tr}_k \left( \text{Log} (I_k - z(\lambda^a_i x_1 + \cdots + \lambda^a_j x_j)) \right) \]

Both sides of this equation can be expressed as power series. A comparison of terms yields that $(*)$ holds for all $m \in \mathbb{N}$.

Now suppose $(*)$ is valid for all natural numbers $m$ less than or equal to $nk$. Then choose diagonal matrices $d_1, \ldots, d_j$ in $M_n(\mathbb{C})$ such that

\[ d_\ell = \Delta(\lambda^a_1, \lambda^a_2, \ldots, \lambda^a_n) \]

Clearly the set $(d_1, \ldots, d_j)$ has Property $kL$ so for any $m \in \mathbb{N}$

\[ \text{tr}_{nk}((x_1 \otimes d_1 + \cdots + x_j \otimes d_j)^m) = \sum_{i=1}^{n} \text{tr}_k((\lambda^a_i x_1 + \cdots + \lambda^a_j x_j)^m) \]

Hence in order to prove that $(*)$ holds for all $m$ we just have to note that by assumption and the computations just made:

for $1 \leq m \leq nk$:

\[ \text{tr}_{nk}((x_1 \otimes a_1 + \cdots + x_j \otimes a_j)^m) = \text{tr}_{nk}((x_1 \otimes d_1 + \cdots + x_j \otimes d_j)^m) \].
This means by Newtons formulae that the characteristic polynomial for \( x_1 \otimes a_1 + \cdots + x_j \otimes a_j \) and \( x_1 \otimes d_1 + \cdots + x_j \otimes d_j \) are identical and hence that for any \( m \) in \( \mathbb{N} \)
\[
\text{tr}_{nk} ((x_1 \otimes a_1 + \cdots + x_j \otimes a_j)^m) = \text{tr}_{nk} ((x_1 \otimes d_1 + \cdots + x_j \otimes d_j)^m) \\
= \sum_{i=1}^{n} \text{tr}_k ((\lambda_i^{a_1} x_1 + \cdots + \lambda_i^{a_j} x_j)^m).
\]

On the other hand if all equations of the type \((*)\) are true for all \( m \in \mathbb{N} \) it follows that \((***)\) holds in some ball around zero and Property \( kL \) is established. \( \square \)

The following lemma is included in order to prepare an application of results from Section 2.

3.5 Lemma. If a set \( S \subseteq M_n(\mathbb{C}) \) has a property \( kL \) then so does \( S \cup I_{M_n(\mathbb{C})} \).

Proof. Let \( a_1, \ldots, a_j \) be in \( S \), \( x_0, x_1, \ldots, x_j \) be in \( M_k(\mathbb{C}) \) and \( z \in \mathbb{C} \) then for \( z \in B(0, r) \) for some \( r > 0 \) we have \( (I_k - z x_0) \) is invertible in \( M_k(\mathbb{C}) \) so
\[
\det (I_k \otimes I_n - (z x_0 \otimes I_n + x_1 \otimes a_1 + \cdots + x_j \otimes a_j))
= \det ((I_k - z x_0) \otimes I_n) \det (I_k \otimes I_n - ((I_k - z x_0)^{-1} x_1 \otimes a_1 + \cdots + (I_k - z x_0)^{-1} x_j \otimes a_j))
= \det(I_k - z x_0)^n \prod_{i=1}^{n} \det (I_k - (I_k - z x_0)^{-1} (\lambda_i^{a_1} x_1 + \cdots + \lambda_i^{a_j} x_j))
= \prod_{i=1}^{n} \det (I_k - (z x_0 + \lambda_i^{a_1} x_1 + \cdots + \lambda_i^{a_j} x_j)).
\]
Hence this identity is true for \( z = 1 \) as well and the lemma follows. \( \square \)

We can now state the main result of this section

3.6 Theorem. Let \( S \subseteq M_n(\mathbb{C}) \) be a set and \( k = \text{sd} (\text{span}(S \cup I)) + 3 \). The matrices in \( S \) are simultaneously triangularizable if and only if \( S \) has Property \( kL \). Let \( s = \dim (\text{span}(S)) \) then \( k \leq n^2 - s + 3 \).

