Quantacell: powerful charging of quantum batteries

Felix C Binder\(^1\), Sai Vinjanampathy\(^2\), Kavan Modi\(^3\) and John Goold\(^4\)

\(^1\) Clarendon Laboratory, Department of Physics, University of Oxford, Oxford OX1 3PU, UK
\(^2\) Center for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore
\(^3\) School of Physics and Astronomy, Monash University, Victoria 3800, Australia
\(^4\) The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy

E-mail: felix.binder@physics.ox.ac.uk and jgoold@ictp.it

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Abstract

We study the problem of charging a quantum battery in finite time. We demonstrate an analytical optimal protocol for the case of a single qubit. Extending this analysis to an array of N qubits, we demonstrate that an N-fold advantage in power per qubit can be achieved when global operations are permitted. The exemplary analytic argument for this quantum advantage in the charging power is backed up by numerical analysis using optimal control techniques. It is demonstrated that the quantum advantage for power holds when, with cyclic operation in mind, initial and final states are required to be separable.

1. Introduction

The maximum work that can be extracted unitarily from a given quantum state \(\rho\) with respect to a reference Hamiltonian \(H_0\) was called ergotropy by Allahverdyan, Balian, and Nieuwenhuizen (Allahverdyan et al 2004). In this extraction scheme the final system state is unique up to degeneracies at the level of the Hamiltonian and belongs to the class of passive states (Lenard 1978, Pusz and Woronowicz 1978). Such states are characterized by being diagonal in the energy eigenbasis of the Hamiltonian and ordered with decreasing eigenvalues corresponding to increasing energies.

This paradigm of unitary work extraction has recently been extended to scenarios where multi-partite systems are considered. It has been shown by Alicki and Fannes (Alicki and Fannes 2013) that the extractable work can be increased by allowing for entangling operations. Subsequently, Hovhannisyan et al (Hovhannisyan et al 2013) demonstrated that, irrespective of an entangling operation being required for optimal work extraction, the entanglement created in the extraction processes can indeed be minimal and even vanish. They argue that the cost of this minimization is a longer process duration and consequently conjecture that there may be a relation between the power of such a process and the entanglement created. This line of reasoning has been extended to explore the work cost in the creation or erasure of more general correlations (Perarnau-Llobet et al 2014, Bruschi et al 2015, Giorgi and Campbell 2015, Huber et al 2015). In this article we are interested in the inverse process of work extraction: charging a quantum battery.

We are motivated by the following question: Are there truly quantum effects or phenomena intrinsic to a specific thermodynamic process that offer an operational advantage over their classical counterpart? After all thermodynamics should be sensitive to the underlying microscopic description. For example, the efficiency of an engine should always be limited by the Carnot bound irrespective of whether the working medium is comprised of quantum or classical components. However for non-equilibrium protocols the question is more subtle. For instance very recently it has been demonstrated that it is possible to use non-equilibrium short cuts to adiabaticity to boost the power of engine cycles without compromising efficiency (Deng et al 2013, del Campo et al 2014). One may then wonder whether it is in the finite time operation of devices that quantum mechanics offers an advantage. The present article provides an intuitive example of one such process where quantum correlations provide an advantage: when taking the finite time nature of a charging process into account. In this case we demonstrate that the non trivial quantum nature of both the Hamiltonian and the state leads to a
substantial advantage. That is, entangling processes, which allow for the creation of quantum correlations, can achieve a higher power than local operations. This is the case even when correlations are only allowed to appear during the process but not in the input and output states.

This paper is structured as follows. In section 2 we outline the general problem of work extraction and powerful driving in a quantum system. We then derive a power-optimal driving for a single qubit—a case where constraints on the driving are unambiguous—in section 3. Section 4 extends the results to an array of N qubits and we demonstrate an advantage in power that stems from the permittance of entangling operations both for an initial pure and an initial thermal state. We conclude in section 5.

