Eliashberg theory with the external pair potential

Konstantin V. Grigorishin
Boholyubov Institute for Theoretical Physics of the National Academy of Sciences of Ukraine, 14-b Metrolohichna str. Kiev-03680, Ukraine.
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Based on BCS model with the external pair potential formulated in a work K.V. Grigorishin arXiv:1605.07080, analogous model with electron-phonon coupling and Coulomb coupling is proposed. The generalized Eliashberg equations in the regime of renormalization of the order parameter are obtained. High temperature asymptotics and influence of Coulomb pseudopotential on them are investigated: as in the BCS model the order parameter asymptotically tends to zero as temperature rises, but the accounting of the Coulomb pseudopotential leads to existence of critical temperature. The effective Ginzburg-Landau theory is formulated for such model, where the temperature dependencies near \( T_c \) of the basic characteristics of a superconductor (coherence length, magnetic penetration depth, GL parameter, the thermodynamical critical field, the first and the second critical fields) recovers to the temperature dependencies as in the ordinary GL theory after the BCS model with the external pair potential.

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I. INTRODUCTION

In a work [1] a hypothetical substance has been considered, where interaction between (within) structural elements of condensed matter (molecules, nanoparticles, clusters, layers, wires etc.) depends on state of Cooper pairs: an additional work \( v \) must be made against this interaction to break a pair. Such a system can be described by BCS Hamiltonian with the external pair potential (EPP) term. In this model the potential essentially renormalizes the order parameter: if the pairing enlarges energy of the structure then suppression of superconductivity and the first order phase transition occur, if the pairing lowers energy of the structure the energy gap is slightly enlarged at zero temperature and asymptotically tends to zero with increasing temperature as \( 1/T \). In the last case the critical temperature is equal to infinity, formally, however the energy gap remains finite quantity. Hence the ratio between the gap and the critical temperature is \( \Delta/T_c = 0 \) instead of a finite value \( 3 \div 7 \) for all known materials. Possible realization of this model has been proposed in [2]. The normal state (when the order parameter is \( \Delta = 0 \)) in this model can be considered as state with a pseudogap: the quasiparticle spectrum has a gap \( \nu/2 \) (hence charge is carried by the pairs of electrons) but this state is not superconducting because the ordering \( \langle a a \rangle, \langle a^+ a^+ \rangle \) is absent. For this case the effective Ginzburg-Landau theory has been formulated, where the coherence length decreases as \( 1/\sqrt{T} \) with temperature, the GL parameter and the second critical field are increasing functions of temperature unlike the ordinary GL theory.

The above results are based on BCS theory, however real electron-electron interaction is due to the exchange of virtual phonons. The potential is described with expression

\[
u_{ph}^2 D(\varepsilon_2 - \varepsilon_1, p_2 - p_1) = u_{ph}^2 \frac{2(\varepsilon_2 - \varepsilon_1) - \omega^2}{\omega^2} (p_2 - p_1),
\]

where, \( u_{ph} \) is an electron-phonon coupling constant, \( \varepsilon_1, \varepsilon_2 \) are energetic parameters of interacting electrons and \( p_2 - p_1 \) is transmitted momentum. Since the electrons interact with small total momentum \( p_2 + p_1 = 0 \), then we can assume that the transmitted momentum is \( \sim 2p_F \), at the same time, near the Fermi surface we have \( \varepsilon_2 \sim \varepsilon_1 \sim 0 \). Therefore the expression (1) can be reduced to

\[
u_{ph}^2 D(\varepsilon_2 - \varepsilon_1 = 0, p_2 - p_1 = 2p_F) = u_{ph}^2 \frac{2}{\omega_{2pF}} \equiv \lambda.
\]
Thus the real interaction is replaced with an effective point attraction, which is nonzero in a layer of width $2\omega_{2pF} \sim 2\omega_D$ (Debye frequency) near Fermi surface [3]. In other words the BCS approximation is the neglecting of retardation $\varepsilon_2 - \varepsilon_1$ in el.-phon. interaction (the field of lattice deformation is supposed without inertia). On the contrary, the Eliashberg model [4, 5] is based on the full interaction [1]. Besides el.-phon. interaction the screened Coulomb interaction (Coulomb pseudopotential $u_c$) between electrons takes place which has width $\sim \epsilon_F$. In metals, as a rule, $\lambda < u_c$ that corresponds to repulsive electron-electron interaction, however in such systems the pairing is possible as result of the second order processes, which lead to effective attraction regardless of the sign of interaction [6]: $\lambda - u_c^*$, where $u_c^* = u_c/(1 + \nu_F u_c \ln \frac{\omega_D}{\epsilon_F})$. A stronger condition $\lambda - u_c > 0$ can occur in nonmetallic superconductors (for example, in alkali-doped fullerides $a_nC_{60}$, where competition between the Jahn-Teller coupling and Hund’s coupling takes place [7, 8]).

Our goal is, to develop model of superconductivity with EPP using the electron-electron interaction in a form [1] and accounting the Coulomb repulsion. In the section II we rewrite BCS theory with EPP, presented in the work [1] with variational method, in the terms of normal and anomalous propagators. In the section III we develop the Eliashberg theory with EPP and investigate high temperature asymptotics in the absence of the Coulomb pseudopotential and in the presence of one. In the section IV, based on effective Ginzburg-Landau theory for the BCS model with EPP developed in [1], we formulate the effective GL theory based on the Eliashberg theory.