Proof. It is obvious that Property \( kL \) is necessary, so let us assume that \( S \) has Property \( kL \) then by Lemma 3.5 \( \{S \cup I\} \) has Property \( kL \). According to Corollary 2.6 it suffices to prove that for any monomial \( a_1 \cdots a_m \) of degree \( k \) or less of elements from \( S \) and any permutation \( \sigma \in \sum_m \) we have
\[
\text{tr}_n (a_1 \cdots a_m) = \text{tr}_n (a_{\sigma(1)} \cdots a_{\sigma(m)}).
\]
Since \( I \in \{S \cup I\} \) we can replace this statement with the following. The set \( S \) is simultaneously triangularizable if for any set \( a_1, \ldots, a_k \) from \( \{S \cup I\} \) and any \( \sigma \) in \( \sum_k \)
\[
\text{tr}_n (a_1, \cdots a_k) = \text{tr}_n (a_{\sigma(1)} \cdots a_{\sigma(k)}).
\]
In order to see that Property $kL$ implies the statement above we let $(e_{ij})$ denote the matrix-units in $M_k(\mathbb{C})$, $\sigma$ a permutation in $\sum_k$ and let $a_1, \ldots, a_k$ be elements from $\{S \cup I\}$. Then we define

$$u = e_{12} \otimes a_1 + e_{23} \otimes a_2 + \cdots + e_{(k-1)k} \otimes a_{k-1} + e_{k1} \otimes a_k$$
$$u^\sigma = e_{12} \otimes a_{\sigma(1)} + e_{23} \otimes a_{\sigma(2)} + \cdots + e_{k1} \otimes a_{\sigma(k)}.$$

A computation – which uses that the trace is invariant under cyclic permutations of products – shows that

$$\text{tr}_nk(u^k) = k \text{tr}_n(a_1a_2 \cdots a_k)$$
$$\text{tr}_nk((u^\sigma)^k) = k \text{tr}_n(a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(k)}).$$

Analogously we define for $1 \leq i \leq n, v_i, v_i^\sigma$ in $M_k(\mathbb{C})$ by

$$v_i = \lambda_i^{a_1} e_{12} + \lambda_i^{a_2} e_{23} + \cdots + \lambda_i^{a_k} e_{k1}$$
$$v_i^\sigma = \lambda_i^{a_{\sigma(1)}} e_{12} + \lambda_i^{a_{\sigma(2)}} e_{23} + \cdots + \lambda_i^{a_{\sigma(k)}} e_{k1}$$

and we find as above that

$$\text{tr}_k(v_i^k) = k \lambda_i^{a_1} \cdots \lambda_i^{a_k} = \text{tr}_k((v_i^\sigma)^k).$$

Since $\{S \cup I\}$ has Property $kL$ we then get

$$\text{tr}_n(a_1a_2 \cdots a_k) = 1 \frac{1}{k} \text{tr}_nk(u^k)$$
$$= \frac{1}{k} \sum_{i=1}^n \text{tr}_k(v_i^k)$$
$$= \frac{1}{k} \sum_{i=1}^n \text{tr}_k((v_i^\sigma)^k)$$
$$= \frac{1}{k} \text{tr}_nk((u^\sigma)^k)$$
$$= \text{tr}_n(a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(k)}),$$

and the theorem follows. \qed

4. Invertibility preserving mappings.

In an earlier article [3] we have obtained a result on continuous linear mappings from a Banach algebra into $M_n(\mathbb{C})$ which preserves invertibility. The result is a generalization of the Gleason-Kahane-Żelazko Theorem [6,8]. In the situation here, we can quote [3] in the following way.
4.1 **Theorem** ([3]). *Let $A \subseteq M_k(\mathbb{C})$ be a unital algebra and $\varphi: A \to M_n(\mathbb{C})$ a linear mapping satisfying $\varphi(I_A) = I_{M_n(\mathbb{C})}$. Let $A_{\text{inv}}$ denote the set of invertibles in $A$ then $\varphi(A_{\text{inv}}) \subseteq \text{GL}_n(\mathbb{C})$ if and only if for any $a$ in $A$ and any $m$ in $\mathbb{N}$

$$\text{tr}_n(\varphi(a^m)) = \text{tr}_n(\varphi(a)^m).$$

There is an immediate corollary which is quite useful.

