Spontaneous Symmetry Breaking of Lorentz and (Galilei) Boosts in (Relativistic) Many-Body Systems

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Abstract

We extend a result by Ojima on spontaneous symmetry breaking of Lorentz boosts in thermal (KMS) states and show that it is in fact a special case in a more general class of examples of spontaneous symmetry breaking of Lorentz symmetry in relativistic many-body systems. Furthermore we analyse the nature of the corresponding Goldstone phenomenon and the type of Goldstone excitations (provided they have particle character).
1 Introduction

Spontaneous symmetry breaking (SSB) and the Goldstone phenomenon, usually accompanying it if interactions are sufficiently short-ranged and if the broken symmetry is a continuous one, which is for example the case if it derives from a conserved current, is a widely studied subject, both in the relativistic and the non-relativistic (typically many-body) regime. It is here not the place to give an exhaustive list of references. We rather choose to cite a few papers which treat the subject in a more rigorous and systematic manner and refer the interested reader to the respective lists of references. As to relativistic quantum field theory (RQFT), just to mention a few, we have e.g. \[1\], \[2\], \[3\], \[4\]. Adopting a slightly more general point of view, covering both the relativistic and the non-relativistic regime, one may mention \[5\] and \[6\]. With more emphasis on the field of non-relativistic many-body physics (ground states and temperature states) see e.g. \[7\], \[8\], \[9\].

In most cases the conserved current, being related to the infinitesimal generator of the continuous symmetry, is translation-covariant, i.e. there do not occur extra and explicit space-time dependent prefactors. But there are a few notable exceptions, for example the currents, being related to the Galilei or Lorentz boosts. Galilei boosts were, to our knowledge, for the first time studied in this context in \[10\] and later in \[11\], a short remark can also be found in \[8\]. Somewhat later, when it became fashionable to study also temperature states in the relativistic regime, a similar analysis was made for the Lorentz group (\[12\]).

The infinitesimal generators of Galilei and Lorentz boosts are special in various respects. First, in the translation-covariant case the Fourier transform (FTr) of

$$
(\Omega | j_0(x,t), A \Omega) \quad \text{or} \quad (\Omega | j_0(x,t) \cdot A \Omega)
$$

(1)

with respect to \((x,t)\) is a measure as has been emphasized and exploited in some of the above mentioned papers (see e.g. \[8\]). This is a useful information as it restricts the degree of possible singularities in the above FTr (in particular near \((\omega,k)=(0,0)\)). In the above expression \(j_0\) is the zero component of the conserved current, the space integral of which (in a certain sense) defines the infinitesimal generator of the conserved or broken symmetry. \(A\) is an observable which possibly breaks the symmetry, \(\Omega\) the vacuum vector, a ground state or temperature state in the so-called KMS- (Kubo-Martin-Schwinger) representation (see below).
Remark: As we are dealing not exclusively with Minkowski-space, we sometimes write \((\omega, \mathbf{k})\) instead of \(p^\mu = (p^0, \mathbf{p})\). We will use the two notions interchangeably.

If explicit prefactors like \(x^\mu\) occur in the expression of \(j_0(x)\), the above mentioned FTr is no longer a measure but consists of derivatives of measures which can behave more singular. Lorentz and Galilei boosts are exactly of this type.

Second, sometimes an important aspect of spontaneous symmetry breaking (SSB) is overlooked or not really exploited, while it is crucial for a Goldstone theorem to hold in the strong form. That is, the (broken) symmetry is usually required to commute with the time evolution. It is this property which leads in a straightforward way to the existence of Goldstone modes of vanishing energy for \(\mathbf{k} \rightarrow 0\) (See however the provisos and remarks in section [3]). Translation covariance plus locality more or less guarantee this in the regime of relativistic quantum field theory (RQFT). In the field of quantum many-body physics, this property is more subtle (see e.g. [6], [9]). Neither Galilei nor Lorentz boosts fulfill this requirement. In order to nevertheless have a Goldstone phenomenon, some extra discussion is necessary (see below).

There is another remark to be made as to possible confusions which may show up if many-body physics and RQFT are merged as in our paper. In the following we emphasize the fact that this fusion has the effect that a variety of new excitation modes are typically expected to occur in the system which are not present in e.g. the vacuum QFT and which are of a more non-relativistic character. To discern them from the original (naked) particles, forming the body of the system we are studying (and being already present in the vacuum QFT), they may be called collective excitations as they are typically excitations of the whole system. In contrast to that the original particles of the relativistic vacuum theory become what one usually calls quasi-particles. (We note however, that there is sometimes no clear distinction of these various modes being made in the literature). These collective excitations include in particular specific Goldstone modes, signalling a non-vanishing particle density. In any case, the corresponding types of SSB are very similar to corresponding phenomena in the non-relativistic many-body regime, and hence have been treated in some detail in the corresponding literature.

On the other hand, one can study the problem what happens for example to (Goldstone) particles which are already present in the vacuum RQFT if they are placed in a temperature state. This is an entirely different question.
It is our impression that it is perhaps this latter row of ideas which provides the motivation for the work in e.g. [13] or [14]. Be that as it may, in our framework Goldstone excitations are not! masked (cf. [13]). Quite to the contrary, they usually give a clear signal of their presence in the form we described already in [8] and in the following. We note that our findings seem to be in accord with the perturbational analysis in the non-relativistic framework (see e.g. [25], [26], [27] or the recent [28] which actually deals with the relativistic case). It can of course happen that the excitations are so short-lived (e.g. for high temperature) so that it make no longer sense to call them particles, but this phenomenon is also well understood (see e.g. the discussion in section 6 or in [8]). We think in fact that the seemingly different approaches are rather complementary and not mutually exclusive.

The paper is organized as follows. In the next section we introduce the notion of spectral support of operators, states etc. as a useful technical tool in the following analysis. We then give a brief account of the notion of SSB and the Goldstone phenomenon in a more general context which is followed by a discussion of the particular case of Lorentz boosts. In sections 5 and 6 we treat the case of temperature states. We provide arguments that the Goldstone excitations belonging to the Lorentz boosts (or Galilei boosts) are of phonon type (provided they have particle character at all) in the interacting case. In section 7 we give a short discussion of the particle-hole picture of KMS-states. Section 8 deals with the quite subtle situation in relativistic many-particle ground states. We show that the emergence of gapless Goldstone excitations even in the presence of a massive (free) theory implies certain delicate modifications of the usual framework, leading to a transmutation of the ordinary energy-momentum spectrum.

2 The Spectral support of Operators

As to FT r our conventions are the following.

\begin{equation}
  f(x) = (2\pi)^{-n/2} \int e^{-ix \cdot p} \cdot f(p) \, dp, \quad f(p) = (2\pi)^{-n/2} \int e^{ix \cdot p} \cdot f(x) \, dx
\end{equation}

with \( n \) the space-time dimension and

\begin{equation}
  x \cdot p = x^0 p^0 - xp = t\omega - xk
\end{equation}

(we set for convenience \( c = h = 1 \)). In the Hilbert space of the respective model theory we are discussing, space-time translations are denoted by

\begin{equation}
  U(x) = e^{ix \cdot P_r}
\end{equation}
and act on operators as

$$\alpha_x(A) =: A(x) = U(x) \cdot A \cdot U^{-1}(x)$$  \hspace{1cm} (5)

Then the spectral support of an operator (field, observable) is defined by

$$\int A(x) \cdot f(x) \, dx =: \int \hat{A}(p) \cdot \hat{f}(p) \, dp$$  \hspace{1cm} (6)

or

$$\hat{A}(p) = (2\pi)^{-n/2} \int dx \, e^{-ix \cdot p} \cdot A(x), \quad A(x) = (2\pi)^{-n/2} \int dp \, e^{ix \cdot p} \cdot \hat{A}(p)$$  \hspace{1cm} (7)

to be understood in the sense of operator-valued distributions. It can also be defined with respect to states, automorphism groups and the like. By mathematicians it is frequently called the Arveson-spectrum. For physicists, who have been using such concepts for quite some time, it is the energy-momentum content of a field or observable. It has for example been systematically used in [15]. Some of its mathematical properties have been discussed in the nice review by Kastler ([16]).

