Private Information Retrieval Over Gaussian MAC

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Abstract—Consider the problem of Private Information Retrieval (PIR) where a user wishes to retrieve a single message from $N$ non-communicating and non-colluding databases (servers). All servers store the same set of $M$ messages and they respond to the user through a block fading Gaussian Multiple Access Channel (MAC). The goal in this setting is to keep the index of the required message private from the servers while minimizing the overall communication overhead.

This work provides joint privacy-channel coding retrieval schemes for the AWGN MAC with and without fading. The schemes exploit the linearity of the channel while using the Compute and Forward (CF) coding scheme. Consequently, single-user encoding and decoding are performed to retrieve the private message. The achievable retrieval rates are shown to outperform a separation-based scheme for which the retrieval and the channel coding are designed separately. Moreover, these rates are asymptotically optimal as the SNR grows and are up to a constant gap of 2 bits per channel use for every SNR.

I. INTRODUCTION

The ability to provide privacy and protection to sensitive data has become a requirement in nowadays communication systems. While cryptography and physical layer security provide various solutions against adversaries which are located outside the system, in some application privacy is required even from the system’s administrators which have access to the data even before transmission. In the most basic setting of PIR, which was first introduced by Chor et al. [1], there are $N$ identical and non-communicating databases (servers) that store $M$ messages. A user, who is interested in a single message yet wishes to keep the servers ignorant about the identity of that message, generates a series of queries to the servers, which answer them truthfully. His goal is to minimize the overhead needed to attain privacy. The problem of PIR was considered by the computer science community extensively, e.g. [2]–[4]. Recently, the problem was considered also by the Information Theory community, which gave it a slightly different interpretation, in the effort to characterize the fundamental limits of the problem. Specifically, in the classic PIR problem, the performance metric, referred to as “communication complexity”, is the sum of the total upload cost (the size of the queries) and the total download cost (the size of the servers’ answers). In the information theoretic formulation, the size of the messages is assumed to be arbitrarily large and thus one may neglect the upload cost. The performance metric in this case is the rate of the PIR scheme, defined as the ratio between the size of the desired message and the total download$^1$ arriving to the user $^5$.

Under such a formulation, the PIR capacity, which is the supremum of PIR rates over all achievable retrieval schemes, was presented in [5] for the classical PIR problem. Specifically, [5] showed that the PIR capacity is $C_{PIR} = (1 + \frac{1}{N} + \frac{1}{N^2} + \ldots + \frac{1}{N^{M-1}})^{-1} = (1 - \frac{1}{N}) / \left(1 - \left(\frac{1}{N}\right)^M\right)$ and provided an achievable retrieval scheme that achieves it.

Naturally, many extensions for the PIR problem were considered. For example, robust PIR with colluding servers was considered in [6], [7] where some of the servers may exchange the queries submitted by the user between them. An extension for Byzantine servers, which respond with erroneous answers, where the errors may be unintentional or even deliberative can be found in [8]. In [9], the case of symmetric PIR was investigated, where the user learns nothing on the other unwanted messages. In [10], the minimum download cost for arbitrary message size $L$ was investigated. PIR with side information was examined in [11], where an additional prefetching phase to the user cache is possible. This phase enables the servers to have partial knowledge on the side information the user has. The above mentioned works assume that the content on the servers is the same (i.e., a repetition code), which on one hand provides the highest resistance against errors but on the other requires extremely large storage cost. Thus, recent works also considered the PIR problem for coded servers, which offers the same amount of data reliability with overall less storage cost [12]–[15]. Interestingly, the capacity of the PIR for coded servers was found to be a function of the coding rate $R_{code}$ and the number of messages $M$ [15]. Specifically, $C_{PIR} = \left(1 - R_{code}\right)/\left(1 - (R_{code})^M\right)$, where one can observe that the case of repetition coding, i.e. $R_{code} = \frac{1}{N}$, assumed in [5] comes as a special case.

In both the classical PIR problem, as well as the extensions mentioned above, it is assumed that the servers answer the user through noiseless orthogonal channels (bit-pipes), which means that the user receives $N$ separate responses, from which it needs to decode the desired message. However, in many practical scenarios, the communication channel endures some kind of noise. For example, random packets are being dropped due to congestion or may be corrupted in some way (e.g., wireless channels). In [16], the PIR problem with noisy orthogonal links was investigated. Therein, the user observes a noisy version of the servers’ responses. [16] provided an upper and a lower bound on the retrieval rate and showed that the channel coding and the retrieval scheme are almost separable. In addition, they considered a variant of the PIR problem for which the responses of the servers are mixed before reaching the user. Such a variant may be represented by the Multiple Access Channel (MAC)$^2$.

$^1$Note that if the communication channel is noisy, the downloaded information contains additional redundancy which must be considered also.

$^2$An example of such a scenario may be when a user is trying to retrieve privately a file from several wireless base-stations.
In the MAC-PIR problem considered in [16], a binary additive MAC and logical conjunction/disjunction MAC were considered. In this case, as opposed to noisy PIR with noisy orthogonal links [16], the channel coding and the retrieval scheme should be designed together. Specifically, the authors provided schemes that can achieve the full channel capacity while still being private. This is done by using the linearity of the MAC and the ability to compute a function of the transmitted servers’ responses. The capacity for the binary additive MAC model and the limits, in general, for computations over MAC were given in [17]. Thus, [16] further enlightens us regarding the channel’s computational capabilities, now working in favor of privacy in the PIR problem.

