ON $L^1$-ESTIMATES OF DERIVATIVES 
OF UNIVALENT RATIONAL FUNCTIONS

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Abstract. We study the growth of the quantity

$$\int_T |R'(z)| \, dm(z)$$

for rational functions $R$ of degree $n$, which are bounded and univalent in the unit disk, and prove that this quantity may grow as $n^\gamma$, $\gamma > 0$, when $n \to \infty$. Some applications of this result to problems of regularity of boundaries of Nevanlinna domains are considered. We also discuss a related result by Dolzhenko which applies to general (non-univalent) rational functions.

1. Introduction

For $n \in \mathbb{N}$, denote by $\mathcal{R}_n$ the set of all rational functions in the complex variable of degree at most $n$. Thus, $\mathcal{R}_n$ consists of all functions of the form $P(z)/Q(z)$, where $P$ and $Q$ are polynomials in the complex variable $z$ such that $\deg P \leq n$ and $\deg Q \leq n$.

Let $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and let $T = \{z \in \mathbb{C}: |z| = 1\}$ be the unit circle. We denote by $\mathcal{RU}_n$ the set of all functions from the class $\mathcal{R}_n$ which have no poles in the closed disk $\mathbb{D}$ and are univalent in $\mathbb{D}$.

In this paper we study the quantity

$$\ell(R) := \int_T |R'(z)| \, dm(z)$$

(1.1)

for $R \in \mathcal{RU}_n$, where $m$ stands for the normalized Lebesgue measure on $T$, that is, $dm(z) = |dz|/(2\pi)$. The quantity $\ell(R)$ is the length of the boundary of the domain $R(\mathbb{D})$ under the mapping by the rational univalent function $R$. Our aim is to determine how $\ell(R)$ may grow when $n \to \infty$. More precisely, we are interested in the following problem:

Problem 1.1. To determine (or estimate) the value of the quantity

$$\gamma_0 := \limsup_{n \to \infty} \sup_{R \in \mathcal{RU}_n, \|R\|_\infty \leq 1} \frac{\log \ell(R)}{\log n},$$

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where for $Y \subset \mathbb{C}$ we set $\|f\|_{\infty, Y} = \sup\{|f(z)| : z \in Y\}$.

It follows from our main result (see Theorem 1.2 below), that $0 < \gamma_0 \leq \frac{1}{2}$.

Let us briefly explain how and where this problem (which is of independent interest) has arisen. The concept of a Nevanlinna domain appeared recently and quite naturally in problems of uniform approximation of functions by polyanalytic polynomials on compact subsets of the complex plane. We formally define and discuss this class of domains and the corresponding approximation problem (with all necessary references) in the last section of the paper. Now we mention only that an intriguing open problem about properties of Nevanlinna domains (posed in [12]) is the question of whether Nevanlinna domains with unrectifiable boundaries exist. Since any function from the class $\mathcal{RU}_n$ maps $\mathbb{D}$ conformally onto some Nevanlinna domain, the fact that $\gamma_0 > 0$ supports the conjecture that Nevanlinna domains with unrectifiable boundaries do exist (see Section 4 for details).

Let us state now the main result of this paper. The estimate of $\gamma_0$ from below is formulated in terms of the number $B_b(1)$, where $B_b(t)$, $t \in \mathbb{R}$, is the integral means spectrum for bounded univalent functions (see Section 3 below). It is worth to mention that, by the known estimates, $0.23 < B_b(1) \leq 0.46$.

**Theorem 1.2.** The following inequalities hold:

\[
B_b(1) \leq \gamma_0 \leq \frac{1}{2}. 
\]  

One should compare Theorem 1.2 with a recent result by I. Kayumov [18] who considered the quantity

\[
\Gamma(t) := \limsup_{n \to \infty} \sup_{P \in \mathcal{PU}_n} \frac{1}{\log n} \log \int_0^{2\pi} |P'(e^{it})|^t d\theta, 
\]

where $\mathcal{PU}_n$ denotes the set of all polynomials of degree at most $n$ which belong to the class $S$ of all univalent functions $f$ in the unit disk $\mathbb{D}$ normalized by conditions $f(0) = 0$, $f'(0) = 1$. It is proved in [18] that $\Gamma(t)$ coincides with $B(t)$, the integral means spectrum for the class $S$, for any $t \in \mathbb{R}$ (let us emphasize that $B(t)$ differs from the integral means spectrum $B_b(t)$ for bounded univalent functions).

We give two proofs for the lower bound in Theorem 1.2. The first is based on the standard Runge approximation scheme, while the second uses an idea from [18]. However, we do not know whether it is true that $\gamma_0 = B_b(1)$. The problem of finding the value of $\gamma_0$ remains open.

The estimate of $\gamma_0$ from above in Theorem 1.2 is the consequence of the following fact.

**Proposition 1.3.** Let $R \in \mathcal{RU}_n$ and $\|R\|_{\infty, T} \leq 1$. Then

\[
\int_T |R'(z)| \, dm(z) \leq 6\pi \sqrt{n}. 
\]  

The estimate (1.3) may be obtained as a consequence of a deep inversion of the Hardy–Littlewood embedding theorem for rational functions which is due to E.M. Dyn’kin [10]. We also give a direct short proof of (1.3). Our main result
is, however, the estimate from below in (1.2). It implies that the length \( \ell(R) \) of the boundary of the domain \( R(\mathbb{D}) \) under the mapping by a function \( R \in \mathcal{RU}_n \) may grow at least as \( n^\gamma \) (with \( 0 < \gamma < B_b(1) \)) when \( n \to \infty \).

It is worth to compare the result of Proposition 1.3 with the known estimates for rational functions which do not satisfy the univalence condition. It follows from a theorem due to E. P. Dolzhenko [8] (see the next section for details) that any function \( R \in \mathcal{R}_n \) which has no poles on \( \mathbb{T} \) admits the estimate
\[
\int_{\mathbb{T}} |R'(z)| \, dm(z) \leq n\|R\|_{\infty, \mathbb{T}}
\]
which is sharp (the equality is attained, e.g., on the function \( R(z) = z^n \) as well as on any finite Blaschke product of degree \( n \)).

