Quasilinear $p(x)$-Laplacian parabolic problem: upper bound for blow-up time

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Abstract. This paper presents a study of blow-up of solutions to a quasilinear $p(x)$-Laplacian problem related to the equation

$$z_t(x, t) = \Delta_{p(x)} z(x, t) + g(z(x, t)).$$

We use a condition on the nonlinear function $g(z)$ given by,

$$\varsigma \int_0^z g(s) ds \leq zg(z) + \eta z^p(x) + \mu, \quad z > 0.$$

We extend the existing results on blow-up for a nonlinear heat equation to variable exponent case and establish an upper bound for the blow-up time with the help of concavity method.

1. Introduction
In this paper we analyze the blow-up of solutions of the following problem

$$\begin{cases}
    z_t(x, t) = \Delta_{p(x)} z(x, t) + g(z(x, t)), & (x, t) \in \Omega \times (0, \infty) \\
    z(x, t) = 0, & (x, t) \in \partial \Omega \times [0, \infty) \\
    z(x, 0) = z_0(x) \geq 0, & x \in \Omega
\end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$ and $\Delta_{p(x)} = \nabla.(|\nabla z|^{p(x)-2} \nabla z)$ is the $p(x)$-Laplacian operator. Which is a generalisation of the $p$–Laplacian operator, $\Delta_p = \nabla.(|\nabla z|^{p-2} \nabla z)$ and is involved in the mathematical models of image processing, elastic mechanics, electro-rheological fluids and the flow in porous media, one can refer [1, 20, 19] for details. The variable exponent $p(x)$ satisfies

$$2 \leq p_- \leq p(x) \leq p_+,$$

where

$$p_- = \min p(x), \quad p_+ = \max p(x).$$

Here we assume that the following condition holds for $g(z)$,

$$\varsigma \int_0^z g(s) ds \leq zg(z) + \eta z^p(x) + \mu, \quad z > 0.$$
for some $\varsigma, \eta, \mu > 0$ with $0 < \eta \leq \frac{(\varsigma - \mu \lambda^* \rho_c)}{\rho_c}$. Where $\lambda^*$ is the first eigenvalue of $\Delta^p_z$.

Existence and nonexistence of solutions to nonlinear problems are always a matter of interest. And occurrence of blow up at finite time ensures the nonexistence of global solution for problems, hence the study of blow up is very important. In 1973, Levine [14] studied the parabolic equations,

$$Pz_t = -Az + f(z), \quad t \in [0, T), \quad f(0) = 0, \quad z(0) = z_0,$$

where $A$ and $P$ are positive linear operators defined on $D$, a dense subdomain of a real or complex Hilbert space. He imposed a condition on $f$ given by

$$2(\varsigma + 1) \int_0^1 (f(\rho x), x) d\rho \leq (x, f(x)), \quad (1.3)$$

where $\varsigma > 0$ and demonstrated the finite time blow up of solutions. Hence for arbitrary initial data, the problem does not possess global solutions. This work is a milestone in the study of blow up since it established the Concavity Method. In [15] Levine and Payne obtained nonexistence results for classes of nonlinear hyperbolic and parabolic equations using concavity method. In particular, finite time blow up of positive solutions of semilinear heat equations were discussed in [13] and the solution behaviour at blow up points was also studied by Friedman and McLeod [11].

The following parabolic problem is studied in [4],

$$\begin{cases}
    z_t = \Delta z + |z|^{\gamma-1}z, & x \in \Omega \\
    z = 0, & x \in \partial \Omega
\end{cases}$$

where $\gamma > 0$. In accordance with the value of $\gamma$ and initial data he established the time for blow up of solutions in various norms. In 1988, Meier [16] discussed about the blow up of solutions of

$$\begin{cases}
    z_t - Lz = h(x, t)f(z), & \text{in } \Omega \times (0, T] \\
    z(x, 0) = z_0(x), & \text{in } \Omega \\
    z(x, t) = 0, & \text{on } \partial \Omega \times (0, T]
\end{cases}$$

where $L$ is a second order linear operator. By assuming certain conditions on $f$ he determined sufficient conditions and bound for blow up in finite time. The blow up of solutions to the problem,

$$\begin{cases}
    z_t - \Delta z = f(z), & x \in \Omega, \quad t > 0 \\
    z = 0, & x \in \partial \Omega, \quad t > 0 \\
    z(x, 0) = g(x), & x \in \Omega
\end{cases} \quad (1.4)$$

