Extended Hadamard expansions for the Airy functions

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Abstract

There are two main power series for the Airy functions, namely the Maclaurin and the asymptotic expansions. The former converges for all finite values of the complex variable, $z$, but it requires a large number of terms for large values of $|z|$, and the latter is a Poincaré-type expansion which is well-suited for such large values and where optimal truncation is possible. The asymptotic series of the Airy function shows a classical example of the Stokes phenomenon where a type of discontinuity occurs for the homonymous multipliers. A new series expansion is presented here that stems from the method of steepest descents and can be related to the Hadamard expansions as presented in [14, 15], and which is convergent for all values of the complex variable. Hadamard expansions were introduced as an extension of the method of steepest descents and are defined in terms of a large number of non-systematic integration path subdivisions. Unlike them, the expansions in the present work originate in the splitting of the steepest descent in a number of segments that is not only finite but very small, and which are defined on the basis of the location of the branch points. One of the segments reaches to infinity and this gives rise to the presence of upper incomplete Gamma functions. This is one of the most important differences with the Hadamard series as defined in the aforementioned references, where all the incomplete Gamma functions are of the lower type. The theoretical interest of the new series expansion is twofold. First of all, it shows how to convert an asymptotic series into a convergent one with a finite splitting of the steepest descent path. Secondly, the inverse of the phase function that is part of the Laplace-type equation is Taylor-expanded around branch points to produce Puiseux series when necessary. In addition to this, the proposed analysis shows again how the Stokes phenomenon for the Airy function is related to the transition of the steepest descent paths at $\arg z = \pm 2\pi/3$ from one to two. In regard to its computational application, these series expansions require a relatively small number of terms for each of them to reach a very high precision.

Keywords — Airy’s integral, Airy functions, asymptotic series, steepest descents method, incomplete Gamma functions.

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\[ \theta = \frac{5\pi}{6} \]
\[ \theta = \frac{\pi}{6} \]
\[ \theta = 5\pi \]
\[ L_{12} \]
\[ L_{31} \]
\[ L_{32} \]
\[ C \]
\[ \text{Im}(u) \]
\[ \text{Re}(u) \]

Figure 1: Integration paths in (2) and (3). The path \( L_{21} \) in equation (5) is the same as \( L_{12} \) but reversed.

1 Introduction

The Airy function was first introduced in 1838 for the calculation of light intensity in a caustic, i.e., a surface where light is focused after reflection or refraction by a curved interface between media [17, 10, 18]. It is defined for real values of \( x \) by the following integral

\[ \text{Ai}(x) = \frac{1}{\pi} \int_{0}^{\infty} du \cos(xu + \frac{1}{3}u^3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{i(xu + 1/3u^3)}. \tag{1} \]

Although a number of other equivalent integral expressions that result from elemental variable changes can be found in [1] and [18], more insight is gained if the integration is shown on the complex plane as in Figure 1 and written as

\[ \text{Ai}(z) = \frac{1}{2\pi i} \int_{L_{32}} du e^{zu - 1/3u^3} \tag{2} \]

where, additionally, a complex argument is considered, or, alternatively, by using Cauchy’s theorem,

\[ \text{Ai}(z) = \frac{1}{2\pi i} \int_{L_{31} + L_{12}} du e^{zu - 1/3u^3}. \tag{3} \]

Integration paths \( L_{m,n} \) \((m, n = 1, 2, 3; m \neq n)\) in Figure 1 are subject to the condition of \(-\pi/6 + 2(k-1)\pi/3 < \text{Arg } z < \pi/6 + 2(k-1)\pi/3\) in each zone \( k = 1, 2, 3 \) as \(|u| \rightarrow \infty\), where \( \text{Arg } z \) refers to the argument of \( z \). Integration in the complex plane facilitates the analytic continuation of the Airy function as given in Equation

\footnote{For the principal value of the argument of \( z \) we will use \text{arg } z.}
(1), which produces the corresponding entire function \( \text{Ai}(z) \), \( z \in \mathbb{C} \). Furthermore, the Airy function is a solution of the differential equation
\[
d^2 y/dz^2 = z y
\]
as can be seen by direct substitution of (2) or (3) as a function of \( z \) in (4), or else by solving the latter with Laplace’s method [5]. Airy equation is a second-order, ordinary differential equation, the general solution of which is a linear combination of two linearly independent solutions that form its functional basis at all points in \( \mathbb{C} \). A second linearly independent solution to (4), known as the Airy function of the second kind, \( \text{Bi}(z) \), is chosen as
\[
\text{Bi}(z) = \frac{1}{2\pi i} \int_{C_{21}+C_{21}} du e^{uz-1/3u^3}.
\]

Any modified Bessel function of order \( \pm 1/3 \), \( Z_{\pm 1/3}(x) \), produces a solution of the Airy equation of the type \( \sqrt{x} Z_{\pm 1/3}(\xi x^{3/2}) \) if \( x > 0 \). Likewise, solutions to the classical Bessel differential equation of the same order, \( R_{\pm 1/3}(x) \), produce Airy equation solutions given by \( \sqrt{-x} R_{\pm 1/3}(\xi (-x)^{3/2}) \) if \( x < 0 \). In particular, \( \text{Ai}(x) \) can be written in terms of Bessel functions of order \( \pm 1/3 \) in \( \xi = \frac{2}{3} x^{3/2} \) as in [18]
\[
\text{Ai}(x) = \left[ I_{-1/3}(\xi) - I_{1/3}(\xi) \right] = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3}(\xi)
\]
\[
\text{Ai}(-x) = \left[ J_{-1/3}(\xi) + J_{1/3}(\xi) \right] = \sqrt{\frac{x}{3}} \text{Re} \left\{ e^{i\pi/6} H_{1/3}(\xi) \right\}
\]
for \( x > 0 \). Analytic continuation guarantees that these identities hold for a complex variable, namely \( z = x + iy \) in (6a) and \( z = -x + iy \) with \( x > 0 \) in (6b).

The role of Airy functions is prominent in the construction of uniform asymptotic expansions for contour integrals in the complex plane with coalescing saddle points [8, 21], as well as in solutions of linear second-order ordinary differential equations with a simple turning point [11]. A large number of applications have been developed in physics: almost in every instance for which wave equations with turning points are relevant [18].

### 2 Maclaurin and asymptotic power expansion of the Airy functions

The expansions of \( \text{Ai}(z) \) and \( \text{Bi}(z) \) near the origin are given by [5] \(^2\)
\[
\text{Ai}(z) = \frac{1}{3^{2/3}\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{3}\right)}{n!} \sin\left\{\frac{2}{3}(n + 1)\pi\right\} \left(3^{1/3}z\right)^n
\]
\[
\text{Bi}(z) = \frac{2}{3^{2/3}\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{3}\right)}{n!} \sin^2\left\{\frac{2}{3}(n + 1)\pi\right\} \left(3^{1/3}z\right)^n.
\]
\(^2\)Equation (8b) is not given in [5] or [18] but results of applying
\[
\text{Bi}(z) = e^{i\pi/6} \text{Ai}(ze^{i2\pi/3}) + e^{-i\pi/6} \text{Ai}(ze^{-i2\pi/3})
\]
to (8a).
These Maclaurin series converge for all finite values of $z$ but can be oscillating and slowly convergent. For large $|z|$, asymptotic expansions of $\text{Ai}(z)$ and $\text{Bi}(z)$ are preferred instead [6, 5, 13, 8, 21]. Asymptotic expansions are not convergent but, for a fixed number of terms, approach the exact function as the variable approaches some distinguished value that defines the asymptotic limit. In particular, $\text{Ai}(z)$ can be approximated by the following asymptotic power expansions [6, 5, 8] for large variable values:

$$
\text{Ai}(z) \sim F(z), \quad |z| \to \infty \quad (|\operatorname{Arg} z| < \pi) \quad (9a)
$$

$$
\text{Ai}(z) \sim F(z) + e^{i\pi/2 \operatorname{sign}(\operatorname{Arg} z)} G(z), \quad |z| \to \infty \quad \left(\frac{\pi}{3} < |\operatorname{Arg} z| < \frac{5\pi}{3}\right) \quad (9b)
$$

$$
F(z) = \frac{1}{2\pi z^{1/4}} e^{-2/3 z^{3/2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(3n + \frac{1}{2})}{3^{2n}(2n)!} z^{-3/2n} \quad (9c)
$$

$$
G(z) = \frac{1}{2\pi z^{1/4}} e^{-2/3 z^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(3n + \frac{1}{2})}{3^{2n}(2n)!} z^{-3/2n}. \quad (9d)
$$

Equation (9a) is often quoted as applicable to $|\operatorname{Arg} z| < \pi/3$ [6, 8] only. The series shown in (9a) is obtained by using Watson’s lemma, whereas the restriction given by $|\operatorname{Arg} z| < \pi/3$ results from the steepest descents method. The former is obviously more general and encompasses the latter. Stokes phenomenon is present in the asymptotic expressions above: the overlapping of the regions in (9) may seem to be inconsistent. The reason for such overlapping is twofold, on the one hand there is a branch cut implicit in $z^{3/2}$ at $\operatorname{Arg} z = \pi$ in (9a), whereas the branch cut is placed at $\operatorname{Arg} z = 0$ for (9b) \(^3\), and, on the other hand, $G(z)$ is subdominant to the asymptotic expansion in (9a) for $|\operatorname{Arg} z| \in (\pi/3, \pi)$ for the corresponding Riemann surface. For the case of $\operatorname{Arg} z = \pm \pi$, which is the anti-Stokes line of the turning point of (4), $z = 0$, we can write, for $x \in \mathbb{R}^+$ \(^4\),

$$
\text{Ai}(-x) \sim P(x) \sin \left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right) - Q(x) \cos \left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right) \quad (10a)
$$

$$
P(x) = \frac{1}{\pi z^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(6n + \frac{1}{2})}{3^{3n}(4n)!} x^{-3n} \quad (10b)
$$

$$
Q(x) = \frac{1}{\pi z^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(6n + \frac{5}{2})}{3^{3n+2}(4n+2)!} x^{-3n}. \quad (10c)
$$

