Stochastic Verification Theorem for Infinite Dimensional Stochastic Control Systems

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Abstract

The verification theorem serving as an optimality condition for the optimal control problem, has been expected and studied for a long time. The purpose of this paper is to establish this theorem for control systems governed by stochastic evolution equations in infinite dimensions, in which both the drift and the diffusion terms depend on the controls.

2010 Mathematics Subject Classification. 93E20.

Key Words: Optimal control, value function, stochastic distributed parameter systems, stochastic verification theorem

1 Introduction

We begin with some notations. Let $T > 0$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, on which a separable Hilbert space $\tilde{H}$-valued cylindrical Brownian motion $W(\cdot)$ is defined. Denote by $\mathcal{F}^n_{t\wedge T}$ the natural filtration generated by $W(\cdot)$ and by $\mathcal{F}$ the progressive $\sigma$-algebra with respect to $\mathcal{F}$.

Let $\mathcal{X}$ be a Banach space. For any $t \in [0, T]$ and $p \in [1, \infty)$, denote by $L^p_{\mathcal{F}_t}(\Omega; \mathcal{X})$ the Banach space of all $\mathcal{F}_t$-measurable random variables $\xi : \Omega \to \mathcal{X}$ such that $\mathbb{E}|\xi|^p_\mathcal{X} < \infty$, with the canonical norm. Denote by $L^p_{\mathcal{F}}(\Omega; C([t, T]; \mathcal{X}))$ the Banach space of all $\mathcal{X}$-valued $\mathcal{F}$-adapted continuous processes $\phi(\cdot)$, with the norm

$$|\phi(\cdot)|_{L^p_{\mathcal{F}}(\Omega; C([t, T]; \mathcal{X}))} \triangleq \left[ \mathbb{E} \sup_{\tau \in [t, T]} |\phi(\tau)|^p_\mathcal{X} \right]^{1/p}.$$

Also, denote by $C_p([t, T]; L^p(\Omega; \mathcal{X}))$ the Banach space of all $\mathcal{X}$-valued $\mathcal{F}$-adapted processes $\phi(\cdot)$ such that $\phi(\cdot) : [t, T] \to L^p_{\mathcal{F}_T}(\Omega; \mathcal{X})$ is continuous, with the norm

$$|\phi(\cdot)|_{C_p([t, T]; L^p(\Omega; \mathcal{X}))} \triangleq \sup_{\tau \in [t, T]} \left[ \mathbb{E}|\phi(\tau)|^p_\mathcal{X} \right]^{1/p}.$$

Fix any $p_1, p_2 \in [1, \infty]$. Put

$$L^{p_2}_{\mathcal{F}}(t, T; L^{p_1}(\Omega; \mathcal{X})) = \left\{ \varphi : (t, T) \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathcal{F}\text{-adapted and } \int_t^T \left( \mathbb{E}|\varphi(\tau)|^p_\mathcal{X} \right)^{p_2} d\tau < \infty \right\}.$$

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Clearly, $L^p_{S,F}(t; L^p(\Omega; \mathcal{X}))$ is a Banach space with the canonical norm. If $p_1 = p_2$, we simply write the above spaces as $L^p_{S,F}(t; \mathcal{X})$. Put

$$L^p_{S,F}(t; L^p(\Omega; \mathcal{X})) = \{ \varphi : (t, T) \times \Omega \to \mathcal{X} \mid |\varphi(\cdot)|_{\mathcal{X}} \in L^p_{S,F}(t, T; L^p(\Omega; \mathbb{R})) \}.$$  

Similarly, if $p_1 = p_2$, we simply write the above spaces as $L^p_{S,F}(t, T; \mathcal{X})$.

For $r \in [0, T]$ and $f \in L^p_{S,F}(\Omega; \mathcal{X})$, denote by $\mathbb{E}(f | \mathcal{F}_t)$ the conditional expectation of $f$ with respect to $\mathcal{F}_t$ and by $\mathbb{E} f$ the mathematical expectation of $f$.

Let $\mathcal{Y}$ be another Banach space. Denote by $\mathcal{L}(\mathcal{X}; \mathcal{Y})$, or $\mathcal{L}(\mathcal{X})$ if $\mathcal{Y} = \mathcal{X}$, the Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ with the usual operator norm. When $\mathcal{X}$ is a Hilbert space, write $\mathcal{S}(\mathcal{X})$ for the space of all bounded linear self-adjoint operators on $\mathcal{X}$.

For $v \in C([0, T] \times \mathcal{X})$ and $(t, \eta) \in [0, T] \times \mathcal{X}$, the second-order parabolic superdifferential of $v$ at $(t, \eta)$ is defined as follows:

$$D^{1, 2+}_{t,x} v(t, \eta) \Delta \left\{ (r, p, P) \in \mathbb{R} \times H \times \mathcal{S}(\mathcal{X}) \mid \lim_{y \to \eta \in [0, T]} \frac{1}{s-t} \left[ v(s, y) - v(t, \eta) - r(s-t) - \langle p, y-\eta \rangle_{\mathcal{X}} - \frac{1}{2} \langle P(y-\eta), y-\eta \rangle_{\mathcal{X}} \right] \leq 0 \right\}.$$  

Note that the limit in $t$ is from the right. This fits the general irreversibility of evolution equations.

Now we can introduce the control problem. Let $H$ be a separable Hilbert space, and $A : D(A) \subset H \to H$ be a linear operator, which generates a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $H$. Write $L^2_0$ for the space of all Hilbert-Schmidt operators from $H$ to $H$, which is also a separable Hilbert space. Let $U$ be a separable metric space with a metric $d(\cdot, \cdot)$. For $t \in [0, T)$, put

$$U[t, T] \Delta \{ u : [t, T] \times \Omega \to U \mid u \text{ is } \mathbb{P}-\text{adapted} \}.$$  

The control system under consideration in this paper is given as follows:

$$\begin{aligned}
\begin{cases}
    dX(t) = (AX(t) + a(t, X(t), u(t)))dt + b(t, X(t), u(t))dW(t), & t \in (0, T], \\
    X(0) = \eta \in H,
\end{cases}
\end{aligned}$$

and the cost functional is

$$J(\eta; u(\cdot)) = \mathbb{E} \left( \int_0^T f(t, X(t), u(t))dt + h(X(T)) \right).$$

We make the following assumptions for the control system (1.1) and the cost functional (1.2):

(S1) Suppose that: i) $a(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H$ is $\mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(U)/\mathcal{B}(H)$-measurable and $b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to L^2_0$ is $\mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(U)/\mathcal{B}(L^2_0)$-measurable; ii) for any $\eta \in H$, the maps $a(\cdot, \eta, \cdot) : [0, T] \times U \to H$ and $b(\cdot, \eta, \cdot) : U \to L^2_0$ are continuous; and iii) for any $(t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U$,

$$\begin{aligned}
    |a(t, \eta_1, u) - a(t, \eta_2, u)|_H \leq C|\eta_1 - \eta_2|_H, \\
    |b(t, \eta_1, u) - b(t, \eta_2, u)|_{L^2_0} \leq C|\eta_1 - \eta_2|_H, \\
    |a(t, 0, u)|_H \leq C, \\
    |b(t, 0, u)|_{L^2_0} \leq C.
\end{aligned}$$
(S2) Suppose that: i) \( f(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathbb{R} \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(U) / \mathcal{B}(\mathbb{R}) \)-measurable and \( h(\cdot) : H \to \mathbb{R} \) is \( \mathcal{B}(H) / \mathcal{B}(\mathbb{R}) \)-measurable; ii) For any \( \eta \in H \), the functional \( f(\cdot, \eta, \cdot) : [0, T] \times U \to \mathbb{R} \) is continuous; and iii) For any \( (t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U \),

\[
\begin{align*}
|f(t, \eta_1, u) - f(t, \eta_2, u)| & \leq C|\eta_1 - \eta_2|_H, \\
|h(\eta_1) - h(\eta_2)| & \leq C|\eta_1 - \eta_2|_H \\
|f(t, 0, u)| & \leq C, \quad |h(0)| \leq C.
\end{align*}
\]

Hereafter, we use \( C \) to denote the generic constant, which may change from line to line.

Remark 1.1 The boundedness condition on \( f \) and \( h \) in (S2) is just for the convenience of computation and to emphasize the main arguments, which can be relaxed in the following manner:

\[
|f(t, \eta_1, u) - f(t, \eta_2, u)| + |h(\eta_1) - h(\eta_2)| \leq C(1 + |\eta_1|_H + |\eta_2|_H)|\eta_1 - \eta_2|_H, \\
\forall (t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U.
\]

Under (S1), for any \( u(\cdot) \in \mathcal{U}[0, T] \), the control system (1.1) has a unique mild solution \( X(\cdot) \in C_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \) (see [17] Theorem 3.14 for example).

Consider the following optimal control problem:

**Problem (S_\eta)**. For any given \( \eta \in H \), find a \( \bar{u}(\cdot) \in \mathcal{U}[0, T] \) such that

\[
\mathcal{J}(u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} \mathcal{J}(u(\cdot)).
\]

Any \( \bar{u}(\cdot) \in \mathcal{U}[0, T] \) satisfying (1.3) is called an optimal control (of **Problem (S_\eta)**). The corresponding state \( \bar{X}(\cdot) \) is called an optimal state, and \((\bar{X}(\cdot), \bar{u}(\cdot))\) is called an optimal pair.

Let us recall the stochastic dynamic programming principle for solving **Problem (S_\eta)**. In the literature, the stochastic dynamic programming principle for **Problem (S_\eta)** in weak formulation is already established. A nice treatise for that is [8]. In that formulation, probability spaces and Brownian motions vary with the controls. In other words, the probability space and Brownian motion are part of the control. Usually, optimal control problems for SEEs are formulated in strong formulation, i.e., the probability space and the Brownian motion are fixed. Hence, it is natural to ask whether the stochastic dynamic programming principle holds in strong formulation. This question is answered in [4].

First, we introduce a family of optimal control problems. For any \( (t, \eta) \in [0, T] \times H \), the control system is

\[
\begin{align*}
\text{d}X(s) &= (AX(s) + a(t, X(s), u(s))) \text{d}t + b(s, X(s), u(s)) \text{d}W(s), \quad s \in (t, T], \\
X(t) &= \eta,
\end{align*}
\]

and the cost functional is

\[
\mathcal{J}(t, \eta; u(\cdot)) = \mathbb{E}\left( \int_t^T f(s, X(s), u(s)) \text{d}s + h(X(T)) \right).
\]

For any \( u(\cdot) \in \mathcal{U}[t, T] \), it follows immediately from the classical well-posedness of SEEs (e.g., [17] Theorem 3.14) that the control system (1.4) has a unique mild solution \( X(\cdot) \in C_{\mathcal{F}}([t, T]; L^2(\Omega; H)) \). Hence, the cost functional (1.5) is well-defined.

Consider the following optimal control problem:
Problem \((S_{t\eta})\). For any given \((t, \eta) \in [0, T] \times H\), find a \(\bar{u}(\cdot) \in \mathcal{U}[t, T]\) such that

\[
\mathcal{J}(t, \eta; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{1.6}
\]

Any \(\bar{u}(\cdot) \in \mathcal{U}[t, T]\) satisfying (1.6) is called an optimal control (of Problem \((S_{t\eta})\)). The corresponding state \(\bar{X}(\cdot)\) is called an optimal state, and \((\bar{X}(\cdot), \bar{u}(\cdot))\) is called an optimal pair.

