E-POLYNOMIALS OF CHARACTER VARIETIES FOR REAL CURVES

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Abstract. We calculate the E-polynomial for a class of (complex) character varieties $\mathcal{M}_n^\tau$ associated to a genus $g$ Riemann surface $\Sigma$ equipped with an orientation reversing involution $\tau$. Our formula expresses the generating function $\sum_{n=1}^\infty E(\mathcal{M}_n^\tau)T^n$ as the plethystic logarithm of a product of sums indexed by Young diagrams. The proof uses point counting over finite fields, emulating Hausel and Rodriguez-Villegas [14].

1. Introduction

Let $\Sigma$ be a compact Riemann surface of genus $g$ and let $\Sigma'' := \Sigma \setminus \{p,p'\}$ be that same surface with two points removed. Given a field $\mathbb{F}$ and a primitive $2n$th root of unity $\xi \in \mathbb{F}$ consider the representation variety $\mathcal{R}_n(\mathbb{F}) := \text{Hom}_\xi(\pi_1(\Sigma''), GL_n(\mathbb{F}))$ of homomorphisms from the fundamental group $\pi_1(\Sigma'')$ to $GL_n(\mathbb{F})$ which sends the positively oriented loops around both $p$ and $p'$ to $\xi I_n$, where $I_n$ is the identity matrix. When $\mathbb{F} = \mathbb{C}$, define the character variety

$$\mathcal{M}_n := \mathcal{R}_n(\mathbb{C})//GL_n(\mathbb{C})$$

(1.1)

to be the GIT quotient of $\mathcal{R}_n(\mathbb{C})$ under the natural conjugation action by $GL_n(\mathbb{C})$. This $\mathcal{M}_n$ is a smooth, affine, complex symplectic variety. If $\xi = e^{2\pi i d / 2n}$, then $\mathcal{M}_n$ is naturally diffeomorphic to the moduli space of Higgs bundles of degree $d$ and rank $n$ over $\Sigma$ via the non-Abelian Hodge correspondence [16, 28].

If $\Sigma$ is endowed with an anti-holomorphic involution $\tau$ that interchanges $p$ and $p'$, then there is an associated character variety $\mathcal{M}_n^\tau$ introduced by Baraglia and Schaposnik [3] and by Biswas, García-Prada, and Hurtubise [4], which embeds as a holomorphic Lagrangian submanifold in $\mathcal{M}_n$ (also known as an ABA-brane [3]). If $\Sigma'' \neq \emptyset$, which we will usually assume, then $\mathcal{M}_n^\tau$ is equal to the the fixed point set of an anti-symplectic involution of $\mathcal{M}_n$. Under the non-Abelian Hodge correspondence, $\mathcal{M}_n^\tau$ is sent to the set of real points in the moduli space of Higgs bundles over the real curve $(\Sigma, \tau)$.

In the present paper, we calculate the E-polynomial of $\mathcal{M}_n^\tau$ for all $n \geq 1$. Our calculation reduces to counting points over finite fields, emulating the calculation of the E-polynomial of $\mathcal{M}_n$ by Hausel and Rodriguez-Villegas [14].

The character variety $\mathcal{M}_n^\tau$ is defined using the orbifold fundamental group $\pi_1(\Sigma'')$, which is the fundamental group of the homotopy quotient of $\Sigma''$ with respect to the $\mathbb{Z}/2$ action generated by $\tau$. This fits into a short exact sequence $1 \to \pi_1(\Sigma'') \to \pi_1(\Sigma'') \to \mathbb{Z}/2 \to 0$ which splits if $\Sigma'' \neq \emptyset$. Let $\tilde{GL}_n(\mathbb{F}) = GL_n(\mathbb{F}) \rtimes \mathbb{Z}/2$ be the semi-direct product determined by the Cartan involution $A \mapsto (A^T)^{-1}$ and let $\mathcal{R}_n(\mathbb{F})$ denote the representation variety of
homomorphisms $\tilde{\phi}$ that extend to a commutative diagram

$$
\begin{array}{c}
1 \longrightarrow \pi_1(\Sigma') \longrightarrow \pi_1(\Sigma') \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\
\downarrow \phi \downarrow \tilde{\phi} \downarrow = \downarrow \\
1 \longrightarrow GL_n(F) \longrightarrow \tilde{GL}_n(F) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.
\end{array}
$$

Define

$$M_{\tau}^n := \mathcal{R}_n(\mathbb{C})//GL_n(\mathbb{C}) \quad (1.2)$$

the GIT quotient under conjugation by $GL_n(\mathbb{C}) \leq \tilde{GL}_n(\mathbb{C})$. The forgetful map $M_{\tau}^n \rightarrow M_n$ is an embedding.

One of the main results of [14] is a formula for the E-polynomial (or Serre characteristic) of $M_n$. They prove the following remarkable generating function identity

$$1 \sum_{n=1}^{\infty} \frac{E(M_n; q) T^n}{(q-1)^{2n^2(g-1)}} = \log \left( \sum_{\lambda \in \mathcal{P}} H_{\lambda}^{2g-2}(q) T^{2\lambda} \right). \quad (1.3)$$

In this expression $E(M_n; q)$ is a polynomial in $q$ from which the E-polynomial of $M_n$ is recovered by setting $q = xy$; the function Log is the plethystic logarithm; and $H_{\lambda}(q)$ is the normalized hook polynomial associated to a partition (or Young diagram) $\lambda \in \mathcal{P}$ (see §10 for details). Our main result is an analogue of this formula for the E-polynomial of $M_{\tau}^n$.

**Theorem 1.1.** Let $(\Sigma, \tau)$ be a genus $g$ Riemann surface equipped with an anti-holomorphic involution such that $\Sigma^\tau$ has $r$-many path components, with $r \geq 1$. Then

$$\frac{2}{q-1} \sum_{n=1}^{\infty} \frac{E(M_{\tau}^n; q) T^n}{(-q^{2k})^{n^2(g-1)}} = \log \prod_{k=0}^{\infty} \left( \frac{\sum_{\lambda \in \mathcal{P}} (a^+_\lambda)^r H_{\lambda}^{-1}(q^{2k}) T^{2\lambda}}{\sum_{\lambda \in \mathcal{P}} (a^-_\lambda)^r H_{\lambda}^{-1}(q^{2k}) T^{2\lambda}} \right)^{\frac{1}{2^g}}. \quad (1.4)$$

where if $\lambda = (s_1^1s_2^2...)$ then

$$a^+_\lambda = (s_1 + 1)(s_2 + 1)... \quad (1.5)$$

$$a^-_\lambda = \begin{cases} 1 & \text{if the conjugate partition } \lambda' \text{ has only even parts} \\ 0 & \text{otherwise.} \end{cases} \quad (1.6)$$

Furthermore, the character variety decomposes into connected components indexed by certain invariants $w$

$$M_{\tau}^n = \coprod_w M_{\tau}^{n,w}$$

(see Corollary 2.3) and we calculate the E-polynomials of these components.

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1The case $\Sigma^\tau = \emptyset$ has been considered by Letellier and Rodriguez-Villegas [20]. They derived a formula for the E-polynomial of $M_{\tau}^n$ in that case, and produced a conjectural formula for the mixed Hodge polynomial which was recently disproven by Scognamiglio [27]. Since we focus on the case $\Sigma^\tau \neq \emptyset$, their work complements ours very nicely.
Theorem 1.1. Our proof of Theorem 1.1, like the proof of (1.3), relies on counting points over finite fields. Hausel and Rodriguez-Villegas [14], construct a polynomial \( p(q) \in \mathbb{Z}[q] \) such that \( |\mathcal{R}_n(\mathbb{F}_q)| = p(q) \) for \( \text{char}(q) \gg 1 \). By Katz' Theorem [14] (alternatively Ito [18, Cor. 6.5]), this determines the E-polynomial by the identity

\[
E(\mathcal{R}_n(\mathbb{C})) = p(xy).
\]

Furthermore, since \( \text{GL}_n(\mathbb{C}) \) acts with constant stabilizer \( \mathbb{C}^\times I_n \) so that \( \text{PGL}_n(\mathbb{C}) := \text{GL}_n(\mathbb{C})/\mathbb{C}^\times I_n \) acts freely, we have

\[
E(\mathcal{M}_n) = p(xy)/E(\text{PGL}_n(\mathbb{C})).
\]

In the present paper, we prove that \( p^r(q) = |\mathcal{R}_n^r(\mathbb{F}_q)| \) is a polynomial function of \( q \) for \( \text{char}(q) \gg 1 \) and deduce similarly

\[
E(\mathcal{M}_n^r) = p^r(xy)/E(\text{GL}_n(\mathbb{C})),
\]

with the difference that \( \text{GL}_n(\mathbb{C}) \) acts on \( \mathcal{R}_n(\mathbb{C}) \) with constant stabilizer group \( \pm I_n \) rather than \( \mathbb{C}^\times I_n \).

To get their point count formula, Hausel and Rodriguez-Villegas use the presentation\(^2\)

\[
\pi_1(\Sigma^\prime) = \langle a_1, b_1, \ldots, a_g, b_g, c, d | \prod_{i=1}^g [a_i, b_i] = cd \rangle,
\]

which determines an isomorphism

\[
\mathcal{R}_n(\mathbb{F}) \cong \{(A_1, B_1, \ldots, A_g, B_g) \in \text{GL}_n(\mathbb{F})^{2g} | \prod_{i=1}^g [A_i, B_i] = \xi^2 I_n \}.
\]

Define the class function \( C : \text{GL}_n(\mathbb{F}_q) \to \mathbb{Z}_{\geq 0} \subseteq \mathbb{C} \) by

\[
C(A) := |\{(X, Y) \in \text{GL}_n(\mathbb{F}_q)^2 | [X, Y] = A\}|.
\]

Then

\[
|\mathcal{R}_n(\mathbb{F}_q)| = (C \ast \cdots \ast C)(\xi^2 I_n) = C^*g(\xi^2 I_n)
\]

where \( \ast \) is the convolution product

\[
(\phi \ast \psi)(A) = \sum_{B \in \text{GL}_n(\mathbb{F}_q)} \phi(B)\psi(B^{-1}A).
\]

In similar fashion, we use an explicit presentation for \( \pi_1(\Sigma^\prime, \tau) \) due to Huisman [17] to derive the identity

\[
|\mathcal{R}_n^r(\mathbb{F}_q)| = (F \ast \cdots \ast F \ast N \ast \cdots \ast N)(\xi I_n) = (F^*r \ast N^*(g-r+1))(\xi I_n)
\]

\[(1.7)\]

\(^2\)In fact Hausel and Rodriguez-Villegas work with a single puncture, but the resulting formula for \( \mathcal{R}_n(\mathbb{F}) \) is identical.
where \( r \) is the number of path components of \( \Sigma^r \cong \prod_r S^1 \). In this expression, \( F, N : GL_n(\mathbb{F}_q) \to \mathbb{Z}_{\geq 0} \subseteq \mathbb{C} \) are class functions defined by

\[
F(A) := |\{ B \in GL_n(\mathbb{F}_q) | B = B^T, ABA^T = B \}|
\]

\[
N(A) := |\{ B \in GL_n(\mathbb{F}_q) | B(B^{-1})^T = A \}|
\]

The function \( N \) was considered by Gow [11] where he proved that

\[
N \ast N = C,
\]

so \( N \) is a “square root” of \( C \). The function \( F(A) \) counts the number of non-degenerate symmetric bilinear form on \( \mathbb{F}_q^n \) for which \( A \) is an isometry.

Convolution products can be understood using harmonic analysis. Recall that the irreducible characters of a finite group \( G \) form an orthonormal basis of the space of class functions, so given a class function \( \phi : G \to \mathbb{C} \) we have,

\[
\phi = \sum_{\chi \in \text{Irr}(G)} \langle \phi, \chi \rangle \chi
\]

where

\[
\langle \phi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \chi(g).
\]

The convolution product satisfies the identity

\[
\langle \phi \ast \psi, \chi \rangle = \frac{|G|}{\chi(1)} \langle \phi, \chi \rangle \langle \psi, \chi \rangle,
\]

so convolution products are easily understood once class functions are expressed in the irreducible character basis. Gow [11] proved that

\[
N = \sum_{\chi \in \text{Irr}(GL_n(\mathbb{F}_q))} \chi
\]

so every irreducible character occurs with multiplicity one.\(^3\) Define \( a_\chi \in \mathbb{Z}_{\geq 0} \) by

\[
F = \sum_{\chi \in \text{Irr}(GL_n(\mathbb{F}_q))} a_\chi \chi.
\]

Applying to (1.7) we get

\[
|R_n^T(\mathbb{F}_q)| = |GL_n(\mathbb{F}_q)|^9 \sum_{\chi \in \text{Irr}(GL_n(\mathbb{F}_q))} a_\chi^r \chi(\xi I_n)^9 \chi(I_n)^9.
\]

The bulk of the current paper is devoted to calculating the coefficients \( a_\chi \). The calculation breaks into three steps:

1. In §5, we compute an explicit formula for \( F \) using Milnor’s classification of orthogonal transformations over perfect fields [25].
2. In §8, §9, we use the formula \( a_\chi = \langle F, \chi \rangle \) (see (1.8)) to calculate \( a_\chi \) in the limit as \( q \to \infty \).
3. In §6, we show that the limit in step 2 actually makes sense, and that the formula remains valid for \( \text{char}(q) \gg 1 \).

\(^3\)The multiplicity in this case can be interpreted as a twisted Frobenius-Schur indicator (see [26]), which helps explain why it always equals one.
Once the multiplicities $a_{\lambda}$ have been determined it is relatively straightforward (emulating [14]) to produce an explicit polynomial expression for (1.7), leading to the generating function in Theorem 1.1. This is carried out in §10. Finally in §11, we calculate the E-polynomial of the connected components $M_{n,w}^r$.

1.2. Further discussion.

1.2.1. Euler characteristic of the $PGL_n$-character variety. The character variety $M_n$ admits an action by the group $A = \text{Hom}(\pi_1(\Sigma), \mathbb{C}^\times) \cong (\mathbb{C}^\times)^g$ via scalar multiplication. The quotient space $\tilde{M}_n := M_n//A$ is identified with the $PGL_n(\mathbb{C})$-character variety. If $g \leq 1$ then $\tilde{M}_n$ is either a point or the empty set. If $g \geq 2$, Hausel and Rodriguez-Villegas proved that the Euler characteristic of $\tilde{M}_n$ is equal to $\mu(n)n^{2g-3}$ where $\mu(n)$ is the Mobius function.

The involution $\tau$ on $\Sigma$ induces an automorphism of $A$ and the invariant subgroup $A^r \cong (\mathbb{C}^\times)^g \times \{\pm 1\}^{r-1}$ acts on $M_n^r$. Define

$$\tilde{M}_n^r := M_n^r//A^r.$$ 

If $n$ is odd then $A$ transitively permutes the set of connected components of $M_n^r$. Consequently if $M_{n,w}^r$ is a particular component, we have

$$\tilde{M}_n^r \cong M_{n,w}^r//A_0^r$$

where $A_0 \cong (\mathbb{C}^\times)^g$ be the identity component of $A$. On the other hand, if $n$ is even, then $A$ does not permute components and we have

$$\tilde{M}_n^r \cong \bigsqcup_w M_{n,w}^r//A^r.$$ 

When $g \geq 2$ and $n$ is odd, we prove (Corollary 11.4) that the Euler characteristic of $\tilde{M}_n^r$ is equal to $\mu(n)n^{g-2}$. When $g \geq 2$ and $n$ is even, we prove that the Euler characteristic of $M_{n,w}^r//A_0^r$ is zero for all $w$, but we do not calculate the Euler characteristic of $\tilde{M}_n^r$.

1.2.2. Mixed Hodge polynomials. The E-polynomial is a specialization of the (compactly supported) mixed Hodge polynomial. Namely, if $Z$ is a complex variety, $E(Z; x, y) := MH(Z; x, y, -1)$ where

$$MH(Z; x, y, t) := \sum h_{i,j,k} x^i y^j t^k$$

and $h_{i,j,k}$ are the dimensions of associated graded components of the mixed Hodge filtration on compactly supported cohomology $H^c(Z; \mathbb{C})$. Hausel and Rodriguez-Villegas conjectured a generating function identity for mixed Hodge polynomial which reduces to (1.3) upon setting $t = -1$:

$$\frac{1}{(xy - 1)(t^2xy - 1)} \sum_{n=1}^{\infty} \frac{MH(M_n; x, y, t)}{(t^2xy)^{n^2(g-1)}} T^n = \text{Log} \left( \sum_{\lambda \in \mathcal{P}} \mathcal{H}_{\lambda}^{2g-2}(xy, t)T^{[\lambda]} \right).$$ (1.12)

for certain two variable rational functions $\mathcal{H}_{\lambda}^{2g-2}(xy, t)$. They verified that (1.12) gives the correct formula for $n = 1$ and $n = 2$. The specialization $x = y = 1$ was later proven by Mellit [24].

It is natural to hope for an analogous conjectural identity for the mixed Hodge polynomial of $M_n^r$. Our efforts in this direction have been hampered by a lack of understanding of the rational cohomology ring of $M_n^r$ when $n > 1$; so far only the $\mathbb{Z}_2$-Betti numbers of $M_2^r$ have been calculated [2]. This is a promising direction for future research.
1.2.3. Curious Poincaré duality. The E-polynomial of $\mathcal{M}_n$ satisfies the so-called curious Poincaré duality property

$$E(\mathcal{M}_n; q) = q^{\dim(\mathcal{M}_n)}E(\mathcal{M}_n; q^{-1}).$$

(1.13)

This is curious because $\mathcal{M}_n$ is non-compact, so topological Poincaré duality does not apply. Curious Poincaré duality is a consequence of the P=W conjecture of de Cataldo, Hausel, and Migliorini [6], which holds that under the non-Abelian Hodge correspondence homeomorphism between $\mathcal{M}_n$ and $\mathcal{M}_{Dol}$, the weight filtration on the cohomology ring $H^*(\mathcal{M}_n; \mathbb{Q})$ is sent to the perverse Leray filtration on $H^*(\mathcal{M}_{Dol}; \mathbb{Q})$ associated to the Hitchin map $\mathcal{M}_{Dol} \to \mathcal{B}$. The $P = W$ conjecture was proven recently by Maulik and Shen [23], and independently by Hausel, Mellit, Minets, and Schiffmann [15].

In the current paper, we find that the E-polynomial of $\mathcal{M}_n^\tau$ generally does not satisfy curious Poincaré duality. This is in keeping with the P=W conjecture, because under the non-Abelian Hodge correspondence $\mathcal{M}_n^\tau$ is sent to a real submanifold $\mathcal{M}_n^{Dol} \subseteq \mathcal{M}_{Dol}$ and we should not expect the real integrable system $\mathcal{M}_n^{Dol} \to \mathcal{B}$ to have a well behaved perverse Leray spectral sequence [8]. As such, our results provided some circumstantial evidence in support of the P=W conjecture (we thank Vivek Shende for emphasizing this point to us).

