Robust Sparse Covariance Estimation by Thresholding Tyler’s M-Estimator

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Abstract: Estimating a high-dimensional sparse covariance matrix from a limited number of samples is a fundamental problem in contemporary data analysis. Most proposals to date, however, are not robust to outliers or heavy tails. Towards bridging this gap, in this work we consider estimating a sparse shape matrix from $n$ samples following a possibly heavy tailed elliptical distribution. We propose estimators based on thresholding either Tyler’s M-estimator or its regularized variant. We derive bounds on the difference in spectral norm between our estimators and the shape matrix in the joint limit as the dimension $p$ and sample size $n$ tend to infinity with $p/n \to \gamma > 0$. These bounds are minimax rate-optimal. Results on simulated data support our theoretical analysis.

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1. Introduction

The covariance matrix $\Sigma$ of a $p$-dimensional random variable $X$ is a central object in statistical data analysis. Given $n$ observations $\{x_i\}_{i=1}^n$, accurately estimating this matrix is of great importance for many tasks including PCA, clustering and discriminant analysis (Anderson, 2003; Mardia, Kent and Bibby, 1979). The sample covariance matrix, which is the standard estimator for $\Sigma$, is quite accurate when the random variable $X$ is sub-Gaussian and $p \ll n$.

In several contemporary applications, however, the number of samples $n$ and the dimension $p$ are comparable, and the data may be heavy tailed. To accurately estimate the covariance matrix when $n$ and $p$ are comparable, additional
assumptions, such as its approximate sparsity are typically made. Over the past decade several sparse covariance matrix estimators were proposed and analyzed (Bickel and Levina, 2008; Cai and Liu, 2011; El Karoui, 2008; Lam and Fan, 2009; Rothman, Levina and Zhu, 2009). In addition, minimax lower bounds for estimating sparse covariance matrices in high-dimensional settings were established (Cai and Zhou, 2012a,b; Cai, Ren and Zhou, 2016).

With respect to heavy tailed data, a popular model which we consider in this work is the generalized elliptical distribution (Cambanis, Huang and Simons, 1981; Fang, Kotz and Ng, 1990; Frahm, 2004; Kelker, 1970). An elliptical distribution is characterized by a \( p \times p \) shape or scatter matrix \( S_p \), which equals a multiple of its population covariance matrix, when the latter exists. However, since an elliptical distribution may be heavy tailed, the classical sample covariance may exhibit large variance and be a poor estimator of the population covariance (Falk, 2002). Moreover, the elliptical distribution might be so heavy tailed as to not even have finite second moments, in which case its population covariance does not exist. Yet due to the structure of the elliptical distribution, even with heavy tails it is nonetheless possible to accurately estimate its shape matrix. This is useful in various applications, since the shape matrix preserves the directional properties of the distribution, such as its principal components.

Following Huber’s pioneering work (Huber and Ronchetti, 2009), over the past decades several robust estimators of the covariance and shape matrix were proposed, and theoretically studied, see Maronna (1976); Maronna and Yohai (2017); Kent and Tyler (1991); Dümbgen, Pauly and Schweizer (2015); Dümbgen, Nordhausen and Schuhmacher (2016) and references therein. For elliptical distributions, Tyler (1987) proposed a robust M-estimator for the scatter matrix \( S_p \) and an iterative scheme to compute it. Tyler’s M-estimator has found widespread use in various applications involving heavy tailed data. However, as it is defined only for \( p < n \), in recent years several regularized variants, applicable also for \( p > n \) were proposed and analyzed (Abramovich and Spencer, 2007; Wiesel, 2012; Sun, Babu and Palomar, 2014; Chen, Wiesel and Hero, 2011; Ollila and Tyler, 2014). The spectral properties of Tyler’s M-estimator and its regularized variants in high dimensions as \( n, p \to \infty \) and \( p/n \to \gamma \) were studied by Zhang, Cheng and Singer (2016); Couillet and McKay (2014); Couillet, Kammoun and Pascal (2016), among others. For a recent survey on Tyler’s M-estimator and its variants, see Wiesel and Zhang (2014).

In this paper we study the combination of heavy tailed data with a “large \( p \) – large \( n \)” setting. As formulated in Section 2, we consider robust estimation of the shape matrix of a generalized elliptical distribution, assuming it is approximately sparse. We address the following two challenges: (i) design a computationally efficient and statistically accurate estimator of the shape matrix \( S_p \), that is adaptive to its unknown sparsity parameters; (ii) provide theoretical guarantees on its accuracy, in the large \( p \) large \( n \) regime.

We make the following contributions. First, in Section 3 we propose simple and computationally efficient estimators for the sparse shape matrix of an elliptical distribution. These are based on thresholding either Tyler’s M-estimator (TME) or its regularized variant. Second, we provide theoretical guarantees on
their accuracy in the limit \(n, p \to \infty\) with \(p/n \to \gamma\). Theorems 1 and 2 show that the estimator \(\hat{E}\) based on thresholding either TME for \(\gamma < 1\) or its regularized variant for any \(\gamma \in (0, \infty)\), converges in spectral norm to the shape matrix \(S_p\) with sparsity parameter \(q\) at rate \(\|\hat{E} - S_p\| = \mathcal{O}_p((\log p/n)^{(1-q)/2})\). Estimating a sparse shape matrix under a heavy tailed elliptical distribution is thus possible with the same asymptotic error rate as estimating a sparse covariance matrix under sub-Gaussian distributions. Moreover, our estimators are rate optimal, as this rate coincides with the minimax rate for sparse covariance estimation with sub-Gaussian data (Cai and Zhou, 2012a)\(^1\).

Our proofs follow the approach of Bickel and Levina (2008), with required modifications given that we analyze Tyler’s M-estimators. Theorem 1, proven in Section 5, relies on Zhang, Cheng and Singer (2016), who studied the spectral properties of Tyler’s M-estimator when \(n, p \to \infty\). The proof of Theorem 2, regarding the thresholded regularized TME, is far more involved. As detailed in Section 6, it combines a careful analysis of the form of the regularized TME together with several results in random matrix theory. Section 7 presents simulation results that support our theoretical analysis. With an eye towards practitioners, given that regularization is common also when \(p < n\), we focus on the regularized TME. With Gaussian data, our thresholded TME estimator is as accurate as thresholding the sample covariance. In contrast, in the presence of heavy tails it is far more accurate. We also illustrate its potential utility in handling outliers. In addition, our estimator is quite fast to compute in practice, requiring only few seconds on a standard PC, say for \(p = n = 1000\).

Our work is related to several recent papers, that also considered sparse shape or covariance matrix estimation with heavy tailed data. Han, Lu and Liu (2014) considered a pair-elliptical distribution, which is a different generalization of the classical elliptical distribution from the one we consider. They assumed moderate tails so the population covariance matrix exists, and proposed an estimator for it. They provided finite sample approximation bounds for their estimator, which depend on various properties of the distribution. For well-behaved elliptical distributions with an exactly sparse covariance matrix, their estimator is minimax rate optimal under the Frobenius norm. Soloveychik and Wiesel (2014) considered estimating a covariance matrix from a convex subset of all positive semidefinite matrices. They added a convex regularization term to the TME and solved the resulting optimization problem by a semidefinite program (SDP). They proved the existence of their estimator and its asymptotic consistency for fixed dimension \(p\) and \(n \to \infty\). However, their SDP-based method is computationally demanding even for moderate values of \(n\) and \(p\). Sun, Babu and Palomar (2014) considered a wider non-convex class of matrices, and derived an SDP-based algorithm with lower time complexity.

Chen, Gao and Ren (2015) considered an elliptical distribution, corrupted by an epsilon-contamination model. They proposed several estimators for the shape

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\(^1\)Technically the minimax rate was proven under the assumption that \(p/n^\beta \to c\) with \(\beta > 1\), see Remark 5 in Cai and Zhou (2012a). However, from personal communication with Profs. Cai and Zhou, the same minimax rate should hold also when \(\beta = 1\).
matrix of the elliptical distribution, based on a generalization of Tukey’s depth function. Under a notion of sparsity different from the one considered here, they proved their estimator is minimax rate optimal when \( n, p \to \infty \) and \( (\log p)/n \to 0 \). However, from a practical perspective this depth function estimator has a significant limitation – it is intractable to compute. Du, Balakrishnan and Singh (2017) considered an epsilon-contamination model for a Gaussian distribution with sparse covariance matrix \( \Sigma \), such that \( \| \Sigma - I \|_0 \leq s \) for a fixed \( s \geq 0 \). They proposed a polynomial-time algorithm for robust covariance estimation under this model and established a suboptimal upper bound on its error under Frobenius norm, assuming \( n, p \to \infty \) and \( (\log p)/n \to c \geq 0 \). Our work in contrast provides a computationally efficient and rate optimal estimator for an approximately sparse shape matrix of a potentially heavy tailed elliptical distribution in the high dimensional setting \( p, n \to \infty \) with \( p/n \to \gamma \). Further discussion and directions for future research appear in Section 8.

2. Problem Setting

With precise definitions below, given \( n \) i.i.d. observations from a generalized elliptical distribution, the problem we study is how to estimate its \( p \times p \) shape matrix \( S_p \). Of particular interest to us is the high-dimensional regime, where both \( p, n \) are large and comparable. Following previous works, to be able to accurately estimate the shape matrix in this regime we assume that it is approximately sparse. For completeness, we first introduce some notation, briefly review the generalized elliptical distribution and the class of approximately sparse shape matrices we consider.

**Notation**  We denote vectors by bold lowercase letters as in \( v \), and matrices by bold uppercase letters as in \( A \). For a vector \( v \in \mathbb{R}^n \), \( \|v\| \) is its Euclidean norm, \( \|v\|_\infty = \max_i |v_i| \), and \( B_R(u) = \{ v \in \mathbb{R}^n \mid \|v - u\|_\infty \leq R \} \). The identity matrix is \( I \) and \( 0 \) and \( 1 \) are the vectors of zeros and ones respectively, with dimensions clear from the context. For a matrix \( A = (a_{ij}) \), \( \|A\| \) denotes its spectral norm, \( \|A\|_F \) its Frobenius norm, \( \|A\|_{\max} = \max_{i,j} |a_{ij}| \) and \( \|A\|_\infty = \max_i \sum_j |a_{ij}| \). We denote the set of \( p \times p \) symmetric positive semidefinite and definite matrices by \( S_p^+ \) and \( S_p^{++} \) respectively. We say that an event occurs with high probability (abbreviated w.h.p.), if its probability is at least \( 1 - C \exp(-cp) \) for constants \( c, C > 0 \) independent of \( p \).

**Generalized Elliptical Distribution and its Shape Matrix**  A random vector \( x \in \mathbb{R}^p \) follows a generalized elliptical distribution if it has the form

\[
x = u S_p^{1/2} \xi = uz,
\]

where \( \xi \sim N(0, I) \), \( S_p \in S_p^{++} \), and \( u \) is an arbitrary random or deterministic nonzero scalar, not necessarily independent of \( \xi \). Eq. (1) generalizes the classical elliptical distribution, in which \( u \) is stochastically independent of \( \xi \), see for example Frahm (2004).
In Eq. (1), \( S_p \) is not unique, as it can be arbitrarily scaled with \( u \) absorbing the inverse scaling factor. Without loss of generality, we thus fix
\[
\text{tr}(S_p) = p,
\]
and refer to \( S_p \) as the *shape matrix*. If the distribution is elliptical and the population covariance \( \Sigma \) exists, then \( \Sigma = cS_p \) for some constant \( c > 0 \), see for example Soloveychik and Wiesel (2014).

**Approximate Sparsity of the Shape Matrix** Following Bickel and Levina (2008), we consider the following class of row/column approximately sparse covariance matrices with fixed parameters \( 0 \leq q \leq 1 \) and \( M > 0 \) and \( s_p > 0 \):
\[
U(q, s_p, M, s_{\text{max}}) = \{ A \in S_{++}^p : a_{ii} \leq M, \sum_{j=1}^{p} |a_{ij}|^q \leq s_p, 1 \leq i \leq p, \|A\| \leq s_{\text{max}} \}.
\]
Bickel and Levina (2008, p. 2580) noted that if \( A \) satisfies the above properties except for \( \|A\| \leq s_{\text{max}} \), then \( \|A\| \leq M^{1-q} s_p \). We explicitly require that \( \|A\| \leq s_{\text{max}} \), since our theorems require a bound on \( \|A\| \) independent of \( s_p \).

**Problem Statement** Let \( \{x_i\}_{i=1}^n \) be \( n \) i.i.d. samples from the model (1) with an approximately sparse shape matrix \( S_p \in U(q, s_p, M, s_{\text{max}}) \). We consider the following two problems: (i) Without explicit knowledge of \( q, s_p, s_{\text{max}} \) and \( M \), design a computationally efficient and statistically accurate estimator of the shape matrix \( S_p \); (ii) Provide theoretical guarantees on its accuracy, in the asymptotic limit as \( p, n \to \infty \) with \( p/n \to \gamma \in (0, \infty) \).

### 3. Sparse Shape Matrix Estimation

If the elliptical distribution is sub-Gaussian, then thresholding the sample covariance matrix, proposed by Bickel and Levina (2008) and El Karoui (2008), yields an accurate estimate of \( S_p \) up to a multiplicative scaling. As illustrated in Section 7, however, in the presence of heavy tails, the individual entries of the sample covariance matrix may be quite far from their population counterparts, and thresholding them may give a poor estimate of the shape matrix.

