Abstract. — We construct a canonical section in derived categories inducing an isomorphism between the log de Rham cohomology of the fiber at the origin of a proper generalized semistable family over a unit polydisc and the stalk of the relative log de Rham cohomology of it at the origin modulo the maximal ideal of the localization of the structure sheaf at the origin. As an application of the existence of this section, we give a proof of the locally freeness of the higher direct image of the log de Rham complex of a locally nilpotent integrable connection on a proper generalized strict semistable analytic family with a horizontal simple normal crossing divisor over a unit polydisc by a purely algebraic method. As another application of the existence, we construct a canonical isomorphism inducing an isomorphism of log de Rham cohomologies of fibers of a proper generalized semistable family over a polydisc with small radiiuses.

1 Introduction

This article is a sequel of my article [N5]. The aim in this article is to construct an ∞-adic analogue of the canonical Hyodo-Kato section in the p-adic case ([HK], [EY]). As an application, we prove the locally freeness of the higher direct image of the log de Rham complex of a locally nilpotent integrable connection on a proper generalized strict semistable analytic family (a generalization of the notion defined in [F1]) over a unit polydisc by a purely algebraic method. First let us recall Steenbrink’s result.

Let \( \Delta := \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the unit disc and let \( f: \tilde{\mathcal{X}} \rightarrow \tilde{\Delta} \) be a proper (not necessarily strict) semistable family of analytic spaces over \( \mathbb{C} \). Endow \( \tilde{\mathcal{X}} \) and \( \tilde{\Delta} \) with the canonical log structures, respectively. Let \( \mathcal{X} \) and \( \Delta \) be the resulting log analytic spaces, respectively. Let \( s \) be the log point of \( \mathcal{X} \): the log analytic space whose underlying analytic space is the origin \( \{ O \} \) and whose log structure is the pull-back of the log structure of \( \Delta \) by the inclusion \( \{ O \} \rightarrow \tilde{\Delta} \). Let \( X/s \) be the log special fiber of \( \mathcal{X}/\Delta \). Let \( \Omega^*_{\mathcal{X}/\Delta} \) and \( \Omega^*_{X/s} \) be the relative log de Rham complexes of \( \mathcal{X}/\Delta \) and \( X/s \), respectively. Set \( H^q_{\text{dR}}(X/s) = H^q(X, \Omega^*_{X/s}) \) (\( q \in \mathbb{N} \)). Then Steenbrink’s result is the following:

**Theorem 1.1** ([S]). The higher direct image \( R^q f_*(\Omega^*_{\mathcal{X}/\Delta}) \) (\( q \in \mathbb{N} \)) is a locally free coherent \( \mathcal{O}_\Delta \)-module (and commutes with base change of a strict morphism of log analytic spaces).

Set \( \mathcal{X}^* := \mathcal{X} \setminus X \). Especially we obtain \( \Delta^* \). Let \( \tilde{\mathcal{Y}} \rightarrow \Delta^* \) be the universal cover of \( \Delta^* \). Set \( \tilde{\mathcal{X}}^* := \mathcal{X}^* \times_{\Delta^*} \tilde{\mathcal{Y}} \) and \( \mathcal{X}_u := \mathcal{X} \times_{\Delta} u \) for a point \( u \in \Delta^* \). The key point of
the proof of (1.1) is to prove that there exists the following isomorphism

\[(1.1.1) \quad H^q(\overline{X}, \mathbb{C}) \overset{\sim}{\longrightarrow} H^q_{\text{dR}}(X/s)\]

depending on a parameter \(z\) of \(\Delta\). By using this isomorphism, we have an isomorphism

\[(1.1.2) \quad \rho_z : H^q_{\text{dR}}(X/s) \overset{\sim}{\longrightarrow} H^q(X_u, \mathbb{C})\]

depending on a parameter \(z\) of \(\Delta\) for a point \(u \in \Delta^*\) and we obtain (1.1). We call this isomorphism Steenbrink’s isomorphism. Let \(R\Psi(\mathcal{C}) = i^{-1}R\overline{j}_*(\mathcal{C})\) be the nearby cycle sheaf on \(X\). To prove (1.1.1) Steenbrink has proved that there exists the following isomorphisms

\[(1.1.3) \quad R\Psi(\mathcal{C}) \overset{\sim}{\longrightarrow} i^{-1}(\Omega^\bullet_{X/C}[\log z]) \overset{\sim}{\longrightarrow} \Omega^\bullet_{X/s}\]

\([\mathbb{S}]\) (cf. \([\text{SGA 7-II}]\)) depending on a parameter \(z\) of \(\Delta\). Here \(\Omega^\bullet_{X/C}[\log z] := \mathbb{C}[\log z] \otimes \Omega^\bullet_{X/C}\) is considered as a dga of \(\overline{j}_*(\Omega^\bullet_{\overline{X}/C})\). The complex \(\Omega^\bullet_{X/C}[\log z]\) is the Hirsch extension of \(\Omega^\bullet_{X/C}\) by a vector space \(\mathbb{C}\log z\) over \(\mathbb{C}\) of rank 1 with respect to a morphism \(\varphi : \mathbb{C}\log z \ni \log z \mapsto d\log z = z^{-1}dz \in \Omega^1_{X/C} = \Omega^1_{\overline{X}/C}(\log \overline{X})\). The notion of the Hirsch extension by a vector space appears in the definition of the minimal model of a dga in Sullivan’s theory ([\(\mathbb{S}\)]). Conversely, if one knew (1.1) a priori, one could construct a non-canonical isomorphism

\[(1.1.4) \quad H^q_{\text{dR}}(X/s) \overset{\sim}{\longrightarrow} H^q(X_u, \mathbb{C})\]

for a close point \(u \in \Delta^*\) from \(O\) and any point \(u \in \Delta^*\).

A more general theorem than the theorem (1.1) has been proved by L. Illusie, K. Kato and C. Nakayama by using the Kato-Nakayama space defined in [\(\text{KN}\) (\([\text{IKN}]\) see also \([\text{KF}]\), \([\mathbb{U}\]) and \([\mathbb{O}]\)):

**Theorem 1.2** ([\(\text{IKN}\) (6.4)]). Let \(f : X \longrightarrow Y\) be a proper log smooth morphism of \(f\) (fine and saturated) log analytic spaces over \(\mathbb{C}\). Assume that \(Y\) is log smooth or \(f\) is exact. Assume also that \(\text{Coker}(\overline{\mathcal{M}}^\text{gp}_{\overline{f}(x)} \longrightarrow \overline{\mathcal{M}}_{\overline{f}(x)})\) is torsion free for any exact point \(x \in X\). Let \(\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/C}\) be a locally nilpotent integrable connection on \(X\) in the sense of [\(\text{KN}\)]. Let \(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}\) be the log de Rham complex obtained by \((\mathcal{F}, \nabla)\). Assume that \(\overline{\mathcal{M}}_Y := \mathcal{M}_Y/\mathcal{O}_Y^\circ\) is locally isomorphic to the finitely many direct sum of \(\mathbb{N}\). Then \(R^qf_*(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}) (q \in \mathbb{N})\) is a locally free coherent \(\mathcal{O}_Y\)-module and commutes with base change.

Let \(V\) be a locally unipotent local system on \(X^\log\) in the sense of [\(\text{KN}\)] corresponding to \((\mathcal{F}, \nabla)\) by the log Riemann-Hilbert correspondence in [loc. cit.]: \(V = \text{Ker}(\nabla : \mathcal{E}^\log \longrightarrow \mathcal{E}^\log \otimes_{\mathcal{O}^\log_{X,C}} \Omega^\log_{X,C})\), where \(\mathcal{E}^\log := \mathcal{O}^\log_{X,C} \otimes_{\mathcal{O}^\log_{X,C}} \epsilon^{-1}(\mathcal{E})\). Here \(\epsilon : X^\log \longrightarrow X\) is the natural morphism of topological spaces, which has been denoted by \(\tau\) in [loc. cit.]. In [loc. cit.] Illusie, K. Kato and Nakayama have also proved that there exists a canonical isomorphism

\[(1.2.1) \quad \mathcal{O}_{Y^\log} \otimes_{\mathcal{O}_Y} Rf_+^\log(V) \overset{\sim}{\longrightarrow} \mathcal{O}_{Y^\log} \otimes_{\epsilon^{-1}(\mathcal{O}_Y)} \epsilon^{-1}Rf_*(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}).\]

Before [\(\text{IKN}\), F. Kato has already proved that there exists the following canonical isomorphism

\[(1.2.2) \quad \mathcal{O}_{Y^\log} \otimes_{\mathcal{O}_Y} Rf_+^\log(\mathcal{C}) \overset{\sim}{\longrightarrow} \mathcal{O}_{Y^\log} \otimes_{\epsilon^{-1}(\mathcal{O}_Y)} \epsilon^{-1}Rf_*(\Omega^\bullet_{X/Y}).\]
isomorphism $\rho$ constructed a canonical contravariantly functorial isomorphism $\rho$ and the isomorphism "the completed one in general, though they are equal in a certain case. Hirsch extension. They have noticed the big difference of the Hirsch extension and this isomorphism (1.2.7) in this case by using the Hirsch extension and the completed Hyodo-Kato isomorphism this isomorphism the

Here (1.2.5) and Yamada have proved this conjecture in a ingenious method when $X$ is a strictly semistable scheme over $S$ ([EY]). They have proved the existence of the Hyodo-Kato isomorphism (1.2.7) in this case by using the Hirsch extension and the completed Hirsch extension. They have noticed the big difference of the Hirsch extension and the completed one in general, though they are equal in a certain case.
Now let us come back to the case where the base field is $\mathbb{C}$. It is a natural problem to ask whether there exists a canonical isomorphism between $R\Gamma_{\text{dR}}(X/s)$ and $R\Gamma(\mathcal{X}_u, \mathbb{C})$ for a sufficiently close point $u \in \Delta^*$ from $O$. Let $\iota_O: \{O\} \hookrightarrow \Delta$ and $\iota: X \hookrightarrow \mathcal{X}$ be the closed immersions. In this article we construct a canonical contravariantly functorial section

\[(1.2.8)\quad \iota_O R\Gamma_{\text{dR}}(X/s) \to Rf_\ast \iota^\ast e^{-1}(\Omega^\bullet_{X/\Delta})\]

of the natural morphism $Rf_\ast \iota^\ast e^{-1}(\Omega^\bullet_{X/\Delta}) \to \iota_O R\Gamma_{\text{dR}}(X/s)$ as an example of a more general isomorphism without using the topological spaces $\overline{X}$ nor $X^{\log}$. Moreover, if $u \in \Delta^*$ is sufficiently close from $O$, then we prove that there exists a canonical isomorphism

\[(1.2.9)\quad \rho: R\Gamma_{\text{dR}}(X/s) \sim \to R\Gamma(\mathcal{X}_u, \mathbb{C}).\]

Let $M_s$ be the log structure of $s$. The isomorphism $(1.2.9)$ is independent of the choice of a parameter $z$ of $\Delta$ in our sense: we prepare a parameter $z$ of $\Delta$ according to a global section of the log structure of $s$ whose image in $M_s := M_s/\mathcal{O}_s^* = \mathbb{N}$ is the generator. (The choice of $z$ is determined up to $\mathbb{C}^*$ not on $\mathcal{O}_s^*$. The existence of the canonical section $(1.2.8)$ and the canonical isomorphism $(1.2.9)$ have not even been conjectured. The hope in [KU, 0.2.23] that the theory of log geometry in the sense of Fontaine-Illusie-Kato is useful for the case where the base field is of characteristic $0$ as in the case of characteristic $p > 0$ or mixed characteristics comes true in our setting. To construct the section $(1.2.8)$, we do not calculate the nearby cycle sheaf $R\Psi(\mathbb{C})$ (especially we do not use the space $\overline{X}$). We use only the (completed) Hirsch extension of a dga as in [EY] (and [N4]) by believing in a mathematical philosophy that dga’s have informations of geometry (modulo torsion).

Our method is nothing but a log analytic version of Ertl-Yamada’s one in the log rigid analytic case ([EY]). In this article we heavily depend on their idea and we show that their method is also applicable for the log analytic case over $\mathbb{C}$ by paying careful new attentions to the proof of the existence of the canonical isomorphism $(1.2.9)$. Though we do not use any result in [EY] and [N3] in this article, we are psychologically supported by the existence of the Hyodo-Kato section $(1.2.4)$.

We prove the following as a special case of our main result:

**Theorem 1.3.** Let $r_1$ be a positive integer and let $r_2$ be a nonnegative integer. Set $B := \Delta^* \times \Delta^{r_2}$. Let $f: \mathcal{X} \to B$ be a (not necessarily proper) $P$-(not necessarily strictly)-semistable analytic space with horizontal NCD (=normal crossing divisor) $\mathcal{D}$. (See the text for the notion of the $P$-semistable analytic space.) Let $(\mathcal{E}, \nabla)$ be a locally nilpotent integrable connection on $(\mathcal{X}, \mathcal{D})$ in the sense of [KN]: $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_B}(\log \mathcal{D})$. Let $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log \mathcal{D})$ be the log de Rham complex obtained by $(\mathcal{E}, \nabla)$. Let $\{O\}$ be the origin of $B$. Endow $\{O\}$ with the pull-back of the log structure of $B$. Let $s$ be the resulting log analytic space. Set $S := s \times \Delta^{r_2}$, $X := \mathcal{X} \times_B S$ and $D := \mathcal{D} \times_B S$. Let $\iota: X \hookrightarrow \mathcal{X}$ be the natural exact closed immersion. Then there exists a canonical contravariantly functorial section

\[(1.3.1)\quad \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log \mathcal{D}) \to \iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log \mathcal{D}))\]

of the natural morphism

\[(1.3.2)\quad \iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log \mathcal{D})) \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log \mathcal{D})\]
in the derived category $D^+(\iota^{-1}f^{-1}(\mathcal{O}_B))$ of bounded below complexes of $\iota^{-1}f^{-1}(\mathcal{O}_B)$-modules. Consequently there exists a canonical contravariantly functorial section

$$\tag{1.3.3} Rf_*\iota^*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D) \longrightarrow Rf_*\iota^*\iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$$

of the natural morphism

$$\tag{1.3.4} Rf_*\iota^*\iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)) \longrightarrow Rf_*\iota^*(E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$$

in the derived category $D^+(\mathcal{O}_B)$ of the bounded below complexes of $\mathcal{O}_B$-modules.