Similar expressions are found for $\text{Bi}(z)$ [5] \(^5\),

$$
\text{Bi}(z) \sim 2G(z), \quad |z| \to \infty \quad (|\operatorname{Arg} z| < \frac{\pi}{3}) \quad (11a)
$$

$$
\text{Bi}(z) \sim 2G(z) + e^{i\pi/2 \operatorname{sign}(\operatorname{Arg} z)} F(z), \quad |z| \to \infty \quad \left(\frac{\pi}{3} < |\operatorname{Arg} z| \leq \pi\right) \quad (11b)
$$

$$
\text{Bi}(-x) \sim P(x) \cos \left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right) + Q(x) \sin \left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right), \quad x \in \mathbb{R}^+. \quad (11c)
$$

\(3\)It can be generally placed at any angle in the range $(-\pi/3, \pi/3)$

\(4\)\text{Ai}(-z)\text{admits an analog expression for }|z| \to \infty, |\operatorname{Arg} z| < 2/3\pi [5]. However, the dominant exponential in the trigonometric functions produce the expansion given in (9a).

\(5\)They can be obtained by using the identity in previous footnote 2.
In (10) and (11c) the oscillatory nature of the Airy functions for real negative values of the variable is revealed.

The asymptotic series must be truncated at some term to produce an acceptable value and its error is bound in magnitude by the first neglected term in absolute value multiplied by a certain factor [13]. There are also some exponentially-improved asymptotic series obtained by further expansion of the remainder term [12]. To overcome the inherent difficulties of a divergent series, techniques such as the Borel summation can be applied to produce a hyperasymptotic series [2, 3]. Notwithstanding the well spread acceptance of these schemes and their extensions, a convergent series expansions alternative to (8a) and (8b) is proposed in the present work. In section 3, the method of steepest descents for a real variable is reviewed in the manner needed in section 4 for producing the new series expansion. Such a new expansion is valid for complex variables and is written in terms of incomplete Gamma functions, as in the method of Hadamard series for steepest descents described in [14, 15]. Here, the integration path is divided in five sections in which the corresponding series present uniform convergence. Unlike in [14, 15], the expansion in two of them is done around the point of the path at infinity. Another difference is the use of a Puiseux series for two of the expansions.

3 Review of the asymptotic expansion of $\text{Ai}(x)$ of a real and positive variable by the method of steepest descents

For the purpose of the demonstration of Theorem 4.6, which is the main result of this work, we will use the already known method that produces the customary series given in (9) for $x \in \mathbb{R}^+$ by changing the integration variable in (1) through $\alpha = i x^{-1/2} u$,

$$
\text{Ai}(x) = \frac{x^{1/2}}{2\pi i} \int_{-i\infty}^{i\infty} d\alpha \, e^{\frac{x^{3/2}(\alpha-1/3\alpha^3)}{2}}.
$$

(12)

The analytic continuation of $\text{Ai}(x)$ to the complex domain is not treated in this section but will be included in the next one for the new series expansion. Equation (12) has the form

$$
I(\sigma) = \int_{C} d\alpha \, F(\alpha) \, e^{\sigma f(\alpha)}
$$

(13)

$$
\sigma = x^{3/2} \in \mathbb{R}^+
$$

(14)

and can be solved by applying Debye’s method of steepest descents [5]. For that purpose the argument of the exponential can be written in terms of the real and imaginary components of $\alpha$ as

$$
f(\alpha) = f_r(\alpha) + if_i(\alpha) = \alpha - \frac{\alpha^3}{3}
$$

$$
f_r(\alpha) = \alpha_r - \frac{\alpha^3}{3} + \alpha_r \alpha_i^2
$$

$$
f_i(\alpha) = \alpha_i + \frac{\alpha^3}{3} - \alpha_r^2 \alpha_i.
$$

(15)
Along the steepest descent path passing through the critical point \( \alpha_s = -1 \) \(^6\) and determined by \( f_i(\alpha) = f_i(\alpha_s) \), that is, by

\[
1 + \frac{\alpha_i^2}{3} - \alpha_r^2 = 0
\]

we can write

\[
f(\alpha) = f(\alpha_s) - s^2, \quad s \in \mathbb{R}
\]

which corresponds to

\[
\begin{align*}
    f_r(\alpha) &= f_r(\alpha_s) - s^2,
    f_i(\alpha) &= f_i(\alpha_s) = 0.
\end{align*}
\]

The change of variable to \( s \)-given in (17) as an implicit function- results in

\[
I(\sigma) = \exp[\sigma f(\alpha_s)] \int_{-\infty}^{\infty} \Phi(s) \exp[-\sigma s^2] ds
\]

where

\[
\Phi(s) = \frac{d\alpha}{ds} F[\alpha(s)].
\]

For the Airy’s integral, \( F[\alpha(s)] = 1 \) and

\[
\Phi(s) = \frac{d\alpha}{ds}.
\]

The function \( \alpha(s) \) can be written through its Maclaurin series with the help of the Lagrange’s inversion theorem \([20]\), that produces

\[
\alpha + 1 = \sum_{n=1}^{\infty} \frac{i^n}{n!} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{3^{n-1}} s^n
\]

where the radius of convergence is given by \(|s| \leq \rho = \frac{2}{\sqrt{3}}\). Uniform convergence in this same region, \(|s| \leq \rho\), follows from the application of Weierstrass’ M-test to (22) and the convergence of the upper bounding series

\[
\sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{n! \Gamma\left(\frac{n}{2}\right)} \frac{1}{3^{n-1}} \left(\frac{2}{\sqrt{3}}\right)^n = 2.
\]

Equation (17) is a complex algebraic curve that results in a single-valued function \( \alpha = \alpha(s) \) on a Riemann surface of three sheets and four branch points, namely \( s = 0, \pm 2i/\sqrt{3}, \infty \). It is usually not mentioned that the series in (22) expands around a branch point. This issue is discussed in the next section. The result of using (22) in (21) and then in (19) beyond such radius of convergence produces the following asymptotic approximation of \( I \) (cf. (9)),

\[
I(\sigma) \sim \exp[-2/3\sigma] \sum_{n=0}^{\infty} \frac{\Phi(2n)(0) \Gamma(n + 1/2)}{(2n)! \sigma^{n+1/2}}
\]

\(^6\)The equivalent saddle point for complex values of the variable of the Airy functions will be used in the next section, as far as the new developments presented here are concerned.
where only even derivatives of $\Phi$,

$$
\Phi^{(2n)}(0) = i^{2n+1} \frac{\Gamma\left(3n + \frac{1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} \frac{1}{3^{2n}}
$$

are present due to

$$
\int_{-\infty}^{\infty} s^{2n+1} \exp[-\sigma s^2] \, ds = 0
$$

$$
\int_{-\infty}^{\infty} s^{2n} \exp[-\sigma s^2] \, ds = \frac{\Gamma(n + 1/2)}{\sigma^{n+1/2}}
$$

with $n \in \mathbb{N}$ and $\sigma \in \mathbb{R}^+$. The lack of convergence of (22) in the domain of integration of the Airy’s integral leads to the lack of convergence in (24). This use of the series beyond its convergence limits is a technique often used in asymptotics, as long as the series can be truncated with a known error bound. The method of steepest descents is not the only manner to obtain (24) with (25): Watson’s lemma [5] can be applied on the first integral expression for the Airy function in (1) with the Maclaurin series for the cosine 7.

However, the method of steepest descents is the starting point for the development presented in the following section.

4 A convergent series expansion for $\text{Ai}(z)$ of complex variable by the method of steepest descents

For the case of complex arguments in (2), the change of variable performed in the previous section is replaced by $\alpha = i \, |z|^{-1/2} u$ where $z$ is the complex variable of the Airy function. Thus,

$$
\text{Ai}(z) = \frac{|z|^{1/2}}{2\pi i} \int_{-i\infty}^{i\infty} \, d\alpha \, e^{\left[3/2 |w\alpha - 1/3\alpha^3|\right]} \\
= e^{i\varphi} \int_{-i\infty}^{i\infty} \, d\alpha \, e^{\left[3/2 (w\alpha - 1/3\alpha^3)\right]} \\
\varphi = \text{arg } z
$$

where $\text{arg } z$ is the principal value of the argument of $z$. The integration path can be deformed to a path of the type $L_{32}$ or $L_{31} + L_{12}$ in Figure 1, or any equivalent homotopic curve.

