Research Article

The Dynamic Response and Failure Model of Thin Plate Rock Mass under Impact Load

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The layered rock mass widely exists in mining, construction, transportation, and water conservancy projects, and the damage phenomena of plate crack and spalling often occurs in the process of coal and rock dynamic disaster in deep mining. Therefore, the rock mass nearby excavation surface is usually considered to be composed of layers of thin plate rock mass to reveal the damage and failure mechanism of rock mass. In the whole dynamic process of mining and coal and rock dynamic disaster, rock mass would bear the dynamic disturbance from mine earthquake, and at present, the mechanical characteristics of rock mass are mainly studied under static load, while dynamic mechanical response characteristics and the mechanisms of dynamic damage, failure, and disaster-causing are still unclear. This study mainly focused on the dynamic response characteristic and failure mechanism of rock mass based on a rectangular thin plate model. The frequency equations and deflection equations of the thin plate rock mass with different boundary conditions (S-F-S-F, S-C-S-C, and C-C-C-C) were established under free vibration by the thin plate model and the dual equation of the Hamilton system, and the deflection equations under impact load were derived based on the Duhamel integral. And then, the effective vibration modes of the thin plate rock mass with different boundary conditions and their natural frequencies were obtained by Newton’s iterative method. Based on the third-strength theory and the numerical simulation results by LS-DYNA, the maximum shear of the effective vibration modes and the processes of damage and failure under impact load were analyzed. The research results showed that the initial position of damage and failure may be determined by effective vibration mode with the lowest frequency; the develop tendency of which by the combined actions of other effective vibration modes and the effective vibration modes with lower frequency could have greater influence on the process of damage and failure of the thin plate rock mass, which are beneficial to revealing the mechanism of coal and rock dynamic disaster.

1. Introduction

Thin plate theory has been widely used and achieved good results in the research of stope movement and the failure mechanism of stope roof. The sedimentary process and the arrangement of mineral particles during the formation of coal measure rock mass make it have obvious bedding structure and show layered structure macroscopically [1–4]. This layered rock mass widely exists in mining, construction, transportation, and water conservancy projects, such as slate [5], shale [6], sandstone [7], and phyllite [8]. During the excavation of underground engineering, a series of layered rock mass failure is a key factor for major geological disasters. For example, the water inrush accident in Chaoyang Tunnel of Luo Tuoshan and Gui Nan High-Speed Railway and the surface collapse in Ma Jiliang Mine of Datong were all caused by the instability of layered rock structure. A group of dominant parallel bedding exists in layered rock mass generally, which are usually regarded as transversely isotropic thin plates in mechanics. Therefore, many scholars have introduced the thin plate theory into the calculation of stope movement and put forward the theory of key strata [9].

and stope thin plate pressure [10], the fracture law of basic roof strata structure [11–13], and the limit analysis method of strata movement [14]. Based on the thin plate theory, the “O-X” and “K” fracture patterns of stope roof were deduced in overlayer strata movement [15–20], and the specific location of crack bifurcation and formation process of spatial “X” crack pattern were obtained [21, 22]. Currently, this theory is widely used in theoretical research, experimental research, and numerical simulation to reveal the laws of plate rock failure and stope movement.

Thin plate rock mass model has been used widely to reveal the mechanism and process of coal and rock dynamic disaster in deep mining where coal and gas outburst, rockburst, rockfall, and chamber instability are prone to occur and usually accompanied by the damage phenomena of plate crack and spalling in stress concentration areas [23–28] (Figure 1).

On the one hand, the excavation of coal and rock mass in heading face of deep mine would lead to stress redistribution in front of heading face, and the distribution of stress concentration state will present layered phenomenon. And the tangential stress increases and the radial stress decreases in the surrounding rock of the heading face. Under the loading, the cracks in the rock expand along the maximum principal stress, which can form a plate failure surface roughly parallel to the heading face [29]. On the other hand, the elastic compressive strain energy stored in the coal and rock mass with high stress in deep area would be released and reflected on the excavation surface to form tensile stress wave because of the unloading of dynamic load caused by excavation. When the tensile stress exceeds the tensile strength of coal and rock mass, the failure surface with plate crack will be formed and roughly parallel to the excavation surface. And the rapid propagation of crack will split the surrounding rock into several rock plates and form thin plate rock finally [30]. Currently, based on the thin plate model, the mechanical behaviors of plate cracking and spalling failure of rock mass were studied deeply and widely [31–34] (Figure 1).

In the whole dynamic process of mining, and during the beginning, formation, development, and termination of coal and rock dynamic disaster, rock mass will bear the dynamic disturbance from mine earthquake, such as blasting, roof and floor breaking, and rock instability. At present, the mechanical characteristics of rock mass are mainly studied under static load, while the research studies on dynamic mechanical response characteristics are only in the stage of experimental exploration, and the mechanisms of dynamic damage, failure, and disaster-causing are still unclear [35–40], because the vibration of rock mass under external force has a significant impact on its mechanical properties [35, 37, 41].

