ON $T-$LOCALLY COMPACT SPACES

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Abstract. The aim of this paper is to introduce and give preliminary investigation of $T-$locally compact spaces. Locally compact and $T-$locally compact are independent of each other. Every Hausdorff, locally compact space is $T-$locally compact. $T-$locally compact is a topological property. $T-$locally compact is not preserved by the product topology.

1. Introduction

By a space, we mean a topological space. A space $X$ will be called $T-$locally compact if for every open set $U$ containing $x$, there exists an open subset $V$ containing $x$ such that $\partial V$ is compact and $V \subseteq U$. The space $X$ is called $T-$locally compact if $X$ is $T-$locally compact at each of points. Locally compact and $T-$locally compact are independent of each other (Examples 2 and Example 3). We show that a Hausdorff, locally compact space is $T-$locally compact (Lemma 4). Open or closed subspace of a $T-$locally compact space is $T-$locally compact (Lemma 6 and Lemma 7). We show that $T-$locally compact is a topological property (Theorem 8). The product of two $T-$locally compact spaces need not to be $T-$locally compact (Example 4). We give some conditions such that the product of two $T-$locally compact spaces is $T-$locally compact (Lemma 9 and Theorem 10).

Throughout, $clA$, $intA$ and $\partial A$ will denote the closure, the interior and the boundary of a set $A$ respectively. Assume that $I$ be a non-empty index set and for every $i \in I$, $X_i$ be a space. We denote by $\prod_{i \in I} X_i$, the cartesian product of $X_i$ with the product topology. For more information on topological spaces, see [1].

2. $T-$LOCALLY COMPACT SPACE

In this section, we introduce the concept and study some properties of $T-$locally compact space.

Definition 1. A space $X$ is called $T-$locally compact at $x$ if for every open set $U$ containing $x$, there exists an open subset $V$ containing $x$ such that $\partial V$...
is compact and \( V \subseteq U \). The space \( X \) is called \( T \)–locally compact if \( X \) is \( T \)–locally compact at each of points.

**Example 1.** Let \( \mathbb{R} \) be the reals with the usual topology. Assume that \( x \in U \) for some open subset \( U \) of \( \mathbb{R} \). Then, \( x \in (a, b) \) for some real numbers \( a \) and \( b \) which \( (a, b) \subseteq U \). It is clear that \( \partial(a, b) = \{ a, b \} \) is compact in \( \mathbb{R} \). So, \( \mathbb{R} \) is \( T \)–locally compact. Also, it is clear that every discrete space is \( T \)–locally compact.

**Definition 2.** A subset \( A \) of a space \( X \) is called nowhere dense if \( X - \text{cl}A \) is dense.

**Lemma 3.** The boundary of an open or closed subsets of a space is nowhere dense.

*Proof.* It is clear. \( \Box \)

Locally compact and \( T \)–locally compact are independent of each other. See the examples 2 and 3.

**Example 2.** Consider \( \mathbb{Q} \) as a subspace of \( \mathbb{R} \) (with the usual topology) and \( U \), be an open subset of \( \mathbb{Q} \). We know that the only compact sets in \( \mathbb{Q} \) are nowhere dense. Hence, by Lemma 3, \( \partial U \) is compact. So, \( \mathbb{Q} \) is \( T \)–locally compact. But, \( \mathbb{Q} \) is not locally compact.

**Example 3.** Let \( X \) be an infinite set and \( p \in X \). Define \( \tau = \phi \cup \{ U; p \in U \} \). Then, \( \tau \) is a topology on \( X \). It is clear that \( (X, \tau) \) is not Hausdorff, locally compact space. Since \( \partial\{ p \} = X - \{ p \} \) is not compact, \( (X, \tau) \) is not a \( T \)–locally compact space.

**Lemma 4.** A Hausdorff, locally compact space is a \( T \)–locally compact space.

*Proof.* Let \( x \in X \) and \( U \) be an open set containing \( x \). Since \( X \) is Hausdorff and locally compact, there is an open set \( V \) containing \( x \) such that \( \text{cl}V \) is compact and \( V \subseteq U \). It is clear that \( \partial V \) is compact as a closed subset of compact set \( \text{cl}V \). \( \Box \)

**Lemma 5.** Every compact space \( X \) is \( T \)–locally compact, even though \( X \) is not Hausdorff.

*Proof.* Let \( X \) be a compact space. Then, for every open subset \( U \) of \( X \), \( \partial U \) is compact and proof is complete. \( \Box \)

**Lemma 6.** An open subspace of a \( T \)–locally compact space is \( T \)–locally compact.

*Proof.* Let \( X \) be a \( T \)–locally compact space and \( Y \), an open subspace of \( X \). Let \( y \in Y \) and \( U \) be an open set in \( Y \) containing \( y \). Then, \( U \) is open in \( X \). Since \( X \) is \( T \)–locally compact, there is an open set \( V \) in \( X \) containing \( y \) such that \( \partial V \) is compact and \( V \subseteq U \). So, \( Y \) os \( T \)–locally compact. \( \Box \)
Remark 1. Let $Y$ be a closed subspace of $X$ and $U$, an open subset in $X$. Then,

$$\partial_Y(U \cap Y) = cl_Y(U \cap Y) - int_Y(U \cap Y)$$

$$= clU \cap Y - U \cap Y$$

$$= \partial U \cap Y$$

Lemma 7. A closed subspace of a $T$–locally compact space is $T$–locally compact.

