Complex Symmetry and Normality of Toeplitz Composition Operators on the Hardy Space

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Abstract. In this paper, we investigate the conditions under which the Toeplitz composition operator on the Hardy space $H^2$ becomes complex symmetric with respect to a certain conjugation. We also study various normality conditions for the Toeplitz composition operator on $H^2$.

1. Introduction and Preliminaries

Let $D$ denote the open unit disc and $T = \{e^{i\theta} : \theta \in [0, 2\pi]\}$ denote the unit circle in the complex plane $\mathbb{C}$. Recall that the Hardy space $H^2$ is a Hilbert space which consists of all those analytic functions $f$ on $D$ having power series representation with square summable complex coefficients. That is,

$$H^2 = \{f : D \to \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \text{ and } \|f\|_2^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$$

or equivalently,

$$H^2 = \{f : D \to \mathbb{C} \text{ analytic} \mid \sup_{0<r<1} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty\}.$$

The evaluation of functions in $H^2$ at each $w \in D$ is a bounded linear functional and for all $f \in H^2$, $f(w) = \langle f, K_w \rangle$ where $K_w(z) = 1/(1 - wz)$. The function $K_w(z)$ is called the reproducing kernel for the Hardy space $H^2$. Consider the Hilbert space $\widetilde{H}^2 = \{f^* : T \to \mathbb{C} \mid f^*(z) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta} \text{ and } \|f^*\|_{\widetilde{H}^2}^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$.

Let $L^2$ denote the Lebesgue (Hilbert) space on the unit circle $T$. It is well known that every function $f \in \mathcal{H}^2$ satisfies the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ for almost every $\theta \in [0, 2\pi)$ and it is obvious that the
correspondence where \( f(z) = \sum_{n=0}^{\infty} f(n)z^n \) is mapped to \( f(e^{i\theta}) = \sum_{n=0}^{\infty} f(n)e^{int} \) is an isometric isomorphism from \( \mathcal{H}^2 \) onto the closed subspace \( \overline{\mathcal{H}^2} \) of \( L^2 \). Since \( \{e_n(z) = z^n : n \in \mathbb{Z} \} \) forms an orthonormal basis for \( L^2 \), every function \( f \in L^2 \) can be expressed as \( f(z) = \sum_{n=0}^{\infty} f(n)z^n \) where \( f(n) \) denotes the \( n \)-th Fourier coefficient of \( f \). Let \( L^\infty \) be the Banach space of all essentially bounded functions on the unit circle \( T \). For any \( \phi \in L^\infty \), the Toeplitz operator \( T_\phi : \mathcal{H}^2 \to \mathcal{H}^2 \) is defined by \( T_\phi f = P(\phi \cdot f) \) for \( f \in \mathcal{H}^2 \) where \( P : L^2 \to \mathcal{H}^2 \) is the orthogonal projection. It can be easily verified that for \( m, n \in \mathbb{Z} \),

\[
P(z^{m} \bar{z}^{n}) = \begin{cases} z^{m-n} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}
\]

For a non-zero bounded analytic function \( u \) on \( \mathbb{D} \) and a self-analytic map \( \phi \) on \( \mathbb{D} \), the weighted composition operator \( W_{u, \phi} \) is defined by \( W_{u, \phi} f = u \cdot f \circ \phi \) for every \( f \in \mathcal{H}^2 \). Over the past several decades, there has been tremendous development in the study of composition operators and weighted composition operators over the Hardy space \( \mathcal{H}^2 \) and various other spaces of analytic functions. Readers may refer [1, 10] for general study and background of the composition operators on the Hardy space \( \mathcal{H}^2 \). In this paper, we introduce the notion of the Toeplitz composition operator on the Hardy space \( \mathcal{H}^2 \) where the symbol \( u \) in \( W_{u, \phi} \) need not necessarily be analytic. For a function \( \psi \in L^\infty \) and a self-analytic map \( \phi \) on \( \mathbb{D} \), the Toeplitz composition operator \( T_{\psi, C_\phi} : \mathcal{H}^2 \to \mathcal{H}^2 \) is defined by \( T_{\psi, C_\phi} f = P(\psi \cdot f \circ \phi) \) for every \( f \in \mathcal{H}^2 \) where \( C_\phi f := f \circ \phi \) is the composition operator on \( \mathcal{H}^2 \). The authors in [5] introduced the concept of the Toeplitz composition operators on the Fock space and also studied its various properties.

Let \( \mathcal{H} \) be a separable Hilbert space. Then a mapping \( S \) on \( \mathcal{H} \) is said to be an anti-linear (also conjugate-linear) if \( S(\alpha x_1 + \beta x_2) = \bar{\alpha} S(x_1) + \bar{\beta} S(x_2) \) for all scalars \( \alpha, \beta \in \mathbb{C} \) and for all \( x_1, x_2 \in \mathcal{H} \).

An anti-linear mapping \( C : \mathcal{H} \to \mathcal{H} \) is said to be a conjugation if it is involutive (i.e., \( C^2 = I \)) and isometric (i.e., \( \|Cx\| = \|x\| \) for every \( x \in \mathcal{H} \). A complex symmetric operator \( S \) on \( \mathcal{H} \) is a bounded linear operator such that \( S = CS^* \) for some conjugation \( C \) on \( \mathcal{H} \). We call such an operator \( S \) to be a C-symmetric operator.

