Non-Euclidean Geometry and Defected Structure for Bodies with Variable Material Composition

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Abstract. In the paper the relationship between pure geometrical concepts of the theory of affine connections, physical concepts related with non-linear theory of distributed defects and concepts of non-linear continuum mechanics for bodies with variable material composition is discussed. Distinguishing feature of the bodies with variable material composition is that their global reference shapes can not be embedded into Euclidean space and have to be represented as smooth manifolds with specific (material) connection and metric. The method for their synthesis based on the modeling of additive process are proposed. It involves specific boundary problem referred to as evolutionary problem. The statement of such problem as well as illustrative exact solutions for it are obtained. Because non-Euclidean connection is rarely used in continuum mechanics, it is illustrated from the perspective of differential geometry as well as from the point of view, adopted in the theory of finite incompatible deformations. In order to compare formal structures defined within the models of solids with variable material composition with their counterpart in non-linear theory of distributed defects, a brief sketch for latter is given. The examples for cylindrical and spherical non-linear problems are presented. The correspondences between geometrical structures that defines material connection, fields of related defect densities and evolutionary problems for bodies with variable material composition are shown.

Introduction

The present paper is intended to illustrate the relationship between pure geometrical concepts adopted in the theory of affine connections, physical concepts related with non-linear theory of distributed defects and concepts of non-linear continuum mechanics for bodies with variable material composition. From abstract viewpoint these theories have much in common. All of them are based on the idea of linear transformations, dependent on spatial points, that being applied to some "conventional" structure, gives principally new one. As regard to the theory of affine connections this idea is implemented in Cartan’s moving frame method [1] that allows one to construct non-Euclidean space upon conventional Euclidean one. It could be roughly described as distortions of global Cartesian frame of Euclidean space, defined individually for each point of new structure. Because these transformations in general are defined by arbitrary smooth functions, the integrability conditions for them are not valid, and there are no curvilinear coordinates, that generate such frames as local bases. Thus, one obtains principally new geometric structure, a space of affine connection, that is no more Euclidean. At the same time, in the framework of non-linear theory of distributed defects a defected body, which is fundamentally different from conventional solid, can be formalized as a result of
individual linear transformations of reference elementary volumes (infinitesimal neighbourhoods of material points constitute the body), that in general are not the same even for two different material points [2]. Such individual transformations (with some smoothness assumptions) can be formalized by smooth field of transformations that brings mentioned above formalism in similarity with Cartan’s moving frame method [3, 4].

At last, non-linear theory of elastic bodies with variable material composition is based on the idea of continuous family of reference configurations [5–8]. This family expresses an evolution of material structure of the body that is going on during some additive or removal process. The elements of this family determine linear transformations regarded as incompatible local configurations of individual elementary volumes. In general these transformations generate non-integrable fields of frames. All of these show the similarity of basic ideas underlie the modeling for solids with variable material composition with fundamentals of the theory of distributed defects and, consequently, with the theory of affine connection. One can see this as a triplicity of closely related viewpoints: affine connection — distributed defects — bodies with variable composition. The discussion of that triplicity is presented below.

The paper is divided into seven sections. In the first section one can find the sketch for the theory of non-Euclidean elastic bodies which are represented as smooth manifolds with specific (material) connection and metric, as well as the ways for their synthesis based on given continuous family of reference configurations. The second section represents general concepts of affine connection from pure geometrical viewpoint. Here one can find the generalization of connection and metric definitions, given in first section, as well as strict definitions for curvature, torsion and nonmetricity of material connection. The third section demonstrates the affinity of geometrical concepts of connection that are rooted in generalization for reparametrization of Cartesian coordinate chart and transition to some curvilinear coordinates with fundamentals of kinematics for simple elastic bodies that leads to the definition of material parallel transport, and, consequently, to the notion of material connection. In the forth section a brief sketch for the geometrical theory of distributed defects is given. The fifth section is devoted to definition and some classification of the bodies with variable material composition and evolutionary problem statements for them. In the last two sections the illustrative examples for cylindrical and spherical non-linear problems are presented. The correspondences between geometrical structures that defines material connection, fields of related defect densities and evolutionary problems for bodies with variable material composition are shown.

Notations

| Symbol  | Notion                                      |
|---------|---------------------------------------------|
| $[\Omega_{m}^{n}]$ | Matrix with elements $\Omega_{m}^{n}$              |
| $\mathcal{E}$ | Three-dimensional Euclidean affine space         |
| $\mathcal{V}$ | Translation vector space, associated with $\mathcal{E}$ |
| $\mathcal{B}$ | Body                                           |
| $g$      | Riemannian metric on $\mathcal{E}$             |
| $\mathcal{G}$ | Riemannian metric on $\mathcal{B}$              |
| $\mathcal{M}$ | Smooth manifolds                               |
| $\mathcal{N}$ | Smooth manifolds                               |
| $T\mathcal{M}$ | Tangent bundle of smooth manifold $\mathcal{M}$ |
| $T^*\mathcal{M}$ | Cotangent bundle of smooth manifold $\mathcal{M}$ |
| Sec($\mathcal{E}$) | Vector space of smooth sections of a vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ |
| Vec($\mathcal{M}$) | Vector space of smooth vector fields on $\mathcal{M}$ |
| $\kappa$ | Configuration                                  |
| $\gamma$ | Deformation                                    |
| $K_x$   | Local configuration at point $x$               |
| $T$     | Cauchy stress tensor                           |
1. Non-Euclidean Solids

1.1. Geometric Formalization of the Body and the Physical Space

1°. Body. In accordance to [3,9] the body \( \mathcal{B} \) is assumed to be a smooth 3-dimensional manifold. Its elements are referred to as material points and are denoted by Fraktur majuscules \( \mathcal{X}, \mathcal{Y}, \ldots \). 

**Remark 1.** Since a body \( \mathcal{B} \) is a smooth 3-dimensional manifold, a smooth maximal atlas is chosen for it [10]. Any atlas \( A \) from the maximal one is a family \( A = \{ (U_\alpha, \varphi_\alpha) \}_{\alpha \in I} \) of ordered pairs \((U_\alpha, \varphi_\alpha)\) called (coordinate) charts. Here each \( U_\alpha \) is an open subset of \( \mathcal{B} \) and \( \bigcup_{\alpha \in I} U_\alpha = \mathcal{B} \), while each \( \varphi_\alpha \) is a homeomorphism between \( U_\alpha \) and open subset \( \mathcal{O}_\alpha \subset \mathbb{R}^3 \). If for some \( \alpha, \beta \in I \) one has \( U_\alpha \cap U_\beta \neq \emptyset \) then smooth compatibility property holds: the transition map 

\[
\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)
\]

is a \( C^\infty \)-diffeomorphism between open subsets of \( \mathbb{R}^3 \). In the paper we refer to each \( U_\alpha \) as a chart’s domain while to each \( \varphi_\alpha \) as a coordinate homeomorphism.

The tangent space to \( \mathcal{B} \) at each point \( \mathcal{X} \in \mathcal{B} \) is denoted by \( T_\mathcal{X} \mathcal{B} \) while the cotangent space, \( \mathcal{Y} \mathcal{B} \), is denoted by \( T^* \mathcal{B} \). Tangent and cotangent bundles are denoted by \( T \mathcal{B} \) and \( T^* \mathcal{B} \) respectively. Any chart \((U, \varphi)\) from the maximal atlas induces local coordinates on \( \mathcal{B} \): there is one-to-one correspondence

\[
\mathcal{X} \mapsto \varphi(\mathcal{X}) = (x_1, x_2, x_3)
\]
on \( U \) between material points \( \mathcal{X} \) and ordered triples \((x_1, x_2, x_3)\) of real numbers. Local coordinates, in turn, define coordinate frame which is denoted by \((\partial x_i)_{i=1}^3\), or, if no ambiguity could arise, simply by \((\partial_i)_{i=1}^3\). The coordinate coframe is denoted by \((dx_i)_{i=1}^3\), i.e., \((dx_i, \partial x_i) := dx_i(\partial x_j) = \delta^i_j\), ♣.

2°. Physical space. A physical space is a set \( \mathcal{Y} \) of places. We denote them by script minuscules \( x, y, \ldots \). In the present study we treat the physical space as a three-dimensional Euclidean affine space \( \mathcal{E} \) with translation vector space \( \mathcal{V} \) and scalar product \((\cdot)\) on \( \mathcal{V} \) [11]. That is,

(i) \( \mathcal{E} \) is a set whose elements are referred to as spatial points,

(ii) \( \mathcal{V} \) is a three-dimensional real vector space whose elements are referred to as translation vectors,

(iii) \( \mathcal{V} \) is endowed by a scalar product \( \cdot : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \),

(iv) there is a mapping \( \psi : \mathcal{E} \times \mathcal{E} \to \mathcal{V} \), which assigns to each ordered pair \((a, b) \in \mathcal{E} \times \mathcal{E}\) some vector from \( \mathcal{V} \) (denoted by \( \overrightarrow{ab} \) or \( b - a \)), such that

\[\text{(W}_1\text{)} \text{ for every points } a, b, c \in \mathcal{E} \text{ the following relation holds ("parallelogram rule")}: \]

\[\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac} = 0 \in \mathcal{V};\]

\[\text{(W}_2\text{)} \text{ for any point } a \in \mathcal{E} \text{ and for any vector } v \in \mathcal{V} \text{ there exists a unique point } b \in \mathcal{E} \text{, such that } \overrightarrow{ab} = v.\]

Vectors (elements of \( \mathcal{V} \)) are denoted by boldface minuscules \( u, v, \ldots \), while tensors (elements of the vector space \( \text{Lin}(\mathcal{V}; \mathcal{V}) \)) of linear mappings from \( \mathcal{V} \) to \( \mathcal{V} \) are denoted by boldface majuscules \( S, T, \ldots \). If \( (e_k)_{k=1}^3 \) is a frame for \( \mathcal{V} \) then we denote by \( (e^k)_{k=1}^3 \) the corresponding dual frame defined by relations \( e^i \cdot e_j = \delta^i_j, \ i, j \in \{1, 2, 3\} \).

Euclidean structure of \( \mathcal{V} \) allows to simplify the notion of abstract tensor product \( \otimes \) to the following definition. If \( u, v \in \mathcal{V} \) then \( u \otimes v \) is element from \( \text{Lin}(\mathcal{V}; \mathcal{V}) \), such that

\[
\forall w \in \mathcal{V} : \quad u \otimes v(w) := (v \cdot w)u
\]

According to axiom \( \text{(W}_2\text{)} \), for any fixed point \( a \in \mathcal{E} \) the mapping

\[
\psi_a : \mathcal{E} \to \mathcal{V}, \quad \psi_a(x) := \overrightarrow{ax},
\]
is a bijection. This allows one to define the external operation
\[ + : \mathcal{E} \times \mathcal{V} \to \mathcal{E}, \quad + : (\mathbf{a}, \mathbf{v}) \mapsto \mathbf{a} + \mathbf{v} := \psi_{\mathbf{a}}^{-1}(\mathbf{v}), \]
which assigns to each ordered pair \((\mathbf{a}, \mathbf{v})\) a unique point \(\mathbf{b} \in \mathcal{E}\), such that \(\mathbf{a} + \mathbf{v} = \mathbf{b}\).

All points from \(\mathcal{E}\) can be uniquely described by triples of real numbers if one chooses some fixed point \(\mathbf{0} \in \mathcal{E}\) (the origin) and frame \((e_k)_{k=1}^3\) in \(\mathcal{V}\). Indeed, introduce the bijective map
\[ p := \psi_{\mathbf{0}} : \mathcal{E} \to \mathcal{V}, \quad p(x) = x - \mathbf{0}, \]
which is referred to as position vector field. Then since \(p(x)\) is a translation vector, there exists a triple \((x^k)_{k=1}^3\) of real numbers such that \(p(x) = x^k e_k\). Thus, \(x = \mathbf{0} + x^k e_k\) and one gets the assignment (arithmetization)
\[ \mathcal{D}_{\mathbf{e}_k} : \mathcal{E} \to \mathbb{R}^3, \quad \mathcal{D}_{\mathbf{e}_k} : x \mapsto (x^1, x^2, x^3), \]
declared as the rule \(\mathcal{D}_{\mathbf{e}_k}(x) := (e^k \cdot p(x))_{k=1}^3\). The tuple \((\mathbf{e}_k)_{k=1}^3\) is referred to as affine coordinate system.

Since \(\mathcal{V}\) is Euclidean space, there exists an orthonormal frame in it. We denote some chosen orthonormal frame by \((i_k)_{k=1}^3\). That is, \(i_k \in \mathcal{V}\) and \(i_k \cdot i_l = \delta_{kl}, \quad k, l \in \{1, 2, 3\}\). The dual frame, \((i^k)_{k=1}^3\), coincides with the original one, i.e., \(i^k = i_k\).

**Remark 2.** The physical space \(\mathcal{E}\) is a metric space with metric \(d\) defined as
\[ d(x, y) := \sqrt{(\dot{x} - \dot{y}) \cdot (\dot{x} - \dot{y})}. \]
Thus, \(\mathcal{E}\) is Hausdorff and second countable space. Since the mapping \(\mathcal{D}_{\mathbf{e}_k} : \mathcal{E} \to \mathbb{R}^3\) is a homeomorphism with respect to the topologies on \(\mathcal{E}\) and \(\mathbb{R}\), it means that \(\{(\mathcal{E}, \mathcal{D}_{\mathbf{e}})\}\) is a trivial smooth atlas on \(\mathcal{E}\). The space \(\mathcal{E}\) is then a smooth 3-manifold with maximal atlas generated by the trivial atlas. By virtue of isomorphism \(T_p \mathcal{E} \cong \mathcal{V}\) [10], \(x \in \mathcal{E}\), the tangent space to \(\mathcal{E}\) is identified with the translation vector space \(\mathcal{V}\).

The scalar product \((\cdot)\) endows \(\mathcal{E}\) with trivial Riemannian metric \(g\) defined as \(g(u, \mathbf{v}) := \mathbf{u} \cdot \mathbf{v}\). In Cartesian chart \(\{(\mathcal{E}, \mathcal{D}_{\mathbf{i}})\}\) this metric is written as the following dyadic decomposition: \(g = \delta_{kl} i^k \otimes i^l\).

**Remark 3.** Together with Cartesian coordinates \((x^k)_{k=1}^3\) established by the mapping \(\mathcal{D}_{\mathbf{i}}\), one can use curvilinear coordinates. They are introduced locally by chart \((U, h)\), where \(U \subset \mathcal{E}\) is an open set and \(h : U \to O, \quad h : x \mapsto (q^1, q^2, q^3)\), is a coordinate homeomorphism between \(U\) and an open set \(O \subset \mathbb{R}^3\). The transition map
\[ \mathcal{D}_{\mathbf{i}} \circ h^{-1} : (q^1, q^2, q^3) \mapsto (x^1, x^2, x^3) \]
is required to be a \(C^\infty\)-diffeomorphism. Denote by \(\tilde{p}\) the following composition:
\[ \tilde{p} = p \circ h^{-1} : O \to \mathcal{V}, \quad \tilde{p}(q^1, q^2, q^3) := x^k(q^1, q^2, q^3) i_k. \]
Then one can define the triple \((e_k)_{k=1}^3\) of vector fields \(e_k : U \to \mathcal{V}\) as follows:
\[ \forall x \in U \quad e_k |_x := \left. \frac{\partial \tilde{p}(q^1, q^2, q^3)}{\partial q^k} \right|_{h(x)} = \left. \frac{\partial x^k(q^1, q^2, q^3)}{\partial q^k} \right|_{h(x)} i_k, \quad k \in \{1, 2, 3\}. \]
The triple \((e_k)_{k=1}^3\) is linearly independent at each point \(x \in U\) and thus constitutes a frame for \(\mathcal{V}\). In this regard, we refer to \((e_k)_{k=1}^3\) as local frame. Its dual frame, \((e^k)_{k=1}^3\), is defined pointwise as \(e^i |_x \cdot e_j |_x = \delta^i_j, \quad i, j \in \{1, 2, 3\}\).

**3° Configurations.** The body and the physical space are related by configurations which are mappings \(\varphi : \mathcal{B} \to \mathcal{E}\). We assume that these mappings are smooth embeddings and the set of all configurations is denoted by \(\mathcal{C}(\mathcal{B})\).

**Remark 4.** Since \(\varphi \in \mathcal{C}(\mathcal{B})\) is a smooth embedding, it satisfies the following conditions:

1 Here and in what follows the vertical bar \(#|_p\) indicates the value of \# at point \(p\). From the context it would be clear when this meaning of notation is distinguished from the restriction of a map.
(a) $\kappa$ is a continuous map,
(b) $\kappa$ is a smooth map,
(c) $\kappa$ has rank 3 at each point of $\mathfrak{B}$,
(d) $\kappa$ is a homeomorphism onto its image.

Let us discuss these items in a moment more. The smoothness of mapping between manifolds is defined by smoothness of its coordinate representation. To construct coordinate representation for $\kappa \in \mathfrak{C}(\mathfrak{B})$ one needs to choose smooth atlases $A = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathbb{C}}$ and $B = \{(V_\beta, \beta)\}_{\beta \in \mathbb{C}}$ from maximal atlases of $\mathfrak{B}$ and $\mathfrak{E}$. Note that $B$ is an atlas of curvilinear coordinates. Now let $x \in \mathfrak{B}$ be some material point. The open set $U_\alpha$ contains it for some $\alpha \in \mathbb{I}$. Similarly, the open set $V_\beta$ contains $x(x) \in \mathfrak{E}$ for some $\beta \in \mathbb{J}$. The composition

$$\tilde{\kappa}_{\alpha, \beta} := h^\beta \circ \kappa \circ \varphi^{-1}_\alpha : \varphi_\alpha(W_\alpha, \beta) \to h_\beta(\kappa(W_\alpha, \beta)),$$

where $W_\alpha, \beta = U_\alpha \cap \kappa^{-1}(V_\beta)$, is called a coordinate representation of $\kappa$ in a neighborhood of $x$. Then the condition (a) signifies that $\tilde{\kappa}_{\alpha, \beta}$ is a mapping between open subsets of $\mathbb{R}^3$. The condition (b) means that for each material point $x$ the coordinate representation of $\kappa$ in a neighborhood of $x$ is a smooth (of class $C^\infty$) map. Next, the condition (c) states that rank of Jacobian matrix of this map at point $\varphi_\alpha(x)$ is equal to 3. Here $\alpha \in \mathbb{I}$ is the appropriate value of index $\alpha$ (personal for each $\kappa$). Finally, (d) requires that the mapping from $\mathfrak{B}$ to $\kappa(\mathfrak{B})$ defined by $\tilde{\kappa} : \mathfrak{B} \to \kappa(\mathfrak{B})$, i.e., the restriction of $\kappa$ to its image $\kappa(\mathfrak{B})$, is a homeomorphism. $$\blacklozenge$$

If $\kappa \in \mathfrak{C}(\mathfrak{B})$ is a configuration then its image, $\kappa(\mathfrak{B})$, is denoted by $\delta_\kappa$ and is referred to as a shape. Any shape of the body $\mathfrak{B}$ is an open submanifold of $\mathfrak{E}$ [10]. That is, $\delta_\kappa \subset \mathfrak{E}$ is an open set. It can be endowed with such smooth structure that the inclusion map $\iota_{\delta_\kappa} : \delta_\kappa \to \mathfrak{E}$, defined as $p \mapsto p$, is a smooth embedding. By virtue of the tangent map $T_p \iota_{\delta_\kappa} : T_p \delta_\kappa \to T_p \mathfrak{E}$, which is isomorphism, we identify tangent spaces $T_p \delta_\kappa$ and $T_p \mathfrak{E}$ at each point $p \in \delta_\kappa$.

In what follows, together with configuration $\kappa$, we will use diffeomorphism $\tilde{\kappa} : \mathfrak{B} \to \delta_\kappa$ defined as $\tilde{\kappa}(\mathfrak{X}) := \kappa(\mathfrak{X})$.

**Remark 5.** Since body is rather abstract entity, one cannot use arbitrary coordinates on it in direct computations. This situation can be fixed in the following way: a coordinate net on some shape may serve as the coordinate net on the body. Let us explain this in more details. Let $\kappa \in \mathfrak{C}(\mathfrak{B})$ be a configuration. Suppose that one had chosen some smooth atlas $\{(V_\alpha, \sigma_\alpha)\}_{\alpha \in \mathbb{C}}$ on $\delta_\kappa$. For each $\alpha \in \mathbb{I}$ let us define $U_\alpha = \tilde{\kappa}^{-1}(V_\alpha)$ and $\varphi_\alpha = \sigma_\alpha \circ \tilde{\kappa}|_{U_\alpha}$. Then each $U_\alpha$ is open in $\mathfrak{B}$, $\bigcup_{\alpha \in \mathbb{C}} U_\alpha = \mathfrak{B}$ and $\varphi_\alpha$ is a homeomorphism between $U_\alpha$ and some open subset of $\mathbb{R}^3$. The family $A = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathbb{C}}$ is the desired smooth atlas. Note that each material point $x$ has similar local coordinates as its image, $x = \kappa(x)$. These reasonings are illustrated on the diagram:

The coordinate representation of $\tilde{\kappa}$ in the neighborhood of each material point $x$ relatively to charts $(U_\alpha, \varphi_\alpha)$ and $(V_\alpha, \sigma_\alpha)$ is simply the inclusion map.

In particular, suppose that $\{(\delta_\kappa, \mathfrak{D}_\kappa, \iota|_{\delta_\kappa})\}$ is atlas on $\delta_\kappa$ induced by Cartesian coordinates. Here $\mathfrak{D}_\kappa, i|_{\delta_\kappa}$ is the restriction of $\mathfrak{D}_\kappa, i$ to $\delta_\kappa$. Then $\mathfrak{B}$ can be covered by trivial smooth atlas $^2 A = \{(U, \varphi)\}$, where $U = \tilde{\kappa}^{-1}(\delta_\kappa) = \mathfrak{B}$ and $\varphi = \mathfrak{D}_\kappa, i \circ \tilde{\kappa}$.

$^2$ Thus, tangent bundle $T\mathfrak{B}$ is diffeomorphic to the product manifold $\mathfrak{B} \times \mathbb{R}^3$. 

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1.2. Material Uniformity and Structural Inhomogeneity

4°. Response. In the framework of nonlinear elasticity the physical state of the body (distribution of elastic energy, stresses, etc.) depends on its particular shape only. If the shape changes, then the physical state changes. To quantify the physical state of the body \( B \) one needs to specify the set \( R \) of ordered tuples of mathematical objects (in particular, \( R = \{ T \mid T \in \text{SymLin}(\mathcal{V}; \mathcal{V})\} \), where \( T \) are symmetric linear mappings from \( \mathcal{V} \) to \( \mathcal{V} \), that represent values of Cauchy stress tensor) that formalize entities which are determined by “measurers”. Following [3] we refer to the elements of the set \( R \) as response descriptors. Generally, the physical response of \( B \) by “measurers”. Following [3] we refer to the elements of the set \( R \) as response descriptors. Following [3] we refer to the elements of the set \( R \) as response descriptors.

Following [3] we refer to the elements of the set \( R \) as response descriptors. Generally, the physical response of \( B \) at any point \( X \in B \) may depend on configuration \( \mathcal{C}(B) \), i.e.,

\[ \mathcal{R}_X : \mathcal{C}(B) \to R, \quad \mathcal{R}_X(\varkappa) = r, \quad r \in R \]  

(2)

This kind of dependence is rather general for the purpose of present paper and in what follows we restrict considerations to some narrower case that corresponds to first order Taylor expansion.

5°. Local configurations. Consider the idea of localization. We assume, that if the “observer” is placed at a material point \( X \), then it can distinguish configurations only within some neighborhood of this point. Restrict our attention to the first order infinitesimal localization: configurations, which have the same values at \( X \) and the same linear approaches, are indistinguishable by the “observer”.

Choose local coordinates on the body and the physical space. Then for the coordinate representations \( \tilde{\kappa}_1, \tilde{\kappa}_2 \) of configurations \( \kappa_1, \kappa_2 \) one has Taylor expansions

\[ \tilde{\kappa}_s^i(\tilde{X} + h) = \tilde{\kappa}_s^i(\tilde{X}) + \frac{\partial \tilde{\kappa}_s^i}{\partial \tilde{X}^j} h^j + \frac{1}{2} \frac{\partial^2 \tilde{\kappa}_s^i}{\partial \tilde{X}^j \partial \tilde{X}^k} h^j h^k + \ldots, \]

for \( i = 1, 2, 3 \) and \( s = 1, 2 \). Here \( \tilde{X} \in \mathbb{R}^3 \) is coordinate triple of \( X \), while \( h \in \mathbb{R}^3 \) is an increment. Then, due to localization principle, configurations \( \kappa_1, \kappa_2 \) are indistinguishable for the “observer”, if and only if

\[ \tilde{\kappa}_1^i(\tilde{X}) = \tilde{\kappa}_2^i(\tilde{X}), \quad \frac{\partial \tilde{\kappa}_1^i}{\partial \tilde{X}^j} \bigg|_{\tilde{X}} = \frac{\partial \tilde{\kappa}_2^j}{\partial \tilde{X}^i} \bigg|_{\tilde{X}}, \quad i, j = 1, 2, 3 \]

Note, that if configurations are indistinguishable with respect to one pair of coordinate systems (on \( B \) and \( \mathcal{S} \)), then they are indistinguishable in another. The relation \( \sim_X \) of being indistinguishable at \( X \) is an equivalence relation on \( \mathcal{C}(B) \) and it can be expressed in coordinate free form as follows [3,8,9,12]:

\[ (\kappa_1 \sim_X \kappa_2) \iff (\kappa_1(X) = \kappa_2(X)) \land (T_X \kappa_1 = T_X \kappa_2), \]  

(3)

where \( T \) denotes the tangent map operation, i.e., if \( F : \mathcal{S} \to \mathcal{R} \) is a smooth map between smooth manifolds \( \mathcal{S}, \mathcal{R} \) then \( T F : T \mathcal{S} \to T \mathcal{R} \) is its tangent map between tangent bundles [10].

We refer to an equivalence class \( \mathcal{K}_X = [\kappa]_X \) as local configuration at \( X \). By the definition, local configuration consists of configurations, which are indistinguishable by the “observer” placed at \( X \). Denote the set of local configurations (the quotient set) by \( \mathcal{C}(B) \).

6°. Local configuration as linear mapping. In continuum mechanics one compares lengths and directions of spatial representations of tangent vectors from tangent space \( T_X B \) in reference and actual states. Thus, the tangent space \( T_X B \) to the body \( B \) can be interpreted as infinitesimal neighborhood of \( X \) and each tangent vector represents differential element [13]. We identify local configuration \( \mathcal{K}_X \) with linear mapping on \( T_X B \) as follows:

\[ \mathcal{K}_X : T_X B \to \mathcal{V}, \quad \mathcal{K}_X u := T_X \kappa(u), \]

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for $\varphi \in \mathcal{X}$. By the definition of equivalence relation (3), such linear map doesn’t depend on representative $\varphi$.

