Approximately Optimal Mechanisms for Strategyproof Facility Location: Minimizing $L_p$ Norm of Costs

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Abstract

We consider the problem of locating a single facility on the real line. This facility serves a set of agents, each of whom is located on the line, and incurs a cost equal to his distance from the facility. An agent’s location is private information that is known only to him. Agents report their location to a central planner who decides where to locate the facility. The planner’s objective is to minimize a "social" cost function that depends on the agent-costs. However, agents might not report truthfully; to address this issue, the planner must restrict himself to strategyproof mechanisms, in which truthful reporting is a dominant strategy for each agent. A mechanism that simply chooses the optimal solution is generally not strategyproof, and so the planner aspires to use a mechanism that effectively approximates his objective function. This general class of problems was first studied by Procaccia and Tennenholtz and has been the subject of much research since then.

In our paper, we study the problem described above with the social cost function being the $L_p$ norm of the vector of agent-costs. We show that the median mechanism (which is known to be strategyproof) provides a $2^{1 - \frac{1}{p}}$ approximation ratio, and that is the optimal approximation ratio among all deterministic strategyproof mechanisms. For randomized mechanisms, we present two results. First, for the case of 2 agents, we show that a mechanism called LRM, first designed by Procaccia and Tennenholtz for the special case of $L_\infty$, provides the optimal approximation ratio among all randomized mechanisms. Second, we present a negative result: we show that for $p > 2$, no mechanism—from a rather large class of randomized mechanisms—has an approximation ratio better than that of the median mechanism. This is in contrast to the case of $p = 2$ and $p = \infty$ where a randomized mechanism provably helps improve the worst case approximation ratio.

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1 Introduction

We consider the problem of locating a single facility on the real line. This facility serves a set of \( n \) agents, each of whom is located somewhere on the line as well. Each agent cares about his distance to the facility, and incurs a disutility (equivalently, cost) that is equal to his distance to access the facility. An agent’s location is assumed to be private information that is known only to him. Agents report their locations to a central planner who decides where to locate the facility based on the reports of the agents. The planner’s objective is to minimize a “social” cost function that depends on the vector of distances that the agents need to travel to access the facility. It is natural for the planner to consider locating the facility at a point that minimizes his objective function, but in that case the agents may not have an incentive to report their locations truthfully. As an example, consider the case of 2 agents located at \( x_1 \) and \( x_2 \) respectively, and suppose the location that optimizes the planner’s objective in this case is the mid-point \( (x_1 + x_2)/2 \). Then, assuming \( x_1 < x_2 \), agent 1 has an incentive to report a location \( x'_1 < x_1 \) so that the planner’s decision results in the facility being located closer to his true location. The planner can address this issue by restricting herself to a strategyproof mechanism: by this we mean that it should be a (weakly) dominant strategy for each agent to report his location truthfully to the central planner. This, of course, is an attractive property, but it comes at a cost: based on the earlier example, it is clear that the planner cannot hope to optimize her objective. One way to avoid this difficulty is to assume an environment in which agents (and the planner) can make or receive payments; in such a case, the planner selects the location of the facility, and also a payment scheme, which specifies the amount of money an agent pays (or receives) as a function of the reported locations of the agents as well as the location of the facility. This option gives the planner the ability to support the “optimal” solution as the outcome of a strategy-proof mechanism by constructing a carefully designed payment scheme in which any potential benefit for a misreporting agent from a change in the location of the facility is offset by an increase in his payment.

There are many settings, however, in which such monetary compensations are either not possible or are undesirable. This motivated Procaccia and Tennenholtz [6] to formulate the notion of Approximate Mechanism Design without Money. In this model the planner restricts herself to strategy-proof mechanisms, but is willing to settle for one that does not necessarily optimize her objective. Instead, the planner’s goal is to find a mechanism that effectively approximates her objective function. This is captured by the standard notion of approximation that is widely used in the CS literature: for a minimization problem, an algorithm is an \( \alpha \)-approximation if the solution it finds is guaranteed to have cost at most \( \alpha \) times that of the optimal cost \( (\alpha \geq 1) \).

Procaccia and Tennenholtz [6] apply the notion of approximate mechanism design without money to the facility location problem considered here for two different objectives: (i) minisum,
where the goal is to minimize the sum of the costs of the agents; and (ii) *minimax*, where the goal is to minimize the maximum agent cost. They show that for the minimax objective choosing any $k$-th median—picking the $k$th largest reported location—is a strategyproof, 2-approximate mechanism. They design a randomized mechanism called LRM (Left-Right-Middle) and show that it is a strategyproof, 3/2-approximate mechanism; furthermore, they show that those mechanisms provide the optimal worst-case approximation ratio possible (among all deterministic and randomized strategyproof mechanisms, respectively). For the *minisum* objective, it is known that choosing the median reported location is optimal and strategyproof [5]. Feldman and Wilf [4] consider the same facility location problem on a line but with the social cost function being the $L_2$ norm of the agents’ costs. They show that the median is a $\sqrt{2}$-approximate strategyproof mechanism for this objective function, and provide a randomized $(1 + \sqrt{2})/2$-approximate strategyproof mechanism. In addition, facility location on other topologies such as circles and trees are considered in Alon et al. [1, 2, 3] as well as in Feldman and Wilf [4].

