Exact Chiral Symmetry on the Lattice.

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Abstract

Developments during the last eight years have refuted the folklore that chiral symmetries cannot be preserved on the lattice. The mechanism that permits chiral symmetry to coexist with the lattice is quite general and may work in Nature as well. The reconciliation between chiral symmetry and the lattice is likely to revolutionize the field of numerical QCD.

1 INTRODUCTION

Chiral symmetries play a central role in Particle and Nuclear Physics, intimately related to such ubiquitous processes as $\beta$-decay. For a while it seemed that one cannot find a non-perturbative regularization of gauge theories with explicit chiral symmetry. There was a folklore that chiral symmetry could not be realized on the lattice. This folklore has been abandoned now. The breakthrough can be traced back to an early paper by Callan and Harvey [1], and to two subsequent papers, one by Kaplan [2] and the other by Frolov and Slavnov [3]. Based on these papers a general way to realize chiral symmetry on the lattice was devised and named the “overlap” [4].

Once the overlap construction was completed and it was established that the required salient features of chiral fermions were indeed successfully reproduced it was realized that one could have arrived at the same construction not by starting from Callan and Harvey’s paper but rather from an older work of Ginsparg and Wilson [5]. While the Callan and Harvey paper pointed the direction towards chiral gauge theories and was a natural outgrowth of the mathematical understanding of anomalies that was developed in the early to mid eighties, Ginsparg and Wilson focused on vector-like gauge theories and took a more traditional approach. It is a surprise that these two lines of thought have merged. This review focuses on the overlap viewpoint because it is the original way the construction was arrived at and because it is simultaneously more physical and more modern mathematically.
Another viewpoint can be found in the recent review of M. Creutz \cite{Creutz}. His review also contains a much more extensive list of references and readers looking for more information than contained in the bibliography of the present review are encouraged to consult this source.

1.1 Chiral Symmetry in Particle and Nuclear Physics

In the minimal standard model the basic building blocks of matter are chiral fermions. If we ignore weak and electromagnetic interactions, we are left with QCD. QCD has almost exact global chiral symmetries and they explain the low energy hadronic spectrum. The weak interactions come from gauged chiral symmetries made consistent by delicate anomaly cancelations ensuring their preservation at the quantum level. Chirality continues to play a central role in supersymmetric extensions of the minimal standard model. Grand unified theories, supersymmetric or not, still must be chiral gauge theories in order to reproduce the standard model.

1.2 The Effective Lagrangian Viewpoint

Relativistic field theory describes Particle Physics simply because it is a unique structure incorporating special relativity, quantum mechanics and the self-consistency of a system consisting of a finite number of fundamental particles in interaction. Because of gravity, it is all but certain that any field theoretical model is not more than an approximation to Nature. This approximation tends to be very good in a certain energy range. The top of this energy range can be viewed as the “ultraviolet cutoff”, $\Lambda$, of the theory. The larger $\Lambda$ is relative to typical masses, $m_{ph}$, in the theory, the more powerful and accurate is the effective model. Since the standard model describes Nature very accurately $\Lambda/m_{ph}$ might be very large. However, for any theory to contain particles of masses much lower than $\Lambda$, one needs a specific mechanism; without one, it is just too unlikely that $m_{ph} \ll \Lambda$. The larger the $\Lambda/m_{ph}$ is, the more awkward is the situation in the absence of a special mechanism. Luckily, mechanisms that produce very light or even massless particles are known for spin 0, 1/2 and 1: Spin zero particles are naturally light if they are Goldstone bosons associated with the spontaneous breakdown of a global symmetry. Spin 1 particles are naturally massless if they are the force carriers in a gauge interaction. Spin 1/2 particles are naturally massless if they are chiral. This takes care of all particles except the yet undiscovered Higgs. One way to also make the Higgs naturally light, although it is not a Goldstone boson, is provided by supersymmetry. Supersymmetry places particles of different spin into equal mass (super)multiplets, so the above mechanisms can also restrict the mass of particles of the “wrong” spin.

One indication for what $\Lambda$ might be is given by the confluence of the three known gauge couplings at a high energy. There may exist a very accurate “Grand Unified” (GU) effective field theory description of Nature with a $\Lambda \sim 10^{16} GeV$! Chiral symmetries continue to play a central role in GU theories, be they supersymmetric or not.
2 CONTINUUM

Chiral fermions in interaction with gauge fields have some peculiar properties in the continuum, at the semiclassical level \( \text{[7]} \). These properties are best expressed in modern mathematical terms. They all deal with chiral fermions in the background provided by a gauge field that is smooth up to gauge transformations. The fermion fields do not need to be smooth; their fluctuations are easily brought under control because their equations of motion are linear. Physically, one expects that fermionic and bosonic fluctuations are controlled by the same ultraviolet cutoff and the quantum fluctuations in the gauge background also have to be taken into account. Perturbatively it is known how to do this, but if the interactions are strong the situation is less certain. Much of this uncertainty has been eliminated by the developments central to this review, but without a good grasp of the continuum semiclassical and perturbative properties neither the problems nor their resolution can be fully appreciated.

The gauge invariant content of a gauge configuration defines a gauge orbit. The multitude of gauge orbits forms a space that has an infinite set of disconnected components, each uniquely identified by a signed integer, \( n_{\text{top}} \), the “topological charge”. This is best understood in the Euclidean formulation of gauge theories, where space-time is replaced by a four dimensional compact manifold. For any fixed gauge background one can calculate expectation values of strings of fermion field operators. Strangely, it turns out that single (or odd numbers of) fermion operators can have non-vanishing expectation values in certain cases, when \( n_{\text{top}} \neq 0 \). This is surprising because in a literally interpreted path integral only products of even numbers of Grassmann variables can integrate to non-zero values. When chiral fermions are integrated out one expects to get a functional of the gauge fields given by the determinant of the Weyl operator. Under a gauge transformation this operator transforms by conjugation, but, sometimes, the determinant unavoidably breaks gauge invariance. This is the single source of anomalies: the expectation value of any fermionic observable transforms as expected under a gauge transformation, up to the chiral determinant which enters as a multiplicative factor. Only the phase of this multiplicative factor can have anomalies. There is therefore no way one can view the chiral determinant as a function on orbit space, because its phase may have to vary along some orbits. There is a mathematical construct which usefully accommodates this situation: One associates with each orbit a one dimensional complex vector space and the collection of these spaces makes up a smooth manifold. The chiral determinant is a section in this “line bundle”, associating one vector to each orbit. The information contained in this vector is that of one complex number since the vector space is one dimensional. The determinant line bundle may be twisted in the sense that it cannot be parameterized globally and smoothly as a direct product of orbit space times a complex line. When there are twists, all sections must have some zeros. In turn, these zeros are related to anomalies. Thus, in the case of compact gauge groups, anomalies have a definite geometric origin which explains why they are irremovable and why they are associated with quantized
numerical coefficients. In addition, orbit space is relatively complicated, and can contain non-contractible loops around which the chiral determinant can have nontrivial monodromies leading to so called global anomalies.

All in all, it is better not to force the determinant to be a function over orbit space, but define it first as a line bundle over orbit space. Avoiding to think about the chiral determinant as a function is compatible with the Weyl operator being a map between spaces that naturally carry distinct spinorial representations of the Euclidean rotation group. There is no natural identification between these spaces, so it is better to stay with a view of the chiral determinant as a map between some derived spaces. This problem disappears when one considers the product of the Weyl operator with its adjoint and this is compatible with the absolute value of the chiral determinant being a naturally defined function over orbit space.

There exist mathematical relationships between anomalies in different dimensions, with distinct roles played by the even and odd dimensional subsets. These relationships are described by so called “descent equations” which can be physically realized by considering defects of dimension one or two embedded in a manifold of higher dimension.

Anomalies can also be understood in the context of local cohomology as breaking gauge invariance by closed but not exact forms that are locally expressed in terms of the gauge field. Both local and global cohomology considerations also enter in the BRS formulation of gauge invariance.

These continuum properties indicate that path integrals should not be taken too literally when one deals with chiral fermions. Nevertheless, the needed generalization is very mild and path integrals work almost perfectly.

3 LATTICE DIFFICULTIES

In lattice field theory one replaces the compact Euclidean manifold by a finite grid of points. Almost all work is restricted to tori. The torus is replaced by a regular lattice preserving a large discrete subgroup of translations. The inverse of the nearest neighbors spacing $a$ in the grid is a physical ultraviolet cutoff and, at distances much larger than $a$ an effective continuum Lagrangian description becomes valid. This effective Lagrangian, so long it is used only in the regime where it applies, can work then just as well as the true effective Lagrangian of Nature does, although nobody thinks that spacetime really is a lattice. The lattice construction is made outside perturbation theory and this is its main advantage. For the lattice to be useful one needs some mechanism on the lattice that would ensure exact chiral symmetry in the effective Lagrangian. A finite grid implies a finite Grassmannian path integral which is well defined and therefore no longer has the flexibility to admit the needed slight generalization mentioned in the previous section. This reflects itself in some difficulties when trying to enforce chiral symmetry on the lattice. In addition, orbit space on the lattice is connected so before one speaks about strange effects at nonzero $n_{\text{top}}$ one needs to give a meaning to $n_{\text{top}}$. To successfully take chirality to the
lattice one must also take gauge field topology to the lattice and do this in a selfconsistent way.