Aiming at the integration along the steepest descent path, we now compute the saddle points of the argument of the exponential function in (27) as a function of $\alpha$,

$$
f(w, \alpha) = f_r(w, \alpha) + if_i(w, \alpha) = w\alpha - \frac{\alpha^3}{3} + \alpha_i \alpha_r
$$

$$
f_r(w, \alpha) = w_r \alpha_r - w_i \alpha_i - \frac{\alpha^3}{3} + \alpha_r \alpha_i^2
$$

$$
f_i(w, \alpha) = w_r \alpha_i + w_i \alpha_r + \frac{\alpha^3}{3} - \alpha_r^2 \alpha_i
$$

7 The cosine series is convergent, unlike (22), for the whole integration domain. However, it is not uniformly convergent and therefore, if integrated term by term, does not produce a convergent series.
where \( w = w_r + iw_i \) and \( \alpha = \alpha_r + i\alpha_i \). These saddle points are given by \( \alpha_s = \pm w^{1/2} \). The steepest descent path is homotopic to \( \mathcal{L}_{32} \) and crosses \( \alpha_s = -w^{1/2} \). By setting

\[
 f(w, \alpha) = f(w, \alpha_s = -w^{1/2}) - s^2 = -\frac{2}{3} w^{3/2} - s^2, \quad s \in \mathbb{R} \quad (29)
\]

so that

\[
 f_r(w, \alpha) = f_r(w, \alpha_s) - s^2 = -\frac{2}{3} \cos \frac{3}{2} \varphi - s^2 \\
 f_i(w, \alpha) = f_i(w, \alpha_s) = -\frac{2}{3} \sin \frac{3}{2} \varphi \quad (30)
\]

we can define the steepest descent path in terms of the parameter \( s \).

Different steepest paths are shown in Figure 6 for different values of \( \arg z \).

The presence of branch points in the algebraic curve given by (29) precludes the analyticity of \( \alpha = \alpha(s) \) in the whole path. The method developed by R. B. Paris in [14, 15, 16] consists in a non-systematic splitting of the integration wherein a large number of segments is produced to achieve analytic continuation. This results in a sum of series containing lower incomplete Gamma functions that Paris names Hadamard expansions. The same idea of dividing the steepest descent path in a set of segments is used in what follows, but with a criterion of bordering the branch points at a certain distance so that analytic continuation is optimally performed along such an integration path. In one case a branch point is found to lie on the steepest descent path or very close to it and a Puiseux series is used\(^8\). In addition to this, expansions around infinity are carried out so that a infinite number of segments as in previously studied Hadamard expansions is avoided. The positions of the branch points is indicated in the following remark.

**Remark 1.** The complex algebraic curve defined by (29) has three branch points in the finite domain of the Riemann surface, whose positions can be dealt with to provide analytic continuation of the multi-valued function \( \alpha = \alpha(s) \). These branch points are readily computed by seeking the roots of the corresponding discriminant and are found at \( s = 0, \pm \frac{2w^{3/2}}{\sqrt{3}} \). The branch point at \( s = 0 \) is of order two but the other ones are simple branch points. In addition to them, the infinity is a branch point of order four.

As a straightforward consequence of Cardano’s formulas for the reduced cubic, the three branches correspond to the following solutions

\[
\begin{align*}
\alpha_1(s) &= \frac{\xi(s, w)}{2^{1/3}} + \frac{2^{1/3} w}{\xi(s, w)} \\
\alpha_2(s) &= e^{i2\pi/3} \frac{\xi(s, w)}{2^{1/3}} + \frac{2^{1/3} e^{-i2\pi/3} w}{\xi(s, w)} \\
\alpha_3(s) &= e^{-i2\pi/3} \frac{\xi(s, w)}{2^{1/3}} + \frac{2^{1/3} e^{i2\pi/3} w}{\xi(s, w)} \\
\xi(s, w) &= [3 s^2 + 2 w^{3/2} + \sqrt{3} \mu(s, w)]^{1/3} \\
\mu(s, w) &= \sqrt{3} s^4 + 4 w^{3/2} s^2.
\end{align*}
\]

\(^8\)In fact, \( s = 0 \) is also a branch point of the algebraic curve defined by (28) and (29), as explained later in footnote\(^{10}\).
Figure 2: Sheets of the Riemann surface for the solutions in Remark 1. a) The three-sheeted Riemann surface has three finite branch points and the branch point at infinity. b) Detail of the branch point at $s = 0$. The series expansion in Lemma 4.1 runs from negative $s$ values on sheet 3, which corresponds to images in the branch given by $\alpha_3(s)$, to positive values on sheet 2, which has its image values in the branch given by $\alpha_2(s)$, as indicated by the dotted line. If the minus sign were selected in equation (36), then the sheets would be traversed in the opposite direction (dashed line). The arrows indicate the fact that the branches are expanded around the central point $s = 0$ and not the eventual direction of the integration through a path, which we will take from $s = -\infty$ to $s = \infty$. 

$$s = -\frac{2iw^{3/4}}{\sqrt{3}} \quad s = 0 \quad s = \frac{2iw^{3/4}}{\sqrt{3}} \quad s = \infty$$
For real values of $s$, functions $\alpha_2(s)$ and $\alpha_3(s)$ define the steepest-descent path through $\alpha_s = -w^{1/2}$. The three sheets of the single-valued description of the algebraic curve (29) are depicted in Figure 2a.

Considering these branch points and the convergence disks that they allow, the steepest descent path will be split in five segments, defined by the points around which each series expansion is computed: $s = 0, \pm \frac{2}{\sqrt{3}}, \infty$.

An extension of (22) for $\alpha_s = -\frac{w}{2}$ is provided by the following Lemma, which will apply to the computation of the central section of the steepest descent path crossing the saddle point at $\alpha_s$.

**Lemma 4.1.** The solution of $\alpha = \alpha(s)$ around $\alpha_s = -\frac{w}{2}$ in the equation

$$w\alpha - \frac{\alpha^3}{3} = -\frac{2}{3} w^{3/2} - s^2$$

is given by the series

$$\alpha + w^{1/2} = \sum_{n=1}^{\infty} \frac{i^n \Gamma \left( \frac{3n}{2} - 1 \right)}{n! \Gamma \left( \frac{5}{2} \right)} w^{-3n/4 + 1/2} \frac{1}{3n-1} s^n$$

which is uniformly convergent for $|s| \leq 2/\sqrt{3}$.

**Proof.** The power series of $\alpha$ as a function of $s$ can be calculated from the function $s = h(\alpha)$ by applying Lagrange’s inverse theorem. Thus [20],

$$\alpha = \alpha_s + \sum_{n=1}^{\infty} g_n \frac{[s - h(\alpha_s)]^n}{n!}$$

where

$$g_n = \lim_{\alpha \to \alpha_s} \left[ \frac{d^{n-1}}{d\alpha^{n-1}} \left( \frac{\alpha - \alpha_s}{h(\alpha) - h(\alpha_s)} \right) \right]^{n}.$$  

(35)

The computation of (35) for $h(\alpha) = \pm i(\alpha + w^{1/2})\sqrt{w^{1/2} - \frac{1}{3}(\alpha + w^{1/2})}$ that defines the steepest descent path through $\alpha_s = -w^{1/2}$ produces the result

$$g_n = (\mp i)^n \frac{\Gamma \left( \frac{3n}{2} - 1 \right)}{\Gamma \left( \frac{5}{2} \right)} \frac{1}{3 \cdot 3n-1} w^{-3n/4 + 1/2}.$$  

(36)

The positive sign will be chosen for the integration between $s = -\infty$ to $s = \infty$ to follow the integration path sense shown in Figure 1. Indeed, the choice of a positive sign in (36) implies that $s < 0$ for the lower half-plane part of the contour $\mathcal{L}_32$ and $s > 0$ for its upper half-plane part. Choosing the negative sign would reverse this correspondence. This is shown in Figure 2b, together with the fact that this series expands around the ramification point at $s = 0$ reaching over $\alpha_2(s)$ and $\alpha_3(s)$. Hence, equation (34) becomes

$$\alpha + w^{1/2} = \sum_{n=1}^{\infty} \frac{i^n \Gamma \left( \frac{3n}{2} - 1 \right)}{n! \Gamma \left( \frac{5}{2} \right)} w^{-3n/4 + 1/2} \frac{1}{3n-1} s^n.$$  

(37)

9The application of Taylor’s theorem directly to the function $\alpha = h^{-1}(s)$ as described in equations (31) is more cumbersome.

10
Equation (37) leads to equation (22) when \( w = 1 \). The upper bounding series in (23) can also be used for the Weierstrass’ M-test on the uniform convergence of (37) in the disk \( |s| \leq \frac{2}{\sqrt{3}} \).

The term-by-term derivative of \( \alpha \) with respect to \( s \) in the series (33) produces a uniform convergent series on every compact subset of the disk \( |s| < \frac{2}{\sqrt{3}} \) that does not reach its boundary [7, pp.326-28, v.I]. Therefore, when integrating the differentiated series we must do it in compact subsets of the path section defined by \( s \in (-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \), which forms the central section of the integration path. The next two integration sections beyond this one are constructed to use the power expansions around \( s = \pm 2/\sqrt{3} \). For the case of \( \arg z = 2\pi/3 \), the point at \( s = 2/\sqrt{3} \) corresponds to a branch point and a Puiseux series is obtained. Indeed, this would have been also the case in Lemma 4.1 should \( s \) have been used instead of \( s^2 \) in equation (32) 10. Furthermore, when \( arg z \) approaches \( 2\pi/3 \), the radius of convergence of the power series around \( s = 2/\sqrt{3} \) tends to zero and the expansion must be made around the branch point outside the path for better convergence. With the next two lemmas, the necessary series for representing the path in the neighborhoods of \( s = \pm 2/\sqrt{3} \) are given, after replacing the term \( s^2 \) by \( \frac{4}{3} + t \) in (32) so that the concerned expansion is done around \( t = 0 \).

