Nonlinear vibrations of an inclined beam subjected to a moving load

Ahmad Mamandi¹, Mohammad H. Kargarnovin² and Davood Younesian*³

¹PhD Student, Mechanical and Aerospace Engineering Department, Science and Research Branch, Islamic Azad University, Tehran, Iran

²Professor, Mechanical and Aerospace Engineering Department, Science and Research Branch, Islamic Azad University, Tehran, Iran and Mechanical Engineering Department, Sharif University of Technology, Tehran, Iran

³Assistant Professor, School of Railway Engineering, Iran University of Science and Technology, Tehran, Iran

*Corresponding Author. E-mail: Younesian@iust.ac.ir

Abstract. In this paper, the nonlinear dynamic responses of an inclined pinned-pinned Euler-Bernoulli beam with a constant cross section and finite length subjected to a concentrated vertical force traveling with constant velocity is investigated by using the mode summation method. Frequency analysis of the PDE’s governing equations of motion for steady-state response is studied by applying multiple scales method. The nonlinear dynamic deflections of the beam are obtained by solving two coupled nonlinear PDE’s governing equations of planar motion for both longitudinal and transverse oscillations of the beam. The dynamic magnification factor and normalized time histories of mid-point of the beam are obtained for various load velocity ratios and the numerical results are compared with those obtained from traditional linear solution. It is found that quadratic nonlinearity renders the softening effect on the dynamic response of the beam under the act of traveling load. Also stability analysis of the steady-state response for the modes equations having quadratic nonlinearity is carried out and it is observed from the amplitude response curves that for the case of internal-external primary resonance, both saturation phenomenon and jump phenomenon are predicted for the longitudinal excitation.

1. Introduction
The linear and nonlinear analysis for forced vibrations of structural elements, such as beams and plates under the act of traveling loads is of considerable practical importance and many analytical and numerical methods have been proposed in the past decades to investigate extensively the dynamic behavior of all customary and non customary shapes of isotropic, composite and FGM¹ made engineering structural elements in the academic literatures. However, until now little attention has been paid to study for dynamic response of the coupled longitudinal and transverse oscillations of an inclined Euler-Bernoulli beam subjected to a vertical concentrated moving load. Bridges on which vehicles or trains travel, and trolleys of overhead traveling cranes that move on their girders may be modeled as moving loads on simply supported beams. Also applications in aerospace industries include the rocket launcher systems and any kinds of firearms in which the inclined beams play an

¹ Functionally Graded Material
important part in the system. A comprehensive treatment of the subject of vibrations of structures due to moving loads, which contains a large number of related cases, is that of which studied in [1]. A common analytical method to tackle with issues of moving force traveling on infinite length beams is the method of integral transforms. On the other hand, in order to deal with problems including finite length beams, the modal analysis is commonly used. Many of researchers have studied the linear dynamic analysis of elastic single-span or multi-span beams traveled by both constant and variable magnitude moving load with various boundary conditions. They usually focus on the effects of the acceleration, deceleration or the dynamical behavior of beams and under one-axle or two-axle load draw some conclusions from structural design point of view [2-8].

The main objective in this study is to focus on the influence of a moving load excitation on the nonlinear coupled PDE’s governing equations of motion for an inclined Euler-Bernoulli beam considering longitudinal and transversal oscillation interactions. Frequency response of the system having quadratic non-linearity and stability analysis of steady-state response in the case of internal-external primary resonance are carried out in this paper.

