Research Article

Weighted Moving Averages for a Series of Fuzzy Numbers Based on Nonadditive Measures with \( \sigma - \lambda \) Rules and Choquet Integral of Fuzzy-Number-Valued Function

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The aim of this study is to generalize moving average by means of Choquet integral. First, by employing nonadditive measures with \( \delta - \lambda \) rules, the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. Finally, some properties are investigated by means of convolution formula of Choquet integral. It shows that the results obtained in this paper extend the previous conclusions.

1. Introduction

The concept of nonadditive measures was originally proposed by Sugeno [1]. It replaces additivity in classical additive measures with monotonicity and can be regarded as an extension of classical additive measures. Indeed, nonadditive measures can be used to describe interdependent or interactive characteristics of information in practical applications. The Choquet integral, initiated by Choquet [2], provides a mechanism to integrate function on the basis of nonadditive measures and is a powerful technique to address interdependence and interaction among information. In fact, the Choquet integral [2] with respect to nonadditive measures has successful application in pattern recognition [3], decision-making [4–7], information fusion [8–10], economic theory [11], and so on.

Another key mathematical structure to cope with imperfect or imprecise information is a fuzzy set, developed by Zadeh [12]. Fuzzy numbers [13], a specific format of fuzzy sets, are utilized to express values in practical situation where the exact values may not be determined because of lack or imperfection of information [14]. That is, fuzzy numbers take into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty and have been used as a way of modeling uncertain and incomplete systems. Fuzzy numbers have been investigated intensively by research studies [15–17] from various aspects since it was introduced.

Motivated by the ability of Choquet integral with respect to nonadditive measures in handling interaction among information and the merit of fuzzy number in depicting uncertainty, it is of both theoretical and practical importance to combine them together and apply the combination to moving average. In this work, we want to give more insight into issues connected with the weighted moving averages for a series of fuzzy numbers based on nonadditive measures with \( \sigma - \lambda \) rules by the new tools, Choquet integral and fuzzy number. This is a new contribution to our previous work [18], in which the moving average for a series of fuzzy numbers based on nonadditive measures with \( \sigma - \lambda \) rules is proposed and discussed. The aim of this paper is to show that the calculation of the moving average for a series of fuzzy
numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. Finally, some properties are investigated by means of the convolution formula of Choquet integral.

The structure of this paper is as follows. In Section 2, we review some basic concepts and properties about nonadditive measure with $\sigma-\lambda$ rules and fuzzy numbers. And the definition of product between a nonnegative matrix and fuzzy number vector is given to make our analysis possible. In Section 3, it shows that the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined and their properties are investigated by means of the convolution formula of Choquet integral. The paper ends with conclusion in Section 4.

2. Preliminaries

In this section, some basic notations and concepts of HFLTS and DTRS are briefly reviewed. Throughout this study, $R^m$ denotes the m-dimension real Euclidean space and $R^+ = (0, \infty)$.

Definition 1 (see [1, 19, 20]). Let $X$ denote a nonempty set and $\mathcal{A}$, a $\sigma-$ algebra on the $X$. A set function $\mu$ is referred to as a regular fuzzy measure if

1. $\mu(\emptyset) = 0$
2. $\mu(X) = 1$
3. For every $A$ and $B \in \mathcal{A}$ such that $A \subseteq B$, $\mu(A) \leq \mu(B)$

Definition 2 (see [1, 19, 20]). $g_\lambda$ is called a fuzzy measure based on $\sigma-\lambda$ rules if it satisfies

$$g_\lambda \left( \bigcup_{i=1}^{\infty} A_i \right) = \begin{cases} \frac{1}{\lambda} \left\{ \prod_{i=1}^{\infty} \left[ 1 + \lambda g_\lambda \left( A_i \right) \right] - 1 \right\}, & \lambda \neq 0, \\ \sum_{i=1}^{\infty} g_\lambda \left( A_i \right), & \lambda = 0, \end{cases}$$

where $\lambda \in (-1/(1/\sup \mu), \infty) \cup \{0\}$, $A_i \in \mathcal{A}$, and $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots$ and $i \neq j$.

Particularly, if $\lambda = 0$, then $g_\lambda$ is a classic probability measure.