Unfortunately this very interesting and important result remained unnoticed till recently whereas several of its particular cases and corollaries were rediscovered by other authors. Most likely this happened because the paper [8] was published in Russian (though in an international journal) and has never been translated into English. Thus, an extra aim of this paper is to set the record straight and present a sketch of the proof of Dolzhenko’s theorem which is much shorter than the original one.

The structure of the paper is as follows. Section 2 is devoted to integral estimates of derivatives for general rational functions. Here we discuss Dolzhenko’s theorem and give a sketch of a proof. Furthermore, we formulate and prove Proposition 2.3 which is the main ingredient of the proof of the first statement in Theorem 1.2. Section 3 consists of five subsections. In the first one we discuss some simple criteria which imply that a given rational function belongs to the class \( \mathcal{RU}_n \). The upper bound from Theorem 1.2 is proved in Subsection 3.2, while in the last three subsections two different proofs of the lower bound will be given. Finally, in Section 4 we discuss the concept of a Nevanlinna domain and the problem about uniform approximation by polyanalytic polynomials. We also give an interpretation of Theorem 1.2 in terms of properties of boundaries of quadrature domains.

2. Dolzhenko’s theorem, Spijker’s lemma and other integral estimates of derivatives of functions from the class \( \mathcal{R}_n \)

For integers \( n \) and \( k \) let \( \mathcal{A}_{n,k} \) be the set of all algebraic functions of order \((n, k)\) which means that each function \( f \in \mathcal{A}_{n,k} \) is an analytic (multivalued) function in \( \mathbb{C} \) except at most finite number of points and satisfies the equation \( P(z, f(z)) = 0 \), where \( P(z, w) \) is some polynomial in two complex variables \( z \) and \( w \) such that \( \deg_z P \leq n \) and \( \deg_w P \leq k \). Thus, any rational function of degree at most \( n \) is algebraic function of order \((n, 1)\), so that \( \mathcal{R}_n = \mathcal{A}_{n,1} \).

For a subset \( G \) of \( \mathbb{C} \) let \( \mathcal{A}_{n,k}(G) \) denote the class of all functions \( f \) which are defined on \( G \) and satisfy the equation \( P(z, f(z)) = 0 \) for every \( z \in G \) (where \( P \) is as above).

In 1978 E. P. Dolzhenko [8] obtained the following remarkable result about the behavior of the integral (1.1) for functions from the class \( \mathcal{A}_{n,k} \):

**Theorem 2.1 (Dolzhenko’s theorem).** Let \( G \) be an open subset of the circle or of the line \( \Gamma \subset \mathbb{C} \) and let a function \( f \in \mathcal{A}_{n,k}(G) \) be continuous on \( G \). Then for any measurable (with respect to the Lebesgue measure \( m_{\Gamma} \) on \( \Gamma \)) subset
\( E \subset G \) one has
\[
\int_E |f'(z)| \, dm_G(z) \leq 2\pi nk \|f\|_{\infty,E}.
\] (2.1)

In particular, for any \( R \in \mathcal{R}_n \) and for any measurable subset \( E \subset \mathbb{T} \) of positive measure the estimate
\[
\int_E |R'(z)| \, dm(z) \leq n \|R\|_{\infty,E}
\] (2.2)
holds and is sharp.

**Sketch of the proof of inequality (2.1).** The original proof of Dolzhenko’s theorem is long and technically involved and, thus, its exposition in corpore is beyond the scope of this paper. Nevertheless, we briefly explain how inequality (2.1) can be obtained by a simpler argument.

Without loss of generality we assume that \( \Gamma = \mathbb{R} \) and that \( E \) consists of finite number of non-overlapping segments (or intervals). Denoting by \( \text{Len}(f(E)) \) the length of the full \( f \)-image of \( E \) (counting multiplicities) and applying the classical Crofton’s formula (see, e.g., [26, Theorem 8]) we obtain that
\[
\int_E |f'(z)| \, dm_G(z) = \text{Len}(f(E)) \leq \frac{1}{4} \int \#(L_{\theta,b} \cap f(E)) \, dM_L,
\] (2.3)
where \( L_{\theta,b} \) denotes the line defined by the equation \( x \cos \theta + y \sin \theta + b = 0 \), \( 0 \leq \theta < 2\pi \), \( b \in \mathbb{R} \), and the integral is taken over the measure \( dM_L = dbd\theta \) on the set of all oriented lines in the plane. By \( \#Y \) we denote the number of elements in the set \( Y \).

Furthermore, it is not difficult to check that the curve \( f(\Gamma) \) intersects each line in the plane at most in \( 2nk \) points. Also note that \( f(E) \subset D(0, \|f\|_{\infty,E}) := \{z: |z| \leq \|f\|_{\infty,E}\} \) and \( L_{\theta,b} \cap D(0, \|f\|_{\infty,E}) = \emptyset \) for \( |b| \geq \|f\|_{\infty,E} \). Hence, the last integral in (2.3) admits the following easy estimate
\[
\int \#(L_{\theta,b} \cap f(E)) \, dM_L \leq 2nk \cdot 2\pi \cdot 2\|f\|_{\infty,E} = 8\pi nk \|f\|_{\infty,E},
\]
which immediately gives (2.1). \( \square \)

**Remark 2.2.** The original proof of Dolzhenko’s theorem in [8] is also based on application of special formulas from integral geometry. It is interesting to note that Crofton’s formula not only implies inequality (2.1), but also gives a sharp constant in it.

One important particular case of the inequality (2.2) was rediscovered in 1991 by M.N. Spijker [25], who proved that
\[
\int_{\mathbb{T}} |R'(z)| \, dm(z) \leq n \|R\|_{\infty,T}
\]
for any \( R \in \mathcal{R}_n \) without poles on \( \mathbb{T} \). The latter result turned out to be very useful in the context of applied matrix theory (for a detailed account on this topic see the survey by N. Nikolski [21]). Now this result is known as Spijker’s lemma, but in all fairness it should be referred to as the Dolzhenko–Spijker lemma in order to underline the crucial contribution of Dolzhenko to the themes under consideration.
Now we give another integral estimate of derivatives of functions from the class $R_n$, which will be used in the proof of Theorem 1.2. Let us denote by $m_2$ the normalized planar Lebesgue measure in $\mathbb{D}$, that is, $dm_2(w) = dudv/\pi$, $w = u + iv$.

