HAUSDORFF DIMENSION OF CANTOR INTERSECTIONS FOR COUPLED HORSESHOE MAPS

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Abstract. As a model to provide a hands-on, elementary understanding of chaotic dynamics in dimension 3, we introduce a $C^2$-open set of diffeomorphisms of $\mathbb{R}^3$ having two horseshoes with different dimensions of instability. We prove that: the unstable set of one horseshoe and the stable set of the other are of Hausdorff dimension nearly 2 whose cross sections are Cantor sets; the intersection of the unstable and stable sets contains a fractal set of Hausdorff dimension nearly 1. Our proof employs the thicknesses of Cantor sets.

1. Introduction

For a diffeomorphism $\varphi$ of a Riemannian manifold $M$ equipped with a distance $d$, the unstable and stable sets of a point $p \in M$ are given by

$$W^u(p) = \{ q \in M : d(\varphi^n(p), \varphi^n(q)) \to 0 \text{ as } n \to -\infty \},$$
$$W^s(p) = \{ q \in M : d(\varphi^n(p), \varphi^n(q)) \to 0 \text{ as } n \to \infty \},$$

respectively. If these sets are submanifolds of $M$, they are called unstable and stable manifolds of $p$. An incredibly rich array of complicated dynamical phenomena is unleashed by a non-transverse intersection between unstable and stable manifolds, see for example [12].

The existence of an intersection of Cantor sets is a fundamental tool to construct examples of open sets of diffeomorphisms which are not structurally stable. Newhouse [9] defined a non-negative quantity called the “thickness” of a Cantor set on the real line, in order to formulate conditions which guarantee that two Cantor sets intersect each other. These conditions have been applied to surface diffeomorphisms to show the robustness of tangencies between unstable and stable manifolds whose cross sections are Cantor sets [9, 10, 11, 12, 14]. In higher dimension, the thickness is still useful for the construction of robust tangencies [5, 8, 13, 15].

Newhouse’s result [9] asserts that two Cantor sets on the real line intersect each other if the product of their thicknesses is greater than one, and neither set lies in a gap of the other. His result does not imply any lower bound of the Hausdorff dimension of the intersection of the two Cantor sets. Indeed, Williams [17] observed that two interleaved Cantor sets can have thicknesses well above 1 and still only intersect at a single point. Therefore, the results [5, 8, 9, 10, 11, 12, 13, 14, 15] mentioned above do not imply any lower bound of the Hausdorff dimension of the sets of tangencies.

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In this paper, using the thickness and thick Cantor sets, we provide an elementary example of diffeomorphisms in dimension 3 that display a new type of robust fractal non-transverse intersection between unstable and stable manifolds. These diffeomorphisms are variants of the piecewise linear map from the unit cube onto itself introduced in [14], as a model to provide a hands-on, elementary understanding of chaotic dynamics in dimension 3. We estimate the Hausdorff dimension of the set of non-transverse intersection in terms of some parameters of the diffeomorphisms.

1.1. Coupled horseshoe maps. A block is a cuboid in the Euclidean space \( \mathbb{R}^3 = \{(x_u, x_c, x_s): x_u, x_c, x_s \in \mathbb{R}\} \) all of whose sides are parallel to one of the axes of coordinates of \( \mathbb{R}^3 \). Given two blocks \( X \) and \( Y \), we say \( X \) penetrates \( Y \) if \( X \) does not intersect the edges of \( Y \) and \( X \setminus Y \) has two connected components.

Let \( A, B, C, D \) be pairwise disjoint blocks in \([0,1]^3\) such that \( D \) is a translate of \( A \) in the \( x_u \)-direction and \( C \) is a translate of \( B \) in the \( x_c \)-direction. Similarly, let \( A^*, B^*, C^*, D^* \) be pairwise disjoint blocks in \([0,1]^3\) such that \( D^* \) is a translate of \( A^* \) in the \( x_c \)-direction and \( C^* \) is a translate of \( B^* \) in the \( x_s \)-direction. We assume:

- (a) \( A^* \) penetrates \( A \) and \( D \), and \( D^* \) penetrates \( A \) and \( D \);
- (a*') \( B \) penetrates \( B^* \) and \( C^* \), and \( C \) penetrates \( B^* \) and \( C^* \);
- (b) \( A^* \) penetrates \( C \), and \( D^* \) penetrates \( B \);
- (c) \( A \) penetrates \( B^* \) and \( C^* \), and \( D \) penetrates \( B^* \) and \( C^* \).

See FIGURE 1 (i), (ii), (iii), (iii) respectively. Set

\[ R_1 = A \cup D \quad \text{and} \quad R_2 = B^* \cup C^* \]

Let \( r \in \{1, 2\} \), and let \( \text{Diff}^r(\mathbb{R}^3) \) denote the set of \( C^r \) diffeomorphisms of \( \mathbb{R}^3 \). We say \( f_0 \in \text{Diff}^1(\mathbb{R}^3) \) is a coupled horseshoe map if it maps \( A, B, C, D \) affinely to \( A^*, B^*, C^*, D^* \) respectively with diagonal Jacobian matrices, see FIGURE 1 (iv). Condition (a) implies that the restriction \( f_0|_{R_1} \) of \( f_0 \) to \( R_1 \) is a horseshoe map whose unstable manifolds are one-dimensional. Condition (a*) implies that \( f_0|_{R_2} \)...

\[ \begin{align*}
\text{(i)} & \quad A \quad B \\
\text{(ii)} & \quad C \quad D \\
\text{(iii)} & \quad \mathcal{L} \quad \mathcal{L} \\
\text{(iv)} & \quad x_u \quad x_s \\
\end{align*} \]
Figure 2. Some coupled horseshoe maps are viewed as compositions of three processes (blocks corresponding to $A$, $B$, $D$ are given the same labels): (i) to (ii); Squeeze and stretch each block as shown, without rotation. Next fix the position of $C^*$. (ii) to (iii); Flip $B$ upward so that it is above $C^*$. Flip $A$ backward so that it is behind $D$. (iii) to (iv); Flip $A$ and $D$ so that they are underneath $C^*$.

is a horseshoe map whose stable manifolds are one-dimensional. Conditions (b) (c) determine how these horseshoe maps are coupled. See FIGURE 2 for one example.

1.2. Statements of results. For $f_1, f_2 \in \text{Diff}^r(\mathbb{R}^3)$, we define

$$\|f_1 - f_2\|_{C^r} = \sum_{j \in \{u,c,s\}} \sum_{0 \leq |\alpha| \leq r} \sup_{[0,1]^3} |\partial^\alpha f_{1,j} - \partial^\alpha f_{2,j}|,$$

where $f_i = (f_{i,u}, f_{i,c}, f_{i,s})$ for $i = 1,2$ and $\alpha = (\alpha_u, \alpha_c, \alpha_s)$ denotes the multi-index: $\partial^\alpha = \partial_{x_u}^{\alpha_u} \partial_{x_c}^{\alpha_c} \partial_{x_s}^{\alpha_s}$. We endow $\text{Diff}^r(\mathbb{R}^3)$ with a $C^r$ topology, a topology that has the collection of sets of the form $\{g \in \text{Diff}^r(\mathbb{R}^3): \|f - g\|_{C^r} < \epsilon\}$ with $f \in \text{Diff}^r(\mathbb{R}^3)$ and $\epsilon > 0$ as a base.

Let $f_0 \in \text{Diff}^2(\mathbb{R}^3)$ be a coupled horseshoe map. We are concerned with a $C^2$ diffeomorphism $f$ which is sufficiently $C^1$-close to $f_0$. The compact $f$-invariant set

$$\Lambda = \bigcap_{n=-\infty}^{n=\infty} f^{-n}(R_1 \cup R_2)$$

contains two compact $f$-invariant sets

$$\Gamma = \bigcap_{n=-\infty}^{n=\infty} f^{-n}(R_1) \quad \text{and} \quad \Sigma = \bigcap_{n=-\infty}^{n=\infty} f^{-n}(R_2),$$

of the horseshoe maps $f|_{R_1}$ and $f|_{R_2}$ respectively. Let $W^u(\Gamma)$ (resp. $W^s(\Sigma)$) denote the unions of the unstable (resp. stable) manifolds of points in $\Gamma$ (resp. $\Sigma$). We will provide a condition on $f_0$ which ensures that the set

$$H(f) = W^u(\Gamma) \cap W^s(\Sigma) \cap \Lambda$$

is non-empty. Under a stronger condition on $f_0$ we will give an estimate of the Hausdorff dimension of $H(f)$. To these ends, we will define thicknesses of cross sections of $W^u(\Gamma)$ and $W^s(\Sigma)$, and relate them to the following numbers

$$a_1 = \frac{3|A^*_c| - |A_c|}{|B^*_c| - 2|A^*_c|} \quad \text{and} \quad a_2 = \frac{3|B_c| - |B^*_c|}{|F_c| - 2|B_c|}.$$
Hereafter, $E^*$ denotes the minimal block containing $A^*$ and $D^*$, and $F$ denotes the minimal block containing $B$ and $C$. We denote by $|I|$ the Euclidean length of a bounded interval $I$. The following two numbers are also in use:

$$b_1 = \frac{|A_c| - |E_c^*|}{|E_c^*| - 2E^*}, \quad b_2 = \frac{|B_c^*| - |F_c|}{|F_c| - 2|B_c|}.$$ 

Let $\dim$ denote the Hausdorff dimension on $\mathbb{R}^3$. The key term *dimension-reducible* will be defined in (3.3) in Section 3.1. Our main results are as follows.

**Theorem A.** Let $f_0 \in \text{Diff}^2(\mathbb{R}^3)$ be a coupled horseshoe map that is dimension-reducible, satisfies $a_1a_2 > 1$ and $\max\{b_1, b_2\} \leq 1$. There exists a $C^2$ neighborhood $V$ of $f_0$ such that for any $f \in V$, $H(f)$ is a non-empty totally disconnected set. Moreover the following estimates hold:

$$\frac{2a_1 + 1}{a_1 + 1} < \dim W^u(\Gamma) < 2, \quad \frac{2a_2 + 1}{a_2 + 1} < \dim W^s(\Sigma) < 2, \quad \dim H(f) < 1.$$ 

The set $W^u(\Gamma)$ and $W^s(\Sigma)$ intersect each other as in FIGURE 3. As $a_1$ and $a_2$ increase, $\dim W^u(\Gamma)$ and $\dim W^s(\Sigma)$ converge to 2. Our next result shows that their intersection contains a set of Hausdorff dimension nearly 1.

