A Q-Polynomial Structure Associated with the Projective Geometry $L_N(q)$

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Abstract
There is a type of distance-regular graph, said to be $Q$-polynomial. In this paper, we investigate a generalized $Q$-polynomial property involving a graph that is not necessarily distance-regular. We give a detailed description of an example associated with the projective geometry $L_N(q)$.

Keywords Adjacency matrix · Dual adjacency matrix · $Q$-polynomial property

Mathematics Subject Classification Primary: 05E30; Secondary: 05C50

1 Introduction

There is a type of finite, undirected, connected graph, said to be distance-regular [1, 2, 4, 7]. These graphs have the sort of combinatorial regularity that can be analyzed using algebraic methods, such as linear algebra (eigenvalues/eigenvectors of the adjacency matrix [4, Section 4.1], tridiagonal pairs [15]); geometry (linear programming bounds [8], root systems [4, Chapter 3]); special functions (orthogonal polynomials [21, 34], hypergeometric series [1, Chapter 3]); and representation theory (the subconstituent algebra [26–28, 35], the $q$-Onsager algebra [13, 18]).

For an integer $N \geq 1$, the $N$-cube $Q_N$ is an attractive example of a distance-regular graph. To define $Q_N$, start with a set $S$ of cardinality $N$. The vertex set of $Q_N$ consists of the subsets of $S$. Vertices $y, z$ of $Q_N$ are adjacent whenever one of $y, z$ contains the other, and their cardinalities differ by one. For each vertex of $Q_N$ the corresponding subconstituent algebra was described by Go [10].
There is a type of distance-regular graph, said to be $Q$-polynomial [4, Chapter 8]. The $Q$-polynomial property was introduced by Delsarte [8] in his study of coding theory, and has been investigated intensely ever since [1, 2, 4, 7, 10, 26–28, 35]. The $N$-cube is $Q$-polynomial; see [8] or [10, Section 12].

We emphasize one feature of a $Q$-polynomial distance-regular graph $\Gamma$. For each vertex $x$ of $\Gamma$ there exists a certain diagonal matrix $A^* = A^*(x)$, called the dual adjacency matrix of $\Gamma$ with respect to $x$ [23, Section 2]. The eigenspaces of $A^*$ are the subconstituents of $\Gamma$ with respect to $x$. The adjacency matrix $A$ of $\Gamma$ is related to $A^*$ by the fact that each of $A$, $A^*$ acts on the eigenspaces of the other one in a (block) tridiagonal fashion [35, Section 13].

Guided by the dual adjacency matrix concept, in [35, Section 20] we generalized the $Q$-polynomial property in three directions: (i) we drop the assumption that $\Gamma$ is distance-regular; (ii) we drop the assumption that every vertex of $\Gamma$ has a dual adjacency matrix, and instead require that one distinguished vertex of $\Gamma$ has a dual adjacency matrix; (iii) we replace the adjacency matrix of $\Gamma$ by a weighted adjacency matrix. The generalized $Q$-polynomial property is described in Definitions 2.2–2.4 below.

Broadly speaking, our goal in this paper is to convince the reader that the generalized $Q$-polynomial property is worth investigating. To this end, we will give a detailed description of one attractive example, associated with the projective geometry $L_N^q$.

As we will see, it is natural to view $L_N^q$ as a $q$-analog of the $N$-cube $Q_N$.

Given a finite field $\text{GF}(q)$ and an integer $N \geq 1$, the graph $L_N^q$ is defined as follows. Start with a vector space $\mathbb{V}$ over $\text{GF}(q)$ that has dimension $N$. The vertex set of $L_N^q$ consists of the subspaces of $\mathbb{V}$. Vertices $y, z$ of $L_N^q$ are adjacent whenever one of $y, z$ contains the other, and their dimensions differ by one. The graph $L_N^q$ is not distance-regular if $N \geq 2$.

Background information about $L_N^q$ can be found in [4, Section 9.3], [5, Chapter 1], [9, 24], [25, Example 3.1(5) with $M = N$], [31, Section 7].

Let $0$ denote the zero subspace of $\mathbb{V}$. We distinguish the vertex $0$ of $L_N^q$. We will work with the weighted adjacency matrix $A$ of $L_N^q$ introduced by Ghosh and Srinivasan in their excellent paper [9]; see Definition 5.1 below. We define a diagonal matrix $A^*$ such that for each vertex $y$ of $L_N^q$, the $(y, y)$-entry of $A^*$ is $q^{-\dim y}$. We remark that $A^*$ is a scalar multiple of the matrix $K$ that appears in [31, Section 7]. By construction, the eigenspaces of $A^*$ are the subconstituents of $L_N^q$ with respect to $0$. By [9, Section 1] the matrix $A$ is diagonalizable over $\mathbb{R}$. The eigenvalues and eigenspaces of $A$ are computed in [9, Section 2]; see Lemma 5.6 below.

We will show that $A^*$ acts on the eigenspaces of $A$ in a (block) tridiagonal fashion. This will imply that $A$ is $Q$-polynomial with respect to $0$, in the sense of Definition 2.4 below. This fact is the main result of the paper.

We just described our main result. We also obtain the following auxiliary results, which may be of independent interest. We display four bases for the standard module of $L_N^q$, said to be split. With respect to each split basis, one of $A, A^*$ acts in an upper triangular fashion, and the other acts in a lower triangular fashion. We compute the actions of $A$ and $A^*$ on each split basis. For the vertex $0$ the associated subconstituent algebra $T$ is generated by $A, A^*$. We show that $A, A^*$ satisfy the tridiagonal

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relations. We show that the standard module of $L_N(q)$ decomposes into a direct sum of irreducible $T$-modules. We describe the irreducible $T$-modules in terms of Leonard systems of dual $q$-Krawtchouk type.

We mentioned above that the weighted adjacency matrix $A$ of $L_N(q)$ was introduced in [9]. We would like to acknowledge that an affine transformation $\alpha A + \beta I$ appears in the work of Bernard, Crampé, Vinet [3, Theorem 7.1] concerning the subconstituent algebra of the symplectic dual polar graph. The spectrum of $A$ can be deduced from [3, Section 7.2], and Leonard systems of dual $q$-Krawtchouk type are alluded to in [3, line (129)].

This paper is organized as follows. In Sect. 2, we discuss the generalized $Q$-polynomial property. In Sect. 3, we define the projective geometry $L_N(q)$, and discuss its basic properties. In Sect. 4, we introduce the four split bases for the standard module of $L_N(q)$. In Sect. 5, we define the matrices $A, A^*$ and discuss their basic properties. In Sects. 6, 7, we compute the actions of $A$ and $A^*$ on the split bases, and consider the implications. In Sect. 8, we show that the matrix $A$ is $Q$-polynomial with respect to $0$. We also show that $A, A^*$ satisfy the tridiagonal relations, and we describe the irreducible $T$-modules.

## 2 The $Q$-Polynomial Property

For distance-regular graphs the $Q$-polynomial property is well known [1, 2, 4, 7, 10, 26–28, 35]. In [35, Section 20] we generalize this $Q$-polynomial property to a setting that involves a weighted adjacency matrix of a graph that is not necessarily distance-regular. In the present section we review the generalized $Q$-polynomial property. Our review involves the adjacency algebra, the dual adjacency algebra, and the subconstituent algebra.