II. BCS THEORY WITH THE EXTERNAL PAIR POTENTIAL

According to BCS theory an electron-electron attraction leads to the appearance of nonzero anomalous averages $\Delta \sim \langle a_{-p}^\dagger a_{p}^\dagger \rangle$ and $\Delta^+ \sim \langle a_{p}^\dagger a_{-p}^\dagger \rangle$, which are the order parameter (pair potential) of the superconducting state. The order parameter is determined with some self-consistency equation $\Delta = I(\Delta)$, which reflects the fact, that superconductivity is a many-particle coherent effect. In this regime the charge is carried by pairs of electrons (current carriers are the pairs with charge $2e$). To break a pair with transfer of its constituents in free quasiparticle states the energy $2|\Delta|$ is needed. We can consider the quantity $2|\Delta|$ as a work against the effective electron-electron attraction given the fact that the quantity $\Delta$ is a collective effect.

In our model we consider a hypothetical substance, where an interaction energy between (within) structural elements of condensed matter (molecules, nanoparticles, clusters, layers, wires etc.) depends on state of Cooper pairs: if the pair is broken, then energy of the molecular system is changed by quantity $u = E_a - E_b$, where $E_a$ and $E_b$ are energies of the system after- and before the breaking of the pair accordingly. Thus to break the Cooper pair we must make the work against the effective electron-electron attraction and must change the energy of the structural elements:

$$2|\Delta| \rightarrow 2|\Delta| + v > 0.$$ (3)

We will call the parameter $v$ as the external pair potential (EPP), since it is imposed on the electron subsystem by the structural elements of a substance, unlike the pair potential $\Delta$, which is result of electron-electron interaction and determined with the self-consistency equation $\Delta = I(\Delta)$. The parameter $v$ can be either $v > 0$ or $v < 0$ and in the simplest case it is not function of the energy gap $|\Delta|$. Moreover we suppose that $v$ does not depend on temperature essentially like the electron-phonon coupling constant $g$. The condition $2|\Delta| + v > 0$ ensures stability of the Cooper pairs (bound state of the electrons is energetically favorable), otherwise transformation (3) has no sense and such superconducting state cannot exist. If $v < 0$ then the breaking of a Cooper pair lowers energy of the molecular structure (or creation of the pair raises the energy). In this case the pairs become less stable. If $v > 0$ then the breaking of the pair increases the energy (or creation of the pair lowers the energy). In this case the pairs become more stable. If $v = 0$ that it is a trivial case corresponding to BCS theory.

Without going into the details of interaction of the structural elements we can write an effective Hamiltonian which takes into account the effect of the structure on the Cooper pairs as some effective external field. The order parameter $\Delta$ is a complex quantity $|\Delta|e^{i\phi}$, where $\phi$ is a phase, and it is the result of a many-particle self-consistent coherent effect. In the same time the field $v$ is an additional parameter imposed on the electron subsystem by the structural elements. As it has been shown in [1] the Hamiltonian corresponding to the transformations [3] is

$$\hat{H} = \hat{H}_{BCS} + \hat{H}_v = \sum_{k, \sigma} \varepsilon(k) a_{k, \sigma}^\dagger a_{k, \sigma} - \frac{\lambda}{V} \sum_{k, p} a_{p, \sigma}^\dagger a_{k, \sigma}^\dagger a_{-k, \sigma} a_{-p, \sigma} - \frac{v}{2} \sum_{k} \left[ \frac{\Delta}{|\Delta|} a_{k, \uparrow}^\dagger a_{-k, \downarrow} + \frac{\Delta^+}{|\Delta^+|} a_{-k, \downarrow}^\dagger a_{k, \uparrow} \right],$$ (4)

where $\hat{H}_{BCS}$ is BCS Hamiltonian: kinetic energy + pairing interaction, energy $\varepsilon(k) \approx v_F(|k| - k_F)$ is counted from Fermi surface. The combinations $a_{k, \uparrow}^\dagger a_{k, \downarrow}^\dagger$ and $a_{-k, \downarrow} a_{k, \uparrow}^\dagger$ are creation and annihilation of Cooper pairs operators, $\Delta$
and $\Delta^+$ are anomalous averages:

$$
\Delta^+ = \frac{\lambda}{V} \sum_p \langle a_{p\uparrow} a_{-p\downarrow} \rangle, \quad \Delta = \frac{\lambda}{V} \sum_p \langle a_{-p\downarrow} a_{p\uparrow} \rangle
$$

which are the complex order parameter $\Delta = |\Delta|e^{i\phi}$. Due to the multipliers $\frac{\Delta}{|\Delta|}$ and $\frac{\Delta^+}{|\Delta|}$ in $\hat{H}_\nu$, the energy does not depend on the phase $\phi$ ($a \to ae^{i\phi}/2, a^+ \to a^+e^{-i\phi}/2 \implies \Delta \to \Delta e^{i\phi}, \Delta^+ \to \Delta^+ e^{-i\phi}$). Thus both $\hat{H}_{BCS}$ and $\hat{H}_\nu$ are invariant under the $U(1)$ transformation. The term $\hat{H}_\nu$ is similar to "source term" in [10], where it means the injection of Cooper pairs into the system. On the other hand, $\hat{H}_\nu$ has a form of an external field acting on a Cooper pairs only, and $\nu$ is energy of a Cooper pair in this field. Using the fermion commutation relations and the anomalous averages (5), Hamiltonian (4) can be rewritten in a form

$$
\hat{H} \approx \sum_{k,\sigma} \varepsilon(k) a_{k,\sigma}^\dagger a_{k,\sigma} + \sum_k \left[ \Delta^+ \left(1 + \frac{\nu}{2|\Delta|}\right) a_{k\uparrow} a_{-k\downarrow} + \Delta \left(1 + \frac{\nu}{2|\Delta|}\right) a_{-k\downarrow} a_{k\uparrow}^\dagger \right] + \frac{1}{\lambda} V|\Delta|^2.
$$

