Non-perturbative Aspects of Schwinger-Dyson Equations

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Abstract.

Schwinger-Dyson equations (SDEs) provide a natural starting point to study non-perturbative phenomena such as dynamical chiral symmetry breaking in gauge field theories. We briefly review this research in the context of quenched quantum electrodynamics (QED) and discuss the advances made in the gradual improvement of the assumptions employed to solve these equations. We argue that these attempts render the corresponding studies more and more reliable and suitable for their future use in the more realistic cases of unquenched QED, quantum chromodynamics (QCD) and models alternative to the standard model of particle physics.

INTRODUCTION

The standard model of particle physics is highly successful in collating experimental information on the basic forces. Yet, its key parameters, the masses of the quarks and leptons, are theoretically undetermined. In the simplest version of the model, these masses are specified by the couplings of the Higgs boson, couplings that are in turn undetermined. However, it could be that it is the dynamics of the fundamental gauge theories themselves that generate the masses of all the matter fields. To explore this possibility, the favorite starting point is to consider quenched QED as the simplest example of a gauge theory and study the behavior of the fermion propagator, using the corresponding SDE. Apart from the fermion propagator itself, the only unknown ingredient in this equation is the fermion-boson vertex. As the SDE of the vertex is quite complicated, a common practice is to start from a suitable construction for it. One should ensure that every ansatz of a non-perturbative fermion-boson interaction must have the following characteristics:

- It should respect the Ward-Green-Takahashi identity (WGTI) which relates it to the fermion propagator. Moreover, in the limit when the fermion momenta are identical, it should also obey the limiting Ward identity (WI).
- In the weak coupling regime, it should match onto its perturbative loop expansion.
- It should transform according to the Landau-Khalatnikov-Fradkin transformations (LKFT) under a variation of gauge. Moreover, it must guarantee that when used in the SDE for the fermion propagator, the resulting propagator also obeys its corresponding LKFT.
- It should not contain any kinematic singularities.
• If we are studying 3+1-dimensional QED, it should ensure that the fermion propagator is multiplicatively renormalizable.
• It should render the physical observables associated with the fermion propagator gauge independent.

In addition to these factors, it is also important to solve the SDE for the fermion propagator by employing a gauge invariant regulator. Since the earliest works on the dynamical breakdown of chiral symmetry through the SDEs, [1], a lot of research has been carried out in order that the above-mentioned goals could be achieved. We review this work in the next sections after a brief overview of the SDE for the fermion propagator.

**SDE FOR THE FERMION PROPAGATOR**

The SDE for the fermion propagator, $S_F(p)$, in QED with a bare coupling, $e$, is displayed in Fig. (1), and is given by:

$$i S_F^{-1}(p) = i S_F^{0-1}(p) - e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S_F(k) \Gamma^\nu(k,p) \Delta_{\mu \nu}(q),$$

where $q = k - p$ and $S_F(p)$ can be expressed in terms of two Lorentz scalar functions, $F(p^2)$, the wavefunction renormalization, and $\mathcal{M}(p^2)$, the mass function, so that

$$S_F(p) = \frac{F(p^2)}{\rho - \mathcal{M}(p^2)}.$$  

(2)

The bare propagator $S_F^0(k) = 1/(\rho - m_0)$, where $m_0$ is the constant (bare) mass. In quenched QED, the photon propagator is unrenormalized and so is given by its bare form:

$$\Delta_{\mu \nu}(q) = \Delta_{\mu \nu}^0(q) = \frac{1}{q^2} \left( g_{\mu \nu} + (\xi - 1) \frac{q_\mu q_\nu}{q^2} \right).$$

$\Gamma^\mu(k,p)$ is the full fermion-boson vertex. Once it is known, one can solve Eq. (1) to fully determine the fermion propagator in terms of $F(p^2)$ and $\mathcal{M}(p^2)$. From these
quantities, one can extract physical observables such as the dynamically generated mass of the fermion, the condensate $< \bar{\psi} \psi >$ and the critical coupling above which the chiral symmetry is broken.