The value function is defined as follows:

\[
\begin{aligned}
V(t, \eta) &= \inf_{u(\cdot) \in \mathcal{U}[t, T]} \mathcal{J}(t, \eta; u(\cdot)), \quad \forall (t, \eta) \in [0, T) \times H, \\
V(T, \eta) &= h(\eta), \quad \forall \eta \in H.
\end{aligned}
\]

It is easy to prove that the value function enjoys the following properties:

**Proposition 1.1** \([4, \text{Proposition 3.1}]\) For each \(t \in [0, T]\), \(\eta\) and \(\eta' \in H\), we have

\[
|V(t, \eta)| \leq C(1 + |\eta|_H) \tag{1.7}
\]

and

\[
|V(t, \eta) - V(t, \eta')| \leq C|\eta - \eta'|_H. \tag{1.8}
\]

**Proposition 1.2** \([4, \text{Proposition 3.2}]\) The function \(V(\cdot, \eta)\) is continuous.

We have the following Dynamic Programming Principle \((4, \text{Theorem 3.1})\):

For any \((t, \eta) \in [0, T) \times H\),

\[
V(t, \eta) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} \mathbb{E}\left(\int_t^T f(s, \bar{X}(s; t, \eta, u(s), \eta), u(s)) ds + V(\bar{X}(s; t, \eta, u(s)))\right), \quad \forall 0 \leq t \leq T. \tag{1.9}
\]

Here \(X(\cdot; t, \eta, u)\) is the mild solution of \((1.4)\). By \((1.3)\), one can derive the HJB equation satisfied by \(V(\cdot, \cdot)\). We do not present that here since we do not use it in this paper.

In this paper, we will investigate the sufficient optimality condition — verification theorem of the Problem \((S_{\eta})\)—via the value function. The verification theorem provides a way of testing whether a given admissible control is optimal and enables one to construct an optimal control via the value function. This theorem was first studied in the 1960s by Pontryagin and his group for the LQ problem of control systems governed by ordinary differential equations (e.g., [20]). The general cases for controlled ordinary differential equations were studied in the sequel by Fleming and Rishel with smooth value function in [12], and by Zhou in [23] under viscosity-solution framework. The succeedent work for control systems governed by stochastic differential equations were studied in [21] [15] [16]. When the value function belongs to \(C^{1,2}([0, T] \times H)\), the verification theorem for Problem \((S_{\eta})\) follows similar standard results for the finite-dimensional case (e.g., [8, Section 2.5]). However, it is well known that the value function does not belong to \(C^{1,2}([0, T] \times H)\) in general. This leads to the study of the verification theorem for Problem \((S_{\eta})\) for nonsmooth value function. Along this line, there are many works when the diffusion term of the control system is independent of the control variable (see [5], [8], [9], [10], [13], [14] and the rich references therein). As far as we know, there is no published work addressing the verification theorem for Problem \((S_{\eta})\) with nonsmooth value function and control dependent diffusion term. This does not mean that such problem is not important. Indeed, the control dependent diffusion term reflects that the control would influence the scale of uncertainty, which is indeed the case in many practical systems. In this paper, we investigate such problem and prove the following result.
Theorem 1.1 Let Assumptions (S1)–(S2) hold. Let \( V \in C([0, T] \times H) \) be the value function of Problem \((S_\eta)\). Let \( \eta \in H \) be fixed, and \((\overline{X}(\cdot), \bar{u}(\cdot))\) be an admissible pair of Problem \((S_\eta)\). Suppose

\[
A\overline{X}(\cdot) \in L^2_{t}(0, T; H),
\]

and for any \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that

\[
|V(t_1, \eta) - V(t_0, \eta)| \leq C_\delta(1 + |\eta|_H^2)|t_1 - t_0|, \quad \forall t_1, t_0 \in [0, T - \delta), \ \eta \in H. \]  

(1.11)

If there exists a triple \((\overline{R}, \bar{p}, \overline{\mathcal{P}}) \in L^2_{t}(0, T; \mathbb{R}) \times L^2_{t}(0, T; H) \times L^2_{t}(0, T; \mathcal{S}(H))\) such that

\[
(\overline{R}, \bar{p}, \overline{\mathcal{P}}) \in D_{t+}^{1,2,+} V(t, \overline{X}(t)), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega (1.12)
\]

and

\[
\mathbb{E} \int_0^T (\overline{R}(t) + \langle \bar{p}(t), A\overline{X}(t) \rangle_H + G(t, \overline{X}(t), \bar{u}(t), \bar{p}(t), \overline{\mathcal{P}}(t))) \, dt \leq 0, \quad (1.13)
\]

where

\[
G(t, \eta, \rho, p, P) = \frac{1}{2} \langle Pb(t, \eta, \rho), b(t, \eta, \rho) \rangle_{L^2_\omega} + \langle p, a(t, \eta, \rho) \rangle_H - f(t, \eta, \rho), \quad (1.14)
\]

\( \forall (t, \eta, \rho, p, P) \in [0, T] \times H \times U \times H \times \mathcal{S}(H) \),

then \((\overline{X}(\cdot), \bar{u}(\cdot))\) is an optimal pair of Problem \((S_\eta)\).

Remark 1.2 Theorem 1.1 is expressed in terms of superdifferential. It is natural to expect that a similar result holds for subdifferential. Unfortunately, the answer is no even for \( H = \mathbb{R} \) (see Example 5.6) for example).

The sufficient condition (1.13) can be replaced by an equivalent condition which looks much stronger.

Proposition 1.3 Condition (1.13) in Theorem 1.1 is equivalent to the following:

\[
\overline{R}(t) + \langle \bar{p}(t), A\overline{X}(t) \rangle_H + G(t, \overline{X}(t), \bar{u}(t), \bar{p}(t), \overline{\mathcal{P}}(t)) = \overline{R}(t) + \langle \bar{p}(t), A\overline{X}(t) \rangle_H + \inf_{u \in U} G(t, \overline{X}(t), u, \bar{p}(t), \overline{\mathcal{P}}(t)), \quad \text{a.e. } t \in [0, T], \ \mathbb{P}\text{-a.s.} (1.15)
\]

Two assumptions, i.e., (1.10) and (1.11), are given in Theorem 1.1. The first one is set for the regularity for the solution of the control system. This depends on the SPDE which governs the control system. The second one, i.e., (1.11), is for the regularity property of the value function. This is very subtle since the value function is not easy to be handled. In fact, in stochastic case, generally the value function associated to a control system is not Lipschitz in \( t \) even when all the coefficients involved are smooth (e.g., \( \| \cdot \| \)). Fortunately, our next result concludes that (1.11) holds under suitable conditions.

(S1)' Suppose that: i) \( a(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(U) / \mathcal{B}(H) \)-measurable and \( b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to L^2_\omega \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(L^2_\omega) \)-measurable; ii) for any \( t \in [0, T], \ \eta \in H, \) the maps \( a(t, \eta, \cdot) : U \to H \) and \( b(t, \eta, \cdot) : U \to L^2_\omega \) are continuous; and iii) for any \( (t, t_1, t_2, \eta_1, \eta_2, u) \in [0, T] \times [0, T] \times \mathbb{R} \times \mathcal{S}(H), \)

\[
\begin{cases}
|a(t_1, \eta_1, u) - a(t_2, \eta_2, u)|_H \leq C(|t_1 - t_2| + |\eta_1 - \eta_2|_H), \\
|b(t_1, \eta_1) - b(t_2, \eta_2, u)|_{L^2_\omega} \leq C(|t_1 - t_2| + |\eta_1 - \eta_2|_H), \\
|a(t, 0, u)|_H \leq C, \quad |b(t, 0, u)|_{L^2_\omega} \leq C.
\end{cases}
\]
\((S2)\) Suppose that: i) \(f(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathbb{R}\) is \(\mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(U) / \mathcal{B}(\mathbb{R})\)-measurable and \(h(\cdot) : H \to \mathbb{R}\) is \(\mathcal{B}(H) / \mathcal{B}(\mathbb{R})\)-measurable; ii) For any \(t \in [0, T], \eta \in H\), the functional \(f(t, \eta, \cdot) : U \to \mathbb{R}\) is continuous; and iii) For any \((t_1, t_2, \eta_1, \eta_2, u) \in [0, T] \times [0, T] \times [0, T] \times H \times H \times U\),

\[
\begin{align*}
|f(t_1, \eta_1, u) - f(t_2, \eta_2, u)| &\leq C(|t_1 - t_2| + |\eta_1 - \eta_2|), \\
h(\eta_1) - h(\eta_2) &\leq C|\eta_1 - \eta_2|H, \\
|f(t, 0, u)| &\leq C, \quad |h(0)| \leq C.
\end{align*}
\]

**Theorem 1.2** Under Assumptions \((S1)'-(S2)'\), the value function \(V\) is Lipschitz continuous in \([0, T - \delta] \times H\) for all \(\delta > 0\), provided that \(A\) generates an analytic semigroup on \(H\).

**Remark 1.3** Generally speaking, the value function is not Lipschitz continuous in \([0, T] \times H\) even for \(H = \mathbb{R}\) (e.g., \([1]\)).

Theorem 1.2 illustrates that the strong restrictions on \(V\) is valid for several important control systems, in particular, those governed by stochastic parabolic equations. What’s more, with the additional assumption that \(A\) is analytic, we can drop the assumption (1.11). Inequality (2.21) can be deduced obviously from Theorem (1.2) and (3.29). Therefore, we have the following result.

**Corollary 1.1** Let Assumptions \((S1)'-(S2)'\) hold. Let \(A\) generate an analytic semigroup, and \(V \in C([0, T] \times H)\) be the value function of Problem (S\(\eta\)). Let \((\overline{X}(\cdot), \bar{u}(\cdot))\) be an admissible pair of Problem (S\(\eta\)). Suppose that

\[A\overline{X}(\cdot) \in L^2_{2}(0, T; H).\]  
(1.16)

If there exists a triple \((\overline{R}, \bar{p}, \overline{P}) \in L^2_{2}(0, T; \mathbb{R}) \times L^2_{2}(0, T; H) \times L^2_{S,2}(0, T; \mathcal{S}(H))\) such that

\[(\overline{R}, \bar{p}, \overline{P}) \in D^{1,2,2}\big, V(t, \overline{X}(t))\big, \quad a.e. \ t \in [0, T], \ \mathbb{P}\text{-a.s.}\]  
(1.17)

and

\[\mathbb{E} \int_{s}^{T} \big(\overline{R}(t) + \langle \bar{p}(t), A\overline{X}(t) \rangle_H + G(t, \overline{X}(t), \bar{u}(t) + \bar{p}(t), \bar{P}(t) \big) \big) dt \leq 0,\]  
(1.18)

then \((\overline{X}(\cdot), \bar{u}(\cdot))\) is an optimal pair of Problem (S\(\eta\)).

The rest of this paper is divided into three sections. Section 2 is devoted to the proof of Theorem 1.1 and Section 3 is addressed to the proof of Theorem 1.2. At last, in Section 4, we provide an illustrative example fitting for the assumptions in Theorem 1.1.

