Computation in Logic and Logic in Computation*

Saeed Salehi
Department of Mathematical Sciences, University of Tabriz, 29 Bahman Blvd., 51666–17766 Tabriz, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395–5746 Tehran, Iran

Abstract: The theory of addition in the domains of natural (\(\mathbb{N}\)), integer (\(\mathbb{Z}\)), rational (\(\mathbb{Q}\)), real (\(\mathbb{R}\)) and complex (\(\mathbb{C}\)) numbers is decidable; so is the theory of multiplication in all those domains. By Gödel’s Incompleteness Theorem the theory of addition and multiplication is undecidable in the domain of natural numbers; though Tarski proved that this theory is decidable in the domains of \(\mathbb{R}\) and \(\mathbb{C}\). The theory of multiplication and order \(\langle\cdot, \leq\rangle\) behaves differently in the above mentioned domains of numbers. By a theorem of Robinson, addition is definable by multiplication and order in the domain of natural numbers; thus the theory \(\langle\mathbb{N}, \cdot, \leq\rangle\) is undecidable. By a classical theorem in mathematical logic, addition is not definable in terms of multiplication and order in \(\mathbb{R}\). In this paper, we extend Robinson’s theorem to the domain of integers (\(\mathbb{Z}\)) by showing the definability of addition in \(\langle\mathbb{Z}, \cdot, \leq\rangle\); this implies that \(\langle\mathbb{Z}, \cdot, \leq\rangle\) is undecidable. We also show the decidability of \(\langle\mathbb{Q}, \cdot, \leq\rangle\) by the method of quantifier elimination. Whence, addition is not definable in \(\langle\mathbb{Q}, \cdot, \leq\rangle\).

Keywords: Decidability; First-Order Logic; Gödel’s Incompleteness Theorems; Church’s Theorem; Presburger Arithmetic; Skolem Arithmetic; Quantifier Elimination.

1 Introduction

The question of the decidability of logical inference has triggered the beginning of computer science. Propositional Logic is decidable, since truth tables provide a finite semantics for it. Aristotle’s Syllogism, or in modern terminology the first-order logic of unary predicates, is decidable, since it has the finite model property. The notion of a Turing Machine was a successful outcome of the struggle to settle the question of the decidability of full First-Order Logic. It is now known that the first-order logic is undecidable if it has a binary relation symbol or a binary function symbol (see [1]). The additive theory of natural numbers \(\langle\mathbb{N}, +\rangle\) was shown to be decidable by Presburger in 1929 (and by Skolem in 1930; see [2]). The additive theories of integer, rational, real and complex numbers \(\langle\mathbb{Z}, +\rangle, \langle\mathbb{Q}, +\rangle, \langle\mathbb{R}, +\rangle\) and \(\langle\mathbb{C}, +\rangle\) are decidable as well. The multiplicative theory of the natural numbers \(\langle\mathbb{N}, \cdot\rangle\) is also shown to be decidable by Skolem in 1930; the theories \(\langle\mathbb{Z}, \cdot\rangle, \langle\mathbb{Q}, \cdot\rangle, \langle\mathbb{R}, \cdot\rangle\) and \(\langle\mathbb{C}, \cdot\rangle\) are also decidable.

Then it was expected that the theory of addition and multiplication of natural numbers would be decidable too; confirming Hilbert’s Program. But the world was shocked in 1931 by Gödel’s Incompleteness Theorem who showed that the theory \(\langle\mathbb{N}, +, \cdot\rangle\) is undecidable (see [3]). The theory \(\langle\mathbb{Z}, +, \cdot\rangle\) is undecidable too, since \(\mathbb{N}\) is definable in this structure: by Lagrange’s Theorem \(k \in \mathbb{N} \iff \exists a, b, c, d \in \mathbb{Z} (k = a^2 + b^2 + c^2 + d^2)\). So is the theory \(\langle\mathbb{Q}, +, \cdot\rangle\) by Robinson’s result [2] which shows that \(\mathbb{N}\) is definable in this structure too. However, Tarski showed that the theories \(\langle\mathbb{R}, +, \cdot\rangle\) and \(\langle\mathbb{C}, +, \cdot\rangle\) are decidable (see [4]). It is worth mentioning that the order relation \(\leq\) is definable by means of addition and multiplication in all the above domains of numbers. For example, the formulas \(\exists z (z + x = y)\) and \(\exists z (z^2 + x = y)\) define the relation \(x \leq y\) in the structures \(\langle\mathbb{N}, +, \cdot\rangle\) and \(\langle\mathbb{R}, +, \cdot\rangle\) respectively. The theory of addition and order \(\langle+, \leq\rangle\) is somehow weak, in all the above number domains, since it cannot define multiplication. The theory of multiplication and order \(\langle\cdot, \leq\rangle\) has not been extensively studied; one reason is that addition is not definable in \(\langle\mathbb{R}, \cdot, \leq\rangle\), since the bijection

