ÉTALE FUNDAMENTAL GROUPS OF STRONGLY $F$-REGULAR SCHEMES

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Abstract. We prove that a strongly $F$-regular scheme $X$ admits a finite, generically Galois, and étale-in-codimension-one cover $\tilde{X} \rightarrow X$ such that the étale fundamental groups of $\tilde{X}$ and $X_{\text{reg}}$ agree. Equivalently, every finite étale cover of $X_{\text{reg}}$ extends to a finite étale cover of $\tilde{X}$. This is analogous to a result for complex klt varieties by Greb, Kebekus and Peternell.

1. Introduction

In [Xu14], Chenyang Xu proved that the algebraic local fundamental group (the profinite completion of the topological fundamental group of the link) of a complex klt singularity is finite. This result was then used by Greb–Kebekus–Peternell [GKP13] to show that for a complex quasi-projective variety $X$ with klt singularities, any sequence of finite quasi-étale (i.e. étale in codimension one) and generically Galois covers eventually becomes étale. In particular, for any quasi-projective klt variety $X/\mathbb{C}$ there exists a finite quasi-étale generically Galois cover $\rho: \tilde{X} \rightarrow X$ such that any further quasi-étale cover $Y \rightarrow \tilde{X}$ is actually étale.

Inspired by the relation between klt singularities and strongly $F$-regular singularities in characteristic $p > 0$ [HW02], a subset of the authors showed that if $(R, m)$ is a strictly Henselian strongly $F$-regular local ring and $U \subseteq \text{Spec } R$ is the regular locus, then $\pi^1(U)$ is finite [CST16], a variant of Xu’s result in characteristic $p > 0$. In this paper we use this result, a theorem of Gabber [Tra14], and recent work on the non-local behavior of $F$-signature [DSPY16], to prove a variant of the result of Greb–Kebekus–Peternell in characteristic $p > 0$.

Philosophically, both these results are studying the obstructions of extending finite étale covers of the regular locus $X_{\text{reg}}$ of a scheme $X$ to the whole scheme. The expected general answer is that for more severe singularities on $X$, there are more obstructions. Analogous to [GKP13], we show that schemes with strongly $F$-regular singularities are mild in this sense.

Main Theorem (Theorem 4.1 Corollary 4.2). Suppose $X$ is an $F$-finite Noetherian integral strongly $F$-regular scheme. Suppose we are given a sequence of finite surjective quasi-étale (étale in codimension one) morphisms of normal integral schemes

$$\xymatrix{X = X_0 & X_1 \ar[l]_{\gamma_1} & X_2 \ar[l]_{\gamma_2} & \cdots & X_i \ar[l]_{\gamma_i} }$$

such that each $X_i/X$ is generically Galois. Then all but finitely many of the $\gamma_i$ are étale.
In particular, there exists a finite quasi-étale generically Galois cover \( \tilde{X} \to X \) so that any further finite quasi-étale cover \( Y \to \tilde{X} \) is actually étale. Equivalently, the map \( \pi_1^{\text{ét}}(\tilde{X}_{\text{reg}}) \to \pi_1^{\text{ét}}(X) \) induced by the inclusion of the regular locus is an isomorphism.

Note that we do not require any quasi-projectivity hypothesis. We actually obtain a slightly stronger version, just as in [GKP13]. As a corollary, we also obtain a variant of [GKP13, Theorem 1.10], see Corollary 4.8, which says in particular that there exists an integer \( N > 0 \) such that for every \( \mathbb{Q} \)-Cartier divisor \( D \) on \( X \) with index not divisible by \( p \), \( N \cdot D \) is in fact Cartier. Finally, we observe that the map \( \tilde{X} \to X \) from our Main Theorem above is tame wherever it is not étale, see Corollary 4.4.

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2. Notation and conventions

Convention 2.1. All rings and schemes considered will be Noetherian. We will typically also be working with normal connected and hence integral schemes.

Notation 2.2. Let \( \rho: Y \to X \) be a morphism of normal schemes.

(a) We say that \( \rho \) is quasi-étale if it is quasi-finite and there exists a closed subset \( Z \subseteq Y \) of codimension \( \geq 2 \) such that \( \rho|_{Y \setminus Z} : Y \setminus Z \to X \) is étale. Notice that if \( \rho \) is further finite, then \( Y \) coincides with the integral closure of \( X \) in \( K(Y) \).

(b) Assume that \( \rho \) is finite and surjective. Let \( W \subseteq X \) be the largest Zariski-open subset such that \( \rho^{-1}(W) \to W \) is étale. The branch locus of \( \rho \) is defined to be \( X \setminus W \). It is worth to remark that by Purity of the Branch Locus, if \( \rho \) is quasi-étale then its branch locus is contained in the non-regular locus of \( X \).

