SUB-CONVEXITY BOUND FOR $GL(3) \times GL(2)$ $L$-FUNCTIONS: THE DEPTH ASPECT

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Abstract. In this article, we will prove the following sub-convex bound

$$L \left( \frac{1}{2}, \pi \times f \times \chi \right) \ll_{f,\pi,\epsilon} (p^r)^{3/2 - 3/20 + \epsilon},$$

for $GL(3) \times GL(2)$ Rankin Selberg $L$-functions.

1. Introduction

Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ with the normalised Fourier coefficients $A(n, k)$ and $f$ be a holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$ with the normalised Fourier coefficients $\lambda_f(n)$. Let $\chi$ be a primitive Dirichlet character modulo $R := p^r$. Rankin-Selberg $L$-series associated to the above objects is given by

$$L(s, \pi \times f \times \chi) = \sum_{n, k=1}^{\infty} \frac{A(n, k)\lambda_f(n)\chi(n)}{(nk^2)^s},$$

(1)

in the half plane $\Re s > 1$. It is well known that this series extends to an entire function to whole of complex plane $\mathbb{C}$ and satisfies a functional equation relating $s$ with $1 - s$. It is an important problem to understand the growth of the $L$-function inside the critical strip. Functional equation and the Phragmen-Lindelöf principle yield the following convex bound

$$L \left( \frac{1}{2}, \pi \times f \times \chi \right) \ll_{f,\pi,\epsilon} R^{\frac{3}{2} + \epsilon},$$

where the implied constant depends on the forms $\pi$, $f$ and $\epsilon > 0$. The Lindelöf hypothesis predicts that such a bound holds with any positive exponent in place of $3/2 + \epsilon$. But even breaking the convexity barrier is difficult and has remained open so far. In this article, our aim is to prove the following subconvex bound in the case when $R = p^r$ tends to infinity (both $p$ or $r$ may vary).

**Theorem 1.** Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$, $f$ be a holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$ and $\chi$ be a primitive Dirichlet character of prime power modulus $p^r$, $r \geq 3$. Then we have

$$L \left( \frac{1}{2}, \pi \times f \times \chi \right) \ll_{f,\pi,\epsilon} (p^r)^{3/2 - 3/20 + \epsilon},$$

where the implied constant depends only on $f$, $\pi$ and $\epsilon$ only.

**Remark 1.** Any unitary Hecke character $\Psi$ on the idele group

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{R}_+ \times \prod_p \mathbb{Z}_p^\times$$

(Date: October 19, 2021.

2010 Mathematics Subject Classification. Primary 11F66, 11M41; Secondary 11F55.

Key words and phrases. Maass forms, subconvexity, Rankin-Selberg $L$-functions.)

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can be decomposed as $\Psi = |.|^d \otimes (\otimes_p \psi_p)$, where $t \in \mathbb{R}$ and $\psi_{\text{finite}} = \otimes_p \psi_p$ corresponds to a Dirichlet character $\chi$, say. Then the twisted $L$-function $L(1/2, \pi \times f \times \Psi)$ corresponds to $L(1/2 + it, \pi \times f \times \chi)$. Thus the $t$-aspect subconvexity bound for $L(1/2 + it, \pi \times f)$ of R. Munshi \[14\] corresponds to bounds for the twisted $L$-values $L(1/2, \pi \times f \times \Psi)$, where the Hecke characters $\Psi$ are ‘supported’ only at the prime at infinity. In the present case, we are considering twists by Hecke characters which are ‘supported’ (ramified) at a fixed finite prime $p$.

In the proof of Theorem \[11\] we do not really require the cuspidality of $\pi$ and $f$, but it only uses summation formulas of $GL(3)$ and $GL(2)$ Fourier coefficients $A(n,1)$ and $\lambda_f(n)$ respectively. We note that the Voronoi summation formula for $d_3(n)$ resembles that of $A(n,1)$, where $d_3(n)$ is the triple divisor function which corresponds to the Fourier coefficients of a $GL(3)$ minimal Eisenstein series $E_{\min}$, see, \[14\] . The only difference is that we get a main term in this case, see Lemma \[11.1\]. We observe that

\[ L(s, E_{\min} \times f \times \chi) \approx L(s, f \times \chi)^3. \]  

(2)

Thus our approach also gives a subconvexity bound for $L(1/2, f \times \chi)$ in the depth aspect, and our bounds are uniform in $p$ and $r$. We give a sketch for this proof in the appendix.

It we take the Eisenstein series, $E = E(\ast, 1/2)$ say, for $SL(2, \mathbb{Z})$ in place of the cusp form $f$, then we have

\[ L(s, \pi \times E \times \chi) \approx L(s, \pi \times \chi)^2. \]  

(3)

Note that the Fourier coefficients of $E(\ast, 1/2)$ are the divisor function $d(n)$, for which we have a Voronoi summation formula similar to that of $\lambda_f(n)$ except for the main term (see Lemma \[22\]). This main term does not appear in our analysis (see Remark \[8\]). Thus in this case, treatment of the remaining terms is very similar to the cusp form $f$ case. Hence we get a subconvexity bound for $L(1/2, \pi \times \chi)$ in the depth aspect and our bounds are uniform in $p$ and $r$.

R. Munshi in \[37\] and \[38\] obtained subconvexity bound for $L(1/2, \pi \times \chi)$ when $\pi$ is a self-dual $GL(3)$ form. In \[37\], he deals with the case when $r$ is fixed and $p \to \infty$ while in \[38\], he considers the case when $p$ is fixed and $r \to \infty$. Recently Q. Sun and Zhao \[49\] extended Munshi’s work \[38\] to any $GL(3)$ form, as $r \to \infty$. Their bounds are not uniform with respect to $p$. Our results extend both the works of Munshi (\[37\] and \[38\]) to any $GL(3)$ form, also removes the dependency of $p$ in the result of Sun and Zhao \[49\].

**Remark 2.** Using the same method along with the amplification trick (see \[38\]), one can also get a similar result for $r = 2$ also.

1.1. A brief history. We will now recall a brief history of the problem. We only focus on the level aspect and the depth aspect. For degree one $L$-functions, $q$-aspect sub-convexity bound was first proved by Burgess \[7\] in 1962 using his ingenious technique of completing short character sums. He proved that

\[ L(s, \chi) \ll_{s,\epsilon} q^{3/16 + \epsilon}, \]  

(4)

for fixed $s$ with $\Re s = 1/2$ and for any $\epsilon > 0$. After Burgess’s result, there was not much progress in the level aspect until 1990. In 2002, Conrey and Iwaniec \[9\], gave a new method to prove the Weyl strength bound $(1/6 + \epsilon)$ for the real characters. Recently I. Petrow and M. Young proved the Weyl-exponent subconvex bound for any
Dirichlet L-function in [45] and [46]. Their work is based on the method of Conrey and Iwaniec [9]. In 2014, Milićević [36] obtained a sub-Weyl subconvex bound for χ primitive Dirichlet character modulo pr.

For GL(2) L-functions, level aspect subconvexity problem was settled by Duke et al. in a series of articles ([11], [12], [13]) using a new form of circle method and amplification technique. Further refined results for GL(2) L-functions have been obtained in [8], [4], [9] and [42]. Extending the above mentioned result of Milićević to GL(2) L-functions, Blomer and Milićević in [6] obtained

$$L(1/2 + it, f \otimes \chi) \ll_{f, \varepsilon} (1 + |t|)^{5/2} p^{7/6} q^{1/3 + \varepsilon},$$

where f is a holomorphic or Maass newform for SL(2, Z), and χ is a primitive character of conductor q = pr, with p an odd prime. Note that the above exponent tends to the Weyl exponent as r → ∞. Using the conductor lower trick introduced by R. Munshi in [40], S. Singh and R. Munshi [43], obtained the following subconvex bound when χ is a primitive Dirichlet character of modulus pr and r ≡ 0 mod 3.

$$L(1/2 + it, f \otimes \chi) \ll_{f, t, \varepsilon} p^{r + \varepsilon}.$$ 

In the case of degree three L-functions, the first sub-convex bound was obtained by V. Blomer [2] for the self-dual forms. He obtained the bound when χ is a quadratic character with prime modulus. For any primitive Dirichlet character of prime power modulus pr, the subconvex estimates were proved by R. Munshi ([37] and [38]). For any genuine GL(3) L-functions, the subconvexity problem was settled by R. Munshi in [39] and [41]. In [39] and [41], he considered the cases of moduli which are product of primes M1M2 with M2^{1/2} < M1 < M2, and prime respectively. Also for any degree three L-functions, Q. Sun and R. Zhao proved subconvex bounds in the depth aspect in [49].

For GL(3) × GL(2) L-functions, the first result in this direction was obtained by V. Blomer ([2]). He proved the subconvex bound for L(s, π × f × χ), where π is a self dual form, f is a GL(2) form and χ is a quadratic character of prime modulus. This result was generalised to any GL(3) form by P. Sharma ([48]) recently.

1.2. **Comment on the method.** Our problem is arithmetic in nature. Using the functional equation (see Lemma 3.1), our proof boils down to getting cancellations in the following sum:

$$\sum_{n \sim p^{3r}} A(n, 1) \lambda_f(n) \chi(n).$$

Note that the ‘arithmetic conductor’ of L(1/2, π × f × χ) is p^{6r}. We compare the above sum with the following sum

$$\sum_{n \sim t^3} A(n, 1) \lambda_f(n)n^{it},$$

to which R. Munshi [44] was seeking cancellations to get the t-aspect subconvexity bound for L(1/2 + it, π × f). His approach was to apply the circle method to separate the oscillations A(n, 1) and λ_f(n)n^{it}. While doing so, he introduced an integral which helps to lower the ‘conductor’. In analytic problems, this trick is not absolutely necessary and can be removed (see [33]). After separating the oscillations, he employs the summations formulas to dualize the sums. While analyzing the resulting character sum, he observes that the character sum boils down to an additive character, which also plays a crucial role in our proof as well as in [26], [27] and [48]. In our
case, following Munshi [44], we also separate the oscillations $A(n, 1)$ and $\lambda_f(n)\chi(n)$. To lower the conductor, we introduce a congruence equation modulo $p^\ell$ (it does not change the conductor of the $L$-function due to the presence of $\chi$), where $\ell$ is some parameter $\ell < r$. It turns out that this congruence equation trick is very crucial in our approach, without which the circle method approach will not work. Our aim in this paper is to show that the circle method approach works equally well to give subconvexity bounds in the depth aspect with the same quality of bounds as in the $t$-aspect.

We now compare our approach with P. Sharma’s recent result [48], in which he considers the twist aspect ($\chi$ modulo $p$, $p$ varying). He also follows Munshi’s method [44] and also uses the congruence equation trick. In his case, he had to transfer some ‘mass’ from the $GL(3)$-coefficients to get required savings in diagonal terms. In our case, we do not need to transfer ‘mass’ from the $GL(3)$-coefficients, as $\chi$ is a character modulo $p^r$, $r \geq 3$. Since, both problems are arithmetic, we end up with a character sum in which we seek square root cancellations. P. Sharma [48] had to appeal to Deligne’s bound (Riemann hypothesis for varieties over finite fields) to show square root cancellations. In our case, we don’t require Deligne’s bound. We achieve the required cancellations by elementary means, albeit tedious.

**Notations.** In this paper, the notation $\alpha \ll A$ will mean that for any $\epsilon > 0$, there is a constant $c$ such that $|\alpha| \leq cA(p^\ell)^\epsilon$. $A \sim B$ will also have the standard meaning, i.e., $B \leq A \leq 2B$. Also, $A \asymp B$ will mean that $(p^\ell)^{-\epsilon}c_1A \leq B \leq Ac_2(p^\ell)^\epsilon$, for some absolute constants $c_1$ and $c_2$. We follow the standard $\epsilon$ convention, i.e., $\epsilon$ may vary from places to places.

**Acknowledgements.** The authors are grateful to Prof. Ritabrata Munshi for sharing his beautiful ideas, explaining his ingenious method in full detail, and his kind support throughout the work. They would also like to thank Prof. Satadal Ganguly for their encouragement and constant support and Stat-Math Unit, Indian Statistical Institute, Kolkata for the excellent research environment. Finally, authors would like to thank the referee for his/her suggestions and comments which really helped to improve the presentation of the article.

2. Preliminaries

In this section, we will recall some known results which we need in the proof.

2.1. **Holomorphic forms on** $GL(2)$. Let $f$ be a holomorphic Hecke eigenform of weight $k_f$ for the full modular group $SL(2, \mathbb{Z})$. The Fourier expansion of $f$ at $\infty$ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k_f-1)/2} e(nz),$$

for $z \in \mathbb{H}$. We have a well-known Deligne’s bound for the Fourier coefficients which says that

$$|\lambda_f(n)| \leq d(n), \quad (7)$$

for $n \geq 1$, where $d(n)$ is the divisor function. We now state the Voronoi summation formula for $f$ in the following lemma.
Lemma 2.1. Let $\lambda_f(n)$ be as above and $g$ be a smooth, compactly supported function on $(0, \infty)$. Let $a$, $q \in \mathbb{Z}$ with $(a, q) = 1$. Then we have

$$
\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) g(n) = \frac{2\pi f}{q} \sum_{n=1}^{\infty} \lambda_f(n) e\left(-\frac{dn}{q}\right) h(n),
$$

where $ad \equiv 1 \pmod{q}$ and

$$
h(y) = \int_{0}^{\infty} g(x) J_{k_f-1}\left(\frac{4\pi \sqrt{x y}}{q}\right) \, dx,
$$

with $J_{k_f-1}$ being the Bessel function of the first kind of order $k_f - 1$.

Proof. See Iwaniec-Kowalski [23].

Next we record the Voronoi summation formula for the Eisenstein series $E(\ast, 1/2)$ for $SL(2, \mathbb{Z})$.

Lemma 2.2. Let $d(n) = \sum_{ab=|n|} 1$ be the Fourier coefficients of $E(\ast, 1/2)$ and $g$ be a smooth, compactly supported function on $(0, \infty)$. Let $a$, $q \in \mathbb{Z}$ with $(a, q) = 1$. Then we have

$$
\sum_{n=1}^{\infty} d(n) e\left(\frac{an}{q}\right) g(n) = I(g, q) + \frac{1}{q} \sum_{\pm} \sum_{n=1}^{\infty} d(n) e\left(\mp \frac{\bar{a} n}{q}\right) h_{\pm}(n),
$$

where

$$
I(g, q) = \frac{1}{q} \int_{0}^{\infty} (\log x + 2\gamma - 2 \log q) g(x) dx,
$$

$$
h_{+}(y) = \int_{0}^{\infty} -2\pi g(x) Y_0\left(\frac{4\pi \sqrt{x y}}{q}\right) \, dx \quad \text{and} \quad h_{-}(y) = \int_{0}^{\infty} 4g(x) K_0\left(\frac{4\pi \sqrt{x y}}{q}\right) \, dx.
$$

In the following lemma, we record some properties of $J_{k_f-1}$.

Lemma 2.3. Let $J_{k_f-1}(2\pi x)$ be the Bessel function of the first kind of integer order $k_f$. Then, for fixed $k_f$, as $x \to \infty$, we have

$$
J_{k_f-1}(2\pi x) = e(x)Z^+(x) + e(-x)Z^-(x),
$$

where $Z^-(x) = \overline{Z^+(x)}$ and $Z^+$ is a smooth function satisfying

$$
x^{j}Z^{+(j)}(x) \ll_{j, k_f} \frac{1}{\sqrt{x}},
$$

for $j \geq 0$.

