FILLINGS METHOD IN NUMBER THEORY

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Abstract. In offered work the series of problems of an analytical number theory is surveyed. This problems have direct and defining reflection in prime numbers distribution. For their solving the new method called as a method of fillings was developed. In basis of its the functional imaging lays of the integrated characteristics and elements of all period of generators - grids on an interval, and the systems of generators appear as the main object of a research. A fillings method separates from a traditional sieving process already on an initial stage because of this fact.

It is necessary to refer possibility of registration and proof of the important state about distribution of the majorizing characteristics of numerical grids on an axe to one of main advantages of a method of fillings. This state is called by the main theorem. Modifications of a relation of the main theorem which is sequentially improved and magnified also were by that source, from which one the concrete outputs about distribution of prime numbers were obtained.

The quite particular versatility of the fillings method has allowed to receive outcomes in address to such different tasks as the evaluation of greatest distance between prime numbers, the Goldbach’s conjecture for even, the increase of the Goldbach’s representations, distribution of twins, asymptotics of distribution of typical configurations primes and some others. Thus there are all basis to guess, that possibilities of a method are not exhausted by considered applications, as the method hold sway over not only primes system.

Key words: Grid, grating and product, sieve, fillings period, system of grids, series and maximum series of zeroes, ineradicable multiplicity of zero, frequency of zeroes in the period, two-dimensional strip region of elements, imaging principle, two-sidedness.
1. Introduction

The concept of an integer with the complete right should be named one of the first abstract object scientific cognition of the world, a tool of which has become the mathematics. The representation about prime number, allocated by distinctive and well by known features has occurred also in antiquity. The further researches have delivered problems of prime numbers distribution, but the progress in the sanction some of them was scheduled in comparison recently.

The successes in an essential degree have appeared connected with development of the sieving process, at one time offered by Eratosthenes. At the same time the series of problems, despite of elementary mathematical statements, has not yielded all efforts and progress to new results was essentially slowed. To the present time is not new directions, leading though presumably to the for a long time scheduled purposes.

The reason of a created situation can be only one: the sieving process as any method has restricted capacity, adequate to limiting level of achievement and conclusions. And this potential, as far as demonstrate real development, is practically exhausted. The absence of fresh ideas in so before a fruitful direction of the number theory also testifies to a limit of research opportunities of the profound antiquity method.

The sieving process has a series essential and in essence ineradicable defects, among which very weak interdependence between base (prime) numbers, forming a sieve. Some characteristics of distribution of primes in a sequence natural is sharp depend from a set of forming values. Therefore there was a idea to study them on classes of sets, being not limited only primes. This idea has resulted to creation of a quite new method of the number theory – fillings method [1, 2].

Beginning of the mathematics development as integer arithmetics is impossible to separate from a problem of divisibility. Probably, definition of prime number was known long before Euclid, but only he proved by the simple and elegant theorem the Cantorian equipotency of countable sets of natural and prime numbers. The representation of actual infinity, necessary for realization of this fact, was produced by Phales school considerably earlier.

The concepts of countability, potential and actual infinity, series of methods and operations over finite and unlimited numerical objects, though and were received in half-intuitional level, have nevertheless found reasonably qualified base for use. Boiling joy of seizing by infinity axiom and unrestrained flight of the idea have not allowed to pay attention for a computability problem, but this permanent sin there were at soul of the mathematics (and not only mathematics) and now.

Ancient Greek approach to registration and definition of number - even integer and infinite, contained in core the ineradicable factor of absolute unrestricted. Naturally, such significant fact has an effect in speed creation of numerical objects of actual infinity. At all development the mathematics (and the science as a whole) have remained at the achieved stage. In particular, one of examples of a extreme
considered unreasonable idealization of a theory has acted Cantorian theory of sets.

Considerable achievement of a modern science including the mathematics cannot, nevertheless, to hide sorrowful facts of deep failures, hopeless deadlocks and insoluble contradictions. Their initial source is the only infinity axiom, transformed a zone of knowledge to field of an authority of an infinity quantifier. The consequences of this phenomenon reach much further problems only sciences.

However all offered development are oriented to the conventional schemes [3] and will be conducted without references to non-foundation, indefiniteness and unprovableness of infinity axiom. Though the series determining and known rules, in particular connected with concept of set of all primes, does not make sense outside of mentioned axiom at all its numerous defects. The ideas, connected with determining, central and paramount role of infinity axiom, are considered and carefully justified in work [4–6].

The Euclidean theorem about infinity of primes set can be considered as the first research, the provable statement of a modern stage of consciousness - stage of infinity axiom. In this sense the theorem has acted as the first-born of a present condition of the science. In a maximum degree is fair, that those there was the theorem about numbers. The modern stage of development and level cognition was determined earlier as time by creation and adaptation of unrestriction idea.

A following major studied stage of primes distribution was the sieve of Eratosthenes, realizing finite algorithm of reception all primes, smaller $p_n^2$, if series primes down to $p_{n-1}$ inclusive is known. It sieve will hereinafter appear great-parent of updating and basis for theoretical research of the various characteristics and features of primes distribution accommodation in the natural sequence [3].

The sieving process as well as any method is exhausted. This cognition law does not know exceptions. The truth, it does not mean, that moment is (can be precisely established), when the reception of new results will become impossible. Is not present, exhaustion has an effect in sharp fall of a level of successes, their quantity and significance. But any results are possible nevertheless. Such process, fundamentally appropriate to reality, is named by a real limit (realimit) and is investigated in works [4, 5].

The offered research essentially uses the theorem about primes. Namely, there is in such known formulation: Function $\pi(x)$ (quantity primes, not superior $x$) is represented by an expression

$$\pi(x) : \frac{x}{\ln x} = C(x), \quad p_i \leq x \quad \text{and} \quad C(x) \xrightarrow{x \to \infty} 1, \quad (1.1)$$

and the borders $C(x)$ are estimated, limited and established. It is thus allowable to approximate the value of prime $p_n$ depending from its index. In turn, it permits to evaluate some important integrated characteristics of the finite sequence $\{p_i\}_{n \to \infty}$, which pursuant to (1.1) will be specified at $n \to \infty$.

All other situations lean on a reasonably known theory or here provable schemes. The elementary character of constructions in determined degree is compensated
by novelty of a method, on principle distinguished from the sieving process – and algorithmically, and main directive purposes, and initial object set.

It is especially necessary to note, that study and the decision of series of problems of primes distribution had by the beginning not given set primes, as for the majority of researches such, but set of natural numbers. The way of research of properties primes through study of natural numbers is not only proven, but also is structurally used with the help of fillings method. The expansion of main object set has basic character and largely predetermines strength of the method.

2. Some definitions

Before to begin a exposition of a method, oriented on research of some specific properties of integers correlation, it is necessary give a series of initial definitions. Them obligation is called by absence of adequate analogues, on which it was possible to refer. If to take into account the general final purpose – distribution of primes, the similar statement appears and characteristic while a un known method.

Postulated integrity of initial values and elementary character of made operations permit to declare, that main relations do not turn out beyond the borders of fraction-rational numbers. Even the concept of numerical infinity at desire can be replaced by estimable finite values, but the traditions and requirements of laconic applications forbid to do it. Unique used function, structurally including representation of infinity is function \( \pi(x) \) from the central primes theorem (1.1).

Idea of the method, number of the proof schemes and majority of the outputs was obtained in 1983 and also the thought arose to use period’s characteristics of some finite numerical forms.

A set \( \mathcal{L} = \{ L_{(s)} \} \) of regulated infinite sequences \( L_{(s)} \) of Boolean elements \( l_i \in \{0; 1\}, \ i \in \mathbb{Z} \) represents itself as the primary source object:

\[
\mathcal{L} : \quad \{ L_{(s)} = \ldots, l_{-m}, l_{-m+1}, \ldots, l_{-1}, l_0, l_1, \ldots, l_n, l_{n+1}, \ldots \}.
\]

Ordering of elements in an assigned sequence \( L_{(s)} \) is possible to fix by correspondence of each elements \( l_i \) to an integer \( i + k_s \in \mathbb{Z} \), where \( |k_s| < \infty \) is some arbitrary constant. Distance \( \rho_{ij} \) is determined for all elements \( l_i, l_j \in L_{(s)} \):

\[
-\infty < i, j < \infty, \quad \rho_{(i+k)(j+k)} \equiv \rho_{ij} \equiv \rho(l_i, l_j) = | i - j |.
\]

It does not depend from \( k_s, k \) or \( s < \infty \). Integer values of distance \( \rho_{ij} \) do not conflict with rules of metric space. The rule of the triangle has a feature. Greatest distance is always equal to sum of others: \( \rho_{ij} = \rho_{ik} + \rho_{kj} \).

Thus it takes place determined isomorphism of sequence \( L_{(s)} \) and vector \( \{\rho_{iw}\} \), \( i \in \mathbb{Z} \) of distances between following one after another next units of sequence \( L_{(s)} \).
If \( l_i = 1 \) for \( -\infty < i < \infty \), that \( i' = \min_{k > i} \{l_k = 1\} \). From here follows, that exact restoration of sequence \( L(a) \) by a vector \( \{\rho_{i'}\} \) (except for constant \( k_a \)) is possible. Certainly, with the same success it is possible to generate and to consider a similar vector of distances between zeroes, instead units.

Now all ready for the direct introduction of definitions of objects, compulsory and constantly used in further constructions and researches.

**Definition 1.** Grid \( S(a) \) of the module \( a \geq 0 \), \( a \in \mathbb{N} \) or \( a \)-grid is named as the sequence \( L_a = \{l_i\}_a \) of periodic elements:

\[
L_a : \quad l_{i+a} = l_i; \quad \text{if } l_j = 0, \quad \text{that } l_{j+1} = l_{j+2} = \ldots = l_{j+a-1} = 1.
\]

The following relations are valid for any grid as periodic object:

\[
\forall i \in \mathbb{Z} : \quad |l_{i+a} - l_i| = 0; \quad \text{if } l_j = 0, \quad \text{that } \exists k : \quad \rho(l_j, l_k) = a.
\]

The sequence from units only (0-grid) is marked \( L_0 \), and sequence from zeroes only (1-grid), is marked as \( L_1 \). The grid \( L_\infty \) has equally one zero for integer axis \( Z \) of indexes. The zeroes and units alternate in first nontrivial 2-grid: \( l_i = 1 - l_{i-1} \).

**Definition 2.** The \( n \)-grating \( V_n \) is named product of \( n \) grids (optionally different) \( S(a_i) \), \( i = 1, 2, \ldots, n \). It is made under the recurrent scheme

\[
V_0 = L_0, \quad V_i = V_{i-1} \& S(a_i), \quad l_j(V_i) = l_j(V_{i-1}) \& l_j(L_{a_i}), \quad j \in \mathbb{Z}.
\]

The itemized logical multiplying of sequences is meant product of grids. We shall give as an example 4-grating for grids \( S(4); S(6); S(12) \):

\[
V_0 = L_0 : \quad \ldots 11111111111111111 \ldots \\
V_1 = L_4 = S(4) : \quad \ldots 101110111011011011011 \ldots \\
V_2 = V_1 \& S(6) : \quad \ldots 10011101100111011001111 \ldots \\
V_3 = V_2 \& S(12) : \quad \ldots 10001011001100101100111 \ldots \\
V_4 = V_3 \& S(12) : \quad \ldots 1000001100111000001100111 \ldots 
\]

Commutability, associativity, symmetry and transitivity of grids products are the direct corollary of these properties for Boolean elements \( l_j \).

**Definition 3.** The \( n \)-filling \( Z_n \) is such \( n \)-grating \( V_n \), \( Z_n \subseteq V_n \), in which one at each index \( i \), \( 1 \leq i \leq n \) is executed

\[
V_n = V_{n-1}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \& S(a_i) \neq V_{n-1}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).
\]
The exception of any grid in \( n \)-filling carries on to the permutation of zeroes and units. It takes place not always for \( n \)-grating.

**Definition 4.** If maximum module \( \max_{1 \leq i \leq n} a_i < \infty \), that anyone \( n \)-filling \( Z_n \) (as well as \( n \)-grating) is periodic, and length of its period \( PZ_n \) does not exceed the least common multiple of modules:

\[
\{ Z_n(a_1, ..., a_n); \quad V_n(a_1, ..., a_n) \} : \quad PZ_n \leq ((a_1, ..., a_n)) .
\]

**Definition 5.** The system of grids \( SS \) is unbounded sequence of grids \( S(a_i) \) of nondecreasing modules selected according to some law, if at each \( n \) the grids \( \{ S(a_i) \}_{1}^{n} \), \( 1 \leq i \leq n \) form \( n \)-filling \( Z_n \).

\[
SS : \quad n \Rightarrow n + 1, \quad a_n \leq a_{n+1}, \quad Z_n \Rightarrow Z_n \& S(a_{n+1}) = Z_{n+1}.
\]

The transition from parameter \( n \) to \( n + 1 \) gives increase of set of nonconterminous fillings \( Z_{n+1} \) in product of prior \( n \)-filling \( Z_n \) with the grid \( S(a_{n+1}) \). A possible incongruity of fillings \( Z_{n+1}(k_{n+1}) \) is determined by change of the shift parameter \( k_{n+1} \) of new grid.

**Definition 6.** The \( q \)-series of zeroes is an interval \( SR_n(q) \) of \( n \)-filling \( Z_n \), restricted by units and containing (including) \( q \) units, where integer \( q \geq 0 \). Length of \( q \)-series \( sr_n(q) \) is distance between initial \( 1^{(0)} \) and final \( 1^{(q+1)} \) units of \( q \)-series:

\[
sr_n(q) = \rho (l_i, l_k) ; \quad l_i = 1^{(0)}, \quad l_k = 1^{(q+1)} .
\]

The irreversible discrepancy between sieving process in all its modifications and fillings method begins with introduction of concept of zeroes series. Series of zeroes \( SR_n(q) \) is the object, unknown for sieving process, where the zeroes and units (that is eliminated and not eliminated numbers), are strongly connected to the value of concrete natural number. Then zeroes and units do not differ as supplemental, passing properties of divisibility of this number. Vice-versa, the role of zeroes becomes defining in the fillings method.

**Definition 7.** The maximum series of zeroes \( MSR_n(q) = MSR_n(Z_n, q) \) of the value \( 0 < msr_n(q) < \infty \) is such \( q \)-series \( SR_n(q) \), length which \( sr_n(q) \) one greatest in the set of \( n \)-fillings \( \{ Z_n \} \):

\[
msr_n(q, a_1, a_2, ..., a_n) = \sup_{\{ Z_n \}} \max_{k, j} \{ sr_n(q) | l_k, l_{k+j} = 1 \} \in Z_n .
\]

The maximum series of zeroes \( MSR_n(q) \) and their value for \( q = 0 \) (without units inside series) are most indispensable for theoretical constructions. These major characteristics of fillings are marked accordingly \( MSR_n \) and \( msr_n \).

**Definition 8.** The regulated \( n \)-filling \( ZU_n = ZU_n(a_1, a_2, ..., a_n) \) distinguishes algorithm of product of grids: after arbitrary fixing of beginning of series,
the zero of each next grid \( S(a_i) \), \( i = 1, \ldots, n \) is multiplied with first right unit of the series. The \textit{unregulated} \( n \)-filling \( ZN_n \) combines all set of possible fillings.

The \textit{semi-regulated} \( n \)-filling \( ZP_n(a_1, \ldots, a_r; a_{r+1}, \ldots, a_n) \), \( n > 3 \),
\[ 1 < r < n-1 \] is an unregulated filling \( ZN_{n-r}(a_{r+1}, \ldots, a_n) \), constructed on regulated filling \( ZU_r(a_1, a_2, \ldots, a_r) \).

The values of the greatest series of zeroes obtained by the filling \( Z_n \) \((ZU_n-ZN_n)\) are marked as \( msr_n(q, Z_n) = msr_n(q) \). It is important characteristics of filling.

**Definition 9.** Zero of \textit{multiplicity} \( k \), \( 1 \leq k \leq n \) is an element of \( n \)-filling \( Z_n \), if it is multiplying of \( k \) zeroes of generating grids \( S(a_i) \).

We shall give by the way for example already considered different fillings \( Z_4 \) with vectors of multiplicity of zeroes

\[
\begin{align*}
...1011101110111011... & \quad \cdots ...1011101110110111... & \quad \cdots ...1011101110111011... \\
...1101111011110111... & \quad \cdots ...1011110111110111... & \quad \cdots ...1011110111111101... \\
...1111011111111101... & \quad \cdots ...1111111101111011... & \quad \cdots ...1111111101111111... \\
...1000011101101000... & \quad \cdots ...1010101010101010... & \quad \cdots ...101110101011011... \\
...v11111vv11vvv11... & \quad \cdots ...v2v1v1v1v1v1v1v1... & \quad \cdots ...v3vvv1v2v1vvv3vv...
\end{align*}
\]

Two 4-fillings and one 4-grating are given here. The grids \( S(4), S(6), S(12), S(12) \) with the shifts are in each column, then the outcome of product is given, and the vectors of multiplicity of zeroes are in the last line. Unit (zero of the multiplicity of zero) is marked by sign \( v \).

The examples demonstrate possibility to construct filling with zeroes only multiplicity 1 (first variant). The second case testifies change of periodicity (from 12 to 2), but thus there is zero of the multiplicity two. The filling with zeroes of multiplicity three and four cannot be constructed for this set of grids, but such it is possible for 4-grating (third variant of product). Zero of multiplicity 3 supplies in the last line for it zeroes of grids \( S(4), S(6) \) and \( S(12) \).

**Definition 10.** The system \( SS = \{ S(a_i) \} \) with modules \( a_i \) as degrees of the same integer \( d \geq 2 \), \( (a_i = d^{k_i}, k_i \geq 1) \), is the degree-system. Integer value \( d \) is named as the \textit{basis} of the degree-system \( SS = SS_d \) and appropriate fillings \( Z_n \).

**Definition 11.** The system \( SS \) is named as the \textit{system without multiple zeroes} \( SS = SS' \), if for anyone \( n \) there are fillings \( Z_n \), in which one there will be no zeroes of the multiplicity above than unit.

**Definition 12.** The system of fillings \( SS = VP \) is named \textit{coprime}, if for anyone \( 1 \leq i \neq j < \infty \) the modules of these grids satisfy to the relation \( (a_i, a_j) = 1 \).

**Definition 13.** The nonsingular system of grids and fillings \( SS \) is named as
mixed \((SS = SM)\), if it does not belong to any circumscribed types.

**Definition 14.** The system \(VP = \{2, 3, 5, \ldots, p_i, \ldots\}\) of modules as primes is named 0-prime system \(SP_0\). The system of the pair primes \(\{S(2), 2 \cdot S(p_i)\}\): \(SS = SM = \{2, 3, 3, 5, 5, \ldots, p_i, p_i, \ldots\}\) is 0-double system of primes \(SW_0\).

**Definition 15.** Some non-singular system of grids \(SS = \{a_1, a_2, \ldots, a_i, \ldots\}\) belongs to first \(SS(I)\), second \(SS(II)\) or third type \(SS(III)\), if accordingly

\[
C_{ss} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{a_i} : \quad C_{ss} \leq 1, \quad 1 < C_{ss} < \infty, \quad C_{ss} = \infty.
\]

For example, the sieve of Eratosthenes on each step is the regulated filling of the system \(SP_0\) for an interval \(I \ll PZ_n\), if the first zeroes of each grid after termination of algorithm are exchanged by units.

But before to address to the main representatives of systems of the third type \(SS(III)\), it is necessary to consider systems without multiple zeroes \(SS'\) (all zeroes have multiplicity 1). And already then to formulate and to present the main statement, in which as evidently as numerically the idea of a fillings method is concentrated. The fact is that without exact formulas of model fillings the transition to more complex and important systems is unreasonable.

### 3. Systems without multiple zeroes

Systems without multiple zeroes \(SS'\) are systems, for which at each \(n\) is present \(n\)-filling \(Z_n\), and in it on a period, hence, on axis there is no zero of multiplicity higher unit. Such systems as model for other more important and necessary systems, nevertheless, have beside features. Just they permit to find their characteristics. For example, the construction of maximum series for fillings of such systems is determined only by ordering of grid modules.

For intervals \(J_n\) about a period and more density for them perfectly is approximated by value \((\gamma_n)\). Decrease of the interval leads to growth of zeroes density because of fall them multiplicity down to \((\gamma_n^*)\), which is limiting. But for systems without multiple zeroes \(SS\) there is the only zeroes frequency, it is density of zeroes \(\gamma_n = (\gamma_n^*)\) for the period and consequently axis for given filling \(Z_n \subset SS\). To the first and the most indicative class of such systems concern degree-systems \(SS_d\).

The zeroes frequency (density) for the period and whole numerical axis of any \(n\)-filling \(Z_n \subset SS_d\) is equal

\[
SS_d: \quad \gamma_n = 1 - \frac{E_n}{PZ_n} = \frac{H_n}{PZ_n} = \sum_{i=1}^{n} \frac{1}{a_i} = \sum_{i=1}^{n} \frac{1}{d_i^*}. \quad (3.1)
\]
As far as research is supposed only non-singular \((msr_n < \infty\) for each \(n\)) systems, that at all \(n\) value \(\gamma_n < 1\). Besides the condition of sequentially non-decreasing modules of grids for a system \(SS_d\) testifies, that according to (3.1) we have obvious relation \(PZ_n \leq a_n\). In such case expediently to consider a number system with the base-radix \(d\) and representation of frequency \(\gamma_n\) in this system.

As is known, number system with basis \(d\) has all digits less \(d : 0 \leq m_i \leq d - 1\). However there is one obstacle on a way of representation of zeroes density \(\alpha\) frequency does not surpass \(d\) multiple. From the definition clearly, that the period of saturated filling for \(\gamma_n\) means the greatest common divisor, and in the second equality – least common multiple. In this connection we shall allocate subsets from the class of degree-systems.

**Definition 16.** The correct system \(SS = SS_d\) of degree grids \(S(a_i)\) with the basis \(d \geq 2\) is named system, for which the quantity of grids of one module \(a_i\) does not surpass \(d - 1\): \(S(a_k) = S(a_{k+1}) = \ldots = S(a_{k+s-1})\), \(s \leq d - 1\).

**Definition 17.** The filling \(Z_n(a_1, \ldots, a_n) \subset SS\) is named as saturated, if filling \(Z_{n+1}\) is singular, that is \(msr_n = msr_n(q) = \infty\),

\[
Z_n(a_1, a_2, \ldots, a_n) \& S(a_{n+1}') = Z_{n+1}(a_1, a_2, \ldots, a_n, a_{n+1}') = L_1 \tag{3.2}
\]

for any grid \(S(a_{n+1}')\), let and not included into the system \(SS\), but satisfying relation for period \(PZ_n\) of \(n\)-filling

\[
(PZ_n, a_{n+1}') = a_{n+1}', \quad PZ_n \leq ((a_1, a_2, \ldots, a_n)). \tag{3.3}
\]

Here, as above as further, the expression in brackets of the first equality (3.3) means the greatest common divisor, and in the second equality – least common multiple. From the definition clearly, that the period of saturated filling for \(n = k_s\) grids contains equally one unit.

**Definition 18.** Sparse frequency \(\gamma\) is value, received by exception from initial frequency \(\alpha\) frequency and \(d\)-th digits of saturated filling:

\[
\alpha = 0.d^*\ldots d^*m_{r+1}m_{r+2}\ldots, \quad d^* = d - 1; \quad \gamma = 0.m_{r+1}m_{r+2}\ldots m_v \ldots \tag{3.4}
\]

where because of correctness of a system value \(m_{r+1} < d - 1\).

**Definition 19.** The grid \(S(a_i), 1 \leq i \leq n\) of filling \(Z_n\) is named as essential, if value of the maximum series is \(msr_n(0) \geq a_i\).

**Theorem 1.** Length \(msr_\alpha\{n\}\) of the maximum series \(MSR_\alpha\{n\}\) for degree \(n\)-filling \(Z_n\) of degree-system \(SS_d\) with the basis \(d \geq 2\) at zeroes frequency \(\alpha\) is expressed by the formulas (here parameter \(q\) is equal 0):

A. Quantity of grids \(n \leq k_s = (d - 1)\lfloor -\log_d(1 - \alpha)\rfloor = (d - 1) r\). Then

\[
MSR_\alpha\{n\} = m_{s\alpha}\{n\} = d^\lceil \frac{n}{d-1} \rceil \left\{ 1 + n - (d - 1) \left\lfloor \frac{n}{d-1} \right\rfloor \right\}. \tag{3.5}
\]
B. Quantity of grids \( n > k_s = (d - 1 \lfloor -\log_d(1 - \alpha) \rfloor) = (d - 1) r \). Then
\[
msr_{\alpha}(k_s) = d^{r'}, \quad k = n - k_s, \quad \gamma = 1 - d^r (1 - \alpha),
\]
\[
MSR_{\alpha}(n) : \quad msr_{\alpha}(n) = d^{r'} \cdot msr_{\gamma}(k).
\] (3.6)

C. Value of the maximum series \( MSR_{\gamma}(k) \) for frequency
\( \gamma = 1 - d^r (1 - \alpha) = 0. m_1 m_2 ... m_v m_{v+1} ... \) is equal
\[
msr_{\gamma}(k) = \frac{k - t_1}{1 - \gamma_v} + \frac{t_1 - t_2}{1 - \gamma_{v-1}} + ... + \frac{t_{v-1} - t_v}{1 - \gamma_1} + t_v + 1,
\] (3.7)
where
\[
t_j = \left( t_{j-1} - \sum_{i=1}^{s-j+1} m_i \right) \mod (d^{v-j+1} - \gamma_{v-j+1} d^{v-j+1}) + \sum_{i=1}^{v-j+1} m_i,
\]
j = 1, 2, ..., v; \( t_0 = k \); \( \gamma_j = \sum_{i=1}^{j} m_i d^{-i} \) under condition
\[
d^v - \sum_{i=1}^{v} m_i (d^{-i} - 1) - 1 < k \leq d^{v+1} - \sum_{i=1}^{v+1} m_i (d^{-i+1} - 1) - 1.
\]

Proof of the formulas we shall give consistently. Let \( n \leq k_s \) and frequency \( \alpha \) is expressed as (3.4). Then
\[
msr_{\alpha}(n) = \begin{cases} 
  d^{r'} & \text{for } n = k_s = r (d - 1), \\
  \frac{n}{d - 1} \left( 1 + n - (d - 1) \left[ \frac{n}{d - 1} \right] \right) & \text{for } n < r (d - 1),
\end{cases}
\] (3.8)
that it is enough clearly from concrete appendices and definition of regulated filling. For example, the second variant (3.8) for \( n = (r - 1) (d - 1) \) is reduced to first. From here expression (3.5) and statement A follows.