In the case of a unitary group action the physical content becomes quite transparent by writing

$$\int A(x) \cdot f(x) \, dx = \int dx \, f(x) \int \int dE_p \, A \, dE_{p'} \cdot e^{ix(p-p')} = (2\pi)^{n/2} \int \int dE_p \, A \, dE_{p'} \cdot \hat{f}(p-p') = (2\pi)^{n/2} \int dq \, \hat{f}(q) \int \int \delta(q-(p-p')) \, dE_p \, A \, dE_{p'} =: \hat{f}(q) \hat{A}(q) dq$$  \hspace{1cm} (8)

with

$$\hat{A}(q) := \int \int \delta(q-(p-p')) \, dE_p \, A \, dE_{p'}$$  \hspace{1cm} (9)

Lorentz boosts, which we will mainly study in the following sections, act on the spectrum in the following way. As the Minkowski scalar product is given by $x \cdot y = (x | \eta y)$ with $\eta$ the Minkowski metric $Diag(1, -1, -1, -1)$ for $n = 4$, and the rhs the ordinary scalar product, we have, because of $\Lambda^T \eta \Lambda = \eta$:

$$\Lambda x \cdot y = (x | \Lambda^T \eta y) = (x | \eta \Lambda^{-1} y) = x \cdot \Lambda^{-1} y$$  \hspace{1cm} (10)
Let us denote the automorphism group, implementing the Lorentz boosts by $\alpha_\Lambda$, we hence have

$$\alpha_\Lambda \hat{A}(q) = (2\pi)^{-n/2} \int e^{-iq \cdot x} \cdot A'(\Lambda^{-1} x) \, d^n x$$  \hspace{1cm} (11)

with

$$\alpha_\Lambda A(x) = A'(\Lambda^{-1} x)$$  \hspace{1cm} (12)

where $A'$ is the image of the direct action of a certain representation of the Lorentz boosts on the operator $A$ (for example some field). This extra representation does however not change the spectral content. This yields

$$\alpha_\Lambda \hat{A}(q) = (2\pi)^{-n/2} \int e^{-i\Lambda^{-1} q \cdot x'} \cdot A'(x') \, d^n x' = \hat{A}(\Lambda^{-1} q)$$  \hspace{1cm} (13)

hence, as expected, the spectral support is simply shifted under the action of $\Lambda$.

### 3 Spontaneous Symmetry Breaking in a Nutshell

We give a very brief outline of the essentials of SSB. Let $\alpha_t$ denote the time-evolution, acting on the algebra of observables or fields (we are not too pedantic in this respect), $\gamma_g$ some symmetry group, $\Omega$ the vacuum state, some other ground state or a temperature state (in the GNS-representation).

We assume that $\Omega$ is the only state, being invariant under the dynamics, i.e.

$$(\Omega | \alpha_t(A) \Omega) = (\Omega | A \Omega)$$  \hspace{1cm} (14)

Furthermore, we assume that the symmetry group, $\gamma_g$, and the time evolution, $\alpha_t$, do commute, i.e.

$$(\Omega | \gamma_g \cdot \alpha_t(A) \Omega) = (\Omega | \alpha_t \cdot \gamma_g(A) \Omega)$$  \hspace{1cm} (15)

Remark: As we already mentioned in the introduction, this is not! fulfilled by the Lorentz boosts.

**Definition 3.1** The symmetry, $\gamma_g$, is called spontaneously broken, if for some $A$ it holds

$$(\Omega | \gamma_g(A) \Omega) \neq (\Omega | A \Omega)$$  \hspace{1cm} (16)

**Observation 3.2** In the case of SSB and uniqueness of $\Omega$ under the time evolution, the group, $\gamma_g$, cannot be unitarily represented.
Proof: We have for some $A$, assuming that $\gamma \cdot A = U_g \cdot A \cdot U_g^{-1}$

$$(U_g^{-1} \Omega | AU_g^{-1} \Omega) \neq (\Omega | A\Omega) = (\Omega | \alpha_t(A) \Omega)$$

(17)

On the other hand, the lhs is equal to

$$(\Omega | \gamma_g(A) \Omega) = (\Omega | \alpha_t \cdot \gamma_g(A) \Omega) = (\Omega | \gamma_g \cdot \alpha_t(A) \Omega) = (U_g^{-1} \Omega | \alpha_t(A) U_g^{-1} \Omega)$$

(18)

We conclude that $U_g^{-1} \Omega$ is also invariant under the dynamics. The assumed uniqueness however implies that

$$U_g^{-1} \Omega = \Omega$$

(19)

hence

$$(\Omega | \gamma_g(A) \Omega) = (\Omega | A\Omega)$$

(20)

for all $A$, which is a contradiction because of the assumed SSB. \hfill \Box

Remark: This result can be extended to certain symmetries, not commuting with the time evolution. See the next section about the Lorentz boosts.

Frequently, the symmetry group is a continuous group, having, at least in a formal sense, infinitesimal generators, which generate the symmetry in the Hilbert space under discussion. For convenience, we choose a one-parameter subgroup, that is, we deal with a single generator. In case the symmetry derives from a conserved current, $j^\mu(x)$ with $\partial_\mu j^\mu(x) = 0$, the formal generator has the form

$$G := \int d^{n-1}x j^0(x,0) = \int d^{n-1}x j^0(x,t)$$

(21)

i.e., it is (formally) time independent.

Remark: Further possible indices of $j$ are suppressed for the moment (there do exist, for example, also higher conserved tensor currents).

In the papers, cited in the introduction, the appropriate way of dealing with such objects is described in more detail. In the following we are a little bit sloppy by suppressing a further smearing with a local test function in the time coordinate. An appropriate definition is then the following:

$$d/ds|_{s=0}(\Omega | \gamma_s(A) \Omega) = \lim_{R \to \infty} (\Omega | [G_R, A] \Omega)$$

(22)
with
\[ G_R := \int d^{n-1}x \cdot j^0(x, 0) \cdot f_R(x) \] (23)
and
\[ f_R(x) := f(|x|/R), \quad f(u) = \begin{cases} 1 & \text{for } |u| \leq 1 \\ 0 & \text{for } |u| \geq 2 \end{cases} \] (24)
and smooth in between. SSB is now expressed via the non-vanishing of the following expression
\[ \lim_{R \to \infty} \langle \Omega \mid [G_R, A(t)] \Omega \rangle = \text{const} \neq 0 \] (25)
(and being time independent!).

Remark: Remember that the symmetry was assumed to commute with the time evolution. This is however not sufficient in all cases. In RQFT we usually assume locality to hold. In this case the conclusion is correct. In quantum many-body theory certain cluster assumptions for the occurring two-point functions have to be made which are however usually fulfilled if interactions are sufficiently short-range. If these assumptions are violated commutators can become time-dependent and the ordinary Goldstone phenomenon does no longer hold, As to the necessary technical details see e.g. [9] section 2 or the recent book by Strocchi ([17]).

We see that the FTr of \( f_R(x) \) is a \( \delta \)-sequence, converging to \( \delta(k) \) for \( R \to \infty \). Furthermore, as the limit in configuration space is time independent, we arrive at the following observation:

**Observation 3.3 (Goldstone theorem)** With \( C(k, \omega) \) the distributional FTr of \( \langle \Omega \mid [j^0(x, 0), A(t)] \Omega \rangle \) we have
\[ \lim_{R \to \infty} \int d^{n-1}k \cdot C(k, \omega) \cdot \hat{f}_R(k) = \text{const} \cdot \delta(\omega) \] (26)
where \( \hat{f}_R(k) \to \delta(k) \) for \( R \to \infty \).

