Solutions to the Yang-Baxter equations with $osp_q(1|2)$ symmetry: Lax operators

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Abstract.

We find a new $4 \times 4$ solution to the $osp_q(1|2)$-invariant Yang-Baxter equation with simple dependence on the spectral parameter and propose $2 \times 2$ matrix expressions for the corresponding Lax operator. The general inhomogeneous universal spectral-parameter dependent $R$-matrix is derived. It is proven, that there are two independent solutions to the homogeneous $osp_q(1|2)$-invariant YBE, defined on the fundamental three dimensional representations. One of them is the particular case of the universal matrix, while the second one does not admit generalization to the higher dimensional cases. Also the $3 \times 3$ matrix expression of the Lax operator is found, which have a well defined limit at $q \to 1$.

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1 Introduction

The simplest orthosymplectic superalgebra $osp(1|2)$ plays the same role in the classification of Lie superalgebras [1] as the algebra $sl(2)$ plays in the classification of Lie algebras. Integrable models with appropriate non-deformed symmetry, based on the solution to the Yang-Baxter equation (YBE) [2], as well as the quantum deformation of the superalgebra, $osp_q(1|2)$, were considered in the articles [3, 4, 5, 6, 5, 7, 8]. In [4] it is presented a fundamental $osp_q(1|2)$-invariant $R(u)$-matrix, found from the classical matrix [3] by replacing the rational factors by trigonometric ones. H. Saleur in 1990 published solution to the spectral parameter dependent YBE for the universal homogeneous $R$ matrix with the quantum deformation of the $osp(1|2)$ symmetry [8]. Since that time different authors turned repeatedly to the study of the $osp_q(1|2)$ superalgebra and integrable models with $osp_q(1|2)$ symmetry (see for example [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]), but some questions are unclear so far. In particular how many different fundamental $R$ matrices with $osp_q(1|2)$ exist? How to build the Lax operator of simple form etc.

In this paper we are constructing the simple form of the Lax operator and are analyzing the all $osp_q(1|2)$ invariant solutions to the spectral parameter dependent YBE. For completeness sake we want to mention the work [19], where the authors connect the quest of the Lax operator of $osp_q(1|2n)$ with the isomorphism existing between the representations of $osp_q(1|2n)$ and $so_q(2n + 1)$ [10]. Our investigation in principle differs from it, as we consider and manipulate directly the $osp_q(1|2)$ symmetric matrices. Though, of course, the resemblance with the $sl_q(2)$ symmetric case can be achieved along the discussion, and we shall turn to this equivalence in the paper (subsection 3.2).

In the next part of the Introduction we give a brief review of the $osp_q(1|2)$ algebra and its finite dimensional irreducible representations (irreps) [5, 7, 8]. In order to determine
the Lax operator, we shall consider YBE defined also for the even dimensional irreps, which have no classical counterparts \([9, 21]\). The irrep with minimal dimension (higher than one) is the two dimensional irrep, and in the second section it is given a simple Lax operator as a \(2 \times 2\) matrix, defined on the two dimensional auxiliary space. Using the YB equations with Lax operators, it is possible to derive determining equations ("Jimbo’s relations") for the universal \(R\)-matrix. It is done in the third section. The results regard general inhomogeneous spectral parameter dependent \(R\)-matrices defined for both the even and odd dimensional irreducible representations. Note, that conventional Lax operator, with three dimensional auxiliary space can be constructed by a descendant procedure (by fusion), from the \(2 \times 2\) matrix operator. We calculated the corresponding \(3 \times 3\) matrix form of the quantum Lax operator (see fifth section), and demonstrated the existence of the appropriate classical limit \((q \rightarrow 1)\).

Another question, which is analyzed in this work, concerns to the existence of two fundamental solutions to the YBE (fourth section). One of them follows from the universal \(R(u)\)-matrix \([8]\) as a particular case. The second solution, existing in the literature \([3, 4, 6]\), seems to be an absolutely separated solution, which does not give rise to descendant solutions and generalization to a universal \(R(u)\) matrix. Here we examined all the possible solutions to the spectral parameter dependent YBE,

\[
R_{ij}(u - v)R_{ik}(u)R_{jk}(v) = R_{jk}(v)R_{ik}(u)R_{ij}(u - v),
\]

with \(osp_q(1|2)\) symmetry. We show, that only for the ordinary fundamental (three-dimensional) representations there are two different solutions for YBE, while when the dimension of one of the (irreducible) representations differs from three, there is one unique solution \(R_{ij}(u)\).

**Quantum \(osp_q(1|2)\) superalgebra and its representations.** The quantum \(osp_q(1|2)\) superalgebra is defined by the simple odd elements \(e, f\) and the even element \(h\), obeying the relations

\[
\{e, f\} = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad [h, e] = e, \quad [h, f] = -f, \quad (1.1)
\]

which at the value \(q = 1\) degenerate to the relations of non-deformed superalgebra \(osp(1|2)\). The even element \(h\) constitutes the Cartan subalgebra.

The definitions of the co-product of the quantum super-algebra are given by the following relations

\[
\Delta(e) = e \otimes q^{-h/2} + q^{h/2} \otimes e, \\
\Delta(f) = f \otimes q^{-h/2} + q^{h/2} \otimes f, \\
\Delta(h) = h \otimes 1 + 1 \otimes h, \quad (1.2)
\]

where \(\otimes\) denotes the graded co-multiplication.

The Casimir operator can be expressed by \(e, f, h\) elements as

\[
c = -(q + 2 + q^{-1})f^2e^2 + (q^{h-1} + q^{1-h})fe + \left[h - \frac{1}{2}\right]_q^2 = \left((q^{\frac{h}{2}} + q^{-\frac{h}{2}})fe - \left[h - \frac{1}{2}\right]_q\right)^2. \quad (1.3)
\]
As usual $[x]_q$ or $[x]$ denotes the expression $\frac{q^x-q^{-x}}{q-q^{-1}}$.

The conventional finite dimensional irreducible representations of this quantum superalgebra, as well as for the non-deformed case, are odd dimensional spin-$j$ irreps $V_{4j+1}$, with integer or half-integer $j$, $\dim[\text{spin-}j] = 4j + 1$;

$$V_{4j+1} = \text{span}\{[j,j] | |j,j, j=1/2\}, \ldots, [j,j] \}, \quad h[j,i] = 2i|j,i|.$$ (1.4)

The tensor product of two irreps splits into the direct sum of irreps

$$V_r \otimes V_k = \bigoplus_{p=|r-k|+1}^{r+k-1} V_p.$$ (1.5)

The quantum superalgebra $osp_q(1|2)$ together with the conventional odd-dimensional spin representations possesses even-dimensional representations as well [9], which have no classical (non-deformed) counterparts, being ill-defined when $q \to 1$. The $r = 2n$, $n \in \mathbb{Z}_+$, dimensional representations form a sequence, labelled by positive even integer $r$ or by ”spin”-$j_r$:

$$2j_r = \frac{r-2}{2} + \lambda = 2j - \frac{1}{2} + \lambda, \quad q^\lambda = iq^{\frac{1}{2}}, \quad \lambda = \frac{i\pi}{2 \log q} + \frac{1}{2}.$$ (1.6)

In particular the ”spin” of two-dimensional ($r = 2$) representation is equal to $\lambda/2$. By means of the quarter-integer numbers $j = j_r - \lambda/2 + \frac{1}{4}$, the $4j + 1$ dimensional representations can be written in the same form as the conventional representations: $V_{4j+1} = \text{span}\{[j,j] | j, j-1/2\}, \ldots, [j,j] \}, h[j,i] = 2(i + \lambda/2 - \frac{1}{4})|j,i|.$

The decomposition of the tensor products, which include even dimensional representations too, remains the same, as in (1.5). Note only, that due to the automorphism of the algebra $e \to -e$, $f \to f$, $q^h \to -q^h$, the definition of the spin-$j$ representations admits an ambiguity: adding the number $2n(\lambda-1/2)$, with arbitrary integer $n$, to the weight $2j$ doesn’t change the irrep [9]. It leads to the multiplication of the eigenvalues of the operator $q^h$ by the extra sign $(-1)^n$. It can be seen in the tensor product (1.5) of the two even dimensional irreps.