4.2 **Corollary** ([3]). *If $\varphi(A_{\text{inv}}) \subseteq \text{GL}_n(\mathbb{C})$ then for any $a, b$ in $A$ and any $k$ in $\mathbb{N}$

(i) $\text{tr}_n(\varphi(ab)) = \text{tr}_n(\varphi(a)\varphi(b)) = \text{tr}_n(\varphi(ba))$.

(ii) $\text{tr}_n(\varphi(a)^k\varphi(b)) = \text{tr}_n(\varphi(a^k)b) = \text{tr}_n(\varphi(a^k)\varphi(b))$

(iii) $\det(\varphi(a)\varphi(b)) = \det(\varphi(ab))$.*

*Proof.* The relation (ii) is not stated explicitly in [3], but follows easily from Theorem 4.1. Let $z \in \mathbb{C}$ and $x = (a + zb)$ then by Theorem 4.1

$$\text{tr}_n(\varphi(x^{k+1})) = \text{tr}_n(\varphi(x)^{k+1}).$$

The relation (i) and the trace properties for $\text{tr}_n$ shows that for the coefficient to $z$ in the identity right above we get

$$k \text{tr}_n(\varphi(a^k)b) = k \text{tr}_n(\varphi(a)^k\varphi(b)).$$

An application of (i) once more gives (ii). □

One of the main motivations to look into the problem of trying to describe invertibility preserving linear mappings was the set of lecture notes [9] where Kaplansky addresses this problem. As mentioned in the introduction Kaplansky had the impression that the – at the time of the notes – quite recent results [6,8] by Gleason and Kahane & Żelazko might be generalized. Of the various articles we have found in this area of research especially the work by Aupetit [1] has been fruitful to us. The content in this article as well as the one in [3] is very much influenced by [1]. As mentioned in the introduction Kaplansky was a bit too optimistic in hoping that invertibility preserving mappings should be Jordan homomorphisms. The following examples demonstrate what sort of obstacles we have found.

4.3 **Examples.**

**A:** *Let $A$ be the diagonal matrices in $M_3(\mathbb{C})$, $T$ be the upper triangular matrices in $M_3(\mathbb{C})$ and let $\varphi: A \to T$ be given by

$$\varphi(\Delta(a, b, c)) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & a - b \\ 0 & 0 & c \end{pmatrix}.$$*
This \( \phi \) is unital and invertibility preserving but \( \phi(\Delta(1,0,0)^2) = \phi(\Delta(1,0,0)) \neq \phi(\Delta(1,0,0))^2 \). On the other hand \( \phi \) is a homomorphism modulo the radical.

**B:** Let \( A \) be the diagonal matrices in \( M_3(\mathbb{C}) \) and \( \phi : A \to M_3(\mathbb{C}) \) be given by

\[
\phi(\Delta(a, b, c)) = \begin{pmatrix}
a & (c-a) & 0 \\
(b-a) & a & (a-c) \\
0 & (b-a) & a
\end{pmatrix},
\]

then \( \det(\phi(\Delta(a, b, c))) = a^3 \) and \( \varphi(I) = I \), but \( \phi \) is not a homomorphism and by Example 3.1 the algebra generated by \( \varphi(A) \) equals \( M_3(\mathbb{C}) \) which is semi-simple and then has no radical. Hence not even modulo the radical do we get a homomorphism.

**C:** Let \( A = M_2(\mathbb{C}) \) and \( \phi : A \to M_6(\mathbb{C}) \) be given by

\[
\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aI_3 & bI_3 \\ cI_3 & \begin{pmatrix} d & b & 0 \\ a-d & d & -b \\ 0 & a-d & d \end{pmatrix} \end{pmatrix}.
\]

Clearly by Example 3.1 for \( x \in M_2(\mathbb{C}) \) \( \det(\phi(x)) = (\det(x))^3 \) and \( \varphi(I_2) = I_6 \). The algebra generated by \( \phi(A) \) equals \( M_6(\mathbb{C}) \), so the Jacobson radical vanishes and \( \varphi \) can not be a Jordan homomorphism modulo the radical.

We will now turn to some positive results. The first is closely related to Aubert's result [1].