To handle heavy tails, we propose the following approach: compute Tyler’s M-estimator (TME) or its regularized variant, and threshold it. In Section 3.1 we review TME and its regularized variant. We prove that computing the latter is computationally efficient. Section 3.2 presents our proposed estimators. A theoretical analysis of their accuracy appears in Section 3.3.
3.1. TME and its Regularized Version

TME, proposed by Tyler (1987) for elliptical distributions, is a $p \times p$ matrix $\hat{\Sigma}$ which satisfies
\[
\frac{p}{n} \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}^{-1} x_i} = \hat{\Sigma}, \quad \text{tr}(\hat{\Sigma}) = 1.
\]  
(2)

Tyler (1987) suggested to solve Eq. (2) by the following iterations, starting from an arbitrary $\hat{\Sigma}_1 \in S^p_{++}$,
\[
\hat{\Sigma}_{k+1} = \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}_k^{-1} x_i} \bigg/ \text{tr} \left( \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}_k^{-1} x_i} \right).
\]

Kent and Tyler (1988)[Theorems 1 and 2] showed that if any linear subspace in $\mathbb{R}^p$ of dimension $1 \leq d \leq p - 1$ contains less than $nd/p$ of the data samples, and no points lie at the origin, then there exists a unique solution to Eq. (2), and the above iterations converge to it. In our setting of i.i.d. observations from a generalized elliptical distribution, these two conditions hold with probability 1.

TME enjoys several important properties: It is the maximum likelihood estimator of the shape matrix of a generalized elliptical distribution (Frahm and Jaekel, 2010). Moreover, it is the “most robust” estimator of the shape matrix with fixed $p$ and $n \to \infty$ for data i.i.d. from a continuous elliptical distribution (Tyler, 1987, Remark 3.1). TME outperforms the sample covariance in a variety of applications, including finance (Frahm and Jaekel, 2007), anomaly detection in wireless sensor networks (Chen, Wiesel and Hero, 2011), antenna array processing (Ollila and Koivunen, 2003) and radar detection (Ollila and Tyler, 2012).

As the TME does not exist when $p > n$, several regularized variants have been proposed and analyzed (Abramovich and Spencer, 2007; Chen, Wiesel and Hero, 2011; Wiesel, 2012; Pascal, Chitour and Quek, 2014; Sun, Babu and Palomar, 2014). Even when $p \leq n$, it is common to add small regularization to the TME. In this paper we consider the following regularized TME $\hat{\Sigma}(\alpha)$, defined as the fixed point solution of
\[
\hat{\Sigma}(\alpha) = \frac{1}{1 + \alpha} \frac{p}{n} \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}(\alpha)^{-1} x_i} + \frac{\alpha}{1 + \alpha} I,
\]  
(3)

where $\alpha > 0$ is a regularization parameter. If $\alpha = 0$, Eq. (3) reverts to Eq. (2).

Sun, Babu and Palomar (2014, Theorem 11) showed that with data drawn i.i.d. from a continuous distribution with no samples at the origin, Eq. (3) has a unique solution for $\alpha > \max(0, p/n - 1)$. In our setting of i.i.d. samples from a generalized elliptical distribution, these conditions hold with probability 1. Sun, Babu and Palomar (2014, Proposition 18) further showed that starting from any positive definite initial guess, the iterations
\[
\hat{\Sigma}_{k+1}(\alpha) = \frac{1}{1 + \alpha} \frac{p}{n} \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}_k(\alpha)^{-1} x_i} + \frac{\alpha}{1 + \alpha} I
\]  
(4)
converge to the unique solution. Various properties of TME and its regularized version, in the limit as $p, n \to \infty$ with $p/n \to \gamma$, were proven by Zhang, Cheng and Singer (2016); Couillet and McKay (2014); Couillet, Kammoun and Pascal (2016).

The following lemma, whose proof appears in the appendix, shows that if $\alpha$ is sufficiently large and $\Sigma(\alpha)$ exists, then the iterations (4), starting from $\hat{\Sigma}_1(\alpha) = \alpha I/(1 + \alpha)$, have a global linear convergence rate. To the best of our knowledge, this result is new and is of independent interest.

To state the lemma, let $e_k = \|\hat{\Sigma}(\alpha) - \Sigma_k(\alpha)\|$ be the error after $k$ iterations, $\hat{\Sigma}$ be the $p \times n$ matrix whose columns are $\{x_i/\|x_i\|\}_{i=1}^p$ and let

$$C(\hat{\Sigma}) = \frac{p}{n} \|\hat{\Sigma} \hat{\Sigma}^T\| = \frac{p}{n} \left\| \sum_{i=1}^n \frac{x_i x_i^T}{\|x_i\|^2} \right\|.$$ 

Note that for a given dataset, $C(\hat{\Sigma})$ is fixed and can be computed a-priori.

**Lemma 1.** Let $\{x_i\}_{i=1}^p$ be a data set in $\mathbb{R}^p$ with constant $C(\hat{\Sigma})$ and let $0 < R < 1$. Suppose that $\alpha > \max(3 + R^{-1})C(\hat{\Sigma}) - 1, 0)$ and let $\hat{\Sigma}(\alpha)$ be a solution of (3). Then, the iterations of Eq. (4), starting from $\hat{\Sigma}_1(\alpha) = \frac{\alpha}{1+\alpha} I$, globally linearly converge to $\hat{\Sigma}(\alpha)$ with the ratio $R$. That is,

$$e_{k+1} \leq Re_k \leq R^k e_1, \text{ for all } k \geq 1. \tag{5}$$

A straightforward calculation yields the bound $C(\hat{\Sigma}) \geq p/n$. Hence, the above assumptions on $\alpha$ imply that $\alpha > \max(0, p/n - 1)$ and consequently guarantee the existence of $\hat{\Sigma}(\alpha)$ in our setting.

Lemma 1 implies that calculating $\hat{\Sigma}(\alpha)$ is computationally efficient, since for accuracy $\epsilon$ and convergence ratio $R$, at most $\lceil \log_{R^{-1}}(\epsilon^{-1}) \rceil$ iterations are needed. At each iteration, the matrix inversion costs $O(p^3)$ operations and the other operations are $O(np^2)$. Therefore, for sufficiently large $\alpha$, the total cost of computing $\hat{\Sigma}(\alpha)$ within accuracy $\epsilon$ is $O(\log(\epsilon^{-1})(n + p)p^2)$.

Our theoretical analysis below studies the regularized TME as $p, n \to \infty$ and $p/n \to \gamma \in (0, \infty)$, but with a fixed value of $\alpha$. We now show that Lemma 1 is useful, since for data sampled from the generalization elliptical distribution, with high probability $C(\hat{\Sigma})$ is bounded by a constant independent of $p, n$.

**Lemma 2.** Let $x_1, \ldots, x_n$ be i.i.d. from the model (1) with shape matrix $S_p$. Then, for a suitable $c = c(\|S_p\|) > 0$, with probability $> 1 - \exp(-cp)$,

$$C(\hat{\Sigma}) \leq 2\|S_p\| \left(1 + 2\sqrt{p/n}\right)^2. \tag{6}$$

### 3.2. TME-Based Thresholding Estimators

For a matrix $A = (a_{ij})$ and threshold $t > 0$, define the hard-thresholding operator by

$$\tau_t(A) = (1(|a_{ij}| > t)a_{ij}).$$
For $n > p$, where the TME $\hat{\Sigma}$ exists and by definition has unit trace, our proposed estimator for the shape matrix $S_p$ takes the form

$$\hat{S}_p = \tau_t \left( p \hat{\Sigma} \right),$$

(7)

where the threshold $t = t(p, n)$ is specified below. Similarly, for general $p, n$, our estimator based on the regularized TME is

$$\hat{S}_p = \tau_t \left( p \frac{\hat{\Sigma}(\alpha) - \frac{\alpha}{1+\alpha} I}{\text{tr}(\hat{\Sigma}(\alpha) - \frac{\alpha}{1+\alpha} I)} \right).$$

(8)

### 3.3. Accuracy of the Thresholded TME

Theorems 1 and 2, proved in Sections 5 and 6, respectively, establish the asymptotic accuracy of Eqs. (7) and (8) as estimates of the shape matrix $S_p$.

**Theorem 1.** Consider a sequence $(n, p, S_p)$ where $n \to \infty$, $p = p_n \to \infty$ with $p/n \to \gamma \in (0, 1)$, and $S_p \in U(q, p, s_{\max})$. For each triplet $(n, p, S_p)$, let $\hat{\Sigma}$ be the TME of $n$ i.i.d. samples $\{x_i\}_{i=1}^n \subset \mathbb{R}^p$ from the generalized elliptical distribution. Then there exists a constant $M'$ depending only on $\gamma$ such that for any fixed $M'' > M'$, the thresholded TME of Eq. (7) with threshold $t_n = M'' \sqrt{\log p/n}$, approaches $S_p$ in spectral norm at a rate

$$\|\tau_t(n \hat{\Sigma}) - S_p\| = O_P \left( s_p \cdot \left( \frac{\log p}{n} \right)^{(1-q)/2} \right).$$

**Theorem 2.** Consider a sequence $(n, p, S_p)$ as in Theorem 1, here with $p/n \to \gamma \in (0, \infty)$ and with the additional assumption that $\lambda_{\min}(S_p) \geq s_{\min} > 0$. For $\alpha > \max(0, \sup_n p/n - 1)$, let $\hat{\Sigma}(\alpha)$ be the regularized TME of $n$ i.i.d. samples $\{x_i\}_{i=1}^n \subset \mathbb{R}^p$ from the generalized elliptical distribution. Then there exists an $M'$ depending only on $\gamma$ and $\alpha$ such that for any fixed $M'' > M'$, the estimator of Eq. (8) with $t_n = M'' \sqrt{\log p/n}$, converges in spectral norm to $S_p$ at rate

$$\|\tau_t \left( p \frac{\hat{\Sigma}(\alpha) - \frac{\alpha}{1+\alpha} I}{\text{tr}(\hat{\Sigma}(\alpha) - \frac{\alpha}{1+\alpha} I)} \right) - S_p\| = O_P \left( s_p \left( \frac{\log p}{n} \right)^{(1-q)/2} \right).$$

The convergence rate in Theorems 1 and 2 coincides with the minimax optimal rate for sparse covariance estimation with sub-Gaussian data, derived by Cai and Zhou (2012a). Since the Gaussian distribution is a particular case of an elliptical distribution, our estimators are thus minimax rate optimal. Furthermore, in light of Lemmas 1 and 2, computing the regularized TME and subsequently thresholding it, is computationally efficient.
4. Preliminaries

In proving Theorems 1 and 2, we shall make frequent use of the following inequality, which is simple to prove. Let $A, B$ be non-negative random variables. Then for any $c > 0$ and $\lambda > 0$,

$$\Pr(AB > c) \leq \Pr(A > \lambda c) + \Pr(B > 1/\lambda). \quad (9)$$

We shall also use the following lemma, proved in Appendix A.2, which shows that TME and regularized TME are unable to distinguish an elliptical distribution from a Gaussian one.

**Lemma 3.** TME or regularized TME with $\alpha > \max(0, p/n - 1)$ under any generalized elliptical distribution with shape matrix $S_p$ has the same distribution as under a Gaussian distribution with covariance $S_p$.

Finally, we shall also make use of the following two results from random matrix theory. The first is a non-asymptotic bound on the spectral norm of a Wishart matrix, and the second on the concentration of quadratic forms.

**Lemma 4.** (Davidson and Szarek, 2001)[Theorem 2.13] Let $\{\xi_i\}_{i=1}^n \subset \mathbb{R}^p$ be i.i.d. $N(0, I)$. Then

$$\Pr \left( \left\| \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T \right\| > \left( 1 + \sqrt{\frac{p}{n}} + t \right)^2 \right) \leq \exp(-nt^2/2).$$

**Lemma 5.** (Rudelson and Vershynin, 2013)[Theorem 1.1] Let $A \in \mathbb{R}^{p \times p}$ and $\xi \sim N(0, I)$. Then, there exist absolute constants $c_1, c_2 > 0$ such that for all $\epsilon > 0$,

$$\Pr \left( \left| \xi^T A \xi - \text{tr}(A) \right| > \epsilon \right) \leq 2 \exp \left( -c_1 \min \left\{ \frac{c_2 \epsilon^2}{\|A\|^2}, \frac{c_2 \epsilon}{\|A\|} \right\} \right).$$

5. Proof of Theorem 1

The proof consists of three main steps: (i) reducing to a bound on $\|p\hat{\Sigma} - \hat{S}\|$; (ii) expressing $\hat{\Sigma}$ as a weighted covariance matrix whose coefficients are all uniformly close to a constant, with high probability; and (iii) bounding $\|p\hat{\Sigma} - \hat{S}\|$.

5.1. Step 1: From $\|\tau_p(p\hat{\Sigma}) - S_p\|$ to $\|p\hat{\Sigma} - \hat{S}\|$

We begin with the following auxiliary lemma, proved in Appendix A.3. It is a slight modification of a result by Bickel and Levina (2008, p. 2583).

