EIGENFUNCTIONS WITH INFINITELY MANY ISOLATED CRITICAL POINTS

LEV BUHOVSKY, ALEXANDER LOGUNOV, MIKHAIL SODIN

Abstract. We construct a Riemannian metric on the 2-dimensional torus, such that for infinitely many eigenvalues of the Laplace-Beltrami operator, a corresponding eigenfunction has infinitely many isolated critical points.

1. Introduction and the main result

Let \((X, g)\) be a compact connected Riemannian manifold without boundary. The Riemannian structure \(g\) on \(X\) defines the Laplace-Beltrami operator \(\Delta_g\) on the space of smooth functions on \(X\). The operator \(\Delta_g\) admits a spectral decomposition via orthonormal basis of \(L^2(X, \Omega_g)\) (where \(\Omega_g\) is the volume density on \(X\) induced by \(g\)), consisting of smooth real-valued functions, and with corresponding real and non-negative eigenvalues:

\[
\Delta_g \varphi_j + \lambda_j \varphi_j = 0, \quad \text{where } \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \text{ and } \varphi_0 \equiv 1.
\]

The number of connected components of the set \(\text{Crit}(\varphi_k)\) of the critical points of \(\varphi_k\) is an important geometric characteristic of the eigenfunction \(\varphi_k\). We denote it by \(N_{\text{crit}}(k)\). In [6], Yau asked whether there exists a non-trivial asymptotic lower bound on \(N_{\text{crit}}(k)\).

The following theorem proven by Jacobson and Nadirashvili in [3] shows the negative answer to this question:

**Theorem** (Jacobson-Nadirashvili). There exists a Riemannian metric on \(\mathbb{T}^2\) and a sequence of eigenfunctions of the corresponding Laplace-Beltrami operator, such that the corresponding eigenvalues converge to infinity, but the number of critical points of the eigenfunctions from the sequence remains bounded.

Our result states that generally one cannot hope for an asymptotic upper bound on \(N_{\text{crit}}(k)\):

**Theorem 1.** On \(\mathbb{T}^2\) there exists a Riemannian metric and a sequence of eigenfunctions of the corresponding Laplace-Beltrami operator, with eigenvalues converging to infinity, such that each one of the eigenfunctions from the sequence has an infinite number of isolated critical points.
Similarly to the construction of Jacobson and Nadirashvili, we also will construct a Riemannian metric on $T^2$ of the Liouville type. The “punch-line argument” in our proof of Theorem 1 invokes the Brower’s fixed point theorem.

Let us mention that Enciso and Peralta-Salas [2] constructed a smooth metric such that the first eigenfunction has arbitrarily large number of isolated critical points. Results of opposite spirit were proven by Polterovich and Sodin [5].

It is worth to point out that, at present, we do not know what happens in the case when the Riemannian metric $g$ is real-analytic. In that case, an eigenfunction must have a finite number of isolated critical points, however, we don’t know whether there exists an asymptotic upper bound for the number of critical points in terms of the corresponding eigenvalue.

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2. Proof of Theorem 1

2.1. The metric and 1-dim reduction. Consider the coordinates $(x, y) \in \mathbb{T}^2 = (\mathbb{R}/2\pi \mathbb{Z})^2$ on the torus. Our Riemannian metric $g$ on $\mathbb{T}^2$ will be of the form $ds^2 = Q(x)(dx^2 + dy^2)$, where $Q$ is a $C^\infty$ smooth, positive, $2\pi$ periodic function. Then the Laplace-Beltrami operator is $\Delta_g = \frac{1}{Q(x)} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. Searching eigenfunctions $\varphi$ of the form $\varphi(x, y) = F(x)G(y)$ with $2\pi$-periodic functions $F$ and $G$, the equation $\Delta_g \varphi + \lambda \varphi = 0$ becomes equivalent to the system

$$
\begin{aligned}
F''(x) + (\lambda Q(x) - \mu)F(x) &= 0 \\
G''(y) + \mu G(y) &= 0
\end{aligned}
$$