Then normal $G$ and anomalous $F$ propagators have forms:

$$
G = i \frac{i\varepsilon_n + \xi}{(i\varepsilon_n)^2 - \xi^2 - |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2},
$$

$$
F = i \frac{\Delta \left(1 + \frac{\nu}{2|\Delta|}\right)}{(i\varepsilon_n)^2 - \xi^2 - |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2},
$$

where $\varepsilon_n = \pi T(2n + 1)$. Thus we can see that the following transformations of the order parameter correspond to the transformation (3):

$$
\Delta \rightarrow \Delta + \frac{\Delta}{|\Delta|} \frac{\nu}{2} = \Delta \left(1 + \frac{\nu}{2|\Delta|}\right), \quad \Delta^+ \rightarrow \Delta^+ + \frac{\Delta^+}{|\Delta|} \frac{\nu}{2} = \Delta^+ \left(1 + \frac{\nu}{2|\Delta|}\right),
$$

and the self-consistency condition for the order parameter is

$$
\Delta = \lambda \nu T \sum_{n=-\infty}^{\infty} \int_{-\omega}^{\omega} d\xi F(\varepsilon_n, \xi) \implies 1 = g \int_{-\omega}^{\omega} d\xi \frac{1 + \frac{\nu}{2|\Delta|}}{2\sqrt{\xi^2 + |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2}} \tanh \frac{\sqrt{\xi^2 + |\Delta|^2 \left(1 + \frac{\nu}{2|\Delta|}\right)^2}}{2T},
$$

Figure 1: Energy gaps $\Delta(T)$ as solution of Eq. (10) for three values of the external pair potential $\nu$. 

...
where $\omega$ is the phonon frequency. We can see that the quasiparticle spectrum has a gap $\nu/2$ ($\nu > 0$) even when $\Delta = 0$. But this state is not superconducting because the ordering $(aa) (a^+ a^+)$ is absent. In our opinion such state can be interpreted as state with a pseudogap due the strong fluctuations of the phase $\phi(r, t)$ of the order parameter $\Delta = |\Delta| e^{i\phi}$ so that $(e^{i\phi(r, t)}) = 0$.

Solutions of Eq. (10) are shown in Fig. If EPP is absent $\nu = 0$ we have an usual self-consistency equation for the gap $\Delta$: the gap is a function of temperature such that $\Delta(T \geq T_c) = 0$. The larger the coupling constant $g = \lambda\nu_F$, the larger $T_c$. If $\nu < 0$ then the pairing of quasiparticles results in increase of the system’s energy that suppresses superconductivity and first order phase transition takes place. If $\nu > 0$ then the pairing results in decrease of the system’s energy. In this case a solution of Eq. (10) is such that the gap $\Delta$ does not vanish at any temperature. At large temperature $T \gg T_c$ the gap is

$$|\Delta(T \to \infty)| = \frac{g\nu^2}{4T}.$$ (11)

Then, formally, the critical temperature is $T_c = \infty$. It should be noted that if $\lambda = 0$ then for any $\nu$ a superconducting state does not exist ($\Delta = 0$ always). This means the electron-electron coupling is the cause of transition to superconducting state only but not EPP.

### III. Generalization of Eliashberg Equations

Let us take into account the fact that the electron-electron interaction is the result of the exchange of virtual phonons and screened of Coulomb interaction: $V_{eff}(q, i\omega) = \nu_{ph}^2(q) iD(q, i\omega) + \tilde{V}(q)$, where $\nu_{ph}(q)$ is an electron-phonon coupling parameter, $D(q, i\varepsilon) = \frac{-i2\omega(q)}{\varepsilon_n - \varepsilon_{\omega} + i\rho_0(q)}$ is a phonon propagator. Eliashberg equations [4], unlike BCS equations, describe renormalization of quasiparticles’ mass: $m = m_0(1 + g)$ at $T \ll \omega_D$, and they describe the decrease of effectiveness of the interaction at phonon energies $\omega(q) \ll T$ (thermal phonons are perceived by electrons as static impurities [5]): as a result $T_c \propto \omega_D^{-4}$ for the el.-phon. model unlike $T_c \propto \omega_D g$ in BCS theory for $g \gg 1$. Moreover at high temperature $T \gg \omega_D$ the renormalization of electron’s mass is absent $m = m_0 (1 + O(1/T^2))$.