3.1 Nielsen-Ninomiya Theorems

The lattice is unique as an ultraviolet cutoff in that it can preserve gauge symmetries. Given a lattice action with finite interaction range and some global internal symmetry it is guaranteed that one can gauge that symmetry, producing a new exactly gauge invariant action. Let us assume that we can write down a finite range lattice field theory describing a single Weyl fermion. Since we could gauge its $U(1)$ symmetry and the resulting theory ought to be anomalous we reached a paradox. The resolution to the paradox is that the toroidal structure of lattice momentum space implies in this case that there will be other massless excitations, which, after gauging, compensate the anomaly [11]. In particular, QCD with a fermion action invariant under chiral transformations (i.e. massless quarks) cannot be put on the lattice because of the absence of obstacles to gauge its chiral symmetries.

3.2 Instantons and Undesired $U(1)$’s

Instantons imply that some of the global symmetries of a fermionic action are destroyed by quantum effects. This is why there is no “ninth Golstone boson” in QCD. However, a global symmetry of a local lattice action is indestructible [12].

3.3 Troubles with BRS Invariance

In the continuum one can replace the principle of gauge invariance by that of BRS symmetry [13]. This might indicate that one could restrict ones attention to gauge fixed actions avoiding the head on collision with the lattice’s ability to easily accommodate gauge invariance. Unfortunately, it seems that BRS symmetry does no generalize to the lattice in an exact way [14].

4 HEURISTIC DESCRIPTION OF NEW APPROACH

Lattices and chiral symmetry have been reconciled by slightly enlarging the concept of fermion path integration in that infinite numbers of fermions are allowed to live on the traditional finite grids.

4.1 Infinite Number of Fermions

If we do not insist on chiral symmetry one can easily put (massive) Dirac fermions on the lattice. The known lattice difficulties imply that the chirality breaking terms cannot be removed without changing the theory in a substantial
way, which goes far beyond rendering the Dirac fermions massless. For several Dirac fermions we can introduce a general mass matrix, but unitary transformations of the fermion fields can be used to make this matrix diagonal with non-negative entries. However, if we make the mass matrix infinite and view flavor space as an infinite dimensional Hilbert space, we can arrange that the above “unitary” operations be not allowed. Moreover, choosing the mass matrix to be a Fredholm operator with nontrivial index we can destroy the balance between the left and right Weyl components of the Dirac fermions. One of the “unitary” operators that would normally rotate one of the fermion fields to produce a diagonal mass matrix becomes a partial isometry, and therefore no longer induces an innocuous change of variables in the fermion integral. In short, the generalization of path integrals we require is that it admit a certain kind of infinite flavor space. In Feynman diagrams one only needs to make sure that all infinite sums one encounters converge. Some transformations of integration variables in the path integral are no longer allowed. The path integral itself no longer is guaranteed to give an unambiguous answer, since the matrix whose determinant we would need to take is infinite.

4.2 Infinite Mass Matrix from Domain Walls

The basic idea described above was motivated by the domain wall setup. With the benefit of hindsight the logical order can be reversed and domain walls can be viewed as one particular realization of a mass matrix operator with index.

5 FUNDAMENTALS

The basic idea is to employ a Lagrangian whose fermionic terms have the following structure:

$$\mathcal{L}_\psi = \bar{\psi}D\psi + \bar{\psi}(P_L\mathcal{M} + P_R\mathcal{M}^\dagger)\psi, \quad P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5).$$

The fermionic fields are Grassmann valued states in the flavor Hilbert space, indexed by a spacetime coordinate, a Dirac index and a group representation index. On a lattice the spacetime coordinate becomes a discrete index of finite range. The mass matrix is an operator in flavor Hilbert space and flavor index is summed over by viewing each term in the Lagrangian as an inner product in flavor Hilbert space. This is possible because the action is bilinear. The main property of the mass operator is that it has unit index and that it is proportional to a dimensionful mass parameter which plays the role of an ultraviolet cutoff in an effective theory working at much lower energy. Chirality is not at all necessarily apparent in the full theory, but emerges in the effective description of low energy physics in a natural way, as a consequence of the structure of the mass matrix in flavor space. If Nature works in a similar way, ordinary chirality has no fundamental meaning. The full theory has a new mechanism which makes chirality appear naturally at low energies.
5.1 Lattice Transfer Matrix and Overlap

A simple choice for the flavor Hilbert space is that it be the linear space of infinite complex sequences with $L^2$ inner product. These sequences can be viewed as living on a one dimensional infinite lattice. It is easy to write down a simple bounded operator with index acting in this space. In the site basis it reads:

$$M_{s,s'} = \delta_{s+1,s'} - a(s)\delta_{s,s'}, \quad s, s' \in \mathbb{Z},$$

$$a(s) = a^0_+ \text{ for } s \geq 0, \quad |a^0_+| < 1, \quad a(s) = a^0_- \text{ for } s < 0, \quad |a^0_-| > 1. \quad (2)$$

One needs to be specific about the form of the other terms in the fermionic Lagrangian, this time specializing to the lattice.

$$L_\psi = \frac{1}{2} \bar{\psi} \gamma_\mu (T_\mu - T^\dagger_\mu) \psi + \bar{\psi} P_L \mathcal{M}(w) \psi + \bar{\psi} P_R \mathcal{M}^\dagger(w) \psi,$$

$$T_\mu(\psi)(x) = U_\mu(x)\psi(x + \mu), \quad w = \sum_\mu \left(1 - \frac{T_\mu + T^\dagger_\mu}{2}\right). \quad (3)$$

To avoid fermion doubling the mass operator must act nontrivially also on indices other than flavor. $\mathcal{M}$ acts trivially on spinor indices. In the absence of gauge fields, we need to make the parameters $a_\pm$ in $\mathcal{M}$ operators in spacetime in such a way that the index of $\mathcal{M}$ be unity only in a small region around the origin of momentum space, but vanish outside this region. Thus, we avoid creating massless fermions at the doubler locations. It is crucial that this can be accomplished with a $\mathcal{M}$ that is totally smooth in its dependence on momenta. Clearly, the spectral properties of $\mathcal{M}$ are not smooth in their dependence on momenta, and this is possible only because $\mathcal{M}$ is an operator in an infinite dimensional space. When the gauge field is turned on, gauge covariance dictates that the parameters $a_\pm$ also act now on the representation index:

$$a_\pm = a^0_\pm + w. \quad (4)$$

The next step is to make sense of the fermionic path integral in this setting. This is done by introducing a finite dimensional fermionic Fock space generated by creation and annihilation operators $\hat{a}^\dagger, \hat{a}$ acting on a vacuum. There is one new pair of $\hat{a}^\dagger, \hat{a}$ for each lattice site $x$, group index $i$ and Dirac index $\alpha$.

Associated with the two semi-infinite lines $s > 0$ and $s < 0$ there are two “evolution” operators, $\hat{T}_\pm = e^{-\hat{a}\hat{a}^\dagger}, \hat{T}_\pm = e^{-H}_\pm$ where $T_\pm = e^{-H}_\pm$ are transfer matrices. The $\hat{T}_\pm$ are smoothly parameterized by the link variables $U_\mu(x)$. The path integrals over the fermion fields located on the semi-infinite lines are interpreted as projectors on the ground states of $\hat{T}_-, \hat{T}_+$ so the entire path integral produces the overlap formula for the lattice regulated chiral determinant [13]:

$$\text{chiral determinant} = \langle -|+ \rangle. \quad (5)$$
5.2 Continuous Flavor Space

Our construction did not make use of the boundedness of the operator \( M^\dagger M \), so we may as well make flavor space continuous and enjoy the simplified form of the single particle Hamiltonian matrices [16]:

\[
H_\pm = \gamma_5[m_\pm + \sum_\mu (1 - V_\mu)], \quad V_\mu = (V_\mu^\dagger)^{-1} = \frac{1 - \gamma_\mu}{2} T_\mu + \frac{1 + \gamma_\mu}{2} T_\mu^\dagger.
\]

The mass parameters are limited by \( m_- > 0 \) and \(-2 < m_+ < 0\). Nothing of importance changes in the limit \( m_- \to \infty \) [17]. Overall scales of \( H_\pm \) do not enter so we can take \( H_- = H' = \gamma_5 \) and \( H_+ = H \). The main point is that \( H \) provides a lattice description of Dirac fermions of negative mass, while \( \gamma_5 \) describes Dirac fermions of positive (actually infinite) mass. The matrix \( m + \sum_\mu (1 - V_\mu) \) is nothing but the Wilson-Dirac operator \( D_W \); it has no chiral symmetry, but by tuning \( m \) to a gauge coupling-dependent value one can recover a single massless Dirac fermion in the continuum limit. It is important to keep in mind that in \( H_\pm \) the mass parameters are not to be tuned.