**4.4 Theorem.** Let \( A \subseteq M_n(\mathbb{C}) \) be a unital algebra and \( \phi : A \to M_n(\mathbb{C}) \) a unital invertibility preserving mapping. If \( \varphi(A) \) is an algebra then \( \phi \) is a Jordan homomorphism modulo the Jacobson radical.

**Proof.** Let \( x \) be in \( B = \text{alg} \varphi(A) = \varphi(A) \) and \( a \) in \( A \) then for \( k \in \mathbb{N} \) we get from Corollary 4.2 (ii), and the assumptions made that there exists \( b \) in \( A \) such that \( x = \varphi(b) \) and hence

\[
\text{tr}_n \left( (\varphi(a)^k - \varphi(a^k))x \right) = \text{tr}_n \left( \varphi((a^k - a^k)b) \right) = 0.
\]

By Lemma 2.1 \( \varphi(a)^k - \varphi(a^k) \) belongs to the Jacobson radical so \( \varphi \) is a Jordan homomorphism modulo the Jacobson radical. \( \Box \)

Analogously to the results in section 3 we can obtain sufficient conditions if we demand that \( \varphi \otimes \text{id} \) on \( A \otimes M_k(\mathbb{C}) \to M_{nk}(\mathbb{C}) \) is invertibility preserving, hence we define.

**4.5 Definition.** Let \( A \) be a unital algebra in \( M_n(\mathbb{C}) \) and \( \varphi : A \to M_n(\mathbb{C}) \) a unital invertibility preserving linear mapping. For \( k \) in \( \mathbb{N} \), \( \varphi \) is said to be \( k \)-invertibility preserving if \( \varphi \otimes \text{id}_k : A \otimes M_k \to M_n \otimes M_k \) preserves invertibility of elements.
4.6 Theorem. Let $A$ be a unital algebra in $M_h(\mathbb{C})$, $\varphi : A \to M_n(\mathbb{C})$ a unital linear mapping, $B$ the algebra generated by $\varphi(A)$, $t = \text{sd}(\varphi(A))$ and $k = t + 3$. The mapping $\varphi$ is $k$-invertibility preserving if and only if $\varphi$ is a homomorphism modulo the Jacobson radical. The semi-simple defect $t$ of $\varphi(A)$ satisfies $t \leq \dim(B) - \dim(\varphi(A))$.

Proof. If $\varphi$ is a homomorphism modulo the radical then so is $\varphi \otimes \text{id}_m$ for any natural number $m$, and $\varphi$ is $m$-invertibility preserving for all $m$.

Let us now assume that $\varphi$ is $k$-invertibility preserving and let $u, v$ be in $A$. In order to prove that $x = (\varphi(uv) - \varphi(u)\varphi(v))$ belongs to Jacobson radical $J$ of $B$ we have by Proposition 2.3 to prove that for any $m \leq k - 2$ and any set $a_1, \ldots, a_m$ from $A$ we have

$$\text{tr}_n(x\varphi(a_1)\cdots\varphi(a_m)) = 0.$$  

Since $I_n = \varphi(I_A)$ it is sufficient to prove that $\text{tr}_n(x\varphi(a_1)\cdots\varphi(a_{k-2})) = 0$ for all sets $a_1, \ldots, a_{k-2}$ from $A$.

We start by proving that for any set $a_1, \ldots, a_k$ from $A$ we have

$$\text{tr}(\varphi(a_1)\cdots\varphi(a_k)) = \text{tr}(\varphi(a_1\cdots a_k))$$

in order for do so we let $(e_{ij})$ denote a set of matrix units in $M_k(\mathbb{C})$ and define

$$u = a_1 \otimes e_{12} + a_2 \otimes e_{23} + \cdots + a_k \otimes e_{k1} \in A \otimes M_k(\mathbb{C}).$$

In analogy with the computations made in the proof of Theorem 3.6 we get via Corollary 4.2 (i) applied to $\varphi$

$$\text{tr}_n(\varphi(a_1 \cdots a_k)) = \frac{1}{k} \text{tr}_n \left( (\varphi \otimes \text{id}_k)(u^k) \right)$$

then since $\varphi$ is $k$ invertibility preserving we get from Theorem 4.1 that

$$\frac{1}{k} \text{tr}_n \left( (\varphi \otimes \text{id}_k)(u^k) \right) = \frac{1}{k} \text{tr}_n \left( ((\varphi \otimes \text{id}_k)(u))^k \right)$$

but the arguments from the proof of Theorem 3.6 applies again and

$$\frac{1}{k} \text{tr}_n \left( ((\varphi \otimes \text{id}_k)(u))^k \right) = \text{tr}_n(\varphi(a_1)\cdots\varphi(a_k)).$$

