Coffee Break
The arbitrage-free equilibrium pricing of liabilities in an incomplete market: application to a South African retirement fund

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Agenda

1. Introduction
2. Liabilities specification & model
3. Pricing method
4. Results and sensitivity
5. Conclusions
Introduction: aim

Apply the pricing method of Thomson (2005) to:
• market-portfolio model (Thomson unpublished);
• equilibrium asset-category model (Thomson & Gott, 2009);
• a DB retirement-fund model;
with a view to operationalising the pricing of such a fund and quantifying the effects of:
• non-additivity due to incompleteness;
• guarantees implicit in reasonable expectations of pension increases; and
• the sensitivity of the price of illustrative liabilities to sources of risk and parameters of the model.
Introduction: method
Introduction: method
Introduction: method
Liabilities specification & model

- no exits before retirement
- mortality only after retirement
- projected unit method
- salaries and pensions expressed in real terms
Liabilities specification & model: salaries model

\[ S_{mt} = S_{m,t-1} \exp(\xi_t + \zeta_t) \]

\[ \xi_t = \mu_\xi + b_{\xi1} \eta_{3t} + b_{\xi2} \eta_{7t} + \sigma_{\xi} \varepsilon_\xi_t \]

\[ \zeta_t = \mu_{\zeta x} + \sigma_{\zeta x} \varepsilon_{\zeta t} \]

\[ \mu_{\zeta x} = \alpha_{\mu_\zeta} + \beta_{\mu_\zeta} \exp(-\lambda_{\mu_\zeta} x) \]

\[ \sigma^2_{\zeta x} = \frac{\sigma^2_{\xi x}}{M_{x,t-1}} \]

\[ \sigma_{\zeta x} = \alpha_{\sigma_\zeta} + \beta_{\sigma_\zeta} \exp(-\lambda_{\sigma_\zeta} x) \]

\( \mu_\xi = 0.01 \)

\( b_{\xi1} = -0.005 \)

\( b_{\xi2} = 0.005 \)

\( \sigma_\zeta = 0.03 \)

\( \alpha_{\mu_\zeta} = 0.016 \)

\( \beta_{\mu_\zeta} = 0.5 \)

\( \lambda_{\mu_\zeta} = 0.1 \)

\( \alpha_{\sigma_\zeta} = 0.042 \)

\( \beta_{\sigma_\zeta} = 0.5 \)

\( \lambda_{\sigma_\zeta} = 0.08 \)
Liabilities specification & model: pension increases

\[ P_t = P_{t-1} \exp\left\{ \max\left(0, -\gamma_t \right) \right\} \]
Liabilities specification & model: pensioner mortality

\[ \nu_{\{x\}}^{SAP98} = \frac{\nu_{\{x\}}^{PNL00}}{\nu_{\{x\}}^{IL00}} \nu_{\{x\}}^{SAIL98} \]

\[ \nu_{\{x\}}^{SAP} = \nu_{\{x\}}^{SAP98} \exp(10\mu_v) \]

\[ \nu_{\{x\}+t}^{SAP} = \nu_{\{x\}+t-1}^{SAP} \exp(\chi_{vt}) \]

where:

\[ \chi_{vt} = \chi_{v,t-1} + \mu_v + b_v \eta_{vt} + \sigma_v \varepsilon_{vt} \]

\[ \mu_v = -0.004 \]

\[ b_v = -0.001 \]

\[ \sigma_v = 0.005 \]
Pricing method: primary & secondary simulations

Time 0  Time 1  Time $t-1$  Time $t$  Time $T-1$  Time $T$

primary simulation  • primary simulation node  ----- secondary simulation
Pricing method: state-space vector

\[ x_t = \begin{pmatrix} P_{lt}(s_1) \\ \vdots \\ P_{lt}(s_u) \\ P_{ct}(s_1) \\ \vdots \\ P_{ct}(s_u) \\ \theta_t \\ P_{x_{1t}} \\ \vdots \\ P_{x_{Nt}} \end{pmatrix} \]

where:

\[ P_{lt}(s) = \exp \{ -Y_{lt}(s) \} \]

\[ P_{ct}(s) = \exp \{ -Y_{ct}(s) \} \]

\[ \theta_t = \exp (\chi_{vt}) \]
Pricing method: primary simulations of the state-space vector

\[ x_0 \rightarrow x_{11} \rightarrow x_{21} \rightarrow x_{T-1,1} \]
\[ x_0 \rightarrow x_{12} \rightarrow x_{22} \rightarrow x_{T-1,2} \]
\[ x_0 \rightarrow x_{1I} \rightarrow x_{2I} \rightarrow x_{T-1,I} \]
Pricing method: secondary simulations

Final year
Pricing method: secondary simulations

year $t$

\[ x_{t-1,1} \rightarrow x^*_{t11} \]
\[ x_{t-1,2} \rightarrow x^*_{t12} \]
\[ x_{t-1,J} \rightarrow x^*_{t1J} \]
\[ x_{t-1,J} \rightarrow x^*_{t2J} \]
\[ x_{t-1,J} \rightarrow x^*_{d1} \]
\[ x_{t-1,J} \rightarrow x^*_{d2} \]
\[ x_{t-1,J} \rightarrow x^*_{dJ} \]

\[ x_{t1} \rightarrow x^*_{t1} \]
\[ x_{t2} \rightarrow x^*_{t2} \]
\[ x_{tJ} \rightarrow x^*_{tJ} \]
\[ x_{d1} \rightarrow x^*_{d1} \]
\[ x_{d2} \rightarrow x^*_{d2} \]
\[ x_{dJ} \rightarrow x^*_{dJ} \]
Pricing method: secondary simulations

