ON EXACT SOLUTIONS TO QUANTUM

$N = 2$ GAUGE THEORIES

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Abstract

Exact solutions to the low-energy effective action (LEEA) of the four-dimensional
$(4d)$, $N = 2$ supersymmetric gauge theories with matter (including $N = 2$ super-
QCD) are discussed from the three different viewpoints: (i) instanton calculus, (ii)
$N = 2$ harmonic superspace, and (iii) M theory. The emphasis is made on the
foundations of all three approaches and their relationship.

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1 Introduction

Quantum Field Theory (QFT) is the theoretical foundation of the elementary particles physics, including the Standard Model (SM). An experimental success of the SM gives some general lessons to field theorists. Among them are: (i) not just an arbitrary QFT is of importance but only those of them which are renormalizable, unitary and asymptotically-free gauge theories, (ii) the crucial role played by various symmetries, including the local (gauge) symmetry, internal (rigid) symmetry and supersymmetry, in a ‘good’ QFT, and (iii) most of the ‘good’ QFT symmetries, however, have to be broken either spontaneously, or quantum-mechanically.

The standard textbook description of quantum gauge theories is often limited to perturbative considerations whereas many physical phenomena (e.g., confinement) are essentially non-perturbative. It is usually straightforward (although, it may be quite non-trivial !) to develop the quantum perturbation theory in which all the fundamental symmetries are manifestly realised. Unfortunately, the perturbative expansion usually does not make sense when the field coupling becomes strong. In other words, the formal generating functional (path integral) of QFT has to be defined in practical terms, and in the past it was actually done in many ways beyond the perturbation theory (e.g., lattice regularization, instantons, duality). Because of this reasoning, until recently, it was common to believe among most field theorists that any non-perturbative gauge QFT is not well-defined enough, in order to allow one to make definitive predictions and non-perturbative calculations ‘from the first principles’.

However, this attitude may have to be revised since the remarkable discovery of Seiberg and Witten \([1]\) of an exact non-perturbative solution to the low-energy effective action (LEEA) in certain \(N = 2\) supersymmetric quantum gauge field theories. Though the non-trivial low-energy solution was found for the certain class of QFTs having no immediate phenomenological applications, it is, nevertheless, of great value since these theories may be a good starting point for further symmetry breaking towards the phenomenologically applicable QFT models at lower energies, including the SM, while maintaining their nice integrability properties at higher energies.

Among the general lessons of the Seiberg-Witten (SW) exact solution for the QFT practitioners are again the same three lessons formulated above in relation to the SM (!), this time as regards the non-perturbative story: (i) in order to be solvable in the low-energy limit, a QFT has to be the ‘good’ one i.e. it should have the \(N = 2\) extended supersymmetry, (ii) the exact symmetries can severely constrain a non-trivial ‘good’ QFT even beyond perturbation theory in such a way that a unique non-perturbative solution may exist in the low-energy limit, the SW solution being
an example, and (iii) many fundamental symmetries (e.g., the non-abelian gauge symmetry and supersymmetry) are either already broken in the full non-perturbative QFT, or they have to be broken further by some dynamical mechanisms, in order to make contact with the low-energy phenomenology to be represented by SM. To achieve the third goal, one may have to go even beyond the framework of a given \( N = 2 \) supersymmetric QFT e.g., by embedding it into a more fundamental superstring theory or M-theory in higher dimensions.

Unlike the SM or its minimal (\( N = 1 \)) supersymmetric extensions, \( N = 2 \) supersymmetric gauge field theories cannot directly serve for phenomenological applications, partly because an \( N = 2 \) matter can only be defined in real representations of the gauge group. Nevertheless, nothing forbids us to think about an \( N = 2 \) gauge theory as the starting point only, or as the intermediate gauge field theory originating from a unified theory (e.g., the M-theory) with even higher symmetry or in higher dimensions at larger energies. At lower energies, however, the \( N = 2 \) supersymmetric gauge theory is supposed to be reproduced while \( N = 2 \) supersymmetry should ultimately be broken at even lower energies.

### 1.1 Motivation

Four-dimensional (4\( d \)), \( N = 2 \) supersymmetric gauge field theories are not integrable, either classically or quantum-mechanically. The full quantum effective action \( \Gamma \) in these theories is highly non-local and intractable. Nevertheless, it can be decomposed to a sum of local terms in powers of space-time derivatives or momenta to be divided by some dynamically generated scale \( \Lambda \) (in components), the leading kinetic terms being called the low-energy effective action (LEEA). Determining the exact LEEA is a great achievement since it provides the information about a non-perturbative spectrum and exact static couplings in the full quantum theory at energies below certain scale \( \Lambda \). Since we are only interested in the 4\( d \), \( N = 2 \) gauge theories with spontaneously broken gauge symmetry via the Higgs mechanism, the effective low-energy field theory may include only abelian massless vector particles. All the massive fields (like the charged \( W \)-bosons) are supposed to be integrated out. This very general concept of LEEA is sometimes called the Wilsonian LEEA since it is familiar from statistical mechanics. There is a difference between the quantum effective action to be defined as the generating functional of the one-particle-irreducible (1PI) Green’s functions or as the Wilsonian effective action, as far as the gauge theories with massless particles are concerned.

\[ \text{It is the self-dual sector of their Euclidean versions that is integrable in the classical sense.} \]
$N = 2$ supersymmetry severely restricts the form of the LEA. The very presence of $N = 2$ supersymmetry in the full non-perturbatively defined quantum $N = 2$ gauge theory follows from the fact that its Witten index \cite{4} does not vanish, $\Delta_W = \text{tr}(-1)^F \neq 0$. It just means that $N = 2$ supersymmetry cannot be dynamically broken.\footnote{Alternatively, one may prove that the whole theory can be consistently defined in a manifestly $N = 2$ supersymmetric way, e.g., in $N = 2$ superspace.}

There are only two basic supermultiplets (modulo classical duality transformations) in the rigid $N = 2$ supersymmetry: an $N = 2$ vector multiplet and a hypermultiplet. The $N = 2$ vector multiplet components (in a WZ-gauge) are

$$\{ A, \lambda^i_\alpha, V_\mu, D^{(ij)} \}, \quad (1.1)$$

where $A$ is a complex Higgs scalar, $\lambda^i$ is a chiral spinor (‘gaugino’) $SU(2)_A$ doublet, $V_\mu$ is a real gauge vector field, and $D^{ij}$ is an auxiliary scalar $SU(2)_A$ triplet.\footnote{The internal symmetry $SU(2)_A$ here is just the automorphism symmetry of the $N = 2$ supersymmetry algebra, that rotates its two spinor supercharges.} Similarly, the on-shell physical components of the Fayet-Sohnius (FS) \cite{5} version of a hypermultiplet are

$$\text{FS} : \quad \{ q^i , \psi_\alpha , \bar{\psi}^*_\alpha \}, \quad (1.2)$$

where $q^i$ is a complex scalar $SU(2)_A$ doublet, and $\psi$ is a Dirac spinor. There exists another (dual) Howe-Stelle-Townsend (HST) version \cite{6} of a hypermultiplet, whose on-shell physical components are

$$\text{HST} : \quad \{ \omega , \omega^{(ij)} , \chi^i_\alpha \}, \quad (1.3)$$

where $\omega$ is a real scalar, $\omega^{(ij)}$ is a scalar $SU(2)_A$ triplet, and $\chi^i$ is a chiral spinor $SU(2)_A$ doublet. The hypermultiplet spinors can be called ‘quarks’, though it would mean an extra ‘mirror’ particle for each ‘true’ quark in the $N = 2$ super-QCD.

The manifestly supersymmetric formulation of supersymmetric gauge theories is provided by superspace \cite{7}. Since superfields are reducible representations of supersymmetry, they have to be restricted by certain superspace constraints. The standard constraints defining the $N = 2$ super-Yang-Mills (SYM) theory in the ordinary $N = 2$ superspace \cite{8} essentially amount to the existence of a restricted chiral $N = 2$ superfield strength $W$, whose leading component is the Higgs field, $W| = A$. The $N = 2$ superfield $W$ contains also the usual Yang-Mills field strength $F_{\mu\nu}(V)$ among its bosonic components, as well as the $SU(2)_A$ auxiliary triplet $D^{(ij)}$. Since the latter has to be real in the sense $D^{(ij)} = \varepsilon_{ik}\varepsilon_{jl}D^{kl}$, it implies certain (non-chiral) $N = 2$
superspace constraints on $W$, which are not easy to solve in terms of unconstrained $N = 2$ superfields in the non-abelian case. The situation is even more dramatic in the case of the FS hypermultiplet whose off-shell formulation does not even exist in the ordinary $N = 2$ superspace. Though the HST hypermultiplet can be defined off-shell in the ordinary $N = 2$ superspace, where it is sometimes called as an $N = 2$ tensor (or linear) multiplet, its self-couplings are very restricted and not universal there. In order to be coupled to the $N = 2$ gauge superfields, the HST hypermultiplet actually has to be generalised to a reducible (relaxed) version that is highly complicated in practice. The most general off-shell formulation of a hypermultiplet is however needed e.g., just in order to write down its couplings which may appear in the LEEA, in a model-independent way.

A universal off-shell solution to all $N = 2$ supersymmetric field theories was proposed in 1984 by Galperin, Ivanov, Kalitzin, Ogievetsky and Sokatchev \[9\]. They introduced the so-called $N = 2$ harmonic superspace (HSS) by adding the extra bosonic variables (=harmonics) parametrizing the sphere $S^2 = SU(2)/U(1)$, to the ordinary $N = 2$ superspace coordinates. It amounts to an introduction of the infinitely many auxiliary fields in terms of the ordinary $N = 2$ superfields. When using the harmonics, one can rewrite the standard $N = 2$ superspace constraints to another form that may be called a ‘zero-curvature representation’ in which the hidden analyticity structure of the constraints becomes manifest. In this reformulation, the harmonics play the role of twistors or spectral parameters that are well-known in the theory of integrable models. As a result, all the $N = 2$ supersymmetric field theories can be naturally formulated in terms of unconstrained so-called analytic $N = 2$ superfields, i.e. fully off-shell. In particular, the off-shell FS hypermultiplet is just described by an analytic superfield $q^+$ of the $U(1)$ charge (+1), whereas the analytic superspace measure has the $U(1)$ charge (−4). A generic hypermultiplet Lagrangian in HSS has to be an analytic function of $q^+$, $\omega$ and the harmonics $u^\pm_i$. In the next subsect. 1.2. we are going to discuss the most general form of LEEA, which is dictated by $N = 2$ supersymmetry alone. The rest of the paper will be devoted to the question how to fix the $N = 2$ supersymmetric Ansatz for the vector and hypermultiplet LEEA completely, when using all the available methods of calculation (Fig. 1 [optional !]).

1.2 Setup

We are now already in a position to formulate the general Ansatz for the $N = 2$ supersymmetric LEEA. As regards the $N = 2$ vector multiplet terms, they can only

\[6\] See subsect. 3.1. for details about the $N = 2$ HSS.
Fig. 1. No comments (special thanks to my children, Denise(6) and Michael(8)).
be of the form
\[ \Gamma_V[W, \bar{W}] = \int_{\text{chiral}} \mathcal{F}(W) + \text{h.c.} + \int_{\text{full}} \mathcal{H}(W, \bar{W}) + \ldots, \] (1.4)
where we have used the fact that the abelian \( N = 2 \) superfield strength \( W \) is an \( N = 2 \) chiral and gauge-invariant superfield. The leading term in eq. (1.4) is given by the chiral \( N = 2 \) superspace integral over a holomorphic function \( \mathcal{F} \) of the \( W \) that is supposed to be valued in the Cartan subalgebra of the gauge group. The next-to-leading-order term is given by the full \( N = 2 \) superspace integral over the real function \( \mathcal{H} \) of \( W \) and \( \bar{W} \). The dots in eq. (1.4) stand for higher-order terms containing the derivatives of \( W \) and \( \bar{W} \).

