Legendre structure of the thermostatistics theory based on the
Sharma-Taneja-Mittal entropy

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Abstract

The statistical properties of complex systems can differ deeply for those of classical systems governed by Boltzmann-Gibbs entropy. In particular, the probability distribution function observed in several complex systems shows a power law behavior in the tail which disagrees with the standard exponential behavior showed by Gibbs distribution. Recently, a two-parameter deformed family of entropies, previously introduced by Sharma, Taneja and Mittal (STM), has been reconsidered in the statistical mechanics framework. Any entropy belonging to this family admits a probability distribution function with an asymptotic power law behavior. In the present work we investigate the Legendre structure of the thermostatistics theory based on this family of entropies. We introduce some generalized thermodynamical potentials, study their relationships with the entropy and discuss their main properties. Specialization of the results to some one-parameter entropies belonging to the STM family are presented.

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1. INTRODUCTION

Complex systems are ubiquitous in nature. Among the many, we can quote high energy and nuclear physics, turbulence, biophysics, geophysics, nano-systems, neural networks, but also earthquakes and volcanic activity, social sciences, economic sciences and others. In general these systems are governed by a nonlinear dynamics which establishes a deep relation among the parts of the system causing a strong correlation between them. As a consequence these systems reach a dynamical equilibrium rather than a statical equilibrium. Such equilibrium configuration (meta equilibrium) changes slowly in time if compared to the time scale of the underlying dynamic governing the system. Remarkably, this dynamical equilibrium can deeply differ from the statical equilibrium, in that the probability distribution function (pdf) shows a behavior different from the standard exponential one, typical of the Gibbs distribution. In particular the statistical distribution observed in several complex systems exhibit a power law behavior in the tail.

A way to study the statistical proprieties of these anomalous systems is based through the replacement of the well-know Boltzmann-Gibbs (BG) entropy with a generalized version of it. Very recently, a generalized entropy with two parameters, previously introduced in literature by Sharma and Taneja and Mittal, in the framework of the information theory, has been reconsidered in from a physical point of view. It has been shown that the formulation of a statistical mechanics based on the Sharma-Taneja-Mittal (STM) entropy preserve many aspects of the theory based on the BG entropy. In particular, it has been shown that it is positive definite, continuous, symmetric, expandable, decisive, maximal, concave, Lesche stable and fulfils a generalized Pesin-like identity. In Ref. the thermodynamics proprieties of this theory has been investigated in the microcanonical picture.

We remark that the STM entropy includes some one-parameter entropies already investigated in literature, like the ones introduced by Tsallis, by Abe and by Kaniadakis. Consequently, it permits us to consider all these one-parameter entropies in a unified scheme.

The purpose of this work it to explore the Legendre structure of the theory based on an entropy belonging to the STM family. In the BG theory, through its Legendre structure, we can introduce and establish some relationships among different thermodynamical potentials like, for instance, the free energy, the Massieu potential and the entropy. It follows that
the whole thermostatistics theory can be formulated equivalently by using one or the other of these quantities. In fact, depending by the physical problem we have to deal with, each of these potentials can be assumed as the more appropriate one for the study of the thermodynamical proprieties of the system \cite{12}.

It is then natural to ask if and how all these features can be generalized in presence of an entropy belonging to the STM family.

II. THE SHARMA-TANEJA-MITTAL ENTROPY

Let us recall the salient features of the STM entropy which can be written in \cite{5}

\[ S_{\kappa, r} = -\sum_i p_i \ln_{\kappa, r}(p_i) , \]

where \( p \equiv \{p_i\}_{i=1, \ldots, N} \) is a discrete pdf. Eq. (1) mimics the BG entropy through the replacement of the standard logarithm with its generalized version

\[ \ln_{\kappa, r}(x) = x^r \frac{x^\kappa - x^{-\kappa}}{2\kappa} , \]

where the deformation parameters \( \kappa \) and \( r \) are restricted to the two dimensional region \( \mathbb{R}^2 \supseteq \mathcal{R} = \{ -|\kappa| \leq r \leq |\kappa|, \text{if } 0 \leq |\kappa| < 1/2 \} \) and \( |\kappa| - 1 \leq r \leq 1 - |\kappa|, \text{if } 1/2 \leq |\kappa| < 1 \} \).