for one dimensional case was analysed by Caffarelli and Friedman [7], assuming $f(z)$ to be $(z + \lambda)^p, \lambda \geq 0, p > 1$ or $e^z$. In 2006, Philippin and Proytcheva [18] examined the solutions of (1.4) by assuming the following condition on $f$,

$$\frac{1}{2} (4 + \varsigma) \int_0^x f(s) ds \leq zf(z), \quad (1.5)$$

and observed that the solutions failed to exist globally because of the blow up at some finite time. Ding and Hu [9] considered the problem,

$$\begin{cases}
    (g(z))_t = \nabla.(\rho(|\nabla z|^2)\nabla z) + k(t)f(z), & x \in \Omega, \quad t > 0 \\
    z = 0, & x \in \partial \Omega, \quad t > 0 \\
    z(x, 0) = z_0(x) \geq 0, & x \in \Omega
\end{cases} \quad (1.6)$$
where $g \in C^2(0, \infty)$, $\rho$ and $k$ are positive $C^1(0, \infty)$ functions and $f$ is a non-negative $C(0, \infty)$ function. They found the lower and upper bounds of blow up time under the assumption on $f(z)$ as
\[
2(1 + \beta) \int_0^z f(\tau)d\tau \leq zf(z), \quad \beta > 0.
\]
In 2002, Messaoudi\cite{17} studied the initial boundary value problem presented below,
\[
\begin{cases}
z_t = \text{div}(|\nabla z|^{p-2}\nabla z) + f(z), & x \in \Omega, t > 0 \\
z = 0, & x \in \partial\Omega, t > 0 \\
z(x, 0) = g(x), & x \in \Omega
\end{cases}
\]
He proved blow up results under the condition
\[
k \int_0^zf(\tau)d\tau \leq zf(z), \quad k > p > 2.
\]
In 2019, finiteness of the time for blow up of the same problem \cite{8} was discussed by Chung and Choi\cite{6}. In fact, the condition was refined to
\[
\alpha \int_0^z f(\tau)d\tau \leq zf(z) + \beta z^p + \gamma,
\]
which depends on the domain and eigenvalue of $\Delta_p = \text{div}(|\nabla z|^{p-2}\nabla z)$. With this motivation of the studies on blow up of solutions, we have studied the blow up of solutions of \cite{11} with new assumptions on the nonlinear term $g(z)$.

This paper contains the following details: In the section 2, we introduce several definitions and notations of variable exponent Lebesgue spaces as well as Sobolev spaces. The 3rd section contains the main content of this paper. It is devoted to the study of finite time blow up of solutions of \cite{11}. Furthermore, we find an upper bound for the time of blow up.

2. Preliminaries
To carry on the problem \cite{11}, first we recall the essential facts about generalized Lebesgue and Sobolev spaces. However, for more details, one can refer \cite{8}.

In this section, we let $p : \Omega \to [1, \infty)$ be a measurable function, where $\Omega$ is a bounded domain in $\mathbb{R}^N$.

**Definition 2.1.** \cite{8} The variable exponent Lebesgue space with exponent $p(x)$ is defined by

\[
L^{p(x)}(\Omega) := \{z : \Omega \to \mathbb{R} | \rho_{p(x)}(\lambda z) < \infty, \text{ for some } \lambda > 0\},
\]

where,

\[
\rho_{p(x)}(z) = \int_\Omega |z(x)|^{p(x)}dx.
\]

**Theorem 2.1.** \cite{8} The space $L^{p(x)}(\Omega)$ set up with the Luxembourg norm

\[
\|z\|_{p(x)} = \inf\{\lambda > 0 | \rho_{p(x)}\left(\frac{z}{\lambda}\right) \leq 1\},
\]

is a Banach space and

\[
\min\{\|z\|_{p(x)}^{p(x)}, \|z\|_{p(x)}^{p(x)}\} \leq \int_\Omega |z|^{p(x)}dx \leq \max\{\|z\|_{p(x)}^{p(x)}, \|z\|_{p(x)}^{p(x)}\}.
\]
Remark 2.1. [3] $L^{p(x)}(\Omega)$ stands for the dual space of $L^{p(x)}(\Omega)$, such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Definition 2.2. [3] The variable exponent Sobolev space is defined as

$$W^{k,p(x)}(\Omega) = \{z \in L^{p(x)}(\Omega)| D^\alpha z \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $k \geq 1$, $D^\alpha z$ is the $\alpha^{th}$ weak partial derivative with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ a multi-index, $|\alpha| = \sum_{j=1}^N \alpha_j$.