7°. Coordinate representations of local configuration. Since local configuration is linear mapping one can use dyadic decompositions for it. Let $\varphi \in \mathcal{X}$ be some configuration. Choose a smooth atlas $A = \{(U_{\alpha}, \varphi_{\alpha})\}_{\alpha \in I}$ on $\mathfrak{B}$ and let $\alpha \in I$ be such index that $\mathfrak{X} \in U_{\alpha}$. Consider two cases:

(a) $\mathfrak{B}$ is endowed by Cartesian coordinates $(x^k)^3_{k=1}$. They are induced by arithmetization $\mathcal{D}_{\alpha, i}$ (1). Then the coordinate representation of $\varphi$ in a neighborhood of $\mathfrak{X}$ is the composition $\tilde{x} = \mathcal{D}_{\alpha, i} \circ \varphi \circ \varphi^{-1} : (\mathfrak{X}) \mapsto (x^k)$, and

$$\mathfrak{X} = K^m_n e_m \otimes dX^n|_{\mathfrak{X}}, \quad K^m_n = \frac{\partial x^m}{\partial X^n}|_{\varphi_{\alpha}(\mathfrak{X})}$$

(b) $\mathfrak{B}$ is endowed by curvilinear coordinates $(q^k)^3_{k=1}$. Let $(V, h)$ be a curvilinear chart such that $x = \varphi(\mathfrak{X}) \in V$. Then the coordinate representation of $\varphi$ in a neighborhood of $\mathfrak{X}$ is the composition $\tilde{x} = h \circ \varphi \circ \varphi^{-1} : (\mathfrak{X}) \mapsto (q^k)$, and if $(e_k)^3_{k=1}$ is the local frame, one gets

$$\mathfrak{X} = K^m_n e_m \otimes dX^n|_{\mathfrak{X}}, \quad K^m_n = \frac{\partial q^m}{\partial X^n}|_{\varphi_{\alpha}(\mathfrak{X})}$$

The local configuration $\mathfrak{X}$, being linear mapping between three-dimensional vector spaces that is generated by some embedding, is isomorphism. Thus, one can consider $\mathfrak{X}^{-1}$. For its coordinate representations consider two cases again:

(a) $\mathfrak{B}$ is endowed by Cartesian coordinates $(x^k)^3_{k=1}$. One gets

$$\mathfrak{X}^{-1} = \Omega^m_n \partial x^m|_{\mathfrak{X}} \otimes i^n, \quad [\Omega^m_n] = [K^m_n]^{-1}$$

(b) $\mathfrak{B}$ is endowed by curvilinear coordinates $(q^k)^3_{k=1}$. Let $(V, h)$ be a curvilinear chart such that $x = \varphi(\mathfrak{X}) \in V$. Then

$$\mathfrak{X}^{-1} = \Omega^m_n \partial x^m|_{\mathfrak{X}} \otimes e^n|_{\mathfrak{X}}, \quad [\Omega^m_n] = [K^m_n]^{-1}$$

8°. Simple body. Assume that the response of the body $\mathfrak{B}$ at $\mathfrak{X}$ depends only on local configurations at $\mathfrak{X}$. That is, instead of (2) one has

$$\mathcal{R}_{\mathfrak{X}} : C_{\mathfrak{X}}(\mathfrak{B}) \rightarrow R, \quad \mathcal{R}_{\mathfrak{X}}(\mathfrak{X}) = r \quad (4)$$

In this case the material at $\mathfrak{X}$ is called simple. Let us give a metaphorical interpretation of (4). Suppose that $\mathcal{R}_{\mathfrak{X}}(\mathfrak{X}) = r$. If one chooses a configuration $\varphi_1 \in \mathcal{X}$ and attaches “measurer” at spatial point $\varphi_1(\mathfrak{X})$ then the measurer will show value $r$ on its scale. Suppose that $\varphi_2 \in \mathcal{X}$ is another configuration, then the measurer will show the similar value $r$ at point $\varphi_2(\mathfrak{X})$. Thus, despite $\varphi_1$ and $\varphi_2$ are different shapes in general, their physical states in infinitesimal neighborhood of $\varphi_1(\mathfrak{X}) = \varphi_2(\mathfrak{X})$ are undistinguishable.

If for all $\mathfrak{X} \in \mathfrak{B}$ the material at $\mathfrak{X}$ is simple then $\mathfrak{B}$ is referred to as a simple body [3]. From now we consider only such bodies.

9°. Material uniformity. Materially uniform body consists of elementary volumes which give the same response on the same deformation. Rigorously speaking, the body $\mathfrak{B}$ is called materially uniform [3] if for each pair $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{B}$ of material points there exists an invertible linear mapping $\Phi_{\mathfrak{X}\mathfrak{Y}} : T_{\mathfrak{Y}}\mathfrak{B} \rightarrow T_{\mathfrak{X}}\mathfrak{B}$ (called material isomorphism) such that the equality

$$\mathcal{R}_{\mathfrak{X}}(\mathfrak{X}) = \mathcal{R}_{\mathfrak{Y}}(\mathfrak{X} \Phi_{\mathfrak{X}\mathfrak{Y}}) \quad (5)$$
holds for every $\mathcal{K}_X \in \mathcal{C}_X(\mathcal{B})$. Suppose that one picked up a point $X_0 \in \mathcal{B}$ and placed it into experimental setup. The setup gave response, i.e., the value of $\mathcal{R}_{X_0}$, on some local configuration $t_{X_0}$. By virtue of material uniformity, any other particle $X$ of the body $\mathcal{B}$ would have the similar response on the local configuration $t_{X_0}$ calibrated by some material isomorphism. Suppose that some family $\mathcal{U} = \{\Phi_{X_0, X}\}_{X \in \mathcal{B}}$ of material isomorphisms is chosen, where $\Phi_{X_0, X} = \text{Id}_{t_{X_0}(\mathcal{B})}$. Then one can define a family $\text{Ref}_B = \{\mathcal{K}_X^R\}_{X \in \mathcal{B}}$, where \[ \mathcal{K}_X^R := t_{X_0} \Phi_{X_0, X} \]

We will refer to the family $\text{Ref}_B$ as an uniform reference [3]. From (5) it follows that \[ \forall X \in \mathcal{B}: \quad \mathcal{R}_X(\mathcal{K}_X^R) = \mathcal{R}_X(t_{X_0}) \]

Note that the family $\text{Ref}_B$ depends on chosen family $\mathcal{U}$ of material isomorphisms. Choosing another family $\mathcal{U}'$ leads to another, in general, family $\text{Ref}'_B$, but the response would be the same as $\mathcal{R}_{X_0}(t_{X_0})$.

Let some uniform reference $\text{Ref}_B = \{\mathcal{K}_X^R\}_{X \in \mathcal{B}}$ for the body $\mathcal{B}$ be chosen. Taking representatives $\mathcal{K}_X^R \in \mathcal{K}_X^R$ from each equivalence class one gets new family $\text{Ref}_B = \{\mathcal{K}_X^R\}_{X \in \mathcal{B}}$. Each element of this family is referred to as a reference configuration. At point $\mathcal{K}_X^R(\mathcal{X})$ of the reference shape $\mathcal{K}_X^R(\mathcal{B})$ there is response similar to $\mathcal{R}_X(t_{X_0})$, while in another point $\mathcal{K}_Y^R(\mathcal{B})$ this is not so, in general.

10°. Crystal reference. One can give the following figurative interpretation to the family $\text{Ref}_B$. Suppose that the body $\mathcal{B}$ is “split” into infinitesimal fragments which may be thought as tangent spaces $T_X \mathcal{B}$. These fragments, being separated from each other, relax to the uniform state. The relaxation of the fragment $T_X \mathcal{B}$ is performed by the linear mapping $\mathcal{K}_X^R : T_X \mathcal{B} \to \mathcal{Y}$. The collection of such relaxed fragments in physical space doesn’t constitute a continuum solid, in general. We will refer to such collection as crystal reference [14].

11°. Intermediate configuration. In previous reasonings we measured response from the body itself. But the body doesn’t subjected to observations directly! Only shapes are observable. Thus, one chooses some shape. This shape can be split into infinitesimal pieces; each of them is independent from others. Any piece from the obtained collection relaxes to the uniform state. This relaxation is accompanied by linear mapping, other than local configuration. The response functional of the body can be calibrated with respect to these linear mappings as it would be shown below.

Let $\mathcal{B}$ be a materially uniform inhomogeneous body and $\text{Ref}_B = \{\mathcal{K}_X^R\}_{X \in \mathcal{B}}$ be its uniform reference. The response on actual configuration $x \in \mathcal{C}(\mathcal{B})$ at a point $X$ is determined by the value $\mathcal{R}_X(T_X x)$. Now choose some other configuration $x_I \in \mathcal{C}(\mathcal{B})$. One gets \[ \mathcal{R}_X(T_X x) = \mathcal{R}_X(F_X G_X^{-1}_X), \]

where $F_X = T_{x_I(x)}(x \circ \mathcal{K}_I^{-1})$, while $G_X = T_{x_I(x)}[\mathcal{K}_X^R]^{-1}$. These tensors have the following interpretation: $F_X$ is just deformation gradient while $G_X^{-1}$ is relaxation of separated infinitesimal volume at $x_I$ to the uniform state. The following commutative diagrams illustrate obtained relations:
12°. Response function relative to intermediate shape. Denote \( \mathbf{H}_X = \mathbf{F}_X \mathbf{G}_X \). Then \( \mathfrak{R}_X(\mathbf{T}_X) = \mathfrak{R}_X(\mathbf{H}_X \mathfrak{R}_X) \). Since \( \mathfrak{R}_X \) is uniform reference, then according to [3], there exists a function \( \delta_{\text{Ref}_\mathfrak{R}} : \text{InvLin}(\mathfrak{Y}; \mathfrak{V}') \to \mathcal{R} \), where \( \text{InvLin}(\mathfrak{Y}; \mathfrak{V}') \) is the vector space of invertible linear mappings from \( \mathfrak{V}' \) to \( \mathfrak{V} \), such that
\[
\forall \mathbf{X} \in \mathfrak{B} \quad \forall \mathbf{A} \in \text{InvLin}(\mathfrak{V}; \mathfrak{V}') : \quad \delta_{\text{Ref}_\mathfrak{R}}(\mathbf{A}) = \mathfrak{R}_X(\mathbf{A} \mathfrak{R}_X)
\]
The function \( \delta_{\text{Ref}_\mathfrak{R}} \) is called the response function of the body relative to the uniform reference \( \text{Ref}_\mathfrak{R} \). Choosing \( \mathbf{A} = \mathbf{H}_X \) one gets
\[
\mathfrak{R}_X(\mathbf{T}_X) = \delta_{\text{Ref}_\mathfrak{R}}(\mathbf{H}_X)
\]  
This equality implies that \( \delta_{\text{Ref}_\mathfrak{R}} \) can be used for measuring response; its argument \( \mathbf{H}_X \) is called the total distortion from the uniform state to the actual state. It is decomposed into “anti-relaxation” \( \mathbf{G}_X \) from crystal reference state into intermediate state and deformation \( \mathbf{F}_X \) from the intermediate state into actual state. In this regard, we refer to \( \mathfrak{X}_I \) as intermediate configuration.

Note that the family \( \{ \mathbf{H}_X \}_{X \in \mathfrak{B}} \), like \( \text{Ref}_\mathfrak{R} \), is incompatible, while \( \{ \mathbf{F}_X \} \) is compatible.

If intermediate configuration \( \mathfrak{X}_I \) is fixed then from the equality (6) it follows that
\[
\mathfrak{R}_X(\mathbf{T}_X) = \mathfrak{R}_X(\mathbf{F}_X),
\]
where \( \mathfrak{R}_X : \mathfrak{B} \times \text{InvLin}(\mathfrak{V}; \mathfrak{V}') \to \mathcal{R} \) is a new response function. We finally calibrated the response functional \( \mathfrak{R}_X \) with respect to the intermediate shape \( \delta_{\mathfrak{X}_I} \), i.e., with respect to linear mappings \( \mathbf{G}_X \). Thus, the shape \( \delta_{\mathfrak{X}_I} \) can be served as a reference. The disadvantage of this approach is that the body, if being isotropic with respect to \( \mathfrak{R} \), may become anisotropic with respect to \( \mathfrak{H} \). To get rid of this disadvantage we use another methodology: the body \( \mathfrak{B} \) is endowed by such metric and connection that turn it into uniform reference shape, non-Euclidean in general. Such construction doesn’t affect on material properties of the body.

13°. Structural inhomogeneity. If one can choose configurations \( \mathfrak{X}_X \in \mathfrak{H}_X \) in such a way that they all coincide with some configuration \( \mathfrak{X} \), then we refer to the body \( \mathfrak{B} \) as structurally homogeneous and to the shape \( \delta_{\mathfrak{X}_I} \) as uniform shape. In this case the response at each point \( x \in \delta_{\mathfrak{X}} \) is similar to \( \mathfrak{R}_X(t_{\mathfrak{X}_I}x) \). If the body \( \mathfrak{B} \) is not homogeneous, it is referred to as structurally inhomogeneous. The inhomogeneity may be caused by a variety of phenomena. Among them are continuously distributed defects and layerwise shrinkage due to AM process.

1.3. Material Connections

14°. Smoothness assumption. For obtaining material connections we will need extra condition on uniform reference, a smoothness condition. At first stage, introduce a special vector bundle. Construct disjoint union
\[
\mathcal{E}(\mathfrak{B}) = \bigsqcup_{X \in \mathfrak{B}} \mathfrak{Y} \otimes T_X^\ast \mathfrak{B},
\]
and define the map \( \pi : \mathcal{E}(\mathfrak{B}) \to \mathfrak{B} \) as follows: \( \pi(\mathbf{X}, \nu) = \mathbf{X} \). Then \( \mathcal{E}(\mathfrak{B}), \mathfrak{B}, \pi \) is a vector bundle [10] with total space \( \mathcal{E}(\mathfrak{B}) \), base \( \mathfrak{B} \) and projection \( \pi \). Now let \( \text{Ref}_\mathfrak{B} = \{ \mathfrak{H}_X \}_{X \in \mathfrak{B}} \) be a uniform reference. It generates a section\(^3\) \( \mathfrak{H}_X : \mathfrak{B} \to \mathcal{E}(\mathfrak{B}) \) of the vector bundle \( \mathcal{E}(\mathfrak{B}), \mathfrak{B}, \pi \); \( \mathfrak{H}_X(x) = \mathfrak{H}_X \). In the present section we require from \( \mathfrak{H}_X \) to be smooth. In particular, it means that if
\[
\mathfrak{H}_X = [\mathfrak{H}_X, \nu]_{e_m} x d\mathbb{X}^m |x,
\]
(see 7°), then \( [[\mathfrak{H}_X]_{n}^m] \) form smooth field of invertible matrices.

\(^3\) If \( \mathcal{E}, \mathfrak{M}, \pi \) is a vector bundle with total space \( \mathcal{E} \), base \( \mathfrak{M} \) and projection \( \pi \), a section of this bundle is a map \( s : \mathfrak{M} \to \mathcal{E} \), such that \( s(p) \in \mathcal{E}_p \) for all \( p \in \mathfrak{M} \). Here \( \mathcal{E}_p = \pi^{-1}(\{p\}) \) is a fiber over \( p \).
Remark 6. The smoothness assumption for the section $\mathcal{K}^R$ may be too restrictive. Wang in [15] proposed weaker assumption, when one has several mappings $\mathcal{K}^R$, each of which is a local section of the vector bundle $(\mathcal{E}(B), B, \pi)$. Let $B$ be a materially uniform body. By a reference element$^4$ we mean an ordered pair $(U, \mathcal{K}^R)$, where $U \subset B$ is an open set, while $\mathcal{K}^R : U \ni X \mapsto \mathcal{K}^R | X \in \mathcal{E}(B)$ is a smooth field of local configurations such that there exists a function $\mathcal{N}_U : \text{InvLin}(\mathcal{F}; \mathcal{F}) \to R$, for which
\[ \forall X \in U \quad \forall A \in \text{InvLin}(\mathcal{F}; \mathcal{F}) : \quad \mathcal{N}_U(A) = \mathcal{K}_X(A \mathcal{K}^R | X) \]
In other words, $\mathcal{K}^R$ is a smooth uniform reference for the body $U$.

Two reference elements $(U, \mathcal{K}^R)$ and $(V, \mathcal{K}^R)$ are said to be compatible if $\mathcal{N}_U = \mathcal{N}_V$. Let $A = \{(U, \mathcal{K}^R)\}_{U \in I}$ be a collection of mutually compatible reference elements such that $\bigcup_{U \in I} U = B$. Denote by $\mathcal{A}_{\text{max}}$ a maximal (relative to subset relation) collection among all such $A$'s. Note, that all response functionals $\mathcal{N}_U$ coincide. One can put
\[ \mathcal{N}_{\text{max}}(A) := \mathcal{N}_U(A) \]
The mapping $\mathcal{N}_{\text{max}}$ is called the response functional relative to $\mathcal{A}_{\text{max}}$.

The generalized smoothness assumption can be formulated as follows: for the body $B$ there exists a family $\mathcal{A}_{\text{max}}$. Note, that by the assumption, given in the main text, the body is covered by one reference element $(B, \mathcal{K}^R)$. In what follows, for simplicity, only such case is considered. $\blacklozenge$

Remark 7. As it was already mentioned, the tangent bundle $T^*B$ has simple structure: it is diffeomorphic to the product manifold $B \times \mathbb{R}^3$. Among all possible smooth global trivializations$^5$ $\Psi : T^*B \to B \times \mathbb{R}^3$ there is a specific one, generated by the smooth field $\mathcal{K}^R$ [15]. Indeed, let $I : \mathcal{F} \to \mathbb{R}^3$ be the vector space isomorphism (one of possible isomorphisms). Then for any $X \in B$ the mapping $\Psi_X : T_XB \to \{X\} \times \mathbb{R}^3$ defined as $\Psi_X(u) := (X, I \circ \mathcal{K}^R | u)$ is a vector space isomorphism. Define the mapping $\Psi : T^*B \to B \times \mathbb{R}^3$ by gluing $\Psi_X$'s together: $\Psi(u) := \Psi_{\sigma(u)}(u)$, where $\pi : T^*B \to B$ is the projection. So obtained mapping $\Psi$ is the desired global trivialization. $\blacklozenge$

15°. Material metric. We intend to turn the body $B$ into reference shape. To this end, one needs a machinery that allows one to measure lengths and angles of differential elements in undeformed, i.e., reference state. From the Calculus on manifolds it is known that such machinery is provided by Riemannian metric [10,16] on $B$. We intend to introduce such metric on $B$, that if $B$ is homogeneous, this metric is just Euclidean one. Otherwise, it is Riemannian and may serve as a some sort of inhomogeneity measure.

Suppose that the family $\mathcal{Ref}_B = \{\mathcal{K}_X^R\}_{X \in B}$ of reference configurations is chosen from $\mathcal{Ref}_B$. Introduce the collection $\{\mathcal{G}(X)\}_{X \in B}$ of Riemannian metrics on $B$ which are pullbacks of the Euclidean metric $g$, relative to the configurations $\mathcal{K}_X^R$:
\[ \mathcal{G}(X) := (\mathcal{K}_X^R)^* g \]
By the definition, one gets
\[ \forall \bar{X} \in B \quad \forall u, v \in T_{\bar{X}}B : \quad \mathcal{G}(X)|_{\bar{X}}(u, v) = \mathcal{T}_{\bar{X}}(u) \cdot \mathcal{T}_{\bar{X}}(v) \]
Here $T_{\bar{X}} : T_{\bar{X}}B \to \mathcal{F}$ is tangent map. The corresponding Levi-Civita connection is Euclidean. Using the family $\{\mathcal{G}(X)\}_{X \in B}$ one can synthesize new Riemannian metric, which is non-Euclidean in general case [3,12]. The section $\mathcal{G} \in \text{Sec}(T^*B \otimes T^*B)$, defined as
\[ \mathcal{G}_X := \mathcal{G}(X)|_X, \]
is of class $C^\infty$ by virtue of the smoothness assumption (see 14°). Each of values $\mathcal{G}$ represents a symmetric positive definite bilinear form. Hence, $\mathcal{G}$ is a Riemannian metric on $B$. We refer to $\mathcal{G}$ as material metric. Thus, by construction,
\[ \forall X \in B \quad \forall u, v \in T_XB : \quad \mathcal{G}_X(u, v) = T_X \mathcal{K}_X^R(u) \cdot T_X \mathcal{K}_X^R(v) \quad (7) \]
$^4$ In terminology of Wang, reference chart [15].

$^5$ A global trivialization of the tangent bundle $T^*B$ is a smooth map $\Psi : T^*B \to B \times \mathbb{R}^3$, such that $pr_1 \circ \Psi = \pi$ and for each $X \in B$ the restricted map $\Psi|_{T_XB} : T_XB \to \{X\} \times \mathbb{R}^3$ is a vector space isomorphism [10]. Here $pr_1 : B \times \mathbb{R}^3 \to B$ and $\pi : T^*B \to B$ are the projections.
In terms of local configurations from $\text{Ref}_\mathfrak{B} = \{ \mathcal{K}_\chi^R \}_{\chi \in \mathfrak{B}}$, the relation (7) turns into [3]

$$\forall \chi \in \mathfrak{B} \, \forall u, v \in T\chi \mathfrak{B} : \quad \mathcal{G}_\chi(u, v) = \mathcal{K}_\chi^R(u) \cdot \mathcal{K}_\chi^R(v)$$ (8)

16°. Connection induced by material metric. The material metric $\mathcal{G}$ defines Levi-Civita material connection. Coefficients $\Gamma^k_{ij}$ of the latter in a coordinate frame $(\partial_i)$ can be calculated using the following expressions:

$$\Gamma^k_{ij} = \frac{G^{kl}}{2} \left( \partial_i G_{jl} + \partial_j G_{il} - \partial_l G_{ij} \right), \quad \text{where} \quad G_{ij} = g_{sp} K^s_k K^p_j$$

Here $[G^{ij}] = [G_{ij}]^{-1}$, while $g_{sp}$ are components of physical metric in curvilinear coordinates. In particular, for the case of Cartesian coordinates one has $G_{ij} = \delta_{sp} K^s_k K^p_j$.

Obtained connection is non-Euclidean in general. Actually, one gets the following representation of Riemann curvature tensor components in coordinate frame:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{jl} \Gamma^j_{ik} - \Gamma^l_{il} \Gamma^j_{jk}$$

The deviation from Euclidean structure can be measured by the values of scalar curvature $\text{Ric}$ which is defined as contraction

$$\text{Ric} = G^{ij} R_{ij}$$

Here $R_{ij} = R^l_{ij}$ are components of Ricci curvature tensor in coordinate frame [17]. Thus, scalar curvature can be considered as the invariant measure of the structural inhomogeneity.

17°. Archetype and implant field. Another way to obtain material connection is to use E. Cartan’s moving frame procedure. The response of the simple body is described by the field of local configurations (linear mappings). At the same time, the moving frame procedure is performed for frame obtained from orthonormal frame by some field of linear mappings. There is a correspondence between local shapes and space of absolute parallelism (Weitzenböck space). Such correspondence is considered below.

Let $\mathfrak{B}$ be materially uniform inhomogeneous simple body with a uniform reference $\text{Ref}_\mathfrak{B} = \{ \mathcal{K}_\chi^R \}_{\chi \in \mathfrak{B}}$ and the corresponding family $\mathcal{U} = \{ \Phi_{\chi_0} \chi \}_{\chi \in \mathfrak{B}}$ of material isomorphisms. Recall that the reference can be represented equivalently as a field $\mathcal{K}_\chi^R : \chi \mapsto \mathcal{K}_\chi^R$ of local configurations where each of them is isomorphism.

Any tangent space $T\chi \mathfrak{B}$ can be obtained from some fixed, $T\chi_0 \mathfrak{B}$, as $T\chi \mathfrak{B} = \Phi_{\chi_0} \mathfrak{B} = \Phi_{\chi_0} \mathcal{K}_\chi^R \mathfrak{B}$. Such relation means that $T\chi_0 \mathfrak{B}$ can be served as a reference (archetype, [18]). Its spatial representation is $\mathcal{K}_\chi^R[T\chi_0 \mathfrak{B}]$ and it coincides with $\mathcal{Y}$. Any infinitesimal neighborhood $T\chi \mathfrak{B}$ can be obtained from the spatial representation of an archetype, that is, from $\mathcal{Y}$, by means of linear mapping

$$\Omega_{\chi} = [\mathcal{K}_\chi^R]^{-1} : \mathcal{Y} \rightarrow T\chi \mathfrak{B}$$

The field $\Omega : \chi \mapsto \Omega_{\chi}$ is referred to as implant [14]. In the present considerations together with notation $\Omega$ we will use the notation $\Omega = \mathcal{K}_\chi^R$.

18°. The moving frame method generated by implant and orthonormal frame. Let $(i_k)_{k=1}^3$ be an orthonormal frame for $\mathcal{Y}$. One gets moving crystallographic frame $(z_k)_{k=1}^3$ by

$$z_k := \Omega(i_k) = \Omega_k^i \partial_i,$$

where $(\partial_i)_{i=1}^3$ is a coordinate frame on the body $\mathfrak{B}$. Due to inhomogeneity of the body, the frame $(z_i)_{i=1}^3$ is nonholonomic, i.e., Lie brackets not vanish:

$$[z_i, z_j] = -c_{ij}^k z_k,$$

\[6\] Here we use dyadic decomposition established in 7°.
where $c_{ij}^k$ are the objects of anholonomity [8]. Parallelize the frame $(z_i)_{i=1}^3$ by introducing affine connection $\nabla$ on $\mathcal{B}$ with the following requirement:

$$\nabla z_i, z_j = 0, \quad i, j = 1, 2, 3$$

Thus, $\nabla$ represents such parallel translation rule, that $z_j$ is parallel translated along the curve generated by $z_i$. Figuratively speaking, vectors of frames $(z_k|x)^3_{k=1}$ and $(z_k|y)^3_{k=1}$, where $x, y \in \mathcal{B}$, are mutually “parallel”. We refer to the connection $\nabla$ as Weitzenböck connection [19,20]. The body $\mathcal{B}$ with geometry induced by $\nabla$ turns into non-Euclidean globally uniform shape.

