\textbf{Abstract}

We define a generalized form of $L_\infty$-algebras called $EL_\infty$-algebras. As we show, these provide the natural algebraic framework for generalized geometry and the symmetries of double field theory as well as the gauge algebras arising in the tensor hierarchies of gauged supergravity. Our perspective shows that the kinematical data of the tensor hierarchy is an adjusted higher gauge theory, which is important for developing finite gauge transformations as well as non-local descriptions. Mathematically, $EL_\infty$-algebras provide small resolutions of the operad $\mathcal{L}ie$, and they shed some light on Loday’s problem of integrating Leibniz algebras.
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1. Introduction and results

In this paper, we define a new kind of homotopy algebras called $EL_8$-algebras, which can be regarded as a weaker yet quasi-isomorphic form of $L_8$-algebras. Our construction vastly generalizes previous work on weaker forms of categorified Lie algebras [1, 2, 3], from where we also borrowed the nomenclature.

1.1. Five questions

Our “everything homotopy” Lie algebras appear naturally in many contexts, and there are five questions that they can answer or, at least, suggest an answer to.

What is the algebraic structure underlying adjusted higher curvatures?

Higher gauge theories, i.e. gauge theories of higher form potentials corresponding to connections on higher or categorified principal bundles, may underlie a future non-abelian M5-brane model [4, 5], but they certainly arise in heterotic supergravity as well as the tensor hierarchies of gauged supergravities, see e.g. [6, 7] for reviews, and of double and exceptional field theories [8]. A proper understanding of the relevant mathematical structures is evidently important: global constructions are only possible with the correct notion of corresponding higher principal bundles.

However, already the appropriate definition of the notion of curvature in a higher gauge theory is not straightforward for non-flat theories. Using categorification, there is a straightforward definition of higher curvature forms, which was also found in the first mathematical papers on non-abelian gerbes, [9] and [10]. Unfortunately, these curvatures are too restrictive for non-flat connections, cf. [11]. A consistency condition known as fake flatness needs to be imposed for the full cocycle data of non-abelian gerbes to glue together consistently (and for gauge or BRST transformations to close). This condition requires all curvature components except for the one of highest form degree to vanish, which in turn implies that all of the components of the gauge potentials except for the one of highest form degree can locally be gauged away. This is readily seen in the strict case [11]; see also [12] for a detailed analytical proof.

In fact, physicists have known for quite some time that coupling abelian 2-form potentials to non-abelian 1-form potentials is best done using a different expression for the 3-form component of the curvature [13, 14]. This modification consists of adding terms proportional to curvature expressions to the flat curvature, which were not visible to the homotopy Maurer–Cartan theory. These necessary modifications were implemented for the string and five-brane structures in [15, 16] by performing a coordinate change on the Weil
algebra of the gauge $L_{\mathcal{X}}$-algebra\(^1\), which induces the necessary change in the definition of curvatures. Considering the curvatures arising in the tensor hierarchy, one encounters similar modifications.

The modifications required for the definitions of non-flat higher curvatures and which lead to a closed BRST complex were dubbed *adjustments* in [11], and a consistent higher parallel transport for adjusted higher curvatures was defined in [17]. An adjustment now requires additional higher products, which are not visible in the original higher gauge algebra. It turns out that they are components of higher products in $EL_{\mathcal{X}}$-algebras that are quasi-isomorphic to the higher gauge algebra.

In particular, we show in theorem 6.2 that there is a particular class of $L_{\mathcal{X}}$-algebras that come with a natural adjustment encoded in an $EL_{\mathcal{X}}$-algebra. This class is precisely the one arising in the tensor hierarchies of gauged supergravity. The latter are thus adjusted higher gauge algebras, employing $EL_{\mathcal{X}}$-algebras in their construction.

**What is the full algebra underlying generalized geometry?**

Generalized geometry in its simplest form is described by an exact Courant algebroid, which captures the infinitesimal gauge symmetries of Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form potential. Roytenberg has shown that the Courant algebroid is best regarded as a symplectic Lie 2-algebroid [18], and more general forms of the generalized tangent bundle can similarly be encoded in higher symplectic Lie algebroids. Dually, they are described by their Chevalley–Eilenberg algebras, which are certain differential graded Poisson algebras. The latter, in turn, give rise to associated $L_{\mathcal{X}}$-algebras via a derived bracket construction [19, 20], describing generalizations of the above mentioned gauge symmetries. In particular, the binary product of the $L_{\mathcal{X}}$-algebra for the Courant algebroid is simply the Courant bracket. This picture has extensions to double field theory, cf. [21].

It is somewhat unsatisfying that the derived bracket construction only reproduces the Courant bracket, while the Dorfman bracket, which antisymmetrizes to the Courant bracket, has to be constructed “by hand”. It turns out that the derived bracket construction has a refinement, in which the differential graded Poisson algebra first gives rise to an $EL_{\mathcal{X}}$-algebra, which can then be antisymmetrized to the $L_{\mathcal{X}}$-algebra obtained in [20]. This construction of an $EL_{\mathcal{X}}$-algebra from a differential graded Lie algebra also underlies the gauge algebras appearing in the tensor hierarchies of gauged supergravity. As mentioned above, the refinement provides here the additional structure constants required for an adjustment.

**What is the higher Poisson algebra arising in multisymplectic geometry?**

Multisymplectic forms are higher, non-degenerate differential forms generalizing symplectic forms. Just as symplectic forms define a Poisson algebra structure on the algebra of functions, multisymplectic forms define higher analogues of Poisson algebras involving functions

\(^1\)At the purely algebraic level, this ensures that one can define invariant polynomials in a way that is compatible with quasi-isomorphisms of $L_{\mathcal{X}}$-algebras, see also [11].

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and differential forms [22]. As known since the work of Rogers [23, 24], these \( L_X \)-algebras can be embedded into the above mentioned \( L_X \)-algebras arising in the case of higher symplectic Lie-algebroids, and so it is not surprising that we again have a refinement which constructs an \( EL_X \)-algebra from the multisymplectic form. For multisymplectic 3-forms, this had already been observed in [22]. With a general definition of \( EL_X \)-algebras, we can now generalize the statement to arbitrary multisymplectic manifolds.

What is a small cofibrant replacement for \( \mathfrak{Lie} \)?

Given a field \( K \), there exists a model category structure on the category of operads\(^2\) in the category of chain complexes (differential graded operads)\(^3\) over \( K \) [26, 25, 27], and in this context one wants to construct a cofibrant replacement \( O_X \) for a symmetric differential graded operad \( O \) over \( K \). Two such constructions are known. First, there is always an \( S \)-cofibrant resolution for any operad (i.e. it need not be \( S \)-cofibrant), obtained via the tensor product with the algebraic Barratt–Eccles operad [28] or, equivalently, through the higher cobar–bar adjunction of [29], but the result is generally “larger” than necessary or expected. Second, one can use classical Koszul duality to construct a candidate \( O_X \), but this only works if the operad \( O \) is \( S \)-cofibrant, which means that \( O(n) \) is a projective \( K[S_n] \)-module for every \( n \), cf. [30, Appendix B.6]. If \( K \) is of characteristic zero, then by Maschke’s theorem every operad is automatically \( S \)-cofibrant. This is not the case if \( K \) is of positive characteristic. If, however, the operations in \( O \) do not have any permutation symmetry of their arguments, then the \( S_n \) action on \( O(n) \) is straightforward, and the operad is again automatically \( S \)-cofibrant.

For example, the operad \( A_X \) is a cofibrant replacement for \( \mathfrak{Ass} \) over any field, since its operations lack any symmetry. However, \( C_X \) and \( L_X \), containing symmetrizations and antisymmetrizations, are cofibrant replacements for \( \mathfrak{Com} \) and \( \mathfrak{Lie} \), respectively, only when \( K \) is of characteristic zero.

Dehling [3], see also [31], raised the question of finding a small cofibrant replacement for \( \mathfrak{Lie} \) over arbitrary characteristic. Recall that \( \mathfrak{Lie} \) is not \( S \)-cofibrant when \( K \) is not of characteristic 0, and the Koszul duality strategy for a small resolution does not go through. Instead, an ad hoc degree-by-degree \( S \)-free resolution over \( \mathbb{Z} \) of \( \mathfrak{Lie} \)\(^1\) was constructed to low degrees in [3]. This partial resolution provides a definition of weak Lie 3-algebras and, equivalently, 3-term \( EL_X \)-algebras.

Lacking any symmetry, the operad \( h\mathfrak{Lie} \) we construct below is automatically \( S \)-cofibrant. Together with the strictification corollary 3.18, which states that any \( EL_X \)-algebra, and thus any \( h\mathfrak{Lie} \)-algebra, is quasi-isomorphic to a differential graded Lie algebra, it follows that \( h\mathfrak{Lie}_X = EL_X \) is a cofibrant replacement for \( \mathfrak{Lie} \), answering Dehling’s question.

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\(^2\)All our operads will be symmetric and differential graded.

\(^3\)There are complications for positive characteristic; see [25].
What is the object integrating Leibniz algebras?

In [32], Loday posed the “coquecigrue problem” of generalizing Lie’s third theorem to Leibniz algebras. A conventional answer to this problem is given in terms of Lie racks, cf. [33]. There is, however, an even simpler answer suggested by $h\mathcal{L}ie$-algebras.

It appears to be a common phenomenon that some forms of integration are only possible after extending to the right cohomology in higher structures. For example, central extensions of Lie algebras do not, in general, integrate to central extensions of Lie groups. This obstruction, however, may be overcome by integrating to a Lie 2-group [34]. Similarly, the general integration of Lie algebroids only becomes possible after they are regarded as Lie $\infty$-algebroids and integrated to Lie $\infty$-groupoids, cf. [35].

We note that any Leibniz algebra canonically gives rise to an $h\mathcal{L}ie$-algebra, a weak Lie 2-algebra, cf. [1, Example 2.22] as well as proposition 4.1. From the standpoint of higher Lie theory, there has to be a natural extension of the usual integration theory of $L_\infty$-algebras, cf. [36, 37], that allows for an integration of this weak Lie 2-algebra.

The higher version of the “coquecigrue problem,” i.e. an integration of $\text{Leib}_\infty$-algebras, would then be similarly resolved: each $\text{Leib}_\infty$-algebra needs to be promoted by additional (higher) alternators to an $EL_\infty$-algebra, which should then be integrable, in principle, by general higher Lie theory.

1.2. Conclusions and outlook

Besides giving the general definition for $EL_\infty$-algebra in a fashion that can be readily used for explicit computations, we also develop the general structure theory to some extent:

- The key to most of our discussion is the notion of $h\mathcal{L}ie$-algebras, which are differential graded Leibniz algebras in which the Leibniz bracket fails to be graded antisymmetric up to a homotopy given by an alternator, whose failure to be graded symmetric is controlled by a higher alternator, and so on ad infinitum.

- Koszul dual to the operad $h\mathcal{L}ie$ is the operad $Eilh$, and we can use semifree $Eilh$-algebras to define the homotopy algebras of $h\mathcal{L}ie$-algebras, which we call $EL_\infty$-algebras.

- $EL_\infty$-algebras come with a good notion of homotopy transfer and, correspondingly, with a minimal model theorem.

- $L_\infty$-algebras are special cases of $EL_\infty$-algebras, and $L_\infty$-algebra morphisms lift to $EL_\infty$-algebra morphisms.

- An $EL_\infty$-algebra can be antisymmetrized to an $L_\infty$-algebra which, when regarded as an $EL_\infty$-algebra, is quasi-isomorphic to the original $EL_\infty$-algebra. Correspondingly, any $EL_\infty$-algebra is quasi-isomorphic to a differential graded Lie algebra.

- Any differential graded Lie algebra gives naturally rise to an $h\mathcal{L}ie$-algebra, i.e. to a hemistrict $EL_\infty$-algebra.
All of the above can be made explicit in terms of multilinear maps, at least order by
order, and we give explicit formulas that should prove useful in future applications.

We can then show that given an $h\mathcal{L}ie$-algebra originating from a differential graded
Lie algebra, adjusted notions of the curvatures of higher gauge theory are naturally found.
These adjusted curvatures are precisely the ones of the tensor hierarchies of gauged super-
gravity for maximal supersymmetry. In the past, these gauge theories have been regarded
as gauge theories of Leibniz algebras or various notions of enhanced Leibniz algebras, see
our discussion in section 8. By the principles of categorification, it is clear that higher
gauge algebras always have to be some higher form of Lie algebras, as these are the ones
integrating to (higher) symmetry group. We show that the various forms of enhanced
Leibniz algebras proposed in the literature are indeed axiomatically incomplete forms of
$h\mathcal{L}ie$-algebras or weaker higher Lie algebras.

There are three main questions that remain or arise from our work. First, it would
be certainly very interesting to explore further the relationship of our constructions to
ones existing in the literature. We feel that e.g. $\mathcal{E}ilh$-algebras should have appeared in
other algebraic contexts; for example, the deformed Leibniz rule arising in $h\mathcal{L}ie$- and $\mathcal{E}ilh$-
algebras is very similar to the formula in [38, Theorem 5.1] for Steenrod’s cup products.4

Second, in the antisymmetrization of an $EL_\infty$-algebra to an $L_\infty$-algebra, we used the
explicit form of a particular contracting homotopy, cf. appendix A, which we had to com-
pute order by order. It would be very nice to have complete analytical control over the
homotopy.

Third, most of our applications of $EL_\infty$-algebras involved them only in their hemistrict
form, namely as $h\mathcal{L}ie$-algebras. This is due to the fact that we were only able to refine
the derived bracket construction to a construction of an $h\mathcal{L}ie$-algebra from a differential
graded Lie algebra. As we explain in section 7.2, there is a clear indication that some
tensor hierarchies originate from $EL_\infty$-algebras that are not $h\mathcal{L}ie$-algebras but that can
be obtained from $L_\infty$-algebras. This suggests a much wider generalization of the derived
bracket construction, which would be certainly very useful to have. In particular, it would
allow us to characterize a very large class of $EL_\infty$-algebras for which the problem of defining
the kinematical data of higher adjusted gauge theories, such as the data arising in the tensor
hierarchies, is fully under control. Altogether, we believe that the general picture, both in
generalized geometry and in higher gauge theory, will ultimately require using fully fledged
$EL_\infty$-algebras.

2. The operads $h\mathcal{L}ie$ and $\mathcal{E}ilh$

We start with the definition of the two Koszul-dual operads $h\mathcal{L}ie$ and $\mathcal{E}ilh$ that underlie
our definition of $EL_\infty$-algebras. The invocation of operads provides us with a solid math-
ematical foundation of our constructions; the algebras over the operad $h\mathcal{L}ie$, however, will
prove to be very interesting in their own right. For background on operads and Koszul

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4We thank Jim Stasheff for pointing out this potential link.
2.1. The operad $h\mathcal{L}ie$

Differential graded Lie algebras are algebras over the differential graded operad $\mathcal{L}ie$. In this section, we define a generalization of this operad whose algebras contain hemistrict Lie 2- and Lie 3-algebras, as defined in [1] and [3]. Besides a differential, Lie 2-algebras contain a binary operator of degree 0 and an alternator, i.e. a binary operator of degree $-1$. In the case of Lie 3-algebras, we also obtain a higher alternator of degree $-2$. It is thus natural to allow for higher alternators of arbitrary degree, and to minimally extend the algebraic relations known from Lie 2- and Lie 3-algebras in such a way that they are compatible with the differential. This leads to the following operad.

**Definition 2.1.** The operad $h\mathcal{L}ie$ is a differential graded operad endowed with binary operations $\varepsilon_i^j$ of degree $i$ for each $i \in \mathbb{N} = \{0, 1, \ldots\}$ that satisfy the following relations:

\[
\begin{align*}
\varepsilon_1(\varepsilon_1(x_1)) &= 0, \\
\varepsilon_1(\varepsilon_1(x_1, x_2)) &= (-1)^i \varepsilon_1^i(\varepsilon_1(x_1), x_2) + (-1)^{|x_1|} \varepsilon_2(x_1, \varepsilon_1(x_2)) \\
&\quad + \varepsilon_2^{-1}(x_1, x_2) - (-1)^{|x_1|+|x_2|} \varepsilon_2^{-1}(x_2, x_1), \\
\varepsilon_2(\varepsilon_2(x_1, x_2), x_3) &= (-1)^{i(|x_1|+1)} \varepsilon_2(x_1, \varepsilon_2(x_2, x_3)) - (-1)^{|x_1|+|x_2|} \varepsilon_2(x_2, \varepsilon_2(x_1, x_3)) \\
&\quad - (-1)^{|x_2|+|x_3|+|x_1|+(i-1)|x_2|} \varepsilon_2^{i+1}(x_2, \varepsilon_2^{-1}(x_3, x_1)), \\
\varepsilon_2(\varepsilon_2(x_1, x_2), x_3) &= (-1)^{1+j+1} \varepsilon_2(x_1, \varepsilon_2(x_2, x_3)) - (-1)^{|x_1|+|x_2|} \varepsilon_2(x_2, \varepsilon_2(x_1, x_3)) \\
&\quad - (-1)^{|x_1|+|x_2|+|x_3|+|x_1|+(i-1)|x_2|} \varepsilon_2^{i+1}(x_3, \varepsilon_2^{-1}(x_2, x_1)) \\
\end{align*}
\]

(2.1)

for all $i, j \in \mathbb{N}$ and $j < i$, where we regard $\varepsilon_2^{-1} = 0$.

An algebra over the operad $h\mathcal{L}ie$, or an $h\mathcal{L}ie$-algebra for short, is then a graded vector space $\mathcal{E}$ together with a differential and a collection of binary products,

\[
\begin{align*}
\varepsilon_1 : \mathcal{E} &\to \mathcal{E}, \quad |\varepsilon_1| = 1, \\
\varepsilon_2^i : \mathcal{E} \otimes \mathcal{E} &\to \mathcal{E}, \quad |\varepsilon_2^i| = -i,
\end{align*}
\]

(2.2)

such that (2.1) are satisfied for all $x_1, x_2, x_3 \in \mathcal{E}$.

We note that the first three relations in (2.1) for $i = 0$ amount to the relations for a differential graded Leibniz algebra with differential $\varepsilon_1$ and Leibniz product $\varepsilon_2^0$. If $\varepsilon_2^i$ vanishes, then $\varepsilon_2^0$ is graded antisymmetric, and the Leibniz bracket becomes Lie. If we restrict to the case $\varepsilon_2^i = 0$ for $i \neq 0$, we simply recover differential graded Lie algebras. If we restrict ourselves to 2-term $h\mathcal{L}ie$-algebras, i.e. $h\mathcal{L}ie$-algebras concentrated in degrees $-1$ and 0, then only $\varepsilon_2^0$ and $\varepsilon_2^1$ can be non-trivial for degree reasons, and we obtain the hemistrict
Lie 2-algebras of [1] with a graded symmetric $\varepsilon_2$. The latter map is a chain homotopy sometimes called the alternator, capturing the failure of $\varepsilon_2^0(x_1, x_2)$ to be antisymmetric:

$$
\varepsilon_2^0(x_1, x_2) + (-1)^{|x_1||x_2|}\varepsilon_2^0(x_2, x_1) = \varepsilon_1(\varepsilon_2^1(x_1, x_2)) + \varepsilon_2^1(\varepsilon_1(x_1), x_2) + (-1)^{|x_1|}\varepsilon_2^0(x_1, \varepsilon_1(x_2)) .
$$

If we restrict to $h\text{Lie}$-algebras concentrated in degrees $-2, -1, 0$, we recover the hemistrict version of the weak Lie 3-algebras of [3], which is again trivially verified. Later, we will see that $h\text{Lie}$-algebras are generally the same as hemistrict $E_{\infty}$-algebras, and every $E_{\infty}$-algebra is equivalent or quasi-isomorphic to an $h\text{Lie}$-algebra. This is the generalization of the statement that any $L_8$-algebra is quasi-isomorphic to a differential graded Lie algebra.

One may be tempted to extend the definition of $h\text{Lie}$-algebras to binary products of arbitrary integer degrees. This, however, would render them essentially trivial from a homotopy algebraic point of view, which we will discuss in section 2.3.

We close this section with two results on $h\text{Lie}$-algebras that allow us to construct new $h\text{Lie}$-algebras from existing ones. First, we can create a larger $h\text{Lie}$-algebra by tensoring $h\text{Lie}$-algebras with differential graded commutative algebras, just as in the case of Lie algebras.

**Proposition 2.2.** The tensor product of a differential graded commutative algebra and an $h\text{Lie}$-algebra carries a natural $h\text{Lie}$-algebra structure.

**Proof.** On the tensor product of a differential graded commutative algebra $A$ and an $h\text{Lie}$-algebra $E$, we define

$$
\mathcal{E} := A \otimes E = \bigoplus_{k \in \mathbb{N}} (A \otimes E)_k , \quad (A \otimes E)_k = \sum_{i+j=k} A_i \otimes E_j ,
$$

$$
\hat{\varepsilon}_1(a_1 \otimes x_1) = (da_1) \otimes x_1 + (-1)^{|a_1|} a_1 \otimes \varepsilon_1(x_1) ,
$$

$$
\hat{\varepsilon}_2^1(a_1 \otimes x_1, a_2 \otimes x_2) = (-1)^{(|a_1|+|a_2|)} (a_1a_2) \otimes \varepsilon_2^1(x_1, x_2) ,
$$

One then readily verifies the axioms (2.1). \qed

Second, we note that the axioms (2.1) of $h\text{Lie}$-algebras have some translation invariance built in, which allows us to construct new $h\text{Lie}$-algebras by shifting degrees.

**Proposition 2.3.** Given an $h\text{Lie}$-algebra $(E, \varepsilon_1, \varepsilon_2)$, there is an $h\text{Lie}$-algebra structure on the grade-shifted complex $\tilde{E} = sE$ with

$$
\tilde{\varepsilon}_1(sa) = -s\varepsilon_1(a) ,
$$

$$
\tilde{\varepsilon}_2^1(sa, sb) = (-1)^{|a|+|b|} s\varepsilon_2^{i-1}(a, b) .
$$

Here, $s$ is the shift isomorphism $s : E \rightarrow E[-1]$ with that $|sa| = |a| + 1$. 

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Proof. The proof is a straightforward check of the relations (2.1) for \( \tilde{\mathcal{E}} \). The fact that \( \tilde{\varepsilon}_1 \) is a differential is evident. The remaining relations follow rather directly by replacing all arguments \( x_i \) in (2.1) by \( sx_i \), shifting all \( |x_i| \) by 1, and pulling out factors of \( s \) using the rules (2.5). The resulting equations are then satisfied due to (2.1).

\[ \square \]

2.2. The operad \( \mathcal{E}^h \)

For our constructions, it will be convenient to have the Koszul-dual operad to \( h\mathcal{L}ie \) at our disposal.

**Definition 2.4.** The operad \( \mathcal{E}^h \) is a differential graded operad endowed with binary operations \( \otimes_i \) of degree \( i \in \mathbb{N} \) that satisfy the following quadratic identities:

\[
\begin{align*}
    a \otimes_i (b \otimes_i c) &= (-1)^{i|a|}|b|(a \otimes_i b) \otimes_i c + (-1)^{|a||b|} (b \otimes_i a) \otimes_i c \\
    a \otimes_i (b \otimes_j c) &= (-1)^{i+j|a|}(a \otimes_i b) \otimes_j c \\
    a \otimes_j (b \otimes_i c) &= \begin{cases} 
        (-1)^{i|a|} |a||b|(b \otimes_i a) \otimes_j c & \text{if } j - i = 1 \\
        (-1)^{i|a|} |a||b|(b \otimes_i a) \otimes_j c & \text{if } j - i = 2 \\
        +(-1)^{i(|a|+j)+(|a|+|b|)}((c \otimes_{j-1} a) \otimes_{i+1} b) & \text{if } j - i > 2 \\
        +(-1)^{i(|a|+|b|+|a|+|b|)}((c \otimes_{j-1} a) \otimes_{i+1} b) \\
        +(-1)^{j+|a||b|+|a|+|b|}((c \otimes_{i+1} b) \otimes_{j-1} a) \\
    \end{cases}
\end{align*}
\]

for \( j > i \). The differential fulfills a deformed Leibniz rule given by

\[
Q(a \otimes_i b) = (-1)^{i} ((Qa) \otimes_i b + (-1)^{|a|} a \otimes_i Qb) + (-1)^{i}(a \otimes_{i+1} b) - (-1)^{|a||b|} (b \otimes_{i+1} a) .
\]

We then have the following relation between \( h\mathcal{L}ie \) and \( \mathcal{E}^h \)

**Proposition 2.5.** The operad \( \mathcal{E}^h \) is the Koszul dual of the operad \( h\mathcal{L}ie \).

This fact can be established by computation, preferably using a computer algebra program, see also [30, Section 7.6.4] for details. Instead of going into further details, let us construct the Chevalley–Eilenberg algebra of an \( h\mathcal{L}ie \)-algebra by constructing the corresponding differential-quadratic operad on a semifree \( \mathcal{E}^h \)-algebra.

\[ \text{Note that } \otimes_i \text{ should really be regarded as a binary function of degree } i \text{ in order to identify the correct Koszul signs. For example,} \]

\[
t_1 \otimes_1 t_2 := \otimes_i(t_1, t_2), \quad (- \otimes_i (- \otimes_j)) (t_1, t_2, t_3) = (-1)^{|i|+|j|} t_1 \otimes_i (t_2 \otimes_j t_3). \quad (2.6)
\]
We start from an \(h\text{-Lie}\)-algebra \(\mathfrak{E}\) which we assume for clarity of the discussion to admit a nice basis \((\tau_\alpha)\) and whose underlying graded vector space can be dualized degree-wise. We will make this assumption for all the graded vector spaces from here on, mostly for pedagogical reasons: it allows us to give very explicit formulas. More generally, one may want to work with (graded) pseudocompact vector spaces, cf. the discussion in [42, §1.1].

