DART: aDaptive Accept RejecT for non-linear top-$K$ subset identification

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Abstract

We consider the bandit problem of selecting $K$ out of $N$ arms at each time step. The reward can be a non-linear function of the rewards of the selected individual arms. The direct use of a multi-armed bandit algorithm requires choosing among $\binom{N}{K}$ options, making the action space large. To simplify the problem, existing works on combinatorial bandits typically assume feedback as a linear function of individual rewards. In this paper, we prove the lower bound for top-$K$ subset selection with bandit feedback with possibly correlated rewards. We present a novel algorithm for the combinatorial setting without using individual arm feedback or requiring linearity of the reward function. Additionally, our algorithm works on correlated rewards of individual arms. Our algorithm, aDaptive Accept RejecT (DART), sequentially finds good arms and eliminates bad arms based on confidence bounds. DART is computationally efficient and uses storage linear in $N$. Further, DART achieves a regret bound of $\tilde{O}(K\sqrt{KNT})$ for a time horizon $T$, which matches the lower bound in bandit feedback up to a factor of $\sqrt{\log 2NT}$. When applied to the problem of cross selling optimization and maximizing the mean of individual rewards, the performance of the proposed algorithm surpasses that of state-of-the-art algorithms. We also show that DART significantly outperforms existing methods for both linear and non-linear joint reward environments.

I. INTRODUCTION

We consider the problem of finding the best subset of $K$ out of $N$ items to optimize a possibly non-linear function of reward of each item. We note that the joint reward as a function of individual rewards is a much natural setting to understand and arises in a number of settings. For example, in the problem of erasure-coded storage [35], the agent chooses $K$ out of $N$ servers to obtain the content for each request; the final reward is the negative of the time taken by the slowest server. A recommendation system agent may present a list of $K$ items out of $N$ items to user for a non-zero reward only if the user selects an item...

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from the list. Similarly, in cross-selling item selection, a retailer creates a bundle with \( K \) items, and the joint reward is a quadratic function of the selected items’ individual rewards \([32]\). The problem of a daily advertising campaign is characterized by a set of sub-campaigns where the aggregate reward is the sum of the rewards of sub-campaigns \([36, 31]\). Combinatorial Multi-Armed Bandit (CMAB) algorithms can solve these problems in an online manner. For many CMAB algorithms, we can bound the regret, that is the loss incurred from accidentally selecting sub-optimal sets some of the time. We aim to find a space and time efficient CMAB algorithm that minimizes cumulative regret.

Existing algorithms for \( K = 1 \) that use Upper Confidence Bound (UCB) or Bayesian resampling methods \([6, 8, 7, 34, 3, 19]\) can bound the regret by \( \tilde{O}(\sqrt{NT}) \). These methods can be naturally extended to the combinatorial setting where \( K \) arms are chosen, treating each of the \( \binom{N}{K} \) possible actions as a distinct ‘arm’. Unfortunately, this approach has two significant drawbacks. First, the regret increases exponentially in \( K \) as the number of total actions to explore has grown from \( N \) to \( \binom{N}{K} \). Second, the time and space complexities increase exponentially in \( K \), requiring storage of values for all actions to find the action with highest UCB \([7, 8]\) or highest sampled rewards \([3]\).

This paper addresses those issues by proposing a novel algorithm called **aDaptive Accept RejecT** (DART). To estimate the “goodness” of an arm, we use the mean of the rewards obtained by playing actions containing that arm. In an adaptive manner, DART moves arms to “accept” or “reject” sets based on those estimates, reducing the number of arms that require further exploration. We assume that the expected joint reward of an arm \( i \) with \( K - 1 \) other arms is better than the expected joint reward of arm \( j \) with the same \( K - 1 \) other arms, if arm \( i \) is individually better than arm \( j \). This assumption is naturally satisfied in many online decision making settings, such as click bandits \([24, 26]\). We then use Lipschitz continuity of the joint reward function to relate orderings between pairs of arms \( i \) and \( j \) to orderings between pairs of actions containing those arms. We construct a martingale sequence to analyze the regret bound of DART. Furthermore, DART achieves a space complexity of \( \mathcal{O}(N) \) and a per-round time complexity of \( \tilde{O}(N) \).

The main contributions of this paper can be summarized as follows:

1) We propose DART - a time and space efficient algorithm for the non-linear top-\( K \) subset selection problem with only the joint reward as feedback. We show that DART has per-step time complexity of \( \tilde{O}(N) \) and space complexity of \( \mathcal{O}(N) \).

2) We prove a lower bound of \( \Omega(K \sqrt{NKT}) \) for the regret of top-\( K \) subset selection problem for a
linear setup where the joint rewards are possibly correlated.

3) We prove that DART achieves a (pseudo-)regret of $\tilde{O}(K \sqrt{NKT})$ over a time horizon $T$ and under certain assumptions. The regret bound matches the lower bound for the bandit setting where individual arm rewards are possibly correlated.

We also empirically evaluate the proposed algorithm DART, comparing it to other, state-of-the-art full-bandit feedback CMAB algorithms. We first consider a linear setting, where the joint reward as simply the mean of individual arm rewards. We also examine the setting where the joint reward is a quadratic function of individual arm rewards, based on the problem of cross-selling item selection [32]. Our algorithm significantly outperforms existing state of the art algorithms, while only using polynomial space and time complexity.

II. RELATED WORKS

[14, 15, 11, 4, 1, 27] consider a linear bandit setup where at time $t$, the agent selects a length $N$ vector $x_t$ from the decision set $D_t \subset \mathbb{R}^N$ and observes a reward $\theta^T x_t$ for an unknown constant vector $\theta \in \mathbb{R}^N$. The algorithms proposed in these works use the linearity of the reward function to estimate rewards of individual arms and achieve a regret of $O(\sqrt{NT})$. [16, 21, 28] studied the problem of generalized linear models (GLM) where the reward $r_t$ is a function $(f(z) : \mathbb{R} \to \mathbb{R})$ of $z = \theta^T x_t$ plus some noise. Generalized linear model algorithms also obtain a regret bound of $O(\sqrt{NT})$. The proposed algorithms can be naturally extended to our setup for linear joint reward functions. However, the space and time complexity remains exponential in $K$ to store all possible $\binom{N}{K}$ actions.

For setting of $K = 1$, [29] reduces the space complexity from $O(N)$ to $O(1)$ at the cost of worse regret bounds. When extended to the combinatorial setting, treating each set of $K$ arms as a distinct ‘arm,’ the regret bound becomes exponential in $K$. Recently, [33] bounded the regret by $O(K \sqrt{NT})$ for identifying the best $K$ subset, in the case when the joint reward is the sum of rewards of independent arms, using $O(N)$ space and per-round time complexity. [30] considered the combinatorial bandit problem with a non-linear reward function and additional feedback, where the feedback is a linear combination of the rewards of the $K$ arms. Such feedback allows for the recovery of individual rewards. [2] proposed a divide and conquer based algorithm for the best $K$ subset problem with a non-linear joint reward with bandit feedback. [2] is the most related work as the setup is similar to ours. Algorithms by [30, 2] achieve $O(T^{2/3})$ regret while the proposed algorithm in this paper achieves $O(T^{1/2})$ regret.
Many works have studied the semi-bandit setting, where individual arm rewards are also available as feedback \cite{23, 12, 23, 26, 18, 17}. \cite{23} provides a UCB-type algorithm for matroid bandits, where the agent selects a maximal independent set of rank \( K \) to maximize the sum of individual arm rewards. \cite{12} considered the combinatorial semi-bandit problem with non-linear rewards using a UCB-type analysis. In contrast to these prior works, we consider the full-bandit setting where individual arm rewards are not available. \cite{25, 26} proved a lower bound of \( \Omega(\sqrt{NKT}) \) for semi-bandit problems where the joint reward is simply the sum of individual arm rewards. \cite{22} also provides a lower bound for the best \( K \) subset problem of \( \Omega\left(\frac{N}{\epsilon^2 \log(\frac{K}{\delta})}\right) \) for any \((\epsilon, \delta)\)-PAC algorithm playing single arm at each time. \cite{5} obtained a lower bound of \( \Omega(K\sqrt{NT}) \) for bandit feedback and provide an algorithm with regret bound of \( \Omega(K\sqrt{NKT}) \) for linear bandits without assuming independence between arms. \cite{13} obtain a tighter lower bound of \( \Omega(K\sqrt{KNT}) \) for a bandit setup where the rewards of individual arms are possibly correlated. Our proposed algorithm achieves the tighter lower bound (ignoring \( \log \) terms) for bandit feedback with possibly correlated rewards.

### III. Problem Formulation

We consider \( N \) “arms” labeled as \( i \in [N] = \{1, 2, \cdots, N\} \). On playing arm \( i \) at time step \( t \), it generates a reward \( X_{i,t} \in [0, 1] \) which is a random variable. We assume that \( X_{i,t} \) are independent across time, and for any arm the distribution is identical at all times. For simplicity, we will use \( X_i \) instead of \( X_{i,t} \) for analysis that holds for any \( t \). The distribution for each arm \( i \)'s rewards \( \{X_{i,t}\}_{t=1}^T \) could be discrete, continuous, or mixed.