**Theorem B.** There exists $\tau_0 > 0$ such that if $f_0 \in \text{Diff}^2(\mathbb{R}^3)$ is a coupled horseshoe map that is dimension-reducible and satisfies $\min\{a_1, a_2\} > \tau_0$ and $\max\{b_1, b_2\} \leq 1$, then there exists a $C^2$ neighborhood $V$ of $f_0$ such that for any $f \in V$,

$$\left(1 + \frac{1}{\sqrt{\min\{a_1, a_2\}}}\right)^{-1} < \dim H(f) < 1.$$ 

The number $\tau_0$ in Theorem B is a large constant determined by a result in [7] which we recall in Section 2. Since $\tau_0$ is large and $a_1 < 3/(|E_c^*| - 2A^*_c|)$ and
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Figure 4. Numerical computations of the sets \( H(f_0) \cap \{x_u = 1\} \) for non-dimension-reducible, coupled horseshoe maps \( f_0 \) with blocks approximately \( A \approx [0, 1/3] \times [0, 1]^2 \), \( B \approx [2/3, 1] \times [0, 1/2] \times [0, 1] \), \( C \approx [2/3, 1] \times [1/2, 1] \times [0, 1] \), \( D \approx [1/3, 2/3] \times [0, 1]^2 \), \( A^* \approx [0, 1] \times [0, 1/2] \times [0, 2/3] \), \( B^* \approx [0, 1]^2 \times [2/3, 5/6] \), \( C^* \approx [0, 1]^2 \times [5/6, 1] \), \( D^* \approx [0, 1] \times [1/2, 1] \times [0, 2/3] \): (left) \( \det Df_0 > 0 \) everywhere as in FIGURE 2; (right) \( \det Df_0 < 0 \) on \( A \) with a flip only of the \( x_s \)-coordinate and \( \det Df_0 > 0 \) on \( B, C, D \) with no flip.

\( a_2 < 3/(|F_c| - 2|B_c|) \), the assumption \( \min \{a_1, a_2\} > \tau_0 \) means that the size of the gap between \( A^* \) and \( D^* \) and that between \( B \) and \( C \) are small.

We do not claim the above theorems give optimal estimates on Hausdorff dimension for general diffeomorphisms in dimension 3. For example, modifying the construction of Asaoka using the Plykin attractor [1], Barrientos and Pérez [2] constructed a \( C^1 \) diffeomorphism having two hyperbolic sets, one of index 1 (the dimension of the unstable subbundle) and the other of index 2, for which the unstable set of the first set and the stable set of the second one contain two-dimensional submanifolds which intersect each other in a smooth curve. As we reiterate, our aim here is to introduce a model and results which provide a hands-on, elementary understanding of chaotic dynamics in dimension 3.

1.3. Outline of proofs of the theorems. Proofs of the main results are briefly outlined as follows. We focus on the dimension estimate of \( H(f) \), as that of \( W^u(\Gamma) \) and \( W^s(\Sigma) \) are by-products. We begin by remarking that geometric structures of the set \( H(f_0) \) for a general coupled horseshoe map \( f_0 \) are rich. Our numerical experiment suggests that there is a case where \( H(f_0) \) becomes a fractal set, as in FIGURE 4, with Hausdorff dimension seemingly exceeding 1. Moreover, these fractal sets appear to persist under small perturbations of the map. The map \( f_0 \) in FIGURE 4 is not dimension reducible. A dimension estimate in such a case is beyond our reach.

Therefore, we perform a dimension reduction. The assumption of dimension reducibility in Theorems A and B allows us to use the theory of normally hyperbolic invariant manifolds [4, 6] to construct two \( C^2 \) surfaces, called graph-invariant surfaces, one of which contains a neighborhood of \( \Gamma \) in \( W^u(\Gamma) \) and the other contains a neighborhood of \( \Sigma \) in \( W^s(\Sigma) \). See Section 6 for relevant definitions and formal statements. The upper bound \( \dim H(f) < 1 \) is a consequence of this construction.

A lower estimate of \( \dim H(f) \) as well as showing \( H(f) \neq \emptyset \) is more delicate. We perform a further reduction to the problem of how two Cantor sets on the real line intersect each other. More precisely, we construct two Cantor sets contained...
in a smooth curve, by projecting the cross sections of $W^u(\Gamma)$ and $W^s(\Sigma)$ along their unstable and stable manifolds. Under the assumption of Theorem A, we show that the product of their thicknesses is at least 1. We then use the result of Newhouse [9] to conclude that the two Cantor sets intersect each other, and $H(f)$ is non-empty. Under the assumption of Theorem B, we show that these two Cantor sets have large thicknesses. By the result of Hunt et al. [7], their intersection contains a Cantor set with large thickness. We then appeal to the general lower bound of Newhouse [11] for the Hausdorff dimension of Cantor sets in terms of their thicknesses. The condition $\max\{b_1, b_2\} \leq 1$ will be used only to facilitate the lower estimates of the thicknesses (see the proof of Lemma 4.2(c)).

The rest of this paper consists of three sections. In Section 2 we introduce key ingredients, and perform main constructions in Section 3. We prove main estimates and complete the proofs of the main results in Section 4.

2. Main Tools

In this section we introduce main tools needed for the proofs of the theorems. In Section 2.1 we state a version of the fundamental theorem of Hirsch et al. [6] on normally hyperbolic invariant manifolds. In Section 2.2 we introduce the thickness of Cantor sets on the real line, and recall the gap lemma of Newhouse [9] and the theorem of Hunt et al. [7] on when two thick Cantor sets intersect thickly.

2.1. Persistence of normally attracting invariant manifolds. Let $\varphi: M \to M$ be a $C^r$ diffeomorphism ($r \geq 1$) of a $C^\infty$ Riemannian manifold $M$. A $C^1$ submanifold $V$ of $M$ is said to be r-normally hyperbolic for $\varphi$ if $\varphi(V) = V$, and there exist a continuous $D\varphi$-invariant splitting $T_V M = TV \oplus N^s \oplus N^u$ and constants $c > 0$, $\nu \in (0, 1)$ such that for any $p \in V$ and all unit vectors $v \in T_p V$, $n^s \in N^s_p$, $n^u \in N^u_p$ and any $n \geq 1$,

$$\frac{\|D\varphi_p^n n^s\|}{\|D\varphi_p^n v\|^r} \leq cn^n \quad \text{and} \quad \frac{\|D\varphi_p^{-n} n^u\|}{\|D\varphi_p^{-n} v\|^r} \leq cn^n.$$

In the case $N^u = \{0\}$, $V$ is called r-normally attracting for $\varphi$.

Normally hyperbolic invariant manifolds are persistent [4, 6]. We recall the precise statement of [6, (4.1) THEOREM] for the normally attracting case. The theorem below asserts that a compact r-normally attracting manifold $V$ for a $C^r$ diffeomorphism $\varphi$ is $C^r$, and the stable set $\bigcup_{p \in V} W^s(p)$ of $V$ is invariantly fibered by $C^r$ submanifolds tangent at $V$ to $N^s$. Moreover, these structures are unique and persist under $C^r$ perturbations of $\varphi$.

**Theorem 2.1 (6, (4.1) THEOREM).** Let $\varphi: M \to M$ be a $C^r$ diffeomorphism, $r \geq 1$, of a $C^\infty$ Riemannian manifold $M$ leaving a compact $C^1$ submanifold $V$ invariant. Assume $\varphi$ is r-normally attracting at $V$ respecting the splitting $T_V M = TV \oplus N^s$. Then $V$ is $C^r$, and the following hold:

(a) For each $p \in V$ there exists a regular $C^r$ submanifold $W^s_p$ contained in the stable set of $p$ and tangent to $N^s_p$, such that $\varphi(W^s_p) \subset W^s_{\varphi(p)}$ for every $p \in V$ and $\bigcup_{p \in V} W^s_p$ contains a neighborhood of $V$;
(b) If $\tilde{\varphi}$ is another diffeomorphism of $M$ which is $C^r$-close to $\varphi$, then $\tilde{\varphi}$ is $r$-normally attracting at some unique $\tilde{V}$, $C^r$-close to $V$.

2.2. **Intersection of two Cantor sets.** We adopt the definition of thickness by Palis and Takens [12] that is equivalent to the one by Newhouse [9]. Let $S$ be a Cantor set in $\mathbb{R}$. A gap of $S$ is a connected component of $\mathbb{R} \setminus S$. A bounded gap is a gap which is bounded. Let $G$ be any bounded gap and $x$ be a boundary point of $G$. Let $I$ be the bridge of $S$ at $x$, i.e., the maximal interval in $\mathbb{R}$ such that:

- $x$ is a boundary point of $I$;
- $I$ contains no point of a gap whose Euclidean length is at least $|G|$.

The thickness of $S$ at $x$ is defined by $\tau(S, x) = |I|/|G|$. The thickness $\tau(S)$ of $S$ is the infimum of $\tau(S, x)$ over all boundary points $x$ of bounded gaps. Two Cantor sets $S_1$, $S_2$ in $\mathbb{R}$ are said to be interleaved if neither set is contained in the closure of a gap of the other set.

**Lemma 2.2** (the gap lemma [9]). Let $S_1, S_2$ be two interleaved Cantor sets in $\mathbb{R}$ such that $\tau(S_1)\tau(S_2) > 1$. Then $S_1 \cap S_2$ is non-empty.

The gap lemma does not imply any lower bound of the Hausdorff dimension of the intersection. The next result in [7] asserts that the intersection of two interleaved Cantor sets with large thicknesses contains a Cantor set with large thickness.

**Theorem 2.3** (in [7, p.881, Remark]). For any $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that if two Cantor sets $S_1, S_2$ in $\mathbb{R}$ with $\tau(S_1) > \delta$, $\tau(S_2) > \delta$ are interleaved, then $S_1 \cap S_2$ contains a Cantor set whose thickness is at least $(1 - \varepsilon)\sqrt{\min\{\tau(S_1), \tau(S_2)\}}$.

We will deal with Cantor sets which are subsets of $C^1$ curves in $\mathbb{R}^3$. Their thicknesses will be defined with respect to the induced metrics on these curves.

### 3. Main constructions

In this section we perform main constructions needed for the proofs of our main results. After some initial setups in Section 3.1, we provide standard hyperbolicity estimates in Section 3.2. In Section 3.3 we construct two graph-invariant surfaces, and in Section 3.4 construct two Cantor sets.

#### 3.1. Initial setups. Let $f_0 \in \text{Diff}^1(\mathbb{R}^3)$ be a coupled horseshoe map. The expansion and contraction rates of $f_0$ on each block are

\begin{align}
\lambda_{0,u} &= \left|\frac{A_u^*}{A_u}\right|, \quad \lambda_{0,c} = \left|\frac{A_c^*}{A_c}\right|, \quad \lambda_{0,s} = \left|\frac{A_s^*}{A_s}\right|, \quad \text{and} \\
\mu_{0,u} &= \left|\frac{B_u}{B_u^*}\right|, \quad \mu_{0,c} = \left|\frac{B_c}{B_c^*}\right|, \quad \mu_{0,s} = \left|\frac{B_s}{B_s^*}\right|.
\end{align}

We say $f_0$ is dimension-reducible if $f_0 \in \text{Diff}^2(\mathbb{R}^3)$ and

\begin{align}
\lambda_{0,c}^2 > \lambda_{0,s} \quad \text{and} \quad \mu_{0,u} > \mu_{0,c}^2.
\end{align}

For simplicity, throughout the rest of this paper we put the following
Standing Assumption: the map \( f_0 \) is a coupled horseshoe map that is dimension-reducible and satisfies \( \max\{b_1, b_2\} \leq 1 \).