The connection $\nabla$ coefficients in the frame $(z_i)_{i=1}^3$ are equal to zero, i.e., $\Gamma_{jk}^i = 0$. Using the relation [8]

$$\Gamma_{jk}^i = \Gamma_{m}^{ij} \Omega_m^i \Omega_k^j + \Omega_j^i \partial_k \Omega_m^m,$$

between functions $\Gamma_{jk}^i$ and $\Gamma_{jk}^i$, where the latter are connection coefficients in coordinate frame $(\partial_X)$, one gets

$$\Gamma_{m}^{ij} \Omega_m^i \Omega_k^j + \Omega_j^i \partial_k \Omega_m^m = 0$$

This expression implies the following formulae for $\Gamma_{jk}^i$:

$$\Gamma_{jk}^i = - \Omega_k^c \partial_j \Omega_c^i + \Omega_j^i \partial_c \Omega_k^c$$ \hspace{1cm} (9)

In terms of $\mathcal{X}^R$, one gets:

$$\Gamma_{jk}^i = - [\mathcal{X}^R]_c^i \partial_j [\mathcal{X}^R]^{-1} c^i = [\mathcal{X}^R]^{-1} c^i \partial_j [\mathcal{X}^R]_c^i$$

19°. Materially constant vector field. Let $u \in \text{Vec}(\mathcal{B})$ be a vector field. Suppose that $\mathcal{X}^R u = a = \text{const} \in \mathcal{Y}$. Pointwisely it signifies that

$$\forall X \in \mathcal{B} : \quad \mathcal{X}^R u|_X = a$$

Following [3], we refer to the field $u$ as materially constant. It has the following sense: the differential elements $u|_X$ have the similar uniform pre-image $a$. In particular, each element of the frame $(z_k|x)^3_{k=1}$ is a materially constant vector field.

In light of the obtained Weitzenböck connection $\nabla$, a materially constant field $u$ has the following property: $\nabla u = 0$ for all vector fields $v \in \text{Vec}(\mathcal{B})$. To establish this, it is sufficient to prove that $\nabla \partial_i u = 0$ for all $i \in \{1, 2, 3\}$. Indeed, one has $\nabla \partial_i u = (\partial_i u^k + u^j \Gamma_{ij}^k) \partial_k$. Using the first expression (with “minus” sign) of (9) for $\Gamma_{ij}^k$, one gets

$$\partial_i u^k + u^j \Gamma_{ij}^k = \partial_i u^k - w^j \Omega_c^j \partial_i \Omega_c^k = a^c \partial_i \Omega_c^k - \partial_i \Omega_c^k = 0,$$

since $w^j \Omega_c^j = a^c$, $u^k = \Omega_c^k a^c$ and $a$ is a constant vector. The established property, in particular, implies that materially constant vector field can be parallel transported along any curve in $\mathcal{B}$.

The connection $\nabla$ is the unique connection for which the property “$\nabla u = 0$ for any materially constant vector $u$” holds. Indeed, suppose that $\nabla$ is an affine connection with such the property and $u$ is a materially constant vector field. Then one has $\nabla \partial_i u = 0$ for all $i \in \{1, 2, 3\}$. This implies $a^c (\partial_i \Omega_c^k + \Omega_c^j \Gamma_{ij}^k) = 0$, where $u^k = \Omega_c^k a^c$ and $\Gamma_{ij}^k$ are connection $\nabla$ coefficients in coordinate frame. Since $u$ is arbitrary materially constant vector field, the vector $a = \mathcal{X}^R u$ is arbitrary as well. It means that $\partial_i \Omega_c^k + \Omega_c^j \Gamma_{ij}^k = 0$ and, finally, $\Gamma_{ij}^k = - \Omega_j^i \partial_i \Omega_c^k$. Thus, $\nabla = \tilde{\nabla}$. 

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[The rest of the text is not included as it continues with more complex mathematical content that is not necessary for this query.]
20°. Torsion of connection. The torsion tensor of obtained connection is defined as

$$\mathbf{T} = (\Gamma^k_{ij} - \Gamma^k_{ji}) \partial_k \otimes dx^i \otimes dx^j$$

Its components are nonzero in general:

$$\mathbf{T}^k_{ij} = \left([\mathcal{R}^R]^{-1}k_r \partial_i [\mathcal{R}^R]_r^j - [\mathcal{R}^R]^{-1}k_r \partial_j [\mathcal{R}^R]_r^i, \right.$$

and serve as measure of inhomogeneity. Indeed, if the body is structurally homogeneous then

$$\mathcal{R}^R = T\mathbf{x}$$

for some $$\mathbf{x} \in \mathcal{C}(\mathcal{B})$$. It means that, in local coordinates, $$[\mathcal{R}^R]_j^i = \frac{\partial x^i}{\partial x^j}$$, and

$$\mathbf{T}^k_{ij} = \left([\mathcal{R}^R]^{-1}k_r \partial_i [\mathcal{R}^R]_r^j - [\mathcal{R}^R]^{-1}k_r \partial_j [\mathcal{R}^R]_r^i, \right.$$

due to the Schwartz theorem of commutativity of the successive partial derivatives.

2. Connections from Geometrical Viewpoint

21°. Affine connection. Suppose that $$\mathcal{M}$$ is a smooth n-manifold. An affine connection is a mapping $$[21, 22]$$

$$\nabla : \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \to \text{Vec}(\mathcal{M}), \ (u, v) \to \nabla_u v,$$

such that

$$(\nabla_a) \forall u, v, w \in \text{Vec}(\mathcal{M}): \quad \nabla_{u+w} v = \nabla_u v + \nabla_w v;$$

$$(\nabla_b) \forall u, v \in \text{Vec}(\mathcal{M}) \quad \forall f \in C^\infty(\mathcal{M}; \mathbb{R}): \quad \nabla_u (fv) = f \nabla_u v;$$

$$(\nabla_c) \forall u, v, w \in \text{Vec}(\mathcal{M}): \quad \nabla_u (v+w) = \nabla_u v + \nabla_u w;$$

$$(\nabla_d) \forall u, v \in \text{Vec}(\mathcal{M}) \quad \forall f \in C^\infty(\mathcal{M}; \mathbb{R}): \quad 
abla_u (fv) = f \nabla_u v + (uf) v.$$

In a fixed local frame $$\mathcal{M} \supset U \ni p \mapsto (e_i|_p)_{i=1}^n$$ and the corresponding local coframe

$$\mathcal{M} \supset U \ni p \mapsto (\partial^i|_p)_{i=1}^n,$$

where $$U$$ is an open set, the mapping $$\nabla$$ is defined by $$n^3$$ functions

$$\Gamma^k_{ij} = \langle \partial^i, \nabla e_j e_k \rangle \in C^\infty(\mathcal{M}; \mathbb{R}).$$

**Remark 8.** In particular, consider coordinate frame. Since $$\nabla_{\partial_u} \partial_v = \Gamma^k_{ij} \partial_k$$, for $$u = u^i \partial_i$$, $$v = v^j \partial_j$$, one gets the following coordinate representation for the covariant derivative $$\nabla_u v$$:

$$\nabla_u v = \sum_{i=1}^n \sum_{k=1}^n u^i \left( \partial_v^k + \sum_{j=1}^n v^j \Gamma^k_{ij} \right) \partial_k = \sum_{i=1}^n \sum_{k=1}^n u^i \nabla_i v^k \partial_k,$$

where $$\nabla_i v^k = (dx^k, \nabla_{\partial_i} v)$$.

22°. Torsion, curvature and nonmetricity. Different affine connections defined on the underlying manifold $$\mathcal{M}$$ with Riemannian metric $$g$$ generate different spaces. To classify them one uses tensor fields of torsion

$$\Theta : \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \to \text{Vec}(\mathcal{M}),$$

curvature

$$\mathcal{R} : \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \to \text{Vec}(\mathcal{M}),$$

and nonmetricity

$$\Omega : \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathbb{R}),$$

which are defined by relations:

$$\Theta(u, v) = \nabla_u v - \nabla_v u - [u, v], \quad (10)$$
\[ \mathcal{R}(u, v, w) = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w, \]  
\[ \Omega(u, v, w) = -\nabla g \]  
(11)  
(12)

Here \([u, v]\) are Lie brackets that act on smooth scalar function \(f \in C^\infty(\mathcal{M}; \mathbb{R})\) as \([u, v]f := u(vf) - v(uf)\).

\textbf{23}. \textit{Autoparallel curves and geodesics.} Let \(\nabla\) and \(g\) be an affine connection and metric on \(\mathcal{M}\). We give some preliminary definitions. A smooth curve \(\chi : I \to \mathcal{M}\), where \(I \subset \mathbb{R}\) is an interval, is called \textit{autoparallel} if

\[ \nabla_{\chi'} \chi' = 0, \]

where \(\chi'\) is velocity vector of the curve \(\chi\). In 4D-formalism (when \(\mathcal{M}\) is considered as a Newtonian space-time) the field \(\nabla_{\chi'} \chi'\) represents acceleration of particle that moves on trajectory \(\chi\). That is, \(\chi\) is a trajectory of particle that moves with influence of no forces \([8]\). In componentwise form,

\[ \dddot{\chi}^q + \Gamma^q_{ij} \chi^i \dot{\chi}^j = 0, \quad q = 1, \ldots, n \]  
(13)

Here \(\chi^i\) are components of the curve \(\chi\) and \(\Gamma^q_{ij}\) are the connection \(\nabla\) coefficients.

\textbf{Remark 9.} Curves generated by materially constant vector fields are autoparallel \([23]\). Indeed, let \(\mathfrak{B}\) be a body equipped with Weitzenböck connection \(\nabla\). Recall that if \(u \in \text{Vec}(\mathfrak{B})\) is a materially constant vector field then \(\nabla_u u = 0\) for every \(v \in \text{Vec}(\mathfrak{B})\) (see \(19^\circ\)). In particular, \(\nabla_u u = 0\). Now let \(\chi : I \to \mathcal{M}\) be a curve generated by \(u\), \(\text{i.e.}, \chi'(t) = u_{\chi(t)}\) for all \(t \in I\). Then \(\nabla_{\chi'} \chi' = 0\) and \(\chi\) is autoparallel curve. \(\blacklozenge\)

Let \(\chi : [a, b] \to \mathcal{M}\) be again a smooth curve. Its length is defined as

\[ l(\chi) = \int_a^b \sqrt{g(\chi'(t), \chi'(t))} \, dt, \]

or, componentwisely, in coordinate frame,

\[ l(\chi) = \int_a^b \sqrt{g_{ij}(\chi^1, \ldots, \chi^n) \dot{\chi}^i \dot{\chi}^j} \, dt \]

\textbf{Remark 10.} Suppose that \(\mathfrak{B}\) is a body equipped with Riemannian metric \(\mathcal{g}\) \((8)\). The length of any smooth curve \(\chi : [a, b] \to \mathfrak{B}\) is then can be calculated as

\[ l(\chi) = \int_a^b \sqrt{\mathcal{R}_{\chi(t)}(\chi'(t)) \cdot \mathcal{R}_{\chi(t)}(\chi'(t))} \, dt \]

Note that for any \(t \in [a, b]\) the translation vector \(\mathcal{R}_{\chi(t)}(\chi'(t))\) represents differential element in uniform state. Denote this spatial counterpart by \(a_{\chi}(t)\). Then the number \(\sqrt{a_{\chi}(t) \cdot a_{\chi}(t)}\) can be interpreted as “undeformed” length of \(\chi'(t)\). Thus, we obtained the vector field \(t \to a_{\chi}(t)\). Let \(\sigma_{\chi} : [c, d] \to \mathfrak{B}\) be a curve generated by this field, \(\text{i.e.}, \sigma_{\chi}'(t) = a_{\chi}(t)\), for all \(t \in [c, d]\). Here \([c, d] \subset [a, b]\) is some segment. Metaphorically speaking, \(\sigma_{\chi}\) is assembled from vectors \(\mathcal{R}_{\chi(t)}(\chi'(t))\) relaxed to uniform state. The number

\[ l(\sigma_{\chi}) = \int_c^d \sqrt{a_{\chi}(t) \cdot a_{\chi}(t)} \, dt, \]

represents the length of the curve \(\sigma_{\chi}\). This length coincides with the length of material curve \(\chi|[c, d]\) with respect to the metric \(\mathcal{g}\). Note that if the body is structurally inhomogeneous, \(\sigma_{\chi}\), as a whole, belongs neither of the body shape. We arrive at the following property of the material metric \(\mathcal{g}\). If one splits a material curve into differential elements, allow them to relax to uniform state and assemble obtained uniform elements into a curve, the lengths of initial and final curves coincide. \(\blacklozenge\)
A curve $\chi$ that is a stationary point of functional $l : \chi \mapsto l(\chi)$ is called geodesics. The equation $\delta l(\chi) = 0$ reduces to equation of geodesics [8]:

$$\ddot{\chi}^q + \frac{g^{qk}}{2} (\partial_k g_{kj} + \partial_j g_{kj} - \partial_k g_{ij}) \dot{\chi}^i \dot{\chi}^j = 0, \quad q = 1, \ldots, n$$  \hspace{1cm} (14)

Here $[g^{ij}] = [g_{ij}]^{-1}$. We will refer to such connection as Levi-Civita connection. Its torsion and nonmetricity vanish, while curvature tensor $\mathcal{R}$ is nonzero and fully characterizes such space. It defines Ricci tensor $\mathcal{R}$ with components $R_{ij} = R^k_{ikj}$ that, in turn, defines scalar curvature $\text{Ric} = g^{ij} R_{ij}$.

24°. Contortion. Over smooth manifold $\mathcal{M}$ with Riemannian metric $g$ one may construct arbitrary connection $\nabla$ with coefficients $\Gamma^i_{jk}$ in coordinate frame and Levi-Civita connection $\tilde{\nabla}$ with coefficients $\tilde{\Gamma}^i_{jk} = g^{im} (\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk})$. Relation between connections $\nabla$ and $\tilde{\nabla}$ is given by the formula [8]:

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + \frac{g^{im}}{2} (\Sigma_{mjk} + \Sigma_{kmj} + \Sigma_{jmk}) + \frac{g^{im}}{2} (Q_{jkm} + Q_{kmj} - Q_{mjk}),$$  \hspace{1cm} (15)

where $\Sigma_{kij} = g_{km} \Sigma^m_{ij}$. Formula (15) shows that it is sufficient to define Riemannian metric and independent tensors $\Omega$ and $\Sigma$ for introducing an affine connection on the manifold $\mathcal{M}$. Thus, $(g, \Sigma, \Omega)$ can be considered as a triple of independent variables. Suppose that $\tilde{\nabla}$ is metric compatible. Thus, $\Omega = 0$ and relation (15) reduces to

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + \mathcal{R}^i_{jk}$$

Here $\mathcal{R}^i_{jk} = g^{im} (\Sigma_{mjk} + \Sigma_{kmj} + \Sigma_{jmk})$ are components of tensor $\mathcal{R}$ called contortion tensor.

25°. Cartan structural equations. In the frame and coframe $(\epsilon_k)_{k=1}^n$ and $(\vartheta^k)_{k=1}^n$ we define connection one-forms $\omega^i_j = \Gamma^i_{kj} \vartheta^k$. Relations between connection, curvature and torsion can be represented in terms of differential forms\footnote{Note that $d\vartheta^i \neq 0$, in general.} [1]:

$$T^i = d\vartheta^i + \omega^i_j \wedge \vartheta^j,$$

$$R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j,$$

where $T^i$ is torsion form, $R^i_j$ is curvature form:

$$T^i = \frac{1}{2} \Sigma^i_{mk} \vartheta^m \wedge \vartheta^k, \quad R^i_j = \frac{1}{2} \Sigma^i_{mkj} \vartheta^m \wedge \vartheta^k$$

26°. Dislocation density tensor as torsion form. Let us consider relation between dislocation mechanics and geometry of affine connections. Suppose that $\mathcal{B}$ is a body endowed
by Weitzenböck connection. The dislocation density tensor \( \alpha \in \text{Sec}(TB \otimes TB) \) is related with torsion two-form \( T^i = d\vartheta^i \) as \([2, 4]\)

\[
\alpha = e_i \otimes (\ast T^i)^2,
\]

where \( \ast \) is the Hodge star operator \([16]\) and \( (\cdot)^2 \) is musical isomorphism that transforms covector field to vector field. In components, relatively to coordinate frame, one gets

\[
\alpha^{ij} = \frac{1}{2} \epsilon_{ijk} \epsilon^{jmk},
\]

(16)

Here \( \epsilon \) is Levi-Civita tensor with components

\[
e^{ijk} = \frac{1}{\sqrt{g}} e^{ijk},
\]

in which \( g = \det[g_{ij}] \) is determinant of Riemannian metric \( g \) and \( e^{ijk} \) is the alternator.

3. Affinity of Geometrical Concepts and Fundamentals of Body Kinematics

27°. Two-dimensional physical space. In the present section we consider an illustrative example for notions related with parallel translation. To this end, let \( \mathcal{E}^2 \) be a two-dimensional Euclidean affine space with translation vector space \( \mathcal{V}^2 \) and scalar product \( (\cdot) \) on \( \mathcal{V}^2 \). Thus, \( \mathcal{E}^2 \) is a set with continuum cardinality and \( \mathcal{V}^2 \) is a two-dimensional real vector space related with \( \mathcal{E}^2 \) by two operations\(^8\):

\[
-: \mathcal{E}^2 \times \mathcal{E}^2 \to \mathcal{V}^2, \quad (a, b) \mapsto b - a,
\]

\[
+: \mathcal{E}^2 \times \mathcal{V}^2 \to \mathcal{E}^2, \quad (a, u) \mapsto a + u.
\]

The metric \( d(a, b) := \sqrt{(b - a) \cdot (b - a)} \) introduces open ball topology in \( \mathcal{E} \). In particular this means that one can consider continuous curves \( c : \mathbb{R} \to \mathcal{E}^2 \). If one chooses some points \( p_1, p_2 \in \mathcal{E}^2 \) then a curve with ends at these points can be defined as \( c : [a, b] \to \mathcal{E}^2, c(a) = p_1, c(b) = p_2 \) (see figure 1). Here \( [a, b] \subset \mathbb{R} \) is a segment. In particular, \( c \) may be a trajectory of some particle, which begins its motion at \( p_1 \) and stops it at \( p_2 \). Then \( [a, b] \) is the time interval.

![Figure 1. A curve in physical space.](Image)

28°. Coordinate systems. Various coordinate systems can be introduced in \( \mathcal{E}^2 \). Among them there is special, Cartesian system \( (o, \{i_k\}_{k=1}^2) \), the existence of which is guaranteed by

\(^8\) In this, the axioms \( W_1 \) and \( W_2 \) from 2° hold.
ordered pair \((\mathbf{p}, \mathbf{q})\) is origin and \(i_k \cdot i_l = \delta_{kl}, k, l \in \{1, 2\}\). Cartesian system generates two families of mutually orthogonal straight lines:

\[
d_1 = \{\mathbf{p} \in \mathbb{R}^2 : \exists \beta \in \mathbb{R} : \mathbf{p} = o + \beta \mathbf{i}_1 + \alpha \mathbf{i}_2\}_{\alpha \in \mathbb{R}},
\]

\[
d_2 = \{\mathbf{p} \in \mathbb{R}^2 : \exists \beta \in \mathbb{R} : \mathbf{p} = o + \alpha \mathbf{i}_1 + \beta \mathbf{i}_2\}_{\alpha \in \mathbb{R}},
\]

and assigns with each point \(\mathbf{p} \in \mathbb{R}^2\) the ordered pair \((x^1, x^2) := (i_1 \cdot (\mathbf{p} - o), i_2 \cdot (\mathbf{p} - o)) \in \mathbb{R}^2\).

Together with Cartesian system we introduce polar coordinate system. It generates two families of curves (circles and rays):

\[
c_1 = \{\mathbf{p} \in \mathbb{R}^2 : \exists r \in \mathbb{R} : \mathbf{p} = o + i_1 r \cos \varphi + i_2 r \sin \varphi\}_r \in \mathbb{R}_+,
\]

\[
c_2 = \{\mathbf{p} \in \mathbb{R}^2 : \exists r \in \mathbb{R}_+ : \mathbf{p} = o + i_1 r \cos \varphi + i_2 r \sin \varphi\}_{\varphi \in [-\pi, \pi]},
\]

where \(\mathbb{R}_+\) is the set of real numbers \(\geq 0\). Polar system associates with each point \(\mathbf{p} \in \mathbb{R}^2\) the ordered pair \((r, \varphi) = (\sqrt{i_1 \cdot (\mathbf{p} - o)^2 + i_2 \cdot (\mathbf{p} - o)^2}, \arctan (i_2 \cdot (\mathbf{p} - o), i_1 \cdot (\mathbf{p} - o))\).

**Remark 11.** The function \(\arctan\) is defined as follows for all \((a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}:

\[
\arctan (a, b) = \begin{cases} 
\arctan (a/b) + \pi \frac{1 - \text{sgn} b}{2} \text{sgn} a, & \text{if } b \neq 0, \\
\frac{\pi}{2} \text{sgn} a, & \text{if } b = 0
\end{cases}
\]

Here \(\arctan\) is the primary branch of inverse to tan function, defined on the set \([-\pi/2, \pi/2]\).

Denote by \(h_1 : \mathbb{R}^2 \to \mathbb{R}^2\) the coordinate homeomorphism \(\mathbf{p} \mapsto (x^1, x^2)\) generated by Cartesian system, and denote by \(h_2 : \mathbb{R}^2 \to \mathbb{D}^2\) the coordinate homeomorphism \(\mathbf{p} \mapsto (r, \varphi)\) generated by polar system. Here \(\mathbb{D}^2 = \{(a, b) \in \mathbb{R}^2 : a > 0, b \in [-\pi, \pi]\}\). The transition map \(h_{12} : \mathbb{R}^2 \to \mathbb{D}^2, (x^1, x^2) \mapsto (r, \varphi)\) is defined as

\[
h_{12}(x^1, x^2) = (\sqrt{(x^1)^2 + (x^2)^2}, \arctan (x^2, x^1)),
\]

while its inverse, \(h_{12}^{-1} : (r, \varphi) \mapsto (x^1, x^2)\), has the following representation:

\[
h_{12}^{-1}(r, \varphi) = (r \cos \varphi, r \sin \varphi)
\]

Coordinate curves, actions of coordinate homeomorphisms and transition map are illustrated in figure 2.

Note that under the actions of coordinate homeomorphisms the curve \(c\) was transformed into curves \(c_1 = h_1 \circ c\) and \(c_2 = h_2 \circ c\).

Polar coordinate system induces field \((e_k)_{k=1}^2\) of local frame and the corresponding field \((e^k)_{k=1}^2\) of local coframe:

\[
e_1 = e_r = \frac{\partial (p \circ h_{12}^{-1})}{\partial r}, \quad e_2 = e_\varphi = \frac{\partial (p \circ h_{12}^{-1})}{\partial \varphi}, \quad e_k \cdot e^l = \delta_k^l, \quad k, l \in \{1, 2\},
\]

where \(p\) is the position vector field. Local frame and coframe are expressed through orthonormal frame \((i_k)_{k=1}^2\) as follows:

\[
e_1 = i_1 \cos \varphi + i_2 \sin \varphi,
\]

\[
e_2 = -i_1 r \sin \varphi + i_2 r \cos \varphi,
\]

\[
e^1 = e_1,
\]

\[
e^2 = -i_1 r^{-1} \sin \varphi + i_2 r^{-1} \cos \varphi
\]
29°. Parallel transport in Euclidean space. In synthetic geometry one uses a ruler and a try square to transport vector, attached at some point of Euclidean plane, to another point in such a way that the result is parallel to original vector. Here vector is represented by a directed line segment, while parallel transport is such operation which result is a directed line segment with same length and direction as the original one. Note that \( \mathcal{E}^2 \) is realization of the Euclidean plane, while \( \mathcal{V}^2 \) is a space of directed line segments. Next, the proposition “a vector is attached at some point” can be formalized as follows. Suppose that some vector field \( f : \mathcal{E}^2 \to \mathcal{V}^2 \) is given. Then for \( p_1 \in \mathcal{E}^2 \) the value \( a = f(p_1) \) can be considered as a vector \( a \) attached at a point \( p_1 \). Now we give the definition of parallel transport of the vector \( a \) along the curve \( c \) from point \( p_1 \) to \( p_2 \). Define a constant vector field \( u : c([a, b]) \to \mathcal{V}^2 \), \( u(p) = a \). Then \( u(p_2) \) is referred to as vector \( a \) parallel transported to point \( p_2 \) along the curve \( c \). Note, that the result of Euclidean parallel transport doesn’t depend on the underlying curve \( c \).

30°. Total derivative vs parallel transport. The notion, closely related to parallel transport, is the operation of differentiation. We are going to talk about it in a moment more in purpose to highlight the specific properties dependent on Euclidean structure. For simplicity we consider mappings \( f : \mathcal{E}^2 \to \mathcal{W} \) defined on the whole set \( \mathcal{E}^2 \). Here \( \mathcal{W} \) is either \( \mathcal{V}^2 \), or \( \mathcal{E}^2 \). The mapping \( f \) is differentiable at a point \( p \in \mathcal{E}^2 \) if there exists a linear map \( D_p f \in \text{Lin}(\mathcal{V}^2, \mathcal{V}^2) \), such that for all \( h \in \mathcal{V}^2 \),

\[
 f(p + h) = f(p) + D_p f[h] + o(\|h\|) \quad \text{as} \quad h \to 0
\]

Here \( \lim_{\|h\| \to 0} \frac{\|o(\|h\|)\|}{\|h\|} = 0 \). The map \( D_p f \) is the total derivative of \( f \) at \( p \). In order to show the implicit dependence of total derivative definition from Euclidean parallel transport rule, consider the case of a vector field \( u \). Let \( h \) be arbitrary translation vector. Using the
In particular this implies \( v \) successively gets \( D_{\rho} w_{\gamma} \) is considered in Cartesian frame. Then \( v \) is attached at point \( \rho \) along some curve \( \gamma \). This results in a vector \( \alpha_{\rho; t} \) at \( \rho \). The vector \( u(\rho) \) is attached at \( \rho \) also. Both vectors belong to \( T_{\rho}\mathbb{S}^2 \) and their difference is well-defined. Thus, by the difference \( u(\rho + \mathbf{t}) - u(\rho) \) one can understand the difference \( \alpha_{\rho; t} - u(\rho) \), where, in Euclidean case, \( \alpha_{\rho; t} = u(\rho + \mathbf{t}) \).

