Functional Calculus for the Series of Semigroup Generators via Transference

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Functional Calculus for the Series of Semigroup Generators via Transference

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Abstract: In this paper, apply an established transference principle to obtain the boundedness of certain functional calculi for the sequence of semigroup generators. It is proved that if \( -A_j \) be the sequence generates \( C_0 \)-semigroups on a Hilbert space, then for each \( \varepsilon > -1 \) the sequence of operators \( A_j \) has bounded calculus for the closed ideal of bounded holomorphic functions on right half-plane. The bounded of this calculus grows at most logarithmically as \((1 + \varepsilon) \downarrow 0\). As a consequence decay at \( \infty \). Then showed that each sequence of semigroup generator has a so-called (strong) m-bounded calculus for all \( m \in \mathbb{N} \), and that this property characterizes the sequence of semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called \( \omega \)-bounded semigroups, the Hilbert space results actually hold in general Banach spaces.

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I. Introduction

Functional calculus for the sequence of operators \( A_j \) on a Banach space \( X \) is a "method" of associating a closed sequence of operators \( f_j(A_j) \) to all \( f_j = f_j(z_j) \) taken from a set of functions in such a way that formulae valid for the functions turn into valid formulae for the operators upon replacing the independent variables \( Z_j \) by \( A_j \). A common way to establish such a calculus is to start with an algebra of "good" functions \( f_j \) where definitions of \( f_j(A_j) \) as bounded sequence of operators are more or less straightforward, and then extend this "primary" or "elementary calculus" by means of multiplicative in [1, Chapter 1] and [2]. It is then natural to ask which of the so constructed closed sequence of operators \( f_j(A_j) \) are actually bounded, a question particularly relevant in applications, e.g., to evolution equations, see, [3,4].

The latter question links functional calculus theory to the theory of vector-valued singular integrals, best seen in the theory of sectorial operators with a bounded \( H^\infty \) -calculus, see, [5]. It appears there that in order to obtain nontrivial results the
underlying Banach space must allow for singular integrals to converge, i.e., be a UMD space. Furthermore, even if the Banach space is a Hilbert space, it turns out that simple resolvent estimates are not enough for the boundedness of an $H^\infty$-calculus.

However, some of the central positive results in that theory — show that the presence of a $C_0$-group of operators does warrant the boundedness of certain $H^\infty$-calculi. In [6], the underlying structure of these results was brought to light, namely a transference principle, a factorization of the sequence of operators $f_j(A_j)$ in terms of vector-valued Fourier multiplier operators. Finally, in [7], it was shown that $C_0$-semigroups also allow for such transference principles.

Markus Haase and Jan Rozendaal [8] developed this approach further. They apply the general form of the transference principle for semigroups given in [9] to obtain bounded functional calculi for the sequence of generators of $C_0$-semigroups. These results, in theorems 3.3, 3.7, and 4.3, are proved for general Banach spaces. However, they make use of the analytic $L^{1+\varepsilon}(\mathbb{R}; X)$ Fourier multiplier algebra, and hence are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In case $X = H$ is a Hilbert space, one obtains particularly nice results, which want to summarize here.

**Theorem 1.1**: Let $-A_j$ be the sequence of generators of bounded $C_0$-semigroups $(T^j(t))_{t \in \mathbb{R}^+}$ on a Hilbert space $H$ with $M := \sup_{t \in \mathbb{R}^+} |T^j(t)|$. Then the following assertions hold.

(a) For $\omega_j < 0$ and $f_j \in H^\infty(R_{\omega_j})$ one has $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(H)$ with

$$\left\| \sum_j f_j(A_j)T^j(1+\varepsilon) \right\| \leq c(1+\varepsilon)M^2 \sum_j \|f_j\|_{H^\infty(R_{\omega_j})}$$

where $c(1+\varepsilon) = O(\|\log(1+\varepsilon)\|) \to 0$, and $c(1+\varepsilon) = O(1)$ as $(1+\varepsilon) \to \infty$.

(b) For $\omega_j < 0 < \beta + \varepsilon$ and $\lambda_j \in \mathbb{C}$ with $\text{Re}\lambda_j < 0$ there is $\varepsilon \geq -1$ such that

$$\left\| \sum_j f_j(A_j)(A_j - \lambda_j)^{-\beta + \varepsilon} \right\| \leq (1+\varepsilon)M^2 \sum_j \|f_j\|_{H^\infty(R_{\omega_j})}$$

For all $f_j \in H^\infty(R_{\omega_j})$. In particular, $\text{dom}(A_j^{\beta+\varepsilon}) \subseteq \text{dom}(f_j(A_j))$.

(c) $A_j$ has strong $m$-bounded $H^\infty$-calculus of type 0 for each $m \in \mathbb{N}$.

When $X$ is a UMD space, one can derive similar results, we extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its $\gamma_j$-boundedness, a concept strongly put forward by Kalton and Weis [9]. In particular, Theorem 1.1 holds true for $\gamma_j$-bounded semigroups on arbitrary Banach spaces with $M$ being the $\gamma_j$-bound of the semigroups.

Stress the fact that in contrast to [1], where sectorial operators and, accordingly, functional calculi on sectors, were considered, deals with general sequence of semigroup generators and with functional calculi on half-planes. The abstract theory of (holomorphic) functional calculi on half-planes can be found in [2 corollaries 6.5 and 7.1].
The starting point of the present work was the article [10] by Hans Zwart. There is shown that one has an estimate (1) with $c(1 + \varepsilon) = O((1 + \varepsilon)^{-1/2})$ as $(1 + \varepsilon) \searrow 0$. (The case $\beta + \varepsilon > 1/2$) in (2) is an immediate consequence, however, that case is essentially trivial)

In [7] and its sequel paper [11], the functional calculus for a semigroup generator is constructed in a rather unconventional way using ideas from systems theory. However, a closer inspection reveals that transference is present there as well, hidden in the very construction of the functional calculus.

a) Notation and terminology

Write $\mathbb{N} := \{1, 2, \ldots \}$ for the natural numbers and $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals. The letters X and Y are used to denote Banach spaces over the complex number field. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For a closed sequence of operators $A_j$ on X their domains are denoted by $\text{dom}(A_j)$ and their ranges by $\text{ran}(A_j)$. The spectrums of $A_j$ are $\sigma(A_j)$ and the resolvent sets $\rho(A_j) := \mathbb{C}^\times \sigma(A_j)$. For all $z_j \in \rho(A_j)$ the operators $R(Z_j, A_j) := (z_j - A_j)^{-1} \in \mathcal{L}(X)$ is the resolvents of $A_j$ at $z_j$.

For $\varepsilon > 0$, $L^{1+\varepsilon} (\mathbb{R}; X)$ is the Bochner space of equivalence classes of X-valued $(1+\varepsilon)$-Lebesgue integrable functions on $\mathbb{R}$. The Hölder conjugate of $(1+\varepsilon)$ is $\left(\frac{1+\varepsilon}{\varepsilon}\right)$. The norm on $L^{1+\varepsilon} (\mathbb{R}, X)$ is usually denoted by $\|\cdot\|_{1+\varepsilon}$.

For $\omega_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$, let $e_{\omega_j}(z_j) := e^{\omega_j z_j}$. By $M(\mathbb{R})$ (resp. $M(\mathbb{R}_+)$), denote the space of complex-valued Borel measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$) with the total variation norm, and write $M_{\omega_j}(\mathbb{R}_+)$ for the distributions $\mu^j$ on $\mathbb{R}_+$ of the form $\mu^j(ds) = e^{\omega_j s} \nu^j(ds)$ for some $\nu^j \in M(\mathbb{R}_+)$. Then $M_{\omega_j}(\mathbb{R}_+)$ is a Banach algebra under convolution with the series of norms

$$\sum_j \|\mu^j\|_{M_{\omega_j}(\mathbb{R}_+)} = \sum_j \|e^{-\omega_j} \mu^j\|_{M(\mathbb{R}_+)}$$

For $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, let $\text{supp}(\mu^j)$ be the topological support of $e^{-\omega_j} \mu^j$, functions $g^j$ such that $e^{-\omega_j} g^j \in L^1(\mathbb{R}_+)$ are usually identified with its associated measures $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$ given by $\mu^j(ds) = g^j(s) ds$. Functions and measures defined on $\mathbb{R}_+$ are identified with their extensions to $\mathbb{R}$ by setting them equal to zero outside $\mathbb{R}_+$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$, let $H^\infty(\Omega)$ be the space of bounded holomorphic functions on $\Omega$, until Banach algebra concerning to the series of norms

$$\sum_j \|f_j\|_\infty = \sum_j \|f_j\|_{H^\infty(\Omega)} = \sup_{z_j \in \Omega} \sum_j |f_j(z_j)| \quad (f_j \in H^\infty(\Omega))$$

Consider the case where $\Omega$ is equal to a right half-planes

$$R_{\omega_j} = \{z_j \in \mathbb{C} | \text{Re}(z_j) > \omega_j \}$$

for some $\omega_j \in \mathbb{R}$ (we write $\mathbb{C}_+$ for $R_0$).
For convenience abbreviate the coordinate functions \( Z_j \mapsto z_j \) simply by the letters \( z_j \). Under this convention, \( f_j = f_j(z_j) \) for functions \( f_j \) defined on some domain \( \Omega \subseteq \mathbb{C} \).

The Fourier transform of an X-valued tempered distribution \( \Phi \) on \( \mathbb{R} \) is denoted by \( \mathcal{F}\Phi \). If \( \mu^j \in \mathcal{M}(\mathbb{R}) \) then \( \mathcal{F}_\mu^j \in L^\infty(\mathbb{R}) \) are given by

\[
\sum_j \mathcal{F}_\mu^j(\xi) = \int \sum_j e^{-i\xi s} \mu^j(ds) \quad (\xi \in \mathbb{R})
\]

For \( \omega_j \in \mathbb{R} \) and \( \mu^j \in M_{\omega_j}(\mathbb{R}_+), \) let \( \tilde{\mu}^j \in H^\infty(R_{\omega_j}) \cap C(\overline{R_{\omega_j}}), \)

\[
\sum_j \tilde{\mu}^j(z_j) = \int \sum_j e^{-z_j s} \mu^j(ds) \quad (z_j \in R_{\omega_j})
\]

Be the Laplace–Stieltjes transforms of \( \mu^j \).

II. Fourier Multipliers and Functional Calculus

Discuss some of the concepts that will be used in what follows (see, e.g., [8]).

a) Fourier multipliers

Fix a Banach space \( X \) and let \( m \in L^\infty(\mathbb{R}; \mathcal{L}(X)) \) and \( \varepsilon \geq 0 \). Then \( m \) is a bounded \( L^{1+\varepsilon}(\mathbb{R}; X) \)-Fourier multiplier if there exists \( \varepsilon \geq -1 \) such that

\[
T_m^j(\varphi_j) = \mathcal{F}^{-1}(m.\mathcal{F}\varphi_j) \in L^{1+\varepsilon}(\mathbb{R}; X) \text{ and } \left\| \sum_j T_m^j(\varphi_j) \right\|_{1+\varepsilon} \leq (1 + \varepsilon) \sum_j \|\varphi_j\|_{1+\varepsilon}
\]

for each \( X \)-valued Schwartz functions \( \varphi_j \). In this case, the mappings \( T_m^j \) extends uniquely to bounded sequence of operators on \( L^{1+\varepsilon}(\mathbb{R}; X) \) if \( \varepsilon < \infty \) and on \( C_0(\mathbb{R}; X) \) if \( \varepsilon = \infty \). Let \( \|m\|_{\mathcal{M}(1+\varepsilon)(X)} \) be the norms of the operators \( T_m^j \) and let \( \mathcal{M}_{1+\varepsilon}(X) \) be the unital Banach algebra of all bounded \( L^{1+\varepsilon}(\mathbb{R}; X) \)-Fourier multipliers, endowed with the norm \( \|\cdot\|_{\mathcal{M}(1+\varepsilon)(X)} \).