In this work, we make another step in this direction, assuming the Gaussian MAC with and without fading; these channel models are the common choices for wireless networks. We provide a joint coding scheme based on lattice codes which exploits the additive nature of the channel. We show that such a joint scheme outperforms separation as the number of servers grows and as the received SNR at the user grows. Finally, to assess the tightness of our results we show that such a joint scheme outperforms separation as the number of servers grows and as the received SNR at the user grows.

II. SYSTEM MODEL AND PROBLEM STATEMENT

A. Notational Conventions

Throughout the paper, we will use boldface lowercase to refer to vectors, e.g., \( \mathbf{h} \in \mathbb{R}^L \), and boldface uppercase to refer to matrices, e.g., \( \mathbf{H} \in \mathbb{R}^{M \times L} \). For a vector \( \mathbf{h} \), we write \( \| \mathbf{h} \| \) for its Euclidean norm, i.e., \( \| \mathbf{h} \| = \sqrt{\sum_i h_i^2} \). We note \( \mathbf{e}_i \) as the unit vector with 1 at the \( i \)th entry and zero elsewhere. We assume that the log operation is with respect to base 2.

B. System Model

Consider the basic setting of the PIR problem with \( N \) identical and non-communicating servers. Each server stores a set of messages \( W_1^M = \{W_1, W_2, ..., W_M\} \) of size \( L \). The messages were drawn uniformly and independently from \( \mathbb{F}_p^L \) where \( p \) is assumed to be prime\(^\text{3} \). i.e., in bit units we have,

\[
H(W_i) = L \log p \quad \text{for} \quad i = 1, ..., M,
\]

\[
H(W_1^M) = ML \log p.
\]

In PIR, a user wishes to privately retrieve the message \( W_i \) while keeping the index \( i \) secret from each server. Accordingly, he generates a set of \( N \) queries \( Q_1(i), Q_2(i), ..., Q_N(i) \), one for each server, which are statistically independent with the messages (as those are not known to him). That is, we have

\[
I(W_1^M; Q_1(i), ..., Q_N(i)) = 0.
\]

The \( k \)th server responds to its query with a massage (or codeword) \( x_k(i) \) of size \( n \). This answer is a deterministic function of the messages and the query. That is, for all \( k \) we have,

\[
H(x_k(i)|W_1^M, Q_k(i)) = 0.
\]

Also, to ensure privacy, the queries should not reveal the desired index \( i \) to the servers. Consequently, this implies that for each server \( j \) the index \( i \) of the desired message is independent of the query and the answer, that is, the privacy constraint is,

\[
I(i; Q_j(i), x_j(i), W_1^M) = 0 \quad \text{for all} \quad j \in \{1, ..., N\}.
\]

We assume the servers receive the queries through independent control channels, and do not have access to each other’s queries or answers.

In this work, we consider the problem of PIR over the Gaussian MAC and the block-fading Gaussian MAC as depicted in Figure 1. Accordingly, the user observes a noisy linear combination of the transmitted signals from the servers through the channel,

\[
y(i) = \sum_{k=1}^{N} h_k x_k(i) + z
\]

where \( h_k \sim \mathcal{N}(0, 1) \) are the real channel coefficients and \( z \) is an i.i.d., Gaussian noise, \( z \sim \mathcal{N}(0, I^{N \times n}) \). Note that the index \( i \) in the received input denotes the private index and not time. Let \( \mathbf{h} = (h_1, h_2, ..., h_N)^T \) denote the vector of channel’s coefficients. We assume a memoryless block-fading channel model, i.e., the channel remains constant during the period of codeword transmission of size \( n \); we assume that in each slot the user knows the channel vector while the servers do not have this information.\(^\text{3} \) When we examine the Gaussian MAC without fading, we fix \( h_k = 1 \) for \( k = 1, ..., N \). In addition, we assume an average power constraint on the codewords, i.e., \( \|x_k\|^2 \leq nP \).\(^\text{4} \)

\(^{3}\) The assumption that the messages’ alphabet is a prime-size finite field generalizes the assumption of binary messages of many PIR works, e.g., [5], [6], [10]. In this work it is a requirement since nested lattice codes are used in the suggested PIR scheme. The construction of such codes require that the original messages alphabet size \( p \) is large.

\(^{4}\) We note that the user actually asks the content of the message located in the \( i \)th place at the server. That is, we assume that there are \( p^L \) possible messages and the number \( M \) of messages each server holds may be much smaller.

\(^{5}\) Our security guarantee does not depend on this assumption at this stage.
Upon receiving the mixed response $y(i)$ from all the servers, the user must be able to decode the required message $W_i$. Let $\hat{W}_i$ denote the decoded message at the user and define the error probability of decoding a message as

$$P_e(L) \triangleq P(\hat{W}_i \neq W_i).$$

(4)

We require that $P_e(L) \to 0$ as $L$ goes to infinity.

C. Performance Metric

In the noiseless, orthogonal case, the PIR (or retrieval) rate (or capacity) [5, 15] is defined by the total desired bits divided by the total received bits. Specifically,

$$R \triangleq \frac{H(W_i)}{\sum_{k=1}^{N} H(x_k(i))} \triangleq \frac{L \log p}{D},$$

(5)

where $D$ is total bits downloaded from the server’s answers. Accordingly, the above retrieval rate definition describes only the coding rate (or redundancy) which is needed to keep the message private.