Theorem 2.2. [3] The space $W^{k,p(x)}(\Omega)$ set up with the norm $\|z\|_{k,p(x)} := \sum_{|\alpha| \leq k} \|D^\alpha z\|_{p(x)}$ is a Banach space.

We denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ by $W_0^{k,p(x)}(\Omega)$.

Definition 2.3. [3] Let $X$ be a Banach space. Then, $L^b(0,T,X)$ is defined as the set of measurable functions $z : [0,T] \rightarrow X$ such that

if $1 \leq b < \infty$,

$$\|z\|_{L^b(0,T,X)} = \left( \int_0^T \|z(t)\|_X^b dt \right)^\frac{1}{b} < \infty,$$

if $b = \infty$,

$$\|z\|_{L^\infty(0,T,X)} = \sup_{0 \leq t \leq T} \|z(t)\|_X < \infty.$$

Remark 2.2. For $1 \leq b \leq \infty$, $L^b(0,T;X)$ endued with the above norms is a Banach space.

Lemma 2.1. [10] There exists $\lambda^* > 0$ and $\phi \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases}
-\text{div}(|\nabla \phi|^{p(x)-2}\nabla \phi) = \lambda |\phi|^{p(x)-2}\phi & \text{in } \Omega \\
\phi = 0 & \text{on } \partial \Omega,
\end{cases}$$

where

$$\lambda^* = \inf_{z \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla z|^{p(x)}dx}{\int_{\Omega} |z|^{p(x)}dx},$$

(2.2)

called the first eigenvalue of $\Delta_{p(x)}$.

Variable order Lebesgue/Sobolev spaces are generalisations of the integer order Lebesgue/Sobolev spaces, so the modelling/generalising the physical/biological process like image processing, electro-rheological fluids, elastic mechanics gives more qualitative information about the behaviour of the system. The analysis of existence and blow up of solution of such variable exponent differential system with different type of source become most active area of research in recent years. Up to our knowledge the existence and blow up of solution of quasilinear parabolic equations involving variable exponent are less studied in literature. Antontsev and Shmarev [2], [3] studied the existence of weak solutions of classes of nonlinear parabolic problems and proved that weak solutions exist in generalized Sobolev spaces. In 2015, Giacomoni et. al. [12] studied the existence and uniqueness of solutions of the problem,

$$\begin{cases}
z_t - \text{div}(|\nabla z|^{p(x)-2}\nabla z) = f(x,z), & \text{in } \Omega \times (0,T) \\
z(x,t) = 0, & \text{on } \partial \Omega \times (0,T) \\
z(x,0) = z_0(x), & \text{in } \Omega
\end{cases}$$

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and observed that the solution $z \in L^\infty(0,T;W^{-1,p(x)}_0(\Omega))$. Simsen [21] examined the existence of the Dirichlet problem of $z_t - \nabla.(|\nabla z|^{p(x)-2} z) = B(z)$. Also, he studied the asymptotic behaviour of the solution. Again, in 2011 [22] he showed that the monotonicity, coercivity and hemicontinuity of $\Delta_{p(x)}$ and proved existence of weak solutions for three classes of $p(x)$-Laplacian parabolic problems, including
\[
\begin{cases}
z_t - \Delta_{p(x)} z = B(z), \\
z(x,t) = 0.
\end{cases}
\]

Simsen et. al. [23] extended their studies to the operator $-\Delta_{p(x)} (z) + |z|^{p(x)-2} z$ and established existence results for certain parabolic equations involving this operator.

3. Main Result: Upper bound for blow-up time

In this segment, our goal is to show that the solutions to the problem (1.1) blow up at finite time and to obtain an upper bound for the time of blow up.

**Theorem 3.1.** Let $z$ be a weak solution of (1.1) and the function $g$ satisfy the condition (1.2). If $z_0$ satisfies,
\[
- \int_\Omega \frac{|\nabla z_0(x)|^{p(x)}}{p(x)} dx + \int_\Omega [G(z_0(x)) - \mu] dx > 0; \quad G(z) := \int_0^z g(s) ds,
\]
then there exists a $T^* > 0$ such that the solution $z$ blows up at finite time $T^*$, that is,
\[
\lim_{t \to T^*} \int_0^t \int_\Omega z^2(x,\tau) d\tau = +\infty.
\]

