BRST Lagrangian construction for spin-2 field in Einstein space.

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Abstract

We explore a new possibility of BRST construction in higher spin field theory to obtain a consistent Lagrangian for massive spin-2 field in Einstein space. Such approach automatically leads to gauge invariant Lagrangian with suitable auxiliary and St"{u}ckelberg fields. It is proved that in this case a propagation of spin-2 field is hyperbolic and causal. Also we extend notion of partial masslessness for spin-2 field in the background under consideration.

1 Introduction

Various aspects of higher spin field theory attract much attention for a long time (see reviews \cite{1} and references therein). One of the modern approaches to deriving the Lagrangians for higher spin fields is based on the use of BRST-BFV construction which was initially developed for quantization of gauge theories \cite{2}. Lagrangian formulation of the higher spin field theories within this approach has been studied in \cite{3, 4, 5, 6}.

At present, all Lagrangian formulations for higher spin fields, including the BRST approach, are given for the background manifolds corresponding to constant curvature spaces (Minkowski or AdS spaces). Problem of finding the more general manifolds admitting the consistent exact or approximate Lagrangian formulation for higher spin fields is open in general. In BRST

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approach theory is considered in terms of Fock vectors and the relations defining on-shell contents are treated as the operator constraints acting on the above vectors. Then, the problem of consistent Lagrangian formulation appears as a problem of closing operator algebra initiated by the constraints and related to both higher and lower spin fields. In particular, the BRST approach in its existing form, yields to Lagrangian formulation only for constant curvature spaces even for spin-1 and spin-2 theories. However, it is evident that spin-1 field can be formulated on the arbitrary curved background. Moreover, it was shown some time ago that there exists a consistent Lagrangian formulation for massive spin-2 field in Einstein space.

In this note we want to pay attention to some new unexplored yet possibility in the BRST approach to higher spin fields which allows us to derive the known Lagrangian for massive spin-1 field without restriction on the background and the Lagrangian for massive spin-2 field without restrictions on Weyl tensor. This possibility is inherent just for BRST Lagrangian construction in higher spin theory and has no counterpart in quantization theory.

The Lagrangian construction in the BRST approach was carried out for fields of all spins simultaneously. The basic notions of this approach are the Fock space vector $|\Psi_s\rangle$, corresponding to spin $s$ and the nilpotent BRST charge $Q$. The equations of motion and gauge transformations are written in the form $Q|\Psi_s\rangle = 0$ and $\delta|\Psi_s\rangle = Q|\Lambda_s\rangle$ respectively, with the BRST operator $Q$ being the same for all spins. Nilpotency of the BRST operator provides us the gauge transformations and fields $|\Psi_s\rangle$ and $|\Psi_s\rangle + Q|\Lambda_s\rangle$ are both physical. Since we consider all spins simultaneously, then from $Q^2|\Lambda_s\rangle = 0$ follows $Q^2 = 0$. But if we want to construct Lagrangian for the field with a given value $s$ of spin, then it is sufficient to require a weaker condition that the BRST operator for given spin $Q_s$ is not nilpotent in operator sense but will be nilpotent only on the specific Fock vector parameter $|\Lambda_s\rangle$ corresponding to a given spin $s$, $Q^2_s|\Lambda_s\rangle = 0$ only and $Q^2_s \neq 0$ on states of general form. This possibility has not been explored in all previous applications of the BRST construction to the higher spin fields. Just this point allows us to construct Lagrangians for spin-1 and spin-2 field in background spaces different from Minkowski and AdS.

The paper is organized as follows. In the next section we describe the BRST procedure of Lagrangian construction for the fields of fixed spin on gravitational background. Then in sections and we apply this method for Lagrangian construction of fields with spin-1 and spin-2 respectively and in section the causal propagation of spin-2 field is proven. In section we summarize the results.

## 2 General scheme of Lagrangian construction

As is well known a completely symmetric bosonic field of spin-$s$ $\varphi_{\mu_1...\mu_s}$ will realize irreducible representation of the Poincare group if the following equations are satisfied

$$\left(\partial^2 - m^2\right)\varphi_{\mu_1...\mu_s} = 0, \quad \partial^{\mu_1}\varphi_{\mu_1...\mu_s} = 0, \quad \eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2...\mu_s} = 0. \tag{1}$$

When we turn to an arbitrary curved spacetime we suppose that conditions on $\varphi_{\mu_1...\mu_s}$ which must be satisfied, tend to in the flat space limit. It tells us that if we don’t consider terms