(1)
for some $\mu \in \mathbb{R}$. Thus we get a sequence of solutions $G_m(y) = e^{imy}$ with $\mu_m = m^2$, where $m \in \mathbb{Z}$, and for each such $m$, the first equation in the system \([1]\) becomes

\begin{equation}
F''(x) + (\lambda Q(x) - m^2)F(x) = 0.
\end{equation}

For each given $m$, the eigen-problem \([2]\) admits a spectral decomposition, where we have a sequence of smooth solutions $F = F_{m,1}, F_{m,2}, \ldots$ with corresponding eigenvalues $\lambda_{m,1} \leq \lambda_{m,2} \leq \ldots$. Going back to our original eigen-problem on the torus, $\Delta g \phi + \lambda \phi = 0$, we have found a sequence of solutions $F_{m,k}(x) \cos my, F_{m,k}(x) \sin my$. It is easy to see that this sequence of solutions is complete in $L^2(T^2, Q dxdy)$, and hence gives a complete spectral decomposition of our original eigen-problem on the torus.

Our goal now is to find a smooth positive function $Q \in C^\infty(\mathbb{R}/2\pi \mathbb{Z})$, such that for infinitely many integers $m$, there will exist a solution $F_m$ of the equation \([2]\) having infinitely many isolated critical points in $\mathbb{R}/2\pi \mathbb{Z}$. For such $Q$ and $F_m$, the eigenfunction $F_m(x) \cos my$ (as well as the eigenfunction $F_m(x) \sin my$) will have infinitely many isolated critical points.

We have reduced 2-dimensional eigen-problem to a 1-dimensional problem, and for convenience, we will consider periodic functions on $\mathbb{R}$, instead of functions on $\mathbb{R}/2\pi \mathbb{Z}$. Moreover, by rescaling, we may assume that the functions are 8-periodic rather than $2\pi$-periodic.

2.2. The main observation. Assume that $u: \mathbb{R} \to \mathbb{R}$ is a non-trivial solution of the differential equation $u''(x) + K(x)u(x) = 0$, where $K: \mathbb{R} \to \mathbb{R}$ is some smooth function. Also assume that for some point $x_* \in \mathbb{R}$, we have $K(x_*) = 0$, and moreover, $K$ makes a large number of oscillations near $x_*$, such that these oscillations decay very fast when we approach $x_*$. To be more precise, we suppose that for some

\[ x_* = x_N < x_{N-1} < \ldots < x_1 < x_* + \varepsilon \]

we have $(-1)^i K|_{(x_{i+1}, x_i)} < 0$, and moreover, for every $1 \leq i \leq N - 2$,

\[ \left| \int_{x_{i+1}}^{x_i} K(x) dx \right| \text{ is much larger than } \sum_{j=i+1}^{N-1} \left| \int_{x_{j+1}}^{x_j} K(x) dx \right|. \]

Furthermore, assume that $u'(x_*) = 0$, that $u$ is positive on the interval $[x_*, x_* + \varepsilon]$, and the values of $u$ on that interval are of the same order of magnitude. We claim that under these assumptions, $u$ has at least $N - 2$ critical points on $(x_*, x_* + \varepsilon)$. 
Indeed, for any $1 \leq i < N$, we have
\[ u'(x_i) - u'(x_{i+1}) = \int_{x_{i+1}}^{x_i} u''(x) \, dx = -\int_{x_{i+1}}^{x_i} K(x)u(x) \, dx, \]
and moreover we have $u'(x_N) = u'(x_1) = 0$. Hence, by our assumptions, for each $1 \leq i < N$, the sign of $u'(x_i) = (u'(x_i) - u'(x_{i+1})) + \ldots + (u'(x_{i-1}) - u'(x_1))$ is positive if $i$ is odd, and is negative when $i$ is even. Therefore for each $1 \leq i < N$, $u$ has a critical point $\xi_i \in (x_{i+1}, x_i)$, which is moreover isolated since $u''(\xi_i) = -K(\xi_i)u(\xi_i) \neq 0$. We conclude that $u$ has at least $N - 2$ isolated critical points on the interval $(x_*, x_* + \varepsilon)$.

