Abstract. Let \( G \) be a finite solvable group. Then \( G \) always has a useful presentation, which we call a “long presentation”. Using a “long presentation” of \( G \), we present an inductive method of constructing the irreducible representations of \( G \) over \( \mathbb{C} \) and computing the primitive central idempotents of the complex group algebra \( \mathbb{C}[G] \). For a finite abelian group, we present a systematic method of constructing the irreducible representations over a field of characteristic either 0 or prime to order of the group and also a systematic method of computing the primitive central idempotents of the semisimple abelian group algebra.

1. Introduction

Let \( G \) be a finite group. Let \( F \) be a field of characteristic either 0 or prime to the order of \( G \). We denote the order of \( G \) by \(|G|\). Let \( \overline{F} \) be the algebraic closure of \( F \). It is well known fact that the primitive central idempotents of the group algebra \( F[G] \) are all elements of the form

\[
\frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g,
\]

where \( \chi \) is an irreducible \( \overline{F} \)-character of \( G \). Using Galois decent, one can obtain all the primitive central idempotents of the semisimple group algebra \( F[G] \).

The known methods to compute the character table of a finite group are tasks of exponential growth with respect to the order of the group. In view of the computational difficulty in this approach, mathematicians are interested in the problem of finding character-free methods for computing the primitive central idempotents of \( F[G] \). In last few years, the problem has been solved for certain classes of groups and for some specific fields.

For a finite abelian group \( G \), an explicit description of the primitive central idempotents of the rational group algebra and Wedderburn decomposition of \( \mathbb{Q}[G] \) was considered by several authors (see [3], [4]). In this paper, for a finite abelian group \( G \) and a field \( F \) of characteristic either 0 or prime to \(|G|\), we present a systematic method of constructing the irreducible \( F \)-representations of \( G \) and a character-free method of computing the primitive central idempotents of the group algebra \( F[G] \).

Key words and phrases. Solvable group, Long presentation, Irreducible representation, Group algebra, Primitive central idempotent.
Among the finite groups, the finite solvable groups surely cover a very large ground. Note that

1. For almost all natural numbers $n$, all finite groups of order $n$ are solvable.
2. Given a natural number $n$, almost all the groups of order $n$ are solvable.

Yet, the ground that is not covered, concentrated on a very small patch of “measure zero”, also contains many beautiful things.

A solvable group always has a useful presentation, which we call a “long presentation”. In this paper, for a finite solvable group $G$, using a “long presentation” of $G$ and the associated “long system of generators”, we inductively construct the irreducible representations of $G$ over $\mathbb{C}$ and also compute the primitive central idempotents of the group algebra $\mathbb{C}[G]$. More precisely, our main concern is avoiding the use of characters by a direct computation of the primitive central idempotents of $\mathbb{C}[G]$ in terms of a “long system of generators” associated with a fixed maximal subnormal series of $G$, which refines it’s derived series.

2. Long Presentation

For a finite solvable group $G$, we can always find a refinement of its derived series, which is a subnormal series of $G$ such that each successive quotients are cyclic groups of prime order. So, for a finite solvable group $G$ of order $N = p_1p_2\ldots p_n$, where $p_i$’s are primes, there is a subnormal series: $\langle e \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ such that for each $i = 1, 2, \ldots, n$, $G_i/G_{i-1} \approx C_{p_i}$, a cyclic group of order $p_i$ after a suitable ordering of $\{p_1, p_2, \ldots, p_n\}$. One can choose an element $x_i$ in $G_i$ s.t. $x_i$ is a $p_i$-element of smallest order $p_i^{n_i}$, and $x_iG_{i-1}$ generates $G_i/G_{i-1}$.

Note that such an element always exists in $G$. Then $G$ has a presentation:

$$\langle x_1, x_2, \ldots, x_n | x_i^{p_i} = 1, x_i^{p_i} = w_i(x_1, \ldots, x_{i-1}), x_i^{-1}x_jx_i = w_{ij}(x_1, \ldots, x_{i-1}) \text{ for } j < i \rangle,$$

where $w_i$ and $w_{ij}$ are certain words in $x_1, x_2, \ldots, x_{i-1}$. We call such a presentation, a long presentation of $G$, and $\{x_1, x_2, \ldots, x_n\}$ is called the associated long system of generators. A remarkable thing is that each element of $G$ can be expressed uniquely as $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$, where $0 \leq a_i < p_i$.

3. Representations and Primitive Central Idempotents of a Cyclic Group over an Algebraically Closed Field

Let $G = C_N$ be a cyclic group of order $N$. Let $N = \prod_{p|N} p^{n_p}$, be its factorisation into primes. Let $G_p = \{y \in G | y^{p^{n_p}} = 1\}$. Then $G = \prod_{p|N} G_p$. Let $y_p$ be a generator of $G_p$. Then $y = \prod_{p|N} y_p$ is a generator of $G$. Choose a system of long generators $G$ in each $G_p$. A system of long generators of $G$ is obtained by various products of long generators in each $G_p$.

**Proposition 3.1.** Let $G$ be a cyclic group of order $N$. Let $F$ be an algebraically closed field of characteristic 0 or prime to $N$. The irreducible representations of $G$ are the tensor products of the irreducible representations of $G_p$’s for $p|N$. 
For simplicity we denote this expression by $\zeta$. Clearly $\zeta$ is an $N^{th}$ root of unity. Clearly $\zeta = \Pi_{p}\zeta_{p}$ and $\rho = \Pi_{p}\rho_{p}$. Hence, $\rho$ is the tensor products of the irreducible representations of $G_{p}$'s for $p|N$.

