General curves on algebraic surfaces

E. SERNESI

Abstract

We give upper bounds on the genus of a curve with general moduli assuming that it can be embedded in a projective nonsingular surface Y so that dim(\(|C|\)) > 0. We find such bounds for all types of surfaces of intermediate Kodaira dimension and, under mild restrictions, for surfaces of general type whose minimal model Z satisfies the Castelnuovo inequality \(K_Z^2 \geq 3\chi(O_Z) - 10\). In this last case we obtain \(g \leq 19\). In the other cases considered the bounds are lower.

Introduction

Notation: \(\overline{M}_g\) is the coarse moduli space of stable curves of genus \(g\), and \(M_g \subset \overline{M}_g\) is the open subset of smooth curves. The corresponding stacks are denoted by \(\overline{M}_g\) and \(M_g\) respectively. We work over \(\mathbb{C}\).

In this paper we study the conditions imposed on \(g\) by assuming that a general projective nonsingular curve \(C\) of genus \(g\) can be embedded in some non-ruled algebraic surface \(Y\) so that \(\dim(|C|) > 0\) on \(Y\). We expect upper bounds on \(g\) depending on the numerical characters of \(Y\), because the assumption made implies that \(\overline{M}_g\) is uniruled. In fact blowing-up the base locus of a linear pencil extracted from \(|C|\) one constructs a non-isotrivial fibration in curves of genus \(g\)

\[ f : X \longrightarrow \mathbb{P}^1 \quad (1) \]

containing \(C\) among its fibres, which defines a non-constant morphism \(\psi_f : \mathbb{P}^1 \longrightarrow \overline{M}_g\) whose image contains \([C]\). Conversely every such fibration determines a pair \(C \subset Y\) as above (in several ways). In general \(f\) is not semistable, and we cannot apply semistable reduction because rationality of the base would be lost after the process. We remedy to this drawback by working directly with an arbitrary non-isotrivial fibration over \(\mathbb{P}^1\) and by studying its deformation theory. The deformation functor of a fibration (1) is controlled by the sheaf \(\text{Ext}_f^1(\Omega_{X/\mathbb{P}^1}^1, O_X)\) on \(\mathbb{P}^1\). A necessary condition for (1) to deform enough to contain a general curve of genus \(g\) as a fibre is that this sheaf is globally generated (Proposition 4.8 and Theorem 4.9), and in this case we call \(f\) a free fibration. We

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prove (Proposition 3.2) that if (1) is free then the following inequality is satisfied:

\[ 11\chi(O_X) - 2K_X^2 \geq 5(g - 1) + 2 \]  

(2)

If we know that the fibration (1) is free and is obtained by blowing-up the base locus of a linear pencil extracted from a linear system \(|C|\) of positive dimension on a surface \(Y\), then the above inequality can be refined as follows:

\[ 10\chi(O_Y) - 2K_Y^2 \geq 4(g - 1) - C^2 - h^0(K_Y - C) \]  

(3)

(Theorem 5.2). This inequality provides a necessary condition for the possibility of embedding a general curve of genus \(g\) in a surface \(Y\) with the given numerical characters so that \(\dim(|C|) > 0\).

In the second part of the paper we apply inequality (3) to give upper bounds on \(g\) depending on the numerical characters of the surface \(Y\), for such an embedding being possible. The case when \(Y\) is rational has been considered classically by B. Segre [19]. His work has been reconsidered and surveyed by Verra in his recent [27]. It gives the bound \(g \leq 10\) for a general curve of genus \(g\) moving in a non-trivial linear system of plane curves, provided the system is regular. It has been conjectured that this condition is always satisfied. We do not consider this case in the present work, referring the reader to [27] for a detailed discussion and to [22] for a general survey.

Recent work by Verra and Bruno-Verra [26, 4], plus experimental evidence in the range \(12 \leq g \leq 15\), seem to indicate that if \(M_g\) is uniruled for some \(g \geq 17\) then a general curve of genus \(g\) should move in a non-trivial linear system on a regular surface of general type. We give evidence for this expectation by proving a series of results. For elliptic surfaces we have:

**Theorem 0.1** Let \(Y\) be a non-ruled and non-rational elliptic surface, \(\pi : Y \to B\) the elliptic fibration onto a nonsingular connected curve \(B\). Assume that there exists a general nonsingular connected curve \(C \subset Y\) of genus \(g \geq 3\) such that \(\dim(|C|) \geq 1\). Then \(B = \mathbb{P}^1\) and \(g \leq 16\).

The following result covers the case of curves on other surfaces of intermediate Kodaira dimension.

**Theorem 0.2** Let \(Y\) be a projective nonsingular surface with \(\kappa = 0\), and let \(C \subset Y\) be a general nonsingular connected curve of genus \(g \geq 3\) moving in a positive dimensional linear system. Then

\[ g \leq 6 + 5p_g(Y) + \frac{1}{2}h^0(K_Y - C) \]

In particular:

\[ g \leq \begin{cases} 6, & \text{if } p_g(Y) = 0 \\ 11, & \text{if } p_g(Y) = 1 \end{cases} \]

This theorem implies in particular the well known bound \(g \leq 11\) for a general nonsingular curve of genus \(g\) on a K3-surface [14], and \(g \leq 6\) for a general nonsingular curve of genus \(g\) on an Enriques surface.

The following result shows that general curves on irregular surfaces of positive geometric genus cannot move in a positive dimensional linear system.
Theorem 0.3 Let $Y$ be a projective nonsingular surface with $p_g > 0$ and $q > 0$, and let $C \subset Y$ be a nonsingular curve of genus $g \geq 3$ such that $\dim(|C|) \geq 1$. Then no pencil $\Lambda \subset |C|$ containing $C$ as a member defines a free fibration.

Theorem 0.2 does not say much when the geometric genus $p_g(Y)$ is large, and therefore it is ineffective for most surfaces of general type. Our main result in this case is the following.

Theorem 0.4 Let $Y$ be a surface of general type and let $Z$ be the minimal model of $Y$. Assume that $K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10$ and that $C \subset Y$ is a general nonsingular connected curve of genus $g \geq 3$. If one of the following holds:

(a) $\dim(|C|) \geq 2$,
(b) $\dim(|C|) = 1$, $h^0(K_Y - C) = 0$ and $C^2 \geq \frac{(g-1)}{2}$,
(c) $\dim(|C|) = 1$, and $h^1(C, \mathcal{O}_C(2C)) = 0$.

then $g \leq 19$.

Recall that $K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10$ is the Castelnuovo inequality and if it is violated then $Z$ is a double cover of a ruled surface $R$. In this case, if $Y$ contains a general curve such that $\dim(|C|) \geq 1$ then $C$ is mapped birationally into $R$ and it follows that $R$ is rational. I have not been able to obtain a meaningful bound for $g$ in this case.

The restrictions (b),(c) in the statement of the theorem might a priori be sharp, and not merely due to the method of proof: as the genus increases, it seems difficult to find a smooth curve with general moduli that moves in a pencil with small self-intersection on a surface.

The paper is divided into sections as follows. In §1 we introduce the terminology and main calculations concerning fibrations. In §2 we develop the deformation theory of fibrations. Free fibrations are introduced in §3. In the next §4 general curves are studied and a characterization of the uniruledness of $\overline{M}_g$ is given. The basic inequality (3) is proved in §5. In §6 we prove Theorems 0.1, 0.2 and 0.3, while in §7 we prove Theorem 0.4.

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1 Fibrations

If

$\varphi : W \to U$

is a morphism of algebraic schemes, and $u \in U$, we denote by $W(u)$ the scheme-theoretic fibre of $\varphi$ over $u$.

A $(-1)$-curve (resp. a $(-2)$-curve) on a projective nonsingular surface $Z$ is a nonsingular connected rational curve with self-intersection $-1$ (resp. $-2$). We will denote by $K_Z$ a canonical divisor on $Z$. By a minimal
surface we mean, as customary, a projective nonsingular connected surface without \((-1)\)-curves.

By a fibration we mean a surjective morphism

\[ f : X \to S \]

with connected fibres from a projective nonsingular surface to a projective nonsingular connected curve. We will always denote by

\[ g = \text{ the genus of the general fibre. } \text{We will always assume } g \geq 2. \]
\[ b = \text{ the genus of } S. \]

A fibration is called:

- relatively minimal if there are no \((-1)\)-curves contained in any of its fibres.
- semistable if it is relatively minimal and every fibre has at most nodes as singularities.
- stable if it is relatively minimal and every fibre is a stable curve (i.e. at most ordinary nodal singularities and with finitely many automorphisms).
- isotrivial if all of its nonsingular fibres are mutually isomorphic; equivalently, if two general nonsingular fibres of \(f\) are mutually isomorphic (the equivalence of the two formulations follows from the separatedness of \(M_g\)).