1We pay attention that not all operators generating this algebra can be considered as the constraints.
2An Einstein space is defined by the relation $R_{\mu\nu} = \text{const} \cdot g_{\mu\nu}$ with Weyl tensor be arbitrary (see e.g. [7]).
3The other differences of BRST approach to higher spin field theory from the BRST approach to quantization of gauge theories are discussed in [4, 5].
4Such a possibility is inherent namely BRST approach to higher spin field theory. In BFV-BRST approach to quantization of gauge theories, unlike the higher spin field theory, one begins with a given classical theory where BRST charge is constructed on the base of given first class constraints and nilpotent by definition.
with the inverse powers of the mass the equations on \( \varphi_{\mu_1 \ldots \mu_s} \) in curved spacetime must be of the form

\[
(\nabla^2 - m^2)\varphi_{\mu_1 \ldots \mu_s} + \text{terms with curvature} = 0, \quad \nabla^{\mu_1} \varphi_{\mu_2 \ldots \mu_s} = 0, \quad g^{\mu_1 \mu_2} \varphi_{\mu_3 \mu_4 \ldots \mu_s} = 0. \tag{2}
\]

Our purpose is to find the “terms with curvature”\(^5\) and restriction on the curvature of the space demanding consistency equations (2) with each other and then to try to construct Lagrangians for the higher spin fields in the background gravity using the BRST method. Let us note that Lagrangian construction can give additional restrictions on the spacetime curvature apart from those which follow from consistency of (2).

To avoid explicit manipulations with a big number of indices it is convenient to introduce the auxiliary Fock space generated by bosonic creation and annihilation operators with tangent space indices \((a, b = 0, 1, \ldots, d - 1)\)

\[
[a_a, a_b^+]=\eta_{ab}, \quad \eta_{ab}=\text{diag}(-, +, \ldots, +). \tag{3}
\]

An arbitrary vector in this Fock space has the form

\[
|\varphi\rangle = \sum_{s=0}^{\infty} \varphi_{a_1 a_2 \ldots a_s}(x) a^{+a_1} \ldots a^{+a_s}|0\rangle = \sum_{s=0}^{\infty} \varphi_{\mu_1 \ldots \mu_s}(x) a^{+\mu_1} \ldots a^{+\mu_s}|0\rangle \equiv \sum_{s=0}^{\infty} |\varphi_s\rangle, \tag{4}
\]

where \(a^+a^=g_{\mu\nu}\). We also suppose the standard relation \(\nabla_\mu e^\nu_a = 0\).

Then one introduces derivative operator

\[
D_\mu = \partial_\mu + \omega^a_\mu a^+_a a_b, \quad D_\mu |0\rangle = 0 \tag{5}
\]

and realize equations (2) in the operator form

\[
l_0|\varphi_s\rangle = l_1|\varphi_s\rangle = l_2|\varphi_s\rangle = 0 \tag{6}
\]

where

\[
l_0 = D^2 - m^2 + \mathcal{X}, \quad l_1 = -ia^\mu D_\mu, \quad l_2 = \frac{1}{2} a^\mu a_\mu, \tag{7}
\]

\[
D^2 = g^{\mu\nu}(D_\mu D_\nu - \Gamma^a_\mu_\nu D_a) \tag{8}
\]

with the operator \(\mathcal{X}\) corresponding to the “terms with curvature” in (2).

Since operators \(l_1\) and \(l_2\) commute \([l_1, l_2] = 0\) the consistency of equations (2) demands

\[
[l_0, l_1]|\varphi_s\rangle \sim 0, \quad [l_0, l_2]|\varphi_s\rangle \sim 0, \tag{9}
\]

where \(\sim\) means “up to equations of motion (2) is equal to”. Equations (9) are the equations on operator \(\mathcal{X}\) and on the background gravity.

Having found operator \(\mathcal{X}\) and the restrictions on the spacetime curvature from (9) we will try to construct Lagrangians for the field with given spin using the BRST method. The procedure of Lagrangian construction is as follows. For the Lagrangian be a real function the BRST operator used for its construction must be a Hermitian operator. It assumes that the set of operators underlying the BRST operator must be invariant under Hermitian conjugation.

\(^5\)Our definition of the curvature tensor is

\[
R^a_{\beta\mu\nu} = \partial_\mu \Gamma^a_{\beta\nu} - \partial_\nu \Gamma^a_{\beta\mu} - \Gamma^a_{\beta\lambda} \Gamma^\lambda_{\mu\nu} + \Gamma^a_{\nu\lambda} \Gamma^\lambda_{\beta\mu},
\]
Thus to have such a set of operators we add to constraints $l_0$, $l_1$, $l_2$ their Hermitian conjugated operators. Since $l_0$ is assumed to be self conjugated we add two operators

\[ l_1^+ = -ia^{\mu+}D_\mu \quad l_2^+ = \frac{1}{2}a^{\mu+}a^{\mu+}. \]  

