Improved Wald formalism and first law of dyonic black strings with mixed Chern-Simons terms

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ABSTRACT: We study the first law of thermodynamics of dyonic black strings carrying a linear momentum in type IIA string theory compactified on K3 with leading order $\alpha'$ corrections. The low energy effective action contains mixed Chern-Simons terms of the form $-2B_{(2)} \wedge \text{tr}(R(\Gamma_\pm) \wedge R(\Gamma_\pm))$ which is equivalent to $2H_{(3)} \wedge \text{CS}_{(3)}(\Gamma_\pm)$ up to a total derivative. We find that the naive application of Wald entropy formula leads to two different answers associated with the two formulations of the mixed Chern-Simons terms. Surprisingly, neither of them satisfies the first law of thermodynamics for other conserved charges computed unambiguously using the standard methods. We resolve this problem by carefully evaluating the full infinitesimal Hamiltonian at both infinity and horizon, including contributions from terms proportional to the Killing vector which turn out to be nonvanishing on the horizon and indispensable to establish the first law. We find that the infinitesimal Hamiltonian associated with $-2B_{(2)} \wedge \text{tr}(R(\Gamma_\pm) \wedge R(\Gamma_\pm))$ requires an improvement via adding a closed but non-exact term, which vanishes when the string does not carry either the magnetic charge or linear momentum. Consequently, both formulations of the mixed Chern-Simons terms yield the same result of the entropy that however does not agree with the Wald entropy formula. In the case of extremal black strings, we also contrast our result with the one obtained from Sen’s approach.

KEYWORDS: Black Holes in String Theory, Extended Supersymmetry, Supergravity Models

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1 Introduction

Since the establishment of black hole mechanics [1] in 1973, many methods have been proposed to compute the thermodynamic quantities, amongst which the most notable ones include the Euclidean action method [2], the construction of the quasi-local conserved charges [3, 4] based on the ADM formalism [5–7] and the ADT method [8, 9]. In these methods, the satisfaction of the first law of mechanics are verified independently after deriving the thermodynamical quantities. It was Wald [10, 11] who first realized that the thermodynamical quantities and the first law of mechanics can be combined into one formula, i.e., through the first law of mechanics, one can identify various thermodynamical quantities. By this way, the first law of mechanics is obeyed automatically. The idea is that given a Killing vector $\xi$ in $D$-dimensional spacetime, one can construct a closed $(D-2)$-form which locally can be written as [10–12]

$$d (\delta Q[\xi] - i_\xi \Theta[\delta \phi]) = 0,$$

(1.1)

where $\delta \phi$ denotes variations of all the fields that satisfy the linearized field equations. Integration of the quantity inside the bracket on the $(D-2)$-dimensional hypersurface defines the infinitesimal Hamiltonian associated with the Killing vector $\xi$, i.e.,

$$\delta H_\Sigma = \int_\Sigma (\delta Q[\xi] - i_\xi \Theta[\delta \phi]).$$

(1.2)

Applying this formalism to black holes, one evaluates (1.1) on the constant time slice sandwiched between the spatial infinity and the bifurcation horizon $\mathcal{B}$ to obtain

$$\delta H_\infty = \delta H_B.$$

(1.3)
Upon substituting a specific black hole solution, one recognizes the equality above gives precisely the first law of mechanics while the integral at infinity yields combinations of conserved charges such as mass, angular momentum; the integral on the horizon is used to define the entropy [10, 11].

However, it was already pointed out in [11] that the density of the infinitesimal Hamiltonian was defined up to an addition of a closed $(D - 2)$-form. In fact, $\delta Q[\xi] - i_\xi \Theta[\delta \phi]$ should be classified by the $(D - 2)$’th cohomology class on the spacetime with a possible gauge bundle structure. In order for (1.3) to hold, one must chose properly the density of the infinitesimal Hamiltonian such that it is globally well defined. A specific example is given by the Reissner-Nordström (RN) dyonic black hole in the $D = 4$ Einstein-Maxwell theory

$$e^{-1}L_{EM} = R - \frac{1}{4}F_{(2)}^2,$$

(1.4)

where $F_{(2)} = dA_{(1)}$. A direct application of the Wald procedure leads to [13]

$$Q[\xi] = -*d\xi - (i_\xi A_{(1)})*F_{(2)}, \quad i_\xi \Theta = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \Theta^\alpha \xi^\beta dx^\mu \wedge dx^\nu$$

$$\Theta^\alpha = g^\alpha\gamma g^{\gamma\rho} \left( \nabla_\rho \delta g_{\nu\sigma} - \nabla_\sigma \delta g_{\nu\rho} \right) - F^{\alpha\rho} \delta A_\rho,$$

(1.5)

where the star denotes the Hodge dual and $\xi$ is the Killing vector vanishing on the bifurcation horizon. In the gauge, $i_\xi A_{(1)}|_{r=r_h} = 0$, one finds the integral at infinity yields $dM - \Phi_e dQ_e$ while the integral at the horizon gives $TdS$. This means the equality (1.3) is not satisfied, since the magnetic contribution is absent. To cure this problem, one has to add a closed form $-d(\Psi \delta A_{(1)})$ to $i_\xi \Theta$ where $\Psi$ is defined by

$$d\Psi = i_\xi * F_{(2)}.$$  

(1.6)

Then the newly defined $i_\xi \Theta$ would contain a term $\Psi \delta F_{(2)}$ whose value on the horizon provides the missing magnetic contribution $\Phi_m dQ_m$ to the first law [13]. Adding to the original infinitesimal Hamiltonian by a closed form $-d(\Psi \delta A_{(1)})$ is also a requirement from regularity. Without such an improvement, it would contain a term proportional $\delta P d\rho \wedge \cos \theta d\phi$ suffering from the Dirac string singularity, when the solution carries the magnetic charge. The magnetic part in the first law can also be introduced via the electromagnetic duality [14] or a careful analysis based on the Hamiltonian formulation [15]. The trick of pulling out a total derivative was also needed in the generalization of the original proof of the first law to Einstein gravity coupled to a non-linear electrodynamic system [16].

In order to see that the closed form $-d(\Psi \delta A_{(1)})$ naturally comes from electromagnetic duality, one can simply repeat the Wald procedure for the dual Lagrangian of Einstein-Maxwell theory $L(g, \tilde{A})$ where $d\tilde{A}_{(1)} = *F_{(2)}$, which is equally good for discussing the on-shell properties such as conserved charges. Using

$$d\tilde{\Psi} = i_\xi * \tilde{F}$$

(1.7)

one finds that

$$\delta H (g, A) - \delta H (g, \tilde{A}) = d (\Psi \delta A_{(1)}) - d \left( \tilde{\Psi} \delta \tilde{A}_{(1)} \right).$$

(1.8)
Equivalently, we have
\[ \delta H (g, A) - d (\Psi \delta A (1)) = \delta H (g, \tilde{A}) - d (\tilde{\Psi} \delta \tilde{A} (1)), \] (1.9)
which means the improved infinitesimal Hamiltonian is invariant under electromagnetic duality. We emphasize that although the Wald formalism has been applied to p-form systems such as [17], the improvement needed to achieve the correct first law in the presence of magnetic charges has not been discussed.

In this paper, we report a novel case where the density of infinitesimal Hamiltonian needs a proper treatment in order for the first law to hold. Built upon our previous work [18], we investigate the first law of thermodynamics for the dyonic strings carrying linear velocity in the context of \(D = 6\) IIA or heterotic string with leading \(\alpha'\) corrections. In the IIA case, there exists a pair of mixed Chern-Simons (CS) terms of form
\[ - 2 B (2) \wedge \text{tr}(R (\Gamma_+ \wedge R (\Gamma_+))) - 2 B (2) \wedge \text{tr}(R (\Gamma_- \wedge R (\Gamma_-))), \] (1.10)
in which \(\Gamma_\pm\) refers to the torsionful connection with the torsion being \(\pm H (3)\). Of course, they can also be recast as
\[ 2 H (3) \wedge \text{CS}_{(3)} (\Gamma_+) + 2 H (3) \wedge \text{CS}_{(3)} (\Gamma_-), \] (1.11)
where \(\text{CS}_{(3)} (\Gamma_\pm)\) is the Chern-Simons form obeying \(d \text{CS}_{(3)} (\Gamma_\pm) = \text{tr}(R (\Gamma_\pm \wedge R (\Gamma_\pm)))\). These two expressions differ by a total derivative term \(d (2 B (2) \wedge \text{CS}_{(3)} (\Gamma_+) + 2 B (2) \wedge \text{CS}_{(3)} (\Gamma_-)) \).

Usually if the Lagrangian is shifted by a total derivative, \(L \to L + d \Lambda\), the \(\Theta\) term also receives a shift
\[ \Theta \to \Theta + \delta \Lambda. \] (1.12)
The Noether current \(J = \Theta - i \xi L\) changes to
\[ J \to J + \delta \xi \Lambda - i \xi d \Lambda. \] (1.13)
If \(\Lambda\) is a covariant quantity, \(\delta \xi \Lambda = L \xi \Lambda\), we then have
\[ J \to J + d i \xi \Lambda, \] (1.14)
which means the Noether charge \(Q\) defined via \(J = d Q\) acquires a shift according to
\[ Q \to Q + i \xi \Lambda. \] (1.15)

Hence the \((D - 2)\)-form \(\delta Q [\xi] - i \xi \Theta\) would appear to be inert. However, since the total differential here is neither gauge invariant nor diffeomorphism invariant, it could have nontrivial effects on the density of infinitesimal Hamiltonian. The density of infinitesimal Hamiltonians resulting from the mixed CS term \(- 2 B (2) \wedge \text{tr}(R (\Gamma_\pm \wedge R (\Gamma_\pm)))\) and \(2 H (3) \wedge \text{CS}_{(3)} (\Gamma_\pm)\) were constructed in [20].\(^1\) Using their results, we identify the possible closed but

\(^1\)The construction of infinitesimal Hamiltonian for CS theories was revisited in [21]. However, both [20] and [21] did not apply their formulas to study the entropy of 6D dyonic strings carrying the linear momentum and thus did not notice the infinitesimal Hamiltonian associated with the \(- 2 B (2) \wedge \text{tr}(R (\Gamma_\pm \wedge R (\Gamma_\pm)))\) term requires an improvement.
topologically nontrivial 4-form that can be inserted into the naive result of $\delta Q[\xi] - i_{\xi} \Theta$ derived from the $-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))$ term.