WARD-GREEN-TAKAHASHI IDENTITY

The simplest ansatz for the full vertex is the bare vertex, i.e., $\Gamma^\mu(k, p) = \gamma^\mu$. Making use of it in Eq. (1), we solve the latter for $F(p^2)$ and $\mathcal{M}(p^2)$ after setting $m_0 = 0$. We find a non-trivial solution for $\mathcal{M}(p^2)$ (different from the trivial solution $\mathcal{M}(p^2) = 0$ obtained in perturbation theory) at a value higher than a critical value of coupling $\alpha = \alpha_c$. In other words, above $\alpha_c$, fermions become massive and below this value, they remain massless. As $\alpha_c$ corresponds to a change of phase, we expect it to be a gauge independent quantity. However, if one solves Eq. (1) for different values of the gauge parameter $\xi$, one finds $\alpha_c$ highly gauge dependent. As a consequence of gauge covariance, Green functions obey certain identities which relate one function to the other. These relations have been named Ward-Green-Takahashi identities (WGTI), [2]. At the level of physical observables, gauge symmetry reflects as the fact that they be independent of the gauge parameter. Owing to the fact that the bare vertex does not satisfy the WGTI $q_\nu \Gamma^\nu(k, p) = S^{-1}_F(k) - S^{-1}_F(p)$ beyond the first order in perturbation theory, we cannot expect the physical observables borne out of this approximation to be gauge independent. In order to incorporate WGTI into the ansatz for the vertex, we follow Ball and Chiu, [3], and write out the vertex as a sum of longitudinal and transverse components:

$$\Gamma^\mu(k, p) = \Gamma^\mu_L(k, p) + \Gamma^\mu_T(k, p). \quad (3)$$

By definition, the transverse part $\Gamma^\mu_T(k, p)$ satisfies $q_\mu \Gamma^\mu_T(k, p) = 0$ and is undetermined the WGTI. It also satisfies $\Gamma^\mu_T(p, p) = 0$. Ball and Chiu suggest the following longitudinal part in order to satisfy WGTI in a manner free of kinematic singularities:

$$\Gamma^\mu_L(k, p) = a(k^2, p^2)\gamma^\mu + b(k^2, p^2)(k + p)(k + p)^\mu - c(k^2, p^2)(k + p)^\mu, \quad (4)$$

where,

$$a(k^2, p^2) = \frac{1}{2} \left( \frac{1}{F(k^2)} + \frac{1}{F(p^2)} \right),$$
$$b(k^2, p^2) = \frac{1}{2} \left( \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right) \frac{1}{k^2 - p^2}, \quad (5)$$
$$c(k^2, p^2) = \left( \frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)} \right) \frac{1}{k^2 - p^2}.$$

Though the choice of the longitudinal part of the vertex is not unique\(^1\), one of the advantages of the ansatz proposed by Ball and Chiu is that it contains no kinematic

\(^1\) Some other attempts to construct the longitudinal vertex can be found in references [4, 5].
singularities. Moreover, they also propose a basis of eight independent tensors to write out the transverse part of the vertex:

\[ \Gamma^\mu_T(k,p) = \sum_{i=1}^{8} \tau_i(k,p) T^\mu_i(k,p). \] (6)

They construct their basis such that the coefficients of each of the basis vectors is independent of kinematic singularities at the one loop level in the Feynman gauge. It is a common practice to use the Ball Chiu vertex as the longitudinal part of the full vertex. In the next section, we discuss the elements which can serve as a guide in our hunt for the transverse piece of the full fermion-boson interaction.