The agent can only play an action \( a \in \mathcal{N} \) where \( \mathcal{N} = \{a \in [N]^K \mid a(i) \neq a(j) \forall i, j: 1 \leq i < j \leq K\} \) is the set of all \( K \) sized tuples created using arms in \([N]\). Thus, the cardinality of \( \mathcal{N} \) is \( \binom{N}{K} \). For an action \( a \), let \( X_{a,t} = (X_{a(1),t}, X_{a(2),t}, \cdots, X_{a(K),t}) \) be the column reward vector of individual arm rewards at time \( t \) from arms in action \( a \). The reward \( r_a(t) \) of an action \( a \) at time \( t \) is a bounded function \( f : [0, 1]^K \rightarrow [0, 1] \) of the individual arm rewards

\[
r_a(t) = f(d_{a,t}). \tag{1}
\]

As \( X_{i,t} \) are i.i.d. across time \( t \), \( d_{a,t} \) are also i.i.d. across time \( t \) for all \( a \in \mathcal{N} \). Later in the text we will skip index \( t \), for brevity, where it is unambiguous. We denote the expected reward of any action \( a \in \mathcal{N} \)
as \( \mu_a = \mathbb{E}[r_a] \). We assume that there is a unique “optimal” action \( a^* \) for which the expected reward \( \mu_a^* \) is highest among all actions,

\[
a^* = \arg \max_{a \in \mathcal{N}} \mu_a. \tag{2}
\]

At time \( t \), the agent plays an action \( a_t \) randomly sampled from an arbitrary distribution over \( \mathcal{N} \) dependent on the history of played actions and observed rewards till time \( t - 1 \). The agent aims to reduce the cumulative (pseudo-)regret \( R \) over time horizon \( T \), defined as the expected difference between the rewards of the best action in hindsight and the actions selected by the agent.

\[
R = \mathbb{E}_{a_1, r_{a_1}(1), \ldots, a_T, r_{a_T}(T)} \left[ \sum_{t=1}^{T} r_{a^*}(t) - r_{a_t}(t) \right] \tag{3}
\]

\[
= T \mu_{a^*} - \mathbb{E}_{a_1, \ldots, a_T} \left[ \sum_{t=1}^{T} \mu_{a_t} \right]. \tag{4}
\]

We define the gap \( \Delta_{i,j} \) between two arms \( i \) and \( j \) as the difference between the expected rewards of arm \( i \) and arm \( j \),

\[
\Delta_{i,j} = \mathbb{E}[X_i] - \mathbb{E}[X_j]. \tag{5}
\]

We now mention the assumptions for this paper. We first assume that the joint reward function \( f \) is permutation invariant. Let \( \Pi \) denote the set of all permutation functions of a length \( K \) vector.

**Assumption 1** (Symmetry). For any permutation \( \pi \in \Pi \) of the vector \( d \) of individual arm rewards,

\[
f(d) = f(\pi(d)) \tag{6}
\]

We also assume that the expected reward of an action with a good arm is higher than the expected reward of action with a bad arm for all possible combinations of the remaining \( K - 1 \) arms. Further, if two arms are equally good, they are indistinguishable in every action and will not contribute to regret. This assumption is similar to Assumption 4 of [26].

**Assumption 2** (Good arms generate good actions). We assume that if the expected reward of arm \( i \) is higher than the expected reward of arm \( j \) (for any given \( i \neq j \)), then for any subset \( S \) of size \( K - 1 \) arms chosen from the remaining \( N - 2 \) arms (arms excluding \( i \) and \( j \)), the expected reward of \( S \cup \{i\} \)
is higher than the expected reward of $S \cup \{j\}$. More precisely, if $\mathbb{E}[X_i] \geq \mathbb{E}[X_j]$ then

$$
\mathbb{E}_{X_i,X_j,d_S} [f(d_{S\cup\{i\}}) - f(d_{S\cup\{j\}})] \geq 0
$$

(7)

for all $S$, where and $d_S \in [0, 1]^{K-1}$ is a random vector of the rewards from the arms in $S$. Further, the equality holds only if $\mathbb{E}[X_i] = \mathbb{E}[X_j]$.

We also assume that $f(\cdot)$ is Bi-Lipschitz continuous (in an expected sense). Let $\mathbb{E}[d_a]$ denote the vector of mean arm rewards for arms in $a$.

**Assumption 3** (Continuity of expected rewards). The expected value of $f(\cdot)$ is bi-Lipschitz continuous with respect to the expected value of the rewards obtained by the individual arms, or there exists a $U \leq \infty$ such that,

$$
\frac{1}{U} \left\| \mathbb{E}[d_{a_1}] - \pi(\mathbb{E}[d_{a_2}]) \right\|_1 \leq |\mu_{a_1} - \mu_{a_2}| \leq U \left\| \mathbb{E}[d_{a_1}] - \pi(\mathbb{E}[d_{a_2}]) \right\|_1
$$

(8)

for any pair of actions $a_1, a_2 \in \mathcal{N}$ and for any permutation $\pi$ of $d$.

**Corollary 1.** For any arms $i, j \in \mathcal{N}$ and any subset $S \subset \mathcal{N} \setminus \{i, j\}$ of size $K - 1$,

$$
|\mathbb{E}[X_i] - \mathbb{E}[X_j]| \leq U|\mu_{S\cup\{i\}} - \mu_{S\cup\{j\}}|.
$$

(9)

**Proof.** We obtain the result by choosing $a_1 = S \cup \{i\}$ and $a_2 = S \cup \{j\}$. \hfill \square

Assumptions 1-3 are satisfied for many problem setups, such as in the cascade model for clicks [24] where a user interacting with a list of documents clicks on the first documents the user likes. The joint reward $r(t) = \max(X_{1:t}, \ldots, X_{K:t})$ is the maximum of individual arm rewards. The corresponding form of Equation (7) is

$$
1 - (\prod_k (1 - \mathbb{E}[X_k])) (1 - \mathbb{E}[X_i]) \geq 1 - (\prod_k (1 - \mathbb{E}[X_k])) (1 - \mathbb{E}[X_j])
$$

which holds when $\mathbb{E}[X_i] > \mathbb{E}[X_j]$, and the Bi-Lipschitz property in individual expected rewards holds too. The assumptions are also satisfied in cross-selling optimization [32], where the reward is a quadratic function of the individual items sold in a bundle $K$. The assumptions are also satisfied for joint reward functions such as the sum or mean of individual arm rewards [33, 12].

**IV. LOWER BOUND ON TOP-K SUBSET IDENTIFICATION**

Given the problem formulation, we now prove a tight lower bound on the subset identification problem for a linear joint reward function with correlated arms. We consider a specific setup. Let $\alpha^* = \ldots
\{1, 2, \ldots, K\} denote the best subset which is initially unknown to the agent. Define the reward function as \(f(d) = \sum_{i=1}^{K} d(i)\), where \(d(i)\) is the \(i^{th}\) entry of \(d\). The individual arm distributions are of the form \(X'_{i,t} = 1/2 + \epsilon 1_{\{i \in a^*\}} + Z_t\), where \(Z_t\) follows a Gaussian distribution with mean 0 and variance \(\sigma^2\) and \(1_{\{\cdot\}}\) denotes the indicator function. The arms are correlated through the shared additive term \(Z_t\).

**Theorem 1.** For \(\epsilon = \frac{\sigma}{\sqrt{NKT}}\), any deterministic player must suffer expected regret of at least \(\Omega(\sigma K \sqrt{KNT})\) against an environment with rewards \(X'_{i,t}\) for \(t = 1, 2, \ldots, T\) for each arm \(i \in \mathcal{N}\).

**Proof.** (Outline:) The proof is based on the proof techniques presented in [13, 5]. We note that if the algorithm plays against a setup where all the arms are identically distributed, then the expected number of times it selects an arm \(i \in a^*\) is \(KT/N\) as the arms are not distinguishable. Using this and the proof of Lemma 4 from [13] we obtain the required result. A detailed proof is reconstructed in Appendix A.

Note that for the lower bound, we considered a general setup with \(X'_{i,t} \in (\infty, \infty)\). However, our setup bounds individual rewards in \([0, 1]\). This can again be managed by the proof technique from [13, Theorem 5], by bounding the probability of \(X'_{i,t} > 1\) for all \(t \leq T\) by choosing \(\sigma^2 = \frac{1}{(4 \log NKT)}\).

**V. PROPOSED DART ALGORITHM**

We first state a relevant lemma to motivate the proposed algorithm. We note that the Assumption 2 and the Assumption 3 allow us to order arms without observing their individual expected rewards.

