Good Reduction of Good Filtrations at Places

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Abstract

We consider filtered or graded algebras $A$ over a field $K$. Assume that there is a discrete valuation $O_v$ of $K$ with $m_v$ its maximal ideal and $k_v := O_v/m_v$ its residue field. Let $\Lambda$ be $O_v$-order such that $\Lambda K = A$ and $\overline{\Lambda} := k_v \otimes_{O_v} \Lambda$ the $\Lambda$-reduction of $A$ at the place $K \rightarrow k_v$. As in many examples of quantized algebras $A$ comes with a specific filtration that reduces well with respect to the valuation filtration defined by $\Lambda$ on $A$ and the reduction relates to the part of degree zero in the associated graded algebra. Hence several lifting properties follow from valuation like theory, also for modules with good filtrations.

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Introduction

One possible arithmetical aspect in the noncommutative geometry of associative algebras may be found in the construction of a noncommutative divisor theory based on noncommutative valuations, e.g. [13]. Reduction of algebras at such valuations have already been investigated in ([9], [13]). Typical algebras considered there are among others: rings of differential operators, certain quantum groups, quantized algebras and regular algebras in the sense of projective noncommutative algebraic geometry. These algebras have a natural gradation or filtration defined in terms of some finite dimensional vector spaces, e.g. the part of degree one is finite dimensional. In this note we study the reduction of

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the filtered or graded structures over a given valuation in the base field, \( K \) say. Its properties relate to certain lattices in the characteristic vector spaces hinted at above. For some filtration \( FA \) on a \( K \)-algebra \( A \) the unramifiedness property of a reduction relates to the induction of good filtrations (cf. \[8\]) in every \( F_n A \). Perhaps the main result in this context is the establishing of a lifting property for unramified reductions from the associated graded ring \( GF(A) \) to the filtered ring \( A \).

Several interesting classes of algebras may be studied via reduction techniques. The color Lie algebras and their enveloping algebras will be separately treated in forthcoming work. An important class of examples consists of generalized Weyl algebras (cf. \[4\]) or generalized crossed products (cf. \[6\]); this class contains popular algebras like : quantum deformed Weyl algebras, the quantum plane, quantum \( U_q(sl_2) \) of \( sl_2 \), the quantum Heisenberg algebra (cf. \[12\]), Witten’s first and Woronowicz’s deformation, the quantum group \( O_q^2 \) of \( so_3 \) (cf. \[17\]) etc... . For algebras in the foregoing class the extension of valuations on the base field to noncommutative valuations on their fields of fractions has been studied (cf. \[13\]) and several lifting results for regularity conditions as well as dimension calculations follow from the reduction properties.

As a general reference for detail on filtered rings and modules we refer to \[9\], full detail on graded ring theory may be found in (\[14\],\[15\]).

1 Preliminaries on Reductions, Filtrations and Gradations

Throughout \( A \) is an associative algebra over a commutative field \( K \). A \( \mathbb{Z} \)-filtration \( FA \) is given by an ascending family \( \{ F_n A, n \in \mathbb{Z} \} \) of additive subgroups such that \( F_n A F_m A \subset F_{n+m} A \) for all \( n, m \in \mathbb{Z}, 1 \in F_0 A \), and we always assume the filtration to be exhaustive, i.e. \( A = \bigcup_{n \in \mathbb{Z}} F_n A \), and separated, i.e. \( 0 = \bigcap_{n \in \mathbb{Z}} F_n A \). We say that \( A \) is a filtered \( \mathbb{K} \)-algebra of \( \mathbb{K} \subset F_0 A \), consequently all \( F_n A \) are \( \mathbb{K} \)-vector spaces. Following conventions and notation of (\[9\], \[14\]), we write \( GF(A) \) for the associated graded ring, or \( \mathbb{K} \)-algebra, with respect to \( F A \) and we let \( \tilde{A} \) be the Rees ring or blow-up ring, respectively \( \mathbb{K} \)-algebra. We write : \( GF(A) = \bigoplus_{n \in \mathbb{Z}} GF(A)_n \) with \( GF(A)_n = F_n A/F_{n-1} A \) for all \( n \in \mathbb{Z} \), \( \tilde{A} = \bigoplus_{n \in \mathbb{Z}} \tilde{A}_n \) with \( \tilde{A}_n = F_n A \) for \( n \in \mathbb{Z} \). It is practical to identify \( \tilde{A} \) with the graded subring \( \sum_{n \in \mathbb{Z}} F_n A T^n \) in \( A[T,T^{-1}] \) where \( T \) is a central variable of degree one. Recall that the so-called principle symbol map \( \sigma_F : A \to GF(A) \) is defined by mapping an \( a \in A \) such that \( a \in F_n A - F_{n-1} A \) to \( a \mod F_{n-1} A \) in \( GF(A)_n \); observe that \( \sigma_F \) is neither additive nor multiplicative in general. If no ambiguity can arise the subscript may be dropped in notation introduced above.

Zariskian filtrations on noncommutative rings have been characterized in several ways (cf. \[9\]) but in any case these filtrations have the property that \( A, GF(A) \)
and $\tilde{A}$ are (twosided) Noetherian rings.

A filtration on a $K$-algebra $A$ is said to be a **finite filtration** if $\dim_K F_n A$ is finite for all $n \in \mathbb{Z}$. Similarly, a graded algebra $R = \oplus_n R_n$ is said to be **finitely graded** if $\dim_K R_n$ is finite for all $n \in \mathbb{Z}$. Obviously, if $FA$ is finite then $G_F(A)$ and $A$ are both finitely graded; if $A$ is finitely graded then $FA$ is finite and $G_F(A)$ is finitely graded. If $G_F(A)$ is finitely graded then $FA$ is finite if and only if at least one $F_n A$ is finite dimensional over $K$. Typical graded algebras appearing in noncommutative projective geometry e.g. regular algebras as studied in [1], [2], are graded algebras appearing in noncommutative projective geometry e.g. regular $K$-algebras as studied in [1], [2], are graded.

Let us recall some definitions and facts concerning valuations of skewfields, the old book of O. Schilling is still a valid basic reference for the general theory, cf. [16]. A subring $\Lambda$ in a skewfield $\Delta$ is said to be a **valuation ring** of $\Delta$ if for every $x \in \Delta - \{0\}$ either $x$ or $x^{-1}$ is in $\Lambda$ and moreover $\Lambda$ is invariant under inner automorphisms of $\Delta$. The unique maximal ideal $P$ of $\Lambda$ given by $P = \{x \in \Lambda, x^{-1} \notin \Lambda\}$ defines the residue field (!) $\Lambda/P$ of $\Delta$; we often write $\Delta_v = \Lambda/P$ (sometimes $\overline{\Delta} = \Lambda/P$). A valuation ring $\Lambda$ of $\Delta$ is said to be **discrete** if $P$ is a principal ideal or equivalently $\Lambda$ is Noetherian and the value group is $\mathbb{Z}$. When $\Delta$ is a $K$-algebra and $\Lambda$ is a valuation ring of $\Delta$ then $\Lambda \cap K$ is a valuation ring of $K$; in case $K \subset \Lambda$ we say that $\Lambda$ is a $K$-valuation ring.

