ON THE GLOBAL BOUNDEDNESS OF FOURIER INTEGRAL OPERATORS

ELENA CORDERO, FABIO NICOLA AND LUIGI RODINO

ABSTRACT. We consider a class of Fourier integral operators, globally defined on \( \mathbb{R}^d \), with symbols and phases satisfying product type estimates (the so-called SG or scattering classes). We prove a sharp continuity result for such operators when acting on the modulation spaces \( M^p \). The minimal loss of derivatives is shown to be \( d(1/2 - 1/p) \). This global perspective produces a loss of decay as well, given by the same order. Strictly related, striking examples of unboundedness on \( L^p \) spaces are presented.

1. INTRODUCTION

The Fourier integral operators (FIOs) of Hörmander \([24, 25, 45]\), in a simplified local version, are operators of the form:

\[
Af(x) = A_{\Phi, \sigma} f(x) = \int e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta) \hat{f}(\eta) \, d\eta.
\]

Here the Fourier transform of \( f \in \mathcal{S}(\mathbb{R}^d) \) is normalized to be \( \hat{f}(\eta) = \int f(t)e^{-2\pi it\eta} \, dt \).

The phase function \( \Phi(x, \eta) \) in \( (1) \) is assumed real-valued, smooth for \( \eta \neq 0 \) and positively homogeneous of degree 1 with respect to \( \eta \); moreover, \( \sigma(x, \eta) \) belongs to Hörmander’s symbol class \( S^m_{1,0} \) of order \( m \in \mathbb{R} \):

\[
|\partial_\eta^\alpha \partial_x^\beta \sigma(x, \eta)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m-|\alpha|}, \quad \forall (x, \eta) \in \mathbb{R}^{2d},
\]

where \( \langle \eta \rangle = (1 + |\eta|^2)^{1/2} \). The definition being local, or localized in a compact manifold, that is, the support of \( \sigma(x, \eta) \) is assumed to have compact projection on the space of the \( x \)-variables, say \( \sigma(x, \eta) = 0 \) for \( |x| \geq R \), for a suitable \( R > 0 \). Moreover, \( \sigma(x, \eta) \) is usually cut to zero near \( \eta = 0 \), that is \( \sigma(x, \eta) = 0 \) in the strip

\[
\{(x, \eta) \in \mathbb{R}^{2d}, \quad |\eta| \leq 1\}.
\]

This eliminates the discontinuity at \( \eta = 0 \) of the phase function \( \Phi(x, \eta) \) without no practical effect on the local behaviour of the operator \( A \), since the eliminated part corresponds to a (locally) regularizing operator.

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Boundedness in $L^2(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ of $A$ have been widely studied, see e.g. [33, 41, 42] and references therein. As basic results, we know that under the non-degeneracy condition

$$\det \left( \frac{\partial^2 \Phi}{\partial x_i \partial \eta_l} (x, \eta) \right) > \delta > 0, \quad \forall (x, \eta) \in \mathbb{R}^{2d},$$

the operator $A$ is $L^2$-bounded for $m = 0$, see [24], as well as $L^p$-bounded, $1 < p < \infty$, if the order $m$ of $\sigma(x, \eta)$ is negative, satisfying

$$m \leq -(d - 1) \left| \frac{1}{2} - \frac{1}{p} \right|,$$

see [40] and references quoted there.

The result cannot be improved in general, as clear from the Fourier integral operator solving the Cauchy problem for the wave equation in space-dimension $d$. See [35, 36] for a precise discussion of the sharpness of (5), depending on the singular support of the kernel of $A$. According to (5), in the one-dimensional case, the assumption $m = 0$ is sufficient to get $L^p$-boundedness for any $p$, $1 < p < \infty$.

In [8] we studied the action of an operator $A$ as above on the spaces $\mathcal{F}L^p$ of temperate distributions whose Fourier transform is in $L^p$ (with the norm $\|f\|_{\mathcal{F}L^p} = \|\hat{f}\|_{L^p}$). There it was shown that $A$ is bounded as an operator $(\mathcal{F}L^p)_{comp} \rightarrow (\mathcal{F}L^p)_{loc}$, $1 \leq p \leq \infty$, if $m \leq -d \left| \frac{1}{2} - \frac{1}{p} \right|$. This is similar to (5), but with the difference of one unit in the dimension. Surprisingly, this threshold was shown to be sharp in any dimension $d \geq 1$, even for phases linear with respect to $\eta$; see [8] (or Section 6 below) for the construction of explicit counterexamples.

In the present paper we want to study the global boundedness of Fourier integral operators as in (1). Namely, we consider the case when the support of $\sigma(x, \eta)$ is not compact with respect to the space variable $x$. In this direction, general $L^2$-boundedness results can be found in [37]; to this paper we address for references on previous $L^2$-global results and for motivations, mainly concerning hyperbolic problems where global-in-space information is needed.

As a preliminary step of our study, we call attention on the following striking, but seemingly unknown, example. In dimension $d = 1$, consider

$$Af(x) = \int_{\mathbb{R}} e^{2\pi i x (\varphi(\eta) x \sigma(x, \eta) \hat{f}(\eta) d\eta),$$

where $\sigma \in S^0_{1,0}$, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism, with $\varphi(\eta) = \eta$ for $|\eta| \geq 1$ and whose restriction to $(-1, 1)$ is non-linear. This can be regarded as a pseudo-differential operator with symbol $e^{2\pi i x (\varphi(\eta) - \eta)} \sigma(x, \eta)$, which satisfies the estimates in (2), with $m = 0$, for $x$ in bounded subsets of $\mathbb{R}^d$. Hence it is bounded as an operator $L^p \rightarrow L^p_{loc}$, $1 < p < \infty$ ([42 page 250]). Naively, one may think that
the uniform bounds (2) for \( \sigma \), with \( m = 0 \), grant global \( L^p \)-boundedness as well. Instead we have:

**Theorem 1.1.** Let \( 2 < p < \infty \). Assume \( \sigma(x, \eta) = 1 \) in (6); then \( A \) is not bounded as an operator from \( L^p(\mathbb{R}) \) to \( L^p(\mathbb{R}) \). More precisely, fix \( \sigma(x, \eta) = \langle x \rangle^{\tilde{m}}, \tilde{m} \in \mathbb{R} \), in (6); then, \( A : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) is bounded if and only if

\[
\tilde{m} \leq -\left(\frac{1}{2} - \frac{1}{p}\right).
\]

Observe that, if \( \sigma(x, \eta) = 1 \), microlocally for \( |\eta| \geq 1 \) the operator \( A \) is the identity operator. Hence the behaviour of \( \Phi(x, \eta) \) in the strip (3) is now crucial and we could as well take

\[
\sigma(x, \eta) = \langle x \rangle^{\tilde{m}} G(\eta), \quad \text{with } G \in C^\infty_0(\mathbb{R}), \ G(\eta) = 1 \text{ for } |\eta| \leq 1,
\]

as symbol in (6), without changing the conclusions.

In the subsequent Proposition 6.1 we present similar examples in every dimension \( d \geq 1 \) and for every \( 1 \leq p \leq \infty \), obtaining the threshold

\[
\tilde{m} \leq -d \left|\frac{1}{2} - \frac{1}{p}\right,
\]

which is the same as that for local \( \mathcal{F}L^p \) spaces; see also Coriasco and Ruzhansky [14] for other examples in this connection.

Results of global \( L^p \)-boundedness, taking simultaneously account of (5) and (9), are given in the forthcoming paper [14].

The approach here will be different. Namely, inspired by our previous papers [7, 8], we replace \( L^p \) by other function spaces, the so-called modulation spaces \( M^p \), introduced by Feichtinger in [16], which will allow us to restore a symmetry between the thresholds (5) and (9). To be definite, let us first be precise about the class of FIOs we consider, and then recall the definition of \( M^p \).

**Global Fourier integral operators.** We will be concerned here with a class of FIOs (1) with phase \( \Phi \) and symbol \( \sigma \) chosen in the so-called SG classes. Namely, keeping locally the Hörmander’s estimates (2), we shall introduce a precise scale for the decay as \( x \to \infty \). The symbol \( \sigma \in C^\infty(\mathbb{R}^{2d}) \) is assumed to belong to the class \( \text{SG}^{m_1, m_2} \) (the so-called class of global symbols, or scattering symbols, of order \( (m_1, m_2) \)), i.e.

\[
|\partial^\alpha_{\eta} \partial^\beta_{x} \sigma(x, \eta)| \leq C_{\alpha, \beta} \langle \eta \rangle^{m_1-|\alpha|} \langle x \rangle^{m_2-|\beta|}, \quad \forall (x, \eta) \in \mathbb{R}^{2d},
\]

see, e.g., Cordes [9], Parenti [32], Melrose [30, 31], Schrohe [38], Schulze [39]. Note that the classes \( \text{SG}^{m_1, m_2} \) are stable under conjugation by Fourier transform, namely \( \mathcal{F}^{-1} \text{SG}^{m_1, m_2} \mathcal{F} = \text{SG}^{m_2, m_1} \). Corresponding FIOs were considered by Coriasco [11, 12, 13], Cappiello [3], Cordes [10], see also Ruzhansky-Sugimoto [37] and references therein. The phase function \( \Phi(x, \eta) \) is real-valued and in the class \( \text{SG}^{1,1} \). We
also assume the non-degeneracy condition (4). The operator $A$ in (6) is of this type, having symbol $\sigma \in \mathbf{SG}^{0,\tilde{m}}$ or, if $\sigma$ is as in (8), $\sigma \in \mathbf{SG}^{-\infty,\tilde{m}}$. Locally, the corresponding FIOs are of the type (2), with a somewhat more general phase function; in particular, for local $L^p$-boundedness the threshold (5) still holds true. Global $L^2$-boundedness follows from [11]; see also [37] for a more general class of FIOs. Finally, we would like to address to the recent monography of Cordes [10] for the role of $\mathbf{SG}$ pseudodifferential operators and FIOs in Dirac’s theory.

**Modulation spaces.** We briefly recall the definition of the modulation spaces $M^p$, $1 \leq p \leq \infty$, which are widely used in time-frequency analysis (see [16, 22] and Section 2 for definition and properties). In short, we say that a temperate distribution $f$ belongs to $M^p(\mathbb{R}^d)$ if its short-time Fourier transform $V_g f(x, \eta)$, defined in (13) below, is in $L^p(\mathbb{R}^{2d})$, namely if

\[
\|f\|_{M^p} := \|\|f(\cdot)g(\cdot - x)\|_{L^p}\|_{L^p_x} < \infty.
\]

Here $g$ is a non-zero (so-called window) function in $\mathcal{S}(\mathbb{R}^d)$, which in (11) is first translated and then multiplied by $f$ to localize $f$ near any point $x$. Changing $g \in \mathcal{S}(\mathbb{R}^d)$ produces equivalent norms. The space $\tilde{M}^\infty(\mathbb{R}^d)$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M^\infty$-norm. For heuristic purposes, distributions in $M^p$ may be regarded as functions which are locally in $\mathcal{F}L^p$ and decay at infinity like functions in $L^p$ (see Lemma 2.1 below for a precise statement). Among their properties, we highlight their stability under Fourier transform: $\mathcal{F}(M^p) = M^p$, $1 \leq p \leq \infty$ (and $\mathcal{F}(\tilde{M}^\infty) = \tilde{M}^\infty$).

We may now state our result.

**Theorem 1.2.** Let $\sigma \in \mathbf{SG}^{m_1, m_2}$ and $\Phi \in \mathbf{SG}^{1,1}$ satisfying (4). If

\[
m_1 \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|, \quad m_2 \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|,
\]

then the corresponding FIO $A$, initially defined on $\mathcal{S}(\mathbb{R}^d)$, extends to a bounded operator on $M^p$, whenever $1 \leq p < \infty$. For $p = \infty$, $A$ extends to a bounded operator on $\tilde{M}^\infty$.