Like Eqs. (7,8) the normal $G$ and anomalous $F$ propagators take forms:

$$G = i \frac{i \varepsilon_n Z(p) + \xi}{(i \varepsilon_n Z(p))^2 - \xi^2 - |W(p)|^2}$$ (12)

$$F = i \frac{\tilde{W}(p)}{(i \varepsilon_n Z(p))^2 - \xi^2 - |W(p)|^2},$$ (13)

where $p \equiv [p, ip = i\pi T(2n + 1)]$ and, analogously to [9], we can write

$$\tilde{W}(p) = W(p) + \frac{W(p) \nu}{|W(p)|^2}$$ (14)

The self-energies are determined by the self-consistency conditions:

$$S(p, ip) = \int \frac{d^3q}{(2\pi)^3} T \sum_{iq} V_{eff}(q, iq) iG(p + q, ip + iq)$$ (15)

$$W(p, ip) = \int \frac{d^3q}{(2\pi)^3} T \sum_{iq} V_{eff}(q, iq) iF(p + q, ip + iq)$$ (16)

The self energy $S(p, ip)$ can be broken up into symmetric and antisymmetric parts: $S(p) = S_s(p, ip) + ipS_0(p, ip)$, where $S_s$ and $S_0$ are both even functions of frequency $ip$. Then renormalization coefficient $Z$ for a single-particle Green function are $Z(p, ip) = 1 - S_0(p, ip)$. Accounting of $S_s$ renormalizes the chemical potential only, that does not influence on the quasiparticles’ specter. Thus we have $ip - \xi - S(p) = ipZ(p) - \tilde{\xi}$, where $\tilde{\xi} = \xi + S_s(p)$. The functions $Z(p)$ and $W(p)$ are functions of $(p, ip)$. For isotropic s-wave superconductor we can suppose $|p| \approx p_F$ and the main dependence of $G(p)$ and $F(p)$ on $p$ is through the factor $\xi \approx \xi$. Then, using method of [4], Eqs. (15,16) can be written in a form of Eliashberg equations:

$$Z(2n + 1) = 1 + \frac{1}{2n + 1} \sum_{m \neq n} \tilde{A}(2m + 1)V(n - m)$$ (17)

$$Z(2n + 1)\Delta(2n + 1) = \pi T \sum_{m \neq n} \tilde{\Phi}(2m + 1)[V(n - m) - \mu_c],$$ (18)

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$$Z(2n + 1)\Delta(2n + 1) = \pi T \sum_{m \neq n} \tilde{\Phi}(2m + 1)[V(n - m) - \mu_c],$$ (18)
where \( \mu_c \) is a Coulomb pseudopotential, \( V(m) = 2 \int \omega d\omega \frac{\alpha^2 F(\omega)}{\varepsilon_n^2 + \varepsilon_0^2} \) is a phonon interaction. Note that \( V(0) = g \), where \( g \) is the dimensionless strength of the electron-phonon interaction. The gap function is \( \Delta(2n+1) = \frac{\tilde{W}(2n+1)}{Z(2n+1)} \).

\[
\tilde{\Lambda}(2n+1) = \frac{\varepsilon_n}{\sqrt{\varepsilon_n^2 + \tilde{\Delta}(2n+1)^2}},
\]

\[
\tilde{\Phi}(2n+1) = \frac{\tilde{\Delta}(2n+1)}{\sqrt{\varepsilon_n^2 + \tilde{\Delta}(2n+1)^2}},
\]

where from Eq. (14) we have

\[
\tilde{\Delta}(2n+1) = \Delta(2n+1) + \nu \frac{\Delta(2n+1)}{2 |\tilde{\Delta}(2n+1)|} \frac{1}{Z(2n+1)}
\]

In Eqs. (17,18) we have excluded the terms with \( m = n \) in the summation, because this term corresponds to elastic scattering of the quasiparticles on thermal phonons, that does not make contribution to quasiparticle’s mass \( m = m_0Z \) and to superconducting order parameter \( \Delta \). Justification of this fact is given in Appendix A.

Let us consider particular cases of Eqs. (17,18). The Einstein model provides a simple example: all of the phonons have the same frequency \( \omega_0 \) and \( \alpha^2 F(\omega) = \omega_0 g \delta(\omega - \omega_0)/2 \). Then

\[
V(m-n) = g \frac{2 \pi T_{\omega_0}}{m-n}.
\]

Let \( \nu = 0 \) and \( T \ll T_c \) so that \( |\Delta| \ll T \). Moreover we suppose \( \mu_c = 0 \) in this case. Then Eq. (18) takes a form:

\[
Z(2n+1)\Delta(2n+1) = \sum_{m \neq n} \frac{\Delta(2m+1)}{|2m+1|} V(n-m).
\]

The asymptotic limit of large \( g \) can be found in a simple way. Assuming that \( \frac{2 \pi T_{\omega_0}}{\omega_0} \) becomes very large, so that \( Z(2n+1) = 1 \), and approximating that \( \Delta(2n+1) = \Delta(1)/|2n+1| \) we can write Eq. (23) in a form:

\[
\Delta(1) = \sum_{m \neq 0} \frac{\Delta(1)}{|2m+1|^2} \frac{g}{\frac{2 \pi T_{\omega_0}}{\omega_0} m^2}.
\]

From this we have critical temperature:

\[
T_c = \frac{\omega_0}{2 \pi} \sqrt{1.16g}
\]

Now let \( \nu > 0 \), temperature is high and the gap is small: \( T \gg \omega_0, T \gg \nu \gg |\Delta| \), hence we have \( \tilde{\Delta}(2n+1) \approx \nu \frac{\Delta(2n+1)}{|\Delta(2n+1)|} \). We can suppose that all \( \Delta(2n+1) \) have the same phase (for all \( n \)). Then Eq. (18) takes a form:

\[
|\Delta(1)| = \sum_{m \neq 0} \frac{\nu/2}{|2m+1|} \left( \frac{2 \pi T_{\omega_0}}{\omega_0} \right)^2 m^2 \frac{1.54 \nu g \omega_0^2}{8 \pi^2 T^2}.
\]

Thus the energy gap \( \Delta \) does not vanish at any temperature, however, unlike result of BCS theory \( (11) \), the gap tends to zero faster (as \( 1/T^2 \)) that is consequence of the ineffectiveness of the el.-phon. interaction for low phonon energies \( \omega(q) \ll T \).