This study mainly focused on the dynamic response characteristic and failure mechanism of rock mass based on a rectangular thin plate model. The Hamilton dual system was introduced to research the dynamic response characteristics of thin plate rock mass under external forced vibration. Besides, according to the characteristics of uniform impact load, the effective vibration mode solution was studied by using Duhamel integral principle. And the process and mechanism of dynamic damage and failure of thin plate rock mass were analyzed, which are beneficial to revealing the mechanism of coal and rock dynamic disaster.

2. Free Vibration Model of the Thin Plate Rock Mass

2.1. Control Equation of Free Vibration of the Thin Plate Rock Mass. In this study, the assumptions were as follows: the rectangular plates of thin rock mass are transversely isotropic in mechanics; the density of the thin plate was \( \rho \); the thickness was \( h \); the size was \( a \times b \); the elastic modulus was \( E \); and Poisson’s ratio was \( v \). Based on the transient equilibrium conditions of the internal mechanics of the thin plate, the differential equation of thin plate rock mass can be derived under forced vibration [42]:

\[
DV^4 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = q(x, y, t),
\]

where \( w(x, y, t) \) is the deflection, \( D \) is the bending stiffness, and \( q(x, y, t) \) is external dynamic load.

For the purpose to solve the homogeneous equation of equation (1), we set \( q(x, y, t) = 0 \) and insert it into (1), and the free vibration differential equation of the thin plate could be obtained:

\[
DV^4 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0.
\]

The vibration of the thin plate is the harmonic oscillator with time. Therefore, \( w(x, y, t) \) could be expressed as follows:

\[
w(x, y, t) = W(x, y)e^{\omega t},
\]

where \( \omega \) is the natural frequency and \( W(x, y) \) is the deflection mode function.

With (2),

\[
\frac{\partial^4 W}{\partial x^4} + \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = k^4 W,
\]

where \( k^4 = \rho \omega^2 / D \).

2.2. Hamilton Dual Vibration Equation of the Thin Plate Rock Mass. In order to decouple the physical quantities in equation (4), it was necessary to introduce the Hamilton dual equations. If we set \( \theta = \partial W / \partial x \), according to the mechanical analysis results of the thin plate, the relationship between the physical quantities may be as follows [43, 44]:

\[
\frac{\partial \theta}{\partial x} = -\frac{M_x}{D} - \nu \frac{\partial^2 W}{\partial y^2},
\]

\[
\frac{\partial V_x}{\partial x} = D(1 - \nu^2) \frac{\partial^4 W}{\partial y^4} - \nu \frac{\partial^2 M_x}{\partial y^2} - \rho \omega^2 W,
\]

\[
\frac{\partial M_x}{\partial x} = V_x + 2D(1 - \nu) \frac{\partial^2 \theta}{\partial y^2},
\]
where \( M_\alpha, M_\beta, M_{xy}, Q_\alpha, Q_\beta, V_\alpha \) and \( V_\beta \) are the bending moment, torque, transverse shear force, and transverse total shear force, respectively. If we set vector \( \nu = [W, \theta, -V_\alpha, M_\alpha] \), then the dual equation of the Hamilton system could be expressed as follows [45]:

\[
\begin{bmatrix}
\frac{\partial W}{\partial x} \\
\frac{\partial \theta}{\partial x} \\
\frac{\partial (-V_x)}{\partial x} \\
\frac{\partial M_\alpha}{\partial x}
\end{bmatrix} = \begin{bmatrix}
0 & -\nu \frac{\partial^2}{\partial y^2} \\
-\nu \frac{\partial^2}{\partial y^2} & 0 & 0 & \frac{1}{D} \\
-D(1-v^2) \frac{\partial^4}{\partial y^4} + \rho \omega^2 h & 0 & 0 & \nu \frac{\partial^2}{\partial y^2} \\
2D(1-v^2) \frac{\partial^2}{\partial y^2} & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
W \\
\theta \\
V_x \\
M_\alpha
\end{bmatrix}
\] 

Then,

\[
\nu' = Hv.
\] 

2.3. Solution of the Hamilton Dual Equations. Based on the symplectic geometry method, the solutions of (6) could be obtained by the separation variable method [43]. And if \( W(x) \) and \( W(y) \) were the deflection modes along the \( x \) and \( y \) directions, respectively, their specific forms may be as follows [46, 47]:

\[
W(x) = a_1 e^{\beta_1 x} + b_1 e^{-\beta_1 x} + c_1 e^{\beta_2 x} + d_1 e^{-\beta_2 x} = A_1 \cos \beta_1 x + B_1 \sin \beta_1 x + C_1 \cosh \beta_2 x + D_1 \sinh \beta_2 x,
\]

