Statistical Behavior of Finite-Size Partially Equilibrated Systems

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Abstract

We examine deviations from Boltzmann-Gibbs statistics for partially equilibrated systems of finite size. We find that such systems are characterized by the Lévy distribution whose non-extensivity parameter is related to the number of internally equilibrated subsystems and to correlations among them. This concept is applied to Quark-Gluon Plasma formation.

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1 Introduction

Non-extensive statistics introduced by Tsallis [1] has become a topic of substantial interest and investigation in many areas of physics including condensed matter, nuclear, and high energy physics. The centerpiece of such considerations is the power-like Lévy distribution

\[ G_q(x) = C_q \left[1 - (1 - q) \frac{x}{\lambda_0} \right]^{\frac{1}{1-q}}, \]

where \(1 \leq q < 2\) and \(C_q\) is the corresponding normalization constant. In the limit \(q \to 1\) this reduces to the standard Boltzmann-Gibbs exponential factor,

\[ g(x) = C \exp \left(-\frac{x}{\lambda_0}\right). \]
In this case the parameter $\lambda_0 \equiv 1/\beta_0$ can be interpreted as the temperature of the system in statistical equilibrium. (We set the Boltzmann constant equal to unity.) On the other hand the meaning of Eq. (1) is still not perfectly clear for $q \neq 1$. In a recent article Wilk and Włodarczyk [2] have shown that the Lévy distribution naturally arises as the mean of the Boltzmann-Gibbs factor over a Gamma distribution of the temperature parameter, $1/\lambda = 1/T$, with mean $1/\lambda_0$. In the same article the authors have shown that the parameter $q$ is determined by the diffusion coefficient and the dumping constant in the Fokker-Planck equation, i.e., the Langevin equation describing gaussian white noise. We wish to further elaborate on this observation by finding other possible origins of this Gamma distribution.

Tsallis’ statistics, initially introduced to describe fractal properties of various systems, maintains the general structure of thermodynamics. Formal analogies between such statistics and the so-called $q$-oscillators have been established by means of $q$-calculus and use of Jakson derivatives [3]. This statistics has found applications in many phenomena including modifications of solar neutrino fluxes [4], fluctuations and correlations in high-energy nuclear collisions [4, 5], the long-flying component of cosmic rays [6], chaotic transport in laminar fluid flow [7], subrecoil laser cooling [8] and others. It has also been incorporated into fractional quantum statistics [9].

Gross and Votyakov [10] have investigated the statistical properties of small systems (systems whose size is comparable with the range of internal interactions) and have concluded that, in this case, the grand-canonical distribution is not appropriate and the thermodynamic limit cannot be taken. Earlier Prosper [11] had already studied temperature fluctuations in finite-size heat baths and shown that the energy distribution of a thermometer is, in fact, binomial. It appears that finiteness of the system size and correlations between its parts due to long range interactions can lead to substantial deviations from the widely-used Boltzmann-Gibbs distribution. An exhaustive coverage of all work related to Tsallis’ statistics can be found in Ref. [12].

In this article we examine these concepts in the case of partially equilibrated systems.

2 Derivation of the Lévy Distribution

First we would like to see how the Lévy distribution can arise for finite-size partially equilibrated systems and to investigate the meaning of its parameter $q$. We consider a Gibbs ensemble of finite-size system replicas each subject to the constraint of fixed total energy, $E$ [13]. Each system, member of the ensemble, is partially equilibrated [14]. It consists of finite-size subsystems each of which is in internal statistical equilibrium corresponding to some temperature, $T$. Since the size of the subsystems and of the whole system is finite the number of subsystems is also finite. For any member of the ensemble there is a number $k$ of subsystems at temperature $T$ (more precisely in the interval between $T$ and $T + dT$). The ensemble mean of the number of subsystems at temperature $T$ is $\langle k \rangle$. We define $T_0$, the ensemble mean of the temperature of all subsystems first averaged over the number of subsystems of each ensemble member.
This is the same as the temperature average over all the subsystems of all ensemble members. The more ensemble members have average temperatures close to \( T_0 \) or the more subsystems over the ensemble have temperatures close to \( T_0 \) the closer to equilibrium the physical system will be. We assume that the energy of the whole system is proportional to the mean temperature,

\[ E = c T_0, \]  

(3)

where \( c \) is a constant. At the same time the ensemble mean energy of all subsystems at temperature \( T \) will be taken equal to the ensemble mean of the number of subsystems at this temperature times the energy of each subsystem, i.e.,

\[ \langle E(T) \rangle = c \langle k \rangle T. \]  

(4)

Under the strong assumption that \( E = \langle E(T) \rangle \) we can easily see that

\[ \langle k \rangle = \frac{T_0}{T} \equiv \lambda_0 \beta, \]  