Let $\mathbb{R}$ denote the field of real numbers. Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with rows and columns indexed by $X$ and all entries in $\mathbb{R}$. Let $I \in \text{Mat}_X(\mathbb{R})$ denote the identity matrix. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with coordinates indexed by $X$ and all entries in $\mathbb{R}$. The algebra $\text{Mat}_X(\mathbb{R})$ acts on $V$ by left multiplication. We call $V$ the standard module. For all $y \in X$, define a vector $\hat{y} \in V$ that has $y$-coordinate 1 and all other coordinates 0. The vectors $\{\hat{y}\}_{y \in X}$ form a basis for $V$.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$, edge set $\mathcal{R}$, and path-length distance function $\partial$. Vertices $y, z \in X$ are said to be adjacent whenever they form an edge. For $y \in X$ and an integer $i \geq 0$ define the set $\Gamma_i(y) = \{z \in X | \partial(y, z) = i\}$. We abbreviate $\Gamma(y) = \Gamma_1(y)$.

The graph $\Gamma$ is said to be bipartite whenever there exists a partition $X = X^+ \cup X^-$ such that $\Gamma(y) \subseteq X^-$ for all $y \in X^+$, and $\Gamma(y) \subseteq X^+$ for all $y \in X^-$. 
Definition 2.1 By a weighted adjacency matrix of $\Gamma$, we mean a matrix $A \in \text{Mat}_X(\mathbb{R})$ that has $(y, z)$-entry
\[
A_{y,z} = \begin{cases} 
\neq 0, & \text{if } y, z \text{ are adjacent}; \\
0, & \text{if } y, z \text{ are not adjacent} 
\end{cases} 
(y, z \in X).
\]

For the rest of this section, we fix a weighted adjacency matrix $A$ of $\Gamma$ that is diagonalizable over $\mathbb{R}$.

Next we discuss the adjacency algebra of $\Gamma$. Let $M$ denote the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by $A$. We call $M$ the adjacency algebra of $\Gamma$ generated by $A$.

Let $D + 1$ denote the dimension of the vector space $M$. Since $A$ is diagonalizable, the vector space $M$ has a basis $\{E_i\}_{i=0}^D$ such that $\sum_{i=0}^D E_i = I$ and $E_i E_j = \delta_{i,j} E_i$ for $0 \leq i, j \leq D$. We call $\{E_i\}_{i=0}^D$ the primitive idempotents of $A$. Since $A \in M$, there exist real numbers $\{\theta_i\}_{i=0}^D$ such that $A = \sum_{i=0}^D \theta_i E_i$. The scalars $\{\theta_i\}_{i=0}^D$ are mutually distinct since $A$ generates $M$. We have $AE_i = \theta_i E_i = E_i A$ for $0 \leq i \leq D$. Note that $V = \sum_{i=0}^D E_i V$ (direct sum).

For $0 \leq i \leq D$ the subspace $E_i V$ is an eigenspace of $A$, and $\theta_i$ is the corresponding eigenvalue. For notational convenience, define $E_{-1} = 0$ and $E_{D+1} = 0$.

Next we discuss the dual adjacency algebras of $\Gamma$. For the rest of this section, fix a vertex $x \in X$. Define the integer $D = D(x)$ by
\[
D = \max\{\partial(x, y) \mid y \in X\}.
\]

We call $D$ the diameter of $\Gamma$ with respect to $x$. We have $D \leq D$, because the matrices $\{A^i\}_{i=0}^D$ are linearly independent. For $0 \leq i \leq D$ we define a diagonal matrix $E^*_i = E^*_i(x)$ in $\text{Mat}_X(\mathbb{R})$ that has $(y, y)$-entry
\[
(E^*_i)_{y,y} = \begin{cases} 
1, & \text{if } \partial(x, y) = i; \\
0, & \text{if } \partial(x, y) \neq i 
\end{cases} 
(y \in X). 
\]

We call $\{E^*_i\}_{i=0}^D$ the dual primitive idempotents of $\Gamma$ with respect to $x$ [26, p. 378]. We have $\sum_{i=0}^D E^*_i = I$ and $E^*_i E^*_j = \delta_{i,j} E^*_i$ for $0 \leq i, j \leq D$. Consequently the matrices $\{E^*_i\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{R})$. We call $M^*$ the dual adjacency algebra of $\Gamma$ with respect to $x$ [26, p. 378].

Next we recall the subconstituents of $\Gamma$ with respect to $x$. From (1) we obtain
\[
E^*_i V = \text{Span} \{\hat{y} \mid y \in \Gamma_i(x)\} \quad (0 \leq i \leq D). 
\]
By (2) and since \( \{ \hat{y} \}_{y \in X} \) is a basis for \( V \), we find

\[
V = \sum_{i=0}^{D} E_i^* V \quad \text{(direct sum)}.
\]

For \( 0 \leq i \leq D \) the subspace \( E_i^* V \) is a common eigenspace for \( M^* \). We call \( E_i^* V \) the \( i^{th} \) subconstituent of \( \Gamma \) with respect to \( x \). For notational convenience, define \( E_{-1}^* = 0 \) and \( E_{D+1}^* = 0 \). By the triangle inequality, for adjacent \( y, z \in X \) the distances \( \partial(x, y) \) and \( \partial(x, z) \) differ by at most one. Consequently

\[
AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \quad (0 \leq i \leq D).
\]

Next we discuss the \( Q \)-polynomial property.

**Definition 2.2** (See [35, Definition 20.6]) A matrix \( A^* \in \text{Mat}_X(\mathbb{R}) \) is called a dual adjacency matrix of \( \Gamma \) (with respect to \( x \) and the ordering \( \{ E_i \}_{i=0}^D \) whenever \( A^* \) generates \( M^* \) and

\[
A^* E_i V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \quad (0 \leq i \leq D).
\]

**Definition 2.3** (See [35, Definition 20.7]) We say that the ordering \( \{ E_i \}_{i=0}^D \) is \( Q \)-polynomial with respect to \( x \) whenever there exists a dual adjacency matrix of \( \Gamma \) with respect to \( x \) and \( \{ E_i \}_{i=0}^D \).

**Definition 2.4** (See [35, Definition 20.8]) We say that \( A \) is \( Q \)-polynomial with respect to \( x \) whenever there exists an ordering of the primitive idempotents of \( A \) that is \( Q \)-polynomial with respect to \( x \).

Assume that \( \Gamma \) has a dual adjacency matrix \( A^* \) with respect to \( x \) and \( \{ E_i \}_{i=0}^D \). Since \( A^* \in M^* \), there exist real numbers \( \{ \theta_i^* \}_{i=0}^D \) such that \( A^* = \sum_{i=0}^{D} \theta_i^* E_i^* \). The scalars \( \{ \theta_i^* \}_{i=0}^D \) are mutually distinct since \( A^* \) generates \( M^* \). We have \( A^* E_i^* = \theta_i^* E_i^* = E_i^* A^* \) for \( 0 \leq i \leq D \). We mentioned earlier that the sum \( V = \sum_{i=0}^{D} E_i^* V \) is direct. For \( 0 \leq i \leq D \) the subspace \( E_i^* V \) is an eigenspace of \( A^* \), and \( \theta_i^* \) is the corresponding eigenvalue.

As we investigate the \( Q \)-polynomial property, it is helpful to bring in the subconstituent algebra [26–28]. The following definition is a variation on [26, Definition 3.3].

**Definition 2.5** Let \( T = T(x, A) \) denote the subalgebra of \( \text{Mat}_X(\mathbb{R}) \) generated by \( M \) and \( M^* \). We call \( T \) the subconstituent algebra of \( \Gamma \) with respect to \( x \) and \( A \).

By construction, the algebra \( T \) has finite dimension.

**Lemma 2.6** Assume that \( \Gamma \) has a dual adjacency matrix \( A^* \) with respect to \( x \) and \( \{ E_i \}_{i=0}^D \). Then the algebra \( T \) is generated by \( A, A^* \).
Proof The algebra $T$ is generated by $M$ and $M^*$. The algebra $M$ is generated by $A$, and the algebra $M^*$ is generated by $A^*$. \hfill \Box

Next we give some relations in $T$.