For \( \omega_j \in \mathbb{R} \) and \( \varepsilon \geq 0 \), we let

\[
A_j M_{1+\varepsilon}^X(R_{\omega_j}) = \left\{ f_j \in H^\infty(R_{\omega_j}) \mid f_j(\omega_j + i \cdot) \in \mathcal{M}_{1+\varepsilon}(X) \right\}
\]  

(3)

be the analytic \( L^\infty(1+\varepsilon)(\mathbb{R}; X) \)-Fourier multiplier algebras on \( R^\prime(\omega_j) \), endowed the series of norms

\[
\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X} = \sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X(R_{\omega_j})} = \sum_j \|f_j(\omega_j + i \cdot)\|_{\mathcal{M}(1+\varepsilon)(X)}
\]

Here \( f_j(\omega_j + i \cdot) \in L^\infty(\mathbb{R}) \) denotes the trace of the holomorphic functions \( f_j \) on the boundary \( \partial R_{\omega_j} = \omega_j + i\mathbb{R} \). By classical Hardy space theories,
\[ f_j(\omega_j + is) = \lim_{\omega_j \to \omega_j} f_j(\omega_j + is) \]  

(4)

Exists for almost all \( s \in \mathbb{R} \), with \( \sum_j \|f_j(\omega_j + i \cdot)\|_{L^\infty(\mathbb{R})} = \sum_j \|f_j\|_{H^\infty(R_{\omega_j})} \).

**Remark 2.1:** (Important!). To simplify notation sometimes omit the reference to the Banach space X and write \( A_j M_1(R_{\omega_j}) \) instead of \( A_j M_1^X(R_{\omega_j}) \), whenever it is convenient.

The spaces \( A_j M_1^{X+\varepsilon}(R_{\omega_j}) \) are until Banach algebra, constructively embedded in \( H^\infty(R_{\omega_j}) \), and \( A_j M_1^{X+\varepsilon}(R_{\omega_j}) = A_j M_\infty^X(R_{\omega_j}) \) are contractively embedded in \( A_j M_1^{X+\varepsilon}(R_{\omega_j}) \) for all \( \varepsilon > 0 \).

Need two lemmas about the analytic multiplier algebra.

**Lemma 2.2:** For every Banach space X, all \((0 \leq \varepsilon \leq \infty)\),

\[
\sum_j A_j M_1^{X+\varepsilon}(R_{\omega_j}) = \left\{ f_j \in H^\infty(R_{\omega_j}) \mid \sup_{\omega_j > \omega_j} \sum_j \|f_j(\omega_j + i \cdot)\|_{M_1^{X+\varepsilon}(X)} < \infty \right\}
\]

With

\[
\sum_j \|f_j\|_{A_j M_1^{X+\varepsilon}(R_{\omega_j})} = \sup_{\omega_j > \omega_j} \sum_j \|f_j(\omega_j + i \cdot)\|_{M_1^{X+\varepsilon}(X)}
\]

for all \( f_j \in A_j M_1^{X+\varepsilon}(R_{\omega_j}) \).

**Proof.** Let \( \omega \in \mathbb{R}, f_j \in A_j M_1^{X+\varepsilon}(R_{\omega_j}) \). For all \( \omega_j > \omega_j \) and \( s \in \mathbb{R} \),

\[
\sum_j f_j(\omega_j + is) = \sum_j \frac{\omega_j - \omega_j}{\pi} \int_{\mathbb{R}} \frac{f_j(\omega_j - ir)}{(s - r)^2 + (\omega_j - \omega_j)^2} \, dr
\]

The right-hand side is the series of the convolutions of \( f_j(\omega_j - i \cdot) \) and the Poisson kernel

\[
P_{\omega_j - \omega_j}(r) = \frac{\omega_j - \omega_j}{\pi(r^2 + (\omega_j - \omega_j)^2)}
\]

Since \( \sum_j \|P_{\omega_j - \omega_j}\|_{L^1(\mathbb{R})} = 1, \)

\[
\left\| \sum_j f_j(\omega_j + i \cdot) \right\|_{M_1^{X+\varepsilon}(X)} \leq \sum_j \|f_j(\omega_j - i \cdot)\|_{M_1^{X+\varepsilon}(X)} = \sum_j \|f_j\|_{A_j M_1^{X+\varepsilon}(R_{\omega_j})}
\]

The converse follows from (4) \( \Box \)

For \( \mu_j \in M(\mathbb{R}) \) and \( \varepsilon \geq 0 \), let \( L_{\mu_j} \in \mathcal{L}(L^{1+\varepsilon}(\mathbb{R}; X)) \),

\[
L_{\mu_j}(f_j) := \mu_j * f_j, \quad (f_j \in L^{1+\varepsilon}(\mathbb{R}; X)),
\]

(5)

be the convolution sequence of operators associated with \( \mu_j \).
Lemma 2.3: For each \( \omega_j \in \mathbb{R} \) the Laplace transform induces an isometric algebra isomorphism from \( M_{\omega_j}(\mathbb{R}_+) \) onto \( A_j M_1^{X}(R_{\omega_j}) = A_j M_1^{X}(R_{\omega_j}) \). Moreover,

\[
\sum_{j} \| \mu^j \| A_j M_1^{X+}(R_{\omega_j}) = \sum_{j} \| L_{e^{-\omega_j}} \mu^j \| \mathcal{L}(l(1+\varepsilon)(X))
\]

for all \( \mu^j \in M_{\omega_j}(\mathbb{R}_+), \varepsilon \geq 0 \).

Proof: The mappings \( \mu^j \rightarrow e^{-\omega_j} \mu^j \) and \( f_j \rightarrow f_j(\cdot + \omega_j) \) are isometric algebra isomorphisms \( M_{\omega_j}(\mathbb{R}_+) \rightarrow M(\mathbb{R}_+) \) and \( A_j M_1^{X+}(R_{\omega_j}) \rightarrow A_j M_1^{X+}(C_+) \), respectively. Hence it suffices to let \( \omega_j = 0 \). The Fourier transform induces an isometric isomorphism from \( M(\mathbb{R}) \) onto \( M_1(X) \). If \( \mu^j \in M(\mathbb{R}_+) \) and \( f_j = \widehat{\mu^j} \in H^\infty(C_+) \) then \( f_j(i \cdot) = \mathcal{F} \mu^j \in M_1(X) \) with \( \sum_j \| f_j(i \cdot) \| M_1(X) = \sum_j \| \mu^j \| M(\mathbb{R}_+) \). Moreover, for \( \varepsilon \geq 0 \),

\[
\sum_j \| f_j(i \cdot) \| M_1(X) = \sum_j \sup \| g_{i \cdot} \|_{1+\varepsilon} = \sup \| g_{i \cdot} \|_{1+\varepsilon} \leq \sum_j \| \mu^j \| \mathcal{F} \mu^j \| M_1(X) = \sum_j \| \mu^j \| \mathcal{F} \mu^j \| M(\mathbb{R}_+). \]

If \( f_j \in A_j M_1(C_+) \) then \( f_j(i \cdot) = \mathcal{F} \mu^j \) for some \( \mu^j \in M(\mathbb{R}) \). An application of Liouville’s theorem shows that \( \text{supp}(\mu^j) \subseteq \mathbb{R}_+ \), hence \( f_j = \widehat{\mu^j} \). \( \blacksquare \)

b) Functional Calculus

Assume that we are familiar with the basic notions and results of the theory of \( C_0 \)-semigroups as developed, e.g., in [5].

All \( C_0 \)-semigroups \( T^j = (T^j(t))_{t \in \mathbb{R}_+} \) on a Banach space \( X \) has the type \( (M, \omega_j) \) for some \( M \geq 1 \) and \( \omega_j \in \mathbb{R} \), which means that \( \| \sum_j T^j(t) \| \leq M \sum_j e^{\omega_j t} \) for all \( t \geq 0 \). The generators of \( T^j \) are the unique closed sequence of operators \( -A_j \) such that

\[
\sum_j (\lambda_j + A_j)^{-1} x = \int_0^\infty \sum_j e^{-\lambda_j t} T^j(t) x dt \quad (x \in X)
\]

for \( \text{Re}(\lambda_j) \) large. The Hille–Phillips (functional) calculus for \( A_j \) are defined as follows. Fix \( M \geq 0 \) and \( (\omega_j)_0 \in \mathbb{R} \) such that \( T^j \) has types \( (M, -(\omega_j)_0) \). For \( \mu^j \in M_{(\omega_j)_0}(\mathbb{R}_+) \) defines \( T^j_{\mu^j} \in L(X) \) by

\[
\sum_j T^j_{\mu^j} x = \int_0^\infty \sum_j T^j(t) x \mu^j dt, \quad (x \in X)
\]

For \( f_j = \widehat{\mu^j} \in A_j M_1 \left( R_{(\omega_j)_0} \right) \) sets \( f_j(A_j) = T^j_{\mu^j} \). The mappings \( f_j \mapsto f_j(A_j) \) is an algebra homomorphism. In a second step the definitions of \( f_j(A_j) \) is extended to a larger class of functions via regularization, i.e.,

\[
f_j(A_j) := e(A_j)^{-1} (ef_j)(A_j)
\]
If there exists \( e \in A_j M_1 \left( R_{(\omega_j)_0} \right) \) such that \( e(A_j) \) is injective and \( ef_j \in A_j M_1 \left( R_{(\omega_j)_0} \right) \). Then \( f_j(A_j) \) is closed and unbounded operator on \( X \) and the definition of \( f_j(A_j) \) are independent of the choice of regularize. The following lemma shows in particular that for \( \omega_j < (\omega_j)_0 \) the sequence of operators \( f_j(A_j) \) are defined for all \( f_j \in H^{\infty}(R_{\omega_j}) \) by virtue of the regularizes \( e(z_j) = (Z_j - \lambda_j)^{-1} \), where \( \text{Re}(A_j) < \omega_j \).

**Lemma 2.4:** Let \( \beta + \varepsilon > \frac{1}{2} \), \( \lambda_j \in \mathbb{C} \) and \( \omega_j , (\omega_j)_0 \in \mathbb{R} , \varepsilon \geq 0 \). Then

\[
f_j (z_j)(z_j - \lambda_j)^{-(\beta + \varepsilon)} \in A_j M_1 (R_{\omega_j})_0 \text{ for all } f_j \in H^{\infty} R_{(\omega_j)_0}
\]

**Proof:** After shifting suppose that \( \omega_j = 0 \). Sets \( h_j(z_j) := f_j(z_j)(z_j - \lambda_j)^{-(\beta + \varepsilon)} \) for \( z_j \in \mathbb{C}_+ \). Then \( h_j(i \cdot) \in L^2(\mathbb{R}) \) with

\[
\left\| \sum_j h_j (i \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \sum_j \left| \frac{f_j(is)}{is - \lambda_j} \right|^2 ds \leq \int_{\mathbb{R}} \sum_j \left| f_j \right|_{M^\infty(\mathbb{C}_+)}^2 ds
\]

Hence \( h_j = \hat{g}_j \) for some \( g_j \in L^2(\mathbb{R}_+) \). Then \( e^{-\omega_0} \hat{g}_j(z_j) = h_j(z_j + (\omega_j)_0) \) for \( z_j \in \mathbb{C}_+ \). Lemma 2.3 yields \( h_j \in A_j M_1 R_{(\omega_j)_0} \) with

\[
\sum_j \left\| h_j \right\|_{A_j M_1 R_{(\omega_j)_0}} = \sum_j \left\| h_j (\cdot + (\omega_j)_0) \right\|_{A_j M_1 (\mathbb{C}_+)} = \sum_j \left\| e^{-\omega_0} \hat{g}_j \right\|_{L^1(\mathbb{R}_+)}
\]

The Hille–Phillips calculus is an extension of the holomorphic functional calculus for the sequence of operators of half-plane type discussed in [2]. The sequence operators of \( A_j \) are of the half-plane types \( (\omega_j)_0 \in \mathbb{R} \) if \( \sigma(A_j) \subseteq \overline{R_{(\omega_j)_0}} \) with

\[
\sup_{\lambda_j \in \mathbb{C} \setminus R_{(\omega_j)_0}} \sum_j \left\| R(\lambda_j, A_j) \right\| < \infty,
\]

for all \( \varepsilon > 0 \)

One can associate the sequence of operators \( f_j(A_j) \in \mathcal{L}(X) \) to certain elementary functions via Cauchy integrals and regularize as above to extend the definitions to all \( f_j \in H^{\infty} (R_{\omega_j}) \). If \( -A_j \) generates \( C_0 \) -semigroups of types \( (M, -(\omega_j)_0) \) then \( A_j \) are of half-plane types \( (\omega_j)_0 \), for \( \omega_j < (\omega_j)_0 , \varepsilon > 0 \) and \( f_j \in H^{\infty}(R_{\omega_j}) \) the definitions of \( f_j(A_j) \) via the Hille–Phillips calculus and the half-plane calculus coincide.