Proof. Let $X$ be a $T$–locally compact space and $Y$, a closed subspace of $X$. Let $y \in Y$ and $U$ be an open set in $Y$ containing $y$. Then, there exists an open set $W$ in $X$ such that $U = W \cap Y$. Since $X$ is $T$–locally compact, there exists an open set $V$ containing $y$ such that $\partial V$ is compact and $V \subseteq W$. By Remark 1, $\partial_Y(U \cap V) = \partial V \cap Y$ which is compact. Also, $V \cap Y \subseteq U$. So, $Y$ is $T$–locally compact. \qed

Theorem 8. $T$–locally compact is a topological property.

Proof. Let $X$ and $Y$ be two spaces and $f : X \to Y$, a homomorphism. Let $X$ be a $T$–locally compact space. We show that $Y$ is $T$–locally compact. Let $y \in Y$ and $V$ be an open subset of $Y$ containing $y$. Then, $f(x) = y$ for some $x \in X$ and $x \in f^{-1}(V)$ is an open subset of $X$. There exists an open subset $U$ of $X$ containing $x$ such that $U \subseteq f^{-1}(V)$ and $\partial U$ is compact. Since $f$ is open, $f(U)$ is open in $Y$ and is contained in $V$. Also,

$$\partial f(U) = cl f(U) - f(U) = f(cl U) - f(U) \subseteq f(\partial U)$$

So, $\partial f(U)$ is compact and $Y$ is $T$–locally compact. \qed

If $X$ and $Y$ be two $T$–locally compact spaces, then, $X \times Y$ need not to be $T$–locally compact. See the Example 4.

Example 4. Let $\mathbb{R}$ be the reals with the usual topology and $\mathbb{Q}$, the rationales with the subspace topology. We show that $\mathbb{Q} \times \mathbb{R}$ is not $T$–locally compact. Let $N = (0, 1)^2 \cap (\mathbb{Q} \times \mathbb{R}) ((0, 1)^2 = (0, 1) \times (0, 1))$. We claim that the boundary of every nonempty open subset of $N$ is not compact. Let $U \subseteq N$ be an open set. First, we show that $\pi_2(U) \subseteq \pi_1(\partial U)$. Let $x \in \pi_1 U$. Assume to contrary, $x \notin \pi_1(\partial U)$. Then, $\{x\} \times \mathbb{R} \subseteq U \subseteq N$. So, $\pi_2(\{x\} \times \mathbb{R}) \subseteq \pi_2(N) = (0, 1)$ which is a contradiction. Now, if $\partial U$ is compact, then $cl \pi_1(U)$ is compact in $\mathbb{Q}$ which is a contradiction (since $int \pi_1(U) \neq \emptyset$). So, $\mathbb{Q} \times \mathbb{R}$ is not $T$–locally compact.

Lemma 9. Let $X$ be a discrete space and $Y$, a $T$–locally compact space. Then, $X \times Y$ is $T$–locally compact.

Proof. Let $N$ be an open subset of $X \times Y$ containing $(x, y)$. Then, there exists an open subset $V$ of $Y$ containing $y$ such that $\partial V$ is compact. It is clear that
$\{x\} \times V \cap N$ is an open subset of $X \times Y$ containing $(x, y)$. Also,
\[
\partial((\{x\} \times V) \cap N) \subseteq (\{x\} \times \partial V) \cap \partial N
\]
So, $\partial((\{x\} \times V) \cap N)$ is compact. Hence, $X \times Y$ is $T$–locally compact.

**Theorem 10.** Let $Y$ be a compact space. Then, $X \times Y$ is $T$–locally compact if and only if $X$ is $T$–locally compact.

**Proof.** First, suppose that $X \times Y$ be $T$–locally compact. Let $U$ be an open subset of $X$ containing $x$. The, $U \times Y$ is an open set in $X \times Y$ containing $(x, y)$ for some $y \in Y$. So, there exists an open subset $N$ of $X \times Y$ containing $(x, y)$ such that $\partial N$ is compact and $N \subseteq X \times Y$. Since $Y$ is compact, $\pi_1$ is a closed map. Hence, $\partial \pi_1(N) \subseteq \pi_1(\partial N)$. So, $\partial \pi_1(N)$ is compact and $\pi_1(N) \subseteq U$.

Conversely, Let $N$ be an open subset of $X \times Y$ containing $(x, y)$. Then, $\pi_1(N)$ is an open set in $X$ containing $x$. Since $X$ is $T$–locally compact, there exists an open set $U$ of $X$ containing $x$ such that $\partial U$ is compact. Clearly, $(x, y) \in (U \times Y) \cap N \neq \emptyset$. Since $\partial((U \times Y) \cap N) \subseteq (\partial U \times Y) \cap \partial N$, so $(U \times Y) \cap N$ is an open set containing $(x, y)$ such that $\partial((U \times Y) \cap N)$ is compact. It shows that $X \times Y$ is $T$–locally compact. \qed

**Corollary 11.** Let $X$ and $Y$ be two Hausdorff spaces. If $X \times Y$ is $T$–locally compact, then $X$ and $Y$ are $T$–locally compact.

**Proof.** Let $y \in Y$. By Lemma 7, $X \times \{y\}$ is $T$–locally compact. Hence, by Theorem 10, $X$ is $T$–locally compact. Similarly, $Y$ is $T$–locally compact. \qed

**Corollary 12.** Let $\{X_i; i \in I\}$ be an arbitrary family of Hausdorff spaces. If $\prod_i X_i$ is a $T$–locally compact space, then each $X_i$ is $T$–locally compact.

**References**

[1] N. Bourbaki, *Elements of mathematics: General topology*, Springer Verlag, Chapters 1-4, Berlin 1995.