**Lemma 6.** Assume $B \in U(q, s_p, M, s_{\text{max}})$. Let $A$ be a matrix such that

$$\|A - B\|_{\text{max}} \leq C_1 \sqrt{\log p/n},$$

...
for some \( C_1 > 0 \). Suppose we threshold \( A \) at level \( t = K \sqrt{\log p/n} \), with \( K > C_1 \). Then, there exists a constant \( C_2 = C_2(C_1, K, q) < \infty \) such that

\[
\| \tau_t(A) - B \| \leq C_2 s_p (\log p/n)^{(1-q)/2}.
\]

Given this lemma, it suffices to prove that \( \| p \hat{\Sigma} - S_p \|_{\max} = O_P \left( \sqrt{\log p/n} \right) \). Let \( \hat{S} \) be the sample covariance of \( \{x_i\}_{i=1}^n \). By the triangle inequality,

\[
\| p \hat{\Sigma} - S_p \|_{\max} \leq \| p \hat{\Sigma} - \hat{S} \|_{\max} + \| \hat{S} - S_p \|_{\max}.
\]

In light of Lemma 3, we may assume that \( x_i \sim N(0, S_p) \). Then, Theorem 1 of Bickel and Levina (2008)\(^2\) implies that

\[
\| \hat{S} - S_p \|_{\max} = O_P \left( \sqrt{\log p/n} \right).
\]

Finally, since \( \| A \|_{\max} \leq \| A \| \), to conclude the proof it suffices to show that

\[
\| p \hat{\Sigma} - \hat{S} \| = O_P \left( \sqrt{\log p/n} \right). \tag{10}
\]

### 5.2. Step 2: The weights of TME

By Zhang, Cheng and Singer (2016, Lemma 2.1), TME has an equivalent definition as a weighted covariance matrix,

\[
\hat{\Sigma} = \sum_{i=1}^n w_i x_i x_i^T / \text{tr} \left( \sum_{i=1}^n w_i x_i x_i^T \right),
\]

where the weights \( w_i \) are the unique solution of

\[
\arg \min_{w_i > 0, \sum w_i = 1} - \sum_{i=1}^n \ln w_i + \frac{n}{p} \ln \det \left( \sum_{i=1}^n w_i x_i x_i^T \right). \tag{11}
\]

This characterization is important because of the following result:

**Lemma 7.** Consider a sequence \( (n, p, S_p) \) where \( n, p \to \infty \) with \( p/n \to \gamma \in (0, 1) \), and \( S_p \in S_{++}^p \). For every triplet \( (n, p, S_p) \), let \( x_i \sim N(0, S_p) \) and let \( \{w_i\}_{i=1}^n \) be the corresponding weights of Eq. (11). Then there exist positive constants \( C, c \) and \( c' \) depending only on \( \gamma \) such that for any \( 0 < \epsilon < c' \), and sufficiently large \( n \),

\[
\Pr \left[ \max_i |n w_i - 1| \geq \epsilon \right] \leq C n e^{-c \epsilon^2 n}. \tag{12}
\]

The case \( S_p = I \) was proved by Zhang, Cheng and Singer (2016, Lemma 2.2). Its generalization to an arbitrary \( S_p \in S_{++}^p \) is proved in Appendix A.4.

---

\(^2\)By a close inspection of their proof, it seems that as stated in their original paper, this theorem is missing a condition that \( \|S_p\| \leq s_{\max} \).
5.3. Step 3: Bounding $\|p\hat{\Sigma} - \hat{S}\|

To conclude the proof of Theorem 1, we apply the following lemma, proven in Appendix A.5.

**Lemma 8.** Let $\hat{\Sigma}$ and $\hat{S}$ be TME and the sample covariance matrix of $x_1, \ldots, x_n$ i.i.d. from $N(0, S_p)$, with $\text{tr}(S_p) = p$. Assume that $p, n \to \infty$, with $p/n \to \gamma \in (0, 1)$. Then there exist positive constants $C, c$ and $c'$ that depend only on $\gamma$, such that for all $\epsilon \in (0, c')$ and $n$ sufficiently large

$$\Pr \left( \|p\hat{\Sigma} - \hat{S}\| \geq \epsilon \right) \leq Cn e^{-c\epsilon^2}.$$

Taking $\epsilon_p = \sqrt{k + 1} \sqrt{(\log p)/n}$ for $k \in \mathbb{N}$ establishes Eq. (10) and concludes the proof.

6. Proof of Theorem 2

We first introduce and prove a slightly modified version of Theorem 2. We then show how Theorem 2 follows from it. The modified theorem uses the following proposition, proved in Appendix A.6.

**Proposition 1.** Let $S_p \in U(q, s_p, M, s_{\max})$ with $\lambda_{\min}(S_p) \geq s_{\min} > 0$. Let $y, \xi_1, \ldots, \xi_{n-1} \in \mathbb{R}^p$ be i.i.d. from $N(0, I)$ and define

$$Q = Q(r) = \frac{1}{p} y^T \left( \frac{1}{n} \sum_{j=1}^{n-1} \xi_j \xi_j^T + \frac{\alpha}{p} r S_p^{-1} \right)^{-1} y.$$

Assume that $\alpha > \max(0, p/n - 1)$, and let $r_{\min} = \frac{\alpha p}{s_{\max} (1 + \alpha p/n)} > 0$. Then, there exists a unique $r \in [r_{\min}, \infty)$, that depends on $p, n, \alpha$ and $S_p$, such that

$$\mathbb{E}[Q(r)] = \frac{1}{1 + \alpha - p/n}, \quad (13)$$

where the expectation is over $y$ and $\xi_1, \ldots, \xi_{n-1}$. Furthermore, along any sequence $(n, p_n, S_p)$ for which $p_n/n \to \gamma \in (0, \infty)$, the sequence $\{r(p_n, n)\}$ is contained in a compact interval in $[r_{\min}, \infty)$.

6.1. A Reformulation of the Main Result

We now introduce the modified theorem.

**Theorem 3.** Consider a sequence $(n, p, S_p)$ satisfying the conditions of Theorem 2. Let $\Sigma(\alpha)$ be the regularized TME of $n$ i.i.d. samples $\{x_i\}_{i=1}^n \subset \mathbb{R}^p$ from the generalized elliptical distribution with $\alpha > \max(0, \sup_n p/n - 1)$. Then there exists an $M'$ depending only on $\gamma$ and $\alpha$ such that for any fixed $M'' > M'$, the
estimator \( \tau_t (\hat{\Sigma}(\alpha) - \alpha I / (1 + \alpha)) \) with \( t_n = M'' \sqrt{\frac{\log p}{n}} \), converges in spectral norm to a multiple of \( S_p \),

\[
\| \tau_t \left( \frac{\alpha}{1 + \alpha} I - \frac{p}{n} S_p \right) \| = O_P \left( \frac{\log p}{n} \right)^{(1 - q)/2},
\]

where the constant \( r = r(p, n, \alpha, S_p) \) is specified in Proposition 1.

### 6.2. Proof of Theorem 3

By Lemma 3, we may assume \( x_i \overset{iid}{\sim} N(0, S_p) \). Following the argument in Section 5.1, it suffices to show that

\[
\left\| \left( \Sigma(\alpha) - \frac{\alpha}{1 + \alpha} I \right) - \frac{p}{n} S_p \right\| = O_P \left( \sqrt{\frac{\log p}{n}} \right).
\]

Our proof proceeds as follows: First, we express \( \hat{\Sigma}(\alpha) \) as the sum of \( \frac{\alpha}{1 + \alpha} I \) and weighted \( x_i x_i^T \) terms, where the weights are the root of some equation. Next, we show that this root is concentrated near the vector \( r 1 / n \), with \( r \) the constant of Proposition 1. Finally, we establish Eq. (14).

Following the definition of the regularized TME, we write \( \hat{\Sigma}(\alpha) \) as

\[
\hat{\Sigma}(\alpha) = \frac{1}{1 + \alpha} \sum_{i=1}^{n} w_i x_i x_i^T + \frac{\alpha}{1 + \alpha} I,
\]

where the weight vector \( w = (w_1, \ldots, w_n)^T \) satisfies

\[
w_i = \frac{1}{x_i^T \Sigma(\alpha)^{-1} x_i} = \frac{1}{x_i^T \left( \frac{1}{1 + \alpha} \sum_{j=1}^{n} w_j x_j x_j^T + \frac{\alpha}{1 + \alpha} I \right)^{-1} x_i}.
\]

By Sun, Babu and Palomar (2014)[Theorem 11], \( \hat{\Sigma}(\alpha) \) is unique.

Next, consider the function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) whose \( n \) components are

\[
g(v)_i = v_i - \frac{1}{x_i^T \left( \frac{1}{1 + \alpha} \sum_{k=1}^{n} v_k x_k x_k^T + \frac{\alpha}{1 + \alpha} n I \right)^{-1} x_i}.
\]

Comparing Eq. (17) to Eq. (16), the \( n \) non-linear equations \( g(v) = 0 \) have a unique solution, which is thus \( \hat{w} \). The next three lemmas state properties of \( g \) used to prove that as \( p, n \rightarrow \infty \), with \( p/n \rightarrow \gamma \), this root concentrates around \( u = r 1 \), with \( r \) given in Proposition 1. The lemmas, proven in Appendices A.7–A.9, assume the setting of Theorem 3, and their generic constants depend only on \( \gamma, \alpha, s_{\min} \) and \( s_{\max} \).

**Lemma 9.** There exist \( C, c > 0 \) such that for any \( \epsilon \in (0, 1) \)

\[
\Pr (\|g(u)\|\infty > \epsilon) < Cpe^{-c\epsilon^2}.
\]
Lemma 10. There exist $c', c_L, C, c > 0$ such that
\[
\Pr (\exists v \in B_{c'}(u), \|\nabla g(v) - \nabla g(u)\|_{\max} > c_L \|v - u\|_\infty) < C \rho^2 e^{-c \epsilon}.
\]

Lemma 11. There exist $c_H, C, c > 0$ such that
\[
\Pr \left( \left\| (\nabla g(u))^{-1} \right\|_\infty > c_H \right) < C \rho e^{-c \epsilon}.
\]  

Lemmas 9 and 10 show that w.h.p. $g(u)$ is small and $\nabla g$ is Lipschitz near $u$. These two properties are consistent with the root of $g$ being close to $u$. To rigorously prove this, following Zhang, Cheng and Singer (2016), we consider the function $f(v) = (\nabla g(u))^{-1} g(v)$). Lemma 11 shows that the matrix $(\nabla g(u))^{-1}$ is w.h.p. not extremely large. Finally, the following lemma combines these properties of $g$ to infer that its root is close to $u$.

Lemma 12. Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $u \in \mathbb{R}^n$ and $C > 0$. Assume that
\begin{enumerate}
  \item $\nabla f(u) = I$;
  \item $\|\nabla f(v) - \nabla f(u)\|_{\max} \leq C \|v - u\|_\infty$ for all $\|v - u\|_\infty \leq 3 \|f(u)\|_\infty$;
  \item $\|f(u)\|_\infty < \min \{1/(9C), 1/3\}$.
\end{enumerate}

Then there exists a $\tilde{v} \in \mathbb{R}^n$ such that $f(\tilde{v}) = 0$ and $\|\tilde{v} - u\|_\infty < 3 \|f(u)\|_\infty$.

Lemma 12 is slightly stronger than Lemma 3.1 of Zhang, Cheng and Singer (2016), as it has a weaker requirement that the Lipschitz condition in (2) holds in a smaller ball $\|v - u\|_\infty \leq 3 \|f(u)\|_\infty$, instead of the original requirement $\|v - u\|_\infty < 1$ in their Lemma 3.1. A careful inspection shows that their original proof is still valid under this weaker assumption.

To apply Lemma 12 to $f(v) = (\nabla g(u))^{-1} g(v)$, we verify that the three conditions of the lemma hold with high probability. The first condition is satisfied trivially. For the other two conditions, by Lemmas 9 and 11, w.h.p.
\[
\|f(u)\|_\infty \leq \| (\nabla g(u))^{-1} \|_\infty \cdot \| g(u) \|_\infty \leq c_H \epsilon.
\]

Similarly, by Lemmas 10 and 11, for all $\|v - u\|_\infty \leq c'$, w.h.p.
\[
\| \nabla f(v) - \nabla f(u) \|_{\max} \leq \|(\nabla g(u))^{-1}\|_\infty \cdot \| \nabla g(v) - \nabla g(u) \|_{\max} \leq c_H c_L \|v - u\|_\infty.
\]

Since for sufficiently small $\epsilon$, $c_H \epsilon < \min \{1/(9c_L c_H), 1/3\}$, both the second and third conditions of Lemma 12 are thus satisfied with constant $C = c_L c_H$.

To conclude, with probability at least $1 - C \rho^2 e^{-c \epsilon}$, all three conditions of Lemma 12 hold, so there exists $\tilde{v} \in \mathbb{R}^n$ such that $f(\tilde{v}) = 0$ and $\|\tilde{v} - u\|_\infty \leq 3 \|f(u)\|_\infty < 3 c_H \epsilon$. Since $nw$ is the unique root of $g(v)$ and also of $f(v)$,
\[
\Pr (\|nw - r1\|_\infty > 3 c_H \epsilon) < C \rho^2 e^{-c \epsilon}.
\]

Next, we use Eq. (19) to bound the LHS of Eq. (14). First, by Eq. (15),
\[
\left\| (\Sigma(\alpha) - \frac{n}{1 + \alpha} I) - \frac{1}{1 + \alpha} \frac{p}{n} \nu S \right\| = \frac{1}{1 + \alpha} \frac{p}{n} \left\| \sum_{i=1}^{n} w_i x_i x_i^T - \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right\|
\leq \frac{1}{1 + \alpha} \frac{p}{n} \|n w - r1\|_\infty \left\| \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right\|.
\]
Using this inequality and Eq. (9) with \( \lambda = 1 / [s_{\text{max}}(1 + 2\sqrt{\gamma})^2] \),

\[
\Pr \left( \left\| \left( \hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I \right) - \frac{1}{1 + \alpha} \frac{p}{n} r \hat{S} \right\| > \epsilon \right) \leq \Pr \left( \frac{1}{1 + \alpha} \frac{p}{n} \| nw - r 1 \| \| \hat{S} \| > \epsilon \right) \\
\leq \Pr \left( \| nw - r 1 \|_\infty > \epsilon \right) \leq \frac{n(1 + \alpha)}{p s_{\text{max}}(1 + 2\sqrt{\gamma})^2} \leq \Pr \left( \| \hat{S} \| > s_{\text{max}}(1 + 2\sqrt{\gamma})^2 \right).
\]

Since \( \hat{S} = S_p^{1/2}(1/n \sum \xi \xi^T) S_p^{1/2} \) with \( \xi_i \sim N(0, I) \), by Lemma 4 the second term is exponentially small in \( p \). By Eq. (19), the first term is bounded by \( C'p^2 e^{-cp\epsilon^2} \). Hence, Eq. (14) holds, which concludes the proof of Theorem 3.

6.3. Concluding the Proof of Theorem 2

As in Theorems 1 and 3, to prove Theorem 2 it suffices to show that

\[
\left\| \frac{p}{n} \left( \hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I \right) - \frac{\alpha}{1 + \alpha} I \right\| = O_P \left( \sqrt{\log p/n} \right). \tag{20}
\]

Eq. (14) combined with Proposition 1 imply that for \( r > r_{\text{min}} > 0 \)

\[
\left\| \frac{n}{p} \frac{1 + \alpha}{r} \left( \hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I \right) - \hat{S} \right\| = O_P \left( \sqrt{\log p/n} \right). \tag{21}
\]

Eq. (20) follows from Eq. (21) by the following lemma, proven in the appendix.