**Lemma 2.5:** (Convergence Lemma). Let \( A_j \) be densely defined sequence of operators of half-plane types \( (\omega_j)_0 \in \mathbb{R} \) on a Banach space \( X \). Let \( \omega_j < (\omega_j)_0 \) and \( (f_j)_{j \in J} \subseteq H^{\infty}(R_{\omega_j}) \) be satisfying the following conditions:

(1) \( \sup \{ \left| (f_j)(z_j) \right| \mid z_j \in R_{\omega_j}, j \in J \} < \infty \);
(2) \((f_j)_j(A_j) \in \mathcal{L}(X)\) for all \(j \in J\) and \(\sup_{j \in J} \| (f_j)_j(A_j) \| < \infty\);

(3) \(f_j(z_i) := \lim_{j \in J} f_j(z_i)\) exists for all \(z_i \in R_{\omega_j}\).

Then \(f_j \in H^\omega(R_{\omega_j})\), \(f_j(A_j) \in \mathcal{L}(X)\), \((f_j)_j(A_j) \to f_j(A_j)\) strongly and

\[
\left\| \sum_j f_j(A_j) \right\| \leq \limsup_{j \in J} \sum_j \| (f_j)_j(A_j) \|
\]

Let \(A_j\) be the sequence of operators of half-plane types \((\omega_j)_0\) and \(\omega_j < (\omega_j)_0\). For a Banach algebra \(F\) of functions continuously embedded in \(H^\omega(R_{\omega_j})\), say that \(A_j\) has bounded \(F\)-calculus if there exists a constant \(\epsilon \geq -1\) such that \(f_j(A_j) \in \mathcal{L}(X)\) with

\[
\left\| \sum_j f_j(A_j) \right\|_{\mathcal{L}(X)} \leq (1 + \epsilon) \sum_j \| f_j \|_F \text{ for all } f_j \in F
\]

The sequence of operators \(-A_j\) generates a \(C_0\)-semigroups \((T^j(t))_{t \in \mathbb{R}_+}\) of types \((M, \omega_j)\) if and only if \(- (A_j + \omega_j)\) generates the semigroups sequence of \((e^{-\omega_j t} T^j(t))_{t \in \mathbb{R}_+}\) of types \((M, 0)\). The functional calculi for \(A_j\) and \(A_j + \omega_j\) are linked by the simple composition rules \(f_j(A_j + \omega_j) = f_j(\omega_j + z_j)(A_j)^n\). Henceforth we shall mainly consider bounded semigroups; all results carry over to general semigroups by shifting.

### III. Functional Calculus for Semigroup Generators

Define the function \(\eta : (0, \infty) \times (0, \infty) \times [1, \infty] \to \mathbb{R}_+\) by

\[
\eta(\beta + \epsilon, t, 1 + \epsilon) = \inf \left\{ \| \psi_j \|_{1+\epsilon} \| \varphi_j \|_{1+\epsilon} \| \psi_j * \varphi_j \equiv e_{-(\beta + \epsilon)} \text{ on } [t, \infty) \right\}
\]

The set on the right-hand side is not empty: choose for instance \(\psi_j := 1_{[\mu_j]} e_{-(\beta + \epsilon)}\) and \(\varphi_j := \frac{1}{t} e_{-(\beta + \epsilon)}\). By Lemma A.1,

\[
\eta(\beta + \epsilon, t, 1 + \epsilon) = O ((\log((\beta + \epsilon)t)) \text{ as } (\beta + \epsilon)t \to 0, \text{ for } \epsilon > 0.
\]

For the following result recall the definitions of the operators \(L_{\mu}^j\) from (5) and \(T_{\mu}^j\) from (6).

**Proposition 3.1:** Let \((T^j(t))_{t \in \mathbb{R}_+}\) be \(C_0\)-semigroup of type \((M, 0)\) on a Banach space \(X\). Let \(\epsilon \geq 0\), \(1 + \epsilon\), \(\omega_j > 0\) and \(\mu^j \in M_{\omega_j}(\mathbb{R}_+)\) with \(\text{supp}(\mu^j) \subseteq [1+\epsilon, \infty)\). Then

\[
\left\| \sum_j T_{\mu}^j \right\|_{\mathcal{L}(X)} \leq M^2 \eta \sum_j (\omega_j, 1 + \epsilon, 1 + \epsilon) \left\| L_{\omega_j} e_{\omega_j} \mu^j \right\|_{\mathcal{L}(L^{1+\epsilon}(X))}
\]

**Proof:** Factorizes \(T_{\mu}^j\) as \(T_{\mu}^j = P \circ L_{\omega_j} e_{\omega_j} \mu^j \circ 1\), where

a) \(1 : X \to L^{1+\epsilon}(\mathbb{R}; X)\) is given by
Theorem 3.1: For every Banach space \( X \), \( \omega_j \in \mathbb{R} \), and \( \varepsilon > -1 \), the spaces

\[
A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) = \left\{ f_j \in A_j M^X_{(1+\varepsilon)}(R_{\omega_j}) \mid f_j(z_j) = 0 \left( e^{-(1+\varepsilon)R\omega(z_j)} \right) \text{ as } |z_j| \to \infty \right\}
\]

endow with the norms of \( A_j M^X_{1+\varepsilon}(R_{\omega_j}) \).

Lemma 3.2: For every Banach space \( X \), \( \omega_j \in \mathbb{R} \), \( 1 \leq \varepsilon \leq \infty \), and \( \varepsilon \leq -1 \)

\[
A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) = A_j M^X_{1+\varepsilon}(R_{\omega_j}) \cap e_{-(1+\varepsilon)H^\infty(R_{\omega_j})} = e_{-(1+\varepsilon)A_j M^X_{1+\varepsilon}(R_{\omega_j})} \tag{10}
\]

In particular, \( A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) \) are closed ideals in \( A_j M^X_{1+\varepsilon}(R_{\omega_j}) \).

Proof: The first equality in (10) is clear, and so are the inclusions

\[
e_{-(1+\varepsilon)A_j M^X_{(1+\varepsilon)}(R_{\omega_j})} \subseteq A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}).
\]

Conversely, if \( f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j}) \cap e_{-(1+\varepsilon)H^\infty(R_{\omega_j})} \) then \( e_{(1+\varepsilon)} f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j}) \), since

\[
\sum_j \left\| e^{(1+\varepsilon)(\omega_j + i \cdot)} f_j(\omega_j + i \cdot) \right\|_{M_{(1+\varepsilon)}(X)} = \sum_j 0^{(1+\varepsilon)\omega_j} \left\| f_j(\omega_j + i \cdot) \right\|_{M_{(1+\varepsilon)}(X)}
\]

Suppose that \( ((f_j)_n)_{n \in \mathbb{N}} \subseteq A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) \) converges to \( f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j}) \).

The Maximum Principle implies

\[
\sum_j \left\| e_{(1+\varepsilon)} f_j_n \right\|_{H^\infty(R_{\omega_j})} = \sum_j e^{(1+\varepsilon)\omega_j} \left\| f_j_n \right\|_{H^\infty(R_{\omega_j})},
\]

hence \( e_{(1+\varepsilon)}(f_j)_n \) is Cauchy in \( H^\infty(R_{\omega_j}) \). Since it converges pointwise to \( e_{(1+\varepsilon)} f_j \), (10) implies \( f_j \in A_j M_{1+\varepsilon}(R_{\omega_j}) \).
bounding the $A_j M^X_{(1+\epsilon)} (R_{\omega_j})$ calculus for all, the Convergence Lemma would imply that $A_j$ has bounded $A_j M^X_{(1+\epsilon)} (R_{\omega_j})$ calculus, but this is known to be false in general [1, Corollary 9.1.8].

**Theorem 3.3:** For each $0 < \epsilon < \infty$, there exists a constant $c_{1 + \epsilon} \geq 0$ such that the following holds. Let $-A_j$ the sequence of generates $C_0$-semigroups $(T^t(t))_{t \in \mathbb{R}_+}$ of type $(M, 0)$ on a Banach space $X$ and let $(1 + \epsilon), \omega_j > 0$. Then $f_j(A_j) \in L(X)$ and

$$
\|\sum_j f_j(A_j)\| \leq \begin{cases} 
\epsilon \sum_j \log (\omega_j (1 + \epsilon)) \|f_j\|_{A_j M^X_{(1+\epsilon)}} & \text{if } \omega_j (1 + \epsilon) \leq \min \left( \frac{1}{1+\epsilon}, \frac{\epsilon}{1+\epsilon} \right) \\
2M^2 \sum_j e^{-\omega_j (1+\epsilon)} \|f_j\|_{A_j M^X_{(1+\epsilon)}} & \text{if } \omega_j (1 + \epsilon) > \min \left( \frac{1}{1+\epsilon}, \frac{\epsilon}{1+\epsilon} \right)
\end{cases}
$$

for all $f_j \in A_j M^X_{(1+\epsilon)} (R_{-\omega_j})$. In particular, $A_j$ has bounded $A_j M^X_{(1+\epsilon)} (R_{-\omega_j})$ calculus.

**Proof:** First consider $f_j \in A_j M_{1,(1+\epsilon)} (R_{-\omega_j})$. Let $\delta_{(1+\epsilon)} \in M_{-\omega_j} (\mathbb{R}_+)$ be the unit point mass at $\epsilon > -1$. By Lemmas 3.2 and 2.3 there exists $\mu^i \in M_{-\omega_j} (\mathbb{R}_+)$ such that $f_j = e_{-(1+\epsilon)\omega_j \mu^i} = \delta_{(1+\epsilon)} * \mu^i$. Since $\delta_{(1+\epsilon)} * \mu^i \in M_{-\omega_j} (\mathbb{R}_+)$ with supp $(\delta_{(1+\epsilon)} * \mu^i) \subseteq [1+\epsilon, \infty)$, Proposition 3.1 and Lemma 2.3 yield

$$
\left\| \sum_j f_j (A_j) \right\| \leq M^2 \eta \sum_j (\omega_j, (1 + \epsilon), (1 + \epsilon)) \|f_j\|_{A_j M^X_{(1+\epsilon)}} \tag{11}
$$

Suppose $f_j \in A_j M_{1,(1+\epsilon)} (R_{-\omega_j})$ are arbitrary. For $\epsilon > 0$, $k \in \mathbb{N}$ and $z_j \in R_{-\omega_j}$

Set $g_k^j (z_j) := \frac{k}{z_j - \omega_j + k}$ and $(f_j)_{k,\epsilon} (z_j) = f_j (z_j + \epsilon) g_k^j (z_j + \epsilon)$. Lemma 2.4 yields $(f_j)_{k,\epsilon} \in A_j M_{1,(1+\epsilon)} (R_{-\omega_j})$, hence, by what have shown,

$$
\left\| \sum_j (f_j)_{k,\epsilon} (A_j) \right\| \leq M^2 \eta \sum_j (\omega_j, 1 + \epsilon, 1 + \epsilon) \|(f_j)_{k,\epsilon}\|_{A_j M^X_{(1+\epsilon)}}
$$

The inclusions $A_j M_{1} (R_{-\omega_j}) \subseteq A_j M_{1+\epsilon} (R_{-\omega_j})$ are contractive, so Lemma 2.3 implies that $g_k^j \in A_j M_{1+\epsilon} (R_{-\omega_j})$ with

$$
\left\| \sum_j g_k^j \right\|_{A_j M^X_{(1+\epsilon)}} \leq \sum_j \|g_k^j\|_{A_j M_1} = k \|e_{-\epsilon k}\|_{L^1 (\mathbb{R}_+)} = 1
$$

Combining this with Lemma 2.2 yields

$$
\left\| \sum_j (f_j)_{k,\epsilon} \right\|_{A_j M^X_{(1+\epsilon)}} \leq \sum_j \|f_j (\cdot + \epsilon)\|_{A_j M^X_{(1+\epsilon)}} \|g_k^j (\cdot + \epsilon)\|_{A_j M^X_{(1+\epsilon)}} \leq \sum_j \|f_j\|_{A_j M^X_{(1+\epsilon)}}
$$
In particular, $\sup_{k, \varepsilon} \| \sum_j (f_j)_{k, \varepsilon} \|_\infty < \infty$ and $\sup_{k, \varepsilon} \| \sum_j (f_j)_{k, \varepsilon} (A_j) \| < \infty$. The Convergence Lemma 2.5 implies that $f_j (A_j) \in \mathcal{L}(X)$ satisfies (11). Lemma A.1 concludes the proof. ■

Remark 3.4: Because $A_j M_1(R_{-\omega_j}) = A_j M_{\infty}(R_{-\omega_j})$ are contractively embedded in $A_j M_{(1+\varepsilon)}(R_{-\omega_j})$ Theorem 3.3 also holds for $\varepsilon \geq 0$. However, $A_j$ trivially has abounded $A_j M_1$-calculus by lemma 2.3 and the Hille-Phillips calculus.

