REVISITING THE DETERMINACY ON NEW KEYNESIAN MODELS: A SURVEY

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Abstract. The goal of this paper is to review some analytic techniques that are potentially useful to shed light on the determinacy question that arises in New Keynesian models as result of a combination of several monetary policy rules; in these models, we provide conditions to guarantee existence and uniqueness of equilibrium by means of results that are obtained from theoretical analysis. In particular, these methods confirm the well known fact that Taylor–like rules in interest rate setting are not the only way to reach determinacy of the rational expectations equilibrium in the New Keynesian setting. The key technical tool we use for that purposes is the so-called Budan–Fourier Theorem, that we review along the paper. All the ideas and techniques presented have been already used, our contribution that might be original here are the organization and emphasis.

1. Introduction

The indeterminacy of the rational expectations equilibrium (REE) poses a complication to the conduct of monetary policy. It is associated with increased volatility as there is uncertainty about which equilibrium will be realized. Hence, it is possible that agents in the economy will produce a second-best outcome in equilibrium. This means that a policy regime in place should not only be consistent with an optimal equilibrium, but also concerned about its uniqueness.

One of the main problems that arises from New Keynesian (NK) models is the so-called multiple equilibria puzzle. This captures the idea that an undesired equilibrium could appear as a result of a specific combination of policies. Cochrane [Coc11] argued that there have been many attempts to tackle this problem. However, practically all of them seem to assume that the government will have the power to blow up the economy if an unexpected equilibrium occurs.

The discussion among academics is still open with new alternative solutions to the dilemma recently proposed. In Bianchi and Nicolò [BN21], the authors used a method that consists of augmenting the original model with auxiliary exogenous equations in order to provide the adequate number of explosive roots. Our paper, addressing the same problem, takes a different direction. We pretend to explore, analytically and numerically, the conditions under which uniqueness and existence are guaranteed in equilibrium. Therefore, the main purpose of our work is to shed light on the determinacy question. We are going to compare computational results (from simulation) with those that are obtained from theoretical analysis; however, we want to single out that we do not produce results based on computational simulation, our path is always, on the one hand, to exhibit statements that are obtained from theoretical analysis and, on the other hand, to illustrate these statements by means of numerical simulation.

The focus of attention in the NK framework has been centered around the determinacy conditions of endogenous interest rate rules of the kind presented by Taylor in [Tay93]. Determinacy there has often been found to depend on the size of the policy response parameters, or more specifically, the

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Taylor principle being followed (see [Woo01] and [BM02]). To guarantee determinacy, the literature has also highlighted the importance of historical feedback in monetary rules, with purely forward-looking policy found to foster non-uniqueness of equilibria [SW05]. Our results here display the well known fact, already pointed out among other by Woodford, Bullard and Mitra, that “Taylor-like” rules in interest rate setting are not the only way to achieve determinacy of the REE in the NK setting.

The importance of our work stems from the fact that we propose a generalization of computational results (applied to finding the roots of a characteristic polynomial) in order to clarify the existence, and potentially, uniqueness of real roots for a linear system of equations. To do so, we use the so-called Budan-Fourier Theorem which is stated in the paper (see Theorem 2.2); this result has been already used for the same purposes in some earlier works to tackle the determinacy issue (e.g. [BM, Proof of Proposition 1]).

The paper is organized as follows: after reviewing the Budan–Fourier Theorem in Section 2 and some determinacy terminology we introduce in Section 3 to shorten and simplify the statement of several of the results, in Section 4, we look at a canonical New Keynesian (NK) model and derive the conditions for determinacy of equilibrium when the money supply follows an exogenous path. In Section 5, we consider a model in which a monetary authority responds to lagged values of output, inflation, and interest rate deviations. In Section 6, we explore stability conditions for a model in which agents do not fully understand future policies. Finally, in Section 7, we explore the potential limitations of these methods to tackle more involved models; even in this case, we illustrate that the Budan–Fourier Theorem is still a useful tool to provide necessary determinacy conditions that are easy to check in practice, because they only require polynomial evaluation. It is well known that, when the characteristic equation is of degree two, there are several more elementary ways to tackle this issue; for instance, Chatelain and Ralf [CR, Proposition 1] use the fact that, when the characteristic equation is of degree two, the eigenvalues are non–linear functions of the trace and the determinant of the corresponding matrix [Aza93, pages 63–67].

We want also to mention here in what linear rational expectational models we are interested to tackle the determinacy issue in this paper; indeed, all our models can be cast in the form

\[ \Gamma_0 y(t) = \Gamma_1 y(t - 1) + \Psi z(t), \]

\((t = 1, \ldots, T)\), where \(z(t)\) is an exogenously evolving, possibly serially correlated, random disturbance, and \(\Gamma_0\) is an invertible matrix. This fits into the framework studied in [BK80], where the determinacy issue boils down to count how many eigenvalues of \(\Gamma_0^{-1}\Gamma_1\) lie inside or outside the complex unit disk. It is worth noting that both in [Sim02] and [LS03] the authors tackle more general models than the ones considered here, in particular they allow \(\Gamma_0\) to be singular; however, in both works they also need to calculate some eigenvectors of certain matrices, and it is well known that, if one wants to do so in practice (e.g. numerically), then one often has to calculate at once both eigenvalues and eigenvectors, for instance using the classical Power method [Hou75, Chapter 7]. For this reason, we hope that the methods we review in this paper will be useful to tackle more complicated models. Again, as we already pointed out, what might be original in this manuscript is the organization of the material and the emphasis, hoping that will be potentially useful for researchers working in this subject. The list of references at the end gives an indication of the provenance of the fundamental ideas and techniques, and might suggest directions for additional research.

All our results will be illustrated through numerical examples that were done with Matlab [Mat15].

2. The Budan–Fourier Theorem

Due to the importance that the Budan–Fourier Theorem plays in this paper, our goal now is to review it for the convenience of the reader, referring to [Akr82, Theorem 1] and the references
given therein for further details (see also [Bar89, page 173]). First of all, we define the notion of sign variation.

**Definition 2.1.** We say that a **sign variation exists** between two nonzero numbers $c_p$ and $c_q$ ($p < q$) in a finite or infinite sequence of real numbers $c_1, c_2, c_3, \ldots$ if the following holds.

(i) If $q = p + 1$, then $c_p$ and $c_q$ have opposite signs.
(ii) If $q \geq p + 2$, then $c_{p+1}, \ldots, c_{q-1}$ are all zero and $c_p$ and $c_q$ have opposite signs.

Keeping in mind this terminology, the Budan–Fourier Theorem can be phrased in the below way.

**Theorem 2.2** (Budan–Fourier). Let $P(x)$ be a polynomial with real coefficients and of degree $d$, and denote by $P^{(i)}$ its $i$th derivative; moreover, set $P_{\text{seq}}(x) := (P(x), P'(x), P''(x), \ldots, P^{(d)}(x))$. Finally, given real numbers $a < b$, denote by $v_a$ (respectively, $v_b$) the number of sign variations of $P_{\text{seq}}(a)$ (respectively, $P_{\text{seq}}(b)$). Then, the following holds.

(i) $v_b \leq v_a$.
(ii) $r \leq v_a - v_b$, where $r$ denotes the number of real roots of the equation $P(x) = 0$ located in the interval $(a, b)$.
(iii) $v_a - v_b - r$ is either zero or an even number.

### 3. Determinacy terminology

In order to simplify and shorten the statements we obtain in this paper about determinacy of several models, our aim now is to introduce a clearer notation on the determinacy question; the interested reader on semi–algebraic sets is referred to [BCR98, Chapter 2] and the references given therein for additional information.

**Definition 3.1.** Let $n \geq 1$ be an integer, and let $S \subseteq \mathbb{R}^n$ be a semi–algebraic set over $\mathbb{R}$ (that is, a subset of $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients). In practice, $S$ will be the space where the parameters of our model lie.