(8)

\[
W(y) = a_2 e^{\alpha_1 y} + b_2 e^{-\alpha_1 y} + c_2 e^{\alpha_2 y} + d_2 e^{-\alpha_2 y} = A_2 \cos \alpha_1 y + B_2 \sin \alpha_1 y + C_2 \cosh \alpha_2 y + D_2 \sinh \alpha_2 y,
\]

(9)

where \( \alpha_1 \) and \( \alpha_2 \) are characteristic values in the \( x \) direction; \( \beta_1 \) and \( \beta_2 \) are characteristic values in the \( y \) direction. The parameters \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) meet the following rules:

\[
\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2 = 2k^2, \quad \alpha_1^2 + \beta_1^2 = k^2, \quad \alpha_2^2 + \beta_2^2 = 3k^2, \quad \alpha_2^2 - \beta_1^2 = k^2.
\]

3. Deflection Equation of the Thin Plate Rock Mass under Free Vibration

It could be considered that the rock mass nearby excavation surface is composed of layers of thin plate rock mass in mining. In order to deduce the deflection equation of thin plate rock mass under free vibration, its boundary conditions need to be introduced, which is fixed (C) on four sides before failure, with the failure of rock mass near boundary caused by external dynamic load, these thin plate rock mass will gradually become simply supported (S) or free (F) state. Therefore, it is necessary to analyze the deflection equations of thin plate rock mass under different boundary conditions.

3.1. Deflection Equation of the Thin Plate Rock Mass S-F-S-F under Free Vibration. The deflection equation of the thin plate rock mass with both sides simply supported (S-S) along the \( x \) direction could be expressed as follows [46–48]:

\[
W(x) = \sin \beta_1 x.
\]

(10)
The boundary conditions of the thin plate rock mass with both free sides (F-F) along the y direction may be as follows:

\[
\frac{\partial^2 W(x, 0)}{\partial y^2} - v \frac{\partial^2 W(x, b)}{\partial x^2} = 0, \quad \frac{\partial^2 W(x, b)}{\partial y^2} - v \frac{\partial^2 W(x, 0)}{\partial x^2} = 0,
\]

\[
\frac{\partial^3 W(x, 0)}{\partial y^3} - (2 - v) \frac{\partial^3 W(x, b)}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 W(x, b)}{\partial y^3} - (2 - v) \frac{\partial^3 W(x, 0)}{\partial x^2 \partial y} = 0.
\]  \hspace{1cm} (11)

With (8) and (9),

\[
\ddot{W}(y) - v \ddot{W}(y) = 0,
\]  \hspace{1cm} (12)

\[
\ddot{W}(y) - (2 - v) \ddot{W}(y) = 0.
\]  \hspace{1cm} (13)

Then, we insert (9) into (12) and (13):

\[
\begin{bmatrix}
 j \cos \alpha_1 b & j \sin \alpha_1 b & f \cos \beta_1 b & f \sin \beta_1 b \\
 0 & m & 0 & n \\
 -m \sin \alpha_1 b & m \cos \alpha_1 b & n \sin \alpha_1 b & n \cos \alpha_1 b
\end{bmatrix}
\begin{bmatrix}
 A_2 \\
 B_2 \\
 C_2 \\
 D_2
\end{bmatrix} = \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix}.
\]  \hspace{1cm} (14)

where \( j = -\alpha_1 - \nu \beta_1^2, \quad f = \alpha_2 - \nu \beta_1^2, \quad m = -\alpha_1^3 - (2 - v) \beta_1^2 \alpha_1, \quad n = \alpha_3^3 - (2 - v) \beta_1^2 \alpha_3^2. \)

Because the above formula has nonzero solutions, the determinant of the coefficient matrix on the left was zero. Therefore, the frequency equation along the y direction could be obtained:

\[
1 - \cos \alpha_1 b \cosh \alpha_2 b = \frac{1}{2} \left( \frac{mf}{jn} - \frac{jn}{mf} \right).
\]  \hspace{1cm} (15)

We set \( C_2 = 1 \) and insert it into (14):

\[
A_2 = -f j, \quad B_2 = -nk2/m, \quad C_2 = 1, \quad D_2 = k3.
\]

Then,

\[
W(y) = -f \cos \alpha_1 y - \frac{n}{m} k_1 \sin \alpha_1 y + \cosh \alpha_2 y - k_3 \sinh \alpha_3 y.
\]  \hspace{1cm} (16)

3.2. Deflection Equation of the Thin Plate Rock Mass C-C-C under Free Vibration. As for the thin plate rock mass with four-sided fixed supported (C-C-C-C), the boundary conditions C-C along the y direction may be as follows:

\[
W(x, 0) = 0, \quad \dot{W}(x, b) = 0; \quad W(x, b) = 0, \quad \dot{W}(x, b) = 0
\]

With (9), the frequency equation along the y direction could be obtained similarly:

\[
1 - \cos \alpha_1 b \cosh \alpha_2 b = \frac{\alpha_2^2 - \alpha_3^2}{2\alpha_1 \alpha_2}.
\]  \hspace{1cm} (17)

