Cooling of a Mechanical Oscillator and Normal Mode Splitting in Optomechanical Systems with Coherent Feedback

Sumei Huang and Aixi Chen *

Department of Physics, Zhejiang Sci-Tech University, Hangzhou 3100018, China

* Correspondence: aixichen@zstu.edu.cn

Received: 26 July 2019; Accepted: 16 August 2019; Published: 19 August 2019

Abstract: The ground state cooling of a mechanical oscillator and strong optomechanical coupling are necessary prerequisites for realizing quantum control of the macroscopic mechanical oscillator. Here, we show that the resolved-sideband cooling of a mechanical oscillator in an optomechanical system can be enhanced by a simple coherent feedback scheme, in which a portion of the output field from the cavity is fed back into the cavity using an asymmetric beam splitter. Moreover, we show that the normal mode splitting in the spectra of the movable mirror and the output field in a weakly coupled optomechanical system can be induced by the feedback scheme due to a reduced effective cavity decay rate. We find that the peak separation becomes larger and two peaks of the spectra become narrower and higher with increasing the reflection coefficient $r$ of the beam splitter.

Keywords: cooling of the movable mirror; normal mode splitting; optomechanical system

1. Introduction

In recent years, optomechanical systems have recently been the focus of extensive investigations due to their potential applications in precision measurement of tiny displacements and forces [1]. Such systems enable us to observe quantum behaviors at the macroscale if the mechanical oscillators can be cooled close to the quantum ground state [2–5]. It has been shown that the radiation pressure acting on the mechanical oscillator exerted by the photons in the cavity can lead to the cooling of the mechanical oscillator [6–9]. It has been demonstrated theoretically [10,11] and experimentally [12–17] that the mechanical oscillator can be cooled close to its quantum ground state in the resolved sideband limit, in which the mechanical frequency is much larger than the cavity decay rate. Moreover, the feedback control has been introduced to the optomechanical systems to effectively cool the mechanical oscillator below the environmental temperature in the unresolved sideband regime [18–24]. The feedback cooling is based on applying a viscous feedback force on the mechanical oscillator to hinder its thermal motion.

On the other hand, the normal mode splitting in the mechanical and optical fluctuation spectra is a signature of the strong coupling between the cavity field and the mechanical oscillator [25]. The strong coupling in the optomechanical system would allow for the preparation of mechanical quantum states [2–4]. Many efforts have been devoted to studying the normal mode splitting of the optomechanical systems theoretically [25–29] and experimentally [30]. It has been investigated that a degenerate optical parametric amplifier in an optomechanical system can produce stronger coupling between the cavity field and the movable mirror so that larger normal mode splitting is observed [26]. Besides, it has been found that the strong coupling can be achieved between two indirectly coupled mechanical and optical modes in an optomechanical system with one mechanical mode and two optical modes [27]. In addition, the two-level atoms embedded into the optomechanical cavity is helpful...
to generate the normal mode splitting [28]. Moreover, the normal mode splitting can be observed in a weakly coupled electromechanical system by modulating the spring constant of the mechanical oscillator [29].

In this paper, we propose a coherent feedback scheme to improve the cooling of a movable mirror in the resolved sideband limit. The feedback scheme is realized by reflecting a part of the output field from the cavity back into the cavity via an asymmetric beam splitter. Moreover, we find that the feedback scheme can lead to the occurrence of the normal mode splitting in the spectra of the movable mirror and the output field when the movable mirror and the cavity field are initially weakly coupled to each other, owing to a decrease of the effective cavity decay rate in the presence of the feedback.

The outline of the paper is as follows. In Section 2, we present the Hamiltonian for the system, give the equations of motion of the system operators, calculate the expectation values of the system operators, obtain the linearized quantum Langevin equations. In Section 3, we analyze how the cooling of the moving mirror can be enhanced in the presence of the feedback. In Section 4, we discuss the normal mode splitting in the mechanical and output optical fluctuation spectra with the feedback.

2. Model

We consider an optical cavity with two mirrors separated by a distance \( L \), as depicted in Figure 1. An input laser at frequency \( \omega_l \) and amplitude \( \varepsilon_l \) is split into two parts by an asymmetric beam splitter which has a \( 2 \times 2 \) scattering matrix [31]

\[
\begin{pmatrix}
t & r \\
r & t
\end{pmatrix}
\quad \text{(1)}
\]

where \( r \) and \( t \) are the amplitude reflection and transmission coefficients of the beam splitter, respectively, they are real, and satisfy the relation \( r^2 + t^2 = 1 \), thus there is no absorption within the beam splitter itself. We assume that \( t \) is positive. Hence the transmitted part of the input laser has the amplitude \( t\varepsilon_l \), the reflected part of the input laser has the amplitude \( -r\varepsilon_l \). A cavity field \( c \) at frequency \( \omega_c \) is pumped by the transmitted part of the input laser. The movable mirror with 100% reflectivity is coupled to the cavity field via radiation pressure. Moreover, a part of the output field from the cavity is fed back into the cavity via a totally reflecting mirror and the asymmetric beam splitter. If \( r = 0 \) and \( t = 1 \), the input laser is totally sent into the cavity, and the output field is not fed back into the cavity.