We can also consider a more general setting where the function $K$ has infinitely many rapidly decaying oscillations near some point $x_*$, and in this case we may conclude that $u$ has infinitely many isolated critical points. More precisely, it is enough to assume the following:

1. We have some $x_* \in \mathbb{R}$ and a strictly decreasing sequence $(x_i)_{i=1}^{\infty}$, such that $\lim_{i \to \infty} x_i = x_*$.
2. We have a smooth function $K : \mathbb{R} \to \mathbb{R}$ such that $(-1)^iK|_{(x_{i+1}, x_i)} < 0$ for every $i$, and
\[ \left| \int_{x_{i+1}}^{x_i} K(x) \, dx \right| \geq C \sum_{j=i+1}^{\infty} \left| \int_{x_j}^{x_{j-1}} K(x) \, dx \right| \]
with some numerical constant $C > 1$.
3. $u : \mathbb{R} \to \mathbb{R}$ is a solution of $u''(x) + K(x)u(x) = 0$ with $u(x_*) > 0$ and $u'(x_*) = 0$.

Then $u$ has infinitely many isolated critical points on a small interval $(x_*, x_* + \delta)$.

2.3. The construction. Here we define several auxiliary functions.

a) Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth even function such that supp $(\psi) \subset (-1, -1/4) \cup (1/4, 1)$ and such that $\psi = 1$ on $[-3/4, -1/2] \cup [1/2, 3/4]$. For every $t \in [0, \infty)$, denote $\psi_t(x) = \psi(4^t x)$ and then define the $2$-periodic function $\varphi_t(x) = 1 - \sum_{n \in \mathbb{Z}} \psi_t(x + 2n)$.

b) Choose a smooth positive function $q : \mathbb{R} \to (0, \infty)$ satisfying:
\begin{itemize}
  \item $q(x) = q(-x)$, $x \in \mathbb{R}$,
  \item $q(x + 2) = q(x)$, $x \in \mathbb{R}$,
  \item $q$ is increasing on $[-1, 0]$ and decreasing on $[0, 1]$.
\end{itemize}

For $t \in [0, \infty)$, denote $q_t(x) = \varphi_t(x)(q(x) - q(4^{-t}))) + q(4^{-t})$. We may assume that the function $q$ is “flat” enough at $x = 0$ so that $q_t \to q$ in the $C^\infty$ topology, when $t \to \infty$. 


c) Fix a smooth even function $h: \mathbb{R} \to \mathbb{R}$ with $\text{supp } h \subset (-3/4, -1/2) \cup (1/2, 3/4)$, such that for some $x_\infty \in (1/2, 3/4)$ and for a strictly decreasing sequence $(x_i)_{i=1}^\infty$, $x_i \in (x_\infty, 3/4)$, with $\lim_{i \to \infty} x_i = x_\infty$, we have $(-1)^i h|_{(x_{i+1}, x_i)} < 0$ for every $i$, and that moreover, for every $i$ we have

$$\left| \int_{x_{i+1}}^{x_i} h(x) \, dx \right| \geq C \sum_{j=i+1}^\infty \left| \int_{x_{j+1}}^{x_j} h(x) \, dx \right|$$

with some numerical constant $C > 1$. For each $t \in [0, \infty)$, denote $h_t(x) = h(4^t x)$, and then $H_t(x) = \sum_{n \in \mathbb{Z}} h_t(x + 2n)$.

d) For any finite 1-separated set $S \subset [0, \infty)$ (1-separation means that $|s-t| \geq 1$ for every $s \neq t \in S$), and for any function $\tau: S \to \mathbb{R}$, define the function $q_{S, \tau}: \mathbb{R} \to \mathbb{R}$ as follows. We set $q_{S, \tau} = q$, and if $S \subset [0, \infty)$ is a non-empty finite set, then we choose some $t \in S$, 
denote $S' = S \setminus \{t\}$ and $\tau' = \tau_{|S'}$, and define $q_{S,\tau}(x) = \varphi_t(x)(q_{S',\tau'}(x) - q_{S',\tau'}(4^{-t})) + q_{S',\tau'}(4^{-t}) + \tau(t)H_t(x)$.