**Lemma 1.** Let \(\mathcal{N}(i)\) and \(\mathcal{N}(j)\) be the set of all actions that contains arms \(i\) and \(j\) respectively (\(|\mathcal{N}(i)| = |\mathcal{N}(j)| = \binom{N-1}{K-1}\)). If the actions are uniformly randomly selected from the sets \(\mathcal{N}(i)\) and \(\mathcal{N}(j)\), then the following holds.

\[
E[X_i] \leq E[X_j]
\]

\[
\Leftrightarrow E_{a_i \sim U(\mathcal{N}(i))}[\mu_{a_i}] \leq E_{a_j \sim U(\mathcal{N}(j))}[\mu_{a_j}]
\]

(10)

where \(U(\cdot)\) is the uniform distribution.

**Proof Sketch:** We take expectation on all the actions over set \(\mathcal{N}(i)\) in the left and over set \(\mathcal{N}(j)\) in the right hand side of the Equation (7) from Assumption 2 to obtain the required result. A detailed proof is presented in Appendix A.
From Lemma \(\boxed{1}\) we note that if we create uniformly random partitions, the expected reward of action containing arm \(i\) will be higher than the expected reward of the action containing arm \(j\) if arm \(i\) is better than arm \(j\). We use this idea to create the proposed DART algorithm in Algorithm \(\boxed{1}\).

The algorithm initializes \(\hat{\mu}_i\) as the estimated mean for actions that contain arm \(i\) and \(n_i\) as the number of times an action containing arm \(i\) is played. The algorithm proceeds in epochs, indexed by \(e\), and maintains three different sets at each epoch. The first set, \(A_e\), contains “good” arms which belong to the top-\(K\) arms found till epoch \(e\). The second set, \(N_e\), contains the arms which the algorithm is still exploring at epoch \(e\). The third set, \(R_e\), contains the arms that are “rejected” and do not belong in the top-\(K\) arms. We let \(K_e\) be the variable that contains the number of spots to fill in the top-\(K\) subset at epoch \(e\). The algorithm maintains a decision variable \(\Delta\) as the concentration bound and a parameter variable \(n\) as the minimum number of samples required for achieving the concentration bound \(\Delta\). Lastly, the algorithm maintains a hyper parameter \(\lambda\) tuned for the value of \(T, N,\) and \(K\). \(\lambda\) is the minimum gap between any two arms the algorithm can resolve within time horizon \(T\).

In Line 5, the algorithm selects a permutation of \(N_e\) uniformly at random and partitions it into sets of size \(K_e\). If \(K_e\) does not divide \(|N_e|\), we repeat arms in the last group (cyclically, so that the last group has \(K_e\) distinct arms). To simplify the bookkeeping, \(\hat{\mu}_i\) and \(n_i\) are not updated if arm \(i\) is repeated in the last group. The algorithm then creates an action \(a_t\) from the partitioned groups and the arms in the good set and plays it to obtain a reward \(r_{a_t}(t)\) at time \(t\) (Line 8-9). DART then updates the estimated mean for all arms played in \(a_t\) with the observed reward and increments the number of counts for the arms played (Line 10-12).

In lines 15-16, the algorithm moves an arm \(i \in N_e\) to \(A_e\) if estimated mean of actions that contain arm \(i\), \(\hat{\mu}_i\), is \(\Delta\) more than the estimated mean of actions that contain arm at \((K + 1)^{th}\) rank, \(\hat{\mu}_{K+1}\). Similarly, the algorithm moves an arm \(i \in N_e\) to \(R_e\) if estimated mean of actions that contain arm \(i\), \(\hat{\mu}_i\), is \(\Delta\) less than the estimated mean of actions that contain arm at \(K^{th}\) rank, \(\hat{\mu}_K\).

The proposed DART algorithm uses a random permutation of \(N_e\). The random permutation can be generated in \(O(N)\) steps. Also after each round, the algorithm finds the \(K^{th}\) and \((K + 1)^{th}\) ranked arms. This operation can be completed in \(\tilde{O}(N)\) time complexity using sorting \(\{\hat{\mu}_i\}_{i=1}^N\). Also going over each arm in \(N_e\) is of linear time complexity. Hence, the per-step time complexity of the algorithm comes out to be \(\tilde{O}(N)\). Also, the proposed DART algorithm only stores the estimates \(\hat{\mu}_i\) for each arm \(i \in [N]\). The resulting storage complexity is \(O(N)\) for maintaining the estimates. To find the top-\(K\) and the top-
(\(K + 1\)) means, the algorithm may use additional space of \(O(N)\) to maintain a heap. Thus, the overall space complexity of the algorithm is only \(O(N)\).

VI. REGRET ANALYSIS

We now analyse the sample complexity and regret of the proposed DART algorithm. To bound the regret, we first bound the number of samples required to move an arm in \(N_e\) to either of \(A_e\) or \(R_e\). Then, we bound the regret from including a sub-optimal arm in the played actions. For the analysis, without loss of generality, we assume that the expected rewards of arms are ranked as \(\mathbb{E}[X_1] > \mathbb{E}[X_2] > \cdots > \mathbb{E}[X_N]\). If the arms are not in the said order, we relabel the arms to obtain the required order. From Assumption \([2]\) we have \(\mathbf{a}^* = \{1, 2, \cdots, K\}\). We refer to arms 1, \(\cdots, K\) as optimal arms and arms \(K + 1, \cdots, N\) as sub-optimal arms.
A. Number of samples to move an arm in $\mathcal{N}_e$ to either of $\mathcal{A}_e$ or $\mathcal{R}_e$

We call two arms $i, j \in \mathcal{N}_e$, $i < j$ separated if the algorithm has high confidence that $\mathbb{E}[X_i] > \mathbb{E}[X_j]$. We first analyze the general conditions to separate any two arms $i, j \in \mathcal{N}_e$ such that $\mathbb{E}[X_i] > \mathbb{E}[X_j]$. Let the epoch where arm $i$ and arm $j$ are separated and the epoch of Algorithm 1 be $e$. We define a filtration $\mathcal{F}_e$ as the history observed by the algorithm till epoch $e$.

For any $u \in \mathcal{N}_e$, let $\mathcal{N}_e(u) = \{a \in [N]^K : u \in a, \mathcal{A}_e \subseteq a, \mathcal{R}_e \cap a = \emptyset, a(i) \neq a(j) \forall i, j : 1 \leq i < j \leq K\}$.

We now define a random variable $Z_{i,j}(e)$ for $i, j \in \mathcal{N}_e$, which denotes the difference between the reward observed from playing an uniform random action from $\mathcal{N}_e(i)$ and an uniform random action from $\mathcal{N}_e(j)$. In other words,

$$Z_{i,j}(e) = r_{a_i}(e) - r_{a_j}(e),$$  \hspace{1cm} (11)

where $a_i \sim \mathbb{U}(\mathcal{N}_e(i))$, $a_j \sim \mathbb{U}(\mathcal{N}_e(j))$ and $\mathbb{U}(\cdot)$ denotes the uniform distribution. Also, $r_{a_i}(e)$ is the reward observed by playing $a_i$ and $r_{a_j}(e)$ is the reward obtained by playing $a_j$ at epoch $e$. Hence, the randomness of $Z_{i,j}(e)$ comes from both the random selection of $a_i$ and $a_j$, and from the reward generated by playing $a_i$ and $a_j$. Let $P_{Z_{i,j}(e)}$ denote the probability distribution of $Z_{i,j}(e)$. We now mention a lemma for bounding the expected value of $Z_{i,j}(e)$ for all epochs $e$.

**Lemma 2.** Let $i, j \in [N]$ be two arms such that $\mathbb{E}[X_i] > \mathbb{E}[X_j]$. Let $Z_{i,j}(e)$ be a random variable denoting the difference between the reward obtained on playing a uniform random action $a_i \sim \mathbb{U}(\mathcal{N}_e(i))$ containing arm $i$ and a randomly selected action $a_j \sim \mathbb{U}(\mathcal{N}_e(j))$ containing arm $j$. Then the expected value of $Z_{i,j}(e)$ is upper bounded by $U \Delta_{i,j}$, and lower bounded by 0, or,

$$\frac{\Delta_{i,j}}{UK} \leq \mathbb{E}[Z_{i,j}(e)] \leq U \Delta_{i,j}$$  \hspace{1cm} (12)

**Proof Sketch.** The upper bound is obtained by calculating the number of possible actions $a_1$ that contain arm $i$ and the number of possible actions $a_2$ that contains arm $j$ and then applying the upper bound on $|\mu_i - \mu_j|$ from Assumption 3. Similarly, we obtain the lower bound by replacing the upper bound by the lower bound on $|\mu_{i} - \mu_{j}|$ from Assumption 3.

The sequence of random variables $Z_{i,j}(e), e = 1, 2, \cdots$ are not independent as the sets $\mathcal{A}_i(e)$ and $\mathcal{A}_j(e)$ are updated as the algorithm proceeds. Hence, we cannot apply Hoeffding’s concentration inequality.