### 31c. Affine connection generated by Euclidean parallel transport

Euclidean parallel transport allows one to calculate values of total derivatives. Total derivatives, in turn, allow one to introduce affine connection on \( \mathbb{S}^2 \). Indeed, define operation \( \nabla \) for smooth vector fields \( \mathbf{u} : \mathbb{S}^2 \to \mathbb{R}^2 \) and \( \mathbf{v} : \mathbb{S}^2 \to \mathbb{R}^2 \) as

\[
\nabla_{\mathbf{u}} \mathbf{v} : \rho \mapsto D_{\rho} \mathbf{v}[u(\rho)]
\]

In order to derive connection coefficients in Cartesian and local frames we consider two cases. In both of them we restrict ourselves at some point \( \rho \in \mathbb{S}^2 \) and take \( \mathbf{t} = u(\rho) \). The first case: \( v \) is considered in Cartesian frame. Then \( v = v_i \mathbf{i}_i \). Decomposing \( h \) by Cartesian frame, one gets \( h = h^k \mathbf{i}_k \). Since \( D_{\rho} \mathbf{v}[h] = h^k D_{\rho} \mathbf{v}[\mathbf{i}_k] \), it is sufficient to deal with \( D_{\rho} \mathbf{v}[\mathbf{i}_k] \). Using (17), one successively gets

\[
D_{\rho} \mathbf{v}[\mathbf{i}_k] = \lim_{t \to 0} \frac{v(\rho + t \mathbf{i}_k) - v(\rho)}{t} = i_i \lim_{t \to 0} \frac{v^l(\rho + t \mathbf{i}_k) - v^l(\rho)}{t} = i_i \partial_k v^l(\rho),
\]

where \( \partial_k v^l(\rho) := \lim_{t \to 0} t^{-1}(v^l(\rho + t \mathbf{i}_k) - v^l(\rho)) \) is partial derivative. Note, that we use absolute quantities like translation vectors and do not appealing to coordinate systems. One finally gets

\[
D_{\rho} \mathbf{v}[h] = i_i h^k \partial_k v^l(\rho).
\]

Thus,

\[
\nabla_{\mathbf{u}} \mathbf{v} = i_i h^k \partial_k v^l.
\]

In particular this implies \( \nabla_{\mathbf{i}_i} \mathbf{i}_i = 0 \) and connection coefficients vanish. Consider the second case, when \( v \) is decomposed over the local frame \( (\mathbf{e}_k)^2_{k=1} \), \( i.e., v = v^l \mathbf{e}_l \). In this, one gets \( h = h^k \mathbf{e}_k|_{\rho} \). As in the first case, \( D_{\rho} \mathbf{v}[h] = h^k D_{\rho} \mathbf{v}[\mathbf{e}_k|_{\rho}] \) and it is sufficient to obtain the value of \( D_{\rho} \mathbf{v}[\mathbf{w}] \), where \( \mathbf{w} = e_k|_{\rho} \). Using (17), one gets

\[
D_{\rho} \mathbf{v}[\mathbf{w}] = \lim_{t \to 0} \frac{v(\rho + tw) - v(\rho)}{t} = \lim_{t \to 0} \frac{v^l(\rho + tw) \mathbf{e}_l|_{\rho + tw} - v^l(\rho) \mathbf{e}_l|_{\rho}}{t}
\]

In contrast to the Cartesian frame, the local basis \( (\mathbf{e}_k)^2_{k=1} \) changes from point to point. Using transformations, similar to those, which allow to obtain product rule, one arrives at:

\[
D_{\rho} \mathbf{v}[\mathbf{w}] = \lim_{t \to 0} \frac{(v^l(\rho + tw) - v^l(\rho)) \mathbf{e}_l|_{\rho + tw} + v^l(\rho) (\mathbf{e}_l|_{\rho + tw} - \mathbf{e}_l|_{\rho})}{t} = \partial_k v^l(\rho) \mathbf{e}_l|_{\rho} + v^l(\rho) \partial_k\mathbf{e}_l(\rho),
\]

where \( \partial_k v^l(\rho) := \lim_{t \to 0} t^{-1}(v^l(\rho + tw) - v^l(\rho)) \), while \( \partial_k \mathbf{e}_l(\rho) := \lim_{t \to 0} t^{-1}(\mathbf{e}_l|_{\rho + tw} - \mathbf{e}_l|_{\rho}) \). Note that \( \partial_k \mathbf{e}_l(\rho) \) is a vector from \( \mathbb{S}^2 \) and it can be decomposed over \( (\mathbf{s}_k)^2_{k=1} \), that is, \( \partial_k \mathbf{e}_l(\rho) = \Gamma^s_{kl} \mathbf{s}_l|_{\rho} \). One finally gets

\[
D_{\rho} \mathbf{v}[h] = h^k \left( \partial_k v^s(\rho) + v^l(\rho) \right) \Gamma^s_{kl} \mathbf{s}_l|_{\rho}
\]
Defining the field $\Gamma^s_{kl} : p \mapsto \Gamma^s_{kl}$, one obtains the following expression for covariant derivative:

$$\nabla_u v = u^k \left( \partial_k v^s + v^l \Gamma^s_{kl} \right) e_s$$

The connection coefficients are exactly $\Gamma^s_{kl}$ and are defined by $\Gamma^s_{kl} = e^s \cdot \partial_k e_l$.

32°. Parallel transport in charts. Consider the sets $\mathbb{R}^2$ and $\mathbb{D}^2$ from the lower half of the figure 2. The left set, $\mathbb{R}^2$, can be endowed by Euclidean affine space structure with translation vector space $\mathbb{R}^2$, equipped with the coordinatewise operations of addition and scalar multiplication, and scalar product defined as $\langle x^1, x^2 \rangle \cdot \langle y^1, y^2 \rangle = x^1 y^1 + x^2 y^2$. The role of orthonormal frame is then played by vectors $i_1 = (1, 0)$ and $i_2 = (0, 1)$. Note, that the space $\mathbb{R}^2$ is connected with $\mathbb{D}^2$ by the mapping $h_1$, which is affine-linear with respect to each argument. In this regard, one may consider $\mathbb{R}^2$ as a “model” of $\mathbb{D}^2$.

Both $\mathbb{R}^2$ and $\mathbb{D}^2$ can be endowed by trivial smooth manifold structures. Denote obtained manifolds by $\mathcal{M}$ and $\mathcal{N}$ respectively. The transition map $h_{12}$ is then diffeomorphism between these manifolds, while $h_1$ and $h_2$ are diffeomorphisms between $\mathbb{D}^2$ and $\mathcal{M}$, $\mathcal{N}$ correspondingly. Tangent spaces to $\mathcal{M}$ and $\mathcal{N}$ are naturally isomorphic to vector space $\mathbb{R}^2$. Denote by $(\partial_{x_1}, \partial_{x_2})$ and $(\partial_{r_1}, \partial_{r_2})$ local frames on $\mathcal{M}$ and $\mathcal{N}$.

The parallel transport rule on $\mathcal{M}$ is induced by Euclidean one. In other words, if $a$ is a vector attached at $p_1$, and $u$ is the vector field of parallel transports along $c$, then vector $Th_1(a)$ is attached at point $h_1(p_1)$ and is parallel transported along the curve $c_1 = h_1 \circ c$ by the vector field $h_1^* u$. There is another characterization of parallel transports in $\mathbb{D}^2$ and $\mathcal{M}$. Indeed, if $u = u^k e_k$ then $Th_1(u) = u^k h_1^* e_k$, and components of $Th_1(u)$ are the same. Since components of the vector field $u$ of parallel transports do not change, so do components of the vector field $h_1^* u$. The converse assertion is also true.

Define the parallel transport rule on $\mathcal{N}$ as follows. If $a$ is attached at $p_1$, $u$ is the vector field of parallel translations of $a$ along $c$, $u = u^k e_k$, then we put $u = u^k \partial_{x^k} + u^2 \partial_{r^2}$ and consider the vector field $u$ as parallel transport vector field of $a = Th_2(a)$ along $c_2 = h_2 \circ c$.

Results of parallel transports on $\mathcal{M}$ and $\mathcal{N}$ and their correspondence with parallel transport on $\mathbb{D}^2$ are illustrated in figure 3. The left side of the lower half corresponds to $\mathcal{M}$, while the right one, to $\mathcal{N}$. Note, that vectors on $c_1$ are drawn parallel, while vectors on $c_2$ are not. Indeed, the parallel transport rule on $\mathcal{N}$ neglects by rotations of local frame $(e_k)$ when it passes along the curve $c$. This leads to rotations of vectors attached at points of the curve $c_2$.

The figure 3 gives rise to another interpretation of parallel transport on $\mathcal{N}$. Let us forget about upper half of the figure and suppose that we defined Euclidean parallel transport on $\mathcal{M}$. Let $a$ be attached at $p_1 = h_2(p_1)$. We refer to vector field $u$ as vector field of parallel transports of $a$ along the curve $c_2$, if pullback $h_{12}^* u$ is the vector field of parallel transports of $Th_{12}^{-1}(a)$ along the curve $c_1$.

33°. Reparametrizations and deformations. In order to give clear mechanical interpretation for reparametrizations of the space $\mathbb{D}^2$, consider two diffeomorphic open sets $\mathcal{D}_1, \mathcal{D}_2 \subset \mathbb{D}^2$. Let $\gamma : \mathcal{D}_1 \to \mathcal{D}_2$ be a diffeomorphism. We interpret sets $\mathcal{D}_1, \mathcal{D}_2$ as shapes of some two-dimensional body $\mathfrak{B}$ and $\gamma$ as a deformation. Suppose that $\gamma$ can be extended to some automorphism $\tau : \mathcal{D} \to \mathcal{D}$ defined on open set $\mathcal{D} \subset \mathbb{D}^2$ that contains $\mathcal{D}_1 \cup \mathcal{D}_2$. That is, $\tau|_{\mathcal{D}_1} = \gamma : \mathcal{D}_1 \to \mathcal{D}_2$. Consider possible cases of relations between parametrizations for $\mathcal{D}$ and $\tau(\mathcal{D}) = \mathcal{D}$. Let $h_1 : \mathcal{D} \to \mathbb{R}^2$ and $h_2 : \tau(\mathcal{D}) \to \mathbb{R}^2$ be coordinate homeomorphisms, $h_{12}$ be transition map, while $\tau$ be coordinate representation of $\tau$ (see figure 4). Then

9 Here $h_1^* u$ denotes the pullback operation which is defined as follows [16]. Suppose that $\mathfrak{B}_1, \mathfrak{B}_2$ are smooth manifolds, while $f : \mathfrak{B}_1 \to \mathfrak{B}_2$ is a diffeomorphism. For a smooth vector field $u : \mathfrak{B} \to T\mathfrak{B}$ one can define smooth vector field $f^* u : \mathfrak{B}_1 \to T\mathfrak{B}_1$ by $f^* u = T f^{-1} \circ u \circ f$. Such definition gives rise to the operation $f^* : \text{Vec}(\mathfrak{B}_2) \to \text{Vec}(\mathfrak{B}_1)$, which is referred to as pullback.
Figure 3. Parallel transports along the curve $\gamma$ and its images.

Case 1. Suppose that parametrizations of $O$ and $\tau(O) = O$ are independent. Then one has $h_{12} = h_2 \circ \text{Id} \circ h_1^{-1}$ for transition map and $\tilde{\tau} = h_2 \circ \tau \circ h_1^{-1}$ for coordinate representation of $\tau$.

Case 2. Let parametrization of $O$ be specified arbitrary while on $\tau(O)$ one established such coordinates that $\tilde{\tau} = \text{Id}$. Then $h_2 = \text{Id} \circ h_1 \circ \tau^{-1}$ and $h_{12} = \text{Id} \circ h_1 \circ \tau^{-1} \circ \text{Id} \circ h_1^{-1}$.

Case 3. Finally, suppose that parametrizations of $O$ and $\tau(O) = O$ are similar, i.e., $h_2 = h_1$. Then $h_{12} = \text{Id}$, while $\tilde{\tau} = h_1 \circ \tau \circ h_1^{-1}$.

34°. Coordinate representations of deformation gradient. Let $\gamma : \delta_1 \to \delta_2$ be a deformation between shapes $\delta_1$, $\delta_2 \subset \mathcal{B}$ of the body $\mathcal{B}$. Since deformation $\gamma$ acts between open subsets of the Euclidean physical space, its deformation gradient $F_\rho = D_\rho \gamma \in \text{Lin}(\mathcal{V}; \mathcal{V})$ at point $\rho \in \delta_1$ can be defined in conventional way as principal part of the function $\gamma$ increment:

$$\gamma(\rho + h) = \gamma(\rho) + F_\rho[h] + o(\|h\|),$$

for $h \in \mathcal{V}$ such that $\rho + h \in \delta_1$. Consider various coordinate representations:

Case 1. Cartesian coordinate system is used. Then deformation $\gamma$ can be represented as $\gamma = \sigma + \gamma_k^i i_k$, where $\gamma_k^i : \delta_1 \to \mathbb{R}$ are component functions. Using reasonings similar to 31° one gets $D_\rho \gamma = \partial_k \gamma^i(\rho) i_i \otimes i^k$, where

$$\partial_k \gamma^i(\rho) := D_\rho \gamma^i[i_k] = \lim_{t \to 0} \frac{\gamma^i(\rho + ti_k) - \gamma^i(\rho)}{t}$$

The definition for $\partial_k \gamma^i(\rho)$ is coordinate-free. At the same time, one can represent it in terms of conventional partial derivatives. Indeed, define $\gamma_d^i := \gamma^i \circ \mathcal{B}^{-1}$, where $\mathcal{B} : \mathcal{B} \to \mathbb{R}^3$ is
Cartesian arithmetization. The mapping $\mathcal{D}$ can be represented as decomposition $\mathcal{D} = \Phi \circ \mathcal{P}$, in which $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{V}$ is position vector field, while $\Phi = I_k \otimes i^k : \mathcal{V} \rightarrow \mathbb{R}^3$ is vector space isomorphism. Symbols $I_k$ stand for triples $I_1 = (1, 0, 0), I_2 = (0, 1, 0)$ and $I_3 = (0, 0, 1)$ from $\mathbb{R}^3$. Applying the chain rule to the decomposition $\mathcal{D} = \Phi \circ \mathcal{P}$ one gets $D_p \mathcal{D} = \Phi$. Thus, from the equality $\gamma^l = \gamma^l \circ \mathcal{D}$ one obtains $\frac{\partial \gamma^l}{\partial X^k} \bigg|_{\mathcal{D}(\mathcal{P})} = D_{\mathcal{D}(\mathcal{P})} \gamma^l[I_k]$. Since $\gamma^l : (X^1, X^2, X^3) \rightarrow (X^1, X^2, X^3)$ is defined on an open subset of $\mathbb{R}^3$ and takes values in $\mathbb{R}$, the number $D_{\mathcal{D}(\mathcal{P})} \gamma^l[I_k]$ is no more than the conventional partial derivative $\frac{\partial \gamma^l}{\partial X^k} \bigg|_{\mathcal{D}(\mathcal{P})}$. Thus,

$$D_p \gamma = \frac{\partial \gamma^l}{\partial X^k} \bigg|_{\mathcal{D}(\mathcal{P})} i_l \otimes i^k$$

It is the conventional representation of the deformation gradient as “matrix of partial derivatives”.

**Case 2.** A curvilinear chart $(\mathcal{E}, h)$ is introduced. Recall that $h : \mathcal{E} \rightarrow \mathbb{R}^3, h : \mathcal{P} \mapsto (q^1, q^2, q^3)$, is a homeomorphism such that $\mathcal{D} \circ h^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a $C^\infty$-diffeomorphism. This implies, in particular, that $h^{-1}$ is a smooth map. Next, local frame is defined as $e_k = \tilde{e}_k \circ h$, where $\tilde{e}_k := \frac{\partial \mathcal{P}}{\partial q^k}$, and $\mathcal{P} = \mathcal{P} \circ h^{-1} : \mathbb{R}^3 \rightarrow \mathcal{V}$. Thus, $e_k$ is represented by conventional partial derivatives. According to the chain rule applied to the decomposition $\mathcal{P} = \mathcal{P} \circ h^{-1}$, the field $e_k$ can be expressed similarly as $e_k = D_h(h^{-1})h^{-1}[I_k]$. Now return to the deformation $\gamma$. It can be represented as $\gamma = \circ + \gamma^k e_k$. Here $\gamma^k : \mathcal{S}_1 \rightarrow \mathbb{R}$ are component functions. One gets

$$D_p \gamma = \left\{ \frac{\partial \gamma^l}{\partial \mathcal{P}}(\mathcal{P}) + \gamma^m(\mathcal{P}) \Gamma^l_{mk}(\mathcal{P}) \right\} e_l \otimes e^k \bigg|_{\mathcal{P}}.$$
where \( \partial_h \gamma^i (p) = \lim_{t \to 0} t^{-1} (\gamma^i (p + te_k |_p) - \gamma^i (p)) \), while \( \Gamma^i_{kl} : \mathcal{S} \to \mathbb{R} \) are functions such that \( \partial_h e_i (p) = \Gamma^i_{kl} (p) e_s |_p \). Here \( \partial_h e_i (p) := \lim_{t \to 0} t^{-1} (e_i |_{p + te_k |_p} - e_i |_p) \). The linear map \( D_{\rho} \gamma \) can be expressed in terms of conventional partial derivatives. In order to do this, let us introduce the mappings \( \gamma^i_h = \gamma^i \circ h^{-1} \). Then one gets \( \partial_h \gamma^i (p) = D_{h(p)} \gamma^i |_h [l_k] = \frac{\partial \gamma^i_h}{\partial q^k} |_{h(p)} \).

Thus,

\[
D_{\rho} \gamma = \left\{ \frac{\partial \gamma^i_h}{\partial q^k} |_{h(p)} + \gamma^m (p) \Gamma^i_{mk} (p) \right\} e_l \otimes e^k |_p
\]

**Case 3.** Shapes \( \delta_1 \) and \( \delta_2 \) are covered by local coordinates \((Q^k)\) and \((q^k)\) which are defined by coordinate homeomorphisms \( H : \delta_1 \to \mathbb{R}^3 \) and \( h : \delta_2 \to \mathbb{R}^3 \). These coordinate homeomorphisms generate local frames \( E_k = DH^{-1} \) and \( e_k = Dh^{-1} \). Let \( \gamma^i_q : (X^1, X^2, X^3) \to x^i \) be coordinate representation of deformation \( \gamma \) in Cartesian frame (see case 1). Then \( D_{\rho} \gamma = \frac{\partial x^i}{\partial X^k} |_{\rho} i_l \otimes \bar{t}^k \). Next, from the chain rule one gets \( \partial x^i / \partial X^k = \partial x^i / \partial q^m \partial Q^m / \partial X^k \). Finally,

\[
D_{\rho} \gamma = \frac{\partial q^m}{\partial Q^k} |_{h(p)} e_m |_{\gamma (\rho)} \otimes E^k |_{\rho}
\]

**Case 4.** Consider the particular situation in the framework of the case 3. Let shape \( \delta_1 \) be covered by local coordinates \((Q^k)\) defined by coordinate homeomorphism \( H : \delta_1 \to \mathbb{R}^3 \). Introduce coordinate homeomorphism \( h : \delta_2 \to \mathbb{R}^3 \) on \( \delta_2 \) as composition \( h = H \circ \gamma^{-1} \).

Then dyadic decomposition for \( D_{\rho} \gamma \) from the case 3 takes simple form \([24]\):

\[
D_{\rho} \gamma = e_k |_{\gamma (\rho)} \otimes E^k |_{\rho}
\]

35\(^{\circ}\). **Deformation gradient as total derivative vs deformation gradient as tangent map.** Let again \( \gamma : \delta_1 \to \delta_2 \) be a deformation between shapes \( \delta_1, \delta_2 \subset \mathcal{S} \) of the body \( \mathcal{B} \). The conventional representation for the deformation gradient at a point \( p \in \delta_1 \) is \( F_p = D_{\rho} \gamma \in \text{Lin}(\mathcal{V}; \mathcal{V}) \). At the same time, shapes \( \delta_1, \delta_2 \) can be considered as smooth manifolds and \( \gamma \) as a smooth map between them. From calculus on manifolds it is known that linear approximation of a smooth mapping between abstract smooth manifolds is provided by tangent map. It allows to propose another representation of the deformation gradient: \( F_p = T_{\rho} \gamma \). In the case of non-Euclidean physical space \( \mathcal{P} \) only such representation can be used. In order to show that this is exactly generalization of the conventional deformation gradient it is sufficient to show that \( D_{\rho} \gamma = T_{\rho} \gamma \), when \( \mathcal{P} = \mathcal{S} \). Indeed, since \( T_{\rho} \delta_1, T_{\gamma (\rho)} \delta_2 \) are naturally isomorphic to the translation space \( \mathcal{V} \), one has that \( T_{\rho} \gamma \in \text{Lin}(\mathcal{V}; \mathcal{V}) \). Next, as it was already shown in 34\(^{\circ}\), if \((Q^k)\) are curvilinear coordinates on the shape \( \delta_1 \), while \((q^k)\) are curvilinear coordinates on the shape \( \delta_2 \), then \( D_{\rho} \gamma = \frac{\partial q^l}{\partial Q^j} |_{\rho} e_l |_{\gamma (\rho)} \otimes E^j |_{\rho} \). But it is exactly \( T_{\rho} \gamma \).

36\(^{\circ}\). **Two dimensional interpretation of materially constant vector field.** Consider the case of compatible local deformations first. Let \( \gamma \) be a deformation between two shapes in \( \mathcal{B}^2 \) (see figure 5). If \( h_1 \) and \( h_2 \) are coordinate homeomorphisms on the shapes (in figure 5 the map \( h_1 \) is illustrated as Cartesian arithmetization), then \( \tilde{\gamma} = h_2 \circ \gamma \circ h_1^{-1} \) is coordinate representation of \( \gamma \) that acts between open subsets of \( \mathbb{R}^2 \). Like in 32\(^{\circ}\), we endow the domain and image of \( \tilde{\gamma} \) by smooth manifold structures (inherited from \( \mathbb{R}^2 \)) and denote the obtained manifolds by \( \mathcal{M} \) and \( \mathcal{R} \). Thus, \( \tilde{\gamma} : \mathcal{M} \to \mathcal{R} \). To the deformation gradient \( F = T \gamma \) there corresponds field \( \tilde{F} = T \tilde{\gamma} \). Note, that if one splits actual shape into infinitesimal fragments, then each individual fragment
would relax to uniform (stress-free) state by means of $F^{-1}$. After relaxation these parts can be assembled into the reference shape. In charts such relaxation is provided by means of $\tilde{F}^{-1}$.

Let $c$ be a curve in reference shape. then $\gamma \circ c$ is deformed curve and $\tilde{c}$, $\tilde{\gamma} \circ \tilde{c}$ are corresponding curves on $\mathfrak{M}$ and $\mathfrak{N}$. Let $u$ be a vector field defined in a neighborhood of $\gamma \circ c$ and $u$ is the corresponding vector field defined in a neighborhood of $\tilde{\gamma} \circ \tilde{c}$. In physical interpretation, values of these fields are stressed differential elements. Then pullbacks $u^* = F^{-1} \circ u \circ \gamma$ and $u^* = \tilde{F}^{-1} \circ u \circ \tilde{\gamma}$ represent vector fields of “relaxed” differential elements in absolute space and in charts. Materially parallel vector field is a field $u$, for which $u^*$ is vector field of parallel transports in sense of $29^\circ$. The chart counterpart, $u$, is materially parallel if $u^*$ is vector field of parallel transports in sense of $32^\circ$.

![Figure 5](image)

Figure 5. Materially constant vector field: case of compatible local deformations.

Now consider the general case, the case of incompatible local deformations. Suppose that we observe some shape $\delta$ of the body in two dimensional space $\mathbb{E}^2$. This shape is stressed and no stress-free shape exists in $\mathbb{E}^2$. If one splits $\delta$ into infinitesimal fragments then these fragments relax and the relaxation of each such fragment $p$ is provided by linear map which we denote by $F_p^{-1}$. Thus, to $\delta$ there corresponds the family $\{F_p^{-1}\}_{p \in \delta}$ of linear maps. In this case there is not exist such shape and such deformation $\gamma$ from it to $\delta$, for which $D_p \gamma = F_p$. At the same time, for each point $p \in \delta$ there exists a shape $\delta^{(p)}$ and deformation $\gamma^{(p)} : \delta^{(p)} \to \delta$, such that $D_p \gamma^{(p)} = F_p$. By its definition, the point $[\gamma^{(p)}]^{-1}(p)$ is stress-free, but other points of the shape $\delta^{(p)}$ are not.

Let $c : I \to \delta$ be a smooth curve in $\delta$. If we translate each point of $c(I)$ into relaxed state then (performing rigid motion, if needed) one would obtain the continual set $\ell = k(I)$ of relaxed fragments. Here $k : I \to \mathbb{E}^2$ is a smooth curve. The set $\ell$ doesn’t lie in one shape: it lies in the “pile” of shapes $\delta^{(p)}$, $p \in c(I)$. Now let us choose a vector field $u$ defined in a neighborhood of $c(I)$. Then for each $p \in c(I)$ one can construct the pullback $(\gamma^{(p)})^* u = T[\gamma^{(p)}]^{-1} \circ u \circ \gamma^{(p)}$. The
family \{((\gamma_\rho)^*)^\ast \mathbf{u})_{\rho \in \delta}\} allows to synthesize the vector field \(\mathbf{u}^* : \rho \mapsto \mathbf{u}^*|_\rho = \mathbf{F}^{-1}_\rho \circ \mathbf{u} \circ \gamma(\rho)\) which is defined on \(\ell\). Note that this field is not pullback of any deformation. In the way analogous to the case of compatible deformations we refer to \(\mathbf{u}\) as materially parallel if \(\mathbf{u}^*\) is parallel in Euclidean sense. This means, in particular, that \(\mathbf{u}^*|_\rho = \mathbf{a} = \text{const}\) for any point \(\rho \in \ell\). The vector \(\mathbf{a}\) plays role of archytypal differential element: the set \(c(\ell)\) consists of deformed copies of \(\mathbf{a}\). To reasonings in absolute space there correspond similar reasonings in charts. In particular, the family \(\{\tilde{\mathbf{F}}^{-1}_\rho\}_{\rho \in \delta}\) corresponds to the family \(\{\mathbf{F}^{-1}_\rho\}_{\rho \in \delta}\). The above reasonings are illustrated in figure 6.

\[\begin{array}{c}
\text{Figure 6. Materially constant vector field: case of incompatible local deformations.}
\end{array}\]

4. Structurally Inhomogeneous Bodies and Defected Structures

4.1. Laminated and Fibrillar Solids

37°. Laminated solids. A laminated body is the example of structurally inhomogeneous body that was proposed by Wang in [25]. It is a body which is composed by locally homogeneous two-dimensional manifolds. We observe this notion in more details.

Let \(\mathcal{B}\) be a body. Suppose that there exists a family \(\{\mathcal{L}_\alpha\}_{\alpha \in I}\) of two-dimensional submanifolds \(\mathcal{L}_\alpha \subset \mathcal{B}\), such that

\((L_1)\) \(\mathcal{B}\) is a disjoint union of \(\{\mathcal{L}_\alpha\}_{\alpha \in I}\):

\[\mathcal{B} = \bigcup_{\alpha \in I} \mathcal{L}_\alpha\]

\((L_2)\) For each \(\alpha \in I\) the set \(\mathcal{L}_\alpha\) is locally homogeneous, i.e., there exists a family \(\{U_{\alpha; \beta}\}_{\beta \in J_\alpha}\) of open subsets\(^{10}\) of \(\mathcal{B}\), that satisfies the following requirements:

\(^{10}\)Thus, each \(U_{\alpha; \beta}\) can be treated as a body in its own manner.
\((U_1)\) \(\mathcal{L}_\alpha \subset \bigcup_{\beta \in J_\alpha} U_{\alpha; \beta}\):

\((U_2)\) for each \(\beta \in J_\alpha\) there exists smooth embedding \(\kappa_{\alpha; \beta} : U_{\alpha; \beta} \to \mathcal{B}\) that sends the set \(U_{\alpha; \beta} \cap \mathcal{L}_\alpha\) to a uniform state.