To prove the theorem (1.3) we use the theory of log geometry essentially and the (completed) Hirsch extension of a dga, more generally the (completed) Hirsch extension of a log integrable connection. As a corollary of (1.4) we prove the following corollary:

**Corollary 1.4** (cf. [4]). Let $\mathcal{Y}$ be a smooth analytic space with an NCD $Y$. Let $\mathcal{Y}$ be the log analytic space obtained by a pair $(\mathcal{Y},Y)$. Let $f: \hat{\mathcal{X}} \longrightarrow \mathcal{Y}$ be a proper morphism of smooth schemes over $K$. Assume that $X := f^*(Y)$ is a normal crossing divisor on $\hat{\mathcal{X}}$. Let $\mathcal{X}$ be the log analytic space obtained by a pair $(\hat{\mathcal{X}},X)$. Let $f: \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper $P$-strictly semistable morphism. Let $D$ be a horizontal NCD on $\mathcal{X}/\mathcal{Y}$. Let $(E,\nabla)$ be a locally nilpotent integrable connection on $(\mathcal{X},D)$ in the sense of [16, 17]. Let $E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)$ be the log de Rham complex obtained by $(E,\nabla)$. Then $R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$ is a coherent locally free $\mathcal{O}_Y$-module and commutes with base change. Here, for a morphism $\hat{\mathcal{Z}} \longrightarrow \mathcal{Y}$, we have endowed $\hat{\mathcal{Z}}$ with the inverse image of the log structure of $\mathcal{Y}$.

Consider the case $\mathcal{Y} = B$. Let $B^*$ be the maximal open log analytic space of $B$ whose log structure is trivial. By (1.4) we see that $R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$ is the canonical extension of $R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$ in the sense of [D]. In the case where $E$ is trivial, this has been already stated in [3] without proof. In order to prove (1.4), we use only log de Rham complexes; we do not use the Kummer étale topos nor $\mathcal{X}^{\log}$ unlike [K] and [IKN]. Hence we can give a purely algebraic proof of (1.4).

Using (1.2.1) and (1.3), we also have the following corollary:

**Corollary 1.5.** Let the notations be as in (1.3). For an open log analytic space $U$ of $\Delta^{r_1}$ including the origin of $\Delta^{r_1}$, set $B_U := U \times \Delta^{r_1}$. Let $\iota_U: X \longrightarrow X^{[1]}(B_U)$ be the natural closed immersion and let $f_U: X^{[1]}(B_U) \longrightarrow B_U$ be the structural morphism. Set $D_U := D_X^{[1]}(B_U)$. Assume that $f: X \longrightarrow B$ is proper. Then there exists an open log analytic space $\mathcal{U}$ of $\Delta^{r_1}$ and there exists a canonical contravariantly functorial section

$$\tag{1.5.1} Rf_*\iota^*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)) \longrightarrow R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)) \quad (q \in \mathbb{N})$$

of the natural morphism

$$\tag{1.5.2} R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)) \longrightarrow R^qf_*\iota^*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$$

This section induces the following isomorphism

$$\tag{1.5.3} \mathcal{O}_B \otimes_{\mathcal{O}_S} R^qf_*\iota^*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D)) \longrightarrow R^qf_*E \otimes_{\mathcal{O}_X} \Omega^*_X/B\log D))$$

at a small open log analytic space $W$ such that $W \cap S \neq \emptyset$. 

5
Consider the case \( r_2 = 0 \). Assume that \( \mathcal{X} \) is proper over \( \mathcal{B} \). Set \( \mathcal{U} := \mathcal{X} \setminus D \).

Let \( u \) be any exact point of \( B \) whose log structure is trivial. As a corollary of the isomorphism (1.5.3), we obtain a canonical contravariantly functorial isomorphism

\[
\DeclareMathOperator{RT}{RT} \RT \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_X/(log D)) \cong \RT \Gamma(U_u, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{U_{d-u}}/(log D)).
\]

if \( u \in \Delta^* \) is close from \( O \). If \( (\mathcal{E}|_{U_u}, \nabla) = (\mathcal{O}_{U_u} \otimes_{\mathcal{O}_X} \Omega^*_{\mathcal{U}_U}, d \otimes \text{id}_V) \) for a local system \( V \) on \( U_u \) of finite dimensional vector spaces over \( \mathbb{C} \), then we obtain a canonical contravariantly functorial isomorphism

\[
\RT \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_X/(log D)) \cong \RT \Gamma(U_u, V).
\]

This is a generalization of the isomorphism (1.2.9). The reader should note that the isomorphism (1.5.6) is independent of the choice of local parameters of \( \Delta^{r_1} \) (up to \((\mathbb{C}^*)^{r_1})\).

We can also obtain the algebraic analogues of (1.3) and (1.4) by algebraic proofs (without using the Lefschetz principle, GAGA nor (1.3)).

As to the locally freeness in (1.4), our method is also applicable for a more general family with worse reduction by virtue of the contravariant functoriality of the section (7.1.3). This general case is not contained in the case (1.2). We would like to discuss this in a future article.

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## 2 Recall of results in \([N5]\)

In [F1] and [F3] Fujisawa has considered the “maximal” logarithmic case for a generalized semistable morphism: the case \( r_2 = 0 \) in the notation (1.3) (without the horizontal log structure). However, even in this case, the “non-maximal” case (the case \( r_2 > 0 \)) arises at points which is different from the “origin”. Hence we consider a general log analytic space as a base log analytic space from the beginning of this section.

In this section we recall results in [N5] which are necessary in this article.

Let \( r \) be a positive integer. Let \( S \) be an analytic family over \( \mathbb{C} \) of log points of virtual dimension \( r \), that is, locally on \( S \), the log structure \( M_S \) of \( S \) is isomorphic to \( \mathbb{N}^r \otimes \mathcal{O}_S^\times \ni (n_1, \ldots, n_r), u \mapsto 0^{\sum_{i=1}^r n_i} u \in \mathcal{O}_S^* \) (cf. [N3 §2]), where \( 0^n = 0 \) when \( n \neq 0 \) and \( 0^0 := 1 \). Let \( g : Y \to S \) be a log smooth morphism of log analytic spaces over \( \mathbb{C} \).

Locally on \( S \), there exists a family \( \{t_i\}_{i=1}^r \) of local sections of \( M_S \) giving a local basis of \( \mathcal{M}_S := M_S/\mathcal{O}_S^* \). Let \( M_i \) be the submonoid sheaf of \( M_S \) generated by \( t_i \). For all \( 1 \leq i \leq r \), the submonoid sheaf \( \bigoplus_{j=1}^i M_j \) of \( M \) and \( \mathcal{S} \) defines a family of log points of virtual dimension \( i \). Let \( S_i := S(M_1, \ldots, M_i) = (\mathcal{S}, (\bigoplus_{j=1}^i M_j \to \mathcal{O}_S)^a) \) be the resulting local log analytic space. Set \( S_0 := \mathcal{S} \). Then we have the following sequence of families of log points of virtual dimensions:

\[
S = S_r \to S_{r-1} \to S_{r-2} \to \cdots \to S_1 \to S_0 = \mathcal{S}.
\]

The one-form \( d\log t_i \in \Omega^1_{S/S} \) is independent of the choice of the generator \( t_i \) of \( M_i \).

Denote also by \( d\log t_i \in \Omega^1_{Y/S} \) the image of \( d\log t_i \in \Omega^1_{S/S} \) in \( \Omega^1_{Y/S} \). Let \( \mathcal{F} \) be a (not
The boundary morphism $\nabla: F \to F \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$ be an integrable connection. Then we have the complex $F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}$ of $g^{-1}(\mathcal{O}_S)$-modules.

**Lemma 2.1 ([N3] (3.1)).** (1) The sheaf $\Omega^i_{Y/S}$ ($i \in \mathbb{N}$) is a locally free $\mathcal{O}_Y$-module.

(2) Locally on $Y$, the following sequence

\[
0 \to F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S_i}[-1] \to F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S_i-1} \to 0 \quad (1 \leq i \leq r).
\]

is exact. (Note that the morphism

\[
\text{id}_F \otimes (d \log t_i \land): F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S_i}[-1] \to F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S_i-1}
\]

is indeed a morphism of complexes of $g^{-1}(\mathcal{O}_S)$-modules.)

Let $\{t_1, \ldots, t_r\}$ be a set of local sections of $M_S$ whose images in $\overline{M}_S$ is a unique system of local generators of $\overline{M}_S$. Set $L_S(t_1, \ldots, t_r) := \bigoplus_{i=1}^r \mathcal{O}_S t_i$ (a free $\mathcal{O}_S$-module with basis $t_1, \ldots, t_r$). This $\mathcal{O}_S$-module patches together to a locally free $\mathcal{O}_S$-module $L_S$ by $[\mathbb{N3}] (3.3)$. In the following we denote $t_i$ in $L_S$ by $u_i$. Because we prefer not to use non-intrinsic variables $u_1, \ldots, u_r$, we consider the following sheaf $U_S$ of sets of $r$-pieces on $\hat{S}$ defined by the following presheaf:

\[
U_S: \{\text{open log analytic spaces of } S\} \ni V \mapsto \{\text{generators of } \Gamma(V, \overline{M}_S)\} \in (\text{Sets}).
\]

We consider the following Hirsch extension

\[
F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}[U_S] := \text{Sym}_{\mathcal{O}_S}(L_S) \otimes_{\mathcal{O}_S} F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}
\]

of $F \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}$ by the morphism

\[
d \log: g^{-1}(L_S) \ni u_i \mapsto d \log t_i \in \text{Ker}(\Omega^1_{Y/S} \to \Omega^2_{Y/S})
\]

([N4] §3). Here $\text{Sym}_{\mathcal{O}_S}(L_S)$ is the symmetric algebra of $L_S$ over $\mathcal{O}_S$. (Here we have omitted to write $g^{-1}$ for $g^{-1}(\text{Sym}_{\mathcal{O}_S}(L_S)) \otimes_{g^{-1}(\mathcal{O}_S)}$ in (2.1.3).) It is easy to check that the morphism $g^{-1}(L_S) \to \text{Ker}(\Omega^1_{Y/S} \to \Omega^2_{Y/S})$ is well-defined (cf. [N3] §2).

The boundary morphism

\[
\nabla: F \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}[U_S] \to F \otimes_{\mathcal{O}_Y} \Omega^{i+1}_{Y/S}[U_S] \quad (i \in \mathbb{Z}_{\geq 0})
\]

is, by definition, the following $\mathcal{O}_S$-linear morphism:

\[
\nabla(m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \omega) = \sum_{j=1}^r m_1^{[e_1]} \cdots m_{j-1}^{[e_{j-1}]} \cdots m_r^{[e_r]} d \log m_j \land \omega + m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \nabla(\omega)
\]

\[
(\omega \in F \otimes_{\mathcal{O}_Y} \Omega^j_{Y/S}, m_1, \ldots, m_r \in L_S, e_1, \ldots, e_r \in \mathbb{Z}_{\geq 1}, m_i^{[e_i]} = ((e_i)!)^{-1} m_i^{e_i}).
\]
We can easily check an equality $\nabla^2 = 0$. Set $\Gamma_{O_S}(L_S) := \text{Sym}_{O_S}(L_S)$. Then we have the following equality

\[(2.1.6)\qquad \Gamma_{O_S}(L_S) = O_S[\overline{M_S}].\]

Let $\epsilon : S^{\log} \to S$ be the real blow up defined in [KN]. Then $\text{Sym}_{O_S}(L_S)$ is nothing but $\epsilon_* (O_{S^{\log}})$. The sheaf $O_S[\overline{M_S}]$ is locally isomorphic to a sheaf $O_S[u_1, \ldots, u_r]$ of polynomial algebras over $O_S$ of independent $r$-variables. The projection $\text{Sym}_{O_S}(L_S) \to \text{Sym}_{O_S}^0(L_S) = O_S$ induces the following natural morphism of complexes:

\[(2.1.7)\quad F \otimes_{O_Y} \Omega^*_{Y/S}[U_S] \to F \otimes_{O_Y} \Omega^*_{Y/S}.\]

(\text{[N4] \S3}). (It may be better to denote $F \otimes_{O_Y} \Omega^*_{Y/S}[U_S]$ by $F \otimes_{O_Y} \Omega^*_{Y/S}[L_S]$.) Let $\text{Sym}_{O_S}^n(L_S) (n \in \mathbb{N})$ be the degree $\geq n$-part of $\text{Sym}_{O_S}(L_S)$. We set

\[(2.1.8)\quad \Gamma_{O_S}(L_S)^\wedge := \lim_{\to n}(\text{Sym}_{O_S}(L_S)/\text{Sym}_{O_S}^n(L_S)).\]

Then we have the following “inclusion morphism”

\[(2.1.9)\quad \Gamma_{O_S}(L_S) \subseteq \Gamma_{O_S}(L_S)^\wedge.\]