Lemma 2.7 We have $E_i^*AE_j^* = 0$ if $|i - j| > 1$ $(0 \leq i, j \leq D)$. Assume that $\Gamma$ has a dual adjacency matrix $A^*$ with respect to $x$ and $\{E_i\}_{i=0}^D$. Then $E_iA^*E_j = 0$ if $|i - j| > 1$ $(0 \leq i, j \leq D)$.

Proof This is a routine consequence of (3) and (4). \hfill \Box

Next we consider the $T$-modules. By a $T$-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. A $T$-module $W$ is called irreducible whenever $W \neq 0$ and $W$ does not contain a $T$-module besides 0 and $W$.

For the rest of this section, we assume that $\Gamma$ has a dual adjacency matrix $A^*$ with respect to $x$ and $\{E_i\}_{i=0}^D$. Let $W$ denote an irreducible $T$-module. Then $W$ is a direct sum of the nonzero subspaces among $\{E_i^*W\}_{i=0}^D$. Similarly, $W$ is a direct sum of the nonzero subspaces among $\{E_iW\}_{i=0}^D$.

Lemma 2.8 Let $W$ denote an irreducible $T$-module. Then

$$AE_i^*W \subseteq E_{i-1}^*W + E_i^*W + E_{i+1}^*W \quad (0 \leq i \leq D),$$

$$A^*E_iW \subseteq E_{i-1}W + E_iW + E_{i+1}W \quad (0 \leq i \leq D).$$

Proof By (3) and (4). \hfill \Box

Let $W$ denote an irreducible $T$-module. By the endpoint of $W$ we mean $\min\{i|0 \leq i \leq D, \ E_i^*W \neq 0\}$. By the diameter of $W$ we mean $\{|i|0 \leq i \leq D, \ E_i^*W \neq 0\} - 1$. By the dual endpoint of $W$ we mean $\min\{i|0 \leq i \leq D, \ E_iW \neq 0\}$. By the dual diameter of $W$ we mean $\{|i|0 \leq i \leq D, \ E_iW \neq 0\} - 1$.

The following result is a variation on [26, Lemma 3.4, Lemma 3.9].

Lemma 2.9 Let $W$ denote an irreducible $T$-module with endpoint $r$ and diameter $d$. Then $r$, $d$ are nonnegative integers such that $r + d \leq D$. Moreover the following (i), (ii) hold:

(i) $E_i^*W \neq 0$ if and only if $r \leq i \leq r + d$ $(0 \leq i \leq D)$;

(ii) $W = \sum_{i=r}^{r+d} E_i^*W$ (direct sum).

Proof (i) By construction $E_r^*W \neq 0$ and $E_r^*W = 0$ for $0 \leq i < r$. Suppose there exists an integer $i (r < i \leq r+d)$ such that $E_i^*W = 0$. Define $W' = E_r^*W + E_{r+1}^*W + \cdots + E_{i-1}^*W$. By construction $0 \neq W' \subseteq W$. Also by construction, $A^*W' \subseteq V$. By Lemma 2.8 and $E_r^*W = 0$ we obtain $A^*W' \subseteq W$. By these comments $W'$ is a $T$-module. We have $W = W'$ since the $T$-module $W$ is irreducible. This contradicts the fact that $d$ is the diameter of $W$. We conclude that $E_i^*W \neq 0$ for $r \leq i \leq r + d$. By the definition of the diameter $d$ we have $E_i^*W = 0$ for $r + d < i \leq D$.

(ii) By (i) and the comments above Lemma 2.8. \hfill \Box
The following is a variation on [26, Lemma 3.4, Lemma 3.12].

**Lemma 2.10** Let $W$ denote an irreducible $T$-module with dual endpoint $t$ and dual diameter $\delta$. Then $t, \delta$ are nonnegative integers such that $t + \delta \leq D$. Moreover the following (i), (ii) hold:

(i) $E_i W \neq 0$ if and only if $t \leq i \leq t + \delta$ ($0 \leq i \leq D$);
(ii) $W = \sum_{i=t}^{t+\delta} E_i W$ (direct sum).

**Proof** Similar to the proof of Lemma 2.9.

The definition of a tridiagonal pair is given in [15, Definition 1.1]. The following is a variation on [15, Example 1.4].

**Proposition 2.11** The pair $A, A^*$ acts on each irreducible $T$-module as a tridiagonal pair.

**Proof** By Lemmas 2.6, 2.8, 2.9, 2.10.

Let $W$ denote an irreducible $T$-module. By Proposition 2.11 and [15, Lemma 4.5], the diameter of $W$ is equal to the dual diameter of $W$.

There is a considerable literature about tridiagonal pairs; see [2, 13–16, 18, 22, 29, 30, 33] and the references therein. This literature motivates us to investigate the $Q$-polynomial property described in Definition 2.4. To begin the investigation, we will examine one example in detail. This example is constructed from the projective geometry $L_N(q)$.

3 The Projective Geometry $L_N(q)$

In Sect. 1, we defined the graph $L_N(q)$. As we will explain in a moment, it is natural to view this graph as the Hasse diagram of a partially ordered set (poset). In order to distinguish between the graph and the poset, we will use the following notation going forward.

Given a finite field $\text{GF}(q)$ and an integer $N \geq 1$, we define a poset $L_N(q)$ as follows. Let $V$ denote a vector space over $\text{GF}(q)$ that has dimension $N$. Let the set $X$ consist of the subspaces of $V$. The set $X$, together with the containment relation, is a poset denoted $L_N(q)$ and called a projective geometry. The partial order is denoted $\leq$. For $y, z \in X$ we write $y < z$ whenever $y \leq z$ and $y \neq z$. We say that $z$ covers $y$ whenever $y < z$ and there does not exist $w \in X$ such that $y < w < z$. Note that $z$ covers $y$ if and only if $y \leq z$ and $\dim z - \dim y = 1$. Next we define a graph $\Gamma$ with vertex set $X$. Vertices $y, z \in X$ are adjacent in $\Gamma$ whenever one of $y, z$ covers the other one. The graph $\Gamma$ is the Hasse diagram of the poset $L_N(q)$. The rest of the paper is about the graph $\Gamma$.

Let $0$ denote the zero subspace of $V$. Recall the vertex $x$ of $\Gamma$ from Sect. 2. For the rest of the paper, we choose $x = 0$.

Recall the path-length distance function $\partial$ for $\Gamma$.

**Lemma 3.1** The following (i)–(iii) hold for the graph $\Gamma$:
(i) for \( y \in X \) we have \( \partial(0, y) = \dim y \);
(ii) \( \Gamma \) has diameter \( N \) with respect to the vertex \( 0 \);
(iii) \( \Gamma \) is bipartite with bipartition \( X = X^+ \cup X^- \), where
\[
X^+ = \{ y \in X | \dim y \text{ is even} \}, \quad X^- = \{ y \in X | \dim y \text{ is odd} \}.
\]

For the rest of this paper we adopt the following notation.

**Definition 3.2** Define \( E^*_i = E^*_i(0) \) for \( 0 \leq i \leq N \). Further define \( M^* = M^*(0) \). By construction the matrices \( \{ E^*_i \}_{i=0}^N \) form a basis for \( M^* \).

Recall the standard module \( V = \mathbb{R}^X \). By Lemma 3.1(i) we obtain
\[
E^*_i V = \text{Span}\{\hat{y} \mid \dim y = i\} \quad (0 \leq i \leq N).
\]

For \( n \in \mathbb{N} \) define
\[
[n]_q = \frac{q^n - 1}{q - 1}.
\]

We further define
\[
[n]_q^1 = [n]_q[n - 1]_q \cdots [2]_q[1]_q.
\]

We interpret \( [0]_q^1 = 1 \). For \( 0 \leq i \leq n \) define
\[
\binom{n}{i}_q = \frac{[n]_q^i}{[i]_q[n - i]_q}.
\]

For notational convenience, define \( \Gamma_{-1}(0) = \emptyset \).