The Hamiltonian of the system in a frame rotating at the laser frequency \( \omega_l \) has a form

\[
H = \hbar (\omega_c - \omega_l) c^\dagger c + \frac{1}{2} \hbar \omega_m (Q^2 + P^2) - \hbar g_0 c^\dagger c Q + i \hbar t \varepsilon_l (c^\dagger - c),
\quad \text{(2)}
\]
where $Q$ and $P$ are the dimensionless position and momentum operators with $Q = \sqrt{\frac{m_0}{\hbar}} q$, $P = \frac{p}{\sqrt{m_0 \hbar}}$, and $[Q, P] = i$. In Equation (2), the first term is the energy of the cavity field, the second term is the energy of the movable mirror with the effective mass $m$ and resonance frequency $\omega_m$, the third term describes the optomechanical interaction between the cavity field and the movable mirror, $g_0 = \frac{\omega}{2 \sqrt{\frac{\hbar}{m_0 \omega_m}}}$ is the single-photon optomechanical coupling strength, the last term represents the cavity field driven by a part of the input laser which is transmitted by the beam splitter, and the laser amplitude $\epsilon_l$ depends on the laser power $\varphi$ by $\epsilon_l = \sqrt{\frac{2 \bar{P}}{\omega_m}}$, where $\kappa$ is the decay rate of the cavity mode due to the light escaping the cavity from the fixed mirror with partial reflectivity.

The time evolution of the system operators can be obtained by using the Heisenberg equation of motion, adding the corresponding damping and noise terms, and taking into account the feedback term. Thus the quantum Langevin equations of the system are given by

$$Q = \omega_m P,$$
$$\dot{P} = -\omega_m Q + g_0 c^\dagger c - \gamma_m P + \xi,$$
$$\dot{\xi} = -[\kappa + i(\omega_c - \omega_l)]c + i g_0 c^\dagger Q + i \epsilon_l + \sqrt{2 \kappa}(1 \epsilon_{in} + r c_{out}).$$  \hspace{1cm} (3)

Here $\gamma_m$ is the damping rate of the movable mirror, $\xi$ is the Brownian stochastic force due to the interaction of the movable mirror with the thermal bath at temperature $T$. The $\xi$ has zero mean value and obeys the two-time non-Markovian correlation function

$$\langle \xi(\tau) \xi'(\tau') \rangle = \frac{1}{2 \pi \omega_m} \int \omega e^{-i \omega (\tau - \tau')} \left[ 1 + \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right] d\omega,$$  \hspace{1cm} (4)

where $k_B$ is the Boltzmann constant. The $c_{in}$ is the optical vacuum noise operator that is incident upon the beam splitter, it has zero mean value and is delta-correlated as

$$\langle c_{in}(\tau) c^\dagger_{in}(\tau') \rangle = \delta(\tau - \tau').$$  \hspace{1cm} (5)

Here, $\epsilon_{in}$ represents a part of the input vacuum noise $c_{in}$ which is transmitted by the beam splitter to enter the cavity. Moreover, $r c_{out}$ represents a part of the output field $c_{out}$ which is reflected by the beam splitter to enter the cavity. The output field $c_{out}$ is related to the cavity field $c$ via the input–output relation $c_{out} = \sqrt{2 \kappa c} - \epsilon_{in}$, which differs from the standard input–output relation [32] due to the presence of the beam splitter. Then the Equation (3) can be written as

$$\dot{Q} = \omega_m P,$$
$$\dot{P} = -\omega_m Q + g_0 c^\dagger c - \gamma_m P + \xi,$$
$$\dot{\xi} = -[\kappa_{eff} + i(\omega_c - \omega_l)]c + i g_0 c^\dagger Q + i \epsilon_l + \sqrt{2 \kappa}(1 \epsilon_{in} + r c_{out}).$$  \hspace{1cm} (6)