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1In chapter 16 of the book by Strocchi we found a comment in a footnote concerning our paper [8]. It is claimed that in theorem 1 of that paper we made some incorrect statements concerning the limit structure of functions like \( C(k, \omega) \) (as to the definition see the following observation) in the limit \( k \to 0 \). We however think, our criticized statements are essentially correct and, given the usual standards of that time, formulated in a rigorous way. We rather think the criticism in [17] is presumably the result of a misunderstanding (for more details see section 6 of the present paper, in particular the remarks after observation 6.1).
Analysing the content of this statement, we are led to the conclusion:

**Conclusion 3.4** The distribution $C(k, \omega)$ necessarily contains a singular contribution, passing through $(k, \omega) = (0, 0)$. The nature of this singular part has to be analysed in the various regimes, mentioned in the introduction. Its character is relatively simple in RQFT in the vacuum sector (existence of massless Goldstone particles). In quantum many-body theory and/or for temperature states the situation is much more complex but also more interesting (cf. e.g. [8], [9], [6]; most of the analysis was already made in [11]).

The following additional observation is perhaps useful. In the relativistic regime we usually have locality. This implies that $[j^0(x, 0), A(t)]$ vanishes for strictly local $A$ and sufficiently large $x$ (which depends of course on $t$) as the two operators become space-like separated. It follows that $(\Omega | [j^0(x, 0), A(t)] | \Omega)$ has a compact support in $x$-space and hence its FT is an analytic function in the variable $k$.

## 4 SSB of the Lorentz Boosts

In contrast to our previous assumption, the automorphisms, implementing the Lorentz boosts, do not commute with the space-time translations, i.e.

$$\alpha_\Lambda \cdot x \neq x \cdot \alpha_\Lambda$$  \hspace{1cm} (27)

with $\Lambda$ a boost (henceforth we use the symbol $\alpha_\Lambda$ instead of $\gamma_\Lambda$). We will however see, that a weaker condition does in general hold which turns out to be sufficient. In the following we will treat systems and states having a non-vanishing energy-density (or mass-density), like e.g. relativistic equilibrium states or many-particle ground states. For convenience we assume that the states are translation invariant (or are invariant under a suitable subgroup like crystals).

In the framework of RQFT each system has an energy-momentum tensor, $T_{\nu\mu}(x)$. With $T_{00}(x)$ the energy density, we hence have

$$(\Omega | T_{00}(x) | \Omega) = \text{const} \neq 0$$  \hspace{1cm} (28)

We assume the system being at rest relative to an inertial frame (IF) $S$, i.e., the expectation of the momentum density, $(\Omega | T_{0j}(x) | \Omega)$, vanishes. For reasons of simplicity we now take an IF, $S'$, moving with velocity $v$ in the negative $x_1$-direction with the coordinate axes of $S, S'$ being parallel and
with \((x_1, t) = 0 \iff (x'_1, t') = 0\). Neglecting the other, transversal, space coordinates in the following, we have \((c = h = 1)\) with \(\gamma_L := (1 - v^2)^{-1/2}\):

\[
x'_0 = \gamma_L \cdot (x_0 + v \cdot x_1), \quad x'_1 = \gamma_L \cdot (x_1 + v \cdot x_0)
\]

(29)

that is \(x' = \Lambda x\) with

\[
\Lambda = \begin{pmatrix} \gamma & v \cdot \gamma \\
                          v \cdot \gamma & \gamma \end{pmatrix}
\]

(30)

The components of the energy-momentum tensor transform like

\[
T'_{\nu\mu}(x) = \alpha_\Lambda (T_{\nu\mu}(x)) = (\Lambda^{-1})'_{\nu} \cdot (\Lambda^{-1})'_{\mu} \cdot T'_{\nu'\mu'}(\Lambda x)
\]

(31)

and the corresponding expectation value in the state \(\Omega\):

\[
\langle \Omega | \alpha_\Lambda(T_{01}(x)) \Omega \rangle = \\
(\Lambda^{-1})_0^0 \cdot (\Lambda^{-1})_0^1 \cdot \langle \Omega | T_{00}(\Lambda x) \Omega \rangle + (\Lambda^{-1})_1^0 \cdot (\Lambda^{-1})_1^1 \cdot \langle \Omega | T_{11}(\Lambda x) \Omega \rangle = \\
\gamma_L^2 \cdot v \cdot ((\Omega | T_{00} \Omega) + (\Omega | T_{11} \Omega))
\]

(32)

and, by assumption, in the rest system

\[
\langle \Omega | T_{01} \Omega \rangle = 0
\]

(33)

Remark: The \(T_{11}\)-contribution represents the possible pressure work term which can occur in the momentum density under a Lorentz boost; cf. e.g. the discussion in the recent [18]. If the above sum accidentally vanishes in some model, we can use a modified observable or a rotated reference frame.

**Conclusion 4.1** (SSB of Lorentz boosts)

\[
\langle \Omega | \alpha_\Lambda(T_{01}(x)) \Omega \rangle = \gamma_L^2 \cdot v \cdot (\langle \Omega | (T_{00} + T_{11}) \Omega \rangle \neq 0 = (\Omega | T_{01} \Omega)
\]

(34)

We see that the lhs is independent of \(x\) and in particular of \(t\). Put differently, the lhs is time independent and non-vanishing. We can hence write

\[
\langle \Omega | \alpha_\Lambda(T_{01}(x)) \Omega \rangle = \langle \Omega | \alpha_\Lambda \cdot \alpha_x(T_{01}(0)) \Omega \rangle = \langle \Omega | \alpha_x \cdot \alpha_\Lambda(T_{01}(0)) \Omega \rangle
\]

(35)

That is, exactly the same proof as in observation 3.2 goes through. The Lorentz boosts cannot be unitarily represented and there are gapless excitations (see below).
This generalizes a result of Ojima ([12]) which was stated for temperature (KMS) states, using a completely different argument (see the next section).

Remark: That Lorentz or Galilei invariance is broken in infinitely extended systems with a non-vanishing mass or energy density, is of course not surprising. What is however interesting is the concrete technical (and not so obvious) implementation of this fact within the given theoretical framework and the row of possible consequences. One should for example remember that e.g. time translation symmetry is a fairly obvious phenomenon but leads to the deep and important consequence of energy conservation.

It is perhaps noticeable that this time independence can be shown to hold for arbitrary observables, $A$. The group law for the Poincaré group reads:

$$\begin{align*}
(a_1, \Lambda_1) \cdot (a_2, \Lambda_2) &= (a_1 + \Lambda_1 \cdot a_2, \Lambda_1 \cdot \Lambda_2)
\end{align*}$$

and a corresponding relation for the respective automorphism group or unitary representation. We have in particular that

$$\begin{align*}
(a, 1) \cdot (0, \Lambda) &= (a, \Lambda) = (0, \Lambda) \cdot (\Lambda^{-1} \cdot a, 1)
\end{align*}$$

and correspondingly for the respective automorphisms. With $a$ a vector so that $\Lambda^{-1} \cdot a = \hat{t}$ is a vector, pointing in the time direction, i.e. $a = \Lambda \cdot \hat{t}$, we hence get:

$$\begin{align*}
\alpha_a \cdot \alpha_\Lambda = \alpha_{(a, \Lambda)} = \alpha_\Lambda \cdot \alpha_{\hat{t}}
\end{align*}$$

with $\hat{t} = \Lambda^{-1} \cdot a$. With $a$ varying we can reach any $t$ on the time axis so that we again arrive at

$$\begin{align*}
(\Omega | \alpha_\Lambda (A(t)) \Omega) = (\Omega | \alpha_\Lambda \cdot \alpha_{\hat{t}} (A) \Omega) = (\Omega | \alpha_\Lambda (A) \Omega)
\end{align*}$$

due to the assumed translation invariance of $\Omega$.