Let \( \epsilon > 0 \) and set \( \lambda_u = \lambda_{0,u} - \epsilon, \lambda_c = \lambda_{0,c} + \epsilon, \lambda_s = \lambda_{0,s} + \epsilon, \) and \( \mu_u = \mu_{0,u} - \epsilon, \mu_c = \mu_{0,c} - \epsilon, \mu_s = \mu_{0,s} + \epsilon. \) We assume \( \epsilon \) is sufficiently small so that the following hold:

\[
\lambda_u > 1 > 1/2 > \lambda_c > \lambda_s > 0 \quad \text{and} \quad \mu_u > \mu_c > 2 > 1 > \mu_s > 0;
\]

\[
\lambda_c^2 > \lambda_s \quad \text{and} \quad \mu_u > \mu_c^2;
\]

\[
\max\{\lambda_c(1 + b_1), \mu_c^{-1}(1 + b_2)\} < 1;
\]

\[
\max\{\lambda_c^2, \mu_c^{-1}, \mu_s^{-1}\} < 1.
\]

The definition of \( f_0 \) implies (3.4) for \( \epsilon > 0 \) small enough. Condition (3.5) and (3.6) can be fulfilled by the dimension reducibility and by the assumption \( \max\{b_1, b_2\} \leq 1 \) respectively, for \( \epsilon \) small enough. Condition (3.7) will facilitate some estimates.

### 3.2. Hyperbolicity estimates

We use the set of the first-order partial derivatives \( \{\partial_{x_u}, \partial_{x_c}, \partial_{x_s}\} \) as a basis for the tangent space at each point of \( \mathbb{R}^3 \). Unstable, center-unstable, stable, and center-stable, cones with angle \( \theta \in (0, 1) \) which are subsets of \( T_p\mathbb{R}^3 \) are defined as follows:

\[
C^u_p(\theta) = \{(\xi, \eta, \zeta) : \eta^2 + \zeta^2 \leq \theta^2 \xi^2\}, \quad C^c_{pu}(\theta) = \{(\xi, \eta, \zeta) : \theta^2(\xi^2 + \eta^2) \geq \zeta^2\}, \quad C^c_{ps}(\theta) = \{(\xi, \eta, \zeta) : \theta^2(\eta^2 + \xi^2) \geq \zeta^2\}.
\]

By a \( C^r \) curve (resp. \( C^r \) surface) we mean a regular one-dimensional (resp. two-dimensional) \( C^r \) submanifold of \( \mathbb{R}^3 \). We say a \( C^1 \) curve \( \gamma \) is tangent to \( C^u(\theta) \) (resp. \( C^{cu}(\theta) \)) if \( T_p\gamma \subset C^u_p(\theta) \) (resp. \( T_p\gamma \subset C^{cu}_p(\theta) \)) holds for every \( p \in \gamma \). Similarly, we say a \( C^1 \) surface \( S \) is tangent to \( C^u(\theta) \) if \( T_pS \subset C^u_p(\theta) \) holds for every \( p \in S \). \( C^1 \) curves and surfaces being tangent to \( C^s(\theta) \) or \( C^{cs}(\theta) \) are defined similarly.

For each \( p \in \mathbb{R}^3 \) let \( ||-|| \) denote the Euclidean norm on \( T_p\mathbb{R}^3 \). The next proposition asserts that the uniform expansion and contraction of \( f_0 \) on a neighborhood of each block is passed on to nearby \( C^1 \) diffeomorphisms.

**Proposition 3.1.** For any sufficiently small \( \theta > 0 \), there exist disjoint open sets \( U, U^* \) containing \( R_1, R_2 \) respectively, such that for any \( f \) which is sufficiently \( C^1 \)-close to \( f_0 \) the following hold:

(a) If \( p \in U \) then \( Df(p)(C^u_p(\theta)) \subset C^u_{f(p)}(\theta), Df(p)(C^{cu}_p(\theta)) \subset C^{cu}_{f(p)}(\theta) \) and \( Df(p)^{-1}(C_s^{cu}_p(\theta)) \subset C^s_{f(p)}(\theta), Df(p)^{-1}(C_s^{cs}_p(\theta)) \subset C^{cs}_{f(p)}(\theta) \);

(b) Let \( \gamma \subset U \) be a \( C^1 \) curve. If \( \gamma \) is tangent to \( C^u(\theta) \) (resp. \( C^{cu}(\theta) \)), then \( f(\gamma) \) is tangent to \( C^u(\theta) \) (resp. \( C^{cu}(\theta) \)). If \( \gamma \) is tangent to \( C^s(\theta) \) (resp. \( C^{cs}(\theta) \)), then \( f^{-1}(\gamma) \) is tangent to \( C^s(\theta) \) (resp. \( C^{cs}(\theta) \)). Analogous statements hold for a \( C^1 \) surface in \( U \);
Proof. All the assertions follow from the assumption that the following hold (see FIGURE 5):

\( (b^*) \) Let \( \gamma \subset U^* \) be a \( C^1 \) curve. If \( \gamma \) is tangent to \( C_s(\theta) \) (resp. \( C_{cs}(\theta) \)), then \( f^{-1}(\gamma) \) is tangent to \( C_s(\theta) \) (resp. \( C_{cs}(\theta) \)). If \( \gamma \) is tangent to \( C_u(\theta) \) (resp. \( C_{cu}(\theta) \)), then \( f(\gamma) \) is tangent to \( C_u(\theta) \) (resp. \( C_{cu}(\theta) \)). Analogous statements hold for a \( C^1 \) surface in \( U^* \).

\( (c) \) Let \( p \in U \). Then \( \|Df_p\|^2 \geq \lambda_u \|v\| \) for all \( v \in C_p^u(\theta) \), \( \|Df_p\|^2 \geq \lambda_c \|v\| \) for all \( v \in C_c^u(\theta) \), \( \|Df_{f(p)}\|^2 \geq \lambda_u \|v\| \) for all \( v \in C_{f(p)}^u(\theta) \), \( \|Df_{f^{-1}(p)}\|^2 \geq \lambda_u \|v\| \) for all \( v \in C_{f(p)}^{cs}(\theta) \).

\( (c^*) \) Let \( q \in U^* \). Then \( \|Df_q\|^2 \geq \mu_s \|v\| \) for all \( v \in C_q^s(\theta) \), \( \|Df_q\|^2 \geq \mu_c \|v\| \) for all \( v \in C_c^s(\theta) \), \( \|Df_{f^{-1}(q)}\|^2 \geq \mu_u \|v\| \) for all \( v \in C_{f(q)}^u(\theta) \), \( \|Df_{f^{-1}(q)}\|^2 \geq \mu_u \|v\| \) for all \( v \in C_{f(q)}^{cs}(\theta) \).

3.3. Construction of graph-invariant surfaces. In this subsection we prove a dimension reduction result announced in Section 1.3. A key concept is that of graph-invariant surfaces. There are two different types of graph-invariant surfaces as we deal with two different horseshoe maps.

Let \( \pi: (x_u, x_c, x_s) \mapsto (x_u, x_c) \), \( \pi^*: (x_u, x_c, x_s) \mapsto (x_c, x_s) \) be the natural projections, and let \( f \in \text{Diff}^2(\mathbb{R}^3) \). A \( C^2 \) surface \( V_{cu} \) is said to be graph-invariant in \( R_1 \) for \( f \) if there exists a \( C^2 \) function \( \phi: \pi(R_1) \to \mathbb{R} \), called the defining function such that the following hold (see FIGURE 5):

\( \circ \) \( V_{cu} = \{ (x_u, x_c, \phi(x_u, x_c)) : (x_u, x_c) \in \pi(R_1) \} \);

\( \circ \) \( V_{cu} \cap f^{-1}(R_1) \subset f^{-1}(V_{cu}) \) and \( V_{cu} \cap f(R_1) \subset f(V_{cu}) \);

\( \circ \) \( \sum_{\alpha:1 \leq |\alpha| \leq 2} \sup_{\pi(R_1)} \|\partial^\alpha \phi\| < 1/10 \).

Similarly, a \( C^2 \) surface \( V_{cs} \) is said to be graph-invariant in \( R_2 \) for \( f^{-1} \) if there exists a \( C^2 \) function \( \phi^*: \pi^*(R_2) \to \mathbb{R} \) such that the following hold:

\( \circ \) \( V_{cs} = \{ (\phi^*(x_c, x_s), x_s) : (x_c, x_s) \in \pi^*(R_2) \} \);

\( \circ \) \( V_{cs} \cap f^{-1}(R_2) \subset f^{-1}(V_{cs}) \) and \( V_{cs} \cap f(R_2) \subset f(V_{cs}) \);

\( \circ \) \( \sum_{\alpha:1 \leq |\alpha| \leq 2} \sup_{\pi^*(R_2)} \|\partial^\alpha \phi^*\| < 1/10 \).

Proposition 3.2. For any sufficiently small \( \theta \in (0, 1/10) \), there exists a \( C^2 \) neighborhood \( V \) of \( f_0 \) in \( \text{Diff}^2(\mathbb{R}^3) \) such that if \( f \in V \) then the following hold:

\( (a) \) There exists a unique graph-invariant surface \( V_{cu} \) in \( R_1 \) for \( f \). Moreover \( \bigcap_{n=0}^{\infty} f^n(R_1) \subset V_{cu} \), and the defining function \( \phi \) of \( V_{cu} \) satisfies \( \sum_{\alpha:1 \leq |\alpha| \leq 2} \sup_{\pi(R_1)} \|\partial^\alpha \phi\| < \theta \);
There exists a unique graph-invariant surface $V^{cs}$ in $R_2$ for $f^{-1}$. Moreover $\bigcap_{n=0}^{\infty} f^{-n}(R_2) \subset V^{cs}$, and the defining function $\phi^*$ of $V^{cs}$ satisfies $\sum_{\alpha: 1 \leq |\alpha| \leq 2} \sup_{(x,y) \in R_2} |\partial^\alpha \phi^*| < \theta$.

