Nonrelativistic Particle in Free Random Gauge Background

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The problem of a nonrelativistic particle with an internal color degree of freedom, with and without spin, moving in a free random gauge background is discussed. Freeness is a concept developed recently in the mathematical literature connected with noncommuting random variables. In the context of large-N hermitian matrices, it means that the multi-matrix model considered contains no bias with respect to the relative orientations of the matrices. In such a gauge background, the spectrum of a colored particle can be solved for analytically. In three dimensions, near zero momentum, the energy distribution for the spinless particle displays a gap, while the energy distribution for the particle with spin does not.

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I. INTRODUCTION

Much of the complexity of strong interaction dynamics derives from the non-commutativity of the gauge fields, encoded in their matrix character. It leads to the gluonic self-interactions, which generate phenomena such as asymptotic freedom, and presumably also confinement due to the constriction of chromoelectric fields into flux tubes. Matrix-valued degrees of freedom are notoriously difficult to handle, even in cases where one has managed to eliminate the space-time dependence of the matrices. One such example may be the large-$N_C$ limit of QCD ($N_C$ denoting the number of colors) [1] [2]. There one argues that gauge field path integrals should be dominated by a large-$N_C$ saddle point, and that there is a gauge where this saddle point, commonly referred to as the “master field”, is constant in space and time. This is plausible in view of the translational invariance in space-time of the underlying action and measure.

In the case of large-$N_C$ QCD, it is not even known how to formulate a potential purely in terms of space-time independent matrices which will reproduce the master field; however, also if one posits some potential of one’s choice, such simplified matrix models generically become intractable as soon as more than one matrix degree of freedom is involved, i.e. as soon as non-commutativity is allowed to play a role. There are notable exceptions, such as the Itzykson-Zuber integral [3], which lies at the heart of the Kazakov-Migdal and Penner models [4] [5]. In these models, due to the special type of potential involved, some selected observables can be calculated.

Recently, there has been some advance in the mathematical literature concerning a type of non-commuting variables called free random variables [6]; it has furthermore been shown (loc.cit.) that in the large-$N$ limit, independent hermitian matrix models describe variables of this type. These are models governed by a partition function of the form

$$Z = \int \prod_{i=1}^D [dA_i] \exp \left( -N \sum_{i=1}^D \text{Tr} V(A_i) \right)$$

(1)

where the integration is to be carried out over the real and imaginary parts of the matrix elements of the $A_i$, subject to the constraints of hermiticity. The potential, consisting of a sum of terms each only involving one of the matrices, thus only determines the eigenvalue distributions of the $A_i$ but there is no bias with respect to the relative “orientations” of the matrix degrees of freedom; each orientation is equally probable. This restriction is quite strong; most interesting matrix models, presumably including the one describing the master field of QCD, contain interactions between the different matrices. It should be noted, though, that the constructions used in the theory of free random variables may be extended to more general models and have recently led to some increased understanding of the functional analytic spaces on which the master field may be formally defined [6] [7] [8].

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However, even noninteracting matrix models constitute interesting laboratories for the study of phenomena associated with non-commutativity. Initially, it is well known how to derive the eigenvalue distributions of the involved matrices from their respective potentials. The eigenvalue distribution is related by a Hilbert transform to the derivative of the potential; i.e. if the hermitian matrix $A$ is governed by a potential term $\text{Tr} V(A)$, then the eigenvalue distribution (normalized to unity) is determined by
\[
\frac{1}{2} V'(z) = \int d\lambda \rho(\lambda) \frac{P_{\lambda}}{z - \lambda} \tag{2}
\]
Then, however, as soon as one attempts to calculate eigenvalue distributions of objects composite in the different matrices contained in the model, one is faced with their non-commutative nature. Here, the methods developed in [6] exhibit their full power: they enable one to convolute the distributions of different free random variables in a quite general and systematic manner, not restricted to a very special form of the objects considered.

In this paper, two not altogether trivial cases of problems which can be solved completely analytically with the techniques developed in [3] are exhibited. These are the problems of nonrelativistic particles with an internal color degree of freedom, with or without spin, moving in a free Gaussian random gauge background. The objective of this work is to develop some intuition regarding the phenomenological consequences of freeness in the context of large-N matrix degrees of freedom. This can be viewed as complementary to the recent more formal investigations mentioned above concerned with the formal definition of master fields and the spaces on which they operate. Here, by contrast, the emphasis lies with the study of some exactly solvable model systems which may serve as paradigms for more complicated realistic cases. The examples treated here, while sufficiently simple to be of pedagogical value, nevertheless seem generic enough to be relevant for applications. Some physics questions for which they may be of relevance will be mentioned at the end of the paper.

It is to be expected that developing some insight into the phenomena resulting from freeness also is of some value from the point of view of the quest to understand interacting matrix models beyond the few existing solvable cases. With the advent of techniques allowing one to deal with free variables, one controls the two limiting cases of vanishing and infinitely strong coupling between matrix degrees of freedom: A system of matrices whose orientations are completely aligned due to some strong coupling is as simple as a one-matrix model, since all matrices can be simultaneously diagonalised; on the other hand, a system of matrices whose relative orientations are subject to no bias at all can now be treated as well. Knowing both limits should provide valuable qualitative predictions for the behaviour of models even with nontrivial couplings.

II. PARTICLE WITHOUT SPIN

Consider the Hamiltonian
\[
H_0 = \sum_{i=1}^{D} (k_i - A_i)^2 \tag{3}
\]
This is the Hamiltonian of a spinless nonrelativistic particle with color in $D$ dimensions, minimally coupled to a background vector field $\vec{A}$. Each of the components of $\vec{A}$ is taken to be an $N \times N$ hermitian matrix (where $N \to \infty$), constant in space and time. Due to this last condition, the Hamiltonian is diagonal in momentum space and the solutions can be classified according to the momentum components $k_i$. For given $\vec{k}$ and $\vec{A}$, there are $N$ different modes of propagation, i.e. color eigenvectors, for the particle. Now, consider the vector field not as fixed, but to be averaged over an ensemble described by an independent matrix model. In other words, any observable $O$ should be averaged as
\[
\langle O \rangle = \frac{1}{Z} \int \prod_{i=1}^{D} [dA_i] O(\vec{A}) \exp \left( -N \sum_{i=1}^{D} \text{Tr} V(A_i) \right) \tag{4}
\]
Note that now the model is invariant under arbitrary unitary rotations of the variables $A_i$.

In short, the problem proposed is the following: Given the eigenvalue distributions of the $A_i$ (which can be easily found from the potential $V(A_i)$, as mentioned in the introduction), find the eigenvalue distribution of $H_0$ for arbitrary $k_i$. Since the model contains no bias with respect to the relative orientations of the $A_i$, the $A_i$ constitute a family of free random variables, and the problem is a fairly straightforward application of the techniques developed in [3]. This will serve as a warmup for the more involved case of the spinning particle, where one will have to deal with more exotic situations, such as the free random variables themselves appearing as components of matrices.
To treat a concrete model, the potential $V(A)$ must be specified. If one wishes to preserve invariance of the model under spatial rotations without destroying the freeness, one is forced to choose a Gaussian potential, $V(A) = \beta A^2$. The eigenvalue distribution for the matrices $A_i$ induced by this potential is the well-known semicircular distribution \cite{1},

$$\rho_A(\lambda) = \frac{2}{\pi a^2} \sqrt{a^2 - \lambda^2}$$

(5)

where $a^2 = 2/\beta$.

As a first step in finding the eigenvalue distribution of the Hamiltonian, one easily finds the eigenvalue distribution of each of the summands $(k_i - A_i)^2$ of $H_0$: If one has $\rho_A(\lambda)d\lambda$ eigenvalues of $A_i$ in the interval $[\lambda, \lambda + d\lambda]$, then the number of eigenvalues of $(k_i - A_i)^2$ to be found in the interval $[(k_i - \lambda)^2, (k_i - \lambda)^2 - 2(k_i - \lambda)d\lambda]$ is $\rho_A(\lambda) + \rho_A(-\lambda)) d\lambda$. Substituting $\epsilon = (k_i - \lambda)^2$, one thus finds $\rho(k_i - A_i)^2(\epsilon)d\epsilon$ eigenvalues of $(k_i - A_i)^2$ in the interval $[\epsilon, \epsilon + d\epsilon]$, where

$$\rho(k_i - A_i)^2(\epsilon) = \frac{1}{2\sqrt{\epsilon}}(\rho_A(\sqrt{\epsilon} + k_i) + \rho_A(\sqrt{\epsilon} - k_i))$$

(6)

Of course now there are only positive eigenvalues, $\epsilon > 0$.

It remains to convolute the eigenvalue distributions of the different summands $(k_i - A_i)^2$. These are hermitian random variables, with eigenvalue distributions given by \cite{1}, and freely rotating with respect to one another, i.e. there is no bias in the relative orientations in color space of the variables when taking the ensemble average over the $A_i$. To convolute these eigenvalue distributions, one must compute their so-called R-transform \cite{6}. The R-transform plays the same role for adding noncommuting free random variables as the logarithm of the Fourier transform does for adding ordinary commuting random variables: Convolution becomes ordinary addition, i.e. adding the R-transforms of the eigenvalue distributions of the summands gives the R-transform of the eigenvalue distribution of the sum. The R-transform is defined by the following algorithm: First, find the generating function of the moments of the distribution:

$$G(\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty} \rho_n \frac{1}{\xi^{n+1}}$$

(7)

$$\rho_n = \int_{-\infty}^{\infty} d\lambda \rho(\lambda)\lambda^n$$

(8)

which can be analytically continued to the entire upper complex half-plane by summing the geometric series:

$$G(\xi) = \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda)}{\xi - \lambda}$$

(9)

This continuation is important for the inverse problem of going from the generating function back to the distribution \cite{12}.

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\eta \rightarrow 0} \text{Im} G(\lambda + i\eta)$$

(10)

Next, find the function inverse to $G$, i.e. $K$ such that $K(G(\xi)) = \xi$. Then, the R-transform is given by $R(z) = K(z) - 1/z$.