We also consider the following PD-Hirsch extension by completed powers:

\[F \otimes_{O_Y} \Omega^*_{Y/S}[[U_S]] := \Gamma_{O_S}(L_S)^\wedge \otimes_{O_S} F \otimes_{O_Y} \Omega^*_{Y/S}\]

of $F \otimes_{O_Y} \Omega^*_{Y/S}$; the boundary morphism

\[(2.1.10)\quad \nabla : F \otimes_{O_Y} \Omega^j_{Y/S}[[U_S]] \to F \otimes_{O_Y} \Omega^{j+1}_{Y/S}[[U_S]] \quad (j \in \mathbb{Z}_{\geq 0})\]

is an $O_S$-linear morphism defined by the following

\[(2.1.11)\quad \nabla(m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \omega) = \sum_{j=1}^r m_1^{[e_1]} \cdots m_j^{[e_j-1]} \cdots m_r^{[e_r]} d \log m_j \wedge \omega + m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \nabla(\omega)\]

as in (2.1.5). We can easily check an equality $\nabla^2 = 0$ for $\nabla$ in (2.1.11). The projection $\text{Sym}_{O_S}(L_S)/\text{Sym}_{O_S}^n(L_S) \to \text{Sym}_{O_S}(L_S)/\text{Sym}_{O_S}^1(L_S) = \text{Sym}_{O_S}^0(L_S) = O_S$ ($n \geq 1$) induces the following natural morphism of complexes:

\[(2.1.12)\quad F \otimes_{O_Y} \Omega^*_{Y/S}[[U_S]] \to F \otimes_{O_Y} \Omega^*_{Y/S}.\]

The morphism (2.1.10) induces the following inclusion morphism

\[(2.1.13)\quad F \otimes_{O_Y} \Omega^*_{Y/S}[[U_S]] \subseteq F \otimes_{O_Y} \Omega^*_{Y/S}[[U_S]].\]
We have the following commutative diagram

\[ \begin{array}{ccc}
\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} [U_S] & \xrightarrow{\cdot c} & \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} [U_S] \\
\downarrow & & \downarrow \\
\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} & = & \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}.
\end{array} \] (2.1.14)

The following is a (simpler) log analytic version of [N3 (3.4)]:

**Definition 2.2** ([N3 (3.4)]). (1) Take a local basis of \( \Omega^1_{Y/S} \) containing \( \{d \log t_1, \ldots, d \log t_r\} \). Let \( (d \log t_i)^*: \Omega^1_{Y/S} \longrightarrow \mathcal{O}_Y \) be the local morphism defined by the local dual basis of \( d \log t_i \). We say that the connection \((\mathcal{F}, \nabla)\) has *no poles along* \( S \) if the composite morphism \( \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_\mathcal{O}_Y \Omega^1_{Y/S} \xrightarrow{\text{id}_\mathcal{F} \otimes (d \log t_i)^*} \mathcal{F} \) vanishes for \( 1 \leq i \leq r \).

(2) If there exists a finite increasing filtration \( \{\mathcal{F}_i\}_{i \in \mathbb{Z}} \) on \( \mathcal{F} \) locally on \( Y \) such that \( \text{gr}_i \mathcal{F} \coloneqq \mathcal{F}_i/\mathcal{F}_{i-1} \) is a locally free \( \mathcal{O}_Y \)-module and \( \nabla \) induces a connection \( \nabla_i: \mathcal{F}_i \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \rightarrow \mathcal{F}_{i+1} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \) whose induced connection \( \text{gr}_i \mathcal{F} \rightarrow (\text{gr}_i \mathcal{F}) \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \) has no poles along \( M_S \), then we say that \((\mathcal{F}, \nabla)\) is a *locally nilpotent integrable connection on* \( Y \) with respect to \( S \).

Let us recall the following result:

**Theorem 2.3** ([N3 (3.5)]). Assume that \((\mathcal{F}, \nabla)\) is a locally nilpotent integrable connection on \( Y \) with respect to \( S \). Then the morphism \((2.1.4)\) is a quasi-isomorphism.
of sheaves of commutative rings of unit elements on $S'$. This morphism satisfies the usual transitive relation

$$\circ (\circ v')^* = (v'^* \circ ')^*.$$  

(2.3.4)

Let $g': Y' \to S'$ be an analogous morphism of log analytic spaces to $g: Y \to S$. Assume that we are given a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{h} & Y' \\
g \downarrow & & \downarrow g' \\
S & \xrightarrow{v} & S'.
\end{array}
$$

(2.3.5)

**Proposition 2.4** (Functoriality). Assume that we are given a commutative diagram

$$
\begin{array}{ccc}
F' \xrightarrow{\nabla'} & F' \otimes_{\mathcal{O}_Y} \Omega^1_{Y'/S'} \\
\downarrow & & \downarrow \\
h_*(F) \xrightarrow{h_*(\nabla)} & h_*(F \otimes_{\mathcal{O}_X} \Omega^1_{Y/S}).
\end{array}
$$

(2.4.1)

where $(F', \nabla')$ is an analogous connection to $(F, \nabla)$ on $Y'$. Then the morphism (2.1.7) is contravariantly functorial for the commutative diagrams (2.3.5) and (2.4.1). This contravariance satisfies the usual transitive relation $\circ (\circ h)^* = h^* \circ i^*$. 

**Proof.** Obvious.

3 Results on completed Hirsch extensions

In this section we give an easy result (3.2) on completed Hirsch extensions. Using this result and the result (2.3) in the previous section, we see that the morphism (2.1.13) is a quasi-isomorphism. Though we do not use this result to obtain the main result in this article, we use the technique of the proof for this result in the next section, which is necessary for the main result. Hence, even if one wants to know the proof of the main result in this article, one cannot skip this section.

**Lemma 3.1.** Let the notations be as in (2.1). Let $\{t_1, \ldots, t_r\}$ be a set of local sections of $M_S$ whose images in $\overline{M}_S$ is a unique system of local generators of $\overline{M}_S$. Denote $t_i$ in $L_S$ by $u_i$. For $1 \leq i \leq r$, let us consider the completed Hirsch extensions $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}[[u_{i+1}, \ldots, u_r]]$ and $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}[[u_{i+1}, \ldots, u_r]]$ of $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}$ and $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}$ by the morphisms $\bigoplus_{j=i+1}^r \mathcal{O}_S u_j \ni u_j \mapsto d \log t_j \in \Omega^1_{Y/S_i}$ and $\bigoplus_{j=i+1}^r \mathcal{O}_S u_j \ni u_j \mapsto d \log t_j \in \Omega^1_{Y/S_i}$, respectively. Then the following sequence

$$
\begin{array}{c}
0 \to \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}[[u_{i+1}, \ldots, u_r]][-1] \xrightarrow{\text{Id}_{\mathcal{F}} \otimes (d \log t_i)} \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}[[u_{i+1}, \ldots, u_r]] \\
\to \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i}[[u_{i+1}, \ldots, u_r]] \to 0 \quad (1 \leq i \leq r)
\end{array}
$$

(3.1.1)

is exact.

**Proof.** This immediately follows from (2.1) by considering “coefficient forms” of each $u_{i+1} \cdots u_r^e_i (e_{i+1}, \ldots, e_r \in \mathbb{N})$. 

\[\square\]
**Proposition 3.2.** The morphism (2.1.12) is a quasi-isomorphism.

**Proof.** The problem is local. We may assume that there exists a sequence (2.0.1) and that $L_S = \oplus_{i=1}^r O_S u_i$. The complex $F \otimes_{O_Y} \Omega^*_{Y/S_{r-1}}[[u_r]]$ is, by definition, the following complex:

\[(3.2.1) \quad \ldots \longrightarrow \quad \ldots \longrightarrow \quad \ldots \]

\[
\begin{array}{c}
\text{id} \otimes \nabla \downarrow \\
O_S u_r^2 \otimes_{O_S} F \xrightarrow{(g) \log t_r \wedge} O_S u_r \otimes_{O_S} F \otimes_{O_Y} \Omega^1_{Y/S_{r-1}} \xrightarrow{(g) \log t_r \wedge} \ldots \longrightarrow \quad \ldots \longrightarrow \quad \ldots \\
\text{id} \otimes \nabla \downarrow \\
\end{array}
\]

where the horizontal arrow $(g) \log t_r \wedge$ is defined by $u_r^{[i]} \otimes f \otimes \omega \mapsto u_r^{[i-1]} \otimes f \otimes (d \log t_r \wedge \omega)$ ($f \in F, \omega \in \Omega^*_{Y/S_{r-1}}$). This is augmented to the complex $F \otimes_{O_Y} \Omega^*_{Y/S}$.

By (2.1.1) we see that this augmentation

\[(3.2.2) \quad F \otimes_{O_Y} \Omega^*_{Y/S_{r-1}}[[u_r]] \longrightarrow F \otimes_{O_Y} \Omega^*_{Y/S}
\]

is an isomorphism. The complex $(F \otimes_{O_Y} \Omega^*_{Y/S_{r-2}}[[u_r, u_{r-1}]] = (F \otimes_{O_Y} \Omega^*_{Y/S_{r-2}}[[u_r]])[[u_{r-1}]]$ is, by definition, the following complex:

\[(3.2.3) \quad \ldots \longrightarrow \quad \ldots \longrightarrow \quad \ldots \]

\[
\begin{array}{c}
\text{id} \otimes \nabla \downarrow \\
O_S u_{r-1}^2 \otimes_{O_S} F[[u_r]] \xrightarrow{(g) \log t_{r-1} \wedge} O_S u_{r-1} \otimes_{O_S} F \otimes_{O_Y} \Omega^1_{Y/S_{r-2}}[[u_r]] \xrightarrow{(g) \log t_{r-1} \wedge} \ldots \longrightarrow \quad \ldots \longrightarrow \quad \ldots \\
\text{id} \otimes \nabla \downarrow \\
\end{array}
\]

The augmentation

\[(3.2.4) \quad (F \otimes_{O_Y} \Omega^*_{Y/S_{r-2}}[[u_r]])[[u_{r-1}]] \longrightarrow F \otimes_{O_Y} \Omega^*_{Y/S_{r-1}}[[u_r]]
\]

is an isomorphism by (3.1). Continuing this argument repeatedly, we see that the morphism (2.1.12) is a quasi-isomorphism.

Though we do not use the following in this article, this is of independent interest as in [PY (3.17) (4)]:

**Corollary 3.3.** The morphism (2.1.13) is a quasi-isomorphism.

**Proof.** This immediately follows from (2.1.14), (2.3) and (3.2).
Proposition 3.4 (Functoriality). Let the notations be as in (2.4). Then the morphism (2.4.1) is contravariantly functorial for the commutative diagram (2.3.5) and (2.4.1). This contravariance satisfies the usual transitive relation \((i \circ h)^* = h^* \circ i^*\).

Proof. Obvious. \qed

4 Log Analytic families

In this section we consider a “thicker” family than the family in the previous two sections.

Let the notations be as in the previous section. Let \( r \) be a positive integer. Set \( \Delta^r_S := \Delta^r \times \hat{S} \). The zero section \( \circ S \subset \Delta^r_S \) defines an fs log structure on \( \Delta^r_S \). Let \( \Delta^r_S \) be the resulting log analytic space and let \( \Delta^r_S \) be the \( r \)-times product of \( \Delta_S \) over \( \circ S \).

Locally on \( S \), we have the following exact closed immersion

\[
S \hookrightarrow \Delta^r_S
\]

of log analytic spaces. This immersion fits into the following commutative diagram

\[
\begin{array}{c}
S \\
\downarrow \\
\circ S
\end{array} \quad \begin{array}{c}
\Delta^r_S \\
\downarrow \\
\hat{S}
\end{array}
\]

where the vertical morphisms are natural morphisms.

By replacing \( k^r_S \) in [N3, (1.1)] by \( \Delta^r_S \), we have a well-defined log analytic space \( \mathcal{S} \) over \( \hat{S} \) with an exact closed immersion \( S \hookrightarrow \mathcal{S} \) fitting into the following commutative diagram over \( \hat{S} \):

\[
\begin{array}{c}
S \\
\downarrow \\
\circ S
\end{array} \quad \begin{array}{c}
\mathcal{S} \\
\downarrow p
\end{array}
\]

(4.0.1)

Here \( p: \mathcal{S} \to \hat{S} \) is the structural morphism. Locally on \( S \), \( \mathcal{S} \) is isomorphic to a unit polydisc \( \Delta^r_S \) over \( \hat{S} \). It is important to note that, by the construction of \( \mathcal{S} \), we obtain well-defined closed 1-forms \( d\log t_1, \ldots, d\log t_r \in \Omega^1_{\mathcal{S}/\hat{S}} \) for local sections \( t_1, \ldots, t_r \) of \( \mathcal{M}_S \) in [2] we do not permit to take other differential forms \( d\log(v_1t_1), \ldots, d\log(v_rt_r) \) for \( v_1, \ldots, v_r \in \mathcal{O}_\hat{S} \) we permit to take only \( d\log(v_it_i) = d\log t_i \in \Omega^1_{\mathcal{S}/\hat{S}} \) for \( v_i \in \mathcal{O}_\hat{S} \). This convention is similar to the convention in [N3] in which to construct the isomorphism \( (1.2.6) \), we use the prime number \( p \in \mathcal{V} \) (note that \( p \) is not necessarily a uniformizer of \( \mathcal{V} ! \)) and the isomorphism \( \rho_p \) in [N3, (6.3.14)].

Remark 4.1. Let \( \epsilon > 0 \) be a real number. Set \( \Delta(\epsilon) := \{ z \in \mathbb{C} \mid |z| < \epsilon \} \). Replacing \( \hat{S} \) by \( \Delta(\epsilon) \), we obtain a log analytic space \( \mathcal{S}(\epsilon) \). If \( S \) is quasi-compact and if \( \epsilon \) is small enough, then we have a natural immersion

\[
\mathcal{S}(\epsilon) \hookrightarrow \mathcal{S}.
\]
In this section we assume that the morphism $g: Y \rightarrow S$ in the previous section fits into the following cartesian diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow g & & \downarrow h \\
S & \longrightarrow & \mathcal{S},
\end{array}
$$

(4.1.2)

where the morphism $h: \mathcal{Y} \rightarrow \mathcal{S}$ is log smooth. Let $U_S$ and $L_S$ be as in \( \mathcal{Y} \). By abuse of notation, denote by $d\log t_i \in \Omega^1_{Y/S}$ the image of $d\log t_i$ by the morphism $h^*(\Omega^1_{\mathcal{Y}/\mathcal{S}}) \longrightarrow \Omega^1_{Y/S}$. Let $\mathcal{F}$ be a locally free $O_Y$-module. Let

$$
\nabla: \mathcal{F} \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^1_{Y/S}^\bullet
$$

be an integrable connection. As in the previous section, we can define the Hirsch extension $\mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}(U_S)$ of the log de Rham complex $\mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}$ by using the well-defined morphism $L_S \ni u_j \mapsto d\log t_j \in \Omega^1_{Y/S}$:

$$
\mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}(U_S) := \text{Sym}_{O_S}(L_S) \otimes_{O_S} \mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}
$$

(4.1.4)

$$
:= (p \circ h)^{-1}(\text{Sym}_{O_S}(L_S)) \otimes (p\circ h)^{-1}(O_S) \mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}.
$$

The following remark is important:

**Remark 4.2.** In the definition \( \mathcal{F} \otimes_{O_Y} \Omega^\bullet_{Y/S}(U_S) \), we consider $L_S$ as a sheaf on $\mathcal{S}$ not on $S$. The reader should also note that we do not consider $\mathcal{M}_\mathcal{S} := M_{\mathcal{S}}/O_{\mathcal{S}}$, $U_{\mathcal{S}}$ nor $L_{\mathcal{S}}$ because the sheaf $\mathcal{M}_\mathcal{S}$ is not locally constant.