## 2 Stochastic Verification Theorem

In this section, we are going to prove the well-known verification theorem for the infinite dimensional stochastic control system. The main idea comes from [15] [22].

Let us first recall the concept of regular conditional probability, which allows us to regard the conditional expectation as merely mathematical expectation taken with respect to the conditional measure. More details can be found in [18] Chapter V, Section 8.
Lemma 2.1 Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then there exists a map $\mathbf{P} : \Omega \times \mathcal{F} \to [0, 1]$, called a regular conditional probability given $\mathcal{G}$, such that

(i) for each $\omega \in \Omega$, $\mathbf{P}(\omega, \cdot)$ is a probability measure on $\mathcal{F}$;
(ii) for each $A \in \mathcal{F}$, the function $\mathbf{P}(\cdot, A)$ is $\mathcal{G}$-measurable;
(iii) for each $B \in \mathcal{F}$, $\mathbf{P}(\omega, B) = \mathbb{P}(B|\mathcal{G})(\omega) = \mathbb{E}(1_B|\mathcal{G})(\omega)$, $\mathbb{P}$-a.s.

We write $\mathbb{P}(\cdot|\mathcal{G})(\omega)$ for $p(\omega, \cdot)$.

The next proposition is taken from [22] with a slight modification.

Proposition 2.1 Let $v \in C([0, T] \times H)$ and $(t_0, x_0) \in [0, T] \times H$ be given. Then $(q, p, P) \in D_{t+}^{1,2,+} v(t_0, x_0)$ if and only if there exists a function $\varphi \in C^{1,2}([0, T] \times H)$ such that

$$
\left\{ \begin{array}{ll}
(\varphi(t_0, x_0), \varphi_t(t_0, x_0), \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) = (v(t_0, x_0), q, p, P), \\
\varphi(t, x) > v(t, x), & (t, x) \in [t_0, T] \times H.
\end{array} \right.
$$

(2.1)

Proof. The “if” part follows directly from the definition of $D_{t+}^{1,2,+} v(t_0, x_0)$.

The “only if” part. Suppose $(q, p, P) \in D_{t+}^{1,2,+} v(t_0, x_0)$. Define a functional on $[t_0, T] \times H$ as follows: if $(t_0, x_0) \neq (t, x) \in [t_0, T] \times H$, then

$$
\Phi(t, x) = \frac{v(t, x) - v(t_0, x_0) - q(t - t_0) - \langle p, x - x_0 \rangle_H}{t - t_0 + |x - x_0|_H^2} + \frac{1}{2} \langle P(x - x_0), x - x_0 \rangle_H.
$$

and if $(t_0, x_0) = (t, x)$, then $\Phi(t, x) = 0$.

Let

$$
\kappa(r) = \begin{cases} 
\sup \left\{ \Phi(t, x) \left| (t, x) \in (t_0, T] \times H, \ |x - x_0|_H^2 \leq r \right. \right\}, & \text{if } r > 0, \\
0, & \text{if } r \leq 0.
\end{cases}
$$

Then $\kappa : \mathbb{R} \to [0, +\infty)$ is a continuous and nondecreasing function with $\kappa(0) = 0$. Further, we have

$$
v(t, x) - \frac{v(t_0, x_0) + q(t - t_0) + \langle p, x - x_0 \rangle_H + \frac{1}{2} \langle P(x - x_0), x - x_0 \rangle_H}{t - t_0 + |x - x_0|_H^2} \leq \kappa(t - t_0 + |x - x_0|_H^2),
$$

\forall (t, x) \in [t_0, T] \times H.

Let

$$
\Psi(\rho) = \frac{2}{\rho} \int_0^{2\rho} \int_0^r \kappa(\theta)d\theta dr, \quad \rho > 0.
$$

Then we have

$$
\Psi_\rho(\rho) = -\frac{2}{\rho^2} \int_0^{2\rho} \int_0^r \kappa(\theta)d\theta dr + \frac{4}{\rho} \int_0^{2\rho} \kappa(\theta)d\theta
$$

and

$$
\Psi_{\rho\rho}(\rho) = \frac{4}{\rho^3} \int_0^{2\rho} \int_0^r \kappa(\theta)d\theta dr - \frac{8}{\rho^2} \int_0^{2\rho} \kappa(\theta)d\theta + \frac{8}{\rho} \kappa(2\rho).
$$

Consequently,

$$
|\Psi(\rho)| \leq 4\rho \kappa(2\rho), \quad |\Psi_\rho(\rho)| \leq 12\kappa(2\rho), \quad |\Psi_{\rho\rho}(\rho)| \leq \frac{32\kappa(2\rho)}{\rho}.
$$
Thus, noting
\[
\psi(t, x) = \begin{cases} 
\Psi(\rho(t, x)) + \rho(t, x)^2, & \text{if } (t_0, x_0) \neq (t, x) \in [t_0, T] \times H, \\
0, & \text{if } (t, x) = (t_0, x_0),
\end{cases}
\tag{2.2}
\]
where \(\rho(t, x) = t - t_0 + |x - x_0|^2_H\). Set
\[
\varphi(t, x) = v(t_0, x_0) + q(t - t_0) + \langle p, x - x_0 \rangle_H
+ \frac{1}{2} P(x - x_0, x - x_0)_H + \psi(t, x), \quad \forall (t, x) \in [0, T] \times H.
\tag{2.3}
\]
We claim that \(\varphi \in C^{1,2}([0, T] \times H)\) satisfies \[2.1\]. First, for any \((t, x) \in [t_0, T] \times H\) with \((t, x) \neq (t_0, x_0)\), we have
\[
\psi(t, x) > \frac{2}{\rho(t, x)} \int_{\rho(t, x)}^{2\rho(t, x)} \kappa(\theta) d\theta dr
\geq \frac{2}{\rho(t, x)} \kappa(\rho(t, x)) \int_{\rho(t, x)}^{2\rho(t, x)} (r - \rho(t, x)) dr
= \rho(t, x) \kappa(\rho(t, x)).
\]
Next, for any \((t, x) \in [t_0, T] \times H\), it follows from \[2.2\] that
\[
\psi_t(t, x) = \Psi(\rho(t, x)) + 2\rho(t, x),
\]
and
\[
\psi_x(t, x) = 2\Psi(\rho(t, x))(x - x_0) + 4\rho(t, x)(x - x_0),
\]
and
\[
\psi_{xx}(t, x) = 4\Psi\rho(\rho(t, x))(x - x_0) + 2\Psi(\rho(t, x))I + 4\rho(t, x)I + 8(x - x_0) \otimes (x - x_0).
\]
Thus, noting \(|x - x_0| \leq \rho(t, x)\), we obtain
\[
\begin{align*}
|\psi(t, x)| & \leq 4\rho(t, x) \kappa(2\rho(t, x)) + \rho(t, x)^2, \\
|\psi_t(t, x)| & \leq 12\rho(t, x) + 2\rho(t, x), \\
|\psi_x(t, x)| & \leq 24|x - x_0| \kappa(2\rho(t, x)) + 4\rho(t, x) |x - x_0|, \\
|\psi_{xx}(t, x)| & \leq \frac{128|x - x_0|^2}{\rho(t, x)} \kappa(2\rho(t, x)) + 24\kappa(2\rho(t, x)) + 12\rho(t, x) \leq 152\kappa(2\rho(t, x)) + 12\rho(t, x).
\end{align*}
\]
Hence, \(\psi \in C^{1,2}([0, T] \times H)\) and
\[
\psi(t_0, x_0) = 0, \quad \psi_t(t_0, x_0) = 0, \quad \psi_x(t_0, x_0) = 0, \quad \psi_{xx}(t_0, x_0) = 0.
\]
This proves our claim. \(\square\)

To continue, we need two known results. The first one is taken from from \[22\], but with a slight modification following the discussion in \[10\] and \[9\] Remark 3.4, Section 3].

**Lemma 2.2** Let \(g \in C[0, T]\). Extend \(g\) to \((-\infty, +\infty)\) with \(g(t) = g(T)\), for \(t > T\), and \(g(t) = g(0)\) for \(t < 0\). Suppose that for each \(\delta \in (0, T)\), there is a \(\rho_0 \in L^1(0, T - \delta)\) such that for some \(\varepsilon_0 > 0\),
\[
\frac{g(t + \varepsilon) - g(t)}{\varepsilon} \leq \rho(t), \quad \forall \varepsilon \leq \varepsilon_0, \text{ a.e. } t \in [0, T - \delta).
\]
Then
\[
g(\beta) - g(\alpha) \leq \int_{\alpha}^{\beta} \lim_{\varepsilon \to 0^+} \frac{g(r + \varepsilon) - g(r)}{\varepsilon} dr, \quad \forall 0 \leq \alpha \leq \beta \leq T - \delta.
\]
Lemma 2.3 \([6\), Theorem 9, Chapter 2\] Let \(Z\) be a Banach space, \([a, b] \subset \mathbb{R}\) and \(z : [a, b] \to Z\) be a Bochner integrable function. Then

\[
\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |z(r) - z(t)|dr \to 0, \quad \text{as } \varepsilon \to 0^+, \quad \text{a.e. } t \in [a, b].
\]

According to \([7\), page 92\], if \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space, \(Z\) is a separable Hilbert space, and \(\mathcal{F}\) is countably generated apart from null sets, then \(L^1(\Omega; Z)\) is separable. Hence, \(L^1(\Omega; H)\) is separable Banach space. Then, following \([15\] Lemma 3.6, Section 3.2\], we have an analogous result.

Lemma 2.4 Let \(z \in L^1_T(0, T; H)\). Then it is Bochner integrable if it is regarded as a map from \([0, T]\) to \(L^1(\Omega; H)\).

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 We divide the proofs into several steps.

Step 1. By Assumption (S1) and (1.13), \(z(\cdot) = \bar{a}(\cdot), \ b(\cdot)\), \(\bar{X}(\cdot)\) can be regarded as Bochner integrable functions from \([0, T]\) to \(L^1(\Omega, H)\) \([15\] Lemma 3.6, Section 3.2\]). Noting that by Lemma 2.3 for \(\varepsilon\) small enough,

\[
\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |z(r) - z(t)|_{L^1(\Omega, H)}dr < \infty,
\]

and

\[
\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |z(r, \cdot) - z(t, \cdot)|_{H}dr \leq C, \quad \mathbb{P}-a.s.
\]

By (2.4), (2.5), Fubini’s Theorem and the Dominated Convergence Theorem, we have

\[
\lim_{\varepsilon \to 0^+} \mathbb{E} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |z(r) - z(t)|_{H}dr = 0, \quad \text{a.e. } t \in [0, T].
\]

Fix \(t_0 \in [0, T]\) such that (1.12) holds at \(t_0\), and (2.6) holds at \(t_0\) for \(z(\cdot) = A\bar{X}(\cdot), \ \bar{a}(\cdot)\) and \(\bar{b}(\cdot)\).

