The Fidelity Alternative and Quantum Identification

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Dedicated to the memory of Rudolph Ahlswede

If a quantum system is subject to noise, it is possible to perform quantum error correction reversing the action of the noise if and only if no information about the system's quantum state leaks to the environment. In this article, we develop an analogous duality in the case that the environment approximately forgets the identity of the quantum state, a weaker condition satisfied by \(\epsilon\)-randomizing maps and approximate unitary designs. Specifically, we show that the environment approximately forgets quantum states if and only if the original channel approximately preserves pairwise fidelities of pure inputs, an observation we call the fidelity alternative. Using this tool, we then go on to study the task of using the output of a channel to simulate restricted classes of measurements on a space of input states. The case of simulating measurements that test whether the input state is an arbitrary pure state is known as equality testing or quantum identification. We establish that the optimal amortized rate at which quantum states can be identified through a noisy quantum channel is equal to the entanglement-assisted classical capacity of the channel, despite the fact that the task is quantum, not classical, and entanglement-assistance is not allowed. In particular, this rate is strictly positive for every quantum channel, including classical channels, even though the ability to identify cannot be cloned.

I. INTRODUCTION

Quantum channels in modern quantum information theory \cite{1} are modeled as completely positive and trace-preserving maps \(N: S(A) \rightarrow S(B)\) between the state spaces of quantum systems with Hilbert spaces \(A\) and \(B\). The requirement of \textit{complete} positivity means that \(N\) is not just \textit{positive}, mapping positive semidefinite operators to positive semidefinite operators, but that \(\text{id} \otimes N\) is positive for the identity map \(\text{id}\) on any \(S(R)\). This distinction plays a central role in the geometry of entanglement because positive but not completely positive maps can be used to identify entangled quantum states \cite{2}. This paper will take as its starting point a similar observation about channel norms.

The Stinespring dilution theorem establishes a fundamental property of quantum channels: for every channel \(N\) there exists an ancilla space \(E\) and an isometry \(V: A \rightarrow B \otimes E\) such that \(N(\rho) = \text{tr}_E V \rho V^\dagger\) \cite{3}. This means that quantum noise can always be interpreted as information loss in an otherwise deterministic evolution. Since \(E\) and \(V\) are essentially unique (up to unitary equivalence), each channel \(N\) also has an associated complementary channel \(N^c: S(A) \rightarrow S(E)\), with \(N^c(\rho) = \text{tr}_B V \rho V^\dagger\), which is uniquely defined up to coordinate changes of \(E\).

In quantum Shannon theoretic error correction we try to find two channels \(\mathcal{E}\) and \(\mathcal{D}\) (an encoder and decoder) such that \(\mathcal{D} \circ N \circ \mathcal{E} \approx \text{id}\). For now we shall consider the encoding \(\mathcal{E}\) fixed, so that \(N \circ \mathcal{E}\) can be treated as a single channel. The central insight of quantum error correction \cite{4-7} is that the existence of a decoding operation \(\mathcal{D}\) for a channel \(N\), i.e.

\[
\forall \rho \in S(RA) \quad \| (\text{id} \otimes N) \rho^{RA} - \rho^{RA} \|_1 \leq \epsilon, \tag{1}
\]

is equivalent to the complementary channel being \textit{completely forgetful}: for all Hilbert spaces \(R\),

\[
\forall \rho, \sigma \in S(RA) \quad \| (\text{id} \otimes N^c) \rho^{RA} - (\text{id} \otimes N^c) \sigma^{RA} \|_1 \leq \delta, \tag{2}
\]

with a universal relation between \(\epsilon\) and \(\delta\).

Here we determine a matching duality for the weaker property of the complementary channel being only (approximately) forgetful:

\[
\forall \rho, \sigma \in S(A) \quad \| N^c(\rho^A) - N^c(\sigma^A) \|_1 \leq \delta. \tag{3}
\]

That this is a much weaker property was noticed in the contexts of approximate encryption and remote state preparation \cite{8, 9}. The difference between Eqs. \(\text{3}\) and \(\text{3}\) is precisely the difference between two norms on superoperators, the naive one inherited from the trace norm, and the so-called completely bounded norm \cite{7, 10, 11}. Not surprisingly, Eq. \(\text{3}\) will hold provided the main channel approximately preserves the pairwise fidelities between input pure states, a property we call \textit{geometry preservation}:

\[
\forall |\psi\rangle, |\varphi\rangle \in A \quad \| |\varphi\rangle - |\psi\rangle \|_1 - \| N(\varphi) - N(\psi) \|_1 \leq \epsilon. \tag{4}
\]

In fact, the reverse is also true. Our investigations will revolve around the \textit{fidelity alternative}, which asserts that a channel \(N\) is geometry-preserving if and only if its complement \(N^c\) is approximately forgetful, with dimension-independent functions relating \(\delta\) and \(\epsilon\). Thus, an isometry with two outputs can preserve geometry to at most one of them. Symmetrically, the isometry can be forgetful to at most one output. A single choice determines which output preserves geometry and which is forgetful; this choice gives the principle its name.

The geometry preservation property, though much weaker than transmission of quantum information, must nonetheless be considered a way of preserving coherence: by virtue of the
fidelity alternative, geometry preservation cannot be cloned. Indeed, if a channel has multiple outputs, one of which is geometry-preserving, then the rest must be forgetful.

Via the fidelity alternative, the many known examples of approximately forgetful channels that are not completely forgetful also provide examples of geometry-preserving channels that are not correctable [8, 9, 12–17]. Most strikingly, it is possible to preserve geometry while almost halving the number of qubits from input to output [18]. In that case, the geometry of the unit sphere in A is necessarily encoded into the eigenvectors and eigenvalues of the much smaller output state on B. In contrast to quantum error correction, dimension counting reveals the mixedness of the output state to be crucial to preserving the geometry. Some of the geometry of the input state space of pure quantum states is thus faithfully encoded as noise in the output state.

Moreover, the analogy with the quantum error correction duality can be made much stronger. There is a channel communication task very similar to quantum state transmission which is intimately related to geometry preservation: quantum identification [18, 19]. Quantum identification is a cooperative communication game between two parties – conventionally called Alice and Bob – where Alice has a given quantum state that she encodes in some way into the channel, and Bob only wants to simulate measurements consisting of an arbitrary pure state projector and its complement, which can interpreted as performing the experiment asking “Is this the state?” [18]. The idea is that Alice has an encoding channel E and Bob has, for every pure state ϕ, a POVM (Dϕ, 1 − Dϕ) such that

\[ \forall |ψ\rangle, |ϕ\rangle, \text{tr}( (N \circ E) ψ Dϕ − tr ψϕ ) ≤ ϵ. \]  

Such an object is called an ϵ-quantum-ID code. (The name is adapted from the classical case [20–22].)

Note that Bob measures the output of the channel, but the quality of the code is measured by how well the statistics of this measurement approximate the statistics of the ideal measurement he wants to perform on the message state. While it may seem that this is an odd way of defining a quantum communication task, normal quantum error correction can also be described this way; namely, Bob wants to be able to simulate all measurements on the message state. Clearly, if he can perform quantum error correction in the usual sense, then he can perform the simulation. But conversely, it follows from the methods of [23–25] that if he only has two measurements approximating generalized X and Z observables sufficiently well, he can build a quantum error correction procedure D. Moreover, a quantum-ID code with ϵ = 0 is itself a quantum error correcting code; there is no difference between error correction and identification if both tasks are to be performed perfectly. Even the task of transmitting classical information is conveniently reflected in this framework. In that case, Bob only wants to simulate the measurement of the generalized Z observable.

With this, one can define in the usual way a quantum-ID capacity Q_{ID}(N) of many uses of the channel as the highest rate at which qubits can be encoded and decoded as in Eq. (5) with vanishing error – see Section III for details. Previously it was only known that for the noiseless qubit channel id₂, Q_{ID}(id₂) = 2, double the value of both the the quantum and classical transmission capacities [18].

While reasoning directly about quantum identification codes has proved challenging, the duality between geometry preservation and approximate forgetfulness provides a new approach to studying them. Up to some technical conditions, geometry preservation is equivalent to the existence of a quantum identification code. It is therefore possible to construct quantum identification codes by finding approximately forgetful maps. This approach is fruitful because destroying information is a comparatively indiscriminate task. Indeed, the analogous strategy has led to a number of straightforward proofs of information the masking bound on the quantum capacity of a quantum channel [24, 26, 28]. Classical data is not immune to analysis by purification either. The duality between privacy amplification and data compression with quantum side information has recently led to a proof in this spirit [29] of the Holevo-Schumacher-Westmoreland theorem on the classical capacity of a quantum channel [5, 20].

With the fidelity alternative in hand, it is even possible to calculate a simple formula for an amortized version of the quantum identification capacity; it is exactly equal to the entanglement-assisted classical capacity of a quantum channel.

A. Structure of the paper

Section II contains the formal statement and proof of the fidelity alternative. The duality is studied in more detail in Section III where forgetfulness is shown to be nearly equivalent to quantum identification. In that section we provide a simple statement whose proof eliminates many technical difficulties, as well as a more flexible version that we prove from first principles. Section IV uses the flexible version of the equivalence to construct quantum identification codes for memoryless quantum channels.

B. Notation

We will restrict our attention throughout to finite dimensional Hilbert spaces. If A is a Hilbert space, we write S(A) for the set of density operators acting on A. Also, if A and B are two finite dimensional Hilbert spaces, we write AB ≡ A ⊗ B for their tensor product. The Hilbert spaces on which linear operators act will be denoted by a superscript. For instance, we write ϕ^{AB} for a density operator on AB. Partial traces will be abbreviated by omitting superscripts, such as ϕ^{A} ≡ tr_{B} ϕ^{AB}. We use a similar notation for pure states, e.g. |ϕ⟩^{AB} ∈ AB, while abbreviating ψ^{AB} ≡ |ψ⟩⟨ψ|^{AB}. We will write id_{A} for the identity map on S(A) and id_{2} for the identity qubit channel. The symbol 1^{A} will be reserved for the identity matrix acting on the Hilbert space A and π^{A} = 1^{A}/|A| for the maximally mixed state on A (where we denote by |A| the dimension of the Hilbert space A).
The trace norm of an operator, $\|X\|_1$ is defined to be $\text{tr}|X| = \text{tr}\sqrt{X^\dagger X}$. The similarity of two density operators $\varphi$ and $\psi$ can be measured by the trace distance $\frac{1}{2}\|\varphi - \psi\|_1$, which is equal to the maximum over all possible measurements of the variational distance between the outcome probabilities for the two states. The trace distance is zero for identical states and one for perfectly distinguishable states.