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2. The character variety

We begin with a review of Hitchin’s equations and the non-abelian Hodge correspondence. Let $P$ be a $U(n)$-principle bundle of degree $d$ over a compact Riemann surface $\Sigma$ and where $d,n$ are coprime, and let $P_c$ be the complexified $GL_n(\mathbb{C})$-bundle. Let $A$ be a connection on $P$ and let $\Phi \in \Omega^{1,0}(\Sigma, adP_c)$. The (inhomogeneous) Hitchin equations are

$$F_A + [\Phi, \Phi^*] = \omega$$

$$d^*_A \Phi = 0$$

where $\omega \in \Omega^2(\Sigma, z)$ is a fixed 2-form with values in the centre $z \subseteq adP_c$. The moduli space of solutions is a manifold we denote $\mathcal{M}_{Hit}(\omega)$. Given any two $\omega, \omega' \in \Omega^2(\Sigma, z)$ we can produce an isomorphism $\mathcal{M}_{Hit}(\omega) \cong \mathcal{M}_{Hit}(\omega')$ by tensoring with an appropriate $U(1)$-bundle connection, so we will abuse notation and denote $\mathcal{M}_{Hit} := \mathcal{M}_{Hit}(\omega)$.

The forgetful map $(A, \Phi) \mapsto (d_A^*, \Phi)$ induces a morphism from $\mathcal{M}_{Hit}$ to the moduli space of stable Higgs pairs $\mathcal{M}_{Dol}$. The forgetful map $(A, \Phi) \mapsto A + \Phi + \Phi^*$ determines a morphism from $\mathcal{M}_{Hit}$ to the moduli space $\mathcal{M}_{DR}$ of $GL_n(\mathbb{C})$-connections with curvature $\omega$. Note in particular that $A + \Phi + \Phi^*$ is projectively flat. The non-Abelian Hodge correspondence says that the forgetful maps defined above determine diffeomorphisms

$$\mathcal{M}_{Dol} \leftarrow \mathcal{M}_{Hit} \to \mathcal{M}_{DR}.$$  

(2.1)
Now consider an anti-holomorphic involution $\tau : \Sigma \to \Sigma$ and suppose $\omega = \tau^*\mathcal{W}$. This determines an involution on $\mathcal{M}_\text{Hit}$ sending the pair $(A, \Phi)$ on $P$ to the pair $\tau(A, \Phi) = (\tau^*A, -\tau^*\Phi)$ on the conjugate pull-back bundle $\tau^*\mathcal{T}$. The involution descends to a holomorphic involution of $\mathcal{M}_{\text{DR}}$ and an anti-holomorphic involution of $\mathcal{M}_{\text{Dol}}$. By Proposition 2.1 we obtain diffeomorphisms of fixed point sets

$$(\mathcal{M}_{\text{Dol}})^\tau \cong (\mathcal{M}_\text{Hit})^\tau \cong (\mathcal{M}_{\text{DR}})^\tau.$$ 

Choose $\omega$ equal to zero except for a pair of delta function singularities at a pair of points $p, p'$ and let $\Sigma' := \Sigma \setminus \{p, p'\}$. Then the Riemann-Hilbert correspondence determines a diffeomorphism to the character variety $(\Sigma^\circ)$

$$(\mathcal{M}_{\text{DR}}) \cong \mathcal{M}_n := \mathcal{R}_n(\mathbb{C})//GL_n(\mathbb{C}).$$

Assume henceforth that $\Sigma^\circ \neq \emptyset$ and choose one of these fixed points as the base point for $\pi_1(\Sigma')$ so that $\tau$ induces an automorphism $\tau_*$ of $\pi_1(\Sigma')$. The following is a minor alteration of ([3], Prop. 15).

**Proposition 2.1.** If $\Sigma^\circ \neq \emptyset$ then the forgetful map $\mathcal{M}_n^\tau \to \mathcal{M}_n$ restricts to a bijection

$$\mathcal{M}_n^\tau \cong (\mathcal{M}_n)^\tau.$$ 

**Proof.** The isomorphism $\pi_1(\Sigma^\circ) = \pi_1(\Sigma') \times \mathbb{Z}_2$ determines an isomorphism

$$\mathcal{R}_n^\tau(\mathbb{C}) \cong \{ (\rho, A) \in \mathcal{R}_n(\mathbb{C}) \times GL_n(\mathbb{C}) | A\theta(\rho)\theta(A) = \rho \circ \tau_*, A\theta(A) = I_n \} \quad (2.2)$$

$$= \{ (\rho, A) \in \mathcal{R}_n(\mathbb{C}) \times GL_n(\mathbb{C}) | A^{-1}\rho A = \theta \circ \rho \circ \tau_*, A = A^T \} \quad (2.3)$$

where $\tau_*$ is the automorphism of $\pi_1(\Sigma')$ induced by $\tau$ and $\theta \in \text{Aut}(GL_n(\mathbb{C}))$ is the Cartan involution $\theta(X) = (X^{-1})^T$.

On the other hand, the involution $\tau$ on $\mathcal{M}_n$ lifts to the involution $\iota$ of $\mathcal{R}_n(\mathbb{C})$

$$\iota(\rho) := \theta \circ \rho \circ \tau_*.$$ 

A homomorphism $\rho \in \mathcal{R}_n(\mathbb{C})$ represents $[\rho] \in (\mathcal{M}_n)^\tau$ if and only if there exists $A \in GL_n(\mathbb{C})$ such that

$$A^{-1}\rho A = \theta \circ \rho \circ \tau_*,$$

which implies, if $B := A^TA^{-1}$, that

$$B\rho B^{-1} = \rho.$$ 

Since $\rho$ is irreducible, $B$ is a scalar matrix, hence $B = \pm I_n$ and $A = \pm A^T$. We call $\rho$ real if $A = A^T$ and quaternionic if $A = -A^T$. It remains to show there are no quaternionic representations.

Under the non-Abelian Hodge correspondence, the real representations are sent to real vector bundles and the quaternionic representations are sent to quaternionic vector bundles in the sense of Atiyah [1]. Therefore by ([5] Prop. 4.2), the quaternionic representations do not exist if $\Sigma^\circ \neq \emptyset$.

□

A presentation of $\pi_1(\Sigma')$ was produced by Huisman [17].

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4In fact Huisman considered the case with no punctures, but the formula for $\pi_1(\Sigma')$ is an immediate corollary.
Proposition 2.2. Let $\Sigma''$ be a twice punctured genus $g$ Riemann surface with orientation reversing involution $\tau$ transposing the punctures. Let $r$ be the number of fixed point components of $\Sigma''$, and let $r + s = g + 1$. Then

$$ \pi_1(\Sigma'') \cong \langle \{a_i, b_i\}_{i=1}^r, \{x_j\}_{j=1}^s, d | b_i^2 = 1, a_i b_i = b_i a_i, \Phi(a, x) = d \rangle $$

where

$$ \Phi = \begin{cases} \prod_{i=1}^r a_i \prod_{j=1}^{s/2} [x_{2j-1}, x_{2j}] & \text{if } \Sigma / \tau \text{ is orientable (} s \text{ is even in this case)} \\ \prod_{i=1}^r a_i \prod_{j=1}^s x_j^2 & \text{if not.} \end{cases} $$

The generators $a_i$ and $d$ lie in the subgroup $\pi_1(\Sigma)$, the $b_i$ do not, and the $x_j$ do if and only if $\Sigma / \tau$ is orientable. The generator $d$ corresponds to a loop around one puncture point.

It follows that $R_n^T(\mathbb{F})$ can be identified with subvariety of $GL_n(\mathbb{F})^{2r+s}$ of tuples $(A_i, B_i, X_j)$ defined by equations

$$ B_i = B_i^T, \text{ and } A_i B_i A_i^T = B_i, \text{ for all } i \in \{1, \ldots, r\} $$

(2.6)

and

$$ \Phi(A, X) = \xi Id_n $$

where

$$ \Phi := \begin{cases} \prod_{i=1}^r A_i \prod_{k=1}^{s/2} [X_{2k-1}, X_{2k}] & \text{if } \Sigma / \tau \text{ is orientable.} \\ \prod_{i=1}^r A_i \prod_{j=1}^s X_j (X_j^2)^{-1} & \text{if not.} \end{cases} $$

(2.7)

Note that (2.6) simply requires $B_i$ to represent a non-degenerate, symmetric bilinear form with respect to which $A_i$ is orthogonal. This implies in particular that $\det(A_i) \in \{\pm 1\}$ for all $i = 1, \ldots, r$, immediately yielding the following.

Corollary 2.3. There are coproduct decompositions

$$ R_n^T(\mathbb{F}) = \bigoplus_w R_n^T(\mathbb{F})_w $$

and

$$ M_n^T = \bigoplus_w M_{n,w}^T $$

indexed by $r$-tuples $w \in \{\pm 1\}^r$ satisfying the condition $\prod_{i=1}^r w(i) = \xi^n = -1$.

The $M_{n,w}^T$ are in fact the connected components of $M_n^T$. This can be proven of the homeomorphic space $(M_{pol})^T$ using Morse theory (see for example [2]). This is also clear from the E-polynomial formulas we calculate in §11.

The group $A := Hom(\pi_1(\Sigma), \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{2g}$ acts on the character variety $M_n$ via the scalar multiplication action of $\mathbb{C}^\times$ on $GL_n(\mathbb{C})$. The quotient

$$ \tilde{M}_n := M_n / A $$

is called the PGL$_n(\mathbb{C})$-character variety.

The subgroup $A^T := \{ \phi \in A | \theta \circ \phi \circ \tau_* = \phi \}$ restricts to an action on $M_n^T$. Denote the quotient

$$ \tilde{M}_n^T := M_n^T / A^T. $$

We have isomorphisms

$$ A^T \cong (\mathbb{C}^\times)^g \times \{\pm 1\}^{r-1} $$

In terms of the presentation above, the action is defined

$$ (\lambda_1, \ldots, \lambda_{g+1}, \epsilon_1, \ldots, \epsilon_r) \cdot (A_i, B_i, X_j) = (\epsilon_i A_i, \lambda_i B_i, \lambda_{j+r} X_j) $$

where $\lambda_i \in \mathbb{C}^\times$, $\epsilon_i \in \{\pm 1\}$ and we impose $\lambda_{g+1} = 1 = \prod_{i=1}^r \epsilon_i$. Under this action $M_1^T$ is an $A^T$-torsor, so
\[ \mathcal{M}_n^r \cong \prod_{2^r-1} (\mathbb{C}^\times)^g. \] (2.8)

**Remark 2.4.** When \( n \) is odd, \( \mathcal{A}^r \) transitively permutes the connected components \( \mathcal{M}_n^r \), so the \( \mathcal{M}_{n,w}^r \) are pair-wise isomorphic. When \( n \) is even \( \mathcal{A}^r \) does not permute components.

The surjective homomorphism \( \widetilde{GL}_n(\mathbb{C}) \to \widetilde{GL}_1(\mathbb{C}) \) defined by sending \((A, \epsilon) \to (\det(A), \epsilon)\) determines a fibre bundle
\[
\det : \mathcal{M}_{n,w}^r \to \mathcal{M}_{1,w}^r \cong (\mathbb{C}^\times)^g.
\]

Given \( \phi \in \mathcal{M}_{1,w}^r \) denote the fibre \( \mathcal{M}_{n,\phi}^r := \det^{-1}(\phi) \). If \( \mathcal{A}_0^r \cong (\mathbb{C}^\times)^g \) is the identity component of \( \mathcal{A} \) then we have an isomorphism
\[
\mathcal{M}_{1,w}^r \cong \mathcal{M}_{1,\phi}^r \times_{\mu_n} \mathcal{A}_0^r
\]
where \( \mu_n \cong (\mathbb{Z}/n)^g \) is the \( n \)-torsion subgroup of \( \mathcal{A}_0^r \). Therefore
\[
H^* (\mathcal{M}_{1,w}^r) \cong H^* (\mathcal{M}_{1,\phi}^r)^{\mu_n} \otimes H^* ((\mathbb{C}^\times)^g),
\]
where \( H^* (\mathcal{M}_{1,\phi}^r)^{\mu_n} \) is the ring of \( \mu_n \)-invariants. In particular, the E-polynomial of \( \mathcal{M}_{1,w}^r \) is divisible by \( E((\mathbb{C}^\times)^g) = (q-1)^g \).

3. **Point counting and the E-polynomial**

**Proposition 3.1.** The conjugation action of \( GL_n(\mathbb{C}) \) on \( \mathcal{R}_n^r(\mathbb{C}) \) is free modulo the centre \( \{ \pm I_n \} \leq GL_n(\mathbb{C}) \). Consequently, the E-polynomials satisfy the identity
\[
E(\mathcal{M}_n^r) = \frac{E(\mathcal{R}_n^r(\mathbb{C})))}{E(GL_n(\mathbb{C})))}.
\]

Similarly
\[
E(\mathcal{M}_{n,w}^r) = \frac{E(\mathcal{R}_{n,w}^r(\mathbb{C})))}{E(GL_n(\mathbb{C})))}.
\]

**Proof.** The forgetful map \( \mathcal{R}_n^r(\mathbb{C}) \to \mathcal{R}_n(\mathbb{C}) \) is \( GL_n(\mathbb{C}) \)-equivariant and it was proven in [14, Lemma 2.2.6] that every point in \( \mathcal{R}_n(\mathbb{C}) \) is stabilized only by scalar matrices. However, the only scalar matrices that centralize elements in the non-identity component of \( \widetilde{GL}_n(\mathbb{C}) \) are \( \{ \pm I_n \} \) so the quotient map
\[
\mathcal{R}_n^r(\mathbb{C}) \to \mathcal{M}_n^r
\]
is a principal \( GL_n(\mathbb{C})/\{ \pm I_n \} \)-bundle. Since \( GL_n(\mathbb{C}) \cong GL_n(\mathbb{C})/\{ \pm I_n \} \) we have
\[
E(\mathcal{R}_n^r(\mathbb{C})) = E(\mathcal{M}_n^r) E(GL_n(\mathbb{C})))
\]
by [21, Remark 2.5].

It remains to calculate \( E(\mathcal{R}_n^r(\mathbb{C})) \). We use a point counting argument analogous to that used by Hausel and Rodriguez-Villegas [14].

**Proposition 3.2.** Suppose \( p^r(t) \in \mathbb{Z}[t] \) is a polynomial such that the cardinality \( p^r(q) = |\mathcal{R}_n^r(\mathbb{F}_q)| \) for \( \text{char}(q) \gg 1 \). Then \( p(xy) = E(\mathcal{R}_n^r(\mathbb{C})) \).

**Proof.** Let \( \Phi_d(x) \) the \( d \)th cyclotomic polynomial and consider the ring \( A := \mathbb{Z}[x]/(\Phi_{2n}(x)) \). We can interpret \( \mathcal{R}_n^r \) as an affine scheme over \( A \) and \( \mathcal{R}_n^r(\mathbb{F}) \) as the variety obtained by an extension of scalars \( \phi : A \to \mathbb{F} \) which sends \( x \) to the chosen primitive \( 2n \)th root of unity \( \xi \in \mathbb{F} \). The result now follows by Katz’ Theorem [14, Thm. 2.1.8].
Next, we want an expression for $|R^\tau_n(F_q)|$. Define functions $N, F, C$ from $G_n := GL_n(F_q)$ to $\mathbb{Z}_{\geq 0}$ as follows:

$$F(A) := \left| \{ B \in G_n : B^T = B, ABA^T = B \} \right|$$
$$N(A) := \left| \{ B \in G_n : B(B^T)^{-1} = A \} \right|$$
$$C(A) := \left| \{ (X, Y) \in G_n^2 : [X, Y] = XYX^{-1}Y^{-1} = A \} \right| .$$

If $\Sigma^\tau$ has $r$-path components, let $r + s = g + 1$. It follows from (2.6) and (2.7) that the cardinality of $R^\tau_n(F_q)$ is equal to the value of the following convolution product at $\xi I_n$:

$$|R^\tau_n(F_q)| = \begin{cases} (F^{sr} \ast C^{ss/2})(\xi Id_n) & \text{if } \Sigma/\tau \text{ is orientable (s is even in this case)} \\ (F^{sr} \ast N^{ss})(\xi Id_n) & \text{if not.} \end{cases}$$

In fact, $N \ast N = C$, (3.1)

so

$$|R^\tau_n(F_q)| = F^{sr} \ast N^{s(g-r+1)}(\xi I_n).$$

(3.2)
is independent of the orientability of $\Sigma/\tau$. Gow [11] proved that $N$ is the sum of the irreducible characters of $G_n$, each with multiplicity one

$$N = \sum_{\chi \in \text{Irr } G_n} \chi.$$ 

The decomposition of $C$ can be found in ([14] (2.3.7)) and the identity (3.1) follows using (1.10).

Observe that $F$ is a character function on $G_n$ because $F(g)$ counts $g$-fixed points for the action of $G_n$ on the set of non-degenerate symmetric bilinear forms on $\mathbb{F}_q^n$. This implies that

$$F = \sum_{\chi \in \text{Irr } G_n} a_{\chi} \chi,$$

where the $a_{\chi} \in \mathbb{Z}_{\geq 0}$ are multiplicities of irreducible characters.