**Proposition 2.3.** Let $R \in R_n$ have no poles in $\mathbb{D}$. Then

$$\int_{\mathbb{T}} |R'(z)| \, dm(z) \leq 6\sqrt{n} \left( \int_{\mathbb{D}} |R'(w)|^2 \, dm_2(w) \right)^{1/2}. \quad (2.4)$$

Proposition 2.3 is a special case of a reverse Hardy–Littlewood embedding theorem by E. Dyn’kin [10]. Let $H^\sigma$, $\sigma > 0$, denote the standard Hardy space in the disk and let $A^\sigma_p$, $1 \leq p < \infty$, $\alpha > -1$, be the weighted Bergman space of functions analytic in $\mathbb{D}$ with finite norm

$$\|f\|_{A^\sigma_p} = \int_{\mathbb{D}} |f(z)|^p (1 - |z|)^\alpha \, dm_2(z).$$

Let $\sigma = \frac{p}{1+\alpha}$. Dyn’kin [10, Theorem 4.1] has shown that for $R \in R_n$ without poles in $\mathbb{D}$ one has

$$\|R\|_{H^\sigma} \leq C_n \|R\|_{A^\sigma_p} \quad (2.5)$$

with some absolute constant $C > 0$. If we put $p = 2$, $\alpha = 0$ (thus, $\sigma = 1$) and apply inequality (2.5) to $R'$ in place of $R$, we obtain (2.4) with some constant. However, the proof of the general result of Dyn’kin is rather complicated (it involves, e.g., the Carleson corona construction). Therefore, we prefer to give a simple direct proof of Proposition 2.3.

**Proof of Proposition 2.3.** Let $z_1, \ldots, z_n$ be such points in $\mathbb{D}$ that $1/z_1, \ldots, 1/z_n$ are all poles of the function $R$ counting with multiplicities (it is clear that without loss of generality we may assume that $R$ has exactly $n$ poles). Let

$$B(z) = \prod_{k=1}^n \frac{\bar{z}_k}{|z_k|} \cdot \frac{z - z_k}{\bar{z}_k z - 1}$$

be the (finite) Blaschke product with zeros at $z_1, \ldots, z_n$. Therefore, there exists $a \in \mathbb{C}$ such that the function $g = R - a$ belongs to the finite-dimensional model space $K_B = H^2 \ominus BH^2$ (for a systematic exposition of the theory of model spaces see, for instance, N. K. Nikolski’s book [20]). Recall that the function

$$k_w(z) = \frac{1 - \overline{B(w)}B(z)}{1 - \overline{w}z}$$

is the reproducing kernel for the space $K_B$, which means that $k_w \in K_B$ and $(f, k_w) = 2\pi i f(w)$ for any $f \in K_B$ (the brackets denote the inner product in $H^2$). It follows from the standard Cauchy formula and from the fact that the function

$$\frac{1}{(1 - \overline{w}z)^2} - \left( \frac{1 - \overline{B(w)}B(z)}{1 - \overline{w}z} \right)^2$$

(as a function in the variable $z$) belongs to the space $BH^2$ that the following integral representation takes place for every $z \in \mathbb{D}$:

$$R'(z) = \int_{\mathbb{T}} R(w) \overline{w} \left( \frac{1 - \overline{B(w)}B(z)}{1 - \overline{w}z} \right)^2 \, dm(w).$$
Since $B$ is a finite Blaschke product (thus, its zeros have no accumulation points on $\mathbb{T}$), the latter representation also takes place for every $z \in \mathbb{D}$.

By a simple version of Green’s formula,
\[
\int_{\mathbb{T}} R(w)\overline{w} \left( \frac{1 - B(w)B(z)}{1 - \overline{w}z} \right)^2 \, dm(w) = \int_{\mathbb{D}} R'(w) \left( \frac{1 - B(w)B(z)}{1 - \overline{w}z} \right)^2 \, dm_2(w).
\]

Therefore, applying the Fubini theorem and the Hölder inequality, we have
\[
\int_{\mathbb{T}} |R'(z)| \, dm(z) \leq \int_{\mathbb{D}} |R'(w)| \int_{\mathbb{T}} \left| \frac{1 - B(w)B(z)}{1 - \overline{w}z} \right|^2 \, dm(z) \, dm_2(w) = \\
= \int_{\mathbb{D}} |R'(w)| \frac{1 - |B(w)|^2}{1 - |w|^2} \, dm_2(w) \leq \\
\leq \left( \int_{\mathbb{D}} |R'(w)|^2 \, dm_2(w) \right)^{1/2} \left( \int_{\mathbb{D}} \left( \frac{1 - |B(w)|^2}{1 - |w|^2} \right)^2 \, dm_2(w) \right)^{1/2}.
\]

It is clear, that
\[
\int_{\mathbb{D}} \left( \frac{1 - |B(w)|^2}{1 - |w|^2} \right)^2 \, dm_2(w) \leq 4 \int_{\mathbb{D}} \left( \frac{1 - |B(w)|}{1 - |w|} \right)^2 \, dm_2(w).
\]

To estimate the last integral we use the following lemma (see Theorem 3.2 in [10]):

Lemma 2.4. For $r \in (0, 1]$ let
\[
L(r) = \int_{\{z : |z| < r\}} \left( \frac{1 - |B(w)|}{1 - |w|} \right)^2 \, dm_2(w).
\]

Then
\[
L(1) \leq 8n + 1. \tag{2.6}
\]