Both the bounds in (12) are sharp. Namely, for any $m_1 > -d\left|\frac{1}{2} - \frac{1}{p}\right|$, or $m_2 > -d\left|\frac{1}{2} - \frac{1}{p}\right|$, there exists $A$ as in (11) with $\sigma \in \mathbf{SG}^{m_1, -\infty}$, $\sigma \in \mathbf{SG}^{-\infty, m_2}$, respectively, ($\sigma$ being compactly supported with respect to $x$ and $\eta$ respectively) which is not bounded on $M^p$.

Let us compare Theorem 1.2 with our preceeding results [7, 8]. In [7] we considered different Fourier integral operators, corresponding to operator solutions to Schrödinger equations, basic example of phase functions being quadratic forms in the $x, \eta$ variables. Such operators were proved to be bounded on $M^p$ without loss of derivatives, i.e., for symbols $\sigma(x, \eta)$ of order zero, see also [2, 4, 5]. In [8] we
considered local Hörmander's FIOs and proved that they are $M^p$ bounded with the sharp loss of regularity $-d|1/2 - 1/p|$, i.e. the same of that of operators acting on local $\mathcal{F}L^p$ spaces. This agrees with the loss for $m_1$ in Theorem 1.2. Moreover, in Theorem 1.2 a further loss of decay (that for $m_2$) appears, which agrees with that of the example in Theorem 1.1 for the action on global $L^p$ spaces. This circle of relationships is well understood by means of the heuristic interpretation, given above, of the modulation spaces. Also, we underline that the invariance under Fourier conjugation of the modulation spaces $M^p$ reveals them to be an appropriate functional framework for global FIOs (this is the insight the reader will catch from the proofs in the sequel).

Finally we observe that these results should extend to the more general class of global FIOs considered in [37]; we plan to devote a subsequent paper to this investigation.

The paper is organized as follows. In Section 2 the definitions and basic properties of the modulation spaces $M^p$ are recalled. Section 3 contains a review of SG FIOs and a boundedness result for a class of SG FIOs whose phases have bounded second derivatives (Proposition 3.4). In Section 4 we prove boundedness results on modulation spaces for SG pseudodifferential operators. Section 5 is devoted to the proof of Theorem 1.2. Finally, Section 6 exhibits the optimality of Theorem 1.2 and shows the negative results for operators acting on $L^p$ spaces, extending the example (6) in Theorem 1.1 above.

Notation. We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $C^\infty_0(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. Translation and modulation operators (time and frequency shifts) are defined, respectively, by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\eta f(t) = e^{2\pi i \eta t} f(t).$$

We have the formulas $(T_x f) \hat{=} M_{-x} \hat{f}$, $(M_\eta f) \hat{=} T_\eta \hat{f}$, and $M_\eta T_x = e^{2\pi i x \eta} T_x M_\eta$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ is $(f, g) = \int_{\mathbb{R}^d} f(t) g(t) \, dt$, and its extension to $\mathcal{S}' \times \mathcal{S}$ will be also denoted by $\langle \cdot, \cdot \rangle$.

Given a weight function $\mu$ defined on some lattice $\Lambda$, the spaces $\ell^p, q_\mu$ are the Banach spaces of sequences $\{a_{m,n}\}_{m,n} \in \Lambda$, such that

$$\|a_{m,n}\|_{\ell^p, q_\mu} := \left( \sum_n \left( \sum_m |a_{m,n}|^p \mu(m, n)^{p} \right)^{q/p} \right)^{1/q} < \infty$$

(with obvious changes when $p = \infty$ or $q = \infty$).
The notation $A \lesssim B$ means $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$, for some $c \geq 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the space $B_1$ into $B_2$.

2. Preliminary results on Time-Frequency methods

First we summarize some concepts and tools of time-frequency analysis, now available in textbooks [21, 22]. We also recall some results from [7, 8].

2.1. Modulation spaces. The short-time Fourier transform (STFT) of a distribution $f \in S'(\mathbb{R}^d)$ with respect to a non-zero window $g \in S(\mathbb{R}^d)$ is

$$V_g f(x, \eta) = \langle f, M_\eta T_x g \rangle = \int_{\mathbb{R}^d} f(t) g(t-x) e^{-2\pi i\eta t} \, dt. \quad (13)$$

The STFT $V_g f$ is defined on many pairs of Banach spaces. For instance, it maps $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $S(\mathbb{R}^d) \times S(\mathbb{R}^d)$ into $S(\mathbb{R}^{2d})$. Furthermore, it can be extended to a map from $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ into $S'(\mathbb{R}^{2d})$.

Recall the inversion formula for the STFT (see e.g. [22, Corollary 3.2.3]): if $\|g\|_{L^2} = 1$ and, for example, $u \in L^2(\mathbb{R}^d)$, it turns out

$$u = \int_{\mathbb{R}^{2d}} V_g u(y, \eta) M_\eta T_y g \, dy \, d\eta. \quad (14)$$

The modulation space norms are a measure of the joint time-frequency distribution of $f \in S'$. For their basic properties we refer, for instance, to [22, Ch. 11-13] and the original literature quoted there.

For the quantitative description of decay and regularity properties, we use weight functions on the time-frequency plane. In the sequel $v$ will always be a continuous, positive, even, submultiplicative weight function (in short, a submultiplicative weight), i.e., $v(0) = 1$, $v(z) = v(-z)$, and $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z, z_1, z_2 \in \mathbb{R}^{2d}$. Associated to every submultiplicative weight we consider the class of so-called $v$-moderate weights $\mathcal{M}_v$. A positive, even weight function $\mu \neq 0$ everywhere on $\mathbb{R}^{2d}$ belongs to $\mathcal{M}_v$ if it satisfies the condition

$$\mu(z_1 + z_2) \leq Cv(z_1)\mu(z_2) \quad \forall z_1, z_2 \in \mathbb{R}^{2d}.\$$

We note that this definition implies that $\frac{1}{v} \lesssim \mu \lesssim v$ and that $1/\mu \in \mathcal{M}_v$.

By abuse of notation, we denote product weights $v_{s_1, s_2}(x, \eta) = \langle x \rangle^{s_1} \langle \eta \rangle^{s_2}$, $s_1, s_2 \in \mathbb{R}$ (the indices’ order follows that of the $\text{SG}^{m_1, m_2}$-classes). Note that $v_{s_1, s_2}$ is submultiplicative only if $s_1, s_2 \geq 0$.

Given a non-zero window $g \in S(\mathbb{R}^d)$, a moderate weight $m \in \mathcal{M}_v$ and $1 \leq p, q \leq \infty$, the modulation space $M^p_q(\mathbb{R}^d)$ consists of all tempered distributions $f \in S'(\mathbb{R}^d)$
such that $V_g f \in L^{p,q}_\mu (\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M^{p,q}_\mu$ is
\begin{equation}
\| f \|_{M^{p,q}_\mu} = \| V_g f \|_{L^{p,q}_\mu} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \eta)|^p \mu(x, \eta)^p \, dx \right)^{q/p} \, d\eta \right)^{1/p},
\end{equation}
with obvious changes if $p = \infty$ or $q = \infty$. If $p = q$, we write $M^p_\mu$ instead of $M^{p,p}_\mu$, and if $\mu(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^p_\mu$ and $M^p$ for $M^{p,q}_\mu$ and $M^{p,p}_\mu$.

Then $M^{p,q}_\mu (\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window $g$. Moreover, if $\mu \in \mathcal{M}_v$ and $g \in M^1_v \setminus \{0\}$, then $\| V_g f \|_{L^{p,q}_\mu}$ is an equivalent norm for $M^{p,q}_\mu (\mathbb{R}^d)$ (see [22 Thm. 11.3.7]). Roughly speaking, a weight in $\eta$ regulates the smoothness of $f \in M^{p,q}_\mu$, whereas a weight in $\mu$ regulates the decay at infinity.

Denote by $\tilde{M}^{p,q}_\mu$ the closure of the Schwartz class in $M^{p,q}_\mu$. We have $\tilde{M}^{p,q}_\mu = M^{p,q}_\mu$ if $p < \infty$ and $q < \infty$ and the duality property for modulation spaces can be stated as follows: if $1 \leq p, q \leq \infty$ and $p', q'$ are the conjugate exponents, then $(\tilde{M}^{p,q}_\mu)^* = \tilde{M}^{p',q'}_\mu$.

For simplicity, we shall write $M^{p,q}_{s_1,s_2}$ for $M^{p,q}_{s_1} \cap M^{p,q}_{s_2}$ and, similarly, for the spaces $\tilde{M}^{p,q}_{s_1,s_2}$.

We also recall from [20] the following useful interpolation relations:
\begin{equation}
(\tilde{M}^{p_1}_{s_1,s_1}, \tilde{M}^{p_2}_{s_2,s_2})_{\theta} = \tilde{M}^{p}_{s,s},
\end{equation}
where $s = (1 - \theta)s_1 + \theta s_2$, $s = (1 - \theta)s_1 + \theta s_2$.

For tempered distributions compactly supported either in time or in frequency, the $M^{p,q}$-norm is equivalent to the $\mathcal{F}L^q$-norm or $L^p$-norm, respectively. This result is well-known ([17, 18]). See also [28] and [6] for a proof.

**Lemma 2.1.** Let $1 \leq p, q \leq \infty$.

(i) For every $u \in \mathcal{S}'(\mathbb{R}^d)$, supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \Leftrightarrow u \in \mathcal{F}L^q$, and
\begin{equation}
C_K^{-1} \| u \|_{M^{p,q}} \leq \| u \|_{\mathcal{F}L^q} \leq C_K \| u \|_{M^{p,q}},
\end{equation}
where $C_K > 0$ depends only on $K$.

(ii) For every $u \in \mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \Leftrightarrow u \in L^p$, and
\begin{equation}
C_K^{-1} \| u \|_{M^{p,q}} \leq \| u \|_{L^p} \leq C_K \| u \|_{M^{p,q}},
\end{equation}
where $C_K > 0$ depends only on $K$. 


In order to state the dilation properties for modulation spaces, we introduce the indices:

\[ \mu_1(p) = \begin{cases} 
-1/p' & \text{if } 1 \leq p \leq 2, \\
-1/p & \text{if } 2 \leq p \leq \infty,
\end{cases} \]

and

\[ \mu_2(p) = \begin{cases} 
-1/p & \text{if } 1 \leq p \leq 2, \\
-1/p' & \text{if } 2 \leq p \leq \infty,
\end{cases} \]

For \( \lambda > 0 \), we define the dilation operator \( U_\lambda f(x) = f(\lambda x) \). Then, the dilation properties of \( M^p \) are as follows (see [43, Theorem 3.1]).

**Theorem 2.1.** We have:

1. \( \|U_\lambda f\|_{M^p} \lesssim \lambda^d \mu_1(p) \|f\|_{M^p}, \quad \forall f \in M^p(\mathbb{R}^d). \)

2. \( \|U_\lambda f\|_{M^p} \lesssim \lambda^d \mu_2(p) \|f\|_{M^p}, \quad \forall f \in M^p(\mathbb{R}^d). \)

These dilation estimates are sharp, as discussed in [43], see also [5].

We also need the following result.

**Lemma 2.2.** Let \( \chi \) be a smooth function supported where \( B_0^{-1} \leq |\eta| \leq B_0 \), for some \( B_0 > 0 \).