2.2. Automorphic forms on $GL(3)$. In this subsection, we will recall some background on the Maass forms for $GL(3)$. This subsection, except for the notations, is taken from [29]. Let $\pi$ be a Hecke-Maass cusp form of type $(\nu_1, \nu_2)$ for $SL(3, \mathbb{Z})$. Let $A(n,k)$ denote the normalized Fourier coefficients of $\pi$. Let

$$
\alpha_1 = -\nu_1 - 2\nu_2 + 1, \quad \alpha_2 = -\nu_1 + \nu_2, \quad \alpha_3 = 2\nu_1 + \nu_2 - 1
$$

be the Langlands parameters for $\pi$ (see Goldfeld [14] for more details). Let $g$ be a compactly supported smooth function on $(0, \infty)$ and $\hat{g}(s) = \int_{0}^{\infty} g(x)x^{s-1} dx$ be its Mellin transform. For $\ell = 0$ and 1, we define

$$
\gamma_\ell(s) := \frac{\pi^{-3s-\frac{3}{2}}}{2} \prod_{i=1}^{3} \Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right) \Gamma\left(\frac{-s-\alpha_i+\ell}{2}\right).
$$
Set \( \gamma_{\pm}(s) = \gamma_0(s) \mp \gamma_1(s) \) and let
\[
G_{\pm}(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_{\pm}(s) \tilde{g}(-s) \, ds,
\]
where \( \sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\} \). With the aid of the above terminology, we now state the \( GL(3) \) Voronoi summation formula in the following lemma:

**Lemma 2.4.** Let \( g(x) \) and \( A(n,k) \) be as above. Let \( a, \bar{a}, q \in \mathbb{Z} \) with \( q \neq 0 \), \((a, q) = 1\), and \( a\bar{a} \equiv 1 \pmod{q}\). Then we have
\[
\sum_{n=1}^{\infty} A(n,k) e\left(\frac{an}{q}\right) g(n) = q \sum_{\pm} \sum_{n_1|qk n_2=1}^{\infty} \frac{A(n_1,n_2)}{n_1n_2} S(k\bar{a}, \pm n_2; qk/n_1) G_{\pm}\left(\frac{n_1^2n_2}{q^3k}\right),
\]
where \( S(a,b;q) \) is the Kloosterman sum which is defined as follows:
\[
S(a,b;q) = \sum_{x \mod q}^* e\left(\frac{ax + b\bar{x}}{q}\right).
\]

**Proof.** See [29] for the proof. \(\square\)

The following lemma, which gives the Ramanujan conjecture on average, is also well-known.

**Lemma 2.5.** We have
\[
\sum_{n_1^2n_2 \leq x} |A(n_1,n_2)|^2 \ll x,
\]
where the implied constant depends on the form \( \pi \).

**Proof.** For the proof, we refer to Goldfeld’s book [14]. \(\square\)

### 2.3. Delta method.

Let \( \delta : \mathbb{Z} \to \{0,1\} \) be defined by
\[
\delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}
\]

The above delta symbol can be used to separate the oscillations involved in a sum. Further, we seek a Fourier expansion of \( \delta(n) \). We mention here an expansion for \( \delta(n) \) which is due to Duke, Friedlander and Iwaniec (see [23]). Let \( L \geq 1 \) be a large number. For \( n \in [-2L, 2L] \), we have
\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq a \leq Q} \frac{1}{q_a \mod q} \sum_{*} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q,x) e\left(\frac{nx}{qQ}\right) \, dx,
\]
where \( Q = 2L^{1/2} \). The * on the \( q \)-sum indicates that the sum over \( a \) is restricted by the condition \((a, q) = 1\). The function \( g \) is the only part in the above formula which is not explicitly given. Nevertheless, we only need the following properties of \( g \) in our
analysis:

1. \( g(q, x) = 1 + h(q, x) \), with \( h(q, x) = O \left( \frac{Q}{q} \left( \frac{q}{Q} + |x| \right)^B \right) \),

2. \( x^j \frac{\partial^j}{\partial x^j} g(q, x) \ll \log Q \min \left\{ \frac{Q}{q}, \frac{1}{|x|} \right\} \),

3. \( g(q, x) \ll |x|^{-B} \),

4. \( \int_{\mathbb{R}} |g(q, x)| \, dx \ll Q^\varepsilon \), \hspace{1cm} (9)

for any \( B > 1 \) and \( j \geq 1 \). Using the above properties of \( g(q, x) \) we observe that the effective range of the above integration over \( x \) is \([-L^\varepsilon, L^\varepsilon]\). We record the above observations in the following lemma.

**Lemma 2.6.** Let \( \delta \) be as above and \( g \) be a function satisfying (9). Let \( L \geq 1 \) be a large parameter. Then, for \( n \in [-2L, 2L] \), we have

\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q^{\ast}} \sum_{a \mod q} e \left( \frac{na}{q} \right) \int_{\mathbb{R}} W_1(x) g(q, x) e \left( \frac{nx}{qQ} \right) \, dx + O(L^{-2020}),
\]

where \( Q = 2L^{1/2} \) and \( W_1(x) \) is a smooth bump function supported in \([-2L^\varepsilon, 2L^\varepsilon]\), with \( W_1(x) = 1 \) for \( x \in [-L^\varepsilon, L^\varepsilon] \) and \( W^{(j)} \ll j \).

**Proof.** For the proof, we refer to Chapter 20 of the book [23] by Iwaniec and Kowalski and Lemma 15 of the article [17] by B. Huang. \( \square \)

### 2.4. Stationary phase method

In this subsection, we will recall some facts about the exponential integrals of the following form:

\[
I = \int_a^b g(x) e(f(x)) \, dx,
\]

where \( f \) and \( g \) are smooth real valued functions on \([a,b]\).

**Lemma 2.7.** Let \( I, f \) and \( g \) be as above. Then, for \( r \geq 1 \), we have

\[
I \ll \frac{\text{Var} \, g}{\min |f'(x)|^{1/r}},
\]

where \( \text{Var} \) is the total variation of \( g \) on \([a,b]\). Moreover, let \( f'(x) \geq B \) and \( f^{(j)}(x) \ll B^{1+j} \) for \( j \geq 2 \) together with \( \text{Supp}(g) \subset (a,b) \) and \( g^{(j)}(x) \ll_{a,b,j} 1 \). Then we have

\[
I \ll_{a,b,j,\varepsilon} B^{-j+\varepsilon}.
\]

**Proof.** Proof of the first part of the lemma is standard. For the second part, we use integration by parts. \( \square \)

The following lemma gives an asymptotic expression for \( I \) when the stationary point exists.

**Lemma 2.8.** Let \( 0 < \delta < 1/10, X, Y, U, Q > 0, Z := Q + X + Y + b - a + 1, \) and assume that

\[
Y \geq Z^{3\delta}, \ b - a \geq U \geq \frac{QZ^{3\delta}}{\sqrt{Y}}.
\]
Assume that $g$ satisfies
\[ g^{(j)}(t) \ll_j \frac{X}{U_j} \text{ for } j = 0, 1, 2, \ldots. \]

Suppose that there exists unique $t_0 \in [a, b]$ such that $h'(t_0) = 0$, and the function $f$ satisfies
\[ f''(t) \gg \frac{Y}{Q^2}, \quad f^{(j)}(t) \ll_j \frac{Y}{Q^2} \quad \text{for } j = 1, 2, 3, \ldots. \]

Then we have
\[
I = e^{if(t_0)} \frac{3\delta^{-1} A}{\sqrt{f''(t_0)}} \sum_{n=0}^{3\delta^{-1} A} p_n(t_0) + O_{A,\delta} \left( Z^{-A} \right), \quad p_n(t_0) = \frac{\sqrt{2\pi e^{\pi i/4}}}{n!} \left( \frac{i}{2f''(t_0)} \right)^n G^{(2n)}(t_0),
\]
where
\[ G(t) = g(t)e^{iH(t)}, \text{ and } H(t) = f(t) - f(t_0) - \frac{1}{2}f''(t_0)(t-t_0)^2. \]

Furthermore, each $p_n$ is a rational function in $h', h'', \ldots$, satisfying the derivative bound
\[
\frac{d^j}{dt_0^j} p_n(t_0) \ll_{j, n} X \left( \frac{1}{U_j} + \frac{1}{Q_j} \right) \left( \frac{U^2Y}{Q^2} \right)^{-n} + Y^{-\frac{n}{2}},
\]

3. The set-up and sketch of the proof

In this section, we will give a set-up to prove Theorem 1. We then give a rough outline for the proof.

3.1. Approximate functional equation. Let $\pi$, $f$ and $\chi$ be as defined in Theorem 1. As a first step, we express $L(1/2, \pi \times f \times \chi)$ in terms of an exponential sum. In fact, we have the following lemma.

Lemma 3.1. Let $\pi$, $f$ and $\chi$ be as defined in Theorem 1. Let $R = p^r$ be the conductor of $\chi$. Then, for any $\epsilon > 0$, as $R \to \infty$, we have
\[
L \left( \frac{1}{2}, \pi \times f \times \chi \right) \ll \sup_{k \ll R^{3/2+\epsilon}} \sup_{N \ll R^{1+\epsilon}} \frac{|S_k(N)|}{k\sqrt{N}} + R^{-2020},
\]

where
\[
S_k(N) := \sum_{n=1}^{\infty} A(n,k) \lambda_f(n) \chi(n) W \left( \frac{n}{N} \right),
\]
and $W$ is a smooth function supported in $[1, 2]$ and satisfying $W^{(j)} \ll R^{\epsilon}$.

Proof. Proof follows by an application of the functional equation of $L \left( \frac{1}{2}, \pi \times f \times \chi \right)$. We refer to Theorem 5.3 and Proposition 5.4 of [23] for more details. □

Thus, to establish Theorem 1 we need to get some cancellations in $S_k(N)$ in (11).
3.2. Application of delta symbol. There are three oscillatory factors in the sum \( S_k(N) \) in [11]. Our next task is to separate these oscillations. We accomplish it using delta method. To this end, we rewrite \( S_k(N) \) as

\[
S_k(N) = \sum_{n,m=1}^{\infty} A(n,k)\lambda_f(m)\chi(m)W\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right),
\]

where \( U \) is a smooth bump function supported in \([1/2, 5/2] \), with \( U(x) = 1 \) for \( x \in [1, 2] \) and \( U^{(j)} \ll R^j \). Now we detect \( n = m \) as fellows:

\[
n = m \iff n \equiv m \mod p^\ell \quad \text{and} \quad \frac{n-m}{p^\ell} = 0,
\]

where \( \ell \) is a positive integer such that \( 1 \leq \ell < r \) (to be chosen optimally later). This is a crucial step to achieve our goal. Thus, \( S_k(N) \) can be rewritten as

\[
S_k(N) = \sum_{n,m=1}^{\infty} \sum_{n \equiv m \mod p^\ell} A(n,k)\lambda_f(m)\chi(m)\delta\left(\frac{n-m}{p^\ell}\right)W\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right),
\]

(12)

where \( \delta \) is the delta symbol defined in Subsection [2.3]. Now, detecting the congruence equation \( n \equiv m \mod p^\ell \) using the additive characters, i.e.

\[
\delta(n \equiv m \mod p^\ell) = \frac{1}{p^\ell} \sum_{b \mod p^\ell} e\left(\frac{b(n-m)}{p^\ell}\right),
\]

and applying Lemma [2.0] with \( Q = R^\epsilon \sqrt{N/p^\ell} \), we arrive at

\[
S_k(N) = \frac{1}{Qp^\ell} \int_{\mathbb{R}} W_1(x) \sum_{1 \leq q \leq Q} \frac{g(q,x)}{q} \sum_{a \mod q} \sum_{b \mod p^\ell} A(n,k)e\left(\frac{(a+bq)n}{p^\ell q}\right) e\left(\frac{nx}{p^\ell qQ}\right) W\left(\frac{n}{N}\right)
\times \sum_{n=1}^{\infty} A(n,k)e\left(\frac{(a+bq)n}{p^\ell q}\right) e\left(\frac{nx}{p^\ell qQ}\right) W\left(\frac{n}{N}\right)
\times \sum_{m=1}^{\infty} \lambda_f(m)\chi(m)e\left(\frac{-(a+bq)m}{p^\ell q}\right) e\left(-\frac{mx}{p^\ell qQ}\right) U\left(\frac{m}{N}\right) dx + O(R^{-2020}).
\]

(13)

3.3. Sketch of the proof. In this subsection, we will give a rough sketch of the proof. For simplicity, let’s consider the generic cases, i.e., \( N = R^5 \), \( n \asymp N \), \( m \asymp N \), \( k = 1 \), \( |x| \asymp 1 \), \( q \asymp Q \) and \( (q,p) = 1 \) in [13]. Thus we essentially have the following expression for \( S_k(N) \):

\[
\frac{1}{Q^2p^\ell} \sum_{b \mod p^\ell} \sum_{q \equiv Qn \mod q} \sum_{n=1}^{\infty} A(n,1)e\left(\frac{(a+bq)n}{p^\ell q}\right) \sum_{m=1}^{\infty} \lambda_f(m)\chi(m)e\left(\frac{-(a+bq)m}{p^\ell q}\right).
\]

(14)

Notice that we have ignored the integral over \( x \), as it has no oscillations in the generic case. On estimating the above expression trivially, we get \( S_k(N) \ll N^2 \). Our aim is show \( S_k(N) \ll \sqrt{NR^{3/2-3/20}} \). In other words, we need to save \( N^2/(\sqrt{NR^{3/2-3/20}}) = NR^{3/20} \) over the trivial bound \( N^2 \) in [13].

