The statistical distribution of curvatures of natural rivers meanders

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Abstract: The rivers meanders are the most common form of natural rivers. In this paper, the shapes of meanders may be characterized in terms of the statistical distribution of curvatures measured along their contours. Deviation degree and span are proposed, in which deviation degree represents the closure degree of the bend and span represents the roundness degree or even degree of curvature distribution of the bend. Using these two indicators, we can effectively distinguish the state of the development of the river bend and thus determine different river regulation schemes. Finally, the method of deducing the reliable curvature of the bend from the satellite pictures of the river is discussed.

1. Introduction
The rivers meanders are the most common form of natural rivers. Its geometric shape of twists and turns and the dynamic characteristics of creeping creep attract the attention of scholars at home and abroad. Many papers have carried out in-depth research on the geometry characteristics of rivers meanders [1-4], self-organization process [5-6] and rivers meanders evolution simulation [7-8], and obtained a series of important research results.

However, in the classification and description of the fractal and developmental state of rivers meanders, they are all qualitative methods, such as the early stage of development and the end stage, which cannot be analyzed more finely. In order to capture the statistical characteristics of the natural development process and evolution of rivers meanders, it is necessary to establish a detailed method for describing the shape of curvatures. This method should be quantified and more closely related to the evolution process. In this paper, we present a key microscopic variable, curvature, using the curvatures distribution of a two-dimensional image of the rivers meanders a tool to describe its shape characteristics.

This paper will detail this method and apply it to 30 natural or artificially rehabilitated river meanders. The first part begins with a quantitative study of two-dimensional shape features, summarizing our curvature-based approach, and details the method of extracting local curvature from river meander satellite photos. The second part is a quantitative description of the application of this method to the development of river meanders, and proposes the possibility of using this method to guide river regulation.
2. Quantification of the two-dimensional shape of river meanders

The shape of the river meanders has long been a concern in the fields of geomorphology, hydraulics and riverbed evolution. There are two basic methods that are often used now. The first method is described using the radius of curvature and the central angle of the river meanders (Fig. 1). In this description, R is the radius of curvatures of the river meander top. There are two problems here: First, the position of the river meanders top is difficult to determine. Second, because the two-dimensional shape function of rivers meanders is unknown, the radius of curvature of the apex is difficult to determine, so this method has certain subjectivity. The second method is to use the degree of curvature to judge [9] (Fig. 2). The degree of curvature C=L/B is more effective in judging the development period of the river meander, but it is described in the same development period of the 2D shape does not apply.

![Fig.1 Definition of shape parameters in method one](image1)

![Fig.2 Definition of shape parameters in method two](image2)

The plane evolution of a river is essentially the constant change of the curvature of each point on the river. The curvature is the rotation rate of the tangential direction of the point on the curve to the length of the arc, which can be given by \( K = d\alpha/ds \). A more vivid understanding is that the curvature is the reciprocal of the radius of the inscribed circle at a point on the curve. Most of the time it is customary to define a contour as convex when \( K > 0 \) and a concave when \( K < 0 \).

To describe the shape of the river meander the probability density function of the curvature distribution is \( \rho(K) \). \( \rho(K) dK \) represents the probability that the curvature is between \( K \) and \( K + dK \). In practice, to distinguish between different distributions, we use the cumulative distribution function of curvature.

\[
f(K) = \int_0^K \rho(K') dK'
\]

Where \( f(K) \) is the ratio of the length of the curve whose curvature is less than \( K \) to the total length. When \( K \) increases from \( 0 \) to \( \infty \), \( f(K) \) from 0 to 1. When \( K \) is the minimum value, \( f(K) \) is 0, when \( K \) is the maximum value, \( f(K) \) is 1, and when \( K \) is the median, \( f(K) \) is 1/2. Unlike \( \rho(K) \), \( f(K) \) does not need to know the function expression of the curve or measure the curvature of all points to derive, because the curve can be discretized and the curvature of the discrete points can be obtained, and they are arranged in order from small to large and record less than this curvature. Arrange them from small to large and record the ratio of the length of the arc segment less than this curvature to the total length. In this paper, we use \( f(K) \) and \( <K> /K \) as the shape function of the river meanders. The \( <K> /K \)-average curvature is dimensionless compared to the curvature of a point, so that the shapes of different sizes of river meanders can be compared. The reason for using \( <K> /K \) is to make the data point fall in a larger space on the right side of the horizontal coordinate "1", and the image is clearer. To build the formula, \( f(K) \) for some simple shape examples will be given below, and lead to two important indicators of the \( f(K) \) graph. Prior to this, we will discuss
the method of extracting local curvature by relying on the images of the river meander satellite.