**Lemma 4.3.** Let the notations be as in \( \mathcal{Y} \). Let \( \{t_1, \ldots, t_r\} \) be a set of local sections of $M_S$ whose images in $\mathcal{M}_S$ is a unique system of local generators of $\mathcal{M}_S$. Denote $t_i$ in $L_S$ by $u_i$. Then the following hold:

1. The sheaf $\Omega^i_{\mathcal{Y}/\mathcal{S}}$ is a locally free $O_{\mathcal{Y}}$-module.

2. Locally on $\mathcal{S}$, the following sequence

$$
0 \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}}[-1] \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_{i-1}} \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_i} \longrightarrow 0 \quad (1 \leq i \leq r),
$$

is exact.

3. For $1 \leq i \leq r$, let us consider the completed Hirsch extensions $\mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_{i-1}}[[u_{i+1}, \ldots, u_r]]$ and $\mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_i}[[u_{i+1}, \ldots, u_r]]$ of $\mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_{i-1}}$ and $\mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_i}$ by the morphisms $\bigoplus_{j=i+1}^r O_S u_j \ni u_j \mapsto d\log t_j \in \Omega^1_{Y/S_{i-1}}$ and $\bigoplus_{j=i+1}^r O_S u_j \ni u_j \mapsto d\log t_j \in \Omega^1_{Y/S_i}$, respectively. The following sequence

$$
0 \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_{i-1}}[[u_{i+1}, \ldots, u_r]][-1] \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_{i-1}}[[u_{i+1}, \ldots, u_r]] \longrightarrow \mathcal{F} \otimes_{O_Y} \Omega^i_{\mathcal{Y}/\mathcal{S}_i}[[u_{i+1}, \ldots, u_r]] \longrightarrow 0 \quad (1 \leq i \leq r)
$$

is exact.
Proof. The proof is the same as that of [N5, (3.1)] and [5.1]; the proof is easy. □

**Proposition 4.4.** The natural morphism

\[(4.4.1) \quad \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}[[U_\lambda]] \to \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}\]

is a quasi-isomorphism. This quasi-isomorphism is contravariantly functorial for a morphism from the commutative diagram (4.1.2) to a similar commutative diagram and a morphism to the integrable connection (4.1.3) from a similar integrable connection on the similar diagram.

Proof. By using (4.3), the proof is the same as that of (3.2). □

**5 P-SNCL analytic spaces**

Let \( r \) be a positive integer. Let \( S \) be a log analytic family of log points of virtual dimension \( r \). In this section we give a definition of a \( P-(S)NCL(=(simple) normal crossing log) analytic space over \( S \). To define it, we have only to mimic the definition of an \( (S)NCL(=(simple) normal crossing log) analytic scheme defined in [N3] suitably.

A \( P-(S)NCL \) analytic space over \( S \) is locally the finitely many product of \( (S)NCL \) analytic spaces in the usual sense. The definition is a generalization of the definition of a generalized semistable analytic space over a log point of \( \mathbb{C} \) virtual dimension \( r \) in [F1] and [F3].

Let \( B \) be an analytic space over \( \mathbb{C} \). Let \( a_1 < \ldots < a_r \) be nonnegative integers. Set \( \alpha := (a_1, \ldots, a_r) \) and let \( d \) be a nonnegative integer such that \( a_r \leq d + r \). Let \( z_1, \ldots, z_{d+r} \) be a system of standard coordinates of \( \hat{\Delta}^{d+r} \). Let \( \hat{\Delta}_B(\alpha, d+r) \) be a closed analytic space of \( \hat{\Delta}^{d+r} \times B \) defined by an ideal sheaf

\[ (z_1 \cdots z_{a_1}, z_{a_1+1} \cdots z_{a_2}, \ldots, z_{a_r-1+1} \cdots z_{a_r}). \]

**Definition 5.1.** Let \( Z \) be an analytic space over \( B \) with structural morphism \( g: Z \to B \). We call \( Z \) a \( P-(S)NCL(=(simple) normal crossing) analytic space over \( B \) if \( Z \) is a union of smooth analytic space \( \{Z_\lambda\}_{\lambda \in \Lambda} \) over \( B \) (\( \Lambda \) is a set) and if, for any point of \( x \in Z \), there exist an open neighborhood \( V \) of \( x \) and an open neighborhood \( W \) of \( g(x) \) such that there exists an étale morphism \( \pi: V \to \hat{\Delta}_B(\alpha, d+r) \) such that

\[(5.1.1) \quad \{Z|_\lambda \}_{\lambda \in \Lambda} = \{\pi^*(z_{b_1}) = \cdots = \pi^*(z_{b_r}) = 0\}_{1 \leq b_1 \leq a_1^r, \ldots, a_{r-1}^r, 1 \leq b_r \leq a_r}.\]

where \( a_1, \ldots, a_r \) and \( d \) are nonnegative integers such that \( \sum_{i=1}^r a_i \leq d + r \), which depend on a local neighborhood of \( z \) in \( Z \) and \( Z|_\lambda := Z_\lambda \cap V \). The strict meaning of the equality (5.1.1) is as follows. There exists a subset \( \Lambda(V) \) of \( \Lambda \) such that there exists a bijection \( i: \Lambda(V) \to \{1, \ldots, a_1\} \times \cdots \times \{a_{r-1} + 1, \ldots, a_r\} \) such that, if \( \lambda \notin \Lambda(V) \), then \( Z|_\lambda |_V = 0 \) and if \( \lambda \in \Lambda(V) \), then \( Z_\lambda \) is a closed analytic space defined by the ideal sheaf \( (\pi^*(z_{b_1}), \ldots, \pi^*(z_{b_r})) \). We call the set \( \{Z_\lambda\}_{\lambda \in \Lambda} \) a decomposition of \( Z \) by smooth components of \( Z \) over \( B \). In this case, we call \( Z_\lambda \) a smooth component of \( Z \) over \( B \).

Let \( Z \) be a \( P-SNC \) analytic space over \( B \). For a nonnegative integer \( k \), set

\[(5.1.2) \quad Z_{(\lambda_0, \lambda_1, \ldots, \lambda_k)} := Z_{\lambda_0} \cap \cdots \cap Z_{\lambda_k} \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j)\]
and set
\[(5.1.3)\quad Z^{(k)} = \prod_{\{\lambda_0, \ldots, \lambda_k \mid \lambda_i \neq \lambda_j \ (i \neq j)\}} Z_{(\lambda_0, \lambda_1, \ldots, \lambda_k)} .\]

For a negative integer \(k\), we set \(Z^{(k)} = \emptyset\). Set \(\Gamma := \{Z_{\lambda}\}_{\lambda \in \Lambda}\). For an open analytic space \(V\) of \(Z\), set \(\Gamma_V := \{Z_{\lambda}|_V\}_{\lambda \in \Lambda}\).

First assume that \(M_S\) is the free hollow log structure \((\mathbb{N}^r)\). We fix an isomorphism
\[(5.1.4)\quad (M_S, \alpha_S) \simeq (\mathbb{N}^r \oplus \mathcal{O}_S^* \longrightarrow \mathcal{O}_S)\]
globally on \(S\). Let \(M_S(a, d + r)\) be the log structure on \(\bar{\Delta}_S(a, d + r)\) associated to the following morphism
\[(5.1.5)\quad \mathbb{N}^{\oplus a_r} \ni (0, \ldots, 0, 1, 0, \ldots, 0) \mapsto z_i \in \mathcal{O}_{\bar{\Delta}_S(a, d + r)} .\]

Let \(\Delta_S(a, d + r)\) be the resulting log analytic space over \(S\). We have the “multi-diagonal” morphism \(\mathbb{N}^r \longrightarrow \mathbb{N}^{a_1} \oplus \mathbb{N}^{a_2 - a_1} \oplus \cdots \oplus \mathbb{N}^{a_r - a_{r-1}} = \mathbb{N}^{a_r}\) induces a morphism \(\Delta_S(a, d + r) \longrightarrow S\) of log analytic spaces. We call \(\Delta_S(a, d + r)\) the standard \(P\text{-NCL}\) analytic space.

Let \(S\) be a family of log points of virtual dimension \(r\) \((M_S\) is not necessarily free).

**Definition 5.2.** Let \(f : X = (\hat{X}, M_X) \longrightarrow S\) be a morphism of log analytic spaces such that \(\hat{X}\) is a \(P\text{-SNC}\) analytic space over \(\hat{S}\) with a decomposition \(\Delta := \{\hat{X}_{\lambda}\}_{\lambda \in \Lambda}\) of \(\hat{X}/\hat{S}\) by its smooth components. We call \(f\) \((or\ X/S)\) a \(P\text{-NCL}(=P\text{-normal crossing log})\) analytic space if, for any point of \(x \in \hat{X}\), there exist an open neighborhood \(\hat{V}\) of \(x\) and an open neighborhood \(\hat{W}\) of \(\hat{f}(x)\) such that \(M_W\) is the free hollow log structure of rank \(r\) and such that \(f|_{\hat{V}}\) factors through a strict étale morphism \(\pi : V \longrightarrow \Delta_S(a, d + r)\) over \(W\) for some \(a, d\) depending on \(V\) such that \(\Delta_{\hat{V}} = \{\pi^*(z_{b_1}) = \cdots = \pi^*(z_{b_s}) = 0\}_{1 \leq b_1 \leq a_1, \ldots, a_{r-1} + 1 \leq b_s \leq a_r}\) in \(\hat{V}\).

**Corollary 5.3.** Let \(f : X \longrightarrow S\) be a \(P\text{-SNCL}\) analytic space with a decomposition \(\Delta := \{\hat{X}_{\lambda}\}_{\lambda \in \Lambda}\) of \(\hat{X}/\hat{S}\) by its smooth components. Then, locally on \(X\), \(X\) is a finitely many product of \(SNCL\) analytic spaces in the usual sense with some decomposition of their smooth components. Consequently \(f\) is log smooth.

**Proof.** Obvious. \(\square\)

The following is an obvious imitation of \([NS\ (2.1.7)]\).

**Definition 5.4.** Let \(f : X \longrightarrow S\) be a \(P\text{-SNCL}\) analytic space with a decomposition \(\Delta := \{\hat{X}_{\lambda}\}_{\lambda \in \Lambda}\) of \(\hat{X}/\hat{S}\) by its smooth components. We call an effective Cartier divisor \(D\) on \(X/S\) a \(\{relatively simple normal crossing divisor\}\) \((=:relatively\ SNCD)\) on \(X/S\) if there exists a set \(\Gamma := \{D_{\mu}\}_{\mu}\) of non-zero effective Cartier divisors on \(X/S\) of locally finite intersection such that
\[(5.4.1)\quad D = \sum_{\mu} D_{\mu} \quad \text{in} \quad \text{Div}(X/S)_{\geq 0},\]
such that $D_{\mu}|_{Z_\lambda}$ is a smooth divisor on $Z_\lambda$ for any $\lambda$ and, for any point $x$ of $D$, there exists an open neighborhood $V \simeq \Delta_S(a, d + r)$ of $x$ in $X$ and $D|_V = \bigcup_{c=a_r,1}\{z_c = 0\}$ for some $a_r + 1 \leq b \leq d + r$.

We call $\Gamma = \{D_\mu\}$ a decomposition of $D$ by smooth components of $D$ over $S$. Note that $D_\mu$ itself is not necessarily smooth over $\overset{\circ}{S}$; $D_\mu$ is a union of smooth analytic space over $\overset{\circ}{S}$.

Let $M(D)'$ be a presheaf of monoids on $X$ defined as follows: for an open subanalytic space $V$ of $X$,

$$\Gamma(V, M(D)') := \{(E, a) \in \text{Div}_{D|_V}(V/S) \times \Gamma(V, \mathcal{O}_X) | a\text{ is a generator of } \Gamma(V, \mathcal{O}_X(-E))\}$$

with a monoid structure defined by $(E, a) \cdot (E', a') := (E + E', aa')$. The natural morphism $M(D)' \to \mathcal{O}_X$ defined by the second projection $(E, a) \mapsto a$ induces a morphism $M(D)' \to (\mathcal{O}_X, *)$ of presheaves of monoids on $X$. The log structure $M(D)$ is, by definition, the associated log structure to the sheafification of $M(D)'$.

Because $\text{Div}_{D|_V}(V/S)_{\geq 0}$ is independent of the choice of the decomposition of $D|_V$ by smooth components, $M(D)$ is independent of the choice of the decomposition of $D$ by smooth components of $D$. Set

$$\Omega^i_{X/S}(\log D) := (\Omega^i_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M(D)))/((da(m), 0) - (0, a(m) \otimes m) \mid m \in M(D))$$

and

$$\Omega^i_{X/T}(\log D) := (\Omega^i_{X/T} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M(D)))/((da(m), 0) - (0, a(m) \otimes m) \mid m \in M(D))$$

for $T = \overset{\circ}{S}$ or $S$. Then we have the log de Rham complexes $\Omega^\bullet_{X/S}(\log D)$ and $\Omega^\bullet_{X/T}(\log D)$ with natural inclusions $\Omega^\bullet_{X/S}(\log D) \subseteq \Omega^\bullet_{X/S}(\log D)$ and $\Omega^\bullet_{X/T} \subseteq \Omega^\bullet_{X/T}(\log D)$.

