Variational and semigroup methods for waves and diffusion in networks

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Abstract

We study diffusion and wave equations in networks. Combining semigroup and variational methods we obtain well-posedness and many nice properties of the solutions in general $L^p$-context. Following earlier articles of other authors, we discuss how the spectrum of the generator can be connected to the structure of the network. We conclude by describing asymptotic behavior of solutions to the diffusion problem.

1 Introduction

In this paper we continue the study of dynamical processes in networks using semigroup methods. While [KS05] and [MS06] studied flow and transport processes, the aim of the present paper is to combine variational and semigroup methods in order to obtain the well-posedness of initial value problems associated with diffusion and wave equations. We thus consider first and second order problems

$$\dot{u}_j(t, x) = (c_j u'_j(t, x))', \quad \ddot{u}_j(t, x) = (c_j u''_j(t, x))', \quad t \geq 0, \quad x \in (0, 1),$$

where $c_j(\cdot)$ and $u_j(t, \cdot)$ are functions on parameterized edges $e_j$ of a finite network. The node conditions in (2.1) and (3.1) below impose continuity and Kirchhoff laws in the ramification vertices. Problems of this kind have already been treated by many authors both from the mathematical and physical communities – among others, we mention the earlier articles of Lumer [Lu80], Ali Mehmeti [Al84], Roth [Ro84], von Below [Be85], Nicaise [Ni85], Exner [Ex89], Cattaneo [Ca97], Kottos and Smilansky [KS97], Kostrykin and Schrader [KS99], and Kuchment [Ku02], as well as the monographs [Ni93], [Al94], and [LLS94], and the proceedings [ABN01].

Since the pioneering work of Beurling and Deny in the 1950s, variational methods have been greatly developed. In combination with the theory of strongly continuous semigroups of operators, they provide a powerful tool to discuss properties of solutions to many parabolic and hyperbolic problems, cf. [Da89], [Ar04], and [Ou04]. While $L^2$-techniques like the lemma of Lax–Milgram have been used in most of the above mentioned papers, our paper seems to be the first applying variational methods to obtain positivity, ultracontractivity, and stability for network equations in a general $L^p$-context. This is the main aim of Sections 2 and 3. We remark that positivity of the semigroup governing the diffusion problem with much more general nodal conditions has been characterized, by algebraic methods, in [KS06].

We then proceed to study the qualitative behavior of the solutions. To that purpose we obtain in Section 4 a characteristic equation for the spectrum of the generator and describe the appropriate eigensolutions. We reprove some results from [Be85], [N85], [N87b], and [Be88b] in our setting with slight generalizations. We see that the spectrum is determined by the structure of the network and corresponds to the spectrum of the Laplacian matrix known from graph theory, see [Mo91]. We give an explicit connection between the two spectra and show the impact of this to our problem. This relates to the well-known question: “Can one hear the shape of a drum?”, first addressed by Kac in [Ka66]. Concerning differential operators on graphs, the analogous question “Can one her the shape of a
network?” has been formulated and answered in the negative by von Below in his contribution to [ABN01]. Quite surprisingly, the same question was raised at the same time in the almost homonymous paper [GS01] by Gutkin and Smilansky. They answered it in the positive, by studying Schrödinger operator on a finite, simple graph with rationally independent arc lengths and imposing some further assumptions on matching conditions at the vertices. In graph theory, however, it is well known that there are many graphs sharing the same spectrum, see [DH03]. Also in our case, the spectrum itself does not determine the network.

In Section 5 we study the asymptotic behavior of solutions to the diffusion problem. To our knowledge this topic has not yet been properly treated by other authors. We show that the solutions always converge towards an equilibrium with rate of convergence depending on the structure of the network. This is discussed for special classes of networks. In similar contexts, convergence to equilibria has already been discussed, e.g., in [BN96].

2 The wave equation on a network

We consider a finite connected network, represented by a finite graph $G$ with $m$ edges $e_1, \ldots, e_m$ and $n$ vertices $v_1, \ldots, v_n$. We assume that all the vertices have degree at least 2, i.e., that each vertex is incident to at least 2 edges. Furthermore, we assume that $G$ is simple, that is it has no multiple edges or loops. We normalize and parameterize the edges on the interval $[0,1]$. The structure of the network is given by the $n \times m$ matrices $\Phi^+ := (\phi^+_{ij})$ and $\Phi^- := (\phi^-_{ij})$ defined by

$$
\phi^+_{ij} := \begin{cases} 1, & \text{if } e_j(0) = v_i, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \phi^-_{ij} := \begin{cases} 1, & \text{if } e_j(1) = v_i, \\ 0, & \text{otherwise}. \end{cases}
$$

We refer to [KS05] for terminology. The $n \times m$ matrix $\Phi := (\phi_{ij})$ defined by $\Phi := \Phi^+ - \Phi^-$ is known in graph theory as incidence matrix of the graph $G$. Further, let $\Gamma(v_i)$ be the set of all the indices of the edges having an endpoint at $v_i$, i.e.,

$\Gamma(v_i) := \{ j \in \{1, \ldots, m \} : e_j(0) = v_i \text{ or } e_j(1) = v_i \}$.

For the sake of simplicity, we denote the value of the functions $c_j(\cdot)$ and $u_j(t, \cdot)$ at 0 or 1 by $c_j(v_i)$ and $u_j(t, v_i)$, if $e_j(0) = v_i$ or $e_j(1) = v_i$, respectively. With an abuse of notation, we also set $u_j(t, v_i) := 0$ whenever $j \notin \Gamma(v_i)$. When convenient, we shall also write the functions $u_j$ in vector form, i.e., $u = (u_1, \ldots, u_m)^\top$.

We start with the second order problem

$$
\begin{cases}
\dot{u}_j(t, x) = (c_j u_j')(t, x), & t \in \mathbb{R}, \ x \in (0,1), \ j = 1, \ldots, m, \\
u_j(t, v_i) = u_d(t, v_i), & t \in \mathbb{R}, \ j, \ell \in \Gamma(v_i), \ i = 1, \ldots, n, \\
\sum_{j=1}^m \phi_{ij} \mu_j c_j u_j(t, v_i) = 0, & t \in \mathbb{R}, \ i = 1, \ldots, n,
\end{cases}
$$

on the network. Note that $c_j(\cdot)$ and $u_j(t, \cdot)$ are functions on the edge $e_j$ of the network, so that the right-hand side of (2.1) reads in fact as

$$
(c_j u_j')(t, \cdot) = \frac{\partial}{\partial x} \left( c_j \frac{\partial}{\partial x} u_j \right)(t, \cdot), \quad t \in \mathbb{R}, \ j = 1, \ldots, m.
$$

The functions $c_1, \ldots, c_m$ are the weights of the edges, and throughout this section we assume that $0 < c_j \in H^1(0,1), \ j = 1, \ldots, m$. They represent the different speeds of propagation along each edge of the network $G$. The equation (2.1) represents the continuity
of the values attained by the system at the vertices. The coefficients \( \mu_j, j = 1, \ldots, m \), are strictly positive constants that influence the distribution of impulse happening in the ramification nodes according to the Kirchhoff-type law \((2.4)\).