2. Mathematical modeling and definitions

A one-dimensional inclined Euler-Bernoulli beam of length $l$ traveled by a concentrated vertical force $F$ is considered (see Fig. 1). In our up-coming analysis, it is assumed that when the force enters the left end of the beam, all initial conditions are zero, i.e. the beam is at rest at time $t = 0$. It is further assumed that the force will be in full contact with the beam during its motion, i.e. no separation occurs. The nonlinear dynamic behavior of the coupled longitudinal and transverse vibrations of a uniform beam under simultaneous axial force and the effect of a moving force is sought. Additionally, it is assumed that the beam's supports are restricted against any translational movement hence, the lateral deflection produces some kind of mid-plane stretching and this one in turn introduces some order of nonlinearity in governing equations of motion. Under these conditions following coupled nonlinear partial differential system of equations are initiated [9]:

$$
\rho A u_t + c_1 u_t - EA u_{tt} = \frac{1}{2}(EA-P)\left[(1-2u_x^2)w_x^2\right]_x - F \sin \theta \delta(x-\zeta)
$$

$$
\rho A w_t + c_2 u_t - P w_{tt} + EI w_{xxxx} = (EA-P)\left[(ew_t^2)_x \right]_x - F \cos \theta \delta(x-\zeta)
$$

where $u$, $w$ and $F$ represent the longitudinal time dependent displacement of a point in the mid-plane, lateral deflection and lateral concentrated moving force, respectively. Moreover, subscribes $nt$ and $nx$ stand for the $n^{th}$ order derivative with respect to time ($t$) and space coordinate ($x$), respectively, and $e = u_x - u_e^2 + \frac{1}{2} w_x^2$. Furthermore, $\rho$, $A$, $EI$ and $\delta$ are the beam's mass density, sectional area, beam's flexural rigidity and Kronecker delta operator, respectively. In addition, $c_1$ and $c_2$ are two constant coefficients corresponding to the structural modal damping of the beam also $P$ is the prescribed either axial tensile or compressive load. Note that the $\delta(x-\zeta)$ is the Dirac’s delta function in which $\zeta$ is the instantaneous position of the moving force with velocity of $v$ on the beam where $\zeta = vt$.

In this study we restrict our attention to the non-rotating planar oscillation and seek only for the first correction which accounts for the beam stretching. This means that to neglect the cubic derivative term $2u_x w_x^2$ in Eq. (1) and the also $u_e^2$ and $\frac{1}{2} w_e^2$ terms in $e$ [9]. Therefore, it is appropriate to consider $e = u_e$ , which is in accordance with the case of thin beam with small radius of gyration. After employing above restrictions on Eqs. (1) and (2), the general governing coupled partial differential equations of planar vibration for an inclined Euler-Bernoulli beam considering linear viscous damping are obtained as:

$$
u_e \frac{c_1}{\rho A} u_e - \frac{E}{\rho} u_{ee} = \frac{(EA-P)}{\rho A} \left[ w_x w_{xx} \right] - F \sin \theta \delta(x-\zeta)
$$

$$
u_e \frac{c_2}{\rho A} w_e - \frac{P}{\rho A} w_{ee} + \frac{EI}{\rho A} w_{xxxx} = \frac{(EA-P)}{\rho A} \left[ u_e w_x + u_x w_{xx} \right] - F \cos \theta \delta(x-\zeta)
$$
According to the separation of variable approach, the response of our continuous system in terms of linear free-oscillation modes can be assumed as follows:

\[ u(x,t) = \sum_{n=1}^{\infty} \xi_n(t) \sin \left( \frac{n\pi x}{l} \right) \text{ and } w(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

The boundary conditions for a pinned-pinned beam with immovable ends are expressed as:

\[ u(0,t) = u(l,t) = 0, \quad w(0,t) = w(l,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(l,t)}{\partial x^2} = 0, \]

and initial conditions are: \( w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0 \).

Substituting Eq. (5) into Eqs (3) and (4) leads to:

\[ \sum_{n=1}^{\infty} \left( \xi_n \right)_0 + 2 \mu \lambda_n \left( \xi_n \right)_0 + \lambda_n^2 \xi_n = -\pi^2 \kappa \sum_{n=1}^{\infty} k^2 m \eta_n \eta_n \cos \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) = \frac{F \sin \theta}{\rho A} \delta(\xi - \xi) \]

