Rough analysis of computation trees

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This paper deals with computation trees over an arbitrary structure consisting of a set along with collections of functions and predicates that are defined on it. It is devoted to the comparative analysis of three parameters of problems with \( n \) input variables over this structure: the complexity of a problem description, the minimum complexity of a computation tree solving this problem deterministically, and the minimum complexity of a computation tree solving this problem nondeterministically. Rough classification of relationships among these parameters is considered and all possible seven types of these relations are enumerated. The changes of relation types with the growth of the number \( n \) of input variables are studied.

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1. Introduction

Computation trees are well known models of algorithms. They are a natural generalization of decision trees: besides one-place operations of predicate type (attributes) which are used in decision trees, in computation trees many-place predicate and functional operations may be used. Just as in general algorithm theory, where both deterministic and nondeterministic algorithms are considered, it is expedient to study not only deterministic but also nondeterministic computation trees.

Linear decision trees and algebraic decision and computation trees were studied most intensively. Lower bounds on the complexity were obtained in [2,4,5] for linear decision trees, in [19–21] for algebraic decision trees, in [1,6] for algebraic computation trees, and in [7] for Pfaffian computation trees. Upper bounds on the complexity were obtained in [3,8,10] for linear decision trees and in [17] for quasilinear decision trees that includes linear decision trees and some kinds of algebraic decision trees. Nondeterministic linear decision and computation trees were studied in [9] and [11,17], respectively.

The complexity of deterministic decision trees over arbitrary infinite sets of \( k \)-valued attributes, \( k \geq 2 \), was studied in [16,17]. Relationships between deterministic and different kinds of nondeterministic decision trees over arbitrary infinite sets of \( k \)-valued attributes were investigated in [12,18].

In this paper, we study computation trees over an arbitrary structure \( U = (A, F, P) \) consisting of a set \( A \) along with a collection of functions \( F \) and a collection of predicates \( P \) that are defined on it.

For each natural \( n \), we describe a set \( \mathcal{P}(U, n) \) of problems over \( U \) with \( n \) input variables. Each such problem with input variables \( x_1, \ldots, x_n \) is given by a finite sequence of functional and predicate expressions over \( U \). This sequence defines \( r \) functions \( \alpha_1, \ldots, \alpha_r \) of the form \( p(t_1, \ldots, t_m) \), where \( p \in P \) and \( t_1, \ldots, t_m \) are functions with variables from the set \( \{x_1, \ldots, x_n\} \) obtained from functions contained in \( F \cup \{x\} \) by the operation of substitution. The functions \( \alpha_1, \ldots, \alpha_r \) divide the set \( A^n \) into areas in which these functions are constant. Each area is labeled with a finite nonempty set of solutions. For
a given n-tuple \( \vec{a} \in A^n \), we should find a solution from the set attached to the area to which \( \vec{a} \) belongs. Various problems of combinatorial optimization, pattern recognition, computational geometry, etc., can be represented in such form.

We define a complexity measure \( \psi \) and, for each problem \( z \in P(U, n) \), we consider three parameters: \( \psi_U(z) \) – the complexity of the problem \( z \) description, \( \psi_U^C(z) \) – the minimum complexity of a computation tree that solves the problem \( z \) deterministically, and \( \psi_U^D(z) \) – the minimum complexity of a computation tree that solves the problem \( z \) nondeterministically. The pair \((U, \psi)\) is called a sm-pair ((structure, measure)-pair).

To study relationships among these parameters, for each \( b, c \in \{i, d, a\} \), we consider two partial functions defined on the set of nonnegative integers:

\[
L_{bc}^U(m) = \max\{\psi_U^b(z) : z \in P(U, n), \psi_U^c(z) \leq m\},
\]

\[
L_{bc}^U(m) = \min\{\psi_U^b(z) : z \in P(U, n), \psi_U^c(z) \geq m\}.
\]

If the value \( L_{bc}^U(m) \) is defined for some \( m \), then it is the unimprovable upper bound on the value \( \psi_U^b(z) \) for problems \( z \in P(U, n) \) such that the inequality \( \psi_U^c(z) \leq m \) holds. If the value \( L_{bc}^U(m) \) is defined for some \( m \), then it is the unimprovable lower bound on the value \( \psi_U^b(z) \) for problems \( z \in P(U, n) \) such that the inequality \( \psi_U^c(z) \geq m \) holds.

The study of the functions \( L_{bc}^U \) and \( U_{bc}^L \) directly is, in general case, too complicated problem. Therefore, instead of the functions \( L_{bc}^U \) and \( U_{bc}^L \), we study their types \( \text{typ}(L_{bc}^U) \) and \( \text{typ}(U_{bc}^L) \) from the set \( \{\alpha, \beta, \gamma, \delta, \varepsilon\} \). In particular, the type \( \alpha \) means that the considered function \( L_{bc}^U \) has infinite domain and is bounded from above, types \( \beta, \gamma, \delta \) describe different variants of the growth of the unbounded function \( L_{bc}^U \) with an infinite domain, and the type \( \varepsilon \) means that the function \( L_{bc}^U \) has a finite domain.

All pairs \( \text{typ}(L_{bc}^U), \text{typ}(U_{bc}^L) \), \( b, c \in \{i, d, a\} \), form the n-type of the sm-pair \((U, \psi)\) that is the table \( \text{typ}(U, \psi, n) \) with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices \( i, d, a \), and the pair \( \text{typ}(L_{bc}^U), \text{typ}(U_{bc}^L) \) is in the intersection of the row with index \( b \in \{i, d, a\} \) and the column with index \( c \in \{i, d, a\} \). We describe all possible seven n-types of sm-pairs. These results are similar to ones obtained for decision trees [12].

For the sm-pair \((U, \psi)\), n-types can change with the increasing of \( n \). To investigate this phenomenon, we study the infinite sequence

\[
\text{type}(U, \psi) = \text{typ}(U, \psi, 1)\text{typ}(U, \psi, 2)\cdots
\]

that is called the dynamic type of the sm-pair \((U, \psi)\). In this paper, we describe all possible dynamic types of sm-pairs.

Some preliminary results in this direction were published without proofs in [13–15]. The publication of the final results with proofs was postponed for years: only in the present paper it was possible to finally resolve the issue of the structure of the set of all possible dynamic types of sm-pairs. This required considering of non-trivial constructions in Section 6.

This section contains also examples of study of different sm-pairs.

The rest of the paper is organized as follows. In Sections 2 and 3, basic notions and main results are considered. Sections 4–7 are devoted to the proofs of auxiliary statements and main theorems. Section 8 contains some explanations of the results of this paper and Section 9 – short conclusions.

2. Basic notions

In this section, we consider the notions of structure, computation tree, problem, complexity measure, sm-pair ((structure, measure)-pair), type of function, n-type of sm-pair, and dynamic type of sm-pair.

2.1. Structures

Let \( \omega = \{0, 1, 2, \ldots\} \) be the set of nonnegative integers, \( E_2 = \{0, 1\} \), and \( X = \{x_i : i \in \omega\} \) be the set of variables.

**Definition 1.** A structure is a triple \( U = (A, F, P) \) such that A is a nonempty set (universe), \( F \) is a set of functions of the kind \( f(x_1, \ldots, x_n) \), where \( f : A^n \to A \) and \( n \in \omega \) (if \( n = 0 \), then \( f \) is a constant), \( P \) is a nonempty set of predicates (relations) of the kind \( p(x_1, \ldots, x_n) \), where \( p : A^n \to E_2 \) and \( n \in \omega \setminus \{0\} \), and \( F \cap P = \emptyset \).

**Example 1.** The structure considered in [1] contains the set of real numbers \( \mathbb{R} \) as the universe, all constants from \( \mathbb{R}, +, \cdot, \times, \div, \sqrt{} \) as functions, and \( =, >, \geq \) as predicates.

We denote by \( [F] \) the set of all functions with variables from \( X \) obtained from functions contained in \( F \cup \{x\} \) by the operation of substitution. We denote by \( P[F] \) the set of all functions of the kind \( p(t_1, \ldots, t_n) \), where \( t_1, \ldots, t_n \in [F] \) and \( p \) is a predicate from \( P \) with \( n \) variables.

An expression \( x_i \leftarrow f(x_{i_1}, \ldots, x_{i_r}) \), where \( f \) is a function from \( F \) with \( r \) variables, will be called a functional expression over \( U \). An expression \( p(x_{i_1}, \ldots, x_{i_k}) \), where \( p \) is a predicate from \( P \) with \( k \geq 1 \) variables, will be called a predicate expression over \( U \).
Consider a pair \((Y, \beta)\), where \(Y\) is a finite nonempty subset of the set \(X\) and \(\beta\) is a finite sequence of functional and predicate expressions over \(U\). We now correspond to each predicate expression from \(\beta\) a function from \(P[F]\). Let \(\beta = \beta_1, \ldots, \beta_m\) and \(s\) be the minimum number of \(w\) such that all variables from \(Y\) and all variables from the expressions \(\beta_1, \ldots, \beta_m\) are contained in the set \(\{x_0, \ldots, x_s\}\). Let \(x_w\) be the variable from the set \(Y\) with the minimum index \(w\).

For \(i = 0, \ldots, m\), we define a sequence \(t_i = t_{i0}, \ldots, t_{is}\) of functions from \(\{F\}\) with variables from \(Y\). Let \(j \in \{0, \ldots, s\}\). If \(x_j \in Y\), then \(t_{ij} = x_j\). If \(x_j \notin Y\), then \(t_{ij} = x_w\). Let sequences \(t_{0i}, \ldots, t_{mi}, 0 \leq i < m\), be already defined. If \(\beta_{i+1}\) is a predicate expression, then \(t_{i+1} = t_i\). If \(\beta_{i+1}\) is a functional expression \(x_j \leftarrow f(x_{i1}, \ldots, x_{im})\), then \(t_{i+1} = t_{i0}, \ldots, t_{ij-1}, f(t_{ij}, \ldots, t_{im}), t_{i+1} = t_{i0}, \ldots, t_{ij-1}, t_{ij+1}, \ldots, t_{im}\).

Let there be exactly \(r > 0\) predicate expressions \(\beta_1, \ldots, \beta_r\) among \(\beta_1, \ldots, \beta_m\), where \(c_1 < \cdots < c_r\). For \(i = 1, \ldots, r\), we associate with the expression \(\beta_i\) a function \(\alpha_i \in P[F]\) with variables from \(Y\). Let \(\beta_i\) be an expression \(p(x_{i1}, \ldots, x_{ik})\). Then \(\alpha_i = p(t_{i1}, \ldots, t_{ik})\). We denote by \(\Pi(Y, \beta)\) the \(r\)-tuple \((\alpha_1, \ldots, \alpha_r)\).

Example 2. To illustrate the definitions, consider the simple structure \(U_0 = (A_0, F_0, P_0)\), where \(A_0 = \mathbb{R}\), \(F_0 = \{f(x_1)\}, f(x_1) = x_1 - 1\), and \(P_0 = \{p(x_1)\}, p(x_1) = 0\) if \(x_1 \leq 0\) and \(p(x_1) = 1\) if \(x_1 > 0\).

For this structure, \([F_0] = \{x_1 - j, j \in \omega\}\) and \(P_0[F_0] = \{l_j(x_1) : j \in \omega\}\), where \(l_j(x_1) = 0\) if \(x_1 \leq j\) and \(l_j(x_1) = 1\) if \(x_1 > j\).

Let us consider a pair \((Y_0, \beta^0)\), where \(Y_0 = \{x_0\}\) and

\[
\beta^0 = p(x_0), x_1 \leftarrow f(x_0), p(x_1).
\]

In the sequence \(\beta^0\), there are two predicate expressions. It is easy to show that \(\Pi(Y_0, \beta^0) = (l_0(x_0), l_1(x_0))\).

2.2. Computation trees

A node in a finite directed tree is called the root, if it is the only node without entering edges. A tree, which has such a node, is called a finite directed tree with the root. The tree nodes without leaving edges are called terminal nodes. The tree nodes, which are neither the root nor terminal, will be called working nodes. A complete path in a finite directed tree with the root is any sequence \(\xi = v_0, d_0, \ldots, v_m, d_m, v_{m+1}\) of nodes and edges of the tree such that \(v_0\) is the root, \(v_{m+1}\) is a terminal node, and the edge \(d_i\) leaves the node \(v_i\) and enters the node \(v_{i+1}\) for \(i = 0, \ldots, m\).

Definition 2. A computation tree over the structure \(U = (A, F, P)\) is a pair \(\Gamma = (Y, G)\), where \(Y\) is a finite nonempty subset of the set of variables \(X\) and \(G\) is a marked finite directed tree with the root, which has at least two nodes and satisfies the following conditions:

- The root and the edges leaving the root are not labeled.
- Each working node is a functional or a predicate node.
- Each functional node is labeled with a functional expression over \(U\), and each edge leaving a functional node is not labeled.
- Each predicate node is labeled with a predicate expression over \(U\), and each edge leaving a predicate node is labeled with a number from \(E_2\).
- Each terminal node is labeled with a number from \(\omega\).

Definition 3. A computation tree is called deterministic if it satisfies the following conditions:

- There is exactly one edge leaving the root.
- Each functional node has exactly one edge leaving it.
- For each predicate node, edges leaving this node are labeled with pairwise different numbers from \(E_2\).