**Lemma 13.** Let \( B \in S_p^+ \) with \( \text{tr}(B) = p \) and \( \| B \| \leq b_{\text{max}} \). Suppose that \( A \in S_p^+ \) satisfies \( \| A - B \| < \epsilon \leq 1/2 \). Then,

\[
\left\| \frac{pA}{\text{tr}(A)} - B \right\| \leq 2(1 + b_{\text{max}})\epsilon. \tag{22}
\]

7. Numerical Experiments

Focusing on the regularized TME, we present simulations that support our theoretical analysis. Section 7.1 compares the regularized TME, the sample covariance and their thresholded versions. Section 7.2 considers the sensitivity of the proposed estimator to \( \alpha \). Section 7.3 demonstrates a simple modification of our estimator in the presence of outliers.

7.1. Comparison of Thresholded TME with Covariance Estimators

We considered the following shape matrix, also used by Bickel and Levina (2008):

\[
S_p = (s_{ij}) = (\tau^{i-j}).
\]

We generated data from Eq. (1) and three different choices for the random variables \( u_i \): (i) \( u_i = 1 \), so \{ \( x_i \) \}_{i=1}^n \) are i.i.d. \( N(0, S_p) \); (ii) \( u_i \sim \text{Laplace}(0, 1) \),
a heavy tailed distribution with finite moments; and (iii) \( u_i \sim \text{Cauchy}(0,1) \), so the distribution does not even have a well-defined mean or covariance.

We computed four estimators for the shape matrix: (i) SampCov: the sample covariance scaled to have trace \( p, \frac{p \hat{S}}{\text{tr}(\hat{S})} \); (ii) th-SampCov: the thresholded version of SampCov, \( \tau_t \left( \frac{p \hat{S}}{\text{tr}(\hat{S})} \right) \); (iii) RegTME: the regularized TME, normalized to have trace \( p, \frac{p(\Sigma(\alpha) - \frac{\alpha}{1+\alpha}I)}{\text{tr}(\Sigma(\alpha) - \frac{\alpha}{1+\alpha}I)} \); and (iv) th-RegTME: the thresholded version of RegTME in Eq. (8). We choose \( \alpha = 10 \), and threshold at level \( t = \sqrt{\log(p)/n} \). The stopping rule for the iterations (4) is \( \|p\Sigma_{k+1}/\text{tr}(\Sigma_{k+1}) - p\Sigma_k/\text{tr}(\Sigma_k)\| < 10^{-8} \).

We measured the accuracy of an estimator \( \hat{S}_p \) by the logarithm of its averaged relative error (abbreviated LRE). That is, for 100 different realizations, we independently generated \( n \) i.i.d. samples in \( \mathbb{R}^p \), and each time estimated \( (\hat{S}_p)_i \), where \( i = 1, \ldots, 100 \). The LRE was then computed as follows

\[
\text{LRE} = \log \left( \frac{1}{100} \sum_{i=1}^{100} \frac{\| (\hat{S}_p)_i - S_p \|}{\| S_p \|} \right).
\]

We considered sample sizes \( n \in [100, 1000] \) and the following three ratios \( p/n \in \{.5, 1, 2\} \). Fig. 1 shows the LRE of the four estimators. As expected theoretically, for \( u_i \equiv 1 \) thresholding the sample covariance or the regularized TME
yield similar errors. In contrast, for heavy-tailed data the thresholded sample covariance performs poorly, whereas the thresholded regularized TME is still an accurate estimate of $S_p$.

7.2. Sensitivity of Regularized TME to Choice of $\alpha$

Next, we study the how the error and run-time of th-RegTME depend on the regularization parameter $\alpha$.

We consider the Gaussian model with covariance $S_p$, and explore the behavior of th-RegTME for the following values of $\alpha$: 0.2, 0.4, 0.6, 0.8, 1, 2, 3, . . . , 20 and the following three cases: $(p, n) = (800, 400)$, $(p, n) = (800, 200)$ and $(p, n) = (400, 200)$. The sufficient condition for RegTME to exist, $\alpha > \max(0, p/n - 1)$, is not necessary as in this example, the iterations (4) converged for all considered values of $\alpha$. The left panel of Fig. 2 shows the LRE of th-RegTME as a function of $\alpha$. The maximal LRE occurs at $p/n - 1$ and larger values of $\alpha$ yield slightly smaller errors. The right panel of Fig. 2 displays the logarithm of the runtime of th-RegTME as a function of $\alpha$, showing a sharp increase in runtime as $p/n - 1$ approaches $\alpha$.

Next, we explore the behavior of th-RegTME for $p = 480$, $\alpha = 1, 2, 3, 4$ and $n = 60, 64, 68, \ldots, 300$. The left panel of Fig. 3 shows the error of th-RegTME as a function of $n$. In accordance with theory, $\alpha$ has little effect on the accuracy. Of particular interest is the runtime, seen in the right panel of Fig. 3. Here we
see a sharp increase in runtime as \( p/n - 1 \) approaches \( \alpha \). For \( n \geq \frac{p}{\alpha + 1} \), the runtime decreases as \( \alpha \) increases.

These experiments indicate that one may generally prefer larger \( \alpha \), particularly for faster runtime. Importantly, for fast runtime one should ensure that \( \alpha \) is not close to \( p/n - 1 \).

7.3. Regularized TME in the presence of outliers

We conclude the numerical section with an illustrative example of the ability of the regularized TME to detect outliers, and upon their removal and thresholding, to provide a robust and accurate estimate of a sparse shape matrix.

To this end, we consider the following \( \epsilon \)-contamination mixture model: \( (1-\epsilon)n \) of the observed data, the inliers, follow an elliptical distribution with the same sparse shape matrix \( S_p \) as above. The remaining \( \epsilon n \) of the samples, the outliers, follow an elliptical distribution with shape matrix \( U(pD/tr(D))U' \), where \( U \) is a unitary matrix, uniformly distributed with Haar measure, and \( D \) is a diagonal matrix. In our first experiment, the diagonal entries \( d_{ii} \) are all i.i.d. uniformly distributed over \([1, 5]\), so the outliers are rather diffuse. In our second experiment \( d_{11} = p, d_{22} = p/2 \) and all other \( d_{ii} = 1 \), so the outliers are nearly on a 2-d randomly rotated subspace.

Given \( n \) samples from this \( \epsilon \)-contamination model, and without knowledge of \( \epsilon \), the task is to accurately estimate the shape matrix \( S_p \). Since both the inliers and outliers have potentially heavy tailed distributions, it might not be possible to detect the outliers by simple schemes, such as those based on the norm of a sample or the number of its neighbors in a given radius. However, recall that by our theoretical analysis, in the absence of outliers (\( \epsilon = 0 \)), for Gaussian data or similarly for elliptical data but normalized to unit norm, \( x_i/\|x_i\| \), the corresponding weights \( w_i \) in the regularized TME are all approximately equal. For \( \epsilon \ll 1 \), with all samples normalized to have unit norm, we thus expect the inliers to still all have similar weights, and the outliers to have quite different weights, hopefully smaller though not necessarily so. With further details in Appendix A.11, our proposed procedure for robustness to outliers is to estimate the mean and standard deviation of the inliers’ weights. Then exclude all samples whose weights are outside, say, the mean plus or minus two standard deviations, recompute the regularized TME on the remaining samples and threshold it.

Fig. 4 illustrates the robustness of this procedure to outliers in two different settings. From left to right, for \( \epsilon = 0.2 \) and 0.4, it shows the weights of the \( n \) normalized samples \( x_i/\|x_i\| \), sorted so the first \( \epsilon n \) of them are the outliers. The blue horizontal line is a robust estimate of the mean weight of the inliers, and the two red lines are this estimated mean plus and minus two standard deviations. The top row corresponds to the first outlier model with \( d_{ii} \sim U[1, 5] \). The second row corresponds to our second outlier model with \( D = \text{diag}(p, p/2, 1, \ldots , 1) \). Note that this outlier shape matrix has a spectral norm \( O(p) \), which does not satisfy our requirement that \( \|D\| \leq s_{\text{max}} \). As indeed observed empirically, the weights of the outliers do not to tightly concentrate around some value. Yet,
our outlier exclusion procedure still succeeds to exclude most of these outliers. The error of the thresholded TME with outliers removed, compared to that of thresholding the original TME is shown in the right column of Fig. 4.

This simple example illustrates the potential ability of TME to screen outliers in high dimensional settings, at least for small contamination levels. A detailed study of this ability is an interesting topic for future research.

8. Summary and Discussion

In this paper we proposed simple estimators for the shape matrix of possibly heavy tailed elliptical distributions, assuming the shape matrix is approximately sparse. We further analyzed their error, showing that under the spectral norm they are minimax rate optimal in a high-dimensional setting with $p/n \rightarrow \gamma$.

There are several directions for future research. One direction is to extend our results to the case $p = n^\beta$, with $\beta > 1$. Our current analysis assumed the regularization parameter $\alpha$ of TME is fixed, whereas if $p = n^\beta$ with $\beta > 1$, just to ensure its existence would require $\alpha \rightarrow \infty$. Handling this case thus requires extending our analysis to allow $\alpha$ to grow with $n$ and $p$.

A question of practical interest is how to set the threshold parameter in a data-driven fashion. Bickel and Levina (2008, Section 3), proposed a cross validation procedure to set the threshold. Rigorously proving that this provides a good estimate in the case of (regularized) TME is an interesting topic for future work.

While our work focused on approximate sparsity of the shape matrix, robust inference under other common assumptions can also be studied. For example, one might assume that the first few leading eigenvectors of $\Sigma$ are sparse, also
known as sparse-PCA, or that \( \Sigma \) is the combination of a low rank and a sparse matrix. In particular, a robust sparse-PCA estimator may be constructed by applying a sparse-PCA procedure to Tyler’s M-estimator.

Finally, another direction for future work is to develop a computationally efficient algorithm for sparse covariance estimation in the presence of a small fraction of arbitrary outliers. This setting was considered in Chen, Gao and Ren (2015), but without a computationally tractable estimator. Our promising preliminary results in Section 7.3 suggest to study whether regularized TME offers such robustness, and under which outlier models.

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Appendix A: Supplementary Details

A.1. Complexity of Calculating the Regularized TME

Proof of Lemma 1. We arbitrarily fix a solution \( \hat{\Sigma}(\alpha) \) of (3). Since \( \hat{\Sigma}(\alpha) \) is invariant to scaling of the data, we assume that \( \|x_i\| = 1, 1 \leq i \leq n \). We first analyze the quantity \( e_1 = \|\hat{\Sigma}(\alpha) - \hat{\Sigma}_1(\alpha)\| \). To this end, let \( \lambda_{\text{max}} = \|\hat{\Sigma}(\alpha)\| \).

Taking the spectral norm in Eq. (3), together with the fact that 
\[
\frac{1}{\lambda_{\text{max}}} = \sqrt{\frac{\alpha}{1 + \alpha}} \frac{1}{\sqrt{1 - \frac{C(\tilde{X})}{1 + \alpha}}}.
\]

Equivalently, for \( 1 + \alpha > C(\tilde{X}) \),

\[
\lambda_{\text{max}} \leq \frac{\alpha}{1 + \alpha} \frac{1}{1 - \frac{C(\tilde{X})}{1 + \alpha}}.
\]

Combining this inequality with the fact that by Eq. (3) \( \hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I \in S^p_+ \),

\[
e_1 = \|\hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I\| = \lambda_{\text{max}} - \frac{\alpha}{1 + \alpha} \leq \frac{C(\tilde{X})}{1 + \alpha} \frac{\alpha}{1 + \alpha} \frac{1}{1 - \frac{C(\tilde{X})}{1 + \alpha}}.
\]  

Next, we analyze the error \( e_k \). We denote \( E_k = \hat{\Sigma}(\alpha) - \hat{\Sigma}_k(\alpha) \) and write 
\[
\hat{\Sigma}_k(\alpha) = \hat{\Sigma}(\alpha) - E_k = \hat{\Sigma}(\alpha)^{1/2}(I - \hat{\Sigma}(\alpha)^{-1/2}E_k)\hat{\Sigma}(\alpha)^{-1/2} \hat{\Sigma}(\alpha)^{1/2}.
\]
Since $\hat{\Sigma}(\alpha)$ and $\hat{\Sigma}_k(\alpha)$ are invertible, so is $I - \hat{\Sigma}(\alpha)^{-1/2}E_k\hat{\Sigma}(\alpha)^{-1/2}$. Let $B_k = I - (I - \hat{\Sigma}(\alpha)^{-1/2}E_k\hat{\Sigma}(\alpha)^{-1/2})^{-1}$ and $R_k = \hat{\Sigma}(\alpha)^{-1/2}B_k\hat{\Sigma}(\alpha)^{-1/2}$. Then, $\hat{\Sigma}_k(\alpha)^{-1} = \hat{\Sigma}(\alpha)^{-1/2}(I - B_k)\hat{\Sigma}(\alpha)^{-1/2} = \hat{\Sigma}(\alpha)^{-1} - R_k$.

Subtracting Eq. (4) from Eq. (3) gives

$$E_{k+1} = \frac{1}{1 + \alpha} \frac{p}{n} \sum_{i} x_i x_i^T \left( \frac{1}{\hat{\Sigma}(\alpha)^{-1} x_i} - \frac{1}{\hat{\Sigma}(\alpha)^{-1} x_i - x_i^T R_k x_i} \right) = \frac{1}{1 + \alpha} \frac{p}{n} \sum_{i} x_i x_i^T \left( \frac{1}{\hat{\Sigma}(\alpha)^{-1} x_i} \left( 1 - \frac{1}{1 - \delta_{ki}} \right) \right),$$

where $\delta_{ki} = x_i^T R_k x_i / x_i^T \hat{\Sigma}(\alpha)^{-1} x_i$.