\[ \sum_{n=1}^{\infty} \left( \eta_n \right)_0 + 2 \mu \omega_n \left( \eta_n \right)_0 + \omega_n^2 \eta_n = -\pi^2 \kappa \sum_{n=1}^{\infty} m \kappa n \eta_n \left( k \cos \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) + m \sin \left( \frac{m\pi x}{l} \right) \cos \left( \frac{n\pi x}{l} \right) \right) - \frac{F \cos \theta}{\rho A} \delta(\eta - \eta) \]

in which \( \kappa = \frac{EA - P}{\rho Al} \), \( \lambda_n = \frac{m\pi}{l(\rho A)^{1/2}} \) and \( \omega_n = \frac{m\pi}{l(\rho A)^{1/2}} \left( \frac{EI}{l^2} + \frac{P}{\rho A} \right)^{1/2} \). Multiplying both sides of Eqs. (6) and (7) by \( \sin \left( \frac{n\pi x}{l} \right) \) and integrating over the interval \([0,l]\) leads to:

\[ \left\{ \begin{aligned} 
(\xi_n)_0 + 2 \mu \lambda_n (\xi_n)_0 + \lambda_n^2 \xi_n &= -\frac{2F \sin \theta}{\rho A} \sin \left( \frac{n\pi vt}{l} \right) - \frac{n^2}{4} \kappa \sum_{n=1}^{\infty} m \eta_n \left[ n - m \right] \eta_{n-m} + (m+n)\eta_{n+m} \\
(\eta_n)_0 + 2 \mu \omega_n (\eta_n)_0 + \omega_n^2 \eta_n &= -\frac{2F \cos \theta}{\rho A} \sin \left( \frac{n\pi vt}{l} \right) - \frac{n^2}{4} \kappa \sum_{n=1}^{\infty} m \xi_n \left[ n - m \right] \xi_{n-m} + (m+n)\xi_{n+m} 
\end{aligned} \]

Eqs. (8) and (9) are numerically solved to determine the values \( \xi_n \) and \( \eta_n \) then by substituting them in Eq. (5) one obtains the longitudinal and transverse modal co-ordinates for the beam, respectively.

4. Numerical results
In order to obtain numerical results, the material properties of the beam are as follows: $E = 200 \times 10^9$ N/m$^2$, $\rho = 7850$ kg/m$^3$ (steel), $l = 8$ m, $\mu = 0.033$; $A = 196 \times 10^{-4}$ m$^2$. It is assumed that the beam has a square cross-section. Results for the dynamic magnification factor $D_d$, which is the ratio of the maximum magnitude of the dynamic deflection at the mid-span of the beam to the maximum static deflection at the same location, are shown in Fig. 2 for different values of velocity ratio $\alpha$, in which

$$\alpha = \frac{T_f}{T} = \frac{v}{v_f},$$

where $T_f$ denotes the first natural period (fundamental period) of the beam while $T$ denotes the total time taken by the moving load to cross from one end to the other and also the velocity of a reference load that would take the time of $T_f$ to traverse the beam of length $l$ is represented by $v_f$.

Fig. 2 shows the effect of the moving load size on the dynamic magnification factor $D_d$ of a horizontal steel beam for various values of load velocity. It can be seen that by increasing the magnitude of the moving load the dynamic displacement grows accordingly in such a way that it does not follow the well-known linear force-deflection relation as in the linear systems. It can be observed that displacement results in the transverse direction of the linear and nonlinear theories are very close to each other up to the $F = 6000$ N and maximum $D_d$ for linear and nonlinear solutions are almost the same and equals to 1.73 at $\alpha = 1.2$. However, this figure further reveals that after this value of $F$, deviation between the linear and the nonlinear theories increases for the considered parameters and the dynamic deflections of the nonlinear cases are larger than those in the linear case. One of the reasons for this difference is mainly due to the effect of larger deflection on the higher values of the transverse traveling force. Furthermore, as seen from this figure, the velocity of the moving load plays an important role on the load-displacement curves and for the considered velocity ranges the difference between the two theories turns to be minimum. It can be seen that the maximum value of dynamic magnification factor $D_d$ is about 1.85 at $F = 20000$ N for a steel beam under the act of moving force and occurs locally at $\alpha = 1.2$. Also, the maximum and minimum $D_d$ occur at $\alpha = 1.2$ and $\alpha = 0.4$, respectively for steel beams under different values of moving loads corresponding to the velocity ratios up to $\alpha = 2$. In most practical applications in the mechanical and civil engineering fields these values play an important role from design point of view. Moreover, as it was seen from the trend of curves in this figure, the dynamic magnification factor for a pinned-pinned beam consists of two regions: under critical region and an overcritical region. In the under critical region the dynamic magnification factor, $D_d$, increases and decreases by increasing $\alpha$. In this region, $D_d$ increases up to $\alpha = 0.3$ and then its variation reverses and it reaches to a minimum value at $\alpha = 0.4$. The main increase in the variation of $D_d$ occurs in the intervals $0.4 \leq \alpha \leq 1.2$. In the overcritical region $D_d$ decreases as $\alpha$ increases. In other words, in the under critical region the beam's dynamic deflection increases by increasing the velocity of the load and in the overcritical region, the beam's dynamic deflection decreases by increasing the velocity of the load, $v$.