When assuming a noisy channel with a certain channel capacity, the servers’ answers should also be resilient to the channel’s errors and the PIR rate should take into account also the redundancy of the channel coding. When the channel coding and the PIR schemes are designed separately such a metric is easy to acquire. In fact, as mentioned in the introduction, the issue of separation between the PIR and the channel coding schemes was addressed in [16] for the case of asymmetric noisy orthogonal channels (i.e., different capacities for each channel) for which the authors showed that the two coding schemes are almost separable and thus applying the capacity-achieving channel code is optimal. As a special case, when all channels are symmetric with a certain channel capacity $C$, each server encodes its $d$ symbols into a $n$-length codeword and sends it to the user. Thus, the PIR rate which includes the noisy channel rate is,

$$R = \frac{L \log p}{Nn} = \frac{L}{Nd}C = \frac{L}{D} C \log p \left[ \frac{\text{Bits}}{\text{Channel use}} \right],$$

where $C = \frac{n}{d} \log p$. Essentially, the above represents the rate one can achieve when there is separation between the PIR and the channel coding schemes.

In this work, we show that for the AWGN MAC one can attain better performance when the PIR scheme and the channel coding are designed together. We provide a joint privacy-channel coding scheme that uses the additive nature of the channel in the design of the queries and answers of the PIR scheme. The achievable rate of this scheme is shown to be up to a constant gap from the full channel capacity without any privacy constraint at all. Furthermore, this rate is asymptotically optimal as the SNR grows.

D. Coding scheme and lattice codes

The seminal work of Erez and Zamir [18] showed that using lattice encoding and decoding, the full capacity of the point to point AWGN channel is achievable. Following this work, several papers, e.g., [17], [19]–[28], considered different channel models with Gaussian noise, all using lattice codes and their structural properties. The most prominent property is the fact that every linear combination of codewords is a codeword itself. We now provide a brief background on lattice codes.

1) Nested Lattice code: An $n$-dimensional lattice $\Lambda$ is a discrete subgroup of the Euclidean space $\mathbb{R}^n$ with the ordinary vector edition operation. This implies that if $\lambda_1, \lambda_2 \in \Lambda$ then $\lambda_1 + \lambda_2 \in \Lambda$. A lattice quantizer is a map $Q_{\Lambda} : \mathbb{R}^n \rightarrow \Lambda$, that sends a point $x$ to the nearest lattice point in Euclidean distance, i.e., $Q_{\Lambda}(x) = \arg \min_{\lambda \in \Lambda} \|x - \lambda\|$. The Voronoi region of $\lambda$, denoted by $V$, is the set of all points in $\mathbb{R}^n$ which are quantized to the zero vector, i.e., $V(\Lambda) = \{x : Q_{\Lambda}(x) = 0\}$. The modulo-$\Lambda$ operation is defined as the quantization error of $x \in \mathbb{R}^n$ with respect to the lattice $\Lambda$, i.e., $x \mod \Lambda = x - Q_{\Lambda}(x)$. The second moment of a lattice $\Lambda$ is defined as

$$\sigma^2_{\Lambda} = \frac{1}{nV(V)} \int_{V(\Lambda)} \|x\|^2 dx,$$

(6)

where $V(V)$ is the volume of the Voronoi region. The normalized second moment of the lattice, is then given by

$$G(\Lambda) \triangleq \frac{\sigma^2_{\Lambda}}{V(\Lambda)^{2/n}}.$$

(7)

Lattice codes are the Euclidean space counterpart of linear codes which provide structure to the codebook. Thus, similar to linear codes, a message $W_n$ with length $L$ is encoded to a codeword with length $n$ using a one-to-one function where, in our case, this codeword is a lattice point. The structure of the lattice (i.e. the positions of the points) and the bounding region, which forms the codebook itself, rule the “goodness” of it to facilitate as a codebook and achieve the limits of the communication channel.

A nested lattice code is a lattice code which its bounding region is the Voronoi region of a sub-lattice. Formally, let $\Lambda_c$ and $\Lambda_f$ be a pair of $n$-dimensional lattices with Voronoi regions $V_c$ and $V_f$ respectively such that $\Lambda_c$ is a subset of $\Lambda_f$, i.e., $\Lambda_c \subset \Lambda_f$. The nested lattice code is thus given by, $\mathcal{C} = \{\Lambda_f \cap V_c\}$, and its rate is equal to,

$$R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log |\Lambda_f \cap V_c| = \frac{1}{n} \log |2^L| = \frac{L}{n}.$$

(8)

Note that this property is true for every linear code and lattice codes are also a linear code.

A lattice is an unbounded set of points. Thus, exploiting lattices for communication problems requires the bounding of the infinite lattice with a finite shaping region, in order to construct a codebook. In [29], it was shown that there is a simple construction for a sequence of lattice codes which achieve the capacity of the AWGN channel. The construction is based on lifting different sub-codes of a linear code to the Euclidean space using Construction A [30] to form a nested lattice code. Additional information on lattices can be found in [30].
A. An Achievable PIR scheme by separation

The PIR scheme presented in [5] requires $N$ noiseless orthogonal channels between the servers and the user. Thus, employ the CF coding scheme with respect to the coefficient vectors which the decoder chooses. We thus assume in this work that $P$ is chosen such that the probability for such a scenario happens is very small and we do not engage it in our analysis.

Remark 2 (Computation of several equations): Since the user is free to choose the coefficient vector, $a$, as he wishes (under the restriction in (13)); he can decode several linear combinations with respect to the chosen coefficient vectors $a_1, a_2, \ldots$ from the same transmission with the expense of reducing the achievable rate. That is, the messages’ rates must comply to the lowest computation rate with respect to $a_1, a_2, \ldots$ . Using this technique the user can acquire enough independent linear combination to retrieve the transmitted message. This technique was shown to achieve the sum capacity of the K-user Gaussian MAC up to a certain gap [25] and was later shown to achieve the entire MAC capacity for the 2-user MAC under specific SNR requirements [31].