**Proof.** We only give a proof of (a) since that of (a*) is analogous. Let $\theta \in (0, 1/10)$ be small enough, $U$ be an open set containing $R_1$, and let $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusion of Proposition 3.2 holds. Let $p_s$ denote the $x_s$-coordinate of the fixed saddle of $f_0$ in $A$. Since $f_0$ is dimension-reducible, there exist a $C^2$ surface $M$ diffeomorphic to the two-dimensional sphere and $\tilde{f}_0 \in \text{Diff}^2(\mathbb{R}^3)$ such that $\pi(R_1) \times \{p_s\} \subset M$, $\tilde{f}_0|_U = f_0|_U$, and $M$ is 2-normally attracting for $\tilde{f}_0$.

By Theorem 2.1 applied to $(\tilde{f}_0, M)$, there exist a $C^2$ neighborhood $\mathcal{Y}$ of $\tilde{f}_0$, an open neighborhood $\Delta$ of $\pi(R_1)$ in $\mathbb{R}^2$, a constant $\delta_0 > 0$, and for each $g \in \mathcal{Y}$ a 2-normally attracting $C^2$ submanifold $V(g)$ for $g$, and a $C^2$ function $\phi(g) : \Delta \to \mathbb{R}$ such that the following hold:

- $\text{graph}(\phi(g)) = \{(x_u, x_c, \phi(g)(x_u, x_c)) : (x_u, x_c) \in \Delta\} \subset V(g)$;
- $\sum_{\alpha: 1 \leq |\alpha| \leq 2} \sup_{(x,y) \in \pi(R_1)} |\partial^\alpha \phi(g)| < \theta$;
- $\forall p \in \text{graph}(\phi(g))$ there exists a $C^2$ curve $W_p^s$ through $p$ which is tangent to $C^s(\theta)$ such that if $g(p) \in \text{graph}(\phi(g))$ then $g(W_p^s) \subset W_p^s$.
- $\forall W(g) = \bigcup\{(W_p^s : p \in \text{graph}(\phi(g)))\}$. Then

$$\pi(R_1) \times (p_s - \delta_0, p_s + \delta_0) \subset W(g) \subset U.$$

For each $f$ which is sufficiently $C^2$-close to $f_0$, we choose $g_f \in \mathcal{Y}$ such that $f|_U = g_f|_U$, and set $V^{cu} = \text{graph}(\phi(g_f)) \cap R_1$. Then $V^{cu}$ is a graph-invariant surface in $R_1$ for $f$ whose defining function is $\phi(g_f)|_{\pi(R_1)}$. Moreover we have

$$\sum_{\alpha: 1 \leq |\alpha| \leq 2} \sup_{(x,y) \in \pi(R_1)} |\partial^\alpha \phi(g_f)|_{\pi(R_1)} | < \theta.$$

**Lemma 3.3.** If $f \in \text{Diff}^2(\mathbb{R}^3)$ is sufficiently $C^2$-close to $f_0$, then $\bigcap_{n=0}^{\infty} f^n(R_1)$ is contained in $W(g_f)$.

**Proof.** Since the block $D$ is a translate of $A$ in the $x_u$-direction, $\bigcap_{n=0}^{\infty} f^n_0(R_1)$ is contained in $\pi(R_1) \times \{p_s\}$. From this and (3.8) the desired inclusion follows. □

**Lemma 3.4.** If $\theta \in (0, 1/10)$ is small enough and $f \in \text{Diff}^1(\mathbb{R}^3)$ is sufficiently $C^1$-close to $f_0$, then any $C^1$ curve which is tangent to $C^s(\theta)$ and intersects $\bigcap_{n=0}^{\infty} f^n(R_1)$ does not intersect $R_1 \setminus f(R_1)$.

**Proof.** If $\theta \in (0, 1/10)$ is small enough, any $C^1$ curve which is tangent to $C^s(\theta/2)$ and intersecting $\bigcap_{n=0}^{2} f^n(R_1)$ does not intersect $R_1 \setminus f(R_1)$. This implies the assertion. □

To complete the proof of Proposition 3.2(a), we prove $\bigcap_{n=0}^{\infty} f^n(R_1) \subset V^{cu}$ by way of contradiction. If $p \notin \bigcap_{n=0}^{\infty} f^n(R_1)$ and $p \notin V^{cu}$, then by Lemma 3.3 there would exist $q \in \text{graph}(\phi(g_f))$ such that $p \in W_q^s \setminus \{q\}$. Let $\sigma$ denote the curve in $W_q^s$ bordered by $p$ and $q$. By Proposition 3.1 the images of $\sigma$ under backward iteration are tangent to $C^s(\theta)$ and expanded by a factor $\lambda^{-1}$ in $W(g_f)$. Hence, we would reach $n_0 \geq 1$ such that $f^{-n}(\sigma) \subset W(g_f)$ for every $0 \leq n \leq n_0 - 1$ and $f^{-n_0}(\sigma) \not\subset W(g_f)$. Since $f^{-n}(p) \in W(g_f)$ for every $n \geq 0$, we would have $f^{-n_0+1}(p) \in \text{graph}(\phi(g_f))$ and $f^{-n_0}(q) \notin \text{graph}(\phi(g_f))$. Therefore $f^{-n_0+1}(\sigma)$ intersects both $\bigcap_{n=0}^{2} f^n(R_1)$ and
Let \( f_{\theta} \) be sufficiently \( C^2 \)-close to \( f_0 \). This yields a contradiction.

To show the uniqueness, let \( V_1, V_2 \) be graph-invariant surfaces in \( R_1 \) for which the conclusion of Proposition 3.2 holds. Let \( P \subseteq A \subseteq C \) denote the fixed saddles of \( f \). Note that \( Q \in C^* \). Let \( W_{\text{loc}}^s(P) \) denote the connected component of \( W^s(P) \cap A \) containing \( P \), and let \( W_{\text{loc}}^u(Q) \) denote the connected component of \( W^u(Q) \cap C^* \) containing \( Q \). Put
\[
\gamma_c = W_{\text{loc}}^s(P) \cap V^{cu} \quad \text{and} \quad \sigma_c = W_{\text{loc}}^u(Q) \cap V^{cs}.
\]
See FIGURE 6. Since \( W_{\text{loc}}^s(P) \) and \( V^{cu} \) are of class \( C^2 \) and intersect each other transversely at any point of \( \gamma_c, \gamma_c \) is a \( C^2 \) curve. For the same reason, \( \sigma_c \) is a \( C^2 \) curve. We take \( C^1 \) curves \( \tilde{\gamma}_c \supset \gamma_c, \tilde{\sigma}_c \supset \sigma_c \) such that \( \tilde{\gamma}_c \setminus \gamma_c \) and \( \tilde{\sigma}_c \setminus \sigma_c \) have two connected components both of infinite length, and view \( \gamma_c \cap \Gamma \) and \( \sigma_c \cap \Sigma \) as Cantor sets with respect to the induced metrics on \( \tilde{\gamma}_c \) and \( \tilde{\sigma}_c \), with thicknesses \( \tau(\gamma_c \cap \Gamma) \) and \( \tau(\sigma_c \cap \Sigma) \) respectively.

For \( p \in \gamma_c \cap \Gamma \), let \( \mathcal{F}_p^u \) denote the connected component of \( W^u(p) \cap f(R_1) \) which contains \( p \). For \( q \in \sigma_c \cap \Sigma \), let \( \mathcal{F}_q^s \) denote the connected component of \( W^s(q) \cap f^{-1}(R_2) \) which contains \( q \). Set
\[
F^u = \bigcup_{p \in \gamma_c \cap \Gamma} \mathcal{F}_p^u \quad \text{and} \quad F^s = \bigcup_{q \in \sigma_c \cap \Sigma} \mathcal{F}_q^s.
\]
The sets \( \mathcal{F}_p^u \) and \( \mathcal{F}_q^s \) are called unstable and stable leaves respectively. Note that \( f^{-1}(\mathcal{F}_p^u) \subset \bigcap_{n=0}^\infty f^n(R_1) \) for \( p \in \gamma_c \cap \Gamma \) and \( f(\mathcal{F}_q^s) \subset \bigcap_{n=0}^\infty f^{-n}(R_2) \) for \( q \in \sigma_c \cap \Sigma \). These properties and Proposition 3.2 imply \( F^u \subset f(V^{cu}) \) and \( F^s \subset f^{-1}(V^{cs}) \).

Our two Cantor sets are subsets of the set
\[
L = f(V^{cu}) \cap f^{-1}(V^{cs}) \subset F^u \cap F^s,
\]
which are obtained by projections along the unstable and stable leaves. The set \( L \) consists of two connected components for any \( f \) which is sufficiently \( C^2 \)-close to \( f_0 \). Since any connected component of \( L \) is a transverse intersection between two \( C^2 \) surfaces, it is a \( C^2 \) curve. By Proposition 3.1, the connected components of \( L \) are tangent to both \( C^{cu}(\theta) \) and \( C^{cs}(\theta) \). For definiteness, we fix a \( C^2 \) curve \( \hat{L} \) which is tangent to both \( C^{cu}(\theta) \) and \( C^{cs}(\theta) \), contains \( L \) and such that \( \hat{L} \setminus L \) consists of two connected components both of infinite length. Note that each unstable or stable leaf intersects \( \hat{L} \) at most at one point.

For \( p \in F^u \) let \( \mathcal{F}_p^u(p) \) denote the unstable leaf containing \( p \), and for \( q \in F^s \) let \( \mathcal{F}_q^s(q) \) denote the stable leaf containing \( q \). We now define projections \( \Pi^u : F^u \to L \) and \( \Pi^s : F^s \to L \) by \( \Pi^u(p) \in L \cap \mathcal{F}_p^u(p) \) and \( \Pi^s(q) \in L \cap \mathcal{F}_q^s(q) \). Set
\[
\Omega_1 = \Pi^u(\gamma_c \cap \Gamma) \quad \text{and} \quad \Omega_2 = \Pi^s(\sigma_c \cap \Sigma).
\]
We view \( \Omega_1 \) and \( \Omega_2 \) as Cantor sets with respect to the induced metric on \( \hat{L} \), with thicknesses \( \tau(\Omega_1) \) and \( \tau(\Omega_2) \). Clearly, \( \Omega_1 \) and \( \Omega_2 \) are interleaved, and satisfy \( \Omega_1 \subset W^u(\Gamma) \), \( \Omega_2 \subset W^s(\Sigma) \) and \( \Omega_1 \cap \Omega_2 \subset \Lambda \).
4. ESTIMATES OF THICKNESSES AND PROOFS OF THE MAIN RESULTS

In this last section we prove main estimates on the previous constructions, and complete the proofs of our main results. Based on preliminary estimates in Sections 4.1 and 4.2, we estimate the thicknesses of the cross sections of the unstable and stable sets in Section 4.3. In Section 4.4 we show that the thicknesses are well-preserved under the projections along the unstable and stable manifolds. In Section 4.5 we estimate the thicknesses of the Cantor sets constructed in Section 3, and complete the proofs of Theorems A and B in Sections 4.6 and 4.7. Computational proofs are postponed to Sections 4.8 and 4.9.