As it was mentioned above, the proof of this lemma may be found in [10]. But we include it here for the sake of completeness and for the reader’s convenience. First of all, it is easy to check that $L(1/2) \leq 1$. Then integrating in polar coordinates $w = re^{it}$ and using the classical Hardy inequality, we have
\[
\frac{1}{\pi} \int_{1/2}^{1} \frac{r \, dr}{(1 - r)^2} \int_{0}^{2\pi} \left( 1 - |B(re^{it})| \right)^2 \, dt \leq \\
\leq \frac{1}{\pi} \int_{1/2}^{1} \frac{r \, dr}{(1 - r)^2} \int_{0}^{2\pi} |B(e^{it}) - B(re^{it})|^2 \, dt \leq \\
\leq \frac{1}{\pi} \int_{0}^{2\pi} dt \int_{1/2}^{1/2} \frac{r \, dr}{(1 - r)^2} \left( \int_{r}^{1} |B'(pe^{it})| \, dp \right)^2 \leq \\
\leq \frac{4}{\pi} \int_{0}^{2\pi} dt \int_{1/2}^{1/2} |B'(pe^{it})|^2 \, dp \leq 8 \int_{\mathbb{D}} |B'(z)|^2 \, dm_2(z) = 8n. \quad \Box
\]

So, in order to complete the proof of Proposition 2.3 it remains to observe that $2\sqrt{8n + 1} \leq 6\sqrt{n}$ for any positive integer $n$. 

3. The class $\mathcal{RU}_n$ and the Proof of Theorem 1.2

Before proving Theorem 1.2 let us give some simple criteria that imply that a given rational function belongs to the class $\mathcal{RU}_n$.

3.1. Univalent functions in the space $\mathcal{RU}_n$. Take a function $R \in \mathcal{R}_n$ having all its poles $b_1, \ldots, b_m$, $m \leq n$, outside $\overline{\mathbb{D}}$. For $j = 1, \ldots, m$ we put $a_j = 1/b_j$, and for $k = 1, \ldots, m$ we define the corresponding (finite) Blaschke products

$$B_k(z) = \prod_{j=1}^k \frac{a_j}{|a_j|}, \quad \frac{z - a_j}{\overline{a_j}z - 1}.$$  

Let, also, $B_0(z) \equiv 1$ and $B = B_m$. Since $R = a + g$, where $a \in \mathbb{C}$ and $g \in K_B$ (the model space $K_B$ is defined in the proof of Proposition 2.3 above), then there exists a set of complex coefficients $\{c_1, \ldots, c_m\}$ such that

$$R = a + \sum_{k=1}^m c_k \frac{\sqrt{1 - |a_k|^2}}{1 - \overline{a_k}z} B_{k-1}(z)$$  

(we recall, that the system of function $\{\sqrt{1 - |a_k|^2} B_{k-1}(z) / (1 - \overline{a_k}z)\}_{k=1}^m$ forms an orthonormal basis in the space $K_B$, see, for instance, [20, Chap. V]).

Assume that $a_1 \in (0, 1/\sqrt{2})$ and $|a_j| \leq |a_k|$ for $j \leq k$. As it was shown in the proof of Theorem 2 in [13], if the set of coefficients $\{c_1, \ldots, c_m\}$ is such that $c_1 = 1$ and

$$\sum_{k=2}^m |c_k| \sqrt{\frac{1 + |a_k|}{1 - |a_k|}} \sum_{j=1}^k \frac{1 + |a_j|}{1 - |a_j|} \leq \frac{a_1 \sqrt{1 - a_1^2(1 - 2a_1^2)}}{(1 + a_1^2)^4},$$

then the function $R$ defined by (3.1) satisfies the condition $\Re R'(z) > 0$ in $\mathbb{D}$ and hence it is univalent in $\mathbb{D}$.

Therefore, for any set of points $\{b_1, \ldots, b_m\} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ there exists a function from $\mathcal{RU}_n$ with poles exactly at these points. Further examples of univalent rational functions were obtained, for instance, in [1].

3.2. Proof of Theorem 1.2: the upper bound. The estimate (1.3) in Proposition 1.3 is a direct corollary of Proposition 2.3. Indeed, if $R \in \mathcal{RU}_n$, then

$$\int_{\mathbb{D}} |R'(z)|^2 dm_2(z) = \pi \text{Area}(R(\mathbb{D})) \leq \pi^2 \|R\|^2_{\infty, \tau},$$

where Area($\Omega$) is the area of the domain $\Omega \subset \mathbb{C}$, and hence

$$6\sqrt{n} \left( \int_{\mathbb{D}} |R'(z)|^2 dm_2(z) \right)^{1/2} \leq 6\pi \sqrt{n} \|R\|_{\infty, \tau}.$$

3.3. Integral means spectrum for bounded univalent functions. Let us recall the definition of the integral means spectrum for bounded univalent functions (for a detailed exposition see [23, Chap. 8] and [14, Chap. VIII]). For a function $f$ and numbers $t \in \mathbb{R}$ and $r$, $0 < r < 1$, let

$$M_t[f^r](r) = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^t \, d\theta,$$
and let
\[ \beta_f(t) = \limsup_{r \to 1^-} \frac{\log M_t[f'](r)}{|\log(1 - r)|}. \]
Thus, \( \beta_f(t) \) is the smallest number \( \tilde{\beta} \) such that for any \( \epsilon > 0 \)
\[ M_t[f'](r) = O\left(\frac{1}{(1 - r)^{\beta + \epsilon}}\right) \]
as \( r \to 1^- \). Furthermore, let \( B_b(t) = \sup_f \beta_f(t) \), where the supremum is taken over all bounded univalent functions.

L. Carleson and P. W. Jones conjectured in [4] that
\[ B_b(1) = \frac{1}{4}. \]
It is worth to mention that in [4] it was proved that
\[ B_b(1) = \sup f \limsup_{n \to \infty} \frac{\log(n|a_n|)}{\log n}, \]
where the supremum is taken over all functions \( f(z) = \sum_{n=0}^\infty a_n z^n \) which are bounded and univalent in \( \mathbb{D} \).

The conjecture of Carleson and Jones remains open. There are several known estimates of the number \( B_b(1) \) from above and from below, see [23, Chap. 8], [14, Chap. VIII], [16] and references therein. The best known estimate of \( B_b(1) \) from below, \( B_b(1) > 0.23 \), was obtained by D. Beliaev and S. Smirnov [3] for the conformal mappings onto domains bounded by some fractal closed curves, while H. Hedenmalm and S. Shimorin showed in [16] that \( B_b(1) \leq 0.46 \). The novel techniques introduced in [16] have been further improved in [17], while in [24] they were applied to give a slightly smaller bound than 0.46. In [2] the estimate of the function \( B_b(t) \) was improved near the point \( t = 2 \), and it also gives slight lowering of \( B_b(1) \).