Consider the graded vector space \(\mathcal{E}(V) := \mathbb{R} \oplus V \oplus \bigoplus_{i \in \mathbb{N}} V \otimes_i V \oplus \bigoplus_{i,j \in \mathbb{N}} (V \otimes_i V) \otimes_j V \oplus \cdots\) (2.9)

We also define the reduced tensor algebra \(\overline{\mathcal{E}(V)} := V \oplus \bigoplus_{i \in \mathbb{N}} V \otimes_i V \oplus \bigoplus_{i,j \in \mathbb{N}} (V \otimes_i V) \otimes_j V \oplus \cdots\) (2.10)

Definition 2.6. We call an \(\text{EiLh}\)-algebra of the form \((\mathcal{E}(V), Q)\) for some graded vector space \(V\) without any restrictions on the products \(\otimes_i\) beyond (2.7) semifree.

Semifree \(\text{EiLh}\)-algebras yield the Chevalley–Eilenberg description of \(h\text{-Lie}\)-algebras:

Definition 2.7. The Chevalley–Eilenberg algebra \(\text{CE}(\mathfrak{E})\) of an \(h\text{-Lie}\)-algebra \(\mathfrak{E}\) whose differential and binary products are given by

\[
\varepsilon_1 : \mathfrak{E} \to \mathbb{R}\ , \quad \tau_\alpha \mapsto m^\beta_\alpha \tau_\beta\ , \quad |\varepsilon_1| = 1 ,
\] (2.11)

\[
\varepsilon^i_2 : \mathfrak{E} \otimes \mathfrak{E} \to \mathfrak{E}\ , \quad \tau_\alpha \otimes \tau_\beta \mapsto m^{i\gamma}_{\alpha\beta} \tau_\gamma\ , \quad |\varepsilon^i_2| = i
\]

for some \(m^\beta_\alpha\) and \(m^{i\gamma}_{\alpha\beta}\) taking values in the underlying ground field is the \(\text{EiLh}\)-algebra \(\mathcal{E}(V)\) with \(V = \mathfrak{E}[1]^*\) and the differential

\[
Qt^\alpha = -(-1)^{|\beta|} m^\alpha_\beta t^\beta - (-1)^{|(\beta|+|\gamma|)+|\beta|-|\gamma|-1}} m^{i\alpha}_{\beta\gamma} t^\beta \otimes_i t^\gamma .
\] (2.12)

Here, \(|\beta|\) is shorthand for \(|t^\beta|\), the degree of \(t^\beta\) in \(V\).

In the case of Lie algebras and \(L\)-algebras, the (homotopy) Jacobi relations are equivalent to a nilquadratic differential in the corresponding Chevalley–Eilenberg algebra, and this is also the case here:

\(^6\)Our convention for degree shift is the common one, \(V[k] := \bigoplus_i V[k]_i\), with \(V[k]_i := V_{k+i}\), implying that \(V[k]\) is the graded vector space \(V\) shifted by \(-k\).
Proposition 2.8. The equation $Q^2 = 0$ for the differential of the Chevalley–Eilenberg algebra of an $\mathcal{E}ilh$-algebra that is of the form (2.12) is equivalent to the relations (2.1).

Proof. The proof is again a straightforward but tedious verification of the axioms, which is better left to a computer algebra program. □

Just as we can consider restricted $h\mathcal{L}ie$-algebras in which only a subset of the binary products $\varepsilon_i^j$ are non-vanishing, we can also restrict the image of $Q^2$ in $\mathcal{E}(V)$. In the case of $\mathcal{E}ilh$-algebras, we can often further restrict the tensor products $m_i$ appearing in $\mathcal{E}(V)$ from $i \in \mathbb{N}$ to $i \in I \subset \mathbb{N}$, resulting in a tensor algebra $\mathcal{E}_I(V)$. We then obtain the following restriction theorem:

Theorem 2.9. The equation $Q^2 = 0$ on a semifree $\mathcal{E}ilh$-algebra $\mathcal{E}(V)$ is equivalent to $Q^2 V = 0$. If the differential $Q$ closes on $\mathcal{E}_I(V)$, then $Q^2 V = 0$ is equivalent to $Q^2 = 0$ on $\mathcal{E}(V)$.

Proof. The condition $Q^2 = 0$ on $\mathcal{E}_I(V)$ is equivalent to the condition $Q^2 = 0$ on $V \subset \mathcal{E}_I(V)$, which follows by direct computation, using the deformed Leibniz rule (2.8):

$$Q^2(a \otimes_i b) = (Q^2 a) \otimes_i b + a \otimes_i (Q^2 b).$$  (2.13)

The same holds for $Q^2 = 0$ on $\mathcal{E}_I(V)$, and combining both we obtain the desired result. □

Note, however, that we cannot simply put the products $\otimes_i = 0$ for $i \notin I$ if $I$ does not contain 0. The deformed Leibniz rule (2.8) would render the lowest non-vanishing product graded symmetric or antisymmetric and the quadratic relations (2.7) would then imply severe restrictions on the products at cubic order. Restrictions of $\mathcal{E}(V)$ to $\mathcal{E}_I(V)$ that are evidently sensible are of the form $I = \{0, \ldots , n\}$.

2.3. Cohomology of semifree $\mathcal{E}ilh$-algebras

An important tool in studying Lie algebras is Lie algebra cohomology, and we consider the generalization to $h\mathcal{L}ie$-algebras in the following. As we saw above, this amounts to the cohomology of semifree $\mathcal{E}ilh$-algebras. Due to the deformation of the usual Leibniz rule to (2.8), a subtle and important point arises. For ordinary differential graded algebras, the cohomology again carries the structure of a differential graded algebra of the same type, with product induced from the product on the original algebra. In particular, the product of two cocycles is again a cocycle. The deformation of the Leibniz rules can now evidently break this connection.

To start, let us consider the cohomology of the semifree $\mathcal{E}ilh$-algebra $(\mathcal{E}(V), Q_0)$ with the most trivial differential $Q_0$ satisfying

$$Q_0(v) = 0,$$

$$Q_0(a \otimes_i b) = (-1)^i((Q_0 a) \otimes_i b + (-1)^{|a|} a \otimes_i Q_0 b)$$  (2.14)

$$+ (-1)^i(a \otimes_{i+1} b) - (-1)^{|a|}|b|(b \otimes_{i+1} a).$$
for all $v \in V$ and $a, b \in \mathcal{E}(V)$.

**Proposition 2.10.** The $Q_0$-cohomology of $(\mathcal{E}(V), Q_0)$ is the vector space $E_0(\bigodot^* V)$, with the embedding map $E_0 : \bigodot^* V \hookrightarrow \mathcal{E}(V)$ defined by

$$E_0(v_1 \odot \cdots \odot v_n) := \sum_{\sigma \in S_n} (\cdots (v_{\sigma(1)} \odot_0 v_{\sigma(2)}) \odot_0 \cdots) \odot_0 v_{\sigma(n)} .$$

(2.15)

This vector space carries the evident structure of a differential graded commutative algebra.

**Proof.** It is clear that the image of $E_0$ forms the kernel of $Q_0$ inside $\mathcal{E}_{[0]}(V)$ and that $\mathcal{E}_{[0]}(V)$ cannot contain any coboundaries. Moreover, it is easy to show that all cocycles are coboundaries in the case of elements of $V \otimes V$: here, the kernel of $Q_0$ consists of elements of the form

$$v_1 \otimes v_2 + (-1)^{|v_1||v_2|} v_2 \otimes v_1 = (-1)^{i-1} Q(v_1 \otimes_{i-1} v_2).$$

(2.16)

For elements that are cubic and of higher order in $V$, the proof is subtle and lengthy, and we only sketch it here. We first restrict to elements $a \in \mathcal{E}(V)$ of a particular degree and order in $V$, as both the deformed Leibniz rule (2.8) and the algebra relations (2.7) preserve these. We then split $Qa \in \mathcal{E}(V)$ into polynomials of the same monomial type, i.e. the same order and type of products $\otimes_i$. These terms have to vanish separately. The condition $Qa = 0$ requires the application of the relations (2.7) $d - 2$ times, where $d$ is the degree of $a$ in $V$. This application enforces a particular symmetry structure on $a$ which forces it to lie in the image of $V$.

We note that from the Koszul-dual perspective, this result is essentially evident since both $(\mathcal{E}(V), Q_0)$ as well as $(\bigodot^* V, d)$ are the Chevalley–Eilenberg algebras of the abelian Lie algebra on $V$.

**Proposition 2.10** together with the usual arguments underlying a general Hodge–Kodaira decomposition then yield the following theorem:

**Theorem 2.11.** Consider the trivial semifree $\mathcal{E}ilh$-algebra from proposition 2.10. Then we have the following contracting homotopy between $\mathcal{E}(V)$ and the differential graded commutative algebra $(\bigodot^* V, 0)$:

$$H_0 \xhookrightarrow{P_0} (\mathcal{E}(V), Q_0) \xrightarrow{E_0} (\bigodot^* V, 0) ,$$

(2.17a)

---

7Here, $\odot$ denotes the symmetrized tensor product.

8see e.g. [43]
with the projection

\[ P_0(v_1) = v_1 \]

\[ P_0(((v_0 \otimes v_1 \otimes v_2) \otimes v_3) \cdots \otimes v_n) = \begin{cases} 
\frac{1}{(n+1)!} v_0 \odot v_1 \odot \cdots \odot v_n & i_1 = \cdots = i_n = 0 , \\
0 & \text{else} 
\end{cases} \]

(2.17b)

for all \( v_0, \ldots, v_n \in V \), such that

\[ H_0 \circ H_0 = H_0 \circ E_0 = 0 , \quad P_0 \circ H_0 = 0 , \]

\[ \text{id}_{E(V)} - E_0 \circ P_0 = H_0 \circ Q_0 + Q_0 \circ H_0 . \]

(2.17c)

As usual, the map \( H_0 \) is not unique. We give the general solution to lowest order in \( V \) and lowest product degrees in appendix A; we shall use these expressions later on.

We note that such a contracting homotopy for ordinary differential graded algebras often induces an algebra morphism \( \Phi := E_0 \circ P_0 \). This is not the case here, as clearly \( \Phi(a) \otimes \Phi(b) \neq \Phi(a \otimes b) \) in general. We shall return to this point in section 3.5.

The contracting homotopy (2.17a) has a number of important generalizations and applications. Here, we note that it evidently extends to differentials \( Q_{\text{lin}} = Q_0 + d \), where \( d : V \to V \) and \( d \) satisfies the ordinary Leibniz rules on \( E(V) \) and \( \otimes^* V \):

\[ H_0 \circ (E(V), Q_0 + d) \xrightarrow{P_0} (\otimes^* V, d) . \]

(2.18)

Moreover, if we have a differential \( d : \otimes^* V \to \otimes^* V \) non-linear in \( \otimes \), we still have a corresponding contracting homotopy

\[ H_0 \circ (E(V), Q_0 + Q_1) \xrightarrow{P_0} (\otimes^* V, d) \]

(2.19)

with

\[ Q_1 = E_0 \circ d \circ P_0 . \]

(2.20)

We therefore arrive at the following statement:

**Theorem 2.12.** Any semifree differential graded commutative algebra \((\otimes^* V, d)\) gives rise to the semifree \( \mathcal{L}ie \)-algebra \((E(V), Q_0 + E_0 \circ d \circ P_0)\).

We will pick up this thread of our discussion again later.

3. **EL\(_{\mathcal{L}ie}\)-algebras**

3.1. **EL\(_{\mathcal{L}ie}\)-algebras and their morphisms**

We define \( EL_{\mathcal{L}ie}\)-algebras to be the homotopy version of \( h\mathcal{L}ie\)-algebras.
Definition 3.1. The differential graded operad $\mathcal{EL}_X = T(s^{-1}\mathcal{hLie}^i)$ is the cobar construction applied to the Koszul-dual cooperad $\mathcal{hLie}^i$. An algebra over it is called an $EL_{X}$-algebra.

This abstract definition needs to be unwrapped to become useful. To get a first explicit handle on $EL_{X}$-algebras, we consider their Chevalley–Eilenberg algebras. For clarity, we restrict ourselves again to the cases where the graded vector space $\mathfrak{E}$ comes with a nice basis and can be dualized degree-wise, cf. section 2.2. Then an $EL_{X}$-algebra structure on a graded vector space $\mathfrak{E}$ is encoded in a nilquadratic differential on the $\mathcal{hEilh}$-algebra $CE(\mathfrak{E}) := \mathcal{E}(V)$ for $V = \mathfrak{E}[1]^\ast$.

The differential $Q$ is fully specified by its action on $V$. With respect to a basis $(t^\alpha)$ on $V$, we encode this action in structure constants $m$ taking values in the ground field as follows:

$$Q t^\alpha = \pm m^{\alpha}_\beta t^\beta + m^{\alpha}_{i_1\beta_2} t^{\beta_1} \varphi_{i_1} t^{\beta_2} + m^{\alpha}_{i_1i_2\beta_3} (t^{\beta_1} \varphi_{i_1} t^{\beta_2} \varphi_{i_2} t^{\beta_3} + \ldots ) . \quad (3.1)$$

Here the choice of signs $\pm$ is a convention\textsuperscript{9} and we shall be more explicit below, cf. also (2.12). The structure constants $m$ define higher products,

$$\varepsilon^I_n : \mathfrak{E}^\otimes n \to \mathfrak{E} , \quad \varepsilon^0 = m^{\alpha}_\beta \tau_\alpha , \quad \varepsilon^1_{1}(\tau_\alpha) = m^{\alpha}_\beta \tau_\beta , \quad \varepsilon^2_{1}(\tau_\alpha, \tau_\beta) = m^{\alpha}_{\beta\gamma} \tau_\gamma , \quad \ldots \quad (3.2)$$

where $I$ is a multi-index consisting of $n - 1$ indices $i_1, i_2, \ldots , \in \mathbb{N}$ and we define $|I| := i_1 + i_2 + \ldots$. The products $\varepsilon^I_n$ have degree $-|I|$.

It is useful to identify the following special cases:

Definition 3.2. Let $(\mathfrak{E}, \varepsilon^I_n)$ be an $EL_{X}$-algebra. If $\varepsilon^0 \neq 0$, we call $\mathfrak{E}$ curved, otherwise $\mathfrak{E}$ is uncurved. An uncurved $EL_{X}$-algebra is called hemistrict, if $\varepsilon^I_n = 0$ for $k \geq 3$. It is called strict if it is hemistrict and $\varepsilon^I_2 = 0$ for $i > 0$. Finally, $\mathfrak{E}$ is called semistrict if $\varepsilon^I_n = 0$ for $I \neq (0, \ldots , 0)$.

Note that in the case of uncurved $EL_{X}$-algebras, we can restrict $\mathcal{E}(V)$ to the reduced tensor product algebra $\mathcal{E}(V)$ defined in (2.10). In the following, all our $EL_{X}$-algebras will be uncurved. Clearly, hemistrict $EL_{X}$-algebras are simply $\mathcal{hLie}$-algebras, and therefore $EL_{X}$-algebras subsume differential graded Lie algebras.

As an example, let us consider a family of weak models of the string Lie 2-algebra which we will use to exemplify many of our constructions in the following. We consider the graded vector space $V = (\mathfrak{g} \oplus \mathbb{R}[1])[1]^\ast$, where $\mathfrak{g}$ is a finite-dimensional quadratic (i.e. metric) Lie algebra with structure constants $f^\beta_{\alpha\gamma}$ and Cartan–Killing form $\kappa_{\alpha\beta}$ with respect to a basis $(\tau_\alpha)$. On $V$, we introduce basis vectors $t^\alpha \in \mathfrak{g}[1]^\ast$ and $r \in \mathbb{R}[2]^\ast$ of degrees 1 and 2, respectively. The differential is given by

$$Q t^\alpha = - f^\beta_{\alpha\gamma} t^\beta \varphi_0 t^\gamma + (1 - \vartheta) \kappa_{\alpha\beta} f^\delta_{\beta\gamma} (t^\beta \varphi_0 t^\gamma) \varphi_0 t^\delta - 2 \vartheta \kappa_{\beta\gamma} t^\beta \varphi_0 t^\gamma , \quad (3.3)$$

\textsuperscript{9}For $L_{X}$-algebras, we follow the conventions of [44] for the structure constants and the differential in the Chevalley–Eilenberg algebra.
with $\vartheta \in \mathbb{R}$, and a direct calculation verifies $Q^2 = 0$. This defines the family of $EL_{x}$-algebras $\text{string}^{\vartheta}_{\mathfrak{g}}(\mathfrak{g})$ with the following underlying graded vector space and higher products:

\[
\begin{align*}
\text{string}^{\vartheta}_{\mathfrak{g}}(\mathfrak{g}) &:= (\mathbb{R}[1] \xrightarrow{\vartheta} \mathfrak{g}) , \\
&\quad \varepsilon_2(x_1, x_2) = [x_1, x_2] , \quad \varepsilon_2^1(x_1, x_2) = 2\vartheta(x_1, x_2) \\
&\quad \varepsilon_3^{00}(x_1, x_2, x_3) = (1 - \vartheta)(x_1, [x_2, x_3]),
\end{align*}
\]  

(3.4)

for $x \in \mathfrak{g}$ and $y \in \mathbb{R}$. All other higher products vanish. We notice that $\text{string}^{\vartheta}_{\mathfrak{g}}(\mathfrak{g})$ is an uncurved $EL_{x}$-algebra for each $\vartheta \in \mathbb{R}$. It becomes semistrict for $\vartheta = 0$ and hemistrict for $\vartheta = 1$.

A second, much more general example is worked out in appendix B.

Another very general and useful example is the $EL_{x}$-algebra of inner derivations $\text{inn}(\mathfrak{e})$ of another $EL_{x}$-algebra $\mathfrak{e}$. This is obtained as a straightforward generalization of the definition of the (unadjusted) Weil algebra of an $L_{8}$-algebra.

**Definition 3.3.** The (unadjusted) Weil algebra of an $EL_{x}$-algebra $\mathfrak{e}$ is the $\mathfrak{e}$-ilh-algebra

\[
\mathcal{W}(\mathfrak{e}) := \left( \bigodot^* (\mathfrak{e}[1]^* \oplus \mathfrak{e}[2]^*) , \; Q_{\mathcal{W}} \right),
\]  

(3.5)

where the Weil differential is defined by the relations

\[
Q_{\mathcal{W}} = Q_{\mathcal{E}} + \sigma , \quad Q_{\mathcal{W}} \sigma = -\sigma Q_{\mathcal{W}}
\]  

(3.6)

with $\sigma : \mathfrak{e}[1]^* \rightarrow \mathfrak{e}[2]^*$ the shift isomorphism, trivially extended to $\mathcal{W}(\mathfrak{e})$ by the (undeformed) Leibniz rule, i.e.

\[
\sigma(a \odot_i b) = (-1)^i (\sigma a \odot_i b + (-1)^{|a|} a \odot_i \sigma b).
\]  

(3.7)

The $EL_{x}$-algebra dual to $\mathcal{W}(\mathfrak{e})$ is the inner derivation $EL_{x}$-algebra $\text{inn}(\mathfrak{e})$ of $\mathfrak{e}$.

In terms of semifree $\mathfrak{e}$-ilh-algebras, the notion of a morphism of $EL_{x}$-algebras becomes evident.

**Definition 3.4.** A morphism of $EL_{x}$-algebras $\phi : \mathfrak{e} \rightarrow \tilde{\mathfrak{e}}$ is a morphism dual to the corresponding morphism $\Phi : \mathcal{E}(\mathfrak{e}) \rightarrow \mathcal{E}(\tilde{\mathfrak{e}})$ of $\mathfrak{e}$-ilh-algebras. For $\mathcal{E}(\mathfrak{e}) = (\mathfrak{e}(V), Q)$ and $\mathcal{E}(\tilde{\mathfrak{e}}) = (\mathfrak{e}(\tilde{V}), \tilde{Q})$, such a morphism is compatible with the differential,

\[
Q \circ \Phi = \Phi \circ \tilde{Q},
\]  

(3.8)

and the product structure,

\[
\Phi(x \odot_i y) = \Phi(x) \odot_i \Phi(y)
\]  

(3.9)

for all $x, y \in \mathcal{E}(\mathfrak{e})$ and $i \in \mathbb{N}$. 

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Recall that the appropriate notion of equivalence for homotopy algebras is that of a quasi-isomorphism, which we make explicit in the following definition.

**Definition 3.5.** Two $EL_{\infty}$-algebras $E$ and $\tilde{E}$ are called quasi-isomorphic if there is a morphism of $EL_{\infty}$-algebras $\phi : E \to \tilde{E}$ such that the contained chain map $\phi_1$, the dual to the linear component of the dual morphism of $Eilh$-algebras $\Phi$, descends to an isomorphism between the cohomologies of $E$ and $\tilde{E}$.

We note that the cohomology $H^*_{\varepsilon_1}(E)$ of an $EL_{\infty}$-algebra $E$ is dual to the cohomology with respect to the linear part of the Chevalley–Eilenberg differential $Q$. In the case of the Weil algebra, the included shift isomorphism $\sigma$ renders the cohomology $H^*_{\varepsilon_1}(\text{inn}(E))$ trivial. We therefore obtain a quasi-isomorphism between $\text{inn}(E)$ and the trivial $EL_{\infty}$-algebra, extending the situation for $L_{\infty}$-algebras.

In the future, it may be useful to have an inner product structure on an $EL_{\infty}$-algebra. The appropriate notion here, which could more formally be derived from lifting our above discussion to cyclic operads, is a simple generalization of cyclic structures on $L_{\infty}$-algebras.

**Definition 3.6.** An $EL_{\infty}$-algebra $E$ is called cyclic if it is equipped with a non-degenerate graded-symmetric bilinear form $\langle - , - \rangle : E \times E \to K$, where $K$ is the ground field, such that

$$\langle e_1, \varepsilon_1(e_2, \ldots, e_{i+1}) \rangle = (-1)^{i+i(|e_1|+|e_{i+1}|)+|e_{i+1}|} \sum_{j=0}^{i} \langle e_{i+1}, \varepsilon_1(e_1, \ldots, e_i) \rangle$$

for all $e_i \in E$.

### 3.2. Homotopy transfer and minimal model theorem

A minimal characteristic of homotopy algebras is that they provide a nice notion of homotopy transfer. The latter will be important for all our future constructions, and we therefore develop a form of the homological perturbation lemma below.

We start from deformation retract between two differential graded complexes $(E, \varepsilon_1)$ and $(\tilde{E}, \tilde{\varepsilon}_1)$. That is, we have chain maps $p$ and $e$, together with a map $h$ of degree $-1$, which fit into the diagram

$$h \circ e = id_{\tilde{E}}, \quad \text{id}_E - e \circ p = h \circ \varepsilon_1 + \varepsilon_1 \circ h,$$

$$h \circ h = 0, \quad h \circ e = 0, \quad p \circ h = 0.$$  

We now want to consider the homological perturbation lemma for the semifree $Eilh$-algebras on $E(V)$ and $\tilde{E}(\tilde{V})$ with differentials $Q$ and $\tilde{Q}$, respectively, defined by

$$V = E[1]^*, \quad Q = \varepsilon_1^* \quad \text{and} \quad \tilde{V} = \tilde{E}[1]^*, \quad \tilde{Q} = \tilde{\varepsilon}_1^*.$$
Theorem 3.7. The deformation retract (3.11) lifts to the deformation retract
\[ H_0 \xrightarrow{\varepsilon} (\mathcal{E}(V), Q) \xrightarrow{P_0 \circ E_0} (\mathcal{E}(\tilde{V}), \tilde{Q}) , \]
with
\[ E_0(v) = p^*(v) , \quad E_0(x \otimes y) = E_0(x) \otimes E_0(y) , \]
\[ P_0(v) = e^*(v) , \quad P_0(x \otimes y) = P_0(x) \otimes P_0(y) , \]
for all \( v \in V \) and \( x, y \in \mathcal{E}(V) \). The dual to the contracting homotopy is continued by a modification of the tensor trick,
\[ H_0 \circ E_0 = \text{id} \]
\[ H_0(x \otimes y) = (-1)^i H_0(x) \otimes E_0(P_0(y)) + (-1)^{i+|x|} x \otimes H_0(y) \]
\[ + (-1)^{|x|+|y|} H_0(x) \otimes H_0(y) . \]
These maps satisfy the relations
\[ P_0 \circ E_0 = \text{id} \]
\[ H_0 \circ H_0 = 0 , \quad H_0 \circ E_0 = 0 , \quad P_0 \circ H_0 = 0 . \]

Proof. The proof of (3.13d) is a straightforward computation for elements in \( \mathcal{E}_2^* \). The general case follows then by iterating the algebra relations and applying (3.11b).

The higher products \( \varepsilon_i \) with \( i > 1 \) on \( \mathcal{E} \) can now be regarded as a perturbation of the differential. Dually, we have a perturbation of \( Q \),
\[ Q \to \hat{Q} := Q + Q_\delta . \]

We can then use the homological perturbation lemma [45, 46, 47] to transfer these structures over to higher products \( \tilde{\varepsilon}_i \) on \( \tilde{\mathcal{E}} \), or, dually, to a perturbed differential \( \hat{Q} \) on \( \mathcal{E}(\tilde{V}) \). The formulas for this are the usual ones, cf. [47].