Expression B and relation (3.6) follow from a obvious conclusion, that the availability of saturated grids increases a maximum series that filling, but without saturated grids, equally in \( d^r \) time, that is increasing of series occurs by value of the period of filling from saturated grids. The particular case of equality (3.6) can be noticed in the second relation (3.8).

The most compound, but also the major variant is submitted by expression C. It finally permits find exact value \( msr_{\gamma}(k) \) of maximum series \( MSR_{\gamma}(n) \) at all parameters of degree-systems \( SS_d \).

Let the basis of correct degree-system \( SS_d \), \( d \geq 2 \) with zeroes density \( \gamma = 0. m_1 m_2 ... m_v m_{v+1} ... \) is given. Then reception of the maximum series \( MSR_{\gamma}(n) \) of filling \( Z_n \subset SS_d \) of length \( msr_{\gamma} = d^v \) will be required equally the quantity \( n \)
\[
\{ MSR_{\gamma}(n) \subset Z_n \} : \quad n = d^v - \sum_{i=1}^{v} m_i (d^{-i} - 1) - 1
\] (3.9)
$d$-th grids. This number is minimum, but from them only $K_1 = \sum_{i=1}^{v} m_i$ grids are essential. Really, quantity of units in the period $PZ = d^v$, formed by grids $K_1 = \sum_{i=1}^{v} m_i$ equally $d^v - \sum_{i=1}^{v} m_i d^{v-i}$. Product executes following grids for the scheme of regulated filling and since these grids will be inessential, that they can be replaced by infinite, we shall receive value $msr(q) = d^v$, where $q$ by one unit (boundary) less than units in the period of filling $Z(K_1)$. Thus, the common quantity of grids is

$$n = d^v - \sum_{i=1}^{v} m_i d^{v-i} + \sum_{i=1}^{v} m_i - 1,$$

that coincides with the formula (3.9). Such quantity of grids is minimum, that it follows from definitions for correct systems $SS_d$ for fixed $d$.

Then we have, after designation of the maximum series length for given frequency and grids quantity $msr_{\gamma} \{n\}$:

$$msr_{\gamma} \left\{ d^v - \sum_{i=1}^{v} m_i (d^{v-i} - 1) \right\} = d^v + msr_{\gamma} \left\{ \sum_{i=1}^{v} m_i \right\}. \quad (3.10)$$

It follows from the formula (3.9), whence we receive immediately the equality $msr_{\gamma} \{d^v - \sum_{i=1}^{v} m_i (d^{v-i} - 1) - 1\} = d^v$. Value and the maximum series, formed by $K_1 = \sum_{i=1}^{v} m_i$ grids, are repeated for an initial interval of the period $d^v$. Moreover it is necessary to note, the grid $S(d^{v+1})$ is inessential for the considered interval, as maximum series $msr_{\gamma} \{\sum_{i=1}^{v} m_i\} < d^v$.

In conditions of relation (3.10) we have equality

$$msr_{\gamma} \left\{ d^v - \sum_{i=1}^{v} m_i (d^{v-i} - 1) + k - 1 \right\} = d^v + msr_{\gamma} \left\{ \sum_{i=1}^{v} m_i + k - 1 \right\}, \quad (3.11)$$

where $1 \leq k \leq \left\{ d^{v+1} - \sum_{i=1}^{v} m_i (d^{v+1-i} - 1) - d^v + \sum_{i=1}^{v} m_i (d^{v-i} - 1) \right\} = (d - 1) \left\{ d^v - \sum_{i=1}^{v} m_i d^{v-i} \right\}$.

Really, there are $d^{v+1} - d \sum_{i=1}^{v} m_i d^{v-i}$ units in the interval by length $d^{v+1} - 1$ as a result of product of $\sum_{i=1}^{v} m_i$ grids, as for this interval already grids of kind $S(d^{v+1})$ are inessential (3.9). Thus, the formula (3.11) will be valid so long as

$$k + d^v - \sum_{i=1}^{v} m_i (d^{v-i} - 1) \leq d^{v+1} - d \sum_{i=1}^{v} m_i d^{v-i} + \sum_{i=1}^{v} m_i,$$

whence we receive the border for values $k$ from above. For large $k$ it should in the expression (3.11) replace parameter $v$ to $v + 1$. At $k = 1$ we are return to
the formula (3.10), and at  \( k = 0 \) (outside of conditions) in a left-hand part (3.11) we receive grids quantity (3.9) and length of series \( msr_\gamma = d^v \).

Let are given again the basis of correct degree filling  \( d \) and density of zeroes \( \gamma = 0. m_1 m_2 ... m_v ... \), but already at  \( 0 \leq m_1 < d - 1 \), that is for sparse. Then

\[
msr_\gamma\{k\} = \frac{k - t_1}{1 - \gamma_v} + msr_\gamma\{t_1\}, \tag{3.12}
\]

where \( t_1 = (k - \sum_{i=1}^{v} m_i) \mod (d^v - \gamma_v d^v) + \sum_{i=1}^{v} m_i \),

\[
\gamma_v = 0, m_1 m_2 ... m_v = \sum_{i=1}^{v} m_id^{-i} \text{ under condition}
\]

\[
d^v - \sum_{i=1}^{v} m_i(d^{v-i} - 1) - 1 < k \leq d^{v+1} - \sum_{i=1}^{v+1} m_i(d^{v-i+1} - 1) - 1, \tag{3.13}
\]

and if  \( k \leq d - 1 \), then \( msr_\gamma\{k\} = k + 1 \).

From expression (3.11) it is possible to conclude, that when the value \( k \) lies in borders, specified by relations (3.10, 3.11), the maximum series is equal

\[
msr_\gamma\{k\} = d^v + msr_\gamma\left\{k - d^v + \sum_{i=1}^{v} m_id^{v-i}\right\} = d^v + msr_\gamma\{k - d^v(1 - \gamma_v)\}. \tag{3.14}
\]

We apply consistently the formula (3.14)  \( j \) time so that the value \( k - j d^v (1 - \gamma_v) \) has not become less \( \sum_{i=1}^{v} m_i \). At the same time  \( j \) should be greatest of possible. The limits of change \( k \) are established in view of product of the first \( \sum_{i=1}^{v} m_i \) grids. Under these conditions the value \( t_1 = k - j d^v (1 - \gamma_v) \) can be found only as

\[
t_1 = \left( k - \sum_{i=1}^{v} m_i \right) \mod (d^v - \gamma_v d^v) + \sum_{i=1}^{v} m_i = \left\{ k - \sum_{i=1}^{v} m_i \right\} d^v (1 - \gamma_v) + \sum_{i=1}^{v} m_i, \tag{3.15}
\]

where  \( \cdot \) means fractional part of function. From here follows, that  \( j = \frac{k - t_1}{d^v(1 - \gamma_v)} \),

and then we shall receive

\[
msr_\gamma\{k\} = jd^v + msr_\gamma\{t_1\} = d^v \frac{k - t_1}{d^v(1 - \gamma_v)} + msr_\gamma\{t_1\}, \tag{3.16}
\]

that coincides with the statement (3.12). The necessity of conditions (3.13) at search of value  \( t_1 \) (3.15) is obvious, as differently becomes impossible filling by \( \sum_{i=1}^{v} m_i \) given grids. Value  \( j \) in (3.16) is common quantity of equality applications (3.14). The last condition (3.13) with the maximum series are also obvious, as far as variant is submitted here, when all grids are inessential.

Now we apply expressions (3.12, 3.13) recursively for reception of maximum series \( MSR_\gamma\{k\} \) value, that is we are addressed to the formulas (3.12, 3.16) at
first at greatest allowable $v$, determined by condition, then at $v-1$, $v-2$, ..., 2, consistently finding values $t_j$ from expression:

$$t_j = \left( t_{j-1} - \sum_{i=1}^{v-j+1} m_i \right) \mod (d^{v-j+1} - \gamma_{v-j+1} d^{v-1} + \sum_{i=1}^{v-j+1} m_i) , \quad (3.17)$$

repeating condition of the theorem. In equality (3.17) the parameters are in the borders $j = 1, 2, ..., v$; $t_0 = k$; $\gamma_j = \sum_{i=1}^{j} m_i d^{-i}$, and initial value $v$ is founded from condition (3.13).

As value $v$ is equal to $(t_{v-1} - m_1) \mod (d - m_1) + m_1 \leq d - 1$, hence maximum series is $msr_\gamma \{ t_v \} = t_v + 1$. The theorem is proven. \hspace{1cm} \Box

As an example we shall consider the degree-system $SS_3$ with given density $\gamma = 5/8 = 0.121212...$. We shall find $msr_\gamma \{ 16 \}$. Thus $k$ value $v = 3$, as with condition (3.13) we receive: $14 < 16 = k < 37$. If to take into account, that from relation (3.17) and for $\gamma_j$ we have: $t_1 = 5$, $1 - \gamma_3 = 11/27$; $t_2 = 5$, $1 - \gamma_2 = 4/9$; $t_3 = 1$, $1 - \gamma_1 = 2/3$, with the help of expression (3.7) we shall receive value of maximum series $msr_\gamma \{ 16 \} = 27 + 6 + 1 + 1 = 35$.

It should note, that in given statement the task of parameter $q$ in inexpedient, as units automatically enter in value $k$ because from inessential grids of next filling. However it do not without this parameter, equivalent to quantity of infinite grids, at the task of zeroes density of filling in a kind of finite fraction.

Thus, if density of zero $\gamma < 1$ and basis $d \geq 2$ of degree-system $SS_d$ are given, intervals of maximum series $MSR_\gamma \{ k \}$ always are determined precisely with the help of all three relations of the theorem 1. Their application does not assume any restrictions relatively included saturated or essential grids in fillings.

The theorem 1 permits to generate the important conclusions.

**Theorem 2.** The value of maximum series $msr_\alpha \{ n + q \}$ with $q$ units allows unimprovable valuation in any degree-system $SS_d$ with zeroes density $0 < \alpha < 1$

$$MSR_\alpha \{ n + q \} \subset SS_d : \quad msr_\alpha \{ n + q \} < \frac{n + q}{1 - \alpha} + 1 . \quad (3.18)$$

*Proof* reasonably transparent follows from the formulas of the theorem 1. Not too complex to show, that availability $k_s \geq d - 1$ grids of saturation only eases the formulation of the theorem. Therefore we shall consider case C. For it

$$msr_\gamma \{ k \} = \frac{k - t_1}{1 - \gamma_v} + \frac{t_1 - t_2}{1 - \gamma_{v-1}} + ... + \frac{t_{v-1} - t_v}{1 - \gamma_1} + t_v + 1 ,$$

the sum of non-negative numerators of fractions is equal $k$, and denominators $1 - \gamma_{v-i} \geq 1 - \gamma$ for all $i$. Thus, the series value (3.7) is the closer to valuation
The closer frequency \( \gamma \) to zero. From here follows unimprovable valuation. A final kind expression (3.18) acquires after replacement \( k \) in (3.7) to \( n + q \). □

The main merit of the theorem 1 consists in important generalization for a class of systems.

Earlier all states and conclusions of this chapter were formulated for class of degree-systems \( SS_d \). At the same time rather easily to look after, that reasoning at designing and algorithmization of constructions of maximum series \( MSR_n(q) \) for system without multiple zeroes are analogous considered by the theorem 1. Similar though naturally little more complex and difficult will be and formula relations of the type (3.4 – 3.7).

**Theorem 3.** The value of maximum series allows absolute unimprovable majorant in any system \( SS \) without multiple zeroes with the density \( 0 < \alpha < 1 \)

\[
MSR_\alpha\{n + q\} \subset SS : \quad msr_\alpha\{n + q\} < \frac{n + q}{1 - \alpha} + 1. \tag{3.19}
\]

**Proof.** A class of systems without multiple zeroes is essentially wider of degree-class: \( SS \supset SS_d \). For example, to such class systems of grids concern:

\[
S(a_i) \subset SS : \quad (a_i, a_{i+1}) = a_i, \quad \forall i \geq 1. \tag{3.20}
\]

At the same time not only the systems, satisfying to relation (3.20), enter in such class. Besides such systems \( SS \) always the first type, but not all systems of the first type are systems without multiple zeroes.

Unimprovable valuation from below, indicated in such transparent form (3.19), is reasonably clear, as far as \( msr_\alpha\{n + q\} \leq n + q + 1 \), and the value \( \alpha \) can be near from zero. It means, that at all inessential grids easily find a border of density \( \alpha < \alpha_0 \), for which

\[
msr_\alpha\{n + q\} = \left\lfloor \frac{n + q}{1 - \alpha} \right\rfloor + 1 = n + q + 1, \quad \left\lfloor \cdot \right\rfloor - integer \ part.
\]

The relations \( A \) and \( B \) of theorem 1 for saturated grids of with evidence are transferred for systems without multiple zeroes. Therefore the special attention is deserved case \( C \) and formula (3.7). However easily to see, that algorithmical features of constructions of maximum series in systems without multiple zeroes and degree-systems coincide. It means, that recurrent formulas for calculation of the maximum series in systems \( SS \) should be the type (3.7) and to differ only reception of values \( t_j \), which we shall designate \( tt_j \).

In a result we shall receive transformed from a relation (3.7) formula, in which the given zeroes density \( \alpha \) is consistently submitted approximations \( \alpha_i \). They are similarly connected by inequalities

\[
\frac{1}{1 - \alpha} > \frac{1}{1 - \alpha_v} \geq \frac{1}{1 - \alpha_{v-1}} \geq ... \geq \frac{1}{1 - \alpha_1},
\]
while corresponding non-negative values $tt_j$ give

$$n + q - tt_1 + tt_1 - tt_2 + \ldots + tt_v = n + q.$$  

If given filling maintains $k_s$ saturated grids, we shall act by analogy with the degree-filling. After substitution of valuation for sparse density $\gamma$, taking into account a period of filling by saturated grids, equal $D = \frac{1-\gamma}{1-\alpha}$, we shall receive:

$$msr_\alpha\{n\} = D \cdot msr_\gamma\{n-k_s\} < \frac{1-\gamma}{1-\alpha} \left(\frac{n-k_s}{1-\gamma} + 1\right),$$

whence inequality (3.19) follows with evidence. □

The estimation of theorems 2 and 3 at availability of saturated grids has some redundancy, which essentially less, if initial zeroes density is sparse. The valuation will be also redundant and in case of incorrect degree-system, that is in variant given non-canonical decomposition of zeroes frequency (density). As follows from appendices [1], the best approximation of valuations (3.18) in the majority of cases is maximum series of correct binary system $SS_2$.

Quite similar state is observed for valuations (3.19) of maximum series and for systems without multiple zeroes $SS$, though the formulas for them are not given here. However role of zeroes density for such systems is same, hence, and the high valuations should coincide, and obstacle can not act concrete formula realization for values of maximum series $MSR_\alpha\{n\}[SS]$.

The reception of algorithms for calculation of exact values of the maximum series (theorem 1) and upper generalized estimation (theorem 3) permit to make conclusions for a specific class of systems without multiple zeroes, which can serve by necessary spring-board at reception essentially important generalization.

**4. Imaging principle and main theorem**

The expressions and high valuations of maximum series of the theorems 1–3 are found for systems without multiple zeroes. However systems with the multiple zeroes present greatest interest just. All systems of the second and third type without fail have zeroes of multiplicity higher unit. In particular, if in filling $Z_n$ there will be though one pair of grids with modules $(a_i, a_j) = 1$, in the period multiple zeroes will meet. At the same time fulfilment of relation

$$S(a_i), S(a_j) \subset Z_n : (a_i, a_j) \neq 1, \quad \forall 1 \leq i \neq j \leq n \quad (4.1)$$

not yet guarantees absence of multiple zeroes. For the most important in appendices systems of the third type the share of zeroes multiplicity higher unit increases to unit at $n \to \infty$.  

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Results of the previous chapter prove, that the problem of search of maximum series of fillings \( Z_n \), their values \( msr_n(q) \) and particularly valuations is decided successfully for systems without multiple zeroes. From here there is idea of approximation of fillings for any systems by fillings without multiple zeroes, let even for level of the majorizing characteristics. This purpose some mental construction, named imaging principle corresponds.

We are addressed to fillings with the extreme characteristics for a given set of \( n \) grids. Naturally, only non-singular systems are implied.

The unregulated fillings are not random, but found during exhaustive search or different way for an evaluation of interesting numerical characteristics. For example, maximum series generally can be guaranteed are found, identified and are appreciated only on the class of unregulated fillings. The modifications of the sieving process do not removal from the numerical nature of the worked up sequence, as against it is in the class \( L \). The call to Boolean elements allows to use completely other constructions.

The main difference of one method from other consists in an evaluation of the majorizing characteristics of the \( n \)-filling. It can and even conveniently be passed from the integer analysis of outcomes of each grid effect to learning cooperative influence of zeroes frequencies of \( n \)-filling. It happens for all period, and not just on an initial interval of length about \( p_{n+1}^2 \), which one restricts itself the sieving process. Therefore fillings method can be interpreted as method of the analysis of frequencies of Boolean zeroes – results of grids products.

**Definition 20.** If \( E_n \) there is quantity of units on period of length \( PZ_n \) for nonsingular \( n \)-filling \( Z_n(a_1, ..., a_n) \) of the system \( SS \), and \( H_n = PZ_n - E_n \) is number of zeroes, the value of total zeroes (sum of multiplicity) \( H^*_n \) is

\[
H^*_n(Z_n) = PZ_n \sum_{i=1}^{n} \frac{1}{a_i}, \quad PZ_n = ((a_1, a_2, ..., a_n)).
\]

**Definition 21.** The main object of the research is two-dimensional strip region of binary values of volume \((n; \infty)\) of grids

\[
S(a_1) : \ldots l_{j-1}^{(1)} l_j^{(1)} l_{j+1}^{(1)} \ldots \quad \forall i \{ l_j^{(i)} = 1 \} : \; t_j = 1;
S(a_2) : \ldots l_{j-1}^{(2)} l_j^{(2)} l_{j+1}^{(2)} \ldots \quad \exists i \{ l_j^{(i)} = 0 \}:
\quad 1 \leq k \leq m \leq n
\]

\[
W_n : \ldots t_{j-1} 0_j^{(1)} 0_j^{(2)} \ldots 0_j^{(k)} t_{j+1} \ldots,
\]

where \( m \) is quantity of zeroes in column \( l_j^{(1)}, l_j^{(2)}, ..., l_j^{(n)} \). The imaging \( W_n \) of
grids $S(a_i)$ on the sequence of the same values (4.2)

$$S(a_1) \& S(a_2) \& \ldots \& S(a_n) \overset{f(k)}{\Rightarrow} W_n; \quad k \leq m$$

is created on rules: the unity element $t_j = 1$ corresponds to unity column; the imaging $\overset{f(k)}{\Rightarrow} W_n$ transforms zeroes to the sequential series from $k$ ($1 \leq k \leq m$) zeroes, if column has $m$ zeroes ($1 \leq m \leq n$).

Variant of four grids of prime modules we shall give as an example of the imaging:

$$
\begin{align*}
S(3) : & \quad \ldots1011011101101101101\ldots \\
S(4) : & \quad \ldots10111011101101101101\ldots \\
S(5) : & \quad \ldots101111011110111011101\ldots \\
S(7) : & \quad \ldots10111111110111110111101\ldots \\
\overset{\Rightarrow}{\Rightarrow} : & \quad \ldots-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-\rightarrow-
W_4 : & \quad \ldots10001100000001001000010100001\ldots.
\end{align*}
$$

Here value $k = 3$ is the imaging of column of four zeroes in first case, where zero of product is substituted by three zeroes. Further product of two zeroes represented by pair of zeroes in imaging $W_4$ in all four cases.

The imaging of multiple zeroes should be realized at the expense of increase of period length, as the quantity of units $E_n$ on period is constant. It in sufficient measure the conditional increase of period is followed to perceive only as the tool of obtaining of necessary numerical characteristics. Length of period of imaging $W_4$ in an example is conditional also. But it exceeds the value $PZ_4$.

Sense of such conditional increase of period at the expense of multiple zeroes consists in an evaluation of the local interval of product of grids. Then the product of grids with multiple zeroes can by suitable shifts be resulted with diminished quantity of such multiple zeroes or even with their complete liquidation. Series and maximum series of zeroes place just on intervals of the similar type, that is highly small length, it is much less value of period $PZ_n$. But the redistribution is possible not always.

**Definition 22.** If the imaging $W_n$ takes into account all $m$ of zeroes, that is in all cases $k = m$, that this complete imaging of zeroes ($W_n^*$). If the value $k > 1$ even in one case, but is not always executed $k = m$, we have incomplete imaging ($W_n^{**}$). Direct imaging of zeroes ($W_n$) is obtained at $k = 1$, given in all period. The imagings are $W_n^* = W_n^{**} = W_n$ for systems $SS'$.

Definitions 20–22 represent itself as the scheme of the imaging principle.

**Definition 23.** The frequency of zeroes $\gamma_n$ on period of imaging – from direct ($W_n$) up to complete ($W_n^*$), serves for the basis adequate ratings of the maximum series $MSR_n(q)$ for all systems with multiple zeroes:

$$1 - \frac{E_n}{PZ_n} = \frac{H_n}{PZ_n} = (\gamma_n) < (\gamma_n^{**}) < (\gamma_n^*) = 1 - \frac{E_n^*}{PZ_n^*} = \frac{H_n^*}{PZ_n^*}.$$
Clearly, the evaluations of the maximum series $MSR_n(q)$ should be constructed because of frequencies of zeroes ($\gamma_n$) of direct imaging for values $q$, near from quantity of units $E_n$ on period. Quite other position develops with evaluations of series $MSR_n(q)$ for small $q$ or even for $q = 0$. The intermediate frequency ($\gamma^{**}_n$) certainly is frequency of incomplete imaging ($W^{**}_n$). It is necessary to mark, if the parameters $\gamma_n(q), n$ and $q$ are given, it is possible to forget about concrete set of grids $\{a_i\}$.

Really, these parameters are sufficient for obtaining unknown quantities, but approximate ratings. However precise definition of maximum series $MSR_n(q)$ requires of greater.

Let's formulate main definition touching the means of learning of introduced systems and fillings.

**Definition 24.** The fillings method is research of properties and characteristics of $n$-fillings and systems $SS$ with multiple zeroes, and also obtaining of the series of fundamental numerical estimations with the help of imaging of zeroes $W_n$ of all types.

Generally filling is reduced to imaging of two-dimensional strip region of elements on one-dimensional with partial or complete conversion of multiple zeroes in single. Such extended filling on changed period is base of learning of properties of source distribution of zeroes and units as products of grids for different classes of systems.

The explained principles of the fillings method have allowed to reveal central relation, all rests are consequences from which. The formula reflects statement, is foolproof enough expressed mathematically and claiming to be main for the rather vast class of the tasks of number theory. The versatility of this relation do not know exceptions on set of nonsingular systems and fillings.

The offered thesis does not imply dependence from principle of imaging. The principle only explains paths and sources of the approach, sense and parents of appearance. At the same time at all riches of applications and importance of the obtained outputs, the thesis can be surveyed in the different forms with direct, incomplete or complete imaging. First of all call to this or that form of the main theorem is determined by the degree of correspondence to content and fundamental essence of the fillings method.

**The main theorem.** Estimation of maximum series $MSR_n(q)$ is valid for anyone $n$-filling $Z_n$ of the arbitrary nontrivial system $SS$:

$$msr_n(q) < \frac{n + q}{1 - \gamma^*_n} + 1 = M_n(q), \quad \gamma^*_n = \frac{H^*_n}{E_n + H^*_n},$$  \hspace{1cm} (4.3)

where $E_n$ is quantity of units of filling's period $PZ_n$ and $H^*_n$ is total
(sum of multiplicity) zeroes. Value of density $\varrho_n$ for each $1 \leq n < \infty$ will be discover always:

$$msr_n(q) \leq \frac{n + q}{1 - \varrho_n} + 1, \quad \sup_{SS} \sup_{(Z_n)} \max_{0 \leq q < \infty} \{\varrho_n(Z_n \subset SS)\} < \gamma_n^*, \quad (4.4)$$

$$\inf_{SS} \sup_{(Z_n)} \max_{0 \leq q < \infty} \{\varrho_n(Z_n \subset SS)\} = \gamma_n, \quad \gamma_n = \frac{H_n}{PZ_n}, \quad (4.5)$$

where $\sup$ and $\inf$ are in the class of nontrivial systems. But concrete kind of $n$-filling can define incomplete imaging and appropriate frequency of zeroes $\gamma_n^{**}$, and consequently unimprovable estimation for some system $SS$

$$msr_n(q) \leq \frac{n + q}{1 - \varrho_n} + 1, \quad \sup_{(Z_n)} \max_{0 \leq q < \infty} \{\varrho_n(Z_n \subset SS)\} < \gamma_n^{**}. \quad (4.6)$$

The following inequalities take place for all classes of nonsingular $n$-fillings $SS$, where $0.5 < C = C(SS) \leq 1$:

$$\gamma_n \in Z_n \subset SS : \quad C \left(\frac{n + q}{1 - \gamma_n}\right) + 1 < msr_n(q) < 2 \frac{n + q}{1 - \gamma_n} + 1. \quad (4.7)$$

The dependence of the zeroes density $\varrho_n$ from $q$ leads to the form

$$\exists q; \varrho_n(q) : \quad msr_n(q) = \frac{1 + q}{1 - \gamma_n} + 1; \quad \lim_{q \to \infty} \varrho_n(q) = \gamma_n. \quad (4.8)$$

The first part (4.3) of theorem states about existence absolute majorizing frequency (density) of zeroes for arbitrary filling of any nonsingular system $SS$ of grids. The density of zeroes of complete imaging $\gamma_n^*$, $(\gamma_n^* \geq \gamma_n^{**} \geq \gamma_n)$ represents itself as such frequency. It determines unconditional and even an inaccessible upper-bound estimate $M_n(q)$ of an appropriate maximum series $MSR_n(q)[SS]$.