(5)

where \( \lambda_0 = T_0 \) and \( \beta = 1/T = 1/\lambda \). This condition correlates the subsystems and may be the result of long-range interactions within the system. The case \( T = T_0 \) is rather special since, then, Eqs. (3,4) imply that \( \langle k \rangle = 1 \) and, in this case, there can be only one subsystem at temperature equal to the average one. The probability distribution for encountering exactly \( k \) subsystems at temperature \( T \) inside one ensemble member can be considered to be Poissonian for those values of \( k \) reached within one ensemble member,

\[ P_k(\beta) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}. \]  

(6)

Consequently, the probability for encountering \( \alpha \) subsystems at temperature \( T \) within the ensemble is given by the Gamma distribution

\[ P_{\alpha}(\beta) = \frac{\lambda_0 (\lambda_0 \beta)^{\alpha-1} e^{-\lambda_0 \beta}}{\Gamma(\alpha)}, \]  

(7)

with first moment \( \alpha/\lambda_0 \). Upon rescaling \( \lambda \) by dividing it by \( \alpha \) and multiplying Eq. (7) by \( \alpha \) we obtain the modified Gamma distribution

\[ f_{\alpha} \left( \frac{1}{\lambda} \right) = \frac{\alpha \lambda_0}{\Gamma(\alpha)} \left( \frac{\alpha \lambda_0}{\lambda} \right)^{\alpha-1} \exp \left( -\frac{\alpha \lambda_0}{\lambda} \right), \]  

(8)

with first moment \( 1/\lambda_0 \). This is the same function as the one obtained by Wilk and Wlodarczyk starting from the Lévy distribution. We may define a parameter \( q \) such that

\[ \alpha = \frac{1}{q - 1}. \]  

(9)
Clearly when $q \to 2$ we have only one subsystem at temperature $T_0$ in the ensemble ($\alpha = 1$). The system is away from equilibrium. We note that if the first moment of the Lévy distribution is to be finite then $q \leq 3/2$ so that complete lack of statistical equilibration cannot be achieved. When $q \to 1$, $\alpha$ tends to infinity; there is an infinite number of subsystems in the ensemble at $T_0$. In this case the system is equilibrated.

To derive the Lévy distribution we follow the inverse steps of Ref. [2]. First we define a variable $\xi$ by means of the equation

$$\xi = \frac{\alpha \lambda_0}{\lambda}.$$  

(10)

If each subsystem is attributed a Boltzmann-Gibbs factor in a variable $x$, $\exp(-x/\lambda)$, then the expectation value of this factor over the distribution given by Eq. (8) is

$$\langle \exp(-x/\lambda) \rangle = \frac{\alpha \lambda_0}{\Gamma(\alpha)} \int_0^\infty \exp(-x/\lambda) \xi^{\alpha-1} \exp \left( -\xi \left( 1 + \frac{x}{\alpha \lambda_0} \right) \right) d\xi.$$  

(11)

Even though for a particular ensemble member $1/\lambda$ cannot cover the entire range $[0, \infty]$ this is certainly possible over the ensemble. We may make the substitution $t = \xi \left( 1 + \frac{x}{\lambda} \right)$. This yields

$$\langle \exp(-x/\lambda) \rangle = \frac{1}{\Gamma(\alpha)} \left( 1 + \frac{x}{\alpha \lambda_0} \right)^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t} dt = \left( 1 + \frac{x}{\alpha \lambda_0} \right)^{-\alpha}.$$  

(12)

and which, by means of Eq. (9), gives the unnormalized equivalent of Eq. (1). In this derivation the finiteness of the system and subsystem sizes has been important in order to assume a (discrete) Poissonian distribution for $k$.

### 3 Entropy Non-Extensivity

We, now, consider the entropy of the system. Each subsystem being in internal equilibrium has a Boltzmann-Gibbs-Shannon entropy,

$$S_i = -\sum_{j=1}^{W_i} p_{ij} \ln p_{ij},$$  

(13)

where $W_i$ is the number of microstates consistent with a given temperature of the subsystem and $p_{ij}$ is the probability for the $j$th microstate of the subsystem of temperature $T_i$. Since each subsystem is characterized by a $q$-value, $q_0$, very close to unity we can apply the approximation $p_{ij}^{q_0-1} \approx 1 + (q_0 - 1) \ln p_{ij}$. In combination with Eq. (13) this yields

$$S_i \approx \frac{1}{q_0 - 1} \left[ \sum_{j=1}^{W_i} p_{ij} - \sum_{j=1}^{W_i} p_{ij}^{q_0} \right] = \frac{1}{q_0 - 1} \left[ 1 - \sum_{j=1}^{W_i} p_{ij}^{q_0} \right].$$  