It turns out that this program can still be carried out analytically for the program at hand; one is led to algebraic equations of at most fourth order and thus just remains within the realm of solvability. The generating function corresponding to the distribution \cite{13} is easily calculated to be

$$G_i(\xi) = \int_{0}^{\infty} d\epsilon \frac{1}{\xi - \epsilon} \frac{\rho_A(\sqrt{\epsilon} + k_i) + \rho_A(\sqrt{\epsilon} - k_i)}{2\sqrt{\epsilon}}$$

(11)

$$= \int_{-\infty}^{\infty} d\epsilon \frac{\rho_A(\epsilon + k_i)}{\xi - \epsilon^2}$$

(12)

\footnote{In the following, a factor $N$ will be taken out of eigenvalue distributions, i.e. $\rho_A$ is normalized to unity.}

\footnote{This problem has recently also been treated from a more physical point of view in \cite{14}.}
\[
\frac{2}{\pi} \int_{-1}^{1} \frac{de}{(\sqrt{\xi} + ki - ae)(\sqrt{\xi} - ki + ae)} = \frac{1}{a^2} \left( 2 - \frac{1}{\sqrt{\xi}}(\sqrt{(\sqrt{\xi} - ki)^2 - a^2} + \sqrt{(\sqrt{\xi} + ki)^2 - a^2}) \right)
\]

(13)

Now, to find the inverse \( K_i \) of \( G_i \), one must solve (14) for \( \xi \) in terms of \( G_i \). By always bringing the terms containing no square roots to the left hand side and squaring the resulting equation (this procedure must be applied twice), one arrives at an equation with no square roots; in this equation, quadratic in \( \xi \), the constant term vanishes, the trivial solution \( \xi = 0 \) is spurious, and thus one arrives at

\[
R_i(z) + \frac{1}{z} \equiv K_i(z) \equiv \xi(z) = \frac{1}{z} + \frac{a^2}{4 - a^2z} + \frac{4k_i^2}{(2 - a^2z)^2}
\]

(15)

The R-transform of the eigenvalue distribution of the Hamiltonian is now simply the sum of the individual R-transforms corresponding to each spatial component:

\[
R_{H_a}(z) = \sum_{i=1}^{D} R_i(z) = \frac{Da^2}{4 - a^2z} + \frac{4k_i^2}{(2 - a^2z)^2}
\]

(16)

where \( k^2 \), the squared Euclidean length of the vector \( \vec{k} \), has been introduced. Note that the \( k_i \) dependence of the individual R-transforms \( R_i \) could have already been predicted from the additivity of the R-transform and the rotational invariance of the Hamiltonian.

From \( \sum_{i=1}^{D} f(k_i) = F(|\vec{k}|) \) it already follows that \( f \) must be of the form \( f(k_i) = c + dak_i^2 \), as is manifest in (13).

It remains to go from the R-transform \( R_{H_0} \) back to the eigenvalue distribution of the Hamiltonian. This is the more arduous task. Inverting \( K_{H_0}(z) \equiv R_{H_0}(z) + 1/z \) corresponds to solving the equation

\[
4(2 - a^2z)^2 + (D - 1)a^2z(2 - a^2z)^2 + 4k^2z(4 - a^2z) = \xi z(4 - a^2z)(2 - a^2z)^2
\]

(17)

for \( z \) in terms of \( \xi \). This is a fourth order equation for \( z \), which via the substitution

\[
z = x - \frac{D - 1}{4a^2} + \frac{2}{a^2}
\]

(18)

can be brought to the normal form

\[
x^4 + (3N - 6M^2)x^2 + (8M^3 - 6NM)x + 3NM^2 - 3M^4 + L/4 = 0
\]

(19)

where the abbreviations

\[
L = \frac{64k^2}{a^8\xi}, \quad M = \frac{D - 1}{4\xi}, \quad N = \frac{4}{3a^2\xi} \left( 1 + \frac{D - 1}{2} - \frac{k^2}{a^2} \right) - \frac{4}{3a^4}
\]

(20)

have been introduced. The analysis of such an equation is quite involved, and for this reason is relegated to the appendix. The final result for the eigenvalue distribution of the Hamiltonian (3), which according to (10) is essentially the imaginary part of the solution of (19), can be stated as follows:

\[
\rho_{H_0}(\xi) = \frac{1}{2\pi} \theta(-d)(1 - \theta(-b_2)\theta(b_1)) \left| \sqrt{|y_2|} + (1 - 2\theta(4M^3 - 3NM))\sqrt{|y_3|} \right|
\]

(21)

where

\[
y_2 = 2\text{Re}(ue^{2\pi i/3}) - 2N + 4M^2 \quad y_3 = 2\text{Re}(ue^{-2\pi i/3}) - 2N + 4M^2
\]

(22)

\[
u = \left( \frac{q}{2} + i\sqrt{-d} \right)^{1/3} \quad d = \frac{p^3 + q^2}{4} \quad b_2 = 6N - 12M^2
\]

(23)

\[
p = -3N^2 - L \quad q = -2N^3 + 2NL - 4M^2L \quad b_1 = 9N^2 - 48NM^2 + 48M^4 - L
\]

(24)

\[3\text{Remember that the original Gaussian potential governing the distribution of the vector field components is invariant under rotation of the vector field in the spatial indices.} \]
The solution $\rho_{\mu}(\lambda) = 2\lambda\rho_{H_0}(\lambda^2)$ is plotted (this representation avoids the trivial square root singularity at zero eigenvalue, especially further below in the case of the spinning particle) for various values of the free parameters $k/a$ and $D$ in figures [1]-[3]. Note that when comparing different $D$, there are two distinct sensible ways of choosing the units. On the one hand, one may take the width $a$ of the distributions of the vector fields as the unit of the square root of energy, as one does when comparing different $k$ (cf. figures [1] and [2]). However, one may also rescale $a^2 = \tilde{a}^2/D$ and use $\tilde{a}$ as the unit of the square root of energy. This corresponds to keeping the norm of the vector field, i.e. $\sum_{i=1}^D A_i^2$, constant as one varies $D$ (just as one automatically does in the case of the momentum $\tilde{k}$ by only talking about its modulus), cf. figure [3].

To characterize the behaviour of the solution more precisely, it is useful to evaluate various moments and limits. The first few moments of the distribution $\rho_{H_0}$ can be obtained without knowledge of the full solution (21). It is sufficient to expand the R-transform (16) in a power series in $1/\xi$ and invert the corresponding series $K_{H_0}(z) \equiv R_{H_0}(z) + 1/z$ to the desired order, yielding the first few terms of $G(\xi)$ expanded in powers of $1/\xi$. The moments are then by definition the coefficients appearing in $G(\xi)$, yielding

$$\langle \frac{1}{N} \text{Tr} H_0 \rangle = \frac{a^2 D}{4} + k^2 = \frac{\tilde{a}^2}{4} + k^2$$

$$\langle \frac{1}{N} \text{Tr} H_0^2 \rangle - \langle \frac{1}{N} \text{Tr} H_0 \rangle^2 = \frac{a^4 D}{16} + a^2 k^2 = \frac{\tilde{a}^4}{16D} + \frac{\tilde{a}^2 k^2}{D}$$

With the help of this information, one can obtain the behaviour of the distribution as $k \to \infty$ or $D \to \infty$.

Consider large $D$, and use $a^2$ as the unit of energy. The distribution is centered around $a^2 D/4$, with a width behaving as $\sqrt{D}$. Substituting therefore

$$\xi = \frac{a^2 D}{4} + \gamma \sqrt{D}$$

in (22), and keeping only leading pieces in $1/\sqrt{D}$, one obtains that $b_2 < 0$ always, and $b_1 < 0$ for $\gamma \in [-a \sqrt{k^2 + a^2 / 2}, a \sqrt{k^2 + a^2 / 2}]$. Also,

$$d = \frac{2^{18} (4\gamma^2 - a^4) k^2}{27 a^{30}} \frac{1}{D^2}$$

i.e. $d < 0$ for $\gamma \in [-a^2 / 2, a^2 / 2]$. The condition on $d$ is more stringent than the one on $b_1$ and thus defines the support of the distribution. Furthermore, one has

$$M^2 - 3N/4 = -\frac{4}{a^8} (a^2 (a^2 - k^2) - 4\gamma^2) \frac{1}{D}$$

$$|y_2| = \frac{16}{a^8} (a^2 (a^2 + k^2) - 4\gamma^2 + 2ak \sqrt{a^4 - 4\gamma^2}) \frac{1}{D}$$

$$|y_3| = \frac{16}{a^8} (a^2 (a^2 + k^2) - 4\gamma^2 - 2ak \sqrt{a^4 - 4\gamma^2}) \frac{1}{D}$$

$$\sqrt{|y_2|} \pm \sqrt{|y_3|} = \sqrt{|y_2| + |y_3|} \pm 2 \sqrt{|y_2||y_3|}$$

$$= \frac{4\sqrt{3}}{a^8} \sqrt{a^2 (a^2 + k^2) - 4\gamma^2 \pm |a^2 (a^2 - k^2) - 4\gamma^2|} \frac{1}{\sqrt{D}}$$

and therefore

$$\rho_{H_0}(\xi) = \frac{4}{\pi a^4 \sqrt{D}} \sqrt{a^4 - 4\gamma^2}$$

In other words, for $D \to \infty$ with fixed $k$ and $a$ one obtains a semicircular distribution centered at $a^2 D/4$ and with radius $a^2 \sqrt{D}/2$.

On the other hand, if one chooses to use $\tilde{a}^2$ as the unit of energy, the limiting behaviour as $D \to \infty$ can already be read off from the first two moments. The width of the distribution vanishes and one obtains a $\delta$-peak at $\tilde{a}^2 / 4 + k^2$.

Finally, taking the limit $k \to \infty$ can be done in complete analogy to the limit $D \to \infty$ above. However, it is not necessary to explicitly go through this: For $k \to \infty$, one can approximate the Hamiltonian by $H_0 \approx k^2 + 2 \sum_{i=1}^D k_i A_i$. This means that one simply has to additively convolute the original semicircle distributions of the $A_i$, with the radii
scaled by a factor $2k$, respectively. It is well known that the semicircle distributions are closed under additive free convolution and that the radii add like components of an Euclidean vector. Therefore one immediately obtains the result that, for $k \to \infty$, the eigenvalue distribution of $H_0$ approaches a semicircle centered at $k^2$ and with radius $2ka$.

