Inducing $\pi$-partial characters with a given vertex

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Abstract

Let $G$ be a solvable group. Let $p$ be a prime and let $Q$ be a $p$-subgroup of a subgroup $V$. Suppose $\varphi \in \text{IBr}(G)$. If either $|G|$ is odd or $p = 2$, we prove that the number of Brauer characters of $H$ inducing $\varphi$ with vertex $Q$ is at most $|N_G(Q) : N_V(Q)|$.

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1 Introduction

Throughout this note, $G$ is a finite group, and $\text{Irr}(G)$ is the set of irreducible characters of $G$. Suppose $\chi \in \text{Irr}(G)$ and $H$ is a subgroup of $G$. It is easy to obtain an upper bound on the number of characters in $\text{Irr}(H)$ that induce $\chi$. Let $\varphi_1, \ldots, \varphi_n \in \text{Irr}(H)$ be the characters so that $\varphi_i^G = \chi$. Evaluating at 1, we obtain $\varphi_i(1) = \chi(1)/|G : H|$ for each $i$. By Frobenius reciprocity (Lemma 5.2 of [5]), each $\varphi_i$ is a constituent of $\chi_H$ with multiplicity 1. Since there are $n$ such characters occurring as constituents of $\chi_H$, it follows that $n(\chi(1)/|G : H|) \leq \chi(1)$. We deduce that $n \leq |G : H|$, and we have an upper bound. If $H$ is normal in $G$, this bound is obtained, and it is not particularly difficult to find nonnormal subgroups where this bound is obtained.

We now turn our attention to Brauer characters. Fix a prime $p$. We will write $\text{IBr}(G)$ for the irreducible $p$-Brauer characters of $G$. If $\varphi \in \text{IBr}(G)$, then it is easy to adapt the above proof to show that $\varphi$ is induced by at most $|G : H|$ Brauer characters of $H$. However, associated with $\varphi$ are certain $p$-subgroups of $G$ called the vertex subgroups. When $G$ is a $p$-solvable group, a $p$-subgroup $Q$ of $G$ is defined to be a vertex for $\varphi$ if there is a subgroup $U$ of $G$ so that $\varphi$ is induced by a Brauer character of $U$ with $p'$-degree and $Q$ is a Sylow subgroup of $U$. It is known that all the vertex subgroups of $\varphi$ are conjugate in $G$. If $\varphi$ is induced from $\tau \in \text{IBr}(H)$, it is easy to see that a vertex for $\tau$ is a vertex for $\varphi$. Thus, $H$ contains some vertex $Q$ for $\varphi$. Now, different Brauer characters of $H$ that induce $\varphi$ may have vertex subgroups that are not conjugate in $H$ but are necessarily conjugate in $G$. Hence, one can ask the following question: Suppose $\varphi \in \text{IBr}(G)$ has vertex $Q$, and $Q \leq H$, how many...
characters in $\text{IBr}(H)$ with vertex $Q$ induce $\varphi$? When either $|G|$ is odd or $G$ is solvable and $p = 2$, we can obtain an upper bound for this question.

**Theorem 1.** Let $G$ be a solvable group and $p$ a prime. Assume either $|G|$ is odd or $p = 2$. Let $Q$ be a $p$-subgroup of $H$. If $\varphi \in \text{IBr}(G)$, then the number of Brauer characters of $H$ with vertex $Q$ that induce $\varphi$ is at most $|N_G(Q) : N_H(Q)|$.

At this time, we are not able to determine whether or not this theorem is true if we loosen the hypothesis that either $|G|$ is odd or $p = 2$. In other words, is the conclusion still true if $G$ is a solvable group of even order and $p$ is an odd prime. This result was motivated by our work with J. P. Cossey. If we could prove the conclusion of Theorem 1 when $p$ is odd, then we would be able to prove J. P. Cossey’s conjecture that the number of lifts of a Brauer character is bounded by the index of a vertex subgroup in the vertex subgroup when $p$ is odd. Our argument can be found in the preprint [1].