We write $O_v \subset K$ for a valuation ring of $K$ and denote its maximal ideal by $m_v$ and its residue field by $k_v = O_v/m_v$. From a valuation ring $O_v \subset K$ we derive a valuation function $v : K^* \to \Gamma$ for a suitable totally ordered abelian group; in the discrete case we are looking at $\Gamma = \mathbb{Z}$. To a noncommutative valuation ring $\Lambda$ in $\Delta$ we may also associate a valuation function $v : \Delta^* \to \Gamma$ where now $\Gamma$ is again totally ordered but not necessarily abelian. In some cases the abelian property of $\Gamma$ is enforced upon us, e.g. noncommutative valuation of the skewfield of the first Weyl algebra are necessarily having an abelian value group. In the sequel, unless otherwise stated, all valuations are supposed to be discrete e.g. in particular we only consider $\mathbb{Z}$-valuations. If $\Lambda$ is a noncommutative discrete valuation ring of $\Delta$ then we define a filtration $F^\nu \Delta$ on $\Delta$, called the **valuation filtrations**, by putting $F^0 \Delta = \Delta$. If $\Delta = O_v, \Delta = K$ then we write $f^\nu K$ for the valuation filtration of $K$. Observe that: $\deg_{F^\nu}(\delta) = -v(\delta)$ for $\delta \in \Delta$, $\deg_{F^\nu}(x) = -v(x)$ for $x \in K$. In the situation $K \subset \Delta$ and for a given valuation ring $\Lambda$ of $\Delta$ with valuation function $\nu$ the valuation ring $\Lambda \cap K$ of $K$ is the **induced valuation ring**, denoted by $O_v$. Of course $P \cap K = m_v$ but it is possible that $P^e \cap K = m_v$ for $e > 1$. Since $\cap_{n \in \mathbb{N}} P^n = 0$ it follows that there is a unique $e_v$ such that $\pi \in P^{e_v}$ but $\pi \notin P^{e_v+1}$ where $m_v = (\pi) \subset O_v$. This $e_v \in \mathbb{N}$ is called the **ramification index of $\Delta$ over $O_v$**. We easily check that $P^{e_v} \cap K = m_v^d$ where $d = \lceil \frac{n}{e} \rceil$ is the smallest integer bigger that or equal to $\frac{n}{e}$. This shows that $e_v$ is in fact the ramification of the valuation filtration $F^{\nu} \Delta$ over $f^{\nu} K$, i.e. $F^{\nu} \Delta \cap K = f^{d_v} K$ where $d_v = \lceil \frac{n}{e} \rceil$ as above. Whereas the $m_v$-adic filtration of $\Lambda$ obviously induces $f^{\nu} K$ or in fact the negative part of it viewed as a filtration on $\Lambda$, such statement is false for $F^\nu$ as noted before. In general
Lemma 1.1 With notation as before, $F^\nu \Delta$ induces in $K$ the scaled filtration with step $e$ associated to $f^\nu K$, where $e$ is in the ramification index of $\Lambda$ over $O_v$.

In the foregoing it is obvious that $m_v$ is contained in the Jacobson radical $J(\Lambda)$ of $\Lambda$; this is so because we assumed that $\Lambda$ is a valuation ring extending $O_v$, e.g. $P \cap K = m_v$. Observe however that for an arbitrary $O_v$-order $\Lambda$ in an infinite dimensional $\Delta$ over $K$ we need not have $m_v$ in the Jacobson radical of $\Lambda$, the latter may even be zero (verify for the Weyl algebra defined over $\mathbb{Z}_p$ as an order in the Weyl field over $Q$)!

Let us recall Proposition 3.1. from [3].

Proposition 1.2 Let $R$ be an Artinian ring with filtration $FR$, then the following statements are equivalent:

i) $G_F(R)$ is a domain

ii) $R$ is a skewfield and every nonzero homogeneous element of $G_F(R)$ is invertible i.e. $G_F(R)$ is a graded-skewfield.

iii) $R$ is a skewfield, $F_0R$ is a discrete valuation ring of $R$ with maximal ideal $F_{-1}R$ and $F_{-ne}R = (F_{-1}R)^n$ for some $e \in \mathbb{N}$.

2 Reductions of Gradations and Filtrations

Again we look either at (separated and exhaustive) filtered $K$-algebras $A$ with a subring $\Lambda$ such that $\Lambda \cap K = O_v$, or else at graded $K$-algebras $\tilde{A}$ with a subring $\tilde{\Lambda}$ that is a graded subring now such that $\tilde{\Lambda} \cap K = O_v$. In the sequel we shall only consider $O_v$-orders $\Lambda$, resp. $\tilde{\Lambda}$, such that $K\Lambda = A$, resp. $K\tilde{\Lambda} = \tilde{A}$. So we have the induced filtration $F\Lambda$ given by $F_n\Lambda = \Lambda \cap F_nA$, or the induced gradation $\Lambda_n = \Lambda \cap \tilde{A}_n$.

Observation 2.1 With notation as before:

i) $m_n^\nu \Lambda \cap F_nA = m_n^\nu (\Lambda \cap F_nA)$, for all $n \in \mathbb{Z}$, $a \in \mathbb{Z}$.

ii) $m_n^\nu \tilde{\Lambda} \cap \tilde{A}_n = m_n^\nu (\tilde{\Lambda} \cap \tilde{A}_n)$, for all $n \in \mathbb{Z}$, $a \in \mathbb{Z}$.

Proof. Let us establish i), the proof of ii) is similar.

i) The inclusion $m_n^\nu (\Lambda \cap F_nA) \subset m_n^\nu \Lambda \cap F_nA$ is trivial. Pick $z \in m_n^\nu \Lambda \cap F_nA$, i.e. $z = \pi^a \lambda$ for some $\lambda \in \Lambda$. Since $F_nA$ is a $K$-space $\lambda = \pi^{-a} Z \in F_nA \cap \Lambda$, hence $Z \in \pi^a (F_nA \cap \Lambda)$ and because $F_nA \cap \Lambda$ is an $O_v$-module the latter equals $m_n^\nu (F_nA \cap \Lambda)$. 

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In the situation as above we call \( \Lambda/m_v\Lambda \) the \( \Lambda \)-reduction of \( A \) at the place \( K \sim k_v \) (or at \( v \)); similarly \( \tilde{\Lambda}/m_v\tilde{\Lambda} \) is the \( \Lambda \)-reduction of \( \tilde{A} \) and it is clearly a graded \( k_v \)-algebra. Faithful to the notation of the residue field we write \( A_v = \Lambda/m_v\Lambda, \tilde{A}_v = \tilde{\Lambda}/m_v\tilde{\Lambda} \) and we write \( \pi : \Lambda \to A_v, \tilde{\pi} : \tilde{\Lambda} \to \tilde{A}_v \) for the corresponding canonical ring epimorphisms. From the observation i) it is clear that \( FA \) defines a filtration \( FA_v \) (in fact i expresses a compatibility relation between \( FA \) and \( F^vA \) defined by \( F^v_nA = m_v^{-n}\Lambda \) i.e. the \( m_v \)-adic \( \mathbb{Z} \)-filtration of \( A \) constructed from \( \Lambda \) given by \( F^v_nA_v = F_n\Lambda/m_vF_n\Lambda \). Moreover we have a graded subring \( G_F(\Lambda) \subset G_F(A) \) such that \( G_F(\Lambda) \cap K = O_v \) and \( KG_F(\Lambda) = G_F(A) \), so this defines an exhaustive (graded) filtration \( F^\Lambda G_F(A) \) by \( F^\Lambda_nG_F(A) = m_v^{-n}G_F(\Lambda) \) On the other hand \( G_F(\Lambda) = \Lambda \otimes_{\mathbb{Z}_v} k_v[t, t^{-1}] \approx A_v[t, t^{-1}] \) has a filtration induced by \( FA \), via \( FA \), let us denote it by \( \overline{FG}(\Lambda) \), then \( \overline{F^\Lambda_nG_F}(A) \cong (F^\Lambda_nA/m_vF^\Lambda_nA)[t, t^{-1}] \) (we may view this as an identification if we identify \( G_F(\Lambda) \) and \( A_v[t, t^{-1}] \). The compatibility between \( FA \) and \( F^vA \) actually establishes : \( G_F(\Lambda) = G_F(\Lambda) \)). In general we do not know that the filtration \( F^vA \) associated to \( \Lambda \) is separated, but when suitable finiteness conditions hold all filtrations constructed before will be separated. Let is first mention a different easy but sometimes interesting good case.