We now introduce weighted incidence matrices \( \Phi^+_w := (\omega^+_ij) \) and \( \Phi^-_w := (\omega^-ij) \) with entries

\[
\omega^+_ij := \begin{cases} 
\mu_j c_j(v_i), & \text{if } e_j(0) = v_i, \\
0, & \text{otherwise}
\end{cases}
\quad \text{and} \quad
\omega^-ij := \begin{cases} 
\mu_j c_j(v_i), & \text{if } e_j(1) = v_i, \\
0, & \text{otherwise}.
\end{cases}
\]

With these notations, equation \((2.1)\) can be rewritten as

\[
\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = u(t, 0) \quad \text{and} \quad (\Phi^-)^\top d = u(t, 1), \quad t \in \mathbb{R},
\]

while the Kirchhoff law \((2.4)\) becomes

\[
\Phi^+_w u'(t, 0) = \Phi^-_w u'(t, 1), \quad t \in \mathbb{R}.
\]

We are now in the position to rewrite our system in form of a second order abstract Cauchy problem. First we consider the (complex) Hilbert space

\[
X_2 := \prod_{j=1}^m L^2(0, 1; \mu_j dx)
\]

endowed with the natural inner product

\[
(f, g)_{X_2} := \sum_{j=1}^m \int_0^1 f_j(x)\overline{g_j(x)}\mu_j dx,
\]

for \( f, g \in X_2 \). Observe that \( X_2 \) is isomorphic to \((L^2(0, 1))^m\) with equivalence of norms. Moreover, \( X_2 \) is in fact a Hilbert lattice whose positive cone consists of \( m \) copies of the positive cone of \( L^2(0, 1; \mu_j dx) \approx L^2(0, 1) \). On \( X_2 \) we define an operator

\[
A := \begin{pmatrix}
\frac{d}{dx} (c_1 \frac{d}{dx}) & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \frac{d}{dx} (c_m \frac{d}{dx})
\end{pmatrix},
\]

with domain

\[
D(A) := \{ f \in (H^2(0, 1))^m : \Phi^+_w f'(0) = \Phi^-_w f'(1) \text{ and } \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \}.
\]

With this notations, we can finally rewrite \((2.1)\) in form of a second order abstract Cauchy problem

\[
\begin{cases}
\ddot{u}(t) = Au(t), & t \in \mathbb{R}, \\
u(0) = f, \\
\dot{u}(0) = g,
\end{cases}
\]

on \( X_2 \). By means of variational techniques, we are going to show that \( A \) enjoys several nice properties. We follow the techniques of \cite{Da89} and \cite[Sec. 7.1]{ABHN01}.

Lemma 2.1. Consider the sesquilinear form

\[
a(f, g) := \sum_{j=1}^m \int_0^1 \mu_j c_j(x)f'_j(x)\overline{g'_j(x)} dx
\]

on the Hilbert space \( X_2 \) with domain

\[
D(a) = V := \{ f \in (H^1(0, 1))^m : \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \}.
\]

Then \( a \) is densely defined and has the following properties:
respect to \( \| \cdot \| \), \( \| \cdot \| \) is symmetric and also positive, since then one has \( V \) is complete for the form norm \( \| f \|_a := \sqrt{a(f,f) + \| f \|_{X_2}^2} \), and positivity: \( |a(f,g)| \leq M \| f \|_a \| g \|_a \) for some \( M > 0 \) and all \( f, g \in D(\alpha) \).

**Proof.** It is apparent that \( V \) is a linear subspace of \( X_2 \). Observe that \( (C^\infty_c(0,1))^m \subset V \). It follows that \( V \) is dense in \( X_2 \), as by definition \( L^2(0,1) \) is the closure of \( C^\infty_c(0,1) \) in the \( L^2 \)-norm. By assumption, the weights \( c_j \) are strictly positive, so that in particular \( \alpha \) is symmetric and also positive, since

\[
a(f,f) = \sum_{j=1}^m \int_0^1 \mu_j c_j(x) |f_j'(x)|^2 dx \geq 0 \quad \text{for all } f \in V.
\]

Furthermore, \( V \) becomes a Hilbert space whenever equipped with the inner product

\[
(f,g)_V := \sum_{j=1}^m \int_0^1 \left( f_j'(x)g_j'(x) + f_j(x)g_j(x) \right) \mu_j dx,
\]

since \( V \) is a closed subspace of \( (H^1(0,1))^m \). Set

\[
c := \min_{1 \leq j \leq m} \min_{x \in [0,1]} c_j(x), \quad C := \max_{1 \leq j \leq m} \max_{x \in [0,1]} c_j(x).
\]

Then one has

\[
(c \land 1) \| f \|_V^2 \leq \| f \|_a^2 \leq (C \lor 1) \| f \|_V^2, \quad f \in V,
\]

so that the form norm \( \| \cdot \|_a \) is equivalent to the norm \( \| \cdot \|_V \). Since \( V \) is complete with respect to \( \| \cdot \|_V \), the closedness of \( \alpha \) follows at once.

Finally, \( \alpha \) is continuous. To see this, take \( f, g \in V \) and observe that

\[
|\alpha(f,g)| \leq C \sum_{j=1}^m \left| \int_0^1 \mu_j f_j'(x)g_j'(x) dx \right|
\]

\[
\leq C \sum_{j=1}^m \| f_j' \|_{L^2(0,1;\mu_j, dx)} \| g_j' \|_{L^2(0,1;\mu_j, dx)}
\]

\[
\leq \frac{C}{2} \left( \sum_{j=1}^m \| f_j' \|_{L^2(0,1;\mu_j, dx)}^2 \right)^{1/2} \left( \sum_{j=1}^m \| g_j' \|_{L^2(0,1;\mu_j, dx)}^2 \right)^{1/2}
\]

\[
\leq \frac{C}{2} \| f \|_a \| g \|_a,
\]

by the Cauchy–Schwartz inequality.

**Definition 2.2.** From the form \( \alpha \) we can obtain a unique operator \( (B, D(B)) \) in the following way:

\[
D(B) := \{ f \in V : \exists g \in X_2 \text{ s.t. } \alpha(f,h) = (g,h)_{X_2} \forall h \in V \},
\]

\[
Bf := -g.
\]

We say that the operator \( (B, D(B)) \) is associated with the form \( \alpha \).

**Lemma 2.3.** The operator associated with the form \( \alpha \) is \( (A, D(A)) \) defined in (2.3)–(2.4).
Proof. Denote by \((B, D(B))\) the operator associated with \(a\). Let us first show that \(A \subset B\). Take \(f \in D(A)\). Then for all \(h \in V\)

\[
\tag{2.6} a(f, h) = \sum_{j=1}^{m} \int_{0}^{1} \mu_{j} c_{j}(x) f_{j}'(x) h_{j}(x) dx = \sum_{j=1}^{m} \left[ \mu_{j} c_{j} f_{j} h_{j} \right]_{0}^{1} - \sum_{j=1}^{m} \int_{0}^{1} \mu_{j} (c_{j} f_{j})' h_{j}(x) dx.
\]

Using the incidence matrix \(\Phi = \Phi^+ - \Phi^-\), the first term above can be written as

\[
\sum_{j=1}^{m} \left[ \mu_{j} c_{j} f_{j} h_{j} \right]_{0}^{1} = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu_{j} c_{j}(\nu_{i}) (\phi_{ij}^{+} - \phi_{ij}^{-}) f_{j}'(\nu_{i}) h_{j}(\nu_{i}).
\]

Observe now that the condition

\[
\exists d \in C^n \text{ s.t. } (\Phi^+) \cdot d = h(0) \text{ and } (\Phi^-) \cdot d = h(1)
\]

in the definition of \(V\) implies that \(h\) is continuous in the vertices, i.e., there exist \(d_{1}, \ldots, d_{n} \in C\) such that \(h_{j}(\nu_{i}) = d_{i}\) for all \(j \in \Gamma(\nu_{i}), i = 1, \ldots, n\). Summing up and using the other condition \(\Phi^+ f(0) = \Phi^- f(1)\) in \(D(A)\) we obtain that

\[
a(f, h) = \sum_{i=1}^{n} d_{i} \sum_{j=1}^{m} (\omega_{ij}^{+} - \omega_{ij}^{-}) f_{j}'(\nu_{i}) - \sum_{j=1}^{m} \int_{0}^{1} (c_{j} f_{j})' h_{j}(x) \mu_{j} dx
\]

\[
= - (Af, h)_{X_2},
\]

which makes sense because \(Af \in X_2\). The proof of the inclusion \(A \subset B\) is completed.