Garcia and Putinar [3, 4] began the general study of complex symmetric operators on Hilbert spaces which are the natural generalizations of complex symmetric matrices. There exist a wide variety of complex symmetric operators which include normal operators, compressed Toeplitz operators, Volterra integration operators etc. Jung et al. [7] studied the complex symmetry of the weighted composition operators on the Hardy space in the unit disc \( \mathbb{D} \). Garcia and Hammond [2] undertook the study of complex symmetry of weighted composition operators on the weighted Hardy spaces. Ko and Lee [8] gave a characterization of the complex symmetric Toeplitz operators on the Hardy space \( \mathcal{H}^2 \) of the unit disc \( \mathbb{D} \). Motivated by this, we study the complex symmetry of the Toeplitz composition operators on the Hardy space \( \mathcal{H}^2 \). In this paper we give a characterization of such types of operators. We also investigate certain conditions under which a complex symmetric operator turns out to be a normal operator. In the concluding section of this article, we discuss the normality of the Toeplitz composition operators on \( \mathcal{H}^2 \).

2. Complex Symmetric Toeplitz Composition Operators

In this section we aim to find the conditions under which a Toeplitz composition operator becomes complex symmetric with respect to a certain fixed conjugation. In order to determine these conditions, we need an explicit formula for the adjoint \( C_\phi^* \) of a composition operator \( C_\phi \) where \( \phi \) is a self-analytic map on the unit disc \( \mathbb{D} \). But there exists no general formula and there are only a few special cases where it is possible to find a formula for \( C_\phi^* \) explicitly. C. Cowen was the first to find the representation for the adjoint of a composition operator \( C_\phi \) on \( \mathcal{H}^2 \), famously known as the Cowen’s Adjoint Formula, where the symbol \( \phi \) is a linear fractional self-map of the unit disc \( \mathbb{D} \). The Cowen’s Adjoint Formula was extended to the Bergman space \( \mathcal{A}^2 \) by P. Hurst [6] and it is stated as follows:

**Theorem 2.1 (1).** (Cowen’s Adjoint Formula) Let \( \phi(z) = \frac{az + b}{cz + d} \) be a linear fractional self-map of the unit disc where \( ad - bc \neq 0 \). Then \( \sigma(\phi) = \frac{\bar{d}z - \bar{b}}{-\bar{c}z + \bar{a}} \) maps disc into itself, \( g(z) = (-\bar{b}z + \bar{a})^{-\nu} \) and \( h(z) = (cz + d)^\nu \) are bounded analytic
functions on the disc and on $\mathcal{H}^2$ or $A^2$, $C_\phi = M_f \circ C_{\psi} M'_n$ where $p = 1$ on $\mathcal{H}^2$ and $p = 2$ on $A^2$. (Note that the operator $M_g$ is the multiplication operator defined by $M_g f = g \cdot f$.)

Next we have the following lemmas which would be instrumental in proving certain results throughout this article:

**Lemma 2.2 [(9)].** A linear fractional map $\phi$, written in the form $\phi(z) = \frac{az + b}{cz + d}$, $ad - bc \neq 0$, maps $D$ into itself if and only if:

$$|a\bar{d} - b\bar{c}| + |ad - bc| \leq |d|^2 - |c|^2. \quad (1)$$

**Lemma 2.3 [(1)].** Let $\phi(z) = \frac{az + b}{cz + d}$ be a linear fractional map and define the associated linear fractional transformation $\phi^*$ by

$$\phi^*(z) = \frac{1}{\phi^{-1}(\overline{b})} = \frac{-\bar{a}z + \bar{c}}{-\bar{b}z + \bar{d}}.$$

Then $\phi$ is a self-map of the disc if and only if $\phi^*$ is also a self-map of the disc.

**Lemma 2.4 [(1)].** If $\phi(z) = \frac{az + b}{cz + d}$ is a linear fractional transformation mapping $D$ into itself where $ad - bc = 1$, then $\sigma(z) = \frac{a - z}{b - z}$ maps $D$ into itself.

In the following lemma, a conjugation on the Hardy space $\mathcal{H}^2$ has been defined with respect to which we will find the complex symmetry of the operator $T_\psi C_\phi$.

**Lemma 2.5 [(8)].** For every $\xi$ and $\theta$, let $C_{\xi,\theta} : \mathcal{H}^2 \to \mathcal{H}^2$ be defined by

$$C_{\xi,\theta} f(z) = e^{i\xi} f(e^{i\theta} z).$$

Then $C_{\xi,\theta}$ is a conjugation on $\mathcal{H}^2$. Moreover, $C_{\xi,\theta}$ and $C_{\xi,\theta}$ are unitarily equivalent where $(\xi, \theta)$ satisfies the equation $\xi - k\theta = -\xi + k\theta - 2\pi n$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

In the next theorem, we determine the conditions under which the Toeplitz composition operator $T_\psi C_\phi$ turns out to be complex symmetric with respect to the conjugation $C_{\xi,\theta}$ on $\mathcal{H}^2$.