Sheaves of differentials will be denoted with the symbol \(\Omega^1\) and dualizing sheaves with the symbol \(\omega\). Let \(f : X \to S\) be any fibration. Since both \(X\) and \(S\) are nonsingular we have:

\[ \wedge^2 \Omega^1_X = \omega_X, \quad \Omega^1_S = \omega_S \]

Moreover \(f\) is a relative complete intersection morphism and it follows that

\[ \omega_{X/S} = \omega_X \otimes f^* \omega_S^{-1} \]  
([11], Corollary (24)). We have an exact sequence

\[ 0 \to f^* \omega_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0 \]  

(5)

(which is exact on the left because the first homomorphism is injective on a dense open set and \(f^* \omega_S\) is locally free). If we dualize the sequence (5) we obtain the exact sequence:

\[ 0 \to T_{X/S} \to T_X \to f^* T_S \to N \to 0 \]  

(6)

where we have denoted

\[ T_{X/S} := \text{Hom}_{X/S}(\Omega^1_{X/S}, \mathcal{O}_X) \]

and

\[ N := \text{Ext}^1_{X/S}(\Omega^1_{X/S}, \mathcal{O}_X) \]  

(7)

The sequence (6) shows, in particular, that \(N\) is the normal sheaf of \(f\), and also the first relative cotangent sheaf \(T^1_{X/S}\) of \(f\) (see [21]). In particular, \(N\) is supported on the set of singular points of the fibres of \(f\).

Moreover, as remarked in [23], p. 408, \(T_{X/S}\) is an invertible sheaf because it is a second syzygy of the \(\mathcal{O}_X\)-module \(N\).
Lemma 1.1 If all the singular fibres of $f$ are reduced then

$$T_{X/S} \cong \omega_{X/S}^{-1}$$  \hspace{1cm} (8)

Proof. Since the locus where $f$ is not smooth is finite, $\Omega^1_{X/S}$ and $\omega_{X/S}$ are isomorphic in codimension one. We thus have

$$c_1(\Omega^1_{X/S}) = c_1(\omega_{X/S})$$

and the conclusion follows because $T_{X/S}$ is invertible. $\square$

Remark 1.2 If $f$ has some non-reduced singular fibre then $T_{X/S}$ and $\omega_{X/S}^{-1}$ are not isomorphic. Precisely we have:

$$T_{X/S} = \omega_{X/S}^{-1}(\sum (\nu_i - 1)E_i)$$

where $\{\nu_iE_i\}$ is the set of all components of the singular fibres of $f$, and where $\nu_i$ denote their corresponding multiplicities. This formula is due to Serrano ([23], Lemma 1.1).

The next proposition will not be applied in the sequel:

Proposition 1.3 If $f$ is any fibration we have:

$$\chi(\omega_{X/S}^{-1}) = \chi(\mathcal{O}_X) + K_X^2 - 6(b - 1)(g - 1)$$

If the singular fibres of $f$ are reduced then

$$\chi(T_{X/S}) = \chi(\mathcal{O}_X) + K_X^2 - 6(b - 1)(g - 1)$$

Proof. By Riemann–Roch:

$$\chi(\omega_{X/S}^{-1}) = \chi(\mathcal{O}_X) + \frac{1}{2}[\omega_{X/S} \cdot \omega_{X/S} + \omega_X \cdot \omega_{X/S}]$$

(replacing $\omega_X \otimes f^*\omega_{S}^{-1}$ for $\omega_{X/S}$)

$$= \chi(\mathcal{O}_X) + \frac{1}{2}[K_X^2 - 2(2b - 2)(2g - 2) + K_X^2 - (2b - 2)(2g - 2)]$$

$$= \chi(\mathcal{O}_X) + K_X^2 - 6(b - 1)(g - 1)$$

The last formula is a consequence of Lemma 1.1. $\square$

The following is a classical result of Arakelov in the semistable case, and it is due to Serrano in the general case (recall that we are assuming $g \geq 2$):

Theorem 1.4 If $f$ is a non-isotrivial fibration then

$$h^0(X, T_{X/S}) = h^0(X, T_X) = 0$$  \hspace{1cm} (9)

If moreover $f$ is relatively minimal then we also have:

$$h^1(X, T_{X/S}) = 0$$  \hspace{1cm} (10)
Proof. $f$ is non-isotrivial if and only if $f_*T_X = 0$ ([24], Lemma 3.2). Since $f_*T_{X/S} \subset f_*T_X$ we also have $f_*T_{X/S} = 0$ if $f$ is non-isotrivial. Thus (9) is a consequence of the Leray spectral sequence. For (10) see [1] or [25] in the semistable case, and [23], Corollary 3.6, in the general case.

Denoting by $\text{Ext}^1_f$ the first derived functor of $f_*\text{Hom}$, we are interested in the sheaf $\text{Ext}^1_f(O^1_{X/S}, O_X)$ because its cohomology controls the deformation theory of $f$ (see Lemma 2.1 below). This sheaf is not locally free in general, but it decomposes as follows:

$$\text{Ext}^1_f(O^1_{X/S}, O_X) = \mathcal{E} \oplus \mathcal{T}$$

where $\mathcal{E}$ is locally free and $\mathcal{T}$ is a torsion sheaf. By the rank of $\text{Ext}^1_f(O^1_{X/S}, O_X)$ we will mean the rank of $\mathcal{E}$.

**Lemma 1.5** For any non-isotrivial fibration $f : X \to S$ the sheaf $\text{Ext}^1_f(O^1_{X/S}, O_X)$ has rank $3g - 3$. Moreover there is an exact sequence of sheaves on $S$:

$$0 \to R^1f_*T_{X/S} \xrightarrow{c_{10}} \text{Ext}^1_f(O^1_{X/S}, O_X) \xrightarrow{c_{13}} f_*\text{Ext}^1_X(O^1_{X/S}, O_X) \to 0$$

If $\text{Ext}^1_f(O^1_{X/S}, O_X)$ is locally free and all the fibres of $f$ are reduced then there is an isomorphism

$$\text{Ext}^1_f(O^1_{X/S}, O_X) \cong \text{Hom}(f_*\omega_{X/S}, O_S)$$

Proof. If $p \in S$ is such that $X(p)$ is smooth then $\text{Ext}^1_f(O^1_{X(p)}, O_{X(p)}) = H^1(T_{X(p)})$ has dimension $3g - 3$. Then, if $U \subset S$ is the open set over which $f$ is smooth, $\text{Ext}^1_f(O^1_{X/S}, O_X)|_U$ is locally free of rank $3g - 3$.

(12) is the sequence associated to the local-to-global spectral sequence for $\text{Ext}^1_f$.

Assume that $\text{Ext}^1_f(O^1_{X/S}, O_X)$ is locally free and that all the fibres of $f$ are reduced. Then, since

$$\text{Ext}^0_f(O^1_{X/S}, O_X) = f_*\text{Hom}(O^1_{X/S}, O_X) = f_*T_{X/S} = 0$$

(see the proof of Theorem 1.4) is locally free as well, and the higher $\text{Ext}^1_f$’s vanish, both sheaves commute with base change ([13], Th. 1.4). The reducedness of the fibres implies that $O^1_{X/S}$ is torsion free, hence flat over $S$. Therefore (13) follows from relative duality for the fibration $f$ ([11], Corollary (24)).

**Remark 1.6** The torsion sheaf $\mathcal{T}$ in (11) is non-zero in general, even if all the fibres of the non-isotrivial fibration $f$ are reduced. Assume for example that for some $p \in S$ there is a $(-2)$-curve in the fibre $X(p)$. Then $\text{Hom}(O^1_{X(p)}, O_X(p)) \neq 0$ because $X(p)$ has non-trivial infinitesimal automorphisms. Since $f_*\text{Hom}(O^1_{X/S}, O_X) = f_*T_{X/S} = 0$ this sheaf does not commute with base change and therefore $\text{Ext}^1_f(O^1_{X/S}, O_X)$ cannot be locally free ([13], Th. 1.4). A fortiori there is torsion if the fibration is
Proposition 1.7 If the fibration \( f \) is non-isotrivial then we have:

\[
\chi(\text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) = 11\chi(\mathcal{O}_X) - 2K_X^2 + 2(b-1)(g-1)
\]  

(14)

Proof. Since the fibres of \( f \) are 1-dimensional we have

\[
R^2f_*T_{X/S} = 0
\]

Moreover \( f_*\text{Ext}^2(\Omega^1_{X/S}, \mathcal{O}_X) = 0 \) because \( \text{Ext}^2(\Omega^1_{X/S}, \mathcal{O}_X) = 0 \) by the exact sequence (5). Therefore, using the local-to-global spectral sequence for \( \text{Ext}_f \) we deduce that

\[
R^1f_*N = \text{Ext}^2_f(\Omega^1_{X/S}, \mathcal{O}_X) = 0
\]

where the last equality is true because the fibres of \( f \) are 1-dimensional. This gives:

\[
\chi(f_*N) = \chi(N)
\]