*This paper is dedicated to Alan Turing, to commemorate the Turing Centenary Year 2012 – his 100th birthyear.
The order relation:
where $x + y = z \iff \lbrack x = y = z = 0 \rbrack \lor \lbrack z \neq 0 \land S(z \cdot x) \cdot S(z \cdot y) = S(z \cdot z) \cdot S(x \cdot y) \rbrack$, where $S(u)$ is the successor of $u$, which is definable by the order relation: $S(u) = v \iff \forall w[u < w \iff v < w]$. The symbol $u < v$ is a shorthand for $u \leq v \land u \neq v$.

The question of the decidability or undecidability of the structures $\langle \mathbb{Z}, \cdot, \leq \rangle$ and $\langle \mathbb{Q}, \cdot, \leq \rangle$ is missing in the literature. In this paper, by modifying Tarski’s identity we show that addition is definable in the structure $\langle \mathbb{Z}, \cdot, \leq \rangle$; this implies the undecidability of $\langle \mathbb{Q}, \cdot, \leq \rangle$. On the contrary, addition is not definable in $\langle \mathbb{Q}, \cdot, \leq \rangle$; here we show a stronger result by the method of quantifier elimination: the theory $\langle \mathbb{Q}, \cdot, \leq \rangle$ is decidable. Whence, by Robinson’s above-mentioned result [2], addition cannot be defined in this structure. An interesting outlook of our results is that though $\langle +, \cdot \rangle$ puts the domains $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ on the undecidable side, and the domains $\mathbb{R}$ and $\mathbb{C}$ on the decidable side, the language $\langle \cdot, \leq \rangle$ puts the domains $\mathbb{N}$ and $\mathbb{Z}$ on the decidable side, but $\mathbb{Q}$ and $\mathbb{R}$ on the decidable side.

2 Multiplication and Order in $\mathbb{Z}$

Tarski’s identity $S(z \cdot x) \cdot S(z \cdot y) = S(z \cdot z) \cdot S(x \cdot y)$ can define the formula $x + y = z$ in $\mathbb{Z}$ when $x + y \neq 0$. The case $x + y = 0$ was easily settled in natural numbers: for any $x, y \in \mathbb{N}$ we have $x + y = 0 \iff x = y = 0$. But this does not hold in $\mathbb{Z}$, and so we have to treat this case differently. Our trick is to define the relation $x = -y$ in terms of multiplication and successor (which is definable by order): $x = -y \iff S(x) \cdot S(y) = S(x \cdot y)$. Thus, the following formula defines addition in terms of multiplication and order in $\mathbb{Z}$: $x + y = z \iff \lbrack z = 0 \land S(x) \cdot S(y) = S(x \cdot y) \rbrack \lor \lbrack z \neq 0 \land S(z \cdot x) \cdot S(z \cdot y) = S(z \cdot z) \cdot S(x \cdot y) \rbrack$. So, the theories $\langle \mathbb{Z}, \cdot, \leq \rangle$ and $\langle \mathbb{Z}, +, \cdot \rangle$ are interdefinable, and hence $\langle \mathbb{Z}, \cdot, \leq \rangle$ is undecidable.

3 Multiplication and Order in $\mathbb{Q}$

Unlike the case of $\mathbb{Z}$, addition is not definable in the structure $\langle \mathbb{Q}, \cdot, \leq \rangle$. In fact, the theory of this structure is decidable. For showing that we use the method of quantifier elimination. First let us note that the language $\langle \cdot, \leq \rangle$ does not allow quantifier elimination for $\langle \mathbb{Q}, \cdot, \leq \rangle$, since e.g. the formula $\exists y [x = y^2]$ is not equivalent to a quantifier-free formula. So, we restrict our attention to $\mathbb{Q}^+ = \{ r \in \mathbb{Q} \mid r > 0 \}$ and extend the language to $\mathcal{L} = \{ 0, 1, \cdot, ^{-1}, <, R_2, R_3, \ldots \}$, where $R_n$ is interpreted as “being the nth power of a rational”; or in other words $R_n(x) \equiv \exists y [y^n = x]$.