e) Choose a smooth positive function $\tilde{q}: \mathbb{R} \to \mathbb{R}$ such that:

- $\tilde{q}(x + 2) = \tilde{q}(x)$, $x \in \mathbb{R}$,
- $\tilde{q}(-x) = \tilde{q}(x)$, $x \in \mathbb{R}$,
- $\tilde{q}(x) = q(5x)$, $x \in [0, 1/5]$,
- $\tilde{q}(x) < q(4x)$, $x \in [1/5, 2/9]$,
- $\tilde{q}(x) < q(1)$, $x \in [2/9, 1]$.

We claim that for any finite 1-separated set $S \subset [1, \infty)$ and for any sufficiently small function $\tau: S \to \mathbb{R}$, we have $\tilde{q} \leq q_{S,\tau} \leq q$. The upper bound is true for $\tau$ small enough by the definition of the function $q_{S,\tau}$. To check the lower bound, it is enough to show
that, for $q_S := q_{S, 0}$ (where $0 : S \to \mathbb{R}$ is the zero function), we have $q_S(x) > \tilde{q}(x)$ for any $x \in (0, 1]$. Let us check this.

Note that, by the construction of $q_S$, we have $q_S(x) \geq q(4x)$ for $x \in [0, 1/4]$, and $q_S(x) \geq q(1)$ for $x \in [1/4, 1]$. Take any $x \in (0, 1]$. Then

**Case 1:** $x \in (0, 1/5]$. In that case we have $q_S(x) - \tilde{q}(x) \geq q(4x) - \tilde{q}(x) = q(4x) - q(5x) > 0$.

**Case 2:** $x \in [1/5, 2/9]$. Then $q_S(x) - \tilde{q}(x) \geq q(4x) - \tilde{q}(x) > 0$.

**Case 3:** $x \in [2/9, 1]$. Then $q_S(x) - \tilde{q}(x) \geq q(1) - \tilde{q}(x) > 0$.

f) Let $Q : \mathbb{R} \to (0, \infty)$ be a 2-periodic positive smooth even function. For every $m \in \mathbb{N}$, consider the Dirichlet problem

\[
\begin{aligned}
\begin{cases}
  u : [0, 4] \to \mathbb{R} \\
  u''(x) + (\lambda Q(x) - m^2)u(x) = 0 \\
  u(0) = u(4) = 0
\end{cases}
\end{aligned}
\]

We denote by $\Lambda_{Q, m}$ the first eigenvalue and by $U_{Q, m}$ the first eigenfunction of this Dirichlet problem. One can extend $U_{Q, m}$ to $[0, 8]$ by $U_{Q, m}(x) = -U_{Q, m}(8 - x)$ for $x \in [4, 8]$, and then extend $U_{Q, m}$ to the whole $\mathbb{R}$ by making it 8-periodic. Then $U_{Q, m} : \mathbb{R} \to \mathbb{R}$ is a
smooth solution of

\[
\begin{aligned}
\left\{
\begin{array}{l}
u : \mathbb{R} \to \mathbb{R} \text{ is } 8\text{-periodic} \\
u''(x) + (\lambda Q(x) - m^2)u(x) = 0
\end{array}
\right.
\end{aligned}
\]

Recall that \(Q\) is even and 2-periodic. We claim that the space of solutions to (4) with \(\lambda = \Lambda_{Q,m}\) is 2-dimensional. We have

\[U'_{Q,m}(2) = 0,\]

which is another solution of (3) with \(\lambda = \Lambda_{Q,m}\), and is linearly independent with \(U_{Q,m}|_{[0,4]}\). Therefore we get:

\[U_{Q,m}(0) = 0, \quad U'_{Q,m}(0) \neq 0, \quad U_{Q,m}(2) \neq 0, \quad U'_{Q,m}(2) = 0.\]

Hence if we define \(V_{Q,m} = U_{Q,m}(x + 2)\), then \(V_{Q,m}(x)\) is a solution of (4) and is linearly independent with \(U_{Q,m}\). We conclude that the space of solutions of (4) with \(\lambda = \Lambda_{Q,m}\) is 2-dimensional (the dimension cannot be greater than 2).