Our next step is to dualize the sum over \( n \) and \( m \) using summation formulae. We
accomplish it in Section 4. In fact, on applying the GL(3) Voronoi formula to the sum over \( n \) in (14), we, roughly, arrive at the following expression:

\[
S_1 := \sum_{n \sim N} A(n, 1) e \left( \frac{(a + bq)n}{p^f q} \right) \approx \frac{N^{2/3}}{Qp^f} \sum_{n_2 \ll \sqrt{np^{3/2}}} \frac{A(1, n_2)}{n_2^{1/3}} S \left( \frac{(a + bq), n_2; qp^f}{} \right).
\]

See Subsection 4.1 for more details. An application of the GL(2) Voronoi formula to the sum over \( n \) which we analyze in Section 7. For

\[
S_2 := \sum_{n \sim N} \lambda_f(m) \chi(m) e \left( \frac{- (a + bq)m}{p^f q} \right) \approx \frac{N^{3/4}}{p^r \sqrt{q}} \sum \sum^* \tilde{\chi}(-\beta) \sum_{m \ll p^{3r-\ell}} \frac{\lambda_f(m)}{m^{1/4}} e \left( \frac{\overline{m} n}{p^f q} \right),
\]

where \( c = p^{r-\ell}(a + bq) + q\beta \). See Subsection 4.2 for full details. Thus, we arrive at the following expression of \( S_k(N) \):

\[
S_k(N) \approx \frac{N^{17/12}}{Q^{11/2} p^{2+2\ell} \sqrt{np^{3/2}}} \sum_{q \leq Q} \sum_{n_2 \ll \sqrt{np^{3/2}}} \frac{A(1, n_2)}{n_2^{1/3}} \sum_{m \ll p^{3r-\ell}} \frac{\lambda_f(m)}{m^{1/4}} C(...),
\]

where

\[
C(...) = \sum^* \sum_{a \mod q} \sum_{b \mod p^{3r-\ell}} \tilde{\chi}(-\beta) S \left( \frac{(a + bq), n_2; qp^f}{} \right) e \left( \frac{\overline{m} n}{p^f q} \right),
\]

in which we seek square root cancellations. We analyze it in Section 5 and we get the following expression:

\[
C(...) \approx p^{(r+\ell)/2} q e \left( \frac{- \overline{m} n_2}{q} \right) C_1(...),
\]

where \( C_1(...) \) is a character sum modulo \( p^f \) in which we still need to get square root cancellations which we get in Section 7. In the next step we apply the Cauchy inequality followed by the Poisson to the sum over \( n_2 \) (See Section 6 for details). The Cauchy inequality transforms \( S_k(N) \) into

\[
S_k(N) \ll \frac{N^{17/12}(np^{3r})^{1/12}}{Q^{11/2} p^{2+2\ell} \sqrt{np^{3/2}}} \left( \sum_{n_2 \ll \sqrt{np^{3/2}}} \sum_{q \leq Q} \sum_{m \ll p^{3r-\ell}} \frac{\lambda_f(m)}{m^{1/4}} e \left( \frac{- \overline{m} n_2}{q} \right) C_1(...) \right)^{1/2}.
\]

Next we apply the Poisson summation formula to the sum over \( n_2 \) (see Subsection 6.2). We observe that the “arithmetic conductor” is of size \( p^f Q^2 \). Thus we see that the sum over \( n_2 \) transfers into

\[
\sum_{q,q' \sim q} \sum_{m,m' \sim p^{3r-\ell}} \sum_{n_2 \ll \sqrt{np^{3/2}}} |C(...)|,
\]

where

\[
C(...) = \sum_{\nu_1 \mod p^f} C_1(...) C_1(...) e \left( \frac{\nu_1 n_2}{p^f} \right) \sum_{\nu_2 \mod q} e \left( \frac{(m' q - m q' + \nu_2) n_2}{qq'} \right),
\]

which we analyze in Section 7. For \( n_2 = 0 \), we get \( q = q' \), \( m = m' \) (essentially) and

\[
C_0(...) \ll p^{2\ell} Q^2.
\]
Lemma 4.1. Let \( n, q \) and consequently, in (13) using the GL \( H \) hence we get Theorem 1. We note that, we are saving \( q \) extra due to the additive character \( e(−nq\overline{m}/q) \) which we gives us a congruence condition modulo \( qq \). In this section, we apply summation formulae to the \( n \) details) \( \ell \) as \( (a, q) = 0 \) or \( (\prime k, q, p) \neq 0 \), we analyze \( \ell \). Thus it follows that \( A(n, k) = \sum_{n=1}^{\infty} A(n, k) e \left( \frac{(a + bq)n}{p^f q} \right) e \left( \frac{nx}{p^f qQ} \right) W \left( \frac{n}{N} \right) \) (17) in (13) using the GL(3) Voronoi summation formula. Let \( q = p^f q' \) with \( (p, q') = 1 \) and \( \ell' \geq 0 \). Thus it follows that \( (a + bq, p^f q) = (a + bq, p^f q') \), as \( (a, q) = 1 \). Moreover, if \( \ell' > 0 \), then \( (a + bq, p^f q') = 1 \). In the other case, i.e., \( \ell' = 0 \) or \( (q, p) = 1 \), let \( (a + bq, p^f q) = p^f \), with \( 0 \leq \ell_1 \leq \ell \). On applying Lemma 2.4 to the \( n \)-sum in (13) with the modulus \( p^f q' \) and \( g(n) = e \left( \frac{nx}{p^f qQ} \right) W \left( \frac{n}{N} \right) \), we arrive at

\[
p^f q' q \sum_{\pm} \sum_{n_1 | p^f q} \sum_{n_2 = 1}^{\infty} \frac{A(n_1, n_2)}{n_1 n_2} S(k((a + bq)/p^f), \pm n_2; q p^f q' k/n_1) G_{\mp} \left( \frac{n_1^2 n_2}{(q p^f q')^3 k} \right).
\]

(18)

Next we will analyze the integral transform \( G_{\pm}(y) \). We have the following lemma.

**Lemma 4.1.** Let \( G_{\pm}(y) \) be the integral transform as defined in (8). Let \( y = \frac{n_1^2 n_2}{(q p^f q')^3 k} \). Then \( G_{\pm}(y) \) is negligibly small unless \( n_1^2 n_2 \ll N^{1/2} p^{3\ell_1/2 - 3\ell_1} k R^e =: N_0 \). In this range, we have

\[
G_{\pm} \left( \frac{n_1^2 n_2}{(q p^f q')^3 k} \right) = \left( \frac{n_1^2 n_2 N}{(q p^f q')^3 k} \right)^{1/2} I_1 \left( n_1^2 n_2, q, x \right),
\]

(19)
where \( I_1(n_1^2 n_2, q, x) \) is a integral transform defined in (25).

Proof. Let’s recall from (8) that

\[
G_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_\pm(s) \int_0^\infty e\left(\frac{xz}{p^f qQ}\right) W\left(\frac{z}{N}\right) z^{-s-1} \, dz \, ds
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} (Ny)^{-\sigma-i\tau} \gamma_\pm(\sigma + i\tau) \int_0^\infty e\left(\frac{Nxyz}{p^f qQ}\right) W(z) z^{-\sigma-1-i\tau} \, dz \, d\tau \quad (20)
\]

On applying integration by parts, we infer that the \( z \)-integral is negligibly small unless

\[
|\tau| \asymp N|x|/(p^f qQ).
\]

Using the Stirling formula, for \( \sigma \geq -1/2 \), we have

\[
\gamma_\pm(\sigma + i\tau) \ll \pi, \sigma (1 + |\tau|)^{3(\sigma+1/2)}.
\]

Thus, on plugging this bound into (20), we get

\[
G_\pm(y) \ll \left(\frac{N}{p^f qQ}\right)^{5/2} \left(\frac{N^3}{Nyp^{3/2}q^3Q^3}\right)^\sigma \ll \left(\frac{Q}{q}\right)^{5/2} \left(\frac{Q^3}{Nyp^3}\right)^\sigma. \quad (22)
\]

Thus, on moving the contour to \( \sigma \) sufficiently large (towards \( \infty \)) and taking \( y = \frac{n_1^2 n_2}{(qp^{f-\ell_1})^3k} \), we see that \( G_\pm(y) \) is negligibly small if

\[
n_1^2 n_2 \gg R^\epsilon \frac{(p^{f-\ell_1}Q)^3k}{N} = N^{1/2}p^{3\ell/2-3\ell_1}kR^\epsilon =: N_0. \quad (23)
\]

In the complimentary range, we move the contour to \( \sigma = -1/2 \), to get

\[
G_\pm\left(\frac{n_1^2 n_2}{(qp^{f-\ell_1})^3k}\right) = \left(\frac{n_1^2 n_2 N}{(qp^{f-\ell_1})^3k}\right)^{1/2} I_1(n_1^2 n_2, q, x), \quad (24)
\]

where

\[
I_1(n_1^2 n_2, q, x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{n_1^2 n_2 N}{(qp^{f-\ell_1})^3k}\right)^{-i\tau} \gamma_\pm(-1/2 + i\tau) \widehat{W}(\tau, x) \, d\tau \quad (25)
\]

and

\[
\widehat{W}(\tau, x) = \int_0^\infty e\left(\frac{Nxyz}{p^f qQ}\right) W(z) z^{-1/2-i\tau} \, dz. \quad (26)
\]

Hence, we have the lemma.

Thus, on applying the above lemma to (13), we arrive at

\[
S_1 = \sqrt{N}p^{f_1/2} \sum_{n_1|p^{f_1}} \sum_{n_2=1}^\infty A(n_1, n_2) \frac{1}{n_2^{1/2}} S(k((a + bq)/p^{f_1}), \pm n_2; qp^{f-\ell_1}k/n_1) I_1(\ldots).
\]

We conclude this subsection by recording the above analysis in the following lemma.

**Lemma 4.2.** Let \( S_1 \) be as in (17). Let \((a + bq, p^f q) = p^{f_1}\). Then, we have

\[
S_1 = \sqrt{N}p^{f_1/2} \sum_{n_1|p^{f_1}} \sum_{n_2=1}^\infty A(n_1, n_2) \frac{1}{n_2^{1/2}} S(k((a + bq)/p^{f_1}), \pm n_2; qp^{f-\ell_1}k/n_1) I_1(\ldots),
\]

where \( N_0 \) and \( I_1(\ldots) \) are as in (23) and (25) respectively.
Lemma 4.3. Let \( S_2 \) denote the sum over \( m \) in (13). Let \( q = q'p' \). Then we have

\[
S_2 = \frac{2\pi i k_j N^{3/4}}{\tau(\bar{\chi}) \sqrt{q p^{r - \ell} q'}} \sum_{\beta \mod p'^r} \bar{\chi}(\beta) \sum_{1 \leq m \leq M_0} \frac{\lambda_f(m)}{m^{1/4}} e\left( \frac{(c/p'uq) m}{p^{r - \ell} q} \right) I_2(q, m, x),
\]  

(27)

where \( c = p^{r - \ell}(a + bq) + q\beta \), \( c, p'^r q = p'^2 \), \( M_0 = R' p^{2r - \ell - 2\ell_2} \) and

\[
I_2(q, m, x) = \int_0^\infty U(y) e\left( -\frac{Nxy}{p'^2 qQ} \right) e\left( \pm \frac{2\sqrt{Nmq}}{p^{r - \ell} q} \right) dy.
\]

(28)

Proof. Firstly, we expand \( \chi(m) \) in terms of additive characters. In fact, we have

\[
\chi(m) = \frac{1}{\tau(\bar{\chi})} \sum_{\beta \mod p'^r} \bar{\chi}(\beta) e(\beta m / p'^r),
\]

where \( \tau(\bar{\chi}) \) is the Gauss sum associated to \( \bar{\chi} \). Therefore the \( m \)-sum in (13) transforms into

\[
S_2 := \sum_{m=1}^\infty \lambda_f(m) \chi(m) e\left( -\frac{(a + bq)m}{p'^2 q} \right) e\left( \frac{-mx}{p'^2 qQ} \right) U\left( \frac{m}{N} \right)
\]

(29)

\[
= \frac{1}{\tau(\bar{\chi})} \sum_{\beta \mod p'^r} \bar{\chi}(\beta) \sum_{m=1}^\infty \lambda_f(m) e\left( -\frac{(a + bq)m}{p'^2 q} \right) e\left( \frac{\beta m}{p'^r} \right) e\left( \frac{-mx}{p'^2 qQ} \right) U\left( \frac{m}{N} \right).
\]

(30)

Observe that, if \( (\beta, p) > 1 \), then \( S_2 = 0 \). Thus we can assume that \( (\beta, p) = 1 \).

Let’s consider \( c = p^{r - \ell}(a + bq) + q\beta \). Let \( (c, p'^r q) = (c, p'^{r + \ell'}) = p'^{\ell_2} \). Note that, if \( \ell' = 0 \), then \( \ell_2 = 0 \). On applying Lemma 2.1 to \( S_2 \) with the modulus \( p^{r - \ell_2} q \) and \( g(m) = e\left( -mx / (p'^2 qQ) \right) U(m/N) \), we arrive at

\[
S_2 = \frac{2\pi i k_j}{\tau(\bar{\chi}) p^{r - \ell_2} q} \sum_{\beta \mod p'^r} \bar{\chi}(\beta) \sum_{m=1}^\infty \lambda_f(m) e\left( \frac{(c/p'^2 q) m}{p^{r - \ell_2} q} \right) h(m),
\]

(31)

where

\[
h(m) = \int_0^\infty U\left( \frac{y}{N} \right) e\left( \frac{yx}{p'^2 qQ} \right) J_{k_1 - 1} \left( \frac{4\pi \sqrt{my}}{p^{r - \ell} q} \right) dy.
\]

(32)

Upon changing the variable \( y \mapsto Ny \) and extracting the oscillations of \( J_{k_1 - 1} \) using Lemma 2.3, we observe that \( h(m) \) has essentially the following expression

\[
h(m) = N \int_0^\infty U(y) Z^\pm \left( \frac{2\sqrt{Nmq}}{p^{r - \ell_2} q} \right) e\left( -\frac{Nxy}{p'^2 qQ} \right) e\left( \pm \frac{2\sqrt{Nmq}}{p^{r - \ell_2} q} \right) dy
\]

\[
= \frac{N^{3/4} p^{r/2} \sqrt{q}}{m^{1/4}} \int_0^\infty U(y) e\left( -\frac{Nxy}{p'^2 qQ} \right) e\left( \pm \frac{2\sqrt{Nmq}}{p^{r - \ell_2} q} \right) dy
\]

\[
:= \frac{N^{3/4} p^{r/2} \sqrt{q}}{m^{1/4}} I_2(q, m, x).
\]

(33)
Note the abuse of notation in here. The weight function $U$ appearing above is different from the one we started with. But the new $U$ still satisfies $U^{(4)}(x) \ll_{j,k} 1/x^j$ and support($U$) $\subset [1/2, 5/2]$. On applying integration by parts, we observe that

$$I_2(q, m, x) \ll_j \left(1 + \frac{N|x|}{p^r q Q}\right)^j \left(\frac{p^{\ell_2} q^j}{\sqrt{N m}}\right) \ll \left(1 + \frac{N R^e}{p^r q Q}\right)^j \left(\frac{p^{\ell_2} q^j}{\sqrt{N m}}\right).$$

Thus the integral $I_2(...)$ is negligibly small if

$$m \gg R^e \max \left(\frac{(p^{\ell_2} q^j)^2}{N}, \frac{p^{2r-\ell_2t_2}}{N}\right) = R^e p^{2r-\ell_2t_2} := M_0.$$

Now plugging the expression (33) of $h(m)$ into (31), we get the lemma. □

**Remark 3.** In the above $m$-sum (29), if we had $d(m)$ instead of $\lambda_l(n)$, then on applying the Voronoi formula for $d(m)$, Lemma 2.2, we get a main term which would vanish, as it does not involve $\beta$ (see Lemma 2.2), and hence the sum over $\beta$ will vanish. The remaining part of the Voronoi summation formula for $d(m)$ is similar to the Voronoi formula for $\lambda_l(m)$. Hence, following the similar arguments, we also get subconvexity bounds for $L(1/2, \pi \times E \times \chi)$.

### 4.3. $S_k(N)$ after summation formulae

We conclude this section by combining Lemma 4.2 and Lemma 4.3.