For any \( (\kappa, r) \in \mathcal{R} \), \( \ln_{\kappa, r}(x) = \ln_{\kappa, -r}(1/x) \) which, for \( r = 0 \), reproduces the well known propriety of the standard logarithm: \( \ln(x) = -\ln(1/x) \).

In Ref. \cite{5}, starting from the following functional-differential equation

\[ \frac{d}{dx} [x \Lambda(x)] = \lambda \Lambda \left( \frac{x}{\alpha} \right) , \]

obtained from certain physically justified assumptions, the two-parameter deformed logarithm \( \Lambda(x) \equiv \ln_{\kappa, r}(x) \) given in Eq. (2) has been derived, where the two constants \( \alpha \) and \( \lambda \) are given by

\[ \alpha = \left( \frac{1 + r - \kappa}{1 + r + \kappa} \right)^{1/2\kappa} , \quad \lambda = \frac{(1 + r - \kappa)(r + \kappa)/2\kappa}{(1 + r + \kappa)(r - \kappa)/2\kappa} . \]
and are related in $\ln_{\kappa, r} (1/\alpha) = 1/\lambda$. The logarithmic solution fulfils the boundary conditions $\Lambda(1) = 0$ and $(d/dx) \Lambda(x) \bigg|_{x=1} = 1$.

Another solution of Eq. (3), $\Lambda(x) \equiv u_{\kappa, r}(x)$, with boundary conditions $\Lambda(1) = 1$ and $(d/dx) \Lambda(x) \bigg|_{x=1} = r$, is given by

$$u_{\kappa, r}(x) = x^r \frac{x^\kappa + x^{-\kappa}}{2}.$$  

(5)

For any $(\kappa, r) \in \mathbb{R}$ the function $u_{\kappa, r}(x) = u_{-\kappa, r}(x)$ is continuous for $x \in \mathbb{R}^+$, with $u_{\kappa, r}(\mathbb{R}^+) \in \mathbb{R}^+$, $u_{\kappa, r}(0) = u_{\kappa, r}(+\infty) = +\infty$ for $r \neq |\kappa|$ and it satisfies the relations $u_{\kappa, r}(x) = u_{\kappa, -r}(1/x)$ and $u_{\kappa, r}(1/\alpha) = (1 + r)/\lambda$. Finally, it reduces to unity in the $(\kappa, r) \to (0, 0)$ limit: $u_{(0,0)}(x) = 1$.

Equations (2) and (5) can be written in the form

$$\ln_{\kappa, r}(x) = \frac{x^r}{\kappa} \sinh(\kappa \ln(x)), \quad u_{\kappa, r}(x) = x^r \cosh(\kappa \ln(x)),$$  

(6)

so that, many properties of $\ln_{\kappa, r}(x)$ and $u_{\kappa, r}(x)$ follow from the corresponding ones of $\sinh(x)$ and $\cosh(x)$. For instance, it is immediate to verify the following relations

$$\ln_{\kappa, r}(x y) = u_{\kappa, r}(x) \ln_{\kappa, r}(y) + \ln_{\kappa, r}(x) u_{\kappa, r}(y),$$  

(7)

$$u_{\kappa, r}(x y) = u_{\kappa, r}(x) u_{\kappa, r}(y) + \kappa^2 \ln_{\kappa, r}(x) \ln_{\kappa, r}(y),$$  

(8)

as a consequence of the additivity formulae of the hyperbolic functions.