(c) If \( X \) and \( Y \) are connected, we say that \( \rho \) finite and surjective is generically Galois if the fraction fields extension \( K(Y)/K(X) \) is Galois. In this case, it follows that \( \rho^{-1}(W) \to W \) is Galois, where \( W \) is as in (b).

(d) By a geometric point \( \bar{x} \in X \) of a scheme \( X \) we mean a morphism \( \text{Spec } L \to X \), where \( L \) is a separably closed field. Note that \( \bar{x} \to X \) factors through \( \text{Spec } k(x) \to X \), where \( x \in X \) is the scheme-theoretic image point.

(e) Suppose that \( \bar{s}, \bar{t} \in X \) are geometric points of a scheme \( X \). A specialization of geometric points \( \bar{s} \to \bar{t} \) is a factorization

\[
\bar{s} \to \text{Spec } O_{\tilde{X}, \bar{t}}^{\text{sh}} \to X
\]

where the composition is the map defining the geometric point \( \bar{s} \to X \). Note that then we get a map \( \text{Spec } O_{\tilde{X}, \bar{t}}^{\text{sh}} \to \text{Spec } O_{\tilde{X}, \bar{t}} \). In particular, the image of \( \bar{t} \) as a scheme theoretic point on \( X \) is contained in the Zariski closure of the image of \( \bar{s} \). Conversely, if \( \bar{s} \in X \) is a geometric point and \( t \in \overline{\{s\}} \), then we can find a geometric point \( \bar{t} \in X \) with scheme-theoretic image point \( t \) such that \( \bar{s} \) specializes to \( \bar{t} \).

3. Gabber’s constructibility and applications to the fundamental group

Recall the following theorem of Gabber.

Theorem 3.1. [Tra14 Exposé XXI, Théorème 1.1, 1.3] Suppose that \( X \) is a normal excellent scheme and \( Z \subseteq X \) a closed subscheme of codimension \( \geq 2 \). Let \( j: U = X \setminus Z \to X \) be
the inclusion. Then for all finite groups \( G \), \( R^i j_* (G_U) \) is constructible for \( i = 0, 1 \). Here \( G_U \) denotes the constant sheaf \( G \) on \( U \) equipped with the étale topology.

We use this to obtain:

**Proposition 3.2.** Let \( i : Z \hookrightarrow X \) be a closed subscheme of codimension \( \geq 2 \) in a normal excellent scheme. Let \( j : U = X \setminus Z \hookrightarrow X \) be the complement. Suppose that

\[
\left| \pi_1^\text{ét}(\text{Spec} \mathcal{O}_{X,x}^\text{sh} \setminus Z^\text{sh}) \right|
\]

is bounded for all geometric points \( x \in X \) (here \( Z^\text{sh} = \mathcal{Z}^\text{sh}_{X,x} \) is the inverse image of \( Z \)). Then there exists a finite stratification \( \{Z_i\}_{i \in I} \) of \( X \) into locally closed subschemes \( Z_i \) with the following property:

Fix a specialization \( s \leadsto t \) of geometric points of \( Z_i \). Consider the map

\[
(3.2.1) \quad \text{Spec}(\mathcal{O}_{X,s}^\text{sh}) \longrightarrow \text{Spec}(\mathcal{O}_{X,t}^\text{sh}).
\]

Remove the inverse images of \( Z \) to obtain

\[
(3.2.2) \quad T_s \xrightarrow{\alpha} T_t.
\]

Then applying \( \pi_1^\text{ét}(\_\_) \) to \((3.2.2)\) yields an isomorphism.

**Proof.** Note there are only finitely many finite groups \( G_x = \pi_1^\text{ét}(\text{Spec} \mathcal{O}_{X,x}^\text{sh} \setminus Z^\text{sh}) \) for geometric points \( x \in X \), since there are only finitely many groups of size less than a given number. Letting \( G_x \) also denote the constant sheaf on \( U^\text{ét} \), since \( R^i j_* G_x \) is constructible by \textbf{Theorem 3.1}, we may choose a stratification \( \{Z_i\} \) of locally closed irreducible subschemes of \( X \) so that \( R^i j_* G_x \) is locally constant on each \( Z_i \) for all geometric points \( x \). Note \( R^1 j_* G_x \) is zero over \( U \) so the induced stratification has the same data as a stratification of \( Z \).