Fix \(\omega_0 \in \Omega\) such that the regular conditional probability \(\mathbb{P}(\cdot \mid \mathcal{F}_{t_0})(\omega_0)\) is well-defined. In the probability space \((\Omega, \mathcal{F}, \mathbb{P}(\cdot \mid \mathcal{F}_{t_0})(\omega_0)\)), the random variables

\[
\bar{X}(t_0), \ \bar{R}(t_0), \ \bar{p}(t_0), \ \bar{R}(t_0)
\]

are almost surely equal to

\[
\bar{X}(t_0, \omega_0), \ \bar{R}(t_0, \omega_0), \ \bar{p}(t_0, \omega_0), \ \bar{R}(t_0, \omega_0),
\]

respectively. Let \(\eta_0 = \bar{X}(t_0, \omega_0)\). Denote by \(\mathbb{E}_{\omega_0}\) the expectation with respect to the probability measure \(\mathbb{P}(\cdot \mid \mathcal{F}_{t_0})(\omega_0)\).

By Proposition 2.4 there exists a function \(\phi \in C^{1,2}([0, T] \times H)\) such that

\[
\phi(t, \eta) > V(t, \eta), \quad \text{for every } (t, \eta) \in (0, T) \times H, \ (t, \eta) \neq (t_0, \eta_0), \quad \text{(2.7)}
\]

\[
(\phi(t_0, \eta_0), \phi_t(t_0, \eta_0), \phi_x(t_0, \eta_0), \phi_{xx}(t_0, \eta_0)) = (V(t_0, \eta_0), \bar{R}(t_0, \omega_0), \bar{p}(t_0, \omega_0), \bar{R}(t_0, \omega_0)), \quad \text{(2.8)}
\]

and that

\[
\phi, \ \phi_t, \ \phi_x, \ \phi_{xx} \text{ are polynomially bounded.} \quad \text{(2.9)}
\]

Then \(\phi\) is a fixed deterministic function if \((t_0, \omega_0)\) is fixed.
Applying Itô’s formula to $\phi$, then for any $\varepsilon > 0$, it follows that

$$
\phi(t_0 + \varepsilon, \overline{X}(t_0 + \varepsilon)) - \phi(t_0, \overline{X}(t_0))
= \int_{t_0}^{t_0+\varepsilon} \left( \phi_t(r, \overline{X}(r)) + \langle \phi_x(r, \overline{X}(r)), \alpha(r) \rangle_H + \langle \phi_{xx}(r, \overline{X}(r)), A\overline{X}(r) \rangle_H + \frac{1}{2} \langle \phi_{xx}(r, \overline{X}(r)) \tilde{b}(r), \tilde{b}(r) \rangle_{L^2_r} \right) dr + \int_{t_0}^{t_0+\varepsilon} \langle \phi_x(r, \overline{X}(r)), \tilde{b}(r)dW(r) \rangle_H.
$$

(2.10)

Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0, +\infty)$ be such that $\lim_{n \to +\infty} \varepsilon_n = 0$. From (2.10), we have

$$
\mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \left( V(t_0 + \varepsilon_n, \overline{X}(t_0 + \varepsilon_n)) - V(t_0, \overline{X}(t_0)) \right)
\leq \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left( \phi_t(r, \overline{X}(r)) + \langle \phi_x(r, \overline{X}(r)), A\overline{X}(r) \rangle_H + \langle \phi_{xx}(r, \overline{X}(r)), \alpha(r) \rangle_H + \frac{1}{2} \langle \phi_{xx}(r, \overline{X}(r)) \tilde{b}(r), \tilde{b}(r) \rangle_{L^2_r} \right) dr.
$$

(2.11)

**Step 2.** In this step, we treat the right hand side of (2.11) term by term.

First, thanks to the continuity of $\overline{X}(\cdot)$ and $\phi_t(\cdot)$, we get

$$
\lim_{\varepsilon_n \to 0^+} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \phi_t(r, \overline{X}(r)) dr = \phi_t(t_0, \overline{X}(t_0)), \quad \mathbb{P}(\cdot | \mathcal{F}_{t_0})(\omega_0)\text{-a.s.}
$$

(2.12)

By the polynomial growth of $\phi_t$, due to the Dominated Convergence Theorem, it holds that

$$
\lim_{\varepsilon_n \to 0^+} \mathbb{E}_{\omega_0} \left| \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \phi_t(r, \overline{X}(r)) dr - \phi_t(t_0, \overline{X}(t_0)) \right|_H = 0.
$$

(2.13)

As for the second term of (2.11), we have

$$
\mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \langle \phi_x(r, \overline{X}(r)), \alpha(r) \rangle_H dr - \langle \phi_{xx}(t_0, \overline{X}(t_0)), \alpha(t_0) \rangle_H
= \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \langle \phi_x(r, \overline{X}(r)) - \phi_{xx}(t_0, \overline{X}(t_0)), \alpha(r) \rangle_H dr
+ \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \langle \phi_x(t_0, \overline{X}(t_0)), \alpha(r) - \alpha(t_0) \rangle_H dr.
$$

(2.14)

Clearly,

$$
\left| \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \langle \phi_x(r, \overline{X}(r)) - \phi_{xx}(t_0, \overline{X}(t_0)), \alpha(r) \rangle_H dr \right|
\leq \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left| \phi_x(r, \overline{X}(r)) - \phi_{xx}(t_0, \overline{X}(t_0)) \right|_H \left| \alpha(r) \right|_H dr
\leq \mathbb{E}_{\omega_0} \left( \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left| \phi_x(r, \overline{X}(r)) - \phi_{xx}(t_0, \overline{X}(t_0)) \right|^2_H dr \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left| \alpha(r) \right|^2_H dr \right)^{\frac{1}{2}}
\leq \left( \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left| \phi_x(r, \overline{X}(r)) - \phi_{xx}(t_0, \overline{X}(t_0)) \right|^2_H dr \right)^{\frac{1}{2}} \left( \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0+\varepsilon_n} \left| \alpha(r) \right|^2_H dr \right)^{\frac{1}{2}}.
$$

(2.15)
By Assumption (S1), it follows that

\[ E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r)|^2_H \, dr \leq C \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} (1 + E_{\omega_0} |\mathcal{X}(r)|^2_H) \, dr \leq C(1 + |\eta|^2_H). \quad (2.16) \]

Arguing as in Step 1 for \( \phi_t \), we get that

\[ \lim_{n \to +\infty} E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\phi_x(t, \mathcal{X}(t)) - \phi_x(t_0, \mathcal{X}(t_0))|_H \, dr = 0. \quad (2.17) \]

By (2.15)–(2.16), we see the first term on the right hand side of (2.14) goes to zero as \( n \to +\infty \).

Now we handle the second term of (2.14). Clearly,

\[ \left| E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(t_0, \mathcal{X}(t_0)), \bar{a}(r) - \bar{a}(t_0) \rangle_H \, dr \right| \]

\[ \leq |\phi_x(t_0, \mathcal{X}(t_0))|_H E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr. \quad (2.18) \]

By the choice of \( t_0 \), we have

\[ 0 = \lim_{n \to +\infty} E \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr \]

\[ = \lim_{n \to +\infty} E \left[ E \left( \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr \bigg| \mathcal{F}_{t_0} \right) \right] \]

\[ = \lim_{n \to +\infty} E \left( E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr \right). \]

This implies that

\[ \lim_{n \to +\infty} E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr = 0 \text{ in } L^1(\Omega; \mathbb{R}). \]

Hence, there is a subsequence \( \{\varepsilon^{(1)}_n\}_{n=1}^\infty \) of \( \{\varepsilon_n\}_{n=1}^\infty \) such that for \( \mathbb{P}\)-a.s. \( \omega_0 \),

\[ \lim_{n \to +\infty} E_{\omega_0} \frac{1}{\varepsilon^{(1)}_n} \int_{t_0}^{t_0 + \varepsilon^{(1)}_n} |\bar{a}(r) - \bar{a}(t_0)|_H \, dr = 0. \]

This, together with (2.18), implies that

\[ \lim_{n \to +\infty} \left| E_{\omega_0} \frac{1}{\varepsilon^{(1)}_n} \int_{t_0}^{t_0 + \varepsilon^{(1)}_n} \langle \phi_x(t_0, \mathcal{X}(t_0)), \bar{a}(r) - \bar{a}(t_0) \rangle_H \, dr \right| = 0. \quad (2.19) \]

Next, we treat the third term of (2.14). Obviously,

\[ E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(r, \mathcal{X}(r)), A\mathcal{X}(r) \rangle_H \, dr - \langle \phi_x(t_0, \mathcal{X}(t_0)), A\mathcal{X}(t_0) \rangle_H \]

\[ = E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(r, \mathcal{X}(r)) - \phi_x(t_0, \mathcal{X}(t_0)), A\mathcal{X}(t_0) \rangle_H \, dr \]

\[ + E_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(r, \mathcal{X}(r)), A\mathcal{X}(r) \rangle_H \, dr. \quad (2.20) \]
The first term of (2.20) reads
\[
\left| \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(r, \overline{X}(r)) - \phi_x(t_0, \overline{X}(t_0)), A\overline{X}(t_0) \rangle_H \, dr \right|
\]
\[
\leq \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\phi_x(r, \overline{X}(r)) - \phi_x(t_0, \overline{X}(t_0))|_H |A\overline{X}(t_0)|_H \, dr
\]
\[
\leq |A\overline{X}(t_0)|_H \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\phi_x(r, \overline{X}(r)) - \phi_x(t_0, \overline{X}(t_0))|_H \, dr.
\]

This, together with (2.17), implies that the first term in (2.20) tends to 0 as \( n \to +\infty \). Now we handle the second term of (2.20). Since \( \phi_x(\cdot) \) and \( \overline{X}(\cdot) \) are continuous, we see that
\[
\mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \langle \phi_x(r, \overline{X}(r)), A\overline{X}(r) - A\overline{X}(t_0) \rangle_H \, dr
\]
\[
\leq \left( \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |\phi_x(r, \overline{X}(r))|^2_H \right)^{1/2} \left( \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |A\overline{X}(r) - A\overline{X}(t_0)|^2_H \, dr \right)^{1/2}
\]
\[
= |\phi_x(t_0, \overline{X}(t_0))|^2_H \left( \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} |A\overline{X}(r) - A\overline{X}(t_0)|^2_H \, dr \right)^{1/2}. \tag{2.21}
\]

By the choice of \( t_0 \), following the same procedure for deducing (2.19), we get from (2.21) that there exists a subsequence \( \{\varepsilon_{n(2)}\}_{j=1}^{\infty} \) of \( \{\varepsilon_{n(1)}\}_{n=1}^{\infty} \) such that
\[
\lim_{n \to \infty} \mathbb{E}_{\omega_0} \frac{1}{\varepsilon_{n(2)}} \int_{t_0}^{t_0 + \varepsilon_{n(2)}} \langle \phi_x(r, \overline{X}(r)), a(r) - \overline{a}(t_0) \rangle_H \, dr = 0. \tag{2.22}
\]

At last, we deal with the forth term of (2.11). We have
\[
\frac{1}{2} \mathbb{E}_{\omega_0} \left[ \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \left( \langle \phi_x(r, \overline{X}(r)) \tilde{b}(r), \tilde{b}(r) \rangle_{L^2_a} - \langle \phi_x(t_0, \overline{X}(t_0)) \tilde{b}(t_0), \tilde{b}(t_0) \rangle_{L^2_a} \right) \, dr \right]
\]
\[
= \frac{1}{2} \mathbb{E}_{\omega_0} \left[ \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \left( \langle \phi_x(r, \overline{X}(r)) - \phi_x(t_0, \overline{X}(t_0)), \tilde{b}(r) \rangle_{L^2_a} - \langle \phi_x(t_0, \overline{X}(t_0)), \tilde{b}(r) \rangle_{L^2_a} \right) \, dr \right]
\]
\[
+ \frac{1}{2} \mathbb{E}_{\omega_0} \left[ \frac{1}{\varepsilon_n} \int_{t_0}^{t_0 + \varepsilon_n} \left( \langle \phi_x(t_0, \overline{X}(t_0)), \tilde{b}(r) \rangle_{L^2_a} - \langle \phi_x(t_0, \overline{X}(t_0)) \tilde{b}(r), \tilde{b}(r) \rangle_{L^2_a} \right) \, dr \right]. \tag{2.23}
\]

Now employing the same arguments used to show the right-hand side of (2.14) approaching zero, we reach that the right-hand side of (2.23) vanishes if we replace \( \{\varepsilon_{n(1)}\}_{n=1}^{\infty} \) by a subsequence \( \{\varepsilon_{n(2)}\}_{n=1}^{\infty} \) of \( \{\varepsilon_{j}\}_{j=1}^{\infty} \) and let \( n \to \infty \).