Hence for any set $a_1, \ldots, a_{k-2}$ we get

$$\text{tr}_n(x\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{k-2}))$$

$$= \text{tr}_n((\varphi(uv)\varphi(I) - \varphi(u)\varphi(v))\varphi(a_2)\cdots\varphi(a_{k-2}))$$

$$= \text{tr}_n(\varphi((uvI - uv)a_2\cdots a_{k-2}))$$

$$= 0,$$

and the theorem follows. \qed
4.7 Remark.

It is well known that transposition on $M_n(\mathbb{C})$ is an anti-automorphism, hence it follows from the theorem that transposition can not be $k$-invertibility preserving for $k \geq 3$. On the other hand the following example for $n = 2$ can be used for all $n \geq 2$ to show that transposition on $M_n(\mathbb{C})$ is never 2-invertibility preserving. Let $x \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ be given by $x = e_{11} \otimes e_{11} + e_{21} \otimes e_{12} + e_{12} \otimes e_{21} + e_{22} \otimes e_{22}$ then $x$ is invertible and for $\varphi$ as transposition we get $\varphi_2(x) = e_{11} \otimes e_{11} + e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{22} \otimes e_{22}$ which is not invertible.

In the proof above we can manage products of length $k$ made by elements from $\varphi(\mathcal{A})$. It is expected that compensation in the size of products manageable when involving $k \times k$ matrices ought to grow like $k^2$ rather than linearly in $k$. For $k = 2$ it is possible to do much better as the following proposition shows. We know that Proposition 4.8 has a generalization to matrices of arbitrary size, but we have not been able to find a nice description of a general result.

4.8 Proposition. Let $\mathcal{A} \subseteq M_n(\mathbb{C})$ and $\varphi : \mathcal{A} \rightarrow M_n(\mathbb{C})$ be a unital linear map. If $\varphi$ is 2-invertibility preserving then for all $\{a, b, c, d\} \subseteq \mathcal{A}$ and all $i, j \in \mathbb{N}$

(i) $\text{tr}_n(\varphi(ab'cd')) = \text{tr}_n((\varphi(a)\varphi(b))^i\varphi(c)\varphi(d))^j$

(ii) $\text{tr}_n(\varphi(ab)^j) = \text{tr}_n((\varphi(a)\varphi(b))^j)$.

Proof. For $s, t \in \mathbb{C}$, $u \in \mathbb{C} \setminus \{0\}$ and $a, b, c, d$ in $\mathcal{A}$ we define $e = u^{-1}(I - td)$ in $\mathcal{A}$ and $x$ in $\mathcal{A} \otimes M_2(\mathbb{C})$ by

$$x = \begin{pmatrix} 1 - sb & c \\ a & e \end{pmatrix}.$$ 

Let $s$ be chosen in a ball $B(0, r)$ such that $(1 - sb)$ is invertible for every $s$ from this ball. We can then define a 2 invertibility preserving unital map $\psi : \mathcal{A} \rightarrow M_n(\mathbb{C})$ by

$$y \in \mathcal{A} : \quad \psi(y) = \varphi(y(1 - sb)) \varphi(1 - sb)^{-1}.$$ 

The following identity is straight forward and verifies that $\psi$ is 2-invertibility preserving since for $\psi_2 = \psi \otimes \text{id}_{M_2(\mathbb{C})}$ we get

$$\det(\varphi_2(x)) = \det(I - s\varphi(b))^2\det(\psi_2(\begin{pmatrix} 1 & c(1 - sb)^{-1} \\ a(1 - sb)^{-1} & e(1 - sb)^{-1} \end{pmatrix})).$$