The logic and constructibility of such evaluation form of maximum series $MSR_n(q)$ is justified by complete coincidence with an evaluation for fillings without multiple zeroes. The transition to the relation (4.3) for arbitrary systems, including with multiple zeroes, becomes well-grounded after operation of complete imaging of multiple zeroes of $n$-dimensional strip region of binary elements (zeroes and units of grids). In an outcome all zeroes on complete (extended) period $PZ_n^*$ have multiplicity of unit.

Each system $SS$, any more not speaking about $n$-filling, has the majorizing density of zeroes, which one here is marked $\varrho_n$. It provides an evaluation of the inequality (4.4), replicated main form (4.3) at all values $n \geq 1; \; q \geq 0$. Nevertheless, top and bottom boundary of densities of zeroes on the class of all acceptable systems of grids are, accordingly, the frequencies $\gamma_n^*$ and $\gamma_n$ of relations (4.4, 4.5).
However redundancy of an evaluation $MSR_n(q)$ for systems without multiple zeroes, especially has an effect in variant of arbitrary fillings. It is explained to the redundancy of an evaluation (4.3) for small intervals (that is $n$ and $q$), in the total reduces in such interval, for which one there is no filling without multiple zeroes. Thereby some multiple zeroes appear superfluous in data conditions and consequently is acceptable to be restricted to incomplete imaging of zeroes. So frequency of zeroes $\gamma_n^{**}$ lesser what $\gamma_n^*$ but exceeding $\gamma_n$ occurs.

All these reasons reduce to appropriate densities of zeroes and evaluations (4.6). The transition to more precise modification of the method of fillings gives detection of multiple zeroes which do not lead to increase of majorizing density. The rather outstanding part of such inefficient multiple zeroes of all period can appear for number of systems. Naturally, it essentially will decrease value $\gamma_n^{**}$ about $\gamma_n^*$. At the same time it is impossible to guess, that the value $\gamma_n^{**}$ will reach value $\gamma_n$ for great many of multiple zeroes.

The unimproving evaluations (4.7) of the maximum series $MSR_n(q)$ was obtained from the study of axis configurations of the prime system $SP_1$.

The realizability of equality (4.8) for some values $q$ (for example, for $q = E_n - 1$) at $\varrho_n = \varrho_n(q) = \gamma_n$ is the quite definite characteristic of majorizing density of zeroes $\varrho_n$ in the expression (4.6). In this case maximum series coincides an evaluation for $n = 1$ and it is equal to length of period $msr_n(E_n - 1) = PZ_n$. From here it is clear, only value $\gamma_n$ can be by limit (4.8) for constant $n (n > 1)$ and $q \to \infty$ for density of zeroes $\varrho_n$ in an estimation (4.6).

Thus, majorizing estimation of the maximum series $MSR_n(q)$ is connected inversely proportional dependence with the density of zeroes in the period. It appears by the adequate characteristic of imaging (complete or incomplete) multiple zeroes of the strip region of elements. Zeroes frequency of complete imaging $\gamma_n^* = \frac{H_{n}}{E_n + H_n}$ in elongated period $PZ_n^* = E_n + H_n^*$ (sometimes $PZ_n^* \gg PZ_n$) is thus natural absolute majorant for any systems and fillings.

So on the first view indisputable on logic and validity the thesis nevertheless, requires the proof of impossibility of sieve substitution of multiple zeroes of units on an interval of the maximum series $MSR_n$. It is really impossible as well as in variant of fillings without multiple zeroes. As the distribution of zeroes of grid is uniform, the transition of units in zeroes is obliged to lead in restoring units on adjacent places. As the frequency $\gamma_n^*$ registers and takes into consideration zeroes of all multiplicity without exception.

At the same time one of proofs of the main theorem is received from the analysis of fixed distributions so named axial series for the period of $n$-fillings. It does not lean on the imaging principle, but confirms legitimacy of its introduction and consideration. Thus the majorizing constant two is found for the third form of the main theorem. This constant is unimproved, as it is given below.

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5. Premises of an evidence of the main theorem

The offered approach is not uniquely possible.

Definition 25. Algorithm of construction of maximum series of zeroes \( MSR_n(q) \) shall be name one-sided \( Z_n^{(1)} \), if the filling is conducted by half-grids from chosen beginnings in one party (for example, right).

Regulatedness, semi-regulatedness and unregulatedness of a mechanism of fillings here remain in complete force at formation of any series of zeroes \( SR_n(q) \), \( q \geq 0 \), including and maximum \( MSR_n(q) \). One-sidedness does not depend from other characteristics of filling algorithm. The half-grids is one-sided infinite grids.

It is possible to remind, at fixing of beginning strict regulatedness means product the first (from the right) unit of current filling \( Z_{n-1} \) and boundary unit of a series \( SR_{n-1} \) with the zero of a grid \( S(a_n) \). The received product then will be filling, but maximality of a formed series is observed not always and requires separate consideration. We shall suggest another scheme of fillings mechanism.

Definition 26. Algorithm of construction of series and maximum \( MSR_n(q) \) series of zeroes shall be name two-sided \( Z_n^{(2)} \), if the filling is conducted by complete grids (two-sided infinite) till both party from a point of beginnings.

The sense of the introduction of one-sidedness of fillings mechanism clears up for regulated and semi-regulated algorithms of search and it is in direct dependence on expressions of next modules of allocated filling.

Consequence 1. The values of maximum series of zeroes \( msr_n(q) \) of unregulated fillings coincide at one-sided and two-sided filling.

\[
MSR_n(q, Z_n^{(1:2)}, ZN_n, SS) : \quad msr_n(q, Z_n^{(1)}, SS) = msr_n(q, Z_n^{(2)}, SS). \quad (5.1)
\]

Proof. At unregulated filling search of maximum series of zeroes equivalent complete selection of all possible products of grids. Then variants of a configuration of zeroes of such series \( MSR_n(q, Z_n^{(k)}) \) can not coincide. Obviously their arrangement on a numerical axis (or half-axis) differs, but the lengths of such series coincide on a sense of construction. Thus the expression (5.1) is executed for any parameters \((n, q)\), fillings \( Z_n \subset SS \) and systems \( SS \).

\( \Box \)

Theorem 4. The construction of maximum series of zeroes \( MSR_n(q) \) is oriented to one-sided filling \( Z_n^{(1)} \) for systems of grids \( SS_2(a_i) \), the unequal next modules of which are connected by the expression \( a_{i+1} \geq 2 \cdot a_i \) (for \( a_{i+1} \neq a_i \)).

\[
MSR_n(q, SS_2) : \quad MSR_n(q, Z_n^{(1)}, SS_2) \Rightarrow msr_n(q), \quad a_i \in SS_2. \quad (5.2)
\]
Proof. It is necessary at once to note, that for grids of equal modules the factor of one-sidedness or two-sideness is away by virtue of them indistinction. Therefore is allowable to be limited to consideration of a system with growth of modules of all grids. Thus there is inessential mutual prime of modules of this system of the first type – that is with a sum of values of reverse modules less unit, if to exclude a trivial binary system.

Let down to some stage, that is up to parameter \( n - 1 \) inclusive, the search of maximum series \( MSR_k(q) \) at \( k \leq n - 1 \) was maintained pursuant to the scheme of one-sided fillings. It means, that all grids down to \( S(a_{n-1}) \) participated in one-sided filling, and concentration of zero from the right from a index point appreciably higher, than at the left, if instead of half-grids of the statement temporarily to consider complete grids.

In such case the second zero of the half-grid \( S(a_{n-1}) \) on a constructed interval of a series will meet earlier, than the first zero of a next half-grid \( S(a_n) \). It means, that the one-sided algorithm of filling contains higher potential of growth of a series, including maximum series \( MSR_n(q) \). Unique difficulty is made in possible earlier occurrence of a zero of multiplicity two, that is in crossing of zero, that can partly deform a reasoning. Just is here allowable some variability in filling, not changing the to common scheme of one-sidedness.

Really, the primary occurrence of zeroes of grids of smaller modules forces to address to the one-sided scheme, as the attempts of two-sided filling are obliged to result in downturn of concentration of zeroes in a interval because of an inequality \( a_n/2 \geq a_{n-1} \). Zeros of a grid \( S(a_n) \) are just on distance \( a_n/2 \) from a central point of a prospective two-sided interval.

But then, if to take into account specific character of such fillings for the first type system, value \( msr_{n-1} < a_{n-1} \), and the second zero of a grid \( S(a_{n-1}) \) by no way can not enter in a series \( SR_{n-1} \) or \( MSR_{n-1} \). For this reason the zero of a grid \( S(a_n) \) as grid of two-sided filling can not already participate in formation of a series of the heaviest length. And on a next step \( (n+1) \) significance of the second zero of a grid \( S(a_{n-1}) \) becomes determining. It liquidates the unjustified claims of grids \( S(a_n) \) and \( S(a_{n+1}) \) for two-sided algorithm of filling.

This conclusion is reflected by expression (5.2). The orientation to one-sided filling is wholly determined by growth of a next module. In such case attempts of two-sided filling \( Z_n^{(2)} \) are inexpedient, as the one-sided filling \( Z_n^{(1)} \) essentially more effective conducts to formation of maximum series \( MSR_n(q) \) for systems of grids of a kind \( SS_{[2]}(a_i) \), \( a_{i+1} \geq 2a_i \), if grids are unequal \( (a_i \neq a_{i+1}) \).

The narrowing of a class of systems \( SS \subset SS_{[2]} \) admits concreteness of the statement. This statement connects important properties of fillings.

**Theorem 5.** Construction of maximum series of zeroes \( MSR_n(q) \) at any \( q \geq 0 \) is provided with regulated one-sided algorithm of filling \( ZU_n^{(1)} \) for all non-singular degree-systems \( SS_d(a_i) \) of any basis \( d \geq 2 \).
\[ MSR_n(q, SS_d) : MSR_n \{ q, ZU_n^{(1)}, SS_d(a_i) \} \Rightarrow msr_n(q), \quad ZU_n^{(1)} \subset Z_n^{(1)}. \quad (5.3) \]

**Proof.** Obviously, the degree-system \( SS_d \) at any basis \( d \geq 2 \) enters in a class of systems of primary growth of modules \( SS_d \subset SS_{[2]} \). Such system has not multiple zeroes at any \( n \) from definition.

But absence of multiple zeroes at such correlation of modules of grids transform one-sidedness and regulatedness of filling into the compulsory rules. Any infringement of such rules conducts to distortion of maximum. The expression (5.3) establishes inevitability of algorithm of regulated one-sided filling \( ZU_n^{(1)} \) at formation of all maximum series \( MSR_n(q) \).

**Theorem 6.** The one-sided filling remains effective for systems without multiple zeroes \( SS' \). Strict regulatedness can be sometimes infringed at realizations of grids of modules \( a_{i+1} < 2 \cdot a_i \), if \( a_{i+1} \neq a_i \).

\[ MSR_n(q, SS') : MSR_n \{ q, SS'(a_i) \} \Rightarrow MSR_n \{ q, Z_n^{(1)} = ZU_n^{(1)}[a_{i+1} \geq 2 \cdot a_i] \}. \quad (5.4) \]

**Proof.** A system of grids \( SS' \), all fillings of which do not contain zeroes of multiplicity above unit, is in the correlation \( SS_d \subset SS' \). But inclusion of a system \( SS' \) in a class of systems \( SS_{[2]} \) optionally, that is probably \( SS' \not\subset SS_{[2]} \). For example, grids \( S(4), S(6) \) will form such 2-filling, which can enter in a system \( SS' \). At the same time fitting to a system of the first type \( \sum 1/a_i \leq 1 \) precisely indicates on basic character of degree-filling at creation of systems \( SS' \) without multiple zeroes.

Really, mutual non-primeness of modules of any grids \( a_i, a_j \neq 1 \) of the systems demonstrates or very the large modules \( a_i \) of all grids as products prime multipliers for initial fillings, or reasonably foreseeable growing similitude with a degree-system. But the first variant can ensure smaller density of zeroes for such fillings, at the same time compelling pass to the degree-schemes at growth of parameter \( n \). Second, more natural and in a limit the same variant is stipulated by conditions of the theorem.

In this case any filling of a system without multiple zeroes \( SS' \) can be submitted on the basis of consecutive representations of some filling of a degree-system. Actually, if the determined filling is realized by grids \( \{ S(a_i) \} \), at first degree, and consequently not possessing multiple zeroes, formation of other filling, in which modules \( a_i \) change by product with any, including identical multipliers \( b_i : a_i \Rightarrow b_i \cdot a_i, \quad b_i \in N \), leaves filling in class without multiple zeroes.

Such way can essentially increase a period of filling and in parallel to decrease density of zeroes. Therefore for growth of initial and received density the introduction of grids is allowable, but already not any degree, but agreed with current modules, not to admit multiple zeroes.
Or else, in the basis of any filling, included in system without multiple zeroes, lies some degree filling (can be, even not unique). The number of modules of its grids (may be, even all) are transformed by product with multipliers \( b_i \), in common case not connected with the basis \( d \) of degree filling. Clearly, that such filling nor will give multiple zeroes. It though because are used \( 1/b_i \) zeroes of each grid \( S(a_i) \).

But the first zero of these two grids of formed series can coincide, and distinction multipliers \( b_i \) in a condition to infringe strict regulatedness of degree filling. Clearly, that such filling nor will give multiple zeroes. It though because are used \( 1/b_i \) zeroes of each grid \( S(a_i) \).

The described way of construction of fillings without multiple zeroes fixes result, lying closer to results for degree-systems, than less various multipliers \( b_i \), different from unit, in a transformed set of modules \( \{ b_i \cdot a_i \} \). Thus there can arise an expression for modules

\[
 b_i \cdot a_i < 2 \cdot b_j \cdot a_j, \quad b_i \cdot a_i > b_j \cdot a_j; \quad \{ S(4), S(6) : 6 < 8; 6 > 4 \},
\]

which partial infringement of regulatedness admits at formation of maximum series. In indicated examples with two grids, in particular, one infringement of regulated algorithm takes place in case \( S(4), S(6) : 8 = m\text{sr}_2[3, ZN^{(1)}] > sr_2[3, ZU^{(1)}] = 7 \).

If to take into account limited opportunities of growth of a set of various multipliers \( b_i \) because of a essential increase of transformed modules of grids (that decreases opportunities of occurrence of close modules), from here and statement of the theorem and expression (5.4) follows. In conditions of a system of the first type infringements of regulated algorithm are reasonably rare on a common background. At primary growth of next modules obviously use of regulated fillings for all variants of series with units \( q \geq 0 : Z_{U_n}^{(1)}[a_i+1 \geq 2 \cdot a_i] \).

\[ \text{Theorem 7.} \]

Just the algorithm of one-sided filling \( Z^{(1)}_{n}, n > n_0 \) is obliged to demonstrate efficiency for systems of grids \( SS_{(I)} = \{ S(a_i) \} \) of the first type, in any case, since some parameter \( n_0 \) for \( n > n_0 \).

\[
 MSR_n[q, SS_{(I)}] : MSR_n \left\{ q, SS_{(I)} \left( \sum \frac{1}{a_i} \leq 1 \right) \right\} \Rightarrow MSR_n \left\{ q, Z_{n}^{(1)} \right\}, n > n_0.
\]

\[ (5.5) \]

\[ \text{Proof.} \]

Degree-systems \( SS_d \) and the systems without multiple zeroes \( SS' \), naturally, satisfy to conditions of the statement, that confirm the proved theorems 5 and 6. About it speaks and theorem 4, considering grids with growing twice and more modules. Systems of the first type \( SS_{(I)} \) include all listed, but wider of them. In particular, this system \( SS_{(I)} \) grids can quite include grids with modules, lying in the borders \( a_i < a_{i+1} < 2 \cdot a_i \).

At the same time relative quantity of such abnormal inequalities for modules of next grids can not be appreciable. Otherwise the sum of reverse modules appears more unit. But in such case there are no essential handicapes for a establishment of
one-sided filling $Z_n^{(1)}$, which can appear effective in general at all $n$, and if similar is not observed, will eventually set in for all $n > n_0$ at some $n_0$. Integrated factor of one-sidedness of algorithm of filling not to overcome by separate infringements of primary increase of modules.

Steady two-sidedness of algorithm of filling requires firm advantage of a set of slowly growing modules of grids. Otherwise the advancing growth of set of units on an interval of a prospective maximum series will transform two-sided algorithm in inefficient. It reflects expression (5.5). The one-sided character of filling at formation of the heaviest series $MSR_n \left\{ q, Z_n^{(1)} \right\}$ is established for all $n > n_0$.

The question should arise about a principle of formation of maximum series for systems with in comparison slow growth of modules of next grids.

**Theorem 8.** The two-sided filling $Z_n^{(2)}$ is obliged to appear effective for systems of grids $\{S(a_i)\}$, the overwhelming part of modules of which are connected by a correlation $a_{i+1} < 2 \cdot a_i$ (not including equal), since some $n_0$.

$$MSR_n(q, SS) : MSR_n \left\{ q, SS(a_{i+1} < 2 \cdot a_i) \right\} \Rightarrow MSR_n \left\{ q, Z_n^{(2)} \right\}, n > n_0.$$ (5.6)

**Proof.** The primary affinity of grids $a_{i+1} < 2 \cdot a_i$ permits to conclude, that such systems concern to second and, mainly, to the third type. The possible grids of an identical module even are not taken into account. Zeroes of multiplicity above unit not only become ineradicable, but also can form a overwhelming set, as far as the talk goes only about non-singular fillings and systems. As an example it is possible to result a system primes $SP_0$. $n$-filling of this system in a condition to have on a period a sum of multiplicities of zeroes, in a many times superior period.

In such case the algorithm of two-sided filling becomes expedient. The high density of zeroes in a period $n$-fillings and, accordingly, in comparison small quantity of units in a vicinity of a formed maximum series $MSR_n(q), q \geq 0$, admit a opportunity some other algorithm of filling $Z_n^{(2)}$. The algorithm extreme effectively uses redundant frequency of zeroes and redistribution of zeroes of high multiplicity.

At the same time particular search in variants of initial values of parameter $n$ as quantities of grids quite we allow one-sided algorithm of filling. However the conditions of the statement prove, that in result moment of saturation of an interval of a series by zeroes will come. Then it is possible to recollect a reserve, about the forgot the second side, not taken into account at one-sided algorithm $Z_n^{(1)}$.

The expression (5.6) emphasizes inevitability of the address to two-sided algorithm of filling $Z_n^{(2)}$. It will be realized for all $n > n_0$ with some $n_0$ by search of maximum series $MSR_n(q)$ in systems of predominy slow growth of modules of grids and, hence, high density of zeroes.

Sometimes the researched system of grids $SS$ extremely slightly differs from
Theorem 9. Inclusion or exception of any finite set of grids, not infringing non-singular system, results in that will be always found such $n_0$, since which is restored one-sidedness or two-sidedness of fillings for any system of grids $SS = \{S(a_i)\}$.

$$MSR_n(q) : MSR_n \{q, SS(a_i), Z_n^{(k)}\} \Rightarrow MSR_n \{q, SS(a_i \cup b_j), Z_n^{(k)}\}, n > n_0.$$  

Proof. The condition of non-singular system and fillings obviously in a result of inclusion of some fixed set of grids. One-sidedness or two-sidedness of fillings is determined by wittingly advantage of inequalities $a_{i+1} \geq 2 \cdot a_i$ or $a_{i+1} < 2 \cdot a_i$ in a chain of comparisons of modules. It remains and after inclusion of any finite set of grids in a system. That is if for one system the kind $Z_n^{(k)}, k = 1, 2$ of fillings algorithm is established at $n > n_0$, for other considered system the same kind of algorithm is observed at other parameters $m > m_0$.

Really, another set of grids $\bigcup_{j \in N} S(b_j)$, let even it and finite, up to some moment, determined by value of parameter $n_0$, in a condition seriously to deform the scheme of algorithm of filling $Z_n^{(k)}$. But the scheme $Z_n^{(k)}$ is restored (5.7) without fail, as far as the system on a condition initially has unlimited total stabilizing effect. □

From here immediately the next conclusion follows for a class of systems of the second type. This class $SS_{(II)}$ is real and spacious in the fillings method.

Theorem 10. The one-sideness of fills $Z_n^{(1)}$ is established always for systems of grids $SS_{(II)} = \{S(a_i)\}$ of the second type, since some $n_0$.

$$MSR_n(q, SS) : MSR_n \{q, SS_{(II)}\} \Rightarrow MSR_n \{q, Z_n^{(1)}, SS_{(II)}\}, n > n_0.$$  

Proof. From definition the system of grids of the second type $SS_{(II)}$ has a finite sum of reverse modules. Such sum is more unit. It obviously means, that multiple zeroes are inevitable for all fillings $Z_m$, since some $m_0$, that is when $m > m_0$.

At the same time finiteness of a sum of reverse modules for a considered system shows, that there will be value $r$, for which certainly correctly this expression

$$SS \subset SS_{(II)} : \sum_{i=1}^{\infty} \frac{1}{a_i} < \infty \Rightarrow \exists (r) \left\{ \sum_{i=r+1}^{\infty} \frac{1}{a_i} < 1, \quad n_0 > 1 \right\}.$$

Or else, the system of grids $\{S(a_i)\}$ at $i > r$ appears by a system of the first type $SS_{(I)}$. But then pursuant to the previous theorem exception the first $r$ grids from the initial system or the inclusion the same $r$ grids in a system of the first type does not change final one-sided algorithms of fillings. That is there will
be such value \( n_0 \), that at all parameters \( n > n_0 \) effective will be, as well as in the expression (5.8), one-sided algorithm of filling \( Z_n^{(1)} \).

Sense of the introduction of one-sided concept of algorithm of filling clears up the following statement, important for the analysis of the characteristics of all systems.

**Theorem 11.** Unit acts by a majorizing coefficient \( \tau_n \) valuations of a series length \( msr_n(q) \) in the main theorem for maximum series \( MSR_n(q) \), generated by algorithm of one-sided filling \( Z_n^{(1)} \subset SS \).

\[
MSR_n(q, Z_n^{(1)}, SS) : \quad msr_n(q) \leq \frac{n + q}{1 - \gamma_n} + 1; \quad \gamma_n = \frac{H_n}{PZ_n}, \quad \tau_n \leq \tau = 1.
\]  

(5.9)

**Proof.** Here \( \gamma_n \) there is the density (average frequency) of zeroes \( H_n \) for a period \( PZ_n \) of filling \( Z_n \). A one-sided algorithm of filling \( Z_n^{(1)} \) at formation of maximum series \( MSR_n(q) \) imposes on a system \( SS \) and concrete modules of grids rather severe restrictions. They are described by conditions of the theorems 4–7. If not dependence between modules of grids, under all other conditions the two-sided filling, would seem, is obliged to ensure just the heaviest series. But the modules of next grids do not permit to run in a double interval of a prospective series of zeroes.

The particular appendix of this situation degree-systems and systems without multiple zeroes follows from the theorems 1–3. The system with primary increase of modules requires the special consideration. We shall offer majorizing variant of such system, providing minimum of mutual prime modules of grids without grids of saturation. Then the modules of similar grids are consistently and unequivocally: \( 3, 7, 16, 37, 79, ... \). Naturally, the equal modules are also excluded, as they do not change the characteristic and properties of a system.

Redundancy of valuation of a maximum series \( MSR_n(q) \) for initial parameters \( n \) at all \( q \geq 0 \) is rather simple directly to establish. Expediently to consider 2-filling and heaviest interval, where will not meet a multiple zero at maximum concentration of zeroes of multiplicity unit. As far as \( \gamma_2 = \frac{3}{7} \), the heaviest growth of a series and valuation will appear at \( msr_2(4) = 11; \quad \frac{6}{1 - \gamma_2} + 1 = 11.5 \). And then on intervals \( I_2 \geq 21 \) zeroes of multiplicity two are inevitable, and the difference between valuation and series grows.

And let such growth is slowed down, but it is inevitable, and transition to next filling \( Z_3 \) with a grid \( S(16) \) at all desire to achieve heaviest concentration of zeroes can grant only variant \( msr_3(8) = 20 \), when \( \frac{11}{1 - \gamma_3} + 1 \sim 21.5 \), as far as \( \gamma_3 = \frac{13}{28} \). The so appreciable difference is connected that on this interval a multiple zero is inevitable. And further the quantity of multiple zeroes accrues, even more increasing gap between estimation and series.

If to take into account, that the relation between modules is saved (about two), occurrence of new multiple zeroes is inevitable also. It promotes a further divergence
between valuation and maximum series of zeroes. Thus completely it is necessary to allocate complete equality of multiple zeroes. They in an identical degree cause fall of value of the heaviest series.

It is impossible not to note, the consideration of maximum series $MSR_n(0)$ without units ($q = 0$) sharply simplifies the proof of the statement. The essential part of grids forms density of zeroes $\gamma_n$, but participates in creation of a series of zeroes $MSR_n(0)$ by only one zero, being compared with an infinite grid. The statement does not deform and occurrence in a system of grids of equal modules, already not speaking about liquidation of the rule of their mutual prime.

Thus, one-sided algorithm of fillings $Z_n^{(1)}$ causes a establishment of a majorizing coefficient $\tau = 1$ in the formula (5.9) of main theorem. It precisely determines thus rather extensive class of systems. □

Now all is prepared for distribution of the statement of the main theorem for all classes of non-singular systems of grids and fillings.

6. Proof of the main theorem

The imaging principle of explains logic and validity of origin of the main theorem in the offered form, but does not remove necessity of its proof. The series of such proofs is found in monograph [1], using various feature of fillings method, including absence of imaging principle’s support. Finishing variant giving the strongest valuation is indicated here.

