POINTED HOPF ALGEBRAS WITH TRIANGULAR DECOMPOSITION
A CHARACTERIZATION OF MULTIPARAMETER QUANTUM GROUPS

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Abstract. In this paper, we present an approach to the definition of multiparameter quantum groups by studying Hopf algebras with triangular decomposition. Classifying all of these Hopf algebras which are of what we call weakly separable type, we obtain a class pointed Hopf algebras which can be viewed as natural generalizations of multiparameter deformations of universal enveloping algebras of Lie algebras. These Hopf algebras are instances of a new version of braided Drinfeld doubles, which we call asymmetric braided Drinfeld doubles. This is a generalization of an earlier result by Benkart and Witherspoon (2004) who showed that two-parameter quantum groups are Drinfeld doubles. It is possible to recover a Lie algebra from these doubles in situations where the braiding parameters can be set equal to one. The Lie algebras arising are generated by Lie subalgebras isomorphic to $\mathfrak{sl}_2$. Weakening these assumptions, one can obtain quantum groups where a certain classical limit is a Lie algebra which is not semisimple.

1. Introduction

1.1. What are Quantum Groups? An important problem in the theory of quantum groups is to give some definition of a class of these objects that captures known series of quantum groups, such as the quantum enveloping algebras $\mathcal{U}_q(g)$ of [Dri86], and their finite-dimensional analogues, as examples. This was for example formulated in [BG02, Problem II.10.2]:

"Given a finite-dimensional Lie algebra $g$, find axioms for Hopf algebras to qualify as quantized enveloping algebras of this particular $g".

A possible hint to the structure of quantum groups is that the quantum enveloping algebras $\mathcal{U}_q(g)$ (as well as the small quantum groups $u_q(g)$ and multiparameter versions) are pointed Hopf algebras. Such Hopf algebras were studied by several authors (see e.g. [AS02]). Classification results as in [AS10] suggest a strong resemblance of all finite-dimensional pointed Hopf algebras over abelian groups with small quantum groups. Another paper [AS04] gives a characterization of quantum groups at generic parameters using pointed Hopf algebras of finite Gelfand-Kirillov-dimension with infinitesimal braiding of positive generic type.

A further hint to the structure of quantum groups is that they can be decomposed in a triangular way (via the PBW theorem) as

$$\mathcal{U}_q(g) = \mathcal{U}_q(n_+) \otimes k\mathbb{Z}^n \otimes \mathcal{U}_q(n_-).$$

Here, the positive and negative part are perfectly paired braided Hopf algebras, and the relation with the group algebra $k\mathbb{Z}^n$ is governed by semidirect product relations. The positive (and negative) part are so-called Nichols algebras.

A third aspect – observed already in the original paper [Dri86] – is that quantum groups are (quotients of) quantum or Drinfeld doubles. It was shown in [Maj99] that $\mathcal{U}_q(g)$ in fact is a braided Drinfeld double (which are referred to as a double bosonization there). It was proved in [BW04] that also two-parameter quantum groups are Drinfeld doubles.

In this paper, we aim to provide an axiomatic approach to the definition of (multiparameter) quantum groups by combining the pointed Hopf algebra and the triangular decomposition approach. Under the additional assumption of what we call a triangular decomposition of weakly separable type, the only indecomposable examples are close generalizations of multiparameter quantum groups. In particular, assuming further non-degeneracy, they are examples of a more general version of braided...
Drinfeld doubles, which we refer to as asymmetric braided Drinfeld doubles. Further, under certain assumptions on the group and the parameters, we can recover Lie algebras from these Hopf algebras, after introducing a suitable integral form.

1.2. This Paper’s Results. This paper starts by recalling the necessary technical background, including a brief overview on classification results of finite-dimensional pointed Hopf algebras, as well as structural results by [BB09] on algebras with triangular decomposition, in Section 2. Next, we give the definition of a bialgebra with a triangular decomposition over a Hopf algebra \( H \) in Section 3. This adapts the two-step approach used for algebras in [BB09] to the study of bialgebras. Namely, we first consider the free case of a bialgebra \( T(V) \otimes H \otimes T(V^*) \) where the positive and negative parts \( (T(V), \text{respectively } T(V^*)) \) are tensor algebras, and then specify by what ideals (called triangular Hopf ideals) we can take the quotient.

The core of this paper is formed by a partial classification of bialgebras with triangular decomposition over a group algebra \( kG \). We again proceed in two steps. First, we determine all pointed bialgebras with free positive and negative part over \( kG \) in Section 4.2, and then look at pairs of ideals \( I, I^* \) such that the quotient \( A/I, I^* \) is still a bialgebra in Section 4.3. We find that indecomposable examples are automatically pointed Hopf algebras, and can only arise over finitely-generated abelian groups. Multiparameter quantum groups share these features. Indeed, the only possible commutator relations (2.10) closely resemble those of multiparameter quantum groups:

\[
[f_i, v_j] = \gamma_{i,j}(k_j - l_i) \in kG, \quad \forall i = 1, \ldots, n.
\]

We further observe that there exists a natural generalization of the definition of a braided Drinfeld double to the setting of primitively generated braided Hopf algebras in the category of Yetter-Drinfeld modules (YD-modules) over \( H \). For this, the base Hopf algebra \( H \) does not need to be quasitriangular. We need two braided Hopf algebras which are only required to be dually paired considered as braided Hopf algebra in the category of modules (rather than YD-modules). That is, the requirement that is weakened compared to the definition of a braided Drinfeld double (as in [Maj99] or [Lau15]) is that the comodule structures do not need to be dually paired. We refer to this generalization as the asymmetric braided Drinfeld double. It gives a natural way of producing Hopf algebras with triangular decomposition – which are not necessarily quasitriangular. We show in Theorem 4.3.2 that the Hopf algebras arising in the classification 4.2.2 are of this form (provided that the parameters \( \gamma_{ii} \) are non-zero).

In Section 4.4 we show that from these asymmetric braided Drinfeld doubles of separable type we can recover Lie algebras provided that there exists a well-defined morphism of rings to \( \mathbb{Z} \) when setting the parameters equal to 1. Hence, in the spirit of the question asked in Section 1.1, we can relate the outcome of our classification back to Lie algebras, which are always generated by Lie subalgebras isomorphic to \( \mathfrak{sl}_2 \).

Here is an overview of the increasingly stronger assumptions on the Hopf algebras \( A \) and \( H \) used in the classification:

- Section 3: \( H \) any Hopf algebra over a field \( k \), \( A \) a bialgebra with triangular decomposition
- Section 4: \( H = kG \), \( A \) a bialgebra with triangular decomposition
  - Section 4.1-4.2: \( A \) is of weakly separable type and indecomposable after 4.1.3
  - Section 4.3: \( A \) is indecomposable of separable type, the scalars \( \gamma_{ii} \) are non-zero.
  - Section 4.4: In addition to the assumptions of 4.3, we require that char \( k = 0 \), and that setting the parameters equal to 1 gives a well-defined homomorphism of rings to \( \mathbb{Z} \).

The final Section 5 contains different classes of indecomposable pointed Hopf algebras with triangular decomposition over a group \( kG \) that arise as examples in the main classification. The first class we discuss are the multiparameter quantum groups \( U_{\lambda, \mu}(\mathfrak{g}_l) \) introduced by [FRT88] (adapting the presentation in [CM96]). They are asymmetric braided Drinfeld doubles, which is a generalization of the result of [BW04] showing that two-parameter quantum groups are Drinfeld doubles. In section 5.2 we bring results of [Ros98] on growth condition (finite Gelfand-Kirillov dimension) and classification of Nichols algebras from [AS04] into the picture. We use these results to characterize the Drinfeld-Jimbo type quantum groups at generic parameters \( q \) within the classification of this paper under the additional assumption that the triangular decomposition is what we call symmetric. Further, classes of finite-dimensional pointed Hopf algebras by Radford can naturally be included as examples in this
framework (Section 5.3). The more flexible approach to defining quantum groups of this paper can be used to construct examples where a certain classical limit is not a semisimple Lie algebra. A small example is given in 5.2.8. This example does not satisfy the assumptions from Section 4.4.

To conclude this paper, we suggest in Section 5.4 that future research could focus on the search for Hopf algebras with triangular decomposition over other Hopf algebras \( H \) (replacing the group algebra \( kG \)). This might give interesting monoidal categories, or even knot invariants in other contexts. As the first – most classical – example, if we take \( H \) to be a polynomial ring \( k[x_1, \ldots, x_n] \). In this case, the only examples are universal enveloping algebras of Lie algebras.

1.3. Notations and Conventions. In this paper, adapted Sweedler’s notation (see e.g. [Swe69]) is used to denote coproducts and coactions omitting sums. Unless otherwise stated, we work with Hopf algebras over an arbitrary field \( k \). A Hopf algebra always has an invertible antipode \( S \). The category of left YD-modules over a Hopf algebra \( H \) is denoted by \( \text{H-Mod} \), while left modules are denoted by \( \text{H-Mod} \).

We denote the module spanned by generators \( S \) over a commutative ring \( R \) as \( R\langle S \rangle \), while \( R[S] \) denotes the \( R \)-algebra generated by elements \( S \) (subject to some specified relations). Groups generated by elements of a set \( S \) are denoted by \( \langle S \rangle \).

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2. Background

2.1. Pointed Hopf Algebras. Let the coproduct \( \Delta: H \to H \otimes H \) make \( H \) a coalgebra over a field \( k \). We can consider simple subcoalgebras \( A \leq H \). That is, \( \Delta(A) \leq A \otimes A \) and there are no proper subobjects of this type in \( A \). A basic observation is that if \( \dim A = 1 \), then \( A \) can be written as \( kg \), for a generator \( g \in H \) such that \( \Delta(g) = g \otimes g \). Such elements are called grouplike. Indeed, if \( H \) is a Hopf algebra, then the set of all grouplike elements \( G(H) \) has a group structure. A Hopf algebra is pointed if all simple subcoalgebras are one-dimensional. This notion can be traced back to [Swe69] and classifying all finite-dimensional pointed Hopf algebras can be taken as a first step in the classification of all finite-dimensional Hopf algebras (see e.g. [And14] for a recent survey).

In the late 1980s and early 1990s, large classes of pointed Hopf algebras have been discovered with the introduction of the quantum groups (and their small analogues). Due to the vast applications of and attention to these Hopf algebras in the literature, the study of pointed Hopf algebras has become an important algebraic question.

2.2. Link-Indecomposability. In the early 1990s, Montgomery asked the question, which groups may occur as \( G(H) \) where \( H \) is an indecomposable pointed Hopf algebras. In [Mon95], an appropriate notion of indecomposability is discussed in different ways. We will briefly recall the description in terms of link-indecomposability which is equivalent to indecomposability as a coalgebra and indecomposability of the Ext-quivier of simple comodules.

Given a pointed Hopf algebra \( H \), we define a graph \( \Gamma_H \) with vertices being the simple subcoalgebras of \( H \) (that is, the grouplike elements). There is an edge \( h \to g \) if there exists a \((g,h)\)-skew-primitive element \( v \in H \), i.e. \( \Delta(v) = v \otimes g + h \otimes v \), which is not contained in \( kG(H) \). We say that \( H \) is indecomposable if \( \Gamma_H \) is connected. As an example, group algebras \( kG \) are only indecomposable if \( G = 1 \). The quantum group \( U_q(\mathfrak{sl}_2) \) is indecomposable if the coproducts are e.g. defined as \( \Delta(E) = E \otimes 1 + K \otimes E \) and \( \Delta(F) = F \otimes 1 + K^{-1} \otimes F \). There are other versions of the coproduct which are not indecomposable (see [Mon95]).

⁴http://gow.epsrc.ac.uk/NGBOViewGrant.aspx?GrantRef=EP/I033343/1
2.3. Classification Results for Pointed Hopf Algebras. It was understood early that pointed Hopf algebras can be obtained as bosonizations $A = B(V) \ltimes kG$ of so-called Nichols (or Nichols-Woronowicz) algebras $\mathcal{B}(V)$ associated to YD-modules over a group $G$ (see e.g. [AS02] for definitions). In this case, the coproducts are given by $\Delta(v) = v^{(0)} \otimes v^{(-1)} + 1 \otimes v$ using Sweedler’s notation. That is, if $v$ is a homogeneous element, then $\Delta(v) = v \otimes g + 1 \otimes v$ for the degree $g \in G(A)$ of $v$ and $A$ is indecomposable over the group generated by $g \in G$ with $V_g = 0$. Thus, the question of finding finite-dimensional pointed Hopf algebras is linked to finding finite-dimensional Nichols algebras$^2$.

Although both questions remain open in general, vast progress has been made in a series of papers by Andruskiewitsch and Schneider (see [AS02, AS10]) for abelian groups $G$, and more recently for symmetric and alternating groups [AFGV11], or groups of Lie type [ACA13, ACA14]. See [And14] for more detailed references.