Then the body \(\mathcal{B}\) is called a laminated body. Each lamina \(\mathcal{L}_\alpha\) can be visualized as infinitesimally thin shell which is locally homogeneous: it has family \(\{\kappa_{\alpha; \beta}\}_{\beta \in J_\alpha}\) of configurations, each of which sends the part of shell to a uniform state.

Let \(\kappa \in \mathcal{C}(\mathcal{B})\) be a configuration. Then, in general, all \(\kappa(U_{\alpha; \beta} \cap \mathcal{L}_\alpha)\) are non-uniform. The deformations from sets \(\kappa_{\alpha; \beta}(U_{\alpha; \beta} \cap \mathcal{L}_\alpha)\) to \(\kappa(\mathcal{B})\) can be considered as assembling of body. Thus, the body \(\mathcal{B}\) can be regarded as being formed by patching lamina together. These lamina are in various deformed states and the whole body is inhomogeneous in general.

There is a particular case of laminated body which is used in the paper. Suppose that body \(\mathcal{B}\) is a total space of some smooth fiber bundle [26] with homogeneous fibers. That is,

(i) There exists a smooth fiber bundle \((\mathcal{B}, \mathcal{J}, \pi, \mathcal{F})\) with total space \(\mathcal{B}\), one-dimensional base \(\mathcal{J}\), projection \(\pi : \mathcal{B} \to \mathcal{J}\) and typical fiber \(\mathcal{F}\), which is a smooth two-dimensional manifold. It means that \(\mathcal{J}\) is a smooth one-dimensional manifold, \(\pi\) is a smooth and surjective mapping.

For each \(\nu \in \mathcal{J}\) the fiber \(\mathcal{F}_\nu := \pi^{-1}(\{\nu\})\) over \(\nu\) is diffeomorphic to \(\mathcal{F}\).

(ii) For each \(\nu \in \mathcal{J}\) there exists a configuration \(\kappa_{\nu} \in \mathcal{C}(\mathcal{B})\), such that \(\kappa_{\nu}(\mathcal{F}_\nu)\) is in uniform state.

From the definition of bundle it follows that fibers are disjoint and \(\mathcal{B} = \bigcup_{\nu \in \mathcal{J}} \mathcal{F}_\nu\). The body is a “pile” of sheets. Thus, we satisfied item \((L_1)\) of the laminated body definition. Next, (ii) is the particular case of \((L_2)\), and then \(\mathcal{B}\) is a laminated body. In what follows we will refer to (ii) as hypothesis of local discharging.

38°. Fibrillar solids. The notion of fiber bundle is flexible and allows one to generate mechanical structures more general than laminated. A fibrillar body may serve as example [27]. It can be viewed as a result of patching together a 2-parameter family of homogeneous chords (called the fibers) in uniform states. Restrict ourselves with the following narrow definition. A body \(\mathcal{B}\) is called the fibrillar body, if

\((F_1)\) There exists a smooth fiber bundle \((\mathcal{B}, \mathcal{P}, \pi, \mathcal{F})\) with total space \(\mathcal{B}\), two-dimensional base \(\mathcal{P}\), projection \(\pi : \mathcal{B} \to \mathcal{P}\) and typical fiber \(\mathcal{F}\), which is a smooth one-dimensional manifold.

Denote \(\mathcal{C}_\alpha := \pi^{-1}(\{\alpha\})\).

\((F_2)\) For each \(\alpha \in \mathcal{P}\) there exists a configuration \(\kappa_{\alpha} \in \mathcal{C}(\mathcal{B})\), such that \(\kappa_{\alpha}(\mathcal{C}_\alpha)\) is in uniform state.

Thus, \(\mathcal{B}\) may be considered as a “seedling” of one-dimensional chords \(\mathcal{C}_\nu\) over two-dimensional “land” \(\mathcal{P}\).

Note that the laminated body is a particular case of fibrillar body. To show it, suppose that \(\mathcal{P} = \mathcal{I} \times \mathcal{J}\), where \(\mathcal{I}\) and \(\mathcal{J}\) are one-dimensional manifolds. Next, assume that chords can be combined into locally homogeneous sheets:

(i) For each \(\nu \in \mathcal{J}\) the union \(\bigcup_{\mu \in \mathcal{I}} \mathcal{C}(\mu, \nu)\) is a two-dimensional submanifold of \(\mathcal{B}\) denoted by \(\mathcal{L}_\nu\).

(ii) Each \(\mathcal{L}_\nu\) is locally homogeneous.

Then, since

\[ \mathcal{B} = \bigcup_{(\mu, \nu) \in \mathcal{I} \times \mathcal{J}} \mathcal{C}(\mu, \nu) = \bigcup_{\nu \in \mathcal{J}} \mathcal{L}_\nu, \]

the body \(\mathcal{B}\) is laminated.
4.2. Edelen Equations of Defects Theory

39°. Distortion one-forms. In conventional continuum mechanics one uses the following assumption: for the body $\mathfrak{B}$ there exists a stress-free shape $\mathcal{S}_R$ in three-dimensional Euclidean space. The response of the body is determined relatively to this shape. If $\gamma: \mathcal{S}_R \rightarrow \mathcal{S}$ is a deformation from stress-free shape $\mathcal{S}_R$ to some other, $\mathcal{S}$, then its localization is represented by three exact one-forms. Indeed, as it is shown in 35°, $F_\mu = T_\mu \gamma$, for $\mu \in \mathcal{S}_R$. Suppose that we stand in conditions of the case 3 of 34° in which $\mathcal{S}_1 = \delta_R$, while $\mathcal{S}_2 = \mathcal{S}$. Then $\gamma$ is represented by scalar functions $\gamma^k_q: \mathcal{S}_R \rightarrow \mathbb{R}$, $k = 1, 2, 3$, such that $\gamma^k_q = h^k \circ \gamma$, where $h^k$ is component function of coordinate homeomorphism $h: \mathcal{S} \rightarrow \mathbb{R}^3$. The composition $\gamma^k_q \circ q = \gamma^k_q \circ H^{-1}$ gives coordinate representation of $\gamma$ with respect to charts on smooth manifolds $\mathcal{S}_R$ and $\mathcal{S}$. The deformation gradient is then has the form

$$F_\mu = \frac{\partial \gamma^k_q}{\partial Q^m} \bigg|_{H(\mu)} \partial_k |_{\gamma(\mu)} \otimes dQ^n|_\mu$$

At the same time $^{12}$,

$$d_\mu \gamma^k_q = \frac{\partial \gamma^k_q}{\partial Q^m} \bigg|_{H(\mu)} dQ^n|_\mu$$

In this regard,

$$F = \partial_t \otimes F^t, \quad F^t = d\gamma^t_q$$

Here we use Fraktur notation $\mathfrak{k}$ for index in $F^t$ in order to emphasize that $F^t$ is not a component of a vector. Since $dF^t = d \circ d\gamma^t_q = 0$, one-forms $F^t$ are exact.

Remark 12. Note, that $F^1 \wedge F^2 \wedge F^3 \neq 0$. Indeed,

$$F^1 \wedge F^2 \wedge F^3 = \det \left[ \frac{\partial q^i}{\partial Q^j} \right] dQ^1 \wedge dQ^2 \wedge dQ^3,$$

where $\det \left[ \frac{\partial q^i}{\partial Q^j} \right] \neq 0$ since $\gamma$ is diffeomorphism.

As it was already mentioned, the condition $dF^i = 0$, $i = 1, 2, 3$, for one-forms $^{13}$ $F^i \in \Omega^1(\mathcal{S}_R)$ is necessary for the existence of deformation $\gamma$. Theory of continuously distributed defects starts with the rejection of this condition [28]. That is, some shape $\mathcal{S}_R$ is fixed and not necessary exact one-forms $^{14}$ $\beta^i \in \Omega^1(\mathcal{S}_R)$ are used instead of exact forms $F^i$. These forms are only required to satisfy the condition $\beta^1 \wedge \beta^2 \wedge \beta^3 \neq 0$ (since the matter doesn’t vanish). We refer to one-forms $\beta^i$ as one-forms of total distortion.

40°. Frames and volume forms. In the present section a three-dimensional Euclidean affine space would be denoted by $\mathcal{S}^3$. Indices, that vary from 1 to 3, are denoted by capital Latin letters. Cartesian coordinates $(x^i)_{i=1}^3$ and orientation are fixed. The volume form on $\mathcal{S}^3$ is designated by $\mu = dx^1 \wedge dx^2 \wedge dx^3$. The coordinate frame $(\partial_A)^3_{A=1}$ on $\mathcal{S}^3$ generates a collection $(\mu_A)_{A=1}^3$ of differential two-forms $\mu_A := \partial_A \wedge \mu$, which constitute a frame for $\Omega^2(\mathcal{S}^3)$. Fields $\mu_A$ can be regarded as oriented surface elements.

Denote $\mathcal{V}^3 \times \mathbb{R}$ by $\mathcal{V}^4$, where $\mathcal{V}^3$ is translation vector space for $\mathcal{S}^3$. The set $\mathcal{V}^4$ can be endowed with a 4-dimensional vector space structure with componentwise operations of addition and scalar multiplication. Then one gets the following direct sum decomposition: $\mathcal{V}^4 = (\mathcal{V}^3 \times \{0\}) \oplus (\{0\} \times \mathbb{R})$. By virtue of isomorphisms $\mathcal{V}^3 \times \{0\} \cong \mathcal{V}^3$ and $\{0\} \times \mathbb{R} \cong \mathbb{R}$, one can write $(\mathbf{h}, t) = h^A \partial_A + t$ for a vector $(\mathbf{h}, t) \in \mathcal{V}^4$.

$^{11}$ That is, $h(x) = (h^1(x), h^2(x), h^3(x))$.

$^{12}$ Here the symbol $d$ stands for the exterior differential [10].

$^{13}$ The designation $T^3(\mathfrak{M})$ stands for differential $k$-forms on smooth manifold $\mathfrak{M}$.

$^{14}$ It means that, in general, $d\beta^i \neq 0$. 

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Euclidean affine space structures of $\mathbb{R}^3$ and $\mathbb{R}$ can be induced to $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. They turn it into four-dimensional Euclidean affine space with translation vector space $\mathbb{V}^4$. Indices, that vary from 1 to 4, are denoted by lowercase Latin letters. The coordinate frame $(\partial_A)_{A=1}^4$ on $\mathbb{R}^3$ induces coordinate frame $(\partial_t)_{t=1}^4 = (\partial t)^3_{t=1} = (\partial t)$ on $\mathbb{R}^4$. Here $t$ stands for natural coordinate on $\mathbb{R}$.

The notation $\partial_t$ for $\partial t$ would be often used. The volume form on $\mathbb{R}^3$ is denoted by $\pi$ and can be represented as $\pi = \mu \wedge dt$. The frame $(\partial_t)_{A=1}^4$ allows to define differential forms $\pi_a = \partial_a \wedge \pi$, $a = 1, 2, 3, 4$. Thus, family $(\tau_a)_{a=1}^4$ constitutes a frame for $\Omega^3(\mathbb{R}^4)$.

We use two exterior derivative operators. The first one, $d$, is defined as $d = dx^i \wedge \partial_i$, while the second one, $\partial$, is defined via $\partial_i = dx^i \wedge \partial_i$. These operators are related as

$$d = \partial + dt \wedge \partial_4,$$

where $dt$ is the coframe for $\partial_4$. The operation $+$ is understood in the sense of isomorphisms $\mathbb{V}^3 \times \{0\} \cong \mathbb{V}^3$, $\{0\} \times \mathbb{R} \cong \mathbb{R}$.

### 41°. Edelen equations

In order to describe density of defects and their currents, one has to introduce the following differential forms [29]:

(a) two-forms of dislocation density: $\alpha_i = \alpha^{AI} \mu_A$,
(b) one-forms of dislocation current: $J^i = J^i_A dx^A$,
(c) two-forms of disclination current: $S^i = S^{AI} \mu_A$,
(d) 3-forms of disclination density: $Q^i = q^i \mu$,

which depend on temporal and spatial variables. Fields $\alpha^i$, $J^i$, $S^i$ and $Q^i$ satisfy the following equations [29]:

$$\partial_t \alpha_i = -dJ_i = -dS^i, \quad \partial_t Q^i = -\bar{d}S^i, \quad \bar{d}\alpha_i = Q^i, \quad \bar{d}Q^i = 0 \quad (18)$$

These equations are represented in componentwise form as (here $e^{ABC}$ is the alternator):

$$\partial_t \alpha^{AI} = -e^{ABC} \partial_B J^i - S^{AI}, \quad \partial_t \alpha^{AI} = q^i, \quad \partial_t q^i = -\partial_A S^{AI}$$

The last equation in (18), $\bar{d}Q^i = 0$, is identically satisfied since $Q^i$ is a 3-form.

Kinematic field equations [29]

$$\partial_t k^i = \bar{d}w^i = S^i, \quad \bar{d}k^i = Q^i, \quad \partial_t \beta_i = \bar{d}V^i - J^i - w^i, \quad \bar{d}\beta^i = \alpha^i - k^i \quad (19)$$

are defined by the first integrals of the system (18). Here

(a) $k^i = k^A \mu_A$ are bend-twist two-forms;
(b) $w^i = w^A dx^A$ are spin one-forms;
(c) $\beta^i = \beta^A dx^A$ are distortion one-forms;
(d) $V^i$ are velocity zero-forms.

Equations (19) have the following representation in components:

$$\partial_t k^A = -S^{AI} + e^{ABC} \partial_B w^i_c, \quad \partial_A k^A = q^i, \quad \partial_t \beta_A = \partial_A V^i - J^i_A - w^i_A, \quad e^{ABC} \partial_B \beta^i_C = \alpha^A - k^A$$

Introduce disclination 3-forms $\Omega^i$ and dislocation 2-forms $D^i$ in $\mathbb{R}^4$ [28, 29]:

$$\Omega^i = -S^i \wedge dt + Q^i = S^{AI} \mu_A \wedge dt + q^i \mu, \quad D^i = J^i \wedge dt + \alpha^i = J^i_A dx^A \wedge dt + \alpha^{AI} \mu_A \quad (20)$$
It allows one to represent the equations (18) in a more concise form:

\[ d\Omega^i = 0, \quad dD^i = \Omega^i \]  
(21)

First integrals of (18), in turn, can be represented as

\[ D^i = dH^i + K^i, \]

where

(a) \( H^i = V^i dt + \beta^i \) are the velocity-distortion one-forms;
(b) \( K^i = -w^i \wedge dt + k^i \) are the spin-twist two-forms.

42°. Balance of momentum. Field equations of defect theory need to be supplemented by the balance of momentum. Let \( \mathcal{F} = \mathcal{F}(H^4) \) be the kinetic energy density and \( \Psi = \Psi(H^A) \) be the potential energy density. Introduce 3-forms \( z_i \) in \( \mathcal{E}^4 \) by [28]

\[ z_i = \frac{\partial(\mathcal{F} - \Psi)}{\partial H^4_a} \pi_a = \frac{\partial\mathcal{F}}{\partial H^4_a} \pi_4 - \frac{\partial\Psi}{\partial H^4_a} \pi_A \]

Using standard terminology, we will refer to \( p_i := \frac{\partial\mathcal{F}}{\partial H^4} \) as components of momentum, and to \( \sigma^A_i := \frac{\partial\Psi}{\partial H^4_A} \) as components of Piola–Kirchhoff stresses. Then one gets

\[ z_i = -\sigma^A_i \pi_A + p_i \pi_4 = -\sigma^A_i \mu_A \wedge dt + p\mu \]

The balance of momentum is equivalent to the assertion that \( z_i \) are exact 3-forms, i.e.,

\[ dz_i = (\partial_4 p_i - \partial_A \sigma^A_i) \pi = 0 \]  
(22)

43°. Star-shaped region. Let \( \mathcal{S}_R \subset \mathcal{E}^3 \) be a shape of a body \( \mathfrak{B} \). This shape is called star-shaped, if there exists a point \( p_0 \in \mathcal{S}_R \) (center), such that the line segment connecting \( p_0 \) and any other point \( p \in \partial\mathcal{S}_R \) intersects \( \partial\mathcal{S}_R \) only at \( p \). In further reasonings we consider only star-shaped set \( \mathcal{S}_R \). In this case the conditions \( dF^i = 0, i = 1, 2, 3, \) and \( F^1 \wedge F^2 \wedge F^3 \neq 0 \) for one-forms \( F^i \in \Omega^1(\mathcal{S}_R) \) become sufficient for the existence of deformation \( \gamma : \mathcal{S}_R \rightarrow \mathcal{E}^3 \), such that \( F^i = d\gamma^i \), due to the Poincaré lemma [10,28].

44°. Homotopy operator. Total distortion can be decomposed into sum of exact and non-exact parts. In order to do this, we introduce a special linear map called homotopy operator [28]. It is a map \( \mathcal{K} : \Omega^p(\mathcal{S}_R \times \mathbb{R}) \rightarrow \Omega^{p-1}(\mathcal{S}_R \times \mathbb{R}) \), \( p = 1, 2, 3, 4 \), such that for any \( p \)-form \( \omega \in \Omega^p(\mathcal{S}_R \times \mathbb{R}) \) one has

\[ \mathcal{K}\omega = \int_0^1 \lambda^{p-1} X \lambda^{\omega}(\lambda) d\lambda, \]

where \( X = (x^i - x_0^i) \partial_i, \) sum over \( i = 1, 2, 3, 4 \), \( (x_0^i) \) are coordinates of the center, and

\[ \lambda(\lambda) = \omega_{i_1...i_p}(x^b_0 + \lambda(x^b - x_0^b)) dx^{i_1} \wedge ... \wedge dx^{i_p} \]

So introduced map \( \mathcal{K} \) has the property [28]:

\[ d\mathcal{K} + \mathcal{K} d = \text{Id}_{\Omega^p(\mathcal{S}_R \times \mathbb{R})} \]

15 Note, that we replace dependence of the potential energy on the deformation gradient by dependence on the distortion.
In this regard, velocity-distortion one-form $H^i$ can be represented as

$$H^i = d\mathcal{H}H^i + \mathcal{H}dH^i$$  \hspace{1cm} (23)

Define a smooth scalar function $\gamma^i = \mathcal{H}H^i + h^i$, where $dh^i = 0$. Then $d\gamma^i = d\mathcal{H}H^i$ and thus the one-form $d\mathcal{H}H^i$ represents an exact part of $H^i$. The second summand in (23), i.e., the one-form $\mathcal{H}dH^i$, is called the antiexact part of $H^i$ and contains all information about incompatibility.

Let $E^1(\delta_R \times \mathbb{R})$ be the space of all exact one-forms and $A^1(\delta_R \times \mathbb{R})$ be the space of all antiexact one-forms. Then one gets the following direct sum [28]:

$$\Omega^1(\delta_R \times \mathbb{R}) = E^1(\delta_R \times \mathbb{R}) \oplus A^1(\delta_R \times \mathbb{R}),$$

and the claim: if $\nu \in A^1(\delta_R \times \mathbb{R})$ then $\nu = \mathcal{H}d\nu$. It means, that $\mathcal{H}d = \text{Id}_{A^1(\delta_R \times \mathbb{R})}$ and the homotopy operator $\mathcal{H}$ is inverse to $d$ on the space $A^1(\delta_R \times \mathbb{R})$.

**45°. Matrix representations.** Equations (21) and (22) can be represented in concise matrix notation. Indeed, introduce matrices

$$H = \begin{pmatrix} H^1 \\ H^2 \\ H^3 \end{pmatrix}, \quad K = \begin{pmatrix} K^1 \\ K^2 \\ K^3 \end{pmatrix}, \quad D = \begin{pmatrix} D^1 \\ D^2 \\ D^3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \end{pmatrix},$$

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Using them, one gets representations

$$d\Omega = 0, \quad \Omega = dD = dK, \quad D = dH + K, \quad dZ = 0,$$

for equations (21) and (22).

**46°. Structural equations and their resolution.** Let $\omega$ be connection one-form on linear frame bundle over $\delta_R$, while $R$, $T$ and $\nu$ be corresponding curvature 2-form, torsion 2-form and soldering one-form respectively. These fields are related by structural equations [30]

$$d\nu = -\omega \wedge \nu + T, \quad dT = -\omega \wedge T + R \wedge \nu, \quad d\omega = -\omega \wedge \omega + R, \quad dR = R \wedge \omega - \omega \wedge R$$  \hspace{1cm} (24)

Applying homotopy operator $\mathcal{H}$ to these equations, one can formally resolve them with respect to fields $\nu$, $T$, $\omega$ and $R$ [28]:

$$\nu = A[d\gamma + \eta - \mathcal{H}(\omega_A \wedge d\gamma)], \quad T = A[d\eta + \omega_A \wedge \eta + \mathcal{H}(d\omega_A \wedge d\gamma) - \omega_A \wedge \mathcal{H}(\omega_A \wedge d\gamma)], \quad \omega = A\omega_A A^{-1} - (dA)A^{-1}, \quad R = A(d\omega_A + \omega_A \wedge \omega_A)A^{-1}$$

Here $A$ is the solution of matrix integral equation $A = I - \mathcal{H}(\omega A)$, while $\gamma$, $\eta$ and $\omega_A$ are defined as

$$\gamma = \gamma_0 + \mathcal{H}(A^{-1}\nu), \quad \eta = \mathcal{H}(A^{-1}T), \quad \omega_A = \mathcal{H}(A^{-1}RA)$$

The 0-form $\gamma_0$ is such that $d\gamma_0 = 0$.

**47°. Equations of defect theory as structural equations.** Identify matrices $H$ and $D$ with forms $\nu$ and $T$ respectively and choose [28]

$$\Omega = R \wedge \nu - \omega \wedge T, \quad K = \omega \wedge \nu$$  \hspace{1cm} (25)

Then kinematic equations of defects are satisfied for any choice of $\omega$ and $R$. Using resolved structural equations, one obtains equivalent representations for $H$ and $D$:

$$H = A(d\gamma + \eta - \mathcal{H}(\omega_A \wedge d\gamma)), \quad D = A(d\eta + \omega_A \wedge \eta + \mathcal{H}(d\omega_A \wedge d\gamma) - \omega_A \wedge \mathcal{H}(\omega_A \wedge d\gamma))$$
Substitution of these expressions into (25) results in
\[
K = A(\omega A - A^{-1}dA) \land (d\gamma + \eta - \nabla (\omega A \land d\gamma)),
\]
\[
\Omega = A[(d\omega A - \omega A \land \omega A) \land (d\gamma + \eta - \nabla (\omega A \land d\gamma)) -
-(\omega A - A^{-1}dA) \land (d\eta + \omega A \land \eta + \nabla (\omega A \land d\gamma)) - \omega A \land \nabla (\omega A \land d\gamma)]
\]
Obtained expressions for \(H, D, K\) and \(\Omega\) are equations relatively to \(\gamma, \eta, \omega A, A\). The problem of solving the kinematic equations is equivalent to the problem of determining these fields.

Suppose that disclinations vanish. Then \(\Omega = 0, K = 0, A = I\) (identity matrix), and expressions, obtained above, reduce to [28]
\[
H = d\gamma + \eta, \quad D = d\eta, \quad \text{or} \quad \eta = \nabla(D)
\]
If dislocations are also vanish, \textit{i.e.}, there are no defects at all, then we get conventional kinematic relations:
\[
D = K = 0, \quad \Omega = 0, \quad \omega A = 0, \quad \eta = 0, \quad A = I, \quad H = d\eta
\]

5. Technological Processes & Evolutionary Problems

48°. Solid with variable material composition. A solid with a fixed composition of material points is formalized by a smooth three-dimensional body \(\mathcal{B}\). At the same time, if the material composition of a solid is not fixed\(^{16}\) \textit{i.e.}, the solid is composed of a set of material points, variable in time, then a body can represent only instantaneous material composition of such a solid. The whole solid with variable material composition is formalized as a family \(\mathcal{G} = \{\mathcal{B}_\alpha\}_{\alpha \in I}\) of smooth three-dimensional bodies over index set \(I\). The index \(\alpha\) serves as a time-like parameter. The cardinal number of \(I\) gives rise to the following classification of solids with variable composition:

- \((\mathcal{G}_1)\) \textit{discrete structure}: the cardinal number of \(I\) is equal to some natural number, \textit{i.e.}, Card\(I = N \in \mathbb{N}\);
- \((\mathcal{G}_2)\) \textit{continuous structure}: the cardinal number of \(I\) is equal to the cardinal number of \(\mathbb{R}\), \textit{i.e.}, Card\(I = \text{Card} \mathbb{R}\).

Introduce the following designations:
\[
\mathcal{B}^* = \bigcup_{\alpha \in I} \mathcal{B}_\alpha, \quad \mathcal{B}_* = \bigcap_{\alpha \in I} \mathcal{B}_\alpha
\]
We refer to the set \(\mathcal{B}_*\) as a kernel and to the set \(\mathcal{B}^*\) as a total body.

49°. Monotonic processes. From the technological viewpoint, one may talk about what was happening with the solid “before” the parameter \(\alpha\) took some value and “after”. In this regard it seems reasonable to suppose \(I\) to be endowed with linearly ordered set structure, \textit{i.e.}, some relation \(\leq\) of linear ordering is chosen on it. Such assumption allows to classify processes, which form structures with variable material composition, as monotonic or non-monotonic. Monotonic processes are described as follows:

- \((m_1)\) \textit{growth}: material can not be removed during the process:
  \[
  \forall \alpha, \beta \in I : \quad (\alpha < \beta) \Rightarrow (\mathcal{B}_\alpha \subset \mathcal{B}_\beta);
  \]
- \((m_2)\) \textit{pure growth}: material is added during the process:
  \[
  \forall \alpha, \beta \in I : \quad (\alpha < \beta) \Rightarrow (\mathcal{B}_\alpha \subset \subset \mathcal{B}_\beta);
  \]

\(^{16}\)This situation can be occured in some LbL (Layer-by-Layer) process.
process is pure growth. In this case we will refer to monotonic processes. Without loss of generality one may assume that elements from $G$ while in case of ($m_3$) and ($m_4$) is satisfied, we refer to the process as non-monotonic.

50°. Initial and final bodies. Since from technological point of view the process of producing the solid has “start” and “stop” points, the index set $I$ is supposed to have minimal and maximal elements $A = \min(I), B = \max(I)$. Then in case of ($m_1$) and ($m_2$),

\[ B_* = B_A, \quad B^* = B_B, \]

while in case of ($m_3$) and ($m_4$),

\[ B_* = B_B, \quad B^* = B_A \]

Thus, $B_*$ and $B^*$ are elements of $G$ and are smooth 3-manifolds therefore. We study only monotonic processes. Without loss of generality one may assume that $\Theta$ satisfies ($m_2$), i.e., the process is pure growth. In this case we will refer to $B_*$ as an initial body (substrate), while to $B^*$ as a final body.