![Fig. 2 Dynamic magnification factor ($D_d$) of an isotropic horizontal steel beam versus $\alpha$ for $l = 8$ m, $P = 0$, $\mu = 0$, $n = 6$; (-----) linear solution [1], (----) nonlinear solution](image-url)
In Fig. 3, the maximum dynamic displacements of the beam under the moving load are obtained for different values of the moving load, which varies from $F = 2000$ N to $F = 30000$ N with an increment of 2000 N for $\alpha = 1$, considering both linear and nonlinear solutions. It can be seen that the dynamic deflection of the nonlinear analysis are higher than those obtained from the linear solution representing a softening behavior because of the existence of quadratic nonlinearity characteristic in the mode equations of motion of the beam. Thus this system is equivalent to a nonlinear soft spring [9]. Also, it is seen from this figure that displacements of the linear and the nonlinear solutions are very close to each other until the value of $F = 10000$ N; however, after this point, displacements shows a sizeable difference between the linear and the nonlinear solutions for the considered parameters.

![Fig. 3 Variation of the beam's mid-point dynamic displacement versus magnitude of the moving load](image1)

In Fig. 4, the dimensionless dynamic deflection of mid-point of the beam versus $vt/l$ for $l = 8$ m, $A = 0.0196$ m$^2$, $\mu = 0$, $\theta = 0^\circ$, $\alpha = 1$, $n = 6$; (-----) linear solution, (---) nonlinear solution.

![Fig. 4 Dimensionless dynamic deflection of mid-point of the beam versus $vt/l$ for $l = 8$ m, $F = 20000$ N, $P = 0$, $\mu = 0$, $\theta = 36^\circ$, $n = 6$](image2)

Fig. 4 shows the results for the normalized time histories of transverse deflection of the center of the beam versus $vt/l$, for different velocity ratios.

To validate the numerical results obtained from presented nonlinear dynamic analysis the outcome answers from nonlinear solution are compared in Fig. 4 with those from the linear solution. As it was seen, the nonlinear solutions give larger deflections compared to those of linear analysis. Also, it can be seen that the peak values do not almost occur simultaneously. In addition, it can be observed that the maximum dynamic displacement initially increases as $vt/l$ increases. However, maximum dynamic
displacement decreases when \( v/t/l \) increases beyond the value of \( \alpha = 1.2 \). Also, it is seen that for higher velocity ratio of \( v/\nu_p \), i.e. \( \alpha \geq 3 \), the time history curves of the beam from both theories overlays. It can be easily seen that the time which the maximum mid-span displacement occurs moves to the right side as \( \alpha \) increases.