2. Charging a quantum battery

A battery is a physical system that stores energy. In this paper we use quantum systems, which store energy in the energy levels and coherences, for this purpose (Alicki and Fannes 2013). The internal energy of a quantum system (the battery in this case) is then given by $\text{tr} \{ \hat{\rho} \hat{H}_0 \}$, where $\hat{\rho}$ is the state of the battery and $\hat{H}_0$ is its internal Hamiltonian. Charging a battery is to change its state from $\hat{\rho}$ to a more energetic state $\hat{\rho}'$ such that $\text{tr} \{ (\hat{\rho}' - \hat{\rho}) \hat{H}_0 \} \geq 0$. Conversely using the battery will take it a lower energy state $\hat{\rho}''$ such that $\text{tr} \{ (\hat{\rho}'' - \hat{\rho}) \hat{H}_0 \} \leq 0$. We restrict both charging and discharging processes to be cyclic unitary processes. This is achieved by applying an external time-dependent potential $V(t)$ for time $T$, in addition to the internal Hamiltonian of the battery $\hat{H}_0$. As a consequence of unitarity the spectrum $\{ p_i \}$ of any accessible battery state is fixed.$^5$

The lowest energy state, $\hat{x}$, is called a passive state, and analogously the highest energy state, $\hat{\omega}$, a maximally active state; both of them are defined with respect to $\hat{H}_0$. Let us express the internal Hamiltonian with increasing energy levels as

$$\hat{H}_0 := \sum e_i \ket{i}_\hat{e} \bra{i} \quad \text{with} \quad e_i \leq e_{i+1}. \quad (1)$$

The passive and active states are respectively

$$\hat{x} := \sum p_i \ket{i}_\hat{e} \bra{i} \quad \text{with} \quad p_i \geq p_{i+1} \quad \text{and} \quad (2)$$

$$\hat{\omega} := \sum p_i \ket{i}_\hat{e} \bra{i} \quad \text{with} \quad p_i \leq p_{i+1}. \quad (3)$$

Note that all thermal states are passive.

If the battery is in a generic state $\hat{\rho} = \sum p_i \ket{i}_\hat{e} \bra{i}$ (with $p_i \geq p_{i+1}$), we can use all available energy and the battery will end up in its passive state. The amount of extractable energy from this battery is called ergotropy $W$.

It is defined as the cyclic work extractable with respect to the internal Hamiltonian, $W := \text{tr} (\hat{H}_0 \hat{\rho}) - \text{tr} (\hat{H}_0 \hat{x})$. Once a quantum state $\hat{\rho}$ has been transformed to a passive state $\hat{x}$, no more work is extractable from that state via cyclic unitary transformations. Therefore ergotropy quantifies the available energy in a battery. Conversely, a fully charged battery is in state $\hat{x}$ and the system cannot be charged further via a cyclic unitary process. Note that states $\hat{x}$, $\hat{\omega}$, and $\hat{\rho}$ are unitarily related to each other.

In practice it is often desirable to charge a battery quickly, or better: with maximal power. Let us consequently address the question of powerful charging of quantum batteries. For the charging protocol we start with a generic state $\hat{\rho}$ and apply cyclic, unitary evolution for some optimal time $T$,

$$\hat{H}_t = \hat{H}_0 + \hat{V}_t \quad \text{with} \quad \hat{V}_t = 0, \quad \text{for} \quad t < 0 \quad \text{and} \quad t > T, \quad (4)$$

where the index $t$ indicates time-dependence. The protocol duration $T$ is not fixed but rather part of the optimal solution. Here we are not particularly interested in the specific output state $\hat{\rho}'$ but rather in the average work $\langle W \rangle$ done in the process and, importantly, the average power $\langle P \rangle = \langle W \rangle / T$. Alternatively, one could study other process-duration dependent quantities. For instance, maximizing $\langle P \rangle \times \langle W \rangle$ will guarantee that power is not optimized at the cost of average work. In order to be fully general we thus consider the family of objective functions

$$F := \langle P \rangle^\alpha \langle W \rangle^{1-\alpha} = \frac{\langle W \rangle}{T^\alpha}, \quad (5)$$

where $T$ is the process duration and $0 \leq \alpha \leq 1$ remains a free parameter. For the extremal case of $\alpha = 1$ the expression for power is recovered. Optimization for $\alpha = 0$, on the other hand, recovers the expression for