The set of computation trees over the structure \(U\) will be denoted by \(\Upsilon(U)\). Let \(\Gamma = (Y, G)\) be a computation tree over \(U\). Nodes, edges and paths in the tree \(G\) are called nodes, edges and paths in the computation tree \(\Gamma\). The set \(Y\) is called the set of input variables for the computation tree \(\Gamma\). Let \(Y = \{x_1, \ldots, x_n\}\) and \(l_1 < \cdots < l_n\). Denote \(x = (x_1, \ldots, x_n)\). We denote by \(\mathcal{S}(\Gamma)\) the set of complete paths in \(\Gamma\). Let \(\xi = v_0, d_0, \ldots, v_m, d_m, v_{m+1}\) be a complete path in \(\Gamma\). We denote by \(s(\xi)\) the number assigned to the node \(v_{m+1}\). We now define a sequence \(\beta(\xi)\) of functional and predicate expressions over \(U\) and a subset \(A(\xi)\) of the set \(A^\theta\) associated with \(\xi\). If \(m = 0\), then \(\beta(\xi)\) is the empty sequence. Let \(m > 0\), and let the expression \(\beta_i\) be assigned to the node \(v_i, j = 1, \ldots, m\). Then \(\beta(\xi) = \beta_1, \ldots, \beta_m\). If there are no predicate expressions in the sequence \(\beta(\xi)\), then \(A(\xi) = A^\theta\). If there are exactly \(r > 0\) predicate expressions \(\beta_1, \ldots, \beta_r\) among \(\beta_1, \ldots, \beta_m\), where \(c_1 < \cdots < c_r\). Let \(\Pi(Y, \beta(\xi)) = (\alpha_1, \ldots, \alpha_r)\), and let \(\delta_i\) be the number assigned to the edge \(d_i, i = 1, \ldots, r\). Then \(A(\xi)\) is the set of solutions on \(A^\theta\) for the system of equations

\[
\{\alpha_i(x) = \delta_1, \ldots, \alpha_r(x) = \delta_r\}.
\]

Example 3 (Continuation of Example 2). Let \(Y_0 = \{x_0\}\). We now consider two computation trees \(\Gamma_0 = (Y_0, G_0)\) and \(\Gamma_d = (Y_0, G_d)\) over the structure \(U_0\) — see Fig. 1. The computation tree \(\Gamma_0\) is not deterministic, but the computation tree \(\Gamma_d\) is deterministic.

Let \(\xi\) be the complete path in the computation tree \(\Gamma_d\), which is finished in the terminal node labeled with the number 2. Then \(\beta(\xi) = x_1 \leftarrow f(x_0), p(x_1), p(x_0), \Pi(Y_0, \beta(\xi)) = (l_0(x_0), l_0(x_0)), \) and \(A(\xi)\) is the set of solutions on \(\mathbb{R}\) for the system of equations \(\{l_1(x_0) = 0, l_0(x_0) = 1\}\), i.e., \(A(\xi) = \{a \in \mathbb{R} : 0 < a < 1\}\).
2.3. Problems

The set of nonempty finite subsets of the set $\omega$ will be denoted by $S(\omega)$.  

**Definition 4.** A problem over the structure $U$ is a tuple of the kind $z = (Y, v, \beta_1, \ldots, \beta_m)$, where $Y$ is a nonempty finite subset of the set $X$, $m \in \omega \setminus \{0\}$, $\beta_1, \ldots, \beta_m$ are functional and predicate expressions over $U$, and there is $r \in \omega \setminus \{0\}$ such that $v : E^r_1 \rightarrow S(\omega)$ and there are exactly $r$ predicate expressions in the sequence $\beta_1, \ldots, \beta_m$.

The set $Y$ is called the set of input variables for the problem $z$. We denote by $\beta(z)$ the sequence $\beta_1, \ldots, \beta_m$. Let $|Y| = n$ and $\Pi(Y, \beta(z)) = (\alpha_1, \ldots, \alpha_r)$. The problem $z$ may be interpreted as a problem of searching for a number from the set $z(\bar{a}) = v(\alpha_1(\bar{a}), \ldots, \alpha_r(\bar{a}))$ for an arbitrary $\bar{a} \in A^n$. Different problems of pattern recognition, combinatorial optimization, and computational geometry can be represented in such form. We denote by $\mathcal{P}(U)$ the set of problems over the structure $U$.

Let $z = (Y_1, v, \beta_1, \ldots, \beta_m) \in \mathcal{P}(U)$ and $\Gamma = (Y_2, G) \in \mathcal{T}(U)$. Let $|Y_1| = n$.

**Definition 5.** We will say that the computation tree $\Gamma$ solves the problem $z$ nondeterministically if the following conditions hold:

- $Y_1 = Y_2$.
- $\bigcup_{\xi \in \mathcal{S}(\Gamma)} A(\xi) = A^n$.
- For any $\bar{a} \in A^n$ and any $\xi \in \mathcal{S}(\Gamma)$ such that $\bar{a} \in A(\xi)$, the relation $\kappa(\xi) \in z(\bar{a})$ holds.

**Definition 6.** We will say that the computation tree $\Gamma$ solves the problem $z$ deterministically if $\Gamma$ is a deterministic computation tree, which solves $z$ nondeterministically.

**Example 4 (Continuation of Examples 2 and 3).** Let us consider a problem $z_0 = (Y_0, v_0, \beta^0)$ over the structure $U_0$, where $Y_0 = \{x_0\}$ and $\beta(x_0) = \beta^0$ is the sequence $p(x_0), x_1 \leftarrow f(x_0), p(x_1)$ considered in Example 2. We know that $\Pi(Y_0, \beta(z_0)) = (l_0(x_0), l_1(x_0))$. The map $v_0$ is defined as follows: $v_0(0, 0) = \{1\}$, $v_0(1, 0) = \{2\}$, $v_0(1, 1) = \{1, 3\}$, and $v_0(0, 1) = \{4\}$.

We can reformulate the problem $z_0$ in the following way: for a given $a \in \mathbb{R}$, we should find a number from the set $z_0(a) = v_0(l_0(a), l_1(a))$. Fig. 2 represents a geometric interpretation of this problem. Two functions $l_0(x_0) = p(x_0)$ and $l_1(x_0) = p(f(x_0))$ divide the set of real numbers $\mathbb{R}$ into three areas that are labeled with the sets $\{1\}$, $\{2\}$ and $\{1, 3\}$, respectively. For a given $a \in \mathbb{R}$, we should recognize a number from the set attached to the area to which $a$ belongs.

One can show that the computation tree $\Gamma_1 = (Y_0, G_0)$ considered in Example 3 (see Fig. 1) solves the problem $z_0$ nondeterministically, and the computation tree $\Gamma_2 = (Y_0, G_2)$ considered in the same example (see Fig. 1) solves the problem $z_0$ deterministically.

2.4. Complexity measures and sm-pairs

Let $U = (A, F, P)$ be a structure. Denote by $(F \cup P)^*$ the set of all finite words over the alphabet $F \cup P$, including the empty word $\lambda$. 

![Fig. 1. Marked directed trees $G_a$ and $G_d$ from the computation trees $\Gamma_a = (Y_0, G_a)$ and $\Gamma_d = (Y_0, G_d)$.](image-url)
**Definition 7.** A complexity measure over the structure $U$ is any map of the kind $\psi : (F \cup P)^* \rightarrow \omega$. The complexity measure $\psi$ is called limited if it has the following properties:

- $\psi(\alpha_1 \alpha_2) \leq \psi(\alpha_1) + \psi(\alpha_2)$ for any $\alpha_1, \alpha_2 \in (F \cup P)^*$.
- $\psi(\alpha_1 \alpha_2 \alpha_3) \geq \psi(\alpha_1 \alpha_3)$ for any $\alpha_1, \alpha_2, \alpha_3 \in (F \cup P)^*$.
- For any $\alpha \in (F \cup P)^*$, the inequality $\psi(\alpha) \geq |\alpha|$ holds, where $|\alpha|$ is the length of $\alpha$.

We extend the complexity measure $\psi$ onto the set of all finite sequences of functional and predicate expressions over $U$ in the following way: $\psi(\beta) = \psi(\lambda)$ if $\beta$ is the empty sequence. Let $\beta$ be a nonempty sequence and $\beta = \beta_1, \ldots, \beta_m$. Then $\psi(\beta) = \psi(\alpha)$, where $\alpha = \beta_1 \cdots \beta_m \in (F \cup P)^*$ and, for $i = 1, \ldots, m$, if $\beta_i$ is a predicate expression $p(x_{\xi_1}, \ldots, x_{\xi_k})$, then $\beta_i = p$, and if $\beta_i$ is a functional expression $x_i \leftarrow f(x_{\xi_1}, \ldots, x_{\xi_k})$, then $\beta_i = f$.

We extend the complexity measure $\psi$ onto the set $T(U)$ of computation trees over $U$ as follows: $\psi(\Gamma) = \max\{\psi(\beta(x)) : \xi \in \mathcal{S}(\Gamma)\}$ for any $\Gamma \in T(U)$. The value $\psi(\Gamma)$ will be called the $\psi$-complexity of a computation tree $\Gamma$.

We now consider some examples of complexity measures. Let $w : (F \cup P) \rightarrow \omega \setminus \{0\}$. We define the function $\psi^w : (F \cup P)^* \rightarrow \omega$ in the following way: for any $\alpha \in (F \cup P)^*$, $\psi^w(\alpha) = 0$ if $\alpha = \lambda$, and $\psi^w(\alpha) = \sum_{i=1}^{m} w(\beta_i)$ if $\alpha = \beta_1 \cdots \beta_m$. The function $\psi^w$ is a limited complexity measure over $U$ and is called a weighted depth. If $w \equiv 1$, then the function $\psi^w$ is called the depth.

Let $\psi$ be a complexity measure over $U$ and $z \in P(U)$. The value $\psi^U(z) = \psi(\beta(z))$ is called the complexity of the problem $z$ description. We denote by $\psi^U_\psi(z)$ the minimum $\psi$-complexity of a computation tree $\Gamma \in T(U)$, which solves the problem $z$ deterministically. We denote by $\psi^U_\psi(z)$ the minimum $\psi$-complexity of a computation tree $\Gamma \in T(U)$, which solves the problem $z$ nondeterministically.

**Definition 8.** A (structure,measure)-pair or, in short, a sm-pair is a pair $(U, \psi)$ such that $U$ is a structure and $\psi$ is a complexity measure over $U$. If $\psi$ is a limited complexity measure, then the pair $(U, \psi)$ will be called a limited sm-pair.

**Example 5 (Continuation of Examples 2-4).** We define a complexity measure $\psi$ over the structure $U_0$ in the following way: for any word $\alpha$ over the alphabet $F_0 \cup P_0 = \{f, p\}$, $\psi(\alpha)$ is equal to the number of occurrences of the letter $p$ in the word $\alpha$. It is clear that the complexity measure $\psi$ is not limited. It is easy to see that, for the computation trees $\Gamma_a$ and $\Gamma_0$ considered in Example 3, $\psi(\Gamma_a) = \psi(\Gamma_0) = 2$. One can show that, for the problem $z_0$ considered in Example 4, $\psi^U_{\Gamma_0}(z_0) = \psi^U_{\Gamma_0}(z_0) = \psi^U_{\Gamma_0}(z_0) = 2$. Later we will study the sm-pair $(U_0, \psi)$.

2.5. n-Types and dynamic types of sm-pairs

Let $(U, \psi)$ be a sm-pair and $n \in \omega \setminus \{0\}$. We denote by $P(U, n)$ the set of problems from $P(U)$ with $n$ input variables.

We have the three parameters $\psi^U_\psi(z)$, $\psi^U_{\psi}(z)$, and $\psi^U_{\psi}(z)$ for any problem $z \in P(U, n)$, and we investigate the relationships between any two such parameters for problems from $P(U, n)$. Let us consider, for example, the parameters $\psi^U_\psi(z)$ and $\psi^U_{\psi}(z)$. Let $m \in \omega$. We will study relations $\psi^U_\psi(z) \leq m \Rightarrow \psi^U_{\psi}(z) \leq l$ for any $z \in P(U, n)$. The minimum value of $l$ is most interesting for us. This value (if exists) is equal to

$$L_{\psi_{\psi}}^U(m) = \max\{\psi^U_{\psi}(z) : z \in P(U, n), \psi^U_\psi(z) \leq m\}.$$ 

We also study relations $\psi^U_\psi(z) \geq m \Rightarrow \psi^U_{\psi}(z) \geq l$. In this case, the maximum value of $l$ is most interesting for us. This value (if exists) is equal to

$$L_{\psi_{\psi}}^U(m) = \min\{\psi^U_{\psi}(z) : z \in P(U, n), \psi^U_\psi(z) \geq m\}.$$ 

The two functions $L_{\psi_{\psi}}^U$ and $L_{\psi_{\psi}}^U$ describe how the behavior of the parameter $\psi^U_\psi(z)$ depends on the behavior of the parameter $\psi^U_{\psi}(z)$.

There are 18 similar functions for all ordered pairs of parameters $\psi^U_\psi(z)$, $\psi^U_{\psi}(z)$, and $\psi^U_{\psi}(z)$. These 18 functions well describe the relationships among the considered parameters. It would be very interesting to enumerate 18-tuples of these functions for all sm-pairs. But this is a very complicated problem.
In this paper, instead of functions we study types of functions. With any function, we associate its type from the set \( \{\alpha, \beta, \gamma, \delta, \varepsilon\} \). For example, if a function has infinite domain of definition, and it is bounded from above, then its type is equal to \( \alpha \). Thus, we enumerate 18-tuples of types of functions. These tuples are represented as tables called the \( n \)-types of sm-pairs. We also consider infinite sequences of the kind 1-type of sm-pair 2-type of sm-pair ... which are called dynamic types of sm-pairs and characterize changes of relationships among the considered parameters with the growth of the number of input variables.

We now give definitions of mentioned above notions. Let \( b, c \in \{i, d, a\} \). We define partial functions \( U_{\psi n}^{bc} : \omega \to \omega \) and \( L_{\psi n}^{bc} : \omega \to \omega \) as follows:

\[
U_{\psi n}^{bc}(m) = \max\{\psi_{\beta}^{b}(z) : z \in \mathcal{P}(U, n), \psi_{\varepsilon}^{b}(z) \leq m\},
\]

\[
L_{\psi n}^{bc}(m) = \min\{\psi_{\beta}^{b}(z) : z \in \mathcal{P}(U, n), \psi_{\varepsilon}^{b}(z) \geq m\}.
\]

If the value \( U_{\psi n}^{bc}(m) \) is defined, then it is the unimprovable upper bound on the value \( \psi_{\beta}^{b}(z) \) for problems \( z \in \mathcal{P}(U, n) \) such that the inequality \( \psi_{\varepsilon}^{b}(z) \leq m \) holds. If the value \( L_{\psi n}^{bc}(m) \) is defined, then it is the unimprovable lower bound on the value \( \psi_{\beta}^{b}(z) \) for problems \( z \in \mathcal{P}(U, n) \) such that the inequality \( \psi_{\varepsilon}^{b}(z) \geq m \) holds.

Let \( g \) be a partial function from \( \omega \) to \( \omega \). We denote by \( \text{Dom}(g) \) the domain of definition (domain for short) of \( g \). Let \( \text{Dom}^{+}(g) = \{ n : n \in \text{Dom}(g), g(n) > n \} \) and \( \text{Dom}^{-}(g) = \{ n : n \in \text{Dom}(g), g(n) \leq n \} \).