For dyonic string solutions carrying linear velocity, we show explicitly that this term takes different values at infinity and horizon. Thus its exclusion in $\delta Q[\xi] - i_{\xi} \Theta$ in literature leads to apparent violation of the first law. It should be noticed that when the solution is static or purely electric, this potential obstruction to the first law vanishes. This explains why in our previous work [18], we had not noticed any problem with the first law derived using the $-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))$ term. The first law derived from the $2H_{(3)} \wedge \text{CS}_{(3)}(\Gamma_{\pm})$ works without any modification. This phenomenon may have to do with the fact that general gauge CS terms are not globally defined on the base space of a principal bundle, while the gravitational CS terms are globally defined in spacetime due to the existence of a natural lift in the frame bundle, see for instance [22].

Once the first law is established, we can read off various thermodynamic quantities. To our surprise, the entropy that satisfies the first law can neither be obtained from the Wald formula applied to the $-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))$ formulation nor Tachikawa [23] formula applied to the $2H_{(3)} \wedge \text{CS}_{(3)}(\Gamma_{\pm})$ formulation. An immediate consequence of our result is that upon taking the BPS limit, we obtain the entropy of the 3-charge BPS string solution in IIA string compactified on K3 that revises the previous results [24–26] obtained by directly applying the Wald-Tachikawa formula or attractor mechanism. Terms proportional to $\xi$ present in the infinitesimal Hamiltonian do not all vanish on the bifurcation horizon, some actually contribute. Also because the BPS string has zero temperature, one cannot verify the validity of its entropy using the first law of thermodynamics as we do here for black strings. The correct entropy for the 3-charge BPS string solution in IIA string compactified on K3 now has the desired property. It matches with the entropy of the 3-charge BPS string solution in heterotic string compactified on 4-torus upon performing the electromagnetic duality.

The outline of the paper is as follows. In section 2, we give a brief review of the 3-charge dyonic string solution carrying the linear momentum in addition to the electric and magnetic charges, in 6D 2-derivative supergravity and its first law of thermodynamics. In section 3, we introduce the $\alpha'$ corrections arising from one loop terms in type IIA string compactified on K3. The $\alpha'$-corrected action contains terms of the form $-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))$ which can also be written as $2H_{(3)} \wedge \text{CS}_{(3)}(\Gamma_{\pm})$. We show how to obtain the correct first law for both formulations by improving the infinitesimal Hamiltonian in the $-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))$ formulation with a topologically nontrivial closed 4-form. We then compute the entropy of the $\alpha'$ corrected 3-charge solution using the first law, for which neither Wald formula nor Tachikawa formula provides the correct answer to the entropy. We also compute the Euclidean action and show it is compatible with the thermodynamic quantities computed from the infinitesimal Hamiltonian. In section 4, we use IIA/heterotic duality to study the leading $\alpha'$ corrections to the thermodynamics of the 3-charge string solutions in 6D heterotic string compactified on 4-torus. We conclude the paper in section 5. In appendix A, we give the $\alpha'$-corrected perturbative solutions to the dyonic black string. In appendix B, we give the higher-derivative corrections to the infinitesimal Hamiltonian.

\footnote{As shown by Sen, attractor mechanism is equivalent to Wald formula for extremal black holes.}
2 3-charge dyonic black string in 6D 2-derivative supergravity

In this section, we review in detail how to interpret the first law of dyonic black string using the Wald procedure. The Lagrangian of $D = 6$ minimal supergravity without higher derivative corrections is given by

$$e^{-1}L_{EH} = L \left( R + L^{-2} \nabla^\mu L \nabla_\mu L - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \quad (2.1)$$

where $e = \sqrt{-\det(g_{\mu\nu})}$, $L$ is the dilaton field, $H_{\mu\nu\rho}$ is a 3-form field strength of the 2-form potential $B_{\mu\nu}$, i.e. $H_{(3)} := dB_{(2)}$. Together, $(g_{\mu\nu}, L, B_{\mu\nu})$ comprise the bosonic part of the 6D dilaton Weyl multiplet [27]. In this paper, we set 6D Newton’s constant $G_6 = 1$. This theory admits a static black dyonic string solution [18] to which we can add the linear momentum by a Lorentz boost

$$t \rightarrow c_3 t + s_3 x, \quad x \rightarrow c_3 x + s_3 t, \quad c_3 \equiv \cosh \delta_3, \quad s_3 \equiv \sinh \delta_3. \quad (2.2)$$

The resulting solution takes the form

$$d\xi_6^2 = D(r) \left( -h_1(r) dt^2 + h_2(r) dx^2 + 2\omega(r) dt dx \right) + H_p(r) \left( \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \right), \quad (2.3)$$

where $\omega_{(2)} = -\frac{1}{4} \cos^2 \theta d\phi \wedge d\chi$, so $d\omega_{(2)} = \text{Vol}(S^3)$. Here the line element on $S^3$, $d\Omega_3^2 = \frac{1}{4}(\sigma_3^2 + d\Omega_2^2)$, is expressed as a $U(1)$ bundle over a $S^2$ in which $\sigma_3 = d\chi - \cos \theta d\phi$, $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

After setting $s_3 = -\sqrt{Q_2/\mu}$, the various functions in the solution are given by

$$L(r) = \frac{1}{D(r) H_p(r)}, \quad D(r) = A(r) = \frac{r^2}{r^2 + Q_1}, \quad H_p(r) = 1 + \frac{P}{r^2}, \quad f(r) = 1 - \frac{\mu}{r^2}, \quad h_1(r) = 1 - \frac{\mu}{r^2} \frac{Q_2}{r^2}, \quad h_2(r) = 1 + \frac{Q_2}{r^2}, \quad \omega(r) = -\sqrt{Q_2(\mu + Q_2)} \frac{1}{r^2}. \quad (2.4)$$

The horizon of the black string is located at $r = r_h$ where the metric in the $dt, dx$ direction degenerates, i.e.

$$\left( g_{tt}g_{xx} - g_{tx}^2 \right)_{r=r_h} = 0 \Rightarrow h_1(r_h) h_2(r_h) + \omega(r_h)^2 = f(r_h) = 0, \quad (2.5)$$

from which we solve $r_h = \sqrt{\mu}$. The Killing vector $\xi$

$$\xi = \partial_t + V_x \partial_x, \quad V_x = \sqrt{\frac{Q_2}{\mu + Q_2}} \quad (2.6)$$

becomes a null vector on the horizon i.e. $\xi^2_{|r=r_h} = 0$. The linear momentum density along $x$-direction can be evaluated from the Komar integral

$$P_x = \frac{1}{16\pi} \int_{S^3} *d\xi_x = \frac{\pi}{4} \sqrt{Q_2} \sqrt{\mu + Q_2}, \quad \xi_x = \partial_x. \quad (2.7)$$
(Note that if \( x \) is compact, \( P_x \) can also interpreted as an angular momentum.) The temperature of the black string is defined through the surface gravity \( \kappa \) using the Killing vector \( \xi \)

\[
\kappa^2 = -\frac{g^{\mu\nu} \partial_\mu \xi^2 \partial_\nu \xi^2}{4\xi^2} \bigg|_{r=r_h}, \quad T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \frac{\mu}{\sqrt{\mu + P\sqrt{\mu + Q_1}\sqrt{\mu + Q_2}}}, \tag{2.8}
\]

The entropy density along the \( x \)-direction is computed using Iyer-Wald formula \[11\]

\[
S = -\frac{1}{8} \int_{S^3} d\Omega_3 \frac{\partial \mathcal{L}_{EH}}{\partial \delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu\rho\sigma} \bigg|_{r=r_h} = \frac{1}{2\pi^2} \frac{\sqrt{\mu + P\sqrt{\mu + Q_1}\sqrt{\mu + Q_2}}}{\sqrt{\mu + P}}, \tag{2.9}
\]

where \( \epsilon_{\mu\nu} \) is the binormal vector of the black string horizon satisfying \( \epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2 \) and \( d\Omega_3 \) is the induced metric on the 3-sphere. The electric and magnetic charges carried by the string are obtained as

\[
Q_e = \frac{1}{16\pi} \int_{S^3} L \ast H_{(3)} = \frac{\pi}{4} \sqrt{Q_1\sqrt{\mu + Q_1}}, \tag{2.10}
\]
\[
Q_m = \frac{1}{16\pi} \int_{S^3} H_{(3)} = \frac{\pi}{4} \sqrt{P\sqrt{\mu + P}}. \tag{2.11}
\]

The corresponding electric and magnetic potential are computed from

\[
\Phi_e = \xi^\mu B_{\mu x}|_{r=\infty} - \xi^\mu B_{\mu x}|_{r=r_h} = \sqrt{\frac{Q_1}{\mu + Q_1}}, \tag{2.12}
\]
\[
\Phi_m = \xi^\mu \tilde{B}_{\mu x}|_{r=\infty} - \xi^\mu \tilde{B}_{\mu x}|_{r=r_h} = \sqrt{\frac{P}{\mu + P}}, \tag{2.13}
\]

where \( \tilde{B}_{\mu\nu} \) is the dual 2-form potential defined via \( L \ast H_{(3)} = d\tilde{B}_{(2)} \). Similar to \[18\] the mass can be computed using Brown-York surface Hamiltonian. The result is

\[
M = \frac{3\pi}{8} \mu + \frac{\pi}{4} (Q_1 + Q_2 + P). \tag{2.14}
\]

One can check that the thermodynamic quantities satisfy the first law.

\[
dM = T dS + \Phi_e dQ_e + \Phi_m dQ_m + V_s dP_s. \tag{2.15}
\]

Below we provide a different perspective based on the Wald procedure \[11\]. For convenience, we fix the gauge \( \xi^\mu B_{\mu x}|_{r=\infty} = 0 \) by shifting \( A(r) \rightarrow A(r) - 1 \). From the Lagrangian \( (2.1) \), one first construct the conserved current \[19\]

\[
J^\mu_{EH} = \frac{1}{5!} \epsilon_{\mu\nu\rho\sigma\delta} J^\mu_{EH} dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\delta, \tag{2.16}
\]
\[
J^\mu_{EH} = \Theta^\mu_{EH} - \xi^\mu e^{-1} \mathcal{L}_{EH} - 2 E^\mu_{EH} \xi_\nu + 2 S^\mu_{EH} B_\nu \xi^\nu, \tag{2.17}
\]
in which
\[ E_{EH}^{\mu\nu} = LR^{\mu\nu} + L^{-1}(\nabla^\nu L)(\nabla^\mu L) - \frac{1}{2}g^{\mu\nu}e^{-1}L_{EH} - \frac{1}{4}LH^{2\mu\nu} + g^{\mu\nu}\Box L - \nabla^\mu \nabla^\nu L, \]
\[ S_{EH}^{\mu\nu} = \frac{1}{2}\nabla_\rho (LH^{\mu\nu}) , \quad \Theta_{EH}^\mu = \Theta_g^\mu + \Theta_B^\mu + \Theta_\Sigma^\mu, \]
\[ \Theta_g^\mu = Lg^{\mu\sigma}g^{\nu\rho}(\nabla_\rho g_{\sigma\nu} - \nabla_\sigma g_{\rho\nu}) + \delta g_{\mu\sigma}(g^{\nu\rho} \nabla_\mu L - g^{\mu\nu} \nabla_\sigma L), \]
\[ \Theta_B^\mu = -\frac{1}{2}LH^{\mu\rho}\delta B_{\rho\nu} , \quad \Theta_\Sigma^\mu = 2L^{-1}\nabla^\mu L\delta L. \] (2.18)