### 5. Frequency analyses of forced oscillations of an inclined beam having quadratic nonlinearities

In this section we study the internal-external primary resonances for the forced response considering coupled longitudinal and transverse oscillations of an inclined Euler-Bernoulli beam with linear viscous damping under the act of traveling load. At this point, it is convenient to introduce the following dimensionless variables:

\[
\text{displacement decreases when } v/t/l \text{ increases beyond the value of } \alpha = 1.2. \text{ Also, it is seen that for higher velocity ratio of } v/\nu_p \text{, i.e. } \alpha \geq 3, \text{ the time history curves of the beam from both theories overlays. It can be easily seen that the time which the maximum mid-span displacement occurs moves to the right side as } \alpha \text{ increases.}
\]

Substituting Eq. (12) into Eqs. (10) and (11) and multiplying both sides by \( \sin(n\pi X) \) and integrating over the interval \([0,1]\) and considering the linear modal damping \( \nu_n \) and \( \mu_n \), leads to the mode equations as:

\[
(\xi_\alpha)_{n\alpha} + \lambda_n^2 \xi_\alpha = \frac{g \sin(n\pi V \tau)}{r^2} - \frac{n \pi \kappa}{4 r^2} \sum_{m=1}^\infty m \eta_m \left[ |n-m|\eta_{m-n} + (m+n)\eta_{m+n} \right] \]

\[
(\eta_\alpha)_{n\alpha} + 2\mu_n \eta_n + \omega_n^2 \eta_n = \frac{f \sin(n\pi V \tau)}{r^2} - \frac{n \pi \kappa}{4 r^2} \sum_{m=1}^\infty m \xi_m \left[ |n-m|\eta_{m-n} + (m+n)\eta_{m+n} \right] \]

where \( f = -\frac{2F \cos \theta}{EA} \) and \( g = -\frac{2F \sin \theta}{EA} \), \( \omega_n = n \pi (n^2 + N) \frac{1}{2} \) and \( \lambda_n = n \pi r \). Let \( \epsilon \) be a small but finite dimensionless parameter in the order as the amplitudes of oscillation, thus we have:

\[
\hat{\xi}_n = \frac{\xi_n}{\epsilon}, \quad \hat{\eta}_n = \frac{\eta_n}{\epsilon}, \quad \hat{\nu}_n = \nu_n, \quad \hat{\mu}_n = \mu_n, \quad \hat{\xi}_m = \xi_m, \quad \hat{\nu}_m = \nu_m, \quad \hat{\mu}_m = \mu_m, \quad \hat{f}_n = f_n, \quad \hat{g}_n = g_n, \quad \hat{\kappa} = \kappa (4r^2)^{-1}
\]

in which \( f_n = f \sin(n\pi V \tau) \) and \( g_n = g \sin(n\pi V \tau) \). Substituting Eq. (15) into Eqs. (13) and (14) and dropping the hats in the results, one would get the quadratic nonlinear mode equations:

\[
(\xi_\alpha)_{n\alpha} + \lambda_n^2 \xi_\alpha = e \left[ -2\nu_n (\xi_\alpha) - nk \sum_{m=1}^\infty m \eta_m (\eta_{m+n} + \eta_{m-n}) \right] - 2eg \cos(n\pi V \tau + \pi / 2)
\]

\[
(\eta_\alpha)_{n\alpha} + \omega_n^2 \eta_n = e \left[ -2\mu_n (\eta_\alpha) - nk \sum_{m=1}^\infty m \xi_m (\eta_{m+n} + \eta_{m-n}) \right] - 2ef \cos(n\pi V \tau + \pi / 2)
\]

where \( p = |n-m| \) and \( q = n+m \). We use the method of multiple scales and seek an approximate solution for Eqs. (16) and (17) for small but finite amplitudes in the form of:
where $\tau \equiv \omega \varepsilon$, $i.e.$, $\tau_0 = \tau$ and $\tau_1 = \omega \varepsilon$ represent different independent time scales.