Consider the following morphism

$$d \log t_1 \wedge \cdots \wedge d \log t_r \wedge: \Omega^\bullet_{X/S}(\log D) \to \Omega^\bullet_{X/S}[r].$$

This is indeed a morphism of complexes. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ be an integrable connection. Then we have the log de Rham complexes $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}$. We also have the following morphism of complexes:

$$d \log t_1 \wedge \cdots \wedge d \log t_r \wedge: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[r].$$

This morphism induces the following morphism of complexes:

$$d \log t_1 \wedge \cdots \wedge d \log t_r \wedge: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[r].$$

Assume that $\mathcal{E}$ is a flat quasi-coherent $\mathcal{O}_X$-module and that $\nabla$ has no poles. Set

$$(5.4.3) \quad \varphi^k_{X/S}(\log D) = \begin{cases} 0 & (k < 0), \\ \text{Im}(\Omega^k_{X/S}(\log D) \otimes_{\mathcal{O}_X} \Omega^{i-k}_{X/S}(\log D) \to \Omega^i_{X/S}(\log D)) & (0 \leq k \leq i), \\ \Omega^i_{X/S}(\log D) & (k > i). \end{cases}$$

(5.4.4)
The filtration $P^X$ induces a filtration on $E \otimes_{O_X} \Omega^*_{X/S}(\log D)$, which we denote $P^X$ again; we obtain the filtered complex $(E \otimes_{O_X} \Omega^*_{X/S}(\log D), P^X)$.

In the rest of this article, we fix total orders on $\lambda$'s and $\mu$'s, respectively.

**Proposition 5.5.** Let $a(k): \bar{X}^{(k)} \rightarrow \bar{X}$ be the natural morphism. Then there exists the following Poincaré residue isomorphism of complexes:

(5.5.1)

$$\text{Res} : \text{gr}_k^{P^X}(E \otimes_{O_X} \Omega^*_{X/S}(\log D)) \sim \bigoplus_{1 \leq \lambda_0 < \cdots < \lambda_{k-1} < \lambda_{k-1} \leq \mu_r} b_{\lambda_0 \cdots \lambda_{k-1}}(\Omega^*_{Y^{\lambda_0 \cdots \lambda_{k-1}/S}(\log D|_{Y^{\lambda_0 \cdots \lambda_{k-1}}})) \otimes \omega^{(k-1)}(Y^{\lambda_0 \cdots \lambda_{k-1}/S})[-k] \quad (k \geq 1).$$

Here

$$\omega^{(k-1)}(X/S) := \bigoplus_{1 \leq \lambda_0 < \cdots < \lambda_{k-1} < \lambda_{k-1} \leq \mu_r} \omega^{(k-1)}(X^{\lambda_0 \cdots \lambda_{k-1}/S}),$$

where $\omega^{(k-1)}(X^{\lambda_0 \cdots \lambda_{k-1}/S})$ is the orientation sheaf of the set $\{X^{\lambda_0}, \ldots, X^{\lambda_{k-1}}\}$.

**Proof.** (Since $r \neq 1$, we need a care for the proof in addition to the usual proof for the usual Poincaré residue isomorphism.) Let $x$ be a point of $\bar{X}$. First assume that there exists an isomorphism $X \sim \Delta_S(a, d + r)$ over $S$. We identify $X$ with $\Delta_S(a, d + r)$ by this isomorphism. Let $k$ be a nonnegative integer. Set $\overline{X} := X^{k-1}$ endowed with a log structure associated to a morphism $\mathbb{N}^{d+r} \ni e \mapsto z \in \mathcal{O}_{S}^{d+r}$

(1 \leq i \leq d + r). Then we have a natural morphism $\overline{X} \rightarrow \overline{S}$ and consider the closed immersion $X \rightarrow \overline{Y}$ of log analytic spaces over $S \rightarrow \overline{S}$. Let $\overline{Y}_{\lambda_0}, \ldots, \overline{Y}_{\lambda_{k-1}}$ be the closed analytic spaces of $\overline{Y}$ defined by $z_1 = 0, \ldots, z_{\lambda_{k-1}} = 0$, respectively. Set $Y := \bigcup_{j=1}^{\lambda_{k-1}} Y_j$. Set $\overline{Y}_{\lambda_0 \cdots \lambda_{k-1}} := \overline{Y}_{\lambda_0} \cap \cdots \cap \overline{Y}_{\lambda_{k-1}}$. For a nonnegative integer $k$, let $b_{\lambda_0 \cdots \lambda_{k-1}} : \overline{Y}_{\lambda_0 \cdots \lambda_{k-1}} \rightarrow \overline{Y}$ be the natural closed immersion. The horizontal divisor $D$ on $\bar{X}$ is naturally extended to an SNCD $D$ on $\overline{X}$. Let $P^Y$ be the filtration on $\Omega^*_{\overline{Y}/\overline{S}}(\log D)$ obtained by the pieces of log poles with respect to $Y$. Then, by the usual Poincaré residue isomorphism [NS] (2.21.3) (with logarithmic forms put on the left), we have the following isomorphism

(5.5.2)

$$\text{Res} : \text{gr}_k^{P^Y}(\Omega^*_{\overline{Y}/\overline{S}}(\log D)) \sim \bigoplus_{1 \leq \lambda_0 < \cdots < \lambda_{k-1} \leq \mu_r} b_{\lambda_0 \cdots \lambda_{k-1}}(\Omega^*_{\overline{Y}_{\lambda_0 \cdots \lambda_{k-1}/\overline{S}}}(\log D|_{\overline{Y}_{\lambda_0 \cdots \lambda_{k-1}}})) \otimes \omega^{(k-1)}(Y_{\lambda_0 \cdots \lambda_{k-1}/\overline{S}})[-k] \quad (k \geq 1).$$

Here $\omega^{(k-1)}(Y_{\lambda_0 \cdots \lambda_{k-1}/\overline{S}})$ is the orientation sheaf associated to the set $\{Y_{\lambda_0}, \ldots, Y_{\lambda_{k-1}}\}$.

Let $a(k) : \bar{X}^{(k)} \rightarrow \overline{Y}$ be the natural morphism and let $\omega^{(k-1)}(\overline{Y}/\overline{S})$ be the orientation sheaf on $\overline{Y}^{(k-1)}$ obtained by $\omega^{(k-1)}(\overline{Y}_{\lambda_0 \cdots \lambda_{k-1}/\overline{S}})$:

$$\omega^{(k-1)}(\overline{Y}/\overline{S}) := \bigoplus_{1 \leq \lambda_0 < \cdots < \lambda_{k-1} \leq \mu_r} \omega^{(k-1)}(Y_{\lambda_0 \cdots \lambda_{k-1}/\overline{S}}).$$
This morphism induces the following morphism of complexes:

\[(5.5.3)\]

\[
\text{gr}_k^{\text{pr}} \Omega^\bullet_{Y/S}(\log D) = (P_k^Y \Omega^\bullet_{Y/S}(\log D)/\Omega^\bullet_{Y/S}(\log D)(-\vec{Y}))/\text{gr}_k^{\text{pr}} \Omega^\bullet_{Y/S}(\log D)(-\vec{Y})
\]

by the log analytic version of [N3, (1.3.12.1)], we have the following isomorphism

\[(5.5.4)\]

\[\text{Res}: \text{gr}_k^{\text{pr}} \Omega^\bullet_{Y/S}(\log D) \sim b_k^{(k-1)}(\Omega^\bullet_{Y/(k-1)/S}(\log D)|_{Y/(k-1)}) \otimes \omega^{(k-1)}(Y/S)[-k] \quad (k \geq 1).\]

By restricting this isomorphism to \(X \subset Y\), we obtain a local isomorphism \((5.5.1)\) in the case \(E = \mathcal{O}_X\). Here note that the closed analytic space \(\hat{\mathcal{O}}\) of \(\hat{\mathcal{X}}\) defined by equations \(z_{c_1} = z_{c_2} = \cdots = z_{c_r} = 0\) \((1 \leq c_1 \leq a_1, \ldots, a_{r-1} + 1 \leq c_r \leq a_r)\) of \(X\) is the intersection of the smooth components \(Y_{c_1}, \ldots, Y_{c_r}\) and that \(\Omega^\bullet_{Y_{c_j}/S}(\log D)|_{Y_{c_j}}\) is \(\Omega^\bullet_{\hat{\mathcal{O}}/S}(\log D)|_{\hat{\mathcal{O}}}\). As in the usual case, we see that the isomorphism is independent of the choice of the isomorphism \(X \sim \Delta_S(a, d+r)\). It is clear that the isomorphism \((5.5.1)\) in the case \(E = \mathcal{O}_X\) can be defined for the general \(E\). This generalized morphism is indeed an isomorphism.

We can complete the proof of this proposition. \(\square\)

Set

\[(5.5.5)\]

\[A^n((X, D)/S, \mathcal{E}) := \bigoplus_{j \in \mathbb{N}}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r}_{X/S}(\log D)/P_j^X) = \bigoplus_{j \in \mathbb{N}}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r}_{X/S}(\log D)/P_j^X(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r}_{X/S}(\log D))) \quad (n \in \mathbb{N}).\]

Consider the following morphisms:

\[(-1)^r \nabla: (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r}_{X/S}(\log D))/P_j^X \rightarrow (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r+1}_{X/S}(\log D))/P_j^X\]

and

\[d \log t_l \wedge: (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r}_{X/S}(\log D))/P_j^X \rightarrow (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{n+r+1}_{X/S}(\log D))/P_{j+1}^X \quad (1 \leq l \leq r).\]

These morphisms induce the following morphism:

\[(-1)^r \nabla + d \log t_1 \wedge + \cdots + d \log t_r \wedge: A^n((X, D)/S, \mathcal{E}) \rightarrow A^{n+1}((X, D)/S, \mathcal{E}) \quad (n \in \mathbb{N})\]

and we obtain a complex \(A^\bullet((X, D)/S, \mathcal{E})\). The morphism \((5.5.3)\) induces the following morphism

\[(5.5.6)\]

\[d \log t_1 \wedge \cdots \wedge d \log t_r \wedge: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^r_{X/S}(\log D) \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{r+1}_{X/S}(\log D).\]

This morphism induces the following morphism of complexes:

\[(5.5.7)\]

\[d \log t_1 \wedge \cdots \wedge d \log t_r \wedge: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^r_{X/S}(\log D) \rightarrow A^\bullet((X, D)/S, \mathcal{E}).\]

The following lemma is well-known:
Lemma 5.6. Assume that \( r = 1 \) in this lemma. Denote \( d \log t_1 \) by \( d \log t \). Then the following hold:

(1) (cf. [38 (1.3.21)]) The following sequence is exact:

\[
0 \longrightarrow P_0^X (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^\bullet (\log D)) \longrightarrow a_*^{(0)} a_!^{(0)} (\mathcal{E} \otimes_{\mathcal{O}_{\tilde{X}(0)}} \Omega_{\tilde{X}(0)/S}^\bullet (\log D_{\tilde{X}(0)}) \otimes_{\mathbb{Z}} \varpi^{(0)} (X/S)) \longrightarrow a_*^{(1)} a_!^{(1)} (\mathcal{E} \otimes_{\mathcal{O}_{\tilde{X}(1)}} \Omega_{\tilde{X}(1)/S}^\bullet (\log D_{\tilde{X}(1)}) \otimes_{\mathbb{Z}} \varpi^{(1)} (X/S)) \longrightarrow \cdots.
\]

(2) (cf. [38 (4.15)], [38 (1.4.3.1)]) The morphism

\[
d \log t |_{\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^\bullet (\log D)} \longrightarrow A^\bullet ((X, D)/S, \mathcal{E})
\]

is a quasi-isomorphism.

Proof. (1): The question is local. We may assume that \( \mathcal{E} = \mathcal{O}_X \). Let the notations be as in the proof of (5.5) in the case \( r = 1 \). In this case we may assume that \( X = Y \).

It is easy to check that the complex \( \Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \) is a subcomplex of \( \Omega_{\tilde{Y}/S}^\bullet (\log D) \) and \( P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D) \) \((k \in \mathbb{N})\). By the analytic version of [41 (4.2.2) (a), (c)], we have the following exact sequence

\[
0 \longrightarrow \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \longrightarrow b_*^{(0)} (\Omega_{Y_S/S}^\bullet (\log D)|_{Y_S(0)}) \otimes_{\mathbb{Z}} \varpi^{(0)} (Y/S)) \longrightarrow \cdots.
\]

It is easy to check that the natural morphism \( \Omega_{\tilde{Y}/S}^\bullet (\log D) \longrightarrow \Omega_{\tilde{Y}/S}^\bullet (\log D) \) is injective. Hence we have an isomorphism \( \Omega_{\tilde{Y}/S}^\bullet (\log D) \sim \sim \sim P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D) \). By the definition of \( P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D) \), we have the following natural surjective morphism

\[
P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \longrightarrow P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D).
\]

The following diagram shows that this morphism is injective:

\[
\begin{array}{ccc}
P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) & \longrightarrow & P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D) \\
\cap & & \cap \\
\Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) & \longrightarrow & \Omega_{\tilde{Y}/S}^\bullet (\log D).
\end{array}
\]

Hence we have an isomorphism

\[
P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \sim P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)
\]

Thus we have the following isomorphism:

\[
\Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \simeq P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) = P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D).
\]

\[
\text{(5.6.5)}
\]

\[
\Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \simeq P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) = P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D).
\]

\[
\text{(5.6.6)}
\]

\[
\Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) \simeq P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D)/\Omega_{\tilde{Y}/S}^\bullet (\log D)(\tilde{Y}) = P_0^Y \Omega_{\tilde{Y}/S}^\bullet (\log D).
\]
By (5.6.3) and (5.6.5) we see that the sequence (5.6.1) is exact. By (5.5.1) for the case \( E \) is exact by (5.6.1). Since \( E \) is a flat \( O_X \)-module, it suffices to prove that the natural morphism

\[
(5.6.6) \quad d \log t : \Omega_{X/S}(\log D) \longrightarrow \{ \Omega_{X/S}^{+1}(\log D)/P_0^X d \log t^+/\Omega_{X/S}^2/P_1^X d \log t^+ \ldots \}
\]

is a quasi-isomorphism.