A regular fuzzy measure $\mu$ is called Sugeno measure based on $\sigma-\lambda$ rules if $\mu$ satisfies $\sigma-\lambda$ rules, briefly denoted as $g_\lambda$. The fuzzy measure denoted in this paper is Sugeno measure.

Remark 1. In Definition 2, if $n = 2$, then

$$\mu(A \cup B) = \begin{cases} \mu(A) + \mu(B) + \lambda \mu(A) \mu(B), & \lambda \neq 0, \\ \mu(A) + \mu(B), & \lambda = 0. \end{cases}$$

Remark 2. If $X$ is a finite set, for any subset $A$ of $X$, then

$$g_\lambda (A) = \begin{cases} \frac{1}{\lambda} \left\{ \prod_{x \in A} \left[ 1 + \lambda g_\lambda \left( \{x\} \right) \right] - 1 \right\}, & \lambda \neq 0, \\ \sum_{i=1}^{\infty} g_\lambda \left( \{x\} \right), & \lambda = 0. \end{cases}$$

Remark 3 (see [19]). If $X$ is a finite set, then the parameter $\lambda$ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the following equation:

$$\prod_{i=1}^{n} \left( 1 + \lambda g_{\lambda_i} \right) = 1 + \lambda.$$  (4)

Let $g_\lambda$ be a fuzzy measure satisfying $\sigma-\lambda$ rules. Denoting $A = \{x_1, x_2, \ldots, x_m\} \in \mathcal{A}$, $f : A \rightarrow R$ be real-valued function, and then, the Choquet integral of $f$ on $A$ is defined as follows [1]:

$$\left( c \right) \int_{A} f(x) g_\lambda (A_i) - g_\lambda (A_{i+1}),$$

where $A = \{x_1, x_{i+1}, \ldots, x_m\}$, $i = 1, 2, \ldots, m$, and $\int f(x_i) \leq f(x_i) \leq f(x_{i+1})$.

Let $g_\lambda (\{x_i\}) = g_{\lambda_i}, i = 1, 2, \ldots, m$; then, $g_\lambda (A_i)$ is obtained from the following recurrence relation:

$$g_\lambda (A_m) = g_{\lambda_m} (x_m) = \mu_{\lambda_m} \mu_\lambda (A_i) = g_\lambda (A_{i+1}) + \lambda \mu_{\lambda_m} g_\lambda (A_{i+1}),$$

where $\lambda 
eq 0$.

Let $\hat{A} (x) \in \hat{E}$, $r \in (0, 1]$ and $[\hat{A}]^r = \{x \in R : u_{\hat{A}} (x) \geq r\}$. $\hat{A}$ satisfies the following:

1. $\hat{A}$ is a normal fuzzy set, i.e., an $x_0 \in R$ exists such that $u_{\hat{A}} (x_0) = 1$
2. $\hat{A}$ is a convex fuzzy set, i.e., $u_{\hat{A}} (\lambda x + (1-\lambda) y) \geq \min \{u_{\hat{A}} (x), u_{\hat{A}} (y)\}$ for any $x, y \in R$ and $\lambda \in (0, 1]$
3. $\hat{A}$ is an upper semi-continuous fuzzy set
4. $[\hat{A}]^0 = X \in R : u_{\hat{A}} (x) > 0 = \bigcup_{r \in (0,1]} [\hat{A}]^r$ is compact, where $\hat{A}$ denotes the closure of $\hat{A}$.

Then, $\hat{A}$ is called a fuzzy number. We use $\hat{E}$ to denote the fuzzy number space [21].

It is clear that each $x \in R$ can be considered as a fuzzy number $\hat{A}$ defined by

$$u_{\hat{A}} (x) = \begin{cases} 1, & x = A, \\ 0, & \text{otherwise}. \end{cases}$$

Given any two fuzzy numbers $\hat{A}_1, \hat{A}_2, k_1, k_2 \in \mathcal{L}$ and $k_2 \geq 0$, the operational rules are as follows:

1. $k_1 (\hat{A}_1 + \hat{A}_2) = k_1 \hat{A}_1 + k_2 \hat{A}_2$
2. $k_1 (k_2 \hat{A}_1) = (k_1 k_2) \hat{A}_1$
3. $(k_1 + k_2) \hat{A}_1 = k_1 \hat{A}_1 + k_2 \hat{A}_1$