On the level of infinitesimal generators we then have

$$\begin{align*}
d/dv|_{v=0} (\Omega | \alpha_\Lambda (T_{01}(x)) \Omega) &= \lim_{R \to \infty} (\Omega | [G_R, T_{01}(x)] \Omega) = (\Omega | (T_{00} + T_{11}) \Omega)
\end{align*}$$

and being independent of $x$. Or

$$\begin{align*}
\lim_{R \to \infty} \int C(k, \omega) \cdot \hat{f}_R(k) \, d^{n-1}k &= (\Omega | (T_{00} + T_{11}) \Omega) \cdot \delta(\omega)
\end{align*}$$

which again shows the existence of gapless Goldstone excitations.
The infinitesimal generator derives from a conserved current
\[ J^{\mu,\nu}(x) := T^{\mu\nu}(x) \cdot x^\rho - T^{\mu\rho}(x) \cdot x^\nu \] (42)
with
\[ \partial_\mu J^{\mu,\nu}(x) = 0 \] (43)
It represents however only the so-called orbital part of the full expression in case the fields carry non-trivial representations of the Lorentz group (cf. e.g. [19], sect.1.2.2). It is a conserved quantity because \( T^{\mu\nu}(x) \) is conserved and provided \( T^{\mu\nu}(x) \) is symmetric. The generator density for boosts in the \( x \)-direction is
\[ j^0(x) = x^1 \cdot T^{00}(x) - x^0 \cdot T^{01}(x) \] (44)
As \( J^{\mu,\nu}(x) \) is conserved
\[ J^{\nu\rho}(t) := \int d^{n-1}x J^{0,\nu\rho}(x, t) \] (45)
is (formally) time independent. For \( t = 0 \) it reads
\[ J^{01}(t) = J^{01}(0) = \int d^{n-1}x T^{00}(x, 0) \cdot x^1 \] (46)
The translation non-covariance has the effect that
\[ U_t \cdot J^{\mu,\nu}(x, 0) \cdot U_t^{-1} \neq J^{\mu,\nu}(x, t) \] (47)
as the operators \( U_t \) act of course only on the operators in \( J^{\mu,\nu} \) and not on the prefactors \( x^i \). This implies for example that
\[ (\Omega \mid [J^{01}(0), A(t)] \Omega) \neq (\Omega \mid [J^{01}(-t), A] \Omega) \] (48)
We have instead
\[ U_t \cdot j^0(x, 0) \cdot U_t^{-1} = x^1 \cdot T^{00}(x, t) = j^0(x, t) + x^0 \cdot T^{01}(x, t) \] (49)
and in integrated form
\[ U_t \cdot J^{01}(0) \cdot U_t^{-1} = J^{01}(t) + x^0 \cdot P^1 \] (50)
with \( P^1 \) the total momentum operator in the \( x \)-direction and \( J^{01}(t) = J^{01}(0) \) due to current conservation.
We hence get
\[ \lim_{R \to \infty} (\Omega \mid [G_R(0), A(t)] \Omega) = \lim_{R \to \infty} (\Omega \mid [G_R(-t), A] \Omega) - t \cdot (\Omega \mid [P^1_R, A] \Omega) = (\Omega \mid [G_R(0), A] \Omega) \] (51)
as the second contribution vanishes in the limit. It follows
Observation 4.2 Due to the time independence of $G(t) = G(0)$ we have

$$\lim_{R \to \infty} (\Omega \mid [G_R(0), A(t)] \Omega) = \lim_{R \to \infty} (\Omega \mid [G_R(0), A] \Omega) = \lim_{R \to \infty} (\Omega \mid [G_R(0), A] \Omega)$$

(52)

That is, on the level of vacuum expectation values we have an analogous result as in the case of a translation covariant current.

5 Lorentz-Invariance for Temperature States

The paper by Ojima ([12]) deals with SSB of Lorentz boosts in KMS-states (i.e. Gibbs-states and their thermodynamic limits). In the case of RQFT such states are still expected to display typical properties of vacuum states on the operator level, for example, commutativity of space-like localized observables. On the other hand, some other properties as e.g. the spectrum condition (joint spectrum of energy-momentum contained in the forward cone) are typically lost. This makes these systems particularly interesting.

The KMS-property is typically expressed as follows:

$$\omega(A(t) \cdot B) = \omega(B \cdot A(t + i\beta))$$

(53)

for $A$ an analytic element so as to allow analytic continuation to $A(i\beta)$. $\omega$ is a KMS-state at inverse temperature $\beta$. In the GNS-representation the corresponding Hilbert space vector is denoted by $\Omega$. In this representation the KMS-property is expressed as

$$(\Omega \mid B \Delta^\beta A \Omega) = (\Omega \mid AB \Omega)$$

(54)

(where we do not discriminate, for convenience, between the elements of the observable algebra and their Hilbert space representatives), and

$$\Delta^\beta = e^{-\beta \hat{H}}$$

(55)

with $\hat{H}$ the KMS-Hamiltonian, $\hat{H} \Omega = 0$. (For technical details see e.g. [20] or [21]).

As to the interplay of KMS-property and possible symmetries of the system the following result is typically invoked.

Theorem 5.1 If a symmetry, $\gamma_g$, leaves the unique KMS-state, $\Omega$, invariant, it commutes with the time evolution, $\alpha_t$, implemented by

$$\alpha_t(A) = \Delta^{it} \cdot A \cdot \Delta^{-it}$$

(56)
in the GNS-representation. Or for the corresponding operator implementation:
\[ U_g \Omega = \Omega \Rightarrow [U_g, \Delta] = 0 = [U_g, J] \quad (57) \]

with \( J \) the Tomita-conjugation (note that \( \Delta \) is unbounded, so that the above statement has to be formulated a little bit more carefully).

These statements are corollaries of slightly more general statements found in e.g. [22] or [23].

In the following we give a short direct proof of the result (for the convenience of the reader). It contains however a certain technical subtlety which seems to have been sometimes overlooked in the literature. We assume that a unitary symmetry, \( U \), leaves the KMS-state invariant, i.e. \( U \Omega = \Omega \). With the KMS-property as in formula (57) and making the following assumption:

**Assumption 5.2** We assume that the symmetry maps analytic elements into analytic elements, where in this context analyticity has in general (e.g. for Lorentz boosts) to be assumed jointly for both \( k \) and \( \omega \)!

we have
\[
(\Omega | B U \Delta^\beta U^{-1} A \Omega) = (\Omega | U^{-1} B U \Delta^\beta U^{-1} A U \Omega) = \\
(\Omega | U^{-1} A U U^{-1} B U \Omega) = (\Omega | A B \Omega) = (\Omega | B \Delta^\beta A \Omega) \quad (58)
\]

**Proposition 5.3** Under the above assumptions and with the necessary technical precautions we have
\[ U \cdot \Delta^\beta \cdot U^{-1} = \Delta^\beta \quad , \quad U \cdot J \cdot U^{-1} = J \quad (59) \]

Proof: The first statement follows from the fact that analytic elements applied to \( \Omega \) generate a dense set. The second property follows from
\[ S = J \cdot \Delta^{1/2} \quad , \quad S A \Omega = A^* \Omega \quad (60) \]

which yields
\[ SU A U^{-1} \Omega = U A^* U^{-1} \Omega \quad (61) \]

or
\[ SU A \Omega = U A^* \Omega = U S A \Omega \quad (62) \]
hence

\[ SU = US \]  \hspace{1cm} (63)

which, together with \( U \Delta = \Delta U \) implies

\[ JU = UJ \]  \hspace{1cm} (64)

Remark: We recently found that a similar idea was presented in [24], sect. 2 without (as far as we can see) the above mentioned necessary technical assumption being made. It would be interesting to investigate which kind of symmetries fulfill or violate this assumption.