**Theorem 2.6.** For $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and for self-analytic linear transformation $\phi(z) = az + b$ ($a \neq 0$) mapping $D$ into itself, let $T_\psi C_\phi$ be a Toeplitz composition operator on $\mathcal{H}^2$. Then $T_\psi C_\phi$ is complex symmetric with the conjugation $C_{\xi,\theta}$ if and only if for each $k, p \in \mathbb{N} \cup \{0\}$ and for every $n \in \mathbb{Z}$, we have:

(i) $\sum_{n=k}^{\infty} \binom{k}{p} \hat{\psi}(n) \overline{\lambda}^{n-k-p} + \lambda^{p-k} \lambda^k$ for $b \neq 0$ and, (ii) $\overline{\psi}(n) \lambda^n = \overline{\psi(-n) \lambda^n}$ for $b = 0$.

**Proof.** If $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\theta}$, then for all $k \in \mathbb{N} \cup \{0\}$ we have

$$C_{\xi,\theta} T_\psi C_\phi z^k = (T_\psi C_\phi) T_{\xi,\theta} z^k.$$

(2)

We take $\mu = e^{i\xi}$ and $\lambda = e^{-i\theta}$ and consider the following two cases:
Case (i) : Let $b \neq 0$. Then

\[
C_{\xi, \theta} T_{\psi} C_{\phi} z^k = C_{\xi, \theta} T_{\psi} (\phi(z))^k
\]
\[
= C_{\xi, \theta} T_{\psi} (az + b)^k
\]
\[
= C_{\xi, \theta} P(\psi(z) \cdot \sum_{m=0}^{k} \left( \sum_{n=-\infty}^{\infty} \binom{k}{m} a^m b^{k-m} z^m + n \right))
\]
\[
= C_{\xi, \theta} P \left( \sum_{m=0}^{k} \left( \sum_{n=-\infty}^{\infty} \binom{k}{m} \psi(n) a^m b^{k-m} z^m + n \right) \right)
\]
\[
= C_{\xi, \theta} P \left( \sum_{m=0}^{k} \left( \sum_{n=-m}^{\infty} \binom{k}{m} \psi(n) a^m b^{k-m} z^m + n \right) \right)
\]
\[
= \sum_{m=0}^{k} C_{\xi, \theta} P \left( \sum_{n=-m}^{\infty} \binom{k}{m} \psi(n) a^m b^{k-m} z^m + n \right)
\]
\[
= e^{ik} \sum_{m=0}^{k} \sum_{n=-m}^{\infty} \binom{k}{m} \overline{\psi(n)} a^m b^{k-m} e^{-im+\mu+n} \lambda^m z^m + n
\]
\[
= \mu \sum_{m=0}^{k} \sum_{n=-m}^{\infty} \binom{k}{m} \overline{\psi(n)} a^m b^{k-m} \lambda^m z^m + n.
\]

(3)

and

\[
(T_{\psi} C_{\phi})^* C_{\xi, \theta} z^k = C_{\phi}^* T_{\psi}^* C_{\xi, \theta} z^k
\]
\[
= C_{\phi}^* T_{\psi}^* (e^{ia \theta} z^k)
\]
\[
= C_{\phi}^* T_{\psi}^* (\mu \lambda z^k)
\]
\[
= C_{\phi}^* P(\mu \lambda z^k \sum_{n=-\infty}^{\infty} \overline{\psi(n)} z^{k-n})
\]
\[
= C_{\phi}^* P(\mu \lambda z^k \sum_{n=-\infty}^{\infty} \overline{\psi(-n)} z^{n+k})
\]
\[
= \mu \lambda C_{\phi}^* \left( \sum_{n=-k}^{\infty} \overline{\psi(-n)} z^{n+k} \right).
\]

(4)

On using Theorem 2.1 for $a \neq 0$, $c = 0$ and $d = 1$, we obtain that $C_{\phi}^* = M_{\phi} C_{\phi}$ where $g(z) = (1 - \bar{b}z)^{-1}$ and $\sigma(z) = \frac{cz}{1 - bz}$. Since $|a| + |b| \leq 1$ from Lemma 2.2, so $|b| < 1$ and hence, $\frac{1}{1 - \bar{b}z} = \sum_{j=0}^{\infty} (i^{j+1}) (\bar{b}z)^j$ for $z \in D$. 
Therefore, from (4) we get that
\[
(T \psi C_\phi) C_{\xi, \theta} z^k = \mu \lambda^k M_\phi C_\phi \left( \sum_{n=-k}^{\infty} \psi(-n) z^{n+k} \right)
\]
\[
= \mu \lambda^k M_\phi \left( \sum_{n=-k}^{\infty} \psi(-n) \left( \frac{-a}{1-bz} \right)^{n+k} \right)
\]
\[
= \mu \lambda^k \left( \sum_{n=-k}^{\infty} \psi(-n)a^{n+k} \left( \frac{1}{1-bz} \right)^{n+k+1} \right) z^{n+k}
\]
\[
= \mu \left( \sum_{j=0}^{\infty} \sum_{n=-k}^{\infty} (n+k+j) \psi(-n)a^{n+k} b^{j} \lambda^k z^{n+k+j} \right).
\tag{5}
\]

It follows from (2) that for each \( k \in \mathbb{N} \cup \{0\} \), we have
\[
\sum_{m=0}^{k} \left( \sum_{n=-m}^{\infty} \left( \frac{k}{m} \right) \frac{\psi(n)a^{m-k} b^{m-n} \lambda^{m+n+n}}{\sum_{n=-k}^{\infty} \left( \sum_{j=0}^{\infty} \left( n+k+j \right) \psi(-n)a^{n+k} b^{j} \lambda^k \right) z^{n+k+j}} \right)
\tag{6}
\]

Thus, the coefficient of \( z^p \) where \( p \in \mathbb{N} \cup \{0\} \) must be equal on the both sides of (6). On comparing the coefficients of \( 1, z, z^2, z^3 \) and so on, on the both sides of (6), we observe that
\[
\sum_{n=-k+p}^{p} \left( \frac{k}{p-n} \right) \frac{\psi(n)a^{n-k-p} \lambda^p}{\sum_{n=-k}^{\infty} \left( \sum_{j=0}^{\infty} \left( n+k+j \right) \psi(-n)a^{n+k} b^{j} \lambda^k \right) z^{n+k+j}} \right)
\tag{7}
\]
for each \( k, p \in \mathbb{N} \cup \{0\} \).