Using our stratification, we know that for any geometric point \( x \in X \), the stalks of \( R^1 j_* G_x \) at \( s \) and \( t \) are the same. These stalks however are \( H^1(T_s, G_x) \) and \( H^1(T_t, G_x) \), respectively \cite[Chapter III, Theorem 1.15]{Mil80}. In other words, the functor of isomorphism classes of \( G_x \)-torsors over \((3.2.2)\) produces isomorphisms. But we have functorial bijections

\[
H^1(T_s, G_x) \cong \text{Hom}_{\text{cont}}(\pi_1^\text{ét}(T_s), G_x) / G_x
\]

and

\[
H^1(T_t, G_x) \cong \text{Hom}_{\text{cont}}(\pi_1^\text{ét}(T_t), G_x) / G_x,
\]

where \( G_x \) acts on \( \text{Hom}_{\text{cont}}(-, G_x) \) by conjugation. For more discussion see for instance \cite[Section 11.5]{GW10}, \cite[Example 11.3]{Mil13} and \cite[Chapter I, Remark 5.4 and Chapter III, Corollary 4.7, Remark 4.8]{Mil80}. In particular, the map

\[
(3.2.3) \quad \text{Hom}(G_t \cong \pi_1^\text{ét}(T_t), G_x) / G_x \longrightarrow \text{Hom}(G_s = \pi_1^\text{ét}(T_s), G_x) / G_x
\]

induced by \( \pi_1^\text{ét}(\alpha) \) is a bijection for any geometric point \( x \in X \).

Recall that we want to prove that \( \pi_1^\text{ét}(\alpha) \) is an isomorphism. We apply \((3.2.3)\) with \( x = s \). Considering the identity \( \pi_1^\text{ét}(T_s) \rightarrow G_s \) as an element of the right-hand side, observe that there exists a homomorphism \( \kappa : G_t \rightarrow G_s \), unique up to conjugacy, so that

\[
(3.2.4) \quad G_s \xrightarrow{\pi_1^\text{ét}(\alpha)} G_t \xrightarrow{\kappa} G_s
\]
is equal to a conjugate of the identity, that is, an inner automorphism. In particular, $\kappa$ is surjective and $\pi_{1}^{\text{ét}}(\alpha)$ is injective. We now apply $\text{Hom}(-, G_{t})/G_{t}$ to (3.2.4). This gives us
\[
\text{Hom}(G_{s}, G_{t})/G_{t} \longrightarrow \text{Hom}(G_{t}, G_{t})/G_{t} \longrightarrow \text{Hom}(G_{s}, G_{t})/G_{t}.
\]
The second map, which is induced from $\pi_{1}^{\text{ét}}(\alpha)$, is a bijection by (3.2.3) applied with $x = t$. The composition is also a bijection, by construction. Hence the first map, which is induced by $\kappa$, is bijective too. By the same argument as above, this implies that there exists a homomorphism $\lambda: G_{s} \to G_{t}$ such that the composition $G_{t} \xrightarrow{\kappa} G_{s} \xrightarrow{\lambda} G_{t}$ is an inner automorphism. In particular, $\kappa$ is injective and hence an isomorphism. By (3.2.4) it follows that $\pi_{1}^{\text{ét}}(\alpha)$ is likewise an isomorphism, which was to be shown. \hfill \square

**Remark 3.3.** The crucial property of the stratification $\{Z_{i}\}$ in Proposition 3.2 is preserved by finite quasi-étale covers:

Suppose that $\rho: X' \to X$ is a separated quasi-finite cover with branch locus contained in $Z$. By Zariski’s Main Theorem, we may assume that $\rho$ is finite since the statement is obvious for open inclusions. Consider a specialization $s' \rightsquigarrow t'$ of geometric points of $\rho^{-1}(Z_{i})$ mapping to a specialization of geometric points of $Z_{i} \subseteq X$, $s \rightsquigarrow t$, under $\rho$. The diagram
\[
(3.3.1) \quad \text{Spec } \mathcal{O}_{X', s'}^{\text{sh}} \longrightarrow \text{Spec } \mathcal{O}_{X', t'}^{\text{sh}}
\]
is a fibre product diagram up to taking a connected component (in the end it will follow that there can be at most one connected component from our assumptions).

Denote $T_{t'} := \text{Spec } \mathcal{O}_{X', t'}^{\text{sh}} \setminus \text{inverse image of } \rho^{-1}(Z)$ and likewise with $T_{s'}$. Then removing the preimages of $Z$ from (3.3.1) yields a fibre product diagram (up to taking a connected component, for now)
\[
T_{s'} \quad \alpha' \longrightarrow T_{t'}
\]
\[
\rho_{s} \downarrow \quad \rho_{t}
\]
\[
T_{s} \quad \alpha \longrightarrow T_{t},
\]
The vertical maps are étale since the branch locus of $\rho$ is contained in $Z$, so the induced maps on $\pi_{1}^{\text{ét}}$ are injective. Since $\pi_{1}^{\text{ét}}(\alpha)$ is an isomorphism, we have that $\pi_{1}^{\text{ét}}(T_{s'})$ and $\pi_{1}^{\text{ét}}(T_{t'})$ define the same subgroups in $\pi_{1}^{\text{ét}}(T_{s}) \cong \pi_{1}^{\text{ét}}(T_{t})$ (up to conjugation). In particular, $\pi_{1}^{\text{ét}}(\alpha')$ is also an isomorphism and so $\{\rho^{-1}(Z_{i})\}_{i \in I}$ stratifies $X'$ in the sense of Proposition 3.2.