Moreover, since \( f \) is non-isotrivial, from (9) and the Leray spectral sequence we get

\[
\chi(R^2f_*T_{X/S}) = -\chi(T_{X/S})
\]

We now use the exact sequence (12) and we deduce that

\[
\chi(\text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) = \chi(f_*N) + \chi(R^1f_*T_{X/S})
\]

\[
= \chi(N) - \chi(T_{X/S})
\]

\[
= \chi(f^*T_S) - \chi(T_X)
\]

(by (6))

Using Riemann-Roch one computes that:

\[
\chi(f^*(T_S)) = \chi(\mathcal{O}_X) + 2(b-1)(g-1)
\]

\[
\chi(T_X) = 2K_X^2 - 10\chi(\mathcal{O}_X)
\]

and by substitution one gets (14). \( \square \)

Remark 1.8 Assume that the fibration \( f \) is stable. Then with \( f \) there is associated a modular morphism

\[
\psi_f : S \rightarrow \overline{\mathcal{M}}_g
\]

to the moduli stack of stable curves of genus \( g \). We know ([8], p. 49) that:

\[
f_*(\Omega^1_{X/S} \otimes \omega_{X/S}) = \psi_f^*\Omega^1_{\overline{\mathcal{M}}_g}
\]

From (13) it follows that:

\[
\text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) = \psi_f^*T_{\overline{\mathcal{M}}_g}
\]

(15)

7
Therefore the identity (14) can be recovered by applying the Riemann-Roch theorem to the vector bundle $\psi^*f^*T_Mg$ as follows. Recall that

$$c_1(\psi^*f^*K_Mg) = 13\lambda(f) - 2\delta(f) = -5(b-1)(g-1) - [11\chi(O_X) - 2K_X^2]$$  \hspace{1cm} (16)$$

where $\lambda(f) = \chi(O_X) - (b-1)(g-1)$ and $\delta(f) = 12\chi(O_X) - K_X^2 - 4(b-1)(g-1)$.

Then:

$$\chi(\psi^*f^*T_Mg) = -c_1(\psi^*f^*K_Mg) + (1-b)(3g-3) = 11\chi(O_X) - 2K_X^2 + 2(b-1)(g-1)$$

that is (14).

In case the fibration $f$ is not stable we still have a non-empty open set $U \subset S$ above which all fibres of $f$ are stable. Therefore we have an induced morphism $U \to Mg$; since $S$ is a nonsingular curve this morphism extends to a morphism:

$$\overline{\psi}_f : S \to Mg$$

with values in the moduli space. Now we no longer have the interpretation (15), even replacing $Mg$ by $Mg$. This will be the reason for some lengthening in the following discussion of the deformation theory of $f$.

## 2 Deformation theory

**Lemma 2.1** Let $f : X \to S$ be a non-isotrivial fibration. Then there is a natural isomorphism

$$\mu : \text{Ext}^1_X(\Omega^1_{X/S}, O_X) \to H^0(S, \text{Ext}^1_f(\Omega^1_{X/S}, O_X))$$

and both spaces are naturally identified with the tangent space of $\text{Def}_f$, the functor of Artin rings of deformations of $f$ leaving the target fixed (see [21], p. 164). Moreover

$$H^1(S, \text{Ext}^1_f(\Omega^1_{X/S}, O_X))$$

is an obstruction space for $\text{Def}_f$.

**Proof.** There is an exact sequence:

$$0 \to H^1(S, f_*T_{X/S}) \to \text{Ext}^1_X(\Omega^1_{X/S}, O_X) \xrightarrow{\mu} H^0(S, \text{Ext}^1_f(\Omega^1_{X/S}, O_X)) \to 0$$

([13], p. 105). Since $f_*T_{X/S} = 0$ (see the proof of Theorem 1.4) $\mu$ is an isomorphism. First order deformations of $f$ are in 1-1 correspondence with isomorphism classes of extensions

$$\zeta : 0 \to O_X \xrightarrow{j} O_X \to O_X \to 0$$

such that $O_X$ is a sheaf of flat $O_S[\epsilon]$-algebras. In [21], Theorem 1.1.10, it is proved that if we only assume that $\zeta$ is an extension of $O_S$-algebras then it has already a structure of deformation by sending $\epsilon \mapsto j(1)$. There is a natural 1-1 correspondence between isomorphism classes of extensions $\zeta$ as above and $\text{Ext}^1_X(\Omega^1_{X/S}, O_X)$ (loc. cit., Theorem 1.1.10).
By comparing the cohomology sequence of (12) with the exact sequence (d) of Lemma 3.4.7 in loc. cit., we see that
\[ H^1(S, \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) \]
is an obstruction space for Def_f. \[ \square \]

Assume given a non-isotrivial fibration \( f : X \to S \). Given \( p \in S \) we have a morphism of functors:
\[ \psi_p : \text{Def}_f \to \text{Def}_C \]
where \( C := X(p) \), defined as follows. Given a local artinian \( \mathbb{C} \)-algebra \( A \) and an element \( \eta_A \in \text{Def}_f(A) \):
\[
\begin{array}{ccc}
X & \xrightarrow{f} & X_f \\
\downarrow & & \downarrow \\
S & \xrightarrow{p} & S \times \text{Spec}(A)
\end{array}
\]
we define \( \psi_p(\eta_A) \in \text{Def}_C \) to be the left square of
\[
\begin{array}{ccc}
C & \xrightarrow{(\eta_A)} & C_f \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \xrightarrow{(p)} & \text{Spec}(A) \xrightarrow{(p) \times 1} S \times \text{Spec}(A)
\end{array}
\]

Lemma 2.2 Let \( f : X \to S \) be a non-isotrivial fibration and \( p \in S \) such that the fibre \( X(p) \) is reduced. Then:
(i) There is an exact sequence of sheaves on \( S \):
\[
0 \to \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)(-p) \to \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \to \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X) \to 0
\]
where \( \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X) \) is the skyscraper sheaf supported on \( p \) with fibre \( \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X) \).
(ii) The homomorphism \( \kappa_f : T_S \to \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \) defined by the exact sequence (5) is injective and the induced map:
\[ \kappa_f(p) : T_pS \to \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X) \]
coincides with the Kodaira-Spencer map of the family \( f \) at \( p \).
(iii) Let
\[ r_p : H^0(S, \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) \to \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X) \]
be the map induced by (17). Then \( r_p \) is the differential of \( \psi_p \).
(iv) If \( S = \mathbb{P}^1 \) then \( \text{Im}(\kappa_f(p)) \subset \text{Im}(r_p) \).

Proof. (i) Consider the base change map:
\[
\tau^1(p) : \text{Ext}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \otimes \mathbb{k}(p) \to \text{Ext}^1_{X(p)}(\Omega^1_{X(p)}, \mathcal{O}_X)
\]
Then the base change theorem for the relative Ext sheaves and the fact that $\text{Ext}_f^2(\Omega^1_{X/S}, \mathcal{O}_X) = 0$ imply that $\tau^1(p)$ is an isomorphism for all $p \in S$ ([13], Th. 1.4). Therefore (17) is the exact sequence obtained by tensoring the sequence

$$0 \to \mathcal{O}_S(-p) \to \mathcal{O}_S \to \mathcal{O}_p \to 0$$

by $\text{Ext}_f^1(\Omega^1_{X/S}, \mathcal{O}_X)$.

(ii) The exact sequence (5) induces an exact sequence on $S$:

$$0 \to f^*T_X \to T_S \to \text{Ext}_f^1(\Omega^1_{X/S}, \mathcal{O}_X)$$

Since $f$ is non-isotrivial we have $f^*T_X = 0$ by Lemma 3.2 of [24], and this proves the first assertion. $\kappa_f(p)$ is the composition

$$T_pS \to \text{Ext}_f^1(\Omega^1_{X/S}, \mathcal{O}_X)_p \otimes \mathbf{k}(p) \to \text{Ext}_f^1(\Omega^1_{X/S}, \mathcal{O}_X)_p$$

of the natural restriction of $\kappa_f$ with the base change map $\tau^1(p)$. Since $X(p)$ is reduced $\text{Ext}_f^1(X(p), \mathcal{O}_X)$ is the vector space of first order deformations of $X(p)$ and $\kappa_f(p)$ is the Kodaira-Spencer map of $f$ at $p$ essentially by definition (see [21], Remark 2.4.4).

(iii) is tautological.