We are addressed to three coprime $VP$ systems of the third type and one mixed $SM$ system, playing central role in some problems of primes distribution:

$$SP_0 = \{2, 3, 5, 7, \ldots, p_i, \ldots\}, \quad SP_1 = \{3, 5, 7, \ldots, p_i, \ldots\},$$
$$SP'_1 = \{3, 4, 5, 7, \ldots, p_i, \ldots\}, \quad SM'_{04} = \{2, 3, 4, 5, \ldots, p_i, \ldots\}. \quad (6.1)$$

All these related systems at any $n > 2$ and united $q \geq 0$ form fillings $Z_n[SS]$, which have the dependent characteristics of series.

**Lemma 1.** The values of all series $SR_n[SS] = SR_n(q)$ in periods of fillings $Z_n[SS]$ of systems of expression (6.1) are connected by next equalities:

$$sr_{n+1}[SP_0] = 2 \cdot sr_n[SP_1], \quad sr_{n+2}[SM'_{04}] = 2 \cdot sr_{n+1}[SP_0] = 4 \cdot sr_n[SP_1]. \quad (6.2)$$

**Proof.** The grid $S(2)$ for system $SP_0$ and grids $S(2), S(4)$ for system $SM'_{04}$ are saturated grids. It means, that all series of their fillings (including period with corresponding number of units $q \gg 0$) are increased, accordingly, in two and four times in relation to series of filling of system $SP_1$. 28
Certainly, this conclusion is valid and for maximum series $MSR_n$. From here borders possible find for values of the maximum series for important coprime system $SP'_1$. This system have grid $S(4)$ and $(4,2) \neq 1$ for grid $S(2) \subset SM'_{04}$.

$$\frac{msr_{n+1}[SP_0]}{msr_n[SP_1]} = 2, \quad \frac{msr_{n+2}[SM'_{04}]}{msr_n[SP_1]} = 4, \quad \frac{msr_n[SP'_1]}{msr_n[SP_1]} < C,$$  \hspace{1cm} (6.3)

where $1 < C < 2$. First two equalities (6.3) follow direct from equalities (6.2), and third inequality follows from obvious reasons, as far as the addition of the same grid $S(2)$ to fillings $Z_n[SP'_1]$ and $Z_n[SP_1]$ leads them into fillings $Z_{n+1}[SM'_{04}]$ and $Z_{n+1}[SP_0]$ accordingly. It means limitation of value $C < 2$. \hfill \Box

**Theorem 12.** If non-singular systems $SS_1, SS_2$ differ by finite set of grids, the correlation of values of their maximum series has constant boundaries:

$$n_0 \geq 1, \quad C_1 < \frac{msr_{n+n_0}[SS_1]}{msr_{n+n_0}[SS_2]} < C_2, \quad \forall \ n > 0.$$  \hspace{1cm} (6.4)

**Proof.** The values of constants $C_1 < C_2$ are determined by concrete set of non-coincide grids. Their quantity gives value $n_0$, that permits some to decrease a difference $C_2 - C_1$. Inequalities (6.4) become clear after addition of grids (optionally coincide) in each of systems $SS_1, SS_2$, that sets of discrepancy have formed saturated fillings. Quantity of such additions is finite and in a result systems will be formed, distinguished only by sets of saturated grids with periods $D_1$ and $D_2$. In such case we receive values of maximum series $D_1 \cdot msr_n[SS]$ and $D_2 \cdot msr_n[SS]$, where $SS = SS_1 \cap SS_2$ is common grids part of initial systems. \hfill \Box

**Definition 27.** Criterion of mixing $K_n(SS)$ for filling $Z_n$ of any non-singular system $SS = \{S(a_i)\}^n_1$, where $(Z_n \subset \{S(a_i)\}^n_1 \subset SS)$ is expression

$$\frac{\gamma^*_n}{\gamma_n} = K_n(SS) \geq 1, \quad \gamma^*_n = \frac{H^*_n}{PZ_n}, \quad \gamma_n = \frac{H_n}{PZ_n}, \quad Z_n \subset \{S(a_i)\}^n_1.$$  \hspace{1cm} (6.5)

One would think, such criterion can act ratio $\frac{1-\gamma_n}{1-\gamma^*_n} = \frac{PZ'_n}{PZ_n}$ or $\frac{H^*_n}{H_n}$, however they are non-informative, as far as they characterize not mixing, but availability of multiple zeroes, their plenty. These values will be maximum for saturated fillings for system $\{(p_i - 1) \cdot S(p_i)\}$, where $p_i$ are primes. The criterion $K_n(SS)$ of expression (6.5) is intended to allocate that fact, that mixing is simplified step to systems $SS'$ without multiple zeroes.

**Lemma 2.** The system $SP'_1 = \{3,4,5,7,...,p_i,...\}$, where $p_i$ are primes, corresponds to majorizing sequence of mixing criteria, that is relation

$$\sup_{SS} \max_{\gamma^*_n \sim \gamma_n(SS)} K_m(SS) = K_n(SP'_1), \quad \gamma^*_n \in Z^1_n \subset SP'_1, \quad \gamma_m \subset SS.$$  \hspace{1cm} (6.6)
The formulation of lemma means, that the upper estimates of maximum series values, found with the help of the analysis of zeroes frequencies (in particular, main theorem) for \( n \)-fillings of the system \( SP'_1 \), will be valid for any other system.

Really, any inclusion in \( n \)-filling of grids with modules \((a_i, a_j) \neq 1\) means decrease of zeroes of multiplicity higher unit, as far as such grids can be considered in this filling as one grid with frequency \((\frac{1}{a_i} + \frac{1}{a_j})\).

The address to the system \( SP'_1 \) is predetermined by step-by-step consideration of \( n \)-fillings. Let \( n = 2 \), and the modules \( a_1, a_2 \) at coprime should be (as well as above) close for maximum of \( K_2(SS) \). In such case setting \( a_1 = m, a_2 = m + 1 \), we shall receive \( K_2 = 1 + \frac{m-1}{2(m^2+m+1)} \), whence follows, that the maximum of this ratio is reached for module \( m = 3 \). The similar maximum for fillings at \( n = 3 \) takes place for grids with initial modules \( 3, 4, 5 \).

Further largely the coprime of modules enters, minimum which is provided by primes \( p_i \). Thus there is generated system \( SP'_1 \), for which relation (6.6) is executed. It is possible to note, that for \( \gamma^*_n \to 1 \) the ratio (5.5) aims to unit. Therefore value \( K_n \) has maximum, which is reached in a system \( SP'_1 \) for \( n = 5 \) and is equal \( K_5 = 1 + \frac{36456}{325367} \sim 1.11205 \). Thus it appears \( K_4 \sim 1.1106, K_6 \sim 1.1116 \).

The existence of maximum \( \max_n K_n \) does not contradict that \( \gamma^*_n \geq \gamma_n \) and \( \gamma^*_{n-1} \geq \gamma_n - \gamma_{n-1} \) for any system, and if at such transitions fresh multiple zero will be fixed, the inequalities should be replaced to strict.

So, according to construction, as well as from definition of mixing criterion \( K_n(SS) \), if \( \gamma^1_n(SP'_1) \sim \gamma_m(SS) \), when are close direct (average) density of zeroes for two systems, the mixing criterion \( K_n(SS) = K_n(SP'_1) \) will be more for \( n \)-filling of system \( SP'_1 \). Clearly, approximation of density nearness dictates and some approximation of criterion advantage, as far as the compared fillings can differ rather slightly for many, if not to all parameters.

Unfortunately, natural requirement of density equality for two fillings incorrectly because from various sets of modules. However for close density and at a essential divergence in sets of grids, advantage of value \( K_n(SP'_1) \) will be without fail displayed. Such advantage will become obvious, if the appreciable part of modules of filling grids for system \( SS \) will appear not coprime.

Therefore further system \( SP'_1 \) will be considered as determining system of limiting concentration of multiple zeroes concerning zeroes \( H_n \) of direct imaging. It is received by withdrawal of saturated grid \( S(2) \) and then inclusion of grid \( S(4) \) with a minimum even module, large two. It should add, that practically such majorizing system (rather close), is standard prime system \( SP_1 \). □

Systems of grids with modules – primes \( SP_0 \) and \( SP_1 \) are determining on a way of the proof of the main theorem. Or else, if the theorem is valid for these systems, it is valid and for a class of all non-singular systems. It is called by that coprime system \( SP'_1 \) appears by a limiting system according to mixing criterion.
(lemma 2), and it only by one grid differs from mentioned prime systems.

**Definition 28.** Next primes, connected by equality \( r \geq 1 \): \( p_n = p_{n-1} + 2r \), are named as *kinsfolk of rank* \( r \in \mathbb{N} \). We designate thus \( p_n = p_n^{(r)} \).

Thus, twins are kinsfolk of first rank. Clearly, that the search kinsfolk very large rank produces to significant difficulties. As the kinsfolk is determined large: \( p_n^{(r)} = BR_r = p_n = p_{n-1} + 2r \), then all primes (except 2 and 3) are kinsfolk of one from ranks. Next kinsfolk of various ranks can be also incorporated and are considered as independent object.

**Definition 29.** *Configuration* of \( m \), \( m \geq 2 \) primes is named group of next primes as vector kinsfolk of ranks \( r_i \) of dimension \( m - 1 : (r_1, r_2, ..., r_{m-1}) \).

\[
p_n^{(r_1)} = p_{n-1} + 2r_1, \quad p_n^{(r_2)} = p_n + 2r_2, ..., \quad p_n^{(r_{m-1})} = p_{n+m-3} + 2r_{m-1}.
\] (6.7)

At the same time quite clearly, any vector \( \{r_i\}_m \) of dimension \( m - 1 \) at \( m \geq 3 \) does not guarantee, that there will be an appropriate configuration. For example, for a vector \( (1, 1, 1) \) the configuration of primes of kind (6.7) does not exist.

The problem about the upper estimate of the maximum series \( MSR_n \) or \( MSR_n(q) \) acts central in the fillings method. Thus majorant of systems \( SP_1 \) and \( SP'_1 \) in class of all systems by mixing criterion acquires decisive character. Therefore main investigated system will become just the system \( SP_1 \), though for the researchers, not aware about fillings method, always unique was the system \( SP_0 \), in which so it is conveniently to build sieve of Eratosthenes.

Object of the fillings method is whole period of grids product, and sometimes the study is not limited even by period. It is testified already repeatedly, that the period of \( n \)-filling in \( SP_1 \) has a length, equal to product of all \( n \) odd prime. Period disintegrates by series of zeroes \( SR_n \), the lengths of which vary from values \( sr_n = 1 \), (there are no zero between units), up to value \( m sr_n \).

In illustrations series \( MSR_n(q) \) and sequences of series \( SR_n \) are submitted and the periodicity has allowed them to close in a ring (Fig. 1). According to lemma 1 transition from a system \( SP_0 \) to \( SP_1 \) means transformation kinsfolk of rank \( r \) in series of this length, and configuration of primes in configuration of ranks.

Fig. 1 from “Graphic Illustrations” clearly demonstrates symmetry of filling series \( Z_3 \), which is present for any \( Z_n \subset SP_1 \). Thus each series of the period of filling has double, except two series \( SR^{(I)}_n, SP^{(II)}_n \), submitted in single specimen – series of length of unit and two. We shall designate symmetric axial configurations of series with these series in center as \( K f^{(I)}_n \) and \( K f^{(II)}_n \).

Each grid \( S(p_k) \in Z_n \) has central series, consisting from two next units, that is length of unit. It clearly, as far as any period of grid \( S(p_k) \) includes \( p_k - 1 \geq 2 \) units. Thus the series \( SR^{(I)}_n \) unit length of the first axial configuration \( K f^{(I)}_n \) will
be saved for each step. It can be concluded from the obvious relation

$$\frac{1}{2} \left\{ p_n \prod_{i=1}^{n-1} p_i \pm 1 \right\} \neq 0 \pmod{p_n} \implies \frac{1}{2} (p_n \pm 1) \neq 0 \pmod{p_n},$$

and serial unit $SR_n^{(I)}$, lying equally in distance of a half-period from central series $SR_n^{(II)}$ of the second configuration, remains in constancy.

In the period $PZ_n$ length $\prod_{i=1}^{n} p_i$, there is unique zero of multiplicity $n$, received by product of zeroes of all grids. It enters in central axial series $SR_n^{(II)}$ of length $sr_n^{(II)} = 2$. Clearly, that all other zeroes of grids lie symmetric concerning mentioned $n$-th zero. Hence, and all remaining series of the configuration $Kf_n^{(I)}$ place symmetric, except one central series $SR_n^{(I)}$.

The clear sense has consideration of those sequences of next series, which will meet too in subsequent fillings at increase of parameter $n$. Such configurations are named typical. The first typical axial configuration $Kf_n^{(I)}(Z_n) = Kf_n^{(I)}$ is submitted uniquely by series in this scheme:

$$Kf_n^{(I)} = Kf \left\{ ..., \frac{p_{n+2} - p_{n+1}}{2}, \frac{p_{n+1} - 1}{2}, 1, \frac{p_{n+1} - 1}{2}, \frac{p_{n+2} - p_{n+1}}{2}, ... \right\},$$

(6.8)

where central series $SR_n^{(I)}$ of unit length ($sr_n^{(I)} = 1$) surround by two series of the greatest length for this configuration $Kf_n^{(I)}$.

We shall put in conformity zero of axis and average point of two central units of each grid. Then representation of configuration (6.8) is proved by that in points of projection $Kf_n^{(I)} \in \mathbb{Z}$: $\pm \left( \frac{p_{n-j} + 1}{2} + k p_{n-j} \right)$ zeroes stand and at $p_{n+i} < p_{n+1}^2$

$$\frac{p_{n-j} + 1}{2} + k p_{n-j} \neq \frac{p_{n+i} + 1}{2}, \quad \forall j, i, k : 0 \leq j \leq n - 1, i, k \geq 1.$$  

(6.9)

The second axial configuration $Kf_n^{(II)}$ is submitted by next series:

$$Kf_n^{(II)}(Z_n) = Kf \{ ..., p_{n+3} - p_{n+2}, p_{n+2} - p_{n+1}, p_{n+1} - 2^m, 2^{m-1}, ...

..., 4, 2, 1, 2, 4, ..., 2^{m-1}, p_{n+1} - 2^m, p_{n+2} - p_{n+1}, p_{n+3} - p_{n+2}, ... \}$$

(6.10)

where $m = \lceil \log_2 p_{n+1} \rceil$, that is $m = \max \{ i : 2^i < p_{n+1} \}$, and typicalness of each configuration, dependent from $n$, should be established especially. Central series $SR_n^{(II)}$ of configuration $Kf_n^{(II)}$ has length two $sr_n^{(II)} = 2$. Besides units, correspond to numbers $2^{m+k}$, $k \geq 1$, will meet in configuration.

We shall put in conformity zero of axis $\mathbb{Z}$ and zero of each grid. Then representation (6.10) is proved by that in points $\pm 2^r$, $r \geq 0$ of projection $Kf_n^{(I)} \in \mathbb{Z}$ units stand and by analogy with (6.9) we have: $k p_{n-j} \neq p_{n+i}$.
The maximum series \( M_{SR_n} \equiv M_{SR_n}(0) \) can enter in a core of typical axial configuration, and it is not (remaining by a typical series), but in virtue of indicated expressions it is possible to make the conclusions about its value.

**Theorem 13.** Values \( m_{sr_n}(0) \) of maximum series \( M_{SR_n}(0) \) in prime systems \( SP_0 \supset \{S(p_i)\}_i^n \) and \( SP_1 \supset \{S(p_i)\}_i^n \) have following lower estimates

\[
\{ m_{sr_n}(SP_1) \geq p_{n-1} \} \Rightarrow \{ m_{sr_{n+1}}(SP_0) \geq 2p_{n-1} \}, \quad n \geq 1.
\]

**Proof.** The second inequality (6.11) is consequence of the first according to lemma 1. It easily be convinced in validity of given equality for parameter \( n \leq 7 \): \( m_{sr_n}(SP_1) = p_{n-1} \). For parameter \( n > 7 \) the equality begins to be infringed. According to representation (6.8) value of series \( SR_n(2) \) for filling \( Z_n \) with two units in system \( SP_1 \) is equal \( s_{r_n}(2) = \frac{p_{n+1}+1}{2} + 1 + \frac{p_{n+1}-1}{2} = p_{n+1} \). It is received from three central series of configuration. As far as for the same series \( s_{r_n}(2) \leq s_{r_{n+2}}(0) \) and the maximum series majorize of any, the statement of the theorem follows from replacement of parameter \( n \) to value \( n - 2 \). \( \square \)

Appeal to second \( K_{f_n}^{(II)} = K_{f}^{(II)}(Z_n) \) axial configuration demonstrates, that lower estimate of maximum series is unjustifiable rough at sufficiently large \( n \).

**Theorem 14.** Lower estimate of maximum series \( M_{SR_n}(0) \) in prime system \( SP_1 \) for \( n \geq 26 \) surpasses estimation \( p_{n-1} \), as it is expressed by the formula

\[
m_{sr_n}(SP_1) \geq 2 \cdot p_{n-2m-1}, \quad m = \lceil \log_2 p_{n-2m-1} \rceil, \quad n \geq 26.
\]

**Proof.** Here the expression for search of intermediate parameter \( m \) provides the decision of a small integer equation, which however can not call difficulties and does not in essence change common kind of estimation. We shall consider expression \( K_{f_n}^{(II)}(Z_n) \) of second axial configuration (6.10). Values \( s_{r_n}(q) \) of central series \( SR_n(q) \) with \( q \) units hence it follow immediately:

\[
s_{r_n}(2m) = 2^{m+1}, \quad m = \lceil \log_2 p_{n+1} \rceil, \quad p_n = \max \{p_i: p_i < 2^m\};
\]

\[
s_{r_n}(2m + 2) = 2 \cdot p_{n+1}; \quad s_{r_{n+k}}(2m + 2) = 2 \cdot p_{n+k+1}, \quad k \geq 1.
\]

From these expressions follows, that as far as any grid can zerofill not less units, than infinite, for \( (n + k + 2m + 2) \)-filling with grids \( S(p_i) \) is always executed

\[
\{ s_{r_{n+k+2m+2}}(0) \geq 2 \cdot p_{n+k+1} \} \Rightarrow \{ s_{r_n}(0) \geq 2 \cdot p_{n-2m-1} \}, \quad (6.14)
\]

and the last inequality is received after replacement of \( (n + k + 2m + 2) \) to \( n \). It is thus necessary to take into account, that value \( m \) is found for parameter \( (n + k + 2m + 2) \) instead of \( n \). It predetermines necessity of the equation decision for search \( m \) in the formulation of theorem.
The concrete check [1] of axial configurations $Kf^{(I)} \subset SP_1$ and $Kf^{(II)} \subset SP_1$ finds out the first and minimum value $n = 26$ for which is executed

$$sr_{n-2q}(2q) [Kf^{(II)}] > sr_{n-2}(2) [Kf^{(I)}], \quad q = q(n)$$

for the greatest series with units of corresponding axial configurations. Then for $n > 26$ the sign of inequality does not already change. Twin quantity of units $q > 1$ is determined from minimum and symmetry conditions according to which the greatest initial series of the configurations should enter in investigated series.

It is simple find, that limit of the lower estimates of theorems 13 and 14 is equal two at increase of filling parameter $n \to \infty$ in prime system $SP_1$:

$$\lim_{n \to \infty} \frac{2 \cdot p_{n-2m-1}}{p_{n-1}} = 2 \lim_{n \to \infty} \left(1 - \frac{c_1 \ln n}{n}\right) \left(1 - \frac{c_2}{n}\right) = 2. \quad (6.15)$$

So, the lower bound (6.12 - 6.15) of maximum $MSR_n(0)$ series for large $n$ qualitatively surpasses similar values for small filling parameters $n$. □

However essentially greater significance for the subsequent research had upper estimates of maximum series $MSR_n$ and $MSR_n(q)$. It is better to have exact values $msr_n(q)$ for all $q \geq 0$. There is, at all complexity and importance of this problem, it is solvable within the framework of the fillings method for system $SP_1$ and the following statements serve necessary step for it.

**Theorem 15.** Greatest series $SR_n(q)$ with $q \geq 0$ units belonging to core of the first $Kf_n^{(I)} = Kf_n^{(I)}(Z_n)$ axial configuration in a system $SP_1$ have the length

$$SR_n(q) \subset Kf_n^{(I)}: \quad sr_n(0) = \frac{p_{n+1} - 1}{2}; \quad sr_n(1) = \frac{p_{n+2} - 1}{2};$$

$$sr_n(2) = p_{n+1}; \quad sr_n(q) = \max_{1 \leq i \leq [q/2]} \left\{p_{n+i} + \frac{p_{n+q-i} - p_{n+i}}{2}\right\}, \quad q \geq 3. \quad (6.16)$$

**Proof.** The representation of expression (6.8) of the first axial configuration $Kf_n^{(I)} = Kf_n^{(I)}(Z_n)$ quite determines and fixes relations of theorem. It should take into account, that values of the first four series are given by a kind of the configuration directly, and their receipt does not require in search, as for $q = 0; 2$ it obviously, for $q = 3$ follows from symmetry, and $sr_n(1) = \frac{p_{n+2} - 1}{2} \geq \frac{p_{n+1} + 1}{2}$. But also linear search during simple selection of maximum does not result to large retrieval of values, as far as it is connected with local non-uniformity of primes distribution for a sequence of indexes from $n + 1$ up to $n + q - 1$.

The length of series with $q$ units of axial configuration is equal to sum of values of making series, separated by $q$ commas. So, the length of series $SR_n(3)$ with three units, which contains both greatest series, is equal $p_{n+1} + \frac{p_{n+2} - p_{n+1}}{2}$. According to condition of the theorem, such series is greatest, as far as its length is unique. Already for $q = 4$ the situation changes.
Really, in this case according to condition has to choose in expression (6.16) from two variants of series \( SR_n(4) \) of lengths \( sr_n(4) \)

\[
SR_n(4) : \quad \left\{ sr_n(4) = p_{n+2}; \quad sr'_n(4) = p_{n+1} + \frac{p_{n+3} - p_{n+1}}{2} \right\}.
\]

Depending on \( n \) the advantage can have this or that variant. For example, for \( n = 2 \) maximum \( SR_2(4) \) determines by first variant (as 11 > 10) and at \( n = 3 \) for \( SR_3(4) \) – by second: (13 < 14). From here we have \( msr_2(4) = 11, \ msr_3(4) = 14. \)

From unique construction of axial configurations for any \( n \) and condition of choice of the greatest series follows, that quantity \( \lfloor q/2 \rfloor \) exists in common case of various series \( SR_n(q) \) as claimants for a role maximum. In common case them lengths of these series can not coincide too. It is thus necessary to take into account typical configurations and value \( q \) is limited naturally. From here search of variants of maximum series is small, estimated and obvious.

The given formula (6.16) acts for values \( q \), satisfying inequality \( p_{n+q-1} < p_{n+1}^2 \). It is obligatory condition for system \( SP_1 \) and fillings method. For large \( q \) there is reminder, that is considered regulated \( n \)-filling, instead of primes distribution. That is in the configuration \( Kf_n^{(II)}(Z_n) \) there will be units, not corresponding primes, and last search it is necessary to change. The theorem is proven. \( \square \)

**Theorem 16.** Greatest series \( SR_n(q) \) with \( q \geq 0 \) units belonging to core of the second \( Kf_n^{(II)} \) axial configuration in a system \( SP_1 \) have the length

\[
SR_n(q) \subset Kf_n^{(II)} : \quad sr_n(q) = \max_k \left\{ \sum_{i=1}^{q} sr_n(0)[k+i] \right\}, \quad (6.17)
\]

where series \( SR_n(0)[j] \subset Kf_n^{(II)} \) of corresponding lengths for consecutive parameters \( j = k+i \) are next series of the configuration \( Kf_n^{(II)} \).

**Proof.** Some indeterminacy of task of parameter \( k \) at search of the greatest series \( SR_n(q) \) is removed by that at small \( q \) in series one of two groups of greatest initial series \( SR_n(0)[k+i] \) is obliged enter and at \( q \geq 2m \) – both groups. It is enough obviously from representation of the configuration. Besides as well as in the theorem 7 search is conducted so long as \( p_{n+q-1} < p_{n+1}^2 \).

From the formula of theorem (6.17) directly follows, that

\[
sr_n(0) = \max\{2^{m-1}, p_{n+1} - 2^m\}; \quad sr_n(1) = \max\{3 \cdot 2^{m-2}, p_{n+1} - 2^m - 1\}; \quad sr_n(2) = \max\{7 \cdot 2^{m-3}, p_{n+1} - 2^m - 2, p_{n+2} - 2^m - 1\}; \quad \ldots
\]

\[
\ldots,\ sr_n(2m) = \max\{2^{m+1}, 2^{m-1} + p_{n+1}\}; \quad sr_n(2m+1) = 2^m + p_{n+1};
\]

\[
sr_n(2m+2) = \max\{2 \cdot p_{n+1}, 2^m + p_{n+2}\}; \quad \ldots
\]

\[
\ldots,\ sr_n(2m+w) = \max_{1 \leq i \leq \lfloor w/2 \rfloor} \{2^m + p_{n+w}, p_{n+w-i} + p_{n+i}\}, \quad w \geq 2.
\]
The decisive significance for further conclusions in research has series $SR_n$ with quantity of units, reached and exceeded border $2m$, when in object both greatest groups of initial series of the configuration are involved. Value $sr_n(2m + 1)$ is deprived of the search factor and is calculated directly after the task of central parameter $n$ of filling. Examples of increasing values of such greatest series for $q = 2m$ and $q = 2m + 2$ are in monograph [1].

In the second axial configuration as regulated filling $Z_n$ all elements corresponding to values $2^s$ for $0 \leq s \in \mathbb{Z}$ are units. It is not taken into account by last relation for series $sr_n(2m + w)$ in which such unit of the greatest number stands in the point $2^m$ near from series bound. Therefore we shall continue representation of the second axial configuration $Kf_n^{(II)}$.