The eigenvalues of the Casimir operator $c$ on the odd and even dimensional irreps $V_p$ are (see [9])

$$c_p = (-1)^p q^{\frac{p}{2}} + 2(-1)^p q^{-\frac{p}{2}} + q^{-p} = \begin{cases} \left[\frac{p}{2}\right]^2, & \text{if } p \text{ is odd}, \\ \left[\frac{p}{2}\right]^2 + \frac{\lambda}{4}, & \text{if } p \text{ is even}. \end{cases}$$ (1.7)

In the space of $V_r \otimes V_k$ (1.5) the Casimir operator is decomposed in terms of the projection operators $P_p$ ($P_p$ acts as unity operator in the space of the irrep $V_p$ and is 0 elsewhere),

$$c = \sum_{p=|r-k|+1}^{r+k-1} c_p P_p, \quad P_p \cdot V_{p'} = \delta_{pp'} V_{p'} .$$ (1.8)

**Intertwiner $R$-matrices and YBE.** Together with the co-product $\Delta$, the operation $\bar{\Delta} = P \Delta P$ is defined for the quantum (super-)algebras, which differs by permutation of the spaces: $P \cdot V \otimes V' = V' \otimes V$. $\Delta$ and $\bar{\Delta}$ are related by an intertwiner matrix $R$

$$R \Delta = \bar{\Delta} R .$$ (1.9)
which satisfies the so called ”constant” Yang-Baxter equations (YBE) \[24]\n
\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \] (1.10)

Here the expressions in the right and left sides of the equation act on the tensor product of three vector spaces \( V_{r_1} \otimes V_{r_2} \otimes V_{r_3} \). The \( R_{ij} \) correspondingly acts on the \( V_{r_i} \otimes V_{r_j} \). By a convention, used in the theory of integrable models, the states \( V_{r_1} \) and \( V_{r_2} \) are regarded as ”auxiliary” states, while the \( V_{r_3} \) state is ”quantum” one.

The YBE with spectral parameters

\[ R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u). \] (1.11)

play a crucial role in the theories of the integrable models \[2, 24\]. If the underlying model has the symmetry of a quantum (super-)algebra, then the \( R(u) \) matrices, with spectral parameter \( u \), satisfy to the relations (1.9). And also the following relations can be considered,

\[ R(u)\Delta_u = \tilde{\Delta}_u R(u), \] (1.12)

where the operation \( \Delta_u \) can be treated as a spectral parameter dependent co-product of the corresponding quantum (super-)algebra. It has not all the properties of the \( \Delta \), it is not co-associative in particular. The structure of the \( \Delta_u \) can be checked as in \[23\] (see the third section).

In the so-called ”check”-formalism, with check \( R \)-matrix \( \tilde{R}(u) = PR(u) \), \( P \) being permutation operator, the following relations are true

\[ \tilde{R}(u)\Delta = \Delta \tilde{R}(u). \] (1.13)

This just means that \( R \)-matrix in ”check”-formalism is commutative with the algebra generators. Hence \( \tilde{R}(u)\Delta(c) = \Delta(c)\tilde{R}(u) \), where \( c \) is Casimir operator, and \( R \)-matrix is diagonalizable simultaneously with Casimir operator and in the space of \( V_k \otimes V_r \) can be represented as a linear combination of the projection operators introduced in the formula (1.8) \[7\]:

\[ \tilde{R}(u) = \sum_p r_p(u)P_p. \] (1.14)

The ”check” \( \tilde{R}(u) \)-matrix satisfies to the ”check” YBE

\[ \tilde{R}_{12}(u)\tilde{R}_{23}(u + v)\tilde{R}_{12}(v) = \tilde{R}_{23}(v)\tilde{R}_{12}(u + v)\tilde{R}_{12}(u). \] (1.15)

As follows from (1.13) and (1.15), the definition of \( R(u) \)-matrix has an ambiguity and \( \tilde{R}(u) \) can be multiplied by an arbitrary function.

Note, that for the super-algebras, when writing the tensor products, co-products and YBE equations in the matrix notations, one must take into account the grading of the states and generators. As example, the matrix representation of the tensor product of two operators \( a \) and \( b \) reads as

\[ (a \otimes b)^{kr}_{ij} = a_i^k b_j^r (-1)^{p_i(p_j + p_r)}. \] (1.16)

Here \( p_i \) is the parity of the state labelled by \( i \) and equals to 0 for the even states and equals to 1 for the odd states.
2 YBE solution of minimal dimension

2.1 The $4 \times 4$ matrix solutions of $RRR = RRR$ YBE

Consider now a homogeneous $\hat{R}(u)$-matrix on the tensor product of two two-dimensional representations of the quantum super-algebra $osp_q(1|2)$. This tensor product is decomposed to the direct sum of the spaces corresponding to the conventional one and three dimensional irreps, which means that the Casimir operator expands into projection operators:

$$c = \left[\frac{3}{2}\right]_q^2 P_3 + \left[\frac{1}{2}\right]_q^2 P_1,$$

where the arbitrary constant $a$ (accurate within multiplication by a function)

$$P_1 = \frac{1}{q^{\frac{1}{4}} - q^{-\frac{1}{4}}} \begin{pmatrix} -q^{-\frac{1}{4}} & i \\ i & q^{\frac{1}{4}} \end{pmatrix}, \quad P_3 = \frac{1}{q^{\frac{1}{4}} - q^{-\frac{1}{4}}} \begin{pmatrix} q^{\frac{1}{4}} & -i \\ -i & -q^{-\frac{1}{4}} \end{pmatrix}.$$

In accordance to (1.14), the same takes place for $\hat{R}(u)$ too:

$$\hat{R}(u) = a(u) P_1 + b(u) P_3,$$

where $a(u)$ and $b(u)$ are some indeterminate functions.

Then we substitute the expression (2.2) into the Yang-Baxter relations (1.15), using that the graded YBE has the same matrix representation as in the non-graded case, i.e.,

$$\sum_{b_1 b_2 e_2} \hat{R}_{a_1 a_2}^{b_1 b_2} (u - v) \hat{R}_{b_2 a_3}^{e_2 e_3} (u) \hat{R}_{e_1 e_2}^{c_1 c_2} (v) = \sum_{b_2 e_2 b_3} \hat{R}_{a_2 a_3}^{b_2 b_3} (v) \hat{R}_{e_1 e_2}^{c_1 c_2} (u) \hat{R}_{e_2 e_3}^{c_2 c_3} (u - v).$$

The equations (2.3) with matrix (2.2) lead to a constraint, fixing the ratio $a(u)/b(u)$:

$$\frac{q^{\frac{1}{4}} b - q^{-\frac{1}{4}} a}{q^{\frac{1}{4}} a - q^{-\frac{1}{4}} b} = q^{ku},$$

where the arbitrary constant $k$ can be eliminated by rescaling of $u$. One can finally set (accurate within multiplication by a function)

$$a(u) = \frac{q^{u\frac{1}{2}} + q^{-u\frac{1}{2}}}{q^{\frac{1}{4}} + q^{-\frac{1}{4}}} = [u] - [u - 1], \quad b(u) = \frac{q^{u+\frac{1}{2}} + q^{-u-\frac{1}{2}}}{q^{\frac{1}{4}} + q^{-\frac{1}{4}}} = [u + 1] - [u].$$

It is easy to check then that all the equations in (2.3) turn to be identities upon substitution (2.4) and one comes to the following matrix elements of the $R$-matrix:

$$R_{11}^{11} = -R_{22}^{22} = \hat{R}_{11}^{11} = \hat{R}_{22}^{22} = [u + 1] - [u], \quad R_{12}^{12} = \hat{R}_{12}^{12} = q^u,$$

$$R_{12}^{12} = R_{21}^{21} = \hat{R}_{21}^{21} = \hat{R}_{12}^{12} = -i(q^{\frac{1}{4}} - q^{-\frac{1}{4}})[u], \quad R_{21}^{12} = \hat{R}_{21}^{21} = q^{-u}.$$

Finally we can state, that the $R(u)$-matrix (2.5) satisfies to graded YBE:

$$(-1)^{p_{a_2} + p_{a_3}} R_{a_1 a_2}^{b_1 b_2} (u) R_{b_1 a_3}^{c_1 c_2} (u + v) R_{a_2 a_3}^{b_2 b_3} (v) = (-1)^{p_{a_2} + p_{a_3}} R_{a_2 a_3}^{b_2 b_3} (v) R_{a_1 a_3}^{b_1 b_2} (u + v) R_{b_1 a_3}^{c_1 c_2} (u).$$

The states with indices 1 and 2 have 0 and 1 parities correspondingly: $p_i = i - 1$. 
2.2 \( RLL = LLR \) YBE: Lax operator

Here we want to consider the YBE with Lax operator \( L(u) \):

\[
R^{(22)}(u - v)L^{(2r)}(u)L^{(2r)}(v) = L^{(2r)}(v)L^{(2r)}(u)R^{(22)}(u - v). \tag{2.6}
\]

The upper indexes show the dimensions of the auxiliary (dim = 2) and quantum (dim = \( r \), arbitrary integer) states on which \( R \)-matrix and \( L \)-operator act. The matrix elements of \( R^{(22)}(u) \) are defined in (2.5).

