SELF-SIMILAR ULTRARELATIVISTIC JETTED BLAST WAVE

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ABSTRACT

Following a suggestion that a directed relativistic explosion may have a universal intermediate asymptotic, we derive a self-similar solution for an ultrarelativistic jetted blast wave. The solution involves three distinct regions: an approximately paraboloid head where the Lorentz factor $\gamma$ exceeds $\sim 1/2$ of its maximal, nose value; a geometrically self-similar, expanding envelope slightly narrower than a paraboloid; and an axial core in which the (cylindrically, henceforth) radial flow $u$ converges inward toward the axis. Most ($\sim 80\%$) of the energy lies well beyond the leading, head region. Here, a radial cross section shows a maximal $\gamma$ (separating the core and the envelope), a sign reversal in $u$, and a minimal $\gamma$, at respectively $\sim 1/6$, $\sim 1/4$, and $\sim 3/4$ of the shock radius. The solution is apparently unique, and approximately agrees with previous simulations, of different initial conditions, that resolved the head. This suggests that unlike a spherical relativistic blast wave, our solution is an attractor, and may thus describe directed blast waves such as in the external shock phase of a $\gamma$-ray burst.

Key words: gamma-ray burst; general – hydrodynamics – ISM: jets and outflows – relativistic processes

1. INTRODUCTION

Relativistic jets give rise to some of the most spectacular astronomical sources, including $\gamma$-ray bursts (GRBs) with Lorentz factors $\Gamma \sim 10^2-10^3$, active galactic nuclei (AGNs) with $\Gamma \sim 10$, and microquasars with $\Gamma \sim$ a few. Such jets are also likely to form in systems currently invisible to us, such as failed supernova explosions. Yet, there is considerable uncertainty regarding the origin, structure, and evolution of relativistic jets.

One might expect that in the limit of an evolved, ultrarelativistic jetted blast wave (henceforth jet), propagating into a sufficiently weakly magnetized, homogeneous medium, the lack of a characteristic length scale would lead the structure of the jet to approach some self-similar attractor. Such a solution may provide a framework for studying jets, and be useful as an approximate description of directed blast waves in astronomical systems, such as in the external shocks stage of a GRB.

It has been argued (Gruzinov 2007, henceforth G07) that a directed expansion, in which the total momentum $\Pi$ is comparable (when multiplied by the speed of light $c$) to the total energy $E_{\text{tot}}$, such that $E_{\text{tot}} - \Pi c \ll E_{\text{tot}}$, may have a well-defined self-similar attractor. This differs from a non-directed ($E_{\text{tot}} - \Pi c \lsim E_{\text{tot}}$) explosion, which is not in full causal contact when highly relativistic, and thus has no universal intermediate asymptotic. Indeed, non-directed explosions asymptote to the Blandford & McKee (1976) self-similar solution in the ultrarelativistic phase only if they are initially spherically symmetric (Gruzinov 2000).

The structure of a relativistic jet was described qualitatively in Rhoads (1999), and quantitatively in G07, where numerical simulations of various initial conditions were argued to approach a unique attractor solution. The reported jet has a nearly paraboloidal shock front, and, with increasing distance from the shock, shows a monotonic decline in the local Lorentz factor $\gamma$, the proper pressure $p$, and the (cylindrically, henceforth) radial velocity $u$.

While these results are promising, the universality, uniqueness, and full structure of the attractor were until now unclear. The jet structure was derived from simulations rather than by directly solving the flow equations. This structure was reported only for the head region, within a distance $1 < \xi \lsim 5$ of the light signal preceding the jet, normalized such that $\xi = 1$ is the nose position (see Equation (18) for a precise definition of $\xi$).

The simulations were reported to approach the attractor very slowly, suggesting that they have not fully converged upon it. A possible sign reversal in $u$ was reported far downstream, probably in the non-converged region, suggesting some deviation from a parabolic profile (G07).

Several questions have thus remained open. What are the full equations describing the self-similar, ultrarelativistic jet? These equations were only partly derived in G07. What is the solution (or solutions) to these equations? What is the corresponding normalization of the self-similarity scaling? Is the solution unique, and if so—what provides closure to the solution system? What is the structure of the jet, in particular far from the nose? Does it show the flow monotonicity reported at the head? Is most of the energy contained in the head, or does it lie beyond it, in regions not yet reported, and probably not converged?

Here we address these questions. First, we derive the equations for an ultrarelativistic, self-similar jet in Section 2, following and supplementing G07. Analytic and semi-analytic constraints on the solution are derived in Section 3. In Section 4, we finally solve the equations numerically, and show that their regular solution is unique and satisfies the constraints obtained in Section 3. The results are summarized and discussed in Section 5. The observational implications and stability of the solution are deferred to a forthcoming publication.

We assume (i) an axisymmetric relativistic flow expanding into a homogeneous medium in flat space; (ii) that the effects of cooling, electromagnetic fields, and self-gravity are negligible; and (iii) self-similarity in time. We normalize the speed of light, $c = 1$.

2. SELF-SIMILAR ULTRARELATIVISTIC EQUATIONS

2.1. Hydrodynamic Equations

Following G07 and using similar notations, we consider a directed flow propagating along the positive $z$ direction, where
\(x^\mu = (t, x, y, z)\) are cartesian coordinates in the upstream frame. The flow downstream of the shock is parameterized by the squared Lorentz factor \(q \equiv \gamma^2\), the proper pressure \(p\), and the cylindrically radial (i.e., perpendicular to the axis of symmetry) velocity \(u\). We thus analyze these downstream properties using upstream frame coordinates, henceforth omitting the terms upstream and downstream in this context.

The flow is governed by the relativistic hydrodynamic equation

\[
\partial_t T^{\mu\nu} = 0, \tag{1}
\]

where \(T^{\mu\nu} = (4u^\mu u^\nu - g^{\mu\nu})p\) is the energy-momentum tensor, \(u^\mu\) is the four-velocity, \(g^{\mu\nu}\) is the Minkowski metric, and we assumed a relativistic fluid with an enthalpy density of \(4p\). Assuming axial symmetry in cylindrical coordinates \((t, r, \phi, z)\), the four-velocity becomes \(u^\mu = \gamma(1, u^t, u^r, 0)\). The \(t, z, and r\) components of Equation (1) are then

\[
\partial_t [(4q - 1)p] + 4\partial_r (qpv) + 4r^{-1}\partial_r (rqup) = 0, \tag{2}
\]

\[
4\partial_r (qpv) + \partial_r \left[ (4q^2 + 1)p \right] + 4r^{-1}\partial_r (rqup) = 0, \tag{3}
\]

and

\[
4\partial_r (qpu) + 4\partial_r (qupu) + 4r^{-1}\partial_r (rqu^2) + \partial_r p = 0. \tag{4}
\]

In the upstream medium (denoted by a tilde), pressure is negligible, so the only non-zero component in its energy-momentum tensor \(\tilde{T}^{\mu\nu}\) is \(\tilde{\gamma} = \tilde{T}^{00}\). Parameterizing the shock (subscript \(s\), henceforth) surface as the zero isosurface of a scalar function

\[
S(x^\mu) = \gamma = z_s(t, r) = 0, \tag{5}
\]

the requirement of continuous energy-momentum fluxes across the shock can be written as

\[
T^{\mu\nu}\partial_\mu S = \tilde{T}^{\mu\nu}\partial_\mu S. \tag{6}
\]

The \(t, z, and r\) components of these constraints become

\[
4q_s p_s (\partial_z z_s - v_z + u_z \partial_r z_s) - p \partial_r z_s = \tilde{\gamma} \partial_r z_s, \tag{7}
\]

\[
4q_s v_s (\partial_z z_s - v_z + u_s \partial_r z_s) - 1 = 0, \tag{8}
\]

and

\[
4q_s u_s (\partial_z z_s - v_z + u_s \partial_r z_s) + \partial_r z_s = 0. \tag{9}
\]

The flow is given by a solution to Equations ((2)–(4)), with shock boundary conditions ((7)–(9)). Notice that in Equation (5) we assumed that \(z_s\) is a function of \(r_s\), and not vice versa, such that the jet is infinitely long, and monotonically widens with \(z\). This excludes possible scenarios in which the jet widens near the head but narrows down farther downstream, as found in some simulations (e.g., Granot et al. 2000; Cannizzo et al. 2004). The assumption can be avoided by parameterizing the shock using \(r_s(t, z)\), instead of \(z_s\); see Section 5.