**Lemma 4.2.** The power series of \( \alpha = \alpha(t) \) around \( t = 0 \) for the solution of the equation

\[
w_\alpha - \frac{\alpha^3}{3} = -\frac{2}{3}w^{3/2} - \frac{4}{3} - t
\]

for the case of \( w \neq e^{i2\pi/3} \) is given by \( \alpha^+ \) and \( \alpha^- \) 11 as below

\[
\alpha^\pm = \alpha_0^\pm + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} a^n (\alpha_0^\pm - \alpha_0^\mp)^{-2n+1} (\alpha_0^\pm - \alpha_0^-)^{-n} \sum_{k=0}^{n-1} \binom{n-1}{k}
\times \frac{\Gamma(2n+1-k)}{\Gamma(n)} \frac{\Gamma(n+k)}{\Gamma(n)} (\alpha_0^\pm - \alpha_0^-)^k (\alpha_0^\mp - \alpha_0^-)^{-k}
\]

where

\[
\alpha_0 = \frac{w}{\left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}} + \left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}
\]

\[
\alpha_0^+ = -\frac{(1 + i\sqrt{3})w}{2 \left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}} - \frac{1}{2} (1 - i\sqrt{3}) \left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}
\]

\[
\alpha_0^- = -\frac{(1 - i\sqrt{3})w}{2 \left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}} - \frac{1}{2} (1 + i\sqrt{3}) \left(2 + w^{3/2} + 2\sqrt{1 + w^{3/2}}\right)^{1/3}
\]

---

10The choice of \( s^2 \) instead of \( s \) in (32) is motivated by the fact that two sheets result from the branching at \( s = 0 \), and it is also convenient for the steepest descents method. The series in (33) is thus sensitive to the sign of \( s \) although the algebraic curve in (32) is not. Had \( s \) been used instead of \( s^2 \), this expansion would have taken the shape of a Puiseux series.

11The superscript + indicates henceforth that the expansion for \( \alpha \) is computed around a point in the second quadrant of the complex plane, where as the superscript – does so for the third quadrant. The same stands for the roots \( \alpha_0^+ \) and \( \alpha_0^- \).
\[ t = -\frac{4}{3} \left(1 + w^{3/2}\right) \quad t = 0 \quad t = -\frac{4}{3} \quad t = \infty \]

Figure 3: Sheets of the Riemann surface for the complex algebraic curve given by (38) and \( \arg z > 0 \). a) This is the case for \( w \neq e^{i\pi/3} \). The series in Lemma 4.2 are expansions around points \( \alpha_0^+ \) and \( \alpha_0^- \). The radius of the convergence disks are given by the distance to the branch points at \( t = -\frac{4}{3} \) and \( t = -\frac{4}{3} \left(1 + w^{3/2}\right) \). b) For the case \( w = e^{i\pi/3} \), \( \alpha_0 = \alpha_0^+ \) is a branch point and we need a Puiseux series around it. The radius of convergence is given then by \( \rho = \frac{4}{3} \). If we were to deal with the case of \( \arg z > 0 \), the labeling of the sheets would change to follow the corresponding branches, but not the topology.

The radius of convergence is given by
\[
\begin{align*}
 r^+ & - \text{sign} \{ \arg z \} = 4 \left( \frac{4}{3}, \frac{4}{3} |1 + w^{3/2}| \right) \\
 r^- & - \text{sign} \{ \arg z \} = \min \left( \frac{4}{3}, \frac{4}{3} |1 + w^{3/2}| \right) \\
\end{align*}
\]

for the corresponding superscripted \( \alpha \) series in (39).

**Proof.** Equation (38) can be seen as \( t = h(\alpha) \) where, as in Lemma 4.1, Lagrange’s inverse theorem can be applied. Thus, we write
\[
 h(\alpha) = \frac{1}{3} (\alpha - \alpha_0)(\alpha - \alpha_0^+)(\alpha - \alpha_0^-) 
\]

where \( \alpha_0, \alpha_0^+ \) and \( \alpha_0^- \) are given in (40). If \( w \neq e^{i\pi/3} \), the three roots are distinct. By applying equations (34) and (35) and the Leibniz rule for product differentiation, we obtain
\[
 g_n = (-1)^{n-1} \frac{4n}{3} \left( \begin{array}{c} 2n - 1 \\ k \end{array} \right) \frac{\Gamma(2n - 1 - k)}{\Gamma(n)} \\
 \times \frac{\Gamma(n + k)}{\Gamma(n)} (\alpha_0^+ - \alpha_0^-)^{-2n+k+1} (\alpha_0^+ - \alpha_0^-)^{-n-k} 
\]

This results in (39). The branch points of (38) are \( t = -4/3, -4/3 \left(1 + w^{3/2}\right), \infty \). As there are no singular points in (38), and by considering Figure 3a, it is straightforward to find that the convergence radius of (39) is \( r^+ \text{sign} \{ \arg z \} = \min \left( \frac{4}{3}, \frac{4}{3} |1 + w^{3/2}| \right) \) and \( r^- \text{sign} \{ \arg z \} = \frac{4}{3} \) for \( \alpha^+ \) and \( \alpha^- \).

If \( w = e^{i\pi/3} \), the roots in (40) are \( \alpha_0 = \alpha_0^+ = \alpha_0^+ = e^{i\pi/3}, \alpha_0^- = -2e^{i\pi/3}, \) and \( t = 0 \) becomes a branch point of the algebraic curve in (38). The following Lemma deals with this case.
Lemma 4.3. The power series of $\alpha = \alpha(t)$ around $t = 0$ for equation (38) and $w = e^{i2\pi/3}$ is given by $\alpha^+$ and $\alpha^-$ as

$$
\alpha^+ = \alpha_0^+ + \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n!} 3^{n/2} \frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{3}{2})} (\alpha_0^+ - \alpha_0^-)^{-3n/2+1} t^{n/2} \big|_b \tag{44a}
$$

$$
\alpha^- = \alpha_0^- + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} 3^n \frac{\Gamma(3n - 1)}{\Gamma(2n)} (\alpha_0^- - \alpha_0^+)^{-3n+1} t^n \tag{44b}
$$

with $\alpha_0^+ = e^{i\pi/3}$ and $\alpha_0^- = -2 e^{i\pi/3}$ and where $|_b$ indicates the branch. The radius of convergence is given by $|t| < \rho = \frac{4}{3}$. For the case of $w = e^{-i2\pi/3}$, the superscripts $\pm$ must be interchanged in both (44a) and (44b) as well as taking $\alpha_0^+ = -2 e^{-i\pi/3}$ and $\alpha_0^- = e^{-i\pi/3}$. The branch noted by $|_b$ is that of $(-1)^{1/2} = \pm i$ for $w = e^{\pm i2\pi/3}$.

**Proof.** The solutions of $h(\alpha) = 0$ as defined in Lemma 4.2 and for $w = e^{i\varphi}$ where $|\varphi| \leq 2\pi/3$ are all simple roots of $h(\alpha)$ except for the case of $\varphi = \pm 2\pi/3$. In the case of $\varphi = 2\pi/3$, we have

$$
h(\alpha) = \frac{1}{3} (\alpha - \alpha_0^+)^2 (\alpha - \alpha_0^-) \tag{45}
$$

with $\alpha_0^+ = e^{i\pi/3}$ and $\alpha_0^- = -2 e^{i\pi/3}$. For $\alpha_0^+ = e^{i\pi/3}$, $t = 0$ is a branch point of order 1, and it is convenient for an expansion around $\alpha_0^+$ to write $t = h(\alpha)$ as

$$
t^{1/2} = -\frac{1}{\sqrt{3}} (\alpha - \alpha_0^+) \sqrt{\alpha - \alpha_0^-} \tag{46}
$$

where the minus sign has been selected as the branch to integrate within so that we remain at the steepest descent path. When $t < 0$, the branch for the square root function in (46) is that of $+i$ as $\alpha_0^+$ is in the second quadrant. Applying the Lagrange inversion theorem for a power expansion around $\alpha_0^+$ in terms of $t^{1/2}$, we obtain

$$
g_n = (-1)^{2n-1} 3^{n/2} \frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{3}{2})} (\alpha_0^+ - \alpha_0^-)^{-3n/2+1} \tag{47}
$$

and the Puiseux series in equation (44a) results from it.

If the power expansion is done around $\alpha_0^-$, instead of (46) we start from

$$
t = \frac{1}{3} (\alpha - \alpha_0^+)^2 (\alpha - \alpha_0^-) \tag{48}
$$

to produce

$$
g_n = (-1)^{n-1} 3^n \frac{\Gamma(3n - 1)}{\Gamma(2n)} (\alpha_0^- - \alpha_0^+)^{-3n+1} \tag{49}
$$

and thus obtain the expansion in (44b). In this case, we have an expansion in powers of $t$ that are positive integers.

As shown in Figure 3b, there is a sheet of the complex values of $t$ where $t = 0$ is not a branch point and that corresponds to the branch where $\alpha_0^- = \alpha(t = 0)$, whereas there is another one that contains both $t = 0$ and $t = 4/3$ as branch points.
Figure 4: Integration segments. The regions of convergence for the series expansions of Lemmas 4.1 to 4.3 are represented in the above part of the figure. The bottom part shows the selection of segments for the integration segments as they are defined to produce the addends in equation (63).
and where $\alpha_0^+ = \alpha(t = 0)$. The series in (44a) and (44b) correspond, respectively, to the power expansions around $t = 0$ on both sheets.

