UNIVERSAL CARTAN-LIE ALGEBROID OF AN ANCHORED BUNDLE WITH CONNECTION AND COMPATIBLE GEOMETRIES

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Abstract. Consider an anchored bundle \((E, \rho)\), i.e. a vector bundle \(E \rightarrow M\) equipped with a bundle map \(\rho: E \rightarrow TM\) covering the identity. M. Kapranov showed in the context of Lie-Rinehard algebras that there exists an extension of this anchored bundle to an infinite rank universal free Lie algebroid \(FR(E) \supset E\). We adapt his construction to the case of an anchored bundle equipped with an arbitrary connection, \((E, \nabla)\), and show that it gives rise to a unique connection \(\tilde{\nabla}\) on \(FR(E)\) which is compatible with its Lie algebroid structure, thus turning \((FR(E), \tilde{\nabla})\) into a Cartan-Lie algebroid. Moreover, this construction is universal: any connection-preserving vector bundle morphism from \((E, \nabla)\) to a Cartan-Lie Algebroid \((A, \bar{\nabla})\) factors through a unique Cartan-Lie algebroid morphism from \((FR(E), \tilde{\nabla})\) to \((A, \bar{\nabla})\).

Suppose that, in addition, \(M\) is equipped with a geometrical structure defined by some tensor field \(t\) which is compatible with \((E, \rho, \nabla)\) in the sense of being annihilated by a natural \(E\)-connection that one can associate to these data. For example, for a Riemannian base \((M, g)\) of an involutive anchored bundle \((E, \rho)\), this condition implies that \(M\) carries a Riemannian foliation. It is shown that every \(E\)-compatible tensor field \(t\) becomes invariant with respect to the Lie algebroid representation associated canonically to the Cartan-Lie algebroid \((FR(E), \tilde{\nabla})\).

Key words and phrases: Universal Lie algebra, Lie algebroids, Cartan connections, Riemannian foliations.

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1. Introduction

Every vector space gives naturally rise to a free infinite-dimensional Lie algebra. Applying the same strategy to an anchored vector bundle

\[ E \xrightarrow{\rho} TM \]

needs some more care due to compatibility with the anchor map \( \rho \), which, as a simple consequence of the Lie algebroid axioms, is required to become a morphism of the brackets. This implies in particular that in general the image of the anchor map will increase within this process. It is shown by M. Kapranov in [3] that any anchored module, a module over a commutative algebra together with a morphism of modules with values in the module of derivations of the algebra, gives rise in a canonical way to a free infinite-dimensional Lie-Rinehart algebra.

A free Lie-Rinehart algebra admits a natural filtration the associated graded algebra to which is the free Lie algebra in the category of modules over the same algebra generated by this module. We apply the construction of Kapranov to the category of smooth real manifolds—the original paper operates with Lie-Rinehart algebras over arbitrary ground fields—and call the resulting Lie algebroid \( FR(E) \rightarrow M \).

The main purpose of this article is, however, to extend this relation between an anchored bundle \( E \) and its free Lie algebroid \( FR(E) \) to the lifting of particular additional structures from \( E \) to \( FR(E) \) such that appropriate compatibility conditions are satisfied. For the case of a vector bundle connection, e.g., there is no natural compatibility condition to be required if the vector bundle is merely an anchored bundle; however, if it is a Lie algebroid, this changes: let \( A \rightarrow M \) be a Lie algebroid and \( \nabla \) a connection on \( A \). Any connection on \( A \) gives rise to a splitting \( \iota_{\nabla} \) of the natural projection map \( J^1(A) \rightarrow A \), where \( J^1(A) \) is the 1-jet bundle of sections of \( A \). On the other hand, \( J^1(A) \) carries a natural Lie algebroid structure itself, induced from the one on \( A \). The compatibility consists of asking that \( \iota_{\nabla} : A \rightarrow J^1(A) \) is a Lie algebroid morphism [1], in which case we call the connection a Cartan connection and the couple \( (A, \nabla) \) a Cartan-Lie algebroid. This compatibility condition can be re-expressed [4] as the vanishing of the following tensor [5]

\[ S := 2\text{Alt} \langle \rho, F_{\nabla} \rangle + \nabla (A^T), \]

where \( F_{\nabla} \in \Gamma(A^* \otimes A \otimes \Lambda^2 T^* M) \) is the curvature of \( \nabla \), the anchor is considered as a section \( \rho \in \Gamma(A^* \otimes TM) \), so that the contraction and skew-symmetrisation are defined in an obvious way, and \( A^T \) is the \( A \)-torsion of the simple \( A \)-connection \( A\nabla \) on \( A \) defined by \( A\nabla_s(s') := \nabla_{\rho(s)} s' \) for all \( s, s' \in \Gamma(A) \).