$^5$ When comparing two different spectra $\{ q_i \}$ and $\{ p_i \}$, we note that $\{ p_i \}$ allows for both a lower minimum-energy state as also a higher maximum-energy state if it majorizes $\{ q_i \}$ (Marshall et al 2011). Consequently, batteries in a pure state have higher capacity than ones in mixed states.
ergotropy as derived in Allahverdyan et al (2004)—this will be demonstrated in a forthcoming publication (Vinjanampathy et al 2015) as part of a general optimal control approach to extremizing $F$.

A few remarks are in order: (1) cyclicity of the process is an important requirement since we are interested in the power. Consider a non-cyclic process such as a sudden quench, $H_0 \rightarrow H'$, which does a finite amount of work on the system in an instant and therefore produces infinite power. Such a scenario renders the problem of studying power trivial. If, on the other hand, the process is required to be cyclic then we must have $H_0 \rightarrow H' \rightarrow H_0$, and, for a sudden quench, the objective function vanishes both in the numerator and denominator.

(2) One could imagine driving the system infinitely fast between two given states and let the denominator of equation (5) approach zero. Such infinitely powerful driving, however, would be in conflict with so-called quantum speed limits which bound the evolution time by the inverse mean energy (Margolus and Levitin 1998), or, alternatively, its variance (Mandelstam and Tamm 1945) and apply to time-dependent unitary evolution between any two quantum states (Defnner and Lutz 2013). These speed limits are not prescriptive for determining optimal driving because they depend on the the system state at each instant of time. Alternative bounds, however, have been developed that depend on norms of the driving Hamiltonian (Uzdin et al 2013).

(3) Such bounds on the Hamiltonian are natural constraints that must be incorporated into the analysis of quantum batteries operating at finite time. In the following sections, we will propose bounds that are external constraints in the sense that they depend on the driving Hamiltonian alone but not the system state. In particular, we will restrict the trace norm of the driving Hamiltonian $\|H_d\| \leq E_{\text{max}}$ with the gauge convention that its lowest eigenvalue is zero. This corresponds to restricting the maximal energy available for external driving.

### 3. Powerful driving for a single qubit

Let us now derive an optimal driving with respect to the objective function, equation (5), for a single qubit. Without loss of generality we take $H_0 = |1\rangle \langle 1|$. We parametrize the driving Hamiltonian $H_d$ with control functions $v_i^x, v_i^y, v_i^z$ for the Pauli operators $\sigma_x, \sigma_y, \sigma_z$. The three control functions can be grouped in a vector $v_i = (v_i^x, v_i^y, v_i^z)^T$ and similarly for the Pauli operators $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$. The instantaneous eigenvalues of $\tilde{V}_i$ are $\lambda_i^\pm = \pm |h_i| = \pm \sqrt{\sum_j (v_j^i)^2}$. A thermodynamically sensible constraint is achieved by bounding the difference between the instantaneous eigenvalues by a maximum value $E_{\text{max}}$—that is, $\lambda_i^+ - \lambda_i^- \leq E_{\text{max}}$ for all times $t$ in accordance with the trace norm introduced above.

We now follow recent results, where full analysis of a general time-optimal control of qubit operations was performed (Hegerfeldt 2013, 2014). More generally this is related to the so-called quantum brachistochrone problem (Carlini et al 2003). These approaches differ from the present analysis in the sense that they are concerned with a process which takes some input state $\rho$ to a specific state $\rho'$ whereas we are concerned with the average work that is done in a driving process. While we are interested in preserving the (qubit) state purity it makes no difference what phase it has. In the Bloch Sphere picture, we are only interested in the state’s $z$ coordinate. In this context, $\rho$ can be any qubit state (active or passive), i.e., parametrized by angles $\theta$ and $\phi$, and a radius $r$ which remains constant during evolution. The value of $\phi$ has no bearing on the result. As will be seen below this leads to zero driving along $\sigma_z$ in the optimal case.