For the time evolution of the cavity field in Equation (6), $\kappa_{eff} = \kappa(1 - 2r)$ is the effective cavity decay rate, we note that the input vacuum noise term always accompanies the cavity damping term, thus $r$ must be less than $\frac{1}{2}$ so that $\kappa_{eff} > 0$. In the steady state, the solution to Equation (6) is given by

$$P_s = \frac{g_0}{\omega_m^2} |c_s|^2,$$
$$Q_s = \frac{g_0}{\omega_m^2} \sqrt{\kappa_{eff} + 2 \kappa} |c_s|^2,$$
$$c_s = \frac{g_0}{\kappa_{eff} + i \kappa},$$  \hspace{1cm} (7)

where $\Delta = \omega_c - \omega_l - g_0 Q_s$ is the effective frequency detuning of the cavity field with respect to the input laser, including the frequency shift $g_0 Q_s$ due to the radiation pressure. It is noted that the steady-state amplitude $c_s$ of the cavity mode is related to the reflection and transmission coefficients $(r, t)$ of the beam splitter, the steady-state displacement $Q_s$ of the movable mirror depends on the photon number $|c_s|^2$ in the cavity, and the steady-state momentum $P_s$ of the movable mirror is zero.
We deal here with the case that the cavity field is driven by a strong laser so that the photon number in the cavity is large ($|c_s|^2 \gg 1$). In this case, the nonlinear quantum Langevin Equation (6) can be linearized by writing each operator of the system in Equation (6) as $Q = Q_0 + \delta Q$, $P = P_0 + \delta P$, $c = c_s + \delta c$, where $\delta Q$, $\delta P$, and $\delta c$ represent the small fluctuations with zero mean value around the steady-state mean values $Q_0$, $P_0$, and $c_s$, respectively. The linearized quantum Langevin equations for the fluctuation operators are found to be

$$
\begin{align*}
\delta Q &= \omega_m \delta P, \\
\delta P &= -\omega_m \delta Q + \frac{1}{\sqrt{2}}(G^\dagger \delta c + G \delta c^\dagger) - \gamma_m \delta P + \xi, \\
\delta c &= -(\kappa_{\text{eff}} + i\Delta)\delta c + \frac{i}{\sqrt{2}}G\delta Q + \sqrt{2}\kappa(1 - r)\delta_{in},
\end{align*}
$$

(8)

where $G = \sqrt{2}G_0c_s$ is the light-enhanced optomechanical coupling strength, being related to the input laser power. By introducing the quadrature operators for the optical mode $\delta x = \frac{1}{\sqrt{2}}(\delta c + \delta c^\dagger)$, $\delta y = \frac{1}{\sqrt{2}}(\delta c - \delta c^\dagger)$ and the quadrature operators for the input vacuum noise $x_{in} = \frac{1}{\sqrt{2}}(c_{in} + c_{in}^\dagger)$, $y_{in} = \frac{1}{\sqrt{2}}(c_{in} - c_{in}^\dagger)$, we can write Equation (8) as

$$
f(\tau) = Mf(\tau) + n(\tau),
$$

(9)

where $f(\tau) = (\delta Q, \delta P, \delta x, \delta y)^T$ is the vector of the fluctuation operators, $n(\tau) = (0, \xi, \sqrt{2}\kappa(1 - r)t_{x, in}, \sqrt{2}\kappa(1 - r)t_{y, in})^T$ is the vector of the noise operators, and $M$ is the $4 \times 4$ matrix given by

$$
M = \begin{pmatrix}
0 & \omega_m & 0 & 0 \\
-\omega_m & -\gamma_m & u & v \\
-v & 0 & -\kappa_{\text{eff}} & \Delta \\
u & 0 & -\Delta & -\kappa_{\text{eff}}
\end{pmatrix}
$$

(10)

with $u = \frac{1}{2}(G + G^\dagger)$, $v = \frac{1}{2}i(G^\dagger - G)$. The system is said to be stable if the real part of each eigenvalue of the matrix $M$ is negative. The stability conditions of the system can be derived by using the Routh–Hurwitz criterion [33], which are found to be

$$
\begin{align*}
2\kappa_{\text{eff}} + \gamma_m &> 0, \\
2\kappa_{\text{eff}}(\Delta^2 + \kappa_{\text{eff}}^2 + 2\kappa_{\text{eff}}\gamma_m) + \gamma_m(\omega_m^2 + 2\kappa_{\text{eff}}\gamma_m) &> 0, \\
[2\kappa_{\text{eff}}(\Delta^2 + \kappa_{\text{eff}}^2 + 2\kappa_{\text{eff}}\gamma_m) + \gamma_m(\omega_m^2 + 2\kappa_{\text{eff}}\gamma_m)]\{\gamma_m(\Delta^2 + \kappa_{\text{eff}}^2) + 2\kappa_{\text{eff}}\omega_m^2\} &> 0, \\
-(\Delta \omega_m[\Delta \omega_m - (u^2 + v^2)] + \kappa_{\text{eff}}\omega_m^2)(2\kappa_{\text{eff}} + \gamma_m)^2 &> 0, \\
\Delta \omega_m[\Delta \omega_m - (u^2 + v^2)] + \kappa_{\text{eff}}\omega_m^2 &> 0.
\end{align*}
$$

(11) (12) (13) (14)

As mentioned before, the reflection coefficient $r$ of the beam splitter must be less than $\frac{1}{2}$. Under this condition ($r < \frac{1}{2}$), the stability conditions Equations (11) and (12) are automatically satisfied. Thus the system stability is determined by $r < \frac{1}{2}$ and Equations (13) and (14). In the following, we choose the appropriate parameters so that the system is operated in the stable regime.

