Non binary LDPC codes over the binary erasure channel: density evolution analysis

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Abstract—In this paper we present a thorough analysis of non binary LDPC codes over the binary erasure channel. First, the decoding of non binary LDPC codes is investigated. The proposed algorithm performs “on-the fly” decoding, i.e. it starts decoding as soon as the first symbols are received, which generalizes the erasure decoding of binary LDPC codes. Next, we evaluate the asymptotical performance of ensembles of non binary LDPC codes, by using the density evolution method. Density evolution equations are derived by taking into consideration both the irregularity of the bipartite graph and the probability distribution of the graph edge labels. Finally, infinite-length performance of some ensembles of non binary LDPC codes for different edge label distributions are shown.

I. INTRODUCTION

Data loss recovery – for instance, for content distribution applications or for distributed storage systems – is widely addressed using FEC (Forward Error Correction) techniques based on error correcting codes. These codes are dealing with erasure channels, i.e. a channel that either transmits the data unit correctly (without error) or erases it completely. In the case of content distribution applications, the potential physical layer CRC, or physical layer FEC codes, or transport level UDP checksums, may lead a receiver to discard erroneous data units. For distributed storage systems, data loss may be due to broken servers, Denial-of-Service (DoS) attacks, etc.

The performance of error correcting codes over erasure channels can be analyzed precisely, and a flurry of research papers have already addressed this issue. Low-density parity-check (LDPC) codes [1], [2] with iterative decoding [3] proved good performance, and as a result, codes with this structure are being considered for a large number of applications and standards. Over the past few years there has been an increased interest in non binary LDPC codes due to their enhanced correction capacity, but at this time only few works are dealing with the BEC [12],[13]. In this paper we give a thorough analysis of non binary LDPC codes over the BEC. The paper is organized as follows: in Section III we review some background on the construction of non binary LDPC codes. The decoding of non binary LDPC codes over the BEC is addressed in Section III. In Section IV we derive the density evolution equations taking into consideration both the irregularity of the bipartite graph and the probability distribution of the graph edge labels. Thresholds of some ensembles of non binary LDPC codes for different edge label distributions are shown in Section V.

II. NON BINARY LDPC CODES

We denote by \( F_q \) the Galois field with \( q \) elements. For practical reasons, we will assume that \( q \) is a power of 2, even if this condition is not always necessary. Thus, we set \( q = 2^p \), where \( p \) is the vector space dimension of \( F_q \) over \( F_2 \) (each time we refer to \( F_q \) as a vector space, we consider its \( F_2 \)-vector space structure). We fix once for all an isomorphism of \( F_2 \)-vector spaces:

\[ F_q^n \cong F_q \]

(1)

and we say that \( (b_0, \ldots, b_{p-1}) \in F_q^p \) are the constituent bits of the symbol \( s \in F_q \), if they correspond to each other by the above isomorphism.

Let \( \mathbb{L} \) be a multiplicative group acting on the vector space \( F_q \). For instance, we may have:

- \( \mathbb{L} = \mathbb{F}_q^* \), acting on \( F_q \) via the internal field multiplication;
- \( \mathbb{L} = \mathbb{M}_p(F_2) \), the multiplicative group of invertible \( p \times p \) matrices, acting on \( F_q \) via the isomorphism \( F_q^p \cong F_q \) from (1).

The action of \( \mathbb{L} \) on \( F_q \) will always be denoted multiplicatively, that is:

\[ \mathbb{L} \times F_q \rightarrow F_q : (h, s) \mapsto hs \]

(2)

For any matrix \( H \in \mathbb{M}_{M,N}(\mathbb{L}) \) one can define a code:

\[ C = \ker(H) \]

(3)

\[ = \{ (s_1, \ldots, s_N) \mid \sum_{n=1}^{N} h_{m,n}s_n = 0, \forall m = 1, \ldots, M \} \]

If \( \mathbb{L} = F_q^* \) acting on \( F_q \) via the internal field multiplication, then \( C \) is a \( F_q \)-linear code, but this does not happen for general \( \mathbb{L} \).

The Tanner graph associated with the code \( C \), denoted by \( \mathcal{H} \), consists of \( N \) variable nodes and \( M \) check nodes representing the \( N \) columns and the \( M \) lines of the matrix \( H \). A variable node and a check node are connected by an edge if the corresponding element of matrix \( H \) is not zero. Each edge of the graph is labeled by the corresponding non zero element of \( H \). Thus, from now on, we refer to the elements of \( \mathbb{L} \) as labels. We also denote \( \mathcal{H}(\ell) \) the set of check nodes connected...
to a given variable node $n \in \{1, 2, \ldots, N\}$, and by $\mathcal{H}(m)$ the set of variable nodes connected to a given check node $m \in \{1, 2, \ldots, M\}$.