(a) For every \( u \in \mathcal{S}(\mathbb{R}^d) \),

\[ \sum_{j=1}^{\infty} \|\chi(2^{-j}D)u\|_{M^1} \lesssim \|u\|_{M^1}, \]

where \( \chi(2^{-j}D)u = \mathcal{F}^{-1}[\chi(2^{-j} \cdot) \hat{u}] \).

(b) For every \( u \in \mathcal{S}(\mathbb{R}^d) \),

\[ \sum_{j=1}^{\infty} \|\chi(2^{-j} \cdot)u\|_{M^1} \lesssim \|u\|_{M^1}. \]

**Proof.** Part (a) was proved in [5, Lemma 5.1], whereas part (b) follows from (a), since the Fourier transform defines an automorphism of any \( M^p \).

Finally we recall the following result.

**Lemma 2.3.** (a) For \( k \geq 0 \), let \( f_k \in \mathcal{S}(\mathbb{R}^d) \) satisfy \( \text{supp} \hat{f}_0 \subset B_2(0) \) and

\[ \text{supp} \hat{f}_k \subset \{ \eta \in \mathbb{R}^d : 2^{k-1} \leq |\eta| \leq 2^{k+1} \}, \quad k \geq 1. \]
Then, if the sequence $f_k$ is bounded in $M^\infty(\mathbb{R}^d)$, the series $\sum_{k=0}^\infty f_k$ converges in $M^\infty(\mathbb{R}^d)$ and

$$\|\sum_{k=0}^\infty f_k\|_{M^\infty} \lesssim \sup_{k \geq 0} \|f_k\|_{M^\infty}. \quad (18)$$

(b) For $k \geq 0$, let $f_k \in S(\mathbb{R}^d)$ satisfy $\text{supp } f_0 \subset B_2(0)$ and

$$\text{supp } f_k \subset \{x \in \mathbb{R}^d : 2^k - 1 \leq |x| \leq 2^k + 1\}, \quad k \geq 1. \quad \text{Then, if the sequence } f_k \text{ is bounded in } M^\infty(\mathbb{R}^d), \text{ the series } \sum_{k=0}^\infty f_k \text{ converges in } M^\infty(\mathbb{R}^d) \text{ and } (18) \text{ holds true.}$$

Proof. Part (a) was proved in [8, Lemma 5.2], whereas part (b) follows from (a), again since the Fourier transform defines an automorphism of any $M^p$. \[ \square \]

2.2. Gabor frames. Fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$, for $\alpha, \beta > 0$. For $(k, n) \in \Lambda$, define $g_{k,n} := M_n T_k g$. The set of time-frequency shifts $\mathcal{G}(g, \alpha, \beta) = \{g_{k,n}, (k, n) \in \Lambda\}$ is called Gabor system. Associated to $\mathcal{G}(g, \alpha, \beta)$ we define the coefficient operator $C_g$, which maps functions to sequences as follows:

$$\begin{align*}
(C_g f)_{k,n} &= (C_g^{\alpha, \beta} f)_{k,n} := \langle f, g_{k,n} \rangle, \quad (k, n) \in \Lambda, \\
\text{the synthesis operator } &D_{g,c} = D_{g}^{\alpha, \beta} c = \sum_{(k,n) \in \Lambda} c_{k,n} M_n T_k g, \quad c = \{c_{k,n}\}_{(k,n) \in \Lambda}
\end{align*}$$

and the Gabor frame operator

$$\begin{align*}
S_g f &= S_{g}^{\alpha, \beta} f := D_g S_g f = \sum_{(k,n) \in \Lambda} \langle f, g_{k,n} \rangle g_{k,n}.
\end{align*} \quad (20)$$

The set $\mathcal{G}(g, \alpha, \beta)$ is called a Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$ if $S_g$ is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. Equivalently, $C_g$ is bounded from $L^2(\mathbb{R}^d)$ to $\ell^2(\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d)$ with closed range, i.e., $\|f\|_{L^2} \asymp \|C_g f\|_{\ell^2}$. If $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then the so-called dual window $\gamma = S_{g}^{-1} g$ is well-defined and the set $\mathcal{G}(\gamma, \alpha, \beta)$ is a frame (the so-called canonical dual frame of $\mathcal{G}(g, \alpha, \beta)$). Every $f \in L^2(\mathbb{R}^d)$ possesses the frame expansion

$$f = \sum_{(k,n) \in \Lambda} \langle f, g_{k,n} \rangle \gamma_{k,n} = \sum_{(k,n) \in \Lambda} \langle f, \gamma_{k,n} \rangle g_{k,n} \quad \text{(21)}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$, and norm equivalence:

$$\|f\|_{L^2} \asymp \|C_g f\|_{\ell^2} \asymp \|C_{\gamma} f\|_{\ell^2}. \quad \text{(18)}$$
This result is contained in [22, Proposition 5.2.1]. In particular, if \( \gamma = g \) and \( \|g\|_{L^2} = 1 \) the frame is called \textit{normalized tight} Gabor frame and the expansion (21) reduces to

\[
(22) \quad f = \sum_{(k,n) \in \Lambda} \langle f, g_{k,n} \rangle g_{k,n}.
\]

If we ask for more regularity on the window \( g \), then the previous result can be extended to suitable Banach spaces, as shown below [19, 23].

**Theorem 2.2.** Let \( \mu \in \mathcal{M}_v \), \( \mathcal{G}(g, \alpha, \beta) \) be a normalized tight Gabor frame for \( L^2(\mathbb{R}^d) \), with lattice \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \), and \( g \in M^1_1 \). Define \( \tilde{\mu} = \mu|_{\Lambda} \).

(i) For every \( 1 \leq p, q \leq \infty \), \( C_g : M^p_q \rightarrow \ell^{p,q}_\mu \) and \( D_g : \ell^{p,q}_\tilde{\mu} \rightarrow M^p_q \) continuously and, if \( f \in M^p_q \), then the Gabor expansions (22) converge unconditionally in \( M^p_q \) for \( 1 \leq p, q < \infty \) and all weight \( \mu \), and weak*-\( M^\infty_q \) unconditionally if \( p = \infty \) or \( q = \infty \).

(ii) The following norms are equivalent on \( M^p_q \):

\[
(23) \quad \|f\|_{M^p_q} \asymp \|C_g f\|_{\ell^{p,q}_\mu}.
\]

We also establish the following property ([17, Theorem 2.3]). Denote by \( \tilde{\ell}^{p,q}_\mu \) the closure of the space of eventually zero sequences in \( \ell^{p,q}_\mu \). Hence \( \tilde{\ell}^{p,q}_\mu = \ell^{p,q}_\mu \) if \( p < \infty \) and \( q < \infty \).

**Theorem 2.3.** Under the assumptions of Theorem 2.2, for every \( 1 \leq p, q \leq \infty \) the operator \( C_g \) is continuous from \( \tilde{M}^p_q \) into \( \tilde{\ell}^{p,q}_\mu \), whereas the operator \( D_g \) is continuous from \( \tilde{\ell}^{p,q}_\mu \) into \( \tilde{M}^p_q \).

3. **Preliminary results on FIOs**

The general theory of SG FIOs was developed by Coriasco in [11], see also [10, 12, 13]. In this section we recall the main properties needed in the sequel. We also present a boundedness result, in the spirit of [7, 8], for FIOs with phases having bounded derivatives of order \( \geq 2 \).

3.1. **SG Fourier integral operators.** First of all we observe that the calculus for SG FIOs was developed in [11] for phases \( \Phi \in \text{SG}^{1,1} \) satisfying the growth condition

\[
(24) \quad \langle \nabla_x \Phi(x, \eta) \rangle \gtrsim \langle \eta \rangle, \quad \langle \nabla_\eta \Phi(x, \eta) \rangle \gtrsim \langle x \rangle.
\]

These conditions are a consequence of our hypotheses, namely \( \Phi \in \text{SG}^{1,1} \) and (4). More precisely, it follows from the estimates from the mixed second derivatives, namely

\[
(25) \quad |\partial^\alpha_\eta \partial^\beta_x \Phi(x, \eta)| \leq C_{\alpha, \beta}, \quad |\alpha| = |\beta| = 1, \quad \forall (x, \eta) \in \mathbb{R}^{2d}.
\]
combined with [4] and the formula for the Jacobian of the inverse function, that Hadamard’s global inverse function theorem ([29][Theorem 6.2.4]) applies to the maps \( x \mapsto \nabla_\eta \Phi(x, \eta) \) and \( \eta \mapsto \nabla_x \Phi(x, \eta) \), which are therefore globally invertible. Moreover these maps have bounded Jacobians uniformly with respect to \( \eta \) and \( x \) respectively, so that they are globally Lipschitz continuous, uniformly with respect to \( \eta \) and \( x \) respectively. The same holds for their inverses, which proves (24).

The first important results is the following formula for the composition of a SG pseudodifferential operator, namely an operator of the form

(26) \[ p(x, D)u = \int e^{2\pi i x \eta} p(x, \eta) \hat{f}(\eta) \, d\eta, \]

with a symbol \( p \in \text{SG}^{t_1, t_2} \), and a SG FIO \( A = A_{\Phi, \sigma} \) as in [1] (Theorem 7); for the case of Hörmander’s symbol classes see [25, 27, 45].

First we recall that a regularizing operator is a pseudodifferential operator \( R = r(x, D) \) with symbol \( r(x, \eta) \) in the Schwartz space \( \mathcal{S}(\mathbb{R}^{2d}) \) (equivalently, an operator with kernel in \( \mathcal{S}'(\mathbb{R}^{2d}) \), which maps \( \mathcal{S}'(\mathbb{R}^{d}) \) into \( \mathcal{S}(\mathbb{R}^{d}) \)).

**Theorem 3.1.** Let the symbol \( \sigma \) and the phase \( \Phi \) satisfy the assumptions in the Introduction. Let \( p(x, \eta) \) be a symbol in \( \text{SG}^{t_1, t_2} \). Then,

\[ p(x, D)A = S + R, \]

where \( S \) is a FIO with the same phase \( \Phi \) and symbols \( s(x, \eta) \) in the class \( \text{SG}^{m_1 + t_1, m_2 + t_2} \), satisfying

\[ \text{supp } s \subset \text{supp } \sigma \cap \{(x, \eta) \in \mathbb{R}^{2d} : (x, \nabla_x \Phi(x, \eta)) \in \text{supp } p\}, \]

and \( R \) is a regularizing operator.

Moreover, the symbol estimates satisfied by \( s \) and the seminorm estimates of \( r \) in the Schwartz space are uniform when \( \sigma \) and \( p \) vary in bounded subsets of \( \text{SG}^{m_1, m_2} \) and \( \text{SG}^{t_1, t_2} \) respectively.

Also, the symbol \( s \) has the following asymptotic expansion:

(27) \[ s(x, \eta) \sim \sum_{\alpha \in \mathbb{Z}^d_+} \frac{1}{\alpha!} \partial^\alpha \psi(x, \eta) \partial_y |_{y = x}, \]

where

\[ \psi(x, y, \eta) = \Phi(y, \eta) - \Phi(x, \eta) - (y - x, \nabla_x \Phi(x, \eta)), \]

and, as usual, \( D^\alpha_y = (-i)^{|\alpha|} \partial^\alpha_y \).

The meaning of the above asymptotic expansion is that the difference between \( s(x, \eta) \) and the partial sum over \( |\alpha| < N \) is a symbol in \( \text{SG}^{m_1 + t_1 - N, m_2 + t_2 - N} \). However, in the next sections only the first part of the statement will be used.
Similarly, one also has the following formula for the composition in the reverse order ([11, Theorem 8]). To state it, we introduce the notation $b(x, \eta) := b(\eta, x)$ for a function $b(x, \eta)$ in $\mathbb{R}^{2d}$.