(i) We say that our model is **unconditionally determined** if, for any $(x_1, \ldots, x_n) \in S$, our model is determined.
(ii) We say that our model is **generically determined** if there exists a semi–algebraic subset $S' \subset S$ such that our model is determined for any $(x_1, \ldots, x_n) \in S'$.

Hereafter, we refer to the set $S$ as the **parameter space**, and to $S'$ as the **determinacy region**.

### 4. A dynamic linear system

In this section we show that a set of non-restrictive assumptions on the structural parameters of the underlying economic model are sufficient for the uniqueness of the equilibrium. The case we consider was presented by Galí in [Gal15, 3.4.2], but in contrast to the numerical methods in the original, here we also show the results analytically. In this particular case, one is interested in the analysis of the so-called timing structure: "cash-when-I’m-done" (CWID). Eventually, we are going to show that an exogenous money growth rule, under this specific setup, is always going to give us unconditional determinacy. In the dynamic linear system considered in the paper, one wants to show that the matrix

$$A := \begin{pmatrix} 1 + \sigma \eta & 0 & 0 \\ -k & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma \eta & \eta & 1 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(where $k > 0$, $\sigma > 0$, $\eta > 0$ and $\beta \in (0,1)$ are real numbers) has two eigenvalues inside the unit disk\(^1\) and the remainder one is outside, because by [BK80] this is equivalent to say that the

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\(^1\)Throughout this paper, by the **unit disk** we mean the set $\{z \in \mathbb{C} : |z| < 1\}$. 

corresponding dynamic linear system has a unique stationary solution; our goal in this section is to show that this is true. Notice that, in this case, our parameter space is

\[ S = \{ (k, \sigma, \eta, \beta) \in \mathbb{R}^4 : k > 0, \sigma > 0, \eta > 0, 0 < \beta < 1 \}. \]

We deduce the unconditional determinacy of this model from the below technical statement.

**Proposition 4.1.** Let \( A \) be a \( 3 \times 3 \) matrix with real entries such that its characteristic polynomial is \( P(x) = x^3 - bx^2 + cx - d \), where we suppose that \( b > 1, c > 0, d \in (0, 1), 1 - d < b - c, \) and \( bc - d > 0 \). Then, the following assertions hold.

(i) All the real roots of \( P \) are located in the interval \((0, b)\).

(ii) \( P \) has at least one real root in the interval \((1, b)\).

(iii) If \( P \) has two complex roots, then both are located in the unit disk.

(iv) If all the roots of \( P \) are real, then \( P \) has at least one real root in the interval \((0, 1)\).

(v) \( P \) has a single root in the interval \((1, b)\).

**Proof.** First of all, given \( x \in [0, +\infty) \) notice that \( P(-x) = -x^3 - bx^2 - cx - d < 0 \), because \( x \geq 0 \) and \( b > 0, c > 0 \) and \( d > 0 \) by our assumptions; this shows that \( P \) has no roots in the interval \((-\infty, 0]\). Moreover, given \( \mu \geq 0 \) a real number, it follows that

\[ P(b + \mu) = \mu(b + \mu)^2 + \mu c + (bc - d) > 0; \]

indeed, \( P(b + \mu) > 0 \) because we know by assumption that \( P(b) = bc - d > 0 \) and the remainder terms of \( P(b + \mu) \) are also non–negative. Summing up, our calculations show that \( P(x) < 0 \) for any \( x \in (-\infty, 0) \), and \( P(x) > 0 \) for any \( x \in [b, +\infty) \). These two facts show that all the roots of \( P \) are located in the interval \((0, b)\), and therefore part (i) holds.

Now, the reader can easily check that \( P(1) = 1 - b - c - d < 0 \) again because of our assumptions; in this way, since \( P(1) < 0 \) and \( P(b) > 0 \) Bolzano’s theorem [BCR98, Proposition 1.2.4] guarantees the existence of at least one real root of \( P \) in the interval \((1, b)\), hence part (ii) is also true. In what follows, we denote by \( \lambda \) this root, and let \( \lambda_2, \lambda_3 \) be the remainder roots of \( P \). Keeping in mind this notation, one has that \( \lambda \cdot \lambda_2 \cdot \lambda_3 = d \) if and only if \( \lambda_2 \cdot \lambda_3 = d/\lambda \). This equality shows that \( \lambda_2 \cdot \lambda_3 < 1 \) because \( d \in (0, 1) \), and \( \lambda > 1 \); now, we want to distinguish two cases.

On the one hand, suppose that \( \lambda_2 \) and \( \lambda_3 \) are complex numbers, then \( \lambda_3 = \bar{\lambda}_2 \) (where \( \bar{\cdot} \) denotes complex conjugation) and therefore \( |\lambda_2|^2 = \bar{\lambda}_2 \cdot \lambda_2 = \lambda_2 \cdot \lambda_3 < 1 \iff |\lambda_2| < 1 \), and this shows that both \( \lambda_2 \) and \( \lambda_3 \) are located in the unit disk, as claimed.

On the other hand, if \( \lambda_2 \) and \( \lambda_3 \) are real numbers, then either \( \lambda_2 \in (0, 1) \) or \( \lambda_3 \in (0, 1) \) because \( \lambda_2 \cdot \lambda_3 < 1 \).

Our final aim is to show that \( P \) has a single root in the interval \((1, b)\), and for this we plan to use the Budan–Fourier Theorem (see Theorem 2.2); first of all, the sequence of \( P \) and all its non–zero derivatives (aka Fourier sequence) turns out to be

\[ P_{\text{seq}}(x) := (x^3 - bx^2 + cx - d, 3x^2 - 2bx + c, 6x - 2b, 6). \]

Now, we evaluate this sequence respectively at \( 1 \) and \( b \); namely,

\[ P_{\text{seq}}(1) := (1 - b + c - d, 3 - 2b + c, 6 - 2b, 6), \quad P_{\text{seq}}(b) := (bc - d, b^2 - c, 4b, 6) \]

Let \( v_1 \) (respectively, \( v_b \)) be the number of signs violations of \( P_{\text{seq}}(1) \) (respectively, \( P_{\text{seq}}(b) \)), and notice that \( v_b = 0 \) because we know by our assumptions that \( P(b) = bc - d > 0, b^2 + c > 0 \) and \( 4b > 0 \).

Remember that the Budan–Fourier Theorem says that the number of real roots of \( P \) located in the open interval \((1, b)\) is less or equal than \( v_1 - v_b = v_1 \); our final aim will be to show that \( v_1 = 1 \). We need to consider four cases

Firstly, if \( 3 - 2b + c \leq 0 \) and \( 6 - 2b \leq 0 \), then clearly \( v_1 = 1 \). Secondly, if \( 3 - 2b + c \leq 0 \) and \( 6 - 2b \geq 0 \), then again \( v_1 = 1 \). Thirdly, if \( 3 - 2b + c \geq 0 \) and \( 6 - 2b \geq 0 \), then once again \( v_1 = 1 \). Finally, assume to reach a contradiction that \( 3 - 2b + c \geq 0 \) and \( 6 - 2b \leq 0 \), so \( v_1 = 3 \). Notice
that the inequality \( 6 - 2b \leq 0 \) is equivalent to say that \( b \geq 3 \). On the other hand, the inequality \( 3 - 2b + c \geq 0 \) is equivalent to \( c \geq 2b - 3 \), and this inequality implies, since \( b > c \), that \( b > 2b - 3 \), hence \( b < 3 \), a contradiction. Therefore, this fourth case can not happen.

Summing up, we have finally checked that \( v_1 = 1 \), which implies that there is at most 1 real root in the interval \((1, b)\) by the Budan–Fourier Theorem, and since we already checked that in this interval \( P \) has at least one real root, we can finally conclude that \( P \) has a single root in the interval \((1, b)\), just what we finally want to show. \(\square\)

Now, building upon Proposition 4.1, we are ready to prove the main result of this section, keeping in mind the notation we introduced at the very beginning.

