Matching van Stockum dust to Papapetrou vacuum*  

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Abstract  
Addressing a long-standing problem, we show that every van Stockum dust can be matched to a 1-parametric family of non-static Papapetrou vacuum metrics, and the converse. The boundary, if existing, is determined by vanishing of certain first-order invariant on the vacuum side. Moreover, we establish a relation to Ehlers and Kramer–Neugebauer transformations, which allows us to look for dust clouds with a prescribed boundary. Explicit examples include the Bonnor metric and a new vacuum exterior to the Lanczos–van Stockum dust metric, as well as dust clouds with nontrivial topology.  

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1. Introduction  
General relativity is an inexhaustible source of mathematical challenges. Although exact vacuum and dust metrics are known in abundance, much less is known about possible dust-vacuum configurations. The central result of this paper is a correspondence between the van Stockum [41] class of dust metrics and the Papapetrou [34] class of vacuum metrics such that the corresponding metrics match, i.e., can be glued together along a common boundary. As an auxiliary result, we prove that the boundary, if existing, is determined by vanishing of certain first-order invariant on the vacuum side (Corollary [1]).  
We also relate the matching van Stockum–Papapetrou pairs to static Weyl vacuum metrics and, in particular, to axisymmetric harmonic functions in $\mathbb{R}^3$. This yields two equivalent ways to obtain dust clouds with a prescribed boundary. Explicit worked-out examples include a vacuum exterior to the Bonnor [3] dust metric, a dust source for the Halilsoy [17] vacuum metric, a new cylindrically non-symmetric vacuum exterior to the Lanczos [24]–van Stockum [41] dust metric, as well as two toroidal dust-vacuum configurations.  

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The problem of matching a general van Stockum dust to vacuum has been open for decades (Bonnor [4] § (3)), Viaggiu [43], Zingg et al. [46], Gürlebeck [14]). The list of previously known van Stockum–vacuum $C^1$-metrics consists of only two items: the van Stockum’s rigidly rotating dust cylinder [41] composed of the Lanczos dust [24] and a Lewis vacuum [28] (three Killing vectors); and the Zsigrai [47] metric composed of the Lukács–Newman–Sparking–Winicour [29] dust metric and the vacuum NUT metric [33] (four Killing vectors).

We only consider glued metrics of class $C^1$ on the boundary in the sense of Lichnerowicz [27], which we call simply $C^1$-metrics. Dust-vacuum $C^1$-metrics model massive dust clouds, leaving the well-known known $C^0$-continuous relativistic analogues of infinitely thin rotating Newtonian discs outside the scope of this paper.

2. Matching

To start with, we present Lichnerowicz’s matching as a particular instance of the contact condition. The latter allows for a simultaneous treatment of a number of derivatives.

Let $M$ be a space-time manifold, partitioned by a two-sided hypersurface $B$. By definition, functions $f^{(I)}, f^{(II)} \in C^\infty M$ satisfy the condition $f^{(I)} \equiv_B f^{(II)}$ of $k$-th order contact along $B$ if and only if, in each coordinate patch,

$$f^{|_{B_{1\ldots l}}} = f^{|_{B_{1\ldots l}}} \text{ for all } 0 \leq l \leq k,$$

where $f_{,i\ldots j}$ denote the partial derivatives and $f|_{B}$ is the restriction to $B$.

The relation $\equiv_B$ is an equivalence relation and, what is of utmost importance, a congruence of the algebra of functions on $M$ with arbitrary $C^k$-continuous multivariate functions as operations. This can be formulated as the following proposition, which can be easily proved by using the chain rule.

**Proposition 1.** Assuming $f_1^{(I)} \equiv_B f_2^{(II)}, \ldots, f_m^{(I)} \equiv_B f_m^{(II)}$, let $F(f_1, \ldots, f_m) \in C^\infty M$ be a $C^k$-continuous function in a neighbourhood of the image $f_1^{(I)} B \times \cdots \times f_m^{(I)} B = f_1^{(II)} B \times \cdots \times f_m^{(II)} B$. Then

$$F(f_1^{(I)}, \ldots, f_m^{(I)}) \equiv_B F(f_1^{(II)}, \ldots, f_m^{(II)})$$

holds.