Then for constructing the BRST operators the underlying set of operators must form an algebra. Note that the nilpotency condition of the BRST operators is needed for existing of gauge symmetry. As it is known if we consider spin-$s$ field and decompose the gauge parameter $|\Lambda_s\rangle$ in series of creation operators, then maximal tensorial rank of gauge parameters $|\lambda_k\rangle = a^{+\mu_1} \cdots a^{+\mu_s} \lambda_{\mu_1 \cdots \mu_k} |0\rangle$, entering in $|\Lambda_s\rangle$ is $k = s - 1$ (see e.g. [4, 5]). Therefore if we want to construct Lagrangian for a particular spin-$s$ field it is enough that this set of operators form algebra only on states $|\lambda_k\rangle$ with $k < s$. To form an algebra we must add to $l_0$, $l_1$, $l_2$, $l_1^+$, $l_2^+$ all the operators generated by the commutators of these operators. But if we want to construct with the help of the obtained algebra Lagrangian for spin-$s$ field, then this algebra must be a deformation of the algebra in Minkowski [4] (or in AdS [5]) space. Thus we can add only two operators which are generalization of operators

\[ g_0 = a^{\mu+}a^{\mu} + \frac{d}{2} \quad g_m = m^2 + \text{const} \]  

(11)

to the case of curved space. Since operator $g_0$ is dimensionless and we do not consider terms with inverse power of the mass then it is impossible to deform operator $g_0$ by terms with curvature. Therefore operator $g_0$ keeps the same form (11) as in the flat case. Let us turn to the operator $g_m$. If we add to it any nonconstant term then the algebra will not be closed since in this case (for example) commutator $[l_1, g_m]$ will not be proportional to the operators of the algebra. Again the form of operators $g_m$ (11) must be kept. Thus we came to the conclusion that in order to construct Lagrangian with the help of the BRST method we must find operators $l_0$ and $g_m$ so that operators $l_0$, $l_1$, $l_2$, $l_1^+$, $l_2^+$, $g_0$, $g_m$ form an algebra on states $|\lambda_k\rangle$ with $k < s$. One should note that the obtained algebra must satisfy Jacobi identity. After the algebra is constructed the Lagrangian construction procedure is the same as in flat or AdS case. It is unclear how the above requirements may be fulfilled in spaces with arbitrary curvature for any spin-$s$. As we show later using this method it is possible to construct Lagrangian for spin-$1$ field in arbitrary curved space and for spin-$2$ case in Einstein space. Let us consider application of the above method.

3 BRST Lagrangian construction for spin-1 field

Let us first consider spin-1 field as an example of the above described procedure of Lagrangian construction. In this case we will be looking for operator $\mathcal{X}$ in the form

\[ \mathcal{X} = X(x) + X^{\mu\nu}(x)a^{\mu+}_\mu a^{\mu+}_\nu \]  

(12)

where yet unknown functions $X(x)$ and $X^{\mu\nu}(x)$ should be defined from the consistency condition (9). Since the tracelessness condition for the spin-1 case is redundant therefore it is sufficient to consider only the following commutator

\[ [l_0, l_1]|\varphi_1\rangle = i(R^\mu_{\mu} + X^{\nu}_{\mu})a^{\mu+}D_\nu|\varphi_1\rangle + i(\frac{1}{2}R_{\mu\nu} + X_{\mu\nu} + X^{\nu}_{\mu\nu})a^{\mu+}|\varphi_1\rangle. \]  

(13)

From the first summand in the right hand side one finds

\[ X^{\mu\nu} = -R^{\mu\nu} + F(x)g^{\mu\nu}, \]  

(14)
where $F(x)$ is an arbitrary scalar function with the dimension of mass squared. Substituting this expression for $X^{\mu\nu}$ into the second summand of the right hand side one gets

$$X = -F(x)$$

(15)

ignoring a possible constant with the dimension of mass squared which can be absorbed by redefinition of the mass term. Collecting together one gets a partial solution to operator $l_0$ in the form

$$\tilde{l}_0 = D^2 - m^2 - R^{\mu\nu} a^+_{\mu} a_{\nu} + F(x)(a^+_{\mu} a_{\mu} - 1).$$

(16)

Since $(a^+_{\mu} a_{\mu} - 1)|\varphi_1\rangle = 0$ the term proportional to $F(x)$ doesn’t influence on the mass-shell equation in the component form. In what follows we put $F(x) = 0$ and take operator $l_0$ in the form

$$l_0 = D^2 - m^2 - R^{\mu\nu} a^+_{\mu} a_{\nu}.$$ 

(17)

Let us turn to Lagrangian construction. To get a set of operators invariant under Hermitian conjugation we add to constraints $l_0, l_1$ one more operator

$$l_1^+ = -ia^{+\mu} D_{\mu}$$

(18)

and close the set of operators $l_0, l_1, l_1^+$ to an algebra adding one more operator

$$g_m = m^2.$$ 

(19)

The algebra has only one non-vanishing commutator

$$[l_1, l_1^+] = -l_0 - g_m.$$ 

(20)

Let us remind that commutator (20) is understood as $[l_1, l_1^+]|\lambda_0\rangle = (-l_0 - g_m)|\lambda_0\rangle$. It is evident that Jacobi identities are satisfied. Thus, the algebra is obtained and further Lagrangian construction goes in the usual way [4, 5].