Remark 3 (AWGN MAC as a special case): We note that the CF coding scheme can be used also in the AWGN MAC model with no fading, where the messages do not endure any attenuation factors. That is, the messages are aligned together (in a trivial linear combination) at the user and we can use the computation rate region defined in (12) to determine the achievable rates.

III. PIR FOR AWGN MAC

In this section, we present a retrieval scheme for the AWGN MAC channel without fading. That is, as mentioned in Section II-B we assume that $h_k = 1$ for $k = 1, \ldots, N$ resulting with the following received signal at the user,

$$y = \sum_{k=1}^{t} x_k + z.$$  

(14)

Note that $t$ is the total number of servers that receive a query and transmitted an answer, as the retrieval scheme may not need all servers. For example, in [19], the capacity of the MAC-PIR was achieved by only using 2 servers out of the possible $N$. This is since in the additive modulo-2 MAC, the transmissions of the servers result in a single bit which then can be flipped with probability $q$. Thus, the additive modulo-2 MAC is equivalent to a Point to Point (P2P) Binary Symmetric Channel (BSC(q)) with capacity $1 - H(q)$ between a single server and the user. This essentially means that the PIR rate in [19] Theorem 3] is optimal since the capacity of the P2P channel and the modulo-2 MAC channel are equal and therefore the privacy is attained for “free”. However, this is not the case in the AWGN MAC, as will be shown below. In fact, the more servers used, the higher is the PIR rate that can be attained. We start our analysis by providing a PIR achievability result using separation between the PIR scheme and the channel coding scheme. This achievable PIR rate will constitute a lower bound in later comparison.

2) Compute-and-Forward: According to our channel model the received answers at the user are attenuated by real (and not integer) attenuations. In [22], the Compute and Forward (CF) coding scheme, which enables receivers to decode “noisy” linear combinations of transmitted messages, was introduced. Specifically, the receiver of the non-integer linear combination seeks a set of integer coefficients, denoted by a vector $a$, to be as close as possible to the true channel coefficients and to serve as the coefficients for the linear combination of the received messages.

The CF scheme uses nested lattice codes for the computation of the linear equation of the transmitted messages. That is, after receiving the noisy linear combination, the user selects a scale coefficient $\alpha \in \mathbb{R}$, an integer coefficient vector $a = (a_1, a_2, \ldots, a_N)^T \in \mathbb{Z}^N$, and attempts to decode the lattice point $\sum_{k=1}^{N} a_k x_k$ from $\alpha y$. Formally, the decoder has

$$\alpha y = \sum_{k=1}^{N} \alpha h_k x_k + \alpha z$$

$$= \sum_{k=1}^{N} a_k x_k + \sum_{k=1}^{N} (\alpha h_k - a_k) x_k + \alpha z.$$  

(9)

Due to the lattice algebraic structure, the relay decodes $\sum_{k=1}^{L} a_k x_k$ as a codeword while enduring the noise of $\sum_{k=1}^{N} (\alpha h_k - a_k) x_k + \alpha z$, namely, the effective noise. The rate of the decoded codeword, i.e., the achievable rate, defines a rate region for which all servers must comply with to correctly decode the linear combination. The achievable rate and the optimal scale coefficient are given in the following theorems.

Theorem 1 ([22] Theorem 1): For real-valued AWGN networks with channel coefficient vectors $h \in \mathbb{R}^N$ and coefficient vector $a \in \mathbb{Z}^N$, the following computation rate region is achievable:

$$R(h, a) = \max_{\alpha \in \mathbb{R}} \frac{1}{2} \log^+ \left( \frac{P}{\alpha^2 + P\|\alpha h - a\|^2} \right),$$  

(10)

where $\log^+(x) \triangleq \max \{\log(x), 0\}$.

Theorem 2 ([22] Theorem 2): The computation rate given in Theorem 1 is uniquely maximized by choosing $\alpha$ to be the MMSE coefficient

$$\alpha_{MMSE} = \frac{P h^T a}{1 + P\|h\|^2},$$  

(11)

which results in a computation rate region of

$$R(h, a) = \frac{1}{2} \log^+ \left( \frac{1 + P\|h\|^2}{\|a\|^2 + P (\|a\|^2 - \|h\|^2 - (h^T a)^2)} \right).$$  

(12)

Note that the above theorems are for real channels and the rate expressions for the complex channel are twice the above ([22] Theorems 3 and 4). In addition, one should note that the coefficient vector $a$ must satisfy,

$$\|a\|^2 \leq 1 + P\|h\|^2,$$  

(13)

so that computation rate in (12) would not be zero ([22] Lemma 1).

Remark 1 (The value of $P$): We note that the restriction in (13) force a minimal value for the transmission power to
using a MAC capacity-achieving code, with which each server encodes his \( d \) symbols, one virtually creates such a setting. Specifically, each server transmits a codeword of length \( n \) (channel uses), the user receives the mixed noisy signal and decodes each server’s answer from it. The PIR rate is thus given by,

\[
R = \frac{L \log p}{n} = \frac{L}{N_d} C_{SR} \log p = C_{PIR} \cdot C_{SR}
\]

\[
= \left(1 - \frac{1}{N}\right) \cdot \frac{1}{2} \log (1 + NP) \left[ \frac{\text{Bits}}{\text{Channel use}} \right],
\]

removing that \( C_{SR} = \frac{1}{2} \log (1 + NP) \) is the sum-capacity of the AWGN MAC and that \( \frac{N_d}{\log p} = D \). We note that due to the additive channel, the rate is measured by the total received bits per channel use at the user, regardless of the fact that each server transmits \( n \) symbols individually. This measure is the acceptable metric for such channels since the number of channel uses (bandwidth) is the resource usually being allocated to a system, hence if multiple servers use the same resource it is natural to count them as one. Note, however, that this depends on the fact that each server has its own, non-transferrable, power constraint. This separation scheme is always achievable, and will constitute a lower bound on the retrieval rate that can be achieved under this model.