For the case of $\varphi = -2\pi/3$, the series for $\alpha^+$ and $\alpha^-$ must be interchanged as well as the values of $\alpha_0^+$ and $\alpha_0^-$, which must be also replaced by their conjugates. The branch for $t^{n/2}$ is such that $t < 0$ produce negative imaginary values for odd values of $n$.

When the distance of the branch point at $t = -4/3 (1 + w^{3/2})$ to $t = 0$ is equal to $2/3$, the circumference centred at this point crosses the $t$ axis at $t = -2/3$ and $t = 1/3$ (see Figure 5). As $\arg z$ approaches $\pm 2\pi/3$, the branch point approaches $t = 0$ and it is more convenient to expand $\alpha = \alpha(t)$ around the branch point for the segment $t \in [-1/3, 1/3]$ than using the expansions of Lemma 4.3.

**Lemma 4.4.** The power series of $\alpha = \alpha(t)$ around $t = -4/3 (1 + w^{3/2})$ for equation (38) and $\arg z > 0$ is given by equation (44a) with $t + 4/3 (1 + w^{3/2})$ replacing $t$ and with $\alpha_0^+ = w^{1/2}$ and $\alpha_0^- = -2 w^{1/2}$. As in Lemma 4.3, the superscripts $+$ must be replaced by $-$ if $\arg z < 0$, so that the series analogous to equation (44a) is now for $\alpha^-$ and the binomial inside is $(\alpha_0^- - \alpha_0^+)$ instead of $(\alpha_0^- - \alpha_0^-)$. The radius of convergence is given by $\rho = 4/3$.

**Proof:** The demonstration is analogous to the one of Lemma 4.3. The radius of convergence of the resulting series around $t = -4/3 (1 + w^{3/2})$ is $\rho = 4/3$, as can be easily seen in Figure 3a). A more detailed view of the convergence disk is given in Figure 5:

The convergence disks of the expansions around $t = 0, -4/3, -4/3 (1 + w^{3/2})$ with $w_b = e^{i\varphi_b}$ and $\varphi_b = 2/3 \arctan(-7/8)$ are shown above. The expansion around $t = -4/3 (s = 0)$ corresponds to the expansion in Lemma 4.1 and has a convergent disk of radius $\rho = 4/3$. The expansion around $t = 0$ has a decreasing radius for its convergent disk as $\varphi \to 2\pi/3$ due to the approach of the branch point at $t = -4/3 (1 + w^{3/2})$. When $2/3 \arctan(-7/8) \leq |\varphi| < 2\pi/3$, the series expansion is done around the branch point instead of around $t = 0$ as described in Lemma 4.4.

Figure 5: The convergence disks of the expansions around $t = 0, -4/3, -4/3 (1 + w^{3/2})$ with $w_b = e^{i\varphi_b}$ and $\varphi_b = 2/3 \arctan(-7/8)$ are shown above. The expansion around $t = -4/3 (s = 0)$ corresponds to the expansion in Lemma 4.1 and has a convergent disk of radius $\rho = 4/3$. The expansion around $t = 0$ has a decreasing radius for its convergent disk as $\varphi \to 2\pi/3$ due to the approach of the branch point at $t = -4/3 (1 + w^{3/2})$. When $2/3 \arctan(-7/8) \leq |\varphi| < 2\pi/3$, the series expansion is done around the branch point instead of around $t = 0$ as described in Lemma 4.4.

and where $\alpha_0^+ = \alpha(t = 0)$. The series in (44a) and (44b) correspond, respectively, to the power expansions around $t = 0$ on both sheets.

For the case of $\varphi = -2\pi/3$, the series for $\alpha^+$ and $\alpha^-$ must be interchanged as well as the values of $\alpha_0^+$ and $\alpha_0^-$, which must be also replaced by their conjugates. The branch for $t^{n/2}$ is such that $t < 0$ produce negative imaginary values for odd values of $n$.

When the distance of the branch point at $t = -4/3 (1 + w^{3/2})$ to $t = 0$ is equal to $2/3$, the circumference centred at this point crosses the $t$ axis at $t = -2/3$ and $t = 1/3$ (see Figure 5). As $\arg z$ approaches $\pm 2\pi/3$, the branch point approaches $t = 0$ and it is more convenient to expand $\alpha = \alpha(t)$ around the branch point for the segment $t \in [-1/3, 1/3]$ than using the expansions of Lemma 4.3.

**Lemma 4.4.** The power series of $\alpha = \alpha(t)$ around $t = -4/3 (1 + w^{3/2})$ for equation (38) and $\arg z > 0$ is given by equation (44a) with $t + 4/3 (1 + w^{3/2})$ replacing $t$ and with $\alpha_0^+ = w^{1/2}$ and $\alpha_0^- = -2 w^{1/2}$. As in Lemma 4.3, the superscripts $+$ must be replaced by $-$ if $\arg z < 0$, so that the series analogous to equation (44a) is now for $\alpha^-$ and the binomial inside is $(\alpha_0^- - \alpha_0^+)$ instead of $(\alpha_0^- - \alpha_0^-)$. The radius of convergence is given by $\rho = 4/3$.

**Proof:** The demonstration is analogous to the one of Lemma 4.3. The radius of convergence of the resulting series around $t = -4/3 (1 + w^{3/2})$ is $\rho = 4/3$, as can be easily seen in Figure 3a). A more detailed view of the convergence disk is given in Figure 5:

The convergence disks of the expansions around $t = 0, -4/3, -4/3 (1 + w^{3/2})$ with $w_b = e^{i\varphi_b}$ and $\varphi_b = 2/3 \arctan(-7/8)$ are shown above. The expansion around $t = -4/3 (s = 0)$ corresponds to the expansion in Lemma 4.1 and has a convergent disk of radius $\rho = 4/3$. The expansion around $t = 0$ has a decreasing radius for its convergent disk as $\varphi \to 2\pi/3$ due to the approach of the branch point at $t = -4/3 (1 + w^{3/2})$. When $2/3 \arctan(-7/8) \leq |\varphi| < 2\pi/3$, the series expansion is done around the branch point instead of around $t = 0$ as described in Lemma 4.4.
Figure 5. An expansion around \( t = 0 \) has a decreasing radius as \(-4/3 (1 + w^{3/2}) \rightarrow 0\) and therefore it is preferable to expand around \( t = -4/3 (1 + w^{3/2}) \) for \( \varphi \) in \( w = e^{i \varphi} \) such that \( \varphi_b \equiv \frac{2}{3} \arctan(-7/8) \leq |\varphi| < \frac{2 \pi}{3} \). In effect, when \( \varphi = \varphi_b \), a circumference of radius \( 1/2 \rho \) centred at \( t = -4/3 (1 + w^{3/2}) \) crosses the real axis of the complex plane for \( t \) at \( t = -2/3 \) and \( t = 1/3 \). If we now consider the angular range given by \( \varphi_b \equiv \frac{2}{3} \arctan(-7/8) \leq |\varphi| < \frac{2 \pi}{3} \), it can be readily seen in Figure 5 that the segment \( t \in [-1/3, 1/3] \) makes use of half the convergence radius at most.

With the preceding four lemmas we have computed series expansions that can be used to integrate the path sections \( \mathcal{L}_{\alpha_n^-}, \mathcal{L}_s \) and \( \mathcal{L}_{\alpha_n^+} \) in Figure 4. For the purpose of integrating along the section of the path that connects the convergence disks of the series in Lemmas 4.3 and 4.4 with \( s = \infty \) in (32), a new variable \( \hat{s} = 1/s^{2/3} \) is introduced,

\[
w\alpha - \frac{\alpha^3}{3} = -\frac{2}{3} w^{3/2} - \frac{1}{\hat{s}^3}, \quad (50)
\]

The resulting complex algebraic curve is analyzed in the following Remark, and the series expansions for \( \alpha = \hat{\alpha}(\hat{s}) \) around \( \hat{s} \) is computed in the subsequent Lemma.

**Remark 2.** The three solutions of (50) in terms of \( \hat{s} \) are given by

\[
\begin{align*}
\hat{\alpha}_1(\hat{s}) &= \chi(\hat{s} \sqrt{w}) \frac{\sqrt[3]{2} \hat{s}}{\hat{s}} + \frac{\sqrt[6]{2} \hat{s} w}{\chi(\hat{s} \sqrt{w})}, \\
\hat{\alpha}_2(\hat{s}) &= e^{i 2\pi/3} \chi(\hat{s} \sqrt{w}) \frac{2^{1/3} \hat{s}}{\hat{s}} + \frac{2^{1/3} e^{-i 2\pi/3} \hat{s} w}{\chi(\hat{s} \sqrt{w})}, \\
\hat{\alpha}_3(\hat{s}) &= e^{-i 2\pi/3} \chi(\hat{s} \sqrt{w}) \frac{2^{1/3} \hat{s}}{\hat{s}} + \frac{2^{1/3} e^{i 2\pi/3} \hat{s} w}{\chi(\hat{s} \sqrt{w})}.
\end{align*}
\]

\[
\chi(t) = [3 + 2 t^3 + \sqrt{3} \eta(t)]^{1/3} \quad \eta(t) = \sqrt{3} + 4 t^3.
\]

(51)

Solutions \( \hat{\alpha}_1(\hat{s}) \) and \( \hat{\alpha}_2(\hat{s}) \) define the integration paths \( \mathcal{L}_{-\infty} \) and \( \mathcal{L}_{\infty} \) for the case of \( |\arg w| = |\arg z| \leq 2 \pi/3 \), respectively.