3. Feedback Cooling of the Movable Mirror

In this section, we show how the feedback scheme affects the cooling of the movable mirror. Fourier transforming of all the fluctuation and noise operators in Equation (8) by $f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)e^{-i\omega\tau}d\omega$ and solving it, we find the position fluctuation of the movable mirror

$$
\delta Q(\omega) = A_1(\omega)c_{in}(\omega) + A_2(\omega)c_{in}^\dagger(-\omega) + A_3(\omega)\xi(\omega),
$$

(15)
where
\[
\begin{align*}
A_1(\omega) &= \frac{\omega_m}{\omega_m^2} \sqrt{(1 - r)} G^* [\kappa_{\text{eff}} - i(\Delta + \omega)], \\
A_2(\omega) &= \frac{\omega_m}{\omega_m^2} \sqrt{(1 - r)} G [\kappa_{\text{eff}} + i(\Delta - \omega)], \\
A_3(\omega) &= \frac{\omega_m}{\omega_m^2} [(\kappa_{\text{eff}} - i\omega)^2 + \Delta^2], \\
d(\omega) &= \frac{\omega_m}{\omega_m^2} [(\omega_m^2 - \omega^2 - i\gamma_m\omega)(\kappa_{\text{eff}} - i\omega)^2 + \Delta^2] - \omega_m\Delta|G|^2.
\end{align*}
\]  

In Equation (15), the term dependent on $\xi(\omega)$ is the contribution of the thermal noise of the movable mirror, the other two terms are the contribution of radiation pressure. In the absence of the optomechanical coupling ($G = 0$), Equation (15) is reduced to $\delta Q(\omega) = \frac{\omega_m}{\omega_m^2 - \omega^2 - i\gamma_m\omega} \xi(\omega)$, the position fluctuation of the movable mirror is only determined by the thermal noise, hence the movable mirror is undergoing Brownian motion due to its contact with the thermal environment. Furthermore, the position spectrum of the movable mirror is defined by
\[
2\pi S_Q(\omega)\delta(\omega + \Omega) = \frac{1}{2}[(\delta Q(\omega)\delta Q(\Omega)) + (\delta Q(\Omega)\delta Q(\omega))].
\]  

Applying the Fourier transform to Equations (4) and (5) yields
\[
\begin{align*}
\langle c_{in}(\omega)c_{in}^\dagger(-\Omega) \rangle &= 2\pi \delta(\omega + \Omega), \\
\langle \xi(\omega)\xi(\Omega) \rangle &= 2\pi \frac{\omega_m}{\omega_m^2}\omega [1 + \coth(\frac{\hbar\omega}{2k_BT})] \delta(\omega + \Omega).
\end{align*}
\]  

Combining Equation (17) with Equation (18), we find the position spectrum of the movable mirror
\[
S_Q(\omega) = \frac{1}{2} A_1(\omega)A_2(-\omega) + \frac{1}{2} A_2(\omega)A_1(-\omega) + \frac{\gamma_m}{\omega_m}\coth(\frac{\hbar\omega}{2k_BT}) A_3(\omega) A_3(-\omega).
\]  

It is noted that the position spectrum $S_Q(\omega)$ of the movable mirror depends on the radiation pressure (the first two terms) and the thermal noise of the movable mirror (the last term). Next we get the momentum fluctuation of the movable mirror $\delta P(\omega) = -\frac{\omega_m}{\omega_m^2}\delta Q(\omega)$ by taking the Fourier transform of the fluctuation operators in Equation (8). Thus the momentum spectrum of the movable mirror is found to be
\[
S_P(\omega) = \frac{\omega_m^2}{\omega_m^2} S_Q(\omega).
\]  

The position and momentum variances $\langle \delta Q^2 \rangle$ and $\langle \delta P^2 \rangle$ of the movable mirror are determined by
\[
\begin{align*}
\langle \delta Q^2 \rangle &= \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega S_Q(\omega), \\
\langle \delta P^2 \rangle &= \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega S_P(\omega).
\end{align*}
\]  

The interesting quantity is the effective mean phonon number $n_{\text{eff}}$ of the movable mirror, which can be obtained from the stationary mean energy of the mechanical oscillator given by
\[
\frac{\hbar\omega_m}{2} [\langle \delta Q^2 \rangle + \langle \delta P^2 \rangle] = \hbar\omega_m(n_{\text{eff}} + \frac{1}{2}).
\]  

We find
\[
n_{\text{eff}} = \frac{1}{2} [\langle \delta Q^2 \rangle + \langle \delta P^2 \rangle - 1].
\]  

When $n_{\text{eff}} = 0$, the movable mirror is in its quantum ground state.