A complementary measure is the mixed state fidelity

$$F(\varphi, \psi) = \left(\text{tr}\sqrt{\varphi \sqrt{\psi} \sqrt{\varphi}}\right)^2,$$

defined such that when one of the states is pure, $F(\varphi, \psi) = \text{tr}\varphi\psi$. More generally, the fidelity is equal to one for identical states and zero for perfectly distinguishable states. We will make frequent use of the following fundamental inequality between fidelity and trace distance of states [31, Prop. 5]:

$$1 - \sqrt{F(\varphi, \psi)} \leq \frac{1}{2}\|\varphi - \psi\|_1 \leq \sqrt{1 - F(\varphi, \psi)}.$$  (7)

Both measures can be extended to unnormalized states, but Eq. (7) need not hold in that case. Further properties of the distance measures are collected in Appendix A.

II. THE FIDELITY ALTERNATIVE

Our investigations will revolve around the duality between geometry preservation and approximate forgetfulness, which we call the fidelity alternative. The rigorous statement is as follows:

**Theorem 1 (Fidelity alternative)** Let $N : S(A) \rightarrow S(B)$ be a quantum channel with complementary channel $N^c : S(A) \rightarrow S(E)$. Approximate geometry preservation on $B$ implies approximate forgetfulness for $E$. That is,

$$\forall |\psi\rangle, |\varphi\rangle \in A \quad \|\varphi - \psi\|_1 - \|N(\varphi) - N(\psi)\|_1 \leq \delta \quad \text{implies} \quad \forall |\psi\rangle, |\varphi\rangle \in A \quad \|N^c(\varphi) - N^c(\psi)\|_1 \leq 4\sqrt{2}\delta^{3/4}.$$

Conversely, approximate forgetfulness for $E$ implies approximate geometry preservation on $B$:

$$\forall |\psi\rangle, |\varphi\rangle \in A \quad \|N^c(\varphi) - N^c(\psi)\|_1 \leq \epsilon \quad \text{implies} \quad \forall |\psi\rangle, |\varphi\rangle \in A \quad \|\varphi - \psi\|_1 - \|N(\varphi) - N(\psi)\|_1 \leq 4\sqrt{2}\epsilon.$$

Note that we have dropped an absolute value sign as compared to Eq. (4) since $\|\varphi - \psi\|_1 \geq \|N(\varphi) - N(\psi)\|_1$ holds automatically for all quantum channels $N$. (See, for example, [32].)

The duality is a straightforward consequence of two basic results in quantum information theory. The first is that the ability to transmit classical data in two conjugate bases is equivalent to the ability to transmit entanglement. That observation is the basis for the stabilizer approach to quantum error correcting codes [33]. Here we will use a clean approximate formulation due to Renes [25]. The second result is the continuity of the Stinespring dilation of a quantum channel, established by Kretschmann et al. [7]. Here we only need a corollary, which can be interpreted as a bound on the information-disturbance trade-off. The theorem is stated in terms of the following norms:

**Definition 2** For a linear superoperator $\Gamma : S(A) \rightarrow S(B)$, let

$$\|\Gamma\|_0^{(k)} = \max_{\|\rho\|_1 \leq 1} \|\text{id}_k \otimes \Gamma \chi\|_1,$$

where $X$ is an operator on $\mathbb{C}^k \otimes A$. Define $\|\Gamma\|_0 = \sup_k \|\Gamma\|_0^{(k)}$, the completely bounded trace norm [26] (also known as diamond norm [17]).

Note that the convexity of the trace norm implies that the supremum is achieved on a rank-one operator (if $\Gamma$ is Hermitian-preserving, in fact on a pure quantum state). Since any operator on $A$ can be “purified” by a system of dimension $|A|$, it follows that the supremum is achieved when $k = |A|$. 

**Theorem 3 (Information-disturbance [7])** Let $V : A \rightarrow B \otimes E$ be an isometric extension of the channel $N : S(A) \rightarrow S(B)$ and let $N^c : S(A) \rightarrow S(E)$ be the complementary channel. Fix a state $\rho \in S(A)$ and let $R : S(A) \rightarrow S(E)$ be the channel taking all inputs to $N^c(\rho)$. Then

$$\frac{1}{4} \inf_D \|D \circ N - \text{id}\|_0^2 \leq \|N^c - R\|_0 \leq 2\inf_D \|D \circ N - \text{id}\|_0^2/2.$$  

Both infimums are over all quantum channels.

The proof of the fidelity alternative is a fairly routine matter of combining these results:

**Proof of Theorem 1** We begin by assuming approximate geometry preservation. Fix $|\varphi\rangle \perp |\psi\rangle$ in $A$ then set $T = \text{span}(|\varphi\rangle, |\psi\rangle)$. Suppose that

$$\|N(\varphi) - N(\xi)\|_1 \geq \|\varphi - \xi\|_1 - \delta$$

for all $|\varphi\rangle, |\xi\rangle \in A$. Then if $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle \pm |\psi\rangle)$, we have

$$\|N(\varphi) - N(\psi)\|_1 \geq 2 - \delta \quad \text{and} \quad \|N(\chi_+ - N(\chi_-)\|_1 \geq 2 - \delta.$$

We can therefore transmit data in two conjugate bases through $N$, which implies that entanglement is also faithfully transmitted. In particular [25, Thm. 1] (with “guessing probability” $1 - \delta/2$) implies that there exists a channel $D : S(B) \rightarrow S(T)$ such that

$$\|\text{id}_2 \otimes D \circ N \Phi - \Phi\|_1 \leq 2\sqrt{\delta},$$

where $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle |\varphi\rangle + |1\rangle |\psi\rangle)$. But trace norm monotonicity with respect to dephasing the first system then gives

$$\|\text{id}_2 \otimes D \circ N \Phi - \Phi\|_1 \geq \frac{1}{2} \|0\rangle 0 \rangle \otimes [(D \circ N) \varphi - \varphi]\|_1$$

$$+ \|1\rangle 1 \rangle \otimes [(D \circ N) \psi - \psi]\|_1$$

$$= \frac{1}{2} \|D \circ N \varphi - \varphi\|_1 \|D \circ N \psi - \psi\|_1.$$
Therefore, \[ \| (D \circ N) \varphi - \varphi \|_1 \leq 4 \sqrt{\delta} \] and by changing the choice of dephasing basis, we can conclude that \[ \| D \circ N - \text{id}_2 \|_1 \leq 4 \sqrt{\delta} \]. Combining this with Lemma 19 in Appendix A implies that \[ \| D \circ N - \text{id}_2 \|_1 \leq 8 \sqrt{\delta} \]. The information-disturbance theorem (Theorem 3) applied with \( R \) the map taking all states to \( N^\circ(\varphi) \) then implies that for all \( |\omega\rangle \in T \),

\[ \| N^\circ(\varphi) - N^\circ(\omega) \|_1 \leq 2(8 \sqrt{\delta})^{1/2} = 4\sqrt{2}\delta^{1/4}. \]

Since \( T \) is an arbitrary two-dimensional subspace of \( A \), however, the inequality must hold for all \( |\varphi\rangle \) and \( |\omega\rangle \) in \( A \).

For the converse, suppose that, for all states \( |\varphi\rangle, |\psi\rangle \in A \), the inequality \( \| N^\circ(\varphi) - N^\circ(\psi) \|_1 \leq \epsilon \) holds. Fix \( |\varphi\rangle \) and \( |\psi\rangle \) then let \( N^\circ \) be the restriction of \( N^\circ \) to states on \( T = \text{span}([|\varphi\rangle, |\psi\rangle]) \). Let \( R \) be the channel on \( S(T) \) that always outputs \( N^\circ(\psi) \). Then once more by Lemma 19 in Appendix A \( \| N^\circ - R \|_1 \leq 2\epsilon \). Using this time the lower bound from Theorem 3 there exists a channel \( D : S(B) \to S(T) \) such that \( \frac{1}{2} \| D \circ N - \text{id}_2 \|_1^2 \leq 2\epsilon \). In particular, for all \( |\omega\rangle \in T \),

\[ \frac{1}{4} \| (D \circ N) \omega - \omega \|_1^2 \leq 2\epsilon. \]

Applying the triangle inequality several more times gives:

\[
4\sqrt{2}\epsilon \geq \| (D \circ N) \varphi - \varphi \|_1 + \| (D \circ N) \psi - \psi \|_1 \\
\geq \| \varphi - \psi \|_1 - \| (D \circ N)(\varphi - \psi) \|_1 \\
\geq \| \varphi - \psi \|_1 - \| N(\varphi - \psi) \|_1,
\]

where the final inequality used that the quantum channel \( D \) cannot increase the trace norm. Rearranging the final expression gives the desired inequality. \( \square \)

### III. QUANTUM IDENTIFICATION

Quantum identification allows a sender to transmit arbitrary quantum states but only allows the receiver to perform a restricted set of measurements, namely tests to determine whether the transmitted state consists of an arbitrary target state. The receiver gets to choose the target state after the sender has transmitted, so the code must work for all targets. If the test can be performed perfectly, then quantum identification is easily seen to be equivalent to quantum state transmission, but in the approximate setting, the tasks are not equivalent.

**Definition 4** [18] An \( \epsilon \)-quantum-ID code for the channel \( N : S(A) \to S(B) \) consists of an encoding map \( E : S(S) \to S(A) \) and, for every pure state \( |\varphi\rangle \in S \), a POVM \( (D_\varphi, 1 - D_\varphi) \) acting on \( S(B) \) such that

\[
\forall |\psi\rangle, |\varphi\rangle \in S \quad |\langle \psi\rangle(\varphi), 1 - |\varphi\rangle(\varphi)| \quad \text{on input state} \quad |\psi\rangle, \text{ then he would have observed outcome} \quad |\varphi\rangle(\varphi) \quad \text{with probability} \quad |(\varphi|\psi)|^2. \]

The definition therefore ensures that the receiver can simulate the measurement for all input and target states.