The physical interpretation of this result is clear: For $k \to \infty$, the quantum mechanical propagation of the particle is dominated by the semiclassical straight trajectories in the direction of $\vec{k}$. The particle does not see the transverse directions and behaves just as it would in a one-dimensional world; correspondingly, the dimension $D$ does not enter the limiting semicircular eigenvalue distribution as $k \to \infty$.

On the other hand, when $k$ is finite, the particle does explore the dimensions available to it to some extent. For large $D$, the width behaves as $a^2\sqrt{D} = a^2/\sqrt{D}$ (and thus does not tend to a finite value when one scales the vector field to have constant norm as $D$ is varied). The semicircular nature of the distribution in the limit of large $D$ is understandable as a manifestation of the central limit theorem: For a large number of variables (matrices), the sum of the random matrices (after separating off the first moments) behaves as if it was distributed according to a Gaussian weight, i.e. as a semicircle.

Most interesting is the soft region around $k = 0$. For $k = 0$, the solution [21] simplifies considerably. In this case, one has

\begin{align}
  d &= 0 \\
  b_2 &= 6N - 12M^2 \\
  b_1 &= (3N - 4M^2)(3N - 12M^2)
\end{align}

and therefore only a nonvanishing eigenvalue distribution when $N > 4M^2/3$. In this case, one has $\sqrt{|y_2|} = \sqrt{|y_3|} = \sqrt{3N - 4M^2}$ and therefore

\begin{equation}
  \rho_{H_0}(\xi) = \frac{1}{\pi} \theta(3N - 4M^2)\sqrt{3N - 4M^2}
\end{equation}

The edges of the distribution are determined by

\begin{equation}
  0 = 3N - 4M^2 = -\frac{(D-1)^2}{4\xi^2} + \frac{2(D+1)}{a^2\xi} - \frac{4}{a^2}
\end{equation}

leading to $\xi = a^2(\sqrt{D} \pm 1)^2/4$, i.e. the support of the distribution is the interval $[a^2(\sqrt{D} - 1)^2/4, a^2(\sqrt{D} + 1)^2/4]$. Thus, in two or more dimensions, the eigenvalue distribution has a gap, even for $k \to 0$, as already evidenced in figures [3]-[6]. Of course, for $D = 1$ one just has for $H_0$ the distribution [8], which behaves proportionally to $1/\sqrt{\xi}$ as $\xi \to 0$. Note also that the above result on the support of the distribution implies that, at $k = 0$, the spectrum of the square root of $H_0$ always has a support of length $a$, for any dimension.

Formally, the emergence of an energy gap does not seem particularly surprising in view of the fact that the Hamiltonian [7] is a sum of squares. Consider the more familiar case of ordinary (commuting) random variables, taking positive values, and with a finite probability density when one approaches zero. Typically, the distribution of a sum of two such variables will vanish linearly at zero, because sampling the two variables, it is improbable to simultaneously find two very small values. There is actually a matrix realisation of this, namely two random matrices with the aforementioned type of distributions for the eigenvalues, but not freely rotating over the whole $U(N)$ group with respect to one another; instead, constrained to commute, i.e. only all permutations of the eigenvalues are allowed. Presumably to realise this, one would have to introduce a strong interaction term between the two matrices, proportional to the square of their commutator.

In the present case, by contrast, one is allowing a much larger class of relative orientations between the matrices, namely an arbitrary $U(N)$ rotation. This larger invariance class evidently enhances the effect of making the occurrence of a small eigenvalue improbable in the sum of the matrices. For free random variables, the effect is so strong that the resulting distribution not only vanishes as one approaches zero, but vanishes on a whole interval between zero and a finite value.

**III. PARTICLE WITH SPIN**

The Hamiltonian of a particle with spin in the same gauge background as was considered in the case of the spinless particle is the $2N \times 2N$ matrix

\begin{equation}
  H_S = (\vec{\sigma}(\vec{k} - \vec{A}))^2 = (\vec{k} - \vec{A})^2 + i\vec{\sigma}(\vec{A} \times \vec{A})
\end{equation}
where the $\sigma_i$ denote the Pauli matrices; the number of space dimensions has now been specialized to $D = 3$. In the last term on the right hand side, one recognizes the coupling of the spin to the nonabelian magnetic field; the latter contains no derivative term since the vector potential is taken to be spatially constant.

In practice, it is not helpful here to separate the Hamiltonian into the spinless part and the spin-magnetic-field coupling. The crucial question when applying the free random variable techniques is whether one can cast the calculation into successive multiplications and additions of mutually free variables. E.g., given two free random variables $X$ and $Y$, it is in general not straightforward to calculate the eigenvalue distribution of $X^2 + XY$; one may accomplish the multiplicative convolution in the second summand (how this is done is explained below), but the two summands do not rotate independently of one another and therefore the free convolution techniques do not apply. On the other hand, the eigenvalue distribution e.g. of $X + XY$ is straightforward to obtain once one writes $X + XY = X(1 + Y)$. Thus, in the case of the Hamiltonian (41), having solved the spinless problem is of no assistance; one must restart and consider the eigenvalue distribution of the object

$$P = \bar{\sigma}(\vec{k} - \vec{A})$$

(obtaining the distribution of $H_S = P^2$ at the end is trivial).

As in the case of the spinless particle, the eigenvalue distributions of the $A_i$ are taken to be semicircular with radius $a$; the harmonic potential generating these distributions is invariant under spatial rotations. Therefore, without loss of generality, one may take $\vec{k}$ in the 3-direction, $\vec{k} = (0, 0, k)$. Being able to do this will be crucial for the developments to follow.

One is faced here with a new situation: Free random variables appearing as components of a matrix. Considering for the moment matrices of finite rank, the eigenvalues $\lambda$ of $P$ for a specific realisation of the gauge matrices $A_i$ are determined by the equation

$$0 = \det(P - \lambda) = \det Q_\lambda$$

with the $N \times N$ matrix

$$Q_\lambda = A_3 - k - \lambda + (A_1 + iA_2)(A_3 - k + \lambda)^{-1}(A_1 - iA_2)$$

Note that $Q_\lambda$ has only half the rank of $P - \lambda$; however, since $Q_\lambda$ is nonlinear in $\lambda$, it can have the same number of singular points in $\lambda$ as $P - \lambda$, as required. The second equality in (43) is only valid as long as $A_3 - k + \lambda$ has no exact zero eigenvalues. However, this happens only on a set of $\lambda$ of zero measure; it suffices here to consider the generic case where the above equality is valid. Also, the same consideration for $P = (i\sigma_2)P(-i\sigma_2)$ shows that the ensemble averaged eigenvalue distribution of $P$ is symmetric about zero if the eigenvalue distributions of the $A_i$ are. Therefore, it will be enough to consider positive $\lambda$.

When one is faced with matrix equations nonlinear in a parameter such as $\lambda$ in the present case, it is customary to go to an equation in larger matrices linear in the parameter $\lambda$; this is also familiar in the context of “supersymmetry tricks”. Here, it is not helpful; it just leads from $Q_\lambda$ back to the original matrix $P$. Instead, here the extended problem

$$\det(Q_\lambda - \mu) = 0$$

will be considered, and at the end specialized to $\mu = 0$. In the large-$N$, ensemble averaged language, the eigenvalue distribution of $Q_\lambda$, parametrically depending on $\lambda$, will be obtained using free convolution techniques. This yields a distribution $\rho_\lambda(\mu)$ which describes how many solutions of (41) one finds at a point in the $\lambda - \mu$ plane if one counts them as they occur on an increment $d\mu$ in $\mu$-direction. In the end, this will have to be translated into how many solutions one finds if one counts them as they occur on an increment $d\lambda$ in $\lambda$-direction; then, one may set $\mu = 0$.

One further issue must be settled before the free convolution techniques can be applied: In general, the two summands $Q_1$ and $Q_2$ appearing in $Q_\lambda = Q_1 + Q_2$,

$$Q_1 = A_3 - k - \lambda$$

$$Q_2 = (A_1 + iA_2)(A_3 - k + \lambda)^{-1}(A_1 - iA_2)$$

do not rotate independently of one another, since they both contain $A_3$. This would thwart attempts to use free convolution. The special case considered here, however, can be rescued: The matrix $A_1 + iA_2$, with $A_1$ and $A_2$ hermitian, represents a general complex matrix, which can be reparametrized using the polar decomposition [14]

$$A_1 + iA_2 = V^\dagger BVU^\dagger$$

(48)
where $U$ and $V$ are unitary and $B$ is diagonal and positive. Integrating over all hermitian $A_1$ and $A_2$ corresponds to integrating over $U$ and $V$ with Haar measure and over the eigenvalues $\eta$ contained in $B$,

$$[dA_1][dA_2] = J(\tilde{\eta})d^N\eta [dU][dV]$$

(49)

where the Jacobian is

$$J(\tilde{\eta}) = \prod_{i=1}^{N} \eta_i \prod_{i<j}(\eta_i^2 - \eta_j^2)^2$$

(50)

To be precise, the integration over $V$ is only over the quotient group $U(N)/(U(1)^N)$; this is no consequence, as will be explained below. Also, note that in (49), the product $VU^\dagger$ is usually written as a single matrix; however, since $U$ is being integrated over with Haar measure, one can always pull out a factor $V$.