We now turn to joint schemes, which outperform separation-based schemes. We start with 2 servers, i.e., \( N=2 \). We then extend the scheme and results to the general case of arbitrary \( N \).

B. Joint PIR scheme non-fading AWGN - MAC, \( N = 2 \)

The following theorem present an achievable retrieval rate for the AWGN MAC.

**Theorem 3:** For the 2 servers AWGN MAC, the following PIR rate is achievable,

\[
R = \frac{1}{2} \log \left( \frac{1}{2} + P \right).
\]

The proof of Theorem 3 which is given below provides a simple and basic scheme for the PIR problem for the 2-servers Gaussian MAC which exploits the additive nature of the channel. Under this scheme, the servers perform a simple task of computation and the user only performs single-user decoding. This is opposed to the separated solution for this problem (5) where the user needs to send complex structured queries and jointly decode all answers. In addition, we would like to point out that the PIR achievable rate in this scheme does not depend on the number of messages \( M \) (a similar observation was made also in [15] for the MAC-PIR).

To gain additional intuition on the above result, we compare it to the separation achievable rate in (15) which transforms the MAC into “bit-pipes” such that the user can decode the servers’ answers separately (i.e., the user disregards the ability of the channel to compute the sum of the servers’ answers) and performs the sum by himself.

The achievable PIR rate for this separation scheme is the sum of any pair of symmetric rates located on the line \( R_1 = R_2 \) inside the MAC capacity region times \( C_{PIR} \) (see Figure 2). This is due to the fact that both servers’ answers are in \( \mathbb{F}_2^L \) with equal number of messages, thus they should have equal rates and since we are restricted to the capacity of the PIR scheme. On the other hand, using the retrieval scheme suggested in Theorem 3, where the user decodes only a function of these answers, the achievable PIR rate can be considered as a point outside this capacity region, on the same symmetric line.

The above comparison shows that for low SNR the separation scheme performs better than the lattice based retrieval scheme. Moreover, considering the restriction in (13), when \( P < 1/2 \) only the separation scheme can achieve a non-zero retrieval rate. Accordingly, we have the following corollary.

**Corollary 1:** For the 2 servers AWGN MAC the following PIR rate is achievable,

\[
R = \max \left\{ C_{PIR} \cdot \frac{1}{2} \log (1 + 2P), \frac{1}{2} \log \left( \frac{1}{2} + P \right) \right\}. \tag{17}
\]

The proof follows immediately if the servers and the users are allowed to choose the coding scheme according to the SNR regime. In Figure 2 we illustrated the above for the symmetric MAC capacity region (assuming \( P > 3.5 \)). Furthermore, we note that in the regime of high SNR, the lattice based retrieval rate given in Theorem 3 is approaching \( \frac{1}{2} \log (1 + P) \), which is the capacity of a P2P AWGN channel.

**Proof of Theorem 3:** The user, which is interested in the message \( W_i \), generates a random vector \( \mathbf{b} \) of length \( M \) such that each entry is either 0 or 1 with equal probability. In addition, the user generates a random variable \( s \) which is either 1 or \(-1\) with equal probability. Then, the user sends the following queries to the two servers,

\[
Q_1(i) = s \mathbf{b}, \quad Q_2(i) = -s(\mathbf{b} \oplus \mathbf{e}_i), \tag{18}
\]

where \( \oplus \) is the XOR operation, i.e., the \( i \)th entry of \( \mathbf{b} \) is flipped.
From the perspective of the servers, each sees a unique random vector with element being 0 or non-zero with equal probability. Thus, the privacy of the index \(i\) is guaranteed. Specifically, following the privacy requirement in [22] for the \(j\)th server we have,

\[
I(i; Q_j(i), x_j(i), W_1^M) = I(i; Q_j(i)) + I(i; x_j(i), W_1^M|Q_j(i))
\]

\[
\geq I(i; Q_j(i)) = H(Q_j(i)) - H(Q_j(i)|i)
\]

\[
\geq M - H(Q_j(i)|i)
\]

\[
\geq M - M = 0,
\]

where (a) follows from \(i \leftrightarrow Q_j(i) \leftrightarrow x_j(i), W_1^M\) which makes \(I(i; x_j(i), W_1^M|Q_j(i)) = 0\). (b) follows since for \(j = 1\), the query \(Q_1(i)\) is an i.i.d. \((\frac{1}{2}, \frac{1}{2})\) random vector \(s_b\), which has entropy equal to \(M\); for \(j = 2\), the distribution of the query \(Q_2(i)\) remains the same, since only the \(i\)th entry of \(b\) is affected, and its value remains independent of the other \(M - 1\) values, with a distribution which is still \((\frac{1}{2}, \frac{1}{2})\). (c) is since for \(j = 1\) the server observes \(s_b\) without a change, while for \(j = 2\), knowing the index \(i\) does not affect the probability of receiving a specific realization of \(Q_2(i)\), which are all equiprobable with probability \(2^{-M}\).