Contrary to the case seen in the previous Lemmas, Taylor’s theorem is preferred over Lagrange inversion theorem for the computation of (21) with (51). To perform the required derivatives, functional composition must be used throughout the whole derivation. In particular, Faà di Bruno’s formula expressed in terms of Bell polynomials \( B_{n,k} \) is recursively used to deal with the functional form of the solutions in (51).

Taylor’s theorem is to be applied separately to each addend of the expressions of \( \hat{\alpha}_i(\hat{s}), i = 1, 2, 3 \), and therefore the branch points that limit the radii of convergence are not the ones corresponding to the complex algebraic curve in (50) but to the one of which they are solutions. It is straightforward to see that such a curve is

\[
\hat{s}^3 z^2 - \left( 2 w^{3/2} \hat{s}^3 - 3 \right) z + s^3 w^3 = 0. \quad (52)
\]

Its branch points are found in \( \hat{s} = \zeta, e^{i2\pi/3} \zeta \) and \( e^{-i2\pi/3} \zeta \) with \( \zeta = \left( \frac{2}{3} \right)^{1/3} w^{-1/2} \). There are only two sheets in \( z = z(\hat{s}) \) for (52) and all the branch points connect
Figure 6: Steepest-descent integration paths for different phases of the $z$ variable. The background colour of the complex plane corresponds to the value of the real part of the argument of the exponential as given in equation (28), $f_r$: the white-to-black scale maps to higher-to-lower values of such argument. The points around which the functions $\alpha(s)$ and $\alpha(t)$ are expanded appear with a black dot. Five sections are distinguished in each integration path when $|\arg z| \leq 2\pi/3$. If $|\arg z| > 2\pi/3$, the steepest descent paths are homotopic to $L_{31}$ and $L_{12}$ in Figure 1.
Figure 6: (Cont.) Steepest-descent integration paths for different phases of the $z$ variable. The background colour of the complex plane corresponds to the value of the real part of the argument of the exponential as given in equation (28), $f$: the white-to-black scale maps to higher-to-lower values of such argument. The points around which the functions $\alpha(s)$ and $\alpha(t)$ are expanded appear with a black dot. Five sections are distinguished in each integration path when $|\arg z| \leq 2\pi/3$. If $|\arg z| > 2\pi/3$, the steepest descent paths are homotopic to $L_{31}$ and $L_{12}$ in Figure 1.

them. Therefore, the power expansions that will result from the application of Taylor’s theorem around $\hat{s} = 0$ will have radii of convergence given by $\rho = \left(\frac{3}{4}\right)^{1/3}$.

**Lemma 4.5.** The $n$-th derivative of $\chi$ at $t = 0$ is given for $n \neq 0$ by

$$
\chi^{(n)}(0) = \begin{cases} 
6^{\frac{n}{3}} \sum_{k=1}^{n/3} \sum_{i=0}^{k-i} \sum_{l=0}^{k-i} (-1)^{k-i-l} \frac{k! \cdot i! \cdot l!}{3^i i! (3-i)! (k-i-l)!} \times \left(\frac{4}{3}\right)_{k} h_{\frac{2}{3}-i} \left(\frac{4}{3}\right) & \text{if } n \equiv 0 \pmod{3} \\
0 & \text{otherwise}
\end{cases} 
$$

(53)

where

$$
h_p(x) = \left(\frac{4}{3}\right)_p (x)_p 
$$

(54)

and $(x)_p$ is the falling factorial of $x$ of order $p$.

**Proof.** The following expression is obtained for the $n$-th derivative of $\chi$ at $t = 0$ after using Faà di Bruno’s formula expressed in terms of Bell polynomials $B_{n,k}$.

$$
\chi^{(n)}(0) = 6^{\frac{n}{3}} \sum_{k=1}^{n} \left(\frac{1}{3}\right)_k \frac{1}{k^2} \times B_{n,k}(0,0,12 + \sqrt{3} \eta^{(3)}(0), 0, 0, \sqrt{3} \eta^{(6)}(0), 0, \ldots)
$$

(55)
expression can be simplified to

\[ \eta^{(3m)}(0) = \sqrt{3}\left(\frac{1}{2}\right)_m \frac{2^{2m} (3m)^{2m}}{3^m} \]  

(56)

and \((x)_n = x(x-1)\cdots(x-n+1)\). By using Bell’s polynomial definition, this expression can be simplified to

\[
B_{n,k}(0,0,\sqrt{3}\eta^{(3)},0,0,\sqrt{3}\eta^{(6)},0,0,\sqrt{3}\eta^{(9)},\ldots) = 3^k (n)_{2n/3}
\]

\[
\times \left\{ \begin{array}{ll}
B_{n/3,k}(h_1(\frac{1}{2}),h_2(\frac{1}{2}),h_3(\frac{1}{2}),\ldots,h_{n/3-k+1}(\frac{1}{2})) & \text{if } n \equiv 0 \pmod{3} \\
0 & \text{otherwise}
\end{array} \right.
\]  

(57)

with

\[ h_p(x) = \left(\frac{4}{3}\right)^p (x)_p \]  

(58)

forming a binomial sequence [9]. Equation (53) is obtained, after some manipulation, by applying the following two properties to (57):

- For a binomial sequence \(\{\varphi_n(x)\}\) and all integers \(m, k \geq 0\), we have [9]

\[
B_{m,k}(\varphi_1(x),\varphi_2(x),\varphi_3(x),\ldots) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j! (k-j)!} \varphi_m(j \frac{1}{2})
\]

(59)

- Sums in the variables of Bell’s polynomials can be written as [4]

\[
B_{m,k}(y_1 + y'_1, y_2 + y'_2, y_3 + y'_3, \ldots) = \sum_{i \leq m} \sum_{j \leq k} \binom{m}{i} B_{i,j}(y_1, y_2, y_3, \ldots) B_{m-i,k-j}(y'_1, y'_2, y'_3, \ldots)
\]

(60)

\[ \square \]

**Corollary 1.** The functions \(\alpha_n(\hat{s})\) \((n = 2, 3)\) can be written as the following power series

\[
\hat{\alpha}_2 = -\frac{1 - i\sqrt{3}}{2^{4/3}} \sum_{m=0}^{\infty} \frac{a_{3m}}{(3m)!} w^{3m/2} s^{3m-1} - \frac{1 + i\sqrt{3}}{2^{2/3}} \sum_{m=0}^{\infty} \frac{b_{3m}}{(3m)!} w^{3m/2} s^{3m+1}
\]

\[
\hat{\alpha}_3 = -\frac{1 + i\sqrt{3}}{2^{4/3}} \sum_{m=0}^{\infty} \frac{a_{3m}}{(3m)!} w^{3m/2} s^{3m-1} - \frac{1 - i\sqrt{3}}{2^{2/3}} \sum_{m=0}^{\infty} \frac{b_{3m}}{(3m)!} w^{3m/2} s^{3m+1}
\]

(61)

with

\[
a_{3m} = \chi^{(3m)}(0)
\]

\[
b_{3m} = 6^{-\frac{1}{3}} \left\{ \begin{array}{l}
\delta_m 0 + \sum_{k=1}^{m} \sum_{i=0}^{k} (-1)^{k-i-l} \frac{2^{i-k}(3m)!}{3^l (m-i)! l! (k-i-l)!} \\
\times \left(\frac{1}{3}\right)_k h_{m-i} \left(\frac{l}{2}\right)
\end{array} \right\}
\]

(62)

which are uniformly convergent for \(|\hat{s}| < \rho = (3/4)^{1/3}\).
Proof. These series result from the use of Faà di Bruno’s formula, Lemma 4.5 and the Remark in (51). They are uniformly convergent for any compact set inside the convergence disk.

After applying Lemmas 4.1 to 4.4, the entire steepest descent path can be segmented in five pieces as shown in Figure 4. These segments are centred at \( s = 0, \pm 2/\sqrt{3}, \) and \( \pm \infty \) and their extremes are given by \( s = -\infty, -\sqrt{5}/3, -1, 0, 1, \sqrt{5}/3, \infty. \) The result of integrating (27) for \( |\arg z| \leq 2\pi/3 \) is given by the following theorem.