Conversely, let us suppose that (7) holds for each \( k, p \in \mathbb{N} \cup \{0\} \). Then from (3) and (5), we have
\[
(C_{\xi, \theta} T_\psi \phi_\psi - (T_\psi C_\phi)^2 C_{\xi, \theta}) z^k = \mu \left( \sum_{m=0}^{k} \left( \sum_{n=-m}^{\infty} \left( \frac{k}{m} \right) \frac{\psi(n)a^{m-k} b^{m-n} \lambda^{m+n+n}}{\sum_{n=-k}^{\infty} \left( \sum_{j=0}^{\infty} \left( n+k+j \right) \psi(-n)a^{n+k} b^{j} \lambda^k \right) z^{n+k+j}} \right) \right)
\]
\[
- \mu \left( \sum_{j=0}^{\infty} \sum_{n=-k}^{\infty} (n+k+j) \psi(-n)a^{n+k} b^{j} \lambda^k \right) z^{n+k+j}
\]
\[
= 0.
\]

**Case (ii):** If \( b = 0 \), then
\[
C_{\xi, \theta} T_\psi \phi_\psi z^k = C_{\xi, \theta} T_\psi \phi_\psi z^k
\]
\[
= C_{\xi, \theta} T_\psi \phi_\psi z^k
\]
\[
= C_{\xi, \theta} T_\psi \phi_\psi z^k
\]
\[
= C_{\xi, \theta} T_\psi \phi_\psi z^k
\]
\[
= C_{\xi, \theta} T_\psi \phi_\psi z^k
\]
\[
= C_{\xi, \theta} \left( \sum_{n=-k}^{\infty} \psi(n) a^k z^{n+k} \right)
\]
\[
= C_{\xi, \theta} \left( \sum_{n=-k}^{\infty} \psi(n) a^k z^{n+k} \right)
\]
\[
= \lambda^k \sum_{n=-k}^{\infty} \psi(n) a^k e^{-i(n+k)\theta} z^{n+k}
\]
\[
= \mu \sum_{n=-k}^{\infty} \psi(n) a^k \lambda^{n+k} z^{n+k}.
\tag{8}
\]
For $a \neq 0, b = c = 0$ and $d = 1$, we get from Theorem 2.1 that $g(z) = h(z) = 1$ and $\sigma(z) = \bar{a}z$. Thus, $C_{\phi} = C_{\omega}$. We compute

\[
(T_{\psi}C_{\phi})^*C_{\mu}z^n = C_{\phi}T_{\psi}^*C_{\mu}z^n = C_{\phi}(\mu^kz^k)
\]

Since the equation (2) holds, on equating the expressions (8) and (9), we obtain that $\overline{\psi(n)}\lambda^n = \overline{\psi(-n)}\lambda^n$ for every $n \in \mathbb{Z}$. Conversely, let us assume that $\overline{\psi(n)}\lambda^n = \overline{\psi(-n)}\lambda^n$ for every $n \in \mathbb{Z}$. Then (8) and (9) implies that $(C_{\mu}T_{\psi}C_{\phi} - (T_{\psi}C_{\phi})^*C_{\mu})z^n = 0$. Thus, $T_{\psi}C_{\phi}$ is complex symmetric with conjugation $C_{\mu}$. $\square$

**Example 2.7.** Let $\psi(z) = z + \bar{z} \in L^\infty$. Then, $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$. Let $\phi(z) = iz$. Then $\phi(z)$ is a self-analytic map on $D$. Consider the conjugation $C_{\mu}$ where we choose $\theta = \pi/2$. Then $\lambda = e^{i\theta} = i$. On taking $a = i, b = 0$ and $\lambda = -i$ in Theorem 2.6, we get that $\overline{\psi(n)}\lambda^n = \overline{\psi(-n)}\lambda^n$ for every $n \in \mathbb{Z}$ and hence, $C_{\mu}T_{\psi}C_{\phi} = (T_{\psi}C_{\phi})^*C_{\mu}$. Therefore, the operator $T_{\psi}C_{\phi}$ is complex symmetric with respect to the conjugation $C_{\mu}$. 

In the light of the above example, an interesting observation has been made in the following Corollary:

**Corollary 2.8.** Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and $\phi(z) = az$ be a self-analytic map on $D$ where $a = e^{i\theta}$ for $\theta \in \mathbb{R}$. Then $T_{\psi}C_{\phi}$ is complex symmetric with respect to the conjugation $C_{\mu}$ if and only if $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$.