**Theorem 4.2.** The following assertions hold.

(i) One has that

\[
A = \begin{pmatrix}
\frac{\sigma \eta}{1+\sigma \eta} & \frac{\eta}{1+\sigma \eta} & \frac{1}{1+\sigma \eta} \\
\frac{k \sigma \eta}{1+\sigma \eta} & \frac{k \eta}{1+\sigma \eta} + \beta & \frac{1}{1+\sigma \eta} \\
\frac{k \sigma \eta}{1+\sigma \eta} & \frac{k \eta}{1+\sigma \eta} + \beta & 1 + \sigma \eta
\end{pmatrix}
\]

(ii) The eigenvalues of \( A \) are exactly the roots of the polynomial \( P(x) = x^3 - bx^2 + cx - d \), where

\[
b = \frac{\sigma \eta + \beta(1 + \eta)}{1 + \sigma \eta} + 1 + \beta, \quad c = (1 + \beta) \frac{\sigma \eta}{1 + \sigma \eta} + \frac{k \eta}{1 + \sigma \eta} + \beta, \quad d = \frac{\beta \sigma \eta}{1 + \sigma \eta}.
\]

(iii) Our model is unconditionally determined.

**Proof.** First of all, part (i) is just an issue of inverting a matrix, and afterwards a multiplication of matrices, and both steps are left to the interested reader. Secondly, it is straightforward to check that \( P(x) \) is the characteristic polynomial of \( A \), hence part (ii) holds too.

To prove the unconditional determinacy of this model, we only need to check that the assumptions of Proposition 4.1 hold; indeed, it is clear that \( b > 1, c > 0 \) and \( d \in (0, 1) \). On the other hand, notice that \( P(1) = 1 - b + c - d = \frac{k}{1+\sigma \eta} < 0 \) again because \( k, \sigma \) and \( \eta \) are strictly positive; finally, one also has that

\[
P(b) = bc - d = \left( \frac{\sigma \eta + \beta(1 + \eta)}{1 + \sigma \eta} + 1 + \beta \right) \left( 1 + \beta \right) \left( \frac{\sigma \eta}{1 + \sigma \eta} + \frac{k \eta}{1 + \sigma \eta} + \beta \right) - \beta \frac{\sigma \eta}{1 + \sigma \eta} > 0.
\]

Summing up, we have checked that we are under the assumptions of Proposition 4.1, and therefore this Proposition implies that our model is unconditionally determined, just what we finally wanted to show. \(\square\)

**Remark 4.3.** We want to single out that Proposition 4.1 does not cover the model analyzed by Bullard and Mitra in [BM02, Proposition 3 and Appendix C] because, in their model, \( d < 0 \). Finally, the Routh–Hurwitz criterion [Mei95, Theorem 1.1], used to prove the stability of one of the models studied by Gabaix (see [Gab20, Proposition 5.3] and [Gab16, Proposition 9.7]), only implies in our case that all the eigenvalues of \( P \) have positive real part, so it is not useful for our purposes. In what follows (see Sections 5 and 6), we analyze these models.

We end this section by exhibiting some numerical examples to illustrate Theorem 4.2; as we already pointed out in the Introduction of this manuscript, the unjustified calculations in all the examples we present in this paper were done with Matlab [Mat15]. Remember that our path here and in the remainder sections is, on the one hand, to provide results that are obtained from theoretical analysis and, on the other hand, to illustrate these results by means of numerical examples.
Example 4.4. First of all, we calibrate our parameters in the following way: \( \beta = 0.99, \sigma = 0.5, \eta = 1.2 \) and \( k = 0.3 \). In this case,

\[
A = \begin{pmatrix}
0.3750 & 0.7500 & 0.6250 \\
0.1125 & 1.2150 & 0.1875 \\
0.1125 & 1.2150 & 1.1875 \\
\end{pmatrix},
\]

its characteristic polynomial is \( x^3 - 2.7775x^2 + 1.9612x - 0.3712 \) (remember that Theorem 4.2 guarantees the existence of a unique eigenvalue in the interval \((1,2.7775)\)), and its eigenvalues are 0.3107, 0.6620 and 1.8048. Therefore, in this case, the unique eigenvalue of \( A \) contained in \((1,2.7775)\) is 1.8048.

Secondly, now we calibrate our parameters following Woodford [Woo99]; indeed, in this case we pick \( \beta = 0.99, \sigma = 0.157, \eta = 1.2 \) and \( k = 0.024 \). In this case,

\[
A = \begin{pmatrix}
0.1585 & 1.0098 & 0.8415 \\
0.0038 & 1.0142 & 0.0202 \\
0.0038 & 1.0142 & 1.0202 \\
\end{pmatrix},
\]

its characteristic polynomial is \( x^3 - 2.1930x^2 + 1.3297x - 0.1569 \) (remember that Theorem 4.2 guarantees the existence of a unique eigenvalue in the interval \((1,2.1930)\)), and its eigenvalues are 0.1547, 0.8634 and 1.1749. Therefore, in this case, the unique eigenvalue of \( A \) contained in \((1,2.1930)\) is 1.1749.

Finally, we calibrate our parameters following Clarida, Galí and Gertler [CGG00]; indeed, in this case we pick \( \beta = 0.99, \sigma = 1, \eta = 1.2 \) and \( k = 0.3 \). In this case,

\[
A = \begin{pmatrix}
0.5455 & 0.5455 & 0.4545 \\
0.1636 & 1.1536 & 0.1364 \\
0.1636 & 1.1536 & 1.1364 \\
\end{pmatrix},
\]

its characteristic polynomial is \( x^3 - 2.8355x^2 + 2.2391x - 0.5400 \) (remember that Theorem 4.2 guarantees the existence of a unique eigenvalue in the interval \((1,2.8355)\)), and its eigenvalues are 0.5455, 0.5784 and 1.7116. Therefore, in this case, the unique eigenvalue of \( A \) contained in \((1,2.8355)\) is 1.7116.

5. Some rules with lagged data

Now, we would like to explore a more realistic version of the model. In what follows, policymakers are assumed to react to changes throughout particular policies, that were recorded in the past. In order to explore this fact, our next goal will be to recover and extend [BM02, Proposition 3 and Appendix C]; before doing so, we want to review the following elementary Linear Algebra technical fact.

Lemma 5.1. Let \( A \) be an invertible matrix with real entries. Then, \( A \) has a unique eigenvalue outside the unit disk if and only if \( A^{-1} \) has a unique eigenvalue inside the unit disk.

Discussion 5.2. For certain non–inertial lagged data rules [BM02, pages 1118–1119], the matrix relevant for uniqueness is the below one:

\[
B = \frac{1}{\varphi_x + k\varphi_\pi} \begin{pmatrix}
0 & -\beta \varphi_x \\
0 & \beta \varphi_x \\
\sigma(\varphi_x + k\varphi_\pi) & \varphi_x + (k + \beta \sigma)\varphi_\pi \\
\end{pmatrix},
\]

where \( k > 0, \sigma > 0, \beta \in (0,1), \varphi_x \geq 0, \varphi_\pi \geq 0 \), and either \( \varphi_x \) or \( \varphi_\pi \) is strictly positive. By Lemma 5.1, \( B \) has two eigenvalues inside the unit disk and one outside if and only if \( B^{-1} \) has one eigenvalue
inside the unit disk and the remainder two ones outside. As pointed out in [BM02, Appendix C], the characteristic polynomial of $B^{-1}$ is $P(x) = x^3 - bx^2 + cx + d$, where

$$b = 1 + \frac{1}{\beta} + \frac{k}{\beta \sigma} > 2, \quad c = \frac{1}{\beta} - \frac{\varphi_x}{\sigma}, \quad d = \frac{\varphi_x + k\varphi_\pi}{\beta \sigma}.$$ 

The reader will easily note that, in this model, our parameter space is

$$S = \{(k, \sigma, \beta, \varphi_x, \varphi_\pi) \in \mathbb{R}^5 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x \geq 0, \varphi_\pi > 0\}$$

$$\cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi) \in \mathbb{R}^5 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x > 0, \varphi_\pi \geq 0\}.$$ 

Motivated by the content of Discussion 5.2, our next goal will be to prove the following:

**Proposition 5.3.** Let $P(x) = x^3 - bx^2 + cx + d \in \mathbb{R}[x]$, where $b > 0$, and $d > 0$. Then, the following assertions hold.