**Lemma 2.2** If \( \Lambda \) and \( A \) have no nonzero ideal in common then \( F^vA \) is separated. A similar statement holds with respect to \( \tilde{\Lambda} \) and \( \tilde{A} \) in the graded case.

**Proof.** Put \( E = \cap \{ n \in \mathbb{N}, m_v^n\Lambda \} \subset \Lambda \). If \( x \in E \) then \( \pi^{-n}x \in E \) for every \( n \in \mathbb{N} \), thus \( Kx \subset E \) and also \( Ax \subset E \), similar for \( A\tilde{x} \subset E \). This leads to a contradiction were \( x \neq 0 \).

**Corollary 2.3** If \( A \) is a simple \( K \)-algebra then \( F^\Lambda A \) is always separated, for every \( O_v \)-order \( \Lambda \).

**Proposition 2.4** With notation as before, if \( F^\Lambda G_F(A) \) is a separated filtration then \( F^\Lambda A \) is separated.

**Proof.** In view of Observation 2.1 we have that \( F^\Lambda_nG_F(A)_d = m_v^{-n}G_F(\Lambda)_d \) for all \( d \in \mathbb{Z} \), and \( m_v^{-n}G_F(\Lambda)_d = G_F(\Lambda)_d \cap m_v^{-n}G_F(\Lambda), G_F(m_v^{-n}G_F(\Lambda)) = m_v^{-n}G_F(\Lambda) \). Hence \( G_F(E) \subset \cap_{n \in \mathbb{N}} m_v^nG_F(\Lambda) = 0 \), the latter following from the assumed separateness of \( f^\Lambda G_F(A) \). Thus \( E \subset \cap_{n \in \mathbb{N}} F^\Lambda_nA \) but as \( FA \) is separated (that was a standing assumption throughout) it follows that \( E = 0 \), hence \( F^\Lambda A \) is separated too.

**Definition 2.5** We say that \( \Lambda \) is \( FA \)-finite if for all \( d \in \mathbb{Z}, \Lambda_d = \Lambda \cap F_dA \) is a finitely generated \( O_v \)-module. In the graded situation \( \tilde{\Lambda} \subset \tilde{A} \) we say that \( \tilde{\Lambda} \) is \( \tilde{A} \)-finite if \( \Lambda_d \cap \tilde{\Lambda} = \Lambda_d \), for all \( d \in \mathbb{Z} \), is a finitely generated \( O_v \)-module. For a finite dimensional vector space \( V \) over \( K \), an \( O_v \)-module \( M \) contained in \( V \) is said to be an \( O_v \)-lattice of \( V \) if \( \text{rank}_{O_v} M = \text{dim}_KV \). Any \( O_v \)-lattice \( M \) of \( V \) defines an unramified reduction \( V_v = M/m_vM = k_v \otimes_{O_v} M \) with \( \text{dim}_{k_v} V_v = \text{dim}_K V \).
Theorem 2.6 With notation and conventions as before:

1. If $G_F(\Lambda)$ is $G_F(A)$-finite then $F^vA$ is separated.

2. (a) If $\Lambda$ is $FA$-finite then $F^vG_F(A)$ and $F^vA$ are both separated filtrations. The restriction of $G_F(A)$ to $A_v = G_F(A)_{d0}$ denoted by $FA_v$, is given by $FA_v = F_nA/\pi F_nA$ which is an unramified reduction of $F_nA$. Moreover $G_F(\Lambda) = A_v[t, t^{-1}]$ and it has the residual filtration given by, $F_nA_v[t, t^{-1}]$ $n \in \mathbb{Z}$

(b) The filtration $F^vA$ induces a good filtration in $F_dA$ for every $d \in \mathbb{Z}$.

Proof. 1. If $G_F(\Lambda)$ is $G_F(A)$-finite then $G_F(A)$ is a finite graded $K$-algebra; since every $G_F(\Lambda)_d$, $d \in \mathbb{Z}$, is a finitely generated and torsion free $O_v$-module it is free of rank $n_d$. Now $E_{gr} = \cap_{n \in \mathbb{N}} m^n_GF(\Lambda)$ is a common graded ideal of $G_F(\Lambda)$ and $G_F(A)$ with $R_{gr,d} \subset G_F(\Lambda)_d$, the latter free of finite rank $n_d$ over $O_v$. As observed for $E$ earlier, also for $E_{gr}$ we do have that $\pi E_{gr} = E_{gr}$ and since $\pi \in G_F(\Lambda)_0$ we also have $\pi E_{gr,d} = E_{gr,d}$ for every $d \in \mathbb{Z}$. Since now we are dealing with finitely generated $O_v$-modules Nakayama’s lemma yields that $E_{gr,d} = 0$ for all $d \in \mathbb{Z}$, hence $E_{gr} = 0$ or $f^vG_F(A)$ is separated. Foregoing proposition 2.a then yields that $F^vA$ is separated.

2.a. If $\Lambda$ is $FA$-finite then $G_F(\Lambda)$ is $G_F(A)$-finite, hence $f^vG_F(A)$ and $F^vA$ are both separated. In view of the finiteness assumption $F_n\Lambda$, for every $d \in \mathbb{Z}$, is an $O_v$-lattice hence a free $O_v$-module of rank $n_d$ say; then $F_d\Lambda/m_\pi F_n\Lambda$ is a $k_v$-vector space of dimension $n_d$, thus $F_nA_v$ is indeed an unramified reduction of $F_nA$. The remaining claims are just reformulations of earlier observations.

2.b. Recall that for a filtered modules $M$, with filtration $FM$, over the filtered ring $A$ we say that $FM$ is a good filtration if there is a finite set $m_1, \ldots, m_s$ in $M$ such that for every $n \in \mathbb{Z}$ we have that $F_nM = \sum_{i=1}^s F_{n-d_i}A.m_i$, for some fixed $d_1, \ldots, d_s$ in $\mathbb{Z}$. Now from $\Lambda m^n_p \cap F_nA = m^n_p F_n\Lambda$ for all $n, p$, it is clear that the filtration induced in $F_nA$ is good (viewed as a filtered $f^vK$-module). Indeed it suffices to pick an $O_v$-basis for the free $O_v$-module $F_n\Lambda$ for the $m_i$ and take each $d_i$ to be zero, then the only way to express an element of $m_vF_n\Lambda$ in the selected basis is by taking coefficients from $m_v$.