In order to optimize the power we now invoke the von Neumann equation for the state’s unitary evolution:

$$i \frac{d}{dt} \rho_t = \left[ H_0 + \hat{V}_t, \rho_t \right] = \frac{1}{2} \left[ v_1, \sigma + a_s, \sigma - a_s, \sigma \right],$$

where $\rho_t$ is parametrized by its Cartesian decomposition $a_s$ in terms of Pauli operators, i.e.

$$a_s = (a_s^x, a_s^y, a_s^z)^T = r (\sin \theta, \cos \phi, \sin \theta, \sin \phi, \cos \theta)^T.$$  

With Einstein summation convention and using the Levi-Civita symbol the evolution becomes

$$\frac{d}{dt} \rho_t = v^i a_i^s \varepsilon_{ijk} \sigma_k.$$

We want to achieve maximum average power $P = \langle W \rangle / T$ (or indeed $F = \langle W \rangle / T^n$) over the duration $T$ of the whole process. To reflect cyclicity, work (and consequently: power) is here defined with respect to the initial Hamiltonian $H_0$: Charging increases the energy in the state $\rho_t$ with respect to the time-independent reference $H_0$. Optimality is hence achieved by first optimizing $\frac{d}{dt} \text{tr}[\rho_t H_0]$ at each instant in time over the driving $v$, and then finding the optimal process duration $T$. Using equation (7) we obtain

$$\text{tr} \left( \frac{d}{dt} \rho_t H_0 \right) = v^i a_i^s - v^j a_j^s.$$
The optimal protocol under the above constraint is found for

\[ v \sin, \cos, 0. \]  

(10)

This implies that in the optimal case no driving happens in the direction of the reference Hamiltonian. The solution corresponds, unsurprisingly perhaps, to driving along the geodesic with \( f_i \) fixed and \( t_0 \phi \) at constant angular speed \( E_{\text{max}} t \). (11)

For a fixed input state we now want to optimize the average power, or more generally the function \( \mathcal{W} \):

\[ \mathcal{W} = \frac{E_{\text{max}}}{T^a} \left( \cos \phi - \sin \phi \right) r \sin \theta_0. \]  

(9)

The optimal protocol under the above constraint is found for

\[ v_1^s = -E_{\text{max}} \sin \phi_1, \quad v_1^t = E_{\text{max}} \cos \phi_1, \quad v_i^s = 0. \]  

(10)

This implies that in the optimal case no driving happens in the direction of the reference Hamiltonian. The solution corresponds, unsurprisingly perhaps, to driving along the geodesic with \( f_i \) fixed and \( t_0 \phi \) at constant angular speed \( E_{\text{max}} t \). (11)

For a fixed input state we now want to optimize the average power, or more generally the function \( \mathcal{F} \):

\[ \mathcal{F} = \frac{\langle W \rangle}{T^a} \left( \cos \phi - \sin \phi \right) r \sin \theta_0. \]  

(12)

(13)

(14)

Linearity of the driving time in \( \theta_T \) (see equation (11)) allows us to choose either variable for optimization. The optimal solution is found for the driving time \( T_m \) that satisfies:

\[ \cos \left( \theta_0 + E_{\text{max}} T_m \right) = \cos \theta_0. \]  

(15)

This equation can be understood geometrically by rewriting it as

\[ \Delta z_T = \frac{E_{\text{max}} T_m}{\alpha} p_{\text{eff}}^Y, \]  

(16)

which is fulfilled when the qubit has traveled a vertical distance \( \Delta z_T \) on the Bloch Sphere which equals the length of its projection onto the \( x-y \)-plane, \( p_{\text{eff}}^Y \), weighed by a factor \( E_{\text{max}} T_m/\alpha \).