Lemma 1 (see [21–23]). For a fuzzy set $\hat{A}$, it satisfies the following equation:
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Definition 5 (see [18]). Let $(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_m)\in E^m, (t_1, t_2, \ldots, t_m)\in\mathbb{R}^m$, and $g_1$ be fuzzy measures satisfying $\delta - \lambda$ rules. Denote $A_i = \{t_i, t_{i+1}, \ldots, t_m\}, i = 1, 2, \ldots, m$, and $A_{m+1} = \emptyset$. Then, the weighted moving averages for fuzzy numbers based on a nonadditive measure with $\sigma - \lambda$ rules is defined as follows:

\[
\bar{x}_n = (g_1(A_1) - g_1(A_2))\bar{x}_{n-m} + (g_1(A_1) - g_1(A_3))\bar{x}_{n-m+1} + \cdots + (g_1(A_m) - g_1(A_{m+1}))\bar{x}_{n-1},
\]

where $n > m$.

Definition 6. Let $(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_m)\in E^m, (t_1, t_2, \ldots, t_m)\in\mathbb{R}^m$, and $g_1$ be fuzzy measures satisfying $\delta - \lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \ldots, t_m\}, i = 1, 2, \ldots, m$, and $A_{m+1} = \emptyset$. Then, for fuzzy number $\bar{x}_n(n > m)$, the Choquet integral of $\bar{x}_n(n > m)$ with respect to fuzzy measure $g_1$ on $A$ is defined as follows:

\[
(C) \int_A \bar{x}_n d\bar{g}_1 = \sum_{i=1}^{m} \bar{x}_{n-i} (g_1(A_i) - g_1(A_{i+1})).
\]

Similarly, for vector $\bar{X}_n = [\bar{x}_m, \bar{x}_{m-1}, \ldots, \bar{x}_{n+m-1}]^T$ $(n > m)$, the Choquet integral of $\bar{X}_n$ with respect to fuzzy measure $g_1$ on $A$ is defined as follows:

\[
(C) \int_A \bar{X}_n d\bar{g}_1 = \left[ (C) \int_A \bar{x}_n d\bar{g}_1, \ldots, (C) \int_A \bar{x}_m d\bar{g}_1 \right]^T.
\]
Theorem 1. Let \( g \) be fuzzy measures satisfying \( \delta - \lambda \) rules. Denote

\[
\bar{x}_n = (x_n - \delta_{n,1}, x_n + \delta_{n,2}).
\]

\( A_i = \{t_0, t_{i+1}, \ldots, t_m\}, i = 1, 2, \ldots, m, A_{m+2} = \emptyset, \) and \( t \) be a positive real number. Then, for vector \( X_n = [\bar{x}_n, \bar{x}_{n+1}, \ldots, \bar{x}_{n+m-1}]^T (n > m) \) and

\[
A_i = \{t_0, t_{i+1}, \ldots, t_m\}, i = 1, 2, \ldots, m, A_{m+2} = \emptyset, \text{ and } t \text{ be a positive real number. Then, for vector } X_n = [\bar{x}_n, \bar{x}_{n+1}, \ldots, \bar{x}_{n+m-1}]^T (n > m) \text{ and}
\]

we have

\[
(C) \int_A \bar{x}_n d g_A = \sum_{i=1}^{m} ((x_{n-m+i-1} - \delta_{n-m+i-1,1})(g(A_i) - g(A_{i+1})), (x_{n-m+i-1} + \delta_{n-m+i-1,2})(g(A_i) - g(A_{i+1})))
\]

\[
= \left( \sum_{i=1}^{m} x_{n-m+i-1}(g(A_i) - g(A_{i+1})) \right)
\]

\[
\sum_{i=1}^{m} x_{n-m+i-1}(g(A_i) - g(A_{i+1})),
\]

\[
\sum_{i=1}^{m} \left( x_{n-m+i-1} + \delta_{n-m+i-1,2} \right)(g(A_i) - g(A_{i+1}))
\]

where \( \bar{x}_n = (x_n - \delta_{n,1}, x_n, x_n + \delta_{n,2}) \).