Now we look at a graded $O_v$-order $\tilde{\Lambda}$ in $\tilde{\Lambda}$ as before and we assume that $\tilde{\Lambda}$ contains a central regular homogeneous element of degree one, $T$ say. Put $A = \tilde{\Lambda}/\tilde{\Lambda}(T-1)$, $\Lambda = \tilde{\Lambda}/\tilde{\Lambda}(T-1)$; then $A$ has a filtration $FA$ given by $F_nA = \tilde{\Lambda}_n/\tilde{\Lambda}(T-1)\cap \tilde{\Lambda}_n$, and $\Lambda$ has a filtration $FA$ given by $F_n\Lambda = \tilde{\Lambda}_n/\tilde{\Lambda}(T-1)\cap \tilde{\Lambda}_n$, for all $n \in \mathbb{Z}$.

Lemma 2.7 With notation as before we obtain:

i) $\tilde{\Lambda}(T-1) \cap \tilde{\Lambda} = \tilde{\Lambda}(T-1)$

ii) $F_n\Lambda = F_nA \cap \Lambda = \tilde{\Lambda}_n/(T-1)\tilde{\Lambda} \cap \tilde{\Lambda}_n$. 

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Proof. i) Obviously \( \Lambda(T - 1) \subset \widetilde{A}(T - 1) \cap \Lambda \). For the converse look at \( \widetilde{z}(T - 1) \subset \Lambda \) with \( \widetilde{z} \in \Lambda \). If \( \widetilde{z} \not\in \Lambda \) then there is a \( d \in \mathbb{Z} \) minimal such that \( \widetilde{z}_d \not\in \Lambda_d \) while on the other hand: \( \widetilde{z}_{d-1}T - \widetilde{z}_d \in \Lambda_d \). In case \( \widetilde{z}_{n-1} \neq 0 \), then the foregoing entails that \( \widetilde{z}_{d-1}T \not\in \Lambda_d \) and thus \( \widetilde{z}_{d-1} \not\in \Lambda_{d-1} \) because \( T \in \Lambda_1 \) but that contradicts minimality of \( d \). Hence \( \widetilde{z}_{d-2} = 0 \), if \( \widetilde{z}_{d-2} \neq 0 \) then \( \widetilde{z}_{d-2}T^2 - \widetilde{z}_d \in \Lambda \), thus as in the first part \( \widetilde{z}_{d-2}T^2 \not\in \Lambda_d \) and certainly \( \widetilde{z}_{d-2} \not\in \Lambda_{d-2} \) again contradicting minimality of \( d \). So we are in the situation where \( \widetilde{z}_d \) is the homogeneous part of lowest degree in the decomposition of \( \widetilde{z} \). They from \( \widetilde{z}(T - 1) \in \Lambda \) we obtain that the homogeneous part of lowest degree in the decomposition of \( \widetilde{z}(T - 1) \), and that is exactly \(-\widetilde{z}_d \), must be in \( \Lambda \) and that leads to a contradiction. Consequently \( \widetilde{z}(T - 1) \in \Lambda \) leads to \( \widetilde{z} \in \Lambda \) and the claim i. follows.

ii) From i) it is clear that \( \Lambda \subset A \) and \( F_n\Lambda \subset F_nA \) for all \( n \in \mathbb{Z} \). If \( a_n \in F_nA \cap \Lambda \) then there exists a \( \bar{\lambda} \in \bar{\Lambda} \) such that \( \bar{\lambda} \mod (\bar{\Lambda} \cap \bar{\Lambda}(T - 1)) = a_n \) but also there is an \( \bar{a}_n \in \bar{\Lambda} \) such that \( \bar{a}_n \mod (\bar{\Lambda} \cap \bar{\Lambda}(T - 1)) = a_n \). Thus \( \bar{a}_n + \bar{b}(T - 1) = \bar{\lambda} \) for some \( \bar{b} \in \bar{\Lambda} \), yields : \((*)\) \( \lambda_n = \bar{a}_n + \bar{b}_{n-1}T - \bar{b}_n \). Also \( \lambda_{n-1} = \bar{b}_{n-2}T - \bar{b}_{n-1}T - \bar{b}_{n-1}T \). Substituting \((*)\) then leads to : \( \bar{a}_n + \bar{b}_{n-1}T = \lambda_{n-1} + \lambda_{n-1}T = \bar{a}_n + \bar{b}_{n-1}T \). If \( \bar{b}_{n-2} \neq 0 \) then we look at \( \lambda_{n-2} = \bar{b}_{n-3}T - \bar{b}_{n-2}T \) and arrive at \( \lambda_n + \lambda_{n-2}T = \bar{b}_{n-3}T^3 \). We repeat this procedure until we obtain \( \bar{a}_n = \lambda_n + \lambda_{n-1}T + \ldots \lambda_{n-d}T^d \) and thus \( \bar{a}_n \in \bar{\Lambda} \). From i. again it is clear that \( a_n \in F_n\Lambda \) follows, and the second equality of ii. also follows.

Returning to the situation of \( \Lambda \in A \) with filtration \( FA \) inducing \( \Lambda A \), then the Rees ring (blow-up ring) of \( A \) with respect to \( FA \), resp. \( \Lambda \) with respect to \( FA \), will be denoted by \( \bar{A} \), resp. \( \bar{\Lambda} \). Applying the foregoing lemma to these graded rings we recover the filtered situation from the Rees ring situation. Now the general theory of filtered rings yields that \( \bar{A}/\bar{T}A \cong G_F(\Lambda) \), \( \bar{\Lambda}/\bar{T}(\Lambda) = G_F(\Lambda) \), with the graduation of \( A \), resp. \( \Lambda \), defining the gradation of \( G_F(\Lambda) \), resp. \( G_F(\Lambda) \). Let us write \( \bar{F}_n \) for the graded filtration of \( \bar{A} \) defined by \( \bar{F}_n = \cap_{n \in \mathbb{N}} m_n^{-n} \bar{A} \). The filtration \( FA \) resp. \( FA \), corresponds to the \( T \)-adic filtration on \( \bar{A} \), resp. \( \bar{\Lambda} \) cf. \([9]\). We obtain the following extension of Proposition\([23]\).

Lemma 2.8 If \( \cap n \Lambda T^n = 0 \), then if \( f^vG_F(\Lambda) \) is separated , recall \( f^v_nG_F(\Lambda) = m_n^{-n}G_F(\Lambda) \) for \( n \in \mathbb{Z} \), then \( \bar{F}_n \bar{A} \) is separated too.