In analogy with Eq. (1), for a given pdf, we introduce the function

$$I_{\kappa, r} = \sum_i p_i u_{\kappa, r}(p_i),$$  

(9)

which can be seen as the linear mean value of the function $u_{\kappa, r}(x)$, according to the relation $I = \langle u_{\kappa, r}(p) \rangle$, as well as the STM entropy can be defined as the linear mean value of $-\ln_{\kappa, r}(x)$, according to the relation $S_{\kappa, r} = -\langle \ln_{\kappa, r}(p) \rangle$.

We remark that Eq. (2) reduces to unity in the $(\kappa, r) \to (0, 0)$ limit: $I_{0,0} = \sum_i p_i = 1$ and likewise for an exact distribution $p^0 = \{0, \ldots, 1, 0, \ldots\}$: $I_{\kappa, r}(p^0) = 1$.

The discrete pdf associated to the entropy (1), under the constraints

$$\sum_i p_i = 1, \quad \sum_i E_i p_i = U,$$  

(10)
on the normalization and on the linear mean energy \( U \), can be obtained through the following variational problem:

\[
\frac{\delta}{\delta p_j} \left[ -\sum_i p_i \ln_{(\kappa, r)}(p_i) - \gamma \sum_i p_i - \beta \sum_i E_i p_i \right] = 0 ,
\]

(11)

where \( \gamma \) and \( \beta \), the Lagrange multipliers associated to the constraints (10), can be, in case, obtained from Eqs. (10) after we know the \( p_i \).

Accounting for Eq. (3), from Eq. (11) we obtain

\[
\lambda \ln_{(\kappa, r)} \left( \frac{p_j}{\alpha} \right) + \gamma + \beta E_j = 0 ,
\]

(12)

which gives the discrete pdf in the form

\[
p_j = \alpha \exp_{(\kappa, r)} \left( -\frac{\gamma + \beta E_j}{\lambda} \right) .
\]

(13)

In Eq. (13) we have introduced the two-parameter deformed exponential function \( \exp_{(\kappa, r)}(x) \), the inverse function of \( \ln_{(\kappa, r)}(x) \). Let us remark that, because \( \ln_{(\kappa, r)}(x) \) is a strictly monotonic function for any \( (\kappa, r) \in \mathbb{R} \), its inverse function certainly exists \( [5] \).

III. MASSIEU FUNCTIONS AND THERMODYNAMICAL POTENTIALS

In the classical thermostatistics, the thermodynamical potentials are defined by means of Legendre transformation on the mean energy. Although less known, another set of functions can be introduced by performing a Legendre transformation on the entropy. Such thermodynamical potentials are named Massieu functions \([12]\). For instance, the free energy \( F = U - S/\beta \) and the Massieu function \( \Phi = S - \beta U \) are obtained by means of Legendre transformation on \( U \) and \( S \), respectively, and are related each other through the relationship \( \Phi = -\beta F \). Let us explore such Legendre structure for a theory based on the STM entropy.

We start by using in Eq. (12) the relation (7), with \( x = p_i \) and \( y = 1/\alpha \), so that it follows

\[
(1 + r) \ln_{(\kappa, r)}(p_j) + u_{(\kappa, r)}(p_j) + \gamma + \beta E_j = 0 .
\]

(14)

By taking the average of Eq. (14) with respect to \( p_i \) we obtain

\[
S_{(\kappa, r)} = \frac{1}{1 + r} \left( T_{(\kappa, r)} + \gamma + \beta U \right),
\]

(15)
which recovers, in the $(\kappa, r) \to (0, 0)$ limit, the classical relationship $S = 1 + \gamma + \beta U$.

From Eq. (13) we can derive some useful properties concerning the STM entropy.

