DEFORMED WOODS-SAXON POTENTIAL IN THE FRAME OF
SUPERSYMMETRIC QUANTUM MECHANICS FOR ANY \( l \)-STATE

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Abstract

A novel analytically solvable deformed Woods–Saxon potential is investigated by means of
the Supersymmetric Quantum Mechanics. The Hamiltonian hierarchy method and the shape
invariance property are used in the calculations. The energy levels are obtained for any \( l \)-state.
The interrelations for some nuclear scattering processes are also discussed.

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1 Introduction

Supersymmetric quantum mechanics (SQM for short) is a framework used to determine energy eigenvalues and eigenfunctions of quantum mechanical problems. SQM appeared 25 years ago and has so far been considered as a new field of research, providing not only a supersymmetric interpretation of the Schrödinger equation, but important results to a variety of non-relativistic quantum mechanical problems [1]. One of the most important works is that Gendenshtein introduced a discrete parametrization invariance labelled “shape invariance” [2]. For the shape invariance potentials, their exact energy levels can be found analytically by making use of the supersymmetry shape invariance approach [3, 4, 5].

The association of the factorization and the hierarchy of the Hamiltonian method with SQM formalism has been introduced to obtain the approximate energy spectra of non-exactly solvable potentials, [6, 7] as well as the partially solvable, [8, 9], the isospectral, [10], the periodic potentials, [11] and the exponential-type potentials [12, 13, 14, 15]. Using the physical arguments, it is possible to make an ansatz for the superpotential which satisfies the Riccati equation by an effective potential. Thus, one can obtain a solution of the Riccati differential equation for the superpotential function. This new scheme can be successfully applied to obtain the spectra of the nuclear systems well fit by the deformed Woods-Saxon potentials.

In this work, a study of such new exactly solvable effective potential through SQM is presented by the hierarchy of the Hamiltonian method for $\ell = 0$ and $\ell \neq 0$ cases. We first introduce SQM briefly, and then show the deformed Woods-Saxon potential hierarchy and its shape invariance property for the $\ell = 0$ states. The deformed Woods-Saxon potential hierarchy is recovered with certain potential constants. The results obtained are improved by means of the shape invariance for the $\ell \neq 0$ states. In this case, the effective potential behaves in the same way as centrifugal part of the Woods-Saxon potential under the condition of the first approximation. Finally, this approximation is applied to the nuclear scattering process.

2 Supersymmetric Quantum Mechanics

The simplest way of generating a new exactly solvable Hamiltonian from a known one is just to consider an invertible bounded operator. In this case, a given Hamiltonian and its supersymmetric partner have identical spectra except the zero energy of ground state. In Supersymmetric Quantum Mechanics (SQM) for $N = 2$ we have two nilpotent operators, $Q$ and $Q^+$, satisfying the algebra

$$\{Q, Q^+\} = H_S, \quad \{Q, Q\} = \{Q^+, Q^+\} = 0,$$  \hspace{1cm} (1)
where $H_S$ is the supersymmetric Hamiltonian. The fact that the supercharges $Q$ and $Q^+$ commute with $H_S$ is responsible for the degeneracy. This algebra can be realized as

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$$

where $A^\pm$ are bosonic operators. With this realization the supersymmetric Hamiltonian $H_S$ is given by

$$H_S = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix}.$$ (3)

$H^\pm$ are called supersymmetric partner Hamiltonians and share the same spectra, apart from the nondegenerate ground state, (see [1] for a review),

$$E_n^{(+)} = E_{n+1}^{(-)}.$$ (4)

For the non-spontaneously broken supersymmetry this lowest level is of zero energy, $E_1^{(-)} = 0$. We have

$$H^\pm = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_{\pm}(r) = A^\pm A^\mp$$

where $V_{\pm}(r)$ are called partner potentials. The operators $A^\pm$ are defined in terms of the superpotential $W(r)$,

$$A^\pm = \mp \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + W(r)$$

which satisfies the Riccati equation

$$W^2 \pm \frac{\hbar}{\sqrt{2m}} W' = V_{\pm}(r)$$

as a consequence of the factorization of the Hamiltonians $H^\pm$.