The solution for \( \mathcal{F} \) has a non-trivial maximum which is different for different \( \alpha \) as can be seen for the example in figure 1(a). The cone of optimal output states for initially thermal states is pictured on the Bloch Sphere in figure 1(b). We point out that for certain initial conditions it is possible to achieve finite power, or

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*For a given input state equation (11) already lets us answer two related questions: what is the maximum work that can be achieved in time \( T \)? What is the minimum time necessary to achieve work \( W \) (parametrized by \( \theta \)?)
indeed $F$, at time $T \to 0^+$. This can be regarded as pathological and it is worth pointing out that for $\alpha < 1$ and $\theta < \pi$ the optimum is always reached for finite $T$.

4. Role of entanglement in charging an array of quantum batteries

Having found the maximally powerful evolution of a single qubit we now proceed by looking at larger systems, in particular arrays of $N$ qubits, i.e., $\hat{\rho}^{(N)} = \hat{\rho}^{\otimes N}$ as shown in figure 2. The aim of the following section is to demonstrate that when global, entangling operations are allowed on the array (rather than local, parallel operations for each qubit) an increase in power linear in $N$ can be achieved.

The reference Hamiltonian is the sum of the local Hamiltonians from the single qubit case

$$H_0^{(N)} = \sum_k \epsilon_k |1_k\rangle \langle 1_k| \otimes |0\rangle \langle 0|$$

where $k$ labels the $k$th qubit. For now, we don’t want to decrease the purity of the local states (with cyclic operation in mind). We define this—the condition that there is no reduction in the purity of the marginals at the end of the protocol—as the condition of no degradation. Restricting the analysis to pure states for now, we are interested in going from $|00\rangle^{(N)} := |0\rangle^{\otimes N}$ to $(a|0\rangle + b|1\rangle)^{\otimes N}$. Since a full optimization over all possible unitary transformations is not analytically feasible we will contend ourselves with finding a specific evolution that shows an improvement in power per qubit compared to the single qubit case.

Let us consider full charging such that $|00\rangle^{(N)} \to |11\rangle^{(N)} := |1\rangle^{\otimes N}$. For parallel driving, e.g. equation (10), the time-independent driving Hamiltonian during $0 \leqslant t \leqslant T$ is

$$\hat{H}_{\text{par}}^{(N)} = \sum_k E_{\text{max}} \left( \cos \phi_k \sigma_x^{(k)} - \sin \phi_k \sigma_y^{(k)} \right) \otimes |0\rangle$$

(17)

It possesses $N + 1$ equidistant eigenvalues with the gap between the largest and the lowest eigenvalue given by $NE_{\text{max}}$. In order to allow for a fair comparison we will hence equally allow for a maximum energy gap of $E_{\text{max}}^{(N)} = NE_{\text{max}}$ in the case of global driving. The work done is trivially $\langle W \rangle = N$ —that is, the average work per qubit is unity and the driving time is $T_{\text{par}} = \pi / E_{\text{max}}$. Global driving is achieved by evolution with

$$\hat{H}_{\text{global}} = E_{\text{max}}^{(N)} \sigma_x^{(N)} := E_{\text{max}}^{(N)} \left( |1\rangle \langle 0|^{(N)} + |0\rangle \langle 1|^{(N)} \right)$$

(18)

for a duration $T = \pi / E_{\text{max}}^{(N)}$. The average work in this process is again $\langle W \rangle = N$. The average power consequently reads

$$\langle P \rangle = \frac{\langle W \rangle}{\pi / E_{\text{max}}^{(N)}}$$

This global driving is in fact optimal for the current scenario. Generally, global driving between any two states $|a\rangle$ and $|b\rangle$ is optimized by evolution with $\hat{H}_g \propto |a\rangle \langle b| + |b\rangle \langle a|$ where the proportionality is given by the external constraint.
and depends on the driving constraint $E_{\text{max}}^{(N)}$ for $\sigma_{\text{max}}^{(N)}$. Permitting for the same constraint as in the parallel case, that is $E_{\text{max}}^{(N)} = NE_{\text{max}}$ an $N$-fold increase in power per qubit is achieved. Interestingly, the advantage from correlations appears even though initial and final states are separable.