Since in the set of operators of the algebra there is operator $g_m$ which is not a constraint neither in the space of ket-vectors nor in the space of bra-vectors one should construct new enlarged expressions for the operators $L_i = l_i + l_i'$, $G_m = g_m + g_m'$ so that $G_m = 0$ with the same algebra as for the initial operators $l_i, g_m$ (see [5] for more details). Introducing new pair of bosonic creation and annihilation operators $b^+, b$ with the standard commutation relations $[b, b^+]=1$ we define the enlarged expressions for the operators as follows

$$L_0 = l_0 \quad L_1 = l_1 + mb \quad L_1^+ = l_1^+ + mb^+ \quad G_m = 0$$

(21)

and then construct the BRST operator

$$Q = \eta_0 l_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_1^+ \eta_1 P_0$$

(22)

with fermionic ghosts

$$\{\eta_1, P_1^+\} = 1, \quad \{\eta_1^+, P_1\} = 1, \quad \{\eta_0, P_0\} = 1$$

(23)

which acts on the vacuum state as

$$\eta_1 |0\rangle = P_1 |0\rangle = P_0 |0\rangle = 0.$$ 

(24)
Finally, Lagrangian (up to an overall factor) and gauge transformations in concise notation are (see \[4, 5\] for details)

\[
\mathcal{L} = \int d\eta_0\langle \Psi_1|Q|\Psi_1\rangle, \quad \delta|\Psi_1\rangle = Q|\Lambda_1\rangle, \quad (25)
\]

where

\[
|\Psi_1\rangle = \left\{-ia^{+\mu}A_\mu(x) + b^+A(x) + \eta_0\mathcal{P}_1^+\varphi(x)\right\}|0\rangle, \quad |\Lambda_1\rangle = \mathcal{P}_1^+\lambda(x)|0\rangle.
\]

(26)

In component form the Lagrangian and gauge transformations are

\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu
\]

(30)

where \(F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\). Thus we have reproduced the Lagrangian for massive spin one field in arbitrary curved space using the BRST method.

4 Lagrangian construction for spin-2 field

In spin-2 case we will search for a partial solution to (9) for operator \(\mathcal{X}\) in the form

\[
\mathcal{X} = X(x) + X^{\mu\nu}(x) a^{+}_\mu a_\nu + X^{\mu\alpha\beta} a^{+\mu}_\alpha a^{+}_\nu a_\alpha a_\beta.
\]

(31)

To find coefficients in (31) we consider commutator

\[
[l_0, l_1]|\varphi_2\rangle = 2i(R^{\mu\alpha\beta\nu} + X^{\mu\alpha\beta\nu}) a^{+\mu}_\alpha a_\beta a_\alpha a_\beta D_\nu|\varphi_2\rangle + i(R^\sigma\alpha + X^\sigma\alpha) a_\alpha D_\sigma|\varphi_2\rangle + iX_{\mu\alpha;\beta} + 2X_{\mu\alpha}\sigma a^{\alpha\beta} + R_{\mu\alpha;\beta} - R_{\alpha\beta;\mu}) a^{\alpha\beta} a^{+\mu} a^{+\nu}|\varphi_2\rangle + iX_{\alpha\beta} + X^\sigma a^{\alpha\beta} + \frac{1}{2} R^\sigma\alpha a^{\alpha}|\varphi_2\rangle.
\]

(32)

From the first two summand of the right hand side of (32) and the last one we find

\[
X_{\mu\alpha\beta} = -R_{\mu(\alpha\beta)\nu} + F_1(x)g_{\mu\nu}g_{\alpha\beta} + F_2(x)(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\alpha\nu})
\]

(33)

\[
X_{\mu\nu} = -R_{\mu\nu} + F_3(x)g_{\mu\nu}
\]

(34)

\[
X = -F_3(x) + C
\]

(35)

where \(F_i(x)\) are arbitrary functions with dimension of mass squared and \(C\) is an arbitrary constant with the same dimension. Let us substitute the found coefficients of the operator \(\mathcal{X}\) (31) into the third summand of the right hand side of (32)

\[
[l_0, l_1]|\varphi_2\rangle = (4F_2 + F_3)\alpha g_{\mu\beta}a^{+\mu}a^{\alpha}a^{\beta}|\varphi_2\rangle + (R_{\alpha\beta;\mu} - 2R_{\mu(\alpha\beta)})a^{+\mu}a^{\alpha}a^{\beta}|\varphi_2\rangle.
\]

(36)
To provide (9) we have to suppose that $\nabla_\alpha (4F_2 + F_3) = 0$ and

$$R_{\alpha\beta\mu} = 2R_{\mu(\alpha\beta)} \Rightarrow R = \text{const.} \quad (37)$$

We see that consistent equations of motion for spin-2 field exist only in space with Ricci and scalar curvature satisfying (37) while Weyl tensor are arbitrary. As a result a partial solution to (9) has the form

$$\tilde{l}_0 = D^2 - m^2 - R^{\mu\alpha\beta\nu}a_\mu^+a^+\alpha a_\beta - R^{\mu\nu}a_\mu^+a_\nu + 4F_1l_1^2l_2 + 2F_2(N-1)(N-2) + \xi_1RN + \xi_2R_0, \quad N = a_\mu^+a_\mu \quad (38)$$

where $F_1$ and $F_2$ are arbitrary functions with dimension of mass squared and $\xi_1$ and $\xi_2$ are arbitrary dimensionless constants. It is easy to check that $[\tilde{l}_0, l_2]|_{\varphi_2} \sim 0$. Now for future convenience we put $F_1 = F_2 = 0$ and constants $\xi_1$ and $\xi_2$ we will fix later.