And the deflection equation could be expressed as follows:

\[
W(x, y, t) = \sum_{n=1}^{\infty} W_n(x, y) \phi_n(t).
\]  \hspace{1cm} (21)

4. Deflection Equation of the Thin Plate Rock Mass under Forced Vibration

4.1. Solution of the Control Equation. The solutions’ form of the nonhomogeneous control equation (1) can be expressed as follows:

\[
w(x, y, t) = \sum_{n=1}^{\infty} W_n(x, y) \phi_n(t).
\]  \hspace{1cm} (21)

We insert (21) into (1):

\[
\sum_{n=1}^{\infty} \left[ D^4 W_n(x, y) \phi_n(t) + \rho \phi W_n(x, y) \phi_n''(t) \right] = q(x, y, t).
\]  \hspace{1cm} (22)
Table 1: Deflection and frequency equations of the thin plate rock mass under different boundary conditions.

| Thin plate rock mass | Frequency equations | Deflection equations \( W(x, y) \) |
|----------------------|---------------------|-------------------------------------|
|                      | \( x \) direction: \( \sin \beta_1 a = 0 \) | \( x \) direction: \( W(x) = \sin \beta_1 x \) |
|                      | \( y \) direction: \( 1 - \cos \alpha_1 b \cosh \alpha_2 b / \sin \alpha_1 b \sinh \alpha_2 b = 1/2 (mf/jn - jn/mf) \) | \( y \) direction: \( W(y) = -f/j \cos \alpha_1 y - n/mk_1 \sin \alpha_1 y + \cosh \alpha_2 y - k_1 \sinh \alpha_2 y \) |
|                      | \( j = -\alpha_1 - \nu \beta_1^2, f = \alpha_2 - \nu \beta_2^2, m = -\alpha_1 - (2-\nu) \beta_1^2 \alpha_1, n = -\alpha_1 - (2-\nu) \beta_2^2 \alpha_2 \) | \( k_1 = \cos \alpha_1 b - \cosh \alpha_2 b / \alpha_1 \sin \alpha_1 b - \sinh \alpha_2 b, \ k_2 = (\cos \beta_1 a - \cosh \beta_2 a) / (\beta_1 / \beta_2 \sin \beta_1 a - \sinh \beta_2 a) \) |
|                      | \( x \) direction: \( 1 - \cos \beta_1 a \cos \beta_2 a / \sin \beta_2 a \sin \beta_2 a = \beta_1^2 - \beta_2^2 / 2 \beta_1 \beta_2 \) | \( x \) direction: \( W(x) = -\cos \beta_1 x + \beta_2 / \beta_1 \sin \beta_1 x + \cosh \beta_2 x - k_2 \sinh \beta_2 x \) |
|                      | \( y \) direction: \( 1 - \cos \alpha_1 b \cosh \alpha_2 b / \sin \alpha_1 b \sinh \alpha_2 b = \alpha_1^2 - \alpha_2^2 / 2 \alpha_1 \alpha_2 \) | \( y \) direction: \( W(y) = -\cos \alpha_1 y + \alpha_2 / \alpha_1 \sin \alpha_1 y + \cosh \alpha_2 y - k_2 \sinh \alpha_2 y \) |
|                      | \( k_1 = \cos \alpha_1 b - \cosh \alpha_2 b / \alpha_1 \sin \alpha_1 b - \sinh \alpha_2 b, \ k_2 = (\cos \beta_1 a - \cosh \beta_2 a) / (\beta_1 / \beta_2 \sin \beta_1 a - \sinh \beta_2 a) \) | \( k_1 = \cos \alpha_1 b - \cosh \alpha_2 b / \alpha_1 \sin \alpha_1 b - \sinh \alpha_2 b \) |

\( \nu \) is Poisson's ratio, \( \alpha_1 \) and \( \alpha_2 \) are coefficients, \( m \) and \( n \) are constants.
With (4),
\[ \text{D}V^4 W_n(x, y) = \rho h^2 W_n(x, y). \]

Then,
\[ \sum_{n=1}^{\infty} \rho h W_n(x, y) \left[ \omega_n^2 \varphi_n(t) + \varphi_n''(t) \right] = q(x, y, t). \]

The orthogonality of the deflection equations may be as follows [48]:
\[ \iint_{\Omega} \rho h W_m(x, y) W_n(x, y) dxdy = 0 \quad (m \neq n). \]

We multiply both sides of (24) by \( W_m(x, y) \) and do integral over the thin plate:
\[ \iint_{\Omega} \rho h W_m(x, y) W_n(x, y) dxdy = \iiint_{\Omega} q(x, y, t) W_n(x, y) dxdy. \]