Theorem 1. Let \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \in E^m, \{t_0, t_1, \ldots, t_m\} \in \mathbb{R}^m, \) and \( g \) be fuzzy measures satisfying \( \delta - \lambda \) rules. Denote

\[
P = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
g_1(A_1) - g_1(A_2) & g_1(A_2) - g_1(A_3) & \cdots & g_1(A_{m-1}) - g_1(A_m)
\end{bmatrix},
\]

we have

\[
(1)
\]

\[
(C) \int_A \bar{x}_n d g_A = P'. \quad (C) \int_A \bar{x}_n d g_A
\]

\[
(2)
\]

\[
\text{especially, if } g_1(A_1) - g_1(A_2) > 0 \text{ and } n - t > m, \text{ then}
\]

\[
(C) \int_A \bar{x}_n d g_A = P^{-t} . \quad (C) \int_A \bar{x}_n d g_A
\]

\[
(3)
\]

\[
(4) \text{ If } \gcd \{i \mid i \in \{1, 2, \ldots, m\}, g_1(A_i) - g_1(A_{i+1}) > 0\} = 1, \text{ then } \lim_{n \to \infty} (C) \int_A \bar{x}_n d g_A \text{ exists and}
\]

\[
\lim_{n \to \infty} (C) \int_A \bar{x}_n d g_A = \lim_{n \to \infty} P^{t-m}. \quad (C) \int_A \bar{x}_n d g_A
\]

\[
(22)
\]

\[
\text{where } e = \sum_{i=1}^{m} e_k = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m \times 1} \text{ and } e_k \text{ is the } i\text{th standard unit column vector}:
\]

\[
e = \begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_m
\end{bmatrix}
\]

\[
(C) \int_A \bar{x}_n d g_A = P^{t-m}. \quad (C) \int_A \bar{x}_n d g_A
\]

\[
(22)
\]
Then, by the expression of (C)\[ \int_A \varphi_n \cdot dg_A \] in (1), we have
\[
(C) \int_A \varphi_{n+1} \cdot dg_A = P \cdot (C) \int_A \varphi_n \cdot dg_A .
\] (27)

By (2), we know that
\[
(C) \int_A \varphi_{n+1} \cdot dg_A = P^r \cdot (C) \int_A \varphi_n \cdot dg_A .
\] (28)

Since \( P \) is an invertible matrix, we have
\[
(C) \int_A \varphi_{n+1} \cdot dg_A = P^{-r} \cdot (C) \int_A \varphi_n \cdot dg_A .
\] (29)

By using Theorem 2 in Reference [18], we note that \( \lim_{n \to \infty} P^n \) exists and
\[
\lim_{n \to \infty} P^n = \frac{ea^T}{a^T e} = er^T .
\] (30)

Combining (3), it follows that
\[
(C) \int_A \varphi_{n+1} \cdot dg_A = P^r \cdot (C) \int_A \varphi_n \cdot dg_A .
\] (31)

Take limit of the above equation, we obtain
\[
\lim_{n \to \infty} (C) \int_A \varphi_n \cdot dg_A = \lim_{n \to \infty} P^{n-1} \cdot (C) \int_A \varphi_1 \cdot dg_A = \frac{ea^T}{a^T e} .
\] (32)

The proof is complete. \( \square \)

**Definition 7.** For vector \( \varphi_n = [\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+m}]^T \) (\( n > m \)) and
\[
\varphi_n^r(r) = [\varphi_n^r(r), \varphi_{n+1}^r(r), \ldots, \varphi_{n+m-1}^r(r)]^T ,
\] (33)

the Choquet integral of \( \varphi_n^r(r) \) with respect to fuzzy measure \( g_A \) on \( A \) is defined as follows:
\[
(C) \int_A \varphi_n^r(r) \cdot dg_A = \left[ (C) \int_A \varphi_n^r(r) \cdot dg_A , \ldots , (C) \int_A \varphi_{n+m-1}^r(r) \cdot dg_A \right] .
\] (34)
Also, the Choquet integral of $\overline{X}_n^+(r)$ with respect to fuzzy measure $g_1$ on $A$ is defined by

\[
(C)\int_A \overline{X}_n^+(r)dg_1 = \left[ (C)\int_A \overline{X}_n^+(r)dg_1, (C)\int_A \overline{X}_{n+1}^+(r)dg_1, \ldots \right]^T.
\]