Let $D_k = \max_{1 \leq i \leq n} |\delta_{ki}/(1 - \delta_{ki})|$. Since all terms $x_i x_i^T / x_i^T \hat{\Sigma}(\alpha)^{-1} x_i$ are positive semidefinite, the above equation implies that

$$\|E_{k+1}\| \leq D_k \left\| \frac{p}{1 + \alpha} \sum_{i} \frac{x_i x_i^T}{x_i^T \hat{\Sigma}(\alpha)^{-1} x_i} \right\| = D_k \left\| \hat{\Sigma}(\alpha) - \frac{\alpha}{1 + \alpha} I \right\| = D_k e_1. \quad (24)$$

Eq. (23) gives a bound on $e_1$. We now bound $D_k$. Since $\hat{\Sigma}(\alpha) \geq \frac{\alpha}{1 + \alpha} I$,

$$\|\hat{\Sigma}(\alpha)^{-1/2} E_k \hat{\Sigma}(\alpha)^{-1/2}\| \leq \|\hat{\Sigma}(\alpha)^{-1}\| e_k \leq \frac{1 + \alpha}{\alpha} e_k.$$  

Assume this quantity is strictly smaller than one, then

$$\|B_k\| = \|I - (I - \hat{\Sigma}(\alpha)^{-1/2} E_k \hat{\Sigma}(\alpha)^{-1/2})^{-1}\| \leq \frac{1 + \alpha}{\alpha} \frac{e_k}{1 - \frac{1 + \alpha}{\alpha} e_k}. \quad (25)$$

Finally, given the relation between $R_k$ and $B_k$,

$$|\delta_{ki}| = \frac{|x_i^T R_k x_i|}{x_i^T \hat{\Sigma}(\alpha)^{-1} x_i} = \frac{|(\hat{\Sigma}(\alpha)^{-1/2} x_i)^T B_k (\hat{\Sigma}(\alpha)^{-1/2} x_i)|}{\|\hat{\Sigma}(\alpha)^{-1/2} x_i\|^2} \leq \|B_k\|.$$  

Thus, assuming $\|B_k\| < 1$,

$$D_k = \max \left\{ \frac{|\delta_{ki}|}{1 - \delta_{ki}} \right\} \leq \frac{\|B_k\|}{1 - \|B_k\|} = \frac{1 + \alpha}{\alpha} e_k \cdot \frac{1}{1 - 2 \frac{1 + \alpha}{\alpha} e_k}. \quad (26)$$

Inserting (26) and (23) into (24) yields that

$$\frac{e_{k+1}}{e_k} \leq \frac{C(\hat{X})}{1 + \alpha} \frac{1}{1 + \frac{1 + \alpha}{\alpha} e_k} \leq \frac{C(\hat{X})}{1 + \alpha} \frac{1}{1 - 2 \frac{1 + \alpha}{\alpha} e_k}. \quad (27)$$

For the proof to hold, we required that $\frac{1 + \alpha}{\alpha} e_k < 1$ and $\|B_k\| < 1$. If $\frac{1 + \alpha}{\alpha} e_k < 0.5$, then the RHS of Eq. (25) is less than one and both assumptions hold. For $0 < R < 1$ and $1 + \alpha > (3 + R^{-1}) C(\hat{X})$, Eq. (23) implies that $\frac{1 + \alpha}{\alpha} e_1 < \frac{1}{2 + R^{-1}}$ and combining this with Eq. (27) results in the estimate $e_2/e_1 < 1/2$. Since $R < 1$, easy induction implies that for $k > 1$, $\frac{1 + \alpha}{\alpha} e_k < \frac{1}{2 + R^{-1}} < 0.5$, as required, and so Eq. (5) holds. Since this convergence holds with any solution of (3), this solution thus has to be unique. \qed
Proof of Lemma 2. Since the regularized TME is invariant to scaling, we assume all \( u_i = 1 \), and express \( x_i = S_p^\frac{1}{2} \xi_i \), where \( \xi_i \sim N(0, I) \). Let \( UDU^T \) be the eigendecomposition of \( S_p \). Then redefining \( \xi = U \xi_i \), \( \| x_i \|^2 = \xi_i^T D \xi_i \) and expressing

\[
C(\tilde{X}) = \left\| S_p^\frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\xi_i \xi_i^T}{x_i^T D x_i} \right) S_p^\frac{1}{2} \right\|.
\]

Combining Lemma 5 with a union bound yields

\[
\Pr \left( \max_{i} \left| \frac{1}{p} \xi_i^T D \xi_i - \frac{1}{p} \tr(D) \right| > \epsilon \right) < 2n \exp \left( -c_1 \min \left\{ \frac{c_2 p^2 \epsilon^2}{\| D \|_F^2}, \frac{c_2 p \epsilon}{\| D \|} \right\} \right).
\]

Since \( \| D \| = \| S_p \| \leq s_{\max} \) and \( \| D \|_F^2 \leq p s_{\max}^2 \), for any fixed \( \epsilon \) the above probability is exponentially small in \( p \). Taking say \( \epsilon = 1/2 \) and recalling that \( \tr(D) = p \), gives that with high probability,

\[
C(\tilde{X}) \leq 2\| S_p \| \cdot \left\| \frac{1}{n} \sum_{i=1}^{n} \xi_i \xi_i^T \right\|.
\]

Eq. (6) follows since by Lemma 4, w.h.p. \( \| \frac{1}{n} \sum_{i=1}^{n} \xi_i \xi_i^T \| \leq (1 + 2 \sqrt{p/n})^2 \). \( \square \)

A.2. Proof of Lemma 3

We prove a more general lemma from which Lemma 3 immediately follows.

Lemma 14. Let \( x_1, \ldots, x_n \) be \( n \) nonzero vectors in \( \mathbb{R}^p \). Let \( z_i = x_i/\| x_i \| \) and \( u_i = \| x_i \| \). If the (regularized) TME has a unique solution, then it depends only on the direction vectors \( \{ z_i \}_{i=1}^{n} \).

Proof. By definition, the TME of the \( n \) samples \( x_i = u_i z_i \) satisfies

\[
\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \Sigma^{-1} x_i} \text{tr} \left( \sum_{i=1}^{n} \frac{x_i x_i^T}{x_i^T \Sigma^{-1} x_i} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{u_i z_i z_i^T u_i}{z_i^T \Sigma^{-1} u_i z_i} \text{tr} \left( \sum_{i=1}^{n} \frac{u_i z_i z_i^T u_i}{z_i^T \Sigma^{-1} u_i z_i} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{z_i z_i^T}{z_i^T \Sigma^{-1} z_i} \text{tr} \left( \sum_{i=1}^{n} \frac{z_i z_i^T}{z_i^T \Sigma^{-1} z_i} \right),
\]

which is the TME of \( \{ z_i \} \). The proof for the regularized TME is similar. \( \square \)

A.3. Proof of Lemma 6

Most of the proof follows Bickel and Levina (2008, p. 2583). By the triangle inequality,

\[
\| \tau_i(A) - B \| \leq \| \tau_i(B) - B \| + \| \tau_i(A) - \tau_i(B) \| = q_1 + q_2.
\]
As in their Eq. (13), \( q_1 \leq t^{1-q_s p} \). For the second term \( q_2 \),

\[
q_2 \leq \max_i \sum_{j=1}^{p} \left| a_{ij} \right| \mathbf{1}(\left| a_{ij} \right| \geq \left| b_{ij} \right| < t) + \max_i \sum_{j=1}^{p} \left| b_{ij} \right| \mathbf{1}(\left| a_{ij} \right| < \left| b_{ij} \right| \geq t) + \max_i \sum_{j=1}^{p} \left| a_{ij} - b_{ij} \right| \mathbf{1}(\left| a_{ij} \right| \geq t, \left| b_{ij} \right| \geq t) = q_3 + q_4 + q_5.
\]

Similarly, \( q_4 \leq C_1 \sqrt{\frac{\log p}{n}} t^{-q_s p} + t^{1-q_s p} \) and \( q_5 \leq C_1 \sqrt{\frac{\log p}{n}} t^{-q_s p} \). For \( q_3 \),

\[
q_3 \leq \max_i \sum_{j=1}^{p} \left| a_{ij} - b_{ij} \right| \mathbf{1}(\left| a_{ij} \right| \geq \left| b_{ij} \right| < t) + \max_i \sum_{j=1}^{p} \left| b_{ij} \right| \mathbf{1}(\left| a_{ij} \right| \geq t, \left| b_{ij} \right| < t).
\]

The second sum is bounded as above by \( t^{1-q_s p} \). For the first sum, we slightly differ from Bickel and Levina (2008). Since \( \left| a_{ij} - b_{ij} \right| \leq C_1 \sqrt{\frac{\log p}{n}} \) and \( t = K \sqrt{\frac{\log p}{n}} \) with \( K > C_1 \) then all terms satisfy \( \left| b_{ij} \right| > t(1 - C_1/K) \). Hence,

\[
q_3 \leq C_1 \sqrt{\frac{\log p}{n}} \max_i \sum_{j=1}^{p} \mathbf{1}(\left| b_{ij} \right| > t \left( 1 - \frac{C_1}{K} \right)) + t^{1-q_s p} \leq C_1 \sqrt{\frac{\log p}{n}} t^{-q_s} (1 - \frac{C_1}{K})^{-q_s p} + t^{1-q_s p}.
\]

Collecting the above inequalities concludes the proof, since

\[
\|\tau_1(\mathbf{A}) - \mathbf{B}\| \leq (3K^{1-q} + C_1K^{-q}(2 + (1 - C_1/K)^{-q})) s_p \left( \frac{\log p}{n} \right)^\frac{1}{2q_s}.
\]

### A.4. Proof of Lemma 7

For \( \mathbf{S}_p = \mathbf{I} \), Eq. (12) was proven by Zhang, Cheng and Singer (2016, Lemma 2.2). We reduce the general case to this case as follows. For \( \mathbf{x}_i \overset{iid}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{S}_p) \), write \( \mathbf{x}_i = \mathbf{S}_p^{1/2} \mathbf{y}_i \), where \( \mathbf{y}_i \) are i.i.d. \( \mathcal{N}(\mathbf{0}, \mathbf{I}) \). Let \( \tilde{\Sigma}_x \) and \( \tilde{\Sigma}_y \) denote the TMEs of \( \{\mathbf{x}_i\}_{i=1}^{n} \) and \( \{\mathbf{y}_i\}_{i=1}^{n} \) respectively. Their weights \( \{w_i^x\}_{i=1}^{n} \) and \( \{w_i^y\}_{i=1}^{n} \) are uniquely determined by Eq. (11). The lemma follows since the two sets of weights are identical. Indeed,

\[
(w_1^x, \ldots, w_n^x)^T = \arg \min_{w_1 > 0, \sum w_i = 1} -\sum_{i=1}^{n} \ln w_i + \frac{n}{p} \ln \det \left( \sum_{i=1}^{n} w_i \mathbf{S}_p^{1/2} \mathbf{y}_i \mathbf{y}_i^T \mathbf{S}_p^{1/2} \right) = \arg \min \left\{ \sum_{i=1}^{n} \ln w_i + \frac{n}{p} \ln \det \left( \sum_{i=1}^{n} w_i \mathbf{y}_i \mathbf{y}_i^T \right) + \frac{n}{p} \ln \det (\mathbf{S}_p) \right\} = (w_1^y, \ldots, w_n^y)^T.
\]
A.5. Proof of Lemma 8

Since \( x_i \sim N(0, S_p) \), by Lemma 7 the TME weights \( w = (w_1, \ldots, w_n)^T \) of Eq. (11) are all concentrated around \( 1/n \). The following lemma shows that \( T_w = \text{tr}(\sum_i w_i x_i x_i^T) \) is close to \( \text{tr}(S_p) = p \).

Lemma 15. Assume the setting of Lemma 8. There exist constants \( C, c \) and \( c' \) depending on \( \gamma \) such that for all \( \epsilon \in (0, c') \) and \( n \) sufficiently large,

\[
\Pr \left( \left| \frac{T_w}{n} - 1 \right| > \epsilon \right) \leq Cne^{-c\epsilon^2}.
\]  

We prove Lemma 8 assuming Lemma 15 holds, and then prove the latter.

Proof of Lemma 8. By definition,

\[
\| p\hat{\Sigma} - \hat{S} \| = \left\| \sum_{i=1}^{n} \left( \frac{p w_i}{T_w} - \frac{1}{n} \right) x_i x_i^T \right\| \leq \left\| \frac{np w}{T_w} - 1 \right\| \cdot \left\| \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right\|.
\]  

Since \( x_i \sim N(0, S_p) \), \( \left\| \frac{1}{n} \sum x_i x_i^T \right\| \leq s_{\text{max}} \left\| \xi \right\| \), where \( \xi \overset{iid}{\sim} N(0, I) \). By Lemma 4, the latter term is bounded w.h.p. by \( (1 + 2\sqrt{\gamma})^2 \).

As for the first term on the RHS of Eq. (23), by the triangle inequality,

\[
\left\| \frac{np w}{T_w} - 1 \right\| \leq \left\| np w - 1 \right\| + \left\| np w - 1 \right\|.
\]

Hence,

\[
\Pr \left( \left\| \frac{np w}{T_w} - 1 \right\| > \epsilon \right) \leq \Pr(\| nw - 1 \| > \epsilon/2) + \Pr(\| nw - 1 \| > \epsilon/2).
\]

Lemma 7 provides an exponential bound on the second term. For the first term, applying Eq. (9) with \( \lambda = 2 \) gives

\[
\Pr(\| nw \| > \epsilon/2) \leq \Pr(\| w \| > 2) + \Pr(\| w \| > \epsilon/2) \leq 2 \Pr(\| w \| > \epsilon/2).
\]

By Lemmas 7 and 15, these two probabilities are exponentially small.