Many variants of the definition have been proposed. In particular, one could imagine drawing a distinction between oblivious ID codes, in which the sender is only given a physical quantum state to send, and visible ID codes, in which the sender knows the identity of the state she is trying to transmit. Entanglement assistance is also interesting and exceptionally powerful in the visible setting [34]. A different task that is nonetheless similar in spirit is to use quantum states as “fingerprints” for identifying classical messages in a model where pairs of messages are to be compared by a referee [35]. For comparing quantum states, however, the simple definition considered here is arguably the most natural.

If we integrate the encoding \( E \) and noisy channel \( N \) from Definition 4 into a single map with output \( B \) and environment \( E \), we may think of the code Hilbert space \( S \) as a subspace of \( B \otimes E \). More formally, if we let \( V \) be the Stinespring dilation of \( N \otimes \epsilon \), then \( V : S \to B \otimes E \) and we can identify the code with a subspace of \( B \otimes E \). This identification simplifies the notation and we will use it for the remainder of the paper.

The main result of this section is a demonstration that a subspace of \( B \otimes E \) is a quantum-ID code for \( B \) iff it is approximately forgetful for \( E \). (There is a small technical caveat to the statement: the reduced states on \( E \) must also obey a regularity condition for the reverse implication to hold, but we will defer discussion of the details.) For the moment, let us begin by considering the relationship between quantum identification and geometry preservation.

**Lemma 5** Let \( S \subseteq B \otimes E \) be a subspace of a tensor product Hilbert space that is an \( \epsilon \)-quantum-ID code for \( B \). In other words, suppose that, for each pure state \( |\varphi\rangle \in S \), there exists an operator \( 0 \leq D_\varphi \leq 1 \) on \( B \) such that for all pure states \( |\varphi\rangle, |\psi\rangle \in S \),

\[
|\text{tr} \psi^B D_\varphi - \text{tr} \psi \varphi\rangle | \leq \epsilon.
\]

Then, for all \( |\varphi\rangle, |\psi\rangle \in S \),

\[
F(\varphi, \psi) \leq F(\varphi^B, \psi^B) \leq F(\varphi, \psi) + 4\sqrt{\epsilon}.
\]

**Proof** Consider the measurement \( (D_\varphi, 1 - D_\varphi) \) and associated channel \( M : \rho \to \text{diag}(\text{tr} \rho D_\varphi, 1 - \text{tr} \rho D_\varphi) \) which acts on \( S(B) \). By applying the monotonicity of the fidelity under quantum channels to \( \text{tr}_E \) and \( M \), we get

\[
F(\psi, \varphi) \leq F(\psi^B, \varphi^B) \leq F(M(\psi^B), M(\varphi^B)) \leq \left( \sqrt{\text{tr} \psi^B D_\varphi} + \sqrt{\epsilon} \right)^2 \leq F(\psi, \varphi) + 2\sqrt{\epsilon} + \epsilon + \epsilon,
\]

which proves the lemma. \( \square \)

The fidelity is therefore approximately preserved by quantum identification codes. Geometry preservation is defined in terms of the trace distance, however, not the fidelity. While it is indeed the case that quantum identification codes preserve
geometry, the argument is somewhat more delicate because applying the measurement \((D_\varphi, \mathbb{I} - D_\varphi)\) causes a significant drop in the trace distance even as it leaves the fidelity nearly unchanged. Instead, Theorem 7 will allow us to infer that quantum identification codes preserve geometry by virtue of the fact that their complementary channels are forgetful.

In order to succeed at quantum identification, the following lemma demonstrates that it is sufficient to be able to identify orthogonal states:

**Lemma 6** Let \(S \subseteq B \otimes E\) be a subspace of a tensor product Hilbert space such that for \(|\varphi\rangle \in S\) there exists \(0 \leq D_\varphi \leq \mathbb{I}\) acting on \(B\) satisfying

\[
\text{tr} \varphi^B D_\varphi \geq 1 - \delta \quad \text{and} \quad \text{tr} \psi^B D_\varphi < \delta
\]

whenever \(|\psi\rangle \in S\) is orthogonal to \(|\varphi\rangle\). Then \(S\) is a quantum identification code with error probability \(\delta + 2\sqrt{\delta}\).

**Proof** Let \(|\varphi\rangle, |\psi\rangle \in S\) be arbitrary and let \(|\varphi'\rangle\) be orthogonal to \(|\varphi\rangle\) in \(\text{span}(|\varphi\rangle, |\psi\rangle)\). Write

\[
|\psi\rangle = \alpha|\varphi\rangle + \beta|\varphi'\rangle.
\]

Expanding shows that \(\text{tr} \psi^B D_\varphi\) is equal to

\[
|\alpha|^2 \text{tr} \varphi^B D_\varphi + |\beta|^2 \text{tr} \varphi'^B D_\varphi
+ \alpha\beta \text{tr} |\varphi\rangle \langle \varphi'|(D_\varphi \otimes \mathbb{I}) + \alpha\beta \text{tr} \langle \varphi'|(D_\varphi \otimes \mathbb{I})|\varphi\rangle,
\]

which results in

\[
|\text{tr} \psi^B D_\varphi - |\alpha|^2| \\
\leq |\alpha|^2 \left(1 - \text{tr} \varphi^B D_\varphi\right) + |\beta|^2 \text{tr} \varphi'^B D_\varphi
+ 2|\alpha\beta||\langle \varphi'|(D_\varphi \otimes \mathbb{I})|\varphi\rangle|
\leq |\alpha|^2 \left(1 - \text{tr} \varphi^B D_\varphi\right) + |\beta|^2 \text{tr} \varphi'^B D_\varphi
+ 2|\alpha\beta|\sqrt{\langle \varphi'|(D_\varphi \otimes \mathbb{I})|\varphi\rangle}
\leq \delta + 2\sqrt{\delta},
\]

where we have used the Cauchy-Schwarz inequality and the assumption that orthogonal states in \(S\) can be well discriminated. \(\square\)

Now we are ready to state and prove our main result on the duality between quantum identification and approximate forgetfulness. As with the fidelity alternative, we have chosen to prove the theorem by composing general purpose results for the purpose of pedagogical clarity, which leads to artificially poor scaling of the parameters. Readers concerned with optimizing the parameters should also consult Theorem 8.

**Theorem 7 (Identification and forgetfulness)** Quantum-ID codes and forgetfulness are dual in the following quantitative sense. If a subspace \(S \subseteq B \otimes E\) is an \(\epsilon\)-quantum-ID code for \(B\), then \(E\) is approximately \(\delta\)-forgetful:

\[
\forall |\varphi\rangle, |\psi\rangle \in S:\quad \frac{1}{2} \|\varphi^E - \psi^E\|_1 \leq \delta := 7\sqrt{\epsilon}.
\]

Conversely, if \(E\) is approximately \(\delta\)-forgetful, then geometry is approximately preserved on \(B\):

\[
\forall |\varphi\rangle, |\psi\rangle \in S:\quad \|\varphi - \psi\|_1 - \|\varphi^B - \psi^B\|_1 \leq \epsilon := 4\sqrt{2\delta}
\]

If, in addition, the nonzero eigenvalues of \(\varphi^B\) lie in the interval \([\mu, \lambda]\) for all \(|\varphi\rangle \in S\), then \(S\) is an \(\eta\)-quantum-ID code for \(\eta := 7\delta^{1/8}\sqrt{\mu}\).

**Remark** While it would be desirable to eliminate the eigenvalue condition at the end of the theorem, the condition is fairly natural in this context. If the reduced states \(\varphi^E\) are very close to a single state \(\sigma^E\) for all \(|\varphi\rangle \in S\), then all the \(|\varphi\rangle\) are very close to being purifications of \(\sigma^E\), meaning that they differ from one another only by a unitary plus a small perturbation. If \(\sigma^E\) is the maximally mixed state or close to it, then the assumption will be satisfied. \(\square\)

**Proof** For the first part, recall that if \(S\) is a quantum-ID code with error probability \(\epsilon\), then for each pure state \(|\varphi\rangle \in S\) there exists an operator \(0 \leq D_\varphi \leq \mathbb{I}\) on \(B\) such that for all pure states \(|\varphi\rangle, |\psi\rangle \in S\),

\[
|\text{tr} \psi^B D_\varphi - \text{tr} \psi^B| \leq \epsilon.
\]

Just as in the proof of Theorem 1, the hypothesis implies that data can be transmitted in two conjugate bases with guessing probability \(1 - \epsilon\). Running exactly the same argument as was made in that proof gives that for all \(|\varphi\rangle, |\psi\rangle \in S\),

\[
\frac{1}{2}\|\varphi^E - \psi^E\|_1 \leq 4\sqrt{2(2\epsilon)^{1/4}} \leq 7\epsilon^{1/4}.
\]

The second part is just a restatement of one direction of the fidelity alternative, but it is a useful step on the way to the third part, which is more challenging since it requires the construction of the decoder, that is, the operators \(D_\varphi\).

Indeed, given \(|\varphi\rangle \in S\), and arbitrary \(|\psi\rangle \perp |\varphi\rangle\) in \(S\), we learn from the second part that

\[
\|\varphi^B - \psi^B\|_1 \geq 2 - 4\sqrt{2\delta}.
\]

By Helstrom’s theorem on the optimal discrimination of \(\varphi^B\) and \(\psi^B\) [36], there exists a projector \(P_{\varphi, \psi}\) on \(B\) such that

\[
\text{tr} \varphi^B P_{\varphi, \psi} \geq 1 - 2\sqrt{2\delta}, \quad \text{tr} \psi^B P_{\varphi, \psi} \leq 2\sqrt{2\delta}.
\]

The problem with using \(P_{\varphi, \psi}\) as the decoding is that this projector may indeed depend not only on \(\varphi\), but also on \(\psi\). Since the goal is to find a single projector that the receiver can use to identify \(\varphi\) that will work regardless of whether the input is \(\varphi\) or \(\psi\), that is unacceptable. Still, let us confirm first that if we manage to find one effect operator \(D_\varphi\) that can deal with all \(\psi\) at once, then by Lemma 6 we’ll be done. Our strategy for doing so will be to first extend Eq. (10) to all mixed states orthogonal to \(|\varphi\rangle\) and supported on \(S\), and then use a minimax argument to extract a single operator independent of \(\psi\).