**Lemma 4.4.** Let $S_k(N)$ be as in (13). Then we have

$$S_k(N) = \frac{2\pi^{k/2} p^{(s+1)/2} N^{5/4}}{\tau(\chi) Qp^{(3s+5)/2} k^{1/2}} \sum_{1 \leq q \leq Q} \frac{1}{q^2} \sum_{n_1 \leq n \leq Q} \sum_{k/n_1 \leq \lambda_l(m) \leq m^{1/4}} C(...) J(n_1^2 n_2, q, m),$$

where

$$J(n_1^2 n_2, q, m) := \int_{\mathbb{R}} W_1(x) g(q, x) I_1(n_1^2 n_2, q, x) I_2(q, m, x) \, dx,$$

and the character sum $C(\ldots)$ is defined as

$$\sum_{a \mod q} \sum_{b \mod p} \sum_{\beta \mod p^r} \chi(-\beta) S\left(k((a + bq)/p^{\ell_1}), \pm n_2; q p^{\ell_1-\ell_2} k/n_1\right) e\left(\frac{(c/p^2)m}{p^{r-\ell_2} q}\right).$$

**Proof.** Proof follows by plugging Lemma 4.2 and Lemma 4.3 into (13). □

### 5. The sum over $a$, $\beta$ and $b$

In this section, we will analyze the character sum $C(\ldots)$ defined in Lemma 4.4. It is given as

$$\sum_{a \mod q} \sum_{b \mod p} \sum_{\beta \mod p^r} \chi(-\beta) S\left(k((a + bq)/p^{\ell_1}), \pm n_2; q p^{\ell_1-\ell_2} k/n_1\right) e\left(\frac{(c/p^2)m}{p^{r-\ell_2} q}\right).$$

Let $k = p^{\ell_3} k'$ and $n_1 = p^{\ell_4} n_1'$ with $(k', p) = (n_1', p) = 1$. Thus $q p^{\ell_1-\ell_2} k/n_1 = (q' k'/n_1') p^{\ell_5}$, where $\ell_5 = \ell + \ell' + \ell_3 - \ell_1 - \ell_4$. On splitting the Kloosterman sum using
As (ℓ, q, p) we will analyze the following lemma.

Let’s consider the sum over \( q, p \) where

\[
\sum_{(\alpha, m) \equiv 0 \pmod{d}} e \left( \frac{\pm n_2\alpha}{p^{\ell_2}q^k/n_1'} \right) \sum_{a \equiv \ell \pmod{q}} e \left( \frac{\alpha k\alpha p^{\ell_1} p^{\ell_2}}{q^k/n_1'} \right) e \left( \frac{(p^{\ell_2} - d_2 - \ell + \ell') a}{q} \right)
\]

We have the following lemma.

**Lemma 5.1.** Let \( \mathcal{C}(\ldots) \) be as in (38). Then, if \((q, p) = 1\), we have

\[
\mathcal{C}(\ldots) = p^{(r + \ell + \ell_1)/2} \chi(-q) \sum_{d | q'} \sigma_0 \left( \frac{q'}{d} \right) \sum_{h_1(\alpha, m) \equiv 0 \pmod{d}} e \left( \frac{\pm n_2\alpha}{p^{\ell_2}q^k/n_1'} \right)
\]

\[
\times \sum_{u \equiv 0 \pmod{p^{(\ell_2')/2}}} \sum_{\nu \equiv 0 \pmod{p^{r/2}}} \chi \left( v - p^{r - \ell - \ell'} u \right) e \left( \frac{\alpha k \alpha q n_{11}}{p^{\ell_1} \ell_1} \right) e \left( \frac{\nu m q}{p^r} \right)
\]

where \( h_1(\alpha, m), h_2(v, u, m) \) and \( h_3(v, u, \alpha, m) \) are defined as in (43), (44) and (45), respectively. More specifically, the above sum vanishes if \((n_1, p) > 1\). In the other case, i.e., for \((q, p) > 1\), we have

\[
\mathcal{C}(\ldots) = p^{(r + \ell + \ell_1)/2} \chi(-q') \sum_{d | q'} \sigma_0 \left( \frac{q'}{d} \right) \sum_{h_1(\alpha, m) \equiv 0 \pmod{d}} e \left( \frac{\pm n_2\alpha}{p^{\ell_2}q^k/n_1'} \right)
\]

\[
\times \sum_{u \equiv 0 \pmod{p^{(\ell_2')/2}}} \sum_{\nu \equiv 0 \pmod{p^{r/2}}} \chi \left( v - p^{r - \ell - \ell'} u \right) e \left( \frac{\alpha k \alpha q n_{11}}{p^{\ell_1} \ell_1} \right) e \left( \frac{\nu m q}{p^r} \right)
\]

**Proof.** We will analyze \( \mathcal{C}(\ldots) \) in two cases.

Case 1. \((q, p) = 1\), i.e., \( \ell = 0 \) and \( q = q' \).

As \((q, p) = 1\), we note that \((c, p^r q) = (p^{\ell_2} - \ell + \ell') a + q\beta, p^r q) = 1\), and hence \( \ell_2 = 0 \). Thus, by changing the variable \( b \mapsto (a + bq) = u \) in (38), we get the following expression for \( \mathcal{C}(\ldots) \):

\[
\mathcal{C}(\ldots) = \sum_{h_1(\alpha, m) \equiv 0 \pmod{d}} e \left( \frac{\pm n_2\alpha}{p^{\ell_2}q^k/n_1'} \right) \sum_{h_2(v, u, m) \equiv 0 \pmod{d}} e \left( \frac{\alpha k \alpha q n_{11}}{p^{\ell_1} \ell_1} \right) e \left( \frac{(p^{\ell_2} - d_2 - \ell + \ell') a}{q} \right)
\]

\[
\times \sum_{u \equiv 0 \pmod{p^{(\ell_2')/2}}} \sum_{\nu \equiv 0 \pmod{p^{r/2}}} \chi \left( v - p^{r - \ell - \ell'} u \right) e \left( \frac{\alpha k \alpha q n_{11}}{p^{\ell_1} \ell_1} \right) e \left( \frac{\nu m q}{p^r} \right)
\]

Let’s consider the sum over \( \beta \) in the above expression. It is given as

\[
\mathcal{C}_\beta(\ldots) := \sum_{\beta \equiv 0 \pmod{p^r}} \chi(-\beta) e \left( \frac{(p^{\ell_2} - d_2 - \ell + \ell') a + q\beta}{p^r} \right)
\]
On changing the variable \( p^{r-\ell+\ell_1} u + q \beta = v \), we arrive at

\[
C_\beta(...) = \chi(-q) \sum_{v \mod p^r} \chi(v - p^{r-\ell+\ell_1} u) e\left(\frac{vmq}{p^r}\right).
\]

Let us first assume that \( r \) is even. Splitting the sum over \( v \) into the residue classes modulo \( p^{r/2} \), i.e., writing \( v = v_1 + v_2 p^{r/2} \), we get the following expression for \( C_\beta(...) \):

\[
\chi(-q) \sum_{v_1 \mod p^{r/2}} \chi(v_1 - p^{r-\ell+\ell_1} u) e\left(\frac{v_1mq}{p^r}\right) \times \sum_{v_2 \mod p^{r/2}} \chi\left(1 + \frac{(v_1 - p^{r-\ell+\ell_1} u)v_2 p^{r/2}}{p^{r/2}}\right) e\left(-\frac{v_1^2 v_2 m q}{p^r}\right).
\]

We observe that \( \chi(1+v_2 p^{r/2}) \) is an additive character modulo \( p^{r/2} \) of order \( p^{r/2} \). More precisely, we have

\[
\chi(1 + v_2 p^{r/2}) = e\left(\frac{A v_2}{p^{r/2}}\right),
\]

for some constant \( A \) such that \( (A, p) = 1 \) which depends only on the character \( \chi \).

Thus, on evaluating the sum over \( v_2 \), we get the following expression for \( C_\beta(...) \):

\[
C_\beta(...) = p^{r/2} \chi(-q) \sum_{v_1 \mod p^{r/2}} \chi(v - p^{r-\ell+\ell_1} u) e\left(\frac{vmq}{p^r}\right),
\]

where

\[
h_2(v, u, m) = A v_2^2 + m v - m p^{r-\ell+\ell_1} u.
\]

For \( r \) odd, a similar analysis can be done. In fact, we get a similar sum as above. We refer to Chapter 12 of [23] for more details. Thus, for simplicity, we will continue the proof for \( r \) even. Next we analyze the sum over \( a \) in (38). It is evaluated as

\[
\sum_{a \mod q} e\left(\frac{\alpha k p^{r/2} u}{q k'/n_1}\right) e\left(\frac{(p^{2r-\ell} u)m}{q}\right) = \sum_{d \mid (q, h_1(\alpha, m))} d \mu\left(\frac{q}{d}\right).
\]

where

\[
h_1(\alpha, m) = n'_1 \alpha p^{2\ell+\ell_1} p^{2r} + m p^{2r}.
\]

Thus, on plugging (42) and (44) into (11), we get the following expression for \( C(...) \):

\[
C(...) = p^{r/2} \chi(-q) \sum_{d \mid q} d \mu\left(\frac{q}{d}\right) \sum_{a \mod p^{s} q k'/n'_1} e\left(\frac{\pm n_2 \alpha}{p^{s} q k'/n'_1}\right) \times \sum_{u \mod p^{r-\ell+\ell_1}} \sum_{v \mod p^{r/2}} \chi\left(v - p^{r-\ell+\ell_1} u\right) e\left(\frac{vmq}{p^r}\right) e\left(\frac{\alpha v q n_1}{p^{r-\ell+\ell_1}}\right).
\]

Next we analyze the sum over \( u \). It is given as

\[
C_u(...) := \sum_{u \mod p^{r-\ell_1}} \chi\left(v - p^{r-\ell+\ell_1} u\right) e\left(\frac{\alpha v q n_1}{p^{r-\ell+\ell_1}}\right).
\]
Let’s assume (for simplicity) that \( \ell - \ell_1 \) is even. On reducing \( u \) modulo \((\ell - \ell_1)/2\), i.e., writing \( u \) as

\[
 u = u_1 + u_2p^{(\ell-\ell_1)/2} \iff \; \bar{u} = \bar{u}_1 - \bar{u}_1^2u_2p^{(\ell-\ell_1)/2},
\]

we get the following expression for \( C_u(\ldots) \):

\[
\sum_{u_1 \mod p^{(\ell-\ell_1)/2}} \sum_{h_2(v, u_1, \alpha) \equiv 0 \mod p^r/2} \chi(v - p^{r-\ell+\ell_1}u_1) e\left( \frac{\alpha u_1 q n_1}{p^{\ell-\ell_1}} \right) \times \sum_{u_2 \mod p^{(\ell-\ell_1)/2}} \chi\left( 1 - (v - p^{r-\ell+\ell_1}u_1)p^{r-(\ell-\ell_1)/2}u_2 \right) e\left( \frac{-\bar{u}_1^2u_2\bar{q}n_1}{p^{(\ell-\ell_1)/2}} \right).
\]

We observe that \( u_2 \mapsto \chi\left( 1 - p^{r-\ell_1/2}u_2 \right) \) is an additive character modulo \( p^{(\ell-\ell_1)/2} \) and of order \( p^{(\ell-\ell_1)/2} \). Thus the sum over \( u_2 \) can be written as

\[
\sum_{u_2 \mod p^{(\ell-\ell_1)/2}} e\left( \frac{B(v - p^{\ell_1}u_1)}{p^{(\ell-\ell_1)/2}} \right) e\left( \frac{-\bar{u}_2^2u_2\bar{q}n_1}{p^{(\ell-\ell_1)/2}} \right),
\]

where \( B \) is an absolute constant such that \((B, p) = 1\) which depends on \( \chi \) only. Evaluating the above sum, we get the following congruence relation

\[
h_3(v, u_1, \alpha) := qB(v - p^{r-\ell_1}u_1) - \bar{u}_1^2n_1\alpha \equiv 0 \mod p^{(\ell-\ell_1)/2},
\]

along with the factor \( p^{(\ell-\ell_1)/2} \). We observe that if \( \ell_4 > 0 \), where \( n_4 = p^\ell n_4' \), then the above equation has no solutions. Hence \( C_u(\ldots) = 0 \). In the other case, i.e., for \((n_1, p) = 1\), or \( \ell_4 = 0 \) and \( n_4' = n_1 \), we get

\[
C_u(\ldots) = p^{(\ell-\ell_1)/2} \sum_{u \mod p^{(\ell-\ell_1)/2}} \sum_{h_2(v, u, \alpha) \equiv 0 \mod p^r/2} \sum_{h_3(v, \alpha) \equiv 0 \mod p^{(\ell-\ell_1)/2}} \chi(v - p^{r-\ell_1}u) e\left( \frac{-\bar{u}_1^2u_2\bar{q}n_1}{p^{(\ell-\ell_1)/2}} \right).
\]

On plugging the above expression into \((46)\), we get the first part of the lemma.

Case 2. \( q = q'p^{\ell'} \), with \( \ell' > 0 \).

In this case, we first observe that \((a + bq, p) = 1\), as \((a, q'p^{\ell'}) = 1\) and hence \( \ell_1 = 0 \). Thus, on splitting the sum over \( a \) in \((38)\) as

\[
a = a_1p^{\ell'} + a_2q'q', \quad a_1 \mod q' \quad \text{and} \quad a_2 \mod p^{\ell'},
\]

we arrive at

\[
C(\ldots) = \sum_{\alpha \mod p^{2\ell'}q'k'/n_1'} e\left( \frac{\pm n_2\bar{\alpha}}{p^{2\ell'}q'k'/n_1'} \right) \sum_{a_1 \mod q'} e\left( \frac{\alpha n_1 a_1 p^{\ell' + \ell_1}}{q'} \right) e\left( \frac{(p^{2\ell-2\ell_1-\ell_1}a_1)m}{q'} \right) \times \sum_{a_2 \mod p^{\ell'}} \sum_{b_2 \mod p^{\ell'}} e\left( \frac{\alpha n_1 ((a + bq))q'}{p^{\ell' + \ell}} \right) \sum_{\beta \mod p^{\ell'}} \chi(-\beta) e\left( \frac{(c/p^{\ell'})m\bar{q}}{p^{\ell_2 + \ell_1}} \right).
\]
On combining the sum over \( a_2 \) and the sum over \( b \), we get the following expression for \( C(\ldots) \):

\[
\sum_{\alpha \mod p^{\ell}q^k/n'_1}^* e\left( \frac{\pm n_2\alpha}{p^{\ell+qk'/n'_1}} \right) \sum_{\alpha_1 \mod q'}^* e\left( \frac{\alpha_1p^{\ell/2}(\alpha_1p^{\ell/2} + p^{2r-2q}/2m)}{q'} \right) \\
\times \sum_{b \mod p^{\ell+\ell'}}^* e\left( \frac{\alpha_1((-a_1p^{\ell'/2} + bq'))q}{p^{\ell+\ell'}} \right) \\
\times \sum_{\beta \mod p^r}^* \bar{\chi}(-\beta) e\left( \frac{((p^{\ell'}(a_1p^{\ell'/2} + bq') + q_2)/p^{\ell'}m)q}{p^{\ell+\ell'}} \right).
\]

(51)

Now changing the variable \( b \mapsto a_1p^{\ell'/2} + bq' = u \), we arrive at

\[
\sum_{\alpha \mod p^{\ell}q^k/n'_1}^* e\left( \frac{\pm n_2\alpha}{p^{\ell+qk'/n'_1}} \right) \sum_{\alpha_1 \mod q'}^* e\left( \frac{\alpha_1p^{\ell/2}(\alpha_1p^{\ell/2} + p^{2r-2q}/2m)}{q'} \right) \\
\times \sum_{u \mod p^r}^* e\left( \frac{\alpha uq}{p^{\ell+\ell'}} \right) \sum_{\beta \mod p^r}^* \bar{\chi}(-\beta) e\left( \frac{((p^{\ell'-\ell}u + q_2)/p^{\ell'})q}{p^{\ell+\ell-\ell'}} \right).
\]

(52)

Subcase 2.1. \( 0 < \ell' < r - \ell \).