(i) $P$ has exactly one negative real root.

(ii) If $b > 2$, then $P$ has at least one root outside the unit disk.

(iii) $P$ has exactly one real root at $(-1,0)$ if and only if $P(-1) < 0$.

(iv) If $P(1) < 0$, then $P$ has exactly one real root at $(0,1)$.

(v) If $P(1) > 0$, and $b > 2$, then $P$ has a single real root at $(-\infty,0)$, and the other two roots are outside the unit disk.

(vi) If $P(-1) < 0$ and $P(1) < 0$, then $P$ has exactly two real roots in the interval $(-1,1)$ and the remainder real root is bigger strictly than 1.

(vii) (Cf. [BM02, Proposition 3]) Suppose that $b > 2$. If $P(-1) < 0$ and $P(1) > 0$, then $P$ has exactly one real root at $(-1,0)$, and the remainder two ones are outside the unit disk.

(viii) Suppose that $b > 2$. If $P(-1) > 0$ and $P(1) < 0$, then $P$ has exactly one root at $(0,1)$, and two roots whose real part is bigger than 1 in absolute value.

(ix) Suppose that $b > 2$. If $P(-1) > 0$ and $P(1) > 0$, then $P$ has all its roots outside the unit disk.

(x) Suppose that $b > 2$. $P$ has exactly one root at the unit disk and the remainder ones outside if and only if one and only one of the following four conditions is satisfied.

- $P(-1) < 0$ and $P(1) > 0$.
- $P(-1) = 0$, $P(-1)' \neq 0$ and $P(1) < 0$.
- $P(-1) > 0$ and $P(1) < 0$.
- $P(-1) > 0$, $P(1) = 0$ and $P(1)' \neq 0$.

**Proof.** First of all, the negative real roots of $P(x)$ are the positive real roots of $Q(x) = P(-x) = -x^3 - bx^2 - cx + d$; let $v_Q$ be the number of sign variations of the coefficients of $Q$. Independently of $c$, one can see that $v_Q = 1$, so Descartes’ rule of signs [BCR98, Proposition 1.2.14] implies that $P$ has exactly one negative real root.

Hereafter, let $\lambda_1, \lambda_2, \lambda_3$ be the roots of $P$. Without loss of generality, suppose that $\lambda_1 < 0$, so it follows that $\lambda_2 + \lambda_3 = b - \lambda_1 > b > 2$. On the one hand, if $\lambda_2$ and $\lambda_3$ are real, then the above upper inequality shows that either $\lambda_2 > 1$ or $\lambda_3 > 1$; on the other hand, if $\lambda_2$ and $\lambda_3$ are complex conjugates, then again the above inequality shows that their real part is bigger strictly than 1. Anyway, this shows that $P$ has at least one root outside the unit disk.

Now, let $r$ be the number of real roots of $P$ at $(-1,0)$; notice that

$$P_{\text{seq}}(x) := (x^3 - bx^2 + cx + d, 3x^2 - 2bx + c, 6x - 2b, 6).$$

Now, we evaluate this sequence respectively at $-1$ and 0; namely,

$$P_{\text{seq}}(-1) := (-1 - b - c + d, 3 + 2b + c, -6 - 2b, 6), \quad P_{\text{seq}}(0) := (d, c, -2b, 6)$$

Let $v_{-1}$ (respectively, $v_0$) be the number of signs variations of $P_{\text{seq}}(-1)$ (respectively, $P_{\text{seq}}(0)$), and remember that $r \leq v_{-1} - v_0$ by the Budan–Fourier Theorem. Moreover, notice also that $v_0 = 2$ because $d > 0$, and $b > 0$. Next, there are four cases to distinguish; first of all, if $-1 - b - c + d < 0$ and $3 + 2b + c < 0$, then $c > -1 - b + d$, and therefore $0 > 3 + 2b + c > 3 + 2b - 1 - b + d = 2 + b + d$, a
contradiction because both $b$ and $d$ are strictly positive, hence this case can not happen. Secondly, if $-1 - b - c + d < 0$ and $3 + 2b + c > 0$, then $v_{-1} = 3$ and therefore, combining Bolzano jointly with Budan–Fourier, we can guarantee that there is a unique real root at $(-1, 0)$. Thirdly, if $-1 - b - c + d > 0$ and $3 + 2b + c < 0$, then $v_{-1} = 2$, hence no real roots at $(-1, 0)$ by Budan–Fourier. Finally, if if $-1 - b - c + d > 0$ and $3 + 2b + c > 0$, then once more $v_{-1} = 2$, so there are no real roots at $(-1, 0)$. Summing up, we have checked that $P$ has exactly one real root at $(-1, 0)$ if and only if $P(-1) < 0$, as claimed.

Next, we looked at the interval $(0, 1)$ assuming that $P(1) < 0$; notice that

$$P_{\text{seq}}(0) := (d, c, -2b, 6), P_{\text{seq}}(1) := (1 - b + c + d, 3 - 2b + c, 6 - 2b, 6).$$

Here, there are three cases to consider, keeping in mind that we are assuming that $1 - b + c + d < 0$; first of all, if $3 - 2b + c < 0$ and either $6 - 2b < 0$ or $6 - 2b > 0$, then $v_1 = 1$, so $v_0 - v_1 = 2 - 1 = 1$, and therefore Bolzano plus Budan–Fourier ensure the existence of a unique real root at $(0, 1)$. Secondly, if $3 - 2b + c > 0$ and $6 - 2b < 0$, then $v_1 = 3$ and thus $v_0 - v_1 = -1$, so this case can not happen because $0 \leq v_0 - v_1$. Finally, if $3 - 2b + c > 0$ and $6 - 2b > 0$, then again $v_1 = 1$, so $v_0 - v_1 = 2 - 1 = 1$, and therefore Bolzano plus Budan–Fourier ensure the existence of a unique real root at $(0, 1)$. Summing up, we have checked that $P$ has exactly one real root at $(0, 1)$ if $P(1) < 0$, as claimed.

Now, assume that $P(1) > 0$, and as above denote by $\lambda_1, \lambda_2, \lambda_3$ the roots of $P$. Without loss of generality, suppose that $\lambda_1 < 0$, before we already saw that, if $\lambda_2$ and $\lambda_3$ are complex, then both have real part strictly bigger than 1, in particular they lie outside the unit disk. Moreover, we also checked that, if $\lambda_2$ and $\lambda_3$ are real and positive, then at least one of them is strictly bigger than 1, without loss of generality suppose that $\lambda_2 > 1$. If $\lambda_2 = \lambda_3$, then we are done, so hereafter we assume that $\lambda_2 \neq \lambda_3$, hence both are simple roots of $P$. Suppose, to reach a contradiction, that $\lambda_3 \in (0, 1)$; since $\lambda_3$ is a simple root of $P$ and $P(0) > 0$, then there is $\varepsilon \in (0, 1)$ such that $\lambda_3 \pm \varepsilon \in (0, 1)$, $P(\lambda_3 - \varepsilon) > 0$ and $P(\lambda_3 + \varepsilon) < 0$. But this implies, since $P(1) > 0$, that there is another real root at $(0, 1)$ by Bolzano, a contradiction by the foregoing. This shows that if $P(1) > 0$, then $P$ has a single real root at $(-\infty, 0)$, and the other two roots are outside the unit disk.

Finally, notice that the remainder items (v)–(x) are immediate consequence of the previous ones, the proof is therefore completed.