Proof of Lemma 15. As \( | T_w - 1 | = | \frac{T_w}{p} - 1 | \), by Eq. (9) with \( \lambda = 2 \)

\[
\Pr \left( \left| \frac{T_w}{p} - 1 \right| > \epsilon \right) \leq \Pr \left( \left| \frac{p}{T_w} \right| > 2 \right) + \Pr \left( \left| \frac{1}{T_w} - \frac{p}{p} \right| > \epsilon/2 \right) \leq 2 \Pr \left( \left| \frac{1}{T_w} - \frac{p}{p} \right| > \epsilon/2 \right).
\]

Next, we relate \( | 1 - \frac{T_w}{p} | \) to \( | 1 - \frac{T}{p} | \), where \( T = \text{tr}(\frac{1}{n} \sum x_i x_i^T) \). Note that

\[
| 1 - \frac{T_w}{p} | \leq | 1 - \frac{T}{p} | + | \frac{T}{p} - \frac{T_w}{p} | = | 1 - \frac{T}{p} | + | \frac{1}{n} \sum_{i=1}^{n} (\frac{1}{n} - w_i) x_i x_i^T |.
\]

By Lemma 15, this term is exponentially small.

\[
| 1 - \frac{T}{p} | \leq | 1 - \frac{T}{p} | + \| nw - 1 \| \leq Cne^{-c\epsilon^2}.
\]
Therefore
\[ \Pr \left( \left| 1 - \frac{T_n}{p} \right| > \frac{\epsilon}{2} \right) \leq \Pr \left( \left| 1 - \frac{T_n}{p} \right| > \frac{\epsilon}{4} \right) + \Pr \left( \|n\mathbf{w} - 1\|_\infty \cdot \left| \frac{T_n}{p} \right| > \frac{\epsilon}{4} \right) = q_1 + q_2. \]

Applying Eq. (9) with \( \lambda = 2 \) to the second term gives
\[ q_2 \leq \Pr \left( \|n\mathbf{w} - 1\|_\infty > \epsilon/8 \right) + \Pr \left( \frac{T_n}{p} > 2 \right) \leq \Pr \left( \|n\mathbf{w} - 1\|_\infty > \epsilon/8 \right) + \Pr \left( \left| 1 - \frac{T_n}{p} \right| > 1 \right). \]

By Lemma 7, the first probability above has the desired exponential decay. To conclude the proof, we thus need to provide an exponential bound on \( q_1 \).

Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \) be the eigenvalues of \( \mathbf{S}_p \). Since \( \mathbf{x}_i \sim N(0, \mathbf{S}_p) \),
\[ T = \text{tr} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T \right) = \sum_{j=1}^{p} \lambda_j \chi_j^2(n)/n, \]
where the \( \chi_j^2(n) \) are i.i.d. chi-square random variables with \( n \) degrees of freedom for \( j = 1, 2, \ldots, p \). Given that \( \text{tr}(\mathbf{S}_p) = \sum_{j=1}^{n} \lambda_j = p \),
\[ \left| 1 - \frac{T_n}{p} \right| = \frac{1}{p} \sum_{j=1}^{p} \lambda_j \left( 1 - \frac{\chi_j^2(n)}{n} \right) \leq \max_j \left| \frac{\chi_j^2(n)}{n} - 1 \right|. \]

Since \( \chi^2 \) random variables are sub-exponential, for a suitable constant \( c > 0 \),
\[ \Pr \left( \left| \frac{\chi_j^2(n)}{n} - 1 \right| > \epsilon \right) < \exp \left( -cn\epsilon^2 \right). \quad (30) \]

Therefore by a union bound, the term \( q_1 \) is also exponentially small. \( \square \)

**A.6. Proof of Proposition 1**

To prove the existence of a unique \( r \) such that Eq. (13) holds, we first show that \( \mathbb{E}[Q(r)] \) is strictly monotone increasing in \( r \) and then use the intermediate value theorem. With some abuse of notation, \( r \) first denotes the variable of \( Q(r) \) and later it is fixed to be the unique solution of Eq. (13).

To simplify notation, let \( \mathbf{T} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{\xi}_j \mathbf{\xi}_j^T \) and \( \beta = \beta(r) = \frac{\alpha}{p} \). Then
\[ \mathbb{E}[Q(r)] = \mathbb{E}_\mathbf{\xi} \left[ \mathbb{E}_\mathbf{w}[Q(r)] \right] = \mathbb{E} \left[ \frac{1}{p} \text{tr} \left( (\mathbf{T} + \beta \mathbf{S}_p^{-1})^{-1} \right) \right], \quad (31) \]
where the expectation is now only over the random variables \( \mathbf{\xi}_i \).

For any \( r > 0 \) and \( 0 < \delta < r \), let \( \mathbf{A} = \mathbf{T} + \beta \mathbf{S}_p^{-1} \) and \( \mathbf{B} = \frac{\delta}{r - \delta} \beta \mathbf{S}_p^{-1} \), so that \( \text{tr}(\mathbf{A}^{-1}) = pQ(r) \) and \( \text{tr}((\mathbf{A} + \mathbf{B})^{-1}) = pQ(r - \delta) \). Since \( \mathbf{A}, \mathbf{B} \in S_{++}^p \),
\[ \mathbf{A}^{-1} - (\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}) = \mathbf{A}^{-1}(\mathbf{A} + \mathbf{B} - \mathbf{A})(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} \mathbf{B}(\mathbf{A} + \mathbf{B})^{-1} = (\mathbf{A}^{-1} \mathbf{B})(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} \mathbf{B} - \mathbf{A}^{-1} \mathbf{A} \in S_{++}^p. \]
Therefore, applying Jensen’s inequality,
\[
\mathbb{E}[Q(r) - Q(r - \delta)] = \frac{1}{p} \mathbb{E}[\text{tr}((AB^{-1}A + A)^{-1})] \geq \mathbb{E} \left[ \frac{1}{\lambda_1(AB^{-1}A + A)} \right] 
\]
\[
\geq \frac{1}{\mathbb{E} \left[ \lambda_1 \left( \frac{1}{p} \frac{(r-\delta)}{\beta} AS_pA + A \right) \right]}. \tag{32}
\]

To prove that \( \mathbb{E}[Q(r)] \) is strictly monotone, consider the eigenvalue \( \lambda_1 \) above. We have that
\[
\lambda_1 \left( \frac{1}{p} \frac{(r-\delta)}{\beta} AS_pA + A \right) \leq \frac{1}{p} \frac{(r-\delta)}{\beta} \cdot \lambda_1(A)^2 \lambda_1(S_pA) + \lambda_1(A). \tag{33}
\]
Since \( A = T + \beta S_p^{-1} \), then \( \lambda_1(A) \leq \lambda_1(T) + \beta/s_{\min} \). Furthermore, upon averaging over the random variables \( \xi_i \), by Lemma 4, there exists a constant \( C_1 = C_1(\gamma) < \infty \) such that \( \mathbb{E}[\lambda_1(T)] \leq C_1 \). Therefore, with a suitable constant \( C = C(\gamma, r, \alpha, s_{\min}) \)
\[
\mathbb{E} \lambda_1(A) \leq C. \tag{34}
\]
Inserting Eqs. (34) and (33) into Eq. (32) yields that for \( r > \delta > 0 \),
\[
\mathbb{E}[Q(r)] - \mathbb{E}[Q(r - \delta)] \geq \frac{1}{p} \frac{(r-\delta)}{\beta} \cdot s_{\max} C^2 + C > 0.
\]
It thus follows that \( \mathbb{E}[Q(r)] \) is strictly monotone increasing in \( r \).

Next, we study the behavior of \( \mathbb{E}[Q(r)] \) as \( r \to 0 \). By definition,
\[
\mathbb{E}[Q(r)] \leq \mathbb{E} \frac{1}{\lambda_p(T + \beta S_p^{-1})} \leq \mathbb{E} \frac{1}{\lambda_p(\beta S_p^{-1})} \leq \frac{s_{\max}}{\beta}. \tag{35}
\]
Since \( \beta = n\alpha/(pr) \), this implies that \( \mathbb{E}[Q(r)] \to 0 \) as \( r \to 0 \).

We now examine the behavior of \( \mathbb{E}[Q(r)] \) when \( r \to \infty \). First of all,
\[
\mathbb{E}[Q(r)] \geq \frac{1}{p} \sum_{j=1}^{p} \frac{1}{\lambda_j(T)} + \frac{1}{\lambda_1(S_p^{-1})}. \tag{36}
\]
We analyze the above quantity separately in two cases: \( p > n - 1 \) and \( p \leq n - 1 \).
In the first case \( T \) has \( p - n + 1 \) eigenvalues equal to zero. Thus,
\[
\mathbb{E}[Q(r)] \geq \frac{p}{p-n+1} \cdot s_{\min} \cdot r, \tag{37}
\]
implying that \( \mathbb{E}[Q(r)] \to \infty \) as \( r \to \infty \).

In the second case, where \( p \leq n - 1 \), \( T \) is invertible almost surely and we define \( q_{\infty} = \lim_{r \to \infty} Q(r) = y^T T^{-1} y / p \). By Anderson (2003, Lemma 7.7.1),
\[
\mathbb{E}[q_{\infty}] = \frac{1}{p} \mathbb{E}[\text{tr} (T^{-1})] = \frac{n}{n-p-2} = \frac{1}{1 - \frac{p+2}{n}}. \tag{38}
\]
Note that $Q(r)$ is dominated by $q_{\infty}$, which is absolutely integrable. Therefore, by the dominated convergence theorem and Eq. (38)

$$\lim_{r \to \infty} \mathbb{E} Q(r) = \mathbb{E} q_{\infty} = \frac{1}{1 - \frac{p + 2}{n}} > \frac{1}{1 - \frac{p}{n} + \alpha}.$$ 

Given the strict monotonicity of $\mathbb{E}[Q(r)]$ and its lower and upper bounds at $r = 0$ and $r = \infty$, the intermediate value theorem implies that for any fixed $p, n$ there is a unique $r = r(p, n, \alpha) \in (0, \infty)$ such that Eq. (13) holds. Furthermore, given the lower bound of Eq. (35), it follows that for any $p, n, \alpha$,

$$r(p, n, \alpha) \geq \frac{\alpha n}{\min p(1 + \alpha - p/n)} = r_{\min}.$$ 

To prove the second part of the proposition, consider a sequence $(n, p_n, S_p)$ with $p/n \to \gamma ∈ (0, \infty)$. From now on $r$ denotes the value $r(p, n, \alpha)$ such that Eq. (13) holds. We bound $r$ from above in three cases: $\gamma < 1$, $\gamma > 1$, and $\gamma = 1$. For $\gamma < 1$, we consider sufficiently large values of $n$ such that $p_n < n - 4$. Using Eqs. (31) and (38),

$$\mathbb{E}[q_{\infty} - Q(r)] = \mathbb{E} \frac{1}{p} \sum_{j=1}^{p} \left( \frac{1}{\lambda_j(T)} - \frac{1}{\lambda_j(T + \beta S_p^{-1})} \right) \leq \mathbb{E} \frac{1}{p} \sum_{j=1}^{p} \frac{\lambda_1(T + \beta S_p^{-1})}{\lambda_j(T) \lambda_j(T + \beta S_p^{-1})} \leq \beta \mathbb{E} \frac{1}{s_{\min}} \frac{1}{\lambda_j(T)^2}.$$ 

The Weyl inequalities $\lambda_j(T) + \lambda_p(\beta S_p^{-1}) \leq \lambda_j(T + \beta S_p^{-1}) \leq \lambda_j(T) + \lambda_1(\beta S_p^{-1})$, imply that

$$\mathbb{E}[q_{\infty} - Q(r)] \leq \mathbb{E} \frac{1}{p} \sum_{j=1}^{p} \frac{\lambda_1(\beta S_p^{-1})}{\lambda_j(T) (\lambda_j(T) + \lambda_p(\beta S_p^{-1}))} \leq \beta \mathbb{E} \frac{1}{s_{\min}} \frac{1}{\lambda_j(T)^2}.$$ 

Since $nT \sim W_p(I, n - 1)$, by Lemmas 2.1(ii) and 2.3(iv) in Das Gupta (1968)

$$\mathbb{E} \frac{1}{p} \sum_{j=1}^{p} \frac{1}{\lambda_j(T)^2} = \frac{(n - 2)(n - 1)}{(n - p - 1)(n - p - 2)(n - p - 4)}. $$

The last two equations imply the following upper bound:

$$\mathbb{E}[q_{\infty} - Q(r)] \leq \frac{\beta}{s_{\min}} \frac{(n - 2)(n - 1)^2}{(n - p - 1)(n - p - 2)(n - p - 4)}. \quad (39)$$

Combining Eqs. (13), (39) and (38),

$$r \leq \alpha n^{-1} \frac{(n - 2)(n - 1)^2(1 - \frac{p}{n} + \alpha)(1 - \frac{p}{n} - \frac{2}{n})}{s_{\min} (n - p - 1)(n - p - 2)(n - p - 4)(\alpha + \frac{2}{n})} < \infty.$$
In the case $\gamma > 1$, for $n$ large enough $p_n > n - 1$ always holds. Then Eqs. (13) and (37) imply that
\[
r \leq \frac{\alpha n}{s_{\min}(p - n + 1)} \frac{1}{1 + \alpha - \frac{E}{n}} < \infty.
\]

Finally, consider the case $p/n \to \gamma = 1$. Here, we use the well-known fact that the empirical spectral distribution of $T$ converges to the Marchenko-Pastur law. Combining this with Eq. (36) and results on the convergence of linear spectral statistics (Bai and Silverstein, 2010, Chapter 9),
\[
E[Q(r)] \geq \mathbb{E} \sum_{j=1}^{p} \frac{1}{\lambda_j(T) + \beta s_{\min}^{-1}} \to \frac{1}{2\pi} \int_0^4 \frac{\sqrt{x(4-x)}}{x(\alpha/(rs_{\min}))} \, dx.
\]

We may bound this integral by
\[
\frac{1}{2\pi} \int_0^4 \frac{\sqrt{x(4-x)}}{x(\alpha/(rs_{\min}))} \, dx \geq \frac{\sqrt{3}}{2\pi} \int_0^1 \frac{1}{x + \alpha/(rs_{\min})} \, dx \geq \frac{\sqrt{3}}{2\pi} \log \left(1 + \frac{rs_{\min}}{\alpha}\right).
\]

The RHS above can be further bounded below by $\log(r_{\min}/\alpha)/5$. Hence, there exists a $N_0 > 0$ such that for $n > N_0$
\[
r \leq \frac{\alpha}{s_{\min}} \exp \left(\frac{10}{1+\alpha-\frac{E}{n}}\right) \leq \infty.
\]

Clearly, $r$ is also bounded for the finite number of values of $n \leq N_0$.