\textbf{Proposition 3.2.} Let $G$ be a cyclic group of order $N$. Let $F$ be a field of characteristic $0$ or prime to $|G|$. Let $\rho$ be an irreducible representation of $G$ over $F$. Let $\rho_{p}$ be an irreducible component of $\rho|_{G_{p}}$, the restriction of $\rho$ to $G_{p}$. Then the primitive central idempotent corresponding to $\rho$ is the product of primitive central idempotents of $\rho_{p}$'s.

\textbf{Proof.} Let $\rho$ be an irreducible representation $\rho$ of $G$ is given by $y \mapsto \zeta$, where $\zeta$ is an $N^{th}$ root of unity. The primitive central idempotent corresponding to $\rho$ in $F[G]$ is:

\begin{equation}
\epsilon_{\rho} = \frac{1}{N} \sum_{i=0}^{N-1} \zeta^{-i} y^{i}.
\end{equation}

For simplicity we denote this expression by $\epsilon_{\zeta}$. The primitive central idempotent of $\rho_{p}$ is obtained by replacing $N$ by $p^{n_{p}}$ in $(\ast)$, and $y$ by $y_{p}$, Clearly the $\epsilon_{\rho} = \epsilon_{\zeta}$ factors as $\prod_{p|N} \epsilon_{\zeta_{p}}$. This completes the proof.

\textbf{Theorem 3.3.} Let $G = C_{p^{n}} = \langle x_{1}, \ldots, x_{n} \mid x_{1}^{p^{n}} = 1, x_{2}^{p} = x_{1}, \ldots, x_{n}^{p} = x_{n-1} \rangle$ be a long presentation of $G$. Let $F$ be an algebraically closed field with characteristic $0$ or prime to $p$. Then every primitive central idempotent of $F[G]$ can be expressed as

\[\epsilon_{\zeta_{1}} x_{1} \epsilon_{\zeta_{2}} x_{2} \cdots \epsilon_{\zeta_{n}} x_{n}\]

with $\zeta_{n}$ is a $p^{n}$th root of unity in $F$ and $\zeta_{1}^{p} = \zeta_{1+1}, \ldots, \zeta_{2}^{p} = \zeta_{2}$, where $e_{X} = (1 + X + \cdots + X^{p-1})/p$ and $X$ is an indeterminate.

\textbf{Proof.} Let $\rho$ be an irreducible representation $\rho$ of $G$, and is given by $x_{n} \mapsto \zeta$, where $\zeta$ is an $N^{th}$ root of unity. By Theorem 3.1, $\rho$ is isomorphic to the tensor product of $\rho_{p}$'s. The primitive central idempotent corresponding to $\rho$ in $F[G]$ is:

\[\epsilon_{\rho} = \frac{1}{N} \sum_{i=0}^{N-1} \zeta^{-i} x_{n}^{i}.\]

For simplicity we denote this expression by $\epsilon_{\zeta}$.

Let $N = mn$, with $m > 1, n > 1$ be a factorisation in natural numbers. Then

\[1 + X + \cdots + X^{mn-1} = (1 + X + X^{2} + \cdots + X^{m-1})(1 + X^{m} + X^{2m} + \cdots + X^{(n-1)m}).\]

We apply this repeatedly to the case: $N = p^{n}$. Write $e_{X} = 1 + X + \cdots + X^{p-1}/p$, and let $x_{1}, x_{2}, \ldots, x_{n}$ be as in the statement, satisfy $x_{1}^{p} = 1, x_{2}^{p} = x_{1}, x_{3}^{p} = x_{2}, \ldots, x_{n}^{p} = x_{n-1}$. Then writing $x_{n} = x$, we have $1 + x + x^{2} + \cdots + x^{p^{n}-1}/p^{n} = e_{x_{1}} e_{x_{2}} \cdots e_{x_{n}}$. 
Replacing \( x \) by \( \zeta^{-1}x \), we obtain the second universal factorisation of
\[
e^{\zeta} = \prod_{i=0}^{n-1} e^{\zeta_{i+1}^{-1}x_1}e^{\zeta_{i+2}^{-1}x_2}...e^{\zeta_{n+1}^{-1}x_n},
\]
where \( \zeta_n = \zeta, \zeta_{n-1} = \zeta^p, \zeta_{n-2} = \zeta^{p^2}, \ldots, \zeta_1 = \zeta_2 = \zeta^{p^{n-1}}. \) This completes the proof. \( \square \)

**Remark 3.4.** The expression of every primitive central idempotents of \( F[C_{pn}] \) has \( p^n \) terms. Such an expression is a product of \( n \) factors, and each factor containing \( p \) terms. So each primitive central idempotent in \( F[C_{pn}] \) has an expression containing \( pn \) terms.

### 4. Representations and Primitive Central idempotents of a Cyclic Group

In this section, we construct the irreducible representations of a cyclic group \( G \) over a field \( F \) of characteristic either 0 or prime to \( |G| \) and compute the primitive central idempotents of \( F[G] \).

#### 4.1. Representations of Cyclic Groups.

**Theorem 4.1.** Let \( G = C_n = \langle x | x^n = 1 \rangle \). Let \( F \) be a field of characteristic 0 or prime to \( n \). Let \( X^n - 1 = \prod_{i=1}^{k} f_i(X) \) be the decomposition into irreducible polynomials over \( F \). Then we have the following.