Let us turn to Lagrangian construction using the BRST method. First, in accordance with this method \[4, 5\] we should add to operators $l_0, l_1, l_2$ their Hermitian conjugated operators. Since $l_0$ is Hermitian operator we add only $l_1^+$ and $l_2^+$

$$l_1^+ = -ia^{+\mu}D_\mu, \quad l_2^+ = \frac{1}{2}a^{+\mu}a^{+\nu}. \quad (39)$$

Next we must add to $l_0, l_1, l_2, l_1^+, l_2^+$ two more operators $g_0, g_m (11)$ to close the algebra, with the “const” in $g_m$ to be defined. To define it we consider commutators $[l_1, l_1^+]$ acting on states $|\lambda_k\rangle$ with $k < 2$

$$[l_1, l_1^+] = -D^2 + R_{\mu\nu\alpha\beta}a^{+\mu}a^{+\nu}a^{+\alpha}a^{+\beta}$$

$$= -\tilde{l}_0 - m^2 + (1 - \frac{d}{2}\xi_1 + \xi_2)R + (\xi_1 - \frac{d}{2})Rg_0$$

$$- 2\tilde{R}^{\mu\nu}a_\mu^+a_\nu - 2R^{\mu\alpha\beta\nu}a_\mu^+a^+\alpha a_\beta + (\xi_1 - \frac{d}{2})Rg_0$$

$$\quad - \frac{1}{d}R(g_0 - \frac{d}{2} - 2) \quad (40)$$

where $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{d}g_{\mu\nu}R$ is the traceless part of the Ricci tensor. Since it is supposed that (40) acts on states $|\lambda_k\rangle$ with $k < 2$ and since $R = \text{const}$ then the right hand side of (40) is expressed through the operators of the algebra only in the case when $\tilde{R}_{\mu\nu} = 0$. Now we fix the arbitrary constants $\xi_1, \xi_2$ so that the mass-shell operators (which we denote $l_0$ instead of $\tilde{l}_0$) becomes

$$l_0 = D^2 - m^2 - R^{\mu\alpha\beta\nu}a_\mu^+a^+\alpha a_\beta - \frac{1}{d}R(g_0 - \frac{d}{2} - 2). \quad (41)$$

Thus commutator $[l_1, l_1^+]$ takes the form

$$[l_1, l_1^+] = -l_0 - m^2 + \frac{d+2}{d}R - \frac{2}{d}Rg_0 - 2R^{\mu\alpha\beta\nu}a_\mu^+a^+\alpha a_\beta. \quad (42)$$

Of course, if we are supposing that the commutator acts on $|\lambda_{k<2}\rangle$ then the last term with Riemann tensor containing two annihilation operators may be discarded completely. But this leads to violation of Jacobi identities. One can check that if we discard only part of the last term with the Weyl tensor $C^{\mu\alpha\beta\nu}a_\mu^+a^+\alpha a_\beta$ then the Jacobi identities will be satisfied. As a result commutator $[l_1, l_1^+]$ should be taken in the form

$$[l_1, l_1^+] = -l_0 - g_m - \frac{2}{d(d-1)}R [g_0^2 - 2g_0 - 4l_2^2l_2] \quad (43)$$

with

$$g_m = m^2 - \frac{d^2-4}{2d(d-1)}R \quad (44)$$
Thus algebra of the operators $l_0, l_1, l_1^+, l_2, l_2^+, g_0, g_m$ has the form given in Table 1. In this Table the first arguments of the commutators are in the first column, the second arguments are in the upper row. One can check that algebra given in Table 1 satisfies Jacobi identities.

Now the procedure of Lagrangian construction is the same as in [5] and we will give only its main steps without details. First, since we have two operators $g_0$ and $g_m$ which are not constraints neither in the bra-vector nor in the ket-vector spaces one should construct extended operators. Introducing two pairs of bosonic creation and annihilation operators $b_1^+, b_1$ and $b_2^+, b_2$ with the standard commutation relations $[b_1, b_1^+] = [b_2, b_2^+] = 1$ the result for the additional parts $l_i'$ has the form

\[
l'_0 = 0 \quad l'_1^+ = m_1 b_1^+ \quad l'_2^+ = b_2^+ \quad g'_0 = b_1^+ b_1 + b_2^+ b_2 - \frac{d-2}{2} \quad g'_m = \frac{d^2-4}{2(d-1)} R - m^2 \quad (45)
\]

\[
l'_1 = m_1 b_1^+ b_2 + \frac{m^2}{m_1} b_1 - \frac{R}{m_1 d} b_1^+ b_1^2 + \ldots \quad l'_2 = -\frac{d-2}{2} b_2 + \frac{m^2}{2m_1} b_2^2 + \ldots \quad (46)
\]

where $m_1$ is an arbitrary constant parameter with dimension of mass which is constructed from $m$ and $R$, $m_1 = f(m, R) \neq 0$ and the dots stand for the terms irrelevant for spin-2 case.