**Theorem 3.2.** Let the symbol $\sigma$ and the phase $\Phi$ satisfy the assumptions in the Introduction. Let $p(x, \eta)$ be a symbol in $SG^{t_1, t_2}$. Then,

$$Ap(x, D) = S + R,$$

where $S$ is a FIO with the same phase $\Phi$ and symbols $s(x, \eta)$ in the class $SG^{m_1 + t_1, m_2 + t_2}$, satisfying

$$\text{supp } s \subset \text{supp } \sigma \cap \{(x, \eta) \in \mathbb{R}^{2d} : (\nabla_\eta \Phi(x, \eta), \eta) \in \text{supp } p\},$$

and $R$ is a regularizing operator.

Moreover, the symbol estimates satisfied by $s$ and the seminorm estimates of $r$ in the Schwartz space are uniform when $\sigma$ and $p$ vary in bounded subsets of $SG^{m_1, m_2}$ and $SG^{t_1, t_2}$ respectively.

Also, the transpose symbol $t's(x, \eta)$ admits the asymptotic expansion in (27), with $p, \sigma$ and $\Phi$ replaced by $t'p, t'\sigma$ and $t'\Phi$ respectively.

This latter result can be proved combining Theorem 3.1 with the following nice formula for the transpose of $A = A_{\Phi, \sigma}$ with respect to the pairing which extends the integral $(u, v) \mapsto \int uv$ ([11, Proposition 9]):

$$tA_{\Phi, \sigma} = \mathcal{F} \circ A_{t'\Phi, t'\sigma} \circ \mathcal{F}^{-1}.$$

Similarly, it is easily verified that the $L^2$ formal adjoint of the FIO $A_{\Phi, \sigma}$ is the operator defined by

$$\widehat{B}f(\eta) = B_{\Phi, \sigma}f(\eta) = \int e^{-2\pi i \Phi(x, \eta)}\overline{\sigma(x, \eta)}f(x) \, dx.$$

namely,

$$A_{\Phi, \sigma}^* = B_{\Phi, \sigma}.$$

In the sequel the operators of the type (29) will be called “type II FIOs”. In contrast, operators of the type (11) will be called “type I FIOs”, or simply “FIOs”.

Another important result that will be used in the sequel is the following one ([11, Theorem 16]).

**Theorem 3.3.** Let $\sigma \in SG^{0, 0}$ and $\Phi$ satisfying the assumptions in the Introduction. Then the corresponding FIO $A_{\Phi, \sigma}$, initially defined on $S(\mathbb{R}^d)$, extends to a bounded operator on $L^2(\mathbb{R}^d)$.

The proof relies on the fact (cf. [24]) that the composition $A^*A$ is a pseudodifferential operator with symbol in $SG^{0, 0}$; therefore it is continuous on $L^2(\mathbb{R}^d)$ (e.g., by [25, Theorem 18.1.11]). So $A$ is. This result was generalized in [37] to FIOs with phases satisfying weaker symbol estimates.
3.2. FIOs with phases having bounded derivatives of order $\geq 2$. We present here a boundedness result of a class of FIOs whose phases have bounded second derivatives (cf. [7, Theorem 4.1] and [8, Proposition 3.3]).

**Proposition 3.4.** Consider a symbol $\sigma \in \mathcal{C}^{\infty}(\mathbb{R}^{2d})$ satisfying the estimates

\[(31) \quad |\partial_{\eta}^{\alpha} \partial_{x}^{\beta} \sigma(x, \eta)| \leq C_{\alpha, \beta}, \quad \forall (x, \eta) \in \mathbb{R}^{2d},\]

and a phase $\Phi \in \mathcal{C}^{\infty}(\mathbb{R}^{2d})$, satisfying

\[(32) \quad |\partial_{\eta}^{\alpha} \partial_{x}^{\beta} \Phi(x, \eta)| \leq C_{\alpha, \beta} \quad \text{for} \quad |\alpha| + |\beta| \geq 2,
\]

for $(x, \eta)$ in an $\epsilon$-neighbourhood of the support of $\sigma$, as well as

\[(33) \quad |\partial_{\eta}^{\alpha} \partial_{x}^{\beta} \Phi(x, \eta)| \leq C_{\alpha, \beta}, \quad |\alpha| = |\beta| = 1, \quad \forall (x, \eta) \in \mathbb{R}^{2d},\]

and

\[(34) \quad \left| \det \left( \frac{\partial^{2} \Phi}{\partial x_{i} \partial \eta_{l}}(x, \eta) \right) \right| > \delta > 0, \quad \forall (x, \eta) \in \mathbb{R}^{2d}.\]

Then, for every $1 \leq p \leq \infty$ it turns out

$$\|Au\|_{M^{p}} \leq C\|u\|_{M^{p}}, \quad \forall u \in \mathcal{S}(\mathbb{R}^{d}),$$

where the constant $C$ depends only on $\delta, \epsilon$ and upper bounds for a finite number of the constants in (31), (32) and (33).

**Proof.** This is a variant of [7 Theorems 3.1, 4.1] and [8 Propositions 3.2, 3.3]. For the sake of completeness we outline the proof.

Let $g, \gamma \in \mathcal{S}(\mathbb{R}^{d}), \|g\|_{L^{2}} = \|\gamma\|_{L^{2}} = 1$, with supp $\gamma \subset B_{\epsilon/4}(0)$, supp $\hat{g} \subset B_{\epsilon/4}(0)$. Let $u \in \mathcal{S}(\mathbb{R}^{d})$. The inversion formula (14) for the STFT gives

$$V_{\gamma}(Au)(y', \omega') = \int_{\mathbb{R}^{2d}} \langle A(M_{\omega}T_{y}g), M_{\omega'}T_{y'}\gamma \rangle V_{g}u(y, \omega) dy d\omega.$$

Hence, it suffices to prove that the map $K_{A}$ defined by

$$K_{A}G(y', \omega') = \int_{\mathbb{R}^{2d}} \langle A(M_{\omega}T_{y}g), M_{\omega'}T_{y'}\gamma \rangle G(y, \omega) dy d\omega$$

is continuous on $L^{p}(\mathbb{R}^{2d})$. By Schur’s test (see e.g. [22 Lemma 6.2.1]) we are reduced to proving that its integral kernel

$$K_{A}(y', \omega'; y, \omega) = \langle A(M_{\omega}T_{y}g), M_{\omega'}T_{y'}\gamma \rangle$$

satisfies

\[(35) \quad K_{A} \in L^{\infty}_{y', \omega'}(L^{1}_{y, \omega}),\]

and

\[(36) \quad K_{A} \in L^{\infty}_{y, \omega}(L^{1}_{y', \omega'}).\]
Now, in view of the hypothesis (31) and (32) we can apply [8 Proposition 3.2], that tells us that for every $N \geq 0$, there exists a constant $C > 0$ such that
\[
|\langle A(M_uT_y g), M_{\omega'}T_y\gamma \rangle| \leq C|\nabla_x \Phi(y', \omega') - \omega'|^{-N}|\nabla_\eta \Phi(y', \omega) - y|^{-N}.
\]
The constant $C$ only depends on $N$, $g$, $\gamma$, and on a finite number of constants in (31) and (32). For $N > d$, $\int_{\mathbb{R}^d} |\nabla_x \Phi(y', \omega) - y|^{-N} = \int |y|^{-N} \, dy < \infty$, hence (35) will be proved if we verify that there exists a constant $C' > 0$ such that
\[
\int_{\mathbb{R}^d} |\nabla_x \Phi(y', \omega) - \omega'|^{-N} \, d\omega \leq C', \quad \forall (y', \omega') \in \mathbb{R}^d \times \mathbb{R}^d.
\]
To this end, we perform the change of variable $\mathbb{R}^d \ni \omega \mapsto \nabla_x \Phi(y', \omega)$ which is a global diffeomorphism of $\mathbb{R}^d$ in view (33) and (34). Moreover the Jacobian determinant of its inverse is uniformly bounded with respect to $y'$ (see the discussion at the beginning of the present section). Hence, the last integral is, for $N > d$,
\[
\lesssim \int_{\mathbb{R}^d} |\omega - \omega'|^{-N} \, d\omega = C'.
\]
The proof of (36) is analogous and left to the reader.

Finally, the uniformity of the norm of $A$ as a bounded operator, established in the last part of the statement, follows from the proof itself. \(\square\)

4. **Sufficient Conditions for Boundedness of Pseudodifferential Operators**

Here we study the boundedness on modulation spaces of pseudodifferential operators, namely operators of the form (28) above, for some symbol classes. First we consider the case of symbols in $SG^{m_1, m_2}$.

We observe that the full pseudodifferential calculus is available for these operators. Indeed, it is a special case of the calculus for general Hörmander’s classes $S(m, g)$ associated with a weight $m$ and a metric $g$ ([25, Chapter XVIII]). Here $m(x, \eta) = \langle x \rangle^{m_2} \langle \eta \rangle^{m_1}$ and $g_{x, \eta}(z, \zeta) = \langle x \rangle^{-2} |dz|^2 + \langle \eta \rangle^{-2} |d\zeta|^2$. In particular the composition of two pseudodifferential operators with symbols in $SG^{m_1, m_2}$ and $SG^{\tilde{m}_1, \tilde{m}_2}$ is a pseudodifferential operators with symbol in $SG^{m_1 + \tilde{m}_1, m_2 + \tilde{m}_2}$.

Now, to show that such an operator, with symbol in $SG^{m_1, m_2}$, extends to a bounded operator $\tilde{M}_{s_1, s_2} \rightarrow \tilde{M}_{p,q}^{\tilde{m}_1 - m_1, \tilde{m}_2 - m_2}$, for every $s_1, s_2, m_1, m_2 \in \mathbb{R}$, $1 \leq p, q \leq \infty$. This was proved in [44 Corollary 4.7] when $m_1 = m_2 = 0$. When, in addition, $s_1 = s_2 = 0$ (the unweighted case) this result is contained in [22, Theorem 14.5.2]. Our claim follows from this latter special case by arguing as follows.

First observe that, for every $s_1, s_2 \in \mathbb{R}$, the SG pseudodifferential operator $\Lambda_{s_1, s_2} = \langle x \rangle^{s_2} \langle D \rangle^{s_1}$, of order $(s_1, s_2)$, is bounded (in fact it defines an isomorphism) from $\mathcal{M}_{p, s_1}^{s_2}$ to $\mathcal{M}_{p}^{s_1}$ [44 Theorem 2.4]. Moreover, $\Lambda_{s_1, s_2}^{-1} u = \langle D \rangle^{-s_1} \langle x \rangle^{-s_2} u$ is a SG
pseudodifferential operator of order \((-s_1, -s_2)\). By the above quoted composition formula,
\[ A = \Lambda_{m_1-s_1,m_2-s_2}A'\Lambda_{s_1,s_2}, \]
for a suitable SG pseudodifferential operator \(A'\) of order \((0, 0)\). As we already observed, \(A'\) is bounded \(\hat{M}^{p,q} \to \hat{M}^{p,q}\) by [22] Theorem 14.5.2], which gives the claim.

We now will show a more general continuity result, for rougher symbols on \(\mathbb{R}^{2d}\) satisfying estimates of the type
\[ |\partial_\eta^\alpha \partial_x^\beta \sigma(x, \eta)| \leq C_{\alpha, \beta} \langle \eta \rangle^{m_1} \langle x \rangle^{m_2}, \quad |\alpha| \leq 2N_2, \quad |\beta| \leq 2N_1, \]
with \(\partial_\eta^\alpha \partial_x^\beta\) standing for distributional derivatives. For \(s_1, s_2 \geq 0\), we recall the definition \(v_{s_1,s_2}(x, \eta) = \langle x \rangle^{s_2} \langle \eta \rangle^{s_1}\). Our result reads as follows.