**Proposition 3.4.** Suppose that $Z, U \subseteq X$ and let $\{Z_{i}\}_{i \in I}$ be such a stratification as in Proposition 3.2 made up of connected $Z_{i}$. Let $\rho: Y \to X$ be a finite surjective map whose branch locus is contained in $Z$ (in particular $\rho$ is quasi-étale), and let $W \subseteq X$ be the maximal Zariski-open subset over which $\rho$ is étale. Then, for every $i \in I$, either $W \cap Z_{i} = \emptyset$ or $Z_{i} \subseteq W$.

Equivalently, either the branch locus of $\rho$ contains $Z_{i}$ or it is disjoint from $Z_{i}$.

**Proof.** We have the following criterion for geometric points of $X$ to belong to $W$.

**Claim 3.5.** A geometric point $t \in X$ is contained in $W$ ($\rho$ is étale over $t$) if and only if the pullback of $\rho: Y \to X$ to $T_{i} := \text{Spec } \mathcal{O}_{X, t}^{\text{sh}} \setminus Z^{\text{sh}}$ is trivial (a finite disjoint union of copies of $T_{i}$).
Proof of Claim 3.5. Since the condition of being étale is Zariski-local, we see that $t \in W$ if and only if $\rho^t_{\text{sh}}: Y^\text{sh}_t \to \text{Spec } \mathcal{O}^\text{sh}_{X,t}$ is étale (here we identify $t$ with its image at $X$ to avoid cumbersome phrasing and notation). However, by faithfully flat descent [Sta16, Tag 02YJ, Lemma 34.20.29] this morphism is étale if and only if its pullback to $\text{Spec } \mathcal{O}^\text{sh}_{X,t}$, say $\rho^t_{\text{sh}}: Y^\text{sh}_t \to \text{Spec } \mathcal{O}^\text{sh}_{X,t}$, is étale. But the target being the spectrum of a strictly henselian local ring, the latter condition implies that the finite étale cover $\rho^t_{\text{sh}}$ is trivial (i.e. a finite disjoint union of copies of $\text{Spec } \mathcal{O}^\text{sh}_{X,t}$). In particular, if $t \in W$ then the pullback $\rho^{-1}(T_t) \to T_t$ is trivial.

Conversely if $\rho^{-1}(T_t) \to T_t$ is trivial, then so is $\rho^t_{\text{sh}}$ since all the schemes are normal and the complement of $T_t \subseteq \text{Spec } \mathcal{O}^\text{sh}_{X,t}$ has codimension $\geq 2$. As above this implies $t \in W$.

Now let $t \in Z_i$ be a geometric point and let $s \in Z_i$ be a generic geometric point with a specialization $s \sim t$. It is sufficient to show that $t \in W$ if and only if $s \in W$. This follows by recalling that we have a canonical map $T_s \to T_i$ which induces an isomorphism on the level of fundamental groups, and moreover a commutative diagram

$$
\begin{array}{ccc}
T_i & \to & W \\
\uparrow & & \downarrow \\
T_s & \to & W \subseteq X.
\end{array}
$$

It follows that the images of $\pi^\text{et}_1(T_i)$ and of $\pi^\text{et}_1(T_s)$ coincide in $\pi^\text{et}_1(W)$ up to conjugation. By [Mil80, Chapter I, Theorem 5.3], the finite étale cover $\rho^{-1}(W) \to W$ corresponds to a continuous $\pi^\text{et}_1(W)$-action on the finite set $Q := F(\rho^{-1}(W))$, where $F$ is the fibre functor.

Now assume $s \in W$, i.e. $\rho$ is étale over $s$. By [Claim 3.5] the induced action of $\pi^\text{et}_1(T_s)$ on $Q$ is trivial. It follows that the induced action of $\pi^\text{et}_1(T_t)$ on $Q$ is also trivial. By [Mil80, Chapter I, Theorem 5.3] again, $\rho^{-1}(T_t) \to T_t$, or more precisely $Y \times_X T_t \to T_t$, is trivial, i.e. the disjoint union of $\text{deg}(\rho)$ copies of $T_t$. Hence by [Claim 3.5] again, we get $t \in W$. Conversely, $t \in W$ implies $s \in W$ by running the argument backwards. \hfill \qed

4. Maximal quasi-étale covers

As mentioned, our result is somewhat more general than the one in the introduction.