Then we construct BRST operator on the base of the extended operators $L_i$ and extract from it the part which is used for Lagrangian construction\footnote{\textit{Q}_2 in (47) is analog of operator \textit{Q}_s in [5] formula (80).}

\[
Q_2 = \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_1^+ \eta_1 P_0 - \eta_1^+ \eta_2 \mathcal{P}_1^+ - \eta_2^+ \eta_1 \mathcal{P}_1 + 4R \frac{1}{d(d-1)} \eta_1^+ \eta_1 \left\{ (L_2^+ - 2l_2^+) \mathcal{P}_2 + (L_2 - 2l_2^+) \mathcal{P}_2^+ \right\} \quad (47)
\]

with the ghosts satisfying the anticommutation relations

\[
\{ \eta_1, \mathcal{P}_1^+ \} = 1, \quad \{ \eta_1^+, \mathcal{P}_1 \} = 1, \quad \{ \eta_2, \mathcal{P}_2^+ \} = 1, \quad \{ \eta_2^+, \mathcal{P}_2 \} = 1, \quad \{ \eta_0, \mathcal{P}_0 \} = 1. \quad (48)
\]
Now Lagrangian and gauge transformations in terms of operator $Q_2$ (49) are written as follows
\[
L = \int d\eta_0 \langle \Psi_2 | K Q_2 | \Psi_2 \rangle \quad \delta | \Psi_2 \rangle = Q_2 | \Lambda_2 \rangle \quad (49)
\]
where operator $K$
\[
K = |0\rangle \langle 0| + \frac{m^2}{m_1} b_1^+ |0\rangle \langle b_1 + \frac{m^2}{2m_1} \left( m^2 - \frac{R}{d} \right) b_1^{+2} |0\rangle \langle b_1^2 \\
- \frac{d-2}{2} b_2^+ |0\rangle \langle b_2 + \frac{m^2}{2m_1^2} \left( b_1^{+2} |0\rangle \langle b_2 b_2^+ |0\rangle \langle b_1^2 \right) + \ldots
\]
(50)
provides the reality of the Lagrangian and states $|\Psi_2\rangle$ and $|\Lambda_2\rangle$ are
\[
|\Psi_2\rangle = |\Phi_2\rangle + \eta_1^+ \mathcal{P}_1^+ |\Phi_0\rangle + \eta_0 \mathcal{P}_2^+ |\Phi_1\rangle + \eta_0 \mathcal{P}_1^+ |\Phi_0\rangle \quad |\Lambda_2\rangle = \mathcal{P}_1^+ |\lambda_1\rangle + \mathcal{P}_2^+ |\lambda_0\rangle \quad (51)
\]
\[
|\Phi_2\rangle = \left[ (-1)^2 \alpha^+ \alpha^+ H_{\mu\nu} (x) + b_1^+ H_1 (x) - \alpha^+ b_1^+ A_\mu (x) + b_1^{+2} \varphi (x) \right] |0\rangle \quad (52)
\]
\[
|\Phi_0\rangle = H (x) |0\rangle, \quad |\Phi_1\rangle = \left[ -\alpha^+ H_\mu (x) + b_1^+ A (x) \right] |0\rangle, \quad |\Phi_0\rangle = H_2 (x) |0\rangle, \quad (53)
\]
\[
|\lambda_1\rangle = \left[ -\alpha^+ \lambda_\mu (x) + b_1^+ \lambda (x) \right] |0\rangle, \quad |\lambda_0\rangle = \lambda_2 (x) |0\rangle. \quad (54)
\]
From (49) one can find Lagrangian in component form
\[
L = \frac{1}{2} H^{\mu\nu} \left\{ (\nabla^2 - m^2) H_{\mu\nu} - 2 R_{\mu\alpha\beta\nu} H^{\alpha\beta} - 2 \nabla_\mu H_\nu + g_{\mu\nu} H_2 \right\} \\
+ \frac{m^2}{m_1} A^\mu \left\{ (\nabla^2 - m^2 + \frac{R}{d}) A_\mu - m_1 H_\mu - \nabla_\mu A \right\} \\
+ \frac{m^2}{m_1} \left\{ H_1 + 2 \varphi \left( \frac{m^2}{m_1^2} - \frac{R}{d m_1^2} \right) \right\} \left\{ (\nabla^2 - m^2 + \frac{2R}{d}) \varphi - m_1 A \right\} \\
+ \left( \frac{m^2}{m_1} \varphi - \frac{d-2}{2} H_1 \right) \left\{ (\nabla^2 - m^2 + \frac{2R}{d}) H_1 - H_2 \right\} \\
- H \left\{ (\nabla^2 - m^2 + \frac{2R}{d}) H + \nabla_\mu H_\mu - \frac{m^2}{m_1} A + H_2 \right\} \\
- H^{\mu} \left\{ -\nabla_\mu H_{\rho\gamma} + \frac{m^2}{m_1} A_\mu - \nabla_\mu H + H_\mu \right\} \\
- \frac{m^2}{m_1} A \left\{ -\nabla_\mu A_\mu + m_1 H_1 + \frac{2m^2}{m_1} \varphi - \frac{2R}{d m_1} \varphi - m_1 H + A \right\} \\
- H_2 \left\{ -\frac{1}{2} H^{\mu} - \frac{d-2}{2} H_1 + \frac{m^2}{m_1} \varphi - H \right\}
\]
(55)
and gauge transformations
\[
\delta H_{\mu\nu} = \nabla_\mu \lambda_\nu + \nabla_\nu \lambda_\mu - g_{\mu\nu} \lambda_2, \quad \delta H_1 = \lambda_2, \quad (56)
\]
\[
\delta A_\mu = \nabla_\mu \lambda + m_1 \lambda_\mu, \quad \delta \varphi = m_1 \lambda, \quad (57)
\]
\[
\delta H = -\nabla_\mu \lambda_\mu + \frac{m^2}{m_1} \lambda + \lambda_2, \quad \delta H_\mu = (\nabla^2 - m^2 + \frac{R}{d}) \lambda_\mu \quad (58)
\]
\[
\delta A = (\nabla^2 - m^2 + \frac{2R}{d}) \lambda \quad \delta H_2 = (\nabla^2 - m^2 + \frac{2R}{d}) \lambda_2. \quad (59)
\]
Thus we have derived Lagrangian for spin-2 field in the Einstein spacetime.
Now we can remove a part or all the auxiliary fields. Doing the transformations analogous to [5] one obtains the following Lagrangian