**Theorem 4.1.** For \(s_1, s_2 \geq 0\), let \(\mu \in \mathcal{M}_{v_{s_1,s_2}}\).

(a) Consider a symbol \(\sigma\) satisfying (37), with \(N_1 > (d + s_1 + |m_1|)/2\), \(N_2 > (d + s_2)/2\). Then, for every \(1 \leq p, q \leq \infty\), \(\sigma(x, D)\) extends to a continuous operator from \(\hat{M}^{p,q}_{\mu}\) to \(\hat{M}^{p,q}_{\mu-m_1,-m_2}\).

(b) Consider a symbol \(\sigma\) satisfying (37), with \(N_1 > (d + s_1)/2\), \(N_2 > (d + s_2 + |m_2|)/2\). Then, for every \(1 \leq p, q \leq \infty\), \(\sigma(x, D)\) extends to a continuous operator from \(\hat{M}^{p,q}_{\mu-m_1,m_2}\) to \(\hat{M}^{p,q}_{\mu}\).

To chase our goal, we first show an approximate diagonalization of \(\sigma(x, D)\) by Gabor frames. In the sequel, we consider a Gabor frame \([g_{k,n}]_{k,n}, (k, n) \in \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d\), with window \(g \in \mathcal{S}(\mathbb{R}^d)\). A small variant of [7] Theorem 3.1, Remark 3.2] (see also [34]) provides the following almost diagonalization.

**Theorem 4.2.** Consider a symbol \(\sigma\) satisfying (37). Then there exists \(C_{N_1,N_2} > 0\) such that
\[ |\langle \sigma(x, D)g_{k,n}, g_{k',n'} \rangle| \leq C_{N_1,N_2} \langle n \rangle^{m_1} \langle k' \rangle^{m_2} \langle n - n' \rangle^{2N_1} \langle k - k' \rangle^{-2N_2}. \]

**Proof.** The proof essentially repeats that of [7] Theorem 3.1, Remark 3.2]. Since that results was actually established for more general classes of FIOs, for the convenience of the reader outline the main ideas. An explicit computation shows that
\[ |\langle \sigma(x, D)(M_n T_k g), M_{n'} T_{k'} \gamma \rangle| \]
\[ = | \int e^{2\pi i (x(n - n') - \eta(k - k'))} [e^{2\pi i \eta n} \sigma(x + k', \eta + n)] \tilde{\gamma}(x) \hat{g}(\eta) \, dx \, d\eta| \]

Then one uses the identity
\[ (1 - \Delta_x)^{N_1} (1 - \Delta_\eta)^{N_2} e^{2\pi i [x(n - n') - \eta(k - k')]} \]
\[ = \langle 2\pi(k - k') \rangle^{2N_2} \langle 2\pi(n - n') \rangle^{2N_1} e^{2\pi i [x(n - n') - \eta(k - k')]}, \]
and integrates by parts. Since \( g \in \mathcal{S} \), the estimates (37) combined with Petree’s inequality \( \langle z + w \rangle^s \leq \langle z \rangle^s \langle w \rangle^s \) give (38).

The proof of the boundedness property of \( \sigma(x, D) \) makes use of the following generalization of the Schur Test, contained in [7, Proposition 5.1].

**Proposition 4.3.** Consider an operator defined on sequences on the lattice \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) by

\[
(Kc)_{m',n'} = \sum_{m,n} K_{m',n',m,n} c_{m,n}.
\]

Assume

\[
\{K_{m',n',m,n} \} \in \ell^\infty_n \ell^1_m \ell^1_{m'} \cap \ell^\infty_{n'} \ell^1_{m} \ell^1_{m'} \quad \text{and} \quad \{K_{m',n',m,n} \} \in \ell^\infty_{m'} \ell^1_{m,n} \cap \ell^\infty_{m,n} \ell^1_{m',n'}.
\]

Then, for every \( 1 \leq p, q \leq \infty \), the operator \( K \) is continuous on \( \ell^{p,q} \) and on \( \tilde{\ell}^{p,q} \) (recall that \( \tilde{\ell}^{p,q} \) is the closure of the space of eventually zero sequences in \( \ell^{p,q} \)).

**Proof of Theorem 4.1** (a) Consider a normalized tight frame \( \mathcal{G}(g, \alpha, \beta) \) with \( g \in \mathcal{S}(\mathbb{R}^d) \). In view of Theorem 2.3 showing the boundedness of \( \sigma(x, D) \) from \( M^{p,q}_{\mu} \) to \( M^{p,q}_{\mu_{\mathbb{Z}^d}} \) is equivalent to proving the boundedness of the infinite matrix

\[
K_{k',n',k,n} = \langle \sigma(x, D)g_{k,n}, g_{k',n'} \rangle \frac{\mu(k', n')}{\langle k' \rangle^{m_2} \langle n' \rangle^{m_1} \mu(k, n)}
\]

from \( \tilde{\ell}^{p,q} \) into itself. We make use of the Schur Test above (Proposition 4.3). The estimate (38) and the assumption \( \mu \in \mathcal{M}_{s_1, s_2} \) combined with Petree’s inequality yield

\[
|K_{k',n',k,n}| \lesssim \langle n - n' \rangle^{s_1 + |m_1| - 2N_1} \langle k - k' \rangle^{s_2 - 2N_2},
\]

so that

\[
\sup_{k',n' \in \mathbb{Z}^d} \sum_{k,n \in \mathbb{Z}^d} |K_{k',n',k,n}| < \infty,
\]

because of the choice of \( N_1, N_2 \). Analogously, one obtains \( \{K_{m',n',m,n} \} \in \ell^\infty_{k,n} \ell^1_{k',n'} \). Similarly one obtains the estimate

\[
\sup_{n \in \mathbb{Z}^d} \sum_{n' \in \mathbb{Z}^d} \sup_{k' \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |K_{k',n',k,n}| \lesssim \sup_{k' \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \langle k - k' \rangle^{s_2 - 2N_2} \sup_{n \in \mathbb{Z}^d} \sup_{n' \in \mathbb{Z}^d} \langle n - n' \rangle^{s_1 + |m_1| - 2N_1} < \infty,
\]

that is, \( \{K_{m',n',m,n} \} \in \ell^\infty_{n'} \ell^1_{k'} \ell^1_{k} \ell^1_{n} \) and also \( \{K_{m',n',m,n} \} \in \ell^\infty_{n'} \ell^1_{n} \ell^1_{k'} \ell^1_{k} \), as desired.

The proof of part (b) is very similar and left to the reader.
Remark 4.4. The formula (37) is not symmetric with respect to the space variables and the dual variables. This is due to the fact that we are using the so called “left” or Kohn-Nirenberg quantization (20). Instead, the Weyl quantization (21 23):

\[ \sigma_w(x, D)u = \int \int e^{2\pi i x \cdot y/2} \sigma \left( \frac{x + y}{2}, \eta \right) f(y) \, dy \, d\eta, \]

combined with properties of the cross-Wigner distribution as in 34, yields

\[ |\langle \sigma_w(x, D)g_{k,n}, g_{k',m'} \rangle| \leq C_{N_1,N_2} \frac{(n + n')^{m_1}}{(n - n')^{2N_1}} \frac{k + k'}{2^{N_2}}. \]

Theorem 4.4 for Weyl operators reads as follows: Let \( s_1, s_2 \geq 0, \mu \in \mathcal{M}_{v_{s_1,s_2}}, \sigma \) be a symbol satisfying (37), with \( N_i > (d + s_i + |m_i|)/2, i = 1, 2 \). Then, for every \( 1 \leq p, q \leq \infty, \sigma(x, D) \) extends to a continuous operator from \( \tilde{M}^p_{\mu} \) to \( \tilde{M}^p_{\mu} \) and from \( \tilde{M}^p_{\mu} \) to \( \tilde{M}^p_{\mu} \).

5. Proof of Theorem 1.2

The claim of Theorem 1.2 will follow if we prove the boundedness of every SG FIO \( A \) of order \( (m_1, m_2) = (-d/2, -d/2) \) on the endpoint cases \( M^1 \) and \( M^\infty \). Indeed, since it is known from Theorem 3.3 that a FIO of order \((0, 0)\) is bounded on \( L^2 = M^2 \), the desired continuity result on \( M^p \), when \( m_1 = m_2 = -d(1/2 - 1/p) \), \( 1 < p < \infty \), is due to complex interpolation, detailed below.

As already observed, for every \( s_1, s_2 \in \mathbb{R} \), the operator \( \Lambda_{s_1,s_2}u = \langle D \rangle^{s_1}((x)^{s_2}u) \) defines an isomorphism from \( \tilde{M}^p_{s_1,s_2} \) to \( M^p \) [24, Theorem 2.4]. Its inverse \( \Lambda_{s_1,s_2}^{-1}u = (x)^{-s_2}((D)^{-s_1}u) \) is a SG pseudodifferential operator of order \((-s_1, -s_2)\). If \( A \) is a SG FIO of order \((m_1, m_2)\), writing

\[ A = T \Lambda_{s_1,s_2}, \quad T := A \Lambda_{s_1,s_2}^{-1}, \]

by Theorem 3.2, the operator \( T \) is a FIO with the same phase as the operator \( A \) and of order \((m_1 - s_1, m_2 - s_2)\). Now, assume that Theorem 1.2 is true for \( p = 1, 2 \). Consider \( 1 < p < 2 \) and let \( A \) be a FIO of order \((m_1, m_2)\), with \( m_1 = m_2 = -d(1/p - 1/2) \). For \( p = 1 \), provided that \( s_i = m_i + d/2, i = 1, 2 \), by (39) the operator \( T \) is of order \((-d/2, -d/2)\), hence bounded on \( M^1 \). As a consequence, the FIO \( A \) extends to a bounded operator from \( M^1_{m_1 + d/2, m_2 + d/2} \) to \( M^1 \).

For \( p = 2 \) and \( s_i = m_i, i = 1, 2 \), the FIO \( T \) is of order \((0, 0)\), so that \( T \) is bounded on \( L^2 \) and \( A \) is bounded from \( M^2_{m_1,m_2} \) to \( M^2 \).

By complex interpolation (see (15)), the operator \( A \) is bounded between the following spaces

\[ A : M^p_{\theta_1, \theta_2} = (M^1_{m_1 + d/2, m_2 + d/2}, M^2_{m_1,m_2})[\theta] \rightarrow M^p = (M^1, M^2)[\theta], \]

with \( 1/p = (1/\theta)(1 + \theta/2), \theta \in (0, 1), \tilde{m}_i = (1-\theta)(m_i + d/2) + \theta m_i = m_i + (1-\theta)d/2, i = 1, 2 \). These equalities yield \( \tilde{m}_i = m_i + d(1/p - 1/2) = 0 \), because \( m_i = \)
−d(1/p − 1/2) by assumption. Hence A is bounded from $M^p$ to $M^p$, as desired. The proof for $2 < p < \infty$ is similar.

Of course, when in one of the inequalities in (12) (or in both) a strict inequality holds, the desired result follows from the equality-case, for an operator of order $(m'_1, m'_2)$ with $m'_1 \leq m_1, m'_2 \leq m_2$, has also order $(m_1, m_2)$.

Hence, from now on, we assume $m_1 = m_2 = −d/2$ and prove the boundedness of $A$ on $M^1$ and on $M^\infty$.