For the ground state of \( \hat{a}^\dagger H \hat{a} \) to be non-degenerate we need to know that \( H^2 \) is bounded away from zero, something we shall assume and come back to later. One can rescale \( H \) not only by a constant but also by any positive function of \( |H| \) which leads to replacing \( H \) by \( \epsilon = \text{sign}(H) \). To unify notation, we denote \( \gamma_5 = \epsilon' \). \( \epsilon' \) and \( \epsilon \) are reflections, and associated with them are hermitian projectors \( P = \frac{1}{2}(1 - \epsilon) \) and \( P' = P_R \). Let \( N \) be the total dimensionality of the space in which these matrices act. The projectors define two orthogonal decompositions of the total space: the one associated with \( P' \) is standard and gauge field independent and the other depends on the gauge field smoothly, so long the sign function \( \epsilon \) is defined. The non-trivial subspaces are spanned by \( \tilde{v}_i \) with \( P\tilde{v}_i = \tilde{v}_i \), \( i = 1, 2, ..., N_v \) and by \( \tilde{w}_i \) with \( P\tilde{w}_i = 0 \), \( i = 1, 2, ..., N - N_v \). The trivial subspaces are associated with \( P' \), and the related vectors are primed. It is convenient to introduce \( N \times N_v \) matrices \( v = (\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{N_v}) \) and do the same with the other collections of vectors. Then \( P = vv^\dagger \), \( v^\dagger v = 1 \), \( 1 - P = ww^\dagger \). If \( N_v = N/2 \) the overlap is

\[
\langle v'|v \rangle = \det(v'^\dagger v),
\]

and if \( n_{\text{top}} \equiv N_v - N/2 \) does not vanish, \( \langle v'|v \rangle = 0 \) [16].

5.3 Effective Low Energy Theory

The effective theory describing only the massless fermion modes is directly obtainable from the propagators in the full theory, because the fermions enter bilinearly. Actually, we don’t even need the action if we have the propagators and the determinant, and, in addition, know how to handle zero modes. Thus, we can eliminate from the picture the entire flavor space, making the domain walls disappear.
Let us first assume that \( n_{\text{top}} = 0 \). Then the propagator is obtained from the matrix element of the creation/annihilation operators between the two vacua \([16]\):

\[
G_{JI}^R = \frac{(v'|\hat{a}_I^\dagger \hat{a}_J^\dagger|v)}{(v'|v)} \Rightarrow G^R = v[v'^\dagger v]^{-1}v'^\dagger. \tag{8}
\]

The matrices \( v \) and \( v' \) are defined up to unitary transformations \( v \rightarrow vU \)
\[ v' \rightarrow v'U' \]. These transformations change the phase of \( \langle v'|v \rangle \) but leave \( G^R \) invariant. Thus, the ambiguity exploited by potential anomalies is restricted to affecting the chiral determinant only. Also, it is clear that

\[
w^\dagger G^R = 0, \quad G^R w' = 0, \tag{9}\]

showing that \( G^R \) is of reduced rank, as it should be since it represents Weyl fermions in a space large enough to accommodate Dirac fermions.

There is an additional freedom in the definition of \( G^R \) stemming from the ordering of \( \hat{a}^\dagger \) and \( \hat{a} \). A more symmetric definition is possible: \( G^R \rightarrow G^R - 1/2 \). With it, defining \( G^L \) from the matrices \( w' \) and \( w \) and again shifting, the new propagators satisfy \( G^R = -G^L \), just like in the continuum.

So long \( n_{\text{top}} = 0 \) only products containing equal numbers of \( \hat{a}^\dagger \) and \( \hat{a} \) have non-zero matrix elements, and these are determined completely by the bilinear we just evaluated as a consequence of a slight generalization of Wick’s theorem. When \( n_{\text{top}} \neq 0 \), to get a non-zero matrix element one needs a product containing unequal numbers of creation and annihilation operators, the imbalance being determined by \( n_{\text{top}} \). It is not hard to derive explicit formulae for this case too \([16]\).

The collection of second quantized state \(|v\rangle\) as a function of the gauge background makes up a line bundle over the space of link variables for which the sign function is defined. The line bundle of the \(|v'\rangle\) is trivial and gauge invariant. This makes the overlap a line bundle over orbit space, just the kind of solution to the problem of chiral fermions the results in the continuum were pointing to. The \(|v\rangle\) line bundle descends from the \( N_{\nu} \) dimensional vector bundle of subspaces defined by \( v \). The latter bundle has a natural complement in the bundle defined by \( w \). The sum of these bundles is obviously trivial, so the twists in each are complementary, explaining the tight relation between right and left handed Weyl fermions. In particular, \( \langle w'|w\rangle^\ast = \langle v'|v\rangle \) \([16]\).

5.4 Perturbative Calculations

To check the formulation in perturbation theory one can proceed in two ways. The quicker way is to calculate the chiral determinant numerically for simple perturbative gauge backgrounds, take the continuum limit and compare to the perturbative continuum result \([16, 18]\). The success of such calculations is strong indication that perturbation theory works. Analytical calculations take more effort and time, but they leave no doubts.

Several types of perturbative calculations have been carried out. The first was an analytical calculation of the gauge anomaly in two dimensions for a chiral
fermion in a $U(1)$ gauge field \[14\]. Soon after, this calculation was followed by numerical calculations in special backgrounds in two and four dimensions, all concentrating on the anomaly and getting the correct consistent coefficient in the abelian case \[10, 13\].

The first comprehensive set of analytical calculations were carried out by S. Randjbar-Daemi and Strathdee \[19\]. All chiral anomalies were calculated in the overlap and shown to come out right, and even Higgs fields were included. It was established that the minimal standard model in the overlap formulation worked correctly to one loop order. This included calculations of the vacuum polarization and of RG functions. Also, both the covariant and the consistent forms of the anomalies were shown to come out correctly. An important extensions of the formalism was the calculation of Lorentz anomalies for chiral fermions coupled to two dimensional Euclidean gravity.

A new wave of perturbative calculations \[20\] focused on the vector-like context, where the overlap simplifies in a manner to be discussed later on. Earlier calculations focused on the abelian axial anomaly. More difficult calculations were carried out for the lattice $\Lambda$-parameter and the renormalization of various operators, quark bilinears and four quark operators. These calculations also dealt with currents. This work will be essential in interpreting numerical QCD data obtained using overlap fermions and extracting phenomenological numbers from it.

Surprisingly, it turned out that an old perturbative calculation of Ginsparg and Wilson implied that the overlap has the right axial anomaly. In the context of domain walls there were early calculations of the covariant axial anomaly \[9\] and later calculations of one loop corrections to the quark mass when the chiral symmetry is only approximate \[21\]. More recently several domain wall calculations were carried out with a QCD orientation \[22\].

5.5 Odd Dimensions

In odd Euclidean dimensions masslessness of fermions is protected by parity \[23\]. There are no perturbative anomalies and no topology induced expectation values of unbalanced products of $\bar{\psi}$ and $\psi$. There is less topological structure in the space of gauge orbits. The single kind of problem one may encounter has to do with global gauge anomalies, which occur only if one insists on parity invariance. The overlap, as a mechanism for making fermions massless, can be used just as in even dimensions, only now we should expect that a natural section choice in the line bundle exist, so long we relax parity requirements.