In such case after the found parameter of degree $m = \lfloor \log_2 p_{n+1} \rfloor$ of filling $Z_n$ for prime system $SP_1$, whence by minimum value $n' \leq n$, at which unit in point $2^m$ becomes the representative of central and already constant core of configuration, will be $n' = \min \{i : p_{i+1} > 2^m\}$. From here under the given scheme we shall find following such unit of the configuration in point $2^{m+1}$, lying directly after unit corresponding to prime $p_u$, $u = \min \{j : p_{j+1} > 2^{n+1}\}$. Now we shall present a half-configuration (for shortening of notation):

$$2, 1, 2, 4, ..., 2^{m-1}, p_{n+1} - 2^m, p_{n+2} - p_{n+1}, ..., 2^{m+1} - p_u, p_{u+1} - 2^{m+1}, ...$$

in which and the subsequent such units are similarly. Received specified kind of configuration should take into account during search of its series.

However this kind of configuration representation $Kf_n^{(II)}$ will be infringed yet earlier than unit corresponding to $p_{n+1}^2$ will meet. Regulated filling of configuration except permanent units corresponding to values $2^k$ has also consistently zerofilled (at $n \to n + s$) units in place $2^k p_{n+s}$, $k, s \geq 1$. Naturally, first such unit the place $2p_{n+1}$ determine. Alongside with taken into account units of representation, these units of configuration play essential role, appreciably complicating formula $Kf_n^{(II)}$ at aspiration to expand observed set $q$.

But additional units of series interval can not be the factor, promoting to increase of the greatest (maximum) series of filling $Z_n$ for any system. □

The theorems 15 and 16 permit to notice important distinction between axial configurations $Kf_n^{(I)}$ and $Kf_n^{(II)}$. In first (I) there is only one permanent series $SR_0$ of length unit, but all units in interval up to $p_{n+1}^2$ correspond exclusively primes. For second (II) quantity of permanent units and hence series increases, but in the same interval additional units corresponding to composite numbers $2^k p_{n+s}$ meet. Thus role as that, as other configuration in formation of the most main objects of fillings is impossible overrate.

**Theorem 17.** In system $SP_1$ for parameter $n \geq 3$ of the filling $Z_n$ the greatest series from considered axial configurations are series $SR_n^{(I)}(q)$ for parameter $0 \leq q < m + 3$, but at exception of series parameter $q = 1 : SR_n(1)$. Then, for
\[ m + 3 < q < 2m - 3 \quad \text{indeterminacy of advantage is accompanied by proximity of series values.} \]

For parameters \( q > 2m - 3 \) the advantage passes to series \( SR_n^{(II)}(q) \) of the second configuration \( K_f^{(II)} \), and it quickly increases with growth \( n \):

\[
\begin{align*}
    sr_n^{(I)}(0) & \geq sr_n^{(I)}(0); \quad sr_n^{(I)}(1) < sr_n^{(I)}(1); \quad sr_n^{(I)}(2) > sr_n^{(I)}(2); \\
    sr_n^{(I)}(3) & \geq sr_n^{(I)}(3); \quad \ldots \quad sr_n^{(I)}(q) > sr_n^{(I)}(q) - \Delta_n, \quad q < m + 3; \\
    n > 15: \quad \ldots \quad sr_n^{(I)}(q) > sr_n^{(I)}(q), \quad 2m - 3 < q < 3n. \quad (6.18)
\end{align*}
\]

**Proof.** The last condition \( q < 3n \) is given as the plenty of units \( q \) loses informative sense at complication of representation of the second configuration. In this case maximum series value begins promptly to approach with average value of such series in period. Intermediate parameters \( m + 3 < q < 2m - 3 \) are omitted from consideration, as for such variants the greatest series of both configurations are far from relative maximum. Certainly, the advantage of series of one of configurations can be established for each concrete \( q \) at increase \( n \), but the special necessity is not present, as far as for \( q = 2m, 2m + 2 \) the series value \( SR_n^{(II)}(q) \), that is value \( sr_n^{(II)}(q) \), becomes determining.

Unsteady advantage of series \( SR_n^{(I)}(q) \) for parameter \( q < m + 3 \) is reflected by the introduction in corresponding relation (6.18) of essentially small \( \Delta_n > 0 \) concerning series value. For parameter \( q \) approaching to bound \( 2m \), when in evaluated series all central units of kind \( 2^k \) enter for \( n > 15 \), advantage of series of the second configuration appears obvious.

For proof of the first inequality it is enough to compare theorems 15 and 16 concerning the greatest series of configurations, designated as \( sr_n^{(I)}(0), sr_n^{(II)}(0) \), and then to see, that

\[
2^{m-1} \leq \frac{p_{n+1} - 1}{2}, \quad p_{n+1} - 2^m \leq \frac{p_{n+1} - 1}{2}.
\]

If to take into account relation \( m = [\log_2 p_{n+1}] \), we shall receive given conditions \( 2^m + 1 \leq p_{n+1} \leq 2^{m+1} - 1 \). Both these variant take place for various primes: \( 5, 17; \ 3, 7, 31, 127 \), that predetermines possible and attainable equality.

Following value \( q = 1 \) results to explicable inequality \( 3 \cdot 2^{m-2} > \frac{p_{n+2} - 1}{2} \), when value \( p_{n+1} \) is reasonably close to lower bound \( 2^m + 1 \). Let it not so and \( 3 \cdot 2^{m-2} < \frac{p_{n+2} - 1}{2} \). Then for \( p_{n+1} = 2^m + R \) and \( \Delta = p_{n+2} - p_{n+1} \) we receive: \( R > 2^{m-1} - \Delta + 1 \). However in this case according to theorem 8 is executed \( p_{n+1} - 2^{m-1} > \frac{p_{n+2} - 1}{2} \), for that it is enough \( R > \Delta - 1 \). This condition follows from the earlier received assumption, as far as \( 2^{m-1} > 2\Delta - 2 \).

The parameter \( q = 2 \) is especially important for the first configuration and whole further as uniting both its greatest series. Thus is reasonably obviously executed

\[
p_{n+1} > \max \{ 7 \cdot 2^{m-3}, p_{n+1} - 2^{m-2}, p_{n+2} - 2^{m-1} \}, \quad p_{n+1} \geq 2^m + 1
\]
because of obligatory last inequality, which and proves advantage of the first configuration for such $q = 2$. It will be saved and for subsequent parameters $q$, the proof of the statement for which similarly.

Such reasonably stable situation begins to change at approach to the dependent value $q = 2m$, when advantage passes to the second configuration $Kf_n^{(II)}$. However this exclusively important fact takes place for rather large values $n$, thus according to large $m$ also. From theorem 6 follows, that the complete definiteness arises for $n + q = 26$ and then in accordance with growth of parameters $n$ and $m$ advantage of series of the second configuration becomes decisive.

The given by theorem condition $n > 15$ is determined by that value $n - 2m$, where $m = \lfloor \log_2 p_{n+1} \rfloor$, there is more unit, that comparison of corresponding series possessed necessary efficiency. In other case advantage of series of the second configuration not so obvious, if is generally present.

Really example without searched determined parameter $q = 2m + 1$ takes place

$$sr_n^{(II)}(2m + 1) = 2^m + p_{n+1}, \quad sr_n^{(I)}(2m + 1) = \max_{1 \leq i \leq m} \left\{ p_{n+i} + \frac{p_{n+q-i} - p_{n+i}}{2} \right\},$$

and $sr_n^{(I)}(2m + 1)$ can approximately estimate by value $p_{n+m+1}$. Then superiority of the second configuration is connected to a obvious inequality $2^m > p_{n+m+1} - p_{n+i}$ growing in accordance with increase $m$ and not always valid for initial $m$. As far as parameter $m$ is connected with $n$ logarithmically, it results in reasonably large values $n + q$, at which firm advantage of series $SR_n^{(II)}(q)$ comes.

The achieved advantage will be saved for reasonably large $q$, but thus the kind of the second configuration $Kf_n^{(II)}$ should be transformed by inclusion of units, according to values $2^k p_{n+s}$. It compels to limit observed set of parameters $q$. □

Fillings method and proven theorems 13 – 17 permit to formulate exclusively important statement concerning maximum series in the system $SP_1$, satisfying to mentioned condition of absolute independence from imaging principle. Nevertheless it does not mean non-necessity or mistake of it, and opposite it independently confirms and it pays attention to universality at appeal to any classes of systems.

**Theorem 18.** Values of maximum series $MSR_n$ and $MSR_n(q), q > 0$ are bending for the greatest series of configurations $Kf_n^{(I)}$ and $Kf_n^{(II)}$. At initial $n$ they are expressed through the first series $SR_n(q)$ of fillings, then they are connected with series of the first configuration, and in result at $q = 2m$ they are already only derivative of greatest series of the second configuration:

$$MSR_n(q) \subset SP_1 : \quad \{ Kf_n^{(II)} \supset SR_n^{(II)}(2m) = MSR_n(2m) \}, \quad (6.19)$$

where quantity of generated grids $n \geq 15$ and $m = \lfloor \log_2 p_{n+1} \rfloor$.

**Proof.** Determining role of greatest series of the second configuration $Kf_n^{(II)}$ becomes absolute only for large $n$. At initial and small $n$ the series of the second
axial configurations accept auxiliary and supporting participation in formation of maximum series. In system $SP_1$ we shall observed formation of initial maximum series $MSR_n = MSR_n(0)$ as the most important for reception of many further conclusions. Moreover their values by simple relation are connected to series $MSR_n(q)$ for reasonably wide spectrum of parameters $q$.

Generation of maximum series of filling $Z_n \subset SP_1$ passes through three stages. At first (the zero stage) consecutive zerofilling of next units in fixed interval results to occurrence of the first series $SR_1^n(0)$, which and become maximum. However this stage is quickly finished. Already for $n = 4$ the first infringement is observed and it appears chronic at growth of parameter $n$.

As was specified above and it is consistently confirmed by examples for various systems, the step-by-step filling next (right) units by zero of following grids is algorithm, realizing one of variants of the greatest series of filling in the given interval. A little that, such process really determines maximum series $MSR_n(0)$ for many systems (without multiple zeroes) and for many fillings in any systems.

Algorithm of sequential filling of units by grids of increasing modules in system $SP_1$ results to maximum series $MSR_n \equiv MSR_n(0)$ for $1 \leq n \leq 3$. And really for these $n$ equality $p_{n-1} = 0.5(p_{n+1} - 1)$ takes place. However then advantage of maximum series over the first series thus regulated filling begins to grow.

It is necessary to specify, that there will be such order of grids product of concrete filling, which occurrence of maximum series as the first series of considered interval provides, if to use described algorithm of formation.

So greatest series $SR_1$ of axial configuration of filling $Z_n$ of length $0.5(p_{n+1} - 1)$ already is enough extended, to claim for the special attention. If to remind, that it is rather close from it (through series $S_0^n$ of length unit) symmetric places such series, the arisen series $SR_n(2)$ with two units ($q = 2$) becomes object, claiming for extremes of characteristics. This moment determines transition to following first stage of generation of maximum series, connected with series of the first axial configuration $Kf_1^n(I)$ in period $PZ_n \subset SP_1$.

According to theorem 15 greatest series $SR_n(I)(2)$ of the first configuration has length $sr_n(I)(2) = p_{n+1}$. It immediately gives the lower estimation of maximum series value $MSR_n$, as far as $msr_{n+s}(q - s) \geq sr_n(q)$, $s \geq 0$. In particular, we receive $msr_{n+2} \geq sr_n(I)(2)$, whence follows that $msr_n \geq p_{n-1}$.

Concrete check has demonstrated, that such lower estimation of maximum series $MSR_n$ is upper for parameters $0 \leq n \leq 7$, that is $msr_n = p_{n-1}$. At the same time maximum series satisfy $msr_8 > p_7$, $msr_{11} > p_{10}$, though for some other $n$ equality $msr_n = p_{n-1}$ is restored.

Theorems 15 and 17 permit to reveal reasons of enough satisfactory approximation of greatest series of the first axial configuration to absolute values of corresponding maximum series. As there is demonstrated above it is explained by interval of series $SR_n(I)(2)$, where each new grid appears generated, differently it zerofills not less than two units. If exactly, it zerofills as time two units, and also does not bring one new multiple zero. Certainly, at increase of series interval occurrence of multiple
zeroes inevitably, but they are formed with the help of the previous grids.

At the same time theorem 15 admits interpretation of separate infringements of equality \( msr_n = p_{n-1} \) for observed values \( n \). The greatest series value of the first configuration with \( q \) units expressed by following formula

\[
\text{sr}_n(q) = \max_{1 \leq i \leq [q/2]} \left\{ p_{n+i} + \frac{p_{n+q-i} - p_{n+i}}{2} \right\}, \\
q \geq 4 ,
\]

can appear by initial decentralized series sum, that is maximum value is reached in the formula for parameter \( i \), not equal to \( [q/2] \). For example, for \( q = 4 \) such parameter \( i = 1 \). It means, that if in set compulsory of non-generated grids for given formula realization there will be such variant, at which one of grids will appear generated (it zerofills two units), that there is filling \( Z_{n+q-1} \), the series of which surpasses centralized series.

In particular, \( \text{sr}_6(2) = 19 = p_7 = \text{sr}_8'(0) \). But \( \text{sr}_4(5) = 20 \), that is explained by advantage \( 0.5 (p_8 - p_7) = 2 \) over \( 0.5 (p_7 - p_6) = 1 \). As far as generated grid is found for filling in variant with five units, it has resulted to value \( msr_8(0) = 20 \). Case is quite analogous: \( \text{sr}_9(2) = 31 = p_{10} = \text{sr}_{11}'(0) \). Here is \( \text{sr}_7(5) = 33 \), that is explained by advantage \( 0.5 (p_{11} - p_{10}) = 3 \) over \( 0.5 (p_{10} - p_9) = 1 \). As generated grid was found and here, \( msr_{11} = 33 \).

Naturally and hereafter for large parameter \( n \) similar effects can be observed. They little decrease maximum series \( MSR_n \subset Kf_n^{(I)} \), that is value \( msr_n \) relatively centralized variant \( \text{sr}_n'(0) = p_{n-1} \) obviously following from relation for the first configuration \( \text{sr}_{n-2}(2) = p_{n-1} \). Searches of such cases of variant estimations would acquire greater sense, if the second configuration \( Kf_n^{(II)} \) has not interfered in generation of maximum series \( MSR_n \) and \( MSR_n(q) \) for \( q = O(n) \).

For first \( Kf_n^{(I)} \subset SP_1 \) configuration of \( n \)-filling (6.8) with two permanent central units each new grid will be generated (it zerofills equally two units) in interval of greatest series \( SR_n^{(I)}(2) \). According to theorems 7 and 9 lengths of its greatest series \( SR_n^{(I)}(q) \) for \( q \geq 2 \) units is equal \( \text{sr}_n^{(I)}(q) = p_{n+q/2} + \Delta \), where \( \Delta = o(p_n) \). Then for the greatest series of this configuration we receive

\[
SR_n^{(I)}(q) \subset Kf_n^{(I)}: \\
\lim_{n,q \to \infty} \sup_q \frac{\text{sr}_n^{(I)}(q)}{p_{n+q}} = \lim_{n \to \infty} \frac{\text{sr}_n^{(I)}(2)}{p_{n+2}} = 1 ,
\]

and by that the ability to be generated for such kind of all grids results to estimation of maximum series with help of the first axial configuration. Received estimation is presented by value \( msr_n \sim p_n \). It is possible once again to note, that such estimation was not surpassed for observed examples of parameter \( n \).

Besides from generation algorithm of the first configuration follows, that zeroes frequency will be upper just in interval \( SR_n^{(I)}(2) \). Its appreciable exceeding inevitably results to occurrence of multiple zeroes for grids of greatest modules. But it causes fall of essential factor of grids. Purely, this phenomenon and quite precisely reflects the last given relation.
However maximum observable series of the first configuration is not occasion for extrapolational conclusions. Attentive consideration of the second configuration $K_{f_n^{(I)}} \subset S_{P_1}$ of expression $(6.10)$ for parameter $m = \lfloor \log_2 p_{n+1} \rfloor$ demonstrates insufficiency of series of the first configuration $K_{f_n^{(I)}}$. Although among series of kind $p_{n+i} - p_{n+i}$ will meet and units, corresponding to numbers $2^{m+s}$, $2^p_{n+j}$.

According to the theorem 16 and formula $(6.10)$ the lengths $(sr_n)$ of greatest series $SR_n^{(II)}(2m)$, $SR_n^{(II)}(2m + 1)$ and $SR_n^{(II)}(2m + 2)$ are equal accordingly

$$sr_n^{(II)}(2m) = \max \{2^{m+1}, 2^{m-1} + p_{n+1}\},$$

$$sr_n^{(II)}(2m + 1) = 2^m + p_{n+1}, \quad sr_n^{(II)}(2m + 2) = \max \{2 \cdot p_{n+1}, 2^m + p_{n+2}\}.$$

The central core as proven here statements, as main numerical characteristic of the second axial configuration $K_{f_n^{(II)}} \subset S_{P_1}$ is following idea:

Series of the second axial configurations $SR_n^{(II)}(q)$ of system $S_{P_1}$ with units $q = 2m$, $2m + 1$, $2m + 2$ act by maximum series $MSR_n(q)$ prime filling $Z_n \subset S_{P_1}$ with corresponding quantity of units for all $n \geq n_0$, since some $n_0$.

This thesis requires careful consideration and confirmation taking into account, that the initial values $n$ of fillings demonstrate advantage of the first series of sequential filling (zero stage) and series with two units for the first axial configuration $K_{f_n^{(I)}} \subset S_{P_1}$, realizing by the first stage.

The second stage of transformation of maximum series coincides with the coming superiority of series of the second axial configuration $SR_n^{(II)}(q)$. It is realized for $n$ practically the depriving researchers of concrete check opportunities, as resources of computer at searching algorithms are rather limited. Nevertheless theorems 5 and 6 grant lower estimations of maximum series, which already permit many.

Lengths of greatest series $SR_n^{(II)}(q) \subset S_{P_1}$ mentioned by last relations contain $2m \leq q \leq 2m + 2$ units. According to the rule of determination of parameter $m$ for small $n$ these values $q$ can even it surpass.

We shall find those values $n$ for which in configuration $K_{f_n^{(II)}}$ there will be series of length $2^{m-1}$ first. According to proven states, the core of configuration with $2m$ units does not already change and will increase by following series of length $2^m$. In such case for each parameter $m$ its value $n = n_m$ will be found by scheme $p_{n+1} = \min \{p_i > 2^m\}$. We shall give initial values $n$:

| m   | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|----|----|----|
| $n = n_m$ | 1 | 3 | 5 | 10 | 17 | 30 | 53 | 96 | 171 | 308 | 559 |
| $p_n$   | 3 | 7 | 13 | 31 | 61 | 127 | 251 | 509 | 1021 | 2039 | 4093 |
| $p_{n+1}$ | 5 | 11 | 17 | 37 | 67 | 131 | 257 | 521 | 1031 | 2053 | 4099 |

From the table it is visible, that only for $m \geq 6$ value $n_m$ confidently surpasses considered parameters $q$. But also there are not enough it, as far as that to take advantage of the second axial configuration for construction of the greatest series,
it is necessary part of grids (naturally greatest modules) to send for liquidation all without exception $2m$ central units of configuration $K f_n^{(I)}$.

From representation of the second configuration (6.10) it is possible to conclude, that for $n = n_m + s$, where $s = 0, 1, \ldots, O(n_m)$, relatively greatest series $SR_n^{(II)}(q)$ contains $q = 2m$ units. For $n = n_m + 1 + s$, where $s = 1, 2, \ldots, O(n_m)$, opposite, relatively greatest series $SR_n^{(II)}(q)$ contains $q = 2m + 2$ units. In intermediate variants there can quite appear most acceptable $q = 2m + 1$.

The fixed places (points), in which units of considered central core of configuration $SR_n^{(II)}(q) \subset SP_1$ stand, permit to estimate opportunity of filling, that is zerofilling of these units by grids of greatest modules, that series without units $sr_n(0)$ to generate by corresponding redistribution.

It is easy to find odd distances between units of central core of configuration $SR_n^{(II)}(q) \subset SP_1$ presented only by numbers of kind $2^k - 1$ or $2^k + 1$, where $k \leq m$. It means, that in considered interval (especially for large value $n$) primes as modules of grids can not find more than one, satisfying these conditions. Really, interval is limited by degree of two and for one $k$ values $2^k - 1$ and $2^k + 1$ can not be primes simultaneously.

Thus practically all $2m$ grids $S(p_i) \subset Z_n$ of large modules $p_i$ should be directed to generation of filling $Z_n$ with series of the second axial configuration $SR_{n-2m}^{(II)}(2m)$ of length $sr_{n-2m}(2m)$ already not containing units. According to given relations this value is equal

$$sr_{n-2m}^{(II)}(2m') = \max\{2^{m'+1}, 2^{m'-1} + p_{n-2m+1}\}, \quad m' = [\log_2 p_{n-2m+1}].$$

For considered parameters $n$ value $m'$ is possible only per unit less $m$, therefore at estimation of greatest series is allowable to consider $n$ such, for which $m = m'$. In view of all these conditions, expressions and conclusions in filling $Z_n$ there will be series without units of length $2p_{n-2m+1} + \Delta$, where $\Delta = O(p_{n-2m+1})$.

However that this series of the second configuration $sr_{n-2m}^{(II)}(2m') \subset K f_n^{(II)}$ exceeds serial structure of the first configuration $K f_n^{(I)} \subset SP_1$, is obviously necessary execution of inequality $2p_{n-2m+1} > p_{n-1}$. According to theorem 6 such inequality will be executed at $n \geq 26$. Advantage of the second configuration series for $n \to \infty$ increases, approached to coefficient two.

Thus maximum series $SR_{n-q}^{(II)}(q)$ with quantity of units $2m \leq q \leq 2m + 2$ automatically means too maximum series without units, constructed by described way in interval of central core of configuration for $n \geq 26$.

For proof of maximum series $SR_{n-q}^{(II)}(q)$ for $2m \leq q \leq 2m + 2$ it is necessary again to address to concept of generated grids, in this case in interval of central core of configuration $K f_n^{(II)} \subset SP_1$. Each new grid $S(p_n)$ in interval of configuration $K f_n^{(II)}$ of length $2p_{n+1}$ is responsible for one zero of multiplicity two (additional multiplicity) and also for zerofilling of two units. The comparison as though for the benefit of series of central core of the first configuration $K f_n^{(I)} \subset SP_1$, where at same two eliminated units does not occur multiple zero.
This conclusion has hurried character. According to it, in general becomes inexplicable occurrence of advantage of series of the second configuration \( K f_n^{(II)} \subset \text{SP}_1 \). The reason that in extended central interval of length \( 4p_{n+1} \) the same grid \( S(p_n) \) zerofills two units, corresponding to values \( 2p_n \). Thus in such extended interval property of grid \( S(p_n) \) to be generated grid reflected by four eliminated (zerofilled) units and one multiple zero.

From elementary product of grid \( S(p_n) \) with grid \( S(p_1 = 3) \) follows, that or such filling \( Z_n \subset \text{SP}_1 \), limiting on efficiency and opportunity of grids to be generated grid in extended interval, exceeds it is impossible, or for even more extended interval it will take place, but in such case it will be executed by the same structure of the second axial configuration \( K f_n^{(II)} \subset \text{SP}_1 \).

There can arise question, how the eliminated second pair of units of extended interval of length \( 4p_{n+1} \) influences to occurrence of series of length \( 2p_{n+1} \). Answer is extremely simple. In series of length \( 2p_{n+1} \) eliminated second pairs of units of grids group of smaller modules have come. It predetermines creation of the second configuration with central series of such extent.

It is necessary to take into account, that the minimum grid \( S(3) \) dictates and determines impossibility of excess of essential coefficient for product with grid \( S(p_n) \) in interval of length \( 4p_{n+1} \) by formula (4 eliminated units – 1 multiple zero). Exactly such quantity of units is zerofilled by the second axial configuration. From here becomes explicable impossibility of excess twice of length of maximum series \( MSR_n(0) \) of greatest grid module: \( msr_n \neq p_n \).

And really, we shall consider occurrence in half-configuration \( K f_n^{(II)} \) of zeroes of multiplicity one and higher, introduced in filling \( Z_n \) by grid \( S(p_n) \). Sign \( \emptyset \) is zero of increased multiplicity:

\[
\begin{array}{cccccccccccc}
0 & p_n & 2p_n & 3p_n & 4p_n & 5p_n & 6p_n & 7p_n & 8p_n & 9p_n & 10p_n & 11p_n & 12p_n & \ldots \\
\emptyset & o & o & \emptyset & o & \emptyset & o & \emptyset & o & \emptyset & \emptyset & \emptyset & \emptyset & \ldots \\
\ldots & \ldots & 4 & 4 & 6 & 6 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & \ldots \\
\end{array}
\]

Here symbol ”o” means zero of one multiplicity of grid \( S(p_n) \) for filling, which is responsible for increase of zero series, as far as zeroes of increased multiplicity can not already play such role. Naturally, zero of multiplicity one of grid \( S(p_n) \) stands in place \( kp_n \) only in that case, when it takes place \( (k, \prod_{i=1}^{n-1} p_i) = 1 \).

Correlation of quantity of generated zero of multiplicity one to zero increased multiplicity in the same interval appreciably decreases with growth of interval \( I_n \). If to take into account half-configuration of given chain, since interval of length \( 4p_{n+1} \) we shall receive the correlation \( 4/1 \), which as will remain greatest in third line of the table. Even appeal to initial filling \( n = 2 \) with grid \( S(p_2) = S(5) \), when only zeroes of kind \( 15k \) will be multiple, does not correct situation, as the subsequent correlation is \( \frac{8}{3} < 4 \).