We use the experimentally accessible parameters given in [34]: the wavelength of the external laser $\lambda = \frac{2\pi}{\omega_l} = 1554.35$ nm, the laser power $P = 0.01$ mW, the decay rate of the cavity $\kappa = 2\pi \times 423$ MHz, the single-photon optomechanical coupling strength $g_0 = 2\pi \times 869 \times \sqrt{2}$ kHz, the resonant frequency
of the movable mirror $\omega_m = 2\pi \times 5.25 \text{ GHz}$, the quality factor of the movable mirror $Q_m = 3.8 \times 10^5$, the mechanical damping rate $\gamma_m = \omega_m / Q_m = 2\pi \times 13.816 \text{ kHz}$. Due to $\kappa / \omega_m \simeq 0.08$, the system is deep in the resolved sideband regime. The environment is initially at the temperature $T = 300 \text{ K}$, thus the corresponding thermal phonon number of the movable mirror is $n_{th}^m = \{\exp[\hbar \omega_m / (k_B T)] - 1\}^{-1} = 1189.61$.

The Figure 2 shows that the effective mean phonon number $n_{eff}$ of the movable mirror versus the normalized effective cavity detuning $\Delta / \omega_m$ for different reflection coefficients $r = 0, 0.2, 0.3, 0.46$. In the absence of the feedback ($r = 0$), the effective mean phonon number $n_{eff}$ has the minimum value 11.97 at $\Delta / \omega_m = 0.994$. In the presence of the feedback ($r \neq 0$), with increasing the reflection coefficient $r$ of the beam splitter, the minimum value of $n_{eff}$ is decreased. For $r = 0.46$, the minimum value of $n_{eff}$ is 2.40 at $\Delta / \omega_m = 1$, thus the cooling of the movable mirror can be improved by a factor of about 5.0 with respect to the case without the feedback. Therefore, the addition of the feedback to the optomechanical system is able to improve the cooling of the movable mirror.

![Figure 2](image-url) The effective mean phonon number $n_{eff}$ of the movable mirror versus the normalized effective cavity detuning $\Delta / \omega_m$ for different reflection coefficients $r = 0$ (black solid), 0.2 (blue dashed), 0.3 (green dot-dashed), 0.46 (red dotted).

4. Normal Mode Splitting in the Spectra of the Movable Mirror and the Output Field in the Presence of the Feedback

In this section, we show the influence of the feedback on the position spectrum of the movable mirror and the spectral density of the output field.

From Equation (8), the fluctuation $\delta c(\omega)$ of the cavity field can be obtained. Then the fluctuation $\delta c_{out}(\omega)$ of the output field can be found according to the input–output relation

$$\delta c_{out}(\omega) = \sqrt{2}\delta c(\omega) - tc_{in}(\omega),$$

which is given by

$$\delta c_{out}(\omega) = \begin{array}{l}
B_1(\omega)c_{in}(\omega) + B_2(\omega)c_{in}^\dagger(-\omega) + B_3(\omega)\xi(\omega),
\end{array} (24)$$

where

$$B_1(\omega) = \frac{2(1-r)\hbar}{\hbar(\omega)} \{ (\omega_m^2 - \omega^2 - i\gamma_m \omega)[\kappa_{eff} - i(\Delta + \omega)] + \frac{1}{2} |G|^2 \omega_m \} - t,$$

$$B_2(\omega) = \frac{1-r}{2}\omega_m |G|^2 \omega_m,$$

$$B_3(\omega) = \frac{\sqrt{\gamma_m}}{\hbar(\omega)} [\kappa_{eff} - i(\Delta + \omega)].$$

In Equation (24), the first two terms arise from the input vacuum noise, the last term originates from the thermal noise of the movable mirror. The spectral density of the output field is defined as

$$\langle \delta c_{out}^\dagger(-\Omega) \delta c_{out}(\omega) \rangle = 2\pi S_{out}(\omega)\delta(\omega + \Omega).$$
Using Equations (18) and (26), we calculate the output spectrum as

\[
S_{out}(\omega) = B_2(\omega)B^*_2(\omega) + B_3(\omega)B^*_3(\omega)\frac{\gamma_m}{\omega_m}\omega\left[-1 + \coth\left(\frac{\hbar\omega}{2k_B T}\right)\right].
\] (27)

It is seen that the output spectrum \(S_{out}(\omega)\) is the sum of two terms associated respectively with the input vacuum noise and the thermal noise of a movable mirror.