On-shell \( dJ_{EH} = 0 \) implies that \( J_{EH} = dQ_{EH} \) where
\[ Q_{EH} = \frac{1}{4!2} \varepsilon_{\alpha\beta\mu\nu\rho\lambda}Q_{EH}^{\alpha\beta}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda, \quad Q_{EH}^{\mu\nu} = Q_g^{\mu\nu} + Q_B^{\mu\nu}, \] (2.19)
where
\[ Q_g^{\mu\nu} = -Lg^{\mu\sigma}g^{\nu\rho}(\nabla_\sigma \xi_\rho - \nabla_\rho \xi_\sigma) - 4\xi^{[\mu}\nabla^{\nu]}L, \quad Q_B^{\mu\nu} = -LH^{\mu\nu}B_{\rho\lambda}\xi^\lambda. \] (2.20)

Using the fact that \( \xi \) is a Killing vector, when the perturbations \( \delta g_{\mu\nu}, \delta B_{\mu\nu}, \delta L \) obey the linearized field equations, one can show that
\[ d(\delta Q_{EH} - i_\xi \Theta_{EH}) = 0, \] (2.21)
where \( \Theta_{EH} = \frac{1}{3!} \varepsilon_{\mu\nu\rho\lambda\delta}Q_{EH}^{\mu\nu}dx^\rho \wedge dx^\lambda \wedge dx^\delta. \)

On substituting the details of the solution, we find that for \( \xi = \partial_t + V_x \partial_x, \int_{r=\infty} \frac{i_\xi \Theta_L}{r} = 0 \) and
\[ \int_{\infty} (\delta Q_B[\xi] - i_\xi \Theta_B) = 0, \quad \int_{r=r_h} (\delta Q_B[\xi] - i_\xi \Theta_B) = \Phi_e Q_e, \]
\[ \int_{\infty} (\delta Q_g[\xi] - i_\xi \Theta_g) = dM - V_x dP_x, \quad \int_{r=r_h} (\delta Q_g[\xi] - i_\xi \Theta_g) = TdS. \] (2.22)

So apparently the infinitesimal Hamiltonian defined in (1.3) does not give rise to correct first law. This issue can be settled by improving the infinitesimal Hamiltonian density \( \delta Q_{EH} - i_\xi \Theta_{EH} \) with the additional term
\[ -d(\Psi_{(1)} \wedge \delta B_{(2)}), \quad d\Psi_{(1)} = i_\xi \star LH_{(3)}. \] (2.23)

For the solution (2.4), we have
\[ \Psi_{(1)} = \frac{\sqrt{P(\mu + P)}}{P + r^2} (V_x dt - dx), \] (2.24)
and
\[ -\int_{\infty} d(\Psi_{(1)} \wedge \delta B_{(2)}) = 0, \quad -\int_{r=r_h} d(\Psi_{(1)} \wedge \delta B_{(2)}) = \Phi_m dQ_m. \] (2.25)

In fact, this can be seen more abstractly. Inclusion of the above total differential brings a term of the form \( d(\Psi_{(1)} \wedge \delta H_{(3)}) \) to the infinitesimal Hamiltonian. Thus with the improvement, we have
\[ \delta H_{\Sigma} = \int_{\Sigma} (\delta Q[\xi] - i_\xi \Theta - d(\Psi_{(1)} \wedge \delta B_{(2)})), \] (2.26)
the first law is indeed implied by
\[ \delta \mathcal{H}_\infty = \delta \mathcal{H}_B. \] (2.27)

It should be emphasized here however that despite the improvement, the Iyer-Wald formula for calculating the entropy is unchanged in this two-derivative theory. This story however no longer holds when we consider the \( \alpha' \) corrections, discussed next.

3 First law of 3-charge dyonic black string with mixed CS term

In this section, we extend the 2-derivative supergravity theory by supersymmetric Gauss-Bonnet term [28, 29] and the Riemann tensor squared [30]

\[ \mathcal{L}_{R+R^2} = \mathcal{L}_{EH} + \frac{\lambda_{\text{GB}}}{16} \mathcal{L}_{GB} + \frac{\lambda_{\text{Riem}}^2}{16} \mathcal{L}_{\text{Riem}}^2, \] (3.1)

\[ e^{-1} \mathcal{L}_{GB} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\nu\rho} R^{\nu\rho} + R^2 + \frac{1}{6} R H^2 - R^{\mu
u} H_{\mu\nu}^{2} + \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\lambda} H^{\rho\sigma\lambda} \]
\[ + \frac{5}{24} H^4 + \frac{1}{144} (H^2)^2 - \frac{1}{8} (H^2)^2 + \frac{1}{4} \epsilon^{\mu
u\rho\sigma\lambda\tau} B_{\mu\nu} R^{\alpha\beta}_{\rho\sigma} (\Gamma_\alpha^\beta) R_{\lambda\tau} (\Gamma_\alpha^\beta), \] (3.2)

\[ e^{-1} \mathcal{L}_{\text{Riem}}^2 = R_{\mu\nu\alpha\beta} (\Gamma_+^\pm) R^{\mu\nu\alpha\beta} (\Gamma_+^\pm) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\lambda\tau} B_{\mu\nu} R^{\alpha\beta}_{\rho\sigma} (\Gamma_+^\pm) R_{\lambda\tau} (\Gamma_+^\pm), \] (3.3)

where in our convention \( \sqrt{-g} \epsilon^{012345} = -1 \). Here \( R^{\alpha\beta}_{\mu\nu}(\Gamma_\pm) \) is the curvature defined with respect to the torsionful spin connection \( \Gamma_{\pm\mu\beta} \)

\[ R^{\alpha\beta}_{\mu\nu}(\Gamma_\pm) = \partial_\mu \Gamma_{\pm\nu}^{\alpha} + \Gamma_{\pm\mu\gamma}^{\alpha} \Gamma_{\pm\nu}^{\gamma} - (\mu \leftrightarrow \nu), \quad \Gamma_{\pm\mu\beta} = \Gamma_{\mu\beta} \pm \frac{1}{2} H_{\mu\beta}. \] (3.4)

The shorthand notations for various contractions of \( H_{\mu\nu\rho} \) are defined as

\[ H^2 = H_{\mu\nu \rho} H_{\mu\nu \rho}, \quad H_{\mu\nu}^2 = H_{\mu\nu \rho} H_{\nu \rho}^{\sigma}, \quad H^4 = H_{\mu\nu\rho\sigma} H^{\mu\rho\lambda} H^{\nu\lambda \delta}. \] (3.5)

The combination with \( \lambda_{\text{GB}} = \lambda_{\text{Riem}}^2 = \alpha' \) describes the leading \( \alpha' \) correction to the NS-NS sector of IIA compactified on K3 [31] and thus is compatible with 6D (1,1) supersymmetry.

After including the leading \( \alpha' \)-correction, all the functions will be perturbed

\[ L(r) = L_0(r) + \delta L(r), \quad D(r) = D_0(r) + \delta D(r), \quad A(r) = D_0(r) + \delta A(r), \]
\[ h_1(r) = h_{1,0}(r) + \delta h_1(r), \quad h_2(r) = h_{2,0}(r) + \delta h_2(r), \quad \omega(r) = \omega_0(r) + \delta \omega(r), \]
\[ f(r) = f_0(r) + \delta f(r), \] (3.6)

where the subscript “0” labels the 2-derivative solution (2.4). Again, the \( \alpha' \) corrected 3-charge dyonic solution can be obtained from the \( \alpha' \) corrected 2-charge solution by the Lorentz boost (2.2). Functions \( \delta L(r), \delta D(r), \delta A(r), \delta f(r) \) remain the same as the unboosted solution, while there is a mixing among the metric components in the \( dt, dx \) direction

\[ \delta h_1(r) = s_3^2 \delta h_3 - 2 c_3 s_3 \delta \bar{\omega}, \quad \delta h_2(r) = - s_3^2 \delta h_3 + 2 c_3 s_3 \delta \bar{\omega}, \quad \delta \omega(r) = - c_3 s_3 \delta h + \left( c_3^2 + s_3^2 \right) \delta \bar{\omega}, \] (3.7)
where $s_3 = -\sqrt{Q_2 \over \mu}$. All the perturbed functions $\delta L(r), \delta D(r), \delta A(r), \delta f(r), \delta h, \delta \omega$ can be found in appendix A. The horizon defined by (2.5) receives correction too

$$r_h \to \sqrt{\mu} + \alpha' \delta r, \quad \delta r = \sqrt{Q_1} \left( -\frac{\mu (\mu + P) - 4\mu Q_1 - 4Q_1^2}{16Q_1(\mu + Q_1)^2} - \frac{\mu^{5/2}(\mu + P)}{16Q_1(\mu + Q_1)^2} \log \left( \frac{1 + Q_1}{\mu} \right) \right). \quad (3.8)$$

Up to $\mathcal{O}(\alpha')$, the Killing vector which becomes null on the horizon is still given by

$$\xi = \partial_t + V_x \partial_x, \quad V_x = \sqrt{Q_2 \over \mu + Q_2}, \quad (3.9)$$

using which we can obtain the temperature according to (2.8)

$$T = \frac{1}{2\pi} \frac{\mu}{\sqrt{\mu + P} \sqrt{\mu + Q_1 \sqrt{\mu + Q_2}} - \frac{\mu Q_1(5\mu + 4Q_1)}{4\pi\sqrt{\mu + P}(\mu + Q_1)^{3/2}(\mu + 2Q_1)\sqrt{\mu + Q_2}} \alpha'}. \quad (3.10)$$