We set
\[ \varphi_n(t) = \frac{1}{M_n \omega_n} \int_0^t P_n(\tau) \sin \omega_n(t-\tau) d\tau = \frac{\iint_{\Omega} \varphi_n W_n(x, y) dxdy}{M_n \omega_n \int_0^t \delta(\tau-t_1) \sin \omega_n(t-\tau) d\tau} = \frac{\iint_{\Omega} \varphi_n W_n(x, y) dxdy}{M_n \omega_n \sin \omega_n(t-t_1)}. \]

Therefore, the solutions of control equation (see (1)) of the thin plate rock mass under impact load can be expressed as follows:
\[ w(x, y, t) = \sum_{n=1}^{\infty} W_n(x, y) \varphi_n(t) = \sum_{n=1}^{\infty} \frac{A_n}{M_n \omega_n} W_n(x, y) \sin \omega_n(t-t_1) = \sum_{n=1}^{\infty} A_n W_n(x, y) \sin \omega_n(t-t_1) \quad (t \geq t_1), \]

where
\[ A_n = \iint_{\Omega} \varphi_n W_n(x, y) dxdy/M_n \omega_n. \]

5. Effective Vibration Modes of the Thin Plate Rock Mass

Taking granite in a mine in Sichuan, the parameter values of the thin plate rock mass were as follows: \( h = 0.06 \text{ m}, \rho = 2800 \text{ kg/m}^3, E = 72 \times 10^3 \text{ Pa}, \nu = 0.3, a \times b = 3 \text{ m} \times 3.6 \text{ m}. \]

5.1. Effective Vibration Modes of the Thin Plate Rock Mass S-F-S-F. Based on the frequency equation of the thin plate rock mass S-F-S-F, the Newton iteration method was introduced to solve the effective vibration modes. The calculated results of the first 10 order vibration parameters under free vibration are shown in Table 2, and the calculated results under impact load are shown in Table 3. According to Table 3, the 1st order, 3rd order, 8th order, and 10th order were effective vibration modes, and the natural frequencies of which were 101 rad/s, 300 rad/s, 909 rad/s, and 989 rad/s, respectively, and we could obtain that the coefficient \( A_9/q_0 \) of the 3rd order, 8th order, and 10th order was about 6.4%, 4.5%, and 0.5% of the 1st order.

5.2. Effective Vibration Modes of the Thin Plate Rock Mass C-C-C-C. Based on the frequency equation of the thin plate rock mass with four-side fixed support C-C-C-C, the effective vibration modes were also solved by Newton’s iterative method. The calculated results of the first 10 order vibration parameters under free vibration are shown in Table 4. Similarly, from the analysis results of the first 10 order vibration parameter of the thin plate rock mass C-C-C-C under impact load, we could obtain that 1st order, 5th order, and 6th order were effective vibration modes, and the natural frequencies of which were 308 rad/s, 975 rad/s, and
Table 2: The first 10 order vibration parameters of the thin plate rock mass under free vibration (S-F-S-F).

| Order   | Parameter | $u_n$ | $M_n$ | $\int_{11} W_n(x,y)dxdy$ | $A_n/q_0$ |
|---------|-----------|-------|-------|--------------------------|-----------|
| 1<sup>st</sup> order | 101 | 2.798 $\times$ 10<sup>4</sup> | 38.097 | 1.348 $\times$ 10<sup>-5</sup> |
| 2<sup>nd</sup> order | 148 | 4.536 $\times$ 10<sup>4</sup> | -4.642 $\times$ 10<sup>-6</sup> | -6.914 $\times$ 10<sup>-12</sup> |
| 3<sup>rd</sup> order | 300 | 1.736 $\times$ 10<sup>4</sup> | -0.451 | -8.663 $\times$ 10<sup>-7</sup> |
| 4<sup>th</sup> order | 404 | 2.197 $\times$ 10<sup>4</sup> | 9.059 $\times$ 10<sup>-10</sup> | 1.020 $\times$ 10<sup>-16</sup> |
| 5<sup>th</sup> order | 456 | 8.646 $\times$ 10<sup>4</sup> | 9.876 $\times$ 10<sup>-20</sup> | 2.505 $\times$ 10<sup>-26</sup> |
| 6<sup>th</sup> order | 575 | 1.208 $\times$ 10<sup>4</sup> | 2.881 $\times$ 10<sup>-6</sup> | 4.148 $\times$ 10<sup>-12</sup> |
| 7<sup>th</sup> order | 633 | 3.684 $\times$ 10<sup>4</sup> | -9.813 $\times$ 10<sup>-15</sup> | -4.208 $\times$ 10<sup>-21</sup> |
| 8<sup>th</sup> order | 909 | 1.786 $\times$ 10<sup>4</sup> | 9.968 | 6.140 $\times$ 10<sup>-7</sup> |
| 9<sup>th</sup> order | 926 | 2.082 $\times$ 10<sup>4</sup> | 6.097 $\times$ 10<sup>-21</sup> | 3.163 $\times$ 10<sup>-27</sup> |
| 10<sup>th</sup> order | 989 | 1.052 $\times$ 10<sup>3</sup> | -0.071 | -6.797 $\times$ 10<sup>-8</sup> |

Based on the third-strength theory, the Cowper–Symonds model is used to describe the yield process of coal and rock, and the impact load acts uniformly on the surface of coal and rock mass. The results could be shown as follows (Figure 6): firstly, plate cracks occurred in the middle of the two short sides and the four concentration regions of the two long sides (Figures 6(a) and 6(b)), which was consistent with Figure 2, then the develop tendency of these cracks was outward along the long central axis of the thin plate (Figure 4), and last, plate cracks would be distributed in any region (Figure 6(d)), which was consistent with Figure 5.