Let account the Coulomb repulsion \( \mu_c \). As in previous consideration we suppose \( \nu = 0 \) and \( g \gg 1 \). For \( g > \mu_c \) the interaction is attractive for small values for \( m \) but it becomes repulsive for large values of \( m \). For such values of \( m \) that \( V(n-m) - \mu_c < 0 \) we suppose \( \Delta(2n+1) = 0 \). Hence it is necessary to retain only the gap components \( \Delta(1) \) and \( \Delta(-1) \), then Eq. (18) has a form:

\[
Z(1)\Delta(1) = \Delta(-1) [V(1) - \mu_c],
\]

\[
Z(-1)\Delta(-1) = \Delta(1) [V(-1) - \mu_c].
\]
Figure 2: Temperature dependencies of the energy gaps: the curve (1) is a dependence when the external pair potential is absent \( \nu = 0 \) (ordinary BCS or Eliashberg theories); the curves (2) and (3) are dependencies at \( \nu > 0 \) and \( \mu_c > \mu_c \neq 0 \); the curve (4) is a dependence when \( \nu > 0 \) but \( \mu_c = 0 \), the critical temperature in this case is equal to infinity because \( \Delta \propto 1/T^2 \).

Setting the determinant equal to zero gives critical temperature:

\[
T_c = \frac{\omega_0}{2\pi} \sqrt{\frac{g}{1 + \mu_c} - 1}. \tag{30}
\]

Thus it must be \( g > 1 + \mu_c \) for such a solution. However in real materials as a rule the relation \( g < \mu_c \) occurs and pairing of electron can be possible due Tolmacev’s mechanism of reduction of the Coulomb repulsion.

Now let \( \nu > 0 \), temperature is hight and the gap is small \( T \gg \nu \gg |\Delta| \). Then we have Eq.(18) in a form:

\[
\begin{align*}
Z(1)\Delta(1) &= \frac{\nu}{2} \frac{\Delta(-1)}{|\Delta(-1)|} Z(-1) [V(1) - \mu_c] \\
Z(-1)\Delta(-1) &= \frac{\nu}{2} \frac{\Delta(1)}{|\Delta(1)|} Z(1) [V(-1) - \mu_c]
\end{align*} \tag{29}
\]

From these equation we find the gap:

\[
\Delta(1) = \Delta(-1) \equiv \Delta, \quad |\Delta| = \frac{\nu}{2} \left[ \frac{g}{1 + \left( \frac{2\pi T}{\omega_0} \right)^2} - \mu_c \right], \tag{30}
\]

from where we get the critical temperature:

\[
\Delta(T_c) = 0 \implies T_c = \frac{\omega_0}{2\pi} \sqrt{\frac{g}{\mu_c} - 1}. \tag{31}
\]

Thus for such a solution it must be \( g > \mu_c \), that is discussed in Appendix B. It can be shown that in such regime the weakening of the Coulomb repulsion by the mechanism like Tolmacev’s one is ineffective. We can see that accounting of the Coulomb pseudopotential leads to existence of critical temperature unlike the result (26) where the gap tends to zero asymptotically. The critical temperature (31) is determined by the coupling constants \( g, \mu_c \) and the frequency \( \omega_0 \) only like ordinary superconductor (28) but not by EPP \( \nu \). However there is a principal difference of Eq.(31) from Eq.(28): if we suppose \( \mu_c = 0 \) then \( T_c(\nu = 0) \sim \sqrt{g} \) but \( T_c(\nu > 0) = \infty \) and the gap (30) passes into the asymptotic (26): \( |\Delta| \propto 1/T^2 \).

The expression (30) can be expanded in the vicinity of \( T_c \):

\[
|\Delta| = \frac{\nu g \omega_0^2}{4\pi^2 T_c^3} (T_c - T). \tag{32}
\]
Thus in this model the gap at $T \to T_c$ linearly depends on the temperature difference $(T_c - T)$ unlike ordinary mean field theory without EPP where the temperature dependence of an order parameter is $(T_c - T)^{1/2}$. Temperature dependencies of the energy gap for different parameters are shown in Fig.2.