**Corollary 3.3.** If the function

$$E_n(q) = |G_n|^g \sum_{\chi \in \text{Irr } G_n} \frac{\chi(\xi)}{\chi(1)^g} a_{\chi}^r$$

is a polynomial function in $q$ for $\text{char}(q) \gg 1$, then

$$E(M^\tau_n) = E_n(xy).$$

**Proof.** Applying (1.10) and (3.2) we get

$$|R^\tau_n(F_q)| = |G_n|^g \sum_{\chi \in \text{Irr } G_n} \frac{\chi(\xi)}{\chi(1)^g} a_{\chi} = |G_n|E_n(q).$$

We have $|G_n| = f(q)$ where $f(t) = \prod_{i=0}^{n-1}(t^n - t^i)$, so if $E_n(q)$ is a polynomial function for $\text{char}(q) \gg 1$, then by Proposition 3.2 we have

$$E(R^\tau_n(\mathbb{C})) = f(xy)E_n(xy).$$

Lastly, note that $E(GL_n(\mathbb{C})) = f(xy)$ and apply Proposition 3.1. □
Suppose now that char \( \mathbb{F} \neq 2 \). Define
\[
F = F_+ + F_-
\]
where \( F_+ \) is supported on the matrices with determinant 1 and \( F_- \) is supported on those of determinant \(-1\). Let \( 1 \leq k \leq r \) be odd and choose \( w \in \{ \pm 1 \}^r \) for which \( k \)-many coordinates equal \(-1\). Then the cardinality of \( \mathcal{R}_n^r(\mathbb{F}_q)^w \) is equal to
\[
|\mathcal{R}_n^r(\mathbb{F}_q)^w| = F_+^{(r-k)} * F_-^k * N^*(g-r+1)(\xi I_n).
\]
If
\[
F_+ := \sum \chi b_\chi \chi \quad F_- := \sum \chi b_\chi^- \chi,
\]
then similar reasoning yields

**Corollary 3.4.** If the function \( E^k_n(q) := |G_n|^{q-1} \sum_{\chi \in \text{Irr} G} \frac{\chi(1)}{\chi \chi}(b_\chi^+)^r(b_\chi^-)^k \)
is a polynomial function of \( q \) for \( \text{char}(q) \gg 1 \), then
\[
E(\mathcal{M}^r_{n,w}) = E^k_n(xy).
\]

Let \( \rho \) be the representation of \( GL_n(\mathbb{F}_q) \) such that \( F = tr(\rho) \), let \( \chi = \iota \circ \det \) the composition of the determinant map \( GL_n(\mathbb{F}_q) \to \mathbb{F}_q^* \) with an injective homomorphism \( \iota : \mathbb{F}_q^* \to \mathbb{C}^* \). Then \( F = tr(\rho \otimes \chi) \) is a character equal to \( F_+ - F_- \). Therefore, if \( F := \sum \tilde{a}_\chi \chi \), then
\[
b_\chi^+ = \frac{1}{2}(a_\chi + \tilde{a}_\chi) \quad b_\chi^- = \frac{1}{2}(a_\chi - \tilde{a}_\chi). \tag{3.3}
\]

4. **Conjugacy classes of \( GL_n(\mathbb{F}_q) \)**

This section establishes notation for finite fields, partitions, and conjugacy classes. We mostly follow [22].

4.1. **Fields.** Denote \( \mathbb{F}_q \) the finite field of order \( q \) with algebraic closure \( \overline{\mathbb{F}}_q \). The Frobenius map \( \text{Frob} : \mathbb{F}_q \to \mathbb{F}_q \) is given by \( x \mapsto x^q \). For \( n \geq 1 \), we identify \( \mathbb{F}_q^n \subset \mathbb{F}_q \) with the fixed point set of \( \text{Frob}^n \). Denote the multiplicative groups by
\[
M_n := \mathbb{F}_q^\times, \quad M := \bigcup_n M_n = \overline{\mathbb{F}}_q^\times.
\]
Denote the orbit set \( \Phi = M/\text{Frob} \), and \( \Phi_d \subset \Phi \) the set of orbits of order \( d \). Each orbit in \( \Phi \) is equal to the set of roots of an monic irreducible polynomial and we represent elements \( f \in \Phi \) by the corresponding polynomial. Write \( d = d_f \) for \( f \in \Phi_d \).

The automorphism \( x \mapsto x^{-1} \) of \( M \) determines the automorphism \( f \mapsto f^* \) of \( \Phi \). Define
\[
\Phi^* := \{ f \in \Phi | f = f^* \} \quad \Phi^p = \{ \{ f, f^* \} | f \neq f^* \}. \tag{4.1}
\]
Observe that
\[
\Phi_1^* = \{ t - 1, t + 1 \}; \tag{4.2}
\]
and
\[
\Phi_{>1}^* := \Phi^* \setminus \Phi_1^* = \bigcup_{d>1} \Phi_{2d}^*.
\]
4.2. **Partitions.** Let

\[ \mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}_n \]

where \( \mathcal{P}_n \) is the set of partitions of \( n \) (note \( \mathcal{P}_0 = \{\emptyset\} \)). If \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell) \) is a partition, define

\[ |\lambda| := \sum_{i=1}^\ell \lambda_i, \quad \ell(\lambda) := \ell, \quad n(\lambda) := \sum_{i=1}^\ell (i-1)\lambda_i. \]

Set \( \ell_{\text{odd}}(\lambda) + \ell_{\text{ev}}(\lambda) = \ell(\lambda) \) where \( \ell_{\text{odd}}(\lambda) := |\{i : \lambda_i \text{ is odd}\}| \) and set \( \text{sgn}(\lambda) = (-1)^{\ell_{\text{ev}}(\lambda)}. \)

For \( d \in \mathbb{Z}_{>0}, \) set

\[ m_d(\lambda) := |\{i : \lambda_i = d\}|, \]

called the multiplicity of \( d \) in \( \lambda \).

We also use notation \( \lambda = (1^m_1 2^{m_2} \ldots) \) for \( m_d = m_d(\lambda) \). If \( s \geq 1 \) is an integer, write

\[ s\lambda = (1^{sm_1}_1 2^{sm_2} \ldots) \quad s \cdot \lambda = (s^{m_1}(2s)^{m_2} \ldots). \]

If \( \mu = (1^{r_1}_1 2^{r_2} \ldots) \) define

\[ \lambda \cup \mu = (1^{m_1+r_1}_1 2^{m_2+r_2} \ldots) \]

so that \( s\lambda = \lambda \cup \ldots \cup \lambda \).

For \( m \in \mathbb{Z}_{>0}, \) set \( \varphi_m(y) := (1-y)(1-y^2) \cdots (1-y^m) \in \mathbb{Z}[y] \). Then for a partition \( \lambda \in \mathcal{P}, \) we set

\[ a_\lambda(y) := y^{|\lambda| + 2n(\lambda)} \prod_{d \geq 1} \varphi_{m_d(\lambda)}(y^{-1}) \]

which is a polynomial in \( y \).

4.3. **Conjugacy classes, types and symmetric types.** Conjugacy classes in \( GL_n(\mathbb{F}_q) \) are classified by rational canonical forms, or equivalently [22, IV.2] by maps \( \mu : \Phi \rightarrow \mathcal{P} \) of norm \( \|\mu\| = n \), where

\[ \|\mu\| := \sum_{f \in \Phi} d_f|\mu(f)|. \]

Write \( c_\mu \) for the conjugacy class corresponding to the map \( \mu \). The support of \( \mu \) is defined as

\[ \text{supp}(\mu) := \{ f \in \Phi : \mu(f) \neq \emptyset \}. \]

Since \( \{t \pm 1\} \) play a special role in this paper, we also make use of the set difference

\[ \text{supp}'(\mu) := \text{supp}(\mu) \setminus \{t \pm 1\}. \]

The type of \( \mu \) is the map \( \rho : \mathcal{P} \setminus \{\emptyset\} \rightarrow \mathcal{P} \) is defined by

\[ \rho(\lambda) = (1^{m_1}_1 2^{m_2} \ldots), \quad m_d = m_{d,\lambda} = m_{d,\lambda}(\rho) := |\{f \in \Phi_d : \mu(f) = \lambda\}|. \]

Define the norm of a type by

\[ \|\rho\| = \sum_{\lambda \in \mathcal{P} \setminus \emptyset} \sum_{d \geq 1} d m_{d,\lambda} |\lambda|. \]

Note that for a given \( n \), the possible types of norm \( n \) are independent of \( q \) for \( q \) sufficiently large.
If $\mu$ has type $\rho$, then [22, IV(2.7)] the order of the centralizer $Z(c_\mu) \leq GL_n(\mathbb{F}_q)$ is

$$|Z(c_\mu)| = a_\mu(q) := \prod_{f \in \Phi} a_{\mu(f)}(q^{d_f}) = \prod_{\lambda \in \mathcal{P}(\emptyset)} \prod_{d \geq 1} a_\lambda(q^d)^{m_{d,\lambda}(\rho)}, \quad (4.3)$$

[22, IV(2.7)]. Notice the order depends only on the type $\rho$.

We call $\mu$ symmetric if $\mu(f) = \mu(f^*)$ for all $f \in \Phi$. The symmetric type of $\mu$ is the tuple $\eta = (\eta_+, \eta_-, \eta_s, \eta_p)$, where $\eta_+, \eta_- \in \mathcal{P}$, and $\eta_s, \eta_p : \mathcal{P} \setminus \{\emptyset\} \to \mathcal{P}$ are defined by

- (i) $\mu(t \mp 1) = \eta_{\mp}$, and
- (ii) for $\lambda \in \mathcal{P} \setminus \{\emptyset\}$, one has

$$\eta_s(\lambda) := (1^m_{\mu,\lambda} 2^m_{\mu,\lambda} \ldots), \quad m_{d,\lambda}^s := |\{f \in \Phi_{d}^s : \mu(f) = \lambda\}|$$

$$\eta_p(\lambda) := (1^m_{\mu,\lambda} 2^m_{\mu,\lambda} \ldots), \quad m_{d,\lambda}^p := |\{f, f^* \in \Phi_{d}^p : \mu(f) = \lambda\}|.$$

We write $\mu \in \eta$ to indicate that $\mu$ has symmetric type $\eta$. We sometimes abuse notation and write $\eta$ for the set of all conjugacy classes $c_\mu$ of symmetric type $\eta$.

5. An Explicit Formula for $F$

In this section, we use Milnor’s classification of orthogonal transformations over perfect fields [25] (following Williamson [30]) to derive explicit formulas for the function $F : GL_n(\mathbb{F}_q) \to \mathbb{Z}_0$. What we use in later sections is the following proposition.

**Proposition 5.1.** The function $F$ vanishes on the conjugacy class $c_\mu$ unless $\mu$ is symmetric (i.e., $\mu(f) = \mu(f^*)$ for all $f \in \Phi$). If $\mu$ has symmetric type $\eta$, then there exists a monic polynomial $b_\eta(y) \in \mathbb{Z}[y]$ depending only on $\eta$, such that

$$F(c_\mu) = b_\eta(q). \quad (5.1)$$

The degree of $b_\eta(y)$ is equal to $\frac{1}{2} \sum_{f = \pm 1} \ell_{odd}(\mu(f)) + \sum_{f \in \text{supp}(\mu)} d_f (n(\mu(f)) + \frac{1}{2}|\mu(f)|).$

Let $V$ be a finite-dimensional vector space over a finite field $\mathbb{F} = \mathbb{F}_q$ and let $t \in GL(V)$ be a linear automorphism. Then there is a natural decomposition

$$V = \bigoplus_{f \in \Phi} V_f, \quad (5.2)$$

where $V_f$ is the $f$-primary component of $V$ with respect to $t$.

**Proposition 5.2.** Suppose $t$ is orthogonal with respect to a non-degenerate symmetric bilinear form (ndsbf) $\langle \cdot, \cdot \rangle$. Then $V_f$ is orthogonal to all components except $V_{f^*}$. Consequently

$$F(t) = \prod_{f \in \Phi^*} F(t|_{V_f}) \cdot \prod_{(f, f^*) \in \Phi^*} F \left( t|_{V_f \oplus V_{f^*}} \right)$$

where the first product is over self-dual irreducible factors and the second is over distinct pairs $f, f^*$.\[Proof.\] Orthogonality is proven in [25, Lemma 3.1]. The consequences are immediate. \[\square\]

**Proposition 5.3.** Suppose that the minimal polynomial of $t \in GL(V)$ has only $f$ and $f^*$ as monic, irreducible factors where $f \neq f^*$ and $d = \deg f$. Then $F(t) = 0$ unless $t|_{V_f}$ is similar to $(t^{-1})^T|_{V_{f^*}}$. If they are similar, then $F(t) = a_{\mu(f)}(q^d)$. In particular, $F(t)$ equals a monic polynomial in $q$ of degree $2d (n(\mu(f)) + \frac{1}{2}|\mu(f)|)$.
Proof. If $F(t) \neq 0$, then $t$ is orthogonal with respect to some ndsbf $\langle \cdot , \cdot \rangle$. By Proposition 5.2, $\langle \cdot , \cdot \rangle$ determines a duality pairing between $V_f$ and $V_{f^*}$. If we choose a basis for $V_f$ and the dual basis in $V_{f^*}$ then $t$ must have the form

$$t = \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix}.$$  

The set of ndsbf$s$ on $V$ compatible with $t$ are those represented by a symmetric matrix of the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where $AB = BA$ and $B$ is invertible. Consequently, $F(t)$ equals the order of the centralizer of $t|_{V_f}$ in $GL(V_f)$, which equals $\alpha_\mu(f)(q^n)$ by (4.3). □

Remark 5.4. Propositions 5.2 and 5.3 imply that if $\mu(f) \neq \mu(f^*)$ for some $f \in \Phi$ then $F(c_\mu) = 0$. This means $F$ is supported on conjugacy classes of symmetric type.

Suppose that $V$ is $f$-primary with $f \in \Phi^s$ (recall (4.1)). By the fundamental theorem of PIDs, there is an isomorphism of $F[t]$-modules

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

for some $k$, where $V_i \cong \frac{F[t]}{f(t)^i F[t]} \otimes F \cdot m_i$ for some sequence of non-negative integers $m_1, \ldots, m_k$.

Proposition 5.5. Suppose $V$ is as above and $d = \text{deg } f$. Then

$$F(t) = q^{d \sum_{1 \leq i < j \leq k} im_i m_j} \prod_{i=1}^k F(t|V_i).$$

Proof. Any ndsbf left invariant by $t$ restricts non-degenerately to $V_k$ [25, Thm. 3.2], so it determines an orthogonal decomposition $V_k \oplus V_k^\perp$ and $V_k^\perp$ is isomorphic as $F[t]$-module to $V_1 \oplus \cdots \oplus V_{k-1}$. The number of complements of $V_k$ in $V$ as a $F[t]$-module is equal to the number $F[t]$-module splittings of $0 \to V_k \to V \to V/V_k \to 0$ which is equal to $q^{d \sum_{i=1}^{k-1} im_i m_k}$. Therefore

$$F(t) = q^{d \sum_{i=1}^{k-1} im_i m_k} F(t|V_k) F(t|V_1 \oplus \cdots \oplus V_{k-1}).$$

The formula follows by induction. □

Proposition 5.6. Suppose that $f \in \Phi_1^s$ and $V = V_i \cong \frac{F[t]}{f(t) F[t]} \otimes F \cdot m$. Then

$$F(t) = \begin{cases} 
q^{(im^2+m)/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is odd and } m \text{ is even} \\
n^{m/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is odd and } m \text{ is odd} \\
n^{im^2/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is even and } m \text{ is even} \\
n^{m/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is even and } m \text{ is odd} \\
0 & \text{if } i \text{ is odd and } m \text{ is even} \\
q^{(im^2+m)/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is odd and } m \text{ is odd} \\
n^{m/2} \prod_{j=1}^m (1 - q^{1-2j}) & \text{if } i \text{ is even and } m \text{ is even} \\
0 & \text{if } i \text{ is even and } m \text{ is odd} \\
\end{cases}$$

Proof. Let $f \in \Phi_1^s = \{ t \pm 1 \}$ and let $\Delta := t - t^{-1} = (t - 1)(t + 1)t^{-1}$. If $\langle \cdot , \cdot \rangle$ is a ndsbf on $V$ for which $t$ is orthogonal then $\Delta$ is skew adjoint in the sense that $\langle \Delta v, w \rangle = -\langle v, \Delta w \rangle$. This determines a non-degenerate bilinear form on $V/f(t)V \cong F^m_q$ defined by

$$(v) \cdot (w) := \langle \Delta^{i-1} v, w \rangle.$$ 

The form $\cdot$ is symmetric if $i$ is odd and antisymmetric if $i$ is even. Therefore

$$F(t) = \alpha \beta$$
where \( \alpha \) is the number of nondegenerate \((-1)^{i-1}\) -symmetric form on \( V/f(t)V \) and \( \beta \) is the number of \( t \) compatible ndsbf on \( V \) associated to a given form on \( V/f(t)V \).

If \( i \) is odd, then \( \alpha \) equals the number of ndsbf on \( \mathbb{F}_q^m \). This set decomposes into a union of two \( GL_m(\mathbb{F}_q) \) orbits, so

\[
\alpha = \sum_B |GL_m(\mathbb{F}_q)|/|O(B)|
\]

summing over representatives \( B \) of the two equivalence classes of ndsbf.

If \( i \) is even, then \( \alpha \) is equal to the number of non-degenerate skew-symmetric bilinear forms on \( \mathbb{F}_q^m \). Therefore

\[
\alpha = \begin{cases} 
|GL_m(\mathbb{F}_q)|/|Sp_m(\mathbb{F}_q)| & \text{if } m \text{ is even} \\
0 & \text{if } m \text{ is odd.}
\end{cases}
\]

Substituting the orders of groups (see e.g. [29]) and we obtain

\[
\alpha = \begin{cases} 
q^{(m^2+m)/2} \prod_{j=1}^{m/2} (1 - q^{1-2j}) & \text{if } i \text{ is odd and } m \text{ is even} \\
q^{(m^2+m)/2} \prod_{j=1}^{(m+1)/2} (1 - q^{1-2j}) & \text{if } i \text{ is odd and } m \text{ is odd} \\
q^{(m^2-m)/2} \prod_{j=1}^{m/2} (1 - q^{1-2j}) & \text{if } i \text{ is even and } m \text{ is even} \\
0 & \text{if } i \text{ is even and } m \text{ is odd.}
\end{cases}
\]

We turn now to \( \beta \). Given an ndsbf \( \langle \cdot \rangle \), choose a basis \((v_1), \ldots, (v_m)\) of \( V/f(t)V \). A choice of representatives \( v_1, \ldots, v_m \in V \) will be called a lift. The number of lifts is \( q^{(i-1)m^2} \). Any lift extends to a basis of \( V \),

\[
\{ \Delta^s v_k \mid s \in \{0, \ldots, i-1\}, k \in \{1, \ldots, m\} \}.
\]

By [25, Thm. 3.4], lifts exist that satisfy

\[
\langle \Delta^s v_k, \Delta^t v_l \rangle = \begin{cases} 
(-1)^t (v_k) \cdot (v_l) & \text{if } s + t = i - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Call such a lift a standard lift for \( \langle \cdot \rangle \). It follows that \( \beta \) equals the number of lifts divided by the number of standard lifts for a given \( \langle \cdot \rangle \). It remains to count the number of standard lifts for a given non-degenerate form \( \cdot \) on \( V/f(t)V \).