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## 2 Results

We will in the more general setting of irreducible $\pi$-partial characters of a $\pi$-separable group $G$. We here briefly mention that if $\pi$ is a set of primes and $G$ is a $\pi$-separable group, one can define (see [10] for more details) a set of class functions $I_{\pi}(G)$ from the set $G^\pi$ (which consists of the elements of $G$ whose order is divisible by only the primes in $\pi$) to $\mathbb{C}$ that plays the role of $\text{IBr}(G)$, and in fact $I_{\pi}(G) = \text{IBr}(G)$ if $\pi = \{p\}'$, the complement of the prime $p$.

We start by considering vertices in Clifford correspondence (see Proposition 3.2 of [9]). Let $G$ be a $\pi$-separable group. Let $N$ be a normal subgroup of $G$. Fix $\varphi \in I_{\pi}(G)$. If $\alpha \in I_{\pi}(N)$ is a constituent of $\varphi_N$, then we write $G_{\alpha}$ for the stabilizer of $\alpha$ in $G$, and we write $\varphi_{\alpha}$ for the Clifford correspondent of $\varphi$ with respect to $\alpha$. In particular, the vertices of the Clifford correspondent form an orbit under the action of the normalizer of a particular vertex.

**Lemma 2.** Let $G$ be a $\pi$-separable group. Let $N$ be a normal subgroup of $G$. Suppose that $\alpha \in I_{\pi}(N)$. Let $\varphi \in I_{\pi}(G)$ and $\hat{\varphi} \in I_{\pi}(G_{\alpha})$ so that $\varphi^G = \varphi$. Suppose that $Q$ is a vertex for $\hat{\varphi}$. Then $Q$ is a vertex for $\hat{\varphi}^\alpha$ if and only if there exists $n \in N_G(Q)$ so that $G_{\alpha}^g = G_{\alpha}n$.

**Proof.** We first suppose that there exists $n \in N_G(Q)$ so that $G_{\alpha}^g = G_{\alpha}n$. Thus, $g = tn$ for some $t \in G_{\alpha}$. We see that $\varphi^g = \varphi^{tn} = \varphi^n$. We see that $Q = Q^n$ is a vertex for $\varphi = \varphi^n$.

Conversely, suppose that $Q$ is a vertex for $\varphi^g$. Then $Q^{g^{-1}}$ is a vertex for $\varphi$. Since $Q$ is also a vertex for $\varphi$, we have $Q^{g^{-1}} = Q^t$ for some $t \in G_{\alpha}$. It follows that $Q = Q^g$, and so, $tg \in N_G(Q)$. This implies that $tg = n$ for some $n \in N_G(Q)$. This implies that $n \in G_{\alpha}^g$, and we conclude that $G_{\alpha}^g = G_{\alpha}n$. 

We continue to work in the context of the Clifford correspondence. In this case, we can get an exact count of the number of partial characters in $N$ whose Clifford correspondent has vertex $Q$. 


Corollary 3. Let $G$ be a $\pi$-separable group. Let $N$ be a normal subgroup of $G$, let $\varphi \in I_\pi(G)$ have vertex $Q$, and suppose that $\beta$ is an irreducible constituent of $\varphi_N$ so that $\varphi_\beta$ has vertex $Q$. Then $|\{ \alpha \in I_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q \}| = |N_G(Q) : N_{G_\beta}(Q)|$.

Proof. By Lemma 2 we see that $\varphi_\alpha$ has $Q$ as a vertex if and only if $\alpha = \beta^g$ where $g \in G$ and $g \in G_\beta n$ for some $n \in N_G(Q)$. Finally, we observe that $G_\beta n_1 = G_\beta n_2$ if and only if $N_{G_\beta}(Q)n_1 = N_{G_\beta}(Q)n_2$ for $n_1, n_2 \in N_G(Q)$. We have $|\{ \alpha \in I_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q \}| = |\{G_\beta n \mid n \in N_G(Q)\}| = |N_G(Q) : N_{G_\beta}(Q)|$. 