**Lemma 3.3** For \( 0 \leq i \leq N \) and \( y \in \Gamma_i(0) \) we have

(i) \( |\Gamma(y) \cap \Gamma_{i-1}(0)| = [i]_q \);
(ii) \( |\Gamma(y) \cap \Gamma_{i+1}(0)| = [N - i]_q \).

**Proof** By elementary counting arguments; see for example [4, Section 9.3]. \( \square \)

Lemma 3.3 implies that the vertex \( 0 \) is distance-regularized in the sense of [11, Section 1.2].

The following result is well known; see for example [4, Theorem 9.3.2].

**Lemma 3.4** For \( 0 \leq i \leq N \),
\[
|\Gamma_i(0)| = \binom{N}{i}_q.
\]

By (2) and Lemma 3.4,
\[
\dim E^*_i V = \binom{N}{i}_q \quad (0 \leq i \leq N).
\]

\( \square \) Springer
4 The Split Bases for the Standard Module $V$

We continue to discuss the graph $\Gamma$ from Sect. 3. Recall that the vectors $\{\hat{y}\}_{y \in X}$ form a basis for the standard module $V$. In this section, we introduce four additional bases for $V$, said to be split.

**Definition 4.1** For $y \in X$ define

\[
y^{\downarrow \downarrow} = \sum_{z \leq y} \hat{z}, \tag{7}
\]

\[
y^{\downarrow \uparrow} = \sum_{z \leq y} \hat{z}(-1)^{\dim z}, \tag{8}
\]

\[
y^{\uparrow \downarrow} = q^{\left(\frac{N-\dim y}{2}\right)} \sum_{y \leq z} \hat{z}q^{(N-\dim z)\dim y}, \tag{9}
\]

\[
y^{\uparrow \uparrow} = q^\left(\frac{N-\dim y}{2}\right) \sum_{y \leq z} \hat{z}q^{(N-\dim z)\dim y}(-1)^{\dim z}. \tag{10}
\]

**Lemma 4.2** Each of following is a basis for the vector space $V$:

\[
\{y^{\downarrow \downarrow}\}_{y \in X}, \quad \{y^{\downarrow \uparrow}\}_{y \in X}, \quad \{y^{\uparrow \downarrow}\}_{y \in X}, \quad \{y^{\uparrow \uparrow}\}_{y \in X}.
\]

**Proof** The vectors $\{y^{\downarrow \downarrow}\}_{y \in X}$ are linearly independent by construction, and hence form a basis for $V$. The remaining assertions are similarly verified. $\square$

**Definition 4.3** The bases for $V$ from Lemma 4.2 are said to be **split**.

We mention how the split bases are related.

**Definition 4.4** Define the diagonal matrix $S \in \text{Mat}_X(\mathbb{R})$ with $(y, y)$-entry

\[
S_{y,y} = (-1)^{\dim y} \quad y \in X.
\]

**Lemma 4.5** For $y \in X$ we have $S\hat{y} = (-1)^{\dim y}\hat{y}$.

**Proof** By Definition 4.4. $\square$

**Lemma 4.6** We have

\[
S = \sum_{i=0}^{N} (-1)^i E_i^*.
\]

Moreover $S \in M^*$ and $S^2 = I$.

**Proof** The first assertion is immediate from Lemma 3.1(i) and Definition 4.4. The other assertions are clear. $\square$
Lemma 4.7  For \( y \in X \) the matrix \( S \) sends
\[
\downarrow \downarrow \leftrightarrow \, \downarrow \uparrow, \quad \uparrow \downarrow \leftrightarrow \, \uparrow \uparrow.
\]

Proof  By Definition 4.1 and Lemma 4.5.

\[ \square \]

5 The Matrices \( A \) and \( A^* \)

We continue to discuss the graph \( \Gamma \) from Sect. 3. In this section, we introduce two matrices \( A \) and \( A^* \) in \( \text{Mat}_X(\mathbb{R}) \). The matrix \( A \) is a weighted adjacency matrix for \( \Gamma \), and the matrix \( A^* \) generates \( M^* \). We investigate the eigenvalues and eigenspaces of \( A \) and \( A^* \). In the next section, we will explain how \( A \) and \( A^* \) act on the vectors in Definition 4.1.

The following matrix \( A \) was introduced by Ghosh and Srinivasan [9].

Definition 5.1 (See [9, Section 1]) Define a matrix \( A \in \text{Mat}_X(\mathbb{R}) \) that has \((y, z)\)-entry
\[
A_{y,z} = \begin{cases} 
1 & \text{if } y \text{ covers } z; \\
q^{\text{dim } y} & \text{if } z \text{ covers } y; \\
0 & \text{if } y, z \text{ are not adjacent}
\end{cases}, \quad y, z \in X.
\]

Note that \( A \) is a weighted adjacency matrix for \( \Gamma \). Next we clarify how \( A \) acts on the vectors \( \{\hat{y}\}_{y \in X} \).

Lemma 5.2  For \( y \in X \),
\[
A\hat{y} = \sum_{y \text{ covers } z} \hat{z} q^{\text{dim } z} + \sum_{z \text{ covers } y} \hat{z}.
\]

Proof  By Definition 5.1.

\[ \square \]

Lemma 5.3  For \( 0 \leq i \leq N \),
\[
AE_i^* V \subseteq E_{i-1}^* V + E_{i+1}^* V.
\]

Proof  By (5) and Lemma 5.2.

\[ \square \]

The following result is due to Ghosh and Srinivasan [9]. We give a short proof for the sake of completeness.

Lemma 5.4 (See [9, Section 1]) The matrix \( A \) is diagonalizable.

Proof  The scalar \( q \) is positive; let \( b \) denote the positive square root of \( q \). Define real numbers \( \{d_i\}_{i=0}^N \) such that \( d_0 = 1 \) and \( d_{i+1}/d_i = b^i \) for \( 0 \leq i \leq N - 1 \). Define the
matrix $\Delta = \sum_{i=0}^{N} d_i E_i^*$. The matrix $\Delta$ is invertible, with inverse $\Delta^{-1} = \sum_{i=0}^{N} d_i^{-1} E_i^*$. The matrix $\Delta A \Delta^{-1}$ has $(y, z)$-entry

$$(\Delta A \Delta^{-1})_{y,z} = \begin{cases} b^{\dim z} & \text{if } y \text{ covers } z; \\ b^{\dim y} & \text{if } z \text{ covers } y; \\ 0 & \text{if } y, z \text{ are not adjacent} \end{cases} \text{ for } y, z \in X.$$ 

The matrix $\Delta A \Delta^{-1}$ is symmetric. By construction $\Delta A \Delta^{-1}$ has real entries, so $\Delta A \Delta^{-1}$ is diagonalizable. The matrices $A$ and $\Delta A \Delta^{-1}$ are similar, so $A$ is diagonalizable. \hfill \Box

Next we compute the eigenvalues of $A$.

**Lemma 5.5** For $y \in X$,

$$A y^\downarrow = \frac{q^{\dim y} - q^{N-\dim y}}{q - 1} y^\downarrow + \sum_{z \text{ covers } y} z^\downarrow. \quad (11)$$

**Proof** This is routinely checked using the combinatorics of $L_N(q)$, see for example [4, Theorem 9.3.2]. \hfill \Box

The following result is due to Ghosh and Srinivasan [9]. We give a short proof for the sake of completeness.