By definition, two partner potentials are called shape invariant if they have the same functional form, differing only by change of parameters, including an additive constant. In this case the partner potentials satisfy

$$V_+(r, a_1) = V_-(r, a_2) + R(a_2),$$

where $a_1$ and $a_2$ denote a set of parameters, with $a_2$ being a function of $a_1$,

$$a_2 = f(a_1)$$

and $R(a_2)$ is independent of $r$.

Through the super-algebra, for a given Hamiltonian $H_1$, factorized in terms of the bosonic operators, it is possible to construct its hierarchy of Hamiltonians. For the general spontaneously broken supersymmetric case we have

$$H_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dv^2} + V_1(r) = A_1^+ A_1^- + E_0^{(1)}$$

(10)
where $E_0^{(1)}$ is the lowest eigenvalue.

The bosonic operators are defined by (6) whereas the superpotential $W_1(r)$ satisfies the Riccati equation

$$W_1^2 - \frac{\hbar}{\sqrt{2m}} W_1' = V_1(r) - E_0^{(1)}.$$  \hfill (11)

The eigenfunction for the lowest state is related to the superpotential $W_1$ by

$$\Psi_0^{(1)}(r) = N \exp \left(-\frac{\sqrt{2m}}{\hbar} \int_0^r W_1(\bar{r}) d\bar{r} \right).$$  \hfill (12)

The supersymmetric partner Hamiltonian is given by

$$H_2 = A_1^+ A_1^- + E_0^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \left(W_1^2 + \frac{\hbar}{\sqrt{2m}} W_1'\right) + E_0^{(1)}.$$  \hfill (13)

Thus, factorizing $H_2$ in terms of a new pair of bosonic operators, $A_2^\pm$ we get

$$H_2 = A_2^+ A_2^- + E_0^{(2)} = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \left(W_2^2 - \frac{\hbar}{\sqrt{2m}} W_2'\right) + E_0^{(2)},$$  \hfill (14)

where $E_0^{(2)}$ is the lowest eigenvalue of $H_2$ and $W_2$ satisfy the Riccati equation,

$$W_2^2 - \frac{\hbar}{\sqrt{2m}} W_2' = V_2(r) - E_0^{(2)}. $$  \hfill (15)

Thus a whole hierarchy of Hamiltonians can be constructed, with simple relations connecting the eigenvalues and eigenfunctions of the $n$-members \[1\]

$$H_n = A_n^+ A_n^- + E_0^{(n)},$$  \hfill (16)

$$A_n^\pm = \pm \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + W_n(r),$$  \hfill (17)

$$\Psi_n^{(1)} = A_1^+ A_2^+ ... A_n^+ \psi_0^{(n+1)}, \quad E_n^{(1)} = E_0^{(n+1)},$$  \hfill (18)

where $\Psi_0^{(1)}(r)$ is given by (12).

### 3 Deformed Woods-Saxon Potential

We consider the following potential which is a generalization of the deformed Woods-Saxon potential

$$V_1(r) = V(r) = -\frac{V_0 e^{-\left(\frac{r-R_0}{a}\right)}}{1 + q e^{-\left(\frac{r-R_0}{a}\right)}} + \frac{C e^{-2 \left(\frac{r-R_0}{a}\right)}}{\left(1 + q e^{-\left(\frac{r-R_0}{a}\right)}\right)^2}. $$  \hfill (19)