Lemma 7 in Appendix A can be used directly to see that for all mixed states \(\sigma\) supported on \(S\) and orthogonal to \(\varphi\),

\[
F(\varphi^B, \sigma^B) \leq \frac{\lambda^2}{\mu^2} \max \{F(\varphi^B, \psi^B) \leq 4\sqrt{2\delta} \frac{\lambda^2}{\mu^2}.
\]
where the maximization is over all $|\psi\rangle \in S$ orthogonal to $|\varphi\rangle$ and the second inequality is an application of Eq. (7) to Eq. (9). Applying Eq. (7) a second time gives

$$\frac{1}{2}\|\varphi^B - \psi^B\|_1 \geq 1 - 2(2\delta)^{1/4}\lambda/\mu.$$  

Applying Helstrom’s theorem to $\varphi^B$ and $\sigma^B$ yields a projector $P_\sigma$ with

$$\text{tr} \varphi^B P_\sigma - \text{tr} \sigma^B P_\sigma \geq 1 - 2(2\delta)^{1/4}\lambda/\mu,$$

Von Neumann’s minimax theorem then ensures the existence of a saddle point in the following two-player game [37] (see Ky Fan [38] for a more general version). One player selects $0 \leq P \leq \mathbb{1}$ while the other player selects a state $\sigma$ supported on $S$ and orthogonal to $\varphi$. The strategy spaces are therefore closed and convex. The payoff function is $1 - \text{tr} \varphi^B P + \text{tr} \sigma^B P$, which is linear in each argument. Thus, the minimax theorem guarantees that there exists an operator $0 \leq D_\varphi \leq \mathbb{1}$ such that for all $\sigma$ supported on $S$ and orthogonal to $\varphi$,

$$\text{tr} \varphi^B D_\varphi \geq 1 - 2(2\delta)^{1/4}\lambda/\mu,$$

$$\text{tr} \sigma^B D_\varphi \leq 2(2\delta)^{1/4}\lambda/\mu,$$

and applying Lemma 10 finishes the proof. \hfill \Box

Unfortunately, Theorem 7 is not quite strong enough to prove our main result on the quantum identification capacity. To control the ratio of the largest to smallest eigenvalues of the coding states, we need to act on them by typical projectors that cause a slight distortion. To accommodate this complication, we will instead use the following slightly more flexible version of the converse that behaves better with respect to the distortion. In particular, the amount of distortion enters the bound on the quality of the quantum-ID code in a term independent of the eigenvalue constraint. That separation proves to be crucial because the eigenvalues cannot be controlled independently of the distortion.

**Theorem 8** Let $S \subset B \otimes E$ be a subspace and $0 \leq X \leq \mathbb{1}$ an operator acting on $B \otimes E$ such that $\text{tr}(X \varphi X^\dagger) \geq 1 - \epsilon$ for all $|\varphi\rangle \in S$. For any state $|\psi\rangle \in S$, write $\varnothing = X \omega X^\dagger$. If there exists a state $\Omega$ such that

$$\forall |\varphi\rangle \in S \quad \|\hat{\Omega}^E - \varphi^E\|_1 \leq \delta$$

with $0 \leq \delta, \epsilon \leq 1/15$ and, in addition, the nonzero eigenvalues of $\hat{\Omega}^E$ lie in the interval $[\mu, \lambda]$, then $S$ is an $\eta$-quantum-ID code for $\eta := 3(30\lambda\delta/\mu + 3\sqrt{\pi} + 4\delta)^{1/2}$. \hfill \Box

**Proof** Let $|\varphi\rangle$ and $|\psi\rangle$ be orthonormal states in $S$. We will begin by showing that $\hat{\varphi}^B$ and $\hat{\psi}^B$ can be effectively distinguished. To this end, consider the states

$$|\theta_\pm\rangle = \frac{1}{\sqrt{2}}|\varphi\rangle \pm \frac{1}{\sqrt{2}}|\psi\rangle,$$

$$|\chi_\pm\rangle = \frac{1}{\sqrt{2}}|\varphi\rangle \pm i\frac{1}{\sqrt{2}}|\psi\rangle,$$

which form two orthogonal pairs. Then

$$\hat{\theta}_\pm = \frac{1}{2}\varphi^E + \frac{1}{2}\psi^E \pm 1/2\left(\text{tr}_B |\varphi\rangle\langle\psi| + \text{tr}_B |\psi\rangle\langle\varphi|\right),$$

$$\hat{\chi}_\pm = \frac{1}{2}\varphi^E + \frac{1}{2}\psi^E \pm i/2\left(\text{tr}_B |\varphi\rangle\langle\psi| - \text{tr}_B |\psi\rangle\langle\varphi|\right),$$

and, by assumption,

$$\frac{1}{2}\|\hat{\theta}_+ - \hat{\theta}_-\|_1 \leq \delta \quad \text{and} \quad \frac{1}{2}\|\hat{\chi}_+ - \hat{\chi}_-\|_1 \leq \delta.$$

Combining these relations reveals that $\|\text{tr}_B |\tilde{\varphi}\rangle\langle\tilde{\psi}| - \text{tr}_B |\tilde{\psi}\rangle\langle\tilde{\varphi}|\|_1 \leq 4\delta$, hence by the triangle inequality, $\|\text{tr}_B |\tilde{\varphi}\rangle\langle\tilde{\psi}|\|_1 \leq 8\delta$. But this gives us, by virtue of Lemma 10

$$F(\hat{\varphi}^B, \hat{\psi}^B) \leq 64\delta^2.$$  

(11)

To proceed as in the proof of Theorem 7 we need to show that any $|\varphi\rangle \in S$ and mixed state $\sigma$ supported on the orthogonal complement of $|\varphi\rangle$ in $S$ can also be distinguished. In order to apply Lemma 10 in Appendix A, we will show that the largest and smallest nonzero eigenvalues of $\varphi^B$, or equivalently, $\varphi^E$, are well-behaved modulo a little bit of truncation. Indeed, let $O = (O_j)$ and $p = (p_j)$ be the eigenvalues of $\hat{\Omega}^E$ and $\varphi^E$, respectively, in nonincreasing order. Then

$$\|O - p\|_1 \leq \|\hat{\Omega}^E - \varphi^E\|_1 \leq \delta.$$

Define the set

$$J = \{j : (1 - \gamma)p_j \leq O_j \leq (1 + \gamma)p_j\}.$$

Then

$$\gamma \sum_{j \notin J} p_j \leq \sum_{j \notin J} |O_j - p_j| \leq \delta.$$

implying that

$$\sum_{j \in J} p_j = \sum_{j} p_j - \sum_{j \notin J} p_j \geq (1 - \epsilon) - \delta/\gamma.$$

Fixing $\gamma = 1/2$ implies that for each $|\varphi\rangle \in S$, there is a positive semidefinite operator $\hat{\varphi}^B \leq \varphi^B$ satisfying $\text{tr} \hat{\varphi}^B \geq 1 - \epsilon - 2\delta$ and whose eigenvalues lie in the interval $[\mu/2, 3\lambda/2]$.

Now let $|\varphi\rangle \in S$ and consider any state $\sigma = \sum_i q_i \psi_i$ whose support lies in the orthogonal complement of $|\varphi\rangle$ in $S$. Since the states $|\psi_i\rangle$ are in $S$, the truncation procedure of the previous paragraph can be used to construct operators $\hat{\psi}_i$. Let $\sigma = \sum_i q_i \psi_i$. Then by Lemma 10

$$F(\hat{\varphi}^B, \hat{\sigma}^B) \leq \frac{9\lambda^2}{\mu^2} \max F(\hat{\varphi}^B, \hat{\psi}^B) \leq \frac{9\lambda^2}{\mu^2} \max F(\hat{\varphi}^B, \hat{\psi}^B) \leq \frac{9\lambda^2}{\mu^2} \cdot 64\delta^2 = \frac{576\lambda^2\delta^2}{\mu^2}.$$
Both maximizations are over states $|\psi\rangle \in S$ such that $\langle \phi | \psi \rangle = 0$. The second inequality follows from the fact that $\hat{\varphi}_B \preceq \hat{\sigma}_B$ (and likewise for $\psi$) along with Lemma [18] while the third arises by substituting in the result of Eq. (11). Introducing one last decoration for our states, let $\hat{\varphi}_B = \hat{\varphi}_B / \text{tr} \hat{\varphi}_B$ and likewise for $\sigma$. Applying Eq. (7) with attention paid to the fact that $\hat{\varphi}_B$ and $\sigma_B$ are not normalized gives

$$\frac{1}{2} \| \hat{\varphi}_B - \sigma_B \|_1 \geq 1 - \frac{24\lambda \delta}{\mu} \left( 1 - \epsilon - 2\delta \right) \geq 1 - \frac{30\lambda \delta}{\mu},$$

where the final inequality uses that $\epsilon \leq 1/15$. Applying Helstrom’s theorem to $\hat{\varphi}_B$ and $\hat{\sigma}_B$ implies that there exists a projector $P_\varphi$ such that

$$\text{tr} \hat{\varphi}_B P_\varphi - \text{tr} \hat{\sigma}_B P_\varphi \geq 1 - \frac{30\lambda \delta}{\mu}.$$

Next we invoke von Neumann’s minimax theorem, just as in the proof of Theorem 7 for the payoff function $1 - \text{tr} \hat{\varphi}_B P + \text{tr} \hat{\sigma}_B P$, with the strategy space of the second player the convex hull of the operators $\hat{\psi}_B$, where $|\psi\rangle \in S$ ranges over states orthogonal to $|\varphi\rangle$. (The operators $\hat{\sigma}_B$ are not normalized but that will not cause any difficulties.) This provides an operator $0 \leq D_\varphi \leq 1$ such that

$$\text{tr} \hat{\varphi}_B D_\varphi \geq 1 - \frac{30\lambda \delta}{\mu} \quad \text{and} \quad (12)$$

$$\text{tr} \hat{\sigma}_B D_\varphi \leq \frac{30\lambda \delta}{\mu}. \quad (13)$$