For instance, by recalling Eq. (3), we can write

$$\frac{d S_{\kappa, r}}{d U} = -\sum_i \frac{d}{dp_i} \left[ p_i \ln_{(\kappa, r)} (p_i) \right] \frac{dp_i}{dU} = -\lambda \sum_i \ln (\kappa, r) \left( \frac{p_i}{\alpha} \right) \frac{dp_i}{dU},$$

and taking into account the expression of the $p_i$ we obtain

$$\frac{d S_{\kappa, r}}{d U} = \sum_i (\gamma + \beta E_i) \frac{dp_i}{dU}.$$  \hspace{1cm} (16)

By assuming the “no work” condition $\sum_i p_i dE_i = 0$, which implies $dU = \sum_i E_i dp_i$, and taking into account that $\sum_i dp_i = 0$, as it follows from the normalization on $p_i$, we obtain

$$\frac{d S_{\kappa, r}}{d U} = \beta.$$  \hspace{1cm} (17)

From this equation we see that $S_{\kappa, r}$ is a function of the mean energy $U$ and that, like in the BG theory, $\beta$ and $U$ are variables canonically conjugated.

The generalized Massieu potential $\Phi_{\kappa, r}$ can be introduced by performing a Legendre transformation on the entropy:

$$\Phi_{\kappa, r} = S_{\kappa, r} - \frac{d S_{\kappa, r}}{d U} U \equiv S_{\kappa, r} - \beta U,$$  \hspace{1cm} (19)

and after using Eq. (13) in Eq. (19) we obtain

$$\Phi_{\kappa, r} = \frac{1}{1 + r} \left( T_{\kappa, r} + \gamma - r \beta U \right).$$  \hspace{1cm} (20)

It is trivial to verify the validity of the relation

$$\frac{d \Phi_{\kappa, r}}{d \beta} = -U,$$  \hspace{1cm} (21)

as it follows readily by using Eqs. (18) and (19). Equation (21) still states that $\beta$ and $U$ are canonically conjugated variables and that $\Phi_{\kappa, r}$ is, as matter of fact, a function of $\beta$.

Finally, we observe that, if one is welling to considers the free energy as a function of $1/\beta$, the generalized free energy $F_{\kappa, r}$ can be introduced, through a Legendre transformation on $U$:

$$F_{\kappa, r} = U - \frac{d U}{d S_{\kappa, r}} S_{\kappa, r} \equiv U - \frac{1}{\beta} S_{\kappa, r}.$$  \hspace{1cm} (22)
By using Eq. (15) in Eq. (22) we obtain

\[ F_{\kappa, r} = -I_{\kappa, r} + \gamma - r \beta U \left(1 + r \right). \]  

Moreover, it results

\[ F_{\kappa, r} = -\Phi_{\kappa, r} / \beta, \]  

like in the standard thermostatistics theory, from which it is easy to show that

\[ \frac{dF_{\kappa, r}}{d(1/\beta)} = -S_{\kappa, r}, \]  

imitating in this way the classical relationships between the free energy and the entropy.

\section*{IV. CANONICAL PARTITION FUNCTION}

Let us introduce the generalized canonical partition function \( Z_{\kappa, r} \) through the relation

\[ \ln \{Z_{\kappa, r}\} = \frac{1}{1 + r} \left( I_{\kappa, r} + \gamma - r \beta U \right), \]  

so that, by inserting Eq. (26) into Eq. (15) we obtain

\[ S_{\kappa, r} = \ln \{Z_{\kappa, r}\} + \beta U, \]  

which mimics the standard relation \( S = \ln(Z) + \beta U \).

We notice that some authors prefer to introduce a partition function \( \overline{Z}_{\kappa, r} \) which refers to the energy levels \( \{E_i\} \) with regards to \( U \). The two different definitions are related each to the other by

\[ \ln \{Z_{\kappa, r}\} = \ln \{\overline{Z}_{\kappa, r}\} + \beta U, \]  

so that Eq. (27) simplifies in \( S_{\kappa, r} = \ln \{\overline{Z}_{\kappa, r}\} \).