1. The set of all isomorphism types of irreducible \( F \)-representations of \( G \) is in a bijective correspondence with the set of all monic irreducible factors of \( X^n - 1 \).
2. Let \( \rho_1, \rho_2, \ldots, \rho_k \) be the \( k \) irreducible \( F \)-representations of \( G \). They are defined by \( \rho_i(x) = C_{f_i(x)} \), where \( C_{f_i(x)} \) denotes the companion matrix of \( f_i(X) \) over \( F \).

**Proof.** (1) Consider the map \( \phi : F[X] \to F[G] \) given by: \( f(X) \mapsto f(x) \). As \( \phi \) is a ring epimorphism, \( F[G] \simeq \frac{F[X]}{\langle \ker(\phi) \rangle} \). So, we get
\[
F[G] \simeq \frac{F[X]}{\langle X^n - 1 \rangle} = \frac{F[X]}{\langle \prod_{i=1}^{k} f_i(X) \rangle}.
\]
Since \( F \) is a field of characteristic 0 or prime to \( n \), \( X^n - 1 \) is a separable polynomial with distinct roots. Using Chinese Remainder Theorem, we can write
\[
F[G] \simeq \bigoplus_{i=1}^{k} \frac{F[X]}{\langle f_i(X) \rangle}.
\]
Under this isomorphism, the generator \( x \) is mapped to the element
\[
(X + \langle f_1(X) \rangle, X + \langle f_2(X) \rangle, \ldots, X + \langle f_k(X) \rangle).
\]
For each \( i, 1 \leq i \leq k \), \( f_i(X) \) is an irreducible polynomial, so \( \frac{F[X]}{\langle f_i(X) \rangle} \) is a field. So, \( G \) has \( k \) irreducible representations.
(2) Let $\rho_i$ be the irreducible representation corresponding to the simple component $\frac{F[X]}{\langle f_i(X) \rangle}$. Then $\rho_i(x) : \frac{F[X]}{\langle f_i(X) \rangle} \rightarrow \frac{F[X]}{\langle f_i(X) \rangle}$ is defined by multiplication by $[X] = X + \langle f_i(X) \rangle$. Let $d$ be the degree of $f_i(X)$. Then $\{[1], [X], \ldots, [X^{d-1}]\}$ is an ordered basis of $\frac{F[X]}{\langle f_i(X) \rangle}$, and w.r.t. this basis the matrix for $\rho_i(x)$ is $C_{f_i(X)}$, where $C_{f_i(X)}$ denotes the companion matrix of $f_i(X)$. This completes the proof. 

Theorem 4.2. Let $G = C_n = \langle x \mid x^n = 1 \rangle$. Let $F$ be a field of characteristic either 0 or prime to $n$. Let $\Phi_n(X)$ denote the $n$-th cyclotomic polynomial over $F$. Let $\Phi_n(X) = \prod_{i=1}^k f_i(X)$ be the factorization into irreducible polynomials over $F$. Then we have the following.

1. The set of all isomorphism types of faithful irreducible $F$-representations of $G$ is in a bijective correspondence with the set of all monic irreducible factors of $\Phi_n(X)$.
2. Let $\rho_i$ be the faithful irreducible $F$-representation corresponding to the irreducible factor $f_i(X)$. Then for each $i$, $\rho_i$ is defined by $x \mapsto C_{f_i(X)}$, where $C_{f_i(X)}$ denotes the companion matrix of $f_i(X)$, and the degree of $\rho_i$'s are same.

Proof. (1) Let $\overline{F}$ be the algebraic closure of $F$. Then

$$\overline{F}[G] \simeq \frac{\overline{F}[X]}{(X^n - 1)} \simeq \bigoplus_{i=0}^{n-1} \frac{\overline{F}[X]}{\langle X - \zeta_n^i \rangle},$$

where $\zeta_n$ is a primitive $n^{th}$ root of unity in $\overline{F}$. Notice that the simple components corresponding to the faithful irreducible $\overline{F}$-representations of $G$ are $\frac{\overline{F}[X]}{\langle X - \zeta_n^i \rangle}$, $(i, n) = 1$. Let $\eta_i$ be the representation corresponding to the component $\frac{\overline{F}[X]}{\langle X - \zeta_n^i \rangle}$. Let $\sigma \eta_i(g) = \sigma(\eta_i(g))$, for all $g \in G$ and $\sigma \in \text{Gal}(\overline{F}(\eta_i)/F)$. Then for each $i$, $\bigoplus_{\sigma \in \text{Gal}(\overline{F}(\eta_i)/F)} \sigma \eta_i$ is an irreducible $F$-representation. As for each $i$, $\sigma \eta_i$ is a faithful irreducible $\overline{F}$-representation, $\bigoplus_{\sigma \in \text{Gal}(\overline{F}(\eta_i)/F)} \sigma \eta_i$ is a faithful irreducible $F$-representation. Also for each $i$, $f_i(X)$ is a separable polynomial, which implies $|\text{Gal}(\overline{F}(\eta_i)/F)| = [\overline{F}(\eta_i) : F] = \deg f_i(X)$. So $G$ has at least $k$ faithful irreducible $F$-representations. Let $\rho_1, \rho_2, \ldots, \rho_k$ be those faithful irreducible $F$-representations of $G$. It is clear that for each $i$, $\rho_i$ is determined by the irreducible factor $f_i(X)$ of $\Phi_n(X)$. It remains to show that these are all the faithful $F$-irreducible representations of $G$. If not, let $\rho$ be a faithful irreducible $F$-representation, which is different from $\rho_1, \rho_2, \ldots, \rho_k$. Then $\rho \otimes_F \overline{F}$ is also a faithful $\overline{F}$-representation of $G$, and is the direct sum of some faithful $\overline{F}$-irreducible representations. So, some of the $\eta_i$’s must occur in the decomposition, and which is a contradiction. Thus $\rho_1, \rho_2, \ldots, \rho_k$ are the only faithful irreducible $F$-representations of $G$. This completes the proof. (2) Follows from Theorem 4.1. 