2.2. Ultrarelativistic Limit

In the ultrarelativistic limit, \(q^{-1}\) can be used as a small expansion parameter, giving

\[
v = \sqrt{1 - q^{-1} - u^2} = 1 - \frac{1}{2} \frac{qu^2}{q} + O(q^{-2}), \tag{10}
\]

where we assumed the scaling \(q u^2 = O(1)\), as confirmed by the results (see G07 and Section 4). We define

\[
w_1 \equiv 1 + q u^2 \text{ and } w_2 \equiv 1 + 2q u^2, \tag{11}
\]

for future convenience.

It is useful to replace \(z\) by a coordinate \(\zeta \equiv t - z\), measuring the distance to a plane perpendicular to the jet and moving ahead of it at the speed of light. Equations ((2)–(4)) then become, to leading order in \(q^{-1}\),

\[
4\partial_r (q p) + \partial_r (w_2 p) + (4/r) \partial_r (r q p u) = 0, \tag{12}
\]

\[
\partial_r (w_2 p) + \partial_r (w_2 p q) + (2/r) \partial_r (r w_2 p u) = 0, \tag{13}
\]

and

\[
4q p \partial_r u + 2p w_1 \partial_r u + u \partial_r p + \partial_r u + 4q p u \partial_r u = 0. \tag{14}
\]

Note that we added the last term in Equation (14) to correct Equation (18) of G07.

In a similar fashion, the shock boundary conditions ((7)–(9)) at \(\zeta = \zeta_s(t, r) = t - z_s\) become, to leading order in \(q^{-1}\),

\[
q_s = \frac{1}{4\partial_s \zeta_s + 2(\partial_s \zeta_s)^2}, \quad p_s = \frac{4}{3} \tilde{e} q_s, \quad \text{and } u_s = \partial_r \zeta_s. \tag{15}
\]

2.3. Similarity

The ultrarelativistic Equations ((12)–(15)) remain invariant if the vector

\[
\Psi \equiv (\zeta, r^2, \zeta q, \zeta p, \zeta^2 / u^2) \tag{16}
\]

is multiplied by an arbitrary constant, and so they admit a self-similar scaling. In \(D\) spatial dimensions, the total energy of a self-similar solution would scale as

\[
E_{\text{tot}} \propto \zeta_s^D \propto q_s^{D-1} p_s q_s \propto \zeta_s^D \propto \zeta_s^{D-3} \frac{D-3}{2}, \tag{17}
\]

where a star denotes a typical value at time \(t\). This energy is constant during the self-similar phase, imposing scalings such as \(q_s \propto \zeta_s^{1/(D-3)}\) and \(r_s \propto \zeta_s^{1/(D-3)}\). These scalings hold for \(D \geq 3\), but in three spatial dimensions they degenerate into exponential functions. Indeed, the equation system remains invariant under rescaling of the vector \(\Psi\) in Equation (16), even if the first, \(t\)-dependent component is eliminated from \(\Psi\).

The resulting exponential scaling motivates the dimensionless, self-similar, axial and radial coordinates

\[
\xi \equiv \Lambda \frac{\zeta}{\tau} \text{ and } \eta \equiv \Lambda \frac{\zeta}{\tau}, \tag{18}
\]

and the self-similar functions describing the Lorentz factor squared, the pressure, and the radial velocity,

\[
Q \equiv \Lambda^{-2} q, \quad P \equiv \Lambda^{-2} \frac{p}{q}, \quad \text{and } U \equiv \Lambda u. \tag{19}
\]

Here, the temporal scaling factor

\[
\tau \equiv \frac{E_{\text{tot}}}{\tilde{e}} \tag{20}
\]

is large in the self-similar regime,

\[
\tau = C \left( \frac{E_{\text{tot}}}{\tilde{e}} \right)^{1/3} \tag{21}
\]
is the e-fold expansion time, $C$ is a dimensionless constant, and we defined a final time $t_f$ when the flow is no longer relativistic. The approximation holds for $q \gg 1$ and $u \ll 1$, so $(t - t_f)$ is assumed to be negative and with respect to $\tau$.

The total energy in the jet is related by $E_{\text{tot}} = \tau^3 E_{\text{tot}}$ to the total self-similar energy

$$E_{\text{tot}} = \int E \, dV = 8\pi \int_0^\infty \eta \, d\eta \int_{\xi(\eta)}^{\infty} d\xi \, Q \, P,$$

where the self-similar energy density is $E = 4Q^2P$. This fixes the constant $C = E_{\text{tot}}^{1/3}$.

Rewriting Equations ((12)–(15)) using these definitions finally gives the $t, z$, and $r$ components of Equation (1) in the self-similar regime.

$$(4 + \partial_\psi)(QP) - \frac{\partial_\psi(w_2P)}{4} - \frac{\partial_\psi QPU}{\eta} = 0, \quad (23)$$

$$(2 + \partial_\psi)(w_2P) - \partial_\psi(w_1^2P/Q) - \frac{\partial_\psi QPU}{\eta} = 0, \quad (24)$$

and

$$(Q_\psi + 1 - \partial_\psi)U + \frac{w_1}{2Q} \psi U + \frac{\partial_\psi + \partial_\psi QPU}{4Q} = 0, \quad (25)$$

where we defined $\psi = \ln((\xi^2/\eta))$ for brevity. The boundary conditions are written in self-similar form on the surface $\xi = \xi_s(\eta) \equiv (N^2/\tau)\xi_s$, as

$$Q_s = \frac{1}{8\xi_s - 4\eta^2} + 2\xi_s^2, \quad P_s = \frac{4}{3}Q_s, \quad \text{and} \quad U_s = \xi_s'.$$

The scaling (19) implies that $w_1 = 1 + QU^2$ and $w_2 = 1 + 2QU^2$ can be written in a self-similar fashion, in a form identical to Equation (11). Note that we added the first term in the parenthesis of Equation (25) to correct Equation (27) in G07.

### 2.4. Equation Properties

The self-similar Equations ((23)–(25)) and boundary conditions (26) remain invariant if $\Lambda$ is multiplied by an arbitrary constant. This constant can in principle be chosen as $(-1)$, indicating that $Q$ and $P$ are symmetric, while $U$ is antisymmetric, under the reflection $\eta \to (-\eta)$ across the axis of symmetry (henceforth, the axis).

Without loss of generality, henceforth we choose this arbitrary constant such that the nose, i.e., the very head (subscript $h$), henceforth) of the jet is at

$$\xi = \xi_h = \xi_s(\eta = 0) = 1.$$  

Equation (26) then implies that at the head, the flow parameters are given by

$$Q_h = \frac{1}{8}, \quad P_h = \frac{1}{6}, \quad U_h = 0, \quad \text{and} \quad E_h = \frac{1}{12}. \quad (28)$$

Here and below, we define $F_0$, $F_h$, and $F_{\psi}$ as the field $F \in \{Q, P, U, E\}$ evaluated at the shock, on the axis, and at the head, respectively.

Notice that Equations ((23)–(25)) are, in addition, unchanged if $P$ is multiplied by an arbitrary constant, but this constant is fixed by the boundary conditions (26).

The equation system involves three first-order partial differential equations, along with three boundary conditions, for the three fields $\{Q, P, U\}$ living in the two-dimensional $\xi - \eta$ space. In addition, the equation system depends on the unknown one-dimensional function $\xi_s(\eta)$, so one additional constraint may appear to be missing. As we show below, the system is closed, and the shock profile $\xi_s(\eta)$ is fixed, by requiring that the solution be regular, even without specifying boundary conditions far downstream.

The oriented derivative along the shock may be defined as

$$\partial_\psi \equiv \partial_\eta + \xi_s'(\eta)\partial_\xi. \quad (29)$$

For a shock profile that is monotonic, in the sense that the function $\xi_s(\eta)$ monotonically increases, one may alternatively write the derivative in Equation (29) and the boundary conditions (26) using the parameterization $\eta_s(\xi)$, instead of $\xi_s(\eta)$. This more general parameterization is used in Section 4.

If the shock profile is known, or is postulated, one may use Equation (29) to turn the self-similar Equations ((23)–(26)) into an infinite series of ordinary differential equations for increasingly high-order field derivatives, in either the $\xi$ or the $\eta$ direction. This is utilized below, in Section 3, and in Appendix D.