Theorem 4.6. The Airy function \( \text{Ai}(z) \) for \( z \in \mathbb{C} \) with \( |\arg z| \leq 2\pi/3 \), is given by the sum of the following convergent series expansions

\[
\text{Ai}(z) = I_{L_{\infty}}(z) + I_{L_{0\alpha}}(z) + I_{L_{\alpha}}(z) + I_{L_{0\alpha}}^{-}(z) + I_{L_{\infty}}^{-}(z)
\]

with

\[
I_{L_{\infty}}(z) = \frac{1}{2\pi i} e^{-2/3 z^{3/2}} \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{\Gamma(3n + \frac{5}{3})}{3^{2n}(2n)!} \frac{\gamma(n + \frac{1}{2}, \frac{3}{2} s_0^2 |z|^{3/2})}{\Gamma(n + \frac{1}{2})} z^{-3/2n}
\]

\[
I_{L_{0\alpha}}(z) = -\frac{\sqrt{3}}{22/3\pi} e^{-2/3 z^{3/2}} \sum_{m=0}^{\infty} \left\{ \frac{1}{3^{2m}} \frac{a_{3m}}{(3m)!} \frac{3^m}{(m-3)!} \Gamma \left( \frac{1}{3} - m, \frac{s_0^2 |z|^{3/2}}{3} \right) z^{3/2m} \right. \nonumber
\]

\[
\left. - \frac{b_{3m}}{(3m)!} \frac{3^m}{(m-3)!} \Gamma \left( \frac{1}{3} - m, \frac{s_0^2 |z|^{3/2}}{3} \right) z^{3/2m+1} \right\}
\]

and

\[
I_{L_{\alpha}}(z) = \pm \frac{|z|^{1/2}}{2\pi i} e^{-2/3 z^{3/2}} e^{-4/3 |z|^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!} 3^n |z|^{-3n/2}
\]

\[
\times \left[ \Gamma(n, \mp t_0 |z|^{3/2}) - \Gamma(n, \pm t_0 |z|^{3/2}) \right] \left( a_{0+}^{\mp} - a_0^{\mp} \right) - 2n + 1 \left( a_{0+}^{\mp} - a_0^{\mp} \right)^{-n}
\]

\[
\times \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\Gamma(2n-1-k)}{\Gamma(n)} \frac{\Gamma(n+k)}{\Gamma(n+k)} \left( a_{0+}^{\mp} - a_0^{\mp} \right)^k
\]

for \( w \neq e^{i2\pi/3} \) and with \( a_{0+}^{\mp} \) as given in Lemma 4.2, or, with \( s = \text{sign} \{ \arg z \} \),

\[
I_{L_{0\alpha}}^{-}(z) = \left| \frac{|z|^{1/2}}{4\pi i} e^{-2/3 z^{3/2}} e^{-4/3 |z|^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^{2n-1}}{(n-1)!} 3^n |z|^{-n/2} \frac{\Gamma(3n - 1)}{\Gamma(2n)} \right.
\]

\[
\times \left( a_{0+}^{s} - a_0^{-s} \right)^{-3n+2} \left[ \Gamma \left( \frac{n}{2}, -s t_0 |z|^{3/2} \right) - \Gamma \left( \frac{n}{2}, -s t_0 |z|^{3/2} \right) \right]
\]

\[
I_{L_{\alpha}}^{-}(z) = \left| \frac{|z|^{1/2}}{2\pi i} e^{-2/3 z^{3/2}} e^{-4/3 |z|^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} 3^n |z|^{-n/2} \frac{\Gamma(3n - 1)}{\Gamma(2n)} \right.
\]

\[
\times \left( a_0^{s} - a_0^{s} \right)^{-3n+1} \left[ \Gamma \left( n, s t_0 |z|^{3/2} \right) - \Gamma \left( n, -s t_0 |z|^{3/2} \right) \right]
\]

for \( w = e^{i2\pi/3} \) and with \( a_{0+}^{\mp} \) as given in Lemma 4.3, where \( b \) is the branch in the case of \( \Gamma \left( \frac{n}{2}, -t_0 |z|^{3/2} \right) \), which implies to take the complex conjugate for the case of
arg $z < 0$. The values of $s_0$, $s_1$ and $t_0$ are given by

$$
\begin{align*}
    s_0 &= 1 \\
    s_1 &= \sqrt{\frac{5}{3}} \\
    t_0 &= \frac{1}{3}.
\end{align*}
$$

(67)

The series for $I_{\frac{3}{\alpha}}(z)$ in (65) is slowly convergent when $|\arg z| < \frac{2}{3}\arctan(-7/8)$ as a consequence of the observations in Lemma 4.4 and can be replaced by

$$
I_{\frac{3}{\alpha}}(z) = \frac{|z|^{1/2}}{4\pi} e^{2/3 z^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^{2n-1}}{(n-1)!} 3^{n/2} \frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{3n}{2})} (\alpha_0^+ - \alpha_0^-)^{-3n/2+1} |z|^{-3n/4}
\times \left[ \Gamma\left(\frac{n}{2}, -s(t_0 + \hat{t})|z|^{3/2}\right) - \Gamma\left(\frac{n}{2}, s(t_0 - \hat{t})|z|^{3/2}\right) \right]
$$

(68)

with $\alpha_0^+$ and $\hat{t} = -4/3(1 + w^{3/2})$ as given in Lemma 4.4.

Functions $\gamma(\nu, x)$ and $\Gamma(\nu, x)$ are the lower and upper incomplete gamma functions, respectively.

For the case where $|\arg z| > 2\pi/3$, the following property is used,

$$
\text{Ai}(z) = -e^{i2\pi/3} \text{Ai}(e^{i2\pi/3} z) - e^{-i2\pi/3} \text{Ai}(e^{-i2\pi/3} z).
$$

(69)

Similarly, the Airy function of the second kind, $\text{Bi}(z)$ can be computed from the series expressions for $\text{Ai}(z)$, with the use of the property (7)

$$
\text{Bi}(z) = e^{i\pi/6} \text{Ai}(z e^{i2\pi/3}) + e^{-i\pi/6} \text{Ai}(z e^{-i2\pi/3}).
$$

(70)

Proof. The integration in (2) can be done through the five sections defined by the expansions around points $s = 0, \pm \infty$ and $t = 0$ or $t = 4/3 (1 + w^{3/2})$, as described in Lemmas 4.1 to 4.4 and Corollary 1. The choice of the limiting points between sections is made to guarantee uniform and fast convergence of the series under integration and also to maintain symmetric integration bounds. In particular, we can see the case of integrating through the segments containing $\alpha_0^+$ and $\alpha_0^-$, as shown in Figure 4 (case of $\varphi = \arg z > 0$): the expansion around $\alpha_0^+$ has a radius of convergence, $\rho = m$, that is more restrictive than the one around $\alpha_0^-$, which is $\rho = \frac{4}{3}$. As $\varphi$ approaches $\frac{2}{3}\pi$, the radius becomes zero. At $\varphi = \frac{2}{3}\arctan(-\frac{7}{8})$, we have $m = \frac{2}{3}$. Therefore, the integration in $t \in [-\frac{1}{3}, \frac{1}{3}]$ uses only half the convergence radius. For values $\frac{2}{3}\arctan(-\frac{7}{8}) < \varphi < \frac{2}{3}\pi$, we use the expansion around $t = -\frac{4}{3}(1 + w^{3/2})$, which has a convergence disk with $\rho = \frac{4}{3}$ independently of $\varphi$. However, this expansion does not properly accommodate the integration segment, $t \in [-\frac{1}{3}, \frac{1}{3}]$, for $\varphi < \frac{2}{3}\arctan(-\frac{7}{8})$: either it brings this segment too close to the convergence disk boundary or out of it. On the other hand, the expansion around $\alpha_0^-$ can include in similar conditions a bigger segment than $t \in [-\frac{1}{3}, \frac{1}{3}]$, such as $t \in [-\frac{2}{3}, \frac{2}{3}]$. The use of such a segment would produce an asymmetry in the integration bounds for the Gamma function that are to be avoided when possible, and, therefore, $t \in [-\frac{1}{3}, \frac{1}{3}]$ is again chosen for integrating the corresponding expansion, as shown in Figure 4.

Therefore, starting with the application of Lemma 4.1 in equation (27) and with
the integration limits given by $-s_0$ and $s_0$, we obtain

$$I_{L_s}(z) = \frac{1}{2\pi i} \int du e^{z u - 1/3 u^3} = \frac{|z|^{1/2}}{2\pi i} \int_{L_s} da e^{z^{3/2}(\omega a - 1/3a^3)}$$

$$= \frac{1}{2\pi |z|^{1/4}} e^{-2/3 z^{3/2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(3n + \frac{1}{2})}{3^{2n}(2n)!} \frac{\gamma(n + \frac{1}{2}, s_0^2 z^{3/2})}{\Gamma(n + \frac{1}{2})} z^{-3/2n}. \quad (71)$$

Function $\gamma(s, t)$ is the lower incomplete gamma function. In effect, this comes from

$$\int_{-a}^{a} s^{2n+1} \exp[-\sigma s^2] ds = 0$$

$$\int_{-a}^{a} s^{2n} \exp[-\sigma s^2] ds = \frac{\gamma(n + \frac{1}{2}, a^2 \sigma)}{\sigma^{n+1/2}} \quad (72)$$

where $n \in \mathbb{N}$, $\sigma \in \mathbb{R}^+$ and $a > 0$.