To check the converse inclusion \(B \subset A\) take \(f \in D(B)\). By definition, there exists \(g \in X_2\) such that

\[
\tag{2.7} \sum_{j=1}^{m} \int_{0}^{1} \mu_{j} c_{j}(x) f_{j}'(x) h_{j}(x) dx = a(f, h) = (g, h)_{X_2} = \sum_{j=1}^{m} \int_{0}^{1} g_{j}(x) h_{j}(x) \mu_{j} dx
\]

for all \(h \in V\), hence in particular for all \(h^{j} \in V\) of the form

\[
h^{j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_{j}^{j} \\ \vdots \\ 0 \end{pmatrix} \text{  \(j^{th}\) row,  \(h_{j} \in H_{0}^{1}(0,1)\).}
\]

From this follows that (2.7) in fact implies

\[
\int_{0}^{1} \mu_{j} c_{j}(x) f_{j}'(x) h_{j}(x) dx = \int_{0}^{1} g_{j}(x) h_{j}(x) \mu_{j} dx \text{ for all } j = 1, \ldots, m, \ h_{j} \in H_{0}^{1}(0,1).
\]

By definition of weak derivative this means that \(c_{j} \cdot f_{j}' \in H^{1}(0,1)\) for all \(j = 1, \ldots, m\). Since \(0 < c_{j} \in H^{1}(0,1)\), it follows that in fact \(f_{j}' \in H^{1}(0,1)\) for all \(j = 1, \ldots, m\). We conclude that \(f \in \left( H^{2}(0,1) \right)^{m}\). Moreover, integrating by parts as in (2.6) we see that if (2.7) holds for some \(h \in V\), then

\[
\sum_{i=1}^{n} d_{i} \sum_{j=1}^{m} (\omega_{ij}^{+} - \omega_{ij}^{-}) f_{j}'(\nu_{i}) = 0,
\]

where \(d_{i}\) is the joint value attained at the vertex \(\nu_{i}\) by all \(h_{j}, j \in \Gamma(\nu_{i})\). Since \(h \in V\) is arbitrary, this means that

\[
\sum_{j=1}^{m} (\omega_{ij}^{+} - \omega_{ij}^{-}) f_{j}'(\nu_{i}) = 0 \quad \text{for all } i = 1, \ldots, n,
\]
that is, $\Phi_w f'(0) = \Phi_w f'(1)$. Therefore $f \in D(A)$ and

$$-\sum_{j=1}^m \int_0^1 \mu_j (c_j f'_j(x) \overline{h_j(x)}) dx = \sum_{j=1}^m \int_0^1 g_j(x) \overline{h_j(x)} \mu_j dx$$

holds for all $h \in V$. This implies that $Af = -g$, and the proof is complete. \hfill \square

We are now able to use some well-known results on sesquilinear forms (cf. ABHN01, Da89, On01) in order to obtain nice properties of our operator $A$.

Proposition 2.4. The operator $(A, D(A))$ defined in (2.3) is self-adjoint and dissipative. Thus, it generates a cosine operator function with associated phase space $V \times X_2$.

Proof. By Lemmas 2.1 and 2.3 we are in the situation described in ABHN01, Sec. 7.1 for $H = X_2$, $V = D(a)$, $(\cdot, \cdot)_V = a(\cdot, \cdot)$, $\omega = 1$, and $A = A_H$. Thus the claim follows by ABHN01, Proposition 7.1.1, ABHN01 Example 7.1.2] and the fact that self-adjoint operators are unitarily equivalent to multiplication operators (see also the remark at p. 413 in ABHN01). \hfill \square

We can now state the main result of this section. This generalizes the well-posedness and regularity results in A84, A86, Be88, and CF03, where only the case of constant or smooth coefficients $c_1, \ldots, c_m$ was considered.

Theorem 2.5. The problem (2.1) is well-posed, i.e., for all $f \in V$ and $g \in X_2$ it admits a unique mild solution that continuously depends on the initial data.

If further $c_j \in C^\infty[0, 1]$, $j = 1, \ldots, m$, and the initial conditions $f, g \in (C^\infty_c[0, 1])^m$, then the solution is of class $(C^\infty_c[0, 1])^m$.

Proof. It is well-known (see e.g. ABHN01 Cor. 3.14.12)] that

$$u(t) := C(t, A)f + S(t, A)g, \quad t \in \mathbb{R},$$

yields the unique mild solution to (2.5) for all initial data $(f, g)$ in the phase space, where we denote by $(C(t, A))_{t \in \mathbb{R}}$ and $(S(t, A))_{t \in \mathbb{R}}$ the cosine and sine operator functions generated by $A$, respectively. The assertion about regularity of solutions follows directly from basic properties of cosine and sine operator functions. \hfill \square

3 The heat equation on a network

We now consider again the same network $G$ and, under the same assumptions and with the same notations of Section 2, we turn our attention to the first order problem

\begin{equation} 
\begin{align*}
\dot{u}_j(t, x) &= (c_j u'_j(t, x), \quad t \geq 0, \ x \in (0, 1), \ j = 1, \ldots, m, \\
\phi_{ij}(t, v_i) &= u_{\ell}(t, v_i), \quad t \geq 0, \ j, \ell \in \Gamma(v_i), \ i = 1, \ldots, n, \\
0 &= \sum_{j=1}^m \mu_j \phi_{ij}(v_i) u'_j(t, v_i), \quad t \geq 0, \ i = 1, \ldots, n, \\
\phi_{ij}(0, x) &= f_j(x), \quad x \in (0, 1), \ j = 1, \ldots, m,
\end{align*}
\end{equation}

This equation describes a diffusion process that takes place in a network and $c_1, \ldots, c_m \in C^1[0, 1]$ are (variable) diffusion coefficients or conductances. Again, we are imposing continuity and Kirchhoff-type conditions in the ramification nodes (controlled by some constants $\mu_1, \ldots, \mu_m$).

It is already known that such a problem is well-posed in an $L^2$-context, cf. Be88. Moreover, at least for the case of constant weights $c_1, \ldots, c_m$ and $\mu_1 = \ldots = \mu_m = 1$ the heat kernel has been computed in Ni87, thus yielding well-posedness in other $L^p$-spaces. We show by variational methods that the semigroup governing (3.1) is $L^\infty$-contractive, and hence we can extend the well-posedness result to an $L^p$-context by interpolation in
the general case of variable diffusion coefficients. In particular, the analyticity of the \( L^p \)-
semigroups seems to be a new result. Also observe that, by the bounded perturbation theorem, this also yields well-posedness for the Cauchy problem associated to the analogous cable equation, cf. [Ni87b].

Let

\[
X_p := \prod_{j=1}^m L^p(0, 1; \mu_j dx), \quad p \in [1, \infty].
\]

We have already seen in Proposition 2.3 that \( A \) is a self-adjoint and dissipative operator on \( X_2 \). By the spectral theorem, this shows that \( A \) generates a contractive, analytic semigroup of angle \( \pi/2 \), and in particular the first order abstract Cauchy problem

\[
\begin{cases}
\dot{u}(t) = Au(t), & t \geq 0, \\
u(0) = f,
\end{cases}
\]

is well-posed in \( X_2 \). In fact, much more can be said.

**Lemma 3.1.** The semigroup \((T_2(t))_{t \geq 0}\) on \( X_2 \), associated with \( a \), is sub-Markovian, i.e., it is real, positive, and contractive on \( X_\infty \).