**PERTURBATION THEORY**

Only a correct choice of the transverse vertex can lead to physically acceptable solutions. How (and if) can one construct such a vertex? The only truncation of the complete set of SDEs known so far that incorporates the key features of a gauge theory such as the WGT identities, LKF transformations and gauge invariance of physical observables, (e.g., the mass and the condensate) at each level of approximation is perturbation theory. Therefore, it is natural to assume that physically meaningful solutions of the SDEs must agree with perturbative results in the weak coupling regime. It requires, e.g., that every non-perturbative *ansatz* chosen for the transverse vertex must reduce to its perturbative counterpart when the interactions are weak. Perturbatively, the transverse vertex is evaluated in the following fashion. One evaluates the fermion propagator to a certain order and hence determines the longitudinal vertex to the same order. One also calculates perturbatively the full vertex, and a mere subtraction of the longitudinal part yields the transverse part, the one which is not fixed by the WGTI. A brief development of work in this direction is outlined below:

- Ball and Chiu, [3], calculate one loop fermion boson vertex in Feynman gauge and hence propose a suitable basis to expand the transverse vertex.
- Curtis and Pennington, [7], calculate one loop fermion boson vertex in an arbitrary covariant gauge in the limit when momentum in one of the fermion legs is much greater than in the other. Using this as a guide, they propose an *ansatz* for the transverse vertex involving just one basis vector and show that the gauge dependence of \( \alpha_c \) is appreciably reduced.
- Following the perturbative calculation in the first article of reference [7], Bashir and Pennington, [8], propose a vertex *ansatz* involving two basis vectors. In terms of gauge invariance of the critical coupling, this *ansatz* works much better than the Curtis-Pennington vertex and in a much wider range of values for the covariant gauge parameter.

\[2\] A complete one loop calculation of the transverse vertex in an arbitrary covariant gauge, [6], slightly modifies this basis.
• Kızılersü et al., [6], calculate complete one loop fermion boson vertex to $\mathcal{O}(\alpha)$ in an arbitrary covariant gauge and modify the basis proposed by Ball and Chiu, [3], to write out the transverse vertex.

• Bashir et al., [9], calculate the perturbative constraint on the fermion boson vertex, imposed by the two loop next to leading log calculation of the wavefunction renormalization.

• Bashir and Raya, [10], calculate one loop fermion boson vertex in an arbitrary covariant gauge in 2+1 dimensions and, guided by it, propose the first ever non-perturbative vertex which agrees with its full one loop expansion in the weak coupling regime. This vertex has an explicit dependence on the gauge parameter $\xi$. They demonstrate, in the massless case, that a vertex cannot be constructed without an explicit dependence on $\xi$. For practical purposes of the numerical study of dynamical chiral symmetry breaking, they also construct an effective vertex which shifts the angular dependence from the unknown fermion propagator functions to the known basic functions, without changing its perturbative properties at the one loop level. This vertex should lead to a more realistic study of the dynamically generated masses through the corresponding SDEs in 2+1 dimensions.

• Davydychev et al., [11], calculate the one loop vertex in an arbitrary covariant gauge in arbitrary dimensions. This may help one to construct a non perturbative vertex in arbitrary dimensions.

A two loop calculation of the transverse vertex would be useful, as it is likely to shed more light on its possible non-perturbative extensions. We believe that a vertex which is reduced to its perturbative expansion in the weak coupling regime stands a better chance to yield gauge invariant results.

LANDAU-KHALATNIKOV-FRADKIN TRANSFORMATIONS

In a gauge field theory, Green functions transform in a specific manner under a variation of gauge, giving rise to LKF transformations in QED, [12]. These were derived also by Johnson and Zumino through functional methods, [13]. LKF transformations are non-perturbative in nature and hence have the potential of playing an important role in addressing the problems of gauge invariance which plague the strong coupling studies of SDEs. In general, the rules governing these transformations are far from simple. The fact that they are written in coordinate space adds to their complexity. As a result, these transformations have played less significant and practical role in the study of SDEs than desired.