3.4. Proof of the lower bound in Theorem 1.2: Runge scheme. The idea of this proof is to use the standard Runge approximation scheme for a bounded univalent function \( f \) with almost extremal growth of \( M_t[f'](r) \) as \( r \to 1^- \).

We start with the following lemma. For \( z_0 \in \mathbb{C} \) and \( \rho > 0 \) let \( D(z_0, \rho) \) denote the open disk \( \{z : |z - z_0| < \rho\} \); we write \( D(\rho) \) in place of \( D(0, \rho) \).

Lemma 3.1. Let \( 0 < \beta < B_b(1) \). Then there exists a sequence of positive numbers \( \delta_k \to 0 \), \( k \to \infty \), and a sequence of functions \( f_k \) such that
(i) \( f_k \) is univalent in the disk \( D(1 + 4\delta_k) \) and \( \|f_k\|_{\infty, D(1+4\delta_k)} \leq 1 \);
(ii) \( \inf_k |f'_k(0)| > 0 \);
(iii) \( \int_0^{2\pi} |f'_k(e^{i\theta})| d\theta \geq c\delta_k^{-\beta} \) for some absolute constant \( c > 0 \).

Proof. By the definition of \( B_b(1) \) there exists a function \( f \) which is bounded and univalent in \( \mathbb{D} \) and such that for some sequence \( r_k > 1/2 \), \( r_k \to 1^- \), \( k \to \infty \), we have
\[ \int_0^{2\pi} |f'(r_k e^{i\theta})| d\theta \geq (1 - r_k)^{-\beta}. \]
Now put \( f_k(z) = f(r_k z) \) and define \( \delta_k \) by \( 1 + 4\delta_k = r_k^{-1} \). Obviously, the functions \( f_k \) have the properties (i)–(iii). \( \square \)
Now we pass to the proof of the lower bound in Theorem 1.2. Let \( f_k \) and \( \delta_k \in (0, 1/2) \) be as in Lemma 3.1. Fix some sufficiently large \( k \). Take the circle \( T_k = \{ z : |z| = 1 + 2\delta_k \} \) and split it into the union of \( N_k \) equal arcs \( I_j, j = 1, \ldots, N_k \), and for each \( j \) take a point \( \zeta_j \in I_j \). The number \( N_k = N_k(\delta_k) \) will be chosen later. For \( |z| < 1 + 2\delta_k \), we have

\[
f_k(z) = \frac{1}{2\pi i} \int_{T_k} \frac{f_k(\zeta)}{\zeta - z} \, d\zeta.
\]

Now, fix a positive integer \( m \) and define the rational function \( R \) of degree at most \( mN_k \) by the formula

\[
R(z) = \frac{1}{2\pi i} \sum_{j=1}^{N_k} \sum_{l=0}^{m} \frac{1}{(\zeta_j - z)^{l+1}} \int_{I_j} (\zeta - \zeta_j)^l f_k(\zeta) \, d\zeta.
\]

Note that, though we omit the index, \( R \) also depends on \( k \) and \( m \). In what follows \( A_1, A_2, \ldots \) will denote positive constants, whose values do not depend on \( \delta_k \) or \( N_k \) (though they may depend on \( m \) and \( \inf_k |f'_k(0)| \) from Lemma 3.1, (ii)).

It follows from the identity

\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta_j - z)(1 - (\zeta_j - \zeta)/(\zeta_j - z))} = \sum_{l=0}^{m} \frac{(\zeta_j - \zeta)^l}{(\zeta_j - z)^{l+1}} + \frac{1}{\zeta - z} \left( \frac{\zeta_j - \zeta}{\zeta_j - z} \right)^{m+1}
\]

that

\[
f_k(z) - R(z) = \frac{1}{2\pi i} \sum_{j=1}^{N_k} \int_{I_j} \frac{f_k(\zeta)}{\zeta - z} \left( \frac{\zeta_j - \zeta}{\zeta_j - z} \right)^{m+1} \, d\zeta. \tag{3.2}
\]

Since \( |\zeta - \zeta_j| \leq 4\pi/N_k \) when \( \zeta \in I_j \) and \( |z - \zeta| \geq \delta_k \) when \( \zeta \in T_k \) and \( |z| \leq 1 + \delta_k \), we conclude that

\[
|f_k(z) - R(z)| \leq \frac{A_1}{N_k^{m+1}\delta_k^{m+2}}, \quad |z| \leq 1 + \delta_k. \tag{3.3}
\]

Also, differentiating (3.2) we obtain that

\[
|f'_k(z) - R'(z)| \leq \frac{A_2}{N_k^{m+1}\delta_k^{m+3}}, \quad |z| \leq 1 + \delta_k. \tag{3.4}
\]

Let us verify now that for sufficiently large \( N_k \) the function \( R \) is univalent in \( \mathbb{D} \). We start with verification of local univalence of \( R \). Since the function \( f_k \) is univalent in \( D(1 + 4\delta_k) \), we have for \( |z| \leq 1 + \delta_k \) by the classical distortion theorem (see, e.g., [23, Chap. 1])

\[
|f'_k(z)| \geq A_3 |f'_k(0)| \delta_k \geq A_4 \delta_k.
\]

Now, fix some \( \varepsilon > 0 \) and take

\[
N_k = \left\lfloor \delta_k^{-m+1} - \varepsilon \right\rfloor + 1,
\]

where \( \lfloor x \rfloor \) stands for the integer part of \( x \). Then \( N_k^{-m} \delta_k^{-m-3} = o(\delta_k) \) when \( k \to \infty \), and so for sufficiently large \( k \) we have

\[
|R'(z)| \geq A_4 \delta_k/2
\]

when \( |z| \leq 1 + \delta_k \). Hence, \( R \) is locally univalent in \( D(1 + \delta_k) \).
As the next step we need to check the injectivity of \( R \) in \( \mathbb{D} \). Assume that there exist two distinct points \( z_1, z_2 \in \mathbb{D} \) such that \( R(z_1) = R(z_2) \). We consider two different cases:

(a) \( |z_1 - z_2| \geq \delta_k / 2 \), that is, the disks \( D(z_1, \delta_k / 4) \) and \( D(z_2, \delta_k / 4) \) are disjoint;

(b) \( |z_1 - z_2| < \delta_k / 2 \).