The experience with solutions to Yang-Baxter equations with \( gl(n) \)-symmetry (or more generally with symmetry with respect to the superalgebra \( gl(m|n) \)) or their quantum deformations suggests us that solution corresponding to the representation with the smallest dimension in auxiliary space has the simplest (linear) dependence on spectral parameter \( u \) (\( [u]_q \) in quantum case). The power of \( u \) grows alone with the growth of dimension of the representation in the auxiliary space. Supposing that this regularity can be continued to the more complicated algebras, one can try the following 2 \( \times \) 2 matrix as a solution to the graded YBE (2.6):

\[
L(u) = \left( \begin{array}{c} [u - \lambda h + 1] - [u - \lambda h] \\ -i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{\frac{1}{2}}q^{-u}f \end{array} \right) \begin{array}{c} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{\frac{1}{2}}q^{-u}f \\ [u - (\lambda - 1)h] - [u - (\lambda - 1)h + 1] \end{array}, \tag{2.7}
\]

where \( e \), \( f \) and \( h \) are the elements of \( osp_q(1|2) \) acting on the \( r \)-dimensional space. The complex number \( \lambda \) is defined by the relation \( [\lambda - 1] = [\lambda] \), or \( q^{\lambda} = iq^{\frac{1}{2}} \).

In terms of matrix elements (2.6) takes the form:

\[
(-1)^{p_1(p_1 + p_2)}R_{a_1a_2}^{b_1b_2}(u - v)L_{b_1}^{a_1}(u)L_{b_2}^{a_2}(v) = (-1)^{p_1(p_1 + p_2)}L_{a_2}^{b_2}(v)L_{a_1}^{b_1}(u)R_{b_1b_2}^{a_1a_2}(u - v). \tag{2.8}
\]

The inspection shows that the equations (2.8) are satisfied based on the algebra relations (1.1) and on two identities:

\[
([u-v+1] - [u-v])([u+x+1] - [u+x]) = [v+x+1] - [v+x] + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2[u-v][u+x+1], \tag{2.9}
\]

and

\[
([u-v+1] - [u-v])([v+x+1] - [v+x]) = [u+x+1] - [u+x] - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2[u-v][v+x]. \tag{2.10}
\]

3 General Lax operator and Jimbo’s relations; connection with the \( sl_q(2) \)

3.1 General Lax operator and Jimbo’s relations; \( \tilde{R}^{(r_1r_2)}(u) \)

In this subsection we want to observe the YBE

\[
R^{(r_1r_2)}(u - v)L^{(2r_2)}(u)L^{(2r_1)}(v) = L^{(2r_1)}(v)L^{(2r_2)}(u)R^{(r_1r_2)}(u - v), \tag{3.1}
\]

with \( L^{(2r)}(u) \) and a general \( R^{(r_1r_2)}(u) \), defined on the \( r_1 \) and \( r_2 \) dimensional spaces.
In terms of the check \( \check{R} \)-matrix elements the equation (3.1) takes the form
\[
(-1)^{p_1(p_2+p_3)} R^a_{b_1b_2}(u-v)(L_\alpha^\beta)^{\gamma_1}_{b_1}(u)(L_\delta^\epsilon)^{\gamma_2}_{b_2}(v) = \\
= (-1)^{p_1(p_2+p_3)}(L_\alpha^\beta)^{b_1}_{b_2}(v)(L_\delta^\epsilon)^{b_2}_{a_2}(u) \check{R}^c_{b_1b_2}(u-v). 
\]

Here it is used the condition \((-1)^{p_1(p_2+p_3)} = 1\) for the \( R^a_{b_1b_2} \). The Greek indexes relate to the 2-dimensional quantum space, while the Latin ones to the \( r_1, r_2 \) dimensional auxiliary spaces. Recalling the definition (1.16), we can trace the graded tensor products in the l.h.s and r.h.s of the equation (3.2).

**General form of the Lax operator.** At first let us present the Lax operator \( L^{(2)}(\tau) \) in rather general form
\[
L(u) = \left( \begin{array}{cc} a_1(u) q^{\lambda h} + a_2(u) q^{-\lambda h} & g_1(u) f \\ g_2(u) e & c_1(u) q^{(\lambda-1)h} + c_2(u) q^{-(\lambda-1)h} \end{array} \right). 
\]

(3.3)

It follows from the YBE (2.6) that the coefficient functions and \( \lambda \) must satisfy the relations
\[
q^\lambda = i q^{1/2}, \quad g_1(u) = a_1(u) g_{10}, \quad g_2(u) = a_1(u) g_{20} q^{-2a_2}, \\
a_2(u) = a_1(u) q^{-2a_2}, \quad c_1(u) = a_1(u) c_{10}, \quad c_2(u) = a_1(u) q^{-2a_2} a_2 c_{10},
\]

(3.4)

with one constraint,
\[
g_{10} g_20 = -i \frac{(1+q)^2(q-1)}{q^{3/2}},
\]

(3.5)

on the constant coefficients \( g_{20}, g_{10}, a_{20}, c_{10} \).

The particular solution given by the matrix (2.7) corresponds to a symmetric choice of the constants \( g_{20}, g_{10} \), with preliminarily change \( q \to q^{-1} \),
\[
a_1(u) = q^{u+1} / (1+q), \quad a_{20} = 1/q, \quad c_{10} = -1, \quad g_{20} = -i g_{10} = i (q-1)^{1/2}(1+q)/q^{1/2}.
\]

So, taking into account all the constraints (3.4, 3.5), the formula (3.3) takes the form
\[
L(u) = a_1(u) q^{-u} \left( \begin{array}{cc} q^{u+\lambda h} + a_2 q^{u-\lambda h} & g_1 f \\ g_2 e & c_1 q^{u+(\lambda-1)h} + c_2 q^{u-(\lambda-1)h} \end{array} \right).
\]

(3.6)

Before proceeding further, let us to analyze the cause of the existence of the arbitrary constants \( g_{10}, a_{20}, c_{10} \). As it was stated before, YBE allow an arbitrariness in solutions in the form of the multiplier functions, hence the appearance of the function \( a_1(u) \). Using the constraint (3.5), and after some manipulations \( L(u) \) takes the form
\[
L(u) = a_1(u) (a_{20})^{1/2} q^{-u} \left( \begin{array}{cc} q^{u+\lambda h} + q^{u-\lambda h} & g_1 f \\ g_2 e & c_1 q^{u+(\lambda-1)h} + c_2 q^{u-(\lambda-1)h} \end{array} \right).
\]

(3.7)

with \( q^{u'} = q^{u}/(a_{20})^{1/2} \). So, the constant \( a_{20} \) can be eliminated by redefinition of the spectral parameter. It is obvious also, that arbitrary constant \( g_{10} \) has appeared due to the known automorphism \( f \to g_{10} f, \quad g \to e / g_{10} \) of the algebra. It remains only one constant, \( c_{10} \), the nature of which will be apparent in the next part of this section (see (3.15)).
Jimbo’s relations. Choosing $a_1(u) = q^u$, let us rewrite (3.6) as
\[
L(u) = q^u L_+ + q^{-u} L_- ,
\]
\[
L_+ = \begin{pmatrix}
q^{\lambda h} & g_{10} f \\
0 & c_{10} q^{(\lambda - 1) h}
\end{pmatrix},
\]
\[
L_- = \begin{pmatrix}
\alpha_{20} q^{-\lambda h} \\
g_{20} e & c_{10} \alpha_{20} q^{-(\lambda - 1) h}
\end{pmatrix}.
\]

This separation transforms the YBE (3.2) to a few relations, below we represent the crucial ones:
\[
\hat{R}(u)(L_\pm)^\beta_\gamma (L_\pm)^\gamma_\beta = (L_\pm)^\beta_\alpha (L_\pm)^\gamma_\beta \hat{R}(u).
\]
(3.9)

\[
\hat{R}(u)\{q^u(L_+)^\beta_\gamma (L_-)^\gamma_\beta + q^{-u}(L_-)^\beta_\gamma (L_+)^\gamma_\beta\} = \{q^{-u}(L_+)^\beta_\alpha (L_-)^\gamma_\beta + q^u(L_-)^\beta_\alpha (L_+)^\gamma_\beta\} \hat{R}(u).
\]
(3.10)

In (3.9, 3.10) the summation of the products by the Latin indexes is replaced by the graded tensor product operation symbol. From the equations (3.9) the symmetry relations (1.2, 1.13) follow, as can be verified. The equations
\[
\hat{R}(u)(q^{\pm \lambda h} \otimes q^{\pm \lambda h}) = (q^{\pm \lambda h} \otimes q^{\pm \lambda h}) \hat{R}(u),
\]
(3.11)

\[
\hat{R}(u)(q^{\pm (\lambda - 1)^h} \otimes q^{\pm (\lambda - 1)^h}) = (q^{\pm (\lambda - 1)^h} \otimes q^{\pm (\lambda - 1)^h}) \hat{R}(u),
\]
appear in the diagonal parts of (3.9). As their consequences the following relations are derived
\[
\hat{R}(u)(q^{\pm h} \otimes q^{\pm h}) = (q^{\pm h} \otimes q^{\pm h}) \hat{R}(u).
\]
(3.12)