(35)

**Theorem 2.** Let $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in \overline{E}^m$, $(t_1, t_2, \ldots, t_m) \in R^m$, and $g_1$ be the fuzzy measure satisfying $\delta - \lambda$ rules. Denote $A_i = [t_i, t_{i+1}, \ldots, t_m]$, $i = 1, 2, \ldots, m$, and $A_{m+1} = \emptyset$, and $t$ be the positive real number. Then, for vector

\[
\overline{X}_n^-(r) = [\overline{x}_1^-(r), \overline{x}_2^-(r), \ldots, \overline{x}_{n+1}^-(r)]^T,
\]

where $P$ is the same matrix in Theorem 1, we have

(1)

\[
(C)\int_A \overline{X}_n^-(r)dg_1 = \left[ \sum_{i=1}^{m} \overline{x}_{n-m+1}^-(r)(g_1(A_i) - g_1(A_{i-1})) \right] + \left[ \sum_{i=1}^{m} \overline{x}_{n-m+1}^-(r)(g_1(A_i) - g_1(A_{i-1})) \right] \ldots + \left[ \sum_{i=1}^{m} \overline{x}_{n-m+1}^-(r)(g_1(A_i) - g_1(A_{i-1})) \right].
\]

(36)

(2)

\[
\begin{aligned}
\text{rpt (35)} \int_A \overline{X}_{n+1}^+(r)dg_1 &= P \cdot (C)\int_A \overline{X}_n^+(r)dg_1 = P^2. \\
(C)\int_A \overline{X}_{n+1}^-(r)dg_1 &= \ldots \\
&= P^{n-m+1} \cdot (C)\int_A \overline{X}_{n+1}^-(r)dg_1 = P^{n-m} \cdot (C)\int_A \overline{X}_{n+1}^+(r)dg_1.
\end{aligned}
\]

(38)

(3)

\[
(C)\int_A \overline{X}_{n-t}^+(r)dg_1 = P^t \cdot (C)\int_A \overline{X}_n^+(r)dg_1,
\]

especially, if $g_1(A_1) - g_1(A_2) > 0$ and $n - t > m$, then

\[
(C)\int_A \overline{X}_{n-t}^-(r)dg_1 = P^{-t} \cdot (C)\int_A \overline{X}_n^-(r)dg_1.
\]

(39)

(4)

If $gcd\{i \in \{1, 2, \ldots, m\}: g_1(A_i) - g_1(A_{i+1}) > 0\} = 1$, then $\lim_{n \to \infty} (C)\int_A \overline{X}_n^+(r)dg_1$ exists and

\[
\lim_{n \to \infty} (C)\int_A \overline{X}_n^+(r)dg_1 = \lim_{n \to \infty} P^{n-1} \cdot (C)\int_A \overline{X}_1^+(r)dg_1 = \frac{ea^T}{a^T e} \cdot (C)\int_A \overline{X}_1^+(r)dg_1,
\]

where $e = \sum_{i=1}^m e_k = [1, 1, \ldots, 1]^T \in R^{m+1}$ and $e_k$ is the $i$th unit standard column vector:

\[
a = [a_1, a_2, \ldots, a_m]^T, \\
b = [b_1, b_2, \ldots, b_m]^T, \\
\]

\[
a_k = \frac{1}{k} \sum_{i=1}^k (g_1(A_i) - g_1(A_{i+1})), \\
b_k = \frac{a^T e_k}{a^T e} = \frac{a_k}{\sum_{i=1}^m a_i} = \frac{g_1(A_i) - g_1(A_{i+1})}{m g_1(A_i) - \sum_{i=2}^m g_1(A_i)}, \quad k = 1, 2, \ldots, m.
\]

(41)

**Proof**

(1) According to Definition 6, we know that

\[
(C)\int_A \overline{X}_n^+(r)dg_1 = \sum_{i=1}^{m} \overline{x}_{n-m+i}^-(r)(g_1(A_i) - g_1(A_{i+1})),
\]