In the new variables, the potential governing $A_1$ and $A_2$ becomes

$$\text{Tr} V(A_1, A_2) = \beta \text{Tr}(A_1^2 + A_2^2) = \beta \text{Tr} B^2$$

(51)

and one has

$$Q_2 = V^\dagger BVU^\dagger(A_3 - k + \lambda)^{-1}UV^\dagger BV$$

(52)

Cast in this form, it becomes clear that the calculation of the $Q_\lambda$ eigenvalue distribution can be carried out using free convolution: The combination $U^\dagger(A_3 - k + \lambda)^{-1}U$ rotates independently of $Q_1 = A_3 - k - \lambda$, and therefore the evaluation of the eigenvalue distribution of $Q_\lambda$ indeed becomes a sequence of free convolutions if one additionally uses that the eigenvalue distribution of $Q_2$ is the same as the one of

$$\tilde{Q}_2 = (V^\dagger BV)^{-1}Q_2(V^\dagger BV) = U^\dagger(A_3 - k + \lambda)^{-1}UV^\dagger B^2V$$

(53)

(up to regions of zero measure in $B$). It is also clear that the restricted integration domain of $V$ is of no consequence: For two variables to be free with respect to one another, it suffices to rotate one of the variables. The problem is in fact invariant under the additional rotations by $V$ and the $V$-integration gives only a normalization factor.

Finally, before applying multiplicative free convolution to obtain the eigenvalue distribution of $\tilde{Q}_2$, it remains to solve the saddle point equation for the eigenvalue distribution of $B^2$, which is controlled by the Jacobian (52) and the potential (51). This standard procedure is carried out in the Appendix, yielding

$$\rho_{B^2}(\eta) = \frac{1}{a^2\pi} \sqrt{\frac{2a^2 - \eta}{\eta}}, \quad 0 \leq \eta \leq 2a^2$$

(54)

Note that $B$ itself is therefore distributed according to a quarter-circle of radius $\sqrt{2}a$.

As explained in connection with the spinless particle, additive free convolution of eigenvalue distributions is accomplished by going to the corresponding R-transforms, which are additive; similarly, for multiplicative free convolution one defines the S-transform, which behaves multiplicatively. The S-transform is defined as follows: Find again the moment generating function $G(z)$ as in (51); solve for the function inverse to $G(1/z)/z - 1$, i.e. $T(y)$ such that

$$\frac{1}{T(y)}G\left(\frac{1}{T(y)}\right) - 1 = y$$

(55)

Then the S-transform is given by

$$S(y) = \frac{1 + y}{y}T(y)$$

(56)

With the help of these techniques, one can now carry out the necessary convolutions leading to the eigenvalue distribution of $Q_\lambda$. Starting with the eigenvalue distributions (54) and (cf. (5),(6))

$$\rho(A_3-k+\lambda)^{-1}(\eta) = \frac{1}{\eta^2}\rho_A\left(\frac{1}{\eta} + k - \lambda\right)$$

(57)

one obtains the corresponding generating functions
where the relevant solution has been picked out by using, in the limit \( T \) to eq. (15). In the case of \( R \), \( R \)-transform with the one of \( Q \), realization of the gauge matrices \( A \) and consequently, \( T \) \( G \) \( \rho \) \( \mu \) \( \lambda \) \( u \) \( v \) \( w \) \( z \). From this, one obtains that the function \( G \) satisfies the equation

\[
G_B^2(z) = \frac{1}{a^2} \left( 1 - \sqrt{\frac{z - 2a^2}{z}} \right)
\]

(58)

\[
G_{(A_3 - k + \lambda)^{-1}}(z) = \int_{-\infty}^{\infty} dx \frac{\rho_A(x)}{z - \frac{1}{x - k + \lambda}}
\]

(59)

\[
= \frac{2}{\pi a^2 z} \int_{-a}^{a} dx \sqrt{a^2 - x^2} \left( 1 + \frac{1}{z - k + \lambda - \frac{x}{z}} \right)
\]

(60)

\[
= \frac{1}{z} - \frac{2}{z^2 a^2} \left( \frac{z}{z} + k - \lambda - \sqrt{\frac{1}{z} + k - \lambda \lambda^2 - a^2} \right)
\]

(61)

In both cases, \( T \) is determined by a quadratic equation after one has applied a procedure analogous to the one leading to eq. \( (15) \). In the case of \( T_B^2 \), the correct solution is easily picked out because there is a spurious zero solution; in the case of \( T_{(A_3 - k + \lambda)^{-1}} \), the correct solution can be picked out by considering the limiting behaviour as \( a \to 0 \) and using that the \( S \)-transform of the unit matrix multiplied by a constant \( w \) is \( 1/w \). Thus one arrives at the \( S \)-transforms

\[
S_B^2(y) = \frac{2}{a^2} \frac{1}{1 + y}
\]

(62)

\[
S_{(A_3 - k + \lambda)^{-1}}(y) = \frac{\lambda - k}{2} \left( 1 + \sqrt{1 - \frac{a^2(1 + y)}{(\lambda - k)^2}} \right)
\]

(63)

\[
S_Q(y) = S_Q^*(y) = S_B^2(y) \cdot S_{(A_3 - k + \lambda)^{-1}}(y) = \frac{\lambda - k}{a^2(1 + y)} \left( 1 + \sqrt{1 - \frac{a^2(1 + y)}{(\lambda - k)^2}} \right)
\]

(64)

From this, one obtains that the function

\[
\frac{G_Q(z)}{z} - 1 \equiv T_{Q_2}^{-1}(z) \equiv u
\]

(65)

satisfies the equation

\[
z^4 u^4(1 + u)^3 - 2za^2(\lambda - k)u(1 + u) + a^2 u^2 = 0
\]

(66)

and consequently, \( G_Q(z) \equiv v \) satisfies

\[
a^4 v^3 z - 2a^2(\lambda - k)v(vz - 1) + a^2(vz - 1)^2 = 0
\]

(67)

There is no need to solve for \( G_Q \) at this point; the eigenvalue distribution of \( Q_2 \) must still be additively convoluted with the one of \( Q_1 \). Therefore, one can solve \( (15) \) directly for \( z = G_Q(v) \),

\[
K_{Q_2}(v) \equiv G_Q^{-1}(v) = \frac{1}{v} - \frac{a^2 v}{2} + \lambda - k - (\lambda - k) \sqrt{\left(1 - \frac{a^2 v}{2(\lambda - k)}\right)^2 - \frac{a^2}{(\lambda - k)^2}}
\]

(68)

where the relevant solution has been picked out by using, in the limit \( a \to 0 \), that \( Q_2 \to 0 \) and therefore the R-transform \( R_{Q_2}(v) = K_{Q_2}(v) - 1/v \to 0 \). Combining this with the R-transform corresponding to \( Q_1 \) (cf. \( (16) \)),

\[
R_{Q_1}(v) = \frac{a^2}{4} v - k - \lambda
\]

(69)

one finally obtains

\[
G_Q^{-1}(v) \equiv K_{Q_3}(v) = \frac{1}{v} - \frac{a^2 v}{4} v - 2k - (\lambda - k) \sqrt{\left(1 - \frac{a^2 v}{2(\lambda - k)}\right)^2 - \frac{a^2}{(\lambda - k)^2}}
\]

(70)

This determines \( G_{Q_3} \) and thus ultimately \( \rho_\lambda(\mu) = -\text{Im} \ G_{Q_3}(\mu)/\pi \). However, before embarking on the arduous task of inverting \( (17) \), it is advantageous at this stage to consider how \( \rho_\lambda(\mu) \) in the end determines the eigenvalue distribution \( \rho_P(\lambda) \) of \( P \) (cf. \( (13) \)). Considering for the moment finite matrices, the solutions of equation \( (15) \) for a specific realization of the gauge matrices \( A \), define continuous trajectories in the \( \lambda - \mu \) plane, cf. figure \( (18) \). Asymptotically,
there are trajectories in the vicinity of the lines defined by $\lambda + k = -\mu$ and $\lambda - k = 0$, where “vicinity” means at a finite distance of roughly up to the semicircle radius $a$. The quantity one is ultimately interested in is the number of trajectories one crosses if one marches from $\lambda = \infty$ in along the $\lambda$-axis to some point $\lambda_0$; this is essentially the integral over $\rho_P(\lambda)$. Consider the conjecture that this is the same as the number of trajectories one crosses marching from $(\mu, \lambda) = (\infty, \infty)$ in $\lambda$-direction to $(\mu, \lambda) = (\infty, \lambda_0)$, and then in $\mu$-direction to $(\mu, \lambda) = (0, \lambda_0)$, cf. figure [4]. There is one point which must be clarified before this statement can be accepted as true: In general, it might happen that a trajectory intersects the integration paths described above more than once, and then the two paths may count a different number of trajectory crossings, as displayed in figure [4] in the right-hand graph. It will now be argued that this is not possible. To begin with, note that

$$Q_\lambda(k) - \mu = Q_{\lambda+\mu/2}(k + \mu/2)$$

(71)

This means that solving the extended problem [45] with general $\mu$ is in fact equivalent to solving the original problem [45] with shifted $k$ and $\lambda$ (conversely, $k$ could be absorbed into $\lambda$ and $\mu$). Now, the solutions $\lambda$ of the original problem $\det(P - \lambda) = 0$ are continuous in $k$, which can be seen as follows: Since the determinant of a Hermitian matrix is real, the characteristic polynomial $\det(P - \lambda)$ is real-valued for real $\lambda$; its coefficients are real polynomials in $k$. Therefore, the graph of $\det(P - \lambda)$ varies continuously with $k$. Now, if one wanted to “annihilate” two zeros of $\det(P - \lambda)$ as $k$ is varied, say by having a minimum of $\det(P - \lambda)$ cross from below the $\lambda$-axis to above the $\lambda$-axis, on would have to “create” two other zeros somewhere else on the $\lambda$-axis in a similar fashion, because $\det(P - \lambda)$ must always have $2N$ zeros ($2N$ denoting the rank of $P$). However, at the point in $k$ where this discontinuity takes place, one would thus have more than $2N$ zeros, which is impossible. Therefore, the eigenvalues are indeed continuous as $k$ is varied. According to [71], this immediately also implies that in the extended problem $\det(Q_\lambda - \mu) = 0$, as $\mu$ is varied, the trajectories $\lambda(\mu)$ must be continuous.

Consider now the possibility that a trajectory in the $\lambda - \mu$ plane crosses a line of constant $\mu$ more than once, as in figure [4] in the right-hand graph. This would imply a discontinuous dependence $\lambda(\mu)$ and can therefore not arise. In conclusion, this shows that marching along any line parallel to the $\lambda$-axis in the $\lambda - \mu$ plane, one can cross any eigenvalue trajectory at most once, as in figure [4] in the left-hand graph.