Upon receiving the queries, the servers perform modulo-\(p\) addition between the messages which got a non-zero in their corresponding entry in \(Q_j(i)\) and form their answers. Specifically,

\[
A_1 = \sum_{m=1}^{M} s_b m W_m \mod p,
\]

\[
A_2 = -\sum_{m=1}^{M} s(b_m + \delta_{m=1}) W_m \mod p,
\]

where \(\delta\) is the Kronecker delta. Note that, \(A_1 + A_2 = s W_i\).

The servers encode their answers \(A_1\) and \(A_2\) using the Compute and Forward (CF) coding scheme presented in [22] which uses nested lattice codebooks. Furthermore, the code structure and the decoder are similar to [22]. Specifically, let \(\Lambda_c\) and \(\Lambda_f\) be a pair of \(n\)-dimensional lattices with Voronoi regions \(\mathcal{V}_c\) and \(\mathcal{V}_f\), respectively, such that \(\Lambda_c\) is a subset of \(\Lambda_f\), i.e., \(\Lambda_c \subset \Lambda_f\). The coarse lattice \(\Lambda_c\) is used as a shaping region which is scaled to suit the power constraint \(P\) and the lattice points from the fine lattice \(\Lambda_f\) contained within \(\mathcal{V}_c\) of \(\Lambda_c\) are used as the codewords. That is, the nested lattice code is given by, \(\mathcal{C} = (\Lambda_f \cap \mathcal{V}_c)\), and its rate is equal to [30].

\[
R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log |\Lambda_f \cap \mathcal{V}_c|.
\]

(20)

Accordingly, each server is equipped with a CF encoder, \(\mathcal{E} : \mathbb{F}_p^L \rightarrow \mathbb{R}^n\), that maps length-\(L\) messages over the finite field to length-\(n\) real-valued codewords, \(x_j = \mathcal{E}(A_j)\). Thus, during the encoding process the answer \(A_j\) is first mapped into a lattice codeword \(v_j\) in \(\mathcal{C}\) then it is dithered to attain \(x_j\) which is uniformly distributed over \(\mathcal{V}_c\). The servers encode their answers using the same lattice codebook and send them on the MAC channel. The received input at the user is thus,

\[
y = x_1 + x_2 + z.
\]

(21)

From the noisy sum \(x_1 + x_2 + z\) the user decodes the sum of the two lattice codewords \(v = v_1 + v_2\) instead of decoding each codeword separately. The decoding of the received noisy sum to \(v\) is successful with probability of error that goes to zero with \(n\) if the lattice rate \(R\), i.e., the transmission rate of each server satisfies

\[
R \leq \frac{1}{2} \log^+ \left( \frac{1}{2} + P \right).
\]

(22)

The above rate was given in [22] Theorem 5, with the channel gains being \((1, 1)\), the coefficient vector \(a\) set to be the all 1 vector and \(\alpha = \frac{1}{2P}\), where \(\alpha\) is a scaling factor used at the decoder. We note that this rate is the rate of the decoded linear combination. Upon successful decoding, the lattice codeword \(v\) is mapped to \(s W_i\) since \(A_1 + A_2 = s W_i\) and since the linear lattice code preserves the linear operations between the codewords and their corresponding messages [22] Lemma 6. Accordingly, the user can retrieve \(W_i\) as he knows the value of \(s\).

The retrieval rate in [16], which is essentially the rate at which one can decode the sum of two lattice codewords, appeared also under different contexts in [20], [21]. Yet, the CF scheme offers a generalization for the computation of any linear combination. We note that the rate in [16] is the best known achievable rate for such a sum computation and although the resulting equivalent channel in (21) is the AWGN channel, i.e., \(y = x^* + z\), where \(x^* = x_1 + x_2\), we cannot achieve a rate of \(\frac{1}{2} \log (1 + P)\) (for further reading see [30]).

C. Joint PIR scheme non-fading AWGN - MAC, arbitrary \(N\)

We now provide an achievable PIR scheme for a general system with \(N\) servers. We show that the retrieval rate is an increasing function of \(N\), similar to the sum-rate of the general Gaussian MAC capacity. The scheme uses the principles of Theorem [3] and its rate is given in the following corollary.

Corollary 2: For the \(N\) servers AWGN MAC, the following PIR rate is achievable,

\[
R = \frac{1}{2} \log \left( \frac{1}{2} + \left( \frac{N}{2} \right)^2 P \right).
\]

(23)

Proof: The retrieval scheme for the general system follows the same steps as the proof of Theorem [3] yet the servers are divided into pairs and are treated as if every pair is a 2-servers Gaussian MAC. Specifically, the user generates a random vector \(b\) and a random variable \(s\) as in the proof of Theorem [3] and sends the following queries to the \(l\)th pair,

\[
Q_l(i) = s_b, \quad Q_{l+1}(i) = -s(b \oplus e_l),
\]

where \(l \in \{1, 3, 5...\}\). In case \(N\) is odd the \(N\)th server is ignored. This extension for the \(N\) servers model does not impair the privacy requirement [2]. This is because from the perspective of the \(j\)th server, it does not matter how many servers are transmitting. The servers form their answers as in
D. Gap from channel capacity

The results above show that privacy comes with a price of rate reduction since one must utilize the servers to provide privacy rather than increasing the transmission rate. That is, privacy does not come for free. To quantify the loss, let us assume that we are not restricted by privacy and compute the capacity in such a model, for sending a single message. For that purpose, let us refine the subtleties in the model assumptions with respect to the original PIR problem. We assume that the servers cannot cooperate, yet cooperation is possible by exploiting the user’s queries. In addition, we assume that all transmitting servers transmit with power $P$, and there is no power allocation between them. That is, we have per-server power constraint with full cooperation. Accordingly, our model matches the Multiple Input Single Output (MISO) channel with per-antenna power constraint, where each transmit antenna has a separate power budget and can fully cooperate with each other.\footnote{For complex channels the pre-log factor is 1 and the sum is on the absolute value of $h_k$.} The MISO sum capacity with per-antenna power constraint is given by

$$C_{SR}^{MISO} = \frac{1}{2} \log \left( 1 + P \sum_{k=1}^{N} h_k^2 \right).$$  \hspace{1em} (25)$$

We note that with per-antenna power constraint we have higher sum capacity than the MAC sum capacity, since in the latter each transmitter acts independently to transmit its own message.