**Theorem.** The structure $\langle \mathbb{Q}^+, \mathcal{L} \rangle$ admits quantifier elimination.

We note that the above main theorem implies that the structure $\langle \mathbb{Q}, \mathcal{L} \rangle$ admits quantifier elimination as well. It is enough to distinguish the signs: for any $x$, either $-x > 0$ or $x = 0$ or $x > 0$; so eliminating the quantifiers in each case, will eliminate all of the quantifiers. Let us also note that the quantifier-free formulas of $\mathcal{L}$ are decidable: for any given rational number $r$ and any natural $n$ one can decide if $r$ is an $n$th power of (an-)other rational number or not. Thus, quantifier elimination in $\langle \mathbb{Q}, \mathcal{L} \rangle$ implies the decidability of the structure $\langle \mathbb{Q}, \mathcal{L} \rangle$, and hence $\langle \mathbb{Q}, \cdot, \leq \rangle$.

The rest of the paper is devoted to proving the main theorem. The folklore technique of quantifier elimination starts from characterizing the terms and atomic formulas, also eliminating negations, implications and universal quantifiers, and then removing the disjunctions from the scopes of existential quantifiers, which leaves the final case to be the existential quantifier with the conjunction of some atomic (or negated atomic) formulas. Removing this one existential quantifier implies the ability to eliminate all the other quantifiers by induction. Let us summarize the first steps:

For a variable $x$ and parameter $a$, all $\mathcal{L}$-terms are equal to $x^k a^l$ for some $k, l \in \mathbb{Z}$. Atomic $\mathcal{L}$-formulas are in the form $u = v$ or $u < v$ or $R_n(u)$ for some terms $u, v$ and $n \geq 2$. Negated atomic $\mathcal{L}$-formulas are thus $u \neq v$, $u \not< v$ and $\neg R_n(u)$; the formulas $u \neq v$ and $u \not< v$ are equivalent to $u < v \lor v < u$ and $u = v \lor v < u$ respectively. By de Morgan’s laws we can assume that the negation appears only behind the atomic formulas of the form $R_n(u)$, and by the equivalences $A \rightarrow B \equiv \neg A \lor B$ and $\forall x \varphi \equiv \neg \exists x \neg \varphi$, we can assume that the implication symbol and universal quantifier do not appear in the formula (whose quantifiers are to eliminated). Finally, the equivalence $\exists x (\varphi \lor \psi) \equiv \exists x \varphi \lor \exists x \psi$ leaves us with the elementary formulas of the form $\exists x (\bigwedge_{\theta_i} \theta_i)$ where each $\theta_i$ is in the form $(x^n = v)$ or $(r < x^{\beta})$ or $(x^\gamma < s)$ or $R_n(tx^\gamma)$ or $\neg R_n(ux^\gamma)$ for some $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}$ and $\mathcal{L}$-terms $r, s, t, u, v$. Whence, it suffices to show that the $\mathcal{L}$-formula $\exists x \{ \bigwedge_{\theta_i} (x^n = v_i) \land \bigwedge_{\theta_i} (r_i < x^{\beta}) \land \bigwedge_{\theta_i} (x^{\gamma_i} < s_i) \land \bigwedge_{\theta_i} (R_{n_i}(tx^{\gamma_i})) \land \bigwedge_{\theta_i} (\neg R_{m_i}(ux^{\gamma_i})) \}$ is equivalent to another $\mathcal{L}$-formula in which $x$ (and so $\exists x$) does not appear. This will finish the proof.

