Internal friction analysis for the viscoelastic constitutive models

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Abstract. In this paper, the internal friction for the viscoelastic materials such as asphalt is analyzed theoretically. The variation of the strain energy and internal friction per period when an alternating stress is applied to the viscoelastic material is discussed which shows that the storage and release of the strain energy are alternately carried out. The internal friction per period is related to the loss modulus. However, it is independent of the energy storage modulus. In addition, the energy storage modulus, loss modulus, tangent of loss angle and internal friction for four viscoelastic constitutive models, i.e., Maxwell model, Kelvin model, Lessersichi model and Jeffreys model, are calculated from which we know that they are all the functions of frequency. When the frequency is zero, the internal friction for the Kelvin model and Jeffreys model are zero, while the Maxwell model and Lessersichi model are infinity.

1. Introduction

There are many forms for the viscoelastic behavior of materials. Under the static condition, the time effect (e.g., creep and stress relaxation) is mainly concerned, while the energy effect (e.g., hysteresis and internal friction) should be concerned under the dynamic condition [1-8]. Under the action of alternating stress, the phenomenon of strain lagging behind stress is called hysteresis. When the change of stress is consistent with strain, there is no hysteresis phenomenon. In this case, the work done by the deformation is equal to the strain energy and there is no energy loss. If the change of strain is lagging behind the stress and the hysteresis phenomenon occurs. In this situation, the energy loss will exist, which is called the internal friction. Internal friction is the reflect of viscoelasticity of materials under alternating stress, and it is the result of strain lagging behind stress. For the ideal elastic solids, stress and strain have the same phase and there is no irreversible energy loss. For the ideal viscous liquids, the phase difference between the stress and strain is 90 degrees, and the total energy is loss during deformation. While for the viscoelastic materials, the phase difference between the stress and strain is from 0 to 90 degrees, part of the work done by the external force is converted into the strain energy stored in the materials, and the other is converted into the thermal energy because of the internal friction [2-4].

Hysteresis is related to the chemical structure of materials. Generally speaking, the hysteresis for the rigid molecular chains is small, while the hysteresis is big for the flexible molecular chains. Hysteresis is also related to the frequency of the external force, temperature and other factors. When the frequency of the external force is high enough, the movement of the molecular chain cannot keep up with the change of the external force, and the hysteresis phenomenon is not obvious. If the frequency of the external force is low enough, the movement of the molecular chain can keep up with
the change of the external force, the hysteresis phenomenon is also not obvious in this situation. Only when the frequency of the external force is not too high or too low, can the obvious hysteresis phenomenon happen. In addition, the factor of temperature has a similar effect on the hysteresis phenomenon. Under the condition that the frequency of the external force is fixed, if the temperature is very high, the deformation will almost not lag behind the stress change because the movement of the molecular chain is active under the action of high temperature. When the temperature is very low, the movement of the molecular chain is inactive. In this situation, the molecular chain totally cannot keep up with the change of stress and there is no delay for this situation. Only in a certain range of temperature can the obvious hysteresis phenomenon happen. Therefore, increasing the frequency of the external force and decreasing temperature have a similar effect on the hysteresis phenomenon. The internal friction caused by the hysteresis under the action of alternating stress, can be explained from the stress-strain curve for the processes of stretching and retracting. If the deformation can completely keep up with the change of stress, the curves for the processes of stretching and retracting will overlap together. If the deformation lags behind the stress, the stress and strain are always in the unbalanced state. In this case, one part of the work done by the external force in the process of stretching is used to change the conformation of the molecular chain in the system, the other part of the work is used to provide the energy consumed by the internal friction. In the process of retracting, the system does work. The extended molecular chain is curled up to the original state and the molecular chain has to overcome the friction resistance during the movement. In the whole processes of stretching and retracting, the conformational change is completely recovered, and no work is consumed, the energy loss is used to overcome the internal friction [2-4].

In practice, the viscoelastic materials such as asphalt are often subjected to the periodic loads. Under this circumstance, mechanical loss, i.e., internal friction is inevitable, which will eventually be converted into the thermal energy. Because the generated heat is not easy to emit, it will cause the temperature rise for the viscoelastic materials and affect the service life. The smaller the internal friction is, the better the temperature stability for the viscoelastic materials is [2-4]. Therefore, the study of energy effect (hysteresis and internal friction) for the viscoelastic materials is of great significance.