There are two common methods for finding the curvature of each point on the curve. One is to use the points on the curve for interpolation, to fit a similar curve, and then calculate the curvature of the point on the interpolation curve instead of the curvature of the original curve point. The advantage of this type of method is that the calculated curvature is relatively accurate and the error is smaller. However, it is difficult to determine the distribution of the interpolation points. The distribution of the points may cause the cumulative distribution function of the curvature to be greatly deformed. Generally, it cannot be used in areas where the curvature changes more severely, especially in areas where the unevenness changes, otherwise there will be a larger error and large amount of calculation.

The second method is to use the known curve to fit the original curve by adjusting the curve parameters to adjust the shape of the curve. The principle is to match as many points as possible on the curve to be sought. This type of method has a small amount of calculation, and the error is visually visible on the graph. It is possible to artificially select a more accurate point, so that the overall error is small. However, the local accuracy is poor, and the adjustment of the fitting curve is subjective. In this paper, two methods are used in combination, and the curve to be obtained is fitted using a known curve. Interpolation fitting is preferred in places where the local error is large.

Systematic errors can be minimized by using cubic spline interpolation to fit the original curve where local errors are large. In order to convert the fitted Cartesian coordinate function to polar coordinates, make sure \( y = y(x) \) is the single-valued function. Therefore, should choose the right origin of the Cartesian coordinate system and coincide with the origin of the polar coordinates. After \( y = y(x) \) is obtained by cubic spline interpolation, \( y = y(x) \) can be converted to \( r = r(\theta) \) directly by \( x = r \cos \theta \) and \( y = r \sin \theta \). Among them \( r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x) \). When converted to polar coordinates, you can find the curvature of the point on the curve by

\[
K = \frac{r^2 + 2r^2_\theta - rr_\theta}{(r^2 + r^2_\theta)^{3/2}} +
\]

Where \( r_\theta = dr/d\theta \), \( r = d r/d \theta \) [10].

The method of fitting a shape of a river meander using a known function in a smoother part. In this paper we choose the elliptic function \( x^2/a^2 + y^2/b^2 = 1 \), where \( a \) is the long axis and \( b \) is the short axis. By adjusting the long and short axes to change the shape of the ellipse, the known river meander is fitted to match as many points as possible on the contour of the river meander. Then you can find the curvature of any point \( P(x, y) \) by

\[
K = \frac{ab^4}{[(a^2 - b^2)y^2 + b^4]^{3/2}}
\]

According to the specific shape of the contour of the river meander, the curvature of each point on the contour curve can be obtained by using these two methods. The following two methods are used to analyze the two practical problems to further explain the curvature distribution method.

Fig.3 Two meanders in different evolution stages, but they have the same curvature in the apex.
Figure 3 is a two-dimensional plane figure of the two rivers meanders on the map scaled down. Currently, it is obviously not advisable to describe the shape of the river meanders according to the above-mentioned method with the curvature radius of the apex of the river meanders as the characteristic value. Because the two river meanders have the same curvature radius \( R_1 = R_2 \) of the river meanders, but at different stages of development. The curvature distribution method is used to obtain the curvature distribution of two rivers, as shown in Fig. 4.

Fig. 4 The statistical distribution of curvatures of two meanders in different evolution stages.

Here, an index deviation \( \gamma \) is defined, which is the degree to which the x coordinate value corresponding to the graph deviates from the scale "1" when \( a = 1/2 \). This index represents the degree of closure of the river meanders, and the larger the value of the deviation \( \gamma \) is the lower the closure of this river meanders. In this case, it can be seen that the development of the river meanders 1 (black solid line) is low and the degree of closure is low, so the deviation degree \( \gamma = 4.8 \) is larger. When the river meander is completely closed to form the oxbow lake, its deviation \( \gamma \) reaches a minimum value, which is close to 0. That is to say, as the river meander continues to develop, its deviation \( \gamma \) is continuously reduced. When the river bends straight, the deviation \( \gamma \) suddenly increases from the minimum value, forming a new round of river meander evolution.