In our paper, we follow the suggestion of Feldman and Wilf [4] and study the problem of locating a single facility on a line, but with the objective function being the $L_p$ norm of the vector of agent-costs (for general $p \geq 1$). We define the problem formally in section 2. In section 3, we show that the median mechanism (which is strategyproof) provides a $2^{1-\frac{1}{p}}$ approximation ratio, and that is the optimal approximation ratio among all deterministic strategyproof mechanisms. We move onto randomized mechanisms in section 4. First, we consider the case of 2 agents, and show that the LRM mechanism provides the optimal approximation ratio among all randomized mechanisms (that satisfy certain mild assumptions) for this special case. Our result for the special case of 2 agents also gives a lower bound on the approximation ratio for all randomized mechanisms. Next, we present a negative result: we show that for $p > 2$, no mechanism—from a rather large class of randomized mechanisms— has an approximation ratio better than that of the median mechanism, as the number of agents goes to infinity. It is worth noting that all the mechanisms proposed in literature so far—for minimax, minisum, as well as quadratic mean social cost functions—belong to this class of mechanisms. We conclude in section 5 with a brief discussion of some directions for further research.

## 2 Model

Let $N = \{1, 2, \ldots, n\}$, $n \geq 2$, be the set of agents. Each agent $i \in N$ reports a location $x_i \in \mathbb{R}$. A *deterministic* mechanism is a collection of functions $f = \{f_n| n \in \mathbb{N}, n \geq 2\}$ such that each $f_n : \mathbb{R}^n \to \mathbb{R}$ is a function that maps each location profile $x = (x_1, x_2, \ldots, x_n)$ to the location

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1Feldman and Wilf actually used the sum of squares of the agents’ costs, but their results can be easily converted to the $L_2$ norm. Of course, the approximation ratios they report need to be adjusted as well.
of a facility. We will abuse notation and let \( f(x) \) denote \( f_n(x) \). Under a similar notational abuse, a randomized mechanism is a collection of functions \( f \) that maps each location profile to a probability distribution over \( \mathbb{R} \): if \( f(x_1, x_2, \ldots, x_n) \) is the distribution \( \pi \), then the facility is located by drawing a single sample from \( \pi \).

Our focus will be on deterministic and randomized mechanisms for the problem of locating a single facility when the location of any agent is private information to that agent and cannot be observed or otherwise verified. It is therefore critical that the mechanism be strategyproof—it should be optimal for each agent \( i \) to report his true location \( x_i \) rather than something else. To that end we assume that if the facility is located at \( y \), an agent’s disutility, equivalently cost, is simply his distance to \( y \). Thus, an agent whose true location is \( x_i \) incurs a cost \( C(x_i, y) = |x_i - y| \).

If the location of the facility is random and according to a distribution \( \pi \), then the cost of agent \( i \) is simply \( C(x_i, \pi) = \mathbb{E}_{y \sim \pi} |x_i - y| \), where \( y \) is a random variable with distribution \( \pi \). The formal definition of strategyproofness is now:

**Definition 1.** A mechanism \( f \) is strategyproof if for each \( i \in N \), each \( x_i, x'_i \in \mathbb{R} \), and for each \( x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \in \mathbb{R}^{n-1} \),

\[
C(x_i, f(x_i, x_{-i})) \leq C(x_i, f(x'_i, x_{-i})),
\]

where \((\alpha, x_{-i})\) denotes a \( n \) vector where the \( i \)-th coordinate of the vector is \( \alpha \) and the \( j \)-th coordinate of the vector is \( x_j \) for all \( j \neq i \).

The class of strategyproof mechanisms is quite large: for example, locating the facility at agent 1’s reported location is strategyproof, but is not particularly appealing because it fails almost every conceivable notion of fairness and could also be highly “inefficient”. To address these issues, and to winnow the class of acceptable mechanisms, we impose additional requirements that often stem from efficiency or fairness considerations. In this paper we assume that locating a facility at \( y \) for the location profile is \( x = (x_1, x_2, \ldots, x_n) \) incurs the social cost

\[
sc(x, y) = \left( \sum_{i \in N} |x_i - y|^p \right)^{1/p}, \quad p \geq 1.
\]

For a randomized mechanism \( f \) that maps \( x \) to a distribution \( \pi \), we define the social cost to be

\[
sc(x, \pi) = \mathbb{E}_{y \sim \pi} \left( \sum_{i \in N} |x_i - y|^p \right)^{1/p}.
\]

For this definition of social cost, our goal now is to find a strategyproof mechanism that does well with respect to minimizing the social cost. A natural mechanism (and this is the approach taken
in the classical literature on facility location) is the “optimal” mechanism: each location profile \( x = (x_1, x_2, \ldots, x_n) \) is mapped to \( OPT(x) \), defined as:

\[
OPT(x) \in \arg\min_{y \in \mathbb{R}} sc(x, y).
\]

This optimal mechanism is not strategyproof as shown in the following example.