2. Theoretical analysis of internal friction

It has been said that the viscoelastic materials such as asphalt are often subjected to the periodic loads. In order to simulate this situation, an alternating stress is applied to the viscoelastic material. The strain response is not synchronized with the stress, but there is a phase difference between the strain and stress responses. Assume the strain response for the viscoelastic material under the alternating stress is $\varepsilon(t) = \varepsilon_0 \sin \omega t$. Because the stress response leads a phase angle $\delta (0 < \delta < 90^\circ)$ compared with the strain response, then we have $\sigma(t) = \sigma_0 \sin (\omega t + \delta)$, which can be expanded as,

$$\sigma(t) = \sigma_0 \sin \omega t \cos \delta + \sigma_0 \cos \omega t \sin \delta$$  \hspace{1cm} (1)

It can be seen from the above equation that the stress is composed of two parts, one part is the same phase with the strain and its amplitude is $\sigma_0 \cos \delta$, the other part has 90 degrees phase difference with the strain and its amplitude is $\sigma_0 \sin \delta$. Introduce the modulus $E'$ and $E''$ as follows,

$$E' = \sigma_0 \cos \delta / \varepsilon_0, \quad E'' = \sigma_0 \sin \delta / \varepsilon_0$$  \hspace{1cm} (2)

Thus, equation (1) can be rewritten as

$$\sigma(t) = \varepsilon_0 E' \sin \omega t + \varepsilon_0 E'' \cos \omega t$$  \hspace{1cm} (3)

The modulus $E'$ and $E''$ can be combined to form a complex modulus,
The real part $E'$ is called energy storage modulus, which reflects the energy stored due to the elastic deformation, while the imaginary part $E''$ is called loss modulus, which reflects energy loss in the form of heat during deformation [3]. The complex modulus is related to the frequency and temperature. When the temperature is fixed, the frequency spectrum can be obtained. The temperature spectrum can be obtained if the frequency is fixed. The temperature spectrum and frequency spectrum are collectively referred to as the mechanical spectrum of materials. By equation (2), we can obtain the tangent of the phase angle,

$$
\tan \delta = \frac{E''}{E'}
$$

(5)

In fact, the strain and stress functions can also be expressed in the complex forms,

$$
\varepsilon(t) = \varepsilon_0 e^{i\omega t}, \quad \sigma(t) = \sigma_0 e^{i(\omega t+\delta)}
$$

(6)

By the above equation, we can prove that the complex modulus $E^*$ can be expressed as $E^* = \sigma(t) / \varepsilon(t)$. In fact,

$$
\frac{\sigma(t)}{\varepsilon(t)} = \frac{\sigma_0 e^{i\omega t}}{\varepsilon_0 e^{i\omega t}} = \frac{\sigma_0}{\varepsilon_0} (\cos \delta + i \sin \delta) = E' + iE'' = E^*
$$

(7)

The Euler formula has been used in the above equation. By the Euler formula, we can also have

$$
\sigma'(t) = \varepsilon_0 E' \sin \omega t, \quad \sigma'(t) = \varepsilon_0 E' \cos \omega t
$$

(9)

For the viscoelastic materials such as asphalt, part of the work done by the external force is stored as strain energy $W_s$, part of which is used to supply the energy loss due to the internal friction $W$. In the following, we will first discuss the variation of the strain energy in a period.

$$
W_s = \int_0^t \sigma'(t) \frac{d\varepsilon(t)}{dt} dt = \int_0^t \varepsilon_0 E' \sin \omega t \varepsilon_0 \omega \cos \omega t dt = \frac{E' \varepsilon_0^2}{4\omega} (1 - \cos 2\omega t), \quad t \in [0, \frac{2\pi}{\omega}]
$$

(10)

$$
\frac{dW_s}{dt} = \frac{E' \varepsilon_0^2}{4\omega} \times 2\omega \times \sin 2\omega t = \frac{E' \varepsilon_0^2}{2} \sin 2\omega t, \quad t \in [0, \frac{2\pi}{\omega}]
$$

(11)

From the properties of the sinusoidal function, we know that, when $t \in \left[0, \frac{\pi}{2\omega}\right), \frac{dW_s}{dt} \geq 0$;

when $t \in \left[\frac{\pi}{2\omega}, \frac{\pi}{\omega}\right), \frac{dW_s}{dt} \leq 0$; when $t \in \left[\frac{\pi}{\omega}, \frac{3\pi}{2\omega}\right), \frac{dW_s}{dt} \geq 0$; when $t \in \left[\frac{3\pi}{2\omega}, \frac{2\pi}{\omega}\right), \frac{dW_s}{dt} \leq 0$.