The electric and magnetic charges are computed by the standard way

$$Q_e = \frac{1}{16\pi} \int_{S^3} \star M_{(3)} = \frac{1}{4} \mu Q_1 \sqrt{\mu + Q_1}, \quad (3.11)$$

$$Q_m = \frac{1}{16\pi} \int_{S^3} H_{(3)} = \frac{1}{4} \mu Q_1 \sqrt{\mu + P}, \quad (3.12)$$

where $d \star M_{(3)} = 0$ is the $B_{(2)}$ field equation with

$$-6M^{\mu\nu\rho} = L H^{\mu\nu\rho} + \frac{\alpha'}{16} \left( 12 \frac{R^{|\mu H_{\lambda}^{\nu\rho}}}{16} - 2RH^{\mu\nu\rho} - \frac{1}{6} H^2 H^{\mu\nu\rho} - 2H^{[\mu \lambda} H^{2\nu|\lambda, \rho]}_{\sigma} + 4\Box H^{\mu\nu\rho} \right) + 6R_{\alpha\beta}^{[\mu\nu}(H^{\sigma]}_{\lambda\rho] \alpha\beta} + 2CS^{\mu\nu\rho}(\Gamma_+) - 6R_{\alpha\beta}^{[\mu\nu}(H^{\sigma]}_{\lambda\rho] \alpha\beta} + 2CS^{\mu\nu\rho}(\Gamma_-). \quad (3.13)$$

The electric and magnetic potential are given by

$$\Phi_e = \xi^\mu B_{\mu x}|_{r=\infty} - \xi^\mu B_{\mu x}|_{r=r_h} = \sqrt{Q_1 \over \mu + Q_1} + \frac{\mu \sqrt{Q_1}(5\mu + 4Q_1)}{2(\mu + Q_1)^{3/2}(\mu + 2Q_1)} \alpha', \quad (3.14)$$

$$\Phi_m = \xi^\mu \tilde{B}_{\mu x}|_{r=\infty} - \xi^\mu \tilde{B}_{\mu x}|_{r=r_h} = \sqrt{P \over \mu + P}. \quad (3.15)$$

Below we compute the mass and the linear momentum by integrating the infinitesimal Hamiltonian associated with Killing vectors $\partial_t$ and $\partial_x$ respectively. Now the charge $Q$ and the surface term $\Theta$ both receive higher derivative corrections. Their expressions can be found in appendix B. The mass can be read off from the gravitational contribution to the infinitesimal Hamiltonian associated with $\partial_t$, namely,

$$\delta M = \delta H_{\infty} [\partial_t] = \int_{\infty} (\delta Q[\partial_t] - i_{\partial_t} \Theta)$$

$$\Rightarrow M = \frac{3\pi}{8} \mu + \frac{\pi}{4} (Q_1 + Q_2 + P) - \frac{3\pi \mu^2(3\mu + 2Q_1)}{32(\mu + Q_1)^2(\mu + 2Q_1)} \alpha'. \quad (3.16)$$
Similarly, we can obtain the momentum by replacing $\partial_t$ with $\partial_x$

$$
\delta P_x = -\delta H_\infty[\partial_x] = - \int_\infty (\delta Q[\partial_x] - i_{\partial_x} \Theta)
$$

$$
\Rightarrow P_x = \frac{\pi}{4} \sqrt{Q^2 \mu + Q^2}.
$$

(3.17)

It appears that the linear momentum does not receive $\alpha'$ correction.

In order to compute the entropy, we now investigate the complete infinitesimal Hamiltonian associated with $\xi = \partial_t + V_x \partial_x$. The higher derivative corrections to the infinitesimal Hamiltonian are given in appendix B. In the presence of the non-diffeomorphism invariant CS term, the infinitesimal Hamiltonian in general takes the form

$$
\delta H_\Sigma = \int_\Sigma (\delta Q[\xi] - i_{\xi} \Theta[\delta \phi] - \Sigma[\xi]).
$$

(3.18)

As showed in the previous section, the above infinitesimal Hamiltonian missed the magnetic contribution to the first law and should be improved by adding $-d(\Psi(1) \wedge \delta B(2))$

$$
\delta H_\Sigma[\xi] = \int_\Sigma (\delta Q[\xi] - i_{\xi} \Theta[\xi] - d(\Psi(1) \wedge \delta B(2))),
$$

(3.19)

where the 1-form $\Psi(1)$ is defined by

$$
d\Psi(1) = i_{\xi} \star M(3).
$$

(3.20)

For the $\alpha'$ corrected solution, it takes the form

$$
\Psi(1) = \frac{\sqrt{P} \sqrt{\mu + P}}{P + r^2} \Upsilon(V_x dt - dx),
$$

(3.21)

where $\Upsilon$ is

$$
\Upsilon = 1 + \frac{\alpha'}{8Q_1^2 (P + r^2) (\mu + Q_1)^2 (Q_1 + r^2)^2}
$$

$$
\times Q_1 \left( \mu^2 r^2 (P + r^2) (2r^2 - \mu) + 4\mu Q_1^2 (2\mu - 5r^2) + \mu^2 Q_1^2 \left( 4\mu + P - 15r^2 \right) + 4Q_1^4 \left( \mu - 2r^2 \right) + \mu Q_1 \left( -\mu P + 3Pr^2 - 5\mu r^2 + 3r^4 \right) \right)
$$

$$
- \mu^2 \left( P + r^2 \right) (Q_1 + r^2)^2 (2r^2 - \mu) \log \left( \frac{Q_1}{r^2} \right) + O \left( \alpha'^2 \right).
$$

(3.22)

At this moment, we need to distinguish the infinitesimal Hamiltonians associated with two formulations of the higher derivative actions. In the original actions (3.2) and (3.3), the mixed CS term is given by $-2B(2) \wedge \text{tr}(R(\Gamma_\pm) \wedge R(\Gamma_\pm))$. We denote the corresponding total infinitesimal Hamiltonian as $\delta H^{(1)}_\Sigma$. By adding a total differential $d(2B(2) \wedge CS_{(3)}(\Gamma_\pm))$, one obtains another formulation in terms of $2H(3) \wedge CS_{(3)}(\Gamma_\pm)$. We denote the corresponding total infinitesimal Hamiltonian as $\delta H^{(2)}_\Sigma$. Here these two infinitesimal Hamiltonians have been improved with the addition of the term $-d(\Psi(3) \wedge \delta B(2))$. 


We now evaluate the infinitesimal Hamiltonian at spatial infinity and horizon using the 3-charge string solution. It turns out that we indeed have
\[
\delta H^{(2)}_{\infty} = \delta H^{(2)}_B, \quad (3.23)
\]
from which we can read off the entropy that satisfies the first law automatically
\[
S_{\text{IIA}} = \frac{\pi^2}{2} \sqrt{\mu + P} \sqrt{\mu + Q_1} \sqrt{\mu + Q_2} + \frac{\pi^2 Q_1 \sqrt{\mu + P} \sqrt{\mu + Q_2} (5 \mu + 4 Q_1)}{4 (\mu + Q_1)^{3/2} (\mu + 2 Q_1)} \alpha', \quad (3.24)
\]
which reproduces the result in [18] when \(Q_2 = 0\). However, for \(\delta H^{(1)}_{\Sigma}\), apparently, its value at the infinity is not equal to its value at the horizon. This means there is a topological obstruction forbidding us to apply the Gauss theorem. After comparing the infinitesimal Hamiltonians associated with two different formulations of the action, we find that the density of \(\delta H^{(1)}_{\Sigma}\) needs a further improvement by adding a term of the form \(\alpha' d\Pi\) where
\[
\Pi = \delta \left( 2 B_{(2)} \wedge \Gamma^a_{+b} \xi^b \Gamma^b_{+a} + 2 B_{(2)} \wedge \Gamma^a_{-b} \xi^b \Gamma^b_{-a} \right) + i \xi \left( 2 B_{(2)} \wedge \Gamma^a_{+b} \wedge \delta \Gamma^b_{+a} + 2 B_{(2)} \wedge \Gamma^a_{-b} \wedge \delta \Gamma^b_{-a} \right). \quad (3.25)
\]
With the second improvement, we find indeed
\[
\delta H^{(1)}_{\infty} = \delta H^{(1)}_B. \quad (3.26)
\]
From this we can read off the same entropy (3.24). By contrast, the Wald entropy formula would lead to different entropies of the black strings for these two different formulation of the theory.

Some remarks need to make here. The entropy formula (3.24) does not coincide with one computed using Wald formula applied to the original formulation of the action with the \(-2 B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))\) term. Also it cannot be obtained using the Tachikawa formula applied to the second formulation with \(2 H_{(3)} \wedge CS_{(3)}(\Gamma_{\pm})\) term. To be specific, we have
\[
S_W = S_{\text{IIA}} - \frac{\pi^2 \sqrt{PQ_2}}{4 \sqrt{\mu + Q_1}} \alpha', \quad S_T = S_{\text{IIA}} + \frac{\pi^2 \sqrt{PQ_2}}{4 \sqrt{\mu + Q_1}} \alpha', \quad (3.27)
\]
from which we see that only when \(P Q_2 = 0\), the Wald-Tachikawa formula yields the right answer in agreement with our previous calculation [18]. We also note that in the extremal case \(\mu = 0\), the entropy obtained from Tachikawa formula equals the one obtained from Sen’s approach. We present the computation based on Sen’s approach in appendix C.

For the 3-charge solution, the improvement term takes the form
\[
d\Pi_{\infty} = 0, \quad d\Pi_{r=r_h} = - \frac{\pi \mu}{\sqrt{P + \mu \sqrt{Q_2} + \mu (Q_1 + \mu)}} \left( \frac{\mu \sqrt{Q_2}}{\sqrt{P (\mu + P)}} \delta P - \frac{2 \sqrt{P} \sqrt{Q_2}}{\mu + Q_1} \delta Q_1 
+ \frac{2 \sqrt{P}}{\sqrt{Q_2}} \delta Q_2 + \frac{\sqrt{P} \sqrt{Q_2} (3 \mu + 2 P + Q_1)}{(\mu + P) (\mu + Q_1)} \delta \mu \right). \quad (3.28)
\]
We see that it vanishes on \( P = 0, \ Q_2 = 0 \) or \( \mu = 0 \). This explains why we had not encountered it in our previous work dealing with the 2-charge string solution without linear momentum.