In summary, for any sequence \( \{\varepsilon_{n}\}_{n=1}^{\infty} \subset (0, +\infty) \) with \( \lim_{n \to \infty} \varepsilon_{n} = 0 \), there exists a subsequence \( \{\varepsilon_{n(2)}\}_{n=1}^{\infty} \) of \( \{\varepsilon_{n}\}_{n=1}^{\infty} \), such that
\[
\lim_{n \to +\infty} \mathbb{E}_{\omega_0} \left[ \frac{1}{\varepsilon_{n(2)}} \int_{t_0}^{t_0 + \varepsilon_{n(2)}} \left( \phi_l(r, \overline{X}(r)) + \langle \phi_x(r, \overline{X}(r)), A\overline{X}(r) \rangle_H + \langle \phi_x(r, \overline{X}(r)), \bar{a}(r) \rangle_H + \frac{1}{2} \langle \phi_x(r, \overline{X}(r)) \tilde{b}(r), \tilde{b}(r) \rangle_{L^2_a} \right) \, dr \right]
\]
\[= 0. \]
\[
= \phi_t(t_0, \overline{X}(t_0)) + \langle \phi_x(t_0, \overline{X}(t_0)), A\overline{X}(t_0) \rangle_H + \langle \phi_x(t_0, \overline{X}(t_0)), \bar{u}(t_0) \rangle_H \\
+ \frac{1}{2} \langle \phi_{xx}(t_0, \overline{X}(t_0)) \bar{b}(t_0), \bar{b}(t_0) \rangle_{L^2}.
\]

**Step 3.** In this step, we are to prove the following claim:

**Claim 1:** For any \( \delta \in (0, T) \), there exists \( \rho_\delta(\cdot) \in L^1(0, T - \delta) \) such that for almost every \( t_0 \in [0, T - \delta) \) chosen as in Step 1 and \( \varepsilon > 0 \) with \( t_0 + \varepsilon \leq T - \delta, \eta \in H \) and \((\overline{X}(\cdot), \bar{u}(\cdot))\) being the admissible pair, it holds that

\[
\frac{1}{\varepsilon} \mathbb{E} \left( V(t_0 + \varepsilon, \overline{X}(t_0 + \varepsilon)) - V(t_0, \overline{X}(t_0)) \right) \leq \rho(t_0).
\]

Indeed, by Assumption (1.11) and the definition of \( D_{t,x}^{1,2,v}(\cdot, \cdot) \), we have

\[
V(t_0 + \varepsilon, \overline{X}(t_0 + \varepsilon)) - V(t_0, \overline{X}(t_0)) \\
\leq C_\delta \varepsilon (1 + |\overline{X}(t_0 + \varepsilon)|_H^2) + \langle p(t_0), \overline{X}(t_0 + \varepsilon) - \overline{X}(t_0) \rangle_H + C_0 |\overline{X}(t_0 + \varepsilon) - \overline{X}(t_0)|_H^2
\]

and

\[
|\bar{p}(t)|_H \leq C (1 + |\overline{X}(t)|_H^2), \quad \forall t \in [0, T].
\]

Here and in what follows, we use \( C_\delta \) to denote a constant depending on \( \delta \), which may vary from line to line.

Now, we begin to estimate the right hand side of (2.25) term by term. Noting the choice of \( t_0 \), we have

\[
\mathbb{E} \left( \bar{p}(t_0), \overline{X}(t_0 + \varepsilon) - \overline{X}(t_0) \right)_H \\
= \mathbb{E} \left( \bar{p}(t_0), (S(\varepsilon) - I)\overline{X}(t_0) + \int_{t_0}^{t_0 + \varepsilon} S(r - t)a(r, \overline{X}(r), \bar{u}(r)) \, dr \\
+ \int_{t_0}^{t_0 + \varepsilon} S(r - t)b(r, \overline{X}(r), \bar{u}(r)) \, dW(r) \right)_H \\
\leq \varepsilon \mathbb{E} \left( \bar{p}(t_0), A\overline{X}(t_0) \right)_H + C \left[ \mathbb{E} (1 + |\overline{X}(t_0)|_H^2)^2 \right]^{1/2} \left[ \mathbb{E} \left( \int_{t_0}^{t_0 + \varepsilon} a(r, \overline{X}(r), \bar{u}(r)) \, dr \right)_H^2 \right]^{1/2} \\
\leq C \varepsilon \left[ \mathbb{E} (1 + |\overline{X}(t_0)|_H^2)^{1/2} \left( \mathbb{E} |A\overline{X}(t_0)|_H^2 \right)^{1/2} + C \varepsilon \left[ \mathbb{E} (1 + |\overline{X}(t_0)|_H^2) \right]^{1/2}.\right.
\]

The third term in (2.25) reads

\[
\mathbb{E} |\overline{X}(t + \varepsilon) - \overline{X}(t)|_H^2 \leq C \left[ \mathbb{E} (S(\varepsilon) - I)\overline{X}(t)|_H^2 + \mathbb{E} \left( \int_{t}^{t + \varepsilon} a(r, \overline{X}(r), \bar{u}(r)) \, dr \right)_H^2 \\
+ \mathbb{E} \left( \int_{t}^{t + \varepsilon} b(r, \overline{X}(r), \bar{u}(r)) \, dW(r) \right)_H^2 \right]^2 \\
\leq C \left[ \mathbb{E} |A\overline{X}(t)|_H^2 \varepsilon^2 + \mathbb{E} (1 + |\overline{X}(t)|_H^2)(\varepsilon^2 + \varepsilon) \right].
\]

Thus, by taking

\[
\rho(t_0) = C_\delta \left[ \mathbb{E} (1 + |\overline{X}(t_0)|_H^2) \right]^{1/2} \left[ \left( \mathbb{E} |A\overline{X}(t_0)|_H^2 \right)^{1/2} + 1 \right] \in L^1(0, T - \delta),
\]

we complete the proof of Claim 1.
Step 4. Applying Claim 1 shown in Step 3 on \((\Omega, \mathcal{F}, \mathbb{P}(\cdot | \mathcal{F}_0)(\omega_0))\), then by Lemma 2.2 and (2.3), we obtain

\[
\lim_{\varepsilon \to 0} \mathbb{E}_{\omega_0} \frac{1}{\varepsilon} (V(t_0 + \varepsilon, X(t_0 + \varepsilon)) - V(t_0, X(t_0))) \leq \mathcal{R}(t_0, \omega_0) + \langle \tilde{p}(t_0, \omega_0), AX(t_0) \rangle_H + \langle \tilde{p}(t_0, \omega_0), \tilde{a}(t_0, \omega_0) \rangle_H + \frac{1}{2} \langle \mathcal{P}(t_0, \omega_0) \tilde{b}(t_0, \omega_0), \tilde{b}(t_0, \omega_0) \rangle_{L_2^2}.
\]  

(2.28)

Using (2.28) and Claim 1 again, by Fatou’s lemma, we get

\[
\lim_{\varepsilon \to 0^+} \mathbb{E}_{\omega_0} \frac{1}{\varepsilon} (V(t_0 + \varepsilon, X(t_0 + \varepsilon)) - V(t_0, X(t_0))) = \lim_{\varepsilon \to 0^+} \mathbb{E} \left[ \mathbb{E}_{\omega_0} \frac{1}{\varepsilon} (V(t_0 + \varepsilon, X(t_0 + \varepsilon)) - V(t_0, X(t_0))) \right]
\]

\[
\leq \mathbb{E} \left\{ \lim_{\varepsilon \to 0^+} \mathbb{E}_{\omega_0} \frac{1}{\varepsilon} (V(t_0 + \varepsilon, X(t_0 + \varepsilon)) - V(t_0, X(t_0))) \right\}
\]

\[
\leq \mathbb{E} \left( \mathcal{R}(t_0) + \langle \tilde{p}(t_0), AX(t_0) \rangle_H + \langle \tilde{p}(t_0), \tilde{a}(t_0) \rangle_H + \frac{1}{2} \langle \mathcal{P}(t_0) \tilde{b}(t_0), \tilde{b}(t_0) \rangle_{L_2^2} \right)
\]

(2.29)

for a.e. \(t_0 \in [0, T - \delta)\). Applying Lemma 2.2 to \(g(t) = \mathbb{E} V(t, X(t))\), and using (2.29) and (1.13), we obtain

\[
\mathbb{E} V(T - \delta, X(T - \delta)) - V(0, \eta) \leq -\mathbb{E} \int_0^{T - \delta} \bar{f}(t)dt.
\]  

(2.30)

Noting that \(V(\cdot, \cdot)\) is continuous and \(V(T, X(T)) = h(X(T))\), letting \(\delta \to 0\) in (2.30), we obtain that

\[
\mathbb{E} \left( \int_0^T f(t, X(t), \bar{u}(t))dt + h(X(T)) \right) \leq V(0, \eta),
\]

which means that the control \(\bar{u}(\cdot)\) is optimal.

Proof of Proposition 1.5: Obviously, (1.15) implies (1.13). We only need to prove that (1.13) implies (1.15).