**Definition 9.** The type of the function \( g \) is the value \( \text{typ}(g) \in \{\alpha, \beta, \gamma, \delta, \varepsilon\} \) defined in the following way:

- If \( \text{Dom}(g) \) is an infinite set and \( g \) is a bounded from above function, then \( \text{typ}(g) = \alpha \).
- If \( \text{Dom}(g) \) is an infinite set, \( \text{Dom}^{+}(g) \) is a finite set, and \( g \) is an unbounded from above function, then \( \text{typ}(g) = \beta \).
- If each of the sets \( \text{Dom}^{+}(g) \) and \( \text{Dom}^{-}(g) \) is an infinite set, then \( \text{typ}(g) = \gamma \).
- If \( \text{Dom}(g) \) is an infinite set and \( \text{Dom}^{-}(g) \) is a finite set, then \( \text{typ}(g) = \delta \).
- If \( \text{Dom}(g) \) is a finite set, then \( \text{typ}(g) = \varepsilon \).

To clarify the notion of a function type, consider in more detail the function \( U_{\psi n}^{ab} \). One can show that \( \psi_{\beta}^{d}(z) \leq \psi_{\varepsilon}^{d}(z) \) for any problem \( z \in \mathcal{P}(U) \). Based on the problem description, it is easy to construct a computation tree, which solves this problem deterministically and which complexity is equal to the complexity of the problem description. Therefore \( \text{typ}(U_{\psi n}^{ab}) \in \{\alpha, \beta, \gamma\} \). If \( \text{typ}(U_{\psi n}^{ab}) = \alpha \), then there is a positive constant \( p \) such that \( \psi_{\beta}^{d}(z) \leq p \) for any problem \( z \in \mathcal{P}(U, n) \). If \( \text{typ}(U_{\psi n}^{ab}) = \gamma \), then there are infinitely many numbers \( m \in \omega \) for each of which there exists a problem \( z \in \mathcal{P}(U, n) \) with \( \psi_{\varepsilon}^{d}(z) = \psi_{\varepsilon}^{d}(z) = m \). The case \( \text{typ}(U_{\psi n}^{ab}) = \beta \) is the most interesting for us: the function \( U_{\psi n}^{ab} \) is not bounded from above and, for each problem with high enough complexity of description, there exists a computation tree, which solves this problem deterministically and which complexity is less than the complexity of the problem description.

**Example 6 (Continuation of Examples 2–5).** Let \( n \in \omega \setminus \{0\} \) and \( \psi \) be the complexity measure over the structure \( U_0 \) defined in Example 5. Denote \( g(m) = U_{\psi n}^{ab}(m) \). We will show that \( \text{typ}(g) = \beta \). It is clear that \( \text{Dom}(g) = \omega \setminus \{0\} \).

Let \( m > 2n \). Then we now show that \( g(m) \in \mathcal{P}(U, n) \) and \( \psi_{\varepsilon}^{d}(z) < m \). Then, as we already mentioned above, \( \psi_{\beta}^{d}(z) \leq \psi_{\varepsilon}^{d}(z) \) \( m \). Let \( \psi_{\varepsilon}^{d}(z) = m \) and \( Y \) be the set of input variables of \( z \). We now show that \( \psi_{\beta}^{d}(z) < m \). It is clear that \( \mathcal{P}(Y, \beta(z)) \) is an \( m \)-tuple of functions of the form \( l_i(x_i) \), where \( x_i \in Y \) and \( j \in \omega \). If in this tuple there are repeated functions, then, as it is easy to show, \( \psi_{\beta}^{d}(z) < m \). Let all functions in the tuple \( \mathcal{P}(Y, \beta(z)) \) are pairwise different. Since \( m > 2n \), in the considered tuple, for some \( x_i \in Y \), there are three functions \( l_j(x_i), l_j(x_i), l_j(x_i) \) such that \( j \leq j \leq j \). If \( l_j(x_i) = 1 \), then \( l_j(x_i) = 1 \). If \( l_j(x_i) = 0 \), then \( l_j(x_i) = 0 \). Therefore to find values of these three functions, it is enough to compute values of two of them. Using this fact, it is easy to show that \( \psi_{\beta}^{d}(z) < m \). As a result, we obtain that \( \text{typ}(g) = \beta \).
3. Main results

The main goal of this paper is to describe the set of all dynamic types of sm-pairs and the set of all dynamic types of limited sm-pairs. The solution of this problem allows us to characterize roughly all possible with the growth of the number of input variables changes of relationships among the complexity of problem description, the minimum complexity of computation trees solving this problem deterministically, and the minimum complexity of computation trees solving this problem nondeterministically.

Define seven tables:

\[
T_1 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & e & e & e & e \\
d & e & e & e & e \\
a & e & e & e & e \\
\hline
\end{array}
\quad T_2 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e & e \\
d & a & a & e & e & e \\
a & a & a & e & e \\
\hline
\end{array}
\]

\[
T_3 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e \\
d & a & b & g & e & e \\
a & a & a & e & e \\
\hline
\end{array}
\quad T_4 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e \\
d & a & a & g & e & e \\
a & a & a & e & e \\
\hline
\end{array}
\]

\[
T_5 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e \\
d & a & a & g & g & e \\
a & a & a & g & g \\
\hline
\end{array}
\quad T_6 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e \\
d & a & a & g & g & e \\
a & a & a & g & g \\
\hline
\end{array}
\]

\[
T_7 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & g & g & e & e \\
d & a & a & g & g & e \\
a & a & a & g & g \\
\hline
\end{array}
\]

Let T be a table and i ∈ ω. We denote by \( T^i \) the sequence \( T \cdots T \), where T is repeated i times (if \( i = 0 \), then \( T^i \) is the empty sequence). We denote by \( T^{\infty} \) the infinite sequence \( TTT \cdots \). Denote \( \Delta = \{ T_1^{\infty}, T_2^{\infty}, T_3^{\infty}, T_4^{\infty}, T_5^{\infty}, T_6^{\infty}, T_7^{\infty}, T_{2t3}^{\infty}, T_{3t2}^{\infty}, T_{2t4}^{\infty}, T_{4t2}^{\infty}, T_{3t5}^{\infty}, T_{5t3}^{\infty}, T_{2t6}^{\infty}, T_{6t2}^{\infty}, T_{4t7}^{\infty}, T_{7t4}^{\infty}, i, j, k ∈ \omega \}. \)

**Theorem 1.** For any sm-pair \((U, ψ)\), the relation \( dtyp(U, ψ) \in \{ T_1^{\infty} \} \cup \Delta \) holds. For any sequence \( σ \in \{ T_1^{\infty} \} \cup \Delta \), there exists a sm-pair \((U, ψ)\) such that \( dtyp(U, ψ) = σ \).

**Theorem 2.** For any limited sm-pair \((U, ψ)\), the relation \( dtyp(U, ψ) \in \Delta \) holds. For any sequence \( σ \in \Delta \), there exists a limited sm-pair \((U, ψ)\) such that \( ψ \) is a weighted depth and \( dtyp(U, ψ) = σ \).

**Example 7 (Continuation of Examples 2–6).** We know (see Example 6) that \( typ(U_{0, ψ}, n) = β \) for any \( n ∈ ω \setminus \{0\} \), where \( ψ \) is the complexity measure over the structure \( U_0 \) defined in Example 5. Using Theorem 1, we obtain that, for any \( n ∈ ω \setminus \{0\} \), \( typ(U_0, ψ, n) = T_3 \). Therefore \( dtyp(U_0, ψ) = T_3^{\infty} \).

4. Possible upper \( n \)-types of sm-pairs

In this section, we will enumerate all possible upper \( n \)-types of sm-pairs.

**Definition 12.** The upper \( n \)-type, \( n ∈ ω \setminus \{0\} \), of a sm-pair \((U, ψ)\) is the table \( typ_n(U, ψ, n) \) with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices \( i, d, a \) and the value \( typ(U_{i,j}^{bc}, n) \) is in the intersection of the row with index \( b ∈ \{i, d, a\} \) and the column with index \( c ∈ \{i, d, a\} \).

We now define seven tables:

\[
t_1 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & a & a & a \\
d & a & a & a \\
a & a & a & a \\
\hline
\end{array}
\quad t_2 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & γ & e & e \\
d & a & a & a & a \\
a & a & a & a \\
\hline
\end{array}
\]

\[
t_3 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & γ & e & e \\
d & a & a & γ & γ \\
a & a & a & γ & γ \\
\hline
\end{array}
\quad t_4 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & e & e \\
d & a & a & a \\
a & a & a & a \\
\hline
\end{array}
\quad t_5 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & e & e \\
d & a & a & γ & γ \\
a & a & a & γ & γ \\
\hline
\end{array}
\]

\[
t_6 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & e & e \\
d & a & a & γ & γ \\
a & a & a & γ & γ \\
\hline
\end{array}
\quad t_7 = \begin{array}{|c|c|c|}
\hline
i & d & a \\
\hline
i & γ & e & e \\
d & a & a & γ & γ \\
a & a & a & γ & γ \\
\hline
\end{array}
\]

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In this section, we will prove the following two propositions.

**Proposition 3.** For any sm-pair \((U, \psi)\) and any \(n \in \omega \setminus \{0\}\), the relation \(\text{typ}_n(U, \psi, n) \in \{t_1, \ldots, t_7\}\) holds.

**Proposition 4.** For any limited sm-pair \((U, \psi)\) and any \(n \in \omega \setminus \{0\}\), the relation \(\text{typ}_n(U, \psi, n) \in \{t_2, \ldots, t_7\}\) holds.

First, we prove some auxiliary statements.

**Lemma 5.** Let \((U, \psi)\) be a sm-pair and \(z \in \mathcal{P}(U)\). Then the inequalities \(\psi_U^0(z) \leq \psi_U^1(z) \leq \psi_U^2(z)\) hold.

**Proof.** Let \(z = (Y, v, \beta_1, \ldots, \beta_m)\). It is not difficult to construct a computation tree \(T_\beta \in \mathcal{T}(U)\), which solves the problem \(z\) deterministically and for which \(\beta(\xi) = \beta(z) = \beta_1, \ldots, \beta_m\) for any complete path \(\xi\) in the computation tree \(T_\beta\). Evidently, if \(\psi(T_\beta) = \psi_U^0(z)\). Therefore \(\psi_U^0(z) \leq \psi_U^1(z)\). If a computation tree \(T \in \mathcal{T}(U)\) solves the problem \(z\) non deterministically, then the computation tree \(T\) solves the problem \(z\) nondeterministically. Therefore \(\psi_U^1(z) \leq \psi_U^2(z)\). \(\square\)

Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), \(m \in \omega\), and \(b, c \in \{i, d, a\}\). The notation \(\mathcal{U}_{\psi, n}^{bc}(m) = \infty\) means that the set \(\{\psi_U^b(z) : z \in \mathcal{P}(U, n)\}\) is infinite. Evidently, if \(\mathcal{U}_{\psi, n}^{bc}(m) = \infty\), then \(\mathcal{U}_{\psi, n}^{bc}(m + 1) = \infty\). It is not difficult to prove the following statement.

**Lemma 6.** Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), \(m \in \omega\), and \(b, c \in \{i, d, a\}\). Then

- (a) \(\text{If there exists } m \in \omega \text{ such that } \mathcal{U}_{\psi, n}^{bc}(m) = \infty\), then \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \varepsilon\), \(\text{Dom}(\mathcal{U}_{\psi, n}^{bc}) = \emptyset\) if \(m_0 = m_1\), and \(\text{Dom}(\mathcal{U}_{\psi, n}^{bc}) = \{m : m \in \omega, m_0 \leq m < m_1\}\) if \(m_0 < m_1\), where \(m_0 = \min(\psi_U^c(z) : z \in \mathcal{P}(U, n))\) and \(m_1 = \min(m : m \in \omega, \mathcal{U}_{\psi, n}^{bc}(m) = \infty)\).
- (b) \(\text{If there is no } m \in \omega \text{ such that } \mathcal{U}_{\psi, n}^{bc}(m) = \infty\), then \(\text{Dom}(\mathcal{U}_{\psi, n}^{bc}) = \{m : m \in \omega, m \geq m_0\}\).

Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), \(b, c, e, f \in \{i, d, a\}\). The notation \(\mathcal{U}_{\psi, n}^{bc} < \mathcal{U}_{\psi, n}^{ef}\) means that, for any \(m \in \omega\), the following conditions hold:

- If the value \(\mathcal{U}_{\psi, n}^{bc}(m)\) is defined, then either \(\mathcal{U}_{\psi, n}^{ef}(m) = \infty\) or the value \(\mathcal{U}_{\psi, n}^{ef}(m)\) is defined and the inequality \(\mathcal{U}_{\psi, n}^{bc}(m) \leq \mathcal{U}_{\psi, n}^{ef}(m)\) holds.
- If \(\mathcal{U}_{\psi, n}^{bc}(m) = \infty\), then \(\mathcal{U}_{\psi, n}^{ef}(m) = \infty\).

We define a linear order \(\preceq\) on the set \(\{\alpha, \beta, \gamma, \delta, \varepsilon\}\) as follows: \(\alpha \preceq \beta \preceq \gamma \preceq \delta \preceq \varepsilon\).

**Lemma 7.** Let \((U, \psi)\) be a sm-pair and \(n \in \omega \setminus \{0\}\). Then \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \text{typ}(\mathcal{U}_{\psi, n}^{bd}) = \text{typ}(\mathcal{U}_{\psi, n}^{ba})\) and \(\text{typ}(\mathcal{U}_{\psi, n}^{ab}) \preceq \text{typ}(\mathcal{U}_{\psi, n}^{bc}) \preceq \text{typ}(\mathcal{U}_{\psi, n}^{bd}) \preceq \text{typ}(\mathcal{U}_{\psi, n}^{ba})\) for any \(b \in \{i, d, a\}\).