Assume $M$ multi-partitioned into $n$ connected manifolds $\bar{U}^{(i)} = U^{(i)} \cup \partial U^{(i)}$ with pairwise disjoint boundaries $\partial U^{(i)}$, separated by a two-sided (discontinuous) hypersurface

$$B = \bigcup_{i=1}^{n} \partial U^{(i)}.
$$

Let $f^{(I)}, f^{(II)} \in C^\infty M$, assuming $f^{(I)} \equiv_B f^{(II)}$. Let $v_i = I$ or $II$ at will, $i = 1, \ldots, n$ (we are free to keep or swap). Then the unique function $f$ on $M$ that coincides with $f^{(v_i)}$ on $U^{(i)}$ is $C^k$-continuous. This way of obtaining a $C^k$-continuous function $f$ is known as gluing along the boundary $B$; there are $2^n$ possible combinations corresponding to $2^n$ possible choices of $\{v_i\}_{i=1,\ldots,n}$. Generalisation to three and more functions $f^{(I)}, f^{(II)}, f^{(III)}, \ldots \in C^\infty M$ is straightforward.

Continuity, contact and gluing extend to tensors in an obvious way. A metric $g$ is said to be of class $C^k$, or a $C^k$-metric, if there exist coordinates $x^\mu$ with respect to which the metric coefficients
\( g_{\mu\nu} \) are of class \( C^k \). The condition of \( C^k \)-continuity of a metric is preserved under all coordinate transformations of class \( C^{k+1} \). Similarly for other tensors.

Given two \( C^\infty \)-metrics \( g^{(I)}, g^{(II)} \) determined by their coordinate components \( g^{(I)}_{\mu\nu}, g^{(II)}_{\mu\nu} \) with respect to coordinates \( x^\mu \), then

\[
\tilde{g}^{(I)}_{\mu\nu} = \tilde{g}^{(II)}_{\mu\nu}
\]

implies that the “piecewise” metric obtained by gluing the coordinate components along \( B \) is \( C^k \)-continuous. As with functions, \( 2^n \) combinations corresponding to \( n \) connected components are possible.

If \( k = 1 \), conditions (2) coincide with the Lichnerowicz matching conditions (“conditions de raccordement” [27, p. 61]) of general relativity. The coordinates \( x^\mu \) with respect to which the matching conditions (2) hold are called admissible. Perhaps more appropriately, we shall call them shared coordinates.

The \( C^1 \)-metrics occur as solutions of the Einstein equations with a discontinuous energy-momentum tensor, of which dust clouds are important examples. Concerning the dust movement, it is required that the trajectories (actually geodesics) the dust particles move along are either disjoint with the joining hypersurface or contained in it [27, p. 63], which is obviously true in our setting.

### 3. The choice of coordinates

Metrics we intend to match should be written in convenient shared coordinates. We do not insist the boundary \( B \) to have any special coordinate expression. Actually, our results would be quite difficult to obtain under any such stipulation.

Both the van Stockum and the Papapetrou metrics admit two commuting and orthogonally transitive Killing vectors, one time-like and one space-like. Assuming shared Killing vectors, the glued metric can be written in the form [30]

\[
g = \tilde{g}_{ij}(t^1, t^2) \, dt^i dt^j + h_{kl}(t^1, t^2) \, dz^k dz^l,
\]

where \( i, j, k, l = 1, 2 \). The Killing vectors are \( \xi^{(k)}_l = \partial / \partial z^k \) and also linear combinations thereof. The term \( \tilde{g}_{ij} \, dt^i dt^j \) represents the metric on the quotient space of Killing orbits [15], referred to as the orbit space, while \( h_{kl} = g(\xi^{(k)}, \xi^{(l)}) \). In what follows, \( \det h < 0 \) everywhere (one of the Killing vectors is time-like) and, consequently, \( \det g > 0 \) (the orbit metric is Riemannian). We also assume that the boundary hypersurface retains the symmetry and projects to a curve in the two-dimensional orbit space. The matching problem reduces to finding that curve.