Theorem 2] for analysis. To use Azuma-Hoeffding’s inequality [9, Chapter 3] (given as Lemma 6 in Appendix for completeness), we need to construct a martingale. For each pair of arms \( i, j \in [N] \) with \( \mathbb{E}[X_i] > \mathbb{E}[X_j] \), we define \( Y_{i,j} \) as a martingale with respect to filtration \( \mathcal{F}_e \),

\[
Y_{i,j}(e) = \sum_{e'=1}^{e} (Z_{i,j}(e') - \mathbb{E}_{e'-1}[Z_{i,j}(e')])
\]

(13)

where \( \mathbb{E}_{e'-1}[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_{e'-1}] \). \( Y_{i,j}(e) \) is a martingale with zero-mean, and \( |Y_{i,j}(e) - Y_{i,j}(e-1)| \leq 2 \), and hence we can apply Azuma-Hoeffding’s inequality to \( Y_{i,j}(e) \) for all \( i, j \in [N] \).

After obtaining the concentration of \( Y_{i,j}(e) \) with respect to the \( e^{th} \) iteration of sample of action with arm \( i \) and arm \( j \), we now obtain the value of \( e \) for which we can consider arm \( i \) and \( j \) to be separated with probability \( 1 - \delta \).

**Lemma 3.** Let arms \( i, j \in [N] \) be two arms such that \( \mathbb{E}[X_i] > \mathbb{E}[X_j] \). Let \( \Delta \) be such that \( \Delta < \hat{\mu}_i - \hat{\mu}_j \leq 2\Delta \). Then, with probability at least \( 1 - \delta \), arm \( i \) and arm \( j \) are separable and \( \frac{32\log 2/\delta}{\Delta^2} \leq e \leq O \left( \frac{288U^2K^2 \log (2/\delta)}{\Delta^2} \right) \).

**Proof Sketch:** At epoch \( e \), \( \hat{\mu}_i - \hat{\mu}_j = \sum_{e'=1}^{e} Z_{i,j}(e')/e \). Using this relation and Azuma-Hoeffding’s inequality on \( Y_{i,j}(e) \), we get the required result. A detailed proof is provided in Appendix A.

We can now bound the number of samples required to move each arm from \( \mathcal{N}_e \) to either the “accept” set \( \mathcal{A}_e \) or the “reject” set \( \mathcal{R}_e \). In the algorithm, arm \( i \) will be moved to the accept set \( \mathcal{A}_e \) when its empirical mean \( \hat{\mu}_i \) is sufficiently larger than that of the \( K + 1 \) ranked arm. Consider an arm \( i \) in the optimal action \( a^* = \{1, \ldots, K\} \). By Lemma 3 with probability \( 1 - \delta \), arms \( i \) and \( K + 1 \) will be separable by epoch

\[
e \leq \frac{288U^2K^2 \log (2/\delta)}{\Delta^2_{i,K+1}}.
\]

(14)

Similarly, arm \( i \) will be moved to the reject set \( \mathcal{R}_e \) when its empirical mean \( \hat{\mu}_i \) is sufficiently less than that of the \( K \)th ranked arm. Consider an arm \( i \in \{K + 1, \ldots, N\} \). By Lemma 3 with probability \( 1 - \delta \), arms \( i \) and \( K \) will be separable by epoch

\[
\frac{288U^2K^2 \log (2/\delta)}{\Delta^2_{K,i}}.
\]

(15)

**B. Regret from sampling sub-optimal arms**

We first bound the regret of playing any action \( a \in \mathcal{N} \) using Assumption 3.
Lemma 4. Let $a = (a_1, a_2, \cdots, a_K)$ be any action. The expected regret suffered from playing action $a$ instead of action $a^* = (1, 2, \cdots, K)$ is bounded as

$$|\mu_a - \mu_{a^*}| \leq U \sum_{i=1}^{K} |\mathbb{E}[X_{a_i}] - \mathbb{E}[X_{\pi(i)}]|,$$

(16)

for any permutation $\pi$ of $\{1, \cdots, K\}$ for which $\pi(i) = a_i$ if $a_i \leq K$.

Proof Sketch. From Assumption 3 we first find a tight upper bound. We finish the proof by using the fact that Assumption 3 selects the permutation which minimizes the bound, hence any other permutation also gives a valid upper bound. A detailed proof is provided in Appendix $\Box$.

We now bound the regret incurred by playing an action $a_t$ at time $t$ containing sub-optimal arm $i \in \{K + 1, \cdots, N\}$ replacing an optimal arm $j \in \{1, \cdots, K\}$ in Lemma 5.

Lemma 5. For any sub-optimal action, the regret it can accumulate by replacing an optimal arm $j \in \{1, \cdots, K\}$ by an arm $i \in K + 1, \cdots, N$ is bounded by

$$\frac{1440U^3K^2 \log \left(\frac{2}{\delta}\right)}{\Delta_{K,i}}$$

(17)

Proof Sketch. The agent suffers from regret if it an action that contains at least one sub-optimal arm. To bound the regret from a sub-optimal action, we use the proof technique of [33] to divide the optimal arms $j \in \{1, \cdots, K\}$ into two groups: first group with $\Delta_{j,K+1} > \Delta_{K,i}$ and second group with $\Delta_{j,K+1} \leq \Delta_{K,i}$. We show that regret from both groups is bounded by $O \left(\frac{1}{\Delta_{K,i}}\right)$ The detailed proof is provided in Appendix $\Box$.

After calculating the regret from individual arms, we now calculate the total regret of the DART algorithm in the following theorem.

Theorem 2. For $\lambda = U \sqrt{\frac{2NKT \log 2NT}{T}}$, the distribution free regret incurred by DART algorithm is bounded by

$$R \leq O \left(U^2K \sqrt{NKT \log 2NT}\right)$$

(18)

Proof Sketch. We use the standard proof technique of bounding regret accumulated while eliminating arms to reject set of a confidence bound based algorithm to tune $\lambda$ and calculate the regret. A detailed proof is provided in Appendix $\Box$. $\square$
We note that the regret bound of DART matches matches the lower bound in Theorem 1 upto the factor of $\log (2NT)$ for bandits with joint reward as sum of rewards of individual arms in an action.

We note that there may be scenarios where an agent does not know the value of $U$ and cannot tune $\lambda$ accordingly. In such a case, the agent increases its regret because of not knowing the joint reward function. For a value of $\lambda = \sqrt{\frac{720NK \log 2NT}{T}}$, which does not use $U$, the regret of the algorithm is bounded as

$$O \left( (U^3 + U) K \sqrt{NKT \log 2NT} \right).$$

Additionally, we note that we can convert DART to an anytime algorithm using doubling trick of restarting algorithm at $T_l = 2^l \forall l = 1, 2, \cdots$ until the unknown time horizon $T$ is reached \[7\]. Using analysis from \[10, Theorem 4\], we show that DART for unknown $T$ achieves a regret bound of $\tilde{O}(\sqrt{T})$.

We present the complete proof in Appendix A.

We also considered a case where the the reward function is indeed linear, but a scaled function of individual rewards such that Equation (8) in the bi-Lipschitz Assumption 2 modifies to

$$|\mu_{a_1} - \mu_{a_2}| = U\|E[d_{a_1}] - \pi(E[d_{a_2}])\|_1$$

for some permutation $\pi$ of $d$. Then, the same analysis holds and for $\lambda = \frac{1}{U} \sqrt{\frac{720NK \log 2NT}{T}}$ the regret bound becomes independent of $U$ as:

$$R \leq \mathcal{O} \left( K \sqrt{NKT \log 2NT} \right).$$

We noted that the regret becoming independent of $U$ is not surprising. The intuition behind this observation is, if $U$ becomes large then it is easy to separate arms and the regret does not grow large as the algorithm quickly finds the good arms. On the other hand, if $U$ becomes small, the cost of choosing a wrong arm is reduced by a factor of $U$. 

---

(a) $K = 2$

(b) $K = 4$

(c) $K = 8$

Fig. 1: Regret plots for joint rewards as mean of individual arm rewards
VI. EXPERIMENTS

We now present comparison results of DART with CSAR [33] and CMAB-SM [2] and UCB [7] algorithms. We used $N = 45$ and $T = 10^6$ for simulations. We chose $K = 2, 4, 8$ for easy construction of Hadamard matrices for CSAR algorithm. We compare for two different reward setups. The first setup has a joint reward that is a linear function of individual rewards. The second setup has a joint reward that is a quadratic function of individual rewards. In each setup, individual arm rewards follow Bernoulli distribution with means sampled from $\mathbb{U}([0, 1])$. We run 25 independent iterations and plot the average regret and the maximum and minimum values of the cumulative regret of each algorithm.

