Lyapunov spectra in fast dynamo Ricci flows of negative sectional curvature

by

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Abstract

Previously Chicone, Latushkin and Montgomery-Smith [Comm. Math. Phys. 173,(1995)] have investigated the spectrum of the dynamo operator for an ideally conducting fluid. More recently, Tang and Boozer [Phys. Plasmas (2000)], have investigated the anisotropies in magnetic field dynamo evolution, from finite-time, Lyapunov exponents, giving rise to a Riemann metric tensor, in the Alfven twist in magnetic flux tubes (MFTs). In this paper one investigate the role of Perelman Ricci flows constraints in twisted magnetic flux tubes, where the Lyapunov eigenvalue spectra for the Ricci tensor associated with the Ricci flow equation in MFTs leads to a finite-time Lyapunov exponential stretching along the toroidal direction of the tube and a contraction along the radial direction of the tube. It is shown that in the case of MFTs, the sectional Ricci curvature of the flow, is negative as happens in geodesic flows of Anosov type. Ricci flows constraints in MFTs substitute the Thiffeault and Boozer [Chaos(2001)] have vanishing of Riemann curvature constraint on the Lyapunov exponential stretching of chaotic flows. Gauss curvature of the twisted MFT is also computed and the contraints on a negative Gauss curvature are obtained.

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I Introduction

After the Perelman’s seminal paper [1], on an explanation of the Poincare conjecture, many applications of the Ricci flow equations to several areas of physics have appeared in the literature [2, 3]. More recently Cao [3] has computed the first eigenvalues, of ”heat flow” operator $-\Delta + \frac{R}{2}$, where $R$ is the Ricci curvature scalar and $\Delta := \nabla^2$ is the Laplacian 3D operator, under Ricci flow. He found that the eigenvalues were non-decreasing as happens in general with Lyapunov exponents [4]. In this paper one investigates a similar problem in the context of the time evolution operator under Ricci flow in the backyard of the twisted MFT. To be able to determine the Lyapunov eigenvalue spectra in twisted MFTs, which is a fundamental problem which helps one to determine the stability of the flows inside flux tubes such as tokamaks or stellarators in plasma physics [5] or even in the context of solar or other stellar plasma loop [6], one computes the eigenvalue spectra of the Ricci tensor under the Ricci flow. This is an important problem for plasma theorists and experimentalists, shall be examined here in the framework of Riemannian geometry [7]. Recently, Thiffeault, Tang and Boozer, investigated Riemannian constraints on Lyapunov exponents [8], based on the relation between the Riemann metric and the finite-time Lyapunov exponential stretching, so fundamental for dynamo action. Note that, here, another sort of constraint is investigated. Instead of the vanishing of the Riemann curvature tensor, called by mathematicians, Riemann-flat space or condition, one use the Lyapunov spectra under Ricci flow. Chaotic flows inside the twisted MFTs are investigated. Anosov diffeomorpism [9], is an important mathematical tool from the theory of dynamical systems, that has often been used, in connection with the investigation of dynamo flows and maps [10] such as the Arnold’s Cat Map [10] on the torus, useful in mixing [11] problems in the physics of fluids [11]. One of the main properties of the Anosov maps is that they yield Lyapunov exponential of the chaotic exponential stretching, which are constant everywhere [12]. This paper is organized as follow: In section 2 the dynamo maps under Ricci flows in MFTs are investigated with the aid of Lyapunov spectra. In section 3, the thin tube perturbations are computed in the negative Ricci sectional curvature assumption. Section 4 addresses discussions and conclusions.
II Ricci fast magnetic dynamo flows in MFTs

Let us start this section, by defining the Ricci flow as:

**Definition 2.1:**

Let us consider a smooth manifold which Ricci tensor $\text{Ric}$ obeys the following equation:

$$\frac{\partial g}{\partial t} = 2\text{Ric} \quad (\text{II.1})$$

Here $g$ is the Riemann metric over the manifold $\mathcal{M}$ where in $g(t)$, $t \in [a, b]$. By choosing a local chart $\mathcal{U}$ on this manifold, the Ricci flow equation may be written as