**Proof.** From the definition of the functions \(\mathcal{U}_{\psi, n}^{bc}\), \(b, c \in \{i, d, a\}\), and from Lemma 5 it follows that \(\mathcal{U}_{\psi, n}^{ab} < \mathcal{U}_{\psi, n}^{bd} < \mathcal{U}_{\psi, n}^{ba}\) and \(\mathcal{U}_{\psi, n}^{ab} < \mathcal{U}_{\psi, n}^{bc} < \mathcal{U}_{\psi, n}^{bd}\) for any \(b \in \{i, d, a\}\). Using these relations and Lemma 6 we obtain the statement of the lemma. \(\square\)

**Lemma 8.** Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), \(b, c \in \{i, d, a\}\). Then

- (a) \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \alpha\) if and only if the function \(\psi_U^0\) is bounded from above on the set \(\mathcal{P}(U, n)\).
- (b) If the function \(\psi_U^0\) is unbounded from above on the set \(\mathcal{P}(U, n)\), then \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \gamma\).

**Proof (a).** The first statement of the lemma is obvious. (b) Let the function \(\psi_U^0\) be unbounded from above on the set \(\mathcal{P}(U, n)\). One can show that in this case the equality \(\mathcal{U}_{\psi, n}^{bc}(m) = m\) holds for infinitely many \(m \in \omega\). Therefore \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \gamma\). \(\square\)

**Corollary 9.** Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), \(b, c \in \{i, d, a\}\). Then \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) \in \{\alpha, \gamma\}\).

Let \(U = (A, F, P)\) be a structure and \(n \in \omega \setminus \{0\}\). We denote by \(P_n[F]\) the set of functions from \(P[F]\) with variables from the set \(\{x_1, \ldots, x_n\}\). It is not difficult to prove the following statement.

**Lemma 10.** Let \((U, \psi)\) be a limited sm-pair, \(U = (A, F, P)\), and \(n \in \omega \setminus \{0\}\). Then the function \(\psi_U^0\) is bounded from above on the set \(\mathcal{P}(U, n)\) if and only if the set \(P_n[F]\) is a finite set.

**Lemma 11.** Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), and \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) \neq \alpha\). Then \(\text{typ}(\mathcal{U}_{\psi, n}^{bc}) = \text{typ}(\mathcal{U}_{\psi, n}^{bd}) = \varepsilon\). 
Proof. By Lemma 8, the function $\psi^t_\delta$ is unbounded from above on $P(U, n)$. Let $r \in \omega$. Then there exists a problem $z = (Y, v, \beta_1, \ldots, \beta_m) \in P(U, n)$ such that $\psi^t_\delta(z) \geq r$. Let us consider the problem $z' = (Y, v', \beta_1, \ldots, \beta_m)$, where $v' \equiv [0]$. It is clear that $\psi^t_\delta(z') \geq r$. Let $\Gamma' = (Y, C)$ be a computation tree which consists of the root, the terminal node labeled with 0 and the edge leaving the root and entering the terminal node. One can show that the computation tree $\Gamma'$ solves the problem $z'$ deterministically. Therefore $\psi^t_\delta(z') \leq \psi^t_\delta(z') \leq \psi(\Gamma') = \psi(\lambda)$. Taking into account that $r$ is an arbitrary number from $\omega$, we obtain $\psi^t_{U\delta}(\psi(\lambda)) = \infty$ and $\psi^t_{U\delta}(\psi(\lambda)) = \infty$. By Lemma 6, $\psi^t_{U\delta}(\psi(\lambda)) = \epsilon$. □

Lemma 12. Let $(U, \psi)$ be a sm-pair and $n \in \omega \setminus \{0\}$. Then $\psi^t_{U\delta}(\psi(\lambda)) \in \{\alpha, \gamma\}$.

Proof. Using Lemma 7 and Corollary 9 we obtain $\psi^t_{U\delta}(\psi(\lambda)) \in \{\alpha, \beta, \gamma\}$. Assume that $\psi^t_{U\delta}(\psi(\lambda)) = \beta$. Then there exists $r \in \omega \setminus \{0\}$ such that $\psi^t_{U\delta}(m) < m$ for any $m \in \omega$, $m > r$. We prove by induction on $m$ that, for any problem $z \in P(U, n)$, if $\psi^t_\delta(z) \leq m$, then $\psi^t_\delta(z) \leq r_0$, where $r_0 = \max(r, \psi(\lambda))$. From Lemma 5 it follows that under the condition $m \leq r$ the considered statement holds. Let it hold for some $m \geq r$. We now show that this statement holds for $m + 1$ too. Let $z \in P(U, n)$ and $\psi^t_\delta(z) \leq m + 1$. Since $m + 1 > r$, $\psi^t_\delta(z) \leq m$. Let $\Gamma' \in T(U)$, $\psi(\Gamma') = \psi(\lambda)$, and the computation tree $\Gamma'$ solves the problem $z$ nondeterministically. Assume that in $\Gamma'$ there exists a complete path $\xi$ such that in the sequence $\beta(\xi)$ there are no predicate expressions. In this case, a computation tree, which has the same set of input variables as $z$ and consists of the root, the terminal node labeled with $k(\xi)$, and the edge leaving the root and entering the terminal node, solves the problem $z$ nondeterministically. Therefore $\psi^t_\delta(z) \leq \psi(\lambda) \leq r_0$. Assume now that, for each complete path $\xi$ in the computation tree $\Gamma'$, the sequence $\beta(\xi)$ contains a predicate expression. Let $\xi \in E(\Gamma')$, $\xi = v_0, v_1, \ldots, v_p, v_{p+1}$ and let the expression $\beta_1$ be assigned to the node $v_1$, $i = 1, \ldots, p$. Let there be exactly $t > 0$ predicate expressions $\beta_1, \ldots, \beta_t$ among the expressions $\beta_1, \ldots, \beta_r$. Let $\delta_1 = \cdots = \delta_t$. For $i = 1, \ldots, t$, let $\delta_i$ be the number from the set $E(\Gamma')$ assigned to the edge $e_i$. Let us consider a problem $z_\delta = (Y, v_\delta, \beta_1, \ldots, \beta_r)$, where $Y$ is the set of input variables for the problem $z$, $v_\delta(\delta_1, \ldots, \delta_t) = \{t(\xi)\}$ and $v_\delta(\sigma) = \{(\xi) + 1\}$ for any $t$-tuple $\sigma \in E(\Gamma')$ such that $\sigma \neq (\delta_1, \ldots, \delta_t)$. It is clear that $\psi^t_\delta(z_\delta) \leq m$. Using the inductive hypothesis we obtain that there exists a computation tree $\Gamma_\xi \in T(U)$, which has the following properties: $\Gamma_\xi$ solves the problem $z_\delta$ nondeterministically and $\psi(\Gamma_\xi) \leq r_0$. Let $A(\xi) \neq \emptyset$. We denote by $\Gamma_{\xi}'$ a computation tree obtained from $\Gamma_\xi$ by removal of all nodes and edges satisfying the following condition: there is no a complete path $\xi' \in \Gamma_{\xi}'$, which contains this node or edge and for which $\psi(\xi') = \psi(\xi)$. Let $\xi' \in E(\Gamma_{\xi})$, $\psi(\xi') \neq \emptyset$ be a limited sm-pair. In this section, we prove the following two propositions.

5. Possible upper dynamic types of sm-pairs

In this section, we begin the study of upper dynamic types of sm-pairs.

Definition 13. The upper dynamic type of a sm-pair $(U, \psi)$ is the infinite sequence $d\psi^{U, \psi} = \psi_{U, \psi}(U, 1)\psi_{U, \psi}(U, 2)\cdots$.

Denote $\Delta_\psi = \{t^{\infty}, t_1t^{\infty}, t_1t_2t^{\infty}, t_1t_2t_3t^{\infty}, t_1t_2t_3t_4t^{\infty}, t_1t_2t_3t_4t_5t^{\infty}, t_1t_2t_3t_4t_5t_6t^{\infty}, t_1t_2t_3t_4t_5t_6t_7t^{\infty} : i, j, k \in \omega\}$. In this section, we prove the following two propositions.
Proposition 14. For any sm-pair \((U, \psi)\), the relation \(dtyp_{\omega}(U, \psi) \in \{t^\infty_1\} \cup \Delta_u\) holds.

Proposition 13. For any sm-pair \((U, \psi)\), the relation \(dtyp_{\omega}(U, \psi) \in \Delta_u\) holds.

Proof. Let \(U = (A, F, P)\). Let \(z = (Y, v, \beta_1, \ldots, \beta_m)\), and \(s\) be the minimum number from \(\omega\) such that all variables from \(Y\) and all variables from the expressions \(\beta_1, \ldots, \beta_m\) belong to the set \(\{x_0, \ldots, x_t\}\). Denote \(r = s + 1\) and \(\tilde{z} = (Y \cup \{x_i\}, v, \beta_1, \ldots, \beta_m)\). We will prove that, for any \(b \in \{i, d, a\}\), the equality \(\psi^U_\omega(z) = \psi^U_\omega(\tilde{z})\) holds. It is clear that the considered equality holds if \(b = i\). We now consider the case \(b = d\).

One can show that there exists a computation tree \(\Gamma_1 = (Y, G_1)\) over \(U\) satisfying the following conditions:

1. \(\Gamma_1\) solves the problem \(z\) nondeterministically.
2. \(\psi(\Gamma_1) = \psi_{\omega}^U(z)\).
3. The variable \(x_i\) is not contained in the expressions assigned to nodes of \(\Gamma_1\).

Denote \(\Gamma_2 = (Y \cup \{x_i\}, G_1)\). Let \(\tilde{a} = (a_1, \ldots, a_n, a_{n+1}) \in A^{n+1}\), \(\tilde{a} = (a_1, \ldots, a_n, a_{n+1})\), \(\xi\) be a complete path in \(\Gamma_2\), and \(\xi\) be the complete path in \(\Gamma_1\), which coincides with \(\xi\). One can show that \(\tilde{z}(\tilde{a}) = \tilde{z}(\tilde{a})\) and \(\tilde{a} \in A(\xi)\) if and only if \(\tilde{a} \in A(\xi)\). Using these relations it is not difficult to show that \(\Gamma_2\) solves the problem \(\tilde{z}\) nondeterministically. Therefore \(\psi^U_\omega(z) \leq \psi^U_\omega(\tilde{z})\).

Denote \(\Gamma_3 = (Y \cup \{x_i\}, G_3)\). Let \(\tilde{a} = (a_1, \ldots, a_n) \in A^n\), \(\tilde{a} = (a_1, \ldots, a_n, a_{n+1})\), \(\xi\) be a complete path in \(\Gamma_3\), and \(\xi\) be the complete path in \(\Gamma_1\), which coincides with \(\tilde{z}\). One can show that \(z(\tilde{a}) = z(\tilde{a})\) and \(\tilde{a} \in A(\xi)\) if and only if \(\tilde{a} \in A(\tilde{z})\). Using these relations one can show that \(\Gamma_3\) solves the problem \(z\) nondeterministically. Therefore \(\psi^U_\omega(z) \leq \psi^U_\omega(\tilde{z})\).

We now define a partial order \(\leq\) on the set of tables \(\{t_1, \ldots, t_7\}\). For \(b, c \in \{i, d, a\}\) and \(i \in \{1, \ldots, 7\}\), let \(t^i\) be the letter from the set \([\alpha, \beta, \gamma, \delta, \varepsilon]\) that is in the table \(t_i\) in the intersection of the row with index \(b\) and the column with index \(c\). Let \(i, j \in \{1, \ldots, 7\}\). Then \(t_i \leq t_j\) if and only if \(t^i \leq t^j\) for any \(b, c \in \{i, d, a\}\). It is easy to check that the graph depicted in Fig. 3 is the Hasse diagram for the partially ordered set \(\{t_1, \ldots, t_7\}\). Nodes of this diagram are tables \(t_1, \ldots, t_7\). An edge goes upward from \(t_i\) to \(t_j\) if \(t_i \leq t_j\) and there is no \(t_k, t_k \notin \{t_i, t_j\}\) such that \(t_i \leq t_k \leq t_j\).

Proof of Proposition 13. Let \((U, \psi)\) be a sm-pair. Assume that there exists \(n_0 \in \omega \setminus \{0\}\) for which \(dtyp_{\omega}(U, \psi, n_0) = t^i_1\). Then, by Lemma 8, the function \(\psi^U_\omega\) is bounded from above on the set \(\{t^i_1\}, \forall n \in \omega \setminus \{0\}\). From here it follows that the function \(\psi^U_\omega\) is bounded from above on the set \(\{t^i_1\}, \forall n \in \omega \setminus \{0\}\). Using Lemmas 5 and 8 we obtain \(dtyp_{\omega}(U, \psi) = t^i_1\).

Let for any \(n \in \omega \setminus \{0\}\), the relation \(dtyp_{\omega}(U, \psi, n) \neq t^i_1\) hold. In this case, by Proposition 3, \(dtyp_{\omega}(U, \psi, n) \in \{t_2, \ldots, t_7\}\) for any \(n \in \omega \setminus \{0\}\). Using Lemma 15, we obtain that \(dtyp_{\omega}(U, \psi, n) \leq dtyp_{\omega}(U, \psi, n+1)\) for any \(n \in \omega \setminus \{0\}\). Simple analysis of the Hasse diagram for the partially ordered set \(\{t_1, \ldots, t_7\}\) depicted in Fig. 3 shows that the set \(\Delta_u\) coincides with the set of infinite sequences \(t_1, t_2, \ldots\) such that \(t_1, t_2, \ldots \in \{t_2, \ldots, t_7\}\) and \(t_1 \leq t_2 \leq \ldots\). Therefore \(dtyp_{\omega}(U, \psi) \in \Delta_u\).

Proof of Proposition 14. Let \((U, \psi)\) be a limited sm-pair. Using Proposition 4 we obtain \(dtyp_{\omega}(U, \psi) \neq t^i_1\). From this relation and Proposition 13 it follows that \(dtyp_{\omega}(U, \psi) \in \Delta_u\).

6. Realizable upper dynamic types of sm-pairs

In this section, we continue the study of upper dynamic types of sm-pairs and prove the following two propositions.

Proposition 16. For any sequence \(\tau \in \{t^i_1\} \cup \Delta_u\), there exists a sm-pair \((U, \psi)\) such that \(dtyp_{\omega}(U, \psi) = \tau\).

Proposition 17. For any sequence \(\tau \in \Delta_u\), there exists a limited sm-pair \((U, \psi)\) such that \(dtyp_{\omega}(U, \psi) = \tau\) and \(\psi\) is a weighted depth.
We now describe a construction that will be used in the proofs of the considered propositions.

