Leonard pairs having specified end-entries

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Abstract
Fix an algebraically closed field $F$ and an integer $d \geq 3$. Let $V$ be a vector space over $F$ with dimension $d + 1$. A Leonard pair on $V$ is an ordered pair of diagonalizable linear transformations $A : V \to V$ and $A^* : V \to V$, each acting in an irreducible tridiagonal fashion on an eigenbasis for the other one. Let $\{v_i\}_{i=0}^d$ (resp. $\{v_i^*\}_{i=0}^d$) be such an eigenbasis for $A$ (resp. $A^*$). For $0 \leq i \leq d$ define a linear transformation $E_i : V \to V$ such that $E_i v_i = v_i$ and $E_i v_j = 0$ if $j \neq i$ ($0 \leq j \leq d$). Define $E_i^* : V \to V$ in a similar way. The sequence $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ is called a Leonard system on $V$ with diameter $d$. With respect to the basis $\{v_i\}_{i=0}^d$, let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be the diagonal entries of the matrix representing $A$ (resp. $A^*$). With respect to the basis $\{v_i\}_{i=0}^d$, let $\{\alpha_i\}_{i=0}^d$ (resp. $\{\alpha_i^*\}_{i=0}^d$) be the diagonal entries of the matrix representing $A^*$ (resp. $A$). It is known that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct, and the expressions $\frac{\theta_{i-1} - \theta_{i+2}}{\theta_{i} - \theta_{i+3}}$ are equal and independent of $i$ for $1 \leq i \leq d - 2$. Write this common value as $\beta + 1$. In the present paper we consider the “end-entries” $\theta_0, \theta_d, \theta_0^*, \theta_d^*$, $a_0, a_d, a_0^*, a_d^*$. We prove that a Leonard system with diameter $d$ is determined up to isomorphism by its end-entries and $\beta$ if and only if either (i) $\beta \neq \pm 2$ and $q^{d-1} \not\equiv -1$, where $\beta = q + q^{-1}$, or (ii) $\beta = \pm 2$ and $\text{Char}(F) \neq 2$.

1 Introduction
Throughout the paper $F$ denotes an algebraically closed field.

We begin by recalling the definition of a Leonard pair. We use the following terms. A square matrix is said to be tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1 (See [5] Definition 1.1.) Let $V$ be a vector space over $F$ with finite positive dimension. By a Leonard pair on $V$ we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

Note 1.2 According to a common notational convention, $A^*$ denotes the conjugate transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$ the matrices $A$ and $A^*$ are arbitrary subject to (i) and (ii) above.

We refer the reader to [3],[5],[8] for background on Leonard pairs.
For the rest of this section, fix an integer $d \geq 0$ and a vector space $V$ over $F$ with dimension $d + 1$. Consider a Leonard pair $A, A^*$ on $V$. By [5, Lemma 1.3] each of $A, A^*$ has mutually distinct $d + 1$ eigenvalues. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be an ordering of the eigenvalues of $A$ (resp. $A^*$), and let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) be the corresponding eigenspaces. For $0 \leq i \leq d$ define a linear transformation $E_i : V \to V$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here $I$ denotes the identity. We call $E_i$ the \textit{primitive idempotent} of $A$ associated with $\theta_i$. The primitive idempotent $E_i^*$ of $A^*$ associated with $\theta_i^*$ is similarly defined. For $0 \leq i \leq d$ pick a nonzero $v_i \in V_i$. We say the ordering $\{E_i\}_{i=0}^d$ is \textit{standard} whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 1.1(ii). A standard ordering of the primitive idempotents of $A^*$ is similarly defined. For a standard ordering $\{E_i\}_{i=0}^d$, the ordering $\{E_{d-j}\}_{j=0}^d$ is also standard and no further ordering is standard. Similar result applies to a standard ordering of the primitive idempotents of $A^*$.

**Definition 1.3** (See [5 Definition 1.4].) By a \textit{Leonard system} on $V$ we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d),$$

where $A, A^*$ is a Leonard pair on $V$, and $\{E_i\}_{i=0}^d$ (resp. $\{E_i^*\}_{i=0}^d$) is a standard ordering of the primitive idempotents of $A$ (resp. $A^*$). We call $d$ the \textit{diameter} of $\Phi$. We say $\Phi$ is \textit{over} $F$.

We recall the notion of an isomorphism of Leonard systems. Consider a Leonard system $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ on $V$ and a Leonard system $\Phi' = (A', \{E_i'\}_{i=0}^d, A'^*, \{E_i'^*\}_{i=0}^d)$ on a vector space $V'$ with dimension $d + 1$. By an \textit{isomorphism of Leonard systems from $\Phi$ to $\Phi'$} we mean a linear bijection $\sigma : V \to V'$ such that $\sigma A = A', \sigma A^* = A'^*$, and $\sigma E_i = E_i'\sigma, \sigma E_i^* = E_i'^*\sigma$ for $0 \leq i \leq d$. Leonard systems $\Phi$ and $\Phi'$ are said to be \textit{isomorphic} whenever there exists an isomorphism of Leonard systems from $\Phi$ to $\Phi'$.

We recall the parameter array of a Leonard system.

**Definition 1.4** (See [7 Section 2], [2 Theorem 4.6].) Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system over $F$. By the \textit{parameter array} of $\Phi$ we mean the sequence

$$\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d, (\varphi_i)_{i=1}^d,(\phi_i)_{i=1}^d\},$$

where $\theta_i$ is the eigenvalue of $A$ associated with $E_i$, $\theta_i^*$ is the eigenvalue of $A^*$ associate with $E_i^*$, and

$$\varphi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^* \prod_{h=0}^{i-1}(A - \theta_h I))}{\text{tr}(E_0^* \prod_{h=0}^{i-2}(A - \theta_h I))},$$

$$\phi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(E_0^{\ast} \prod_{h=0}^{i-1}(A - \theta_{d-h} I))}{\text{tr}(E_0^{\ast} \prod_{h=0}^{i-2}(A - \theta_{d-h} I))},$$

where tr means trace. In the above expressions, the denominators are nonzero by [2 Corollary 4.5].
Lemma 1.5 (See [5, Theorem 1.9].) A Leonard system is determined up to isomorphism by its parameter array.

Lemma 1.6 (See [5, Theorem 1.9].) Consider a sequence (1) consisting of scalars taken from $F$. Then there exists a Leonard system over $F$ with parameter array (1) if and only if (i)-(v) hold below:

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ ($0 \leq i < j \leq d$).

(ii) $\varphi_i \neq 0$, $\phi_i \neq 0$ ($1 \leq i \leq d$).

(iii) $\varphi_i = \phi_1 \sum_{\ell=0}^{i-1} \frac{\theta_i - \theta_{d-\ell}}{\theta_0 - \theta_d}\theta_{i-1} - \theta_d)(1 \leq i \leq d)$.

(iv) $\phi_i = \varphi_1 \sum_{\ell=0}^{i-1} \frac{\theta_i - \theta_{d-\ell}}{\theta_0 - \theta_d}\theta_{d-i+1} - \theta_0(1 \leq i \leq d)$.

(v) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

(2)

are equal and independent of $i$ for $2 \leq i \leq d - 1$.

Definition 1.7 By a parameter array over $F$ we mean a sequence (1) consisting of scalars taken from $F$ that satisfy conditions (i)-(v) in Lemma 1.6.

Definition 1.8 Let $\Phi = (A_i, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system over $F$ with diameter $d \geq 3$. Let (1) be the parameter array of $\Phi$. By the fundamental parameter of $\Phi$ we mean one less than the common value of (2).

Let $\Phi = (A_i, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be a Leonard system over $F$ with parameter array (1). In (1) we considered the end-parameters:

$$\theta_0, \theta_d, \theta_0^*, \theta_d^*, \varphi_1, \phi_d.$$

We proved that a Leonard system with diameter $d \geq 3$ is determined up to isomorphism by its fundamental parameter and its end-parameters (see Lemma 6.1). In the present paper we consider another set of parameters.

Definition 1.9 [5, Definition 2.5] For $0 \leq i \leq d$ define

$$a_i = \text{tr}(AE_i^*), \quad a_i^* = \text{tr}(A^*E_i).$$

We call $\{a_i\}_{i=0}^d$ (resp. $\{a_i^*\}_{i=0}^d$) the principal sequence (resp. dual principal sequence) of $\Phi$.