Local coordinate transformations

\[
\tilde{t}^i = \Phi^i(t^1, t^2), \quad \tilde{z}^k = A^k_{\ell} \xi^\ell
\]

preserve the form (3) of the metric. Here \( \Phi^i(t^1, t^2) \) are local coordinate transformations in the orbit space, while \( A = (A^m_i) \in \text{GL}_2 \) are constant matrices acting on the components \( h_{kl} \) by

\[
\tilde{h}_{kl} = A^m_{\ell} h_{mn} A^n_{\gamma},
\]

(linear transformation of the Killing vectors).

Furthermore, the glued orbit space admits isothermal coordinates, i.e., coordinates \( x, y \) in which the orbit metric assumes the form \( p(x, y) \, (dx^2 + dy^2) \). This follows from the Korn–Lichtenstein
 theorem [22, 28] (see also [18] p. 262 or [10] p. 772), according to which local isothermal co-
ordinates exist under the assumption of the Hőlder $C^{0,\alpha}$-continuity of degree $0 < \alpha \leq 1$. Since
the $C^1$-continuity required by the Lichnerowicz conditions implies the Hőlder $C^{0,\alpha}$-continuity,
shared isothermal coordinates $x, y$ are guaranteed to exist on the orbit space. This means that the
plued space-time metric can be written in the Lewis–Papapetrou [26] form

$$g = e^\rho (dx^2 + dy^2) + h_{ij} dx^i dy^j. \tag{6}$$

The remaining freedom to transform $z^1, z^2$ is represented by the GL2-action [5]. Thus, two Lewis–
Papapetrou metrics $g^{(1)} = e^{\rho^{(1)}} (dx^2 + dy^2) + h^{(1)}_{ij} dx^i dy^j$, $g^{(2)} = e^{\rho^{(2)}} (dx^2 + dy^2) + h^{(2)}_{ij} dx^i dy^j$
match if and only if

$$\rho^{(1)} = \rho^{(2)}, \quad h^{(1)}_{ij} = A^{\mu}_{\nu} h^{(2)}_{\mu\nu}. \tag{7}$$

Here $A \in \text{GL}_2$ is an arbitrary constant matrix.

4. A useful invariant

According to [31], metrics (5) possess four algebraically independent first-order scalar invariants
$C_\rho, C_X, Q_X, Q_T$ with respect to coordinate transformations [4]. Given a Lewis–Papapetrou
metric (6), the scalar invariant $Q_X(g)$ is

$$Q_X(g) = \frac{\det \chi}{e^{2\rho}}, \quad \chi = \frac{1}{\det h} \begin{vmatrix} dh_{11} & dh_{12} \\ dh_{12} & dh_{22} \end{vmatrix}. \tag{8}$$

Being a first-order invariant, $Q_X(g)$ is continuous for every $C^1$-metric $g$. Therefore, the matching
conditions (2) imply

$$Q_X(g^{(1)})|_B = Q_X(g^{(2)})|_B. \tag{9}$$

Condition (3) provides a tool to locate the boundary $B$, while avoiding the use of the unknown
matrix $A$. An analogous condition can be written for every first-order scalar invariant. However, $Q_X$
is special in that it vanishes for dust metrics of interest in this paper, see Proposition 2.

A dust metric of the Lewis–Papapetrou class is said to be circular if the 4-velocity $U$ belongs
to the distribution spanned by the Killing vectors, i.e., can be written as $U = U^k \partial / \partial z^k$, $k = 1, 2$.