As immediate consequence of Proposition 5.3, we obtained our promised generalization of [BM02, Proposition 3], namely the below:

**Theorem 5.4.** Preserving the notations of Discussion 5.2, $B$ has two eigenvalues inside the unit disk if and only if one and only one of the following four conditions is satisfied.

1. $k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) < 0$ and $k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) > 0$.
2. $k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) = 0$, $\beta\varphi_x \neq \sigma(3 + 5\beta) + 2k$ and $k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) < 0$.
3. $k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) > 0$ and $k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) < 0$.
4. $k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) > 0$, $k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) = 0$, and $\beta\varphi_x \neq \sigma(\beta - 1) - 2k$.

Therefore, in this case our model is generically determined, and the determinacy region is

$$S' = \{ (k, \sigma, \beta, \varphi_x, \varphi_{\pi}) \in S : k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) < 0, k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) > 0 \} \cup$$

$$\{ (k, \sigma, \beta, \varphi_x, \varphi_{\pi}) \in S : k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) = 0, \beta\varphi_x \neq \sigma(3 + 5\beta) + 2k, k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) < 0 \} \cup$$

$$\{ (k, \sigma, \beta, \varphi_x, \varphi_{\pi}) \in S : k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) > 0, k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) < 0 \} \cup$$

$$\{ (k, \sigma, \beta, \varphi_x, \varphi_{\pi}) \in S : k(\varphi_{\pi} - 1) + (\varphi_x - 2\sigma)(1 + \beta) > 0, \beta\varphi_x \neq \sigma(\beta - 1) - 2k, k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) = 0 \}.$$

**Remark 5.5.** Notice that, in Theorem 5.4, the condition $k(\varphi_{\pi} - 1) + \varphi_x(1 - \beta) > 0$ is what Woodford calls the Taylor principle (see [Woo01] and [Woo03]); Theorem 5.4 shows, in particular, that the
Taylor principle is neither sufficient, nor necessary to guarantee determinacy. This fact was already pointed out by Bullard and Mitra [BM07, Propositions 1, 2 and 11].

Before moving to the next model, we want to consider the below:

Example 5.6. As we already proved, if all the conditions appearing in Theorem 5.4 are not satisfied, then we can not expect determinacy. As example, suppose that \( \varphi_x = 2.4, \varphi_\pi = 3.2, \sigma = 1, \beta = 0.99 \) and \( k = 0.3 \); in this case,

\[
B^{-1} = \begin{pmatrix}
1.3030 & -1.0101 & 1 \\
-0.3030 & 1.0101 & 0 \\
2.4 & 3.2 & 0
\end{pmatrix},
\]

its characteristic polynomial is \( P(x) = x^3 - 2.3131 x^2 - 1.3899 x + 3.3939 \) and its eigenvalues are \( -1.2003, 1.2482 \) and \( 2.2653 \). Indeed, this is because \( P(-1) > 0 \) and \( P(1) > 0 \) (cf. Proposition 5.3 (ix)).

Discussion 5.7. Now, we want to consider an inertial lagged data rule studied by Woodford [Woo03] and Bullard and Mitra [BM07, page 1183]; in this specific model, the matrix which is relevant to study determinacy is the below one:

\[
B = \begin{pmatrix}
1 + \frac{k \sigma}{\beta} & -\frac{\sigma}{\beta} & \sigma \\
-\frac{k \sigma}{\beta} & \frac{1}{\beta} & 0 \\
\varphi_x & \varphi_\pi & \varphi_r
\end{pmatrix},
\]

where \( k > 0, \sigma > 0, \beta \in (0, 1), \varphi_x \geq 0, \varphi_\pi \geq 0, \varphi_r \geq 0 \) and at least one among \( \varphi_x, \varphi_\pi \) and \( \varphi_r \) is strictly positive. In this case, building upon [Woo03, Appendix C, Proposition C.2], Bullard and Mitra [BM07, Propositions 1, 2 and 11] gave necessary and sufficient conditions to ensure determinacy; in this case, determinacy holds if and only if \( B \) has a single eigenvalue inside the unit disk. It is known [BM07, page 1198] that the characteristic polynomial of \( B \) is \( P(x) = x^3 - bx^2 + cx + d \), where

\[
b = 1 + \frac{1}{\beta} + \frac{k \sigma}{\beta} + \varphi_r > 2, \quad c = \frac{1}{\beta} + \left(1 + \frac{1}{\beta} + \frac{k \sigma}{\beta}\right) \varphi_r - \sigma \varphi_x, \quad d = \frac{\sigma (\varphi_x + k \varphi_\pi - \sigma^{-1} \varphi_r)}{\beta}.
\]

Notice that, in this case, our parameter space is

\[
S = \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x \geq 0, \varphi_\pi \geq 0, \varphi_r > 0\},
\]

\[
\cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x \geq 0, \varphi_\pi > 0, \varphi_r \geq 0\},
\]

\[
\cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x > 0, \varphi_\pi \geq 0, \varphi_r \geq 0\}.
\]

Once more, as immediate consequence of Proposition 5.3, we obtain the below:

Theorem 5.8. Preserving the notations of Discussion 5.7, if \( \varphi_r < \sigma (k \varphi_\pi + \varphi_x) \), then \( B \) has a single eigenvalue inside the unit disk if and only if one of the following conditions is satisfied.

(i) \( k \sigma (\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r (2 \beta + k \sigma + 1) < 0 \) and \( k (\varphi_\pi - 1 + \varphi_r) + \varphi_x (1 - \beta) > 0 \).

(ii) \( k \sigma (\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r (2 \beta + k \sigma + 1) = 0, \sigma \beta \varphi_x \neq (3 + 5 \beta + \varphi_r (1 + 3 \beta + k \sigma)) \) and \( k (\varphi_\pi - 1 + \varphi_r) + \varphi_x (1 - \beta) < 0 \).

(iii) \( k \sigma (\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r (2 \beta + k \sigma + 1) > 0 \) and \( k (\varphi_\pi - 1 + \varphi_r) + \varphi_x (1 - \beta) < 0 \).

(iv) \( k \sigma (\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r (2 \beta + k \sigma + 1) > 0, \sigma \beta \varphi_x \neq (\beta - 2 k \sigma - 1 + \varphi_r (1 + k \sigma - \beta)) \).

\[\]
Notice that, in this case, one has generic determinacy in

\[ S' = \{(k, \sigma, \beta, \varphi_x, \varphi_r) \in S : k\sigma(\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r(2\beta + k\sigma + 1) < 0, \]

\[ k(\varphi_\pi - 1 + \varphi_r) + \varphi_x(1 - \beta) > 0 \}

\[ \cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in S : k\sigma(\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r(2\beta + k\sigma + 1) = 0, \]

\[ k(\varphi_\pi - 1 + \varphi_r) + \varphi_x(1 - \beta) < 0, \quad \sigma \beta \varphi_x \neq (3 + 5\beta + \varphi_r(1 + 3\beta + k\sigma)) \}

\[ \cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in S : k\sigma(\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r(2\beta + k\sigma + 1) > 0, \]

\[ k(\varphi_\pi - 1 + \varphi_r) + \varphi_x(1 - \beta) = 0, \quad \sigma \beta \varphi_x \neq (\beta - 2k\sigma - 1 + \varphi_r(1 + k\sigma - \beta)) \} \}

\[ \cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi, \varphi_r) \in S : k\sigma(\varphi_\pi - 1) + (\sigma \varphi_x - 2)(1 + \beta) - \varphi_r(2\beta + k\sigma + 1) > 0, \]

\[ k(\varphi_\pi - 1 + \varphi_r) + \varphi_x(1 - \beta) = 0, \quad \sigma \beta \varphi_x \neq (\beta - 2k\sigma - 1 + \varphi_r(1 + k\sigma - \beta)) \} \}

**Remark 5.9.** Notice that both Theorem 5.4 and Theorem 5.8 deal even with non-generic boundary cases; in case of Theorem 5.8, Woodford already observed [Woo03, footnote of page 672] that his determinacy conditions are sufficient but not generically necessary, whereas the ones we are providing in our results work with full generality. Finally, observe that the determinacy conditions obtained in Theorem 5.8 only work, roughly speaking, for bounded values of inertia, whereas Bullard and Mitra’s ones [BM07, Proposition 2] work for unbounded inertia.

