Effect of Intermediate Elastic Support on Vibration of Functionally Graded Euler-Bernoulli Beams Excited by a Moving Point Load

Buntara Sthenly Gan*, Nguyen Dinh Kien2 and Le Thi Ha3

1 Professor, Department of Architecture, College of Engineering, Nihon University, Japan
2 Research Associate, Institute of Mechanics, Vietnam Academy of Science and Technology, Vietnam
3 Ph.D. Student, Theoretical Group, Hanoi University of Transport and Communications, Vietnam

Abstract
The effect of intermediate elastic support on the vibration of functionally graded Euler-Bernoulli beams excited by a moving point load is studied. The material properties of the beams are assumed to vary in the thickness direction of the beams by a power function. Governing equations of motion that take into account the shift in the physically neutral surface position, are constructed using Hamilton's principle. A finite element model is developed and employed in computing the dynamic response of the beams. The influence of the stiffness and position of the elastic support on the dynamic characteristics of the FGM beams is examined and highlighted.

Keywords: functionally graded beam; elastic support; physically neutral surface; moving force; dynamic response

1. Introduction
The problems of elastic beams traversed by moving loads are often met in the design of structures such as overhead cranes, runways, and sleepers, and have been investigated for a long time. A large number of studies on beams excited by moving loads can be found in the literature, amongst which the monograph of Frýba (1999) is an early and excellent reference with a number of closed-form solutions. In addition to the analytical methods, the finite element method, which has versatility in the spatial discretization, has also been widely employed in solving the moving load problems (Hino et al., 1983; Lin and Trethewey, 1990; Thambiranam and Zhuge, 1996).

Functionally graded materials (FGMs) have received much attention from researchers since they were first introduced by Japanese scientists in Sendai in 1984 (Koizumi, 1997). FGMs are produced by continuously varying the volume fraction of the constituent materials, usually ceramics and metals, in one or more desired spatial directions. The effective properties of the formed material exhibit continuous change and therefore eliminate interface problems and reduce thermal stress concentrations. FGMs have extensive applications in industry and civil engineering, especially where the operating conditions are severe (Jha et al., 2013). Many investigations carrying out analysis of FGM structures under different loadings are summarized by Birman and Byrd (2007); only the contributions that are most relevant to the present work are briefly discussed herein. Chakraborty et al. (2003) formulated a first-order shear deformation beam element for studying the thermoelastic behavior of FGM beams. Based on a co-rotational finite element formulation, Nguyen and Gan (2014) studied the large deflections of tapered FGM beams subjected to end forces. Gan and Nguyen (2014) formulated a total Lagrange finite element formulation for investigating the large displacement behavior of FGM beams resting on a Pasternak foundation. Şimşek and Kocatürk (2009) and Şimşek (2010) considered the moving load problems of FGM beams by using polynomials to approximate the variables in solving the governing equations of motion. Using the finite element method, Nguyen et al. (2013) and Le et al. (2014) investigated the dynamic response of non-uniform and multi-span FGM Timoshenko beams subjected to a moving load, respectively.

In an industrial building structure, load-moving facilities such as overhead cranes can be considered as the moving load on a beam. The beam that supports the crane can be designed either rigid or semi-rigid depending on the designer. The example of elastic support in a seismic resistant building could be a rubber type damper against earthquake.

There are two kinds of potential benefit from applying FGM technology to building structures. First, the real condition of the strengths gradation...
in a building can be investigated for retrofitting or rehabilitation purposes. Second, by using the FGM technology a building can be built to be functionally graded in strength for optimal design purposes.

Currently, in building structures, the application of FGM to the beam or column has not been realized. Research concerns the methodology of producing the FGM in concrete are still under investigation (Han et al., 2016, Han et al., 2015, Gan et al., 2015).

The author’s aim is to study how the dynamic characteristics of the FGM beams depend on the stiffness and position of the elastic support. To this end, the governing equations of motion based on the Euler-Bernoulli beam theory of the beam-elastic support system are derived using Hamilton’s principle. The effect of the shift in the physically neutral surface is taken into account in deriving the governing equations. The finite element model is developed for solving the equations of motion. Newmark’s direct integration method is then employed in computing the dynamic deflection of the beams. The influence of the elastic support parameters on the dynamic response of the beams is examined and highlighted.