Note that the exponential decays of $\sum_j |f_j(z_j)|$ are only required as the real parts of $z_j$ tends to infinity. If $\sum_j |f_j(z_j)|$ decays exponentially as $|z_j| \to \infty$ the result is not interesting by lemma 2.4.

Equivalently formulate Theorem 3.3 as a statement about composition with sequence semigroupoperators.

Corollary 3.5: Under the assumptions of Theorem 3.3, $f_j (A_j) T^j(1 + \varepsilon) \in \mathcal{L}(X)$ and

$$\left\| \sum_j f_j (A_j) T^j (1 + \varepsilon) \right\| \leq \begin{cases} c_{1+\varepsilon} M^2 \sum_j \left| \log (\omega_j (1 + \varepsilon)) \right| |e^{\omega_j (1+\varepsilon)}| \| f_j \|_{A_j M_{1+\varepsilon}}, & \text{if } \omega_j (1 + \varepsilon) \leq \min \left( \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right) \\ 2M^2 \sum_j \| f_j \|_{A_j M_{1+\varepsilon}}, & \text{if } \omega_j (1 + \varepsilon) > \min \left( \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right) \end{cases}$$

For all $f_j \in A_j M_{1+\varepsilon}^X(R_{-\omega_j})$.

Proof: Note that $\sum_j f_j (A_j) T^j (1 + \varepsilon) = \sum_j (e^{-(1+\varepsilon)} f_j) (A_j)$ and $\sum_j \| e^{-(1+\varepsilon)} f_j \|_{A_j M_{1+\varepsilon}}$

$$= \sum_j e^{\omega_j (1+\varepsilon)} \| f_j \|_{A_j M_{1+\varepsilon}} \quad \blacksquare$$

a) Additional results

As the first corollary of Theorem 3.3 we obtain a sufficient condition for a semigroupgenerator to have a bounded $A_j M_{1+\varepsilon}$-calculus (see,e.g.,[8]).

Corollary 3.6: Let $-A_j$ be the sequence of generates bounded $C_0$ -semigroups $(T^j(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X)$ with

$$\bigcup_{\varepsilon > 1} \sum_j \mathrm{ran}(T^j(1 + \varepsilon)) = X$$

Then $A_j$ has bounded $A_j M_{1+\varepsilon}(R_{\omega_j})$-calculus for all $\omega_j \geq 0, \varepsilon \geq 0$.

Proof: Using Corollary 3.5 note that $f_j (A_j) T^j(1 + \varepsilon) \in \mathcal{L}(X)$ implies ran $(T^j(1 + \varepsilon)) \subseteq \text{dom}(f_j (A_j))$. An application of the Closed Graph Theorem (using the Convergence Lemma) yields (7). ■

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**Theorem 3.7:** Let $0 < \varepsilon < \infty$, $\omega_j > 0$ and $\beta + \varepsilon$, $\lambda_j \in \mathbb{C}$ with $\text{Re}(\lambda_j) < 0 < \text{Re}(\beta + \varepsilon)$. There exists a constant $C = C(1 + \varepsilon, \beta + \varepsilon, \lambda_j, \omega_j) \geq 0$ such that the following holds. Let $-A_j$ be the sequence of generators $C_0-$ semigroups $(T^j(t))_{t \in \mathbb{R}^+}$ of type $(M,0)$ on a Banach space $X$. Then $\text{dom}((A_j - \lambda_j)^{(\beta + \varepsilon)}) \subseteq \text{dom} (f_j(A_j))$ and
\[
\left\| \sum_j f_j(A_j) (A_j - \lambda_j)^{- (\beta + \varepsilon)} \right\| \leq (1 + \varepsilon)M^2 \sum_j \| f_j \|_{A_jM^{1+\varepsilon}}
\]
for all $f_j \in A_jM^{X} (R - \omega_j)$. 

**Proof:** First note that $-(A_j - \lambda_j)$ generates the exponentially stable semigroups $(e^{\lambda_j t} (T^j(t)))_{t \in \mathbb{R}^+}$. Hence to write
\[
\sum_j (A_j - \lambda_j)^{- (\beta + \varepsilon)} x = \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^\infty t^{(\beta + \varepsilon) - 1} \sum_j e^{\lambda_j t} T^j(t) x dt \quad (x \in X)
\]
Fix $f_j \in A_jM^{1+\varepsilon} (R - \omega_j)$ and set $a := \frac{1}{\omega_j} \min \left\{ \frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon} \right\}$. By Corollary 3.5,
\[
\int_0^\infty t^{\text{Re}(\beta + \varepsilon) - 1} e^{\text{Re}(\lambda_j) t} \left\| \sum_j f_j (A_j) T^j(t)(x) \right\| dt \leq (1 + \varepsilon)M^2 \sum_j \| f_j \|_{A_jM^{1+\varepsilon}} \| x \| < \infty
\]
for all $x \in X$, where
\[
C = c_{1+\varepsilon} \int_0^a t^{\text{Re}(\beta + \varepsilon) - 1} \sum_j |\log(\omega_j t)| e^{(\text{Re}(\lambda_j) + \omega_j) t} dt + 2 \int_a^\infty t^{\text{Re}(\beta + \varepsilon) - 1} \sum_j e^{(\text{Re}(\lambda_j)) t} dt
\]
are independents of $f_j$, $M$, and $x$. Since $f_j(A_j)$ are closed operators, this implies that $(A_j - \lambda_j)^{- (\beta + \varepsilon)}$ maps into $\text{dom} f_j(A_j)$ with
\[
\sum_j f_j(A_j)(A_j - \lambda_j)^{- (\beta + \varepsilon)} = \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^\infty t^{(\beta + \varepsilon) - 1} \sum_j e^{\lambda_j t} f_j(A_j) T^j(t) dt
\]
as a strong integral. 

**Remark 3.8:** Theorem 3.7 shows that for all analytic multiplier functions $f_j$ the domains $\text{dom} (f_j(A_j))$ are relatively large, it contains the real interpolation spaces $(X, \text{dom}(A_j))_{(\theta,1+\varepsilon)}$ and the complex interpolation spaces $[X, \text{dom}(A_j)]_{\theta}$ for all $\theta \in (0, 1)$ and $\varepsilon \geq 0$.

**Remark 3.9:** Describe the ranges of $f_j(A_j)(A_j - \lambda_j)^{- (\beta + \varepsilon)}$ in Theorem 3.7. More explicitly. In fact
\[
\text{ran} (f_j(A_j)(A_j - \lambda_j)^{- (\beta + \varepsilon)}) \subseteq \text{dom}(A_j - \lambda_j)^{\beta}
\]
for all $\text{Re}(\beta)< \text{Re}(\beta + \varepsilon)$. Indeed, this follows if show that
ran\(\left((A_j - \lambda_j)^{-\beta+\varepsilon}\right) \subseteq \text{dom}\left((A_j - \lambda_j)^\beta f_j(A_j)\right)\) implies
\[
dom\left((A_j - \lambda_j)^\beta f_j(A_j)\right) = \text{dom}(f_j(A_j)) \cap \text{dom}\left(\left[(z_j - \lambda_j)^-\varepsilon f_j(z_j)\right](A_j)\right)
\]
The inclusion ran\(\left((A_j - \lambda_j)^{-\beta+\varepsilon}\right) \subseteq \text{dom}(f_j(A_j))\) follows from Theorem 3.7. Since
\[
\left[(z_j - \lambda_j)^\beta f_j(z_j)\right](A_j)(A_j - \lambda_j)^{-\varepsilon} = \left[(z_j - \lambda_j)^-\varepsilon f_j(z_j)\right](A_j) = f_j(A_j)(A_j - \lambda_j)^{-\varepsilon}
\]
The same holds for the inclusion ran\(\left((A_j - \lambda_j)^{-\beta+\varepsilon}\right) \subseteq \text{dom}\left(\left[(z_j - \lambda_j)^\beta f_j(z_j)\right](A_j)\right)\)

\(b)\) **Semigroups on Hilbert spaces**

If \(X = H\) is a Hilbert space, Plancherel’s Theorem implies \(A_j M^H_2 = H^\infty\) with equality of norms. Hence the theory above specializes to the following result, implying (a) and (b) of Theorem (1.1),

**Corollary 3.10:** Let \(-A_j\) be the sequence of generators bounded \(C_0\)-semigroups \((T^j(t))_{t \in \mathbb{R}_+}\) of type \((M, 0)\) on a Hilbert space \(H\). Then the following assertions hold.

(a) There exists a universal constant \(c \geq 0\) such that the following holds. Let \(1 + \varepsilon, \omega_j > 0\). Then\(f_j(A_j) \in \mathcal{L}(H)\) and

\[
\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} 
2M^2 \sum_j \left\|\log(\omega_j(1 + \varepsilon))\right\| f_j \|f_j\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) \leq \frac{1}{2} \\
2M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \left\|f_j\right\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) > \frac{1}{2} 
\end{cases}
\]

for all \(f_j \in e_{-(1+\varepsilon)H^\infty(R_{-\omega_j})}\). Moreover, \(f_j(A_j)T^j(1 + \varepsilon) \in \mathcal{L}(H)\) with

\[
\left\|\sum_j f_j(A_j) T^j(1 + \varepsilon)\right\| \leq \begin{cases} 
2M^2 \sum_j \left\|\log(\omega_j(1 + \varepsilon))\right\| e^{\omega_j(1+\varepsilon)} \left\|f_j\right\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) \leq \frac{1}{2} \\
2M^2 \sum_j \left\|f_j\right\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) > \frac{1}{2} 
\end{cases}
\]

for all \(f_j \in H^\infty(R_{-\omega_j})\).

(b) If

\[
\bigcup_{\varepsilon > -1} \sum_j \text{ran}\left( T^j(1 + \varepsilon) \right) = H
\]

then \(A_j\) has bounded \(H^\infty(R_{\omega_j})\)-calculus for all \(\omega_j < 0\).

(c) For \(\omega_j < 0\) and \(\beta + \varepsilon, \lambda_j \in \mathbb{C}\) with \(\text{Re}(\lambda_j) < 0 < \text{Re}(\beta + \varepsilon)\) there is \(C = C(\beta + \varepsilon, \lambda_j, \omega_j)\) such that

\[
\left\|\sum_j f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right\| \leq CM^2 \sum_j \left\|f_j\right\|_{\infty}
\]
for all \( f_j \in H^\infty(R_{\omega_j}) \). In particular, \( \text{dom}(A_j^{\beta+\epsilon}) \subseteq \text{dom}(f_j(A_j)) \).

**Note:** We can deduce that:

\[
C \sum_j \|f_j\|_\infty \leq \frac{(1 + \epsilon)}{C} \sum_j \|f_j\|_{A_j M_{1+\epsilon}^X},
\]

From Theorem 3.7 and Corollary 3.10 Part (c).

Part (c) shows that, even though the sequence of semigroup generators on Hilbert spaces do not have abounded \( H^\infty \)-calculus in general, each functions \( f_j \) that decays with polynomial rate \( \epsilon > 0 \) at infinity yields bounded sequence of operators \( f_j(A_j) \). For \( \beta + \epsilon > \frac{1}{2} \) this is already covered by Lemma 2.4, but for \( \beta + \epsilon \in (0, \frac{1}{2}] \) it appears to be new.