**Theorem 4.1.** (cf. [GKP13, Theorem 2.1]) Suppose $X$ is an $F$-finite Noetherian integral strongly $F$-regular scheme. Suppose that we have a commutative diagram of separated quasi-finite maps between normal $F$-finite Noetherian integral schemes

$$
\begin{array}{ccc}
Y_1 & \leftarrow & Y_2 \leftarrow Y_3 \leftarrow \ldots \\
\gamma_1 & & \gamma_2 & & \gamma_3 \\
X & \leftarrow & X_1 \leftarrow X_2 \leftarrow \ldots \\
\eta_0 & & \eta_1 & & \eta_2
\end{array}
$$

such that the following conditions hold.

(i) The maps $j_i$ are inclusions of open sets.

(ii) The maps $\eta_i$ are finite, quasi-étale and generically Galois.

Then all but finitely many of the $\gamma_j$ are étale.

To recover the statement in the introduction, simply set all the $X_i = X$. 

\footnote{Note finite maps between normal schemes are determined outside a set of codimension 2 [Har94].}
Proof. By [DPSY16, Theorem B] (see also the semi-continuity of Hilbert-Kunz multiplicity and $F$-signature [Smi16, Pol15, PT16]) and the fact that $X$ is quasi-compact, we know there exists a uniform lower bound $\delta > 0$ on $s(\mathcal{O}_{X,x}^\text{sh} : R_x)$ for each scheme-theoretic point $x$ of $X$. Let $R_x^\text{sh}$ denote the strict henselization of the local ring $R_x$. Since $R_x \subseteq R_x^\text{sh}$ is flat with regular closed fiber, we know that $s(R_x^\text{sh}) = s(R_x) \geq \delta$ by [Yao06, Theorem 5.6]. Also $R_x^\text{sh}$ is $F$-finite since quite generally, the strict henselization of a normal $F$-finite local ring is again $F$-finite. In other words, we see that $s(\mathcal{O}_{X,x}^\text{sh}) \geq \delta$ and $\mathcal{O}_{X,x}^\text{sh}$ is $F$-finite for every geometric point $x$ of $X$. By [CST16, Theorem A], we thus know that

$$|\pi_1^\text{et}(\text{Spec } \mathcal{O}_{X,x}^\text{sh} \setminus Z^\text{sh})| \leq 1/\delta$$

for each geometric point $x$ of $X$, where $Z$ is the singular locus of $X$.

Construct the stratification $\{Z_i\}_{i \in I}$ as in Proposition 3.2 where $Z = X_{\text{sing}}$; assume the $Z_i$ are irreducible. Observe that the $Z_i \cap X_k$ also stratify the open sets $X_k \subseteq X$ (although we may lose some pieces of the stratification). Let $s_i$ be a geometric generic point of $Z_i$ and let $T_{s_i} := \text{Spec } \mathcal{O}_{X,s_i}^\text{sh} \setminus Z^\text{sh}$ be its “tubular neighborhood”. Pull back the whole sequence $X \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$ to $T_{s_i}$ to obtain

$$T_{s_i} \leftarrow T_{1,s_i} \leftarrow T_{2,s_i} \leftarrow \cdots.$$

Then all but finitely many of the $\gamma_{k,i}$ become trivial, as $\pi_{1,i}^\text{et}(T_{s_i})$ is finite (of course, it is possible that $T_{k,s_i}$ becomes empty for $k \gg 0$ if $s_i \notin X_k$). Note here is where we use the hypothesis that each $\eta_k : Y_k \rightarrow X_k \subseteq X$ is generically Galois, in particular Galois over the regular locus $U_k$ of $X_k$.

Now, given that $I$ is finite, one can pick $N \gg 0$ such that $\gamma_{n,s_i}$ is trivial for all $i \in I$ and all $n \geq N$. But by Remark 3.3, the inverse images of the generic points $s_i$ of the $Z_i$ in $Y_n$ are exactly the generic points $s_{n,j}$ of a stratification $\{Z_{n,j}\}_{j \in J_n}$ of $Y_n$ satisfying the conclusion of Proposition 3.2 with the $Z_{n,j}$ irreducible. Since

$$\gamma_n : Y_{n+1} \xrightarrow{\text{finite}} \eta_n^{-1}(X_{n+1}) \xrightarrow{\text{open}} Y_n$$

is trivial after base changing with

$$T_{s_{n,j}} := \text{Spec } \mathcal{O}_{Y_{n,s_{n,j}}}^\text{sh} \setminus \text{preimage of } \eta_n^{-1}(Z)$$

for $n \geq N$ and $j \in J_n$, by Claim 3.5, we see that $\gamma_n$ is étale over the geometric generic point of every stratum $Z_{n,j}$ (of course, it might miss some completely). Now applying Proposition 3.4 shows that $\gamma_n$ is finite étale over every point of $\eta_n^{-1}(X_{n+1})$, which completes the proof. \qed

The following corollary follows immediately.