Thus correlation \( 4/1 \) of quantity of unitary zero to multiple quantity for interval of length \( I_n = 4p_{n+1} \) is impossible to exceed, and it is impossible and in extended
interval, including structure of the second configuration \( Kf_{n}^{(II)} \subset SP_1 \). But if second axial configuration \( Kf_{n}^{(II)} \) realizes exactly such limiting generation, which restricts length of series by value \( sr_{n}^{(II)}(q) \) for \( 2m \leq q \leq 2m+2 \), it and will be maximum in period \( PZ_n \) of filling \( Z_n \subset SP_1 \).

Assuming that common problem is fixed to create product of grids in interval under condition of greatest zerofilling of units set. In such case it is impossible to escape product of next grid \( S(p_n) \) with first and basic grid \( S(3) \). First stage of creation of such product without multiple zeroes is characterized by reception of the first axial configuration \( Kf_{n}^{(I)} \), in which each new grid appears generated that is it zerofills two units in interval of greatest series \( SR_n^{(I)}(2) \).

But we shall notice, the length of such series \( sr_{n}^{(I)}(2) = p_{n+1} \) appears not limiting for large \( n \). Therefore has to consider other product, other configuration, already with multiple zero in interval of series. The product of grids \( S(3), S(p_n) \) in interval of length \( 4p_{n+1} \), when central zero of multiplicity two appears common, permits in given variant to zerofill at once four units. Easily to see, such construction will be executed for each new grid.

Above is proven that product of grids \( S(3), S(p_n) \) in any interval of greater length not capable to ensure large concentration of zeroes of multiplicity one. It is necessary to note, for all that effect of intermediate grids is not taken into account. It does not touch on all four new essential zero, but results in additional multiple zeroes in interval, exceeding of length \( 4p_{n+1} \).

Thus described structure of sequential product of grids provides maximum concentration of unitary zeroes in interval with length \( 4p_{n+1} \). But this structure and is the second axial configuration \( Kf_{n}^{(II)} \). If to take into account, that its greatest series with central units \( SR_n^{(II)}(2m+2) \) has length about \( 2p_{n+1} \), that is far from \( 4p_{n+1} \), it and means that the series with \( (2m+2) \) units of considered second axial configuration are maximum in all period of filling.

The outstripped interval of extreme property \( 4p_{n+1} \) relatively maximum series, not exceeding \( 2p_{n+1} \), consists that zeroes of grid \( S(p_n) \) form the basis for the subsequent maximum series, though they do not include in series, remained unitary. Thus the realization of maximum series permanently overtakes interval of extreme properties of grids to be generated. It creates preconditions of successful preservation of established relation of intervals (\( \approx 2 \)).

The second axial configuration realizes ineradicable condition of preservation of property of grids to be generated in interval of length \( 2p_{n+1} \) and increased property – in interval of length \( 4p_{n+1} \). As far as at increase \( n \) this extremity will also find reflection in generated the greatest series \( SR_{n}^{(II)}(q) \), that such property can not be already abolished. Interval of length \( 4p_{n+1} \) is least with such high essential coefficient (remaining all appreciably more length of greatest series) and the received series of the second axial configuration \( SR_n^{(II)}(q) \) for \( 2m \leq q \leq 2m+2 \) is maximum.

If to take into account, that the third axial configuration is not present, the central thesis about maximum series \( SR_{n}^{(II)}(q) \) is proven. For the proof of theorem
and expression (6.19) now it is enough to remind, that grids of the greatest modules, sent for elimination of units of central core of configuration, at the best once (that is one grid) can zerofill two units (in the other cases – only one).

Thus, reception of maximum series \( MSR_n(q) \) for \( q < 2m \) or even for \( q = 0 \) for large \( n \geq 26 \) is directly connected with series \( SR_{n+2}^{(II)}(q) \) for \( 2m \leq q \leq 2m + 2 \) and maximum objects in period of filling are generated with the help of typical series of the second configuration of previous \( n' < n \) by the same way, which they were submitted in model. It is redistribution of zeroes of grids in filling \( Z_n \).

Let series \( SR_{n+2}^{(II)}(q) \) is maximum \( MSR_n(q) \) series and

\[
MSR_n(q) : sr_n^{(II)}(q) = mr_n(q), \quad q = 2m \text{ and } q = 2m + 2,
\]

that is maximum series \( m_{sr_{n+2m}}(0) \) is created in its basis. Then at transition from \( n \) to \( n + 2 \) series \( SR_{n+2}^{(II)}(2m) \) arises without fail for creation maximum series \( m_{sr_{n+2m+2}}(0) \). This is connected with grids \( S(p_{n+1}) \) and \( S(p_{n+2}) \). They are used ineffectively at generation of maximum series \( m_{sr_{n+2m}}(0) \). Moreover let is formed maximum series \( MSR_{n+2m}(0) \) on basis of the greatest series \( SR_{n+2}^{(II)}(2m) \) with help \( 2m \) grids \( S(p_{n+i}) \) for \( 1 \leq i \leq 2m \).

Then series \( SR_{n+2m}^{(II)}(q) \), where \( q = 2m \) or \( q = 2m + 2 \) is source of generation of next maximum series \( MSR_{n+2m+q}(0) \). It follows from inequality

\[
sr_{n+2m+q}(0) < sr_n^{(II)}(q), \quad SR_{n+2m+q}(0) \supset MSR_{n+2m}(0),
\]

so far as grids \( S(p_{n+j}), 1 \leq j \leq 2m + q \) can zerofill only \( q + 2m + \epsilon \), where \( \epsilon = O(m) \) units. Decentralized series on basis of the maximum series \( MSR_{n+2m}(0) \) can direct to addition of a little more one zerofilling of units by each grid. For all that series \( SR_{n+2m+q}(0) \) increases by only average series of filling in period.

Axial configurations \( K_f_{n}^{(I)} \) and especially \( K_f_{n}^{(II)} \) are natural, unique and active factories of maximum series. Natural packing of grids in the second axial configuration maximum effectively and clearly sequentially realizes activity of every new grid at generation of the maximum series \( MSR_n(2m) \). Any infringement of packing leads to loss of maximum.

The proven theorem permits quite essentially and considerably more precisely to judge distribution of maximum series with units and without them in given filling.

**Theorem 19.** Maximum series \( MSR_n \) and \( MSR_n(q) \) for all \( n, q \) in prime system \( SP_1 \) have the upper estimation

\[
MSR_n(q) : m_{sr_n}(q) < C_n p_{n+q}; \quad C_n < 1 : n + q < 25 ; \quad C_n > 1 : \ n > 15 \text{ and } n + q > 28 ; \quad 2 < C_n \overset{n \to \infty}{\to} 2. \quad (6.20)
\]

**Proof.** For initial \( n \leq 3 \) value of maximum series \( MSR_n \equiv MSR_n(0) \) coincides with the first series \( SR_n^{(I)} \) of sequential filling:

\[
MSR_n \subset SP_1 : \ sr_1^+(0) = p_{n-1} = \frac{1}{2}(p_{n+1} - 1), \quad 1 \leq n \leq 3.
\]
But already for \( n = 4 \) this equality is infringed and advantage goes to series of the first axial configuration \( K_f^{(I)} \). According to theorems 17 and 18 as well as with given calculations, for \( n < 25 \) equality is valid

\[
MSR_n(2) \subset SP: \quad msr_n(2) = sr_n^{(I)}(2), \quad 3 \leq n < 25,
\]

whence follows \( msr_n(0) < p_n \). If to take into account, that parameter of units of series \( q > 0 \) does not change estimations, only sharply them easing at increase, the first part of the theorem is proven.

However the further increase of main parameter of filling \( n \) results in loss of advantage of series of the first configuration \( K_f^{(I)} \), which goes to second \( K_f^{(II)} \). From the same theorems 9 and 10 it is possible to receive equality, especially not pay attention to details to extreme exact expression, that is having chosen one of variants of maximum series \( MSR_n(2m + 2) \):

\[
msr_n(2m + 2) = sr_n^{(II)}(2m + 2) = 2 \cdot p_{n+1}; \quad n > 15, \quad m = [\log_2 p_{n+1}].
\]

For \( n \) close 12 – 15 the values of corresponding maximum series of both configurations are reasonably close one another, that is connected with proximity of values \( n \) and \( 2m \). Thus and coefficient \( C_n \) of theorem will be close to 1. But from the last equality clearly, that at the increase \( n \) value \( 2m \) as \( 2m = O(n) \) all less influences for estimation of maximum series \( MSR_n \subset SP_1 \):

\[
MSR_n(0): \quad msr_n < 2 \cdot p_{n-2m}; \quad n + 2m > 28, \quad m = [\log_2 p_{n+2m'}],
\]

as well as \( msr_n > p_n \), and where value \( m' \) not more, than per unit differs from \( m \). To the point from connection of systems \( SP_1 \) and \( SP_0 \) and received inequality immediately follows

\[
MSR_n(0) \subset SP_0: \quad msr_n < 4 \cdot p_{n-2m-1}
\]

under the same easily attainable conditions. Taking into account, that parameter \( q > 0 \) inequality do not infringed, from ultimate expression the proof of the second part of the theorem and relation (6.20) follows.

**Theorem 20.** The validity of theorem 19 for system \( SP_1 \) means validity of the main theorem for any non-singular system \( SS \), and with the same upper coefficient two, as well as in unimproved expression (4.7).

**Proof.** As far as in system \( SP_1 \) there is expression

\[
\frac{n}{1 - \gamma_n} = n \prod_{i=1}^{n} \left(1 + \frac{1}{p_i - 1}\right) = C_n p_n, \quad C_n < 1, \quad (6.21)
\]

from theorems 13–18 and especially from theorem 19 follow validity of the main theorem in the form (4.7) for this system. According to lemma 2 if the main theorem
is valid in system $SP_1'$, it is fair and in any other system, and its estimations will remain by majorizing. As far as under theorem 4 systems $SP_1'$ and $SP_1$ differ whole by one grid, the characteristics of their maximum series are rather close. Taking into account, that $C_n$ in (6.21) does not reach unit, the upper bound 2 in estimation (4.7) will remain and for system $SP_1'$. Besides possibility of exception for system $SP_1'$ is refuted by inequality $msr_{n+s}[SP_1] > msr_n[SP_1']$, where $s = O(n)$.

It is necessary to note, that in monograph [1] proofs of the main theorem are placed also, giving for system $SP_1$ estimations more weak, than value of expression (6.21): $C_n(1)p_n \ln \ln n; \ C_n(2)p_n \ln \ln n \ln n; \ C_n(3)p_n, \ C_n(3) < 3$. □

7. Two-sided fillings and main theorem

The statements of 5 chapter grant some other approach to the proof of the finishing form of the main theorem produced by theorem 20. It is thus necessary to note a decisive role of complex (summarized) characteristics $n$-fillings in estimation of its major numerical parameters.

**Theorem 21.** Upper estimations of maximum series $MSR_n(q)$ depend on density of zero of filling $\gamma_n$ and parameters $(n, q)$ for all classes of systems.

$$MSR_n(q, SS) : msr_n(q) \leq FF\{\gamma_n, SS, Z_n\}, \ \forall \{Z_n \subset SS\}. \quad (7.1)$$

**Proof.** Certainly, only non-singular fillings and systems are meant. The statement and formula estimations would not have equally the price, if not unity of law $FF$ outside of dependence from a class of system $SS$. Reasonably the weak difference of concrete functions $FF_1(SS_1), FF_2(SS_2)$ for the most various systems is meant too. The discrepancy of functions $FF_1 \neq FF_2$ is in complete dependence from availability whether or not representative sets of multiple zeroes in a appropriate system. The systems from first to third types are various.

Such function $FF$ (7.1) does not depend on a particular set of grids, if given $n$-filling does not leave the same class of systems. And if other set of grids has same or though close density of zeroes $\gamma_n$, estimations of maximum series will coincide or are close. Such unity permits to consider a number of problems without concreteness of this or that system, as far as for them it should expect uniform conclusions.

The function of the main theorem $MT$ from parameters $(n, q)$ and density (average frequency) of zeroes for the period of filling in the formula of the main theorem acts unified majorant of values of maximum series $MSR_n(q)$. And irrespective of availability of multiple zeroes as in period, as in series. But if multiple zeroes have managed to avoid or any image pass to variant without multiple zeroes in period (even only in interval), the global problem estimate of maximum series
can be considered permitted. □

Will not hinder to specify occurrence of the universal formula (5.9, 7.1) of main theorem \( MT \), playing by a determining role in the fillings method.

**Theorem 22.** The coefficient \( \tau_n \) of the main theorem \( MT \) does not exceed unit \( \tau_n \leq \tau = 1 \) for degree-systems, without multiple zeroes and with primary growth of modules. It determines one-sideness of their fillings \( Z_n^{(1)} \).

\[ \{SS_d, SS', SS[2]\} : \quad msr_n(q) \leq \tau \frac{n + q}{1 - \gamma_n} + 1; \quad \gamma_n = \frac{H_n}{PZ_n}, \quad (\tau = 1) \Rightarrow Z_n^{(1)}. \]

(7.2)

**Proof.** First occurrence of the formula of the main theorem on the basis of exact expression of value of a maximum series \( MSR_n(q) \) for a class of degree-systems \( SS_d \) can be complemented a little by other reasons. We shall consistently consider some turning-points for the clearing of exposition.

1. We shall evaluate important value, if frequency (density) of zeroes \( \gamma_n \) in period \( PZ_n \), period and other characteristics of \( n \)-fillings are

\[ PZ_n = H_n + E_n, \quad \gamma_n = \frac{H_n}{PZ_n} : \quad r_n = \frac{1}{1 - \gamma_n}; \quad r_n = \frac{PZ_n}{E_n} = \frac{H_n}{E_n} + 1. \]  

(7.3)

Value \( r_n \) from expressions (7.3) it is possible to interpret by quantity of zeroes per unit of a period, summarized with unit. Thus \( r_n \) according to definition is interval, stipulated by zeroes of average grid.

2. We shall consider examples. The value \( r_n \) equally to unit at \( n \) infinite grids, and value of maximum series coincides with the upper estimation in the main theorem \( msr_n = \frac{n}{1 - \gamma_n} + 1 = n + 1 \). Thus estimation of a series with coefficient \( \tau = 1 \) is unimproved. At the same time the value of maximum series \( msr_n \) will remain same, that is \( n + 1 \) and at \( a_1 \geq n + 2 \). Though in such case density of zeroes, obviously, different from zero (\( \gamma_n' \neq 0 \)).

The interval of a series with zeroes of one average grid is equal two at \( \gamma_n = \frac{1}{2} \), that is at \( r_n = \frac{1}{1 - \gamma_n} = 2 \). It means, that the estimation of maximum series at such density \( \gamma_n \) is equal \( 2n + 1 \), and \( msr_1(q - 1) = 2q \), that is estimation more length of series whole per unit (the affinity of values is doubtless) at identical quantity forming grids or grid with units.

Interval of a series with zeroes of one average grid fractional at density of zeroes \( \gamma_n = \frac{1}{3} \), that is at \( r_n = \frac{1}{1 - \gamma_n} = \frac{3}{2} \). It does not interfere formation of the upper estimation of maximum series \( \frac{3n}{2} + 1 \) or \( \frac{3(1+q)}{2} + 1 \), and it more values \( msr_1(q) \) for 1-filling – grid \( S(3) \).

3. The considered examples demonstrate and confirm a role of value \( r_n \), zero reflecting zeroes of an averaging grid of \( n \)-filling. Then the value \( r_n + 1 \) will appear by the upper estimation of maximum series \( MSR_1 \), found with the help of such
grid. The estimation is well grounded as far as from expression (7.3) follows: the quantity of zeroes per unit of filling takes into account the frequent contribution of all grids of \( n \)-filling. From here the numerical value \( r_n + 1 \) is absolute majorant of series, formed by one grid. This estimation is achievable \((r_n + 1 = 2)\) for infinite grid. In other cases the estimation is not achievable.

4. Attraction of the second unit or value \( n = 2 \) results in summarized estimation \( 2r_n + 1 \) of the heaviest series by two grids. This estimation of maximum series with the help \( r_n \) zeroes precisely such a conditional grid takes into account summarized density of all grids of filling. The estimation of maximum series \( MSR_n \) by all \( n \) grids arises completely similarly. Taking into account, that the frequent contribution of each initial grid \( S(a_i) \) is included in values \( r_n \), received estimation \( nr_n + 1 \) is obliged to surpass actual value \( msr_n \) of the most maximum series \( MSR_n \). It occurs in a reality.

5. The address to variant \( q > 0 \), certainly, does not reduce efficiency of estimations of maximum series of the main theorem. Any growth \( q \) provides linear increase of estimation, which can become only superfluous. The classes of systems, for which indicated reasons (for coefficient \( \tau = 1 \) of the main theorem), are described by the theorem 11.

6. If to consider conditional grids from \( r_n \) intervals was held, there is the question about parameters provided that all \( n \) such grids will form conditional filling with the same density of zeroes. There is expression immediately following from sum of zeroes of all \( n \) grids, as far as these grids are identical:

\[
\left\{ \frac{n r_n}{PZ_n'} = \gamma_n \right\} \Rightarrow \frac{n}{(1 - \gamma_n) PZ_n'} = \gamma_n; \quad PZ_n' = \frac{n}{\gamma_n (1 - \gamma_n)} ; \quad a'_i = PZ_n',
\]

that is modules of such grids \( a'_i = PZ_n' > nr_n \) coincide with the conditional period.

7. Creation of conditional \( n \)-filling with conditional grids of \( r_n \) zero for a conditional period not only results to majorizing upper estimations of maximum series, but it also explains sources of origins of the main theorem.

Alongside with the theorem 11 statement and expression (7.2) prove, that by initial premises of a coefficient \( \tau = 1 \) in the formulation of the main theorem and one-sidedness of filling \( Z_n^{(1)} \) a system correlation between modules acts. It is realized in classes of systems of grids \( SS_d, SS', SS_{[2]} \). The interdependence of the described characteristics is reasonably obvious. \( \square \)

Rather in detail the investigated systems of grids of the previous theorems can not affect central interest, which cause systems of the third type with other components – grids of modules of completely other kind.

We shall remind about achieved. Estimate possibility of the upper values of maximum series common (summarized) density of zeroes is the main purpose of the fillings method. The theory of numbers in sieving process tries to operate with modules of separate grids \( S(a_i) \). Besides the fillings method is oriented to a period
of products of grids, and other methods consider at the best interval of length as a square of the heaviest module of filling. Would seem, unremovability of multiple zeroes in somehow appreciable interval is a insuperable obstacle to further reasons and conclusions. Especially, for major systems (for example, $SP_1$), the summarize set of all multiplicities in many times surpasses a initial period.

On a way of the decision of the problem of essential set of multiple zeroes, so characteristic for many important systems and fillings, it is necessary to specify the rule of transformation once more.

**Definition 30.** If $PZ_n$ is period of filling $Z_n$, $H_n$ is quantity of zeroes and $E_n = PZ_n - H_n$ is units for this period at density of zeroes $\gamma_n$, transition to $n$-fillings with majorizing frequency of zeroes $\gamma^*_n$ of multiplicity unit for that a set of grids $\{S(a_i)\}$ is executed under the scheme

$$PZ_n = H_n + E_n, \quad \gamma_n = \frac{H_n}{PZ_n} : \quad H_n^* = PZ_n \sum_{i=1}^{n} \frac{1}{a_i}, \quad \gamma^*_n = \frac{H_n^*}{H_n^* + E_n}.$$ (7.4)

Completely obviously, that $\gamma_n \leq \gamma^*_n$, and the equality can be observed only and only in case, when initial $n$-filling has not multiple zeroes.

**Theorem 23.** The density of zeroes $\gamma^*_n$ in the formulation of the main theorem provides absolute majorizing estimation $msr_n(q)$ of maximum series with a coefficient $\tau = 1$ in the class of any non-singular systems.

$$MSR_n(q) \subset \forall SS : \left\{ \gamma = \gamma^*_n = \frac{H_n^*}{H_n^* + E_n} \right\} \Rightarrow \left\{ msr_n(q) \leq \frac{n + q}{1 - \gamma^*_n} + 1 \right\}.$$ (7.5)

**Proof.** The statement reflects idea of a opportunity of local redistribution of zeroes high multiplicity in lowered multiplicity, down to unitary. The expansion of a allocated interval thus occurs, and the quantity of units remains constant. The decrease of number of units in a interval would mean not redistribution of zeroes, but product with unknown new grid. It contradicts the principle of filling. Really the limited opportunities of similar redistribution (for the interval whole period they are reduce to zero), can not be obstacle to idea of reception of imaginary filling without multiple zeroes in period.

The disposal from zeroes of multiplicity higher unit is, purely, decision of the problem of upper estimation of maximum series. From here a reason about replacement of actual density of zero for a period, where zero of multiplicity $k > 1$ is one zero. The former frequency $\gamma_n$ varies other $\gamma^*_n$, and at its formation multiple zeroes are redistributed by a set of zeroes of multiplicity unit. Received in a result obviously the higher conditional density appears by majorizing frequency of zero in the formulas of estimation of a maximum series. Thus the theorem 3 about
the upper estimation of a series $MSR_n(q)$ is applicable to such conditional filling (already without multiple zeroes).

However the density (frequency of zeroes) found thus appears obviously redundant for practically important systems in applications (especially for systems of third type). It is explained unremovability of multiple zeroes in intervals of maximum series, and even at reasonably small $n$. For example, for a system of primes $SP_1$ achievement rather small value $msr_n \sim 15$ during growth $n$ not in forces to avoid multiple zero in interval of such length. Product of modules of the first two grids equally 15, and zero of multiplicity two is unremoved.

From here supervision follows. The increase of main parameter $n$ of filling and consequently value $msr_n(q)$ (even at $q = 0$) in systems of the third type provides a fast increase of a set of those multiple zeroes, which can not participate in redistribution. Though it is conditional mental operation.

Let any non-singular system $SS$ and fillings $Z_n$ are given. The construction of majorizing density of zeroes at preservation of other parameters of filling $(n, q)$ consists of expansion number of zeroes by them multiplicities for increasing period. The quantity of units remains constant at such transformation. Then the received density reflects density of filling without multiple zeroes or even degree-filling.

The deep sense of the generalized formula of the main theorem consists of confirmation of a decisive role of density of zeroes $\gamma_n$ for the upper estimation of maximum series $MSR_n(q)$. Such estimation does not depend already from a class of a system and particular grids $\{S(a_i)\}$, which enter in filling. The idea of the formula of the upper estimation is incorporated in reasonably transparent reasons.

The variants considered above of systems $SS_d$, $SS'$ and their fillings have not zeroes of multiplicity higher unit. Then from definition 28 and expressions (7.4, 7.5) coincidence $\gamma_n = \gamma^*_n$ and already proven case of the main theorem follows.

If initial grids $\{S(a_i)\}$ are infinite (infinite module $a_i = \infty$), the upper estimation of series coincides their length at $\tau = 1$. The variant of density of zeroes, different from a zero $\gamma_n > 0$, is also considered by the theorem 22. The theorem talks, that on each unit of filling drops $r_n = \frac{1}{1-\gamma_n} > 1$ – interval with zeroes. So conditional filling of $n$ equal grids creates, and value $r_n$ is the main characteristic of a compact arrangement of zeroes, including all frequent components of former $n$ grids. From here the simple summation of values $r_n$ provides majorizing character of numerical estimations of maximum series.

Or else, the created conditional grid in difference from standard contains not one zero in a own period, equal to module, but $r_n$ at unified averaged period. Then the heaviest series is formed by simple association of zeroes each from $n$ equal grids. Such process of formation conducts to creation of majorizing estimation of maximum series, and at all not to creation of series of zeroes $MSR_n$ of initial filling. Certainly, in common case the number $r_n$ is not integer, but also it not a obstacle to conditional construction of a redundant modernized series, consequently and upper estimation.
Now simply to notice, that the described mechanism of the account of multiple zeroes and density of zeroes of multiplicity not above unit is situated also in a scheme of redistribution of zeroes at formation of the upper estimation of maximum series \( MSR_n(q) \). The count of quantity of zeroes in the period of \( n \)-filling can be considered by arithmetic consecutive summation of zeroes of all grids. It reduces process to the scheme (7.4), to a sum of multiplicities of zeroes, and to density \( \gamma_n^* \). And the discrepancy with \( \gamma_n \) is not the certificate of falsehood.

The received estimation is absolutely upper for a class of any non-singular systems and fillings. Certainly, such conclusion does not remove the subsequent conclusions concerning redundancy of the last found estimation (7.5) for systems with abundance of multiplicities. This consequence of growth of a set of unremovable multiple zeroes and parallel increase of a set of grids of large modules, participating in creation of density \( \gamma_n^* \). But they introduce in comparison weak contribution to formation of a concrete series \( MSR_n \).

The obvious incompleteness of effect of multiple zeroes, even promptly growing, compels critical revision of idea of frequency \( \gamma_n^* \) as absolute majorant of the theorem 23 and expression (7.5). Too frank redundancy of density \( \gamma_n^* \) forces to resort to restrictions. As a example it is possible to result a extreme system \( SS \) with abundance of multiple zeroes \( \{(p_i - 1) \cdot S(p_i)\} \) at \( i \geq 0 \).

Theorem 24. The obvious redundancy of a set of multiple zeroes \( H_n^* \), that is density of zeroes \( \gamma_n^* \) at creation of majorizing estimation \( msr_n(q) \) maximum series, returns the basic characteristic to frequency of zeroes \( \gamma_n \).

\[
\{ H_n^* \gg H_n^{RR} > H_n; \gamma_n^* \gg \gamma_n^{RR} > \gamma_n \Rightarrow MSR_n(q) \} : \quad \left[ msr_n(q) \leq \tau_n \frac{n + q}{1 - \gamma_n} + 1 \right].
\] (7.6)

Proof. Even the fluent sight on advancing growth of a set of passive (at formation of maximum series) multiple zeroes in relation to value \( H_n^* \) specifies insecure of density of zeroes \( \gamma_n^* \) as the candidate of basic value. The fact is that for any system valid density \( \alpha_n \), received with the help of \( msr_n \) from the formula of the main theorem, all further deviates from \( \gamma_n^* \) with growth \( n \).