5.1. **Boundedness on $M^1$.** Consider now the usual Littlewood-Paley decomposition of the frequency domain. Namely, fix a smooth function $\psi_0(\eta)$ such that $\psi_0(\eta) = 1$ for $|\eta| \leq 1$ and $\psi_0(\eta) = 0$ for $|\eta| \geq 2$. Set $\psi(\eta) = \psi_0(\eta) - \psi_0(2\eta)$, $\psi_j(\eta) = \psi(2^{-j} \eta), j \geq 1$. Then

$$1 = \sum_{j=0}^\infty \psi_j(\eta), \quad \forall \eta \in \mathbb{R}^d.$$ 

Following the general philosophy of [15], we perform a dyadic decomposition of the symbol $\sigma$ on boxes of size $2^k \times 2^j$, $k, j \geq 0$, hence tailored to the SG symbol estimates; then we conjugate each dyadic operator with dilations in such a way to transform any box into a cube of size $2^{(j+k)/2} \times 2^{(j+k)/2}$. Finally we will apply Proposition 3.4 to these transformed operators.

Namely, consider the decomposition

$$A = \sum_{j,k \geq 0} A_{j,k} = \sum_{0 \leq j < k} A_{j,k} + \sum_{0 \leq k \leq j} A_{j,k},$$

where $A_{j,k}$ is the FIO with the same phase $\Phi$ as $A$ and symbol

$$\sigma_{j,k}(x, \eta) := \psi_k(x) \sigma(x, \eta) \psi_j(\eta).$$

The key point here is that the symbols $\sigma_{j,k}$ are supported where $\langle \eta \rangle \asymp 2^j, \langle x \rangle \asymp 2^k$ and, for any $\alpha, \beta \in \mathbb{Z}^d$,

$$|\partial_\eta^\alpha \partial_x^\beta \sigma_{j,k}(x, \eta)| \leq C_{\alpha, \beta} \langle \eta \rangle^{-\frac{d}{2} + |\alpha|} \langle x \rangle^{-\frac{d}{2} + |\beta|}, \quad \forall j, k \geq 0,$$

i.e., $\sigma_{j,k}$ lie in a bounded subset of $\text{SG}^{-d/2, -d/2}$.

Moreover, we observe that

$$A_{j,k} = U_{2^{-j-k}} \tilde{A}_{j,k} U_{2^{j+k}},$$

where $\tilde{A}_{j,k}$ is the FIO with phase

$$\Phi_{j,k}(x, \eta) := \Phi(2^{-j-k} x, 2^{-j-k} \eta),$$

and symbol

$$\tilde{\sigma}_{j,k}(x, \eta) := \sigma_{j,k}(2^{-j-k} x, 2^{-j-k} \eta),$$
and $U_\lambda f(y) = f(\lambda y)$, $\lambda > 0$, is the dilation operator.

Notice that $\tilde{\sigma}_{j,k}$ is supported in the set
\begin{equation}
V_C = \{(x, \eta) \in \mathbb{R}^{2d} : C^{-1} 2^j \leq \langle 2^{\frac{d}{2}j} \eta \rangle \leq C 2^j, \ C^{-1} 2^k \leq \langle 2^{\frac{d}{2}k} x \rangle \leq C 2^k \},
\end{equation}
for some $C > 0$.

We first consider the sum over $k \leq j$ in (40).

Assume for a moment that the following estimate holds
\begin{equation}
\| \tilde{A}_{j,k} u \|_{M^1} \lesssim 2^{-(j+k)d/2} \| u \|_{M^1},
\end{equation}
and recall the dilation properties for modulation spaces (Theorem 2.1), for $p = 1$:
\begin{equation}
\| U_\lambda f \|_{M^1} \lesssim \| f \|_{M^1}, \ \lambda \geq 1,
\end{equation}
and
\begin{equation}
\| U_\lambda f \|_{M^1} \lesssim \lambda^{-d} \| f \|_{M^1}, \ \ 0 < \lambda \leq 1.
\end{equation}

Then, using (46) (with $\lambda = 2^{(j-k)/2}$) and (47) (with $\lambda = 2^{-(j-k)/2}$) we obtain
\begin{equation}
\| A_{j,k} u \|_{M^1} \lesssim 2^{-kd} \| u \|_{M^1}.
\end{equation}

Actually, for the frequency localization of $A_{j,k}$, the following finer estimate holds:
\begin{equation}
\| A_{j,k} u \|_{M^1} = \| A_{j,k} (\chi(2^{-j} D) u) \|_{M^1} \lesssim 2^{-kd} \| \chi(2^{-j} D) u \|_{M^1}, \ j \geq 1,
\end{equation}
where $\chi$ is a smooth function satisfying $\chi(\eta) = 1$ for $1/2 \leq |\eta| \leq 2$ and $\chi(\eta) = 0$ for $|\eta| \leq 1/4$ and $|\eta| \geq 4$ (so that $\chi \psi = \psi$). Summing on $j, k$ this last estimate, with the aid of Lemma 2.2 (a), we obtain
\begin{equation}
\| \sum_{0 \leq k \leq j} A_{j,k} u \|_{M^1} \leq \sum_{j=0}^{\infty} \sum_{k=0}^{j} \| A_{j,k} u \|_{M^1}
\lesssim \| u \|_{M^1} + \sum_{j=1}^{\infty} \sum_{k=0}^{j} 2^{-kd} \| \chi(2^{-j} D) u \|_{M^1}
\lesssim \| u \|_{M^1} + \sum_{j=1}^{\infty} \| \chi(2^{-j} D) u \|_{M^1}
\lesssim \| u \|_{M^1},
\end{equation}
which is the desired estimate for the sum over $k \leq j$.

It remains to prove (45). This follows from Proposition 3.4 applied to the operator $2^{(j+k)d/2} \tilde{A}_{j,k}$. Indeed, it is easy to see that the hypotheses are satisfied uniformly with respect to $j, k$. Precisely, the chain rule gives, for every $j, k \geq 0$,
\begin{equation}
| \partial_\eta^a \partial_x^\beta \tilde{\sigma}_{j,k}(x, \eta) | \lesssim 2^{(j+k)\left( -\frac{\alpha}{a} - \frac{\beta}{b} \right)}.
\end{equation}
Similarly, we also have
\[ |\partial_\eta^\alpha \partial_x^\beta \Phi_{j,k}(x, \eta)| \lesssim 2^{(j+k)(1 - \frac{|\alpha| + |\beta|}{2})}, \quad \text{for every } (x, \eta) \in \mathcal{V}_{C'} \]
for every fixed \(C' > 0\) (see (44) and notice that \(\mathcal{V}_{C'}\) contains an \(\epsilon\)-neighborhood of the support of \(\tilde{\sigma}_{j,k}\) if \(C'\) is large and \(\epsilon\) small enough). Clearly we also have
\[ |\partial_\eta^\alpha \partial_x^\beta \Phi_{j,k}(x, \eta)| \leq C_{\alpha, \beta}, \quad |\alpha| = |\beta| = 1, \quad \forall (x, \eta) \in \mathbb{R}^{2d}, \]
and
\[ \left| \det \left( \frac{\partial^2 \Phi_{j,k}}{\partial x_i \partial \eta_l} (x, \eta) \right) \right| > \delta > 0, \quad \forall (x, \eta) \in \mathbb{R}^{2d}. \]
Hence Proposition 3.4 applies and gives, for \(1 \leq p \leq \infty\),
\[ \| \widetilde{A}_{j,k} u \|_{M_p} \lesssim 2^{-(j+k)d/2} \| u \|_{M_p}. \]
For \(p = 1\) this is (45).

We now consider the sum over \(j < k\) in (40). Namely, we prove that
\[ \| \sum_{0 \leq j < k} A_{j,k} u \|_{M^1} \lesssim \| u \|_{M^1}. \]
Using the Littlewood-Paley decomposition \(\sum_{l \geq 0} \psi_l(x) = 1\), we write
\[ \sum_{0 \leq j < k} A_{j,k} u = \sum_{l=0}^{\infty} \sum_{0 \leq j < k} A_{j,k} (\psi_l u). \]
By the triangle inequality and Lemma 2.2 \((b)\), it suffices to prove
\[ \| \sum_{0 \leq j < k} A_{j,k} (\psi_l u) \|_{M^1} \lesssim \| u \|_{M^1}. \]
More precisely, one should apply this estimate with \(u\) replaced by \(\chi(2^{-l} \cdot) u\), with \(\chi\) as in (48) above and then use Lemma 2.2 \((b)\).

Applying Theorem 3.2 with \(p(x, D) = \psi_l(x)\) (notice that the multiplicative operators \(\psi_l(x)\) are pseudodifferential operators with symbols in a bounded subset of \(\mathcal{SG}^{(0,0)}\)) to each composition \(2^{(j+k)d/2} A_{j,k} \psi_l\), we obtain
\[ A_{j,k} \psi_l = 2^{-(j+k)d/2} S_{j,k}^{(l)} + 2^{-(j+k)d/2} R_{j,k}^{(l)} \]
where \(S_{j,k}^{(l)}\) are FIOs with the same phase \(\Phi_{j,k}\) in (43) and symbols \(\sigma_{j,k}^{(l)}\) belonging to bounded subset of \(\mathcal{SG}^{0,0}\), supported in
\[ \{(x, \eta) \in \mathbb{R}^{2d} : \langle \eta \rangle \simeq 2^j, \langle \nabla_\eta \Phi(x, \eta) \rangle \simeq 2^l, \langle x \rangle \simeq \langle \eta \rangle \}. \]
The operators \(R_{j,k}^{(l)}\) are smoothing operators whose symbols \(r_{j,k,l}\) lie in a bounded subset of \(\mathcal{S}(\mathbb{R}^{2d})\).
Observe that, by (24),
\[ \langle \nabla_y \Phi(x, \eta) \rangle \asymp \langle x \rangle. \]
Inserting this equivalence in (55), we deduce that there exists \( N_0 > 0 \) such that \( \sigma_{j,k}^{(l)} \) vanishes identically if \( |k - l| > N_0 \). Whence, the left-hand side in (54) is seen to be
\[ \leq \sum_{k \geq 0; |k - l| \leq N_0} \sum_{j=0}^{k-1} 2^{-j+k} \| S_{j,k}^{(l)} u \|_{M^1} + \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} 2^{-(j+k)d/2} \| R_{j,k}^{(l)} u \|_{M^1}. \]
Since
\[ \| R_{j,k}^{(l)} u \|_{M^1} \lesssim \| u \|_{M^1}, \]
(54) will follow from
\[ \| S_{j,k}^{(l)} u \|_{M^1} \lesssim 2^d u \|_{M^1}. \]
In order to prove this estimate, we write
\[ S_{j,k}^{(l)} := U_{2^{j-k}} \tilde{S}_{j,k}^{(l)} U_{2^{j-k}}, \]
where \( \tilde{S}_{j,k}^{(l)} \) is the FIO with phase \( \Phi_{j,k}(x, \eta) \) defined in (43), and symbol
\[ \tilde{\sigma}_{j,k}^{(l)}(x, \eta) := \sigma_{j,k}^{(l)}(2^{j-k} x, 2^{j-k} \eta), \]
supported in a set \( \mathcal{V}_C \) of the type (44).
Now, taking into account (46), (47), we see that (56) will follow (with an additional factor \( 2^{-jd/2} \)) from
\[ \| \tilde{S}_{j,k}^{(l)} u \|_{M^1} \lesssim \| u \|_{M^1}. \]
Precisely, if (57) holds true, we have
\[ \| \tilde{S}_{j,k}^{(l)} u \|_{M^1} \lesssim 2^{-d/2} \| \tilde{S}_{j,k}^{(l)} U_{2^{j-k}} u \|_{M^1} \lesssim 2^{-d/2} \| U_{2^{j-k}} u \|_{M^1} \lesssim 2^{-d/2} \| u \|_{M^1}. \]
The estimate (57) is a consequence of Proposition 3.4 applied to \( \tilde{S}_{j,k} \). Indeed, since the symbols \( \sigma_{j,k}^{(l)} \) belong to a bounded subset of \( \Sigma_{G}^{0,0} \) and are supported where \( \langle \eta \rangle \asymp 2^j, \langle x \rangle \asymp 2^k \), it turns out
\[ | \partial^\alpha_x \partial^\beta_y \sigma_{j,k,l}(x, \eta) | \lesssim 2^{-(j+k) \frac{|\alpha| + |\beta|}{2}}. \]
On the other hand, we already observed that (49), (50) and (51) hold true. Hence Proposition 3.4 gives (57).
5.2. **Boundedness on $\tilde{M}^\infty$.** We now show the boundedness of $A$ (of order $(m_1, m_2) = \left(-\frac{d}{2}, -\frac{d}{2}\right)$) on $\tilde{M}^\infty$, using the notations of the previous subsection. Our arguments in this case reflect the symmetry of the $\mathbf{SG}$ symbol classes with respect to the exchange $x \leftrightarrow \eta$.