One could argue in a similar way for paths in the $\mu$-direction, except for having to be more careful due to the poles in $Q_\lambda$. There is an easier way to handle paths in the $\mu$-direction once one has treated the case of the $\lambda$-direction: Since one crosses each trajectory at most once when integrating in $\lambda$-direction, it is necessary to cross all of them to saturate the normalisation condition in $\lambda$-direction. Then, however, one must also cross all of them if one marches in straight lines between the points $(\mu, \lambda) = (\infty, \infty) \to (\infty, \lambda_0) \to (-\infty, \lambda_0) \to (-\infty, -\infty)$. In other words, in this way one counts at least $2N$ trajectories, and more if there are multiple intersections when marching in $\mu$-direction. However, one can easily check at the end using the large-$N$ ensemble averaged result that the integral is exactly 2 (remember that the factor $N$ has been scaled out everywhere in the eigenvalue distributions) and therefore realisations of the $A_0$ which generate multiple intersections when marching in $\mu$-direction give vanishing contributions to the large-$N$ eigenvalue distributions.

In essence, the basic property which makes these arguments work is that the trajectories in the $\lambda - \mu$ plane can be interpreted in two different ways. Obviously, when considering the extended problem [45], one is considering the behaviour of the eigenvalues $\mu$ under changes of the parameter $\lambda$. On the other hand, due to the equivalence [71], one can also interpret the trajectories as describing the behaviour of the eigenvalues $\lambda$ under changes of the parameter $\mu$ (up to the trivial additional linear shift in $\lambda$). These two interpretations taken together ensure the monotonicity of the trajectories, i.e. that they can cross each line of constant $\lambda$ or $\mu$ only once.

As a result of these observations one can now indeed relate $\rho_\lambda(\mu)$ to $\rho_P(\lambda)$:

$$\int_{\lambda_0}^\infty d\lambda \rho_P(\lambda) = \int_0^\infty d\mu \rho_{\lambda_0}(\mu) + \int_{\lambda_0}^\infty d\lambda \rho_{\text{extra}}(\lambda)$$

(72)

which by differentiation with respect to $\lambda_0$ gives $\rho_P$. The second contribution on the right hand side arises as follows: For $|\lambda - k| < a$, there is a nonvanishing density of trajectories as $\mu \to \infty$ due to the poles in $Q_\lambda$ (cf. figure [4]). These trajectories asymptotically become parallel to the $\mu$-axis, and therefore $\rho_\lambda(\mu)$ vanishes (as it must due to normalisation) as $\mu \to \infty$, since $\rho_\lambda(\mu) d\mu$ measures the number of trajectories encountered when marching in $\mu$-direction. On the other hand, marching in $\lambda$-direction, one crosses these trajectories perpendicularly. For a concrete realization of the $A_0$, they are determined as follows: In the complete problem $\det(Q_1 + Q_2 - \mu) = 0$, one can neglect $Q_1$ for $\mu \to \infty$, i.e. one is looking for the infinite eigenvalues of $Q_2$; these occur exactly when $A_3 - k + \lambda$ has zero eigenvalue, i.e. when $-\lambda$ is an eigenvalue of $A_3 - k$. Therefore $\rho_{\text{extra}}(\lambda)$ is just the (shifted) original semicircle distribution,

$$\rho_{\text{extra}}(\lambda) = \rho_A(k - \lambda)$$

(73)
One can therefore now specify $\rho_P(\lambda)$:

$$
\rho_P(\lambda) = -\frac{\partial}{\partial \lambda} \int_0^\infty d\mu \rho_\lambda(\mu) + \rho_A(k - \lambda) \\
= \frac{1}{\pi} \text{Im} \frac{\partial}{\partial \lambda} \int_0^\infty d\mu G_{Q_\lambda}(\mu) + \rho_A(k - \lambda) \\
= \frac{1}{\pi} \text{Im} \int_{G_{Q_\lambda}(0)}^0 dx \frac{\partial}{\partial x} G_{Q_\lambda}^{-1}(x) + \rho_A(k - \lambda) \\
= \frac{1}{\pi} \int_{G_{Q_\lambda}(0)}^0 dx G_{Q_\lambda}^{-1}(x) + \rho_A(k - \lambda) \\
= \frac{1}{\pi} \frac{\partial}{\partial \lambda} \int_{G_{Q_\lambda}(0)}^0 dx G_{Q_\lambda}^{-1}(x) + \rho_A(k - \lambda) \\
= \frac{2}{a^2\pi} \text{Im} \left[ (\lambda - k) \sqrt{\left(1 - \frac{a^2 G_{Q_\lambda}(0)}{2(\lambda - k)}\right)^2 - \frac{a^2}{(\lambda - k)^2} - \sqrt{(\lambda - k)^2 - a^2}} \right] + \rho_A(k - \lambda)
$$

(74, 75, 76, 77, 78, 79)

(in this derivation it has been used several times that $\text{Im} G_{Q_\lambda}(0)$ never diverges, which will be verified below). The last two terms in (79) cancel, and using $0 = G_{Q_\lambda}^{-1}(G_{Q_\lambda}(0))$ together with (77), one can additionally eliminate the square root to arrive at

$$
\rho_P(\lambda) = \frac{2}{a^2\pi} \text{Im} \left( \frac{1}{G_{Q_\lambda}(0)} - \frac{a^2}{4} G_{Q_\lambda}(0) \right) = \frac{2}{a^2\pi} \text{Im} G_{Q_\lambda}(0) \left( \frac{1}{(\text{Re} G_{Q_\lambda}(0))^2 + (\text{Im} G_{Q_\lambda}(0))^2} + \frac{a^2}{4} \right)
$$

(80)

It only remains thus to solve (77) for $G_{Q_\lambda}(0)$. For $v \equiv G_{Q_\lambda}(0)$ in (77), the left hand side vanishes and the square root can be eliminated by squaring the equation. Eliminating the cubic term in the resulting quartic equation by the substitution

$$
v = \frac{x}{a} + \frac{4\lambda}{3a^2}
$$

(81)

one arrives at

$$
x^4 + c_2x^2 + c_1x + c_0 = 0
$$

(82)

with

$$
c_2 = -\frac{8}{3}(2L^2 + 4KL + 6K^2 + 1) \\
c_1 = -\frac{64}{27}(2L^3 + 12KL^2 + 18K^2L + 3L - 9K) \\
c_0 = -\frac{16}{27}(32KL^3 + 48K^2L^2 + 8L^2 - 48KL + 9)
$$

(83, 84, 85)

given in terms of the dimensionless $K = k/a$ and $L = \lambda/a$. The analysis of this equation is relegated to the appendix; the final result for $G_{Q_\lambda}(0)$ can be stated as follows:

$$
\text{Im} G_{Q_\lambda}(0) = -\frac{1}{2a} \theta(d)|i(\sqrt{y_1} - \sqrt{y_2})| \\
\text{Re} G_{Q_\lambda}(0) = \frac{1}{2a} \left(2\theta(9K - 2L^2 - 12KL^2 - 18K^2L - 3L) - 1\right) \sqrt{d} + \frac{4L}{3a} \text{ for } d > 0
$$

(86, 87)

where

$$
y_1 = e^{2\pi i/3}u + e^{-2\pi i/3}u' - b_2/3 \\
y_2 = e^{-2\pi i/3}u + e^{2\pi i/3}u' - b_2/3 \\
y_3 = u + u' - b_2/3 \\
b_2 = \frac{16}{3} \left(1 + 2K^2 + L^2 + (2K + L)^2\right) \\
u = -\frac{|p|}{q} \sqrt{\frac{a}{d}} \left[\frac{q}{2} + \sqrt{d}\right]^{1/3} \\
u' = -\frac{|p|}{q} \sqrt{\frac{a}{d}} \left[\frac{q}{2} + \sqrt{d}\right]^{-1/3}
$$

(88 - 90)

11
and
\[ d = \frac{p^3}{27} + \frac{q^2}{4} \]  
(91)
\[ p = \frac{256}{27} (2 - 3K^2 - 9K^4 - 14KL - 12K^3L + L^2 + 2K^2L^2 + 4KL^3 - L^4) \]  
(92)
\[ q = \frac{4096}{729} (7 - 36K^2 + 27K^4 + 54K^6 + 48KL + 144K^3L + 108K^5L + 12L^2 + 66K^2L^2 - 18K^4L^2 - 48KL^3 - 56K^3L^3 + 3L^4 - 6K^2L^4 + 12KL^5 - 2L^6) \]  
(93)
The solution \( \rho_{\sqrt{P_{\lambda}}}(\lambda) = 2\lambda\rho_{H_S}(\lambda^2) = 2\rho_P(\lambda)|_{\lambda > 0} \) is plotted for various values of the free parameter \( K = k/a \) in figure [3]. In the limit \( k \to \infty \), the coupling to the magnetic field in the Hamiltonian (41) can be neglected and the energy spectrum is therefore identical to the spinless case: Semicircular with radius \( 2ka \), centered at \( k^2 \). On the other hand, in the soft region around \( k = 0 \), the two cases differ qualitatively: There is no gap in the spectrum at \( k = 0 \) for the particle with spin, since the discriminant (91) is positive for \( k = \lambda = 0 \), giving a nonzero result for (86). Only at a certain nonzero value of \( k \) does the distribution detach from the origin.

