Quantum wells and the generalized uncertainty principle

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Received 19 May 2014, revised 18 July 2014
Accepted for publication 1 August 2014
Published 9 September 2014

Abstract
The finite and infinite square wells are potentials typically discussed in undergraduate quantum mechanics courses. In this paper, we discuss these potentials in the light of the recent studies of the modification of the Heisenberg uncertainty principle into a generalized uncertainty principle (GUP) as a consequence of attempts to formulate a quantum theory of gravity. The fundamental concepts of the minimal length scale and the GUP are discussed and the modified energy eigenvalues and transmission coefficient are derived.

Keywords: square well, generalized uncertainty principle, minimal length

1. Introduction
The prediction by quantum gravity theories such as string theory and loop quantum gravity, of a minimum length scale has paved the way to discuss quantum gravity effects (which used to only be accessible to students with backgrounds in general relativity and quantum field theory) in ordinary non-relativistic quantum mechanics at the level studied in undergraduate quantum mechanics. In string theory for instance, this minimum measurable length is of the order of the dimension of the fundamental string. A number of papers have been written discussing the phenomenological consequences of the minimal length scale [1–5]. A considerable number of references can be found in a recent review by Hossenfelder [6].

This minimal length can be shown to arise from the modification of the Heisenberg uncertainty principle (UP) to a generalized uncertainty principle (GUP). The UP is given by $\Delta x_i \Delta p_i \geq \frac{\hbar}{2}$, where the index $i = 1, 2, 3$ represent the $x$, $y$, and $z$ directions of the position and

\[ (\Delta x_i)(\Delta p_i) \geq \frac{\hbar}{2} \]

with $\langle q \rangle = \langle p \rangle = \langle \tilde{p} \rangle$, see [7].

1 A ‘tighter’ form of the uncertainty principle is given by $(\Delta q)^2(\Delta p)^2 - \left(\frac{\langle q \rangle - \langle \tilde{q} \rangle}{2} - \langle \tilde{q} \rangle\right)^2 \geq \frac{\hbar^2}{4}$.
momentum of a particle, with \[ \hat{x}_i, \hat{p}_j = i\hbar \delta_{ij}. \] Quantum gravity theories however, modify this commutation relation to [3]

\[
\left[ \hat{x}_i, \hat{p}_j \right] = i\hbar \left[ \delta_{ij} + \beta \delta_{ij} p^2 + 2\beta p_i p_j \right],
\]

where \( \beta \) is a small parameter which depends on the Planck length and Planck’s constant. This modification in the commutator gives rise to a GUP [3]

\[
\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \left[ 1 + \beta \left( (\Delta p)^2 + (\langle p \rangle)^2 \right) + 2\beta \left( \Delta p_i^2 + \langle p_i \rangle^2 \right) \right].
\]

Confining the discussion to one dimension, we see that for the UP, \( \Delta x \geq \frac{\hbar}{2\Delta p} \). Hence, one can in principle decrease \( \Delta x \) arbitrarily by increasing \( \Delta p \). We show this behaviour in figure 1 which gives a schematic plot of \( \Delta x \sim 1/\Delta p \) (blue graph). However for the GUP in equation (2), \( \Delta x \sim \frac{\hbar}{2\Delta p} + \frac{\hbar \beta}{2} \Delta p \), and increasing \( \Delta p \) does not arbitrarily decrease \( \Delta x \) because the term \( \frac{\hbar \beta}{2} \Delta p \) dominates. Instead, this yields a minimum value for \( \Delta x = \Delta x_0 \), which is the minimum length. This is also shown schematically in figure 1 by the graph of \( \Delta x \sim 1/\Delta p + \Delta p \) (red graph). It can be shown that equation (1) can be realized by setting [3]

\[
\Delta x_0 \Delta p = \frac{\hbar}{\beta} + \frac{\hbar \beta}{2},
\]

with

\[
p_0^2 = \sum_{j=0}^3 \hat{p}_{0j} \hat{p}_{0j}, \quad \text{and} \quad \hat{x}_0, \hat{p}_0 \text{ satisfying the usual commutation relation} \]

\[
[\hat{x}_0, \hat{p}_0] = i\hbar \delta_{ij} \quad \text{and with} \quad \hat{p}_0 = \frac{\hbar}{i} \frac{d}{dx}. \]

In this paper we will illustrate how equation (3) will modify the quantum mechanics of the one-dimensional finite square well and the infinite square well. We chose to study these familiar quantum wells because they are discussed in a typical undergraduate quantum mechanics class.