In order to verify the consistence of the definition (26) let us evaluate the following derivative

\[ \frac{dI_{\kappa, r}}{d\beta} = \sum_i \frac{d}{dp_i} \left[ u_{\kappa, r} \left( p_i \right) \right] \frac{dp_i}{d\beta} = \lambda \sum_i u_{\kappa, r} \left( \frac{p(x_i)}{\alpha} \right) \frac{d(x_i)}{d\beta}, \]  

where we have posed \( p(x_i) \equiv p_i \) and \( x_i = \gamma + \beta E_i \). By using the relation

\[ \lambda u_{\kappa, r} \left( \frac{p(x)}{\alpha} \right) dp(x) = r x_i dp(x_i) - p(x_i) dx_i, \]
as it follows by deriving Eq. (13), we can rewrite
\[
\frac{d \mathcal{I}_{\kappa, \gamma}}{d \beta} = r \sum_i x_i \frac{dp(x_i)}{d \beta} - \sum_i p(x_i) \frac{dx_i}{d \beta} = 
\]
\[
= r \frac{d}{d \beta} \left( \sum_i p(x_i) x_i \right) - (1 + r) \sum_i p(x_i) \frac{dx_i}{d \beta} = 
\]
\[
= r \frac{d}{d \beta} (\gamma + \beta U) - (1 + r) \frac{d \gamma}{d \beta} - (1 + r) U = 
\]
\[
= \beta \frac{d U}{d \beta} - \frac{d \gamma}{d \beta} - U . 
\] (31)

On the other hand, by taking the derivative of Eq. (26) with respect to \( \beta \), we obtain
\[
\frac{d}{d \beta} \ln \left( \kappa, r \right) \left( Z_{\kappa, r} \right) = \frac{1}{1 + r} \left( \frac{d \mathcal{I}_{\kappa, r}}{d \beta} + \frac{d \gamma}{d \beta} - r U - r \beta \frac{d U}{d \beta} \right) , 
\] (32)
and accounting for Eq. (31) it follows
\[
\frac{d}{d \beta} \ln \left( \kappa, r \right) \left( Z_{\kappa, r} \right) = -U , 
\] (33)
according to the classical relationship \( d \ln(Z)/d \beta = -U \).

We remark that, by comparing Eq. (20) with Eq. (26) it follows
\[
\Phi_{\kappa, r} = \ln \left( \kappa, r \right) \left( Z_{\kappa, r} \right) 
\] (34)
so that, accounting for Eq. (33), we recover again Eq. (21).

Finally, we recall that in the classical statistical mechanics the canonical partition function encodes all the statistical proprieties of the system. This feature also holds in the generalized theory under investigation. In fact, by assuming \( \beta \simeq constant \) for a long period of time, we have
\[
\frac{d}{d E_i} \ln \left( \kappa, r \right) \left( Z_{\kappa, r} \right) = \frac{1}{1 + r} \left( \frac{d \mathcal{I}_{\kappa, r}}{d E_i} + \frac{d \gamma}{d E_i} - r \beta \frac{d U}{d E_i} \right) , 
\] (35)
and following the same argument used in Eq. (31) it follows
\[
\frac{d \mathcal{I}_{\kappa, r}}{d E_i} = -(1 + r) \beta p_i - \frac{d \gamma}{d E_i} + r \beta \frac{d U}{d E_i} , 
\] (36)
so that, from Eq. (35), we obtain
\[
p_i = -\frac{1}{\beta \frac{d}{d E_i} \ln \left( \kappa, r \right) \left( Z_{\kappa, r} \right) } , 
\] (37)
i.e., the equilibrium distribution can be derived equivalently through the generalized canonical partition function.
V. PARTICULAR CASES

Let us specify our results to some relevant one-parameter deformed entropies belonging to the STM family.

As a first example, we choose \( r = 0 \). From Eq. (1) we obtain the entropy proposed by Kaniadakis [11]:

\[
S_\kappa = - \sum_i p_i^\kappa - p_i^{-\kappa}.
\]  (38)

It was conjectured that this entropy emerges naturally in the context of the special relativity. Among the many possible applications, the entropy (38) has been employed in the reproduction of the energy distribution of the fluxes of cosmic rays [11] and in the study of the fracture propagation in brittle materials [14], showing a good agreement with the data observed both experimentally and through numerical simulation.