4.1. Distortion estimates and symbolic coding. Let $\theta \in (0, 1/10)$ be small enough and let $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusion of Proposition 3.2 holds. We establish bounded distortions on iterations of curves on the graph-invariant surfaces $V_{cu}$, $V_{cs}$ in Proposition 3.2. Since these surfaces are graphs of functions with two-variables, it is convenient to treat them through the natural projections $\pi$ and $\pi^*$ in Section 3.3. A $C^2$ curve $\gamma$ in $V_{cu} \cup f(V_{cu})$ is called $cu$-admissible if it is tangent to $C^{cs}(\theta)$ and the curvature of $\pi(\gamma)$ is everywhere at most $\theta$. Similarly, a $C^2$ curve $\sigma$ in $V_{cs} \cup f^{-1}(V_{cs})$ is called $cs$-admissible if it is tangent to $C^{cu}(\theta)$ and the curvature of $\pi^*(\sigma)$ is everywhere at most $\theta$. Since $\gamma_c$ and $\sigma_c$ are transverse intersections of two $C^2$ surfaces with small derivatives, the implicit function theorem shows that $\gamma_c$ is $cu$-admissible and $\sigma_c$ is $cs$-admissible. A proof of the next lemma is given in Section 4.8.

Lemma 4.1. For any $K > 0$ there exists $\theta \in (0, 1/10)$ such that if $f \in \text{Diff}^2(\mathbb{R}^3)$ is sufficiently $C^2$-close to $f_0$, then the following hold:

(a) For any $n \geq 1$ and any $cu$-admissible curve $\gamma$ in $\bigcap_{k=0}^n f^k(R_1)$, $f^{-n}(\gamma)$ is $cu$-admissible. Moreover, for all $p, q \in \gamma$,

$$\frac{\|Df^{-n}|_{T_p\gamma}\|}{\|Df^{-n}|_{T_q\gamma}\|} \leq e^K.$$
Since \( \gamma \) on two symbols \( \gamma \) those points not contained in \( \gamma \) over the set \( \{ \) and \( n \) components (see FIGURE 7). For convenience, let us call \( \gamma \) the set \( \gamma \). For each word \( C \) Cantor set: at step 3.2 holds. We view \( \gamma \) enough and let \( \gamma \) f from \( \gamma \) we define \( \sigma \). For convenience, let us call \( \gamma \) for any \( n \geq 1 \) and any cs-admissible curve \( \sigma \) in \( \bigcap_{k=0}^{n} f^{-k}(R_2) \), \( f^n(\sigma) \) is cs-admissible. Moreover, for all \( p, q \in \sigma \),

\[
\| Df^n | T_{p\sigma} \| \leq e^K.
\]

To describe the structure of the Cantor set \( \gamma \cap \Gamma \), it is convenient to use a coding on two symbols \( f(A), f(D) \). For each integer \( n \geq 0 \) define

\[
\gamma_c^{(n)} = \gamma_c \cap \bigcap_{k=0}^{n} f^k(f(R_1)).
\]

The set \( \gamma_c^{(n)} \) with \( n \geq 1 \) is obtained from \( \gamma_c^{(n-1)} \) by deleting from \( f^{-n}(\gamma_c^{(n-1)}) \) all those points not contained in \( f(R_1) \), and then applying \( f^n \) to the remaining set. Since \( \gamma_c \subset \bigcap_{n=0}^{\infty} f^{-n}(A) \), \( \bigcap_{n=0}^{\infty} \gamma_c^{(n)} = \gamma_c \cap \Gamma \) holds. Each point in \( \gamma_c \cap \Gamma \) is uniquely coded by a sequence of the symbols which records the history of its backward orbit over the set \( f(R_1) = f(A) \cup f(D) \). For each word \( \omega = \omega_1 \ldots \omega_{n} \) of elements of \( \{ f(A), f(D) \} \) of length \( n \geq 1 \), we define an \( n \)-cylinder in \( \gamma_c \) by

\[
[\omega^-] = \{ p \in \gamma_c : f^{-i}(p) \in \omega_{i-1} \text{ for all } 0 \leq i \leq n - 1 \}.
\]

The set \( \gamma_c^{(n-1)} \) with \( n \geq 1 \) consists of \( 2^n \) connected components all of which are \( n \)-cylinders. Each \( n \)-cylinder \( [\omega^-] \) contains exactly two \( (n+1) \)-cylinders \( [f(A)\omega^-] \) and \( [f(D)\omega^-] \), and the set \( [\omega^-] \setminus ([f(A)\omega^-] \cup [f(D)\omega^-]) \) has exactly three connected components (see FIGURE 7). For convenience, let us call \( \gamma_c \) the 0-cylinder in \( \gamma_c \).

Similarly, each point in \( \sigma_c \cap \Sigma \) is uniquely coded by a sequence of two symbols. For each word \( \omega^+ = \omega_1 \ldots \omega_n \) of elements of \( \{ f^{-1}(B^*), f^{-1}(C^*) \} \) of length \( n \geq 1 \), we define an \( n \)-cylinder in \( \sigma_c \) by

\[
[\omega^+] = \{ q \in \sigma_c : f^i(q) \in \omega_{i+1} \text{ for all } 0 \leq i \leq n - 1 \}.
\]

For convenience, let us call \( \sigma_c \) the 0-cylinder in \( \sigma_c \).

4.2. Estimates on bounded gaps. As in Section 4.1 let \( \theta \in (0, 1/10) \) be small enough and let \( f \) be sufficiently \( C^2 \)-close to \( f_0 \) for which the conclusion of Proposition 3.2 holds. We view \( \gamma_c \cap \Gamma \) as constructed inductively like the middle third Cantor set: at step \( n \geq 1 \) we construct the \( n \)-th approximation \( \gamma_c^{(n)} \) by deleting from \( f^{-n}(\gamma_c^{(n-1)}) \) all those points not contained in \( f(R_1) \), and then applying \( f^n \) to the remaining set. An important difference from the middle third Cantor set is
that we must delete from each connected component of $\gamma_{c}^{(n-1)}$ some neighborhoods of its boundary points. Hence, any gap of $\gamma_{c} \cap \Gamma$ is not created in a finite step of the induction, and a careful analysis is required to estimate the thickness of $\gamma_{c} \cap \Gamma$.

Let $G$ be a bounded gap of $\gamma_{c} \cap \Gamma$. Define

$$t_{G} = \max\{n \geq 0: \text{there exists an } n\text{-cylinder in } \gamma_{c} \text{ which contains } G\}.$$  

Let $G$ denote the maximal curve in $G$ such that $f^{-t_{G}}(G)$ does not intersect $f(R_{1})$. The set $G \setminus G$ has exactly two connected components. If $t_{G} \geq 1$, let $\omega(G) = \omega_{-t_{G}} \cdots \omega_{-1}$ denote the word of length $t_{G}$ that defines the $t_{G}$-cylinder in $\gamma_{c}$ which contains $G$. If $t_{G} = 0$, then put $[\omega(G)] = \gamma_{c}$. Any bounded gap $G'$ of $\gamma_{c} \cap \Gamma$ other than $G$ contained in $[\omega(G)]$ satisfies $t_{G'} > t_{G}$. If $t_{G} \geq 1$, $f^{-t_{G}}([\omega(G)])$ stretches across $A$ or $D$, depending on whether $\omega_{-t_{G}} = f(A)$ or $\omega_{-t_{G}} = f(D)$.

Similarly, for a bounded gap $J$ of $\sigma_{c} \cap \Sigma$ define

$$t_{J} = \max\{n \geq 0: \text{there exists an } n\text{-cylinder in } \sigma_{c} \text{ which contains } J\}.$$  

Let $J$ denote the maximal curve in $J$ such that $f^{t_{J}}(J)$ does not intersect $f^{-1}(R_{2})$. Let $\omega^{+}(J) = \omega_{1} \cdots \omega_{t_{J}}$ denote the word of length $t_{J}$ that defines the $t_{J}$-cylinder in $\sigma_{c}$ containing $J$. If $t_{J} = 0$, then put $[\omega(J)] = \sigma_{c}$. The next lemma provides preliminary estimates on the gaps of $\gamma_{c} \cap \Gamma$ and $\sigma_{c} \cap \Sigma$. Let $\ell(\gamma)$ denote the length of a $C^{1}$ curve $\gamma$.

**Lemma 4.2.** For any $K > 0$ there exists $\theta \in (0, 1/10)$ such that if $f \in \text{Diff}^{2}(\mathbb{R}^{3})$ is sufficiently $C^{2}$-close to $f_{0}$, then the following hold:

(a) For any bounded gap $G$ of $\gamma_{c} \cap \Gamma$,

$$e^{-K} \leq \frac{\ell(f^{-t_{G}}(G))}{|E^{*}_{c}| - 2|A^{*}_{c}|} \leq e^{K} \quad \text{and} \quad \frac{\ell(f^{-t_{G}}(G))}{\ell(f^{-t_{G}}(G))} \leq 1 + e^{K}(1 + \theta)b_{1};$$

(a') For any bounded gap $J$ of $\sigma_{c} \cap \Sigma$,

$$e^{-K} \leq \frac{\ell(f^{t_{J}}(J))}{|F_{c}| - 2|B_{c}|} \leq e^{K} \quad \text{and} \quad \frac{\ell(f^{t_{J}}(J))}{\ell(f^{t_{J}}(J))} \leq 1 + e^{K}(1 + \theta)b_{2};$$

(b) Let $G_{1}$, $G_{2}$ be bounded gaps of $\gamma_{c} \cap \Gamma$ such that $0 \leq t_{G_{1}} < t_{G_{2}}$ and $[\omega^{-}(G_{1})] \supseteq G_{2}$. Then $\ell(G_{1}) > \ell(G_{2})$;

(b') Let $J_{1}$, $J_{2}$ be bounded gaps of $\sigma_{c} \cap \Sigma$ such that $0 \leq t_{J_{1}} < t_{J_{2}}$ and $[\omega^{+}(J_{1})] \supseteq J_{2}$.

Then $\ell(J_{1}) > \ell(J_{2})$.

**Proof.** Let $K > 0$, and let $\theta \in (0, 1/10)$ be small enough and let $f$ be sufficiently $C^{2}$-close to $f_{0}$ for which the conclusion of Lemma 4.1 holds. For any bounded gap $G$ of $\gamma_{c} \cap \Gamma$, $f^{-t_{G}}(G)$ is $cu$-admissible by Lemma 1.1. Hence, the first inequality in Lemma 4.2(a) holds for $\theta$ small enough.

Let $G$ be a bounded gap of $\gamma_{c} \cap \Gamma$. There is a decomposition of $f^{-t_{G}}(G \setminus G)$ into a countably infinite number of $cu$-admissible curves

$$f^{-t_{G}}(G \setminus G) = \bigcup_{n=1}^{\infty} G_{+}^{n} \cup \bigcup_{n=1}^{\infty} G_{-}^{n},$$

with the following properties:
The sets $G^1_+$ and $G^1_-$ sandwich $f^{-tG(G)}$. For each $n \geq 2$, $G^1_+$ and $G^1_-$ sandwich $f^{-tG(G)} \cup \bigcup_{k=1}^{n-1} (G^k_+ \cup G^k_-)$.