Let us briefly recall the classification results of [AS10] here in order to provide the basis for comparison to our own classification in Section 4 later. To fix notation, let $\mathcal{D}$ denote a finite Cartan datum. That is, a finite abelian group $\Gamma$, a Cartan matrix $A = (a_{ij})$ of dimension $n \times n$ with a choice of group elements $g_i$, characters $\chi_i$ for $i = 1, \ldots, n$. Then define $q_{ij} := \chi_j(g_i)$ and impose the conditions that

\[
q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad q_{ii} \neq 1.
\]

We can associate to the Cartan matrix $A$ a root system $\Phi$ (with positive roots $\Phi^+$). The simple roots $\alpha_i$ of $\Phi$ can be indexed by $i = 1, \ldots, n$. Denote by $\chi$ the set of connected components of the corresponding diagram, and by $\Phi_J$ the root system restricted to the component $J \in \chi$, and write $i \sim j$ if $i$ and $j$ are in the same connected component. Denote further

\[
g_\alpha := \prod_{i=1}^n g_i^{n_i}, \quad \chi_\alpha := \prod_{i=1}^n \chi_i^{n_i}, \quad \text{for a root } \alpha = \sum_{i=1}^n n_i \alpha_i.
\]

To state the classification of finite-dimensional pointed Hopf algebras, some technical assumptions need to be made.

(a) Assume that the parameters $q_{ii}$ are roots of odd order $N_i$.
(b) $N_i$ is constant on each connected component, $i \in J$.
(c) If $J \in \chi$ is of type $G_2$, then 3 does not divide $N_J$.

To construct pointed Hopf algebra from a Cartan datum $\mathcal{D}$, we need two families of parameter.

(d) Let $\lambda = (\lambda_{ij})$ be a $n \times n$-matrix of elements in $k$ such that for all $i \neq j$, $g_ig_j = 1$ or $\chi_i\chi_j \neq \varepsilon$ implies $\lambda_{ij} = 0$.
(e) Further let $\mu = (\mu_\alpha)_{\Phi^+}$ be elements in $k$ such that for any $\alpha \in \Phi^+$, for $J \in \chi$.

**Definition 2.3.1** ([AS10]). Given the a Cartan datum $\mathcal{D}$ with families of parameters $\lambda, \mu$ as above, there is a Hopf algebra $u = u(\mathcal{D}, \lambda, \mu)$. The algebra $u$ is generated by elements $g \in \Gamma$ (define $u_\alpha(\mu) \in k\Gamma$, see [AS10, 2.14] for $\alpha \in \Phi^+$), and $x_i$ for $i = 1, \ldots, n$, subject to the relations

\[
(2.2) \quad gx_i = \chi_i(\mu)x_i g,
\]
\[
(2.3) \quad \text{ad}(x_i)x_i = 0, \quad \text{for } i \neq j, \quad i \sim j,
\]
\[
(2.4) \quad \text{ad}(x_i)x_j = \lambda_{ij}(1 - g_ig_j), \quad \text{for } i < j, \quad i \neq j,
\]
\[
(2.5) \quad x_i^{N_j} = u_{\alpha_j}(\mu), \quad \text{for all } \alpha \in \Phi^+_J, \quad J \in \chi.
\]

Here, \text{ad}(x)(y) is the braided commutator $xy - m \circ \Psi(x \otimes y)$ where $m$ denotes multiplication and $\Psi$ is the YD-braiding. The comultiplication is given by $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$.

**Theorem 2.3.2** ([AS02, 0.1]). Under the above assumptions (a)-(e) on a Cartan datum $\mathcal{D}$ with parameters $\lambda, \mu$, the Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is pointed with $G(u) = \Gamma$ and of finite dimension. Moreover, if $|G|$ is not divisible by 2, 3, 5 or 7, then any finite-dimensional pointed Hopf algebra is of this form.

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$^2$However, a pointed Hopf algebra is not necessary bosonizations of this form. Important tools available are the coradical filtration (see e.g. [Mon93]) and the lifting method of Andruskiewitsch and Schneider [AS02].

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2.4. Algebras with Triangular Decomposition (Free Case). A triangular decomposition of algebras means that an intrinsic PBW decomposition exists, similar to universal enveloping algebras of Lie algebras. This is a common feature of quantum groups and rational Cherednik algebras, but more generally shared by all braided Drinfeld or Heisenberg doubles (cf. [Lau15, 3.4]). Here, we are using the definitions introduced in [BB09] to study such algebras with triangular decomposition (so-called braided doubles).

From a deformation theoretic point of view, triangular decomposition can be viewed as follows. Let $V, V^*$ be dual pairs of finite-dimensional vector spaces and $H$ a Hopf algebra over a field $k$, such that $V$ is a left $H$-module, and $V^*$ carries the dual right $H$-action. That is, for the evaluation map $\langle \cdot, \cdot \rangle : V^* \otimes V \to k$, we have

$$\langle f \otimes h, v \rangle = \langle f, h \mapsto v \rangle, \quad \forall f \in V^*, v \in V, h \in H.$$  

(2.6)

Now define $A_0(V, V^*)$ to be the algebra on $T(V) \otimes H \otimes T(V^*)$ with relations

$$fh = h(1)(f \otimes h(2)), \quad hv = (h(1) \mapsto v)h(2).$$  

(2.7)

(i.e. the bosonizations $T(V) \otimes H$ and $H \otimes T(V^*)$ are subalgebras), and $[f, v] = 0$.

In [BB09], a family of deformations of $A_0(V, V^*)$ over Hom$_k(V \otimes V^*, H)$ is defined. The algebra $A_\beta(V, V^*)$ over a parameter $\beta : V \otimes V^* \to H$ is defined using the same generators in $V, V^*$ and $H$ with the same bosonization relations, but the commutator relation

$$[f, v] = \beta(f, v).$$  

(2.8)

In order to obtain flat deformations we restrict to maps $\beta$ such that the multiplication

$$m : T(V) \otimes H \otimes T(V) \to A_\beta(V, V^*), \quad v \otimes h \otimes f \mapsto vhf,$$

is an isomorphism of $k$-vector spaces.

Definition 2.4.1. In the case where $m$ is an isomorphism of $k$-vector spaces, we say that $A_\beta(V, V^*)$ is a free braided double.

Theorem 2.4.2 ([BB09]). The algebra $A_\beta(V, V^*)$ is a free braided double if and only if there exists a $k$-linear map $\delta : V \to H \otimes V$, $\delta(v) = v^{-1}[0] \otimes v[0]$ which is YD-compatible with the $H$-action on $V$, i.e. for any $h \in H$

$$h(1)v^{-1} \otimes (h(2) \mapsto v[0]) = (h(1) \mapsto v)^{-1}h(2) \otimes (h(1) \mapsto v)[0].$$  

(2.9)

In this case, we call $(V, \delta)$ a quasi-YD-module and we have

$$[f, v] = \beta(f \otimes v) = v^{-1}\langle f, v[0] \rangle.$$  

(2.10)

2.5. Triangular Ideals. So far, the braided Hopf algebras $T(V)$ and $T(V^*)$ were assumed to be free. We can add additional relations into the picture, defining braided double that are not necessarily free. Let $I \lhd T(V)$ and $I^* \lhd T(V^*)$ be ideals. We want to determine when the quotient map

$$m : T(V)/I \otimes H \otimes T(V^*)/I^* \to A_\beta(V, V^*)/\langle I, I^* \rangle$$

is still an isomorphism of $k$-vector spaces. In [BB09] it is show that this is the case if and only if $J := \langle I, I^* \rangle$ is a so-called triangular ideal. That is, $J = I \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^*$, where $I \lhd T^{>0}(V)$, $I^* \lhd T^{>0}(V^*)$ such that $I$ and $I^*$ are $H$-invariant and

$$T(V^*)I \subseteq J, \quad I^*T(V) \subseteq J.$$  

(2.11)

This is equivalent to the commutator $[f, I]$ and $[I^*, v]$ being contained in $J$ for all degree one elements $v \in V, f \in V^*$. For each quasi-YD-module, there exists a unique largest triangular ideal $I_{\text{max}}$, and thus a unique maximal quotient referred to as a minimal braided double.

If $\delta$ is a YD-module, then the maximal quotient $T(V)/I_{\text{max}}$ is the Nichols algebra $B(V)$ of $V$, and the braided double on $B(V) \otimes H \otimes B(V^*)$ is a generalization of the Heisenberg double, a so-called braided Heisenberg double.

For the purpose of this paper, we need ideals $I$ such that $T(V)/I$ is a braided bialgebra, where $V$ is a YD-module. That is, not a bialgebra object in the category of $k$-vector spaces but in the category of YD-modules over $kG$ (see e.g. [AS02]). However, if $I$ is a homogeneous ideal in $T^{>1}(V)$ which is a coideal and YD-submodule, then $T(V)/I$ is a braided Hopf algebra. We denote the collection of such ideals by $Z_V$. In fact $I_{\text{max}} \in Z_V$ as the Nichols algebra $B(V)$ is a braided Hopf algebra.
3. HOPF ALGEBRAS WITH TRIANGULAR DECOMPOSITION

In this section, we let $k$ be a field of arbitrary characteristic and $H$ a Hopf algebra over $k$. We introduce a notion of a Hopf algebra with triangular decomposition.

3.1. Definitions. We refer to the grading of a braided double $T(V)/I \otimes H \otimes T(V^*)/I^*$ given by

$$\deg v = 1, \quad \deg f = -1, \quad \deg h = 0, \quad \forall v \in V, \ f \in V^*, \ h \in H,$$

as the natural grading. We want to study Hopf algebras with triangular decomposition preserving this grading.

Definition 3.1.1. A bialgebra (or Hopf algebra) $A$ with triangular decomposition over a Hopf algebra $H$ is a braided double $H = T(V)/I \otimes H \otimes T(V^*)/I^*$ which is a bialgebra (respectively Hopf algebra) such that

\begin{align}
(3.1) & \quad H \text{ is a subcoalgebra of } A \text{ with respect to the original coproduct of } H, \\
(3.2) & \quad \text{the subspaces } T(V) \otimes H \text{ and } H \otimes T(V^*) \text{ closed under the coproduct of } A, \\
(3.3) & \quad \text{the coproduct and counit are morphisms of graded algebras for the natural grading.}
\end{align}

(In the Hopf case, the antipode $S$ is required to preserve the natural grading and the subspaces $T(V) \otimes H$ and $H \otimes T(V^*)$.)

The coalgebra axioms imply that $\delta_l$ and $\delta_r$ are left (respectively right) $H$-coactions. In particular, as the semidirect product relations in $A$ are preserved by $\Delta$, $\delta_l$ and $\delta_r$ are left (respectively right) YD-compatible with the given action of $H$ on $V$. Similarly, we can obtain a left and right YD-module structure over $H$ on the dual $V^*$ from the coproduct. These are denoted by $\delta^*_l$ and $\delta^*_r$.

Definition 3.1.2. Given a bialgebra $A$ with triangular decomposition over $H$, we define the right (respectively, left) YD-structure of $A$ to be $\delta_r$ (respectively, $\delta_l$). We refer to $\delta^*_r$ and $\delta^*_l$ as the right and left dual YD-structures.

To fix Sweedler’s notation for the different coactions, denote $\delta_r(v) = v^{(0)} \otimes v^{(-1)}$ and $\delta_l(v) = v^{(-1)} \otimes v^{(0)}$ and use similar notations for $f \in V^*$. We will reformulate the definition of a bialgebra with triangular decomposition in terms of conditions on the YD-structures of $A$ in (3.6)-(3.10) in the free case first.

Lemma 3.1.3. A bialgebra with triangular decomposition $A$ is a Hopf algebra with triangular decomposition if and only if

\begin{align}
(3.4) & \quad S(v^{(0)}v^{(-1)}) + (S(v^{(-1)})v^{(0)})S(v^{(-1)}) = 0, \quad \forall v \in V, \\
(3.5) & \quad f^{(-1)}S(f^{(0)}) + S(f^{(-1)})f^{(0)} - S(f^{(-1)})f^{(0)} = 0, \quad \forall f \in V^*.
\end{align}

In this case, the antipode extends uniquely to all of $A$.

Proof. This follows (under use of the semidirect product relations) by restating the antipode axioms for the coproduct of a Hopf algebra with triangular decomposition, which has the form $\Delta(v) = v^{(0)} \otimes v^{(-1)} + v^{(-1)} \otimes v^{(0)}$. Note that $\varepsilon(v) = 0$ as we require the counit to be a morphism of graded algebras. \(\square\)
3.2. The Free Case. Let $A$ be a free braided double, i.e. $A = T(V) \otimes H \otimes T(V^*)$. We can now state necessary and sufficient conditions on the YD-structures of $A$ to make the algebra $A$ a bialgebra with triangular decomposition. In the following, we stick to the notation of [BB09] denoting the quasi-coaction determining the commutator relation between elements of $V$ and $V^*$ by $\delta(v) = v^{[-1]} \otimes v^{[0]}$, for $v \in V$.