IV. EFFECTIVE GINZBURG-LANDAU THEORY

In a work [1] the effective Ginzburg-Landau theory for the BCS model with EPP has been formulated. Corresponding free energy functional has a form:

$$F_s - F_n = V \sum_{\mathbf{q}} \left[ -A|\Delta_{\mathbf{q}}| + \frac{B}{2} |\Delta_{\mathbf{q}}|^2 + C (q - 2e\mathbf{a}_q)^2 |\Delta_{\mathbf{q}}| + \frac{1}{2\mu_0 \hbar^2} \left( q^2 a_{\mathbf{q}}^2 - (q \mathbf{a}_q)^2 \right) \right],$$

where the last term is energy of the magnetic field $\frac{1}{2\mu_0} (\mathbf{curl} \mathbf{A})^2$, the coefficients are

$$A = \nu_F \left( \frac{\hbar \omega_0}{2T} \right)^2 v, \quad B = \nu_F \frac{\omega_0}{T}, \quad C = \nu_F \frac{\omega_0}{144 T^2 m^2 v}.$$  

From the free energy functional we can obtain an equilibrium value of the gap, value of the free energy in this point and the critical momentum of a Cooper pair accordingly:

$$\frac{\delta F}{\delta |\Delta|} = 0 \implies |\Delta|_{\text{min}} = \frac{A}{B} \left( 1 - \frac{C}{A} q^2 \right)$$

$$\left( F_s - F_n \right)_{\text{min}} = \frac{A^2}{2B} \left( 1 - \frac{C}{A} q^2 \right)^2 \implies q_c^2 = \frac{A}{C}$$

If $q = 0$ then we obtain Eq.[11] for the equilibrium value of the gap. Using the transformations $\Delta(\mathbf{r}) = \sum_{\mathbf{q}} \Delta_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}, \mathbf{A}(\mathbf{r}) = \sum_{\mathbf{q}} \mathbf{a}_q e^{i\mathbf{q}\cdot\mathbf{r}}$ and the inverse transformations $\Delta_{\mathbf{q}} = \frac{1}{V} \int \Delta(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}, \mathbf{a}_q = \frac{1}{V} \int \mathbf{A}(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}$ we can write the functional [33] in real space, however the functional will have a complicated and inconvenient form due to terms $|\Delta_{\mathbf{q}}|$ and $q^2 |\Delta_{\mathbf{q}}|$. The functional [33] can be replaced by an effective GL functional, which has the same symmetry, the same extremes and the same values in these extremes. The effective GL functional has a form:

$$F_s - F_n = \int \left[ -B|\Delta(\mathbf{r})|^2 + \frac{B^3}{2A^2} |\Delta(\mathbf{r})|^4 + \frac{BC}{A} \left[ (-i\mathbf{\nabla} - 2e\mathbf{A}) \Delta(\mathbf{r}) \right]^2 + \frac{(\mathbf{curl} \mathbf{A})^2}{2\mu_0} \right] d\mathbf{r},$$

Unlike BCS theory with EPP, accounting of the Coulomb pseudopotential leads to the gap in a form [30] that provides existence of critical temperature [31]. In order to account this facts we should to write the functional [33] in a form

$$F_s - F_n = V \sum_{\mathbf{q}} \left[ -A\Xi |\Delta_{\mathbf{q}}| + \frac{B}{2} |\Delta_{\mathbf{q}}|^2 + C (q - 2e\mathbf{a}_q)^2 |\Delta_{\mathbf{q}}| + \frac{1}{2\mu_0 \hbar^2} \left( q^2 a_{\mathbf{q}}^2 - (q \mathbf{a}_q)^2 \right) \right].$$

where the coefficient $\Xi$ is such to obtain the gap [32]:

$$|\Delta(T \to T_c)| = \Xi \frac{A}{B} = \Xi \frac{\omega_0 \hbar v}{4T_c} = \frac{\nu v q_0^2}{4\pi^2 T_c^3} (T_c - T) \implies \Xi = \frac{\omega_0}{\pi^2 T_c^3} (T_c - T).$$

Then the critical momentum of a pair is

$$q_c = \sqrt{\frac{\Xi A}{C}} \propto (T_c - T)^{1/2}.$$

and the gain in free energy (at $q = 0$) is

$$(F_s - F_n)_{\text{min}} = -\frac{\Xi^2 A^2}{2B} \propto v^2 (T_c - T)^2$$

(41)
Following the above method we can write the effective GL functional:

\[
F_s - F_n = \int \left[ \text{sign}(T - T_c) B |\Delta(r)|^2 + \frac{B^3}{2A^2 \Xi^2} |\Delta(r)|^4 + \frac{BC}{A \Xi} \left| (-i \nabla - 2eA) \Delta(r) \right|^2 + \frac{(\text{curl}A)^2}{2\mu_0} \right] \text{d}r, \tag{42}
\]

Basic characteristics of a superconductor (coherence length $\xi$, magnetic penetration depth $\lambda$, GL parameter $\kappa$, thermodynamical critical field $H_{cm}$, the first $H_{c1}$ and the second $H_{c2}$ critical fields) for ordinary GL theory at $T \to T_c$, effective GL theory based on BCS theory with EPP at $T \to \infty$ and effective GL theory based on Eliashberg theory with EPP at $T \to T_c$ are presented in the following table:

| $\xi = h \sqrt{\frac{c}{4\pi}} \propto$ | GL theory ($\nu = 0$) | BCS theory with $\nu > 0$ | Eliashberg theory with $\nu > 0$, $\mu_c \neq 0$ |
|------------------------------------------|-----------------|-----------------|-----------------|
| $\lambda = \frac{\Phi_0}{2\sqrt{2\pi\mu_0}H_{cm} \xi} \propto$ | $(T_c - T)^{-1/2}$ | $(T_c - T)^{-1/2}$ | $(T_c - T)^{-1/2}$ |
| $\kappa = \lambda \xi \propto$ | $\text{const}$ | $\frac{1}{\sqrt{T}}$ | $\frac{1}{\nu}$ |
| $H_{cm} = \frac{\Phi_0}{\sqrt{\mu_0 B}} \propto$ | $(T_c - T)$ | $\frac{T^2}{\nu}$ | $v(T_c - T)$ |
| $H_{c1} = \frac{\Phi_0}{\sqrt{\mu_0 \lambda^2}} \ln \kappa \propto$ | $(T_c - T)$ | $\frac{v^2}{T^2}$ | $v^2(T_c - T)$ |
| $H_{c2} = \frac{\Phi_0}{2\mu_0 \kappa^2} = \sqrt{2\kappa H_{cm}} \propto$ | $(T_c - T)$ | $T$ | $(T_c - T)$ |

