OPTIMAL INVESTMENT FOR AN INSURER UNDER LIQUID RESERVES

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Abstract. In this paper, we study the optimal investment problem for an insurer, who is allowed to invest in a financial market which consists of \( N \) risky securities modeled by an \( N \)-dimensional Itô process. The surplus of the insurer is modeled by a general risk model. For the insurer’s wealth, some money (called liquid reserves) can only be used to cope with risk, and cannot be invested in the financial market. We suggest that the liquid reserve is a proportion of the total claim amount. By the martingale approach, we derive the optimal strategies for the CARA and the quadratic utilities, respectively.

1. Introduction. Classical insurance risk models model an insurer’s surplus over time. Research in this area usually focuses on the time of ruin when the surplus becomes negative for the first time, and the evaluation of the surplus and deficit at the time of ruin. See Gerber and Shiu [10, 11], Lin and Willmot [15], Cai et al. [5, 6], Yuan and Hu [23], Asmussen and Albrecher [2], Cheung and Landriault [7], Albrecher et al. [1] and references therein.

More general insurance risk models, where the investment of the surplus in the financial market are incorporated, are considered by many researchers. See Browne [4], Hipp and Plum [13], Liu and Yang [16], Luo [17], Zhang et al. [24], Belkina et al. [3] and references therein. In this case, portfolio selection plays an important role in maximizing the surplus over time or minimizing the risk associated with the surplus. Among these optimal investment studies, various optimization criteria have been proposed. One common and popular optimization criterion is the so-called utility criterion, in which the expected utility of the terminal wealth is maximized. See Zhou [26], Perera [19], Zou and Cadenillas [27], Guo [12] and references therein.

In this direction, two common and popular utility functions are constant absolute risk aversion (CARA) utilities and quadratic utilities in the economics, finance and insurance literature. For example, see Zhou [26], Perera [19] and Zou and Cadenillas [27]. In the present, we will also employ these two kinds of utility functions.

Another important optimization criterion is the mean-variance criterion. The mean-variance criterion was initiated by Markowitz [18], in which the mean and variance of the portfolio return are used to measure and quantify the trade-off between the investment return and risk. It is well known that to find the optimal

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investment strategy with mean-variance criterion can be converted to find the optimal strategy for the quadratic utility for the terminal wealth. For example, see Schweizer [20], Wang et al. [21], Xie [22] and Zhou et al. [25] and references therein.

In this paper, we will revisit the optimal investment for an insurer with utility criteria. Namely, we will study the optimal strategies for the CARA utility and the quadratic utility. The surplus was described by a general risk model, and his money can be invested to the financial market. In the financial market, there are multiple risk securities whose price follows an Itô process. The appreciation rate and volatility of risk securities depend on time, which are assumed to be continuous (deterministic) functions of time. We also assume that the risk-free rate is a continuous function of time. The liquid constraint is incorporated into the wealth process, while the liquid reserve is described by a proportion of the total claim amount. The liquid reserves can not be invested in the financial market, and can only be used to deal with the risk or be considered as the capital requirement of the regulators. By the martingale approach, we derive the optimal investment strategies for the CARA utility and the quadratic utility, respectively.

It should be mentioned that the liquid constraint was first introduced into insurance risk models by Embrechts and Schmidli [9], who considered that an insurer is allowed to make investment only if the insurer’s surplus attains a constant level. Cai et al. [5, 6] considered the insurance risk model with the constant liquid reserves.

It should be also mentioned that without the liquid constraint, the optimal investment strategies similar to this paper have been studied in this literature. Wang et al. [21] considered an optimal investment for an insurer using the martingale approach. They considered that the financial market has two tradeable securities, and obtained the optimal strategies. Zhou [26] also considered that the financial market has two tradeable securities and obtained an optimal strategy for the CARA utility. Perera [19] studied the consumption and investment problem in a general Lévy market, and obtained the optimal strategy for the CARA utility. Zou and Cadenillas [27] studied the optimal investment and risk control strategies for various utilities. Guo [12] considered optimal portfolio choice for an insurer with loss aversion under a financial market, which has multiple risk securities, and obtained the optimal strategy for an S-shaped utility function.

The rest of the paper is organized as follows. In Section 2, the model and problem are formulated, and some important exiting results are listed. In Section 3, by the martingale approach, the optimal strategy for the CARA utility is given. In Section 4, the optimal strategy for the quadratic utility is obtained.