But

$$\left| \text{tr} \hat{\varphi}_B D_\varphi - \text{tr} \hat{\sigma}_B D_\varphi \right| \leq \| \hat{\varphi}_B - \hat{\sigma}_B \|_1$$

$$\leq \| \hat{\varphi}_B - \hat{\sigma}_B \|_1 + \| \hat{\sigma}_B - \hat{\varphi}_B \|_1 \leq 2\sqrt{\epsilon} + 2\delta + \left| 1 - (1 - \epsilon - 2\delta) \right| \leq 3\sqrt{\epsilon} + 4\delta,$$

where the fourth line follows from the gentle measurement lemma (Appendix A, Lemma 20), the definition of $\hat{\varphi}_B$, and the fact that $\hat{\varphi}_B = \hat{\varphi}_B / (\text{tr} \hat{\varphi}_B)$. Similarly, for any $\hat{\sigma}_B = \sum_i q_i \hat{\psi}_i$ a convex combination of states arising from $|\psi_i\rangle \in S$ perpendicular to $|\varphi\rangle$,

$$\left| \text{tr} \hat{\sigma}_B D_\varphi - \text{tr} \hat{\sigma}_B D_\varphi \right| \leq \| \hat{\sigma}_B - \hat{\sigma}_B \|_1$$

$$\leq \sum_i q_i \| \hat{\psi}_i - \hat{\varphi}_i \|_1 \leq \sum_i q_i \left( \| \hat{\psi}_i - \hat{\varphi}_i \|_1 + \| \hat{\varphi}_i - \hat{\varphi}_i \|_1 \right) \leq 2\sqrt{\epsilon} + 2\delta.$$

Combining these estimates with the outcome of the minimax theorem in Eq. (12) and Lemma 6 completes the proof. □

**IV. QUANTUM IDENTIFICATION CAPACITY**

While it might not be possible to design low error quantum-ID codes for any given channel, the situation becomes more promising if many uses of the channel are allowed. In analogy with classical and quantum data transmission, we can define asymptotic quantum-ID codes as follows.

**Definition 9 (Quantum-ID capacity [18])** A rate $Q$ is said to be achievable for quantum identification over $N$ if for all $\epsilon > 0$ and sufficiently large $n$, there are $\epsilon$-quantum-ID codes for $N \otimes n$ with encoding domain $S$ of dimension at least $2^nQ$. The quantum identification capacity $Q_{ID}(N)$ is defined as the supremum of the achievable rates.

The capacity should be interpreted as the number of qubits that can be identified per use of the channel $N$ in the limit of many uses of the channel. The only nontrivial channel for which the quantum identification capacity was known prior to this paper was the identity channel: asymptotically, a noiseless qubit channel can be used to identify two qubits. That is, $Q_{ID}(id_2) = 2$ [18]. As we will see below, the theory of the quantum identification capacity is considerably simpler when the given channel $N$ can be used in conjunction with noiseless channels to the receiver. This obviously increases the capacity, so the interesting question is how much the use of $N$ increases the quantum identification capacity over what would have been achievable with the noiseless channels alone. When defining the achievable amortized rates it is therefore necessary to subtract off two qubits for every noiseless qubit channel used per copy of $N$.

**Definition 10 (Amortized quantum-ID capacity)** A rate $Q$ is said to be achievable for amortized quantum identification over $N$ if for all $\epsilon > 0$ and sufficiently large $n$, there are $\epsilon$-quantum-ID codes for $id_1 \otimes N \otimes n$ with encoding domain $S$ such that $Q \leq \frac{1}{n} \log |S| - 2 \log |C|$), where $\log = \log_2$ is the binary logarithm throughout this paper. The amortized quantum identification capacity $Q_{ID}^{am}(N)$ is defined as the supremum of the achievable rates.

Readers familiar with the identification capacities of classical channels might be surprised to see that the dimension of a quantum identification code scales only exponentially with the number of channel uses, as opposed to doubly exponentially. The essential difference between the classical and quantum settings is that the number of distinguishable quantum states in dimension $d$ already scales exponentially with $d$, which makes quantum identification a much more demanding task. Nonetheless, as we will see below, the amortized quantum identification capacity can be positive for some channels with zero quantum capacity, like the noiseless bit channel. One then finds that the dimension of the quantum identification code can scale super-exponentially with the number of qubits used to supplement the classical channel.

The fidelity alternative is a very effective tool for studying the quantum-ID capacities. As a warm-up, the fact that the complements of quantum-ID codes are forgetful supplies a quick answer to an open question from [18].
Theorem 11 If $\mathcal{N}$ is an antidegradable channel, that is, if there exists channel $\mathcal{T}$ such that $\mathcal{N} = \mathcal{T} \circ \mathcal{N}^c$, then $Q_{ID}(\mathcal{N}) = 0$. This is true in particular for the noiseless cbit channel $\mathbb{I}_2$. More generally, if the quantum capacity of the channel vanishes, $Q(\mathcal{N}) = 0$, then so does the quantum-ID capacity, $Q_{ID}(\mathcal{N}) = 0$.

Proof Given a quantum-ID code for the channel $\mathcal{N}$ that encodes as little as one qubit, the channel $\mathcal{N} \circ \mathcal{E}$ will be geometry-preserving if $\mathcal{E}$ is the encoding map. Hence, by the fidelity alternative, the channel complementary to $\mathcal{N} \circ \mathcal{E}$ will be approximately forgetful. But if $\mathcal{N}$ is antidegradable, then so is $\mathcal{N} \circ \mathcal{E}$, meaning that the channel complementary to $\mathcal{N} \circ \mathcal{E}$ can simulate $\mathcal{N} \circ \mathcal{E}$. But then the complementary channel would be simultaneously forgetful and geometry-preserving, a contradiction.

For the more general statement, we show the contrapositive: assume $Q_{ID}(\mathcal{N}) > 0$, then for all $\epsilon > 0$ and sufficiently large $n$, $\mathcal{N}^{\otimes n}$ has in particular a 2-dimensional quantum-ID code $S$ which is $\epsilon$-close to being forgetful for the environment, by Lemma 5. But by Lemma 19 this means that the channel from the code qubit to the environment is arbitrarily close to a constant map in the diamond norm. At this point we can then invoke Theorem 3 of [11] on information-disturbance to conclude that the channel from the code qubit to $B^n$ can be arbitrarily well error-corrected. (Note that this argument is following our proof of the fidelity alternative; in particular, any 2-dimensional subspace of a quantum-ID code is a quantum error-correcting code!) By the Lloyd-Shor-Devetak theorem on the quantum capacity (see [6]), this implies that there exists an input state for which the coherent information $I(A^n)B^n) > 0$ is positive, and hence $Q(\mathcal{N}) > 0$. \hfill $\Box$

As usual, quantitative statements about asymptotically achievable rates and upper bounds on the capacities are naturally expressed in terms of entropies. For a bipartite density matrix $\varphi^{AB}$, we write

$$H(A)_\varphi \equiv H(\varphi^A) \equiv - \text{tr} \varphi^A \log \varphi^A$$

for the von Neumann entropy of $\varphi^A$. The mutual information of the state $\varphi^{AB}$ is defined to be

$$I(A:B)_\varphi = H(A)_\varphi + H(B)_\varphi - H(AB)_\varphi$$

while the coherent information and the conditional entropy are, respectively,

$$I(A|B)_\varphi = H(B)_\varphi - H(AB)_\varphi$$

$$H(A|B)_\varphi = H(AB)_\varphi - H(B)_\varphi.$$

Our main theorem on the quantum identification capacities includes a concise formula for $Q_{ID}^{\text{am}}$ that eliminates the optimization over multiple channel uses.

Theorem 12 (Quantum identification capacity) For any quantum channel $\mathcal{N}$, its quantum-ID capacity is given by $Q_{ID}(\mathcal{N}) = \sup_{\epsilon > 0} n \log Q_{ID}^{(n)}(\mathcal{N}^{\otimes n})$, where $Q_{ID}^{(n)}(\mathcal{N}) = \sup_{\varphi} \{I(A:B)_\rho \text{ s.t. } I(A|B)_\rho > 0\}$,

where $|\varphi\rangle$ is the purification of any input state to $\mathcal{N}$ and $\rho^{AB} = (\text{id} \otimes \mathcal{N})|\varphi\rangle$, and where we declare the sup to be 0 if the set above is empty. Furthermore, the amortized quantum-ID capacity equals

$$Q_{ID}^{am}(\mathcal{N}) = \sup I(A:B)_\rho = C_E(\mathcal{N}),$$

the entanglement-assisted classical capacity of $\mathcal{N}$.

Remark It follows from Theorem 12 that the amortized quantum-ID capacity of a noiseless cbit channel is one. Reconciling this observation with Theorem 11 which asserts this channel’s unamortized quantum-ID capacity is zero, reveals that some amortized noiseless quantum communication is necessary to achieve $Q_{ID}^{am}$ without determining how much. In fact, inspection of the proof of Theorem 12 reveals that, for the noiseless cbit channel $\mathbb{I}_2$, a zero rate of noiseless side qubits is sufficient to achieve the maximum value of one. These observations extend to cq-channels, so named because they consist of a destructive measurement resulting in classical information, followed by the preparation of a state conditioned on the measurement outcome. For these channels, the entanglement-assisted capacity $C_E$ is equal to the unassisted classical capacity $C$, also known as the Holevo capacity $\mathcal{R}$. As a result, $Q_{ID}(\mathcal{N}) = 0$ for all such channels even as $Q_{ID}^{am}(\mathcal{N}) = C(\mathcal{N})$, the latter strictly positive for all nontrivial channels. The difference in all cases can be traced to a sublinear amount of free quantum communication in the amortized setting.