We end the discussion of this model with the below:

**Example 5.10.** We want to single out that, of course, the assumption \( \varphi_r < \sigma(k\varphi_\pi + \varphi_x) \) is not solely enough to ensure determinacy. As example, suppose that \( \beta = 0.99, k = 0.3, \sigma = 1, \varphi_x = 4.3, \varphi_\pi = 1.82 \) and \( \varphi_r = 0.5 \); in this case,

\[ B = \begin{pmatrix} 1.3030 & -1.0101 & 1 \\ -0.3030 & 1.0101 & 0 \\ 2.4 & 3.2 & 0.5 \end{pmatrix}, \]

its characteristic polynomial is \( P(x) = x^3 - 2.8131x^2 - 2.1333x + 4.3899 \) and its eigenvalues are \(-1.3204, 1.0937 \) and \(3.0399\). Indeed, this is because \( P(-1) = 2.7101 > 0 \) and \( P(1) = 0.4434 > 0 \) (cf. Proposition 5.3 (ix)).

6. Studying a behavioral New Keynesian model

As consequence of the Routh-Hurwitz criterion [Mei95, Theorem 1.1], Gabaix (see [Gab20, Proposition 5.3] and [Gab16, Proposition 9.7]) obtained the below:

**Proposition 6.1** (Gabaix). Let \( P(x) = x^3 - bx^2 + cx - d \in \mathbb{R}[x] \), where \( b > 0, c > 0, d > 0 \) and \( P(1) \neq 0 \). Then, the following statements are equivalent.

(i) \( P \) has exactly one root at \((0, 1)\) and the remainder ones are outside the complex unit disk.

(ii) The sequence \((e_3, e_2, (e_2e_1 - e_3e_0)/e_2, e_0)\) contains exactly two sign changes, where \( e_3 = 1 - b + c - d, e_2 = 3 - b - c + 3d, e_1 = 3 + b + c - 3d \) and \( e_0 = 1 + b + c + d \).

(iii) Either \( e_2 \leq 0 \) or \( e_2e_1 - e_3e_0 \leq 0 \).

Let us briefly review what was the original motivation for Gabaix to look at Proposition 6.1; indeed, building upon a Taylor stability criterion which includes behavioral agents [Gab20, Proposition 3.1], Gabaix introduced a behavioral New Keynesian Model with backward looking terms (see [Gab20, Proposition 5.3] and [Gab16, Proposition 9.7]). In this extended model, the relevant matrix to ensure determinacy is the below one:

\[ B = \begin{pmatrix} \frac{\sigma \phi_\pi \beta + \beta f + k\sigma}{M\beta^f} & \frac{\sigma(\beta \phi_\pi - \alpha f \eta \phi_\pi - 1)}{M\beta^f} & \frac{\alpha f(\eta - 1)\rho + 1)}{M\beta^f} \\ \frac{-k}{\beta^f} & \frac{\alpha f\eta \phi_\pi + 1}{\beta^f} & \eta \chi \\ 0 & \frac{\alpha f \eta \phi_\pi + 1}{\beta^f} & 1 - \eta \end{pmatrix}. \]
In this case, since in this model there is a single predetermined variable and the remainder two ones are jump variables, again using [BK80], determinacy holds if and only if $B$ has a single real
eigenvalue less than 1 in absolute value, and the remainder two ones are complex number with modulus greater than one.

We also want to single out that, among all the parameters involved in the above matrix, both $\phi_x$ and $\phi_\pi$ are non–negative, and both $M, M^f \in [0,1]$ represent a degree of behavioralism in the models studied by Gabaix, as already observed by Cochrane in [Coc16].

Going back to Proposition 6.1, notice that the expression $e_2e_1 - e_3e_0$ is quadratic in terms of the coefficients of our polynomial; our next goal will be to provide a sufficient condition to guarantee stability that only involves a linear expression in the coefficients of the polynomial; namely:

**Proposition 6.2.** Let $P(x) = x^3 - bx^2 + cx - d \in \mathbb{R}[x]$, where $b > 0$, $c > 0$, $d > 0$ and $P(1) \neq 0$. Then, the following assertions hold.

(i) All the real roots of $P$ are contained in the interval $(0,1 + M)$, where $M = \max\{b, c, d\}$.

(ii) If $P$ has only one real root at $(0,1)$, then $P(1) \geq 0$.

(iii) If either $3 - 2b + c \leq 0$ or $b \geq 3$, and $P(1) > 0$, then $P$ has a single real root at $(0,1)$.

(iv) If $b - c > 0$ and $P(1) > 0$, then $P$ has a single real root at $(0,1)$.

**Proof.** Let $x_0 \in [0, +\infty)$, and notice that $P(-x_0) = -x_0^3 - bx_0^2 - cx_0 - d < 0$; this shows that all the real roots of $P$ are strictly positive. The fact that all of them are less than $1 + \max\{b, c, d\}$ is just by the classical Cauchy bound [IM97, Theorem 1]; this proves part (i).

Now, assume, to reach a contradiction, that $P(1) < 0$; keeping in mind that $P(0) = -d < 0$, we have to distinguish two cases. On the one hand, if $P(x_0) > 0$ for some $x_0 \in (0,1)$, then Bolzano’s Theorem implies that there are at least two real roots at $(0,1)$ (indeed, because $P(0) < 0$, $P(x_0) > 0$ and $P(1) < 0$), so we get a contradiction. On the other hand, if $P(x_0) \leq 0$ for all $x_0 \in (0,1)$ and $P(\lambda) = 0$ for some $\lambda \in (0,1)$, then $\lambda$ has to be of multiplicity two, and this is again a contradiction.

Finally, notice that parts (iii) and (iv) were already shown in the course of the proofs of Propositions 4.2 and 5.3; the proof is therefore completed.

**Discussion 6.3.** Our plan here is to use Proposition 6.2 to partially describe the sets where determinacy holds in Gabaix model; indeed, remember that in his model, the relevant matrix to ensure determinacy is the below one:

$$B = \begin{pmatrix}
\frac{\sigma \phi_x + \beta \eta + k \xi}{M^{\beta}} & \frac{\sigma (\beta \phi_x - \alpha \eta \chi - 1)}{M^{\beta}} & \frac{\alpha \eta (\eta - 1) \rho + 1)}{M^{\beta}} \\
\frac{-k}{\beta} & \frac{\alpha \eta \chi + 1}{\eta \chi} & \frac{\alpha (-\eta + 1)}{\beta} \\
0 & \frac{1}{\eta \chi} & 1 - \eta
\end{pmatrix}.$$ 

One can check, using the expression of the characteristic polynomial of $B$ written down by Gabaix [Gab16, page 64] jointly with the value of this polynomial evaluated at 1 [Gab16, page 66] the below facts.