In the first setup, we have the joint reward of the form $r(t) = \theta^T d_\alpha t$, where $\theta \in \mathbb{R}^K$ is a vector with all entries as $1/K$. From Figure 1, we note that the performance of DART is significantly better than both CSAR and CMAB-SM for joint reward as the mean of the individual arm rewards. For CSAR algorithm in [33], this is because after updating $\Delta$, it generates fresh $K^2/\Delta^2$ samples instead of using previous samples to improve estimates. Hence, although the algorithm is order optimal, we observed the performance deteriorates in practice. We only compare with UCB [7] for $K = 2$ as the action space became too large for $K = 4, 8$. Also, we note that LinUCB algorithm [27] for linear bandits runs extremely slow even for $K = 2$ and we show comparison for $N = 15, K = 2$ in Appendix A.

We also simulate the joint reward of the form $r(t) = d_{\alpha t}^T A d_\alpha t$, where $A \in \mathbb{R}^{K \times K}$ is an upper triangular matrix with all entries as $2/K(K + 1)$. A quadratic reward function is used in cross-selling optimization to quantify the total profit from selling a bundle of items compared to the profit from selling the items in the bundle separately [32]. From Figure 2, we note that DART significantly outperforms CSAR and CMAB-SM algorithm for quadratic function of individual rewards as well. We note that CSAR, though designed for linear setup, is able to model the ranking of the arms from quadratic rewards and beat CMAB-SM algorithm. However, we note that CSAR is not able to outperform when the joint reward is
the max of individual rewards as noted in Appendix B.

VIII. CONCLUSION

We considered the problem of combinatorial multi-armed bandits with non-linear rewards, where the agent chooses $K$ out of $N$ arms in each time-step and receives an aggregate reward. We obtained a lower bound of $\Omega(K \sqrt{NKT})$ for the linear case with possibly correlated rewards. We proposed a novel algorithm, called DART, which is computationally efficient and has a space complexity which is linear in number of base arms. We analyzed the algorithm in terms of regret bound, and show that it is upper bounded by $\tilde{O}(K \sqrt{NKT})$, which matches the lower bound of $\Omega(K \sqrt{NKT})$ for bandit setup with correlated rewards. DART works efficiently for large $N$ and $K$ and outperforms existing methods empirically.
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APPENDIX

We start by noting that the reward function \( f(d) = \sum_i^K d(i) \) satisfies Assumptions 1 – 3. Assumption 1 holds from the fact that addition is commutative. Assumption 2 holds from the fact that Expectation is linear. Lastly, the Corollary from the Assumption 3 follows from the linearity of the the sum. The next part of proof follows on the lines of the proof of Lemma 4 in [13], with relevant modifications based on the change in problem setup.

Proof of Theorem 1. We first convert the rewards \( X'_{i,t} \) of each arms into losses as \( L'_{i,t} = -X'_{i,t} = -1/2 - \epsilon 1_{i \in a^*} \), where \( a^* \) is the best subset. Let \( a^* = (i_1^*, i_2^*, \ldots, i_K^*) \in \mathcal{N} \) such that \( i_j^* < i_k^* \forall 1 \leq j < k \leq K \) be chosen uniformly randomly from \( \mathcal{N} \). Let \( T_1, \ldots, T_k \) be random variables such that \( T_j \) is the number of times the agent plays \( a_t \) such that \( i_j^* \in a_t \). For each \( a \in \mathcal{N} \), let \( \mathbb{P}_a \) and \( \mathbb{E}_a \) be the probability distribution and the expectation with respect to the marginal distributions under which \( a^* = a \). Then,

\[
R \geq \mathbb{E} \left[ \sum_{t=1}^T \left( \sum_{i=1}^K X'_{a^*(i)} - X'_{a_t(i)} \right) \right] \quad (22)
\]

\[
= \mathbb{E} \left[ \sum_{t=1}^T \left( \sum_{i=1}^K L_{a_t(i)} - L_{a^*(i)} \right) \right] \quad (23)
\]

\[
= \frac{1}{\binom{K}{N}} \sum_{a \in \mathcal{N}} \mathbb{E}_a \left[ \sum_{t=1}^T \left( \sum_{i=1}^K L_{a_t(i)} - L_{a(i)} \right) \right] \quad (24)
\]

\[
= \frac{1}{\binom{K}{N}} \sum_{a \in \mathcal{N}} \epsilon \mathbb{E}_a \left[ \sum_{j=1}^K (T - T_j) \right] \quad (25)
\]

\[
= \epsilon \left( KT - \sum_{j=1}^K \frac{1}{\binom{K}{N}} \sum_{a \in \mathcal{N}} \mathbb{E}_a[T_j] \right). \quad (26)
\]

We now need to upper bound \( \mathbb{E}_a[T_j] \) for each \( j \).

For every \( a \in \mathcal{N} \) and \( j \in [K] \), we introduce a new distribution, which is same as \( \mathbb{P}_a \) except that the loss of \( i_j^* \) is also \(-1/2 - Z_t\). We refer to these laws by \( \mathbb{P}_{a,-j} \) and \( \mathbb{E}_{a,-j} \). Let \( \lambda_t \) be the loss observed at time \( t \), and \( \lambda^{(t)} = (\lambda_1, \ldots, \lambda_t) \) be the sequence of losses observed up to and including time \( t \). Then, since \( \lambda^{(T)} \) determines the actions of the learner over the entire game, and using Pinsker’s inequality,

\[
\mathbb{E}_a[T_j] - \mathbb{E}_{a,-j}[T_j] \leq T \cdot D_{TV}(\mathbb{P}_{a,-j}[\lambda^{(T)}], \mathbb{P}_a[\lambda^{(T)}]) \quad (27)
\]

\[
\leq T \sqrt{\frac{1}{2} D_{KL}(\mathbb{P}_{a,-j}[\lambda^{(T)}] || \mathbb{E}_a[\lambda^{(T)}])} \quad (28)
\]

\( \square \)
Now, from the chain rule of KL-divergence, $D_{KL} \left( \mathbb{P}_{a,-j}^{\lambda(T)} || \mathbb{E}_a \left[ \lambda^{(T)} \right] \right)$ becomes

$$
\sum_{t=1}^{T} \mathbb{E}_{\lambda^{(t-1)} \sim \mathbb{P}_{a,-j}} \left[ D_{KL} \left( \mathbb{P}_{a,-j} \left[ \lambda_t | \lambda^{(t-1)} \right] || \mathbb{P}_a \left[ \lambda_t | \lambda^{(t-1)} \right] \right) \right] 
$$

(29)

Consider a single term in the sum in Equation (29). If $i_j^* \notin a_t$, then the loss observed under $P_a$ and $P_{a,-j}$ are the same, and the KL divergence is 0. If $i_j^* \in a_t$, then the losses under $P_a$ and $P_{a,-j}$ are both Gaussian with $\epsilon$ far means and variance of $\sigma^2 K^2$. Hence, we have

$$
D_{KL} \left( \mathbb{P}_{a,-j} \left[ \lambda_t | \lambda^{(t-1)} \right] || \mathbb{P}_a \left[ \lambda_t | \lambda^{(t-1)} \right] \right) \leq \frac{\epsilon^2}{2K^2 \sigma^2}.
$$

(30)

Using the obtained KL-divergence in Equation (29) and subsequently in Equation (28) we get,

$$
D_{KL} \left( \mathbb{P}_{a,-j}^{\lambda(T)} || \mathbb{E}_a \left[ \lambda^{(T)} \right] \right) \leq \sum_{t=1}^{T} \mathbb{P}_{a,-j} \left[ i_j^* \in a_t \right] \frac{\epsilon^2}{2K^2 \sigma^2} = \frac{\epsilon^2}{2K^2 \sigma^2} \mathbb{E}_{a,-j} \left[ T_j \right], \quad \text{and (31)}
$$

$$
\mathbb{E}_a \left[ T_j \right] \leq \mathbb{E}_{a,-j} \left[ T_j \right] + \frac{\epsilon}{2K \sigma} \sqrt{\mathbb{E}_{a,-j} \left[ T_j \right]}.
$$

(32)

We now want to upper bound $\mathbb{E}_{a \sim U(N)} \left[ \mathbb{E}_{a,-j} \left[ T_j \right] \right]$ to proceed from Equation (26).