(43)

\[
(C)\int_A \overline{X}_{n+1}^+(r)dg_1 = \sum_{i=1}^{m} \overline{x}_{n-m+i}^-(r)(g_1(A_i) - g_1(A_{i+1})),
\]

Furthermore,

\[
(C)\int_A \overline{X}_n^-(r)dg_1 = \left[ (C)\int_A \overline{X}_n^+(r)dg_1, (C)\int_A \overline{X}_{n+1}^+(r)dg_1, \ldots \right]^T.
\]

(42)

Thus, we have

\[
(C)\int_A \overline{X}_n^-(r)dg_1 = \left[ (C)\int_A \overline{X}_n^+(r)dg_1, (C)\int_A \overline{X}_{n+1}^+(r)dg_1, \ldots \right]^T.
\]

(44)
(2) According to Definition 7, we can obtain

\[
P \cdot (C) \int_A \overline{X}_n^* (r) \, dg_\lambda = \begin{bmatrix}
\sum_{i=1}^m \overline{x}_{n-m+i-1} (r) (g_\lambda (A_i) - g_\lambda (A_{i+1})) \\
\sum_{i=1}^m \overline{x}_{n-m+i+1} (r) (g_\lambda (A_i) - g_\lambda (A_{i+1})) \\
\vdots \\
\sum_{i=1}^m \overline{x}_{n-i} (r) (g_\lambda (A_i) - g_\lambda (A_{i+1}))
\end{bmatrix}.
\]

(46)

Then, by the expression of \((C) \int_A \overline{X}_{n-1} (r) \, dg_\lambda\) in (1), we have

\[
(C) \int_A \overline{X}_{n-1} (r) \, dg_\lambda = P \cdot (C) \int_A \overline{X}_n (r) \, dg_\lambda.
\]

(47)

(3) By (2), we know that

\[
(C) \int_A \overline{X}_{n-t} (r) \, dg_\lambda = P^{-t} \cdot (C) \int_A \overline{X}_n (r) \, dg_\lambda.
\]

(48)

Since \(P\) is an invertible matrix, we have

\[
(C) \int_A \overline{X}_{n-t} (r) \, dg_\lambda = P^{-t} \cdot (C) \int_A \overline{X}_n (r) \, dg_\lambda.
\]

(49)

(4) By using Theorem 2 in Reference [18], we note that \(\lim_{n \to \infty} P^n\) exists and

\[
\lim_{n \to \infty} P^n = \frac{ea^T}{a^T e} = \alpha r^T.
\]

(50)

Combining (3), it follows that

\[
\lim_{n \to \infty} P^n \cdot \frac{ea^T}{a^T e} = \alpha r^T.
\]

(51)

Taking limit of the above equation, we obtain

\[
\lim_{n \to \infty} (C) \int_A \overline{X}_n^* (r) \, dg_\lambda = \lim_{n \to \infty} P^n \cdot (C) \int_A \overline{X}_1 (r) \, dg_\lambda = \frac{ea^T}{a^T e} \cdot (C) \int_A \overline{X}_1 (r) \, dg_\lambda = \alpha r^T \cdot (C) \int_A \overline{X}_1 (r) \, dg_\lambda.
\]

(52)

The proof is complete. \(\square\)

Theorem 3. Let \((\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in \overline{E}^m, (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m,\) and \(g_\lambda\) be a fuzzy measure satisfying \(\delta - \lambda\) rules. Denote \(A_i = \{t_1, t_2, \ldots, t_m\}, i = 1, 2, \ldots, m,\) and \(A_{m+1} = \emptyset,\) and \(t\) be a positive real number. For vector

\[
\overline{X}_n^* (r) = [\overline{x}_1^* (r), \overline{x}_2^* (r), \ldots, \overline{x}_{m-1}^* (r)]^T,
\]

we have

\[
(1)
\]

(53)

(2)

(54)

(3)