Fix a particular standard lift \( v_1, \ldots, v_m \). If \( \cdot \) is described by a \((-1)^{i-1}\) -symmetric \( m \times m \)-matrix \( A \), then in terms of the basis (5.4), \( \langle \cdot \rangle \) is described by the symmetric \((im) \times (im)\)-matrix

\[
X := \begin{bmatrix}
0 & 0 & \ldots & ( -1)^{i-1} A \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & -A & \ldots & 0 \\
A & 0 & \ldots & 0
\end{bmatrix}.
\]

(5.5)
Then another lift $v'_1, ..., v'_m$ will be standard if and only if the change of basis matrix sending $\Delta^k v_k \mapsto \Delta^k v'_k$ is a lower triangular matrix of the form

$$Y := \begin{bmatrix} I & 0 & 0 & \ldots & 0 \\ B_1 & I & 0 & \ldots & 0 \\ B_2 & B_1 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{i-1} & B_{i-2} & B_{i-3} & \ldots & I \end{bmatrix}.$$ 

and satisfies $Y^TXY = X$, or equivalently

$$B_1^T A - AB_1 = 0,$$

$$B_2^T A - B_1^T AB_1 + AB_2 = 0,$$

$$\vdots$$

$$B_{i-1}^T A - B_{i-2}^T AB_1 + \ldots + (-1)^{i-1} AB_{i-1} = 0.$$

Using the fact that $A$ is $(-1)^{i-1}$-symmetric, these can be rewritten

$$(-1)^{i-1}(AB_1)^T - AB_1 = 0,$$

$$(-1)^{i-1}(AB_2)^T + AB_2 = B_1^T AB_1,$$

$$\vdots$$

$$(-1)^{i-1}(AB_{i-1})^T + (-1)^{i-1} AB_{i-1} = B_{i-2}^T AB_1 + \ldots$$

The first equation gives $m(m - (-1)^{i+1})/2$ independent linear equations for entries of $B_1$. Given a solution to this, the second equation gives $m(m - (-1)^{i+2})/2$ independent linear equations for entries of $B_2$, and so on. Altogether, the number of independent linear equations for $Y$ is $m^2(i - 1)/2$ if $i$ is odd and $(m^2(i - 1) + m)/2$ if $i$ is even. It follows that

$$\beta = \begin{cases} 
q^{m^2(i-1)/2} & \text{if } i \text{ is odd} \\
q^{(m^2(i-1)+m)/2} & \text{if } i \text{ is even}.
\end{cases}$$

\[ \square \]

If $f \in \Phi^s_{2d}$, then $\mathbb{E} := \mathbb{F}[t]/f(t)\mathbb{F}[t]$ is a field extension of degree $2d$ over $\mathbb{F}$. There is a unique automorphism $\sigma$ of $\mathbb{E}$ over $\mathbb{F}$ which sends the class of $t$ to its multiplicative inverse. A Hermitian form on an $\mathbb{E}$-vector space $W$ is a non-degenerate $\mathbb{F}$-bilinear pairing $h : W \times W \to \mathbb{E}$ which is $\mathbb{E}$-linear in the first entry and satisfies $h(v, w) = \overline{h(v, u)}$.

**Proposition 5.7.** Suppose that $f \in \Phi^s_{2d}$, and $V = V_i \cong \mathbb{F}[t]/t^{2d} \otimes_{\mathbb{F}} \mathbb{F}^m$. Then

$$F(t) = q^{idm^2} \prod_{j=1}^m (1 + (-1)^j q^{-dj}).$$

**Proof.** By [25, Thm. 3.3], the ndsbf on $V$ which are compatible with $t$ are classified as follows. If $\langle \cdot, \cdot \rangle$ is fixed by $t$ then the operator $s(t) := f(t)/t^d$ is adjoint in the sense that $\langle s(t)v, w \rangle = \langle v, s(t)w \rangle$.
\[ \langle v, s(t)w \rangle. \] The quotient space \( V/f(t)V \) is isomorphic to \( \mathbb{E}^m \) where \( \mathbb{E} = \mathbb{F}[t]/f(t)\mathbb{F}[t] \). The quotient \( V/f(t)V \) admits one and only one Hermitian inner product \( \cdot \) such that
\[ \text{Tr}_{\mathbb{E}/\mathbb{F}}((v) \cdot (w)) = \langle s(t)^{i-1}v, w \rangle, \]
where \( \text{Tr}_{\mathbb{E}/\mathbb{F}} \) is the field trace.

Thus
\[ F(t) = \alpha \beta \]
where \( \alpha \) is the number of Hermitian inner products on \( \mathbb{E}^m \) and \( \beta \) is the number of \( t \)-compatible ndsbf associated to each Hermitian inner product. Since \( \mathbb{E}^m \) only admits one Hermitian form up to change of basis ([25] Example 1), we deduce using [29] that
\[ \alpha = \frac{|GL_m(\mathbb{E})|}{|U_m(\mathbb{E})|} = q^{2m^2} \prod_{\gamma=1}^m (1 + (-1)^\gamma q^{-d\gamma}). \]

To calculate \( \beta \), let \( \langle \cdot, \cdot \rangle \) be given and choose an \( \mathbb{E} \)-basis \( (v_1), ..., (v_m) \in V/f(t)V \) which is orthonormal with respect to \( \cdot \). There are \( q^{2md^2(i-1)} \) possible choices of representatives \( v_1, ..., v_m \in V \) and each such choice determines a basis \( \{t^a s(t)^b v_k | a \in \{0, ..., 2d-1\}, b \in \{0, ..., i-1\}, k \in \{1, ..., m\} \} \) for \( V \). The representatives \( v_1, ..., v_m \in V \) can be chosen so that \( \langle \cdot, \cdot \rangle \) has standard form
\[ \langle t^a s(t)^b v_k, t^{a'} s(t)^{b'} v_l \rangle = \begin{cases} \text{Tr}_{\mathbb{E}/\mathbb{F}}((t^a v_k) \cdot (t^{a'} v_l)) & \text{if } b + b' = i - 1 \\ 0 & \text{if } |a - a'| < d \text{ and } b + b' \neq i - 1 \end{cases} \]
with the remaining pairings determined uniquely from these using an induction argument. We call such a choice of representatives \( v_1, ..., v_m \) a standard lift with respect to \( \langle \cdot, \cdot \rangle \). Therefore \( \beta \) equals \( q^{2md^2(i-1)} \) divided by the number of standard lifts for a given \( \langle \cdot, \cdot \rangle \).

Let \( v_1, ..., v_m \) be a choice of standard lift for \( \langle \cdot, \cdot \rangle \). Any other choice of representative \( v'_1 \) for \( (v_1) \) must be of the form
\[ v'_1 = v_1 + \sum_{k=1}^m \sum_{j=1}^{i-1} a_{j,k}(t) s(t)^j v_k \]
where \( a_{j,k}(t) \) is a polynomial of degree at most \( 2d-1 \). In order for \( v'_1 \) to extend to a standard lift, Milnor shows that \( a_{1,1}(t) + a_{1,1}(t^{-1}) \) must descend to zero in \( \mathbb{E} = \mathbb{F}[t]/f(t) \). Since the kernel of the map \( \mathbb{E} \to \mathbb{E} \) sending \( e \mapsto e + \bar{e} \) has order \( q^d \), this means there are \( q^d \) possible choices for \( a_{1,1}(t) \). Similarly, once \( a_{1,1}(t) \) is chosen, there are \( q^d \) choices for \( a_{2,1}(t) \) and so on. For \( k > 1 \) the polynomials \( a_{j,k}(t) \) can be chosen arbitrarily so there are \( q^{2d} \) choices each. Altogether there are
\[ q^{(d+2d(m-1)(i-1))} = q^{d(i-1)(2m-1)} \]
choices of \( v'_1 \) that extend to a standard lift. The ndsbf \( \langle \cdot, \cdot \rangle \) restricts to a non-degenerate form on \( \text{span} \{t^a s(t)^b v'_1 \} \) and the remaining representatives \( v'_2, ..., v'_m \) must be chosen from its orthogonal complement. Using induction we deduce that there are \( q^{d(i-1)((2m-1)+2m-3+...+1)} = q^{d(i-1)m^2} \) choices of standard lift. It follows then that \( \beta = q^{dm^2(i-1)} \) which concludes the proof.

Proof of Proposition 5.1. Assemble the results of this section to identify the leading order term in \( F(e_\mu) \). □
6. Characters of $GL_n(\mathbb{F}_q)$

In this section we recall the classification of irreducible characters of $GL_n(\mathbb{F}_q)$ due to Green [10] and prove Proposition 6.1 concerning sums of character values over conjugacy classes of fixed symmetric type. Our presentation borrows from [9, 10, 22].

It is helpful to first consider the symmetric group $S_n$, which can morally be thought of as $GL_n(\mathbb{F}_1)$. The conjugacy classes $c_\mu$ are classified by partitions $\mu \in \mathcal{P}_n$ determined by the disjoint cycle decomposition. If $\mu = (1^{r_1}2^{r_2}...k^{r_k})$ then the stabilizer of any representative of $c_\mu$ has order
\[ z_\mu := r_1! \ldots r_k! 1^{r_1} \ldots k^{r_k}. \]
The irreducible characters $\chi_\lambda$ of $S_n$ are also classified by partitions $\lambda \in \mathcal{P}_n$. We write
\[ \chi_\mu^\lambda := \chi_\lambda(c_\mu). \]

Now suppose that $q$ is a prime power, and recall that $M_n := \mathbb{F}_q^n$. When $n|m$, the norm map $Nm_{m,n} : M_m \to M_n$ is defined $Nm_{m,n}(x) = x \cdot x^{q^n} \ldots x^{q^{(m-1)n}} = x^{(q^m-1)/(q^n-1)}$. Denote the character groups $L_n := \text{Hom}(M_n, \mathbb{C}^\times)$ and define the direct limit
\[ L := \lim_{\to} L_n = \bigcup_n L_n, \]
using the dual maps $Nm_{m,n}^* : L_n \hookrightarrow L_m$. These are injective and we treat them as subset inclusions. If $x \in M_n$ and $\gamma \in L_n$, we define
\[ \langle \gamma, x \rangle_n := \gamma(x). \]
Note that if $n|m$ and $\gamma \in L_n \subset L_m$ and $x \in M_m$, we have $\langle \gamma, x \rangle_m = \langle \gamma, x^{(q^m-1)/(q^n-1)} \rangle_n$. The Frobenius automorphism of $M$ induces one on $L$ and we write
\[ \Theta = \bigcup_{d \geq 1} \Theta_d := L/Frob \]
where $\Theta_d$ is the set of orbits $\theta \subseteq L$ of order $d$. Observe
\[ \theta = \{ \gamma, \gamma^q, \ldots, \gamma^{q^{d-1}} \} = \{ \gamma^q, \ldots, \gamma^{qd} \} \]
for any $\gamma \in \Theta_d \setminus (\bigcup_{i \neq d} L_i)$. We say $\theta$ has degree $d_\theta := |\theta|$. Given a map $\Lambda : \Theta \to \mathcal{P}$ of finite support define the norm
\[ ||\Lambda|| := \sum_{\theta \in \Theta} d_\theta |\Lambda(\theta)|. \]
The irreducible characters of $GL_n(\mathbb{F}_q)$ are in one-to-one correspondence maps $\Lambda$ of norm $n$. We use notation $\Lambda = (\theta_1^{\lambda_1} \theta_2^{\lambda_2} \ldots)$ to mean $\Lambda(\theta_i) = \lambda_i$. A character $\chi_\Lambda$ is called primary if $\Lambda = (\theta^\lambda)$ is supported on a single $\theta \in \Theta$. The type of $\Lambda$ is the map $\tau : \mathcal{P} \setminus \{ \emptyset \} \to \mathcal{P}$ defined by
\[ \tau(\lambda) = (1^{m_{1,\lambda}} 2^{m_{2,\lambda}} \ldots), \quad m_{d,\lambda} = m_{d,\lambda}(\tau) := |\{ \theta \in \Theta_d : \Lambda(\theta) = \lambda \}|. \]
We will later use a more refined notion of type valid if $q$ is odd. If $\theta \in \Theta_d$ and $\gamma \in \theta$, set
\[ \langle \theta, -1 \rangle_d := \langle \gamma, -1 \rangle_d \in \{ \pm 1 \} \]
which is well-defined independently of the choice of $\gamma \in \theta$. For each $d \geq 1$ partition $\Theta_d = \Theta_d^+ \cup \Theta_d^-$ where
\[ \Theta_d^+ := \{ \theta \in \Theta_d | \langle \theta, -1 \rangle_d = -1 \}. \]
The signed type of \( \Lambda \) is the pair \( \sigma^+, \sigma^- : \mathcal{P} \setminus \{\emptyset\} \to \mathcal{P} \) defined by
\[
\sigma^\pm(\lambda) = (1^{m^\pm_1, \lambda} 2^{m^\pm_2, \lambda} \ldots), \quad m^\pm_{d, \lambda} := |\{\theta \in \Theta^\pm : \Lambda(\theta) = \lambda\}|. \tag{6.1}
\]

Our main technical result in this section is the following.

**Proposition 6.1.** Given an irreducible character \( \chi_\Lambda \) and a symmetric type \( \eta \), there is an equality
\[
h_{\Lambda, \eta}(q) = \sum_{\mu \in \eta} \chi_\Lambda(c_\mu) \tag{6.2}
\]
where \( h_{\Lambda, \eta}(y) \in \mathbb{C}[y] \) is a polynomial of degree no greater than \(^5\)
\[
n(\mu(t + 1)) + n(\mu(t - 1)) + \sum_{f \in \text{supp}(\mu)} d_f \left(n(\mu(f)) + \frac{1}{2}\right), \tag{6.3}
\]
whose coefficients are uniformly bounded by a constant that depends only on \( \eta \) and the type of \( \Lambda \).

**Remark 6.2.** The point of Proposition 6.1 is that for large \( q \) the sum (6.2) is dominated by the leading order term. This will permit us to calculate multiplicities by taking limits \( q \to \infty \).

**Remark 6.3.** The informal explanation for (6.3) is that each \( \chi_\Lambda(c_\mu) \) equals a polynomial expression in \( q \) of degree bounded by \( \sum d_f n(\mu(f)) \) (see Lemma 6.8), while the number of terms in the sum is a polynomial of degree \( \frac{1}{2} \sum_{f \in \text{supp}(\mu)} d_f \). The full proof involves an inclusion-exclusion argument.

We begin with some preliminaries before stating Green’s character formula for \( \chi_\Lambda(c_\mu) \).

Let \( \theta \in \Theta \), let \( f \in \Phi \), let \( x \in M \) be a root of \( f \), and let \( e \) be a positive integer such that \( d_f d_e \). Define
\[
S^\theta_e(f) = S^\theta_e(x) := \sum_{\gamma \in \theta} \langle \gamma, x \rangle_{d_f d_e}
\]
(the sum is independent of the choice of root \( x \) of \( f \)). If \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \in \mathcal{P} \) is a partition such that every block of \( d_\theta \cdot \lambda \) is divisible by \( d_f \), then define
\[
S^\theta_\lambda(f) = S^\theta_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} S^\theta_{\lambda_i}(f).
\]
For example, one can check the identities
\[
S^\theta_\lambda(t - 1) = d_\theta^{\ell(\lambda)} \quad S^\theta_\lambda(t + 1) = d_\theta^{\ell(\lambda)} \langle \theta, -1 \rangle_{d_\theta}. \tag{6.4}
\]

**Lemma 6.4.** Let \( g = \gcd(\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \). Then \( S^\theta_\lambda \) determines a character
\[
M_{d_\theta g} \to \mathbb{C}^\times
\]
of degree \( d_\theta \ell(\lambda) \) which is constant along Frobenius orbits.

**Proof.** If \( \gamma \in \theta \) and \( x \in M_{d_\theta g} \) and \( g \) divides \( e \), then
\[
S^\theta_e(x) = \sum_{i=1}^{d_\theta} \langle \gamma, q^i, x \rangle_{d_f d_e} = \sum_{i=1}^{d_\theta} \langle \gamma, x^{q^{i(q^{d_\theta e} - 1)/(q^d - 1)}} \rangle_{d_\theta}
\]
\(^5\)We define the degree bound in terms of a representative \( \mu \in \eta \) instead of \( \eta \) directly since this is more convenient for later application.
which is a Frobenius invariant character of degree $d_\theta$. Since $M_{d_\theta g}$ is an abelian group, the product

$$S^\theta_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} S^\theta_{\lambda_i}(x)$$

is a Frobenius invariant character of degree $d_\theta \ell(\lambda)$. \qed

Given partitions $\lambda, \mu \in \mathcal{P}_n$, the Green polynomial $Q^\lambda_\mu(q) \in \mathbb{Z}[q]$ was defined by Green [10, §4].

**Lemma 6.5.** Let $\mu$ and $\lambda$ be partitions of $n$ with $\mu = \{1^{r_1}2^{r_2}\}$. The Green polynomial $Q^\lambda_\mu(q)$ has degree less than or equal to $n(\lambda)$. If equal, then the leading coefficient is $(-1)^{\epsilon_{ev}(\mu)}c^\lambda_{\pi_1,\ldots,\pi_2}(1^{r_1},(1^2),\ldots,1^2)$ where $(1^1)$ occurs $r_i$ times in the subscript. Here the $c^\lambda_{\pi_1,\ldots,\pi_2}(1^{r_1},(1^2),\ldots,1^2)$ is a Littlewood-Richardson coefficient.

**Proof.** The degree bound is stated in [10, Lemma 4.3], but it is convenient to recall the proof. By definition

$$Q^\lambda_\mu(q) = \sum g^\lambda_{\rho_1,\ldots,\rho_2}(q)k(\rho_1, q)k(\rho_2, q)$$

summed over sequences $(\rho_i)$ where $\rho_1, \ldots, \rho_1$ are partitions of 1, $\rho_1, \ldots, \rho_1$ are partitions of 2 and so on. Here

$$k(\rho_i, q) = (1 - q)\cdots(1 - q^{l(\rho_i) - 1})$$

which implies $\deg(k(\rho_i, q)) \leq n(\rho_i)$ with equality if only if $\rho_i = (1^{\lfloor \rho_i \rfloor})$. The $g^\lambda_{\rho_1,\ldots,\rho_2}(q)$ are Hall polynomials and by Hall’s Theorem [10, Thm. 4]

$$g^\lambda_{\rho_1,\ldots,\rho_2}(q) = c^\lambda_{\rho_1,\ldots,\rho_2}q^{n(\lambda) - n(\rho_1)} + \text{lower order terms} \quad (6.5)$$

from which the result follows. \qed

**Proposition 6.6** (Green’s Character Formula [10]). If $\Lambda = (\theta^\lambda)$ is a primary character and $c_\mu$ is a conjugacy class then

$$\chi^\Lambda(c_\mu) = (-1)^{n-|\lambda|} \sum_{\pi \approx |\lambda|} \chi^\lambda_{\pi} \sum_{|\rho| = |\mu|, \forall f \in \Phi, Part(\rho) = d_\theta \pi} S^\theta(\rho)Q^\mu_\rho. \quad (6.6)$$

where

$$Part(\rho) := \cup_{f \in \Phi} d_f \cdot \rho(f) \quad (6.7)$$

$$Q^\mu_\rho := \prod_{f \in \Phi} \frac{1}{\varphi(\rho(f))} Q^\mu(f) \left( q^{d_f} \right) \quad (6.8)$$

$$S^\theta(\rho) := \prod_{f \in \Phi} S^\theta_{d_f}(\rho(f)). \quad (6.9)$$

For a general irreducible character $\Lambda = (\theta_1^{\lambda_1}, \ldots, \theta_k^{\lambda_k})$ and conjugacy class $c_\mu$ we have

$$\chi^\Lambda(c_\mu) := \sum_{|\mu_i| = |\lambda|_i \phi_\lambda_i} g^\mu_{\mu_1,\ldots,\mu_k}(q) \prod_{i=1}^{k} \chi^{\lambda_i}_{\lambda_i}(c_\mu_i) \quad (6.10)$$

where

$$g^\mu_{\mu_1,\ldots,\mu_k}(q) = \prod_{f \in \Phi} g^\mu_{\mu_1(f),\ldots,\mu_k(f)} \left( q^{d_f} \right) \quad (6.11)$$
counts flags in $F^n$ which are invariant under a fixed representative of $c_\mu$ and whose subquotients are isomorphic to $(c_{\mu_1}, \ldots, c_{\mu_l})$.