First of all, part (ii) of Proposition 6.2 shows that the determinacy region must be contained inside

$$\tilde{S} := \{(k, \sigma, \alpha, \alpha^f, \beta, \beta^f, M, M^f, \eta, \rho, \chi, \phi_x, \phi_\pi) \in \mathbb{R}^{13} : (1 - \beta^f - \alpha \chi (1 - \rho))(1 - M + \sigma \phi_x) + k \sigma (\phi_\pi - 1) \geq 0\}.$$
shows that the determinacy region must contain the subset of $\tilde{S}$ given by the inequality
\[
\left(\frac{\sigma((\beta - 1) + \beta(\eta - 1) - \eta \alpha \rho \chi)}{M^{\beta}}\right) - \left(\frac{k \sigma}{M^{\beta}}\right) \phi_x
+ \frac{(\eta - 1)(k \sigma + \beta + M - M^{\beta}) + \eta(\alpha \chi(\rho(M - 1) - M) - 1) + M + \beta^{\beta} + k \sigma}{M^{\beta}} > 0.
\]

Finally, part (iii) of Proposition 6.2 shows that the determinacy region must contain, on the one hand, the subset of $\tilde{S}$ given by the inequality
\[
\left(\frac{\sigma}{M} \phi_x + \frac{(1 - \eta)M^{\beta} + \eta \alpha \rho \chi M + M + \beta^{\beta} + k \sigma}{M^{\beta}}\right) \geq 3,
\]
and, on the other hand, the subset of $\tilde{S}$ given by the inequality
\[
\left(\frac{\sigma}{M} \phi_x + \frac{(1 - \eta)M^{\beta} + \eta \alpha \rho \chi M + M + \beta^{\beta} + k \sigma}{M^{\beta}}\right) \geq 3.
\]

One can easily check that Proposition 6.1 and Proposition 6.2 can be applied to obtain necessary and sufficient (respectively, sufficient) conditions to guarantee the determinacy of the model studied by Bullard and Mitra in [BM07, page 1185]; here, we only write down the sufficient conditions of determinacy given by Proposition 6.2 in their specific model (cf. [BM07, Propositions 3 and 4]).

**Theorem 6.4.** The matrix
\[
\frac{1}{1 - \varphi_x \sigma} \begin{pmatrix}
1 - \beta^{-1} k \sigma (\varphi - 1) & \beta^{-1} \sigma (\varphi - 1) & \sigma \varphi \\
- k \beta^{-1} (1 - \varphi \sigma) & \beta^{-1} (1 - \varphi \sigma) & 0 \\
(1 - \beta^{-1} k \sigma) - \beta^{-1} k \varphi & \beta^{-1} (\varphi - \varphi \sigma) & \varphi_r
\end{pmatrix},
\]
(where $k > 0$, $\sigma > 0$, $\beta > 0$, $\varphi_x \geq 0$, $\varphi \geq 0$, $\varphi_r \geq 0$, and at least one among $\varphi_x, \varphi, \varphi_r$ strictly positive) has exactly one eigenvalue at $(0, 1)$ if $\varphi_x < \sigma^{-1}$, $\varphi \leq 1$,
\[
(1 - \varphi_x \sigma) (\beta(1 - \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (\beta + \varphi_r (\beta + 1) + k \sigma (1 - \varphi)) < 0,
\]
and, in addition, at least one of the below inequalities holds:
\[
\begin{cases}
(1 - \varphi_x \sigma) (\beta(2 - 3 \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (2 \beta(1 + \varphi_r) + 2 k \sigma (1 - \varphi)) \geq 0, \medskip \\
(1 - \varphi_x \sigma)(1 - 3 \beta) + \beta (1 + \varphi_r) + k \sigma (1 - \varphi) \geq 0, \medskip \\
(1 - \varphi_x \sigma)(\beta - \varphi_r (1 + \varphi_r) (\beta + k \sigma)) + \beta (\beta(1 + \varphi_r) + k \sigma (1 - \varphi)) > 0.
\end{cases}
\]
In this case, our parameter space is
\[
S = \{ (k, \sigma, \beta, \varphi_x, \varphi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x \geq 0, \varphi \geq 0, \varphi_r > 0 \},
\]
\[
\cup \{ (k, \sigma, \beta, \varphi_x, \varphi, \varphi_r) \in \mathbb{R}^6 : (k, \sigma, \beta, \varphi_x, \varphi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x \geq 0, \varphi \geq 0, \varphi_r \geq 0 \},
\]
\[
\cup \{ (k, \sigma, \beta, \varphi_x, \varphi, \varphi_r) \in \mathbb{R}^6 : k > 0, \sigma > 0, 0 < \beta < 1, \varphi_x > 0, \varphi_r \geq 0 \}.
\]
and generic determinacy holds in the below subset:

\[
S' = \{(k, \sigma, \beta, \varphi_x, \varphi_\pi) \in S : \sigma \varphi_x < 1, \varphi_\pi \leq 1,
(1 - \varphi_x \sigma) (\beta (1 - \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (\beta + \varphi_r (\beta + 1) + k \sigma (1 - \varphi_\pi)) < 0,
(1 - \varphi_x \sigma) (\beta (2 - 3 \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (2 \beta (1 + \varphi_r) + 2 k \sigma (1 - \varphi_\pi)) \geq 0\},
\]

\[
\cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi) \in S : \sigma \varphi_x < 1, \varphi_\pi \leq 1,
(1 - \varphi_x \sigma) (\beta (1 - \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (\beta + \varphi_r (\beta + 1) + k \sigma (1 - \varphi_\pi)) < 0,
(1 - \varphi_x \sigma) (1 - 3 \beta) + \beta (1 + \varphi_r) + k \sigma (1 - \varphi_\pi) \geq 0\}
\]

\[
\cup \{(k, \sigma, \beta, \varphi_x, \varphi_\pi) \in S : \sigma \varphi_x < 1, \varphi_\pi \leq 1,
(1 - \varphi_x \sigma) (\beta (1 - \beta) - \varphi_r (1 + (1 + \varphi_r) (\beta + k \sigma))) + \beta (\beta + \varphi_r (\beta + 1) + k \sigma (1 - \varphi_\pi)) < 0,
(1 - \varphi_x \sigma) (\beta - \varphi_r - (1 + \varphi_r) (\beta + k \sigma)) + \beta (\beta (1 + \varphi_r) + k \sigma (1 - \varphi_\pi)) > 0\}.
\]

We also want to illustrate this model with the below:

**Example 6.5.** Suppose that \(\beta = 0.99, k = 0.3, \sigma = 1, \varphi_x = 4.3, \varphi_\pi = 1.82\) and \(\varphi_r = 0.5\); in this case,

\[
\begin{pmatrix}
-0.2277 & -0.2510 & -0.1515 \\
-0.3030 & 1.0101 & 0 \\
-1.5308 & 0.7591 & -0.1515
\end{pmatrix},
\]

its characteristic polynomial is \(P(x) = x^3 - 0.6309 x^2 - 0.6566 x + 0.1530\) and its eigenvalues are \(-0.6758, 0.2057\) and \(1.1010\). Indeed, this is because \(P(-1) = -0.8212 < 0\) and \(P(1) = -0.1344 < 0\) (cf. Proposition 5.3 (vi)). Notice that, in this case, \(0.6309 < 2\).

We want to conclude this section with the below:

**Example 6.6.** We want to single out that the assumption \(P(1) \geq 0\) is necessary (but not sufficient) to ensure determinacy. As example, suppose that \(\phi_x = 1, \phi_\pi = 2, \alpha = 0.5, \rho = 0.35, \beta = 0.99, \eta = 0.05, \chi = 0.3, m = 0.85, \sigma = 0.2,\) and \(k = 0.053\). In this case,

\[
B = \begin{pmatrix}
1.4244 & 0.2323 & 1.1488 \\
-0.0535 & 1.0177 & -0.3371 \\
0 & 0.0150 & 0.9500
\end{pmatrix},
\]

its characteristic polynomial is \(P(x) = x^3 - 3.3920 x^2 + 3.7870 x - 1.3952\) and its eigenvalues are \(1.3861, 1.0030 + 0.0240i\) and \(1.0030 - 0.0240i\). Indeed, this is because \(P(1) = -2.2650e - 04 < 0\). Notice that, even in this case, one has that \(b - c < 0\).

7. Potential limitations

So far in this paper, the use of the Budan–Fourier Theorem to address the determinacy issue has been restricted to models where the characteristic equation one has to deal with is of degree three, so one natural question to ask is the potential use of this result to tackle models where the characteristic equation has higher degree; our goal in this section will be to briefly explain what might be the potential limitations of doing so. We illustrate it in what follows.