**Proof.** By [On94] Prop. 2.5, Thm. 2.7, and Cor. 2.17, we need to check that the following criteria are verified for the domain \( V \) of \( a \):

- \( f \in V \Rightarrow \overline{f} \in V \) and \( a(\text{Re} f, \text{Im} f) \in \mathbb{R} \),
- \( f \in V, f \) real-valued \( \Rightarrow |f| \in V \) and \( a(|f|, |f|) \leq a(f, f) \),
- \( 0 \leq f \in V \Rightarrow 1 \wedge f \in V \) and \( a(1 \wedge f, (f - 1)^+) \geq 0 \).

It is clear that \( \overline{f} \in H^1(0, 1) \) if \( k \in H^1(0, 1) \). Further, if \( k \) is real valued, then \( |k| \in H^1(0, 1) \) and \( |k'| = \text{sign} k \cdot k' \), and if \( 0 \leq k \), then \( 1 \wedge k \in H^1(0, 1) \) with \( (1 \wedge k)' = k'1_{\{k<1\}} \) and \( ((k - 1)^+) = k'1_{\{k>1\}} \).

By definition, the subspace \( V \) contains exactly those functions on the network that are continuous in the vertices (see [2.2]). Take any \( f \in V \). By definition we have \( \overline{f_j} = (\overline{f})_j \), \( 1 \leq j \leq m \). It follows from the above arguments that \( \overline{f} \in (H^1(0, 1))^m \), and one can see that the continuity of the values attained by \( f \) in the vertices is preserved after taking the complex conjugate \( \overline{f} \). Hence, \( \overline{f} \in V \). Moreover, \( a(\text{Re} f, \text{Im} f) \) is the sum of \( m \) integrals. Recall that the weights are real-valued, positive functions. Since all the integrated functions are real-valued, it follows that \( a(\text{Re} f, \text{Im} f) \in \mathbb{R} \). Thus, the first criterion has been checked.

Moreover, if \( f \) is a real-valued function in \( V \), then \( |f_j| = |f_j|, 1 \leq j \leq m \), and one sees as above that \( |f| \in V \). In particular, \( ||f||^2 = |f'|^2 \), and there holds

\[
a(|f|, |f|) = \sum_{j=1}^m \int_0^1 \mu_j c_j(x)|f'_j(x)|^2 dx = a(f, f).
\]

This shows that the second criterion applies.

Finally, take \( 0 \leq f \in V \). Then

\[
1 \wedge f = 1 \wedge \left( \begin{array}{c} f_1 \\ \vdots \\ f_m \end{array} \right) = \left( \begin{array}{c} 1 \wedge f_1 \\ \vdots \\ 1 \wedge f_m \end{array} \right),
\]

with all the functions \( 1 \wedge f_j \in H^1(0, 1) \), hence \( 1 \wedge f \in (H^1(0, 1))^m \). Again, the continuity of \( f \) in the vertices imposes the same property to the function \( 1 \wedge f \), i.e., \( 1 \wedge f \in V \). Further, there holds

\[
a(1 \wedge f, (f - 1)^+) = \sum_{j=1}^m \int_0^1 \mu_j c_j(1 \wedge f_j)'(x)((f_j - 1)^+)'(x) dx
\]

\[
= \sum_{j=1}^m \int_0^1 \mu_j c_j f'_j(x)1_{\{f_j<1\}}(x)f'_j(x)1_{\{f_j>1\}}(x) dx = 0.
\]
We have checked also the third criterion, thus the claim follows.

**Lemma 3.2.** The semigroup \((T_2(t))_{t \geq 0}\) on \(X_2\) associated with \(a\) is ultracontractive. In particular, it satisfies the estimate

\[
\|T_2(t)f\|_{X_\infty} \leq Mt^{-\frac{k}{2}}\|f\|_{X_2} \quad \text{for all } t \in (0,1], f \in X_2,
\]

for some constant \(M\).

**Proof.** The form norm \(|\cdot|_a\) on \(V\) is equivalent to the norm \(|\cdot|_V\), cf. the proof of Lemma \[2.1\]. Thus, by \[Ou04\] Thm. 6.3 and following remark it suffices to show that there holds

\[
\|f\|_{X_2} \leq M\|f\|_V^{\frac{1}{2}}\|f\|_{X_1}^{\frac{1}{2}} \quad \text{for all } f \in V,
\]

for some constant \(M\). Recall the Nash inequality

\[
\|k\|_{L^2(0,1)} \leq M_1 \left(\|k'\|_{L^2(0,1)} + \|k'\|_{L^1(0,1)}\right)^{\frac{1}{2}} \|k\|_{L^1(0,1)}^{\frac{3}{2}} \|k\|_{L^1(0,1)}^{\frac{1}{2}},
\]

which is valid for all \(k \in H^1(0,1)\) and some constant \(M_1\), cf. \[Ma85\] Thm. 1.4.8.1.

Take finally \(f \in V\) and observe that by (3.3)

\[
\|f\|^2_{X_2} = \sum_{j=1}^m \|f_j\|^2_{L^2(0,1;\mu_j, dx)} \leq M_1^2 \sum_{j=1}^m \|f_j\|_{H^1(0,1;\mu_j, dx)} \|f_j\|_{L^2(0,1;\mu_j, dx)} \leq M_2 \left(\sum_{j=1}^m \|f_j\|_{H^1(0,1;\mu_j, dx)}\right)^{\frac{3}{2}} \left(\sum_{j=1}^m \|f_j\|_{L^2(0,1;\mu_j, dx)}\right)^{\frac{1}{2}},
\]

using the Hölder inequality. Thus, the claim follows.

The following now holds by \[Da89\] Thm. 1.4.1, Thm. 1.6.4, and Thm. 2.1.5] and \[Ou04\] Thm. 3.13.

**Corollary 3.3.** The semigroup \((T_2(t))_{t \geq 0}\) extends to a family of compact, contractive, positive one-parameter semigroups \((T_p(t))_{t \geq 0}\) on \(X_p\), \(1 \leq p \leq \infty\). Such semigroups are strongly continuous if \(p \in [1, \infty)\), and analytic of angle \(\frac{\pi}{2} - \arctan \frac{p-2}{2p-1}\) for \(p \in (1, \infty)\).

Moreover, the spectrum of \(A_p\) is independent of \(p\), where \(A_p\) denotes the generator of \((T_p(t))_{t \geq 0}\), \(1 \leq p \leq \infty\).

The estimate on the analyticity angle of \((T_p(t))_{t \geq 0}\) is not sharp, cf. \[Mu03\] for details.

**Remark 3.4.** Consider the part \(\hat{A}\) of \(A\) in \((C[0,1])^m\), whose domain is given by

\[
D(\hat{A}) = \{ f \in (C^2(0,1))^m : \Phi^+_w f'(0) = \Phi^-_w f'(1) \text{ and } \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+_w)^\top d = f(0) \text{ and } (\Phi^-_w)^\top d = f(1) \}.
\]

Define

\[
C(G) := \{ f \in (C[0,1])^m : \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+_w)^\top d = f(0) \text{ and } (\Phi^-_w)^\top d = f(1) \},
\]

which can be looked at as the space of all continuous functions on the graph \(G\). It is easy to see that \(D(\hat{A}) = C(G)\). By Corollary \[4.3\] \(\hat{A}\) has positive resolvent, and it follows by \[ABHN01\] Thm. 3.11.9] that its part in \(C(G)\) generates a positive strongly continuous semigroup.
In the next lemma we show that the generators of the semigroups in the spaces $X_p$, $1 \leq p \leq \infty$ (see Corollary 3.3) have in fact the same form as in $X_2$, with appropriate domain.