This effect can be viewed as an instance of (un-)locking since the quantum-ID rate increases from strictly 0 to an arbitrarily large amount by the addition of any positive rate of quantum communication, cf. [23, 41, 42]. Unlike the previously known examples where a certain finite rate is always required, however, here an arbitrarily small rate of extra quantum communication is sufficient to bring about an unbounded increase in the capacity. \hfill $\Box$

The intuition behind the achievability of the rates in Theorem 12 is quite simple. The structure of an amortized code is illustrated in Figure 1. Fix a state $|\varphi\rangle$ purifying any input to the channel $\mathcal{N}$ and let $|\rho^{A|B}|(\mathcal{N})$ be $(\text{id} \otimes U_N)|\varphi\rangle$, where $U_N$ is the Stinespring extension of $\mathcal{N}$. The encoding will embed the input into a random subspace of a typical subspace of $A^n$ tensored with ancillary spaces $C$ and $F$, where $C$ will consist of the amortized quantum communication and $F$ the environment for the encoding. Since the encoding is into a random subspace, it will produce states highly entangled between $B^nC$ and $E^nF$. By arranging for $E^nF$ to be slightly smaller than $B^nC$ in the appropriate sense, one ensures that the states are indistinguishable on the environment $E^nF$. By the fidelity alternative, they can therefore be identified by Bob. Letting $R = \frac{1}{n} \log |C|$ and $f = \frac{1}{n} \log |F|$, the condition ensuring that $E^nF$ is “smaller” than $B^nC$ is roughly

$$H(E)_\rho + f < H(B)_\rho + R,$$

so $f - R$ is chosen to be very slightly less than $H(B)_\rho - H(E)_\rho$. Moreover, measure concentration for the choice of
random subspace will make it possible to choose the subspace almost as large as the ambient space $A^n CF$, which in qubit terms has effective size

$$nH(A)_\rho + nR + nf.$$  

The rate of the amortized code will therefore be

$$H(A)_\rho + R + f - 2R = H(A)_\rho + f - R$$

$$\approx H(A)_\rho + H(B)_\rho - H(E)_\rho.$$  

Since $\rho$ is pure, $H(E)_\rho = H(AB)_\rho$ which means that the rate is precisely the mutual information.

The detailed proof of the achievability of the rates in Theorem 12 builds on the techniques developed in Refs. 13 and 43 analyzing the properties of generic quantum states. The proof will combine the following theorem, originally motivated by the foundations of statistical mechanics, with the duality between quantum identification and approximate forgetfulness formulated in Theorem 7 or, more precisely, its technical variant Theorem 8.

**Theorem 13 (Random versus average states [43])** Let $S$ be a subspace of $B \otimes E$, $\Omega$ be the maximally mixed state on $S$, and $X$ any operator acting on $B \otimes E$ with $\|X\|_\infty \leq 1$. If $|\varphi\rangle \in S$ is chosen according to the unitarily invariant measure, then for all $\epsilon > 0$

$$\Pr \left\{ \|\text{tr}_B X \rho X^\dagger - \text{tr}_B X \varphi X^\dagger\|_1 \geq \eta \right\} \leq \eta'$$

where

$$\eta = \epsilon + \sqrt{\frac{d_E}{d_B}}$$

and

$$\eta' = 2 \exp(-C\eta^2 |S|).$$

Here $C > 0$ is a constant, $d_E = |\text{supp tr}_B X X^\dagger|$ is an upper bound on the dimension of the support of $\text{tr}_B X \rho X^\dagger$, and $d_B = 1/\text{tr}((\text{tr}_E X \rho X^\dagger)^2)$ can be thought of as the effective dimension of $B$.

**Proof** This is a slight modification of [43, Thm. 2]. In the original, the theorem bounds $\|\text{tr}_B \Omega - \text{tr}_B \varphi\|_1$ under similar hypotheses but $\eta$ includes a correction dependent on $\text{tr}_B X \rho X^\dagger$. The correction disappears if the argument is applied to $\|\text{tr}_B X \rho X^\dagger - \text{tr}_B X \varphi X^\dagger\|_1$ instead under the assumption that $\|X\|_\infty \leq 1$, which ensures that the map $\rho \mapsto X \rho X^\dagger$ is 1-Lipschitz.

In order to use Theorem 13 to make statements about random subspaces, we will use the following lemma.

**Lemma 14** Let $f$ be an real-valued function on $\mathbb{C}^d$ (identified with rank one projectors acting on $\mathbb{C}^d$) and suppose that $f$ is $\alpha$-Lipschitz with respect to the trace norm. Let $\mu$ be the unitarily invariant measure on $\mathbb{C}^d$ and $\hat{\mu}$ the unitarily invariant measure on the space of $k$-dimensional subspaces of $\mathbb{C}^d$. If

$$\mu \left\{ |\xi\rangle : f(\xi) > \eta \right\} \leq g(d)$$

then

$$\hat{\mu} \left\{ S : \max_{|\xi\rangle \in S} |f(\xi)| > (1 + \alpha)\eta \right\} \leq \left( \frac{5}{\eta} \right)^{2k} g(d).$$

**Proof** This is a standard discretization argument. Fix a $k$-dimensional subspace $S_0 \subseteq \mathbb{C}^d$. According to Ref. 8, there is a trace norm $\eta$-net $M$ for the rank one projectors on $S_0$ of cardinality no more than $(5/\eta)^{2k}$. If $U$ is distributed according to the Haar measure $\nu$, then $US_0$ is distributed according to the unitarily invariant measure. So, we find by the triangle inequality that

$$\hat{\mu} \left\{ S : \max_{|\xi\rangle \in S} |f(\xi)| > (1 + \alpha)\eta \right\}$$

$$\leq \nu \left\{ U : \max_{|\xi\rangle \in S_0} |f(U|\xi\rangle)| > (1 + \alpha)\eta \right\}$$

$$\leq \nu \left\{ U : \max_{|\xi\rangle \in M} |f(U|\xi\rangle)| > \eta \right\}$$

$$\leq \left( \frac{5}{\eta} \right)^{2k} \mu \left\{ |\xi\rangle : f(\xi) > \eta \right\},$$

where the second inequality is just the union bound over elements of the net.

The following theorem collects the facts we will need about type and typical projectors. We omit their definitions, which will not be required here and can be found in Ref. 44.

**Theorem 15 (Typicality)** Let $|\rho\rangle \in A \otimes B \otimes E$ and set $|\psi\rangle = |\rho\rangle^\otimes n$. For any $\delta, \epsilon > 0$ sufficiently small there exist projectors $\Pi_B^n$, $\Pi_B^1$ and $\Pi_E^n$ on $P^n_0$ and $E^n_0$, respectively, and a projection $\Pi_t^A$ onto a fixed type subspace of $A^n$ such that the states

$$|\psi_t\rangle = \frac{\Pi_A^t \otimes I_B^t \otimes I_E^t |\psi\rangle}{\sqrt{|\langle \psi | \Pi_B^t \otimes I_B^t \otimes I_E^t |\psi\rangle|}}$$

and

$$|\tilde{\psi}_t\rangle = \frac{\Pi_A^t \otimes \Pi_B^t \otimes \Pi_E^t |\psi\rangle}{\sqrt{|\langle \psi | \Pi_B^t \otimes \Pi_B^t \otimes \Pi_E^t |\psi\rangle|}}$$

FIG. 1. Structure of a quantum identification code. $U_{X^\otimes n}$ and $V_\epsilon$ are the Stinespring extensions of the noisy channel $X^\otimes n$ and the encoding operation $E$. The receiver, Bob, has access to the channel $B^n$ as well as $C$, which consists of $nR$ qubits transmitted noiselessly from the receiver. (In the non-amortized setting, there is no $C$.) The encoding map $E$ is generally noisy, so part of its output is transmitted to the environment as $F$.
satisfy the following conditions for $X = A^n$, $B^n$, $E^n$ and sufficiently large $n$:

1. $\psi^A_t = \Pi^A_t / \text{rank} \Pi^A_t$.
2. $\|\psi_t - \bar{\psi}_t\|_1 \leq \epsilon$.
3. $\text{tr}[(\bar{\psi}X)^2] \leq 3(1 - 3\epsilon)^{-1} 2^{-n[H(X)|_\nu - \delta]}$.
4. $2^n[H(X)|_\nu - \delta] \leq \text{rank} \Pi^X \leq 2^n[H(X)|_\nu + \delta]$.
5. The largest eigenvalue of $\bar{\psi}^E_t$ is bounded above by $(1 - 3\epsilon)^{-1} 2^{-n[H(\bar{E})|_\nu - \delta]}$.
6. The ratio of the largest to the smallest nonzero eigenvalue of $\bar{\psi}^E_t$ is at most $2^{2n\delta}$.

where $\Pi^A$ and $\Pi^E$ should respectively be understood to be $\Pi^A_t$ and $\Pi^E_t$ in property 4, and $\epsilon > 0$ is a constant.