For a monotonic shock profile, $\xi_s'(\eta) > 0$, Equation (26) implies that $U_s > 0$, i.e., near the shock, the radial flow is always directed outward. Combining Equations (25), (26), and (29) indicates that

$$\partial_\eta U_s = 0, \quad (30)$$

which may be interpreted as a vanishing self-similar radial acceleration. Note that the term we added to Equation (27) of G07, in order to obtain Equation (25), is essential for recovering this property. In particular it implies, using Equation (29), that $\partial_\eta U_s = \xi_s'$.

### 3. SEMI-ANALYTIC CONSTRAINTS

#### 3.1. Overview of the Solution

Expand the shock profile above the nose of the jet, $\xi_s = 1 + \xi_s\eta^2 + \xi_s\eta^4 + O(\eta^6)$, where $\xi_s$ and $\xi_s$ are numerical constants, and we used reflection symmetry to omit odd powers of $\eta$. The coefficient $\xi_s$ is unlikely to vanish, as confirmed numerically below, so the head of the jet is approximately a paraboloid. Define the logarithmic shock profile slope,

$$\beta \equiv \frac{d \ln \xi_s}{d \ln \eta},$$

such that at the nose $\beta \to 2$.

A parabolic, $\beta = 2$ shock profile is stationary, in the sense that a point $\{r, \zeta_s(r) \sim r^2\}$ is mapped at a time $\Delta t$ later onto $\{\Lambda r, \Lambda^2 \zeta_s(r) \sim (\Lambda r)^2\}$, with $\Lambda = e^{\Delta t/\tau}$. In non self-similar coordinates, a self-similar $\beta < 2$ (i.e., wide jet) profile narrows down in time, eventually approaching $\zeta_s \sim r^2$, whereas a $\beta > 2$ (narrow) shock gradually widens toward $\zeta_s \sim r^2$.

Far from the head, the self-similar shock profile must become narrower than a paraboloid, i.e., $\beta > 2$, because $\beta < 2$ leads to nonphysical results, in particular a divergence of the energy in Equation (22). Moreover, $\beta < 2$ leads to only non-
real valued solutions (see Appendix A), while $\beta = 2$ leads to a divergence of $Q$ (see Section 3.3 and Figure 1), and cannot be matched to the axis region (see Appendix B).

At large distances from the head, one may approximate $\beta$ as a constant, up to logarithmic corrections. For such power-law behavior, $\xi _{s} \propto \eta ^{4}$, an additional, geometrical self-similarity (GSS) in the $\xi - \eta$ plane may emerge far from the head and from the axis, with the self-similar parameter (G07)

$$\chi _{0} \equiv \xi / \eta ^{3},$$

as discussed in Section 3.2.

A GSS scaling in the relevant, $\beta > 2$ regime (demonstrated in Section 3.4 and in Figure 2) breaks down near the head and near the axis, as equation terms that do not follow the GSS scaling become large and can no longer be neglected. Denote the surface along which the axial terms break the GSS scaling as $\xi = \xi _{s}(\eta )$. One may define this surface as the boundary between an axial, or core, region at $\xi > \xi _{s}$, and the GSS region at $\xi < \xi _{s}$. In Section 4 we show numerically that this boundary can be associated with a maximal value of $Q(\eta )$, giving $\xi _{s} \approx 20\eta ^{3}$.

In our $\beta > 2$ regime, Equations ((26)–(30)) imply that $\partial _{\eta }U = \eta ^{\beta } > 0$, so near the shock the (positive) radial velocity increases radially outwards. At large radii, $\beta > 2$ implies that $Q$ and $P$ similarly become monotonically larger as one approaches the shock, such that $\partial _{\eta }[Q_{s}, P_{s}] < 0$ and $\partial _{\eta }[Q_{s}, P_{s}] > 0$, as seen by solving Equations (23–26), with the aid of Equation (29), for these derivatives at the shock.

This behavior, in which $F \in \{Q, P, U, E\} > 0$ increases monotonically toward the shock, namely

$$\partial _{\eta }F \leq 0 \quad \text{and} \quad \partial _{\eta }F > 0,$$

is henceforth referred to as flow monotonicity. The simulations of G07 show this type of behavior throughout the reported, $1 \leq \xi < 5$ regime. It is therefore natural to ask if flow monotonicity persists throughout the jet.

Within the GSS regime, $\beta > 2$ solutions are found to transition at some $x_{0} = x_{0}$ from $U(x_{0} < 1) > 0$ toward the shock, to $U(x_{0} > 1) < 0$ toward the axis (see Section 3.4). Indeed, far from the head, $Q_{s}(\xi \gg \xi _{s}) \propto \xi ^{-\beta }$ approaches a power-law along the axis, with $\alpha \leq 1$, which, as we show, implies that $\partial _{\eta }U(\eta = 0) \to (\alpha - 1)$ asymptotes to a negative constant (except in the special case $\alpha = 1$; see Section 3.5). This is consistent with a $U < 0$ inflow emanating at $\chi _{s}$ within the GSS regime, and extending all the way to the axis. Thus, the radial velocity reverses sign, corresponding to a radial inflow near the axis, and a radial outflow near the shock. As the radial velocity vanishes along the axis, $U_{a} = 0$, this indicates that $U$ cannot be monotonic.

To see that $Q$ cannot be monotonic either, recall that along the axis, $Q_{a} \propto \xi ^{-\alpha }$ declines with increasing $\xi$ no faster than $\xi ^{-1}$. In contrast, along the shock, Equation (26) with $\beta > 2$ implies that $Q_{s} \propto \xi ^{-2(1-\beta ^{-1})}$ declines faster than $\xi ^{-1}$. Hence, there must be a region where $Q$ declines as $\eta$ increases toward the shock, ruling out monotonicity in $Q$. This behavior is confirmed in the GSS regime (see Sections 3.3 and 3.4).

Nevertheless, flow monotonicity does manifest in the head region. Requiring such monotonicity near the head implies that the jet cannot be too narrow, constrains the flow in the head region, and limits the extent of this region to $\xi < \xi _{g} \approx 5$ (see Section 3.6 and Figure 3).

Although the shock profile is not directly constrained by the system of equations, and no far downstream boundary condition is imposed, we find that only a unique profile avoids a divergence of the flow. For example, if the jet is too wide, $q = \gamma ^{2}$ becomes negative on the axis (see Appendix C), which is both nonphysical and leads to divergencies. Indeed, as shown in Section 4, numerically solving the equations and requiring a regular jet picks out a unique flow solution (see Figures 4–7), which agrees with all the features mentioned above and derived quantitatively below.

The above arguments indicate that a regular self-similar jet has a unique solution, composed of three distinct flow regions: a monotonic head region ($\xi \leq \xi _{g}$), an effectively one-dimensional, GSS envelope ($\xi _{g} \leq \xi \leq \xi _{s}$), and an axial, or core, region ($\xi \geq \xi _{s}$). A radial inflow encompasses the core and the inner envelope regimes. Only the head and the outer part of the envelope, which harbor a radial outflow, show a monotonic flow.

### 3.2. Geometric Self-similarity

At large distances from the head of the jet, the shock profile may be approximated by a power law, in which case the temporally self-similar Equations (23)–(26) may become further simplified. Using a scaling parameter $\chi _{0}$ such as that defined in Equation (32), these equations can be cast in a geometrically, and not only temporally, self-similar form.

It is useful to introduce a slightly different GSS parameter,

$$\chi = \frac{\xi - \xi _{h}}{A\eta ^{3}},$$

where the unknown normalization constant $A$ is introduced such that the position of the shock in the GSS regime can be taken as $\chi = \chi _{s}$, with a constant $\chi _{s}$ which we choose as $\chi _{s} \equiv 1$.

The case $\beta < 2$ does not lead to physical GSS solutions, as mentioned above and discussed in Appendix A. Therefore, here we focus on $\beta \geq 2$.