51°. Smooth structure of solid with variable material composition. Although all elements from $\Theta$ are subsets of the final body $B^*$, their topological and smooth manifold structures are not compatible with each other and with those of $B^*$. That is, bodies $B_{\alpha}, B_{\beta}, \alpha \neq \beta$, may have slightly different topologies and smooth manifold structures [31]. In particular, it means that the topological invariants of solid with variable material composition change due to attaching of extra material. Thus, the mathematical description of such process requires to specify among independent variables the following quantities: 1) Topological invariants $I_{1;\alpha}, \ldots, I_{k;\alpha}$ of each body $B_{\alpha}$; 2) Material metric $\mathcal{G}_{\alpha}$ of each body $B_{\alpha}$; 3) Torsion, curvature and nonmetricity of affine connection on each body $B_{\alpha}$.

In wide variety of technological processes the above consideration, when topological and smooth structures are independent, is superfluous. Assume that for $\alpha < \beta$ the body $B_{\alpha}$ is a part of $B_{\beta}$. It can be formalized as follows: $B_{\alpha}$ is a smooth submanifold of $B_{\beta}$. This implies, in particular, that for any $\alpha \in I$ the space $B_{\alpha}$ is a smooth submanifold of $B^*$.

Remark 13. The fact that some space $\mathcal{U}$ is a smooth submanifold of $B_{\alpha}$ (in particular, of $B^*$) means that $\mathcal{U} \subset B_{\alpha}$, $\mathcal{U}$ is a smooth manifold with topology induced from $B_{\alpha}$, and smooth structures of $\mathcal{U}$ and $B_{\alpha}$ are related by the following condition: the inclusion map $\iota_\mathcal{U}: \mathcal{U} \to B_{\alpha}$, defined as $x \mapsto x$, is a smooth embedding [10].

52°. Final body as material manifold and joining operation. Since for each $\alpha \in I$ the dimensions of the bodies $B_{\alpha}$ and $B^*$ are equal, i.e., $\dim B_{\alpha} = \dim B^*$, the body $B_{\alpha}$ is an open subset of $B^*$. Thus, $B^*$ represents a material universal set for bodies from $\Theta$. It contains all information about material points that were influxed during the pure growth process.

The representation of each body $B_{\alpha}$ as a decomposition of mutually disjoint parts (layers) seems natural. At the same time, the formal realization of such decomposition has technical issues. Indeed, the set difference between open subsets of $B^*$ results in a set which may not be open and therefore doesn’t represent a part. In addition, the set union of two adjacent parts doesn’t contain their common boundary. These issues can be solved, for example, in the following way [8, 12].

We introduce the operation $\mathcal{J}$ of layer partitioning. For $\mathcal{O}_1 \subset \mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are open subsets of $B^*$, it returns the set

\[ \mathcal{J}(\mathcal{O}_2, \mathcal{O}_1) := \text{Int}(\mathcal{O}_2 \setminus \mathcal{O}_1) \]
We refer to this set as the layer or extra body (in the continuous case). Here \( \text{Int} \) is the operation of interior in the topology of \( \mathbb{B}^* \).

The operation \( \mathcal{J} \) allows one to split any three-dimensional body \( \mathcal{B} \subset \mathbb{B}^* \) into part \( \mathcal{A} \subset \mathcal{B} \) and part \( \mathcal{J}(\mathcal{B}, \mathcal{A}) \). Then the operation of joining, \( \mathcal{V} \), can be introduced as follows:

\[
\mathcal{B} = \mathcal{A} \cup \mathcal{J}(\mathcal{B}, \mathcal{A}) := \mathcal{A} \cup \text{Int}(\mathcal{B} \setminus \mathcal{A}) \cup \partial \mathcal{B}(\mathcal{B} \setminus \mathcal{A}),
\]

where \( \partial \mathcal{B} \) is the boundary operation relatively to the topology of \( \mathcal{B} \). Therefore, we get the following decompositions:

(i) In the case of a continuous process one has \( \mathcal{B}_{\alpha} = \mathcal{B}_s \cup \mathcal{B}_{\alpha} \), where \( \mathcal{B}_{\alpha} := \mathcal{J}(\mathcal{B}_{\alpha}, \mathcal{B}_s) \) corresponds to extra material adhered to the moment \( \alpha \). Thus, one arrives at the family \( \mathcal{G} = \{ \mathcal{B}_{\alpha} \}_{\alpha \in I} \) of extra bodies. Here\(^{\ref{footnote17}} \ I := I \setminus \{ \emptyset \} \). This family is also ordered by the subset relation\(^{\ref{footnote18}} \subset \). Note, that for the whole final body one has \( \mathcal{B}_s = \mathcal{B}_s \cup \mathcal{B}_s, \mathcal{B}_s = \mathcal{B}_B \) represents the totality of adhered material. For each \( \alpha \in I \) one has \( \mathcal{B}_{\alpha} \subset \mathcal{B}^* \).

(ii) In the case of a discrete process, if \( I = \{ 1, \ldots, N \}, \ N \geq 1 \), then \( \mathcal{B}_{n+1} = \mathcal{B}_n \cup \mathcal{L}_n \), in which \( \mathcal{L}_n := \mathcal{J}(\mathcal{B}_{n+1}, \mathcal{B}_n) \) is a layer. The body \( \mathcal{B}_n \) can be represented as

\[
\mathcal{B}_n = (\ldots (\mathcal{B}_s \cup \mathcal{L}_1) \cup \ldots \cup \mathcal{L}_{n-2}) \cup \mathcal{L}_{n-1}, \quad n \in I, \ \mathcal{L}_0 := \mathcal{B}_s.
\]

53°. Fiber bundle structure of body with variable material composition. Consider a process of continuous adhering of material to a substrate\(^{\ref{footnote19}} \). In order to give mathematical description for such phenomena, one has to introduce a family \( \mathcal{G} = \{ \mathcal{B}_{\alpha} \}_{\alpha \in I} \) of bodies with continual index set. This family is required to satisfy the property \( (m_2) \) of pure growth (see 49°).

At the same time, one more requirement is needed. Indeed, the extra body \( \mathcal{B}^* \) is assembled by topologically equivalent disjoint foils (laminae). This property can be formalized by using the notion of fiber bundle. That is, we posit\(^{\ref{footnote20}} \):

(a1) There exists a smooth fiber bundle \( (\mathcal{B}^*, \mathcal{J}, \pi, \mathfrak{F}) \), where \( \mathcal{B}^* \) is total space, \( \mathcal{J} \) is one-dimensional base, \( \pi : \mathcal{B}^* \to \mathcal{J} \) is a smooth and surjective map (called projection) and \( \mathfrak{F} \) is a smooth two-dimensional manifold (called typical fiber).

(a2) For each \( \alpha \in I \) the fiber bundle \( (\mathcal{B}_\alpha, \mathcal{J}_\alpha, \pi, \mathfrak{F}) \) induces fiber bundle \( (\mathcal{B}_\alpha, \mathcal{J}_\alpha, \pi|\alpha, \mathfrak{F}) \). Here \( \mathcal{J}_\alpha \subset \mathcal{J} \) is a submanifold\(^{\ref{footnote21}} \), while \( \pi_\alpha = \pi|\mathcal{B}_\alpha : \mathcal{B}_\alpha \to \mathcal{J}_\alpha \).

Fiber bundle structure implies that for each \( \nu \in \mathcal{J} \) the fiber \( \mathcal{G}_\nu := \pi^{-1}(\{ \nu \}) \) is diffeomorphic to \( \mathfrak{F} \). Moreover, fibers \( \mathcal{G}_\nu \) are nonempty, disjoint and \( \mathcal{B}_\alpha = \bigcup_{\nu \in \mathcal{J}_\alpha} \mathcal{G}_\nu \).

54°. Local discharging. At this stage and in what follows by a uniform state we mean a stress-free state. The set of response descriptors is then the set \( R = \{ T | T \in \text{SymLin}(\mathcal{Y}; \mathcal{Y}) \} \). From technological viewpoint one may expect that each of the bodies from the family \( \mathcal{G} \) is inhomogeneous, despite, maybe, the first body \( \mathcal{B}_s \) which represents a substrate. Indeed, the process of layer (in this indent we use the term “layer” as technological one) joining is accompanied by its solidification (in case of polymer material), or by melting (in case of metal powder). Due to these processes the layer undergoes shrinkage. It causes structural inhomogeneity in the obtained body.

We use the hypothesis of local discharging. Its formalization in continuous case is given below:

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\(^{\ref{footnote17}}\) Note, that \( \mathcal{J}_1 = \emptyset \).

\(^{\ref{footnote18}}\) Indeed, for \( \alpha < \beta \) one has \( \mathcal{B}_\alpha \subset \mathcal{B}_\beta \). The set difference gives \( \mathcal{B}_\alpha \setminus \mathcal{B}_\beta \subset \mathcal{B}_\beta \setminus \mathcal{B}_\alpha \). Since the interior operation preserves subset relation, one finally gets \( \mathcal{B}_\alpha \subset \mathcal{B}_\beta \).

\(^{\ref{footnote19}}\) Such the process can be considered as a limit case of layer-by-layer technological process, in which the deposition of sufficiently big amount of thin layers is performed.

\(^{\ref{footnote20}}\) The item (a1) says that \( \mathcal{B}^* \) is laminated body in the sense of 37°.

\(^{\ref{footnote21}}\) Note, that \( \mathcal{J}_B := \mathcal{J} \).
(b₁) There exists a configuration \( \mathcal{X}_k^e \in \mathcal{E}(\mathcal{B}^*) \) for which the shape \( \mathcal{X}_k^e(\mathcal{B}_n) \) is stress-free.

(b₂) For each \( \nu \in \mathbb{J} \) there exists a configuration \( \mathcal{X}_\nu^e \in \mathcal{E}(\mathcal{B}^*) \), such that \( \mathcal{X}_\nu^e(\mathcal{S}_\nu) \) is stress-free.

These assumptions are illustrated by the following diagram:

\[
\begin{array}{ccc}
\mathcal{B}_n & \cdots \subset \cdots & \mathcal{B}^* & \cdots \subset \cdots & \mathcal{S}_\nu \\
\mathcal{X}_k^e(\mathcal{B}_n) & \cdots \subset \cdots & \mathcal{X}_k^e(\mathcal{B}^*) & \cdots \subset \cdots & \mathcal{X}_\nu^e(\mathcal{S}_\nu) \\
\text{stress-free shape} & & & & \text{stress-free shape}
\end{array}
\]

**Remark 14.** In the case of a discrete process, the hypothesis of local discharging can be stated as follows. Let solid with variable material composition be represented by a family \( \mathcal{S} = \{ \mathcal{B}_n \}_{n=1}^N \) of bodies. Then for each \( k \in \{0, \ldots, N-1\} \) there exists a configuration \( \mathcal{X}_k^e \in \mathcal{E}(\mathcal{B}^*) \) for each \( \mathcal{X} \in \mathcal{B}_n \). Then, one can endow \( \mathcal{B}^* \) with material metric and material connection by formulae (8) and (9). Restricting obtained mappings to the bodies \( \mathcal{B}_n \) one can get the corresponding metric and connection on \( \mathcal{B}_n \). At the same time, the reference configurations may not be known. Hence, the field \( \mathcal{F}^R : \mathcal{X} \mapsto \mathcal{F}^R_\mathcal{X} \) is unknown. In order to obtain it, one needs to consider the consecutive joining of laminae to the substrate, \( \mathcal{J} \), i.e., the body evolution. This evolution is subordinated to the requirement: in infinitesimal time interval a lamina adheres to the body. We formalize it as follows. Denote \( \tilde{\mathcal{I}} = \mathbb{I} \setminus \{ A, B \} \) and suppose that this set is endowed by topological space structure, \( \tilde{\mathcal{B}^*} = \tilde{\mathcal{B}}_B = \bigcup_{\alpha \in \tilde{\mathcal{I}}} \mathcal{B}_\alpha \) and there exists a homeomorphism \( \psi : \tilde{\mathcal{I}} \to \mathbb{I} \). This imply the equalities:

\[
\bigcup_{\alpha \in \tilde{\mathcal{I}}} \tilde{\mathcal{B}}_\alpha = \bigcup_{\alpha \in \tilde{\mathcal{I}}} \mathcal{S}_{\psi(\alpha)} = \tilde{\mathcal{B}^*}
\]

These relations reflect the fact that the body \( \tilde{\mathcal{B}^*} \) can be obtained either as a union of elements from the collection \( \{ \mathcal{B}_\alpha \}_{\alpha \in \tilde{\mathcal{I}}} \) or as a union of fibers (laminae) over the set \( \tilde{\mathcal{I}} \). Consider each lamina \( \mathcal{S}_{\psi(\alpha)} \) as a part of the boundary of \( \mathcal{B}_\alpha \) which corresponds to an infinitesimally thin layer adhered to the body. Formally,

\[
\forall \alpha \in \tilde{\mathcal{I}} : \quad \mathcal{S}_{\psi(\alpha)} \subset \partial_{\tilde{\mathcal{B}^*}} \mathcal{B}_\alpha,
\]

where \( \partial_{\tilde{\mathcal{B}^*}} \) is the boundary operation relatively to the topology of \( \tilde{\mathcal{B}^*} \).

Let \( \mathcal{F}^R_\alpha := \mathcal{F}^R|_{\mathcal{B}_\alpha} \). Then one gets

\[
\mathcal{F}^R_\beta|_{\mathcal{B}_\alpha} = \mathcal{F}^R_\alpha
\]

Here \( | \) denotes the restriction of field \( | \) to \( \mathcal{B}_\alpha \). Obtained equality (26) can be interpreted as follows: the structural inhomogeneity in already adhered material is lacked. Let \( \mathcal{G}_\alpha \) and \( \nabla_\alpha \) be material metric and connection on the body \( \mathcal{B}_\alpha \). The equality (26) imposes restrictions on possible relations between material metrics and connections on bodies that constitute the family \( \mathcal{S} \). These restrictions are formalized as follows:

\[
\forall \alpha, \beta \in \mathbb{I} : \quad (\alpha < \beta) \Rightarrow (\mathcal{G}_\beta|_{\mathcal{B}_\alpha} = \mathcal{G}_\alpha) \land (\nabla_\beta|_{\mathcal{B}_\alpha} = \nabla_\alpha)
\]

22 Recall that \( A = \min(\mathbb{I}) \), \( B = \max(\mathbb{I}) \).
6. Cylindrical Problem

A model problem for cylindrical body with axial symmetry is considered as an extension of (a) from 53° and (b) from 54°, assume that

(i) Each set $\mathbb{B}_\alpha$ is a cylinder in $\mathcal{S}$ with height $h$, and each of shapes $\mathbb{X}_\alpha(\mathbb{B}_\alpha)$, $\mathbb{X}_*^e(\mathbb{B}_*)$, $\mathbb{X}_*^o(\mathbb{B}_*)$ is a hollow cylinder. All cylindrical bodies have the same height, $h$.

(ii) For any $\alpha \in \mathcal{I}$ the actual configuration $\mathbb{X}_\alpha \in \mathcal{C}(\mathbb{B}_\alpha)$ is such that the actual shape is a hollow cylinder with height $h$.

57°. Intermediate configuration. In order to introduce local coordinates on $\mathbb{B}_*$, which are suitable for computations, we follow the reasonings of Remark 5. Therefore, choose a configuration $\mathbb{X}_I \in \mathcal{C}(\mathbb{B}_*)$ such that $\mathbb{X}_I(\mathbb{B}_*)$ and $\mathbb{X}_I(\mathbb{B}_*)$ are hollow cylinders with the height $h$. We will refer to $\mathbb{X}_I$ as intermediate configuration and to $\mathbb{X}_I(\mathbb{B}_*)$ as intermediate shape. Denote by $C$ the coordinate homeomorphism which associates points of the intermediate shape with their cylindrical coordinates, i.e.,

$$C : \mathbb{X}_I(\mathbb{B}_*) \ni x \mapsto (R, \Theta, Z) \in \mathbb{R}^3$$

Then the composition

$$C \circ \tilde{\mathbb{X}}_I : \mathbb{B}_* \ni x \mapsto (R, \Theta, Z) \in \mathbb{R}^3$$

is a coordinate homeomorphism on $\mathbb{B}_*$.

Since, according to 51°, all elements of the collection $\mathcal{S} = \{\mathbb{B}_\alpha\}_{\alpha \in \mathcal{I}}$ are submanifolds of $\mathbb{B}_*$, the coordinate mapping on every $\mathbb{B}_\alpha$ can be defined as restriction of $C \circ \tilde{\mathbb{X}}_I$ to $\mathbb{B}_\alpha$. Similarly, one can introduce the coordinate homeomorphism on the cylinder $\mathbb{X}_I(\mathbb{S}_\nu)$. Denote cylindrical coordinates of its points by $(R_\nu, \Theta_\nu, Z_\nu)$.

58°. Deformation ansatz. In view of the axial symmetry of the intermediate shape and local discharge shapes it seems justified to specify the deformation

$$\gamma_\nu = \tilde{\mathbb{X}}_\nu \circ \mathbb{X}_I^{-1} : \mathbb{X}_I(\mathbb{B}_*) \to \mathbb{X}_\nu(\mathbb{B}_*)$$  \hspace{1cm} \nu \in \mathcal{I}, \hspace{1cm} (27)$$

from intermediate shape into discharged one as a bloating. However, in general, we come to some complication related with possible violation of non-penetration principle. This difficulty can be resolved as follows. Since we intend to obtain local configuration, only local behaviour of the mapping (27) is needed to be taken into account. Thus, assume that there exists a number $\delta_\nu > 0$ such that the deformation $\gamma_\nu|_{D_\nu}$ of hollow cylinder $D_\nu \subset \mathbb{X}_I(\mathbb{B}_*)$,

$$D_\nu = \{x \in \mathbb{X}_I(\mathbb{B}_*) \mid C(x) = (R, \Theta, Z), \hspace{0.5cm} R_\nu - \delta_\nu < R < R_\nu + \delta_\nu\},$$

is a bloating. This means, that the coordinate representation of $\gamma_\nu|_{D_\nu}$ is of the form

$$\tau_\nu(R, \Theta, Z) = [R^2 - a(\nu)]^{1/2},$$

$$t_\nu(R, \Theta, Z) = \Theta,$n

$$\hat{z}_\nu(R, \Theta, Z) = Z,$$

(28)

where $a(\nu) \in \mathbb{R}$ is a number, while $(\tau_\nu, t_\nu, \hat{z}_\nu)$ are cylindrical coordinates on $\mathbb{X}_\nu(\mathbb{B}_*)$. To (28) there corresponds the deformation gradient at point $p \in D_\nu$:

$$\mathbf{F}_\nu(p) = \left. \frac{R}{\sqrt{R^2 - a(\nu)}} \right|_{C(p)} \mathbf{E}_{\tau_\nu} \otimes e^R + \mathbf{E}_{t_\nu} \otimes e^\Theta + \mathbf{E}_{\hat{z}_\nu} \otimes e^Z$$

(29)
Here \((e_R, e_θ, e_Z)\), \((e^R, e^θ, e^Z)\) are local frame and coframe on \(x_ν(\mathcal{B}^*)\) and \((E_{νr}, E_{νθ}, E_{νz})\), \((E^r, E^θ, E^z)\) are local frame and coframe on \(x_ν^*(\mathcal{B}^*)\) generated by coordinates \((r_ν, θ_ν, z_ν)\).

Since the substrate \(\mathcal{B}_s\) has global stress free shape \(x_s^*(\mathcal{B}_s)\), its case is needed to be considered individually. The deformation of the shape \(x_s(\mathcal{B}^*)\) into the shape \(x_s^*(\mathcal{B}^*)\) is represented by mapping

\[ γ_s = \hat{\nu}_s \circ [\hat{x}_s]^{-1} : x_s(\mathcal{B}^*) \rightarrow x_s^*(\mathcal{B}^*) \]

Denote \(D_s = x_s(\mathcal{B}_s)\). The deformation \(γ_s|_{D_s}\) of the shape \(D_s\) is a bloating,

\[
\begin{align*}
  r_s(R, θ, Z) &= \left[R^2 - a_s^2\right]^{1/2}, \\
  t_s(R, θ, Z) &= θ, \\
  \hat{z}_s(R, θ, Z) &= Z,
\end{align*}
\]

where \(a_s \in \mathbb{R}\) is deformation parameter, while \((r_s, t_s, \hat{z}_s)\) are cylindrical coordinates on \(x_s^*(\mathcal{B}^*)\).

Then the gradient of \(γ_s|_{D_s}\) at point \(p \in D_s\) has dyadic representation

\[
F_s(p) = \left. \frac{R}{\sqrt{R^2 - a_s^2}} \right|_{C(p)} E_{r_ν} \otimes e^R + E_{t_ν} \otimes e^θ + E_{z_ν} \otimes e^Z,
\]

in which \((E_{r_ν}, E_{t_ν}, E_{z_ν})\), \((E^r, E^θ, E^z)\) are consequently local frame and coframe on \(x_s^*(\mathcal{B}^*)\) that correspond to coordinates \((r_s, t_s, \hat{z}_s)\).

From the above considerations it follows that local distortions of the intermediate shape are represented by deformation parameters \(a_s\) and \(a(ν)\), \(ν \in J\). These distortions can be expressed in terms of mapping \(b : \mathcal{B}^* \rightarrow \mathbb{R}\) which is defined as a piecewise-continuous function with points of discontinuity that correspond to the interface surface between substrate and deposited material:

\[
b(\mathcal{X}) = \begin{cases} 
  a_s, & \mathcal{X} \in \mathcal{B}_s \cup \partial \mathcal{B}_s(\mathcal{B}^* \setminus \mathcal{B}_s), \\
  a(π(\mathcal{X})), & \mathcal{X} \in \hat{\mathcal{B}}^*,
\end{cases}
\]

On purpose to represent formulae (29) and (31) in unified form, denote \(K = \{\ast\} \cup J\) and define the map \(.: \mathcal{B}^* \rightarrow K\) as

\[
\Pi(\mathcal{X}) = \begin{cases} 
  \ast, & \mathcal{X} \in \mathcal{B}_s \cup \partial \mathcal{B}_s(\mathcal{B}^* \setminus \mathcal{B}_s), \\
  π(\mathcal{X}), & \mathcal{X} \in \hat{\mathcal{B}}^*.
\end{cases}
\]

Then from (29) and (31) it follows that for \(\mathcal{X} \in \mathcal{B}^*, \ k = \Pi(\mathcal{X})\) and \(p \in D_k\) one has

\[
F_k(p) = \left. \frac{R}{\sqrt{R^2 - b(\mathcal{X})}} \right|_{C(p)} E_{r_k} \otimes e^R + E_{t_k} \otimes e^θ + E_{z_k} \otimes e^Z.
\]

59°. **Local configuration**. Denote coordinate frame and coframe on \(\mathcal{B}^*\) by \((∂R, ∂θ, ∂Z)\) and \((dR, dθ, dZ)\). Then considerations of 57° imply that

\[
T_{\mathcal{Y}}x_ν = e_R|_{x_ν(\mathcal{Y})} \otimes dR|_\mathcal{Y} + e_θ|_{x_ν(\mathcal{Y})} \otimes dθ|_\mathcal{Y} + e_Z|_{x_ν(\mathcal{Y})} \otimes dZ|_\mathcal{Y},
\]

for each point \(\mathcal{Y} \in \mathcal{B}^*\). From the equality \(\hat{\nu}_k = γ_k \circ \hat{ν}_1\), the chain rule and (33) it follows that for \(\mathcal{X} \in \mathcal{B}^*, \ k = \Pi(\mathcal{X})\) and \(\mathcal{Y} \in x_1^{-1}(D_k)\),

\[
T_{\mathcal{Y}}x_k = \left. \frac{R}{\sqrt{R^2 - b(\mathcal{X})}} \right|_{C(\mathcal{Y})} E_{r_k}|_{γ_k(\mathcal{Y})} \otimes dR|_\mathcal{Y} + E_{t_k}|_{γ_k(\mathcal{Y})} \otimes dθ|_\mathcal{Y} + E_{z_k}|_{γ_k(\mathcal{Y})} \otimes dZ|_\mathcal{Y}.
\]
Here $y = \kappa_I(y)$. Finally, local configuration at $x \in \mathcal{B}^*$ can be established as

$$
\mathcal{K}_x^R = T^I_{\mathcal{B}_k(y)|y=x}, \quad k = \Pi(x)
$$

Thus, one gets

$$
\mathcal{K}_x^R = \left. \frac{R}{\sqrt{R^2 - b(x)}} \right|_{C(x)} \begin{bmatrix}
E_{t_k}|_{\gamma_k(x)} \otimes dR|x + E_{t_k}|_{\gamma_k(x)} \otimes d\Theta|x + E_{t_k}|_{\gamma_k(x)} \otimes dZ|x
\end{bmatrix},
$$

where $x = \kappa_I(x)$, $k = \Pi(x)$.

The composition

$$
b = b \circ [C \circ \kappa_I]^{-1} : (R, \Theta, Z) \mapsto b(R, \Theta, Z)
$$

is the coordinate representation of $b$. Since distortions of $D_s$ and $D_v$ don’t depend on angular and axial coordinates $\Theta, Z$, one can assume that $b$ depends only on $R$. Denote internal and external radii of the hollow cylinder $\kappa_I(\mathcal{B}_s)$ (shape of the substrate) by $R^i$ and $R^e$, and denote external radius of the hollow cylinder $\kappa_I(\mathcal{B}_x)$ (intermediate shape) by $R^I$. Then one gets

$$
b(R) = \begin{cases}
  a_s, & R^i < R < R^e, \\
  \tilde{b}(R), & R^e < R < R^I
\end{cases}
$$

Here $\tilde{b} = a \circ \pi \circ [C \circ \kappa_I]^{-1}$. Finally, one arrives at the following dyadic representation for local configuration $\mathcal{K}_x^R$ at $x \in \mathcal{B}^*$:

$$
\mathcal{K}_x^R = \left. \frac{R}{\sqrt{R^2 - b(R)}} \right|_{C(x)} \begin{bmatrix}
E_{t_k}|_{\gamma_k(x)} \otimes dR|x + E_{t_k}|_{\gamma_k(x)} \otimes d\Theta|x + E_{t_k}|_{\gamma_k(x)} \otimes dZ|x
\end{bmatrix},
$$

where $x = \kappa_I(x)$, $k = \Pi(x)$.