4.2. Primitive Central idempotents of Cyclic Groups. Let $G = \langle x \mid x^n = 1 \rangle$ be a cyclic group of order $n$. Let $F$ be a field of characteristic 0 or prime to $n$. Then
$F[G]$ is isomorphic to $F[X]/\langle X^n - 1 \rangle$. Let $X^n - 1 = \prod_{d|n} \Phi_d(X)$ be the "universal" decomposition into cyclotomic polynomials. For $\mathbb{Q}$, it is the factorization into monic irreducible polynomials. In general, $\Phi_d(X)$ will decompose further. Let
\[ X^n - 1 = (X - 1)f_2(X) \ldots f_k(X), \]
with $f_1(X) = X - 1$, be the factorization into monic irreducible polynomials over $F$. Note that each $f_i(X)$ is a monic irreducible factor of $\Phi_d(X)$ for some $d|n$. Then $F[G]$ is abstractly isomorphic to the direct sum of $F$-algebras $F[X]/\langle f_i(X) \rangle, i = 1, 2, \ldots, k$. For a fixed $i$, let $\zeta_1, \zeta_2, \ldots, \zeta_d$ be the roots of $f_i(X)$ in $\overline{F}$. Then the primitive central idempotent corresponding to $f_i(X)$ is
\[ \sum_{j=1}^k e_{\zeta_j^{-1}X}, \]
where $e_X = (1 + X + \cdots + X^{n-1})/n$ and $X$ is an indeterminate.

By Newton's formulae, power sums can be expressed in terms of elementary symmetric functions, which implies the coefficients of $x, x^2, \ldots, x^{n-1}$ can be expressed in terms of the coefficients of the $f_i(X)$. So, computation of the primitive central idempotents of $F[G]$ reduces to factorization of $X^n - 1$ into irreducible polynomials over $F$.

## 5. Representations and Primitive Central Idempotents of an Abelian Group

Let $A$ be a finite dimensional semisimple $F$-algebra. By Artin-Wedderburn structure theorem on semisimple algebras, we have $A$ is abstractly isomorphic to direct sum of matrix algebras over finite dimensional division algebras over $F$. We define the abelian part of $A$ as the inverse image of direct sum of commutative simple components. We now state the following two propositions without proof.

**Proposition 5.1.** Let $G$ be a finite group. Let $H \leq G$. Let $F$ be a field of characteristic does not divide $|G|$. Let $e_H = \frac{1}{|H|} \sum_{h \in H} h$. Then $e_H$ is an idempotent in $F[G]$. Moreover, if $H \triangleleft G$, then $e_H$ is a central idempotent in $F[G]$.

For the proof (see [4], Lemma 3.3.6).

**Proposition 5.2.** Let $G$ be a finite group. Let $F$ be a field of characteristic 0 or prime to $|G|$. Let $\Delta(G, G') = \sum_{\alpha \in G'} \alpha(g - 1)$, where $\alpha \in F$. Then $\Delta(G, G') = F[G] \Delta(G, G')$, where $F[G] \Delta(G, G')$ is the sum of all commutative simple components of $F[G]$ and $\Delta(G, G')$ sum of all the others.

For the proof (see [4], Proposition 3.6.11).

### 5.1. Representations of abelian groups

Let $G$ be a finite group. Let $F$ be a field of characteristic 0 or prime to $|G|$. Let $\Omega_{G,F}$ be the set of all the inequivalent irreducible $F$-representations of $G$. Let $\Omega^0_G = \{ \rho \in \Omega_{G,F} \mid \text{the irreducible } F\text{-representation } \rho \text{ of } G \text{ s.t. } \text{Im} \rho \text{ is cyclic} \}$. We divide the set $\Omega_{G,F}$ into two parts:

1. The irreducible $F$-representations with image is abelian.
2. The irreducible $F$-representations with image is non-abelian.

Let $\Omega^a_G = \{ \rho \in \Omega_{G,F} \mid \text{the irreducible } F\text{-representation } \rho \text{ of } G \text{ s.t. } \text{Im} \rho \text{ is abelian} \}$. 
Proposition 5.3. $\Omega_G^0 = \Omega_G^{\text{ab}}$.

Proof. Let $\rho$ be an irreducible $F$-representation of $G$ with abelian image. Then $\rho$ factors through an irreducible $F$-representation of $G/G'$. By Proposition 5.2, $\text{Im}\, \rho$ is finite subgroup of a field, implies that $\text{Im}\, \rho$ is cyclic. So, $\Omega_G^{\text{ab}} \subseteq \Omega_G^0$. Clearly, if $\rho \in \Omega_G^0$, then $\text{Im}\, \rho$ is abelian, which implies that $\Omega_G^0 \subseteq \Omega_G^{\text{ab}}$. This completes the proof.