A. The binary image of a non binary code

Any sequence $(s_1, \ldots, s_N) \in \mathbb{F}_q^N$ may be mapped into a binary sequence of length $Np$ via the isomorphism $\phi$. The binary sequences associated with the codewords $(s_1, \ldots, s_N) \in \mathcal{C}$ constitute a linear binary code $\mathcal{C}_{\text{bin}} \subseteq \mathbb{F}_2^{Np}$, which is called the binary image of $\mathcal{C}$. Moreover, the action $\phi$ of the multiplicative label group $L$ on $\mathbb{F}_q$ induces a group morphism from $L$ into the group of vector space endomorphisms $L_{\phi}(\mathbb{F}_q, \mathbb{F}_q)$, and identifying $\mathbb{F}_q$ and $\mathbb{F}_2^p$ via $\phi$, we get a morphism:

$$L \to L_{\phi}(\mathbb{F}_q, \mathbb{F}_q) \cong L_{\phi}(\mathbb{F}_2^p, \mathbb{F}_2^p) = M_p(\mathbb{F}_2)$$

Replacing each coefficient of the matrix $H \in M_{M,N}(L)$ with its image under the above morphism, we obtain a binary matrix $H_{\text{bin}} \in M_{M,p,Np}(\mathbb{F}_2)$, which is simply the parity check matrix of the binary code $\mathcal{C}_{\text{bin}}$. While the encoding may be performed using either the non binary code or its binary image, the iterative decoding of a non binary code on its binary image generally yields very poor performance.

III. Decoding non binary LDPC codes

For general channels, several decoding algorithms for non binary LDPC codes were proposed in the literature [14], [15], [16]. Because of the BEC specificity, these algorithms are all equivalent over the BEC, and they can be described in a slightly different manner, as presented below.

A. Decoding over the BEC

In this section we assume that a non binary LDPC code is used over BEC$(\epsilon)$ – the binary erasure channel with erasure probability $\epsilon$. Thus, the length $N$ sequence of encoded $\mathbb{F}_q$-symbols is mapped into the corresponding binary sequence of length $Np$, which is transmitted over the BEC, each bit from the binary sequence being erased with probability $\epsilon$. We say that a $\mathbb{F}_q$-symbol is:

- received, if all of its constituent bits are received;
- erased, if all of its constituent bits are erased by the channel;
- partially erased, if some of its constituent bits are erased by the channel and some others are received.

At the receiver part, the received bits are used to reconstruct the corresponding $\mathbb{F}_q$-symbols. The reconstruction may be complete, partial, or lacking, according to whenever the corresponding symbol is received, partially erased, or erased.

Let $n$ be a variable node of the Tanner graph and $s \in \mathbb{F}_q$. We say that the symbol $s$ is eligible at the variable node $n$, if the probability of the $n$th transmitted symbol being $s$ is non zero. Tacking into consideration the channel output, the a priori set of eligible symbols, denoted by $\mathcal{E}_n$, consists of the symbols that fit with the received constituent bits (if any) of the $n$th transmitted symbol. Thus:

- $\mathcal{E}_n = \mathbb{F}_q$, if the symbol is erased,
- $\mathcal{E}_n \subseteq \mathbb{F}_q$, if the symbol is partially erased,
- $\mathcal{E}_n \neq \mathbb{F}_q$, if the symbol is received.

These sets constitute the a priori information of the decoder. They are iteratively updated by exchanging extrinsic messages between variable and check nodes in the graph. Each message is a subset of $\mathbb{F}_q$, representing a set of eligible symbols.

Precisely, the message sent by a graph node on an outgoing edge is a set of eligible symbols, which is computed according to messages received by the same node on the incoming edges.

We use the following notation:

- $\mathcal{A}_{m,n}$ the set of eligible symbols sent by the variable node $n$ to the check node $m$;
- $\mathcal{B}_{m,n}$ the set of eligible symbols sent by the check node $m$ to the variable node $n$.