**Proof.** It follows from Theorem 2.6 that $T_{\psi}C_{\phi}$ is complex symmetric with respect to the conjugation $C_{\mu}$ if and only if $\hat{\psi}(n)\lambda^n = \hat{\psi}(-n)\lambda^n$ if and only if $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$ where $a = e^{i\theta}$ and $\lambda = e^{i\theta}$. $\square$

An operator $T : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H}$ denotes a Hilbert space is said to be hyponormal if $TT^* \geq TT^*$ or equivalently, $|\langle Tx, x \rangle| \geq |\langle T^*x, x \rangle|$ for every $x \in \mathcal{H}$. Our next goal is to find out the conditions under which a Toeplitz composition operator $T_{\psi}C_{\phi}$ becomes a normal operator. The proof involves the technique followed in [Proposition 2.2, [2]].

**Theorem 2.9.** Let $\psi \in L^\infty$ and let $\phi$ be any self-analytic mapping from $D$ into itself. If the operator $T_{\psi}C_{\phi} : \mathcal{H}^2 \to \mathcal{H}^2$ is hyponormal and complex symmetric with conjugation $C_{\mu}$, then $T_{\psi}C_{\phi}$ is a normal operator on $\mathcal{H}^2$.

**Proof.** Since $T_{\psi}C_{\phi}$ is complex symmetric with respect to the conjugation $C_{\mu}$, this gives that $(T_{\psi}C_{\phi})^* = C_{\mu}T_{\psi}C_{\phi}C_{\mu}$. On using the isometry of $C_{\mu}$, we obtain that

$$
\|(T_{\psi}C_{\phi})^*f\| = \|C_{\mu}T_{\psi}C_{\phi}C_{\mu}f\| = \|T_{\psi}C_{\phi}C_{\mu}f\| \text{ for every } f \in \mathcal{H}^2.
$$

By hypothesis, $T_{\psi}C_{\phi}$ is a hyponormal operator on $\mathcal{H}^2$ and thus, $\|T_{\psi}C_{\phi}f\| \geq \|(T_{\psi}C_{\phi})^*f\|$ for every $f \in \mathcal{H}^2$. Therefore, $\|(T_{\psi}C_{\phi})^*f\| = \|T_{\psi}C_{\phi}C_{\mu}f\| \geq \|T_{\psi}C_{\phi}f\| = \|T_{\psi}C_{\phi}C_{\mu}f\| \text{ for every } f \in \mathcal{H}^2$. Hence, $\|(T_{\psi}C_{\phi})^*f\| \geq \|T_{\psi}C_{\phi}f\|$ and this together with the hyponormality of $T_{\psi}C_{\phi}$ implies that $\|(T_{\psi}C_{\phi})^*f\| = \|T_{\psi}C_{\phi}f\|$ for every $f \in \mathcal{H}^2$ which proves that $T_{\psi}C_{\phi}$ is a normal operator. $\square$
In the following theorem, the conditions under which the Toeplitz composition operator $T_{\varphi}C_{\psi}$ commutes with the conjugation $C_{\xi,\theta}$ has been investigated which further provides us with a criteria which together with the complex symmetry of $T_{\varphi}C_{\psi}$ makes the operator $T_{\varphi}C_{\psi}$ a normal operator.

**Theorem 2.10.** Let $\psi(z) = \sum_{m=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and $\phi(z) = az + b$ ($a \neq 0$) be a linear fractional transformation mapping $D$ into itself. Then the Toeplitz composition operator $T_{\varphi}C_{\psi}$ commutes with the conjugation $C_{\xi,\theta}$ on $H^2$ if and only if for each $m, k \in \mathbb{N} \cup \{0\} (0 \leq m \leq k)$ and $n \in \mathbb{Z}$, we have:

(i) $\hat{\psi}(n)a^mb^{k-m}\lambda^k = \overline{\hat{\psi}(n)a^mb^{k-m}\lambda^m}$ if $b \neq 0$ and,

(ii) $\hat{\psi}(n)a^k = \overline{\hat{\psi}(n)a^k}$ if $b = 0$.

**Proof.** If the operator $T_{\varphi}C_{\psi}$ commutes with $C_{\xi,\theta}$, then for each $k \in \mathbb{N} \cup \{0\}$, we have $T_{\varphi}C_{\psi}C_{\xi,\theta}z^k = C_{\xi,\theta}T_{\varphi}C_{\psi}z^k$. We consider the following two cases:

**Case (i):** Let us suppose that $b \neq 0$. Since for each $k \in \mathbb{N} \cup \{0\}$,

$$T_{\varphi}C_{\psi}C_{\xi,\theta}z^k = T_{\varphi}C_{\psi}(e^{2\pi i k}\zeta^k) = P(\psi(z) \cdot \mu\lambda^k az + b) = \mu\lambda^k \sum_{m=0}^{k} \left( \sum_{n=-\infty}^{\infty} \left(\frac{k}{m}\right) \hat{\psi}(n)a^mb^{k-m}\lambda^{m+n}\right)$$

and

$$C_{\xi,\theta}T_{\varphi}C_{\psi}z^k = C_{\xi,\theta}P((az + b)^k) = C_{\xi,\theta}P\left( \sum_{m=0}^{k} \left( \sum_{n=-\infty}^{\infty} \left(\frac{k}{m}\right) \hat{\psi}(n)a^mb^{k-m}\lambda^{m+n}\right) \right)$$

we obtain that $\hat{\psi}(n)a^mb^{k-m}\lambda^k = \overline{\hat{\psi}(n)a^mb^{k-m}\lambda^m}$ for each $n \in \mathbb{Z}$ and $m \in \mathbb{N} \cup \{0\} (0 \leq m \leq k)$.