Similarly, the leading non-trivial term in the hypermultiplet LEA takes the general form
\[ \Gamma_H[q^+, \bar{q}^+; \omega] = \int_{\text{analytic}} \mathcal{K}^{(+4)}(q^+, \bar{q}^+; \omega; u_i^\pm) + \ldots, \] (1.5)
where \( \mathcal{K}^{(+4)} \) is a function of the FS analytic superfield \( q^+ \), its conjugate \( \bar{q}^+ \), the HST analytic superfield \( \omega \) and the harmonics \( u_i^\pm \). The action (1.5) is supposed to be added to the kinetic hypermultiplet action whose analytic Lagrangian is quadratic in \( q^+ \) or \( \omega \), and of \( U(1) \)-charge \((+4)\). A function \( \mathcal{K} \) is called the hyper-Kähler potential. When being arbitrarily chosen in eq. (1.5), it automatically leads to the \( N = 2 \) supersymmetric non-linear sigma-model (NLSM) with a hyper-Kähler metric, just because of the \( N = 2 \) supersymmetry by construction (see an example in subsect. 3.2).

When being expanded in components, the first term in eq. (1.4) also leads, in particular, to the certain Kähler NLSM in the Higgs sector \((A, \bar{A})\). The corresponding NLSM Kähler potential \( K_{\mathcal{F}}(A, \bar{A}) \) is dictated by the holomorphic function \( \mathcal{F} \) as \( K_{\mathcal{F}} = \text{Im}[\bar{A}\mathcal{F}'(A)] \), so that the function \( \mathcal{F} \) plays the role of a potential for this special NLSM Kähler \((\text{not hyper-Kähler})\) geometry \( K_{\mathcal{F}}(A, \bar{A}) \). As regards the hypermultiplet NLSM of eq. (1.5), a relation between the hyper-Kähler potential \( \mathcal{K} \) and the corresponding Kähler potential \( K_{\mathcal{K}} \) of the same NLSM is much more involved. Indeed, it is easy to see that the hyper-Kähler condition on a Kähler potential amounts to a non-linear (Monge-Ampere) partial differential equation which is not easy to solve. It is remarkable that the HSS approach allows one to formally get a ‘solution’ to any hyper-Kähler geometry in terms of the analytic scalar potential \( \mathcal{K} \). Of course, the real problem is now being translated into the precise relation between \( \mathcal{K} \) and the corresponding Kähler potential (or metric) in components, whose determination amounts to solving infinitely many linear differential equations altogether, just in order to eliminate the infinite number of the auxiliary fields (see subsect. 3.2. for an
example). Nevertheless, the hyper-Kähler potential turns out to be an extremely useful notion in dealing with the hypermultiplet LEEA (see sect. 3).

The LEEA gauge-invariant functions $\mathcal{F}(W)$ and $\mathcal{H}(W, \bar{W})$ generically receive both perturbative and non-perturbative contributions,

$$
\mathcal{F} = \mathcal{F}_{\text{per.}} + \mathcal{F}_{\text{inst.}}, \quad \mathcal{H} = \mathcal{H}_{\text{per.}} + \mathcal{H}_{\text{non-per.}},
$$

while the non-perturbative corrections to the holomorphic function $\mathcal{F}$ are entirely due to instantons. This is an important difference from the (bosonic) non-perturbative QCD whose LEEA is dominated by instanton-antiinstanton contributions.

Unlike the vector LEEA, the exact (charged) hypermultiplet LEEA is essentially a perturbative one (see sects. 3 and 4), i.e. it does not receive any instanton corrections,

$$
\mathcal{K}[q^+] = \mathcal{K}_{\text{per.}}[q^+].
$$

It is quite remarkable that the perturbative contributions to the leading and sub-leading terms in the $N = 2$ LEEA entirely come from the one loop only. As regards the leading holomorphic contribution, a standard argument goes as follows: $N = 2$ supersymmetry puts the trace of the energy-momentum tensor $T_{\mu}{}^\mu$ and the axial or chiral anomaly $\partial_{\mu}j_{R}{}^{\mu}$ of the abelian $R$-symmetry into one $N = 2$ supermultiplet. The $T_{\mu}{}^\mu$ is essentially determined by the perturbative renormalization group $\beta$-function, $T_{\mu}{}^\mu \sim \beta(g)FF$, whereas the one-loop contribution to the chiral anomaly, $\partial\cdot j_{R} \sim C_{1-\text{loop}}F^{*}F$, is known to saturate the exact solution to the Wess-Zumino consistency condition for the same anomaly (e.g. the one to be obtained from the index theorem). Hence, $\beta_{\text{per.}}(g) = \beta_{1-\text{loop}}(g)$ by $N = 2$ supersymmetry also. Finally, since the $\beta_{\text{per.}}(g)$ is effectively determined by the second derivative of $\mathcal{F}_{\text{per.}}$, one concludes that $\mathcal{F}_{\text{per.}} = \mathcal{F}_{1-\text{loop}}$ too. This simple component argument can be extended to a proof in the manifestly $N = 2$ supersymmetric way, while the whole chiral perturbative contribution $\int_{\text{chiral}}\mathcal{F}_{\text{per.}}(W)$ arises in $N = 2$ HSS as an anomaly. The non-vanishing central charges of the $N = 2$ supersymmetry algebra turn out to be of crucial importance for the non-vanishing leading holomorphic contribution to the gauge LEEA. Its perturbative part takes, therefore, the form $\mathcal{F}_{\text{per.}}(W) \sim W^2 \log(W^2/\mu^2)$, where $\mu$ is the renormalization group parameter, with the coefficient being fixed by the one-loop $\beta$-function (see sect. 5).

The usual strategy in determining the exact LEEA exploits exact symmetries of a given $N = 2$ quantum gauge theory together with a certain physical input. As the particular important example of $N = 2$ supersymmetric gauge theory, one can use

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7Here and in what follows $g$ denotes the gauge coupling constant.
the \( N = 2 \) supersymmetric QCD with the gauge group \( G_c = SU(N_c) \) and \( N = 2 \) matter to be described by some number \( (N_f) \) of hypermultiplets in the fundamental representation \( N_c + N_c^* \) of the gauge group. Asymptotic freedom then requires that \( N_f < 2N_c \).

All possible \( N = 2 \) supersymmetric vacua can be classified as follows:

- **Coulomb branch**: \( \langle q \rangle = 0 \), while \( \langle A \rangle \neq 0 \); the gauge group \( G_c \) is broken to its abelian subgroup \( U(1)^{\text{rank } G_c} \); non-vanishing 'quark' masses are allowed;

- **Higgs branch**: \( \langle q \rangle \neq 0 \) for some hypermultiplets, while \( \langle A \rangle = 0 \), and all the 'quark' masses vanish; the gauge group \( G_c \) is completely broken;

- **mixed (Coulomb-Higgs) branch**: some \( \langle q \rangle \neq 0 \) and \( \langle A \rangle \neq 0 \); it requires \( N_c > 2 \), in particular.

In the Coulomb branch, one has to specify the both equations (1.6) and (1.7), whereas in the Higgs branch only eq. (1.7) has to be determined. In addition, there may be less symmetric vacua when e.g., a non-vanishing Fayet-Iliopoulos (FI) term, \( \langle D^{ij} \rangle = \xi^{ij} \neq 0 \), is present. \( N = 2 \) supersymmetry may be (spontaneously) broken by the FI-term too.

## 2 Gauge LEEA in the Coulomb branch

Seiberg and Witten [1] gave a full solution to the holomorphic function \( \mathcal{F}(W) \) by using certain physical assumptions (about the global structure of the quantum moduli space \( \mathcal{M}_{qu} \) of vacua) and electric-magnetic duality, i.e. not from the first principles. Their main assumption was the precise value of the Witten index [1] \( \Delta_W = 2 \), i.e. exactly two physical singularities in \( \mathcal{M}_{qu} \). It is the electric-magnetic duality (=S-duality) that was used to connect the weak and strong coupling regions in \( \mathcal{M}_{qu} \).

The Seiberg-Witten solution [1] in the simplest case of the \( SU(2) \) gauge group (no fundamental \( N = 2 \) matter) reads

\[
\begin{align*}
a_D(u) &= \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx \sqrt{x - u}}{\sqrt{x^2 - 1}}, \\
a(u) &= \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx \sqrt{x - u}}{\sqrt{x^2 - 1}},
\end{align*}
\]

where the renormalization-group independent (Seiberg-Witten) scale \( \Lambda^2 = 1 \) and, by definition,

\[
a_D = \frac{\partial \mathcal{F}(a)}{\partial a}.
\]

\(^{8}\)A formal derivation of Witten’s index \( \Delta_W \) from the path integral is plagued with ambiguities.
The solution (2.1) is thus written down in the parametric form. Its holomorphic parameter can be identified with the second Casimir eigenvalue, \( u = \langle \text{tr} A^2 \rangle \), that parametrizes \( M_{\text{qu}} \). The holomorphic function \( F \) can therefore be considered as the one over the quantum moduli space of vacua, while the S-duality can be identified with the action of a modular group. The monodromies of the multi-valued function \( F \) around the singularities are supplied by the perturbative \( \beta \)-functions, whereas the whole function \( F \) is a (unique) solution of the corresponding Riemann-Hilbert problem.

In order to make contact with our general discussion in sect. 1, let’s consider the expansion of the SW solution in the semiclassical region, i.e. when \(|W| \gg \Lambda\),

\[
F(W) = \frac{i}{2\pi} W^2 \log \frac{W^2}{\Lambda^2} + \frac{1}{4\pi i} W^2 \sum_{m=1}^{\infty} c_m \left( \frac{\Lambda^2}{W^2} \right)^{2m},
\]

where we restored the dependence on \( \Lambda \) and written down the interacting terms only. It is now obvious that the first term in eq. (2.3) represents the perturbative (one-loop) contribution whereas the rest is just the sum over the non-perturbative instanton contributions (see subsect. 2.1). It is straightforward to calculate the numerical coefficients \( \{c_m\} \) from the explicit solution (2.1) and (2.2):

\[
\begin{array}{c|cccccc}
 m & 1 & 2^{1/2} & 3 & 4 & 5 & \ldots \\
 c_m & 1/2^5 & 5/2^{14} & 3/2^{18} & 1469/2^{31} & 4471/5 \cdot 2^{34} & \ldots \\
\end{array}
\]

From the technical point of view, the SW solution is nothing but eq. (2.4). It is a challenge for field theorists to reproduce this solution from the first principles.

### 2.1 On the instanton calculations

The SW solution predicts that the non-perturbative holomorphic contributions to the \( N = 2 \) vector gauge LEHA are entirely due to instantons. It is therefore quite natural to try to reproduce them ‘from the first principles’, i.e. from the path integral approach. The \( N = 2 \) supersymmetric instantons are solutions of the classical self-duality equations

\[
F = \ast F, \quad i\gamma^\mu D_\mu \lambda = 0, \quad D^\mu D_\mu A = [\lambda, \lambda],
\]

whose Higgs scalar \( A \) approaches a non-vanishing constant \( (a) \) at the spacial infinity so that the whole configuration has a non-vanishing topological charge \( m \in \mathbb{Z} \).

\[\text{9See refs. [1] for a review.}\]
From the path-integral point of view, the sum over instantons should be of the form

$$F_{\text{inst}} = \sum_{m=1}^{\infty} F_m$$

where

$$F_m = \int d\mu^{(m)}_{\text{inst.}} \exp \left[ -S_{(m)-\text{inst.}} \right] .$$

(2.6)

Each term $F_m$ in this sum can be interpreted as the partition function in the multi($m$)-instanton background. The non-trivial measure $d\mu^{(m)}_{\text{inst.}}$ in eq. (2.6) appears as the result of changing variables from the original fields to the collective instanton coordinates in the path integral, whereas the $S_{(m)-\text{inst.}}$ is just the Euclidean action of the $N = 2$ superinstanton configuration of charge $(m)$. The details about the instanton calculus can be found e.g., in ref. [13]. One usually assumes that the scalar surface term ($\sim \text{tr} \int dS^\mu A^\dagger D_\mu A$) is the only relevant term in the action $S_{(m)-\text{inst.}}$ that contributes. In particular, the bosonic and fermionic determinants, that always appear in the saddle-point expansion and describe small fluctuations of the fields, actually cancel in a supersymmetric self-dual gauge background [14]. Supersymmetry is thus also in charge for the absence of infra-red divergences present in the determinants.