We now look at the conditions of a minimal counterexample. For this we need to review and develop more notation. We make use of the canonical set of $\pi$-lifts, $B_\pi(G)$, that was defined in [5] by Isaacs. In other words, $B_\pi(G) \subseteq \text{Irr}(G)$ and the map $\chi \mapsto \chi^o$ is a bijection from $B_\pi(G)$ to $I_\pi(G)$. Closely related to this set is the subnormal nucleus which also was defined in [5]. To define the subnormal nucleus, we need the $\pi$-special characters. Let $G$ be a $\pi$-separable group. A character $\chi \in \text{Irr}(G)$ is $\pi$-special if $\chi(1)$ is a $\pi$-number and for every subnormal group $M$ of $G$, the irreducible constituents of $\chi_M$ have determinants that have $\pi$-order. Many of the basic results of $\pi$-special characters can be found in Section 40 of [2] and Chapter VI of [13]. One result that is proved is that if $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special, then $\alpha \beta$ is necessarily irreducible. We say that $\chi$ is factored if $\chi = \alpha \beta$ where $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special. We also note that if $\chi \in B_\pi(G)$ and $N$ is normal in $G$, then the irreducible constituents of $\chi_N$ lie in $B_\pi(N)$.

If $\chi \in \text{Irr}(G)$, Isaacs constructs the subnormal vertex as follows. Let $M$ be maximal so that $M$ is subnormal in $G$ and the irreducible constituents of $\chi_M$ are factored. Let $\mu$ be an irreducible constituent of $\chi_M$ and let $T$ be the stabilizer of $(M, \mu)$ in $G$. Isaacs proved in [5] that there is a Clifford theorem for $T$. In other words, there is a unique character $\tau \in \text{Irr}(T \mid \mu)$ so that $\tau^G = \chi$. He also proved that $(M, \mu)$ is unique up to conjugacy, and so, $(T, \tau)$ is unique up to conjugacy. If $T = G$, then $\chi$ is $\pi$-factored and we take $(G, \chi)$ to be the subnormal nucleus of $\chi$. If $T < G$, then inductively, the subnormal nucleus for $\tau$ is the subnormal nucleus for $\chi$. We write $(W, \gamma)$ for the subnormal nucleus of $\chi$, and Isaacs showed that $\gamma^G = \chi$, $\gamma$ is factored, and $(W, \gamma)$ is unique up to conjugacy. A character $\chi \in \text{Irr}(G)$ is in $B_\pi(G)$ if and only if the character of its nucleus is $\pi$-special.

If $Q$ is a $\pi'$-subgroup of $G$, then we use $I_\pi(G \mid Q)$ to denote the $\pi$-partial characters in $I_\pi(G)$ that have vertex $Q$. If $\varphi \in I_\pi(G)$ and $V \leq G$, then we write $I_\pi(V \mid Q) = \{ \eta \in I_\pi(V \mid Q) \mid \eta^G = \varphi \}$. We now find details about properties of a minimal counterexample. We will see that a counterexample cannot occur when either $|G|$ is odd or 2 is not in $\pi$. Our goal is find enough information so that we can either find a contradiction or build an example when $|G|$ is even and $2 \in \pi$.

Theorem 4. Let $G$ be a solvable group. Assume $\varphi \in I_\pi(G)$ has vertex $Q$, let $V$ be a subgroup of $G$, and let $N$ be the core of $V$ in $G$. If $G$ and $V$ are chosen so that $|G| + |G : V|$ is minimal subject to the condition that $|I_\pi(V \mid Q)| > |N_G(Q) : N_{V}(Q)|$, then the following are true:

1. $V$ is a nonnormal maximal subgroup of $G$,
2. $|G : V|$ is a power of 2,
3. $2 \in \pi$,
4. $Q \leq V$,
5. $\varphi_N = a\alpha$ for some $\alpha \in \text{IBr}(N)$,
6. $\alpha(1)$ is a $\pi$-number,
7. if $K$ is normal in $G$ so that $K/N$ is a chief factor for $G$, then $\alpha$ is fully ramified with respect to $K/N$.