By embedding the odd dimensional torus in a degenerate one of one dimension higher we can use the even dimensional overlap to define the odd dimensional one \[24\]. Making some global basis choices one learns that the overlap hamiltonians in odd dimensions have a simple structure

$$H = \begin{pmatrix} 0 & D_W \\ D_W^\dagger & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (10)$$

where $D_W$ is the Wilson-Dirac operator in odd dimensions $d$, using one of the
two inequivalent $2^{d-1}$ dimensional representation of the $\gamma$-matrices. We are assuming that the gauge background is such that $D_W$ is invertible. The following identity is derived by manipulations familiar from finding solutions to the Dirac equation in the continuum:

$$H \left( V \xi_i - \xi_i \right) = - \left( \sqrt{D_W D_W^\dagger} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{D_W^\dagger D_W} \end{pmatrix} \right) \left( V \xi_i - \xi_i \right).$$  \quad (11)$$

$V$ is a unitary matrix,

$$V = D_W \frac{1}{\sqrt{D_W^\dagger D_W}} = \frac{1}{\sqrt{D_W^\dagger D_W}} D_W$$  \quad (12)$$

and $\xi_i$ is an arbitrary normalized vector. There are $N/2$ independent vectors $\xi_i$ and the above identities show that they can build a basis of the negative eigenspace of $\text{sign}(H)$, trivializing the vector bundle of negative eigenspaces over gauge backgrounds. Let $\xi = (\xi_1, \xi_2, ..., \xi_{N/2})$. Similarly, we get a gauge field independent basis for the negative eigenspace of $\text{sign}(H')$, $\xi'$ producing

$$v^\dagger v = \xi'^\dagger \frac{1 + V}{2} \xi.$$  \quad (13)$$

$\xi$ and $\xi'$ are gauge field independent and can be chosen to be $N/2 \times N/2$ unit matrices. Then, $v^\dagger v = \frac{1 + V}{2}$. This expression is obviously gauge invariant. It is easy to also derive explicit expressions for the fermionic propagators. The odd dimensional descendants of the other handedness correspond to the other inequivalent representation of odd dimensional $\gamma$-matrices.

In odd dimensions one can defined parity as a flip of all coordinates. If the gauge field in $V$ is replaced by its parity transform $V$ changes to $V^\dagger$. We see therefore that the fermion determinant conserves parity only if $\det(V) = 1$, and parity will end up necessarily sacrificed when the potential for global anomalies arises. Note that $\det(V)$ is the exponent of a local gauge invariant functional of the gauge field

$$\det(V) = \det(D_W)/|\det(D_W)|,$$  \quad (14)$$

so that parity could be restored if the spectrum of $V$ could be shown to have a gap (for all gauge fields) at some point on the unit circle. Then, it would be possible to define smoothly $\sqrt{\det(V)}$ and use this quantity to redefine the overlap phase, so that both gauge invariance and parity be preserved. If there is a global anomaly one can show that a gap at a fixed location on the unit circle is excluded because some loops of gauge orbits make $\det(V)$ wind round the circle.

We learn that the overlap is flexible enough to work across all dimensions. Also, there is great benefit in exploiting the relations between consecutive dimensions, as indicated by experience in the continuum.
5.6 Spectral Properties of $H$

For the sign function to be defined we need to avoid situations where $H$ has an exact zero mode. This can be assured by a local, gauge invariant constraint on the gauge fields: The spectrum of $H$ clearly is gauge invariant, so any condition on it is gauge invariant too. For a trivial background the matrices $T_\mu$ can be simultaneously diagonalized, and the spectrum of $H$ has a finite gap around zero. By continuity, this gap cannot close when the matrices $T_\mu$ are slightly deformed by turning on a nontrivial background, so that they no longer commute. The norms of all $T_\mu$ commutators are gauge invariant and controlled by the norms of the plaquette variables, because the $T_\mu$’s are elementary parallel transporters. When the plaquette variables are close to unity the commutators are small in norm and the gap is present.

This can be made rigorous and one can prove the following inequality about the smallest eigenvalue $\lambda_{\text{min}}$:

$$\left[\lambda_{\text{min}} \left(D_W^\dagger D_W\right)\right]^{\frac{1}{2}} \geq 1 - \left(2 + \sqrt{2}\right) \sum_{\mu>\nu} \epsilon_{\mu\nu} - |1 + m|. \quad (15)$$

$-2 < m < 0$ is the mass parameter, and every plaquette variable $U_{\mu\nu}(x)$ obeys

$$\|1 - U_{\mu\nu}(x)\| \leq \epsilon_{\mu\nu}. \quad (16)$$

So long the $\epsilon_{\mu\nu}$’s are small enough a gap is assured. When the continuum limit is approached $\|1 - U_{\mu\nu}(x)\| \to 0$. Thus, if we chose to restrict each plaquette variable by equation (16) the continuum limit would remain intact.

5.7 Global Anomalies in 4D

In the continuum the space of gauge fields is contractible, but the space of gauge orbits may not be so. An important example, due to Witten [26], occurs for $SU(2)$ and has far reaching consequences when we have a single Weyl fermion in the representation $I = 1/2$. The space of gauge transformations is disconnected and this makes the space of gauge orbits multiply connected. Although the chiral determinant is real, it has a twist and gauge invariance cannot be maintained. The existence of the twist can be established by using a spectral flow argument to show that every section will have a net zero of degree one on some gauge orbit.

On the lattice the space of gauge transformations is connected, but the space of orbits still can be multiply connected because one needs to excise gauge configurations for which the spectral gap at zero in $H$ closes. For $I = 1/2$ there is a global basis choice for which the matrix $H$ is real [27], and hence the overlap has no local phase ambiguity. Globally however, the overlap again defines a twisted bundle (this time it is a $Z(2)$ bundle) and Witten’s anomaly is reproduced as the sign switch associated with conic degeneracies in real hamiltonians [28]. This generic type of sign switch has been discovered by Herzberg and Longuet-Higgins [29].
5.8 4D Noncompact Chiral $U(1)$

The first four dimensional anomaly ever understood was for $U(1)$. This is a perturbative anomaly and in perturbation theory it does not matter whether the $U(1)$ is compact or not. The much more recent topological understanding of anomalies does not really apply to noncompact $U(1)$ on infinite $R^4$. The $U(1)$ anomaly can be understood in terms of local cohomology. By starting from some good regularization scheme one finds a violation of gauge invariance in the phase of the chiral determinant. The question then arises whether one could add to the original action a local functional of the gauge field so that its gauge variation cancel the anomaly. It turns out that the anomaly is inevitable because it cannot be obtained from the gauge variation of any local functional. The anomaly itself is a local gauge invariant functional of the gauge field, proportional to $\epsilon_{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}$. $F$ denotes the field strength, which is a two form, $F = dA$, where the gauge potential $A$ is a one form. The anomaly, $(dA)^2$ obviously is closed, meaning $d(F^2) = 0$. If there were a local, gauge invariant, three form $\omega$ such that $F^2 = d\omega$ our counterterm would be $\int A \omega \propto \int \epsilon_{\mu \nu \rho \sigma} A_\mu \omega_{\nu \rho \sigma}$. There seems to be nothing wrong with the equation $F^2 = d\omega$: it is consistent both with $d(F^2) = 0$ and with $\omega$ being gauge invariant. One is free to choose the four unknown components of $\omega$ to reproduce the single component $F^2$. It is obvious that one can find some solutions, however, all will be nonlocal functionals of the field strength. One can classify candidates for anomaly functionals like $F^2$ by looking for local, gauge invariant, functionals of the gauge field that are closed but not exact. They define the local cohomology of the system. In four dimensions there is only one bad functional, $F^2$. So, if we can eliminate it, any other “anomaly” will be removable.

For noncompact chiral $U(1)$ gauge theory the above logic also works on the lattice [30]. If the charges of all Weyl fermions do not satisfy the well known continuum anomaly cancelation condition $\sum q^3 = 0$ one can convince oneself that gauge invariance cannot be restored.

A key role in the argument is played by another geometrical object, the Berry connection associated with the $|v\rangle$ bundle over gauge fields, $A = i\langle v | dv\rangle$. This connection has a field strength, Berry’s curvature, and the latter is gauge invariant because it is independent of the phase of $|v\rangle$. Viewing the space of gauge fields as the base manifold, $F = dA$ is obviously exact. The curvature is defined also over orbit space, but there, while still closed, it is not necessarily exact if space time is compact. In the case when $\sum q^3 = 0$ the total $F$ should be exact, since it would be such if $A$ were gauge invariant, and this would be true if a local gauge invariant phase choice were possible. If anomalies do not cancel it has been shown that there are noncontractible closed two dimensional manifolds in the space of gauge orbits over which the integral of $F$ is a non-zero integer. This effect takes place only when we work on a finite torus. It proves that in the case of the torus a gauge invariant phase choice is impossible if anomalies do not cancel.

However, if the anomaly cancelation condition is met, a gauge invariant choice for the product of chiral determinants associated with the individual
fermions is possible. The construction is quite technical and works completely only on an infinite lattice. One needs to start from an initial phase choice which gives an anomaly for each fermion. These anomalies do not cancel even if \( \sum q^3 = 0 \). The initial phase choice best suited for this problem is an adiabatic phase choice \[^{[32]}\]. It produces relatively simple formulae for the anomalies. One can show then that the condition \( \sum q^3 = 0 \) eliminates the one irremovable lattice version of \( F^2 \), and the rest of the total anomaly can be canceled by a local functional, which enters as an improvement over the original phase choice. Note the similarity of this procedure to the situation in odd dimensions.