Proposition 5.4. $\Omega_G^0$ parametrizes all the irreducible representation of $G/G'$.

Proof. Let $\rho \in \Omega_G^0$. Then $\rho$ factors through an irreducible $F$-representation of $G/G'$. On the other hand, let $\rho$ be an irreducible $F$-representation of $G/G'$, by Proposition 5.3, $\rho$ induces an irreducible $F$-representation of $G$ with cyclic image. Clearly, this is a bijective correspondence. This completes the proof.

Theorem 5.5. Let $G$ be a finite group of order $N$. For a divisor $d$ of $N$, let $H_d = \{H \leq G|G/H \cong \mathbb{C}_d\}$, and let $\Omega^0_d = \{\text{faithful irreducible } F\text{-representations of } \mathbb{C}_d\}$. Then $\cup_{d|N}(H_d \times \Omega^0_d)$ is in a bijective correspondence with $\Omega^0_G$.

Proof. Let $\rho \in \Omega^0_G$. Then $\rho$ factors through a faithful irreducible $F$-representation of cyclic quotient. So, $\rho$ corresponds to an element in $\cup_{d|N}(H_d \times \Omega^0_d)$. On the other hand, take an element in $\cup_{d|N}(H_d \times \Omega^0_d)$, then corresponding to that there is an element in $\Omega^0_G$. Clearly, this correspondence is bijective. This completes the proof.

Corollary 5.6. Let $G$ be a finite abelian group. Then $\Omega_{G,F} = \Omega_G^0$.

Proof. By Proposition 5.3, $\Omega_G^0 = \Omega_G^{\text{ab}}$. As $G$ is abelian, $\Omega_{G,F} = \Omega_G^{\text{ab}}$, which implies that $\Omega_{G,F} = \Omega_G^0$.

5.2. Primitive central idempotents of abelian group algebras.

Definition 5.7. Let $N \triangleleft G$. Let $\phi : F[G] \mapsto F[G/N]$ be the projection map. Let $\overline{e} \in F[G/N]$, then $e \in F[G]$ is called a lift of the element $\overline{e}$ if $\phi(e) = \overline{e}$, and $e_N \in F[G]$ is called the pull back of $\overline{e}$.

Let $G$ be an abelian group. By Corollary 5.6, we get that every primitive central idempotent of $F[G]$ is pull back of the primitive central idempotent in $F[G/K]$ corresponding to a faithful irreducible representation of $G/K$, where $G/K$ is isomorphic to a cyclic group. Let $G/K = \langle x | x^n = 1 \rangle \cong \mathbb{C}_n$, where $x = xK$. Let $e = \sum_{i=0}^{n-1} a_i x^i$ be a primitive central idempotent in $F[G/K]$ corresponding to a faithful irreducible representation of $G/K$. Then the pull back of $e = \sum_{i=0}^{n-1} a_i x^i$, that is, $e_K \sum_{i=0}^{n-1} a_i x_i$, where $e_K = \frac{1}{|K|} \sum_{k \in K} k$, is a primitive central idempotent of $F[G]$.

Later using long generators, we shall give further factorization of $e_K \sum_{i=0}^{n-1} a_i x_i$. Thus the problem reduces to computing the primitive central idempotents of a cyclic group as we have described in Section 4.
6. Representations of a finite solvable group over \( \mathbb{C} \)

The following theorem is known, we call this theorem index-\( p \) theorem. Using this theorem and a long presentation, one can inductively construct the irreducible representations of a finite solvable group over \( \mathbb{C} \).

**Theorem 6.1** ([2], Theorem 13.52). (Index-\( p \) theorem) Let \( G \) be a group, \( H \) a normal subgroup of index \( p \), \( p \) a prime. Let \( \eta \) be an irreducible representation of \( H \) over \( \mathbb{C} \).

1. If the \( G/H \)-orbit of \( \eta \) is a singleton, then \( \eta \) extends to \( p \) mutually inequivalent representations \( \rho_1, \rho_2, \ldots, \rho_p \) of \( G \).
2. If the \( G/H \)-orbit of \( \eta \) consists of \( p \) points \( \eta = \eta_1, \eta_2, \ldots, \eta_p \), then the induced representations \( \eta_1 \uparrow_G^H, \eta_2 \uparrow_G^H, \ldots, \eta_p \uparrow_G^H \), are equivalent, say \( \rho \) and \( \rho \) is irreducible.

**Corollary 6.2.** Let \( G \) be a solvable group. Then \( G \) has a maximal subnormal series such that the successive quotients are isomorphic to cyclic groups of prime order. The last but one term in the maximal subnormal series is a cyclic group of prime order. So all its irreducible \( F \)-representations are of degree one. So starting with degree one representations of subgroups, we can build all the irreducible representations of \( G \) by the processes of extension and induction.

7. Primitive central idempotents of complex group algebra of a finite solvable group

The following theorem is due to Berman (see [11]). Here, we give an elegant proof of the theorem. For a finite solvable group \( G \), using the following theorem and a long system of generators, one can inductively construct the primitive central idempotents of \( \mathbb{C}[G] \).