### 2.4. The main argument.

**Proposition 2.** There exist sequences: \(m_1 < m_2 < \ldots\) of positive integers, \(\varepsilon_1, \varepsilon_2, \ldots \in (0, \infty)\), and \(M_0 = 0, M_1, M_2, \ldots \in [0, \infty)\), with \(M_{i+1} > M_{i} + 1\) such that, for any \(k \in \mathbb{N}\) and any \(\tau_1, \tau_2, \ldots, \tau_k \in \mathbb{R}\) with \(|\tau_i| \leq \varepsilon_i\), one can find \(t_1, \ldots, t_k\) with \(t_i \in [M_{i-1} + 1, M_i]\), \(i = 1, 2, \ldots, k\), with the following property:

For \(S = \{t_1, t_2, \ldots, t_k\}\) and for the function \(\tau : S \to \mathbb{R}, \tau(t_i) = \tau_i, i = 1, 2, \ldots, k\), the function \(Q = q_{S,\tau}\) satisfies \(\Lambda_{Q,m,\tau} Q(4^{-t_i}x_\infty) - m_i^2 = 0\) for \(i = 1, 2, \ldots, k\), where \(\Lambda_{Q,m,\tau}\) is the first eigenvalue of the problem (3) with \(m = m_i\).

**Remark 3.** Note that in the setting of the proposition we always have \(Q(4^{-t_i}x_\infty) = q(4^{-t_i})\) for \(i = 1, 2, \ldots, k\).

We will start with the following lemma.

**Lemma 4.** Let \(Q : \mathbb{R} \to (0, \infty)\) be a 2-periodic positive smooth function with \(Q(0) > Q(x), x \notin 2\mathbb{Z}\). Then we have

\[\Lambda_{Q,m} > \frac{m^2}{Q(0)}\]

for each \(m\), and for every \(\varepsilon > 0\), we have

\[\Lambda_{Q,m} < \frac{m^2}{Q(0) - \varepsilon}\]

when \(m\) is large enough.
Proof of Lemma 4. The variational principle says that
\[ \Lambda_{Q,m} = \min \frac{\int_0^4 (u'(x))^2 + m^2(u(x))^2 \, dx}{\int_0^4 Q(x)(u(x))^2 \, dx}, \]
where the minimum is taken over all smooth functions \( u: [0, 4] \to \mathbb{R} \) with \( u(0) = u(4) = 0 \).

From here we clearly have
\[ \Lambda_{Q,m} > m^2 \max Q = \frac{m^2}{Q(0)}. \]

Now, choose \( \delta > 0 \) small enough, and pick a smooth function \( u: [0, 4] \to \mathbb{R} \) with \( \text{supp} (u) \subset (2 - \delta, 2 + \delta) \) and \( u \not\equiv 0 \). Then for every \( m \) we have
\[ \Lambda_{Q,m} \leq \frac{\int_0^4 (u'(x))^2 + m^2(u(x))^2 \, dx}{\int_0^4 Q(x)(u(x))^2 \, dx} \leq \frac{\int_0^4 (u'(x))^2 + m^2(u(x))^2 \, dx}{(\min_{[-\delta, 2+\delta]} Q) \cdot \int_0^4 (u(x))^2 \, dx} = \frac{C + m^2}{\min_{[-\delta, \delta]} Q}, \]
where
\[ C = \frac{\int_0^4 (u'(x))^2 \, dx}{\int_0^4 (u(x))^2 \, dx}. \]

From here we see that by choosing \( \delta > 0 \) so small that \( \min_{[-\delta, \delta]} Q > Q(0) - \varepsilon \), we obtain
\[ \Lambda_{Q,m} < \frac{m^2}{Q(0) - \varepsilon}, \]
for large enough \( m \).

Proof of Proposition 2. First, we choose sequences \( (m_k), (\varepsilon_k), (M_k) \) inductively as follows.