6. Dynamic Damage and Failure of the Thin Plate Rock Mass

According to the deflection equations of the thin plate rock mass under different boundary conditions shown in Table 1, it could be obtained that the solutions’ form of the thin plate rock mass S-C-S-C was the simplest and that of the thin plate rock mass S-F-S-F was the most complex. And then, we would analyze the process of dynamic damage and failure of thin plate rock mass in order of the complexity of their solutions’ form.

6.1. Dynamic Damage and Failure Process of the Thin Plate Rock Mass S-C-S-C. Based on the third-strength theory, the maximum shear of effective vibration modes of the thin plate rock mass S-C-S-C under impact load is shown in Figures 2–5. It can be seen from Figure 2 that the maximum shear of the 1<sup>st</sup> order was mainly distributed in the middle of the two short sides and four concentration regions of the two long sides, that of the 5<sup>th</sup> order in the middle of the two short sides and the region along the long central axis of the thin plate (Figure 3), that of the 6<sup>th</sup> order in the region along the short center axis of the thin plate (Figure 4), and that of the 10<sup>th</sup> order in any region (Figure 5).

Based on LS-DYNA, the PLASTIC_KINEMATIC material model was used to simulate and analyze the process of damage and failure of the thin plate rock mass S-C-S-C under impact load. The long side of the model is horizontal, and the short side is vertical. The model size is 3 m × 3.6 m, the Cowper–Symonds model is used to describe the yield process of coal and rock, and the impact load acts uniformly on the surface of coal and rock mass. The results could be shown as follows (Figure 6): firstly, plate cracks occurred in the middle of the two short sides and the four concentration regions of the two long sides (Figures 6(a) and 6(b)), which was consistent with Figure 2, then the develop tendency of plate cracks was along the long and short central axes toward the center of the thin plate rock mass (Figures 6(c) and 6(d)), which was consistent with Figures 3 and 4, and last, plate cracks would be distributed in any region (Figure 6(d)), which was consistent with Figure 5.

6.2. Dynamic Damage and Failure Process of the Thin Plate Rock Mass C-C-C-C. Based on the third-strength theory, the maximum shear of effective vibration modes of the thin plate rock mass C-C-C-C under impact load is shown in Figures 7–9. It can be seen from Figure 7 that the maximum shear of the 1<sup>st</sup> order was mainly distributed in the middle of the four sides, that of the 5<sup>th</sup> order in the middle of the two short sides and the region along the long central axis of the thin plate (Figure 8), and that of the 6<sup>th</sup> order in the middle of the two long sides and the region along the short central axis of the thin plate (Figure 9).

Similarly, based on LS-DYNA, the PLASTIC_KINEMATIC material model was used to simulate and analyze the process of damage and failure of the thin plate rock mass C-C-C-C under impact load, which could be shown as follows (Figure 10): firstly, plate cracks occurred in the middle of the two short sides and two long sides and these cracks would propagate rapidly along the boundary of the thin plate rock mass(Figures 10(a) and 10(b)), which was consistent with Figure 7, then plate cracks occurred at the central of the thin plate and the main develop tendency of these cracks was outward along the long central axis (Figures 10(c) and 10(d)), which was consistent with Figure 8, and these cracks also tended to expand outward along the direction of short side (Figure 10(d)), which was consistent with Figure 9.

1309 rad/s, respectively (Table 5). And the results showed that the coefficient $A_n/q_0$ of the 5<sup>th</sup> order and 6<sup>th</sup> order were about 9.4% and 4.9% of the 1<sup>st</sup> order.

5.3. Effective Vibration Modes of the Thin Plate Rock Mass S-C-S-C. The calculated results of the first 10 order vibration parameters of the thin plate rock mass S-C-S-C under free vibration are shown in Table 6. Similarly, according to the first 10 order vibration parameter values of the thin plate rock mass S-C-S-C under impact load, it could be obtained that the 1<sup>st</sup>, 5<sup>th</sup>, 6<sup>th</sup>, and 10<sup>th</sup> order were effective vibration modes, and the natural frequencies of which were 230 rad/s, 943 rad/s, 999 rad/s, and 1680 rad/s, respectively (Table 7). And the results showed that the coefficient $A_n/q_0$ of the 5<sup>th</sup> order, 6<sup>th</sup> order, and 10<sup>th</sup> order were about 10.6%, 2.8%, and 1.3% of the 1<sup>st</sup> order.