In this situation, we observe that \( \ell_2 = \ell' \). Thus the sum over \( \beta \) becomes

\[
C_\beta(\ell' > 0) := \sum_{\beta \mod p^r}^* \bar{\chi}(-\beta) e\left( \frac{((p^{\ell'-\ell}u + q_2)/p^{\ell'})q}{p^{\ell+\ell-\ell'}} \right) \\
= \sum_{\beta \mod p^r}^* \bar{\chi}(-\beta) e\left( \frac{((p^{\ell'-\ell}u + q_2)/p^{\ell'})q}{p^{\ell+\ell-\ell'}} \right).
\]

This sum is similar to the sum over \( \beta \) in Case 1. Thus analyzing it in a similar way, we arrive at

\[
C_\beta(\ell' > 0) = p^{r/2}\chi(-q') \sum_{v \mod p^{r/2}}^* \bar{\chi}(v - p^{\ell'-\ell'}u) e\left( \frac{vq}{p^r} \right),
\]

(53)

where

\[
h'_2(v, u, m) = Aq'v^2 + mv - mp^{\ell'-\ell'}u.
\]

On analyzing the sum over \( a_1 \) and the sum over \( u \) as in Case 1, we get the following expression for \( C(\ldots) \):

\[
C(\ldots) = p^{(r+\ell+\ell')/2}\chi(-q') \sum_{d \mid q'} \frac{d'}{d} \sum_{\alpha \mod p^{\ell}q^k/n'_1}^* e\left( \frac{\pm n_2\alpha}{p^{\ell+qk'/n'_1}} \right) \\
\times \sum_{u \mod p^{(\ell+\ell')/2}}^* \sum_{\beta \mod p^r}^* \bar{\chi}(v - p^{\ell'-\ell'}u) e\left( \frac{\alpha uq}{p^{\ell+\ell'}} \right) e\left( \frac{vq}{p^r} \right).
\]

(54)
where \( h_1(\alpha, m) = n_1 \Omega p^{2+2r} + mp^{2r} \) and
\[
h_3(v, u, \alpha) := q'B(v - p^{r-r'-\ell}u) - u^2n_1\alpha \equiv 0 \mod p^{(\ell+\ell')/2}.
\]

Subcase 2.2. \( r - \ell < \ell' \).
In the case, we have \( \ell_2 = r - \ell \). Thus the sum over \( \beta \) in (52) is given by
\[
C_\beta(\ell' > r - \ell) = \sum_{\beta \mod p^r}^* \chi(-\beta) e\left(\frac{(u + q'\beta p^{\ell-r+\ell})m\eta}{p^{\ell+\ell'}}\right).
\]

We first assume that \((m, p) = 1\). On extending this as a sum modulo \( p^{\ell+\ell'} \), we see that
\[
C_\beta(\ell' > r - \ell) = \frac{1}{p^{\ell+\ell'-r}} \sum_{\chi_1 \mod p^{\ell+\ell'}}^* \chi_1(-v) e\left(\frac{(u + q'vp^{\ell-r+\ell})m\eta}{p^{\ell+\ell'}}\right),
\]
where \( \chi_1 \) is the character modulo \( p^{\ell+\ell'} \) which is induced from \( \chi \). The above equality follows upon writing \( v = \beta + v_1p^r \) and realizing the sum over \( v_1 \) as a free sum. Now, upon changing the variable \( v \mapsto vu \), we arrive at
\[
C_\beta(\ell' > r - \ell) = \frac{\chi_1(-u)}{p^{\ell+\ell'-r}} \sum_{\chi \mod p^{\ell+\ell'}}^* \chi_1(v) e\left(\frac{\overline{\tau}\theta}{p^{\ell+\ell'}}\right),
\]
where \( \theta = (u + q'vp^{\ell-r+\ell})m\eta \). Note that the sum over \( v \) can be written as
\[
\sum_{\chi \mod p^{\ell+\ell'}}^* \chi_1(v) e\left(\frac{\overline{\tau}\theta}{p^{\ell+\ell'}}\right) = \sum_{\chi \mod p^{\ell+\ell'}} \chi_1(v) e\left(\frac{\overline{\tau}\theta}{p^{\ell+\ell'}}\right) = \overline{\chi_1(\theta)} \tau(\chi),
\]
as \((\theta, p) = 1\). We observe that \( \tau(\chi_1) = 0 \), as \( \chi_1 \) is an imprimitive character modulo \( p^{\ell+\ell'} \). See Davenport, [10], Chapter 9. Hence, the character sum vanishes. When \((m, p) > 1\), we can carry out the same analysis by extracting powers of \( p \) from \( m \), yielding the same result.

**Remark 4.** We note that the expressions of \( C(\ldots) \) for \((q, p) = 1\) and \((q, p) > 1\) in the above lemma are structurally similar. So their further analysis will be along the similar lines. For simplicity, we will continue with the expression (39) of \( C(\ldots) \).

## 6. Applying Cauchy and Poisson

In this section, we apply the Cauchy’s inequality followed by the Poisson summation formula to the sum over \( n_2 \) in (34). The aim to apply Cauchy is to get rid of \( GL(3) \) Fourier coefficients \( A(n_1, n_2) \).

### 6.1. Cauchy’s inequality.

Splitting the sum over \( q \) in dyadic blocks \( q \sim C \), with \( C \ll Q \), and writing \( q = p^{r}q' = p^{r'}q_1q_2 \) with \((q', p) = 1\), \( q_1' \mid (n'_1k')^\infty \) and \((n'_1k', q_2') = 1\), we see that \( S_k(N) \) in (34) is dominated by
\[
\sup_{C \ll Q} \frac{p^{\ell_1/2+\ell_2/2}N^{5/4}}{Q^3p^{3+2r}k^{1/2}C^2} \sum_{n_1 \mid n'_1} \sum_{n''_1 \mid (n'_1k')^\infty} \sum_{n_2 \ll N_0/n_1^2} |A(n_1, n_2)| \frac{n_1n_2}{n_1^2} \times \left| \sum_{q_1' \sim C/(p^{r}q_1')} \sum_{m \ll M_0} \lambda_f(m) m^{1/4} C(\ldots) \mathcal{J} \left( n_1^2n_2, p^{r}q', m \right) \right|.
\]
On splitting the sum over \( m \) into dyadic blocks \( m \sim M_1, M_1 \ll M_0 \), and applying the Cauchy’s inequality to the sum over \( n_2 \), we arrive at

\[
S_k(N) \ll \sup_{C \ll Q, M_1 \ll M_0} \frac{p^{\ell/2+\ell/2}N^{5/4}}{Qp^{\ell/2+\ell/2/2}N^{5/4}C^2} \sum_{n_2} \sum_{n_1} \Theta^{1/2} \Omega^{1/2},
\]

where

\[
\Theta = \sum_{n_2 \ll N_0/n_2^2} \frac{|A(n_1,n_2)|^2}{n_2},
\]

and

\[
\Omega = \sum_{n_2 \ll N_0/n_2^2} \sum_{q_{2,0} \sim C/(p'^e q'_1)} \sum_{m \sim M_1} \frac{\lambda_f(m)}{m^{1/4}} C(...) \mathcal{J}\left( \frac{n_1^2 n_2}{m}, p'^e q', m \right)^2,
\]

with

\[
M_1 \leq M_0 = p^{2r-\ell-2\ell^2} R^e \quad \text{and} \quad N_0 = N^{1/2} p^{3/2-3\ell}, k R^e.
\]

### 6.2. Poisson summation.

In this step, we will apply the Poisson summation formula to the sum over \( n_2 \) in (60). To this end, we first smooth out the sum over \( n_2 \) using a smooth bump function \( V \). In fact, splitting \( n_2 \) into dyadic blocks \( n_2 \sim N' \), \( N' \ll N_0/n_2^2 \), we arrive at the following expression:

\[
\Omega \ll \sup_{N' \ll N_0/n_2^2} \sum_{q_{2,0} \sim C/(p'^e q'_1)} \sum_{m \sim M_1} \frac{|\lambda_f(m)\lambda_f(m')|}{(mm')^{1/4}} |L(...)|,
\]

where

\[
L(...) = \sum_{n_2 \in \mathbb{Z}} V\left( \frac{n_2}{N'} \right) C(...) \mathcal{C}(...) \mathcal{J}\left( \frac{n_1^2 n_2}{m}, p'^e q', m \right) \mathcal{J}\left( \frac{n_2^2 n_2}{p'^e q''}, m' \right),
\]

\( q'' = q'_1 q'_2 \) and \( V \) is a smooth bump function supported on \([1, 2]\). Reducing \( n_2 \) modulo \( Q \) : \( p^{\ell} q'_1 q'_2 k'/n'_1 \), i.e., changing the variable \( n_2 \mapsto \nu + n_2 Q \) in the expression of \( L(...) \), we arrive at

\[
\sum_{\nu \mod Q} C(...) \mathcal{C}(...) \sum_{n_2 \in \mathbb{Z}} V\left( \frac{n_2 Q + \nu}{N'} \right) \times \mathcal{J}\left( \frac{n_1^2 (n_2 Q + \nu)}{n_2 Q}, p'^e q', m \right) \mathcal{J}\left( \frac{n_2^2 (n_2 Q + \nu)}{p'^e q''}, m' \right).
\]

Now on applying the Poisson summation formula to the sum over \( n_2 \), we get

\[
L(...) = \frac{N'}{Q} \sum_{\nu \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} C(...) \mathcal{C}(...) e\left( \frac{n_2 \nu}{Q} \right) \mathcal{I}(...) ,
\]

where the integral \( \mathcal{I}(...) \) is given by

\[
\mathcal{I}(...) = \int_{\mathbb{R}} V(w) \mathcal{J}\left( \frac{n_2^2 N' w}{n_1^2}, p'^e q', m \right) \mathcal{J}\left( \frac{n_2^2 N' w}{n_2^2}, p'^e q'', m' \right) e\left( \frac{-n_2 N' w}{Q} \right) dw.
\]

Finally, on plugging (63) in (61), we get

\[
\Omega \ll \frac{1}{M_1^{1/2}} \sup_{N' \ll N_0/n_2^2} \sum_{q_{2,0} \sim C/(p'^e q'_1)} \sum_{m \sim M_1} \sum_{n_2 \in \mathbb{Z}} |\mathcal{C}(...)||\mathcal{I}(...)|,
\]
where
\[ C(... := \frac{1}{Q} \sum_{\nu \mod Q} C(...) \overline{C}(...) e\left(\frac{\nu n_2}{Q}\right). \] (66)

6.3. The sum over \( \nu \). Following Remark 41, we assume that \((q, p) = 1\). Thus \( \ell' = 0 \) and \( q = q' \). Also, we can take \( n_1 = n'_1 \), as otherwise, the character sum in Lemma (6.1) vanishes. On plugging the expression of \( C(...) \) from (66) into (66), we see that the sum over \( \nu \) is given by
\[ \frac{1}{Q} \sum_{\nu \mod Q} e\left(\frac{\pm \nu \alpha}{p^{\ell_2} q'_1 q'_2 k'/n'_1}\right) e\left(\frac{\mp \nu \alpha'}{p^{\ell_2} q'_1 q'_2 k'/n'_1}\right) e\left(\frac{\nu n_2}{Q}\right). \]

It’s evaluation gives us the following congruence relation:
\[ \pm \alpha q'' + \alpha' q'_2 + n_2 \equiv 0 \mod Q. \]

Thus we are left with the following expression of \( C(...) \):
\[
\chi(qq'')p^{r+\ell-\ell_1} \sum_{dd' \mu} \left(\frac{q'}{d'}\right) \mu\left(\frac{q''}{d'}\right) \sum_{\alpha \mod p^{\ell_2}q'_1 q'_2 k'/n'_1} \sum_{\alpha' \mod p^{\ell_2}q'_1 q'_2 k'/n'_1} \sum_{h_1(a, m) \equiv 0 \mod d' h_1(a', m') \equiv 0 \mod d'}
\]
\[ \times \sum_{u \mod p^{(\ell-\ell_1)/2}} \sum_{v \mod p^{r/2}} \sum_{h_2(v, u, m) \equiv 0 \mod p^{r/2}} \sum_{h_3(v, u, \alpha) \equiv 0 \mod p^{(\ell-\ell_1)/2}} \chi\left(v - p^{r-\ell+\ell_1} u\right) e\left(\frac{\alpha m q_n}{p^{\ell_2}}\right) e\left(\frac{\pi m q}{p^{r}}\right). \] (67)

7. Final estimates for the character sum \( C(...) \)

In this section, we will give final estimates for the character sum \( C(...) \) given in (67).

Lemma 7.1. Let \( C_0(...) \) and \( C_{\neq 0}(...) \) denote the contributions of \( n_2 = 0 \) and \( n_2 \neq 0 \) respectively to \( C(...) \) in (67). Then we have
\[ C_0(...) \ll p^{r+2(\ell-\ell_1)} \sum_{dd'[q]} qk \left(\frac{d,d'}{d',d'}\right)^{m-m'} \]
and
\[ C_{\neq 0}(...) \ll \frac{q^2 k(m, m')}{n_1} p^{r+3(\ell-\ell_1)/2} \sum_{dd'[(q'_1, q'_2, n_1)]} \sum_{dd'[(q'_1, q'_2, n_1)]} d_2 d'_2, \]
where \( \ell_6 = 2(r - \ell + \ell_1) \).
Proof. Splitting \( \alpha \) and \( \alpha' \) as

\[
\alpha = \alpha_1 p^{\ell^5} + \alpha_2 q^k/n_1 q^k/n'_1, \quad \alpha_1 \text{ mod } q^k / n'_1 \text{ and } \alpha_2 \text{ mod } p^{\ell^5},
\]

\[
\alpha' = \alpha'_1 p^{\ell^5} + \alpha'_2 q'^k/n_1 q'^k/n'_1, \quad \alpha'_1 \text{ mod } q'^k / n'_1 \text{ and } \alpha'_2 \text{ mod } p^{\ell^5},
\]

we observe that the sum over \( \alpha_2 \) and \( \alpha'_2 \) is given by

\[
\mathcal{C}_{\alpha_2, \alpha'_2}(...) := \sum_{\alpha_2 \text{ mod } p^{\ell^5}} \sum_{\alpha'_2 \text{ mod } p^{\ell^5}} e \left( \frac{n \alpha_2 q n_1}{p^{\ell^5}} \right) e \left( \frac{u \alpha'_2 q' n_1}{p^{\ell^5}} \right),
\]

and the sum over \( \alpha_1 \) and \( \alpha'_1 \) is given by

\[
\mathcal{C}_{\alpha_1, \alpha'_1}(...) := \sum_{\alpha_1 \text{ mod } q^k / n'_1} \sum_{\alpha'_1 \text{ mod } q'^k / n'_1} 1,
\]

which we will analyze further.