Finally, if $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2 \subseteq \mathbb{F}_q$ and $h \in L$ we define:

$$h.\mathcal{I} = \{hs \mid s \in \mathcal{I}\}$$

$$\mathcal{I}_1 + \mathcal{I}_2 = \{s_1 + s_2 \mid s_1 \in \mathcal{I}_1, s_2 \in \mathcal{I}_2\}$$

The iterative decoder for the BEC can be expressed as follows:

**Initialization step**

- variable-to-check messages initialization $\mathcal{A}_{m,n} = \mathcal{E}_n$

**Iteration step**

- check-to-variable messages $\mathcal{B}_{m,n} = \sum_{n' \in \mathcal{H}(m) \setminus \{n\}} h_{m,n'} \mathcal{A}_{m,n'}$
- variable-to-check messages $\mathcal{A}_{m,n} = \mathcal{E}_n \cap \left( \bigcap_{m' \in \mathcal{H}(n) \setminus \{m\}} h^{-1}_{m',n} \mathcal{B}_{m',n} \right)$
- a posteriori sets of eligible symbols $\overline{\mathcal{E}}_n = \mathcal{E}_n \cap \left( \bigcap_{m \in \mathcal{H}(n)} h^{-1}_{m,n} \mathcal{B}_{m,n} \right)$

The decoder stops when all the a posteriori sets of eligible symbols $\overline{\mathcal{E}}_n$ are of cardinality 1, or when a maximum number of iterations is reached. It is important to note that any set of eligible symbols $(\mathcal{E}_n, \mathcal{A}_{n,m}, \mathcal{B}_{m,n}, \text{ or } \overline{\mathcal{E}}_n)$ is a $\mathbb{F}_2$-affine sub-space of $\mathbb{F}_q$; in particular, its cardinal is a power of 2.

B. Minimum-delay decoding

In this section we propose a minimum-delay decoding algorithm over the BEC, in the sense that the decoding starts since the reception of the first bits, which is suited for Upper-Layer Forward Error Correction (UL-FEC).

The minimum-delay decoding of non binary codes consists of removing symbols from the sets of eligible symbols:

- initialize $\mathcal{E}_n = \mathbb{F}_q$, $n = 1, \ldots, N$
- each time a new bit is received, identify the variable node $n$ of which the received bit is a constituent bit, and then:
  - A($n$): remove symbols from $\mathcal{E}_n$ whose corresponding constituent bit is different from the received bit
B(n): process the check nodes \( m \in \mathcal{H}(n) \), then update the sets of eligible symbols \( \mathcal{E}_{n'} \leftarrow \mathcal{E}_{n}, \) for each \( n' \in \mathcal{H}(m) \setminus \{ n \} \).

C(n): For each of the above \( n' \)’s, if by updating \( \mathcal{E}_{n'} \) its cardinal has been reduced, go to B(n ← n').

The decoder stops when all the sets \( \mathcal{E}_n \) are of cardinality 1.

1) Decoding inefficiency: It follows that the minimum-delay decoding is actually an on-the-fly implementation of the previous iterative decoding. A performance metric that is often associated with on-the-fly decoding is the decoding inefficiency, defined as the ratio between the number of received bits before decoding stops and the number of information bits. Let \( K_{\text{bin}} \) be the binary dimension of the code, and \( K_{\text{received}} \) be the number of received bits before decoding stops. Then the inefficiency \( \mu \) is defined as:

\[
\mu = \frac{K_{\text{received}}}{K_{\text{bin}}} \tag{5}
\]

The expectation of the random variable \( \mu \), denoted by \( \mu_m \), is called average inefficiency. In practice \( \mu_m \) can be estimated by Monte-Carlo simulation.

The average inefficiency of the on-the-fly decoding can be related to the failure probability of the iterative decoding (section III). Precisely, for any \( \epsilon \in [0, 1] \), let \( p(\epsilon) \) be the failure probability of the iterative decoding assuming that \( \epsilon \) is the channel erasure probability. Assuming that the function \( p \) is integrable on [0, 1], we have:

\[
\mu_m - 1 = \int_0^1 p(\epsilon) d\epsilon \tag{6}
\]

IV. DENSITY EVOLUTION

Density evolution for non-binary LDPC codes over the BEC was already derived in [12], assuming an uniform distribution on the edge labels. In \textit{loc. cit.}, the authors suggest that the distribution of the edge labels represents a degree of freedom that should be integrated to our understanding of capacity approaching iterative coding schemes. To do so, we derive the density evolution of non-binary codes taking into consideration the variable and check nodes degree distributions, but also the probability distribution of the edge labels. We use the following notation:

- \( \lambda_d \) is the fraction of edges connected to variable nodes of degree \( d \), \( \lambda(X) = \sum_{d=1}^{d_v} \lambda_d X^{d-1} \) is the polynomial of variable node degree distribution;
- \( \rho_d \) is the fraction of edges connected to check nodes of degree \( d \), \( \rho(X) = \sum_{d=1}^{d_c} \rho_d X^{d-1} \) is the polynomial of check node degree distribution;
- \( f: \mathbb{L} \rightarrow [0, 1] \) the probability distribution function defined by \( f(h) = \) fraction of edges with label \( h \in \mathbb{L}. \)

By extending the notation, for a given sequence \( h = (h_1, \ldots, h_I) \) we define \( f(h) = \prod_{i=1}^{I} f(h_i). \)

Without losing generality, we may assume that the all-zero codeword is transmitted. Thus, any set of eligible symbols \( (\mathcal{E}_{n}, \mathcal{S}_{m,n}, \mathcal{P}_{m,n}, \text{or } \mathcal{E}_{n}) \) is a \( \mathbb{F}_2 \)-linear subspaces of \( \mathbb{F}_q \).

Table I gives the list of the possible values of the a priori sets of eligible symbols \( \mathcal{E}_n \) for the case of a \( \mathbb{F}_8 \)-code, according to the received binary sequence.

| received bits \( \mathcal{E}_n \) | \( \mathcal{E}_{n} \) | \( \text{Pr}(\mathcal{E}_n) \) |
|---|---|---|
| XXX | \( \mathbb{F}_8 \) | \( \epsilon^3 \) |
| x0x | \( \{0, 1, 2, 3\} \) | \( \epsilon^4(1-\epsilon) \) |
| x00 | \( \{0, 1, 4, 5\} \) | \( \epsilon^4(1-\epsilon) \) |
| xx0 | \( \{0, 2, 4, 6\} \) | \( \epsilon^4(1-\epsilon) \) |
| x00 | \( \{0, 3\} \) | \( \epsilon(1-\epsilon)^2 \) |
| 0x0 | \( \{0, 2\} \) | \( \epsilon(1-\epsilon)^2 \) |
| 00x | \( \{0, 1\} \) | \( \epsilon(1-\epsilon)^2 \) |
| 000 | \( \{0\} \) | \( (1-\epsilon)^2 \) |

* Symbol x denotes an erased bit

Let \( \text{Gr}(\mathbb{F}_q) \) be the Grassmannian of \( \mathbb{F}_q \), that is the set of all \( \mathbb{F}_2 \)-linear subspaces of \( \mathbb{F}_q \). For \( V \in \text{Gr}(\mathbb{F}_q) \), we note:

\[
P_I(V) = \text{Pr}(\mathcal{S}^{(I)}_V = V) \tag{7}
\]

\[
Q_I(V) = \text{Pr}(\mathcal{P}^{(I)}_V = V) \tag{8}
\]

where superscript \( (I) \) is used to denote sets of eligible symbols computed at the \( I \)th iteration. Thus, the decoding is successfully if and only if:

\[
\lim_{\ell \to +\infty} P_I(\{0\}) = 1 \tag{9}
\]

In order to simplify the notation, we define:

- For any \( V \in \text{Gr}(\mathbb{F}_q) \):

\[
\gamma(V) := P_0(V) = \text{Pr}(\mathcal{E}_n = V)
\]

\[
\mathcal{S}^{(I)}_V := \{ V = (V_1, \ldots, V_I) \mid \bigcap_{i=1}^{I} V_i = V \} \subseteq \text{Gr}(\mathbb{F}_q)^I
\]

\[
\mathcal{P}^{(I)}_V := \{ (V_0, V) = (V_0, V_1, \ldots, V_I) \mid \bigcap_{i=0}^{I} V_i = V \} \subseteq \text{Gr}(\mathbb{F}_q)^{I+1}
\]

- For any \( h = (h_1, \ldots, h_I) \in \mathbb{L}^I \) and \( V = (V_1, \ldots, V_I) \in \text{Gr}(\mathbb{F}_q)^I \):

\[
h^{-1} := (h_1^{-1}, \ldots, h_I^{-1}), \quad h \cdot V := (h_1 V_1, \ldots, h_I V_I)
\]

- For any \( V = (V_1, \ldots, V_I) \in \text{Gr}(\mathbb{F}_q)^I \):

\[
P_I(V) := \prod_{i=1}^{I} P_1(V_i), \quad Q_I(V) := \prod_{i=1}^{I} Q_1(V_i)
\]

Let \((m, n)\) be an edge of the Tanner graph. Assume that \( \mathcal{H}(m) = \{ n, n_1, \ldots, n_d \} \), where \( d \) is the degree of the check node \( m \). To simplify the notation, we set \( h_i = h_{m,n_i} \), the non zero label of the edge \((m, n_i)\), for \( i = 1, \ldots, d - 1 \).