The functional dependence $F_m(a)$ easily follows from the integrated renormalization group (RG) equation for the one-loop $\beta$-function,

$$\exp \left( -\frac{8\pi^2 m}{g^2} \right) = \left( \frac{\Lambda}{a} \right)^4 m,$$

(2.7)

and dimensional reasons as follows:

$$F_m(a) = \frac{a^2}{4\pi i} \left( \frac{\Lambda}{a} \right)^4 m c_m,$$

(2.8)

as it should have been expected, up to a numerical coefficient $c_m$. It is therefore the exact values of the coefficients $\{c_m\}$ that is the issue here, as was already noticed above. Their evaluation can thus be reduced to the problem of calculating the finite-dimensional multi-instanton measure $\{d\mu_{\text{inst.}}^{(m)}\}$.

A straightforward computation of the measure naively amounts to an explicit solution of the $N = 2$ supersymmetric self-duality equations in terms of the collective $N = 2$ instanton coordinates for any positive integer instanton charge. As is well known, the Yang-Mills self-duality differential equations of motion (as well as their supersymmetric counterparts) can be reduced to the purely algebraic (though highly non-trivial) set of equations when using the standard ADHM construction [15]. Unfortunately, an explicit solution to the algebraic ADHM equations is known for only $m = 1$ [16] and $m = 2$ [17], but it is unknown for $m > 2$. Nevertheless, as was recently demonstrated by Dorey, Khoze and Mattis [18], the correct multi-instanton measure for any instanton number can be fixed indirectly, by imposing $N = 2$ supersymmetry
and the cluster decomposition requirements alone, without using the electric-magnetic duality! It is closed enough to a derivation 'from the first principles'. In particular, in the Seiberg-Witten model with the $SU(2)$ gauge group considered above, there exists an instanton solution for $\{c_m\}$ in quadratures \[18\]. The leading instanton corrections for $m = 1, 2$ do agree with the exact Seiberg-Witten solution \[19\].

### 2.2 Seiberg-Witten curve

From the mathematical point of view, the Seiberg-Witten exact solution (2.1) is a solution to the standard Riemann-Hilbert problem of fixing a holomorphic multi-valued function $F$ by its given monodromy and singularities. The number (and nature) of the singularities is the physical input: they are identified with the appearance of massless non-perturbative BPS-like physical states (dyons) like the famous t’Hooft-Polyakov magnetic monopole. The monodromies are supplied by perturbative beta-functions and S-duality.

The solution (2.1) can be nicely encoded in terms of the auxiliary (Seiberg-Witten) elliptic curve $\Sigma_{SW}$ defined by the algebraic equation \[1\]:

$$\Sigma_{SW} : \quad y^2 = (v^2 - u)^2 - \Lambda^4 . \quad (2.9)$$

The multi-valued functions $a_D(u)$ and $a(u)$ now appear by integration of a certain abelian differential $\lambda$ (of the 3rd kind) over the torus periods $\alpha$ and $\beta$ of $\Sigma_{SW}$:

$$a_D(u) = \oint_\beta \lambda , \quad a(u) = \oint_\alpha \lambda , \quad \text{where} \quad \lambda = v^2 \frac{dv}{y(v, u)} . \quad (2.10)$$

This fundamental relation to the theory of Riemann surfaces can be generalized further to the other simply-laced gauge groups and $N = 2$ super-QCD as well \[20\] \[21\]. For instance, the solution to the LEEA of the pure $N = 2$ gauge theory with the gauge group $SU(N_c)$ is encoded in terms of the hyperelliptic curve of genus $(N_c - 1)$, whose algebraic equation reads \[20\]

$$\Sigma_{SW} : \quad y^2 = W_{A_{N_c-1}}^2(v, \vec{u}) - \Lambda^{2N_c} . \quad (2.11)$$

The polynomial $W_{A_{N_c-1}}(v, \vec{u})$ introduced above is known in mathematics \[22\] as the simple singularity associated with $A_{N_c-1} \sim SU(N_c)$, or the Landau-Ginzburg superpotential in the $N = 2$ supersymmetric $2d$ conformal field theory \[23\]. It is given by

$$W_{A_{N_c-1}}(v, \vec{u}) = \sum_{l=1}^{N_c} \left( v - \vec{\lambda}_l \cdot \vec{u} \right) = v^{N_c} - \sum_{l=0}^{N_c-2} u_{l+2}(\vec{a})v^{N_c-2-l} , \quad (2.12)$$
where $\tilde{\lambda}_l$ are the weights of $SU(N_c)$ in the fundamental representation, and $\tilde{u}$ are the Casimir eigenvalues, i.e. the Weyl group-invariant polynomials in $\tilde{a}$ to be constructed by a standard (classical) Miura transformation. The simple singularity is the only trace of the fundamental non-abelian gauge symmetry in the Coulomb branch.

Adding the (fundamental) $N = 2$ matter does not pose a problem in calculating the corresponding Seiberg-Witten curve. It reads \cite{21}

\[ \Sigma_{SW} : \quad y^2 = W_{A_{N_c-1}}^2(v, \tilde{u}) - A^{2N_c-N_f} \prod_{j=1}^{N_f}(v - m_j), \]  

(2.13)

where $\{m_j\}$ are the bare hypermultiplet masses of $N_f$ hypermultiplets ($N_f < N_c$), in the fundamental representation of the gauge group $SU(N_c)$.

The minimal data $(\Sigma_{SW}, \lambda)$ needed to reproduce the Seiberg-Witten exact solution can be associated with a certain two-dimensional (2d) integrable system \cite{24}. In addition, the SW potential $F$ is a solution to the Dijkgraaf-Verlinde-Verlinde-Witten-type \cite{23} non-linear differential equations known in the 2d (conformal) topological field theory \cite{20}:

\[ F_i F^{-1}_j = F_j F^{-1}_k F_i, \quad \text{where} \quad (F_i)_{jk} \equiv \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}. \]  

(2.14)

There also exists another non-trivial equation for $F$ which is a consequence of the anomalous (chiral) $N = 2$ superconformal Ward identities in 4d \cite{27}.

Though the mathematical relevance of the Seiberg-Witten curve is quite clear from what was already written above, its geometrical origin and physical interpretation are still obscure at this point. It is the issue that can be understood in the context of M theory (see sect. 4).

### 3 Hypermultiplet LEEA in the Coulomb branch

The previous sect. 2 was entirely devoted to the holomorphic function $F$ appearing in the gauge LEEA (1.4) in the Coulomb branch. In this sect. 3 we are going to discuss another analytic function $K$ dictating the hypermultiplet LEEA (1.5). The function $K$ is known as a hyper-Kähler potential, \footnote{Any 4d, globally $N = 2$ supersymmetric NLSM with the highest physical spin 1/2 necessarily has a hyper-Kähler metric in its kinetic terms.} and it plays the role in the hypermultiplet LEEA similar to that of $F$ in the vector gauge LEEA. Since the very notion of the hyper-Kähler potential, in fact, requires an introduction of the harmonic superspace.
(HSS), in the next subsect. 3.1 a brief introduction into the \( N = 2 \) HSS is provided (see refs. [9, 29] for more details).

### 3.1 \( N = 2 \) harmonic superspace

The \( N = 2 \) supersymmetric 4d field theories can be formulated in a manifestly \( N = 2 \) supersymmetric way in \( N = 2 \) superspace, in terms of certain constraints. Unfortunately, the constraints defining a (non-abelian) \( N = 2 \) vector multiplet or a hypermultiplet in the ordinary \( N = 2 \) superspace do not have a manifestly holomorphic (or analytic) structure, and they do not have a simple solution in terms of unconstrained \( N = 2 \) superfields which are needed for quantization. The situation is even more dramatic for the hypermultiplets whose known off-shell formulations in the ordinary \( N = 2 \) superspace are not universal so that their practical meaning is very limited.

In the HSS formalism, the standard \( N=2 \) superspace \( \mathbb{Z}^M = (x^m, \theta^\alpha_i, \bar{\theta}^{\alpha i}) \), \( \alpha = 1, 2, \text{ and } i = 1, 2 \), is extended by adding the bosonic variables (or ‘zweibeins’) \( u^{\pm i} \) parameterizing the sphere \( S^2 \sim SU(2)/U(1) \). By using these extra variables one can make manifest the hidden analyticity structure of all the standard \( N = 2 \) superspace constraints as well as find their solutions in terms of unconstrained (analytic) superfields. The harmonic variables have the following fundamental properties:

\[
\begin{pmatrix}
u^{+i} \\ u^{-i}
\end{pmatrix} \in SU(2), \quad \text{so that } \quad u^{+i}u^{-i} = 1, \quad \text{and } \quad u^{+i}u_{-i}^{+} = u^{-i}u_{-i}^{-} = 0. \tag{3.1}
\]

Instead of using an explicit parameterization of the sphere \( S^2 \), it is convenient to deal with functions of zweibeins, that carry a definite \( U(1) \) charge \( q \) to be defined by \( q(u^{\pm i}) = \pm 1 \), and use the following integration rules [9]:

\[
\int du = 1, \quad \int du u^{+(i_1} u^{j_1} u^{+(i_2} u^{j_2} \cdots u^{-(j_n)} = 0, \quad \text{when } \quad m + n > 0. \tag{3.2}
\]

It is obvious that any integral over a \( U(1) \)-charged quantity vanishes.

The usual complex conjugation does not preserve analyticity (see below). However, when being combined with another (star) conjugation that only acts on the \( U(1) \) indices as \( (u_{-i}^{+})^* = u_{-i}^{-} \) and \( (u_{-i}^{-})^* = -u_{-i}^{+} \), it does preserve analyticity. One easily finds [2]

\[
\begin{align*}
\star u^{\pm i} &= -u_{-i}^{\pm}, \\
\star u_{-i}^{\pm} &= u_{-i}^{\pm}.
\end{align*}
\tag{3.3}
\]

\footnote{Since our spacetime is flat we identify the flat \((m = 0, 1, 2, 3)\) and curved \((\mu = 0, 1, 2, 3)\) spacetime vector indices.}
The covariant derivatives with respect to the zweibeins, that preserve the defining conditions (3.1), are given by

\[ D^{++} = u^i \frac{\partial}{\partial u^- i} , \quad D^{--} = u^{-i} \frac{\partial}{\partial u^+ i} , \quad D^0 = u^i \frac{\partial}{\partial u^+ i} - u^{-i} \frac{\partial}{\partial u^- i} . \] (3.4)

It is easy to check that they satisfy the \( SU(2) \) algebra,

\[ [D^{++}, D^{--}] = D^0 , \quad [D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm} . \] (3.5)

The key feature of the \( N = 2 \) HSS is the existence of the so-called analytic subspace parameterized by the coordinates

\[ (\zeta, u) = \{ x_A^m = x^m - 2i \theta^i (\sigma^m \bar{\theta}^j) u^+_i u^-_j , \quad \theta^+_\alpha = \theta^i \alpha u^+_i , \quad \bar{\theta}^+_\alpha = \bar{\theta}^i \alpha u^+_i , \quad u^\pm_i \} , \] (3.6)

which is invariant under \( N = 2 \) supersymmetry, and is closed under the combined conjugation of eq. (3.3) [9]. It allows one to define the analytic superfields of any \( U(1) \) charge \( q \), by the analyticity conditions

\[ D^+_\alpha \phi(q) = \bar{D}^+_\alpha \phi(q) = 0 , \quad \text{where} \quad D^+_\alpha = D^i_\alpha u^+_i \quad \text{and} \quad \bar{D}^+_\alpha = \bar{D}^i_\alpha u^+_i , \] (3.7)

and introduce the analytic measure \( d\zeta^{(-4)} du \equiv d^4 x_A d^2 \theta^+ d^2 \bar{\theta}^+ du \) of charge \((-4)\), so that the full measure in the \( N = 2 \) HSS can be written down as

\[ d^4 x d^4 \theta d^4 \bar{\theta} du = d\zeta^{(-4)} du (D^+)^4 , \] (3.8)

where

\[ (D^+)^4 = \frac{1}{16} (D^+)^2 (\bar{D}^+)^2 = \frac{1}{16} (D^{+\alpha} D^{+}_\alpha)(\bar{D}^{+\dot{\alpha}} \bar{D}^{+\dot{\alpha}}) . \] (3.9)

In the analytic subspace, the harmonic derivative \( D^{++} \) takes the form

\[ D^{++}_{\text{analytic}} = D^{++} - 2i \theta^+ \sigma^m \bar{\theta}^+ \partial_m , \] (3.10)

it preserves analyticity, and it allows one to integrate by parts. Both the original (central) basis and the analytic one can be used on equal footing in the HSS. In what follows we omit the subscript \( \text{analytic} \) at the covariant derivatives in the analytic basis, in order to simplify the notation.