Fig. 5 Two meanders which in same evolution stage have the same \( C = L/B \), and their shapes are completely different.

Figure 5 is a two-dimensional plan image of two rivers on the map scaled down according to different scales. It can be clearly seen that the two rivers and meanders are in the middle and late stages of development, with the same curvature, \( C_1 = L_{m1}/B_{m1} = C_2 = L_{m2}/B_{m2} \), but the shape of the river meander is very different, and the shape of the first river meander is closer to a circle. The curvature distribution method is used to obtain the curvature distribution of two river meanders, as shown in Fig. 6.
Here this paper define another indicator span $\beta$, which is the span of the graph on the abscissa $x$. This indicator represents the roundness of the river meanders. The smaller the span $\beta$ is, the more uniform the curvature distribution is, and the shape of the river meander is closer to a part of a circle or a circle. It can be seen from Figure 7 that although the two rivers are in the middle and late stages of development, their spans are significantly different, and the span of the circular river meander (black solid line) is much smaller than that closer to the ellipse river meander (black dotted line).

3. Two-dimensional graphic curvature distribution of natural rivers

In order to test whether the method of curvature distribution can find some rules in natural rivers to distinguish different river meander shapes, this paper selects ten river meanders at different scales in the early, middle and late stages of development as statistical sample space. The Google Earth satellite map is combined with AutoCAD to extract the geometric parameters of the selected river meander. In order to be able to be used as a standard and more clear, we should try to choose a typical river meander, avoiding the controversial transition period, but one thing to be clear is that river meander development is a continuous process, there is no obvious node.

First, select ten river meanders in the early and upper-middle-lower reaches of the Yangtze River. For the sake of clarity, take the first five to draw their curvature distribution (Figure 7), and the other river meander's deviation and span list (table 1).

Figure 7 shows the curvature distribution of five rivers at the beginning of development. The deviation $\gamma$ is in the range of 2.5~3.5, which is far from the scale "1", indicating that the degree of closure is relatively low. The spans of the first four rivers are similar, with a length of 3 units. The span of the blue river meander is smaller, indicating that the curvature of the blue river meander is more evenly distributed. The shape of the river meander is closer to a circle or arc.

In the upper, middle and lower reaches of the Yangtze River, ten river meanders of the Yangtze
River are selected respectively. For the sake of clarity, use the first five to plot its curvature distribution (Figure 8), and the other river meanders are shown in the deviation and span list (Table 1).

Fig. 8 The statistical distribution of evolution of meanders in mid-term stage

Figure 8 shows the curvature distribution of five rivers in the middle of development, which the deviation $\gamma$ is in the range of 0.5~1.5. The deviation $\gamma$ of the river meander relative to the early development is closer to the scale “1”, indicating that the degree of closure in the mid-development river meander is much higher than that in the early development. However, the span $\beta$ of the river meander in the mid-development period is irregular and not comparable to the early river meander. It is indicated that the span $\beta$ is an index for judging roundness, and it can be judged which river meander curvature distribution is more uniform, and it is impossible to judge the development of the river meander.

Finally, ten river meanders in the middle and lower reaches of the Yangtze River were selected. For the clarity of the map, the first five strips were taken to plot their curvature distribution (Fig. 9), and the other river meanders were shown in the deviation and span list (Table 1). Figure 9 shows the curvature distribution in the late development of the river meander. The deviation $\gamma$ is between 0 and 0.5, which indicates that it is closer to the scale “1” than the former two groups of river meanders, and the closure degree is much higher in the late development river meander. It is also easy to see from the two-dimensional shape of the river meander.

Fig. 9 The statistical distribution of evolution of meanders in later stage

Thirty different river meanders are listed in Table 1, all located on the Yangtze River and its tributaries. The statistics of these thirty rivers further indicate that the developmental stage of the river meander can be distinguished from the deviation $\gamma$. The early stage of development, the deviation of $\gamma$ is located between (2.5, 3.5) the deviation of $\gamma$ in the middle of development is located between (0.5, 1.0), and the deviation of $\gamma$ in the late development is located between (0, 0.5). However, the data of span $\beta$ has no obvious law, but it can be seen that the span $\beta$ of the river meander which is close to a circle or arc is small, and the span $\beta$ is a sign of whether the curvature distribution is uniform.
Therefore, the span $\beta$ can judge the shape of the river meander at the same development stage and distinguish it.