**A.7. Proof of Lemma 9**

Let $\hat{S} = \frac{1}{n} \sum_{k=1}^{n} x_k x_k^T$ and $\hat{T} = \frac{1}{n} \sum_{k=1}^{n} \xi_k \xi_k^T$, where $x_i = S_p^{1/2} \xi_i$ and $\xi_i \sim N(0, I)$. Then, Eq. (17) may be written as
\[
1 - \frac{1/(1 + \alpha)}{\hat{p} x_i^T (\hat{S} + \beta I)^{-1} x_i} = 1 - \frac{1/(1 + \alpha)}{\hat{p}^2 \xi_i^T (S_p^{-1/2} S_p S_p^{-1/2} + \beta S_p^{-1})^{-1} \xi_i}
\]
\[
= 1 - \frac{1}{1 + \frac{\alpha}{\hat{p}^2} \xi_i^T E \xi_i},
\]

where $E = (\hat{T} + \beta S_p^{-1})^{-1}$ and $\beta = \alpha n \frac{1}{\hat{p}}$. The quadratic form $\frac{1}{\hat{p}^2} \xi_i^T E \xi_i$ is difficult to analyze directly because $E$ depends on $\xi_i$. To disentangle this dependency, let $T_{\cdot \cdot i} = \frac{1}{\hat{p}} \sum_{k \neq i} \xi_k \xi_k^T$, and $E_{\cdot \cdot i} = (T_{\cdot \cdot i} + \beta S_p^{-1})^{-1}$. As $E^{-1}$ and $E^{-1}_{\cdot \cdot i}$ differ by a rank-one matrix $\frac{1}{\hat{p}^2} \xi_i \xi_i^T$, by the Sherman-Morrison formula,
\[
E = E_{\cdot \cdot i} - \frac{1}{\hat{p}^2} \xi_i \xi_i^T E_{\cdot \cdot i}.
\]
Therefore, denoting by $Q_i$ the quadratic form
\[ Q_i(r) \equiv Q_i = \frac{1}{p} \xi_i^T E_{-i} \xi_i, \quad (41) \]
it follows that
\[ \frac{1}{p} \xi_i^T E \xi_i = Q_i - \frac{\xi_i^2 Q_i^2}{1 + \xi_i^2 Q_i} = \frac{Q_i}{1 + \frac{\xi_i^2 Q_i}{n}}. \quad (42) \]

Plugging this expression into Eq. (40) gives
\[ \frac{1}{r} g(u)_i = Q_i \left( 1 + \frac{\alpha}{p} - \frac{1}{p} \right) - \frac{1}{(1 + \alpha)Q_i}. \quad (43) \]

Next, to establish a concentration bound for $g(u)_i/r$, we study the concentration of $Q_i$. Since $\xi_i \sim N(0, I)$ and is independent of $E_{-i}$,
\[ \mathbb{E}Q_i = \mathbb{E} \text{tr}(E_{-i})/p. \]
We first show that $Q_i$ concentrates tightly around $\text{tr}(E_{-i})/p$ in view of concentration of quadratic forms. We then show that $\text{tr}(E_{-i})$ concentrates tightly around its mean using results about the concentration of certain functions of the eigenvalues of random matrices.

Applying Lemma 5 with $\xi = \xi_i$ and viewing the matrix $E_{-i}$ as fixed,
\[ \Pr \left( \left| Q_i - \frac{1}{p} \text{tr} (E_{-i}) \right| > \epsilon \right) \leq 2 \exp \left( -c_1 \min \left\{ \frac{c_2^2 \beta^2}{\|E_{-i}\|^2}, \frac{c_2 \beta \epsilon}{\|E_{-i}\|} \right\} \right), \]
where the above probability is only w.r.t. $\xi_i$. Next, given that $E_{-i} = (T_{-i} + \beta S_p^{-1})^{-1}$, then $\|E_{-i}\| \leq \frac{\epsilon}{\sqrt{\beta}}$ and $\|E_{-i}\|^2 \leq \frac{\epsilon^2}{\sqrt{\beta}}$. Thus,
\[ \Pr \left( \left| Q_i - \frac{1}{p} \text{tr} (E_{-i}) \right| > \epsilon \right) \leq C \exp \left( -c_2 \epsilon^2 \right), \quad (44) \]
where now the probability is over all of the $\xi_i$’s.

It remains to obtain a concentration inequality for $\text{tr} (E_{-i})/p$. To this end, consider the following $p \times (n - 1 + p)$ matrix,
\[ Y = \begin{pmatrix} \xi_1 & \cdots & \xi_{i-1} & \xi_{i+1} & \cdots & \xi_n \end{pmatrix} \sqrt{n \beta S_p^{-1/2}}. \]
By definition, all entries of $Y$ are independent, the first $p \times (n - 1)$ are standard Gaussian random variables and the rest deterministic. Then, by Guionnet and Zeitouni (2000)[Corollary 1.8b]3, for any function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x^2)$ is Lipschitz with constant $L$, for any $\delta > 0$
\[ \Pr \left( \left| \frac{1}{K} \text{tr} (Y Y^T) - \mathbb{E} \text{tr} (h(Y Y^T)) \right| > \delta \right) \leq 2 \exp \left( -\frac{\delta^2 K^2}{2L^2} \right) \quad (45) \]

\[ ^3 \text{There is a typo in the original paper. In the notation of their Corollary 1.8, } Z \text{ should be replaced with } Z/(M + N). \]
where $K = 2p + n - 1$ and for a symmetric matrix $A$ with eigenvalues $\lambda_j$, $\text{tr} h(A) = \sum_j h(\lambda_j)$.

Since $YY^T = n(\hat{T}_{-i} + \beta S_p^{-1}) = nE^{-1}_i$, consider the function

$$h(x) = \frac{n}{p} \cdot \frac{1}{x}$$

for which $\frac{1}{2p + n - 1} \text{tr} h(YY^T/(2p + n - 1)) = \text{tr}(E_{-i})/p$. Next, note that for sufficiently large $n$ and sufficiently small $\epsilon$

$$\lambda_{\min} \left( \frac{YY^T}{2p + n - 1} \right) = \frac{n}{2p + n - 1} \lambda_{\min}(\hat{T}_{-i} + \beta S_p^{-1}) \geq \frac{1}{2 \gamma + 1 + \epsilon} s_{\max} = x_0.$$ 

We thus apply the function $h$ only in the interval $x \geq x_0$. The Lipschitz constant of $h(x^2)$ for $n$ sufficiently large is bounded by

$$L \leq \left| \frac{d}{dx} h(x^2) \right| \leq 16 (\gamma + 0.5 + \epsilon)^3 \left( \frac{s_{\max}}{\beta} \right)^3 \leq 16 (\gamma + 1)^3 \left( \frac{s_{\max}}{\beta} \right)^3.$$ 

Hence, applying (45), there exists a positive constant $c$ that depends on $\gamma, \alpha, r$ and $s_{\max}$ such that

$$\Pr \left( \frac{1}{\lambda} |\text{tr}(E_{-i}) - \mathbb{E} \text{tr}(E_{-i})| > \delta \right) \leq 2 \exp \left( -c p^2 \delta^2 \right). \quad (46)$$

Next, by the triangle inequality

$$\Pr(|Q_i - \frac{\mathbb{E} \text{tr}(E_{-i})}{p}| > \epsilon) \leq \Pr(|Q_i - \frac{\text{tr}(E_{-i})}{p}| > \frac{\epsilon}{2}) + \Pr(|\frac{\text{tr}(E_{-i})}{p} - \frac{\mathbb{E} \text{tr}(E_{-i})}{p}| > \frac{\epsilon}{2})$$

Thus, at the value of $r$ specified in Proposition 1, for which $\mathbb{E} |\text{tr}(E_{-i})/p| = \frac{1}{1 + \alpha - \frac{p}{n}}$, combining the above with Eqs. (44) and (46),

$$\Pr \left( |Q_i - \frac{1}{1 + \alpha - \frac{p}{n}}| > \epsilon \right) < C e^{-cp^2}.$$ \quad (47)

We are finally ready to establish a concentration result for $\frac{1}{\gamma}g(u)$. Combining Eq. (43) and a union bound over all $p$ coordinates of $g$,

$$\Pr \left( \left\| \frac{1}{\gamma} g(u) \right\|_\infty > \epsilon \right) \leq p \Pr \left( \left\| \frac{1}{\gamma} g(u) \right\| > \epsilon \right) \leq p \Pr \left( \left| Q_i(1 + \alpha - \frac{p}{n}) - 1 \right| > \epsilon \right).$$

Applying Eq. (9) with $\lambda = 1$ to the equation above gives

$$\Pr \left( \left\| \frac{1}{\gamma} g(u) \right\|_\infty > \epsilon \right) < p \Pr \left( \left| Q_i(1 + \alpha - \frac{p}{n}) - 1 \right| > \epsilon \right) + p \Pr \left( (1 + \alpha)Q_i < 1 \right).$$

By Eq. (47), the first term on the RHS is exponentially small in $p$. As for the second term, since $(1 + \alpha)^{-1} < (1 + \alpha - p/n)^{-1}$, then again by Eq. (47), $\Pr(Q_i < 1/(1 + \alpha))$ is also exponentially small in $p$. The lemma thus follows from the boundedness of $r$ from above, as established in Proposition 1.
A.8. Proof of Lemma 10

For any $v \in \mathbb{R}^n$, denote $\hat{S}(v) = \frac{1}{n} \sum_k v_k x_k x_k^T$ and

$$ F(v) = \left( \frac{1}{\tau} \hat{S}(v) + \beta I \right)^{-1}. \quad (48) $$

We prove Lemma 10 using the following Lemma, which is proved later.

**Lemma 16.** Assume the setting of Theorem 3 and let $u = r1$, with $r$ defined in Proposition 1. The matrix $F$ of Eq. (48) satisfies:

1. For all $v \in \mathbb{R}^n$ with $\|v - u\|_\infty \leq r$, $\|F(v)\| \leq 1/\beta$.
2. There exists $c > 0$ such that with probability at least $1 - \exp(-cp)$, for all $v \in \mathbb{R}^n$ with $\|v - u\|_\infty \leq r$,

$$ \lambda_{\min}(F(v)) \geq c_F = \frac{1}{2s_{\max}(1 + 2\sqrt{q})^2 + \beta}. \quad (49) $$

3. With the same constant $c > 0$ above, there exist $c', C > 0$ such that

$$ \Pr (\forall v \in B_r(u), \|F(u) - F(v)\| < C\|v - u\|_\infty \geq 1 - \exp(-cp)). \quad (50) $$

**Proof of Lemma 10.** Recall that for an invertible matrix $A(v)$ that depends on a vector $v$, $\frac{\partial(A^{-1})}{\partial v_i} = -A^{-1} \frac{\partial A}{\partial v_i} A^{-1}$. Then, differentiating $g(v)$, in Eq. (17) with respect to $v_i$ gives that $\nabla g(v) = I - B(v)$, where

$$ B(v)_{ij} = \frac{1}{1 + \alpha n} \frac{p}{F(v)_{ii}} \frac{F(v)_{ij}^2}{F(v)_{ii}^2}, \quad 1 \leq i, j \leq n \quad (51) $$

and $F(v)_{ij} = x_i^T F(v) x_j$. With this expression for $\nabla g(v)$,

$$ \|\nabla g(u) - \nabla g(v)\|_{\max} = \max_{i,j} \frac{1}{1 + \alpha n} \left| \frac{F(v)_{ij}^2}{F(v)_{ii}^2} - \frac{F(u)_{ij}^2}{F(u)_{ii}^2} \right|. \quad (52) $$

By the triangle inequality,

$$ \frac{F(v)_{ij}^2}{F(v)_{ii}^2} - \frac{F(u)_{ij}^2}{F(u)_{ii}^2} \leq \frac{F(v)_{ij}^2}{F(v)_{ii}^2} \left| \frac{1}{F(v)_{ii}^2} - \frac{1}{F(u)_{ii}^2} \right| + \left| \frac{F(v)_{ij}^2}{F(v)_{ii}^2} - \frac{F(u)_{ij}^2}{F(u)_{ii}^2} \right| = q_1 + q_2. $$

We now bound each of these two terms. For the first one,

$$ q_1 \leq \frac{F(v)_{ij}^2}{F(u)_{ii}^2} \frac{\|F(u)_{ii} - F(v)_{ii}\| \cdot (F(u)_{ii} + F(v)_{ii})}{F(u)_{ii}^2 F(v)_{ii}^2}. $$

For any $v$ for which $F(v)$ is defined, $|F(v)_{ij}| \leq \|F(v)||x_i||x_j|$ and $F(v)_{ii} \geq \lambda_{\min}(F(v))|x_i|^2$. Combining these with parts 1 and 2 of Lemma 16,

$$ q_1 \leq \frac{1}{c_F} \frac{|x_j|^4}{|x_i|^4} (\|F(u)\| + \|F(v)\|) \|F(u) - F(v)\| \leq \frac{2}{\beta c_F^2} \frac{|x_j|^4}{|x_i|^4} \|F(u) - F(v)\|.$$
and similarly
\[ q^2 \leq \frac{2}{\beta c_F^2} \frac{\|x_j\|^2}{\|x_i\|^2} \|F(u) - F(v)\|. \]

Finally, we write \( x_j = S_p^{1/2} \xi_j \) with \( \xi_j \sim N(0, I) \). Since \( \|\xi_j\|^2 \sim \chi^2(p) \), by Eq. (30), we conclude that w.h.p., \( \|x_j\|^2/\|x_i\|^2 \leq 2s_{\text{max}}/s_{\text{min}} \). Next, Eq. (50) implies that w.h.p. \( \|F(u) - F(v)\| \leq C\|v - u\|_\infty \). A union bound on all \( p^2 \) terms in Eq. (52) concludes the proof of the lemma. \( \square \)

**Proof of Lemma 16.** Part 1: For any \( v \in \mathbb{R}^n \) with \( \|v - u\|_\infty \leq r \), all entries \( v_j \geq 0 \), so \( \hat{S}(v) \in S_+^p \) and thus \( \|F(v)\| \leq 1/\beta \).