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (\text{II.2})$$

with this equation in hand let us now compute the eigenvalue spectra of the Ricci tensor as

$$R_{ij}\chi^j = \lambda\chi_i \quad (\text{II.3})$$

where here $(i, j = 1, 2, 3)$. Substitution of the Ricci flow equation (II.2) into the eigenvalue equation (II.3) one obtains an eigenvalue equation for the metric itself, as

$$\frac{\partial g_{ij}}{\partial t}\chi^j = -2\lambda g_{ij}\chi^j \quad (\text{II.4})$$

and since the eigenvector $\chi^k$ is in principle arbitrary, one can reduce this equation to

$$\frac{\partial g_{ij}}{\partial t} = -2\lambda g_{ij} \quad (\text{II.5})$$

which reduces to the solution

$$g_{ij} = \exp[-2\lambda t]\delta_{ij} \quad (\text{II.6})$$

one notes that the, $\delta_{ij}$ is the Kroenecker delta diagonal unity matrix. No Einstein sum convention is being used here. By considering the Tang-Boozer relation between the metric $g_{ij}$ components and the Lyapunov exponents

$$g_{ij} = \Lambda_1 e_1 e_1 + \Lambda_2 e_2 e_2 + \Lambda_3 e_3 e_3 \quad (\text{II.7})$$
the following lemma, can be proved:

**Lemma 1:**

If \( \lambda_i \) is the eigenvalue spectra of the Ricci tensor \( \textbf{Ric} \) under Ricci flow equation, the Lyapunov spectra is given by the following relations:

\[
\lambda_i = -\gamma_i \leq 0 \tag{II.8}
\]

where \( \lambda_i \) are the finite-time Lyapunov numbers. The infinite or true Lyapunov number is given by

\[
\lambda_i^\infty = \lim_{t \to \infty} \left( \frac{ln \Lambda_i}{2t} \right) \tag{II.9}
\]

Here one has used the finite-time Lyapunov exponent given by

\[
\lambda_i = \left( \frac{ln \Lambda_i}{2t} \right) \tag{II.10}
\]

Note that one of the interesting features of the Ricci flow method is that one may find the eigenvalue Lyapunov spectra without computing the Ricci tensor, of the flux tube, for example. Actually the twisted flux tube Riemannian line element [13]

\[
dl^2 = dr^2 + r^2 d\theta^2 + K^2(r, s) ds^2 \tag{II.11}
\]

can now be used, to compute the Lyapunov exponential stretching of the flow as in Friedlander and Vishik [14] "dynamo flow" with the Ricci flow technique above. Here twist transformation angle is given by

\[
\theta(s) := \theta_R - \int \tau(s) ds \tag{II.12}
\]

One another advantage of the method used here is that this allows us to compute the finite-time Lyapunov exponential, without the need of recurring to the non-Anosov maps [15]. The Lyapunov exponents here are naturally non-Anosov since the exponents are non-homogeneous. Here \( K(r, s) := (1 - r\kappa(s, t)\cos\theta) \), is the stretching in the metric. Let us now compute the eigenvalue spectra for the MFTs. Note that the eigenvalue problem, can be solved by the 3D matrix

\[
\textbf{M}_{3D} = \begin{pmatrix}
2\lambda g_{11} & 0 & 0 \\
0 & \partial_t g_{22} + \lambda g_{22} & 0 \\
0 & 0 & \partial_t g_{33} + \lambda g_{33}
\end{pmatrix} \tag{II.13}
\]
The eigenvalue equation

\[ \text{Det}[\mathbf{M}_{3D}] = 0 \]  \hspace{1cm} (II.14)

the following eigenvalue Lyapunov spectra for thick tubes, where \( K \approx -\kappa \theta r \cos \theta(s) \) are

\[ \lambda_1 = 0, \lambda_2 = 2 \frac{v_r(r)}{r}, \lambda_3 = \frac{1}{2} \lambda_2 + \omega_1 \tan \theta(s) \]  \hspace{1cm} (II.15)

Therefore, since for the existence of dynamo action, at least two of the Lyapunov exponents have to have opposite signs [15] in order to obey the stretching (\( \lambda_3 > 0 \)) and contracting (\( \lambda_2 < 0 \)), in order that the radial flow \( v_r \) be negative, \( \lambda_3 > 0 \). Therefore from the above expressions the following constraint is obtained:

\[ |\omega_1 \tan \theta(s)| \geq |\lambda_2| = \left| \frac{v_r}{r} \right| \]  \hspace{1cm} (II.16)