**Proposition 2.** Every circular Lewis–Papapetrou dust metric satisfies

$$Q_X(g) = 0. \tag{10}$$

**Proof.** The energy–momentum tensor is $T^{\mu\nu} = \rho^{\text{dust}} U^\mu U^\nu$, where $\rho^{\text{dust}}$ is the density. The van-
ishing of the divergence, $T^{\mu\nu}_{\nu} = 0$, implies $h_{ij} U^i U^j = 0 = h_{ij} U^i U^j$, which is a system of
homogeneous equations on $U^1, U^2$. This system has a nonzero solution if and only if its resultant
is zero, which is equivalent to $Q_X(g) = 0$.

It easily follows that the boundary between vacuum and circular dust, if existing, is almost
uniquely determined on the vacuum side. By almost uniquely we mean that for each boundary
separating two connected components we can choose to perform gluing or not, see Sect. 2.

**Corollary 1.** If a Lewis–Papapetrou metric $g$ matches a van Stockum dust metric along a bound-
dary $B$, then $Q_X(g)|_B = 0$. 

4
5. Van Stockum metrics

Henceforth we restrict attention to dust metrics of the van Stockum class [41]. Van Stockum’s dust is isometrically flowing [44], meaning that the 4-velocity $U^a$ is a Killing vector.

We briefly recall the derivation of van Stockum equations [41, Eqs. (5.5)–(5.10)]. Renaming the coordinates as $t^1 = x, t^2 = y, t^3 = z$, where $x, y$ are isothermal (see Section 2), coordinates $\phi, t$ can be chosen as comoving, i.e., in such a way that the 4-velocity $U^a$ equals $\partial / \partial t$. Then the coefficient at $dt^2$, which is the magnitude $g^{\mu \nu} U^\mu U^\nu$ of the 4-velocity, equals $-1$.

Consequently, a general van Stockum dust metric can be written in the form

$$g^d = e^\rho (dx^2 + dy^2) + r^2 d\phi^2 - (f d\phi + dt)^2.$$ (9)

The dust moving circularly, the Einstein equations

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = T_{\mu \nu}$$

imply

$$R_{33} + R_{44} = 0.$$ (10)

This gives $r_{xx} + r_{yy} = 0$. Then $r$ is a harmonic function that, if non-constant, can serve as one of the isothermal coordinate functions (Weyl’s canonical coordinates [45, p. 137] or [26]). With $r = x$, the Einstein equations reduce to

$$f_{xx} + f_{yy} - f_x = 0, \quad p_x = \frac{f_y^2 - f_x^2}{2x}, \quad p_y = -f_x f_y,$$ (11)

while the dust density turns out to be

$$\rho^d = \frac{f_x^2 + f_y^2}{x^2 e^\rho}.$$ (12)

The case of constant $r$ (or, invariantly, $C_\rho = 0$ by [9]) has been studied by Hoenselaers and Vishveshwara [19], who also found a matching vacuum metric.

6. Papapetrou metrics

The vacuum part is chosen to be the Papapetrou metric [34]. The necessary details are set forth below with the aim to expose similarity to the dust case. In isothermal coordinates,

$$g^v = e^q (dx^2 + dy^2) + \frac{r^2}{v} d\phi^2 - v(w d\phi + dr)^2.$$ (13)

by appropriate choice of the field variables in eq. [3]. Again, we have $R^3 + R^4 = 0$, giving $r_{xx} + r_{yy} = 0$, i.e., $r$ is a harmonic function. Since $\operatorname{det} h = -x^2$ on the dust side and $\operatorname{det} h = -r^2$ on the vacuum side are required to have a first-order contact on the boundary, we have simultaneous Dirichlet and Cauchy boundary conditions $r = \pm x, r_x = \pm 1, r_y = 0$ and we are left with $r = x$ everywhere. This also restricts the choice of the matrix $A$ in the matching condition (7) to SL-2.