We can see from the table that in the effective GL theory based on the Eliashberg approach the temperature dependencies of the basic characteristics of a superconductor is similar to the ordinary GL theory unlike the approach based on BCS theory. In particular, this model restores ordinary temperature dependence of the coherence length ($T_c - T)^{-1/2}$ after the BCS model with EPP where it decreases as $1/T$ at large $T$ that corresponds extremely small value of order of interatomic distances. However the effective free energy functional (42) has an extraordinary form due temperature dependence of the gap as $\Delta \propto (T_c - T)$ unlike the ordinary GL theory where $\Delta \propto (T_c - T)^{1/2}$. We cannot write the effective functional as $-(T_c - T)^2 a \Delta^2 + b \Delta^4$ (that gives the desired temperature dependence of the gap too) because we will obtain the free energy as $F_{min} \propto (T_c - T)^4$, that means the phase transition to superconducting state will not be second-order phase transition. Although the effective GL functional (42) has the same symmetry, extremes and values in these extremes with the initial free energy functional (38), it is not suitable for studying of fluctuations because nonequilibrium quantities corresponding to these functionals differ radically, for this problem the initial functional (38) must be used.

V. SUMMARY

Based on BCS model with EPP formulated in [1] in this work we have proposed analogous model with electron-phonon coupling and Coulomb coupling. We have obtained the generalized Eliashberg equations (17,18,19,20,21) to the case of the external pair potential. As in BCS theory the electron-electron coupling is the cause of superconductivity, but not EPP, however the potential essentially renormalizes the order parameter. Solving these equations for the case $\nu > 0$ (that is the pairing lowers the energy of the molecular structure, that supports superconductivity) we have obtained the following asymptotic solutions.

If electron-phonon interaction is present only (the Coulomb pseudopotential is absent $\mu_c = 0$) then the energy gap $\Delta$ does not vanish at any temperature, however the gap tends to zero faster (as $1/T^2$ - Eq.(26)) than the result of BCS theory (as $1/T$ - Eq.(11)) that is consequence of drop in efficiency of the el.-phon. interaction with increasing temperature. On the other hand the accounting of the Coulomb pseudopotential $\mu_c \neq 0$ leads to existence of critical temperature (31). The critical temperature is determined by the coupling constants $g > \mu_c$ and the phonon frequency $\omega_0$ only but not by the potential $\nu$. The gap at $T \to T_c$ linearly depends on temperature difference $T_c - T$ - Eq.(32), unlike the ordinary mean field theory without EPP where the temperature dependence of an order parameter is $(T_c - T)^{1/2}$.

Based on a free energy functional for BCS model with EPP obtained in [1] we have written free energy functional (38) for our model which takes into account above-mentioned critical temperature and the linear dependency of the order parameter on the temperature difference $T_c - T$. Following [1] we have obtained the effective Ginzburg-Landau functional, which has the same symmetry, the same extremes and the same values in these extremes as in the initial functional. The temperature dependencies near $T_c$ of the basic characteristics of a superconductor (coherence length, magnetic penetration depth, GL parameter, thermodynamical critical field, the first and the second critical fields) recovers to the temperature dependencies as in the ordinary GL theory after the BCS model with EPP.
Appendix A: Scattering on thermal phonons

Let us consider Eq. (17) with the symmetrical addendum \( m = n \) and with \( \Delta = 0 \):

\[
Z(2n + 1) = 1 + \frac{1}{2n + 1} \sum_{m=-\infty}^{+\infty} \frac{\varepsilon_m}{|\varepsilon_m|} V(n - m) \tag{A1}
\]

Then we have

\[
Z(1) = Z(-1) = 1 + V(0), \quad Z(3) = Z(-3) = 1 + \frac{1}{3} [V(0) + 2V(1)], \ldots, \tag{A2}
\]

where \( V(0) = 2 \int \omega d\omega \frac{\alpha^2 F(\omega)}{\omega^2} \equiv g \), \( V(l) = 2 \int \omega d\omega \frac{\alpha^2 F(\omega)}{\omega^2} + (2\pi Tl)^2 \). We can see that at \( T \to \infty \) (this means \( T \gg \omega \)) we have \( V(|l| \geq 1) = 0 \). Thus \( Z(T \to \infty) = 1 + g \). However it must be \( Z(T \to 0) = 1 + g \) but \( Z(T \to \infty) = 1 \) [5]: electron’s mass \( m = m_0 \) is renormalized due electron-phonon interaction (an electron is being followed by cloud of virtual phonons), but at high temperatures \( T \gg \omega \) the renormalization is absent, that underlies the experimental method of finding of the constants \( g \).