We now show the following:

**Observation 5.4** The Lorentz boosts fulfill the above assumption.

Proof: We use the results on the spectral support developed in sect. 2. A Lorentz boost simply transforms a point of the spectrum in the following way

\[ \Lambda : (\omega, k) \rightarrow \Lambda \cdot (\omega, k) \]  \hspace{1cm} (65)

Analyticity of an element, \( A \), means that \( \hat{A}(\omega, k) \) has a certain decay property with respect to the energy, \( \omega \) and the momentum \( k \). This decay property at infinity is evidently preserved by the above geometric operation, effected by \( \Lambda \), which is essentially a finite shift of the spectral support of \( A \).

The reasoning of Ojima now is the following. If the symmetry, induced by the Lorentz boosts, is conserved, implying that it is unitarily implemented with \( U \Omega = \Omega \), the above result shows that \( U \) has to commute with the time evolution, provided by the KMS-evolution. On the other hand, the time evolution is part of the unitary representation of the Poincaré group. Therefore it cannot commute with the Lorentz boosts as it is already forbidden algebraically. Hence we have

**Theorem 5.5** In a KMS-state, representing a pure phase, the Lorentz boosts are spontaneously broken.

We note that our more general result, derived in the preceding section, comprises this particular result. We have seen that it is already sufficient that the energy density or some other appropriate observable does not vanish in the state \( \Omega \).
6 The Goldstone Theorem for Lorentz Boosts – the Temperature State

This topic was briefly discussed by Ojima in [12]. As we do not restrict our investigation to temperature (KMS) states, it appears to be reasonable to approach this question a little bit more systematically, in particular as we already made a quite detailed analysis of the general situation in [8] (and in our PhD-thesis [11]), which, apparently, Ojima was not aware of.

In the case of KMS-states we have a relative transparent situation in Fourier-space. We begin with the translation covariant case. With

\[ J(k, \omega) := F.Tr. \text{ of } (\Omega | j^0(x,t) \cdot A \Omega), \quad C(k, \omega) := F.Tr. \text{ of } (\Omega | [j^0(x,t), A] \Omega) \]

the Fourier-transformed KMS-condition reads

\[ C(k, \omega) = (1 - e^{-\beta \omega}) \cdot J(k, \omega) \]  

In [8] or [11] we exhibited the natural two-sidedness of the Fourier spectrum of \( J(k, \omega) \) by using the relation:

\[ C(k, \omega) = J(k, \omega) - J(-k, -\omega) = (1 - e^{-\beta \omega}) \cdot J(k, \omega) \]

which yields the relations

\[ \text{Re} J(-k, -\omega) = e^{-\beta \omega} \cdot \text{Re} J(k, \omega), \quad \text{Im} J(-k, -\omega) = -e^{-\beta \omega} \cdot \text{Im} J(k, \omega) \]

and with \( J(k, \omega) \) being a measure.

The infinitesimal generator of the Lorentz boosts in the \( x^1 \)-direction reads (cf. section 4)

\[ j^0(x) = x^1 T^{00}(x) - x^0 T^{01}(x) \]

with \( T^{\nu \mu}(x) \) translation covariant. Hence

\[ J_\Lambda(k, \omega) = \partial_{k^1} J^{00}(k, \omega) - \partial_{\omega} J^{01}(k, \omega) \]

with \( J^{00}, J^{01} \) measures.

**Observation 6.1** \( J_\Lambda(k, \omega) \) consists of terms which are in a distributional sense first derivatives of measures.
The Goldstone theorem was formulated with the help of $J(k, \omega)$ or $C(k, \omega)$ in the following way (cf. section 3):

$$\lim_{R \to \infty} \int C(k, \omega) \cdot \hat{f}_R(k) \, d^{n-1}k = \text{const} \cdot \delta \omega$$

(72)

with $\text{const} \neq 0$. In our particular case this reads

$$\lim_{R \to \infty} (1 - e^{-\beta \omega}) \cdot \int J(k, \omega) \cdot \hat{f}_R(k) \, d^{n-1}k = \text{const} \cdot \delta \omega$$

(73)

with

$$J(k, \omega) = \partial_k J^{00}(k, \omega) - \partial_\omega J^{01}(k, \omega)$$

(74)

Performing the above limit sloppily we would get a non-sensical result, i.e.

$$(1 - e^{-\beta \omega}) \cdot J(0, \omega) = \text{const} \cdot \delta \omega$$

(75)

or

$$J(0, \omega) = \text{const} \cdot (1 - e^{-\beta \omega})^{-1} \cdot \delta \omega$$

(76)

which is not well-defined.

It turns out that it is better to include a $\omega$-integration in the limit calculation, or, as we said already in section 3, it is advisable to include an extra time smearing in the original operator expression. I.e. we will study

$$\lim_{R \to \infty} (\Omega | [Q_R(t), A] \Omega) = \lim_{R \to \infty} \int (1 - e^{-\beta \omega}) \cdot J(k, \omega) \cdot \hat{f}_R(k) \, d\omega \, d^{n-1}k = \text{const}$$

(77)

(the limit being independent of $t$!)

We will now analyse the singularity structure of $J(k, \omega)$ in the vicinity of $(k, \omega) = (0, 0)$. We made a detailed (but of course not exhaustive) analysis in [8] and [11] and isolated a so-called singular contribution in $J(k, \omega)$. More precisely, we showed in [8] that it is the imaginary part of $J(k, \omega)$ which contains the information of the Goldstone phenomenon and it is exactly this imaginary part we are mainly dealing with in this context (it may however be true that we sometimes forgot to make this sufficiently explicit). In chapter 16 of [17] it was shown that the real part contains in the limit $k \to 0$ the derivative of a $\delta$-function in $\omega$. As we already proved in [8] that this real part does not! contribute in the limits we are taking, we think, the criticism in [17] is a little bit beside the point, in particular as, in our view, most of what has been said in chapter 16 of [17] (and more) can already be found in [8] and [9]. Furthermore, we constantly emphasized in our contributions...
dealing with this topic and in particular in the criticized theorem 1 that taking limits carelessly in the Goldstone context is usually dangerous.

Making now a simplifying model assumption (only to get an idea what is happening) and assuming that there exists a sharp excitation branch in the spectrum of $J(k,\omega)$, which is however a certain idealisation, we inferred that $J(k,\omega)$ contains a contribution of the form

$$J_S(k,\omega) = J(k)\delta(\omega - \sigma(k)) - e^{-\beta|\omega|} J(-k)\delta(\omega + \sigma(k))$$  \hspace{1cm} (78)

with $J(k)$ a function which becomes singular in $k = 0$, the first term on the rhs being the particle-excitation branch and the second term being the hole-excitation branch.

Time independence of the above limit allows us to set $t = 0$ which further simplifies the expression. For our Lorentz boost this yields

$$\text{const} = \lim_{R \to \infty} \int (1 - e^{-\beta\omega}) \cdot \partial_{k^1} J^{00}(k,\omega) \cdot \hat{f}_R(k) d\omega d^{n-1}k$$  \hspace{1cm} (79)

with $J^{00}(k,\omega)$ being a measure. In the following we deal with the contribution having positive $\omega$. We hence have to analyse the expression

$$\int \lim_{R \to \infty} (1 - e^{-\beta\omega}) \cdot \partial_{k^1} (J^{00}(k) \cdot \delta(\omega - \sigma(k)) \cdot \hat{f}_R(k)) d\omega d^{n-1}k =$$

$$\lim_{R \to \infty} \int d^{n-1}k (1 - e^{-\beta\sigma(k)}) \cdot \partial_{k^1} J^{00}(k) \cdot \hat{f}_R(k)$$  \hspace{1cm} (80)

and get again

$$\partial_{k^1} J^{00}(k) \sim \sigma(k)^{-1} \text{ for } k \to 0$$  \hspace{1cm} (81)

Ojima in [12] mentioned the possibility of a Goldstone spectrum without particle structure. This was already discussed in greater generality in [8] and [11] and is in fact quite a delicate point. It is difficult to discuss this matter from a general point of view. Below we are making certain, in our view reasonable and physically motivated, model assumptions. We however emphasize that, depending on the particular context, i.e. type of models, value of thermodynamic parameters as e.g. the temperature, completely different situations may prevail.