As in [31] 3.15 (cf. [31] (6.28) (9), (6.29) (1)), it suffices to prove that the sequence

\[
(5.6.7) \quad 0 \longrightarrow \text{gr}_0^{P^X}(\Omega^\bullet_{X/S}(\log D)) \xrightarrow{d \log t} \text{gr}_1^{P^X}(\Omega^\bullet_{X/S}(\log D))[1] \xrightarrow{d \log t} \text{gr}_2^{P^X}(\Omega^\bullet_{X/S}(\log D))[2] \xrightarrow{d \log t} \cdots
\]

is exact. By (5.5.1) for the case \( r = 1 \), this sequence is equal to the exact sequence (5.6.1) for the case \( E = O_X \).

The following is a generalization of (5.6):

**Theorem 5.7.** Let \( T_i (1 \leq i \leq r) \) be a family of log points of virtual dimension 1. If \( S = T_1 \times \cdots \times T_r \), if \( X = Y_1 \times \cdots \times Y_r \), where \( Y_i \) is an SNCL analytic space and if \( D \) is a relative horizontal SNCD on \( Y_r/T \), then the morphism (5.5.6) is a quasi-isomorphism.

**Proof.** The problem is local. We may assume that \( E = O_X \). Let \( U_j \) be \( T_j \) or \( Y_j \). To prove that (5.5.6) for the case in (5.7) is a quasi-isomorphism, consider an increasing filtration \( P_{Y_1 \times \cdots \times Y_r/U_1 \times \cdots \times U_r}(\log F) \) defined by the following: if \( U_j = T_j \), then \( k_j = \infty \) and if \( U_j = Y_j \), then \( k_j = \infty \) for \( j \leq 1 \leq r \), and \( P_{Y_1 \times \cdots \times Y_r/U_1 \times \cdots \times U_r}(\log F) \) is defined to be a sub \( O_{Y_1 \times \cdots \times Y_r}/U_1 \times \cdots \times U_r \)-module of \( \Omega_{Y_1 \times \cdots \times Y_r/U_1 \times \cdots \times U_r}(\log F) \) generated by differential forms of at most \( k_j \)-pieces of log poles with respect to \( Y_j \) for \( 1 \leq j \leq r \). Let \( p_j : Y_{r-1} \times Y_r \longrightarrow Y_j \) be the projection. Because \( \mathbb{C}\{x_1, \ldots, x_r\} \) is a Zariski ring, \( \mathbb{C}\{x_1, \ldots, x_r\} \) is (faithfully) flat over \( \mathbb{C}\{x_1, \ldots, x_r\} \) for any ideal \( I \) of \( \mathbb{C}\{x_1, \ldots, x_r\} \). Hence the following sequence

\[
(5.7.2) \quad 0 \longrightarrow P_0^{Y_r}(p_r^* \Omega^\bullet_{Y_r/S_r}(\log D)) \longrightarrow p_r^* a_0^{(0)}(\Omega_{Y_r/S_r}^\bullet(\log D))[0] \otimes \mathbb{Z}[\omega](Y_r/T_r) \longrightarrow \cdots
\]

is exact by (5.6.1). Since

\[
p_{Y_{r-1}}^{\times Y_r} \Omega_{Y_{r-1} \times Y_r/S_{r-1} \times S_r}^1(\log D) = p_{r-1}^* \Omega_{Y_{r-1}/S_{r-1}}^1 \oplus p_r^* P_{Y_r}^{Y_r} \Omega_{Y_r/S_r}^1(\log D)
\]

for \( k = 0, 1 \),

\[
p_{Y_{r-1}}^{\times Y_r} \Omega_p^{Y_r} Y_{r-1 \times Y_r/S_{r-1} \times S_r}(\log D) = \bigoplus_{p_1+p_2=p} p_{r-1}^* \Omega_{Y_{r-1}/S_{r-1}}^{p_1} \oplus p_r^* P_{k}^{Y_r} \Omega_{Y_r/S_r}^{p_2}(\log D).
\]

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Because $\Omega^p_{Y-1/S-1}$ is a locally free $\mathcal{O}_{Y-1}$-module, we see that the following morphism
\[(5.7.3) \quad d \log t_r \land: \Omega^i_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D) \to \{ \Omega^{i+1}_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D) / \mathcal{P}^0_{\infty,0} \to \Omega^{i+2}_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D) / \mathcal{P}^1_{\infty,1} \to \ldots \} \]
is a quasi-isomorphism by using the exact sequence \[(5.7.2)\).

Next consider the sheaf $\mathcal{P}^1_{\infty,1} (\log D)$ $(k \in \mathbb{Z} \cup \{ \infty \})$. We claim that the following morphism
\[(5.7.4) \quad d \log t_r \land: \mathcal{P}^1_{\infty,1} (\log D) \to \{ \mathcal{P}^0_{\infty,0} (\log D) / \mathcal{P}^1_{0,0} \to \mathcal{P}^1_{\infty,1} (\log D) / \mathcal{P}^1_{1,1} \to \ldots \} \]
It suffices to prove that the sequence
\[(5.7.5) \quad 0 \to \mathcal{G}^1_{0,\infty} (\Omega^i_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D)) \to \mathcal{G}^1_{1,\infty} (\Omega^i_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D)) \to \mathcal{G}^1_{2,\infty} (\Omega^i_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D)) \to \ldots \]
is exact. This exactness follows from the argument in the previous paragraph. Hence the morphism \[(5.7.3)\] is a quasi-isomorphism. By using the quasi-isomorphisms \[(5.7.3)\] and \[(5.7.4)\], we see that the morphism $d \log t_r \land d \log t_r \land$ from $\Omega^i_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D)$ to
\[
\begin{align*}
\Omega^{i+3}_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D) / \mathcal{P}^1_{0,0} & \to \Omega^{i+3}_{Y_{r-1} \times Y_r/S_{r-1} \times S_r} (\log D) / \mathcal{P}^1_{1,1} \to \ldots \\
d \log t_r \land & \to d \log t_r \land
\end{align*}
\]
is a quasi-isomorphism. Continuing this argument repeatedly, we see that \[(5.7.5)\] for the general $r$ is a quasi-isomorphism.

The following is a generalization of \[(5.7.6)\] (3.26):\[
\textbf{Corollary 5.8.} (\text{cf. } [F3, (3.26)]) \quad \text{The morphism } \mathcal{P}^i_{1,1} \text{ is a quasi-isomorphism.}
\]
\textbf{Proof.} The problem is local. Hence we may assume that $S = S_1 \times \ldots \times S_r$ and $X = \Delta_{S_1} (a_1, d_1 + 1) \times \cdots \times \Delta_{S_r} (a_r, d_r + 1)$ with $\sum_{i=1}^r d_i = d$. We may assume that $D$ is a horizontal SNCD on $\Delta_{S_r} (a_r, d_r + 1)$. Hence \[(5.8)\] follows from \[(5.7)\]. \hfill \qed
Remark 5.9. Our proof of (5.8) is simpler and more direct than those of [F1, (5.13)] and [F3, (3.26)].

Endow $A^n((X, D)/S, \mathcal{E})$ with the following filtration $P^X$ defined by the following:

$$P^X_k A^n((X, D)/S, \mathcal{E}) := \bigoplus_{j \in \mathbb{N}} ((P^X_{2j+k+r} + P_j)(\mathcal{E} \otimes \mathcal{O}_X \Omega^{n+r}_X(\log D)/P^X_j)).$$

Then we have the filtered complex $(A^\bullet((X, D)/S, \mathcal{E}), P^X)$.

Proposition 5.10.

(5.10.1)

$$\text{gr}_k^P A^\bullet((X, D)/S, \mathcal{E}) = \bigoplus_{j \in \max \{0, -k-(r-1)\}} \left( \mathcal{E} \otimes \mathcal{O}_X a_*^{(2j+k+r-1)}(\Omega^\bullet_{\mathcal{X}(2j+k+r-1)/S}(\log D) \times \bigotimes Z (2j+k+r-1)(\mathcal{X}(2j+k+r-1)/S))^r \right)[-2j-k].$$

Proof. We have the following equalities by (5.5.1):

$$\text{gr}_k^P A^\bullet((X, D)/S, \mathcal{E}) = \bigoplus_{j \in \max \{0, -k-(r-1)\}} \left( \mathcal{E} \otimes \mathcal{O}_X a_*^{(2j+k+r-1)}(\Omega^\bullet_{\mathcal{X}(2j+k+r-1)/S}(\log D) \times \bigotimes Z (2j+k+r-1)(\mathcal{X}(2j+k+r-1)/S))^r \right)[-2j-k].$$

Theorem 5.11. There exists the following spectral sequence:

(5.11.1)

$$E_1^{k,q+k} = \bigoplus_{j \in \max \{0, -k-(r-1)\}} R^q f_{\mathcal{X}(2j+k+r-1)/S}(\mathcal{E} \otimes \mathcal{O}_X) \Omega^\bullet_{\mathcal{X}(2j+k+r-1)/S}(\log D) \times \bigotimes Z (2j+k+r-1)(\mathcal{X}(2j+k+r-1)/S))^r \Rightarrow R^q f_{X/S}(\mathcal{E} \otimes \mathcal{O}_X \Omega^\bullet_{X/S}(\log D)).$$

Proof. (5.11.1) follows from (5.7) and (5.10.1).

Proposition 5.12. Let $j$ be a nonnegative integer. Then there exists the following spectral sequence

(5.12.1)

$$E_1^{k,q+k} = R^q f_{\mathcal{X}(j)/\tilde{X}(j)}(\mathcal{E} \otimes \mathcal{O}_X) \Omega^\bullet_{\mathcal{X}(j)/\tilde{X}(j)}(\log D) \times \bigotimes Z (k)(\mathcal{D}(k)/\tilde{S}) \Rightarrow R^q f_{\mathcal{X}(j)/S}(\mathcal{E} \otimes \mathcal{O}_X) \Omega^\bullet_{\mathcal{X}(j)/S}(\log D).$$

Proof. This is obtained by the standard Poincaré residue isomorphism of complexes:

$$\text{Res} : \text{gr}_k^P \Omega^\bullet_{\mathcal{X}(j)/S}(\log D) \xrightarrow{\sim} \Omega^\bullet_{\mathcal{X}(j)/S} \times \bigotimes Z (k)(D/\tilde{D})[-k],$$

where $P^D$ is the filtration obtained by the pieces of log poles with respect to $D$. □
Remark 5.13. For the proof of the main result in the next section, we need only the following: there exists the spectral sequence converging to $R^i f_{X/S_*} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D))$ whose $E_1$-terms are the direct sums of cohomology sheaves on $S$ such that there exist spectral sequences converging to these cohomology sheaves whose $E_1$-terms are the direct sums of cohomology sheaves $R^s f_{a_j,k_*} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{D^{(k)}}|_{\tilde{X}(j)/\tilde{S}})$ for various $j$'s and $k$'s, where $a^{(j)(k)} : D^{(k)}|_{\tilde{X}(j)} \rightarrow \tilde{X}$ is the natural morphism.

In the following corollary, for a morphism $\tilde{S}' \rightarrow \tilde{S}$ of log analytic spaces, we endow $\tilde{S}'$ with the inverse image of the log structure of $S$.

Corollary 5.14. Assume that $\tilde{S}$ is smooth and that the structural morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$ is proper. Then $R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D))$ is a locally free $\mathcal{O}_S$-module and commutes with base change.

Proof. Let $y$ be an exact point of $S$. By Grauert’s theorem ([G, p. 62 Satz 3]), we have only to prove that, for any point $y \in S$, the natural morphism

$$R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D)) \rightarrow R^q f_{t_y*} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/y}(\log D))$$

is surjective. Here $t_y : X_y \rightarrow X$ is the natural inclusion. By ([11.4]) and ([12.4]) and by the analogous spectral sequences for $R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/y}(\log D))$, we have only to prove that the natural morphism

$$R^q f_* a_*^{(j)(k)} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{D^{(k)}}|_{\tilde{X}(j)/\tilde{S}}) \rightarrow R^q f_{a_*^{(j)(k)} t_y*} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{D^{(k)}}|_{\tilde{X}(j)/\tilde{S}})$$

is surjective. Here $t_y^{(j)(k)} : \tilde{D}^{(k)}_{\tilde{y}(j)} \rightarrow \tilde{D}^{(k)}|_{\tilde{X}(j)/\tilde{S}}$ is the natural inclusion. Because we have the Gauss-Manin connection for $R^q f_* a_*^{(j)(k)} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{D^{(k)}}|_{\tilde{X}(j)/\tilde{S}})$ and since $\Omega^\bullet_{S/C}$ has no log poles, $R^q f_* a_*^{(j)(k)} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{D^{(k)}}|_{\tilde{X}(j)/\tilde{S}})$ is a locally free $\mathcal{O}_S$-modules ([D] (2.17)) and commutes with base change. Hence the morphism ([14.1]) is surjective.

6 Main result

In [F1] Fujisawa has introduced the notion of a generalized semistable family over a unit polydisc. In this article we give a generalization of this notion and we call this a P-semistable family. In this section we give a main result in this article.