It is known that the normal mode splitting is observable when the optomechanical system enters the strong coupling regime \(|G| > \kappa_{eff}\) [25,27]. For \(0 \leq r < \frac{1}{2}\), it is noted that the effective cavity decay rate \(\kappa_{eff} = \kappa(1 - 2r)\) is decreased with increasing the reflective coefficient \(r\) of the beam splitter. Therefore, in the presence of the feedback (\(0 \leq r < \frac{1}{2}\)), it is possible to make the value of \(|G|/\kappa_{eff}\) larger than unity even when the value of \(|G|/\kappa_{eff}\) is less than unity in the absence of the feedback.

We still use the parameters from the recent experiment [34]: \(\lambda = \frac{2\pi}{\omega}\) = 1554.35 nm, \(\kappa = 2\pi \times 423\) MHz, \(g_0 = 2\pi \times 869 \times \sqrt{2}\) kHz, \(\omega_m = 2\pi \times 5.25\) GHz, \(\gamma_m = 2\pi \times 13.816\) kHz. Moreover, we choose the effective cavity detuning \(\Delta = \omega_m\), the initial temperature of thermal environment \(T = 30\) mK, and the power of the input laser \(\phi = 1.3\) mW.

The structures of the spectra of \(S_Q(\omega)\) and \(S_{out}(\omega)\) are determined by the solutions of the equation \(d(\omega) = 0\). The real and imaginary parts of the solutions of \(d(\omega) = 0\) determine the locations and linewidths of the normal modes of the optomechanical system, respectively. We plot the real parts \(\text{Re}(\omega)\) and the imaginary parts \(\text{Im}(\omega)\) of the solutions of \(d(\omega) = 0\) versus the reflection coefficient \(r\) of the beam splitter in Figures 3 and 4. When \(r \leq 0.043\), \(\text{Re}(\omega)\) has two identical values, so there is no normal mode splitting in \(S_Q(\omega)\) and \(S_{out}(\omega)\), however, \(\text{Im}(\omega)\) has two different values, thus the lifetime splitting appears [35]. When \(r > 0.043\), \(\text{Re}(\omega)\) has two different values, and the difference between them is increased with increasing \(r\), thus the normal mode splitting occurs in \(S_Q(\omega)\) and \(S_{out}(\omega)\), and the peak separation becomes larger with increasing \(r\), whereas \(\text{Im}(\omega)\) has two equal values, the opposite of \(\text{Im}(\omega)\) is decreased with increasing \(r\), which implies two normal modes in \(S_Q(\omega)\) and \(S_{out}(\omega)\) have the same linewidths, and their linewidths are reduced with increasing \(r\).

**Figure 3.** The real parts of the solutions of \(d(\omega) = 0\) in the domain \(\text{Re}(\omega) > 0\) versus the reflection coefficient \(r\) of the beam splitter.

**Figure 4.** The imaginary parts of the solutions of \(d(\omega) = 0\) versus the reflection coefficient \(r\) of the beam splitter.
Figures 5 and 6 show the spectra $S_Q(\omega)$ and $S_{\text{cout}}(\omega)$ versus the normalized frequency $\omega/\omega_m$ for different reflection coefficients $r$ of the beam splitter. In the absence of the feedback ($r = 0$), it is seen that the spectra do not show the normal-mode splitting. The reason is that the value of $|G|/\kappa$ is 0.91, which is less than unity, thus the system is in the weak coupling regime. In the presence of the feedback, two peaks become more observable in the spectra with increasing the reflection coefficient $r$ of the beam splitter, thus the normal mode splitting occurs in the spectra. For $r = 0.2, 0.3, 0.35$, the values of $|G|/\kappa_{\text{eff}}$ are 1.49, 2.18, 2.86, respectively, which are larger than unity, thus the system is effectively promoted to the strong coupling regime. In addition, when the reflection coefficient $r$ of the beam splitter is increased, the separation between two peaks becomes larger, and two peaks of the spectra become narrower. These results are consistent with those in Figures 3 and 4. We also note that the two peaks of the spectra become higher with increasing the reflection coefficient $r$ of the beam splitter.