**60°. Material metric.** The material metric $\mathcal{G}$ of the body $\mathcal{B}^*$ is defined by (36), (8) and has the following dyadic decomposition:

$$
\mathcal{G} = \frac{R^2}{R^2 - b(R)} dR \otimes dR + \left( \frac{R^2 - b(R)}{R^2} \right) d\Theta \otimes d\Theta + dZ \otimes dZ
$$

The scalar curvature $\text{Ric}$, which corresponds to such metric, has the form

$$
\text{Ric} = \frac{Rb''(R) - b'(R)}{R^3}
$$

**61°. Weitzenböck connection.** To introduce Weitzenböck connection on $\mathcal{B}^*$ one needs to use formulae (9) and (36). Nonzero connection coefficients have the form:

$$
\Gamma^1_{11} = \frac{Rb'(R) - 2b(R)}{2R(R^2 - b(R))}, \quad \Gamma^1_{22} = \frac{b(R)}{R} - R, \quad \Gamma^2_{12} = \frac{2R - b'(R)}{2(R^2 - b(R))}, \quad \Gamma^2_{21} = \frac{R}{R^2 - b(R)}
$$

Nonzero components of torsion $\Xi$ of this connection are given below:

$$
\Xi^2_{12} = -\Xi^2_{21} = -\frac{b'(R)}{2(R^2 - b(R))}
$$
62°. Dislocation density tensor. The components of dislocation density tensor\(^ {23} \mathfrak{N} \) are defined by (16) and thus \( \mathfrak{N} \) has the following dyadic decomposition:

\[
\mathfrak{N} = - \frac{b'(R)}{2R(R^2 - b(R))} \partial_\Theta \otimes \partial_Z
\]

63°. Body evolution. In the framework of reasonings from 55°, the evolution of the body can be considered as a motion of a surface (growing surface) over the shape \( \mathfrak{X}_I(\mathfrak{B}^+) \). It can be formalized by introducing a function \( R^c : \mathcal{I} \rightarrow \mathbb{R}_+ \), which for each value of evolution parameter \( \alpha \) returns the radius of the exterior boundary of \( \mathfrak{X}_I(\mathfrak{B}_\alpha) \), i.e.,

\[
\forall \alpha \in \mathcal{I} : \quad \mathfrak{X}_I(\mathfrak{B}_\alpha) = \{ x \in \mathfrak{X}_I(\mathfrak{B}^+) \mid C(x) = (R, \Theta, Z), \ R^i \leq R < R^c(\alpha) \},
\]

and \( R^c(A) = R^c \) (recall that \( A = \min(\mathcal{I}) \)). The function \( R^c \) is assumed to be increasing, since material cannot be removed during the evolutionary process.

For each \( \alpha \in \mathcal{I} \) introduce the function \( b_\alpha \) as the restriction of distortion function \( b \) to the interval \( [R^i, R^c(\alpha)] \):

\[
b_\alpha := b|_{\{ R^i < R < R^c(\alpha) \}}
\]

So defined function \( b_\alpha \) represents the distortion function over laminae which constitute the body \( \mathfrak{B}_\alpha \). The definition implies that if \( \alpha < \beta \), then

\[
b_\beta|_{\{ R^i < R < R^c(\alpha) \}} = b_\alpha
\]

It reflects the condition (26) in terms of distortion functions.

64°. Actual configuration. By assumption (ii) from 56°, the body \( \mathfrak{B}_\alpha \) in actual configuration \( \mathfrak{X}_\alpha \) is a hollow cylinder with the height \( h \). Designate \( \mathfrak{X}_{I; \alpha} = \mathfrak{X}_I|_{\mathfrak{B}_\alpha} \) and assume that deformation

\[
\chi_\alpha = \tilde{\mathfrak{X}}_\alpha \circ (\mathfrak{X}_I; \alpha)^{-1} : \mathfrak{X}_{I; \alpha}(\mathfrak{B}_\alpha) \rightarrow \mathfrak{X}_\alpha(\mathfrak{B}_\alpha)
\]

is a bloating. Introducing cylindrical coordinates \( (r_\alpha, \theta_\alpha, z_\alpha) \) on \( \mathfrak{X}_\alpha(\mathfrak{B}_\alpha) \), we define

\[
\begin{align*}
    r_\alpha(R, \Theta, Z) &= [R^2 + A_\alpha]^{1/2}, \\
    \theta_\alpha(R, \Theta, Z) &= \Theta, \\
    z_\alpha(R, \Theta, Z) &= Z,
\end{align*}
\]

for the coordinate representation of \( \chi_\alpha \). Here \( R \in [R^i, R^c(\alpha)] \), while \( A_\alpha \in \mathbb{R} \) is parameter of the deformation. The deformation gradient \( F_\alpha = D\chi_\alpha \) has the following representation:

\[
F_\alpha = \frac{R}{\sqrt{R^2 + A_\alpha}} e_{r_\alpha} \otimes e^R + e_{\theta_\alpha} \otimes e^\Theta + e_{z_\alpha} \otimes e^Z
\]

In order to obtain dyadic decomposition for \( T\mathfrak{X}_\alpha \), we note that \( T\mathfrak{X}_\alpha = F_\alpha \circ T\tilde{\mathfrak{X}}_I; \alpha \). Thus,

\[
T\mathfrak{X}_\alpha = \frac{R}{\sqrt{R^2 + A_\alpha}} e_{r_\alpha} \otimes dR + e_{\theta_\alpha} \otimes d\Theta + e_{z_\alpha} \otimes dZ
\]

Here \( (dR, d\Theta, dZ) \) is the coframe on \( \mathfrak{B}^+ \) considered at points of \( \mathfrak{B}_\alpha \subset \mathfrak{B}^+ \).

65°. Strain measures. The embedding \( \mathfrak{X}_\alpha \) induces strains which can be measured by left Cauchy–Green strain measure \( B_\alpha = T\mathfrak{X}_\alpha T\mathfrak{X}_\alpha^T \). Here \( T\mathfrak{X}_\alpha^T \) is the transpose of \( T\mathfrak{X}_\alpha \) that is defined by the relation:

\[
\forall u \in T\mathfrak{X}\mathfrak{B} \quad \forall v \in \mathfrak{V} : \quad g_\alpha(T\mathfrak{X}_\alpha(u), v) = \mathfrak{F}_\alpha(u, T\mathfrak{X}_\alpha^T(v))
\]

\( ^{23} \) We use designation \( \mathfrak{N} \) instead of \( \alpha \) in order to avoid ambiguity with evolution parameter \( \alpha \in \mathcal{I} \).
The material metric $\mathcal{g}_\alpha$ is the restriction of material metric $\mathcal{g}$, defined by (37), to $\mathcal{B}_\alpha$. The spatial metric $g_\alpha$ is Euclidean and can be represented in local basis of cylindrical coordinates as

$$g_\alpha = e^{r_\alpha} \otimes e^{r_\alpha} + (r_\alpha)^2 e^{\theta_\alpha} \otimes e^{\theta_\alpha} + e^{z_\alpha} \otimes e^{z_\alpha}.$$

One obtains the following dyadic representation for $B_\alpha$:

$$B_\alpha = \frac{\tau(r_\alpha)}{(r_\alpha)^2} e_{r_\alpha} \otimes e^{r_\alpha} + \frac{(r_\alpha)^2}{\tau(r_\alpha)} e_{\theta_\alpha} \otimes e^{\theta_\alpha} + e_{z_\alpha} \otimes e^{z_\alpha}.$$

Hereinafter $\tau(r_\alpha) = (r_\alpha)^2 - A_\alpha - b_\alpha(\sqrt{(r_\alpha)^2 - A_\alpha})$ is the deformation parameter which determines the axially symmetric transformation that corresponds to the embedding $\kappa_\alpha$.

**66°. Hyperelastic potential.** With the purpose to perform further calculations, specify in more details the mechanical features for the material of solid with variable material composition. Suppose that material of $\mathcal{B}_\alpha$ is incompressible, homogeneous (materially uniform [3]) and isotropic, and elastic energy density is defined by Mooney–Rivlin potential [32,33]

$$W_\alpha = C_1 (I_1(B_\alpha) - 3) + C_2 (I_2(B_\alpha) - 3)$$

(42)

Here $I_1(B) = \text{tr } B$ and $I_2(B) = (I_1^2(B) - I_1(B^2))/2$ are principal invariants of $B$, while $C_1$, $C_2$ are material constants. Note that constants $C_1$, $C_2$ (elastic moduli) can be expressed through another pair of moduli $\beta$ and $\mu$ as follows: $C_1 = (1 + \beta)\mu/4$, $C_2 = (1 - \beta)\mu/4$. Moduli $\mu$ and $\beta$ point to the relationship between the non-linear hyperelastic material model and the linear-elastic model: for infinitesimal deformations from a stress-free state $\mu$ coincides with shear modulus of linearly elastic incompressible material.

**67°. Explicit formulae for stresses.** Since actual shape of $\mathcal{B}_\alpha$ dwells in conventional Euclidean physical space $\mathcal{E}$, Cauchy stress tensor $T_\alpha$ can be determined within the framework of conventional continuum mechanics. Thus, it can be derived from the potential (42) with Doyle–Ericksen relation [34]

$$T_\alpha = -p_\alpha I + 2B_\alpha \frac{\partial W_\alpha}{\partial B_\alpha} = -p_\alpha I + 2C_1 B_\alpha - 2C_2 B_\alpha^{-1}$$

Here $I$ is the identity tensor and $p_\alpha$ is a hydrostatic loading.

The hydrostatic loading $p_\alpha$ can be obtained from the equilibrium equation $\nabla \cdot T_\alpha = 0$ with the condition $T_{(r_\alpha r_\alpha)}(r_\alpha) = p_\alpha$ on the actual inner boundary $r_\alpha = r^I$ as follows. In physical components the equilibrium equation $\nabla \cdot T_\alpha = 0$ reduces to three equations:

$$\frac{\partial T_{(r_\alpha r_\alpha)}}{\partial r_\alpha} + \frac{1}{r_\alpha} (T_{(r_\alpha r_\alpha)} - T_{(\theta_\alpha \theta_\alpha)}) = 0, \quad \frac{\partial T_{(\theta_\alpha \theta_\alpha)}}{\partial \theta_\alpha} = 0, \quad \frac{\partial T_{(z_\alpha z_\alpha)}}{\partial z_\alpha} = 0$$

Here $T_{(r_\alpha r_\alpha)}$, $T_{(\theta_\alpha \theta_\alpha)}$ and $T_{(z_\alpha z_\alpha)}$ are components of Cauchy stress tensor in physical (normalized) basis. From the latter two equations it follows that hydrostatic loading $p_\alpha$ depends only on radial coordinate $r_\alpha$. Thus, it can be obtained from the ordinary differential equation

$$\frac{dT_{(r_\alpha r_\alpha)}}{dr_\alpha} + \frac{1}{r_\alpha} (T_{(r_\alpha r_\alpha)} - T_{(\theta_\alpha \theta_\alpha)}) = 0,$$

after performing integration by $r_\alpha$. Substitution of so obtained $p_\alpha$ to the Cauchy stress tensor $T_\alpha$ gives the following expressions:

$$T_{(r_\alpha r_\alpha)}(r_\alpha) = p_\alpha^i + \mu \int_{r_\alpha}^{r^I} \left[ \frac{\xi}{\xi^2 - A_\alpha - b_\alpha(\sqrt{\xi^2 - A_\alpha})} - \frac{\xi^2 - A_\alpha - b_\alpha(\sqrt{\xi^2 - A_\alpha})}{\xi^3} \right] d\xi, \quad (43)$$

(43)
\[ T(\theta_\alpha, \rho_\alpha)(r_\alpha) = T(\rho_\alpha, \theta_\alpha)(r_\alpha) + \mu \left( \frac{r_\alpha^2}{r_\alpha^2 - A_\alpha - b_\alpha(\sqrt{r_\alpha^2 - A_\alpha})} - \frac{r_\alpha^2 - A_\alpha - b_\alpha(\sqrt{r_\alpha^2 - A_\alpha})}{r_\alpha^2} \right), \]

\[ T(\xi_\alpha, \rho_\alpha)(r_\alpha) = T(\rho_\alpha, \xi_\alpha)(r_\alpha) + \mu \left\{ A_\alpha + b_\alpha(\sqrt{r_\alpha^2 - A_\alpha}) \right\} \frac{r_\alpha^2 - (1 + \beta)\left( A_\alpha + b_\alpha(\sqrt{r_\alpha^2 - A_\alpha}) \right)/2}{r_\alpha^2(r_\alpha^2 - A_\alpha - b_\alpha(\sqrt{r_\alpha^2 - A_\alpha}))}. \]

**Remark 15.** The normalized basis for \((r_\alpha, \theta_\alpha, \xi_\alpha)\) can be defined as follows:

\[ e_{(c)} = \frac{1}{\sqrt{g_{c\alpha}}} e_c, \quad c \in \{ r_\alpha, \theta_\alpha, \xi_\alpha \}, \]

where \(g_{ij}\) are components of metric tensor

\[ g_\alpha = e^{\alpha\alpha} \otimes e^{\alpha\alpha} + (r_\alpha)^2 e^{\theta\alpha} \otimes e^{\theta\alpha} + e^{\xi\alpha} \otimes e^{\xi\alpha}. \]

**68°. Evolutionary problem.** All calculations were carried out above in assumption that the family \(\mathcal{B}_\alpha\) of configurations is given. Equivalently, the family of distortion functions \(\{b_\alpha\}_{\alpha \in I}\) is known. At the same time, as a rule, they need to be found from the problem which takes into account the specificity of the assembly process. Let us consider this in detail.

Recall that \(R^i\) and \(R^e\) are internal and external radii of the intermediate shape of \(\mathcal{B}_\alpha\). These radii, like the intermediate shape, don’t depend on evolutionary parameter \(\alpha\). Suppose that the following data are given: radius function \(R^e : I \to \mathbb{R}_+\) of the growing surface, internal \(p^i : I \to \mathbb{R}\) and external \(p^e : I \to \mathbb{R}\) hydrostatic loadings in actual configuration. On the interior and exterior boundaries of \(\mathcal{B}_\alpha\) consider the conditions:

\[ T(\rho_\alpha, \theta_\alpha)(r_\alpha) = p^f(\alpha), \quad T(\rho_\alpha, \theta_\alpha)(r_\alpha) = p^e(\alpha), \]

where \(T(\rho_\alpha, \theta_\alpha)(r)\) is defined by the (43). Since in the relation (35) one can set \(a_\alpha = 0\) without loss of generality, i.e.,

\[ b_\alpha = \begin{cases} 0, & R^i < R < R^e, \\ \tilde{b}_\alpha(R), & R^e < R < R^e(\alpha), \end{cases} \]

then one arrives at the following integral equation\(^{24}\):

\[ \int_{R^e}^{R^i} \left[ \frac{1}{\xi^2} - \frac{\xi^2}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi + \int_{R^e}^{R^e(\alpha)} \left[ \frac{1}{\xi^2 - \tilde{b}_\alpha(\xi)} - \frac{\xi^2 - \tilde{b}_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p^f(\alpha) - p^e(\alpha)}{\mu}, \]

or, since the first integral can be expressed in elementary functions,

\[ \ln \frac{R^e \sqrt{(R^e)^2 + A_\alpha}}{R^i \sqrt{(R^i)^2 + A_\alpha}} - \frac{A_\alpha}{2} \left( \frac{1}{(R^e)^2 + A_\alpha} - \frac{1}{(R^i)^2 + A_\alpha} \right) + \int_{R^e}^{R^e(\alpha)} \left[ \frac{1}{\xi^2 - \tilde{b}_\alpha(\xi)} - \frac{\xi^2 - \tilde{b}_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p^f(\alpha) - p^e(\alpha)}{\mu} \quad (44) \]

The nonlinear integral equation (44) contains two unknown functions \(b_\alpha\) and \(A_\alpha\). The additional equation, which completes the evolutionary problem, may be formulated in terms of shrinkage coefficient \(S : I \to [0, 1]_*\),

\[ r^e(\alpha) = S(\alpha)r^f(\alpha), \]

\(^{24}\)In course of obtaining this equation, the integral in (43) was rewritten in intermediate shape variables.
where \( r^e(\alpha) \) is “crystal reference radius” \([14]\) which corresponds to the exterior boundary of \( \zeta_I(\mathfrak{B}_\alpha) \), while \( r^e(\alpha) \) is exterior radius of \( \zeta_\alpha(\mathfrak{B}_\alpha) \). Thus, with (28) and (40),

\[
\begin{align*}
\text{r}^e(\alpha) &= \sqrt{R^e(\alpha)^2 - b_\alpha(R^e(\alpha))}, \\
\text{r}^e(\alpha) &= \sqrt{R^e(\alpha)^2 + A_\alpha}
\end{align*}
\]

Finally, one gets the relation

\[
[1 - S(\alpha)^2]R^e(\alpha)^2 - \tilde{b}_\alpha(R^e(\alpha)) = S(\alpha)^2 A_\alpha
\]

We arrive at the statement for evolutionary problem given by equations (44) and (45).

**69°. Exact particular solution.** Consider particular case, in which \( \mathfrak{B}_x \) degenerates into a material surface. Then \( R^t = R^s \) and consequently, the equation (44) reduces to

\[
\begin{align*}
\int_{R^t}^{R^e(\alpha)} \left[ \frac{1}{\xi^2 - b_\alpha(\xi)} - \frac{\xi^2 - \tilde{b}_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p^e(\alpha) - p^i(\alpha)}{\mu}
\end{align*}
\]

Fix \( \alpha \in I \) and introduce a new variable \( \eta = \vartheta^2 + A_\alpha \). Define functions \( B_\alpha : \eta \mapsto B_\alpha(\eta), \eta^i \) and \( \eta^e(\alpha) \) as follows:

\[
\begin{align*}
B_\alpha(\eta) &= A_\alpha + \tilde{b}_\alpha(\sqrt{\eta - A_\alpha}), \\
\eta^i &= (r^i)^2 = (R^t)^2 + A_\alpha, \\
\eta^e(\alpha) &= r^e(\alpha)^2 = R^e(\alpha)^2 + A_\alpha
\end{align*}
\]

Note that the mapping \( B_\alpha \) characterizes local deformation of “crystal reference” elementary volume that brings it into actual shape. Equations (46) and (45) in new variable take the form

\[
\begin{align*}
\frac{1}{2} \int_{\eta^i}^{\eta^e} \left( \frac{1}{\eta - B_\alpha(\eta)} - \frac{\eta - B_\alpha(\eta)}{\eta^2} \right) d\eta &= \frac{p^e(\alpha) - p^i(\alpha)}{\mu}, \\
B_\alpha(R^e(\alpha)^2 + A_\alpha) &= (1 - S(\alpha)^2)(R^e(\alpha)^2 + A_\alpha)
\end{align*}
\]

The integral

\[
I = \frac{1}{2} \int_{\eta^i}^{\eta^e} \left( \frac{1}{\eta - B_\alpha(\eta)} - \frac{\eta - B_\alpha(\eta)}{\eta^2} \right) d\eta
\]

can be calculated exactly for several particular expressions for \( B_\alpha(\eta) \). For example, if we put

\[
B_\alpha(\eta) = f + g\eta + h\eta^2,
\]

where \( f, g, h \in \mathbb{R} \) depend on \( \alpha \) only, then the integral \( I \) results in

\[
I = \frac{1}{2} \left( -\frac{2 \arctan \left( \frac{g + 2gh - 1}{\sqrt{4fh - g^2 + 2g - 1}} \right)}{\sqrt{4fh - g^2 + 2g - 1}} - \frac{f}{\eta} + (g - 1) \ln \eta + \eta h \right)
\]

Since

\[
\tilde{b}_\alpha(R) = B_\alpha(R^2 + A_\alpha) - A_\alpha
\]
then by virtue of (48), one gets
\[ \tilde{b}_\alpha(R) = (f + gA_\alpha + hA_\alpha^2 - A_\alpha) + (g + 2hA_\alpha)R^2 + hR^4 \]
To satisfy the condition (39), it is sufficient to choose \( f, g, h \) as
\[ h = \tilde{h}, \quad g = \tilde{g} - 2hA_\alpha, \quad f = \tilde{f} + A_\alpha - gA_\alpha - hA_\alpha^2, \]
where \( \tilde{f}, \tilde{g}, \tilde{h} \) are arbitrary real constants. With such the choice of \( f, g, h \), one has
\[ \tilde{b}_\alpha(R) = \tilde{f} + \tilde{g}R^2 + \tilde{h}R^4 \] (49)
With (49) one can determine \( A_\alpha \) from (45):
\[ A_\alpha = \left[ 1 - S(\alpha)^2 \right] R^2(\alpha)^2 - \tilde{f} - \tilde{g}R^2(\alpha)^2 - \tilde{h}R^4(\alpha)^4 \]

Using above obtained results one can derive the expression for the relative difference \( (p^e(\alpha) - p^i(\alpha))/\mu \) of hydrostatic loadings in internal and external boundaries:
\[ \frac{p^e(\alpha) - p^i(\alpha)}{\mu} = \frac{1}{2} \left( - \frac{2 \arctan \frac{g + 2\eta h - 1}{\sqrt{4fh - g^2 + 2g - 1}}}{\sqrt{4fh - g^2 + 2g - 1}} - \frac{f}{\eta} + (g - 1) \ln \eta + \eta h \right) \]
It is the relative loading difference, to which the solution (49) corresponds.

7. Spherical Problem

70°. Shapes. As a second example, we consider model problem for spherical shell. Complete items (a1), (a2) from 53° and (b1), (b2) from 54°, by assumptions:

(i) Each of shapes \( \kappa^i_r(\mathbb{B}^*) \), \( \kappa^r_\alpha(B_\alpha) \), \( \kappa^r_\alpha(B^*_\alpha) \) and \( \kappa^r_\alpha(B^*_{\alpha}) \) is a hollow ball, while each set \( \kappa^r_\alpha(\mathcal{S}_\nu) \) is a sphere in \( \mathbb{S} \).

(ii) For any \( \alpha \in I \) the actual configuration \( \kappa^o_\alpha \in \mathcal{C}(\mathbb{B}_\alpha) \) is such that the actual shape is a hollow ball.

71°. Intermediate configuration. By reasonings, similar to the previous example, we choose an intermediate configuration \( \kappa^I \in \mathcal{C}(\mathbb{B}^*) \) such that \( \kappa^I(\mathbb{B}^*) \) and \( \kappa^I(\mathbb{B}_\alpha) \) are hollow balls. Designate the coordinate homeomorphism which associates points of the intermediate shape with their spherical coordinates by
\[ C : \kappa^I(\mathbb{B}^*) \ni x \mapsto (R, \Theta, \Phi) \in \mathbb{R}^3 \]
This map induces coordinate homeomorphism
\[ C \circ \tilde{\kappa}^I : \mathbb{B}^* \ni x \mapsto (R, \Theta, \Phi) \in \mathbb{R}^3 \]
on \( \mathbb{B}^* \). Denote spherical coordinates on the surface \( \kappa^I(\mathcal{S}_\nu) \) (which is a sphere) by \( (R_\nu, \Theta_\nu, \Phi_\nu) \).

72°. Deformation ansatz. The transformation
\[ \gamma^I_\nu = \tilde{\gamma}^I_\nu \circ \tilde{\kappa}^{-1}^I : \kappa^I(\mathbb{B}^*) \to \kappa^I_\nu(\mathbb{B}^*), \quad \nu \in \tilde{J}, \quad (50) \]
represents deformation from intermediate shape into discharged one. By reasons similar to the cylindrical problem, we need to take into account only local behaviour of (50). To this end, assume that there exists a number \( \delta_\nu > 0 \) such that the deformation \( \gamma_\nu|_{D_\nu} \) of hollow ball \( D_\nu \subset \mathcal{X}_I(\mathcal{B}^*) \),

\[
D_\nu = \{ x \in \mathcal{B}^* \mid R_\nu - \delta_\nu < \| x - \sigma \| < R_\nu + \delta_\nu \},
\]

is a uniform inflation. That is, the coordinate representation of \( \gamma_\nu|_{D_\nu} \) has the form

\[
\begin{align*}
\tau_\nu(R, \Theta, \Phi) &= \left[ R^3 - a(\nu) \right]^{1/3}, \\
t_\nu(R, \Theta, \Phi) &= \Theta, \\
f_\nu(R, \Theta, \Phi) &= \Phi,
\end{align*}
\]

(51)

where \( a(\nu) \in \mathbb{R} \) is a number, while \( (\tau_\nu, t_\nu, f_\nu) \) are spherical coordinates on \( \mathcal{X}_I(\mathcal{B}^*) \). By virtue of (51) the deformation gradient \( F_\nu(\rho) \) at point \( \rho \in D_\nu \) has the following dyadic representation:

\[
F_\nu(\rho) = \frac{R^2}{[R^3 - a(\nu)]^{2/3}} E_{\tau_\nu} \otimes e^R + E_{t_\nu} \otimes e^\Theta + E_{f_\nu} \otimes e^\Phi,
\]

(52)

where \( (e_R, e_\Theta, e_\Phi), (e^R, e^\Theta, e^\Phi) \) are local frame and coframe on \( \mathcal{X}_I(\mathcal{B}^*) \), while \( (E_{\tau_\nu}, E_{t_\nu}, E_{f_\nu}) \), \( (E^{\tau_\nu}, E^{t_\nu}, E^{f_\nu}) \) are local frame and coframe on \( \mathcal{X}_I^*(\mathcal{B}^*) \) with respect to coordinates \( (\tau_\nu, t_\nu, f_\nu) \).

As it was already noted, the case of substrate \( \mathcal{B}_s \) requires individual consideration. The mapping

\[
\gamma_s = \tilde{\gamma}_s \circ \hat{\mathcal{X}}^{-1} : \mathcal{X}_I(\mathcal{B}^*) \rightarrow \mathcal{X}_I^*(\mathcal{B}^*)
\]

is a deformation of the shape \( \mathcal{X}_I(\mathcal{B}^*) \) into the shape \( \mathcal{X}_I^*(\mathcal{B}^*) \). Denote \( D_s = \mathcal{X}_I(\mathcal{B}_s) \). The deformation \( \gamma_s|_{D_s} \) of the shape \( D_s \) is supposed to be a uniform inflation,

\[
\begin{align*}
\tau_s(R, \Theta, \Phi) &= \left[ R^3 - a_s \right]^{1/3}, \\
t_s(R, \Theta, \Phi) &= \Theta, \\
f_s(R, \Theta, \Phi) &= \Phi,
\end{align*}
\]

(53)

where \( a_s \in \mathbb{R} \) is deformation parameter, while \( (\tau_s, t_s, f_s) \) are spherical coordinates on \( \mathcal{X}_I^*(\mathcal{B}^*) \). Then the gradient of \( \gamma_s|_{D_s} \) at point \( \rho \in D_s \) is represented in the dyadic form as

\[
F_s(\rho) = \frac{R^2}{[R^3 - a_s]^{2/3}} E_{\tau_s} \otimes e^R + E_{t_s} \otimes e^\Theta + E_{f_s} \otimes e^\Phi
\]

(54)

Here \( (E_{\tau_s}, E_{t_s}, E_{f_s}), (E^{\tau_s}, E^{t_s}, E^{f_s}) \) are consequently local frame and coframe on \( \mathcal{X}_I^*(\mathcal{B}^*) \) generated by coordinates \( (\tau_s, t_s, f_s) \).