The flow equations can then be written approximately as functions of $\chi$, if the parameters are rescaled as

$$Q(\chi ) \equiv A^{2}\eta ^{2(\beta - 1)}Q,$$

$$P(\chi ) \equiv A^{2}\eta ^{2(\beta - 1)}P,$$

$$U(\chi ) \equiv A^{2}\eta ^{2(\beta - 1)}U.$$

Equations (23)–(25) now become, respectively,

$$\frac{(w_{2}P')'}{4} - \beta \chi (QP')' - \frac{2\xi _{h}}{\eta ^{2(\beta - 1)}A^{2}}(QP')' + \beta - 2\frac{Q}{\eta ^{2(\beta - 1)}} = 0,$$

$$\left(\frac{w_{2}P}{Q}\right)' - 2\beta \chi (w_{1}P')' - 2(\beta - 2)w_{1}P' = 0.$$
and
\[
(w_1 - 2\beta \chi Q U')U' - (2 - w_2)(\beta - 1) + (U - \beta \chi) \frac{P'}{2P} - \frac{4\xi h}{\eta^{2(\beta - 1)}A^2} Q U' + \frac{2(\beta - 2)\chi^2 Q}{\eta^{\beta - 2}A} \left( \frac{U'}{\chi} \right)' = 0,
\]
where the scaling (35) maintains \( w_1 = 1 + Q U'^2 \) and \( w_2 = 1 + 2Q U'^2 \) in GSS form. Note that in Equation (38), we added the second parenthesis, and the second term in the first parenthesis, to correct Equation (36) of G07.

The last two terms in each of the above three Equations ((36)–(38)) do not follow the GSS scaling. However, the first of the two terms is negligible far from the head, and the second is negligible far from the axis. Note that the last term in each equation vanishes in the special case \( \beta = 2 \). GSS behavior is therefore expected to emerge far from the axis (for \( \beta \geq 2 \)), or even on the axis but far from the head (for \( \beta = 2 \)). Far from the axis (and thus also from the head), the shock boundary conditions on \( \chi = 1 \) asymptote to the GSS form
\[
Q_{\chi} = \frac{1}{2\beta}, \quad P_{\chi} = \frac{2}{3\beta^2}, \quad \text{and} \quad U_{\chi} = \beta.
\]

Finite energy solutions exist only for \( \beta > 2 \). Indeed, in Section 4, we numerically find that the full (self-similar but generally non-GSS) solution asymptotes to \( \chi \approx 2.02 \) far from the axis. Before addressing such \( \beta > 2 \) solutions, we first discuss the special case \( \beta = 2 \), for which an analytic flow solution can be found. Although the GSS equations are in principle valid in this case even as \( \eta \to 0 \), the solution itself is shown to diverge near the axis.

3.3. Analytic GSS Solution for \( \beta = 2 \)

As mentioned above, in the special case \( \beta = 2 \), the last terms in Equations (36)–(38) vanish. The equations can then be solved analytically far from the head, where the \( \xi \) terms are negligible. For the boundary conditions (39), the solution is
\[
U = 6\chi - g_1 - \sqrt{4g_1^2\chi(4\chi^2 + 27)} + 12\chi^2 - 54 - g_1^2,
\]
\[
Q = (8\chi U - 2U^2)^{-1},
\]
\[
P = \frac{1}{6} \exp \left[ \int_1^{\infty} \left( \frac{4}{U - 4\chi} + \frac{U'}{2\chi - U} \right) d\chi \right],
\]
where for brevity we defined \( g_1 = [4\chi^2 + 3(g_2 - 3)^2/g_2]^{1/2} \) and \( g_2 = [27 + 32\chi^4 + 8\chi^2(27 + 16\chi^4)^{1/2}]^{1/3} \). This solution is shown in Figure 1, as an azimuthal cross section through the jet.

Far from the shock front (\( \chi \gg 1 \)), the leading terms in this solution are
\[
Q \approx \frac{1}{27} + \frac{3}{64\chi^2} \quad \rightarrow \quad Q \approx \frac{1}{27\eta^2A^2},
\]
\[
P \approx \frac{0.046}{\chi} \quad \rightarrow \quad P \approx \frac{0.046}{A\xi},
\]
and
\[
U \approx \frac{27}{8\chi} \quad \rightarrow \quad U \approx \frac{27\eta^3A^2}{8\xi}.
\]
Hence, although Equation (40) provides an exact solution to the GSS equations when \( \xi_{\eta} \to 0 \), and so gives an asymptotic solution for \( \xi \gg \xi_{\eta} \), this solution is not physical on the axis, where \( Q \) diverges. This local divergence is more severe than, and is not directly related to, the global logarithmic divergence of the total energy, associated with the marginally wide \( \beta = 2 \) shock profile.

Nevertheless, the \( \beta = 2 \) solution does provide an adequate analytic approximation of the jet in the outer envelope region, where \( \chi \) is not too large. For example, in a radial cross section, it shows a minimal \( Q \) where \( \theta = \partial_\theta Q \propto \beta \chi Q^2 + 2(\beta - 1)Q \). This occurs at \( \chi \approx 1.61 \), or equivalently at a fraction
\[
f \equiv \frac{\eta}{\eta_{\chi}} = \chi^{-1/\beta} \approx 0.79
\]
of the shock radius, close to the numerical value found for the full solution in Section 4. Moreover, its \( P \) and \( U \) profiles (see Figure 1) agree qualitatively with the full solution even in the head (Figures 3 and 6) region, as well as with the simulated (G07) head.

3.4. GSS with \( \beta = 2.02 \)

As the \( \beta \leq 2 \) GSS regime is ruled out in Section 3.3 and in Appendix A, here we consider \( \beta \) slightly larger than 2. The numerical solution in Section 4 indicates that far from the head, the shock indeed approaches a \( \beta \approx 2.02 \) profile. Accordingly, we now derive the GSS solution for \( \beta = 2.02 \), but point out that the qualitative features of the solution are not sensitive to the precise value of \( \beta \) in the \( 2 < \beta < 3 \) range.

In this range of \( \beta \), the GSS Equations ((36)–(39)) form a closed system of ordinary differential equations, which can only be solved numerically. The solution for \( \beta = 2.02 \) is shown in Figure 2.

As the figure shows, while the \( P \) profile is not qualitatively changed with respect to the analytic \( \beta = 2 \) case, the profiles of \( Q \) and \( U \), and subsequently also of \( E \), are significantly altered. Most importantly, \( Q \) no longer diverges near the axis, and never becomes a function of \( \eta \) alone. Such \( \beta > 2 \) solutions are therefore physical, unlike the diverging \( \beta = 2 \) profile, and can in principle be matched to the near-axis solution.

Interestingly, the \( U \) profile is not everywhere positive, i.e., the radial flow is not everywhere an outflow. Unlike the \( \beta = 2 \) case, \( U \) becomes negative at large \( \chi \), corresponding to a radial inflow converging on the axis. For \( \beta = 2.02 \), the transition occurs at \( \chi \approx 20.3 \), or equivalently at \( f \approx 0.23 \). The exact location of the transition is sensitive to the precise value of \( \beta \). For example, \( \beta = 2.04 \) gives \( f \approx 0.32 \).

The minimum of \( Q \) along a radial cross section is found at \( f \approx 0.77 \), not far from the corresponding minimum Equation (42) in the \( \beta = 2 \) case. This result is less sensitive than the \( U = 0 \) contour to the precise value of \( \beta \), giving, for example, a similar \( f = 0.75 \) value for \( \beta = 2.04 \).