In case (a) we show that the equation \( f_k(z) - w \), where \( w = R(z_1) \), has roots in both disks \( D(z_1, \delta_k / 4) \) and \( D(z_2, \delta_k / 4) \) which contradicts the univalence of \( f_k \). Take \( z \in D(1 + \delta_k) \). We have

\[
|R(z) - R(z_1)| \geq |f_k(z) - f_k(z_1)| - |f_k(z) - R(z)| - |f_k(z_1) - R(z_1)| \geq
\]

\[
\geq |f_k(z) - f_k(z_1)| - \frac{2A_1}{N_k^{m+1} \delta_k^{m+2}}.
\]

It follows from the Koebe theorem (see [23, Chap. 1]) and from the property (ii) of Lemma 3.1 that for \( z \) with \( |z - z_1| = \delta_k / 4 \) we have

\[
|f_k(z) - f_k(z_1)| \geq \frac{1}{4}|z - z_1| \cdot |f'_k(z_1)| \geq A_5 \delta_k^2.
\]

By the choice of \( N_k \) we have \( 4A_1 N_k^{-m-1} \delta_k^{-m-2} < A_5 \delta_k^2 \) for sufficiently large \( k \), and so

\[
|R(z) - R(z_1)| \geq A_6 \delta_k^2
\]

as \( |z - z_1| = \delta_k / 4 \). At the same time, by (3.3), we have \( |f_k(z) - R(z)| \leq A_7 \delta_k^{2+(m+1)\varepsilon} \). As a standard application of the Rouché theorem we conclude that the equation \( f_k(z) - w \) has a root in the disk \( D(z_1, \delta_k / 4) \) if \( k \) is sufficiently large. But the same arguments show that this equation also has a root in the disk \( D(z_2, \delta_k / 4) \), which is clearly impossible since \( f_k \) is univalent.

In the case (b) an application of the Rouché theorem together with above estimates shows that the function \( f_k - w \) has two zeros in the disk \( D(z_1, \delta_k) \), again a contradiction. Thus, \( R \) is univalent in \( \mathbb{D} \) for sufficiently large \( k \). Also it is clear from (3.3) that the norms \( \|R\|_{\infty, \mathbb{D}} \) are bounded uniformly with respect to \( k \).

To complete the proof, notice that, by (3.4),

\[
\int_{\mathbb{T}} |f'_k(z) - R'(z)| \, dm(z) \leq \frac{A_2}{N_k^{m+1} \delta_k^{m+3}} \leq A_8 \delta_k^{1+(m+1)\varepsilon}.
\]

Then, using Lemma 3.1, (iii), we finally obtain that for any \( \varepsilon > 0 \)

\[
\int_{\mathbb{T}} |R'(z)| \, dm(z) \geq A_9 \delta_k^{-\beta} \geq A_{10}(mN_k)^{\frac{m+1}{m+4+(m+1)\varepsilon}}
\]

for sufficiently large \( k \). Recall that the degree of \( R \) is at most \( mN_k \). We conclude that

\[
\gamma_0 \geq \beta \frac{m + 1}{m + 4 + (m + 1)\varepsilon}
\]

for any \( \varepsilon > 0 \) and any \( m \in \mathbb{N} \). Hence, \( \gamma_0 \geq \beta \). Since \( \beta \) in Lemma 3.1 was an arbitrary number less than \( B_b(1) \) we finally get \( \gamma_0 \geq B_b(1) \). \( \square \)
3.5. Proof of the lower bound in Theorem 1.2: Kayumov’s approach.

In this subsection we present another proof of Theorem 1.2. Take some \( \varepsilon > 0 \) and a function \( f \in H^\infty \) such that \( \beta_f(1) > B_b(1) - \varepsilon \). Without loss of generality we may assume that \( f(0) = 0 \) and \( f'(0) = 1 \), so that the function \( f \) belongs to the class \( S \).

From now on we will follow the idea proposed by I. Kayumov in [18]. Let us choose \( r_k \rightarrow 1 \) such that
\[
\beta_f(1) = \lim_{k \to \infty} \frac{\log M_1[f'](r_k)}{|\log(1 - r_k)|},
\]
and such that
\[
r_k = 1 - \frac{5 \log n_k}{n_k}
\]
for some integers \( n_k \). Assume that \( f(z) = \sum_{j=1}^{\infty} a_j z^j \) is the Taylor expansion of \( f \) at the origin and consider the sequence of polynomials \( (P_k)_{k=1}^{\infty} \) such that
\[
P_k(z) := \sum_{j=1}^{n_k} a_j r_k^j z^j.
\]
In [18] it was proved that these polynomials are univalent in \( \mathbb{D} \). For the polynomials \( P_k \) we have
\[
\limsup_{k \to \infty} \frac{\log \ell(P_k)}{\log n_k} \geq \beta_f(1) > B_b(1) - \varepsilon.
\]

Note that the norms \( \|P_k\|_{\infty,T} \) of the polynomials \( P_k \) are not necessarily bounded. However, by a classical result of E. Landau, the norm of the projector
\[
\Pi_m: \sum_{j=0}^{\infty} b_j z^j \to \sum_{j=0}^{m} b_j z^j
\]
from \( H^\infty \) to the space of all analytic polynomials of degree at most \( m \) equipped with \( L^\infty \)-norm satisfies \( \|\Pi_m\| \sim \frac{1}{\pi} \log m \) as \( m \to \infty \) (see [9, Section 8.5] for details). Thus,
\[
\|P_k\|_{\infty,T} \leq C \log n_k
\]
for some absolute positive constant \( C \). Let us define the polynomials \( Q_k(z) := (\log n_k)^{-1} P_k(z) \). We have
\[
\gamma_0 \geq \limsup_{k \to \infty} \frac{\log \ell(Q_k)}{\log n_k} = \limsup_{k \to \infty} \frac{\log \ell(P_k)}{\log n_k} - \lim_{k \to \infty} \frac{\log \log n_k}{\log n_k} = \limsup_{k \to \infty} \frac{\log \ell(P_k)}{\log n_k} > B_b(1) - \varepsilon,
\]
whence \( \gamma_0 \geq B_b(1) \). \( \square \)
4. **Theorem 1.2, Nevanlinna Domains and Related Topics**

As it was mentioned in Introduction, Problem 1.1 and our main result (Theorem 1.2) are related with the concept of a Nevanlinna domains and therefore, with the problem of uniform approximability of functions by polyanalytic polynomials on compact subsets of the complex plane.