The quantum speed-up observed here can be understood in part as a consequence of the shorter distance that has to be traveled through state space when entangling operations are permitted. To calculate and compare the path length in state space between the global and local pure state case (i.e., $|0^{(N)}\rangle \rightarrow |1^{(N)}\rangle$) we first note that in the global case the proposed evolution does in fact prescribe a path along a geodesic. As a consequence the path length is directly given by the Bures angle $\theta$ between the input and the output state. For pure states the Bures angle is equal to the Fubini-Study distance and we directly (see Bengtsson and Zyczkowski (2006))

$$\theta^{(N)} = \arccos \left( \langle 0^{(N)} | 1^{(N)} \rangle \right) = \frac{\pi}{2}. \tag{20}$$

In the case of parallel driving we note that the square of the line element is additive under tensor product. That is, for $|\psi_{\text{global}}\rangle = \bigotimes_j |\psi_j\rangle$, $ds_{\text{global}}^2 = \sum_j ds_j^2$. Accordingly the path length in this case scales with $\sqrt{N}$.

For comparison, let us numerically study the same charging process, i.e., namely going from $|0^{(N)}\rangle \rightarrow |1^{(N)}\rangle$. Figure 3 shows results of a numerical optimization of full charging between the number of qubits in (a), plotted on the $x$-axis, and the time that the quantum system takes to rotate between the states. The minimum time represents an optimization over all possible time-independent Hamiltonians such that their eigenvalues are bounded by the same constraint as above (a factor linear in $N$). We see that the charging time is commensurate with $N^{1/2}$ behaviour. The solid line represents using the single qubit charging protocol for all $N$ qubits. The dashed line represents the $N^{1/2}$ line, whereas the circles represent the solution to numerical optimization. The entanglement dynamics for the four qubit numerical optimization is shown in (b), by plotting the entropy of the reduced state obtained by tracing over the last two qubits. We note that at the beginning and at the end of the dynamics, the qubits are disentangled, indicated by vanishing entropies. For the numerical optimization the eigenvalues of the instantaneous Hamiltonian were bounded between 0 and $E_{\text{max}}=1$. Since the absolute scaling $E_{\text{max}}$ of the Hamiltonian is arbitrary, the single qubit–time serves as a normalization to see the behavior exhibited.

Figure 3. Numerical optimization results for minimum time $t_\perp$ for complete charging of $N$ qubits, where the state of the battery rotates from $|0^{(N)}\rangle$ to $|1^{(N)}\rangle$. The maximum eigenvalue of the Hamiltonian is scales linearly with the number of qubits, and we note that the full charging time $t_\perp$ is commensurate with $1/N$. The solid line represents using the single qubit charging protocol for all $N$ qubits. The dashed line represents the $1/N$ line, whereas the circles represent the solution to numerical optimization. The entanglement dynamics for the four qubit numerical optimization is shown in (b), by plotting the entropy of the reduced state obtained by tracing over the last two qubits. We note that at the beginning and at the end of the dynamics, the qubits are disentangled, indicated by vanishing entropies. For the numerical optimization the eigenvalues of the instantaneous Hamiltonian were bounded between 0 and $E_{\text{max}}=1$. Since the absolute scaling $E_{\text{max}}$ of the Hamiltonian is arbitrary, the single qubit–time serves as a normalization to see the behavior exhibited.
5. Conclusion

In this article we have given a full derivation of a qubit protocol that achieves maximum power when charging a quantum battery—a work qubit—under constrained driving. With cyclic operation in mind we allowed for unitary evolution, hence keeping the state’s purity constant. In extension we then examined the charging of an array of N work qubits. Using a specific exemplary evolution we demonstrated N-fold advantage in power per work qubit. This example was presented under the operational constraint that the purity of the state must again be conserved. This highlights the potential for quantum enhancement of devices working under non-equilibrium conditions. For future work it will be interesting to extend this scenario and allow for degradation—that is, an extra purifying stage is added after the discharging of the battery array to make up for any loss in purity during the global charging process. Furthermore, in order to allow for realistic noise and decoherence processes it will be of paramount importance to generalize the approach to open systems (Binder et al 2015). We hope that this work will inspire further investigations into the advantages of quantum effects in finite time thermodynamic processes.

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