The principal sequence and the dual principal sequence have the following geometric interpretation. For $0 \leq i \leq d$ pick a nonzero $v_i \in E_iV$ and a nonzero $v_i^* \in E_i^*V$. Note that
each of \( \{ v_i \}_{i=0}^d \) and \( \{ v_i^* \}_{i=0}^d \) is a basis for \( V \). As easily observed, with respect to the basis \( \{ v_i \}_{i=0}^d \) the matrix representing \( A \) has diagonal entries \( \{ \theta_i \}_{i=0}^d \) and the matrix representing \( A^* \) has diagonal entries \( \{ a_i^* \}_{i=0}^d \). Similarly, with respect to the basis \( \{ v_i^* \}_{i=0}^d \) the matrix representing \( A^* \) has diagonal entries \( \{ \theta^*_i \}_{i=0}^d \) and the matrix representing \( A \) has diagonal entries \( \{ a_i \}_{i=0}^d \).

We now state our main results. Let \( \Phi = (A, \{ E_i \}_{i=0}^d, A^*, \{ E_i^* \}_{i=0}^d) \) be a Leonard system over \( \mathbb{F} \) with parameter array (1). Let \( \{ a_i \}_{i=0}^d \) (resp. \( \{ a_i^* \}_{i=0}^d \)) be the principal sequence (resp. dual principal sequence) of \( \Phi \). We consider the end-entries:

\[ \theta_0, \theta_d, \theta^*_0, \theta^*_d, a_0, a_d, a^*_0, a^*_d. \]

The end-entries are algebraically dependent:

**Proposition 1.10** Assume \( d \geq 1 \). Then

\[
\frac{(a_0 - \theta_0)(a_d - \theta_d)}{(a_0 - \theta_d)(a_d - \theta_0)} = \frac{(a^*_0 - \theta^*_0)(a^*_d - \theta^*_d)}{(a^*_0 - \theta^*_d)(a^*_d - \theta^*_0)}
\]  

(3)

In (3), all the denominators are nonzero. Once we fix \( \theta_0, \theta_d, \theta^*_0, \theta^*_d \), any one of \( a_0, a_d, a^*_0, a^*_d \) is determined by the remaining three.

For the rest of this section, assume \( d \geq 3 \).

**Theorem 1.11** Assume \( \varphi_1 + \varphi_d \neq \varphi_1 + \varphi_d \). Let \( \Phi' \) be a Leonard system over \( \mathbb{F} \) with diameter \( d \) that has the same fundamental parameter and the same end-entries as \( \Phi \). Then \( \Phi' \) is isomorphic to \( \Phi \).

In Appendix A, we display formulas that represent the parameter array in terms of the fundamental parameter and the end-entries.

**Theorem 1.12** Assume \( \varphi_1 + \varphi_d = \varphi_1 + \varphi_d \). Then there exist infinitely many mutually non-isomorphic Leonard systems over \( \mathbb{F} \) with diameter \( d \) that have the same fundamental parameter and the same end-entries as \( \Phi \).

In our proof of Theorem 1.12 we construct infinitely many Leonard systems that has the same fundamental parameter and the same end-entries as \( \Phi \). The condition \( \varphi_1 + \varphi_d = \varphi_1 + \varphi_d \) is interpreted in terms of the fundamental parameter as follows:

**Proposition 1.13** Let \( \beta \) be the fundamental parameter of \( \Phi \), and pick a nonzero \( q \in \mathbb{F} \) such that \( \beta = q + q^{-1} \). Then \( \varphi_1 + \varphi_d = \varphi_1 + \varphi_d \) if and only if one of the following (i), (ii) holds:

(i) \( \beta \neq \pm 2 \) and \( q^{d-1} = -1 \).

(ii) \( \beta = 0 \) and \( \text{Char}(\mathbb{F}) = 2 \).

By Theorems 1.11 1.12 and Proposition 1.13 we obtain:
Corollary 1.14 A Leonard system with diameter \( d \) is determined up to isomorphism by its fundamental parameter \( \beta \) and its end-entries if and only if one of the following (i), (ii) holds:

(i) \( \beta \neq \pm 2 \) and \( q^{d-1} \neq -1 \), where \( \beta = q + q^{-1} \).

(ii) \( \beta = \pm 2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \).

The paper is organized as follows. In Section 2, we recall the action of the dihedral group \( D_4 \) on the set of all Leonard systems. In Section 3, we collect some formulas concerning end-entries. In Section 4, we prove Proposition 1.10. In Section 5, we recall the notion of the type of a Leonard system. In Section 6, we recall some results from [1], and prove Proposition 1.13. In Section 7, we prove Theorem 1.11. In Section 8, we prove a lemma for later use. In Sections 9 and 10, we prove Theorem 1.12.

2 The \( D_4 \) action

For a Leonard system \( \Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d) \) over \( \mathbb{F} \), each of the following is a Leonard system over \( \mathbb{F} \):

\[
\begin{align*}
\Phi^* & := (A^*, \{E^*_i\}_{i=0}^d, A, \{E_i\}_{i=0}^d), \\
\Phi^\downarrow & := (A, \{E_i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d), \\
\Phi^\downarrow & := (A, \{E_d-i\}_{i=0}^d, A^*, \{E^*_i\}_{i=0}^d).
\end{align*}
\]

Viewing \( * \), \( \downarrow \), \( \downarrow \) as permutations on the set of all Leonard systems,

\[
*^2 = \downarrow^2 = \downarrow^2 = 1, \quad \downarrow * = * \downarrow, \quad \downarrow * = * \downarrow, \quad \downarrow \downarrow = \downarrow \downarrow.
\] (4)

The group generated by symbols \(*\), \( \downarrow\), \( \downarrow\) subject to the relations (4) is the dihedral group \( D_4 \). We recall \( D_4 \) is the group of symmetries of a square, and has 8 elements. For an element \( g \in D_4 \), and for an object \( f \) associated with \( \Phi \), let \( f^g \) denote the corresponding object associated with \( \Phi^g \). The \( D_4 \) action affects the parameter array as follows:

Lemma 2.1 (See [5] Theorem 1.11). Let \( \Phi \) be a Leonard system over \( \mathbb{F} \) with parameter array \( (\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d) \). Then for \( g \in \{\downarrow, \downarrow, \downarrow\} \) the scalars \( \theta^g \), \( \theta^*_g \), \( \varphi^g \), \( \phi^g \) are as follows:

\[
\begin{array}{c|cccc}
 g & \theta^g_i & \theta^*_g_i & \varphi^g_i & \phi^g_i \\
\hline
 \downarrow & \theta_i & \theta_{d-i}^* & \phi_{d-i+1} & \varphi_{d-i+1} \\
\downarrow & \theta_{d-i} & \theta_i^* & \phi_i & \varphi_i \\
\uparrow & \theta_i^* & \theta_i & \varphi_i & \phi_{d-i+1}
\end{array}
\]

Lemma 2.2 Let \( \Phi \) be a Leonard system over \( \mathbb{F} \) with principal sequence \( \{a_i\}_{i=0}^d \) and dual principal sequence \( \{a^*_i\}_{i=0}^d \). Then for \( 0 \leq i \leq d \)

\[
a_i^\downarrow = \alpha_{d-i}, \quad a_i^* = a_i^*, \quad a_i^\uparrow = a_i, \quad a_i^\downarrow = a_{d-i}^*.
\]

Proof. Immediate from Definition 1.9. \( \square \)
3 The end-entries

In this section we recall some formulas concerning the end-entries. Fix an integer \(d \geq 1\). Let \(\Phi = (A, \{E_i\}^d_{i=0}, A^*, \{E_i^*\}^d_{i=0})\) be a Leonard system over \(F\) with parameter array \((\{\theta_i\}^d_{i=0}, \{\theta_i^*\}^d_{i=0}, \{\varphi_i\}^d_{i=1}, \{\phi_i\}^d_{i=1})\). Let \({a_i}^d_{i=0}\) (resp. \({a_i^*}^d_{i=0}\)) be the principal sequence (resp. dual principal sequence) of \(\Phi\).