For each $n \geq 1$, $f^{-tG(G)} \cup G^1_+ \subset f(R_1)$ for all $0 \leq i \leq n-1$ and $f^{-n}(G^n_+ \cup G^n_-) \subset R_1 \setminus f(R_1)$.

By Lemma 4.4, $f^{-n}(G^n_+)$ and $f^{-n}(G^n_-)$ are $cu$-admissible. Proposition 3.1 gives

$$
\ell(G^n_+) + \ell(G^n_-) \leq (1 + \theta)(|A_c| - |E^c_e|)\lambda^n_c.
$$

Using this estimate, $\lambda_c < 1/2$ and the first inequality in Lemma 4.2(a) we have

$$
\frac{\ell(f^{-tG(G)})}{\ell(f^{-tG(G)})} = 1 + \frac{\ell(f^{-tG(G)} \setminus G)}{\ell(f^{-tG(G)})} = 1 + \sum_{n=1}^{\infty} (\ell(G^n_+) + \ell(G^n_-))
$$

$$
\leq \left(1 + e^K (1 + \theta) \frac{|A_c| - |E^c_e|}{|E^c_e| - 2|A^e_c|} \sum_{n=1}^{\infty} \frac{1}{2^n} \right) = 1 + e^K (1 + \theta) b_1,
$$

as required in the second inequality in Lemma 4.2(a). A proof of Lemma 4.2(b) is completely analogous.

Let $G_1$, $G_2$ be bounded gaps of $\gamma_c \cap \Gamma$ as in Lemma 4.2(b). Then

$$
\frac{\ell(G_1)}{\ell(G_2)} \geq e^{-K} \frac{\ell(f^{-tG_1}(G_1))}{\ell(f^{-tG_2}(G_2))} = e^{-K} \frac{\ell(f^{-tG_1}(G_1))}{\lambda_c^{tG_2 - tG_1} \ell(f^{-tG_2}(G_2))} \geq \frac{e^{-K}}{\lambda_c (1 + e^K(1 + \theta) b_1) \ell(f^{-tG_2}(G_2))} \geq \frac{e^{-3K}}{\lambda_c (1 + e^K(1 + \theta) b_1)} > 1.
$$

We have used Lemma 4.4(a) for the first inequality, and Proposition 3.1(c) for the second one. The third and fourth inequalities follow from Lemma 4.2(a). By (3.6), the last inequality holds if $K$ and $\theta$ are small enough. A proof of Lemma 4.2(b) is completely analogous.

Let $G$ be a bounded gap of $\gamma_c \cap \Gamma$. Let $x \in G$ and let $I$ be the bridge of $\gamma_c \cap \Gamma$ at $x$. The maximal curve $\tilde{I}$ in $I$ such that $f^{-tG(\tilde{I})}$ is contained in the minimal curve $\gamma_c'$ containing $\gamma_c \cap \Gamma$ is called the core of $I$. Similarly, let $J$ be a bounded gap of $\sigma_c \cap \Sigma$. Let $x \in J$ and let $L$ be the bridge of $\sigma_c \cap \Sigma$ at $x$. The maximal curve $\tilde{L}$ in $L$ such that $f^{-tJ(\tilde{L})}$ is contained in the minimal curve $\sigma_c'$ containing $\sigma_c \cap \Sigma$ is called the core of $L$. Instead of directly estimating the length of a bridge from below, we estimate the length of its core from below.

Lemma 4.3. For any $\delta > 0$ there exists a $C^2$ neighborhood $\mathcal{V}$ of $f_0$ such that if $f \in \mathcal{V}$ then the following hold:

(a) Let $G$ be a bounded gap of $\gamma_c \cap \Gamma$, $x$ be a boundary point of $G$, and let $\tilde{I}$ be the core of the bridge of $\gamma_c \cap \Gamma$ at $x$. Then

$$
\frac{\ell(\tilde{I})}{\ell(G)} > a_1 - \delta;
$$

(a*) Let $J$ be a bounded gap of $\sigma_c \cap \Sigma$, $x$ be a boundary point of $J$, and let $\tilde{L}$ be the core of the bridge of $\sigma_c \cap \Sigma$ at $x$. Then

$$
\frac{\ell(\tilde{L})}{\ell(J)} > a_2 - \delta.
$$

□
Proof. Let \( K > 0 \) and let \( \theta \in (0, 1/10) \) be small enough and let \( f \) be sufficiently \( C^2 \)-close to \( f_0 \) for which the conclusion of Lemma \ref{lem:4.2} holds. Let \( G \) be a bounded gap of \( \gamma_c \cap \Gamma \) and let \( x \) be a boundary point of \( G \). Let \( \tilde{I} \) denote the core of the bridge at \( x \). Lemma \ref{lem:4.2}(b) implies that the endpoint of \( f^{-t\sigma}(\tilde{I}) \) is \( f^{-t\sigma}(x) \) and one of the endpoints of \( \gamma_c \), see FIGURE 8. For any \( f \) sufficiently \( C^2 \)-close to \( f_0 \), the sum of the lengths of the two connected components of \( (\gamma_c \setminus \gamma_c') \cap f(R_1) \) is at most \( e^K(|A_c| - |E^*_c|) \). Moreover, Lemma \ref{lem:4.2}(a) yields

\[
\ell(f^{-t\sigma}(G \setminus G)) \leq \ell(f^{-t\sigma}(G)) \leq e^K(|E^*_c| - 2|A^*_c|). \tag{4.1}
\]

Hence

\[
\ell(f^{-t\sigma}(\tilde{I})) \geq e^{-K}|A^*_c| - e^K(|E^*_c| - 2|A^*_c|) - e^K(|A_c| - |E^*_c|). \tag{4.2}
\]

Using Lemma \ref{lem:4.2}(a) and \ref{lem:4.1} and \ref{lem:4.2}, we obtain

\[
\frac{\ell(\tilde{I})}{\ell(G)} \geq \frac{e^{-K}}{e^{-K} + e^K(1 + \theta)b_1} \geq \frac{1}{1 + e^K(1 + \theta)b_1} - e^{-2K} \frac{e^{-K}|A^*_c| - e^K(|E^*_c| - 2|A^*_c|) - e^K(|A_c| - |E^*_c|)}{|E^*_c| - 2|A^*_c|} > a_1 - \delta.
\]

Since \( b_1 \leq 1 \), the last inequality holds for \( K \) and \( \theta \) small enough. A proof of Lemma \ref{lem:4.3}(a*) is completely analogous. \( \square \)

4.3. **Main estimates on thicknesses I.** Using the results in the previous sections, we obtain the following lower bounds of thicknesses.

**Proposition 4.4.** For any \( \delta > 0 \) there exists a \( C^2 \) neighborhood \( \mathcal{V} \) of \( f_0 \) such that if \( f \in \mathcal{V} \) then

\[
\tau(\gamma_c \cap \Gamma) > a_1 - \delta \quad \text{and} \quad \tau(\sigma_c \cap \Sigma) > a_2 - \delta.
\]

**Proof.** Let \( \delta > 0 \). Let \( \mathcal{V} \) be a \( C^2 \)-neighborhood of \( f_0 \) as in Lemma \ref{lem:4.3}. Let \( f \in \mathcal{V} \). Let \( G \) be a bounded gap of \( \gamma_c \cap \Gamma \) and let \( x \) be a boundary point of \( G \). Let \( I \) denote the bridge at \( x \) and \( \tilde{I} \) the core of \( I \). Then

\[
\tau(\gamma_c \cap \Gamma, x) = \frac{\ell(I)}{\ell(G)} \geq \frac{\ell(\tilde{I})}{\ell(G)} > \frac{a_1 - \delta}{2}.
\]

Since \( G \) is an arbitrary bounded gap of \( \gamma_c \cap \Gamma \) and \( x \) is an arbitrary boundary point of \( G \), we obtain \( \tau(\gamma_c \cap \Gamma) > a_1 - \delta \). A proof of the other bound is analogous. \( \square \)

4.4. **Lipschitz continuity of projections.** Let \( \theta \in (0, 1/10) \) be small enough and let \( f \) be sufficiently \( C^2 \)-close to \( f_0 \) for which the conclusion of Proposition \ref{prop:3.2} holds. For two \( cu \)-admissible curves \( \gamma_1, \gamma_2 \) we write \( \gamma_1 \sim \gamma_2 \) if \( \Pi^u(\gamma_1 \cap F^u) = \Pi^u(\gamma_2 \cap F^u) \), and define a projection \( \Pi^u_{\gamma_1 \gamma_2} : \gamma_1 \cap F^u \to \gamma_2 \cap F^u \) by \( \Pi^u_{\gamma_1 \gamma_2}(p) \in \gamma_2 \cap F^u(p) \). Note that \( \Pi^u_{\gamma_1 \gamma_2} \) is invertible and the inverse is \( \Pi^s_{\gamma_2 \gamma_1} \). Reciprocally, for two \( cs \)-admissible curves \( \sigma_1, \sigma_2 \) we write \( \sigma_1 \sim \sigma_2 \) if \( \Pi^s(\sigma_1 \cap F^s) = \Pi^s(\sigma_2 \cap F^s) \), and define a projection \( \Pi^s_{\sigma_1 \sigma_2} : \sigma_1 \cap F^s \to \sigma_2 \cap F^s \) by \( \Pi^s_{\sigma_1 \sigma_2}(q) \in \sigma_2 \cap F^s(q) \).
For a $C^1$ curve $\gamma$ and $p, q \in \gamma$, let $\ell_\gamma(p, q)$ denote the length of the curve in $\gamma$ bordered by $p$ and $q$. The next lemma proved in Section 4.9 asserts that the projections defined as above are bi-Lipschitz continuous, and the Lipschitz constants can be made arbitrarily small for $f$ which is sufficiently $C^2$-close to $f_0$.

**Lemma 4.5.** For any $K > 0$ there exists $\theta \in (0, 1/10)$ such that if $f \in \text{Diff}^2(\mathbb{R}^3)$ is sufficiently $C^2$-close to $f_0$, then the following hold:

(a) Let $\gamma_1, \gamma_2$ be $cu$-admissible such that $\gamma_1 \sim \gamma_2$. For all $p, q \in \gamma_1 \cap F^u$,
$$
\ell_{\gamma_2}(\Pi_{\gamma_1 \gamma_2}u(p), \Pi_{\gamma_1 \gamma_2}u(q)) \leq e^K \ell_{\gamma_1}(p, q);
$$

(a*) Let $\sigma_1, \sigma_2$ be $cs$-admissible such that $\sigma_1 \sim \sigma_2$. For all $p, q \in \sigma_1 \cap F^s$,
$$
\ell_{\sigma_2}(\Pi_{\sigma_1 \sigma_2}^s(p), \Pi_{\sigma_1 \sigma_2}^s(q)) \leq e^K \ell_{\sigma_1}(p, q).
$$

4.5. Main estimates on thicknesses II. By Lemma 4.4 the lower bounds in Proposition 4.4 transfer to that of the thicknesses of $\Omega_1$ and $\Omega_2$ as follows.