It can be seen that when an alternating stress is applied to the viscoelastic material, the strain energy of the material is not increased all the time. In a period, the strain energy increases first and then decreases, and then increases, and then decreases. That is to say, the storage and release of the strain energy are alternately carried out. At the 1/4 period, i.e., $t = \pi / 2\omega$, the strain energy reach the maximum. Under the action of the alternating stress, the deformation for the ideal elastic body in the first 1/4 period is consistent with the external force and there is no phase difference. All the work done by the external force is stored in the form of potential energy. In the second 1/4 period, the
deformation is restored and the stored energy is released completely, and the above process is repeated in the third and fourth 1/4 period. Under the action of the alternating stress, the phase difference between stress and strain is 90 degrees for the ideal viscous liquids, and the work done by the external force is completely loss in the form of heat. When the viscoelastic material is subjected to the action of the alternating stress, the strain lags behind stress in a phase angle. In each period, part of the energy is stored and released, and the other part of the energy is loss in the form of heat [2-4].

In the next, we will discuss the variation of internal friction $W$ in a period.

$$W = \int_0^\tau \sigma(t) \frac{d\varepsilon(t)}{dt} dt = \int_0^\tau \varepsilon_0 E \varepsilon_0 \omega d\omega \cos \omega t \cos \omega t dt = \frac{1}{4} E^\prime \varepsilon_0^2 (2\omega t - \sin 2\omega t), \quad t \in [0, \frac{2\pi}{\omega}]$$

Equation (12)

$$\frac{dW}{dt} = \frac{1}{4} E^\prime \varepsilon_0^2 (2\omega - 2\omega \cos 2\omega t) = \frac{1}{2} E^\prime \varepsilon_0^2 \omega (1 - \cos 2\omega t), \quad t \in [0, \frac{2\pi}{\omega}]$$

Equation (13)

Because $1 - \cos 2\omega t \geq 0$, so we have $\frac{dW}{dt} \geq 0$. That is to say, the internal friction $W$ is always increasing in a period.

$$W \mid_{t=2\pi/\omega} = \pi E^\prime \varepsilon_0^2$$

Equation (14)

Therefore, the internal friction $W$ per period is related to the loss modulus $E^\prime$. However, it is independent of the energy storage modulus $E^\prime$.

3. Internal friction analysis for the viscoelastic constitutive model

According to the theory of rheological mechanics, it is considered that elasticity, viscosity and plasticity are the basic elements to understand the mechanical properties of materials. New models can be formed from the three basic mechanical elements through the series and (or) parallel connection. When the elements are in series, the total stress is equal to the stress of each element, and the total strain is equal to the sum of each strain. When the elements are in parallel, the total stress is equal to the sum of each stress, and the total strain is equal to the strain of each element. In order to describe the mechanical behavior of the viscoelastic material, various viscoelastic constitutive models have been put forward, such as the Maxwell model, Kelvin model, Lessersichi model, Jeffreys model and so on [9-12]. The more complex the model is, the more accurate the mechanical behavior of materials can be. However, if the model is too complex, it will increase the difficulty of the mathematical analysis. In the following, we will discuss the internal friction properties concretely for the four viscoelastic constitutive models, i.e., Maxwell model, Kelvin model, Lessersichi model and Jeffreys model.

- Maxwell model

When the Maxwell model is subjected to the action of an alternating stress $\sigma(t) = \sigma_0 e^{i\omega t}$, its constitutive equation can be written as [9-12],

$$\frac{d\varepsilon(t)}{dt} = \frac{\sigma_0}{E} i\omega e^{i\omega t} + \frac{\sigma_0}{\eta} e^{i\omega t}$$

Equation (15)

Integrating the above equation, we have

$$\varepsilon(t) = \left( \frac{\sigma_0}{E} + \frac{\sigma_0}{i\eta\omega} \right) e^{i\omega t} = \left( \frac{\sigma_0}{E} - i\frac{\sigma_0}{\eta\omega} \right) e^{i\omega t}$$

Equation (16)