6.6. Dynamic Damage and Failure Process of the Thin Plate Rock Mass C-C-C-C. Based on the third-strength theory, the maximum shear of effective vibration modes of the thin plate rock mass C-C-C-C under impact load is shown in Figures 7–9. It can be seen from Figure 7 that the maximum shear of the 1<sup>st</sup> order was mainly distributed in the middle of the four sides, that of the 5<sup>th</sup> order in the middle of the two short sides and the region along the long central axis of the thin plate (Figure 8), and that of the 6<sup>th</sup> order in the middle of the two long sides and the region along the short central axis of the thin plate (Figure 9).

Similarly, based on LS-DYNA, the PLASTIC_KINEMATIC material model was used to simulate and analyze the process of damage and failure of the thin plate rock mass C-C-C-C under impact load, which could be shown as follows (Figure 10): firstly, plate cracks occurred in the middle of the two short sides and two long sides and these cracks would propagate rapidly along the boundary of the thin plate rock mass(Figures 10(a) and 10(b)), which was consistent with Figure 7, then plate cracks occurred at the central of the thin plate and the main develop tendency of these cracks was outward along the long central axis (Figures 10(c) and 10(d)), which was consistent with Figure 8, and these cracks also tended to expand outward along the direction of short side (Figure 10(d)), which was consistent with Figure 9.
6.3. Dynamic Damage and Failure Process of the Thin Plate Rock Mass S-F-S-F. Similarly, based on the third-strength theory, the maximum shear of effective vibration modes of the thin plate rock mass S-F-S-F under impact load is shown in Figures 11–14. It can be seen from Figure 11 that the maximum shear of the 1st order was mainly distributed in the middle of the two short sides, that of the 3rd order at the four corners of the thin plate (Figure 12), that of the 8th

| Parameter | 1st order | 2nd order | 3rd order | 4th order | 5th order | 6th order | 7th order | 8th order | 9th order | 10th order |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|------------|
| $\beta_1$ | 1.443     | 1.301     | 2.573     | 2.483     | 1.227     | 3.643     | 2.404     | 3.589     | 2.346     | 2.346      |
| $\beta_2$ | 2.145     | 3.247     | 2.949     | 3.765     | 4.437     | 3.895     | 4.802     | 4.514     | 5.930     | 5.930      |
| $\alpha_1$ | 1.122    | 2.104     | 1.019     | 2.001     | 3.015     | 0.975     | 2.940     | 1.935     | 3.904     | 3.851      |
| $\alpha_2$ | 2.329    | 2.795     | 3.778     | 4.042     | 3.479     | 5.243     | 4.494     | 5.433     | 4.248     | 5.083      |
| $\kappa$ | 1.828    | 2.474     | 2.767     | 3.189     | 3.255     | 3.771     | 3.797     | 4.078     | 4.079     | 4.509      |

Table 4: The first 10 vibration parameters of the thin plate rock mass under free vibration (C-C-C-C).

| Order | Parameter | $w_n$ | $M_n$ | $\int_\Omega W_n(x, y) dxdy$ | $A_n/\eta_0$ |
|-------|-----------|-------|-------|-----------------------------|--------------|
| 1st order | $\beta_1$ | 308   | 5.787,710$^3$ | 13.589 | 7.624,610$^-6$ |
| 5th order | $\alpha_2$ | 975   | 1.267,210$^5$ | 8.503 | 6.883,810$^-7$ |
| 6th order | $\kappa$ | 1309  | 2.587,510$^4$ | 12.565 | 3.71,710$^-7$ |

Table 5: The effective vibration modes of the thin plate rock mass under impact load (C-C-C-C).

| Parameter | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------|---|---|---|---|---|---|---|---|---|----|
| $\beta_1$ | 1.047 | 2.094 | 1.047 | 2.094 | 1.047 | 3.142 | 2.094 | 3.142 | 1.047 | 3.142 |
| $\alpha_1$ | 1.183 | 1.052 | 2.127 | 2.031 | 3.025 | 0.992 | 2.960 | 1.959 | 3.908 | 2.896 |
| $\alpha_2$ | 1.896 | 3.143 | 2.592 | 3.591 | 3.368 | 4.552 | 4.187 | 4.855 | 4.180 | 5.303 |
| $\kappa$ | 1.580 | 2.343 | 2.371 | 2.917 | 3.201 | 3.294 | 3.626 | 3.702 | 4.046 | 4.273 |

Table 6: The first 10 order vibration parameters of the thin plate rock mass under free vibration (S-C-S-C).

| Order | Parameter | $w_n$ | $M_n$ | $\int_\Omega W_n(x, y) dxdy$ | $A_n/\eta_0$ |
|-------|-----------|-------|-------|-----------------------------|--------------|
| 1st order | $\beta_1$ | 230   | 1.407,410$^3$ | 7.171 | 2.511,510$^-3$ |
| 5th order | $\alpha_2$ | 943   | 9.980,910$^2$ | 2.540 | 2.699,9610$^-6$ |
| 6th order | $\kappa$ | 999   | 8.898,810$^3$ | 6.171 | 6.943,910$^-7$ |
| 10th order | $\beta_1$ | 1680  | 1.856,810$^3$ | 1.045 | 3.352,310$^-7$ |

Table 7: The effective vibration modes of the thin plate rock mass under impact load (S-C-S-C).