And similarly, from the next equations,
\[
\hat{R}(u)(q^{\lambda h} \otimes f + c_{10} f \otimes q^{(\lambda - 1)^h}) = (q^{\lambda h} \otimes f + c_{10} f \otimes q^{(\lambda - 1)^h}) \hat{R}(u),
\]
(3.13)

\[
\hat{R}(u)(e \otimes q^{-\lambda h} + c_{10} q^{-(\lambda - 1)^h} \otimes e) = (e \otimes q^{-\lambda h} + c_{10} q^{-(\lambda - 1)^h} \otimes e) \hat{R}(u),
\]
constituting the non-diagonal parts of the (3.9), the one can verify, using (3.11), that the following relations are derived,
\[
\hat{R}(u)(q^{h/2} \otimes f' + c_{10} f' \otimes q^{-h/2}) = (q^{h/2} \otimes f' + c_{10} f' \otimes q^{-h/2}) \hat{R}(u),
\]
(3.14)

\[
\hat{R}(u)(e' \otimes q^{-h/2} + c_{10} q^{h/2} \otimes e') = (e' \otimes q^{-h/2} + c_{10} q^{h/2} \otimes e') \hat{R}(u),
\]
with corresponding redefinitions $f' = q^{-(\lambda - 1/2)^h} f$, $e' = e q^{(\lambda - 1/2)^h}$, which are possible due to the automorphisms of the algebra. The number $c_{10}$ is arbitrary. When it equals to 1, the relations (3.14) are equivalent to (1.2). Though the new definitions for the co-products
\[
\Delta^c(f') = \frac{1}{\sqrt{c_{10}}} \left(q^{h/2} \otimes f' + c_{10} f' \otimes q^{-h/2}\right), \quad \Delta^c(e') = \frac{1}{\sqrt{c_{10}}} \left(e' \otimes q^{-h/2} + c_{10} q^{h/2} \otimes e'\right),
\]
(3.15)

where $c_{10}$ is arbitrary, are allowed.

From equations (3.10) we obtain the following relations
\[
\hat{R}(u)\left(c_{10} q^u f \otimes q^{-(\lambda - 1)^h} + q^{-u} q^{-\lambda h} \otimes f\right) = (c_{10} q^{-u} f \otimes q^{-(\lambda - 1)^h} + q^u q^{-\lambda h} \otimes f) \hat{R}(u),
\]
\[
\hat{R}(u)\left(c_{10} q^u q^{(\lambda - 1)^h} \otimes e + q^{-u} e \otimes q^{\lambda h}\right) = (c_{10} q^{-u} q^{(\lambda - 1)^h} \otimes e + q^u f \otimes q^{\lambda h}) \hat{R}(u).
\]
After some easy calculations, taking into account, that $q^{(2\lambda-1)h} f' = (-1)^h f'$ and $e' q^{-(2\lambda+1)h} = e'(-1)^h$, we come to the following relations (called ”Jimbo’s relations”, [23]),

$$
\tilde{R}(u) \left( c_{10} q^u (-1)^h f' \otimes q^{h/2} + q^{-u} q^{-h/2} \otimes (-1)^h f' \right) = \\
= \left( c_{10} q^{-u} (-1)^h f' \otimes q^{h/2} + q^u q^{-h/2} \otimes (-1)^h f' \right) \tilde{R}(u),
$$

$$
\tilde{R}(u) \left( c_{10} q^u q^{-h/2} \otimes e'(-1)^h + q^{-u} e'(-1)^h \otimes q^{h/2} \right) = \\
= \left( c_{10} q^{-u} q^{-h/2} \otimes e'(-1)^h + e'(-1)^h \otimes q^{h/2} \right) \tilde{R}(u).
$$

(3.16)

The operators in the brackets can be treated as definitions of the co-products, dependent on the continuous parameter $u$:

$$
\Delta^c_u(f') = \frac{1}{\sqrt{c_{10}}} \left( c_{10} q^u (-1)^h f' \otimes q^{h/2} + q^{-u} q^{-h/2} \otimes (-1)^h f' \right),
$$

$$
\Delta^c_u(e') = \frac{1}{\sqrt{c_{10}}} \left( c_{10} q^u q^{-h/2} \otimes e'(-1)^h + q^{-u} e'(-1)^h \otimes q^{h/2} \right).
$$

(3.17)

With these definitions the formula (1.12) is valid, as it follows from (3.16).

In the homogeneous case, when two irreps in the tensor product have the same dimension, the generators can be redefined such that $e'(-1)^h$, $(-1)^h f'$ operators are replaced by $e'(-1)^p$, $(-1)^p f'$, where $p$ is to define the parity of the state.

One can verify, that the following equations hold

$$
\{ (-1)^h f' \otimes q^{h/2}, \Delta^c(f') \} = 0, \quad \{ q^{-h/2} \otimes (-1)^h f', \Delta^c(f') \} = 0,
$$

$$
\{ q^{-h/2} \otimes e'(-1)^h, \Delta^c(e') \} = 0, \quad \{ e'(-1)^h \otimes q^{h/2}, \Delta^c(e') \} = 0.
$$

(3.18)

(3.19)

The consequences of the above relations are particularly the observations:

$$
e'(-1)^h \otimes q^{h/2} |j, j\rangle = \gamma(j) |j + \frac{1}{2}, j + \frac{1}{2}\rangle, \quad q^{-h/2} \otimes e'(-1)^h |j, j\rangle = \tilde{\gamma}(j) |j + \frac{1}{2}, j + \frac{1}{2}\rangle.
$$

(3.20)

Let us follow the procedure proposed in the paper [23] and find the matrix $\tilde{R}^{(r_1 r_2)}(u)$, which acts on the product of the spaces $V_{4i_1+1} \otimes V_{4i_2+1}$, $r_k = 4i_k + 1$. The homogeneous case restricted to the odd dimensional conventional irreps was calculated in the work [8]. First we write $\tilde{R}^{(r_1 r_2)}(u)$ in the general form

$$
\tilde{R}^{(r_1 r_2)}(u) = \sum_{j=|i_1-i_2|}^{i_1+i_2} r_j(u) \tilde{P}_{d_{ij+1}},
$$

(3.21)

with projector operators $\tilde{P}_r$, $\tilde{P}_r \cdot V_g = \delta_{rg} V_g$, acting as map $V_{4i_1+1} \otimes V_{4i_2+1} \rightarrow V_{4i_1+1} \otimes V_{4i_2+1}$. When $r_1 = r_2$, then $\tilde{P}_r = P_r$. And then we place $\tilde{R}(u)$ in (3.16). The action of the r.h.s and the l.h.s of the equation (3.16) on the state $|j, j\rangle$ gives the wanted recurrent equation for the $r_j(u)$ functions:

$$
r_{j + \frac{1}{2}}(u) \left( c_{10} q^u \tilde{\gamma}_{i_1 i_2}(j) + q^{-u} \gamma_{i_1 i_2}(j) \right) = \left( c_{10} q^{-u} \gamma_{i_1 i_2}(j) + q^u \gamma_{i_1 i_2}(j) \right) r_j(u).
$$

(3.22)

The appearance of the indexes $i_1 i_2$ and $i_2 i_1$ in the relation shows, that there is a difference between the action of an operator product $a \otimes b$ on the space $V_{r_1} \otimes V_{r_2}$ and the action on the space $V_{r_2} \otimes V_{r_1}$, when $r_1 \neq r_2$. 

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For finding a connection between the $\gamma$ coefficients we are using the relations

$$\left((-1)^h \otimes q^h\right)\Delta^c(e')|j, j\rangle = \frac{1}{\sqrt{e_{10}}} \left((-1)^h e' \otimes q^{h/2} + c_{10}((-1)^h q^h \otimes (-1)^h q^{h/2} \otimes (-1)^h e')\right)|j, j\rangle = \frac{1}{\sqrt{e_{10}}} \left(\gamma_{i_1 i_2}(j) + (-1)p_0^{i_1 i_2} c_{10}(-q)^{h_j+1}\gamma_{i_1 i_2}(j) \right) |j, j\rangle = 0. \tag{3.23}$$

Here the values of $h_j$ are the eigenvalues of the $h$ operator on the highest vector of an irrep; when dimension $r$ is odd, they coincide with $2j$, $h_j \equiv \frac{r-1}{2}$ (1.4), and have an additional complex summand in the even case, as we have seen in (1.6), i.e $h_j = 2j_r \equiv 2j + \lambda - 1/2$. $p_0^{i_1 i_2} = 0$ for the case, when both of $r_1$ and $r_2$ are either even or odd numbers, i.e., when $|r_1 - r_2|$ is an even number. Otherwise, when the second irrep in the tensor product is even dimensional, $p_0^{i_1 i_2} = -2\lambda$. So, the recurrent relations are solved as

$$r_{j'}(u) = \prod_{j=j'}^{i_1+i_2-1/2} \left(\frac{c_{10}q^u\gamma_{i_1 i_2}(j)}{c_{10}q^{-u}\gamma_{i_1 i_2}(j)}\right) r_{i_1+i_2}(u) = \prod_{j=j'}^{i_1+i_2-1/2} \frac{\gamma_{i_1 i_2}(j)}{\gamma_{i_2 i_1}(j)} \left(\frac{q^u - q^{-u}(-1)p_0^{i_1 i_2}(-q)^{2h_j+1}}{q^{-u} - q^u(-1)p_0^{i_1 i_2}(-q)^{2h_j+1}}\right) r_{i_1+i_2-1}(u). \tag{3.24}$$