(iv) If $S = \mathbb{P}^1$ then every $\theta \in T_p\mathbb{P}^1$ comes from a global vector field $\Theta \in H^0(T_{\mathbb{P}^1})$. Then the assertion follows from the commutativity of the diagram:

$$
\begin{array}{ccc}
H^0(T_{\mathbb{P}^1}) & \to & H^0(\text{Ext}_f^1(\Omega^1_{X/S}, \mathcal{O}_X)) \\
\uparrow & & \uparrow \\
T_p\mathbb{P}^1 & \to & \text{Ext}_f^1(X(p), \mathcal{O}_X)
\end{array}
$$

Every non-isotrivial fibration $f$ has a semiuniversal formal deformation $(R, \{\eta_n\})$, where $\eta_n \in \text{Def}_f(R/m^{n+1})$, $n \geq 1$, and $m \subset R$ is the maximal ideal of the local complete $\mathbb{C}$-algebra $R$ ([21], Th. 3.4.8). The next result states the existence of an algebraic semiuniversal deformation in the relatively minimal case.

**Theorem 2.3** Let $f : X \to S$ be a non-isotrivial relatively minimal fibration. Then $f$ has a semiuniversal algebraic deformation, i.e. a deformation:

$$
\begin{array}{ccc}
X & \to & X \\
\downarrow f & & \downarrow F \\
S & \to & S \times V
\end{array}
$$

parametrized by a pointed algebraic scheme $(V, v)$ which is formally semiuniversal at $v$. 
Proof. The functor \( \text{Def}_{(X, \omega_{X/S})} \) of deformations of the pair \((X, \omega_{X/S})\) has a formal semiuniversal deformation \((O, \{(X_n, L_n)\})\), where \(O\) is a complete local \(C\)-algebra with maximal ideal \(I\) and \((X_n, L_n)\) is a deformation of \((X, \omega_{X/S})\) over \(O/I^{n+1}\) ([21], Th. 3.3.11). Let \(Y\) be the surface obtained by contracting all the \((-2)\)-curves of \(X\) contained in the fibres of \(f\). From Theorem 2’ of [25] and the Nakai-Moisezon-Kleiman criterion it follows that \(\omega_{X/S}\) is the pullback of an ample invertible sheaf on \(Y\). Therefore a positive power \(\omega_{X/S}^k\) is globally generated and maps \(X\) birationally to a projective surface whose only singularities are rational double points. Arguing as in [3], Example 5.5 one deduces the existence of a semiuniversal algebraic deformation of the pair \((X, \omega_{X/S})\), i.e. of a pair consisting of a deformation of \(X\):

\[
\begin{array}{ccc}
X' & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\text{Spec}(C) & \rightarrow & W \\
\end{array}
\]

parametrized by a pointed algebraic scheme \((W, w)\) and of an invertible sheaf \(L\) on \(Z\) such that the pair \((\zeta, L)\) is a deformation of \((X, \omega_{X/S})\) which is formally semiuniversal at \(w\). Now let \((V, v)\) be the relative local Hom-scheme \(\text{Hom}(Z/W, S \times W/W)\) around the point \([f]\) representing \(f : X = \mathbb{Z}(w) \to S \times \{w\}\) (see [12], Ch. 1.1). Then it is straightforward to check that the family of deformations of \(f\) parametrized by \((V, v)\) has the required properties. \(\square\)

3 Free fibrations

From now on we will only consider fibrations parametrized by \(\mathbb{P}^1\). All examples of such fibrations are obtained as follows.

Let \(Y\) be a projective nonsingular surface, and let \(C \subset Y\) be a projective nonsingular connected curve of genus \(g\) such that

\[\dim(|C|) \geq 1\] (18)

Consider a linear pencil \(\Lambda\) contained in \(|C|\) whose general member is nonsingular and let \(\sigma : X \to Y\) be the blow-up at its base points (including the infinitely near ones). We obtain a fibration

\[f : X \to \mathbb{P}^1\]

by choosing an isomorphism \(\Lambda^\vee \cong \mathbb{P}^1\) and taking the composition:

\[X \xrightarrow{\sigma} Y \rightarrow \Lambda^\vee \cong \mathbb{P}^1\]

We will call \(f\) the fibration defined by the pencil \(\Lambda\).

In the decomposition (11) we have

\[E = \bigoplus_{i=1}^{3g-1} \mathcal{O}_{\mathbb{P}^1}(a_i)\] (19)

for some integers \(a_i\).
**Definition 3.1** Let \( f : X \to \mathbb{P}^1 \) be a fibration. We call \( f \) free if it is non-isotrivial and \( \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X) \) is globally generated, i.e. if in (19) we have \( a_i \geq 0 \) for all \( i \).

**Proposition 3.2** Suppose that \( f : X \to \mathbb{P}^1 \) is a free fibration. Then:

(i) The functor \( \text{Def}_f \) is smooth of dimension \( \geq 3g - 1 \).

(ii) \( 11\chi(\mathcal{O}_X) - 2K^2_X \geq 5(g - 1) + 2 \) \hspace{1cm} (20)

Proof. (i) Since \( a_i \geq 0 \) for all \( i \) in (19), we have \( h^1(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) = 0 \). Therefore \( \text{Def}_f \) is smooth, by Lemma 2.1. Since \( f \) is non-isotrivial the homomorphism

\[
\kappa_f : T_{\mathbb{P}^1} \to \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)
\]

induced by the exact sequence (5) is injective, by Lemma 2.2. It follows that in (19) we have \( a_i \geq 2 \) for some \( i \) and therefore \( \text{Def}_f \) has dimension \( h^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \geq 3g - 1 \) \hspace{1cm} (21)

(ii) Recalling Proposition 1.7 we obtain:

\[
11\chi(\mathcal{O}_X) - 2K^2_X - 2(g - 1) = \chi(\text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) = h^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \geq 3g - 1
\]

and this is equivalent to (20).

\[\square\]

**Remark 3.3** If \( f \) is stable then recalling (16) we see that (20) is equivalent to the condition

\[
c_1(\psi^*K_{\mathbb{P}^1}) \leq -2
\]

Consider now a non-isotrivial fibration \( f : X \to \mathbb{P}^1 \) (not necessarily stable anymore) and the associated exact sequence:

\[
0 \longrightarrow f^*\omega_{\mathbb{P}^1} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/\mathbb{P}^1} \longrightarrow 0
\]

Taking \( f_*\text{Hom}(\cdot, \mathcal{O}_X) \) of it and recalling that \( f_*T_X = 0 \) ([24], Lemma 3.2) we obtain the following 4-term exact sequence of sheaves on \( \mathbb{P}^1 \):

\[
0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow \text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X) \longrightarrow R^1f_*T_X \longrightarrow T_{\mathbb{P}^1} \otimes R^1f_*\mathcal{O}_X \longrightarrow 0
\]

We thus obtain a long exact cohomology sequence as follows:

\[
\begin{align*}
& \longrightarrow H^0(T_{\mathbb{P}^1}) \longrightarrow H^0(\text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \longrightarrow H^0(R^1f_*T_X) \rightarrow \\
& \longrightarrow H^0(T_{\mathbb{P}^1} \otimes R^1f_*\mathcal{O}_X) \longrightarrow H^1(\text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \longrightarrow H^1(R^1f_*T_X) \longrightarrow H^1(T_{\mathbb{P}^1} \otimes R^1f_*\mathcal{O}_X) \longrightarrow 0 \\
& \longrightarrow H^2(T_X) \longrightarrow H^2(\text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \longrightarrow H^2(R^1f_*T_X) \rightarrow \\
& \longrightarrow H^2(T_{\mathbb{P}^1} \otimes R^1f_*\mathcal{O}_X) \longrightarrow H^3(T_{\mathbb{P}^1}) \longrightarrow H^3(\text{Ext}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \longrightarrow H^3(R^1f_*T_X) \longrightarrow 0
\end{align*}
\]
**Lemma 3.4** Let \( g : Y \rightarrow \mathbb{P}^1 \) be a non-isotrivial fibration, \( \pi : X \rightarrow Y \) the blow-up of a point \( y \in Y \) and \( f = g \cdot \pi : X \rightarrow \mathbb{P}^1 \). Then \( f \) is free if and only if \( g \) is free.