In order to characterize this spin-induced effect more precisely, it is useful to evaluate the first two moments of the energy distribution. Since in the present case, the full generating function \( G_{H_S} \) corresponding to the distribution \( \rho_{H_S} \) is not explicitly available, one cannot directly read off the moments. However, they are easily calculated using the axioms of freeness \( [3] \), which the random matrices \( A_i \) obey. First, one trivially has that the first moments of the eigenvalue distributions of \( H_0 \) and \( H_S \) are the same,
\[ \left\langle \frac{1}{2N} \text{Tr} H_S \right\rangle = \left\langle \frac{1}{N} \text{Tr} H_0 \right\rangle = k^2 + \frac{3}{4} a^2 \]  
(94)
since the additional spin-magnetic field coupling in \( H_S \) obeys \( \text{Tr} (i\vec{\sigma} (\vec{A} \times \vec{A})) = 0 \) already in every fixed realization of the vector potential \( \vec{A} \). Then, one has
\[ \left\langle \frac{1}{2N} \text{Tr} H_S^2 \right\rangle - \left\langle \frac{1}{2N} \text{Tr} H_S \right\rangle^2 = \left\langle \frac{1}{N} \text{Tr} H_0^2 \right\rangle - \left\langle \frac{1}{N} \text{Tr} H_0 \right\rangle^2 - \left\langle \frac{1}{2N} \text{Tr} (\vec{\sigma} (\vec{A} \times \vec{A}) \vec{\sigma} (\vec{A} \times \vec{A})) \right\rangle \]  
(95)
again having used the tracelessness of the \( \sigma \)-matrices). Now,
\[ \left\langle \frac{1}{2N} \text{Tr} (\vec{\sigma} (\vec{A} \times \vec{A}) \vec{\sigma} (\vec{A} \times \vec{A})) \right\rangle = \left\langle \frac{6}{N} \text{Tr} (A_1 A_2 A_1 A_2 - A^2_1 A^2_2) \right\rangle = -6 \left( \frac{a^2}{4} \right)^2 \]  
(96)
Here, in the first equality it has been used that the matrices \( A_i \) obey identical distributions; in the second equality, the axioms of freeness for \( A_1 \) and \( A_2 \) have been utilized along with \( \langle (1/N) \text{Tr} A_i \rangle = 0 \) and \( \langle (1/N) \text{Tr} A_i^2 \rangle = a^2/4 \). Therefore,
\[ \left\langle \frac{1}{2N} \text{Tr} H_S^2 \right\rangle - \left\langle \frac{1}{2N} \text{Tr} H_S \right\rangle^2 = \left\langle \frac{1}{N} \text{Tr} H_0^2 \right\rangle - \left\langle \frac{1}{N} \text{Tr} H_0 \right\rangle^2 + \frac{3}{8} a^4 = k^2 a^2 + \frac{9}{16} a^4 \]  
(97)
Thus, the effect of the additional interaction of the spin with the magnetic field is merely to broaden, but not to shift, the eigenvalue distribution of the Hamiltonian. It does this sufficiently strongly to make the energy gap arising in the case of a spinless particle disappear at small momenta \( k \). Figure [3] compares \( \rho_{\sqrt{P_{\lambda}}} \) and \( \rho_{\sqrt{P_{\lambda}}}/2 \) for different \( k \).

Evidently, it is possible for the particle to align its spin with the nonabelian magnetic field in ways which allow it to lower the energy associated with some of the modes of propagation to values near zero. There are sufficiently many such possibilities to make the gap in the energy distribution disappear for small momenta \( k \). A calculation of more specific spin-color magnetic field correlation functions, which would give a more detailed description of this effect, will be foregone here.

\[^4\text{A factor two is divided out of } \rho_{\sqrt{P_{\lambda}}} \text{ in order to compare two distributions normalised to unity.}\]
IV. SUMMARY

In this work, the spectrum of a nonrelativistic particle with internal colour degree of freedom moving in a constant Gaussian free random gauge background was discussed. Both the case of a spinless particle and the case of a spin-1/2 particle were considered. The limit of large momenta \( k \) is easily understood perturbatively; the energy distribution becomes semicircular. Also, in the spinless case for a large number of spatial dimensions, one obtains a semicircle as a consequence of the central limit theorem. On the other hand, the nonperturbative regime near \( k = 0 \) displays very interesting features. The spectrum of the spinless particle exhibits a finite gap in two or more dimensions. Evidently, for free random variables, the suppression of small eigenvalues in sums of squares (such as the spinless Hamiltonian) is stronger than in the case of ordinary commuting random variables, for which only a linear vanishing of the distribution at zero occurs. By contrast, when the particle possesses a spin degree of freedom, the gap in the energy distribution disappears at small momenta. Formally, this is understandable since the Hamiltonian ceases to be a sum of squares; physically, the particle can align its spin with the background color magnetic field such as to lower the energies of some of the modes. This effect is sufficiently strong to obliterate the gap in the energy spectrum occurring in the spinless case.

On a technical level, the treatment depended crucially on the fact that the Hamiltonians considered were block-diagonal in momentum space. Thus, it would be very hard e.g. to combine a gauge background such as the one used here with a spatial harmonic oscillator potential for the particle. On the other hand, one can easily envision introducing such a gauge background into a bag model. In such a framework, the gauge background would provide a mechanism for generating masses for the constituent quarks. It would be interesting to pinpoint the differences between a bag model with a random gauge background and a conventional bag model with constituent quark masses introduced by hand.

A related issue is the question of chiral symmetry breaking in a free random gauge background. The chiral condensate can be related via the Casher-Banks formula \[15\] to the spectrum of the Dirac operator \[16\] \[17\] \[18\]. The methods developed here for the \( 2 \times 2 \) spin structure occurring in the Pauli Hamiltonian would seem to constitute a first step towards also evaluating the eigenvalue distribution of the \( 4 \times 4 \) Dirac operator in a random gauge background.

Finally, a different physical way to view the average over an ensemble of spatially constant gauge configurations is to identify the ensemble averaging with a domain or time averaging. If the gauge background through which the particle is propagating consists of domains of approximately constant color magnetic fields with random orientations, then the wave packet will feel different realisations of the gauge matrices as it evolves. Averaging over such a background can be replaced by averaging over one domain with all possible background gauge configurations if one argues the effects of the domain walls to be small.

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APPENDIX A: SOLUTION OF FOURTH ORDER EQUATION DETERMINING EIGENVALUE DISTRIBUTION OF SPINLESS PARTICLE

The solutions of the equation \[19\]

\[ x^4 + (3N - 6M^2)x^2 + (8M^3 - 6NM)x + 3NM^2 - 3M^4 + L/4 = 0 \]  

(A1)

are determined by the solutions of the corresponding cubic resolvent equation \[19\]

\[ y^3 + b_2 y^2 + b_1 y + b_0 = 0 \]  

(A2)

with

\[ b_2 = 6N - 12M^2 \quad b_1 = 9N^2 - 48NM^2 + 48M^4 - L \quad b_0 = -(8M^3 - 6NM)^2 \]  

(A3)

as
\[
x_1 = \frac{1}{2}(\sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}) \quad x_2 = \frac{1}{2}(\sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3})
\]
\[
x_3 = \frac{1}{2}(-\sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}) \quad x_4 = \frac{1}{2}(-\sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3})
\] (A4)

where in addition to the signs given, the signs of the square roots must be adjusted so as to guarantee

\[
\sqrt{y_1}\sqrt{y_2}\sqrt{y_3} = 8M^3 - 6NM
\] (A5)

In order to preserve continuity of the solutions in the parameters \(L, M, N\), this latter condition in practice is handled as follows: Choose the square roots in the conventional way at some initial point in parameter space where this is allowed. Points in parameter space where the right hand side of (A5) changes sign coincide with one of the solutions \(y_i\) of (A2) touching zero; give the square root of this solution an additional minus sign when crossing such a point to maintain (A5). Note that after two such sign changes one may have switched between solutions in (A4).

The third order equation (A2) in turn can be simplified as follows. Via the substitution

\[
x = \frac{w}{\sqrt{a}} \quad y = \frac{w^2}{a^2}
\] (A6)

it reduces to the normal form

\[
w^3 + pw + q = 0
\] (A7)

with

\[
p = -3N^2 - L \quad q = -2N^3 + 2NL - 4M^2L
\] (A8)

and the associated discriminant

\[
d = \frac{q^2}{4} + \frac{p^3}{27} = 4L^2(M^2 - 3N/4)^2 + (4N^3L + 2NL^2)(M^2 - 3N/4) - \frac{N^2L^2}{12} - \frac{L^3}{27}
\] (A9)

The latter form will be useful below.

As a next step, it is necessary to pick out which of the solutions of the original fourth order equation is the relevant one. The only place where this question is easily answered is for \(\xi \to \infty\). This will be chosen as the starting point and the solution will then be followed through parameter space using continuity. At \(\xi \to \infty\), using the definitions (24), eq. (A2) simplifies to \(y(y - 4/a^4)^2 = 0\) and thus two of the solutions are \(y_{1,2} = 4/a^4\). For the third solution, one needs to go to order \(O(1/\xi^2)\). Writing \(y_3 = s/\xi^2\), and keeping only the leading order in \(1/\xi\) in (A3), one obtains \(s = (D - 1)^2/4\). Note thus that (A3) is fulfilled to leading order in \(1/\xi\) with the conventional choice of the square roots. Now, since the moment generating function \(G_{H_0}(\xi)\) must vanish as \(\xi \to \infty\), and

\[
G_{H_0}(\xi) = x(\xi) - \frac{D - 1}{4\xi} + \frac{2}{a^2}
\] (A10)

(cf. (18)), one easily picks out the solution \(x_4\) in (A4) as the correct one for \(\xi \to \infty\).

Now, to follow the solution through parameter space, it is necessary to collect some facts about its behaviour. First, note that the constant term in the cubic resolvent equation (A2) is negative. By Vieta’s theorem it thus follows that the product of its three solutions is always positive. Therefore, the solutions of the equation may behave in three qualitatively distinct ways:

1. Three positive real solutions: \(d < 0 \land b_2 < 0 \land b_1 > 0\).

2. One positive real solution, two complex conjugate solutions: \(d > 0\).

3. One positive real solution, two negative real solutions: \(d < 0 \land (b_2 > 0 \lor b_1 < 0)\).

where in the characterization of alternatives 1.) and 3.), Descartes’ sign rule for the coefficients in (A2) has been used.

Further to this, one can show that, for \(\xi > 0\), changes of sign of square roots according to condition (A5) can only occur in regions of type 3.) in parameter space. The argument goes as follows: The right hand side of (A5) being zero implies \(4M^2 = 3N\) (note that \(M > 0\) for \(\xi > 0\)). In these circumstances, the cubic resolvent eq. (A2) simplifies to \(y^3 - 4M^2y^2 - Ly = 0\). Noting that \(L > 0\) for \(\xi > 0\), this equation has one positive and one negative solution apart
from the one which is zero. One is therefore indeed in a region of type 3.) in parameter space; it is one of the two negative solutions which touches zero and whose square root therefore acquires a minus sign.