**Lemma 5.6** (See [9, Section 2]) The eigenvalues of $A$ are $\{\theta_i\}_{i=0}^{N}$, where

$$\theta_i = \frac{q^{N-i} - q^i}{q - 1} \quad (0 \leq i \leq N). \quad (12)$$

Moreover, for $0 \leq i \leq N$ the dimension of the $\theta_i$-eigenspace of $A$ is equal to $\binom{N}{i}_q$.

**Proof** Linearly order the elements of $X$, such that for all $y, z \in X$ the element $y$ comes before $z$ if $\dim y < \dim z$. Consider the matrix in $\text{Mat}_X(\mathbb{R})$ that represents $A$ with respect to the basis $\{y^\downarrow\}_{y \in X}$. This matrix is lower triangular by Lemma 5.5. Also by Lemma 5.5, this matrix has diagonal entries $\{\theta_i\}_{i=0}^{N}$, with $\theta_i$ appearing $\binom{N}{i}_q$ times for $0 \leq i \leq N$. The result follows. \hfill \Box

We remark that

$$\theta_{N-i} = -\theta_i \quad (0 \leq i \leq N). \quad (13)$$

Recall from Sect. 2 the adjacency algebra $M$ generated by $A$. By Lemmas 5.4, 5.6 the vector space $M$ has a basis $\{E_i\}_{i=0}^{N}$ of primitive idempotents, labelled such that $A = \sum_{i=0}^{N} \theta_i E_i$. The subspace $E_i V$ is the $\theta_i$-eigenspace of $A$ for $0 \leq i \leq N$. By this and Lemma 5.6,

$$\dim E_i V = \binom{N}{i}_q \quad (0 \leq i \leq N). \quad (14)$$
Recall the integer $D$ from below Definition 2.1. We have $D = N$.

**Definition 5.7** Define a diagonal matrix $A^* \in \text{Mat}_X(\mathbb{F})$ with $(y, y)$-entry

$$A^*_{y, y} = q^{-\dim y} \quad y \in X.$$

**Lemma 5.8** We have

$$A^* = \sum_{i=0}^{N} q^{-i} E^*_i.$$

Moreover, $A^*$ generates $M^*$.

**Proof** The first assertion follows from Lemma 3.1(i) and Definition 5.7. The second assertion follows from the first assertion and the fact that $\{q^{-i}\}_{i=0}^{N}$ are mutually distinct. \qed

**Lemma 5.9** The eigenvalues of $A^*$ are $\{\theta^*_i\}_{i=0}^{N}$, where

$$\theta^*_i = q^{-i} \quad (0 \leq i \leq N). \quad (15)$$

Moreover, for $0 \leq i \leq N$ the $\theta^*_i$-eigenspace of $A^*$ is equal to $E^*_i V$.

**Proof** By Lemma 5.8. \qed

Recall the matrix $S$ from Definition 4.4.

**Lemma 5.10** We have

$$SAS^{-1} = -A, \quad SA^*S^{-1} = A^*.$$

**Proof** The first equation is verified using Lemmas 4.6, 5.3. The second equation holds since $S$ and $A^*$ are diagonal. \qed

**Lemma 5.11** For $0 \leq i \leq N$ we have

$$SE_i S^{-1} = E_{N-i}, \quad SE^*_i S^{-1} = E^*_i. \quad (16)$$

Moreover

$$SE_i V = E_{N-i} V, \quad SE^*_i V = E^*_i V. \quad (17)$$

**Proof** By (13) and Lemma 5.10. \qed

\[ Springer \]
6 The Action of $A$ and $A^*$ on the Split Bases

We continue to discuss the graph $\Gamma_1$ from Sect. 3. Recall the four split bases of $V$, from Lemma 4.2. In this section, we compute the action of $A$ and $A^*$ on these bases.

Lemma 6.1 For $0 \leq i \leq N$ and $y \in \Gamma_i(0)$ we have

$$Ay_{\downarrow\downarrow} = \theta_{N-i}y_{\downarrow\downarrow} + \sum_{z \text{ covers } y} z_{\downarrow\downarrow},$$

$$A^*y_{\downarrow\downarrow} = \theta_i^*y_{\downarrow\downarrow} + (q - 1)q^{-i} \sum_{y \text{ covers } z} z_{\downarrow\downarrow}.$$

Proof The first equation is a reformulation of (11). The second equation is routinely obtained using (7) and Definition 5.7.

Lemma 6.2 For $0 \leq i \leq N$ and $y \in \Gamma_i(0)$ we have

$$Ay_{\downarrow\uparrow} = \theta_iy_{\downarrow\uparrow} - \sum_{z \text{ covers } y} z_{\downarrow\uparrow},$$

$$A^*y_{\downarrow\uparrow} = \theta_i^*y_{\downarrow\uparrow} + (q - 1)q^{-i} \sum_{y \text{ covers } z} z_{\downarrow\uparrow}.$$

Proof For the equations in Lemma 6.1, apply $S$ to each side and evaluate the result using (13) along with Lemmas 4.7, 5.10.

Lemma 6.3 For $0 \leq i \leq N$ and $y \in \Gamma_i(0)$ we have

$$Ay_{\uparrow\downarrow} = \theta_i^*y_{\uparrow\downarrow} + \sum_{y \text{ covers } z} z_{\uparrow\downarrow},$$

$$A^*y_{\uparrow\downarrow} = \theta_i^*y_{\uparrow\downarrow} + (q^{-1} - 1)q^{-i} \sum_{z \text{ covers } y} z_{\uparrow\downarrow}.$$

Proof The equations are routinely verified using the combinatorics of $L_N(q)$, see for example [4, Theorem 9.3.2].

Lemma 6.4 For $0 \leq i \leq N$ and $y \in \Gamma_i(0)$ we have

$$Ay_{\uparrow\uparrow} = \theta_{N-i}y_{\uparrow\uparrow} - \sum_{y \text{ covers } z} z_{\uparrow\uparrow},$$

$$A^*y_{\uparrow\uparrow} = \theta_i^*y_{\uparrow\uparrow} + (q^{-1} - 1)q^{-i} \sum_{z \text{ covers } y} z_{\uparrow\uparrow}.$$

Proof For the equations in Lemma 6.3, apply $S$ to each side and evaluate the result using (13) along with Lemmas 4.7, 5.10.
7 The Split Decompositions of the Standard Module $V$

We continue to discuss the graph $\Gamma$ from Sect. 3. Recall the standard module $V$. By a decomposition of $V$ we mean a sequence of nonzero subspaces $\{U_i\}_{i=0}^{N}$ whose direct sum is equal to $V$. For example, the sequences $\{E_i V\}_{i=0}^{N}$ and $\{E^*_i V\}_{i=0}^{N}$ are decompositions of $V$. In this section, we introduce four additional decompositions of $V$, said to be split. We discuss how the split decompositions are related to $\{E_i V\}_{i=0}^{N}$ and $\{E^*_i V\}_{i=0}^{N}$. We also discuss how $A$ and $A^*$ act on the split decompositions.

Lemma 7.1 For following hold for $0 \leq i \leq N$.

(i) The vectors $\{y \downarrow \uparrow | y \in X, \dim y \leq i\}$ form a basis for $E^*_0 V + E^*_1 V + \cdots + E^*_i V$.

(ii) The vectors $\{y \uparrow \downarrow | y \in X, \dim y \geq i\}$ form a basis for $E^*_N V + E^*_N V + \cdots + E^*_i V$.

Proof (i) Use (5) along with Definition 4.1 and Lemma 4.2.

(ii) We claim that the given vectors are contained in the given subspace. To prove the claim, pick $y \in X$ such that $\dim y \geq i$. For notational convenience, define $j = \dim y$.