**Example.** Suppose there are two agents located at the points 0 and 1 respectively on the real line. If they report their locations truthfully, the optimal mechanism will locate the facility at \( y = 0.5 \), for any \( p > 1 \). Assuming agent 2 reports \( x_2 = 1 \), if agent 1 reports \( x_1' = -1 \) instead, the facility will be located at 0, which is best for agent 1.

Given that strategyproofness and optimality cannot be achieved simultaneously, it is necessary to find a tradeoff. In this paper we shall restrict ourselves to strategyproof mechanisms that approximate the optimal social cost as best as possible. The notion of approximation that we use is standard in computer science: an \( \alpha \)-approximation algorithm is one that is guaranteed to have cost no more than \( \alpha \) times the optimal social cost. Formally, the approximation ratio of an algorithm \( A \) is \( \sup_I \{ A(I)/OPT(I) \} \), where the supremum is taken over all possible instances \( I \) of the problem; and \( A(I) \) and \( OPT(I) \) are, respectively, the costs incurred by algorithm \( A \) and the optimal algorithm on the instance \( I \).

Our goal then is to design strategyproof (deterministic or randomized) mechanisms whose approximation ratio is as close to 1 as possible.

## 3 The Median Mechanism

For the location profile \( x = (x_1, x_2, \ldots, x_n) \), the median mechanism is a deterministic mechanism that locates the facility at the “median” of the reported locations. The median is unique if \( n \) is odd, but not when \( n \) is even, so we need to be more specific in describing the mechanism. For odd \( n \), say \( n = 2k - 1 \) for some \( k \geq 1 \), the facility is located at \( x[k] \), where \( x[k] \) is the \( k \)th largest component of the location profile. For even \( n \), say \( n = 2k \), the “median” can be any point in the interval \([x[k], x[k+1]]\); to ensure strategyproofness, we need to pick either \( x[k] \) or \( x[k+1] \), and as a matter of convention we take the median to be \( x[k] \). It is well known that the median mechanism is strategyproof\(^2\). Furthermore, the median mechanism is anonymous\(^4\). Thus we may assume,
without loss of generality, that each agent reports her location truthfully.

Our main result in this section is that, for any \( p \geq 1 \), the median mechanism uniformly achieves the best possible approximation ratio among all deterministic strategyproof mechanisms. We start with two simple observations, which will be used repeatedly in the proof of this main result.

**Lemma 1.** For any real numbers \( a, b, c \) with \( a \leq b \leq c \), and any \( p \geq 1 \),

\[
(c - a)^p \leq 2^{p-1}[(c - b)^p + (b - a)^p].
\]

*Proof.* For any \( p \geq 1 \), \( f(x) = x^p \) is a convex function on \([0, \infty)\), and so for any \( \lambda \in [0, 1] \) and \( x, y \geq 0 \),

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).
\]

Setting \( \lambda = 1/2, x = c - b, \) and \( y = b - a \), and multiplying both sides of the inequality by \( 2^p \) gives the result. \( \square \)

**Lemma 2.** For any non-negative real numbers \( a \) and \( b \), and any \( p \geq 1 \),

\[
(a + b)^p \geq a^p + b^p.
\]

*Proof.* For integer \( p \), the result is a direct consequence of the binomial theorem; the same argument covers the case of rational \( p \) as well. Continuity implies the result for all \( p \). \( \square \)

**Theorem 1.** Suppose there are \( n \) agents with the location profile \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \). Define the social cost of locating a facility at \( y \) as \( \left( \sum_{i=1}^{n} |y - x_i|^p \right)^{\frac{1}{p}} \) for \( p \geq 1 \). The social cost incurred by the median mechanism is at most \( 2^{\frac{1}{p}} - 1 \) times the optimal social cost.

*Proof.* We may assume that \( x_1 \leq \ldots \leq x_n \). Let \( \text{OPT} \) be a facility location that minimizes the social cost, and let \( m \) be the median. The inequality we need to prove is

\[
\sum_{i=1}^{n} |m - x_i|^p \leq 2^{p-1} \sum_{i=1}^{n} |\text{OPT} - x_i|^p.
\]

We do this by pairing each location \( x_i \) with its “symmetric” location \( x_{n+1-i} \) and arguing that the total cost of these two locations in the median mechanism is within the required bound of their total cost in an optimal solution. For even \( n \), this completes the argument; for odd \( n \) the only location without such a pair is the median itself, which incurs zero cost in the median mechanism, and so the argument is complete. Formally, the result follows if we can show

\[
|m - x_i|^p + |x_{n+1-i} - m|^p \leq 2^{p-1}(|\text{OPT} - x_i|^p + |\text{OPT} - x_{n+1-i}|^p), \text{ for all } i \leq \lfloor n/2 \rfloor.
\]

When \( p = \infty \), the median mechanism provides a 2-approximation, as shown in Procaccia and Tennenholtz [6].
We consider two cases, depending on whether \( OPT \) is in the interval \([x_i, x_{n+1-i}]\) or not. In each of these cases, \( OPT \) may be above the median or below, but the proof remains identical in each subcase, so we give only one.