2. Model formulation. In the absence of investment, the surplus of the insurer at time $t$ is

$$R(t) = R(0) + \int_0^t \alpha(s)ds - \sum_{i=1}^{N(t)} Y_i,$$

where $\alpha(t) (\alpha(t) > 0)$ is a deterministic function, representing the premium or gain rate at time $t$. $\{\sum_{i=1}^{N(t)} Y_i; t \geq 0\}$ is a compound Poisson process, $\sum_{i=1}^{N(t)} Y_i$ representing the aggregate losses or total claim amount up to time $t$. $\{N(t); t \geq 0\}$ is a Poisson process with parameter $\lambda$, representing the number of claims. The claims $\{Y_i; i \geq 1\}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common distribution function $F_Y(x)$, probability density function $f_Y(x)$ and finite expectation $m_Y$. It is also assumed that $\{Y_i; i \geq 1\}$ and $\{N(t); t \geq 0\}$ are independent.
If the function $\alpha(s)$ is identically a constant, then the risk model (1) becomes the classical compound Poisson risk model.

Remark 1. If the function $\alpha(s)$ is identically a constant, then the risk model (1) becomes the classical compound Poisson risk model.

Denote $L(t) := \sum_{i \leq 1} Y_i - \lambda Y t$, then $\{L(t); t \geq 0\}$ is an one-dimensional compensated compound Poisson process, and is a pure jump Lévy process. Let $\mu$ denote the jump measure of $\{L(t); t \geq 0\}$, then its dual predictable projection $\nu$ is $\nu(dt, dx) = dt \times m(dx)$, where $m(dx) = \lambda F_Y(x)$ with $m(\{0\}) = 0$.

In this paper, we assume that

$$\int_{\mathbb{R}} x^2 m(dx) < \infty. \quad (2)$$

Under assumption (2), the process $\{L(t); t \geq 0\}$ is a square-integrable martingale and has the following Lévy decomposition (see Cont and Tankov [8]):

$$L(t) = \int_0^t \int_{\mathbb{R}} x(\mu - \nu)(ds, dx),$$
or equivalently,

$$dL(t) = \int_{\mathbb{R}} x(\mu - \nu)(dt, dx).$$

Based on the Lévy decomposition for $L(t)$, the risk model (1) can be rewritten as

$$dR(t) = c(t)dt - dL(t) = c(t)dt - \int_{\mathbb{R}} x(\mu - \nu)(dt, dx), \quad (3)$$

where $c(t) := \alpha(t) - \lambda Y t$ is a deterministic function.

We assume that the financial market consists of $N + 1$ securities, indexed by $n = 0, 1, 2, \ldots, N$. The 0th security is risk-free with price

$$B(t) = \exp \left\{ \int_0^t r(s)ds \right\}, \quad (4)$$

where $r(s)$ is the risk-free interest rate, depends on the time. Naturally, we further assume that $r(s)$ is a continuous function. The $n$th security is risky with price $S_n(t)$ at time $t$, $n = 1, 2, \ldots, N$. Denote $S(t) := (S_1(t), S_2(t), \ldots, S_N(t))^T$, where $T$ means the transpose of a vector. The price process $S(t)$ of risk securities satisfies the following Itô process:

$$S(t) = S(0) + \int_0^t I_S(s)\mu(s)ds + \int_0^t I_S(s)\sigma(s)dW(s), \quad t \in [0, T], \quad a.s., \quad (5)$$

where $I_S(t)$ is a diagonal matrix with diagonal elements $S_n(t)$, i.e. $I_S(t) := \text{diag} (S_1(t), S_2(t), \ldots, S_N(t))$. $\mu(t) := (\mu_1(t), \mu_2(t), \ldots, \mu_N(t))^T$ represents the appreciation rate, $\sigma(t)$ is an $N \times N$ nonsingular matrix which represents the volatility, and $\{W(t); t \geq 0\}$ is a $N$-dimensional standard Brownian motion defined on the filtered completed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. We assume that $\{W(t); t \geq 0\}$, $\{N(t); t \geq 0\}$ and $\{Y_i; i \geq 1\}$ are mutually independent. This independence assumption is reasonable in the following sense. The Brownian motion $\{W(t); t \geq 0\}$ affecting the volatility of the prices of securities in the financial market should have nothing to do with the occurring of the claims in the insurance market.