Consider again the decomposition in (40). Hence, $A_{j,k}$ is the FIO with phase $\Phi_{j,k}$ in (43) and symbol $\sigma_{j,k}$ in (41). We first test the sum over $k \leq j$. By Lemma 2.3 (a) we have

$$\| \sum_{0 \leq k \leq j} A_{j,k}u \|_{M^\infty} = \| \sum_{l \geq 0} \psi_l(D) \sum_{0 \leq k \leq j} A_{j,k}u \|_{M^\infty} \leq \sup_{l \geq 0} \| \psi_l(D) \sum_{0 \leq k \leq j} A_{j,k}u \|_{M^\infty} \leq \sup_{l \geq 0} \sum_{0 \leq k \leq j} \| \psi_l(D) A_{j,k}u \|_{M^\infty}. $$

(58)

Applying Theorem 3.1 to each product $\psi_l(D)2^{(j+k)d/2}A_{j,k}$, we have

$$\psi_l(D)A_{j,k} = 2^{-(j+k)d/2}S_{j,k,l} + 2^{-(j+k)d/2}R_{j,k,l},$$

where $S_{j,k,l}$ are FIOs with the same phase $\Phi_{j,k}$ and symbols $\sigma_{j,k,l}$ belonging to a bounded subset of $\mathbf{SG}^{0,0}$, supported in

$$(59) \quad \{(x, \eta) \in \mathbb{R}^{2d} : \langle x \rangle \asymp 2^k, \langle \nabla_x \Phi(x, \eta) \rangle \asymp 2^l, \langle \eta \rangle \asymp 2^j\}.$$ 

The operators $R_{j,k,l}$ are smoothing operators whose symbols $r_{j,k,l}$ are in a bounded subset of $\mathcal{S}(\mathbb{R}^{2d})$.

Observe that, by (24),

$$\langle \nabla_x \Phi(x, \eta) \rangle \asymp \langle \eta \rangle.$$

Inserting this equivalence in (59), we obtain that there exists $N_0 > 0$ such that $\sigma_{j,k,l}$ vanishes identically if $|j - l| > N_0$. Whence, the right-hand side in (58) is seen to be

$$\leq \sup_{l \geq 0} \sum_{j \geq 0, |j - l| \leq N_0} \sum_{k=0}^{j} 2^{-(j+k)d/2} \| S_{j,k,l}u \|_{M^\infty} + \sup_{l \geq 0} \sum_{j=0}^{\infty} \sum_{k=0}^{j} 2^{-(j+k)d/2} \| R_{j,k,l}u \|_{M^\infty}. $$

This expression will be dominated by the $M^\infty$ norm of $u$ if we prove that

$$\| S_{j,k,l}u \|_{M^\infty} \lesssim 2^{j \frac{d}{2}} \| u \|_{M^\infty},$$

(60)

because clearly

$$\| R_{j,k,l}u \|_{M^\infty} \lesssim \| u \|_{M^\infty}. $$

(61)

To prove (60) we recall from Theorem 2.1 that

$$\| U_\lambda f \|_{M^\infty} \lesssim \| f \|_{M^\infty}, \quad \lambda \geq 1,$$

(62)
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and

\begin{equation}
\|U_\lambda f\|_{M^\infty} \lesssim \lambda^{-d}\|f\|_{M^\infty}, \quad 0 < \lambda \leq 1.
\end{equation}

Then we write

\[ S_{j,k,l} := U_{2^{-j-k}} \tilde{S}_{j,k,l} U_{2^{-j-k}}, \]

where \( \tilde{S}_{j,k,l} \) is the FIO with phase \( \Phi_{j,k}(x, \eta) \) in (43), and symbol

\[ \tilde{\sigma}_{j,k,l}(x, \eta) := \sigma_{j,k,l}(2^{-j-k} x, 2^{j-k} \eta), \]

supported in a set \( V_C \) of the type (44).

Now, taking into account (62), (63), we see that (60) will follow (with an additional factor \( 2^{-kd/2} \)) from

\begin{equation}
\|\tilde{S}_{j,k,l} u\|_{M^\infty} \lesssim \|u\|_{M^\infty}.
\end{equation}

This last estimate is a consequence of Proposition 3.4 applied to \( \tilde{S}_{k,j} \). Indeed, since the symbols \( \sigma_{j,k,l} \) belong to a bounded subset of \( \mathcal{S}\mathcal{G}^{0,0} \) and are supported where \( \langle \eta \rangle \asymp 2^j, \langle x \rangle \asymp 2^k \), it turns out

\[ |\partial_\eta^\alpha \partial_x^\beta \tilde{\sigma}_{j,k,l}(x, \eta)| \lesssim 2^{-(j+k)|\alpha|/2}. \]

Again, (49), (50) and (51) have already been verified. Hence Proposition 3.4 gives (64).

We now treat the sum over \( j < k \) in (40).

By Lemma 2.3 (b) and the triangle inequality we have

\[ \| \sum_{0 \leq j < k} A_{j,k} u\|_{M^\infty} = \| \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} A_{j,k} u\|_{M^\infty} \]

\[ \lesssim \sup_{k \geq 0} \sum_{j=0}^{k-1} \| A_{j,k} u\|_{M^\infty}. \]

Hence, the desired result will follow from the estimate

\[ \| A_{j,k} u\|_{M^\infty} \lesssim 2^{-jd}\|u\|_{M^\infty}. \]

Using (42) and (62), (63), we see that it suffices to prove

\[ \| \tilde{A}_{j,k} u\|_{M^\infty} \lesssim 2^{-(j+k)d/2}\|u\|_{M^\infty}, \]

but this is (52) for \( p = \infty \).

This concludes the proof of Theorem 1.2.

Remark 5.1. Theorem 1.2 holds true for operators of Type II as well (see (29)), as one sees by using (30).
6. Sharpness of the results and negative results for $L^p$

In this section we prove the sharpness of Theorems [1,2]. Precisely, if one of the index pairing $m_1, m_2$ fulfills $m_i > -d \left\lfloor \frac{1}{2} - \frac{1}{p} \right\rfloor$, $1 \leq p \leq \infty$, $(i = 1, 2)$, there are FIOs of the type (1) and order $(m_1, m_2)$, satisfying the assumptions in the Introduction, which do not extend to bounded operators on $M^p$, $1 \leq p < \infty$, nor on $M^\infty$.

In fact in [8] we exhibited, for every $1 \leq p \leq \infty$, $m > -d[1/2 - 1/p]$, a FIO which does not extend to a bounded operator on $M^p$, with the following features. The phase $\Phi(x, \eta) = \sum_{j=1}^{\infty} \varphi(x_j)\eta_j$ is linear in $\eta$, where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism satisfying (70) below. The symbol $\sigma(x, \eta)$ belongs to Hörmander’s class $S^m_{1,0}$ and is compactly supported with respect to $x$. In particular we see that $\Phi \in \text{SG}^{1,1}$ and satisfies (1), and $\sigma \in \text{SG}^{m,-\infty}_1$. This shows that the threshold for the index $m_1$ in Theorem [1,2] is sharp, even for symbols compactly supported in $x$. For the sake of completeness we briefly recall the construction of such an operator. Then we show that the threshold for the index $m_2$ is sharp as well, even for symbols which are compactly supported with respect to $\eta$. Finally we show how the example in this latter case gives the following negative result for $L^p$ spaces.

**Proposition 6.1.** For every $1 \leq p \leq \infty$, $m > -d[1/2 - 1/p]$, there exists a FIO having phase $\Phi(x, \eta) = \sum_{k=1}^{d} \varphi(\eta_k)x_k$, where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism satisfying (70) below, and symbol compactly supported with respect to $\eta$ and in the class $\text{SG}^{m,\infty}_1$, which does not extend to a bounded operator on $L^p$, $1 \leq p < \infty$, nor on the closure of the Schwartz space in $L^\infty$, if $p = \infty$.

We first recall some results of [8].

**Proposition 6.2.** Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^\infty$ diffeomorphism, whose restriction to the interval $(0, 1)$ is a non-linear diffeomorphism on $(0, 1)$. This means that there exists a point $t_0 \in (0, 1)$ such that $\varphi''(t_0) \neq 0$. Let $\chi \in C^\infty_0(\mathbb{R})$, $\chi \geq 0$, with $\chi(\varphi(t_0)) \neq 0$. Then, if we set

\begin{equation}
(65) \quad f_n(t) = \chi(t)e^{2\pi int}, \quad n \in \mathbb{N},
\end{equation}

for $1 \leq p \leq 2$, we have

\begin{equation}
(66) \quad \|f_n \circ \varphi\|_{F^p_L} \geq c n^{1/p-1/2}, \quad \forall n \in \mathbb{N}.
\end{equation}

The generalization to dimension $d \geq 1$ reads as follows.

**Corollary 6.3.** Let $\varphi$ be as in Proposition 6.2 and $f_n$ defined in (65). We define

\begin{equation}
(67) \quad \tilde{f}_n(t_1, \ldots, t_d) = f_n(t_1) \cdots f_n(t_d), \quad \tilde{\varphi}(t_1, \ldots, t_d) = (\varphi(t_1), \ldots, \varphi(t_d)),
\end{equation}

then

\begin{equation}
(68) \quad \|\tilde{f}_n \circ \tilde{\varphi}\|_{F^p_L(\mathbb{R}^d)} \geq c n^{d(1/p-1/2)},
\end{equation}

where $c$ is a constant independent of $n$. 

This shows that the threshold for the index $m_2$ in Theorem [1,2] is sharp as well, even for symbols which are compactly supported with respect to $\eta$. Finally we show how the example in this latter case gives the following negative result for $L^p$ spaces.
for $1 \leq p \leq 2$.

The action of the multiplier $\langle D \rangle^m$ on the functions $\tilde{f}_n$ is the following.

**Lemma 6.1.** Let $m \in \mathbb{R}$ and $\tilde{f}_n$ defined in (67). Then,

\begin{equation}
\|\langle D \rangle^m \tilde{f}_n\|_{L^p} \leq C n^m.
\end{equation}

We can now prove the sharpness of Theorem 1.2. The key idea is that a $C^1$ change of variables that leaves the $FL^p$ spaces invariant must be affine (the so-called Beurling-Helson Theorem [1] 26, 28).

**Sharpness of the threshold for the frequency index** $m_1$ (see [8] for details). We first study the case $1 \leq p \leq 2$. Consider the FIO

\[ T_{\tilde{\phi}} f(x) = f \circ \tilde{\phi}(x) = \int_{\mathbb{R}^d} e^{2\pi i \tilde{\phi}(x)\eta} \hat{f}(\eta) \, d\eta, \]

where $\tilde{\phi}$ is defined in (67). We require that the one-dimensional diffeomorphism $\phi$ satisfies the assumptions of Proposition 6.2 and the additional hypothesis

\begin{equation}
\phi(x) = x, \quad \text{for } |x| \geq 1.
\end{equation}

Then, the phase $\Phi(x, \eta) = \tilde{\phi}(x)\eta$ fulfills $\Phi \in SG^{1,1}$ and is non-degenerate. Notice that $T_{\tilde{\phi}}$ maps $C_0^\infty(\mathbb{R}^d)$ into itself and $supp T_{\tilde{\phi}} f \subset (0,1)^d$ if $supp f \subset (0,1)^d$.