In section 2, we discuss how the GUP modifies the Hamiltonian in the time-independent Schrödinger equation (TISE). In section 3, we derive the GUP-modified results for the finite and infinite square well potentials. Some conclusions are discussed in section 4.

2. Gravitational effects in quantum mechanics

Given a particle of mass \( m \) subjected to a potential \( V(x) \), one can describe its quantum mechanical behaviour by solving the wavefunctions \( \psi(x) \) and its energies \( \epsilon \) from the TISE

\[
\hat{H} \psi = \epsilon \psi \Rightarrow \left( \frac{\hat{\rho}^2}{2m} + V \right) \psi = \epsilon \psi,
\]

where \( \hat{H} \) is the Hamiltonian operator given by \( \frac{\hat{\rho}^2}{2m} + V \). In ordinary quantum mechanics

\[
\hat{\rho} = \frac{\hbar}{i} \frac{d}{dx}.
\]

Plugging in equation (5) into equation (4)

\[
\epsilon \psi = \left( \frac{1}{2m} \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 + V \right) \psi = - \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V \psi,
\]

which is the familiar form of the TISE in ordinary quantum mechanics. The Schrödinger equation is modified by the GUP by substituting equation (3) with \( \hat{\rho}_0 = \frac{\hbar}{i} \frac{d}{dx} \) into equation (4). Keeping only terms up to the order of \( \beta \), we get
\[ \varepsilon \psi = \left( \frac{p_x^2}{2m} \left( 1 + \beta p_0^2 \right)^2 + V \right) \psi = \left( \frac{p_x^2}{2m} \left( 1 + 2 \beta p_0^2 \right) + V \right) \psi \]

or

\[ \left( \frac{p_x^2}{2m} + \frac{\beta}{m} p_0^4 + V \right) \psi = \varepsilon \psi. \]

Substituting \( \hat{p}_0 = \frac{\hbar}{i \partial} \) into the preceding equation gives

\[ \varepsilon \psi = \left( - \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\beta \hbar^4}{m} \frac{d^4}{dx^4} + V \right) \psi. \]

Equation (7) is the GUP version of equation (6) with the extra term \( \frac{\beta \hbar^4}{m} \frac{d^4}{dx^4} \). It is interesting to note that the extra term is similar to the first correction (quartic in \( p \)) that arises when the classical kinetic energy term is replaced by the special relativistic kinetic energy expression in the TISE (see 268 of [8]).

3. GUP effects in the finite and infinite square wells

The finite square well potential is given by

\[ V(x) = \begin{cases} -V_0 & \text{for } -a < x < a, \\ 0 & \text{for } |x| > a \end{cases} \]  

(8)

With the modified Schrödinger equation due to GUP, we have to solve equation (7) with \( V(x) \) given by equation (8).

For the bound states with \( \varepsilon = -E \), where \( V_0 > E > 0 \), we get the following physical solutions (we follow the method of [3] in solving the fourth order differential equation)
where \( \psi_1(x), \psi_2(x), \) and \( \psi_3(x) \) are the solutions for the regions \( x < -a, \) \( -a < x < a, \) and \( x > a \) respectively, \( k' = k + \hbar^2k^3\beta, \) \( k = \sqrt{\frac{2mE}{\hbar^2}} \) and \( l' = l - \hbar^2k\beta, \) \( l = \sqrt{\frac{2m}{\hbar^2}}(V_0 - E). \)

Applying the boundary conditions

\[
\begin{align*}
\psi_1(-a) &= \psi_2(-a) \\
\psi_2(a) &= \psi_3(a) \\
\frac{d\psi_1}{dx} \bigg|_{x=-a} &= \frac{d\psi_2}{dx} \bigg|_{x=-a} \\
\frac{d\psi_2}{dx} \bigg|_{x=a} &= \frac{d\psi_3}{dx} \bigg|_{x=a},
\end{align*}
\]

we get the following system of equations

\[
\begin{align*}
Be^{-kx} - Fe^{-il'x} - Ge^{-ilx} &= 0 \\
Bk'e^{-k'a} - IF'e^{-il'x} + iGl'e^{-ilx} &= 0 \\
F'e^{ilx} + Ge^{-ilx} - Jk'e^{-kax} &= 0 \\
iF'e^{-ilx} - iGl'e^{-ilx} + Jk'e^{-kax} &= 0.
\end{align*}
\]