Lemma 3.8. Starting from the deformation retract (3.13), we can construct another deformation retract
\[ H \xrightarrow{\varepsilon} (\mathcal{E}(V), \hat{Q}) \xrightarrow{P \circ E} (\mathcal{E}(\tilde{V}), \tilde{Q}) , \]
with
\[ \hat{Q} := Q + Q_\delta , \quad \tilde{Q} := \hat{Q} + P \circ Q_\delta \circ E_0 , \]
\[ P := P_0 \circ (\text{id} + Q_\delta \circ H_0)^{-1} , \quad E := E_0 - H \circ Q_\delta \circ E_0 , \quad H = H_0 \circ (\text{id} + Q_\delta \circ H_0)^{-1} . \]

\[ 10^\text{th} \text{The notation is chosen to match more closely the formulas of section 2.3.} \]
where inverse operators are defined via the evident geometric series, such that
\[ P \circ E = \text{id}_{\hat{\mathcal{E}}(\hat{\mathcal{V}})} , \quad \text{id}_{\hat{\mathcal{E}}(\hat{\mathcal{V}})} - E \circ P = H \circ \hat{Q} + \hat{Q} \circ H , \]
\[ H_0 \circ H_0 = H_0 \circ E = 0 , \quad P \circ H = 0 , \] (3.15c)

**Proof.** The lemma follows from the usual perturbation lemma, cf. [47], with the specialization that \( E \) and \( P \) are here morphisms of \( \hat{\mathcal{E}}l\mathcal{H} \)-algebras. To see this, we note that \( Q_\delta \) acts as a derivation,
\[ Q_\delta(x \otimes_i y) = (-1)^i ((Q_\delta x) \otimes_i y + (-1)^{|x|} x \otimes_i Q_\delta y) . \] (3.16)
Moreover, in the non-vanishing terms of
\[ P(x \otimes_i y) = (P_0 \circ (\text{id} - Q_\delta \circ H_0 + Q_\delta \circ H_0 \circ Q_\delta \circ H_0 - \ldots))(x \otimes_i y) \]
\[ = P_0(x) \otimes_i P_0(y) - P_0(Q_\delta(H_0(x)) \otimes_i P_0(y) - P_0(x) \otimes_i P_0(Q_\delta(H_0(x))) + \ldots , \] (3.17)
all the \( H_0 \) are precomposed by a \( Q_\delta \), as otherwise the map \( P_0 \), which is precomposed to all summands, would annihilate the term due to \( P_0 \circ H_0 = 0 \). The relation
\[ P(x \otimes_i y) = P(x) \otimes_i P(y) \] (3.18)
follows then by a direct computation. The same holds for \( E \).

We note that for small perturbations \( Q_\delta \), the homological perturbation lemma 3.8 implies that
\[ \hat{\tilde{Q}} = \hat{Q} + P_0 \circ Q_\delta \circ E_0 - P_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ E_0 + P_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ E_0 - \ldots . \] (3.19)

A direct consequence of homotopy transfer is the existence of minimal models for homotopy algebras. Consider the deformation retract \((3.11)\) with \((\mathcal{E}, \bar{\varepsilon}_1 = 0) = H^{\star}_1(\mathcal{E})\) the cohomology of \((\mathcal{E}, \varepsilon_1)\) as well as \( p \) and \( e \) the projection onto the cohomology and a choice of embedding, respectively. Then the homotopy transfer yields the structure of a quasi-isomorphic \( E_{LX} \)-algebra on the cohomology of \((\mathcal{E}, \varepsilon_1)\). This implies the minimal model theorem.

**Theorem 3.9.** Any \( E_{LX} \)-algebra \( \mathcal{E} \) comes with a quasi-isomorphic \( E_{LX} \)-algebra structure on its cohomology \( H^{\star}_1(\mathcal{E}) \). We call this a minimal model of \( \mathcal{E} \).

### 3.3. \( L_{LX} \)-algebras as \( E_{LX} \)-algebras

As expected, \( L_{LX} \)-algebras are special cases of \( E_{LX} \)-algebras.

**Theorem 3.10.** A semistrict \( E_{LX} \)-algebra \( \mathcal{E} \) is an \( L_{LX} \)-algebra. Conversely, any \( L_{LX} \)-algebra is a (semistrict) \( E_{LX} \)-algebra. Dually, the data contained in a differential \( Q \) in a semifree \( \hat{\mathcal{E}}l\mathcal{H} \)-algebra \((\mathcal{E}(\mathcal{V}), Q)\) with \( \text{Im}(Q|_\mathcal{V}) \subset \mathcal{O}^*_0 \mathcal{V} \) is equivalent to the data contained in a differential \( \hat{Q} \) on the semifree differential graded commutative algebra \((\mathcal{O}^*_0 \mathcal{V}, \hat{Q})\).
Proof. It suffices to show the dual statement, which is a direct consequence of theorem 2.12.

Concretely, given an $L_{\infty}$-algebra $\mathfrak{L}$ with higher products $\mu_k$ this yields a semistrict $EL_{\infty}$-algebra with higher products

$$\varepsilon'_k = \begin{cases} 
\mu_k & \text{for } |I| = 0, \\
0 & \text{else}.
\end{cases}$$ (3.20)

Dually, we have agreement in the structure constants in the corresponding Chevalley–Eilenberg differentials $Q_L$ and $Q_{EL}$ for the $L_{\infty}$-algebra and its trivial lift to $EL_{\infty}$-algebra up to combinatorial factors:

$$Q_LT^\alpha = q_\beta^\alpha t^\beta + \frac{1}{2} q_\beta^\gamma q_\gamma^\delta t^\beta t^\gamma + \frac{1}{3!} q_\beta^\gamma q_\gamma^\delta q_\delta^\rho t^\beta t^\gamma t^\rho + \ldots ,$$

$$Q_{EL}T^\alpha = q_\beta^\alpha t^\beta + q_\beta^\alpha t^\beta \circ_0 t^\gamma + q_\beta^\alpha \circ_0 (t^\beta \circ_0 t^\gamma) \circ_0 t^\delta + \ldots .$$ (3.21)

Conversely, any semistrict $EL_{\infty}$-algebra $\mathfrak{E}$ is an $L_{\infty}$-algebra with higher products $\mu_k = \varepsilon'_k$. As an example, consider the $\vartheta = 0$ case of the family of weak string Lie 2-algebra models (3.4). This is a semistrict $EL_{\infty}$-algebra and therefore an $L_{\infty}$-algebra.

For consistency, we obviously expect the following.

**Theorem 3.11.** Any $L_{\infty}$-algebra morphism $\phi: \mathfrak{L} \to \tilde{\mathfrak{L}}$ lifts to an $EL_{\infty}$-algebra morphism $\hat{\phi}: \hat{\mathfrak{L}} \to \hat{\tilde{\mathfrak{L}}}$, where $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are the $L_{\infty}$-algebras $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$, regarded as $EL_{\infty}$-algebras.

**Proof.** We prove this statement again from the dual perspective. Let $(\bigodot^* V, Q)$ and $(\bigodot^* \hat{V}, \hat{Q})$ be the Chevalley–Eilenberg algebras of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$, respectively. The Chevalley–Eilenberg algebras of $\hat{\mathfrak{L}}$ and $\hat{\tilde{\mathfrak{L}}}$ are then

$$(\mathcal{E}(V), \hat{Q} = Q_0 + E \circ Q \circ P) \text{ and } (\mathcal{E}(\hat{V}), \hat{\hat{Q}} = Q_0 + E \circ \hat{Q} \circ P),$$ (3.22)

cf. theorem 2.12. The dual of the morphism $\phi$,

$$\Phi: CE(\mathfrak{L}) \to CE(\tilde{\mathfrak{L}}),$$ (3.23)

trivially lifts to the following dual of an $EL_{\infty}$-algebra morphism

$$\hat{\Phi}: CE(\hat{\mathfrak{L}}) \to CE(\hat{\tilde{\mathfrak{L}}}) \text{ with } \hat{\Phi}(v) := E(\Phi(v)),$$ (3.24)

and we note that $\hat{\Phi} \circ E = E \circ \Phi$. It then follows that

$$(\hat{\Phi} \circ E \circ Q)(v) = (E \circ \Phi \circ Q)(v)$$

$$\hat{\Phi}(\hat{Q}v) = (E \circ \hat{Q} \circ P \circ E \circ \Phi)(v),$$ (3.25)

and $\hat{\Phi}$ is the dual to the desired morphism of $EL_{\infty}$-algebras $\hat{\phi}$. \qed
3.4. 2- and 3-term $EL_{\infty}$-algebras

Having identified $L_{8}$-algebras within $EL_{\infty}$-algebras, let us also make contact with the 2- and 3-term $EL_{\infty}$-algebras of [1] and [3], starting with the former.

In [48], Baez–Crans introduced semistrict Lie 2-algebras: linear categories equipped with a strictly antisymmetric bilinear functor, the categorified Lie bracket, that is only required to satisfy the Jacobi identity up to a coherent trilinear natural transformation, the Jacobiator. In [1] semistrict Lie 2-algebras were fully categorified to weak Lie 2-algebras by also relaxing antisymmetry to hold only up to a coherent natural transformation, the alternator. Of course, a weak Lie 2-algebra with trivial alternator is a semistrict Lie 2-algebra. Similarly, a weak Lie 2-algebra with trivial Jacobiator is referred to as a hemi-strict Lie 2-algebra. If both the alternator and Jacobiator are trivial, it is a strict Lie 2-algebra.

By passing to its normalized chain complex, we transition from the categorical description containing many redundancies to a more convenient description in terms of differential graded algebras. In particular, a semistrict Lie 2-algebra is seen to be equivalent to a 2-term $L_{\infty}$-algebra [48], i.e. an $L_{\infty}$-algebra with underlying graded vector space concentrated in degrees $-1$ and $0$. Analogously, by passing to its normalized chain complex, any weak Lie 2-algebra is seen to be equivalent to a 2-term $EL_{\infty}$-algebra in the sense of [1], where the letter $E$ was added to indicate that ‘everything’ is relaxed up to homotopy.

Theorem 3.12. An $EL_{\infty}$-algebra structure on a two-term complex concentrated in degrees $-1$ and $0$, $\mathcal{E} : \mathcal{E}_{-1} \xrightarrow{\varepsilon_{1}} \mathcal{E}_{0}$ has only three non-trivial higher products,

\[
\begin{align*}
\varepsilon_{2} & : \mathcal{E}_{i} \otimes \mathcal{E}_{j} \rightarrow \mathcal{E}_{i+j} , \\
\varepsilon_{3} & : \mathcal{E}_{0} \otimes \mathcal{E}_{0} \otimes \mathcal{E}_{0} \rightarrow \mathcal{E}_{-1} , \\
alt & : \mathcal{E}_{0} \otimes \mathcal{E}_{0} \rightarrow \mathcal{E}_{-1} .
\end{align*}
\]  

The map $\varepsilon_{2}$ is a chain map, and the maps $\alt$ and $\varepsilon_{3}$ are chain homotopies$^{\dagger}$

\[
\begin{align*}
\alt & : \varepsilon_{2}(\cdot, -) + \varepsilon_{2}(\cdot, -) \circ \sigma_{12} \rightarrow 0 , \\
\varepsilon_{3} & : \varepsilon_{2}(\cdot, \varepsilon_{2}(\cdot, -)) - \varepsilon_{2}(\varepsilon_{2}(\cdot, -), -) - \varepsilon_{2}(\varepsilon_{2}(\cdot, -), \cdot) \circ \sigma_{12} \rightarrow 0 .
\end{align*}
\]

$^{\dagger}$Here, $\sigma_{12}$ denotes the obvious permutation.
In addition, the higher products satisfy the relations
\[
\text{alt}(x_1, x_2) = \text{alt}(x_2, x_1),
\]
\[
\varepsilon_3(x_1, x_2, x_3) + \varepsilon_3(x_2, x_1, x_3) = \varepsilon_2(\text{alt}(x_1, x_2), x_3),
\]
\[
\varepsilon_3(x_1, x_2, x_3) + \varepsilon_3(x_1, x_3, x_2) = \text{alt}(\varepsilon_2(x_1, x_2), x_3) + \text{alt}(x_2, \varepsilon_2(x_1, x_3))
\]
\[
- \varepsilon_2(x_1, \text{alt}(x_2, x_3)),
\]
\[
\varepsilon_2(x_1, \varepsilon_3(x_2, x_3, x_4)) + \varepsilon_3(x_1, \varepsilon_2(x_2, x_3), x_4) + \varepsilon_3(x_1, x_3, \varepsilon_2(x_2, x_4)) + \varepsilon_2(x_3, \varepsilon_3(x_1, x_2, x_4)) = \varepsilon_3(x_1, x_2, \varepsilon_2(x_1, x_3), x_4) + \varepsilon_3(\varepsilon_2(x_1, x_2), x_3, x_4) + \varepsilon_2(x_2, \varepsilon_3(x_1, x_3, x_4)) + \varepsilon_3(x_2, \varepsilon_2(x_1, x_3), x_4) + \varepsilon_3(x_2, x_3, \varepsilon_2(x_1, x_4))
\] (3.26c)
for all \(x_i \in \mathcal{C}_0\).

**Proof.** Perhaps the easiest way of proving the above relations is to consider the corresponding Chevalley–Eilenberg algebra (and we again assume, for simplicity, that the degree-wise duals of \(\mathcal{E}\) are nice, cf. section 2.2). That is, we consider the tensor algebra \(\mathcal{T}(V)\) for \(V = \mathcal{E}[1]^*\). The differential \(Q\) is determined by its action on the basis elements \(t^\alpha\). Since the latter can be of degree 1 or 2, it follows that \(Qt^\alpha\) is of degree 2 or 3, and therefore it has to be of the form
\[
Qt^\alpha = -(-1)^{|\beta|} m^\beta t^\alpha - (-1)^{|\gamma|(|\beta|-1)} m^0_\beta t^\alpha \otimes_0 t^\gamma
\]
\[
- (-1)^{|\beta|+|\gamma|+|\delta|+|\gamma|(|\beta|-1)+|\delta|(|\beta|-|\gamma|-1)+|\delta|(|\beta|-|\gamma|-2)} m^0_\delta t^\beta \otimes_0 t^\gamma \otimes_0 t^\delta
\]
\[
- (-1)^{|\beta|+|\gamma|+|\delta|(|\beta|-1)} m^1_\gamma t^\beta \otimes_1 t^\gamma,
\]
where the \((t^\alpha)\) form a basis on \(V\). The above formula can be reduced further when we split \((t^\alpha) = (r^a, s^i)\), where \(|r^a| = 1\) and \(|s^i| = 2\):
\[
Qr^a = -m^a_0 s^a - m^a_{bc} r^b \otimes_0 r^c,
\]
\[
Qs^i = -m^i_{aj} r^a \otimes_0 s^j + m^a_{ja} s^j \otimes_0 r^a
\]
\[
+ m^i_{abc} r^a \otimes_0 r^b \otimes_0 r^c - \delta^i_{ab} r^a \otimes_1 r^b.
\]
(3.28)

Defining
\[
\varepsilon_1(\tau_\alpha) = m^\beta_\alpha \tau_\beta,
\]
\[
\varepsilon_2(\tau_\alpha, \tau_\beta) = m^0_\alpha \tau_\gamma,
\]
\[
\varepsilon_3(\tau_\alpha, \tau_\beta, \tau_\gamma) = m^0_\alpha \tau_\gamma,
\]
\[
\text{alt}(\tau_\alpha, \tau_\beta) = m^1_\alpha \tau_\gamma
\]
with respect to the basis \((\tau_\alpha)\) of \(\mathcal{E}\), which is shifted-dual to the basis \((t^\alpha)\) of \(V = \mathcal{E}[1]^*\), we readily verify that \(Q^2 = 0\) corresponds to the equations (3.26).

We note that this 2-term \(EL_{x^*}\)-algebra is a degree-restricted version of the more general \(EL_{x^*}\)-algebra given as an example in appendix B.
The properties (3.26), with a slightly weaker condition on alt, were given as the defining axioms in the definition of a 2-term $EL_{\mathcal{X}}$-algebra in the sense of [1].

**Corollary 3.13.** An $EL_{\mathcal{X}}$-algebra of the form considered in theorem 3.12 is a 2-term $EL_{\mathcal{X}}$-algebra in the sense of [1] with a graded symmetric alternator.

Our additional condition of graded symmetric alternator is, in fact, very natural. It guarantees a rectification to semistrict Lie 2-algebras, and we shall return to this point in section 3.5.

Let us also consider the Koszul-dual picture, which we have already encountered in the proof of theorem 3.12, in more detail. We note that in the Chevalley–Eilenberg algebra $(\mathcal{E}(V), Q)$ for 2-term $EL_{\mathcal{X}}$-algebras, we can restrict the tensor products $\otimes_i$ to $i \in I = \{0, 1\}$ by the restriction theorem 2.9. The corresponding restricted operad $\mathcal{Eilh}_{\{0,1\}}$ consists of two products $\otimes_0$ and $\otimes_1$, satisfying the relations

\[
\begin{align*}
a \otimes_0 (b \otimes_0 c) &= (a \otimes_0 b + (-1)^{|a||b|} b \otimes_0 a) \otimes_0 c, \\
-(-1)^{|a|} a \otimes_1 (b \otimes_1 c) &= (a \otimes_1 b + (-1)^{|a||b|} b \otimes_1 a) \otimes_1 c, \\
a \otimes_0 (b \otimes_1 c) &= (-1)^{|a|} (a \otimes_0 b) \otimes_1 c, \\
a \otimes_1 (b \otimes_0 c) &= (-1)^{|a|} (b \otimes_0 a) \otimes_1 c,
\end{align*}
\]

(3.30a)

and the differential satisfies the evident deformed Leibniz rule given by

\[
\begin{align*}
Q(a \otimes_0 b) &= (Qa) \otimes_0 b + (-1)^{|a|} a \otimes_0 Qb + a \otimes_1 b - (-1)^{|a||b|} b \otimes_1 a, \\
Q(a \otimes_1 b) &= -(Qa) \otimes_1 b + (-1)^{|a|} a \otimes_1 Qb.
\end{align*}
\]

(3.30b)

The relations (3.30a) and (3.30b) are equivalent (up to some signs) to those defining the operad $\mathcal{R}^l$ introduced in Squires [2] to capture 2-term $EL_{\mathcal{X}}$-algebras. The free algebra over $\mathcal{R}^l$ is dual to our notion of 2-term $EL_{\mathcal{X}}$-algebra with a graded symmetric alternator $\text{alt}(x, y) = (-1)^{|x||y|}\text{alt}(y, x)$.

In [3], Dehling generalized Roytenberg’s 2-term $L_{\mathcal{X}}$-algebras in order to find a small cofibrant resolution of the operad $\mathcal{Lie}$, as mentioned in section 1.1. The partial resolution of $\mathcal{Lie}^l$ was given up to degree three, yielding explicit formulas for 3-term $EL_{\mathcal{X}}$-algebras which were shown to be equivalent to weak Lie 3-algebras.

To make contact with Dehling’s discussion, we consider a general $EL_{\mathcal{X}}$-algebra $\mathcal{E}$ concentrated in degrees $-2$, $-1$, and 0, and dualize immediately to a graded vector space $V = \mathcal{E}[1]^*$ concentrated in degrees $1, 2, 3$, $V = V_1 \oplus V_2 \oplus V_3$. The limited degrees restrict the structure constants in the differential (3.1) as well as the resulting $EL_{\mathcal{X}}$-products:

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Correspondingly, we can restrict $E(V)$ to $E_{[0,1,2]}(V)$, according to the restriction theorem 2.9. Verifying the relations that $Q^2 = 0$ imposes on the higher products, we arrive at the following theorem.

**Theorem 3.14.** An $EL_{X}$-algebra $E$ concentrated in degrees $-2$, $-1$, and $0$ is the same as weak Lie 3-algebra in the sense of [3].

### 3.5. $EL_{X}$-algebras as $L_{X}$-algebras and the rectification theorem

In section 3.3, we identified $L_{X}$-algebras with semistrict $EL_{X}$-algebras, which suggests that there should be a rectification of $EL_{X}$-algebras to $L_{X}$-algebras. This is indeed the case, as we show below.

We start with a projection of $EL_{X}$-algebras onto $L_{X}$-algebras, extending results of [1, 3].

**Theorem 3.15.** Any $EL_{X}$-algebra $(E, \mu_3)$ induces an $L_{X}$-algebra structure on the graded vector space $E$. This $L_{X}$-algebra structure is induced by homotopy transfer using the homotopy $H_0$.

**Proof.** The proof is readily obtained by applying the homological perturbation lemma to the contracting homotopy

\[
H_0 \begin{array}{c} (E(V), Q_0 + Q_1 + Q_\delta) \end{array} \xrightarrow{P_0} (\otimes^* V, Q_L),
\]

(3.32)

cf. (2.18). Consider the Chevalley–Eilenberg algebra $CE(E)$ of $E$, and split the differential $Q = Q_0 + Q_1 + Q_\delta$ into $Q_0$, a linear part $Q_1$, and a perturbation $Q_\delta$. Then homotopy transfer yields a differential

\[
Q_L = Q_1 + P_0 \circ Q_\delta \circ E_0 - P_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ E_0 + P_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ H_0 \circ Q_\delta \circ E_0 + \ldots
\]

(3.33)

on $\otimes^*(E[1]^*)$. By construction, $Q_L^2 = 0$. Moreover, $Q_L$ satisfies the Leibniz rule on $\otimes^*(E[1]^*)$: the deformation terms in the Leibniz rule (2.8) are graded antisymmetric, and this graded antisymmetry is preserved by subsequent applications of $H$ and $Q_\delta$. The final projector $P_0$ then eliminates these terms.

As an example, we can compute the antisymmetrization of a 2-term $EL_{X}$-algebra and reproduce\textsuperscript{12} [1, Proposition 3.1].

\textsuperscript{12}Due to different conventions, there is a relative minus sign between our $\mu_3$ and that of [1].
Corollary 3.16. Given a 2-term $EL_{\mathcal{E}}$-algebra $\mathcal{E}$, there is an $L_{\mathcal{E}}$-algebra structure on the graded vector space $\mathcal{E}$ with higher products

\[
\begin{align*}
\mu_1(y) &= \varepsilon_1(y), \\
\mu_2(x_1, x_2) &= \frac{1}{2}(\varepsilon_2(x_1, x_2) - \varepsilon_2(x_2, x_1)), \\
\mu_2(x_1, y) &= -\mu_2(y, x_1) = \frac{1}{2}(\varepsilon_2(x_1, y) - \varepsilon_2(y, x_1)), \\
\mu_3(x_1, x_2, x_3) &= \frac{1}{3!} \sum_{\sigma \in S_3} \chi(\sigma; x_1, x_2, x_3)
\left(\varepsilon_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})
\right.
\left.+
\frac{1}{2} \text{alt}(\varepsilon_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})\right)
\end{align*}
\]

for all $x \in \mathcal{E}_0$ and $y \in \mathcal{E}_{-1}$.

Proof. We start from the Chevalley–Eilenberg algebra $CE(\mathcal{E})$. As always, we assume for convenience that there is a basis, explicitly given by elements $r^a, s^a$ of degrees 1 and 2 of $\mathcal{E}$. The differential then reads as

\[
\begin{align*}
Qr^a &= \sum_{b, c} m^a_{bc} r^b \otimes s^c, \\
Qs^i &= \sum_{a, j} m^i_{aj} r^a \otimes s^j + m^a_{ja} s^j \otimes r^a, \\
&\quad + m^i_{abc} (r^a \otimes s^b) \otimes r^c - n^i_{ab} r^a \otimes r^b \otimes r^c,
\end{align*}
\]

cf. (3.28). We then evaluate formula (3.33) using the formulas (A.1) for the homotopy $H_0$:

\[
\begin{align*}
Q_Lr^a &= -m^a_{ij} s^i - \frac{1}{2} m^a_{bc} r^b \otimes r^c, \\
Q_Ls^i &= -(m^i_{aj} + m^i_{ja} r^a \otimes s^j + \frac{1}{3} (m^i_{abc} + \frac{1}{2} n^i_{dc} m^d_{ab}) r^a \otimes r^b \otimes r^c.
\end{align*}
\]

This is the differential for the Chevalley–Eilenberg algebra of $(\mathcal{E}, \mu_i)$ with the higher products (3.34).

Similarly, we reproduce the antisymmetrization of 3-term $EL_{\mathcal{E}}$-algebras described in [3, Section 4] using the homotopy $H_0$ given in (A.1) with the choice $\alpha_1 = \alpha_2 = 0$; a more comprehensive case will be discussed in section 4.2.

As observed already in [1, 3], the antisymmetrization map is functorial for 2-term $EL_{\mathcal{E}}$-algebras, and any morphism of $EL_{\mathcal{E}}$-algebra induces a morphism between the corresponding antisymmetrized $L_{\mathcal{E}}$-algebras. The antisymmetrization map fails, however, to be functorial for 3-term $EL_{\mathcal{E}}$-algebras. It is natural to conjecture that the antisymmetrization is $n$-functorial for an $n$-term $EL_{\mathcal{E}}$-algebra. We will not need this result and refrain from going further into these technicalities.

A new result is that this antisymmetrization map lifts indeed to a quasi-isomorphism of $EL_{\mathcal{E}}$-algebras.