Substituting Eq. (18) into Eqs. (16) and (17) and noting that $d / d \tau = D_0 + \varepsilon D_1 + \ldots$, $d^2 / d \tau^2 = D_0^2 + 2\varepsilon D_0 D_1 + \ldots$, where $D_0 = d / d \tau$, $D_1 = d / d \tau$, and equating coefficients of like powers of $\varepsilon$, and considering $\xi_{i\pi} = \lambda_0(\tau_1)\exp(i\lambda_1 \tau_1) + c.c.$ and $\eta_{i\pi} = B_0(\tau_1)\exp(i\omega_0 \tau_1) + c.c.$, we obtain following set of second order ordinary differential equations:

$$O(\varepsilon): \quad D_0^2 \xi_{i\pi} + \lambda_0^2 \xi_{i\pi} = -2i\lambda_0(D_0 A + \nu A_0)\exp(i\lambda_1 \tau_1) - n\sum_{n=1}^{\infty} mB_0 \exp[i(\omega_n + \omega_0)\tau_1]$$

$$\quad + pB_0 \exp[i(\omega_n - \omega_0)\tau_1] + qB_0 \exp[i(\omega_n + \omega_0)\tau_1] + qB_0 \exp[i(\omega_n - \omega_0)\tau_1] - \frac{f}{2} \exp[i(\Omega_\pi + \pi / 2)] + c.c.$$ (19)

$$D_0^2 \eta_{i\pi} + \omega_n^2 \eta_{i\pi} = -2i\omega_n(D_0 B + \mu B_0)\exp(i\omega_n \tau_1) - n\sum_{n=1}^{\infty} mA_0 \exp[i(\lambda_n + \omega_0)\tau_1]$$

$$\quad + pB_0 \exp[i(\lambda_n - \omega_0)\tau_1] + qB_0 \exp[i(\lambda_n + \omega_0)\tau_1] + qB_0 \exp[i(\lambda_n - \omega_0)\tau_1] - \frac{f}{2} \exp[i(\Omega_\pi + \pi / 2)] + c.c.$$ (20)

where $\Omega_\pi = n\pi V$, $(n = 1, 2, 3, \ldots)$. The bar sign over any parameters and $c.c.$ stand for the complex conjugates terms. Inspection of Eqs. (19) and (20) shows that when $\lambda_0 = \omega_n \pm \omega_0$ or $\lambda_n = \omega_n \pm \omega_q$ there is an extra link, or term, connecting $\xi_{i\pi}$ and $\eta_{i\pi}$. This is referred to as an internal resonance.

5.1. Internal-external resonance in longitudinal excitation

Internal resonance could be described by these conditions: $\lambda_0 = \omega_n \pm \omega_0$ and or $\lambda_n = \omega_n \pm \omega_q$ and longitudinal primary external resonance condition would be defined when $\Omega_\pi = \lambda_n$. As a case study for the internal resonance portion we consider the case in which $\lambda_0$ is near $\omega_n + \omega_0$, other cases could be treated as following. The solvability conditions of Eqs. (19) and (20) are:

$$-2i\lambda_0(D_0 A + \nu A_0) - n\sum_{n=1}^{\infty} mB_0 \exp(-i\sigma_0 \tau_1) - g \exp(i\sigma_0 \tau_1) = 0$$

$$-2i\omega_n(D_0 B + \mu B_0) - n\sum_{n=1}^{\infty} mA_0 \exp(i\sigma_0 \tau_1) = 0$$ (21)

in which $\lambda_0 = \omega_n + \omega_0 + \varepsilon \sigma_0$ and hence $\Omega_\pi = \lambda_0 + \varepsilon \sigma_0$. We introduce the polar notation for amplitudes $A_0 = 1/2 a_0 \exp(i\alpha_0)$ and $B_0 = 1/2 b_0 \exp(i\beta_0)$, where $a_0$, $b_0$, $\alpha_0$ and $\beta_0$ are all real function of $\tau_1$. Also to obtain numerical solutions, we consider $n = 1$, $m = 2$ and $p = 3$, thus for the steady-state response we need to solve the following differential algebraic equations (DAE’s) simultaneously:

$$a'_0 = -\nu_0 a_0 + \frac{3\kappa_0}{2\lambda_0} b_0 \sin \gamma_0 + \frac{\kappa_0}{\lambda_0} \gamma' \gamma_0, \quad a'_1 = a_0 \gamma_0 - \frac{3\kappa_0}{2\lambda_0} b_0 \cos \gamma_0 + \frac{\kappa_0}{\lambda_0} \cos \gamma_0, \quad b'_0 = -\mu_0 b_0 + \frac{3\kappa_0}{2\omega_0^2} a_0 b_0 \sin \gamma_0$$