**Lemma 3.1** (See [5, Lemma 5.1], [6, Lemma 10.3].) With the above notation,

\[
\begin{align*}
a_0 - \theta_0 &= \frac{\varphi_1}{\theta_0^* - \theta_1^*}, & a_0 - \theta_d &= \frac{\varphi_d}{\theta_0^* - \theta_d^*}, \\
a_0 - \theta_d &= \frac{\phi_1}{\theta_0^* - \theta_1^*}, & a_0 - \theta_0 &= \frac{\phi_d}{\theta_0^* - \theta_d^*}, \\
a_d^* - \theta_0^* &= \frac{\varphi_1}{\theta_0 - \theta_1}, & a_d^* - \theta_d^* &= \frac{\varphi_d}{\theta_0 - \theta_d}, \\
a_d^* - \theta_d^* &= \frac{\phi_1}{\theta_0 - \theta_1}, & a_d^* - \theta_0^* &= \frac{\phi_d}{\theta_0 - \theta_d}.
\end{align*}
\]

**Note 3.2** In Lemma 3.1, the eight equations are obtained from one of them by applying \(D_4\).

The following lemmas are well-known. We give a short proof based on Lemma 3.1 for convenience of the reader.

**Lemma 3.3** With the above notation,

\[
\begin{align*}
a_0 &\neq \theta_0, & a_0 &\neq \theta_d, & a_d &\neq \theta_0, & a_d &\neq \theta_d, \\
a_0^* &\neq \theta_0^*, & a_0^* &\neq \theta_d^*, & a_d^* &\neq \theta_0, & a_d^* &\neq \theta_d.
\end{align*}
\]

**Proof.** Follows from Lemma 1.6(ii) and Lemma 3.1. \(\square\)

**Lemma 3.4** With the above notation,

\[
\begin{align*}
a_0 - \theta_0 &= \frac{\varphi_1}{\theta_0^* - \theta_1^*}, & a_0 - \theta_d &= \frac{\varphi_d}{\theta_0^* - \theta_d^*}, \\
a_0^* - \theta_0^* &= \frac{\varphi_1}{\theta_0 - \theta_1}, & a_0^* - \theta_d^* &= \frac{\varphi_d}{\theta_0 - \theta_d}.
\end{align*}
\]

**Proof.** Follows from Lemma 3.1. \(\square\)

**Lemma 3.5** With the above notation,

\[
\begin{align*}
\theta_0 - \theta_1 &= \frac{\phi_1^* - \varphi_1}{\theta_0^* - \theta_d^*}, & \theta_d - \theta_{d-1} &= \frac{\phi_d^* - \varphi_1}{\theta_0^* - \theta_d^*}, \\
\theta_0^* - \theta_1^* &= \frac{\phi_1 - \varphi_1}{\theta_0 - \theta_d}, & \theta_d^* - \theta_{d-1}^* &= \frac{\phi_d - \varphi_1}{\theta_0 - \theta_d}.
\end{align*}
\]

**Proof.** The equation on the left in (11) is obtained from the equations on the left in (7) and (8). The remaining equations can be obtained in a similar way. \(\square\)
Note 3.6 Lemma 3.5 can be obtained also from Lemma 1.6 (iii), (iv).

Lemma 3.7 With the above notation,

\[ \phi_1 \neq \varphi_1, \quad \phi_1 \neq \varphi_d, \quad \phi_d \neq \varphi_1, \quad \phi_d \neq \varphi_d. \]

Proof. Follows from Lemma 3.5 and Lemma 1.6 (i). \(\square\)

Lemma 3.8 With the above notation,

\[ a_0 - \theta_0 = \frac{\varphi_1(\theta_0 - \theta_d)}{\phi_1 - \varphi_1}, \quad a_d - \theta_d = \frac{\varphi_d(\theta_0 - \theta_d)}{\varphi_d - \phi_d}, \]
\[ a_0 - \theta_d = \frac{\phi_1(\theta_0 - \theta_d)}{\phi_1 - \varphi_1}, \quad a_d - \theta_0 = \frac{\varphi_d(\theta_0 - \theta_d)}{\varphi_d - \phi_d}, \]
\[ a_0^* - \theta_0^* = \frac{\varphi_1(\theta_0^* - \theta_d^*)}{\phi_d - \varphi_1}, \quad a_d^* - \theta_d^* = \frac{\varphi_d(\theta_0^* - \theta_d^*)}{\phi_1 - \varphi_1}, \]
\[ a_0^* - \theta_d^* = \frac{\phi_1(\theta_0^* - \theta_d^*)}{\phi_d - \varphi_1}, \quad a_d^* - \theta_0^* = \frac{\varphi_d(\theta_0^* - \theta_d^*)}{\phi_1 - \varphi_1}. \]

Proof. Use Lemmas 3.1 and 3.5 \(\square\)

Lemma 3.9 With the above notation,

\[ a_0 = \frac{\theta_0 \phi_1 - \theta_d \varphi_1}{\phi_1 - \varphi_1}, \quad a_d = \frac{\theta_d \phi_d - \theta_0 \varphi_d}{\phi_d - \varphi_d}, \]
\[ a_0^* = \frac{\theta_0^* \phi_d - \theta_d^* \varphi_1}{\phi_d - \varphi_1}, \quad a_d^* = \frac{\theta_d^* \phi_1 - \theta_0^* \varphi_d}{\phi_1 - \varphi_1}. \]

Proof. Follows from Lemma 3.8 \(\square\)

4 Proof of Proposition 1.10

In this section we prove Proposition 1.10. Fix an integer \(d \geq 1\). Let \(\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)\) be a Leonard system over \(F\) with parameter array \((\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)\). Let \(\{a_i\}_{i=0}^d\) (resp. \(\{a_i^*\}_{i=0}^d\)) be the principal sequence (resp. dual principal sequence) of \(\Phi\).

Proof of Proposition 1.10 In (3), all the denominators are nonzero by Lemma 3.3. Using Lemma 3.4 one checks that each side of (3) is equal to \(\varphi_1 \varphi_d (\phi_1 \phi_d)^{-1}\), so (3) holds. Rewrite (3) as

\[ (a_0 - \theta_0)(a_d - \theta_d)(a_0^* - \theta_d^*)(a_d^* - \theta_0^*) - (a_0 - \theta_d)(a_d - \theta_0)(a_0^* - \theta_0)(a_d^* - \theta_d) = 0. \]
Viewing (13) as a linear equation in \( a_0 \), the coefficient of \( a_0 \) is

\[
(a_d - \theta_d)(a^+_0 - \theta^+_0)(a^+ - \theta^+_0) - (a_d - \theta_0)(a^+_0 - \theta^+_0)(a^+ - \theta^+_0).
\]

Using Lemma 3.8 one checks that the above coefficient is equal to

\[
\frac{\varphi_d \phi_d (\theta_0 - \theta_d)(\theta^+_0 - \theta^+_d)^2(\phi_1 - \varphi_1)}{(\phi_1 - \varphi_d)(\phi_d - \varphi_1)(\phi_d - \varphi_d)}.
\]

This is nonzero by Lemma 1.6(i), (ii) and Lemma 3.7. So one can solve (13) in \( a_0 \). Thus \( a_0 \) is determined by \( \theta_0, \theta_d, \theta^+_0, \theta^+_d, a_d, a^+_0, a^+_d \). Concerning \( a_d, a^+_0, a^+_d \), apply the above arguments to \( \Phi \downarrow, \Phi^*, \Phi \downarrow^* \).  

\[\Box\]

5 The type of a Leonard system

In this section we recall the type of a Leonard system. Let \( \Phi \) be a Leonard system over \( F \) with diameter \( d \geq 3 \). Let \( \beta \) be the fundamental parameter of \( \Phi \), and pick a nonzero \( q \in F \) such that \( \beta = q + q^{-1} \).

Definition 5.1 We define the type of \( \Phi \) as follows:

| Type of \( \Phi \) | Description |
|------------------|-------------|
| I                | \( \beta \neq 2 \), \( \beta \neq -2 \) |
| II               | \( \beta = 2 \), \( \text{Char}(F) \neq 2 \) |
| III\(^+\)        | \( \beta = -2 \), \( \text{Char}(F) \neq 2 \), \( d \) is even |
| III\(^-\)        | \( \beta = -2 \), \( \text{Char}(F) \neq 2 \), \( d \) is odd |
| IV               | \( \beta = 0 \), \( \text{Char}(F) = 2 \) |

Lemma 5.2 (See [4, Sections 13–17].) The following hold:

(i) Assume \( \Phi \) has type I. Then \( q^i \neq 1 \) for \( 1 \leq i \leq d \).