3.5. Axial Expansion and its Monotonicity Constraint

As shown in Sections 3.3 and 3.4 above, the axial region of the jet is distinct from the GSS envelope. It is also distinguishable from the head region, as shown in Section 3.6 below. Indeed, the axial structure shows that the monotonic nature of the flow near the head, in which the fields \( F \) increase
monotonically toward the shock in both the \((-\xi)\) and \(\eta\) directions, cannot hold far beyond the head region. To see this, expand \(Q\) and \(P\) near the axis in even powers of \(\eta\),

\[
Q(\xi, \eta) = Q_a(\xi) + Q_2(\xi) \eta^2 + Q_4(\xi) \eta^4 + \ldots \tag{43}
\]

and

\[
P(\xi, \eta) = P_a(\xi) + P_3(\xi) \eta^2 + P_4(\xi) \eta^4 + \ldots, \tag{44}
\]

and expand \(U\) in odd powers of \(\eta\),

\[
U(\xi, \eta) = U_1(\xi) \eta + U_3(\xi) \eta^3 + U_5(\xi) \eta^5 + \ldots. \tag{45}
\]

Here, the \(F_n\) are numerical factors. An \(O(\eta)\) expansion of the flow equations near the axis now shows that

\[
U_1(\xi) = \partial_\eta U(\xi, \eta = 0) = \frac{Q_a'}{Q_a^2} - 1 + \frac{9}{2} - \frac{Q_a'}{Q_a^2} \left(4\xi^2 + \frac{5}{4Q_a^2}\right). \tag{46}
\]

For large \(\xi \gg \xi_h\), one can approximate the axial behavior of \(Q\) as a power-law, \(Q_a \simeq Q_{a0} \xi^{-\alpha}\), where \(Q_{a0}\) is a constant, so Equation (46) gives

\[
U_1(\xi \gg \xi_h) \simeq \frac{7}{2} - 5\alpha + \frac{\alpha Q_{a0}^{\alpha-1}}{4Q_{a0}} + \frac{6(4\alpha - 3)Q_{a0}^{\alpha}}{4Q_{a0}^2\xi + \xi^\alpha}. \tag{47}
\]

Figure 1. Self-similar jet in the GSS regime, for the special, analytically solvable case \(\beta = 2\). The azimuthal cross section shows the self-similarly scaled Lorentz factor squared \(Q\), proper pressure \(P\), radial velocity \(U\), and energy density \(E\) (see labels). Color maps (cubehelix; Green 2011) and contours (in interval factors of 2 for \(Q\), \(P\) and \(U\), and in factors of 4 for \(E\)) show each quantity, scaled by the shock width normalization \(A\) (see Equation 34), with \(P\), \(Q\), and \(E\) also normalized by their head values. Note the divergence of \(Q\) (and so, also of \(E\)) toward the axis.
If \( \alpha > 1 \), then \( U_1(\xi \gg \xi_h) \) diverges due to the third term. In addition to being nonphysical, this also breaks monotonicity, as at the head \( U_1 \) is finite. If \( \alpha < 1 \), then \( U_1(\xi \to \infty) = \alpha - 1 < 0 \). This rules out monotonicity as well, and is consistent with the \( U < 0 \) inflow inferred near the axis from the GSS analysis (see Section 3.4).

A fully monotonic flow near the axis is thus possible only for the special case \( \alpha = 1 \), such that

\[
U_1 = \frac{1 - 2Q_{a0}}{4Q_{a0} + 16Q_{a0}^2} = \text{const.} \geq 0,
\]

which also requires \( Q_{a0} \leq 1/2 \). We conclude that only a \( Q_a \) profile with \( \alpha = 1 \) and \( Q_{a0} \leq 1/2 \) can yield a fully monotonic behavior near the axis. However, as mentioned in Section 3.1, such monotonicity cannot persist radially out to the shock, as this would contradict the faster decline of \( Q_1(\xi) \) according to the \( \beta > 2 \) shock boundary conditions.

### 3.6. Monotonic Head Region

Near the head, previous simulations (G07) and our numerical solution (Section 4) suggest a monotonic behavior. This has several interesting implications.

Near the head, the shock profile is nearly parabolic, so one may approximate \( \xi_s \approx \xi_h + A\eta^2 \). Here, the arbitrary constant \( A \) coincides with that defined in Equation (34), for \( \beta \to 2 \). Equations ((23), (26), (29)) then yield

\[
\partial_\eta Q \approx \frac{A(2 - 3A)}{4\eta} \quad \text{and} \quad \partial_\xi Q \approx \frac{A - 1}{4},
\]

so imposing monotonicity would imply that the jet cannot be too narrow, i.e., that \( A < 2/3 \). The jet cannot be too wide, either; \( A \lesssim 0.1 \) (\( A \lesssim 0.3 \)) can be shown semi-analytically (numerically) to lead \( Q \) to vanish near the head; see Appendix C.

One may attempt to solve the self-similar equations for a monotonic flow. To do so, we numerically minimize the sum of
the squares of the left hand sides of Equations ((23)–(25)),
while imposing the boundary conditions (26), and, in addition,
constraining the fields to be monotonic. This leads to an
approximate solution, illustrated in Figure 3.

The resulting monotonic profiles qualitatively resemble the
structure of the head found numerically and in G07, as shown
in the figure. We find such solutions only for \( \xi < 5 \), beyond
which the monotonic fields diverge. This suggests that \( \xi \approx 5 \)
roughly marks the edge of the monotonic head region.

An approximate, monotonic description of the flow in the
head region may be found using the axial expansion in
Equations (43–45); see Appendix D. The resulting approxima-
tion is plotted on top of the numerical solution of the head
region in Figure 6, using an approximate shock profile \( \xi(\eta) \)
inferred from the numerical solution of Section 4. As the figure
shows, the expansion fits the results rather well near the head.

4. NUMERICAL SOLUTION

In order to solve the self-similar flow Equations (23–26)
numerically, we expand the shock profile \( \eta_\beta(\xi) \) to increasingly
high order, and find the optimal jet solution at each order, in
two different methods. Following the arguments of Section 3,
an optimal solution is defined as the most regular solution to
the flow equations, i.e., the solution diverging farthest either
from the head or from the shock. The results of this procedure
indicate that a unique solution, which remains regular infinitely
far from the head and from the shock, indeed exists.

4.1. Method

First, we map the (azimuthal cross section of the) jet onto the
unit square, \( 0 \leq \rho, \sigma \leq 1 \), through the transformation
\[
\rho \equiv 1 - \frac{\eta}{\eta_\infty} \quad \text{and} \quad \sigma \equiv 1 - \frac{\xi}{\xi_\infty}.
\]
This maps the shock and the axis, respectively, onto \( \rho = 0 \) and
\( \rho = 1 \). The head and infinite downstream (\( \xi \rightarrow \infty \)) are
similarly mapped onto \( \sigma = 0 \) and \( \sigma = 1 \).

Next, the shock profile is parameterized as
\[
\xi_s = (\xi_\infty + A\eta^2)^{\beta(\sigma)/2},
\]
where we choose this functional form, using \( \eta_\beta^2 \) instead of \( \eta \) in
the parenthesis in order to obtain better behaved functions. We
expand the a priori unknown function \( \beta(\sigma) \) to order \( n \), and
switch from a \( \xi(\eta) \) parameterization to the more general, \( \eta(\xi) \)
description of the shock, such that
\[
\xi = \left(1 + A\eta^2\right)^{1+c_0+c_1\sigma^2+c_2\sigma^4+...+c_n\sigma^n}.
\]
(52)

Consider the solution for a given order \( n \). The shock profile
is defined by the \((n+2)\) undetermined parameters \( A \) and
\( c_0, c_1, \ldots, c_n \). Given some choice of these parameters, one can
integrate the equations in two different methods:

1. Start from the \((\sigma = 0)\) head boundary conditions, and
   advance toward \((\sigma = 1)\) downstream infinity.
2. Start from the \((\rho = 0)\) shock boundary conditions, and
   advance toward the \((\rho = 1)\) axis.

For finite \( n \), both methods eventually fail, as the fields
diverge at some finite \( \sigma_{\text{max}} \) (or equivalently \( \xi_{\text{max}} \)) in method 1,
and at some finite \( \rho_{\text{max}} \) (or equivalently \( \eta_{\text{max}} \)) in method 2.
Indeed, the shock profile must be fine-tuned in order to delay
the divergence, and uncover a larger fraction of the jet.

We use both methods, independently maximizing \( \xi_{\text{max}} \) and
\( \rho_{\text{max}} \) at each order \( n \) by scanning the \((n+2)\) dimensional phase
space of the shock profile parameters. Thus, we identify the
best approximation to the shock profile at every order.

The results of an order \( n = 2 \) scan are presented in Figure 4,
for the maximization of both \( \xi_{\text{max}} \) (left panel) and \( \rho_{\text{max}} \) (right
panel). In order to project our four-dimensional scan onto a
two-dimensional figure, here we set \( c_0 = c_1 = 0 \), and vary only
\( A \) and \( c_2 \). It is useful to define the parameter \( \beta_2 \) through
\[
c_2 = (-1 + \beta_2/2); \quad \text{this corresponds to a paraboloid}, \quad \xi_s \approx \xi_\infty + A\eta^2 \quad \text{profile for} \quad \xi \rightarrow \xi_\infty.
\]

Both methods show that the parameters \( A \approx (0.53–0.54) \)
and \( \beta_2 \approx (2.02–2.03) \) provide the most accurate shock profile
at this order, in the sense that the divergence of the flow takes
place farthest from the head or from the shock. The full \((n+2)\)
dimensional optimization process converges with \( n \) on a unique
shock profile, with similar \( A \) and \( \beta_2 \) values, as discussed next.
is maximized, is much smaller than the $\rho_{\text{max}}$ maximization of method 2. Therefore, after demonstrating that both methods give similar results at low orders, we pursue high orders using only the $\xi_{\text{max}}$ maximization method 1. Table 1 summarizes the results of this method, providing the parameters of the optimal shock profile at each order $n$.