In four dimensions one faces an additional complication in the \( U(1) \) case \[^{[32]}\]. The theory does not have a selfconsistent interacting continuum limit due to ultraviolet “triviality”. In the more interesting non-abelian case this problem would not arise, but also the simplification of a noncompact version would be unavailable. For \( U(1) \), whether anomalies cancel or not, the theory only exists (even on paper) as an effective theory, with a physical scale of energies \( E_{\text{ph}} \) and a cutoff \( \Lambda \). Triviality means that the physical coupling constant \( e_{\text{ph}} \) has to vanish as \( \Lambda \) is taken to infinity. The distinction between the gauge invariant and gauge noninvariant cases now is quantitative. If anomalies do not cancel we have

\[
e_{\text{ph}} \leq c_1 \left( \frac{E_{\text{ph}}}{\Lambda} \right)^{\frac{1}{3}},
\]

but if they do, we have a much weaker restriction:

\[
e_{\text{ph}} \leq \frac{c_2}{\log \left( \frac{\Lambda}{E_{\text{ph}}} \right)}.
\]

### 5.9 Majorana-Weyl Fermions

In two and ten dimensional Minkowski space one can impose on Dirac fermions simultaneously a Majorana and a Weyl condition, and be left with propagating fermionic degrees of freedom. The Euclidean version of theories containing Majorana-Weyl fermions in interaction with gauge fields can also be regulated on the lattice using the overlap. In the appropriate dimensions, if one starts with Weyl fermions in a real representation of the gauge group, one can factorize the overlap into two factors, each representing a single Majorana-Weyl fermion. This certainly is important for ten dimensional supersymmetric models, but to keep things simple, we shall discuss two dimensions.

Starting from a real set of link matrices one can rewrite \( \hat{H} \) as

\[
\hat{H} = \frac{1}{2} \hat{\alpha}^\dagger H \hat{\alpha} + \frac{1}{2} \hat{\beta}^\dagger H \hat{\beta},
\]

where the spinorial components of \( \hat{\alpha}, \hat{\beta} \) no longer are independent, but are one the hermitian conjugate of the other. A similar representation exists for \( \hat{H}' \). The total Fock space naturally factorizes, and so does the overlap. Keeping one of the factors it is convenient to introduce hermitian operators, \( \gamma \), replacing \( \hat{\alpha} \) and \( \hat{\alpha}^\dagger \), and generating a Clifford algebra.
The overlap description of the Weyl-Majorana \cite{33} case is based on the many-body hamiltonian
\[
\hat{H} = \frac{1}{2} \hat{\gamma} H \hat{\gamma}, \quad H = \Gamma_3 D_W, \quad D_W = m + \sum_\mu (1 - V_\mu), \quad \Gamma_3 = -\sigma_2,
\]
\[
V_\mu = V_\mu^* = (V_\mu^T)^{-1} = \frac{1 - \Gamma_\mu}{2} V_\mu + \frac{1 + \Gamma_\mu}{2} V_\mu^T, \quad \Gamma_1 = \sigma_3, \quad \Gamma_2 = \sigma_1.
\] (20)

with \(-2 < m < 0\) and \(H' = \Gamma_3\). \(H\) and \(H'\) are hermitian and antisymmetric. By conjugation with a real orthogonal matrix \(O\), \(H\) can be brought to an elementary block diagonal form with \(\sigma_2\) matrices times non-negative numbers strung along the diagonal. \(H'\) is already in this form.

Multiplying all \(\hat{\gamma}\) operators one obtains a “parity” operator \(\hat{\gamma}_{\infty+1}\). \(\hat{\gamma}_{\infty+1}\) squares to unity and commutes with \(\hat{H}\) and \(\hat{H}'\). If \(\det O = -1\), \(|v\rangle\) and \(|v'\rangle\) are eigenvectors of \(\hat{\gamma}_{\infty+1}\) of opposite sign and the overlap vanishes, reflecting a mod(2) continuum index theorem.

Once again we see that the overlap is able to accommodate rather subtle continuum features. The mod(2) index theorem means that an infinite antisymmetric matrix can still be meaningfully odd or even dimensional, and which it is can depend on the gauge field background. No approach based on a finite number of fermions per site could reproduce this.

6 TWO DIMENSIONAL ABELIAN MODELS

Theoretically, the overlap construction has overcome major obstacles and dispelled the folklore that chiral symmetry cannot be realized on the lattice. But, there still remains the question whether this progress can be put to use in numerical simulations that would be practicable at least in the foreseeable future. In addition, in the chiral context, some issues of principle regarding the effects of gauge violating phase choices can be tested only by computer simulation.

One should start gaining numerical experience from the simplest models that have chiral symmetry. This points us to models with chiral fermions in two dimensions.

6.1 Vector-like Case

One of the most important results of the interplay between chirality and gauge fields is the explanation why the \(\eta'\) cannot be thought of as an approximate Goldstone boson, while the \(\eta\) can. A similar effect occurs in the massless Schwinger model in two dimensions, where an apparent global chiral \(U(1)\) symmetry is invalidated by fluctuations in the topology of the gauge field background. This also happens in a generalization of the model to several flavors. With one flavor \(\bar{\psi}\psi\) acquires an expectation value, also at finite Euclidean volumes - for more flavors, \(N_f\), the condensate consists of \(N_f\) factors of \(\psi\) and \(N_f\) factors of \(\bar{\psi}\). These condensates can be exactly calculated in the continuum for

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any size and shape of the Euclidean two-torus and they are a direct consequence of instantons.

On a finite lattice this effect cannot be recovered from a quenched simulations, where fermion loops are ignored. (Actually, the quenched model is theoretically more subtle than the original one [34].) In the course of a Monte-Carlo computation one can identify which gauge field configurations are responsible for the condensate. Thus, we see the Atiyah-Singer theorem at work on the lattice, in a dynamical context.

The simulation produced results entirely consistent with the continuum calculation and with quite high accuracy [35], so that the agreement could not be fortuitous. The parameters of the simulation were chosen so that one could simultaneously get close enough to continuum and still keep the fluctuation in the overlap representing the fermion determinant sufficiently small to permit the inclusion of the determinant in the observable. The fluctuations of the determinant were further diminished by exploiting continuum exact results which made it possible to multiply the determinant entering the observable by a certain function of the gauge background and correcting the action accordingly.

The success of this first simulation showed that quantum fluctuations in the gauge fields are not so violent as to erase topological effects expected from the continuum, semiclassical analysis.

The Schwinger model has been studied more recently [36] using a simpler representation of the overlap which became available in the meantime.

6.2 Chiral Case

In the chiral version of the Schwinger model one has several right-handed and several left-handed Weyl fermions interacting through a \( U(1) \)-gauge field. Perturbative anomalies would cancel if \( \sum_R q_R^2 = \sum_L q_L^2 \). If we were working on infinite Euclidean space and the \( U(1) \) gauge field action were of the non-compact type, starting from an adiabatic phase choice, we could pick an improved phase convention so that the total chiral determinant is gauge invariant.

But, to be practical on the lattice, we need to work on a finite toroidal lattice. Also, it is natural to make the gauge dynamics that of compact \( U(1) \). This induces some complications. The adiabatic phase choice is no longer universally applicable, because it would connect gauge field sectors of different topology by going through gauge field configurations that have \( \text{det}(H) = 0 \). One needs to resort to some other gauge choice; we chose a Wigner-Brillouin phase convention [16].

\[
\langle v(U) = 1 | e^{\hat{v}_{\text{thooft}}} | v(U) \rangle > 0. \tag{21}
\]

Here, \( \hat{v}_{\text{thooft}} \) is a 't Hooft local vertex operator summed over the lattice. \( \hat{v}_{\text{thooft}} \) is the lowest dimension local Lorentz scalar made out of \( \hat{a} \) and \( \hat{a}^\dagger \) carrying the minimal amount of \( U(1) \) charge that can be violated in an elementary topology induced process. Under a gauge transformation \( U \rightarrow U^g \), which acts on \( H \) by \( H(U) = G(g)H(U^g)G^\dagger(g) \), and is implemented in Fock space by \( \hat{G}(g) \),

\[
| v(U) \rangle = e^{iS_{WZ}(U,g)} \hat{G}(g) | v(U^g) \rangle \tag{22}
\]
The phase prefactor is the lattice expression of the Wess-Zumino functional with the Brillouin-Wigner phase choice because

\[
\langle v' | v(U) \rangle = e^{i S_{WZ}(U,g)} \langle v' | v(Ug) \rangle.
\]

(23)

The Brillouin-Wigner phase choice was chosen because it can be shown analytically to preserve many discrete symmetries, restricting \(S_{WZ}\) in a manner consistent with continuum, and because it can be implemented numerically in a straightforward manner [16]. The main disadvantage of this phase choice is that the definition does not hold on all gauge field configurations; however, the subset on which it fails to hold has zero measure in the path integral. The Brillouin-Wigner phase choice is also well suited to perturbative calculations.