**Remark 3.11:** Part (c) of Corollary 3.10 yields a statement about stability of numerical methods. Let \( -A_j \) be the sequence generates an exponentially stable semigroups \( (T^j(t))_{t \geq 0} \) on a Hilbert space,

Let \( r \in H^\infty(\mathbb{C}_+) \) be such that \( \|r\|_{H^\infty(\mathbb{C}_+)} \leq 1 \), and let \( \beta + \epsilon, h_j > 0 \). Then

\[
\sup \left\{ \|r(h_j A_j)^n x\| | n \in \mathbb{N}, x \in \text{dom}(A_j^{\beta+\epsilon}) \right\} < \infty \quad (12)
\]

Follows from (c) in Corollary 3.10 after shifting the generator. Elements of the form \( r(h_j A_j)^n x \) are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to \( -A_j \) with initial value \( x \), and (12) shows that such approximations are stable whenever \( x \in \text{dom}(A_j^{\beta+\epsilon}) \).

**IV. M-Bounded Functional Calculus**

Describe another transference principle for semigroups, one that provides estimates for the norms of the sequence of operators of the form \( f_j^{(m)}(A_j) \) for \( f_j \) analytic multiplier functions and \( f_j^{(m)} \) its \( m \)-th derivatives, \( m \in \mathbb{N} \). Moreover, recall our notational simplifications \( A_j M_{1+\epsilon}(R_{\omega_j}) := A_j M_{1+\epsilon}^X(R_{\omega_j}) \) (Remark 2.1).

Let \( \omega_j < (\omega_j)_0 \) be real numbers. The sequence operators of \( A_j \) of half-plane types \( (\omega_j)_0 \) a Banach space \( X \), has an \( m \)-bounded \( A_j M_{1+\epsilon}^X(R_{\omega_j}) \)-calculus if there exists \( \epsilon \geq -1 \), such that \( f_j^{(m)}(A_j) \in \mathcal{L}(X) \) with

\[
\left\| \sum_j f_j^{(m)}(A_j) \right\| \leq (1 + \epsilon) \sum_j \|f_j\|_{A_j M_{1+\epsilon}^X} \quad \text{for all } f_j \in A_j M_{1+\epsilon}^X(R_{\omega_j})
\]

This is well defined since the Cauchy integral formula implies that \( f_j^{(m)} \) is bounded on every half-planes \( R_{\omega_j} \) with \( \omega_j > \omega_j \).
Say that \( A_j \) has a strongm-bounded \( A_j M_{1+\varepsilon}^X \)-calculus of types \((\omega_j)_0\) if \( A_j \) has an m-bounded \( A_j M_{1+\varepsilon}^X (R_{\omega_j}) \)-calculus for every \( \omega_j < (\omega_j)_0 \) such that for some \( \varepsilon \geq 0 \) one has

\[
\left\| \sum_j f_j^{(m)} (A_j) \right\| \leq (1 + \varepsilon) \sum_j \frac{1}{((\omega_j)_0 - \omega_j)^m} \| f_j \|_{A_j M_{1+\varepsilon}^X (R_{\omega_j})} \quad (13)
\]

for all \( f_j \in A_j M_{1+\varepsilon}^X (R_{\omega_j}) \) and \( \omega_j i(\omega_j)_0 \).

**Lemma 4.1:** Let \( A_j \) be the sequence of operators of half-plane types \((\omega_j)_0 \in \mathbb{R} \) on a Banach space \( X \), and let \( 0 \leq \varepsilon \leq \infty \), and \( m \in \mathbb{N} \). If \( A_j \) has a strong m-bounded \( A_j M_{1+\varepsilon}^X \)-calculus of types\((\omega_j)_0\), then \( A_j \) has a strong n-bounded \( A_j M_{1+\varepsilon}^X \)-calculus of types\((\omega_j)_0\) for all \( n, \varepsilon > 0 \).

**Proof:** Let \( \omega_j < \beta + \varepsilon < (\omega_j)_0 \). \( f_j \in A_j M_{1+\varepsilon} (R_{\omega_j}) \) and \( n \in \mathbb{N} \). Then

\[
\sum_j f_j^{(n)} (\beta + is) = \frac{(n)!}{2\pi i} \int_{\mathbb{R}} \frac{f_j ((\beta + \varepsilon) + ir)}{((\beta + \varepsilon) + ir) - (\beta + is)^{n+1}} dr
\]

\[
= \frac{(n)!}{2\pi i} \sum_j f_j ((\beta + \varepsilon) + i \cdot ((\varepsilon - i \cdot) - n^{-1}) (s)
\]

For some \( s \in \mathbb{R} \), by the Cauchy Integral formula. Hence, using lemma 2.2,

\[
\left\| \sum_j f_j^{(n)} (\beta + i \cdot) \right\|_{M(1+\varepsilon)(X)} \leq \frac{(n)!}{2\pi i} \| \varepsilon - i \cdot - n^{-1} \|_{L^1(\mathbb{R})} \sum_j \| f_j ((\beta + \varepsilon) + i \cdot) \|_{M(1+\varepsilon)(X)}
\]

\[
\leq \frac{C}{(-\varepsilon)^n} \sum_j \| f_j \|_{A_j M_{1+\varepsilon} (R_{\omega_j})}
\]

for some \( C = C(n) \geq 0 \) independents of \( f_j \), \( \beta \), \( \beta + \varepsilon \) and \( \omega_j \). Letting \( \beta + \varepsilon \) tend to \( \omega_j \) yields

\[
\left\| \sum_j f_j^{(n)} \right\|_{A_j M_{1+\varepsilon} (R_{\beta})} = \left\| \sum_j f_j^{(n)} (\beta + i \cdot) \right\|_{M(1+\varepsilon)(X)} \leq C \sum_j \frac{1}{(\beta - \omega_j)^n} \| f_j \|_{A_j M_{1+\varepsilon} (R_{\omega_j})} \quad (14)
\]

Let \( \varepsilon \geq 0 \). Applying (14) with \( n - m \) in place of \( n \) shows that \( f_j^{(n-m)} \in A_j M_{1+\varepsilon} (R_{\beta}) \) with

\[
\left\| \sum_j f_j^{(n)} (A_j) \right\| \leq C' \sum_j \frac{1}{((\omega_j)_0 - \beta)^m} \| f_j^{(n-m)} \|_{A_j M_{1+\varepsilon} (R_{\beta})}
\]

\[
\leq C C' \sum_j \frac{1}{((\omega_j)_0 - \beta)^m (\beta - \omega_j)^{n-m}} \| f_j \|_{A_j M_{1+\varepsilon} (R_{\omega_j})}
\]

Finally, letting \( \beta + \varepsilon = \frac{1}{2} ((\omega_j) + (\omega_j)_0) \),
\[ \left\| \sum_j f_j^{(n)} (A_j) \right\| \leq C^* \sum_j \frac{1}{(\omega_j)^n} \left\| f_j \right\|_{A_j M(1+\epsilon)(R\omega_j)} \]

for some \( C^* \geq 0 \) independents of \( f_j \) and \( \omega_j \).

For the transference principle in Proposition 3.1 it is essential that the support of \( \mu_j \in M_{\omega_j}(\mathbb{R}_+) \) are contained in some interval \([1+\epsilon, \infty)\) with \( \epsilon > -1 \). One cannot expect to find such a transference principle for arbitrary \( \mu_j \), as this would allow one to prove that the sequence of semigroup generators has a bounded analytic multiplier calculus. However, if we let \( \mu_j \) be given by \( (t \mu_j)(dt) = t \mu_j (dt) \) then we can deduce the following transference principle. Use the conventions \( 1/\infty := 0, \infty^0 := 1 \).

**Proposition 4.2.** Let \(-A_j\) be the sequence of generators of a \( C_0 \) -semigroups \((T^j(t))_{t \in \mathbb{R}^+}\) of type \((M, 0)\) on a Banach space \( X \). Let \( 0 \leq \epsilon < \infty \), \( \omega_j > 0 \) and \( \mu_j \in M_{\omega_j}(\mathbb{R}_+) \). Then

\[ \left\| \sum_j T^j_{\epsilon \mu_j} \right\| \leq M^2 \sum_j \frac{1}{|\omega_j|} (1 + \epsilon)^{-\left(1 + \epsilon\right)} \left( \frac{1}{\epsilon} \right)^{-\left(1 + \epsilon\right)} \left\| L_{e^{-\omega_j \mu_j}} \right\|_{L(1+\epsilon(X))} \]

**Proof:** Factorizes \( T^j_{\epsilon \mu_j} \) as \( T^j_{\epsilon \mu_j} = P \circ L_{e^{-\omega_j \mu_j}} \circ 1 \), where \( 1 \) and \( P \) are as in the proof of Proposition 3.1 with \( \psi_j, \varphi_j := 1_{\mathbb{R}_+} e_{\omega_j} \). Since \((\psi_j * \varphi_j) e^{-\omega_j \mu_j} = t \mu_j \). Then

\[ \left\| \sum_j T^j_{\epsilon \mu_j} \right\| \leq M^2 \sum_j \left\| e_{\omega_j} \right\| \left\| L_{e^{-\omega_j \mu_j}} \right\|_{L(1+\epsilon(X))} \left\| e_{\omega_j} \right\| \left(1 + \epsilon\right) \]

\[ = M^2 \sum_j \frac{1}{|\omega_j|} (1 + \epsilon)^{-\left(1 + \epsilon\right)} \left( \frac{1}{\epsilon} \right)^{-\left(1 + \epsilon\right)} \left\| L_{e^{-\omega_j \mu_j}} \right\|_{L(1+\epsilon(X))} \]

by Holder’s inequality.

To prove our main result \( m \) - bounded functional calculus, a generalization of theorem 7.1 in [2] to arbitrary Banach spaces.

**Theorem 4.3.** Let \( A_j \) be densely defined sequence of operators of half-plane type 0 on a Banach space \( X \). Then the following assertions are equivalent:

(i) \(-A_j\) is the sequence of generators of bounded \( C_0 \) -semigroup on \( X \).

(ii) \( A_j \) has a strong \( m \)-bounded \( A_j M_{\omega_j}^{X+\epsilon} \) -calculus of type 0 for some/all \( \epsilon \geq 0 \) and some/all \( m \in \mathbb{N} \).

If \(-A_j\) be the sequence of generates bounded \( C_0 \) -semigroup then \( A_j \) has an \( m \)-bounded \( A_j M_{\omega_j}^{X+\epsilon} (R\omega_j) \)-calculus for all \( \omega_j < 0, \epsilon \geq 0 \) and \( m \in \mathbb{N} \).

**Proof.** (i) \( \Rightarrow \) (ii) By Lemma 4.1 it suffices to let \( m = 1 \). Proceed along the same lines as the proof of Theorem 3.3. Let \((T^j(t))_{t \in \mathbb{R}_+} \subseteq L(X)\) be the sequence semigroups generated by \(-A_j\) and fix \( \omega_j < 0, \epsilon \geq 0 \) and \( f_j \in A_j M_{1+\epsilon} (R\omega_j) \). Define the functions \((f_j)_k(\epsilon) := f_j (\epsilon + \epsilon) g_k^{(j)} (\epsilon + \epsilon) \) for \( k \in \mathbb{N} \) and \( \epsilon > 0 \), where \( g^{(j)}_k(z_j) := \frac{k}{z_j - \omega_j + k} \) for \( z_j \in R\omega_j \). Then
\((f_j)_{k, \varepsilon} \in A_j M_1(R_{\omega_j})\) by Lemma 2.4, and Lemma 2.3 yields \((\mu^j)_{k, \varepsilon} \in M_{\omega_j} (\mathbb{R}_+)\) with \((f_j)_{k, \varepsilon} = \mu^j_{k, \varepsilon}\). Then
\[
\sum_j (f_j)_{k, \varepsilon}(z_j) = \lim_{h_j \to 0} \sum_j (f_j)_{k, \varepsilon}(z_j + h_j) - (f_j)_{k, \varepsilon}(z_j) = \lim_{h_j \to 0} \int_0^\infty \sum_j \frac{e^{-(z_j + h_j)t} - e^{-z_j t}}{h_j} (\mu^j)_{k, \varepsilon}(dt) = -\sum_j \int_0^\infty t e^{-z_j t} \mu^j_{k, \varepsilon}(dt)
\]
for \(z_j \in R_{\omega_j}\), by the Dominated Convergence Theorem. Hence \((f_j)_{k, \varepsilon}(A_j) = -T^j_{\mu^j_{k, \varepsilon}}\), and Proposition 4.2 and Lemma 2.3 imply
\[
\left\| \sum_j (f_j)_{k, \varepsilon}(A_j) \right\| \leq (1 + \varepsilon)^{-\frac{1}{1+\varepsilon}} \left(1 + \varepsilon \right)^{-\frac{\varepsilon}{1+\varepsilon}} M^2 \sum_j \frac{\left\| (f_j)_{k, \varepsilon} \right\|_{A_j M_1^{\frac{\varepsilon}{1+\varepsilon}}}}{|\omega_j|}
\]
which is (4.1) for \(m = 1\).