Below, slightly more than this will be needed, namely that when the solutions go from a region of type 2.) to a region of type 3.) and back to a region of type 2.), either none or both the solutions which go from being complex conjugate to negative and back to complex conjugate must have acquired a minus sign for their square root. This is due to the fact that in regions of type 2.), i.e. for \( d > 0 \), the right hand side of (\( A_3 \)) is always positive. To see this, consider the converse proposition, namely that \( 4M^2 - 3N < 0 \) implies \( d < 0 \). First, note that \( 4M^2 - 3N < 0 \Rightarrow N > 0 \) and \( 4M^2 - 3N > -3N \). Check that at \( 4M^2 - 3N = 0 \) and at \( 4M^2 - 3N = -3N \), \( d \) is manifestly negative; furthermore, regarding (\( A_3 \)) as a function of \( 4M^2 - 3N \) and finding the lone extremum at \( 4M^2 - 3N = -N - 2N^3/L \), one verifies that also there, \( d < 0 \), completing the argument.

Apart from sign changes in accordance with condition (\( A_3 \)), there is one other possibility for a continuous switch between the different solutions in (\( A_4 \)), namely that two of these solutions may become degenerate (this is the generic case; higher degeneracies occur only at exceptional points in parameter space). This happens precisely when \( d = 0 \), i.e. when there is a concomitant change of region. One can make this more precise: When starting with the conventional choice of square roots in (\( A_4 \) in a region of type 1.), upon reaching a region of type 2.), \( x_4 \) always remains unique, only two of the other three solutions may become degenerate. This remains true as long as one only alternates between regions of type 1.) and type 2.). On the other hand, if one continues from a region of type 2.) to a region of type 3.) (and back), \( x_4 \) becomes degenerate with one of the other solutions; if, without loss of generality, one denotes as \( y_1 \) the solution of (\( A_3 \)) which has remained real and positive, then it is \( x_4 \) and \( x_3 \) which become degenerate. This is the same switch which may also be effected by condition (\( A_2 \)) as shown further above.

Armed with these properties, one can now argue that, despite all switches of branch which may happen as one tracks the solution through parameter space, it is only in regions of type 3.) that it may have a nonzero imaginary part (which up to trivial factors gives the sought-after eigenvalue distribution of the Hamiltonian (\( \mathcal{H} \)). As elucidated above, at \( \xi \rightarrow \infty \) one starts in a region of type 1.) with the solution \( x_4 \) in (\( A_4 \)). Now, changes of region occur precisely where \( d = 0 \). This latter condition is a fifth order equation in \( \xi \), so there can be at most five changes of region as \( \xi \) is varied, i.e. the following sequences are possible:

a. type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (1) \( \rightarrow \) type (2)

b. type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (3) \( \rightarrow \) type (2)

c. type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (3) \( \rightarrow \) type (2) \( \rightarrow \) type (1) \( \rightarrow \) type (2)

d. type (1) \( \rightarrow \) type (2) \( \rightarrow \) type (3) \( \rightarrow \) type (2) \( \rightarrow \) type (3) \( \rightarrow \) type (2)

Of course, the sequences may be shorter if \( d \) has less than five real zeros as a function of \( \xi \). Now, according to the above, in sequences a.), b.), and d.) one always remains with solutions \( x_4 \) and \( x_3 \) (with the convention that \( y_1 \) is the solution which has remained real and positive throughout, otherwise \( x_3 \) is replaced by one of the other two solutions). However, both in \( x_4 \) and \( x_3 \) in regions of type 2.) the imaginary parts of \( \sqrt{y_2} \) and \( \sqrt{y_3} \) cancel; thus one indeed only has a nonzero imaginary part of the solution in regions of type 3.). The remaining sequence c.) is more complicated. There, one may emerge in the second region of type 1.) with the solution \( x_3 \) and thus, going to the last region of type 2.), switch to a solution in which the imaginary parts do not cancel anymore. Here, this last scenario will not be analyzed in detail. In practice, it suffices to note that the region of type 3.) already saturates the normalization of the eigenvalue distribution and that therefore there can be any additional nonzero contribution from the last region of type 2.).

Summing up, the imaginary part of \( G_{H_0}(\xi) \) is only nonzero in regions of type 3.), i.e. formally one can write

\[
\text{Im} G_{H_0}(\xi) = -\frac{1}{2} \theta(-d)(1 - \theta(-b_2)\theta(b_1))\text{Im}(\pm \sqrt{y_1} \pm \sqrt{y_2} \pm \sqrt{y_3})
\]

where \( \theta \) denotes the step function. The remaining ambiguity will be settled presently using condition (\( A_3 \)) and the positivity of the eigenvalue distribution. To this end, one must specify more explicitly the solutions \( w_i \) of (\( A_7 \)). In the case \( d < 0 \), which is the only one of interest here, (\( A_3 \)) is solved by

\[
\begin{align*}
w_1 &= 2\text{Re}(u) \\
w_2 &= 2\text{Re}(ue^{2\pi i/3}) \\
w_3 &= 2\text{Re}(ue^{-2\pi i/3})
\end{align*}
\]

\( ^5 \)Note that a direct transition between regions of type 1.) and type 3.) can only occur if \( b_0 = b_1 = 0 \) in (\( A_2 \)) for some value of \( \xi \), which is only possible at the exceptional point \( k = 0 \).
\[ u = \left(-\frac{q}{2} + \sqrt{q}\right)^{1/3} \] \hfill (A13)

Choosing the roots in (A13) in the conventional way (i.e. \( \sqrt{-1} = i \), any other choice merely permutes the \( w_i \)), one has \( w_1 \geq w_3 \geq w_2 \), and consequently \( y_1 \geq 0 \geq y_3 \geq y_2 \), i.e. \( \operatorname{Im} \sqrt{y_1} = 0 \), the convention which was already used above. Furthermore, upon entering a region of type 3.), \( \sqrt{y_2} \) and \( \sqrt{y_3} \) must be chosen to lie in different half-planes (upper or lower) in accordance with continuity and condition (A5), the right hand side of which is positive at the interface, \( 4M^3 - 3NM > 0 \). Only when within the region of type 3.), \( 4M^3 - 3NM \) becomes negative, the root which touches zero acquires an additional minus sign; this is \( \sqrt{y_3} \) according to the above conventions. Therefore, one can now specify \( \operatorname{Im} G_{H_0} \) up to an overall sign,

\[
\operatorname{Im} G_{H_0}(\xi) = \pm \frac{1}{2} \theta(-d)(1 - \theta(-b_2)\theta(b_1)) \left[ \sqrt{|y_2|} + (1 - 2\theta(4M^3 - 3NM))\sqrt{|y_3|} \right] \hfill (A14)
\]

and this last ambiguity is resolved using the positivity of the eigenvalue distribution, which thus reads

\[
\rho_{H_0}(\xi) = \frac{1}{\pi} \operatorname{Im} G_{H_0}(\xi) \hfill (A15)
\]

\[
= \frac{1}{2\pi} \theta(-d)(1 - \theta(-b_2)\theta(b_1)) \left[ \sqrt{|y_2|} + (1 - 2\theta(4M^3 - 3NM))\sqrt{|y_3|} \right] \hfill (A16)
\]

**APPENDIX B: EIGENVALUE DISTRIBUTION IN THE POLAR REPRESENTATION**

In the polar representation, expectation values

\[ \langle O \rangle = \frac{1}{2^{N}} \int \frac{d^N \eta}{\sqrt{2\pi}} \frac{J(\tilde{\eta})O(\tilde{\eta})}{\exp(-N\beta\tilde{\eta}^2)} \hfill (B1) \]

\[ = \int d^N \eta \frac{O(\tilde{\eta})}{\exp\left(-N\beta \sum_{i=1}^{N} \eta_i^2 + \sum_{i<j} \ln(\eta_i^2 - \eta_j^2)^2 + \sum_{i=1}^{N} \ln \eta_i \right)} \hfill (B2) \]

are dominated by the saddle points of the exponent for large \( N \). Note that the last term in the exponent in eq. (B2) is only of order \( O(N) \) as opposed to \( O(N^2) \) for the other two terms. Thus, going to a continuous eigenvalue distribution\( ^6 \) \( \rho_B(\eta) \) for \( N \to \infty \), the saddle point distribution is determined by

\[ 0 = \frac{\delta}{\delta \rho_B(\eta)} \left[ \frac{1}{2} \int_c^b dx dy \rho_B(x)\rho_B(y) \ln(x^2 - \eta^2)^2 - \beta \int_c^b dx x^2 \rho_B(x) \right] \hfill (B3) \]

\[ = \int_c^b dx \rho_B(x) \ln(x^2 - \eta^2)^2 - \beta \eta^2 \hfill (B4) \]

or, taking the derivative with respect to \( \eta \) and dividing by \( \eta \),

\[ 0 = 2 \int_c^b dx \frac{\rho_B(x)}{x^2 - \eta^2} + \beta \hfill (B5) \]

Here, the (as yet undetermined) edges of the support of \( \rho_B \) have been made explicit. Note that \( c \) and \( b \) are positive because the eigenvalues \( \eta_i \) are by definition positive. Substituting \( x^2 = s \) and \( \eta^2 = t \), one obtains

\[ - \beta = 2 \int_{c^2}^{b^2} ds \frac{\rho_B(s)}{s - t} \hfill (B6) \]