**Remark 6.7.** Except for the factor $S^q(\rho)$ appearing in (6.6), Green’s formula for $\chi_\Lambda$ depends only on the type of $\mu$.

**Lemma 6.8.** Let $c_\mu$ be a conjugacy class of $GL_n(\mathbb{F}_q)$ and let $\chi_\Lambda$ be an irreducible character. Green’s formula (6.10) expresses $\chi_\Lambda(c_\mu)$ as a polynomial in $q$ of degree bounded above by

$$
\sum_{f \in \text{supp } \mu} d_{fn}(\mu(f)). \tag{6.12}
$$

**Proof.** From (6.8) and Lemma 6.5 it follows that

$$
\deg(Q^\mu_{\rho}(q)) \leq \sum_{f \in \text{supp } \mu} d_{fn}(\mu(f))
$$

which by (6.6) takes care of the primary case. For the general case, note from (6.5) and (6.11) that

$$
\deg(g^\mu_{\mu_1, \ldots, \mu_k}(q)) \leq \sum_f d_f (n(\mu(f)) - n(\mu_1(f)) - \ldots - n(\mu_k(f))). \tag{6.13}
$$

Applying this to (6.10) completes the proof. \hfill \square

**Proof of Proposition 6.1.** We use an inclusion-exclusion argument to reduce to cyclic character sums (compare [14, §3.3]). Let $\mu : M \to \mathcal{P}$ be a Frobenius invariant map representing the symmetric type $\eta$. Introduce a finite set $I$ and an injective map $\zeta_0 : I \hookrightarrow M$ onto the support of $\mu$. Define permutations $\rho$ and $\iota$ of $I$ such that

$$
\zeta_0 \circ \rho = \text{Frob} \circ \zeta_0 \quad \zeta_0 \circ \iota = \text{inv} \circ \zeta_0 \tag{6.14}
$$

where $\text{inv}(x) = x^{-1}$. Denote by $I/\rho$ the set of $\rho$-cycles. Consider the set of equivariant maps

$$(I, M)_{eq} := \{ \zeta : I \to M | \zeta \circ \rho = \text{Frob} \circ \zeta, \zeta \circ \iota = \text{inv} \circ \zeta \}
$$

and denote $(I, M)'_{eq}$ the subset of injective maps. Then there is a natural $z_\eta$-to-one surjective map from $(I, M)_{eq}$ to the set of maps of symmetric type $\eta$, where

$$
z_\eta = \prod_{d \geq 1} \prod_{\lambda \in \mathcal{P}\setminus\emptyset} 2^{m_{d,\lambda}^p} d^{m_{d,\lambda}^s + m_{d,\lambda}^p} (m_{d,\lambda}^s m_{d,\lambda}^p!)
$$

where $m_{d,\lambda}^s, m_{d,\lambda}^p \in \mathbb{Z}_{\geq 0}$ are as in §4.3. It follows from Proposition 6.6 and Lemma 6.8 that

$$
\sum_{\mu \in \eta} \chi_\Lambda(c_\mu) = \frac{1}{z_\eta} \sum_{\zeta \in (I, M)'_{eq}} \varphi(\zeta)
$$

where $\varphi(\zeta)$ is the value of $\chi_\Lambda$ on the conjugacy class determined by $\zeta$. By Remark 6.7, $\varphi(\zeta)$ is a sum of terms of the form

$$
f(q) \prod_{c \in (I/\rho)} S^q_{\lambda(c)}(\zeta(c))
$$

with the $f(q) \in \mathbb{C}[q]$ are polynomials depending only on the type $\eta$, of degree no greater than $\sum_{f \in \text{supp } \mu} d_{fn}(\mu(f))$, and where $\Theta : I/\rho \to \Theta$, and $\lambda : I/\rho \to \mathcal{P}$ satisfy

$$
\sum_{c \in (I/\rho)} |\lambda(c)| d_{\theta(c)} = n.
To complete the proof it remains to show that the sums
\[ \sum_{\zeta \in (I, M)_{eq}} \prod_{c \in I/\rho} S_{\lambda(c)}^{\theta(c)}(\zeta(c)) \]
equal polynomial expressions in \( q \) with degree no greater than \( \frac{1}{2} \sum_{f \in \text{supp}'(\mu)} d_f \) which have uniformly bounded coefficients.

A partition of \( I \) describes a surjective map \( I \to J \) onto the set of blocks. Let \( \Pi(I)_{eq} \) be the poset of partitions of \( I \) for which \( \theta \) is constant on blocks and both \( \iota \) and \( \rho \) permute blocks. Let \((J, M)_{eq} \subseteq (I, M)_{eq} \) be the subset of maps which are constant on blocks. Then by Moebius inversion we obtain the equality
\[ \sum_{\zeta \in (J, M)_{eq}} \prod_{c \in I/\rho} S_{\lambda(c)}^{\theta(c)}(\zeta(c)) = \sum_{J \in \Pi(I)_{eq}} \mu_{eq}(J) S(J) \] (6.15)
where \( \mu \) is the Mobius function for the poset \( \Pi(I)_{eq} \) and
\[ S(J) := \sum_{\zeta \in (J, M)_{eq}} \prod_{c \in I/\rho} S_{\lambda(c)}^{\theta(c)}(\zeta(c)) \]
where \( \lambda(c) = \cup_{c' \in c} \lambda(c') \). Next we prove that the \( S(J) \) are polynomial functions of \( q \) with uniformly bounded coefficients. Define \( M^s_n := \{ x \in M_n \mid x^{-1} = x^{q^i} \text{ for some } i \} \). This has cardinality
\[ |M^s_n| = \begin{cases} 2 & \text{if } n \text{ is odd} \\ q^{n/2} + 1 & \text{if } n \text{ is even.} \end{cases} \] (6.16)
Then
\[ S(J) = \prod_{c \in I/\rho} \left( \sum_{x \in M^s_{|c|}} S_{\lambda(c)}^{\theta(c)}(x) \right) \times \prod_{\{c, \iota(c), \theta(c)\}} \left( \sum_{x \in M_{|c|}} S_{\lambda(c)}^{\theta(c)}(x) S_{\lambda(c)}^{\theta(\iota(c))}(x^{-1}) \right) \] (6.17)

By Lemma 6.4, \( S_{\lambda(c)}^{\theta(c)} \) restricts to a character of degree \( d_{\theta(c)} \ell(\lambda(c)) \) on \( M^s_{|c|} \), so
\[ \sum_{x \in M^s_{|c|}} S_{\lambda(c)}^{\theta(c)}(x) = C|M^s_{|c|}| = \begin{cases} 2C & \text{if } |c| \text{ is odd} \\ C(q^{\lfloor |c|/2 \rfloor} + 1) & \text{if } |c| \text{ is even.} \end{cases} \]
where \( 0 \leq C \leq d_{\theta(c)} \ell(\lambda(c)) \) is the multiplicity of the trivial character in \( S_{\lambda(c)}^{\theta(c)} \). On the other hand
\[ \sum_{x \in M_{|c|}} S_{\lambda(c)}^{\theta(c)}(x) S_{\lambda(c)}^{\theta(\iota(c))}(x^{-1}) = \sum_{x \in M_{|c|}} S_{\lambda(c)}^{\theta(c)}(x) \overline{S_{\lambda(c)}^{\theta(\iota(c))}(x)} \]
\[ = (q^{|c|} - 1) \langle S_{\lambda(c)}^{\theta(c)}, S_{\lambda(c)}^{\theta(\iota(c))} \rangle_{M_{|c|}} \]
\[ = C'(q^{|c|} - 1) \]
where \( 0 \leq C' \leq d_{\theta(c)} d_{\theta(\iota(c))} \ell(\lambda(c)) \ell(\lambda(\iota(c))) \) by Lemma 6.4. Taken together, this implies that \( S(J) \) is polynomial whose coefficients are bounded by \( \prod_{c \in I/\rho} d_{\theta(c)} \ell(\lambda(c)) \) and whose degree at most
\[ \frac{1}{2} |\{ j \in J | \rho(j) \neq j \text{ or } \iota(j) \neq j \}| \]
which is bounded above by
\[
\frac{1}{2}|\{i \in I| \rho(i) \neq i \text{ or } \iota(i) \neq i\}| = \frac{1}{2} \sum_{f \in \mathrm{supp}'(\mu)} d_f.
\]

7. Leading coefficients of $h_{\Lambda, \eta}$ when $\eta$ is of restricted symmetric type

It will be important in the next section to calculate the leading coefficient of $h_{\Lambda, \eta}(y)$ some special cases of $\eta$.

We say a symmetric map $\mu : \Phi \to \mathcal{P}$ has restricted symmetric type if $\mu(f) = 1^{||f||}$ for $f \in \{t \pm 1\}$ and $\mu(f) = 1^{+}$ for $f \in \mathrm{supp}'(\mu) := \mathrm{supp} \mu \setminus \{t \pm 1\}$.

A restricted symmetric type can be encoded as a tuple $\tau = (n_+, n_-, \tau_s, \tau_p)$, where $n_+, n_- \in \mathbb{Z}_{\geq 0}$ and $\tau_s, \tau_p \in \mathcal{P}$ and $n_+ + n_- + 2|\tau_s| + 2|\tau_p| = n$. This $\tau$ determines the symmetric type $\eta = (\eta_+, \eta_-, \eta_s, \eta_p)$

\[
\eta_{\pm} := (1^{n_{\pm}}) \quad \eta_{s}(\lambda) := \begin{cases} 2 \cdot \tau_s & \lambda = (1) \\ 0 & \text{otherwise} \end{cases} \quad \eta_{p}(\lambda) := \begin{cases} 2 \tau_p & \lambda = (1) \\ 0 & \text{otherwise} \end{cases}
\]

By (6.3), when $\tau$ is of restricted symmetric type, the degree of $h_{\Lambda, \tau}$ is bounded by
\[
\deg h_{\Lambda, \tau}(y) \leq \frac{1}{2} \left( (n_+ - 1)^2 + (n_- - 1)^2 + n - 2 \right).
\]

and we will call the coefficient of this term the leading coefficient of $h_{\Lambda, \tau}$, even if it is zero.

Given a restricted symmetric type $\tau = (n_+, n_-, \tau_s, \tau_p)$ and $d \geq 1$, we may construct another restricted symmetric type $d \cdot \tau = (dn_+, dn_-, d \cdot \tau_s, d \cdot \tau_p)$ and say that $d$ divides $d \cdot \tau$.

**Lemma 7.1.** Let $\Lambda = (\theta^\lambda)$ be primary, where $\theta \in \Theta_d$ and $v := |\lambda|$. Let $\tau = (n_+, n_-, \tau_s, \tau_p)$ be a restricted symmetric type of norm $n = dv$. Then the leading coefficient of $h_{\Lambda, \tau}$ is zero, unless $d$ divides $(n_+, n_-, \tau_s, \tau_p)$.

Suppose that $(n_+, n_-, \tau_s, \tau_p) = d \cdot (\tilde{n}^+, \tilde{n}^-, \tilde{\tau}_s, \tilde{\tau}_p)$ and let $\tilde{\rho} := 2 \cdot \tilde{\tau}_s \cup 2 \tilde{\tau}_p$. Then the leading coefficient of $h_{\Lambda, \tau}$ is equal to

\[
\langle \theta, -1 \rangle_d^{n_-} \frac{(-1)^{\ell(\tau_s)}}{z_{\tilde{\tau}_s} \cdot z_{\tilde{\tau}_p}} \sum_{\pi \vdash v} \mathrm{sgn}(\pi) \chi_{\pi}^\lambda \sum_{\tilde{\rho}_+ \cup \tilde{\rho}_- = \pi} \frac{1}{z_{\tilde{\rho}_+} \cdot z_{\tilde{\rho}_-}}.
\]

**Proof.** Let $c_\mu$ be a conjugacy class having a restricted symmetric type $(n_+, n_-, \tau_s, \tau_p)$ and let $\rho = (2 \cdot \tau_s) \cup 2 \tau_p$. Applying Green’s character formula (Proposition 6.6) and identity (6.4) we have

\[
\chi_\Lambda(c_\mu) = (-1)^{n-v} \langle \theta, -1 \rangle_d^{n_-} \left( \prod_{f \in \mathrm{supp}'(\mu)} S_{d/f(d)}(f) \right) \times \sum_{\pi \vdash v} \chi_{\pi}^\lambda \sum_{\rho_+ \cup \rho_- = \pi} \frac{d^{\ell(\rho_+) + \ell(\rho_-)}}{z_{\rho_+} \cdot z_{\rho_-}} Q_{\rho_+}^{(1^{n_+})}(q) Q_{\rho_-}^{(1^{n_-})}(q).
\]

This expression is trivial unless $d$ divides $(\tilde{n}^+, \tilde{n}^-, 2 \cdot \tau_s, \tau_p)$, which we assume for the rest of the proof. By Lemma 6.5, the Green polynomial

\[
Q_{\rho_+}^{(1^{n_+})}(q) = (-1)^{f_{\ell_0}(\rho)} q^{(\alpha)} + \text{lower order terms}.
\]
because $c_{\{1\} \ldots \{12\}} = 1$. Therefore the leading coefficient of $\chi_{\lambda}(c_{\mu})$ as a polynomial in $q$
eq\sum_{\nu=0}^{n} (-1)^{n-v} \langle \theta_{\nu}, -1 \rangle_{d}^{n-v/d} \left( \prod_{f \in \text{supp}(\mu)} S_{\nu/d}(f) \right) \sum_{\pi \vdash v} \sum_{\rho_{+} + \rho_{-} = d} d(\rho_{+}) \prod_{\rho_{+} + n_{+}} \prod_{\rho_{-} + n_{-}} \sum_{f_{\pi} z_{f_{\rho_{+}}} z_{f_{\rho_{-}}}} \langle -1 \rangle_{d}^{\nu}(\rho_{+} \cup \rho_{-}) \pi_{\rho_{+}} \pi_{\rho_{-}}.

(7.2)

The only part of (7.2) that varies with $\mu \in \tau$ is $S_{\nu/d}(f)$. Comparing with (6.15) and (6.17), if $r_{s} = (1 \rho_{1} 2 \rho_{2} \ldots)$ and $r_{p} = (1 \rho_{1} 2 \rho_{2} \ldots)$, then

$$\sum_{\mu \in \tau} \prod_{f \in \text{supp}(\mu)} S_{\nu/d}(f) = \sum_{x \in M_{\nu}} \left( \sum_{x \in M_{\nu}} S_{\nu/d}(x) \right)^{r_{s}} \left( \sum_{x \in M_{\nu}} S_{\nu/d}(x) S_{\nu/d}(x^{-1}) \right)^{s_{i}} + \text{lower order terms in } q.$$  

(7.3)

Following (6.18) we have

$$\sum_{x \in M_{\nu}} S_{\nu/d}(x) S_{\nu/d}(x^{-1}) = (q^{i} - 1) \langle S_{\nu/d}, S_{\nu/d} \rangle_{M_{\nu}} = (q^{i} - 1) d.$$  

Furthermore, $M_{\nu} = \{ x \in M | x^{d+1} = 1 \}$ by (6.16), so we have

$$\sum_{x \in M_{\nu}} S_{\nu/d}(x) = \sum_{x \in M_{\nu}} \sum_{\gamma \in \theta} \langle \gamma, x \rangle_{2i} = \sum_{\gamma \in \theta} \sum_{x \in M_{\nu}} \langle \gamma, x(q^{2i-1})/(q^{d-1}) \rangle_{d} = \begin{cases} d(q^{i} + 1) & \text{if } d|i \\ 0 & \text{otherwise} \end{cases}$$

since

$$\frac{(q^{2i}-1)/(q^{d}-1)}{q^{i} + 1} = \frac{q^{i} - 1}{q^{d} - 1} \in \mathbb{Z} \iff \frac{i}{d} \in \mathbb{Z}.$$  

The leading coefficient of $h_{\lambda, \tau}$ is therefore zero unless $d$ divides $(n_{+}, n_{-}, r_{s}, r_{p}) = d \cdot (n_{+}, n_{-}, r_{s}, r_{p})$, the leading coefficient equals

$$(-1)^{n-v} \langle \theta_{\nu}, -1 \rangle_{d}^{n-v/d} \frac{d(\tau_{s})}{z_{\nu/d} z_{\nu/d}} \sum_{\pi \vdash v} \sum_{\rho_{+} + \rho_{-} = d} d(\rho_{+}) \prod_{\rho_{+} + n_{+}} \prod_{\rho_{-} + n_{-}} \sum_{f_{\pi} z_{f_{\rho_{+}}} z_{f_{\rho_{-}}}} \langle -1 \rangle_{d}^{\nu}(\rho_{+} \cup \rho_{-}) \pi_{\rho_{+}} \pi_{\rho_{-}}.$$

$$= (-1)^{n-v} \langle \theta_{\nu}, -1 \rangle_{d}^{n-v/d} \frac{1}{z_{\nu/d} z_{\nu/d}} \sum_{\pi \vdash v} \sum_{\rho_{+} + \rho_{-} = d} d(\rho_{+}) \prod_{\rho_{+} + n_{+}} \prod_{\rho_{-} + n_{-}} \sum_{f_{\pi} z_{f_{\rho_{+}}} z_{f_{\rho_{-}}}} \langle -1 \rangle_{d}^{\nu}(\rho_{+} \cup \rho_{-}) \pi_{\rho_{+}} \pi_{\rho_{-}}.$$

For the sign, note

$$\ell_{ev}(\rho_{+}) + \ell_{ev}(\rho_{-}) \equiv \ell_{ev}(d \cdot \pi) + \ell(\tau_{s}) \mod 2.$$  

If $d$ is odd, then $n - v$ is even and $\ell_{ev}(d \cdot \pi) = \ell_{ev}(\pi)$. If $d$ is even, then $n$ is even and $\ell_{ev}(d \cdot \pi) \equiv \ell_{ev}(\pi) + v \mod 2$. Both cases yield (7.1), using $sgn(\pi) = (-1)^{\ell_{ev}(\pi)}$.  

\[ \square \]
Proposition 7.2. If \( \Lambda = (\theta_1^{\lambda_1}, \ldots, \theta_k^{\lambda_k}) \) and \( \tau \) is restricted symmetric type then

\[
h_{\Lambda, \tau}(y) = \text{e.o.t} \sum_{\tau_1 \cup \ldots \cup \tau_k = \tau} g_{\tau_1, \ldots, \tau_k}^T(y) \prod_{i=1}^k h_{(\theta_i^{\lambda_i}), \tau_i}(y).
\]

where \( \text{e.o.t} \) means equal modulo terms of less than leading order.