**Theorem 7.1.** Let \( G \) be a finite group and \( H \) be a normal subgroup of index \( p \), a prime. Let \( G/H = \langle x \rangle \), for some \( x \) in \( G \). Let \( \overline{C}(x) \) be the conjugacy class sum of \( x \) in \( \mathbb{C}[G] \). Let \( (\eta, W) \) be an irreducible representation of \( H \) over \( \mathbb{C} \) and \( e_\eta \) be its corresponding primitive central idempotent in \( \mathbb{C}[H] \). We distinguish two cases:

1. If \( e_\eta \) is a central idempotent in \( \mathbb{C}[G] \), then \( \eta \) extends to \( p \) distinct irreducible representations \( \rho_0, \rho_1, \ldots, \rho_{p-1} \) (say) of \( G \). Moreover, \( x \) can be chosen s.t. \( (\overline{C}(x))^p e_\eta = \lambda e_\eta \), where \( \lambda \neq 0 \). For each \( i, 0 \leq i \leq p-1 \), let \( e_\rho_i \) be the primitive central idempotent corresponding to the representation \( \rho_i \). Then
   \[
eq \frac{1}{p} \left( 1 + \zeta^i c + \zeta^{2i} c^2 + \cdots + \zeta^{i(p-1)} c^{p-1} \right) e_\eta,\]
   where \( c = \frac{\overline{C}(x)e_\eta}{\sqrt{\lambda}} \) and \( \zeta \) is a primitive \( p \)th root of unity in \( F \). Moreover,
   \[
eq e_\eta = e_\rho_0 + e_\rho_1 + \cdots + e_\rho_{p-1}.
   
2. If \( e_\eta \) is not a central idempotent in \( \mathbb{C}[G] \), then \( \eta \uparrow_{G/H}^G, \eta^x \uparrow_{G/H}^G, \ldots, \eta^{x^{p-1}} \uparrow_{G/H}^G \) are all equivalent to an irreducible representation \( \rho \) (say) of \( G \) over \( \mathbb{C} \) and in this case,
   \[
eq e_\rho = e_\eta + e_\eta^x + \cdots + e_\eta^{x^{p-1}}.
   

Proof. (1) Since \( e_\eta \) is a central idempotent in \( \mathbb{C}[G] \), which implies that \( \eta \cong \eta^x \). By Theorem [6.1], \( \eta \) extends to \( p \) distinct irreducible representations \( \rho_0, \rho_1, \ldots, \rho_{p-1} \) (say) of \( G \). By Frobenius reciprocity theorem, the induced representation \( \eta \uparrow^G_H \cong \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_{p-1} \), which implies that \( \mathbb{C}[G]e_\eta = \bigoplus_{i=0}^{p-1} \mathbb{C}[ho_i] \) and \( e_\eta = \sum_{i=0}^{p-1} e_{\rho_i} \). Note that \( \mathbb{C}[G]e_\eta \) is a ring with identity \( e_\eta \). Notice that \( \mathbb{C}[G]e_\eta \) is the direct sum of \( p \) minimal two-sided ideals \( \mathbb{C}[\rho_i], i = 0, 1, \ldots, p-1 \), and therefore \( \mathbb{C}[G]e_\eta \) contains \( p \) primitive central idempotents.

Since \( G/H = \langle xH \rangle \), for some \( x \in G \), this implies that \( x^p \in H \). Then one can show that \( \mathcal{C}_G(x)^p = x^{p}a \) for some \( a \in \mathbb{C}[H] \). It follows that \( \mathcal{C}_G(x)^p \) is central element in \( \mathbb{C}[H] \). Hence, by Schur’s Lemma,

\[
(\mathcal{C}_G(x)^p e_\eta = \lambda e_\eta, \quad \text{for some } \lambda \in \mathbb{C}.)
\]

Note that \( \lambda \) depends on chosen \( x \).

Claim: The element \( x \in G - H \) can be chosen in such a way that \( \lambda \neq 0 \).

To prove the claim, let \( \rho \) be an extension of \( \eta \), with corresponding character \( \chi_\rho \) and the primitive central idempotent \( e_\rho \). If \( \chi_\rho \) vanishes on \( G - H \), then

\[
e_\rho = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^{-1}) e_\rho = \frac{1}{p |H|} \sum_{h \in H} \chi_\eta(h^{-1}) e_\eta = \frac{1}{p} e_\eta,
\]

and this implies that \( e_\rho \) is not an idempotent, a contradiction. Thus there exists \( x \in G - H \) such that \( \chi_\rho(x) \neq 0 \).

If \( V \) is the representation space corresponding to \( \eta \), then it is also a representation space for the extension \( \rho \). Extend \( \rho : G \to \text{GL}(V) \) linearly to algebra homomorphism \( \mathbb{C}[G] \to \text{End}(V) \), which we still denote by \( \rho \). Similarly, extend \( \eta \) to the algebra homomorphism \( \eta : \mathbb{C}[H] \to \text{End}(V) \). Now by Schur’s Lemma, \( \rho(\mathcal{C}_G(x)) = \mu I \) for some \( \mu \in \mathbb{C} \). Taking traces of both sides, we get

\[
|\mathcal{C}_G(x)| \chi_\rho(x) = \mu \deg \rho.
\]

Since \( \chi_\rho(x) \neq 0 \), we get \( \mu \neq 0 \). Then

\[
\lambda I = \rho(\lambda e_\eta) = \rho((\mathcal{C}_G(x)^p e_\eta) = \rho((\mathcal{C}_G(x)e_\eta)^p) = \mu^p I \neq 0,
\]

hence \( \lambda \neq 0 \). This proves the Claim.