Even if perturbative anomalies cancel, the sum of all \(S_{WZ}\) contributions on the lattice will generically not. So long the gauge action is compact, gauge invariance can be restored on the lattice by gauge averaging. In a good theory, gauge invariance could have been restored even without gauge averaging, by adding to the phase of the product of all overlaps a certain local functional of the link variables. It is difficult to construct the required functional, but if it is not too large, gauge averaging would only produce answers that differ by terms that vanish in the continuum limit, by a mechanism due to Förster Nielsen and Ninomiya [37].

The toroidal structure of space-time in conjunction with the compactness of the gauge group induces some complications in the chiral case. These complications seem to imply that there are two dimensional chiral models that, although consistent at infinite Euclidean volume, nevertheless are not entirely consistent on a compact torus. Luckily, there are chiral models that are free of this difficulty, and a good example is the 11112 model [38].

On a torus there are two angular degrees of freedom associated with elementary “Polyakov loops”. These degrees of freedom can be turned on in a background that has no field strength and can be traded for twists in the boundary conditions for the fermions. Even though there is no background field strength, anomalies still need to cancel in the continuum to make the total chiral determinant gauge invariant. The overlap with the Brillouin-Wigner phase choice was shown to reproduce the correct continuum \(\theta\)-function structure of the chiral determinant [39]. Studies of gauge invariance restoration showed that the cases with problems in the continuum behaved badly on the lattice and ought to be avoided. On the other hand, the 11112 model for example, works fine. Thus, this model was selected for a full dynamical simulation.

### 6.3 Massless Composite Fermions

The 11112 model exhibits one of the most interesting applications of chirality: when chiral symmetries are combined with confining forces one often ends up producing massless fermions that are bound states of other massless fermions. In addition, in the continuum the 11112 model has a combined chiral determinant that is positive. On the lattice, before gauge averaging, the product of overlaps
can have a phase, and to again include the chiral determinant in the observable it is important that also the fluctuations in the phase be small. This can be arranged for because the phase vanishes in the continuum limit.

The continuum action of the 11112 model is given by

\[ S = \frac{1}{4e_0} \int d^2 x F_{\mu \nu}^2 - \sum_{k=1}^{4} \int d^2 x \bar{\chi}_k \sigma_\mu (\partial_\mu + iA_\mu) \chi_k - \int d^2 x \bar{\psi}_\sigma^\dagger (\partial_\mu + 2iA_\mu) \psi, \]

where \( \sigma_1 = 1, \sigma_2 = i \). Massless neutral bound states are created by \( \bar{\eta}_{ij} = \bar{\chi}_i \chi_j \psi \) and \( \eta_{ij} = \chi_i \chi_j \bar{\psi} \) and each provide a six dimensional representation of the global SU(4) acting on the \( \chi \)-fields. At first one would guess that the \( \bar{\eta} \) and \( \eta \) fields create massless left-handed Weyl fermions. But, the SU(4) anomaly carried by the left-handed Weyl \( \chi \)-particles would not match unless the \( \eta \)-particles are left-handed Majorana Weyl rather than just Weyl. This requires additional mixing between the \( \eta \) particles, possible only if fermion number is not conserved. Instantons are present in the models and they indeed induce the necessary violation and mixing. The instanton effect makes the 't Hooft vertex,

\[ v = \pi^2 e_0 \chi_1 \chi_2 \chi_3 \chi_4 \bar{\psi} (\sigma \cdot \partial) \bar{\psi} \]

and its conjugate get an expectation value. In terms of \( \eta, v \) looks like an off-diagonal kinetic energy term, \( \epsilon_{ijkl} \eta_{ij}(\sigma \cdot \partial) \eta_{kl} \). The massless Majorana Weyl composites are \( \rho_{ij} \propto [\eta_{ij} - \frac{1}{4} \epsilon_{ijkl} \eta_{kl}] \); the other linear combination has no massless poles in its propagator [40].

The expectation value of the 't Hooft vertex operator can be calculated in the continuum both at infinite volume and on a torus [41]. The continuum number has been obtained on the lattice using the Brillouin-Wigner phase definition and gauge averaging in a full dynamical simulation [42].

7 QCD

Lattice techniques are unique in that they provide, in principle, a method of calculating to any accuracy various properties of a strongly interacting field theory like QCD. This is important not only in itself, but, QCD effects mask almost all minimal standard model predictions and need to be “peeled off” before one can extract from experiment the numerical value of quark masses and mixings. Steady progress over the last twenty years has made numerical QCD into a competitive quantitative method in estimating strong interaction effects. Among other problems, numerical QCD also suffered from the difficulty to make global chiral symmetries only weakly broken on the lattice, just as it is in the continuum. This problem has been solved in principle and the solution is already marginally practicable. Further advances, including the certain increase in computational power, will eventually make it possible to incorporate the new lattice chirality into most numerical QCD projects.
7.1 Instantons

One of the first areas where the new formalism has an advantage is in identifying instanton effects on the lattice. In QCD instanton effects are important mainly because of their impact on fermions and the key ingredient is chirality. In the background of an instanton the Weyl-Dirac operator governing the dynamics of, say, the right handed component of a massless quark, should be thought of as an infinite matrix with one more row than columns. This situation is represented by having $N_v$, the dimension of the negative eigenspace of $H$, differ from $N/2$ by one unit. This happens when the mass parameter in $H$, $m$, is around $-1$. For positive $m$, $N_v = N/2$ always. Therefore, as $m$ is decreased from a slightly positive value to about $-1$, the lowest positive eigenstate of $H$ crosses zero and becomes negative. The same is true if the background is more general, but still carries the same amount of topological charge as an instanton would.

The flow of eigenvalues as a function of the mass parameter $m$ provides a relatively inexpensive method for numerically finding the topological charge, $n_{\text{top}}$, in the overlap definition. Several studies in two and four dimensions have been made using this method and the results were quite useful [16, 18, 43]. In principle, the main advantage of the method is that it uses a definition of topological charge on the lattice that is entirely consistent with the response of the overlap lattice fermions. Thus, even if the gauge background is quite “rough”, and a precise association with any topological number somewhat questionable, there is no question that precisely this background contributes to the $\eta-\eta'$ mass difference, on the lattice, before any continuum limit is taken.

Superficially, the flow method is similar to a method used before [44] to estimate topological charge. There, one used traditional Wilson fermions with a mass $m$ tuned to the vicinity of its critical value. Instantons were related to approximate zero modes of the hermitian Wilson-Dirac operator with substantial chirality. A fermion state was considered an approximate zero mode if it was the eigenstate of the hermitian Wilson Dirac operator that would cross zero for some $m$ numerically close to the critical $m$ and had substantial chirality. In the overlap we look at the same eigenvalue flows, but not only in the vicinity of the critical $m$.

7.2 The Overlap Dirac Operator

There were two essential reasons against having a traditional lattice action for Weyl fermions. The first was that the global $U(1)$ associated with the Weyl fermions could be gauged and no room for anomaly was left. The second was that an instanton background changed the effective shape of the Weyl-Dirac operator from square to rectangular. In a vector like theory we have a pair of conjugate Weyl fermions. There still exists an objection against a traditional action, but it is weaker now: One should still be prohibited from gauging the axial $U(1)$. On the other hand, the pair of Weyl fermions are controlled by Weyl-Dirac matrices of complementary rectangular shapes, so that, when packed together to make the Dirac operator, they produce a square matrix of fixed size.
From the overlap we know that the global $U(1)$ is avoided since the fermionic number of $|v\rangle$ can be unequal to that of $|v'\rangle$. This means that the global chiral $U(1)$ is not quite realized. We conclude that it should be possible to have a more traditional formulation in the vector-like case than in the chiral case. All that has to happen is that the global $U(1)$ must not quite hold, but appear in a way realized by the overlap. So long there is no obvious way to gauge this $U(1)$ we should be fine.