Theorem 3.17. Let $\mathcal{E}$ be an $EL_{\mathcal{E}}$-algebra, and let $\mathcal{E}'$ be the $L_{\mathcal{E}}$-algebra induced by theorem 3.15, regarded as an $EL_{\mathcal{E}}$-algebra. Then there is a quasi-isomorphism $\phi : \mathcal{E} \to \mathcal{E}'$. 24
Proof. We prove this statement again using the Chevalley–Eilenberg algebras $CE(\mathcal{E})$ and $CE(\mathcal{E}')$. Note that as graded vector spaces, $\mathcal{E}[1]^* = \mathcal{E}'[1]^*$. If $Q = Q_0 + Q_1 + Q_\delta$ is the differential on $CE(\mathcal{E})$, then the differential on $CE(\mathcal{E}')$ reads as

$$Q'v = Q_1v + E_0 \circ P_0 \circ (id + Q_\delta \circ H_0)^{-1} \circ Q_\delta v$$

(3.37)

for all $v \in \mathcal{E}[1]^*$. We need to construct an invertible $\mathcal{E}_{ilh}$-algebra morphism $\Phi : \mathcal{E}(\mathcal{E}[1]^*) \to \mathcal{E}(\mathcal{E}[1]^*)$ satisfying $Q\Phi = \Phi Q'$. The desired morphism on $\mathcal{E}[1]^*$ is

$$\Phi(v) = (id - H_0 \circ Q_\delta + H_0 \circ Q_\delta \circ H_0 \circ Q_\delta - \ldots)(v) = (id + H_0 \circ Q_\delta)^{-1}(v)$$

and using

$$Q_0 \circ H_0 = id - E_0 \circ P_0 - H_0 \circ Q_0$$

(3.39)

$$Q_0Q_\delta = -Q_\delta^2 - Q_\delta Q_0$$

one readily verifies that $Q\Phi v = \Phi Q'v$ for all $v \in \mathcal{E}[1]^*$, which is sufficient. Since the morphism is clearly invertible, this is a quasi-isomorphism.

We can now combine theorem 3.17, theorem 3.11 as well as the strictification theorem for $L_{\infty}$-algebras to obtain the following.

**Corollary 3.18.** Any $EL_{\infty}$-algebra is quasi-isomorphic to a differential graded Lie algebra, trivially regarded as an $EL_{\infty}$-algebra.

More directly, this follows from the strictification theorem for generic homotopy algebras, see e.g. [30, Proposition 11.4.9].

Theorem 3.17 also shows that the choice of [1] not to symmetrize the alternator was perhaps not the best. It leads to a classification of 2-term $EL_{\infty}$-algebras which is generally larger than that of $L_{\infty}$-algebras [1, Theorem 4.5], contradicting the rectification theorem expected in line with the situation for $L_{\infty}$-algebras.

A consequence of the strictification theorem and homotopy transfer is the following. Just as for $h\mathcal{L}ie$-algebras, we can also tensor an $EL_{\infty}$-algebra by a differential graded commutative algebra.

**Theorem 3.19.** The tensor product of an $EL_{\infty}$-algebra and a differential graded commutative algebra carries a natural $EL_{\infty}$-algebra structure.

13This morphism implements a coordinate transformation such that the image of $Q$ on $\tilde{v} = \Phi(v)$ has no component in the subspace $Q_0H_0\mathcal{E}(\mathcal{E}[1]^*)$. This then implies that it has no component in $H_0Q_0\mathcal{E}(\mathcal{E}[1]^*)$ either. The only remaining component is in $E_0P_0\mathcal{E}(\mathcal{E}[1]^*)$, which implies that $Q$ is the Chevalley–Eilenberg differential of an $L_{\infty}$-algebra, trivially regarded as an $EL_{\infty}$-algebra.

14One may be tempted to replace the differential graded commutative algebra with an $\mathcal{E}_{ilh}$-algebra, but already the product between an $\mathcal{E}_{ilh}$-algebra and an $h\mathcal{L}ie$-algebra does not carry a natural $h\mathcal{L}ie$-algebra structure.
Proof. We can invoke the argument presented in [49] for the existence of general tensor products between certain homotopy algebras. That is, by corollary 3.18, $E$ is quasi-isomorphic to a hemistrict $EL_{\omega}$-algebra $E_{hst}^{list}$, and the chain complexes $A \otimes E_{hst}$ and $A \otimes E$ are quasi-isomorphic. By proposition 2.2, $A \otimes E_{hst}$ carries an $hLie$-algebra structure, and the homological perturbation lemma allows us to perform a homotopy transfer from $A \otimes E_{hst}$ to $A \otimes E$, leading to the desired $EL_{\omega}$-algebra structure.

Instead of using the above elegant but abstract argument, we can also perform a direct computation in the dual Chevalley–Eilenberg picture. This leads to the following explicit formulas for the tensor product $\hat{E}$ of a differential graded commutative algebra $A$ and an $EL_{\omega}$-algebra $E$:

$$\hat{E} := A \otimes E = \bigoplus_{k \in \mathbb{Z}} (A \otimes E)_k,$$

$$(A \otimes E)_k = \sum_{i+j=k} A_i \otimes E_j,$$

$$\hat{\varepsilon}_1(a_1 \otimes x_1) = (da_1) \otimes x_1 + (-1)^{|a_1|} a_1 \otimes \varepsilon_1(x_1),$$

$$\hat{\varepsilon}_k^l(a_1 \otimes x_1, \ldots, a_k \otimes x_k) = (-1)^{|\sum_{i=1}^k |a_i| |\varepsilon_k^l(x_1, \ldots, x_k)|} (a_1 \ldots a_k) \otimes \varepsilon_k^l(x_1, \ldots, x_k).$$

3.6. Examples: String Lie algebra models

Let us illustrate the above structure theorems using the important and archetypal examples of 2-term $EL_{\omega}$-algebra models for the string Lie algebra. We have already encountered the $EL_{\omega}$-algebras $\text{string}_{sk}^{wk,d}(g)$ in (3.4). A short computation using formulas (3.34) shows that these all antisymmetrize to the following 2-term $L_{\omega}$-algebra:

$$\text{string}_{sk}(g) := (\mathbb{R} \xrightarrow{0} g),$$

$$\mu_2(x_1, x_2) = [x_1, x_2], \quad \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3]).$$

It turns out that this $L_{\omega}$-algebra (which is a minimal model for its quasi-isomorphism class) is quasi-isomorphic to a strict one,

$$\text{string}_{lp}(g) := (L_0 g \oplus \mathbb{R} g \xrightarrow{\mu_1} P_0 g),$$

$$\mu_1(\lambda, r) = \lambda,$$

$$\mu_2(\gamma_1, \gamma_2) = [\gamma_1, \gamma_2], \quad \mu_2(\gamma_1, (\lambda, r)) = ([\gamma_1, \lambda], 2 \int_0^1 d\tau (\gamma_1, \lambda),$$

$$\mu_3(\gamma_1, \gamma_2, \gamma_3) = 0,$$

where $L_0 g$ and $P_0 g$ are the based path and based loop spaces of $g$, cf. [50]. There are two quasi-isomorphisms,

$$\text{string}_{sk}(g) \xrightarrow{\psi} \text{string}_{lp}(g),$$

$$\text{string}_{lp}(g) \xleftarrow{\phi} \text{string}_{sk}(g).$$
and their explicit forms are found e.g. in [11]. This implies that there is a quasi-isomorphic family of $\text{EL}_8$-algebras that antisymmetrize to $\text{string}_{\text{lp}}(\mathfrak{g})$, which is readily found:

$$\begin{align*}
\text{string}^{\text{wk, } \vartheta}_{\text{lp}}(\mathfrak{g}) &:= (L_0 \mathfrak{g} \oplus \mathbb{R} \xrightarrow{\varepsilon_1} F_0 \mathfrak{g}) , \\
\varepsilon_1(\lambda, r) &= \lambda , \\
\varepsilon_2^0(\gamma_1, \gamma_2) &= [\gamma_1, \gamma_2] , \\
\varepsilon_2^0(\gamma_1, (\lambda, r)) &= \left([\gamma_1, \lambda], 2 \int_0^1 \mathrm{d}\tau \left(\gamma_1, \lambda\right)\right) , \\
\varepsilon_2^1(\gamma_1, \gamma_2) &= (0, 2\vartheta(\partial \gamma_1, \partial \gamma_2)) , \\
\varepsilon_3^0(\gamma_1, \gamma_2, \gamma_3) &= \vartheta(\partial \gamma_1, [\partial \gamma_2, \partial \gamma_3]) .
\end{align*}$$

(3.44)

Altogether, we can summarize the situation in the following commutative diagram:

$$
\begin{array}{ccc}
\text{string}^{\text{wk, } \vartheta}_{\text{sk}}(\mathfrak{g}) & \xrightarrow{\hat{\psi}} & \text{string}^{\text{wk, } \vartheta}_{\text{lp}}(\mathfrak{g}) \\
\downarrow \text{asym} & & \downarrow \text{asym} \\
\text{string}_{\text{sk}}(\mathfrak{g}) & \xleftarrow{\hat{\phi}} & \text{string}_{\text{lp}}(\mathfrak{g})
\end{array}
$$

(3.45)

The morphisms $\text{asym}$ are special cases of the antisymmetrization map (3.34), and the morphisms $\hat{\phi}$ and $\hat{\psi}$ are formed by lifts of the morphisms $\phi$ and $\psi$ as given by theorem 3.11.

Generically, on top of every $L_8$-algebra, there is a family of $\text{EL}_8$-algebras that antisymmetrize to it. The additional structure constants contained in the alternators of the $\text{EL}_8$-algebra will turn out to be crucial in the construction of higher gauge theories.

4. Relations to other algebras

In the following, we explain the relation between $\text{EL}_8$-algebras and homotopy Leibniz algebras and, in particular, to differential graded Lie algebras. The latter prepares our interpretation of generalized geometry and the tensor hierarchies.

4.1. Relation to homotopy Leibniz algebras

Just as Lie algebras are Leibniz algebras that happen to have an antisymmetric Leibniz bracket, $\text{EL}_8$-algebras are $\text{Leib}_8$-algebras whose higher Leibniz brackets are antisymmetric up to homotopies. Homotopy Leibniz algebras were defined in [51, 52], and they are the homotopy algebras over the Zinbiel operad $\text{Zinb}$ [53, 39] which, as suggested by the name\footnote{This nomenclature is a successful joke suggested by J. M. Lemaire. Zinbiel algebras are also known as (commutative) shuffle algebras, and the free Zinbiel algebra over a vector space is the shuffle algebra on its tensor algebra.}, is the Koszul-dual to the Leibniz operad $\text{Leib}$.

Explicitly, consider the semifree non-associative tensor algebra $\mathcal{O}_0^* V$ for a graded vector space $V$ with only the first relation of (2.7) imposed. A (nilquadratic) differential $Q$ on this algebra which satisfies the ordinary Leibniz rule then defines a homotopy Leibniz algebra. \footnote{This nomenclature is a successful joke suggested by J. M. Lemaire. Zinbiel algebras are also known as (commutative) shuffle algebras, and the free Zinbiel algebra over a vector space is the shuffle algebra on its tensor algebra.}

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All the additional structure in $Eilh$ (as well as the resulting additional structure in $EL_{x}$-algebras) capture the appropriate notion of symmetry up to homotopy of the higher Leibniz brackets.

Ordinary Leibniz algebras form an interesting source of 2-term $hLie$-algebras, which had been observed before:

**Proposition 4.1** ([1]). *Any Leibniz algebra induces canonically a hemistrict 2-term $EL_{x}$-algebra concentrated in degrees $-1$ and $0$.*

Explicitly, let $\mathfrak{g}$ be a Leibniz algebra, and write $\mathfrak{g}^{\text{ann}} = [\mathfrak{g},\mathfrak{g}]$. Then

$$\mathcal{E}(\mathfrak{g}) = (\mathcal{E}(\mathfrak{g})_{-1} \xrightarrow{e_{1}} \mathcal{E}(\mathfrak{g})_{0}) := (\mathfrak{g}^{\text{ann}} \xrightarrow{e_{1}} \mathfrak{g})$$

(4.1)

is a differential graded Leibniz algebra, and we promote it to a 2-term $EL_{x}$-algebra by

$$\text{alt}(e_{1}, e_{2}) := [e_{1}, e_{2}] + [e_{2}, e_{1}] \in \mathfrak{g}^{\text{ann}}$$

(4.2)

for all $e_{1}, e_{2} \in \mathfrak{g}$.$^{16}$

**4.2. $hLie$-algebras from differential graded Lie algebras and derived brackets**

Given a differential graded Lie algebra $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{k}$, one can construct an associated $L_{x}$-algebra on the grade-shifted partial complex $\mathfrak{L} = \bigoplus_{k \geq 0} \mathfrak{g}[1]$. As explained in [20], this is a corollary to the result of [19] that the mapping cone of a morphism between two differential graded Lie algebras carries a natural $L_{x}$-algebra structure. In this section, we present a refinement of this associated $L_{x}$-algebra to an $hLie$-algebra. The existence of the $L_{x}$-algebra is then a corollary to the antisymmetrization theorem 3.15. Our construction extends the construction of $\text{Leib}_{x}$-algebras from $\text{Leib}$-algebras in [55] as well as the construction of 2-term $EL_{x}$-algebras from 3-term differential graded Lie algebras in [1].

Given a differential graded Lie algebra, we readily construct a grade-shifted $hLie$-algebra.

**Theorem 4.2.** *Given a differential graded Lie algebra $(\mathfrak{g}, d, \{-,-\})$ with $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{k}$, we have an associated $hLie$-algebra

$$\mathcal{E} = \bigoplus_{k \leq 0} \mathcal{E}_{k} , \quad \mathcal{E}_{k} = \mathfrak{g}_{k-1}$$

(4.3)

with higher products

$$\varepsilon_{1}(x_{1}) := \begin{cases} d_{\mathfrak{g}}x & \text{for } |x|_{\mathcal{E}} < 0 , \\ 0 & \text{else} \end{cases}$$

$$\varepsilon_{2}^{i}(x_{1}, x_{2}) := \begin{cases} \delta x_{1}, x_{2} & \text{for } i = 0 , \\ (-1)^{|x_{1}|_{\mathcal{E}}} \{x_{1}, x_{2}\} & \text{for } i = 1 , \\ 0 & \text{else} \end{cases}$$

(4.4)

We note that this result, together with theorem 3.15, immediately implies that any Leibniz algebra gives rise to a 2-term $L_{x}$-algebra as shown separately in [54].

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for all \( x_1, x_2 \in \mathcal{E} \). Here, \( \delta := d_0|_{\mathcal{E}[-1]} \) and \( |x_1| \) denotes the degree of \( x_1 \) in \( \mathcal{E} \).

**Proof.** The proof is a straightforward verification of the axioms of an \( h\text{Lie} \)-algebra (2.1), which is most conveniently done again with a computer algebra program.

Let us discuss the explicit form of the antisymmetrization in some more detail. We assume, as usual, that \( \mathcal{E} \) admits a nice basis \( (\tau_\alpha) \), so that \( \mathcal{E}[1]^* \) has a dual basis \( (t^\alpha) \). The Chevalley–Eilenberg differential then reads as

\[
Q t^\alpha = -(1)^{|\beta|} m_{\beta}^\alpha t^\beta - (1)^{i(|\beta|+|\gamma|)+|\gamma|(|\beta|-1)} m_{\beta}^{i\alpha} t^\beta \odot t^\gamma,
\]

and we have the following theorem.

**Theorem 4.3.** For each \( h\text{Lie} \)-algebra \( (\mathcal{E}, \varepsilon^1) \) (with the above mentioned restrictions), there is an \( L_\infty \)-algebra \( (\mathcal{E}, \mu_4) \) with first four higher products reading as

\[
\begin{align*}
\mu_1(x_1) &:= \varepsilon_1(x_1), \\
\mu_2(x_1, x_2) &:= \frac{1}{2} \varepsilon_2^0(x_1, x_2) - \varepsilon_2^0(x_2, x_1), \\
\mu_3(x_1, x_2, x_3) &:= \frac{1}{6} \sum_{\sigma \in S_3} \chi(\sigma; x_1, x_2, x_3) \left( \varepsilon_3^0(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \\
&\quad + \frac{1}{4} \left( \varepsilon_2^1(\varepsilon_2^0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + \varepsilon_2^1(\varepsilon_2^0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \right) \right), \\
\mu_4(x_1, x_2, x_3, x_4) &:= 0,
\end{align*}
\]

(4.6)

for all \( x_i \in \mathcal{E} \).

**Proof.** We use again theorem 3.15 and determine the Chevalley–Eilenberg differential (3.33) of \( (\mathcal{E}, \mu_4) \) using the explicit form of the homotopy (A.1), which allows us to compute \( Q_L \) up to quartic order. This produces the higher products (4.6) for \( \alpha_1 = \alpha_2 = 0 \).

We note that our choice \( \alpha_1 = \alpha_2 = 0 \) is, in fact, not the most natural one. One gets a nicer pattern in the expressions for \( H_0 \) if one puts \( \alpha_1 = \alpha_2 = -\frac{1}{2} \), and this results in an expression for \( \mu_4 \) which does not vanish but involves nestings of two maps \( \varepsilon_2^0 \) and one map \( \varepsilon_2^2 \). In the case of \( h\text{Lie} \)-algebras obtained from differential graded Lie algebras, we have \( \varepsilon_2^2 = 0 \), and therefore the distinction is irrelevant.

We can now compose the map from differential graded Lie algebras to \( h\text{Lie} \)-algebras with the antisymmetrization theorem 3.15. This reproduces the following proposition of [20], which in turn is a specialization of [19]:

**Proposition 4.4.** Given a differential graded Lie algebra \( (\mathfrak{g}, d, [\cdot, \cdot]) \), we have an \( L_\infty \)-algebra structure on the truncated complex

\[
\mathfrak{g}_{\leq 0} = ( \ldots \xrightarrow{d} \mathfrak{g}_{-2} \xrightarrow{d} \mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{g}_0 \xrightarrow{0} \ast \xrightarrow{0} \ldots )
\]

(4.7)
with
\[
\mu_1(x_1) = \begin{cases} 
  \mathrm{d}x_1 & \text{for } |x_1| < 0 \\
  0 & \text{for } |x_1| = 0
\end{cases}
\]
\[
\mu_k(x_1, \ldots, x_k) = \frac{(-1)^k}{(k-1)!} \sum_{\sigma \in S_k} \chi(\sigma; x_1, \ldots, x_k)[[\ldots[[\delta x_{\sigma(1)}], x_{\sigma(2)}], \ldots], x_{\sigma(k)}]
\]
where
\[
\delta(x_1) = \begin{cases} 
  \mathrm{d}x_1 & \text{for } |x_1| = 0 \\
  0 & \text{else}
\end{cases}
\]
for all $x_i \in \mathfrak{g}_{\leq 0}$. Here, $B_k$ are the (first) Bernoulli numbers$^{17}$.

Altogether, our above constructions suggest the following picture:

\[
\begin{array}{ccc}
\text{dg Lie algebra} & \Rightarrow & \text{hLie-algebra} \\
\Rightarrow & & \Rightarrow \\
\text{Proposition 4.4} & & \text{Theorem 4.3} \\
& & \text{Theorem 4.2}
\end{array}
\]

(4.10)

Our formulas (3.33) show that this picture is true for differential graded Lie algebras concentrated in degrees $d \geq -3$. For more general differential graded Lie algebras, this picture is still very plausible from the expression for (3.33). A complete proof, however, would require an explicit expression of the homotopy $H_0$ to all orders, which is currently beyond our technical capabilities.

From proposition 4.4 it is also clear that $\mu_4$ in (4.6) vanishes because $B_3 = 0$. Similarly, all even higher brackets $\mu_{2i}$ with $i \geq 1$ vanish, as the odd Bernoulli numbers $B_k$ for $k \geq 3$ vanish.

As a simple example, consider a quadratic Lie algebra $\mathfrak{g}$, and construct the differential graded Lie algebra
\[
\mathcal{G} = ( \ldots \xrightarrow{0} \xrightarrow{0} \mathbb{R} \xrightarrow{0} \mathfrak{g} \xrightarrow{0} \mathfrak{g} \xrightarrow{id} \mathfrak{g} \xrightarrow{0} \mathfrak{g} \xrightarrow{0} \mathfrak{g} \xrightarrow{0} \mathfrak{g} \xrightarrow{0} \ldots ),
\]
(4.11)

concentrated in degrees $-2, -1, 0$ with differential and Lie brackets
\[
[x_1, x_2]_\mathfrak{g} = 2(x_1, x_2), \quad [y_1, x_1]_\mathfrak{g} = -[x_1, y_1]_\mathfrak{g} = y_1(x_1), \quad [y_1, y_2]_\mathfrak{g} = (y_1, y_2)
\]
(4.12)
for all $x_1, x_2 \in \mathfrak{g}_0 \cong \mathfrak{g}$ and $y_1, y_2 \in \mathfrak{g}_{-1} \cong \mathfrak{g}$, where $[\cdot, \cdot]$ and $(\cdot, \cdot)$ are the Lie bracket and the Cartan–Killing form on $\mathfrak{g}$. Then the associated $hLlie$-algebra is
\[
\mathcal{E} = ( \mathbb{R} \xrightarrow{0} \mathfrak{g} ) ,
\]
\[
\varepsilon_1(r) := 0,
\]
\[
\varepsilon_2^0(x_1, x_2) = [x_1, x_2] , \quad \varepsilon_2^1(x_1, x_2) = 2(x_1, x_2) .
\]

$^{17}$i.e. $B_0, B_1, \ldots = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots$
We thus recover the hemistrict $EL_{\infty}$-algebra $\text{string}_{sk}^{wk,1}(g)$ introduced in section 3.6. The antisymmetrization of this $h\text{Lie}$-algebra then yields the skeletal string Lie 2-algebra model $\text{string}_{sk}(g)$. Interestingly, a quick consideration of the case leads to the conclusion that there is no differential graded Lie algebra that reproduces the strict string Lie 2-algebra model $\text{string}_{sk}^{0,0}(g) = \text{string}_{sk}^{wk,0}(g)$. This points towards a possible extension of theorem 4.2 producing $EL_{\infty}$-algebras from certain $L_{\infty}$-algebras.

5. Generalized and multisymplectic geometry

We now come to our first two applications of $EL_{\infty}$-algebras, or rather $h\text{Lie}$-algebras: the symmetry algebras of symplectic $L_{\infty}$-algebroids and, in a closely related way, a categorified version of higher Poisson algebras.

5.1. Generalized geometry from symplectic $L_{\infty}$-algebroids

The string Lie 2-algebra $\text{string}_{sk}(\text{spin}(n))$ is a finite-dimensional $L_{\infty}$-subalgebra of the 2-term $L_{\infty}$-algebra of symmetries associated to the Courant algebroid [56] over $\text{Spin}(n)$. It is therefore not surprising that the symmetries of symplectic $L_{\infty}$-algebroids, are important sources for examples of $EL_{\infty}$-algebras. This link was noticed before in [1] and [3], who constructed 2- and 3-term $EL_{\infty}$-algebras. Here, we can present the general picture. We shall follow the conventions of [21].

Theorem 5.1. The symmetry algebra of a symplectic $L_{\infty}$-algebroid is naturally an $h\text{Lie}$-algebra.

Proof. The Chevalley–Eilenberg algebra of a symplectic $L_{\infty}$-algebroids $L$ is a differential graded Lie algebra. The differential is the Chevalley–Eilenberg differential, encoding the anchor and the higher maps on sections of $L$, and it is given by a vector field $Q$ on $L$. The Lie bracket is the Poisson bracket induced by the symplectic form $\omega$. Compatibility of the differential with the Lie bracket amounts to the condition $L_Q \omega = 0$, which is part of the definition of a symplectic $L_{\infty}$-algebroid. The $h\text{Lie}$-algebra of this differential graded algebra is a refined version of the symmetry algebra of the $L_{\infty}$-algebroid, which is the $L_{\infty}$-algebra obtained from the original differential graded Lie algebra via proposition 4.4.

This theorem explains the interest in extension of Leibniz algebras in the context of generalized geometry and double field theory. The generalized tangent bundles used there are indeed symplectic $L_{\infty}$-algebroids (or symplectic pre-$NQ$-manifolds, as explained in [21]). Therefore, the relevant symmetry algebras are $h\text{Lie}$-algebras, and the most prominently visible feature of them in all construction is their Leibniz brackets $\varepsilon^0_2$.