For (ii) \(\Rightarrow\) (i) assume that \(A_j\) has a strong m-bounded \(A_j M_1^{m+\varepsilon}\)-calculus of type 0 for some \(\varepsilon \geq 0\) and some \(m \in \mathbb{N}\). Then
\[
e^{-t} \in A_j M_1 (R_{\omega_j}) \subseteq A_j M_1 +\varepsilon (R_{\omega_j})
\]
for all \(t > 0\) and \(\omega_j < 0\), with
\[
\left\| \sum_j e^{-t A_j} \right\|_{A_j M_1^{m+\varepsilon}(R_{\omega_j})} \leq \sum_j \| e^{-t} \|_{A_j M_1^{m+\varepsilon}(R_{\omega_j})} = \sum_j e^{-t \omega_j}
\]
Then, \((e^{-t})^m = (-t)^m e^{-t}\) implies
\[
t^m \left\| \sum_j e^{-t A_j} \right\| \leq C \sum_j \frac{1}{|\omega_j|^m} e^{-t \omega_j}
\]
Letting \(\omega_j := -\frac{1}{t} \) yields the required statement. \(\blacksquare\)
If \( X = H \) is a Hilbert space then Plancherel’s theorem yields the following result.

**Corollary 4.4:** Let \( A_j \) be densely defined sequence of operators of half-plane type 0 on a Hilbert space \( H \). Then the following assertions are equivalent:

(i) \( -A_j \) is the sequence of generators of a bounded \( C_0 \) -semigroup on \( H \).

(ii) \( A_j \) has strong \( m \)-bounded \( H^\infty \) -calculus of type 0 for some/all \( m \in \mathbb{N} \).

In particular, if \( -A_j \) be the sequence of generated bounded \( C_0 \)-semigroup then \( A_j \) has \( m \)-bounded \( H^\infty (R_{\omega_j}) \)-calculus for all \( \omega_j < 0 \) and \( m \in \mathbb{N} \).

### V. Semigroups on UMD Spaces

A Banach space \( X \) is a UMD space if the function \( t \mapsto \text{sgn}(t) \) is a bounded \( L^2(X) \)-Fourier multiplier. For \( \omega_j \in \mathbb{R} \), let

\[
H^\infty_1 (R_{\omega_j}) = \{ f_j \in H^\infty (R_{\omega_j}) \mid (Z_j - \omega_j) \hat{f}_j (Z_j) \in H^\infty (R_{\omega_j}) \}
\]

be the analytic Mikhlin algebras on \( R_{\omega_j} \), a Banach algebra endowed with the series of norms

\[
\sum_j \| f_j \|_{H^\infty_1} = \sum_j \| f_j \|_{H^\infty_1 (R_{\omega_j})} = \sup_{Z_j \in \mathbb{R} \omega_j} \sum_j |f_j (Z_j)| + \sum_j |(Z_j - \omega_j) \hat{f}_j (Z_j)| (f_j \in H^\infty_1 (R_{\omega_j}))
\]

Lemma 2.2 yield the continuous inclusion

\[
H^\infty_1 (R_{\omega_j}) \hookrightarrow A_j M^X_{1+\varepsilon} (R_{\omega_j})
\]

For each \( \varepsilon > 0 \), if \( X \) is a UMD space. Combining this with Theorems 3.3 and 4.3 and Corollaries 3.5 and 3.6 proves the following theorem (see, e.g., [8]).

**Theorem 5.1:** Let \( -A_j \) be the sequence of generates \( C_0 \)-semigroups \( (T^j (t))_{t \in \mathbb{R}^+} \) of type \((M, 0)\) on a UMD space \( X \). Then the following assertions hold.

(a) For each \( \varepsilon > 0 \), there exists a constant \( c_{\varepsilon+1} = c(1+\varepsilon, X) \geq 0 \) such that the following holds.

Let \( 1 + \varepsilon, \omega_j > 0 \). Then \( f_j (A_j) \in \mathcal{L}(X) \) with

\[
\left\| \sum_j f_j (A_j) \right\| \leq \begin{cases} c_{\varepsilon+1} M^2 \sum_j \log (\omega_j (1+\varepsilon)) \| f_j \|_{H^\infty_1} & \text{if } \omega_j (1+\varepsilon) \leq \min \left\{ \frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon} \right\} \\ 2c_{\varepsilon+1} M^2 \sum_j e^{-\omega_j (1+\varepsilon)} \| f_j \|_{H^\infty_1} & \text{if } \omega_j (1+\varepsilon) > \min \left\{ \frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon} \right\} \end{cases}
\]

for all \( f_j \in H^\infty_1 (R_{-\omega_j}) \cap e_{-(1+\varepsilon)H^\infty} (R_{-\omega_j}) \), and \( f_j (A_j) T^j (1+\varepsilon) \in \mathcal{L}(X) \) with
\[
\left\| \sum_j f_j(A_j) T^j (1 + \varepsilon) \right\| \\
\leq \begin{cases} 
2 c_{\varepsilon+1} M^2 \sum_j |\log(\omega_j (1 + \varepsilon))| e^{\omega_j (1+\varepsilon)} \left\| f_j \right\|_{H_1^\omega} & \text{if } \omega_j (1 + \varepsilon) \leq \min \left\{ \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right\} \\
2 c_{\varepsilon+1} M^2 \sum_j \left\| f_j \right\|_{H_1^\omega} & \text{if } \omega_j (1 + \varepsilon) > \min \left\{ \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right\}
\end{cases}
\]

for all \( f_j \in H_1^\omega \left( R_{-\omega_j} \right) \).

(b) If
\[
\bigcup_{\varepsilon > -1} \sum_j \text{ran} \left( T^j (1 + \varepsilon) \right) = X
\]

then \( A_j \) has bounded \( H_1^\omega \left( R_{\omega_j} \right) \)-calculus for all \( \omega_j < 0 \).

(c) \( A_j \) has a strong \( m \)-bounded \( H_1^\omega \)-calculus of type 0 for all \( m \in \mathbb{N} \).

**Remark 5.2:** Theorem 3.7 yields the domain inclusions \( \text{dom} \left( A_j^{\beta+\varepsilon} \right) \subseteq \text{dom} \left( f_j (A_j) \right) \) for all \( \beta + \varepsilon \in \mathbb{C}+, \omega_j < 0 \) and \( f_j \in H_1^\omega \left( R_{\omega_j} \right) \), on a UMD space X. However, this inclusion in fact, holds true on a general Banach space X. Indeed, for \( \lambda_j \in \mathbb{C} \) with \( \text{Re}(\lambda_j) < 0 \), Bernstein’s Lemma [12, Proposition 8.2.3] implies \( \frac{f_j(z_j)}{(\lambda_j - z_j)^{\beta+\varepsilon}} \in A_j M_1 \left( \mathbb{C}_+ \right) \), hence \( f_j(A_j)(\lambda_j - A_j)^{-(\beta+\varepsilon)} \in \mathcal{L}(X) \) and \( \text{dom}(A_j^{\beta+\varepsilon}) \subseteq \text{dom}(f_j (A_j)) \). Series estimates
\[
\left\| \sum_j f_j \left( A_j \right) (\lambda_j - A_j)^{-(\beta+\varepsilon)} \right\| \leq (1 + \varepsilon) \sum_j \left\| f_j \right\|_{H_1^\omega \left( R_{\omega_j} \right)}
\]

then follows from an application of the Closed Graph Theorem and the Convergence Lemma.

**Remark 5.3:** To apply Theorem 5.1 one can use the continuous inclusion
\[
H_1^\omega \left( R_{\omega_j} \cup (S_{\varphi_j} + a) \right) \subseteq H_1^\omega \left( R_{\omega_j} \right) \tag{15}
\]

For \( \omega_j > \omega_j \), \( a \in \mathbb{R} \) and \( \varphi_j \in [\frac{\pi}{2}, \pi] \). Here \( R_{\omega_j} \cup \left( S_{\varphi_j} + a \right) \) are the union of \( R_{\omega_j} \) and the translated sectors \( S_{\varphi_j} + a \), where
\[
S_{\varphi_j} = \left\{ z_j \in \mathbb{C} \left| \left| \arg(z_j) \right| < \varphi_j \right. \right\}
\]

Indeed, to derive (15) it suffices to let \( a = 0 \), yields the desired result.

**VI. \( \gamma_j \) – Bounded Semigroups**

The geometry of the underlying Banach space X played an essential role in the results of properties of the analytic multiplier algebras \( A_j M_1^{X+\varepsilon} \). To wit, in to identify
nontrivial functions in $A_1^+ M^X_{1+\varepsilon}$ one needs a geometric assumption on $X$, for instance that it is a Hilbert or a UMD space. Take a different approach and make additional assumptions on the semigroup instead of the underlying space. Show that if the semigroups in question are $\gamma_j$-bounded then one can recover the Hilbert space results on an arbitrary Banach space $X$.

Assume to be familiar with the basics of the theory of $\gamma_j$-radonifying sequence of operators and $\gamma_j$-boundedness as collected in the survey article by van Neerven [13].

Let $H$ be a Hilbert space and $X$ a Banach space. The linear sequence of operators $T_j : H \to X$ is $\gamma_j$-summing if

$$\sum_j \|T_j\|_{\gamma_j} = \sup_F \sum_j \left( \mathbb{E} \left\| \sum_{h \in F} (\gamma_j)_h T_j h_j \right\|^2_X \right)^{1/2} < \infty,$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $((\gamma_j)_h)_{h \in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$(\gamma_j)_\infty (H; X) := \{ T_j : H \to X \mid T_j \text{ are } \gamma_j \text{-summing} \}$$

with the norms $\| \cdot \|_{\gamma_j}$ and let the spaces $\gamma_j (H; X)$ of all $\gamma_j$-radonifying sequence of operators be the closure in $(\gamma_j)_\infty (H; X)$ of the finite-rank sequence of operators $H \otimes X$.

For a measure space $(\Omega, \mu^j)$ let $\gamma_j (\Omega; X)$ (resp. $(\gamma_j)_\infty (\Omega; X)$) be the space of all weakly $L^2$-functions $f_j : \Omega \to X$ for which the integration sequence of operators of $(J)_{f_j} : L^2(\Omega) \to X$,

$$\sum_j (J)_{f_j} (g^j) = \int \Omega \sum_j g^j \cdot f_j \ d\mu^j \quad (g^j \in L^2(\Omega))$$

is $\gamma_j$-radonifying ($\gamma_j$-summing), and endow it with the norms $\| f_j \|_{\gamma_j} = \| (J)_{f_j} \|_{\gamma_j}$. Collections $T_j \subseteq \mathcal{L}(X)$ are $\gamma_j$-bounded if there exists a constant $C \geq 0$ such that

$$\left( \mathbb{E} \left\| \sum_{T_j \in \hat{T}} \sum_{x_{T_j}} (\gamma_j)_{T_j} T_j x_{T_j} \right\|^2 \right)^{1/2} \leq C \sum_j \left( \mathbb{E} \left\| \sum_{T_j \in \hat{T}} (\gamma_j)_{T_j} x_{T_j} \right\|^2 \right)^{1/2}$$

for all finite subsets $\hat{T} \subseteq T_j$, sequences $((x)_{T_j})_{T_j \in \hat{T}} \subseteq X$ and independent complex-valued standard Gaussian random variables $((\gamma_j)_{T_j})_{T_j \in \hat{T}}$. The smallest such $C$ is the $\gamma_j$-bound of $T_j$ and is denoted by $\| T_j \|_{\gamma_j}$. Every $\gamma_j$-bounded collections are uniformly bounded with supremum boundless than or equal to the $\gamma_j$-bound, and the converse holds if $X$ is a Hilbert space.