(ii) Assume \( \Phi \) has type II. Then \( \text{Char}(F) \neq i \) for any prime \( i \) such that \( i \leq d \).

(iii) Assume \( \Phi \) has type III\(^+\). Then \( \text{Char}(F) \neq i \) for any prime \( i \) such that \( i \leq d/2 \). In particular, neither of \( d, d - 2 \) vanish in \( F \).

(iv) Assume \( \Phi \) has type III\(^-\). Then \( \text{Char}(F) \neq i \) for any prime \( i \) such that \( i \leq (d - 1)/2 \). In particular, \( d - 1 \) does not vanish in \( F \).

(v) Assume \( \Phi \) has type IV. Then \( d = 3 \).
6 The end-parameters

In this section, we first recall some results from [1]. We then prove Proposition 1.13. Fix an integer \( d \geq 3 \). Let \( \Phi \) be a Leonard system over \( \mathbb{F} \) with diameter \( d \) that has parameter array \( \{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d, \{ \varphi_i \}_{i=1}^d, \{ \phi_i \}_{i=1}^d \). Consider the end-parameters:

\[ \theta_0, \theta_d, \theta_0^*, \theta_d^*, \varphi_1, \varphi_d, \phi_1, \phi_d. \]

**Lemma 6.1** (See [1, Theorem 1.9].) A Leonard system over \( \mathbb{F} \) with diameter \( d \) is determined up to isomorphism by its fundamental parameter and its end-parameters.

We recall a relation between the end-parameters and the fundamental parameter.

**Lemma 6.2** (See [1, Proposition 1.11].) Let \( \beta \) be the fundamental parameter of \( \Phi \), and pick a nonzero \( q \in \mathbb{F} \) such that \( \beta = q + q^{-1} \). Then the scalar

\[ \Omega = \frac{\phi_1 + \phi_d - \varphi_1 - \varphi_d}{(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)} \]  \hspace{1cm} (14)

is as follows:

| Type of \( \Phi \) | \( \Omega \) |
|---------------------|----------------|
| I                   | \( \frac{(q - 1)(q^d - 1)}{q^d - 1} \) |
| II                  | \( \frac{2}{d} \) |
| III\(^+\)           | \( \frac{2(d - 1)}{d} \) |
| III\(^-\)           | \( 2 \) |
| IV                  | \( 0 \) |

**Lemma 6.3** With reference to Lemma 6.2, the following (i) and (ii) are equivalent:

(i) \( \Omega = 0 \).

(ii) \( \varphi_1 + \varphi_d = \phi_1 + \phi_d \).

**Proof.** Immediate from (14). \( \square \)

**Lemma 6.4** With reference to Lemma 6.2, \( \Omega = 0 \) if and only if one of the following (i), (ii) holds:

(i) \( \beta \neq \pm 2 \) and \( q^d - 1 = -1 \).

(ii) \( \beta = 0 \) and \( \text{Char}(\mathbb{F}) = 2 \).

**Proof.** First assume \( \beta \neq \pm 2 \). Then \( \Phi \) has type I. Now by Lemma 6.2, \( \Omega = 0 \) if and only if \( q^d - 1 = -1 \). Next assume \( \beta = \pm 2 \). Then \( \Phi \) has one of types II, III\(^+\), III\(^-\), IV. By Lemma 6.2, \( \Omega \neq 0 \) for types II, III\(^+\), III\(^-\), and \( \Omega = 0 \) for type IV. The result follows. \( \square \)

**Proof of Proposition 1.13.** Follows from Lemmas 6.3 and 6.4 \( \square \)
7 Proof of Theorem 1.11

Fix an integer $d \geq 3$. Let $\Phi$ be a Leonard system over $\mathbb{F}$ with diameter $d$ that has parameter array $\{(\theta_i)_i, (\theta_i^*), (\varphi_i)_i, (\phi_i)_i\}$. Let $\{a_i\}_i$ (resp. $\{a_i^*\}_i$) be the principal sequence (resp. dual principal sequence) of $\Phi$. Let the scalar $\Omega$ be from (14).

Define

$$\Delta = (a_0 - \theta_0)(a_0^* - \theta_0^*) - (a_d - \theta_0)(a_0^* - \theta_0^*).$$

(15)

Lemma 7.1 With the above notation,

$$\Delta = \frac{\varphi_1 \phi_d(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*)(\phi_1 + \phi_d - \varphi_1 - \varphi_d)}{(\phi_1 - \varphi_1)(\phi_d - \varphi_1)(\phi_d - \varphi_d)}.$$  

(16)

Proof. Routine verification using Lemma 3.8.

Lemma 7.2 With the above notation, the following (i)–(ii) are equivalent:

(i) $\Delta = 0$.

(ii) $\varphi_1 + \varphi_d = \phi_1 + \phi_d$.

Proof. Follows from (16) and Lemma 1.6(i), (ii).

Consider the following expressions:

$$\Gamma_1 = (a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0 - \theta_d),$$

$$\Gamma_2 = (a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_d^*)(\theta_0^* - \theta_d),$$

$$\Gamma_3 = (a_0 - \theta_d)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0^* - \theta_d^*),$$

$$\Gamma_4 = (a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_d^*)(\theta_0 - \theta_0^*).$$

Lemma 7.3 With the above notation,

$$\Gamma_1 = -\frac{\varphi_1 \phi_d(\theta_0 - \theta_d)^2(\theta_0^* - \theta_d^*)^2}{(\phi_1 - \varphi_1)(\phi_d - \varphi_1)(\phi_d - \varphi_d)},$$

$$\Gamma_2 = -\frac{\varphi_1 \phi_1 \phi_d(\theta_0 - \theta_d)^2(\theta_0^* - \theta_d^*)^2}{(\phi_1 - \varphi_1)(\phi_d - \varphi_1)(\phi_d - \varphi_d)},$$

$$\Gamma_3 = -\frac{\varphi_1 \phi_1 \phi_d(\theta_0 - \theta_d)^2(\theta_0^* - \theta_d^*)^2}{(\phi_1 - \varphi_1)(\phi_d - \varphi_1)(\phi_d - \varphi_d)},$$

$$\Gamma_4 = -\frac{\varphi_1 \phi_d^2(\theta_0 - \theta_d)^2(\theta_0^* - \theta_d^*)^2}{(\phi_1 - \varphi_1)(\phi_d - \varphi_1)(\phi_d - \varphi_d)}.$$  

Proof. Routine verification using Lemma 3.8.
Lemma 7.4  With the above notation,
\[ \frac{\Omega_{\Gamma}}{\varphi_1} = \frac{\Omega_{\Gamma}}{\varphi_d} = \frac{\Omega_{\Gamma}}{\phi_1} = \frac{\Omega_{\Gamma}}{\phi_d} = -\Delta. \]

**Proof.** Follows from (14), (16) and Lemma 7.3. □

Lemma 7.5  With the above notation, assume \( \Delta \neq 0 \). Then
\[ \varphi_1 = -\frac{\Omega_{\Gamma}}{\Delta}, \quad \varphi_d = -\frac{\Omega_{\Gamma}}{\Delta}, \quad \phi_1 = -\frac{\Omega_{\Gamma}}{\Delta}, \quad \phi_d = -\frac{\Omega_{\Gamma}}{\Delta}. \]

**Proof.** Immediate from Lemma 7.4. □

**Proof of Theorem 1.11.** Assume \( \varphi_1 + \varphi_d \neq \phi_1 + \phi_d \). Note that \( \Delta \neq 0 \) by Lemma 7.2. Observe that each of \( \Delta, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) is determined by the end-entries. By this and Lemma 7.5 each of \( \varphi_1, \varphi_d, \phi_1, \phi_d \) is determined by the end-entries and \( \Omega \). By Lemma 6.2 \( \Omega \) is determined by the fundamental parameter \( \beta \). By these comments, the end-parameters are determined by the end-entries and \( \beta \). Now the result follows by Lemma 6.1. □

8  A lemma

Fix an integer \( d \geq 3 \). Let \( \Phi \) be a Leonard system over \( \mathbb{F} \) with diameter \( d \) that has parameter array \( \{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d \). Let \( \{a_i\}_{i=0}^d \) (resp. \( \{a^*_i\}_{i=0}^d \)) be the principal sequence (resp. dual principal sequence) of \( \Phi \).