Theorem [1] proven in this paper, is a refinement of the above-mentioned result of Kapranov: given any anchored bundle \( E \) equipped with an arbitrary connection \( \nabla \) there is a unique Cartan connection \( \tilde{\nabla} \) on the corresponding free Lie algebroid \( FR(E) \) which extends the one on \( E \subset FR(E) \). It is interesting to see that it is precisely the compatibility condition \( S = 0 \) which fixes the extension to all of \( FR(E) \) uniquely. We call \( (FR(E), \tilde{\nabla}) \)

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1 A Lie-Rinehart algebra is an algebraic counterpart of a Lie algebroid, cf. [7].
the free Cartan-Lie algebroid generated by the anchored bundle with connection \((E, \nabla)\). \((FR(E), \bar{\nabla})\) has a universality property, moreover, which we will specify further below. We mention as an aside that albeit we deal only with smooth manifolds in this paper, a purely algebraic version of this theorem in the spirit of \[3\] is quite obvious.

For an anchored bundle with connection \((E, \nabla)\) there is a natural compatibility with any tensor field \(t\) defined over its base \(M\): define the \(E\)-connection \(E\nabla\) when acting on vector fields \(v \in \Gamma(TM)\) by means of \(E\nabla_s v := [\rho(s), v] - \rho(\nabla_v s)\) for all \(s \in \Gamma(E)\) and extend this canonically to all tensor fields over \(M\). It is natural to ask that \(t\) should be annihilated by this \(E\)-derivative:

\[(2)\]
\[E\nabla t = 0.\]

The meaning of this condition becomes clearer with an example: suppose the image of the anchor map is involutive, \([\rho(\Gamma(E)), \rho(\Gamma(E))] \subset \rho(\Gamma(E))\), then \(\rho(\Gamma(E))\) defines a singular foliation on \(M\). If \(M\) is equipped with a metric \(g\) satisfying \(E\nabla g = 0\), then this singular foliation is Riemannian, and in particular transversally invariant with respect to the foliation. Similar statements hold true for other geometrical structures defined by means of a tensor field \(t\) satisfying Equation \(\text{(2)}\). We note in parenthesis, if \(E\) carries in addition a Lie algebroid structure and the connection \(\nabla\) is compatible with it in the sense of \(S = 0\), then \(E\nabla\) as defined above provides an honest Lie algebroid representation on \(TM\), \(T^*M\), and its tensor powers, and the compatibility with \(t\) then simply implies that this tensor is invariant under this canonical representation.

The second result of the present paper, formulated in some generalisation in Proposition \[4\] below, is that geometrical structures \(t\) on \(M\) which are compatible with the anchored bundle with connection \((E, \nabla)\) in the sense of Equation \(\text{(2)}\) remain compatible also with respect to \((FR(E), \bar{\nabla})\), i.e. they become invariant with respect to the canonical representation of the universal free Cartan-Lie algebroid \((FR(E), \bar{\nabla})\) on \(TM\). Moreover, this construction is universal: every connection-preserving morphism from \((E, \nabla, t)\) to a \(t\)-compatible Cartan-Lie algebroid \((A, \nabla, t)\) such that the base map is the identity factors through this free Cartan Killing Lie algebroid.

Although it would be desirable to find conditions under which the free (Cartan-)Lie algebroid admits a finite-dimensional reduction, for the moment we leave this problem open. A necessary condition for such a reduction is that the—unmodified finite dimensional—base \(M\) of \(FR(E)\) carries a singular foliation: in the infinite rank setting, involutivity of the image of \(\rho_{FR(E)}\) is not sufficient for its integrability. In addition, even if \(FR(E)\) admits a finite rank reduction, there are in general additional obstructions for the Cartan structure to reduce to the quotient Lie algebroid, since not every finite rank Lie algebroid even admits a compatible connection.