An important result involving $\gamma_j$-boundedness is the multiplier theorem. State a version that is tailored to the purposes. Given a Banach space $Y$, a function $g^j : \mathbb{R} \to Y$
is piecewise $W^{1,\infty}$ if $g^i \in W^{1,\infty}(\mathbb{R}^n \setminus \{a_1, \ldots, a_n\}; Y)$ for some finite set $\{a_1, \ldots, a_n\} \subseteq \mathbb{R}.$

**Theorem 6.1 (Multiplier Theorem):** Let $X$ and $Y$ be Banach spaces and $T^j : \mathbb{R} \to \mathcal{L}(X, Y)$ a strongly measurable mappings such that

\[ T^j = -T^j(s) \mid s \in \mathbb{R} \}

are $\gamma_j$-bounded. Suppose furthermore that there exists a dense subset $D \subseteq X$ such that $s \mapsto T^j(s)x$ is piecewise $W^{1,\infty}$ for all $x \in D$. Then the multiplication sequence of operators

\[ \mathcal{M}_{T^j} : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R}; Y), \mathcal{M}_{T^j} (f_j \otimes x) = f_j(\cdot) T^j(\cdot)x \]

Extends uniquely to bounded sequence of operators

\[ \mathcal{M}_{T^j} : \gamma_j(L^2(\mathbb{R}); X) \to \gamma_j(L^2(\mathbb{R}); Y) \]

with

\[ \left\| \sum_j \mathcal{M}_{T^j} \right\| \leq \sum_j \left\| T^j \right\|^{\gamma_j} \]

**Proof:** That $\mathcal{M}_{T^j}$ extends uniquely to bounded sequence of operators into $(\gamma_j)_\infty(L^2(\mathbb{R}); Y)$ with $\left\| \sum_j \mathcal{M}_{T^j} \right\| \leq \sum_j \left\| T^j \right\|^{\gamma_j}$. To see that in facts $\text{ran}(\mathcal{M}_{T^j}) \subseteq \gamma_j(\mathbb{R}; Y)$, employ a density argument. For $x \in D$ let $a_1, \ldots, a_n \in \mathbb{R}$ be such that $s \mapsto T^j(s)x \in W^{1,\infty}(\mathbb{R}^n \setminus \{a_1, \ldots, a_n\}; Y)$, and set $a_0 := -\infty, a_{n+1} := \infty$. Let $f_j \in C_c(\mathbb{R})$ be given and note that

\[ \sum_j \int_{a_j}^{a_{j+1}} \left\| f_j \right\|_{L^2(s,a_{j+1})} \left\| T^j(s)^* x \right\| ds < \infty \]

for all $j \in \{1, \ldots, n\}$. Furthermore,

\[ \int_{-\infty}^{a_1} \sum_j \left\| f_j \right\|_{L^2(-\infty,s)} \left\| T^j(s)^* x \right\| ds < \infty \]

yields $(1_{(a_j,a_{j+1})} f_j)(\cdot)T^j(\cdot)x \in \gamma_j(\mathbb{R}; Y)$ for all $0 \leq j \leq n$, hence $f_j(\cdot)T^j(\cdot)x \in \gamma_j(\mathbb{R}; Y)$. Since $C_c(\mathbb{R}) \otimes D$ is dense in $L^2(\mathbb{R}) \otimes X$, which in turn is dense in $\gamma_j(L^2(\mathbb{R}); X)$, the result follows.

To prove a generalization of part (a) of Corollary 3.10, recall that

\[ e_{-(1+\varepsilon)} H^\infty(R_{\omega_j}) = \{ f_j \in H^\infty(R_{\omega_j}) \mid f_j(z_j) = O(e^{-(1+\varepsilon)R(z_j)}) \text{ as } |z_j| \to \infty \} \]

for $\varepsilon > -1, \omega_j \in \mathbb{R}$. 

\[ F \]

Notes
Theorem 6.2: There exists a universal constant $c \geq 0$ such that the following holds. Let $A_j$ be sequence of generator $\gamma_j$- bounded $C_0$-semigroups $(T^j(t))_{t \in \mathbb{R}_+}$ with $M := \|\|T^j\|\|_{\gamma_j}$ on Banach space $X$, and let $1 + \varepsilon$, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ with

$$
\left\| \sum_j f_j(A_j) \right\| \leq \begin{cases} 
M^2 \sum_j \|\log(\omega_j(1 + \varepsilon))\|f_j\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) \leq \frac{1}{2} \\
2M^2 \sum_j e^{-\omega_j(1 + \varepsilon)} \|f_j\|_{\infty} & \text{if } \omega_j(1 + \varepsilon) > \frac{1}{2}
\end{cases}
$$

for all $f_j \in e_{-(1 + \varepsilon)}H^\omega(R - \omega_j)$. In particular, $A_j$ has a bounded $e_{-(1 + \varepsilon)}H^\omega(R - \omega_j)$-calculus.

Proof: Only need to show that the estimate (9) in Proposition 3.1 can be refined to

$$
\left\| \sum_j T^j_{\mu} \right\| \leq M^2 \eta \sum_j (\omega_j, 1 + \varepsilon, 2) \left\| L_{e_{\omega_j}} \mu \right\|_{\mathcal{L}(\gamma_j(\mathbb{R}, X))}
$$

for $\mu^j \in \mathcal{M}_{\omega_j}(\mathbb{R}_+)$ with $\text{supp}\mu^j \subseteq [1 + \varepsilon, \infty)$. Then one uses that

$$
\left\| \sum_j L_{e_{\omega_j}} \mu^j \right\|_{\mathcal{L}(\gamma_j(\mathbb{R}, X), E)} \leq \sum_j \left\| \gamma_j \mu^j \right\|_{H^\omega(\mathbb{C}_+)} = \sum_j \left\| \mu^j \right\|_{H^\omega(R - \omega_j)}
$$

by the ideal properties of $\gamma_j(L^2(\mathbb{R}; X))$ [13, Theorem 6.2], and proceeds as in the proof of Theorem 3.3 to deduce the desired result.

To obtain (16) we factorizes $T^j_{\mu}$ as $T^j_{\mu} = \Pi \circ L_{e_{-\omega_j}} \mu \circ 1$, where $\Pi : \gamma_j(\mathbb{R}; X) \to X$ are given by

$$
\Pi x(s) := \psi_j(-s)T^j(-s)x \quad (x \in X, s \in \mathbb{R}),
$$

$$
\sum_j P g^j = \int_0^\infty \sum_j \varphi_j(t)T^j(t)g^j(t)dt \quad (g^j \in \gamma_j(\mathbb{R}, X))
$$

for $\psi_j$, $\varphi_j \in L^2(\mathbb{R}_+)$ such that $\psi_j * \varphi_j \equiv e_{-\omega_j}$ on $[1 + \varepsilon, \infty)$. Show that the maps $\Pi$ and $P$ are well-defined and bounded. To this end, first note that $s \mapsto T^j(-s)x$ is piecewise $W^{1, \infty}$ for all $x$ in the dense subset $\text{dom}(A_j) \subseteq X$ and that

$$
\psi_j(-s) \otimes x \in L^2(-\infty, 0) \otimes X \subseteq \gamma_j(L^2(\mathbb{R}; X)).
$$

Therefore Theorem 6.1 yields $\Pi x \in \gamma_j(\mathbb{R}, X)$ with

$$
\left\| \sum_j l x \right\|_{\gamma_j} = \left\| \sum_j J_{l,x} \right\|_{\gamma_j} \leq M \sum_j \left\| \psi_j(-\cdot) \otimes x \right\|_{\gamma_j} = M \sum_j \left\| \psi_j \right\|_{L^2(\mathbb{R}_+)} \|x\|_X
$$

As for $P$, write
\[ \sum_{j} P g^{j} = \int_{0}^{\infty} \sum_{j} \varphi_j(t) T^{j} (t) g^{j}(t) \, dt = \sum_{j} J_{T^j g^{j}} (\varphi_j) \]

And use Theorem 6.1 once again to see that \( T^{j} g^{j} \in \gamma_{j} (\mathbb{R}; X) \). Hence

\[ \left\| \sum_{j} P g^{j} \right\|_{X} \leq \sum_{j} \left\| J_{T^{j} g^{j}} \right\|_{\gamma_{j}} \left\| \varphi_j \right\|_{L^2 (\mathbb{R}^+)} \leq M \sum_{j} \left\| \varphi_j \right\|_{L^2 (\mathbb{R}^+)} \left\| g^{j} \right\|_{\gamma_{j}} \]

Finally, estimating the norms of \( T^{j}_{\mu^{j}} \) through this factorization and taking the infimum over all \( \psi_j \) and \( \varphi_j \) yields (16).

Note: In putting \( \mu^{j} \) by \( t \mu^{j} \) in the proof of Theorem 6.2 we have,

\[ \sum_{j} (\omega_j, 1 + \varepsilon, 2) \left\| L_{\omega_j \mu^{j}} \right\|_{L_{(\gamma_j (\mathbb{R}; X))}} \leq \frac{1}{n} \sum_{j} \frac{1}{|\omega_j|} (1 + \varepsilon)^{-\left(1 + \frac{1}{1 + \varepsilon}\right)} \left\| L_{\omega_j \mu^{j}} \right\|_{L_{(1 + \varepsilon) (X)}} \]

Corollary 6.3: Corollary 3.10 generalizes to \( \gamma_{j} \) -bounded semigroups on arbitrary Banach spaces upon replacing the uniform bound \( M \) of \( T^{j} \) by \([T^{j}]^{\mu^{j}}\).

Theorem 4.3 can be extended in an almost identical manner to \( \gamma_{j} \) -versions (see, e.g., [8]).

**Theorem 4.4:** Let \(-A_{j}\) be the sequence generates \( \gamma_{j} \) -bounded \( C_{0} \)-semigroup on a Banach \( X \). Then \( A_{j} \) has a strongly-bounded \( H^{\infty} \)-calculus of type 0 for all \( m \in \mathbb{N} \).

**Appendix A. Growth estimates**

In this appendix we examine the function \( \eta: (0, \infty) \times (0, \infty) \times [1, \infty] \to \mathbb{R}_{+} \) from (3.1):

\[ \eta (\beta + \varepsilon, t, 1 + \varepsilon) = \inf \left\{ \left\| \psi_j \right\|_{1 + \varepsilon} \left\| \varphi_j \right\|_{1 + \varepsilon} \left| \psi_j \ast \varphi_j = e_{-(\beta + \varepsilon)} \right. \text{ on } [t, \infty) \right\} \]

Use the notation \( f_{j} \leq g^{j} \) for real-valued functions \( f_{j} , g^{j} : Z \to \mathbb{R} \) on some set \( Z \) to indicate that there exists a constant \( c \geq 0 \) such that \( f_{j} (z_{j}) \leq cg^{j} (z_{j}) \) for all \( z_{j} \in Z \).

**Lemma A.1:** For each \( \varepsilon > 0 \) there exist constants \( c_{1 + \varepsilon}, d_{1 + \varepsilon} \geq 0 \) such that

\[ d_{1 + \varepsilon} |\log (\varepsilon) (\beta + \varepsilon) t)| \leq \eta (\beta + \varepsilon, t, 1 + \varepsilon) \leq c_{1 + \varepsilon} |\log (\varepsilon) (\beta + \varepsilon) t)| \]  

(A.1)

If \( (\beta + \varepsilon) t \leq \min \left\{ \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right\} \) \( \text{if } (\beta + \varepsilon) t > \min \left\{ \frac{1}{1 + \varepsilon}, \frac{\varepsilon}{1 + \varepsilon} \right\} \text{ then} \)

\[ e^{-(\beta + \varepsilon) t} \leq \eta (\beta + \varepsilon, t, 1 + \varepsilon) \leq 2e^{-(\beta + \varepsilon) t} \]  

(A.2)

**Proof:** First note that \( \eta (\beta + \varepsilon, t, 1 + \varepsilon) = \eta ((\beta + \varepsilon) t, 1, 1 + \varepsilon) = \eta (1, (\beta + \varepsilon) t, 1 + \varepsilon) \), for all \( \beta + \varepsilon, t \) and \( 1 + \varepsilon \). Indeed, for \( \psi_j \in L^{1 + \varepsilon} (\mathbb{R}_{+}), \varphi_j \in L^{-1 + \varepsilon} (\mathbb{R}_{+}) \) with \( \psi_j \ast \varphi_j \equiv e_{-(\beta + \varepsilon)} \) on \([1, \infty)\) defines \( (\psi_j)_1 (s) := t^{-(\frac{1}{1 + \varepsilon})} \psi_j \left( \frac{s}{t} \right) \) and \( (\varphi_j)_1 (s) := t^{-(\frac{1}{1 + \varepsilon})} \varphi_j (s/t) \) for some \( s \geq 0 \). Then
\[
\sum_j (\psi_j)_t * (\varphi_j)_t(r) = \int_0^\infty \sum_j \psi_j \left( \frac{r-s}{t} \right) \varphi_j \left( \frac{s}{t} \right) \frac{ds}{t} = \sum_j \psi_j * \varphi_j \left( \frac{r}{t} \right)
\]
for all \( r \geq 0 \), so \((\psi_j)_t * (\varphi_j)_t \equiv e_{-(\beta+\varepsilon)} \) on \([t, \infty)\). Moreover,

\[
\sum_j \|(\psi_j)_t\|_{1+\varepsilon}^{1+\varepsilon} = \int_0^\infty \sum_j \left| \psi_j \left( \frac{s}{t} \right) \right|^{1+\varepsilon} \frac{ds}{t} = \int_0^\infty \sum_j \left| \psi_j(s) \right|^{1+\varepsilon} ds = \sum_j \|\psi_j\|_{1+\varepsilon}^{1+\varepsilon}
\]

and similarly \( \sum_j \|(\varphi_j)_t\|_{1+\varepsilon}^{1+\varepsilon} = \sum_j \|\varphi_j\|_{1+\varepsilon}^{1+\varepsilon} \). Hence \( \eta(\beta + \varepsilon, t, 1 + \varepsilon) \leq \eta((\beta + \varepsilon)t, 1, 1 + \varepsilon) \).