Take an integer \( m \geq 1 \). Recall that a function \( f \) is said to be polyanalytic of order \( m \) (or, shortly, \( m \)-analytic) on an open set \( U \subset \mathbb{C} \) if it is of the form

\[
f(z) = \overline{\varphi}^{m-1}f_{m-1}(z) + \cdots + \overline{\varphi}f_1(z) + f_0(z),
\]

where \( f_0, \ldots, f_{m-1} \) are holomorphic functions in \( U \); one denotes by \( \text{Hol}_m(U) \) the class of all \( m \)-analytic functions on \( U \). It is clear, that any function \( f \in \text{Hol}_m(U) \) satisfies in \( U \) the elliptic partial differential equation \( \overline{\partial}^m f = 0 \), where \( \overline{\partial} \) is the standard Cauchy–Riemann operator in \( \mathbb{C} \), and therefore, the uniform convergence preserves the polyanalyticity property. Furthermore, by polyanalytic polynomials we mean polyanalytic functions such that all functions \( f_0, \ldots, f_{m-1} \) from the representation (4.1) are polynomials in the complex variable.

For a compact set \( X \subset \mathbb{C} \) let us denote by \( C(X) \) the space of all continuous complex valued functions on \( X \) endowed with the standard uniform norm \( \|f\|_X = \max_{z \in X} |f(z)|, f \in C(X) \). Moreover, let \( A_m(X) = C(X) \cap \text{Hol}_m(X^o) \), where \( X^o \) stands for the interior of \( X \), and \( P_m(X) = \{ f \in C(X) : \forall \varepsilon > 0 \text{ there exists } m\text{-analytic polynomial } P \text{ such that } \|f - P\|_X < \varepsilon \} \). It is clear, that \( P_m(X) \subset A_m(X) \). Now we are able to state the approximation problem mentioned above.

**Problem 4.1.** Let \( m \geq 2 \). What conditions on \( X \) are necessary and sufficient in order that

\[
A_m(X) = P_m(X)?
\]

We restrict ourselves to the case \( m \geq 2 \) and exclude the case \( m = 1 \) from the consideration since, by the remarkable Mergelyan theorem, \( A_1(X) = P_1(X) \) if and only if the set \( \mathbb{C} \setminus X \) is connected.

The study of Problem 4.1 started in the middle of 1970s as the study of the problem about uniform approximation by polyanalytic rational functions (i.e., by functions of the form (4.1) such that all functions \( f_0, \ldots, f_{m-1} \) are rational functions) having their poles outside \( X \). Several necessary and sufficient conditions in the latter problem were obtained in [22], [27] and [5]. For instance, the following result is a direct consequence of Theorem 2 in [5] and the well-known Runge method of ‘pole-pushing’: if the set \( \mathbb{C} \setminus X \) is connected, then \( A_m(X) = P_m(X) \) for any \( m \geq 2 \). However, as was shown in [11], even in the simplest case when the compact set \( X \) has disconnected complement (namely, when \( X \) is a rectifiable contour in \( \mathbb{C} \)), the solution of Problem 4.1 is formulated in terms of special analytic properties of \( X \). In particular, if \( X \) is an arbitrary circle in \( \mathbb{C} \), then the equality (4.2) fails, but it is satisfied (for any integer \( m \geq 2 \)) for any closed polygonal Jordan curve as well as for any non-degenerate ellipse (which is not a circle) \( X \) in \( \mathbb{C} \). In 1990s several results on approximability of functions by polyanalytic polynomials have been obtained using the concept of a Nevanlinna domain which was introduced in [11] and [6]. For instance, Theorem 2.2 in [6] says that \( A_m(X) = P_m(X) \) for a Carathéodory compact set \( X \) if and only if any bounded connected component of the set \( \mathbb{C} \setminus X \) is not a Nevanlinna domain. We
recall that $X$ is a Carathéodory compact set if $\partial X = \hat{\partial} \hat{X}$, where $\hat{X}$ is the union of $X$ and all bounded connected components of $\mathbb{C} \setminus X$. An interested reader can find a comprehensive survey about Problem 4.1 and certain related problems in [19].

Let us now formulate the definition of a Nevanlinna domain (see [11, Definition 3] and [6, Definition 2.1]). For an open set $U \subset \mathbb{C}$ let $H^\infty(U)$ be the space of all bounded holomorphic functions in $U$. In view of the classical Fatou theorem, every function $f \in H^\infty(\mathbb{D})$ has finite angular boundary values $f(\zeta)$ for almost all (with respect to the Lebesgue measure on $T$) $\zeta \in T$.

**Definition 4.2.** A bounded simply connected domain $\Omega$ is called a Nevanlinna domain if there exists two functions $u, v \in H^\infty(\Omega)$, $v \not\equiv 0$, such that the equality

$$z = \frac{u(z)}{v(z)}$$

holds almost everywhere on $\partial \Omega$ in the sense of conformal mapping, which means that the equality of angular boundary values

$$\varphi(\zeta) = \frac{(u \circ \varphi)(\zeta)}{(v \circ \varphi)(\zeta)}$$

holds for almost all points $\zeta \in T$, where $\varphi$ is some conformal mapping from $\mathbb{D}$ onto $\Omega$.

Let $ND$ be the class of all Nevanlinna domains. Note that the definition of Nevanlinna domain is consistent since it does not depend on the choice of $\varphi$. Furthermore, in view of Luzin–Privalov boundary uniqueness theorem, the quotient $u/v$ is uniquely defined in (a Nevanlinna domain) $\Omega$.