2. Equations of Motion

A simply supported beam with an intermediate elastic support of stiffness $k$ subjected to a load $Q$ moving at speed $v$ from left to right, as depicted in Fig.1., is considered. The cross-section is assumed to be rectangular with width $b$ and height $h$. In the figure, a Cartesian co-ordinate system, $(x, z)$, is introduced with the $x$-axis lying on the mid-plane, and the $z$-axis directed upward. Denoting the total beam length as $L$, $aL$, with $0 \leq a \leq 1$ ($a$ is called the position parameter below), is the distance from the elastic support to the left end of the beam.

![Fig.1. A Simply Supported Beam with Intermediate Elastic Support Under a Moving Concentrated Load Q](image)

Assuming that the beam material is composed of metal and ceramic phases whose volume fraction varies in the $z$ direction only, the effective material properties $P_i$, such as the Young’s modulus $E$ and the mass density $\rho$, are distributed in the thickness direction by power-law functions of $z$, measured from the mid-plane, as

$$P(z) = (P_e - P_m) \left( \frac{z}{h} + \frac{1}{2} \right)^n + P_m$$

(1)

where the subscripts ‘$c$’ and ‘$m$’ stand for ceramic and metal, respectively, and $n$ is the non-negative power-law index dictating the distribution of the material constituents.

The present analysis relies heavily on the concept of a physically neutral surface, which is defined as a surface of the zero axial train of a beam subjected to pure bending (Levyakov, 2013; Zhang, 2013). Based on the Euler-Bernoulli beam theory, the displacements $u_i$, $w_i$, and $u_i$ in the $x$, $y$, and $z$ directions, respectively, are given by

$$u_i(x, z, t) = u(x, t) - (z - h_0)w_i(x, t)$$

$$u_i(x, z, t) = 0, \ u_i(x, z, t) = w(x, t)$$

(2)

where $u (x, t)$ and $w(x, t)$ are the axial and transverse displacements of a point on the neutral surface, $h_0$ is the distance from the neutral surface to the mid-plane, which will be determined later, and $z$ is the distance from the considered point to the neutral surface. In Eq. (2) and hereafter, a subscripted comma is used to indicate the differentiation with respect to the spatial variable $x$.

The axial strain $\varepsilon$ resulting from Eq. (2) is as follows:

$$\varepsilon = u_x - (z - h_0)w_{zz} = \varepsilon_0 + (z - h_0)\chi$$

(3)

where $\varepsilon_0 = u_x$ is the axial strain of the physically neutral surface, and $\chi = -w_{zz}$ is the beam curvature. Using Hook’s law, Eq. (3) gives the axial stress $\sigma$ in the form:

$$\sigma = E(z) \left[ \varepsilon_0 + (z - h_0)\chi \right]$$

(4)

where $E$ is the effective Young’s modulus, defined by Eq. (1).

Based on Eq. (4), we can determine the distance from the physically neutral surface to the mid-plane by using the equilibrium condition for the beam subjected to pure bending. Since the strain of the neutral surface $\varepsilon_0$ and the resultant axial stress should vanish in pure bending, one has

$$\int_A E(z)(z - h_0) dA \chi = 0$$

(5)

where $A$ denotes the beam cross-sectional area. Substituting $E(z)$ from Eq. (1) into Eq. (5), one gets

$$h_0 = \frac{n E_c (E_c - E_m)}{2(n + 2)(E_c + nE_m)}$$

(6)

The physically neutral surface, as seen from Eq. (6), shifts upward from the mid-plane when $E_c > E_m$. In order to derive the governing equations of motion, Hamilton’s principle is used:

$$\int_A \left[ T - (U + V) \right] dt = 0$$

(7)
where \( U, T, \) and \( V \) denote the strain, kinetic energies for the beam with the elastic support, and potential of the moving load. From Eq. (4), one has

\[
U = \frac{1}{2} \left[ \left( A_{11} u_x^2 - 2 A_{12} u_x w_x + A_{22} w_x^2 \right) + \frac{1}{2} k w^2 \right]_{x=aL}
\]

where \( A_{11}, A_{12}, \) and \( A_{22} \) are the axial, axial-bending coupling, and bending rigidities and are defined as

\[
(A_{11}, A_{12}, A_{22}) = \frac{h^2}{12 b} \left( E(1, z-h_0), (z-h_0)^2 \right) dz
\]