Proof. Put \( \bar{E} = \cap_{n \in \mathbb{Z}} \bar{F}_n \bar{A} = \cap_{n \in \mathbb{N}} m_n^{-n} \bar{A} \). Then (compare to Proposition\([23]\), first part of proof) : \( \bar{E}_{\text{mod} TA} \subset \cap_{n \in \mathbb{N}} \pi^n \Lambda / \Lambda T \Lambda = 0 \), i.e. \( \bar{E} \subset \bar{T} \Lambda \). Pick \( \bar{e} \in \bar{E} \), then we have \( \bar{e} = \bar{T} \Lambda \) but also from \( \pi \bar{E} = \bar{E} \) it follows that \( \bar{e} = \pi \bar{x} \) for some \( \bar{x} \in \bar{E} \) i.e. \( \bar{x} = T \bar{y} \) for some \( \bar{y} \in \bar{E} \). Consequently : \( \pi \bar{x} = \pi T \bar{y} = \bar{T} \lambda \) and since \( T \) is regular in \( \Lambda \), \( \pi \bar{y} = \lambda \) holds i.e. \( \lambda \in \Lambda \pi \). From \( e = T \pi \lambda_1 \) it follows that \( T \lambda_1 = \bar{e} \in \bar{E} \) (note : \( \pi^{-1} \bar{E} = \bar{E} \)). Repetition of the foregoing argument leads to \( \lambda_1 \in \Lambda \pi \) etc... until we arrive at \( \bar{e} = \pi \bar{x} = T \bar{y} \) with \( \bar{y} \in \bar{E} \), while \( e = T^2 \lambda_1 \)
follows from $\tilde{\mu} = T\tilde{\lambda}_1$ for some $\tilde{\lambda}_1 \in \tilde{E}$, etc... Finally we obtain:

$$\tilde{e} = T\tilde{\lambda} = T^2\tilde{\lambda}_1 = T^3\tilde{\lambda}_2 = \ldots = T^n\tilde{\lambda}_n = \ldots \in \cap_{n \in \mathbb{N}} T^n\tilde{\lambda} = 0$$

The proof is thus finished as $E = 0$ follows. \hfill\Box

The Rees ring of the valuation filtration $f^vK$ is $\tilde{K} = O_v[\pi t^{-1}, \pi^{-1}t]$, where we now write $t$ for the regular homogeneous element of degree one in $\tilde{K}$. The ring $\tilde{K}$ is in fact a gr-valuation ring in the field $K(t)$ of rational functions in $T$. When calculating the Rees ring of $A$ with respect to $F^vA$, $A^{(v)}$ say, we may take $T = t \in \tilde{A}_1^{(v)}$ and moreover $A^{(v)}$ is a $\tilde{K}$-algebra and it is strongly graded (recall that a graded ring $R$ is strongly graded if $R_n R_{-n} = R_0$ for all $n \in \mathbb{Z}$, equivalently when $R_1R_{-1} = R_0$). Note that the Rees ring of $A$ with respect to $FA$ need not have $t$ in degree one, in fact one has to use another $T \in \tilde{A}_1$ which relates to $t$ in some specific way reflecting the ramification of $FA$ over $f^vK$.

In particular $\tilde{A}$ is not necessarily strongly graded (but it contains a strongly $e\mathbb{Z}$-graded subring where $e$ is the ramification index of $FA$ over $f^vK$). We have a Rees version of Theorem 2.6.

**Proposition 2.9** If $\Lambda$ is $FA$-finite then $\tilde{\Lambda}$, respectively $\Lambda$, are defined with respect to $F\Lambda$, respectively $FA$; the converse holds too. Any of the aforementioned properties entails that $G_F(\Lambda)$ is $G_F(A)$ finite.

If $G_F(\Lambda)$ is $G_F(A)$-finite then both $\tilde{F}^{(v)}$ and $F^vA$ are separated and so is $f^vG_F(A)$.

Conversely if $\tilde{\Lambda} \subset \tilde{A}$ are given graded rings having a regular central homogeneous degree of element one $T \in \tilde{A}_1 \subset \tilde{A}_1$, then $A = \tilde{A}/(-T)\tilde{A}, \Lambda = \tilde{\Lambda} / (T)\tilde{\Lambda}$ have filtrations $FA$, resp. $FA$ such that $\tilde{\Lambda}$, respectively $\tilde{A}$, are indeed the Rees rings with respect to those filtrations $F\Lambda$, respectively $FA$, and moreover $G_F(\Lambda) = \tilde{A}/T\tilde{A}, G_F(A) = \tilde{A}/T\tilde{A}$.

The statements concerning (unramified) reductions as in Theorem 2.6 shift from filtered to Rees level or back.

**Proof.** All statements are consequence of earlier observations and results; let us just point out that the property in Theorem 2.6, 2.6.b., i.e. $F^vA$ inducing a good filtration in $F_dA$, for every $d \in \mathbb{Z}$, viewed as a filtered $K$-module with respect to $f^vK$, is just the finite generation property for the Rees module of $F_dA$ with respect to $\tilde{K}$ which is exactly $\tilde{K} \otimes_{O_v} F_d\Lambda = \tilde{F}_d\Lambda^{(v)}$. \hfill\Box

Let us finish this section by mentioning some further remarks about strong filtrations, relating to valuations. In general a filtration on a ring $A$ is said to be a strong filtration if $F_n A F_m A = F_{n+m} A$ for all $n, m \in \mathbb{Z}$, equivalently if $G_F(A)$ is a strongly graded ring, i.e. $G_F(A)_n G_F(A)_m = G_F(A)_{n+m}$. By definition $F^vA$ is a strong filtration.

**Lemma 2.10** Let $A$ and $\Lambda$ be as before and consider an Ore set (left, right, left and right) $S$ in $A$, then there is an Ore set $S_\Lambda$ in $\Lambda$ (left, right, let and right) such that $S_\Lambda^{-1}A = S^{-1}A$. 

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Proof. Take $S_\Lambda = K^* S \cap \Lambda, K^* = K - \{0\}$. It is straightforward to check the (left, right, left and right) Ore conditions in $\Lambda$ for $S_\Lambda$ and (obviously) $S_\Lambda^{-1} A = S^{-1} A$.

Corollary 2.11 The localized filtration derived from $F^n A$ is exactly the localized filtration derived from the $m_v$-adic filtration of $\Lambda$; it is a strong filtration denoted by $F^v S^{-1} A$.

Let us say that $A$ is an order in an Artinian ring if its set of regular elements $S_0$ is a left and right Ore set such that $S_0^{-1} A$ is Artinian (one sided left or right statements may be formulated similarly).

Proposition 2.12 Let $A$ be an order in an Artinian ring $Q = S_0^{-1} A$ and $\Lambda \subset A$ as before. If $A_v$ is a domain then $Q$ is a skewfield and $F^v A$ extends to a strong filtration of $Q$ such that $F^v_v Q$ is a valuation ring of $Q$ extending $v$ from $K$ to $Q$. Then in the situation where $\Lambda \subset A$ are graded, $A$ a graded $K$-algebra such that each $A_n$ has finite $R$-dimension, the set of homogeneous elements of $A$ is an Ore set too and the graded ring of fractions $Q^g$ is a gr-skewfield with $F^v Q$ inducing a graded valuation in $Q^g$.

Proof. Since $A_v$ is a domain and $G_{F^v}(A) = A_v[t, t^{-1}], \sigma_v(S_0)$ is a multiplicatively closed set, where $\sigma_v$ is the principal symbol map for $F^n A$, and in fact $\sigma_v$ is a multiplicative map. It follows that :

$$G_{F^v}(S_0^{-1} A) = \sigma_0(S_0)^{-1} G_{F^v}(A) = O_{cl}(A_v)[t, t^{-1}]$$

the latter equality holds because the gradation is strong, hence localization happens completely in degree zero. Since the latter ring is a domain we may apply Observation 2.11 and following to $S_0^{-1} A$. Note that in the finite case we have $e = 1$ because $\pi^n \Lambda \cap A_m = \pi^n \Lambda_m \neq \pi^{n'} \Lambda_m$ for $n \neq n'$ in view of the Nakayama lemma.