**Lemma 3.5.** For all $p \in [1, \infty]$ the generator $A_p$ of the semigroup $(T_p(t))_{t \geq 0}$ is given by the operator defined in (2.3) with domain

$$D(A_p) = \left\{ f \in \prod_{j=1}^{m} W^{2,p}(0,1; \mu_j dx) : \Phi^+_w f'(0) = \Phi^-_w f'(1) \text{ and } \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \right\}.$$ 

**Proof.** Let us prove the claim for $p > 2$. We have already remarked that $X_p \hookrightarrow X_q$ for all $1 \leq q \leq p \leq \infty$. Moreover, it follows by the ultracontractivity of $(T_2(t))_{t \geq 0}$ (see Lemma 3.2) that $X_p$ is invariant under $(T_2(t))_{t \geq 0}$ for all $p > 2$ because if $f \in X_p$ then $f \in X_2$, and by (3.2),

$$\|T_2(t)f\|_{X_p} \leq C \cdot \|T_2(t)f\|_{X_2} \leq C \cdot Mt^{-\frac{q}{p}} \|f\|_{X_2} \leq C' \cdot Mt^{-\frac{q}{p}} \|f\|_{X_p}.$$ 

Thus, by [EN00, Prop. II.2.3] the generator of $(T_p(t))_{t \geq 0}$ is the part of $A$ in $X_p$. A direct computations yields the claim.

For $1 \leq p < 2$ the claim can be proven by duality, mimicking the proof of [Mu05, Lemma 4.11].

**Theorem 3.6.** The first order problem (3.1) is well-posed on $X_p$, $p \in [1, \infty)$, as well as on $C(G)$, i.e., for all initial data $f \in X_p$ or $f \in C(G)$ the problem (3.1) admits a unique mild solution that continuously depends on the initial data.

Such a solution is bounded in the time as well as (if $p > 1$) in the space variables. If further $f \in X_p$, $1 < p < \infty$, and $c_j \in C^\infty(0,1)$, $j = 1, \ldots, m$, then the solution $u(t, \cdot)$ is of class $(C^\infty[0,1])^m$ for all $t > 0$, and in particular the problem is solved pointwise for $t > 0$.

**Proof.** The well-posedness and boundedness results follow from the fact that the operators $A_p$ generate ultracontractive analytic semigroups. If $c_j \in C^\infty(0,1)$, $j = 1, \ldots, m$, then we can show as in the proof of Theorem 2.5 that $D(A_p^\infty) \subset (C^\infty[0,1])^m$ for all $p \in [1, \infty]$. Since the semigroup $(T_p(t))_{t \geq 0}$ is analytic, $1 < p < \infty$, it maps $X_p$ into $D(A_p^\infty)$, and the claim follows.

**4 A characteristic equation**

Having proved that the Cauchy problem (3.1) is well-posed, we want to study the qualitative behavior of its solutions. To this end we investigate the spectrum of the generator $(A, D(A))$. Since by Corollary 3.3 the spectra of all $A_p$ on $X_p$, $1 \leq p \leq \infty$, coincide, it suffices to study the operator $A = A_2$ on $X_2$. Hence, we are interested in the spectrum of the operator

$$A := \begin{pmatrix} c_1 \frac{d^2}{dx^2} & 0 \\ 0 & \cdots & \cdots \\ 0 & \cdots & c_m \frac{d^2}{dx^2} \end{pmatrix}$$

with domain

$$D(A) := \left\{ f \in \prod_{j=1}^{m} H^2(0,1; \mu_j dx) : \Phi^+_w f'(0) = \Phi^-_w f'(1) \text{ and } \exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^\top d = f(0) \text{ and } (\Phi^-)^\top d = f(1) \right\}.$$ 

Recalling properties of $(A, D(A))$ already yields some information on its spectrum.

**Lemma 4.1.** The spectrum of $(A, D(A))$ already yields some information on its spectrum.
Proof. First note that \( 1 \in D(A) \) and \( A1 = 0 \), thus \( A \) is not invertible and \( 0 \in \sigma(A) \). Since \( A \) generates a contractive semigroup (cf. Proposition \[2,4\], \( s(A) = 0 \)). It follows from \[EN00\] II.4.30.(4)] that, since \( D(A) \) is contained in \((H^2(0,1);\mu_j dx)^\ell\), the resolvent of \( A \) is compact. Therefore the operator \( A \) only has point spectrum. Recall that by Proposition \[2,4\] the operator \( A \) is self-adjoint, hence all its eigenvalues are real.

From now on we will assume that all the weights \( c_i, i = 1, \ldots, m \), are constant. Our aim is to find a ‘characteristic equation’ for the spectrum of \( A \). In particular, we will be able to connect the eigenvalues of the operator \( A \) to the eigenvalues of the Laplacian or admittance matrix of the corresponding graph. This is the \( n \times n \) matrix

\[
L := D - A,
\]

where \( A \) is the standard \( 0 - 1 \) adjacency matrix of the graph and \( D \) the diagonal matrix of vertex degrees. It is well known that its spectrum reveals many properties of the graph, hence it is used in many applications (see e.g. \[Ch97, Me94, Mo97\]).

We further define the generalized weighted adjacency matrix of the graph \( G \) in the case \( 0 < \lambda \neq c_j l^2 \pi^2, j = 1, \ldots, m, l \in \mathbb{Z} \), as

\[
(A_C(\lambda))_{ik} := \begin{cases} 0, & \text{if } \Gamma(v_i) \cap \Gamma(v_k) = \emptyset, \\ \frac{\mu_j}{\sqrt{c_j}} \sin^{-1} \sqrt{\lambda/c_j} & \text{if } j \in \Gamma(v_i) \cap \Gamma(v_k). \end{cases}
\]

By \( D_C(\lambda) \) we denote the \( n \times n \) diagonal matrix (again for \( 0 < \lambda \neq c_j l^2 \pi^2, j = 1, \ldots, m, l \in \mathbb{Z} \)) defined as

\[
D_C(\lambda) := \text{diag} \left( \sum_{j \in \Gamma(v_i)} \frac{\mu_j}{\sqrt{c_j}} \cot \sqrt{\lambda/c_j} \right)_{i=1,\ldots,n}.
\]

Finally, we define the generalized weighted Laplacian matrix as

\[
L_C(\lambda) := D_C(\lambda) - A_C(\lambda).
\]

We will now express the above matrices using the weighted incidence matrices. For this purpose we define diagonal matrices

\[
\begin{align*}
\text{Sin} x & := \text{diag} \left( \sin \frac{x}{\sqrt{c_1}}, \ldots, \sin \frac{x}{\sqrt{c_m}} \right), \\
\text{Cos} x & := \text{diag} \left( \cos \frac{x}{\sqrt{c_1}}, \ldots, \cos \frac{x}{\sqrt{c_m}} \right), \\
\text{Cot} x & := \text{Sin}^{-1} x \cdot \text{Cos} x, \quad \text{and} \\
C & := \text{diag}(1/\sqrt{c_1}, \ldots, 1/\sqrt{c_m}).
\end{align*}
\]

Lemma 4.2. For \( 0 < \lambda \neq c_r l^2 \pi^2, r = 1, \ldots, m, l \in \mathbb{Z} \) we have

\[
\begin{align*}
A_C(\lambda) & = \Phi^+_w \cdot C \cdot \text{Sin}^{-1} \sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi^-_w \cdot C \cdot \text{Sin}^{-1} \sqrt{\lambda} \cdot (\Phi^+)^\top \quad \text{and} \\
D_C(\lambda) & = \Phi^+_w \cdot C \cdot \text{Cot} \sqrt{\lambda} \cdot (\Phi^+)^\top + \Phi^-_w \cdot C \cdot \text{Cot} \sqrt{\lambda} \cdot (\Phi^-)^\top.
\end{align*}
\]

We are now able to describe the spectrum of our operator \( A \) in terms of spectral values of \( L_C(\lambda) \). Similar results have already been obtained in much the same way as in \[Be85, Ni83, Ni87, Ni87b, Be88b, \] and \[Ca97\] for the cases \( \mu_j = 1 \) and/or \( c_j = 1 \).