1Here we identify \( \mathbb{F}_8 = \{0, 1, 2, \ldots, 7\} \), and the constituent bits of a given symbol correspond to the binary decomposition.
The probability of $\mathcal{E}_{m,n}^{(t+1)}$ being equal to $V$, conditioned on $h = (h_1, \ldots, h_{d-1})$, may be computed as:

$$\Pr(\mathcal{E}_{m,n}^{(t+1)} = V \mid h) = \sum_{V \in \mathcal{S}_V^{(d-1)}} \left( \prod_{i=1}^{d-1} P_t(h_i^{-1} V_i) \right)$$

(10)

Averaging over all possible label sequences $h$ we get:

$$Q_{t+1}^{(d-1)}(V) := \Pr(\mathcal{E}_{m,n}^{(t+1)} = V) = \sum_{h \in \mathbb{L}^{d-1}} \left( f(h) \cdot \sum_{V \in \mathcal{S}_V^{(d-1)}} \gamma(V_0) \prod_{i=1}^{d-1} Q_{t+1}(h_i V_i) \right)$$

(11)

Averaging over all possible check node degrees $d$, we obtain:

$$Q_{t+1}(V) = \sum_{d=1}^{d_c} \rho_d \cdot Q_{t+1}^{(d-1)}(V)$$

(12)

Now, consider an edge $(n,m)$ of the Tanner graph, and let the variable node $n$ be of degree $d$ and $H(n) = \{m, m_1, \ldots, m_{d-1}\}$. To simplify notation, we set $h_i = h_{mi}$, the non zero label of the edge $(n,m_i)$ for $i = 1, \ldots, d-1$. The probability of $\mathcal{E}_{m,n}^{(t+1)}$ being equal to $V$, conditioned on $h = (h_1, \ldots, h_{d-1})$, may be computed as:

$$\Pr(\mathcal{E}_{m,n}^{(t+1)} = V \mid h) = \sum_{(V_0, V) \in \mathcal{I}_{0,V}^{(d-1)}} \left( \gamma(V_0) \prod_{i=1}^{d-1} Q_{t+1}(h_i V_i) \right)$$

(13)

Again, by averaging over all possible label sequences $h$, it follows that:

$$P_{t+1}^{(d-1)}(V) := \Pr(\mathcal{E}_{m,n}^{(t+1)} = V) = \sum_{h \in \mathbb{L}^{d-1}} \left( f(h) \cdot \sum_{(V_0, V) \in \mathcal{I}_{0,V}^{(d-1)}} \gamma(V_0) Q_{t+1}(h \cdot V) \right)$$

(14)

Finally, averaging over all possible variable node degrees $d$, we obtain:

$$P_{t+1}(V) = \sum_{d=1}^{d_c} \left( \lambda_d \cdot P_{t+1}^{(d-1)}(V) \right)$$

(15)

**Proposition 1:** Let $V, W \in \text{Gr}(\mathbb{F}_q)$ and $h \in \mathbb{L}$ such that $W = hV$. Then:

$$Q_{t+1}^{(d-1)}(W) = \sum_{h \in \mathbb{L}^{d-1}} \left( f(h \cdot h) \cdot \sum_{V \in \mathcal{S}_V^{(d-1)}} P_t(h^{-1} \cdot V) \right)$$

(16)

$$P_{t+1}^{(d-1)}(W) = \sum_{h \in \mathbb{L}^{d-1}} \left( f(h \cdot h) \cdot \sum_{(V_0, V) \in \mathcal{I}_{0,V}^{(d-1)}} \gamma(V_0) Q_{t+1}(h \cdot V) \right)$$

(17)

where $h \cdot (h_1, \ldots, h_{d-1}) = (hh_1, \ldots, hh_{d-1})$. In particular, if $f$ is the uniform distribution, then $Q_{t+1}^{(d-1)}(W) = Q_{t+1}^{(d-1)}(V)$ and $P_{t+1}^{(d-1)}(W) = P_{t+1}^{(d-1)}(V)$.

We say that $V$ and $W$ are conjugate if there exists $h \in \mathbb{L}$ such that $W = hV$ and denote by $\text{Gr}(\mathbb{F}_q)/\mathbb{L}$ the quotient set of conjugation classes.