| River name | Deviate degree | Span | River name | Deviate degree | Span |
|------------|----------------|------|------------|----------------|------|
| Qingjiang | 3.49           | 3.25 | Yangtze    | 3.44           | 3.10 |
| Yangtze   | 3.57           | 2.65 | Zishui     | 3.57           | 2.65 |
| Yangtze   | 3.62           | 2.22 | Mangdao    | 3.62           | 2.22 |
| Jinshui    | 3.12           | 0.60 | Qingsgejiang| 3.12           | 0.60 |
| Yangtze   | 2.95           | 2.55 | Yangtze    | 2.95           | 2.55 |
| Yangtze   | 3.45           | 3.07 | Yangtze    | 3.45           | 3.07 |
| Yangtze   | 2.50           | 1.20 | Yangtze    | 2.50           | 1.20 |
| Jinjiang  | 3.43           | 1.05 | Yangtze    | 3.43           | 1.05 |
| Qingjiang | 1.23           | 3.10 | Huangbai   | 1.23           | 3.10 |
| Yangtze   | 1.33           | 0.92 | Mangdao    | 1.33           | 0.92 |
| Yangtze   | 0.54           | 2.05 | Xiushui    | 0.54           | 2.05 |
| Yangtze   | 1.35           | 2.65 | Qingshang` | 1.35           | 2.65 |
| Yangtze   | 1.52           | 3.25 | Yangtze    | 1.52           | 3.25 |

Many of the rivers have been analyzed using the curvature distribution method above, but why should we choose the curvature distribution for the river meander fractal? How is the curvature distribution of the river meander and the river meander development process linked together? It is only when the curvature distribution can reflect the development mechanism of the river meander that the curvature distribution is appropriate to describe the shape of the river meander. The meaning of curvature in mathematics is the rate of rotation of the tangential direction of a point on the curve to the length of the arc. If it is used in the development of the river meander, assuming that the flow velocity of the river is $v$, the curvature can be described as the rate at which the angle of the river meander changes. Because if the flow rate is assumed, the time and the arc length can be linked together ($t=L/v$), and the rotation rate of the tangential direction angle to the arc length can be converted into the tangential direction angle versus time, that is, the tangent the rate of change of the direction angle. At the same flow rate and flow rate, the greater the curvature, the greater the rate of change of the tangential direction angle, indicating that the greater the acceleration perpendicular to the flow direction, so the greater the pressure perpendicular to the river bank received by the bank. This explains why the straightening of the curve occurs where the curvature is greatest.

River meander curvature distribution and river management are also closely linked. Above we have linked the curvature to the force per unit volume of water. If we know the flow and the pressure on the river bank, we will naturally think that the actual pressure on the river bank is equal to the pressure on the river bank. At this time, this curvature is the ideal curvature of the river meander. In fact, we have some empirical understandings in this respect. For example, the lower reaches of the Yellow River proposed to rectify the river meander with continuous curves. The purpose is to make every point on the river meander at the best curvature, so that the river meander is at in this relatively balanced state, research in this area will be carried out in the future.
Discussion and conclusions

In this paper, thirty river meanders in different development stages of the Yangtze River and its tributaries were analyzed using a method describing the curvature distribution of the river. It is found that for the complex alluvial river meander, the traditional method cannot distinguish its shape and determine its development state, but the method of curvature distribution can find obvious laws. As the degree of development increases, the degree of deviation $\gamma$ decreases continuously, approaching zero. The $\gamma$ change interval of the river meander at the beginning of development is $[2.5, 3.5]$, the deviation range of $\gamma$ deviation in the mid-development period is $[0.5, 1.0]$, and the deviation range of $\gamma$ in the late development is $[0, 0.5]$. In the same developmental period, the contours of the river meander are still diverse, some like ellipse, and some similar to a circle.

Because the time scale of river meander evolution is large, this paper only analyzes the different river meander patterns at the same time, but cannot describe the evolution of a river meander, or explain the change process of curvature distribution. In the subsequent research, many experiments will be carried out, and the longitudinal analysis of multiple river meanders will be carried out at the same time. Further study the complexity and variability of the river meander geometry in the evolution of the river meander.

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