1. \( x_i \leq m \leq OPT \leq x_{n+1-i} \) or \( x_i \leq OPT \leq m \leq x_{n+1-i} \). We will prove the first of these subcases; the proof of the second is identical. Applying Lemma 1 by setting \( a = m, b = OPT, \) and \( c = x_{n+1-i} \), we get
   \[
   |x_{n+1-i} - m| \leq 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - m|^p).
   \]
Thus,
   \[
   |m - x_i|^p + |x_{n+1-i} - m|^p \leq |m - x_i|^p + 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - m|^p) \\
   \leq 2^{p-1}(|m - x_i|^p + |x_{n+1-i} - OPT|^p + |OPT - m|^p) \\
   \leq 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - x_i|^p),
   \]
where the last inequality is obtained by applying Lemma 2 to the terms \(|m - x_i|^p\) and \(|OPT - m|^p\).

2. \( OPT \leq x_i \leq m \leq x_{n+1-i} \) or \( x_i \leq m \leq x_{n+1-i} \leq OPT \). Again, we prove only the first subcase. Note that
   \[
   |x_{n+1-i} - m|^p + |m - x_i|^p \leq |x_{n+1-i} - x_i|^p \\
   \leq |OPT - x_{n+1-i}|^p \\
   \leq 2^{p-1}(|OPT - x_i|^p + |OPT - x_{n+1-i}|^p)
   \]
   where the first inequality follows from Lemma 2 (Note that Lemma 1 is not used in the proof of this case.)

We end this section by showing that no deterministic and strategyproof mechanism can give a better approximation to the social cost.

**Lemma 3.** Consider the case of two agents and suppose the location profile is \((x_1, x_2)\) with \( x_1 < x_2 \). For \( p \geq 1 \), suppose the social cost of locating a facility at \( y \) is \((|x_1 - y|^p + |x_2 - y|^p)^{1/p}\). Any deterministic mechanism whose approximation ratio is better than \(2^{1 - \frac{1}{p}}\) for \( p > 1 \) must locate the facility at \( y \) for some \( y \in (x_1, x_2) \).

**Proof.** The function \( f(y) = |y - x_1|^p + |y - x_2|^p \) is strictly convex, and its unique minimizer is \( y^* = (x_1 + x_2)/2 \), with the corresponding value \( f(y^*) = |x_2 - x_1|^p/2^{p-1} \). Moreover \( f(x_1) = f(x_2) = |x_2 - x_1|^p = 2^{p-1}f(y^*) \). It follows that for the deterministic mechanism to do strictly better than the stated ratio, the facility cannot be located at the reported locations; locating the facility to the left of \( x_1 \) or to the right of \( x_2 \) only increases the cost of the mechanism, so the only option left for a mechanism to do better is to locate the facility in the interior, i.e., in \((x_1, x_2)\). \( \square \)
Theorem 2. Any strategyproof deterministic mechanism has an approximation ratio of at least \(2^{1-\frac{1}{p}}\) for the \(L_p\) social cost function for any \(p \geq 1\).

Proof. The bound holds trivially for \(p = 1\). Suppose \(p > 1\), and suppose a deterministic strategyproof mechanism yields an approximation ratio strictly better than \(2^{1-\frac{1}{p}}\) to the \(L_p\) social cost. For the two-agent location profile \(x_1 = 0, x_2 = 1\), Lemma 3 implies the facility is located at some \(y \in (0, 1)\). Now consider the location profile \(x_1 = 0, x_2 = y\). Again, by Lemma 3, the mechanism locate the facility at \(y' \in (0, y)\) to guarantee the improved approximation. But if agent 2 is located at \(y < 1\), he can misreport his location as 1, forcing the mechanism to locate the facility at \(y\), his true location; this violates strategyproofness.

4 Randomized Mechanisms

Recall that when the social cost is measured by the \(L_2\) norm or the \(L_\infty\) norm, randomization provably improves the approximation ratio. In the former case, Feldman and Wilf [4] describe an algorithm whose approximation ratio is \((\sqrt{2} + 1)/2\); for the latter, Procaccia and Tennenholtz [6] design an algorithm with an approximation ratio of \(3/2\). The mechanisms in both cases are simple and somewhat similar, placing non-negative probabilities only on the optimal and reported locations, where these probabilities are independent of the reported location profile. There are two reasonable ways of choosing the reported locations: one is via dictatorships and the other is via generalized medians. In this section we show that the former is not enough; namely, randomizing over the dictatorships and the optimal location does not improve the approximation ratio of the median mechanism for all \(p \in (2, \infty)\). For the case of 2 agents we show that the best approximation ratio is given by the LRM mechanism among all strategyproof mechanisms. Extending this analysis even to the case of 3 agents appears to be highly non-trivial.