We expect for example, that the existence of an interaction favors the emergence of relatively long-lived Goldstone excitations (as e.g. phonons). In the free case (i.e. no interaction) there is no reason that relatively stable compressional modes should exist. So, in this situation, we expect that the singular contribution, $J_s$, does not correspond to typical long-lived excitations. However, in the appendix of [8] we made an explicit calculation for
a free Bose-gas in the temperature regime where a Bose-condensate does exist. The calculation clearly shows that the density-density correlation function consists of a normal term, $J_n$, containing no pronounced particle excitation structure, and what we called a singular contribution, $J_s$, which exactly displays the sharp excitation branch of the underlying Bose particles as intermediate states with exactly the diverging weight function, we predicted for vanishing energy-momentum. The underlying physical origin of this divergence is the vanishing of the chemical potential in the presence of a condensate in the usual expression for the $k$-dependent occupation number of the Bose particles. In that paper the Bose particles are associated with the Goldstone particles of the SSB of the phase invariance (Condensation of zero-modes)!

In real many-body systems, excitations usually have a finite lifetime. The deeper apriori reasons for this phenomenon were brought to light in [15]. A Goldstone excitation branch having a finite lifetime was modelled in the above mentioned papers as

$$J_S(k, \omega) = J(k) \cdot \chi(k)^{-1} \cdot \phi(\omega - \sigma(k)/\chi(k))$$  \hspace{1cm} (82)

with $\phi(s)$ a smooth function of compact support, normalized so that $\int \phi(s) \, ds = 1$, $\sigma(k)$ and $\chi(k)$ are smooth functions for $k \neq 0$ with $\sigma(0) = \chi(0) = 0$, positive elsewhere.

**Observation 6.2** Note that the $\phi$-term is essentially a $\delta$-sequence for $k \to 0$ with respect to $\omega$. $\chi(k)^{-1}$ plays the role of the lifetime of the respective excitation with energy $\sigma(k)$. We see that by construction the lifetime becomes infinite for $k \to 0$.

Proof: see [11] or [8]. The general many-body philosophy is roughly the following (see e.g. [25]). These elementary excitations do not! correspond to exact statistical (or micro) states of the system, but represent superpositions (wave packets) of a large number of such micro states with a narrow spread in energy.

In the above mentioned papers we showed that under such conditions we get a Goldstone phenomenon of essentially the same type as in the case of a sharp excitation branch. If the energy uncertainty, $\chi(k)$, fulfills

$$\chi(k) \ll \sigma(k) \quad \text{for} \quad k \to 0$$  \hspace{1cm} (83)

one can speak of a particle-like excitation. If, on the other hand,

$$\chi(k) \gg \sigma(k) \quad \text{for} \quad k \to 0$$  \hspace{1cm} (84)
holds, the Goldstone excitation has no longer a particle-character but rather resembles a resonance.

**Observation 6.3** *Note that in both cases the contribution, $J_S(k,\omega)$, contracts to a sharp singularity for $k \to 0$, which we showed, was crucial for the Goldstone phenomenon. That is, both situations are logically possible and will in fact occur in nature.*

Remark: In the more standard Green’s-function approach (see e.g. [25], [27]) the lifetime is given by the distance of the respective poles from the real axis in the complex plane (typically the second sheet).

The question is now, what is the character of the Goldstone excitations in the case of the SSB of the Lorentz boosts? Before we discuss this question in more detail a general remark is in order. In contrast to the non-relativistic regime, where, typically, the energy-momentum dispersion law of the constituents naturally passes through $(\omega,k) = (0,0)$, the relativistic constituents frequently have a non-zero mass and a dispersion law of the type

$$\sigma(k) = \sqrt{m^2 + k^2}$$  \hspace{1cm} (85)

i.e., in most of the cases we should expect a gap in the energy spectrum. Then the (naive) question poses itself of the consistency of this observation with the predictions of the Goldstone theorem we derived in the preceding sections. Ojima calls this a paradox, which is perhaps an exaggeration. In the case of temperature states he provides an explanation using the double-sidedness of the spectrum of $\hat{H}$ (see the next section). Much more interesting is however the case of relativistic many-body ground states, to which we come later. In general the following observation is crucial.

**Observation 6.4** *In contrast to naive expectation (resulting perhaps from experience with RQFT in the vacuum sector), the excitation spectrum of energy-momentum in relativistic states having a non-vanishing energy or particle density is richer compared to the situation in ordinary RQFT and contains additional excitations which are entirely new and rather of a non-relativistic type (phonons, magnons etc.).*

Therefore representations, which essentially take only into account contributions in for example so-called Green’s functions coming from the original particle modes of vacuum field theory, are usually incomplete to a certain degree. One should in this context mention the ingenious idea of elementary excitations developed by Landau ([29]; see also the remarks in [15]).
According to this philosophy it may be much more convenient to represent the Hamiltonian as an assembly of weakly interacting elementary excitations (a kind of canonical transformation) which can considerably differ from the type of particles one starts from.

The analysis shows that in general the typical low-lying excitations in quantum liquids of e.g. Bose-type are phonons (see [29] loc. cit. or [30]). For an interacting Bose system Landau conjectures an excitation branch with a linear dispersion law for small $k$ which goes over into another law for larger $k$. As to a more rigorous quantitative treatment in form of perturbation theory see [25], chapt. 25, in particular chapt. 25.5. A similar discussion can be found in [27], chapt. 21. What is particularly remarkable is that the phonon excitation branch shows up as intermediate states in the perturbational calculation of the 2-point Green’s function of the original massive Bose particles. Usually one would expect them to occur as intermediate states in density-density correlations which are composed of 4 field operators. While these calculations are of course made for non-relativistic many-body systems, there is no reason to expect a different situation in the relativistic case. These theoretical observations are further corroborated experimentally by measuring the specific heat, $c_V$, which shows a $T^3$-behavior for small $T$, which is typical for a phonon gas. It is notable in this respect that the free (non-relativistic) Bose gas shows a behavior $\sim T^{3/2}$ which is typical for massive particles.

As these collective excitations show up if we are in a state of non-vanishing particle density, these phonon-type excitations have to be regarded as the Goldstone modes belonging to the SSB of Galilei or Lorentz boosts in real (i.e. interacting) quantum many-body systems.

Conclusion 6.5 In interacting quantum many-body systems the Goldstone modes, belonging to the SSB of Galilei or Lorentz boosts are of phonon type.

Remark: For Galilei boosts this was already discussed in [8], sect. 4 or [11]. Note that in the case of the free Bose gas with a condensate the Bose particles themselves are also Goldstone particles of the spontaneously broken phase transformation belonging to the number operator.