**Corollary 4.2.** (cf. [GKPT13, Theorem 1.5]) Suppose that $X$ is an $F$-finite Noetherian integral strongly $F$-regular scheme. Then there exists a finite quasi-étale generically Galois cover $\rho : \tilde{X} \rightarrow X$ with $\tilde{X}$ normal and which satisfies the following property:

Every finite étale cover of the regular locus of $\tilde{X}$ extends to a finite étale cover of $\tilde{X}$. Equivalently, the map $\pi_1^\text{et}(\tilde{X}_\text{reg}) \rightarrow \pi_1^\text{et}(\tilde{X})$ induced by the inclusion of the regular locus is an isomorphism.

We give two proofs of this result. The first one uses Theorem 4.1 while the second one is a direct proof which emphasizes the Galois correspondence.

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1 Indeed, if $R$ is an $F$-finite ring and $S$ is a direct limit of étale maps (i.e. an ind-étale map), then $R^{1/p} \otimes_R S = S^{1/p}$ so that $S \subseteq S^{1/p}$ is finite.
First proof of Corollary 4.2. Suppose not, then for every finite quasi-étale generically Galois cover \( Y \to X \) there exists a further finite quasi-étale cover \( Y' \to Y \), which we may also assume to be generically Galois over \( X \), such that \( Y' \to Y \) is not étale. Repeating this process now with \( Y' \) taking the role of \( Y \), we obtain a sequence of covers contradicting the conclusion of Theorem 4.1.

The proof of the statement about the map \( \pi_1^{\text{ét}}(\tilde{X}_{\text{reg}}) \to \pi_1^{\text{ét}}(\tilde{X}) \) is the same as \([\text{GKP13}]\). Step 2 of the proof of Thm. 1.5, and is thus omitted. \( \square \)

Second proof of Corollary 4.2. Consider the stratification \( \{Z_i\}_{i \in I} \) as in Proposition 3.2 where each \( Z_i \) is connected. Set \( \tilde{Z} = X_{\text{sing}} \). As before, for every \( Z_i \) choose \( s_i \) to be a geometric generic point. As observed in the proof of Proposition 3.4, if \( t \) is a geometric point of \( Z_i \) generizing to \( s_i \) (as in Proposition 3.2), then \( \pi_1^{\text{ét}}(T_t) \) and \( \pi_1^{\text{ét}}(T_s) \) have a common image in \( \pi_1^{\text{ét}}(U) \), where \( U = X \setminus Z \). Let then \( G_i \subseteq \pi_1^{\text{ét}}(U) \) denote this common image. Note this common image is only unique up to conjugation. In particular, since the index set \( I \) is finite, there are only finitely many \( G_i \) up to conjugation.

Claim 4.3. There is an open normal subgroup \( H \leq \pi_1^{\text{ét}}(U) \) that intersects \( G_i \) trivially for every \( i \).

Proof of claim. Note that \( \pi_1^{\text{ét}}(U) \) is a profinite group

\[
\pi_1^{\text{ét}}(U) = \lim_{\leftarrow j} Q_j
\]

where the \( Q_j \) are quotients of \( \pi_1^{\text{ét}}(U) \). Thus each \( G_i \) maps injectively to \( Q_j \) for large enough \( j \gg 0 \). Choose a large enough \( j \) that works for all \( G_i \) and let \( H = \ker(\pi_1^{\text{ét}}(U) \to Q_j) \). In particular, \( H \) is an open normal subgroup that intersects each \( G_i \) trivially and hence by normality, it also intersects any conjugate trivially. This proves the claim. \( \square \)

Returning to the proof, by the Galois correspondence, there is a Galois cover \( \rho : \tilde{U} \to U \) such that \( \text{Aut}_U(\tilde{U}) = Q_j \) and \( \pi_1^{\text{ét}}(\tilde{U}) = H \leq \pi_1^{\text{ét}}(U) \). Let \( \rho : \tilde{X} \to X \) be the integral closure of \( X \) in \( K(\tilde{U}) \), which is finite and its pullback to \( U \) is exactly \( \rho \). In particular, \( \rho \) is quasi-étale and generically Galois. In what remains, we prove that \( \rho \) has the desired property.

Let \( V \to \tilde{U} \) be a finite étale cover in \( \text{FÉt}(\tilde{U}) \), we will extend it across \( \tilde{X} \). Take \( \sigma : Y \to \tilde{X} \) to be the integral closure of \( \tilde{X} \) in \( K(V) \). It suffices to prove that \( Y \to \tilde{X} \) is étale.