Case 1. Let’s first assume that \( n_2 = 0 \). In this case, the congruence

\[
\pm \overline{\alpha_1 q'^n} + \overline{\alpha'_1 q'^n} + n_2 \equiv 0 \text{ mod } q'_2 q'^n k'/n'_1
\]

implies that \( q'_2 = q'^n \) and hence \( \alpha = \alpha'_1 \text{ mod } q^k / n'_1 \). Furthermore, the congruence

\[
\pm \overline{\alpha_2 q'^n} + \overline{\alpha'_2 q'^n} + n_2 \equiv 0 \text{ mod } p^{\ell^5}
\]

yields \( \alpha_2 \equiv \alpha'_2 \text{ mod } p^{\ell^5} \). Thus \( \mathcal{C}_{\alpha_2, \alpha'_2}(...) \) transforms into

\[
\mathcal{C}_{\alpha_2, \alpha'_2}(...) = \sum_{\alpha_2 \text{ mod } p^{\ell^5}} e \left( \frac{-(u - u') \alpha_2 q n_1}{p^{\ell^5}} \right)
\]

Recall that \( \ell_5 = \ell - \ell_3 + \ell_3 \). On reducing \( \alpha_2 \) modulo \( p^{(\ell - \ell_3)/2} \), we get

\[
\mathcal{C}_{\alpha_2, \alpha'_2}(...) = p^{\ell_5 + (\ell - \ell_3)/2} \sum_{\alpha_2 \text{ mod } p^{(\ell - \ell_3)/2}} e \left( \frac{-(u - u') \alpha_2 q n_1}{p^{\ell^5}} \right),
\]

along with the congruence \((u - u') \overline{q} p^{\ell_5} n_1 \equiv 0 \text{ mod } p^{\ell_5 + (\ell - \ell_3)/2}\) which implies that \( u \equiv u' \text{ mod } p^{(\ell - \ell_3)/2} \). Hence,

\[
h_3(v, u, \alpha_2) - h_3(v', u', \alpha_2) \equiv q'B(v - p^{\ell - \ell_3} u) - q'B(v' - p^{\ell - \ell_3} u) \equiv 0 \text{ mod } p^{(\ell - \ell_3)/2},
\]

giving us \( v \equiv v' \text{ mod } p^{(\ell - \ell_3)/2} \) and consequently,

\[
h_2(v, u, m) - h_2(v', u', m') \equiv (m - m') v \equiv 0 \text{ mod } p^{\ell - \ell_3},
\]
which yields the restriction $p^{r - \ell + \ell_1} | (m - m')$. Using $\alpha_1 \equiv \alpha'_1 \mod qk'/n'_1$, we arrive at the following expression of $C_{\alpha_1, \alpha'_1}(\ldots)$:

$$C_{\alpha_1, \alpha'_1}(\ldots) = \sum_{\alpha_1 \mod qk'/n'_1} \sum_{h_1(\alpha_1, m) \equiv 0 \mod d} \sum_{h_1(\alpha_1, m') \equiv 0 \mod d'} 1.$$ 

Hence, combining all the above observations together, we see that $C_0(\ldots)$ is dominated by

$$C_0(\ldots) \ll p^{r + \ell - \ell_1} p^{\ell_3 + (\ell - \ell_1)/2} \sum_{d, d' | q} \sum_{(d, d') \equiv (m - m') \mod p^{\ell_1}} \sum_{u \equiv u' \mod p^{(\ell - \ell_1)/2}} \sum_{\alpha_2 \mod p^{(\ell - \ell_1)/2}} \sum_{\alpha_1 \mod qk' \mod d} \sum_{\alpha_2 \mod p^{(\ell - \ell_1)/2}} 1.$$ 

Now we count the number of tuples $(u, u', v, v', \alpha_2)$. We observe that, given $u$ and $v$, the congruence

$$h_3(v, u, \alpha_2) = q^r B(v - p^{r - \ell + \ell_1} u) - u^2 n'_1 \alpha_2 \equiv 0 \mod p^{(\ell - \ell_1)/2}$$

determines $\alpha_2$ uniquely. Next we count the number of $v$ and $v'$ using the Hensel’s lemma. Let’s consider

$$h_2(v, u, m) = Aq' v^2 + mv - mp^{r - \ell + \ell_1} u \equiv 0 \mod p^{r/2}, \quad (68)$$

in which we want to count the number of $v$’s modulo $p^{r/2}$ (keeping $u$ fixed) satisfying $h_2(v, u, m)$. We may assume that $(m, p) = 1$, as otherwise $h_2(v, u, m)$ has no solutions. Let $v_0$ be a solution of $h_2(v, u, m)$ modulo $p^{r/2}$. We observe that

$$v_0 \equiv -m Aq' \mod p^{r - \ell + \ell_1}.$$ 

By the Hensel’s lemma, it can be lifted uniquely modulo $p^{r/2}$. Hence $v_0$ is determined uniquely modulo $p^{r/2}$. Similar arguments apply to $h_2(v', u', m')$ as well. Hence, on estimating the sum over $u$ trivially, we arrive at

$$C_0(\ldots) \ll p^{r + \ell - \ell_1} p^{\ell_3 + (\ell - \ell_1)/2} \sum_{d, d' | q} \sum_{(d, d') \equiv (m - m') \mod p^{\ell_1}} \sum_{u \equiv u' \mod p^{(\ell - \ell_1)/2}} \sum_{\alpha_2 \mod p^{(\ell - \ell_1)/2}} \sum_{\alpha_1 \mod qk' \mod d} \sum_{\alpha_2 \mod p^{(\ell - \ell_1)/2}} 1.$$ 

Hence we have the first part of the lemma.

Case 2. Now we will analyze $C(\ldots)$ for $n_2 \neq 0$. The congruence

$$\pm \alpha_{2} q''_2 \equiv \alpha''_2 q'_2 + n_2 \equiv 0 \mod p^{(\ell_3 - \ell_1 + \ell_3)}$$

determines $\alpha''_2$ in terms of $\alpha_2$. In fact, taking $+$ sign for simplicity, we have

$$\alpha''_2 \equiv q'_2 (n_2 + \alpha q''_2) \equiv q'_2 \alpha_2 (n_2 \alpha_2 + q''_2) \mod p^{\ell - \ell_1 + \ell_3}.$$
Thus, upon changing the variable $\gamma_2 = n_2 \alpha_2 + q_2''$, we get the following expression of $\mathcal{C}_{\alpha_2, \alpha_2'}(...)$:

$$
\mathcal{C}_{\alpha_2, \alpha_2'}(...) = e\left(\frac{(-\overline{u_2'q_2'' - u_2'q_2''})n_2\overline{q_1'}}{p^{\ell_1}}\right)
\times \sum_{\gamma_2 \mod p^{(\ell_1-1)}} \sum_{*}^{*} e\left(\frac{\overline{u_2'\gamma_2 + u_2'q_2''})n_2\overline{q_1'}}{p^{(\ell_1)}}\right)
$$

Note that we have assumed $(n_2, p) = 1$. In the other case, on extracting the $p$-powers from $n_2$, we can analyze similarly. Now reducing $\gamma_2$ modulo $(\ell - \ell_1)/2$, we get the following sum over $\gamma_2$:

$$p^{(\ell_1+\ell_1)/2} \sum_{\gamma_2 \mod p^{(\ell_1)/2}} \sum_{*}^{*} e\left(\frac{\overline{u_2'\gamma_2 + u_2'q_2''})n_2\overline{q_1'}}{p^{\ell_1}}\right),$$

where

$$h_4(u, u'; \gamma_2) := \overline{u_2'q_2''}. $$

Hence, we have

$$\mathcal{C}_{\alpha_2, \alpha_2'}(...) \ll p^{(\ell_1+\ell_1)/2} \sum_{\gamma_2 \mod p^{(\ell_1)/2}} 1.$$ 

On plugging the above expression into (31), we arrive at

$$\mathcal{C}_{v, u, u', \gamma_2}(...),$$

where

$$\mathcal{C}_{v, u, u', \gamma_2}(...) := \sum_{v \mod p^{(\ell_1)/2}} \sum_{u' \mod p^{(\ell_1)/2}} \sum_{u'' \mod p^{(\ell_1)/2}} \sum_{*}^{*} 1.$$

Our next step is to analyze $\mathcal{C}_{v, u, u', \gamma_2}(...)$. We will prove that

$$\mathcal{C}_{v, u, u', \gamma_2}(...) \ll p^{\ell_1}.$$ 

We have five variables $v$, $v'$, $u$, $u'$ and $\gamma_2$ with the following five congruences:

$$h_2(v, u, m) = Aq''v^2 + mv - mp^{r-\ell_1}u \equiv 0 \mod p^{r/2}, \quad (70)$$

$$h_2(v', u') = Aq''v'^2 + mv' - mp^{r-\ell_1}u' \equiv 0 \mod p^{r/2}, \quad (71)$$

$$h_3(v, u, ...) = q'B(v - p^{r-\ell_1}u) - \overline{u_2'q_2''} \equiv 0 \mod p^{(\ell_1)/2}, \quad (72)$$
Reducing (76) and (77) modulo $p$ by the change of variable $\gamma_2 = u_2 \ell_2 (\gamma_2 - q_2'' q_2') \equiv 0 \mod p^{(\ell - \ell_1)/2}$, we get

\[ h_4(u, u', \gamma_2) = \overline{u q_2'} - \overline{u_2' q_2'' \gamma_2^2} \equiv 0 \mod p^{(\ell - \ell_1)/2}. \]  

We observe from (74) that, fixing $u$ and $u'$, $\gamma_2$ has at most two choices. In fact,

\[ \gamma_2^2 \equiv u/\overline{u q_2'} \mod p^{(\ell - \ell_1)/2}. \]  

Let’s now consider (70), in which we want to find the number of solutions (keeping $u$ fixed) of $h_2(v, u, m)$ modulo $p^{r/2}$. We argue as in the zero frequency case. Let $v_0$ be a solution of $h_2(v, u, m)$ modulo $p^{r/2}$. We observe that

\[ v_0 \equiv -m Aq' \mod p^{r - \ell + \ell_1}. \]

By Hensel’s lemma, it can be lifted uniquely modulo $p^{r/2}$. Hence $v_0$ is determined uniquely modulo $p^{r/2}$. The same arguments can be applied to (71). Thus, $h_2(v', u', m')$ also has a unique solution, say, $v_0'$ modulo $p^{r/2}$ such that

\[ v_0' \equiv -m Aq'' \mod p^{r - \ell + \ell_1}. \]

Now it remains to count the number of pairs $(u, u')$. On substituting (75) and $v_0 - p^{r - \ell + \ell_1} u = -m Aq' v_0^2$ into (72), we get

\[ u q_2'' \left( -n_1 n_2 B m \overline{A v_0^2 u^2 + q_2''} \right)^2 \equiv u' \mod p^{(\ell - \ell_1)/2}, \]  

which determines $u'$ uniquely in terms of $u$, as $v_0$ is depending only on $u$. Now substituting (75) in place of $\gamma_2^2$ and $-m Aq''$ in place of $v_0 - p^{r - \ell + \ell_1} u'$ into (73), we arrive at

\[ \left( u' \left( q_2'' q_2' \right)^2 \right)^2 \left( -n_1 q_2'' n_2 B m \overline{A v_0^2 u^2 + 1} \right)^2 \equiv u \mod p^{(\ell - \ell_1)/2}. \]  

Reducing (76) and (77) modulo $p^{r - \ell + \ell_1}$, we get

\[ u \left( B_2 u^2 + q_2'' q_2' \right)^2 \equiv u' \mod p^{(r - \ell + \ell_1)}, \]

\[ u' \left( B_3 u^2 + q_2'' q_2' \right)^2 \equiv u \mod p^{(r - \ell + \ell_1)}, \]

where $B_2 = -n_1 n_2 B m Aq'' q_2'$ and $B_3 = -n_1 q_2'' n_2 B m Aq'' q_2'' q_2' q_2' q_2'$. On plugging (75) into (79), we get

\[ u \left( B_2 u^2 + q_2'' q_2' \right)^2 \left( B_3 u^2 \left( B_2 u^2 + q_2'' q_2' \right)^4 + \overline{q_2'' q_2'} \right)^2 \equiv u \mod p^{(r - \ell + \ell_1)}. \]

Reducing it modulo $p$, we get

\[ \left( B_2 u^2 + q_2'' q_2' \right) \left( B_3 u^2 \left( B_2 u^2 + q_2'' q_2' \right)^4 + \overline{q_2'' q_2'} \right) \equiv \pm 1 \mod p. \]

By the change of variable $B_2 u^2 + q_2'' q_2' \mapsto u_3$, we arrive at

\[ h_4(u_3) := B_2 \overline{B_2 u_3^5} - B_3 \overline{B_2 q_2'' q_2' u_3^5} + q_2'' q_2' u_3 \equiv 1 \equiv 0 \mod p. \]

Let’s consider the negative sign (similar arguments hold true for the positive sign). In this case, we have

\[ h_4(u_3) = (q_2'' q_2' B_2 B_2 u_3^5 + 1) \overline{(q_2'' q_2' u_3 - 1)} \equiv 0 \mod p. \]

Thus either $u_3 \equiv q_2'' q_2' \mod p$ or $(q_2'' q_2' B_2 B_2 u_3^5 + 1) \equiv 0 \mod p$. Thus, $h_4(u_3)$ has at most 6 solutions modulo $p$. Let $u_0$ be a solution of $h_4(u_3)$ modulo $p$. If $(\frac{u_0}{u_3})(u_0, p) = 1,$
then, using the Hensel’s lemma, we get a unique lift. Moreover, it has \( p \)-many lifts if 
\[
\frac{dh_3}{du_3}(u_0) \equiv 0 \pmod{p} \text{ and } h_3(u_0) \equiv 0 \pmod{p^2}.
\]
Let’s first take \( u_0 = q_2''q_2' \). Thus we get 
\[
\frac{dh_3}{du_3}(u_0) \equiv q_3''q_2'P_3B_2(q_2''q_2')^5 + 1 = m\overline{m}q_2''q_2' + 1 \equiv 0 \pmod{p}.
\]
Thus \( m' \equiv -mq_2''q_2' \pmod{p} \), from which we save \( p \) which analyzing the sum over \( m' \).
This tells us that, if we loose \( p \) while lifting modulo \( p^2 \), it is gained back from the \( m' \) sum. Similar arguments holds for other solutions as well. Hence, on applying the Hensel lemma repeatedly, we get the desired claim (69). Thus we have

\[
C_{\not=0}(...) \ll \sum_{d' | q''} \prod_{l_2 | d'} 1,
\]

which we will analyze now. The above sum can be dominated by a product of two sums \( C_{\not=0}(...) \ll C_{\not=0,1}C_{\not=0,2} \), where

\[
C_{\not=0,1} \ll \sum_{d_1, d_1' | q''} \prod_{l_2 | d_1} 1,
\]

and

\[
C_{\not=0,2} \ll \sum_{d_2, d_2' | q''} \prod_{l_2 | d_2} 1.
\]

In the second sum, since \( (n', q_2q_2') = 1 \), we get \( \alpha_1 \equiv -mn_12^{(r+\ell_1)}p^{2r} \pmod{d_2} \) and \( \alpha_1' \equiv -m'm_12^{(r+\ell_1)}p^{2r} \pmod{d_2} \). Now using the congruence modulo \( q_2'q_2'' \), we infer that

\[
C_{\not=0,2} \ll \sum_{d_2 | (q'', n_1m_1p^{2r} + q_2''n_1p^{2r} + q_2'n_1m_1p^{2r})} \prod_{l_2 | d_2} 1.
\]

In the first sum \( C_{\not=0,2} \), the congruence condition modulo \( q_1'k' / n_1' \) determines \( \alpha_1' \) uniquely in terms of \( \alpha_1 \), and hence

\[
\sum_{d_1, d_1' | q''} \prod_{l_2 | d_1} 1 \ll \frac{q_1'^2k'(m, n_1')}{{n_1'}},
\]

as \( h_1(\alpha_1, m) = n_1' \alpha_1p^{2r_1}p^{-2r} \equiv -mp^{2r} \pmod{d_1} \) has \( (n_1', m) \) many solutions modulo \( d_1 \). Finally combining estimates from (83) and (84), we get the lemma.

\[\square\]

8. Estimates for the integral \( \mathcal{J}(...) \)

In this section, we will analyze the integral transform \( \mathcal{I}(...) \) given in (65). We have the following lemma.
Lemma 8.1. Let $I(...)$ be as in (64). Then we have

$$I(...) \ll \frac{M_1}{p^{2r-2\ell-2}} \frac{C^2Q^c}{Q^2}. $$

Moreover, if

$$|n_2| \gg R^c \frac{Q}{C N'},$$

then $I(...)$ is negligibly small.