Using Lemma 6.1 repeatedly, we obtain

$$(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{N-j} I) y \downarrow \uparrow = 0.$$ 

Consequently

$$y \downarrow \uparrow \in E_0 V + E_1 V + \cdots + E_{N-j} V \subseteq E_0 V + E_1 V + \cdots + E_{N-i} V.$$ 

The claim is proved. By Lemma 4.2 the given vectors are linearly independent. By Lemma 3.4 and (14), the number of given vectors is equal to the dimension of the given subspace. The result follows from these comments and linear algebra.

Lemma 7.2 For following hold for $0 \leq i \leq N$.

(i) The vectors $\{y \uparrow \downarrow | y \in X, \dim y \leq i\}$ form a basis for $E^*_N V + E^*_N V + \cdots + E^*_i V$.

(ii) The vectors $\{y \downarrow \uparrow | y \in X, \dim y \geq i\}$ form a basis for $E^*_0 V + E^*_1 V + \cdots + E^*_i V$.

Proof Apply $S$ to everything in Lemma 7.1, and evaluate the result using Lemma 4.7 and (17).

Lemma 7.3 For following hold for $0 \leq i \leq N$.

(i) The vectors $\{y \uparrow \downarrow | y \in X, \dim y \geq i\}$ form a basis for $E^*_N V + E^*_N V + \cdots + E^*_i V$.

(ii) The vectors $\{y \uparrow \downarrow | y \in X, \dim y \leq i\}$ form a basis for $E^*_0 V + E^*_1 V + \cdots + E^*_i V$.

Proof (i) Use (5) along with Definition 4.1 and Lemma 4.2.

(ii) We claim that the given vectors are contained in the given subspace. To prove the claim, pick $y \in X$ such that $\dim y \leq i$. For notational convenience, define $j = \dim y$.

Using Lemma 6.3 repeatedly, we obtain

$$(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_j I) y \uparrow \downarrow = 0.$$
Consequently
\[ y^\updownarrow \in E_0 V + E_1 V + \cdots + E_j V \subseteq E_0 V + E_1 V + \cdots + E_i V. \]

The claim is proved. By Lemma 4.2 the given vectors are linearly independent. By Lemma 3.4 and (14), the number of given vectors is equal to the dimension of the given subspace. The result follows from these comments and linear algebra. \(\square\)

**Lemma 7.4** For following hold for \(0 \leq i \leq N\).

(i) The vectors \(\{y^\updownarrow | y \in X, \dim y \geq i\}\) form a basis for \(E_N^* V + E_{N-1}^* V + \cdots + E_i^* V\).

(ii) The vectors \(\{y^\uparrow | y \in X, \dim y \leq i\}\) form a basis for \(E_N V + E_{N-1} V + \cdots + E_{N-i} V\).

**Proof** Apply \(S\) to everything in Lemma 7.3, and evaluate the result using Lemma 4.7 and (17). \(\square\)

**Definition 7.5** For \(0 \leq i \leq N\) we define

\[
\begin{align*}
U_i^{\updownarrow} &= (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_{N-i} V), \\
U_i^{\uparrow} &= (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_N V + E_{N-1} V + \cdots + E_i V), \\
U_i^{\downarrow} &= (E_N^* V + E_{N-1}^* V + \cdots + E_{N-i}^* V) \cap (E_0 V + E_1 V + \cdots + E_{N-i} V), \\
U_i^{\uparrow\downarrow} &= (E_N^* V + E_{N-1}^* V + \cdots + E_{N-i}^* V) \cap (E_N V + E_{N-1} V + \cdots + E_i V).
\end{align*}
\]

**Lemma 7.6** The following (i)–(iv) hold for \(0 \leq i \leq N\):

(i) the vectors \(\{y^{\downarrow\downarrow} | y \in \Gamma_1(0)\}\) form a basis for \(U_i^{\downarrow\downarrow}\);

(ii) the vectors \(\{y^{\downarrow\uparrow} | y \in \Gamma_1(0)\}\) form a basis for \(U_i^{\downarrow\uparrow}\);

(iii) the vectors \(\{y^{\uparrow\downarrow} | y \in \Gamma_{N-i}(0)\}\) form a basis for \(U_i^{\uparrow\downarrow}\);

(iv) the vectors \(\{y^{\uparrow\uparrow} | y \in \Gamma_{N-i}(0)\}\) form a basis for \(U_i^{\uparrow\uparrow}\).

**Proof** Recall Lemmas 3.1(i) and 4.2.

(i) Compare the two assertions in Lemma 7.1.

(ii) Compare the two assertions in Lemma 7.2.

(iii) Compare the two assertions in Lemma 7.3.

(iv) Compare the two assertions in Lemma 7.4. \(\square\)

**Lemma 7.7** For \(0 \leq i \leq N\) the following subspaces have dimension \(\binom{N}{i}_q\):

\[ U_i^{\downarrow\downarrow}, \quad U_i^{\downarrow\uparrow}, \quad U_i^{\uparrow\downarrow}, \quad U_i^{\uparrow\uparrow}. \]

**Proof** By Lemmas 3.4 and 7.6 along with \(\binom{N}{i}_q = \binom{N}{N-i}_q\). \(\square\)
Lemma 7.8 Each of the following is a decomposition of $V$:

$$\{U_i^{↓↓}\}_{i=0}^N, \quad \{U_i^{↑↓}\}_{i=0}^N, \quad \{U_i^{↑↑}\}_{i=0}^N, \quad \{U_i^{↓↑}\}_{i=0}^N.$$  

Proof By Lemmas 4.2, 7.6.

The following definition is motivated by [15, Section 4] and [32, Section 5]; see also [17, 19, 20].

Definition 7.9 The decompositions of $V$ from Lemma 7.8 are said to be split.

Lemma 7.10 For $0 \leq i \leq N$ we have

$$SU_i^{↓↓} = U_i^{↑↓}, \quad SU_i^{↑↑} = U_i^{↓↓}, \quad SU_i^{↑↓} = U_i^{↑↑}, \quad SU_i^{↓↑} = U_i^{↓↓}.$$  

Proof By Lemmas 4.7, 7.6.

Lemma 7.11 For $0 \leq i \leq N$ the following are equal:

$$E_0^*V + E_1^*V + \cdots + E_i^*V, \quad U_0^{↓↓} + U_1^{↓↓} + \cdots + U_i^{↓↓}, \quad U_0^{↑↑} + U_1^{↑↑} + \cdots + U_i^{↑↑}.$$  

Proof By Lemma 7.1(i) the vectors $\{y^{↑↓}|\dim y \leq i\}$ form a basis for $E_0^*V + \cdots + E_i^*V$. By Lemma 7.6(i) the vectors $\{y^{↑↓}|\dim y \leq i\}$ form a basis for $U_0^{↓↓} + \cdots + U_i^{↓↓}$. By these comments $E_0^*V + \cdots + E_i^*V$ is a basis for $U_0^{↓↓} + \cdots + U_i^{↓↓}$. By Lemma 7.2(i) the vectors $\{y^{↓↓}|\dim y \leq i\}$ form a basis for $E_0^*V + \cdots + E_i^*V$. By Lemma 7.6(ii) the vectors $\{y^{↓↓}|\dim y \leq i\}$ form a basis for $U_0^{↓↓} + \cdots + U_i^{↓↓}$. By these comments $E_0^*V + \cdots + E_i^*V = U_0^{↓↓} + \cdots + U_i^{↓↓}$. □

Lemma 7.12 For $0 \leq i \leq N$ the following are equal:

$$E_N^*V + E_{N-1}^*V + \cdots + E_i^*V, \quad U_0^{↑↑} + U_1^{↑↑} + \cdots + U_i^{↑↑}, \quad U_0^{↓↑} + U_1^{↓↑} + \cdots + U_i^{↓↑}.$$  