year 1
Pricing method

\[ \hat{\sigma}_{\varepsilon t}^2 = \hat{\sigma}_{Ft}^2 - \hat{\sigma}_{FVt}' \hat{\Sigma}_{Vt}^{-1} \hat{\sigma}_{FVt} \]

\[ z_t = \hat{\Sigma}_{Vt}^{-1} (\hat{\mu}_{Vt} - f_t^{1}) \quad m_t = \frac{1}{z_t'1} z_t \]

\[ \hat{\mu}_{Mt} = m_t' \hat{\mu}_{Vt} \quad \hat{\sigma}_{Mt}^2 = m_t' \hat{\Sigma}_{Vt} m_t \quad \hat{\sigma}_{HMt} = m_t' \hat{\sigma}_{FVt} \]

\[ \hat{\beta}_{Ft}^* = \frac{\hat{\sigma}_{HMt} + \hat{\sigma}_{\varepsilon t} \hat{\sigma}_{Mt}}{\hat{\sigma}_{Mt}^2} \]

\[ P_{L,t-1} = \frac{1}{f_t} \left\{ \hat{\mu}_{Ft} - \hat{\beta}_{Ft}^* (\hat{\mu}_{Mt} - f_t) \right\} \]
## Results and sensitivity

| Sex   | Age | Value per unit accrued pension | Aggregate value |
|-------|-----|--------------------------------|-----------------|
|       |     | deterministic valuation | stochastic price | deterministic valuation | stochastic price |
|       | 1 member | entire cohort | % incr | % incr | R\’million | % incr |
| Female | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 25    | 14,11 | 16,00 | 13,4 | 15,84 | -1,0 | 17 | 19 | 12,2 |
| 35    | 11,94 | 13,46 | 12,8 | 13,36 | -0,8 | 172 | 193 | 11,8 |
| 45    | 12,00 | 13,49 | 12,4 | 13,39 | -0,7 | 311 | 347 | 11,6 |
| 55    | 12,67 | 13,70 | 8,1 | 13,62 | -0,6 | 347 | 372 | 7,5 |
| 65    | 13,36 | 13,78 | 3,1 | 13,78 | 0,0 | 434 | 447 | 3,1 |
| 75    | 9,53  | 9,69  | 1,7 | 9,69  | 0,0 | 236 | 240 | 1,7 |
| 85    | 5,96  | 6,01  | 0,9 | 6,01  | 0,0 | 62  | 63  | 0,9 |
| total | 1 579 | 1 681 | 6,5 |  |  |  |  |  |
# Results and sensitivity

|                  | Deterministic valuation | Stochastic price  |
|------------------|-------------------------|-------------------|
|                  | R’million | % incr |
| Female           | 1 579     | 1 681  | 6,5  |
| Male             | 1 351     | 1 427  | 5,6  |
| Total            | 2 930     | 3 109  | 6,1  |
| Aggregate        | 2 930     | 3 094  | 5,6  |
| Adjusted to fund data | 2 912     | 3 074  | 5,6  |
### Results and sensitivity

| Description                                                                 | Value  |
|-----------------------------------------------------------------------------|--------|
| Deterministic valuation of model-point data                                | 2,930  |
| Difference due to risk-free stochastic pricing                             | 11     |
| Risk-free stochastic price                                                  | 2,941  |
| Hedge-portfolio risks                                                       | -19    |
| Stochastic price with hedge-portfolio risks                                 | 2,922  |
| Residual risks                                                              | -20    |
| Stochastic price: hedge-portfolio & residual risks                          | 2,902  |
| Cost of guarantee                                                           | 192    |
| Stochastic price based on model-point data                                  | 3,094  |
| Adjustment to fund data                                                     | -20    |
| Stochastic price based on fund data                                         | 3,074  |
## Results and sensitivity: major effects

| Parameter | Test result |
|-----------|-------------|
| name      | description | standard value | test value | price (R’million) | change in price (%) |
|           |             |                |            |                  |                    |
| standard values | | 3 094 | N/A | |
| $b_{\xi_{1}}$ | general salary increase: sensitivity to inflation | $-0.005$ | $0$ | $3 098$ | $0.14$ |
| $g$ | return on market portfolio: sensitivity to risk-free rate | $1.39$ | $1.2$ | $3 082$ | $-0.37$ |
| $\sigma_M$ | return on market portfolio: residual volatility | $0.159$ | $0.1$ | $3 108$ | $0.45$ |
| $b_\gamma$ | force of inflation: residual volatility | $-0.01379$ | $0$ | $3 061$ | $-1.07$ |
Conclusions

- Method computationally demanding, but not impossible: 41 hours.
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- Convergence complicated, but Sobol numbers expedite it.
- Stochastic price 5.6% higher than deterministic.
- Without guarantee, stochastic price only 1% less than deterministic: *If the valuation of the liabilities should allow for a risk premium only to the extent that the trustees are unable to avoid risk, then the valuation basis must be much closer to a risk-free basis than that produced by the risk premiums typically used.*
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- Effects of non-additivity: intra-cohort 0–1% plus inter-cohort 0.5%.
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• Convergence complicated, but Sobol numbers expedite it.
• Stochastic price 5.6% higher than deterministic: because of pension guarantee.
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• Effects of non-additivity: intra-cohort 0–1% plus inter-cohort 0.5%.
• Major sensitivities:
  volatility of force of inflation in excess of conditional ex-ante expected inflation;
  sensitivity of ex-ante expected returns on the market portfolio to positive risk-free returns;
  residual volatility of the return on the market portfolio.
Conclusions

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• Effects of non-additivity: intra-cohort 0–1% plus inter-cohort 0.5%.
• Major sensitivities.
• Overall effect: Excluding uncertainties common to deterministic and stochastic valuations, an error of about 5.6% is reduced to uncertainty of about 1%.
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