Here comes the next steps of quantifier elima-
The powers of $x$ can be unified: let $p$ be the least common multiplier of the $\alpha_i$’s, $\beta_i$’s, $\gamma_i$’s, $\delta_k$’s and $e_i$’s. From the $(Q^+ \cup L)$-equivalences $a = b \leftrightarrow a^p = b^p$, $a < b \leftrightarrow a^p < b^p$ and $R_n(a) \leftrightarrow R_n(a^p)$, we infer that the above formula can be re-written equivalently as

$$\exists x [\bigwedge_k (R_n(x \cdot x^p) \land \bigwedge_k (R_m(t_k \cdot x))) \land \bigwedge_k (R_m(t_k \cdot x)) \land \bigwedge_k (R_m(t_k \cdot x))]$$

for possibly new $v_n$’s, $r_i$’s, $s_j$’s, $n_k$’s, $t_k$’s, $m_i$’s and $u_i$’s. This formula is in turn equivalent to

$$\exists y [\bigwedge_k (y = v_n) \land \bigwedge_k (y = v_n) \land \bigwedge_k (y = v_n) \land R_i(y)] \land \bigwedge_k (R_i(y))$$

(with the substitution $y = x^p$). Thus it suffices to show that the following formula

$$\exists x [\bigwedge_k (x = v_n) \land \bigwedge_k (y < x) \land \bigwedge_k (R_i(y)) \land \bigwedge_k (R_i(y))]$$

is equivalent to a quantifier-free formula. The conjunction $\bigwedge_k (x = v_n)$ is not empty, then the above formula is equivalent to the quantifier-free formula

$$\bigwedge_k (v_0 = v_n) \land \bigwedge_k (y = v_n) \land \bigwedge_k (y = v_n) \land R_i(y)$$

for some term $v_0$. So, let us assume that the conjunction $\bigwedge_k (x = v_n)$ is empty, and thus we are to eliminate the quantifier of the formula

$$\exists x [\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_i(x \cdot x)) \land \bigwedge_k (R_i(x \cdot x))]$$

for some fixed $v_n$. The formula $\exists x [\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_i(x \cdot x)) \land \bigwedge_k (R_i(x \cdot x))]$ is equivalent to the quantifier-free formula $\bigwedge_k (r_i < x)$ (that is $\max_i (r_i) < \min_j (s_j)$), since $Q^+ \cup L$ is finite.

For the formula $\exists x [\bigwedge_k (R_n(x \cdot x)) \land \bigwedge_k (R_n(x \cdot x)) \land \bigwedge_k (R_n(x \cdot x))]$, let $p$ be a prime number, and put $t'^p_k$ be the greatest number such that $p^x$ divides $t_k$; similarly $x'$ is the greatest number such that $p^x$ divides $x$. Then $\bigwedge_k (R_n(x \cdot x))$ is equivalent to $\bigwedge_k (t'_k, x' \neq 0)$. By a generalized form of the Chinese Remainder Theorem (4) the existence of such an $x'$ is equivalent to $\bigwedge_{x' \neq 0} (x' \equiv (n_k, m_i) t'_k$; here $a, b$ is the greatest common divisor of $a$ and $b$. That is equivalent to $\bigwedge_{x' \neq 0} R_n(a \cdot m_i, x' \equiv (n_k, m_i) t'_k$. We further note that in case of $\bigwedge_{x' \neq 0} t'_k \equiv (n_k, m_i) t'_k$ there are infinitely many solutions for $\bigwedge_k (v_0 = v_n) \land \bigwedge_k (v_0 = v_n) \land R_i(y)$ which are in the form $x' = N y - \sum_k n_k v_k t'_k$ for some fixed integers $N$ and $v_k$’s; $y'$ is arbitrary. In fact $N$ is the least common multiplier of $n_k$’s, and $v_k$’s are $v_k = c_k \cdot N/n_k$ where $\sum_k c_k N = 1$. Moreover, the solution $x'$ is unique up to the module $N$. So, if there exists some $x \in Q^+$ which satisfies $\bigwedge_k (R_n(t_k \cdot x))$ for some $t_k \in Q^+$, then it must be of the form $x = \gamma N \cdot \prod_k (t_k)^{-v_k}$ for some (arbitrary) $\gamma \in Q^+$. Thus, the formula $\exists x [\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_n(x \cdot x))]$ is equivalent to the (quantifier-free formula) $\bigwedge_k (r_i < s_j) \land \bigwedge_k (R_n(x \cdot x)) (t_k, t'^p_k)$, since the solution $x = \gamma N \cdot \prod_k (t_k)^{-v_k}$ for $\bigwedge_k (R_n(t_k \cdot x))$ can be chosen to satisfy $\max_i (r_i) < x < \min_j (s_j)$, choose a rational number $\gamma \in Q^+$ between the positive real numbers $\alpha = (\max_i (r_i) \cdot (\prod_k (t_k)^{-v_k})^{1/N}$ and $\beta = (\min_j (s_j) \cdot (\prod_k (t_k)^{-v_k})^{1/N}$. Since the set $Q$ is dense in $R$, there exists such a rational number $\gamma$. Then $x = \gamma N \cdot \prod_k (t_k)^{-v_k}$ is the desired solution.