**Proof.** Let \( \pi^{-1}(y) = E \) be the exceptional curve. Then there is an exact sequence (\([21]\), p. 172)

\[
0 \rightarrow T_X \rightarrow \pi^*T_Y \rightarrow \mathcal{O}_E(1) \rightarrow 0
\]

which implies that we have a surjection \( \epsilon : R^1f_*T_X \rightarrow R^1g_*T_Y \). Now compare the exact sequence (23) with the analogous one for \( g \):

\[
0 \rightarrow T_{\mathbb{P}^1} \rightarrow Ext^1_\mathbb{P}^1(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X) \rightarrow R^1f_*T_X \rightarrow T_{\mathbb{P}^1} \otimes R^1f_*\mathcal{O}_X \rightarrow 0
\]

\[
0 \rightarrow T_{\mathbb{P}^1} \rightarrow Ext^1_\mathbb{P}^1(\Omega^1_{Y/\mathbb{P}^1}, \mathcal{O}_Y) \rightarrow R^1g_*T_Y \rightarrow T_{\mathbb{P}^1} \otimes R^1g_*\mathcal{O}_Y \rightarrow 0
\]

This diagram implies the existence of the dotted arrow, which is necessarily surjective. Then \( Ext^1_\mathbb{P}^1(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X) \) is globally generated if and only if \( Ext^1_\mathbb{P}^1(\Omega^1_{Y/\mathbb{P}^1}, \mathcal{O}_Y) \) is. \( \Box \)

### 4 Free fibrations and general curves

We need to have geometrical criteria to verify that a given fibration is free. For this purpose we introduce some definition. We start from the following well known one:

**Definition 4.1** A connected projective nonsingular curve \( C \) of genus \( g \) has general moduli, or is a general curve of genus \( g \), if it is a general fibre in a smooth projective family \( C \rightarrow V \) of curves of genus \( g \), parametrized by a nonsingular connected algebraic scheme \( V \), and such that the morphism \( V \rightarrow M_g \) induced by functoriality is dominant. Such a family will be said to have general moduli.

Whether a given family has general moduli can be checked in practice by means of the Kodaira-Spencer map. The following are some well known properties of a general curve that we will use:

**Proposition 4.2** Let \( C \) be a general curve of genus \( g \geq 3 \). Then:

(i) \( C \) has Clifford index

\[
\text{Cliff}(C) = \left[ \frac{g - 1}{2} \right]
\]

(ii) For any invertible sheaf \( L \) on \( C \), of degree \( d \geq 0 \) and with \( h^0(L) = r + 1 \geq 1 \) we have

\[
\rho(g, r, d) \geq 0
\]

where \( \rho(g, r, d) := g - (r + 1)(g - d + r) \) is the Brill-Noether number. In particular:

\[
d \geq \frac{1}{2}g + 1 \quad \text{if} \ r \geq 1
\]

\[
d \geq \frac{2}{3}g + 2 \quad \text{if} \ r \geq 2
\]
(iii) $H^1(L^2) = 0$ for any invertible sheaf $L$ on $C$ such that $h^0(L) \geq 2$.

(iv) $C$ does not possess irrational involutions, i.e. non-constant morphisms of degree $\geq 2$ onto a curve of positive genus.

(v) $C$ does not possess non-trivial automorphisms.

Proof. (i) follows from the main result of [5] and from Brill-Noether theory. (ii) is proved in [7]. For (iii) see [2], p. 22, and references therein. (iv) is classical and follows from Riemann’s existence theorem and a dimension count. For (v) see [18].

The next definition we need is the following one:

**Definition 4.3** Let $Y$ be a projective nonsingular surface, and $C \subset Y$ a projective nonsingular connected curve of genus $g \geq 3$. Let $r := \dim(|C|)$ and denote by $j : C \hookrightarrow Y$ the embedding. We say that $C$ is a general curve moving in an $r$-dimensional linear system on $Y$ if there is a family of deformations of $j$:

![Diagram](25)

parametrized by a pointed connected nonsingular algebraic scheme $(B, b)$ such that

1. the family of deformations of $C$ given by the upper square has general moduli;
2. $\dim(|C(u)|) \geq r$ on the surface $Y(u)$ for all closed points $u \in B$.

In the case $r = 0$ we will just say that $C$ is a curve with general moduli in $Y$.

**Remark 4.4** In the case $r = 0$ Definition 4.3 is equivalent to the notion of costability of $C$ in $Y$. (see [10] and [21], Def. 3.4.22).

**Example 4.5** Let $C \subset \mathbb{P}^3$ be a nonsingular curve of type $(3,3)$ on a nonsingular quadric $Q$. It is easy to show that there is a nonsingular quintic surface $Y$ containing $C$. On $Y$ we have $C^2 = 0$ and $\dim(|C|) = 1$. Since $C$ is a canonical curve of genus 4 one can construct a family of deformations of the pair $(C, Y)$ so that $C$ has general moduli. It follows that $C$ is a general curve of genus 4 moving in a 1-dimensional linear system on $Y$. This is a special case of a class of examples that can be constructed in a similar way. See also Example 5.4 and [27], Example 2.3.

**Example 4.6** Let $Y \subset \mathbb{P}^3$ be a nonsingular surface of degree $n \geq 6$, $C \subset Y$ a nonsingular curve of genus $g \geq 3$ such that $\dim(|C|) \geq 1$. Then $L = O_C(1)$ satisfies $h^0(L) \geq 3$ ($\geq 4$ if $C$ is non-degenerate) and $L^{n-4} = \omega_Y|C|$ is special. Therefore, by Proposition 4.2(iii), $C$ cannot be a general curve. We therefore see that a general curve of genus $g \geq 3$ cannot
move in a positive dimensional linear system on a nonsingular surface of degree $n \geq 6$ in $\mathbb{P}^3$.

**Proposition 4.7** Assume that $C$ is a general nonsingular curve of genus $g \geq 3$ moving in a positive dimensional linear system on a projective nonsingular surface $Y$. Then the following conditions are equivalent:

(i) Each linear pencil $\Lambda \subset |C|$ containing $C$ as a member defines an isotrivial fibration.

(ii) There is a linear pencil $\Lambda \subset |C|$ containing $C$ as a member which defines an isotrivial fibration.

(iii) $Y$ is birationally equivalent to $C \times \mathbb{P}^1$.

(iv) $Y$ is a non-rational birationally ruled surface.

**Proof.** (i) $\implies$ (ii) is trivial.

(ii) $\implies$ (iii). Let $f : X \to \mathbb{P}^1$ be the fibration defined by the pencil $\Lambda$. Assume that $f$ is isotrivial. Then, by the structure theorem for isotrivial fibrations [24], there is a nonsingular curve $\Gamma$ and a finite group $G$ acting on both $C$ and $\Gamma$ such that there is a birational isomorphism $X \to (C \times \Gamma)/G$ and a commutative diagram:

$$
\begin{array}{ccc}
X & \to & (C \times \Gamma)/G \\
f \downarrow & & \downarrow \\
\mathbb{P}^1 & \to & \Gamma/G
\end{array}
$$

where the right vertical arrow is the projection. But since $C$ is general, it has no non-trivial automorphisms (Prop. 4.2(v)), and therefore $G$ acts trivially on $C$: thus $X$ is birational to $C \times (\Gamma/G) = C \times \mathbb{P}^1$.

(iii) $\implies$ (iv) is obvious.

(iv) $\implies$ (i). By hypothesis there is a birational isomorphism $\xi : Y \to \Gamma \times \mathbb{P}^1$ for some projective nonsingular curve $\Gamma$ of positive genus. Let $D$ be a general member of $\Lambda$. The composition

$$
h : D \xleftarrow{\iota} Y \xrightarrow{\xi} \Gamma \times \mathbb{P}^1 \xrightarrow{\text{pr}} \Gamma
$$

where the last morphism is the projection, is non-constant. Since $D$ is a general curve, it does not possess irrational involutions (Prop. 4.2(iv)), thus $h$ must be an isomorphism. Therefore all general fibres of $f$ are mutually isomorphic, and this means that $f$ is isotrivial.

The next proposition relates the notion of free fibration with Definition 4.3.

**Proposition 4.8** Assume that $C$ is a general nonsingular curve of genus $g \geq 3$ moving in a positive-dimensional linear system on a projective nonsingular surface $Y$ which is not irrational ruled. Then a general pencil $\Lambda \subset |C|$ containing $C$ as a member defines a free fibration.
Proof. Let \( f : X \rightarrow \mathbb{P}^1 \) be the fibration defined by \( \Lambda \) and let \( p \in \mathbb{P}^1 \) be such that \( C = X(p) \). By Proposition 4.7, \( f \) is non-isotrivial. By hypothesis there is a pointed nonsingular algebraic scheme \((B,b)\) and a commutative diagram as follows:

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & \mathcal{Y} \\
\downarrow && \downarrow \\
\text{Spec}(\mathbb{C}) & \xrightarrow{(b)} & B
\end{array}
\]

such that the left square is a smooth family of deformations of \( C \) having surjective Kodaira-Spencer map at \( p \); \( \beta \) is a smooth family of projective surfaces and the upper right inclusion restricts over \( b \) to the inclusion \( C \subset \mathcal{Y} \); moreover \( |C(u)| \) is a positive-dimensional linear system on the surface \( \mathcal{Y}(u) \) for all closed points \( u \in B \).