**Proposition 4.6.** For any $\delta > 0$ there exists a $C^2$ neighborhood $\mathcal{V}$ of $f_0$ such that if $f \in \mathcal{V}$ then
$$
\tau(\Omega_1) > a_1 - \delta \quad \text{and} \quad \tau(\Omega_2) > a_2 - \delta.
$$

**Proof.** Let $\delta > 0$. Let $K > 0$ be such that $e^{2K} < 1 + \delta$ and $e^{-2K} > 1 - \delta$. Let $\theta \in (0, 1/10)$ be small enough, and $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusion of Lemma [4.1] holds. There is a one-to-one correspondence between the bounded gaps of $\Omega_1$ and the bounded gaps of $\gamma_c \cap \Gamma$: a bounded gap $G$ of $\Omega_1$ and the corresponding bounded gap $G'$ of $\gamma_c \cap \Gamma$ satisfy the relation $\partial G = \Pi^u(\partial G')$. There is also a one-to-one correspondence between the cores of bridges of $\Omega_1$ and the cores of bridges $\gamma_c \cap \Gamma$: the core $I$ of a bridge of $\Omega_1$ and the corresponding core $\tilde{I}'$ of a bridge of $\gamma_c \cap \Gamma$ satisfy the relation $\partial I = \Pi^u(\partial \tilde{I})$.

Let $G$ be a bounded gap of $\Omega_1$ and let $x$ be a boundary point of $G$. Let $I$ be the bridge of $\Omega_1$ at $x$. Let $G'$ be the bounded gap of $\gamma_c \cap \Gamma$ such that $\partial G = \Pi^u(\partial G')$. Let $y$ be the boundary point of $G'$ such that $\Pi^u(y) = x$. Let $I'$ be the bridge of $\gamma_c \cap \Gamma$ at $y$. By Lemmas 4.3(a) and 4.5(a) we have
$$
\frac{\ell(I)}{\ell(G)} \geq \frac{\ell(\tilde{I})}{\ell(G)} \geq e^{-2K} \frac{\ell(\tilde{I}')}{\ell(G')} \geq e^{-2K} \left( a_1 - \frac{\delta}{2} \right) > a_1 - \delta.
$$
Since $G$ is an arbitrary bounded gap of $\Omega_1$ and $x$ is an arbitrary boundary point of $G$, it follows that $\tau(\Omega_1) \geq a_1 - \delta$. Hence the first inequality in Proposition 3.6 holds. A proof of the second one is completely analogous. □

4.6. **Proof of Theorem A.** From Propositions 4.4, 4.6, Lemma 2.2 and the assumption $a_1 a_2 > 1$, $H(f)$ is non-empty for any $f$ which is $C^2$-close to $f_0$. The lower bounds of $\dim W^u(\Gamma)$ and $\dim W^s(\Sigma)$ are consequences of Proposition 4.4 and Lemma 4.5. The upper bounds of these follow from Lemma 4.5.

Let $\theta \in (0, 1/10)$ be small enough and let $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusions of Propositions 3.1 and 3.2 hold. We claim that there exists a countable family $\{\gamma_k\}_{k=1}^{\infty}$ of $C^1$ curves contained in $V^{cu} \cap \bigcup_{n=0}^{\infty} f^{-n}(V^{cs})$ and tangent to $C^{cs}(\theta)$ such that for each $p \in H(f)$ there exist integers $n < 0$ and $k \geq 1$ such that $f^n(p) \in \gamma_k \cap F^u$. Since each $\gamma_k \cap F^u$ is covered by $2^n$ curves in $\gamma_k$ of length at most $\lambda_0^2$, for all $n \geq 1$, it follows that $\dim(\gamma_k \cap F^u) \leq -\ln 2/\ln \lambda_c < 1$. By the countable stability and the invariance of Hausdorff dimension under the action of bi-Lipschitz homeomorphisms, we obtain $\dim H(f) < 1$. Hence, $H(f)$ is totally disconnected [13].

It is left to prove the claim. If $p \in H(f)$ then $p \in L$, and there exist integers $n_- < 0 \leq n_+$ such that $f^n(p) \in V^{cu} \cap F^u$ for all $n \leq n_-$ and $f^n(p) \in V^{cs} \cap F^s$ for all $n \geq n_+$. Set $N = n_+ - n_-$. By Proposition 3.1, $f^N(V^{cu})$ intersects $V^{cs}$ transversely at $f^{n_+}(p)$, and so $f^N(V^{cu}) \cap V^{cs}$ contains a curve which contains $f^{n_+}(p)$ and is tangent to $C^{cs}(\theta)$. We apply $f^{-N}$ and use Proposition 3.1 to obtain a curve in $V^{cu} \cap f^{-N}(V^{cs})$ which contains $f^{n_-}(p)$ and is tangent to $C^{cs}(\theta)$. This implies the claim, and the proof of Theorem A is complete. □

4.7. **Proof of Theorem B.** In view of Theorem 2.3, we fix $\tau_0 > 0$ such that if $S_1$, $S_2$ are two interleaved Cantor sets in $\mathbb{R}$ with $\tau(S_1) > \tau_0/2$, $\tau(S_2) > \tau_0/2$, then $S_1 \cap S_2$ contains a Cantor set whose thickness is at least $(6/7)\sqrt{\min\{\tau(S_1), \tau(S_2)\}}$. Assume $\min\{a_1, a_2\} > \tau_0$. Combining Propositions 4.4 and 4.6 we obtain $\tau(\Omega_1) > \tau_0/2$ and $\tau(\Omega_2) > \tau_0/2$ for all $f$ that is sufficiently $C^2$-close to $f_0$. Then

$$\tau(\Omega_1 \cap \Omega_2) \geq \frac{6}{7} \sqrt{\min\{\tau(\Omega_1), \tau(\Omega_2)\}} > \frac{5}{6} \sqrt{\min\{a_1, a_2\}}.$$  

By [13] (see also [12, p.77, Proposition 5]), the Hausdorff dimension of a Cantor set with thickness $\tau$ is at least $\ln 2/\ln(2 + 1/\tau)$. Hence we obtain

$$\dim H(f) - 1 < \frac{1}{2 \ln 2 \tau(\Omega_1 \cap \Omega_2)} < \frac{1}{\sqrt{\min\{a_1, a_2\}},}$$

as required. This completes the proof of Theorem B. □

4.8. **Proof of Lemma 4.1** Let $K > 0$, and let $\theta \in (0, 1/10)$ be small enough, and let $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusion of Proposition 3.2 holds. To obtain distortion bounds on $V^{cs}$ and $V^{cs}$, we consider iterations of planar maps $\varphi: \pi(R_1) \to \pi(R_1)$ and $\varphi^*: \pi^*(R_2) \to \pi^*(R_2)$ given by

$$\varphi = \pi \circ f^{-1} \circ (\pi|_{V^{cs}})^{-1} \quad \text{and} \quad \varphi^* = \pi^* \circ f \circ (\pi^*|_{V^{cs}})^{-1}.$$
Since $V^{cu}$ and $V^{cs}$ are $C^2$ submanifolds, $\varphi$ and $\varphi^*$ are of class $C^2$. Moreover

\begin{equation}
\sup_{\pi(R_1)} |\det D\varphi| < \lambda_{0,c}^{-1} \quad \text{and} \quad \sup_{\pi^*(R_2)} |\det D\varphi^*| < \mu_{0,c}.
\end{equation}

Furthermore, Proposition 3.2 implies

\begin{equation}
\sum_{i,j \in \{u, c\}} \sup_{\pi(R_1)} \|\nabla \Phi_{ij}\| < \theta^2 \quad \text{and} \quad \sum_{i,j \in \{c, s\}} \sup_{\pi^*(R_2)} \|\nabla \Phi_{ij}\| < \theta^2,
\end{equation}

where $\nabla = (\partial_{x_u}, \partial_{x_c})$ and

\[ D\varphi = \begin{pmatrix} \Phi_{uu} & \Phi_{uc} \\ \Phi_{cu} & \Phi_{cc} \end{pmatrix} \quad \text{and} \quad D\varphi^* = \begin{pmatrix} \Phi_{cc} & \Phi_{cs} \\ \Phi_{sc} & \Phi_{ss} \end{pmatrix}. \]

Let $n \geq 1$ and let $\gamma$ be $cu$-admissible as in Lemma 4.1(a). By Proposition 3.1, $f^{-n}(\gamma)$ is tangent to $C^s(\theta)$. By the graph invariance of $V^{cu}$, $f^{-n}(\gamma)$ is tangent to $C^{cu}(\theta)$. We parametrize the curve $\gamma_0 = \pi(\gamma)$ by arc length $s$, and put $\gamma_{i+1}(s) = \varphi \circ \gamma_i(s)$ inductively for $0 \leq i \leq n-1$. We have

\begin{equation}
\dot{\gamma}_{i+1}(s) = D\varphi_{\gamma_i(s)} \dot{\gamma}_i(s) \quad \text{and} \quad \ddot{\gamma}_{i+1}(s) = M_i \dot{\gamma}_i(s) + D\varphi_{\gamma_i(s)} \ddot{\gamma}_i(s),
\end{equation}

where the single and double dots denote the first- and second-order derivatives on $s$ respectively, and

\[ M_i = \begin{pmatrix} \langle \nabla \Phi_{uu}, \dot{\gamma}_i(s) \rangle & \langle \nabla \Phi_{uc}, \dot{\gamma}_i(s) \rangle \\ \langle \nabla \Phi_{cu}, \dot{\gamma}_i(s) \rangle & \langle \nabla \Phi_{cc}, \dot{\gamma}_i(s) \rangle \end{pmatrix}, \]

and the brackets in $M_i$ denote the inner products. Let $\kappa_i(s)$ denote the curvature of the curve $\gamma_i$ at $\gamma_i(s)$. Using the above formulas we have

\begin{equation}
\kappa_{i+1}(s) \leq \frac{1}{\|\dot{\gamma}_{i+1}(s)\|^3} \left( \|D\varphi_{\gamma_i(s)} \dot{\gamma}_i(s) \times M_i \dot{\gamma}_i(s)\| + \|D\varphi_{\gamma_i(s)}\| \cdot \|\dot{\gamma}_i(s) \times \ddot{\gamma}_i(s)\| \right) \leq \frac{1}{\|\dot{\gamma}_{i+1}(s)\|^3} \left( \|D\varphi_{\gamma_i(s)}\| \cdot \|M_i\| \frac{1}{\|\dot{\gamma}_i(s)\|^3} + \|D\varphi_{\gamma_i(s)}\| \kappa_i(s) \right).
\end{equation}

Using $\|\dot{\gamma}_i(s)\| \geq \lambda_c^{-i}$ and $\|\dot{\gamma}_{i+1}(s)\| \geq \lambda_c^{-i-1}\|\dot{\gamma}_i(s)\|$ from Proposition 3.1, (4.3), and (4.4) we have $\kappa_{i+1}(s) \leq \lambda_c^3 (\theta + \lambda_{0,c}^{-1} \kappa_i(s)) \leq \lambda_c^3 \theta + \lambda_c \kappa_i(s)$. Using this inequality inductively and then $\kappa_0 \leq \theta$, we have

\begin{equation}
\kappa_n(s) \leq \lambda_c^3 \theta \sum_{i=0}^{n-1} \lambda_c^i + \lambda_c^n \kappa_0(s) \leq 2 \lambda_c^3 \theta + \lambda_c^n \theta \leq \frac{\theta}{4} + \frac{\theta}{2} = \frac{3\theta}{4}.
\end{equation}

Since $\pi(f^{-n}(\gamma(s))) = \gamma_n(s)$, from (4.5) it follows that $f^{-n}(\gamma)$ is $cu$-admissible.