Proof. If either $V = G$ or $I_\varphi(V \mid Q)$ is empty, then $|I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|$ contradicting the hypotheses. Thus, $V < G$ and $I_\varphi(V \mid Q)$ is not empty, and so, $Q \leq V$ and there exist characters in $I_\varphi(V)$ that induce $\varphi$ and have vertex $Q$.

We begin by showing that $V$ is a maximal subgroup. Suppose that $V < M < G$ for some subgroup $M$. Let $I_\varphi(M \mid Q) = \{\eta_1, \ldots, \eta_m\}$. Using minimality, we have $m = |I_\varphi(M \mid Q)| \leq |N_G(Q) : N_M(Q)|$. Suppose that $\zeta \in I_\varphi(V \mid Q)$, then $\zeta^M \in I_\varphi(M)$ and $\zeta^M$ has $Q$ as a vertex. Since $(\zeta^M)^G = \zeta^G = \varphi$, we see that $\zeta^M \in I_\varphi(M \mid Q)$. It follows that $\zeta^M = \eta_i$ for some $i$. We conclude that $|I_\varphi(V \mid Q)| \leq \sum_{i=1}^m |I_{\eta_i}(V \mid Q)|$. Since this contradicts our hypothesis, we obtain $|I_{\eta_i}(V \mid Q)| \leq |N_M(Q) : N_V(Q)|$. We deduce that

$$|I_\varphi(V \mid Q)| \leq m|N_M(Q) : N_V(Q)| \leq |N_G(Q) : N_M(Q)||N_M(Q) : N_V(Q)| = |N_G(Q) : N_V(Q)|.$$

Since this violates the hypotheses, $V$ is maximal in $G$.

If $V$ is normal in $G$, then either $\varphi$ is induced from $V$ or $\varphi$ restricts irreducibly to $V$. If $\varphi$ is induced from $V$, then we can apply Corollary to see that $|I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|$ in violation of the hypotheses. If $\varphi$ restricts irreducibly, then it cannot be induced from $V$, and we have seen that this is also a contradiction. We conclude that $V$ is not normal in $G$.

Suppose $\alpha \in I_\varphi(N)$ is a constituent of $\varphi_N$. We use $\varphi_o \in I_\varphi(G \alpha \mid \alpha)$ to denote the Clifford correspondent for $\varphi$ with respect to $\alpha$ (see Proposition 3.2 of [9] again). Write $\{\alpha \in I_\varphi(N) \mid \varphi_o \text{ has vertex } Q\} = \{\alpha_1, \ldots, \alpha_k\}$, and let $\varphi_i = \varphi_o$, and $G_i = G_{\alpha_i}$. By Lemma we know that $k = |N_G(Q) : N_{G_i}(Q)|$.

Suppose $\eta \in I_\varphi(V \mid Q)$. Denote $\{\beta \in I_\varphi(N) \mid \eta\text{ has vertex } Q\} = \{\beta_1, \ldots, \beta_l\}$, and let $\eta_j = \eta_{\beta_j}$ and $V_j = V_{\beta_j}$. By Lemma $l = |N_V(Q) : N_V(Q)|$.

We see that $(\eta_j)^G = ((\eta_j)^V)^G = \eta^G = \varphi$. This implies that $(\eta_j)^{G_{\beta_j}}$ is irreducible and has vertex $Q$. It follows that $\beta_j = \alpha_{i_j}$ for some $i_j$. We obtain $G_{\beta_j} = G_{i_j}$ and $(\beta_j)^{G_{i_j}} = \alpha_{i_j}$. Observe that $V_j = G_{i_j} \cap V$, and we denote this subgroup by $V_{i_j}^*$.