- For freeing the formulas in the brackets (3.24) from the unwanted factors like $(-1)^{i_2}$, which arise when $|r_1 - r_2|$ is odd number, we can redefine the spectral parameter $u$ by an appropriate shift. Suppose $i_1$-irrep has odd dimension, while the second irrep, with "spin" $i_2$, has even dimension. Then $p_0^{i_1 i_2} \equiv p_0 = -2\lambda$, and $p_0^{i_2 i_1} = 0$. Now all the $h_j$-s represent even dimensional irreps and are complex. Using the quarter integer numbers $j$ introduced in (1.6) and also recalling that $(-q) = q^{2\lambda}$, $(-1) = q^{2\lambda-1}$, it is possible to redefine the formulas in (3.24) as follows

$$r_{j'}(u') = \prod_{j=j'}^{i_1+i_2-1/2} \frac{\gamma_{i_1 i_2}(j)}{\gamma_{i_2 i_1}(j)} \left(\frac{q^{u'} - (-q)^{2i_2}q^{-u'}}{q^{-u'} - (-q)^{2i_2}q^{u'}}\right) r_{i_1+i_2}(u'), \tag{3.25}$$

where $u' = u + \lambda(\lambda - 1/2)$. 

In the homogeneous case, when $i_1 = i_2$, and hence $\frac{\gamma_{i_1 i_2}(j)}{\gamma_{i_2 i_1}(j)} = 1$, the expression (3.24) coincides with a similar expression, derived in [8] for odd dimensional irreps. Now, for obtaining the fraction $\frac{\gamma_{i_1 i_2}(j)}{\gamma_{i_2 i_1}(j)}$ in general inhomogeneous cases, we shall express the coefficients $\gamma_{i_1 i_2}(j)$ by means of the Clebsh-Gordan coefficients [21, 8]. Suppose (see [21])

$$|j, j\rangle_{12} = \sum_{j_1 + j_2 = j} C^{(i_1 i_2)}_{(j_1 j_2)} |i_1, j_1\rangle \otimes |i_2, j_2\rangle, \tag{3.26}$$

$$|j, j\rangle_{21} = \sum_{j_1 + j_2 = j} C^{(i_2 i_1)}_{(j_1 j_2)} |i_2, j_2\rangle \otimes |i_1, j_1\rangle, \tag{3.27}$$

$$e'(i, j) = \alpha^{(j+1/2)}_{i_2} \gamma^{(i_1 i_2)}_{(j+1/2)} \tag{3.28}.$$ 

Then we have (we assume, that $i_2 \geq (j + \frac{1}{2} - i_2)$)

$$e'(-1)^h \otimes q^{h/2}|j, j\rangle_{12} = \alpha^{2i_1-1}_{i_2}(-1)^{2i_1-1}q^{j-i_2+1/2} \frac{C^{(i_1^1)}_{(i_1 + i_2)}(j)}{C^{(i_1^2)}_{(i_1 + i_2)}(j)} |j + \frac{1}{2}, j + \frac{1}{2}\rangle_{12}, \tag{3.29}$$
\[ q^{-h/2} \otimes e'(-1)^h |j,j\rangle_{12} = a_{i_2}^{2(j-i_1)}(-1)^{2(j-i_1)}p_{i_1}q^{-i_1} \frac{C(i_1,j_1, i_2, j_2)}{C(i_1, j_1, i_2, j_2)} |j + \frac{1}{2}, j + \frac{1}{2}\rangle_{12}. \] (3.30)

\[ \frac{\gamma_{i_2,i_1}(j)}{\gamma_{i_1,i_2}(j)} = (-1)^{2(i_2-i_1)}q^{i_1-i_2} \frac{\alpha_{i_2}^{2(j-i_1)} C(i_1,j_1, i_2, j_2)}{\alpha_{i_1}^{2(j-i_1)} C(i_1, j_1, i_2, j_2)} \frac{C(i_2-j_1)}{C(i_1-j_2)}. \] (3.31)

**Some examples.** Here we represent the exact non-homogeneous check \( R \)-matrices for some simple cases.

\[ \tilde{R}^{(23)}(u) = \tilde{P}_4 + \left( \frac{1}{q^{1/2} - q^{-1/2}} \right) \frac{(q^{-u} + q^{3/2} + u)}{(q^{-u} - q^{3/2} - u)} \tilde{P}_2, \] (3.32)

\[ \tilde{R}^{(24)}(u) = \tilde{P}_5 + \left( \frac{q^{1/2} + q^{-1/2}}{q^{3/2} - q^{-3/2}} \right) \frac{(q^{-u} + q^{3/2} + u)}{(q^{-u} - q^{3/2} - u)} \tilde{P}_3, \] (3.33)

\[ \tilde{R}^{(35)}(u) = \tilde{P}_7 + \left( \frac{1}{q^{1/2} - q^{-1/2}} \right) \frac{(q^{-u} - q^{3/2} + u)}{(q^{-u} - q^{3/2} + u)} \tilde{P}_5 + \left( \frac{1}{q^{1/2} + q^{-1/2}} \right) \frac{(q^{-u} - q^{3/2} + u)}{(q^{-u} - q^{3/2} + u)} \tilde{P}_3. \] (3.34)

### 3.2 Connection with the quantum algebra \( sl_q(2) \)

**Representations and universal \( R \)-matrices.** In [9] the author has demonstrated a correspondence between \( sl_q(2) \) and \( osp_q(1|2) \) algebras at the level of finite dimensional representations and universal \( R \)-matrices (see also [7, 10]). Below we give a brief overview of that study.

If \( e, f, h \) are the two-dimensional \( osp_q(1|2) \) generators, then \( E, F, H \)

\[ E = -i(q^{1/2} - q^{-1/2})e, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f, \quad H = 2h - 2(\lambda - 1/2). \] (3.35)

generate the quantum algebra \( sl_l(2) \), with deformation parameter \( t = iq^{1/2} \).

For the general even dimensional representations the correspondence is given by the relations:

\[ E = -(t + t^{-1})e, \quad F = (-i)2^{(h-\lambda+1/2)}f, \quad H = 2h - 2(\lambda - 1/2). \] (3.36)

In the odd dimensional case the correspondence is stated by the formulæ

\[ E = -(t + t^{-1})e, \quad F = (-1)^{h}f, \quad H = 2h. \] (3.37)

The universal \( R \)-matrix for the \( sl_l(2) \) algebra is found by simple replacements in the \( osp_q(1|2) \) invariant universal \( R \)-matrix

\[ R_{osp_q(1|2)} = q^{h \otimes h} \sum_{n=0}^{\infty} \frac{(-1)^{1/2}2(n-1)(q - q^{-1})^n}{[n]_+!} q^{-nH/2}e^n \otimes f^n q^{nH/2}, \] (3.38)

\[ [n]_+ = \frac{(-1)^{n-1}q^{h/2} + q^{-n/2}}{q^{1/2} + q^{-1/2}}, \] (3.39)

after inserting the expressions in (3.36) or (3.37), and replacing \( iq^{1/2} \) factors by \( t \).

The resulting matrix \( R_{sl_l(2)} \) is understood as matrix representation. If to take into account the fermionic character of the tensor products by using appropriate signs for each
representation, the extra factors \((-1)^{H\otimes H}\) and \((-1)^{nH}\) would be cancelled in the \(R_{sl_t(2)}\) and it will be the universal \(R\)-matrix of \(sl_t(2)\).

It is noted in [9], that there are slightly different formulae for stating the correspondence (3.36, 3.37) (see references in [9]). A correspondence of \(R\)-matrices for the odd dimensional irreps is discussed in [8], there the correspondent matrices are differing by a gauge transformation. The [10] devotes to the correspondence of the conventional odd dimensional (non-spinorial) representations of the \(osp_q(1|2n)\) and \(so_q(2n + 1)\).

\(R(u)\) matrices and Lax operators. Now a correspondence with \(sl_t(2)\) case can be stated also for the Lax operators and \(R\)-matrices (2.5, 3.6, 3.21), constructed in the first sections.

For example, let us represent a general form of the Lax operator in terms of \(E, F, H\) generators (3.37), defined for conventional irreps.

\[
L_t(u) = \begin{pmatrix}
  a_1(u,t) t^{H/2} + a_2(u,t) t^{-H/2} \\
  g_2(u,t)(-1)^{-h} E
\end{pmatrix}
\begin{pmatrix}
  -g_1(u,t) t^{F/2 - H/2} \\
  c_1(u,t) t^{H/2} (-t)^{-2h} + c_2(u,t) t^{-H/2} (-t)^{2h}
\end{pmatrix}.
\] (3.40)

There are relations among the functions \(a_i(u,t), c_i(u,t), g_i(u,t)\), given in the equations (3.4, 3.5). The operator (3.40) satisfies the graded Yang-Baxter relations, with the four dimensional \(R\)-matrix, where one also has to replace the \(q\) parameter by \(t\). One can arrive at the ordinary \(sl_t(2)\) Lax operators if one attributes the \((-1)^{h}\) type factors in (3.40) to the graded character of the original operator and YB equations.