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\[2\]We refer the reader to \[4\] for proofs and further details about the statements in this paragraph.
2. Anchored bundles and free Cartan-Lie algebroids

Let us denote by $\text{Anch}_c(M)$ the category whose objects are anchored bundles with connections and morphisms are connection-preserving bundle morphisms commuting with the anchor maps. Let $\text{CLie}(M)$ be the category of Cartan-Lie algebroids over $M$. Every Cartan-Lie algebroid is an anchored bundle and every connection-preserving Lie algebroid morphism is a morphism of the underlying anchored bundle structures, thus there is a natural forgetful functor $\text{CLie}(M) \to \text{Anch}_c(M)$.

(3)

Theorem 1. The functor (3) admits a left-adjoint functor

(4) $F_R : \text{Anch}_c(M) \to \text{CLie}(M)$

whose value at an anchored bundle with connection $(E, \rho, \nabla)$ is a Lie algebroid $F_R(E)$ together with a Cartan connection and an embedding of anchored bundles $i : E \to F_R(E)$, called the free Lie algebroid generated by $E$. Thus we have a natural isomorphism

(5) $\text{Hom}_{\text{CLie}(M)}(F_R(E), A) = \text{Hom}_{\text{Anch}_c(M)}(E, A)$

for every Cartan-Lie algebroid $A$.

In other words, for every connection-preserving morphism of anchored bundles $\phi : E \to A$ there exists a unique Cartan-Lie algebroid morphism $\tilde{\phi} : F_R(E) \to A$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
E & \xrightarrow{i} & F_R(E) \\
\downarrow{\phi} & & \downarrow{\tilde{\phi}} \\
A & &
\end{array}
\]

Proof. The proof will consist of several consecutive steps. First, we construct the free almost Lie algebroid $F_R^{\text{alm}}(E)$ generated by an anchored bundle $E$. This follows by the same method as in [3]. By an almost Lie algebroid we shall mean an anchored bundle equipped with a skew-symmetric bilinear operation satisfying the Leibniz rule with respect to the anchor map $\rho$, and $\rho$ is, moreover, a morphism of brackets.

We start with $F L^{\text{alm}}(E/\mathbb{R})$, the free almost Lie $\mathbb{R}$-algebra generated by the real vector space of sections of $E$, along with $F L^{\text{alm}}(E)$, the bundle of free almost Lie algebras generated by $E$ as a bundle over $M$. Recall that an almost Lie algebra is like a Lie algebra except that the bracket operation does not necessarily respect the Jacobi identity. Both $F L^{\text{alm}}(E/\mathbb{R})$ and $F L^{\text{alm}}(E)$ are naturally graded, such that the degree $d$ factors of $F L^{\text{alm}}(E/\mathbb{R})$ and $\Gamma(F L^{\text{alm}}(E))$ are spanned by brackets involving exactly $d$ elements. It is easily seen that all homogeneous factors of $\Gamma(F L^{\text{alm}}(E))$ are finite-rank projective modules, hence we obtain the required grading of $F L^{\text{alm}}(E)$ as a vector bundle whose degree $d$ factors are finite-dimensional vector bundles, so that the fiber at $x \in M$ is naturally isomorphic to the free almost Lie algebra generated by $E_x$. The free almost
Lie algebroid $FR^{alm}(E)$, which we now want to construct, is the union of an increasing sequence of anchored finite-rank bundles

$$FR_{\leq 1}^{alm}(E) \subset FR_{\leq 2}^{alm}(E) \subset \ldots,$$

which are defined inductively starting from $FR_{\leq 0}^{alm}(E) = E$ as follows: suppose $FR_{\leq d}^{alm}(E)$ is constructed as an anchored bundle with the anchor $\rho_d$ together with a surjective homomorphism of real vector spaces

$$FL_{\leq d}^{alm}(E/\mathbb{R}) \equiv \bigoplus_{i=1}^{d} FL_i^{alm}(E/\mathbb{R}) \xrightarrow{q_d} \Gamma(FR_{\leq d}^{alm}(E)),$$

where $q_1$ is the identity map. Then we define $FR_{\leq d+1}^{alm}(E)$ as an anchored bundle whose space of sections is the quotient of $FL_{d+1}^{alm}(E/\mathbb{R})$ by the following relations:

$$[s, r] = 0, \quad s \in \Gamma(E), \quad r \in \text{Ker}(q_d),$$

$$[fs, s'] - [s, fs'] = \rho(s)(f)s' - \rho_d(s')(fs), \quad s \in \Gamma(E), \quad s' \in \Gamma(FR_{\leq d}^{alm}(E)).$$