Let the notations be as in the previous section. Let $f : X \rightarrow S$ be a log smooth morphism. We assume that $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$ is smooth and that, locally on $\tilde{X}$, this is a product of semistable families of analytic spaces in the usual sense. That is, locally on $\tilde{X}$, there exist a system of local coordinates $t_1, \ldots, t_r$ of $\tilde{S}$ over $S$ and a system of local coordinates $x_1, \ldots, x_d+r$ such that $x_1 \cdots x_{a_1} = t_1, x_{a_1+1} \cdots x_{a_2} = t_2, \ldots, x_{a_{r-1}+1} \cdots x_{a_r} = t_r \ (0 \leq i_1 < i_2 < \cdots < i_r \leq d+r)$ over $\tilde{S}$ ([F1] (6.2), (6.5)). We also assume that the morphism $f^{-1}(M_\Sigma) \rightarrow M_X$ of log structures of $f$
is locally isomorphic to the associated log structure to the multi-diagonal morphism
\[ \mathbb{N}^r \to \bigoplus_{i=1}^r \mathbb{N}^{a_i} \to \bigoplus_{i=1}^r \mathbb{N}^{a_i} = \mathbb{N}^r \], where \( M_S \) and \( M_X \) are the associated log structures to the morphism \( \mathbb{N}^r \to e_i \to t_i \in \mathcal{O}_{\mathcal{X}} \) (1 \( \leq i \leq r \) and \( \mathbb{N}^{a_i} \to e_j \to x_{j+a_i+1} \in \mathcal{O}_{\mathcal{X}} \) (1 \( \leq i \leq r \) in the local situation above. We call \( \mathcal{X}/\mathcal{S} \) a \( P \)-semistable family over \( \mathcal{S} \). Let \( \mathcal{D} \) be a horizontal NCD (=normal crossing divisor) on \( \mathcal{X}/\mathcal{S} \). That is, it is a relative NCD on \( \mathcal{X}/\mathcal{S} \) which is locally defined by an equation \( x_{a_i+1} \cdots x_b = 0 \) for some \( a_r+1 \leq b \leq d+r \) in the local situation above. Set \( \mathcal{X} := \mathcal{X} \times_{\mathcal{S}} S \) and \( D := \mathcal{D} \times_{\mathcal{S}} S \). Let \( \psi : \mathcal{X} \to S \) be the structural morphism.

**Lemma 6.1.** Let \( \mathcal{E} \) be a coherent locally free \( \mathcal{O}_X \)-module and let \( \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/C}(\log D) \) be a locally nilpotent integrable connection on \( \mathcal{X} \) and let \( \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/C}(\log D) \) be the associated log de Rham complex to \( \nabla \). Let \( \nu : X \to \mathcal{X} \) be the natural exact closed immersion. Then the following natural morphism

\[
\iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}(\log D)) \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}(\log D)
\]

is a quasi-isomorphism. This quasi-isomorphism is contravariantly functorial.

**Proof.** (cf. [S (2.10)], [EMN (3.17) (2)]) Since the problem is local, we may assume that \( \nabla \) is nilpotent by the five lemma. In this case, by [KN (4.6) (2)], there exists a local system \( \mathcal{V} \) of finite dimensional \( \mathbb{C} \)-vector spaces on \( \mathcal{X} \) such that \( \mathcal{E} := \mathcal{O}_X \otimes \mathcal{V} \) and \( \nabla := d \otimes \text{id} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/C}(\log D) \). Let \( \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/C}(\log D) \) be the associated log de Rham complex to \( \nabla \). We may assume that \( \mathcal{V} = \mathbb{C} \). Moreover we may assume that \( S = \{O\} \) since the family \( \mathcal{X}/\mathcal{S} \) is locally obtained by the base change of a semistable family in the case \( \mathcal{S} = \{O\} \). Since \( \Omega^*_{X/C}(\log D) = \Omega^*_{X/C}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X \), it suffices to prove that the natural morphism

\[
\iota^{-1}(\Omega^*_{X/C}(\log D)) \to \Omega^*_{X/C}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X
\]

is a quasi-isomorphism.

Let \( (z_1, \ldots, z_{m+1}) \) \((|z_i| < 1) \) be the local coordinate of a point of \( \mathcal{X} \). We may assume that \( \mathcal{X} = \Delta^{m+1} \) and that the morphism \( f : \mathcal{X} \to \Delta \) is defined by the following equation

\[
f(z_1, \ldots, z_m) = (z_1 \cdots z_{a_1}, z_{a_1+1} \cdots z_{a_2}, \ldots, z_{a_r-1+1} \cdots z_{a_r}) = (t_1, \ldots, t_r) \in \Delta^r.
\]

We may also assume that \( \mathcal{D} \) is a closed analytic space of \( \mathcal{X} \) defined by an ideal sheaf \( (z_{a_i+1} \cdots z_b) \) \((a_r+1 \leq b \leq m) \).

To prove that the morphism \( \iota^{-1}(t_1 \cdots t_r \Omega^*_{X/C}(\log D)) \) is acyclic. (This is indeed a complex.) Indeed, the stalk of \( \iota^{-1}(t_1 \cdots t_r \Omega^*_{X/C}(\log D)) \) at \( S \) is the Koszul complex on \( \mathbb{C}\{z_0, \ldots, z_m\} \) with respect to the operators \( z_i \frac{d}{dz_i} + 1 \) \((1 \leq i \leq a_r) \), \( z_i \frac{d}{dz_i} (a_r + 1 \leq i \leq b) \) and \( \frac{d}{dz_i} \)

\((b+1 \leq i \leq m) \) Since the operator \( z_i \frac{d}{dz_i} + 1 \) \((1 \leq i \leq a_r) \) is bijective, we see that \( \iota^{-1}(t_1 \cdots t_r \Omega^*_{X/C}(\log D)) \) is acyclic by [S (1.12)].

We can complete the proof of (6.1).

**Corollary 6.2.** The natural morphism

\[
\iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}(\log D))[U_S] \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}(\log D))[U_S]
\]

is a quasi-isomorphism. This quasi-isomorphism is contravariantly functorial.
Proof. In [Na (3.14)] we have proved that the (PD-)Hirsch extension preserves the quasi-isomorphism. Hence (6.2) follows from (6.1).

Theorem 6.3. Let the notations be as in (6.1). Then there exists a canonical section
\[(6.3.1) \rho: E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D) \rightarrow \iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D))\]
of the natural morphism
\[(6.3.2) \iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)) \rightarrow E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)\]
in the derived category $D^+(\iota^{-1}f^{-1}(O_S))$ of bounded below complexes of $\iota^{-1}f^{-1}(O_S)$-modules. The section (6.3.1) is contravariantly functorial with respect to a morphism $(X, D) \rightarrow (X', D')$ over $S \rightarrow S'$, where $(X', D')/S'$ is an analogous log analytic spaces over $\mathbb{C}$.

Proof. By (2.4), (4.4) and (6.2) we obtain the following commutative diagram (cf. [EY (3.17)]):
\[
\begin{array}{ccc}
\iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D))[U_S] & \xrightarrow{\sim} & \iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)) \\
\iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D))[U_S] & \xrightarrow{\sim} & \iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)) \\
E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)[U_S] & \xrightarrow{\sim} & E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D) \\
E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D) & \xrightarrow{\sim} & E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D).
\end{array}
\]

(This commutative diagram is the crux in this article.)
The contravariantly functoriality follows from (2.4), (4.4) and (6.2). □

Corollary 6.4. Let $i_S: S \rightarrow S$ be the natural inclusion. Then there exists a canonical section
\[(6.4.1) \rho: i_S^*Rf_*\iota^*(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)) \rightarrow Rf_*i_\ast\iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D))\]
of the natural morphism
\[(6.4.2) Rf_*i_\ast\iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)) \rightarrow i_S^*Rf_*i_\ast\iota^{-1}(E \otimes_{O_X} \Omega_{X/S}^\bullet(\log D)).\]
The section (6.4.1) is contravariantly functorial with respect to a morphism $(X, D) \rightarrow (X', D')$ over $S \rightarrow S'$, where $(X', D')/S'$ is an analogous log analytic spaces over $\mathbb{C}$.

For $(X, D)$, we can reprove [IKN (6.4)] when $\tilde{S}$ is smooth.

Theorem 6.5. Assume that $\tilde{S}$ is smooth and that $\tilde{f}: \tilde{X} \rightarrow \tilde{S}$ is proper. Then the higher direct image $R^d\tilde{f}_*\iota^*(E \otimes_{\tilde{O}_X} \Omega_{X/S}^\bullet(\log D))$ is a locally free $\tilde{O}_{\tilde{S}}$-module and commutes with base change.
Proof. By Grauert’s theorem ([8, p. 62 Satz 3]), we have only to prove that, for any point \( y \in \mathcal{S} \), the natural morphism

\[
(6.5.1) \quad R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D) \rightarrow R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X_{y/S})(\log D_y)
\]

is surjective. Because we have the Gauss-Manin connection on \( R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D) \)
and because \( \mathcal{S} \) is smooth, we may assume that \( M_y \neq \mathcal{O}_y^* \) by [8] (2.17)]. Furthermore we may assume that \( y \in S \) because, if \( y \not\in S \), then we have only to consider the smaller \( r \). In the following we assume that \( y \in S \). In this case, the morphism \((6.5.1)\) factors through the natural morphism

\[
(6.5.2) \quad R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/\mathcal{S})(\log D) \rightarrow R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)).
\]

We claim that this morphism is surjective at \( y \in S \). By \((6.5.1)\) we see that the following natural morphism

\[
(6.5.3) \quad R^q f_*(\iota_*\iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D))) \rightarrow R^q f_*(\iota_* (E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)))
\]

is surjective. Set

\[
\mathcal{K}^* := \text{Ker}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D) \rightarrow \iota_* \iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D))).
\]

Then we have the following exact sequence

\[
0 \rightarrow \mathcal{K}^* \rightarrow E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D) \rightarrow \iota_* \iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D))) \rightarrow 0.
\]

Hence we have the following exact sequence

\[
\cdots \rightarrow R^q f_*(\mathcal{K}^*) \rightarrow R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)) \rightarrow R^q f_*(\iota_* \iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D))) \rightarrow R^{q+1} f_*(\mathcal{K}^*) \rightarrow \cdots.
\]

Since \( \mathcal{K}^*|_{f^{-1}(\mathcal{S}) \setminus f^{-1}(S)} = 0 \), \( R^q f_*(\mathcal{K}^*)|_S = 0 \) by the proper base change theorem. Hence

\[
R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D))|_y = R^q f_*(\iota_* \iota^{-1}(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)))|_y.
\]

Now we have only to prove that the morphism

\[
(6.5.4) \quad R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D) \rightarrow R^q f_*(E \otimes_{\mathcal{O}_X} \Omega^*_X_{y/S})(\log D_y))
\]

is surjective. In \((6.4)\) we have already proved this surjectivity.

We complete the proof of this theorem. \(\square\)

Lemma 6.6. Assume that \( \mathcal{S} \) is quasi-compact. Then the following equality holds:

\[
(6.6.1) \quad H^q(X, E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)) = \lim_{S \subset W \subset \mathcal{S}} H^q(f^{-1}(W), E \otimes_{\mathcal{O}_X} \Omega^*_X/S)(\log D)),
\]

where \( W \) runs over a fundamental system of neighborhoods of \( S \).
**Proof.** Consider the morphism $f^{\log} : \mathcal{X}^{\log} \rightarrow \mathcal{S}^{\log}$. If $\epsilon$ is small enough, then we have a natural immersion $\mathcal{S}(\epsilon) \subseteq \mathcal{S}$ (13.1). Let $f^{\log} \circ \mathcal{X}^{\log}(\epsilon) \rightarrow \mathcal{S}^{\log}(\epsilon)$ be the induced morphism by $f$. Because the homotopy types of $\mathcal{X}^{\log}(\epsilon)$ and $\mathcal{X}^{\log}$ are the same, the equality (6.6.1) follows from (13.1) and the faithfully flatness of $\mathcal{O}_{\mathcal{S}^{\log}}$ over $\mathcal{O}_{\mathcal{S}}$ (KN (3.3)).

**Corollary 6.7.** Then there exists a canonical isomorphism

\[
(6.7.1) \quad \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}^{-1}(\mathcal{S}) \rightarrow \mathcal{O}^{1}(\mathcal{S}) \rightarrow \mathcal{O}^{1}(\mathcal{S}^{\log}(\log(D))) \rightarrow \mathcal{O}^{1}(\mathcal{S}^{\log}(\log(D)))
\]

at a small open log analytic space $W$ such that $W \cap S \neq \emptyset$.