4 All solutions to YBE with higher dimensional irreps

4.1 Customary (standard) three dimensional fundamental representations

The fundamental representation of the \(osp(1|2)\) is the three dimensional one. There are two known solutions to the spectral parameter dependent YBE equations (1.11), with \(osp_q(1|2)\) symmetry, which can be found from the articles [4] and [8].

Let us find all the solutions to (1.11) with \(9 \times 9\) \(R\)-matrices, acting on the product of the fundamental irreps \(V_3 \otimes V_3\). Remind, that the \(R\)-matrices, being the intertwiner matrices for the \(osp_q(1|2)\) superalgebra, satisfy to (1.13), which implies the decomposition (1.14). So we are looking for \(\tilde{R}\) in the form

\[
\tilde{R}^{(33)}(u) = P_3 + f(u) P_3 + g(u) P_1,
\] (4.1)

where \(P_i\)-s are the projectors acting on the decomposition \(V_3 \otimes V_3 = V_1 \oplus V_3 \oplus V_5\).

\(R\) is defined within the multiplication by a function of the spectral parameter. Here and henceforth we choose that function so, that \(\tilde{R}(0) = 1\).

Now let us to turn to the investigation of the equations (2.3). It turns out, that all the possible solutions can be found by considering only three independent equations from the set (2.3). Inserting (4.1) representation in the mentioned equations and taking at first the equation with elements

\[\{a_1, a_2, a_3, c_1, c_2, c_3\} = \{1, 2, 1, 2, 1, 1\},\]
we found a functional relation for the $f(u)$,

$$f(v) = \frac{(1 + q^4)f(u) + q^2(1 - f(u - v) + f(u) + f(u - v)f(u))}{(1 + q^4)f(u - v) + q^2(1 + f(u - v) - f(u) + f(u - v)f(u))}.$$  \hspace{1cm} (4.2)

Solving this relation, we found that

$$f(0) = 1, \quad f(u)f(-u) = 1, \quad f(u) = \frac{q^{up}q^2 - 1}{q^2 - q^{up}}.$$  \hspace{1cm} (4.3)

The constant $p$ is arbitrary, since the transformations $u \rightarrow pu, \ v \rightarrow pv$ leave YBE invariant.

As the next equations we take

$$\{a_1, a_2, a_3, c_1, c_2, c_3\} = \{2, 2, 1, 1, 2, 2\} \quad \text{and} \quad \{a_1, a_2, a_3, c_1, c_2, c_3\} = \{2, 2, 1, 2, 2, 1\}.$$  

The first of them gives an expression for the $g(u)$ function, $g(0) = 1$, without fixing $g'(0)$,

$$g(u) = \frac{g'(0)(q-1)(q^{up}-1)(q^{4+up}-1) - p(1 - q + q^2)(1 + (q + q^2 - 2 - 2q^3)q^{up} + q^{2up+3})}{(q^2 - q^{up})(g'(0)q(q-1)(q^{up}-1) - p(1 - q + q^2)(q^{up}-q))},$$  \hspace{1cm} (4.4)

and the second equation gives two possibilities for the $g'(0)$,

$$g'(0) = \frac{2p(1 - q + q^2)}{q^2 - 1} \quad (a) \quad \text{and} \quad g'(0) = p\frac{1 + q^3}{q^3 - 1} \quad (b),$$  \hspace{1cm} (4.5)

corresponding to the two known solutions. After substitution the expressions (4.5) into the formula of $g(u)$ and fixing the values of $p$, we recover Saleur’s solution, (4.5, a),

$$\hat{R}_a(u) = P_5 + \frac{1 - q^2y}{y - q^2}P_3 + \frac{(1 + qy)(1 - q^2y)}{(y + q)(y - q^2)}P_1, \quad y = q^{-2a},$$  \hspace{1cm} (4.6)

and Kulish’s solution, (4.5, b),

$$\hat{R}_b(u) = P_5 + \frac{q^2y^2 - 1}{q^2 - y^2}P_3 - \frac{1 - q^3y^2}{q^3 - y^2}P_1, \quad y = q^{-u}.$$  \hspace{1cm} (4.7)

For some special values of $q$, namely, if $q^3 = 1$, there is another solution too

$$g(u) = \frac{-1 + q^4q^{up}}{q^3 - qq^{up}}, \quad q^3 = 1.$$  \hspace{1cm} (4.8)

These all solutions satisfy the remaining equations also, which don’t give additional constraints.

The solution (4.6) is given in H. Saleur’s paper [8]. This formula can be checked also via the general solution, defined by (3.24). In the classical limit $q \rightarrow 1$ it can be represented in the common form

$$\hat{R}_a(u) = a(u)I + b(u)P,$$  \hspace{1cm} (4.9)

with $I, \ P$ unit and permutation operators. The other solution (4.7), which we called as Kulish’s solution, as it corresponds to the $R$ operator discussed in the articles [4, 14], in the classical limit $q \rightarrow 1$ have the form [3]

$$\hat{R}_b(u) = a(u)I + b(u)P + c(u)K,$$  \hspace{1cm} (4.10)
where $K$ is an operator, which together with unit and permutation operators, commute with $osp(1|2)$ superalgebra generators.

Two solutions have the same braid group limit $R^\pm = \lim_{u \to \pm \infty} R(u)$. $R^+$ corresponds to the formula (3.38), derived for the fundamental representation. $R^- = P(R^+)^{-1}P$.

At the point $q^{2u_0} = -q^{-1}$ the solution (4.6) reduces to the higher spin projector, $\tilde{R}_{a}(u_0) \approx P_5$. Therefore via a "fusion" procedure, developed in [22], it is possible to generate the solution of the YBE for the higher dimensional representations of $osp_q(1|2)$ superalgebra, using solution $\tilde{R}_{a}(u)$ (4.6). The $\tilde{R}_{a}(u)$ matrix itself could be obtained by the fusion from the $4 \times 4$ matrices $R(u)$, defined by (2.5) ($\tilde{R}(u_0 = \lambda) \approx P_3$).

For the solution (4.7) there is no such point $u_0$, for which $\tilde{R}_b(u_0)$ would become proportional to the maximal $P_5$ projection operator, so it is impossible to apply the fusion method for finding the solutions for YBE with $R$ defined on the higher spin representations. We shall discuss the consequences of it in more detail in the subsection 4.2. Note, that at the poles $q^{u''} = q^{-1}$ and $q^{u'''} = q^{-3/2}$ of (4.7) the performing of the "fusion" procedure would reproduce the fundamental $\tilde{R}_b$ or would decrease the dimension of the quantum space to 1.

Let us recall now the exceptional solution (4.8). Such solutions to YBE for the cases when $q$ is a root of unity can meet also for the higher dimensional representations (see the subsection 4.2). As it is known from the analysis of the quantum (super-)algebras [20, 21], the structure of the set of the non reducible representations and their fusion rules are deformed, when $q$ is a root of unity, so the cases, like to the (4.8) require separate investigation.

### 4.2 Uniqueness of the solutions

We have seen that there are two general solutions to the Yang-Baxter equation (1.11), when 1, 2, 3 refer to the 3-dimensional fundamental irreps. Only one of them (4.6) allows to construct descendant solutions for higher spin representations and belongs to the class, defined by equation (3.24). But is it enough to conclude that there are no other solutions for higher spin representations, besides them?

We can try to check it for some simple cases. At first let us write the most general form of $\tilde{R}^{(44)}(u)$, satisfying (1.11), if 1, 2, 3 are 4 dimensional irreps:

$$\tilde{R}^{(44)}(u) = P_7 + f(u)P_5 + g(u)P_3 + h(u)P_1.$$  

(4.11)

Inserting it in the YBE, via not so hard analysis, we can separate equations, where there are only $f(u)$, $f(v)$, $f(u + v)$ functions. Actually there is one independent equation of this kind, namely

$$f(v) = \frac{(f(u) - 1)q^3 + f(u + v) (1 - (1 + f(u)q^3 + q^6))}{(f(u + v) - 1)q^3 + f(u) (1 - (1 + f(u + v)q^3 + q^6))}. \quad (4.12)$$

And we can find the solution for $f(u)$, which is unique (more precisely there is always freedom of the rescaling of the spectral parameter $u \to pu$, with arbitrary $p$)

$$f(u) = \frac{1 + q^{3+u}}{q^3 + q^u}. \quad (4.13)$$

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Then discussing the equations, where only $f(u)$, $f(v)$, $f(u+v)$ and $g(u)$, $g(v)$, $g(u+v)$ are involved, we find two possible solutions for the $g(u)$

$$g(u) = \frac{(1 + q^{3+u})(q^{2+u} - 1)}{(q^3 + q^u)(q^2 - q^u)} \quad \text{(a),} \quad g(u) = \frac{q^{5+u} - 1}{q^5 - q^u} \quad \text{(b).} \quad (4.14)$$

After analyzing the equations, which contain also $h(u)$, $h(v)$, $h(u+v)$, we find that only one solution $g(u)$ (4.14, a) is consistent with YBE. And $h(u)$ function turns out to be

$$h(u) = \frac{(q^{3+u} + 1)(q^{2+u} - 1)(q^{u+1} + 1)}{(q^3 + q^u)(q^2 - q^u)(q + q^u)} \quad (4.15)$$

And certainly, this is the solution belonging to the class of mentioned solutions, i.e. it is possible by fusion (descendant) procedure to obtain this solution from the product of the $R^{(22)}$ matrices (2.5), or by (3.24) relations.