Starting from Eq. (38) and from its related function

\[
\mathcal{I}_\kappa = \sum_i p_i^\kappa + p_i^{-\kappa},
\]  (39)

we can define the canonical \( \kappa \)-partition function \( Z_\kappa \) as

\[
\ln(Z_\kappa) = S_\kappa - \beta U \equiv \mathcal{I}_\kappa + \gamma.
\]  (40)

Remark that, by introducing the function \( Z_\kappa \), by means of Eq. (28), we obtain the relation

\[
(Z_\kappa)^\kappa = \kappa S_\kappa + \sqrt{1 + \kappa^2 S_\kappa^2}.
\]  (41)

Finally, the expressions of the \( \kappa \)-Massieu potential \( \Phi_\kappa \) and the \( \kappa \)-free energy \( F_\kappa \) are given, respectively, by

\[
\Phi_\kappa = \ln(Z_\kappa) \equiv \mathcal{I}_\kappa + \gamma,
\]  (42)

\[
F_\kappa = -\frac{1}{\beta} \ln(Z_\kappa) \equiv -\frac{1}{\beta}(\mathcal{I}_\kappa + \gamma).
\]  (43)

As a second example we pose in Eq. (1) \( r = \pm |\kappa| \) and after introducing the parameter \( q = 1 \mp 2 |\kappa| \) we obtain

\[
S_{2-q} = \sum_i \frac{p_i^{2-q} - p_i}{q - 1},
\]  (44)

which coincides with the Tsallis entropy in the “2 – q formalism” [15].

After its introduction, entropy (44) has been widely applied, as a paradigm, in the study of
the statistical proprieties of complex systems showing a pdf with a power law behavior in
the tail $q$.

The canonical $q$-partition function $Z_{2-q}$ associated to the entropy $S_{2-q}$ is defined by

$$\ln_{2-q} \left( Z_{2-q} \right) = S_{2-q} - \beta U \equiv \frac{2}{3-q} \left( \mathcal{I}_{2-q} + \gamma \right) + \frac{q-1}{3-q} \beta U ,$$

(45)

where the function $\mathcal{I}_{2-q}$ takes the expression

$$\mathcal{I}_{2-q} = \sum_i p_i^{2-q} + p_i ,$$

(46)

and by introducing the function $\overline{Z}_{2-q}$, through Eq. (28), we obtain the relation

$$\overline{Z}_{2-q}^{q-1} = 1 + (q-1) S_{2-q} ,$$

(47)

according to the results reported in [15].

From definition (45) we readily obtain the $q$-deformed thermodynamical potentials corresponding to the Massieu potential $\Phi_{2-q}$ and to the free energy $F_{2-q}$, given respectively by

$$\Phi_{2-q} = \ln_{2-q} \left( Z_{2-q} \right) \equiv \frac{2}{3-q} \left( \mathcal{I}_{2-q} + \gamma \right) + \frac{q-1}{3-q} \beta U ,$$

(48)

$$F_{2-q} = -\frac{1}{\beta} \ln_{2-q} \left( Z_{2-q} \right) \equiv -\frac{2}{(3-q) \beta} \left( \mathcal{I}_{2-q} + \gamma \right) - \frac{q-1}{3-q} U .$$

(49)

VI. CONCLUSIONS

In the present work we have analyzed some aspects of the thermostatistics theory based on the two-parameter deformed Sharma-Taneja-Mittal entropy. In particular, we have studied the Legendre structure of the theory by introducing consistently some generalized thermodynamical functions like the canonical partition function, the free energy and the Massieu potential, and we have analyzed the relationships among these functions and the entropy. All the theoretical structure collapse, in the $(\kappa, r) \to (0, 0)$ limit, to the standard theory based on the BG entropy.

We recall that the pdf associated to the thermodynamical potentials introduced in this paper is characterized by an asymptotic power law behavior. For this reason, the theory under scrutiny is expected to be relevant in the study of the thermostatistics proprieties of those complex systems exhibiting such behavior in the observed pdf.
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