**Proof** If $\Pi^E_t$ is removed and property 6 omitted, the theorem is precisely a result proved in Ref. [44], with $\Pi^E_t$ the typical projector for $\rho$ on $E^n$. $\Pi^E_t$ will be a projector that removes all eigenvalues of the reduced density operator on $E^n$ below the stated threshold. Let

$$|\xi\rangle = \frac{\Pi^A \otimes \Pi^B \otimes \Pi^E |\psi\rangle}{\sqrt{\langle\psi|\Pi^A \otimes \Pi^B \otimes \Pi^E |\psi\rangle}}$$

The largest eigenvalue of $\xi^E_t$ is bounded above by $(1 - 3\epsilon)^{-1} 2^{-n[H(\xi)|_\nu - \delta]}$ according to property 5 as stated above and the state’s rank is at most $2^n[H(\xi)|_\nu + \delta]$ by property 4. Applying Lemma 21 to the eigenvalues of $\xi^E_t$, reveals that the sum of all eigenvalues less than or equal to $2^{-2n\delta}$ is at most

$$\frac{2^{-2n\delta}}{1 - 3\epsilon} \leq 2^{-n\delta}$$

for sufficiently large $n$. We can therefore let $\Pi^E_t$ be the orthogonal projection onto the direct sum of the eigenspaces of $\xi^E_t$ corresponding to eigenvalues larger than $2^{-2n\delta}$ and $\Pi^E_t$ is bounded above by $2^{-n\delta}$ and $\text{rank} \xi^E_t$ is at most

$$\frac{\lambda}{2^{-2n\delta} / \text{rank} \xi^E_t} \leq \frac{\lambda}{2^{-2n\delta} \cdot \lambda} = 2^{n\delta}.$$

A redefinition of $\epsilon$ completes the proof. □

**Proof (Direct coding part of Theorem 12)** The regular and amortized cases can be handled simultaneously. Fix an input state $\varphi$ as in Theorem 12 let $|\rho\rangle_{ABE}$ be a purification of $(id \otimes \mathcal{N})|\varphi\rangle$ and let $|\psi\rangle = |\rho\rangle^\otimes n$. To construct the code, we will need to project $\psi^A$ to a type subspace having favorable properties. $\psi^A_t\otimes B^n$ is the Choi-Jamiołkowski state for the channel $\mathcal{N}^\otimes n$ restricted to the type subspace $A_t$ defined by the projector $\Pi^A_t$. Call this channel $\mathcal{N}_t$, write $U_t$ for its Stinespring dilation, and consider $\mathcal{N}_t \otimes \text{id} \otimes \text{id}^F$. $C$ will play the role of the noiseless channel from Alice to Bob in the case of the amortized capacity and $F$ will represent quantum information discarded by Alice at the encoding stage. Our code will consist of a subspace of $S$ of $A_t \otimes C \otimes F$ selected according to the unitarily invariant measure, which then defines a subspace $S$ of $(B^n \otimes C) \otimes (E^n \otimes F)$. Our aim will be to show that $S$ is likely to be approximately forgetful for $E^n \otimes F$ when $C$ and $F$ are chosen appropriately, allowing for an application of Theorem 9.

Let $\Omega = \psi^B_t E^n \otimes \pi Z \otimes \pi F$ be the image under $U_t \otimes \text{id}^C \otimes \text{id}^F$ of the maximally mixed state on $A_t \otimes C \otimes F$. (Recall that $\pi Z$ denotes the maximally mixed state on $Z$.) Define $|\psi_t\rangle$ as in Theorem 13 and let $\Omega = \psi^B_t E^n \otimes \pi Z \otimes \pi F$. Then

$$\psi^B_t E^n = (\Pi^B \otimes \Pi^E) |\psi_t E^n (\Pi^B \otimes \Pi^E)|$$

so for $X = \Pi^B \otimes \Pi^E$, Theorem 13 states that a randomly chosen state $|\omega\rangle$ in $U_t (A_t) \otimes C \otimes F$ will satisfy

$$\mathbb{P}_r \left[ \left\| \Omega^{E^n F} - \bar{\omega}^{E^n F} \right\|_1 \geq \frac{\eta}{2} \right] \leq \eta'$$

for $\bar{\omega} = X \omega X^\dagger$ and where, for any $\nu > 0$,

$$\eta = \nu + \sqrt{\text{rank}(\Pi^E \Pi^E) \cdot \text{tr}[(\psi^B_t \otimes \pi C)^2]}$$

$$\eta' = 2 \exp \left( - C t^2 |A_t \otimes C \otimes F| \right).$$

We will fix $\nu$ to be $\nu = 2^{-3n\delta}$. So by Lemma 13 a random $S$ in $U_t (A_t) \otimes C \otimes F$ chosen according to the unitarily invariant measure will satisfy

$$\mathbb{P}_r \left[ \max_{|\omega\rangle \in S} \left\| \Omega^{E^n F} - \bar{\omega}^{E^n F} \right\|_1 \geq \eta \right] \leq 2 \left( \frac{10}{n} \right)^{2|S|} \exp \left( - C t^2 |A_t \otimes C \otimes F| \right)$$

since the function $\omega \mapsto \| \Omega^{E^n F} - \bar{\omega}^{E^n F} \|_1$ is 1-Lipschitz with respect to the trace norm. For convenience, let $|F| = 2^n R$ and $|C| = 2^n R$. Since $|A_t| \geq 2^n[H(A)|_\nu - \delta]$, choosing $|S|$ to be $2^n[H(A)|_\nu + R + f - 3\delta]$ will lead to

$$\max_{|\omega\rangle \in S} \left\| \Omega^{E^n F} - \bar{\omega}^{E^n F} \right\|_1 < \eta$$

(14)

with high probability for sufficiently large $n$ provided $\eta$ decays at most exponentially with $n$.

Now let us determine how to choose $f$ and $R$ in order to ensure a small value for $\eta$. Observe that by properties 3 and 4 in Theorem 13

$$\text{rank} \Pi^E \Pi^E \otimes \text{id}^C \otimes \text{id}^F \leq 2^n[H(E)|_\nu + 3\delta + f]$$

$$\text{tr}[(\psi^B_t \otimes \text{id}^C \otimes \text{id}^F)^2] \leq 3(1 - 3\epsilon)^{-1} 2^{-n[H(B)|_\nu - \delta - R]}.$$ 

Therefore,

$$\eta \leq \nu + 3 \cdot 2^n[H(E)|_\nu - H(B)|_\nu + f - R + (1 + c\delta)] / 2$$

provided $\epsilon$ is chosen smaller than $1 / 15$. There are two cases to consider:

...
Case 1. First suppose that $I(A: B)_{\rho} > 0$ or, equivalently, that $H(E)_{\rho} < H(B)_{\rho}$. Under these circumstances, amortization is not required. Choosing $R = 0$ and $f = H(B)_{\rho} - H(E)_{\rho} - (7 + c)\delta$ leads to $\eta \leq \nu + 3 \cdot 2^{-3n\delta} \leq 4 \cdot 2^{-5n\delta}$.

Thus, the rate of the associated code will be

$$Q = \frac{1}{n} \log |S|$$

$$= H(A)_{\rho} + R + f - 8\delta$$

$$= H(A)_{\rho} + H(B)_{\rho} - H(E)_{\rho} - (7 + c)\delta - 8\delta$$

$$= I(A : B)_{\rho} - (15 + c)\delta.$$  

Case 2. Now suppose that $I(A : B)_{\rho} \leq 0$ so that $H(E)_{\rho} \geq H(B)_{\rho}$. In this case we set $R = H(E)_{\rho} - H(B)_{\rho} + (7 + c)\delta$ and $f = 0$ to again achieve $\eta \leq 4 \cdot 2^{-5n\delta}$. This time, the rate of the code will be

$$Q = \frac{1}{n} \log |S| - 2R$$

$$= H(A)_{\rho} + R + f - 8\delta - 2R$$

$$= H(A)_{\rho} + H(B)_{\rho} - H(E)_{\rho} - (15 + c)\delta$$

$$= I(A : B)_{\rho} - (15 + c)\delta.$$  

We have established that the subspace $S$ corresponds to a code of the correct rate. Applying Theorem 8 to $\widehat{\Omega}$ and the states in $S$ with $X = \Pi^{B} \otimes 1^{C} \otimes \Pi^{E}_{1} \otimes 1^{F}$ will complete the proof. Recalling that the ratio of the largest to the smallest nonzero eigenvalues of $\Omega^{E \otimes F}$ is at most $2^{2n\delta}$, the theorem asserts that $S$ is a quantum-ID code with error probability at most

$$3 \cdot 30 \cdot 2^{2n\delta} \cdot (4 \cdot 2^{-3n\delta}) + 4\sqrt{\epsilon} \leq 1/2,$$

which can be made arbitrarily small for sufficiently large $n$. \hfill $\Box$

Proof (Converse for Theorem 12) We will address the regular and amortized capacities at the same time. Consider an amortized quantum-ID code for $n$ copies of $N$ as illustrated in Figure 1. The Stinespring dilations of $N^{\otimes n}$ and $E$ together have three output registers: one for the channel input, one for the transmission to Bob and one going to the environment. Abbreviating $\widehat{B} = B^{n}C$ and $\widehat{E} = E^{n}F$ in Figure 1 the quantum-ID code is equivalent to a subspace $S \subseteq \widehat{B} \otimes \widehat{E}$, and we can apply our lemmas.

A key observation is that for any pure state ensemble $\{p_{x}, \varphi_{x}\}$ on $S$ decomposing the maximally mixed state,

$$H(\widehat{B}) \geq H(\widehat{B}|X) = H(\widehat{E}|X) = H(\widehat{E}) - o(n). \quad (15)$$

The first inequality is just the concavity of the entropy function while the first equality follows from the fact that $\varphi_{x}$ is pure on $\widehat{B}\widehat{E}$. The final relation is a consequence of Theorem 7; the fidelity alternative implies that if states can be identified on $\widehat{B}$ then they must be indistinguishable on $\widehat{E}$. Continuity of the von Neumann entropy in the form of the Fannes inequality [43] shows the correction to be $o(n)$. Thus, sending one half of a maximally entangled state $\Phi^{AB}$ between $S$ and an auxiliary space named $A$ into the circuit of Figure 1 we obtain a multipartite pure state $\Psi^{AB\widehat{E}}$ with respect to which

$$\log |A| = H(A) \leq H(A) + H(\widehat{B}) - H(\widehat{E}) + o(n)$$

$$= I(A : B) + o(n)$$

$$= I(A : B^{n}) + I(A : C|B^{n}) + o(n)$$

$$\leq I(A : B^{n}) + 2 \log |C| + o(n).$$

Therefore, the amortized quantum identification capacity is bounded above by $\lim_{n \to \infty} \frac{1}{n} g(N^{\otimes n})$ where $g(N) = \max_{|\phi>} I(A : B)_{\rho}$ for $\rho = (|\phi> \otimes \phi>)$. It is well-known, however, that $g(N^{\otimes n}) = ng(N)$ so the limit is not necessary [39].