There can be some special solutions also, for the exceptional values of $q$, such as (4.8), but we shall not concentrate our attention on them now.

We can go further and try to find solution for 5-dimensional representations also, looking for the solution in the form

$$\hat{R}^{(55)}(u) = P_9 + f(u)P_7 + g(u)P_5 + h(u)P_3 + e(u)P_1. \quad (4.16)$$

And quite similarly to the previous analysis, we can separately discuss four kind of the equations, which contain correspondingly only the functions $f$, the functions $f$, $g$, the functions $f$, $g$, $h$ and at last all functions $f$, $g$, $h$. The first type of equations contain only one independent equation:

$$f(v) = \frac{f(u + v)(q^8 + (1 + f(u)q^4 + 1) + (1 - f(u))q^4}{f(u)(q^8 + (1 + f(u + v)q^4 + 1) + (1 - f(u + v))q^4. \quad (4.17)$$

It has solution

$$f(u) = \frac{q^4 - q^u}{q^{4+u} - 1}. \quad (4.18)$$

The only arbitrariness here is regarding to the rescaling of the spectral parameter $u - u \to (pu)$, which leaves the YBE invariant.

Then the second type of the equations gives two possible solutions for $g(u)$.

$$g(u) = \frac{(q^4 - q^u)(q^3 + q^u)}{(q^{4+u} - 1)(1 + q^{3+u})} \quad \text{(a),} \quad g(u) = \frac{q^7 - q^u}{q^{7+u} - 1} \quad \text{(b).} \quad (4.19)$$

And the third kind of the equations gives the unique solution for $h(u)$ with general $q$,

$$h(u) = \frac{(q^4 - q^u)(q^3 + q^u)(q^2 - q^u)}{(q^{4+u} - 1)(1 + q^{3+u})(q^{2+u} - 1)}. \quad (4.20)$$

corresponding to the one of the $g(u)$ solutions, namely to (4.19, a).
For the second one, (4.19, b), there is no \( h(u) \) function, which could satisfy to all the equations. So from the three kind of the equations we find out the \( f(u), g(u), h(u) \) functions, and the fourth kind of the equations gives us the unique solution of the function \( e(u) \),

\[
e(u) = \frac{(q^4 - q^u)(q^3 + q^u)(q^2 - q^u)(q + q^u)}{(q^{1+u} - 1)(1 + q^{3+u})(q^{2+u} - 1)(q^{1+u} + 1)},
\]

which together with the other functions satisfy to all the YBE equations. This solution is also consistent with the formula (3.24).

Note, that in the case of the fundamental 3-dimensional irreps, there are only two kind of the equations in the YBE set, i.e., equations which contain only the \( f(u) \) function, and the equations with both of the \( f(u) \) and \( g(u) \) functions. So, after finding two possible solutions for the \( g(u) \) function, there are no additional equations, which could exclude the second solution of \( g(u) \). Hence both of the two solutions (4.6, 4.7) are valid.

And at the end we would like to clarify, whether there is a \( R^{(3r)} \) generalization for the (4.7) \( R_0 \)-matrix. We can try to find solution (if there is any) for the YBE (1.15), where the 1,2 spaces are the fundamental three dimensional irreps, and the space 3 is the five-dimensional irrep, when the scattering of the three-dimensional irreps is described by the solution (4.7).

\[
\tilde{R}_{0}^{(33)}(u - v)\tilde{R}_{0}^{(35)}(u)\tilde{R}_{0}^{(35)}(v) = \tilde{R}_{0}^{(35)}(v)\tilde{R}_{0}^{(35)}(u)\tilde{R}_{0}^{(33)}(u - v).
\]

We write the \( \tilde{R}_{0}^{(35)} \) in this form

\[
\tilde{R}_{0}^{(35)} = \hat{P}_7 + f(u)\hat{P}_5 + g(u)\hat{P}_3,
\]

where the projectors \( \hat{P}_r \) are acting on the \( V_3 \otimes V_5 \rightarrow V_5 \otimes V_3 \). And we can verify by straightforward calculations that there is no solution \( \tilde{R}_{0}^{(35)} \) for the YBE of this kind. So, we can conclude, that there is no general universal \( R^{(rp)}(u) \)-matrix, which would satisfy to the YBE, being \( osp_q(1|2) \)-invariant, with \( R^{(33)}(u) \) defined in (4.7) as it’s particular case.

5 \hspace{1em} RLL = LLR relations with 9 × 9 \( R \)-matrices; the rational limit.

Relying on the fusion procedure [22], as it is stated already, it is possible to obtain the solution (4.6) to the YBE for the three dimensional representations of \( osp_q(1|2) \) superalgebra, starting from the solution (2.5) for the two-dimensional ones. Similarly we can try to find the Lax operator, with the three dimensional auxiliary space from the fusion of two Lax operators with two-dimensional auxiliary spaces, defined in the previous sections.

The operator \( P_3L^{(2r)}(u)L^{(2r)}(u - u_0) \) with the point \( u_0 = \lambda \), where \( \hat{R}(u_0) \approx P_3 \) (2.5), serves as a such operator. It can be proved by repeatedly applying the YB equations. Then by some reformulations we can find the (3 × 3) matrix representation of the Lax operator. Let us demonstrate it step by step.

From the YBE (2.8) it follows that

\[
(-1)^{p_1 + p_2 + p_3} \left( P_3 \right)_{b_1 b_2} L_{c_1}^{a_1} L_{c_2}^{b_1}(u)(u - u_0) = L_{b_2}^{a_2}(u - u_0)L_{b_1}^{a_1}(u) \left( P_3 \right)_{c_1 c_2}^{b_1 b_2} (-1)^{p_1 + p_3}.
\]

(5.1)
So the operator-matrix
\[
(PLL)_{c_1c_2}^{a_1a_2}(u) = (-1)^{p_{c_1}(p_{a_2}+p_{c_2})} (P_3)_{b_1b_2}^{a_1a_2} L_{c_1}^{b_1}(u) L_{c_2}^{b_2}(u-u_0),
\]
which acts as matrix on \( V_2 \otimes V_2 = V_1 + V_3 \) auxiliary space, really is nonzero only in the three dimensional space \( V_3 \). The orthonormalized eigenvectors of the projector \( P_3 \) are \( \{1,0,0\}, \{0,i\sqrt{q},i\sqrt{1-q}\}, \{0,0,1\} \). By means of the \( 3 \times 4 \) matrix constructed by them
\[
\mathbb{V} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{i\sqrt{q}}{1-q} & \frac{1}{1-q} & 0 \\
0 & \frac{i\sqrt{q}}{1-q} & \frac{1}{1-q} & 1
\end{pmatrix},
\]
and by the following transformation
\[
L^{(3)}(u) = \mathbb{V} \cdot (PLL)(u) \cdot \mathbb{V}^t,
\]
we arrive at an three dimensional matrix-operator \( L^{(3)}(u) \), which satisfies to the YBE
\[
R^{(33)}(u-v)L^{(3)}(u)L^{(3)}(v) = L^{(3)}(v)L^{(3)}(u)R^{(33)}(u-v)
\]
defined on the spaces \( V_3 \otimes V_3 \otimes V_{\text{arbitrary}} \), with \( 9 \times 9 \) matrix \( R^{(33)}(u) \). It is obtained by fusion from the tensor product of the \( 4 \times 4 \) matrices \( R(u) \) \((2.5)\): on the space \( V_2^a \otimes V_2^d \otimes V_2^b \otimes V_2^0 \) it is
\[
R^{(33)}(u) = (\mathbb{V} P_3^{aa'} \mathbb{V} P_3^{bb'}) \cdot R_{ab'}(u+u_0) R_{a'b'}(u) R_{ab}(u+u_0) \cdot (\mathbb{V}^t P_3^{aa'} \mathbb{V}^t P_3^{bb'}).
\]
The last matrix coincides with the solution \((4.6)\) given in the previous Section. There can be some uncertainties, caused by the matrix representations of the projector operators. It is obtained by the redefinition of the matrix representations of the algebra generators, taking place during the fusion procedure. Note, that in general we can define a matrix
\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & x_1 & 0 \\
0 & 0 & -x_1 x_2
\end{pmatrix},
\]
which transforms the three dimensional representations of the algebra generators \( a \) to \( U^{-1} \cdot a \cdot U \), particularly \( f = \begin{pmatrix}
0 & 0 & 0 \\
x_1 & 0 & 0 \\
0 & x_2 & 0
\end{pmatrix} \) generator transforms into
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}. \]
It is equivalent to the automorphism \( f \rightarrow k^{-x} f k^y, \quad e \rightarrow k^{-y} e k^x \), with \( q^x = x_2, \quad q^y = x_1 \). So the more general form of the \( 3 \times 3 \) Lax operator is defined by
\[
U^{-1} \cdot L^{(3)}(u) \cdot U,
\]
which is
\[
L(u) = \begin{pmatrix}
L_1^1 & L_2^1 & L_3^1 \\
L_1^2 & L_2^2 & L_3^2 \\
L_1^3 & L_2^3 & L_3^3
\end{pmatrix}(u),
\]
with matrix elements (as the finding of the Lax operator with conventional fundamental auxiliary space is reasonable, when the quantum spaces are also conventional odd dimensional representations, with with eigenvalues of $h$ being integers, below we take everywhere $(-1)^{2h} = 1$)