Lemma 8.1  Assume \( \varphi_1 + \varphi_d = \phi_1 + \phi_d \). Then
\[ \begin{align*}
    a_d - \theta_0 &= a_0^* - \theta^*_0, \\
    a_0 - \theta_0 &= a_0^* - \theta^*_0, \\
    a_0 - \theta_d &= a_d^* - \theta^*_d, \\
    a_0 - \theta_0 &= a_0^* - \theta^*_0, \\
    a_d - \theta_d &= a_d^* - \theta^*_d, \\
    a_0 - \theta_0 &= a_0^* - \theta^*_0. 
\end{align*} \tag{17} \tag{18} \tag{19} \tag{20} \]

**Proof.** Let the scalar \( \Delta \) be from (15). By Lemma 7.2 \( \Delta = 0 \), and (17) follows. Line (18) is obtained from (17) by applying \( * \). Combining (3) and (17),
\[ \begin{align*}
    a_d - \theta_d &= a_d^* - \theta^*_d, \\
    a_0 - \theta_d &= a_0^* - \theta^*_0. 
\end{align*} \]

By this and (18) we get (19). By (17) minus (19),
\[ \begin{align*}
    \theta_d - \theta_0 &= a_0^* - a_d^*, \\
    a_0 - \theta_0 &= a_0^* - \theta^*_0. 
\end{align*} \]
Applying \( * \) to this, we obtain (20). □
9 Proof of Theorem 1.12: part 1

In this and the next section we prove Theorem 1.12. Fix an integer \( d \geq 3 \). Let \( \Phi \) be a Leonard system over \( F \) with diameter \( d \), and let \( \{(\theta_i)^d_{i=0},(\theta^*_i)^d_{i=0},(\varphi_i)^d_{i=1},(\phi_i)^d_{i=1}\} \) be the parameter array of \( \Phi \). Let \( \beta \) be the fundamental parameter of \( \Phi \), and pick a nonzero \( q \in F \) such that \( \beta = q + q^{-1} \). Let \( \{a_i^0\}_{i=0}^d \) (resp. \( \{a_i^1\}_{i=0}^d \)) be the principal sequence (resp. dual principal sequence) of \( \Phi \). Assume \( \varphi_1 + \varphi_d = \phi_1 + \phi_d \). By Proposition 1.13 we have two cases:

Case (i): \( \Phi \) has type I and \( q^{d-1} = -1 \).

Case (ii): \( \Phi \) has type IV.

In this section we consider Case (i). Note that \( q^i \neq 1 \) for \( 1 \leq i \leq d \) by Lemma 5.2(i). Also note that \( \text{Char}(F) \neq 2 \); otherwise \( q^{d-1} = 1 \).

For a nonzero scalar \( \zeta \in F \) we define a sequence

\[
\tilde{p}(\zeta) = (\{(\tilde{\theta}_i(\zeta))^d_{i=0},(\tilde{\theta}^*_i(\zeta))^d_{i=0},(\tilde{\varphi}_i(\zeta))^d_{i=1},(\tilde{\phi}_i(\zeta))^d_{i=1}\})
\]

as follows. For \( 0 \leq i \leq d \) define

\[
L_i(\zeta) = (q+1)(q^{i-1}+1)\zeta - (q-1)(q^{i-1} - 1)(a_0^* - \theta_0^*)(\theta_0 - \theta_d), \quad (21)
\]

\[
K_i(\zeta) = -\frac{q^i - 1}{2q^{i-1}(q^2 - 1)(a_0^* - \theta_0^*)}L_i(\zeta). \quad (22)
\]

For \( 0 \leq i \leq d \) define \( L_i^*(\zeta) \) and \( K_i^*(\zeta) \) by viewing \( \zeta^* = \zeta \):

\[
L_i^*(\zeta) = (q+1)(q^{i-1}+1)\zeta - (q-1)(q^{i-1} - 1)(a_0^* - \theta_0)(\theta_0^* - \theta_d^*), \quad (23)
\]

\[
K_i^*(\zeta) = -\frac{q^i - 1}{2q^{i-1}(q^2 - 1)(a_0^* - \theta_0^*)}L_i^*(\zeta). \quad (24)
\]

For \( 0 \leq i \leq d \) define \( L_i^\#(\zeta) \) and \( K_i^\#(\zeta) \) by viewing \( \zeta^\# = (a_0 - \theta_d)\zeta/(a_0 - \theta_0) \):

\[
L_i^\#(\zeta) = \frac{(q+1)(q^{i-1}+1)(a_0 - \theta_d)}{a_0 - \theta_0} \zeta + (q-1)(q^{i-1} - 1)(a_d^* - \theta_0^*)(\theta_0 - \theta_d), \quad (25)
\]

\[
K_i^\#(\zeta) = -\frac{q^i - 1}{2q^{i-1}(q^2 - 1)(a_d^* - \theta_0^*)}L_i^\#(\zeta). \quad (26)
\]

Now for \( 0 \leq i \leq d \) define

\[
\tilde{\theta}_i(\zeta) = \theta_0 + K_i(\zeta), \quad \tilde{\theta}^*_i(\zeta) = \theta_0^* + K_i^*(\zeta), \quad (27)
\]

and for \( 1 \leq i \leq d \) define

\[
\tilde{\varphi}_i(\zeta) = K_{d-i+1}^\#(\zeta)K_i^*(\zeta) + \frac{(q^i - 1)(q^{i-2}+1)(a_0 - \theta_d)}{q^{i-2}(q^2 - 1)(a_0 - \theta_0)}\zeta, \quad (28)
\]

\[
\tilde{\phi}_i(\zeta) = K_{d-i+1}(\zeta)K_i^*(\zeta) + \frac{(q^i - 1)(q^{i-2}+1)}{q^{i-2}(q^2 - 1)}\zeta. \quad (29)
\]

We have defined \( \tilde{p}(\zeta) \). We prove the following result:
Proposition 9.1 The sequence $\tilde{p}(\zeta)$ is a parameter array over $\mathbb{F}$ for infinitely many values of $\zeta$. Assume $\check{p}(\zeta)$ is the parameter array of a Leonard system $\Phi(\zeta)$. Then $\check{\Phi}(\zeta)$ has the same fundamental parameter and the same end-entries as $\Phi(\zeta)$.

The following three lemmas can be routinely verified.

Lemma 9.2 We have
$$\tilde{\theta}_0(\zeta) = \theta_0, \quad \tilde{\theta}_d(\zeta) = \theta_d, \quad \tilde{\theta}_0^*(\zeta) = \theta_0^*, \quad \tilde{\theta}_d^*(\zeta) = \theta_d^*.$$

Lemma 9.3 For $0 \leq i, j \leq d$
$$\tilde{\theta}_i(\zeta) - \tilde{\theta}_j(\zeta) = \frac{q^j - q^i}{2q^{i+j}-1} \left( \frac{q^{i+j-1} + 1}{(q-1)(a_0^* - \theta_0^*)} \zeta - \frac{(q^{i+j-1} - 1)(\theta_0 - \theta_d)}{q+1} \right), \quad (30)$$
$$\tilde{\theta}_i^*(\zeta) - \tilde{\theta}_j^*(\zeta) = \frac{q^j - q^i}{2q^{i+j}-1} \left( \frac{q^{i+j-1} + 1}{(q-1)(a_0 - \theta_0)} \zeta - \frac{(q^{i+j-1} - 1)(\theta_0^* - \theta_d^*)}{q+1} \right). \quad (31)$$

Lemma 9.4 For $2 \leq i \leq d - 1$ each of the expressions
$$\frac{\tilde{\theta}_{i-2}(\zeta) - \tilde{\theta}_{i+1}(\zeta)}{\tilde{\theta}_{i-1}(\zeta) - \tilde{\theta}_i(\zeta)}, \quad \frac{\tilde{\theta}_{i-2}^*(\zeta) - \tilde{\theta}_{i+1}^*(\zeta)}{\tilde{\theta}_{i-1}^*(\zeta) - \tilde{\theta}_i^*(\zeta)}$$
is equal to $q + q^{-1} + 1$.

Lemma 9.5 For $0 \leq i < j \leq d$ the following hold:
(i) $\tilde{\theta}_i(\zeta) = \tilde{\theta}_j(\zeta)$ holds for at most one value of $\zeta$.
(ii) $\tilde{\theta}_i^*(\zeta) = \tilde{\theta}_j^*(\zeta)$ holds for at most one value of $\zeta$.