\(^6\)The subscript makes explicit that \( \rho_B \) is the eigenvalue distribution of the matrix \( B \) with eigenvalues \( \eta_i \). Below, also \( \rho_B(x) = \rho_B(\sqrt{x})/2\sqrt{x} \) will be considered.
After defining
\[ \tilde{\rho}_{B^2}(s) = \rho_{B^2}(s) \sqrt{\frac{s-c^2}{b^2-s}} \]  \hspace{1cm} (B7)

one can use the inversion formulae on a finite interval \[ g(x) = \frac{1}{\pi} \int_{c^2}^{b^2} dy \sqrt{\frac{b^2-y}{y-c^2}} f(y) \] to obtain
\[ \tilde{\rho}_{B^2}(s) = \frac{\beta}{2\pi} \rho_{B^2}(s) = \frac{\beta}{2\pi} \sqrt{\frac{b^2-s}{s-c^2}} \rho_B(\eta) = \frac{\beta \eta}{\pi} \sqrt{\frac{b^2-\eta^2}{\eta^2-c^2}} \] \hspace{1cm} (B9)

Now, the condition that \( \rho_{B^2} \) must be normalized to one can be used to determine \( b \),
\[ b^2 = \frac{4}{\beta} + c^2 \] \hspace{1cm} (B10)

On the other hand, the lower bound \( c \) is not determined by any auxiliary condition and should also be varied when determining the saddle point distribution. In other words, up to now, not the full space of possible \( \rho_B \) has been explored, but only the subspace with fixed, albeit arbitrary, lower bound \( c \). To determine \( c \), one inserts the form (B9) into the “action” in the square brackets in (B3) and now varies with respect to \( c \). The first term in the action can be made manifestly independent of \( c \) by again substituting as in eq. (B6) and a shift in the integration variables; thus one very easily obtains \( c = 0 \), as one would expect. Finally, inserting the radius of the original semicircular distributions of the gauge fields, \( a^2 = 2/\beta \), one has
\[ \rho_{B^2}(s) = \frac{1}{\pi a^2} \sqrt{\frac{2a^2-s}{s}} \] \hspace{1cm} (B11)
on the interval \([0, 2a^2]\) for the eigenvalue distribution of \( B^2 \).

**APPENDIX C: SOLUTION OF FOURTH ORDER EQUATION DETERMINING EIGENVALUE DISTRIBUTION OF PARTICLE WITH SPIN**

The solutions of the equation (B2)
\[ x^4 + c_2 x^2 + c_1 x + c_0 = 0 \] \hspace{1cm} (C1)

with
\[ c_2 = -\frac{8}{3} (2L^2 + 4KL + 6K^2 + 1) \] \hspace{1cm} (C2)
\[ c_1 = -\frac{64}{27} (2L^3 + 12KL^2 + 18K^2L + 3L - 9K) \] \hspace{1cm} (C3)
\[ c_0 = -\frac{16}{27} (32KL^3 + 48K^2L^2 + 8L^2 - 48KL + 9) \] \hspace{1cm} (C4)
are determined by the solutions of the corresponding cubic resolvent equation
\[ y^3 + b_2 y^2 + b_1 y + b_0 = 0 \] \hspace{1cm} (C5)
with
\[ b_2 = -\frac{16}{3} (1 + 2K^2 + L^2 + (2K + L)^2) < 0 \] \hspace{1cm} (C6)
\[ b_1 = \frac{256}{27} (3 + 4L^2 + (3K - L)^2 + 24L^2(K + L/3)^2 + (3K + L)^4/3) > 0 \] \hspace{1cm} (C7)
\[ b_0 = -\frac{4096}{729} (2L^3 + 12KL^2 + 18K^2L + 3L - 9K)^2 < 0 \] \hspace{1cm} (C8)
as in eq. (A4). Additionally, the signs of the square roots of the \( y_i \) in (A4) must be adjusted so as to guarantee

\[
\sqrt{y_1}\sqrt{y_2}\sqrt{y_3} = c_1 = \frac{64}{27}(2L^3 + 12KL^2 + 18K^2L + 3L - 9K) \tag{C9}
\]

The third order equation (C3) in turn can be simplified by the substitution \( y = w - b_2/3 \) to

\[
w^3 + pw + q = 0 \tag{C10}
\]

with

\[
p = \frac{256}{27}(2 - 3K^2 - 9K^4 - 14KL - 12K^3L + L^2 + 2K^2L^2 + 4KL^3 - L^4) \tag{C11}
\]

\[
q = \frac{4096}{729}(7 - 36K^2 + 27K^4 + 54K^6 + 48KL + 144K^3L + 108K^5L + 12L^2 + 66K^2L^2 + 18K^4L^2 - 48K^3L^3 - 56K^3L^3 + 3L^4 - 6K^2L^4 + 12KL^5 - 2L^6) \tag{C12}
\]

and the associated discriminant

\[
d = \frac{p^3}{27} + \frac{q^2}{4} = \frac{4194304}{19683}\left(3 - 24K^2 + 54K^4 - 81K^8 + 8L^2 + 260K^2L^2 + 384K^4L^2 + 180K^6L^2 + 6L^4 - 128K^2L^4 - 118K^4L^4 + 20K^2L^6 - L^8\right) \tag{C13}
\]

The analysis of these equations turns out to be considerably simpler than for the equations describing the case of the spinless particle. There is only one property which is not immediately obvious, namely that the discriminant (C14) has at most two zeros as a function of \( L \) for \( L \in [0, \infty) \) (remember that the whole problem is symmetrical about \( L = 0 \) and it is therefore sufficient to consider positive \( L \)). One proves this by regarding \( d \) as a polynomial in \( L^2 \) and constructing the Sturm chain \( \text{[19]} \) of \( d(L^2) \) at \( L^2 = 0 \) and for \( L^2 \to \infty \). The absolute difference between the number of sign changes occurring in the one or the other chain gives the number of zeros of \( d(L^2) \). One can check that this is less than or equal to two for any \( K \), where it turns out to be advantageous to distinguish between the cases \( K^2 \in [0, 1/60], K^2 \in [1/60, 1/3] \), and \( K^2 \in [1/3, \infty] \).

Now, just as in the case of the spinless particle (cf. Appendix A), the solutions of the cubic resolvent equation (C5) can in principle display three qualitatively different types of behaviour, partitioning the space of parameters \( K \) and \( L \) into regions corresponding to the different types. Here, however, there are no regions of type 3.), since one always has \( b_2 < 0 \) and \( b_1 > 0 \). Therefore, the support of the eigenvalue distribution of the Pauli Hamiltonian must come from a region of type 2.) in which the imaginary parts of the two complex conjugate solutions do not cancel. Moreover, due to the fact that \( d \) has at most two zeros as \( L \) is varied over all positive values, and \( d < 0 \) for \( L \to \infty \), there is only one unique region of type 2.) for \( L > 0 \) (and its mirror image for \( L < 0 \)). Denoting without loss of generality as \( y_1 \) and \( y_2 \) the two complex conjugate solutions and as \( y_3 \) the real positive solution, the form of the relevant solution \( x \equiv aQ_\lambda(0) - 4L/3 \) of (C1) on the region of type 2.) is therefore already almost uniquely determined:

\[
\text{Im} x = -\frac{1}{2}|\lambda(\sqrt{y_1} - \sqrt{y_2})| \quad \text{Re} x = \pm\frac{1}{2}\sqrt{y_3} \tag{C15}
\]

where in \( \text{Im} x \) it has additionally been used that due to the positivity of the eigenvalue distribution, \( \text{Im} x = a\text{Im} G_{Q_\lambda}(0) \) must be chosen such that it is negative (cf. eq. (80)). The ambiguity in \( \text{Re} x \) on the other hand is resolved with the help of condition (C3). When the right hand side of (C3) is positive, it is consistent to choose the square roots \( \sqrt{y_1} \) in the conventional way and (C13) can then be identified with one of the two solutions \( x_2 \) or \( x_3 \) in (A4), i.e. \( \sqrt{y_3} \) then comes with the positive sign in (C13). On the other hand, if a sign change occurs in condition (29) within the region of type 2.), then it can only be the real positive solution \( y_3 \) which touches zero, i.e. it is \( \sqrt{y_3} \) which acquires an additional minus sign. In other words, one has

\[
\text{Re} x = \frac{1}{2}(4\theta(9K - 2L^3 - 12KL^2 - 18K^2L - 3L) - 1)\sqrt{y_3} \tag{C16}
\]

It remains to give the solutions \( w_i \) of equation (C14) explicitly. In a region of type 2.), i.e. for \( d > 0 \), they are

\[
w_1 = e^{2\pi i/3}u + e^{-2\pi i/3}u' \quad w_2 = e^{-2\pi i/3}u + e^{2\pi i/3}u' \quad w_3 = u + u' \tag{C17}
\]
where
\[ u = \frac{p}{|p|} \left| -\frac{q}{2} + \sqrt{d} \right|^{1/3} \quad \text{and} \quad u' = -\frac{|p|}{3} \left| -\frac{q}{2} + \sqrt{d} \right|^{-1/3} \] (C18)

This fixes the solutions \( y_i = w_i - b_2/3 \) of the cubic resolvent equation (C5), in terms of which the solution for \( G_{Q_\lambda}(0) \) is now finally given as
\[ \text{Im} \, G_{Q_\lambda}(0) = -\frac{1}{2a} \theta(d)|i(\sqrt{y_1} - \sqrt{y_2})| \] (C19)
\[ \text{Re} \, G_{Q_\lambda}(0) = \frac{1}{2a} (2\theta(9K - 2L^3 - 12KL^2 - 18K^2L - 3L) - 1)\sqrt{y_3} + \frac{4L}{3a} \quad \text{for} \; d > 0 \] (C20)

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FIG. 6. $\rho \sqrt{\pi_0}$ (solid lines) and $\rho \sqrt{\pi_0} / 2$ (dashed lines) in three dimensions for (from left to right) $k/a = 0, \frac{1}{2}, 1, 2, 5$ (note the shift in the $\lambda$-axis for the last two plots).
$a \cdot \rho_{\sqrt{H_0}}[\lambda]$, $a \cdot \rho_{\sqrt{H_S}}[\lambda]/2$

M. Engelhardt, fig. 6
\[ a \cdot \rho_{\sqrt{H_0}}[\lambda] \]
\( \tilde{a} \cdot \rho_{\sqrt{H_0}}[\lambda] \)

M. Engelhardt, fig. 3
$a \cdot \rho_{\sqrt{H_0}}[\lambda]$
M. Engelhardt, fig. 5
M. Engelhardt, fig. 4