Theorem 4.3. For the spectrum of the operator \((A, D(A))\), defined in \[4.1\], \[4.2\], we obtain

\[
\sigma(A) = \{0\} \cup \sigma_C \cup \sigma_L,
\]
where
\[ \sigma_C \subseteq \{ -c_i k^2 \pi^2 : k \in \mathbb{Z} \setminus \{0\}, \ i = 1, \ldots, m \} \quad \text{and} \quad \sigma_L = \{ -\lambda \in \mathbb{R} : \lambda \neq c_i k^2 \pi^2, \ \det \mathcal{L}_C(\lambda) = 0 \} . \]

Furthermore,

1. \( \lambda = 0 \in \sigma(A) \) is always an eigenvalue of (geometric and algebraic) multiplicity 1 with an eigenvector \( f(x) \equiv 1 \), the constant 1 function.

2. \(-\lambda \in \sigma_L \) is an eigenvalue of \( A \) with corresponding eigenvector
\[
 f(x) = \cos \sqrt{\lambda} x \cdot (\Phi^+)^T d + \sin^{-1} \sqrt{\lambda} \cdot \sin \sqrt{\lambda} x \cdot \left( (\Phi^-)^T - \cos \sqrt{\lambda} \cdot (\Phi^+)^T \right) d
\]
where \( d \in \ker \mathcal{L}_C(\lambda) \), and so the multiplicity \( m(-\lambda) \) of this eigenvalue is equal to \( \dim \ker \mathcal{L}_C(\lambda) \);

3. \(-c_i k^2 \pi^2 \in \sigma_C \) is an eigenvalue of \( A \) if and only if there exist \( b \in \mathbb{C}^m \) and \( d \in \mathbb{C}^n \) such that whenever \( j \in \Gamma(v_r) \cap \Gamma(v_s), \ j \in \{1, \ldots, m\} \), we have
\[
(4.4) \quad \begin{cases} 
  d_r = (-1)^{c_i k / c_j} d_s, & \text{if } \sqrt{\frac{c_i}{c_j}} k \in \mathbb{Z}, \\
  b_j = \sin^{-1} \sqrt{\frac{c_i}{c_j}} k \pi \cdot d_r - \cot^{-1} \sqrt{\frac{c_i}{c_j}} k \pi \cdot d_s, & \text{otherwise}.
\end{cases}
\]
These vectors further satisfy the equation
\[
(4.5) \quad \Phi^- \cdot C \cdot \sin \sqrt{c_i k \pi} \cdot (\Phi^+)^T d = (\Phi^- \cdot C \cdot \cos \sqrt{c_i k \pi} - \Phi^+ \cdot C) \cdot b.
\]
If the eigenvector exists, then it has the form
\[
f(x) = \cos \sqrt{c_i k \pi} x \cdot (\Phi^+)^T d + \sin \sqrt{c_i k \pi} x \cdot b.
\]

Proof. By Lemma 4.1, we need to solve the equation
\[
Af = -\lambda f \quad \text{for} \quad f \in D(A) \quad \text{and} \quad \lambda \geq 0.
\]

We will distinguish three cases.

**Case 1:** Assume that \( \lambda \neq c_i k^2 \pi^2 \) for all \( k \in \mathbb{Z}, \ i = 1, \ldots, m \).

In this case the eigenfunctions of \( A \) are of the form
\[
f(x) = \cos \sqrt{\lambda} x \cdot a + \sin \sqrt{\lambda} x \cdot b \quad \text{for some} \quad a, b \in \mathbb{C}^m.
\]
From the continuity assumption in the domain of \( A \) (see 4.2),
\[
\exists d \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^T d = f(0) \text{ and } (\Phi^-)^T d = f(1)
\]
we obtain
\[
f(x) = \cos \sqrt{\lambda} x \cdot (\Phi^+)^T d + \sin^{-1} \sqrt{\lambda} \cdot \sin \sqrt{\lambda} x \cdot \left( (\Phi^-)^T - \cos \sqrt{\lambda} \cdot (\Phi^+)^T \right) d
\]
for some \( d \in \mathbb{C}^n \). The other condition \( \Phi^- f'(0) = \Phi^- f'(1) \) in the domain \( D(A) \) (i.e. the Kirchhoff law) yields that \( f \in \ker (\lambda - A) \) if and only if the vector \( d \in \mathbb{C}^n \) satisfies
\[
\Phi^- \cdot C \cdot \sin^{-1} \sqrt{\lambda} \cdot \left( (\Phi^-)^T - \cos \sqrt{\lambda} \cdot (\Phi^+)^T \right) d = \Phi^- \cdot C \cdot \left( \cot \sqrt{\lambda} \cdot (\Phi^-)^T - \cos \sqrt{\lambda} \cdot (\Phi^+)^T \right) - \sin \sqrt{\lambda} \cdot (\Phi^+)^T d.
\]
Observe now that, by Lemma 4.2, the following two terms are the previously defined diagonal matrix
\[ \Phi_w^+ \cdot C \cdot \cot \sqrt{\lambda} \cdot (\Phi^+)^\top + \Phi_w^- \cdot C \cdot \cot \sqrt{\lambda} \cdot (\Phi^-)^\top = D_C(\lambda), \]
while rearranging the remaining terms yields the weighted adjacency matrix
\[ \Phi_w^+ \cdot C \cdot \sin^{-1} \sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi_w^- \cdot C \cdot \left( \sin^{-1} \sqrt{\lambda} \cdot \cos^2 \sqrt{\lambda} + \sin \sqrt{\lambda} \right) \cdot (\Phi^+)^\top = \]
\[ = \Phi_w^+ \cdot C \cdot \sin^{-1} \sqrt{\lambda} \cdot (\Phi^-)^\top + \Phi_w^- \cdot C \cdot \sin^{-1} \sqrt{\lambda} \cdot (\Phi^+)^\top = A_C(\lambda). \]
Summing up, the condition for \( d \in \mathbb{C}^n \) becomes
\[ (A_C(\lambda) - D_C(\lambda)) d = 0, \quad \text{that is} \quad d \in \ker L_C(\lambda). \]

**Case 2:** \( \lambda = 0 \).
The eigenfunctions of \( A \), corresponding to \( \lambda = 0 \), are of the form
\[ f(x) = x \cdot a + b \quad \text{for some} \quad a, b \in \mathbb{C}^m. \]
We repeat the above procedure and the conditions in the domain \([2.4]\) yield
\[ f(x) = (\Phi^+)^\top d - x \cdot \Phi^\top d \quad \text{for} \quad d \in \ker \Phi_w \Phi^\top \quad \text{with} \quad \Phi_w = \Phi^+_w - \Phi^-_w. \]
Since our graph is connected, the multiplicity of 0 in \( \sigma(\Phi_w \Phi^\top) \) is 1 (cf. [Ch97, Lemma 1.7.(iv)] or [Mo97, Prop. 2.3]). It is easy to see that the corresponding eigenvector equals \( d = 1 := (1, \ldots, 1)^\top \). Now compute
\[ f(x) = (\Phi^+)^\top 1 - x \cdot \Phi^\top 1 \equiv 1 \quad \text{for all} \quad x. \]