$$L_1^1(u) = (-1)^h[-1/2 + a_x + h + 2u]_q - \frac{\alpha \sqrt{q}}{1 + q},$$

$$L_1^2(u) = \frac{\sqrt{1-q}g_{10}x_1}{q^2 - 1} \left((i)^h q^{\frac{b}{2} + \frac{1}{2} + a_x + 2u} + (i)^{-h} q^{\frac{b}{2} - \frac{1}{2}}\alpha \right) f,$$

$$L_1^3(u) = q^{1/2 + a_x + 2u} x_1 x_2 g_{10}^2 f^2,$$

$$L_2^1(u) = \frac{-i\sqrt{1-q}(1+q)c_{10}}{g_{10}x_1} \left((i)^h q^{\frac{b}{2} - 1}\alpha - (i)^{-h} q^{a_x + 2u} - (i)^h q^{-a_x} - (i)^{-h} q^{a_x}ight) e,$$

$$L_2^2(u) = \left((-1)^h[-1/2 + a_x + 2u]_q - \frac{1+q}{\sqrt{q}}\alpha f e + \alpha[h - 1/2]_q \right) c_{10},$$

$$L_2^3(u) = \frac{i\sqrt{1-q}c_{10}g_{10}x_2}{q^2 - 1} \left((i)^{-h} q^{1 + \frac{b}{2}}\alpha - (i)^h q^{2 + a_x + 2u} - (i)^{-h} q^{a_x - 1} - (i)^h q^{a_x}ight) f,$$

$$L_3^1(u) = \frac{(1-q)(1+q)^{\frac{3}{2}}c_{10}^2 q^{-3/2 - a_x - 2u}}{x_1 x_2 g_{10}^2 e^2},$$

$$L_3^2(u) = \frac{\sqrt{1-q}(1+q)c_{10}^2}{g_{10}x_2} \left((i)^h q^{1 + \frac{b}{2}}\alpha - (i)^{-h} q^{2 + a_x + 2u} - (i)^{-h} q^{a_x - 1} - (i)^h q^{a_x}ight) e,$$

$$L_3^3(u) = \left((-1)^{-h}[-1/2 + a_x - h + 2u]_q - \frac{\alpha \sqrt{q}}{1 + q} \right) c_{10}^2,$$

where $a_x$ is a number, defined by $a_{20} = \alpha q^{-a_x}$, $\alpha = \pm 1$. The $a_{20}$, $c_{10}$, $g_{10}$ arbitrary constants come from the definition of $2 \times 2$ operator $(3.6)$. Although, as it was mentioned before, by using the super-algebra's automorphism, shift of the spectral parameter and redefinition of the co-products, it is possible to fix all the constants, we preferred to represent more general form of the Lax operator. Remind, that there is also an arbitrariness of multiplication by a function, which in above formulas is fixed, by choosing appropriate $a_1(u)$ function.

As we see this matrix-operator has a well defined classical limit, $q \to 1$, with additional requirements

$$x_1 = x_{10} \sqrt{1 - q}, \quad x_2 = x_{20} \sqrt{1 - q}. \quad (5.8)$$

If define $q = e^h$, then at the limit $h \to 0$, we arrive at

$$L_1^1(u) = (-1)^h(h - \frac{1}{2} + a_x + 2u) - \frac{\alpha}{2}, \quad L_2^2(u) = \frac{-g_{10}x_{10}}{2} \left((i)^h + (i)^{-h}\alpha \right) f,$$

$$L_1^2(u) = \frac{-x_{10}x_{20}g_{10}^2}{2} f^2, \quad L_2^1(u) = \frac{-2ic_{10}}{g_{10}x_{10}} \left((i)^h\alpha - (i)^{-h}\right) e,$$

$$L_2^2(u) = \left((-1)^h(a_x - \frac{1}{2} + 2u) - 2\alpha f e + \alpha(h - 1/2) \right) c_{10},$$

$$L_3^2(u) = \frac{-ic_{10}g_{10}x_{20}}{2} \left((i)^{-h}\alpha - (i)^h\right) f, \quad L_3^1(u) = \frac{8c_{10}^2}{x_{10}x_{20}g_{10}^2} e^2.$$
\[
\mathbf{L}^2(u) = \frac{2c_{10}^2}{g_{10}x_{20}} \left( (i)^h \alpha + (i)^{-h} \right) e, \quad \mathbf{L}^3(u) = \left( (-1)^{-h}(a_x - h - \frac{1}{2} + 2u) - \frac{\alpha}{2} \right) c_{10}^2.
\]

Let us fix \( a_x = \frac{1}{2} \) and \( \alpha = 1 \), \( c_{10} = 1 \), \( g_{10} = 2 \), \( x_{10} = x_{20} = 1 \).

\[
\mathbf{L}(u) = \left( \begin{array}{ccc}
(-1)^h(h + 2u) - \frac{1}{2} & -2 \cos \left[ \frac{\pi}{2} h \right] f & -2f^2 \\
2 \sin \left[ \frac{\pi}{2} h \right] e & (-1)^h(2u) - 2f e + (h - \frac{1}{2}) & -2 \sin \left[ \frac{\pi}{2} h \right] f \\
2e^2 & 2 \cos \left[ \frac{\pi}{2} h \right] e & (-1)^{-h}(2u - h) - \frac{1}{2}
\end{array} \right). \tag{5.10}
\]

In the case of choice \( x_{10} = \sqrt{-i} \), \( x_{20} = -\sqrt{-i} \), which corresponds to a real matrix representations of the \( 9 \times 9 \) \( R(u) \) matrix, we have

\[
\mathbf{L}(u) = \left( \begin{array}{ccc}
(-1)^h(h + 2u) - \frac{1}{2} & -2\sqrt{-i} \cos \left[ \frac{\pi}{2} h \right] f & 2f^2 \\
2\sqrt{i} \sin \left[ \frac{\pi}{2} h \right] e & (-1)^h(2u) - 2f e + (h - \frac{1}{2}) & \sqrt{-i} \sin \left[ \frac{\pi}{2} h \right] f \\
-2e^2 & 2\sqrt{-i} \cos \left[ \frac{\pi}{2} h \right] e & (-1)^{-h}(2u - h) - \frac{1}{2}
\end{array} \right). \tag{5.11}
\]

6 Discussion

The investigation of the quantum extension of the super-algebra \( osp(1|4) \) (may be, it is true for the general case \( osp(1|2n) \), too), shows, that in the quantum deformation case, besides of the conventional irreducible representations, there are irreps, which don’t appear in the classical case. Suppose the irrep with minimal dimension in the quantum case don’t coincide with the conventional fundamental representation. We can find the solutions to YBE for that minimal representation, and then, from the fusion procedure, find all the solutions of YBE with arbitrary representations. As we have checked, the minimal representation of the \( osp_q(1|4) \) has four dimension, while the fundamental representation is a five-dimensional one.

We expect that analogous relations hold in the general case \( osp_q(1|2n) \) too. There are works (see [10] and references therein), which show connection between the finite dimensional representations of the \( osp_q(1|2n) \) and the non-spinorial representations of the quantum deformation of \( so(2n+1) \) algebra. It is expected that there are also non conventional ”spinorial” representations for the \( osp_q(1|2n) \) algebra, with complex eigenvalues of the \( h_i \) Cartan matrices, as it was in the case of the \( osp_q(1|2) \) algebra (the even dimensional representations). And surely then the minimal dimensional irreducible representations belong to the ”spinorial” kind. Although such representations have no classical limit (when \( q \to 1 \)), it is sensible to construct Lax operator defined on the auxiliary space with such minimal dimensional representation. The Lax operator with auxiliary space being the conventional fundamental representation can be constructed by a fusion procedure from the simpler operators, like in the case \( n = 1 \) discussed in this paper.

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