It is worth noting that because of Eq. (6), the coupling rigidity \( A_{12} \) defined in Eq. (8) vanishes. The kinetic energy for the beam resulting from the displacement field (2) is as follows:

\[
T = \frac{1}{2} \int_0^l \left[ I_{11} (\dot{u}_x^2 + \dot{w}_x^2) - 2 I_{12} \dot{u}_x \dot{w}_x + I_{22} \dot{w}_x^2 \right] dx
\]

in which an overdot is used to denote the differentiation with respect to time \( t \), and

\[
(I_{11}, I_{12}, I_{22}) = \frac{h^2}{12 b} \left( \rho(z)(1, z-h_0), (z-h_0)^2 \right) dz
\]

are the mass moments. It should be noted that unlike the coupling rigidity, the coupling moment \( I_{12} \) does not vanish. The potential of the moving load is simply given by

\[
V = -Q \delta(x-v t)
\]

where \( \delta \) is the delta Dirac function, and \( v \) is the current traveling time of load \( Q \). The moving speed \( v \) is assumed to be constant in the present work.

From Eqs. (8), (10), and (12), Hamilton’s principle stated by Eq. (7) gives the equations of motion in the following forms:

\[
\begin{align*}
I_{11} \ddot{u}_x - I_{12} \ddot{w}_x - A_{11} \dot{u}_x &= 0 \\
I_{11} \ddot{w}_x + I_{12} \ddot{u}_x - I_{22} \ddot{w}_x - A_{12} \dot{w}_x + k \dot{w} \bigg|_{x=aL} &= Q
\end{align*}
\]

It should be noted that the effect of the shift in the neutral surface position which leads to \( A_{12} = 0 \) has been taken into consideration in the above governing equations of motion.

3. Finite Element Formulation

The theory for FGM beams with an intermediate elastic support described in the previous section is implemented via a displacement-based finite element method. To this end, the beam is assumed to be divided into a number of two-node beam elements with length \( l \). There are axial and transverse displacements and a rotation at each node, and thus the vector of nodal displacements, \( \mathbf{d} \), for the element contains six components:

\[
\mathbf{d} = [u_1, w_1, \theta_1, u_2, w_2, \theta_2]^T
\]

where the superscript ‘\( T \)’ is used to denote a transpose of a vector or a matrix. The axial and transverse displacements inside the element are interpolated from their nodal values as

\[
\mathbf{u} = \mathbf{N}_u^T \mathbf{d}, \quad \mathbf{w} = \mathbf{N}_w^T \mathbf{d}
\]

where \( \mathbf{N}_u \) and \( \mathbf{N}_w \) denote the matrices of interpolation functions for \( u \) and \( w \), respectively. For the present Euler-Bernoulli finite element model, the following linear and Hermite cubic polynomials are employed to interpolate the axial displacement \( u \) and transverse displacement \( w \) (Cook et al., 2002):

\[
\begin{align*}
N_{u1} &= \frac{l-x}{l}, & N_{u2} &= N_{u3} = 0, \\
N_{u4} &= \frac{x}{l}, & N_{u5} &= N_{u6} = 0
\end{align*}
\]

and

\[
\begin{align*}
N_{w1} &= 0, & N_{w2} &= 1 - 3 \frac{x^2}{l^2} + 2 \frac{x^3}{l^3}, \\
N_{w3} &= x - 2 \frac{x^2}{l} + \frac{x^3}{l^2}, & N_{w4} &= 0, \\
N_{w5} &= 3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3}, & N_{w6} &= -\frac{x^2}{l} + \frac{x^3}{l^2}
\end{align*}
\]