5. Conclusions

In conclusion, we have shown that coherent feedback via the asymmetric beam splitter in the optomechanical system could improve the resolved-sideband cooling of the movable mirror. We also have shown that the feedback could make the normal mode splitting occur in the spectra of the movable mirror and the output field even when the movable mirror is initially weakly coupled to the cavity mode because the feedback makes the effective cavity decay rate smaller than that without the feedback. We have found that increasing the reflection coefficient $r$ of the beam splitter makes the peak separation larger and two peaks of the spectra narrower and higher. Therefore, the feedback scheme offers an alternative way for quantum manipulation of the macroscopic mechanical oscillator.
Author Contributions: Conceptualization, S.H. and A.C.; methodology, S.H. and A.C.; software, S.H.; formal analysis, S.H.; writing—original draft preparation, S.H.; writing—review and editing, S.H. and A.C.

Funding: This research was funded by Science Foundation of Zhejiang Sci-Tech University (grant numbers 18062121-Y, 17062071-Y), and by the National Natural Science Foundation of China [grant number 11775190].

Acknowledgments: S. Huang would like to thank G.S. Agarwal for helpful discussions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Aspelmeyer, M.; Kippenberg, T.J.; Marquardt, F. Cavity optomechanics. Rev. Mod. Phys. 2014, 86, 1391. [CrossRef]
2. Bose, S.; Jacobs, K.; Knight, P.L. Preparation of nonclassical states in cavities with a moving mirror. Phys. Rev. A 1997, 56, 4175–4186. [CrossRef]
3. Marshall, W.; Simon, C.; Penrose, R.; Bouwmeester, D. Towards quantum superpositions of a mirror. Phys. Rev. Lett. 2003, 91, 130401. [CrossRef] [PubMed]
4. Huang, S.; Agarwal, G.S. Entangling nanomechanical oscillators in a ring cavity by feeding squeezed light. New J. Phys. 2009, 11, 103044. [CrossRef]
5. Lü, X.Y.; Wu, Y.; Johansson, J.R.; Jing, H.; Zhang, J.; Nori, F. Squeezed optomechanics with phase-matched amplification and dissipation. Phys. Rev. Lett. 2015, 114, 093602. [CrossRef] [PubMed]
6. Huang, S.; Agarwal, G.S. Enhancement of cavity cooling of a micromechanical mirror using parametric interactions. Phys. Rev. A 2009, 79, 013821. [CrossRef]
7. Liu, Y.C.; Shen, Y.F.; Gong, Q.H.; Xiao, Y.F. Optimal limits of cavity optomechanical cooling in the strong-coupling regime. Phys. Rev. A 2014, 89, 053821. [CrossRef]
8. Liu, Y.C.; Liu, R.S.; Dong, C.H.; Li, Y.; Gong, Q.H.; Xiao, Y.F. Cooling mechanical resonators to the quantum ground state from room temperature. Phys. Rev. A 2015, 91, 013824. [CrossRef]
9. Guo, Y.J.; Li, K.; Nie, W.J.; Li, Y. Electromagnetically-induced-transparency-like ground-state cooling in a double-cavity optomechanical system. Phys. Rev. A 2014, 90, 053841. [CrossRef]
10. Wilson-Rae, I.; Nooshi, N.; Zwerger, W.; Kippenberg, T.J. Theory of ground state cooling of a mechanical oscillator using dynamical backaction. Phys. Rev. Lett. 2007, 99, 093901. [CrossRef]
11. Marquardt, F.; Chen, J.P.; Clerk, A.A.; Girvin, S.M. Quantum theory of cavity-assisted sideband cooling of mechanical motion. Phys. Rev. Lett. 2007, 99, 093902. [CrossRef]
12. Schliesser, A.; Arcizet, O.; Rivièrè, R.; Anetsberger, G.; Kippenberg, T.J. Resolved-sideband cooling and position measurement of a micromechanical oscillator close to the Heisenberg uncertainty limit. Nat. Phys. 2009, 5, 509–514. [CrossRef]
13. Park, Y.; Wang, H. Resolved-sideband and cryogenic cooling of an optomechanical resonator. Nat. Phys. 2009, 5, 489–493. [CrossRef]
14. Gröblacher, S.; Hertzberg, J.B.; Vanner, M.R.; Cole, G.D.; Gigan, S.; Schwab, K.C.; Aspelmeyer, M. Demonstration of an ultracold micro-optomechanical oscillator in a cryogenic cavity. Nat. Phys. 2009, 5, 485–488. [CrossRef]
15. Rocheleau, T.; Ndukum, T.; Macklin, C.; Hertzberg, J.B.; Clerk, A.A.; Schwab, K.C. Preparation and detection of a mechanical resonator near the ground state of motion. Nature 2010, 463, 72–75. [CrossRef]
16. Chan, J.; Mayer, A.T.P.; Safavi-Naeini, A.H.; Hill, J.T.; Krause, A.; Gröblacher, S.; Aspelmeyer, M.; Painter, O. Laser cooling of a nanomechanical oscillator into its quantum ground state. Nature 2011, 478, 89–92. [CrossRef]
17. Teufel, J.D.; Donner, T.; Li, D.; Harlow, J.W.; Allman, M.S.; Cicak, K.; Sirosi, A.J.; Whittaker, J.D.; Lehnert, K.W.; Simmonds, R.W. Sideband cooling of micromechanical motion to the quantum ground state. Nature 2011, 475, 359–363. [CrossRef]
18. Cohadon, P.F.; Heidmann, A.; Pinard, M. Cooling of a mirror by radiation pressure. Phys. Rev. Lett. 1999, 83, 3174. [CrossRef]
19. Arcizet, O.; Cohadon, P.F.; Briant, T.; Pinard, M.; Heidmann, A.; Mackowski, J.M.; Michel, C.; Pinard, L.; François, O.; Rousseau, L. High-sensitivity optical monitoring of a micromechanical resonator with a quantum-limited optomechanical sensor. Phys. Rev. Lett. 2006, 97, 133601. [CrossRef]
20. Kleckner, D.; Bouwmeester, D. Sub-kelvin optical cooling of a micromechanical resonator. *Nature* **2006**, *444*, 75–78. [CrossRef]