Following the proof of Lemma 8 from [13], after conditioning on $i_1^*, \ldots, i_{j-1}^*, i_{j+1}^*, \ldots, i_K^*$ we have only $N - K + 1$ choices available for $i_j^*$. This gives

$$
\mathbb{E}_{a^* \sim U(N)} \left[ \mathbb{E}_{a^*, -j} \left[ T_j \right] \right] = \frac{1}{\binom{N}{K}} \sum_{i_1^* \ldots i_{j-1}^* i_{j+1}^* \ldots i_K^*} \mathbb{E}_{a^*, -j} \left[ T_j \right] \leq \frac{TK}{N}
$$

(33)
Since $N > 2K$, we have:

\[
\frac{1}{(N/K)} \sum_{a \in \mathcal{N}} \mathbb{E}_{a \in \mathcal{N}}[T_j] \leq \frac{1}{(N/K)} \sum_{a \in \mathcal{N}} \mathbb{E}_{a,j}[T_j] + \frac{1}{(N/K)} \sum_{a \in \mathcal{N}} \frac{\epsilon}{2K} \sqrt{\mathbb{E}_{a,j}[T_j]} \tag{36}
\]

\[
\leq \frac{1}{(N/K)} \sum_{a \in \mathcal{N}} \mathbb{E}_{a,j}[T_j] + \frac{\epsilon}{2K} \sqrt{\frac{1}{(N/K)} \sum_{a \in \mathcal{N}} \mathbb{E}_{a,j}[T_j]} \tag{37}
\]

\[
\leq \frac{TK}{N} + \frac{\epsilon}{2K} \sqrt{\frac{TK}{N}} \tag{38}
\]

\[
\leq \frac{T}{2} + \frac{\epsilon}{2K} \sqrt{\frac{2TK}{N}} \tag{39}
\]

\[
\leq \frac{T}{2} + \frac{\epsilon}{2} \sqrt{\frac{2T}{NK}} \tag{40}
\]

where Equation (36) is obtained from the Cauchy Schwarz inequality.

Substituting this in Equation (26), we get

\[
R \geq \epsilon KT \left( \frac{1}{2} - \frac{\epsilon}{2} \sqrt{\frac{2T}{NK}} \right) \tag{42}
\]

for all values of $\epsilon$. Choosing $\epsilon = \frac{\sigma}{2} \sqrt{\frac{NK}{2T}}$, we have

\[
R \geq \frac{\sigma K}{8} \sqrt{\frac{NKT}{2}} \tag{43}
\]

From this lower bound, we note that the regret bound of CSAR algorithm by [33] is tight for the case of top-$K$ subset selection with independent rewards.

**Proof of Lemma 1** We begin for the $\implies$ part first. From Assumption 2 we take $\mathbb{E}[X_i] > \mathbb{E}[X_j]$, then
for a subset $S$ of size $K - 1$ we have the following:

$$
\mathbb{E}_{X_i,d_S}[f(h(X_i, d_S))] > \mathbb{E}_{X_j,d_S}[f(h(X_j, d_S))]
$$  \hspace{1cm} (44)

$$
\implies \mathbb{E}_{d_{a_i}}[f(d_{a_i})] > \mathbb{E}_{d_{a_j}}[f(d_{a_j})]
$$  \hspace{1cm} (45)

$$
\iff \mu_{a_i} > \mu_{a_j}
$$  \hspace{1cm} (46)

$$
\iff \frac{1}{\binom{N-2}{K-1}} \mu_{a_i} > \frac{1}{\binom{N-2}{K-1}} \mu_{a_j}
$$  \hspace{1cm} (47)

$$
\iff \frac{1}{|N(i)|} \mu_{a_i} > \frac{1}{|N(j)|} \mu_{a_j}
$$  \hspace{1cm} (48)

$$
\implies \sum_{a_i \in N(i) \setminus (N(i) \cap N(j))} \frac{\mu_{a_i}}{|N(i)|} > \sum_{a_j \in N(j) \setminus (N(i) \cap N(j))} \frac{\mu_{a_j}}{|N(j)|}
$$  \hspace{1cm} (49)

$$
\implies \mathbb{E}_{a_i \sim U(N(i))}[\mu_{a_i}] > \mathbb{E}_{a_j \sim U(N(j))}[\mu_{a_j}].
$$  \hspace{1cm} (50)

Equation (44) comes from the linearity of the expectation from Assumption 2 and taking expectations on $X_j$ and $X_i$. Equation (45) is obtained by creating action $a_i = S \cup \{i\}$ and $a_j = S \cup \{j\}$. Equation (46) is obtained by using the definition of $\mu_{a_i}$ and $\mu_{a_j}$. Equation (48) is obtained by noting that $|N(i)| = |N(j)| = \binom{N-1}{K-1}$. Equation (49) follows by summing over all the actions that contain either arm $i$ or arm $j$ on either side of the equality. Equation (50) is obtained by adding the expected rewards of actions that contain both arm $i$ and arm $j$. All the actions that contains both arm $i$ and arm $j$ belong to set $N(i) \cap N(j)$. This gives the required expectation.

We prove the $\iff$ part by contradiction. Assume that $\mathbb{E}[X_j] > \mathbb{E}[X_i]$ and $\mathbb{E}[\mu_{a_i}] < \mathbb{E}[\mu_{a_j}]$ holds. Then from $\mathbb{E}[X_j] > \mathbb{E}[X_i]$ and the $\implies$ part, we obtain $\mathbb{E}[\mu_{a_i}] > \mathbb{E}[\mu_{a_j}]$ which results in a contradiction. $\square$

**Proof.** We first show the upper bound. The cardinality of both $N_e(i) \cap N_e(j)$ and $N_e(j)$ is $\binom{|N_e| - 1}{K_e - 1}$ as we have fixed one of the $K_e$ places for arm $i$ and now we can fill only $K_e - 1$ places from the available $|N_e| - 1$ arms. Algorithm 1 partitions a random, uniformly distributed permutation over $N_e$, so all actions $a \in N_e(i)$ are equally likely, and likewise for $a \in N_e(j)$. Taking the expectation over the actions played and the reward
obtained, we get the expected value of $Z_{i,j}(e)$ as

$$
\mathbb{E} [Z_{i,j}(e)] = \mathbb{E} [r_{a_i}(e) - r_{a_j}(e)]
$$

(53)

$$
= \frac{1}{(|\mathcal{N}_e|-1)} \left( \sum_{a \in \mathcal{N}_e(i)} \mu_a - \sum_{a \in \mathcal{N}_e(j)} \mu_a \right)
$$

(54)

$$
\leq \frac{1}{(|\mathcal{N}_e|-1)} \left( \frac{|\mathcal{N}_e| - 2}{K_e - 1} \right) U \Delta_{i,j}
$$

(55)

$$
= \frac{|\mathcal{N}_e| - K_e}{|\mathcal{N}_e| - 1} U \Delta_{i,j} \leq U \Delta_{i,j}.
$$

(56)

Equation (54) is obtained by linearity of expectation and taking the expectation over rewards of uniformly distributed actions $a_i$ and $a_j$. Equation (55) is obtained by noting that there exist exactly $\left( \frac{|\mathcal{N}_e| - 2}{K_e - 1} \right)$ actions where arm $i$ is replaced by arm $j$. From Assumption 3 of Lipschitz continuity, the difference between the expected reward of those actions is bounded by $U \Delta_{i,j}$. The remaining actions contain both arms $i$ and $j$, thus are in both $\mathcal{N}_e(i)$ and $\mathcal{N}_e(j)$, and so cancel out. Equation (56) comes from simplifying the fraction with binomial and noticing that $K_e \geq 1$. This proves the upper bound.

Similarly we obtain the lower bound using Assumption 1

$$
\mathbb{E} [Z_{i,j}(e)] = \mathbb{E} [r_{a_i}(e) - r_{a_j}(e)]
$$

(57)

$$
= \frac{1}{(|\mathcal{N}_e|-1)} \left( \sum_{a \in \mathcal{N}_e(i)} \mu_a - \sum_{a \in \mathcal{N}_e(j)} \mu_a \right)
$$

(58)

$$
\geq \frac{\Delta_{i,j}}{U} \frac{1}{(|\mathcal{N}_e|-1)} \left( \frac{|\mathcal{N}_e| - 2}{K_e - 1} \right)
$$

(59)

$$
= \frac{\Delta_{i,j} |\mathcal{N}_e| - K_e}{U |\mathcal{N}_e| - 1} \geq \frac{\Delta_{i,j}}{UK} \geq \frac{\Delta_{i,j}}{UK}.
$$

(60)

Equation (55) is obtained from Assumption 1. The difference between the expected reward of the actions are lower bounded by $\frac{\Delta_{i,j}}{U}$. Equation (60) comes from noting that $K_e(|\mathcal{N}_e| - K_e) \geq |\mathcal{N}_e| - 1$. This proves the lower bound.

Before proving the result, we first state the Azuma-Hoeffding Lemma which we use to calculate the concentration inequalities.

**Lemma 6** (Azuma-Hoeffding [9, Chapter 3]). If $\{W_n\}$ is a zero-mean martingale process with almost surely bounded increments $|W_n - W_{n-1}| \leq C$, then for any $\delta > 0$ with probability at least $1 - \delta$, $|W_n| \leq C \sqrt{2n \log(2/\delta)}$.  