**Proof.** First we construct a canonical morphism

\[
(6.7.2) \quad p^{-1} \mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D)) \rightarrow \mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D)).
\]

Let $W$ be any open log subanalytic space of $\mathcal{S}$. The sheaf $\mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))$ is associated to the following presheaf:

\[
W \mapsto \mathcal{H}^q(f^{-1}(W), \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D)).
\]

Because the homotopy types of $f^{-1}(\mathcal{S}(\epsilon))$ and $\mathcal{X}$ are the same for any small $\epsilon > 0$, we have the following equality by (6.6.1):

\[
\mathcal{H}^q(f^{-1}(W), \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D)) = \lim_{(S \cap W) \subseteq W} \mathcal{H}^q(f^{-1}(W_1), \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D)).
\]

Since $f : \mathcal{X} \rightarrow \mathcal{S}$ is proper, the last group is isomorphic to

\[
\lim_{X_{S \cap W} \subseteq V \subseteq \mathcal{X}, f^{-1}(W)} \mathcal{H}^q(V, \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_{V} = \mathcal{H}^q(X_{S \cap W}, \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_{X_{S \cap W}}.
\]

Here we have used the Godement resolution to calculate the cohomologies above. By (6.7.1) we have a canonical morphism

\[
(6.7.3) \quad \mathcal{H}^q(\rho) : \mathcal{H}^q(X_{S \cap W}, \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_{X_{S \cap W}} \rightarrow \mathcal{H}^q(X_{S \cap W}, \mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_{X_{S \cap W}}.
\]

Hence we have a morphism (6.7.2). By (5.14) and (6.5) the source and the target of the morphism (6.7.1) are locally free $\mathcal{O}_{\mathcal{S}}$-modules and commutes with base change. Hence the morphism (6.7.1) induces the following isomorphism

\[
(6.7.4) \quad \mathcal{O} \otimes \mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/s}(\log D_s)) = \mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/s}(\log D_s))
\]

at any point $s \in S$. Take an open log analytic space $W$ such that $W \cap S \neq \emptyset$. Take $s \in W \cap S$. Take an open log analytic space $V$ such that $\mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_V$ is free. Let $\{v_i\}_{i=1}^r$ be a basis of $\mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/S}(\log D))|_V$. Let $\{w_{i,s}\}_{i=1}^r$ be a basis of $\mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/s}(\log D_s))$. Then there exist an open log analytic space $V_1$ and a set $\{w_i\}_{i=1}^r$ of local sections of $\mathcal{R}^q f_*(\mathcal{E} \otimes \mathcal{O}_X \Omega^i_{\mathcal{X}/s}(\log D))|_{V_1}$ which gives $\{w_{i,s}\}_{i=1}^r$ at $s$. Then $v_{ij,s} = \sum_{i=1}^r f_{ij,s} w_i(s)$ at $s$ for some $f_{ij,s} \in \mathcal{O}_{\mathcal{S},s}$. Hence there exist an open log analytic space $V_2$ and a set $\{f_{ij}\}_{i,j=1}^r$ of local sections of $\mathcal{O}_{\mathcal{S},s}(V_2)$ such that $v_j = \sum_{i=1}^r f_{ij} w_i$ at $V_2$. Hence we obtain (6.7).
Example 6.8. (1) Consider the simplest case $\mathcal{O} = \{O\}$. Then $\mathcal{O}$ is the polydisc $\Delta^r$. In this case $R^q f_{t*}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_X/S(\log D))$ is a finite dimensional $\mathbb{C}$-vector space. Set $(\Delta^r)^* := \{u \in \Delta^r \mid M_u = \mathcal{O}^*_u\}$ and let $u$ be a point of $(\Delta^r)^*$. Let $O$ be the origin of $\Delta^r$. Then $S$ is the log point of virtual dimension $r$ whose underlying analytic space is $\{O\}$. Set $U := X \setminus \Delta^r$. If $u$ is sufficiently close from $O$, then we have the following isomorphisms

$$H^q(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_X/S(\log D)) \cong H^q(X_u, \mathcal{E} \otimes_{\mathcal{O}_{X_u}} \Omega^+_u/S_u(\log D_u))$$

$$\cong H^q(U_u, \mathcal{E} \otimes_{\mathcal{O}_{U_u}} \Omega^+_u/S_u(\log D_u)) \cong H^q(U_u, \omega).$$

Here $V := \text{Ker}(\nabla: \mathcal{E}|_{U_u} \rightarrow \mathcal{E}|_{U_u} \otimes_{\mathcal{O}_{U_u}} \Omega^+_u/S_u(\log D_u))$. (We obtain the first isomorphism in (6.8.1) by (6.7.1), the second isomorphism in (6.8.1) by [EGA III-2 (7.7.5)] and the third isomorphism in (6.8.1) by [loc. cit., I (2.27.2)].) This is nothing but the generalization of the isomorphism (1.2.9).

(2) Let $f: E \rightarrow \Delta$ be a log elliptic curve (Tate curve) described in [KU 0.2.10]: for $q \in \Delta^r$, $f^{-1}(q) = \mathbb{C}^*/q^\mathbb{Z}$ and $f^{-1}(0) = \mathbb{P}^1_\mathbb{C}(\mathbb{C})/(0 \sim \infty)$. Let $e_1$ and $\omega = (2\pi \sqrt{-1})^{-1}d\log t_1$ be the global sections of $R^1f_*(\Omega^+_{E/\Delta})$ defined in [loc. cit. 0.2.15]. Then $e_1$ and $\omega$ define an isomorphism

$$R^1f_*(\Omega^+_{E/\Delta}) = \mathcal{O}_{\Delta} \otimes_{\mathbb{C}} H^1_{dR}(E/s).$$

Corollary 6.9. (1.4) holds.

Proof. The problem is local. Hence (1.4) follows from (6.5). \hfill \square

7 Algebraic case

In this section we give algebraic analogues of results in the previous section. Because we have only to use [EGA III-2 (7.7.5)] (resp. [KZ (8.8)]) instead of Grauert’s theorem (resp. [D (2.17)]) used in the proofs of (5.14) and (5.5), we give only statements without proofs.

Theorem 7.1. Let $K$ be a field of characteristic 0. Endow the affine line $\mathcal{A}_K = \text{Spec}(K[t])$ over $K$ with the log structure associated to a morphism $\mathbb{N} \ni 1 \mapsto t \in K[t]$. Let $\mathcal{A}_K$ be the resulting log scheme. Let $r_1$ be a positive integer and let $r_2$ be a nonnegative integer. Set $B := \mathcal{A}_K^{r_1} \times \mathcal{A}_K^{r_2}$. Let $f: X \rightarrow B$ be a (not necessarily proper) P-(not necessarily strictly)-semistable scheme with horizontal NCDS (normal crossing divisors) $D$. (See the text for the notion of the P-semistable scheme.) Let $(\mathcal{E}, \nabla)$ be a locally nilpotent integrable connection on $(X, D): \nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/C}(\log D)$. Let $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_{X/B}(\log D)$ be the log de Rham complex obtained by $(\mathcal{E}, \nabla)$. Let $\{O\}$ be the origin of $B$. Endow $\{O\}$ with the pull-back of the log structure of $B$. Let $s$ be the resulting log analytic space. Set $S := s \times \mathcal{A}_K^{r_2}$, $X := X \times_BS$ and $D := D \times_BS$. Let $\iota: X \rightarrow X$ be the natural exact closed immersion. Then there exists a canonical contravariantly functorial section

$$\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_{X/S}(\log D) \rightarrow \iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_{X/B}(\log D))$$

of the natural morphism

$$\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_{X/B}(\log D) \rightarrow \iota^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^+_{X/S}(\log D)).$$
in the derived category $D^+(\iota^{-1} f^{-1}(\mathcal{O}_B))$ of bounded below complexes of $\iota^{-1} f^{-1}(\mathcal{O}_B)$-modules. If $\mathcal{X}$ is proper over $\mathcal{B}$, then there exists a canonical contravariantly functorial section

(7.1.3) \[ R_f(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D)) \rightarrow R_f(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log D)) \]

of the natural morphism

(7.1.4) \[ R_f(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log D)) \rightarrow R_f(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D)) \]

in the derived category $D^+(\mathcal{O}_B)$ of the bounded below complexes of $\mathcal{O}_B$-modules. This section induces the following isomorphism

(7.1.5) \[ \mathcal{O}_B \otimes_{\mathcal{O}_S} R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log D)) \cong R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/B}(\log D)) \quad (q \in \mathbb{N}) \]

at a small open log analytic space $W$ such that $W \cap S \neq \emptyset$.

**Proof.** We have only to use [EGA III-2 (7.7.5)] (resp. [Kz (8.8)]) instead of Grauert’s theorem (resp. [1] (2.17)) used in the proofs of (5.14) and (5.15). \[ \square \]

**Corollary 7.2.** Let $\mathcal{Y}$ be a smooth scheme with an NCD $Y$ over a field $K$ of characteristic 0. Let $\mathcal{Y}$ be a resulting log scheme. Let $\overline{g}: \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth schemes over $K$. Assume that $X := g(Y)$ is a normal crossing divisor on $\mathcal{X}$. Let $g: X \rightarrow Y$ be a $P$-semistable morphism over $K$. Let $\mathcal{D}$ be a horizontal NCD on $X$. Let $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}(\log \mathcal{D})$ be a locally nilpotent integrable connection on $X$. Then the sheaf $R^q f_* (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}(\log \mathcal{D}))$ $(q \in \mathbb{N})$ is a coherent locally free $\mathcal{O}_Y$-module and commutes with base change.

**Remark 7.3.** In the algebraic case, we can prove the analogue of (5.7) by a simpler method as follows.

We may assume that $\mathcal{E} = \mathcal{O}_X$. We may assume that $S = S_1 \times_K \cdots \times_K S_r$ and $X = X_1 \times_K \cdots \times_K X_r$. We may assume that $D$ is a horizontal NCD on $X_r$. Denote $D$ by $D_1$ and set $D_i := 0$ for $i \geq 2$. Then $\Omega^\bullet_{X/S}(\log D) = \bigoplus_{i=1}^r \Omega^\bullet_{X_i/S_i}(\log D_i)$. Set $I_n := \{(n_1, \ldots, n_r) \in \mathbb{N}^r \mid \sum_{i=1}^r n_i = n\}$. We have the following equality:

(7.3.1) \[ A^n((X, D)/S, \mathcal{O}_X) := \bigoplus_{j \in \mathbb{N}} (\Omega^{n+r}_{X/S}(\log D)/P^{X_j}_j) \]

is an algebraic analogue of (5.6) the morphism

(7.3.2) \[ d\log t_i \wedge: \Omega^\bullet_{X_i/S_i}(\log D_i) \rightarrow A^\bullet((X_i, D_i)/S_i, \mathcal{O}_{X_i}) \]

is a quasi-isomorphism. By the K"unneth formula we see that the morphism (5.5.6) is a quasi-isomorphism.
References

[B] Bourbaki, N. *Algèbre commutative*. Masson, Paris, (1985).

[D] Deligne, P. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Math. 163, Springer-Verlag (1970).

[DI] Deligne, P., Illusie, L. *Relèvements modulo $p^2$ et décomposition du complexe de de Rham*. Invent. Math. 89 (1987), 247–270.

[EGA III-2] Grothendieck, A., Dieudonné, J. *Éléments de géométrie algébrique III-2*. Publ. Math. IHÉS 17 (1963).

[EY] Ertl, V., Yamada, K. *Rigid analytic reconstruction of Hyodo-Kato theory*. Preprint: available from https://arxiv.org/abs/1907.10964.

[F1] Fujisawa, T. *Limits of Hodge structures in several variables*. Comp. Math. 115 (1999), 129–183.

[F2] Fujisawa, T. *Limits of Hodge structures in several variables, II*. Preprint: available from https://arxiv.org/pdf/1506.02271.pdf.

[F3] Fujisawa, T. *Limiting mixed Hodge structures on the relative log de Rham cohomology groups of a projective semistable log smooth degeneration*. Preprint: available from https://arxiv.org/pdf/2011.00926.pdf.

[G] Grauert, H. *Ein Theorem der analytischen Garbentheorie und die Modulkörper komplexer Strukturen*. IHÉS Publ. Math. 5 (1960), 5–64.

[HK] Hyodo, O., Kato, K. *Semi-stable reduction and crystalline cohomology with logarithmic poles*. In: *Périodes p-adiques*, Seminaire de Bures, 1988. Astérisque 223, Soc. Math. de France (1994), 221–268.

[IKN] Illusie, L., Kato, K., Nakayama, C. *Quasi-unipotent Logarithmic Riemann-Hilbert Correspondences*. J. Math. Sci. Univ. Tokyo 12 (2005), 1–66.

[KF] Kato, F., *The relative log Poincaré lemma and relative log de Rham theory*. Duke Math. J. 93 (1998), 179–206.

[KK] Kato, K. *Logarithmic structures of Fontaine-Illusie*. In: *Algebraic analysis, geometry, and number theory*, Johns Hopkins Univ. Press (1989), 191–224.

[KN] Kato, K., Nakayama, C. *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over $\mathbb{C}$. Kodai Math. J. 22* (1999), 161–186.

[KU] Kato, K., Usui, S. *Classifying spaces of degenerating polarized Hodge structures*. Annals of Mathematics Studies 169, Princeton Univ. Press (2009).

[Kz] Katz, N. M. *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*. IHÉS Publ. Math. 39 (1970), 175–232.

[Mo] Mokrane, A. *La suite spectrale des poids en cohomologie de Hyodo-Kato*. Duke Math. J. 72 (1993), 301–336.
[N1] Nakkajima, Y.  *p*-adic weight spectral sequences of log varieties. J. Math. Sci. Univ. Tokyo 12 (2005), 513–661.

[N2] Nakkajima, Y. Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic $p > 0$. Mém. Soc. Math. France 130–131 (2012).

[N3] Nakkajima, Y. Limits of weight filtrations and limits of slope filtrations on infinitesimal cohomologies in mixed characteristics I. Preprint: available from https://arxiv.org/abs/1902.00182.

[N4] Nakkajima, Y. Derived PD-Hirsch extensions of filtered crystalline complexes and filtered crystalline dga’s. Preprint: available from http://arxiv.org/abs/2012.12981.

[N5] Nakkajima, Y. An ideal proof for Fujisawa’s result and its generalization. Preprint: available from https://arxiv.org/abs/2012.12985.

[NS] Nakkajima, Y., Shiho, A. Weight filtrations on log crystalline cohomologies of families of open smooth varieties. Lecture Notes in Math. 1959, Springer-Verlag (2008).

[O] Ogus, A. *On the logarithmic Riemann-Hilbert correspondence*. Documenta Math. Extra Volume: Kazuya Kato’s Fiftieth Birthday (2003), 655–724.

[S] Steenbrink, J. H. M. *Limits of Hodge structures*. Invent. Math. 31 (1976), 229–257.

[SGA 7-II] Deligne, P., Katz, N. *Groupes de monodromie en géométrie algébrique*. Lecture Notes in Math. 340 (1973), Springer-Verlag.

[Su] Sullivan, D. *Infinitesimal computations in topology*. Publ. Math. IHÉS 47 (1977), 269–331.

[SZ] Steenbrink, J. H. M., Zucker, S. Variation of mixed Hodge structure. I. Invent. Math. 80 (1985), 489–542.

[U] Usui, S. Recovery of vanishing cycles by log geometry: Case of several variables. In: Proceeding of International Conference “Commutative Algebra and Algebraic Geometry and Computational Methods”, Hanoi 1996, Springer-Verlag (1999), 135–144.

[W] Weibel, C. A. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge (1994).

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