**Case 3:** \( \lambda = c_i k^2 \pi^2 \) for some \( k \in \mathbb{Z} \setminus \{0\} \) and some \( i \in \{1, \ldots, m\} \).
We proceed as before while some care need to be taken with zero entries that arise. Before applying the inverse of \( \sin \sqrt{\lambda} \), the continuity condition in \([4.2]\) implies
\[ f(x) = \cos \sqrt{c_i k \pi x} \cdot (\Phi^+)^\top d + \sin \sqrt{c_i k \pi x} \cdot b, \]
where \( b \) satisfies the equation
\[ (4.6) \quad \sin \sqrt{c_i k \pi} \cdot b = \left( (\Phi^-)^\top - \cos \sqrt{c_i k \pi} \cdot (\Phi^+)^\top \right) d. \]
Since the \( i \)-th entry on the left-hand side equals 0, the vector \( d \) should satisfy the condition
\[ d_p = (-1)^k d_q \quad \text{for} \quad i \in \Gamma(v_p) \cap \Gamma(v_q). \]
Moreover, if for any other \( j \in \{1, \ldots, m\} \) we have \( \sqrt{c_j} k \in \mathbb{Z} \), then also
\[ d_r = (-1)^k \sqrt{c_j} d_s \quad \text{for} \quad j \in \Gamma(v_r) \cap \Gamma(v_s). \]
For each of these cases we have no conditions on \( b_j \). If on the other hand \( \sqrt{c_j} k \notin \mathbb{Z} \), equation \([4.6]\) yields
\[ b_j = \sin^{-1} \sqrt{c_j} k \pi \cdot d_r - \cot^{-1} \sqrt{c_j} k \pi \cdot d_s, \quad j \in \Gamma(v_r) \cap \Gamma(v_s). \]
Furthermore, the condition \( \Phi_w^+ f'(0) = \Phi_w^- f'(1) \) in the domain \( D(A) \) (i.e. the Kirchhoff law) implies that above vectors \( d \) and \( b \) have to satisfy the equation \([4.5]\).
Let us emphasize that the condition 3 in the above theorem is not always satisfied, therefore the spectrum of our operator $A$ strongly relies on the underlying graph and on the weights $c_j$.

From now on we assume $c_j = 1$, $j = 1, \ldots, m$. In this case we are able to connect the spectrum of the operator $A$ to the spectrum of yet another matrix known in graph theory. The transition matrix is defined as

$$
P := D^{-1}A$$

and is studied in connections with random walks on graphs (see e.g. [Mo97 Sec. 5.2] or [Ch97 Sec. 1.5]). The matrix $P$ is always positive, symmetric, row stochastic matrix with eigenvalues

$$
\sigma(P) = \{\alpha_1, \ldots, \alpha_n\} \text{ where } -1 \leq \alpha_n \leq \cdots \leq \alpha_2 < \alpha_1 = 1.
$$

By [Mo97 Claim 5.3], 1 is a simple eigenvalue whenever $G$ is connected (what we assumed at the beginning) and $-1$ is an eigenvalue if and only if $G$ is bipartite (see also Lemma 1 in Section 5 of [Be85]). It turns out that an important subset of the spectrum depends on the fact whether the graph $G$ is bipartite or not. This property means that the set of vertices can be divided into two disjoint subsets $V_1$ and $V_2$ such that any edge of $G$ has one endpoint in one and the other endpoint in the other subset. Note that $G$ is bipartite if and only if it does not have any odd cycle.

The following characteristic equation has already been proved by von Below [Be85]. We state and sketch the proof in our context for the convenience of the reader.

**Theorem 4.4.** Let $(A, D(A))$ be the operator defined in (4.1)–(4.2), with $c_j = 1$, $j = 1, \ldots, m$. Then for the spectrum of $A$ we have

$$
\sigma(A) = \{0\} \cup \sigma_p \cup \sigma_k,
$$

where

$$
\sigma_p = \left\{ -(2l\pi \pm \arccos \alpha)^2 : \alpha \in \sigma(P) \setminus \{-1, 1\} \text{ and } l \in \mathbb{Z} \right\},
$$

and

$$
\sigma_k = \{ -k^2\pi^2 : k \in \mathbb{Z} \setminus \{0\} \}.
$$

For the multiplicities of the eigenvalues we have:

1. $m(0) = 1$;
2. $m(-\lambda) = \dim \ker (\cos \sqrt{\lambda} \cdot I - P)$ for $-\lambda \in \sigma_p$;
3. $m(-k^2\pi^2) = m - n + 2$, if $G$ is bipartite;
4. $m(-4l^2\pi^2) = m - n + 2$ and $m(-(2l + 1)^2\pi^2) = m - n$, if $G$ is not bipartite.

**Proof.** We use Theorem 4.3 for $C = I$. For the spectral point $\lambda = 0$ the statement follows directly from Theorem 4.3. Assume first that $\lambda \neq k^2\pi^2$ for any $k \in \mathbb{Z}$. Then

$$
L_I(\lambda) = \sin^{-1} \sqrt{\lambda} \left(\cos \sqrt{\lambda} \cdot D - \mathbb{1}\right),
$$

and the characteristic equation becomes

$$
-\lambda \in \sigma(A) \iff \det \left(\cos \sqrt{\lambda} \cdot D - \mathbb{1}\right) = 0 \iff \det \left(\cos \sqrt{\lambda} \cdot I - P\right) = 0
$$

for the transition matrix $D^{-1}A = P$. The last equivalence says that

$$
-\lambda \in \sigma(A) \iff \cos \sqrt{\lambda} \in \sigma(P) \iff \lambda = (2l\pi \pm \arccos \alpha)^2
$$

for some $\alpha \in \sigma(P)$, $-1 < \alpha < 1$, and $l \in \mathbb{Z}$. Since $\dim \ker(\cos \sqrt{\lambda} \cdot D - \mathbb{1}) = \dim \ker(\cos \sqrt{\lambda} \cdot I - P)$, statement 2 also follows by Theorem 4.3.
Now, let $\lambda = k^2 \pi^2$ for some $k \in \mathbb{Z} \setminus \{0\}$. Observe that condition (4.4) in Theorem 4.3 becomes
\[ d_v = (-1)^k d_s \quad \text{whenever} \quad \Gamma(v_r) \cap \Gamma(v_s) \neq \emptyset. \]
If $k$ is even this condition is always fulfilled for $d = c \cdot 1$ for any $c \in \mathbb{R}$. Because the network is assumed to be connected, there is no other solution. For odd $k$ we can always choose $d = 0$. However, we can find a nonzero $d$ only in the case $G$ does not have any odd cycles, that is when $G$ is bipartite – hence, when its set of vertices can be divided into two disjoint subsets $V_1$ and $V_2$ such that any edge of $G$ has one endpoint in one and the other endpoint in the other subset. If this holds, the coordinates of $d$ can be chosen in such a way that at the places of vertex indices belonging to $V_1$ we set $c$ and at the places of vertex indices belonging to $V_2$ we set $-c$, $c \in \mathbb{R}$. By connectivity this are again all possible solutions.

Since $\sin k \pi = 0$ and $\cos k \pi = (-1)^k$, the other condition (4.5) in Theorem 4.3 becomes
\[ (\Phi_w^r - (-1)^k \Phi_w^s) b = 0. \]

Using the proof of [Be85, Theorem 5 (17)] we obtain 3. and 4.

5 Stability results for the diffusion problem

In the last section we are interested in the asymptotic behavior of solutions to the problem (3.1). By Corollary 3.3 the corresponding semigroup $(T_p(t))_{t \geq 0}$ on $X_p$, $p \in [1, \infty)$, has many nice properties: it is contractive, compact, positive. These properties already yield norm convergence of the solutions to an equilibrium (cf. [EN00, Cor. V.2.15]).

From the connectedness of our graph, used in the proof of Thm. 4.3.1, we obtain another useful property of the semigroup.