The bracket on $FL_{\leq d}^{alm}(E/\mathbb{R})$ descends to a bracket on smooth sections of $FR_{\leq d}^{alm}(E)$. The anchor map is uniquely determined by requiring the morphism property. The multiplication on smooth functions is given by the formula

$$f[s, s'] = [fs, s'] + \rho_d(s')(fs) = [s, fs'] - \rho_d(s)(fs'),$$

where $s$ and $s'$ are arbitrary sections of $FR_{\leq d}^{alm}(E)$ and $FR_{\leq d'}^{alm}(E)$, respectively. The filtration $\{FR_{\leq d}^{alm}(E)\}$ makes $FR_{\leq d}^{alm}(E)$ into a filtered almost Lie algebroid, and the associated graded almost Lie algebroid is isomorphic to $FL_{\leq d}^{alm}(E)$ with trivial anchor map.

It is worth mentioning that in the end the image of the anchor becomes involutive and although the original anchored bundle does not necessarily carry a (singular) foliation, one now obtains an involutive (singular) tangent distribution which contains the image of $\rho$ of the original $E$. If it were just for this involutive tangent distribution, it could have been obtained equally, and more easily, by completing $\rho(E)$ by means of iterated Lie brackets of $\rho(\Gamma(E))$.

Now we extend the connection on $E$ to the free almost Lie algebroid obtained above. For any $s, s' \in \Gamma(E)$ we claim

$$\nabla[s, s'] = \mathcal{L}_s(\nabla s') - \mathcal{L}_{s'}(\nabla s) - \nabla_{\rho(\nabla s)}s' + \nabla_{\rho(\nabla s')}s,$$

where $\mathcal{L}_s$ acts on $\Gamma(T^*M \otimes E) \ni \omega \otimes s'$ by means of

$$\mathcal{L}_s(\omega \otimes s') := \mathcal{L}_{\rho(s)}\omega \otimes s' + \omega \otimes [s, s'],$$

where in the last equation $\mathcal{L}$ denotes the standard Lie derivative. The expression (7) is well-defined as from its definition it follows that

$$\nabla((f[s, s'] - \rho(s)(fs') - f[s, s'])) = 0$$

for any smooth function $f$. This gives rise to the connection on $FR_{\leq 2}^{alm}(E)$ for $d = 2$. Now we proceed analogously by induction for all $d$. Let us notice that in each step we automatically obtain $S = 0$, where $S$ is defined by the same formula (7) as for a Lie algebroid. This becomes obvious by rewriting (7) $S$ according to the following formula: $S(s, s') = \mathcal{L}_s(\nabla s') - \mathcal{L}_{s'}(\nabla s) - \nabla_{\rho(\nabla s)}s' + \nabla_{\rho(\nabla s')}s - \nabla[s, s']$, cf. also (7).
The last task is to define a Cartan structure on the free almost Lie algebroid and finally show that it descends to the associated free Lie algebroid.

Given an almost Lie algebroid $L$, there is a unique almost Lie algebroid structure on $J^1(L)$ compatible with prolongations $j_1: \Gamma(L) \to J^1(L)$, namely which satisfies $\rho(j_1(s)) = \rho(s)$ as well as

$$[j_1(s), j_1(s')] = j_1([s, s']) \quad (9)$$

for every $s, s' \in \Gamma(L)$. The notion of a Cartan connection along with the formula for the compatibility tensor $\Omega$ does not need the bracket to obey the Jacobi identity. Thus starting with a connection $\nabla$ which satisfies $S = 0$, we obtain a morphism of almost Lie algebroids $L \to J^1(L)$ which is determined by the corresponding splitting $\iota_\nabla: L \to J^1(L)$ defined by

$$\iota_\nabla = j_1(s) + \nabla(s) \quad (10)$$

In this equation, $\nabla(s)$ is viewed as a section of $J^1(L)$ by means of Bott’s exact sequence $0 \to T^*M \otimes L \to J^1(L) \to L \to 0$.