Suppose (1.15) holds. By Proposition 2.1 for a.e. \((t, \omega) \in [0, T] \times H\) such that \((\mathcal{R}(t, \omega), \tilde{p}(t, \omega), \bar{P}(t, \omega)) \in D^{1,2,+}_{t,x} V(t, X(t)), X(t, \omega) = x\), there exists a function \(\varphi \in C^{1,2}(0, T] \times H\) so that

\[
\begin{align*}
\left\{ (\varphi(t, x), \varphi_t(t, x), \varphi_x(t, x), \varphi_{xx}(t, x)) = (V(t, x), \mathcal{R}(t, \omega), \tilde{p}(t, \omega), \bar{P}(t, \omega)), \\
\varphi(s, y) > v(t, x), \quad \forall (t, x) \neq (s, y) \in [t, T] \times H.
\end{align*}
\]

(2.31)

Fix a \(u \in U\). Let \(X(\cdot) = X(\cdot; t, x, u)\) be the trajectory with the control \(u(r) \equiv u\). Then by Itô’s formula, for any \(s > t\) with \(s - t > 0\) small enough, we have

\[
\begin{align*}
0 & \leq \frac{1}{s - t} \mathbb{E}(V(t, x) - \varphi(t, x) - V(s, X(s)) + \varphi(s, X(s))) \\
& \leq \frac{1}{s - t} \left( \mathbb{E} \int_t^s f(r, X(r), u)dr - \varphi(t, x) + \varphi(s, X(s)) \right) \\
& = \frac{1}{s - t} \mathbb{E} \int_t^s \left( \varphi_t(r, X(r)) + \langle \varphi_x(r, X(r)), AX(r) \rangle_H \\
& \quad + G(r, X(r), u, \varphi_x(r, X(r)), \varphi_{xx}(r, X(r))) \right)dr
\end{align*}
\]

(2.32)
This leads to
\[ \varphi_t(t,x) + \langle \varphi_x(t,x), Ax \rangle_H + G(t,x,u,\varphi_x(t,x),\varphi_{xx}(t,x)) \geq 0, \quad \forall u \in U. \]

Hence
\[ \varphi_t(t,x) + \langle \varphi_x(t,x), Ax \rangle_H + \inf_{u \in U} G(t,x,u,\varphi_x(t,x),\varphi_{xx}(t,x)) \geq 0. \]

This, together with (2.31), implies that
\[ \varphi_t(t,x) + \langle \varphi_x(t,x), Ax \rangle_H + \inf_{u \in U} G(t,x,u,\varphi_x(t,x),\varphi_{xx}(t,x)) \geq 0, \]

which yields
\[ \bar{R}(t,\omega) + \langle \bar{p}(t,\omega), AX(t,\omega) \rangle_H + \inf_{u \in U} G(t,\bar{X}(t,\omega),u,\varphi_x(t,\bar{X}(t,\omega)),\varphi_{xx}(t,\bar{X}(t,\omega))) \geq 0. \]

This combining with (1.13) gives (1.15).

3 Lipschitz continuity of the value function

In this section, we are to prove Theorem 1.2. We first introduce an auxiliary control problem to be used in the sequel.

3.1 An auxiliary control problem

Recall that for Problem (S\textsubscript{th}), both the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the Brownian motion \(W(\cdot)\) on \((\Omega, \mathcal{F}, \mathbb{P})\) are given a priori, and our controls are \(F\)-adapted processes. In this subsection, we introduce a family of auxiliary control problems in which only the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is fixed and the Brownian motion is part of the controls. We will see this newly introduced admissible control is closely related to our original one under the original strong formulation and plays an important role in the proof of the Lipschitz continuity of the value function.

Let \(t \in [0,T)\), denote by \(\bar{\mathcal{W}}_t\) the set of all cylindrical Brownian motions \(\bar{W}_t(\cdot)\) on \((\Omega, \mathcal{F}, \mathbb{P})\) over \([t,T]\) (with \(\bar{W}_t(t) = 0\) almost surely). It is well known that \(\bar{W}_t(\cdot)\) is continuous, \(\mathbb{P}\)-a.s. and admits a modification which is continuous for all \(\omega \in \Omega\). In what follows, we always take the continuous modification of cylindrical Brownian motion.

For a given \(\bar{W}_t(\cdot) \in \bar{\mathcal{W}}_t\), write \(\mathcal{F}_{\bar{W}_t}\) for the natural filtration generated by \(\bar{W}_t(\cdot)\). Let
\[ \tilde{\mathcal{U}}_{\bar{W}_t}[t,T] \triangleq \left\{ u : [t,T] \times \Omega \to U | \text{u is } \mathcal{F}_{\bar{W}_t} \text{-adapted} \right\}. \]

Clearly, both \(\bar{\mathcal{W}}_t\) and \(\tilde{\mathcal{U}}_{\bar{W}_t}[t,T]\) depend on the Brownian motion \(\bar{W}_t(\cdot)\).

The admissible control set is
\[ \tilde{\mathcal{U}}_{EX}[t,T] \triangleq \left\{ (\tilde{u}(\cdot), \bar{W}_t(\cdot)) : \bar{W}_t(\cdot) \in \bar{\mathcal{W}}_t, \tilde{u}(\cdot) \in \tilde{\mathcal{U}}_{\bar{W}_t}[t,T] \right\}. \]

Consider the following control system:
\[
\begin{align*}
  d\bar{X}(s) &= \left( A\bar{X}(s) + a(s,\bar{X}(s),\tilde{u}(s)) \right) ds + b(s,\bar{X}(s),\tilde{u}(s)) d\bar{W}_t(s), \quad s \in (t,T], \\
  \bar{X}(t) &= \eta,
\end{align*}
\]

\[ (3.1) \]
where \( \eta \in H \), and \((\tilde{u}(\cdot), \tilde{W}(\cdot)) \in \tilde{U}_{EX}[t, T]\).

Under Assumption (S1), for any \( \eta \in H \), (3.1) admits a unique mild solution \( \tilde{X}(\cdot) \) (e.g., [17, Theorem 3.14]). Then for any \((t, \eta) \in [0, T] \times H \) and \((\tilde{u}(\cdot), \tilde{W}(\cdot)) \in \tilde{U}_{EX}[t, T]\), the cost functional

\[
\tilde{J}(t, \eta; \tilde{u}(\cdot)) = \mathbb{E}\left( \int_t^T f(s, \tilde{X}(s), \tilde{u}(s))ds + h(\tilde{X}(T)) \right)
\]

is well-defined. So does the corresponding value function

\[
\tilde{V}(t, \eta) \triangleq \inf_{(\tilde{u}(\cdot), \tilde{W}(\cdot)) \in \tilde{U}_{EX}[t, T]} \tilde{J}(t, \eta; \tilde{u}(\cdot)), \quad \forall \, (t, \eta) \in [0, T] \times H.
\]

**Remark 3.1** Compared with Problem (S\(f\)), we enlarge the admissible control set \( \tilde{U}_{EX}[t, T] \) to admit the Brownian as part of the control. But the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is fixed. Recall that there is another formulation (which is called weak formulation) in the literature, in which the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is also part of the control (e.g., [8, Subsection 2.1.2] or [22, Subsection 4.2, Chapter 2]). For the weak formulation, one can also define the value function \( V_w(t, \cdot) \) (see [8, Subsection 2.1.2] for the details).

By the definition of \( V(\cdot, \cdot) \) and \( \tilde{V}(\cdot, \cdot) \), it is clear that

\[
V(t, \eta) \geq \tilde{V}(t, \eta), \quad \forall \, (t, \eta) \in [0, T] \times H.
\]  
(3.4)

On the other hand, by [8, Theorem 2.22, Chapter 2], for all \((t, \eta) \in [0, T] \times H \), \( V(t, \eta) \) equals the value function under the weak formulation mentioned in Remark 3.1. Consequently, we have

\[
V(t, \eta) = \tilde{V}(t, \eta), \quad \forall \, (t, \eta) \in [0, T] \times H.
\]

(3.5)

Next, we introduce a special case of the auxiliary control problem presented above, which plays a major role in the next subsection.

For each \( t > 0 \), denote by \( \mathcal{F}_t^t \) the \( \sigma \)-algebra generated by \( \{W(t) - W(t) \}_{t \leq \tau \leq T} \) and by \( \mathcal{F}_t^T \) the natural filtration of the Brownian motion \( \{W(t) - W(t) \}_{t \leq \tau \leq T} \). Write \( \mathbb{F}^t \) for the progressive \( \sigma \)-algebra with respect to \( \mathcal{F}_t^t \). Let

\[
U^t[t, T] \triangleq \{ u(\cdot) \in U[t, T] \mid u(r) \text{ is } \mathcal{F}_r^t \text{-adapted}, \forall \, t \leq r \leq T \}\;.
\]

For \( u(\cdot) \in U^t[t, T] \) and any \( s \in [0, T) \), let

\[
\tilde{u}(\cdot) = u(\tau(\cdot)), \quad \tilde{W}(\cdot) = \sqrt{1/\tau}W(\tau(\cdot)) - W(t).
\]

where \( \tau(r) = \frac{T(t-s)+T-t}{T-s} \).

We claim that \((\tilde{u}(\cdot), \tilde{W}(\cdot)) \in \tilde{U}_{EX}[s, T]\). Indeed, it is clear that

(i) \( \tilde{W}(s) = \sqrt{1/\tau}W(t) - W(t) = 0 \), for that \( W(\cdot) - W(t) \) is cylindrical Brownian motion;

(ii) For all \( n \in \mathbb{N}^+ \) and \( s = s_1 < s_2 < \cdots < s_n \leq T \), we have

\[
\left( \tilde{W}(s_1), \tilde{W}(s_2) - \tilde{W}(s_1), \cdots, \tilde{W}(s_n) - \tilde{W}(s_{n-1}) \right) = \left( \sqrt{1/\tau}W(r_1) - W(t), \sqrt{1/\tau}W(r_2) - \sqrt{1/\tau}W(r_1), \cdots, \sqrt{1/\tau}W(r_n) - \sqrt{1/\tau}W(r_{n-1}) \right)
\]

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are independent, where \( r_i = \tau(s_i), \ \forall \ 1 \leq i \leq n. \)

(iii) For any \( s \leq r < l \leq T, \)

\[
\tilde{W}(r) - \tilde{W}(l) = \sqrt{1/\tau}(W(\tau(r)) - W(\tau(l))) \sim N(0, (1/\tau(\tau(r) - \tau(l)))I) = N(0, (r - l)I),
\]

where \( I \) is the identity operator in \( \tilde{H}. \) On the other hand,

\[
\tilde{F}_s^r \triangleq \sigma \{ \tilde{W}(l) : s \leq l \leq r \} = \sigma \{ \tilde{W}^{-1}(B) : B \in \mathcal{B}(\mathbb{R}), s \leq l \leq r \} = \sigma \{ (\sqrt{\tau}W^{-1}(\tau(l)) - W(t)) (B) : B \in \mathcal{B}(\mathbb{R}), s \leq l \leq r \} = \sigma \{ (W^{-1}(\rho) - W(t)) (B) : B \in \mathcal{B}(\mathbb{R}), t \leq \rho \leq \tau(r) \} = F_s^r(t).
\]

Since \( \tilde{u}(r) = u(\tau(r)) \) is \( F_s^r(t) \)-measurable, it follows from (3.6) that \( \tilde{u}(r) = u(\tau(r)) \) is \( F_s^r(t) \)-measurable.

We end up this subsection with the following result.

**Proposition 3.1** For any \( \eta \in H, \ t \in [0, T), \)

\[
\inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.7}
\]

The proof of Proposition 3.1 is very similar to the one for Subsection 4.1, Proposition 4.3]. We provide it here for the convenience of readers.