In [11] we provided an entirely different (qualitative) argument why the Goldstone particles should be of phonon type. It goes as follows. In contrast to most cases of SSB which are usually related to the occurrence of certain phase transitions, and which depend on the space dimension, Galilei or Lorentz boosts can evidently be broken in all space dimensions. Assuming then that the dispersion law is independent of the space dimension, we
can argue in the following way. Making the simplification of a Goldstone
excitation of infinite lifetime, we know from our preceding analysis that in
a KMS-state it holds
\[ \partial_k J(k) \sim \sigma(k)^{-1} \quad \text{for} \quad k \to 0 \quad (86) \]
On the other hand, \( J(k) \) has to be a locally integrable function as \( J(k, \omega) \)
is a measure. Making the further reasonable assumption that for \( k \to 0 \)
\[ \sigma(k) \sim k^{\eta} \quad (87) \]
we have
\[ \partial_k J(k) \sim k^{-\eta} \quad \text{and} \quad J(k) \sim k^{\eta-1} \quad (88) \]
for \( k \to 0 \). We hence can conclude that
\[ \eta - 1 < d \quad \text{or} \quad \eta < d + 1 \quad (89) \]
with \( d \) the space dimension. Choosing now \( d = 1 \) we get

**Conclusion 6.6** We conclude from out above qualitative argument that the
Goldstone mode belonging to SSB of the Galilei or Lorentz boosts is expected
to have a dispersion law for small \( k \), \( \sigma(k) \sim k^{\eta} \) with \( \eta < 2 \). If the exponent
is an integer (which is the case in most examples), the only possibility which
remains is a linear law, \( \sigma(k) \sim k \), i.e. of phonon type.

### 7 The Particle-Hole Picture of KMS-States

Ojima explained the occurrence of zero-energy Goldstone excitations in the
presence of a massive free Bose field with the help of the two-sidedness of
the spectrum of the KMS-Hamiltonian. For a free Bose field in a KMS-state
(without condensate) the annihilation part of the Bose field reads (without
giving all the technical prerequisites)
\[ \psi(f) = \psi_F((1 + \rho^{1/2}f) \otimes 1 + 1 \otimes \psi_F^\dagger(\rho^{1/2}f)) \quad (90) \]
with \( \psi(f) \) affiliated with the observable algebra, \( \mathcal{A} \). The dual field, affiliated
with the commutant, \( \mathcal{A}' \), reads:
\[ \tilde{\psi}(f) = \psi_F^\dagger(\rho^{1/2}f) \otimes 1 + 1 \otimes \psi_F((1 + \rho)^{1/2}f) \quad (91) \]
The whole construction is performed over the tensor product of two Fock
spaces. The F.Tr. of \( \rho \), i.e. \( \rho(k) \), is the occupation density of the Bose
particles in thermal equilibrium at inverse temperature \( \beta \).
This representation was (to our knowledge) first given by Araki and Woods for the free non-relativistic Bose field in [31]. Some years later (and seemingly independently) it was rediscovered in the framework of thermo field theory (cf. e.g. [32] or [33] and references therein). One should note that such a tensor product representation is in fact rather natural. Assuming for convenience that the Hamiltonian of a system (e.g. enclosed in a finite box) has discrete spectrum, it is well-known that the canonical equilibrium state over the observable algebra can be extended to a vector state in a larger Hilbert space having such a tensor product structure with the observable algebra, \( \mathcal{A} \), acting in the one Hilbert space of the product, the commutant, \( \mathcal{A}' \) in the other one. This vector state is however not annihilated by the original annihilation operators belonging to the Bose field (it is in fact a cyclic and separating vector). But a so-called Bogoliubov-transformation leads to the usual Fock-space creation and annihilation operators which enter in the above formula.

In [15] and in more detail in [34], it was shown that a similar structure is present in essence also in the case of a general interacting field theory over a KMS-state. More specifically, it was rigorously shown that in an approximative sense one can confirm the Landau picture of elementary excitations in general KMS-states, in particular the particle-hole picture. This means for example that the annihilation part of an interacting quantum field consists approximately of the weighted superposition of the annihilation of an underlying (quasi) particle mode and the creation of a respective hole contribution being immersed in the infamous 'Dirac sea'. A corresponding relation holds for the creation part. Furthermore, the duality between \( \mathcal{A} \) and \( \mathcal{A}' \) is established via corresponding dual fields.

In e.g. [11] the consequences of this picture for SSB were analysed in quite some detail. In the free case the full Hamiltonian reads

\[
\hat{H} = H_F \otimes 1 - 1 \otimes H_F
\]

and a similar relation holds for the full symmetry generator (corresponding relations hold for general finite (volume) systems and typically, the thermodynamic-limit KMS-Hamiltonian in general is a certain limit of such expressions). One can now employ the rigorous results about the spectral support (Arveson spectrum) given for example in [16] which we briefly arrange in the following for the convenience of the reader. The following spectral results hold for a cyclic state, \( \Omega \):

- With \( \lambda \in \text{Spec}(\hat{H}) \) there exists an \( A \in \mathcal{A} \) for each neighborhood \( V(\lambda) \) s.t. \( \text{Spec}(A) \subset V(\lambda) \) and \( A\Omega \neq 0 \).
• We have in general

\[ \text{Spec}(A_1 \cdot A_2) \subset \text{Spec}(A_1) \cup \text{Spec}(A_2) \] (93)

• If \( \Omega \) fulfills a certain plausible cluster property under e.g. the space translations, the spectrum is even additive, i.e.

\[ \lambda_1, \lambda_2 \in \text{Spec}(\hat{H}) \Rightarrow \lambda_1 + \lambda_2 \in \text{Spec}(\hat{H}) \] (94)

The same holds for the joint spectrum of energy-momentum.

All this shows that it is in fact easy to construct low-lying excitation modes from modes which are a distance away from energy equals zero. I.e., by using the first and the third statement, we can get a \((k, \omega) \approx (0, 0)\) by composing two observables, \(A_1, A_2\), with energy-momentum concentrated around

\[ (k, \omega), (-k + \delta(k), -\omega + \delta(\omega)) \] (95)

i.e., by taking

\[ A_1 \cdot A_2 \Omega \text{ with } (k_1, \omega_1) + (k_2, \omega_2) = (\delta(k), \delta(\omega)) \approx (0, 0) \] (96)

8 The Goldstone Theorem for Lorentz-Boosts – the Many-Body Ground State

As a consequence of the strong implications coming from the KMS-structure of temperature states, the explanation of the occurrence of gapless excitations was relatively straightforward. The situation for relativistic many-body ground states, on the other hand, is surprisingly subtle. The reason is the following. In the case of KMS-states the existence of mirror-excitations, lying in the Dirac-sea, or, stated less poetically, the natural two-sidedness of the joint energy-momentum spectrum, makes it quite easy to construct particle-hole states with arbitrarily small energy or energy-momentum. This construction is not at our disposal in the case of ground states, which, by definition, have a one-sided energy-spectrum. By scanning the accessible literature, we found practically nothing in this direction we can rely on. So, as this is apparently sort of uncharted territory and, on the other hand, turns out to be quite intricate, it would be necessary to treat the subject, i.e. the case of relativistic ground-state models, in considerably more detail which we would prefer to do elsewhere in order not to blow up the present
paper to much. In the following we rather give a brief motivation and try to provide some insights.

In order to better understand the impending problems, we analyse the massive free Klein-Gordon field. To get a better feeling for the underlying physics, we do not proceed in the most abstract way (like e.g. in [31]) and attempt to construct abstract (infinitely extended) states over the given observable algebra and try to verify their properties afterwards. We think it is rather advisable to start from finite volume systems and perform the thermodynamic limit at finite fixed density. Furthermore, for the sake of physical intuition, we use the creation-annihilation-operator framework, as it is done in most of the more physically oriented treatments. While they are unbounded we are nevertheless not aware of any real problem stemming from this fact. On the other hand, the bounded Weyl-operator framework has certain unphysical features as these Weyl-operators contain an arbitrary number of particle creation and annihilation operators. In the finite volume case this is a little bit nasty. This effect becomes only negligible in the infinite-volume limit.