In expression (7.6) effective multiple zeroes \( H_n^{RR} \) and density \( \gamma_n^{RR} \) can appreciably concede to number \( H_n^* \) and frequency \( \gamma_n^* \). It restores support of density of zeroes \( \gamma_n \) in the formula of estimation of maximum series with some factor \( \tau_n \), it is possible, different from unit.

It is necessary to take into account an alternate kind of algorithm of filling for confirmation of a determining role of the characteristic \( n \)-filling – frequency \( \gamma_n \).

Theorem 25. Supportness of density of zeroes \( \gamma_n \) results to to the majorizing coefficient two \( (\tau = 2) \) of the main theorem at frequency of zeroes \( \gamma_n \). It reflects two-sided algorithm of \( n \)-filling at formation of the upper estimation of maximum
series \( MSR_n(q) \) in a class of all non-singular systems \( \{SS\} \).

\[
\{\gamma_n^* \gg \gamma_n^{RR} \Rightarrow \gamma_n \Rightarrow MSR_n[q, Z_n^{(2)} \subset SS_{III}]\} : \quad \text{msr}_n(q) \leq 2 \frac{n + q}{1 - \gamma_n} + 1. \tag{7.7}
\]

**Proof.** This major result is in detail discussed and submitted above. Here the coefficient two is specified as a natural border, predetermined by two-sided algorithm of filling. But the statement cannot be perceived by the declaration of intentions. The two-sidedness of filling is destroyed by equivalent replacement of one grid \( S(a_{i+1}) \) by two grids \( S(2a_{i+1}) \), each of which and both are together the components of one-sided scheme. But the one-sided algorithm of filling permits to construct a series of zeroes twice smaller distance. From here and there is the coefficient two.

Argument about replacement of one grid \( S(a_{i+1}) \) by two grids \( S(2a_{i+1}) \) should perceive by the only explanation of a qualitative difference between one-sided and two-sided fillings algorithms. Really, the continuation of such scheme results to replacement of a next grid \( S(a_{i+2}) \) by four grids \( S(4a_{i+2}) \) with preservation of former zeroes frequency \( \gamma_n \). Thus one-sided algorithm of filling is saved, but parameter of quantity of grids of filling \( n' \) appreciably grows. Further growth \( n' \) results in essential distortion of estimation \( \text{msr}_{n'} < \frac{n'}{1 - \gamma_n} + 1 \) of maximum series \( MSR_{n'} \). Its value becomes more \( 2 \frac{n}{1 - \gamma_n} + 1 \) even at preservation of one-sided fillings algorithm \( Z_n^{(1)} \).

Let the inequality \( a_{i+1} < 2a_i \) takes place for all grids of \( n \)-filling of described system \( SS \) for \( i = 1, 2, ..., n - 1 \). Then consecutive growth of parameter \( n : n = 2, 3, ... \) according to consequence 1 for each stage permits find alternate one-sided and two-sided algorithms of fillings. At the same time renewed for each step the replacement of the grid \( S(a_{i+1}) \) by two grids \( S(2a_{i+1}) \) results to estimation \( \text{msr}_n : (i \Rightarrow 2i \Rightarrow 2n - 1) \) of maximum series \( MSR_n \).

The coefficient \( \tau = 2 \) appears by limit (at \( i = 1, 2, ..., n - 1 \)) of sequential reference to one-sided algorithm in a result of replacement of the senior grid of current \((i + 1)\)-filling by two grids.

According to the previous theorem, excessive redundancy of zeroes density \( \gamma_n^* \) concerning real effective density \( \gamma_n^{RR} \) conducts to supported frequency \( \gamma_n \). For systems of the third type \( SS_{III} \) it means two-sided algorithm of filling \( Z_n^{(2)} \) at formation of maximum series \( MSR_n(q) \) in the expression (7.7). In turn, such fact predetermines establishment of the majorizing coefficient \( (\tau = 2) \) in the formulation of the main theorem.

Following statement specifies deep connections between imaging principle, two-sided character of fillings algorithm and formulation of the main theorem within the
framework of fillings method for systems of all types, among which it is necessary to allocate systems of the third type $SS_{(III)}$.

**Theorem 26.** The imaging principle meets with unremoval redundancy of multiple zeroes in systems the second and especially the third type $SS_{(III)}$ and is transformed to two-sided algorithm of $n$-fillings, that results to majorizing coefficient $\tau = 2$ of the main theorem $MT$.

$$\{MSR_n[q, (W^*)] \Leftrightarrow MSR_n[q, Z_{n}^{(2)} \subset SS_{(III)}]\} : \left[msr_n(q) \leq \frac{n + q}{1 - \gamma_n} + 1\right].$$

(7.8)

**Proof.** All results of the previous chapters and fillings method as a whole specify validity of a put forward statement. It is thus necessary to take into account majorizing character of related systems of primes $SP_0$, $SP'_1$, $SP_1$ on mixing criterion. It guarantees maximum of a constant $\tau$ in the formulation of the main theorem. Such determined by two different ways value of the coefficient is equal $\tau = 2$.

In the expression (7.8) process of creation of maximum series $MSR_n(q)$ with the help of imagings $(W^*)$ and then account of their redundancy, results together with the introduction of algorithm of two-sided filling, to occurrence of majorizing coefficient $\tau = 2$ in the formula of the main theorem. \(\square\)

8. **The central theses of research and method**

We shall move a intermediate result. We shall allocate and shall remind the most important and turning points of the offered fillings method, directly leading as to the formulation of the main theorem, as to its proof. The determining character of consequences from it for number theory is obvious.

Let’s restore turning-points of the fillings method immediately carrying on to the statement of the main theorem and to the proof of it.

1. The major numerical characteristics of $n$-fillings and fillings method directly depend from zeroes distribution of given grids and from correlation (on base of divisibility) of grid’s modules.
2. The source object is the strip region of binary elements. It consists from $n$ of grids and has volume $(n; \infty)$.
3. The strip region is periodic, and the period of this strip is equal $PZ_n$.
4. The main numerical characteristics of period of elements strip region (quantity of units $E_n$ and zeroes $H^*_n$) are constants.
5. The imaging of the strip region on line inevitable erases constancy of zeroes quantity (from $H^*_n$ up to $H_n$) on mapped period and length of period, accordingly from $PZ^*_n$ up to $PZ_n$. 

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6. The numerical relations of filling, including extremal properties, accessible on some interval, depend from zeroes and units of imagined period. The received estimations are corrected by length of an interval.

7. The imaging \( f(k) \) of multiple zeroes of the elements strip region are transformed in the density of zeroes \( (\gamma_n - \gamma_n^{**} - \gamma_n^*) \).

8. The precise values in expressions for the maximum series of zeroes are found for fillings without multiple zeroes.

9. The upper-bound estimates of the maximum series depend from density of zeroes in the interval of updated length.

10. The majorizing density of zeroes \( \varrho_n \), not exceeding \( \gamma_n^* \), leads to an evaluation \( msr_n(q) < (n + q)/(1 - \varrho_n) + 1 \).

11. The proof of the main theorem is carried in classes of systems and fillings - from degree-system up to systems of the third type.

12. The precise values of all numerical characteristics of fillings, including maximum series, are found for all class of degree-systems.

13. The main theorem is proved for arbitrary systems of the first and second type on the basis of outcomes for systems without multiple zeroes.

14. The transition to systems of the third type is carried out because of redundancy of the array of multiple zeroes.

15. The possibilities of the fillings method allow to construct number of the independent proofs of the main theorem.

16. On the basis of the fillings method the proof of the main theorem in three forms is created, including strong decisive form without the support of the imaging principle with majorizing multiplicative constant two for all classes of systems.

17. The direct application of the main theorem leads in fillings and systems of prime numbers \( SP_0 \) and \( SW_0 \) to outcomes noticeably majorizing reached earlier.

18. The constant two \( (\tau = 2) \) of the main theorem \( MT \) is confirmed by a principle of two-sided filling for all systems of the third type, and also does not lean obviously on the imaging principle.

19. At the same time unremoval connection between the imaging principle and two-sided scheme of filling the main theorem reveals.

In the basis both fillings method and imaging principle are incorporated rule of constancy of main numerical characteristics of period and rule of adequacy of carry of possibilities of an interval to the same period of filling.

The quantity of units remains by the absolutely invariable value of mapped period. But imaging of the strip region can not save simultaneously length of period and quantity of zeroes in systems with multiple zeroes.

Whole way lead to the proof of the main theorem, reasonably convincingly testifies to independence of the fillings method and originality of problem statement. Nevertheless it is necessary precisely to separate offered from known sieving process.
Already comparison quite not neutral names of methods specifies cardinal difference of initial points at formation of research domain. If sieving process picks out primes rejecting other numbers as rubbish, fillings method concentrates attention as time on these rejected numbers, which are reflected by zeroes of grids – major elements of subsequent constructions.

Just zeroes, deleted elements, them multiplicity and frequency in period of fillings appear by determining objects of further structures and numerical analysis. This initial divergence in determination of main object of fixed attention is ineradicable reason of basic difference of methods.

There are all basis to consider, that fatal unimprovement of many received classical estimations and it is enough easily guessed proximity to unimproved many other, is explained by absolute non-removal of basic value \( x \) as bound of sieving process algorithm. And really, any sieve by fixing of \( x \) concludes self in rigid cage.

Two compared methods are not reduced one to other, though at the stage of initial model sieve is some fragment of filling. There are especially far from fillings method of trigonometrical sums and group of methods, connected with research of famous Riemannian zeta-function \( \zeta(s) \).

All numerous differences of sieving process and fillings method, considered in monograph [1], result in the following conclusion:

*The fillings method in essence and even in private variants do not reduce to the sieving process because of discrepancy of their premises and initial objects.*

### 9. Main achievements of the fillings method

All without exception described and proved statements lead to the main theorem. The constants enter into their formulas. They are different, but they are limited by several units. All received results by fillings method follow qualitatively and sometimes even quantitatively from first (weak) form of the main theorem with complete imaging of zeroes. However given results of number theory strengthen by unimproved form of the main theorem with coefficient two.

Fillings method is oriented to study of connection and dependence of natural numbers as modules of grids in systems, however its achievements, presented here, primes distributions concern. As the first step we are addressed to the statement, antiquity which permits to name its by mathematical symbol.

*Confirmation of statement.* The series of primes \( p_i \) has not completion.

*Third proof of the famous theorem* (after Euclid and Euler), given in [1] is received not so much because from its independence, necessity or special significance for further, how many with purposes of demonstration of force and efficiency of the
fillings method. The address to primes system \( SP_0 \) demonstrates increased set of units in interval \([p_n, p^2_p]\) of regulated filling. They indicate primes above \( p_n \).

Concrete objects of the study usually lie on fixed intervals regulated or semi-regulated fillings. Therefore all further proofs base on statement, validity of which is clear from definitions and whole fillings method:

Values \( msr_n(q) \) of maximum series \( MSR_n(q) \) of unregulated fillings majorize series values of other kinds of fillings algorithms.

**A.** The maximum series in prime systems \( SP_1 \) and \( SP_0 \) are

\[
msr_n(SP_1) < 2p_{n-s},
msr_n(SP_0) < 4p_{n-s-1},
\]

where \( 1 < s = O(n) \).

(9.1)

*Proof* follows from the main theorem and theorem 19, it is whence possible to receive reasonably exact estimation for parameter \( s \).

**B.** The distance between prime numbers satisfies to an estimation:

\[
p_n, p_{n-1} \in \{P\}, \quad p_n - p_{n-1} < C_n \sqrt{p_n}, \quad C_n < 4.
\]

(9.2)

*Proof.* Let some integer \( N \in \mathbb{N}, N \geq 9 \) is given. Proceeding from this value, we shall find quite uniquely index of prime \( p_m \) and corresponding interval \( I_m = I_m(N) \) under the offered simple scheme:

\[
m = 1 + \max_{p_i \leq \sqrt{N}} i, \quad I_m = [p_{m-1} + 1, p^2_m - 1],
\]

(9.3)

whence \( m \geq 2 \). We shall consider sieve of Eratosthenes and corresponding regulated \( m \)-filling, generated by grids \( S(p_0), S(p_1), \ldots, S(p_{m-1}) \). According to expression (9.3) for given dependent intervals

\[
[p_{m-1} + 1, p^2_m - 1] = I_m \supseteq I'_N = [p_{m-1} + 1, N - 1]
\]

(9.4)

all units and only they correspond to primes. According to received relations (9.3,9.4), upper bounds of intervals \( I_m \) and \( I'_N \) are connected by inequality \( p^2_m \geq N \). It compels to address to the principal statement \( MT \).

According to the main theorem greatest possible distance between units (that is between primes) in interval \( I_m \) always can not exceed maximum series \( msr_m \) of \( m \)-fillings zeroes for system \( SP_0 \). Then we shall receive from expressions (9.1) as estimates of maximum series \( MSR_n(q) \) for all primes \( p_n < p^2_m - 1 \):

\[
p_n - p_{n-1} < C_m p_m, \quad m = 1 + \max_{p_i \leq \sqrt{N}} i, \quad C_m < 4.
\]

(9.5)
Now we shall distinguish responsible moment. At reception of decisive conclusion there is no necessity to consider whole interval \( m = [p_{m-1} + 1, p_m^2 - 1] \), though just for it the inference is valid concerning maximum series as possible distances between units. But regulation of \( m \)-filling (sieve) and way of construction of interval \( I_m \) from (9.3) demonstrate, that for \( N < p_{m-1}^2 \) found parameter \( m \) decreases. Thus the constructive interval of \( N \) estimation, on which is searched of greatest distance between primes, has bounds \( p_{m-1}^2 + 1 \leq N \leq p_m^2 - 1 \).

We shall take into account received conclusion, let \( s = 0 \) in expression (9.1) little coarsened estimation, and as far as with sufficient precision

\[
p_m = \sqrt{N} + O(m), \quad N = p_n + o(N),
\]

we receive inequality (9.2).

The opportunity of decrease of coefficient \( C_n < 4 \) of inequality (9.2) is problematic. Within the framework of filling method and support on estimation of maximum series in filling period (9.1) it is impossible, as far as according to theorem 14 limit (for \( n \to \infty \)) of coefficient \( C_n \) is just 4. Reserve of estimation decrease can consist in concrete distribution of series in interval \( J_m = [2, p_m^2] \).

From inequality (9.2) also follows, that between squares of numbers \( (M + 2)^2 \) and \( M^2 \) there is minimum one prime. Clearly, this statement is valid and for squares of next primes. Thus sequence of primes isn’t limited.

C. There is such integer \( n_0 \gg 25 \), that for \( n > n_0 \) the inequality \( p_{n+1} - p_n > \sqrt{p_n} \) is valid unlimited quantity in spite of Legendre’s hypothesis.

**Proof.** According to theorem 14 for reasonably large \( m (m \gg 26) \) inequality

\[
msr_m(SP_0) \geq 4 \cdot p_{m-2s-2}^2 \quad \text{where} \quad s = O(m)
\]

is executed. Thus in period of \( m \)-filling extensive file of series \( SR_m \) meet without fail, length of which lies in bounds \( p_m < sr_m < C p_m \), where \( C = 4 - \varepsilon_m \). Already one such case of a long series, fallen in initial interval \( I_m \) (9.4, 9.6) of filling, refutes Legendre’s hypothesis.

However the reasons, which have resulted to inequality (9.2) are invalid at establishment of upper bound of maximum series \( MSR_n \) in interval \( I_m \). So long series is not obliged to meet in interval \( I_m \). At the same time series of practically such length really lies in initial segment of extended interval \( J_m = [2, N] \), but already regulated filling, instead of prime numbers.

This fact of regulated filling in \( SP_0 \) sharply limits opportunity of achievement of series such length in interval \( I_m \) with upper bound \( p_m^2 - 1 \) even for enough large \( m \). For confirmation we shall notice, that according to theorem 11 we have \( msr_m \approx 2 \cdot p_m \) for \( m \sim 25 \), but series of length \( sr_n > p_m \) place on distance about share of period from beginnings of fixed filling. And only at inclusion of such series in following intervals \( I_n \) it is possible to expect qualitative change of estimations.

We shall consider major relations, describing condition of Legendre’s hypothesis. As basic Legendre’s bound acts value \( p_m \sim \sqrt{p_n} \):

\[
\frac{p_m - 1}{p_m}, \quad \frac{p_{n+1} - p_n}{\sqrt{p_n}}, \quad \frac{p_m - p_m^2}{p_m}, \quad \frac{msr_m}{C_m p_m}.
\]

(9.7)

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The first expression (9.7) of length of the first series of current $m$-filling to bound demonstrates firm proximity to unit. The second expression gives checked values. Except the first infringements, explicable by fillings method, for all $n > 8$ this value less unit. The third expression demonstrates growth of length of filling interval and search of infringement of hypothesis. Special attention is deserved fourth value (9.7).

According to important theorem 19 expression of length of maximum series $msr_m$ to Legendre’s bound already for small $m$ retires from unit and up to $m \sim 26$ saves stable proximity to 2. It is explained by role of the first axial configuration $Kf_m^{(I)}$ at generation of maximum series $MSR_n$. But only at $m > 27$ qualitative transition to basic role of the second axial configuration $Kf_m^{(II)}$ occurs, then last expression (9.7) aspires to constant 4.

As far as checked values of the second expression (9.7) demonstrates striking proximity to unit for some parameters $n$, for guaranteed change of situation qualitative leap is necessary. Such leap determines analysis of last correlation (9.7). It consists in transition to the second axial configuration $Kf_m^{(II)}$ at generation of maximum series $MSR_n \rightarrow MSR_n(q)$.

In such case it is allowable to estimate order of those values, since which it is possible to expect infringement of Legendre’s bound. As far as $p_{25}^2 > 10000$, it is necessary find domain, near to length of period and $R$ as product of primes

$$R \sim \prod_{i=0}^{nn} p_i : \quad nn \sim \max\{j : p_j < 10000\} \sim 1085.$$ 

Checks for such intervals $R$ of regulated filling for primes, limited by value $p_{mn} \sim R$, are presented impossible for observed time. At the same time and they are only beginning, as coefficient $C_m$ in expression (9.7) begins essentially to surpass insufficient value 2 only for parameters $m > 28$. Nevertheless, fillings method immediately specifies the thesis, refutes Legendre’s hypothesis.

From here directly follows, that such series $SR_n$, superior by length $sr_n$ initial series of filling, in result is met in observed interval, that determines availability of pair of primes connected by relation $p_{N+1} - p_N > \sqrt{p_N}$ at natural condition $p_{N+1}, p_N \in I_{nn}$. Taking into account further increase of coefficient $C_m \rightarrow 4$ in expressions (9.7), the quantity of such cases is not limited.

□

D. The sequence of twins $B_s$ has not completion on the numerical axis.

E. The distance between twins satisfies to an evaluation with the other constant:

$$B_{s+1} - B_s < C_s \sqrt{B_s} \ln B_s, \quad (C_s < C' , C > 1). \quad (9.8)$$

F. The quantity of twins $\beta i \left( N \right)$ of smaller value $N$ lays in boundaries

$$\frac{C' N}{\ln^2 N} < \beta i \left( N \right) < \frac{C'' N}{\ln^2 N}, \quad (C' < C''). \quad (9.9)$$

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G. The infinite series, introduced by reverse values of twins $B_s$, is convergent sequence.

$$B_s \in \{B_s\} \subset \{P\} : \sum_{s=1}^{\infty} \frac{1}{B_s} = C_B < \infty ,$$

(9.10)

and value of constant $C_B$ accessible to numerical estimation.

Proof of connected statements $D - G$ we shall conduct in common. We shall consider $(2n + 1)$-filling in system $SW_0$ and frequency of zeroes $\gamma_{2n+1}$ for it:

$$Z_{2n+1} : \quad \gamma_{2n+1} = \frac{H_{2n+1}}{PZ_{2n+1}} = 1 - \frac{1}{2} \prod_{i=1}^{n} \left(1 - \frac{2}{p_i}\right).$$

(9.11)

Final estimation of maximum series for double prime system $SW_0$ according to the main theorem and by analogy to theorem 20 and formula (6.21) is

$$MSR_{2n+1}(q) : \quad msr_{2n+1}(q) < C (2n + q + 1) \ln^2 n ,$$

(9.12)

where value $C$ is limited and is reasonably well estimated lower and upper with help of frequency from expression (9.11).

Now fillings method permits proceed to practical estimation of the numerical characteristics of double systems. At first we shall consider kinsfolk of rank one.

Sieve for picking out of twins is regulated $(2n + 1)$-filling in system $SW_0$, received by product of two regulated $(n + 1)$-fillings (sieves of Eratosthenes) in system $SP_0$, shifted relatively one another by two positions (then grid $S(2)$ for them common). All units in interval $I_n = [p_n + 1, p_{2n+1}^2 - 1]$ and only they correspond senior from twins $B_s = p_k$.

The statement $D$ immediately follows from here and from expression (9.12) as far as $p_n \sim c n \ln n$ and even for small $n$ in interval $I_n$ is always executed

$$B_s \in I_n : \quad p_{n+1}^2 - p_{n+1} - 2 > C (2n + q + 1) \ln^2 n$$

(9.13)

for some $q \geq 0$. At the same time any $q$ even $q = 0$ means availability $B_s \in I_n$. Then infinity of twins is consequence of relation (9.13) and infinity primes. \[\square\]

The statement $F$ also follows from expressions (9.12, 9.13). Estimations of maximum series values $msr_{2n+1}(q, SW_0)$ permit to find bounds of $q = \beta i(N)$ for interval $I_n$ after equalization $p_{n+1}^2 = N$. Precision of approximation is corrected by constants $C', C''$ from correlation (9.9). \[\square\]

Statement $G$ and estimation $C_B$ from expression (9.10) follow from expression (9.9) of statement $F$ as far as twins $B_s = c_s S \ln^2 S$, where factor $c_s$ is limited.
and coefficient \( c_s \to c \) at parameter \( s \to \infty \). \( \square \)

Proof of statement \textbf{E} basically repeats \textbf{B}. For estimation of greatest distance between next twins \( B_{s+1}, B_s \) it is necessary to apply to interval \( I_n \). If put \( B_s \sim p_n^2 \), the inequality \( (9.8) \) follows from \( (9.12) \) for \( q = 0 \) as \( \ln B_s \sim 2 \ln n \). \( \square \)

\textbf{H}. The distance between next Smith’s numbers satisfies to the following estimation
\[ S_m - S_{m-1} < C \sqrt{S_m} \ln \sqrt{S_m} \]  
Smith’s number is prime \( S_m = p_k \) and number \( 0.5(p_k - 1) = p_l \) is prime too.

\begin{equation}
S_m \in \{S_m\} \subset \{P\} : \quad S_m - S_{m-1} < C \sqrt{S_m} \ln S_m . \tag{9.14}
\end{equation}

\textbf{I}. The sequence of Smith’s numbers \( S_m \) is not limited on an axis. The infinite series, introduced by reverse Smith’s numbers \( S_m \), is convergent sequence.

Proof of related statements \textbf{H} and \textbf{I} preferably to unite. The Smith’s numbers important for many appendices are offered and justified in [7].

That to pick out numbers of Smith it is necessary to generate system \( S2P \) as association of grid \( S(2) \) and grids \( S(2p_i) \) of double modules of system \( SP_0 \). In system \( S2P \) regulated \( (n+1) \)-filling repeats regulated \( n \)-filling of prime system \( SP_0 \) with doubled lengths of all zeroes series. Product of two regulated fillings: \( Z_n \in SP_0 \) and \( Z_{n+1} \in S2P \), shifted by two positions, determines \( 2n \)-filling (initial grid \( S(2) \) thus common) of system \( S3W \), for which units in interval \( I_n \) correspond to Smith’s numbers and only to them.

In result frequency of zeroes \( \gamma_{2n} \) of filling \( Z_{2n} \subset S3W \) is equal
\begin{equation}
S3W : \quad \gamma_{2n} = \frac{H_{2n}}{PZ_{2n}} = 1 - \frac{1}{4} \prod_{i=1}^{n-1} \left( 1 - \frac{2}{p_i} \right) . \tag{9.15}
\end{equation}

The comparison of expressions \( (9.11) \) and \( (9.15) \) permits to conclude, that all inferences concerning Smith’s numbers including expression \( (9.14) \), can be transferred from statements \textbf{D}–\textbf{G} and expressions \( (9.8–9.10) \) about twins distribution. \( \square \)

\textbf{J}. (Goldbach’s Conjecture). Even though one pair of prime numbers \( p_k, p_i \) (sometimes they can coincide) satisfy to the equality \( 2J = p_k + p_i \) for each integer \( J \geq 2 \), that is when \( (J \in \mathbb{N}) \).

\textbf{K}. The lower bound of Goldbach’s different representations \( G(2J) \) for numbers \( 2J \) and \( (J \in \mathbb{N}) \) does not exist at increase of argument \( J \):
\begin{equation}
\lim_{J \to \infty} \inf_{J \in [J_k, J_{k+1}]} G(2J) = \infty ; \quad \forall (J_k < J_{k+1}) . \tag{9.16}
\end{equation}

\textbf{L}. The quantity of Goldbach’s different representations \( G(2J) \) for the even value \( 2J \gg 12 \) lays in enough narrow limits for some constants:
\begin{equation}
\frac{C'}{\ln^2 J} < G(2J) < \frac{C'' J \ln \ln J}{\ln^2 J} , \quad (C' < C'') . \tag{9.17}
\end{equation}
Proof of statements J – L concerning representation of any even number $2J$ as sum of two primes $p_k + p_i$ is also better to unite.

We shall consider sieve of Eratosthenes as regulated variant of $(n + 1)$-filling with one difference: the first zeroes each from $n$ odd grids corresponding to number (and module) $p_i, 1 \leq i \leq n$ are replaced by units. Then for received structure, we shall designate which as $(n + 1)^*$-filling, all units in interval $I_n = [2, p_{n+1}^2 - 1]$, and only they, are primes. For the offered interval $I_n$ it is usual classical sieve of Eratosthenes. This object is initial for next construction.