We have checked that the near horizon geometry for the non-extremal black string is perfectly smooth describing \( \mathbb{R} \times \text{Rindler}_2 \times S^3 \). The reason that Wald or Tachikawa formula does not apply to our case is due to the fact on the horizon not only terms proportional \( \nabla_a \xi_b \) contribute, but also terms proportional to Killing vector \( \xi \) have nonvanishing contributions. After a careful calculation we find that in the improved infinitesimal Hamiltonian evaluated on the horizon \( \delta \mathcal{H}_B \), terms proportional to \( \delta B \) contribute to \( \Phi_e dQ_e + \Phi_m dQ_m \), which is known previously. However, terms proportional to \( \xi, \delta g, \delta L \) which one would naively think to vanish on the horizon also contribute. We denote these contributions as \( \delta \mathcal{H}_B[\xi, \delta g, \delta L] \). Together with the variation of the Wald charge term, they yield \( T dS \) term in the first law. As a concrete example, in the formulation of the 4-derivative action with \( -2B_{(2)} \wedge \text{tr}(R(\Gamma_\pm) \wedge R(\Gamma_\pm)) \), we obtain

\[
\delta \mathcal{H}_B[\xi, \delta g, \delta L] = \frac{\mu(P - Q_1)}{4(\mu + P)(\mu + Q_1)} \delta \mu
+ \frac{\alpha'}{32Q_1^2(\mu + P)(\mu + Q_1)^2} (A_1 \delta \mu + A_2 \delta Q_1 + A_3 \delta Q_2 + A_4 \delta P,)
\]

(3.29)

where

\[
A_1 = -\frac{Q_1^2}{(\mu + P)^{3/2}(\mu + 2Q_1)} \sqrt{\mu + Q_2} \left( 2Q_1 \sqrt{PQ_2}(\mu + P)(\mu + Q_1)^2 (\mu + 2Q_1)
\times (\mu(6\mu + Q_1) - P(Q_1 - 4\mu)) + \mu P \sqrt{\mu + Q_2} \left( \mu^5(P - 13Q_1)
+ \mu^4(P - 7Q_1)(2P + Q_1) + \mu^2 Q_1^2 \left( 46P^2 + 133PQ_1 - 29Q_1^2 \right)
+ 12PQ_1^2(5P - 2Q_1) + 2\mu Q_1^2 \left( 59P^2Q_1 - 2P^3 + 19PQ_1^2 - 13Q_1^3 \right)
+ \mu^3(-P^2Q_1 + P^3 + 48PQ_1^2 + 6Q_1^3) \right) \right),
\]

\[
A_2 = -\mu^2Q_1 \left( \mu P^2(2\mu + 3Q_1) + P \left( 2\mu^3 + 7\mu Q_1^2 - 4Q_1^3 \right)
- Q_1 \left( 2\mu^3 + 3\mu^2 Q_1 - 4\mu Q_1^2 - 4Q_1^3 \right) \right)
+ 2\mu^4(\mu + P)(P - Q_1)(\mu + 2Q_1) \log \left( \frac{\mu + Q_1}{\mu} \right),
\]

\[
A_3 = \frac{2\mu \sqrt{PQ_1} \sqrt{\mu + P}(\mu + Q_1)^2}{\sqrt{\mu + Q_2}},
\]

\[
A_4 = \mu^3(Q_1(P - Q_1)(\mu + Q_1) \left( Q_1 - \mu \log \left( \frac{\mu + Q_1}{\mu} \right) \right) \right). \]

(3.30)

In the BPS limit \( \mu \to 0 \), the entropy density and various charges become

\[
S = \frac{\pi^2}{2} \sqrt{PQ_1Q_2} + \frac{\pi^2}{2} \sqrt{\frac{PQ_2}{Q_1} \alpha'} = \frac{\pi^2}{2} \sqrt{P(Q_1 + 2\alpha')Q_2 + \mathcal{O}(\alpha')^2},
\]

\[
Q_e = \frac{\pi}{4} Q_1, \quad Q_m = \frac{\pi}{4} P, \quad P_x = \frac{\pi}{4} Q_2, \quad M = \frac{\pi}{4}(Q_1 + Q_2 + P).
\]

(3.31)
The near horizon limit of the BPS solution is given by an extremal BTZ $\times S^3$ whose entropy is the same as the entropy of the full BPS string solution, since the entropy is determined by the geometry of the horizon. Thus from the entropy of the extremal BTZ black hole, one can read off one of the central charges in the dual 2D CFT. In string unit, $\ell_s = 1$ $G_6 = \frac{\pi^2}{T}$, the entropy density can be expressed as\(^3\)

$$S = \sqrt{P(Q_1 + 2)Q_2}, \quad (3.32)$$

where parameters $P, Q_1, Q_2$ are integer valued corresponding the number of NS5 compactified on K3, number of fundamental string and excitation level of momentum modes respectively. The formula (3.32) implies that one of central charge in the 2D CFT dual to IIA string on $AdS_3 \times S^3 \times K3$ is given by

$$c = 6P(Q_1 + 2). \quad (3.33)$$

The finite shift in $Q_1$ is reminiscent of the central charge in the CFT dual to heterotic string in the $AdS_3 \times S^3 \times T^4$ background [32]. In fact, since IIA string on K3 is dual to heterotic string on 4-torus, it is natural to expect that the CFT duals of two scenarios should have the same central charge. As we will see, this is indeed the case in the next section.

We end this section by computing the Euclidean action of the $\alpha'$-corrected 3-charge string solution. Similar to the 2-charge string solution, we evaluate the Euclidean action using the background substraction method. The total action is thus given by

$$I_E = I_0 + I_{\text{hd}} + I_{\text{GHY}} - I_c, \quad (3.34)$$

where $I_0$ and $I_{\text{hd}}$ are the leading and subleading bulk actions, while $I_{\text{GHY}}$ and $I_c$ are the string frame analogue of Gibbons-Hawking-York boundary term and background subtraction term given below

$$I_{\text{GHY}} = \frac{1}{8\pi} \int_{\partial M} d^5x\sqrt{-hLK}, \quad I_c = \frac{1}{8\pi} \int_{\partial M} d^5x\sqrt{-h\bar{L}\bar{K}}. \quad (3.35)$$

Here we emphasize that Euclidean action seems to be insensitive to the choice of $-2B(2) \wedge \text{tr}(R(\Gamma_+) \wedge R(\Gamma_+))$ or $2H(3) \wedge CS(3)(\Gamma+)$. Both formulations yield the same answer. The reason is that the contribution from $-2B(2) \wedge \text{tr}(R(\Gamma_+) \wedge R(\Gamma_+))(2H(3) \wedge CS(3)(\Gamma+))$ cancels with that from $-2B(2) \wedge \text{tr}(R(\Gamma_-) \wedge R(\Gamma_-))(2H(3) \wedge CS(3)(\Gamma-))$.

In the intermediate steps, the boundary is located at some large value of the radial coordinate $r = r_c$ which will be taken to infinity eventually. $K$ is the trace of the extrinsic curvature of the $r = r_c$ hypersurface embedded in $\alpha'$-corrected string solution. $\bar{K}$ is the trace of the extrinsic curvature of the $r = r_c$ hypersurface embedded in the flat background metric below

$$ds_6^2 = D(r_c) \left( h_1(r_c) d\tau^2 + h_2(r_c) dx^2 \right) + dR^2 + R^2d\Omega_3^2, \quad R^2 = r^2H_p(r). \quad (3.35)$$

After substituting the solution in (3.33), we obtain

$$I_E = \beta G, \quad \beta = T^{-1}, \quad G = \frac{\pi}{8}(\mu + 2P) - \frac{\pi\mu(9\mu + 8Q_1)}{32(\mu + Q_1)^2}\alpha', \quad (3.36)$$

\(^3\)The factor of $2\pi$ in the entropy formula is recovered had we compactified the $x$-direction with period $2\pi$ and calculated the entropy instead of its density.
which indeed satisfies

\[ G = M - TS = \Phi_e Q_e - V_x P_x. \] (3.37)

Instead we can also use Reall-Santos method [33]. A consequence of [33] is that at fixed
conserved charges, the entropy is given by \(-I_{\text{hd}}\) evaluated on the leading order solution.
For the 3-charge string solution, we have

\[ I_{\text{hd}} = -\frac{\pi \beta \mu (9 \mu + 8 Q_1)}{32 (\mu + Q_1)^2} \alpha', \] (3.38)

which does match with \(\Delta S(M, Q_e, Q_m)\).

4 Six-dimensional heterotic/Type IIA duality

In this section, we use the IIA/heterotic duality to study the leading \(\alpha'\) corrections to the
thermodynamics of the 3-charge string solutions in 6D heterotic string compactified on
4-torus. In the heterotic string, the leading \(\alpha'\) corrections arise at the tree level [27, 35].
When compactified on 4-torus, the heterotic string is dual to IIA string compactified on
K3 [36]. After discarding the Yang-Mills field, the bosonic Lagrangian takes the form

\[ L = L \left( R + L^{-2} \nabla^\mu L \nabla_\mu L - \frac{1}{12} \tilde{H}_{\mu
u\rho} \tilde{H}^{\mu\nu\rho} + \frac{\alpha'}{8} R_{\mu\nu\alpha\beta} (\Gamma_+) R^{\mu\nu\alpha\beta} (\Gamma_+) \right), \] (4.1)

where the 3-form field strength

\[ \tilde{H}_{(3)} = H_{(3)} + \frac{1}{4} \alpha' CS_{(3)} (\Gamma_+) \] (4.2)
satisfies a non-trivial Bianchi identity

\[ d\tilde{H}_{(3)} = \frac{1}{4} \alpha' R^a_b (\Gamma_+) \wedge R^b_a (\Gamma_+). \] (4.3)

The equations of motion in linear order of \(\alpha'\) are

\[ 0 = R_{\mu\nu} - L^{-2} \nabla^\mu L \nabla_\nu L - \frac{1}{12} \tilde{H}_{\mu
u\rho} \tilde{H}^{\mu\nu\rho} + \frac{\alpha'}{8} R_{\mu\nu\alpha\beta} (\Gamma_+) R^{\mu\nu\alpha\beta} (\Gamma_+) \],

\[ 0 = R_{\mu\nu} - L^{-2} \nabla^\mu L \nabla_\nu L + L^{-2} (\nabla_\mu L) (\nabla_\nu L) - \frac{1}{4} \tilde{H}_{\mu\nu}^2 + \frac{1}{4} \alpha' R_{\mu\nu\alpha\beta} (\Gamma_+) R^{\alpha\beta} (\Gamma_+) \],

\[ 0 = d \left( L \star \tilde{H}_{(3)} \right). \] (4.4)