It is the advantage of the analytic \( N = 2 \) HSS compared to the ordinary \( N = 2 \) superspace that both an off-shell \( N = 2 \) vector multiplet and an off-shell hypermultiplet can be introduced there on equal footing. There exist two off-shell hypermultiplet versions in HSS, which are dual to each other. The so-called \( \text{Fayet-Sohnius (FS)} \) hypermultiplet is defined as an unconstrained complex analytic superfield \( q^+ \) of the
$U(1)$-charge (+1), whereas its dual, called the **Howe-Stelle-Townsend** (HST) hypermultiplet, is a real unconstrained analytic superfield $\omega$ with the vanishing $U(1)$-charge. The on-shell physical components of the FS hypermultiplet comprise an $SU(2)_A$ doublet of complex scalars and a Dirac spinor which is a singlet w.r.t. the $SU(2)_A$. The on-shell physical components of the HST hypermultiplet comprise real singlet and triplet of scalars, and a doublet of chiral spinors. The FS hypermultiplet is natural for describing a charged $N = 2$ matter (e.g. in the Coulomb branch), whereas the HST hypermultiplet is natural for describing a neutral $N = 2$ matter or the Higgs branch. Similarly, an $N = 2$ vector multiplet is described by an unconstrained analytic superfield $V^{++}$ of the $U(1)$-charge (+2). The $V^{++}$ is real in the sense $V^{++} = V^{++}$, and it can be naturally introduced as a connection to the harmonic derivative $D^{++}$.

A free FS hypermultiplet action is given by

$$S[q] = -\int d\zeta (-4) du \overline{q}^+ D^{++} q^+ ,$$

whereas its minimal coupling to an $N = 2$ gauge superfield reads

$$S[q, V] = -\text{tr} \int d\zeta (-4) du \overline{q}^+ (D^{++} + iV^{++}) q^+ ,$$

where the both superfields, $q^+$ and $V^{++}$, are now Lie algebra-valued.

It is not difficult to check that the free FS hypermultiplet equations of motion, $D^{++} q^+ = 0$, imply $q^+ = q'(z) u_i^+$ as well as the usual (on-shell) Fayet-Sohnius constraints \[3\] in the ordinary $N = 2$ superspace,

$$D^{(i} q^{j)}(z) = D^{(i} (q^{j)}(z) = 0 .$$

Similarly, a free action of the HST hypermultiplet is given by

$$S[\omega] = -\frac{1}{2} \int d\zeta (-4) du (D^{++} \omega)^2 ,$$

and it is equivalent on-shell to the standard $N = 2$ tensor (or linear) multiplet (see subsect. 3.3).

The standard Grimm-Sohnius-Wess (GSW) constraints \[8\] defining the $N = 2$ super-Yang-Mills theory in the ordinary $N = 2$ superspace imply the existence of a (covariantly) chiral \[4\] and gauge-covariant $N = 2$ SYM field strength $W$ satisfying, in addition, the reality condition (or the Bianchi ‘identity’)

$$D^{(i} \overline{D}_{j)\alpha} W = \overline{D}^{(i} \cdot \overline{D}_{j)\alpha} \overline{W} .$$

\[12\] A covariantly-chiral superfield can be transformed into a chiral superfield by field redefinition.
Unlike the $N = 1$ SYM theory, an $N = 2$ supersymmetric solution to the non-abelian $N = 2$ SYM constraints in the ordinary $N = 2$ superspace is not known in an analytic form. It is the $N = 2$ HSS reformulation of the $N = 2$ SYM theory that makes it possible [9]. The exact non-abelian relation between the constrained, harmonic-independent superfield strength $W$ and the unconstrained analytic (harmonic-dependent) superfield $V^{++}$ is given in refs. [9, 29], and it is highly nonlinear. It is merely its abelian version that is needed for calculating the perturbative LEEA in the Coulomb branch. The abelian relation is given by

$$W = \frac{1}{4} \{D^+_{\alpha}, D^{-\alpha}\} = -\frac{1}{4} (\bar{D}^+)^2 A^{--} , \quad (3.16)$$

where the non-analytic harmonic superfield connection $A^{--}(z, u)$ to the derivative $D^{--}$ has been introduced, $D^{--} = D^{--} + i A^{--}$. As a consequence of the $N = 2$ HSS abelian constraint $[D^{++}, D^{--}] = D^0 = D^0$, the connection $A^{--}$ satisfies the relation

$$D^{++} A^{--} = D^{--} V^{++} , \quad (3.17)$$

whereas eq. (3.15) can be rewritten to the form

$$(D^+)^2 W = (\bar{D}^+)^2 \bar{W} . \quad (3.18)$$

A solution to the $A^{--}$ in terms of the analytic unconstrained superfield $V^{++}$ easily follows from eq. (3.17) when using the identity [29]

$$D^{++}_{1+} (u_1^+ u_2^+)^{-2} = D^{--}_{1-} \delta^{(2,-2)}(u_1, u_2) , \quad (3.19)$$

where we have introduced the harmonic delta-function $\delta^{(2,-2)}(u_1, u_2)$ and the harmonic distribution $(u_1^+ u_2^+)^{-2}$ according to their definitions in refs. [9, 29], hopefully, in the self-explaining notation. One finds [30]

$$A^{--}(z, u) = \int dv \frac{V^{++}(z, v)}{(u^+ v^+)^2} , \quad (3.20)$$

and

$$W(z) = -\frac{1}{4} \int du (\bar{D}^-)^2 V^{++}(z, u) , \quad \bar{W}(z) = -\frac{1}{4} \int du (D^-)^2 V^{++}(z, u) , \quad (3.21)$$

by using the identity

$$u_i^+ = v_i^+(v^--u^+) - v_i^-(u^+ v^+) , \quad (3.22)$$

which is the obvious consequence of the definitions (3.1).

The equations of motion are given by the vanishing analytic superfield

$$(D^+)^4 A^{--}(z, u) = 0 , \quad (3.23)$$
while the corresponding action reads

\[ S[V] = \frac{1}{4} \int d^4xd^4\theta W^2 + \text{h.c.} = \frac{1}{2} \int d^4xd^4\theta d^4\bar{\theta} du V^{++}(z,u) A^-(z,u) \]

\[ = \frac{1}{2} \int d^4xd^4\theta d^4\bar{\theta} d\bar{u}_1 du_2 \frac{V^{++}(z,u_1) V^{++}(z,u_2)}{(u_1^+ u_2^+)^2} . \]  

(3.24)

In a WZ-like gauge, the abelian analytic pre-potential \( V^{++} \) amounts to the expression

\[
V^{++}(x_A, \theta^+, \bar{\theta}^+, u) = \bar{\theta}^+ \theta^+ a(x_A) + \bar{a}(x_A) \theta^+ \bar{\theta}^+ - 2i \theta^+ \sigma^m \bar{\theta}^+ V_m(x_A) \\
+ \bar{\theta}^+ \theta^+ \theta^\alpha \psi^i(x_A) u_i^- + \theta^+ \bar{\theta}^\alpha \bar{\psi}^i(x_A) u_i^- \\
+ \theta^+ \theta^+ \bar{\theta}^+ D^{(ij)}(x_A) u_i^- u_j^- ,
\]

(3.25)

where \((a, \psi^i, V_m, D^{ij})\) are the usual \(N=2\) vector multiplet components.

The (BPS) mass of a hypermultiplet can only come from the central charges of the \(N=2\) SUSY algebra since, otherwise, the number of the massive hypermultiplet components has to be increased. The most natural way to introduce central charges \((Z, \bar{Z})\) is to identify them with spontaneously broken \(U(1)\) generators of dimensional reduction from six dimensions via the Scherk-Schwarz mechanism. Being rewritten to six dimensions, eq. (3.10) implies the additional ‘connection’ term in the associated four-dimensional harmonic derivative

\[
D^{++} = D^{++} + v^{++}, \quad \text{where} \quad v^{++} = i(\theta^+ \theta^+) \bar{Z} + i(\bar{\theta}^+ \bar{\theta}^+) Z .
\]

(3.26)

Comparing eq. (3.26) with eqs. (3.12) and (3.21) clearly shows that the \(N=2\) central charges can be equivalently treated as a non-trivial \(N=2\) gauge background, with the covariantly constant chiral superfield strength

\[
\langle W \rangle = \langle a \rangle = Z ,
\]

(3.27)

where eq. (3.25) has been used too. See refs. for more details.

### 3.2 Taub-NUT metric or KK-monopole

Since the HSS formulation of hypermultiplets has the manifest off-shell \(N=2\) supersymmetry, it is perfectly suitable for discussing possible hypermultiplet self-interactions which are highly restricted by \(N=2\) supersymmetry. Moreover, the manifestly \(N=2\) supersymmetric Feynman rules can be derived in HSS. The latter can be used to actually calculate the hypermultiplet LEEA (see below).
To illustrate the power of HSS, let’s consider a single FS hypermultiplet for simplicity. Its free action in HSS can be rewritten in the pseudo-real notation, \( q^a = (q^+, q^-) \), \( q^a = \varepsilon^{ab} q_b \), \( a = 1, 2 \), as follows:

\[
S[q] = -\frac{1}{2} \int_{\text{analytic}} q^a \mathcal{D}^{++} q^+_a ,
\]

where the derivative \( \mathcal{D}^{++} \) (in the analytic basis) includes central charges in accordance with eq. (3.26). It is obvious from eq. (3.28) that the action \( S[q] \) has the extended internal symmetry given by

\[
SU(2)_A \otimes SU(2)_{\text{extra}} ,
\]

where the \( SU(2)_A \) is the automorphism symmetry of the \( N = 2 \) supersymmetry algebra and the \( SU(2)_{\text{extra}} \) acts on the extra indices \( a \) only. Adding a minimal interaction with an abelian \( N = 2 \) vector superfield \( V^{++} \) in eq. (3.28) obviously breaks the internal symmetry (3.29) down to a subgroup

\[
SU(2)_A \otimes U(1)_{e} ,
\]

It is now easy to see that the \textit{only} FS hypermultiplet self-interaction, that is consistent with the internal symmetry (3.30), is given by the hyper-Kähler potential

\[
\mathcal{K}^{(+4)} = \frac{\lambda}{2} \left( \mathbf{q}^+ q^+_+ \right)^2 ,
\]

since it is the only admissible term of the \( U(1)-\text{charge} \ (+4) \) which can be added to the FS hypermultiplet action (3.28). We thus get an answer for the hypermultiplet LEEA in the Coulomb branch almost for free, up to the induced NLSM coupling constant \( \lambda \).

Similarly, the unique FS hypermultiplet self-interaction in the \( N = 2 \) super-QCD with \( N_c = 3 \) colors and \( N_f \) flavors, and vanishing bare hypermultiplet masses, that is consistent with the \( SU(N_f) \otimes SU(2)_A \otimes U(1)^2 \) internal symmetry, reads

\[
\mathcal{K}^{(+4)}_{\text{QCD}} = \frac{\lambda}{2} \sum_{i,j=1}^{N_f} \left( \mathbf{7}^+, \mathbf{q}_i^+ \right) \left( \mathbf{7}^+, \mathbf{q}_j^+ \right) ,
\]

where the dots stand for contractions of color indices.