Let $w \in \mathcal{A} \otimes M_2(\mathbb{C})$ be given by

$$w = \begin{pmatrix} 1 & 0 \\ -a(1 - sb)^{-1} & 1 \end{pmatrix},$$

then by Corollary 4.2 (iii) applied to multiplication by $\psi_2(w)$ from the left gives

$$\det(\varphi_2(x)) = \det(I - s\varphi(b))^2\det(\psi_2(\begin{pmatrix} 1 & c(1 - sb)^{-1} \\ 0 & (e - a(1 - sb)^{-1}c)(1 - sb)^{-1} \end{pmatrix}))) = \det(1 - s\varphi(b))\det(\varphi(e - a(1 - sb)^{-1}c)).$$
The same type of row operations applied to $\varphi_2(x)$ yields

\[
\det (\varphi_2(x)) = \det \left( \begin{pmatrix}
1 - s\varphi(b) & \varphi(c) \\
0 & \varphi(e) - \varphi(a)(I - s\varphi(b))^{-1}\varphi(c)
\end{pmatrix} \right),
\]

and therefore

\[
\det \left( u^{-1}(I - t\varphi(d)) - \varphi(a(1 - sb)^{-1}c) \right) = \det \left( u^{-1}(I - t\varphi(d)) - \varphi(a)(I - s\varphi(b))^{-1}\varphi(c) \right).
\]

So

\[
\det \left( I - u\varphi(a(1 - sb)^{-1}c)(1 - t\varphi(d))^{-1} \right) = \det \left( I - u\varphi(a)(I - s\varphi(b))^{-1}\varphi(c)(I - t\varphi(d))^{-1} \right).
\]

This identity can be extended to $u = 0$ and we find by differentiation and evaluation at $u = 0$ that

\[
\text{tr}_n \left( \varphi(a(1 - sb)^{-1}c)(1 - t\varphi(d))^{-1} \right) = \text{tr}_n \left( \varphi(a)(I - s\varphi(b))^{-1}\varphi(c)(I - t\varphi(d))^{-1} \right).
\]

The relation (i) now follows from Corollary 4.2 (ii) and the Neumann series for $(I - x)^{-1}$ applied 4 times.

When computing coefficients for higher powers of $u$ one can see that much more is true, but the combinatorics becomes very complicated.

The relation (ii) is a lot easier to prove, let

\[
x = \begin{pmatrix}
0 & a \\
b & 0
\end{pmatrix},
\]

then

\[
\varphi_2(x^{2i}) = \begin{pmatrix}
\varphi((ab)^i) & 0 \\
0 & \varphi((ba)^i)
\end{pmatrix},
\]

\[
\varphi_2(x)^{2i} = \begin{pmatrix}
(\varphi(a)\varphi(b))^{i} & 0 \\
0 & (\varphi(b)\varphi(a))^{i}
\end{pmatrix},
\]

but

\[
\text{tr}_n \left( (\varphi(a)\varphi(b))^{i} \right) = \text{tr}_n \left( \varphi(a)(\varphi(b)\varphi(a))^{i-1}\varphi(b) \right) = \text{tr}_n \left( (\varphi(b)\varphi(a))^{i} \right)
\]
and

\[ \text{tr}_n \left( \varphi((ab)^i) \right) = \text{tr}_n \left( \varphi(a) \varphi((ba)^{i-1} b) \right) = \text{tr}_n \left( \varphi((ba)^i) \right), \]

so

\[ \text{tr}_n \left( (\varphi(a) \varphi(b))^i \right) = \frac{1}{2} \text{tr}_n \left( \varphi_2(x)^{2i} \right) = \frac{1}{2} \text{tr}_n \left( \varphi_2(x^{2i}) \right) = \text{tr}_n \left( \varphi(ab)^i \right) \]

and the proposition follows. \( \square \)

**4.9 Corollary.** Let \( \varphi \) be a linear unital mapping of a unital matrix algebra \( A \) onto a matrix algebra \( B \) contained in \( M_n(\mathbb{C}) \).

If \( \varphi \) is 2-invertibility preserving, then it is a homomorphism modulo the Jacobson radical.

**Proof.** By Proposition 4.8 we have for \( a, b, c \in A \)

\[ \text{tr}_n \left( (\varphi(a) \varphi(b) - \varphi(ab)) \varphi(c) \right) = \text{tr}_n \left( \varphi(abc) - \varphi(abc) \right) = 0. \]

Since \( \varphi \) is surjective the corollary follows from Lemma 2.1. \( \square \)

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