From a risk management viewpoint, the insurer usually keeps some liquid reserves in order to pay for the potential claims. In the present paper, we suggest that the
liquid reserve up to time $t$ can be chosen as a proportion of the total claim amount up to time $t$. The reason for this kind of suggestion is as follows. On one hand, the total claim amount up to time $t$ is $\sum_{i=1}^{N(t)} Y_i$. On the other hand, if one overly keeps the liquid reserves, then the profit of investment will be lowered. If one keeps insufficient liquid reserves, then the insurer may fail to pay for the potential claims.

In summary, the liquid reserve up to time $t$, denoted by $\Delta(t)$, can be chosen as $k \sum_{i=1}^{N(t)} Y_i$, where $k \in (0,1)$ is a constant, that is $\Delta(t) = k \sum_{i=1}^{N(t)} Y_i$. Under this liquid constraint, the wealth process $X_{x_0, \pi} := \{X_{x_0, \pi}^t, t \in [0,T]\}$ of the insurer is governed by the following equation

$$
\begin{align}
\left\{ 
\begin{array}{l}
\frac{dX_{t}^{x_0, \pi}}{dt} = & \pi(t)^T [\mu(t)dt + \sigma(t)dW(t)] + [X_{t}^{x_0, \pi} - \pi(t)^T 1 - \Delta(t)] v(t)dt \\
& + c(t)dt - dL(t),
\end{array}
\right.
\end{align}
$$

where $1 := (1, 1, \cdots, 1)^T$, $\pi(t) := (\pi_1(t), \pi_2(t), \cdots, \pi_N(t))^T$. Here $\pi_n(t)$ represents the amount invested in the $n$th security at time $t$, $n = 1, 2, \cdots, N$. Define $\pi_0(t) := X_{t}^{x_0, \pi} - \pi(t)^T 1 - \Delta(t)$, representing the amount invested in the 0th security.

Since $\pi_0(t)$ is controlled by $X_{t}^{x_0, \pi} - \pi(t)^T 1$ and $\Delta(t)$, we call $\pi(t)$ a trading strategy at time $t$. If a trading strategy $\pi$ satisfies $\mathbb{E} \left[ \int_{0}^{T} \pi(t)^T \pi(t) dt \right] = \sum_{i=1}^{N} \mathbb{E} \left( \int_{0}^{T} \pi_i(t)^2 dt \right) < \infty$, then $\pi$ is called an admissible strategy. Denote by $\mathcal{A}$ the set of all admissible trading strategies.

**Remark 2.** Note that $\pi(t)$ and $\pi_0(t)$ could be negative. If $\pi(t)$ is negative, it can be considered as short selling. If $\pi_0(t)$ is negative, it means that insurer can borrow money to invest. For convenience, let the debit rate be equal to interest rate $r(t)$.

**Remark 3.** If $N = 1$, $k = 0$, $\mu(t) = \mu_1$, $\sigma(t) = \sigma_1$, $\alpha(t) = \alpha$ and $r(t) = r$, then $c(t) = \alpha - \lambda m Y$. In this case, the model coincides with the one in the case of $\beta = 0$ of Wang et al [21]. If $k = 0$, the model coincides with the one of Guo [12].

Let

$$
\tilde{X}_{t}^{x_0, \pi} := X_{t}^{x_0, \pi} - k \sum_{i=1}^{N(t)} Y_i, \quad t \in [0, T],
$$

then $\{\tilde{X}_{t}^{x_0, \pi}, t \in [0, T]\}$ satisfies the following equation

$$
\begin{align}
\frac{d}{dt} \left( \frac{\tilde{X}_{t}^{x_0, \pi}}{B(t)} \right) &= \frac{\pi(t)^T (\mu(t) - r(t) 1)}{B(t)} + \frac{c(t) - \lambda k m Y}{B(t)} dt + \frac{\pi(t)^T \sigma(t)}{B(t)} dW_t \\
&- \int_{\mathbb{R}} \frac{1+k}{B(t)} (\mu - \nu)(dt, dx),
\end{align}
$$

or equivalently,

$$
\tilde{X}_{t}^{x_0, \pi} = B(t) x_0 + B(t) \int_{0}^{t} \frac{\pi(s)^T (\mu(s) - r(s) 1)}{B(s)} ds + \int_{0}^{t} \frac{c(s) - \lambda k m Y}{B(s)} B(t) ds \\
+ B(t) \int_{0}^{t} \frac{\pi(s)^T \sigma(s)}{B(s)} dW_s - B(t)(1 + k) \int_{\mathbb{R}} \frac{x}{B(s)} (\mu - \nu)(ds, dx).
$$
Hence the wealth process $X^x_0,\pi$ can be expressed as follows:

$$
X^x_0,\pi = B(t)x_0 + B(t)\int_0^t \frac{\pi(s)^T(\mu(s) - r(s)1)}{B(s)}ds + \int_0^t \frac{c(s) - \lambda km\sigma}{B(s)}B(t)ds \\
+ B(t)\int_0^t \frac{\pi(s)^T\sigma(s)}{B(s)}dW_s - B(t)(1 + k)\int_0^T \frac{x}{B(s)}(\mu - \nu)(ds, dx) \\
+ k\int_0^t x(\mu - \nu)(ds, dx) + \lambda km\nu t.
$$

Next, we introduce the optimal investment problem. Let $U$ be a utility function. Then the optimal investment problem is to maximize the utility of the terminal wealth of the insurer, that is

$$
\max_{\pi \in A} \mathbb{E}[U(X^x_0,\pi_T)].
$$

We end this section with two lemmas, which are crucial for our main result. The first lemma is from Wang et al. (see [21], Proposition 2.1). The second lemma is from Cont and Tankov (see [8], Proposition 9.4), which is the martingale representation.

**Lemma 2.1.** If there exists a strategy $\pi^* \in A$ such that

$$
\mathbb{E} \left[ U'(X^x_0,\pi^*)X^x_0,\pi^*_T \right] \text{ is constant for any } \pi \in A,
$$

then $\pi^*$ is the optimal strategy for (11).

**Lemma 2.2.** For any local (resp. square-integrable) martingale $\{Z_t; t \in [0, T]\}$, there exist some $\theta = (\theta_1, \theta_2, \cdots, \theta_N)^T$ and $\theta_0$, such that

$$
Z_t = Z_0 + \int_0^t \theta(s)^T dW_s + \int_0^t \theta_0(s, x)(\mu - \nu)(ds, dx),
$$

for all $t \in [0, T]$.

3. **Optimal strategy for CARA utility.** In this present paper, we will consider two utility functions: CARA utility and quadratic utility functions. In this section, we discuss the optimal investment problem (11) with $U$ being CARA utility function, that is $U(x) = 1 - \frac{1}{\gamma}e^{-\gamma x}$, where $\gamma > 0$.

Note that $U'(x) = e^{-\gamma x}$. By Lemma 2.1, we know that if a strategy $\pi^* \in A$ such that

$$
\mathbb{E} \left[ e^{-\gamma X^x_0,\pi^*_T}X^x_0,\pi^*_T \right] \text{ is constant for every } \pi \in A,
$$

then $\pi^*$ is the optimal strategy for (11).

It follows from (10) and (13) that if the special strategy $\pi^*$ satisfies

$$
\mathbb{E} \left[ e^{-\gamma X^x_0,\pi^*_T} \left( \int_0^T \frac{\pi(s)^T(\mu(s) - r(s)1)}{B(s)}ds + \int_0^T \frac{\pi(s)^T\sigma(s)}{B(s)}dW_s \right) \right] \text{ is constant},
$$

for every $\pi \in A$, then $\pi^*$ is the optimal strategy.

The next theorem is the main result of this section.
Theorem 3.1. Assume that the utility function \( U(x) = 1 - \frac{1}{\gamma}e^{-\gamma x} \), then 
\[ \pi^* := \{ \pi^*(t); \ t \in [0, T] \} \] 
with \( \pi^*(t) := \frac{B(t)[\sigma(t)^T - \gamma B(t)]^{-1}(\mu(t) - r(t)1)}{\gamma} \) is the optimal strategy for (11).

Proof. First, we will find a candidate strategy (also denoted by \( \pi^* \)) by martingale approach (Karatzas et al. [14]).

Let 
\[ M^*_t := \frac{e^{-\gamma X^{T0, \pi^*}_t}}{\mathbb{E}[e^{-\gamma X^{T0, \pi^*}_T}]} \] 
and \( M^*_t := \mathbb{E}[M^*_T | \mathcal{F}_t] \), for all \( t \in [0, T] \), then \{\( M^*_t \)\} is a martingale under \( P \).