The vacuum Einstein equations reduce to

$$w_{xx} + w_{yy} - \frac{w_x}{x} = -2 \frac{v_x w_x + v_y w_y}{v},$$
$$v_{xx} + v_{yy} + \frac{v_x}{x} = \frac{v^2 + v_y^2}{v} - \frac{v^2 w_x^2 + w_y^2}{x^2},$$
$$q_x = -\frac{v_x}{v} + \frac{x}{2v^2} (v_y^2 - v_x^2) - \frac{v^2 (w_x^2 - w_y^2)}{2x},$$
$$q_y = -\frac{v_y}{v} + \frac{x v_x v_y}{v^2} - \frac{v^2 w_x w_y}{x}.$$ (14)
The Papapetrou class of metrics is determined by the condition
\[ v_x w_x + v_y w_y = 0 \]  
(eq. (3.1)). If \( w = \text{const} \), then the Riemann tensor becomes zero and the metric is flat. Otherwise \( w_x \neq 0 \) or \( w_y \neq 0 \). We work out the case \( w_y \neq 0 \) (the other one leads to the same result).

Denoting
\[ c^2 = \frac{x^2 v_x^2}{v^2 w_y^2} + v^2 > 0, \]
\[ (15) \]

it is easily checked that \( c_x = c_y = 0 \) in consequence of equations (13) and (14). Hence, expression (15) is a first integral, meaning that solutions are classified by \( c > 0 \). However, system (13), the Papapetrou condition (14) and formula (15) are preserved under a three-dimensional Lie group of coordinate transformations (4), one of the generators being
\[ \mathcal{S}_a \]

Using transformation \( \mathcal{S}_a \), one can always normalise \( c \) to 1, which we assume henceforth. Setting
\[ v = 1 / \cosh u, \quad e^s = \frac{e^u}{v} = e^u \cosh u, \]
the whole system (13), (14) simplifies to
\[ w_{xx} + w_{yy} - \frac{w_x}{x} = 0, \]
\[ u_x = -\frac{w_y}{x}, \quad u_y = \frac{w_x}{x}, \quad s_x = -\frac{w_x^2 - w_y^2}{2x}, \quad s_y = -\frac{w_x w_y}{x}. \]  
(17)

7. Locating the boundary

According to Corollary [14], the boundary \( B \) must satisfy \( Q(\mathcal{g}^\ast)|_B = 0 \). Computing the invariant \( Q_\chi \) of a general Papapetrou metric \( \mathcal{g}^\ast \) given by eq. (12) under identification (16), we obtain
\[ Q_\chi(\mathcal{g}^\ast) = \frac{e^u \sinh u}{e^u \cosh u} P(\sinh u, \cosh u, w_x, w_y), \]
where \( P \) is a polynomial. Consequently, the boundary is either \( u = 0 \) or \( P = 0 \). We continue with \( u = 0 \), which leads to the general result presented below.

Definition 1. A dust metric \( \mathcal{g}^d \) determined by field variables \( f, p \) satisfying equations (10) and a normalised non-flat Papapetrou metric \( \mathcal{g}^\ast \) determined by field variables \( w = f, s = p \) and \( u \) satisfying equations (17) are called companions.

Under relations (15), the companion dust and vacuum metrics can be written as
\[ \mathcal{g}^d = e^u (dx^2 + dy^2) + x^2 d\phi^2 - (f d\phi + dt)^2 \]
and
\[ \mathcal{g}^\ast = e^u \cosh u (dx^2 + dy^2) + x^2 \cosh u d\phi^2 - \frac{(f d\phi + dt)^2}{\cosh u}, \]
respectively.

The companion correspondence between the dust and the vacuum metrics is one to continuum, since system (17) determines \( u \) uniquely up to an integration constant.
**Theorem 1.** The companion dust and vacuum metrics match along the boundary located at $u = 0$.

**Proof.** Obviously from Proposition 1 and formulas (18) and (19),

$$g_{dij} \equiv \begin{cases} u = 0 \\ g_{vij} \end{cases},$$

since $\cosh u \equiv \begin{cases} u = 0 \\ 1 \end{cases}$ (omitting the superscript 1).

A $C^1$-metric obtained by gluing according to Theorem 1 will be called the van Stockum–Papapetrou $C^1$-metric or the van Stockum–Papapetrou dust cloud.