For the AWGN with no fading the MISO sum capacity reduces to $\frac{1}{2} \log (1 + N^2P)$ which is the maximal rate of transmitting a single message with $N$ servers. Thus, the PIR rate for the $N$ servers AWGN MAC given in Corollary \ref{corollary:PIR_MAC} has a finite gap from channel capacity. Namely,

$$C_{SR}^{MISO} - R = \frac{1}{2} \log \left( 1 + N^2P \right) - \frac{1}{2} \log \left( \frac{1}{2} + \left( \frac{N}{2} \right)^2 P \right) \leq \frac{1}{2} \log \left( \frac{1}{2} + \left( \frac{N}{2} \right)^2 P \right) \leq \frac{1}{2} \log \left( \frac{1}{2} + \left( \frac{N-1}{2} \right)^2 P \right) = \frac{1}{2} \log \left( \frac{1 + N^2P}{2 + (N-1)^2 P} \right) \leq 2,$$

where the last line follows since $\frac{1 + N^2P}{2 + (N-1)^2 P} < 4$ for $N \geq 2$ for all $P$. Moreover, we note that in the limit of $P \to \infty$ and a fixed $N$, the PIR rate is asymptotically optimal as it achieves the capacity.

IV. JOINT PIR SCHEME FADING AWGN - MAC

When the channel suffers from fading, assuming a 2-server model, the transmitted codewords do not align trivially to form...
the integer linear combination $x_1 + x_2$. Yet, the CF coding scheme used in Theorem 3 still enables the user to decode the sum $x_1 + x_2$ from the noisy version of $h_1x_1 + h_2x_2$ as if no fading existed, at the price of a possible rate loss. This is done by approximating the channel vector $h$ by an integer coefficient vector $a$. In our case, this coefficient vector will belong to the set, $a = \{(1, 1), (-1, 1), (1, -1), (-1, -1)\}$, depending on the channel coefficients’ signs which are known to the user. That is, we force the decoding of a sum with the specific coefficients, although they might be far from the true $h_1, h_2$.

The following corollary provides an achievable PIR rate for the case of 2 servers.

**Corollary 3:** For the 2 servers block-fading AWGN MAC, the following PIR rate is achievable,

$$R = \frac{1}{2} \log \left( \frac{1 + P||h||^2}{2 + P(|h_1| - |h_2|)^2} \right).$$

(27)

The proof is identical to the proof of Theorem 3 except now, in the rate expression, which is given in previous Theorem 5, we set the corresponding $h$ and the $a \in a^*$ that matches the signs of $h$. Note that if there is no fading, i.e., $h = (1, 1)$ the rate in (27) reduces to (16).

As a direct consequence of the PIR scheme suggested in Corollary 2 along with the rate in Corollary 3 we provide the following achievable rate for the general case of $N$ server system.

**Corollary 4:** For the $N$ servers block-fading AWGN MAC, the following PIR rate is achievable,

$$R = \frac{1}{2} \log \left( \frac{1 + P||\tilde{h}||^2}{2 + P(|\tilde{h}_1| - |\tilde{h}_2|)^2} \right).$$

(28)

where, $	ilde{h} = (\sum_{i \in S_1} h_i, \sum_{i \in S_2} h_i)$ such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 \subseteq \{1, ..., N\}$ and $|S_1| = |S_2| = [N/2]$.

**Proof:** The PIR scheme is similar to the scheme suggested in the proof of Corollary 2 however, now the user divides the servers into 2 sets of size $[N/2]$, denoted as $S_1$ and $S_2$, for which he sends the queries $Q_1(i)$ and $Q_2(i)$, as in (18), respectively. Hence, the received input at the user is thus,

$$y = \sum_{i \in S_1} h_1x_1 + \sum_{i \in S_2} h_2x_2 + z = \tilde{h}_1x_1 + \tilde{h}_2x_2 + z.$$

Applying Corollary 3 provides the achievable rate. 

The user may choose $S_1$ and $S_2$ to maximize the achievable rate in (28). To assess the results, we provide simulation results in Figure 4 for the average rate as a function of the SNR for different values of $N$. The PIR rate (dashed line) is compared to the MISO capacity (solid lines) as given in Equations (28) and (25), respectively. The simulation was performed with fading coefficients distributed as a standard normal random variable and the sets $S_1$ and $S_2$ were chosen to maximize the rate. Figure 4 depicts that the PIR is an increasing function of the SNR and as the number of servers grows the PIR rate approaches the capacity of the channel. Moreover, one may observe that once the user may optimize the rate according to $S_1$ and $S_2$, significant improvement may be achieved.