Using Eqs. (15)-(17), the strain energy (8) can be written in the form

\[
U = \frac{1}{2} \sum_{n=ELE} \mathbf{d}^T \left( \mathbf{k}_{nw} + \mathbf{k}_{nw} \right) \mathbf{d} + \frac{1}{2} \mathbf{d}^T \mathbf{k}_{nw} \mathbf{d}
\]

where \( nELE \) denotes the total number of elements, and

\[
\begin{align*}
\mathbf{k}_{uw} &= \int_0^l \mathbf{N}_u^T \mathbf{A}_u \mathbf{N}_w^T dx, \\
\mathbf{k}_{ww} &= \int_0^l \mathbf{N}_w^T \mathbf{A}_w \mathbf{N}_w^T dx, \\
\mathbf{k}_{ww} &= \int_0^l \mathbf{N}_w^T k \mathbf{N}_w^T \bigg|_{x=aL}
\end{align*}
\]

are the element stiffness matrices stemming from stretching, bending, and elastic support, respectively. In a similar way, one can write the kinetic energy for the beam in the form

\[
T = \frac{1}{2} \sum_{n=ELE} \mathbf{d}^T \left( \mathbf{m}_{nw} + \mathbf{m}_{nw} + \mathbf{m}_{nw} + \mathbf{m}_{nw} \right) \mathbf{d}
\]
where

\[ m_{ww} = \int_0^l N_{ww} I_{11} N^T_{ww} dx, \quad m_{ww} = \int_0^l N_{ww} I_{11} N^T_{ww} dx, \]

\[ m_{oo} = -2 \int_0^l N_{oo} I_{12} N^T_{oo} dx, \quad m_{oo} = -2 \int_0^l N_{oo} I_{12} N^T_{oo} dx \]

are the element mass matrices resulting from the axial and transverse translations, axial-rotation coupling, and bending of the beam. The potential of the moving load \( Q \) is written in terms of the nodal displacements as

\[ V = -Q N^T \delta (x - vt) \]  

(22)

Substituting Eqs. (18), (20), and (22) into Eq. (7), one can express the equations of motion in terms of the finite element method as

\[ MD + KD = F \]

(23)

where \( D \) denotes the structural nodal displacements, \( M \) and \( K \) are the structural mass and stiffness matrices, respectively, which are obtained by assembling the derived element matrices in the standard way for the finite element method,

\[ M = \sum_{\text{ELE}} (m_{ww} + m_{ww} + m_{oo} + m_{oo}) \]

(24)

and

\[ K = \sum_{\text{ELE}} (k_{ww} + k_{ww}) + k_{oo} \]

(25)

and \( F \) is the structural nodal force vector which consists of all zero components except for the element under the moving load

\[ F = Q(0 \ldots N_{u1} N_{u2} N_{u3} N_{u4} N_{u5} N_{u6} \ldots 0)^T \]

(26)

The equations of motion (23) can be solved by Newmark’s direct integration method. The average constant acceleration method, which ensures numerically unconditional stability (Gérardin and Rixen, 1997) is employed in the present work. For free vibration analysis, the right hand side of Eq. (23) is set to zero, and thus Eq. (23) leads to an eigenvalue problem which can be solved by a standard method described by Gérardin and Rixen (1997).

4. Numerical Results and Discussion

A simply supported FGM beam composed of stainless steel (SUS304) and alumina (Al₂O₃) is considered. The Young’s modulus and mass density of the constituent materials are as follows: \( E_o = 210 \) GPa and \( \rho_o = 7800 \) kg/m³ for steel; \( E_i = 390 \) GPa and \( \rho_i = 3960 \) kg/m³ for alumina (Şimşek and Kocaturk, 2009). Unless otherwise stated, the geometric data of the beam are as follows: \( b = 0.4 \) m, \( h = 0.9 \) m, and \( L = 20 \) m. The amplitude of the moving load is assumed to be \( Q = 100 \) kN. A total of 500 time steps are employed in the computations for the Newmark method.

4.1 Fundamental Frequency

The accuracy of the derived formulation in evaluating the fundamental frequency is verified first. To this end, the fundamental frequency of a homogeneous steel beam is computed for various values of the elastic support position and stiffness, and the results obtained are listed in Table 1.