The normalisation $c = 1$ we made in Section 6 ensures that the coefficients $h^d_{ij} \equiv h^v_{ij}$ match without the need for an adjustment by transformation (5).

**8. Electrostatic analogy**

Dust clouds of a prescribed shape can be obtained in terms of the classical potential theory. Recall that companion metrics are determined by system (17). Eliminating $w$, one obtains the equivalent system

$$u_{xx} + u_{yy} + \frac{u_x}{x} = 0,$$

$$w_x = xu_y, \quad w_y = -xu_x, \quad s_x = x \frac{u_x^2 - u_y^2}{2}, \quad s_y = xu_xu_y.$$

Here $u_{xx} + u_{yy} + u_x/x = 0$ is the cylindrical Laplace equation. As is well known, its solutions correspond to axisymmetric solutions of the three-dimensional Laplace equation. Consequently, the admissible dust-vacuum boundaries, which are represented by the levels of $u$ by Theorem 1, correspond to equipotential surfaces of axisymmetric electrostatic potentials in dimension three. Thus, the problem of finding van Stockum–Papapetrou dust clouds of a prescribed shape reduces to that of finding electrostatic fields with a prescribed equipotential surface, which is a classical boundary problem in electrostatics.

Since $u$ is a harmonic function, it is either constant or unbounded, and the boundary curve $u = u_0 = \text{const}$ is nonempty and regular for the continuum of values of $u_0$ in the interval $(\liminf u, \limsup u) = (-\infty, \infty)$. Observe that $u = \text{const}$ if and only if $w = \text{const}$, in which case the Papapetrou metric is flat and has no van Stockum companion.

Choosing two different values for $u_0$, we obtain a dust layer sandwiched between two Papapetrou vacua. Moreover, disconnected as well as self-intersecting boundaries can occur. Examples are presented in Section 10. In the same vein, recent work [12] on the topology of level sets of harmonic functions in three dimensions reveals a rich topology of possible boundaries.

We end this section with some elementary facts related to the electrostatic picture.

**Proposition 3.** Raising indices with $g^d_{ij}$, we have

$$u^i u_j = \rho^d, \quad f^i u_j = 0.$$

**Proof.** By straightforward computation, using eq. (20).

Since $u$ and $f$ depend only on the orbit space coordinates, and the orbit space and the cut plane are locally conformally diffeomorphic, we have the following corollary.
Corollary 2. Level sets of functions $u$ and $f$ intersect orthogonally in both the orbit space and the cut plane $\mathbb{R}^2$.

Corollary 3. In the electrostatic picture, the level sets of $f$ coincide with the electric field lines corresponding to the potential $u$.

9. Relation to static Weyl vacuum metrics

The following proposition can be easily verified by straightforward computation.

Proposition 4. The dust and vacuum metrics (18) and (19) are, respectively, the Ehlers and the Neugebauer–Kramer transform of the static Weyl vacuum metric

$$e^{u+p}(dx^2 + dy^2) + x^2 e^u d\phi^2 - e^{-u} dt^2.$$ (21)

For the Neugebauer–Kramer transform of a static metric see [23, § 4.1], for the Ehlers transform of a static metric see [40, Theorem 21.1] or the original paper [11].

Thus, the companion correspondence can be decomposed as follows:

The last proposition opens a way to reuse static Weyl metrics and obtain dust clouds of a required shape $u = \text{const}$. The coefficient $e^{-u}$ at $dt^2$ is the relativistic analogue of the Newtonian gravitational potential (possibly somewhat distorted [25]). Otherwise said, boundaries of van Stockum dust clouds in Papapetrou vacuum correspond to Newtonian equipotential surfaces in static Weyl space-times, even unphysical ones (permitting negative masses).