**Lemma 3.2.1.** A free braided double $A$ on $T(V) \otimes H \otimes T(V^*)$ is a bialgebra with triangular decomposition if and only if there exist YD-structures $\delta_l$, $\delta_r$, $\delta_l^*$, and $\delta_r^*$ such that the following conditions hold for $v \in V$, $f \in V^*$:

\[
\begin{align*}
(3.6) & \quad (f^{(0)} \triangleleft v^{[-1]}) \otimes (f^{(-1)} \triangleright v^{[0]}) = f \otimes v, \\
(3.7) & \quad (f^{(-1)} \triangleright v^{[0]}) \otimes (f^{(0)} \triangleleft v^{[-1]}) = v \otimes f, \\
(3.8) & \quad v^{(0)} f^{(0)} \otimes (f^{(-1)} v^{(-1)} - v^{(-1)} f^{(-1)}) = 0, \\
(3.9) & \quad (f^{(-1)} v^{[-1]} - v^{[-1]} f^{(-1)} \otimes v^{[0]}) f^{(0)} = 0, \\
(3.10) & \quad v^{(-1)} f^{(0)} v^{(-1)} + f^{(-1)} v^{[-1]} f^{(0)} = v^{[-1]} f^{(0)} v^{[0]}. 
\end{align*}
\]

**Proof.** The conditions are easily checked to be equivalent – under use of the relations in $A$ and the PBW theorem – to the requirement that (2.10) is preserved by $\Delta$. This gives the relations (3.8)-(3.10), as well as

\[
\begin{align*}
\Delta(v) &= v^{(0)} \otimes v^{(-1)} + v^{(-1)} \otimes v^{[0]}, \\
\Delta(f) &= f^{(0)} \otimes f^{(-1)} + f^{(-1)} \otimes f^{[0]},
\end{align*}
\]

for $v \in V$, $f \in V^*$ by YD-compatibility. \hfill \Box

It will become apparent in Section 4 what constraints on the structure of $A$ conditions (3.6)-(3.10) give working over a group, and over a polynomial ring in Section 5.4.

3.3. Triangular Hopf ideals. We are looking for triangular ideals $J = I \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^*$ (cf. [BB09, Appendix A] or Section 2.5) which are also coideals, and hence $A/J$ is a triangular bialgebra or Hopf algebra.

Using the description of the coproduct $\Delta$ in terms of the left and right YD-structures on $A$, the triangular ideals $J$ that are also coideals are simply those triangular ideals for which $I$ (and $I^*$) are YD-submodules for both $\delta_l$ and $\delta_r$ (respectively, $\delta_l^*$ and $\delta_r^*$).

If $A$ is a triangular Hopf algebra with antipode given as in Lemma 3.1.3, then every triangular ideal which is also a coideal is automatically a Hopf ideal.

**Definition 3.3.1.** We denote the collection of ideals of the form

\[
J = I \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I^*
\]

for $I \triangleleft T(V)$ and $I^* \triangleleft T(V^*)$ which are also YD-submodules for $\delta_r$, $\delta_l$ (respectively for $\delta_r^*$, $\delta_l^*$) by $\mathcal{J}_A(A)$. Such ideals $J$ are called triangular Hopf ideals.

3.4. Asymmetric Braided Drinfeld doubles. A special class of Hopf algebras with triangular decomposition can be provided by braided Drinfeld doubles of primitively generated Hopf algebras over a quasitriangular base Hopf algebra $H$. This form of the Drinfeld double was introduced as the double bosonization in [Maj95, Maj99], see also [Lau15] for the presentation used here. We now give a more general definition of an asymmetric braided Drinfeld double which is suitable to capture the more general class of Hopf algebras that we find in Section 4, including multiparameter quantum groups, as examples. In this construction, the base Hopf algebra $H$ need not be quasitriangular, and the asymmetric braided Drinfeld double is also not quasitriangular in general.

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To define the braided Drinfeld double of dually paired braided Hopf algebras $C$ and $B$ in the category $\text{Drin}(H)\text{-Mod} = H^H\text{YD}$ we require that $\langle \ , \rangle : B \otimes C \to k$ is a morphism of YD-modules. This implies that the actions and coactions on $C$ and $B$ are dual to one-another (by means of the antipode of $H$). A weaker requirement is that we consider the images of $C$ and $D$ under the forgetful functor

$$F : H^H\text{YD} \to H\text{-Mod},$$

and require that $F(C)$ and $F(B)$ are dually paired Hopf algebras in $H\text{-Mod}$, while $C$ and $B$ may not be dually paired in $H^H\text{YD}$. Hence the coactions on $C$ and $B$ do not necessarily have to be related via the antipode, but the actions and resulting braidings need to be related by duality. In this case, we say that $C, B$ are weakly dually paired braided Hopf algebras in $H^H\text{YD}$.

This weaker duality is equivalent to an analogue of condition (3.7). Assuming this axiom, we can define an analogue of the braided Drinfeld double on the $k$-vector space $B \otimes H \otimes C$ (rather than using $B \otimes \text{Drin}(H) \otimes C$) with this weaker requirement of duality on $C$ and $B$. The definition of the asymmetric braided Drinfeld double can be given using Tannakian reconstruction theory by describing their category of modules. This is similar to the approach used for the braided Drinfeld double in [Maj99, Lau15].

**Notation 3.4.1.** Let $C, B$ be weakly dually paired. We denote the left coactions by $c \mapsto c^{(-1)} \otimes c^{(0)}$ and $b \mapsto b^{(-1)} \otimes b^{(0)}$ respectively. We can regard the left $H$-coaction on $B$ as a right $H^\text{cop}$-coaction. Given a left $H$-action $\triangleright$, we define a right $H^\text{cop}$-action $\langle := \triangleright (S^{-1} \otimes \text{Id})\tau$ (where $\tau$ denotes the $\otimes$-symmetry in $\text{Vect}_k$). The resulting structures make $B$ a right YD-module over $H^\text{cop}$.

The condition (3.7) can be rephrased as

$$b^{(0)}c^{(-1)} \otimes b^{(-1)}c^{(0)} = c^{(-1)}b^{(0)} \otimes c^{(0)}b^{(-1)}, \quad \forall b \in B, c \in C,$$

$$\iff \quad b_c = (c^{(-1)} \triangleright b^{(0)})(c^{(0)} \triangleright b^{(-1)}),$$

$$\iff \quad (c^{(-1)} \triangleright b)c^{(0)} = b^{(0)}(c \triangleright S(b^{(-1)})) = b^{(0)}(S(b^{(-1)}) \triangleright c).$$

We can visualize condition (3.12) using graphical calculus (using the conventions from [Lau15]):

\[
\begin{array}{cccc}
C & B & \Rightarrow & C & B \\
B & H & H & C & B & H & H & C \\
\end{array}
\]

**Definition 3.4.2.** Let $C, B$ be weakly dually paired braided Hopf algebras in $H^H\text{YD}$. We define the category $^B\text{YD}^{C}_{\text{asy}}(H)$ of asymmetric YD-modules over $B, C$ as having objects $V$ which are left $H$-modules (also viewed as right modules by means of the inverse antipode), equipped with a left $C$-action and a right $B$-action (by morphisms of $H$-modules) which satisfy the compatibility condition

$$((c_2) \triangleright v) \triangleleft b^{(-1)} = b^{(-1)}((c_1) \triangleright b^{(0)}) = c^{(-1)} \triangleright b^{(0)}\langle (c_2) \triangleright v \triangleleft b^{(-1)} \triangleright (v \triangleleft b^{(0)}),$$

for all $v \in V, b \in B, c \in C$. Morphisms in $^B\text{YD}^{C}_{\text{asy}}(H)$ are required to commute with the actions of $H$, $B$ and $C$. Note that on $V$ the right action is induced from the left action via $v \triangleleft b = S^{-1}(b) \triangleright v$.

It may help to visualize the condition (3.13) using graphical notation:
Proposition 3.4.3. The category $^B\mathcal{YD}_{asy}^C(H)$ is monoidal, with a monoidal fiber functors.

$$\text{Mod}^B(\text{H-Mod}) \xrightarrow{\Delta} H-\text{Mod} \xrightarrow{\text{Vect}_k} \text{C-Mod}(\text{H-Mod})$$

Proof. This monadicity statement can for example be checked directly using graphical calculus. Note that condition (3.12) is crucial. The fiber functors simply forget the additional structure at each step. □

Definition 3.4.4. The asymmetric braided Drinfeld double $\text{Drin}_H(B,C)$ is defined as the algebra obtained by Tannakian reconstruction on $B \otimes H \otimes C$ applied to the functor $^B\mathcal{YD}_{asy}^C(H) \rightarrow \text{Vect}_k$. Hence $\text{Drin}_H(B,C)$-Mod and $^B\mathcal{YD}_{asy}^C(H)$ are canonically equivalent as categories.

Proposition 3.4.5. An explicit presentation for the asymmetric braided Drinfeld double $\text{Drin}_H(B,C)$ on the $k$-vector space $B \otimes H \otimes C$ can be given as follows: the multiplication on $B$ is opposite, and for $c \in C$, $b \in B$ and $h \in H$ we have

(3.14) $hb = (h_{(2)} \triangleright b) h_{(1)}$,
(3.15) $hc = (h_{(1)} \triangleright c) h_{(2)}$,
(3.16) $b_{(2)} S^{-1}(b_{(1)}(-1)) c_{(2)}(c_{(1)} \otimes b_{(1)}(0)) = c_{1(2)}(-1) b_{(1)}(c_{(2)}(0) \otimes b_{(2)})$.

The coproducts are given by

(3.17) $\Delta(h) = h_{(1)} \otimes h_{(2)}$,
(3.18) $\Delta(b) = b_{(1)}(0) \otimes b_{(2)} S^{-1}(b_{(1)}(-1))$,
(3.19) $\Delta(c) = c_{(1)} c_{(2)}(-1) \otimes c_{(2)}(0)$,

and the antipode is

(3.20) $S(h) = S(h), \quad S(b) = S^{-1}(b(0)) b(-1), \quad S(c) = S(c(-1)) S(c(0))$.

Proof. This follows under application of reconstruction (in $\text{Vect}_k$) applied to $^B\mathcal{YD}_{asy}^C(H)$. See e.g. [Lau15, 2.3] for formulas on how to obtain the structures, including the antipode (Figure 2.1). □

An important feature of the braided Drinfeld double is that it has braided categories of representations. For the asymmetric braided Drinfeld double to be quasitriangular, we either $H$ to be quasitriangular. If $H$ is not quasitriangular, this can be achieved by working with over $\text{Drin}(H)$ instead of $H$ as a base Hopf algebra.

From now on, we restrict to the important special case where $B$ and $C$ are primitive generated by finite-dimensional YD-modules. This way, we obtain examples of Hopf algebras with a triangular decomposition over $H$.

Lemma 3.4.6. Let $V$, $V^*$ be left YD-modules over $H$, such that the action on $V^*$ is dual to the action on $V$. Then the algebras $T(V)^{op}$ and $T(V^*)^{cop}$ are dually paired Hopf algebras in the monoidal category of right modules over $H$. Further assume that the compatibility condition (3.12) holds.

Then the asymmetric braided Drinfeld double $\text{Drin}_H(T(V)^{op}, T(V^*)^{cop})$ is given on $A = T(V) \otimes H \otimes T(V^*)$ subject to the usual bosonization relations (2.7) and the cross relation

(3.21) $[f, c] = S^{-1}(f(-1)) \langle f, v(0) \rangle - f(-1) \langle f(0), v \rangle$.

---

3See e.g. [Lau15, 2.3].
4We choose the opposite $T(V)^{op}$ and coopposite $T(V^*)^{cop}$ in order to avoid having to take the opposite multiplication in the resulting double (cf. 3.4.5). As tensor algebras are cocommutative, this choice does not affect the formulas for the coproduct.
The coalgebra structure is given by
\[ \Delta(v) = v^{(0)} \otimes S^{-1}(v^{(-1)}) + 1 \otimes v, \quad \Delta(f) = f \otimes 1 + f^{(-1)} \otimes f^{(0)}. \]
The counit is given by \( \varepsilon(v) = \varepsilon(f) = 0 \) and the antipode can be computed using the conditions from equations (3.4) and (3.5) as
\[ S(v) = -v^{(0)}v^{(-1)}, \quad S(f) = -S(f^{(-1)})f^{(0)}. \]
We can also consider quotients of the form \( A/J \) for any triangular Hopf ideal \( J \in \mathcal{I}_\Delta(A) \). The quotient of \( A \) by the maximal triangular Hopf ideal in \( \mathcal{I}_\Delta(A) \) is denoted by \( \text{Drin}_H(V, V^*) \).

**Lemma 3.4.7.** Let \( A = \text{Drin}_H(T(V)^{op}, T(V^*)^{cop}) \) for \( V, V^* \) as in Lemma 3.4.6. Then the maximal ideal \( I_{\text{max}}(A) \) in \( \mathcal{I}_\Delta(A) \) is given by
\[ I_{\text{max}}(A) = I_{\text{max}}(V) \otimes H \otimes T(V^*) + T(V) \otimes H \otimes I_{\text{max}}(V^*), \]
where \( I_{\text{max}}(V) \) is the maximal ideal for the left coaction on \( V \), and \( I_{\text{max}}(V^*) \) is the maximal ideal for the left coaction on \( V^* \). Hence
\[ m: B(V) \otimes H \otimes B(V^*) \longrightarrow \text{Drin}_H(V, V^*) \]
is an isomorphism of \( k \)-vector spaces (PBW theorem).