On the other hand, in the non-amortized case, $|C| = 1$, and the rate of the code is bounded above by $\frac{1}{n} I(A : B^{n}) + o(1)$. On the other hand, Eq. (15) above yields

$$I(A : B^{n}) = I(A : \widehat{B}) = H(\widehat{B}) - H(\widehat{E}) \geq -o(n), \quad (16)$$

which is almost what we need, except that the claim of Theorem 12 requires strictly positive coherent information. We will achieve this by modifying the input state in such a way that the coherent information becomes strictly positive and all other entropic quantities change only by a sublinear amount (in $n$).

To this end, note that if $Q_{ID}(N) = 0$ there is nothing to prove, so we shall assume $Q_{ID}(N) > 0$, in which case $Q(N) > 0$ by Theorem 11. Hence, fix a $k$ and the purification $|\phi>$ of an appropriate input state to $N^{\otimes k}$, such that with respect to $A^{k}B^{k} = (|\phi> \otimes N^{\otimes k}) |\phi>$, $I(A^{k}B^{k})_{\sigma} \geq 1$, and let $\ell = \max \{0, -I(A^{k}B^{k})_{\rho} \} > 1$. Hence, considering block length $N = n + k\ell$ and the input state $\Phi \otimes \phi^{\otimes \ell} \to N^{\otimes N}$, resulting in the state $\omega^{AA^{k}B^{k}N} = (|\Phi> \otimes N^{\otimes N})(\Psi \otimes \phi^{\otimes \ell})$, with respect to which we have

$$\log |A| \leq I(A : B^{n}) + o(n) \leq I(AA^{k} : B^{N}) + o(N), \quad I(AA^{k}B^{N}) \geq 1 > 0.$$

As $N = n + o(n)$, this shows indeed that $\sup_{n} \frac{1}{n} Q_{ID}(1)(N^{\otimes n})$ is an upper bound on all achievable rates. \hfill $\Box$

V. CONCLUSION AND OPEN QUESTIONS

The fidelity alternative states that geometry preservation and approximate forgetfulness are complementary properties, much like quantum data transmission and complete forgetfulness. Subject to some technical conditions, geometry preservation is itself equivalent to quantum identification, an operational task very much in the spirit of quantum data transmission but strictly weaker. Just as analyzing complete forgetfulness has proved a versatile and effective tool for studying asymptotic quantum error correction, approximate forgetfulness provides a new approach to asymptotic quantum identification. Indeed, by focusing on approximate forgetfulness of the complementary channel, we have established that the
amortized quantum identification capacity is exactly equal to the entanglement-assisted capacity.

The fidelity alternative suggests a number of possible extensions, such as asking what happens if geometry is preserved not only for pure states but for higher rank mixed states. Would such a property have an operational interpretation and corresponding interpretation in terms of a form of forgetfulness intermediate between the weak form studied here and complete forgetfulness? It would also be interesting to understand geometry preservation as a type of pseudo-isometry [46] from projective space to the Grassmannian of subspaces corresponding to the supports of the mixed output states.

Meanwhile, Theorem [12] poses an entertaining and potentially deep puzzle: why do amortized quantum identification and entanglement-assisted classical communication result in the same capacity in the absence of any known operational relationship between these tasks? The theorem also leaves open the important problem of evaluating the quantum identification capacity formula in the unamortized case. Similarly, the theorem fails to determine how much extra quantum communication is necessary to achieve the amortized capacity. In particular, does there exist a channel where the required rate is strictly positive? We do expect this to be true, but have been unable to establish it rigorously.

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Appendix A: Miscellaneous Facts

The following results were used in various proofs but have been collected here so as not to distract from the main line of argument in the paper. This first relation provides a convenient way to calculate mixed state fidelity:

**Lemma 16** For pure states φ, ψ on a bipartite system B ⊗ E,

\[ F(\varphi^B, \psi^B) = \| \text{tr}_B [\varphi|\psi] \|_1^2, \]  \hspace{1cm} (A1)  

**Proof** This is a straightforward calculation:

\[
\| \text{tr}_B [\varphi|\psi] \|_1 = \max_{\|X\|_1 \leq 1} | \text{tr} (\text{tr}_B [\varphi|\psi]) X | \\
= \max_{U \text{ unitary}} | \text{tr} (\text{tr}_B [\varphi|\psi]) U | \\
= \max_{U \text{ unitary}} | \text{tr} [\varphi|\psi](I \otimes U) | \\
= \max_{U \text{ unitary}} \sqrt{F((I \otimes U)\varphi(I \otimes U^\dagger), \psi)} \\
= \sqrt{F(\varphi^B, \psi^B)},
\]

invoking, successively, the duality between trace and sup norm, the fact that the maximum is always attained at a unitary, the defining property of the partial trace, and in the last line Uhlmann’s relation [47, 48]. 

The following lemma provides conditions under which mixing preserves near-orthogonality.

**Lemma 17** Let ρ and σ_i, for all i, be states on the same Hilbert space such that there exist projectors P and Q_i of rank ≤ r, and μP ≤ ρ ≤ λ P, μQ_i ≤ σ_i ≤ λ Q_i such that μ r ≤ 1. If furthermore for all i, F(ρ, σ_i) ≤ ε, then

\[ F(\rho, \sigma) \leq \delta := \frac{\lambda^2}{\mu^2} \]

for every \( \sigma = \sum_i p_i \sigma_i \) in the convex hull of the \( \sigma_i \).

**Proof** We use the definition of the fidelity to first obtain

\[ \epsilon \geq \left( \text{tr} \sqrt{\rho \sigma_i \sqrt{\rho}} \right)^2 \geq \mu^2 \left( \text{tr} P Q_i P \right)^2. \]

Invoking the definition again, we now get from this

\[ \sqrt{F(\rho, \sigma)} = \| \sqrt{\rho \sigma} \|_1 \leq \lambda \text{tr} \sum_i p_i P Q_i P \]

\[ \leq \lambda r \sqrt{ \sum_i p_i \frac{1}{\mu r} \mu \text{tr} P Q_i P } \]

\[ \leq \lambda r \sqrt{ \frac{\epsilon}{\mu r} } \leq \sqrt{\epsilon \frac{\lambda}{\mu}}, \]

using the concavity of the square root twice in turn [49].

**Lemma 18** Let \( 0 \leq \hat{\rho} \leq \rho \) and \( 0 \leq \hat{\sigma} \leq \sigma \). Then \( F(\hat{\rho}, \hat{\sigma}) \leq F(\rho, \sigma) \).

**Proof** Denoting unitary congruence of matrices (in particular having the same spectrum) by ~, we have

\[ \sqrt{\rho \sigma} \sqrt{\hat{\rho}} \sim \sqrt{\hat{\rho} \sigma} \sqrt{\hat{\rho}} \sim \sqrt{\hat{\rho} \sigma} \sqrt{\hat{\rho}} \sim \sqrt{\rho \sigma} \sqrt{\hat{\rho}}. \]

Hence, since the square root is operator monotone [49] and the trace is invariant under unitary basis change, \( \text{tr} \sqrt{\rho \sigma} \sqrt{\hat{\rho}} \leq \text{tr} \sqrt{\rho \sigma} \sqrt{\hat{\rho}}, \) completing the proof.

The next lemma constrains the increase of the maximal output trace norm when tensoring with a fixed-size identity transformation:
Lemma 19 Let $\Gamma : S(A) \to S(B)$ be a linear superoperator.
Then for any $t$ any positive integer,
\[ \|\Gamma\|_1^{(t)} \leq t \|\Gamma\|_1^{(1)}. \]

Proof Write $X$, an operator on $C^t \otimes A$ such that $\|X\|_1 \leq 1$, in its singular value decomposition as $\sum j s_j |v_j\rangle \langle w_j|$, with $0 \leq s_j \leq 1$ and $\langle v_j| = \langle w_j| \delta_{jk}$. By convexity (triangle inequality), $\|\Gamma\|_1^{(t)}$ is attained with a rank-one $X = |v\rangle \langle w|$, and for the following fix Schmidt decompositions $|v\rangle = \sum k \alpha_k |e_k\rangle |f_k\rangle$ and $|w\rangle = \sum \ell \beta_\ell |g_\ell\rangle |h_\ell\rangle$. Then,
\[ \| (\text{id}_t \otimes \Gamma) X \|_1 = \left\| (\text{id}_t \otimes \Gamma) \left( \sum k \alpha_k \beta_\ell |e_k\rangle |g_\ell\rangle \otimes |f_k\rangle |h_\ell\rangle \right) \right\|_1 \]
\[ \leq \sum k \alpha_k \beta_\ell \left\| \left( \text{id}_t \otimes \Gamma \right) \left( |e_k\rangle \otimes |f_k\rangle |h_\ell\rangle \right) \right\|_1 \]
\[ = \sum k \alpha_k \beta_\ell \| (f_k) (h_\ell) \|_1 \leq t \|\Gamma\|_1^{(1)}, \]
where the first step is just the triangle inequality and the next follows from the fact that $\|X\|_1 = \sum j s_j \leq 1$. The final inequality uses the fact that $\sum k=1 \alpha_k$ and $\sum \ell=1 \beta_\ell$ are both bounded above by $\sqrt{t}$ since $\|\alpha\|_2 = \|\beta\|_2 = 1$. \qed

Remark The factor $t$ is optimal, as the example of the matrix transposition shows where the bound of the lemma becomes an equality. \qed

Lemma 20 (Gentle measurement [50–52]) Let $\rho$ be a state, and $0 \leq X \leq \mathbb{I}$ be an operator on some Hilbert space, such that $\text{tr} \rho X \geq 1 - \epsilon$. Then, $\| \rho - \sqrt{X \rho \sqrt{X}} \|_1 \leq 2 \sqrt{\epsilon}$. \qed

The following, final, lemma is used to argue that the small eigenvalues of a density operator can be discarded without causing much disturbance.

Lemma 21 Let $(p_1, p_2, \ldots, p_r)$ be a probability density with $p_i \geq p_{i+1}$ for all $i$ and let $\chi = \{i; p_i \leq D/r\}$ for some $0 \leq D \leq 1$. Then, $\sum_{i \in \chi} p_i \leq (D/r) = D$.

Proof Since evidently $|\chi| \leq r$,
\[ \sum_{i \in \chi} p_i \leq |\chi| \frac{D}{r} \leq r \frac{D}{r} = D, \]
and that’s it. \qed

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