Up to this \(\alpha'\) order, the above field equations are mapped to those in IIA case via [31]

\[ L_{\text{IIA}} \star H_{(3)}^{\text{IIA}} = \tilde{H}_{(3)}^{\text{het}}, \quad L_{\text{IIA}} g_{\mu\nu}^{\text{IIA}} = g_{\mu\nu}^{\text{het}}, \quad L_{\text{IIA}} = \frac{1}{L_{\text{het}}}. \] (4.5)

We have checked that the 3-charge solution in IIA indeed maps to a 3-charge string solution
in heterotic side. Similar to the 2-charge case studied in [18], one needs to perform a shift
on the \(B_{(2)}\)

\[ B_{(2)} \rightarrow B_{(2)} - \Lambda_{(2)}, \quad \Lambda_{(2)} = \frac{8 P}{r^2 + P} \sqrt{1 + \frac{\mu}{P}} \alpha' (\Gamma_{(2)}). \] (4.6)
so that the ansatz (2.3) is still applicable. As pointed out in [18], that for the 2-charge solution, the thermodynamic quantities in the heterotic side is related to those in the IIA side by the change of variables
\[ P \rightarrow Q_1, \quad Q_1 \rightarrow P. \] (4.7)

Now since the map (4.5) commutes with the Lorentz boost, we can obtain the thermodynamic quantities for the heterotic string. The conserved charges can also be computed independently using other methods except for the entropy which can only be derived from the first law. The results are listed below

\[ M^{(\text{het})} = \frac{3\pi}{8} \mu + \frac{\pi}{4} (Q_1 + P + Q_2) - \frac{3\pi \mu^2 (3\mu + 2P)}{32(\mu + P)^2(\mu + 2P)} \alpha', \]
\[ T^{(\text{het})} = \frac{1}{2\pi} \sqrt{\mu + P} \sqrt{\mu + Q_1} \sqrt{\mu + Q_2} - \frac{\mu P (5\mu + 4P)}{4\pi \sqrt{\mu + Q_1} \sqrt{\mu + Q_2} (\mu + P)^{5/2} (\mu + 2P)} \alpha', \]
\[ S^{(\text{het})} = \frac{1}{2\pi} \sqrt{\mu + P} \sqrt{\mu + Q_1} \sqrt{\mu + Q_2} + \frac{\mu \sqrt{P} \sqrt{Q_1} \sqrt{\mu + Q_2}}{4(\mu + P)^{3/2} (\mu + 2P)} \alpha', \]
\[ Q_e^{(\text{het})} = \frac{\pi}{4} \sqrt{Q_1} \sqrt{\mu + Q_1}, \quad \Phi_e^{(\text{het})} = \sqrt{\frac{Q_1}{\mu + Q_1}}, \]
\[ Q_m^{(\text{het})} = \frac{\pi}{4} \sqrt{P} \sqrt{\mu + P}, \quad \Phi_m^{(\text{het})} = \sqrt{\frac{P}{\mu + P} + \frac{\mu \sqrt{P} (5\mu + 4P)}{2(\mu + P)^{5/2} (\mu + 2P)}} \alpha', \]
\[ P_x^{(\text{het})} = \frac{\pi}{4} \sqrt{Q_2} \sqrt{\mu + Q_2}, \quad V_x^{(\text{het})} = \sqrt{\frac{Q_2}{\mu + Q_2}} \] (4.8)

We find that the Wald-Tachikawa entropy formula and the recently proposed covariantized Wald-Tachikawa entropy [37] yield
\[ S_{\text{WT}} = S^{(\text{het})} + \frac{\pi^2 \sqrt{Q_1 Q_2}}{4\sqrt{\mu + P}} \alpha', \] (4.9)

which does not satisfy the first law of thermodynamics. In the BPS limit \( \mu \rightarrow 0 \), the entropy becomes to
\[ S^{(\text{het})} = \frac{\pi^2}{2} \sqrt{P Q_1 Q_2} + \frac{\pi^2}{2} \sqrt{\frac{Q_1 Q_2}{P}} \alpha' = \frac{\pi^2}{2} \sqrt{(P + 2\alpha')Q_1 Q_2}. \] (4.10)

5 Conclusion

In this work, using the Wald procedure, we performed a careful study on the leading \( \alpha' \) corrections to the first law of thermodynamics for 3-charge string solutions in 6D supergravity arising from IIA string compactified on K3. The low energy effective action contains mixed CS terms of the form \(-2B_{(2)} \wedge \text{tr}(R(\Gamma_{\pm}) \wedge R(\Gamma_{\pm}))\) which can also be recast as \(2H_{(3)} \wedge \text{CS}_{(3)}(\Gamma_{\pm})\) up to a total derivative. We found that the infinitesimal Hamiltonian derived from the former formulation does not lead to the desired equality \( \delta H_{\infty} = \delta H_B \) that implies the first law of thermodynamics. Thus it cannot be used to define the entropy.
consistently. On the other hand, the infinitesimal Hamiltonian derived from the second formulation involving $2H_{(3)} \wedge CS_{(3)}(\Gamma_\pm)$ gives straightly the desired equality $\delta H_\infty = \delta H_B$. By comparing the infinitesimal Hamiltonians associated with the 2 formulations, we realized that the density of the infinitesimal Hamiltonian associated with the first formulation can be improved by adding a closed but non-exact 4-form (65,67) whose value at spatial infinity and horizon is just right to restore the first law. With this improvement, both formulations yield infinitesimal Hamiltonians obeying the equality $\delta H_\infty = \delta H_B$. From the first law of thermodynamics we read off the $\alpha'$-corrected entropy of the 3-charge black string solution. Taking the extremal limit, we obtained the $\alpha'$-corrected entropy for BPS 3-charge string solution (3.31) that is different from the previous results [24, 25] obtained by directly applying the Wald-Tachikawa formula or attractor mechanism. We also reproduced our entropy result using another method proposed by Reall and Santos [33]. Thus we have found a case where the Wald-Tachikawa formula does not apply and developed a procedure to find the entropy consistent with the first law.

To our surprise, the entropy cannot be obtained directly from either Wald formula or Tachikawa formula. Our results illustrate a danger of using either formulae to compute directly the entropy of extremal black holes, where the first-law of black hole thermodynamics cannot provide a consistent check at zero temperature. In our computation, we noticed that on the horizon terms proportional to $\xi$ can also contribute to the first law instead of being vanishing. One possible reason is that we usually define the linearized solution by

$$\delta(c_i)\phi = \delta c_i \partial_{c_i} \phi(c_i) \quad (5.1)$$

where $\phi$ is a shorthand notation for all the fields in the theory and $c_i$ denotes the physical parameters. This seems to be a natural way to generate the linearized solutions. However, it is likely that for generic $c_i$, the linearized solutions obtained this way do not satisfy the smoothness condition needed in the abstract proof given by [38–40] where terms proportional to $\xi$ all vanish on the horizon.

Our way of improving the infinitesimal Hamiltonian can be readily generalized to other mixed CS terms in diverse dimensions, for instance the $A \wedge R \wedge R$ and $A \wedge F \wedge F$ terms. These terms appear in 5D 4-derivative supergravity actions together with the curvature squared terms [41, 42] and are relevant in the precision test of AdS$_5$/CFT$_4$ correspondence [43–46]. Compared to the Euclidean action method [33], the improved infinitesimal Hamiltonian is applicable to all solutions regardless of its asymptotic structure since it is insensitive to the asymptotics of the spacetime metric. Finally, it would be interesting to apply our procedure to study first law of thermodynamics for other black objects such as black rings in models with Chern-Simons interactions [47].

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A Perturbative solutions in IIA

In this appendix section, we present the perturbations to the 2-charge string solution in the type IIA case which are used to generate the perturbative solutions to the 3-charge string solutions. We first define

\[
C_0 = 2C_1 Q_1 + C_3 Q_1 (\mu + 2 Q_1) - 2 (C_4 + C_7) Q_1 (\mu + Q_1) + 2C_5 (\mu + Q_1),
\]

\[
X(r) = \left(1 + \frac{Q_1}{\mu}\right)^{-2} \left(3 + \frac{2Q_1}{\mu}\right) \log \left(1 - \frac{\mu}{r^2}\right) - \frac{\mu^2}{Q_1^2} \log \left(1 + \frac{Q_1}{r^2}\right).
\]

(A.1)

Solutions to various perturbations are given by

\[
\delta \tilde{\omega} = \frac{\mu \sqrt{P \sqrt{\mu - P}}}{2 \pi r (P + r^2)(Q_1 + r^2)} \alpha' \sqrt{\frac{r^2}{\mu}} \alpha' \mu,
\]

\[
\delta f = \frac{C_1}{r^2} \left(1 + \frac{P}{r^2}\right) - \frac{\mu^2 + 4 P (\mu + 2 Q_1) + 4Q_1^2 + 4 \mu Q_1}{\mu^2 + 4P (\mu + 2 Q_1) + 4Q_1^2 + 4 \mu Q_1 + (1 + \frac{Q_1}{\mu})^{-1} \left(2 (\mu + 2 Q_1)ight)} + 2P (\mu + 2 Q_1) - 2 \mu P (\mu + 2 Q_1) - 4 \mu Q_1 (\mu + Q_1) + \frac{P}{r^2} \left(2 (\mu + 2 Q_1) - 2 \mu P (\mu + Q_1) + \mu (2 \mu - Q_1) \right) \left(1 + \frac{Q_1}{r^2}\right)^{-1} \left(1 + \frac{P}{r^2}\right)^{-1} \left(1 + \frac{Q_1}{r^2}\right) \log \left(1 - \frac{\mu}{r^2}\right) - \frac{\alpha'}{8 \mu} \left(2 - \frac{\mu}{r^2}\right) \left(1 + \frac{P}{r^2}\right) X(r),
\]

\[
\delta h = \frac{C_1}{r^2} \left(1 + \frac{P}{r^2}\right) + \frac{\mu^2 + 4 P (\mu + 2 Q_1) + 4Q_1^2 + 4 \mu Q_1}{\mu^2 + 4P (\mu + 2 Q_1) + 4Q_1^2 + 4 \mu Q_1 + (1 + \frac{Q_1}{\mu})^{-1} \left(2 (\mu + 2 Q_1)ight)} + 2P (\mu + 2 Q_1) - 2 \mu P (\mu + 2 Q_1) - 4 \mu Q_1 (\mu + Q_1) + \frac{P}{r^2} \left(2 (\mu + 2 Q_1) - 2 \mu P (\mu + Q_1) + \mu (2 \mu - Q_1) \right) \left(1 + \frac{Q_1}{r^2}\right)^{-1} \left(1 + \frac{P}{r^2}\right)^{-1} \left(1 + \frac{Q_1}{r^2}\right) \log \left(1 - \frac{\mu}{r^2}\right) - \frac{\alpha'}{8 \mu} \left(2 - \frac{\mu}{r^2}\right) \left(1 + \frac{P}{r^2}\right) X(r),
\]