Conversely, if for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\}$, we have $\hat{\psi}(n)a^mb^{k-m}\lambda^k = \overline{\hat{\psi}(n)a^mb^{k-m}\lambda^m}$, then $(T_{\varphi}C_{\psi}C_{\xi,\theta} - C_{\xi,\theta}T_{\varphi}C_{\psi})z^k = 0$ which proves that $T_{\varphi}C_{\psi}$ commutes with $C_{\xi,\theta}$.

**Case (ii):** Let $b = 0$. Then $T_{\varphi}C_{\psi}C_{\xi,\theta}z^k = C_{\xi,\theta}T_{\varphi}C_{\psi}z^k$ if and only if $P(\psi(z) \cdot \mu\lambda^k (az)^k) = C_{\xi,\theta}P(\psi(z) \cdot (az)^k)$ if and only if $P(\sum_{m=-\infty}^{\infty} \hat{\psi}(n)a^mb^{k-m}\lambda^{m+n+k}) = e^{2\pi i k} \sum_{m=-\infty}^{\infty} \overline{\hat{\psi}(n)a^mb^{k-m}\lambda^{m+n+k}}$ if and only if $\sum_{n=-k}^{\infty} \hat{\psi}(n)a^k\lambda^{-n-k}z^{n+k}$ if and only if $\hat{\psi}(n)a^k = \overline{\hat{\psi}(n)a^k}$$ \lambda^{-n-k}$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. 

**Corollary 2.11.** Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and $\phi(z) = az + b$ ($a \neq 0$) be a linear fractional transformation mapping $D$ into itself. Then the Toeplitz composition operator $T_{\varphi}C_{\psi}$ commutes with the conjugation $C_{0,0}$ on $H^2$ if and only if for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\} (0 \leq m \leq k)$, we have:
The following theorem is in general valid for any linear operator $T$ on a Hilbert space $\mathcal{H}$ which is complex symmetric with respect to any conjugation $C$ defined on $\mathcal{H}$ such that $T$ commutes with $C$.

**Theorem 2.12.** Let $\psi \in L^\infty$ and let $\phi$ be any self-analytic mapping from $\mathbb{D}$ into itself. Suppose that $T_\psi C_\phi$ is a complex symmetric operator with conjugation $C_{\xi,\theta}$ on $\mathcal{H}_2$ and further, suppose that $T_\psi C_\phi$ commutes with $C_{\xi,\theta}$. Then $T_\psi C_\phi$ is a normal operator on $\mathcal{H}_2$.

**Proof.** By hypothesis, $T_\psi C_\phi$ is a complex symmetric operator with conjugation $C_{\xi,\theta}$ such that it commutes with $C_{\xi,\theta}$ which implies that $T_\psi C_\phi$ is a self-adjoint operator. That is,

$$
(T_\psi C_\phi)^* = C_{\xi,\theta} T_\psi C_\phi C_{\xi,\theta} = C_{\xi,\theta} C_{\xi,\theta} T_\psi C_\phi = T_\psi C_\phi.
$$

Hence, $T_\psi C_\phi$ is a normal operator on $\mathcal{H}_2$. □

**Corollary 2.13.** Let $(\psi(z) = \sum_{n=-\infty}^{\infty} \psi(n)z^n \in L^\infty$ and $\phi(z) = az + b$ ($a \neq 0$) be a linear fractional transformation mapping $\mathbb{D}$ into itself. Suppose that $T_\psi C_\phi : \mathcal{H}_2 \to \mathcal{H}_2$ is a complex symmetric operator with conjugation $C_{0,0}$ and $\psi(n)a^m b^{-n} \in \mathbb{R}$ for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\} (0 \leq m \leq k)$. Then $T_\psi C_\phi$ is a normal operator on $\mathcal{H}_2$.

**Proof.** From Corollary 2.11, we obtain that $T_\psi C_\phi$ commutes with the conjugation $C_{0,0}$ such that $\psi(n)a^m b^{-n} \in \mathbb{R}$ for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\} (0 \leq m \leq k)$. Thus, we get that $T_\psi C_\phi$ is a normal operator on $\mathcal{H}_2$ by Theorem 2.12. □

### 3. Normality Of Toeplitz Composition Operators

In this section we discuss the normality of the Toeplitz composition operators on $\mathcal{H}_2$. We explore the conditions under which the operator $T_\psi C_\phi$ becomes normal and further we discover the necessary and sufficient conditions for the operator $T_\psi C_\phi$ to be Hermitian.

**Theorem 3.1.** Let $(\psi(z) = \sum_{n=-\infty}^{\infty} \psi(n)z^n \in L^\infty$ and $\phi(z) = az + b$ ($a \neq 0$) be a linear fractional transformation mapping $\mathbb{D}$ into itself. Let the operator $T_\psi C_\phi$ on $\mathcal{H}_2$ be hyponormal. Then we have the following:

(i) If $b \neq 0$, then $\sum_{n=0}^{\infty} (n+1)^2 |\hat{\psi}(n)|^2 - \sum_{n=0}^{\infty} (n+1) |\hat{\psi}(n)|^2 |b^{n+1}|^2 \geq 0$.