Essentially, all static Weyl vacua are known and many particular cases have been studied (e.g., [7, 25, 38, 39, 40] and references therein). According to Proposition 4, each yields a matching van Stockum–Papapetrou pair along with an explicit expression for the field variables $u$ and $p$, sufficient to compute the dust density and admissible borders. By contrast, computing the coefficients $h_{ij}$ requires a closed-form representation for $f$, which is not always available, because $f$ is determined by a path integral which is not known analytically in many cases (see Examples 2 and 3), although its levels can be inferred from the levels of $u$ by orthogonality (Corollary 2). Yet all invariant quantities can be computed without explicit knowledge of $f$, including the Petrov type [40, Ch. 4] and the curvature invariants [40, Ch. 9], as well as the first-order invariants [31]. Obviously from (20), the derivatives of $f$ and $p$ are expressible via derivatives of $u$, although $f$ itself is not. This reflects the fact that $f \mapsto f + c$ corresponds to a coordinate transformation.

10. Examples

Example 1. Table 1 provides information about four cases when the dust part or the vacuum part or both have been studied earlier.

From left to right, the columns harbour the axisymmetric charge geometry (see Section 8); the potential $u$ the levels of which determine the boundaries according to Theorem 1; the field
Let us comment on individual rows of Table 1. For visualisations see the end of this section.

1. The vacuum is the spinning metric due to Halilsoy [17], which, contrary to its name, is not rotating [32]. For this reason, the matching surfaces \( R = \text{const} \) are not interpretable as spheres. The static seed (see Section 9) is the Chazy–Curzon metric.

2. The Bonnor–Bonnor cloud. The dust part is the well-known Bonnor metric [3]. The vacuum is due to Bonnor [5] as well, but the finding that both metrics match is new. All boundaries traverse the singularity located at the centre.

3. The dust part is the Lanczos [24] metric (the “cylindrical world”), which has been rediscovered by van Stockum [41] and matched to a cylindrically symmetric Lewis [26] vacuum metric along \( x = \text{const} \), yielding an infinite dust cylinder. Our matching along \( y = \text{const} \) breaks the cylindrical symmetry and results in a thick wall of rigidly rotating dust extending to infinity (ignoring what happens when the density becomes too high). This confirms that one and the same dust solution can match to different vacua along different boundaries (which is not true for perfect fluids, where the boundary occurs at the zero level of pressure).

4. The Zsigrai cloud. The Zsigrai [47] metric results from gluing the Lukács–Newman–Sparling–Winicour dust metric [29] to the vacuum NUT metric [33]. Matching is along isodensity surfaces since

\[
\rho = \frac{2}{b^2} \sinh^4 \frac{u}{R^2},
\]

in this case.
In the following two examples we reuse known toroidal static Weyl metrics to produce axisymmetric van Stockum–Papapetrou dust clouds (they satisfy the regular axis condition \[40, \S\ 19.1\]). Although closed-form representations for \(f\) are not available, we are able to understand the topologies the clouds can have. All clouds are named after the static vacuum seed.

**Example 2. Rotating Bach–Weyl cloud.** The seed is the static Bach–Weyl solution [2]. In Weyl coordinates, we can write (Semerák [39, III.B])

\[
\begin{align*}
  u &= \frac{4mK(\Omega)}{\sqrt{(x+a)^2 + y^2}}, \\
  \Omega &= 2\sqrt{\frac{ax}{(x+a)^2 + y^2}}, \\
  p &= -\frac{m}{a^2} \left( \frac{x^2 + y^2 + 3a^2}{(x+a)^2 + y^2} K(\Omega)^2 - 2K(\Omega)E(\Omega) + \frac{x^2 + y^2 - a^2}{(x-a)^2 + y^2} E(\Omega)^2 \right),
\end{align*}
\]

where \(K, E\) denote the complete elliptic functions.

Originally obtained as the gravitational field of a static ring, it can be also obtained as an invariant solution with respect to the Lie symmetry \(2\sqrt{xy_a + (a^2 - x^2 + y^2)}u_y + yu\), as can be easily checked. The density \([22]\) is

\[
\rho = \frac{4m^2}{x^2}\sqrt{\frac{K(\Omega)^2}{(x+a)^2 + y^2} - 2(a^2 - x^2 + y^2)} \frac{K(\Omega)}{(x+a)^2 + y^2} \frac{E(\Omega)}{(x-a)^2 + y^2} + \frac{E(\Omega)^2}{(x-a)^2 + y^2}.
\]

Boundaries can have one or two components, with a transient eightlike boundary passing through the saddle point \(2m\pi/a\).