**Proof.** This is clear as we know that \( T(V)^{op}/I_{\text{max}}(V) \) and \( T(V^*)^{cop}/I_{\text{max}}(V^*) \) are weakly dually paired braided Hopf algebras and their asymmetric braided Drinfeld double is given by the quotient \( \text{Drin}_H(T(V)^{op}, T(V^*)^{cop})/I_{\text{max}}(A) \), which must be the minimal double \( \text{Drin}_H(V, V^*) \). \( \square \)

A perfect pairing between the positive and negative part of \( \text{Drin}_H(V, V^*) \) implies the existence of a formal power series coev satisfying the axioms of coevaluation. This can be used to give a braiding on a suitable category of modules over \( \text{Drin}_H(V, V^*) \) (where \( B(V) \) acts integrally), and all modules have the structure of being YD-modules over \( H \).

### 3.5. Symmetric Triangular Decompositions
The rest of this section will be devoted to the question of recovering the braided Drinfeld double over a quasitriangular base Hopf algebra \( H \) as a special case of the asymmetric braided Drinfeld double. For this, we introduce the idea of a Hopf algebra with a symmetric triangular decomposition:

**Definition 3.5.1.** Given a bialgebra with triangular decomposition. If the associated coactions satisfy that the right coaction \( \delta^*_r \) of \( V^* \) is the dual coaction to \( \delta_l \), i.e.
\[ \langle f^{(0)} \otimes v \rangle f^{(-1)} = \langle f \otimes v^{(0)} \rangle v^{(-1)}, \]
and the coactions \( \delta_l \) and \( \delta^*_r \) are compatible in the same way, then we call the triangular decomposition symmetric.

In the case where \( H \) is a quasitriangular Hopf algebra, we can recover a special case of the definition of the braided Drinfeld double given in [Lau15, 3.5.6] from the more general form given in Definition 3.4.4, and the resulting triangular decomposition will be symmetric. For this, note that the universal \( R \)-matrix and its inverse give functors
\[ R^{-1}: H\text{-Mod} \longrightarrow H\text{-YD}, \quad (V, \triangleright) \longmapsto (V, \triangleright, (\text{Id}_H \otimes \triangleright)(R^{-1} \otimes \text{Id}_V)), \]
\[ R: \text{Mod}-H \longrightarrow H\text{-YD}, \quad (V, \triangleleft) \longmapsto (V, \triangleleft, (\triangleleft \otimes \text{Id}_H)(\text{Id}_V \otimes R)). \]
Given a right \( H \)-module \( V \), we can hence give \( V \) the left YD-module structure using \( R^{-1} \), and \( V^* \) the dual YD-module structure. Note that (3.12) is satisfied in this case. With these structures, the relation (3.21) becomes
\[ [f, c] = S^{-1}(R^{-2})(f, v \triangleleft R^{-1}) - R^{-1}(f, v) = R^{-2}([R, \triangleright], f, v) - R^{-1}([R, \triangleright], f, v). \]
This is precisely the condition of [Lau15, 3.5.6]. Note that we use \( R = (S^{-1} \otimes \text{Id}_H)R^{-1} \). This proves the following:

**Proposition 3.5.2.** Braided Drinfeld doubles of braided Hopf algebras over a quasitriangular Hopf algebras are a special case of Definition 3.4.4 with symmetric triangular decomposition.
Note that a partial converse statement also holds: Given an asymmetric braided Drinfeld double that is symmetric, then it can be displayed as a braided Drinfeld double in the sense of [Lau15,Maj99], but unless \( H \) is quasitriangular (and the coaction induced by the \( R \)-matrix), we need to view if over the base Hopf algebra \( \text{Drin}(H) \). If the positive and negative part are perfectly paired, then we can give a formal power series describing the \( R \)-matrix and an appropriate subcategory (corresponding to the Drinfeld center) is braided.

Particularly interesting examples of such braided Drinfeld doubles include the quantum groups \( U_q(g) \) for generic \( q \), and the small quantum groups \( u_q(g) \) (see [Maj99]). Their construction uses the concept of a weak quasitriangular structure for which a similar statement to 3.5.2 can be made. We will see in Section 5 that multiparameter quantum groups can be viewed as examples of asymmetric braided Drinfeld doubles that are not symmetric. Further, all the pointed Hopf algebras classified in the main result of this paper (Theorem 4.2.2), under the additional assumption that the braiding is of separable type and some commutators do not vanish, are asymmetric braided Drinfeld doubles.

4. Classification over a Group

In this section, we denote by \( A = T(V) \otimes kG \otimes T(V^*) \) a bialgebra with triangular decomposition over a group algebra \( kG \). Note that we do not assume \( G \) to be finite.

4.1. Preliminary Observations. Hopf algebras that are generated by grouplike and skew primitive elements are always pointed. We show that assuming a Hopf algebra has triangular decomposition over a group and is of what we call weakly separable type, it is generated by skew-primitive elements and hence pointed.

Lemma 4.1.1. For a bialgebra \( A \) with triangular decomposition over \( kG \) as above, there exists a basis \( v_1, \ldots, v_n \) of \( V \) and \( f_1, \ldots, f_n \) of \( V^* \), as well as invertible matrices \( M \) and \( N \) such that

\[
\Delta(v_i) = v_i \otimes g_i + \sum_j M_{i,j} h_j \otimes v_j', \quad \Delta(f_i) = f_i \otimes a_i + \sum_j N_{i,j} b_j \otimes f_j',
\]

where \( v_1', \ldots, v_n' \) is another basis of \( V \), and \( f_1', \ldots, f_n' \) of \( V^* \).

Proof. Let \( v_1, \ldots, v_n \) be a homogeneous basis for the \( YD \)-compatible grading \( \delta \), and \( v_1', \ldots, v_n' \) a homogeneous basis for \( \delta' \). The form (4.1) of the coproducts is obtained by letting \( M \) be the base change matrix from \( \{v_i\} \) to \( \{v_i'\} \). The same argument works for the dual \( V^* \), denoting the base change matrix from \( \{f_i\} \) to \( \{f_i'\} \) by \( N \).

Lemma 4.1.2. A bialgebra \( A \) with a triangular decomposition over \( kG \) as above is a Hopf algebra, with antipode \( S \) given on generators of the form \( v_i, f_i \) as in (4.1) by

\[
S(v_i) = -\sum_j M_{i,j} (h_j^{-1} \otimes v_j' ) h_j^{-1} g_i^{-1}, \quad S(f_i) = -\sum_j N_{i,j} (f_j' \otimes b_j) h_j^{-1} a_i^{-1}.
\]

Proof. The antipode axioms require that \( S \) is of the form stated, using that \( kG \) is a Hopf subalgebra, cf. (3.4)-(3.5). As \( T(V) \) and \( T(V^*) \) are free, defining \( S \) on the generators extends uniquely to an antialgebra and coalgebra map on all of \( A \).

Definition 4.1.3. A Hopf algebra with triangular decomposition \( A \) is called of weakly separable type if the degrees right \( g_1, \ldots, g_n \) of \( V \) are pairwise distinct group elements, and the same holds for the left degrees \( h_1, \ldots, h_n \) of \( V \) as well as the dual degrees.

We observe that being of weakly separable type over a group implies that \( V \) and \( V^* \) have 1-dimensional homogeneous components. This gives that for a homogeneous basis element \( v_i \) of degree \( a_i \), \( g \triangleright v_i \neq 0 \) is homogeneous of degree \( g a_i g^{-1} \) which hence has to be a scalar multiple of a basis element \( v_{g(i)} \) where \( g(i) \) is an index \( 1, \ldots, n \). Hence we obtain an action of \( G \) on \( \{1, \ldots, n\} \). To fix notation, we write

\[
g \triangleright v_i = \lambda_i(g) v_{g(i)}, \quad f_i \triangleleft g = \mu_i(g) f_{g(i)}.
\]

We will see that for \( A \) of weakly separable type, the bases change matrices \( M, N \) are diagonal matrices and can be chosen to be the identity matrix by rescaling of the diagonal bases. This implies that \( A \) is generated by primitive and group-like elements and hence pointed. It is a conjecture in [AS02] that


all finite-dimensional pointed Hopf algebras over a field of characteristic zero are in fact generated by skew-primitive and grouplike elements.

**Proposition 4.1.4.** If $A$ is of weakly separable type, then there exists a basis $\{v_i\}$ of $V$ and $\{f_i\}$ of $V^*$ consisting of $(g_i, h_i)$-skew primitive elements, i.e.,

\begin{equation}
\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i, \quad \Delta(f_i) = f_i \otimes a_i + b_i \otimes f_i,
\end{equation}

and the antipode on these skew-primitive elements is given by $S(v_i) = (h_i^{-1} \triangleright v_i) h_i^{-1} g_i^{-1}$, $S(f_i) = (f_i \lhd b_i) b_i^{-1} a_i^{-1}$.

**Proof.** Consider the right and left coactions $\delta_r$ and $\delta_l$ from Section 3.1. Choosing a basis $v_1, \ldots, v_n$ homogeneous for $\delta_l$ and $v'_1, \ldots, v'_n$ homogeneous for $\delta_r$, (4.1) gives

\begin{equation}
\Delta(v_i) = v_i \otimes g_i + \sum_j M_{i,j} h_j \otimes v'_j,
\end{equation}

where $M = (M_{i,j})$ is the base change matrix. By coassociativity, we find that

\begin{equation}
\sum_{j,k} M_{i,j} (M^{-1})_{j,k} h_j \otimes v_k \otimes g_k = \sum_j M_{i,j} h_j \otimes v'_j \otimes g_i.
\end{equation}

By weak separability of $\delta_r$ and $\delta_l$ we now have for each $j = 1, \ldots, n$:

$$\sum_k M_{i,j} (M^{-1})_{j,k} v_k \otimes g_k = M_{i,j} v'_j \otimes g_i.$$

Note that $M_{i,j} \neq 0$ for at least some $i$. This implies that $(M^{-1})_{j,k} = 0$ unless $k = i$ as the $g_i$ are all distinct. Further, if $M_{i,j} \neq 0$, then $v_i$ and $v'_j$ are proportional. This can only be true for at most one $i$ for given index $j$ by weak separability. Hence by reordering the basis $v'_1, \ldots, v'_n$ we find that $M$ is a diagonal matrix and can rescale the basis $\{v'_j\}$ such that $M$ is the identity matrix. Hence we have $\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i$. The antipode conditions for $A$ give (using Lemma 3.1.3) that $S$ is of the form claimed. \hfill $\square$

**Remark 4.1.5.** The bases $\{v_i\}$ and $\{f_i\}$ do not necessarily need to be orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle$. We will see in Theorem 4.2.2 that if the characters $\lambda_i$ are all distinct, then the bases can be chosen to be dual bases.

**Notation 4.1.6.** In the following, we fix a basis $v_1, \ldots, v_n$ for $V$ and $f_1, \ldots, f_n$ for $V^*$ such that

\begin{equation}
\Delta(v_i) = v_i \otimes g_i + h_i \otimes v_i, \quad \Delta(f_i) = f_i \otimes a_i + b_i \otimes f_i, \quad i = 1, \ldots, n.
\end{equation}

A direct observation from Proposition 4.1.4 is that the algebra $A$ is generated by primitive and grouplike elements (which are precisely the group $G$) and hence pointed. Even in the general case (not assuming that $A$ is of weakly separable type), we have the following restrictions on the group structure.

**Proposition 4.1.7.** In the group $G$, the relations $[g_i, a_j] = [h_i, a_j] = 1$ and $[h_i, b_j] = [g_i, b_j] = 1$ hold for all $i, j = 1, \ldots, n$. In particular, if $A$ has a symmetric triangular decomposition, then the subgroup of $G$ generated by all degrees is abelian.

Further, the following identities for the characters of the group action hold:

\begin{equation}
\mu_j(h_i) = \lambda_i(a_j)^{-1}, \quad \mu_j(g_i) = \lambda_i(b_j)^{-1}.
\end{equation}

**Proof.** The commutator relations follow by applying (3.8) and (3.9) to a pair of homogeneous basis elements of $V$ and $V^*$ with respect to $\delta_l, \delta_r$ (or $\delta_r^*, \delta_l^*$). Then, even without weak separability, it follows from (3.6) and (3.7) that $h_i(j) = j$, $a_j(i) = i$, $g_i(j) = j$ and $b_j(i) = i$ by the PBW theorem. This implies the relations (4.8). In the symmetric case, $a_i = g_i^{-1}$ and $b_i = h_i^{-1}$ which forces the subgroup generated by all degrees to be abelian. \hfill $\square$
4.2. Classification in the Free Case of Weakly Separable Type. We are now in the position, that we can classify all Hopf algebras $A$ with triangular decomposition of weakly separable type (cf. Definition 4.1.3). This will enable us view the Hopf algebras arising from this classification as analogues of multiparameter quantum groups in Section 5. We start by considering the case $A = T(V) \otimes kG \otimes T(V^*)$ which is referred to as the free case.