\[
\delta D = - \left(\frac{C_5}{2 \mu^2} + \frac{C_1 P Q_1}{\mu^2 + 2 P (\mu + 2 Q_1)} \left(2 + \frac{P}{r^2}\right) - \frac{C_2 Q_1}{\mu^2 + 2 P (\mu + 2 Q_1)} \left(2 - \frac{P}{r^2}\right) - \frac{C_3 Q_1}{\mu^2 + 2 P (\mu + 2 Q_1)} \left(1 + \frac{Q_1}{r^2}\right) \right)^{-2} \left(\mu + 2 Q_1\right)^{-1} \left(3 + \frac{4 Q_1}{\mu^2} \left(P + 4 Q_1\right) + \mu^2 \left(3 P + 7 Q_1\right) + 4 \mu Q_1 \left(2 P + 5 Q_1\right)\right) + \frac{P Q_1}{r^2} \left(P (4 Q_1^2 + 2 \mu Q_1 - \mu^2) + 4 \mu Q_1 (\mu + Q_1)\right) \left(1 + \frac{Q_1}{r^2}\right)^{-1} \left(1 + \frac{P}{r^2}\right)^{-1} \left(1 + \frac{Q_1}{r^2}\right) \log \left(1 - \frac{\mu}{r^2}\right) - \frac{\alpha'}{8 \mu^2} \left(3 \mu + 4 Q_1 - \frac{2 Q_1 (\mu + P)}{r^2} + \frac{P Q_1}{r^2}\right) \left(1 + \frac{Q_1}{r^2}\right)^{-2} X(r),
\]
\[ \delta A = C_6 - \left( \frac{C_0 Q_2}{2 \mu^2 r^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 1 - \frac{\mu}{r^2} \right) + \frac{C_1 P Q_1}{\mu^2 r^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 2Q_1 + \frac{\mu (\mu - Q_1)}{r^2} \right) \right) \]
\[ + \frac{C_2 Q_1}{2 \mu^2 r^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( \mu + 4Q_1 - \frac{2P (\mu + Q_1) + 3\mu Q_1}{r^2} \right) \]
\[ + \frac{C_3 Q_2}{2 \mu^2 r^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 1 + \frac{Q_1}{r^2} \right)^{-2} \]
\[ + \frac{\alpha'}{8 \mu^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 4 \left( \mu + 2Q_1 \right) - \frac{5\mu^2 + 2P (\mu + 2Q_1) - 16Q_1 + 2\mu Q_1}{r^2} \right) \]
\[ - \frac{1}{r^4} \left( P \left( -\mu^2 + 8Q_1^2 + 4\mu Q_1 + 5\mu Q_1 (3\mu + 4Q_1) \right) - \frac{Q_1}{r^2} \left( P \left( -\mu^2 + 4Q_1^2 + 2\mu Q_1 \right) \right) \]
\[ + \frac{2\mu Q_1 (\mu + Q_1)}{r^4} \right) - \frac{Q_1}{8 \mu^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 4C_0 \mu \left( 1 - \frac{\mu}{r^2} \right) + 16C_1 P Q_1 \left( 1 - \frac{\mu}{r^2} \right) \right) \]
\[ + \frac{Q_1 \alpha'}{8 \mu^2} \left( 4 \left( \frac{5\mu + 2P}{r^2} + \frac{4P}{r^4} \right) \right) \left( 1 + \frac{Q_1}{r^2} \right)^{-2} X (r) , \]
\[ \delta L = \left( \frac{C_0 P (2Q_1 - \mu) + 2C_2 (P + 2Q_1) - C_3 \mu^{2} Q_1}{2 \mu^2 r^2} \right) + C_8 \left( 1 + \frac{Q_1}{r^2} \right) \left( 1 + \frac{P}{r^2} \right)^{-1} \]
\[ - \frac{\alpha'}{8 \mu^2 r^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( \mu + 2Q_1 (3 \mu + 2P + 4Q_1) - \frac{1}{r^4} \left( -8Q_1^2 (P + 2Q_1) + \mu^2 (P - 7Q_1) \right) \right) \]
\[ - \frac{4\mu Q_1 (P + 5Q_1)}{r^4} \left( \frac{Q_1}{r^2} \right) \left( P \left( \mu^2 - 4Q_1^2 - 2\mu Q_1 + 2\mu Q_1 (2\mu + 3Q_1) \right) \right) \left( 1 + \frac{Q_1}{r^2} \right)^{-2} \left( 1 + \frac{P}{r^2} \right)^{-1} \]
\[ + \frac{1}{8 \mu^2} \left( 1 + \frac{Q_1}{\mu} \right)^{-1} \left( 2 \left( C_0 \mu + 4C_1 P Q_1 \right) \left( \mu + 2Q_1 - \frac{\mu Q_1}{r^2} \right) \right) + 2C_2 \left( 3\mu^2 + 4P (\mu + Q_1) + 8Q_1^2 \right) \]
\[ + \frac{8\mu Q_1 - \mu 2P (P + 2Q_1) + 16Q_1 (\mu + 4Q_1)}{r^4} \right) \left( 1 + \frac{P}{r^2} \right)^{-1} X (r) . \]  
(A.2)

The physical solution, with an appropriate horizon and asymptotic falloffs, corresponds to the parameter choice

\[ \{ C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8 \} \]
\[ = \left\{ 0, \frac{\mu^2 (3\mu + 2Q_1)}{4(\mu + Q_1)^2}, \frac{3\mu^2 (3\mu + 2Q_1)}{4Q_1 (\mu + Q_1)^2 (\mu + 2Q_1)}, 0, 0, 0, 0 \right\} . \]  
(A.3)

### B α′ corrections to infinitesimal Hamiltonian

In this appendix section, we present the contributions to the Noether charge \( Q \) and surface term \( \Theta \) from the higher derivative action. In the presence of non-diffeomorphism invariant action such as the \( 2H_{(3)} \wedge CS_{(3)} (\Gamma_+) \) term, there is a third contribution to the infinitesimal Hamiltonian denoted by \( \Sigma [\xi ] \) [20]. Thus the most general form of the infinitesimal Hamiltonian in a theory with CS type interaction is given by

\[ \delta H_{\Sigma} = \int_{\Sigma} (\delta Q [\xi ] - i_x \Theta [\delta \phi ] - \Sigma [\xi ]) . \]  
(B.1)
Due to the CS terms, the Noether charge and surface term contain covariant part and non-covariant part

\[
Q^\mu = Q^\mu_{\text{cov}} + Q^\mu_{\text{nc}}, \quad \Theta^\mu = \Theta^\mu_{\text{cov}} + \Theta^\mu_{\text{nc}}.
\]  
(B.2)

The covariant parts in \( Q \) and \( \Theta \) consist of several pieces

\[
Q^\mu_{\text{cov}} = Q^\mu_{\text{cov}1} + \frac{\lambda_{\text{GB}}}{16} Q^\mu_{\text{cov}2+} + \frac{\lambda_{\text{Riem}}^2}{16} Q^\mu_{\text{cov}2-},
\]  
(B.3)

\[
\Theta^\mu_{\text{cov}} = \Theta^\mu_{\text{cov}1} + \frac{\lambda_{\text{GB}}}{16} \Theta^\mu_{\text{cov}2+} + \frac{\lambda_{\text{Riem}}^2}{16} \Theta^\mu_{\text{cov}2-},
\]  
(B.4)

in which

\[
Q^\mu_{\text{cov}1} = -2P^{\mu\nu\gamma\delta}\nabla_\gamma \xi_\delta + 4\xi_\delta \nabla_\gamma P^{\mu\nu\gamma\delta} + 6M^{\mu\nu\rho\sigma} B_{\nu\rho\sigma} \xi^\gamma,
\]

\[
+ \frac{\lambda_{\text{Riem}}^2}{16} \left( -2\nabla_\nu H_{\alpha\beta\rho} H^{\nu\beta\rho\sigma} \xi^\alpha + 2\nabla_\nu H^{\nu\beta\rho\sigma} H_{\alpha\beta\rho} \xi^\alpha - 2\nabla_\alpha H_{\beta\gamma}^{\mu} H^{\mu\beta\gamma} \xi^\alpha \right),
\]  
(B.5)

\[
\Theta^\mu_{\text{cov}1} = 2P^{\mu\nu\rho\sigma}\nabla_\nu \delta g_{\beta\gamma} - 2\delta g_{\beta\gamma} \nabla_\nu P^{\mu\nu\rho\sigma} + 3M^{\mu\beta\gamma} \delta B_{\beta\gamma}
\]

\[
+ \frac{\lambda_{\text{Riem}}^2}{16} \left( \left( \nabla_\alpha H^{\beta\nu\rho} \right) H^\mu_{\nu\rho} - \left( \nabla_\beta H^{\beta\nu\rho} \right) H^\alpha_{\nu\rho} - \left( \nabla_\gamma H^{\mu\beta\gamma} \right) H^\alpha_{\nu\rho} \right) \delta g_{\alpha\beta}
\]

\[+ \frac{2}{3} \left( \nabla_\alpha H^{\alpha\beta\gamma} \right) \delta H_{\alpha\beta\gamma},
\]  
(B.6)

with

\[
P_{\mu\nu\rho\sigma} = \frac{\lambda_{\text{GB}}}{16} \left( 2R_{\mu\nu\rho\sigma} + 2(g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho}) R_{\nu\rho} + \frac{1}{4} H_{\mu\nu\rho\sigma} + \frac{15}{4} \left( g_{\mu\rho} H_{\nu\sigma} - g_{\mu\sigma} H_{\nu\rho} \right) \right)
\]

\[
+ \frac{1}{12} \left( g_{\mu\rho} g_{\nu\sigma} - 2g_{\mu\sigma} g_{\nu\rho} \right) H^2 + \frac{\lambda_{\text{Riem}}^2}{16} \left( 2R_{\mu\nu\rho\sigma} - \frac{1}{2} H^2_{\mu\nu\rho\sigma} \right),
\]  
(B.7)