The Jacobiator, which is a measure of the failure of the bracket to satisfy the Jacobi identity, is defined as $\text{Jac}(s_1, s_2, s_3) = [s_1, [s_2, s_3]] + \text{cycl}(s_1, s_2, s_3)$ for every triple of sections of $L$. From the definition of an almost Lie algebroid, it follows that $\text{Jac}$ is totally anti-symmetric and $C^\infty(M)$-linear in all its arguments. Since $\iota_\nabla$ is in particular a morphism of the brackets, one has

$$\iota_\nabla \circ \text{Jac} = \text{Jac} \circ \iota_\nabla \otimes \iota_\nabla \otimes \iota_\nabla \quad (11)$$

**Lemma 1.** $\text{Jac}$ is a covariantly constant map.

**Proof.** From $(9)$ we obtain

$$\text{Jac} (j_1(s_1), j_1(s_2), j_1(s_3)) = j_1 (\text{Jac} (s_1, s_2, s_3)) \quad (12)$$

for all $s_1, s_2, s_3 \in \Gamma(L)$. After establishing that

$$[j_1(s), \omega' \otimes s'] = \mathcal{L}_s (\omega' \otimes s') \quad (13)$$

and using that $\rho$ is a morphism of the brackets, one finds for all $\omega \in \Omega^1(M)$

$$\text{Jac} (j_1(s_1), j_1(s_2), \omega \otimes s_3)) = \omega \otimes \text{Jac} (s_1, s_2, s_3) \quad (14)$$

From $(13)$ and the fact that $[\omega \otimes s, \omega' \otimes s'] = \iota_{\rho(s')}(\omega' \otimes s) - \iota_{\rho(s)}(\omega' \otimes s)$, we conclude that for all sections $s_1, s_2, s_3$ and 1-forms $\omega_1, \omega_2, \omega_3$

$$\text{Jac} (j_1(s_1), \omega_2 \otimes s_2, \omega_3 \otimes s_3)) = 0 \quad ,$$

$$\text{Jac} (\omega_1 \otimes s_1, \omega_2 \otimes s_2, \omega_3 \otimes s_3) = 0 \quad .$$

So we see that the only non-vanishing term of $(11)$ is of the form $(14)$. To be more explicit, let $b_i$ be a local basis of sections of $L$ and denote the corresponding connection coefficients by $\omega'_i$. Then according to $(11)$ and $(10)$,

$$\nabla \text{Jac}(b_i, b_j, b_k) = \text{Jac}(j_1(b_i), j_2(b_j), \nabla b_k) + \text{cycl}(ijk) = \omega'_k \otimes \text{Jac}(b_i, b_j, b_l) + \text{cycl}(ijk) .$$

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$^3$Alternatively, there is a (somewhat sophisticated) description of the canonical almost Lie algebroid structure on the bundle of $k$-jets of $L$ in terms of supergeometry, see Remark below.
But this equation implies indeed $\nabla(\text{Jac}) = 0$, or, equivalently, that the following diagram is commutative:

$$
\begin{array}{ccc}
\Gamma(L^3) & \xrightarrow{\text{Jac}} & \Gamma(L) \\
\downarrow & & \downarrow \\
\Gamma(T^*M \otimes L^3) & \xrightarrow{\text{id} \otimes \text{Jac}} & \Gamma(T^*M \otimes L)
\end{array}
$$

Here the connection on $L^3$ is extended by the Leibniz rule. This completes the proof of Lemma 1.

**Corollary 1.** Any Cartan connection on an almost Lie algebroid preserves the Jacobi ideal of $L$, i.e. the ideal of sections generated by the image of the Jacobiator, thus it gives rise to a Cartan connection on the quotient Lie algebroid whenever it exists.