For example, for $N = 2$, where such optimization is not possible the performance is far from the capacity. We note that comparison with separation based scheme is an ongoing work. Such a comparison should consider the fact that by using MAC capacity achieving codes, the “virtual” orthogonal channels between the servers and the user are not symmetric due to the fading. For such a scenario the exact description of the $C_{PIR}$ is not known in general and only upper and lower bounds are known [16]. We further note that asymmetric channels are not an issue in our suggested joint PIR scheme.

**APPENDIX A**

**DECODING SUM OF TWO LATTICE CODEWORDS**

We describe the coding scheme and analyze its achievable rate for decoding a sum of 2 lattice codewords for a general system consisting $N$ servers. As described in the proof of Corollary 2 the queries sent by the user were designed so that each pair of servers will transmit the sum $x_1 + x_2$. This coordination between the servers provides an increase for the computation rate since the user knows in advance that he needs to decode a sum of only two codewords which is amplified by a constant instead of a sum of $N$ codewords. This observation is important since the rate for decoding a sum of $N$ codewords, which is given by the general expression of CF in (12), is significantly lower than the rate in the proof of Corollary 2. The reason lies in the number of self noise penalties as will be shown below.

Let $(\Lambda_1, \Lambda_c)$ be a pair of lattices which form a nested lattice code with rate $R$. The coarse lattice $\Lambda_c$ was chosen with second moment $P$ to meet the power constraint. The code is known to the user and all the servers. In addition, let $d_1$ and $d_2$ be two mutually independent dithers which are uniformly distributed over the Voronoi region $V_c$. The dithers are known in advanced also to the servers and the user. The servers are divided into pairs where each server in a certain pair $k$ maps his $L$ bit long answer to a lattice codewords $v_{k,1}$ and $v_{k,2}$ respectively. The transmitted signals by each pair of servers is
where $k \in \{1, 3, 5, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1\}$. Note that since the queries are similar between each pair we get that, $v_{k,1} = v_{1,1}$ and $v_{k,2} = v_{1,2}$. Thus, $[v_{k,1} + v_{k,2}] \mod \Lambda_c = v^*$ for all $k$, where $v^*$ is the lattice codeword for the private message $W_1$. The received input at the user is thus,

$$y = \sum_k (x_{k,1} + x_{k,2}) + z.$$ 

In order to decode $v^*$ the user preforms the following

$$\hat{v} = [\alpha' y + d_1 + d_2] \mod \Lambda_c$$

where $\alpha' = \alpha \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^{-1} = \left( \frac{2P}{\left\lfloor \frac{N}{2} \right\rfloor^2 + 2P} \right) \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^{-1}$. The above reduces to the Modulo-Lattice Additive Noise (MLAN) channel \(\left[18\right]\) as follows,

$$\hat{v} = [\alpha' y + d_1 + d_2] \mod \Lambda_c = [\alpha' \sum_k (x_{k,1} + x_{k,2}) + \alpha' z + d_1 + d_2] \mod \Lambda_c = [\alpha' \sum_k (v_{k,1} - d_1) \mod \Lambda_c + ] \mod \Lambda_c$$

$$\mod \Lambda_c$$

$$\mod \Lambda_c$$

$$\mod \Lambda_c$$

where (a) and (c) follows from the distributive property of the mod $\Lambda_c$ operation. In (b), the sum of codewords of each pair equals to $v^*$ and in (d) we replace the term in the right parenthesis with an equivalent term sent by the first pair of servers. Lastly we define the equivalent noise term $z_{eq} \triangleq \alpha' z - (1 - \alpha') (x_{1,1} + x_{1,2})$. This noise term and $v^*$ are independent of each other since by definition $v^*$ is a fine lattice point in $\Lambda_f \cap \mathcal{V}(\Lambda_c)$. In addition, it is uniformly distributed in $\mathcal{V}(\Lambda_c)$ according to the crypto lemma \(\left[18\right]\). From the same reason, $v^*$ is independent with $x_{1,1}$ and $x_{1,2}$. Finally we note that $z$ is independent from all. Accordingly, the second moment of $z_{eq}$ is given by

$$\sigma_{z_{eq}}^2 = \mathbb{E}[z_{eq}^2] = \alpha^2 \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^{-2} + (1 - \alpha')^2 2P$$

where we can optimize it on $\alpha$. Specifically, let us denote $\sigma_{z^*}^2 = \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^{-2}$ we get that, $\alpha_{opt} = \frac{2P}{\sigma_{z^*}^2 + 2P}$ and the resulting optimal second moment $\sigma_{eq, opt}^2 = \frac{2P \sigma_{z^*}^2}{2P + \sigma_{z^*}^2}$. The decoding is successful if $Q_{\Lambda_c}(\hat{v}) = v^*$ which will happen with the probability that the effective noise vector is inside the Voronio region $\mathcal{V}(\Lambda_f)$.

Accordingly, we are left to show the coding rate and the existence of appropriately nested lattices, $(\Lambda_f, \Lambda_c)$, so that $v^*$ is decoded correctly with arbitrarily low probability of error. For that manner, we can use \(\left[21\right]\) Theorem 1, which is a modified version of \(\left[18\right]\) Theorem 5, by just setting the channel noise in their result. Specifically, we may write $z_{eq} \triangleq \alpha' z - (1 - \alpha') (x_{1,1} + x_{1,2})$, where $z'$ is a Gaussian noise with variance $\sigma_{z^*}^2 = \left( \left\lfloor \frac{N}{2} \right\rfloor \right)^{-2}$ and set it in the rate term in \(\left[21\right]\) Theorem 1 which provide us our PIR rate,

$$R = \frac{1}{2} \log \left( 1 + \left( \frac{\left\lfloor \frac{N}{2} \right\rfloor^2}{2P} \right) \right).$$

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