For $i = 2, \ldots, 7$, we define a sm-pair $\pi_i = (U_i, \psi_i)$, where $U_i = (A_i, F_i, P_i)$, $F_i = \emptyset$, $P_i$ is a set of one-place predicates, and $\psi_i$ is a weighted depth. It is clear that for the definition of the function $\psi_i$ it is enough to define values of $\psi_i$ on elements of the set $P_i$.

Define the sm-pair $\pi_2$ as follows: $A_2 = \{0\}$, $P_2 = \{q_1\}$, $q_1(0) = 0$, and $\psi_2(q_1) = 1$.

Define the sm-pair $\pi_3$ as follows: $A_3 = \omega$, $P_3 = \{l_i : i \in \omega\}$,

$$l_i(j) = \begin{cases} 0, & j \leq i, \\ 1, & j > i, \end{cases}$$

and $\psi_3(l_i) = 1$ for any $i, j \in \omega$, i.e., $\psi_3 = h$.

Define the sm-pair $\pi_4$ as follows: $A_4 = \omega$, $P_4$ is the set of mappings of the kind $f : \omega \to \{0, 1\}$, and $\psi_4(f) = 1$ for any $f \in P_4$, i.e., $\psi_4 = h$.

Define the sm-pair $\pi_5$ as follows: $A_5 = \omega$, $P_5 = \{q_i : i \in \omega\}$,

$$q_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i \end{cases}$$

for any $i, j \in \omega$, $\psi_5(q_0) = 1$, and $\psi_5(q_i) = i \sum_{j=0}^{i-1} \psi_5(q_j)$ for $i \geq 1$.

Define the sm-pair $\pi_6$ as follows: $A_6 = \omega$, $P_6 = \{q_{2i}, q_{2i+1}, p_{2i} : i \in \omega \setminus \{0\}\}$, $q_{2i}(j) = 1$ if and only if $j = 2i$, $q_{2i+1}(j) = 1$ if and only if $j = 2i+1$, $p_{2i}(j) = 1$ if and only if $j = 2i$ or $j = 2i+1$, and $\psi_6(q_{2i}) = \psi_6(q_{2i+1}) = \psi_6(p_{2i}) = i$ for any $i \in \omega \setminus \{0\}$ and $j \in \omega$.

Define the sm-pair $\pi_7$ as follows: $A_7 = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, $P_7 = \{l_i : i \in \omega\} \cup \{q_{-k} : k \in \omega \setminus \{0\}\}$,

$$l_i(j) = \begin{cases} 0, & j \leq i, \\ 1, & j > i, \end{cases} \quad q_{-k}(j) = \begin{cases} 0, & j \neq -k, \\ 1, & j = -k. \end{cases}$$

$\psi_7(l_i) = 1$, and $\psi_7(q_{-k}) = k$ for any $i \in \omega$, $k \in \omega \setminus \{0\}$, and $j \in \mathbb{Z}$.

Let $r \in \{2, \ldots, 7\}$ and $n \in \omega \setminus \{0\}$. Define a sm-pair $\pi_r^{(n)} = (U_r^{(n)}, \psi_r^{(n)})$, where $U_r^{(n)} = (A_r^{(n)}, F_r^{(n)}, P_r^{(n)})$ and $\psi_r^{(n)}$ is a weighted depth. Let $K = \{k_i : i \in \omega\}$ be a set such that $k_i \neq k_j$ if $i \neq j$ and $K \cap \mathbb{Z} = \emptyset$. For any $c \in A_r \cup K$, denote
Let for any different sorts of such functions. Using an approach similar to the binary search algorithm it is not difficult to show that, expression of the kind problem typ

\[ n \in A \]

\[ \tau \in \Delta_u \]

\[ \pi_{\tau}(n) = \psi_{\pi}(g) \]

for any \( g \in P \).

Let \( t_1, \ldots, t_m \in \{2, \ldots, 7\} \), \( n_1, \ldots, n_m \in \omega \setminus \{0\} \), and \( n_1 < \cdots < n_m \). Define the sm-pair \( \pi_{t_1}^{(n_1)} \oplus \cdots \oplus \pi_{t_m}^{(n_m)} \) as follows:

\[ \pi_{t_1}^{(n_1)} \oplus \cdots \oplus \pi_{t_m}^{(n_m)} = (U, \psi), U = (A, F, P), A = \pi_{t_1}^{(n_1)} \cup \cdots \cup \pi_{t_m}^{(n_m)}, F = \emptyset, P = P_{t_1}^{(n_1)} \cup \cdots \cup P_{t_m}^{(n_m)}, \]

and \( \psi_{\pi}(g) = \psi_{\pi}(g) \) for any \( g \in P \).

Using Proposition 3 we obtain

\[ \gamma \]

where

\[ g \in P \]

and if \( r < m \), then \( n < n_{r+1} \). Let \( U_r = (A, F, P) \). It is clear that all functions from \( P[F] \) are of the kind \( g(x_1, \ldots, x_m) \), where \( g \in P \). Since \( n < n_{r+1} \), any function from \( P[F] \) depends on \( n \) variables and does not equal to \( 0 \) identically to \( 0 \) of the kind \( g(x_1, \ldots, x_n) \), where \( j \leq r, g \in P \), and \( x_1, \ldots, x_n \) are pairwise different variables. The tuple \( x_1, \ldots, x_n \) will be called the sort of the considered function. Let \( Y \) be a sort of variables and \( |Y| = n \). Then a function depending on \( n \) pairwise different variables from \( Y \) may have exactly \( (n)_m = n(n-1) \cdots (n-m+1) \) different sorts.

We now prove some statements about properties of sm-pairs \((U, \psi, r)\), where \( r \in \Delta_u \).

**Lemma 18.** Let \( r = 2 \), \( r = 1 \). Denote \( (U, \psi) = (U, \psi, r) \). It is clear that the function \( \psi_{\pi}^{(n)} \) is unbounded from above on the set \( \mathcal{P}(U) \). Using Lemma 8 we obtain that \( \psi_{\pi}^{(n)} = \gamma \). Let \( U = (A, F, P) \). Taking into account that in the case \( m \geq 2 \) the inequality \( n < n_2 \) holds, one can show that all functions from \( P[F] \) depending on \( n \) variables are equal identical to \( 0 \). Therefore the function \( \psi_{\pi}^{(n)} \) is bounded from above on the set \( \mathcal{P}(U) \). From here and from Lemma 8 it follows that \( \psi_{\pi}^{(n)} = \alpha \). Using Proposition 3 we obtain \( \psi_{\pi}^{(n)} = \gamma \).

**Proof.** Since \( n = 2, n = 1 \). In the latter case, \( n = 2 \). Denote \( (U, \psi) = (U, \psi, r) \). Let \( U = (A, F, P) \). Since \( n < n_{r+1} \), all functions from \( P[F] \), which depend on \( n \) variables and do not equal to \( 0 \) identically to \( 0 \) are of the kind \( \pi_{t_1}^{(n_1)} \cup \cdots \cup \pi_{t_m}^{(n_m)} \), where \( x_1, \ldots, x_m \) are pairwise different variables.

We now prove that \( \psi_{\pi}^{(n)} \neq \gamma \). Let \( m \in \omega \setminus \{0\} \), \( z = (Y, v, \beta_1, \ldots, \beta_t) \in \mathcal{P}(U, n) \), and \( t \leq m \). One can transform the problem into a problem \( z' = (Y, v', \alpha_1, \ldots, \alpha_t) \) such that \( z(\tilde{a}) = z'(\tilde{a}) \) for any \( \tilde{a} \in A \), \( t' \leq t \) and, for \( s = 1, \ldots, t ' \), the expression \( \alpha_t \) is an expression of the kind \( \pi_{t_1}^{(n_1)} \cup \cdots \cup \pi_{t_m}^{(n_m)} \), where \( x_1, \ldots, x_m \) are pairwise different variables from \( Y \). There are exactly \( (n)_m \) different sorts of such functions. Using an approach similar to the binary search algorithm it is not difficult to show that, for any \( c \) functions of the same sort, there exists a computation tree over \( U \), which computes values of the considered \( c \) functions and which depth is at most \( 1 + \log_2 c \). Using this fact it is not difficult to show that \( h_{d_1}^{(\gamma)}(\tilde{z}') \leq (n)_m (1 + \log_2 t') \) and \( h_{d_2}^{(\gamma)}(\tilde{z}) \leq (n)_m (1 + \log_2 t) \). Taking into account that \( z \) is an arbitrary problem from \( \mathcal{P}(U, n) \) such that \( \psi_{\pi}^{(n)}(z) \leq m \), we obtain \( \psi_{\pi}^{(n)}(m) \leq (n)_m (1 + \log_2 m) \). Therefore \( \psi_{\pi}^{(n)}(m) < m \) for large enough \( m \) and \( \psi_{\pi}^{(n)} \neq \gamma \).

We now prove that \( \psi_{\pi}^{(n)} \neq \alpha \). Assume the contrary. Using Lemma 8 we obtain that there exists a number \( p \in \omega \) such that \( \psi_{\pi}^{(n)}(z) \leq p \) for any problem \( z \in \mathcal{P}(U, n) \). Therefore there exists a number \( q \in \omega \setminus \{0\} \) which satisfies the following condition: for any problem \( z \in \mathcal{P}(U, n) \), there exists a computation tree \( \Gamma' \in \mathcal{P}(U) \), which solves the problem \( z \) deterministically and has at most \( q \) terminal nodes. Let us consider a problem

\[ z' = (Y, v, \pi_{t_1}^{(n_1)}(x_1, \ldots, x_m), \ldots, \pi_{t_m}^{(n_m)}(x_1, \ldots, x_m)) \]

from \( \mathcal{P}(U, n) \), where \( Y = (x_1, \ldots, x_n), v : E_2 \to S(\omega) \) and \( v(\delta_1) \cap v(\delta_2) = \emptyset \) for any \( \delta_1, \delta_2 \in E_2 \) such that \( \delta_1 \neq \delta_2 \). It is easy to check that there are \( q + 1 \) tuples \( \delta_1, \ldots, \delta_{q+1} \in A^n \), such that \( z(\tilde{a}) \cap z(\tilde{a}) = \emptyset \) for any \( i, j \in \{1, \ldots, q + 1\}, i \neq j \). Let \( \Gamma' \) be an arbitrary computation tree over \( U \), which solves the problem \( z' \) deterministically. It is not difficult to show
that the tree $I$ should have at least $q + 1$ terminal nodes. We obtain a contradiction. Therefore \( \text{typ}(I_{U_{\psi}|aN}) = \beta \). From this equality and Proposition 3 it follows that \( \text{typ}_0(U_\tau, \psi_\tau, n) = t_3 \). □

**Lemma 20.** Let \( \tau \in \Delta_{U_\tau}, \left[U_\tau, \psi_\tau \right] = \pi_1^{(n_1)} \oplus \cdots \oplus \pi_\nu^{(n_\nu)}, \ r \in \{1, \ldots, m\}, \ v_r = 4, \ n \in \omega \ \{0\}, \ n \geq n_r \) and if \( r < m \), then \( n \leq n_{r+1} \). Then \( \text{typ}_0(U_\tau, \psi_\tau, n) = t_4 \).

**Proof.** One can show that if \( r > 1 \), then \( \{v_1, \ldots, v_{r-1}\} \subseteq \{2, 3\} \). Denote \( (U, \psi) = (U_\tau, \psi_\tau) \). Let \( U = (A, F, P) \).

We show that \( \text{typ}(I_{U_{\psi}|n}) = \alpha \). Let \( Y \) be a set of \( n \) variables. A system of equations

\[
\begin{align*}
\{g_1 = \delta_1, \ldots, g_m = \delta_m\},
\end{align*}
\]

where \( g_1, \ldots, g_m \) are functions from \( P \) depending on \( Y \) and \( \delta_1, \ldots, \delta_m \in E_2 \), is called a system of equations over \( Y \). We show that, for any consistent on \( A^\alpha \) system of equations over \( Y \), there exists a system of equations over \( Y \), which has the same set of solutions from \( A^\alpha \) and contains at most \( m_0 = 2 \sum_{\omega=0}^{n} \) equations. Consider a consistent on \( A^\alpha \) system of equations (2). Remove from this system all equations \( g_i = \delta_i \) such that \( g \equiv 0 \). Denote the obtained system by \( S \). This system can contain functions of \( (n)_n \) sorts from \( P_{\omega} \) and if \( v_j = 3 \) for some \( t \in \{1, \ldots, r-1\} \), then this system can contain functions of \( (n)_n \) sorts from \( P_{\omega} \) too.

Divide the system \( S \) into at most \( (n)_n \) subsystems each of which contains equations composed from functions of the same sort and from the same set \( P_{\omega} \) or \( P_{\omega} \). Let \( S \) be one of such subsystems. Let \( \Sigma \) consist of equations with the left parts from \( P_{\omega} \). One can show that there exists a function \( g \in \lambda(\omega) \) depending on variables from \( Y \) and a number \( \delta \in E_2 \) such that the set of solutions \( \delta \) of the equation system \( \Sigma \) coincides with the set of solutions of \( A^\alpha \) of the equation system \( \Sigma' = (g = \delta) \). Let now \( \Sigma' \) consist of equations with the left parts from \( P_{\omega} \). One can show that there exists a subsystem \( \Sigma' \) of the system \( \Sigma \), which has the same set of solutions as the system \( \Sigma \) and contains at most two equations. Replace in the system \( \Sigma \) each subsystem \( \Sigma' \) with the corresponding subsystem \( \Sigma' \) and denote the obtained system by \( S' \). It's clear that \( S' \) has the same set of solutions as \( S \) and contains at most \( m_0 \) equations.

We now show that, for any problem \( z = (Y, \nu, \beta_1, \ldots, \beta_m) \in \mathcal{P}(U, n) \), the inequality \( \psi_\omega(z) \leq m_0 \) holds. It is clear that there exists a problem \( z' = (Y, \nu, \alpha_1, \ldots, \alpha_m) \in \mathcal{P}(U, n) \) such that \( \alpha_1, \ldots, \alpha_m \) are expressions depending on variables \( Y \) and \( z(\bar{a}) = z'(\bar{a}) \) for any \( \bar{a} \). For any \( \delta = (\delta_1, \ldots, \delta_m) \in E_2^m \), denote by \( S(\delta) \) the system of equations

\[
\begin{align*}
\{\alpha_1 = \delta_1, \ldots, \alpha_m = \delta_m\}
\end{align*}
\]

Let the system \( S(\delta) \) be consistent on the set \( A^\alpha \). Then there exists a system of equations

\[
\{\gamma_1 = \sigma_1, \ldots, \gamma_t = \sigma_t\}
\]

over \( Y \) such that \( t \leq m_0 \) and the set of solutions of this system coincides with the set of solutions of the system \( S(\delta) \).