Proof. (i): First assume $q^{i+j-1} = -1$. Then by (30)
$$\tilde{\theta}_i(\zeta) - \tilde{\theta}_j(\zeta) = \frac{(q^j - q^i)(\theta_0 - \theta_d)}{q+1} \neq 0.$$  
Next assume $q^{i+j-1} \neq -1$. Then by (30), $\tilde{\theta}_i(\zeta) - \tilde{\theta}_j(\zeta)$ is a polynomial in $\zeta$ with degree 1. The result follows.
(ii): Similar to the proof of (i). \hfill \Box

Lemma 9.6 We have
$$\check{\varphi}_1(\zeta) = \zeta, \quad \check{\phi}_1(\zeta) = \frac{a_0 - \theta_d}{a_0 - \theta_0} \zeta,$$
$$\check{\varphi}_d(\zeta) = \frac{a_d - \theta_d}{a_0 - \theta_0} \zeta, \quad \check{\phi}_d(\zeta) = \frac{a_d - \theta_0}{a_0 - \theta_0} \zeta.$$  

Proof. Routine verification using $q^{d-1} = -1$. \hfill \Box

For $1 \leq i \leq d$ define
$$\vartheta_i = \sum_{\ell=0}^{i-1} \frac{\tilde{\varphi}_\ell(\zeta) - \tilde{\phi}_{d-\ell}(\zeta)}{\theta_0(\zeta) - \tilde{\theta}_d(\zeta)}.$$
Lemma 9.7 For $1 \leq i \leq d$

$$\vartheta_i = \frac{(q^i - 1)(q^{i-2} + 1)}{q^{i-2}(q^2 - 1)}. \quad (32)$$

**Proof.** By (30),

$$\tilde{\theta}_\ell(\zeta) - \tilde{\theta}_{d-\ell}(\zeta) \quad \tilde{\theta}_0(\zeta) - \tilde{\theta}_d(\zeta) = \frac{q^{d-\ell} - q^\ell}{2q^{d-1}(\theta_0 - \theta_d)} \left( \frac{q^{d-1} + 1}{(q - 1)(a_0^* - \theta_0^*)} - \frac{(q^{d-1} - 1)(\theta_0 - \theta_d)}{q + 1} \right).$$

Simplify this using $q^{d-1} = -1$ to find

$$\frac{\tilde{\theta}_\ell(\zeta) - \tilde{\theta}_{d-\ell}(\zeta)}{\tilde{\theta}_0(\zeta) - \tilde{\theta}_d(\zeta)} = \frac{q^\ell + q^{1-\ell}}{q + 1}.$$  

Now one routinely verifies (32).  \hfill \Box

Lemma 9.8 For $1 \leq i \leq d$

$$L_{d-1-i}(\zeta) = q^{1-i}(q + 1)(q^{i-1} - 1)\zeta + q^{1-i}(q - 1)(q^{i-1} + 1)(a_0^* - \theta_0^*)(\theta_0 - \theta_d),$$

$$K_{d-1-i}(\zeta) = -\frac{q(q^{i-2} + 1)}{2(q^2 - 1)(a_0^* - \theta_0^*)} L_{d-i+1}(\zeta).$$

**Proof.** Follows from (21), (22) and $q^{d-1} = -1$.  \hfill \Box

Lemma 9.9 For $1 \leq i \leq d$

$$L_{d-i+1}^\dagger(\zeta) = \frac{(q + 1)(q^{i-1} - 1)(a_0 - \theta_0)}{q^{i-2}(a_0 - \theta_0)} \zeta - q^{1-i}(q - 1)(q^{i-1} + 1)(a_d^* - \theta_0^*)(\theta_0 - \theta_d),$$

$$K_{d-i+1}^\dagger(\zeta) = -\frac{q(q^{i-2} + 1)}{2(q^2 - 1)(a_d^* - \theta_0^*)} L_{d-i+1}^\dagger(\zeta).$$

**Proof.** Follows from (25), (28) and $q^{d-1} = -1$.  \hfill \Box

Lemma 9.10 For $1 \leq i \leq d$

$$\tilde{\varphi}_i(\zeta) = \tilde{\phi}_1(\zeta)\vartheta_i + (\tilde{\theta}_i^*(\zeta) - \tilde{\theta}_0^*(\zeta)) (\tilde{\theta}_{i-1}(\zeta) - \tilde{\theta}_d(\zeta)), $$

$$\check{\varphi}_i(\zeta) = \check{\phi}_1(\zeta)\vartheta_i + (\check{\theta}_i^*(\zeta) - \check{\theta}_0^*(\zeta)) (\check{\theta}_{d-i+1}(\zeta) - \check{\theta}_0(\zeta)).$$

**Proof.** Routine verification using Lemmas 9.3 and 9.6–9.9.  \hfill \Box
Lemma 9.11 For $1 \leq i \leq d$ the following hold:

(i) $\tilde{\varphi}_i(\zeta) = 0$ holds for at most two values of $\zeta$.

(ii) $\tilde{\phi}_i(\zeta) = 0$ holds for at most two values of $\zeta$.

Proof. (i): We show that $\tilde{\varphi}_i(\zeta)$ is a polynomial in $\zeta$ with degree 1 or 2. By Lemma 9.6 $\tilde{\varphi}_1(\zeta) = \zeta$. So we assume $2 \leq i \leq d$. In view of (28), first consider $K^\|_i(\zeta)$. Note that $q^{i-1} + 1 \neq 0$; otherwise $q^{d-i} = q^{d-1}q^{1-i} = (-1)(-1) = 1$. By this and (24), $K^\|_i(\zeta)$ is a polynomial in $\zeta$ with degree 1. Next consider $K^\|_{d-i+1}(\zeta)$. Note that $q^{d-i} + 1 \neq 0$; otherwise $q^{d-i+1} = q^{d-1}q^{2-i} = (-1)(-1) = 1$. By this and Lemma 9.9, $K^\|_{d-i+1}(\zeta)$ is a polynomial in $\zeta$ with degree 1. By these comments and (28), $\tilde{\varphi}_i(\zeta)$ is a polynomial in $\zeta$ with degree 1 or 2. The result follows.

(ii): Similar to the proof of (i). \hfill \Box

Lemma 9.12 The sequence $\tilde{p}(\zeta)$ is a parameter array over $\mathbb{F}$ for infinitely many values of $\zeta$.

Proof. We verify conditions (i)–(v) in Lemma 1.6. Conditions (iii) and (iv) are satisfied by Lemma 9.10. Condition (v) is satisfied by Lemma 9.4. Note that $\mathbb{F}$ has infinitely many elements since $\mathbb{F}$ is algebraically closed. By this and Lemmas 9.5, 9.11 conditions (i) and (ii) are satisfied for infinitely many values of $\zeta$. The result follows. \hfill \Box

Lemma 9.13 We have

$$
\tilde{\varphi}_d(\zeta) = \frac{a_d^* - \theta_d}{a_0^* - \theta_0} \zeta, \\
\tilde{\phi}_1(\zeta) = \frac{a_0^* - \theta_0}{a_0^*} \zeta, \\
\tilde{\phi}_d(\zeta) = \frac{a_d^* - \theta_d}{a_0^* - \theta_0} \zeta.
$$

Proof. Follows from Lemma 9.6 using (17)–(19). \hfill \Box

Lemma 9.14 Assume $\tilde{p}(\zeta)$ is a parameter array of a Leonard system $\tilde{\Phi}(\zeta)$. Let $\{\tilde{a}_i(\zeta)\}_{i=0}^d$ (resp. $\{\tilde{a}_i^*(\zeta)\}_{i=0}^d$) be the principal sequence (resp. dual principal sequence) of $\tilde{\Phi}(\zeta)$. Then

$$
\tilde{a}_0(\zeta) = a_0, \\
\tilde{a}_d(\zeta) = a_d, \\
\tilde{a}_0^*(\zeta) = a_0^*, \\
\tilde{a}_d^*(\zeta) = a_d^*.
$$