In general for given $A, FA$ (or $\tilde{A}$) the construction of $\Lambda, FA$ such that $\Lambda$ is $FA$-finite is not so easy, this problem is related to the existence of discrete valuations having certain unramifiedness properties. In case the algebra is given by a finite number of generators and finitely many relations between these, properties of so-called good reduction will allow certain constructions of suitable $O_v$-orders.

3 Positively Graded Connected Algebras

A connected positively graded $K$-algebra is given as $A = K \oplus A_1 \oplus A_2 \oplus \ldots$, where $A_1$ is a finitely dimensional $K$-vector space $A$ is generated as a $K$-algebra by $A_1$. We may view $A$ as a $K$-algebra given by generators and homogeneous relations as follows :

$$0 \rightarrow R \rightarrow K\langle X \rangle \xrightarrow{\pi} A \rightarrow 0$$

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where \( K(\underline{X}) \) is the free \( K \)-algebra on \( \underline{X} = \{X_1, \ldots, X_n\} \) and \( \pi \) is defined by \( \pi(X_i) = a_i \) where \( \{a_1, \ldots, a_n\} \) is a preselected \( K \)-basis of \( A_1 \). The ideal \( \mathcal{R} \) is the ideal of relations. By restricting \( \pi \) to \( O_v(\underline{X}) \) we obtain a graded subring \( \Lambda \) of \( A \) with \( \Lambda_0 = O_v \) as follows:

\[
0 \rightarrow \mathcal{R} \cap O_v(\underline{X}) \rightarrow O_v(\underline{X}) \rightarrow \Lambda \rightarrow 0
\]

It is clear that \( \pi \) maps \( m_v(\underline{X}) \) to \( m_v \Lambda \) which is a graded ideal of \( \Lambda \). So we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{R}_v & \rightarrow & k_v(\underline{X}) & \rightarrow & A_v & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{R} \cap O_v(\underline{X}) & \rightarrow & O_v(\underline{X}) & \rightarrow & \Lambda & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{R} & \rightarrow & K(\underline{X}) & \rightarrow & A & \rightarrow & 0
\end{array}
\]

where \( A_v = \Lambda/m_v \Lambda \) as usual and \( \mathcal{R}_v = \mathcal{R} \cap O_v(\underline{X}) + m_v(\underline{X})/m_v(\underline{X}) \). When \( \mathcal{R} \) is generated by \( \{p_1(\underline{X}), \ldots, p_d(\underline{X})\} \) as a two-sided ideal then, without loss of generality, we may assume that \( p_1(\underline{X}) \in O_v(\underline{X}) \) (up to multiplying by a suitable constant) but such that not all of them are in \( m_v(\underline{X}) \). However the foregoing does not imply that \( \mathcal{R} \cap O_v(\underline{X}) = O_v(\underline{X})p_1(\underline{X}) + \ldots + O_v(\underline{X})p_d(\underline{X}) \), nor that \( \mathcal{R}_v = k_v(\underline{X})p_1(\underline{X}) + \ldots + k_v(\underline{X})p_d(\underline{X}) \), where \( p_i(\underline{X}) \) is the image of \( p_i(\underline{X}) \) under reduction.

**Definition 3.1** With conventions and notation as before, we say that \( \mathcal{R} \) (or \( A \)) reduces well or that \( \Lambda \) defines a good reduction of \( A \) whenever \( \mathcal{R}_v \) is generated by the residues \( p_i(\underline{X}), i = 1, \ldots, d \), i.e. whenever \( \mathcal{R} \cap O_v(\underline{X}) \) is generated by the \( p_i(\underline{X}), i = 1, \ldots, d \), as a two-sided ideal of \( O_v(\underline{X}) \). Since \( \pi \) is a graded morphism, the fact that \( \dim_K A_1 = n \) entails that \( \mathcal{R}_\infty = 0 \); it follows that \( \dim_K (\Lambda/m_v \Lambda)_1 = n \) but \( \dim_K A_n \) and \( \dim_K A_{v,n} \) may be different from \( n > 1 \).

**Proposition 3.2** Let \( A = K[A_1] \) be a connected affine prime finite graded \( K \)-algebra and \( \pi : K(\underline{X}) \rightarrow A \) a presentation of the \( K \)-algebra \( A \) as in the above diagram. Then \( F^n A \) is separated and \( \cap \) is \( FA \)-finite if and only if for all \( n \in \mathbb{N} \), \( \dim_K A_n = \dim_K A_{v,n} \). Moreover, if \( A_v \) is a Goldie domain then \( A \) is a domain and the \( m_v \Lambda \)-adic filtration \( F^n A \) is induced by a valuation filtration on the skewfield of microfractions \( \Lambda \) of \( A \).

**Proof.** From Proposition 2.10 we retain that \( \Lambda \) is \( FA \)-finite. Suppose that \( A_v \) is a domain then we claim that \( \Lambda \) and \( A \) are domains too (we cannot use Proposition 2.12 here because here \( A \) is not necessarily an order in a semisimple Artinian ring, in other words the Goldie ring property does not follow from our assumptions unless we start from a Noetherian \( A \)). It will be sufficient to check that there are no homogeneous zero-divisors. Take \( a \in \Lambda_\alpha, b \in \Lambda_\alpha \) such that
When considering a filtered $K$-algebra $A$ with a finite filtration $FA$, we observe that there is an $n_0 \in \mathbb{Z}$ such that for $n \leq n_0$, $F_n A = F_{n_0} A$. Since we restricted attention to separated filtrations this means that $F_n A = 0$ for all $n \leq n_0$ i.e. the filtration is left limited, $F_{-1} A$ is a nilpotent ideal of $F_0 A$. Therefore, when dealing with finite filtrations, it is not really restrictive to restrict attention to positively filtered rings as we will do. Moreover when domains have to be considered, $F_0 A$ will be an algebraic field extension of $K$ and so $O_v$ may be replaced by a discrete valuation ring of $F_0 A$ lying over $O_v \subset K$. In other words we are lead to consider the case of a positively filtered domain $K = F_0 A \subset \ldots \subset F_n A \subset \ldots \subset A$ and a discrete valuation ring $O_v$ of $K$ with an $O_v$-order $\Lambda$ in $A$ such that $\Lambda \cap K = O_v$, $K \Lambda = A$, equipped with the induced filtration $FA$ and $G_F(\Lambda) \subset G_F(A)$ a graded $O_v$-order. In the “positive” situation we have the following lifting result.

**Proposition 3.3** If $G_F(\Lambda)$ is $G_F(A)$-finite then $\Lambda$ is $FA$-finite.

**Proof.** One easily establishes that $rk(F_q \Lambda) = \dim_K F_q A$ by induction on $q$. The case $q = 0$ is trivial enough. Assume that the equality holds for $q - 1$. From $F_q \Lambda / F_{q-1} \Lambda = G(\Lambda)_q$ it follows that:

$$rk(F_q \Lambda) = rk(F_{q-1} \Lambda) + \dim_K G(\Lambda)_q = \dim_K (F_{q-1} A) + \dim_K (G(\Lambda)_q).$$

$$= \dim_K (F_q A).$$
Corollary 3.4 Under the hypothesis of the foregoing Proposition 3.3, the results of Theorem 2.6 are valid; in particular \( G_fvG_f(A) = G(A)v[t, t^{-1}] = G_fG_f(A) \) where \( f_nG_f(A) = F_nA_v[t, t^{-1}] \) is the filtration induced by \( F \) in \( G_f(A) \) (this is a version of a general compatibility result for arbitrary filtrations, cf. (13), Proposition 2.4).