### Table 1. Frequency Parameter \( \mu_0 \) of the Homogeneous Beam

| \( \alpha \) | \( k_o \) |
|---|---|
| 0.2 | 10.22 |
| Present | 10.2077 |
| 11.47 | 12.78 |
| 10.2077 | 12.5332 |
| 0.3 | 10.51 |
| Present | 10.5027 |
| 12.67 | 14.77 |
| 10.5027 | 14.5515 |
| 0.4 | 10.74 |
| Present | 10.7406 |
| 13.62 | 16.39 |
| 10.7406 | 16.3112 |
| 0.5 | 10.83 |
| Present | 10.8319 |
| 14.05 | 17.18 |
| 10.8319 | 17.0682 |

Fig.2. Position Parameter \( \alpha \) versus Frequency Parameter \( \mu_0 \)

The stiffness parameter of the elastic support \( k_o \) and the frequency parameter \( \mu_0 \) in Table 1. are defined as follows:

\[ k_o = k_s \frac{I^3}{E_n I}, \quad \mu_0 = \omega_1^2 \frac{\rho A}{E_n I} \]

(27)

where \( I = bh^3/12 \) is the moment of inertia, and \( \omega_1 \) is the fundamental of the steel beam. The frequency parameters shown in Table 1. of the present work are in good agreement with those of Irassar et al. (1984), regardless of the elastic support position and stiffness. It should be noted that the sixth-order polynomials have been used by Irassar et al. in approximating the transverse displacement, and a slenderness ratio of \( L/h = 50 \) has been employed in computing the frequency in Table 1.
The position parameter $\alpha$ of the elastic support versus the frequency parameter $\mu_0$ of the FGM beam is shown in Fig.2. for various values of the index $n$ and for $k_0 = 50$ and 100. The fundamental frequency, as seen from the figure, is clearly governed by the stiffness and the position of the elastic support. The fundamental frequency of the beam increases when raising the position parameter $\alpha$ and the stiffness parameter $k_0$ of the elastic support, regardless of the index $n$.

4.2 Dynamic Deflection

Since no data on the dynamic deflection of a beam with an intermediate elastic support under a moving load are found in the literature, the dynamic response of a homogeneous steel beam without elastic support is computed first to verify the formulation. In Fig.3., the normalized deflection under the moving load of a homogeneous steel beam is shown for a moving speed of $v = 20$ m/s, where the analytical solution derived by Timoshenko et al. (1974) is also given. In the figure and hereafter, $w_0 = PL^3/48EI$ is the mid-span deflection of the steel beam under a static load $Q$. A good agreement between the numerical result of the present work and the analytical solution derived by Timoshenko et al. (1974) is noted.

![Fig.3. Normalized Deflection Under Moving Load of Homogeneous Beam without Elastic Support ($v = 20$ m/s)](image1)

Table 2. Maximum Normalized Mid-span Deflection and Corresponding Speed of FGM Beam

| $n$   | $\max \left( w(L/2,t)/w_0 \right)$ | $v$ (m/s) | $\max \left( w(L/2,t)/w_0 \right)$ | $v$ (m/s) |
|-------|-----------------------------------|-----------|-----------------------------------|-----------|
|       | Present & Şimşek & Kocatürk (2009) | Present & Şimşek & Kocatürk (2009) | Present & Şimşek & Kocatürk (2009) | Present & Şimşek & Kocatürk (2009) |
| 0.2   | 1.0347                            | 1.0344    | 222                               | 222       |
| 0.5   | 1.1445                            | 1.1444    | 197                               | 198       |
| 1     | 1.2504                            | 1.2503    | 179                               | 179       |
| 2     | 1.3377                            | 1.3376    | 164                               | 164       |
| SUS304| 1.7326                            | 1.7324    | 132                               | 132       |
| Al$_2$O$_3$| 0.9329                            | 0.9328    | 252                               | 252       |

To verify the derived formulation in computing the dynamic response of the FGM beam, the maximum normalized deflection at the mid-span and the corresponding speed of the steel-alumina beam with various values of the index $n$ are evaluated and the results are given in Table 2. The numerical results listed in Table 2 have been obtained by computing the deflection while steadily raising the moving speed with an increment of 1 m/s, as used by Şimşek and Kocatürk (2009). As seen from Table 2, the results of the present work are in good agreement with those of Şimşek and Kocatürk (2009).