Proof The proof is similar to the proof of Lemma 7.11. By Lemma 7.3(i) the vectors $\{y^{↑↓}|\dim y \geq N - i\}$ form a basis for $E_i^*V + \cdots + E_{N-i}^*V$. By Lemma 7.6(iii) the vectors $\{y^{↑↓}|\dim y \geq N - i\}$ form a basis for $U_0^{↑↑} + \cdots + U_i^{↑↑}$. By these comments $E_i^*V + \cdots + E_{N-i}^*V$ is a basis for $U_0^{↑↑} + \cdots + U_i^{↑↑}$. By Lemma 7.4(i) the vectors $\{y^{↑↓}|\dim y \geq N - i\}$ form a basis for $E_i^*V + \cdots + E_{N-i}^*V$. By Lemma 7.6(iv) the vectors $\{y^{↑↓}|\dim y \geq N - i\}$ form a basis for $U_0^{↑↑} + \cdots + U_i^{↑↑}$. By these comments $E_i^*V + \cdots + E_{N-i}^*V = U_0^{↑↑} + \cdots + U_i^{↑↑}$. □

Lemma 7.13 For $0 \leq i \leq N$ the following are equal:

$$E_0V + E_1V + \cdots + E_iV, \quad U_0^{↑↓} + U_1^{↑↓} + \cdots + U_i^{↑↓}, \quad U_0^{↑↑} + U_1^{↑↑} + \cdots + U_i^{↑↑}.$$
Lemma 7.10. For $0 \leq i \leq N$ the following are equal:

$$E_N V + E_{N-1} V + \cdots + E_{N-i} V, \quad U_N^\uparrow + U_{N-1}^\uparrow + \cdots + U_{N-i}^\uparrow, \quad U_N^{\uparrow \downarrow} + U_{N-1}^{\uparrow \downarrow} + \cdots + U_{N-i}^{\uparrow \downarrow}.$$  

$\square$

Proof. The proof is similar to the proof of Lemma 7.11. By Lemma 7.1(ii) the vectors $\{y^\uparrow\downarrow|\dim y \geq N-i\}$ form a basis for $E_0 V + \cdots + E_i V$. By Lemma 7.6(i) the vectors $\{y^\uparrow\downarrow|\dim y \geq N-i\}$ form a basis for $U_N^\uparrow + \cdots + U_{N-i}^\uparrow$. By these comments $E_0 V + \cdots + E_i V = U_N^\uparrow + \cdots + U_{N-i}^\uparrow$.

$\square$

Lemma 7.14 For $0 \leq i \leq N$ the following are equal:

$$E_N V + E_{N-1} V + \cdots + E_{N-i} V, \quad U_N^\uparrow + U_{N-1}^\uparrow + \cdots + U_{N-i}^\uparrow, \quad U_N^{\uparrow \downarrow} + U_{N-1}^{\uparrow \downarrow} + \cdots + U_{N-i}^{\uparrow \downarrow}.$$  

$\square$

Proof. Apply $S$ to everything in Lemma 7.13, and evaluate the result using (17) and Lemma 7.10.

We make a definition for notational convenience. For a decomposition $\{U_i\}_{i=0}^N$ of $V$, define $U_{-1} = 0$ and $U_{N+1} = 0$.

Lemma 7.15 For $0 \leq i \leq N$ we have

$$(A - \theta_{N-i} I)U_i^{\uparrow \downarrow} \subseteq U_{i+1}^{\uparrow \downarrow}, \quad (A^* - \theta_i^* I)U_i^{\uparrow \downarrow} \subseteq U_{i-1}^{\uparrow \downarrow}, \quad (A - \theta_i I)U_i^\uparrow \subseteq U_{i+1}^\uparrow, \quad (A^* - \theta_i^* I)U_i^\uparrow \subseteq U_{i-1}^\uparrow, \quad (A - \theta_{N-i} I)U_i^{\uparrow \downarrow} \subseteq U_{i+1}^{\uparrow \downarrow}, \quad (A^* - \theta_{N-i}^* I)U_i^{\uparrow \downarrow} \subseteq U_{i-1}^{\uparrow \downarrow}. \quad (18)$$

$$(A - \theta_i I)U_i^\uparrow \subseteq U_{i+1}^\uparrow, \quad (A^* - \theta_i^* I)U_i^\uparrow \subseteq U_{i-1}^\uparrow, \quad (A - \theta_{N-i} I)U_i^{\uparrow \downarrow} \subseteq U_{i+1}^{\uparrow \downarrow}, \quad (A^* - \theta_{N-i}^* I)U_i^{\uparrow \downarrow} \subseteq U_{i-1}^{\uparrow \downarrow}. \quad (19)$$

$$(A - \theta_{N-i} I)U_i^{\uparrow \downarrow} \subseteq U_{i+1}^{\uparrow \downarrow}, \quad (A^* - \theta_{N-i}^* I)U_i^{\uparrow \downarrow} \subseteq U_{i-1}^{\uparrow \downarrow}. \quad (20)$$

$$(A - \theta_i I)U_i^\uparrow \subseteq U_{i+1}^\uparrow, \quad (A^* - \theta_i^* I)U_i^\uparrow \subseteq U_{i-1}^\uparrow. \quad (21)$$

Proof. The inclusions (18) follow from Lemmas 6.1 and 7.6(i). The inclusions (19) follow from Lemmas 6.2 and 7.6(ii). The inclusions (20) follow from Lemma 6.3 and Lemma 7.6(iii). The inclusions (21) follow from Lemmas 6.4 and 7.6(iv).

We finish this section with a comment. Let $0 \leq i \leq N$. In (5), we gave a basis for $E_i^* V$. In Lemma 7.6, we gave a basis for each of $U_i^\uparrow, U_i^{\uparrow \downarrow}, U_i^{\downarrow \uparrow}, U_i^\downarrow$. A basis for $E_i^* V$ can be found in [9, Section 4].

8 The Matrix $A$ is Q-Polynomial with Respect to 0

We continue to discuss the graph $\Gamma$ from Sect. 3. Recall the weighted adjacency matrix $A$ from Definition 5.1, and the diagonal matrix $A^*$ from Definition 5.7. In this section, we show that $A^*$ is a dual adjacency matrix of $\Gamma$ with respect to 0 and the ordering $\{E_i\}_{i=0}^N$. Using this fact, we show that $A$ is $Q$-polynomial with respect to 0. We consider the subconstitutent algebra $T = T(0, A)$. We show that the generators $A, A^*$ satisfy a pair of relations called the tridiagonal relations. We describe the irreducible $T$-modules.

Recall the standard module $V$. 

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Proposition 8.1  For $0 \leq i \leq N$,
\[
A^*E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V. \quad (22)
\]

Proof  Using Lemma 7.13 and (18), we obtain
\[
A^*E_i V \subseteq A^*(E_0V + \cdots + E_iV)
= A^*(U_{N-1}^{\uparrow\downarrow} + \cdots + U_N^{\uparrow\downarrow})
\subseteq U_{N-1}^{\uparrow\downarrow} + \cdots + U_N^{\uparrow\downarrow}
= E_0V + \cdots + E_{i+1}V.
\]

Using Lemma 7.14 and (21), we obtain
\[
A^*E_i V \subseteq A^*(E_iV + \cdots + E_NV)
= A^*(U_{i-1}^{\uparrow\downarrow} + \cdots + U_N^{\uparrow\downarrow})
\subseteq U_{i-1}^{\uparrow\downarrow} + \cdots + U_N^{\uparrow\downarrow}
= E_{i-1}V + \cdots + E_NV.
\]

By the above comments
\[
A^*E_i V \subseteq (E_0V + \cdots + E_{i+1}V) \cap (E_{i-1}V + \cdots + E_NV)
= E_{i-1}V + E_iV + E_{i+1}V.
\]

Corollary 8.2  The matrix $A^*$ is a dual adjacency matrix of $\Gamma$ with respect to the vertex $0$ and the ordering $\{E_i\}_{i=0}^N$.