Thus the \( \frac{v_r}{r} < 0 \) yields a compression on the flux tube which induces the in the tube a stretch along the toroidal direction-s, by the stretch-twist and fold dynamo generation method of Vainshtein and Zeldovich [16]. In the next section one shall compute the relation between twist or vorticity, and the when the sectional curvature of the thin tube is negative. Actually this leads us to the following time dependence of the magnetic field components, as

\[ B_\theta \approx e^{2\lambda_3 t} = e^{\left( \frac{vr}{r} \right) t} \]  \hspace{1cm} (II.17)

\[ B_\phi \approx e^{\lambda_3 t} = e^{\left( \frac{vr}{r} + \omega_1 \tan \theta \right) t} \]  \hspace{1cm} (II.18)

where \( \omega_1 \) is a constant vorticity inside the dynamo flux tube.

### III Magnetic dynamo flows of negative sectional curvature

In this section the Ricci sectional curvature [17] in the case of a radial perturbation of a thin twisted MFT. This is justified since, as one has seen in the last section, the radial flow is fundamental for the existence of non-vanishing Lyapunov exponential stretching, which in turn are fundamental for the existence of dynamo action. On the other hand, following work by D. Anosov [9], Chicone and Latushkin [18] have previously shown that geodesic
flows, which possesses negative Riemannian curvature has a fast dynamo action. Let \( X \) and \( Y \) be vectors laying in tangent manifolds \( T \mathcal{M} \) to a Riemannian manifold \( \mathcal{M} \subset \mathcal{N} \) where \( \mathcal{N} \) is an Euclidean three dimensional space. The Ricci sectional curvature is given by
\[
K(X, Y) := \frac{\langle R(X, Y)Y, X \rangle}{S(X, Y)} \quad \text{(III.19)}
\]
where \( R(X, Y)Z \) is the Riemann curvature given by
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{(III.20)}
\]
where
\[
S(X, Y) := ||X||^2||Y||^2 - \langle X, Y \rangle^2 \quad \text{(III.21)}
\]
As usual \( \nabla_X Y \) is the Riemannian covariant derivative given by
\[
\nabla_X Y = (X, \nabla)Y \quad \text{(III.22)}
\]
Expression \([X, Y]\) is the commutator, where which on the vector frame \( e_l, (l = 1, 2, 3) \) in \( \mathbb{R}^3 \), as
\[
X = X_k e_k \quad \text{(III.23)}
\]
or its dual basis
\[
X = X^k \partial_k \quad \text{(III.24)}
\]
where Einstein summation convention is used. Thus the commutator is written as
\[
[X, Y] = [X, Y]^k \partial_k \quad \text{(III.25)}
\]
Thus the Riemann curvature tensor becomes
\[
R(X, Y)Z = [R^l_{jkp}Z^j X^k Y^p] \partial_l \quad \text{(III.26)}
\]
By considering the thin tube approximation \( K \approx 1 \) where the gradient is given by
\[
\nabla = [\partial_r, r^{-1} \partial_{\theta_k}, \partial_\theta] \quad \text{(III.27)}
\]
Let the radial perturbation in the manifold of flux tube be given by
\[
X = v^i_r e_r \quad \text{(III.28)}
\]
where it was considered that the background radial flow vanishes. The vector $Y$ is given by

$$Y = v_\theta e_\theta + v_s t$$  \hspace{1cm} (III.29)

where the Frenet vector $t$ is tangent to the magnetic tube axis, and the perturbation is chosen along the radial direction, because in general flux tubes initially possesses a strongly confined magnetic field along the tube. In the way of computing the Ric, the covariant derivative is

$$\nabla_X Y = v^1_r \partial_r [v_\theta e_\theta + v_s t]$$ (III.30)

which, under the approximations

$$v^1_r \partial_r v_\theta \approx 0$$ (III.31)
$$v^1_r \partial_r v_s \approx 0$$ (III.32)

The MFT relation

$$\nabla_Y \nabla_X Y \approx 0$$ (III.33)

Since the term

$$\nabla_{[X,Y]} Y = O(v^3)$$ (III.34)

and one is assuming that the velocities involved in the plasma dynamo flow are nor small neither turbulent, this term can be dropped and the Ric can be expressed as

$$R(X,Y)Y \approx \nabla_X \nabla_Y Y$$ (III.35)

This yields

$$R(X,Y)Y \approx [v_s - \tau(s)^{-1}] [v_\theta \kappa \tau \sin \theta e_\theta - \tau v_\theta \sin \theta t + v_s \kappa n]$$ (III.36)