(ii) If $b = 0$, then $\sum_{n=0}^{\infty} |\hat{\psi}(n)|^2 - |\hat{\psi}(0)|^2 |a|^2 \geq 0$.

**Proof.** By the hyponormality of $T_\psi C_\phi$ on $\mathcal{H}_2$, we have $\|T_\psi C_\phi f\|^2 \geq \|(T_\psi C_\phi)^* f\|^2$ for every $f \in \mathcal{H}_2$. In particular, on taking $f \equiv 1$, we obtain that

$$
\|T_\psi C_\phi(1)\|^2 \geq \|(T_\psi C_\phi)^*(1)\|^2.
$$

Then $\|T_\psi C_\phi(1)\|^2 = \|P(\sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n)\|^2 = \|\sum_{n=0}^{\infty} \hat{\psi}(n)z^n\|^2 = \sum_{n=0}^{\infty} |\hat{\psi}(n)|^2$. It can be noted that the function $\psi(z)$ can be expressed as

$$
\psi(z) = \psi_+(z) + \psi_0(z) + \overline{\psi_-(z)}
$$

where $\psi_+(z) = \sum_{n=0}^{\infty} \hat{\psi}(n)z^n$, $\psi_0(z) = \sum_{n=1}^{\infty} \hat{\psi}(-n)z^n$ and $\psi_0(z) = \hat{\psi}(0)$. It follows that $P(\overline{\psi(z)}) = P(\overline{\psi_+(z)} + \overline{\psi_0(z)} + \overline{\psi_-(z)}) = \sum_{n=0}^{\infty} \hat{\psi}(-n)z^n$. 


Let us first assume that $b \neq 0$. Since $C_{\hat{\phi}}^* = M_{\hat{\phi}}C_{\rho}$ where $g(z) = (1 - \bar{b}z)^{-1}$ and $\sigma(z) = \frac{\bar{b}z}{1-bz}$, it is obtained that

$$||T_{\hat{\psi}}C_{\hat{\phi}}^*(1)||^2 = ||C_{\hat{\phi}}T_{\hat{\psi}}(1)||^2 = ||M_{\hat{\phi}}C_{\rho}P(\bar{\psi}(z))||^2$$

$$= ||M_{\hat{\phi}}C_{\rho}(\sum_{n=0}^{\infty} \bar{\psi}(-n)z^n)||^2$$

$$= ||\sum_{n=0}^{\infty} \bar{\psi}(-n)(\frac{-\bar{b}z^n}{1-bz^{n+1}})||^2$$

$$= ||\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+n)}{m} \bar{\psi}(-n)a^m b^n z^{m+n}||^2$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{(m+n)}{m}\right)^2 ||\bar{\psi}(-n)||^2 |a|^m |b|^n.$$

Hence, it follows from (11) that $\sum_{n=0}^{\infty} ||\bar{\psi}(n)||^2 - \sum_{m=0}^{\infty} (\sum_{n=m}^{\infty} ||\bar{\psi}(-n)||^2 |a|^m |b|^n)^2 \geq 0$.

If $b \neq 0$, then $C_{\hat{\phi}}^* = C_{\phi}$, where $\sigma(z) = \frac{\bar{b}z}{1-bz}$. This implies that $||T_{\hat{\psi}}C_{\hat{\phi}}^*(1)||^2 = ||C_{\phi}T_{\hat{\psi}}(1)||^2 = \sum_{n=0}^{\infty} ||\bar{\psi}(-n)||^2 |a|^2$. Thus, from (11), we get that $\sum_{n=0}^{\infty} ||\bar{\psi}(n)||^2 - ||\bar{\psi}(-n)||^2 |a|^2 \geq 0$. \hspace{1cm} \Box

**Corollary 3.2.** Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and $\phi(z) = az + b (a \neq 0)$ be a linear fractional transformation mapping $\mathcal{D}$ into itself. Let the operator $T_{\hat{\psi}}C_{\hat{\phi}}$ on $\mathcal{H}^2$ be normal. Then we have the following:

(i) If $b \neq 0$, then $\sum_{n=0}^{\infty} ||\hat{\psi}(n)||^2 - \sum_{m=0}^{\infty} (\sum_{n=m}^{\infty} ||\hat{\psi}(-n)||^2 |a|^m |b|^n)^2 \geq 0$.

(ii) If $b = 0$, then $\sum_{n=0}^{\infty} ||\hat{\psi}(n)||^2 - ||\hat{\psi}(-n)||^2 |a|^2 \geq 0$.

The condition obtained above in Corollary 3.2 is necessary but not sufficient which can be observed through the following example:

**Example 3.3.** Let $\psi(z) = z + \bar{z}$ and $\phi(z) = iz$. Then, for $a = i$, $b = 0$, $\hat{\psi}(1) = \hat{\psi}(1) = 1$ and $\hat{\psi}(-n) = \hat{\psi}(n) = 0$ where $n \in \mathbb{Z} - \{0\}$, the condition $\sum_{n=0}^{\infty} ||\hat{\psi}(n)||^2 - ||\hat{\psi}(-n)||^2 |a|^2 \geq 0$ is satisfied. But the Toeplitz composition operator $T_{\hat{\psi}}C_{\hat{\phi}}$ is not normal as $(T_{\hat{\psi}}C_{\hat{\phi}})(T_{\hat{\psi}}C_{\hat{\phi}})^*(z) = z^3 + 2z$ whereas $(T_{\hat{\psi}}C_{\hat{\phi}})^*(T_{\hat{\psi}}C_{\hat{\phi}})(z) = -z^3 + 2z$.