Part 2: If \( \|v - u\|_\infty \leq r \), then \( v_j \leq 2r \) for all \( 1 \leq j \leq n \). Thus,
\[ \lambda_{\text{min}}(F(v)) \geq \frac{1}{\lambda_{\text{max}}(\frac{1}{n^2} \sum_k v_k x_k x_k^T) + \beta} \geq \frac{1}{2\lambda_{\text{max}}(\frac{1}{n^2} \sum_k x_k x_k^T) + \beta}. \]

Eq. (49) follows since by Lemma 4, with probability at least \( 1 - \exp(-c\beta) \), the largest eigenvalue is smaller than \( s_{\text{max}}(1 + 2\sqrt{\gamma})^2 \).

Part 3: Using the Hadamard product \( \circ \), \( d_v \in \{-1, 1\}^n \) and \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) with \( \epsilon_1, \ldots, \epsilon_p \geq 0 \), we express \( v \) as \( v = u + r d_v \circ \epsilon \). Next, we apply the following classical perturbation result (Stewart, 1990, Eq. (1.2)): Let \( M \) be an invertible matrix \( M \), then for any matrix \( \Delta M \) with \( \|M^{-1}\|\|\Delta M\| < 1 \),
\[ \|M + \Delta M\|^{-1} - M^{-1} \| \leq \frac{\|M^{-1}\|\|\Delta M\|}{1 - \|M^{-1}\|\|\Delta M\|}. \]

We use this inequality with \( M = F(u)^{-1} = \hat{S}(1) + \beta I \) and \( \Delta M = \hat{S}(d_v \circ \epsilon) \), so that \( F(v) - F(u) = (M + \Delta M)^{-1} - M^{-1} \).

We first verify that the condition \( \|M^{-1}\|\|\Delta M\| < 1 \) holds. Combining the non-negativity of the elements of \( \epsilon \), the fact that for any \( P, Q \in S_+^p \), \( \|P - Q\| \leq \|P + Q\| \) and Lemma 4, we conclude that with probability \( \geq 1 - \exp(-c\beta) \),
\[ \|\Delta M\| = \|\hat{S}(d_v \circ \epsilon)\| \leq \|\hat{S}(\epsilon)\| \leq \|\hat{S}(1)\| \cdot \|\epsilon\|_\infty \leq s_{\text{max}}(1 + 2\sqrt{\gamma})^2 \|\epsilon\|_\infty. \]

Next, by definition \( \|M^{-1}\| = \|F(u)\| \leq 1/\beta \). Thus, \( \|M^{-1}\|\|\Delta M\| \leq \frac{1}{\beta} s_{\text{max}}(1 + 2\sqrt{\gamma})^2 \|\epsilon\|_\infty. \)

So, there is a constant \( c' \leq 1 \) such that with probability \( 1 - \exp(-c\beta) \), for all \( \|v - u\|_\infty \leq c' \), \( \|M^{-1}\|\|\Delta M\| < 1 \). Eq. (53) and the definitions of \( M \) and \( \Delta M \) imply the desired bound. \( \square \)

**A.9. Proof of Lemma 11**

Recall that \( \nabla g(u) = I - B \), with \( B \) given in Eq. (51). Since \( 1 + \alpha > \sup_n p/n \), then \( \text{diag}(B) = \kappa I \), with \( \kappa = p/(n(1 + \alpha)) \in (0, 1) \). Therefore, \( \nabla g(u) = (1 - \kappa) I - B_0 \), where \( \text{diag}(B_0) = 0 \), and
\[ \| (\nabla g(u))^{-1} \|_\infty = \left\| \sum_{k=0}^{\infty} \left( \frac{1}{1 - \kappa} \right)^k B_0^k \right\|_\infty \leq \sum_{k=0}^{\infty} \left( \frac{1}{1 - \kappa} \right)^{k+1} {\|B_0\|_\infty}^k, \]
Suppose that for some fixed $\lambda \in (0, 1)$
\[
\Pr (\|B_0\|_\infty > \lambda (1 - \kappa)) \leq Cpe^{-\kappa},
\] (54)
then the lemma follows, since with probability at least $1 - Cpe^{-\kappa}$
\[
\| (\nabla g(u))^{-1} \|_\infty \leq \sum_{k=0}^{\infty} \left( \frac{1}{1 - \kappa} \right)^k (1 - \kappa)^k \lambda^k = \frac{1}{(1 - \lambda)(1 - \kappa)}.
\]

It suffices to prove Eq. (54). To this end, from Eq. (51), with $F = F(u)$ and
\[
\hat{S}_{-i} = \frac{1}{n} \sum_{j \neq i} x_j x_j^T,
\]
\[
\sum_{j=1}^{n} (B_0)_{ij} = \frac{1}{1 + \alpha} \frac{p}{n} \sum_{j \neq i} \left( \frac{x_i^T F x_j}{(x_i^T F x_i)^2} \right) = \frac{1}{1 + \alpha} \frac{p}{n} \frac{x_i^T F \left( \sum_{j \neq i} x_j x_j^T \right) F x_i}{(x_i^T F x_i)^2} = \frac{p}{1 + \alpha} \frac{A_1}{A_2}.
\] (55)

Recall that $F = F(u) = (\hat{S} + \beta I)^{-1}$ and denote $F_{-i} = (\hat{S}_{-i} + \beta I)^{-1}$. By the Sherman-Morrison formula, the numerator $A_1$ may be rewritten as
\[
A_1 = x_i^T \left( F_{-i} - \frac{F_{-i} x_i x_i^T F_{-i}}{n + \frac{p}{n} x_i^T F_{-i} x_i} \right) \hat{S}_{-i} \left( F_{-i} - \frac{F_{-i} x_i x_i^T F_{-i}}{n + \frac{p}{n} x_i^T F_{-i} x_i} \right) x_i
\]
\[
= x_i^T F_{-i} \hat{S}_{-i} F_{-i} x_i - \frac{2 \left( x_i^T F_{-i} x_i \right) \left( x_i^T F_{-i} \hat{S}_{-i} F_{-i} x_i \right)}{1 + \frac{p}{n} x_i^T F_{-i} x_i} + \frac{1}{n^2} \left( \frac{1}{n + \frac{p}{n} x_i^T F_{-i} x_i} \right)^2 \left( x_i^T F_{-i} x_i \right)^2 \left( x_i^T F_{-i} \hat{S}_{-i} F_{-i} x_i \right).
\]

Next, recall that by Eq. (41), with $x_i = S_{1/2} x_i$, it follows that $Q_i = \frac{1}{p} x_i^T F_{-i} x_i$. With $R = \frac{1}{p} x_i^T F_{-i} \hat{S}_{-i} F_{-i} x_i$, the term $A_1$ can be simplified to
\[
A_1 = pR \left( 1 - \frac{2}{n + \frac{p}{n} Q_i} + \frac{1}{n^2} \left( \frac{pQ_i}{1 + \frac{p}{n} Q_i} \right)^2 \right).
\]

Similarly, by Eq. (42), $A_2 = \left( x_i^T F x_i \right)^2 = (\xi_i^T E \xi_i)^2 = \left( \frac{pQ_i}{Q_i + pQ_i} \right)^2$. Thus,
\[
\frac{p}{1 + \alpha} A_1 = \frac{(1 + \frac{p}{n} Q_i)^2 p^2 R \left( 1 - \frac{2}{n + \frac{p}{n} Q_i} + \frac{1}{n} \left( \frac{1}{1 + \frac{p}{n} Q_i} \right)^2 p^2 Q_i^2 \right)}{(1 + \alpha)p^2 Q_i^2}
\]
\[
= \frac{R}{Q_i} \frac{1}{1 + \alpha} \left( \left( 1 + \frac{p}{n} Q_i \right)^2 - \frac{2}{n} pQ_i \left( 1 + \frac{p}{n} Q_i \right) + \frac{p^2 Q_i^2}{n^2} \right)
\]
\[
= \frac{R}{Q_i^2} \frac{1}{1 + \alpha} \left( \left( 1 + \frac{p}{n} Q_i \right) - \frac{p}{n} Q_i \right)^2 = \frac{R}{Q_i^2} \frac{1}{1 + \alpha}.
\] (56)
Eqs. (55) and (56) give that \( \sum_{j=1}^{n}(B_0)_{ij} = \frac{R}{Q_i^{1+\alpha}_i} \). Taking a union bound,
\[
\Pr \left( \|B_0\|_\infty > \lambda(1 - \kappa) \right) \leq p \Pr \left( \frac{1}{1 + \alpha} \frac{R}{Q_i^{1+\alpha}_i} > \lambda(1 - \kappa) \right).
\]

(57)

To estimate the RHS of Eq. (57) we first show that \( R/Q_i < 1 \). Let \( UDU^T \), with \( D = \text{diag}(d_1, \ldots, d_p) \) be the eigendecomposition of \( \hat{S}_- \). Then
\[
\frac{R}{Q_i} = \frac{x_i^T F_{-i} \hat{S}_{-i} F_{-i} x_i}{x_i^T F_{-i} x_i} = \frac{x_i^T (UDU^T + \beta I)^{-1}U DU^T (UDU^T + \beta I)^{-1} x_i}{x_i^T (UDU^T + \beta I)^{-1} x_i}
\]
\[
= \frac{(U^T x_i)^T (D + \beta I)^{-1} D (D + \beta I)^{-1} (U^T x_i)}{(U^T x_i)^T (D + \beta I)^{-1} (U^T x_i)}
\]
\[
= \frac{(U^T x_i)^T \text{diag} \left( \frac{d_i}{(d_i + \beta)} \right) (U^T x_i)}{(U^T x_i)^T \text{diag} \left( \frac{1}{d_i + \beta} \right) (U^T x_i)} \leq \frac{d_1}{d_1 + \beta},
\]

where \( d_1 = \|S_-\| \). By Lemma 4, with high probability \( d_1 < s_{\text{max}}(1 + 2\sqrt{n})^2 \).

Hence, there exists a \( \delta > 0 \), so that w.h.p. \( R/Q_i < 1/(1 + \delta) \). Let \( \lambda = \left( \frac{1}{1 + \delta} \right)^2 < 1 \), then by Eq. (47),
\[
\Pr \left( \frac{R}{(1 + \alpha)Q_i^{1+\alpha}_i} > \lambda(1 - \kappa) \right) \leq \Pr \left( Q_i(1 + \alpha) < \frac{1 + \delta}{1 - \kappa} \right)
\]
\[
\leq \Pr \left( Q_i(1 + \alpha) - \frac{1}{1 - \kappa} < \frac{\delta}{1 - \kappa} \right) \leq Ce^{-cp}
\]

Combining the above with Eq. (57) implies that Eq. (54) holds, as desired.

**A.10. Proof of Lemma 13**

By the triangle inequality,
\[
\left\| \frac{pA}{\text{tr}(A)} - B \right\| \leq \|A - B\| + \|A\| \left| \frac{1 - \text{tr}(A)/p}{\text{tr}(A)/p} \right|.
\]

Next, observe that \( \|A\| \leq \|B\| + \|A - B\| \leq b_{\text{max}} + 1/2 \), and since \( \text{tr}(B) = p \)
\[
\left| 1 - \text{tr}(A)/p \right| = \frac{1}{p} |\text{tr}(B) - \text{tr}(A)| = \frac{1}{p} |\text{tr}(B - A)| \leq \|A - B\|.
\]

Hence, \( \frac{1 - \text{tr}(A)/p}{\text{tr}(A)/p} \leq \frac{1}{1 - \|A - B\|} \leq 2 \). Combining these proves the lemma.

**A.11. TME with outliers**

Consider an \( \epsilon \)-contamination model, where \((1 - \epsilon)n\) of the data come from an elliptical distribution with shape matrix \( S_{in} \), and the remaining \( \epsilon n \) from an
elliptical distribution with matrix $S_{\text{out}}$. We conjecture that under suitable assumptions, for $p, n \gg 1$, the weights of the TME concentrate around two values, $w_{\text{in}}$ and $w_{\text{out}}$, for the inliers and outliers, respectively.

For our procedure to select the inliers, we further assume that the inlier weights are approximately Gaussian distributed around $w_{\text{in}}$ with an unknown standard deviation $\sigma_{\text{in}}$. To estimate $w_{\text{in}}$ and $\sigma_{\text{in}}$ we compute a non-parametric density estimate $\hat{f}(w)$ of all $n$ weights (using MATLAB’s ksdensity procedure). Then $w_{\text{in}} = \arg \max_w \hat{f}(w)$ is the weight with highest estimated density. Next, for some $r$ we find the largest interval $[w_L, w_R]$ around $w_{\text{in}}$ so that $\hat{f}(w) \geq r \max \hat{f}(w_{\text{in}})$. Then, given our assumption that the weights are Gaussian distributed, $\sigma_{\text{in}} = \frac{1}{2} (w_R - w_L) / \sqrt{-2 \log(r)}$. In our simulations we used $r = 0.7$.

Of course, one might obtain improved estimates of these quantities, as well as the unknown $\epsilon$, for example by fitting a mixture of two Gaussians to the vector of weights. However, for our illustrative example, we opted for the above simpler procedure.

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