(14)
This power-law entropy was first introduced by Tsallis [15] and upon optimization under the constraints of normalizability and given second moment produces the Lévy distribution. Even though this equation was derived as an approximation its applicability is more general than that of the Boltzmann-Gibbs-Shannon entropy [12]. This form of the entropy is, generally, non-extensive as can be seen by calculating the combined entropy of two stochastically independent (sub)systems,

\[
S_{\tau\nu} = \frac{1}{q_0 - 1} \left[1 - \sum_{l=1}^{W_{\tau}} \sum_{k=1}^{W_{\nu}} p_{\tau l}^{q_0} p_{\nu k}^{q_0}\right] = S_\tau + S_\nu - (q_0 - 1) S_\tau S_\nu,
\]

(15)

where the combined system is assigned the same \(q\)-value as its components. The last equation can be proven by considering the product of the two individual entropies. Clearly as \(q_0 \to 1\) extensivity is recovered. If all subsystems are equally close to internal equilibrium and are stochastically independent the entropy of the whole system is

\[
S = \sum_i S_i - (q_0 - 1) \sum_i \sum_{j>i} S_i S_j - (q_0 - 1)^2 \sum_i \sum_{j>i} \sum_{k>j>i} S_i S_j S_k - \ldots
\]

(16)

A microstate of a subsystem is part of a microstate of the system. If a particular subsystem exists in one ensemble member but does not exist in another then the contribution to the probability of the subsystem microstate is zero for the latter member. Therefore, the above summations extend over all the subsystems that may appear in any ensemble member. We observe that in the case of stochastically independent subsystems the total entropy is smaller than or equal to the sum of the individual entropies, the equality always holding when the subsystems are internally equilibrated. However, Eq. (15) induces correlations among subsystems which remove the stochastic independence. It is due to this effect that the total entropy of the system is non-extensive even though its parts are in (or close to) internal equilibrium. This conclusion extends the one reached by Landau and Lifshitz [14] on partially equilibrated systems.

In the presence of long-range correlations and if the stochastic dependence is relatively weak we may approximate the sum of the \(q\)-power products of the microstate probabilities of two (sub)systems by

\[
\sum_{l=1}^{W_{\tau}} \sum_{k=1}^{W_{\nu}} p_{\tau l}^{q_0} p_{\nu k}^{q_0} \approx \sum_{\rho=1}^{W_{\tau\nu}} p_{\rho}^{q_0} + C_{\tau\nu},
\]

(17)

where \(W_{\tau\nu}\) is the number of microstates of the combined system, \(p_{\rho}\) the probability for one such microstate, \(C_{\tau\nu}\) is a function of \(q_0\) that incorporates the correlations and \(q\) is a parameter defined below. We express the functions \(C_{\tau\nu}\) in terms of the individual entropy product as

\[
C_{\tau\nu} = \kappa S_\tau S_\nu,
\]

(18)
where $\kappa = \kappa(q_0)$. Using the last two equations we find that the total entropy is

$$S_{\tau\nu} = S_{\tau} + S_{\nu} - [q_0 - 1 + \kappa/(q_0 - 1)] S_{\tau} S_{\nu} = S_{\tau} + S_{\nu} - (q - 1) S_{\tau} S_{\nu},$$

(19)

with $q \equiv q_0 + \kappa/(q_0 - 1)$. When $\kappa \neq 0$ the total entropy is not extensive even though the individual subsystems may be close to internal equilibrium. We note that $\kappa$ must be an increasing function of $q_0 - 1$ to prevent the paradoxical situation of $q$ becoming larger the closer $q_0$ is to unity.

4 Perspectives for Heavy Ion Physics

We have seen that the condition of partial statistical equilibration in a finite-size system leads to a Lévy distribution that generalizes the Boltzmann-Gibbs statistics. Long-range correlations among the internally equilibrated subsystems are important and give rise to the non-extensivity of the entropy. The parameter $q$ that characterizes this situation is related to the correlations and to the number of internally equilibrated subsystems. It is certainly interesting that the very complicated dynamics of such systems could be effectively described by a single-parameter statistical distribution. One possible application of these ideas is in the formation of Quark-Gluon Plasma (QGP) in relativistic heavy ion collisions. If such an unconfined state is actually produced at high temperatures but relatively low densities then the fragmentation time of the quarks and gluons to hadrons may be comparable to the relaxation time of the whole system. In this case the plasma is at most only partially equilibrated. Thermal photon emission from it will correspond to a distribution of temperatures and will effectively be described by Tsallis’ statistics that corresponds to the average temperature of the system and a $q$-value that exceeds 1. It is particularly interesting that as Walton and Rafelski have shown [16] diffusion of the charmed quarks in a plasma in equilibrium can be described well only by the non-extensive Tsallis’ statistics. On the other hand if the plasma is not formed but photons are emitted by a perhaps partially equilibrated hadron system the resulting enhancement of photon production (for $q > 1$) as the photon transverse momentum increases may mimic photon production from a QGP.

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