Figure 2: The maximum shear of the 1st order mode (S-C-S-C).
Figure 3: The maximum shear of the 5th order mode (S-C-S-C).

Figure 4: The maximum shear of the 6th order mode (S-C-S-C).

Figure 5: The maximum shear of the 10th order mode (S-C-S-C).
order in the six concentration regions of the two short sides of the thin plate (Figure 13), and that of the 10th order at the four corners of the thin plate, in the region along the long center axis and in the four concentration regions of the two long sides (Figure 14).

It could be analyzed that the process of damage and failure of the thin plate rock mass S-F-S-F under impact load may be as follows: firstly, plate cracks occurred in the middle of the two short sides and the four corner regions (Figures 11 and 12), and then these cracks would propagate rapidly along the...
Figure 9: The maximum shear of the 6th order mode (C-C-C-C).

(a) (b) (c) (d)

Figure 10: Dynamic damage and failure of thin plate rock mass C-C-C-C under impact load.

Figure 11: The maximum shear of the 1st order mode (S-F-S-F).
Figure 12: The maximum shear of the 3rd order mode (S-F-S-F).

Figure 13: The maximum shear of the 8th order mode (S-F-S-F).

Figure 14: The maximum shear of the 10th order mode (S-F-S-F).
two short sides (Figure 13) and also tended to expand inward along the direction of long side (Figures 13 and 14).

7. Discussion on Dynamic Failure Mechanism of the Thin Plate Rock Mass

According to Table 7, the vibration modes of the thin plate rock mass S-C-S-C under impact load were dominated by the 1st order (main frequency), following by the 5th, 6th, and 10th order. From Figures 2–6, the maximum shear of the 1st order was mainly distributed in the middle of the two short sides and four concentration regions of the two long sides where plate cracks occurred firstly and that of the 5th and 6th order in the regions along the long and central axes of the thin plate rock mass where the plate cracks tended to expand. Similarly, according to Table 5, the vibration modes of the thin plate rock mass C-C-C-C under impact load were dominated by the 1st order (main frequency), following by the 5th and 6th order. From Figures 7–10, the maximum shear of the 1st order was mainly distributed in the middle of the four sides where plate cracks occurred firstly and that of the 5th and 6th order in the regions along the long and central axes of the thin plate rock mass where the plate cracks tended to expand.

According to Table 3, the vibration modes of the thin plate rock mass S-F-S-F under impact load were dominated by the 1st order (main frequency), following by the 3rd, 8th, and 10th order. From Figures 11–14, it could be concluded by the maximum shear of the 1st order that plate cracks occurred firstly in the middle of the two short sides and the four corner regions, and these cracks would propagate rapidly along the two short sides and also tended to expand inward along the direction of long side based on that of the 3rd, 8th, and 10th order.

According to the above discussion, it could be concluded that the initial position of damage and failure of the thin plate rock mass may be determined by the effective vibration mode with the lowest frequency, and the develop tendency of damage and failure may be determined by the combined actions of other effective vibration modes. Furthermore, we could obtain that the coefficient $A_n/q_0$ of 1st order is the largest (Tables 3, 5 and 7), and it could be concluded that the effective vibration modes with lower frequency could have greater influence on the process of damage and failure of the thin plate rock mass.

8. Conclusions

(1) Based on the thin plate model and the dual equation of the Hamilton system, the frequency equations and deflection equations of the thin plate rock mass with different boundary conditions (S-F-S-F, S-C-S-C, and C-C-C-C) were established under free vibration, and the deflection equations under impact load were derived by the orthogonality of vibration modes and Duhamel integral.

(2) Taking granite in a mine in Si Chuan, the effective vibration modes of the thin plate rock mass with different boundary conditions and their natural frequencies were obtained by the frequency equations and Newton’s iterative method.

(3) Based on the third-strength theory, the maximum shear of the effective vibration modes was obtained, and the processes of damage and failure of the thin plate rock mass under impact load were analyzed with the numerical simulation results by LS-DYNA.

(4) Based on the analysis of damage and failure of the thin plate rock mass under impact load, we could conclude that the initial position of damage and failure may be determined by the combined actions of other effective vibration modes. In addition, the effective vibration modes with lower frequency could have greater influence on the process of damage and failure of the thin plate rock mass.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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