Here $r$ denotes the center-of-mass distance between the projectile nucleus and the target nucleus. Other parameters in the potential, $q$ is the deformation or range parameter ($q > 0$), $R_0 = r_0 A^{1/3}$ is the radius of the corresponding spherical nucleus or the width of the potential, $A$ is the target mass number, $r_0$ is the radius parameter, $V_0$ is the the potential depth, $a$ is the
diffuseness of the nuclear surface and lastly $C$ is the setting parameter which is proposed by us. Therefore, we can construct the hierarchy of Hamiltonians for the original Schrödinger equation

$$-\frac{h^2}{2m}\nabla^2\Psi_{nlm}(\mathbf{r}) + V(r)\Psi_{nlm}(\mathbf{r}) = E\Psi_{nlm}(\mathbf{r}), \quad (20)$$

and find a solution of Eq.(20) by separating variables in spherical coordinates, putting

$$\Psi_{nlm}(\mathbf{r}) = \frac{1}{r}\chi_{nl}(r)Y_{lm}(\theta, \phi). \quad (21)$$

Then, we get the radial Schrödinger equation for all of the angular momentum states

$$-\frac{h^2}{2m}\frac{d^2}{dr^2}\chi_{nl}(r) + \left(V(r) + \frac{(l+1)h^2}{2mr^2}\right)\chi_{nl}(r) = E\chi_{nl}(r). \quad (22)$$

### 3.1 Solution for the $l = 0$ case

For a generalization of the deformed Woods-Saxon potential given by Eq.(19), it is substituted into the Schrödinger equation for the zero angular momentum states,

$$-\frac{h^2}{2m}\frac{d^2}{dr^2}\chi_{n}(r) + V(r)\chi_{n}(r) = E\chi_{n}(r), \quad (23)$$

The ground state eigenfunction $\chi_0(r)$ can be written as

$$\chi_0(r) = N\exp\left(-\frac{\sqrt{2m}}{h}\int W_1(r)dr\right), \quad (24)$$

where $N$ is the normalized constant. Substituting Eq.(24) into Eq.(23), we obtain

$$W_1^2 - \frac{h}{\sqrt{2m}}\frac{dW_1}{dr} = V(r) - E^{(1)}_0, \quad (25)$$

where $E^{(1)}_0$ is the lowest energy-eigenvalue or the ground state energy. Through the superalgebra we take superpotential

$$W_1 = -\frac{h}{\sqrt{2m}}\left(S_1 + S_2\frac{e^{-\alpha(r-R_0)}}{1 + qe^{-\alpha(r-R_0)}}\right), \quad (26)$$

which satisfies the associated Riccati equation (Eq.(11)) and substituting this expression into Eq.(25), we find the following identity

$$\frac{\hbar^2 S_1^2}{2m} + \frac{\hbar^2(2S_1S_2 - \alpha S_2)}{2m(q + e^{\alpha(r-R_0)})} + \frac{\hbar^2(S_2^2 + \alpha q S_2)}{2m(q + e^{\alpha(r-R_0)})^2} = V_1 - E^{(1)}_0 \quad (27)$$

$$= -\frac{V_0}{q + e^{\alpha(r-R_0)}} + \frac{C}{(q + e^{\alpha(r-R_0)})^2} - E^{(1)}_0.$$

With the comparison of each side of Eq.(27), we obtain

$$\alpha = 1/a, \quad \frac{\hbar^2 S_1^2}{2m} = -E^{(1)}_0, \quad \frac{\hbar^2}{2m}(2S_1S_2 - \alpha S_2) = -V_0, \quad \frac{\hbar^2}{2m}(S_2^2 + \alpha q S_2) = C. \quad (28)$$
The eigenfunction $\chi(r)$ for the ground state can be expressed as

$$\chi_0(r) = N \exp \left[ \int \left( S_1 + \frac{S_2 e^{-\alpha(r-R_0)}}{1 + q e^{-\alpha(r-R_0)}} \right) dr \right],$$

$$= N e^{S_1r} \left( \frac{e^{\alpha(r-R_0)}}{e^{\alpha(r-R_0)} + q} \right)^{S_2/q}.$$  \hspace{1cm} (29)