Let $G \in C_0^\infty(\mathbb{R}^d)$, $G \geq 0$ and $G \equiv 1$ on $[0,1]^d$. For $m_1 \in \mathbb{R}$, the symbol $a(x, \eta) = G(x)\langle \eta \rangle^{m_1}$ satisfies $a \in SG^{m_1, -\infty}$, and the related FIO is given by

\begin{equation}
A f(x) = \int_{\mathbb{R}^d} e^{2\pi i \tilde{\phi}(x)\eta} G(x)\langle \eta \rangle^{m_1} \hat{f}(\eta) \, d\eta = G(x)[(T_{\tilde{\phi}} \langle D \rangle^{m_1}) f](x).
\end{equation}

If $m_1 \leq -d[1/2 - 1/p]$, Theorem 1.2 assures the boundedness of $A$ on $M^p$. We now show that this threshold is sharp for $1 \leq p \leq 2$. Indeed, consider the functions $\tilde{f}_n$ in (67). They are supported in $(0,1)^d$, so $T_{\tilde{\phi}} \tilde{f}_n$ are. Hence, applying the estimate (68) and Lemma 2.1, we obtain

\[ n^{d(1/p - 1/2)} \lesssim \| \tilde{f}_n \circ \tilde{\phi} \|_{L^p(\mathbb{R}^d)} = \| T_{\tilde{\phi}} \tilde{f}_n \|_{L^p(\mathbb{R}^d)} = \| G T_{\tilde{\phi}} \tilde{f}_n \|_{L^p(\mathbb{R}^d)} \]

\[ \lesssim \| G T_{\tilde{\phi}} \tilde{f}_n \|_{M^p(\mathbb{R}^d)} \| \langle D \rangle^{m_1} \|_{L^p(\mathbb{R}^d)} \]

\[ \lesssim \| F \|_{M^p \to M^p} \| \langle D \rangle^{-m_1} \tilde{f}_n \|_{M^p(\mathbb{R}^d)} \lesssim \| F \|_{M^p \to M^p} n^{-m_1}, \]

where the last inequality is due to (69). For $n \to \infty$, we obtain $-m_1 \geq d(1/p - 1/2)$, i.e., (12).

We now study the case $2 < p \leq \infty$. Observe that the adjoint operator $T_{\tilde{\phi}}^*$ of the above FIO $T_{\tilde{\phi}}$ is still a FIO given by

\[ T_{\tilde{\phi}}^* f(x) = \frac{1}{|J_{\tilde{\phi}}(\tilde{\phi}^{-1}(x))|} \int_{\mathbb{R}^d} e^{2\pi i \tilde{\phi}^{-1}(x)\eta} f(\eta) \, d\eta, \]
with \( \widetilde{\varphi}^{-1}(x_1, \ldots, x_d) = (\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_d)) \) and \( |J_\varphi| \) the Jacobian of \( \varphi \). Its phase \( \Phi(x, \eta) = \widetilde{\varphi}^{-1}(x)\eta \) still fulfills \( \Phi \in \text{SG}^{1,1} \) and the standard assumptions.

Now, let \( H \in C^\infty_0(\mathbb{R}^d) \), \( H \geq 0 \), and \( H(x) \equiv 1 \) on supp \( (G \circ \varphi^{-1}) \). For \( m_1 \in \mathbb{R} \), we define the operator

\[
\tilde{A} f(x) = H(x) |(D)^{m_1} T_{\tilde{\varphi}}(Gf)|(x).
\]

Using Theorem 3.1, it is easily seen that \( \tilde{A} \) is a FIO with symbol in \( \text{SG}^{m_1, -\infty} \) (the symbol is compactly supported in the \( x \)-variable). Its adjoint is given by

\[
\tilde{A}^* = GT_{\varphi}(D)^{m_1} H = A + R,
\]

where \( A \) is defined in (71) and the remainder \( R \) is given by

\[
Rf(x) = G(x)[T_{\varphi}(D)^{m_1}((H - 1)f)](x).
\]

If we choose a function \( \tilde{G} \in C^\infty_0(\mathbb{R}^d) \), \( \tilde{G} \equiv 1 \) on supp \( G \) we can write

\[
Rf = \tilde{G}(x) G(x)[T_{\varphi}(D)^{m_1}((H - 1)f)](x)
\]

\[
= \tilde{G}(x) T_{\varphi}[(G \circ \varphi^{-1})(D)^{m_1}((H - 1)f)](x).
\]

By assumptions, supp \( (G \circ \varphi^{-1}) \cap \text{supp} \ (H - 1) = \emptyset \), so that the pseudodifferential operator

\[
f \mapsto (G \circ \varphi^{-1})(D)^{m_1}((H - 1)f)
\]

is a regularizing operator (it immediately follows by the composition formula of pseudodifferential operators, see e.g. [25, Theorem 18.1.8, Vol. III]): this means that it maps \( S'(\mathbb{R}^d) \) into \( S(\mathbb{R}^d) \). The operator \( T_{\varphi} \) is a smooth change of variables, so \( \tilde{G}(x) T_{\varphi} \) maps \( S(\mathbb{R}^d) \) into itself. To sum up, the remainder operator \( R \) maps \( S'(\mathbb{R}^d) \) into \( S(\mathbb{R}^d) \), hence it is bounded on \( M^p \). This means that \( \tilde{A}^* \) is continuous on some \( M^p \) iff \( A \) is.

The operator \( \tilde{A} \) is a FIO, with symbol in \( \text{SG}^{m_1, -\infty} \) (compactly supported in the \( x \) variable). Hence it is bounded on \( M^p \) if \( m_1 \leq -d|1/2 - 1/p| \) fulfills (12). We now show that this threshold is sharp for \( 2 < p < \infty \). Indeed, if \( \tilde{A} \) were bounded on \( M^p \), then its adjoint \( \tilde{A}^* \) would be bounded on \( (M^p)' = M^{p'} \), with \( 1 < p' < 2 \), and the same for \( A \). But the former case gives the boundedness of \( A \) on \( M^{p'} \) iff \( -m_1 \geq d(1/p' - 1/2) = d(1/2 - 1/p) \), that is the desired threshold. For \( p = \infty \), if \( \tilde{A} \) were bounded on \( M^\infty \), its adjoint \( \tilde{A}^* \) would be bounded on \( (M^\infty)' = M^1 \) and the former argument applies.

**Sharpness of the threshold for the space index \( m_2 \).** The argument rely on the previous counterexample, combined with the Fourier invariance of \( M^p \) and a duality trick.

Consider, for \( 1 \leq p \leq \infty \), \( m > -d|1/2 - 1/p| \) the type I FIO \( A = A_{\Phi, \sigma} \) constructed in the previous subsection. Hence, \( \Phi(x, \eta) = \sum_{k=1}^d \varphi(x_k)\eta_k \), for a diffeomorphism \( \varphi : \mathbb{R} \to \mathbb{R} \) satisfying (70), \( \sigma \in \text{SG}^{m, -\infty} \) is compactly supported with
respect to $x$, and $A$ does not extends to a bounded operator on $\tilde{M}^p$.

Let us set
\[ ^t\Phi(x, \eta) = \Phi(\eta, x) \quad \sigma^*(x, \eta) = \sigma(\eta, x). \]

Then, by comparing the two definitions (11) and (29), we have
\[ (74) \quad B_{-^t\Phi, \sigma^*} = \mathcal{F} \circ A_{\Phi, \sigma} \circ \mathcal{F}^{-1}, \]
where $B_{-^t\Phi, \sigma^*}$ is the type II operator in (29) having phase $-^t\Phi$ and symbol $\sigma^*$. Using (74) and the fact that the Fourier transform defines an isomorphism of any $\tilde{M}^p$ we see that the operator $B_{-^t\Phi, \sigma^*}$ does not extends to a bounded operator on $\tilde{M}^p$. The same therefore holds for $(B_{-^t\Phi, \sigma^*})^*$ on $\tilde{M}^p$, since $\tilde{M}^p = (\tilde{M}^p)'$. On the other hand, by (30) we have $(B_{-^t\Phi, \sigma^*})^* = A_{-^t\Phi, \sigma^*}$. The last operator possesses symbol $\sigma^* \in \text{SG}^{-\infty, m}$, compactly supported with respect to $\eta$, and gives the desired counterexample.

Proof of Proposition 6.1. We start with an elementary remark. Consider a FIO $A$ and suppose that it does not satisfy an estimate of the type
\[ \|Au\|_{M^p} \leq C\|u\|_{M^p}, \quad \forall u \in \mathcal{S}(\mathbb{R}^d). \]

Suppose, in addition, that the distribution kernel of $K(x, y)$ of $T$ has the property that the two projections of $\text{supp} \, K$ on $\mathbb{R}_x^d$ and $\mathbb{R}_y^d$ are bounded sets. Then, it follows by Lemma 2.1 (i) that $A$ does not extend to a bounded operator on $\mathcal{F}L^p$, if $1 \leq p < \infty$, nor on the closure of the Schwartz functions in $\mathcal{F}L^\infty$, if $p = \infty$.

Taking this fact into account, we see that the operator $\hat{A}$ in (72), if $m_1 > -d|1/2 - 1/p|$, does not extend to a bounded operator on $\mathcal{F}L^p$, $2 < p' < \infty$, nor on the closure of the Schwartz space in $\mathcal{F}L^\infty$, if $p' = \infty$. This operator has a phase $\Phi(x, \eta) = \sum_{k=1}^d \varphi(x_k)\eta_k$, for a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying (70), and a symbol $\tau \in \text{SG}^{m_1, -\infty}$, compactly supported with respect to $x$. By repeating the same arguments as in the proof of the sharpness of the space index $m_2$ we see that the operator $A_{-^t\Phi, \tau^*}$, with phase $^t\Phi(x, \eta) = \Phi(\eta, x)$ and symbol $\tau^*(x, \eta) = \tau(\eta, x)$, does not extend to a bounded operator on $L^p$, $1 \leq p < 2$, and gives the desired counterexample in that case.

Similarly, for $2 \leq p \leq \infty$, we consider the operator $\hat{A}^*$ in (73) which, for the same reason, does not extend to a bounded operator on $\mathcal{F}L^{p'}$, $1 \leq p' \leq 2$, if $m_1 > -d|1/2 - 1/p|$. The same holds for $A$ in (71), because of the second inequality in (73). By arguing as before, we obtain the requested counterexample, having the desired phase and symbol $G(\eta)(\langle x \rangle)^{m_1}$.

Observe that this latter example is precisely that observed in the Introduction (see (8)). Actually, in Theorem 1.1 the cut off function $G(\eta)$ was removed, because the eliminated part is a pseudodifferential operator which is bounded on any $L^p$, when $\tilde{m} \leq 0$. 


As an alternative, one could also use the failure of the boundedness on $M^p$ combined with Lemma 2.1(ii), but the above approach seems a little bit shorter.

We observe that the operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^*$ above allowed us to prove in [8] the sharpness of the threshold $-d[1/2 - 1/p]$ for FIOs acting on local $\mathcal{F}L^p$ spaces.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORINO, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORINO, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY

*E-mail address:* elena.cordero@unito.it

*E-mail address:* fabio.nicola@polito.it

*E-mail address:* luigi.rodino@unito.it