Now, we assume that $k > 1$, and we start to count. We see that $\eta \in I_\varphi(G \mid Q)$ is induced by $|N_V(Q) : N_{V_{i_j}^*}(Q)|$ partial characters in $\bigcup I_{\varphi_i}(V_{i_j}^* \mid Q)$. Because $G_i < G$, we may use minimality of $|G| + |G : V|$ to deduce $|I_{\varphi_i}(V_{i_j}^* \mid Q)| \leq |N_{G_i}(Q) : N_{V_{i_j}^*}(Q)|$. We compute

$$|I_\varphi(V \mid Q)| = \sum_{i=1}^k \frac{1}{|N_V(Q) : N_{V_{i_j}^*}(Q)|}|I_{\varphi_i}(V_{i_j}^* \mid Q)| \leq \sum_{i=1}^k \frac{1}{|N_V(Q) : N_{V_{i_j}^*}(Q)|}|N_{G_i}(Q) : N_{V_{i_j}^*}(Q)|.$$

We determine that

$$\frac{1}{|N_V(Q) : N_{V_{i_j}^*}(Q)|}|N_{G_i}(Q) : N_{V_{i_j}^*}(Q)| = \frac{|N_{G_i}(Q)|}{|N_V(Q)|},$$
for each $i$. Notice that $|N_{G_i}(Q)| = |N_{G_i}(Q)|$ for all $i$ and $k = |N_G(Q) : N_{G_i}(Q)|$. This yields

$$|I_\varphi(V \mid Q)| \leq \sum_{i=1}^k \frac{|N_{G_i}(Q)|}{N_V(Q)} = \frac{|N_G(Q) : N_{G_i}(Q)||N_{G_i}(Q)|}{|N_V(Q)|} = |N_G(Q) : N_V(Q)|.$$  

This contradicts the hypothesis. We deduce that $k = 1$, and $\alpha$ is invariant in $G$.

Set $\alpha = \alpha_1$, and let $\alpha^*$ be the character in $B_\pi(N)$ satisfying $(\alpha^*)^\circ = \alpha$. Write $(W, \hat{\alpha})$ for the nucleus of $\alpha^*$, and take $T$ to be the stabilizer of $(W, \hat{\alpha})$ in $G$. By Lemma 2.3 of \cite{11}, there is a unique character $\hat{\varphi} \in I(T \mid \hat{\alpha})$ so that $\varphi^G = \varphi$ and $Q$ is a vertex for $\hat{\varphi}$. Similarly, if $\eta \in I_\varphi(V \mid Q)$, then there is a unique character $\hat{\eta} \in I(T \cap V \mid \hat{\alpha})$ so that $\hat{\eta}^V = \eta$ and $Q$ is a vertex for $\hat{\eta}$. Observe that $\eta^T \in I(T \mid \hat{\alpha})$ and induces $\varphi$, so $\hat{\eta}^T = \hat{\varphi}$. It follows that $|I_\varphi(V \mid Q)| = |I_{\hat{\varphi}}(T \cap V \mid Q)|$. If $T < G$, then we can use the minimality of $|G| + |G : V|$ to see that $|I_{\hat{\varphi}}(T \cap V \mid Q)| \leq |N_T(Q) : N_{V \cap T}(Q)|$. By the diamond lemma, we have $|N_T(Q) : N_{V \cap T}(Q)| = |N_T(Q) : V \cap N_T(Q)| \leq |N_G(Q) : N_V(Q)|$. This contradicts the hypotheses, and so $T = G$.

We now have that $(W, \hat{\alpha})$ is $G$-invariant. By the construction of the subnormal, this implies that $W = N$. Since $\alpha^* \in B_\pi(N)$, the nucleus for $\alpha^*$ has a character that is $\pi$-special. Thus, $\hat{\alpha}$ is $\pi$-special, and since $W = N$, we see that $\hat{\alpha} = \alpha^*$. In particular, $\hat{\alpha}$ is $\pi$-special. We deduce that $\alpha(1)$ is a $\pi$-number.

Take $K$ normal in $G$ so that $K/N$ is a chief factor for $G$. This is the point where we use the fact that $G$ is solvable to see that $G = VK$ and $V \cap K = N$ where $K/N$ is an elementary abelian $p$-group for some prime $p$. (This is the only place we use the hypothesis that $G$ is solvable in place of $G$ being $\pi$-separable.) Let $L/K$ be a chief factor for $G$. We know that $([L : K], |K : N|) = 1$ and $C_{L/N}(K/N)$. (See Lemma 5.1 of \cite{12} for a proof of this.) By Problem 6.12 of \cite{9}, either $\alpha^*$ extends to $K$ or $\alpha^*$ is fully-ramified with respect to $K/N$.