Let us come back to the free almost Lie algebroid generated by $E$; it is obvious that the Jacobi ideal of $FR^\text{alm}(E)$, i.e. the ideal generated by the Jacobiator, inherits the filtration from the free almost Lie algebroid, such that the degree $d$ factors are finite-rank modules over the algebra of smooth functions. Hence the quotient of $FR^\text{alm}(E)$ by the Jacobi ideal is a Lie algebroid, which we denote by $FR(E)$. In [3] it is called the free Lie algebroid generated by $E$. By Lemma 1 and Corollary 1, we obtain the unique Cartan connection on $FR(E)$ which is compatible with the inclusion $E \hookrightarrow FR(E)$. We leave it to the reader to verify functorial properties of this construction.

**Remark 1.** Let $L$ be an almost Lie algebroid. Consider $L[1] \to M$ as a graded superbundle with the degree 1 odd fibers. In the way similar to the Lie algebroid case [8], an almost Lie algebroid structure is in one-to-one correspondence with a degree 1 vector field $Q$ on $L[1]$, defined by the Cartan’s formula, such that $Q^2$ commutes with all smooth functions on the base $M$. However, in contrast to Lie algebroids, the odd vector field $Q$ is not necessarily homological, as $Q^2 = 0$ is equivalent to the Jacobi identity for the almost Lie algebroid structure on $L$. Now the canonical prolongation of $Q$ to the total space of $J^k(L[1]) = J^k(L)[1]$ determines an almost Lie algebroid structure on the space of $k$-jets of $L$ compatible with the given one on $L$.

3. **Compatible tensor fields on the base manifold**

For every anchored bundle $E$ equipped with a vector bundle connection $\nabla$, there is a natural compatibility of $(E, \rho, \nabla)$ with any tensor field $t$ defined over the base $M$. Define $E\nabla$ by $E\nabla_s v = \mathcal{L}_\rho(s)v - \rho(\nabla v s)$ for arbitrary sections $s \in \Gamma(E)$ when acting on vector fields $v$ and extend it by duality and the Leibniz rule to all tensors: so, e.g., for 1-forms $\omega$, one obtains $E\nabla_s \omega = \mathcal{L}_\rho(s)\omega + t_{\rho(\nabla(s))}\omega$, where, by definition, $t_{\omega \otimes v} \omega = (t_v \omega)\omega'$. Then the compatibility of the tensor $t$ with the anchored bundle with connection is provided by the condition (2). Somewhat more generally:

**Definition 1.** Let $(E, \rho)$ be an anchored bundle on $M$ and $\nabla^1, \ldots, \nabla^m$ be vector bundle connections on $E$ which give rise to $E\nabla^1, \ldots, E\nabla^m$ on tensors on $M$ as defined above for a

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4*For a motivation and consequences of the compatibility condition see [4].
single connection \( \nabla \). A tensor \( t \) on \( \mathcal{M} \) of type \((r, s)\) with \( r + s = m \) is called compatible with \((\mathcal{E}, \nabla^1, \ldots, \nabla^m)\) if \( E\nabla^\text{comb}(t) = 0 \), where \( E\nabla^\text{comb} = E\nabla^1 \otimes \text{Id} \otimes \ldots \otimes \text{Id} + \ldots + \text{Id} \otimes E\nabla^m \).

**Remark 2.** In the particular case that \( E \) is an almost Lie algebroid and \( \nabla \) a Cartan connection on it—as defined in the previous subsection—any \( E \)-connection as above is an almost Lie algebroid representation. In other words, its \( E \)-curvature vanishes or, equivalently, \( E\nabla_{[s, s']} = [E\nabla_s, E\nabla_s] \). This then also applies to the combined \( E \)-covariant derivative \( E\nabla^\text{comb} \) above, provided, certainly, that \( \nabla^1, \ldots, \nabla^m \) are Cartan connections for the almost Lie algebroid structure \((\mathcal{E}, \rho, [\cdot, \cdot])\).

**Proposition 1.** Let \((\mathcal{E}, \rho, \nabla^1, \ldots, \nabla^m)\) be an anchored bundle with \( m \) connections and \( t \) be a tensor field on the base \( \mathcal{M} \) of type \((r, s)\) with \( r + s = m \), such that the compatibility condition \( E\nabla^\text{comb}(t) = 0 \) holds true. Let us extend \( \nabla^1, \ldots, \nabla^m \) to the corresponding Cartan connections \( \tilde{\nabla}^1, \ldots, \tilde{\nabla}^m \) on \( FR(\mathcal{E}) \). Then \((FR(\mathcal{E}), \tilde{\nabla}^1, \ldots, \tilde{\nabla}^m)) \) is compatible with \( t \), or, equivalently, \( t \) is invariant with respect to the natural Lie algebroid representation of this \( m \)-fold free Cartan Lie algebroid.