Let $0 \leq i \leq n-1$. For all arc length parameters $s, t \in [0, \ell(\gamma_0)]$ we have

\[ \frac{\|\dot{\gamma}_{i+1}(s)\|}{\|\dot{\gamma}_i(s)\|} - \frac{\|\dot{\gamma}_{i+1}(t)\|}{\|\dot{\gamma}_i(t)\|} \leq \frac{3\theta}{4} \|D\varphi_{\gamma_i(t)}\| \ell_{\gamma_i(s)}(\gamma_i(s), \gamma_i(t)) + \|D\varphi_{\gamma_i(s)} - D\varphi_{\gamma_i(t)}\| \leq \sqrt{\theta} \ell_{\gamma_i(s)}(\gamma_i(s), \gamma_i(t)). \]

The second inequality is from the upper bound on the curvature of $\gamma_i$ in (4.3). The last one is from (4.3) and (4.4). Summing this estimate over all $0 \leq i \leq n-1$,
and then using $\ell_{\gamma}(\gamma_i(s), \gamma_i(t)) \leq \ell_{\gamma_n}(\gamma_n(s), \gamma_n(t)) \lambda_c^{n-i}$ from Proposition 3.1 and $\ell_{\gamma_n}(\gamma_n(s), \gamma_n(t)) < 2$, we obtain

$$\ln \left( \frac{\|\dot{\gamma}(t)\|}{\|\dot{\gamma}(s)\|} \right) \leq \sum_{i=0}^{n-1} \left( \frac{\|\dot{\gamma}_{i+1}\|}{\|\dot{\gamma}_i\|} - \frac{\|\dot{\gamma}_{i+1}(t)\|}{\|\dot{\gamma}_i(t)\|} \right) \leq 2\sqrt{\theta} \sum_{i=0}^{n-1} \lambda_c^{n-i} < 2\sqrt{\theta} < K_3.$$  

The last inequality holds for $\theta$ small enough. Since $f^{-n}(\gamma) = (\pi|_{V^{cu}})^{-1} \circ \varphi^n \circ \pi(\gamma)$, for all points $p = \gamma(s)$ and $q = \gamma(t)$ in $\gamma$, the chain rule yields

$$\frac{\|Df^{-n}|_{T_{\pi\gamma}}\|}{\|Df^{-n}|_{T_{\eta\gamma}}\|} = \frac{\|D(\pi|_{V^{cu}})^{-1}|_{T_{\pi(f^{-n}(p))}\gamma_n}\| \|\dot{\gamma}_n(s)\| \|D\pi|_{T_{\pi\gamma}}\|}{\|D(\pi|_{V^{cu}})^{-1}|_{T_{\pi(f^{-n}(q))}\gamma_n}\| \|\dot{\gamma}_n(t)\| \|D\pi|_{T_{\pi\gamma}}\|}.$$  

The first and last fractions of the right-hand side are at most $e^{K_3}$ since $\gamma$ and $f^{-n}(\gamma)$ are tangent to $C^{cu}(\theta)$. The proof of Lemma 4.4(a) is complete. A proof of Lemma 4.4(a*) is done by exchanging the roles of $\varphi$, $\varphi^*$ and proceeding in the same way.

4.9. **Proof of Lemma 4.5.** Let $K > 0$, and let $\theta \in (0, 1/10)$ be small enough, and let $f$ be sufficiently $C^2$-close to $f_0$ for which the conclusion of Lemma 4.4 with $K$ replaced by $K/5$, and the following hold: for all $cu$-admissible curves $\gamma_1, \gamma_2$ such that $\gamma_1 \sim \gamma_2$ and for all $p, q \in \gamma_1 \cap F^u$,

$$\ell_{f^{-n}(\gamma_2)}(f^{-n}(\Pi_{\gamma_1\gamma_2}(p)), f^{-n}(\Pi_{\gamma_1\gamma_2}(q))) \leq e^{K_3} \ell_{f^{-n}(\gamma_1)}(f^{-n}(p), f^{-n}(q)) \tag{4.6}$$

where $n = n(p, q) \geq 0$ is the integer such that $f^{-k}(p)$ and $f^{-k}(q)$ belong to the same connected component of $f(V^{cu})$ for all $0 \leq k \leq n$, and $f^{-n}(p)$ and $f^{-n}(q)$ do not belong to the same connected component of $f(V^{cu})$.

Let $\gamma_1, \gamma_2$ be $cu$-admissible curves such that $\gamma_1 \sim \gamma_2$. Let $p, q \in \gamma_1 \cap F^u$. By Proposition 3.1 and the mean value theorem, there exist $\xi \in \gamma_1$ in between $p$ and $q$, and $\eta \in \gamma_2$ in between $\Pi_{\gamma_1\gamma_2}(p)$ and $\Pi_{\gamma_1\gamma_2}(q)$ such that

$$\ell_{\gamma_1}(\Pi_{\gamma_1\gamma_2}(p), \Pi_{\gamma_1\gamma_2}(q)) \leq e^{K_3} \frac{\|\dot{\gamma}_n\|_{\gamma_1\gamma_2}}{\|\dot{\gamma}(n, p, q)\|_{\gamma_1\gamma_2}} \leq e^{K_3} \frac{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|}{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|}.$$  

For the fraction of the right-hand side, Lemma 4.4(a) gives

$$\frac{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|}{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|} \leq \frac{3K}{\|D(\pi|_{V^{cu}})^{-1}|_{T_{\pi(f^{-n}(p))}\gamma_n}\| \|\dot{\gamma}_n(s)\| \|D\pi|_{T_{\pi\gamma}}\|} \leq e^{K_3} \frac{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|}{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|},$$

where $p' = \Pi_{\gamma_1\gamma_2}(p)$. From this estimate and (4.6), it is enough to show

$$\frac{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|}{\|Df^{-n}|_{\Pi_{\gamma_1\gamma_2}}\|} \leq e^{K_3}. \tag{4.7}$$

To this end, we use the planar map $\varphi$ introduced in the proof of Lemma 4.4. For $0 \leq i \leq n$ put $p_i = \pi(f^{-i}(p))$ and $p'_i = \pi(f^{-i}(p'))$. Let $v_0, v'_0$ be unit vectors at $p_0$ and $p'_0$ which are tangent to $\pi(\gamma_1)$ and $\pi(\gamma_2)$ respectively. Put $v_i = D_{\varphi^i}v_0$.
and \(v'_i = D\varphi^i v'_0\). Let \(\angle(v_i, v'_i) \in [0, \pi)\) be the angle between \(v_i\) and \(v'_i\), and put \(s_i = \sin \angle(v_i, v'_i)\). Set \(c_0 = \lambda_0^{-1}\). For \(0 \leq i \leq n - 1\) we have

\[
s_{i+1} = \frac{1}{\|v_{i+1}\| \cdot \|v'_{i+1}\|} \|D\varphi_{p_i} v_i \times D\varphi_{p_i} v'_i + D\varphi_{p_i} v_i \times (D\varphi_{p_i} - D\varphi_{p_i}) v'_i\|
\leq \frac{\|v_i\| \cdot \|v'_i\|}{\|v_{i+1}\| \cdot \|v'_{i+1}\|} \left( |\det D\varphi_{p_i}| s_i + \theta \ell_{F_{u}}(f^{-i}(p))(f^{-i}(p), f^{-i}(p')) \right)
\leq \lambda_0^2(\lambda_0 s_i + 2\theta \lambda^{-i}) \leq c_0 s_i + \sqrt{\theta} \lambda^{-i}.
\]

To estimate the second cross product we have used (4.3). For the last inequality we have used (4.4). For the last inequality we have used (4.3). For the last inequality we have used (4.3). For the last inequality we have used (4.3). For the last inequality we have used (4.3). For the last inequality we have used (4.3). For the last inequality we have used (4.3). For the last inequality we have used (4.3).

\[
\sum_{i=0}^{\infty} (c_0 \lambda_i)^k
\]

Summing this inequality over all \(0 \leq i \leq n - 1\) yields

\[
\ln \frac{\|v_n\|}{\|v'_n\|} \leq 4\theta + \frac{2\theta}{1 - c_0} + \sqrt{\theta} \sum_{i=0}^{\infty} \lambda_i^{-i+1} \sum_{k=0}^{i-1} (c_0 \lambda_k)^k \leq K.
\]

Since \(0 < c_0 < 1\) by (3.7), the series indeed converges, and the last inequality holds for \(\theta > 0\) small enough. Moreover, the chain rule gives

\[
\frac{\|Df^{-n}_{|T_g \gamma_1}\|}{\|Df^{-n}_{|T_g \gamma_2}\|} = \frac{\|D(\pi|_{C^{cu}})^{-1} |_{\partial^c(f^{-n}(\gamma_1))} \| v_n \| \| D\pi|_{T_g \gamma_1} \|}{\|D(\pi|_{C^{cu}})^{-1} |_{\partial^c(f^{-n}(\gamma_2))} \| v'_n \| \| D\pi|_{T_g \gamma_2} \|}.
\]

The first and last fractions of the right-hand side are at most \(e^K\) since \(\gamma_1, \gamma_2\) and \(f^{-n}(\gamma_1), f^{-n}(\gamma_2)\) are tangent to \(C^{cu}(\theta)\). Hence (1.7) holds. A proof of Lemma 4.5(a*) is done by exchanging the roles of \(\varphi, \varphi^*\) and proceeding in the same way.

\[
\text{References}
\]

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