Solving Eq.(28) yields

$$S_1 = \frac{2m}{\hbar^2} \left(-V_0 + \frac{C}{q}\right) - \frac{1}{2S_2} - \frac{S_2}{2q},$$

$$S_2 = -\frac{\alpha q}{2} \pm \sqrt{\left(\frac{\alpha q}{2}\right)^2 + \frac{2mC}{\hbar^2}}.$$  \hspace{1cm} (30)

At this point, using Eqs.(26) and (30), the supersymmetric partner potentials can be expressed as

$$V_+(r) = \frac{\hbar^2}{2m} \left[ S_1^2 + \frac{2m}{\hbar^2} \left(-V_0 + \frac{C}{q}\right) - \frac{2S_2^2}{q} \right] - \frac{\alpha S_2}{q + e^{\alpha(r-R_0)}} + \frac{\alpha q S_2}{(q + e^{\alpha(r-R_0)})^2},$$

$$V_-(r) = \frac{\hbar^2}{2m} \left[ S_1^2 + \frac{2m}{\hbar^2} \left(-V_0 + \frac{C}{q}\right) - \frac{2S_2^2}{q} \right] + \frac{\alpha S_2}{q + e^{\alpha(r-R_0)}} + \frac{\alpha q S_2}{(q + e^{\alpha(r-R_0)})^2}.\hspace{1cm} (31)$$

Clearly, one can write

$$V_+(r, S_2) = V_-(r, S_2 - \alpha q)$$

$$+ \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left(-V_0 + \frac{C}{q}\right) \frac{1}{2S_2} - \frac{S_2}{2q} \right]^2 - \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left(-V_0 + \frac{C}{q}\right) \frac{1}{2(S_2 - \alpha q)} - \frac{S_2 - \alpha q}{2q} \right]^2,\hspace{1cm} (33)$$

which is precisely the requirement for the shape invariance. The shape invariant concept was introduced by Gendenshtein [2]. In fact, comparing Eqs. (33) and (8) yield

$$S_2 \rightarrow a_1, \hspace{0.5cm} S_2 - \alpha q \rightarrow a_2,$$

$$R(a_2) = \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left( \frac{C}{q} - V_0 \right) \frac{1}{2a_1} - \frac{a_1}{2q} \right]^2 - \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left( \frac{C}{q} - V_0 \right) \frac{1}{2a_2} - \frac{a_2}{2q} \right]^2,\hspace{1cm} (34)$$

where the remainder $R(a_2)$ is independent of $r$. On repeatedly using the shape invariance condition Eq.(8), it is then clear that

$$H^{(k)} = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_-(r; a_k) + \sum_{s=1}^{k} R(a_s),\hspace{1cm} (35)$$

where $H^{(k)}$ is a series of Hamiltonians, $k=1, 2, 3,...,$ and $H^{(1)} = H_-$. Thus the bound-state energy spectrum of $H^{(k)}$ is obtained

$$E^{(k)}_0 = \sum_{s=1}^{k} R(a_s),\hspace{1cm} (36)$$
and its n-th level coincides with the ground state of the Hamiltonian $H_n \ (n = 0, 1, 2, \ldots)$. The energy eigenvalues of Hamiltonian are given by

$$E_0^{(-)} = 0,$$

$$E_n^{(-)} = \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left( \frac{C}{q} - V_0 \right) \frac{1}{2S_2} - S_2 \right] - \frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left( \frac{C}{q} - V_0 \right) \frac{1}{2(S_2 - naq) - \frac{S_2 - naq}{2q}} \right]^2.$$  

(38)

Hence, the energy levels of the deformed Woods-Saxon potential in Eq.(19) for the zero angular momentum states are found as

$$E_n = E_n^{(-)} + E_0 = -\frac{\hbar^2}{2m} \left[ \frac{2m}{\hbar^2} \left( \frac{C}{q} - V_0 \right) \frac{1}{2(S_2 - naq) - \frac{S_2 - naq}{2q}} \right]^2,$$