\[
M^{\alpha\beta\gamma} = \frac{\lambda_{\text{GB}}}{16} \left( R^{\alpha\beta\gamma} - 2R^{\alpha}_{\lambda\gamma} H^\lambda_{\mu} - 2R^{\alpha}_{\lambda\gamma} H^\lambda_{\mu} + \frac{1}{3} R H^{\alpha\beta\gamma} + \frac{1}{36} H^2 H^{\alpha\beta\gamma}
\]

\[- \frac{1}{2} H^2 \frac{\lambda^{\alpha}_{\lambda}}{\mu\rho\sigma} + \frac{5}{6} H^{\alpha}_{\lambda\gamma} H^{2\beta\gamma}_{\lambda\gamma} + \frac{1}{3} H^2 H^{\alpha\beta\gamma}
\]

\[- R^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} + \frac{1}{2} R^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} + \frac{1}{2} H^2 \frac{\lambda^{\alpha}_{\lambda}}{\mu\rho\sigma} - \frac{1}{2} H^{\alpha}_{\lambda\gamma} H^{2\beta\gamma}_{\lambda\gamma} \right),
\]  
(B.8)

where \( H^{2\alpha\beta\gamma}_{\mu\nu\rho\sigma} = H^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} \). The second covariant part come from the mixed CS action and is universal for \(-2B_{\alpha\beta\gamma} \wedge \text{tr}(R(G^-) \wedge R(G^-)) \) or \(2H_{\alpha\beta\gamma} \wedge \text{CS}_{\beta\gamma}(G^\pm)\)

\[
Q^{\alpha\beta}_{\text{cov}2\pm} = 2 \mu_{\nu\rho\sigma} (G^\pm) \star H^{\mu\nu\rho\sigma} \xi_\sigma + 4 \mu_{\nu\rho\sigma} \rho_{\alpha\beta} (G^\pm) \star H^{\beta\mu\nu\rho\sigma} \xi_\sigma + 6R_{\mu\nu\rho\sigma} \delta g_{\beta\gamma} \xi_\sigma + 6 \mu_{\nu\rho\sigma} \rho_{\alpha\beta} (G^\pm) \star H^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} B_{\nu\rho\sigma} \xi_\sigma,
\]  
(B.9)

\[
\Theta^{\mu}_{\text{cov}2\pm} = 2 \alpha_{\nu\rho\sigma} \rho_{\mu\beta} (G^\pm) \star H^{\gamma\beta\alpha\beta\gamma}_{\mu\nu\rho\sigma} \delta g_{\beta\gamma} + 3 \mu_{\nu\rho\sigma} \rho_{\alpha\beta} (G^\pm) \star H^{\alpha\beta\gamma}_{\mu\nu\rho\sigma} B_{\nu\rho\sigma} \xi_\sigma.
\]  
(B.10)
Although the difference between \( \frac{2}{3} \) and \( \frac{2}{3} \) where locally one can write

\[
\delta = 2B_\text{(2)} \wedge R^a_b (\Gamma_\pm) \delta a \wedge R^a_b (\Gamma_\pm) i \xi a \pm \delta a,
\]

where \( \delta a = \partial a \xi b \). Here due to the noncovariant nature, it is more convenient to present these terms in differential form. For \( 2H_\text{(3)} \wedge CS_\text{(3)} (\Gamma_\pm) + 2H_\text{(3)} \wedge CS_\text{(3)} (\Gamma_\pm) \) term, one obtains [20]

\[
Q_{\text{HCS}}^{\pm} [\xi] = 4H_\text{(3)} \wedge \Gamma^a_b a \Lambda b + 2H_\text{(3)} \wedge \Gamma^a_b a b + 2CS_\text{(3)} (\Gamma_\pm) \wedge i \xi B_\text{(2)} ,
\]

\[
\Theta_{\text{HCS}}^{\pm} = 2H_\text{(3)} \wedge \Gamma^a_b a b + 2CS_\text{(3)} (\Gamma_\pm) \wedge \delta B_\text{(2)} ,
\]

Although the difference between \( 2B_\text{(2)} \wedge R (\Gamma_\pm) \wedge R (\Gamma_\pm) \) and \( 2H_\text{(3)} \wedge CS_\text{(3)} (\Gamma_\pm) \) is just \( d(2B_\text{(2)} \wedge CS_\text{(3)} (\Gamma_\pm)) \), the difference between the Noether charge and the surface term is more than what’s been discussed in the introduction for the covariant total derivative term (1.12), (1.15). Instead, we find

\[
Q_{\text{HCS}}^{\pm} [\xi] = i \xi (2B_\text{(2)} \wedge CS_\text{(3)} (\Gamma_\pm)) + d \Pi_\pm = Q_{\text{HCS}}^{\pm} [\xi] ,
\]

\[
\Theta_{\text{HCS}}^{\pm} + d (2B_\text{(2)} \wedge CS_\text{(3)} (\Gamma_\pm)) + d \Pi_\pm = \Theta_{\text{HCS}}^{\pm} ,
\]

where locally one can write

\[
\Pi_\pm = 2B_\text{(2)} \wedge \Gamma^a_b a b , \quad \Pi_\pm = 2B_\text{(2)} \wedge \Gamma^a_b a b + \delta a b .
\]

In our derivation of (B.13), we have used the fact that \( \xi \) is Killing vector and in the coordinate system adopted in the computation, its components are constant. Thus

\[
\mathcal{L}_\xi \Gamma^a_b = 0 , \quad \Lambda b = 0 .
\]

Using the fact that \( \mathcal{L}_\xi \Pi_\pm = 0 \) on shell, we obtain

\[
\delta Q_{\text{HCS}}^{\pm} [\xi] = \delta Q_{\text{HCS}}^{\pm} [\xi] ,
\]

\[
\delta Q_{\text{HCS}}^{\pm} [\xi] = \delta Q_{\text{HCS}}^{\pm} [\xi] - \delta \Pi_\pm + d \Pi_\pm .
\]

Finally, we give the expression of last noncovariant term arising only from

\[
\Sigma_{\text{HCS}} = 2 \delta B_\text{(2)} \wedge \Gamma^a b a b \wedge d \Lambda a .
\]

C Entropy of the extremal black string obtained from Sen’s approach

In this section, we demonstrate that Sen’ approach to the computation of the entropy of the extremal black string yields the same answer as the Tachikawa formula for the case of IIA string compactified on K3. We first study the near horizon geometry of the extremal black string corresponding to \( \mu = 0 \) in the black string solution, as the of \( \alpha' \)-corrected temperature eq. (52) goes to 0. According to eq. (50), the horizon is located at \( r = 0 \) and...
thus the near horizon limit is achieved by zooming in the region near \( r = 0 \). We define the new variables

\[
\rho = \frac{r^2}{Q_1'}, \quad a^2 = \frac{Q_2'}{Q_1'}, \quad t' = \frac{2t}{a\sqrt{P}},
\]

\[
Q_1' = Q_1 + \alpha', \quad Q_2' = Q_2 \left(1 - 2\frac{\alpha'}{Q_1}\right),
\]

in terms of which, the near horizon geometry becomes

\[
ds_{\text{NH}}^2 = \frac{P}{4} \left(-\rho^2 dt'^2 + \frac{d\rho^2}{\rho^2}\right) + a^2 \left(dx + \frac{\sqrt{P}}{2a} \rho dt'\right)^2 + P d\Omega_3^2,
\]

\[
H_{(3)} = P \text{Vol}(S^3) + \frac{a\sqrt{P}}{2} dt' \wedge dx \wedge d\rho, \quad L = \frac{Q_1'}{P},
\]

which is \( U(1) \times \text{AdS}_2 \times S^3 \) or extremal BTZ \( \times S^3 \).

To construct the entropy function, we then make the ansatz based on the near horizon geometry

\[
ds_{\text{NH}}^2 = \ell^2 \left(-\rho^2 dt'^2 + \frac{d\rho^2}{\rho^2}\right) + \left(\frac{\ell}{2e_2}\right)^2 \left(dx + e_2 \rho dt'\right)^2 + P d\Omega_3^2,
\]

\[
H_{(3)} = P \text{Vol}(S^3) + e_1 dt' \wedge dx \wedge d\rho, \quad L = L_h,
\]

where \( \ell, e_1, e_2, L_h \) are constants to be determined from extremizing the entropy function. Then we can write down the entropy density function using eq. (3.17) given in [26] by setting \( \lambda_1 = \lambda_2 = \alpha' \). If we do not compactify \( x \) to have \( 2\pi \) period, the entropy density function is given by

\[
E = e_1 q_1 + e_2 q_2 - f(\ell, e_1, e_2, L_h),
\]

\[
f(\ell, e_1, e_2, L_h) = \frac{P^3}{2} \left[ L_h \left(\frac{4e_2^3 e_1^2}{\ell^3} + \frac{\ell^3}{2e_2^3 P} - \frac{3\ell}{4e_2}\right) + \alpha' \left(\frac{24e_2^3 e_1^4}{\ell^6} - \frac{3e_2^2 e_1^2}{\ell^5}\right)\right. - \frac{4e_2^2 e_1^2}{\ell^3 P^2} - \frac{3e_1}{2\ell^2 P} + \frac{3}{32e_2^4} - \frac{1}{2e_1 \sqrt{P}} - \frac{3\ell}{4e_2^2 P}\right],
\]

where \( q_1, q_2 \) are conjugate variables of \( e_1, e_2 \) and we have set \( 1/G_3 = 4P^3/2 \) according to the convention of [26]. Extremizing \( E \) with respect to \( \ell, e_1, e_2, L_h \) leads to

\[
\ell = \sqrt{P}, \quad e_2 = \frac{1}{2} \sqrt{\frac{P}{q_2}} \frac{(L_h P + 2\alpha')}{q_2}, \quad L_h = \frac{q_1 + \alpha'}{P}, \quad 4e_1 e_2 = P.
\]

Comparing solutions above for \( \ell, e_1, e_2, L_h \) to those arising from the near horizon geometry of the asymptotically flat extremal black string, we find that

\[
q_1 = Q_1, \quad q_2 = Q_2,
\]
which is reasonable since $Q_1, Q_2$ are the conserved charges carried by the black string even with $\alpha'$ corrections. Finally, after substituting (C.5) and (C.6) to the entropy density function, we obtain

$$E = \sqrt{PQ_2 (Q_1 + 3\alpha')}.$$ (C.7)

To see this matched with the one obtained from Tachikawa formula, we take the second formula in eq. (69) and set $\mu = 0$. Using eq. (66), we see that the result obtained from Tachikawa formula coincides with the one derived from Sen’s approach after recovering the $G_6$ dependence in the denominator and using the fact that in string unit $\ell_s = 1$, $G_6 = \frac{\pi^2}{2}$.

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