\( k = 1 \): By Lemma 4 there exists \( m_1 \in \mathbb{N} \) such that
\[ \frac{m_1^2}{q(0)} < \Lambda_{\widetilde{q},m_1} < \frac{m_1^2}{q(1/4)}. \]

(Recall that \( \max \widetilde{q} = \widetilde{q}(0) = q(0) \)). Then again by the lemma and by the monotonicity property for eigenvalues \( (q \geq \widetilde{q}) \), we have
\[ \frac{m_1^2}{q(0)} < \Lambda_{q,m_1} \leq \Lambda_{\widetilde{q},m_1} < \frac{m_1^2}{q(1/4)}. \]

Let \( M_1 > 1 \) be such that
\[ q(4^{-M_1}) = \frac{m_1^2}{\Lambda_{q,m_1}}. \]

Now choose \( \varepsilon_1 \) such that for any \( t_1 \in [1, M_1] \) and \( \tau_1 \in [-\varepsilon_1, \varepsilon_1] \) we have \( \widetilde{q} \leq q_{S,\tau} \leq q \) on \( \mathbb{R} \), where \( S = \{t_1\}, \tau : S \to \mathbb{R}, \tau(t_1) = \tau_1 \).

The inductive step: Assume that we have chosen
\[ m_1, \ldots, m_{k-1}, \varepsilon_1, \ldots, \varepsilon_{k-1}, M_1, \ldots, M_{k-1}, \]
and now choose \( m_k, \varepsilon_k, M_k \). By Lemma 4, there exists \( m_k \in \mathbb{N} \) such that
\[
\frac{m_k^2}{q(0)} < \Lambda_{q,m_k} < \frac{m_k^2}{q(4^{-1-M_{k-1}})}.
\]
Then, again by the lemma and by the monotonicity property for eigenvalues, we have
\[
\frac{m_k^2}{q(0)} < \Lambda_{q,m_k} \leq \Lambda_{\tilde{q},m_k} < \frac{m_k^2}{q(4^{-1-M_{k-1}})}.
\]
Now let \( M_k > M_{k-1} + 1 \) be such that
\[
q(4^{-M_k}) = \frac{m_k^2}{\Lambda_{q,m_k}},
\]
and let \( \varepsilon_k > 0 \) be such that for any \( t_k \in [M_{k-1} + 1, M_k] \) and any \( \tau_k \in [-\varepsilon_k, \varepsilon_k] \) we have \( \tilde{q} \leq q_{S,\tau} \leq q \) on \( \mathbb{R} \), where \( S = \{ t_k \} \) and \( \tau : S \to \mathbb{R}, \tau(t_k) = \tau_k \).

Recall that \( M_0 = 0 \). Let \( k \in \mathbb{N} \), and let
\[
P := [M_0 + 1, M_1] \times [M_1 + 1, M_2] \times \cdots \times [M_{k-1} + 1, M_k] \subset \mathbb{R}^k.
\]
Let \( \tau_1, \ldots, \tau_k \in \mathbb{R} \) be such that \( |\tau_i| \leq \varepsilon_i, \ i = 1, 2, \ldots, k \). Now define the map \( F : P \to \mathbb{R}^k \) as follows. For any \( (t_1, \ldots, t_k) \in P \), set \( S := \{ t_1, \ldots, t_k \} \), let \( \tau : S \to \mathbb{R} \) be the function such that \( \tau(t_i) = \tau_i, \ i = 1, 2, \ldots, k \), and denote \( Q = q_{S,\tau} \). We have \( \tilde{q} \leq Q \leq q \), hence for each \( 1 \leq i \leq k \),
\[
\frac{m_i^2}{q(4^{-M_i})} = \Lambda_{q,m_i} \leq \Lambda_{Q,m_i} \leq \Lambda_{\tilde{q},m_i} < \frac{m_i^2}{q(4^{-1-M_{i-1}})};
\]
and therefore there exists a unique \( s_i \in (M_{i-1} + 1, M_i] \) such that
\[
q(4^{-s_i}) = \frac{m_i^2}{\Lambda_{Q,m_i}}.
\]
Then we put \( F(t_1, \ldots, t_k) = (s_1, \ldots, s_k) \).