4.1 Mixing Dictatorships with the Optimal Location

We first show that any strategyproof randomized mechanism that places location-independent probabilities on dictatorships and the location of \(OPT\) (the unique minimizer of the \(L_p\) social cost function) cannot have an approximation ratio better than that of the median mechanism for any \(p \in (2, \infty)\).

Lemma 4. Consider the location profile with \(n = 2k\) agents in which \(k\) agents are located at 0 and another \(k\) agents located at 1. Suppose agent \(i\) is located at 1. If agent \(i\) is the only misreporting agent, and wishes to change \(OPT\) from 0.5 to 1, then \(i\) should report his location to be \(1 + k^{\frac{1}{p-1}}\).

6The lower bound of 2 on the approximation ratio also holds when \(p = \infty\), see Procaccia and Tennenholtz [6].
Proof. Let $y$ denote the facility’s location and $x_i' \geq 1$ denote the reported location of agent $i$. When all other agents report their locations truthfully, the social cost as a function of $y \in [0,1]$ given the location profile $x' = (x'_i, x_{-i})$ is

$\text{sc}(x', y) = (ky^p + (k - 1)(1 - y)^p + (x'_i - y)^p)^\frac{1}{p}$

$f(y) = \text{sc}(x', y)^p$ is differentiable on $(0, 1)$ with $f'(y)$ on $(0, 1)$ being:

$f'(y) = p(ky^{p-1} - (1 - y)^{p-1}(k - 1) - (x'_i - y)^{p-1})$

For $OPT$ to be located at $1$, $f'(1^-)$ must equal $0$. By the convexity of $f$, it suffices to find $x'_i$ such that $f'(1^-) = k - (x'_i - 1)^{p-1} = 0$. Solving for $x'_i$ and we get the desired result. \qed

Lemma 5. Let $A$ be any strategyproof randomized mechanism that places location-independent probabilities on dictatorships and $OPT$. Then, for the location profile with $n = 2k$ agents in which $k$ agents are at zero and $k$ agents are at $1$, the probability that $A$ locates the facility at $OPT$ tends to zero as $k \to \infty$.

Proof. Let $i$ be an agent located at $1$, and let $x_{-i}$ denote the location profile of all other agents. Let $\bar{\rho}$ denote the probability $A$ locates the facility at $OPT$; and for $j = 1, \ldots, n$, let $\bar{q}_j$ denote the probability that $A$ locates the facility at the reported location of agent $j$. Then the strategyproofness of $A$ implies that the difference between the cost to agent $i$ when he reports truthfully versus when he misreports $x'_i$, or $C(1, f(1, x_{-i})) - C(1, f(x'_i, x_{-i}))$, is non-positive for all $x'_i \in \mathcal{R}$. Plug in $x'_i = 1 + k\frac{1}{p-1}$ and Lemma \ref{lemma:1} implies that $OPT$ is located at $1$ in the reported profile. The facility location does not change except when $OPT$ or agent $i$’s location is chosen by the mechanism. Hence, the only two terms that survive in the difference of the cost functions are the term corresponding to $OPT$ and the term corresponding to agent $i$’s location. Consequently, we have that $\bar{\rho}(1 - 0.5) - \bar{q}_i(1 + k\frac{1}{p-1} - 1) \leq 0$, which implies that $\bar{q}_i \geq \bar{\rho}/2k\frac{1}{p-1}$. By symmetry of the location profile, for any agent $i$ located at $0$, strategyproofness again implies that $\bar{q}_i \geq \bar{\rho}/2k\frac{1}{p-1}$. Thus, we have that

$$1 - \bar{\rho} = \sum_{i=1}^{2k} \bar{q}_i \geq 2k(\frac{\bar{\rho}}{2k\frac{1}{p-1}}) = (k\frac{1}{p-1})\bar{\rho}.$$ 

Solving for $\bar{\rho}$, we observe that $\bar{\rho} \leq \frac{1}{1+k\frac{1}{p-1}}$, which goes to $0$ as $k$ goes to infinity for any $p > 2$. \qed

Lemma 6. Let $A$ be any random (not necessarily uniform) dictator mechanism. Consider the location profile with $n = 2k$ agents, in which $k$ agents are located at $0$ and $k$ agents are located at $1$. Then the approximation ratio of $A$ is at least $2^{1-\frac{1}{p}}$. 