For \( i = 0, 1, \ldots, p-1 \), let

\[
f_i = \frac{1}{p} \left( 1 + \zeta^i \epsilon + \zeta^{2i} \epsilon^2 + \cdots + \zeta^{(p-1)i} \epsilon^{p-1} \right) e_\eta,
\]

where \( c = \frac{\mathcal{C}_G(x)e_\eta}{\eta} \) and \( \zeta \) is a primitive \( p^th \) root of unity in \( \mathbb{C} \). It is clear that all \( f_i \)’s are non zero. Since \( e_\eta \) and \( \mathcal{C}_G(x) \) are central in \( \mathbb{C}[G] \), then all \( f_i \)’s are central in \( \mathbb{C}[G] \). It is clear that

\[(*) \quad c^p e_\eta = e_\eta.
\]
Suppressing the index $i$, let us write
\[ f = \frac{1}{p}(1 + \zeta c + \cdots + \zeta^{p-1}c^{p-1})e_\eta. \]

Then
\[
 cf = \frac{1}{p}(ce_\eta + \zeta^2c^2e_\eta + \cdots + \zeta^{p-2}c^{p-2}e_\eta + \zeta^{p-1}c^{p-1}e_\eta) \\
= \frac{1}{p}(ce_\eta + \zeta^2c^2e_\eta + \cdots + \zeta^{p-2}c^{p-2}e_\eta + \zeta^{p-1}e_\eta) \text{ (by (*)}) \\
= \frac{\zeta^{-1}}{p}(1 + \zeta c + \cdots + \zeta^{p-1}c^{p-1})e_\eta \\
= \zeta^{-1}f.
\]

Thus for $0 \leq i, k < p$, we can deduce from above calculation that
\[(**)
 c^k f_i = \zeta^{-ik} f_i.
\]

Now we show that $f_i f_j = \delta_{i,j} f_j$.

\[
f_i f_j = \frac{1}{p} \left( \sum_{k=0}^{p-1} (\zeta^i c)^k \right) e_\eta f_j \\
= \frac{1}{p} \left( \sum_{k=0}^{p-1} \zeta^{ik} c^k f_j \right) e_\eta \\
= \frac{1}{p} \left( \sum_{k=0}^{p-1} \zeta^{ik} \zeta^{-jk} f_j \right) e_\eta \text{ (by (**))} \\
= \frac{1}{p} \left( \sum_{k=0}^{p-1} \zeta^{(i-j)k} f_j e_\eta \right) \\
= \delta_{i,j} (f_j e_\eta) = \delta_{i,j} f_j.
\]

We now show that $\sum_{i=0}^{p-1} f_i = e_\eta$.

\[
\sum_{i=0}^{p-1} f_i = \sum_{i=0}^{p-1} \left( \frac{1}{p} \sum_{k=0}^{p-1} (\zeta^i c)^k \right) e_\eta \\
= \frac{1}{p} \left\{ p + \left( \sum_{i=0}^{p-1} \zeta^i \right) c + \left( \sum_{i=0}^{p-1} \zeta^{2i} \right) c^2 + \cdots + \left( \sum_{i=0}^{p-1} \zeta^{(p-1)i} \right) c^{p-1} \right\} e_\eta \\
= e_\eta
\]

Notice that each $f_i$ belongs to $\mathbb{C}[G]e_\eta$. Therefore, $\{f_i \mid i = 0, 1, \ldots, p-1\}$ are mutually pairwise orthogonal central idempotents of the ring $\mathbb{C}[G]e_\eta$, and whose sum is $e_\eta$. Since $\mathbb{C}[G]e_\eta$ contains $p$ primitive central idempotents, $\{f_i \mid i = 0, 1, \ldots, p-1\}$
1\} are all the primitive central idempotents of $C[G]e_{\eta}$. Hence, for each $i = 0, 1, \ldots, p - 1$, 
\[ e_{\rho_i} = \frac{1}{p} \left( 1 + \zeta^i c + \zeta^{2i} c^2 + \cdots + \zeta^{ip(p-1)} c^{p-1} \right) e_{\eta}, \]
where $c = \frac{\overline{\chi_G(x)e_{\eta}}}{\sqrt{d}}$ and $\zeta$ is a primitive $p^{th}$ root of unity in $C$. Moreover, 
\[ e_{\eta} = e_{\rho_0} + e_{\rho_1} + \cdots + e_{\rho_{p-1}}. \]

This completes the proof of (1).

(2) If $e_{\eta}$ is not a central idempotent in $C[H]$, then $\eta$ is not equivalent to $\eta^x$. By Theorem [6.1] induced representations $\eta \uparrow_H^G, \eta^x \uparrow_H^G, \ldots, \eta^{x^{p-1}} \uparrow_H^G$ are all equivalent to $\rho$ (say) and $\rho$ is irreducible. Since the character $\chi_{\rho}$ of $\rho$ vanishes outside normal subgroup $H$ and $\chi_{\rho} \downarrow_H^G = \chi_{\eta} + \chi_{\eta^x} + \cdots + \chi_{\eta^{x^{p-1}}}$, we have $e_{\rho} = e_{\eta} + e_{\eta^x} + \cdots + e_{\eta^{x^{p-1}}}$. This completes the proof of (2). \hfill \square 

**Example 7.2.** (Special linear group $SL_2(3)$) Using a long presentation of $SL_2(3)$, we construct the irreducible representations of $SL_2(3)$ over $C$ and the primitive central idempotents of $C[SL_2(3)]$. Let us consider the maximal subnormal series 
\[ (7.1) \quad \langle e \rangle = G_0 < C_2 = G_1 < C_4 = G_2 < Q_8 = G_3 < SL_2(3) = G_4. \]
A long presentation of $SL_2(3)$ associated with the series \[ (7.1) \] is 
\[ \langle x, y, z, t \mid x^2 = 1, y^2 = x, y^2 = z^2, z^{-1}yz = xy, t^3 = 1, t^{-1}yt = z, t^{-1}zt = yz \rangle. \]

The group $Q_8 = G_3$ is a normal subgroup of index 3 in $SL_2(3)$. Now we construct the complex irreducible representations of $SL_2(3)$ starting with the complex irreducible representations of $Q_8$.