Note first, that \( F \) is continuous. Secondly, for any \( (t_1, \ldots, t_k) \in P \) we have \( (s_1, \ldots, s_k) = F(t_1, \ldots, t_k) \in P \). That is, \( F(P) \subset P \). Therefore, by the Brower fixed point theorem, there exists a point \( (t_1, \ldots, t_k) \in P \) for which \( F(t_1, \ldots, t_k) = (t_1, \ldots, t_k) \), and hence by Remark 3 we have
\[
Q(4^{-t_i}x_\infty) = q(4^{-t_i}) = \frac{m_i^2}{\Lambda_{Q,m_i}}
\]
for \( i = 1, 2, \ldots, k \). \( \square \)

For an infinite subset \( S \subset [1, \infty) \), \( S = \{ t_1, t_2, \ldots \} \), such that \( t_{i+1} \geq t_i + 1 \ \forall i \), and for any function \( \tau : S \to \mathbb{R} \), define the function \( q_{S,\tau} \) as follows: for each \( k \geq 1 \), put \( S_k := \{ t_1, \ldots, t_k \} \), and \( \tau_k := \tau|_{S_k} : S_k \to \mathbb{R} \). Then for each \( x \in \mathbb{R} \), the sequence \( (q_{S_k,\tau_k}(x)) \)
stabilises for large \( k \), and we define \( q_{S,\tau}(x) \) to be equal to \( q_{S,k}(x) \) for \( k \) large enough. If \( \tau(t_k) \) converges to 0 sufficiently fast, then the function \( q_{S,\tau} \) is \( C^\infty \)-smooth (here we use the remark made at the end of Section 2.3).\( \)

From Proposition 2, by applying a compactness argument, we get the following corollary:

**Corollary 5.** There exists a sequence \( m_1, m_2, \ldots \in \mathbb{N} \), an infinite set \( S = \{t_1, t_2, \ldots\} \subset [1, \infty) \), with \( t_{i+1} \geq t_i + 1 \), and a function \( \tau : S \to \mathbb{R} \setminus \{0\} \), such that the function \( Q := q_{S,\tau} : \mathbb{R} \to \mathbb{R} \) is \( C^\infty \)-smooth and satisfies

\[
\Lambda_{Q,m_i} Q(4^{-t_i} x_\infty) - m_i^2 = 0, \quad i \in \mathbb{N}.
\]

Now we will readily finish off the proof of Theorem 1. Let \( Q \) be as in the corollary. Recall that the function \( h \) makes infinitely many rapidly decreasing oscillations near \( x_\infty \) (the function \( h \) and the point \( x_\infty \) were chosen in Section 2.3). Hence for each \( i \), the function \( Q(x) \) makes infinitely many rapidly decreasing oscillations near \( 4^{-t_i} x_\infty \).

Since the space of solutions of (4) is 2-dimensional (see Section 2.3), we can always find a solution of

\[
\begin{aligned}
\left\{ \begin{array}{l}
u : \mathbb{R} \to \mathbb{R} \text{ is 8-periodic} \\
u''(x) + \left(\Lambda_{Q,m_i} Q(x) - m_i^2\right)u(x) = 0,
\end{array} \right.
\end{aligned}
\]

denoted by \( u_i(x) \), such that \( u_i(4^{-t_i} x_\infty) > 0 \) and \( u'_i(4^{-t_i} x_\infty) = 0 \). Applying our main observation (Section 2.2) with \( K(x) = \Lambda_{Q,m_i} Q(x) - m_i^2 \) and \( x_* = 4^{-t_i} x_\infty \), we conclude that \( u_i \) has infinitely many isolated critical points near \( 4^{-t_i} x_\infty \). This finishes the proof of Theorem 1.

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Lev Buhovski,
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
Israel
E-mail: levbuh@post.tau.ac.il

Alexander Logunov,
Department of Mathematics
Princeton University
Princeton, NJ, 08544
E-mail: log239@yandex.ru

Mikhail Sodin,
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
Israel
E-mail: sodin@post.tau.ac.il