The induced coupling constant \( \lambda \) in eq. (3.31) is entirely determined by the one-loop HSS graph shown in Fig. 2. Since the result vanishes \( (\lambda = 0) \) in the absence of

\footnote{It is easy to keep track of the \( SU(2)_A \) symmetry in the \( N = 2 \) HSS where this symmetry amounts to the absence of an explicit dependence of a HSS Lagrangian upon the harmonic variables \( u_i^\pm \).}
Fig. 2. The one-loop harmonic supergraph contributing to the induced hypermultiplet self-interaction.

central charges, let’s assume that \( Z = a \neq 0 \), i.e. we are in the Coulomb branch. The free HSS actions of an \( N = 2 \) vector multiplet and a hypermultiplet given above are enough to compute the corresponding \( N = 2 \) superpropagators. The \( N = 2 \) vector multiplet action takes the particularly simple form in the \( N = 2 \) super-Feynman gauge (there are no central charges for the \( N = 2 \) vector multiplet),

\[
S[V]_{\text{Feynman}} = \frac{1}{2} \int_{\text{analytic}} V^{++} \Box V^{++} ,
\]

so that the corresponding analytic HSS propagator (the wave lines in Fig. 2) reads

\[
i \left\langle V^{++}(1) V^{++}(2) \right\rangle = \frac{1}{\Box_1} (D_1^{++})^4 \delta^{12}(Z_1 - Z_2) \delta^{(-2,2)}(u_1, u_2) ,
\]

where the harmonic delta-function \( \delta^{(-2,2)}(u_1, u_2) \) has been introduced \[29\]. The FS hypermultiplet HSS propagator (solid lines in Fig. 2) with non-vanishing central charges is more complicated \[30, 35\]:

\[
i \left\langle q^{+}(1) q^{+}(2) \right\rangle = \frac{-1}{\Box_1 + \bar{a}a} \frac{(D_1^{++})^4 (D_2^{++})^4}{(u_1^+ u_2^+)^3} e^{\tau_3 [v(2) - v(1)]} \delta^{12}(Z_1 - Z_2) ,
\]

\[14\] The same conclusion also follows from the \( N = 1 \) superspace calculations \[8\].

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where \( v \) is the so-called ‘bridge’ satisfying the equation \( D^{++}e^v = 0 \). One easily finds that
\[
iv = -a(\bar{\theta}^+ \theta^-) - \bar{a}(\theta^+ \theta^-) .
\] (3.36)
The rest of the \( N = 2 \) HSS Feynman rules is very similar to that of the ordinary \((N = 0)\) Quantum Electrodynamics (QED).

A calculation of the HSS graph in Fig. 2 is now straightforward, while the calculational details are given in ref. [35]. One finds the predicted form of the induced hyper-Kähler potential as in eq. (3.31) indeed, with the induced NLSM coupling constant given by
\[
\lambda = \frac{g^4}{\pi^2} \left[ \frac{1}{m^2} \ln \left( 1 + \frac{m^2}{\Lambda^2} \right) - \frac{1}{\Lambda^2 + m^2} \right] ,
\] (3.37)
where \( g \) is the gauge coupling constant, \( m^2 = |a|^2 \) is the hypermultiplet BPS mass, and \( \Lambda \) is the IR-cutoff parameter. Note that \( \lambda \to 0 \) when the central charge \( a \to 0 \).

Yet another technical problem is how to decode the HSS result (3.31) in the conventional component form. In other words, one still has to find an explicit hyper-Kähler metric that corresponds to the hyper-Kähler potential (3.31). The general procedure of getting the component form of the bosonic NLSM from a hypermultiplet self-interaction in HSS consists of the following steps:

- expand the equations of motion in the Grassmann (anticommuting) coordinates, and ignore all the fermionic field components,
- solve the kinematical linear differential equations for all the auxiliary fields, thus eliminating the infinite tower of them in the harmonic expansion of the hypermultiplet HSS analytic superfields;
- substitute the solution back into the HSS hypermultiplet action, and integrate over all the anitcommuting and harmonic HSS coordinates.

Of course, it is not always possible to actually perform this procedure. For instance, just the second step above would amount to solving infinitely many linear differential equations altogether. However, just in the case of eq. (3.31), the explicit solution exists \([37, 35]\). When using the parametrization
\[
q^+ \big|_{\theta = 0} = f^i(x)u^+_i \exp \left[ \lambda f^{ij}(x)f^k(x)u^+_j u^-_k \right] ,
\] (3.38)
one finds the 4d bosonic NLSM action
\[
S_{\text{NLSM}} = \int d^4x \left\{ g_{ij} \partial_m f^i \partial^m f^j + \bar{g}^{ij} \partial_m \bar{f}_i \partial^m \bar{f}_j + h^{ij} \partial_m f^j \partial^m \bar{f}_i - V(f) \right\} ,
\] (3.39)
whose metric is given by
\[ g_{ij} = \frac{\lambda(2 + \lambda f \bar{f})}{4(1 + \lambda f \bar{f})} f_i f_j, \quad \bar{g}^{ij} = \frac{\lambda(2 + \lambda f \bar{f})}{4(1 + \lambda f \bar{f})} f^i f^j, \]
\[ h^i{}_j = \delta^i{}_j (1 + \lambda f \bar{f}) - \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} f^i f_j, \quad f \bar{f} \equiv f^i \bar{f}^i, \] (3.40)
whereas the induced scalar potential reads
\[ V(f) = |Z|^2 \frac{f \bar{f}}{1 + \lambda f \bar{f}}. \] (3.41)

In the form (3.40) the induced metric is apparently free from any singularities. It is generically non-trivial to compare the induced NLSM metric with any standard hyper-Kähler metric since the NLSM metrics themselves are defined modulo field redefinitions, i.e. modulo four-dimensional diffeomorphisms in the case under consideration. Fortunately, it is known how to transform the metric (3.40) to the standard Taub-NUT form:
\[ ds^2 = \frac{r + M}{2(r - M)} dr^2 + \frac{1}{2} (r^2 - M^2) (d\varphi^2 + \sin^2 \varphi d\psi^2) + 2M^2 \left( \frac{r - M}{r + M} \right) (d\psi + \cos \varphi d\varphi)^2, \] (3.42)
by using the following change of variables [37]:
\[ f^1 = \sqrt{2M(r - M)} \cos \frac{i}{2} (\psi + \varphi), \]
\[ f^2 = \sqrt{2M(r - M)} \sin \frac{i}{2} (\psi - \varphi), \] (3.43)
\[ f \bar{f} = 2M(r - M), \quad r \geq M = \frac{1}{2\sqrt{\lambda}}, \]
where \( M = \frac{1}{2} \lambda^{-1/2} \sim g^{-2} \) is the mass of the Taub-NUT instanton, also known as the KK-instanton [38].

Therefore, the induced metric of the hypermultiplet LEA in the Coulomb branch is generated in one loop, and it is given by the Taub-NUT or its higher-dimensional generalizations. The non-trivial scalar potential is also generated by quantum corrections, with the BPS mass being unrenormalized as it should.

### 3.3 Duality transformation and \( N = 2 \) tensor multiplet

There exists an interesting connection between the FS hypermultiplet Taub-NUT self-interaction in the \( N = 2 \) harmonic superspace and the \( N = 2 \) tensor (or linear)
multiplet self-interaction in the ordinary \( N = 2 \) superspace. Namely, the \( N = 2 \) supersymmetric Taub-NUT NLSM is equivalent to a sum of the naive (quadratic in the fields and non-conformal) and improved (non-polynomial in the fields and \( N = 2 \) superconformally invariant) actions for the \( N = 2 \) tensor multiplet in the ordinary \( N = 2 \) superspace!

It is worth mentioning here that the \( N = 2 \) tensor multiplet in the ordinary \( N = 2 \) superspace is defined by the constraints

\[
D_\alpha (iL^{ik})(Z) = \bar{D}_\alpha (iL^{jk})(Z) = 0 , \quad (3.44)
\]

and the reality condition

\[
\overline{L^{ij}} = \varepsilon_{ik} \varepsilon_{jl} L^{kl} . \quad (3.45)
\]

Unlike the FS hypermultiplet in the ordinary \( N = 2 \) superspace, the constraints (3.44) are off-shell, i.e. they do not imply the equations of motion for the components of the \( N = 2 \) tensor multiplet. The \( N = 2 \) tensor multiplet itself can be identified as a restricted HST hypermultiplet (i.e. as an analytic \( \omega \) superfield subject to extra off-shell constraints), while its \( N = 2 \) supersymmetric self-interactions are a subclass of those for \( \omega \) [39]. The \( N = 2 \) tensor multiplet has \( 8_B \oplus 8_F \) off-shell components:

\[
\bar{L}, \quad \zeta^i_\alpha , \quad B , \quad E'_m = \frac{1}{2} \varepsilon_{mpq} \partial_n E_{pq}, \quad (3.46)
\]

where \( \bar{L} \) is the scalar \( SU(2)_A \) triplet, \( \bar{L} = \text{tr}(\overline{\tau} L) \) and \( \overline{\tau} \) are Pauli matrices, \( \zeta^i_\alpha \) is a chiral spinor doublet, \( B \) is a complex auxiliary scalar, and \( E_{mn} \) is a gauge antisymmetric tensor whose field strength is \( E'_m \).

Let’s start with our induced hypermultiplet LEEA

\[
S[q^+]_{\text{Taub–NUT}} = \int_{\text{analytic}} \left[ \sqrt{\frac{\lambda}{q^+}} D^{++} q^+ + \frac{\lambda}{2} (q^+) (\sqrt{\frac{q^+}{\lambda}})^2 \right] , \quad (3.47)
\]

and make the following substitution of the HSS superfield variables [33]:

\[
\sqrt{\lambda} q^+ = -i \left( 2u_1^+ + ig^{++} u_1^- \right) e^{-i\overline{\omega}/2} , \quad \sqrt{\lambda} q^- = i \left( 2u_2^+ - ig^{++} u_2^- \right) e^{i\overline{\omega}/2} , \quad (3.48)
\]

where

\[
g^{++}(l, u) \equiv \frac{2(l^{++} - 2iu_1^+ u_2^+)}{1 + \sqrt{1 - 4u_1^+ u_2^+ u_1^- u_2^- - 2il^{++} u_1^- u_2^-}} , \quad (3.49)
\]

and \( (l^{++}, \omega) \) are the new dimensionless analytic superfields. It is not difficult to check that eqs. (3.48) and (3.49) imply, in particular, that

\[
\lambda \sqrt{q^+ q^+} = 2il^{++} , \quad (3.50)
\]
whereas the action (3.47) takes the form (after the rescaling \( t^{++} \equiv \sqrt{\lambda} L^{++} \) and \( \tilde{\omega} = \sqrt{\lambda} \omega \)):

\[
S[L^{++}; \omega]_{\text{Taub-NUT}} = S_{\text{free}}[L^{++}; \omega] + S_{\text{impr.}}[L^{++}; \omega] ,
\]

(3.51)

where

\[
S_{\text{free}}[L^{++}; \omega] = \frac{1}{2} \int_{\text{analytic}} \left[ (L^{++})^2 + \omega D^{++}L^{++} \right] ,
\]

(3.52)

and

\[
S_{\text{impr.}}[L^{++}; \omega] = \frac{1}{2\lambda} \int_{\text{analytic}} \left[ g^{++}(L; u) \right]^2 .
\]

(3.53)

The action (3.51) or (3.52) contains \( \omega \) as a Lagrange multiplier. On the one hand, varying it w.r.t. \( \omega \) yields the constraint

\[
D^{++}L^{++} = 0 ,
\]

(3.54)

that, in its turn, implies \( L^{++} = u_i^+ u_j^+ L^{ij}(Z) \) and eq. (3.44). Therefore, the actions (3.51) and (3.52) describe an \( N = 2 \) tensor multiplet in the \( N = 2 \) HSS. On the other hand, one can vary the action (3.52) w.r.t. \( L^{++} \) first. One finds that

\[
L^{++} = D^{++} \omega .
\]

(3.55)

Hence, \( L^{++} \) can be removed altogether in favor of \( \omega \). It is just a manifestation of the existing classical duality between the FS hypermultiplet \( q^+ \) and the HST hypermultiplet \( \omega \) in the \( N = 2 \) HSS.