Let \( \mathcal{L} = \mathcal{O}_\mathcal{Y}(C) \). After possibly performing an etale base change, we can find a trivial free subsheaf of rank two \( \mathcal{O}_B^2 \subset \beta^* \mathcal{L} \) which defines a rational \( B \)-map

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\chi} & \mathbb{P}^1 \times B \\
\downarrow \beta && \downarrow \\
B & \xrightarrow{} & \mathbb{P}^1 \times B
\end{array}
\]

whose restriction over \( b \) is the rational map defined by the pencil \( \Lambda \). Let \( Z \subset \mathcal{Y} \) be the scheme of indeterminacy of \( \chi \) and let \( \theta : \mathcal{X} \rightarrow \mathcal{Y} \) be the blow-up with center \( Z \). Composing with \( \theta \) we obtain from (26) a family of deformations of \( f \):

\[
\begin{array}{ccc}
X & \xrightarrow{\chi} & \mathcal{X} \\
\downarrow \beta \theta && \downarrow \\
\mathbb{P}^1 \times \{b\} & \xrightarrow{} & \mathbb{P}^1 \times B
\end{array}
\]

After restricting over \((p,b)\) we obtain an induced family of deformations of \( C \):

\[
\begin{array}{ccc}
C & \xrightarrow{} & \mathcal{X} \\
\downarrow \beta \theta && \downarrow \\
\text{Spec}(\mathbb{C}) & \xrightarrow{(p,b)} & \mathbb{P}^1 \times B
\end{array}
\]

Its Kodaira-Spencer map:

\[
\kappa(p,b) : T_p \mathbb{P}^1 \times T_b B \longrightarrow \text{Ext}^1_C(\Omega^1_C, \mathcal{O}_C)
\]

is surjective because, by construction, it contains the image of the Kodaira-Spencer map of \( \alpha \). Now, recalling Lemma 2.2, we deduce that \( \text{Im}(\kappa(p,b)) \) is contained in the image of the restriction map

\[
H^0(\mathbb{P}^1, \text{Ext}^1(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) \longrightarrow \text{Ext}^1_b(\Omega^1_C, \mathcal{O}_C)
\]
and therefore this map is surjective. Therefore $\text{Ext}_1^f(\Omega_{X/P}^1, \mathcal{O}_X)$ is generated by its global sections at $p$, thus $f$ is free.

\textbf{Theorem 4.9} The following conditions are equivalent for an integer $g \geq 3$:

(i) $\overline{M}_g$ is uniruled.

(ii) There exists a free fibration $f : X \rightarrow \mathbb{P}^1$ with fibres of genus $g$.

(iii) A general curve of genus $g$ moves in a positive-dimensional linear system on some nonsingular projective surface which is not irrational ruled.

\textit{Proof.} (i) $\Rightarrow$ (ii). By assumption there is an algebraic integral scheme $M$ of dimension $3g - 4$ and a dominant rational map

$$\Psi : \mathbb{P}^1 \times M \twoheadrightarrow \overline{M}_g$$

Since $\overline{M}_g$ is projective we can take $M$ projective and normalize it. Therefore $\Psi$ is defined on the complement of a codimension two closed subset $Z$. Then the image $\pi(Z) \subset M$ under the projection $\pi : \mathbb{P}^1 \times M \rightarrow M$ is a proper closed subset. This means that, modulo replacing $M$ by an open subset, we may assume that $\Psi$ is a morphism and that $M$ is nonsingular of dimension $3g - 4$. Let $(p, m) \in \mathbb{P}^1 \times M$ be a general point. Then $\Psi(p, m) \in \overline{M}_g \subset \overline{M}_g$, the open set of curves without automorphisms. The universal family over $\overline{M}_g$ pulls back to a family over an open subset of $\mathbb{P}^1 = \mathbb{P}^1 \times \{m\}$ containing $p$. After embedding the total space into a projective surface and desingularizing it we obtain a relatively minimal fibration $f : X \rightarrow \mathbb{P}^1$ containing $C$ among its fibres. Consider a semiumiversal deformation of $f$:

$$
\begin{array}{ccc}
\mathbb{P}^1 \times V & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
\mathbb{P}^1 \times \{m\} & \xrightarrow{\text{id} \times \{v\}} & \mathbb{P}^1 \times V
\end{array}
$$

and the induced rational map

$$\psi_F : \mathbb{P}^1 \times V \twoheadrightarrow \overline{M}_g$$

Since $\psi_F$ is well defined at the point $(p, v)$ and $\psi_F(p, v) = \Psi(p, m)$ we see that $\psi_F$ is dominant because $\Psi(p, m)$ is a general point of $\overline{M}_g$. It follows that the differential of $\psi_F$ at $(p, v)$ is surjective. Recalling Lemma 2.2(iv) we deduce that the map:

$$r_p : H^0(\mathbb{P}^1, \text{Ext}_1^f(\Omega_{X/P}^1, \mathcal{O}_X)) \longrightarrow \text{Ext}_1^{f}(\Omega_{X(p)}^1, \mathcal{O}_{X(p)})$$

is surjective. But this means that $\text{Ext}_1^f(\Omega_{X/P}^1, \mathcal{O}_X)$ is generated at $p$, and therefore it is globally generated, i.e. $f$ is free.
(ii) ⇒ (iii). By Lemma 3.4 we may assume that \( f \) is relatively minimal. By Theorem 2.3 it has a semiuniversal algebraic deformation

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times \{v\} & \xrightarrow{\text{id} \times \{v\}} & \mathbb{P}^1 \times V
\end{array}
\]

Since \( f \) is free, \( V \) is nonsingular at \( v \). Let \( p \in \mathbb{P}^1 \) be such that the fibre \( C = X(p) \) is nonsingular. Then we obtain a family of deformations of \( C \):

\[
\begin{array}{ccc}
C & \xrightarrow{\mathcal{X}} & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \xrightarrow{(p,v)} & \mathbb{P}^1 \times V
\end{array}
\]

and from Lemma 2.2 it follows that this family has surjective Kodaira-Spencer map at \( v \). Consider the composition

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathbb{P}^1 \times V \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times V & \xrightarrow{V} & V
\end{array}
\]

and let \( \mathcal{Y} := (\mathbb{P}^1 \times V) \times_V \mathcal{X} \). Then we have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathbb{P}^1 \times V \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times V & \xrightarrow{V} & V
\end{array}
\]

showing that \( F \) defines a family of general curves over \( \mathbb{P}^1 \times V \) moving in a linear pencil on a surface.

(iii) ⇒ (i). With the same notations as in the proof of Proposition 4.8, the hypothesis implies that, for a general curve \( C \) of genus \( g \), there a family of deformations \( (27) \). The induced functorial rational map \( \mathbb{P}^1 \times B \to \mathcal{M}_g \) is not constant along \( \mathbb{P}^1 \times \{b\} \) because the fibration \( f \) is not isotrivial. Therefore \( \mathcal{M}_g \) is uniruled. \( \square \)

The equivalence of conditions (i) and (iii) of the Theorem is well-known (see the Proposition on p. 25 of [8]).

5 The main estimate

We will need the following:

Lemma 5.1 Let \( C \) be a general curve of genus \( g \geq 3 \) contained in a projective nonsingular surface \( Y \) which is not irrational ruled, and such that

\[
r := \dim(|C|) \geq 1
\]

Let \( f : X \to \mathbb{P}^1 \) be the (free) fibration defined by a general pencil contained in \( |C| \) and containing \( C \) as a fibre. Then:

\[
h^0(\mathbb{P}^1, \text{Ext}^1_{\mathbb{P}^1} (\Omega^1_X, \mathcal{O}_X)) \geq 3(g - 1) + r + 1
\]

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Proof. There is an \((r - 1)\)-dimensional family of pencils contained in \(|C|\) and containing the curve \(C\) as a fibre. This family of pencils defines a family of free fibrations which has dimension \(r - 1 + 2 = r + 1\) and is effectively parametrized, i.e. it has injective Kodaira-Spencer map, at \(f\). In fact the pencils containing \(C\) as a member are parametrized by the \((r - 1)\)-dimensional characteristic linear series, defined by the image of the restriction:
\[ H^0(O_Y(C)) \to H^0(O_C(C)) \]
At most finitely many fibrations coming from different pencils can coincide, because the automorphism groups of the fibres are finite. Moreover the image of the Kodaira-Spencer map is contained in
\[ H^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{\mathbb{P}^1}, O_X)(-1)) \]
by the exact sequence (17) and therefore each pencil gives rise to a two-dimensional family of fibrations parametrized by the projectivities of \(\mathbb{P}^1\) fixing \(p\). Therefore from the freeness of \(f\) it follows that:
\[ h^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{\mathbb{P}^1}, O_X)) \geq r + 1 + 3(g - 1) \]
\[ \blacksquare \]
We can now prove the following refinement of the estimate (20).