Proof. Applying Lemma 3.9 to $\tilde{\Phi}(\zeta)$ and using Lemma 9.2

$$
\tilde{a}_0(\zeta) = \frac{\theta_0\tilde{\varphi}_1(\zeta) - \theta_d\tilde{\varphi}_1(\zeta)}{\tilde{\phi}_1(\zeta) - \tilde{\varphi}_1(\zeta)}.
$$

In this equation, eliminate $\tilde{\varphi}_1(\zeta)$ and $\tilde{\phi}_1(\zeta)$ using Lemma 9.6 to get $\tilde{a}_0(\zeta) = a_0$. The remaining three equations are obtained in a similar way using Lemmas 9.6 and 9.13. \hfill \Box

Proof of Proposition 9.1. By Lemma 9.12 $\tilde{p}(\zeta)$ is a parameter array over $\mathbb{F}$ for infinitely many values of $\zeta$. Assume $\tilde{p}(\zeta)$ is the parameter array of a Leonard system $\Phi(\zeta)$. By Lemma 9.4 $\tilde{\Phi}(\zeta)$ has the same fundamental parameter as $\Phi$. By Lemmas 9.2 and 9.14 $\tilde{\Phi}(\zeta)$ has the same end-entries as $\Phi$. \hfill \Box

Proof of Theorem 1.12 case (i). Follows from Proposition 9.1 and Lemmas 1.5, 1.6 \hfill \Box
10 Proof of Theorem 1.12; part 2

In this section we prove Theorem 1.12 for Case (ii). Let $\Phi$ be a Leonard system of type IV. By Lemma 5.2(v) $\Phi$ has diameter $d = 3$. Note that $\text{Char}(F) = 2$. Let $(\{\theta_i\}^3_{i=0}, \{\theta^*_i\}^3_{i=0}, \{\varphi_i\}^3_{i=1}, \{\phi_i\}^3_{i=1})$ be the parameter array of $\Phi$. Let $(a_i)_{i=0}^3$ (resp. $(a_i^*)_{i=0}^3$) be the principal sequence (resp. dual principal sequence) of $\Phi$.

For a nonzero scalar $\zeta \in F$ we define a sequence

$$\tilde{p}(\zeta) = (\{\tilde{\theta}_i(\zeta)\}_{i=0}^3, \{\tilde{\theta}_i^*(\zeta)\}_{i=0}^3, \{\tilde{\varphi}_i(\zeta)\}_{i=1}^3, \{\tilde{\phi}_i(\zeta)\}_{i=1}^3)$$

as follows. Define

$$\tilde{\theta}_0(\zeta) = \theta_0, \quad \tilde{\theta}_1(\zeta) = \theta_0 + \frac{\zeta}{a_0^* - \theta_0}, \quad \tilde{\theta}_2(\zeta) = \theta_3 + \frac{\zeta}{a_0^* - \theta_0}, \quad \tilde{\theta}_3(\zeta) = \theta_3,$$

$$\tilde{\theta}_0^*(\zeta) = \theta_0^*, \quad \tilde{\theta}_1^*(\zeta) = \theta_0^* + \frac{\zeta}{a_0 - \theta_0}, \quad \tilde{\theta}_2^*(\zeta) = \theta_3^* + \frac{\zeta}{a_0 - \theta_0}, \quad \tilde{\theta}_3^*(\zeta) = \theta_3^*,$$

and define

$$\tilde{\varphi}_1(\zeta) = \zeta,$$

$$\tilde{\varphi}_2(\zeta) = \left(\theta_0^* - \theta_3^* + \frac{\zeta}{a_0^* - \theta_0}\right) \left(\theta_0^* - \theta_3^* + \frac{\zeta}{a_0 - \theta_0}\right),$$

$$\tilde{\varphi}_3(\zeta) = \frac{a_3 - \theta_3}{a_0 - \theta_0} \zeta,$$

$$\tilde{\varphi}_1(\zeta) = \frac{a_0 - \theta_0}{a_0 - \theta_3} \zeta,$$

$$\tilde{\varphi}_2(\zeta) = \left(\theta_0^* - \theta_3^* + \frac{\zeta}{a_0^* - \theta_0}\right) \left(\theta_0^* - \theta_3^* + \frac{\zeta}{a_0 - \theta_0}\right),$$

$$\tilde{\varphi}_3(\zeta) = \frac{a_3 - \theta_0}{a_0 - \theta_0} \zeta.$$

We have defined $\tilde{p}(\zeta)$. We prove the following result:

**Proposition 10.1** The sequence $\tilde{p}(\zeta)$ is the parameter array over $F$ for infinitely many values of $\zeta$. Assume $\tilde{p}(\zeta)$ is the parameter array of a Leonard system $\tilde{\Phi}(\zeta)$. Then $\tilde{\Phi}(\zeta)$ has the same fundamental parameter and the same end-entries as $\Phi$.

The following three lemmas can be routinely verified.
Lemma 10.2  We have
\[
\tilde{\theta}_0(\zeta) - \tilde{\theta}_1(\zeta) = \frac{\zeta}{a_0 - \theta_0}, \quad \tilde{\theta}_0(\zeta) - \tilde{\theta}_2(\zeta) = \theta_0 - \theta_3 + \frac{\zeta}{a_0 - \theta_0},
\]
\[
\tilde{\theta}_0(\zeta) - \tilde{\theta}_3(\zeta) = \theta_0 - \theta_3,
\]
\[
\tilde{\theta}_1(\zeta) - \tilde{\theta}_3(\zeta) = \theta_0 - \theta_3 + \frac{\zeta}{a_0 - \theta_0},
\]
\[
\tilde{\theta}_0^*(\zeta) - \tilde{\theta}_1^*(\zeta) = \frac{\zeta}{a_0 - \theta_0}, \quad \tilde{\theta}_0^*(\zeta) - \tilde{\theta}_2^*(\zeta) = \theta_0^* - \theta_3^* + \frac{\zeta}{a_0 - \theta_0},
\]
\[
\tilde{\theta}_0^*(\zeta) - \tilde{\theta}_3^*(\zeta) = \theta_0^* - \theta_3^*,
\]
\[
\tilde{\theta}_1^*(\zeta) - \tilde{\theta}_3^*(\zeta) = \theta_0^* - \theta_3^* + \frac{\zeta}{a_0 - \theta_0}.
\]

Lemma 10.3  Each of the expressions
\[
\frac{\tilde{\theta}_0(\zeta) - \tilde{\theta}_3(\zeta)}{\tilde{\theta}_1(\zeta) - \tilde{\theta}_2(\zeta)}, \quad \frac{\tilde{\theta}_0^*(\zeta) - \tilde{\theta}_3^*(\zeta)}{\tilde{\theta}_1^*(\zeta) - \tilde{\theta}_2^*(\zeta)}
\]
is equal to 1.

For \(1 \leq i \leq 3\) define
\[
\varphi_i = \sum_{\ell=0}^{i-1} \frac{\tilde{\theta}_\ell(\zeta) - \tilde{\theta}_{d-\ell}(\zeta)}{\tilde{\theta}_0(\zeta) - \tilde{\theta}_d(\zeta)}.
\]

Lemma 10.4  We have
\[
\varphi_1 = 1, \quad \varphi_2 = 0, \quad \varphi_3 = 1.
\]

Lemma 10.5  For \(1 \leq i \leq 3\)
\[
\tilde{\varphi}_1(\zeta) = \tilde{\phi}_1(\zeta)\varphi_i + (\tilde{\theta}_1^*(\zeta) - \tilde{\theta}_0^*(\zeta))\tilde{\theta}_{i-1}(\zeta) - \tilde{\theta}_3(\zeta)),
\]
\[
\tilde{\varphi}_1(\zeta) = \tilde{\phi}_1(\zeta)\varphi_i + (\tilde{\theta}_1^*(\zeta) - \tilde{\theta}_0^*(\zeta))\tilde{\theta}_{i+1}(\zeta) - \tilde{\theta}_0(\zeta))
\]

Proof. Routine verification using Lemmas 10.2, 10.4 and (17). □

Lemma 10.6  The sequence \(\tilde{\varphi}(\zeta)\) is a parameter array over \(\mathbb{F}\) for infinitely many values of \(\zeta\).