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Proof. Let \( \bar{q}_i \) be the probability that the mechanism locates the facility at \( x_i \). The expected social cost under the random dictator mechanism is 
\[
\sum_{i=1}^{2k} \bar{q}_i \left( \sum_{j:x_j \neq x_i} 1^p \right) = \frac{k}{2^p},
\]
whereas the minimum social cost is \( (2k(0.5)^p) \frac{1}{2^1} = \frac{k}{2^1} \). Thus we get the approximation ratio of \( 2^{1-\frac{1}{p}} \). □

The following theorem is immediate from Lemmas 4-6.

**Theorem 3.** Let \( A \) any strategyproof randomized mechanism that places location-independent non-negative probabilities only on dictatorships and the optimal location, then \( A \) gives at least an approximation ratio of \( 2^{1-\frac{1}{p}} \) in the worst case.

### 4.2 Optimality of the LRM Mechanism for 2 Agents

Our next result shows that the LRM mechanism provides the best possible approximation ratio among all shift and scale invariant (defined below) strategyproof mechanisms for the case of 2 agents for all \( L_p \) social cost functions for \( p \geq 1 \).

We begin with some definitions: we say that a mechanism \( f \) is *shift and scale invariant* if for every location profile \( x = \{x_1, x_2\} \) s.t. \( x_1 \leq x_2 \) and every \( c \in \mathbb{R} \), the following two properties are satisfied:

1. \( f(\{x_1 + c, x_2 + c\}) = f(x) + c \).
2. When \( c \geq 0 \), we have \( f(\{cx_1, cx_2\}) = cf(x) \), and when \( c < 0 \), we have \( f(\{cx_1, cx_2\}) = -cf(\{-x_2, -x_1\}) \).

A convenient notation for a given location profile \( x \) is to denote its midpoint as \( m_x = \frac{x_1 + x_2}{2} \). We say that a mechanism \( f \) is *symmetric* if for all location profile \( x \) and for any \( y \in \mathbb{R} \), \( P(f(x) \geq m_x + y) = P(f(x) \leq m_x - y) \).

The following lemma allows us to convert any strategyproof mechanism into a symmetric mechanism.

**Lemma 7.** Given any strategyproof mechanism, there exists another symmetric strategyproof mechanism with the same approximation ratio.

**Proof.** Given a mechanism \( f \), we define the *mirror mechanism* of \( f \), \( f_{\text{mirror}} \), to be such that for any profile \( x \), we have that \( P(f_{\text{mirror}}(x) \geq m_x + b) = P(f(x) \leq m_x - b) \) for all \( b \in \mathbb{R} \) and location profiles \( x \).

Assume \( f \) is a strategyproof mechanism. Symmetry dictates that \( f_{\text{mirror}} \) must also be strategyproof, since any misreport of the right agent with respect to \( f \) induces the same cost as that of an equivalent a misreport of the left agent with respect to \( f_{\text{mirror}} \) and vice versa. Moreover, since composing two strategyproof mechanisms yields a strategyproof mechanism, the mechanism...
\[ g = \frac{1}{2}f + \frac{1}{2}f_{\text{mirror}} \] is a strategyproof mechanism which is also symmetric. Finally, note that \( g \) has the same approximation ratio as \( f \) for all location profiles, since \( f_{\text{mirror}} \) has the same approximation ratio as \( f \).

From now on, whenever we talk about a shift and scale invariant mechanism, we will also assume that it is symmetric. To simplify our proof of the main result, we will assume in addition that given a reported profile \( x = \{x_1, x_2\} \), the mechanism will only assign a facility location that lies in between \( x_1 \) and \( x_2 \), i.e. \( \mathbb{P}(y \in [x_1, x_2]) = 1 \), where \( y \) is a random variable representing the facility location assigned by the mechanism. It is worth noting that the main result remains true even without this assumption, although the complete proof is somewhat long and cumbersome, so we will omit it. The next lemma deals with an equivalent condition for strategyproofness with respect to a shift, scale invariant and symmetric mechanism.

**Lemma 8.** A shift, scale invariant, and symmetric mechanism \( f \) is strategyproof if and only if for any profile \( x = \{x_1, x_2\} \) with \( x_1 = 0 < x_2 \), the following condition hold:

\[
\int_{(-\infty,x_2)} ydF(y) + \int_{(x_2,\infty)} ydF(y) + x_2\mathbb{P}(Y = x_2) \geq 0,
\]

where \( Y = f(x) \) with c.d.f. \( F \).

**Proof.** By shift invariance, it suffices to check strategyproofness for profiles where \( x_1 = 0 \) and by symmetry, we can assume without lost of generality that \( x_2 \geq 0 \). Moreover, any shift, scale invariant mechanism is trivially strategyproof with respect to the profile \( \{0,0\} \) since by definition, the mechanism would place all of the probability mass on 0, which means that no agent has incentive to misreport his location. Thus, we can assume that \( x_2 > 0 \).