Let $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ be the five inequivalent complex irreducible representations $Q_8$, and are defined by 
\[ \rho_1(y) = 1, \quad \rho_1(z) = 1, \]
\[ \rho_2(y) = 1, \quad \rho_1(z) = -1, \]
\[ \rho_3(y) = -1, \quad \rho_1(z) = 1, \]
\[ \rho_4(y) = -1, \quad \rho_1(z) = -1, \]
and 
\[ [\rho_5(y)] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad [\rho_5(z)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

The trivial representation $\rho_1$ extends to three irreducible representations of $SL_2(3)$, and are given by 
\[ \theta_1(t) = 1, \quad \theta_2(t) = \omega, \quad \theta_3(t) = \omega^2, \quad \text{where } \omega = e^{2 \pi i / 3}. \]

Notice that $\rho_2, \rho_3, \rho_4$ are conjugate to each other, so they induce the same irreducible representation $\theta_4$ of degree 3, and is given by
\[
\begin{align*}
[\theta_4(x)] &= I_3, \quad [\theta_4(y)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\theta_4(z)] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
[\theta_4(t)] &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\end{align*}
\]

\(\rho_5\) extends to three irreducible representations of \(\text{SL}_2(3)\), let these extensions be \(\theta_5, \theta_6, \theta_7\). The matrix representations of \(\theta_5, \theta_6, \theta_7\) are given by

\[
\begin{align*}
\theta_k(x) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \theta_k(y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \theta_k(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (k = 5, 6, 7),
\end{align*}
\]

and

\[
\theta_5(t) = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad \theta_6(t) = \omega \left[ \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \right], \quad \theta_7(t) = \omega^2 \left[ \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \right].
\]

The primitive central idempotents of \(\mathbb{C}[Q_8]\) are

1. \(e_x e_y e_z\),
2. \(e_x e_y e_{-z}\),
3. \(e_x e_{-y} e_z\),
4. \(e_x e_{-y} e_{-z}\),
5. \(e_{-x} e_{iy} + e_{-x} e_{-iy} = 1 - e_x\),

where \(e_X = \frac{1+X}{2}, \ X \in \{\pm x, \pm y, \pm z\}\).

Let \(\overline{C}(t)\) denotes the conjugacy class sum of \(t\) as an element of \(\mathbb{C}[\text{SL}_2(3)]\). Then

\[
\begin{align*}
(1) \quad u_1 &= \frac{1}{3} \left\{ 1 + \left( \frac{\overline{C}(t)}{2} \right) + \left( \frac{\overline{C}(t)}{2} \right)^2 \right\} e_{-x}, \\
(2) \quad u_2 &= \frac{1}{3} \left\{ 1 + \omega^2 \left( \frac{\overline{C}(t)}{2} \right) + \omega \left( \frac{\overline{C}(t)}{2} \right)^2 \right\} e_{-x}, \\
(3) \quad u_3 &= \frac{1}{3} \left\{ 1 + \omega \left( \frac{\overline{C}(t)}{2} \right) + \omega^2 \left( \frac{\overline{C}(t)}{2} \right)^2 \right\} e_{-x}, \\
(4) \quad u_4 &= e_x e_y e_{-z} + e_x e_{-y} e_z + e_x e_{-y} e_{-z}, \\
(5) \quad u_5 &= e_x e_y e_{zt}, \\
(6) \quad u_6 &= e_x e_y e_{z^2t}, \\
(7) \quad u_7 &= e_x e_y e_{z^3t},
\end{align*}
\]

where \(\omega = e^{\frac{2\pi i}{3}}\) and \(e_X = \frac{1+X}{2}\) for \(X \in \{\pm x, \pm y, \pm z\}\); \(e_Y = \frac{1+Y+Y^2}{3}\) for \(Y \in \{t, \omega t, \omega^2 t\}\), are the primitive central idempotents of \(\mathbb{C}[\text{SL}_2(3)]\).

REFERENCES

[1] S. D. Berman, Group algebras of Abelian extensions of finite groups (Russian), Dokl. Akad. Nauk. SSSR (1955), Vol. 102, page 431-434.

[2] P. Bürgisser, M. Clausen and A. Shokrollahi, Algebraic Complexity Theory, Grundlehren der mathematischen Wissenschaften, Vol. 315, Springer-Verlag (1997), Berlin Heidelberg GmbH.

[3] E. G. Goodaire, E. Jespers, C. Polcino Milies, Alternative Loop Rings, Math.Studies No. 184 (North-Holland, 1996).

[4] C. P. Milies and S. K. Sehgal, An Introduction to Group Rings, Kluwer Academic Publishers (2002), Dordrecht.
REPRESENTATIONS AND PRIMITIVE CENTRAL IDEMPOTENTS OF A FINITE SOLVABLE GROUP

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