The odd $n$ terms do not significantly modify the solution or increase the values of $\xi_{\text{max}}$. We therefore set these terms to zero, and use only the even $n$ terms. This is equivalent to taking $\beta(\sigma^2)$ instead of $\beta(\sigma)$ in the shock parameterization Equation (51).

For convergence tests, we thus define $N = 1 + n/2$ as the effective order of the shock expansion. Our highest order, $n = 16$, corresponds to $N = 9$, and thus involves searching for the maximum of $\xi_{\text{max}}$ in a 10 dimensional parameter space. Due to the high dimensionality, for orders $n = 6$ and above, our maximal $\xi_{\text{max}}$ values should be considered as lower limits, as noted in the table.

Figure 5 shows a convergence plot of our results with the inverse effective order $N^{-1}$ of the expansion. As the table and figure show, the maximal $\xi_{\text{max}}$ (disks in the figure) monotonically increases with $n$, and suggests convergence near our resolution limit (anticipated at $\xi_{\text{max}} \approx 50$). The shock profile is well behaved, in the sense that low order $e_n$ terms do not change significantly as higher order terms are added.

As the order $n$ increases, $e_0 \rightarrow 0$, corresponding to the expected paraboloid head. The largest coefficient in the high order $\beta(\sigma)$ expansion is $e_2 \approx 0.011$, corresponding to $\beta_2 \approx 2.022$. This dominates the deviation from a parabolic profile, and was therefore chosen as the term scanned in Figure 4.

Far downstream, as $\sigma \rightarrow 1$, the shock profile (52) approaches a power law, $\xi \propto \eta^{\beta_\infty}$, suggesting a GSS scaling with

$$\beta = \beta_{\infty} = 2 + \sum_{j=0}^{n} e_j .$$

4.2. Convergence

Method number 1, in which $\xi_{\text{max}}$ is maximized, is much simpler and faster computationally than the $\rho_{\text{max}}$ maximization of method 2. Therefore, after demonstrating that both methods give similar results at low orders, we pursue high orders using only the $\xi_{\text{max}}$ maximization method 1. Table 1 summarizes the results of this method, providing the parameters of the optimal shock profile at each order $n$.

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$$\beta = \beta_{\infty} = 2 + \sum_{j=0}^{n} e_j .$$

The values of $\beta_{\infty}$ found at different orders $n$ are shown in the table and in Figure 5 (diamonds). These results suggest that the shock profile converges, as $n \rightarrow \infty$, at $\beta_{\infty} \approx 2.02$.

The energy $E(<\xi_{\text{max}})$ is shown in Figure 5 (squares plotted against left and bottom axes) to converge slowly. The figure also shows (solid curve with right and upper axes) a better method to estimate the total energy of the jet, by extrapolating to $\xi \rightarrow \infty$ the $E(<\xi)$ profile of a solution with a fixed high $n$. The extrapolated result of this (good) fit is $E_{\text{extr}} \approx 2.1$; most ($\approx 80\%$) of this energy lies well beyond the head region.

4.3. Solution Existence and Uniqueness

Our convergence tests suggest that the procedure outlined above can in principle be continued indefinitely, to arbitrarily high order, such that a solution in the $n \rightarrow \infty$ limit exists. Namely, given sufficient computational power, the shock
profile could be adjusted such that the integration be successfully carried out to arbitrarily large $\xi_{\text{max}}$ and $p_{\text{max}}$.

The existence of a solution extending infinitely far downstream is further supported by the GSS analysis of Sections 3.2–3.4, in which a solution for the envelope was indeed shown to exist out to $\xi \to \infty$. However, in the absence of a matched solution for the core, we cannot rigorously prove the existence of the full jet solution much beyond our present $\xi_{\text{max}} \sim 25$ limit.

Even if we assume that the solution exists, there is no a priori guarantee that it is physically meaningful, because in realistic scenarios the assumption $q \gg 1$ of an ultrarelativistic flow eventually breaks down far downstream. As the downstream flow is subsonic with respect to the shock, the solution could in principle be completely altered once the unavoidable subsonic transition is incorporated. The physical relevance of the solution would be assured, however, if it is a sufficiently strong attractor.

Assuming that the solution exists, it does appear to be unique, because (i) we do not identify any other solution at low order $n \leq 4$, where the phase space can be throughly searched for additional solutions, as illustrated in Figure 4; (ii) such a search is double-checked using the two scanning methods described in Section 4.1, found to agree well with each other; (iii) high dimensional scans, although not as complete, show no evidence of other solutions; and (iv) the semi-analytic constraints of Section 3 limit the phase space of possible solutions.

Numerical simulations (e.g., G07) provide the best evidence that the solution indeed exists, is unique, is physically meaningful, and even behaves as a strong attractor and is likely to be stable, at least against axisymmetric perturbations. For, in such simulations, roughly tuned for external GRB shock parameters, the self-similar solution is found to emerge from different initial configurations, involving various relativistically moving blobs.

Finally, note that although in Sections 2 and 3 we assumed an infinite jet with $z_0 = z_0(t, r)$, the numerical solution here was derived using Equation (52), thus essentially assuming an $r_0 = r_0(t, z)$ shock profile. Hence, we also search for, and rule out, self-similar jet (i.e., relativistic, directed blast wave) solutions with non-monotonic $n_0(\xi)$ profiles, such as pinched jets.

### 4.4. Jet Structure

The numerical solution to the self-similar structure of the jet is shown, for the head region ($\xi < 5$) in Figure 6, and for the full range available ($1 < \xi < \xi_{\text{max}}$) in Figure 7, using a high ($n = 10$) order expansion of the shock profile, optimized in method 1. The corresponding shock profile parameters are listed in Table 1.

As Figure 6 shows, the numerical results in the head region are fairly well fit by the monotonic head model derived in Section 3.6 (dashed curves). It is also in qualitative agreement with the G07 simulations (dotted–dashed curves), although the latter correspond to a somewhat narrower shock profile.

In Figure 7, the slight deviation of the shock profile from a paraboloid shape (thick dashed curve) becomes apparent far from the head, as $\beta$ approaches its asymptotic value $\beta_{\infty} \approx 2.0$. Indeed, here the envelope of the jet qualitatively resembles the non-monotonic, $\beta = 2.02$ GSS profile of Figure 2.

More quantitatively, a radial cross section at large $\xi$ shows the minimum of $Q$ at $\xi = \eta_{b}/\eta_{c} \approx 0.75$ (dotted curve), and the sign flip of $U$ at $\xi = 0.33$ (dotted–dashed curve in the $U$ panel). These values deviate somewhat from those expected for $\beta = 2.02$; both would agree with $\beta = 2.04$ (see Section 3.4). This is not surprising, as our numerical solution only reaches $\eta_{\text{max}} \approx 6.5$, which may not be sufficiently large to show the asymptotic GSS behavior. Moreover, the $U = 0$ contour emerges from the axis only at $\xi = \xi_{U} \approx 5.5$; so it may not have converged by $\xi_{\text{max}} \approx 25$.

The breakdown of $Q$ monotonicity is already evident around $\xi = \xi_{Q} \approx 4$, and perhaps even closer to the head, but the precise location at which the “shoulder” in $Q$ (e.g., the wiggle in the $Q/Q_{b} = 0.3$ contour shown in Figures 6 and 7) emerges from the axis is not well converged.

At large $\xi$, this feature corresponds to a maximum of $Q$ in a radial cross section, located roughly at $f = f_{c} \approx 0.17$ (dotted–dashed curve in the $Q$ panel of Figure 7). This may be regarded as the boundary between the core and envelope regions. Note that, although the transition appears to take place at a constant $f_{c}$, this did not have to be the case, as the GSS scaling does not apply in the core.