Denote by \( G(\delta) \) the computation tree \( (Y, G(\delta)) \), where \( G(\delta) \) is a trie with the root consisting of unique complete path \( v_0, d_0, v_1, d_1, v_{r+1} \) in which, for \( i = 1, \ldots, t \), the node \( v_i \) is labeled with the expression \( \gamma_i \) and the edge \( d_i \) is labeled with the number \( \alpha_i \) and the node \( v_{r+1} \) is labeled with the minimum number from the set \( \nu(\delta) \).

Identify the roots of the trees \( G(\delta) \), where \( \delta \in E_2^m \) and the system of equations \( S(\delta) \) is consistent. Denote the obtained tree by \( G \). By \( G \) we denote the pair \( (Y, G) \). It is not difficult to prove that \( G \) is a computation tree over \( U \), which solves the problem \( z \) nondeterministically and for which \( \nu(\gamma) \leq m_0 \). Thus, \( \psi_\omega(z) \leq m_0 \) for any problem \( z \in \mathcal{P}(U, n) \). From here and from Lemma 8 it follows that \( \text{typ}(I_{U_{\psi}|n}) = \alpha \).

We now show that \( \text{typ}(I_{U_{\psi}|n}) = \gamma \). Using Lemmas 6 and 7, and Corollary 9 we obtain \( \text{Dom}(I_{U_{\psi}|n}) = \omega \ \{0\} \). Let \( m \in \omega \ \{0\} \). One can show that there exist functions \( f_1, \ldots, f_m \in P_{\omega}^m \) such that, for any \( \delta_1, \ldots, \delta_m \in E_2 \), the system of equations

\[
\{f_i(\bar{x}) = \delta_i, \ldots, f_m(\bar{x}) = \delta_m\},
\]

where \( \bar{x} = (x_1, \ldots, x_n) \), is consistent on \( A^\alpha \).

Consider the problem \( z = (Y, \nu, f_1(\bar{x}), \ldots, f_m(\bar{x})) \), where \( Y = \{x_1, \ldots, x_n\}, \nu : E^m \rightarrow \mathcal{S}(\omega) \) and \( \nu(\delta_1) \cap \nu(\delta_2) \neq \emptyset \) for any \( \delta_1, \delta_2 \in E_2^m \) such that \( \delta_1 \neq \delta_2 \). It is clear that \( \psi_\omega(z) = m \). Let \( I' \) be a computation tree over \( U \), which solves the problem \( z \) deterministically and for which \( \psi(\gamma) = \psi_\omega(z) \). Evidently, the computation tree \( I' \) must have at least \( 2^m \) terminal nodes. Therefore \( \nu(\gamma) \geq m \) and \( \psi_\omega(z) \geq m \). Thus, \( I_{U_{\psi}|n}(m) \geq m \). By Lemma 5, \( I_{U_{\psi}|n}(m) = m \). Taking into account that \( m \) is an arbitrary number from \( \omega \ \{0\} \), we obtain \( \text{typ}(I_{U_{\psi}|n}) = \gamma \). From this equality, from equality \( \text{typ}(I_{U_{\psi}|n}) = \alpha \), and from Proposition 3 it follows that \( \text{typ}_0(U, \psi, n) = t_4 \). □

**Lemma 21.** Let \( \tau \in \Delta_{U_\tau}, \left[U_\tau, \psi_\tau \right] = \pi_1^{(n_1)} \oplus \cdots \oplus \pi_\nu^{(n_\nu)}, \ r \in \{1, \ldots, m\}, \ v_r = 5, \ n \in \omega \ \{0\}, \ n \geq n_r \) and if \( r < m \), then \( n \leq n_{r+1} \). Then \( \text{typ}_0(U_\tau, \psi_\tau, n) = t_5 \).

**Proof.** One can show that \( r = 1 \) or \( r = 2 \). In the latter case, \( v_1 = 2 \). Denote \( (U, \psi) = (U_\tau, \psi_\tau) \). Let \( U = (A, F, P) \).

We now show that \( \text{typ}(I_{U_{\psi}|n}) = \gamma \). For an arbitrary \( i \in \omega \), denote \( c_i = \psi(q^{(\omega)}) \). Remind that \( c_0 = 1 \) and \( c_i = 1 \sum_{j=0}^{i-1} c_j + 2 \) for \( i \geq 1 \). Consider a problem \( z_i = (Y, \nu, q_i^{(\omega)}(x_1, \ldots, x_n)), \) where \( Y = \{x_1, \ldots, x_n\}, \nu(0) = \{0\} \) and \( \nu(1) = \{1\} \). It
is clear that $\psi'(z) = c_i$. We now show that $\psi'(z) = \psi'(z) = c_i$. One can prove that there exists a computation tree $\Gamma$ over $U$, which solves the problem $z_i$ nondeterministically and satisfies the following conditions: $\psi(\Gamma) = \psi'(\Gamma)$ and all expressions attached to nodes of $\Gamma$ depend on variables from $Y$ only. We now show that the expression $q^{(i)}_i(x_1, \ldots, x_n)$ is attached to a node $\Gamma$ of $\Gamma$. Assume the contrary. Consider two $n$-tuples from $A^n$: $\bar{\alpha} = (k_0^{(i)}, \ldots, k_{n-1}^{(i)})$ and $\bar{\beta} = (k_0^{(i)}, \ldots, k_{n-1}^{(i)}, k_0^{(i)}, \ldots, k_0^{(i)})$. Note that all functions from $P$ depending on variables from $Y$ take the value 0 on the tuple $\bar{\alpha}$, and all functions from $P$ depending on variables from $Y$ with the exception of $q^{(i)}_i(x_1, \ldots, x_n)$ take the value 0 on the tuple $\bar{\beta}$. It is clear from $z_i(\bar{\alpha}) = \emptyset$ and $z_i(\bar{\beta}) = \{1\}$. Let $\xi$ be a complete path in $\Gamma$ such that $\bar{\alpha} \in A(\xi)$. Evidently, all edges of this path with the exception of the first one are labeled with the number 0. Therefore $\beta \in A(\xi)$, but this is impossible since $z_i(\bar{\alpha}) \cap z_i(\bar{\beta}) = \emptyset$. Hence, the expression $q^{(i)}_i(x_1, \ldots, x_n)$ is attached to at least one node of $\Gamma$. Taking into account that $\psi'(z) = c_i$ and using Lemma 5 we obtain that $\psi'(z) = \psi'(z) = c_i$. Hence if the value $\psi'(z) = c_i$ is definite, then it satisfies the inequality $\psi'(z) \geq c_i$.

Let $i \in \omega \setminus \{0\}$ and $i \geq (n)x_n$. We now prove that the value $\psi'(z) = c_i$ is definite and $\psi'(z) = c_i < c_i - 1$. One can show that the set $\{z \in P(U, n) \mid \psi'(z) = c_i - 1\}$ is not empty. Let $z = (Y, v, \beta, \ldots, \beta_k)$ be an arbitrary problem from this set. One can show that there exists a computation tree $\Gamma'$, which solves the problem $\psi'(z)$ deterministically by sequential computing values of all functions from $P(\Gamma')$. Since all functions from $P(\Gamma')$ are not identically equal to 0, there are of the kind $\psi(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$.

Lemma 6. Dom($\psi'(z)$) = $\omega$. Since, for any $i \in \omega$, the inequality $\psi'(z) = c_i$ holds, Dom($\psi'(z)$) is an infinite set. Since, for any $i \in \omega$, $i \geq (n)x_n$, the inequality $\psi'(z) = c_i - 1$ holds, the set Dom($\psi'(z)$) is infinite. Therefore $\psi'(z) = c_i$ is definite and satisfies the inequality $\psi'(z) = c_i - 1$. Therefore Dom($\psi'(z)$) is an infinite set.

By Lemma 6, Dom($\psi'(z)$) = $\omega$. Since, for any $i \in \omega$, the inequality $\psi'(z) = c_i$ holds, Dom($\psi'(z)$) is an infinite set. Since, for any $i \in \omega$, $i \geq (n)x_n$, the inequality $\psi'(z) = c_i - 1$ holds, the set Dom($\psi'(z)$) is infinite. Therefore $\psi'(z) = c_i$ is definite and satisfies the inequality $\psi'(z) = c_i - 1$. Therefore Dom($\psi'(z)$) is an infinite set.

Proof. One can show that if $r > 1$, then $\{s_1, \ldots, s_{r-1}\} \subseteq \{2, 5\}$. Denote $\psi(U, \psi) = \psi(U, \psi)$. Let $U = (A, F, P)$.

Show that, for any $m \in \omega$, the value $\psi'(z) = m$ is definite. One can prove that the set $\{z \in P(U, n) \mid \psi'(z) = m\}$ is not empty. Let $z = (Y, v, \beta, \ldots, \beta_k)$ be an arbitrary problem from this set. Let $\Phi(z)$ be the set of all functions from $P(\Gamma')$ with variables from $Y$, which are not identically equal to 0 on $A^n$. One can show that the set $\Phi(z)$ contains only functions of the kind $q^{(i)}_i(x_1, x_2, \ldots, x_m)$, $q^{(i)}_i(x_1, x_2, \ldots, x_m)$, and $q^{(i)}_i(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are pairwise different variables from $Y$, and if $s = 5$ for some $t \in \{1, \ldots, r - 1\}$, then the set $\Phi(z)$ contains also functions of the kind $q^{(i)}_i(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are pairwise different variables from $Y$. Denote $\Phi(z) = \{f : f \in P(\Gamma'), \psi(f) = m\}$. One can show that $\Phi(z)$ is a finite set and $\psi'(z) = \sum_{f \in \Phi(z)} \psi(f)$. Taking into account that $z$ is an arbitrary problem from $P(U, n)$ such that $\psi'(z) = m$, we obtain that the value $\psi'(z)$ is definite and satisfies the inequality $\psi'(z) = c_i - 1$. Therefore Dom($\psi'(z)$) is an infinite set.

We now show that, for any $m \in \omega \setminus \{0\}$, the inequality $\psi'(z) = m$ holds. Consider the problem $z_m = (Y, v, q_1, q_2, q_3, \ldots, q_n)$, where $Y = (x_1, \ldots, x_n)$, $x = (x_1, \ldots, x_n)$, $v(1, 0, 1) = \{\emptyset\}$, $\psi(0, 1, 1) = \{2\}$, and $\psi(\emptyset) = \emptyset$. For any 3-tuple $t \in E^3_1 \{1, 0, 1\}, (0, 1, 1)$, consider the computation tree $\Gamma(z)$ containing exactly three complete paths $\xi_0$, $\xi_1$, and $\xi_2$, where $\xi_0 = v_0$, $\xi_1 = v_{i0}$, $\xi_2 = v_{i2}$, for $i = 0, 1, 2$. For $i = 0, 1, 2$, the node $v_0$ and the edge $\bar{\beta}_0$ are not labeled, and the node $v_0$ is labeled with the number 1. The node $v_{i0}$ is labeled with the expression $P_{i0}(\bar{\alpha}_i)$ and the edge $d_{i1}$ is labeled with the number 0. The node $v_{i1}$ is labeled with the expression $q_{i1}^{(n)}(\bar{\alpha}_i)$ and the edge $d_{i1}$ is labeled with the number 1. The node $v_{i2}$ is labeled with the expression $q_{i2}^{(n)}(\bar{\alpha}_i)$ and the edge $d_{i2}$ is labeled with the number 1. One can show that $\Gamma(z)$ solves the problem $z_m$ nondeterministically and satisfies the following conditions: $\psi(\Gamma(z)) = \psi'(z)$ and all functions attached to nodes of $\Gamma$ depend on variables from the set $Y$ only. Define three $n$-tuples $\bar{\alpha}_i, \bar{\alpha}_1, \bar{\alpha}_2$ from $A^n$: $\bar{\alpha}_0 = (k_0^{(i)}, \ldots, k_0^{(i)})$, $\bar{\alpha}_1 = (k_0^{(i)}, \ldots, k_{n-1}^{(i)}, 2m + 1, k_0^{(i)}, \ldots, k_0^{(i)})$, and $\bar{\alpha}_2 = (k_0^{(i)}, \ldots, k_{n-1}^{(i)}, 2m + 1, y_n, k_0^{(i)}, \ldots, k_0^{(i)})$. It is clear that $\psi(\bar{\alpha}_0) = \emptyset$, $\psi(\bar{\alpha}_1) = \emptyset$, $\psi(\bar{\alpha}_2) = \emptyset$, and $\psi(\bar{\alpha}_2) = \emptyset$. For $i = 0, 1, 2$, denote by $\psi_i$ the set of all functions from $\Phi(z)$, which take value 1 on the tuple $\bar{\alpha}_i$. One can show that $\psi_0 = \emptyset$, $\psi_1 = \{q_1^{(n)}(\bar{\alpha}_1), q_2^{(n)}(\bar{\alpha}_1), q_3^{(n)}(\bar{\alpha}_1)\}$, and $\psi_2 = \{q_3^{(n)}(\bar{\alpha}_2), q_4^{(n)}(\bar{\alpha}_2), q_5^{(n)}(\bar{\alpha}_2)\}$.