Next we investigate the necessary and sufficient conditions under which the operator $T_{\hat{\psi}}C_{\hat{\phi}}$ becomes Hermitian.

**Theorem 3.4.** Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^\infty$ and $\phi(z) = az + b (a \neq 0)$ be a linear fractional transformation mapping $\mathcal{D}$ into itself. Then the Toeplitz composition operator $T_{\hat{\psi}}C_{\hat{\phi}}$ on $\mathcal{H}^2$ is Hermitian if and only if for each $k, p \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$, we have:

(i) $\sum_{n=-\infty}^{\infty} (\sum_{p=0}^{n+k} (\sum_{m=0}^{n+p} a^{m+p-k} b^{n-p} \bar{\psi}(-n)a^m b^n)) = 0$ when $b \neq 0$

and, (ii) $a^k \hat{\psi}(n) = \bar{a}^{-k} \hat{\psi}(n)$ when $b = 0$.

**Proof.** Let us suppose that the operator $T_{\hat{\psi}}C_{\hat{\phi}}$ is Hermitian on $\mathcal{H}^2$. This implies that $T_{\hat{\psi}}C_{\hat{\phi}} z^k = (T_{\hat{\psi}}C_{\hat{\phi}})^* z^k$ for
every $k \in \mathbb{N} \cup \{0\}$. Let us suppose $b \neq 0$. Since
\[
T_{\psi}C_{\phi}z^k = T_{\psi}(\phi(z))^k
\]
\[
= P(\psi(z) \cdot \sum_{m=0}^{k} \binom{k}{m} g^{m}b^{k-m}z^{m})
\]
\[
= P\left(\sum_{m=0}^{k} \left(\sum_{n=-\infty}^{\infty} \binom{k}{m} \tilde{\psi}(n)g^{m}b^{k-m}z^{n+m}\right)\right)
\]
\[
= \sum_{m=0}^{k} P\left(\sum_{n=-\infty}^{\infty} \binom{k}{m} \tilde{\psi}(n)g^{m}b^{k-m}z^{n+m}\right)
\]
\[
= \sum_{m=0}^{k} \sum_{n=-m}^{m} \binom{k}{m} \tilde{\psi}(n)g^{m}b^{k-m}z^{n+m+n}
\]
and
\[
(T_{\psi}C_{\phi})^*z^k = C_{\phi}^*T_{\psi}^*z^k
\]
\[
= C_{\phi}^*P\left(\sum_{n=-\infty}^{\infty} \overline{\psi(-n)}z^{n+k}\right)
\]
\[
= M_{\phi}C_{\phi} \sum_{n=-k}^{\infty} \overline{\psi(-n)}z^{n+k}
\]
\[
= \sum_{n=-k}^{\infty} \overline{\psi(-n)}a^{n+k} \left(\frac{1}{1 - Bz}\right)^{n+k+1}z^{n+k}
\]
\[
= \sum_{n=-k}^{\infty} \left(\sum_{j=0}^{\infty} \binom{n+k+j}{j} \overline{\psi(-n)}a^{n+k+1}b^j z^{n+k+j}\right)
\]
where $g(z) = (1 - \bar{B}z)^{-1}$ and $\sigma(z) = \frac{\sigma_{\psi}}{1 - B}$; it follows that the coefficient of $z^p$ for $p \in \mathbb{N} \cup \{0\}$ in the expressions for $T_{\psi}C_{\phi}z^k$ and $(T_{\psi}C_{\phi})^*z^k$ are equal for each $k \in \mathbb{N} \cup \{0\}$. Therefore, on comparing the coefficients of $1$, $z$, $z^2$, $z^3$ and so on in the expressions of $T_{\psi}C_{\phi}z^k$ and $(T_{\psi}C_{\phi})^*z^k$, we obtain that for each $k, p \in \mathbb{N} \cup \{0\}$,
\[
\sum_{n=-k}^{\infty} \binom{k}{p-n}\tilde{\psi}(n)a^{p-n+k-p} = \sum_{n=-k}^{\infty} \binom{k}{p-k}\overline{\psi(-n)}a^{p-k} = \sum_{n=-k}^{\infty} \binom{k}{p-k}\overline{\psi(-n)}a^{p-k}.
\]
(12)
Conversely, let us assume that for each $k, p \in \mathbb{N} \cup \{0\}$, equation (12) holds. Then evaluating the expression $(T_{\psi}C_{\phi} - (T_{\psi}C_{\phi})^*)z^k$ for each $k \in \mathbb{N} \cup \{0\}$ gives the value as zero. Hence, we obtain that the operator $T_{\psi}C_{\phi}$ is Hermitian on $H^2$.

Now we take $b = 0$. Then it can be easily evaluated that $(T_{\psi}C_{\phi} - (T_{\psi}C_{\phi})^*)z^k = 0$ if and only if $\sum_{n=-k}^{\infty} \tilde{\psi}(n)a^{n+k} = 0$ if and only if $\sum_{n=-k}^{\infty} \overline{\psi(-n)}a^{n+k} = 0$ if and only if $a^k\tilde{\psi}(n) = a^k\overline{\psi(-n)}$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$.

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