\[ \mathcal{L} = \frac{1}{2} H_{\mu \nu} \left\{ (\nabla^2 - m^2 - \frac{2R}{d(d-1)}) H_{\mu \nu} - 2 \nabla_{\mu} \nabla_{\nu} H_{\sigma \kappa} - 2 C_{\alpha \beta \mu} H^{\alpha \beta} + 2 \nabla_{\mu} \nabla_{\nu} H \right\} \]

\[ + \frac{m^2}{m_1^2} A_{\mu} \left\{ (\nabla^2 + \frac{R}{d}) A_{\mu} - \nabla_{\mu} \nabla_{\nu} A_{\nu} - 2 m_1 \nabla_{\nu} H_{\mu \nu} + 2 m_1 \nabla_{\mu} H \right\} \]

\[ + \frac{m^2}{m_1^2} 2 \varphi \left( \frac{d-1}{d-2} \frac{m^2}{m_1^2} - \frac{R}{d(d-1)} \right) \left\{ (\nabla^2 + \frac{d m^2}{d-2}) \varphi - 2 m_1 \nabla_{\nu} A_{\mu} + m_1^2 H \right\} \]

\[ - \frac{1}{2} H \left( \nabla^2 - m^2 + \frac{R(d-3)}{d(d-1)} \right) H \]  

(60)

\[ \delta H_{\mu \nu} = \nabla_{\mu} \lambda_{\nu} + \nabla_{\nu} \lambda_{\mu} - g_{\mu \nu} \lambda \left( \frac{2 m^2}{m_1 d-2} \right), \quad \delta A_{\mu} = \nabla_{\mu} \lambda + m_1 \lambda_{\mu}, \quad \delta \varphi = m_1 \lambda, \]  

(61)

with \( H = H_{\mu}^{\mu} \). The Lagrangian (60) contains Weyl tensor \( C_{\mu \alpha \beta \nu} \) in explicit form. If the \( C_{\mu \alpha \beta \nu} = 0 \) one gets, after some field redefinition, the Lagrangian given in [9] for the constant curvature background space.

It is easy to see that if \( m^2 = \frac{d-2}{d(d-1)} R \), then scalar field \( \varphi \) disappears from Lagrangian (60). As a result such a Lagrangian in \( d = 4 \) describes propagation of the helicities \( \pm 2, \pm 1 \), thus the field \( H_{\mu \nu} \) becomes partial massless in terminology [10], although in (60) we did not assume that Weyl tensor vanished.

Finally one can write Lagrangian in terms of physical field alone

\[ \mathcal{L} = \frac{1}{2} H_{\mu \nu} \left\{ (\nabla^2 - m^2 + \frac{2R}{d}) H_{\mu \nu} - 2 \nabla_{\mu} \nabla_{\nu} H_{\sigma \kappa} + 2 \nabla_{\mu} \nabla_{\nu} H \right\} \]

\[ - \frac{1}{2} H \left( \nabla^2 - m^2 + \frac{R}{d} \right) H \]  

(62)

with no gauge symmetry.