Proof of Lemma 3. Notice that the difference of estimates of arms $i$ and $j$, $\hat{\mu}_i - \hat{\mu}_j$, at epoch $e$ is $(\sum_{e'=1}^{e} Z_{i,j}(e'))/e$. Also, the number of times arm $i$ and arm $j$ are sampled is the same as the epoch counter $e$ as each arm is sampled only once in an epoch. Using the concentration bound in Lemma 6 on $|Y_{i,j}(e)|$ with $C = 2$, then with probability at least $1 - \delta$, we get

$$\left| \frac{Y_{i,j}(e)}{e} \right| \leq \frac{2}{e} \sqrt{\frac{2e \log (2/\delta)}{e}}$$

$$= \sqrt{\frac{8 \log (2/\delta)}{e}}$$

$$\leq \sqrt{\frac{8 \log (2/\delta) \Delta^2}{32 \log (2/\delta)}}$$

$$\leq \Delta,$$

(61)

where equation (61) follows from the condition in the statement of Lemma 3 that $\frac{32 \log 2/\delta}{\Delta^2} \leq e$.

Plugging in the value of $Y_{i,j}(e)$ from Equation (13), we have

$$\frac{1}{e} \left| \sum_{e'=1}^{e} Z_{i,j}(e') - \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')] \right| \leq \Delta$$

(62)

Taking the positive part of the left hand side and rearranging, we have

$$\frac{1}{e} \sum_{e'=1}^{e} Z_{i,j}(e') - \Delta \leq \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')]$$

(63)

Since $\hat{\mu}_i - \hat{\mu}_j$ is $(\sum_{e'=1}^{e} Z_{i,j}(e'))/e$, we have

$$0 < \hat{\mu}_i - \hat{\mu}_j - \Delta \leq \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')]$$

(64)

Also, from $\hat{\mu}_i - \hat{\mu}_j < 2\Delta$ (from statement of Lemma 3) and from the definition of $Y_{i,j}(e)$ in Equation (13), we have

$$\hat{\mu}_i - \hat{\mu}_j < 2\Delta$$

(65)

Subtracting $\frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')]$ from both sides, we have

$$\hat{\mu}_i - \hat{\mu}_j - \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')] < 2\Delta - \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')]$$

(66)
Again using \( \hat{\mu}_i - \hat{\mu}_j = \left( \sum_{e'=1}^{e} Z_{i,j}(e') \right)/e \),
\[
\frac{1}{e} \sum_{e'=1}^{e} Z_{i,j}(e') - \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')] < 2\Delta - \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')] \tag{67}
\]

By definition of \( Y_{i,j}(e) \), we obtain
\[
\frac{Y_{i,j}(e)}{e} < 2\Delta - \frac{1}{e} \sum_{e'=1}^{e} \mathbb{E}[Z_{i,j}(e')] \tag{68}
\]
\[
< 2\Delta - \frac{\Delta_{i,j}}{UK} \tag{69}
\]

Combining Equation (69) and the negative part of left hand side of Equation (62), we have
\[
-\Delta \leq \frac{Y_{i,j}(e)}{e} < 2\Delta - \frac{\Delta_{i,j}}{UK} \tag{70}
\]
\[
\Rightarrow \frac{\Delta_{i,j}}{UK} < 3\Delta \tag{71}
\]
\[
\Rightarrow \Delta > \frac{\Delta_{i,j}}{3UK} \tag{72}
\]

Combining Lemma 6 and the minimum value of \( \Delta \) we obtain the required upper bound on the number of epochs to separate arm \( i \) and arm \( j \).

**Proof.** Let \( d_a = (X_{a1}, \cdots, X_{aK}) \) be the random vector of the rewards of the arms in action \( a \). Also, let \( d_{a^*} = (X_1, \cdots, X_K) \) be the random vector of the rewards of the optimal arms. Then from Assumption 3 we have

\[
\left| \mu_a - \mu_{a^*} \right| \leq U \min_{\pi \in \Pi} \left\| \mathbb{E}[d_a] - \pi \left( \mathbb{E}[d_{a^*}] \right) \right\|_2 \leq U \min_{\pi \in \Pi} \left\| \mathbb{E}[d_a] - \pi \left( \mathbb{E}[d_{a^*}] \right) \right\|_1 \tag{73}
\]
\[
= U \min_{\pi \in \Pi} \sum_{i=1}^{K} \left| \mathbb{E}[X_{a_i}] - \mathbb{E}[X_{\pi(i)}] \right| \tag{74}
\]
\[
\leq U \sum_{i=1}^{K} \left| \mathbb{E}[X_{a_i}] - \mathbb{E}[X_{\pi'(i)}] \right|, \tag{75}
\]

where (73) uses the property that the \( \ell_2 \) norm is upper bounded by the \( \ell_1 \) norm, (74) evaluates the \( \ell_1 \) norm and uses the property that \( a^* \) contains arms 1 through \( K \), and (75) holds for any permutation \( \pi' \) of \( \{1, \cdots, K\} \) which matches arms in \( a^* \) with corresponding \( a \), so \( \pi'(i) = a_i \) for any arm \( a_i \leq K \).
Proof. We begin by counting the cumulative pseudo-regret incurred from actions involving sub-optimal arms. Let \( i \geq K+1 \) denote a sub-optimal arm. Similar to (4), let \( R_i \) denote the cumulative pseudo-regret incurred from all of the actions \( a_t \) with sub-optimal arm \( i \in a_t \),

\[
R_i = \mathbb{E}_{a_1, \ldots, a_T} \left[ \sum_{t=1}^{T} (\mu_{a^*} - \mu_{a_t}) 1_{i \in a_t} \right].
\]  

(76)

Similarly, let \( R_{i,j} \) denote the cumulative pseudo-regret incurred from all of the actions \( a_t \) with sub-optimal arm \( i \in a_t \), with respect to the action where an optimal arm \( j \notin a_t \) is swapped with arm \( i \),

\[
R_{i,j} = \mathbb{E}_{a_1, \ldots, a_T} \left[ \sum_{t=1}^{T} (\mu_{\{j\} \cup a_t \setminus \{i\}} - \mu_{a_t}) 1_{i \in a_t} 1_{j \notin a_t} \right].
\]  

(77)

Similar to [33], we group the top-\( K \) arms into two groups based on how easy it is to separate them from the \((K+1)\)th arm, \( R^< = \{ j \mid 1 \leq j \leq K, \Delta_{j,K+1} \leq \Delta_{K,i} \} \) and \( R^> = \{ j \mid 1 \leq j \leq K, \Delta_{j,K+1} > \Delta_{K,i} \} \). This gives the bound on the regret from arm \( i \), \( R_i \) as

\[
R_i \leq \sum_{j \in R^<} R_{i,j} + \sum_{j \in R^>} R_{i,j}
\]  

(78)

We now calculate the regret for both groups \( R^< \) and \( R^> \) separately in the following cases:

1) **Regret by replacing arm** \( j \in R^< \): Note that \( \Delta_{j,i} < \Delta_{j,K+1} + \Delta_{K,i} \) as the gap for \( \Delta_{K,K+1} \) is counted twice. Hence we get the upper bound on regret as,

\[
\sum_{j \in R^<} R_{i,j} \leq \frac{288U^2K^2 \log (2/\delta)}{\Delta_{K,i}^2} \max_{j \in R^<} U \Delta_{j,i}
\]

(79)

\[
\leq \frac{288U^2K^2 \log (2/\delta)}{\Delta_{K,i}^2} \max_{j \in R^<} U (\Delta_{j,K+1} + \Delta_{K,i})
\]

(80)

\[
\leq \frac{288U^2K^2 \log (2/\delta)}{\Delta_{K,i}^2} U (2\Delta_{K,i}) = \frac{576U^3K^2 \log (2/\delta)}{\Delta_{K,i}}
\]

(81)

2) **Regret by replacing arm** \( j \in R^> \): Note that since \( \Delta_{j,K+1} > \Delta_{K,i} \), arm \( j \) will move to the accept set before arm \( i \) is rejected with probability at least \( 1 - 2\delta \). Once the algorithm moves arm \( j \) to the accept set \( A_e \), it will not suffer any regret from replacing arm \( j \). Let, \( l = \arg \min_{j \in R^> \Delta_j} \), then we
can bound the regret from arms in $R^>$ using following inequalities.

\[
\sum_{j \in R^>} R_{i,j} \leq \frac{288U^2K^2 \log(2/\delta)U \Delta_{1,i}}{\Delta_{1,K+1}^2} + \sum_{j=2}^{l} 288U^2K^2 \log(2/\delta) \left( \frac{U \Delta_{j,i}}{\Delta_{j-1,K+1}^2} - \frac{U \Delta_{j,K+1}}{\Delta_{j-1,K+1}^2} \right) \tag{82}
\]

\[
\leq 288U^3K^2 \log(2/\delta) \left( \sum_{j=1}^{l-1} \frac{\Delta_{j,i} - \Delta_{j+1,i}}{\Delta_{j,K+1}^2} \right) + \left( \frac{\Delta_{l,i}}{\Delta_{l,K+1}^2} \right) \tag{83}
\]

\[
\leq 288U^3K^2 \log(2/\delta) \left( \sum_{j=1}^{l-1} \frac{\Delta_{j,K+1} - \Delta_{j+1,K+1}}{\Delta_{j,K+1}^2} \right) + \left( \frac{\Delta_{l,K+1} + \Delta_{K,i}}{\Delta_{l,K+1}^2} \right) \tag{84}
\]

\[
\leq 288U^3K^2 \log(2/\delta) \left( \int_{\Delta_{l,K+1}}^{\Delta_{l,K+1}+1} x^{-2} \, dx + \frac{2\Delta_{l,K+1}}{\Delta_{l,K+1}^2} \right) \tag{85}
\]

\[
= 288U^3K^2 \log(2/\delta) \left( \frac{1}{\Delta_{l,K+1}^2} - \frac{1}{\Delta_{l,K+1}} + \frac{2}{\Delta_{l,K+1}} \right) \tag{86}
\]

\[
\leq 288U^3K^2 \log(2/\delta) \left( \frac{3}{\Delta_{l,K+1}} \right) \leq \frac{864K^2U^3 \log(2/\delta)}{\Delta_{K,i}} \tag{87}
\]

Summing up the regrets for $R^<$ and $R^>$, we get total regret for sampling arm $i$ to be bounded as

\[
R_i = \frac{1440U^3K^2 \log(2/\delta)}{\Delta_{K,i}} \tag{88}
\]
Proof. We note that there are three sources of regret for the DART.