**Proposition 4.2.1.** For the Hopf algebra $A$ with triangular decomposition of weakly separable type to be indecomposable as a coalgebra it is necessary that $G$ is generated by elements $k_1, \ldots, k_n, l_1, \ldots, l_n$ such that there exist generators $v_i$ of $V$ and $f_i$ of $V^*$ which are skew-primitive of the form

$$\Delta(v_i) = v_i \otimes k_i + 1 \otimes v_i,$$

$$\Delta(f_i) = f_i \otimes 1 + l_i \otimes f_i,$$

with $[k_i, l_j] = 1$ for all $i, j$. For the characters of the actions on the homogeneous components of $V$ and $V^*$ we require that

$$\mu_j(k_i) = \lambda_i(l_j)^{-1}.$$

**Proof.** To determine when pointed Hopf algebras are indecomposable as coalgebras, consider the graph $\Gamma_A$ described in 2.2. Assume that $A$ has generators given as in 4.1.6. We claim that the connected components of $\Gamma_A$ are in bijection with the double cosets of the subgroup

$$Z := \langle g_1^{-1} h_1, \ldots, g_n^{-1} h_n, a_1^{-1} b_1, \ldots, a_n^{-1} b_n \rangle$$

in $G$ which partition $G$. Indeed, using that the elements $g v_i$ and $g f_i$ are skew-primitive of type $(g g_i, g h_i)$ and $(g a_i, g b_i)$, we find that the connected component of $g$ contains, for $i = 1, \ldots, n$, the strands

$$\ldots \rightarrow g(g_i^{-1} h_i)^{-2} \rightarrow g(g_i^{-1} h_i)^{-1} \rightarrow g \rightarrow g(g_i^{-1} h_i)^{1} \rightarrow g(g_i^{-1} h_i)^{2} \rightarrow \ldots$$

for $i = 1, \ldots, n$ and the same strand with $a_i^{-1} b_i$ instead of $g_i^{-1} h_i$ (and with $g$ multiplied on the right). Moreover, as the elements $g v_i$, $g f_i$, $v_i, f_i$ (and possibly linear combinations of products of them, which would again be of type given by elements in $Z$) are the only skew-primitive elements in $A$, and thus give the only arrows in $\Gamma_A$, two elements $g$ and $h$ are in the same connected component if and only if $z_1 g z_2 = z_2 h z_4$, for some $z_i \in Z$. Thus, $A$ is indecomposable if and only if $G$ equals the connected component of $1$ in the graph $\Gamma_A$, hence if $G = Z$ which is the finitely-generated group generated by the elements $k_i := h_i^{-1} g_i$, $l_i := a_i^{-1} b_i$ for $i = 1, \ldots, n$. Hence, in order to obtain indecomposability, the coproducts are of the form as stated in (4.9). This is achieved by replacing the generators $v_i$ by $v_i h_i^{-1}$ and $f_i$ by $a_i^{-1} f_i$. The rest of the statements follow directly from Proposition 4.1.7.

**Theorem 4.2.2.** For an indecomposable pointed Hopf algebra $A$ as in Theorem 4.2.1 of weakly separable type, the commutator relation (2.10) is of the form

$$[f_i, v_j] = \gamma_{ij}(k_j - l_i)$$

for all $i, j \leq n$, where $\gamma_{ij}$ are scalars in $k$ such that $\gamma_{ij} = 0$ whenever $\lambda_j \neq \lambda_i$ in which case also $\langle f_i, v_j \rangle = 0$. Conversely, any choice of such scalars gives a pointed Hopf algebra of this form.

**Proof.** With the work done in Proposition 4.1.3, it remains to verify that the form of the commutator relation (2.10) is as stated. Recall that in [BB09], the commutator relation is given by means of a quasi-coaction. That is a morphism $\delta : V \rightarrow kG \otimes V$ satisfying (2.9) and (2.10). Such a morphism has the general form

$$\delta(v_j) = v_j^{[1]} \otimes v_j^{[0]} = \sum_{k,g} \alpha_{k,g}^{(j)} g \otimes v_k,$$

where $\alpha_{k,g}^{(j)} \in k$, on the basis elements. Then (3.10), which is required for $A$ to be a bialgebra, rewrites as

$$\sum_{k,g} \alpha_{k,g}^{(j)} (g \otimes k_j + l_i \otimes g)(f_i, v_k) = \sum_{k,g} \alpha_{k,g}^{(j)} g \otimes g(f_i, v_k).$$

For each $i$, there exists $k$ such that $\langle f_i, v_k \rangle \neq 0$. For given $i$, we denote the set of indices such that $\langle f_i, v_k \rangle \neq 0$ by $I_i$. For such $k \in I_i$, we find that $\alpha_{k,g}^{(j)} = 0$ for $g \neq k$, $l_i$, and $\alpha_{k,k_i}^{(j)} = -\alpha_{k,l_i}^{(j)}$. Thus, we obtain that $\delta$ is of the form

$$\delta(v_j) = v_j^{[1]} \otimes v_j^{[0]} = \sum_{i=1}^{n} \gamma_{ij}(k_j - l_i) \otimes v'_i,$$

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where $\gamma_{ij} = \sum_{k \in I} \alpha_{i,k}^j 1/(f_i, v_k) |I|_i$ and $\{v_i^j\}$ is the dual basis of $V$ to $\{f_i\}$. Conversely, given arbitrary scalars $\gamma_{ij}$ for $i, j = 1, \ldots, n$, we can define a quasi-coaction by the same formula (4.13). Then $\delta$ is YD-compatible with the given action of $G$ on $V$ if and only if (cf. to condition (A) in [BB09, Theorem A])

$$\gamma_{ij} \mu_i(g)(g k_j - g l_i) = g[f_i < g, v_j] \overset{\text{(A)}}{=} [f_i, g \rightharpoonup v_j] g = \gamma_{ij} \lambda_j(g)(l_j g - l_i g).$$

As $A$ is indecomposable of weakly separable type, $G$ is abelian and hence this condition is equivalent to $\lambda_1 = \mu_1$ whenever $\gamma_{ij} \neq 0$. But by duality of the action, if $\langle f_i, v_j \rangle \neq 0$ then $\lambda_1 = \mu_1$.

As for given $i = 1, \ldots, n$, $\langle f_i, v_j \rangle \neq 0$ for some $j$ we have that $\lambda_1 = \mu_j$ for at least some $j$, and vice versa. Hence, the set of characters and dual characters are in bijection. We can change the numbering and assume without loss of generality (recall that we are in the weakly separable case) to obtain

$$\lambda_i = \mu_i. \tag{4.14}$$

From now on, we will hence only use the notation $\lambda_i$.

The situation, where $\{v_i\}$ and $\{f_i\}$ are orthogonal bases deserves particular attention. In this case, the scalars $\gamma_{ij} = 0$ for $i \neq j$. The following concept of separability ensure this.

**Definition 4.2.3.** If the characters $\lambda_1, \ldots, \lambda_n$ are distinct for different indices, we will speak of a triangular decomposition of separable type in this case.

**Remark 4.2.4.** At this point, to [AS10] and [AS04] seems appropriate. The condition (4.11) is equivalent to the so-called linking relation (2.4) after a change of generators $f_i \leftrightarrow l_i^{-1} f_i$, since in the form of 2.3.1 all generators have coproducts $\delta(v_i) = v_i \otimes 1 + g_i \otimes v_i$. Such a change of generators causes the commutators $ad = [ , ]$ to become braided commutators $ad = Id_{V \otimes \mathbb{C}} - \Psi$. The scalars $\lambda_{ij}$ satisfy the condition (d) in 2.3, where for the characters $\chi_i \chi_j \neq \varepsilon$ implies $\lambda_{ij} = 0$. This is the analogue of our condition $\lambda_1 = \lambda_j$, implying $\gamma_{ij} = 0$.

This linking relation also appears in the quantum group characterization of [AS04, Theorem 4.3]. Hence we can conclude that the classification in this section gives Hopf algebras with similar relations as appearing in the work of Andruskiewitsch and Schneider.

**Example 4.2.5.** The most degenerate case, where $\gamma_{ij} = 0$, gives the Hopf algebra $(T(V) \oplus T(V^*)) \otimes kG$ where the tensor algebras are again computed in the category of YD-modules over $kG$.

Assuming the non-degeneracy that $\gamma_{ii} \neq 0$, we can adapt the terminology of [BB09] that the braided doubles in this case come from mixed YD-structures. A mixed YD-structure is a quasi-coaction $\delta$ that is a weighted sum $\sum_i t_i \delta_i$, where $\delta_i$ are YD-modules compatible with the same action, and $t_i$ are generic scalars. The quasi YD-module in the theorem is the sum $\delta = \delta_+ - (\delta_-)^*$, where $(\delta_+)^*$ is the YD-module given by $v_j \mapsto l_j \otimes v_j$, which is dual to $\delta_-$. We will see that in this case all the Hopf algebras arising are certain asymmetric braided Drinfeld doubles (as defined in 3.4).

In the symmetric case, these algebras are in fact braided Drinfeld doubles. In particular, their adequately defined module categories (resembling the category $O$, see [Lau15, 3.9]) are braided.

### 4.3. Interpretation as Asymmetric Braided Drinfeld Doubles

So far, we have only classified free braided doubles over $kG$. That is, as a $k$-vector space $A \cong T(V) \otimes kG \otimes T(V^*)$ via the multiplication map. To capture examples such as quantum groups, it is necessary to consider quotients of $A$ by ideals $J = \langle I, I^* \rangle$ such that $A/J \cong T(V)/I \otimes kG \otimes T(V^*)/I^*$ is still a Hopf algebra (and thus pointed). Here $I \triangleleft T(V)$ and $I^* \triangleleft T(V^*)$ are ideals and also coideals, and $J \in \mathcal{I}_\Delta(A)$. We will now refine our considerations from Section 3.3 to find for what ideals $I$ and $I^*$ this is the case. We will use the notation

$$q_{ij} := \lambda_j(k_i). \tag{4.15}$$

Then, by (4.10), we have that $\lambda_j(l_i) = q_{ij}^{-1}$, and the matrix $q = (q_{ij})$ describes the braiding on $V$ fully, i.e. it is of diagonal type.

The collection of triangular Hopf ideals $\mathcal{I}_\Delta(A)$ introduced in Section 2.5 can be described more concretely for $A$ satisfying the following restrictions: we assume that the parameters $\gamma_{ii} \neq 0$ for all $i$ and that $V$ (and hence $V^*$) are of separable type, and that $k_i \neq l_i$. Recall that in this situation, the algebras of the classification 4.2.2 are displayed as what is referred to in [BB09] as arising from mixed.
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YD-structures. More specifically, the quasi-coaction \( \delta = \delta_l - (\delta^*)^* \), where \( \delta^* \) denotes the coaction on \( V \) that is obtained by dualizing the left coaction \( \delta^l \) on \( V^* \) (this is possible as \( G \) is abelian).

By Lemma 3.4.7, the ideals in \( \mathcal{I}_A(A) \) are of the form \( I = I \otimes kG \otimes T(V^*) + T(V) \otimes kG \otimes I^* \) where \( I \) is an ideal in the collection \( \mathcal{I}_l(V, \delta_i) \) for \( V \) with the right coaction given by \( \delta_r \) and \( I^* \) is in \( T(V^*, \delta^r) \) for the left dual coaction \( \delta^r \) on \( V^* \).

Note that by (4.10) we the braiding \( \Psi_r \) coming from \( \delta_r \) and \( \Psi_l \) from \( (\delta^r)^* \) on \( V \) are given by

\[
\Psi_r (v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \Psi_l (v_i \otimes v_j) = q^{-1}_{ji} v_j \otimes v_i.
\]

Hence \( \Psi_l = \Psi_r^{-1} \), the inverse braiding. We hence drop the subscripts \( l,r \).

**Example 4.3.1.** In the quantum groups \( A = \mathcal{U}_q(\mathfrak{g}) \), the braiding satisfies the symmetry \( q_{ij} = q^{ij} = q^{ji} \) as the Cartan datum is symmetric. This implies that the relations in \( I \) are symmetric under reversing the order of tensors \( v_1 \otimes \ldots \otimes v_n \leftrightarrow v_n \otimes \ldots \otimes v_1 \). This can be verified explicitly by observing that in \( \mathcal{U}_q(\mathfrak{g}) \) the ideal \( I \) is generated by \( q \)-Serre relations, which carry such a symmetry.

**Theorem 4.3.2.** All quotients by triangular ideals \( J \in \mathcal{I}_A(A) \) of algebras \( A \) occurring in the classification 4.2.2, where \( A \) is of separable type with \( \gamma_{ii} \neq 0 \) for all \( i \), are asymmetric braided Drinfeld doubles. If \( J \) is maximal of this form, then \( A/J \cong \text{Drin}_{kG}(V, V^*) \).