5 Causality of massive spin-2 field propagation

Now we turn to the problem of causality of massive spin-2 field with Lagrangian (62) in Einstein space. It is well known that the higher spin field theory faces a problem of inconsistency, in particular, coupling to external field can violate a causality of free theory\[7\]. Our consideration is based on the method of Velo and Zwanziger [11] adapted to the theories in curved spacetime.

We begin with a brief outline of the method. If one has a system of the second order differential equations for a set of fields \( \varphi^B \)

\[ M_B^{A \mu \nu} \partial_{\mu} \partial_{\nu} \varphi^B + \ldots = 0, \quad \mu, \nu = 0, \ldots, d-1 \]  

(63)

then to verify that the system (63) describes hyperbolic propagation one should check that all solutions \( n_0(n_i), (i = 1, \ldots d-1) \) of the algebraic equation

\[ \det(M_B^{A \mu \nu} n_\mu n_\nu) = 0 \]  

(64)

are real for any given real set of \( n_i \). The hyperbolic system is called causal if there are no timelike vectors among solutions \( n_\mu \) of (64).

In many physical cases (including spin-2 field) equation (64) fulfills identically. In this case one should replace the initial system of equations by another equivalent system of equations

\[ \text{Aspects of consistency and causality for spin-2 field in external background are discussed in [11], [8], [12].} \]
supplemented by constraints at a given initial time. Then the above analysis should be applied to this new system.

Let us turn to our spin-2 field described by Lagrangian \((62)\). The equations of motion are

\[
E_{\mu \nu} \equiv (\nabla^2 - m^2 + \frac{2R}{d})H_{\mu \nu} - 2\nabla^\sigma \nabla_{(\mu} H_{\nu)\sigma} + \nabla_\mu \nabla_\nu H - g_{\mu \nu}(\nabla^2 - m^2 + \frac{R}{d})H + g_{\mu \nu}\nabla^\alpha \nabla^\beta H_{\alpha \beta} = 0. \tag{65}
\]

If we consider equation \((64)\) for equations \((65)\) then we find that it fulfils identically. Therefore one should replace equations \((65)\) by another equivalent system of equations with constraints on initial data. It can be done by the same method as in \([11]\) and we will not repeat all the steps and proofs. The system of equations equivalent to \((65)\) are

\[
E_{\mu \nu} + \nabla_\mu C_\nu + \nabla_\nu C_\mu + \nabla_\mu \nabla_\nu D = 0, \tag{66}
\]

where

\[
C_\nu = -\frac{1}{m^2} \nabla^\nu E_{\mu \nu} = \nabla^\nu H_{\mu \nu} - \nabla_\nu H, \tag{67}
\]

\[
D = \frac{d - 2}{m^2((d - 1)m^2 + R)} \left( \nabla^\mu \nabla^\nu E_{\mu \nu} + \frac{m^2}{d - 2} E_{\mu}^\mu \right) = H \tag{68}
\]

and it is supplemented by the constraints at an initial time (say \(t = 0\))

\[
E_{\mu 0}|_{t=0} = 0, \quad C_\nu|_{t=0} = 0, \quad D|_{t=0} = 0, \quad (\nabla_0 D)|_{t=0} = 0. \tag{69}
\]

Then one can find that the kinetic part of \((66)\) has the form

\[
\nabla^2 H_{\mu \nu} + g_{\mu \nu}(\nabla^\alpha \nabla^\beta H_{\alpha \beta} - \nabla^2 H) + \ldots = 0 \tag{70}
\]

and coincides at any point \(x_0\) with its analog in the flat space if we choose locally around \(x_0\) the Riemann normal coordinates. Thus like in the flat case, the equations for spin-2 field followed from Lagrangian \((62)\) are hyperbolic and causal for the background under consideration.

6 Summary

We have studied a new aspect of BRST approach to Lagrangian construction for higher spin fields. The approach efficiently works if we require that BRST operator is nilpotent only in weak sense, i.e. \(Q_s^2 |\Lambda_s\rangle = 0\) for some spin-s but in general as the operator \(Q_s^2 \neq 0\). In all previous applications of BRST approach to higher spin theory the operator equality \(Q^2 = 0\) was valid only in constant curvature spaces for any spin, even for spin-1 field. We have shown that nilpotency of BRST charge in weak sense leads to standard spin-1 field Lagrangian in arbitrary Riemann spacetime and to consistent spin-2 field Lagrangian in Einstein spacetime. As a result we resolved an old enough puzzle that BRST approach to field theory with arbitrary spin is unable to reproduce the known Lagrangian for spin-1 field in curved spacetime.

As usual in BRST approach, spin-1 and spin-2 Lagrangians are obtained in gauge invariant form with suitable St"uckelberg auxiliary fields. In particular, we see that the Lagrangian found in \([9]\) for constant curvature space is consistent in more general spaces with nontrivial Weyl tensor. Also we extended the notion of partial masslessness, formulated in \([10]\) in constant curvature space, for spin-2 field in above background with non-zero Weyl tensor. And at last, we proved that the Lagrangian under consideration describes a hyperbolic and causal propagation of spin-2 field in the above spacetime.
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