Proof. Set \( n_i := \|\theta_i^{\lambda_i}\| = d_{\theta_i}|\lambda_i| \) and \( n := \|\Lambda\| = n_1 + \ldots + n_k \). Recall (6.10) that for \( \Lambda = (\theta_1^{\lambda_1}, \ldots, \theta_k^{\lambda_k}) \) we have

\[
\chi_\Lambda(c) := \sum_{(c_1, \ldots, c_k) \in (c_1, \ldots, c_k)} g_{c_1, \ldots, c_k}(q) \prod_{i=1}^k \chi_{\theta_i^{\lambda_i}}(c_i).
\]

where \( c_i \) are conjugacy classes of rank \( n_i \). If \( c = c_\mu \) has restricted symmetric type \( \tau = (n_1, n_\ldots, \tau_s, \tau_p) \), then it is principal and cannot be produced using non-trivial extensions. Therefore \( g_{c_1, \ldots, c_k}(q) = 0 \) unless \( c_\mu \cong c_1 \oplus \ldots \oplus c_k \) and we may write

\[
\chi_\Lambda(c_\mu) := \sum_{\mu_1 \cup \ldots \cup \mu_k = \mu} g_{\mu_1, \ldots, \mu_k}(q) \prod_{i=1}^k \chi_{\theta_i^{\lambda_i}}(c_i).
\]

By (6.5) and (6.11), \( g_{\mu_1, \ldots, \mu_k}(q) \) is monic of degree

\[
\deg(g_{\mu_1, \ldots, \mu_k}(q)) := n(1^{n_+}) + n(1^{n_-}) - \sum_{i=1}^k (n(1^{n_i,+}) + n(1^{n_i,-}))
\]

where \( n_i,\pm \) is the multiplicity of the eigenvalue \( \pm 1 \) in \( c_i \).

Summing (7.5) over \( \mu \) gives

\[
h_{\Lambda, \tau}(q) = \sum_{\mu \in \tau} \chi_\Lambda(c_\mu) = \sum_{\mu_1 \cup \ldots \cup \mu_k = \mu} g_{\mu_1, \ldots, \mu_k}(q) \prod_{i=1}^k \chi_{\theta_i^{\lambda_i}}(c_i).
\]

This sum includes tuples \( (\mu_1, \ldots, \mu_k) \) that send polynomials in a symmetric pair \( \{f, f^\ast\} \) to different blocks \( \theta_i^{\lambda_i}, \theta_j^{\lambda_j} \) where \( \theta_i \neq \theta_j \). But the contribution of these tuples to the calculation of the leading term involves a factor equal to the character sum

\[
\sum_{x \in M_{d_f}} S_{d_f/d_{\theta_i}}^{\theta_i}(x) S_{d_f/d_{\theta_j}}^{\theta_j}(x^{-1}) = \sum_{x \in M_{d_f}} \sum_{\gamma \in \theta_i, \gamma' \in \theta_j} \langle \gamma, x \rangle_{d_f} \langle \gamma', x \rangle_{d_f} = 0
\]

(compare (7.3)). To leading order, it is therefore enough sum over \( (\mu_1, \ldots, \mu_k) \) for which each \( \mu_i \) has restricted symmetric type \( \tau_i \) of norm \( \|\tau_i\| = n_i \). Since \( g_{\mu_1, \ldots, \mu_k}(q) \) depends only on
type, we can write $g_{\tau_1 \cdots \tau_k}(q) = g_{\mu_1 \cdots \mu_k}^{\mu}(q)$ when $\mu \in \tau, \mu_i \in \tau_i$. Then

\[
\sum_{\mu_1 \cup \cdots \cup \mu_k = \mu \in \tau} g_{\mu_1 \cdots \mu_k}^{\mu}(q) \prod_{i=1}^{k} \chi_{\theta_i}^{\lambda_i}(c_{\mu_i}) = \text{i.o.t.} \sum_{\tau_1 \cup \cdots \cup \tau_k = \tau} g_{\tau_1 \cdots \tau_k}(q) \sum_{\mu_i \in \tau_i, i \in \{1, \ldots, k\}} \left( \prod_{i=1}^{k} \chi_{\theta_i}^{\lambda_i}(c_{\mu_i}) \right)
\]

\[
= \text{i.o.t.} \sum_{\tau_1 \cup \cdots \cup \tau_k = \tau} g_{\tau_1 \cdots \tau_k}(q) \prod_{i=1}^{k} \left( \sum_{\mu_i \in \tau_i} \chi_{\theta_i}^{\lambda_i}(c_{\mu_i}) \right)
\]

\[
= \sum_{\tau_1 \cup \cdots \cup \tau_k = \tau} g_{\tau_1 \cdots \tau_k}(q) \prod_{i=1}^{k} h_{(\theta_i), \tau_i}(q)
\]

where we get further lower order terms accounting for when $\text{supp}'(\mu_i) \cap \text{supp}'(\mu_j) \neq \emptyset$ for $i \neq j$.

\[\square\]

8. Irreducible Character Decomposition of $F$

Consider the character $F : GL_n(\mathbb{F}_q) \to \mathbb{Z}_{\geq 0} \subseteq \mathbb{C}$ of Section 5. In this section, we compute the multiplicity in $F$ of irreducible characters of $GL_n(\mathbb{F}_q)$, for $\text{char}(q) \gg 1$. That is, we determine the coefficients $a_{\Lambda}$ in the decomposition

\[
F = \sum_{\Lambda : \theta \to \mathcal{P}} a_{\Lambda} \chi_{\Lambda}.
\]

The conjugacy classes of $GL_n(\mathbb{F}_q)$ are parametrized by functions $\mu : \Phi \to \mathcal{P}$ of degree $n$ (see §4.3). By (1.8) we have

\[
a_{\Lambda} = \sum_{||\mu|| = n} \frac{F(c_{\mu})}{|Z(c_{\mu})|} \chi_{\Lambda}(c_{\mu}). \quad (8.1)
\]

Since $F$ is supported on conjugacy classes of symmetric type, substituting (4.3), (5.1), (6.2), we deduce that

\[
a_{\Lambda} = \sum_{||\eta|| = n} \frac{b_{\eta}(q)}{a_{\eta}(q)} h_{\Lambda, \eta}(q) \quad (8.2)
\]

where the sum is over symmetric types $\eta$ with $||\eta|| = n$ and $a_{\eta}(y), b_{\eta}(y), h_{\Lambda, \eta}(y) \in \mathbb{C}[y]$ are certain polynomials introduced earlier.

Recall that a symmetric map $\mu : \Phi \to \mathcal{P}$ has restricted symmetric type if $\mu(f) = 1^{||\mu(f)||}$ for $f \in \{t \pm 1\}$ and $\mu(f) = 1^1$ for $f \in \text{supp}'(\mu) := \text{supp} \mu \setminus \{t \pm 1\}$. The following lemma implies that only restricted symmetric types contribute to (8.2) in the limit $q \to \infty$.

**Lemma 8.1.** If $\eta$ be a symmetric type with $||\eta|| = n$, then

\[
\frac{b_{\eta}(y)}{a_{\eta}(y)} h_{\Lambda, \eta}(y)
\]

is rational function in $y$ of non-positive degree and has degree zero only if $\eta$ is restricted. If $\eta$ is restricted, the constant term of (8.3) agrees with the leading coefficient of $h_{\Lambda, \eta}(y)$.

**Proof.** From (4.3), Proposition 5.1, and Proposition 6.1, each of $a_{\eta}(y), b_{\eta}(y),$ and $h_{\Lambda, \eta}(y)$ are equal to polynomials in $y$, so (8.3) is a rational function in $y$. Both $a_{\eta}(y)$ and $b_{\eta}(y)$ are monic, so the leading coefficient is same as $h_{\Lambda, \eta}(y)$. 
The degrees of the polynomials satisfy (in)equalities
\[
\deg a_\eta(y) = 2 \sum_{f \in \text{supp}(\mu)} df \left(n(\mu(f)) + \frac{1}{2}|\mu(f)|\right), \quad (8.4)
\]
\[
\deg b_\eta(y) = \frac{1}{2} (\ell_{\text{odd}}(\mu(t + 1)) + \ell_{\text{odd}}(\mu(t - 1))) + \sum_{f \in \text{supp}(\mu)} df \left(n(\mu(f)) + \frac{1}{2}|\mu(f)|\right), \quad (8.5)
\]
\[
\deg h_{\Lambda, \eta}(y) \leq n(\mu(t + 1)) + n(\mu(t - 1)) + \sum_{f \in \text{supp}'(\mu)} df \left(n(\mu(f)) + \frac{1}{2}\right). \quad (8.6)
\]
Therefore the degree of (8.3) bounded above by
\[
\frac{1}{2} \left( \sum_{f=t\pm 1} (\ell_{\text{odd}}(\mu(f)) - |\mu(f)|) + \sum_{f \in \text{supp}'(\mu)} df (1 - |\mu(f)|) \right). \quad (8.7)
\]
which is a sum of non-positive terms, hence non-positive. The upperbound (8.7) is zero if and only if \(\mu(f) = 1|\mu(f)|\) for \(f \in \{t \pm 1\}\) and \(\mu(f) = 1\) for \(f \in \text{supp}'(\mu)\).

For \(\pi \in \mathcal{P}\), define \(C^+_{\pi}, C^-_{\pi} \in \mathbb{C}\) by
\[
C^+_{\pi} := \sum_{\rho^+, \rho^-, \tau_s, \tau_p \in \mathcal{P}, \rho^+ \cup \rho^- \cup 2\tau_s \cup 2\tau_p = \pi, |\rho^-| \equiv 0 \text{ mod } 2} \frac{(-1)^{f(\tau_s)}}{z_{\rho^+} z_{\rho^-} z_{2\tau_s} z_{2\tau_p}} C^-_{\pi} := \sum_{\rho^+, \rho^-, \tau_s, \tau_p \in \mathcal{P}, \rho^+ \cup \rho^- \cup 2\tau_s \cup 2\tau_p = \pi, |\rho^-| \equiv 1 \text{ mod } 2} \frac{(-1)^{f(\tau_s)}}{z_{\rho^+} z_{\rho^-} z_{2\tau_s} z_{2\tau_p}}. \quad (8.8)
\]
and define
\[
C_{\pi} := C^+_{\pi} + C^-_{\pi} \quad D_{\pi} := C^+_{\pi} - C^-_{\pi}. \quad (8.9)
\]

**Proposition 8.2.** Let \(\theta \in \Theta_d\) let \(\lambda \vdash v = n/d\) be a partition and let \(\text{char}(q) \gg 1\). Then for the primary character \(\theta^\lambda\) we have
\[
a_{\theta^\lambda} = \begin{cases} a_\pi^+: \sum_{\tau} sgn(\pi) C_{\pi, \lambda} & \text{if } \langle \theta, -1 \rangle_d = 1 \\ a_\pi^-: \sum_{\tau} sgn(\pi) D_{\pi, \lambda} & \text{if } \langle \theta, -1 \rangle_d = -1. \end{cases}
\]

**Proof.** By (8.2) we have
\[
a_{\theta^\lambda} = \sum_{\eta \text{ symmetric type } ||\eta|| = n} \sum_{a_\eta(q) h_{\Lambda, \eta}(q)}
\]
By Lemma 8.1 we know that
\[
a_{\theta^\lambda} + O(q^{-1}) = \sum_{\eta \text{ restricted symmetric type } ||\eta|| = n} \sum_{a_\eta(q) h_{\Lambda, \eta}(q)}.
\]
By Remark 6.2 we know that constant terms must match for \(\text{char}(q) \gg 1\). Applying Lemma 7.1, and taking a limit \(q \to \infty\) we have
\[
a_{\theta^\lambda} = \sum_{n, n- \geq 0, \tau_s, \tau_p \in \mathcal{P}, |n, n-| + |n+| + 2|\tau_s| + 2|\tau_p| = v} \langle \theta, -1 \rangle_d^n \frac{(-1)^{f(\tau_s)}}{z_{2\tau_s} z_{2\tau_p}} \sum_{\pi \vdash v} sgn(\pi) \chi_{\pi}^\lambda \sum_{\rho^+, \rho^- \cup 2\tau_s \cup 2\tau_p = \pi, |\rho^-| = n, |\rho^+| = n-} \frac{1}{z_{\rho^+} z_{\rho^-}}
\]
\[
= \sum_{\pi \vdash v} sgn(\pi) \chi_{\pi}^\lambda (C^+_{\pi} + \langle \theta, -1 \rangle_d C^-_{\pi})).
\]

\[\square\]
Remark 8.3. It may seem strange not to absorb the $\sgn(\pi)$ into the definition of $C_\pi$ and $D_\pi$. However we show in §9 that $C_\pi$ and $D_\pi$ are always non-negative.

Proposition 8.4. If $\Lambda = (\theta^\lambda_1, \ldots, \theta^\lambda_k)$ then

$$a_\Lambda = \prod_{i=1}^k a_{\theta^\lambda_i}.$$ 

Proof. By Lemma 8.1 we know that

$$a_\Lambda + O(q^{-1}) = \sum_{\tau \text{ restricted symmetric type}} b_\eta(q) a_\eta(q) h_{\Lambda,\eta}(q). \quad (8.11)$$

Substituting (7.4) we get

$$a_\Lambda = \text{l.o.t.} \sum_{\|\tau_i\|=n_i} \frac{b_{\tau_1 \cup \ldots \cup \tau_k}(q)}{a_{\tau_1 \cup \ldots \cup \tau_k}(q)} g^\tau_{\tau_1, \ldots, \tau_k}(q) \prod_{i=1}^k h_{(\theta^\lambda_i)_{\tau_i}}(q)$$

$$= \text{l.o.t.} \prod_{i=1}^k \left( \sum_{\|\tau_i\|=n_i} \frac{b_{\tau_i}(q)}{a_{\tau_i}(q)} h_{(\theta^\lambda_i),\tau_i}(q) \right) = \text{l.o.t.} \prod_{i=1}^k a_{\theta^\lambda_i}$$

where the second equivalence is a consequence of the two rational functions in $q$

$$\frac{b_{\tau_1 \cup \ldots \cup \tau_k}(q)}{a_{\tau_1 \cup \ldots \cup \tau_k}(q)} g^\tau_{\tau_1, \ldots, \tau_k}(q) \prod_{i=1}^k \frac{b_{\tau_i}(q)}{a_{\tau_i}(q)}$$

being monic of the same degree, which is readily verified from (7.6), (8.4), and (8.5). It follows that $a_\Lambda = \prod_i a_{\theta^\lambda_i}$ for $\text{char}(q) \gg 1$. \hfill \Box

9. SCHUR FUNCTION FORMULAS

Proposition 8.2 can be reformulated in terms of symmetric functions. Let $\Lambda_\mathbb{Z}$ be the ring of symmetric functions in variables $\{x_1, x_2, \ldots\}$. For each $\pi \in \mathcal{P}$ define the power function $p_\pi = \prod_i p_{\pi_i}$ where $p_\alpha := x_1^n + x_2^n + \ldots \in \Lambda_\mathbb{Z}$. These are related to the Schur functions $s_\lambda \in \Lambda_\mathbb{Z}$ by

$$p_\pi = \sum_{\pi \rightarrow \lambda} \lambda p_\lambda. \quad (9.1)$$

Both $\{p_\pi | \pi \in \mathcal{P}\}$ and $\{s_\lambda | \lambda \in \mathcal{P}\}$ form $\mathbb{Z}$-bases of $\Lambda_\mathbb{Z}$.

Corollary 9.1. We have an equality of symmetric functions

$$\sum_{\lambda \in \mathcal{P}} a_\lambda^+ s_\lambda = \sum_{\pi \in \mathcal{P}} C_\pi p_\pi \quad \sum_{\lambda \in \mathcal{P}} a_\lambda^- s_\lambda = \sum_{\pi \in \mathcal{P}} D_\pi p_\pi \quad (9.2)$$

Proof. Using (9.1), Proposition 8.2 is equivalent to the identities

$$\sum_{\lambda \in \mathcal{P}} a_\lambda^+ s_\lambda = \sum_{\pi \in \mathcal{P}} \sgn(\pi) C_\pi p_\pi \quad \sum_{\lambda \in \mathcal{P}} a_\lambda^- s_\lambda = \sum_{\pi \in \mathcal{P}} \sgn(\pi) D_\pi p_\pi$$

and (9.2) follows by multiplying the degree $n$ summands on both sides by $s_1^n$, which corresponds to tensoring by the alternating representation of the symmetric group and satisfies $s_\lambda s_1^n = s_\lambda'$ and $p_\pi s_1^n = \sgn(\pi) p_\pi$. \hfill \Box
Proposition 9.2. We have equalities of symmetric functions
\[
\sum_{\pi \in \mathcal{P}} C_{\pi p_{\pi}} = \left( \prod_{m \text{ odd}} e^{\frac{2m}{m}} \right) \left( \prod_{m \text{ even}} e^{\frac{m}{m} + \frac{2m}{2m}} \right).
\]
\[
\sum_{\pi \in \mathcal{P}} D_{\pi p_{\pi}} = \left( \prod_{m \text{ odd}} e^{\frac{2m}{m}} \right) \left( \prod_{m \text{ even}} e^{\frac{m}{m} + \frac{2m}{2m}} \right).
\]

Proof. Because \(z_{1^2, 2^2, \ldots} = \prod_m z_{m^m}\) we deduce from (8.9) that
\[
C_{1^2, 2^2, \ldots} = \prod_m C_{m^m} \quad \text{and} \quad D_{1^2, 2^2, \ldots} = \prod_m D_{m^m}.
\]
so
\[
\sum_{\pi \in \mathcal{P}} C_{\pi p_{\pi}} = \prod_m \left( \sum_{n=0}^{\infty} C_{n^m} p_{n^m} \right) \quad \text{and} \quad \sum_{\pi \in \mathcal{P}} D_{\pi p_{\pi}} = \prod_m \left( \sum_{n=0}^{\infty} D_{n^m} p_{n^m} \right).
\]
We are reduced to identifying the generating functions \(\sum_{n \geq 1} C_{m^n} t^n\) and \(\sum_{n \geq 1} D_{m^n} t^n\).