Proof. Let’s first recall from (64) that

$$I(...) = \int_{\mathbb{R}} V(w) J \left( n_1^2N'w, p^\ell q', m \right) J \left( n_1^2N'w, p^\ell q'', m' \right) e \left( \frac{-n_2N'w}{Q} \right) \, dw, \quad (85)$$

where

$$J \left( n_1^2N'w, p^\ell q', m \right) = \int_{\mathbb{R}} W_1(x) g(p^\ell q', x) I_1(n_1^2N'w, p^\ell q', x) I_2(p^\ell q', m, x) \, dx,$$

which we will analyze now. Let’s write $q = p^\ell q'$ for simplicity. On plugging the expressions of $I_1(n_1^2N'w, q, x)$ and $I_2(q, m, x)$ from (25) and (28) respectively in the above expression, we arrive at the following expression of $J \left( n_1^2N'w, q, m \right)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} U(y) \int_{0}^{\infty} \frac{W(z)}{\sqrt{z}} \int_{\mathbb{R}} W_1(x) g(q, x) e \left( \frac{Nx(z - y)}{p^\ell qQ} \right) \gamma_\pm(-1/2 + i\tau) \times z^{-i\tau} e \left( \pm \frac{2\sqrt{Nmz}}{p^\ell qQ} \right) \left( \frac{n_1^2N'w}{(q p^\ell')^{3k}} \right)^{-i\tau} \, dx \, dz \, dy \, d\tau. \quad (86)$$

On differentiating the above expression with respect to $w$, we see that

$$\frac{\partial^j}{\partial w^j} J \left( n_1^2N'w, q, m \right) \ll \left( \frac{N}{p^\ell qQ} \right)^{j+1} \ll \left( \frac{Q}{C} \right)^{j+1},$$

as $|\tau| \asymp N|x|/(p^\ell qQ)$ from (21), and $\gamma_\pm(-1/2 + i\tau) \ll 1$. Thus, on applying integration by parts repeatedly on the $w$-integral in (85), we see that $I(...)$ is negligibly small unless

$$|n_2| \ll R^c \frac{Q}{C N'}.$$

This gives the second part of the lemma. On considering the $y$-integral in (86), we observe that

$$\frac{N|x|}{p^\ell qQ} \asymp \frac{\sqrt{Nm}}{p^\ell qQ},$$

as otherwise, using the first derivative bound, Lemma 2.7 the $y$-integral $I_2(q, m, x)$ will be negligibly small. Hence, we can assume that

$$|x| \asymp \sqrt{m}/p^\ell qQ. \quad (87)$$

Case 1. $q, p^\ell q'' \sim C \gtrsim Q^{1-\epsilon}$.

In this case, we estimate $J \left( n_1^2N'w, q, m \right)$ trivially. In fact, we have

$$J \left( n_1^2N'w, q, m \right) \ll \int_{\mathbb{R}} |W_1(x)||g(q, x)| \, dx \ll Q^c \sqrt{M_1/p^\ell qQ},$$

where we used (9) and (87) for the second inequality. On analyzing $J \left( n_1^2N'w, q', m' \right)$ similarly and plugging the corresponding estimates into (64), we get

$$I(...) \ll M_1/p^{2r-2\ell-2}. $$
Case 2. $q, p'q'' \sim C \ll Q^{1-\epsilon}$. Let’s consider the $x$-integral in (86)

\[ I_{z-y} := \int_{\mathbb{R}} W_1(x)g(q,x)e\left(\frac{Nx(z-y)}{p'qQ}\right) \, dx. \]

We will analyze it in two subcases.

Subcase 2.1. $m \sim M_1 \asymp M_0 = R^r p^{2r-2\ell-\epsilon}$.
In this situation, we have

\[ I_{z-y} = \int_{|x| > 1} W_1(x)g(q,x)e\left(\frac{Nx(z-y)}{p'qQ}\right) \, dx. \]

Using the second property (9) of $g(q,x)$, we observe that

\[ \frac{\partial^j}{\partial x^j}g(q,x) \ll \log Q \frac{Q}{|x|^j} \min\left\{ \frac{Q}{q}, \frac{1}{|x|}\right\} \ll Q^j. \]

Thus, using integration by parts repeatedly, the above integral $I_{z-y}$ is negligibly small unless $|z - y| \ll Q'q/Q$.

Subcase 2.2. $m \sim M_1 \ll M_0^{1-\epsilon}$.
In this case, we have the following $x$-integral

\[ I_{z-y} = \int_{|x| \ll R^{-\epsilon}} W_1(x)g(q,x)e\left(\frac{Nx(z-y)}{p'qQ}\right) \, dx. \]

Using the first property (see (9)) of $g(q,x)$, we observe that

\[ g(q,x) - 1 = O\left(\frac{Q}{q} \left(\frac{q}{Q} + |x|\right)^B\right) \ll R^{-2020}. \]

Thus we can replace $g(q,x)$ by 1 at the cost of a negligible error term so that we essentially have

\[ \int_{|x| \ll R^{-\epsilon}} W_1(x)e\left(\frac{Nx(z-y)}{p'qQ}\right) \, dx. \]

Now using integration by parts, we observe that the above integral is negligibly small unless $|z - y| \ll Q'q/Q$. Thus combining Subcases 2.1 and 2.2, we conclude that the $x$-integral $I_{z-y}$ is negligibly small unless $|z - y| \ll Q'q/Q$. Now writing $z = y + u$, with $|u| \ll Q'q/Q$, we get the following expression for $J\left(n_1^2 N'w, q, m\right)$:

\[ \frac{1}{2\pi i} \int_0^\infty I_u \int_{-\infty}^\infty e^{-1/2 + i\tau} \left(\frac{n_1^2 N'Nw}{qp'^{\ell_1}}\right)^{-i\tau} \times \int_{-\infty}^\infty U(y)W(y+u)\sqrt{y+u}^\tau e^\left(\pm \frac{2\sqrt{Nmy}}{p'^{\ell_1}q}\right) dy \, d\tau \, du. \]

Next we analyze the integral over $y$. To this end, we will employ the stationary phase expansion, Lemma 2.8, to it. We first observe that $|u| \ll C/Q \ll Q^{-\epsilon}$. Hence, on writing

\[ (y + u)^{-i\tau} = e^{-i\tau \log y}e^{-i\tau \log (1+u/y)}, \]

we note that $e^{-i\tau \log (1+u/y)}$ can be inserted into the weight function $U$, as

\[ \frac{\partial^j}{\partial y^j}e^{-i\tau \log (1+u/y)} \ll Q^j. \]
Thus the $y$-integral in (88) looks like

$$I(\tau, u) := \int_0^\infty U_u(y) y^{-ir} e^{\left( \pm \frac{2\sqrt{Nm\gamma y}}{p^{r-\ell_2q}} \right)} dy,$$

where $U_u(y)$ is the new weight function

$$U_u(y) := 2yU(y)W(y + u)(y + u)^{-1/2}e^{-ir\log(1 + u/y)}.$$

On taking $+$ sign, and using the change of variable $y \mapsto y^2$, we see that the stationary point of the phase function is given by $y_0 = \frac{pr - \ell_2q}{2\pi\sqrt{Nm}}$. Thus on applying Lemma 2.8, we arrive at

$$I(\tau, u) = U_{y_0}(y_0)e^{(\tau \log(e/y_0)/\pi + 1/8)} \sqrt{\tau/(\pi y_0^2)} + \text{lower order terms}.$$

On plugging the above expression in (88) and proceeding with the main term, we arrive at

$$e^{(1/8)} \frac{2\sqrt{\pi}}{2\sqrt{\pi}} \int_0^\infty I_u \int_{-\infty}^{\infty} \gamma_\pm(-1/2 + i\tau)\frac{\sqrt{\pi}}{\tau} y_0U_y(y_0)e^{(\tau \log(e/y_0)/\pi)} \left( \frac{n_1^2N'Nw}{(q_{p'\ell_1}q_1)^3k} \right)^{-i\tau} d\tau du.$$

On using the following expansion (due to Stirling formula) (see [40])

$$\gamma_\pm(-1/2 + i\tau) = e^{3i\tau \log(\tau/e\pi)}\Phi_\pm(\tau), \quad \Phi^{(j)}_\pm(\tau) \ll 1/\tau^j$$

and applying the second derivative bound, Lemma 2.7, on the $\tau$-integral, we see that it is bounded by 1. Hence

$$J(n_1^2N'w, q, m) \ll \int_0^\infty |I_u| du \ll \sqrt{M_1} CQe^{p^{r-\ell_2-\ell_2/2}},$$

where we used

$$|I_u| \ll \sqrt{M_1}/p^{r-\ell_2-\ell_2/2},$$

which follows using (87) and (9). On analyzing $J(n_1^2N'w, q', m')$ in a similar fashion, we get the first part of the lemma.

\[ \square \]

9. Final estimates for $\Omega$ and $S_k(N)$

In this section, we will estimate $\Omega$ given in (65). We will analyze it in two cases.

9.1. The zero frequency. Let $\Omega_0$ denote the contribution of $n_2 = 0$ to $\Omega$, and let $S_{k,0}(N)$ be its contribution to $S_k(N)$ in (68).

**Lemma 9.1.** For $\ell' = 0$, we have

$$\Omega_0 \ll \frac{N_0 p^{r+2} R^2 C^4 M_0^{1/2} k}{n_1^2 p^{2\ell_1} Q q_1'},$$

and

$$S_{k,0}(N) \ll R^e N^{1/2} p^{3r/4 + 3\ell'/4}.$$  

**Proof.** Let's recall from (65) that

$$\Omega_0 \ll \frac{1}{M_1^{1/2}} \sup_{N' \ll N_0/n_1^2} \sum_{q_2' q_2''} \sum_{q_1'} \sum_{m,m'} \sum_{M_1} |\mathfrak{C}_0(\ldots)| |\mathcal{I}(\ldots)|.$$
On plugging the bound for $C_0(...)$ from Lemma 7.1, the following bound for $I(...)$

$$I(...) \leq \frac{M_1}{p^{2r-2\ell-\ell}} C^2 Q^e \leq \frac{C^2 Q^e}{Q^2}$$

from Lemma 8.1 and using the fact $q''_2 = q'_2$ in the above expression, we get

$$\Omega_0 \leq \frac{p^{r+2\ell} R' C^2 N_0}{p^{2t+1} M_1^{1/2} Q^2 n_1^2} \sum_{q_2' \sim C/q_1'} qk \sum_{d,d' \mid q} (M_1(d,d') + \frac{M_1^2}{p^{(r-\ell+t_1)}})$$

Now estimating the sum over $m$ and $m'$, we arrive at

$$\Omega_0 \leq \frac{N_0 p^{r+2\ell} R' C^2}{n_1^2 p^{2t+1} M_1^{1/2} Q^2} \sum_{q_2' \sim C/q_1'} qk \left( q + \frac{M_1}{p^{(r-\ell+t_1)}} \right)$$

$$\leq \frac{N_0 p^{r+2\ell} R' C^2 M_1^{1/2} C^2 k}{n_1^2 p^{2t+1} Q^2} \left( Q + \frac{M_0}{p^{(r-\ell+t_1)}} \right)$$

$$\leq \frac{N_0 p^{r+2\ell} R' C^2 M_0^{1/2} C^2 Q k}{n_1^2 p^{2t+1} Q^2 q_1'}$$

as $M_0/p^{(r-\ell+t_1)} = R' p^{r-\ell_1} \leq R' p^{3r/2-\ell/2} = Q$. Hence we have the first part of the lemma. To prove the second part, we substitute the above expression of $\Omega_0$ in (58). Thus we see that $S_{k,0}(N)$ is dominated by

$$\sup_{C \ll Q} \frac{p^{t_1/2} N^{5/4}}{Q p^{3/2+\ell k_1/2} C^2} \sum_{n_1' \in \mathbb{A}'} n_1^{1/2} \sum_{q_1' \mid q_1'} \Theta^{1/2} \left( \frac{N_0 p^{r+2\ell} R' C^4 M_0^{1/2} k}{n_1^2 p^{2t+1} Q q_1'} \right)^{1/2}$$

$$\leq \sup_{C \ll Q} \frac{p^{t_1/2} N^{5/4}}{Q p^{3/2+\ell k_1/2}} \left( \frac{N_0 p^{r+2\ell} M_0^{1/2} k}{p^{2t+1} Q} \right)^{1/2} \sum_{n_1' \in \mathbb{A}'} \Theta^{1/2} \sum_{n_1' \in \mathbb{A}'} \frac{1}{n_1' \sqrt{q_1'}}$$

$$\leq \sup_{C \ll Q} \frac{p^{t_1/2} N^{5/4}}{Q p^{3/2+\ell k_1/2}} \left( \frac{N_0 p^{r+2\ell} M_0^{1/2} k}{p^{2t+1} Q} \right)^{1/2} \sum_{n_1' \in \mathbb{A}'} \frac{1}{n_1' \sqrt{q_1'}} \Theta^{1/2},$$

as $n_1 = n_1'$. Note that

$$\sum_{n_1' \in \mathbb{A}'} \frac{(n_1', k')^{1/2} \Theta^{1/2}}{n_1'^{1/2}} \leq \left[ \sum_{n_1' \in \mathbb{A}'} \frac{(n_1', k')^{1/2}}{n_1'} \right]^{1/2} \left[ \sum_{n_1'^2 n_2 \leq N_0} \frac{|A(n_1', n_2)|^2}{n_1'^2 n_2} \right]^{1/2} \leq R'. \quad (90)$$

On using this bound, we arrive at

$$S_{k,0}(N) \leq \frac{p^{t_1/2} N^{5/4}}{Q p^{3/2+\ell k_1/2}} \left( \frac{N_0 p^{r+2\ell} M_0^{1/2} k}{p^{2t+1} Q} \right)^{1/2} \leq R' N^{1/2} p^{3r/4+3\ell/4}.$$
9.2. The non-zero frequencies. Now it remains to estimate $\Omega$ for non-zero values of $n_2$. Let $\Omega_{\neq 0}$ denote the contribution of $n_2 \neq 0$ to $\Omega$ in (65), and let $S_{k, \neq 0}(N)$ be its contribution to $S_k(N)$ in (58).