**Proof.** Multiply (1.1) with $z_t$ and integrate over $\Omega$. Applying integration by parts we get,
\[
\int_\Omega |z_t|^2 dx = - \int_\Omega |\nabla z|^{p(x)-2} \nabla z. \nabla z_t + \int_\Omega g(z(x,t)) z_t dx
\]
\[= - \frac{d}{dt} \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)} dx + \frac{d}{dt} \int_\Omega [G(z(x,t)) - \mu] dx.
\]

Now, define a functional
\[
\mathcal{R}(t) = - \int_\Omega \frac{|\nabla z|^{p(x)}}{p(x)} dx + \int_\Omega [G(z(x,t)) - \mu] dx.
\]

By equation (3.1),
\[
\mathcal{R}(0) = - \int_\Omega \frac{|\nabla z_0|^{p(x)}}{p(x)} dx + \int_\Omega [G(z_0(x,t)) - \mu] dx > 0,
\]
by the equations (3.4) and (3.5) we can observe that,
\[
\mathcal{R}(t) = \mathcal{R}(0) + \int_0^t \frac{d}{dt} \mathcal{R}(\tau) d\tau = \mathcal{R}(0) + \int_0^t \int_\Omega z^2_t(x,\tau) d\tau.
\]

Now, introduce
\[
\mathcal{N}(t) = \int_0^t \int_\Omega z^2(x,\tau) d\tau + K,
\]

\[\text{(3.8)}\]
Let $0 < \eta \leq \frac{(\varepsilon - p_+)^2}{p_+}$. Thus,

$$N''(t) \geq 2\varsigma R(t). \tag{3.10}$$

Now consider,

$$N'(t)^2 \leq 4(1 + \theta) \left[ \int_0^t z(x, \tau)z_t(x, \tau) d\tau dx \right]^2 + \left( 1 + \frac{1}{\theta} \right) \left[ \int_\Omega z_0^2(x) dx \right]^2. \tag{3.11}$$

Then, Holder’s inequality is used to acquire

$$N'^2 \leq 4(1 + \theta) \left( \int_\Omega \int_0^t z^2(x, \tau) d\tau dx \right) \left( \int_\Omega \int_0^t z_t^2(x, \tau) d\tau dx \right) + \left( 1 + \frac{1}{\theta} \right) \left[ \int_\Omega z_0^2(x) dx \right]^2. \tag{3.11}$$

The value of $\theta > 0$ is arbitrary. Now, for $\vartheta = \theta = \sqrt{2} - 1 > 0$,

$$N''(t)N(t) - (1 + \vartheta)N'(t)^2 \geq 2\varsigma KR(0) - (1 + \vartheta) \left( 1 + \frac{1}{\theta} \right) \left[ \int_\Omega z_0^2(x) dx \right]^2. \tag{3.12}$$

Since we have (3.7), $K > 0$ can be chosen large enough in order to get

$$N''(t)N(t) - (1 + \vartheta)N'(t)^2 > 0. \tag{3.13}$$
For $t \geq 0$ equation (3.13) gives,

$$N'(t) \geq \left[ \frac{N'(0)}{N^{1+\vartheta}(0)} \right] N^{1+\vartheta}(t).$$

(3.14)

This implies that $N(t)$ cannot remain finite for all $t > 0$. That is, $z$ blow up at a finite time $T^\star$.

Now we are interested in finding an upper bound for the blow up time of solutions. At this point we choose $K$ as,

$$K = \frac{(1 + \vartheta)(1 + \frac{1}{\vartheta})}{2\vartheta R(0)} \left[ \int_\Omega z_0^2(x)dx \right]^2.$$

Since we have $N(0) = K$, equation (3.14) will give

$$N(t) \geq \left[ \frac{1}{K^{\vartheta}} - \frac{\vartheta}{K^{1+\vartheta}} t \right]^{-\frac{1}{\vartheta}},$$

which gives

$$K - \left( \vartheta \int_\Omega z_0^2(x)dx \right) T^\star \geq 0.$$

Hence an upper bound for finite time of blow up $T^\star$ is given by

$$0 < T^\star \leq \frac{K}{\vartheta \int_\Omega z_0^2(x)dx}.$$  

(3.15)

Conclusion

In general, nonlinear parabolic equations do not possess global solutions for arbitrary initial data. And occurrence of blow up at finite time ensures the nonexistence of global solutions. In this paper, we found a criterion for blow up for the problem (1.1), which tells us where not to look for a global solution. Also we found an upper bound for the blow up time.

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