Proof  By Definition 2.2, Lemma 5.8, and Proposition 8.1.

Corollary 8.3  The ordering $\{E_i\}_{i=0}^N$ is $Q$-polynomial with respect to the vertex $0$.

Proof  By Definition 2.3 and Corollary 8.2.

The following is the main result of the paper.

Theorem 8.4  The matrix $A$ is $Q$-polynomial with respect to the vertex $0$.

Proof  By Definition 2.4 and Corollary 8.3.

For the rest of the paper we adopt the following notation.

Definition 8.5  Let $T = T(0, A)$ denote the subconstituent algebra of $\Gamma$ with respect to $0$ and $A$.

By Lemma 2.6, the algebra $T$ is generated by $A, A^*$. Next we will display two relations satisfied by these generators.
Proposition 8.6  The matrices $A$ and $A^*$ satisfy

$$A^3 A^* - (q + q^{-1} + 1)A^2 A^* A + (q + q^{-1} + 1)AA^* A^2 - A^* A^3$$

$$= qN^{-2}(q + 1)^2(AA^* - A^* A),$$

$$A^* A - (q + q^{-1} + 1)A^* A^2 AA^* + (q + q^{-1} + 1)A^* AA^* A^2 - AA^* A^3 = 0.$$ 

Proof  Concerning the first equation, let $C_1$ denote the left-hand side minus the right-hand side. We show that $C_1 = 0.$ We have

$$C_1 = IC_1 I = \sum_{i=0}^{N} \sum_{j=0}^{N} E_i C_1 E_j.$$  

For $0 \leq i, j \leq N$ we show that $E_i C_1 E_j = 0.$ Using $E_i A = \theta_i E_i$ and $AE_j = \theta_j E_j,$ we obtain

$$E_i C_1 E_j = E_i A^* E_j \left( \theta^3_i - (q + q^{-1} + 1)\theta^2_i \theta_j + (q + q^{-1} + 1)\theta_i \theta^2_j - \theta^3_j - qN^{-2}(q + 1)^2(\theta_i - \theta_j) \right)$$

$$= E_i A^* E_j (\theta^2_i - (q + q^{-1})\theta_i \theta_j + \theta^2_j - qN^{-2}(q + 1)^2).$$

We examine the factors in the above line. By Lemma 2.7 we have $E_i A^* E_j = 0$ if $|i - j| > 1.$ Of course $\theta_i - \theta_j = 0$ if $i = j.$ Using (12) we obtain

$$\theta^2_i - (q + q^{-1})\theta_i \theta_j + \theta^2_j - qN^{-2}(q + 1)^2 = 0$$

if $|i - j| = 1.$

By these comments $E_i C_1 E_j = 0.$ We have shown that $C_1 = 0,$ so the first equation holds. Concerning the second equation, let $C_2$ denote the left-hand side. We have

$$C_2 = IC_2 I = \sum_{i=0}^{N} \sum_{j=0}^{N} E_i^* C_2 E^*_j.$$  

For $0 \leq i, j \leq N$ we show that $E_i^* C_2 E^*_j = 0.$ Using $E_i^* A^* = \theta^*_i E_i^*$ and $A^* E^*_j = \theta^*_j E^*_j,$ we obtain

$$E_i^* C_2 E^*_j = E_i^* A E^*_j \left( \theta^*_{i} - (q + q^{-1} + 1)\theta^*_{i} \theta^*_j + (q + q^{-1} + 1)\theta^*_i \theta^*_{j}^2 - \theta^*_{j}^3 \right)$$

$$= E_i^* A E^*_j (\theta^*_{i} - \theta^*_{j})(\theta^*_{i} - q\theta^*_{j})(\theta^*_{i} - q^{-1}\theta^*_{j}).$$

We examine the factors in the above line. By Lemma 2.7 we have $E_i^* A E^*_j = 0$ if $|i - j| > 1.$ Of course $\theta^*_{i} - \theta^*_{j} = 0$ if $i = j.$ By (15) we have $\theta^*_{i} - q\theta^*_{j} = 0$ if $j - i = 1$ and $\theta^*_{i} - q^{-1}\theta^*_{j} = 0$ if $i - j = 1.$ By these comments $E_i^* C_2 E^*_j = 0.$ We have shown that $C_2 = 0,$ so the second equation holds.  

$\square$
Note 8.7. The equations in Proposition 8.6 are an instance of the tridiagonal relations [30].

Next we discuss the representation theory of \( T \). Before we go into detail, we have some remarks about \( L_N(q) \). The poset \( L_N(q) \) is uniform in the sense of [25]. Indeed \( L_N(q) \) appears as the special case \( M = N \) of [25, Example 3.1(5)]. In [25, p. 195] we defined the incidence algebra of \( L_N(q) \). Comparing that definition with Definitions 2.5, 8.5 we find that the incidence algebra of \( L_N(q) \) is equal to \( T \). By [25, Theorem 2.5], the standard module \( V \) is a direct sum of irreducible \( T \)-modules. In Lemmas 2.9, 2.10 and Proposition 2.11 we described the irreducible \( T \)-modules. We now give a more detailed description using [25, Theorem 2.5, Theorem 3.3(5)] along with [9, Section 2]. Let \( W \) denote an irreducible \( T \)-module, with endpoint \( r \), dual endpoint \( t \), and diameter \( d \). Then \( 0 \leq r \leq N/2 \) and \( t = r \) and \( d = N - 2r \). Moreover, the subspaces \( E_i W \) and \( E^*_i W \) have dimension one for \( r \leq i \leq N - r \). By these comments the sequence \( (A; \{ E_i \}_{i=r}^{N-r}; A^*; \{ E^*_i \}_{i=r}^{N-r}) \) acts on \( W \) as a Leonard system in the sense of [29, Definition 1.4]. For notational convenience, let \( \Phi \) denote this Leonard system. By Lemma 5.3 we see that \( \Phi \) is bipartite in the sense of [12, Definition 5.1]. Using the bipartite feature and the form of the eigenvalues (12), (15) we find that \( \Phi \) has dual \( q \)-Krawtchouk type [34, Example 20.7] with parameters

\[
\begin{align*}
    d(\Phi) &= N - 2r, \\
    h(\Phi) &= \frac{q^{N-r}}{q-1}, \\
    h^*(\Phi) &= q^{-r}, \\
    s(\Phi) &= -q^{2r-N-1}, \\
    \theta_0(\Phi) &= q^r [N - 2r]_q, \\
    \theta^*_0(\Phi) &= q^{-r}.
\end{align*}
\]

The Leonard system \( \Phi \) is determined up to isomorphism by the above six parameters [34, Proposition 9.8], and these parameters are determined by \( r, N, q \). By these comments and [6, Lemma 9.7], a pair of irreducible \( T \)-modules are isomorphic if and only if they have the same endpoint. For \( 0 \leq r \leq N/2 \) let \( \mu_r \) denote the multiplicity with which the irreducible \( T \)-module with endpoint \( r \) appears in the standard module \( V \). It follows from [25, Theorem 3.3(5)] that \( \mu_0 = 1 \) and \( \mu_r = \binom{N}{r}_q - \binom{N}{r-1}_q \) for \( 1 \leq r \leq N/2 \).

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Declarations

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