If \(m\) is odd, we have
\[
C_{m^n} := C_{m^n}^+ + C_{m^n}^- = \sum_{i+j+2k=n} \frac{1}{i!j!k!} 2^k m^i j^j k^k,
\]
\[
D_{m^n} := C_{m^n}^+ - C_{m^n}^- = \sum_{i+j+2k=n} \frac{(-1)^j}{i!j!k!} 2^k m^i j^j k^k,
\]
so the generating functions satisfies
\[
\sum_{n \geq 0} C_{m^n} t^n = \left( \sum_i \frac{(t/m)^i}{i!} \right) \left( \sum_j \frac{(t/m)^j}{j!} \right) \left( \sum_k \frac{(t^2/2m)^k}{k!} \right) = e^{2m} t^{2m}.
\]
\[
\sum_{n \geq 0} D_{m^n} t^n = \left( \sum_i \frac{(t/m)^i}{i!} \right) \left( \sum_j \frac{(-t/m)^j}{j!} \right) \left( \sum_k \frac{(t^2/2m)^k}{k!} \right) = e^{2m}. \]

If \(m\) is even, then
\[
C_{m^n} = C_{m^n}^+ = D_{m^n} = \sum_{i+j+2k=n} \frac{(-1)^j}{i!j!k!} 2^k m^i j^j k^k + l^l
\]
so
\[
\sum_{n \geq 0} C_{m^n} t^n = \sum_{n \geq 0} D_{m^n} t^n = \left( \sum_i \frac{(t/m)^i}{i!} \right)^2 \left( \sum_k \frac{(t^2/2m)^k}{k!} \right) \left( \sum_l \frac{(-t/m)^l}{l!} \right) = e^{2x} t^{2m}. \]

It remains to express these symmetric polynomials in the Schur function basis. This makes use of Pieri’s rule. Let \(\pi \in \mathcal{P}\) be a partition thought of as Young diagram. Pieri’s rule states that
\[
s_{\pi} s_{\lambda} = \sum_{\lambda} s_{\lambda}
\]
summed over \(\lambda\) obtained from \(\pi\) by adding \(n\) blocks with at most one block per column. Dually
\[
s_{\pi} s_{1^n} = \sum_{\lambda} s_{\lambda}
\]
summed over $\lambda$ obtained from $\pi$ by adding $n$ blocks with at most one block per row.

**Theorem 9.3.** Let $\lambda \in \mathcal{P}$ be a partition with transpose $\lambda^t = (1^{r_1} 2^{r_2} \ldots)$. Then

$$a^+_{\lambda} = (r_1 + 1)(r_2 + 1)\ldots$$

and

$$a^-_{\lambda} = \begin{cases} 1 & \text{if $\lambda'$ has only even parts} \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** By Macdonald ([22] §I.7, Example 11), we have an equality of symmetric functions

$$\sum_{\lambda \in \mathcal{P}} s_{\lambda} = \left( \prod_{n \text{ odd}} e^{\frac{p_n}{n}} \right) \left( \prod_{n \text{ even}} e^{\frac{p_n}{2n}} \right). \quad (9.4)$$

The Schur function $s_n = h_n$ is the complete sum of monomials of degree $n$, so

$$\sum_{n=0}^{\infty} s_n = \prod_i (1 - x_i)^{-1} = \prod_{n} e^{\frac{p_n}{n}} \quad (9.5)$$

where the second equality is verified by taking logarithms. It follows that

$$\left( \sum_{\lambda \in \mathcal{P}} s_{\lambda} \right) \left( \sum_{n=0}^{\infty} s_{1^n} \right) = \prod_{n \text{ odd}} e^{\frac{p_n+\frac{p_n}{n}}{2n}} \cdot \prod_{n \text{ even}} e^{\frac{p_n}{2n} + \frac{p_n}{n}} = \sum_{\pi \in \mathcal{P}} C_{\pi} p_{\pi} = \sum_{\lambda \in \mathcal{P}} a^+_{\lambda} s_{\lambda}. \quad (9.6)$$

from which (9.3) is deduced using Pieri’s rule.

For any Young diagram, there is a unique way to remove at most one block from each row to get another diagram with only even length rows. It therefore follows by Pieri’s rule that

$$\left( \sum_{\lambda \in \mathcal{P}, \lambda \text{ is even}} s_{\lambda} \right) \left( \sum_{n} s_{1^n} \right) = \sum_{\lambda \in \mathcal{P}} s_{\lambda},$$

where $s_{1^n} = e_n$ is the $n$-th elementary symmetric polynomial. Here

$$\sum_{n} s_{1^n} = \prod_i (1 + x_i) = \left( \prod_{n \text{ odd}} e^{\frac{p_n}{n}} \right) \left( \prod_{n \text{ even}} e^{\frac{p_n}{n}} \right) \quad (9.7)$$

where the second equality is verified by applying logarithms. Combined with (9.4) we deduce

$$\sum_{\lambda \in \mathcal{P}, \lambda \text{ is even}} s_{\lambda} = \left( \prod_{n \text{ odd}} e^{\frac{p_n}{2n}} \right) \left( \prod_{n \text{ even}} e^{\frac{p_n}{2n} + \frac{p_n}{n}} \right) = \sum_{\pi \in \mathcal{P}} D_{\pi} p_{\pi} = \sum_{\lambda \in \mathcal{P}} a^-_{\lambda} s_{\lambda}. \quad \blacksquare$$

10. **The E-polynomial of $\mathcal{M}_n^\tau$**

Recall Corollary 3.4 that we are considering

$$E_n(q) := \left| G_n \right|^{g-1} \sum_{\chi \in \text{Irr} G} \frac{\chi(\xi)}{\chi(1)} (a_\lambda)^r = \sum_{\|\lambda\| = n} (a_\lambda)^r \left( \frac{|G_n|}{\chi(1)} \right)^{g-1} \frac{\chi_\lambda(\xi)}{\chi_\lambda(1)}$$

where $G_n := GL_n(\mathbb{F}_q)$ and we use shorthand $aI_n = \alpha$ for $\alpha \in \mathbb{F}_q^\times$. 

Note that \( l \) where \( I/\rho \) denote by \( \rho \) function, where \( \rho \) = \( \lambda \). Denote by \( \rho \) the product indexed by boxes in the Young diagram of \( \lambda \) and \( h \) is the hook length of the box (see [14] (2.47)). Despite the fractional exponents, (10.2) is a polynomial in \( q \).

Both (10.2) and \( a_\Lambda \) depend only on the signed type \( \sigma = (\sigma^+, \sigma^-) \) of \( \Lambda \) (6.1) so we can rearrange our formula

\[
E_n(q) = (-q^{\frac{1}{2}})^n q^ {\frac{1}{2}} \sum_{\text{signed types } \sigma} (a_\sigma)^{\tau} \mathcal{H}_{\sigma'}(q)^{\tau-1} \sum_{\Lambda \in \sigma} \Delta_\Lambda(\xi). \tag{10.4}
\]

**Lemma 10.1.** Let \( \sigma = (\sigma^+, \sigma^-) \) be a signed type for which \( m_{d,\pm}^\pm \) is the number of Frobenius orbits \( \theta \in \Theta_d^{\pm} \) that are sent to the partition \( \lambda \). Set \( m_{d,\pm}^\pm := \sum_{\lambda} m_{d,\pm}^\pm \), \( m_{d,\lambda} = m_{d,\pm}^+ + m_{d,\pm}^- \), \( m_d := m_{1,\pm}^+ + m_{1,\pm}^- \), and \( m := \sum_d m_d \). Then

\[
\sum_{\Lambda \in \sigma} \Delta_\Lambda(\xi) = \begin{cases} (-1)^{m-1} \mu(d) \frac{(m-1)!}{2} & \text{if } m = m_d^+ \\ 0 & \text{otherwise} \end{cases}
\]

where \( \mu \) is the classical Möbius function

\[
\mu(d) = \begin{cases} 1 & \text{if } d \text{ is square free and has an even number of prime factors} \\ -1 & \text{if } d \text{ is square free and has an odd number of prime factors} \\ 0 & \text{if } d \text{ is not square free} \end{cases}
\]

**Proof.** We use an inclusion-exclusion argument that modifies the proof of [14, §3.4]. Let \( \Lambda_0 : L \to \mathcal{P} \) represent the signed type \( \sigma \). Choose a finite set \( I \) with an injective map \( \zeta_0 : I \to L \) onto the support of \( \Lambda_0 \). Define the permutation \( \rho \) of \( I \) by \( \zeta \circ \rho = \text{Frob} \circ \zeta \) and define functions

\[
l, n : I \to \mathbb{Z}_{\geq 0}
\]

where \( l(i) \) equals to the length of the \( \rho \)-cycle containing \( i \), and \( n(i) = |\Lambda_0(\zeta_0(i))| \). Note that

\[
\sum_{i \in I} n(i) = n.
\]

Denote by \( I/\rho \) the set of \( \rho \)-cycles.

For \( a \geq 1 \), define

\[
L_a^\pm := \{ \gamma \in L_a | \langle \gamma, -1 \rangle_a = \pm 1 \}.
\]

Note that \( L_a^+ \leq L_a \) is an index two cyclic subgroup and \( (L_a^+/\text{Frob}) \cap \Theta_a = \Theta_a^{\pm} \). Partition \( I = I^+ \cup I^- \) where

\[
I^\pm := \{ i \in I | \zeta_0(i) \in L_{l(i)}^\pm \}.
\]

Note \( \rho \) preserves both \( I^+ \) and \( I^- \). Consider the set of maps

\[
(I, L)_\rho := \{ \zeta : I \to L | \zeta \circ \rho = \text{Frob} \circ \zeta, \text{ and } \zeta(i) \in L_{l(i)}^\pm \text{ for } i \in I^\pm \}
\]

From [14, (3.1.1)] we know that

\[
\Delta_\Lambda(\xi) := \frac{\chi_\Lambda(\xi)}{\chi_\Lambda(1)} = \langle \prod_{\gamma \in L} \gamma^{\langle \Lambda(\gamma) \rangle}, \xi \rangle_1. \tag{10.1}
\]

From [14, (3.1.5)] we know

\[
\frac{|G_n|}{\chi_\Lambda(I_n)} = (-q^{\frac{1}{2}})^n \mathcal{H}_\Lambda(q) = (-q^{\frac{1}{2}})^n \prod_{\theta \in \Theta} \mathcal{H}_{\Lambda(\theta)}(q^{d_{\theta}}) \tag{10.2}
\]

where for \( \lambda \in \mathcal{P} \), the \( \mathcal{H}_\Lambda(t) \) is the normalized hook polynomial

\[
\mathcal{H}_\Lambda(t) = t^{-(n(\lambda) + \frac{|\Delta|}{2})} \prod (1 - t^h) \tag{10.3}
\]

the product indexed by boxes in the Young diagram of \( \lambda \) and \( h \) is the hook length of the box (see [14] (2.47)). Despite the fractional exponents, (10.2) is a polynomial in \( q \).
and \((I, L)_{\rho}\)' the subset of injective maps. There is a natural \(z_{\sigma}\)-to-1 surjective map from \((I, L)_{\rho}\)' onto the set \(\Lambda\) of signed type \(\sigma\), sending \(\zeta_0\) to \(\Lambda_0\), where
\[
z_{\sigma} := \prod_d (d^{m_d} \prod_\lambda m^+_{d,\lambda}!m^-_{d,\lambda}!). \tag{10.5}\]

Consider the function \(\varphi : (I, L)_{\rho} \to \mathbb{C}^\times\) by
\[
\varphi(\zeta) = \prod_i \zeta(i)^{n(i)} \xi_1
\]
If we define \(\text{Nm}_{a,1} : L_a \to L_1\) by \(\text{Nm}_{a,1}(\gamma) = \gamma^{1+q+\ldots+q^{a-1}}\), then
\[
\varphi(\zeta) = \prod_{c \in I/\rho} \langle \text{Nm}_{l(c),1}(\zeta(c)), \xi_1^{n(c)}\rangle
\]
where \(l(c) = l(i), n(c) = n(i)\), and \(\zeta(c) = \zeta(i)\) for some \(i \in c\) (the value of \(\varphi\) is independent of this choice). If
\[
S(I)' := \sum_{\zeta \in (I, L)_{\rho}'} \varphi(\zeta)
\]
then by construction we have an equality
\[
\sum_{\Lambda \in \sigma} \Delta_{\Lambda}(\xi) = \frac{1}{z_{\sigma}} S(I)'. \tag{10.6}\]

A partition of \(I\) describes a surjective map \(I \to J\) onto the set of blocks. Consider \(\Pi(I)\) the poset of partitions of \(I\) whose blocks are permuted by \(\rho\). Denote \((J, L)_{\rho} \subseteq (I, L)_{\rho}\) the subset of maps which are constant on the blocks of \(J\) and set \(n(j) = \sum_{i \in j} n(i)\) for \(j \in J\). By the Moebius inversion formula we have
\[
S(I)' = \sum_{J \in \Pi(I)} \mu_{\rho}(J) S(J)
\]
where \(\mu_{\rho}\) is the Moebius function for the poset \(\Pi(I)\) and
\[
S(J) := \sum_{\zeta \in (J, L)_{\rho}} \varphi(\zeta).
\]
For \(\zeta \in (J, L)_{\rho}\) we have
\[
\varphi(\zeta) = \prod_{c \in J/\rho} \langle \text{Nm}_{l(c),1}(\zeta(c)), \xi_1^{n(c)}\rangle_1,
\]
where for \(c \in J/\rho\), \(l(c)\) is the length of the cycle in \(J\). We can therefore commute the product through the sum to get
\[
S(J) = \prod_{c \in J/\rho} \left( \sum_{\gamma \in L(c)} \langle \text{Nm}_{l(c),1}(\gamma), \xi_1^{n(c)}\rangle_1 \right) \tag{10.7}
\]
where
\[
L(c) := \left( \bigcap_{\tilde{c} \in I^+/\rho} L^+_{l(\tilde{c})} \right) \cap \left( \bigcap_{\tilde{c} \in I^-/\rho} L^-_{l(\tilde{c})} \right) \cap L_l(c).
\]
If \(a\) divides \(a'\), then
\[
L^\pm_{a'/a} \cap L_a = L^\pm_a \quad \text{if } a'/a \text{ is odd} \tag{10.8}
\]
\[
L^\pm_{a'/a} \cap L_a \quad \text{and} \quad L^\pm_{a'/a} \cap L_a = \emptyset \quad \text{if } a'/a \text{ is even} \tag{10.9}
\]
so each \( L(c) \) must equal one of \( L_{(c)}, L_{(c)}^\pm \), or \( \emptyset \). Since \( \xi \) is a primitive \( 2n \)-root of unity, the character \( \langle \text{Nm}_{a,1}(\gamma), \xi^{n(c)} \rangle_1 \) restricts to the trivial character on \( L_a \) if and only if \( 2n|n(c) \). But this never happens because \( n(c) \leq n \). It follows that for all \( a \)
\[
\sum_{\gamma \in L_a} \langle \text{Nm}_{a,1}(\gamma), \xi^{n(c)} \rangle_1 = 0
\]
so
\[
\sum_{\gamma \in L_a^+} \langle \text{Nm}_{a,1}(\gamma), \xi^{n(c)} \rangle_1 = - \sum_{\gamma \in L_a^-} \langle \text{Nm}_{a,1}(\gamma), \xi^{n(c)} \rangle_1.
\]

The restriction of \( \langle \text{Nm}_{a,1}(\gamma), \xi^{n(c)} \rangle_1 \) to \( L_a^\pm \) is the trivial character if and only if \( n|n(c) \). But \( n(c) \leq n \) is an equality if and only if \( J \) is a singleton, so \( S(J) = 0 \) unless \( J = \{ I \} \). We deduce
\[
S(I') = \mu_\rho(\{ I \} )S(\{ I \} ) = (10.10)
\]
The poset in question is identical with the one considered in \([14]\) so we recover the formula (from Hanlon \([12]\])
\[
\mu_\rho(\{ I \} ) = \begin{cases} 
\mu(d)(-d)^{m_d-1}(m_d-1)! & \text{if } \rho \text{ has cycle type } (d^{m_d}) \\
0 & \text{otherwise}.
\end{cases}
\]

Comparing with (10.8) and (10.9) we see that if \( \rho = (d^{m_d}) \) then
\[
S(\{ I \} ) = \begin{cases} 
\pm(q-1)/2 & \text{if } d \text{ is odd and } m = m_d = m_d^\pm \\
0 & \text{otherwise}
\end{cases}
\]
which combined with (10.5), (10.6), (10.10) completes the proof. \( \square \)

**Theorem 10.2.** The \( E \)-polynomial \( E(Tx \cdot y) \) equals \( E_n(xy) \) where
\[
E_n(q) := \frac{1}{2}(q-1)(-q^{\frac{1}{2}})^{n^2(q-1)}V_n(q)
\]
and
\[
V_n(q) := \sum_{\substack{d|n \text{ odd} \ d \text{ odd} \ d\ |
\sum m_\lambda | \lambda | = \frac{n}{2} \text{}}}
(-1)^{m-1} \mu(d) \frac{d}{(m-1)!} \prod m_\lambda \left( \left( a_\sigma^+ \right)^r - \left( a_\sigma^- \right)^r \right) \mathcal{H}_\sigma^{q^{-1}}(q^d)
\]

where the sum taken over odd divisors \( d \) of \( n \) and non-negative integers \( m_\lambda \) for \( \lambda \in \mathcal{P} \) such that \( \sum m_\lambda | \lambda | = n/d \) and
\[
m := \sum \lambda m_\lambda \quad a_\sigma^\pm := \prod_{\lambda} (a_\lambda^\pm)^{m_\lambda} \quad \mathcal{H}_\sigma(q) := \prod \mathcal{H}_\lambda(q)^{m_\lambda}.
\]

**Proof.** The formula for \( E_n(q) \) is immediate from substituting Lemma 10.1 into (10.4). The formula for \( a_\sigma^\pm \) is found in Propositions 8.2 and 8.4.