**Proof.** Given that each \( \tilde{\nabla}^i \) is a Cartan connection on \( FR(\mathcal{E}) \), i.e. it respects the Lie algebroid structure in the sense of \( S = 0 \), we can make use of the observation made in Remark 2 it follows that \( FR(\mathcal{E})\tilde{\nabla}^i \) as well as \( FR(\mathcal{E})\tilde{\nabla}^\text{comb} \) are Lie algebroid representations of \( FR(\mathcal{E}) \). Since the free Cartan Lie algebroid is generated by \( E \), i.e. the space of its sections is spanned over smooth functions by all multiple brackets of sections of \( E \), the identity \( E\nabla^\text{comb}(t) = 0 \) for all \( s \in \Gamma(\mathcal{E}) \) implies \( FR(\mathcal{E})\nabla^\text{comb}(t) = 0 \) for all sections \( \xi \) of \( FR(\mathcal{E}) \), which proves the desired property. ■

We add two examples.

**Example 1** (cf. [9, 3.29, example 2]). Consider the trivial rank two bundle \( E \) with fiber generators \( e_1, e_2 \) over the base \( \mathbb{R}^2 \) parametrised by \( x \) and \( y \). Define the anchor map by means of \( \rho(e_1) = \partial_x \) and \( \rho(e_2) = \chi \partial_y \), where the function \( \chi \) is identically zero for \( x \leq 0 \) and equal to \( \exp(-1/x^2) \) for \( x > 0 \). Since the repeated commutators of these vector fields produce higher and higher powers of \( 1/x \) in front of \( \rho(e_2) \), this is not finitely generated. Moreover, we see that for non-positive \( x \) the integral curves are straight horizontal lines, while for strictly positive \( x \) there is only one two-dimensional leaf. Since these two foliations are glued together along \( x = 0 \), however, one does not obtain a singular foliation in the sense of a partition of \( \mathcal{M} \), which is \( \mathbb{R}^2 \) here, into embedded submanifolds. Correspondingly, there can be also no finite rank quotient of \( FR(\mathcal{E}) \), since any finite rank Lie algebroid carries a singular foliation. The situation certainly does not improve, if additional structures are added, like a connection on \( \mathcal{E} \) and a compatible tensor on the base.

**Example 2.** Consider the trivial rank three bundle \( E \) with fiber generators \( e_1, e_2, e_3 \) over the base \( \mathbb{R}^3 \) parametrised by \( x, y, z \). Define \( \rho \) by means of the following equations: \( \rho(e_1) = \partial_x \), \( \rho(e_2) = x \partial_y - y \partial_x \), and \( \rho(e_3) = x \partial_z - z \partial_x \). As any trivial bundle, \( E \) carries a canonical flat connection, \( \nabla(e_i) = 0 \) for all \( i = 1, 2, 3 \). If we equip the base with \( g = dx^2 + dy^2 + dz^2 \), \((E, \rho, \nabla, g)\) are compatible in the sense of Definition 1 with \( \nabla^a := \nabla \) for \( a = 1, 2 \). As is evident from Equation 2, the extension of \( \nabla \) to the connection \( \nabla \) defined on all of the free Cartan-Lie algebroid \( FR(\mathcal{E}) \) is flat as well. And due to Proposition 1, the data \((FR(\mathcal{E}), \nabla, g)\) are compatible, i.e. they form an (infinite-rank) Killing Lie algebroid in the
nomenclature of [4]. This free Cartan-Lie algebroid does permit a finite quotient: factoring $FR(E)$ by the ideal generated by triple brackets, $FR(E)_{\geq 3}$, one obtains the action Killing Lie algebroid $(g \times \mathbb{R}^3, \nabla^{can}, g)$, where $g$ is the 6-dimensional isometry Lie algebra of $g$ and $\nabla^{can}$ is the canonical flat connection on the action Lie algebroid.

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