![Fig.4. Time-histories for Normalized Deflection Under the Moving Load of an FGM Beam with Various Values of $\alpha$ ($n = 5, v = 50$ m/s)](image2)

![Fig.5. Time-histories for Normalized Deflection Under the Moving Load of an FGM Beam with Various Values of $k_0$ ($n = 5, v = 25$ m/s)](image3)

![Fig.6. Time-histories for Normalized Deflection Under the Moving Load of an FGM Beam with Various Values of $n$ ($n = 5, v = 25$ m/s)](image4)

In Figs.4.-7., the times-histories for the normalized deflection under the moving load of the FGM beam
are depicted for various values of \( k_0 \), \( \alpha \), and \( n \). The figures show that dynamic behavior of the beam is strongly dependent on the position of the elastic support, material parameter, and speed of movement. The maximum deflection, as seen from Fig.4., is smaller and occurs at an earlier time for a large position parameter \( \alpha \). Fig.5. shows the sensitivity of the deflection under the moving load to the stiffness of the elastic support, and the reduction in the deflection is clearly seen on raising the stiffness. The material index \( n \), as seen from Fig.6., also affects the dynamic response, and the maximum deflection decreases when the index \( n \) increases, regardless of the position parameter \( \alpha \). This can be explained by the fact that the beam, as seen from Eq. (1), contains more steel for a higher value of the index \( n \) and is thus softer. The dynamic response of the beam, as seen from Fig.7., is also influenced by the moving speed. The beam tends to execute more vibration cycles when it is subjected to a moving load of lower speed, regardless of the position parameter \( \alpha \).

Table 3. Maximum Normalized Deflection Under the Moving Load of the FGM Beam (\( n = 5, v = 50 \text{ m/s} \))

| \( \alpha \) | 10   | 20   | 50   | 100  |
|-------------|------|------|------|------|
| 0           | 0.8873 | 0.8873 | 0.8873 | 0.8873 |
| 0.1         | 0.8809 | 0.8743 | 0.8533 | 0.8176 |
| 0.2         | 0.8640 | 0.8383 | 0.7596 | 0.6470 |
| 0.3         | 0.8408 | 0.7886 | 0.6402 | 0.4622 |
| 0.4         | 0.8159 | 0.7382 | 0.5409 | 0.3714 |
| 0.5         | 0.8068 | 0.7123 | 0.5109 | 0.3329 |

In order to study the effect of the stiffness and position of the elastic support in more detail, the maximum normalized deflection of the beam with \( n = 5 \) under a load with speed \( v = 50 \text{ m/s} \) is evaluated for different values of the parameters \( \alpha \) and \( k_0 \), and the results are listed in Table 3. The maximum normalized deflection in the table steadily decreases with increases in the stiffness \( k_0 \) and/or the position parameter \( \alpha \). On examining the table in more detail, one can see that the position of the elastic support strongly affects the change of the maximum deflection with the increase of the stiffness \( k_0 \). For \( \alpha = 0.1 \), the maximum deflection decreases by 7.19% when \( k_0 \) increases from 10 to 100, but this value is 45.03 and 58.74% for \( \alpha = 0.3 \) and \( \alpha = 0.5 \), respectively.

4.3 Axial Stress Distribution

The thickness distributions of the normalized axial stress at the mid-span of the FGB beam subjected to the moving load are depicted in Figs.8. and 9. for different values of \( \alpha, k_0, \) and \( n \). The axial stress in the figures was computed at the time when the load arrived at the beam midpoint, and was normalized by the maximum stress of a beam under a static load \( Q \) acting at the midpoint, \( \sigma_0 = QLh/8I \).

The axial stress of the FGM beam shown in the figures is very different from that of a homogeneous beam. The stress of the FGM beam does not vanish at the mid-plane, and it is not symmetric with respect to the coordinate origin. At the given values of the index \( n \) and the moving speed, the amplitude of the axial stress, as seen from Fig.8., decreases with increases in the position parameter \( \alpha \), and the reduction is more pronounced for a higher stiffness of the elastic support. The maximum compressive stress, as seen from Fig.9., increases slightly on raising the index \( n \), but this tendency lessens at higher values of the position parameter \( \alpha \).

6. Conclusions

The effect of the intermediate elastic support on the dynamic response of GM Euler-Bernoulli beams has been studied. Based on Hamilton's principle, the governing equations of motion for the beam-elastic support system were derived. The shift of the physically neutral surface position has been taken into account in the derivation of the governing equations. The obtained numerical results have shown that the dynamic characteristics of the beams are governed by the parameters of the elastic support. The fundamental of the beams increases with increases in the stiffness and position parameter of the elastic support. The time-histories of the dynamic deflection under the moving load and the axial stress distribution of the FGM beams are also strongly dependent on the elastic support parameters. The numerical results of the present work will be useful for engineers when choosing appropriate parameters of the elastic support in designing FGM beams subjected to a moving load.
Acknowledgement

The support from the Vietnam National Foundation for Science and Technology Development (NAFOSTED), Grant Number 107.02-2015.02 to the last two authors is gratefully acknowledged.