Proposition 3.5 If \( G(A)_v \) is a domain then also \( G_F(A) \), \( G_Fv(A) \) and \( A \) are domains.

Proof. Easy from the compatibility result for filtrations applied to \( F_v \) and \( F \), i.e. \( G_fvG_f(A) = G_fG_f(A) \).

Corollary 3.6 If \( G_F(A) \) is \( G_F(A) \)-finite then:

1) \( \dim_k(A_{v,n}) = \text{rk} G_F(A)_n = \dim_K G_F(A)_n \)

2) \( \sum_{m=1}^n \dim_k(A_{v,m}) = \sum_{m=1}^n \text{rk} G_F(A)_m \text{rk}(F_n A) = \dim_K F_n A = \sum_{m=1}^n \dim_K(G(A)_m). \)

Assuming that \( A = K[F_1A] \) then \( A \) may be obtained as an epimorphic image of the free \( K \)-algebra \( K\langle X \rangle \) in \( \dim_K F_1A \)-letters, say \( X_1, \ldots, X_d \), letting \( \{x_1, \ldots, x_d\} \) be a \( K \)-basis for \( F_1A \).

\[ \pi : K\langle X_1, \ldots, X_d \rangle \to A, \quad X_i \mapsto x_i, \quad i = 1, \ldots, d \]

The filtration on \( K\langle X_1, \ldots, X_d \rangle \) is the degree filtration and this makes \( \pi \) a strict filtered morphism in the sense of [9]. Writing \( \mathcal{R} = \text{Ker} \pi \), we have a strict exact sequence of filtered objects:

\[ (\star) \quad 0 \to \mathcal{R} \to K\langle X_1, \ldots, X_d \rangle \to A \to 0 \]

Strict exactness of \((\star)\) entails that by passing to Rees objects one obtains an exact sequence of graded \( K\langle X_1, \ldots, X_d \rangle \)-modules:

\[ 0 \to \tilde{\mathcal{R}} \to K\langle X_1, \ldots, X_d \rangle^\sim \to \tilde{A} \to 0 \]

Again from strict exactness it follows that \( G_A(A) = G_F(A) \) where \( G_A(A) \) is the associated graded of a \( A \) as a filtered \( \mathcal{F} \)-module, writing \( \mathcal{F} = K\langle X_1, \ldots, X_d \rangle \). From \((\star)\) we thus derive an exact sequence in \( G(\mathcal{F}) \)-gr:

\[ 0 \to G(\mathcal{R}) \to G(\mathcal{F}) \to G(A) \to 0 \]

The filtration on \( \mathcal{F} \) is exactly the gradation filtration it follows that \( G(\mathcal{F}) \cong \mathcal{F} \) and under this isomorphism \( G(\mathcal{R}) \) corresponds to the ideal \( \mathcal{R} \) in \( \mathcal{F} \) being the graded ideal generated by the homogeneous components of highest degree in the homogeneous decompositions of elements of \( \mathcal{R} \). The following is a version of Theorem 2.13 in [13].

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Theorem 3.7 Assume that $A$ is given by finitely many generators and relations, where $F\,A$ is a described before Proposition 3.3. If $G_F(A)$, which is a connected positively graded $R$-algebra, reduces well with respect to $O_v$ then $G_F(A) = F = F/\mathcal{R}$, where $\mathcal{R}$ is generated as a two-sided ideal by $p_1(X), \ldots, p_s(X)$ having as homogeneous parts having highest degree $q_1(X), \ldots, q_s(X)$ that generate $\mathcal{R}$ as a two-sided ideal. Moreover $A$ reduces well at $O_v$, in other words:

$$\mathcal{R} \cap O_v(X) = \sum_i O_v(X)p_i(X)O_v(X)$$

and $A_v$ is defined by the relations $p_1^v(X), \ldots, p_s^v(X)$.

Foregoing theorem completes information about lifting properties of $G_F(A)$ to $A$ connected to the existence of valuation rings extending $O_v$ in either $Q_{cl}(A)$ if this exists (Noetherian or Goldie ring situation) or else in a corresponding micro-localization $Q^\mu_{cl}(A)$. The finiteness properties with respect to $\Lambda$ then provide the unramifiedness of the extension of the valuation. The latter unramified situation has been observed in several independent interesting examples e.g.:

i) $\Delta(g) = Q_{cl}(U(g))$ for finite dimensional Lie algebras, Weyl algebras $A_n(C)$, cf. [19].

ii) Sklyanin algebras, cf. [13].

iii) Generalized gauge algebras including Witten algebras, cf. [9]. The problem of finding an extending noncommutative valuation has been reduced to finding an $O_v$-order in an associated graded algebra having the finiteness property we discussed and having a domain for its reduction.

4 Another Example: Generalization Weyl Algebras

A generalized crossed product $A$ is a $\mathbb{Z}$-graded ring such that $A = A_0 v_1$ is a free left $A_0$-module of rank one, and $v_0 = l_A$ identifying $A_0$ as the subring $A_0 1_A$ in $A$. Multiplication of $A$ is defined by:

$$a v_i b v_j = a \sigma^i(b)c(i, j)v_{i+j} \text{ for } i, j \in \mathbb{Z}, a \text{ and } b \in A_0$$

where $\sigma$ is an automorphism of $A_0$ and $c : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}(A_0)$ is a 2-cocycle satisfying:

$$c(i, j)c(i + j, k) = \sigma^i(c(j, k))c(i, j + k) \text{ for } i, j, k \in \mathbb{Z}. A \text{ generalized Weyl algebra in the sense of } ([11], [14], [15], [17]), \text{ is as before but now letting } A \text{ be generated over } A_0 \text{ by two indeterminates } X = v_1 \text{ and } Y = v_{-1} \text{ such that :}$$

$$Xa = \sigma(a)X, \quad Ya = \sigma^{-1}(a)Y \text{ for } a \in A_0 \quad YX = a, \quad XY = \sigma(a)$$
If $A_0 = D$ is a commutative ring, e.g. a Dedekind domain, then these rings have now been extensively studied. Even over a Dedekind domain the class of generalized Weyl algebras contains many popular algebras: the first Weyl algebra and its quantum deformation, the quantum plane, the quantum 2-dimensional sphere, $U(sl_2)$ and its quantum version $U_q(sl_2)$ Witten’s first deformation and Woronowicz’s deformation, the quantum Heisemberg algebra, the Virasaro algebra. We write $D(a, \sigma)$ for generalized Weyl algebra as above with $A_0 = D$.

We now consider $K \subset D$ a fixed base field invariant under $\sigma$. Write $D(\sigma)$ for the invariant algebra with respect to $\sigma$. We may restrict attention to affine $K$-algebra $D$ but the results can be generalized to the consideration of Noetherian integrally closed domains (localization at height one ideals then yields discrete valuation rings). Localizing $D(\sigma, a)$ at $D - \{0\}$ yields $K \otimes_D D(\sigma, a) \simeq K[t, t^{-1}\sigma]$, where $K = Q_\mathfrak{d}(D)$, $\sigma$ the extended automorphisms of $K$.

**Lemma 4.1** If $P$ is a $\sigma$-invariant prime ideal of $D$ then $D(\sigma, a) P$ is a two-sided ideal such that $D(\sigma, a)/D(\sigma, a) P$ is of type $\mathcal{D}(\sigma, \pi) P$ where $\mathcal{D} = D/P, \sigma$ is induced by $\sigma$ on $\mathcal{D}$ and $\pi = a \mod P$ ($\pi = 0$ is allowed).