**Proof.** We have seen that the commutator relations are of the form \( [f_i, v_j] = \delta_{ij} \gamma_{ii}(k_i - l_j) \). This is precisely the form of the asymptotic braided Drinfeld double of \( V \) with right YD-module structure given by the right grading, and \( V^* \) with left YD-module structure given by the left dual grading. The pairing is given by \( \langle f_i, v_j \rangle = \delta_{ij} \gamma_{ii} \) here. We have to check that the braided Hopf algebras \( T(V) \) and \( T(V^*) \) of YD-modules over \( G \) are dually paired when viewed in the category of left \( kG \)-modules. This however follows from condition (4.10). Taking the maximal quotient by a triangular ideal (or the left and right radical of the pairing) gives the asymmetric braided Drinfeld double \( \text{Drin}_{kG}(V, V^*) \).

If some of the parameters \( \gamma_{ii} \) are zero, then the pointed Hopf algebras obtained are not an asymmetric braided Drinfeld double any more (in the sense of Definition 3.4.4).

4.4. Recovering a Lie Algebra. We assume that char \( k = 0 \) in this section and study Hopf algebras with triangular decomposition of separable type which are of the form \( \text{Drin}_{kG}(V, V^*) \) (see Theorem 4.3.2). The aim is to set the characters \( \lambda_i \) and the group elements \( k_i, l_i \) equal to 1. This way, we want to recover a Lie algebra \( \mathfrak{g} \) for any of the indecomposable pointed Hopf algebras of the form \( \text{Drin}_{kG}(V, V^*) \), relating back to the question asked in the introduction of finding quantum groups for a given Lie algebra. The tool available for this is the Milnor-Moore theorem from [MM65] (see also [Mon93, Theorem 5.6.5]) which shows that any cocommutative connected Hopf algebras is of the form \( \mathcal{U}(\mathfrak{g}) \) for a (possibly infinite-dimensional) Lie algebra \( \mathfrak{g} \).

There are technical problems with this naive approach. To set the elements \( q_{ij} \) – which will be replaced by formal parameters – equal to one, we need to give an appropriate integral form to avoid that the modules collapse to zero. This rules out examples like e.g. \( k[x]/(x^n) \) (and, more generally, the small quantum groups) which are braided Hopf algebras in the category of YD-modules over \( k\mathbb{Z} \), as here a generator of the group acts by a primitive \( n \)th root of unity \( q \) on \( x \), and \( \mathbb{Z}[q] \cap k \) is a cyclotomic ring.

As a first step, we introduce appropriate integral forms of \( \text{Drin}_{kG}(V, V^*) \), for which we need square roots of \( q_{ij} \). We consider the subring \( Z := \mathbb{Z}[q_{ij}^{\pm 1/2}]_{i,j} \subset k \) adjoining all square roots of the numbers \( q_{ij} \) and their inverses. This will now be treated as formal parameters with certain relations between them, coming from the relations we have among them in \( k \).

**Assumption 4.4.1.** In this section, we assume that the ideal \( \langle q_{ij}^{\pm 1/2} - 1 \mid i, j = 1, \ldots, n \rangle \) in \( Z \) is a proper ideal, and hence \( p: Z \to \mathbb{Z}, q_{ij}^{\pm 1/2} \to 1 \) is a well-defined morphism of rings.

This assumption is crucial in the formal limiting process. It, for example, prevents examples in which \( q^a + q^{a-1} + \ldots + q + 1 = 0 \) as in cyclotomic rings.

To produce an integral form, we replace a given YD-module \( V \) over \( kG \) of separable type as in the previous sections, by a YD-module over \( ZG \). For this, we can choose a \( G \)-homogeneous basis
\(v_1, \ldots, v_n\) and a homogeneoue dual basis \(f_1, \ldots, f_n\) such that (possibly after rescaling)
\[
\langle f_i, v_j \rangle = \frac{1}{q_{ii}^{1/2} - q_{jj}^{1/2}} \delta_{ij}, \quad \forall i, j.
\]

An important observation is that the Woronowicz symmetrizers, which are used to compute the Nichols ideal \(I_{\text{max}}(V)\), have coefficients in \(Z\). Hence their kernels will be \(Z\)-modules. That is, for \(V^{\text{int}}\) defined as \(Z\langle v_1, \ldots, v_n\rangle\), which is a YD-module over the group ring \(ZG\), the Woronowicz symmetrizer \(\Psi \) is a \(Z\)-linear map \(V^{\text{int}} \otimes Z \rightarrow V^{\text{int}} \otimes Z\). Hence \(I_{\text{max}}(V^{\text{int}}) := \ker \Psi \) is an ideal in \(T(V^{\text{int}})\), the tensor algebra over \(Z\).

In order to provide an integral form of \(\text{Drin}_{kG}(V, V^*)\), we will change the presentation by introducing new commuting generators, namely \([f_i, v_j] =: t_i\). One verifies that the following commutator relations hold over \(k\), as we are given the relation \(t_i = \frac{1}{q_{ii}^{1/2} - q_{ii}^{-1/2}} (k_i - l_i)\) when working over the field:
\[
[f_i, t_j] = \delta_{i,j} (q_{ii}^{1/2} k_i f_i + q_{ii}^{-1/2} f_i k_i),
\]
\[
[v_i, t_j] = -\delta_{i,j} (q_{ii}^{1/2} f_i v_i + q_{ii}^{-1/2} v_i f_i).
\]

**Definition 4.4.2.** The integral form \(\text{Drin}_{kG}(V^{\text{int}}, V^{\text{int}})\) of \(\text{Drin}_{kG}(V, V^*)\) is defined as the graded Hopf algebra over the ring \(Z\) generated by \(v_1, \ldots, v_n\), of degree 1, \(f_1, \ldots, f_n\) of degree \(-1\), and the group elements \(k_1, \ldots, k_n, l_1, \ldots, l_n \in G\), and additional elements \(t_1, \ldots, t_n\) of degree 0, subject to the relations of \(I_{\text{max}}(V^{\text{int}})\) and \(I_{\text{max}}(V^{\text{int}})\), bosonization relations
\[
g v_i = (g \mapsto v_i) g, \quad f_i g = g (f_i \mapsto g),
\]
and the relations (4.18), (4.19) and
\[
g v_i = (g \mapsto v_i) g, \quad f_i g = g (f_i \mapsto g),
\]
\[
q_{ii}^{1/2} (k_i - l_i) = (q_{ii} - 1) t_i,
\]
\[
[f_i, v_j] = \delta_{i,j} t_i,
\]
\[
[t_i, t_j] = 0.
\]

The coproducts are given as before on the generators \(f_i, v_i, k_i, l_i\) and \(\Delta(t_i) = t_i \otimes k_i + l_i \otimes t_i\).

Note that as \(A = \text{Drin}_{kG}(V^{\text{int}}, V^{\text{int}})\) is a Hopf algebra over the commutative ring \(Z\), the coproduct is a map \(A \rightarrow A \otimes_Z A\). For the quantum groups \(U_q(G)\) at generic parameter, the integral form in this case is so-called non-restricted integral form (see e.g. [CP95, 9.2]) which goes back to De Concini-Kac [DCK90]. To set the parameters equal to one, and to consider extensions of Hopf algebras to fields, we use the following Lemma:

**Lemma 4.4.3.** Let \(\phi: R \rightarrow S\) be a morphism of commutative algebras. We denote the category of Hopf algebra over \(R\) by \(\text{Hopf}_R\). Then base change along \(\phi\) induces a functor
\[
\text{Hopf}_\phi: \text{Hopf}_R \rightarrow \text{Hopf}_S, \quad A \mapsto A \otimes_R S.
\]

**Proof.** Given a Hopf algebra \(A\) which is an \(R\)-algebra, i.e. there is a morphism \(R \rightarrow A\), we induce the multiplication and comultiplication on \(S \rightarrow A \otimes R S\) using the isomorphism
\[
(A \otimes R S) \otimes S (A \otimes R S) \cong (A \otimes R A) \otimes R S.
\]
It is easy to check that the Hopf algebra axioms are preserved under base change. \(\square\)

**Proposition 4.4.4.** There is an isomorphism of graded Hopf algebras
\[
\text{Drin}_{kG}(V^{\text{int}}, V^{\text{int}}) \otimes Z \cong \text{Drin}_{kG}(V, V^*).
\]

**Proof.** Recall that \(Z \leq k\) by construction. Extending to \(k\), we are able to divide by \(q_{ii} - 1\) in (4.22), and recover the original commutator and bosonization relations in \(\text{Drin}_{kG}(V, V^*)\). It remains to verify that
\[
I_{\text{max}}(V^{\text{int}}) \otimes Z \cong \ker \Psi \otimes Z \cong \ker \Psi = I_{\text{max}}(V).
\]
This follows by noting that \(k\) is flat as a \(Z\)-module (since the function field \(K(Z)\) is flat over \(Z\) as a localization, and \(k\) is free over \(K(Z)\)), and \(V^{\text{int}} \otimes Z \cong V\) as \(k\)-vector spaces. \(\square\)
Definition 4.4.5. We define the classical limit of Drin$_{kG}(V,V^\ast)$ as the algebra
\[
\text{Drin}^\text{cl}_k(V,V^\ast) := (\text{Drin}_{ZG}(V^\text{int},V^\text{int}^\ast) \otimes \mathbb{Z} \otimes \mathbb{Z} k) \otimes \ker \varepsilon_G,
\]
using the morphism $p: \mathbb{Z} \to \mathbb{Z}$ mapping all $q_{ij}^{\pm 1/2}$ equal to 1, and the two sided ideal $\langle \ker \varepsilon_G \rangle$ generated by the kernel of the augmentation map $\varepsilon_G: kG \to k$ mapping all group elements to 1. Note that this ideal is a Hopf ideal.

That is, to obtain the classical form we first set the parameters $q_{ij}^{\pm 1/2}$ equal to 1 in the integral form and then extend the resulting $\mathbb{Z}$-module to a $k$-vector space, and finally set the group elements equal to 1 along the counit $\varepsilon_G: kG \to k$. We obtain a primitively generated Hopf algebra, and hence a Lie algebra, this way:

Proposition 4.4.6. The classical limit Drin$_{k}^\text{cl}(V,V^\ast)$ is a connected Hopf algebra, generated by primitive elements. Hence, for the Lie algebra $p_V$ of primitive elements, $U(p_V) = \text{Drin}^\text{cl}_k(V,V^\ast)$. This algebra is generated by triples $f_i,v_i,t_i$, which form a subalgebra isomorphic to $U(\mathfrak{sl}_2)$.

Proof. Lemma 4.4.3 ensures that Drin$_{k}^\text{cl}(V,V^\ast)$ is a Hopf algebra over $k$, and freeness of $V^\text{int}$ over $\mathbb{Z}$ ensures that the positive and negative part do not collapse to the zero space. In particular, the $k$-vector space $V^\text{int} \otimes V^\text{int}^\ast$ embeds into the Lie algebra $p_V$ of primitive elements. In the classical limit, we obtain the relations
\[
[f_i,v_j] = \delta_{i,j} t_i, \quad [f_i,t_j] = 2\delta_{i,j} f_i, \quad [v_i,t_i] = -2\delta_{i,j} v_i.
\]
Hence every triple $f_i,v_i,t_i$ generates a Lie subalgebra of $p_V$ isomorphic to $\mathfrak{sl}_2$. Note that Drin$_{k}^\text{cl}(V,V^\ast)$ is generated by primitive elements:
\[
\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i, \quad \Delta(v_i) = v_i \otimes 1 + 1 \otimes v_i.
\]
We also compute
\[
\Delta(t_i) = \Delta([f_i,v_i]) = [f_i,v_i] \otimes k_i + l_i \otimes [f_i,v_i] = t_i \otimes k_i + l_i \otimes t_i.
\]
Hence, $t_i$ is skew primitive in Drin$_{kG}(V,V^\ast)$ and primitive in the classical limit. Thus, Drin$_{k}^\text{cl}(V)$ is a pointed Hopf algebra over the trivial group. That is, a connected pointed Hopf algebra. It is further cocommutative and Theorem 5.6.5 in [Mon93] implies that such a Hopf algebra is of the form $U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of primitive elements as char $k = 0$.

Example 4.4.7. For $U_q(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, viewed as a braided Drinfeld double, the classical limit is $U(\mathfrak{g})$.

We can also compute examples that do not give finite-dimensional semisimple Lie algebras. As a general rule, the relations between the parameters $q_{ij}$ determine the relations in the Lie algebra. It is easy to construct free examples, for which there are no relations between the $v_1,\ldots,v_n$ by choosing algebraically independent parameters $q_{ij}$. The work of [Ros98] and [AS04] give restrictions on examples satisfying the growth condition of finite Gelfand-Kirillov dimension. We will view their results in the setting of this paper in Section 5.2.