Let \( W \subseteq \tilde{X} \) be the complement of the branch locus of \( \sigma \). Let \( \tilde{Z} \) denote the inverse image of \( Z \) in \( \tilde{X} \) and note that \( \tilde{U} \) is the inverse image of \( U \) in \( \tilde{X} \), so that \( \tilde{U} = \tilde{X} \setminus \tilde{Z} \). Now \( \tilde{U} \subseteq W \) and we want to show that \( W = \tilde{X} \). By Claim 3.5, a geometric point \( x \in \tilde{X} \) belongs to \( W \) if and only if the pullback of \( \sigma : Y \to \tilde{X} \) to \( \tilde{T}_x := \text{Spec} \mathcal{O}^{sh}_{\tilde{X},x} \setminus \tilde{Z}^{sh} \) is trivial, where \( \tilde{Z}^{sh} \) is the preimage of \( \tilde{Z} \).

For any geometric point \( x \in \tilde{X} \), \( \tilde{T}_x \to \tilde{X} \) factors through \( \tilde{U} \). Thus the pullback of \( \sigma : Y \to \tilde{X} \) to \( \tilde{T}_x \) coincides with the pullback of \( V \to \tilde{U} \) to \( \tilde{T}_x \). We see that \( x \in W \) if and only if the pullback of \( V \to \tilde{U} \) to \( \tilde{T}_x \) is trivial. Hence it suffices to show that for all geometric points \( x \in \tilde{X} \) and all \( V/\tilde{U} \in \text{FÉt}(\tilde{U}) \), the pullback of \( V \to \tilde{U} \) to \( \tilde{T}_x \) is trivial. Equivalently, we want the induced homomorphism of fundamental groups \( \pi_1^{\text{ét}}(\tilde{T}_x) \to \pi_1^{\text{ét}}(\tilde{U}) \) to be zero \([\text{Mur67}] 5.2.3\). This is argued below.
Let \( t \in X \) be the image of the geometric point \( x \in \tilde{X} \). We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{T}_x & \longrightarrow & \tilde{U} \\
\downarrow & & \downarrow \\
T_t & \longrightarrow & U
\end{array}
\]

with finite étale vertical morphisms. By applying \( \pi_1^{\text{ét}}(-) \) we obtain the commutative square

\[
\begin{array}{ccc}
\pi_1^{\text{ét}}(\tilde{T}_x) & \longrightarrow & H_t \\
\downarrow & & \downarrow \\
\pi_1^{\text{ét}}(T_t) & \longrightarrow & \pi_1^{\text{ét}}(U)
\end{array}
\]

However, by construction \( H \) meets the image of \( \pi_1^{\text{ét}}(T_t) \) in \( \pi_1^{\text{ét}}(U) \) trivially, which forces the top map in this square to be zero, as required.

If \( X \) is a strongly \( F \)-regular proper variety over an \( F \)-finite field \( k \), and \( U = X_{\text{reg}} \), then it follows from [ST14, Theorem 7.6] that any étale cover \( U \) is cohomologically tame and hence tame in all the senses of [KS10]. From this we immediately obtain the following corollary.

**Corollary 4.4.** If \( X \) is a strongly \( F \)-regular proper variety over an \( F \)-finite field \( k \) with \( U = X_{\text{reg}} \), then the map \( \tilde{X} \rightarrow X \) from Corollary 4.2 restricts to a tame étale cover of \( U \).

We also obtain a local version of Corollary 4.2.

**Corollary 4.5.** (cf. [GKP13, Theorem 1.9]) Suppose that \( X \) is an \( F \)-finite Noetherian integral strongly \( F \)-regular scheme and that \( x \in X \) is a point. Then there exists a Zariski-open neighborhood \( x \in X^o \subseteq X \) and a finite quasi-étale generically Galois cover \( \rho: \tilde{X}^o \rightarrow X^o \) with \( \tilde{X}^o \) normal and which satisfies the following property:

For every further Zariski-open neighborhood \( x \in W \subseteq X^o \), with \( \tilde{W} = \rho^{-1}(W) \), any further finite quasi-étale cover \( \tilde{W} \rightarrow \tilde{W} \) is étale. Equivalently, \( \pi_1^{\text{ét}}(W) \rightarrow \tilde{Y} \) is an isomorphism.

**Proof.** Suppose not, i.e. assume that for every \( x \in X^o \subseteq X \) and every finite quasi-étale generically Galois \( \tilde{X}^o \rightarrow X^o \), there is \( x \in W \subseteq X^o \) and a finite quasi-étale \( \tilde{W} \rightarrow \rho^{-1}(W) \) that is not étale. By taking Galois closure, we may assume that \( \tilde{W} \rightarrow W \) is Galois.

Apply this assumption with \( X^o := X \) and \( \rho := \text{id}_X \). We obtain a map \( \tilde{W} \rightarrow W \), which we denote \( \eta_1: Y_1 \rightarrow X_1 \). Applying the assumption again, this time to \( \eta_1 \), we get a map \( \eta_2: Y_2 \rightarrow X_2 \) together with maps \( \gamma_1: Y_2 \rightarrow Y_1 \) and \( X_2 \rightarrow X_1 \). Inductively, we construct a diagram as in Theorem 4.1 but where none of the \( \gamma_i \) are étale. This contradicts Theorem 4.1.