Consider the complete path $\xi_0$ in $\Gamma$ such that $\alpha_0 \in A(\xi_0)$. It is clear that $\alpha_1 \notin A(\xi_0)$ and $\alpha_2 \notin A(\xi_0)$. Therefore the expression $P_{i0}(\bar{\alpha}_i)$ or both expressions $q_{i1}^{(n)}(\bar{\alpha}_i)$ and $q_{i2}^{(n)}(\bar{\alpha}_i)$ are among expressions attached to the nodes of $\xi_0$. If nodes of $\xi_0$ are labeled with at least two expressions from the set $B = \{q_2^{(n)}(\bar{\alpha}_2), q_3^{(n)}(\bar{\alpha}_2), q_4^{(n)}(\bar{\alpha}_2), q_5^{(n)}(\bar{\alpha}_2)\}$ then $\psi(\bar{\alpha}_2) = \emptyset$. Let only one expression from the set $B$ be attached to the nodes of $\xi_0$. Then this is $P_{i0}(\bar{\alpha}_i)$. Let the considered expression be attached to the node $\nu$ of the path $\xi_0$. Consider the complete path $\xi_1$ in $\Gamma$ such that $\alpha_1 \in A(\xi_1)$. Since $\Gamma$ is a deterministic computation
proof. Denote \((U, \psi) = (U, \psi_\tau)\). Let \(U = (A, F, P)\). Show that \(\text{typ}(U_{\psi_{\tau}}) = \epsilon\). Let \(t \in \omega \setminus \{0\}\). Consider the problem \(z_t = (Y, v, p_{\ell}(\alpha), \ldots, p_{\ell}(\alpha))\) from \(P(U, n)\), where \(Y = \{x_1, \ldots, x_n\}\), \(v : E^\tau \to S(\omega)\), and \(v(\delta_1) \cap v(\delta_2) = \emptyset\) for any \(\delta_1, \delta_2 \in E^\tau\) such that \(\delta_1 \neq \delta_2\). It is not difficult to show that, for any consistent on \(A^\tau\) system of equations

\[
\{\ell_1(\alpha) = \delta_1, \ldots, \ell_k(\alpha) = \delta_k\},
\]

where \(\delta_1, \ldots, \delta_k \in E^\tau\), there exists a subsystem, which has the same set of solutions and which contains at most two equations. Using this fact it is not difficult to show that \(\psi^a_{\tau}(z_t) \leq 2\).

Let us prove that there is no \(c \in \omega\) such that \(\psi^a_{\tau}(z_t) \leq c\) for any \(t \in \omega \setminus \{0\}\). Assume the contrary. Then there exists a number \(w \in \omega \setminus \{0\}\) satisfying the following condition: for any \(t \in \omega \setminus \{0\}\), there exists a computation tree \(\Gamma^t\) over \(U\), which solves the problem \(z_t\) deterministically and has at most \(w\) terminal nodes. Let us consider the problem \(z_w\). Let \(\Gamma^t\) be an arbitrary computation tree over \(U\), which solves the problem \(z_w\) deterministically. It is not difficult to show that \(\Gamma^t\) must have at least \(w + 1\) terminal nodes. We obtain a contradiction. Thus, \(\psi^a_{\tau}(z_t) \leq 2\) for any \(t \in \omega \setminus \{0\}\) and there is no \(c \in \omega\) such that \(\psi^a_{\tau}(z_t) \leq c\) for any \(t \in \omega \setminus \{0\}\). Therefore \(\text{typ}(U_{\psi_{\tau}}(2)) = \infty\). Using Lemma 6 we obtain \(\text{typ}(U_{\psi_{\tau}}) = \epsilon\).

We now prove that the function \(\psi^a_{\tau}\) is unbounded from above on the set \(P(U, n)\). Let \(t \in \omega \setminus \{0\}\). Consider the problem \(\eta_t = (Y, v, q^a_{\tau}(\alpha))\) over \(U\), where \(Y = \{x_1, \ldots, x_n\}\), \(v : E^\tau \to S(\omega)\), and \(v(0) = \emptyset\), and \(v(1) = \{1\}\). Show that \(\psi^a_{\tau}(\eta_t) \geq 1\).

One can prove that there exists a computation tree \(\Gamma^t\) over \(U\), which solves the problem \(\eta_t\) nondeterministically and satisfies the following conditions: \(\psi(\Gamma) = \psi^a_{\tau}(\eta_t)\), and all expressions attached to nodes of \(\Gamma^t\) depend on variables from \(Y\) only. Show that the expression \(q^a_{\tau}(\alpha)\) is attached to a node of \(\Gamma^t\). Assume the contrary. Consider two \(n\)-tuples from \(A^n\):

\[
\bar{\alpha} = (\underbrace{\alpha_0, \ldots, \alpha_0}_{n}), \quad \bar{\beta} = (\underbrace{\alpha_1, \ldots, \alpha_1}_{n}).
\]

It is clear that \(\eta_t(\bar{\alpha}) = \{0\}\) and \(\eta_t(\bar{\beta}) = \{1\}\). One can show that all functions from \(P\) depending on variables from \(Y\) take the value \(0\) on the tuple \(\bar{\alpha}\), and all functions from \(P\), depending on variables from \(Y\), with the exception of \(q^a_{\tau}(\alpha)\), take the value \(0\) on the tuple \(\bar{\beta}\). Let \(\varepsilon\) be a complete path in \(\Gamma^t\) such that \(\bar{\alpha} \in \mathcal{A}(\varepsilon)\). Evidently, all edges of this path with the exception of the first one are labeled with \(0\). Therefore \(\bar{\beta} \in \mathcal{A}(\varepsilon)\) but this is impossible since \(\eta_t(\bar{\alpha}) \cap \eta_t(\bar{\beta}) = \emptyset\). Hence the expression \(q^a_{\tau}(\alpha)\) is attached to a node of \(\Gamma^t\). Therefore \(\psi(\Gamma) \geq 1\) and \(\psi^a_{\tau}(\eta_t) \geq 1\). Thus, the function \(\psi^a_{\tau}\) is unbounded from above on the set \(P(U, n)\). Using Lemma 8 we obtain \(\text{typ}(U_{\psi_{\tau}}) = \gamma\). From this equality, the equality \(\text{typ}(U_{\psi_{\tau}}) = \epsilon\), and from Proposition 3 it follows that \(\text{typ}(U, \psi, n) = t_\gamma\). \(\Box\)

Lemma 24. Let \((U, \psi)\) be a sm-pair such that \(\psi \equiv 0\). Then \(\text{typ}(U, \psi) = t_1^\infty\).

Proof. Let \(n \in \omega \setminus \{0\}\). It is clear that the function \(\psi^a\) is bounded from above on the set \(P(U, n)\). Using Lemma 8 we obtain \(\text{typ}(U_{\psi^a}(n)) = \alpha\). From this equality and from Proposition 3 it follows that \(\text{typ}(U, \psi, n) = t_1\). \(\Box\)

Proof of Proposition 16. Let \(t \in \{t_1^\infty\} \cup \Delta_u\). Assume that \(t = t_1^\infty\). Consider an arbitrary sm-pair \((U, \psi)\) such that \(\psi \equiv 0\). By Lemma 24, \(\text{typ}(U, \psi) = t_1^\infty\). Assume now that \(t \in \Delta_u\). Consider the sm-pair \((U, \psi_t)\). Using Lemmas 18–23 we obtain \(\text{typ}(U, \psi_t) = t_1^\infty\). \(\Box\)

Proof of Proposition 17. Let \(t \in \Delta_u\). Consider the sm-pair \((U, \psi_t)\). By construction, \(\psi_t\) is a weighted depth. Therefore \((U, \psi_t)\) is a limited sm-pair. From Lemma 18–23 it follows that \(\text{typ}(U, \psi_t) = t_1\). \(\Box\)

7. Proofs of Theorems 1 and 2.

Let us define a function \(\rho : [\alpha, \beta, \gamma, \delta, \varepsilon] \to [\alpha, \beta, \gamma, \delta, \varepsilon]\) as follows: \(\rho(\alpha) = \varepsilon, \rho(\beta) = \delta, \rho(\gamma) = \gamma, \rho(\delta) = \beta, \rho(\varepsilon) = \alpha\). The following statement (Proposition 5 from [12]) allows us to analyze the relationships between upper types and types of sm-pairs.

Proposition 25. Let \(B\) be a nonempty set, \(f : B \to \omega, g : B \to \omega, \ell^{(k)}(n) = \max(f(b) : b \in B, g(b) \leq n), \text{ and } \epsilon^{(k)}(n) = \min(g(b) : b \in B, f(b) \geq n)\) for any \(n \in \omega\). Then \(\text{typ}(\epsilon^{(k)}) = \rho(\text{typ}(\ell^{(k)}))\).

Using Proposition 25 we obtain the following statement.

Proposition 26. Let \((U, \psi)\) be a sm-pair, \(n \in \omega \setminus \{0\}\), and \(\text{typ}(U, \psi, n) = t_i\) for some \(i \in \{1, \ldots, 7\}\). Then \(\text{typ}(U, \psi, n) = T_i\).
Proof of Theorem 1. The statement of the theorem follows from Propositions 13, 16, and 26. □

Proof of Theorem 2. The statement of the theorem follows from Propositions 14, 17, and 26. □

8. Explanations of results

Rather unusual formulations of the results (in particular, the use of the types of functions) require additional explanations.

Let \((U, \psi)\) be a sm-pair, and \(n \in \omega \setminus \{0\}\). We will start by considering the upper \(n\)-type \(\mathcal{U}_{U, \psi}^b\) of the sm-pair \((U, \psi)\). After that, discuss the lower \(n\)-type \(\mathcal{U}_{U, \psi}^a\) of the sm-pair \((U, \psi)\) and the upper \(n\)-type \(\mathcal{U}_{U, \psi}^a\) of the sm-pair \((U, \psi)\). We end by looking at the dynamic type \(d\mathcal{U}_{U, \psi}\) of the sm-pair \((U, \psi)\).

8.1. Upper \(n\)-types of sm-pairs

For any \(b, c \in [i, d, a]\), the matrix \(\mathcal{U}_{U, \psi}^a\) (the upper \(n\)-type of the sm-pair \((U, \psi)\)) contains the value \(\mathcal{U}_{U, \psi}^b(m) \in [\alpha, \beta, \gamma, \delta, \epsilon]\) of the intersection of the row \(b\) and the column \(c\), where

\[
\mathcal{U}_{U, \psi}^b(m) = \max\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \leq m\}
\]

for any \(m \in \omega\). If the value \(\mathcal{U}_{U, \psi}^b(m)\) is defined for some \(m \in \omega\), then it is the unimprovable upper bound on the value \(\psi_U^b(z)\) for problems \(z \in \mathcal{P}(U, n)\) such that the inequality \(\psi_U^c(z) \leq m\) holds. From Propositions 13, 14, 16, and 17 it follows that \(\{t_1, \ldots, t_1\}\) is the set of all possible upper \(n\)-types of sm-pairs and \(\{t_1, \ldots, t_1\}\) is the set of all possible upper \(n\)-types of limited sm-pairs.

Let us remind that \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is the domain of \(\mathcal{U}_{U, \psi}^b\), \(\mathcal{D}^+(\mathcal{U}_{U, \psi}^b) = \{m : m \in \mathcal{D}(\mathcal{U}_{U, \psi}^b), \mathcal{U}_{U, \psi}^b(m) \geq m\}\), and \(\mathcal{D}^-(\mathcal{U}_{U, \psi}^b) = \{m : m \in \mathcal{D}(\mathcal{U}_{U, \psi}^b), \mathcal{U}_{U, \psi}^b(m) \leq m\}\).

By Lemma 6, the set \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is finite if and only if there exists \(m \in \omega\) such that \(\mathcal{U}_{U, \psi}^b(m) = \infty\), i.e., the set \(\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^c(z) \leq m\}\) is infinite. In this case, \(\mathcal{D}^+(\mathcal{U}_{U, \psi}^b) = \emptyset\) if \(m_0 = m_1\) and \(\mathcal{D}^+(\mathcal{U}_{U, \psi}^b) = \{m : m \in \omega, m_0 \leq m < m_1\}\) if \(m_0 < m_1\), where \(m_0 = \min\{\psi_U^c(z) : z \in \mathcal{P}(U, n)\}\) and \(m_1 = \min\{m : m \in \omega, \mathcal{U}_{U, \psi}^b(m) = \infty\}\). If the set \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is infinite, then \(\mathcal{D}^-(\mathcal{U}_{U, \psi}^b) = \{m : m \in \omega, m \geq m_0\}\).

Let us remind that, by Lemma 5, \(\psi_U^b(z) \leq \psi_U^c(z) \leq \psi_U^d(z)\) for any problem \(z \in \mathcal{P}(U, n)\).

The equality \(\mathcal{U}_{U, \psi}^b = \alpha\) means that the set \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is infinite and the function \(\mathcal{U}_{U, \psi}^b\) is bounded from above. This equality can hold for any pair \(b, c \in [i, d, a]\). By Lemma 8, \(\mathcal{U}_{U, \psi}^b = \alpha\) if and only if the function \(\psi_U^b\) is bounded from above on the set \(\mathcal{P}(U, n)\). Later we will often omit words “from above on the set \(\mathcal{P}(U, n)\)” and write that the function \(\psi_U^b\) is bounded or that the function \(\psi_U^b\) is unbounded. The function \(\psi_U^b\) can be bounded only for sm-pairs that are not limited. The function \(\psi_U^b\) can be bounded for sm-pairs that are limited, but this case is in some sense degenerate — see Lemma 10.

The equality \(\mathcal{U}_{U, \psi}^b = \beta\) means that \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is an infinite set, \(\mathcal{D}^-(\mathcal{U}_{U, \psi}^b)\) is a finite set, and \(\mathcal{U}_{U, \psi}^b\) is an unbounded from above function. This equality can hold only if \(bc = di\). One can show that \(\mathcal{U}_{U, \psi}^b = \gamma\) if and only if the function \(\psi_U^b\) is bounded and there exists \(p \in \omega\) such that \(\psi_U^b(z) < \psi_U^b(z)\) for any problem \(z \in \mathcal{P}(U, n)\) such that \(\psi_U^b(z) \geq p\).

The equality \(\mathcal{U}_{U, \psi}^b = \gamma\) means that each of the sets \(\mathcal{D}^+(\mathcal{U}_{U, \psi}^b)\) and \(\mathcal{D}^-(\mathcal{U}_{U, \psi}^b)\) is an infinite set. This equality can hold for any pair \(bc, c \in [i, d, a]\), with the exception of \(id\) and \(ia\). One can show that, for \(bc \in \{di, ai, ad\}\), \(\mathcal{U}_{U, \psi}^b = \gamma\) if and only if, for any \(q \in \omega\), there exists a problem \(z \in \mathcal{P}(U, n)\) such that \(\psi_U^b(z) \geq q\). If \(bc \in \{ii, da, a\}\), then \(\mathcal{U}_{U, \psi}^b = \gamma\) if and only if the function \(\psi_U^b\) is unbounded — see Lemma 8. One can show that \(\mathcal{U}_{U, \psi}^b = \gamma\) if and only if the set \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is infinite, the function \(\psi_U^b\) is unbounded and, for any \(q \in \omega\), there exists a number \(m \in \omega, m \geq q\), such that, for any problem \(z \in \mathcal{P}(U, n)\) with \(\psi_U^b(z) \leq m\), the inequality \(\psi_U^b(z) \leq m\) holds.