5. Classes of Quantum groups

In this section, we relate the classification from Section 4 to various classes of examples which are often regarded as quantum groups. This includes the multiparameter quantum groups studied by [Res90, FRT88, AST91, Sud90] and others in Section 5.1, a characterization of Drinfeld-Jimbo quantum groups in Section 5.2, and classes of examples of pointed Hopf algebras from the work of Radford in Section 5.3. The classification in Theorem 4.2.2 points out natural generalizations of these classes of examples. We finally sketch how one can define analogues of quantum groups using triangular decompositions over other Hopf algebras than $kG$. 

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5.1. Multiparameter Quantum Groups. Let $k$ be a field of characteristic zero. For the purpose of this section, let $\lambda \in k$ be generic, and $p_{ij} \in k$ for $1 \leq i < j \leq n$. Assume that $p_{ii} = 1$ and $p_{ji} = p_{ij}^{-1}$. Following [AST91, CM96] and to fix notation, we set

$$
\kappa_j^{(i)} = \begin{cases} 
p_{ij}, & \text{if } i < j, \\
\lambda, & \text{if } i = j, \\
\frac{\lambda}{p_{ji}}, & \text{if } i > j.
\end{cases}
$$

We prove the theorem by first considering the morphism

$$
\phi(\lambda) = \begin{cases} 
p_{ij}, & \text{if } i < j, \\
\lambda, & \text{if } i = j, \\
\frac{\lambda}{p_{ji}}, & \text{if } i > j.
\end{cases}
$$

We will provide a variation of the presentation of [AST91, CM96] in order to display multiparameter quantum groups as a Hopf algebra with triangular decomposition.

Example 5.1.1 (Multiparameter quantum groups). Let $F = k(f_1, \ldots, f_{n-1})$ be the YD-module over a group algebra $G$ with commuting generators $k_1, \ldots, k_{n-1}, l_1, \ldots, l_{n-1}$. Denote the dual by $E = k(e_1, \ldots, e_{n-1})$, where the pairing is given by $\langle e_i, f_j \rangle = (1 - \lambda)\delta_{ij}$. The YD-structure is of separable type, given by assigning the right degree $k_i$ to $f_i$, and the left degree $l_i$ to $e_i$, and actions

$$
k_i \triangleright f_j = \lambda_j(k_i)f_j = \frac{\lambda^{(i)}_{j+1}\lambda^{(i+1)}_{j}}{\lambda^{(j+1)}_{j}\lambda^{(j)}_{j+1}}f_j,
$$

$$
l_i \triangleright f_j = \lambda_j(l_i)f_j = \frac{\lambda^{(i)}_{j+1}k_{j+1}^{(i+1)}}{\lambda^{(j)}_{j+1}k_{j+1}^{(i)}}f_j,
$$

for $i, j = 1, \ldots, n-1$. We define the multiparameter quantum group $U_{\lambda,p}^{kG}(\mathfrak{gl}_n)$ to be the asymmetric braided Drinfeld double $Drin_{kG}(F, E)$.

Note that the definition of Drin$_{kG}(F, E)$ is possible as (4.10) holds, i.e.

$$
q_{ij} := \lambda_j(k_i) = \lambda^{(i)}_{j+1}\lambda^{(i+1)}_{j} = \frac{\lambda^{(j)}_{j+1}k_{j+1}^{(i+1)}}{\lambda^{(i)}_{j+1}k_{j+1}^{(i)}} = \lambda_l(k_i)^{-1}.
$$

The commutator relation in Drin$_{kG}(F, E)$ is given by

$$
[E_i, F_j] = (1 - \lambda)\delta_{ij}(k_i - l_i).
$$

Our definition of the multiparameter quantum group is justified by the following isomorphism to an indecomposable subalgebra of the multiparameter quantum group considered in the literature:

**Proposition 5.1.2.** There is an isomorphism of Hopf algebras $U_{\lambda,p}^{kG}(\mathfrak{gl}_n) = Drin_{kG}(F, E) \cong U'$ where $U'$ is a subalgebra of the multiparameter quantum group $U$ (as defined in the literature).

**Proof.** We prove the theorem by first considering the morphism

$$
\phi: T(E) \otimes kG \otimes T(F) \longrightarrow U.
$$

Such a morphism will descent to an injective morphism $\bar{\phi}$: Drin$_{kG}(F, E) \to U$ by the following Lemma 5.1.3. We further note that the image $\text{Im} \bar{\phi} = U'$ is a Hopf subalgebra isomorphic to Drin$_{kG}(F, E)$. Denote the generators of $U$ by $E_i, F_i$ for $i = 1, \ldots, n$ and group elements $K_i, L_i$ for $i = 1, \ldots, n$ (see [CM96, 4.8]). The map $\phi$ is defined by $\phi(e_i) = \lambda E_i K_{i+1}^{-1} K_i, \phi(f_i) := F_i, \phi(k_i) = L_{i+1}L_i^{-1}$, and $\phi(l_i) := K_{i+1}^{-1} K_i$. One checks directly that the relations in the free braided double $T(E) \otimes kG \otimes T(F)$ are preserved under this map, using the presentation in [CM96, 4.8] for $U$. □

**Lemma 5.1.3.** The quantum Serre relations in the positive part of $A = U_{\lambda,p}^{kG}(\mathfrak{gl}_n)$ are given by the largest ideal in $\mathcal{I}_A(A)$, making the positive part a Nichols algebra. This ideal is generated by the braided commutators

$$
\text{ad}(E_i)^{1-a_{ij}}(E_j) = \text{ad}(F_i)^{1-a_{ij}}(F_j) = 0,
$$

where $\text{ad}(E_i)(E_j) = E_iE_j - q_{ij}E_jE_i$. 

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Proof. It follows from Lemma 3.4.7 that the maximal ideal \( J \) in \( I_\Delta(A) \) is given by \( J = \langle I, I^* \rangle \) where \( I \) is the Nichols ideal of the YD-module \( F \).

In \( U \), the explicit description of the ideal is the quotient of the positive (respectively negative) part is generated by quantum Serre relations. This follows from Lemma 4.5 in [CM96]. For this, it is crucial that \( \lambda \) is not a root of unity. The proof uses the observation in [Res90], or [AST91] for the deformed function algebra, that multiparameter quantum groups, using quantum coordinate rings, can be obtained via a 2-cocycle from a one-parameter quantum groups. The fact that the quantum Serre relations generate the ideal \( J \) follows from Theorem 4.4 in [CM96] where it is shown that these relations generate the radical of the pairing of \( T(F) \) with \( T(E) \) extending the pairing of \( E \) and \( F \). □

The result that the multiparameter quantum group \( U_{\lambda, p}(g_{t_0}) \) is the asymmetric braided Drinfeld double \( \text{Drin}_G(F, E) \) can be seen as a generalization of the result in [BW04] where the two-parameter quantum groups were shown to be Drinfeld doubles.

5.2. Characterizations of Quantum Groups. Let \( \text{char} k = 0 \) in this section. In Section 4 we observed that for an algebra \( A \) with triangular decomposition to be an indecomposable pointed Hopf algebra, \( G(A) \) needs to be abelian acting on \( V \) by scalars. That means, in the terminology of [AS02] that the YD-braiding \( \Psi(v \otimes w) = v^{(-1)} \Rightarrow w \otimes v^{(0)} \) is of diagonal type, i.e. there exist non-zero scalars \( q_{ij} \) such that \( \Psi(v_i \otimes v_j) = q_{ij} v_j \otimes v_i \) for a basis \( \{v_1, \ldots, v_n\} \).

We assume that the braiding arise from YD-module structures over an abelian group \( G \) in this section. That is, \( q_{ij} = \lambda_i(k_j) \) for the characters \( \lambda \) by which \( G \) acts on \( kv_i \) and group elements \( k_j \) such that \( \delta(v_i) = v_i \otimes k_i \). It is a basic observation that the braided Hopf algebras \( T(V)/I \) for \( I \in I_V \), including the Nichols algebras for \( V \), only depend on the braiding on \( V \) (rather than the concrete choice of \( \lambda_i, k_i \)). However, different diagonal braidings \( (V, \Psi) \) and \( (V, \Psi') \) give isomorphic braided Hopf algebras \( T(V)/I \). Such isomorphisms can be obtained using the notion of twist equivalence for diagonal braidings (which is a special case of the more general concept of twisting an algebra by a 2-cocycle).

Definition 5.2.1. Two braided \( k \)-vector spaces of diagonal type \( (V, \Psi), (V', \Psi') \) (given by scalars \( q_{ij}, q_{ij}' \)) twist equivalent if \( V \cong V' \), \( q_{ij} = q_{ij}' \), and \( q_{ij}q_{ji} = q_{ij}'q_{ji}' \).

Lemma 5.2.2. If \( (V, \Psi), (V', \Psi') \) are twist equivalent of diagonal type, then \( T(V) \cong T(V') \) as braided Hopf algebras in the category of braided \( k \)-vector spaces, preserving the natural grading.

Proof. For a proof see e.g. [AS02, 3.9–3.10]. We can find generators \( v_i \) of \( V \) and \( v'_j \) of \( V' \) such that the isomorphism \( \phi \) is determined by \( v_i \mapsto v'_j \). Defining a 2-cocycle \( \sigma \) by \( \sigma(v_i \otimes v_j) = q_{ij}^{-1}q_{ji}' \) for \( i < j \) and 1 otherwise, we find that the product \( v_iv_j \) maps to the product twisted by \( \sigma \). Note that the isomorphism is not an isomorphism in the category of YD-modules over \( kG \) unless \( (V', \Psi') = (V, \Psi) \). □

For an ideal \( I \in I_V \), denote the corresponding ideal under the isomorphism \( T(V) \cong T(V') \) from Lemma 5.2.2 by \( I' \). Then we conclude that \( T(V)/I \cong T(V')/I' \) is also an isomorphism of braided Hopf algebras. In particular, \( B(V) \cong B(V') \) for the corresponding Nichols algebras.

Lemma 5.2.3. If \( (V, \Psi) \) and \( (V', \Psi') \) are twist equivalent, such that \( G = \langle k_1, \ldots, k_n \rangle \cong \langle k'_1, \ldots, k'_n \rangle = G' \) via \( k_i \mapsto k'_i \), then \( \text{Drin}_G(V, V^*) \cong \text{Drin}_{G'}(V', V'^*) \) as Hopf algebras.

Proof. By Lemma 5.2.2, \( T(V)/I \cong T(V')/I' \) and \( T(V^*)/I^* \cong T(V'^*)/I'^* \). By the assumptions on the group generators, \( k_i \mapsto k'_i \) extends to an isomorphism \( kG \cong kG' \). Thus we can define a morphism \( \text{Drin}_G(V, V^*) \to \text{Drin}_{G'}(V', V'^*) \) which is an isomorphism of \( k \)-vector spaces. Further, preservation of the bosonization condition can be checked on generators using the isomorphism \( \phi \) from Lemma 5.2.2. Finally, the commutator relation (4.11) is preserved using the isomorphism on \( kG \). □

Diagonal braidings are a very general class of braidings. Quantized enveloping algebras at generic parameters however are based on braidings of specific type, called Drinfeld-Jimbo type. Following [AS04], there are different classes of braidings which we distinguish:

Definition 5.2.4 ([AS04]). Let \( (q_{ij}) \) be the \( n \times n \)-matrix of a braiding of diagonal type.

(a) The braiding given by \( (q_{ij}) \) is generic if \( q_{ii} \) is not a root of unity for any \( i = 1, \ldots, n \).

(b) If the case \( k = \mathbb{C} \) we say the braiding \( (q_{ij}) \) is positive if it is generic and all diagonal elements \( q_{ii} \) are positive real numbers.
(c) The braiding \((q_{ij})\) is of Cartan type if \(q_{ii} \neq 1\) for all \(i\) and there exists a \(\mathbb{Z}\)-valued \(n \times n\)-matrix \((a_{ij})\) with values \(q_{ii} = 2\) on the diagonal and \(0 \leq -a_{ij} < \text{ord } q_{ii}\) for \(i \neq j\), such that
\[
q_{ij}q_{ji} = q_{ij}^{a_{ij}} \quad \text{for all } i, j.
\]
That implies that \((a_{ij})\) is a generalized Cartan matrix which may have several connected components. We denote the collection of these by \(\chi\).

(d) The braiding \((q_{ij})\) is of Drinfeld-Jimbo type (DJ-type) if \(q_{ij}\) are generic (no roots of unity) and there exist positive integers \(d_1, \ldots, d_n\) such that for all \(i, j\), \(d_i a_{ij} = d_j a_{ji}\) (hence the matrix \((a_{ij})\) is symmetrizable, and for any \(J \in \chi\), there exists a scalar \(q_J \neq 0\) in \(k\) such that \(q_{ij} = q_J^{d_{ij}}\) for any \(i \in I\), and \(j = 1, \ldots, n\).