**Theorem 5.2** Let \(C\) be a general nonsingular curve of genus \(g \geq 3\) moving in a positive dimensional linear system in a projective nonsingular surface \(Y\) which is not irrational ruled. Then:
\[ 10\chi(O_Y) - 2K_Y^2 \geq 4(g - 1) - C^2 - h^0(K_Y - C) \]  
\[ (28) \]
If moreover \(\dim(|C|) \geq 2\) or \(h^1(O_C(2C)) = 0\) then \(H^0(Y, K_Y - C) = 0\).

Proof. Let \(\Lambda \subset |C|\) be a general pencil and let \(f : X \to \mathbb{P}^1\) be the fibration defined by \(\Lambda\). Then we have:
\[ K_X^2 = K_Y^2 - C^2, \quad \chi(O_X) = \chi(O_Y) \]
Proposition 1.7 gives:
\[ h^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{\mathbb{P}^1}, O_X)) = 11\chi(O_Y) - 2K_Y^2 + 2C^2 - 2(g - 1) \]  
\[ (29) \]
because \(h^1(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{\mathbb{P}^1}, O_X)) = 0\), since \(f\) is free. From Lemma (5.1) we obtain:
\[ 3(g - 1) + h^0(O_Y(C)) \leq h^0(\mathbb{P}^1, \text{Ext}^1_f(\Omega^1_{\mathbb{P}^1}, O_X)) \]  
\[ (30) \]
Putting (29) and (30) together we get:
\[ 11\chi(O_Y) - 2K_Y^2 + 2C^2 \geq 5(g - 1) + h^0(Y, O_Y(C)) \]  
\[ (31) \]
By applying the Riemann–Roch theorem we have:
\[ \chi(O_Y(C)) = \chi(O_Y) - (g - 1 - C^2) \]
and therefore:

$$h^0(\mathcal{O}_Y(C)) \geq \chi(\mathcal{O}_Y) - (g - 1 - C^2) - h^0(Y, K_Y - C)$$

Substituting in (31) we get (28).

Assume that $\dim(|C|) \geq 2$. Then $h^0(C, \mathcal{O}_C(C)) \geq 2$ and therefore $H^1(\mathcal{O}_C(2C)) = 0$, by Proposition 4.2(iii). If $H^0(Y, K_Y - C) \neq 0$ then $\mathcal{O}_Y(2C) \subset \mathcal{O}_Y(K_Y + C)$, and therefore $\mathcal{O}_C(2C) \subset \omega_C$, i.e. $H^1(\mathcal{O}_C(2C)) \neq 0$. This is a contradiction. 

Theorem (5.2) will be applied to show that, for certain algebraic surfaces $Y$, there is an upper bound on the genus $g$ of a general curve which moves in a positive dimensional linear system on $Y$.

**Remark 5.3** Note that if $h^0(T_Y) = 0$ then the left hand side of (28) is

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = -\chi(T_Y) = h^1(T_Y) - h^2(T_Y) =: \mu(Y)$$

the expected number of moduli of the surface $Y$.

**Example 5.4** In [4], §3, it is shown that a general curve $C$ of genus 15 can be embedded as a non-degenerate nonsingular curve of degree 19 in $\mathbb{P}^6$, lying on a nonsingular canonical surface $Y \subset \mathbb{P}^6$ which is a complete intersection of 4 quadrics, and that $\dim(|C|) = 2$ on $Y$. Therefore, by Theorem 5.2, inequality (28) holds and $H^0(K_Y - C) = 0$. Let’s check.

The relevant numbers are in this case:

$$C^2 = 9, \quad K_Y^2 = 16, \quad \chi(\mathcal{O}_Y) = 8$$

We find:

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = 48 > 47 = 4(g - 1) - C^2$$

and this shows that (28) is sharp.

### 6 General curves on surfaces of non-negative Kodaira dimension

In this section we start analyzing the case of general curves on non-rational surfaces. The following elementary lemma will be useful.

**Lemma 6.1** Let $\sigma : Y \to Z$ be a birational morphism of projective nonsingular surfaces and let $D_0 \subset Z$ be an irreducible and reduced curve. Factor $\sigma$ as a sequence of blow-ups:

$$Y = Z_n \xrightarrow{\sigma_n} Z_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} Z_1 \xrightarrow{\sigma_1} Z \quad (32)$$

and denote by $D_i \subset Z_i$ the proper transform of $D_0$ under

$$\pi_i := \sigma_1 \cdots \sigma_i : Z_i \to Z_i, \quad i = 1, \ldots, n$$

Assume that the center of $\sigma_i$ is a point of $D_{i-1}$ of multiplicity $\nu_{i-1} \geq 1$, for each $i = 1, \ldots, n$. Then

$$K_YD_n = K_ZD_0 + \sum_{i=1}^n \nu_{i-1}$$
In particular, if the center of $\sigma_i$ is a singular point of $D_{i-1}$ for all $i$, then

$$K_Y D_n \geq K_Z D_0 + 2n$$

Proof. The last assertion is an obvious consequence of the first. It suffices to prove the first assertion in the case $n = 1$. We have $K_Y = \sigma^* K_Z + E$ where $E$ is the exceptional curve. We have:

$$K_Y D_1 = (\sigma^* K_Z + E)(\sigma^* D_0 - \nu_0 E) = K_Z D_0 + \nu_0$$

Theorem 6.2 Let $Y$ be a projective nonsingular surface with $\kappa$-dim$(Y) \geq 0$, and let $C \subset Y$ be a general nonsingular curve of genus $g \geq 3$ moving in a positive dimensional linear system. Then

$$g \leq 5p_g(Y) + 6 + \frac{1}{2} h^0(K_Y - C)$$

In particular:

$$g \leq \begin{cases} 
6, & \text{if } p_g = 0 \\
11, & \text{if } p_g = 1
\end{cases}$$

Proof. Let $\sigma : Y \rightarrow Z$ be the birational morphism onto the minimal model of $Y$. After possibly contracting finitely many $(-1)$-curves on $Y$ we may assume that $\sigma$ can be factored as a sequence of $\delta \geq 0$ blow-ups $Y \rightarrow Z_{\delta} \rightarrow \cdots \rightarrow Z_1 \rightarrow Z$ so that, for each $i = 1, \ldots, \delta$, letting $D_i = (\sigma \cdots \sigma)(C) \subset Z_i$, the center of $\sigma_i : Z_i \rightarrow Z_{i-1}$ is a singular point of $D_{i-1}$.

In all three cases inequality (28) holds. From Lemma 6.1 it follows that

$$C^2 = 2(g - 1) - K_Y C \leq 2(g - 1) - [\sigma(C)K_Z + 2\delta] \leq 2(g - 1) - 2\delta$$

because $\sigma(C)K_Z \geq 0$ by the assumption on the Kodaira dimension of $Z$.

Therefore, since $K_Y^2 = K_Z^2 - \delta$, using (28) we obtain:

$$10\chi(O_Z) - 2K_Z^2 + 2\delta = 10\chi(O_Y) - 2K_Y^2 \geq 4(g - 1) - C^2 - h^0(K_Y - C) \geq 4(g - 1) - 2(g - 1) - 2\delta - h^0(K_Y - C)$$

This gives

$$10\chi(O_Z) - 2K_Z^2 \geq 2(g - 1) - h^0(K_Y - C)$$

The conclusion now follows because $\chi(O_Z) \leq 1 + p_g$ and $K_Z^2 \geq 0$. \[\square\]

Theorem 6.3 Let $Y$ be a non-ruled and non-rational elliptic surface, $\pi : Y \rightarrow B$ the elliptic fibration onto a nonsingular connected curve $B$. Assume that there exists a general nonsingular connected curve $C \subset Y$ of genus $g \geq 3$ such that dim$(|C|) \geq 1$. Then $B = \mathbb{P}^1$ and $g \leq 16$. 21
Proof. Let \(|K| = |M| + D\), where \(|M|\) is the mobile part and \(D\) is the fixed divisor. Then \(M = \sum_{i=1}^{n} F_{b_i}\), where \(b_1, \ldots, b_m \in B\) and \(F_{b_i} = \pi^{-1}(b_i)\). It follows that \(h^0(K_Y - C) = 0\) because all fibres of \(\pi\) have arithmetic genus 1.

Let \(F\) be a general fibre of \(\pi\). We have \(C \cdot F = k \geq 1\). If \(k = 1\) then \(\pi|_C : C \rightarrow B\) is an isomorphism, and it follows that every linear pencil \(\Lambda\) containing \(C\) is isotrivial. Since \(C\) is general, it follows that \(Y\) is ruled (Proposition 4.7), which is a contradiction.