\footnote{cf. [21] for a definition}
As a short example, let us work out the case of Vinogradov Lie \( n \)-algebroids, which generalize the Courant algebroid. The latter case, i.e. the case \( n = 2 \), was sketched in [1, Example 5.4]. The Vinogradov Lie \( n \)-algebroids are given as the graded vector bundles
\[
\mathcal{V}_n(M) := T^*[n]T[1]M
\] (5.1)
over some manifold \( M \). We introduce local coordinates \( x^\mu \) on the base \( M \) and extend these to Darboux coordinates \((x^\mu, \xi^\mu, \zeta^\mu, p_\mu)\) of degrees \( 0, 1, n - 1, n \), leading to the canonical symplectic form
\[
\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta^\mu \quad \text{and} \quad Q = \xi^\mu p_\mu .
\] (5.2)
This symplectic form induces the Poisson bracket
\[
\{f, g\} := \left( \frac{\partial}{\partial p_\mu} f \right) \left( \frac{\partial}{\partial x^\mu} g \right) - \left( \frac{\partial}{\partial x^\mu} f \right) \left( \frac{\partial}{\partial p_\mu} g \right) - (-1)^{|f|} \left( \frac{\partial}{\partial \zeta^\mu} f \right) \left( \frac{\partial}{\partial \xi^\mu} g \right) - (-1)^{|f|} \left( \frac{\partial}{\partial \xi^\mu} f \right) \left( \frac{\partial}{\partial \zeta^\mu} g \right),
\] (5.3)
and we have a Hamiltonian vector field \( Q \) given by
\[
Q = \{Q, -\} = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta^\mu} \quad \text{for} \quad Q = \xi^\mu p_\mu .
\] (5.4)
The algebra of functions \( C^{\infty}(\mathcal{V}_n(M)) \) is identified with the smooth functions in \( x^\mu \) and the analytical functions in the remaining coordinates, and it receives a grading from the grading of the coordinates. The vector field \( Q \) is a natural differential on \( C^{\infty}(\mathcal{V}_n(M)) \), and \( Q^2 = 0 \) is equivalent to \( \{Q, Q\} = 0 \).

We note that the Poisson bracket (5.3) is a Poisson bracket of degree \(-n\). We can now shift the grading in the algebra of functions by \(+n\) to obtain the differential graded Lie algebra
\[
\mathfrak{L}(M) := C^{\infty}(\mathcal{V}_n(M))[-n]
\] (5.5)
with differential \( Q \) and Lie bracket \( \{-, -\} \). The \( hLie \)-algebra associated to \( \mathfrak{L}(M) \) (and thus to \( \mathcal{V}_n(M) \)) by theorem 4.2 is then
\[
\mathfrak{E} = \left( \frac{C^{\infty}(M)}{\mathfrak{e}_{-n+1}} \right) \xrightarrow{Q} \left( \frac{C^{\infty}(M)}{\mathfrak{e}_{-n+2}} \right) \xrightarrow{Q} \cdots \xrightarrow{Q} \left( \frac{C^{\infty}(M)}{\mathfrak{e}_0} \right) ,
\]

\[
\mathfrak{e}_1(f_1) = \begin{cases} Qf_1 & \text{for } |f_1|_{\epsilon} < 0 , \\ 0 & \text{else} , \end{cases}
\]
\[
\mathfrak{e}_i(f_1, f_2) := \begin{cases} \{Qf_1, f_2\} & \text{for } i = 0 \text{ and } |f_1|_{\epsilon} = 0 , \\ (-1)^{|f_1|_{\epsilon}} \{f_1, f_2\} & \text{for } i = 1 , \\ 0 & \text{else} \end{cases}
\] (5.6)
for all \( f_1, f_2 \in \mathfrak{E} \). We can identify the elements of \( \mathfrak{E}_k \) with \( \Omega^{k+n-1}(M) \) for \( k < 0 \) and \( \mathfrak{E}_0 \cong \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \), where \( \mathfrak{X}(M) \) and \( \Omega^{k}(M) \) are the vector fields and differential \( k \)-forms.
on $M$, respectively. The latter are the generalized vector fields on $\mathcal{V}_n(M)$. Restricted to these, $\varepsilon^2_0$ is (a generalization of the) Dorfman bracket, whose antisymmetrization yields the Courant bracket, and $\varepsilon^2_1$ is a natural contraction $(\mathfrak{X}(M) \oplus \Omega^{n-1}(M)) \times (\mathfrak{X}(M) \oplus \Omega^{n-1}(M)) \to \Omega^{n-2}$.

As an explicit example, let us briefly present the case $n = 2$ for some manifold $M$. Here, we have the 2-term $h\mathfrak{Lie}$-algebra $\mathfrak{E}$ with underlying differential complex

$$\mathfrak{E} = (\mathfrak{E}-1 \xrightarrow{\varepsilon_1} \mathfrak{E}_0) = (C^\infty(M) \xrightarrow{d} \mathfrak{X}(M) \oplus \Omega^1(M)) .$$

The binary brackets are the Dorfman bracket, the evident action of $\mathfrak{E}_0$ on $\mathfrak{E}^{-1}$, and the evident dual pairing on $\mathfrak{E}_0$:

$$\varepsilon^0_2(X + \alpha, Y + \beta) = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha ,$$

$$\varepsilon^0_2(X + \alpha, f) = \mathcal{L}_X f = \iota_X df ,$$

$$\varepsilon^1_2(X + \alpha, Y + \beta) = \iota_X \beta + \iota_Y \alpha$$

for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, and $\alpha, \beta \in \Omega^1(M)$. The corresponding $L_{\pi}$-algebra obtained from theorem 4.3 yields the well known $L_{\pi}$-algebra of the Courant algebroid, cf. e.g. [21]. This $L_{\pi}$-algebra has the same differential complex as $\mathfrak{E}$, but with higher brackets

$$\mu_1(f) = df ,$$

$$\mu_2(X + \alpha, Y + \beta) = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2}d(\iota_X \beta - \iota_Y \alpha) ,$$

$$\mu_3(X + \alpha, f) = \frac{1}{2}\mathcal{L}_X f ,$$

$$\mu_3(X + \alpha, Y + \beta, Z + \gamma) = \frac{1}{3!}(\iota_X \iota_Y d\gamma + \frac{2}{3}\iota_X d\iota_Y \gamma \pm \text{perm.}) .$$

(5.8)

Following [21], one readily extends this discussion to the pre $\mathcal{NQ}$-manifolds underlying double field theory to reproduce the $D$- and $C$-brackets there.

Another specialization of symplectic $L_{\pi}$-algebroids is that described by the differential graded algebra given by the Batalin–Vilkovisky (BV) complex of a classical field theory, cf. [44] for definitions and conventions. Here, we have a Poisson bracket of degree $-1$ and a BV complex

$$C^\infty(\mathfrak{F}) := \left( \ldots \xrightarrow{Q} C^\infty_{-2}(\mathfrak{F}) \xrightarrow{Q} C^\infty_{-1}(\mathfrak{F}) \xrightarrow{Q} C^\infty_0(\mathfrak{F}) \xrightarrow{Q} C^\infty_1(\mathfrak{F}) \xrightarrow{Q} \ldots \right) ,$$

(5.9)

where $\mathfrak{F}$ is the full BV field space and the $C^\infty_i$ contains (the coordinate functions for) ghosts or gauge parameters for $i = 1$, fields for $i = 0$, antifields for $i = -1$, and antifields of ghosts for $i = -2$. If we shift this complex by $-1$, we obtain a differential graded Lie algebra, which then gives rise to an $h\mathfrak{Lie}$-algebra. At the moment, we do not have a concrete interpretation of this $EL_{\pi}$-algebra.

**5.2. Multisymplectic geometry**

There is a close relation between the associated $L_{\pi}$-algebras of $L_{\pi}$-algebroids and multisymplectic geometry, as explained in [23] and [57].
A multisymplectic manifold \((M, \omega)\) of degree \(p\), or a \(p\)-plectic manifold, is a manifold \(M\) with a closed differential form \(\omega \in \Omega^{p+1}(M)\) which is non-degenerate in the sense that \(\iota_X \omega = 0\) implies \(X = 0\) for all \(X \in \mathfrak{X}(M)\).

Any multisymplectic manifold \((M, \omega)\) comes with a differential complex

\[
\mathcal{L}(M, \omega) = \left( \frac{\Omega^0(M)}{\mathcal{L}(M, \omega)_{-n}}, \frac{\Omega^1(M)}{\mathcal{L}(M, \omega)_{-n}}, \ldots, \frac{\Omega^{p-1}(M)}{\mathcal{L}(M, \omega)_{-n}}, \frac{\Omega^{p}(M)}{\mathcal{E}(M, \omega)_0} \right),
\]

where \(\Omega^{p-1}(M)\) are the Hamiltonian \(n - 1\)-forms, i.e. differential forms \(\alpha\) for which there are vector fields \(\delta(\alpha)\) such that

\[
\iota_{\delta(\alpha)} \omega = d\alpha.
\]

In previous work [23, 24], it was realized that the shifted complex \(\mathcal{L}(M, \omega)[-1]\) restricted to non-positive degrees carries an \(L_\omega\)-algebra as well as a differential graded Leibniz algebra. The situation is, in fact, a bit richer.

**Theorem 5.2.** The complex \(\mathcal{L}(M, \omega)\) carries a natural differential graded Lie algebra structure with the Lie bracket \(\{-, -\}\) given by

\[
\begin{align*}
\{X_1, X_2\} &:= \{X_1, X_2\}, \\
\{X_1, \alpha_1\} &:= \mathcal{L}_{X_1} \alpha_1, \\
\{\alpha_1, \alpha_2\} &:= \iota_{\delta(\alpha_1)} \alpha_2 - (-1)^{|\alpha_1| |\alpha_2|} \iota_{\delta(\alpha_2)} \alpha_1
\end{align*}
\]

for all \(X_1, X_2 \in \mathfrak{X}(M)\) and \(\alpha_1, \alpha_2 \in \mathcal{L}(M, \omega)\) with \(|\alpha_1, \alpha_2|_{\mathcal{L}(M, \omega)} < 0\).

**Proof.** The proof is a straightforward verification of the axioms of a differential graded Lie algebra. \(\square\)

Via theorem 4.2, the above theorem has the following corollary.

**Corollary 5.3.** Any multisymplectic manifold \((M, \omega)\) comes with an \(h\text{Lie}\)-algebra

\[
\mathcal{E}(M, \omega) = \left( \frac{\Omega^0(M)}{\mathcal{E}(M, \omega)_{-n+1}}, \frac{\Omega^1(M)}{\mathcal{E}(M, \omega)_{-n}}, \ldots, \frac{\Omega^{p-1}(M)}{\mathcal{E}(M, \omega)_0} \right)
\]

with nonvanishing binary products

\[
\begin{align*}
\varepsilon_{\frac{p}{2}}^0(\alpha, \beta_1) &= \{\delta(\alpha), \beta_1\} = \mathcal{L}_{\delta(\alpha)} \beta_1, \\
\varepsilon_{\frac{p}{2}}^1(\beta_1, \beta_2) &= (-1)^{|\beta_1|} \varepsilon_{\frac{p}{2}}^0(\beta_1, \beta_2) = \iota_{\delta(\beta_1)} \beta_2 - (-1)^{|\beta_1| |\beta_2|} \iota_{\delta(\beta_2)} \beta_1.
\end{align*}
\]

for all \(\alpha \in \mathcal{E}(M, \omega)_0\) and \(\beta_1, \beta_2 \in \mathcal{E}(M, \omega)\).

The antisymmetrization of this \(h\text{Lie}\)-algebra is the \(L_\omega\)-algebra described in [23, 24]. Note that the special case \(M = S^3\) and \(\omega = \text{vol}_{S^3}\), upon restricting to left-invariant objects, yields another derivation of the hemistrict string \(EL_\omega\)-algebra model \(\text{string}_\omega^{wk,1}(g)\).
6. Higher gauge theory with $EL_{x^*}$-algebras

In this section, we develop and explore the generalities of higher gauge theory using $EL_{x^*}$-algebras as higher gauge algebras.

6.1. Homotopy Maurer–Cartan theory for $EL_{x^*}$-algebras

Recall that given an $L_{x^*}$-algebra $L$ with higher products $\mu_i$, there is a functor $\text{MC}(L, -)$ taking a differential graded commutative algebra $a$ to Maurer–Cartan elements with values in $a$, cf. e.g. [58]. This functor is represented by the Chevalley–Eilenberg algebra $CE(L)$ of the $L_{x^*}$-algebra.

What we usually call Maurer–Cartan elements in $L$ are Maurer–Cartan elements with values in $\mathbb{R}$, where the latter is regarded as a trivial differential graded algebra $\mathbb{R}_a$ with underlying vector space $\mathbb{R}$, spanned by a generator $w$ subject to the relation $w^2 = w$, and trivial differential.

For concreteness sake, let us assume that $L$ is degree-wise finite, and let $(t^A)$ be the generators of $L[1]^*$ dual to some basis $(\tau_A)$ of $L$. A Maurer–Cartan element is encoded in a morphism of differential graded commutative algebras $a : CE(L) \to \mathbb{R}_a$, which is fully determined by the image of the generators $(t^A)$ of degree 0,

$$a : CE(L) \to \mathbb{R}, \quad t^A \mapsto a^\alpha w$$  \hspace{1cm} (6.1)

for $a^\alpha \in \mathbb{R}$. Dually, we have an element $a := a^\alpha \tau_A \in L_1$, the gauge potential. Compatibility with the differential requires the curvature

$$f := \mu_1(a) + \frac{1}{2!} \mu_2(a, a) + \frac{1}{3!} \mu_3(a, a, a) + \cdots \in L_2$$  \hspace{1cm} (6.2)

to vanish, and the equation $f = 0$ is called the homotopy Maurer–Cartan equation. This curvature satisfies the Bianchi identity

$$\sum_{k \geq 0} \frac{1}{k!} \mu_{k+1}(a, \ldots, a, f) = 0.$$  \hspace{1cm} (6.3)

Infinitesimal gauge transformations are obtained from infinitesimal homotopies between morphisms from $CE(L)$ to $\mathbb{R}$. They are parameterized by elements $c \in g_0$ and act according to

$$\delta_c a = \sum_{i \geq 0} \frac{1}{i!} \mu_{i+1}(a, \ldots, a, c).$$  \hspace{1cm} (6.4)

Higher homotopies yield higher gauge transformations.

Similarly, one defines Maurer–Cartan elements of an $A_{x^*}$-algebra with values in a differential graded algebra.

In the case of $EL_{x^*}$-algebras, we can still consider tensor products of a base $EL_{x^*}$-algebra $\mathcal{E}$ and a differential graded commutative algebra $\mathfrak{A}$. However, the Chevalley–Eilenberg algebra $CE(\mathcal{E})$ is an $\mathcal{Elh}$-algebra and not a differential graded commutative algebra. Therefore the homotopy Maurer–Cartan functor cannot be represented by it directly.
There are two loopholes to this obstruction. First, we can lift the differential graded commutative algebra $\mathfrak{A}$, if it is semifree, to an $Eilh$-algebra $\tilde{\mathfrak{A}}$ as explained in theorem 2.12. We can then consider $Eilh$-algebra morphisms

$$a : CE(\mathfrak{E}) \to \tilde{\mathfrak{A}}.$$  

(6.5)

Second, we can project $CE(\mathfrak{E})$ to the Chevalley–Eilenberg algebra of the $L_{\mathfrak{E}}$-algebra $\mathfrak{L}$ induced by $\mathfrak{E}$ and consider the usual morphisms

$$a : CE(\mathfrak{L}) \to \mathfrak{A}.$$  

(6.6)

A third approach is simply to consider general morphisms of $Eilh$-algebras. In particular, one may want to replace differential forms with more general objects, cf. also [59].

We note that, in general, the three different types of morphism will give rise to different sets of Maurer–Cartan elements with the first one encompassing the second one. In all the applications we are aware of, however, the second approach is the appropriate one. While the difference between an $EL_{\mathfrak{E}}$-algebra and the corresponding $L_{\mathfrak{E}}$-algebra obtained by antisymmetrization is then invisible at the level of homotopy Maurer–Cartan theory, the additional algebraic structure in an $EL_{\mathfrak{E}}$-algebra is important in adjusting non-flat higher gauge theories.

### 6.2. Adjustment of higher gauge theory

In the construction of a higher gauge theory from an $EL_{\mathfrak{E}}$-algebra $\mathfrak{E}$, we will always employ the corresponding $L_{\mathfrak{E}}$-algebra $\mathfrak{L}$ obtained from theorem 3.15. We then consider its Weil algebra, which is the Chevalley–Eilenberg algebra of the inner derivations of $\mathfrak{L}$,

$$W(\mathfrak{L}) = \left( \circ^* (\mathfrak{L}[1]^* \oplus \mathfrak{L}[2]^*), Q_{W} \right), \quad Q_{W} = Q_{CE} + \sigma,$$

(6.7)

where $Q_{CE}$ is the Chevalley–Eilenberg differential of $\mathfrak{L}$ and $\sigma$ is the shift isomorphism $\sigma : \mathfrak{L}[1]^* \to \mathfrak{L}[2]^*$, extended to a morphism of differential graded commutative algebras.

The local kinematical data of an unadjusted higher gauge theory over a patch $U$ of some manifold $M$ is given by a differential graded algebra morphism

$$\mathcal{A} : W(\mathfrak{L}) \to \Omega^*(M).$$

(6.8)

This yields the definition of gauge potentials (the images of $\mathfrak{L}[1]^*$), curvatures (the images of $\mathfrak{L}[2]^*$ together with compatibility of $\mathcal{A}$ with the differentials on $\mathfrak{L}[1]^*$) and Bianchi identities (compatibility of $\mathcal{A}$ with the differentials on $\mathfrak{L}[2]^*$). Infinitesimal gauge transformations are given as partially flat homotopies between two such morphisms, and they are therefore determined by the form of the curvatures. For details, see the original discussion in [16]; the worked examples in [11] may also be helpful.

One severe issue with this direct definition of higher gauge theory is that consistency of the gauge algebroid (read: closure of the BRST differential) requires the so-called fake curvature condition, which is highly restrictive [11], as mentioned in the introduction.
Within supergravity, this problem had been solved in a special case corresponding to the string Lie 2-algebra (3.41) by working with different curvatures [13, 14]. As shown in [15], this kinematical data can be obtained from a morphism (6.8) after a modification of the Weil algebra, which also results in nicer mathematical properties. Such a modification can be performed for a large class of higher gauge theories, and an appropriately modified Weil algebra was termed adjusted Weil algebra in [11], where also a number of examples were worked out that are relevant to the (1,0) tensor hierarchies of gauged supergravity. In fact, all the kinematical data arising within the tensor hierarchies seem to be adjusted higher gauge theories, and we shall return to them in section 7. Moreover, the additional structure constants arising in the adjustment seem to originate from the higher products contained in $EL_{\infty}$-algebras that antisymmetrize to the gauge $L_{\infty}$-algebra. While we do not have a complete picture of the situation yet, we develop a partial one in the next section, which is sufficient for the treatment of tensor hierarchies in maximally supersymmetric gauged supergravities.

6.3. Firmly adjusted Weil algebras from $h\mathfrak{lie}$-algebras

Special cases of Weil algebras that are adjusted and whose corresponding morphisms (6.8) into differential forms yield adjusted higher gauge theories with closed BRST complex are the following ones:

Definition 6.1. A firmly adjusted Weil algebra of an $L_{\infty}$-algebra $\mathfrak{L}$ is a differential graded commutative algebra obtained from the Weil algebra $\mathcal{W}(\mathfrak{L})$ by a coordinate change

$$\hat{t}^A \mapsto \hat{t}^A := \hat{t}^A + p_{B_1 B_2 \ldots B_m} B^A C_1 \ldots C_n \hat{t}^{B_1} \ldots \hat{t}^{B_m} \hat{t}^{C_1} \ldots \hat{t}^{C_n},$$

(6.9)

where $t^A \in \mathfrak{L}[1]^*$, $\hat{t}^A \in \mathfrak{L}[2]^*$, $m \geq 1$, and $n \geq 0$, such that the image of the resulting differential $Q_{\text{adj}}$ on generators in $\mathfrak{L}[2]^*$ contains no generator in $\mathfrak{L}[1]^*$ except for at most one of degree 1.

We note that putting the generators $(\sigma t^A)$ to zero still recovers the Chevalley–Eilenberg algebra $CE(\mathfrak{L})$ of $\mathfrak{L}$. In this sense, the coordinate change has not changed the underlying $L_{\infty}$-algebra. Moreover, note that any Weil algebra is fully contractible in the sense that the cohomology of its linearized differential is trivial. Dually, it is the Chevalley–Eilenberg algebra of an $L_{\infty}$-algebra which is quasi-isomorphic to the trivial $L_{\infty}$-algebra. The non-trivial information contained in the Weil algebra is the relation between the generators $(t^A)$ and $(\sigma t^A)$, which translates under the morphism (6.8) into the relation between gauge potentials and their curvatures. Our coordinate change thus changes the definition of the curvatures and, as partially flat homotopies describe gauge transformations, also the gauge transformations. Firmly adjusted Weil algebras ensure that the corresponding BRST complex closes: the restricted terms govern the Bianchi identities, which fix the gauge transformations of the curvatures. Closure of the latter is what induces the fake curvature conditions, cf. the discussion in [11, section 4.4]. Thus, firmly adjusted Weil algebras are adjusted Weil algebras in the sense of [11].
As an example, consider the following firmly adjusted Weil algebra of the string Lie 2-algebra (3.41):

\[
Q_{\text{fadj}}: \quad t^\alpha \mapsto -\frac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha, \quad r \mapsto \frac{1}{3} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma - \kappa_{\alpha\beta} \kappa_\delta t^\beta + \hat{r}^\alpha, \quad \hat{r}^\beta = \kappa_{\alpha\beta} \hat{t}^\alpha, \quad (6.10)
\]

which is obtained from the coordinate transformation \( \hat{r} \mapsto \hat{r}' = \hat{r} + \kappa_{\alpha\beta} \hat{t}^\alpha t^\beta \). Here, \( t \in \mathfrak{g}[1]^* \), \( r \in \mathbb{R}[2]^* \) and \( \hat{t} = \sigma t, \hat{r} = \sigma r \). Under the morphism (6.8), this firmly adjusted Weil algebra gives rise to the usual string connections

\[
a = A + B \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^2(M, \mathbb{R}) ,
\]

\[
f = F + H \in \Omega^2(M, \mathfrak{g}) \oplus \Omega^3(M, \mathbb{R}) ,
\]

\[
F = dA + \frac{1}{2} [A, A] ,
\]

\[
H = dB - \frac{1}{3} (A, [A, A]) + (A, F) = dB + cs(A) .
\]

More generally, consider an \( L_{\mathfrak{x}} \)-algebra obtained from an \( h\mathcal{L}ie \)-algebra by antisymmetrization. For simplicity, we also assume that the \( L_{\mathfrak{x}} \)-algebra has maximally ternary brackets. Its Weil algebra then reads as

\[
Q_{W} t^\alpha = -(-1)^{|\beta|} m^\alpha_{\beta\gamma} t^\beta t^\gamma - (-1)^{|\gamma|} m^\alpha_{\gamma\delta} t^\gamma t^\delta + \hat{t}^\alpha ,
\]

\[
Q_{W} \hat{t}^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} \hat{t}^\beta t^\gamma + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} \hat{t}^\gamma t^\delta + \hat{\hat{t}}^\alpha ,
\]

\[
Q_{W} r^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} r^\beta t^\gamma + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} r^\gamma t^\delta + \hat{r}^\alpha ,
\]

\[
Q_{W} \hat{r}^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} \hat{r}^\beta t^\gamma + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} \hat{r}^\gamma t^\delta + \hat{\hat{r}}^\alpha .
\]

In general, this Weil algebra is clearly not firmly adjusted because of the explicit form of \( Q_{W} \hat{t}^\alpha \). Let us therefore perform the coordinate change

\[
\hat{t}^\alpha \mapsto \hat{\hat{t}}^\alpha := \hat{t}^\alpha + s^\alpha_{\beta\gamma} \hat{t}^\beta t^\gamma .
\]

The new Weil differential then reads as follows.

\[
Q'_{W} \hat{t}^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} \hat{\hat{t}}^\beta t^\gamma + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} \hat{\hat{t}}^\gamma t^\delta + \hat{\hat{t}}^\alpha ,
\]

\[
Q'_{W} \hat{r}^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} \hat{\hat{r}}^\beta t^\gamma + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} \hat{\hat{r}}^\gamma t^\delta + \hat{\hat{r}}^\alpha .
\]

where the ellipsis denotes cubic and higher terms. Let us now further restrict to \( h\mathcal{L}ie \)-algebras obtained from a differential graded algebra via theorem 4.3 with differential \( \Theta^\alpha_\beta \) and structure constants \( f^\alpha_{\beta\gamma} \). In this case, we have

\[
m^\alpha_{\beta\gamma} = \Theta^\alpha_\beta , \quad m^\alpha_{\gamma\delta} = \left\{ \begin{array}{ll}
\frac{1}{2} f^\delta_{\beta\gamma} \Theta^\delta_\beta & \text{if } |\beta| = 1 , \\
0 & \text{else} ;
\end{array} \right.
\]

we also put

\[
s^\alpha_{\beta\gamma} = \frac{1}{2} (-1)^{|\beta|(|\gamma|+1)} f^\alpha_{\beta\gamma} .
\]

In the above formulas, \( |\alpha|, |\beta|, |\gamma| \geq 1 \), and \( |\delta| = 0 \). Together with the Jacobi identity for the \( f^\alpha_{\beta\gamma} \), one can then easily verify that \( Q' \) becomes a firmly adjusted Weil differential,

\[
Q_{\text{fadj}} \hat{t}^\alpha = (-1)^{|\beta|} m^\alpha_{\beta\gamma} \hat{t}^\beta + (-1)^{|\gamma|} m^\alpha_{\gamma\delta} \hat{t}^\gamma t^\delta + \hat{\hat{t}}^\alpha .
\]

We conclude with the following theorem.
Theorem 6.2. Given an $L_\infty$-algebra with maximally ternary brackets that is obtained from the antisymmetrization of a differential graded Lie algebra by proposition 4.4, then there is a corresponding firmly adjusted Weil algebra. The data necessary for an adjustment arises from the alternators in the corresponding $hLie$-algebra.