Indeed, we consider one of the models studied by Bhattarai, Lee and Park in [BLP14]; in such a model, the parameter space is

\[
S = \{(\beta, \eta, \gamma, \rho_R, k, \varphi, \phi_Y, \phi_\pi) \in \mathbb{R}^8 : \beta \in (0, 1), \eta \in [0, 1), \rho_R \in [0, 1), \gamma \in [0, 1],
\]

\[
k > 0, \varphi > 0, \phi_Y > 0, \phi_\pi > 0\}.
\]
and one has to look at the polynomial $P(x) = a_5 x^5 - a_4 x^4 + a_3 x^3 - a_2 x^2 + a_1 x - a_0$, where

\[
a_5 = \beta, \ a_4 = \beta + 1 + \beta(\eta + \gamma + \rho_R) + (1 - \eta)k \left( \varphi + \frac{1}{1 - \eta} + (1 - \rho_R)\phi_Y k^{-1}\beta \right),
\]

\[
a_3 = 1 + (\beta + 1)(\eta + \gamma + \rho_R) + \beta(\eta\gamma + \eta\rho_R + \gamma\rho_R)
\]

\[
+ (1 - \eta)(1 - \rho_R)k \left( \left( \phi_\pi + \frac{\rho_R}{1 - \rho_R} \right) \left( \varphi + \frac{1}{1 - \eta} \right) + (1 + \beta\gamma)\phi_Y k^{-1} + \frac{1}{1 - \rho_R} \left( \frac{\eta}{1 - \eta} \right) \right),
\]

\[
a_2 = (\eta + \gamma + \rho_R) + (\beta + 1)(\eta\gamma + \eta\rho_R + \gamma\rho_R) + \beta\gamma\rho_R
\]

\[
+ (1 - \eta)(1 - \rho_R)k \left( \left( \phi_\pi + \frac{\rho_R}{1 - \rho_R} \right) \left( \varphi + \frac{1}{1 - \eta} \right) + \phi_Y \gamma k^{-1} \right),
\]

\[
a_1 = \eta\gamma + \rho_R(\eta + \gamma + \eta\gamma + \beta\eta\gamma), \ a_0 = \eta\gamma\rho_R.
\]

Since $P(-x) \leq 0$ for any $x \in [0, +\infty)$, all the real roots of $P$ need to be strictly positive; moreover, since the degree of $P$ is odd, Bolzano’s Theorem guarantees that $P$ has at least one real root. On the other hand, since in this case determinacy holds if and only if $P$ has exactly three roots inside the unit disk, at least one of them has to be real, hence contained at $(0,1)$ (indeed, otherwise $P$ would have 0, 2 or 4 roots inside the unit disk); summing up, as observed in [BLP14], a necessary condition for determinacy is $P(1) > 0$. Therefore, in this case, the determinacy region must be contained inside

\[
\left\{(\beta, \eta, \gamma, \rho_R, k, \varphi, \phi_Y, \phi_\pi) \in S : \phi_\pi + \frac{(1 - \gamma)(1 - \beta)}{k(\varphi + 1)}\phi_Y > 1 \right\}.
\]

Our next goal will be to show that the Budan–Fourier Theorem provide other necessary conditions for determinacy; indeed, let $v_1$ be the number of sign variations of

\[
P_{\text{seq}}(1) = (a_5 - a_4 + a_3 - a_2 + a_1 - a_0, 5a_5 - 4a_4 + 3a_3 - 2a_2 + a_1, 2(10a_5 - 6a_4 + 3a_3 - a_2), 6(10a_5 - 4a_4 + a_3), 24(5a_5 - 4a_4), 120a_5).
\]

Moreover, since one can easily check that $v_0 = 5$, the Budan–Fourier Theorem tells us that the number $r$ of real roots of $P$ at $(0,1)$ is less or equal than $5 - v_1$, and that $5 - v_1 - r$ is either zero, two or four. In this case, since determinacy holds if and only if there are exactly three roots inside the unit circle, it follows that $r$ can only be either one or three, which implies that $5 - v_1$ must be zero, one or three, which is equivalent to say that $v_1$ can only be zero, two or four. Summing up, this shows that the determinacy region must be contained inside

\[
\bar{S} := \left\{(\beta, \eta, \gamma, \rho_R, k, \varphi, \phi_Y, \phi_\pi) \in S : \phi_\pi + \frac{(1 - \gamma)(1 - \beta)}{k(\varphi + 1)}\phi_Y > 1, \ v_1 = 0 \right\}
\]

\[
\cup \left\{(\beta, \eta, \gamma, \rho_R, k, \varphi, \phi_Y, \phi_\pi) \in S : \phi_\pi + \frac{(1 - \gamma)(1 - \beta)}{k(\varphi + 1)}\phi_Y > 1, \ v_1 = 2 \right\}
\]

\[
\cup \left\{(\beta, \eta, \gamma, \rho_R, k, \varphi, \phi_Y, \phi_\pi) \in S : \phi_\pi + \frac{(1 - \gamma)(1 - \beta)}{k(\varphi + 1)}\phi_Y > 1, \ v_1 = 4 \right\}.
\]

The reader will easily note that, while the necessary and sufficient conditions obtained in [BLP14, Proposition of page 222] by means of a stronger version of the Rouché Theorem [Llo79, Theorem 2] require the evaluation of transcendental functions, the necessary conditions we give through Budan–Fourier just involve polynomial evaluation.

**Conclusion**

By means of the Budan–Fourier Theorem [Akr82, Theorem 1], we have shown in a completely analytical way the existence and uniqueness of real roots for several linear systems of equations arising from New Keynesian models; indeed, we have done so, first of all, for a model when the money
supply follows an exogenous path [Gal15, 3.4.2], secondly when a monetary authority responds to lagged values of output (see [BM02, Proposition 3 and Appendix C] and [BM07, Propositions 1, 2 and 11]), and finally when agents do not fully understand future policies (see [Gab20, Proposition 5.3] and [Gab16, Proposition 9.7]). We also pinpoint the potential limitations of these methods to tackle models where the characteristic equation is of high degree. It is well known that, when the characteristic equation is of degree two, there are several more elementary ways to tackle this issue; for instance, Chatelain and Ralf [CR, Proposition 1] use the fact that, when the characteristic equation is of degree two, the eigenvalues are non–linear functions of the trace and the determinant of the corresponding matrix [Aza93, pages 63–67].

One thing the reader may ask is why we have only used the Budan–Fourier Theorem to estimate the number of real roots of a polynomial in a given interval; this is because the models studied in this paper involved between four and thirteen parameters, and from our perspective it is not obvious, neither to evaluate polynomials in the whole interval we are interested in, nor to make too many manipulations with them. This prevented us to employ other techniques, like Sturm sequences [BCR98, Corollary 1.2.10], to tackle this complicated issue; of course, obtaining complete necessary and sufficient determinacy conditions require, in general, not only to look at the real roots, but also at the complex ones. This explains why in recent works (e.g. [Lub07], [Gab20], [BLP14]) authors dealing with the determinacy issue use more sophisticated tools, like the Routh–Hurwitz criterion [Mei95, Theorem 1.1], the Schur–Cohn criterion (see [Mar66, page 198, Th. (43,1)] or [LaS86, page 27, 5.3.]) or the Rouche Theorem [Llo79, Theorem 2].

We would like to single out that our motivation comes from the issue of indeterminacy of the rational expectations equilibrium that complicated the conduct of monetary policy, and also with the multiple equilibria puzzle problem arising from New Keynesian models; we hope the techniques used throughout this paper can help to tackle, not only these issues, but also others appearing in models different from the ones considered here. Once again, as we already mentioned in the Introduction, we repeat that all the techniques and most of the examples presented here are not new, what might be original in this manuscript is the organization of the material and the emphasis, hoping that will be potentially useful for researchers working in this subject. The list of references at the end gives an indication of the provenance of the fundamental ideas and techniques, and might suggest directions for additional research.

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