$$b'_1 = \frac{3\kappa_0}{2\omega_0^2} a_0 b_0 \cos \gamma_0, \quad b'_2 = -\mu_0 b_0 + \frac{3\kappa_0}{2\omega_0^2} a_0 b_0 \sin \gamma_0, \quad b'_3 = \frac{3\kappa_0}{2\omega_0^2} a_0 b_0 \cos \gamma_0$$ (22)

where $\gamma_0 = \sigma_0 \tau_1$, $\gamma_0 = \beta_0$, $\beta_1 = \gamma_0 = 0$, $\gamma_0 = \sigma_0 \tau_1 - \gamma_0 = \pi / 2$. For the steady-state response we will note that $a'_0 = b'_2 = b'_3 = \gamma'_0 = 0$, where the prime over parameters denotes the first derivative with respect to $\tau_1$. In Figs. 5, the amplitudes $a_0$, $b_2$ and $b_3$ have been plotted as functions of $g$, the amplitude of longitudinal excitation. The so called saturation phenomenon and jump phenomenon are illustrated in this figure. In Fig. 5-a there is a small detuning of the external resonance corresponded to the moving load velocity, while the internal resonance is perfectly tuned. Fig. 5-b identifies the situation in which the external resonance and internal resonance are perfectly tuned. As illustrated in Fig. 5-a in addition to the saturation phenomenon, there is a jump phenomenon associated with the variation of the
amplitude of the excitation of \( g \). When \( g \) increases slowly from zero, \( a_1 \) follows increases linearly and \( b_2 \) and \( b_3 \) are zero and more increases in the value of \( g \) renders \( a_1 \) to have the saturated value of \( a_{1s} \) and \( b_2 \) and \( b_3 \) suddenly jump from zero to their upper branch of corresponded curves, accordingly. Further increases in \( g \), \( b_2 \) and \( b_3 \) follow to the right direction along their corresponded curves. When \( g \) decreases slowly from a large value, \( a_1 \) follows along straight line which has a constant value and \( b_2 \) and \( b_3 \) follow along their upper branch curves. Further decrease in the value of \( g \), causes \( b_2 \) and \( b_3 \) jump down from their bend points and follow the linear solution back to the origin. These processes are shown graphically by the bilateral arrows in the Fig. 5-a and b.

**Fig. 5** Variation of \( a_1 \), \( b_2 \) and \( b_3 \) (amplitude response) vs. \( g \) for an inclined pinned-pinned beam-longitudinal primary resonant case: \( \mu_i = \mu_1 = 0.165, \nu_1 = 0.1, r = 0.0245, \epsilon = 0.2, \lambda_1 = \pi \sqrt{\nu_1} \)

(a) \( \sigma_1 = 0, \sigma_2 = 1 \)  
(b) \( \sigma_1 = \sigma_2 = 0 \)

### 6. Conclusions

Nonlinear dynamic analysis of an inclined Euler-Bernoulli beam under the action of a concentrated moving load was carried out. Numerical results were compared with the linear solution and frequency analysis was performed by using multiple scales method. It was found that because of having cubic nonlinearity in mode equations, the system behaves like a nonlinear soft spring; it means that by increasing of magnitude of moving load, the dynamic deflections become larger than those obtained by linear solution. Steady-state solutions were obtained and it was found that in the case of internal-external longitudinal primary resonance, there are two possible solutions: either longitudinal amplitude is nonzero and transversal amplitudes are zero, or both longitudinal and transversal amplitudes are all nonzero. That is we have a jump phenomenon associated with this motion. In addition to the jump phenomenon, the saturation phenomenon is also associated with this motion. This means that by increasing either of external detuning or internal detuning parameters the stable region corresponded to the zero solutions for transverse amplitudes will be extended accordingly.

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