The above results support the partition of the jet into three distinct regions, as anticipated in Section 3: a head region for $\xi < \xi_{c}$, a core region for $\xi_{Q} \approx \xi < \xi_{c}$, and a GSS envelope for $\xi > \xi_{c}$. The boundary of the head region corresponds to the breakdown of monotonicity, $\xi_{Q} \approx 5$, and is related to the onset of non-monotonic $Q$ and negative $U$ behavior, at $\xi_{Q}$ and $\xi_{c}$. The boundary between the core and the envelope can be identified as $f_{c} \approx 1/6$, corresponding to $\chi_{c} \approx 36$ or equivalently $\xi_{c} \approx 20\eta_{c}^{3}$.

### 5. SUMMARY AND DISCUSSION

The equations governing the structure of a self-similar, ultrarelativistic, directed blast wave are derived (Equations (23)–(26)), following and correcting G07. A numerical
analysis (Section 4) suggests a converging (Figure 5), unique (Figure 4) jet solution (Figures 6 and 7), which qualitatively agrees with previous simulations (G07) of the head region, and with our semi-analytic study (Section 3). The jet can be broadly partitioned into three distinct regimes: a head region ($\xi \lesssim \xi_{h} \approx 5$), an axial core ($\xi_{g} \lesssim \xi \lesssim \xi_{c} \approx 20 \eta^{1/3}$), and an envelope ($\xi \gtrsim \xi_{c}$). The highest Lorentz factors, $\gamma \gtrsim \gamma_{h}/2$, are found in the head, most of the energy lies in the envelope, and the core contains an axial inflow originating from the envelope.

In the head region, $Q \gtrsim (Q_{h}/4)$, $P \gtrsim (P_{h}/10)$, $U > 0$, the shock profile is very close to a paraboloid, and the flow is monotonic, in the sense that $Q$, $P$, and $U$ monotonically decrease away from the shock in both the $\xi$ and the $(-\eta)$ directions. This region, analyzed in Section 3.6 and in Appendix D, is strongly constrained by monotonicity (e.g., Figure 3); it qualitatively agrees with the head structure reported based on simulations in G07.

The core, analyzed in Section 3.5, shows a monotonic behavior in $Q$ and $P$. The radial velocity $U$ is negative here, corresponding to a radial inflow toward the axis; thus $(-U)$, rather than $U$, diminishes toward the axis. A radial cross section shows $Q$ increasing outward from the axis, until it reaches a maximum at $\xi = 16/3$; this marks the transition into the envelope region.

The envelope region follows an additional, geometric self-similarity (GSS), such that the two-dimensional flow in the $\xi-\eta$ plane becomes essentially one-dimensional when written in terms of the similarity variable $\chi \sim \xi/\eta^{1/3}$ (defined more precisely in Equation (34)). Far from the head, the shock power-law index $\beta = d(\ln \xi)/d(\ln \eta)$ asymptotes to $\beta \approx 2.02$; solving the GSS Equations (36–39) for this case (Figure 2) reproduces the envelope structure. Moreover, the $P$
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Figure 7. Full numerical solution for a high \((n = 10)\) order expansion of the shock profile. The difference between the shock front (thick solid curved) and a paraboloid (thick dashed blue) is small, but noticeable. Far from the head, the minimal \(Q\) is found in the envelope at \(f \approx 0.75\) (dotted cyan curve). Curves with non-monotonic \(Q\) first appear at \(\xi_q \approx 3.8\), roughly located at \(f \approx 0.17\) (dotted–dashed red curve); this may be regarded as the boundary between the core and envelope regions. An inflow, \(U < 0\) region appears at \(\xi_i \approx 5.5\), confined to \(f \leq 0.33\) (dotted–dashed yellow curve; the narrow \(U > 0\) stripe along the axis is probably a numerical artifact). The dotted–dashed curves are plotted using Equation (34), with \(\xi_q\) replaced by \(\xi_q\) or \(\xi_i\).

Our numeric solution appears to be converged, physical, and unique, because (i) it qualitatively agrees with the simulations of G07, as provided for the head region (Figure 6), and with the \(U\) sign change reported far downstream; (ii) it agrees qualitatively with the semi-analytic constraints of Section 3; (iii) it quantitatively agrees with the GSS envelope solution, which extends to \(\xi \to \infty\); (iv) convergence tests suggest that it extends to large \(\xi\), out to our resolution limit (Table 1 and Figure 5); and (v) no other solution is found in systematic scans of the shock profile at low order (e.g., Figure 4), performed in two different methods, nor in high order scans.

Nevertheless, we cannot prove that the solution is an attractor, or that it even survives in physical situations in which the far downstream transitions into the sub-relativistic regime, where our \(q \gg 1\) assumption fails. However, the simulations of G07 suggest
that a self-similar solution indeed exists, and is a physically relevant, strong attractor. Under this assumption, our results substantiate the existence of the solution, uncover its structure, in particular far from the head, and show that it is unique.

Moreover, the G07 simulations then imply that the solution is relevant for the typical parameters of external GRB shocks, and is probably stable, at least under axisymmetric perturbations. While computing the observational implications is straightforward, and a stability analysis of the self-similar solution appears feasible, we defer these to a forthcoming publication.

The self-similar regime is strictly valid in the limit of an initially extremely narrow and ultrarelativistic jet, such that causal contact is reached when the shock is still highly relativistic, i.e., the opening angle $\theta \lesssim \Gamma^{-1} \ll 1$. For finite initial opening angle $\theta_i$ and Lorentz factor $\Gamma_i$, the attractor is only partly filled. The simulations of G07 show that explosions with $\theta_i \sim 0.01 - 0.1$ and $\Gamma_i \sim 20$ indeed produce only a partial attractor, gradually filling up throughout the entire quasi-self-similar stage.

For GRB afterglow jets, simulations with $\theta_i \gtrsim 0.05$ typically show a slow, non-exponential deceleration and widening.

APPENDIX A

NO REAL-VALUED GSS SOLUTION FOR $\beta < 2$

Consider the case where $\xi_i \propto \eta^3$ is far from the axis, with $\beta < 2$. The GSS scaling is constrained here by the boundary conditions (26), giving rise to modified GSS coordinates $Q$, $P$, and $U$, defined by

$$Q = A^{-1}\eta^{-\beta}Q(\chi), \quad P = A^{-1}\eta^{-\beta}P(\chi), \quad U = A\eta^{\beta-1}U(\chi), \quad \chi \equiv \xi/(A\eta^3),$$

such that Equation (26) becomes

$$Q(1) = \frac{1}{8-4\beta}, \quad P(1) = \frac{1}{6-3\beta}, \quad \text{and} \quad U(1) = \beta.$$  

Far from the axis, the hydrodynamic Equations ((23)–(25)) now become

$$[1 + 4(\beta - 2)\chi Q]P'' + 4(\beta - 2)(2Q + \chi Q')P = 0,$$

$$[1 + (\beta - 2)\chi Q]Q'P' + [(\beta - 2)Q^2 - Q']P = 0,$$

and

$$(U - \chi\beta)P' - [(\beta - 2)U' + 4(\beta - 2)(U - \chi U')Q]P = 0.$$  

The first two equations are independent of $U$, and may be solved analytically; $Q$ is then given by the transcendental equation

$$[1 - (2 - \beta)\chi Q]^BQ = \frac{27e^{-1}}{256(2 - \beta)}e^{4(2 - \beta)\chi Q}.$$  

For finite values of $\chi$, this equation has no real-valued solutions for $Q$.

APPENDIX B

A GSS ENVELOPE WITH $\beta = 2$ CANNOT BE MATCHED TO THE AXIS

For $\beta = 2$, the boundary conditions give $U_i \propto \eta$. In the regime $1/4 \lesssim Q_{\theta 0} < 1/2$, this can be matched to an expansion of $U$ along the axis,

$$U(\xi, \eta) = c_1\eta + c_2\eta^3\xi^{-1} + c_3\eta^5\xi^{-2} + \ldots,$$

where $c_n$ are constants that depend only on $Q_{\theta 0}$. In this regime, the expansion leads to

$$Q(\xi, \eta) = \xi^{-1}Q(\chi) \quad \text{and} \quad P(\xi, \eta) = \xi^{-2(2+Q_{\theta 0})}Q(\chi).$$  

where \( Q \) is an unknown function that satisfies \( Q(\chi \to \infty) = Q_a0 \). As \( \chi \) is constant on the shock, \( Q \) and \( P \) scale differently there, contradicting the boundary conditions (26). This rules out a monotonic jet with a \( \beta = 2 \) GSS envelope.