**Lemma 9.2.** For $\ell' = 0$, we have

$$
\Omega_{\neq 0} \ll \frac{C^5 k^2 p^{\ell - 5\ell_1/2} p^r}{n_1 Q d_1^2},
$$

and

$$
S_{k, \neq 0}(N) \ll N^\alpha p^{-\ell/2} k N^{3/4}.
$$

**Proof.** Let’s recall from (65) that

$$
\Omega_{\neq 0} \ll \frac{1}{M_1^{1/2}} \sup_{N' \leq N_0/n_2^2} N'
$$

$$
\sum_{q_1' \sim C/\sqrt{q_1'}} \sum_{d_1' \sim \sqrt{d_1}} \sum_{d_2 \sim d_1'} \sum_{d_2' \sim d_1'} \sum_{m, m' \sim M_1} \sum_{0 < |n_2| \leq N_2} |\xi_{\neq 0}(\ldots)| |I(\ldots)|.
$$

On plugging bounds for $\xi_{\neq 0}(\ldots)$ and $I(\ldots)$ from Lemma 7.1 and Lemma 8.1 respectively in the above expression, we arrive at

$$
\frac{C^2 q_1'^2 k p^{\ell - \ell_1/2} M_1^{1/2}}{p^{\ell/2} n_1 Q^2} \sup_{N' \leq N_0/n_1^2} N'
$$

$$
\sum_{q_2, q_2' \sim C/\sqrt{q_1'}} \sum_{d_2, d_2' \sim \sqrt{d_1}} \sum_{d_2' \sim d_1'} \sum_{m, m' \sim M_1} \sum_{0 < |n_2| \leq N_2} (m, n_1').
$$

By the change of variable $q_2' \mapsto q_2 d_2$ and $q_2'' \mapsto q_2' d_2'$, we arrive at

$$
\frac{C^2 q_1'^2 k p^{\ell - \ell_1/2} M_1^{1/2}}{p^{\ell/2} n_1 Q^2} \sup_{N' \leq N_0/n_1^2} N'^2/3
$$

$$
\sum_{d_2, d_2' \sim \sqrt{d_1}} \sum_{d_2' \sim d_1'} \sum_{m, m' \sim M_1} \sum_{0 < |n_2| \leq N_2} (m, n_1').
$$

Next, we count the number of $m$ in the above expression as follows:

$$
\sum_{m \sim M_1} (n_1', m) = \sum_{\delta(n_1')} \sum_{m \sim M_1/\delta} \sum_{-n_2 m \pm \delta q_2 d_2' n_1' p^6 \equiv 0 \mod d_2} 1 \ll (d_2, n_2) \left( n_1' + \frac{M_1}{d_2} \right),
$$

Recall that $(n_1', d_2) = 1$. Counting the number of $m'$ in a similar fashion we get that the number of $(m, m')$ pairs is dominated by

$$
O((d_2', q_2' d_2 n_1') (d_2, n_2) (n_1' + M_1/d_2)(1 + M_1/d_2')).
$$

It follows that the contribution of this to $\Omega_{\neq 0}$ is dominated by

$$
\frac{C^2 q_1'^2 k p^{\ell - \ell_1/2} M_1^{1/2}}{p^{\ell/2} n_1 Q^2} \sup_{N' \leq N_0/n_1^2} N'
$$

$$
\sum_{d_2, d_2' \sim \sqrt{d_1}} \sum_{d_2' \sim d_1'} \sum_{q_2' \sim C/d_2 q_1'} \sum_{q_2'' \sim C/d_2 q_1'}
$$

$$
\times \sum_{1 \leq n_2 \leq N_2} (d_2', q_2' d_2 n_1') (d_2, n_2) \left( n_1' + \frac{M_1}{d_2} \right) \left( 1 + \frac{M_1}{d_2'} \right).
$$
Summing over \(n_2\) and \(q_2'\) we arrive at
\[
\frac{C^3 q_1' k p^{3(\ell-\ell_1)/2} M_1^{1/2}}{p^{r-\ell_{1}^1} n_1 Q^2} \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1} \sum_{d_2, d_2' \leq \frac{C}{q_1'}} \sum_{q_2' \sim \frac{C}{d_2 q_1'}} (d_2', q_2' d_2 n_1') \left( n_1' + \frac{M_1}{d_2} \right) \left( 1 + \frac{M_1}{d_2} \right),
\]
where we used
\[
\sup_{N' \ll \frac{CQ}{M}} N_1' \ll \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1}.
\]
Next summing over \(d_2'\) we get
\[
\frac{C^3 q_1' k p^{3(\ell-\ell_1)/2} M_1^{1/2}}{p^{r-\ell_{1}^1} n_1 Q^2} \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1} \sum_{d_2, d_2' \leq \frac{C}{q_1'}} \sum_{q_2' \sim \frac{C}{d_2 q_1'}} (n_1' + \frac{M_1}{d_2} \left( C' + M_1 \right) \left( C' + M_1 \right).
\]
Executing the remaining sums we get
\[
\frac{C^3 q_1' k p^{3(\ell-\ell_1)/2} M_1^{1/2}}{p^{r-\ell_{1}^1} n_1 Q^2} \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1} C \left( \frac{Cn_1' + M_1}{q_1'} \right) \left( \frac{C + M_1}{q_1'} \right).
\]
Now bounding \(M_1\) by \(M_0\) and using \(C < Q < M_0\), we get
\[
\Omega_{\neq 0} \ll \frac{C^3 q_1' k p^{3(\ell-\ell_1)/2} M_1^{1/2}}{p^{r-\ell_{1}^1} n_1 Q^2} \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1} C \frac{Cn_1' + M_1}{q_1'} \frac{C + M_1}{q_1'}
\]
\[
\ll \frac{C^5 k^2 p^{5(\ell-\ell_1)/2} M_0^{5/2}}{p^{r-\ell_{1}^1} n_1 Qq_1'} \frac{kCQp^{\ell_{1}-\ell}}{q_1' n_1} C \frac{Cn_1' + M_1}{q_1'} \frac{C + M_1}{q_1'}
\]
where we used \(M_0 \ll R^e p^{2r-\ell}\). On using the above bound in (88), we arrive at
\[
S_k(N) \ll \sup_{C < Q} \frac{p^{\ell_{1}/2} N^{5/4}}{Q p^{3r/2 + \epsilon} k^{1/2} C^2} \sum_{n_1' \leq C} \Theta^{1/2} \sum_{n_1' \leq C} \sum_{n_1' \leq C} \frac{C^{5/2} k p^{\ell_{1}/2 - 5\ell_1/4} p^{2r}}{n_1' q_1'} \sqrt{n_1 q_1'}
\]
\[
\ll \frac{p^{r-\ell - 3\ell_1/4} N^{5/4} k^{1/2}}{Q} \sum_{n_1' \leq C} \Theta^{1/2} \sum_{n_1' \leq C} \sum_{n_1' \leq C} \sum_{n_1' \leq C} \frac{1}{n_1' q_1'} \sqrt{n_1 q_1'} \Theta^{1/2}. \tag{91}
\]
Note that
\[
\sum_{n_1' \leq C} \frac{(n_1', k') \Theta^{1/2}}{n_1'^{3/2}} \leq \left[ \sum_{n_1' \leq C} \frac{(n_1', k')^2}{n_1'^{2}} \right]^{1/2} \left[ \sum_{n_1'^{2} n_2 \leq N_0} \frac{|A(n_1', n_2)|^2}{(n_1'^2 n_2)} \right]^{1/2} \ll \sqrt{k' R^e}. \tag{92}
\]
On plugging the above bound in (91), we get
\[
S_{k, \neq 0}(N) \ll \frac{p^{r-\ell - 3\ell_1/4} N^{5/4}}{Q} \ll p^{r/2} k N^{3/4}, \tag{93}
\]
where we used \(Q = \sqrt{N/p^e}\). Hence the lemma follows. \(\Box\)
10. CONCLUSION: PROOF OF THEOREM 1

In this section, we will conclude the proof of Theorem 1. Using bounds from Lemma 9.1 and Lemma 9.2, we see that

\[ S_k(N) \ll |S_{k,0}(N)| + |S_{k,\neq 0}(N)| \ll R_k N^{1/2} p^{3\ell/4 + 3\ell/4} + R_k k N^{3/4} p^{r-\ell/2}. \]

On dividing the above equation by \( k \sqrt{N} \), we get

\[ \frac{S_k(N)}{k \sqrt{N}} \ll R_k p^{3\ell/4 + 3\ell/4} + R_k N^{1/4} p^{r-\ell/2} \ll R_k p^{3\ell/4 + 3\ell/4} + R_k^2 p^{7\ell/4 - \ell/2}. \]

Optimizing the above bound by equating the terms on the right side, we get

\[ p^{3\ell/4 + 3\ell/4} = p^{7\ell/4 - \ell/2} \iff p^{5\ell/4} = p^{7\ell/4} \iff \ell = \lfloor 4 \ell / 5 \rfloor. \]

On plugging this in (10), we get

\[ L \left( \frac{1}{2}, \pi \times f \times \chi \right) \ll R_k p^{3\ell/4 + 3\ell/5} \ll R_k R^{3/2 - 3/20}. \] (94)

Hence Theorem 1 follows.

11. APPENDIX

In this section, we will give a rough sketch of the proof of sub-convexity of

\[ L(1/2, E_{\min} \times f \times \chi). \]

Following Lemma 3.1, the problem boils down to getting cancellations in the following sum:

\[ S(N) = \sum_{n=1}^{\infty} d_3(n) \lambda_f(n) \chi(n) W \left( \frac{n}{N} \right). \] (95)

After applying DFI and congruence equation trick (see Subsection 3.2), we arrive at (upto negligible error terms)

\[ S(N) = \frac{1}{Q p^f} \int_\mathbb{R} W_1(x) \sum_{1 \leq q \leq Q} \frac{g(q, x)}{q} \sum_{a \mod q \ b \mod p^f} \sum_{d_1 | k_2} d_3(n) e \left( \frac{(a + bq)n}{p^f q} \right) e \left( \frac{nx}{p^f q Q} \right) W \left( \frac{n}{N} \right) \]

\[ \times \lambda_f(m) \chi(m) e \left( -\frac{(a + bq)m}{p^f q} \right) e \left( \frac{-mx}{p^f q Q} \right) U \left( \frac{m}{N} \right). \] (96)

Next, we apply summation formulae to the above \( n \)-sum and \( m \)-sum. We first recall Voronoi formula for \( d_3(n) \). Set

\[ \sigma_{0,0}(k_1, k_2) = \sum_{d_1 | k_2} \sum_{d_2 d_3 | k_2} 1. \]

Let \( g \) be a compactly supported smooth function on \((0, \infty)\) and \( \tilde{g}(s) = \int_0^\infty g(x) x^{s-1} dx \) be its Mellin transform. For \( \ell = 0 \) and 1, we define

\[ \gamma_\ell(s) := \pi^{-3s-3/2} \left( \frac{\Gamma \left( \frac{1+s+\ell}{2} \right)}{\Gamma \left( \frac{3+2\ell}{4} \right)} \right)^3. \]
Set $\gamma_\pm(s) = \gamma_0(s) \mp \gamma_1(s)$ and let
\[
G_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_\pm(s) \tilde{g}(-s) \, ds,
\]
where $\sigma > -1$. With the aid of the above terminology, we now state the $GL(3)$ Voronoi summation formula in the following lemma:

**Lemma 11.1.** Let $g(x)$ and $d_3(n)$ be as above. Let $a, \bar{a}, q \in \mathbb{Z}$ with $c \neq 0, (a, c) = 1$, and $a\bar{a} \equiv 1(\text{mod } q)$. Then we have
\[
\sum_{n=1}^{\infty} d_3(n) e\left(\frac{an}{c}\right) g(n) = \frac{1}{2c^2} \tilde{g}(1) \sum_{n_1 | c} n_1 \tau(n_1) P_2(n_1, c) S \left( \tilde{a}, 0; \frac{c}{n_1} \right) + \frac{1}{2c^2} \tilde{g}'(1) \sum_{n_1 | c} n_1 \tau(n_1) P_1(n_1, c) S \left( \tilde{a}, 0; \frac{c}{n_1} \right) + \frac{1}{4c^2} \tilde{g}''(1) \sum_{n_1 | c} n_1 \tau(n_1) S \left( \tilde{a}, 0; \frac{c}{n_1} \right) + c \sum_{n_1 | c} \sum_{n_2 = 1}^{\infty} \sum_{n_3 | n_1 n_2} \sum_{n_4 | n_3} \sigma_{0,0} \left( \frac{n_1}{n_3 n_4}, n_2 \right) S \left( \tilde{a}, \pm n_2; \frac{c}{n_1} \right) G_\pm \left( \frac{n_2^2 n_3}{c^3} \right)
\]
where $P_1(n_1, c) = \frac{5}{3} \log n_1 - 3 \log c + 3 \gamma - \frac{1}{3\tau(n)} \sum_{d|n} \log d$, where $\gamma$ is the Euler constant and $P_2(n_1, c)$ is also some polynomial similar to $P_1(n_1, c)$ in $\log n_1$ and $\log c$.

**Proof.** See [30] for the proof. \qed

On applying the above lemma to the $n$-sum and the $GL(2)$ Voronoi formula to the $m$-sum in [39], we arrive at
\[
S(N) = S_{\text{error}}(N) + S_{\text{main},1}(N) + S_{\text{main},2}(N) + S_{\text{main},3}(N),
\]
where $S_{\text{error}}(N)$ is the expression of $S(N)$ (after the Voronoi formulæ) corresponding to last line of [39], $S_{\text{main},j}(N)$ is the expression of $S(N)$ corresponding to $(j+1)$-th line of [39] for $j = 1$, $2$ and $3$. We observe that the analysis of $S_{\text{error}}(N)$ is exactly similar to that of $S(N)$ in [39]. Thus, we will analyze $S_{\text{main},j}(N)$ only. Let’s consider $S_{\text{main},1}(N)$. Let’s assume $q \sim Q$ for simplicity (generic case). After Voronoi formula, $n$-sum has transfered to
\[
\frac{1}{2(p^f q)^2} \tilde{g}(1) \sum_{n_1 | p^f q} n_1 \tau(n_1) P_2(n_1, p^f q) S \left( \frac{a + bq}{p^f}, 0; \frac{p^f q}{n_1} \right).
\]
Assuming square root cancellations (which we will get on average over $a$) in the Kloosterman sum, we see the the above sum is bounded by $N/(p^f q)$, as $\tilde{g}(1) \ll N$. Thus we save $p^f q$ over the trivial bound which is $N$. Analysis of $GL(2)$ Voronoi formula will give us a saving of size $N/(p^f q)$ over the trivial bound $N$. Moreover, on analysing the sum over $a$ and $b$ like before, we save $\sqrt{q} \sqrt{p^f}$. Thus, in total, we have saved
\[
p^f q \times \frac{N}{p^f q} \times \sqrt{q} \sqrt{p^f} = \frac{N \sqrt{qp}^{3f/2}}{p^f}
\]
over the trivial bound $N^2$. This is sufficient as long as

$$\frac{N\sqrt{q}p^{3\ell/2}}{p^r} > N \iff \frac{\sqrt{N}p^{3\ell}}{p^{r/2}} > p^{2r} \iff p^{5\ell/2} > p^{r/2}. $$

By our choice of $\ell$ which is $\ell = 4r/5$, we get the subconvexity.

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