Below, we shall give examples motivated from higher gauge theory. We stress, however, that the definition of an adjustment is also interesting for purely algebraic considerations, as it allows for the definition of a differential graded algebra of invariant polynomials for an $L_\infty$-algebra which is compatible with quasi-isomorphisms of this $L_\infty$-algebra, cf. the discussion in [11].

We also note that our construction highlights the features needed for obtaining a firmly adjusted Weil algebra. In particular, it is not necessary that the $hLie$-algebra was obtained from a differential graded Lie algebra; it was sufficient that there be a relation between the parameters $s^\alpha_{\beta\gamma}$ of the coordinate change and the structure constants $f^\alpha_{\beta\gamma}$ of the Lie algebra to ensure that (6.14) reduces to (6.16). This is the case, for example, in the tensor hierarchies in non-maximally supersymmetric gauged supergravity.

6.4. Example: (1,0)-gauge structures

As a first more involved example of $EL_\infty$-algebras arising in the context of higher gauge theory, let us consider the higher gauge algebra defined in [60], see also [4, 11, 61]. This algebra is a specialization of the general non-abelian algebraic structure identified in [62] and can be derived from tensor hierarchies, to which we shall return shortly. The latter had received an interpretation as an $L_\infty$-algebra with some “extra structure” before, cf. [63] as well as [64]. Here, we show that it is, in fact an $EL_\infty$-algebra.

The higher gauge algebra $\hat{\mathfrak{g}}^\omega_{sk}$ for $\mathfrak{g}$ a quadratic Lie algebra has underlying graded complex

$$\hat{\mathfrak{g}}^\omega_{sk} = \left( \begin{array}{c}
\mathfrak{g}^+_{v} \oplus \mathfrak{g}^+_{u} \oplus \mathfrak{g}^+_{t} \\
\mathfrak{g}^-_{sk, -3} \oplus \mathfrak{g}^-_{sk, -2} \oplus \mathfrak{g}^-_{sk, -1} \oplus \mathfrak{g}^-_{sk, 0} \\
\mathbb{R}^*_{s} \oplus \mathbb{R}^*_{p} \oplus \mathbb{R}^*_{r} \\
\mathbb{R}^*_{v} \oplus \mathbb{R}^*_{u} \oplus \mathbb{R}^*_{t}
\end{array} \right), \quad (6.17)$$

where the subscripts merely help to distinguish between isomorphic subspaces. In [4], this differential complex was extended to an $L_\infty$-algebra $\hat{\mathfrak{g}}^\omega_{sk}$ with higher products

$$\begin{align*}
\mu_2(t_1, t_2) &= [t_1, t_2] \in \mathfrak{g}_t, \\
\mu_2(t, u) &= u[-, t] \in \mathfrak{g}^+_{u}, \quad \mu_2(t, v) = v[-, t] \in \mathfrak{g}^+_{v}, \\
\mu_3(t_1, t_2, t_3) &= (t_1, [t_2, t_3]) \in \mathbb{R}_r, \quad \mu_3(t_1, t_2, s) = s(-, [t_1, t_2]) \in \mathfrak{g}^+_{u},
\end{align*} \quad (6.18)$$

where $t \in \mathfrak{g}_t$, etc. Moreover, $[-, -]$ and $(-, -)$ denote the Lie bracket and the quadratic form in $\mathfrak{g}$, respectively. When constructing gauge field strengths based on this $L_\infty$-algebra,
the following, additional maps feature:

\[ \nu_2(t_1, t_2) = -2(t_1, t_2) \in \mathbb{R}_r, \quad \nu_2(t, s) = 2s(-, t) \in \mathfrak{g}_u^*, \quad \nu_2(t_1, u_1) = u_1([-t_1]) \in \mathfrak{g}_u^*. \]  

(6.19)

As motivated in more detail later, it is useful to first perform a quasi-isomorphism on \( \hat{\mathfrak{g}}_{sk}^* \) leading to the higher brackets

\[
\begin{align*}
\mu_2(t_1, t_2) &= [t_1, t_2] \in \mathfrak{g}_t, \\
\mu_2(t, u) &= \frac{1}{2} u([-t, t]) \in \mathfrak{g}_u^*, \\
\mu_2(t, v) &= \frac{1}{2} v([-t, t]) \in \mathfrak{g}_v^*, \\
\mu_3(t_1, t_2, t_3) &= (t_1, [t_2, t_3]) \in \mathbb{R}_r, \\
\mu_3(t_1, t_2, s) &= s([-t_1, t_2]) \in \mathfrak{g}_u^*, \\
\mu_3(t_1, t_2, u) &= \frac{1}{2} v([-t_1, t_2]) \in \mathfrak{g}_v^*. 
\end{align*}
\]

(6.20)

This is the \( L_\infty \)-algebra obtained by theorem 4.3 from the \( h\text{Lie} \)-algebra \( \mathfrak{c} \) with differential complex (6.17) with \( \varepsilon_1 = \mu_1 \) and the additional binary products

\[
\begin{align*}
\varepsilon_1 &= \mu_1, \\
\varepsilon_2(t_1, t_2) &= -\varepsilon_2(2t_2, t_1) = [t_1, t_2] \in \mathfrak{g}_t, \\
\varepsilon_2(t, u) &= u([-t, t]) \in \mathfrak{g}_u^*, \\
\varepsilon_2(t, v) &= v([-t, t]) \in \mathfrak{g}_v^*, \\
\varepsilon_3(t_1, t_2) &= \varepsilon_3(2t_2, t_1) = 2(t_1, t_2) \in \mathbb{R}_r, \\
\varepsilon_3(t, s) &= 3s(-, t) \in \mathfrak{g}_u^*, \\
\varepsilon_3(t, u) &= u([-t, t]) \in \mathfrak{g}_u^*, \\
\varepsilon_3(t, v) &= v([-t, t]) \in \mathfrak{g}_v^*, \\
\varepsilon_3(u, t) &= \varepsilon_3(t, u) = u([-t, t]) \in \mathfrak{g}_v^*, \\
\varepsilon_3(v, t) &= \varepsilon_3(t, v) = v([-t, t]) \in \mathfrak{g}_v^*, \\
\varepsilon_3(t, s, t) &= \varepsilon_3(3t, s) = 3! s(-, t) \in \mathfrak{g}_u^*, \\
\varepsilon_3(t, u, t) &= \varepsilon_3(t, u) = u([-t, t]) \in \mathfrak{g}_u^*, \\
\varepsilon_3(t, v, t) &= \varepsilon_3(t, v) = v([-t, t]) \in \mathfrak{g}_v^*, \\
\varepsilon_3(u, t, u) &= \varepsilon_3(t, u, t) = u([-t, t]) \in \mathfrak{g}_u^*. 
\end{align*}
\]

as one verifies by direct computation. This \( h\text{Lie} \)-algebra is obtained from a differential graded Lie algebra \( \mathfrak{g} \) by theorem 4.2, and we have

\[
\mathfrak{g} = \begin{pmatrix}
\mathfrak{g}_u & \mathbb{R}_s \\
\mathfrak{g}_t & 0
\end{pmatrix}
\]

with the non-trivial Lie brackets \([-\cdot, \cdot]_{\mathfrak{g}}\) fixed by

\[
\begin{align*}
[t_1, t_2]_{\mathfrak{g}} &= [t_1, t_2] \in \mathfrak{g}_t, \\
[t_1, u]_{\mathfrak{g}} &= u([-t_1, t_2]) \in \mathfrak{g}_u^*, \\
[t_1, v]_{\mathfrak{g}} &= v([-t_1, t_2]) \in \mathfrak{g}_v^*, \\
[t_1, t_2]_{\mathfrak{g}} &= (t_1, t_2) \in \mathbb{R}_r, \\
[t_1, s]_{\mathfrak{g}} &= \alpha_2 s([-t_1, t_2]) \in \mathfrak{g}_u^*. 
\end{align*}
\]

(6.23)

This is an extension of the example presented at the end of section 4.2.

We thus see that we have the following sequence that leads to a construction of \( \hat{\mathfrak{g}}_{sk}^* \):

\[
\text{dg Lie algebra } \mathfrak{g} \xrightarrow{\text{Theorem 4.2}} \text{hLie-algebra } \mathfrak{c} \xrightarrow{\text{Theorem 4.3}} \text{L}_\infty\text{-algebra } \hat{\mathfrak{g}}_{sk}^*, 
\]

(6.24)
specializing the picture (4.10). The additional information (i.e. structure constants) contained in the \( EL_8 \)-algebra are vital for constructing the adjusted form of the curvatures.

A corresponding adjusted Weil algebra was found in [11], and it agrees with the one obtained from our construction of a firmly adjusted Weil algebra from section 6.3:

\[
Q_{\text{fadj}} : \quad t^\alpha \mapsto -\frac{1}{2} f^\alpha_{\beta\gamma} t^\beta r^\gamma + \hat{t}^\alpha , \quad p \mapsto -s + \hat{p} , \\
\hat{t}^\alpha \mapsto -f^\alpha_{\beta\gamma} t^\beta r^\gamma , \quad \hat{p} \mapsto \hat{s} , \\
r \mapsto \frac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma + \kappa_{\alpha\beta} \hat{t}^\alpha + q + \hat{r} , \\
\hat{r} \mapsto \kappa_{\alpha\beta} \hat{r}^\alpha - \hat{q} , \quad s \mapsto \hat{s} , \\
u_\alpha \mapsto -f^\gamma_{\alpha\beta} t^\beta u_\gamma - \frac{1}{2} f_{\alpha\beta\gamma} t^\beta \hat{t}^\gamma s - v_\alpha + \hat{u}_\alpha , \\
v_\alpha \mapsto -f^\gamma_{\alpha\beta} \hat{t}^\beta v_\gamma + f_{\alpha\beta\gamma} t^\beta \hat{r}^\gamma v_\gamma , \\
u_\alpha \mapsto -f^\gamma_{\alpha\beta} \hat{t}^\beta u_\gamma + f_{\alpha\beta\gamma} t^\beta \hat{t}^\gamma u_\gamma + \hat{v}_\alpha , \\
\hat{u}_\alpha \mapsto -f^\gamma_{\alpha\beta} \hat{t}^\beta \hat{v}_\gamma + f_{\alpha\beta\gamma} \hat{t}^\beta \hat{r}^\gamma \hat{v}_\alpha , \\
\hat{v}_\alpha \mapsto -f^\gamma_{\alpha\beta} \hat{t}^\beta \hat{v}_\gamma + f_{\alpha\beta\gamma} \hat{t}^\beta \hat{t}^\gamma \hat{v}_\alpha .
\]

(6.25)

7. Tensor hierarchies

Tensor hierarchies are particular forms of higher gauge theories that were introduced in the context of gauging maximal supergravity theories [65, 66, 67, 68, 69]. They are constructed using the embedding tensor formalism, introduced in [70, 71, 72, 73]. For comprehensive reviews see [6, 7]. Tensor hierarchies are also crucial to, for example, conformal field theories such as the \( \mathcal{N} = (1, 0) \) superconformal models of [62, 74, 75].

Although initially applied to gauged supergravity theories, tensor hierarchies do not require supersymmetry and appear through the embedding tensor formalism applied to the gauging of a broader class of Einstein–Maxwell-matter theories, as discussed in [69, 76].

7.1. Physical context

Before analyzing the algebraic structure underlying tensor hierarchies in more detail, let us briefly review the physical context. Consider the Lagrangian of ungauged Einstein–Maxwell-scalar theory in \( d \) dimensions,

\[
\mathcal{L}_{\text{ungauged}} = \bullet R + \frac{1}{2} g_{xy} d\varphi^x \wedge \bullet d\varphi^y - \frac{1}{2} a_{ij} F^i \wedge \bullet F^j + \cdots
\]

(7.1)

with scalars \( \varphi^x \) mapping spacetime to a scalar manifold \( \mathcal{M} \) and 1-form abelian gauge potentials \( A^i \) with field strengths \( F^i = dA^i \). Here, \( g_{xy}(\varphi) \) and \( a_{ij}(\varphi) \) are symmetric and positive-definite on the entire scalar manifold \( \mathcal{M} \). The ellipsis denotes possible deformations, such as a scalar potential \( \mathcal{V}(\varphi) \) or topological terms such as, e.g., \( d_{ijk} F^i \wedge F^j \wedge A^k \) for \( d = 5 \), as familiar from supergravity. The set of constant tensors controlling these deformation terms, which includes those appearing in the tensor hierarchies that do not enter (7.1), will be referred to as deformation tensors.

We assume that there is a global symmetry group \( \mathbf{G} \) acting on scalars and 1-form potentials such that the undeformed action (7.1) is invariant under this action. In particular,
the total gauge potential one-form $A$ takes value in a representation $V_{-1}$ of $G$, which is isomorphic to the fibers of the tangent space of the scalar manifold. In the presence of deformations, we assume that there is a non-abelian subgroup $K \subseteq G$ leaving the full action invariant.

Infinitesimally, the corresponding Lie algebra actions of $g = \text{Lie}(G)$ on the scalars and the gauge potential are given by

$$\delta \lambda \varphi^x = \lambda^\alpha k_\alpha^x(\varphi), \quad \delta \lambda A^i = \lambda^\alpha t^i_{\alpha} A^j,$$ (7.2)

where $t^i_{\alpha}, \alpha = 1, 2, \ldots, \dim g$, are the generators of $g$ in the representation $V_{-1}$ with respect to some bases $e_\alpha$ of $g$ and $e_i$ of $V_{-1}$. Invariance under $G$ requires that $k(\phi)$ be Killing vectors of the scalar manifold and $L_{k^a} a_{ij} = -2 t^k_{(a)j}.k$.

In order to gauge\(^\text{19}\) a subgroup $H \subseteq K \subseteq G$ with Lie algebra $\mathfrak{h} = \text{Lie}(H)$, we first note that we can trivially regard the pair $(V_{-1}, g)$ as a differential graded Lie algebra

$$V = ( V_{-1} \xrightarrow{0} V_0 = g )$$ (7.3)

with evident Lie bracket on $V_0$ and the Lie bracket $[-,-]: V_0 \times V_{-1} \to V_{-1}$ given by the action of $g$ on $V_{-1}$. Because the gauge potential takes values in $V_{-1}$, it does not make sense to gauge a Lie subalgebra of $g$ which is larger than $V_{-1}$. Therefore, we can identify the subalgebra $\mathfrak{h}$ with the image of a linear map

$$\Theta : V_{-1} \to \mathfrak{h} \subseteq g$$ (7.4)

The $(\Theta^\alpha e_\alpha)$ then form a spanning set\(^\text{20}\) of the Lie algebra $\mathfrak{h}$. Moreover, we have an induced action of $\mathfrak{h}$ on $V_{-1}$, given by

$$(\Theta^\alpha e_\alpha) \triangleright e_j = \Theta^\alpha t^{a}_{\alpha} e^{k} e_k =: X^{k}_{ij} e_k$$ (7.5)

with

$$t^{k}_{ij} = -t^{k}_{ij} \quad \text{and} \quad t^{a}_{\alpha} t^b_{\beta} t^c_{\alpha} = -f^{a}_{\alpha \beta \gamma} t^{c}_{\gamma k}.$$ (7.6)

In order to guarantee closure of the Lie bracket on $\mathfrak{h}$ and consistency of the action, we can assume that we can incorporate $\Theta$ into (7.3) such that

$$V_\Theta = ( V_{-1} \xrightarrow{\Theta} V_0 = g )$$ (7.7)

is again a differential graded Lie algebra. To jump ahead of the story, note that this guarantees the existence of a higher gauge algebra via proposition 4.4, which we anticipate as part of the construction of a higher gauge theory. The fact that $\Theta$ is a derivation for the graded Lie bracket then implies the quadratic closure constraint

$$f^{\alpha \beta}_{\gamma} \Theta^\alpha_i X^{k}_{ij} = \Theta^a_k X^{a}_{ki} \Leftrightarrow X_{im}^{\ell} X^{jn}_{it} - X_{jm}^{\ell} X^{in}_{it} = -X^{j}_{ij} X_{\ell m}^{n}.$$ (7.8)

\(^\text{19}\)That is, we promote a global symmetry $\mathbb{H}$ to a local one by adding a principal $\mathbb{H}$-bundle on our spacetime and consider (a part of) the one-form potential as a connection on this bundle.

\(^\text{20}\)but not necessarily a basis
It is well known that this quadratic closure constraint implies that the $X_{ij}^k$ form the structure constants of a Leibniz algebra on $V_{-1}$,

$$e_i \circ e_j := X_{ij}^k e_k , \quad (7.9)$$

cf. e.g. [77, 78, 79]. This is unsatisfactory given that the 1-form gauge potentials $A$ will take values in $V_{-1}$ and $V_{-1}$ should therefore have some Lie structure. As noted in [11], and as evident from proposition 4.1, this Leibniz algebra can be promoted to an $h\mathit{Lie}$-algebra. Moreover, the fact that we have the differential graded algebra (7.7) guarantees that we will have an $h\mathit{Lie}$-algebra via theorem 4.3 (or, if preferred, the corresponding $L_\infty$-algebra obtained from theorem 4.3). This will turn out to be indeed the higher gauge algebra underlying the tensor hierarchies.

But let us continue with the tensor hierarchy from the physicists’ perspective. The quadratic closure constraint (7.8) allows us to introduce a consistent combination of a covariant derivative on the scalar fields and local transformations parameterized by $\Lambda^p_0 e^q_0 r$:

$$\begin{align*}
\nabla \phi^i := & \text{d} \phi^i + \Theta^i_j \Lambda^k_0 A^j_0 \kappa^i_0 (\phi), \\
\delta \Lambda^p_0 \phi^i := & \Lambda^p_0 (\%Lambda^i_j) \kappa^i_0 (\phi), \\
\delta \Lambda^p_0 A^i := & \text{d} \Lambda^p_0 A^i + X_{jk}^i A^j \Lambda^k_0 ,
\end{align*} \quad (7.10)$$

Note that the action (7.9) of $V_{-1}$ on $V_{-1}$ is usually not faithful, and the parameterization by $\Lambda^i$ is thus usually highly degenerate.

In light of our above discussion of the higher Lie algebra arising from the Leibniz algebra (7.9), it is not surprising that the naive gauge transformation (7.10) of the gauge potential $A$ does not render the naive definition of curvature $\text{d} A^i + \frac{1}{2} X_{jk}^i A^j \wedge A^k$ covariant. This is remedied by introducing a second $\mathfrak{g}$-module $V_{-2} \subset \text{Sym}^2 (V_{-1})$, where $r = 1, 2 \ldots \dim V_{-2}$ for some basis $(e_r)$ together with a map

$$Z : V_{-2} \rightarrow V_{-1} ,
\quad e_r \mapsto Z^r e_i . \quad (7.11)$$

This allows us to introduce a $V_{-2}$-valued 2-form potential $B$ and a $V_{-2}$-valued 1-form gauge parameter $\Lambda^r (0)$ to generalized gauge transformations and curvatures as usual in higher gauge theory:

$$\begin{align*}
\delta A^i := & \text{d} \Lambda^i_0 + X_{jk}^i A^j \Lambda^k_0 + Z^i_r \Lambda^r_0 , \\
\delta B^r := & \nabla \Lambda^r_0 + \ldots , \\
F^i := & \text{d} A^i + \frac{1}{2} X_{jk}^i A^j \wedge A^k + Z^i_r B^r , \\
H^r := & \nabla B^r + \ldots ,
\end{align*} \quad (7.12)$$

where here $\nabla$ is the covariant derivative given by the natural action of $\mathfrak{h}$ on $V_{-2}$ and the ellipses refer to covariantizing terms that are needed to complete the kinematical data to that of an adjusted higher gauge theory. In particular, the latter will include terms involving the various deformation tensors. This process is then iterated in a reasonably obvious fashion until the full kinematical data of an adjusted higher gauge theory is obtained$^{21}$.

$^{21}$The fact that this iteration terminates is guaranteed because spacetime is finite-dimensional.
In the gauged supergravity literature there is also often a linear representation constraint
\[ P_\Theta \Theta = \Theta , \]  
where \( P_\Theta \) is the projector onto the representation contained in \( V_1^* \otimes g \) carried by \( \Theta \), which will be denoted \( \rho_\Theta \). This can be understood as a requirement of supersymmetry \([72, 65]\), the mutual locality of the action \([66]\) or anomaly cancellation \([80]\).

A final important ingredient is now that the electromagnetic duality contained in U-duality needs to link potential \( p \)-forms to potential \( d - p - 2 \)-forms, and correspondingly the \( G \)-modules \( V_{-p} \) and \( V_{p+2-d} \) have to be dual to each other in the lowest degrees that involve physical gauge potentials.

The above constraints restrict severely the choices of representations \( V_{-2}, V_{-3} \). In table 1 we listed some important concrete examples of maximal supergravities, in which \( K = G \). In this case, there is a tensor hierarchy \( \text{dgLa} \) determined by the U-duality group \([81, 82]\), with graded vector space described in Table 1 and derivation given by the action of \( \Theta \). Also, the electromagnetic duality is visibly reflected in the duality of representations in the cases \( d = 5, 6, 7 \).

| \( d \) | \( G \) | \( V_{-1} \) | \( V_{-2} \) | \( V_{-3} \) | \( V_{-4} \) | \( V_{-5} \) | \( V_{-6} \) |
|---|---|---|---|---|---|---|---|
| 7 | \( \text{SL}(5, \mathbb{R}) \) | 10\(_c\) | 5 | 5\(_c\) | 10 | 24 | 15\(_c\) \( \oplus \) 40 |
| 6 | \( \text{SO}(5, 5) \) | 16\(_c\) | 10 | 16 | 45 | 144 | 10 \( \oplus \) 126 \( \oplus \) 320 |
| 5 | \( \text{E}_6(6) \) | 27\(_c\) | 27 | 78 | 351\(_c\) | 27 \( \oplus \) 1728 |
| 4 | \( \text{E}_7(7) \) | 56 | 133 | 912 | 133 \( \oplus \) 8645 |
| 3 | \( \text{E}_8(8) \) | 248 | 1 \( \oplus \) 3875 | 3875 \( \oplus \) 147250 |

Table 1: Global symmetry groups \( G \) of maximal supergravity in \( 3 \leq d \leq 7 \) spacetime dimensions and their maximal compact subgroups (ignoring discrete factors). The \( G \) representations \( V_{-p} \) are carried by \( p \)-forms in the tensor hierarchy. The scalars (0-forms) are valued in \( \mathcal{M} := G/G_0 \), where \( G_0 \subset G \) is the maximal compact subgroup.

We note that in the presence of generic deformations, the differential graded Lie algebra constructed in the maximally supersymmetric case is actually insufficient and needs to be extended further by at least one step in both directions. We shall explain this below, when discussing the example \( d = 5 \).

### 7.2. Generic tensor hierarchies

Let us ignore the link between tensor hierarchies and gauged supergravity for a moment; clearly, the resulting kinematical data is potentially of interest in higher gauge theory in a much wider context.

The construction prescription is rather straightforward. We consider a Lie algebra \( g \), which we enlarge to a differential graded Lie algebra
\[ V = \left( \cdots \xrightarrow{d} V_{-2} \xrightarrow{d} V_{-1} \xrightarrow{d} V_0 = g \xrightarrow{d} V_1 \xrightarrow{d} \cdots \right) , \]
(7.14)
where we allowed for additional vector spaces $V_i$ with $i > 0$. All vector spaces $V_i$ are $g$-modules, and the Lie bracket on $V_0$ as well as the Lie brackets on $V_0 \otimes V_i$ are given. Further Lie brackets $[-,-] : V_i \otimes V_j \rightarrow V_{i+j}$ can be introduced, but due to the Jacobi identity, the underlying structure constants have to be invariant tensors of $g$ (as we shall also see below in an example). The differentials do not have to satisfy this restriction. As an additional constraint, we can also impose the condition that $V_p^* = V_{p+2-d}$ as required by the U-duality condition from supergravity. This can be useful in the construction of action principles.