Finally, we show that the formula

$$\exists x [\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_n(x \cdot x)) \land \bigwedge_k (R_n(x \cdot x))]$$

is equivalent to the following quantifier-free formula $\bigwedge_k (r_i < x) \land \bigwedge_k (R_n(x \cdot x))$, holds, clearly $\bigwedge_k (r_i < s_j)$ is true, and it can be easily seen that we also have $\bigwedge_k (R_n(x \cdot x))$. Assume $m_i \not| N$; we show that $\bigwedge_k (R_n(x \cdot x))$. Note that there exists some $y$ such that $x = \gamma N \cdot t$. Now if $\bigwedge_k (u_i \cdot t)$, then $u_i \cdot t = \gamma N \cdot t$, and so by $m_i \not| N$ we have $\bigwedge_k (u_i \cdot t)$ which contradicts the assumption $\bigwedge_k (u_i \cdot t)$.

Whence, $\bigwedge_k (u_i \not| N) \bigwedge_k (u_i \cdot t)$ holds.

Conversely, if we have $\bigwedge_k (r_i < s_j) \land \bigwedge_k (R_n(x \cdot x))$, then by the above arguments there exist some positive real numbers $\alpha < \beta$ such that for any rational number $\gamma$ with $\alpha < \gamma < \beta$, the number $z = \gamma N \cdot t$ satisfies the formula $\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_n(x \cdot x))$ where $N$ and $t$ are as above. Let $P$ be a sufficiently large prime number which does not divide any of the numerators or denominators of (the reduced fractions of) $t_k$’s or $u_i$’s. Let $M = \prod_k m_i$ and let $\delta$ be a positive rational number such that $(\alpha/\beta)^{1/M} < \delta < (\beta/\alpha)^{1/M}$. We show that $x = \gamma N \cdot t$ satisfies $\bigwedge_k (u_i \cdot t)$. Note that since $\alpha < P < \delta^M < \beta$ we already have $\bigwedge_k (r_i < x) \land \bigwedge_k (r_i < x) \land \bigwedge_k (R_n(x \cdot x))$. For showing $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$ we distinguish two cases. (1) If $m_i | N$ then $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$. For showing $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$ we distinguish two cases. (1) If $m_i | N$ then $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$. (2) If $m_i \not| N$, then $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$ since $m_i \not| M$. Since $P$ does not divide any of the numerators or denominators of (the reduced fractions of) $t_k$’s or $t_l$’s, then we must have $\bigwedge_k (P \not\mid N)$ which holds if and only if $m_i | N$; this contradicts our assumption $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$. Whence, all in all we showed that $\bigwedge_k (u_i \cdot t) \land \bigwedge_k (R_n(x \cdot x))$ holds.
Acknowledgements This research was partially supported by grant No. 90030053 of the Institute for Research in Fundamental Sciences (IPM), Tehran.

References

[1] E. Börger, E. Grädel, and Y. Gurevich, The Classical Decision Problem, Springer-Verlag, Berlin, 2001.

[2] J. Robinson, Definability and Decision Problems in Arithmetic, The Journal of Symbolic Logic 14 (1949), 98–114.

[3] D. Marker, Model Theory: An Introduction, Springer-Verlag, Berlin, 2002.

[4] C. Smoryński, Logical Number Theory I: An Introduction, Springer-Verlag, Berlin, 1991.

Saeed Salehi, “Computation in Logic and Logic in Computation”, Invited Paper in: Bahram B. Sadeghi (editor), Proceedings of the Third International Conference on Contemporary Issues in Computer and Information Sciences (CICIS 2012), Institute for Advanced Studies in Basic Sciences, Gavazangh, Zanjan, Iran, Brown Walker Press 2012, USA, ISBN 9781612336237 (624 pages), pp. 580–583. http://www.universal-publishers.com/book.php?method=ISBN&book=161233623X