Proof. We verify conditions (i)–(v) in Lemma 1.6. Conditions (iii) and (iv) are satisfied by Lemma 10.5. Condition (v) is satisfied by Lemma 10.3. Note that \(\mathbb{F}\) has infinitely many elements since \(\mathbb{F}\) is algebraically closed. Observe that for \(1 \leq i \leq 3\), \(\tilde{\varphi}_i(\zeta)\) and \(\tilde{\phi}_i(\zeta)\) vanish for at most two values of \(\zeta\). Observe by Lemma 10.2 that for \(0 \leq i < j \leq 3\), \(\tilde{\theta}_i(\zeta) - \tilde{\theta}_j(\zeta)\) and \(\tilde{\theta}_i^*(\zeta) - \tilde{\theta}_j^*(\zeta)\) vanish for at most one value of \(\zeta\). Thus conditions (i) and (ii) are satisfied for infinitely many values of \(\zeta\). The result follows. □
Lemma 10.7 Assume \( \tilde{p}(\zeta) \) is a parameter array of a Leonard system \( \tilde{\Phi}(\zeta) \). Let \( \{\tilde{a}_i(\zeta)\}_{i=0}^{3} \) (resp. \( \{\tilde{a}_i^*(\zeta)\}_{i=0}^{3} \)) be the principal sequence (resp. dual principal sequence) of \( \tilde{\Phi}(\zeta) \). Then
\[ \tilde{a}_0(\zeta) = a_0, \quad \tilde{a}_3(\zeta) = a_3, \quad \tilde{a}_0^*(\zeta) = a_0^*, \quad \tilde{a}_3^*(\zeta) = a_3^*. \]

Proof. Similar to the proof of Lemma 9.14. □

Proof of Proposition 10.1 By Lemma 10.6 \( \tilde{p}(\zeta) \) is a parameter array over \( F \) for infinitely many values of \( \zeta \). Assume \( \tilde{p}(\zeta) \) is the parameter array of a Leonard system \( \tilde{\Phi}(\zeta) \). By Lemma 10.3 \( \tilde{\Phi}(\zeta) \) has fundamental parameter 0. By the construction and Lemma 10.7 \( \tilde{\Phi}(\zeta) \) has the same end-entries as \( \Phi \). □

Proof of Theorem 1.12 case (ii). Follows from Proposition 10.1 and Lemmas 1.5, 1.6. □

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Appendix A
Let \( \Phi \) be a Leonard system over \( F \) with diameter \( d \geq 3 \). Let
\[ (\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\varphi_i\}_{i=1}^{d}, \{\phi_i\}_{i=1}^{d}) \]
be the parameter array of \( \Phi \), and let \( \beta \) be the fundamental parameter of \( \Phi \). Let \( \{a_i\}_{i=0}^{d} \) (resp. \( \{a_i^*\}_{i=0}^{d} \)) be the principal parameter (resp. dual principal parameter) of \( \Phi \). Let the scalar \( \Delta \) be from (15):
\[ \Delta = (a_0 - \theta_0)(a_0^* - \theta_0^*) - (a_d - \theta_0)(a_0^* - \theta_0^*). \]
Assume \( \varphi_1 + \varphi_d \neq \phi_1 + \phi_d \). Note that \( \Delta \neq 0 \) by Lemma 7.2. By Proposition 1.13 the type of \( \Phi \) is one of I, II, III\(^+\), III\(^-\). For each type, we display formulas that represent the parameter array in terms of \( \beta \) and the end-entries.

Type I.
Pick a nonzero \( q \in F \) such that \( \beta = q + q^{-1} \).

For \( 0 \leq i \leq d \) define
\[ K_i = \frac{(q^i - 1)(\theta_0 - \theta_d)}{(q^{d-1} - 1)(q^d - 1)\Delta^*} L_i, \]
where
\[ L_i = (q^{2d-i-1} - 1)(a_0 - \theta_0)(a_d^* - \theta_0^*) - q^{d-i}(q^{i-1} - 1)(a_0 - \theta_d)(a_0^* - \theta_0^*). \]
Then for $0 \leq i \leq d$
\[ \theta_i = \theta_0 + K_i, \quad \theta_i^* = \theta_0^* + K_i^*, \]
and for $1 \leq i \leq d$
\[ \phi_i = K_{d-i+1}^\parallel K_i^* = \frac{(q^i - 1)(q^{d-i+1} - 1)(a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{(q^d - 1)^2 \Delta}, \]
\[ \phi_i = K_{d-i+1}^\parallel K_i^* = \frac{(q^i - 1)(q^{d-i+1} - 1)(a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{(q^d - 1)^2 \Delta}. \]

**Type II.**

For $0 \leq i \leq d$ define
\[ K_i = \frac{i(\theta_0 - \theta_d)}{d(d - 1) \Delta^*} L_i, \]
where
\[ L_i = (2d - i - 1)(a_0 - \theta_0)(a_d^* - \theta_0^*) - (i - 1)(a_0 - \theta_d)(a_0^* - \theta_0^*). \]
Then for $0 \leq i \leq d$
\[ \theta_i = \theta_0 + K_i, \quad \theta_i^* = \theta_0^* + K_i^*, \]
and for $1 \leq i \leq d$
\[ \phi_i = K_{d-i+1}^\parallel K_i^* = \frac{2i(d - i + 1)(a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta}, \]
\[ \phi_i = K_{d-i+1}^\parallel K_i^* = \frac{2i(d - i + 1)(a_0 - \theta_0)(a_d - \theta_0)(a_0^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta}. \]

**Type III+**.

For $0 \leq i \leq d$ define
\[ K_i = \begin{cases} -\frac{i(\theta_0 - \theta_d)}{d} & \text{if } i \text{ is even}, \\ \frac{d}{d(\theta_0 - \theta_d)} L_i & \text{if } i \text{ is odd}, \end{cases} \]
where
\[ L_i = (2d - i - 1)(a_0 - \theta_0)(a_d^* - \theta_0^*) + (i - 1)(a_0 - \theta_d)(a_0^* - \theta_0^*). \]
Then for $0 \leq i \leq d$
\[ \theta_i = \theta_0 + K_i, \quad \theta_i^* = \theta_0^* + K_i^*. \]
and for $1 \leq i \leq d$

$$\varphi_i = \begin{cases} K_{d-i+1}^* K_i^* - \frac{2(d-1)(a_0 - \theta_d)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta} & \text{if } i \text{ is even}, \\
K_{d-i+1}^* K_i^* - \frac{2(d-1)(d-1)(a_0 - \theta_d)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta} & \text{if } i \text{ is odd}, \end{cases}$$

$$\phi_i = \begin{cases} K_{d-i+1}^* K_i^* - \frac{2(i-1)(a_0 - \theta_0)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta} & \text{if } i \text{ is even}, \\
K_{d-i+1}^* K_i^* - \frac{2(i-1)(d-1)(a_0 - \theta_0)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{d^2 \Delta} & \text{if } i \text{ is odd}. \end{cases}$$

**Type III**−.

For $0 \leq i \leq d$ define

$$K_i = \frac{\theta_0 - \theta_d}{(d - 1) \Delta^*} L_i$$

where

$$L_i = \begin{cases} i(a_0 - \theta_0)(a_d^* - \theta_0^*) + i(a_0 - \theta_d)(a_d - \theta_0) & \text{if } i \text{ is even}, \\
(2d - i - 1)(a_0 - \theta_0)(a_d - \theta_0) - (i - 1)(a_0 - \theta_d)(a_d^* - \theta_0^*) & \text{if } i \text{ is odd}. \end{cases}$$

Then for $0 \leq i \leq d$

$$\theta_i = \theta_0 + K_i, \quad \theta_i^* = \theta_0^* + K_i^*,$$

and for $1 \leq i \leq d$

$$\varphi_i = \begin{cases} K_{d-i+1}^* K_i^* - \frac{2(a_0 - \theta_d)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{\Delta^*} & \text{if } i \text{ is even}, \\
K_{d-i+1}^* K_i^* - \frac{2(a_0 - \theta_0)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{\Delta^*} & \text{if } i \text{ is odd}, \end{cases}$$

$$\phi_i = \begin{cases} K_{d-i+1}^* K_i^* - \frac{2(a_0 - \theta_0)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{\Delta^*} & \text{if } i \text{ is even}, \\
K_{d-i+1}^* K_i^* - \frac{2(a_0 - \theta_0)(a_d - \theta_0)(a_d^* - \theta_0^*)(\theta_0^* - \theta_d^*)}{\Delta^*} & \text{if } i \text{ is odd}. \end{cases}$$

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