(55)

especially, if \(g_\lambda (A_i) - g_\lambda (A_2) > 0\) and \(n - t > m,\) then
where \( e = \sum_{i=1}^{m} e_k = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m \times 1} \) and \( e_k \) is the \( i \)th standard unit column vector:

\[
ap = [a_1, a_2, \ldots, a_m]^T,
\]

\[
b = [b_1, b_2, \ldots, b_m]^T,
\]

\[
a_k = \sum_{i=1}^{k} (g_i(A_i) - g_i(A_{i+1})),
\]

\[
b_k = \frac{a^T e}{a^T e} \sum_{i=1}^{m} a_i = \frac{g(A_i) - g(A_{k+1})}{mg(A_i) - \sum_{i=2}^{m} g(A_i)}, \quad k = 1, 2, 3, \ldots, m.
\]

\[
\lim_{n \to \infty} \left( \frac{e a^T}{a^T e} \right) \int_A \chi_n^+(r) d\lambda = \lim_{n \to \infty} P^n \cdot \left( \frac{e a^T}{a^T e} \right) \int_A \chi_{n+1}^+(r) d\lambda
\]

\[
= \frac{e a^T}{a^T e} \cdot \left( \frac{e b^T}{b^T e} \right) \cdot \left( \frac{e a^T}{a^T e} \right) \int_A \chi_{n+1}^+(r) d\lambda.
\]

\[
\lim_{n \to \infty} \left( \frac{e a^T}{a^T e} \right) \int_A \chi_n^+(r) d\lambda = \lim_{n \to \infty} P^n \cdot \left( \frac{e a^T}{a^T e} \right) \int_A \chi_{n+1}^+(r) d\lambda
\]

\[
= \frac{e a^T}{a^T e} \cdot \left( \frac{e b^T}{b^T e} \right) \cdot \left( \frac{e a^T}{a^T e} \right) \int_A \chi_{n+1}^+(r) d\lambda.
\]

Proof. Theorem 1 implies.

\[
(C) \int_A \chi_n d\lambda = \left( \sum_{i=1}^{m} x_{n-m+i-1} - \delta_{n-m+i-1,1} \right) (g(A_i) - g(A_{i+1})),
\]

\[
\sum_{i=1}^{m} x_{n-m+i-1} (g(A_i) - g(A_{i+1})), \sum_{i=1}^{m} x_{n-m+i-1} + \delta_{n-m+i-1,2} \left( g(A_i) - g(A_{i+1}) \right),
\]

\[
\sum_{i=1}^{m} x_{n-m+i} (g(A_i) - g(A_{i+1})), \sum_{i=1}^{m} x_{n-m+i} + \delta_{n-m+i,2} \left( g(A_i) - g(A_{i+1}) \right),
\]

\[
\cdots,
\]

\[
\sum_{i=1}^{m} x_{n+2i-2} - \delta_{n+2i-2,1} \left( g(A_i) - g(A_{i+1}) \right), \sum_{i=1}^{m} x_{n+2i-2} + \delta_{n+2i-2,2} \left( g(A_i) - g(A_{i+1}) \right) \right]^{T}.
\]
\[ (2) \quad \int_A \bar{X}_n^{a} (r) dg_1 = \begin{bmatrix} \sum_{i=1}^{m} (\delta_{n-m+i-1} r + x_{n-m+i-1} - \delta_{n-m+i-1,1}) \\ \sum_{i=1}^{m} (\delta_{n-m+i} r + x_{n-m+i} - \delta_{n-m+i,1}) \\ \sum_{i=1}^{m} (\delta_{n-m+i+1} r + x_{n-m+i+1} - \delta_{n-m+i+1,1}) \\ \vdots \\ \sum_{i=1}^{m} (\delta_{n-m+i-2} r + x_{n-m+i-2} - \delta_{n-m+i-2,1}) \end{bmatrix}. \]

and by Remark 4 and Theorem 3, we have
\[ (C) \quad \int_A \bar{X}_n^{a} (r) dg_1 = \begin{bmatrix} \sum_{i=1}^{m} (\delta_{n-m+i-1} r + x_{n-m+i-1} - \delta_{n-m+i-1,1}) \\ \sum_{i=1}^{m} (\delta_{n-m+i} r + x_{n-m+i} - \delta_{n-m+i,1}) \\ \sum_{i=1}^{m} (\delta_{n-m+i+1} r + x_{n-m+i+1} - \delta_{n-m+i+1,1}) \\ \vdots \\ \sum_{i=1}^{m} (\delta_{n-m+i-2} r + x_{n-m+i-2} - \delta_{n-m+i-2,1}) \end{bmatrix}. \]

(3) (2) implies.