Solving this system of equations for the constants \( B, F, G, \) and \( J, \) yields a trivial solution in which the constants are zero which yields non-normalizable solutions. Hence we set the determinant of the matrix with elements consisting of the coefficients of \( B, F, G, \) and \( J, \) to zero. This yields

\[
e^{-4ai(-1+\hbar^2\beta)} = \frac{(il - k)(2i\hbar^2l^3\beta - il + k + 2\hbar^2k^3\beta)}{(il + k)(2i\hbar^2l^3\beta - il - k - 2\hbar^2k^3\beta)}.
\]

Using the formula, \( e^{i\theta} = \cos \theta + i\sin \theta, \) we get from the preceding equation,

\[
\tan \left( 2al \left( -1 + \hbar^2l^2\beta \right) \right) = \frac{2k \left( -\hbar^2l^2\beta + 1 + \hbar^2k^2\beta \right)}{k^2 + \hbar^2k^4\beta + 2l^4\hbar^2\beta - l^2}.
\]

Using the trigonometric identity \( \tan 2\gamma = \frac{2\tan \gamma}{1 - \tan^2\gamma}, \) we get

\[
\tan \left( al \left( 1 - \hbar^2l^2\beta \right) \right) = \frac{k}{l} \left( 1 + \hbar^2\beta \right) \left( k^2 + l^2 \right) \quad \text{(even solutions)}
\]

\[
\tan \left( al \left( 1 - \hbar^2l^2\beta \right) \right) = -\frac{l}{k} \left( 1 - \hbar^2\beta \right) \left( k^2 + l^2 \right) \quad \text{(odd solutions)}. \quad (11)
\]

It is important to note that the equations in (11) reduce to the non-GUP case \( (\beta = 0), \) \( \tan la = kl \) (even solutions) and \( \tan la = -l/k \) or \( \cot la = -k/l \) (odd solutions), which are the well-known results found in textbooks [8]. The preceding results agree with the paper by Vahedi [5] which used a different approach less familiar to students in undergraduate quantum mechanics class.

To see the relative behaviour of the energy eigenvalues, we plot both sides of equation (11) in figures 2 and 3 (with the energy in the horizontal axis) using Planck units.
with $\hbar = 1$, $V_0 = 8$, $a = \pi$ and $m = 1/2$. The energies are given by the intersections of the graphs (shown by the red and blue dots). The $\beta = 0$ graphs (blue graphs) correspond to the usual finite square well solutions (non-GUP) while the $\beta = 0.01$ (red graph) corresponds to the GUP result. It is clear that the energy eigenvalues are shifted in the GUP solutions.

Looking next at the scattering states, with $\varepsilon = E$, where $E > 0$, we get the following physical solutions, assuming a particle incident from the left ($x < -a$)

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**Figure 2.** Graphs of equation (11) for $f(E) = \tan \left( aE \left( 1 - \hbar^2 \beta \right) \right)$ and $\frac{1}{E} \left( 1 + \hbar^2 \beta \left( k^2 + F^2 \right) \right)$ with $\beta = 0$ (blue) and $\beta = 0.01$ (red), (even solutions).

**Figure 3.** Graphs of equation (11) for $f(E) = \tan \left( aE \left( 1 - \hbar^2 \beta \right) \right)$ and $\frac{1}{E} \left( 1 - \hbar^2 \beta \left( k^2 + F^2 \right) \right)$ with $\beta = 0$ (blue) and $\beta = 0.01$ (red), (odd solutions).
where $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$ are the solutions for the regions $x < -a$, $-a < x < a$, and $x > a$ respectively, $k^* = k - k^3 \hbar^2 \beta$ and $\sigma' = \sigma - \sigma^3 \hbar^2 \beta$. $\sigma = \sqrt{\frac{2m}{\hbar^2}} (V_0 + E)$. Applying once more the boundary conditions of equation (10), we get the following system of equations

\[
\begin{align*}
\psi_1(x) &= Ae^{ik^*x} + Be^{-ik^*x} \\
\psi_2(x) &= He^{ik^*x} + J e^{-ik^*x} \\
\psi_3(x) &= Ce^{ik^*x},
\end{align*}
\]

\[
(12)
\]