The action (3.53) describes the so-called improved \( N = 2 \) tensor multiplet \[40\]. It can be shown that it is fully invariant under the rigid \( N = 2 \) superconformal symmetry, while the associated hyper-Kähler metric is equivalent to the flat metric up to a 4d diffeomorphism \[40\]. However, the sum of the actions (3.52) and (3.53) describes an interacting theory, and it is just the Taub-NUT or the KK-monopole.

Because of this connection between certain \( N = 2 \) supermultiplets and their self-interactions in the HSS, it should not be very surprising that the Taub-NUT self-interaction can also be reformulated in the ordinary \( N = 2 \) superspace in terms of the \( N = 2 \) tensor multiplet alone, just as a sum of its naive and improved actions. The most elegant formulation of the latter exists in the projective \( N = 2 \) superspace \[11, 42\] in which the harmonic variables are replaced by a single complex projective variable \( \xi \in CP(1) \). Unlike the \( N = 2 \) HSS, the projective \( N = 2 \) superspace does not introduce any extra auxiliary fields beyond those already present in the off-shell \( N = 2 \) tensor multiplet. The starting point now are the defining constraints (3.44) for the
\(N = 2\) tensor multiplet in the ordinary \(N = 2\) superspace. It is not difficult to check that they imply (see ref. \([42]\) for more details and generalizations)

\[
\nabla_\alpha G \equiv (D^1_\alpha + \xi D^2_\alpha)G = 0, \quad \Delta_\alpha G \equiv (\bar{D}^1_\alpha + \xi \bar{D}^2_\alpha)G = 0,
\]

for any function \(G(Q(\xi), \xi)\) that is a function of \(Q(\xi) \equiv \xi_i \xi_j L_{ij}(Z)\) and \(\xi_i \equiv (1, \xi)\) only.

It follows that we can build an \(N = 2\) superinvariant just by integrating \(G\) over the rest of the \(N = 2\) superspace coordinates in the directions which are 'orthogonal' to those in eq. (3.56), namely,

\[
S_{inv}[L] = \int d^4x \frac{1}{2\pi i} \oint_C d\xi \tilde{\nabla}^2 \tilde{\Delta}^2 G(Q, \xi), \quad (3.57)
\]

where we have introduced the new derivatives

\[
\tilde{\nabla}_\alpha = \xi D^1_\alpha - D^2_\alpha, \quad \tilde{\Delta}_\alpha = \xi \bar{D}^1_\alpha - \bar{D}^2_\alpha.
\]

The choice of the function \(G(Q, \xi)\) and the contour \(C\) in the complex \(\xi\)-plane, which yields the Taub-NUT self-interaction in eq. (3.57), is given by \([11, 42]\)

\[
S_{Taub-NUT}[L] = \int d^4x \tilde{\nabla}^2 \tilde{\Delta}^2 \left\{ \int_{C_1} d\xi \frac{Q^2}{2\xi} + \frac{1}{\sqrt{\lambda}} \int_{C_2} d\xi Q \ln(\sqrt{\lambda}Q) \right\}, \quad (3.59)
\]

where the contour \(C_1\) goes around the origin, whereas the contour \(C_2\) encircles the roots of the quadratic equation \(Q(\xi) = 0\) in the complex \(\xi\)-plane.

Finally, one may wonder, in which sense an \(N = 2\) tensor multiplet action describes a 4d, \(N = 2\) supersymmetric NLSM with the highest physical spin 1/2, because of the apparent presence of the gauge antisymmetric tensor \(E_{mn}\) among the \(N = 2\) tensor multiplet components — see eq. (3.46). A detailed investigation of the component action, that follows from the superspace action (3.59), shows that the tensor \(E_{mn}\) and its field strength \(E'_m\) enter the action only in the combination

\[
\left(1 + \frac{1}{\lambda L^2}\right) (E'_m)^2 + \frac{1}{2} \varepsilon_{mnpq} E_{pq} F_{mn}(L), \quad (3.60)
\]

where the tensor

\[
F_{mn}(L) \equiv \left(\partial_m \vec{L} \times \partial_n \vec{L}\right) \cdot \frac{\vec{L}}{|\vec{L}|^3} \quad (3.61)
\]

is formally identical to the electromagnetic field strength of a magnetic monopole. Therefore, there exists a vector potential \(A_m\) such that \(F_{mn}(L) = \partial_m A_n - \partial_n A_m\). An explicit magnetic monopole solution for the locally defined potential \(A_m(\vec{L})\) cannot
be 'rotationally' invariant w.r.t. the $SO(3) \sim SU(2)/Z_2$ symmetry, but it can be written down as a function of the $SO(2)$-irreducible $L^{ij}$-components to be defined by $L^{ij} = \delta^{ij}S + P^{(ij)}_{\text{traceless}}$. After integrating by parts and introducing a Lagrange multiplier $V$ as

$$*EF = *EdA \rightarrow -d^*EA = -E_m^\prime A_m \rightarrow -E_m A_m - E_m \partial_m V,$$

we can integrate out the full vector $E_m$. It leaves us with the bosonic NLSM action in terms of the four real scalars $(S, P^{(ij)}_{\text{traceless}}, V)$ only.

## 4 Brane technology

The exact solutions to the LEEA of $4d$, $N = 2$ supersymmetric gauge theories can be interpreted in a nice geometrical way, when considering these quantum supersymmetric field theories in the *common world-volume* of the (type IIA superstring or M-theory) branes [43, 44, 45].

The relevant brane configuration in the type IIA picture, in ten dimensions:

$$R^{1+9} \sim (x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9),$$

is schematically pictured in Fig. 3. It consists of two solitonic (NS) 5-branes carrying no RR-charges, $N_c$ Dirichlet-4-branes stretching between the 5-branes, and $N_f$ Dirichlet-6-branes [45].

The two parallel 5-branes are located at $\vec{w} = (x^7, x^8, x^9) = 0$ and (classically) fixed $x^6$ values. Their world-volumes are $R^{1+3} \otimes \Sigma_0$, where $R^{1+3}$ is the effective (uncompactified) 4d spacetime parametrized by $(x^0, x^1, x^2, x^3)$, and $\Sigma_0$ is a (singular) Riemann surface of genus $N_c - 1$, parametrized by $(x^4, x^5)$; $v \equiv x^4 + ix^5$.

The D-4-branes are also parallel to each other, but are orthogonal to the 5-branes. Their world-volumes are parametrized by $(x^0, x^1, x^2, x^3) \sim R^{1+3}$ and $x^6$.

The D-6-branes are orthogonal to both 5-branes and D-4-branes, they are located at fixed values of $(x^4, x^5, x^6)$, and their world-volumes are parametrized by $(x^0, x^1, x^2, x^3) \sim R^{1+3}$ and $\vec{w} \in B$, where $B$ is the internal space of the D-6-branes.

An $SU(N_c)$ gauge $N = 2$ vector multiplet in $R^{1+3}$ appears as the massless mode of an open $(4-4)$ string stretching between the D-4-branes, whereas $N_f$ hypermultiplets in $R^{1+3}$ come from $N_f$ open $(6-4)$ strings connecting the D-6-branes to the D-4-branes. Their BPS masses are fixed by the distance (in $x^{4,5}$) between the D-6-branes and D-4-branes.
The configuration of NS-5-branes (two vertical lines), D-4-branes (horizontal lines and dots), and D-6-branes (encircled crosses) in type-IIA picture.

The effective 4\(d\) coupling constant \(g\) is determined by the distance between the 5-branes,
\[
\frac{1}{g^2} = \frac{x_1^6 - x_2^6}{2\lambda},
\]
where \(\lambda\) is the type-IIA superstring coupling constant.

The whole brane configuration schematically pictured in Fig. 3 is supposed to be 'blown up' in order to accommodate the non-perturbative physics. Still, in this type-IIA picture, it inevitably suffers from quantum singularities to be associated with the intersections of the NS 5-branes with the D-4-branes. These singularities cannot be described semi-classically so that one needs yet another resolution that is going to be provided by reinterpreting the brane configuration of Fig. 3 in M theory (see the next subsect. 4.1).

Among the basic properties of the brane configuration under consideration, let’s emphasize the following ones:

- macroscopically, it is \((1 + 3)\)-dimensional;
- it is a BPS-like solution to the eleven-dimensional supergravity;
- it is invariant under \(\frac{1}{2} \cdot \frac{1}{2} \cdot 32 = 8\) supercharges, since the type-IIA superstring has 32 supercharges, while a half of them is conserved by the 5-branes, whereas a half of the remaining 16 supercharges is still conserved by the D-4-branes;
being orthogonal to the 5-branes and D-4-branes, the D-6-branes do not break any more supercharges; the eight supercharges in four dimensions imply the $N = 8/4 = 2$ extended supersymmetry in the effective spacetime $R^{1+3}$;

- the ten-dimensional Lorentz group is spontaneously broken to

$$SO(1, 3) \otimes SU(2)_A \otimes U(1)_{c.c.}$$

These are just the properties that actually determine the brane configuration of Fig. 3, driven by a desire to have the $N = 2$ extended supersymmetry in the effective 4d quantum field theory in the common brane world-volume as the effective spacetime $R^{1+3}$.

### 4.1 M-theory resolution

It is obvious from eq. (4.1) that one can keep the effective 4d gauge coupling constant $g$ fixed while increasing the distance $L = x_1^6 - x_2^6$ between the two 5-branes as well as the type-IIA superstring coupling constant $\lambda$. At strong coupling, the type-IIA superstring becomes the M-theory [46]: one extra dimension ($x^{10}$) to be represented by a circle $S^1$ of radius $R \sim \lambda^{2/3}$ shows up, as well as the non-perturbative $U(1)_M$ gauge symmetry appears. The latter is associated with the $S^1$-rotations.

As a result, the low-energy description of M-theory and its branes suffices for a geometrical interpretation of the exact solutions to the four-dimensional LEAE of the effective $N = 2$ supersymmetric gauge theories in the M-theory brane world-volume, just because all the relevant distances in the non-perturbative eleven-dimensional brane configuration become large while no singularity appears unlike that in the type-IIA picture. In particular, the D-4-branes and NS-5-branes in the type-IIA picture are now replaced by a single and smooth M-theory 5-brane whose world-volume is given by $R^{1+3} \otimes \Sigma$, where $\Sigma$ is a genus $(N_c - 1)$ Riemann surface holomorphically embedded into a four-dimensional hyper-Kähler manifold $Q$. The manifold $Q$ is topologically a bundle $Q \sim B \times S^1$ parametrized by the coordinates $x^4, x^5, x^6$ and $x^{10}$, whose base $B$ is the hidden part of the D-6-brane in the type-IIA picture and whose fiber $S^1$ is the hidden eleventh dimension of M theory [38].

The origin of the abelian gauge fields in the Coulomb branch of the 4d gauge theory also becomes more transparent from the M-theory point of view [43]. As is

$^{15}$The hyper-Kähler geometry of $Q$ is, in fact, required by $N = 2$ supersymmetry in the effective (macroscopic) spacetime $R^{1+3}$.  