We begin with some notations. We denote the Fock-vacuum by $|0>$ . The particle dispersion law is $\omega_k = (k^2 + m^2)^{1/2}$. The commutation relations between the annihilation and creation operators read $[a(k), a^\dagger(k')] = \delta(k, k')$. Note that we are in the regime of finite volumes, $V$, with periodic boundary conditions tacitly assumed. That is, the $k$'s are actually taken from a certain discrete set, which depends on the volume $V$. As we are in the regime of Bose-statistics, the normalized $n$-Boson ground state is denoted by $|n>$. It contains $n$ modes of energy

$$\omega_0 = m \quad , \quad k = 0$$

(97)

It is created by $a^\dagger(0)$ from the Fock-vacuum:

$$(a^\dagger(0))^n |0> = (n!)^{1/2} |n> , \quad <n|n> = 1$$

(98)

with

$$a^\dagger(0) |n - 1> = n^{1/2} |n> , \quad a(0) |n> = n^{1/2} |n - 1>$$

(99)

From this we see that the $n$-particle ground state has an energy gap

$$E_n = n \cdot \omega_0$$

(100)

with respect to the Fock vacuum. We start from a $n$-particle system in a box of volume $V$ (periodic boundary conditions) and density $\rho = n/V$. If we now try to perform the thermodynamic limit, the ground state energy $E_n$
wanders away towards infinity (in contrast to the non-relativistic scenario!). If we want to arrive at a definite limit theory with a well-defined Hamiltonian we have, among other things, to renormalize the energy. While this seems to be pretty obvious there are nevertheless quite a few technical problems lurking in the background, for example the loss of relativistic covariance due to the change of spectrum.

At zero temperature we have in the limit a model system with an infinite occupation of the ground state, i.e. what is called a condensate. This is known from non-relativistic many-body theory (cf. e.g. [26], [27] or [25]; one of the early references is [35]) and the methods to correctly deal with such a phenomenon should be similar in principle in the relativistic realm apart from the different energy-momentum dispersion law which makes the treatment more complex (note that in the non-relativistic case energy is proportional to $k^2$).

The following observation is crucial.

**Observation 8.1** \(a(0)\) and \(a^\dagger(0)\) trivially commute with all the other \(a(k)\) and \(a^\dagger(k)\) for \(k \neq 0\). Furthermore it holds

\[
[a(0)/n^{1/2}, a^\dagger(0)/n^{1/2}] = n^{-1} \to 0
\]

for \(n \to \infty\). Hence, in the limit, the operators \(a(0)/n^{1/2}, a^\dagger(0)/n^{1/2}\) commute with all the elements of the algebra \(A\).

We now construct excitations which remain within the \(n\)-particle Hilbert space \(\mathcal{H}_n\). We define

\[
b^\dagger(k) := a^\dagger(k) \cdot a(0)/n^{1/2} \quad \text{for} \quad k \neq 0
\]

and

\[
b^\dagger(k) |n> = |n-1, k> \in \mathcal{H}_n
\]

The ordinary Hamiltonian reads

\[
H = \sum_k \omega_k \cdot a^\dagger(k)a(k)
\]

We have

\[
H |n-1, k> = (\omega_k + (n-1) \cdot \omega_0) |n-1, k> = ((\omega_k - \omega_0) + n \cdot \omega_0) |n-1, k>
\]
**Definition 8.2** In the $n$-particle Hilbert space we define the renormalized Hamiltonian

\[ K := H - n \cdot \omega_0 \]  
(106)

and get

\[ K \vert_{n-1,k} = (\omega_k - \omega_0) \vert_{n-1,k} \]  
(107)

**Observation 8.3** We have in $\mathcal{H}_n$

\[ K = H - n \cdot \omega_0 = \sum_k (\omega_k - \omega_0) \cdot a^\dagger(k)a(k) \]  
(108)

as at most $n$ terms in the above sum can contribute in $\mathcal{H}_n$. In general we get

\[ K = H - \mu \cdot \hat{N} \quad \text{with} \quad \mu := \omega_0 \]  
(109)

and $\hat{N}$ the particle number operator in the finite volume Fock-space.

We now have

\[ e^{iKt} \cdot b(k) \cdot e^{-iKt} = e^{-i(\omega_k - \omega_0)t} \cdot b(k) \]  
(110)

\[ e^{iKt} \cdot b^\dagger(k) \cdot e^{-iKt} = e^{i(\omega_k - \omega_0)t} \cdot b^\dagger(k) \]  
(111)

and a corresponding result for the $a^\dagger(k), a(k)$ if $k \neq 0$. For $k = 0$ we have

\[ e^{iKt} \cdot a(0)/n^{1/2} \cdot e^{-iKt} = a(0)/n^{1/2} \]  
(112)

\[ e^{iKt} \cdot a^\dagger(0)/n^{1/2} \cdot e^{-iKt} = a^\dagger(0)/n^{1/2} \]  
(113)

We see that in the thermodynamic limit we can treat the zero modes as (singular!) c-numbers with

\[ \lim_{n \to \infty} a_0/n^{1/2} = e^{i\alpha} \cdot \delta k \]  
(114)

More specifically, if we smear the respective fields with a test function $\hat{f}(k)$, the zero-component becomes $\hat{f}(0) \cdot e^{i\alpha} \cdot 1$. As to the particular phase factor see [30] or [31]. It comes from the additional breaking of gauge invariance due to the existence of a condensate.

Along these lines one can proceed further, the main task being to cope properly with the many subtle effects coming from the now missing relativistic covariance induced by the change in the spectrum of the Hamiltonian. Note for example that the corresponding redefined quantum field, based on the $b(k)^{(1)}$, does no longer fulfill the ordinary Klein-Gordon equation but
rather some kind of pseudo-differential equation. In a sense one can however reconstruct the original covariant Klein-Gordon field. We want to postpone such an analysis for the reasons mentioned above and instead indicate the consequences for the Goldstone phenomenon, provided a satisfactory limit theory can be constructed.

Crucial in this respect are the excitations coming from $b(k) \dagger$ applied to the finite volume $n$-particle ground state. We saw that $b(k) \dagger$ create gapless excitations for $k \to 0$ with respect to the redefined Hamiltonian $K$,

$$K \cdot b(k) \dagger |n> = (\omega_k - \omega_0) \cdot b(k) \dagger |n> \quad (115)$$

with

$$(\omega_k - \omega_0) \approx 1/2m \cdot k^2 \quad \text{for} \quad k \to 0 \quad (116)$$

In order to show that the joint energy-momentum spectrum in the thermodynamic limit is expected to cover the full half-space of positive energies and arbitrary momenta, we study particular $N$-particle excitations. We take $N$ excitations, $(\omega_{k_i} - \omega_0)$, with $|k_i| = \varepsilon_i$. We get approximately for $\varepsilon_i$ small:

$$E_N = \sum_{i=1}^{N} (\omega_{k_i} - \omega_0) \approx 1/2m \sum_{i=1}^{N} \varepsilon_i^2 \quad (117)$$

Choosing now $\sum_{i=1}^{N} \varepsilon_i = k$ fixed but arbitrary with

$$\varepsilon \geq \varepsilon_i \geq \varepsilon/4 \quad , \quad N \cdot \varepsilon /2 = k \quad (118)$$

we have the simple estimate

$$\sum_{i=1}^{N} \varepsilon_i^2 \leq N \cdot \varepsilon^2 = N^{-1} \cdot (2k)^2 \quad (119)$$

**Observation 8.4** We can arrange the $k_i$ in such a way that for $N$ large

$$\sum_{i=1}^{N} k_i = k \quad \text{but} \quad E_N \to 0 \quad (120)$$

Remark: In a slightly different context the interrelation of the form of the energy-momentum spectrum and broken Lorentz symmetry is discussed in [37].
9 Commentary

We have seen that the SSB of the Lorentz-boosts (and the Galilei-boosts) is the consequence of the non-vanishing of the energy or particle density or perhaps some other density. In so far our results generalize the result of Ojima which exploited some particular property of KMS-states. We analyzed in quite some detail the nature of the Goldstone excitations in the various regimes and showed that in the interacting case they are of phonon-type. What is remarkable in this context is the transmutation of the original energy-momentum spectrum into a gapless spectrum. The gapless excitations are of particle-hole type. In the case of relativistic many-body ground states the emergence of gapless excitations is particularly subtle.

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