Similarly to the cylindrical problem, we represent local distortions, represented by \( a(\nu) \) and \( a_s \), in terms of the mapping \( b : \mathcal{B}^* \rightarrow \mathbb{R} \) which is defined by (32). In order to represent deformation gradients (52) and (54) in unified form, denote \( \mathcal{K} = \{*\} \cup \mathbb{J} \) and introduce the map \( \Pi : \mathcal{B}^* \rightarrow \mathcal{K} \) as

\[
\Pi(\mathcal{X}) = \begin{cases} 
*, & \mathcal{X} \in \mathcal{B}_s \cup \partial_{\mathcal{B}^*}(\mathcal{B}^* \setminus \mathcal{B}_s), \\
\pi(\mathcal{X}), & \mathcal{X} \in \mathcal{B}^*
\end{cases}
\]

Then for \( \mathcal{X} \in \mathcal{B}^*, \ k = \Pi(\mathcal{X}) \) and \( \rho \in D_k \), one gets

\[
F_k(\rho) = \frac{R^2}{[R^3 - b(\mathcal{X})]^{2/3}} E_{t_k} \otimes e^R + E_{l_k} \otimes e^\Theta + E_{f_k} \otimes e^\Phi
\]

(55)
73°. Local configuration. Let \((\partial R, \partial \Theta, \partial k)\) and \((dR, d\Theta, d\Phi)\) be coordinate frame and coframe on \(\mathfrak{B}^*\). Then for each point \(\mathfrak{y} \in \mathfrak{B}^*\) one obtains

\[
T_\mathfrak{y} \kappa I = e_\mathfrak{y} |_{\kappa I} (\mathfrak{y}) \otimes dR|_{\mathfrak{y}} + e_\mathfrak{y} |_{\kappa I} (\mathfrak{y}) \otimes d\Theta|_{\mathfrak{y}} + e_\mathfrak{y} |_{\kappa I} (\mathfrak{y}) \otimes d\Phi|_{\mathfrak{y}}.
\]

From the chain rule, applied to the equality \(\kappa I_k = \gamma_k \circ \kappa I\), and expression for (55) it follows that for \(X \in \mathfrak{B}^*, \ k = \Pi(X)\) and \(\mathfrak{y} \in \kappa I^{-1}(D_k)\),

\[
T_\mathfrak{y} \kappa I_k = \frac{R^2}{[R^3 - b(X)]^{2/3}} \left|_{C(x)} \right. \left. E_{ik} |_{\gamma_k(x)} \otimes dR|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Theta|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Phi|_{X}, \quad (56)
\]

where \(X = \kappa I(\mathfrak{y})\). Local configuration at \(X \in \mathfrak{B}^*\) is defined by relation \(\mathcal{X}_X^R = T_\mathfrak{y} \kappa I_k|_{\mathfrak{y} = X}\):

\[
\mathcal{X}_X^R = \frac{R^2}{[R^3 - b(X)]^{2/3}} \left|_{C(x)} \right. \left. E_{ik} |_{\gamma_k(x)} \otimes dR|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Theta|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Phi|_{X},
\]

where \(x = \kappa I(X), \ k = \Pi(X)\).

The coordinate representation of \(b\) is provided by the mapping

\[
b = b \circ [C \circ \kappa I]^{-1} : (R, \Theta, \Phi) \mapsto b(R, \Theta, \Phi)
\]

Since shapes \(D_a\) and \(D_\nu\) are deformed centrally symmetrical, it seems reasonable to suppose that \(b\) depends only on \(R\). In this regard, one gets

\[
b(R) = \begin{cases} a_s, & R_i < R < R^s, \\ \bar{b}(R), & R^s < R < R^j, \end{cases} \quad (57)
\]

where \(\bar{b} = a \circ \pi \circ [C \circ \kappa I]^{-1}\). Here \(R^i\) and \(R^s\) are internal and external radii of the hollow ball \(\kappa I(\mathfrak{B}_s)\), while \(R^j\) is external radius of the hollow ball \(\kappa I(\mathfrak{B}^*)\). Finally, one arrives at the following expression for local configuration \(\mathcal{X}_X^R\) at \(X \in \mathfrak{B}^*\):

\[
\mathcal{X}_X^R = \frac{R^2}{[R^3 - b(R)]^{2/3}} \left|_{C(x)} \right. \left. E_{ik} |_{\gamma_k(x)} \otimes dR|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Theta|_{X} + E_{ik} |_{\gamma_k(x)} \otimes d\Phi|_{X},
\]

in which \(x = \kappa I(X), \ k = \Pi(X)\).

74°. Material metric. The material metric \(\mathcal{E}\) of the body \(\mathfrak{B}^*\) is defined by (8), (58) and has the following dyadic decomposition:

\[
\mathcal{E} = \frac{R^4}{[R^3 - b(R)]^{4/3}} dR \otimes dR + [R^3 - b(R)]^{2/3} d\Theta \otimes d\Theta + [R^3 - b(R)]^{2/3} \sin^2 \Theta d\Phi \otimes d\Phi \quad (59)
\]

Consequently, scalar curvature \(\text{Ric}\) is represented by the expression

\[
\text{Ric}(R) = 2 \frac{-b'(R)(6R^3 - 12b(R) + Rb'(R)) + 6R(R^3 - b(R))b''(R)}{9R^3(R^3 - b(R))^{2/3}} \quad (60)
\]

75°. Weitzenböck connection. To introduce Weitzenböck connection on \(\mathfrak{B}^*\) one needs to use formulae (9) and (58). Nonzero connection coefficients have the form:

\[
\Gamma^1_{11} = \frac{2b'(R)}{3(R^3 - b(R))}, \quad \Gamma^1_{12} = \frac{b'(R)}{R^2} - R, \quad \Gamma^1_{33} = \frac{(b(R) - R^3) \sin^2 \Theta}{R^2},
\]

\[
\Gamma^2_{12} = \Gamma^3_{13} = \frac{3R^3 - b'(R)}{3(R^3 - b(R))}, \quad \Gamma^2_{21} = \Gamma^3_{31} = \frac{R^2}{R^3 - b(R)},
\]

\[
\Gamma^2_{33} = - \sin \Theta \cos \Theta, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot \Theta
\]
Nonzero components of torsion are
\[ \Xi_{12} = \Xi_{33} = -\Xi_{21} = -\Xi_{31} = -\frac{b'(R)}{3(R^3 - b(R))}. \]

**76°. Dislocation density tensor.** The dislocation density tensor \( \mathcal{N} \) components are defined by (16). Thus,
\[ \mathcal{N} = -\frac{b'(R) \csc \Theta}{3R^2(R^3 - b(R))} (\partial_\theta \otimes \partial_\phi - \partial_\phi \otimes \partial_\theta) \]

**77°. Body evolution.** By continuing reasonings from 55°, one can represent the evolution of the material composition of the solid as a motion of the growing surface over the intermediate shape of \( \mathcal{M}^* \). To this end we introduce an increasing function \( R^e : I \to \mathbb{R}_+ \), such that
\[ \forall \alpha \in I : \mathcal{N}_I(\mathcal{M}_\alpha) = \{ x \in \mathcal{M} : R^i < ||x - \phi|| < R^e(\alpha) \}, \]
and \( R^e(A) = R^e \).

For each \( \alpha \in I \) define the function \( b_\alpha \) as
\[ b_\alpha := b_{|\{R^i < R < R^e(\alpha)\}} \]
That is, \( b_\alpha \) is equal to the restriction of the global distortion function \( b \) to laminae which constitute the body \( \mathcal{M}_\alpha \). From the definition it follows that if \( \alpha < \beta \) then, like for the cylindrical problem, the equality (39) holds.

**78°. Actual configuration.** By assumption (ii) of 70°, the body \( \mathcal{M}_\alpha \) in actual configuration \( \mathcal{N}_\alpha \) is also a hollow ball. Denote \( \mathcal{N}_I : \mathcal{M}_\alpha \to \mathcal{M}_\alpha \) and suppose that deformation
\[ \chi_\alpha = \hat{\mathcal{N}}_\alpha \circ (\mathcal{N}_I)^{-1} : \mathcal{N}_I(\mathcal{M}_\alpha) \to \mathcal{M}_\alpha(\mathcal{M}_\alpha) \]
is a uniform inflation. Introducing spherical coordinates \((r_\alpha, \theta_\alpha, \varphi_\alpha)\) on \( \mathcal{M}_\alpha(\mathcal{M}_\alpha) \), we set
\[ r_\alpha(R, \Theta, \Phi) = [R^3 + A_\alpha]^{1/3}, \]
\[ \theta_\alpha(R, \Theta, \Phi) = \Theta, \]
\[ \varphi_\alpha(R, \Theta, \Phi) = \Phi, \]
for the coordinate representation of \( \chi_\alpha \). Here \( R \in [R^i, R^e(\alpha)] \), while \( A_\alpha \in \mathbb{R} \) is the deformation parameter. The deformation gradient \( F_\alpha \), which corresponds to \( \chi_\alpha \), has the following dyadic decomposition:
\[ F_\alpha = \frac{R^2}{[R^3 + A_\alpha]^{2/3}} e_{r_\alpha} \otimes e^R + e_{\theta_\alpha} \otimes e^\Theta + e_{\varphi_\alpha} \otimes e^\Phi \]
Noticing that \( T \chi_\alpha = F_\alpha \circ T \mathcal{N}_I : \alpha, \) one obtains
\[ T \chi_\alpha = \frac{R^2}{[R^3 + A_\alpha]^{2/3}} e_{r_\alpha} \otimes dR + e_{\theta_\alpha} \otimes d\Theta + e_{\varphi_\alpha} \otimes d\Phi \]
Here \((dR, d\Theta, d\Phi)\) is the coframe on \( \mathcal{M}^* \) considered at points of \( \mathcal{M}_\alpha \subset \mathcal{M}^* \).

**79°. Strain measures.** The embedding \( \mathcal{N}_\alpha \) results in strains which can be measured by left Cauchy–Green strain measure \( B_\alpha = T \chi_\alpha T \chi_\alpha^T \). Here \( T \chi_\alpha^T \) is defined by formula (41), in which the material metric \( G_\alpha \) is represented by (59). The spatial metric \( g_\alpha \) is Euclidean and its dyadic decomposition in local basis of spherical coordinates has the form
\[ g_\alpha = e^{r_\alpha} \otimes e^{r_\alpha} + (r_\alpha)^2 e^{\theta_\alpha} \otimes e^{\theta_\alpha} + (r_\alpha)^2 \sin^2 \theta_\alpha e^{\varphi_\alpha} \otimes e^{\varphi_\alpha} \]
In course of deriving (63), the integral (62) was expressed in intermediate shape variables. Hereinafter we use designation \( \tau(r_\alpha) = (r_\alpha)^3 - A_\alpha - b_\alpha (\sqrt[3]{(r_\alpha)^3} - A_\alpha) \).

80°. Explicit formulae for stresses. In order to obtain explicit formulae for stresses, suppose that, like in the cylindrical problem, the material is hyperelastic, isotropic and its stored elastic energy can be obtained by Mooney–Rivlin potential (42). Due to the central symmetry of deformations, the equilibrium equation \( \nabla \cdot \mathbf{T}_\alpha = 0 \) reduces to

\[
\frac{\partial T_{(r_\alpha r_\alpha)}}{\partial r_\alpha} + \frac{2}{r_\alpha}(T_{(r_\alpha r_\alpha)} - T_{(\theta_\alpha \theta_\alpha)}) = 0, \quad \frac{\partial T_{(\theta_\alpha \theta_\alpha)}}{\partial \theta_\alpha} = 0, \quad \frac{\partial T_{(\varphi_\alpha \varphi_\alpha)}}{\partial \varphi_\alpha} = 0.
\]

In these three equations the designations \( T_{(r_\alpha r_\alpha)} \), \( T_{(\theta_\alpha \theta_\alpha)} \) and \( T_{(\varphi_\alpha \varphi_\alpha)} \) stand for components of Cauchy stress tensor in physical (normalized) basis. Then the solution of the obtained system with the condition \( T_{(r_\alpha r_\alpha)}(r_\alpha^i) = p_\alpha^i \) on the actual inner boundary \( r_\alpha = r_\alpha^i \) gives the following expressions for components of Cauchy stress tensor:

\[
T_{(r_\alpha r_\alpha)} = p_\alpha^i + \mu \int_{r_\alpha^i}^{r_\alpha} \left\{ (1 + \beta) \left( \frac{\xi^2}{\tau^{2/3}(\xi)} - \frac{\tau^{1/3}(\xi)}{\xi^4} \right) + (\beta - 1) \left( \frac{\tau^{2/3}(\xi)}{\xi^2} - \frac{\xi^4}{\tau^{4/3}(\xi)} \right) \right\} d\xi, \quad (62)
\]

\[
T_{(\theta_\alpha \theta_\alpha)} = \frac{1 + \beta}{2} \mu \left( \frac{(r_\alpha)^2}{\tau^{2/3}(r_\alpha)} - \frac{\tau^{4/3}(r_\alpha)}{r_\alpha^4} \right) + \frac{\beta - 1}{2} \mu \left( \frac{\tau^{2/3}(r_\alpha)}{(r_\alpha)^2} - \frac{(r_\alpha)^4}{\tau^{4/3}(r_\alpha)} \right),
\]

and \( T_{(\varphi_\alpha \varphi_\alpha)} = T_{(\theta_\alpha \theta_\alpha)} \).

Remark 16. The normalized basis for \( (r_\alpha, \theta_\alpha, \varphi_\alpha) \) is defined as follows:

\[
e_{(a)} = \frac{1}{\sqrt{g_{cc}}} e_c, \quad c \in \{r_\alpha, \theta_\alpha, \varphi_\alpha\},
\]

where \( g_{ij} \) are components of metric tensor

\[
g_{\alpha} = e^{\alpha} \otimes e^{\alpha} + (r_\alpha)^2 e^{\theta}\otimes e^{\theta} + (r_\alpha)^2 \sin^2 \theta_\alpha e^{\varphi}\otimes e^{\varphi}
\]

81°. Evolutionary problem. Suppose that the family \( \{b_\alpha\}_{\alpha \in I} \) of distortion functions is unknown. In order to formulate the evolutionary problem for its determination, let the following data be given: external radius \( R^e : \mathbb{I} \rightarrow \mathbb{R}_+ \) of the intermediate shape, internal \( p^i : \mathbb{I} \rightarrow \mathbb{R} \) and external \( p^e : \mathbb{I} \rightarrow \mathbb{R} \) hydrostatic loadings in actual configuration. On the interior and exterior boundaries of \( \mathcal{X}_\alpha(\mathcal{B}_\alpha) \) consider the conditions:

\[
T_{(r_\alpha r_\alpha)}(r_\alpha^i) = p^i(\alpha), \quad T_{(r_\alpha r_\alpha)}(r_\alpha^e) = p^e(\alpha),
\]

where \( T_{(r_\alpha r_\alpha)}(r) \) is defined by the (62). In the relation (57) one can put \( a_\alpha = 0 \) without loss of generality. Thus,

\[
b_\alpha(R) = \begin{cases} 0, & R^i < R < R^e, \\ \tilde{b}_\alpha(R), & R^e < R < R^e(\alpha), \end{cases}
\]

and one gets:

\[
I_0 + (1 + \beta)(I_1 - I_2) + (\beta - 1)(I_3 - I_4) = \frac{p^e(\alpha) - p^i(\alpha)}{\mu}, \quad (63)
\]

\(^{25}\)In course of deriving (63), the integral (62) was expressed in intermediate shape variables.
where \( I_k, k \in \{0, \ldots, 4\} \), denote the following integrals:

\[
I_0 = \int_{R^s} \left\{ (1 + \beta) \left( \frac{(\vartheta^3 + A_\alpha)^{2/3}}{\vartheta^2} - \frac{\vartheta^4}{(\vartheta^3 + A_\alpha)^{4/3}} \right) + 
(\beta - 1) \left( \frac{\vartheta^2}{(\vartheta^3 + A_\alpha)^{2/3}} - \frac{(\vartheta^3 + A_\alpha)^{4/3}}{\vartheta^4} \right) \right\} \frac{\vartheta^2 d\vartheta}{\vartheta^3 + A_\alpha},
\]

\[
I_1 = \int_{R^s} \frac{\vartheta^2}{(\vartheta^3 + A_\alpha)^{2/3}(\vartheta^3 - \tilde{b}_\alpha(\vartheta))^2/3} d\vartheta, \quad I_2 = \int_{R^s} \frac{\vartheta^2}{(\vartheta^3 + A_\alpha)^{4/3}} \frac{(\vartheta^3 - \tilde{b}_\alpha(\vartheta))^{4/3} \vartheta^2}{(\vartheta^3 + A_\alpha)^{7/3}} d\vartheta,
\]

\[
I_3 = \int_{R^s} \frac{(\vartheta^3 - \tilde{b}_\alpha(\vartheta))^{2/3} \vartheta^2}{(\vartheta^3 + A_\alpha)^{5/3}} d\vartheta, \quad I_4 = \int_{R^s} \frac{(\vartheta^3 + A_\alpha)^{1/3} \vartheta^2}{(\vartheta^3 - \tilde{b}_\alpha(\vartheta))^{4/3}} d\vartheta
\]

Note, that value of \( I_0 \) in (63) can be exactly expressed in elementary functions:

\[
I_0 = \frac{4A_\alpha^2(\beta - 1) + \vartheta^5(5(\beta + 1)(\vartheta^3 + A_\alpha)^{1/3} + 2(\beta - 1)\vartheta)}{4\vartheta(\vartheta^3 + A_\alpha)^{5/3}} \left|_{\vartheta=R^s}^{\vartheta=R^e} \right.
\]

\[
+ \frac{A_\alpha \vartheta(2(\beta + 1)(\vartheta^3 + A_\alpha)^{1/3} + 3(\beta - 1)\vartheta)}{2(\vartheta^3 + A_\alpha)^{5/3}} \left|_{\vartheta=R^e}^{\vartheta=R^s} \right.
\]

The nonlinear integral equation (63) contains two unknown functions \( b_\alpha \) and \( A_\alpha \). The additional equation, which completes the evolutionary problem, is formulated in terms of shrinkage coefficient \( S : I \rightarrow [0, 1[ \),

\[
\tau^e(\alpha) = S(\alpha) r^e(\alpha),
\]

where \( \tau^e(\alpha) \) is “crystal reference radius” that corresponds to exterior boundary of the intermediate shape and \( r^e(\alpha) \) is exterior radius of actual shape. Thus, with (51) and (61),

\[
\tau^e(\alpha) = [R^e(\alpha)^3 - b_\alpha(R^e(\alpha))]^{1/3}, \quad r^e(\alpha) = [R^e(\alpha)^3 + A_\alpha]^{1/3}
\]

Finally, one gets the relation

\[
[1 - S(\alpha)^3]R^e(\alpha)^3 - b_\alpha(R^e(\alpha)) = S(\alpha)^3 A_\alpha
\]

(64)

We arrive at the statement for evolutionary problem given by equations (63) and (64).

**82°. Exact particular solution.** Here and throughout we suppose that \( \mathcal{B}_\alpha \) degenerates into a material surface. We put \( R^e = R^s \) and consequently, the term \( I_0 \) in equation (63) vanishes. In order to provide further simplification, fix \( \alpha \in I \) and introduce a new variable \( \eta = \vartheta^3 + A_\alpha \). Define functions \( B_\alpha : \eta \mapsto B_\alpha(\eta), \eta^j, \eta^j(\alpha) \) as follows:

\[
B_\alpha(\eta) := A_\alpha + \tilde{b}_\alpha(\sqrt[3]{\eta - A_\alpha}), \quad \eta^j := (r^i)^3 = (R^i)^3 + A_\alpha, \quad \eta^j(\alpha) := r^i(\alpha)^3 = R^e(\alpha)^3 + A_\alpha
\]

Equations (63) and (64) in new variable take the form

\[
(1 + \beta)(\tilde{I}_1 - \tilde{I}_2) + (\beta - 1)(\tilde{I}_3 - \tilde{I}_4) = \frac{p^f(\alpha) - p^i(\alpha)}{\mu}, \quad B_\alpha(R^e(\alpha)^3 + A_\alpha) = (1 - S(\alpha)^3)(R^e(\alpha)^3 + A_\alpha),
\]

(65)
where \( \tilde{I}_k \), \( k \in \{1, \ldots, 4\} \), are given by equalities

\[
\begin{align*}
\tilde{I}_1 &= \frac{1}{3} \int_{\eta'}^{\eta''} \frac{w^2 F_1\left(\frac{3}{5}; \frac{2}{5}; \frac{2}{5}; p, q\right)}{\eta^{1/3}} \eta^{5/3} d\eta, \\
\tilde{I}_2 &= \frac{1}{3} \int_{\eta'}^{\eta''} \frac{(\eta - B_\alpha(\eta))^{4/3}}{\eta^{7/3}} d\eta, \\
\tilde{I}_3 &= \frac{1}{3} \int_{\eta'}^{\eta''} \frac{w^2 F_1\left(\frac{2}{5}; \frac{3}{5}; \frac{2}{5}; p, q\right)}{\eta^{5/3}} d\eta, \\
\tilde{I}_4 &= \frac{1}{3} \int_{\eta'}^{\eta''} \frac{\eta^{1/3}}{\eta - B_\alpha(\eta)} d\eta.
\end{align*}
\]

(66)

The integrals (66) and (67) can be calculated exactly for several particular expressions for \( B_\alpha(\eta) \). For example, if we take

\[ B_\alpha(\eta) = f + g\eta + h\eta^2, \]

where \( f, g, h \in \mathbb{R} \) depend on \( \alpha \) only, then the integrals (66), (67) result in:

\[
\begin{align*}
\tilde{I}_1 &= \frac{1}{2} \left(\frac{\eta}{7}\right)^{2/3} w^2 F_1\left(\frac{3}{5}; \frac{2}{5}; \frac{2}{5}; p, q\right) \bigg|_{\eta = \eta''}^{\eta = \eta''}, \\
\tilde{I}_2 &= \frac{1}{2} \left(\frac{\eta}{7}\right)^{2/3} w^2 F_1\left(\frac{2}{5}; \frac{3}{5}; \frac{2}{5}; p, q\right) \left(fh + (g + 1)^2\right) \bigg|_{\eta = \eta''}^{\eta = \eta''}, \\
\tilde{I}_3 &= - \left(\frac{\eta}{7}\right)^{1/3} (g - 1) w F_1\left(\frac{1}{3}; \frac{1}{5}; \frac{4}{5}; p, q\right) \bigg|_{\eta = \eta''}^{\eta = \eta''} - \frac{1}{2\eta^{3/3}} (h^2 w F_1\left(\frac{4}{5}; \frac{1}{5}; \frac{1}{5}; p, q\right) + l) \bigg|_{\eta = \eta''}^{\eta = \eta''}, \\
\tilde{I}_4 &= - \left(\frac{\eta}{7}\right)^{1/3} \frac{g - 1}{1 - 4fh + g^2 - 2g} w F_1\left(\frac{1}{5}; \frac{1}{5}; \frac{4}{5}; p, q\right) \bigg|_{\eta = \eta''}^{\eta = \eta''} - \eta^{4/3} w F_1\left(\frac{1}{5}; \frac{1}{5}; \frac{1}{5}; \frac{4}{5}; p, q\right) + \eta^{1/3} \left(1 - 2\eta h\right) \bigg|_{\eta = \eta''}^{\eta = \eta''}.
\end{align*}
\]

Here \( F_1(a; b; c; d; p, q) \) is first Appell function \([35]\), and symbols \( p, q, l \) denote the following expressions

\[
\begin{align*}
p &= \frac{2h\eta}{-g + \sqrt{g^2 - 2g - 4fh + 1} + 1}, \\
q &= \frac{2h\eta}{g + \sqrt{g^2 - 2g - 4fh + 1} + 1}, \\
w &= \left[(1 - p)(1 - q)\right]^{1/3}, \\
l &= -f - \eta(g + \eta h - 1)
\end{align*}
\]

Remark 17. Appell function can be expressed as hypergeometric function of two variables:

\[
F_1(a; b; c; d; p, q) = \sum_{m, n \geq 0} \frac{(a)_{m+n}(b)_m(c)_n p^m q^n}{(d)_{m+n} m! n!},
\]

where \((x)_n\) is Pochhammer symbol:

\[
(x)_0 = 1, \quad (x)_n = \prod_{k=0}^{n-1} (x + k), \quad n = 1, 2, \ldots
\]
Since
\[
\tilde{b}_\alpha(R) = B_\alpha(R^3 + A_\alpha) - A_\alpha,
\]
then according to (68) one has
\[
\tilde{b}_\alpha(R) = (f + gA_\alpha + hA_\alpha^2 - A_\alpha) + (g + 2hA_\alpha)R^3 + hR^6
\]
To satisfy the condition (39), it is sufficient to choose \(f, g, h\) as
\[
h = \tilde{h}, \quad g = \tilde{g} - 2hA_\alpha, \quad f = \tilde{f} + A_\alpha - gA_\alpha - hA_\alpha^2,
\]
where \(\tilde{f}, \tilde{g}, \tilde{h}\) are arbitrary real constants. With such the choice of \(f, g, h\), one gets
\[
\tilde{b}_\alpha(R) = \tilde{f} + \tilde{g}R^3 + \tilde{h}R^6
\]
(69)
With (69) one can determine \(A_\alpha\) from (64):
\[
A_\alpha = \frac{[1 - S(\alpha)]^3 R^\alpha(\alpha)^3 - \tilde{f} - \tilde{g}R^\alpha(\alpha)^3 - \tilde{h}R^\alpha(\alpha)^6}{S(\alpha)^3}
\]
Using above obtained results one can get the expression for the relative difference \((p^\alpha(\alpha) - p^\alpha(\alpha))/\mu\) of hydrostatic loadings in internal and external boundaries. Indeed, from (65) it follows that
\[
\frac{p^\alpha(\alpha) - p^\alpha(\alpha)}{\mu} = (1 + \beta)\tilde{I}_1 + (\beta - 1)\tilde{I}_2;
\]
where
\[
\tilde{I}_1 = - \frac{1}{2} w^2 \left( \frac{\eta}{7} \right)^{2/3} F_1 \left( \frac{5}{3}, \frac{2}{3}, \frac{2}{3}; \frac{2}{3}; p, q ; (fh + g^2 - 2g) \right) |_{\eta = \eta^\prime} - \frac{1}{2} (g - 1) \eta^{5/3} \eta^{1/3} h w^2 F_1 \left( \frac{5}{3}, \frac{2}{3}, \frac{2}{3}; \frac{2}{3}; p, q ; \right) - \frac{1}{4} \frac{1}{\eta^{1/3}} (f - \eta (-5g + \eta h + 5)) |_{\eta = \eta^\prime},
\]
\[
\tilde{I}_2 = - (g - 1) w \left( 1 - \frac{1}{z} \right) \left( \frac{\eta}{7} \right)^{1/3} F_1 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}; p, q ; \right) |_{\eta = \eta^\prime} - \frac{\eta^{4/3}}{\eta^{1/3}} h w \left( \frac{1}{2} - \frac{1}{z} \right) F_1 \left( \frac{4}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}; p, q ; \right) |_{\eta = \eta^\prime} + \left( \frac{\eta}{7} \right)^{1/3} \frac{1}{z} - \frac{1}{2} \left( \frac{1}{\eta} \right)^{2/3} \left|_{\eta = \eta^\prime},
\]
and \(z = (g - 1)^2 - 4fh\). The solution (69) corresponds to such the difference.

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