One can solve for the coefficients $B$, $H$, $J$, and $C$ in terms of $A$ in the above equation. Computing for the transmission coefficient $T_{\text{GUP}} = \left( \frac{\psi_1^*}{\psi_2} \right) \left( \frac{\psi_3}{\psi_4} \right)$, we get the following equation:

\[
T_{\text{GUP}} = -\left( \sigma' \right)^2 \left( k^* \right)^2 \left\{ -2 \left( k^* \right)^2 \left( \sigma' \right)^2 \cos^2(\sigma'a) \\
- \left( k^* \right)^2 \left( \sigma' \right)^4 + 2 \left( k^* \right)^2 \left( \sigma' \right)^4 \cos^2(\sigma'a) \\
- \cos^2(\sigma'a) \left( k^* \right)^4 + \cos^4(\sigma'a) k^4 \\
- \cos^2(\sigma'a) \left( \sigma' \right)^4 + \cos^4(\sigma'a) \left( \sigma' \right)^4 \right\}^{-1},
\]

\[
(13)
\]

The above equation can be shown to reduce to the non-GUP ($\beta = 0$) result [8]:

\[
T = \left\{ 1 + \frac{V_0^2}{4E(V_0 + E)} \cdot \sin^2 \left( \frac{2\sqrt{2m(V_0 + E)}}{\hbar} a \right) \right\}^{-1}.
\]

We compare $T$ and $T_{\text{GUP}}$ in figure 4, where we plot the transmission coefficients using equation (13) (with the energy in the horizontal axis) using Planck units with
\( h = 1, \ V_0 = 8, \ a = \pi \) and \( m = 1/2 \). The \( \beta = 0 \) graphs (blue graph) correspond to the usual finite square well transmission coefficient \([8]\) (non-GUP) while the \( \beta = 0.01 \) (red graph) corresponds to the GUP result. The energies at which the transmission coefficient are equal to one are shifted in the GUP result.

As another example of the effect of the GUP, let us discuss briefly the infinite square well or the particle-in-a-box potential. The potential is given by

\[
V(x) = \begin{cases} 
0 & \text{for } 0 < x < a \\
\infty & \text{otherwise}.
\end{cases}
\]

Of course, \( \psi = 0 \) outside \( 0 < x < a \) since \( V \) is infinite in these regions. For the region inside the box, \( 0 < x < a \), in which \( V = 0 \), we have to solve (from equation \((7)\))

\[
-\frac{\hbar^2 \psi''}{2m} + \frac{\hbar^2 \psi'}{m} = E\psi.
\]

The physical solutions are given by \( \psi(x) = Ae^{ikx} + Be^{-ikx} \). From the boundary condition \( \psi(0) = 0 \) and \( \psi(a) = 0 \), we get the usual form of the solution, \( \psi(x) = C \sin kx \) with \( k = \frac{n \pi}{a} \), where \( n = 1, 2, 3, \ldots \). One can solve for \( k \) and then the energies to be

\[
E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} + \beta \frac{n^4\pi^4\hbar^4}{ma^4}.
\]

The preceding equation agrees with a previous result \([4]\) and clearly reduces to the ordinary (non-GUP) energies of the particle-in-a-box when \( \beta = 0 \).

4. Conclusions

This paper demonstrates that quantum gravity effects can be discussed at the level of ordinary quantum mechanics without the need for a background in general relativity and quantum field theory. By a straightforward revision of the time-independent Schrödinger equation using the modified momentum operator in equation \((3)\), one can solve for the shifted energies of the bound states of the finite and infinite square wells and the shifted values of the transmission coefficient of the scattering states of the finite square well.

It is not surprising that several papers have been written discussing GUP effects in potentials typically discussed in undergraduate quantum mechanics classes such as the infinite square well \([1, 4]\), free particle \([9]\), harmonic oscillator \([10, 11]\), and the hydrogen atom \([12]\). Reference \([3]\) also discussed the potential step and potential barrier functions. Besides describing the quantum mechanics of point-like objects, the minimal length can also be applied in describing non-pointlike systems such as nucleons, quasi-particles and collective excitations. Reference \([13]\) works out the application of the minimal length to a non-pointlike particle in a \( d \)-dimensional isotropic harmonic oscillator. Another interesting potential that will be the subject of a future paper is the double square well potential \([14]\), where tunnelling is an important phenomenon. The methods discussed here can be readily applied to this potential. We believe that research on the phenomenological consequences of the minimal length is accessible to undergraduate students with a background in quantum mechanics.

Acknowledgements

The authors would like to thank the referees for their enlightening suggestions to improve the presentation of the manuscript.
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