(39)

where one can write the relationship between $V(r)$ and $V_{-}(r)$ is $V(r) = V_{-}(r) + E_0$. Substituting Eq.(30) into Eq.(39) and setting $C = 0$, we can immediately obtain the energy eigenvalues

$$E_n = -\frac{\hbar^2}{2ma^2} \left[ \left( \frac{2ma^2V_0}{\hbar^2q(n + 1)} \right)^2 + \left( \frac{n + 1}{2} \right)^2 + \frac{4maV_0^2}{\hbar^2q^2} \right],$$  

(40)

where the minus sign of $S_2$ is used corresponding to $q > 0$. This result is in good agreement with the ones obtained before [16, 17].

### 3.2 Solution for the $l \neq 0$ case

The Hamiltonian for the deformed Woods-Saxon potential for the $l \neq 0$ case is written as

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - \frac{V_0 e^{-\left(\frac{r-R_0}{\alpha}\right)}}{1 + q e^{-\left(\frac{r-R_0}{\alpha}\right)}} + \frac{\hbar^2 l(l + 1)}{2mr^2}.  

(41)

The second term is taken from the deformed Woods-Saxon potential and the third term comes from the potential barrier. The last term prevents us to build the superfamily as in the $l \neq 0$ case, since the full potential is not exactly solvable. However, several numerical approaches have been utilized in order to evaluate the spectra of energy eigenvalues and eigenfunctions. Now, we introduce a new effective potential whose functional form is given as follows

$$V_{\text{eff}} = -\frac{V_0}{q + e^{\left(\frac{r-R_0}{\alpha}\right)}} + \frac{\hbar^2 l(l + 1)\alpha^2}{2m \left(q + e^{\left(\frac{r-R_0}{\alpha}\right)}\right)^2},  

(42)

In the case of $\alpha = (1 + q)/R_0$ and for small $\alpha$, the second term of Eq.(42) behaves as a potential barrier term of Eq.(41) in first approximation and it has the advantage that the Schrödinger equation for this potential is solvable analytically. As the effective potential given by Eq.(42) has the same functional form as Eq.(19), we can solve the Schrödinger problem by the factorisation method of SQM and find the whole super family. Comparing Eqs.(42) and (19), one can see the transformation of $C \rightarrow \hbar^2 l(l + 1)\alpha^2/2m$. Substituting this parameter into Eq.(30), we arrive at

$$S_2 = -\frac{\alpha q}{2} \pm \sqrt{\left(\frac{\alpha q}{2}\right)^2 + l(l + 1)\alpha^2}.  

(43)
The lowest energy levels of the potential in Eq.(41) are given by

\[ E_{nl} = -\frac{\hbar^2}{2m} \left[ \frac{2mV_0a^2}{\sqrt{q^2}} - \frac{l(l+1)\alpha}{q^2} \right] + \frac{\alpha}{4} \left( 1 + 2n + \sqrt{1 + \frac{4l(l+1)}{q^2}} \right)^2. \]  

(44)

Taking \( \alpha = 1/a \), we can get the result as

\[ E_{nl} = -\frac{\hbar^2}{2ma^2} \left[ \left( \frac{2mV_0a^2}{q\hbar^2} - \frac{l(l+1)}{q^2} \right) \right]^2 + \frac{1}{2} \left( \frac{2mV_0a^2}{q\hbar^2} - \frac{l(l+1)}{q^2} \right)^2 + \frac{1}{16} \left( 1 + 2n + \sqrt{1 + \frac{4l(l+1)}{q^2}} \right)^2. \]  

(45)

This result is exactly the same with the ones obtained before for the \( l = 0 \) states [16, 17]. In our case, quantum numbers take \( l = 1, 2, \ldots \) and \( n = 0, 1, 2, \ldots \). Thus, \( n = 0 \) and \( l = 1 \) correspond to the state \( 2p \); \( n = 1 \) and \( l = 1 \) correspond to the state \( 3p \) and so on.