To illustrate the above, let us construct a generic example in $d = 5$. Let $g$ be a Lie algebra and $V_{-1}$ any representation. Imposing the duality constraint and allowing for an extension in one degree on either side leads to the differential complex

$$V = \left( V_{-4} \cong \text{coker}(\Theta)^* \xrightarrow{d} V_{-3} \cong g^* \xrightarrow{d} V_{-2} \cong V_1^* \xrightarrow{d} V_{-1} \xrightarrow{\Theta} V_0 \xrightarrow{d} V_1 \cong \text{coker}(\Theta) \right).$$

(7.15)

Let us now switch to the Chevalley–Eilenberg description $\text{CE}(V)$ of the differential graded Lie algebra $V$ we want to construct, which is generated by coordinates $r^\mu$, $r^\alpha$, $r^a$, $r_{\alpha}$, $r_{\mu}$ of degrees $0, 1, 2, 3, 4, 5$, respectively. We note that we have a natural symplectic form on $V[1]^*$ of degree 5,

$$\omega = dr^\alpha \wedge dr_{\alpha} + dr^a \wedge dr_{a} + dr^\mu \wedge dr_{\mu}.$$

(7.16)

Compatibility of the Lie algebra action with the duality pairing amounts to the fact that the Chevalley–Eilenberg differential $Q$ is Hamiltonian for the Poisson bracket of degree $-5$,

$$\{f, g\} := \frac{\partial f}{\partial r^\alpha} \frac{\partial g}{\partial r^\alpha} + (-1)^{|f|+1} \frac{\partial f}{\partial r^a} \frac{\partial g}{\partial r^a} + (-1)^{|g|+1} \frac{\partial f}{\partial r^\mu} \frac{\partial g}{\partial r^\mu}$$

(7.17)

induced by $\omega$. That is,

$$Q = \{Q, -\}, \quad |Q| = 6.$$

(7.18)

The most generic Hamiltonian $Q$ of degree 6 that is at most cubic in the generators\(^{22}\) is

$$Q = \frac{1}{2} f_{\beta \gamma}^\alpha r^\alpha r^\beta r^\gamma + \Theta_{\alpha}^a r^a r_{\alpha} + \frac{1}{3} Z_{ab} r^a r^b r_{\alpha} + \frac{1}{2} g_{\alpha \mu} r_{\alpha} r_{\mu} + g_{ab} r^a r^b$$

(7.19)

where besides the structure constants $f_{\beta \gamma}^\alpha$ and the embedding tensor $\Theta_{\alpha}^a$ we have the deformation tensors $d_{abc}$ and $Z^{ab}$, which are totally symmetric and antisymmetric, respectively, due to the grading of the generators. The remaining structure constants will be called auxiliary. For $Q$ to give rise to a Chevalley–Eilenberg differential, we have to impose

$$Q^2 = 0 \Leftrightarrow \{Q, Q\} = 0.$$

(7.20)

---

\(^{22}\)This restriction is required to obtain a differential graded Lie algebra, as opposed to an $L_\infty$-algebra.
This equation imposes conditions on the structure constants. For example, we have

$$\Theta_a^\gamma f_{\beta\gamma}^\alpha + t_{\beta\alpha}^b \Theta_b^\alpha - g_{a\beta}^\mu g_{\mu}^\alpha = 0 .$$  \hspace{1cm} (7.21)

For $g_1 = g_3 = 0$, this implies that the embedding tensor is an invariant tensor, which is clearly too strong a condition. We can make a non-canonical choice of an embedding

$$i : \text{coker}(\Theta) \hookrightarrow \mathfrak{g} ,$$  \hspace{1cm} (7.22)

which is given by structure constants $i^\alpha_\mu$ such that

$$i^\alpha_\mu g_{1\alpha}^\mu = \delta^\nu_\mu .$$  \hspace{1cm} (7.23)

With this choice, we can split the condition (7.21) into

$$\Theta^\beta_\gamma f_{\beta\gamma}^\alpha + X^b_{\alpha\beta} \Theta_b^\alpha = 0 ,$$

$$i^\alpha_\mu (\Theta^\beta_\gamma f_{\beta\gamma}^\alpha + t_{\beta\alpha}^b \Theta_b^\alpha) = g^\alpha_3 g_{\mu}^\alpha ,$$  \hspace{1cm} (7.24)

and the first condition is the usual one encountered in the $d = 5$ tensor hierarchy, while the second condition fixes one of the auxiliary structure constants. Besides the above condition and the fact that $f_{\beta\gamma}^\alpha$ and $t_{\alpha\beta}^a$ are the structure constants of the Lie algebra $\mathfrak{g}$ and a representation of $\mathfrak{g}$, we also have

$$Z^{ab} \Theta_b^\alpha = 0 ,$$

$$Z^{ab} d_{ac} - 2 X^a_{(cd)} = 0 ,$$

$$Z^{a[b} t_{c]}^c + 2 g^\alpha_1 g_{3\mu}^\alpha = 0 ,$$

$$t_{\alpha(a} d_{bc)d} = 0 ,$$  \hspace{1cm} (7.25)

as well as a number of conditions for the auxiliary structure constants. As expected, the tensor $d_{abc}$ capturing the Lie bracket $\mathcal{V}_- \otimes \mathcal{V}_- \rightarrow \mathcal{V}_- \otimes \mathcal{V}_-$ has to be an invariant tensor.

The kinematical data of a generic tensor hierarchy can then be constructed from the firmly adjusted Weil algebra of the corresponding $L_\infty$-algebra as described in detail in section 6.3.

We note that the condition that $d_{abc}$ be an invariant tensor is too strong a constraint, e.g. for the non-maximally supersymmetric case. From the formulas of the curvatures, it is clear that there is no differential graded Lie algebra underlying this case, if the higher gauge algebra is constructed using the formulas of theorem 4.2. This observation strongly suggests that there are generalizations of these derived bracket constructions, but this is beyond the scope of this paper.

### 7.3. Example: $d = 5$ maximal supergravity

Let us give a concrete and complete picture of the interpretation of a tensor hierarchy using $h\mathcal{L}_\infty$-algebras, including the construction of curvatures. We choose the case $d = 5$, which allows us to recycle observations made in section 7.2. For a detailed discussion of this theory, see [83].
Maximal supergravity in $d = 5$ dimensions has the non-compact global symmetry group $E_{6(6)}(\mathbb{R})$ [84]. When dimensionally reducing from $d = 11$, in order to make manifest the $\mathfrak{e}_{6(6)}$ structure of the scalar sector in $d = 5$, one must first dualize the 3-form potential, as described in detail in [85]. This gives a total of 42 scalars parameterizing $E_{6(6)}(\mathbb{R})/USp(8)$.

The fully dualized bosonic Lagrangian with manifest $E_{6(6)}(\mathbb{R})$-invariance can be written as
\[
\mathcal{L}_5 = R \ast 1 + \frac{1}{2} g_{xy} d \varphi^x \wedge \cdots d \varphi^y - \frac{1}{2} a_{ab} F^a \wedge F^b - \frac{1}{6} d_{abc} F^a_{(2)} \wedge F^b_{(2)} \wedge A^c_{(1)} , \tag{7.26}
\]
The 1-form potentials transform linearly in the $27_c$ of $\mathfrak{e}_{6(6)}$, and $a, b, c \in \{1, \ldots, 27\}$. In addition to the singlet $1 \in 27_c \otimes 27$, used to construct the 1-form kinetic term, there is a singlet in the totally symmetric 3-fold tensor product $1 \in \otimes^3 (27_c)$, which is used to construct the topological cubic term.

For the construction of the tensor hierarchy, we shall need the following $E_6$-invariant tensors:
\[
f_{\alpha \beta \gamma} \in \bigwedge^3 78 , \quad t_{ab}^b \in 78 \otimes 27 \otimes 27_c , \quad d_{abc} \in \bigwedge^3 27 , \quad d_{abc} \in \bigwedge^3 27_c . \tag{7.27}
\]
To optimize our notation, we also introduce the following tensors:
\[
X_{ab}^c = \Theta_a^\alpha t_{ab}^\alpha c , \quad Y_{\alpha \beta} = \Theta_{a \beta} f_{\gamma \alpha \beta} + t_{a a}^b \Theta_b^\beta \equiv \delta_a \Theta_b^\beta , \quad X_{a \alpha \beta} = \Theta_a^\gamma f_{\gamma \alpha \beta} , \quad Z^{ab} = \Theta_{c a}^\alpha t_{\alpha d}^d d_{b d e} = X_{c d}^a d_{b c e} = Z^{[ab]} , \tag{7.28}
\]
The above tensors satisfy the following identities [83]:
\[
d_{acb} d_{bce} = \delta_a^b , \quad X_{(ab)}^c = d_{abc} Z^{cd} , \quad X_{[ab]}^c = 10 d_{abc} d_{a e} d_{b d e} Z^{f g} , \tag{7.29a}
\]
and in addition, we have the following three equivalent forms of the closure constraints:
\[
2 X_{[a | c}^d X_{b]}^e X_{d e} = 0 , \quad Z^{ab} X_{bc}^d = 0 , \quad X_{d e}^a Z^{b c} = 0 . \tag{7.29b}
\]
Using these, we can now apply the formalism of section 7.2 and construct the differential graded Lie algebra. It helps to broaden the perspective a bit and derive the latter from a graded Lie algebra $V$, with underlying vector space consisting of $\mathfrak{e}_{6(6)}$-modules:
\[
V_{\mathfrak{e}_{6(6)}} = V_{-5} \oplus V_{-4} \oplus V_{-3} \oplus V_{-2} \oplus V_{-1} \oplus V_0 \oplus V_1 \quad \rho_{(k)} = 27 \oplus 1728 \oplus 351_c \oplus 78 \oplus 27 \oplus 27_c \oplus 78 \oplus 351 \quad e_{(k)} = (e^a, e_{ab}^a) , \quad e_{a}^\alpha, e^a , \quad e_a , \quad e_\alpha , \quad e_{a \alpha}^a \tag{7.30}
\]
We have indicated the $\mathfrak{e}_{6(6)}$-representations $\rho_{(k)}$ carried by each $V_{\mathfrak{e}_{6(6)}}$-degree $k$ summand, $V_k$, and their corresponding basis elements $e_{(k)}$, e.g. $(e_{(0)})_\alpha = e_\alpha$, where $(e_\alpha)$ is some basis for the exceptional Lie algebra $\mathfrak{e}_{6(6)}$. Note that the embedding tensor $\Theta = \Theta_a^\alpha e_\alpha^a$ is an element of $V_1$ and $e_\alpha^a = P_{351} e_{a \alpha}^a \otimes e^a$.

The graded Lie bracket on $V$ is now given mostly by the obvious projections of the graded tensor products,
\[
[e_\alpha, e_\beta] = f_{\alpha \beta}^\gamma e_\gamma , \quad [e_\alpha, e_{(k)}] = \rho_{(k)}(e_\alpha) e_{(k)} , \quad [e_{(k)}, e_{(l)}] = T_{k,l} e_{(k+l)} . \tag{7.31}
\]
Here, \( T_{k,l} \) are the intertwiners dual to the projectors \( P_{k,l} : V_k \wedge V_l \to V_{k+l} \). For example,

\[
[e_a, e_b] = 2d_{abc}e^c, \quad [e_a, e^b] = (t_a)_b^c e^c, \quad \tag{7.32}
\]

where

\[
e^a := \frac{1}{2}d^{abc}[e_b, e_c], \quad e_\alpha := (t_\alpha)_b^a[e_a, e^b]. \quad \tag{7.33}
\]

The adjoint indices are raised/lowered with \( \eta_{\alpha\beta} = \text{tr}(t_\alpha t_\beta) \), which is proportional to the Cartan–Killing form.

Selecting an element \( \Theta = \Theta_\alpha^\alpha e_\alpha^a \in V_1 \) now defines a differential

\[
dv := [\Theta, v] \quad \tag{7.34}
\]

for \( v \in V \), and we note that \([\Theta, \Theta] = 0\) for degree reasons. The explicit action of \( d e_{(k)} := [\Theta, e_{(k)}] \) can be determined using the graded Jacobi identity from the initial condition

\[
[e^a_\alpha, e_b] = P_{351} \delta^a_b e_\alpha, \quad \tag{7.35}
\]

where \( P_{351} \) is the projector \( P_{351} : 78 \otimes 27_c \to 351_c \).

We thus obtain a differential graded Lie algebra, and this is a special case of the algebra called dgLie (THA') in section 8.3.

Let us now construct the \( h\)Lie-algebra \( \mathcal{E} \) of this differential graded Lie algebra using theorem 4.2. We arrive at the graded vector space

\[
\mathcal{E}_{e^0(\Theta)} = \bigoplus_{\ell} \mathcal{E}_\ell = 27 \oplus 1728 \oplus 351_c \oplus 78 \oplus 27 \oplus 27_c, \quad \tag{7.37}
\]

with non-trivial products

\[
\varepsilon_1(x) := [\Theta, x], \quad \varepsilon^0_1(x, y) := [[\Theta, x], y], \quad \varepsilon^1_2(x, y) := (-1)^{|x|}|x, y|. \quad \tag{7.38}
\]

Explicitly, we have the differentials

\[
\varepsilon^1(e^a) = \Theta_\alpha^\alpha t_{\alpha\beta} d_{\beta\gamma} e_\gamma \equiv X_{bc} d_{\beta\gamma} e_\gamma = -Z_{\alpha\beta} e_\gamma, \quad \varepsilon^1(e^a) = \Theta_\alpha^\alpha e^a, \quad \varepsilon^1(e_\alpha^a) = -\delta_{\beta\gamma} \Theta_\alpha^\alpha e_{\gamma} = -Y_{\alpha\beta} e_{\gamma}, \quad \tag{7.39a}
\]

the Leibniz-like products

\[
\varepsilon^0_2(e_a, e_b) = [\Theta_a^\alpha e_\alpha^a, e_b] = \Theta_a^\alpha t_{\alpha\beta}^c e_c = X_{ab} e^c, \quad \varepsilon^0_2(e^a, e^b) = [\Theta_\alpha^\alpha e_\alpha^a, e^b] = -\Theta_a^\alpha t_{\alpha\beta}^c e_c = -X_{ac} e^c, \quad \varepsilon^0_2(e_\alpha^a, e_\beta^b) = [\Theta_\alpha^\alpha e_\alpha^a, e_\beta^b] = -\Theta_a^\alpha f_{\alpha\gamma}^b e_{\gamma} = -X_{\alpha\gamma} e_{\gamma}, \quad \varepsilon^0_2(e_a, e_\beta^b) = [\Theta_a^\alpha e_\alpha^a, e_\beta^b] = -\Theta_a^\alpha f_{\alpha\gamma}^b e_{\gamma} + \Theta_a^\alpha t_{\alpha\beta}^c e_{\gamma} = -X_{\alpha\gamma} e_\gamma + X_{ab} e_{\gamma}, \quad \tag{7.39b}
\]

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as well as the alternator-type products

\[ \varepsilon^2_{2}(e_a, e_b) = 2d_{abc}e^c, \]
\[ \varepsilon^2_{2}(e_a, e^b) = t_{ab}^a b e^a = \varepsilon^2_{2}(e_b, e_a), \]
\[ \varepsilon^2_{2}(e_a, e^a) = e_a^a = P_{351}^a e_b^b = -\varepsilon^2_{2}(e_a, e_a), \]
\[ \varepsilon^2_{2}(e^a, e^b) = -e^{[ab]} = t_{ac}^a [d^b]_{cd} e_d^d, \] (7.39c)

where we used that \( t_{ac}^a [d^b]_{cd} \) is the intertwiner between the \( 351_c \in 27_c \otimes 78 \) and \( 351_c \cong \wedge^2 27 \).

We can now construct the corresponding curvatures. We start from the Chevalley–Eilenberg algebra of \( \mathcal{E}_{\alpha(6)} \) with the following generators \((\tilde{r}^A)\) spanning \( \mathcal{E}_{\alpha(6)}[1]^* \):

\[
\begin{array}{cccccc}
\text{degree} & 1 & 2 & 3 & 4 & 5 \\
\mathcal{E}_{\alpha(6)}[1]^* &=& 27 & 27_c & 78 & 351 \oplus 27_c \oplus 1728_c.
\end{array}
\]

Consider now the Weil algebra \( \mathcal{W}(\mathcal{E}_{\alpha(6)}) \), cf. definition 3.3. Here, we introduce a second copy of shifted generators \((\hat{r}^A)\) spanning \( \mathcal{E}_{\alpha(6)}[2]^* \) with \(|\hat{r}^A| = |r^A| + 1\). The usual Weil differential up to degree 3 elements, dual to scalars in \( d = 5 \), then reads as

\[
\begin{align*}
Q_W r^a &= -Z^{ab} r_b - X_{bc}^a r^b \otimes_0 r^c + \tilde{r}^a, \\
Q_W r_a &= \Theta_a^\beta r_a + X_{ba}^\beta r_a \otimes_0 r_c - d_{abc}^b \hat{\otimes}_1 r^c + \hat{r}_a, \\
Q_W r_a &= Y_{aa}^\beta r_\beta + X_{aa}^\beta r_a \otimes_0 r_\beta + t_{aa}^b r_a \hat{\otimes}_1 r_b + r_a, \\
Q_W \hat{r}^a &= Z^{ab} \hat{r}_b + X_{bc}^a r^b \otimes_0 r^c - X_{bc}^a r^b \otimes_0 \hat{r}^c, \\
Q_W \hat{r}_a &= -\Theta_a^\alpha \hat{r}_a - X_{ba}^\alpha r_a \otimes_0 r_c + X_{ba}^\alpha r^b \otimes_0 \hat{r}_c + 2d_{abc}^b \hat{\otimes}_1 r^c, \\
Q_W \hat{r}_a &= -Y_{aa}^\beta \hat{r}_\beta + X_{aa}^\beta \hat{r}_a \otimes_0 r_\beta + X_{aa}^\beta \hat{r}_a \otimes_0 \hat{r}_\beta - t_{aa}^b \hat{r}_a \hat{\otimes}_1 r_b + t_{aa}^b \hat{r}_b \hat{\otimes}_1 \hat{r}_a.
\end{align*}
\] (7.41)

where we have introduced the notation

\[
\begin{align*}
a \hat{\otimes}_1 b &= a \hat{\otimes}_1 b + (-1)^{|a|+|b|}|b| \hat{\otimes}_1 a, \\
a \hat{\otimes}_1 b &= a \hat{\otimes}_1 b - (-1)^{|a|+|b|}|b| \hat{\otimes}_1 a.
\end{align*}
\] (7.42)

The deformed Leibniz rule (2.8), together with the remaining \( Eilh \)-relations (2.7) and the identities (7.29), then imply \( Q_W^2 = 0 \) as one can check by direct computation.

In order to define the curvatures of the tensor hierarchy, we symmetrize to an \( L_x \)-algebra using theorem 4.3. We can then use the formalism of section 6.3 to construct an adjusted Weil algebra in the sense of [11], ensuring closure of the gauge algebra without any further constraints on the field strengths.

To illustrate in more detail the procedure and what it achieves, we can perform the coordinate change already at the level of the Weil algebra of the \( h\mathcal{L}ie \)-algebra. This coordinate change yields a symmetrized and firmly adjusted Weil algebra through an evident coordinate change, \( r^A \rightarrow \tilde{r}^A \), which removes all appearances of \( \otimes_1 \) in \( Q_W \tilde{r}^A \) via the deformed
Leibniz rule (2.8). Hence, by theorem 3.10 we are left with an $L_{\mathcal{X}}$-algebra. Explicitly, the following coordinate change manifestly removes all appearances of $\mathcal{O}_1$:

\[
\begin{align*}
  r^a &\to a^a := r^a, \\
  r^a &\to b_a := r_a + \frac{1}{2}d_{abc}r^b \mathcal{O}_0 r^c, \\
  r^a &\to c_a := r_a - \frac{1}{2}t_{\alpha a} b^a \mathcal{O}_0 r_b, \\
  r^a &\to d_a^a := r^a_a + \frac{1}{2}P_{351} r^b \mathcal{O}_0 r^c + \frac{1}{2}t_{\alpha a} [b^c d^e] r^c \mathcal{O}_0 r^e,
\end{align*}
\]

(7.43)

where $d_a^a$ is included as it is needed for $Q_{\mathcal{W}}\hat{r}_a$. The corresponding coordinate change on $\hat{r}^A$ is firmly adjusted by simply first ordering the occurrences of $\hat{r}^B$ in $\hat{r}^A$ to the left (which is permitted by the appearance of only $\mathcal{O}_0$ in $\hat{r}^A$) and then sending $\mathcal{O}_1$ to $\mathcal{O}_i + \mathcal{O}_i = 2\mathcal{O}_1$. The choice of left ordering follows from the choice of left Leibniz rule, which is a matter of convention. Applied to (7.43) this yields

\[
\begin{align*}
  \hat{r}^a &\to f^a := \hat{r}^a, \\
  \hat{r}_a &\to h^a := \hat{r}_a + 2d_{abc}r^b \mathcal{O}_0 r^c, \\
  \hat{r}_a &\to g_\alpha := \hat{r}_a - t_{\alpha a} b^a \mathcal{O}_0 r_b - \hat{r}_b \mathcal{O}_0 r^a, \\
  \hat{r}_a &\to k^a := \hat{r}_a + P_{351}(r^b \mathcal{O}_0 r^c + \hat{r}_b \mathcal{O}_0 r^b) + 2t_{\alpha a} [b^c d^e] r^c \mathcal{O}_0 r^e. \\
\end{align*}
\]

(7.44)

Note, this is a special case of the transformation (6.9) for a firm adjustment.

The result of this coordinate change is the differential graded commutative algebra $\mathcal{W}_{\text{adj}}(\mathcal{E}_{t\alpha(6)})$ generated by $\mathcal{E}_{t\alpha(6)}/[1] \oplus \mathcal{E}_{t\alpha(6)}/[2]$ and differential

\[
\begin{align*}
  Q_{\mathcal{W}_{\text{adj}}} a^a &:= -Z_{ab} b^b - \frac{1}{2}X_{bc} a^b a^c + f^a, \\
  Q_{\mathcal{W}_{\text{adj}}} b_a &:= \Theta_{\alpha a} c_a + \frac{1}{2}X_{\alpha b} a^b b_c + \frac{1}{2}d_{abc}X_{\alpha e} b^e a^d a^e - d_{abc}f^b a^c + h_a, \\
  Q_{\mathcal{W}_{\text{adj}}} c_\alpha &:= Y_{\alpha a} \beta d_\beta a^a + \frac{1}{2}X_{\alpha b} \beta a^b c_\beta + (\frac{1}{2}X_{\alpha a} \beta t_{\beta b} c + \frac{1}{3}t_{\alpha a} \beta X_{\alpha b} c^d) a^a a^b b_c \\
  &\hspace{1cm} + \frac{1}{2}t_{\alpha a} b^a f^b b^b - \frac{1}{2}t_{\alpha a} b^a h_b a^c - \frac{1}{2}t_{\alpha a} b^a d_{bc a} a^c f^d + g_\alpha, \\
  Q_{\mathcal{W}_{\text{adj}}} f^a &:= Z_{ab} h_b + X_{bc} a^b f^c, \\
  Q_{\mathcal{W}_{\text{adj}}} h_a &:= -\Theta_{\alpha a} g_a + X_{\alpha a} c_\alpha d_{b c} h_c + d_{abc}f^b f^c, \\
  Q_{\mathcal{W}_{\text{adj}}} g_\alpha &:= -Y_{\alpha a} \beta b^a + X_{\alpha a} \beta a^a g_\beta - t_{\alpha a} b^a h_b f^a. \\
\end{align*}
\]

We can now define the corresponding curvatures in the adjusted higher gauge theory as usual as a morphism of differential graded algebras

\[
(A, F) : \mathcal{W}_{\text{adj}}(\mathcal{E}_{t\alpha(6)}) \to \Omega^*(M),
\]

(7.46)

where

\[
\begin{align*}
  (a^a, b_a, c_\alpha, d_a^a) &\to (A^a, B_a, -C_\alpha, -D_a^a), \\
  (f^a, h_a, g_\alpha, k^a) &\to (F^a, H_a, -G_\alpha, -K^a). \\
\end{align*}
\]

(7.47)

The additional signs here follow from the choice of sign convention in (7.33).
This indeed yields the gauge potentials and curvatures of the $d = 5$ tensor hierarchy:

$$
F^a = dA^a + \frac{1}{2}X_{bc}^a A^b \wedge A^c + Z^{ab} B_b ,
$$  \hspace{1cm} (7.48a)

$$
H_a = dB_a - \frac{1}{2}X_{ac}^b A^c \wedge B_b - \frac{1}{6}d_{abc}^d X_{de}^b A^c \wedge A^d \wedge A^e + d_{abc} A^b \wedge F^c + \Theta_a^\alpha C_\alpha ,
$$  \hspace{1cm} (7.48b)

$$
G_\alpha = dC_\alpha - \frac{1}{2}X_{\alpha \beta}^\gamma A_\gamma \wedge C_\beta + \left( \frac{1}{4}X_{\alpha \beta}^\gamma t_{\beta \gamma}^c + \frac{1}{3}X_{(db)}^{-1}d_{abc} A^a \wedge A^b \wedge B_c
\right)
+ \frac{1}{2}t_{\alpha a}^b F^a \wedge B_b - \frac{1}{2}t_{\alpha a}^b H_b \wedge A^a - \frac{1}{6}t_{\alpha a}^b d_{bed} A^a \wedge A^c \wedge F^d - Y_{\alpha a}^\beta D_{\beta}^a ,
$$  \hspace{1cm} (7.48c)

along with the corresponding Bianchi identities,

$$
0 = dF^a - X_{bc}^a A^b \wedge F^c - Z^{ab} H_b ,
$$  \hspace{1cm} (7.49a)

$$
0 = dH_a - X_{ab}^c A^b \wedge H_c - d_{abc} F^b \wedge F^c - \Theta_a^\alpha G_\alpha ,
$$  \hspace{1cm} (7.49b)

$$
0 = dG_\alpha - X_{\alpha \beta}^\gamma A_\gamma \wedge G_\beta - t_{\alpha a}^b H_b \wedge F^a - Y_{\alpha a}^\beta K_{\beta}^a .
$$  \hspace{1cm} (7.49c)

We note that the full kinematical data is determined in this way: the Bianchi identities are implied by compatibility of the morphism (7.46) with the differential, and the gauge transformations are constructed as infinitesimal partially flat homotopies, cf. e.g. [11] for details.