**Remark 4.6.** Note that one cannot always take \( X^o = X \) in the statement of Corollary 4.5 even in characteristic zero. Indeed let \( X \) be the projective quadric cone \( V(x^2 - yz) \subseteq \mathbb{P}^3_\mathbb{C} \). Then \( X_{\text{reg}} \) is simply connected (it is an \( \mathbb{A}^1 \)-bundle over \( \mathbb{P}^1_\mathbb{C} \)), so every finite quasi-étale cover of \( X \) is trivial. On the other hand, the affine quadric cone \( U \subseteq X \) does have a finite quasi-étale cover that is not étale, corresponding to \( k[x^2, xy, y^2] \subseteq k[x, y] \).

In characteristic 2, \( \pi_1^{\text{ét}} \) of the punctured quadric cone singularity is trivial and hence this computation does not work (the local cover above is in fact inseparable). However, one can
obtain the same conclusion over an algebraically closed field of characteristic $p > 2$. Let $X' \to X$ be a finite quasi-étale generically Galois cover of $X$ that is not étale. Let $O \in X$ be the cone point. It follows that

$$X' \times_X \text{Spec } \mathcal{O}_{X,O}^{\text{sh}}$$

is a disjoint union of copies of $V \to \mathcal{O}_{X,O}^{\text{sh}}$ where $V$ is the regular 2-to-1 cover of $\text{Spec } \mathcal{O}_{X,O}^{\text{sh}}$ (corresponding to $k[x^2, xy, y^2] \subseteq k[x, y]$).

Let $L$ be a ruling of the cone and consider $X' \times_X L \to L$. This map is étale except over the cone point $O \in L \subseteq X$, and over that point $X' \times_X L$ is not even reduced. However, if one takes $L' = (X' \times_X L)_{\text{red}}$ then the computation $k[[x^2, xy, y^2]]/(x^2, xy) \subseteq k[[x, y]]/(x^2, xy)$ shows that $L'$ is normal and furthermore that $L' \to L$ is ramified of order 2 over $O$ on each connected component. In particular, $L' \to L$ is ramified over a single point and that ramification is tame (since $p > 2$), but $L \cong \mathbb{P}^1$ and so this is a contradiction. Hence any quasi-étale cover of $X$ is in fact étale and so we cannot take $X^\circ = X$ in Corollary 4.5 just as in characteristic zero.

**Remark 4.7.** We believe one can prove Corollary 4.5 using a strategy similar to Corollary 4.2. In particular, we can first assume that $x$ is in the closure of any stratum $Z_i$ (if not, shrink $X$ to remove those strata). Then use

$$\pi_{1,\text{ét}}(\{\text{Spec } \mathcal{O}_{X,x}\} \setminus \text{inverse image of } Z_i)$$

as the replacement for $\pi_{1,\text{ét}}(U)$ in the proof of Corollary 4.5. The $H$ we obtain produces a cover that is quasi-étale over a neighborhood $X^\circ$ of $x$ and which satisfies the desired property.

Just as in [GKP13, Theorem 1.10], we obtain a result on simultaneous index-one covers.

**Corollary 4.8.** Suppose that $x \in X$ and $\tilde{X}^\circ \to X^\circ \subseteq X$ is as in Corollary 4.5. Suppose further that $X^\circ$ is chosen to be affine (or quasi-projective over an affine scheme).

(a) If $\tilde{D}$ is any $\mathbb{Z}_{(p)}$-Cartier divisor on $\tilde{X}^\circ$, then $\tilde{D}$ is Cartier.

(b) There exists an integer $N > 0$ so that if $D$ is any $\mathbb{Z}_{(p)}$-Cartier divisor on $X^\circ$, then $N \cdot D$ is Cartier.

**Proof.** The proof is exactly the same as the proof of [GKP13, Theorem 1.10], and so we omit it. Of course, the main idea is to take a cyclic cover. The hypothesis that $X^\circ$ is affine (or more generally quasi-projective over an affine) is assumed so that for any finite collection of points $S \subseteq X^\circ$ and for any line bundle $\mathcal{L}$ on $X$, that there exists an open neighborhood of $S$ which trivializes $\mathcal{L}$.

Just as in [GKP13, Corollary 4.8] implies that for any $F$-finite Noetherian integral strongly $F$-regular scheme, there exists an integer $N > 0$ such that if $D$ is $\mathbb{Z}_{(p)}$-Cartier, then $N \cdot D$ is Cartier. The point is that by quasi-compactness, $X$ can be covered by finitely many open $X^\circ$ satisfying Corollary 4.8.

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³This just means it is $\mathbb{Q}$-Cartier with index not divisible by $p$. 

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