As for the integrals in $L_{-\infty}$ and $L_{\infty}$, they are computed here through the change of variable given in (50). This makes it possible to compute an expansion of the solutions of $\alpha = \alpha(s)$ for $|s| < (4/3)^{1/3}$ as corresponding to $s \in (-\infty, -2/\sqrt{3})$ and $s \in (2/\sqrt{3}, \infty)$. As stated in Remark 2, the solutions $\alpha_2(s)$ and $\alpha_3(s)$ correspond to the curves $L_{-\infty}$ and $L_{\infty}$, respectively. Corollary 1 produces

$$\Phi(s)|_{L_{-\infty}} = \frac{\sqrt{3}}{2^{4/3} \pi} \sum_{m=0}^{\infty} \frac{a_{3m}}{(3m)!} w^{3m/2} \left(2m - \frac{2}{3}\right) s^{-2m-\frac{1}{2}}$$

$$- \frac{1}{2^{2/3}} w \sum_{m=0}^{\infty} \frac{b_{3m}}{(3m)!} w^{3m/2} \left(2m + \frac{2}{3}\right) s^{-2m-\frac{1}{2}}$$

$$\Phi(s)|_{L_{\infty}} = \frac{\sqrt{3}}{2^{4/3} \pi} \sum_{m=0}^{\infty} \frac{a_{3m}}{(3m)!} w^{3m/2} \left(2m - \frac{2}{3}\right) s^{-2m-\frac{1}{2}}$$

$$+ \frac{1}{2^{2/3}} w \sum_{m=0}^{\infty} \frac{b_{3m}}{(3m)!} w^{3m/2} \left(2m + \frac{2}{3}\right) s^{-2m-\frac{1}{2}}. \quad (73)$$

Therefore, the contribution to the Airy’s integral by $L_{-\infty}$ and $L_{\infty}$ is

$$I_{L_{-\infty}}(z) + I_{L_{\infty}}(z) = \frac{\sqrt{3}}{2^{4/3} \pi} e^{-2/3 z^{3/2}}$$

$$\times \sum_{m=0}^{\infty} \left\{ \frac{1}{2^{2/3}} \frac{a_{3m}}{(3m)!} \left(m - \frac{1}{3}\right) \Gamma \left(1 - m, \frac{4}{3} |z|^{3/2}\right) z^{3/2m} \right.$$ 

$$\left. - \frac{b_{3m}}{(3m)!} \left(m + \frac{1}{3}\right) \Gamma \left(1 - m, \frac{4}{3} |z|^{3/2}\right) z^{3/2m+1} \right\} \quad (74)$$

since

$$\int_{-a}^{a} s^{-2n-k/3} \exp[-\sigma s^2] ds = \frac{\Gamma(\frac{1}{2} - \frac{k}{2} - n, a^2 \sigma)}{2 \sigma^{\frac{k}{2} - \frac{n}{2}} \sigma^{n}} \quad (75)$$

with $n, k \in \mathbb{N}$, $\sigma \in \mathbb{R}^+$ and $a > 0$. 22
In a similar manner, Lemmas 4.2, 4.3 and 4.4 produce equations (65), (66) and (68) by substituting the corresponding $\alpha$-series into equation (21) and by using

$$
\int_{-t_0}^{t_0} t^{n-1} \exp[-\sigma t] \, dt = \sigma^{-n} \left[ \Gamma(n, -t_0\sigma) - \Gamma(n, t_0\sigma) \right]
$$

$$
\int_{-t_0}^{t_0} t^{n/2-1} \exp[-\sigma t] \, dt = \sigma^{-n/2} \left[ \Gamma\left(\frac{n}{2}, -t_0\sigma\right) \right. \left|_b - \Gamma\left(\frac{n}{2}, t_0\sigma\right) \right]
$$

$$
\int_{-t_0}^{t_0} (t - \hat{t})^{n/2-1} \exp[-\sigma t] \, dt = \sigma^{-n/2} e^{-\sigma \hat{t}} \left[ \Gamma\left(\frac{n}{2}, -(t_0 + \hat{t})\sigma\right) - \Gamma\left(\frac{n}{2}, (t_0 - \hat{t})\sigma\right) \right]
$$

with $|_b$ being the branch of $t^{n/2}$, which implies to take the complex conjugate for the case of arg $z < 0$ as results from Lemma 4.3.

The five-section decomposition is valid for $|\text{arg } z| \leq 2\pi/3$ as has just been described, and it is illustrated in Figures 6(a) to 6(f), except in Figure 6(d). The latter shows the case of $|\text{arg } z| > 2\pi/3$, for which the integration path splits in two separate paths. Integration through the two paths would require a different set of integrals. However, the well known property of the Airy function given by equation (69) allows to identify the mapping between the lower path in Figure 6(d) with the first term in (69) and the upper path with its second term$^{12}$. A similar correspondence is found for the case of Bi($z$), where we have (70) to relate it to our results for Ai($z$)$^{13}$.

The splitting of the integration path for $|\text{arg } z| > 2\pi/3$ is related to the discontinuity of the Stokes’ multipliers as described in [3]. The transition from Figure 2c to Figure 2d illustrates such a discontinuity in a visual way. As also stated in [3], the Stokes phenomenon can be analyzed from a topological point of view: the steepest descent contour moves from including only one saddle point to both of them at $|\text{arg } z| = 2\pi/3$, and then to split into two different paths, now homotopic to $\mathcal{L}_{31}$ and $\mathcal{L}_{12}$. Equation (69) reflects this splitting, and $-(e^{i2\pi/3} z)^{3/2} = z^{3/2} e^{-i\pi/3}$ allows us to recognize the presence of a positive exponential after the splitting. However, Theorem 4.6 goes beyond [3] in this sense and shows the oscillatory behavior of the Airy function for the anti-Stokes line for small, non-asymptotic values of $|z|$.

Theorem 4.6 produces a Hadamard expansion of a type that is more general than those presented in [14, 15]. It also reveals that the method originally devised by Paris in them inherently contains a degree of complexity when dealing with the branch points of the involved complex algebraic curves that has not been studied before.

$^{12}$This can be checked by realizing that the $\alpha$-solutions of the steepest descent paths for the saddle point $\alpha_s = w^{1/2}$ are given by analogous solutions to (51) but with

$$
\chi(t) = [3 - 2 t^3 + \sqrt{3} \eta(t)]^{1/3}
$$

$$
\eta(t) = \sqrt{3} - 4 t^3
$$

replacing $\chi(t)$ and $\eta(t)$.

$^{13}$In fact, if the change of variable applied to (1) to become (12) is used in (3) or (5), it is easy to see that either (69) or (70) are produced.
Figure 7: The accuracy of the different expansions as defined in (78) is shown in a), b) and d). The Maclaurin series is computed with its first 500 terms. In c), the accuracy of MATHEMATICA® with regard to MATLAB® is shown.

5 Numerical analysis of the new series expansion and its comparison to Maclaurin and the asymptotic expansions

In this section, the new expansions of Theorem 4.6 are analyzed numerically for \( \{ z = x + iy : -10 \leq x \leq 10, -10 \leq y \leq 10 \} \). This domain contains both the unit circle, where the Maclaurin series is expected to perform best, and large enough values of \( |z| \) where the asymptotic series is applicable. The series of the new expansion are truncated when the difference between two consecutive terms is less than the machine epsilon, which, for the double precision floating-point format in use, is of \( 2^{-53} \).

The accuracy of any expansion method truncated at \( n = N \) with respect to the benchmark is defined as

\[
\text{Accuracy} = \log_{10} \left| \frac{[\text{Ai}(z)]_N^{\text{method}} - [\text{Ai}(z)]^{\text{benchmark}}}{[\text{Ai}(z)]^{\text{benchmark}}} \right|. \quad (78)
\]
Convergence of the series expansions

Figure 8: Histogram of the number of terms that are required by each one of the series expansions in equation (63) to reach convergence within the machine precision for double precision floating-point numbers.

As a benchmark, we use the routine for the integral of Ai(z) provided by MATLAB®, which is also fully compliant with both Scipy and Maxima for the aforementioned domain. Figures 7(a) to 7(d) show the accuracy of the Maclaurin series (N = 500), the classical asymptotic expansion, MATHEMATICA®’s Airy function and the new set of expansions. The Maclaurin series has a bad performance for \(\text{Re}(z) > 5.5\), whereas the classical asymptotic series shows this level of inaccuracy for \(|z| \leq 3.5\). The asymptotic series is truncated when the error given by the first neglected term starts rising. The values for the function given by MATHEMATICA 9.0® are also compared with the benchmark and some differences are found, due to a different implementation. The new expansion has an accuracy better than \(10^{-12.78}\) for the complex plane zone under study.

The number of terms needed for each one of the series expansions of Theorem 4.6 in the numerical test of this section are shown in Figure 8. The convergence of the expansions for \(I_{\mathcal{L}_s}\), \(I_{\mathcal{L}_{\infty}}\) and \(I_{\mathcal{L}_{\infty}+}\), as defined above, occurs for a statistical mode of 20 terms. The largest number of terms is required in the case of the computation of the sum \(I_{\mathcal{L}_{\infty}} + I_{\mathcal{L}_{\infty}}\). The coefficients \(a_n\) and \(b_n\) in equation (74) can computed from equation (62) once and for all values of the variable \(z\).

6 Conclusions

A new convergent series expansion has been obtained for the Airy function Ai(z) of complex argument, and also for Bi(z) as a consequence of equation (70). This new expansion includes incomplete Gamma functions of \(|z|\) in its five series. In this
They are a type of Hadamard expansions as defined in [14, 15]. However, the new series are different to the ones introduced in these references in the sense that they include upper incomplete Gamma functions, and also in the manner of selecting the points around which the series are computed. The developments that are performed in the current analysis show the complexity of the original idea of segmenting the steepest-descent path in pieces where uniform convergence occurs and where the integration is done avoiding an extreme proximity to the convergence disk boundaries. The use of Puiseux series in this context for expansions around the branch points is an additional new ingredient of the treatment done here, as it is the fact that only a small and finite number of segments is necessary for a full and exact splitting of the integration path. These recourses had not been studied in [14, 15]. The new approach could be extended to other Laplace-type integrals by using the same procedure.

Theorem 4.6 also provides a clear picture of the Stokes phenomenon present in the classical asymptotic series. It shows the impact of the splitting of the steepest descent path in the series of equation (64), and the correspondence with the well known property given in (69).

The convergence of the new expansions has also been studied in section 5. It produces a level of accuracy that improves the performance of the asymptotic expansion and equals that of the Maclaurin series. Therefore, the new expansion is a candidate to replace under a single framework the computation algorithm for the Airy functions, which is currently based on the combined use of the Maclaurin series, the asymptotic expansion and usually a Gauss-Laguerre quadrature method for the corresponding integral where the other two series are inadequate [22].

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