Looking for a more traditional action for the vector-like case we observe that in one dimension higher the overlap formulation simplifies as we have seen and one has a simple natural formula for the five dimensional kernel $v'^\dagger v$. Dimensionally reducing this formula produces a field with the right number of components to be a Dirac field in even dimensions. We therefore are led trivially to the following action (for definiteness in four dimensions) [45]:

$$\bar{\psi}D_o\psi, \quad D_o = \frac{1 + V}{2}, \quad V = D_W \frac{1}{\sqrt{D_W^\dagger D_W}} = \gamma_5 \text{sign}(H).$$

Actually, recalling that the five dimensional case was derived from the six dimensional overlap, we can dimensionally reduce by 2 from six dimensions directly and obtain the same formula. A Weyl fermion in six dimension has the same number of components as a Dirac fermion in four. The overlap Dirac operator $D_o$ does not anticommute with $\gamma_5$, but clearly represents massless fermions because it comes from Weyl fermions in six dimensions. $D_o$ does have a special relation with $\gamma_5$: The propagator $D_o^{-1}$ anticommutes with $\gamma_5$ up to a local contact term, as a result of $\gamma_5 V \gamma_5 = V^\dagger$. While the contact term has no influence on low energy consequences of chirality, it prohibits the construction of an exactly axial-$U(1)$ gauge invariant theory.

$$\{D_o^{-1}, \gamma_5\} = \gamma_5 \left[ \frac{2}{1 + V} + \frac{2}{1 + V^\dagger} \right] = 2 \gamma_5, \quad \{D_o^{-1} - 1, \gamma_5\} = 0.$$  

Unexpectedly, the above relations turned out [46, 47] to be a particular case of a proposal by Ginsparg and Wilson made already in 1982 [5]. This time there is no doubt that the desired $D_o$ exists, so long $H$ is gapped around zero. More recently, it was suggested that another action satisfying the Ginsparg Wilson criterion is provided by a fermionic “perfect” action [48]. This action is allegedly obtainable as a limit of an iteration process that acts also on the gauge field background. There is no explicit formula, and the iteration process involves a non-analytic step of maximization. It is unclear that the iteration process has a limit, that such a limit, if it exists, is unique and it is worrisome that analyticity in the gauge field background is sacrificed in each iteration. Moreover, the precise set of requirements, beyond that of satisfying the Ginsparg Wilson relation, a fermionic action needs to obey in order to really represent one massless Dirac fermion on the lattice is not fully understood. There are known actions that do satisfy the Ginsparg Wilson relation but do not represent massless Dirac fermions [49].
With the help of $D_o$ the formula $n_{\text{top}} = -\frac{1}{2} Tr \epsilon$ can be rewritten as $n_{\text{top}} = -Tr \gamma_5 D_o$ and the $Tr$ operation can be rewritten as $Tr O = \sum_x tr O_{xx}$ where $tr$ denotes a trace over spinorial and group indices and $x$ is a lattice site. The quantity $\gamma_5 D_{\text{cov}}$ is the divergence of the covariant current defined in \[31\]. The manipulations required to see this explicitly can be found in \[30\].

### 7.3 Truncation I: Domain Walls

While $D_o$ looks simple and is a concrete resolution to the chirality problem in the vector-like context, its practical usefulness seems unclear: the sign function of $H$ is not easy to compute numerically, and it is not a sparse matrix. Somehow, this exact expression has to be truncated to make it practical for use. Since we got to the overlap by making the fifth direction in Kaplan’s setup (modified by taking the gauge fields to be truly four dimensional) infinite the first thing to analyze is the case when we make the extent of the fifth dimension finite again. As is clear from the original paper by Kaplan and from other work, one gets a system containing many Dirac fermions, with one of them very light. The mass of the light fermion goes to zero exponentially with the length of the fifth dimension.

A more precise description of the domain wall setup is that it describes a finite, but large, number of Dirac flavors, whose action is local and represented by a sparse matrix, but who are mixed by a gauge field dependent mass matrix in such a way that when the number of flavors goes to infinity, the mass of one of the Dirac fermions goes to zero in a way that requires no fine tuning, while all the rest of the Dirac fermions (including doublers) keep a mass of the order of the inverse lattice spacing. This must mean that the effective action for the light fermion is a truncation of some version of the overlap Dirac operator, a truncation that becomes exact in the limit of an infinite number of flavors. It is important to understand the mechanism that ensures masslessness in the limit, without fine tuning. To this end we need some notation:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \quad C = \sum_\mu \sigma_\mu (T_\mu - T_\mu^\dagger), \quad B = m + \sum_\mu (1 - \frac{T_\mu + T_\mu^\dagger}{2}). \quad (28)$$

The number of flavors is $k$. The matrices $B$, $C$ are $q \times q$ where, for $SU(3)$, $q = 6V_l$, with $V_l$ being the total number of sites in the four-dimensional space-time lattice. The action is

$$S = \bar{\psi} D(0, 0) \psi, \quad (29)$$

where the matrix $D(0, 0)$ is a special case of $D(X, Y)$ with $X$ and $Y$ general $q \times q$ matrices.

$$D(X, Y) = \begin{pmatrix} C^\dagger & B & 0 & 0 & 0 & \ldots & \ldots & 0 & X \\ B & -C & -1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & -1 & C^\dagger & B & 0 & \ldots & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ Y & 0 & 0 & 0 & 0 & \ldots & \ldots & B & -C \end{pmatrix}. \quad (30)$$
A key formula is

$$\det D(X,Y) = (-)^{q(k-1)}(\det B)^k \det \left[ \begin{pmatrix} -X & 0 \\ 0 & 1 \end{pmatrix} - T^{-k} \begin{pmatrix} 1 & 0 \\ 0 & -Y \end{pmatrix} \right]$$  \hspace{1cm} (31)$$

with the transfer matrix $T$ given by

$$T = e^{-H} = \left( \begin{array}{cc} 1 & B \\ C^T & C \end{array} \right) C^{\dagger} (C C^{\dagger} + B).$$  \hspace{1cm} (32)$$

The determinant of $D(0,0)$ contains contributions from the light fermion and from all heavy ones. Setting $X = Y = \mu$ with $\mu << 1$ gives the light fermion a mass of order $\mu$ at $k = \infty$. Setting $X = Y = 1$ makes all fermions heavy. Thus, one can associate with the massless fermion the ratio

$$\frac{\det D(0,0)}{\det D(1,1)} = \det \left[ \frac{1}{2} \left( 1 + \gamma_5 \tanh\left( \frac{1}{2} kH \right) \right) \right].$$  \hspace{1cm} (33)$$

The division by $D(1,1)$ can be viewed as the consequence of additional $k$ Dirac fields of wrong statistics.

So long $H$ has no zero eigenstates the $k = \infty$ limit is seen to be given by the determinant of an overlap-type operator, only the form of $H$ is different (here $H = -\log(T)$). The difference is irrelevant to the continuum limit. By varying with respect to $X$ and $Y$ one can obtain light fermion propagators. Setting $X = Y = \mu$ and differentiating produces a formula for the physical condensate. Repeating the construction for several physical flavors one gets:

$$\frac{1}{N_f} \sum_{f=1}^{N_f} \langle \bar{\psi}_f \psi_f \rangle_{\text{physical}} = \frac{Z}{L^d} \langle \det N_f \left[ \begin{array}{c} 1 + V \\ 2 \end{array} \right] \rangle T^* \left[ \left( \frac{1}{1 + V} \right) \text{Tr} \left( \frac{1}{1 + V} \right) \right] U.$$  \hspace{1cm} (34)$$

Here $\langle \ldots \rangle_U$ means an average with respect to the pure gauge action and $Z$ is a renormalization constant. An instanton produces an eigenstate of $V$ with eigenvalue $-1$. We see how single instantons would give a nonzero contribution for one flavor, but no contribution for more flavors, just as expected from the more formal continuum expressions. If topology is trivial, conjugation by $\gamma_5$ produces a sign switch, as a consequence of $\gamma_5 \frac{1-V}{1+V} \gamma_5 = - \frac{1-V}{1+V}$. Connection to the Ginsparg Wilson relation is made through $\frac{1-V}{1+V} = \frac{2}{1+V} - 1$.

This analysis easily generalizes to the supersymmetric case where the gauge link variables can be chosen real. Global choices of basis exist then that make the $D$ matrices antisymmetric and, roughly, the determinants get replaced by pfaffians [44, 46].

The mechanism assuring masslessness without fine tuning is a generalized see-saw [44]. Making some further basis changes, the mass matrix can be brought to a hermitian form:

$$M = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & B \\ 0 & 0 & \ldots & 0 & B & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B & -1 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (35)$$
In the absence of gauge fields, $B$ is small for momenta near zero but exceeds unity in the vicinity of the doublers. When $|B|$ is smaller than unity $M$ has one eigenvalue of order $|B|^k$ and all other eigenvalues of order 1. One does not need to fine tune; so long $|B|$ is smaller than one and $k$ is large enough a very light Dirac fermion is present, at least for weak gauge fields. It is not entirely trivial, in this way of looking at things, why the mechanism is stable under radiative corrections. To one loop this has been checked, and at most the radiative correction changes the light fermion mass from $\sim e^{-k} \text{Const}^1$ to $\sim k\text{Const}^2 e^{-k} \text{Const}^1$, where the constants are positive [21].

The main advantage of using domain walls is that one has a local action of traditional form and tested numerical methods can be applied. Theoretically, the domain wall setup is appealing because the truncation has a physical interpretation: the two walls on which the two Weyl components reside are separated by a finite rather than infinite distance. The approximation is limited by how small the eigenvalues of $H^2$ can get in typical gauge fields. Perturbative calculations at finite $k$ are tedious. One cannot focus on the light fermion with ease because the structure of $H$ is awkward.