Now, we calculate the energy spectrum taking the deformed \( q = 1 \) and then \( \alpha = 2/R_0 \) or \( a = R_0/2 \). If we take \( q = 2 \) and then \( \alpha = 3/R_0 \) or \( a = R_0/3 \). For the scattering processes, it has been well accepted that the surface diffuseness parameter \( a \) is ranging between 0.8 and 1.4 fm [18, 19]. Here, we use the empirical value of \( r_0 = 0.90 \) fm, from which we only retain the Woods-Saxon potential which yields ”satisfactory” fits to the experimental data, for projectile particles and targets with mass number \( A \). It is shown that the mass number is in the domains \( 6 \leq A \leq 30 \) for \( q = 1 \) and \( 19 \leq A \leq 101 \) for \( q = 2 \). Our numerical results are listed for the \( l = 1 \) and \( l = 3 \) in Table 1 and Table 2, respectively. They are also compared with exact numerical values.

4 Conclusions

We have applied the hierarchy of the Hamiltonian method in the context of SQM to get energy spectra of the deformed Woods-Saxon potential. We have used a new effective potential depending on the diffuseness parameter \( a \) in the calculations. It is a generalization of the deformed Woods-Saxon potential. We have obtained the exact analytical eigenfunction and eigenvalue for the \( l = 0 \) case. In addition, we have also derived the solutions for the \( l \neq 0 \) case, using an effective potential suggested by the \( l = 0 \) case. We have also noticed that \( \alpha = (1+q)/R_0 \) should be taken for a nuclear scattering process. Finally, we would like to point out that although the SQM scheme works quite well for the deformed Woods-Saxon potential, extensive applications to other effective Woods-Saxon-like potentials are needed to test the credibility of the method.

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Table 1: Energy eigenvalues as a function of the diffuseness parameter for the state $2p \ (n = 0, \ l = 1)$ and $q = 2$.

| State | $a$ (fm) | $A$ | $E_{01}^{(\text{analytical})}$ (MeV) | $E_{01}^{(\text{numerical})}$ (MeV) |
|-------|---------|-----|-------------------------------------|-------------------------------------|
| $2p$  | 0.814   | 20.0| -17.5727121                         | -17.5700005                        |
|       | 1.026   | 40.0| -16.6731389                         | -16.6701250                        |
|       | 1.148   | 56.0| -17.3509649                         | -17.3500205                        |
|       | 1.293   | 80.0| -19.1773309                         | -19.1800050                        |
|       | 1.392   | 100.0| -20.2905543                       | -20.3004000                        |

Table 2: Energy eigenvalues as a function of the diffuseness parameter for the states $4p \ (n = 2, \ l = 1), \ 4d \ (n = 1, \ l = 2), \ 4f \ (n = 0, \ l = 3)$ and $q = 1$.

| States | $a$ | $A$ | $E_{nl}^{(\text{analytical})}$ (MeV) | $E_{nl}^{(\text{numerical})}$ (MeV) |
|--------|-----|-----|-------------------------------------|-------------------------------------|
| $4p$   | 0.818| 6.0 | -113.754997                         | -113.7550000                        |
|        | 0.969| 10.0| -87.027172                          | -87.0270405                        |
|        | 1.084| 14.0| -73.982269                          | -73.9810000                        |
|        | 1.179| 18.0| -66.165173                          | -66.1555005                        |
| $4d$   | 1.001| 11.0| -46.633933                          | -46.6290000                        |
|        | 1.109| 15.0| -41.128693                          | -41.1250005                        |
|        | 1.201| 19.0| -37.731683                          | -37.7312450                        |
|        | 1.279| 23.0| -35.550305                          | -                         |
| $4f$   | 1.084| 14.0| -11.191323                          | -11.1900050                        |
|        | 1.201| 19.0| -10.866104                          | -10.8625400                        |
|        | 1.298| 24.0| -10.828699                          | -                         |
|        | 1.398| 30.0| -10.977817                          | -                         |