**Proof.** The $\sigma$-invariance of $P$ yields that $D(\sigma, a) P$ is two-sided. Since $D$ is Dedekind, $\mathcal{D}$ is a field. If $a \notin P$ then $\mathcal{D}(\sigma, \pi)$ is again a generalized Weyl algebra and a domain. If $a \in P$, then $\mathcal{D}(\sigma, a)$ is not a domain. If $D = O_v \subset K$ the maximal ideal is necessarily $\sigma$-invariant and, $\mathcal{D}(\sigma, \pi) = K(\sigma, \pi)$. If $\pi \neq 0$ is necessarily a unit of $K$. □

More generally, If $P$ is $\sigma$-invariant in $D$ then $D - P$ is also $\sigma$-invariant hence an Ore set in $D(\sigma, a)$. Localizing $D(\sigma, a)$ at $D - P$ then yields $D(\sigma, a) P = D_P(\sigma, a)$. If $P \neq 0$ then $D_P$ is a discrete valuation ring of $K$ and $D_P(\sigma, a)$ is a gr-valuation ring in $K[t, t^{-1}\sigma]$ (the latter being a graded-skewfield). The corresponding valuation filtration on $K[t, t^{-1}, \sigma]$ is compatible with $\mathbb{Z}$-grading and the associated graded ring for the valuation filtration is exactly $K[\sigma, t^{-1}, \sigma]$. It follows that $D_P(\sigma, a)$ is an intersection of $K[t, t^{-1}, \sigma]$ and a discrete valuation on $Q_\mathfrak{d}(D(\sigma, a)) = K[t, \sigma]$. So we have proved.

**Proposition 4.2** If $P$ is a $\sigma$-invariant prime ideal of $D$ such that $a \notin P$ then $P$ determines a noncommutative valuation of $Q_\mathfrak{d}(D(\sigma, a)) \cong K(t, \sigma)$ with ring $\Lambda_P$ say, and maximal ideal $w$, such that $D(\sigma, a) = \Lambda_P \cap K[t, t^{-1}, \sigma]$, and $P = w \cap D(\sigma, a)$. The residue skew field of this discrete valuation is $Q_\mathfrak{d}(\mathcal{D}(\sigma, \pi)) = K[t, \pi]$.

Note that for the Weyl algebra

$$A_1(\mathbb{C}) = \mathbb{C}[X, Y] / (XY - YX - 1), \quad D_1 = \mathbb{C}[X, Y], \quad \sigma(XY) = XY + 1$$

there are not nontrivial $\sigma$-invariant prime ideals in $D_1$. In a sense the prime at $\infty$ is an invariant prime (corresponding to $\mathbb{C}[(XY)^{-1}](\sigma_Y)^{-1}$) and it is the valuation ring in $D_1(\mathbb{C}) = Q_\mathfrak{d}(A_1(\mathbb{C}))$ corresponding to the quotient filtration.
of the Bernstein filtration on $A_1(\mathbb{C})$ that represents this prime at $\infty$ (we refer to [19] for some results on valuations of $D_1(\mathbb{C})$).

Look at $D_1$, the coordinate ring of curve $C$ in affine $n$-space over $\mathbb{K}$, in particular $\mathbb{K}$ is algebraically closed in the field of fractions of $D_1$, $\mathbb{K}$ say.

Let $O_v \subset \mathbb{K}$ be such that $C$ has good reduction at $O_v$ i.e. the reduced equations of $C$ define a nonsingular curve over the residue field $\mathbb{K}$, or equivalently $\overline{D} = D_{O_v}/m_vD_{O_v}$ is a Dedekind domain, where $D_{O_v} = O_v[X_1, \ldots, X_n]/I \cap O_v[X_1, \ldots, X_n]$, $I$ is an ideal of $C$. If $a \in D_{O_v} - m_vD_{O_v}$ then $D_{O_v}(\sigma,a) = \Lambda \subset D(\sigma,\Lambda)$. It is clear that $m_v D_{O_v}$ is $\sigma$-invariant, therefore the left ideal $D_{O_v}(\sigma,a)m_v$ is two-sided and we have:

**Lemma 4.3** as before : $D_{O_v}(\sigma,a)/m_v D_{O_v}(\sigma,a) \cong \overline{D}(\sigma,\pi)$. If the curve given by $D$ over $\mathbb{K}$ has good reduction at $O_v$ then for $a \notin m_v D$ we obtain an $m_v$-adic filtration on $D(\sigma,a)$ with $F_n D(\sigma,a) = D_{O_v}(\sigma,a)\pi^{-n}$ where $m_v = (\pi)$ for $n \in \mathbb{Z}$ such that the associated graded ring is $\overline{D}(\sigma,\pi)[t, t^{-1}]$, hence a domain.

**Remark 4.4** The restriction to $O_v$ defining good reduction for $C$ can be avoided. In the above result it is only important to have $\overline{D}(\sigma,\pi)$ to be a domain and the assumption on a shows that it is enough to have $\overline{D}(\sigma,\pi)$ to be a domain. However from the point of view of the theory of generalized Weyl algebras it is nice to have a residue algebra again being a generalized Weyl algebra of the same type. Therefore “good reduction” is an interesting condition. Combining this with the results of section 1, we may phrase all this as follows.

**Theorem 4.5** Let $K \subset D^{(\sigma)} \subset D$ where $D$ is the coordinate ring of a nonsingular curve $C$ in $n$-space. Let $m_b \subset O_v \subset K$ define a discrete valuation of $K$ such that $a \in D_{O_v} - m_v D_{O_v}$. The filtration $FD(\sigma,a)$ defined by $F_n D(\sigma,a)$ extends to a valuation filtration on the skewfield $Q_d(D(\sigma,a)) \cong K(t,\sigma)$ with residue skewfield $A_d(\overline{D}(\sigma,\pi))$.

If $C$ has good reduction at $O_v$ then $\overline{D}(\sigma,\pi)$ is a generalized Weyl algebra over the Dedekind domain $\overline{D}$. If $\sigma$ has infinite order then $D^{(\sigma)} = K$.

**Proof.** Only the final statement has not yet been fully established. If $D^{(\sigma)}$ is not algebraic over $K$ then $K$ must be algebraic over $Q_d(D^{(\sigma)}) = K^{(\sigma)}$. Since $D$ is affine over $K$, $K$ is finitely generated as a field over $K$ hence over $K^{(\sigma)}$. It follows that $[K : K] \leq \infty$ but then $\langle \sigma \rangle$ is a finite group, a contradiction. \qed

**Corollary 4.6** Certain discrete valuations of the base field $K$ extend to noncommutative discrete valuations (unramified extension) on the skewfield of fractions of quantum enveloping algebras, the quantum plane, the quantum $O_{q,2}$ of $so(K,3)$ [17], the quantum Heisenberg algebra [12], generalized gauge algebra of [11]. The condition on discrete valuation of the base field is given in terms of good reduction of some constant e.g. $q$ or $a$. For example in case of the quantum Weyl algebra $A = K[t]|(\sigma,t)$ where $\sigma(t) = q^{-1}(t - 1)$ it is clear that $\pi$ explodes when one allows an $O_v$ of $K$ containing $q$ in $m_v$. 

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