References

1) Birman, V. and Byrd, L. W. (2007) Modeling and analysis of functionally graded materials and structures. Applied Mechanics Reviews, 60 (5), pp.195-216.
2) Chakraborty, A., Gopalakrishnan, S., and Reddy, J.R. (2003) A new beam finite element for the analysis of functionally graded materials. International Journal of Mechanical Science, 45 (3), pp.519-539.
3) Cook, R.D., Malkus, D.S., Plesha, M.E. and Witt R.J. (2002) Concepts and applications of finite element analysis, 4th ed. New York: John Wiley & Sons.
4) Frýba, L. (1972) Vibration of solids and structures under moving loads. Prague: Academia.
5) Gan, B.S. and Nguyen, D.K. (2014) Large deflection analysis of functionally graded beams resting on a two-parameter elastic foundation. Journal of Asian Architecture and Building Engineering, 13 (3), pp.649-656.
6) Gan, B.S., Han, A.L. and Pratama, M.M.A. (2015) Procedia Engineering, 125, pp.885-891.
7) Gérardin, M. and Rixen, R. (1997) Mechanical vibrations. Theory and application to structural dynamics, 2nd ed. Chichester, UK: John Wiley & Sons.
8) Han, A.L., Gan, B.S., As’ad, S. and Pratama, M.M.A. (2015) International Journal of Engineering and Technology Innovation, 5 (4), pp.233-241.
9) Han, A.L., Gan, B.S. and Pratama, M.M.A. (2016) Effects of graded concrete on compressive strengths. International Journal of Technology, 5, pp.732-740.
10) Hino, J., Yoshimura, T., Konishi, K. and Ananthanarayana, N. (1984) A finite element method prediction of the vibration of a bridge subjected to a moving vehicle load. Journal of Sound and Vibration, 96 (1), pp.45-53.
11) Jha, D.K., Kant, T. and Singh, R.K. (2013) A critical review of recent research on functionally graded plates. Composite Structures, 96, pp.833-849.
12) Koizumi, M. (1997) FGM activities in Japan. Composites Part B: Engineering, 28 (1/2), pp.1-4.
13) Le, T.H., Gan, B.S. and Nguyen, D.K. (2014) Finite element analysis of multi-span functionally graded beams under a moving harmonic load. Mechanical Engineering, Bulletin of the JSME, 1 (3), pp.1-13.
14) Levyakov, S.V. (2013) Elastica solution for thermal bending of a functionally graded beam. Acta Mechanica, 224, pp.1731-1740.
15) Lin, W.H. and Trethewey, M.W. (1990) Finite element analysis of elastic beams subjected to moving dynamic loads. Journal of Sound and Vibration, 136 (2), pp.323-342.
16) Nguyen, D.K. and Gan, B.S. (2014) Large deflections of tapered functionally graded beams subjected to end forces. Applied Mathematical Modelling, 38, pp.3054-3066.
17) Nguyen, D.K., Gan, B.S. and Le, T.H. (2013) Dynamic response of non-uniform functionally graded beams subjected to a variable speed-moving load. Journal of Computational Material Science and Technology, JSME, 7 (1), pp.12-27.
18) Şimşek, M. (2010) Vibration analysis of a functionally graded beam under a moving mass by using different beam theories. Composite Structures, 92 (4), pp.904-917.
19) Şimşek, M. and Kocatürk, T. (2009) Free and forced vibration of a functionally graded beam subjected to a concentrated moving harmonic load. Composite Structures, 90 (4), pp.465-473.
20) Thambiranam, D. and Zhuge, Y. (1996) Dynamic analysis of beams on elastic foundation subjected to moving loads. Journal of Sound and Vibration, 198 (2), pp.149-169.
21) Timoshenko, S.P., Young, D.H. and Weaver, W. (1974) Vibration problems in engineering. 4th ed. New York: John Wiley.
22) Zhang, D-G. (2013) Nonlinear bending analysis of FGM beams based on physical neutral surface and high order shear deformation theory. Composite Structures, 100, pp.121-126.