Let us consider the term with \( m = n \) in Eqs. (15, A1). This term corresponds to elastic interaction because the energetic parameters of electron and phonon do not change \( \varepsilon_n = \varepsilon_m \) but the momentum changes as \( p \to p - q \). Let us consider elastic scattering of an electron on impurities of concentration \( \rho \) using diagrammatics for disordered systems [3] - Fig. (3). The self-energy has a form:

\[
S(k, \varepsilon_n) = -\rho U^2 \nu F \int_{-\infty}^{+\infty} \frac{i\tilde{\varepsilon}_n + \xi}{\varepsilon_n^2 + \xi^2} = -i \frac{\varepsilon_n}{|\varepsilon_n|} \pi \rho U^2 \nu F \equiv -i\gamma \text{sign} \varepsilon_n. \tag{A3}
\]

Thus the elastic impurities do not influence upon effective mass of quasi-particles but they stipulate for a quasi-particles’ damping \( \gamma \text{sign} \varepsilon_n \) (the mean free time and the free length are determined as \( \tau = \frac{1}{2\gamma}, \quad l = v_F \tau \)).

Now let us consider the self-energy [15] with the symmetrical addendum \( m = n \) only using (A1)

\[
S = i\varepsilon_n S_0 = i\varepsilon_n (1 - Z) = -i\varepsilon_n \frac{1}{2n + 1} \frac{\varepsilon_n}{|\varepsilon_n|} V(0) = -i \frac{\varepsilon_n}{|\varepsilon_n|} \pi T g. \tag{A4}
\]

Comparing Eq. (A4) and Eq. (A3) we can see that elastic scattering of the quasiparticles on thermal phonons is equivalent to the elastic scattering on impurities, and it does not influence upon effective mass of quasi-particles but stipulates for a quasi-particles’ damping. Hence the term with \( m = n \) must be omitted in the equation for \( Z \).

Let us consider Eq. (18) with the symmetrical addendum \( m = n \) when \( T \gg v \gg |\Delta| \):

\[
Z(2n + 1)\Delta(2n + 1) = \sum_{m=-\infty}^{+\infty} \frac{\nu}{2} \frac{\Delta(2m + 1)}{|\Delta(2m + 1)|} \frac{1}{|2m + 1|} [V(n - m) - \mu_c]. \tag{A5}
\]

For \( m = n \) we have nonzero gap at \( T \to \infty \):

\[
|\Delta(2n + 1)| = \frac{\nu/2}{|2n + 1|} (g - \mu_c). \tag{A6}
\]
This gap, like the renormalization factor $Z$ at $m = n$, is result of scattering on thermal phonons. Above we have seen that this scattering stipulates a quasi-particles’ damping but it does not make contribution to the effective mass. It can be assumed, that this scattering cannot lead to coherent assemble of Cooper pairs so that for the gap $(\Delta)$ we have $\langle \Delta(r, t) \rangle = |\Delta(r)| \langle e^{i\phi(r, t)} \rangle = 0$ although it can be $|\Delta(r)| \neq 0$, i.e. the superconducting ordering is destroyed by phase fluctuations as described in a work [11]. Thus the term with $m = n$ must be omitted in the equation for $\Delta$.

Appendix B: The ratio between $g$ and $\mu_c$

The electron-electron interaction in a metal has a form:

$$V_{\text{eff}}(q, \omega) = \frac{4\pi e^2}{q^2 \varepsilon(q, \omega)} + \frac{\lambda^2(q)}{\varepsilon^2(q, \omega)} \frac{\Omega^2_q}{\omega^2 - \omega^2(q)},$$

(B1)

where $\varepsilon(q, \omega)$ is a dielectric permittivity, $\omega(q)$ is frequency of renormalized phonons $\omega^2(q) \approx \Omega^2_q / \varepsilon(q, 0)$. The first term is a screened Coulomb interaction, the second term is an interaction via phonons. The renormalized el.-phon. coupling constant $g$ is connected with the bare constant $\lambda^2(q)$ by a following expression [5]:

$$g = \nu_F \int_{0}^{2k_F} dq \frac{\lambda^2(q)}{2k_F \varepsilon^2(q, 0)} \frac{\Omega^2_q}{\omega^2(q)} = \nu_F \int_{0}^{2k_F} dq \frac{\lambda^2(q)}{2k_F \varepsilon(q, 0)},$$

(B2)

The Coulomb coupling constant $\mu_c$ is

$$\mu_c = \nu_F \int_{0}^{2k_F} dq \frac{4\pi e^2}{2k_F q^2 \varepsilon(q, 0)}.$$  

(B3)

Comparing these equations we can see that to be $g > \mu_c$ it is necessary the bare el.-phon. interaction exceeds the direct Coulomb interaction $V(q) \equiv \frac{4\pi e^2}{q^2}$:

$$\frac{\lambda^2(q)}{V(q)} > 1,$$

(B4)

moreover in the jelly model $\lim_{q\to0} \frac{\lambda^2(q)}{V(q)} = 1$. This means that $g > \mu_c$ can occur without Tolmachev’s reduction $\mu_c^* = \mu_c / \left(1 + \mu_c \ln \left(\frac{\varepsilon_F}{\omega_0}\right)\right) \ll \mu_c$. Thus superconducting phase can be in the materials with narrow conduction band $\varepsilon_F \gtrsim \omega_0$, for example in alkali-doped fullerides [7-9].

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