Considering \((\psi_j)_{(1/t)}\) and \((\varphi_j)_{(1/t)}\) yields \( \eta(\beta + \varepsilon, t, 1 + \varepsilon) = \eta((\beta + \varepsilon)t, 1, 1 + \varepsilon) \). The other equality follows immediately. Hence, to prove all of the inequalities in (A.1) or (A.2), assume either that \( \beta + \varepsilon = 1 \) or that \( t = 1 \) (but not both).

For the left-hand inequalities, assume that \( \beta + \varepsilon = 1 \) and first consider the left-hand inequality of (A.1). Let \( t < 1 \) and \( \psi_j \in L^{1+\varepsilon}(\mathbb{R}_+) \), \( \varphi_j \in L^{1+\varepsilon}(\mathbb{R}_+) \) such that \( \psi_j * \varphi_j \equiv e_{-1} \) on \([t, \infty)\). Then

\[
|\log(t)| = -\log(t) = \int_t^1 \frac{ds}{s} \leq e \int_t^1 e^{-\frac{ds}{s}} ds = e \int_t^1 \sum_j |\psi_j * \varphi_j(s)| \frac{ds}{s}
\]

\[
\leq e \int_0^\infty \int_r^\infty \sum_j |\psi_j(s-r)| |\varphi_j(r)| dr \frac{ds}{s}
\]

\[
\leq e \int_0^\infty \int_0^\infty \sum_j |\varphi_j(s-r)| s |\psi_j(r)| dr ds
\]

\[
eq e \int_0^\infty \int_0^\infty \sum_j |\psi_j(s)| |\varphi_j(r)| \frac{ds}{s} dr = \frac{e\pi}{\sin(\pi/1+\varepsilon)} \sum_j \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{1+\varepsilon}
\]

where used Hilbert’s absolute inequality [14, Theorem 5.10.1]. It follows that

\[
\eta(1, t, 1 + \varepsilon) \geq \frac{\sin(\pi/1+\varepsilon)}{e\pi} |\log(t)|
\]

For the left-hand inequality of (A.2), assume that \( \beta + \varepsilon = 1 \) and let \( t > 0 \) be arbitrary. Then

\[
e^{-t} = \sum_j (\psi_j * \varphi_j)(t) \leq \int_0^t \sum_j |\psi_j(t-s)| |\varphi_j(s)| ds \leq \sum_j \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{1+\varepsilon}
\]

By Hölder’s inequality, hence \( e^{-t} \leq \eta(1, t, 1 + \varepsilon) \).

For the right-hand inequalities in (A.1) and (A.2), assume that \( t = 1 \) and first consider the right-hand inequality in (A.1) for \( \beta + \varepsilon \leq \min \left\{ \frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon} \right\} \). In the proof of Lemma A.1, it is shown that

\[
((\psi_j)_0 * (\varphi_j)_0)(s) = \begin{cases} s, & s \in [0,1) \\ 1, & s \geq 1 \end{cases}
\]
for

\[ \sum_j (\psi_j)_0 = \sum_{j=0}^{\infty} \beta_j 1_{(j,j+1)} \quad \text{and} \quad (\varphi_j)_0 = \sum_{j=0}^{\infty} \hat{\beta}_j 1_{(j,j+1)} \]

where \((\beta_j)_j\) and \((\hat{\beta}_j)_j\) are sequences of positive scalars such that \(\beta_j = O((1 + j)^{-\frac{1}{1+\varepsilon}})\) and \(\hat{\beta}_j = O((1 + j)^{-\frac{1}{1+\varepsilon}})\) as \(j \to \infty\). Let \(\psi_j := e^{-(\beta + \varepsilon)}(\psi_j)_0\) and \(\varphi_j := e^{-(\beta + \varepsilon)}(\varphi_j)_0\). Then \(\psi_j \ast \varphi_j \equiv e^{-(\beta + \varepsilon)}\) on \([1, \infty)\) and

\[
\left\| \sum_j \psi_j \right\|_{1+\varepsilon}^{1+\varepsilon} = \left\| \sum_j e^{-(\beta + \varepsilon)}(\psi_j)_0 \right\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{j=0}^{\infty} \beta_j^{1+\varepsilon} \int_j^{j+1} e^{-(\beta + \varepsilon)(1+\varepsilon)s} ds \leq \sum_{j=0}^{\infty} \frac{e^{-(\beta^2 + \beta (1+\varepsilon) + \varepsilon + j)}}{1 + j} \\
\leq 1 + \int_0^{\infty} \frac{e^{-(\beta^2 + \beta (1+\varepsilon) + \varepsilon)s}}{1 + s} ds = 1 + e^{(\beta^2 + \beta (1+\varepsilon) + \varepsilon)} \int_0^{\infty} \frac{e^{-s}}{s} ds 
\]

The constant in the first inequality depends only on \(1 + \varepsilon\). Since \((\beta^2 + \beta(1+\varepsilon) + \varepsilon) \leq 1\),

\[
\left\| \sum_j \psi_j \right\|_{1+\varepsilon}^{1+\varepsilon} \leq 1 + e^{\beta^2 + \beta (1+\varepsilon) + \varepsilon} \left( \int_1^{\infty} \frac{e^{-s}}{s} ds + \int_1^{\infty} \frac{e^{-s}}{s} ds \right) \\
\leq 1 + \int_1^{\infty} \frac{1}{s} ds + e^{\beta^2 + \beta (1+\varepsilon) + \varepsilon} \int_1^{\infty} e^{-s} ds \\
= 1 - \log(\beta^2 + \beta (1+\varepsilon) + \varepsilon) + e^{(\beta^2 + \beta (1+\varepsilon) + \varepsilon)} - 1 \leq \log \left( \frac{1}{(\beta + \varepsilon) + \varepsilon} \right) + 2
\]

Moreover, \(\frac{1}{(\beta + \varepsilon) + \varepsilon} \geq 1 + \varepsilon > 1\) hence \(\log \left( \frac{1}{\beta + \varepsilon} \right) \geq \log(1 + \varepsilon) > 0\) and

\[
\log \left( \frac{1}{\beta + \varepsilon} \right) + 2 \leq \left( 1 + \frac{2}{\log(1 + \varepsilon)} \right) \log \left( \frac{1}{\beta + \varepsilon} \right) 
\]

Therefore

\[
\left\| \sum_j \psi_j \right\|_{1+\varepsilon}^{1+\varepsilon} \leq \log \left( \frac{1}{\beta + \varepsilon} \right)^{1+\varepsilon} = |\log (\beta + \varepsilon)|^{1+\varepsilon} 
\]

For a constant depending only on \(1 + \varepsilon\). Similarly deduce

\[
\left\| \sum_j \varphi_j \right\|_{1+\varepsilon}^{1+\varepsilon} \leq |\log (\beta + \varepsilon)|^{\frac{1+\varepsilon}{\varepsilon}} 
\]

for a constant depending only on \(\frac{1+\varepsilon}{\varepsilon}\) (and thus on \(1+\varepsilon\)). This yields (A.1).
For the right-hand side of (A.2) we assume that \( t = 1 \) and, without loss of generality (Since \( (\beta + \epsilon, t, 1 + \epsilon) = \eta(\beta + \epsilon, t, 1 + \epsilon) \)), that \( \beta + \epsilon > \frac{1}{1 + \epsilon} \) let \( \varphi_j = 1_{[0,1]} e^{(\beta + \epsilon)}(\epsilon) \) and \( \psi_j = \frac{(\beta^2 + \beta(1 + \epsilon) + \epsilon)}{e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)} - 1} 1_{\mathbb{R}^+} e^{-(\beta + \epsilon)}. \) Then

\[
\sum_j \psi_j * \varphi_j(r) = \frac{(\beta^2 + \beta(1 + \epsilon) + \epsilon)}{e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)} - 1} \int_0^1 e^{(\beta + \epsilon)(s)e^{-(\beta + \epsilon)(r-s)}} ds = e^{-(\beta + \epsilon)r}
\]

For \( r \geq 1. \) Hence

\[
\eta(\beta + \epsilon, 1, 1 + \epsilon) \leq \sum_j \|\psi_j\|_{1+\epsilon} \|\varphi_j\|_{1+\epsilon}
\]

\[
= \frac{(\beta^2 + \beta(1 + \epsilon) + \epsilon)}{e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)} - 1} \left( \int_0^1 e^{-(\beta^2 + \beta(1 + \epsilon) + \epsilon)s} ds \right) \left( \int_0^1 e^{(\beta + \epsilon)(s)(1 + \epsilon)} ds \right) \left( \frac{\epsilon}{1 + \epsilon} \right)
\]

\[
= \frac{(\beta^2 + \beta(1 + \epsilon) + \epsilon)}{e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)} - 1} \left( \int_0^1 e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)s} ds \right) \left( \frac{\epsilon}{1 + \epsilon} \right) = (e^{(\beta^2 + \beta(1 + \epsilon) + \epsilon)} - 1)^{-\left( \frac{1}{1 + \epsilon} \right)}
\]

\[
\leq 2 (\frac{1}{1 + \epsilon}) e^{-(\beta + \epsilon)} \leq 2 e^{-(\beta + \epsilon)}
\]

Where have used the assumption \( (\beta^2 + \beta(1 + \epsilon) + \epsilon) > 1 \) in the penultimate inequality. ■

Note: Deduce that:

(1) \( \|\sum_j \psi_j\|_{1+\epsilon} \leq M_{\beta, \epsilon} \sum_j \|\varphi_j\|_{1+\epsilon} \)

(2) \( e^{-t} \leq \|\varphi_j\|_{1+\epsilon} \|\varphi_j\|_{1+\epsilon} \leq 2 e^{-(\beta + \epsilon)} \)

When \( \beta + \epsilon = 1, t > 0 \)

Proof. (1) Since

\[
\left\| \sum_j \psi_j \right\|_{1+\epsilon} \leq |\log(\beta + \epsilon)| \frac{1}{1 + \epsilon} \tag{a}
\]

And

\[
\left\| \sum_j \varphi_j \right\|_{1+\epsilon} \leq |\log(\beta + \epsilon)| \frac{\epsilon}{1 + \epsilon} \tag{b}
\]

Divide we have

\[
\left\| \sum_j \psi_j \right\|_{1+\epsilon} \leq M_{\beta, \epsilon} \sum_j \|\varphi_j\|_{1+\epsilon}
\]
Where we have $M_{\beta, \epsilon} = |\log(\beta + \epsilon)|^{\frac{1}{1+\epsilon}}$

(2) From (A.2), we can get

$$e^{-t} \leq \sum_j \|\psi_j\|_{1+\epsilon} \|\varphi_j\|_{1+\epsilon} \leq \frac{1}{2} \left( \sum_j \|\psi_j\|^2_{1+\epsilon} \right) \left( \sum_j \|\varphi_j\|^2_{1+\epsilon} \right)^{\frac{1}{2}} = \|\psi_j\|_{1+\epsilon} \|\varphi_j\|_{1+\epsilon} \leq 2e^{-(\beta+\epsilon)}. \tag{c}$$

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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