Some observations can be made about the Nichols algebras associated to braided vector spaces of DJ-type. First, observe that for a braiding of Cartan type with connected components \(I_1, \ldots, I_n \in \chi\), we have that \(B(V)\) is the braided tensor product \(B(V_{I_1}) \otimes \cdots \otimes B(V_{I_n})\) ([AS00, Lemma 4.2]). Further, for \(V\) with braiding \((q_{ij})\) of DJ-type, the Nichols algebra can be computed explicitly as the quantum Serre relations ([Ros98, Theorem 15]):
\[
B(V) = k\langle x_1, \ldots, x_n \mid \text{ad}(x_i)^{1-a_{ii}}(x_j) = 0, \forall i \neq j \rangle.
\]

We now bring the growth condition of finite Gelfand-Kirillov dimension (GK dimension) into the picture, using characterization results of [Ros98] of Nichols algebras with this property.

**Lemma 5.2.5** ([Ros98]). Let \(k = \mathbb{C}\). Let \((q_{ij})\) be the matrix of a braiding of diagonal type which is generic such that the Nichols algebra \(B(V)\) has finite Gelfand-Kirillov dimension. Then \((q_{ij})\) is of Cartan type.

Moreover, if the braiding is positive then the braiding is twist equivalent to a braiding of DJ-type, and this condition is equivalent to finite GK dimension.

**Proof.** See [AS04], Corollary 2.12 and Theorem 2.13. \(\square\)

**Corollary 5.2.6.** Let \(A = \text{Drin}_{CC}(V, V^*)\), for \(V\) or separable type, with generic positive braiding \((q_{ij})\). Then the following are equivalent

(i) \(A \cong U_q(\mathfrak{g})\) for \(\mathfrak{g}\) a semisimple Lie algebra.

(ii) The braided \(\mathbb{C}\)-vector space \(V\) with braiding \((q_{ij})\) is twist equivalent to a braiding of DJ-type with finite type Cartan matrix.

(iii) \(B(V)\) has finite Gelfand-Kirillov dimension.

**Proof.** The equivalence of (ii) and (iii) is the statement of Lemma 5.2.5 due to [Ros98]. Using Lemma 5.2.3 we find that (ii) implies (i), while it is clear that (i) implies (ii). In fact, the GK dimension of \(B(V)\) for \(V\) of DJ-type equals the number of positive roots [AS04, 2.10(ii)]. \(\square\)

**Corollary 5.2.7.** The only indecomposable bialgebras with a symmetric triangular decomposition on \(B(V) \otimes k\mathbb{Z}^n \otimes B(V^*)\) of separable type, such that \(V = \mathbb{C}\langle v_1, \ldots, v_n \rangle\) is of positive diagonal type, and that no \(v_i\) commutes with all of \(V^*\) are isomorphic to \(U_q(\mathfrak{g})\) for some semisimple Lie algebra \(\mathfrak{g}\).

**Proof.** This follows from the classification 4.2.2, combined with the result of Rosso. The Lie algebra \(\mathfrak{g}\) is determined by the Cartan matrix one obtains under twist equivalence in Lemma 5.2.5. The technical condition that no \(v_i\) commutes with all of \(V^*\) ensures that \(\{f_i, v_i\} \neq 0\) for a dual basis \(f_1, \ldots, f_n\) of \(V^*\), resembling the so-called non-degeneracy condition that the scalars \(\gamma_{ii} \neq 0\) in Theorem 4.3.2. \(\square\)

This is a characterization for quantum groups at generic parameters. The work surveyed in [AS02, AS10] on pointed Hopf algebras over finite-dimensional Hopf algebras can be viewed as a characterization of small quantum groups. The triangular decomposition can be view as the case where the graph \(\Gamma\) described in 2.3 has two connected components, such that the corresponding generators for the two components give dually paired braided Hopf algebras.

The characterization suggests that if we are looking for examples outside of DJ-type, we can consider braidings of generic Cartan type which are not positive. In fact, [AS04, 2.6] gives an example that is generic of Cartan type, but not of DJ-type. We compute the associated quantum group here:
Example 5.2.8. Let \( G = \langle k_1, k_2 \rangle \cong \mathbb{Z}_x \times \mathbb{Z}_x \) be a free abelian group with two generators. We define a two-dimensional YD-module \( V \) over \( G \) on generators \( v_1 \) of degree \( k_1 \), \( v_2 \) of degree \( k_2 \) via

\[
  k_1 \triangleright v_1 = qv_1, \quad k_2 \triangleright v_2 = q^{-1}v_2, \quad k_1 \triangleright v_1 = q^{-1}v_1, \quad k_2 \triangleright v_2 = -qv_2.
\]

Lemma 2.1 in [AS04] shows that

\[
  \mathcal{B}(V) = \langle v_1, v_2 \mid \text{ad}(v_1)^3(v_2) = \text{ad}(v_2)^3(v_1) = 0 \rangle.
\]

The asymmetric braided Drinfeld double \( \text{Drin}_{\mathbb{C}G}(V, V^*) \) is in fact a braided Drinfeld double if we define \( V^* \) to be the dual YD-module. It is the Hopf algebra given on \( \mathcal{B}(V) \otimes \mathbb{C}G \otimes \mathcal{B}(V^*) \), subject to the relations

\[
  [f_1, v_1] = \frac{k_1 - k_1^{-1}}{q^{1/2} - q^{-1/2}}, \quad [f_2, v_1] = \frac{k_2 - k_2^{-1}}{iq^{1/2} + iq^{-1/2}},
\]

\[
  k_1v_2 = q^{-1}v_2k_1, \quad k_2v_1 = q^{-1}v_1k_2, \quad k_1v_1 = qv_1k_1, \quad k_2v_2 = -qv_2k_2,
\]

\[
  k_1f_2 = qf_2k_1, \quad k_2f_1 = qf_1k_2, \quad k_1f_1 = q^{-1}f_1k_1, \quad k_2f_2 = -q^{-1}f_2k_2,
\]

and with coproducts

\[
  \Delta(v_i) = v_i \otimes k_i + 1 \otimes v_i, \quad \Delta(f_i) = f_i \otimes 1 + g_i^{-1} \otimes f_i.
\]

In this example, we cannot apply the procedure from Section 4.4 to recover a Lie algebra because \( q_{11} = q, q_{22} = -q \) cannot both be set equal to 1. Instead, we can try just setting \( q = 1 \). This gives the following Lie algebra \( g \):

\[
  [f_1, v_1] = t_1, \quad [f_2, v_2] = t_2, \quad [f_1, v_2] = 0, \quad [f_2, v_1] = 0,
\]

\[
  [v_1, t_1] = -2f_1, \quad [v_2, t_2] = 0, \quad [v_1, t_2] = 0, \quad [v_2, t_1] = 0,
\]

\[
  \text{ad}(v_1)^3(v_2) = 0, \quad \text{ad}(v_2)^3(v_1) = 0, \quad \text{ad}(f_1)^3(f_2) = 0, \quad \text{ad}(f_2)^3(f_1) = 0.
\]

Note that \( \mathbb{C}t_2 \) is an abelian Lie ideal and hence \( g \) is not semisimple. In fact, considering the smaller quantum group given by the Cartan datum \((-q\rangle \) already gives an example of a quantum group where the "classical limit" (setting \( q = 1 \)) is not semisimple.

Apart from such examples, we can also include examples where free and nilpotent generators are combined, hence capturing features of both small and generic quantum groups. Here is such an example of minimal size:

Example 5.2.9. Let \( G = \mathbb{Z}_x \times \mathbb{Z}_p = \langle g_x \rangle \times \langle g_p \rangle \) be the product of an infinite cyclic group and one of order \( p \). We define 2-dimensional YD-module over \( G \) as \( \mathbb{C}v_x \oplus \mathbb{C}v_p \), where \( v_x \) has degree \( g_x \), and \( v_p \) has degree \( g_p \). The group action is given by

\[
  g_p \triangleright v_p = \xi_p v_p, \quad g_p \triangleright v_x = \eta_p v_x, \quad g_x \triangleright v_p = \xi_x v_p, \quad g_x \triangleright v_x = \eta_x v_x,
\]

where scalars with a subscript \( p \) are primitive \( p \)-th roots of unity, and scalars with subscript \( \infty \) are no roots of unity. We can now compute the Nichols algebra with generators \( v_p \) and \( v_x \). It is given by \( \mathcal{B}(V) = \mathbb{C}\langle v_p, v_x \rangle/\langle v_p^p \rangle \). We denote the YD-dual by \( V^* \) with generators \( f_p, f_x \). The braided Drinfeld double on \( \mathcal{B}(V) \otimes kG \otimes \mathcal{B}(V^*) \) of the braided Hopf algebra \( \mathcal{B}(V) \) is a quantum group that combines both \( u_q(\mathfrak{sl}_2) \) and \( U_q(\mathfrak{sl}_2) \):

\[
  [f_p, v_1] = \delta_{1,p} \frac{g_p - g_p^{-1}}{\xi_p^{-1/2} - \xi_p^{1/2}}, \quad [f_x, v_1] = \delta_{x,1} \frac{g_x - g_x^{-1}}{\eta_x^{-1/2} - \eta_x^{1/2}}, \quad v_p^p = 0, \quad f_p^p = 0,
\]

\[
  g_p v_p = \xi_p v_p g_p, \quad g_p v_x = \eta_p v_x g_p, \quad g_x v_p = \xi_x v_p g_x, \quad g_x v_x = \eta_x v_x g_x,
\]

\[
  g_p f_p = \xi_p^{-1} f_p g_p, \quad g_p f_x = \eta_p^{-1} f_x g_p, \quad g_x f_p = \xi_x^{-1} f_p g_x, \quad g_x f_x = \eta_x^{-1} f_x g_x.
\]

and with coproducts

\[
  \Delta(v_i) = v_i \otimes g_i + 1 \otimes v_i, \quad \Delta(f_i) = f_i \otimes 1 + g_i^{-1} \otimes f_i, \quad \text{for } i = p, \infty.
5.3. Classes of Pointed Hopf Algebras by Radford. In [Rad94], a class of pointed Hopf algebras $U_{(N, \nu, \omega)}$ was introduced (see also [Gel98] for generalizations). These Hopf algebras are associated to the datum of a positive integer $N$ and $1 \leq \nu < N$ such that $N$ does not divide $\nu^2$, and $\omega \in \mathbb{k}$ is a primitive $N$th root of unity in a field $\mathbb{k}$. Denote $q := \omega^\nu$ and $r = \left| q^r \right| = |\omega^{r^2}|$. We let $C_N$ denote a cyclic group of order $N$ generated by an element $a$.

The algebra $U_{(N, \nu, \omega)}$ is the braided Drinfeld double of the $\mathbb{YD}$-module Hopf algebra $U_{\nu} := k[x]/(x^r)$ over $\mathbb{C}$, with grading given by $x \mapsto a^\nu \otimes x$ and action $a \ast x = q^{-1}x$. Note that $U_\nu$ is the Nichols algebra of the one-dimensional $\mathbb{YD}$-module $kk$. The coalgebra structure is given by $\Delta(x) = x \otimes a^\nu + 1 \otimes x$, and $\Delta(y) = y \otimes 1 + a^{-\nu} \otimes y$ for the dual generator $y$. Note further that the other Hopf algebra $H_{(N, \nu, \omega)}$ introduced by Radford is simply the bosonization $U_\nu \otimes kC_N$ in this set-up. The algebras $U_{(N, \nu, \omega)}$ and $H_{(N, \nu, \omega)}$ are not indecomposable unless $\nu = 1$. To obtain indecomposable pointed Hopf algebras, we can consider the subalgebras generated by $x, y$, and $a^\nu$ (respectively, $x$ and $a^\nu$). Since these only depend on the choices of $r$ and $q$ we denote these Hopf algebras by $U_{(r, q)}$ (respectively, $H_{(r, q)}$). Note that $U_{(r, 1, q)} = U_{(r, q)}$.

5.4. Quantum Group Analogues in Other Contexts. To conclude this paper, we would like to adapt the point of view that quantum groups can also be studied over other Hopf algebras $H$ than the group algebra. For this, one can, motivated by the results of this paper, look for Hopf algebras $A$ with triangular decomposition over $H$. The property over a group that $A$ is of separable type can be generalized by requiring that the $\mathbb{YD}$-modules $V$ with respect to the left and right coactions $\delta_\epsilon$ and $\delta_\delta$ are a direct sum of distinct one-dimensional simples.

As a first example, we can consider the case where $H$ itself is primitively generated, i.e. $H = k[x_1, \ldots, x_n]$ over a field of characteristic zero. If $A$ is a bialgebra with triangular decomposition over $H$, then for $v \in V$, $\Delta(v) = v \otimes H + H \otimes V$ implies that $\Delta(v)$ in fact equals $v \otimes 1 + 1 \otimes v$ using the comutative condition. This gives that $A$ is generated by primitive elements and hence is a pointed Hopf algebra that is connected (i.e. the group like elements are the trivial group). Now $A$ is in particular cocommutative, so Theorem 5.6.5 in [Mon93] implies (for char $k = 0$) that $A = U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of primitive elements in $A$. From this point of view, all quantum groups over $H = k[x_1, \ldots, x_n]$ are simply the classical universal enveloping algebras. Investigating Hopf algebras with triangular decomposition over other Hopf algebras $H$ can be the subject of future research.

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