21. Poggio, M.; Degen, C.L.; Mamin, H.J.; Rugar, D. Feedback cooling of a cantilever’s fundamental mode below 5 mK. *Phys. Rev. Lett.* **2007**, *99*, 017201. [CrossRef]

22. Corbitt, T.; Wipf, C.; Bodiya, T.; Ottaway, D.; Sigg, D.; Smith, N.; Whitcomb, S.; Mavalvala, N. Optical dilution and feedback cooling of a gram-scale oscillator to 6.9 mK. *Phys. Rev. Lett.* **2007**, *99*, 160801. [CrossRef]

23. Zhang, J.; Liu, Y.X.; Nori, F. Cooling and squeezing the fluctuations of a nanomechanical beam by indirect quantum feedback control. *Phys. Rev. A* **2009**, *79*, 052102. [CrossRef]

24. Schäfermeier, C.; Kerdoncuff, H.; Hoff, U.B.; Fu, H.; Huck, A.; Bilek, J.; Harris, G.I.; Bowen, W.P.; Gehring, T.; Andersen, U.L. Quantum enhanced feedback cooling of a mechanical oscillator using nonclassical light. *Nat. Commun.* **2016**, *7*, 13628. [CrossRef]

25. Dobrindt, J.M.; Wilson-Rae, I.; Kippenberg, T.J. Parametric normal-mode splitting in cavity optomechanics. *Phys. Rev. Lett.* **2008**, *101*, 263602. [CrossRef]

26. Huang, S.; Agarwal, G.S. Normal-mode splitting in a coupled system of a nanomechanical oscillator and a parametric amplifier cavity. *Phys. Rev. A* **2009**, *80*, 033807. [CrossRef]

27. Liu, Y.C.; Xiao, Y.F.; Luan, X.S.; Gong, Q.H.; Wong, C.W. Coupled cavities for motional ground-state cooling and strong optomechanical coupling. *Phys. Rev. A* **2015**, *91*, 033818. [CrossRef]

28. Han, Y.; Cheng, J.; Zhou, L. Normal-mode splitting in the atom-assisted optomechanical cavity. *Phys. Scr.* **2013**, *88*, 065401. [CrossRef]

29. Zhang, Z.C.; Wang, Y.P.; Yu, Y.F.; Zhang, Z.M. Normal-mode splitting in a weakly coupled electromechanical system with a mechanical modulation. *Ann. Phys.* **2019**, *531*, 1800461. [CrossRef]

30. Gröblacher, S.; Hammerer, K.; Varner, M.R.; Aspelmeyer, M. Observation of strong coupling between a micromechanical resonator and an optical cavity field. *Nature* **2009**, *460*, 724–727. [CrossRef]

31. Mandel, L.; Wolf, E. Effect of an Attenuator or Beam Splitter on the Quantum Field. In *Optical Coherence and Quantum Optics*; Cambridge University Press: Cambridge, UK, 1995; pp. 639–640.

32. DeJesus, E.X.; Kaufman, C. Routh-Hurwitz criterion in the examination of eigenvalues of a system of nonlinear ordinary differential equations. *Phys. Rev. A* **1987**, *35*, 5288. [CrossRef]

33. Gupta, S.D.; Agarwal, G.S. Strong coupling cavity physics in microspheres with whispering gallery modes. *Opt. Commun.* **1995**, *115*, 597–605. [CrossRef]