Since the mechanism is symmetric, it suffices to show that agent 2 cannot benefit by deviating from his true location if and only if the aforementioned condition hold. Since \( x_2 > 0 \), we can denote agent 2’s deviation \( x'_2 \) as \( cx_2 \) for some \( c \in \mathbb{R} \). Moreover, since \( \mathbb{P}(Y \in [x_1, x_2]) = 1 \) by assumption, we can further restrict ourselves to the case where \( c > 1 \) because agent 2 has no incentive to deviate to a location \( cx_2 \) where \( cx_2 < x_2 \) as the mechanism is scale invariant.

When agent 2 reports his location to be \( cx_2 \), where \( c > 1 \), the change in cost incurred by agent
2 is:

\[ C_{\text{dev}} - C_{\text{orig}} = -(c - 1) \int_{(-\infty, \frac{2x_2}{c})} ydF(y) + \int_{[\frac{2x_2}{c}, x_2)} ((c + 1)y - 2x_2)dF(y) + (c - 1) \int_{(x_2, \infty)} ydF(y) \]
\[ + (c - 1)x_2P(Y = x_2) \]
\[ = -(c - 1) \int_{(-\infty, x_2)} ydF(y) + \int_{[\frac{2x_2}{c}, x_2)} (2cy - 2x_2)dF(y) + (c - 1) \int_{(x_2, \infty)} ydF(y) + \]
\[ (c - 1)x_2P(Y = x_2) \]
\[ \geq -(c - 1) \int_{(-\infty, x_2)} ydF(y) + (c - 1) \int_{(x_2, \infty)} ydF(y) + (c - 1)x_2P(Y = x_2) \]

Hence, when condition 1 holds, we have that \(- (c - 1) \int_{(-\infty, x_2)} ydF(y) + (c - 1) \int_{(x_2, \infty)} ydF(y) + (c - 1)x_2P(Y = x_2) \geq 0\), which means that \(C_{\text{dev}} - C_{\text{orig}} \geq 0\).

To prove the other direction, suppose the condition does not hold, then there exists \(\epsilon > 0\) small enough such that \(- \int_{(-\infty, x_2)} ydF(y) + \int_{(x_2, \infty)} ydF(y) + x_2P(Y = x_2) \leq -\epsilon\) for some \(x_2 > 0\). We choose \(c > 1\) s.t. \(P(Y \in [\frac{2x_2}{c}, x_2)) < \frac{\epsilon}{4x_2}\), then we have that

\[ C_{\text{dev}} - C_{\text{orig}} = -(c - 1) \int_{(-\infty, x_2)} ydF(y) + \int_{[\frac{2x_2}{c}, x_2)} (2cy - 2x_2)dF(y) + (c - 1) \int_{(x_2, \infty)} ydF(y) \]
\[ + (c - 1)x_2P(Y = x_2) \]
\[ \leq (c - 1)(- \int_{(-\infty, x_2)} ydF(y) + \int_{[\frac{2x_2}{c}, x_2)} (2x_2)dF(y) + \int_{(x_2, \infty)} ydF(y) + x_2P(Y = x_2)) \]
\[ < -(c - 1)\frac{\epsilon}{2} < 0 \]

which contradicts strategyproofness of the mechanism.

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Notice that given any shift, scale invariant, and symmetric mechanism \(f\), in order to check whether \(f\) is a strategyproof mechanism, it suffices to check whether \(f\) is strategyproof for one particular profile. Without lost of generality, we can assume that \(x_1 = 0\) and \(x_2 = 1\). Here is a short proof of the claim. By the same argument as before, it suffices to check strategyproofness for all profiles \(\{x_1, x_2\}\), where \(x_2 > x_1 = 0\). Let \(Y = f(\{0, 1\})\), then \(f(\{0, x_2\}) = x_2Y\). The mechanism is strategyproof with respective to the profile \(\{x_1, x_2\}\) if and only if for all \(c \in \mathbb{R}\), we have that

\[ E[|cx_2Y - x_2|] \geq E[|x_2Y - x_2|]. \]

Since \(x_2 > 0\), this follows directly from the strategyproofness condition for the profile \(\{0, 1\}\):

\[ E[|cY - 1|] \geq E[|Y - 1|] \forall c \in \mathbb{R}. \]
restricts the probability assignment to \(x_1, x_2\), and \(m_x\) for all profile \(x\) and simultaneously gives a better approximation than the original mechanism.

**Lemma 9.** Let \(f\) be a strategyproof shift, scale invariant and symmetric mechanism, where \(P(f(x) \in [x_1, x_2]) = 1\) for location profile \(x = \{x_1, x_2\}\) with \(x_2 > x_1\), then there exists another strategyproof mechanism \(g\) such that \(P(g(x) \in \{x_1, x_2, m_x\}) = 1\) for the (and thus every) location profile \(x\) and that \(E[sc(g(x), x)] \leq E[sc(f(x), x)]\). Furthermore, \(g\) satisfies shift, scale invariance and symmetry.