To make contact with the expressions in the supergravity literature, cf. [83, 76], one must make the field redefinitions

$$
C_\alpha \mapsto C_\alpha + \frac{1}{2}t_{\alpha a}^b A^a \wedge B_b ,
$$

$$
D_{\alpha}^a \mapsto D_{\alpha}^a - \frac{1}{2}P^{51a}_{ABC} A^a \wedge C_\alpha .
$$  \hspace{1cm} (7.50)

Similar field redefinitions were also used in [86] to link another elegant derivation of the curvature forms (in which, however, the link to higher gauge algebras also is somewhat obscured) to the supergravity literature. We stress that from the higher gauge algebra point of view, the form (7.48) is special in the sense that all exterior derivatives of gauge potentials in non-linear terms have been absorbed in field strengths. This makes (7.48) particularly useful, as it exposes cleanly the separation of unadjusted curvature and adjustment. From the former, one can straightforwardly identify the higher Lie algebra of the structure group of the underlying higher principal bundle. Moreover, gauge transformations are readily derived from partially flat homotopies, as mentioned above. As a side effect, it is interesting to note that the arising higher products are at most ternary.

An interesting aspect of (7.48) is the fact that the covariantizations of the differentials $dB$ and $dC$ contain a perhaps unexpected factor of $\frac{1}{2}$. This factor is a clear indication that the origin of the gauge $L_{\alpha \beta}$-algebra is indeed an $h\Lie$-algebra: the action $\Rightarrow$ of $A$ on $B$ and $C$ is encoded in an $h\Lie$-algebra with

$$
\varepsilon^0_2(A, B) := A \Rightarrow B \quad \text{and} \quad \varepsilon^0_2(A, C) := A \Rightarrow C ,
$$  \hspace{1cm} (7.51)

which is then antisymmetrized by theorem 4.3 to

$$
\mu_2(A, B) := \frac{1}{2}\varepsilon^0_2(A, B) \quad \text{and} \quad \mu_2(A, C) := \frac{1}{2}\varepsilon^0_2(A, C) ,
$$  \hspace{1cm} (7.52)

at the cost of introducing non-trivial higher products $\mu_3$, cf. (4.6). This is fully analogous to the situation in generalized geometry, cf. e.g. the Dorfman and Courant brackets (5.8) and (5.9).
8. Comparison to the literature

We conclude by comparing our results with algebraic structures previously introduced in the literature to capture the gauge structure underlying the higher gauge theories obtained in the tensor hierarchies of gauged supergravity. We shall focus on the particularities of the gauge algebraic structures of the tensor hierarchies; for other work linking the tensor hierarchy to ordinary $L_{\infty}$-algebras, see also [87].

8.1. Enhanced Leibniz algebras

A notion of enhanced Leibniz algebras was introduced in [88, 89] to capture the parts of the higher gauge algebraic structures appearing in the tensor hierarchy. See also [79] for a discussion of the higher gauge theory employing these enhanced Leibniz algebras and the link to the tensor hierarchy.

**Definition 8.1 ([89]).** An enhanced Leibniz algebra is a Leibniz algebra $pV$, $r,q$ together with a vector space $W$ and a linear map $t : W \to V$ as well as a binary operation $\circ : V \otimes V \to W$ such that

$$
\begin{align*}
[t(w), v] &= 0, & u \hat{\circ} [v, v] &= v \hat{\circ} [u, v] \\
t(w) \circ t(w) &= 0, & [v, v] &= t(v \circ v)
\end{align*}
$$

(8.1)

for all $u, v \in V$ and $w \in W$, where $u \hat{\circ} v$ denotes the symmetric part of $u \circ v$.

A symmetric enhanced Leibniz algebra additionally satisfies the condition that

$$
\begin{align*}
u \circ v &= v \circ u
\end{align*}
$$

(8.2)

for all $u, v \in V$.

A symmetric enhanced Leibniz algebra is an $h\text{Lie}$-algebra concentrated in degrees $-1$ and 0 with a few axioms missing. We can identify the structure maps as follows.

$$
\begin{align*}
\mathfrak{e} &= (\mathfrak{e}_{-1} \xrightarrow{\varepsilon_1} \mathfrak{e}_0) = (W \xrightarrow{t} V), \\
\varepsilon_2(v_1, v_2) &= [v_1, v_2], & \varepsilon_2(v, w) &= 0, & \text{alt}(v_1, v_2) &= v_1 \circ v_2,
\end{align*}
$$

(8.3)

for $v, v_1, v_2 \in V$ and $w \in W$. The $h\text{Lie}$-algebra relations (2.1) are trivially satisfied since $\varepsilon_2$ is a Leibniz bracket. Moreover, $\varepsilon_1$ is trivially a differential and a derivation of $\varepsilon_2$. The relation $\varepsilon_2(v_1, v_2) + \varepsilon_2(v_2, v_1) = \varepsilon_1(\text{alt}(v_1, v_2))$ is the polarization of $[v, v] = t(v \circ v)$. The relation $u \hat{\circ} [v, v] = v \hat{\circ} [u, v]$ fails to accurately reproduce the relation between $\varepsilon_2$ and the alternator, $\text{alt}(v_1, \varepsilon_2(v_2, v_3)) = \text{alt}(\varepsilon_2(v_2, v_3), v_1)$. Moreover, the relation $t(w) \circ t(w) = 0$ fails to reproduce the appropriate relation for the alternator, $\text{alt}(v_1, t(w_1)) = \text{alt}(t(w_1), v_1) = 0$.

The original definition in [89] of a (not necessarily symmetric) “enhanced Leibniz algebra” is slightly more general, allowing for the operation $\circ$ to be not symmetric. However, this is not very natural, as discussed in sections 3.4 and 3.5. Moreover, the algebraic structure underlying the tensor hierarchy is an $h\text{Lie}$-algebra, so enhanced Leibniz algebras require axiomatic completion.
8.2. $\infty$-Enhanced Leibniz algebras

A similar notion of extended Leibniz algebras was formulated in [90], see also [91] as well as the previous work on Leibniz algebra gauge theories [78].

Definition 8.2 ([90]). An $\infty$-enhanced Leibniz algebra is an $\mathbb{N}$-graded differential complex $(X = \oplus_{i \in \mathbb{N}} X_i, d)$ with differential of degree $-1$, endowed with two binary operations

\[
\circ : X_0 \otimes X_0 \to X_0 , \\
\bullet : X_i \otimes X_j \to X_{i+j+1} ,
\]

satisfying the following relations:

\[
\begin{align*}
(x \circ y) \circ z &= x \circ (y \circ z) - y \circ (x \circ z) , \\
\varepsilon^{0} a \bullet b &= (-1)^{|a||b|}(b \bullet a) , \\
(dw) \circ x &= 0 , \\
d(x \bullet y) &= x \circ y + y \circ x , \\
d(u \bullet v) &= -(du) \bullet v + (-1)^{|u|+1}u \bullet dv , \\
(\varepsilon^{0} a \bullet b) \bullet c &= (-1)^{|a|+1}a \bullet (b \bullet c) - (-1)^{|a|+1}|b|b \bullet (a \bullet c) , \\
d(x \bullet (y\bullet z)) &= (x \circ y) \bullet z + (x \circ z) \bullet y - (y \circ z \circ y) \bullet x , \\
[d(x \bullet (y\bullet z))]_{x \leftrightarrow y} &= [(x \circ y) \bullet u - 2x \bullet d(y \bullet u) - x \bullet (y \bullet du)]_{x \leftrightarrow y} ,
\end{align*}
\]

where $x, y, z$ range over degree 0 elements, $w$ ranges over degree 1 elements, $u, v$ range over positive degree elements, and $a, b, c$ over arbitrary elements of homogeneous degrees, and where $[\cdots]_{x \leftrightarrow y}$ signifies that the enclosed expression is antisymmetrized with respect to the permutation between $x$ and $y$.

An $\infty$-enhanced Leibniz algebra is a particular type of $h\text{Lie}$-algebra with some axioms missing. Clearly, to compare the axioms, we have to invert the sign of the degree. We thus consider an $h\text{Lie}$-algebra $E$ concentrated in non-positive degrees with $\varepsilon^{0} = \circ$ non-trivial only on elements of degree 0. Moreover, we are led to identify $\varepsilon^{1} 2$ with with $\bullet$; all other $\varepsilon^{i} 2$ are trivial. Then we have the following relations between the axioms of an $\infty$-enhanced Leibniz algebra and an $h\text{Lie}$-algebra:

(8.4b) is simply the Leibniz identity and follows from the quadratic relation for $\varepsilon^{0}$.  

(8.4c) amounts to $\varepsilon^{1} 2$ being graded symmetric and follows from the modified Leibniz rule, as do (8.4d)–(8.4f).  

(8.4g) follows from the $h\text{Lie}$-axiom for $\varepsilon^{1} 2 \circ \varepsilon^{1} 2$.  

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(8.4h) follows from the modified Leibniz rule together with the ℋLie-axioms for $\varepsilon^1_2 \circ \varepsilon^0_2$ and $\varepsilon^0_2 \circ \varepsilon^1_2$:
\[
\varepsilon_1(\varepsilon^1_2(x, \varepsilon^1_2(y, z))) = \varepsilon^0_2(x, \varepsilon^1_2(y, z)) + \varepsilon^0_2(\varepsilon^1_2(y, z), x) - \varepsilon^1_2(x, \varepsilon^1_2(y, z) + \varepsilon^0_2(z, y)) \\
= \varepsilon^1_2(\varepsilon^0_2(x, y), z) + \varepsilon^1_2(y, \varepsilon^0_2(x, z)) - \varepsilon^1_2(x, \varepsilon^0_2(y, z) + \varepsilon^0_2(z, y)),
\]
(8.5)
as does (8.4i): we have:
\[
\varepsilon^0_2(\varepsilon^1_2(-, -, -)) = 0,
\]
\[
\varepsilon_1(\varepsilon^1_2(x, \varepsilon^1_2(y, u))) = \varepsilon^1_2(x, \varepsilon^1_2(y, \varepsilon_1(u))) + \varepsilon^1_2(\varepsilon^1_2(x, y), u) + \varepsilon^1_2(y, \varepsilon^1_2(x, u)) \\
- \varepsilon^1_2(x, \varepsilon^0_2(y, u)) - \varepsilon^1_2(x, \varepsilon^0_2(u, y)) ,
\]
\[
\varepsilon^1_2(x, \varepsilon^1_2(y, u))) = -\varepsilon^1_2(x, \varepsilon^1_2(y, \varepsilon_1(u))) + \varepsilon^1_2(x, \varepsilon^0_2(y, u)) + \varepsilon^1_2(u, \varepsilon^1_2(y, u)) ,
\]
(8.6)
and putting this together, we obtain
\[
[d(\varepsilon^1_2(x, \varepsilon^1_2(y, z))) + 2\varepsilon^1_2(x, d(\varepsilon^1_2(y, u)))]_{x \rightarrow y} \\
= [-\varepsilon^1_2(x, \varepsilon^1_2(y, du)) + \varepsilon^1_2(\varepsilon^1_2(x, y), u) + \varepsilon^1_2(x, \varepsilon^0_2(y, u))]_{x \rightarrow y} .
\]
(8.7)
Note, however, that while the ℋLie-algebra axioms imply the axioms of an $\infty$-enhanced
Leibniz algebra, the reverse statement is not true, even for $\infty$-enhanced Leibniz algebras
concentrated in degrees 0 and 1. The latter essentially implies that $\infty$-enhanced Leibniz
algebras are an incomplete abstraction of the operad $Lie$ and thus do not give the full
picture. Altogether, we arrive at the same conclusion as for enhanced Leibniz algebras.

As a side remark, we note that in the outlook of [90], the authors mentioned the desire
for the interpretation of $\infty$-enhanced Leibniz algebras as the homotopy algebras of some
simpler algebraic structure. Our discussion suggests that this is not possible; instead, the
axiomatic completion of $\infty$-enhanced Leibniz algebras yields ℋLie-algebras whose homotopy
algebras form $EL_\infty$-algebras, a much weaker version of $L_\infty$-algebras.

8.3. Algebras producing the tensor hierarchies

We now come to larger picture of algebras that lead to the gauge structures visible in
the tensor hierarchies, see figure 1. Note that this picture has only been applied in the
context of the tensor hierarchy for maximal supersymmetry. We shall be less detailed in
the following.

In [82], Palmkvist constructs an infinite-dimensional $\mathbb{Z}/2$-graded Lie algebra, which he
calls the tensor hierarchy algebra, “gLie (THA)” in figure 1. For further work on the tensor
hierarchy algebra, see also [93, 94, 95]. As observed in [86], see also [82], this $\mathbb{Z}/2$-grading
can be naturally refined into a $\mathbb{Z}$-grading, and picking an element of degree 1 and subse-
quint restriction induces the structure of a differential graded Lie algebra, “dgLie (THA)”
in figure 1. In [77], Lavau called this differential graded Lie algebra the “tensor hierarchy
algebra” (not to be confused with Palmkvist’s larger graded Lie algebra), and derived it from a further algebraic structure called *Lie–Leibniz triples*, “LieLeibTriple” in figure 1. This differential graded Lie algebra then naturally gives rise to $\infty$-enhanced Leibniz algebras, as described in [92, Section 3]. As explained above, the $\infty$-enhanced Leibniz algebra were an incomplete “guess” of the axioms of an $h\text{Lie}$-algebra with $\varepsilon_i^2 = 0$ for $i \geq 2$. Thus, from our perspective, $\infty$-enhanced Leibniz algebras are appropriately replaced by these, and we then have the construction of the gauge $L_\infty$-algebra via the picture (4.10), which is refined in figure 1. We note that the composition of the arrows “complete axioms” and “antisym.”, which produces an $L_\infty$-algebra from an $\infty$-enhanced Leibniz algebra, is found in [90, Appendix B]. As indicated in figure 1, the direct construction of an $L_\infty$-algebra from a differential graded Lie algebra is the Fiorenza–Manetti–Getzler construction Proposition 4.4, as pointed out in [92], where Getzler’s formulas were specialized to the tensor hierarchy differential graded Lie algebra.

For prior relations amongst tensor hierarchies, the embedding tensor formalism and (homotopy) algebras see also [78, 79]. We again stress that from our point of view, it is not natural to consider gauge theories with infinitesimal symmetries that are not (weaker forms of) Lie algebras. Axiomatically completing the various forms of Leibniz algebras to $h\text{Lie}$-algebras solves this issue.

As a side remark, let us note that the fact that Leibniz algebras naturally produce $L_\infty$-algebras has been pointed out in [96]. This is proposition 4.1 stating that any Leibniz algebra naturally extends to an $h\text{Lie}$-algebra combined with theorem 4.3 antisymmetrizing this $h\text{Lie}$-algebra to an $L_\infty$-algebra.
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Appendix

A. The homotopy $H_0$

Here, we concisely list the explicit form of the homotopy $H_0$ appearing in (2.17a). These are used in the computation of the antisymmetrization of an $EL_x$-algebra. To lowest order in $V$, we have

\[
H_0(v_1) = 0 ,
\]

\[
H_0(v_1 \otimes_i v_2) = \begin{cases} 0 & \text{for } i = 0 \\ \frac{(-1)^{i-1}}{4} v_1 \otimes_{i-1} v_2 - \frac{(-1)^{|v_1||v_2|}}{4} v_2 \otimes_{i-1} v_1 & \text{else} \end{cases} \tag{A.1a}
\]

for all $v_1, v_2 \in V$. To cubic order in $V$, we obtain the general results

\[
H_0((v_1 \otimes_i v_2) \otimes_i v_3) = \frac{1}{18} \left( - (v_1 \otimes_{i-1} v_2) \otimes_i v_3 + (-1)^{i+|v_2||v_3|} (v_1 \otimes_{i-1} v_3) \otimes_i v_2 \\
- (-1)^{|v_1|+|v_2||v_3|} (v_3 \otimes_{i-1} v_1) \otimes_i v_2 - (-1)^{|v_1|(|v_2|+|v_3|)} (v_2 \otimes_{i-1} v_3) \otimes_i v_1 \\
+ (-1)^{i+|v_1||v_2|} (v_2 \otimes_{i-1} v_1) \otimes_i v_3 - (-1)^{i+|v_1|(|v_2|+|v_3|)+|v_2||v_3|} (v_2 \otimes_{i-1} v_3) \otimes_i v_1 \right) , \tag{A.1b}
\]

\[
H_0((v_1 \otimes_i v_2) \otimes_i v_3) = \left( \frac{1}{4} + \frac{3}{2} \alpha_1 \right) (v_1 \otimes_0 v_2) \otimes_0 v_3 + (-1)^{|v_2||v_3|} \left( \frac{1}{12} + \frac{3}{4} \alpha_1 \right) (v_1 \otimes_0 v_3) \otimes_0 v_2 \\
+ (-1)^{|v_1||v_2|} \left( - \frac{1}{4} + \frac{3}{2} \alpha_1 \right) (v_2 \otimes_0 v_1) \otimes_0 v_3 \\
+ (-1)^{|v_1|(|v_2|+|v_3|)} \left( - \frac{1}{12} + \frac{3}{4} \alpha_1 \right) (v_2 \otimes_0 v_3) \otimes_0 v_1 \\
+ (-1)^{|v_1|(|v_2|+|v_3|)+|v_2||v_3|} \left( - \frac{1}{12} - \frac{4}{3} \alpha_1 \right) (v_3 \otimes_0 v_2) \otimes_0 v_1 \right) \tag{A.1c}
\]

\[
H_0((v_1 \otimes_i v_2) \otimes_i v_3) = \left( \frac{1}{6} - \frac{3}{2} \alpha_2 \right) (v_1 \otimes_0 v_2) \otimes_0 v_3 + (-1)^{|v_2||v_3|} \left( - \frac{1}{6} - \frac{3}{2} \alpha_2 \right) (v_1 \otimes_0 v_3) \otimes_0 v_2 \\
+ (-1)^{|v_1||v_2|} \left( \frac{1}{6} + \frac{3}{2} \alpha_2 \right) (v_2 \otimes_0 v_1) \otimes_0 v_3 + \frac{4}{3} (-1)^{|v_1|(|v_2|+|v_3|)} \alpha_2 (v_2 \otimes_0 v_3) \otimes_0 v_1 \\
+ (-1)^{|v_1|(|v_2|+|v_3|)+|v_2||v_3|} \alpha_2 (v_3 \otimes_0 v_1) \otimes_0 v_2 \\
+ \frac{4}{3} (-1)^{|v_1|(|v_2|+|v_3|)+|v_2||v_3|} \alpha_2 (v_3 \otimes_0 v_2) \otimes_0 v_1 \right) , \tag{A.1d}
\]
\[
H_0((v_1 \odot_0 v_2) \odot_2 v_3) \\
= -\frac{1}{4} (v_1 \odot_1 v_2) \odot_0 v_3 + (-1)^{|v_2||v_3|} \alpha_1(v_1 \odot_1 v_3) \odot_0 v_2 \\
- \frac{1}{4} (-1)^{|v_1||v_2|} (v_2 \odot_1 v_1) \odot_0 v_3 + (-1)^{|v_1||v_2|+|v_3|} \alpha_1(v_2 \odot_1 v_3) \odot_0 v_1 \\
- (-1)^{|v_3||v_3|} \alpha_3(v_3 \odot_1 v_1) \odot_0 v_2 \\
- (-1)^{|v_1||(v_2|+|v_3)|+|v_2||v_3|} \alpha_1(v_3 \odot_1 v_2) \odot_0 v_1,
\]
\[
H_0((v_1 \odot_2 v_2) \odot_0 v_3) \\
= -\frac{1}{4} (v_1 \odot_0 v_2) \odot_1 v_3 - \frac{1}{4} (-1)^{|v_2||v_3|} (v_1 \odot_0 v_3) \odot_1 v_2 - (-1)^{|v_1||v_2|} \alpha_1(v_2 \odot_0 v_3) \odot_1 v_1 \\
+ (-1)^{|v_1||(v_2|+|v_3)|} \alpha_1(v_2 \odot_0 v_3) \odot_1 v_1 - (-1)^{|v_1||v_2|+|v_3|} \alpha_1(v_2 \odot_0 v_1) \odot_1 v_1 \\
+ (-1)^{|v_1||(v_2|+|v_3)|+|v_2||v_3|} \alpha_1(v_2 \odot_0 v_2) \odot_1 v_1
\]

\text{(A.1e)}

\text{(A.1f)}

for \( v_1, v_2, v_3 \in V \) and generic constants \( \alpha_1, \alpha_2 \) in the ground field.

**B. Example of an \( EL_\infty \)-algebra**

Let us give the explicit form of an \( EL_\infty \)-algebra \( \mathfrak{C} \), in which the products \( \varepsilon_1, \varepsilon_2^0, \varepsilon_2^1, \) and \( \varepsilon_3^{00} \) are generic while all other products are trivial. We do not impose any conditions on the underlying differential complex. The compatibility relations are readily computed in the Chevalley–Eilenberg picture, and they read as follows:

\[
\varepsilon_1(\varepsilon_1(1)) = 0, \\
\varepsilon_1(\varepsilon_2^0(1, 2)) = \varepsilon_2^0(\varepsilon_1(1), e_2) + (-1)^{|\varepsilon_1||\varepsilon_2^0|} \varepsilon_2^0(\varepsilon_1, \varepsilon_2) \\
\varepsilon_1(\varepsilon_2^1(1, 2)) = \varepsilon_2^1(\varepsilon_1, e_2) + (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_2^1(\varepsilon_2, e_1) - \varepsilon_2^1(\varepsilon_1, e_2) \\
- (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_2^1(\varepsilon_1, \varepsilon_2) \\
\varepsilon_2^1(1, e_2) = (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_2^1(e_2, e_1)
\]
\[
\varepsilon_1(\varepsilon_3^{00}(1, e_2, e_3)) = \varepsilon_3^{00}(\varepsilon_2^0(1, e_2), e_3) + (-1)^{|\varepsilon_1||\varepsilon_2^0|} \varepsilon_3^{00}(\varepsilon_2^0, \varepsilon_3) - \varepsilon_3^{00}(\varepsilon_1, \varepsilon_2^0, e_3) \\
- \varepsilon_3^{00}(\varepsilon_1(1), e_2, e_3) - (-1)^{|\varepsilon_1||\varepsilon_3^{00}|} \varepsilon_3^{00}(\varepsilon_1, \varepsilon_2, e_3) \\
- (-1)^{|\varepsilon_1||\varepsilon_2^0||\varepsilon_3^{00}|} \varepsilon_3^{00}(\varepsilon_1, e_2, \varepsilon_3) \\
\varepsilon_2^0(\varepsilon_2^1(1, e_2), e_3) = \varepsilon_3^{00}(\varepsilon_1, e_2, e_3) + (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_3^{00}(\varepsilon_1, e_2, e_3) \\
\varepsilon_2^0(1, \varepsilon_2^1(e_2, e_3)) = (-1)^{|\varepsilon_1||\varepsilon_2^0|} \varepsilon_3^{00}(\varepsilon_1, e_2, e_3) - (-1)^{|\varepsilon_2^0||\varepsilon_3^{00}|} \varepsilon_3^{00}(\varepsilon_1, e_2, e_3) \\
+ (-1)^{|\varepsilon_1||\varepsilon_2^0||\varepsilon_3^{00}|} \varepsilon_3^{00}(\varepsilon_1, e_2, e_3) \\
\varepsilon_2^1(1, \varepsilon_2^1(e_2, e_3)) = (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_2^1(\varepsilon_2^1(1, e_2), e_3) + (-1)^{|\varepsilon_1||\varepsilon_2^1|} \varepsilon_2^1(\varepsilon_2^1, e_2, e_3)
\]
\text{(B.1a)}

\text{(B.1b)}
\[
\begin{align*}
\varepsilon^0_2(e_1, \varepsilon^0_3(e_2, e_3, e_4)) + (-1)^{[\varepsilon_1+\varepsilon_2]} \varepsilon^0_3(e_1, \varepsilon^0_2(e_2, e_3), e_4) + (-1)^{[\varepsilon_1+\varepsilon_2]} \varepsilon^0_3(e_1, e_3, \varepsilon^0_2(e_2, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_1, e_2, \varepsilon^0_3(e_2, e_3), e_4) + (-1)^{[\varepsilon_1+\varepsilon_2]} \varepsilon^0_3(e_1, e_2, e_3, e_4)) \\
= (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_1, e_2, e_3, e_4)) \\
- (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_2(e_1, e_2, e_3, e_4)) + (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_2(e_1, e_3, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_2(e_1, e_3, e_4)) ,
\end{align*}
\]

(B.1c)

\[
\begin{align*}
\varepsilon^0_3(e^1_2(e_1, e_2), e_3, e_4)) = 0 , \\
\varepsilon^0_3(e_1, \varepsilon^1_2(e_2, e_3), e_4)) = 0 , \\
\varepsilon^1_2(e^0_3(e_1, e_2, e_3), e_4)) = -(-1)^{[\varepsilon_1+\varepsilon_2]} \varepsilon^0_3(e_1, e_2, \varepsilon^0_1(e_3, e_4)) \\
- (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_1, e_2, \varepsilon^0_3(e_1, e_2, e_4))
\end{align*}
\]

(B.1d) and

\[
0 = \varepsilon^0_3(e^0_3(e_1, e_2, e_3), e_4, e_5)) + (-1)^{[\varepsilon_1+\varepsilon_2]} \varepsilon^0_3(e_1, e_2, \varepsilon^0_3(e_3, e_4, e_5)) \\
- (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_1, e_2, \varepsilon^0_3(e_3, e_4, e_5)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_2(e_1, e_2, e_3, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_3(e_1, e_2, e_3, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_3(e_1, e_2, e_3, e_4)) \\
- (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_3(e_1, e_2, e_3, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_3(e_1, e_2, e_3, e_4)) \\
+ (-1)^{[\varepsilon_1]} \varepsilon^0_3(e_3(e_1, e_2, e_3, e_4))
\]

(B.1e)

for all \(e_i \in \mathcal{E}\).

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