**Proposition 5.1.** The semigroup $(T_p(t))_{t \geq 0}$ on $X_p$, $p \in [1, \infty)$, is irreducible.

**Proof.** It is enough to prove the statement for $p = 2$, because the irreducibility is inherited for the extrapolation semigroups in $X_p$, $1 \leq p < \infty$, using Corollary 3.3 and [Ar04, Theorem 7.2.2]. Since $X_2$ is reflexive and $(T_2(t))_{t \geq 0}$ is bounded, by [EN00, Example V.4.7] we have that the semigroup is mean ergodic (see [EN00, Definition V.4.3]). By Theorem 4.3.4, we obtain that the corresponding mean ergodic projection $P$ defined by
\[ Px := \lim_{r \to \infty} \frac{1}{r} \int_0^r T(s)xds \]
is the projection $1 \otimes 1$ onto the subspace $\langle 1 \rangle$. Let $\{0\} \neq J \subset X_2$ be a closed invariant ideal for $(T(t))$, that is $T(t)J \subset J$, $t \geq 0$. Then also $PJ \subset J$ holds. By definition, $PJ \subset \langle 1 \rangle$ and so $J$ contains the closed ideal generated by a constant function – hence the whole space $X_2$. From this follows that the semigroup is irreducible.

Knowing that our semigroup is irreducible we can now show its norm convergence towards a projection of rank one.

**Corollary 5.2.** For the semigroup $(T_p(t))_{t \geq 0}$ on $X_p$, $p \in [1, \infty)$, the following hold.

1. The limit $Pf := \lim_{t \to \infty} T_p(t)f$ exists for every $f \in X_p$.
2. $P$ is a strictly positive projection onto $\langle 1 \rangle = \text{Ker}A$, the one-dimensional subspace of $X_p$ spanned by the constant function $1$.
3. For every $\epsilon > 0$ there exists $M > 0$ such that
\[
\|T_p(t) - P\| \leq Me^{(\epsilon + \lambda_2)t} \quad \text{for all} \quad t \geq 0,
\]
where $\lambda_2$ is the largest nonzero eigenvalue of the generator $A$.
Proof. The first assertions follows from [Na86, Cor.C-IV.2.10] and the second one from [Na86, C-III.3.5.(d)] and Theorem 4.3. Since $P$ is the residue corresponding to the spectral value $\lambda_1 = 0$ and 0 is a first order pole of the resolvent, estimate (3.2) in [EN00, Cor.V.3.2] yields the third assertion.

Note that the property of converging to an equilibrium does not depend on the structure of the network. However, the speed of the convergence towards a projection is determined by the second largest eigenvalue $\lambda_2$ of $A$ and thus relies on the network.

Combining graph theory and results about the spectrum of $A$, obtained in Section 4, we can now draw some further estimates containing graph parameters for the speed of convergence in (5.1). Let us demonstrate this for the case when all $c_j = 1$ and the graph is regular, that is, every vertex has the same degree. This means that

$$|\Gamma(v_i)| = \gamma \text{ for all } i = 1, \ldots, n, \text{ and } D = \gamma \cdot I.$$ 

Two generic examples of regular graphs are the $n$-cycle $C_n$ and the complete graph $K_n$ on $n$ vertices (in the latter, every two vertices are connected by an edge).

In this case the characteristic equation becomes

$$-\lambda \in \sigma(A) \iff \det \left( \cos \sqrt{\lambda} \cdot \gamma \cdot I - \mathbb{A} \right) = 0 \iff \cos \sqrt{\lambda} \cdot \gamma \in \sigma(A),$$

see the proof of Theorem [4.3] or [Bo85 Sec. 6]. Using the Laplacian of the graph, defined in [4.3], we obtain for its spectrum

$$\nu \in \sigma(L) \iff \det (\nu \cdot I - (\gamma \cdot I - \mathbb{A})) = 0 \iff -\nu + \gamma \in \sigma(A).$$

Hence, investigating the spectrum of the generator $A$, we are looking for $\lambda$’s such that

$$\lambda = - \left( 2l \pi \pm \arccos \left( 1 - \frac{\nu}{\gamma} \right) \right)^2, \quad l \in \mathbb{Z} \tag{5.2}$$

where $\nu \in \sigma(L)$. The spectrum of $L$ is sometimes also called the spectrum of the graph and is well investigated in graph theory. In the following we always refer to the survey paper [Mo91].

Example 5.3. In the case when $G = C_n$, the eigenvalues of $L$ are precisely the numbers

$$\nu_k = 2 - 2 \cos \left( \frac{2(k-1)\pi}{n} \right), \quad k = 1, \ldots, n,$$

while for $G = K_n$ we have

$$\nu_1 = 0 \text{ and } \nu_k = n \text{ for } 2 \leq k \leq n.$$

As explained above we are interested in the second-smallest eigenvalue $\nu_2 \in \sigma(L)$, which is also called the algebraic connectivity of the graph, see [Mo91]. It is related to the classical connectivity parameters of the graph (see below). If we look at (5.2) we can easily conclude that

$$\lambda_2 = - \left( \arccos \left( 1 - \frac{\nu_2}{\gamma} \right) \right)^2, \tag{5.3}$$

since the function $\arccos$ is strictly monotone decreasing and assumes its values between 0 and $\pi$. 

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Example 5.4. By increasing the number of vertices \( n \), the convergence gets slower on the cycle \( C_n \) and on the complete graph \( K_n \). In fact, for \( C_n \) we have \( \gamma = 2 \), hence by Example 5.3 we obtain that

\[
\lambda_2 = - \left( \arccos \left( 1 - \left( 1 - \cos \frac{2\pi}{n} \right) \right) \right)^2 = - \frac{4\pi^2}{n^2}.
\]

For \( K_n \) we have \( \gamma = n - 1 \), hence by Example 5.3,

\[
\lambda_2 = - \left( \arccos \left( 1 - \frac{n}{n-1} \right) \right)^2 = - \left( \arccos \left( - \frac{1}{n-1} \right) \right)^2.
\]

If we have an estimate from below for \( \nu_2 \), using (5.1) and formula (5.3), we obtain an upper estimate for the convergence speed of the semigroup to the one-dimensional projection.

In [Mo91] we find many estimates for \( \nu_2 \) that use several graph parameters. As an example we mention the next.

Definition 5.5. The edge connectivity parameter \( \eta = \eta(G) \) of the graph \( G \) is defined as the minimum number of edges whose deletion from \( G \) disconnects it.

Example 5.6. By [Mo91, Theorem 6.2(b)],

\[
\nu_2 \geq 2\eta \left( 1 - \cos \frac{\pi}{n} \right)
\]

holds, where \( \eta \) is the edge connectivity parameter of the graph. From this we obtain for (5.1) that for every \( \varepsilon > 0 \) there exists an \( M > 0 \) such that

\[
\| T_p(t) - P \| \leq Me^{\varepsilon \left( \arccos (1 - 2n(1 - \cos \frac{\pi}{n})) \right)^2} t \quad \text{for all } t \geq 0.
\]

Another estimate can be obtained using the diameter of \( G \).

Definition 5.7. The diameter of \( G \) denoted by \( \text{diam}(G) \) is the largest number of vertices which must be traversed in order to travel from one vertex to another when paths which backtrack, detour, or loop are excluded from consideration.

Example 5.8. By [Mo91 (6.10)], we have

\[
\nu_2 \geq \frac{4}{n \cdot \text{diam}(G)}.
\]

Hence, again we can conclude that for every \( \varepsilon > 0 \) there exists an \( M > 0 \) such that

\[
\| T_p(t) - P \| \leq Me^{\varepsilon \left( \arccos (1 - \frac{4}{n \cdot \text{diam}(G)}) \right)^2} t \quad \text{for all } t \geq 0.
\]

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