**APPENDIX C**

A WIDE JET LEADS TO A NEAR-HEAD DIVERGENCE

The uniqueness of the numerical solution derived in Section 4 suggests that the shock profile is fixed by the regularity of the flow. To demonstrate how the flow diverges for any deviation from the true shock profile, we consider a wide jet, in which an axial expansion series converges rapidly near the head, in the region where the shock is nearly parabolic, \( \xi_v \simeq \xi_h + A\eta^2 \).

We expand \( Q_s(\xi) \) to orders as high as \( n = 5 \) near the head. For wide jets with \( A \lesssim 0.1 \), this expansion converges rapidly with \( n \) near the shock, and shows that \( Q_s \) becomes non-physically zero and subsequently negative near the head. Moreover, at the point where \( Q_s \to 0 \), \( P_s \) and its derivative must also vanish, and \( U_1 \) diverges. This vanishing of \( Q \) near the head of a wide jet is verified numerically, and traced even for \( A \lesssim 0.3 \).

**APPENDIX D**

MONOTONIC HEAD MODEL

An approximate, monotonic head description may be found using the axial expansion ((43)–(45)), truncated beyond order \( \eta^7 \). A monotonic profile far from the head requires \( Q_s(\xi \gg 1) \propto \xi^{-1} \) (see Section 3.5). For simplicity, we assume that \( Q_s \propto \xi^{-1} \) even near the head, so the boundary conditions fix \( Q_s = 1/(8\xi) \). The axial analysis then implies that \( P_s = (1/6)\xi^{-5/3} \). The axial analysis also implies that \( U_1(\xi) = 1 \), but it is more accurate to determine \( U_1 \) directly from the shock boundary conditions, as shown below.

Combining Equations ((23)–(26)) and (29), the expansion coefficients of \( P \) and \( Q \) are fixed up to order \( \eta^3 \), and the coefficients of \( U \) are fixed up to order \( \eta^5 \), by the shock boundary conditions on \( Q \), \( P \), and \( U \), their first derivatives, and the second derivative of \( U \).

This can be done for any shock profile, but the resulting coefficient expressions are in general lengthy. For brevity, we provide the expansion coefficients for the parabolic profile \( \xi_v = \xi_h + A\eta^2 \), accurate close to the nose of the jet. Here, the boundary conditions are explicitly given by

\[
Q_s = \frac{1}{8A^2\eta^2 + 8(A\eta^2 + \xi_h) - 8A\eta^2} \quad \text{and} \quad Q_s' = \frac{A\eta\left(A^3\eta^2 - 3A\xi_h + 2\xi_h\right)}{4\left(A^2\eta^2 + \xi_h\right)^3},
\]

\[
P_s = \frac{4}{3}Q_s \quad \text{and} \quad P_s' = \frac{7A\eta\left(A^3\eta^2 - A\xi_h + \xi_h\right)}{9\left(A^2\eta^2 + \xi_h\right)^3},
\]

and

\[
U_s = 2A\eta, \quad U_s' = 2A, \quad \text{and} \quad U_s'' = -\frac{32A^2\eta^3\left(A^3\eta^2 - 3A\xi_h + 2\xi_h\right)}{3\left(A^2\eta^2 + \xi_h\right)^3}.
\]

The expansion coefficient solutions are then

\[
Q_s(\xi) = \frac{(1 - 2A)A^3\xi^2 + A(A(4A - 7) + 7) - 2\xi_h) - 2A(A - 1)^2\xi_h^2}{8\xi(A(\xi - \xi_h) + \xi_h)^3},
\]

\[
Q_s(\xi) = \frac{A^2(A^3\xi^2 + (A((3 - 2A)A - 5) + 2\xi_h) + (A - 1)^2\xi_h^2)}{8\xi(A(\xi - \xi_h) + \xi_h)^3},
\]

\[
P_s(\xi) = \frac{A\left[-6A^3(\xi - \xi_h)^3 - A^2(\xi^{5/3} + 18\xi_h)(\xi - \xi_h)^2 + A\xi_h(19\xi^{5/3} - 18\xi_h)(\xi - \xi_h)^2\right]}{18\xi^{5/3}(\xi - \xi_h)(A(\xi - \xi_h) + \xi_h)^3} + \frac{A\xi_h(13\xi^{5/3} - 18\xi_h)}{18\xi^{5/3}(\xi - \xi_h)(A(\xi - \xi_h) + \xi_h)^3}.
\]
\[
P_\lambda(\xi) = \frac{A^2 \left(3A^3 (\xi - \xi_h)^3 + A^2 \left(4\xi^{5/3} + 9\xi_h\right)(\xi - \xi_h)^2 - A\xi_h \left(13\xi^{5/3} - 9\xi_h\right)(\xi - \xi_h)\right)}{18\xi^{5/3} (\xi - \xi_h)^3 \left(A(\xi - \xi_h) + \xi_h\right)^3}
+ \frac{A^2 \xi_h \left(-10\xi^{4/3}\xi_h + 7\xi^{8/3} + 3\xi_h^2\right)}{18\xi^{5/3} (\xi - \xi_h)^3 \left(A(\xi - \xi_h) + \xi_h\right)^3},
\]
\[
U_1(\xi) = \frac{2A \left(3(\xi - \xi_h)^3 + 15A^2\xi_h(\xi - \xi_h)^2 - A\xi_h \left(4\xi - 13\xi_h\right)(\xi - \xi_h) + 3\xi_h^3\right)}{3 \left(A(\xi - \xi_h) + \xi_h\right)^3},
\]
\[
U_3(\xi) = \frac{8A^3 (\xi - \xi_h)^2 \left(A(\xi - (A + 3)\xi_h) + 2\xi_h\right)}{3 \left(A(\xi - \xi_h) + \xi_h\right)^3},
\]
and
\[
U_5(\xi) = \frac{4A^4 \left(A(A + 3) - 2\right)\xi_h - 4A^6\xi}{3 \left(A(\xi - \xi_h) + \xi_h\right)^3}.
\]

(65)

REFERENCES

Blandford, R. D., & McKee, C. F. 1976, PhFl, 19, 1130
Cannizzo, J. K., Gehrels, N., & Vishniac, E. T. 2004, ApJ, 601, 380
De Colle, F., Ramirez-Ruiz, E., Granot, J., & Lopez-Camara, D. 2012, ApJ, 751, 57
Granot, J. 2007, RMxAA, 27, 140
Granot, J., Miller, M., Piran, T., & Suen, W.-M. 2000, in AIP Conf. Ser. 526, Gamma-ray Bursts, 5th Huntsville Symp., ed. R. M. Kippen, R. S. Mallozzi & G. J. Fishman (Melville, NY: AIP), 540
Granot, J., Miller, M., Piran, T., Suen, W. M., & Hughes, P. A. 2001, in Gamma-ray Bursts in the Afterglow Era, ed. E. Costa, F. Frontera & J. Hjorth (Berlin: Springer-Verlag), 312
Granot, J., & Piran, T. 2012, MNRAS, 421, 570
Green, D. A. 2011, BASI, 39, 289
Gruzinov, A. 2000, arXiv:astro-ph/0012364
Gruzinov, A. 2007, arXiv:0704.3081
Meliani, Z., & Keppens, R. 2010, in ASP Conf. Ser. 429, Numerical Modeling of Space Plasma Flows, Astronomum-2009, ed. N. V. Pogorelov, E. Audit & G. P. Zank (San Francisco, CA: ASP), 121
Meliani, Z., Keppens, R., Casse, F., & Giannios, D. 2007, MNRAS, 376, 1189
Rhoads, J. E. 1999, ApJ, 525, 737
van Eerten, H. 2013, arXiv:1309.3869
van Eerten, H., Zhang, W., & MacFadyen, A. 2010, ApJ, 722, 235
van Eerten, H. J., & MacFadyen, A. I. 2011, ApJL, 733, L37
Wygoda, N., Waxman, E., & Frail, D. A. 2011, ApJL, 738, L23
Zhang, W., & MacFadyen, A. 2009, ApJ, 698, 1261