We shall consider units of product of two regulated fillings (sieves of Eratosthenes): $(n + 1)^*$-filling of system $SP_0$ and such $(n + 1)^*$-filling of the same system, constructed on the first filling, up to from value $2J$ in opposite direction. Units of such product and only units correspond to Goldbach’s representations for even numbers (concrete for number $2J \leq p_{n+1}^2 + 1$) in interval $[2, 2J - 2]$. The quantity of various decompositions of Goldbach for number $2J$ is designated $G(2J)$. Now it is necessary find unique suitable parameter $n$ of double filling:

$$n \equiv n(2J) = \max_{i \geq 1} \left\{ i : p_i \leq \sqrt{2J - 2} \right\}, \quad p_n^2 < 2J - 1 \leq p_{n+1}^2. \quad (9.18)$$

Statement J (that is Goldbach’s Conjecture in the initial known form of uniqueness) follows from expressions (7.12) and (7.18) as far as maximum series value appreciably less $J$: $msr_{2n+1} < J$ or even $msr_{2n+1} \ll J$. It means, that in fixed interval $[2, J]$ units will be always found out, which correspond to Goldbach’s decompositions of number $2J$. It proves the given statement. □

Statement K (Goldbach’s Conjecture in strengthened formulation) follows from advancing growth of parameter $J \sim p_n^2$ in relation to value of maximum series $msr_{2n+1} \sim C_w p_n \ln n$. Thus quantity of decompositions $G(2J)$ grows. □

Statement L in essential degree repeats the statement F for twins, but divisibility $J$ by primes decreases frequency of zeroes and increases the upper bound of quantity of Goldbach’s decompositions $G(2J)$. □

M. The kinsfolk of all ranks $r, (1 \leq r < \infty)$ without exception there are on the numerical axis (that is integers N).

N. The distance between next kinsfolk of rank $r : (BR_r^r, BR_{r-1}^r$ are their senior prime numbers), satisfies to an evaluation ($C^r$ is some constant):

$$BR_r^r - BR_{r-1}^r < C^r \sqrt{BR_r^r} \ln \sqrt{BR_r^r}, \quad r \geq 1. \quad (9.19)$$

O. The sequence of kinsfolk of rank $r : \{BR_s^r\}_s$ is infinite (not completion)
for all $r \geq 1$. For example, twins.

**P.** The quantity of kinsfolk of rank $r$: $\{\beta r i(N) = s, BR_s^r \leq N\}$ lays in boundaries ($C_1^r < C_2^r$ - some constants):

$$\frac{C_1^r N}{\ln^2 N} < \beta r i(N) < \frac{C_2^r N}{\ln^2 N}, \quad r \geq 1. \quad (9.20)$$

**Q.** The series of numbers, introduced by reverse kinsfolk of rank $r$, is convergent sequence. It includes and variant of twins.

$$BR_s^r \in \{BR_s^r\} \subset \{P\} : \sum_{s=1}^{\infty} \frac{1}{BR_s^r} = C_r < \infty, \quad (9.21)$$

where value (constant) $C_r$ at concrete and not larger $r \geq 1$ quite accessible to numerical estimation. There is example of twins.

*Proof of statements $M - Q$ concerning distribution kinsfolk of rank $r \geq 1$ (definition 28) in numerical axis we shall present consistently.*

Statement $M$ follows from the analysis of generation of configurations (definition 29) of type $Kf(i, r - i), \ r \geq 3, \ 1 \leq i \leq r - 1$ in systems $SP_0$ and $SP_1$ in period of fillings. Thus the existence $BR_s^r$ - kinsfolk of rank $r$, that is configurations $Kf(r)$ for initial $r$ is known.

Already the first grid $S(2)$ results to generation of important and typical configurations $Kf(1, 2), Kf(2, 1)$. But not all configurations $Kf(i, r - i)$ are possible. For example, configurations $Kf(1, 3v + 1), v \geq 0$ can not meet. However if kinsfolk of rank $r$ have already arisen in period, according to divisibility there will be configurations $Kf(i, r - i + 1)$ though for some $i$. After this configuration $Kf(r + 1)$ will be generated by next grid.

If to take into account, that some required configurations $Kf(r)$ are generated from configurations of kind $Kf(i, r - i)$, provided that quantity of configurations was $K_n$ in period $PZ_n$, in period $PZ_{n+1}$ them will become

$$(p_{n+1} - 2)K_n \leq K_{n+1} \leq (p_{n+1} - 2 + \epsilon)K_n, \quad 0 < \epsilon \leq 1, \quad (9.22)$$

whence occurrence of values $BR_s^r$ (kinsfolk of rank $r$) in interval of primes follows. There is variant of fillings and system $SW_0$. □

Statement $N$ repeats the statement $E$ for twins. At the same time it is necessary to take into account, that not all units of double filling (the second is shifted on $2r$), correspond kinsfolk of rank $r$. However capacity of set of exceptions relatively small. □

Statement $O$ repeats $D$ and follows from expression (9.22). □

Statement $P$ reminds the statement $F$ for twins, but it should take into
account the remark in N. □

Statement Q similarly G and follows from P. □

R. If given object $BK_s$ is the typical $s$-th configuration of $m$ primes of length $M = 2 \sum_{i=1}^{m-1} r_i$, that the configurations $BK_s$ of primes lay in boundaries, $(C' < C'')$ are some constants:

$$C' s \ln^m s < BK_s < C'' s \ln^m s, \quad s \gg M. \quad (9.23)$$

Proof. Picking out of configuration of $m - 1$ ranks, that is $m$ primes, requires product of $m$ fillings $Z_{n+1}$ of system $SP_0$, consistently shifted by $2r_i$ positions. So complex system including at $m$ grids $S(p_i)$ is generated. For each $n$ under the formulas, similar (9.11) it is possible find corresponding frequencies and estimations, whence relation (9.23) follows immediately. □

S. The distance between next configurations of $m$ primes ($BK_s$, $BK_{s-1}$ are their senior prime numbers), satisfies to an estimation, $C_b$ is some constant:

$$BK_s - BK_{s-1} < C_b \sqrt{BK_s} \ln^{m-1} \sqrt{BK_s}, \quad s \gg m. \quad (9.24)$$

Proof. Distance between configurations is estimated as well as between twins (E), if to take into account transition to system with $m$ grids $S(p_i)$ and bounds (9.23). From here inequality (9.24) follows, partial case of which relations (9.2), (9.8) and (9.19) act. This result is in boundaries of fillings method. □

Purely, all indicated conclusions follow already from the first form of the main theorem, however they are specified by third, strongest form. The level of results achieved earlier with the help of sieving process and some other methods can be in works of the number theory, for example, in monograph [3].

10. Graphic illustrations

(Diagrams don’t insert in text because large volume. They can be produce always)

All enclosed pictures and diagrams have especially illustrative purpose. They do not claim for other role, and consequently it should not try on their basis to make far conclusions. At the same time and such demonstrations can put an idea into some change of a researched direction. In the similar plan exhibition patterns of the fillings method can bring doubtless advantage.
For this reason first of all the necessary explanation becomes obligatory for creation regulated and unregulated fillings with occurrence of zeroes series of a various length, including maximum. As initial filling we shall graphically present complete period of 3-filling in system $SP_1$, that is product of three grids with series of zeroes from unit up to five.

Strict cycle of such filling $Z_3 \subset SP_1$ and independence of arrangement of series $SR_3(0)$ from the order of appearance of grids in the period $PZ_3$ at product causes coincidence regulated and unregulated fillings. It permits to present the period on a circle, consistently having allocated for each series sector, proportional its length. In such case of allocated series of a length four ($sr_3 = 4$) and five (maximum, $msr_3 = 5$) will be till two specimens in the period.

Fig. 1. Ring of 3-filling series in system $SP_1$.

Grids $S(3), S(5)$ and $S(7)$ form sole filling with the united cyclic set of series on period $PZ_3$ of length 105:

$$PZ_3 \subset SP_1 : 51212313213234212124323123133212313212151$$

This period includes first axis configuration of series $Kf_3^{(I)}(1, 5, 1, 5, 1)$ and second axis configuration $Kf_3^{(II)}(4, 2, 1, 2, 1, 2, 4)$ of zeroes series. The axis (symmetric) configurations are important for evaluation of maximum series $MSR_n = MSR_n(0)$ for system $SP_1$ and other systems.

Fig. 2. The regulated 4-filling in system $SP_0$.

The grid $S(0) = L_0$ is line of units. $S(p_i)$ is grid of the module $p_i$. $V(2), V(3), V(4)$ are products of 2, 3, 4 grids accordingly. $V(4) = Z_4$ for fixed part of filling period. The multiplicity of zeroes in $Z_4$ is marked by additional dots. One dot signifies multiplicity of two zeroes. The line $Z_4^*$ is conditional enlargement of filling by multiplicity of zeroes. This line is given for explanation of imaging principle.

Fig. 3. Series of zeroes and maximum series $MSR_n$ in system $SP_0$.

The grid $S(p_i)$ is grid of the module $p_i$. The period is $PZ_5 = 2310$. The regulated $n$-fillings $Z_2 - Z_5$ are given and series $SR_5(0), sr_5(0) = 12$ (not maximum) is distinguished in line $Z_5$. The first maximum series $MSR_5^0 = MSR_5(0)$ at regulated 5-filling $Z_5$ is situated on interval under numbers $113 - 127$ and it is equal $msr_5^0 = 14$. The grid $S_w(p_i)$ has displacement $w$ of the first unit.
Other displacements lead to maximum series with one unit \( MSR^1_5 = MSR_5(1) \). The length of this maximum series is value \( msr^1_5 = 24 \).

Fig. 4. Maximum series \( MSR_{11} \) in system \( SP_1 \).

The grids \( S_w(p_i) \) of the module \( p_i \) have displacements \( w = w_i \) at product. Period of this filling is \( PZ_{11} \sim 3.71 \cdot 10^{12} \). The maximum series of zeroes \( MSR = MSR^0_{11} = MSR_{11}(0) \) has length \( msr^0_{11} = 33 \). This series has 5 zeroes of two multiplicity and such zeroes are marked by additional dots.

Fig. 5. Maximum series \( MSR_{12} \) in system \( SP_1 \).

This illustration repeats scheme of Fig. 4. The grids \( S_w(p_i) \) of the module \( p_i \) have other displacements \( w = w_i \) at product. Period of this filling is \( PZ_{12} \sim 1.52 \cdot 10^{14} \). The maximum series of zeroes \( MSR = MSR^0_{12} \) has length \( msr^0_{12} = 37 \). This series has 8 zeroes of two multiplicity and such zeroes are marked by dots.

Fig. 6. Twins, kinsfolk of rank two and numbers of Smith.

There are grids \( S(2), S(3), S(5), S(7) \) in lines 1 – 4. This grids form regulated 4-filling in system \( SP_0 \). Sieve of Eratosthenes without elements 2, 3, 5, 7, that is as zeroes, is presented by line 5.

The product of two sieves of Eratosthenes \( Z^*_4 \) with displacement \( w = 2 \) is presented by lines 6 and 7. Twins as corresponding units are marked by dark color. For example \( (5, 3), (7, 5), ... , (61, 59) \). This product is variant of double prime 7-filling of system \( SW_0 \).

The product of two sieves of Eratosthenes \( Z^*_4 \) with displacement \( w = 4 \) is presented by lines 8 and 9. Kinsfolk of rank two (primes \( p_i + 4 = p_{i+1} \)) as corresponding units are marked by dark color. For example \( (11, 7), (17, 13), ... , (47, 43) \). However primes \( (7, 3) \) are not kinsfolk of rank two. This product is variant of double prime 7-filling also.

Smith’s number is prime \( S_m = p_k \) at prime \( 0.5(p_k - 1) = p_r \). The product of sieve of Eratosthenes \( (Z^*_4, \text{ line } 10) \) and filling \( ZZ^*_4 \) (line 11) pick out numbers of Smith by corresponding units. These units are marked by dark color. The filling \( ZZ^*_4 \) has elements: \( l_{2s+1}(ZZ^*_4) = l_s(Z^*_4), l_{2s}(ZZ^*_4) = 0 \). For example, numbers of Smith are 5 (units on places 5 and 2), 7 (7 and 3), ... , 59 (59 and 29). This
product of two prime fillings is mixed filling.

Fig. 7. Twins and kinsfolk of rank two in intervals.

The coincidence of number of series in the period does not mean such coincidence twins and kinsfolk of rank two in an interval. A little that, there can arise suspicion, that abundance of objects of one kind necessarily forces out other objects, in particular if the interval is not greater. To this question and next diagram is devoted.

Boundaries of interval \( I_n \) are given by formula \( \lfloor 1.2(n+1)\ln(n+1) \rfloor^2 \) at \((n, n+1)\), where sign \( \lfloor \cdot \rfloor \) means nearest integer. Quantity \( B_n \) of twins (twin) and kinsfolk of rank two (primes \( p_i + 4 = p_{i+1} \)) as (qvad) in interval \( I_n \) are presented by two broken lines. In spite of it, these functions have strong correlation tie.

Fig. 8. Increase of total quantity of kinsfolk of three ranks.

Intervals \( I_n \) are the same for Fig. 7. Total quantity \( B_n \) of kinsfolk of rank 1 (twins, sumtw), rank 2 (sumqv) and rank 3 (primes \( p_i + 6 = p_{i+1} \)) as (sumsx) for interval \( \sum I_k \) are presented. Quantity of kinsfolk of rank 3 is essentially more than kinsfolk of rank 2 or 1. Diagrams of number of twins and kinsfolk of rank 2 (sumtw, sumqv), not superior borders \( N \), practically coincide on all interval of supervision.

Here is submitted (curve, instead of graph) advancing growth of number of kinsfolk of rank 3, reflected by function (sumsx). It is impossible not to note high smoothness of this function, monotone growing concerning function of twins quantity and kinsfolk of rank two, not superior value \( \sum R_m \).

Fig. 9. Number of kinsfolk of three ranks on axis.

At the same time the interval representation of common number of kinsfolk not absolutely precisely reflects character and features about growth of these functions. Therefore the following drawing offers three curve distributions of the same objects depending on a growing border \( N \), instead of from the number of the heaviest interval, included in area \([1, N]\).

Quantity \( B_n \) of kinsfolk of rank 1 (twins), rank 2 (primes \( p_i + 4 = p_{i+1} \)) and rank 3 (primes \( p_i + 6 = p_{i+1} \)) in interval \((1, N)\) are presented. Difference with
Fig. 8 is concluded in axis of abscissa.

Fig. 10. Kinsfolk of rank 3 in intervals.

There is difference between kinsfolk of rank 3 \((p_{k+1} - p_k = 6)\) with any such primes (optionally next). For fillings in systems \(SP_0\) and \(SW_0\) they mean, accordingly, series of a length six and units of product of two identical fillings of a unary system \(SP_0: Z_{2n}(SW_0) = Z_{n+1} & Z_{n-1}\), shifted to six points.

This diagram is analogy of Fig. 7. Quantity \(B_n\) of kinsfolk of rank 3 (primes \(p_i + 6 = p_{i+1}\)) in interval \(I_n\) is presented by broken line as (single). Function (double) is unification and averaging of function (single): \(B'_{(4n-1)/2}(\text{double}) = \frac{1}{2}\{B_{2n-1}(\text{single}) + B_{2n}(\text{single})\}\).

Fig. 11. Relation between quantity of objects of rank 3 and number of twins.

Certainly, with the same success it was possible to replace twins by kinsfolk of rank 3 two. In the diagram two functions of aspiration to various asymptotes of the relation of quantity of kinsfolk of rank three (sumsx) and number of units of product of fillings \(ZZ^*_n\) with shift six (sums6) to the same number of twins in interval \([3, N]\) are submitted at \(N < 50000\). One asymptote is \(y = 2\), a situation with second a little more difficult.

Functions sums6 and sumsx reflect relations between primes:

\[
\text{sums6}(N) = \frac{S6(N)}{TW(N)}, \quad \text{sumsx}(N) = \frac{K6(N)}{TW(N)}, \quad p_i \leq N,
\]

where function \(S6(N)\) is quantity of pairs primes: \(p_i - 6 = p_k\) and \(k = i - 1\) or \(k = i - 2\), \(p_i \leq N\); function \(K6(N)\) is quantity of pairs primes (kinsfolk of rank 3): \(p_i - 6 = p_{i-1}\); function \(TW(N)\) is quantity of pairs primes (kinsfolk of rank 1, twins): \(p_i - 2 = p_{i-1}\). Asymptotes are \(\lim_{N \to \infty} \text{sums6}(N) = 2\), \(\lim_{N \to \infty} \text{sumsx}(N) = 1.96683\). However estimation of function \(\text{sumsx}(N)\) gives value \(\sim 1.56\) for \(N \sim 50000\) or \(n \sim 50\), since \(p_{50}^2 > 50000\). Diagram confirms this estimation.

Fig. 12. Goldbach’s Conjecture and number of decompositions \(G(2J)\).

Goldbach’s Conjecture about representation of even number \(2J\) in a kind of a sum two primes \(2J = p_k + p_m\) also a subject of fixed attention of the fillings method. For a illustration how by a specific image filling in the double prime system \(SW_0\) acts we shall again consider initial regulated filling in system \(SP_0\). First 5 lines of this diagram are grids \(S(2), S(3), S(5), S(7)\) in lines 1 – 4, and they form regulated 4-filling \(Z_4\) in system \(SP_0\) (sieve of Eratosthenes without
The product of two sieves of Eratosthenes $Z_4^{**}$ is presented in lines 6 and 7. The mirror reflection of sieve from point 60 is given in line 7. Then units and only units correspond to decompositions of Goldbach for product of sieves in interval $[2, 2J - 2]$, where $2J = 60$. The red units correspond to decompositions of Goldbach and dark units correspond to commutants of these decompositions.

The lines 8, 9 and 10, 11 are analogous to lines 6, 7 with $2J = 58$ and $2J = 56$. Quantities $G(2J)$ of Goldbach’s decompositions are various: $G(60) = 6$, $G(58) = 4$, $G(56) = 3$. Only one decomposition $(58 = 29 + 29)$ at $2J = 58$ has not commutant on this diagram.

Fig. 13. Number of Goldbach’s decompositions in intervals $J_n$.

The diagram reflects behavior of three functions, connected with quantity of Goldbach’s representations for each even number $2J$ in an interval $J_n \supseteq 2J$. Heaviest, the average and least values from number of decompositions in interval set three growing curves.

Values (max$G$), (mean$G$), (min$G$) depending from parameter of the number of an interval $n$ are submitted here. According to theoretical development, smoothness of maximum value of Goldbach’s decompositions max$G$ appreciably below similar functions of average and minimum value. It is explained specific divisibility of value $2J_{max}$ by a number of primes, unique in each interval. So, for example, $60060 = 3 \cdot 4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Intervals $J_n$ must have even boundaries, therefore they several differ from intervals $I_n$ of Fig. 7: $A_n = 2 \cdot 0.5 \cdot 1.2(n+1) \ln(n+1) \{1 \ln(n+1)\}^2$, $J_n = [A_{n-1} + 2, A_n]$. There are maximum of Goldbach’s decompositions $maxG = maxG_n = \max_{2J \in J_n} G(2J)$, minimum of Goldbach’s decompositions $minG = minG_n = \min_{2J \in J_n} G(2J)$ and mean mean$G$:

$$meanG_n = \frac{1}{A_n - A_{n-1}} \sum_{2J \in J_n} G(2J), \quad minG < meanG < maxG$$

for $n > 1$. Values are for $n = 1$: $meanG_1 = maxG_1 = 2$, $minG_1 = 1$. Functions min$G_n$ and mean$G_n$ are monotonically increasing sequences for all $1 \leq n \leq 50$, and smoothness of function mean$G_n$ is essentially higher, than functions min$G_n$ and max$G_n$.

Fig. 14. Extremal and mean data of number of Goldbach’s decompositions in segments of axis $(2J \in J_n)$.

This diagram repeats scheme of Fig. 13 with the exception of abscissa axis.
Arguments $2J$ of functions $\min G(2J)$ and $\max G(2J)$ are found and belong to intervals $J_n \ni 2J$. Argument $2J(n)$ of function $\mean G_n[2J(n)]$ is $2J(n) = 0.5 \left[ A_{n-1} + A_n \right]$. Linearity of functions especially $\mean G$ clearly visible and justifies choice of intervals $J_n$.

Fig. 15. Approximation of minimum quantity of Goldbach’s decompositions in intervals $J_n$ by normalizing function.

The diagram reflects perfect approximation of all three considered functions. Broken lines $\max G$, $\mean G$ and $\min G$ on this diagram signify correlations of functions from Fig. 13 and 14 to one and the same increasing function $2J/\ln^2(2J)$. In particular importance has approximation

$$\min G = \frac{\min G_n(2J)}{2J} \ln^2(2J), \quad 2J \in J_n.$$  

The others functions have secondary significance. These broken lines have asymptotes: line $y = \sim 0.74$ for main function $\min G$, $y = \sim 1.22$ for function $\mean G$ and $y = \sim 3$ for function $\max G$. There are empirical data, are confirmed by the theory.

All given illustrations are applications of the fillings method and do not contradict to these conclusions.

11. Conclusion

After whole stated it follows to accent attention to sources, given such appreciable advantage of developed method before sieving process and some other methods having application in number theory. This superiority can not be explained in essence, if initially to assume the fillings method affiliated.

The fillings method, described in [1] and briefly in [2], variant of which is offered here, it is far before completion still. It is faster allowable to say about initial stage of research. At the same time and considered states, leaning on main theorem, permit to answer rather wide range of problems of primes distribution, is not solved by other methods, including various modifications of sieving process, exhaustion which less and less doubts causes.

Results of the fillings method already have found practical application at creation of algorithms of randomness simulating [6,7]. A little that just use of a sequence of Smith’s numbers has allowed to generate algorithms, in a limit ensuring random simulated sequences [8]. Research of theoretic-numerical congruent pseudorandom numbers generator has resulted in a little unexpected conclusion about necessity of modernization of ancient Euclid’s algorithm [9].
It is possible interesting, how there were ideas about other approach to problem. The fact is that sieving of numbers, multiple to some, should be optionally take place to multiple numbers. Only linear connection is quite necessary, but in such variant concept of grid and its shift is generated.

There was representation about zeroes series, greatest series and its determining role in forthcoming appendices. All this appears in overlapping strips with put grids, that is in product of grids. So purely practically was born concept of filling in unregulated and regulated form. Naturally, from here there was equally step before picking out classes of systems.

The fillings method has become to crystallize little late [2], when the exact formulas for degree systems, then exact estimations for systems without multiple zero and some other were received. Then and there was idea about imaging of strip region of multiple zeroes to extended period and imaging leads to variant of filling without multiple zeroes. Qualitative might of such idea was immediately confirmed by all accessible examples.

Particularly all this has found reflection in formulation of the main theorem of fillings method. Its validity and logic, clear from preamble, all have passed through numerous stage of proof, including even not leaning on main idea. Such increased attention to the central theorem is explained by that just here unique turning point of fillings method lies, which offers to accept idea, and not just naked mathematical transformation.

It should note, that at reception of main conclusions, concerning primes distribution and corresponding numerical objects, only classical theorem about primes is used. And the necessity arises only in it is possible to sharper estimation of sum reverse primes for finite and fixed value \( n \). Summation permits sharply to specify integrated estimations of corresponding constants, which in turn find reflection in all received expressions.

In either case fillings method does not use which was the obliging assumptions or doubtful postulates at one of stages. Its sharp mathematization is based on finiteness of all initial objects and clear logic of finitely observed conclusions. In this method essentially differs from other extreme formalized methods, for example inevitably leaning on such concept as ”set of all primes”.

For this reason rather the plenty of problems, enabling to hope on progress in solving, always it is not limited only to classical problems of primes distribution. The fillings method admits research of any integer objects, as elements of constructed system. At the same time arbitrary modules vector of filling grids can not be abstract from their coprime characteristics, that it is reflected in next values of zeroes frequency. From here follows, that aspects of primes distribution will be always exclusively important and for the fillings method.

Achieved results, enumeration of which is not limited by classical problems of primes distribution, specifies high efficiency of the fillings method. Thus it is impossible to underestimate significance of the main theorem as base of all research. The fillings method or method of imaging of frequent functions in period to interval
within the framework of system of generated grids permits in another way to present many problems of the number theory.

The number of considered problems has received due base support, published only in monograph [4]. On this basis insufficiency of mathematically-logic constructions was proven at creation of defended statements [10]. In particular, it concerns so of not clear (imaginary) objects as infinite or continuum set.

Reference

[1] Antipov M.V. The Fillings Method and Problems of Prime Numbers Distribution. – Novosibirsk, – Printing-house of Siberian Stateservice Academy, 2002, 503 pp.
[2] Antipov M.V. The fillings method and some problems of number theory. – Novosibirsk, 1984. – 19 pp. – (Preprint / AN USSR, Sib. Br., Comp. Cent.; N 528).
[3] Prachar K. Primzahlverteilung. Wien, Springer - Verlag, 1957, 512 pp.
[4] Antipov M.V. The Restriction Principle. – Novosibirsk, – Printing-house of Siberian Branch of Russian Academy of Sciences, 1998, 444 pp.
[5] Antipov M.V. The restriction principle and foundation of mathematics. – Novosibirsk, 1997. – 112 pp. – (Preprint / RAN, Siberian Branch, Inst. of Comp. Math. and Math. Geoph.; N 1100).
[6] Antipov M.V. Reality and Pseudorandomness. – Novosibirsk, Manuscript of monograph, 1992, 420 pp.
[7] Antipov M.V. Sequences of Numbers for the Monte Carlo Methods // Monte Carlo Methods and Appl., Vol. 2, N 3, pp. 219 – 236 (1996) VSP, Utrecht, Tokyo.
[8] Antipov M.V. Congruent Operator Simulation of Continuous Distributions // Computational Mathematics and Mathematical Physics, Vol. 42, N 11, 2002, pp. 1572 – 1580.
[9] Antipov M.V. Congruence operator of the pseudo-random numbers generator and a modification of Euclidean decomposition // Monte Carlo Methods and Appl., Vol. 1, N 3, pp. 203 – 219 (1995), Utrecht, Tokyo.
[10] Antipov M.V. Mirages of Evidence. – Novosibirsk, – OOO ”Omega Print”, 2006, 120 pp.

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