The LKF transformation for the three-point vertex is complicated and hampers direct extraction of analytical restrictions on its structure. Burden and Roberts, [14], carried out a numerical analysis to compare the self-consistency of various ansatze for the vertex, [3, 7, 5], by means of its LKF transformation. In addition to these numerical constraints, indirect analytical insight can be obtained on the non-perturbative structure of the vertex by demanding correct gauge covariance properties of the fermion propagator. References [7, 9, 15, 16] employ this idea. However, the inclusion of LKF transformations has
been restricted to massless fermions alone. The masslessness of the fermions implies that the fermion propagator can be written only in terms of one function, the so-called wavefunction renormalization, $F(p)$. In order to apply the LKF transformation, one needs to know a Green function at least in one particular gauge. This is a formidable task. However, one can rely on approximations based on perturbation theory. It is customary to take $F(p) = 1$ in the Landau gauge, an approximation justified by one loop calculation of the massless fermion propagator in arbitrary dimensions, see for example, [11]. The LKF transformation then implies a power law for $F(p)$ in QED4 and a simple trigonometric function in QED3. To improve upon these results, one can take two paths: (i) incorporate the information contained in higher orders of perturbation theory and (ii) study the massive theory. As pointed out in [9], in QED4, the power law structure of the wavefunction renormalization remains intact by increasing order of approximation in perturbation theory although the exponent of course gets contribution from next to leading logarithms and so on. In [9], constraint was obtained on the 3-point vertex by considering a power law where the exponent of this power law was not restricted only to the one loop fermion propagator. In QED3, the two loop fermion propagator was evaluated in [18], where it was explicitly shown that the the approximation $F(p) = 1$ is only valid up to one loop, thus violating the transversality condition advocated in the second article of reference [16]. The result found there was used used in [19] to find the improved LKF transform. Later on, in reference [20], the LKF transformed fermion propagator in massive QED3 and QED4 was evaluated with the simplest input which corresponds to the lowest order of perturbation theory, i.e., the propagator being bare in the Landau gauge. We believe that the incorporation of LKF transformations, along with WGT identities, in the SDE can play a key role in addressing the problems of gauge invariance. For example, only those assumptions should be permissible which keep intact the correct behavior of the Green functions under the LKF transformations, in addition to ensuring that the WGTI is satisfied. It makes it vital to explore how two and three-point Green functions transform in a gauge covariant fashion.

**OTHER KEY FACTORS**

We now comment on other key factors which should be taken into account while studying SDEs:

- The transverse vertex should be free of any kinematic singularities. Within the framework of the basis proposed by Kızılersü et al., [6], it amounts to saying that every coefficient of the basis vectors itself should be free of kinematic singularities.
- As discovered by Curtis and Pennington, [7], multiplicative renormalizability of the fermion propagator plays an important role in the restoration of gauge invariance of the critical coupling above which masses are generated for fundamental fermions. However, their work as well as the one presented in reference [15], only incorporates the leading log behavior of the propagator in the construction of the

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3 For the two loop calculation of the fermion propagator, see for example [17].
non-perturbative vertex. However, in a subsequent work, Bashir *et. al.* presented the most general construction of the transverse vertex required by multiplicative renormalizability of the fermion propagator to all orders, [9].

- If one takes into account all the relevant features mentioned so far, one is likely to acquire gauge invariance of all the physical observables. However, it is a prohibitively difficult to implement all the constraints to the required degree. Therefore, a direct requirement of the gauge invariance of the physical observables can serve as an additional driving force to constrain the fermion-boson vertex. One such attempt is made by Bashir *et. al.*, [15]. They hold the critical coupling to be gauge invariant and obtain constraints on the transverse vertex.

- The works described so far use cut-off regularization scheme to study the gauge dependence of the physical observables related to the fermion propagator. As the cut-off method in general does not respect gauge symmetry, a criticism of these works has been raised recently, [21]. They suggest dimensional regularization scheme to study the chirally asymmetric phase of QED so that the possible gauge dependence coming from the inappropriate regulator could be filtered out. However, implementation of dimensional regularization leads to complicated kernels in the coupled integral equations which are then hard to solve, [21, 22].

**CONCLUSIONS**

We summarize the attempts made so far to make the study of Schwinger-Dyson equation for the fermion propagator in QED more realistic by constructing an ansatz for the fermion boson interaction in such a fashion that it can effectively recuperate the necessary information lost on truncating the infinite tower of these equations. Although a lot of work has been carried out in this direction, more work is needed to make these studies fully reliable. One should then embark on the studies of unquenched QED and move on to consider more realistic cases such as QCD and the improved versions of top quark condensation.

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