Suppose first that $\alpha^*$ extends to $K$. Notice that multiplication by $\text{Irr}(K/L)$ is a transitive action on the irreducible constituents of $(\alpha^*)^K$. Also, $(V \cap K)/L$ acts on compatibly on the irreducible constituents of $(\alpha^*)^K$ and on $\text{Irr}(K/L)$ where the action on $\text{Irr}(K/L)$ is coprime. We can use Glauberman’s lemma (Lemma 13.8 of \cite{13}) to see that $\alpha^*$ has a $V \cap L$-invariant extension. The corollary to Glauberman’s lemma (Corollary 13.9 of \cite{13}) can be applied to see that $\alpha^*$ has a unique $V \cap L$-invariant extension $\delta$. Since $V$ permutes the $V \cap L$-extensions of $\alpha^*$, it follows that $\delta$ is $V$-invariant. We now use Corollary 4.2 of \cite{13} to see that restriction is a bijection from $\text{Irr}(G \mid \beta)$ to $\text{Irr}(V \mid \alpha^*)$.

Let $\eta \in I_\varphi(V \mid Q)$ so that $\hat{\varphi}^G = \varphi$. We can find $\eta^* \in B_\pi(V)$ so that $(\eta^*)^\circ = \eta$. Since $(\eta^*)^G = (\eta^*)^G = \eta^G = \varphi \in \text{IBr}(G)$, we see that $\eta^G$ is irreducible. On the other hand, $(\eta^*)^N = (\eta^*)^N = \beta a$ for some integer $b$. Since the irreducible constituents of $\eta^*_N$ lie in $B_\pi(N)$, we deduce that $\eta^* \in \text{Irr}(V \mid \alpha^*)$. But we saw that this implies that $\eta^*$ extends to $G$. Since $V < G$, it is not possible for $\eta^*$ to both extend to $G$ and induce irreducibly. Therefore, we have a contradiction. We see that $\alpha^*$ (and hence, $\alpha$) is fully-ramified with respect to $K/N$. Notice that if $p$ is not in $\pi$, then Corollary 6.28 of \cite{13} applies and $\alpha^*$ extends to $K$. Therefore, $p \in \pi$.

We suppose that $p$ is odd, and we work for a contradiction. Since $\alpha^*$ is fully-ramified with respect to $K/N$ and $|K : N|$ has odd order, main theorem of \cite{7} implies that no character in $\text{Irr}(V \mid \alpha)$ induces irreducibly to $G$. (A stronger theorem is proved in \cite{11}.) As in the previous paragraph, this implies that $\varphi$ is not induced from $V$ which contradicts the assumption that
I_\varphi(V \mid Q) is not empty. (This strongly uses the fact that p is odd. When p = 2, it is tempting to try use the correspondence in [6], but that correspondence does not preclude inducing characters in Irr(G \mid \alpha) from V. In fact, GL_2(3) is an example where this occurs.) We conclude that p = 2. Since \(|G : V| = |K : N|\), we see that |G : V| is a power of 2. This proves the theorem.

As a corollary, we obtain Theorem 1 stated for \(\pi\)-partial characters.

**Corollary 5.** Let G be a solvable group. Assume either |G| is odd or 2 \(\not\in\) \(\pi\). Let Q be a \(\pi'\)-subgroup of G and suppose that Q \(\leq\) V. If \(\varphi \in I_{\pi}(G)\), then |I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|.

**Proof.** We suppose the result is not true. Let G be a counterexample with |G| + |G : V| as in Theorem 1. By that result, we have that |G : V| is a nontrivial power of 2 which is a contradiction if |G| is odd. We also have 2 \(\in\) \(\pi\) which is a contradiction to 2 \(\not\in\) \(\pi\). This proves the corollary.

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