**Proof.** We divide the proof into three steps.

**Step 1.** Let

\[
\mathcal{U}_D^t \triangleq \left\{ u(s) = \sum_{j=1}^{N} u^j(s) 1_{\Omega_j} \left| u^j(s) \in \mathcal{U}[t,T], \ \{\Omega_j\}_{j=1}^{N} \subset \mathcal{F}_t \text{ is a partition of } \Omega \right. \right\}.
\]

Since \( \mathcal{U}_D^t \subset \mathcal{U}[t,T] \), we have

\[
\inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)) \leq \inf_{u(\cdot) \in \mathcal{U}_D^t} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.8}
\]

On the other hand, from [21] Lemma 4.12], we know that \( \mathcal{U}_D^t \) is dense in \( \mathcal{U}[t,T]. \) Consequently,

\[
\text{ess inf}_{u(\cdot) \in \mathcal{U}_D^t} \mathcal{J}(t, \eta; u(\cdot)) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.9}
\]

From (3.8) and (3.9), we see that

\[
\inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_D^t} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.10}
\]

**Step 2.** In this step, we prove

\[
\inf_{u(\cdot) \in \mathcal{U}_D^t} \mathcal{J}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.11}
\]

Since \( \mathcal{U} \subset \mathcal{U}_D^t, \) we have

\[
\inf_{u(\cdot) \in \mathcal{U}_D^t} \mathcal{J}(t, \eta; u(\cdot)) \leq \inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{3.12}
\]
Now we show the inverse inequality of (3.12). For all \( u(\cdot) \in \mathcal{U}_{D}^{t} \), we have
\[
\mathcal{J}(t, \eta; u(\cdot)) = \mathcal{J}(t, \eta; \sum_{j=1}^{N} \chi_{\Omega_{j}} u_{j}(\cdot)) = \sum_{j=1}^{N} \chi_{\Omega_{j}} \mathcal{J}(t, \eta; u_{j}(\cdot)).
\]
For \( j = 1, 2, \cdots, N \), noting that \( u_{j}(\cdot) \) is \( \mathbb{F}^{t} \) measurable, we find that \( \mathcal{J}(t, \eta; u_{j}(\cdot)) \) is deterministic. Without loss of generality, we assume that
\[
\mathcal{J}(t, \eta; u_{1}(\cdot)) \leq \mathcal{J}(t, \eta; u_{j}(\cdot)), \quad \forall j = 2, 3, \cdots, N.
\]
Thus, it holds that
\[
\mathcal{J}(t, \eta; u(\cdot)) \geq \mathcal{J}(t, \eta; u_{1}(\cdot)) \geq \inf_{u(\cdot) \in \mathcal{U}_{D}^{t}} \mathcal{J}(t, \eta; u(\cdot)).
\]
For \( u(\cdot) \in \mathcal{U}_{D}^{t} \) is arbitrarily chosen, we get that
\[
\inf_{u(\cdot) \in \mathcal{U}_{D}^{t}} \mathcal{J}(t, \eta; u(\cdot)) \geq \inf_{u(\cdot) \in \mathcal{U}_{D}^{t}[t,T]} \mathcal{J}(t, \eta; u(\cdot)).
\]

**Step 3.** We finish the proof in this step.
From (3.10) and (3.11), we see that
\[
\inf_{u(\cdot) \in \mathcal{U}_{D}^{t}} \mathcal{J}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{D}^{t}[t,T]} \mathcal{J}(t, \eta; u(\cdot)).
\]
The right hand side of (3.13) is deterministic, hence (3.13) can be simplified as
\[
\inf_{u(\cdot) \in \mathcal{U}_{D}^{t}[t,T]} \mathcal{J}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{D}^{t}[t,T]} \mathcal{J}(t, \eta; u(\cdot)).
\]

### 3.2 Proof of Theorem 1.2

Before completing the proof of Theorem 1.2, we present some preliminaries to be used latter, which will play a crucial role in the proof of the regularity result. More details can be found in [19].

**Proposition 3.2** Suppose \( 0 < t_{1} < t_{2} \), \( \eta \in H \). Then
\[
|S(t_{2})\eta - S(t_{1})\eta|_{H} \leq (t_{2} - t_{1})|AS(t_{1})|_{\mathcal{L}(H)}|\eta|_{H}.
\]

Proposition 3.2 should be a well-known result and its proof is very easy. However, we do not find an exact reference for it. For the convenience of readers, we provide a proof below.

**Proof.** Since \( \{S(t)\}_{t \geq 0} \) is analytic and contractive, the map \( t \rightarrow S(t)\eta, \forall \eta \in H \), is differentiable for \( t > 0 \) and \( |S(t)| \leq 1 \). Then \( AS(t_{1}) \) is a bounded linear operator and
\[
|S(t_{2})\eta - S(t_{1})\eta|_{H} = \left| \int_{t_{1}}^{t_{2}} AS(t)\eta dt \right|_{H} = \left| \int_{t_{1}}^{t_{2}} S(t - t_{1})AS(t_{1})\eta dt \right|_{H} \leq (t_{2} - t_{1})|AS(t_{1})|_{\mathcal{L}(H)}|\eta|_{H}.
\]

The next proposition is taken from [19] Theorem 6.13, Chapter 2].
Proposition 3.3 There exists a constant $C > 0$ such that
\[
|AS(t)|_{L(H)} \leq \frac{C}{t}, \quad \forall \ t > 0.
\]

Now let us prove Theorem 1.2.

Proof of Theorem 1.2 Let us fix $\delta > 0$, and let $(t_1, \eta_1)$ and $(t_0, \eta_0) \in [0, T) \times H$ be such that
\[
\min\{T - t_1, T - t_0\} > \delta.
\]
Without loss of generality, assume that $t_0 > t_1$.

For any $\varepsilon > 0$, by Proposition 3.1, there exists $u_0(\cdot) \in \mathcal{U}^{t_0}[t_0, T]$ such that
\[
\mathcal{J}(t_0, \eta_0; u_0(\cdot)) < V(t_0, \eta_0) + \varepsilon.
\]

Let $X_0(\cdot)$ be the solution of
\[
\begin{cases}
    dX_0(t) = (AX_0(t) + a(t, X_0(t), u_0(t)))dt + b(t, X_0(t), u_0(t))dW(t), & t \in (t_0, T], \\
    X(t_0) = \eta_0.
\end{cases}
\]

Consider the change of time as follows:
\[
\tau : [t_1, T] \to [t_0, T], \quad \tau(t) = \frac{T(t_0 - t_1) + (T - t_0)t}{T - t_1}.
\]

Then it holds that
\[
\dot{\tau}(t) = \frac{T - t_0}{T - t_1}, \quad \tau^{-1}(t) = \frac{t(T - t_1) - T(t_0 - t_1)}{T - t_0},
\]
and that
\[
\tau(t_1) = t_0, \quad \tau(T) = T.
\]

Let $\tilde{u}(t) = u_0(\tau(t))$, and denote by $\tilde{X}(t)$ the solution of
\[
\begin{cases}
    d\tilde{X}(t) = (A\tilde{X}(t) + a(t, \tilde{X}(t), \tilde{u}(t)))dt + b(t, \tilde{X}(t), \tilde{u}(t))d\tilde{W}(t), & t \in (t_1, T], \\
    \tilde{X}(t_1) = \eta_1,
\end{cases}
\]
where $\tilde{W}(t) = \sqrt{\frac{1}{\tau}}W(\tau(t))$. Obviously, $(\tilde{W}(\cdot), \tilde{u}(\cdot)) \in \tilde{\mathcal{U}}_{EX}[t_1, T]$. Thus,
\[
\tilde{X}(t) = S(t - t_1)\eta_1 + \int_{t_1}^{t} S(t - s)a(s, \tilde{X}(s), \tilde{u}(s))ds + \int_{t_1}^{t} S(t - s)b(s, \tilde{X}(s), \tilde{u}(s))d\tilde{W}(s)
\]
\[
= S(t - t_1)\eta_1 + \int_{t_1}^{t} S(t - s)a(s, \tilde{X}(s), u_0(\tau(s)))ds + \int_{t_1}^{t} \sqrt{1/\tau}S(t - s)b(s, \tilde{X}(s), u_0(\tau(s)))dW(\tau(s)).
\]

Next, by (3.15) and (3.15), we have
\[
V(t_1, \eta_1) - V(t_0, \eta_0) - \varepsilon = \tilde{V}(t_1, \eta_1) - V(t_0, \eta_0) - \varepsilon
\]
\[
\leq \mathcal{J}(t_1, \eta_1; \tilde{u}) - \mathcal{J}(t_0, \eta_0; u_0)
\]
\[
= \mathbb{E}\left(\int_{t_1}^{T} f(t, \tilde{X}(t), \tilde{u}(t))dt - \int_{t_0}^{T} f(t, X_0(t), u_0(t))dt + h(\tilde{X}(T)) - h(X_0(T))\right).
\]
Set $X_1(r) = \bar{X}(\tau^{-1}(r))$ for $r \in [t_1, T]$. By changing the variable $r = \tau(t)$ in the first integral of (3.22), we obtain that

$$V(t_1, \eta_1) - V(t_0, \eta_0) - \varepsilon \leq \mathbb{E} \int_{t_0}^{T} \left( \frac{1}{\tau} f(\tau^{-1}(r), X_1(r), u_0(r)) - f(r, X_0(r), u_0(r)) \right) dr + \mathbb{E} (h(X_1(T)) - h(X_0(T))).$$

By Assumption (S2)', we have

$$V(t_1, \eta_1) - V(t_0, \eta_0) - \varepsilon \leq \mathbb{E} \int_{t_0}^{T} \left( \frac{1}{\tau} f(\tau^{-1}(r), X_1(r), u_0(r)) - f(\tau^{-1}(r), X_1(r), u_0(r)) \right) dr \quad (3.23)$$

$$+ \mathbb{E} \int_{t_0}^{T} (f(\tau^{-1}(r), X_1(r), u_0(r)) - f(r, X_0(r), u_0(r))) dr + \mathbb{E} (h(X_1(T)) - h(X_0(T)))$$

$$\leq C \mathbb{E} \int_{t_0}^{T} \left( \left| 1 - \frac{1}{\tau} \right| + |\tau^{-1}(r) - r| + |X_1(r) - X_0(r)|_H \right) dr + C \mathbb{E} |X_1(T) - X_0(T)|_H.$$

We claim the following estimates hold:

$$\left| 1 - \frac{1}{\tau} \right| \leq C |t_1 - t_0|, \quad (3.24)$$

$$|\tau^{-1}(r) - r| \leq C |t_1 - t_0|, \quad (3.25)$$

$$\mathbb{E} \int_{t_0}^{T} |X_1(r) - X_0(r)|_H \, dr \leq C (|t_1 - t_0| + |\eta_1 - \eta_0|_H) (1 + |\eta_1|_H), \quad (3.26)$$

$$\mathbb{E} |X_1(T) - X_0(T)|_H \leq C (|t_1 - t_0| + |\eta_1 - \eta_0|_H) (1 + |\eta_1|_H). \quad (3.27)$$

**Proof of (3.24).** By the definition of $\tau(\cdot)$,

$$\left| 1 - \frac{1}{\tau} \right| = \left| 1 - \frac{T - t_1}{T - t_0} \right| = \left| \frac{t_1 - t_0}{T - t_0} \right| \leq \frac{1}{\delta} |t_1 - t_0| \leq C |t_1 - t_0|.$$

**Proof of (3.25).** By the definition of $\tau(\cdot)$ and (3.14), we obtain

$$|\tau^{-1}(r) - r| = \left| \frac{(T - t_1)r - (t_0 - t_1)T}{T - t_0} - r \right| = \left| \frac{(T - t_1)(T - r)}{T - t_0} \right| \leq C |t_1 - t_0|.$$