**Example 3. Rotating Appell–Gleiser–Pullin cloud.** The seed is the static axisymmetric solution introduced by Gleiser and Pullin [16], which incorporates Appell’s [11] harmonic function possessing a circular singularity. In Weyl’s coordinates, the solution is determined by

\[
\begin{align*}
  u &= \sqrt{4a^2y^2 + (x^2 + y^2 - a^2)^2 + x^2 + y^2 - a^2}, \\
  p &= -\frac{x^2(x^2 + y^2 - 2ay - a^2)(x^2 + y^2 + 2ay - a^2)}{4((x+a)^2 + y^2)^2((x-a)^2 + y^2)^2} - \frac{x^2 + y^2 + a^2}{8a^2\sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2}}.
\end{align*}
\]

(Semerák [39, III.C]). The density \([22]\) is

\[
\rho = \frac{a^2 - x^2 - y^2 + \sqrt{(a+x)^2 + y^2}}{2a^2y^2((a+x)^2 + y^2)^2((a-x)^2 + y^2)^2} e^p \times \left( (x^2 + y^2)(x^2 + y^2 - a^2) - 2a^2y^2 \sqrt{(a+x)^2 + y^2} + \sqrt{(a-x)^2 + y^2} + (x^2 + y^2)(x^2 + y^2 - a^2) + 4a^2y^2 \right).
\]

The function \(u\) has two saddle points \((0, a), (0, -a)\), where it assumes the value \(u(0, \pm a) = \sqrt{3}/2a\). All level curves (boundaries) have two components, except the transient one, which is self-intersecting. Curves entering the singularity have cusps there.

For visualisation see Appendix A. A rich variety of admissible shapes can be seen. We already noted the possibility of hollow Bonnor–Bonnor clouds. The Bach–Weyl clouds can be also toroids, ovaloids containing an ovaloidal or toroidal hole, toroids containing a toroidal hole. Finally, the Appell–Gleiser–Pullin clouds can be also two disjoint (nested) hollow ovaloids, corresponding to two different two-component level sets of \(u\).
11. Discussion

What is really surprising is that the problem of matching van Stockum dust to vacuum has been waiting for solution so long, considering the simplicity of the answer and the demand for it \[4, 14, 43, 46\]. Not only are the Papapetrou and van Stockum metrics widely known, they also turn out to be rather natural candidates for matching. A hint from physics is that asymptotically flat rotating Papapetrou metrics require a zero-mass source, see \[21 \S 2.5\] or \[40 \S 20.3\], while the overall mass of the van Stockum dust is zero, since a negative mass singularity balances the positive mass of the dust (Bonnor \[3\], Bratek et al. \[6\]).

That said, we must also admit that rotating Papapetrou vacua have no known physical interpretation other than being a zero-mass limit \[5, 37\], while serious doubts persist about whether van Stockum dust can exist in nature \[8, 13, 14, 35, 36, 46\]. It is, however, no less true that negative masses have been admitted as constituents of relativistic models repeatedly during the last decades, suggesting that van Stockum metrics can avoid the fate of being unphysical. As a case in point, Ilyas et al. \[20\] proposed a measurement to identify possible occurrence of the Bonnor dust \[3\] in a galaxy centre. Anyhow, investigation of wider classes of dust-vacuum \(C^1\)-metrics, of which our Papapetrou–van Stockum class would be a limiting case, is under way.

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Appendix A. Visualisation

Figure A.1 shows three examples in axial section.

Figure A.1: Axisymmetric dust clouds examples in Weyl’s coordinates. (1) Densities (increasing from dark to light). (2) Admissible boundaries. (3) Example clouds.

The eightlike voids (to be viewed zoomed in) reveal directional singularities, which are artifacts of the Weyl coordinates (Taylor [42]).