Therefore \(k \geq 2\). If \(B\) has genus \(g(B) \geq 1\) then \(\pi|_C : C \rightarrow B\) is an irrational involution on \(C\), and this contradicts the fact that \(C\) has general moduli (Proposition 4.2(iv)). Therefore we may assume that \(p_g \geq 3\). Since \(p_g \geq 3\), we have \(m \geq 2\), and it follows that \(\mathcal{O}_C(mF) \subset \mathcal{O}_C(K_Y) \subset \mathcal{O}_C(K_Y + C) = \omega_C\). This means that \(h^1(C, \mathcal{O}_C(mF)) \neq 0\), while \(h^0(C, \mathcal{O}_C(F)) \geq 2\), contradicting the generality of \(C\) (Proposition 4.2(iii)).

Remarks 6.4 The well-known bound \(g \leq 11\) for the genus of a nonsingular curve with general moduli on a K3 surface can be deduced from a dimension count, as shown in [14]. In [15] and [16] it is proved that a general nonsingular curve of genus \(g\) can be embedded in a K3 surface if and only if \(g \leq 11\) and \(g \neq 10\). In particular the bound of Theorem 6.2 is sharp for the case \(p_g = 1\). In [6] it is shown that even a general curve of genus 10 can be birationally embedded in a K3 surface provided one allows it to have one node (hence arithmetic genus 11). I don’t know if the bound \(g \leq 16\) in Theorems 6.2 and 6.3 is sharp.

The following result takes care of irregular surfaces of positive geometric genus, showing the impossibility for a general curve of genus \(g \geq 3\) to move in a positive dimensional linear system on such a surface. The case \(p_g = 0\) is already included in previous results of this section.

Theorem 6.5 Let \(Y\) be a projective nonsingular surface with \(p_g > 0\) and \(q > 0\), and let \(C \subset Y\) be a nonsingular curve of genus \(g \geq 3\) such that \(\dim(C) \geq 1\). Then no pencil \(\Lambda \subset |C|\) containing \(C\) as a member defines a free fibration.

Proof. Let \(f : X \rightarrow \mathbb{P}^1\) be the fibration associated to a pencil \(\Lambda\) as in the statement. We have:

\[ h^2(TX) = h^2(T_Y) \geq p_g + q - 1 \geq p_g \]

The equality is a standard computation (see e.g. [21], Example 3.4.13(iv)), and the first inequality is a classical result of B. Segre [20] (see [17], p. 127 for a modern proof). Now consider the exact sequence (24) associated to \(f\).
If \( f \) is free then \( H^1(\mathbb{P}^1, Ext^1_1(\Omega_{X/\mathbb{P}^1}, \mathcal{O}_X)) = 0 \), and therefore \( H^2(T_X) \cong H^1(T_{\mathbb{P}^1} \otimes R^1 f_* \mathcal{O}_X) \). But
\[
p_g = h^1(R^1 f_* \mathcal{O}_X) > h^1(T_{\mathbb{P}^1} \otimes R^1 f_* \mathcal{O}_X) = h^2(T_X)
\]
Here the inequality is a consequence of the assumption \( p_g > 0 \) and the last equality follows from the exact sequence (24). Therefore we get a contradiction. \( \Box \)

7 General curves on surfaces of general type

The following is our main result on general curves on surfaces of general type.

**Theorem 7.1** Let \( Y \) be a surface of general type and let \( Z \) be the minimal model of \( Y \). Assume that \( K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10 \) and that \( C \subset Y \) is a general nonsingular connected curve of genus \( g \geq 3 \). If one of the following holds:

(a) \( \dim(|C|) \geq 2 \),
(b) \( \dim(|C|) = 1 \), \( h^0(K_Y - C) = 0 \) and \( C^2 \geq \frac{(g - 1)}{2} \),
(c) \( \dim(|C|) = 1 \), and \( h^1(C, \mathcal{O}_C(2C)) = 0 \).

Then \( g \leq 19 \).

**Proof.** By Theorem 6.2 we may assume that \( Y \) has geometric genus \( p_g \geq 2 \). As in the proof of Theorem 6.2 we may assume that the birational morphism

\[
\sigma : Y \longrightarrow Z
\]
can be factored as a sequence of \( \delta \geq 0 \) blow-ups

\[
Y = Z_\delta \xrightarrow{\sigma_\delta} Z_{\delta - 1} \xrightarrow{\sigma_{\delta - 1}} \cdots \xrightarrow{\sigma_2} Z_1 \xrightarrow{\sigma_1} Z
\]
so that, for each \( i = 1, \ldots, \delta \), letting \( D_i = (\sigma_{i+1} \cdots \sigma_\delta)(C) \subset Z_i \), the center of \( \sigma_i : Z_i \rightarrow Z_{i-1} \) is a singular point of \( D_{i-1} \). We have \( K_Y = \sigma^* K_Z (\sum r_i E_i) \), where the \( E_i \)'s are the irreducible components of the exceptional locus of \( \sigma \) and \( r_i > 0 \) for all \( i \).

Let \( L = \mathcal{O}_C(\sigma^* K_Z) \). Then, since \( H^0(K_Y - C) = 0 \) (by Theorem 5.2) we have \( H^0(\sigma^* K_Z - C) = 0 \) and therefore
\[
h^0(L) \geq h^0(\sigma^* K_Z) = p_g \geq 2
\]
(33)

Inequality (28) can be written under the form:
\[
10\chi(\mathcal{O}_Z) - 2K_Z^2 + 2\delta \geq 4(g - 1) - C^2
\]
After substitution of the inequality \( K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10 \) we obtain:
\[
4\chi(\mathcal{O}_Z) + 20 + 2\delta \geq 4(g - 1) - C^2
\]
(34)

which implies:
\[
28 + 2\delta \geq 4(g - 1) - C^2 - 4p_g + 4
\]

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From Lemma 6.1 we deduce
\[ 2(g - 1) - C^2 = \deg(O_C(K_Y)) \geq \deg(L) + 2\delta \]
and therefore we obtain:
\[ 28 + 2 \delta \geq 2[\deg(L) - 2\rho_g + 2 + 2\delta] + C^2 \tag{35} \]
Assume that we are in case a). Then
\[ h^0(\omega_C L^{-1}) = h^0(O_C(C + \sum r_i E_i)) \geq h^0(O_C(C)) \geq \dim(|C|) \geq 2 \tag{36} \]
Inequalities (36) and (33) imply that \( L \) contributes to the Clifford index of \( C \). Since \( C \) is general, recalling that
\[ \text{Cliff}(L) = \deg(L) - 2h^0(L) + 2 \]
we have the following relations satisfied by the Clifford indices:
\[ \left[ \frac{g - 1}{2} \right] = \text{Cliff}(C) \leq \text{Cliff}(L) \leq \deg(L) - 2\rho_g + 2 \tag{37} \]
For the equality see Proposition 4.2(i). The first inequality follows from the very definition of Cliff\((C)\), and the last is by (33). Substituting in (35) we obtain:
\[ 28 + 2 \delta \geq 2[\deg(L) - 2\rho_g + 2 + 2\delta] + C^2 \]
\[ \geq 2[\text{Cliff}(C) + 2\delta] + C^2 \]
\[ \geq g - 2 + 4\delta + C^2 \]
Since \( h^0(O_C(C)) \geq 2 \), from Proposition 4.2(ii) we have \( C^2 \geq \frac{1}{2}g + 1 \). Therefore:
\[ 28 - 2\delta \geq \frac{3}{2}g - 1 \]
which implies \( g \leq 19 \).

Assume now that we are in case b). We may assume that \( h^0(\omega_C L^{-1}) = 1 \), because otherwise \( L \) contributes to Cliff\((C)\) and we conclude as before. Then:
\[ \deg(L) - 2\rho_g + 2 \geq \deg(L) - 2h^0(L) + 2 = \deg(\omega_C L^{-1}) \geq C^2 \]
Substituting in (35) we obtain:
\[ 28 + 2 \delta \geq 2[\deg(L) - 2\rho_g + 2 + 2\delta] + C^2 \]
\[ \geq 2(C^2 + 2\delta) + C^2 = 3C^2 + 4\delta \geq \frac{3}{2}(g - 1) + 4\delta \]
because of the assumptions. Therefore we obtain \( g \leq 19 \) again.

If we are in case c) then \( C^2 \geq g/2 \), since \( O_C(2C) \) is non-special, and \( H^0(K_Y - C) = 0 \) by Theorem 5.2. Therefore we reduce to case b). \( \square \)
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ADDRESS OF THE AUTHOR:
Dipartimento di Matematica e Fisica, Università Roma Tre
Largo S. L. Murialdo 1, 00146 Roma, Italy.
sernesi@mat.uniroma3.it