Since \( E_n(q) \) is a polynomial expression in \( q \), this establishes the hypothesis of Corollary 3.3 and determines the \( E \)-polynomial. \( \square \)

**Example 10.1.** \( E_1(q) \) is a single term, indexed by \( (d, \lambda^{m_\lambda} ) = (1, (1)^1) \)
\[
E_1(q) = \frac{(q-1)}{2} |G_1|^{q-1}(2^r) = 2^{r-1}(q-1)^g
\]
in agreement with Example 2.8.
Example 10.2. $E_2(q)$ is a sum of three terms, indexed by 
$$(d, \lambda^m) \in \{(1, (2)^1), (1, (1^2)^1), (1, (1)^2)^1\}.$$
Explicitly
$$E_2(q) = \frac{(q-1)}{2} G_2^{g-1} \left( 2^r + \frac{3^r - 1}{q^{g-1}} - \frac{2^{2r-1}}{(q+1)^{g-1}} \right)$$
$$= \frac{1}{2} (q-1)^g \left( 2^r (q^3 - q)^{g-1} + (3^r - 1)(q^2 - 1)^{g-1} - 2^{2r-1}(q^2 - q)^{g-1} \right).$$

Example 10.3. $E_3(q)$ is a sum of seven terms, indexed by 
$$(d, \lambda^m) \in \{(3, (1)^1), (1, (1^3)^1), (1, (3)^1), (1, (21)^1), (1, (1^1(2)^1)), (1, (1^1(1^2)^1)), (1, (1)(1^1(1^1)^1))\}.$$
Explicitly
$$E_3(q) = \frac{(q-1)}{2} G_3^{g-1} \left( 2^r + \frac{4^r - 1}{q^{g-3}} + \frac{4^r}{(q^2 + q)^{g-1}} - \frac{4^r}{(q^2 + q + 1)^{g-1}} - \frac{6^r}{(q^3 + q^2 + q + 1)^{g-1}} \right)$$
$$+ \frac{8^r}{3(q+1)^{g-1}(q^2 + q + 1)^{g-1}} - \frac{2^r}{3(q-1)^{g-1}(q^2 - 1)^{g-1}}.$$

10.1. The generating function. The expression (10.11) has a beautiful interpretation using plethystic algebra. Let $K = \mathbb{Q}(x)$ be the ring of rational functions and consider $K[[T]]$ the ring of formal power series in $T$. The plethystic exponential $\text{Exp} : TK[[T]] \to 1 + TK[[T]]$ is defined by the rule $\text{Exp}(V + W) = \text{Exp}(V)\text{Exp}(W)$ and
$$\text{Exp}(ax^m T^n) = (1 - x^m T^n)^{-a}$$
for $a \in \mathbb{Q}$. The inverse function, $\text{Log} : 1 + TK[[T]] \to TK[[T]]$ is called the plethystic logarithm. For each $\lambda \in \mathcal{P}$ choose $A_{\lambda} \in K$, setting $A_{\emptyset} = 1$. By [13] §2.3,
$$\text{Log} \left( \sum_{\lambda \in \mathcal{P}} A_{\lambda} T^{|\lambda|} \right) := \sum_{n \geq 1} U_n T^n$$
where
$$U_n(x) = \sum (-1)^{md-1} \frac{\mu(d)}{d} (m_d - 1)! \prod_{\lambda} \frac{A_{\lambda}(x^d)^{md}_{m_d,\lambda}}{m_d,\lambda !} \quad (10.12)$$
and the sum is indexed by the set of functions
$$m : \mathbb{Z}_{>0} \times (\mathcal{P} \setminus \{\emptyset\}) \to \mathbb{Z}_{\geq 0}, \quad (d, \lambda) \mapsto m_{d, \lambda}$$
satisfying $\sum_{\lambda \in \mathcal{P}} d|\lambda|m_{d, \lambda} = n$ and we set $m_d := \sum_{\lambda} m_{d, \lambda}$.

Proof of Theorem 1.1. From Theorem 10.2 we are reduced to proving
$$\sum_{n \geq 1} V_n(q) T^n = \text{Log} \prod_{i=0}^{\infty} \left( \frac{\sum_{\lambda} (a_{\lambda}^+) \beta^{g-1}_{\lambda^i}(q^2)^i T^{2|\lambda|}}{\sum_{\lambda} (a_{\lambda}^-) \beta^{g-1}_{\lambda^i}(q^2)^i T^{2|\lambda|}} \right)^{\frac{1}{2^i}}. \quad (10.13)$$
Rewrite the right hand side of (10.13) as
$$\sum_{i=0}^{\infty} \frac{1}{2^i} \text{Log} \left( \sum_{\lambda} (a_{\lambda}^+) \beta^{g-1}_{\lambda^i}(q^2)^i T^{2|\lambda|} \right) - \sum_{i=0}^{\infty} \frac{1}{2^i} \text{Log} \left( \sum_{\lambda} (a_{\lambda}^-) \beta^{g-1}_{\lambda^i}(q^2)^i T^{2|\lambda|} \right)$$
Define $V_{n,+}^i(q)$ and $V_{n,-}^i(q)$ by the formula
\[
\sum_n V_{n,\pm}^i(q) T^n := \log \left( \sum_{\lambda} (a_\lambda^\pm)^r H_{\lambda'}^{-1}(q^{2^i}) T^{2^i|\lambda|} \right).
\]
Our task is to prove that for $n \geq 1$,
\[
V_n(q) = \sum_{i=0}^{\infty} \frac{1}{2^i} (V_{n,+}^i(q) - V_{n,-}^i(q))
\]  
(10.14)
where the left hand side satisfies (10.11). Applying (10.12) we see that
\[
V_{n,\pm}^0(q) = \sum_{d|n} \sum_{\lambda|\lambda| \neq \frac{d}{2}} (-1)^{m-1} \frac{\mu(d)}{d} (m - 1)! \prod_{\lambda} (a_\lambda^\pm)^r H_{\lambda'}^{-1}(q^{2d}) q^{\mu(m|\lambda|)} m_\lambda!.
\]
More generally $V_{n,\pm}^i(q) = 0$ unless $2^i | n$, in which case
\[
V_{n,\pm}^i(q) = \sum_{d|n} \sum_{\lambda|\lambda| \neq \frac{d}{2}} (-1)^{m-1} \frac{\mu(d/2^i)}{d} (m - 1)! \prod_{\lambda} (a_\lambda^\pm)^r H_{\lambda'}^{-1}(q^{2^id}) q^{\mu(m|\lambda|)} m_\lambda!.
\]
where in the second expression we reindex replacing $2^d$ with $d$. We deduce (10.14) from the identity
\[
\mu(d) + \mu(d/2) + \ldots + \mu(d/2^k) = 0
\]
that holds when $d$ is even and $d/2^k$ is odd.

10.2. **Combinatorial verification for special cases** $g = 0, 1$. If $g = 0$ and $r = 1$, then $\mathcal{M}_1$ is a point and $\mathcal{M}_n = \emptyset$ so (1.4) reduces to the identity
\[
\frac{2q^d}{1 - q} T = \log \prod_{k=0}^{\infty} \left( \sum_{\lambda} a_\lambda^+ H_{\lambda'}^{-1}(q^{2k}) T^{2^k|\lambda|} / \sum_{\lambda} a_\lambda^- H_{\lambda'}^{-1}(q^{2k}) T^{2^k|\lambda|} \right)^{\frac{1}{2^k}}.
\]
Applying the plethystic exponential, this is equivalent to
\[
\prod_{j \geq 1} \left( 1 - q^{2^j-1} T \right)^{-2} = \prod_{k=0}^{\infty} \left( \sum_{\lambda} a_\lambda^+ H_{\lambda'}^{-1}(q^{2^k}) T^{2^k|\lambda|} / \sum_{\lambda} a_\lambda^- H_{\lambda'}^{-1}(q^{2^k}) T^{2^k|\lambda|} \right)^{\frac{1}{2^k}}.
\]  
(10.15)
According to ([22] I.3 ex.2) we have $H_{\lambda}(q)^{-1} = q^{\lambda|\lambda|/2} s_\lambda(q)$ where $s_\lambda(x_1, x_2, \ldots)$ is the Schur function and $\tilde{s}_\lambda(q) := s_\lambda(1, q, q^2, \ldots)$ is the standard Schur function.

\[
\sum_{\lambda} a_\lambda^+ H_{\lambda'}^{-1}(q^{2^k}) T^{2^k|\lambda|} = \sum_{\lambda} a_\lambda^+ \tilde{s}_\lambda(q) (q^{2^k} T)^{|\lambda|}
\]
\[
= \left( \sum_{\lambda} \tilde{s}_\lambda(q) (q^{2^k} T)^{|\lambda|} \right) \left( \sum_{n=0}^{\infty} \tilde{s}_n(q) (q^{2^k} T)^n \right)
\]
Similar considerations apply to the $a^-_\lambda$ series and we deduce using (9.5) and (9.7)

$$\frac{\sum_{\lambda} a^+_\lambda \mathcal{H}_{\lambda}^{-1}(q) T^{\mid \lambda \mid}}{\sum_{\lambda} a^-_\lambda \mathcal{H}_{\lambda}^{-1}(q) T^{\mid \lambda \mid}} = \left( \sum_{n=0}^{\infty} \bar{s}_n(q) (q^{1/2} T)^n \right) \left( \sum_{n=0}^{\infty} \bar{s}_1 n(q) (q^{1/2} T)^n \right)$$

$$= \left( \prod_{i=1}^{\infty} (1 - q^{-1/2} T)^{-1} \right) \left( \prod_{i=1}^{\infty} (1 + q^{-1/2} T) \right)$$

$$= \prod_{i=1}^{\infty} \frac{(1 - q^{2i-1} T)}{(1 - q^{2i-1} T)^2}.$$

Consequently the right hand side of (10.15) is a telescoping product verifying the identity.

When $g = 1$, we have an isomorphism

$$\mathcal{M}_n \cong \mathbb{C}^\times \times \mathbb{C}^\times$$

for all $n \geq 1$ (see [14] Theorem 2.2.17). When $r = 1$, this isomorphism can be chosen so that $\tau(\alpha, \beta) = (\beta^{-1}, \alpha^{-1})$ so that

$$\mathcal{M}_n^r \cong \mathbb{C}^\times,$$

and $E(\mathcal{M}_n^r; q) = q - 1$. When $r = 2$, this isomorphism can be chosen so that $\tau(\alpha, \beta) = (\alpha, \beta^{-1})$ so that

$$\mathcal{M}_n^r \cong \mathbb{C}^\times \times \{\pm 1\},$$

and $E(\mathcal{M}_n^r; q) = 2(q - 1)$. Substituting into (1.4) and exponentiating, we get identities

$$\prod_{n=1}^{\infty} (1 - T^n)^{-2} = \prod_{k=0}^{\infty} \left( \sum_{\lambda} a^+_\lambda \mathcal{T}_{2k}^{\mid \lambda \mid} \right) \frac{1}{\sum_{\lambda} a^-_\lambda \mathcal{T}_{2k}^{\mid \lambda \mid}},$$

$$\tag{10.16}$$

$$\prod_{n=1}^{\infty} (1 - T^n)^{-4} = \prod_{k=0}^{\infty} \left( \sum_{\lambda} (a^+_\lambda)^2 \mathcal{T}_{2k}^{\mid \lambda \mid} \right) \frac{1}{\sum_{\lambda} (a^-_\lambda)^2 \mathcal{T}_{2k}^{\mid \lambda \mid}},$$

$$\tag{10.17}$$

We have

$$\sum_{\lambda} a^-_\lambda T^{\mid \lambda \mid} = \sum_{\lambda} (a^-_\lambda)^2 T^{\mid \lambda \mid} = \prod_{n=1}^{\infty} \left( \sum_{k=0}^{\infty} T^{2kn} \right) = \prod_{n \geq 1} (1 - T^{2n})^{-1}$$

and

$$\sum_{\lambda} a^+_\lambda T^{\mid \lambda \mid} = \prod_{n \geq 1} \left( \sum_{k=0}^{\infty} (k + 1) T^{kn} \right) = \prod_{n \geq 1} (1 - T^n)^{-2}$$

so that

$$\frac{\sum_{\lambda} a^+_\lambda T^{\mid \lambda \mid}}{\sum_{\lambda} a^-_\lambda T^{\mid \lambda \mid}} = \prod_{n \geq 1} (1 - T^n)^{-2} (1 - T^{2n})$$

and the right hand side of (10.16) is a telescoping product verifying the identity. Similarly,

$$\sum_{\lambda} (a^+_\lambda)^2 T^{\mid \lambda \mid} = \prod_{n \geq 1} \left( \sum_{k=0}^{\infty} (k + 1)^2 T^{kn} \right) = \prod_{n \geq 1} (1 - T^n)^{-4} (1 - T^{2n})$$

so

$$\frac{\sum_{\lambda} (a^+_\lambda)^2 T^{\mid \lambda \mid}}{\sum_{\lambda} (a^-_\lambda)^2 T^{\mid \lambda \mid}} = \prod_{n \geq 1} (1 - T^n)^{-4} (1 - T^{2n})^2$$

and the right hand side of (10.17) is a telescoping product verifying the identity.
11. The E-polynomial of $\mathcal{M}_{n,w}^\tau$

Recall from Corollary 3.4 that
\[ E_n^k(q) := |G_n|^{g-1} \sum_{\chi \in \text{Int}G} \frac{\chi\Lambda(\xi)}{\chi\Lambda(1)q^{g-1}} (b_+^\Lambda)^{r-k}(b_-^\Lambda)^k. \]

**Lemma 11.1.** Suppose $\Lambda = (\theta_1^{\lambda_1}, \ldots, \theta_m^{\lambda_m})$ where $d_{\theta_1} = \ldots = d_{\theta_m} = d \equiv 1 \mod 2$, and $\langle \theta_1, -1 \rangle_d = \ldots = \langle \theta_m, -1 \rangle_d = \pm 1$ are both constant for all $i = 1, \ldots, m$. Then
\[ b_+^\Lambda = \frac{1}{2} \left( \prod_{i=1}^m a_{\lambda_i}^+ + \prod_{i=1}^m a_{\lambda_i}^- \right) \quad \text{and} \quad b_-^\Lambda = \frac{1}{2} \left( \prod_{i=1}^m a_{\lambda_i}^+ + \prod_{i=1}^m a_{\lambda_i}^- \right) \]

**Proof.** Choose $\alpha \in L_1$ such that $\langle \alpha, -1 \rangle_1 = -1$. Tensoring by $\chi = \chi_{\alpha^n}$ sends $(\theta_1^{\lambda_1}, \ldots, \theta_m^{\lambda_m})$ to $((\alpha\theta_1)^{\lambda_1}, \ldots, (\alpha\theta_m)^{\lambda_m})$. Since $d_{\theta_i} = d$ is odd, we have (see [19])
\[ \langle \alpha\theta_i, -1 \rangle_d = \langle \alpha, -1 \rangle_d (\theta_i, -1 \rangle_d = \langle \alpha, -1 \rangle_d (\theta_i, -1 \rangle_d = -\langle \theta_i, -1 \rangle_d. \]

The formula follows from (3.3) and our earlier calculation of $a_\Lambda$. 

The following is proven in similar fashion to Theorem 10.2.

**Theorem 11.2.** Suppose that $(\Sigma, \tau)$ is genus $g$ Riemann surface with anti-holomorphic involution $\tau$ such that $\Sigma^\tau = \prod_i S^1$ is non-empty. Let $w : \pi_0(\Sigma^\tau) \to \{\pm 1\}$ send $k$ many elements to $-1$ where $k$ is odd. Then the E-polynomial of the path component $\mathcal{M}_{n,w}^\tau$ equals $E(\mathcal{M}_{n,w}^\tau; x, y) = E_n^k(xy)$ where
\[ E_n^k(q) := \frac{1}{2^r}(q - 1)(-q^{\frac{1}{2}})^{n^2(g-1)}V_n^k(q) \]

and
\[ V_n^k(q) := \sum_{\text{odd } d|n} (-1)^{m-1} \frac{\mu(d)}{d} \left( \sum_{m_\lambda|\lambda} \frac{m_\lambda!}{\prod_{\lambda} m_\lambda!} \right) \left( a_\sigma^+ + a_\sigma^- \right)^{r-k} \left( a_\sigma^+ - a_\sigma^- \right)^k H_{\sigma^r}^{q-1}(q^d) \]

where
\[ m := \sum_{\lambda} m_\lambda \quad a_\sigma^\pm := \prod_{\lambda} (a_{\lambda\sigma}^\pm)^{m_\lambda} \quad H_{\sigma^r}(q) := \prod_{\lambda} H_{\lambda}(q^{m_\lambda}). \]

Using the identity
\[ (x + y)^{r-k}(x - y)^k = \sum_{j=0}^r c_j x^{r-j} y^j \]
\[ c_j := \sum_{l=0}^j (-1)^l \binom{k}{l} \binom{r-k}{j-l} \]
we derive the generating function.

**Corollary 11.3.** With $V_n^k(q)$ as above we have
\[ \sum_{n=1}^\infty V_n^k(q) T^n = \Log \prod_{i=0}^r \prod_{j=0}^r \left( \sum_{\lambda} (a_{\lambda\sigma}^+)^{r-j}(a_{\lambda\sigma}^-)^j H_{\lambda}(q^2) T^{2|\lambda|} \right)^{c_j/2} \]

Let $A_0^g \cong (\mathbb{C}^*)^g$ be the identity component of $A^\tau$ (see §1.2.1).
Corollary 11.4. If $g \geq 2$, then the Euler characteristic of $\mathcal{M}_{n,w}/\mathcal{A}_0^g$ is zero if $n$ is even and $\mu(n)n^{g-2}$ if $n$ is odd.

Proof. The Euler characteristic equals

$$\chi(\mathcal{M}_{n,w}/\mathcal{A}_0^g) = \frac{E^k_n(q)}{(q-1)^g} |_{q=1}.$$

It is clear that from (10.3) that $H_\lambda(q^d)$ is divisible by $(q-1)$ for all partitions $\lambda$ with $|\lambda| \geq 1$ and is divisible by $(q-1)^2$ unless $\lambda = (1)$. Comparing with Theorem 11.2 we see that if $n$ is odd, and $g \geq 2$ then

$$\frac{E^k_n(q)}{(q-1)^g} |_{q=1} = \frac{1}{2^r(q-1)}(-q^{\frac{1}{2}}n^2(q-1))\mu(n)^n \left(a_1^+ + a_1^-(1)\right)^{r-k} \frac{H^{g-1}(q^n)}{(q-1)^g} |_{q=1} = \mu(n)n^{g-2}$$

whereas if $n$ is even then

$$\frac{E^k_n(q)}{(q-1)^g} |_{q=1} = 0.\quad \square$$

Example 11.1. We have

$$E^k_1(q) = (q-1)|G_1|^{g-1} = (q-1)^g.$$

$$\frac{E^k_2(q)}{(q-1)^g} = \frac{1}{2^r \left|G_2\right|} (2^r + (3+1)r-k(3-1)^k) \frac{q^{g-1}}{2(q+1)^{g-1}}$$

$$= (q^3 - q)^{g-1} + 2^{r-k}(q^2 - 1)^{g-1} - 2^{r-1}(q^2 - q)^{g-1}.$$

$$\frac{E^k_3(q)}{(q-1)^g} = (q^6 - q^3)^{g-1} + 2^r(q^3 - 1)^{g-1}(q^2 - 1)^{g-1}$$

$$+ 2^r(q^5 - q^2)^{g-1}(q+1)^{g-1} + \frac{4^r}{3}q^{3g-3}(q-1)^{2g-2}$$

$$- \frac{1}{3}(q^5 + q^4 + q^3)^{g-1} - 2^r(q^4 - q^3)^{g-1}(q^2 - 1)^{g-1} - 3^r(q^3 - q^2)^{g-1}(q^2 - 1)^{g-1}.$$
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