The proof is complete.

Example 1. We choose the same example in Reference [18] to illustrate our study and make comparison. Given a closing stock price system over 5 days, the closing prices of each day are denoted as \( \bar{x}_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_5 \) \( \in \mathbb{R}^5 \), and every \( \bar{x}_i \) is a triangle fuzzy number, \( \bar{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2}) \), \( i = 1, 2, \ldots, 5 \). Suppose \( (t_1, t_2, \ldots, t_5) \) \( \in \mathbb{R}^5 \), \( A_1 = \{t_1, t_2, \ldots, t_5\}, i = 1, 2, \ldots, 5, \) and \( A_6 = \emptyset \). The value and weight of each \( \bar{x}_i, i = 1, 2, \ldots, 5, \) are shown in Table 1. Then, we can obtain the closing stock price over 10 days and some relevant results. According to Remark 3 in Reference [18], we can obtain
\[
\begin{align*}
g_1 (A_1) &= 1, \\
g_1 (A_2) &= 0.88, \\
g_1 (A_3) &= 0.65, \\
g_1 (A_4) &= 0.33, \\
g_1 (A_5) &= 0.175, \\
g_1 (A_6) &= 0.
\end{align*}
\]

By Definition 6 and Remark 4, the Choquet integral of \( \bar{x}_6 \) with respect to fuzzy measure \( g_1 \) on \( A \) is determined as follows:
\[
\bar{x}_6 = (C) \int_A \bar{x}_6 dg_1 = (22.04, 23.04, 24.04).
\]

Similarly, we can also calculate the Choquet integral of \( \bar{x}_n, n = 7, 8, 9, 10, \) with respect to fuzzy measure \( g_1 \) on \( A \), as shown in Table 2.

And according to Definition 6 and Theorem 4, the Choquet integral of fuzzy number vector \( \bar{x}_6 = \left[ \bar{x}_{61}, \bar{x}_{71}, \ldots, \bar{x}_{101} \right]^{T} \) with respect to fuzzy measure \( g_1 \) on \( A \) is determined as follows:
\[
(C) \int_A \bar{x}_6 dg_1 = [(C) \int_A \bar{x}_{61} dg_1, (C) \int_A \bar{x}_{71} dg_1, \ldots, (C) \int_A \bar{x}_{101} dg_1]^{T} \\
= [(22.04, 23.04, 24.04), (22.76, 23.76, 24.76), \\
\quad (22.72, 23.72, 24.72), \\
\quad (22.5, 23.5, 24.5), (22.45, 23.45, 24.45)]^{T}.
\]
of fuzzy number vector, containing Choquet integral of fuzzy number and the Choquet integral in Reference [18], we introduce the new concepts: the work. More specifically, compared with our previous work fuzzy-number-valued function under discrete case in this numbers in [18] is transformed into Choquet integration of calculation of the moving average for a series of fuzzy as a generalization of the previous method [18]. kU_his the method presented in this article can be regarded as a generalization of the previous method [18]. That is, the calculation of the moving average for a series of fuzzy numbers in [18] is transformed into Choquet integration of fuzzy-number-valued function under discrete case in this work. More specifically, compared with our previous work in Reference [18], we introduce the new concepts: the Choquet integral of fuzzy number and the Choquet integral of fuzzy number vector, containing m elements needed to make forecasting of the m +1th element. These new concepts provide a possibility to dealing with the moving average from vector integral, which could describe the moving average of time series in a more intuitive perspective using an important mathematical tool.

Meanwhile, when the data degenerate into distinct data and the nonadditive measure degenerates into probability measure, our method will degenerate into the classical moving weighted average method. Therefore, this method is the extension of the classical method. In this paper, we consider the mutual influence and connection of time nodes, while in the classical method, time nodes are independent of each other. Moreover, the classical time series cannot deal with problems of natural language assignment, Internet language assignment, qualitative description, etc. So, the advantage of this method is obvious.

### 4. Conclusion

In this paper, on the combination of Choquet integral and fuzzy number, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. And it shows that the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Subsequently, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined, respectively. Finally, by means of the convolution formula of Choquet integral, some properties of the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are also investigated.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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