The equality \(\mathcal{U}_{U, \psi}^b = \delta\) means that \(\mathcal{D}(\mathcal{U}_{U, \psi}^b)\) is an infinite set and \(\mathcal{D}^+(\mathcal{U}_{U, \psi}^b)\) is a finite set. This equality can hold only if \(bc = da\). One can show that \(\mathcal{U}_{U, \psi}^b = \delta\) if and only if there exists \(m \in \omega\) such that \(\mathcal{U}_{U, \psi}^b(m) = \infty\), i.e., the set \(\{\psi_U^b(z) : z \in \mathcal{P}(U, n), \psi_U^b(z) \leq m\}\) is infinite. The case \(\mathcal{U}_{U, \psi}^b = \epsilon\) is not very informative: for large enough \(m\), there are problems \(z \in \mathcal{P}(U, n)\) for which \(\psi_U^b(z) \leq m\) but we cannot derive any upper bound on the value \(\psi_U^b(z)\) for these problems.

Let \(\mathcal{U}_{U, \psi}^a = t \in \{t_1, \ldots, t_1\}\). We now consider possible behavior of the rows in the table \(t\), which is closely related to the boundedness of the functions \(\psi_U^a\) and \(\psi_U^b\).

The row \(i\) in the matrix \(t\) is equal to \((\alpha, \alpha, \alpha)\) if the function \(\psi_U^a\) is bounded (if \(t = t_1\)) and is equal to \((\gamma, \epsilon, \epsilon)\) if the function \(\psi_U^a\) is unbounded (if \(t = \{t_2, \ldots, t_1\}\)).

The row \(a\) in the matrix \(t\) is equal to \((\alpha, \alpha, \alpha)\) if the function \(\psi_U^a\) is bounded (if \(t \in \{t_1, t_2, t_3, t_4\}\)) and is equal to \((\gamma, \gamma, \gamma)\) if the function \(\psi_U^a\) is unbounded (if \(t \in \{t_5, t_6, t_7\}\)).
The behavior of the row $d$ in the matrix $t$ is more complicated. If the function $\psi_0^d$ is bounded (if $t \in \{t_1, t_2\}$), then this row is equal to $(\alpha, \alpha, \alpha)$. If the function $\psi_0^d$ is unbounded and the function $\psi_0^a$ is bounded (if $t \in \{t_3, t_4\}$), then the row $d$ is equal to $(x, \alpha, \varepsilon)$, where $x \in \{\beta, \gamma\}$. If each of the functions $\psi_1^a, \psi_1^d$, and $\psi_0^a$ is unbounded (if $t \in \{t_5, t_6, t_7\}$), then the row $d$ is equal to $(\gamma, \gamma, x)$, where $x \in \{\gamma, \delta, \varepsilon\}$.

8.2. Lower $n$-types of sm-pairs

In this section, we consider lower $n$-types of sm-pairs.

**Definition 14.** The lower $n$-type, $n \in \omega \setminus \{0\}$, of a sm-pair $(U, \psi)$ is the table $\text{typ}_t(U, \psi, n)$ with three rows and three columns in which rows from top to bottom and columns from the left to the right are labeled with indices $i, d, a$ and the value $\text{typ}_t(U, \psi, n)$ is in the intersection of the row with index $b \in \{i, d, a\}$ and the column with index $c \in \{i, d, a\}$.

For any $b, c \in \{i, d, a\}$ and $m \in \omega$,

$$L_b^{bc}(m) = \min\{\psi_0^b(z) : z \in \mathcal{P}(U, n), \psi_0^b(z) \geq m\}.$$  

If the value $L_b^{bc}(m)$ is defined for some $m \in \omega$, then it is the unimprovable lower bound on the value $\psi_0^b(z)$ for problems $z \in \mathcal{P}(U, n)$ such that the inequality $\psi_0^b(z) \geq m$ holds.

One can show that the set $\text{Dom}(L_b^{bc}(m))$ is finite if and only if the function $\psi_0^b$ is bounded. In this case, $\text{Dom}(L_b^{bc}(m)) = \{0, \ldots, M\}$, where $M = \max\{\psi_0^b(z) : z \in \mathcal{P}(U, n)\}$. If the function $\psi_0^b$ is unbounded, then $\text{Dom}(L_b^{bc}(m)) = \omega$.

Define seven tables:

$$L_1 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_2 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_3 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_4 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_5 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_6 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

$$L_7 = \begin{bmatrix} i & d & a \\ i & d & a \\ i & d & a \end{bmatrix}$$

Using Proposition 25 we obtain that $\{l_1, \ldots, l_7\}$ is the set of all possible lower $n$-types of sm-pairs and $\{l_1, \ldots, l_7\}$ is the set of all possible lower $n$-types of limited sm-pairs. Moreover, $\text{typ}_t(U, \psi, n) = l_i, i \in \{1, \ldots, 7\}$, if and only if $\text{typ}_t(U, \psi, n) = l_i$.

From Proposition 25 it follows that, for any pair $bc, b, c \in \{i, d, a\}$.

$$\text{typ}_t(L_b^{bc}(m)) = \rho(\text{typ}_t(L_b^{bc}(m)))$$

where $\rho(\alpha) = \varepsilon$, $\rho(\beta) = \delta$, $\rho(\gamma) = \gamma$, $\rho(\delta) = \beta$, and $\rho(\varepsilon) = \alpha$. Using this equality and criteria of the behavior of the value $\text{typ}_t(L_b^{bc}(m))$ described in the previous section we can obtain the criteria of the behavior of the value $\text{typ}_t(L_b^{bc}(m))$. As examples, we consider the criteria for equalities $\text{typ}_t(L_b^{bc}(m)) = \alpha$ and $\text{typ}_t(L_b^{bc}(m)) = \varepsilon$.

The equality $\text{typ}_t(L_b^{bc}(m)) = \alpha$ means that the set $\text{Dom}(L_b^{bc}(m))$ is infinite and the function $L_b^{bc}(m)$ is bounded from above. This equality can hold if $bc \in \{di, ai, ad\}$. One can show that $\text{typ}_t(L_b^{bc}(m)) = \alpha$ if and only if there exists $p \in \omega$ such that $L_b^{bc}(m)(p) = \infty$, i.e., the set $\{\psi_0^b(z) : z \in \mathcal{P}(U, n), \psi_0^b(z) \leq p\}$ is infinite. In this case, $L_b^{bc}(m)(p) \leq p$ for any $m \in \omega$.

The equality $\text{typ}_t(L_b^{bc}(m)) = \varepsilon$ means that $\text{Dom}(L_b^{bc}(m))$ is a finite set. This equality can hold for any pair $bc, b, c \in \{i, d, a\}$. One can show that $\text{typ}_t(L_b^{bc}(m)) = \varepsilon$ if and only if the function $\psi_0^b$ is bounded. The case $\text{typ}_t(L_b^{bc}(m)) = \varepsilon$ is not as bad as the case $\text{typ}_t(L_b^{bc}(m)) = \alpha$. In the former case, the value $L_b^{bc}(m)(m)$ is undefined only if there are no problems $z \in \mathcal{P}(U, n)$ such that $\psi_0^b(z) \geq m$. In the latter case, if the value $L_b^{bc}(m)(m)$ is undefined, then there exist problems $z \in \mathcal{P}(U, n)$ with $\psi_0^b(z) \leq m$.

Let $\text{typ}_t(U, \psi, n) = l_i \in \{l_1, \ldots, l_7\}$. We now consider possible behavior of the columns in the table $l_i$.

The column $i$ in the matrix $l_i$ is equal to $(\varepsilon, \varepsilon, \varepsilon)^T$ if the function $\psi_0^i$ is bounded (if $l_i = l_1$) and is equal to $(\gamma, \gamma, \gamma)^T$ if the function $\psi_0^i$ is unbounded and the function $\psi_0^d$ is bounded (if $l_i \in \{l_1, l_4\}$), then the column $d$ is equal to $(x, \gamma, \gamma)^T$, where $x \in \{\delta, \gamma\}$. If each of the functions $\psi_0^i, \psi_0^d$, and $\psi_0^a$ is unbounded (if $l_i \in \{l_5, l_6, l_7\}$), then the column $d$ is equal to $(\gamma, \gamma, x)^T$, where $x \in \{\gamma, \beta, \alpha\}$. 


8.3. n-Types of sm-pairs

From Theorems 1 and 2 it follows that \( \{T_1, \ldots, T_7\} \) is the set of all possible n-types of sm-pairs and \( \{T_2, \ldots, T_7\} \) is the set of all possible n-types of limited sm-pairs.

For \( i \in \{1, \ldots, n\} \) and \( b, c \in \{i, d, a\} \), we denote by \( t_{i}^{bc} \) the value in the intersection of the row \( b \) and the column \( c \) in the matrix \( t_{i} \) and by \( \delta_{i}^{bc} \) the value in the intersection of the row \( b \) and the column \( c \) in the matrix \( \delta_{i} \). Then the matrix \( T_{i} \) has the pair \( \delta_{i}^{bc}, t_{i}^{bc} \) in the intersection of the row \( b \) and the column \( c \).

In the table:

| \( i \) | \( d \) | \( a \) |
|---|---|---|
| \( 1 \) | \( \gamma \gamma, \varepsilon \alpha \) | \( \gamma \varepsilon, \delta \varepsilon, \varepsilon \alpha, \varepsilon \varepsilon \) | \( \gamma \varepsilon, \varepsilon \alpha, \varepsilon \varepsilon \) |
| \( 2 \) | \( \alpha \alpha, \alpha \beta, \alpha \gamma, \varepsilon \alpha \) | \( \gamma \gamma, \varepsilon \alpha \) | \( \gamma \gamma, \gamma \delta, \varepsilon \gamma, \varepsilon \alpha, \varepsilon \varepsilon \) |
| \( 3 \) | \( \alpha \alpha, \alpha \gamma, \varepsilon \alpha \) | \( \alpha \alpha, \alpha \gamma, \beta \gamma, \gamma \gamma, \varepsilon \alpha \) | \( \gamma \gamma, \varepsilon \alpha \) |

In the intersection of the row with index \( b \in \{i, d, a\} \) and the column with index \( c \in \{i, d, a\} \), we have all possible pairs that appear in tables \( T_{1}, \ldots, T_{7} \) in the intersection of the row and the column with the same indices. Note that out of 25 different pairs \( pq, p, q \in \{\alpha, \beta, \gamma, \delta, \varepsilon\} \), only ten pairs \( \alpha \alpha, \alpha \beta, \alpha \gamma, \beta \gamma, \gamma \gamma, \gamma \delta, \varepsilon \gamma, \delta \alpha, \varepsilon \varepsilon, \varepsilon \varepsilon \) are present in tables \( T_{1}, \ldots, T_{7} \).

The situation, when \( \text{typ}(\mathcal{L}_{U_{\varepsilon}}^{bc}) \), \( \text{typ}(\mathcal{L}_{U_{\delta}}^{bc}) \) \( \in \{\alpha \alpha, \alpha \beta, \alpha \gamma, \beta \gamma, \gamma \gamma, \gamma \delta, \varepsilon \gamma, \delta \alpha, \varepsilon \varepsilon, \varepsilon \varepsilon\} \) is good enough: the difference between the lower and upper bounds \( \mathcal{L}_{U_{\varepsilon}}^{bc} \) and \( \mathcal{L}_{U_{\delta}}^{bc} \) is reasonable (they have infinite domains and are either both bounded from above or both unbounded from above). For the rest of the cases, the situation is worse. For pairs \( \alpha \beta \) and \( \alpha \gamma \), there is too big a gap between lower and upper bounds \( \mathcal{L}_{U_{\varepsilon}}^{bc} \) and \( \mathcal{L}_{U_{\delta}}^{bc} \): the lower bound is bounded from above and the upper bound is unbounded from above. For pairs \( \gamma \varepsilon, \delta \varepsilon, \varepsilon \alpha, \varepsilon \varepsilon \), at least one of the bounds has finite domain.

8.4. Dynamic types for sm-pairs

We defined the linear order \( \preceq \) on the set \( \{\alpha, \beta, \gamma, \delta, \varepsilon\} \) as follows: \( \alpha \preceq \beta \preceq \gamma \preceq \delta \preceq \varepsilon \). Based on this order, we defined the partial order \( \preceq \) on the set of tables \( \{T_{1}, \ldots, T_{7}\} \). Let \( i, j \in \{1, \ldots, 7\} \). Then \( T_{i} \preceq T_{j} \) if and only if \( \delta_{i}^{bc} \preceq \delta_{j}^{bc} \) for any \( b, c \in \{i, d, a\} \). The graph depicted in Fig. 3 is the Hasse diagram for the partially ordered set \( \{T_{1}, \ldots, T_{7}\} \). Nodes of this diagram are tables \( T_{1}, \ldots, T_{7} \). An edge goes upward from \( T_{i} \) to \( T_{j} \) if \( T_{i} \preceq T_{j} \) and there is no \( T_{k} \) \( \notin \{T_{i}, T_{j}\} \), such that \( T_{i} \preceq T_{k} \preceq T_{j} \). We now define a partial order \( \preceq \) on the set of tables \( \{T_{1}, \ldots, T_{7}\} \) for any \( i, j \in \{1, \ldots, 7\} \) if and only if \( T_{i} \preceq T_{j} \).

From Theorems 1 and 2 it follows that, for limited sm-pairs, the set of all possible dynamic types coincides with the set \( \Delta \) of infinite sequences \( T_{1}, T_{2}, \ldots \) such that \( T_{1}, T_{2}, \ldots \in \{T_{2}, \ldots, T_{7}\} \) and \( T_{1} \preceq T_{2} \preceq \cdots \). For arbitrary sm-pairs the set of all possible dynamic types coincides with the set \( \Delta \cup T_{1}^{\infty} \).

9. Conclusions

In this paper, we studied computation trees over arbitrary structures. We described the set of all possible dynamic types for (i) arbitrary sm-pairs (structure, measure)-pairs), (ii) for limited sm-pairs, and, in fact, (iii) for limited sm-pairs with a weighted depth as the complexity measure. The question about the set of all possible dynamic types for sm-pairs that have the depth as the complexity measure is open. In the future, we are planning to consider both this issue and various issues related to computational trees over structures with finite collections of predicates and functions.

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