Therefore after some algebraic manipulation one obtains the sectional curvature as

$$K(X,Y) := \frac{\kappa(s) \cos \theta}{v^1_r v_s}$$ (III.37)

Where one has considered the approximation that due to dynamo action $v_s >> v_\theta$ which is usual, for example, in solar physics. In the above computations one has considered the flow as incompressible, $\nabla . v = 0$, or obeying the equation

$$\partial_s v_\theta := \kappa \tau r \sin \theta v_\theta$$ (III.38)
The Gauss curvature is given by

$$K_G := \frac{R_{1212}}{g}$$

(III.39)

Here $g := \det g_{ij}$ and $R_{1212}$ is the Riemann curvature of the twisted MFT surface, given by the line element

$$dl^2 = r_0^2 d\theta R^2 + K^2(s) ds^2$$

(III.40)

This metric form has been obtained from expression (II.11) by simply consider the Riemann curved twisted flux tube surface of constant cross-section of radius, $r = r_0 = constant$. The Riemann curvature $R_{1212}$ of this Riemannian line element is

$$R_{1212} = -\kappa(s)K(s)\cos\theta$$

(III.41)

Here one have considered that the stretching metric coefficient is $K(s) = (1 - r_0\kappa(s)\cos\theta)$. Thus since $g = r_0^2$, substitution of this values into the $K_G$ Gauss curvature above yields

$$K_G = -\frac{\kappa(s)K(s)\cos\theta}{r_0}$$

(III.42)

This is a general form of the Gaussian curvature, which can now be applied to the case of thin constant Frenet curvature MFTs, by making $K \approx 1$ and $\kappa = \kappa_0 = constant$, which yields

$$K_G = -\frac{\kappa_0\cos\theta}{r_0}$$

(III.43)

Note from this expression that, for it be negative, the $\cos\theta > 0$ and the Frenet curvature of the magnetic flux tube axis, positive, or they have to have the same sign at all. Just for comparison one mention here the fast kinematic dynamo eigenvalue spectrum, in diffusive media obtained by Chicone and Latushikin [18], which is

$$\lambda_\epsilon = \frac{1}{2}[-\epsilon(1 + \kappa^2) + \sqrt{\epsilon^2(1 - \kappa^2)^2 - 4\kappa^2}]$$

(III.44)

where $\epsilon$ is the resistive plasma coefficient of the diffusive fast kinematic dynamo represents the growth rate of the magnetized plasma dynamo [19], and $\lambda_\epsilon$ In the case of diffusive-free ($\epsilon = 0$) ideal plasma one obtains

$$\lambda_0 = i|\sqrt{\kappa}|$$

(III.45)

which is a pure magnetic unity. This is similar to the $\lambda_3$ one has obtained from the eigenvalue matrix above that the $\lambda_3 = 2\kappa r_0\cos\theta$ it is also proportional to the Frenet curvature.
IV Conclusions

Let us state here for discussion the Chicone-Latushkin fast dynamo in geodesic flow theorem [18] as:

**Theorem**: If $\mathbf{v}$ is the vector field that generates the geodesic flow for a closed two dimensional Riemannian manifold $\mathcal{M}$, then $\mathbf{v}$ is a steady solution of Euler's equation on $\mathcal{M}$. In addition if $\mathcal{M}$ has constant negative curvature $\kappa$, then for each magnetic Reynolds number $Re_m > \sqrt{-\kappa}$, the corresponding dynamo operator has a positive eigenvalue given by $\lambda_\epsilon$ above.

Therefore one may say that in this paper we give an example of the validity of Chicone-Latushkin theorem on geodesic dynamo flows in the case of flux tubes where a containing Ricci plasma flow. Also in this paper the importance of investigation the Ricci flows as a constraint to dynamo flows inside twisted MFTs is stressed, given examples of the dynamo action existence in negative sectional curvature. This can be basically done with the help of the Lyapunov spectra of the Ricci fast dynamo flows, which are fundamental for the exponential stretching which are in turn so important for dynamo action. An instability of the Lyapunov exponential stretching influence on the instability of the Euler equations has been discussed by Friedlander and Vishik [14]. Physical implications to the ideas discussed here to the dynamo magnetic flux tube discussed and developed by Schuessler [?] may appear elsewhere.

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