**Corollary 2:** Assume that $f$ is the uniform distribution and let $V \in \text{Gr}(\mathbb{F}_q)$. Then $Q_t(V)$ and $P_t(V)$ depend only on the conjugation class of $V$ in $\text{Gr}(\mathbb{F}_q)/\mathbb{L}$.

**Corollary 3:** Assume that $f$ is the uniform distribution and that $L = \mathbb{M}_q^p(F_2)$, the multiplicative group of invertible $p \times p$ matrices, acting on $\mathbb{F}_q$ via the isomorphism $F_2^p \cong \mathbb{F}_q$ from [1]. Let $V \in \text{Gr}(\mathbb{F}_q)$. Then $Q_t(V)$ and $P_t(V)$ depend only on the dimension of the vector space $V$.

The above corollaries may be used to simplify the density evolution formulas, assuming a uniform distribution of the graph edge labels. For instance, if $\mathbb{L} = \mathbb{M}_q^p(F_2)$, one can derive the same formulas as in [12].

**V. Thresholds**

We denote by $\mathcal{E}_{\mathbb{L},\mathbb{L}}(\lambda, \rho, f)$ the ensemble of LDPC codes over $\mathbb{F}_q$, with labels group $\mathbb{L}$, distribution degree polynomials $\lambda$ and $\rho$, and probability distribution of edge labels $f$. Whenever the Galois group $\mathbb{F}_q$ and the labels group $\mathbb{L}$ are subunderstood, we will simply use $E(\lambda, \rho, f)$. We also denote by $p_{h\mathcal{E}_{\mathbb{L},\mathbb{L}}}(\lambda, \rho, f)$ (or simply $p_{h\mathcal{E}}(\lambda, \rho, f)$) the threshold probability of the above ensemble, that is (see also [9]):

$$p_{h\mathcal{E}}(\lambda, \rho, f) = \max\{\epsilon \mid \lim_{\ell \to +\infty} P_{\ell}(\{0\}) = 1 \text{ on BEC}(\epsilon)\} \quad (18)$$

By fixing the polynomials of degree distribution $\lambda$ and $\rho$, the probability threshold $p_{h\mathcal{E}}$ may be seen as a function of the probability distribution $f$. This is illustrated in Fig. 1 and Fig. 2. The Galois field is $\mathbb{F}_4$ and the labels group $\mathbb{L} = \mathbb{F}_4^*$, acting on $\mathbb{F}_4$ by the internal field multiplication. The horizontal axes $f(1)$ and $f(2)$ represent the probabilities of edge labels being 1 and 2, respectively. Thus, the probability of edge labels being 3 is given by $f(3) = 1 - f(1) - f(2)$. We draw the surface representing $p_{h\mathcal{E}}$ as function of $f(1)$ and $f(2)$. The top of the surface is plotted in red, the middle in green, and the bottom in blue. The two figures correspond to two couples $(\lambda, \rho)$ of degree distributions that were also considered in [12].

In Fig. 1 we fix $\lambda = X$ and $\rho = X^2$. The maximum $p_{h\mathcal{E}}$ is obtained for the uniform distribution $f(1) = f(2) = f(3) = 1/3$ and its value is equal to 0.5772. The minimum $p_{h\mathcal{E}} = 0.5$ is obtained for the three distributions concentrated in a single label (such codes are equivalent to binary codes). In Fig. 2 we fix $\lambda(X) = X^2$ and $\rho(X) = X^3$. For the uniform distribution $f(1) = f(2) = f(3) = 1/3$, the threshold $p_{h\mathcal{E}} = 0.6348$. The minimum $p_{h\mathcal{E}} = 0.6348$. The maximum $p_{h\mathcal{E}} = 0.6474$ is obtained for the three distributions concentrated in a single label.

These two examples highlight a more general phenomenon that we observed for other ensembles of codes, as shown for instance in Fig. 3. For a given Galois field $\mathbb{F}_q$, and given polynomials $\lambda$, and $\rho$, it is possible to find a probability distribution $\tilde{f}$ of edge labels, such that:

- edge labels are equal to 1 with high probability (meaning that $\tilde{f}(1)$ is close to 1)
- $p_{h\mathcal{E}_q,\tilde{f}}(\lambda, \rho, \tilde{f}) \approx \max_f p_{h\mathcal{E}_q,f}(\lambda, \rho, f)$
performance, but the most important advantage is that the decoder complexity can be significantly reduced. The design of capacity approaching non binary LDPC codes will be addressed in future works.

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