1) The first is that regret will accumulate while eliminating sub-optimal arms. From Lemma 5, the regret accumulated while eliminating sub-optimal arms is bounded as

$$\sum_{j=K+1}^{N} \frac{1440K^2U^3 \log 2/\delta}{\Delta_{K,i}}$$

(89)

2) The second is when the algorithm is not able to move optimal arms from $N_e$ to the accept set $A_e$ or move sub-optimal arms to the reject set $R_e$ because of separability. That is, $\Delta_{i,K+1} < \lambda$ for some optimal arm $i : 1 \leq i \leq K$, or $\Delta_{K,i}$ for some sub-optimal arm $i : K + 1 \leq i \leq N$.

To bound the regret, we will apply Lemma 4 with a sub-optimal action from a “worst-case” scenario where the top $K$ arms are not separable from the $(K + 1)$st, so $\Delta_{i,K+1} < \lambda$ for $i : 1 \leq i \leq K$, and the $(K + 1)$st through $(2K + 1)$st arms are not separable from the $K$th, so $\Delta_{K,i} < \lambda$ for $i : K + 2 \leq i \leq 2K + 1$. In this scenario, the accept set remains empty. Consider the action $a = (K + 2, \ldots, 2K + 1)$ formed by using only sub-optimal arms. Note we can make the regret largest with $\Delta_{K,K+1} \approx 0$. By construction, we have $\Delta_{1,2K+1} < 2\lambda$, e.g. those two arms cannot have means that are too far apart. Consequently, using Lemma 4 for this action $a$, the expected (instantaneous) regret can be bounded as $UK(2\lambda)$. Thus, we can bound the overall cumulative regret the algorithm will suffer from this issue as

$$2TU K\lambda$$

(90)

3) The third source of regret is due to either if a sub-optimal arm $i : K + 1 \leq i \leq N$ is not moved to “reject” set despite $\Delta_{K,i} \geq \lambda$ or if an optimal arm $i : 1 \leq i \leq K$ is not moved to “accept” set despite $\Delta_{i,K+1} > \lambda$, in the number of rounds calculated in Lemma 3. Using the union bound and Lemma 6, the probability of this event can be bounded using $N\delta$ for each arm moved to corresponding incorrect set. For $\delta = \frac{1}{NT}$, using Lemma 4 with the loose upper bound of $UK$ (since the difference in means of any pair of arms is at most 1), the expected cumulative regret from this situation can be bounded as

$$TU K \times N \frac{1}{NT} = UK$$

(91)
Thus, the total regret of DART algorithm can be bounded as
\[
R \leq \sum_{j=K+1}^{N} \frac{1440U^3K^2 \log 2NT}{\Delta_{K,i}} + 2TUK\lambda + UK
\] (92)
\[
\leq \frac{1440NU^3K^2 \log 2NT}{\lambda} + 2TUK\lambda + UK.
\] (93)

Equation (93) is obtained from the fact that the algorithm stops sampling arms if the gap is small and cannot be resolved.

Choosing \(\lambda = U\sqrt{\frac{720N K \log 2NT}{T}}\), we get the required regret bound.

Our proposed algorithm DART requires the time horizon \(T\) as an input. However, DART can be modified to not require knowledge of \(T\). We use the standard doubling trick from Multi-Armed Bandit literature [7, 10]. To use the doubling trick, we start the algorithm from \(T_0 = 0\). We restart the algorithm after every \(T_l = 2^l\), \(l = 1, 2, \cdots\) time steps, till the algorithm reaches the unknown \(T\). Each restart of the algorithm runs for \(T_l - T_{l-1}\) steps with \(T_0 = 0\) with \(\lambda_l = \sqrt{\frac{1440N \log 2N(T_l - T_{l-1})}{K(T_l - T_{l-1})}}\).

To show that the regret is bounded by \(T^{1/2}\) for the doubling algorithm, we use Theorem 4 from [10] which we state in the following lemma.

**Lemma 7.** [10, Theorem 4] If an algorithm \(A\) satisfies \(R_T(A_T) \leq c T^\gamma (\log T)^\delta + f(T),\) for \(0 < \gamma < 1, \delta \geq 0\) and for \(c > 0,\) and an increasing function \(f(t) = o(t^\gamma (\log t)^\delta)\) (at \(t \to \infty\)), then anytime version \(A' := DT(A, (T_i)_{i \in \mathbb{N}})\) with geometric sequence \((T_i)_{i \in \mathbb{N}}\) of parameters \(T_0 \in \mathbb{N}^*, b > 1, (i.e., T_i = \lfloor T_0 b^i \rfloor)\) with the condition \(T_0(b-1) > 1\) if \(\delta > 0\) satisfies,
\[R_T(A') \leq l(\gamma, \delta, T_0, b) c T^\gamma (\log T)^\delta + g(T),\] (94)

with a increasing function \(g(t) = o(t^\gamma (\log t)^\delta)\) and a constant loss \(l(\gamma, \delta, T_0, b) > 1,\)
\[l(\gamma, \delta, T_0, b) := \left(\frac{\log(T_0(b-1) + 1)}{\log(T_0(b-1))}\right)^\delta \times \frac{b^\gamma(b-1)^\gamma}{b^\gamma - 1}\] (95)

Using Lemma 7 for \(b = 2, \gamma = 1/2, \delta = 1/2,\) we can convert our algorithm to an anytime algorithm.

A. Comparisons with LinUCB algorithm for linear case

We compare DART with CSAR [33], CMAB-SM [2] algorithms, UCB [7], and LinUCB [27]. We choose \(T = 5 \times 10^4, N = 15,\) and \(K = 2\) to keep the number of actions manageable. For the joint reward function, we use the mean of the rewards of the individual arms. We run 25 independent iterations
to plot the average cumulative regret, where the average is over the 25 runs. The error bars indicate the maximum and minimum values of the cumulative regret of each algorithm. Individual arm rewards followed the Bernoulli distribution with means sampled from $\mathbb{U}(0, 1)$. The arm reward means were kept constant across the runs.

![Regret plots for joint rewards as mean of individual arm rewards for $N = 15, K = 2$.](image)

Fig. 3: Regret plots for joint rewards as mean of individual arm rewards for $N = 15, K = 2$.

The results are shown in Figure 3. DART outperforms all the algorithms significantly. Since LinUCB algorithm selects an action with the highest upper confidence bound at any time $t$ and never removes sub-optimal actions, it converges slowly compared to the other algorithms. Also, the CSAR algorithm attempts to estimate individual arm rewards and then find the best action; it suffers the highest regret among UCB based algorithms. CMAB-SM performs better than CSAR. It eliminates actions faster because of the divide-and-conquer strategy which reduces arms directly instead of estimating rewards first. For small $K$, $\binom{N}{2} \approx N^2$; consequently eliminating actions is faster compared to estimating individual arm rewards and then finding best actions. This accounts for the reason of the performance of UCB. Lastly, DART uses a similar strategy to UCB. However, DART does not eliminate actions by finding the arm estimates but directly removing arms using action rewards.

**B. Joint reward as $\max$ of arm rewards**

This experimental setup was the same as Appendix A, except the joint reward function is of the form $r(t) = \max_{i \in a_t} X_{i,t}$. The cumulative regrets of DART and other algorithms, averaged over 20 runs, are shown in Figure 4. We note that the performance of DART is significantly better than all other algorithms considered for $K = 2$. We note that LinUCB and CSAR algorithms perform the worse. We suspect this is because the algorithms are not able to approximate the $\max$ function with a linear model. For $K = 4$, we note that CMAB-SM algorithm performs the best. We suspect that is because CMAB-SM is able to eliminate arms faster for small $U$ with a divide and conquer approach.
Fig. 4: Regret plots for joint rewards as max of individual arm rewards

(a) $K = 2$

(b) $K = 4$