29
well-known, there exists a two-form in the M-theory 5-brane world-volume, whose
field strength (3-form) $T$ is self-dual (see e.g., ref. [43]). Since the M-theory 5-
brane is wrapped over the Riemann surface, i.e. its world-volume is locally a product
$R^{1+3} \otimes \Sigma_{N_c-1}$, one can decompose the self-dual 3-form $T$ as

$$T = F \wedge \omega + * F \wedge \omega \ , \quad (4.3)$$

where $F$ is a two-form on $R^{1+3}$, and $\omega$ is a one-form on the Riemann surface $\Sigma_{N_c-1}$
of genus $N_c - 1$. The equations of motion $dT = 0$ now imply

$$dF = d^* F = 0 \ , \quad (4.4)$$

and

$$d\omega = d^* \omega = 0 \ . \quad (4.5)$$

Eq. (4.5) means that the one-form $\omega$ is harmonic on $\Sigma_{N_c-1}$. Since the number of
independent harmonic one-forms on a Riemann surface exactly equals to its genus [47],
one also has $(N_c-1)$ two-forms $F$, while each of them satisfies eq. (4.4). But eq. (4.4)
is nothing but the Maxwell equations for an electro-magnetic field strength $F$. It
explains the origin of the gauge group $U(1)^{N_c-1}$ in the Coulomb branch of the effective
4d gauge theory.

The geometrical interpretation of the $N = 2$ gauge LEEA in the Coulomb branch
of the effective 4d gauge field theory is provided by the identification [43, 45]

$$\Sigma_{N_c-1} = \Sigma_{SW} \ . \quad (4.6)$$

It order to understand the hypermultiplet LEEA in a similar way, one first notices
that the D-6-branes [17] are magnetically charged w.r.t. the non-perturbative $U(1)_M$
symmetry. Hence, the fiber $S^1$ of $Q$ has to be non-trivial, i.e. of non-vanishing
magnetic charge (or the first Chern class) $m \neq 0$. When taking into account the $U(2)$
isometry of the hyper-Kähler manifold $Q_{(m)}$, one concludes that $Q_{(m)}$ is necessarily a
multi-Taub-NUT space or a (multi)-KK monopole, just because it is the only space
among the asymptotically locally flat (ALF) spaces $B \otimes S^1$, whose fibration $S^1$ is
non-trivial. In particular, when $m = 1$, one gets the Taub-NUT space whose metric
was already described previously in subsect. 3.2.

\footnote{The eight conserved supercharges then imply the existence of a six-dimensional self-dual massless
(tensor) supermultiplet in the 5-brane world-volume.}

\footnote{The D-6-branes and their dual D-0-branes in the type-IIA picture are of Kaluza-Klein origin.}
4.2 Relation to the HSS results and S duality

The relation between the HSS results of subsect. 3.2 and the brane technology of subsect. 4.1 towards the hypermultiplet LEEA (in fact, their equivalence) is provided by the S-duality in field theory (Fig. 4).

Consider, for simplicity, the famous Seiberg-Witten model \([1]\) whose fundamental action describes the purely gauge \(N = 2\) super-Yang-Mills theory with the \(SU(2)\) gauge group spontaneously broken to its \(U(1)_c\) subgroup. In the strong coupling region of the Coulomb branch, near a singularity in the quantum moduli space where a BPS-like (t’Hooft-Polyakov) monopole becomes massless, the Seiberg-Witten theory is just described by the S-dual \(N = 2\) supersymmetric QED. In particular, the t’Hooft-Polyakov monopole belongs to a magnetically charged (HP) hypermultiplet \(q^+_{HP}\) that represents the non-perturbative degrees of freedom in the theory (Fig. 4). The HSS results of subsect. 3.2 imply that the HP hypermultiplet self-interaction in the vicinity of the monopole singularity is regular in terms of the magnetically dual variables,

\[
\mathcal{K}^{(+4)}_{\text{Taub-NUT}}(q^+_{HP}) = \frac{\lambda_{\text{dual}}}{2} \left( \frac{q^+_{HP} q^+_{HP}}{q^+_{HP} q^+_{HP}} \right)^2 ,
\]

i.e. it is given by the Taub-NUT (or KK-monopole).

On the other hand, from the type-IIA superstring (or M-theory) point of view, the HP-hypermultiplet is just the zero mode of the open superstring stretching between a magnetically charged D-6-brane and a D-4-brane. Therefore, it is the magnetically charged (HP) hypermultiplet that only survives in the effective 4\(d\), \(N = 2\) gauge
Fig. 5. The one-loop series for the N=2 vector gauge LEEA. The internal loop is made out of the hypermultiplet or N=2 superghost propagators.

theory, after taking the local limit $\alpha' \to 0$ of the brane configuration. According to the preceding subsect. 4.1, the target space (NLSM) geometry governing the HP hypermultiplet self-interaction has to be the Taub-NUT (or KK-monopole) again!

5 On the next-to-leading-order correction to the gauge LEEA

The next-to-leading-order correction to the $N = 2$ gauge LEEA in the Coulomb branch is governed by a real function of $W$ and $\bar{W}$ only, i.e. without any dependence upon their $N = 2$ superspace derivatives [48],

$$H(W, \bar{W}) = H_{\text{per}}(W, \bar{W}) + H_{\text{non-per}}(W, \bar{W}),$$

(5.1)

The exact function $H$ has to be S-duality invariant [19].

The one-loop contribution to $H_{\text{per}}$ is given by a sum of the $N = 2$ HSS graphs schematically pictured in Fig. 5. The sum goes over the external $V^{++}$-legs, whereas the loop consists of the $N = 2$ matter (and $N = 2$ ghost) superpropagators (sect. 3). $N = 2$ ghosts contribute in very much the same way as $N = 2$ matter does, since the $N = 2$ ghosts are also described in terms of the FS and HST hypermultiplets (with the opposite statistics of components) in the $N = 2$ HSS [31]. Because of the (abelian) gauge invariance, the result can only depend upon the abelian $N = 2$ superfield
strength $W$ and its conjugate $\bar{W}$ via eq. (3.21). In fact, Fig. 5 also determines the one-loop perturbative contribution to the leading holomorphic LEEA, which appears as the anomaly associated with the non-vanishing central charges [10]. The self-energy HSS supergraph with two external legs is the only one which is UV-divergent in Fig. 5. The IR-divergences of all the HSS graphs are supposed to be regulated by an IR-cutoff $\Lambda$, which is proportional to the Seiberg-Witten scale introduced in sect. 2 (the relative coefficient depends on the calculational scheme used, see e.g., ref.[50]).

One finds [33, 50]

$$F_{\text{1-loop}} \sim W^2 \ln \frac{W^2}{\Lambda^2}, \quad H_{\text{1-loop}} \sim \ln^2 \left( \frac{W\bar{W}}{\Lambda^2} \right), \quad |W| \gg \Lambda,$$

where we have identified the renormalization scale with $\Lambda$. The numerical coefficients in front of the logarithms in eq. (5.2) depend upon the content of the $N = 2$ gauge theory under consideration, i.e. upon the data $(N_c, N_f)$. The coefficient in front of the holomorphic contribution is fixed by the perturbative (one-loop) RG beta-function, i.e. it is proportional to $(-2N_c + N_f)$ and is gauge-invariant. The coefficient in front of the logarithm squared is proportional to $(-N_c + 2N_f)$, in the $N = 2$ super-Feynman gauge which was actually used above [50].

Eq. (5.2) is the result of straightforward and manifestly $N = 2$ supersymmetric calculations in the $N = 2$ HSS [33, 34, 50], and it agrees with the standard arguments based on the perturbative $R$-symmetry and the integration of the associated chiral anomaly [51, 52]. It is also straightforward to check (as I did) that there are no two-loop contributions to both $\mathcal{F}_{\text{per.}}$ and $\mathcal{H}_{\text{per.}}$, since all the relevant HSS graphs shown in Fig. 6 do not actually contribute in the local limit. This conclusion is also in agreement with some recent calculations in terms of the $N = 1$ superfields [33], as well as the general perturbative structure of the $N = 2$ supersymmetric gauge field theories within the background-field method in $N = 2$ HSS [10]. It does therefore seem to be conceivable that all the higher-loop contributions to $\mathcal{H}_{\text{per.}}(W, \bar{W})$ are absent too.

The exact result for the real function $\mathcal{H}(W, \bar{W})$ is still unknown. There is, however, an interesting proposal [54] that the exact function $\mathcal{H}(W, \bar{W})$ should satisfy a non-linear differential equation

$$\partial_W \partial_{\bar{W}} \ln \left[ \mathcal{H} \partial_W \partial_{\bar{W}} \ln \mathcal{H} \right] = 0,$$

which may be interpreted as a fully non-perturbative non-chiral superconformal ‘Ward identity’. For instance, the leading one-instanton contribution in the pure $N = 2$
Fig. 6. A two-loop N=2 HSS graph for the N=2 vector gauge LEEA in the N=2 HSS.

gauge theory was already calculated in ref. [55], and it does not vanish. The full non-perturbative contribution is not going to be given by a sum over instanton contributions only, but it should also include (multi)anti-instanton and mixed (instanton-anti-instanton) contributions. The brane technology of sect. 4 might offer a direct procedure of calculating the exact next-to-leading-order contribution, by using the covariant action describing the M-theory 5-brane dynamics. The manifestly (world-volume) general coordinate invariant and supersymmetric action of the M-theory 5-brane is known [56], and it contains, in particular, a Born-Infeld (BI) -type term and a Wess-Zumino (WZ) -type term. After being expanded in powers of derivatives, they yield the higher-derivative terms (in components). The latter are responsible for the exact form of the function \( \mathcal{H}(W,\bar{W}) \) in the effective LEEA of the M-theory 5-brane that should be related to the effective \( N = 2 \) gauge field theory action. However, as was argued in ref. [58], the actual results to be obtained from the brane technology may differ from that in the field theory, as regards the non-holomorphic terms which are not fully protected by symmetries. The only alternative seems to be the use of the \( N = 2 \) HSS in the instanton-type calculations, which is yet to be developed.

There are, however, some special cases when the non-perturbative corrections to \( \mathcal{H}(W,\bar{W}) \) vanish altogether. It just happens in the \textit{scale invariant} \( N = 2 \) supersym-
metric gauge field theories that cannot be (scale) \( \Lambda \)-dependent \[52\]. This proposal is supported by the instanton calculus \[59\]. In the scale-invariant case, it is the one-loop perturbative contribution to \( \mathcal{H}(W,\bar{W}) \) that is exact. It is easy to check that the \( \mathcal{H}_{\text{per}}(W,\bar{W}) \) of eq. (5.2), in fact, does not depend upon \( \Lambda \), since the real function \( \mathcal{H}(W,\bar{W}) \) itself is defined modulo the Kähler transformations

\[
\mathcal{H}(W,\bar{W}) \rightarrow \mathcal{H}(W,\bar{W}) + f(W) + \bar{f}(\bar{W}) ,
\]

with an arbitrary holomorphic function \( f(W) \) as a parameter.

In yet another scale-invariant \( N = 4 \) supersymmetric Yang-Mills theory, that amounts to the \( N = 2 \) super-Yang-Mills multiplet minimally coupled to a hypermultiplet in the \textit{adjoint} representation of the gauge group, both functions \( F_{\text{int}}(W) \) and \( \mathcal{H}(W,\bar{W}) \) vanish \[50\].

6 Hypermultiplet LEEA in the Higgs branch

As was already mentioned in sect. 3, the most natural and manifestly \( N = 2 \) supersymmetric description of hypermultiplets in the Higgs branch is provided by HSS in terms of the HST-type analytic superfield \( \omega \) of vanishing \( U(1) \) charge. The \( N = 2 \) HSS is also the quite natural framework to address all possible symmetry breakings.

The free action of a single \( \omega \) superfield reads

\[
S[\omega] = -\frac{1}{2} \int_{\text{analytic}} \left( D_{A}^{\dagger +} \omega \right)^{2} .
\]

Similarly to the free action (3.11) for a \( q^{+} \)-type analytic superfield, the action (6.1) also possesses the extended internal symmetry

\[
SU(2)_{A} \otimes SU(2)_{\text{extra}} ,
\]

where \( SU(2)_{A} \) is the automorphism symmetry of \( N = 2 \) supersymmetry algebra (sometimes also called the \( SU(2)_{R} \) symmetry). The extra \( SU(2) \) symmetry of eq. (6.1) is a bit less obvious \[35\] :

\[
\delta \omega = c^{--} D_{A}^{\dagger +} \omega - c^{+-} \omega ,
\]

where \( c^{--} = c^{(ij)} u_{i}^{-} u_{j}^{-} \) and \( c^{+-} = c^{(ij)} u_{i}^{+} u_{j}^{-} \), and \( c^{(ij)} \) are the infinitesimal parameters of \( SU(2)_{\text{extra}} \).