### 7.4 Truncation II: Overlap

It makes sense to look for other truncations, which are closer to the overlap with the $H$ given by the hermitian Wilson-Dirac matrix. There is no reason to seek a physical interpretation of the truncation, like in the domain wall context.

We are simply faced with a problem in numerical analysis: how do we calculate to machine accuracy the action of $\text{sign}(H)$ on an arbitrary vector when $H$ is sparse and $\text{sign}(H)$ is too large to fit in the computer’s memory. The first observation is that achieving machine accuracy will be very expensive if $H$ is ill conditioned. Only the condition number enters because obviously the sign function is invariant under a positive rescaling of $H$. The most direct way is to look at the sign function as any other special function that we need to implement on the machine. This suggests a rational approximation. Since we have to deal with a possibly ill conditioned matrix, we would like to exploit one of the better tested numerical methods, which is the conjugate gradient algorithm (CG). Thus, we look for a rational approximation to the sign function that can be written as a simple sum of pole terms. Such an approximation can be obtained starting from a Newton iteration algorithm for the sign function [51, 52]:

$$
\epsilon_n(z) = (1 + z)^{2n} - (1 - z)^{2n} = z^n \sum_{s=1}^{n} \frac{z^2 \cos^2 \frac{\pi}{2n} (s - \frac{1}{2}) + \sin^2 \frac{\pi}{2n} (s - \frac{1}{2})}{s^2}.
$$

Clearly, so long $z \neq 0$, $\epsilon_n(z)$ approaches the sign of $z$. The approach is exponential in $n$, but slows when $|z|$ is very small or very large. All needed inversions would involve $H^2$ shifted by varying positive constants. These inversions can all be done simultaneously at a total computational cost equal to that of the slowest inversion. Only memory usage grows linearly with $k$. $k$ also controls numerical accuracy.
The linear growth of memory may be a problem because even if main memory suffices, the faster accessible memory in caches is always severely limited. One can trade memory usage for operations. After all, the CG is a numerically stable application of the Lanczos algorithm and it is known that in the latter one can resort to a “two pass version” in situations where a single pass approach would have a large memory cost. Thus, at the cost of about a factor of two in operations, also the memory demands are limited and \( k \) independent. The difficulty of the problem has now been reduced to its essential part, namely the condition number of \( H \). [53]

If one applies the above method in a dynamical fermion simulation one needs to compute the inverse of \( D \) very often. This results in a two level nested CG procedure, the outer CG computing the inverse of \( D_0^a D_0 \) and the inner the sign of \( H \). This seems cumbersome when compared to domain wall fermions, where a single CG is needed, albeit on a sparse matrix which is of order \( k \) larger than \( H \). Whether there is real cost associated with this complication or not is somewhat unclear. However, even if one is loath of CG nesting, one can still go about implementing the sign function by a rational approximation because any such approximation is equivalent to an extended system governed by a sparse matrix of larger size [54].

For example [55], the system appropriate to the particular representation used above is:

\[
S = \bar{\Psi}\gamma_5 K\Psi. \tag{37}
\]

Introducing also a small quark mass \( \mu \) and the notation

\[
c_s = \cos \theta_s, \quad s_s = \sin \theta_s, \quad \theta_s = \frac{\pi}{2n}(s - \frac{1}{2}), \quad s = 1, 2, ..., n, \tag{38}
\]

the \( K \) matrix can be written as follows:

\[
K = \begin{pmatrix}
-\frac{1+\mu}{2} \gamma_5 & \sqrt{\frac{1-\mu}{2n}} & 0 & 0 & \ldots & 0 \\
\sqrt{\frac{1-\mu}{2n}} & c_1^2 H & s_1 & 0 & 0 & \ldots & 0 \\
0 & s_1 & -H & 0 & 0 & \ldots & 0 \\
\sqrt{\frac{1-\mu}{2n}} & 0 & 0 & c_2^2 H & s_2 & \ldots & 0 \\
0 & 0 & 0 & s_2 & -H & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
\sqrt{\frac{1-\mu}{2n}} & 0 & 0 & 0 & 0 & \ldots & c_n^2 H \\
0 & 0 & 0 & 0 & 0 & \ldots & s_n & -H
\end{pmatrix}. \tag{39}
\]

Detailed study indicates that the above system is as efficient numerically as domain walls, but has the advantage that the effective action for the light degrees of freedom is relatively simple:

\[
D_{\text{eff}} = \frac{1+\mu}{2} + \frac{1-\mu}{2} \gamma_5 \epsilon_n(H). \tag{40}
\]

It is also possible that the nested CG versions, with one or two passes are actually more efficient numerically. Anyhow, the indications are that domain walls...
will be eventually replaced in numerical QCD by a more direct implementation of the overlap Dirac operator, $D_o$.

### 7.5 The Problem of Zero Mode Artifacts

Any numerical method approximating the sign function of some $H$ will face difficulties when $H$ has very small eigenvalues relative to 1 (this presupposes that the largest eigenvalues of $H^2$ are always of order unity). Although the $H$ in the overlap and the one in domain walls are different, their small eigenvalues are closely related, so the same gauge configurations will be problematic in either approach. At numerically feasible QCD gauge couplings it is a fact that there are quite a few states that have very small $H$-eigenvalues $\delta$.

A first worry about these states is whether they signal that somehow the lattice has outwitted us again, and has found a way to avoid chirality. The exact lower bounds on $H^2$ which we discussed before void this worry. What is left to understand is why these modes occur, how they affect QCD observables and what can be done to eliminate them from the numerical process.

That there are some gauge backgrounds that will produce such modes is easy to understand as a result of topology. We know that any lattice gauge configuration can be smoothly deformed to any other so a background that has non-zero lattice topology $n_{\text{top}}$ can be deformed to unit link matrices everywhere. Necessarily, some eigenvalues of $H$ will have to cross zero during the deformation, producing zero modes at any $m$ we happened to choose. We also know that taking $m$ from 0 to a little less than $-2$ must produce several crossings if the background has, say one relatively smooth instanton. This is so because, say, at $m = -1$ we have one massless Dirac fermion, so $n_{\text{top}}$ will change from zero to one, requiring a zero mode at some negative $m$ close to zero. Further, after $m = -2$ four more doublers come into play. Since their frames are reversed in orientation relatively to the fermion at zero momentum, they see an anti-instanton, and would produce, by themselves, a $n_{\text{top}} = -4$. Taking into account all massless Dirac particles we should produce a total $n_{\text{top}} = -3$ which requires four modes to head in the opposite direction and cross zero before $m$ reaches, say $m = -3$. Typically, the mode that crossed first turns around and crosses again, being joined by three more modes. All in all, there is extensive traffic across zero in the spectrum of $H$ as $m$ is varied throughout the region of interest. This is further complicated by the effect of the fluctuations in the gauge fields, which through the Wilson mass term contribute a stochastic component to the mass parameter $m$ moving the crossing points around, and effectively covering the entire range of $m$. The exact upper bound for $0 < m < -4$, $8 + m$ is saturated in typical gauge configurations. When $m$ decreases, the spectrum of $H$ is squeezed and level repulsion has a tendency to push more states towards zero eigenvalue.

The zero modes are a lattice artifact; their appearance results from the substantial difference between the spaces of lattice and continuum gauge fields. The zero modes are well localized and are likely to make only small contributions to correlations at large distances. Hence, they ought to have little numerical
effect on physical observables. Explicit examples of localized zero modes and the localized gauge field configurations that support them are presented in [57].

There are many ways to ameliorate the numerical pain caused by approximate zero modes to $H$. This is an area of intensive current research.

8 SUMMARY

The folklore that chiral symmetries cannot be realized on the lattice has been almost rigorously proven to be false. Chiral symmetry and the lattice get reconciled by a mechanism that employs an infinite number of fermions per unit Euclidean four volume. This mechanism extends beyond lattice regulators, and provides a generic way for effective Lagrangians with chiral fermions to appear out of more fundamental theories which may contain no concept of chirality. The solution to the problem of chiral symmetry on the lattice is almost complete and the remaining open issues are mostly technical. Perhaps the most physical of the remaining open problems is to construct a Hamiltonian version of the overlap.

The key to the breakthrough was to take seriously the indications from the mathematical insights gained during the mid eighties about anomalies and about the chiral determinant. Crucial was the acceptance of the viewpoint that the chiral determinant is a section in a line bundle over orbit space and that the response of fermions to topology indeed is as semiclassical computations indicated.

With the benefit of hindsight it appears that one could have solved the problem of lattice chirality much earlier had the suggestion of Ginsparg and Wilson been scrutinized more thoroughly. Historically, this can be viewed either as the case of a paper before its time, or assign blame to the many lattice papers in the decade between the early eighties to the early nineties that produced failure after failure, ignored the mathematical progress that was taking place in the continuum, and ended up reinforcing what now is widely considered a false folklore.

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