**Proof.** Now, let \(g\) be the mechanism that satisfies \(P(g(x) = x_1) = P(f(x) = x_1), P(g(x) = x_2) = P(f(x) = x_2), P(g(x) = m_x) = 1 - P(g(x) = x_1) - P(g(x) = x_2)\). Note that since \(m_x\) minimizes the social cost function for the profile \(x\), \(g\) certainly provides a weakly better approximation ratio than \(f\). By shift invariance, we can assume wlog that \(0 = x_1 \leq x_2\), then an alternative way to show strategyproofness is to check to see that the condition of the lemma 1 is satisfied by \(g\). Since \(f\) is a strategyproof mechanism, the condition implies that

\[
0 \leq -\int_{(0,x_2)} yd(F(y)) + x_2P(f(x) = x_2) \\
= -\int_{\left(\frac{x_2-x_1}{x_2}, \frac{x_2-x_1}{x_1}\right)} (m_x + u)d(F(m_x + u)) + x_2P(f(x) = x_2) \\
= -m_xP(f(x) \in (x_1, x_2)) + \int_{\left(\frac{x_2-x_1}{x_2}, \frac{x_2-x_1}{x_1}\right)} u d(F(m_x + u)) + x_2P(f(x) = x_2) \\
= -m_x(1 - P(g(x) = x_1) - P(g(x) = x_2)) + x_2P(g(x) = x_2)
\]

Note, \(-\int_{\left(\frac{x_2-x_1}{x_2}, \frac{x_2-x_1}{x_1}\right)} u d(F(m_x + u)) = 0\) because the distribution is symmetric around \(m_x\). Hence, the condition is satisfied for the mechanism \(g\).

Thus, \(g\) is a symmetric strategyproof mechanism that provides a weakly better approximation ratio than \(f\) and which satisfies \(P(g(x) \in \{x_1, x_2, m_x\}) = 1\) for every location profile \(x\).

Now we are ready to prove the main theorem.

**Theorem 4.** The LRM mechanism gives the best approximation ratio among all strategyproof mechanisms that are shift and scale invariant.

**Proof.** By the previous lemma, it suffices to search among the class of strategyproof shift, scale invariant and symmetric mechanisms where any element \(f\) of the class satisfies the property that \(P(f(x) \in \{x_1, x_2, m_x\}) = 1\). It is not difficult to see that in order to enforce strategyproofness, we must have that \(P(f(x) \in \{x_1, x_2\}) \geq 0.5\), which implies that among all such mechanisms, LRM provides the best approximation ratio of \(0.5(2^{1-\frac{1}{\theta}} + 1)\).

\[\square\]
An immediate consequence of Theorem 4 is the following corollary.

**Corollary 1.** Any strategyproof shift and scale invariant mechanism has an approximation of at least $0.5(2^{1-p} - 1^p + 1)$ in the worst case.

## 5 Discussion

The most important open question in our view is whether or not randomization can help improve the worst-case approximation ratio for general $L_p$ norm cost functions. The case of $p = 1$ is uninteresting because there is an optimal deterministic mechanism; for $p = 2$ and $p = \infty$ we already saw that randomization improves the worst-case approximation ratio, but we do not know if this is simply a happy coincidence, or if one can obtain similar results for all $p > 2$.

There are many other natural questions as well: for instance, what happens for more general topologies such as trees or cycles? Is it possible to characterize all randomized strategy-proof mechanisms on specific topologies?

Finally, we believe it is of interest to consider more general cost functions for the individual agents. The properties established for the LRM and many other mechanisms depend on the assumption that agents incur costs that are exactly equal to the distance to access the facility. Clearly, this is a very restrictive assumption, and working with more general individual agent costs is a promising direction to broaden the applicability of this class of models.

## References

[1] Noga Alon, Michal Feldman, Ariel D. Procaccia, and Moshe Tennenholtz. Strategyproof approximation mechanisms for location on networks. *CoRR*, abs/0907.2049:3432–3435, 2009.

[2] Noga Alon, Michal Feldman, Ariel D. Procaccia, and Moshe Tennenholtz. Strategyproof approximation of the minimax on networks. *Math. Oper. Res.*, 35(3):513–526, 2010.

[3] Noga Alon, Michal Feldman, Ariel D. Procaccia, and Moshe Tennenholtz. Walking in circles. *Discrete Mathematics*, 310(23):3432–3435, 2010.

[4] Michal Feldman and Yoav Wilf. Randomized strategyproof mechanisms for facility location and the mini-sum-of-squares objective. *CoRR*, abs/1108.1762, 2011.

[5] Herve Moulin. On strategy-proofness and single-peakedness. *Public Choice*, 35:437–455, 1980.

[6] Ariel D. Procaccia and Moshe Tennenholtz. Approximate mechanism design without money. In John Chuang, Lance Fortnow, and Pearl Pu, editors, *ACM Conference on Electronic Commerce*, pages 177–186. ACM, 2009.