The complex modulus $E^\ast$ is equal to
\[ E' = \frac{\sigma(t)}{\varepsilon(t)} = \frac{E \omega^2 \eta^2}{E^2 + \omega^2 \eta^2} + i \left( \frac{E^2 \eta \omega}{E^2 + \omega^2 \eta^2} \right) \]  

(17)

By the above equation, we have

\[ E' = \frac{E \omega^2 \eta^2}{E^2 + \omega^2 \eta^2}, \]
\[ E'' = \frac{E^2 \eta \omega}{E^2 + \omega^2 \eta^2}, \]  

(18)

\[ \tan \delta = \frac{E''}{E'} = \frac{E}{\omega \eta}. \]

From the above three equations, we can see that the storage modulus, loss modulus and tangent of phase angle are all the functions of frequency.

\[ W = \pi \varepsilon_0^2 E'' = \pi \left[ \left( \frac{\sigma_0}{E} \right)^2 + \left( \frac{\sigma_0}{\eta \omega} \right)^2 \right] \frac{E^2 \eta \omega}{E^2 + \omega^2 \eta^2} = \frac{\pi \sigma_0^2}{\eta \omega} \]  

(19)

Therefore, the internal friction obtained by Maxwell model decreases with the increase of frequency. If \( \omega = 0 \), then \( \sigma = \sigma_0 = \text{const} \). The phenomenon of strain variation with time under the action of constant stress is called creep. Substituting the condition \( \omega = 0 \) into equation (19), we have \( W = \infty \). That is to say, the energy loss for the Maxwell model in the creep process is infinite.

● Kelvin model

When the Kelvin model is subjected to the action of an alternating strain \( \varepsilon(t) = \varepsilon_0 e^{i\omega t} \), its constitutive equation can be written as [9-12],

\[ \sigma(t) = E \varepsilon + \eta \frac{d \varepsilon}{dt} = E \varepsilon_0 e^{i\omega t} + i \omega \eta \varepsilon_0 e^{i\omega t} \]  

(20)

The complex modulus \( E' \) can be expressed as,

\[ E' = \frac{\sigma(t)}{\varepsilon(t)} = \frac{E \varepsilon_0 e^{i\omega t} + i \omega \eta \varepsilon_0 e^{i\omega t}}{\varepsilon_0 e^{i\omega t}} = E + i \omega \eta \]  

(21)

By the above equation, we have

\[ E' = E, \]
\[ E'' = \omega \eta. \]  

(22)

For the Kelvin model, the loss modulus and tangent of phase angle are the linear functions of frequency, while the storage modulus is independent of frequency.

If the Kelvin model is subjected to the action of an alternating stress \( \sigma(t) = \sigma_0 e^{i\omega t} \), then we have

\[ \sigma_0 e^{i\omega t} = E \dot{\varepsilon} + \eta \ddot{\varepsilon} \]  

(23)

Solving the above ordinary differential equation, we can get
The internal friction for the Kelvin model can be expressed as,

\[
W = \pi \varepsilon_0^2 E^\prime = \frac{\pi \sigma_0^2 \omega \eta}{E^2 + \eta^2 \omega^2}
\]  

When \( \omega = 0 \), we have \( W = 0 \), i.e., there is no energy loss for the Kelvin model in the creep process.

- **Lessersichi model**

When the Lessersichi model is subjected to the action of an alternating stress \( \sigma(t) = \sigma_0 e^{iat} \), we have,

\[
\varepsilon(t) = \frac{E \sigma_0}{E^2 + \eta^2 \omega^2} \left[ 1 + i \frac{\omega \eta \omega^2 (\eta + \eta_2)}{E^2 + \eta^2 \omega^2} \right]
\]  

By the definition of the complex modulus,

\[
E^\prime = \frac{E \eta_1^2 \omega}{E^2 + (\eta_1 + \eta_2)^2 \omega^2}, \quad E^\prime = \frac{E \eta_1^2 \omega + \omega^3 \eta \eta_2 (\eta_1 + \eta_2)}{E^2 + (\eta_1 + \eta_2)^2 \omega^2}
\]

By the above equation, we have

\[
\tan \delta = \frac{E^\prime}{E^\prime} = \frac{E^2 + \omega^3 \eta_2 (\eta_1 + \eta_2)}{E \eta_1 \omega}
\]