**Proof of (3.26).** Recalling the definition of $X_1(r)$, we conclude that

$$\mathbb{E} |X_1(r) - X_0(r)|_H^2 = \mathbb{E} |\bar{X}(\tau^{-1}(r)) - X_0(r)|_H^2$$

$$= \mathbb{E} \left| S(\tau^{-1}(r) - t_1)\eta_1 - S(r - t_0)\eta_0 + \int_{t_1}^{\tau^{-1}(r)} S(\tau^{-1}(r) - \rho)a(\rho, \bar{X}(\rho), u_0(\tau(\rho))) \, d\rho \right.$$

$$- \int_{t_0}^{r} S(\rho)a(\rho, X_0(\rho), u_0(\rho)) \, d\rho + \int_{t_1}^{\tau^{-1}(r)} \sqrt{\frac{1}{\tau}} S(\tau^{-1}(r) - \rho)b(\rho, \bar{X}(\rho), u_0(\tau(\rho))) \, dW(\tau(\rho))$$

$$- \int_{t_0}^{r} S(\rho)b(\rho, X_0(\rho), u_0(\rho)) \, dW(\rho) \left|_H^2 \right.$$}

$$= \mathbb{E} \left( \left| S(\tau^{-1}(r) - t_1)\eta_1 - S(r - t_0)\eta_0 \right.$$

$$+ \int_{t_0}^{r} \frac{1}{\tau} S(\tau^{-1}(r) - \tau^{-1}(\rho))a(\tau^{-1}(\rho), X_1(\rho), u_0(\rho)) - S(\rho)a(\rho, X_0(\rho), u_0(\rho)) \right) \, d\rho$$
\begin{align*}
+ \int_{t_0}^{r} \left[ \frac{\tau^{-1} S(\tau^{-1}(r) - \tau^{-1}(\rho)) b(\tau^{-1}(\rho), X_1(\rho), u_0(\rho))}{\tau} \right] dW(\rho) \bigg|_{H}^2 \\
-S(r - \rho) b(\rho, X_0(\rho), u_0(\rho)) \bigg| dW(\rho) \bigg|_{H}^2 \right}.
\end{align*}

By Assumption (S1)', we have

\begin{align*}
\mathbb{E} \left| X_1(r) - X_0(r) \right|_{H}^2 \\
\leq C \mathbb{E} \left[ |S(\tau^{-1}(r) - t_1) \eta_1 - S(r - t_0) \eta_1|_{H}^2 + |S(r - t_0)|_{L(H)}^2 |\eta_1 - \eta_0|_{H}^2 \right] \\
+ C \mathbb{E} \left[ \int_{t_0}^{r} \left( \frac{\tau^{-1}}{\tau} \right) |S(\tau^{-1}(r) - \tau^{-1}(\rho))|_{\mathcal{L}(H)} |a(\tau^{-1}(\rho), X_1(\rho), u_0(\rho))|_{H} \\
+ |S(\tau^{-1}(r) - \tau^{-1}(\rho))|_{\mathcal{L}(H)} |a(\tau^{-1}(\rho), X_1(\rho), u_0(\rho)) - a(\rho, X_0(\rho), u_0(\rho))|_{H} \\
+ |S(\tau^{-1}(r) - \tau^{-1}(\rho)) - S(r - \rho)|_{\mathcal{L}(H)} |a(\rho, X_0(\rho), u_0(\rho))|_{H} \right] d\rho \right] \\
+ C \mathbb{E} \left[ \int_{t_0}^{r} \left( \frac{\tau^{-1/2}}{\tau} \right) |S(\tau^{-1}(r) - \tau^{-1}(\rho))|_{\mathcal{L}(H)}^2 |b(\tau^{-1}(\rho), X_1(\rho), u_0(\rho))|_{H} \\
+ |S(\tau^{-1}(r) - \tau^{-1}(\rho))|_{\mathcal{L}(H)} |b(\tau^{-1}(\rho), X_1(\rho), u_0(\rho)) - b(\rho, X_0(\rho), u_0(\rho))|_{H} \\
+ |S(\tau^{-1}(r) - \tau^{-1}(\rho)) - S(r - \rho)|_{\mathcal{L}(H)} |b(\rho, X_0(\rho), u_0(\rho))|_{H}^2 \right] d\rho.
\end{align*}

Recalling that $t_0 > t_1$, we have $(r - \rho) \frac{T - t_1}{T - t_0} > r - \rho$. Noting that $\{S(t)\}_{t \geq 0}$ is an analytic semigroup, for any $\eta \in H$, we reach that

\begin{align*}
\left| S \left( (r - \rho) \frac{T - t_1}{T - t_0} \right) \eta - S(r - \rho) \eta \right|_{H} \\
\leq C(r - \rho) \frac{t_0 - t_1}{T - t_0} |S(r - \rho)|_{\mathcal{L}(H)} |\eta|_{H} \\
\leq C(r - \rho) \frac{t_0 - t_1}{T - t_0} \frac{1}{r - \rho} |\eta|_{H} \leq C(t_0 - t_1) |\eta|_{H}.
\end{align*}

From (3.28), we see that

\begin{align*}
\mathbb{E} \left| X_1(r) - X_0(r) \right|_{H}^2 \\
\leq C \left( |\eta_1 - \eta_0|_{H}^2 + |t_1 - t_0|^2 |\eta_1|_{H}^2 \right) + C \mathbb{E} \left[ |t_1 - t_0|^2 + \int_{t_0}^{r} |X_1(\rho) - X_0(\rho)|_{H}^2 d\rho \right] \\
\leq C \left( |\eta_1 - \eta_0|_{H}^2 + |t_1 - t_0|^2 (|\eta_1|_{H}^2 + 1) \right) + C \mathbb{E} \int_{t_0}^{r} |X_1(\rho) - X_0(\rho)|_{H}^2 d\rho.
\end{align*}

This, along with Gronwall’s inequality, implies that

\begin{align*}
\mathbb{E} \left| X_1(r) - X_0(r) \right|_{H}^2 \leq C \left( |\eta_1 - \eta_0|_{H}^2 + |t_1 - t_0|^2 (|\eta_1|_{H}^2 + 1) \right), \tag{3.29}
\end{align*}

\text{21}
which yields
\[ \mathbb{E}|X_1(r) - X_0(r)|_H \leq C(|\eta_1 - \eta_0|_H + |t_1 - t_0|)(|\eta_1|_H + 1). \] (3.30)

**Proof of (3.27).** Owing to (3.26),
\[ \mathbb{E}|X_1(T) - X_0(T)|_H = \mathbb{E}|\tilde{X}(T) - X_0(T)|_H \leq C(|\eta_1 - \eta_0|_H + |t_1 - t_0|)(|\eta_1|_H + 1). \]
Combining (3.23)–(3.27), we conclude that
\[ V(t_1, \eta_1) - V(t_0, \eta_0) - \varepsilon \leq C(|\eta_1|_H + 1)(|t_1 - t_0| + |\eta_1 - \eta_0|_H). \]
From the arbitrariness of \(\varepsilon\), we obtain that
\[ V(t_1, \eta_1) - V(t_0, \eta_0) \leq C(|\eta_1|_H + 1)(|t_1 - t_0| + |\eta_1 - \eta_0|_H). \]
Similarly, we can prove that
\[ V(t_0, \eta_0) - V(t_1, \eta_1) \leq C(|\eta_1|_H + 1)(|t_1 - t_0| + |\eta_1 - \eta_0|_H). \]
Therefore, we conclude that
\[ |V(t_1, \eta_1) - V(t_0, \eta_0)| \leq C(|\eta_1|_H + 1)(|t_1 - t_0| + |\eta_1 - \eta_0|_H). \]
This completes the proof.

**Remark 3.2** In the proof of Theorem 1.2, we use the weakly formulated admissible control \((\tilde{u}(\cdot), \tilde{W}(\cdot))\) and the control system driven by it as an auxiliary tool to legitimate our “change of time” strategy applied here. We borrow this idea from [1].

4 **An Illustrative Example**

In this section, we present an illustrative example which fulfill the assumptions in Theorem 1.1 and Corollary 1.1.

Let \(O \subset \mathbb{R}^n\) be a bounded domain with the smooth boundary \(\partial O\). Let \(H = L^2(O)\) and \(U\) be a bounded closed subset of \(L^2(O)\). Consider the following stochastic parabolic equation:
\[
\begin{cases}
  dy = (\Delta y + \tilde{a}(t, y, u))dt + \tilde{b}(t, y, u)dW(t) & \text{in } (0, T] \times O, \\
  y = 0 & \text{on } (0, T] \times \partial O, \\
  y(0) = \eta & \text{in } O,
\end{cases}
\] (4.1)
where \(\eta \in L^2(O)\), \(u(\cdot) \in U[0, T]\), and \(\tilde{a}\) and \(\tilde{b}\) satisfy the following condition:

(B1) For \(r = \tilde{a}, \tilde{b}\), suppose that \(\varphi(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) satisfies:

i) For any \((r, u) \in \mathbb{R} \times \mathbb{R}\), the function \(\varphi(\cdot, r, u) : [0, T] \to \mathbb{R}\) is Lebesgue measurable;

ii) For any \(t \in [0, T]\), \(r \in \mathbb{R}\), the function \(\varphi(t, r, \cdot) : \mathbb{R} \to \mathbb{R}\) is continuous; and

iii) For all \((t, t_1, t_2, r_1, r_2, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\),
\[
\begin{align*}
  &|\varphi(t, r_1, u) - \varphi(t, r_2, u)| \leq C(|t_1 - t_2| + |r_1 - r_2|), \\
  &|\varphi(t, 0, u)| \leq C;
\end{align*}
\] (4.2)

iv) For all \((t, u) \in [0, T] \times \mathbb{R}\), \(\varphi(t, \cdot, u)\) are \(C^2\), and for any \((r, u) \in \mathbb{R} \times \mathbb{R}\) and a.e. \(t \in [0, T]\),
\[ |\varphi_r(t, r, u)| \leq C. \]
Consider the following cost functional:

\[ J(\eta; u(\cdot)) = \mathbb{E}\left( \int_0^T \int_G \tilde{f}(t, y(t), u(t)) \, dx \, dt + \int_G \tilde{h}(y(T)) \, dx \right), \quad (4.3) \]

where \( \tilde{f} \) and \( \tilde{h} \) satisfy the following condition:

**(B2)** \( \tilde{f}(\cdot, r, u) \) is Lipschitz, \( \tilde{f}(t, \cdot, u) \) and \( \tilde{h}(\cdot) \) are \( C^2 \), such that \( \tilde{f}_r(t, r, \cdot) \) and \( \tilde{f}_{rr}(t, r, \cdot) \) are continuous, and for any \( (r, u) \in \mathbb{R} \times \mathbb{R} \) and a.e. \( t \in [0, T] \),

\[ |\tilde{f}_r(t, r, u)| + |\tilde{h}_r(r)| \leq C. \]

**(B3)** For \( \varphi = \tilde{a}, \tilde{b}, \varphi(\cdot, 0, \cdot) = 0 \).

Under **(B1)** and **(B2)**, it is easy to see that **(S1)**–**(S2)** hold. Under **(B1)**–**(B3)**, by the regularity theory of stochastic parabolic equations (e.g.,[11]), we know that \( \Delta y \in L^2_B(0, T; H) \), namely, **(L.10)** holds. Hence, all assumptions in Theorem **1.1** and Corollary **1.1** are fulfilled.

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