It can be easily verified that $\mathbb{D} \in ND$ but any domain bounded by any closed polygonal Jordan curve as well as by any non-degenerate ellipse is not in $ND$.

As it was proved in [6], the property $\Omega \in ND$ is equivalent to the following property of conformal mapping $\varphi$ from $\mathbb{D}$ onto $\Omega$: there exist two functions $u_1, v_1 \in H^\infty(\mathbb{C} \setminus \mathbb{D})$ with $v_1 \not\equiv 0$ such that the equality of angular boundary values

$$\varphi(\zeta) = \frac{u_1(\zeta)}{v_1(\zeta)}$$

holds for almost all points $\zeta \in T$, where $u_1(\zeta)$ and $v_1(\zeta)$ are angular boundary values of $u_1, v_1$, evaluated from the domain $\mathbb{C} \setminus \mathbb{D}$. In this case one says that $\varphi$ admits a pseudocontinuation (or pseudocontinuation of Nevanlinna type, because the quotient $u_1/v_1$ belongs to the Nevanlinna class in $\mathbb{C} \setminus \mathbb{D}$).

This property implies the following useful description of Nevanlinna domains (see Theorem 1 in [12]): $\Omega \in ND$ if and only if $\varphi \in K_{\Theta} := H^2 \ominus (\Theta H^2)$ for some inner function $\Theta$ (i.e., $\Theta \in H^\infty(\mathbb{D})$ and $|\Theta(\zeta)| = 1$ for a.e. $\zeta \in T$).

Therefore, if $R$ is a rational functions having their poles outside $\overline{\mathbb{D}}$ and if $R$ is univalent in $\mathbb{D}$, then $R(\mathbb{D}) \in ND$. Of course, the boundary of the domain $R(\mathbb{D})$ is analytic.

Several interesting and important problems arise in connection with the concept of a Nevanlinna domain. One of them is the problem of description of possible regularity (or irregularity) of boundaries of Nevanlinna domains. The following question was posed in [12] and remains open:
Problem 4.3. Do there exist Nevanlinna domains with unrectifiable boundaries?

Notice that several results about regularity of boundaries of Nevanlinna domains were recently obtained in [13] and [1]. It was proved that there exist Nevanlinna domains with $C^1$, but not $C^{1,\alpha}$, $\alpha \in (0,1)$, boundaries as well as that Nevanlinna domains may have 'almost' unrectifiable boundaries. The latter means that there exists such bounded univalent in $D$ function $f$ that $f$ admits a pseudocontinuation and $f' \notin H^p$ for any $p > 1$ (as usual, $H^p$ stands for the Hardy space in $D$).

Let us show how Theorem 1.2 suggests that the answer to the question stated in Problem 4.3 is positive. By a contour we will mean the boundary of some Jordan domain (not necessarily rectifiable) and by Nevanlinna contours we mean boundaries of Jordan Nevanlinna domains.

It is not difficult to prove that analytic Nevanlinna contours are dense (in the sense of Hausdorff metric) in the set of all contours in $C$, so that in any neighborhood of an arbitrary contour in $C$ there exists an analytic Nevanlinna contour. Indeed, let $G$ be a Jordan domain in $C$ and let $f$ be some conformal mapping from $D$ onto $G$. Approximating $f$ by appropriate univalent in $D$ polynomial (which is clearly possible) we obtain (in view of the aforesaid) some analytic Nevanlinna contour lying in a given $\varepsilon$-neighborhood of $\partial G$. It is clear that the degrees $n$ of the corresponding polynomials should grow to $\infty$ whenever $\varepsilon \to 0$. In view of Theorem 1.2 the lengths of boundaries of corresponding analytic Nevanlinna domains may grow at least as $n^\gamma$ when $n \to \infty$. This observation supports the conjecture that Nevanlinna domains with unrectifiable boundaries do exist.

The observation that the lengths of the boundaries of rational images of the unit disk may grow by a power low with the degree of the corresponding mapping function has one more interesting interpretation related to the concept of a quadrature domain.

Let us briefly recall this notion. For a bounded domain $\Omega$ let $A^2(\Omega)$ be the standard Bergman space in $\Omega$ (it means that $A^2(\Omega)$ consists of all holomorphic functions $f$ in $\Omega$ such that $|f|^2$ is integrable with respect to the planar Lebesgue measure in $\Omega$). The following definition may be found in many sources (see, for instance, [15] and references therein).

Definition 4.4. A bounded domain $\Omega$ is called a (classical) quadrature domain if there exist a finite set of points $\{z_1, \ldots, z_k\} \subset \Omega$ and a set of complex numbers $\{a_{js}: j = 1, \ldots, k; s = 1, \ldots, n_j\}$ (where $k, n_1, \ldots, n_k$ are some positive integers and $a_{jn_{j-1}} \neq 0$) such that the equality

$$\int_\Omega f(z) \, dxdy = \sum_{j=1}^{k} \sum_{s=0}^{n_j-1} a_{js} f^{(s)}(z_j) \quad (4.3)$$

is satisfied for every function $f \in A^2(\Omega)$.

Equality (4.3) is traditionally called a quadrature identity and the number $n = \sum_{j=1}^{k} n_j$ is the order of this quadrature identity. It is also appropriate to refer to the points $\{z_1, \ldots, z_k\}$ as to the nodes of the quadrature identity (4.3).

Let now $R \in RH_n$, $R(0) = 0$, and let $a_1, \ldots, a_k \in \mathbb{C} \setminus \mathbb{D}$ be all poles of $R$ of multiplicities $n_1, \ldots, n_k$. As it was shown in [7, Chap. 14], the domain $R(D)$
is in this case a quadrature domain and the set of nodes of the corresponding quadrature identity is $1/\pi_1, \ldots, 1/\pi_k$. Moreover, any quadrature domain with analytic boundary is of this form.

In view of Theorem 1.2 the lengths of the boundaries of quadrature domains may grow by a power law $k^\gamma$ with the order $k$ of the corresponding quadrature identity.

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