The internal friction for the Lessersichi model can be expressed as,

\[
W = \pi \varepsilon_0^2 E^\prime = \pi \sigma_0^2 \left( \frac{\eta_2 \omega E^2}{\eta \omega (E^2 + \eta_1^2 \omega^2)} + \frac{[E^2 + \omega \eta \omega^2 (\eta_1 + \eta_2)]^2}{\eta \omega (E^2 + \eta_2^2 \omega^2)} \right) \times \frac{E^2 + \omega \eta \omega^2 (\eta_1 + \eta_2)}{E^2 + (\eta_1 + \eta_2)^2 \omega^2}.
\]  

When \( \omega = 0 \), we have \( W = \infty \). That is to say, the energy loss for the Lessersichi model in the creep process is infinite.

- **Jeffreys model**

When the Jeffreys model is subjected to the action of an alternating stress \( \sigma(t) = \sigma_0 e^{iat} \), we have,

\[
\varepsilon(t) = \frac{\sigma_0 E \eta^2 \omega^4}{E^2 (\eta_1 + \eta_2)^2 + \eta_1^2 \eta_2^2 \omega^4} \left[ 1 + i \frac{\omega \eta \omega^2 (\eta_1 + \eta_2)}{E^2 (\eta_1 + \eta_2)^2 + \eta_1^2 \eta_2^2 \omega^4} \right]
\]

By the definition of the complex modulus,

\[
E^\prime = \frac{\sigma(t)}{\varepsilon(t)} = \frac{E \eta_1^2 \omega^2}{E^2 + \eta_1^2 \omega^2} + i \frac{E^2 (\eta_1 + \eta_2) \omega + \omega^3 \eta_2^2 \omega}{E^2 + \eta_1^2 \omega^2}
\]

By the above equation, we have
\[ E' = \frac{E\eta^2 \omega^2}{E^2 + \eta^2 \omega^2}, \]
\[ E'' = \frac{E^2(\eta_1 + \eta_2)\omega + \omega^3\eta_1^2\eta_2^2}{E^2 + \eta^2 \omega^2}, \]
\[ \tan \delta = \frac{E''}{E'} = \frac{E^2(\eta_1 + \eta_2) + \eta_1^2\eta_2^2\omega^2}{E\eta^2 \omega}. \]  

When \( \omega = 0 \), the internal friction for the Jeffreys model \( W = 0 \). That is to say, there is no energy loss for the Jeffreys model in the creep process.

4. Conclusions
In this paper, the internal friction properties for the viscoelastic materials such as asphalt are analyzed theoretically. When an alternating stress is applied to the viscoelastic material, part of the work done by the external force is stored as strain energy, part of which is used to supply the energy loss due to the internal friction. The storage and release of the strain energy are alternately carried out. The internal friction per period is related to the loss modulus. However, it is independent of the energy storage modulus.

The energy storage modulus, loss modulus, tangent of loss angle and internal friction for four viscoelastic constitutive models, i.e., Maxwell model, Kelvin model, Lessersichi model and Jeffreys model, are calculated, and from the calculation we know that the energy storage modulus, loss modulus, tangent of phase angle and internal friction are all the functions of frequency. When the frequency is zero, the internal friction for the Kelvin model and Jeffreys model are zero, while the Maxwell model and Lessersichi model are infinity.

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References
[1] Ge T S 1997 *J. Metal* **1** 9-21
[2] Fang Q F 2001 *J. Sunyatsen. Univ.* **3** 211-3
[3] Cheng K J 1955 *J. Phys.* **2** 163-77
[4] Lu C X, Wang Z X and Shi Z Z 1987 *J. Wuhan. Univ.* **4** 31-42
[5] Yin H and Chen N 2012 *J. Computational Mechanics* **6** 966-71
[6] Shui J P and Liu Y S 1999 *J. Phys.* **4** 135-41
[7] Kimball A L and Lovell D E 1997 *Phys. Rev.* **30** 948-56
[8] Read T A 1940 *Phys. Rev.* **58** 371-8
[9] Kong Q P and Shan B 1989 *J. Phys.* **8** 1299-305
[10] Jin F N and Pu K Y 1995 *J. Rock Mechanics and Engineering* **4** 355-61
[11] Janete A 2015 *Mechanism and Machine Theory* **85** 172-88
[12] Bonet J 2001 *International J. Solids and Structures* **38** 2953-68