It is quite clear now that it is not possible to construct any non-trivial self-interaction in terms of the \( U(1) \)-chargeless superfield \( \omega \) alone, simply because any
hyper-Kähler potential has $U(1)$ charge (+4). Hence, when $N = 2$ supersymmetry and the $SU(2)_A$ internal symmetry are not broken, one gets the well-known result \[ \mathcal{K}^{(+4)}_{\text{Higgs}}(\omega) = 0 \quad (6.3) \] i.e. the induced hyper-Kähler metric in the fully $N = 2$ supersymmetric Higgs branch is flat.

It is, however, possible to break the internal symmetry (6.2) down to \[ U(1)_A \otimes SU(2)_{\text{extra}}, \quad (6.4) \] by introducing the so-called Fayet-Iliopoulos (FI) term \[ \langle D^{ij} \rangle = \xi^{ij} = \text{const.} \neq 0. \quad (6.5) \] This way of symmetry breaking still allows us to maintain control over the quantum hypermultiplet LEEA because of the non-abelian internal symmetry (6.4). The only non-trivial hyper-Kähler potential, that is invariant w.r.t. the symmetry (6.4) is given by \[ \mathcal{K}^{(+4)}_{\text{EH}}(\omega) = -\frac{(\xi^{++})^2}{\omega^2}, \quad (6.6) \] where $\xi^{++} = \xi^{ij} u^+_i u^+_j$. It is straightforward to deduce the corresponding hyper-Kähler metric from eq. (6.6) by using the procedure already described in subsect. 3.2. One finds that the metric is equivalent to the standard Eguchi-Hanson (EH) instanton metric in four dimensions \[ [60]. \] The induced scalar potential was calculated in ref. \[ [61]. \]

It should be noticed that the hyper-Kähler potential (6.6) already implies that $\langle \omega \rangle \neq 0$, so that we are in the Higgs branch indeed. Therefore, we now have to understand how a FI-term could be generated. Let’s slightly generalize this problem by allowing non-vanishing vacuum expectation values for all the gauge-invariant bosonic components of the abelian $N = 2$ superfield strength $W$,

\[ \langle W \rangle = \left\{ \begin{array}{l} \langle A \rangle = Z, \quad \langle F_{\mu \nu} \rangle = n_{\mu \nu}, \quad \langle \vec{D} \rangle = \vec{\xi} \end{array} \right\}, \quad (6.7) \]

where all the parameters $(Z, n_{\mu \nu}, \vec{\xi})$ are constants. Generally speaking, it amounts to the soft $N = 2$ supersymmetry breaking \[ [22]. \] We already know about the physical meaning of $Z$ — it is just the central charge or the related gauge-invariant quantity $u \sim \langle \text{tr} A^2 \rangle$, that parametrize the quantum moduli space of vacua. The central charge can be naturally generated via the standard Scherk-Schwarz mechanism of dimensional reduction from six dimensions \[ [35]. \] Similarly, $n_{\mu \nu} \neq 0$ can be interpreted as a toron background after replacing the effective spacetime $R^{1+3}$ by a hypertorus.
$T^{1+3}$ and imposing t’Hooft’s twisted boundary conditions \[63\]. The $\vec{\xi} \neq 0$ is just a FI term.

The brane technology helps us to address the question of dynamical generation of both $n_{\mu\nu}$ and $\vec{\xi}$ in a very geometrical way: namely, one should deform the brane configuration of Fig. 3 by allowing the branes to intersect at angles instead of being parallel! Indeed, the vector $\vec{w} = (x^7, x^8, x^9)$ is the same in Fig. 3 for both (NS) solitonic 5-branes. Its non-vanishing value

$$\vec{\xi} = \vec{w}_1 - \vec{w}_2 \neq 0 \quad (6.8)$$

effectively generates the FI term. Similarly, when allowing the D-4-branes to intersect at angles, some non-trivial values of $\langle F_{\mu\nu} \rangle = n_{\mu\nu} \neq 0$ are generated \[64\].

Since the LEEA of BPS branes is governed by a gauge theory, it does not seem to be very surprising that torons can also be understood as the BPS bound states of certain D-branes in the field theory limit $\alpha' \to 0$ (or $M_{\text{Planck}} \to \infty$) \[64\]. Moreover, torons generate a gluino condensate \[65\]

$$\langle \lambda^i \lambda^j \rangle = \Lambda^3 (\xi^2)^{ij} , \quad \xi^{ij} \sim \delta^{ij} \exp \left( - \frac{2\pi^2}{g^2} \right) , \quad (6.9)$$

where $\vec{\xi} \sim \{\xi^{ij}\}$ have to be constant \[66\], and they can be identified with the FI term by $N = 2$ supersymmetry.

Finally, it is also quite useful to understand the origin of the hypermultiplet EH-type self-interaction in the Higgs branch from the viewpoint of brane technology. It is worth mentioning here that the D-4-branes can also end on the D-6-branes (in the type-IIA picture) so that these D-4-branes actually support hypermultiplets, not $N = 2$ vector multiplets \[45\]. It results in another hyper-Kähler manifold $Q$ that has different topology $\sim S^3/Z_2$ in its spacial infinity. It is now enough to mention that the EH-instanton is the only hyper-Kähler manifold having this topology among the four-dimensional ALF spaces!
7 Conclusion

Though being very different, all the main three approaches considered above and depicted in Fig. 1, namely,

(i) instanton calculus,

(ii) Seiberg-Witten approach and M theory (=brane technology),

(iii) harmonic superspace,

lead to the consistent results, as regards the leading terms in the LEEA of the 4d, N = 2 supersymmetric gauge theories. The third (superspace) approach was mostly discussed in this paper, since it seems to be underrepresented in the current literature. There is no unique universal method to handle all the problems associated with the 4d gauge theories in the most natural and easy way; in fact, each approach has its own advantages and disadvantages. For example, in the Seiberg-Witten approach, the physical information is encoded in terms of functions defined over the quantum moduli space whose modular group is identified with the duality group. The very existence of this approach is crucially dependent upon knowing exactly the perturbative limits of the gauge theory where, in its turn, the HSS approach is very efficient. At the same time, the HSS approach itself cannot be directly applied to address truly non-perturbative phenomena yet. It can, however, when being combined with the strong-weak coupling duality (=S-duality). In its turn, the instanton calculus is very much dependent upon applicability of its own basic assumptions. It is not manifestly supersymmetric at any rate, if it is supersymmetric at all, and it sometimes needs an additional input too. On the other hand, though being geometrically very transparent, the recently developed (M theory) brane technology has a rather limited analytic support by now, and its applications are limited so far to those terms in the LEEA which are protected by N = 2 supersymmetry, i.e. either holomorphic or analytic ones. Hence, a care should be exercised in order to play safely with it. I believe, it is a combination of all the methods available that has the strongest potential for a further progress, and that simultaneously teaches us how to proceed with each particular approach.

I would like to conclude with a few comments about N = 2 supersymmetry breaking and confinement, in order to indicate on a possible importance of the exact hypermultiplet low-energy effective action towards a solution to these problems. Indeed, it seems to be quite natural to take advantage of the existence of exact solutions to the low-energy effective action in N = 2 supersymmetric gauge field theories,
and apply them to the old problem of color confinement in QCD. In fact, it was one of the main motivations in the original work of Seiberg and Witten [1]. The most attractive mechanism for color confinement is known to be the dual Meissner effect or the dual (Type II) superconductivity [67]. It takes three major steps to connect an ordinary BCS superconductor to the simplest Seiberg-Witten model in quantum field theory: first, define a relativistic version of the superconductor, known as the (abelian) Higgs model in field theory, second, introduce a non-abelian version of the Higgs model, known as the Georgi-Glashow model, and, third, $N = 2$ supersymmetrize the Georgi-Glashow model in order to get the Seiberg-Witten model [1]. Since the t’Hooft-Polyakov monopole of the Georgi-Glashow model belongs to a (HP) hypermultiplet in its $N = 2$ supersymmetric (Seiberg-Witten) generalisation, it is quite natural to explain confinement as the result of a monopole condensation (= the dual Meissner effect as a consequence of the dual Higgs effect), i.e. a non-vanishing vacuum expectation value for the magnetically charged (dual Higgs) scalars belonging to the HP hypermultiplet. Of course, it is only possible after (or simultaneously with) $N = 2$ supersymmetry breaking.

Exact solutions to the low-energy effective action in quantum gauge field theories are only available in $N = 2$ supersymmetry, and neither in $N = 1$ supersymmetry nor in the bosonic QCD. Hence, on the one side, it is the $N = 2$ supersymmetry that crucially simplifies an evaluation of the low-energy effective action. However, on the other side, it is the same $N = 2$ supersymmetry that is so obviously incompatible with phenomenology e.g., because of equal masses of bosons and fermions inside $N = 2$ supermultiplets (it also applies to any $N \geq 1$ supersymmetry), and the non-chiral nature of $N = 2$ supersymmetry (e.g. ‘quarks’ appear in real representations of the gauge group). Therefore, if we believe in the $N = 2$ supersymmetry, we should find a way of judicious $N = 2$ supersymmetry breaking. The same dual Higgs mechanism may also be responsible for the chiral symmetry breaking and the appearance of the pion effective Lagrangian if the dual Higgs field has flavor charges also [1]. In fact, Seiberg and Witten used a mass term for the $N = 1$ chiral multiplet, which is a part of the $N = 2$ vector multiplet, in order to softly break $N = 2$ supersymmetry to $N = 1$ supersymmetry. As a result, they found a non-trivial vacuum solution with a monopole condensation and, hence, a confinement. The weak point of their approach is an ad hoc assumption about the existence of the mass gap, i.e. the mass term itself. It would be nice to derive the mass gap from the fundamental theory instead of postulating it.

The $N = 2$ supersymmetry can be broken either softly or spontaneously, if one wants to preserve the benefits of its presence (e.g. for the full control over the low-
energy effective action) at high energies. The general analysis of the soft $N = 2$ supersymmetry breakings in the $N = 2$ supersymmetric QCD was given by Alvarez-Gaumé, Mariño and Zamora [62]. The soft $N = 2$ supersymmetry breaking is most naturally done by using the FI-terms, as in ref. [62]. Though being pragmatic, the soft $N = 2$ supersymmetry breaking has a limited predictive power because of many parameters, whose number, however, is significantly less than that in the $N = 1$ case. Hence, it should make sense to search for the patterns of spontaneous $N = 2$ supersymmetry breaking, where the non-vanishing FI-terms would appear as stationary solutions to the dynamically generated scalar potential. This would mean the existence of a non-supersymmetric vacuum solution for the $N = 2$ supersymmetric scalar potential at the level of the low-energy effective action in $N = 2$ gauge theories. Since the $N = 2$ supersymmetry remains unbroken for any exact Seiberg-Witten solution in the gauge sector, we should consider the induced scalar potentials in the hypermultiplet sector of an $N = 2$ gauge theory. Indeed, given the non-trivial kinetic terms in the hypermultiplet low-energy effective action to be represented by the (hyper-Kähler) non-linear sigma-model, in a presence of non-vanishing central charges they also imply a non-trivial hypermultiplet scalar potential whose form is entirely determined by the hyper-Kähler metric of the kinetic terms and $N = 2$ supersymmetry. Though it is not easy to search for the most general solutions with spontaneously broken $N = 2$ supersymmetry because of complications associated with HSS and hyper-K"ahler geometry, our examples demonstrate the richness of possible solutions.

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20 See e.g., ref. [68] for a similar analysis in $N = 1$ supersymmetric gauge field theories.
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