Define 
\[ V_t := \int_0^t \frac{1}{M^*_s}dM^*_s, \ t \in [0, T], \] 
then \{\( V_t \)\} is a local martingale under \( P \). By Lemma 2.2, we know that 
\[ dV_t = \theta(t)^T dW_t + \int_{\mathbb{R}} \theta_0(t, x)(\mu - \nu)(dt, dx), \] 
where \( \theta(t) := (\theta_1(t), \theta_2(t), \cdots, \theta_N(t))^T \). From (16) and (17) it follows that 
\[ dM^*_t = M^*_s \left[ \theta(t)^T dW_t + \int_{\mathbb{R}} \theta_0(t, x)(\mu - \nu)(dt, dx) \right]. \] 

By Doléans-Dade exponential formula we have that 
\[ M^*_t = \exp \left\{ \int_0^t \theta(s)^T dW_s - \frac{1}{2} \int_0^t \theta(s)^T \theta(s) ds \\ + \int_0^t \int_{\mathbb{R}} \theta_0(s, x)(\mu - \nu)(ds, dx) + \int_0^t \left[ \log(1 + \theta_0(s, x)) - \theta_0(s, x) \mu(ds, dx) \right] \right\}. \] 

Let \( Q \) be a probability measure on \((\Omega, \mathcal{F})\) such that \( \frac{dQ}{dP} = M^*_T \). By the Girsanov’s Theorem, we know that \( W_t - \int_0^t \theta(s) ds \) is an \( N \)-dimensional standard Brownian motion under \( Q \).

For any stopping time \( \tau \leq T \), let \( \pi^{\tau, i}_t := I_{[0, \tau]}(t)e_i, \ i = 1, 2, \cdots, N \), where \( e_i \) is the unit vector with the \( i \)th element being 1 and other elements being 0, and \( I_{[0, \tau]}(t) \) is the indicator function of \([0, \tau]\), then \( \pi^{\tau, i}_t \in \mathcal{A} \). Substituting \( \pi^{\tau, i}_t \in \mathcal{A} \) into (14) yields 
\[ E \left[ e^{-\gamma X^{T0, \pi^*}_T} \left( \int_0^T \frac{(\pi^{\tau, i}_t)^T(\mu(s) - r(s)1)}{B(s)} ds + \int_0^T \frac{(\pi^{\tau, i}_t)^T \sigma(s)}{B(s)} dW_s \right) \right] \]
\[ = E \left[ e^{-\gamma X^{T0, \pi^*}_T} \left( \int_0^T \frac{e_i^T(\mu(s) - r(s)1)}{B(s)} ds + \int_0^T \frac{e_i^T \sigma(s)}{B(s)} dW_s \right) \right] \]
\[ = E \left[ M^*_T \left( \int_0^T \frac{e_i^T(\mu(s) - r(s)1)}{B(s)} ds + \int_0^T \frac{e_i^T \sigma(s)}{B(s)} dW_s \right) \right] \]
is a constant, which results in
\[
\int_0^t \frac{e_i^T (\mu(s) - r(s) \mathbf{1})}{B(s)} ds + \int_0^t \frac{e_i^T \sigma(s)}{B(s)} dW_s
\]
is a martingale under $Q$, $i = 1, 2, \cdots, N$.\)

On one hand, $W_t - \int_0^t \theta(s) ds$ is an $N$-dimension Brownian motion under $Q$. On the other hand, for any $i = 1, 2, \cdots, N$, \( \int_0^t \frac{e_i^T (\mu(s) - r(s) \mathbf{1})}{B(s)} ds + \int_0^t \frac{e_i^T \sigma(s)}{B(s)} dW_s \) is a $Q$-martingale. Hence for any $t \in [0, T]$, the following equalities

\[
e_i^T[(\mu(t) - r(t) \mathbf{1}) + \sigma(t) \theta(t)] = 0, \quad i = 1, 2, \cdots, N,
\]
hold. Therefore

\[
\theta(t) = -\sigma(t)^{-1}(\mu(t) - r(t) \mathbf{1}).
\]

Now, we discuss the expressions of $e^{-\gamma X_T^{\pi_0}}$. By (15) and (19),

\[
e^{-\gamma X_T^{\pi_0}} = \mathbb{E} \left[ e^{-\gamma X_T^{\pi_0}} \right] M_T^I = \mathbb{E} \left[ e^{-\gamma X_T^{\pi_0}} \right] \exp \left\{ \int_0^T \theta(s)^T dW_s + \int_0^T \theta_0(s, x)(\mu - \nu)(ds, dx) \right. \]

\[
- \frac{1}{2} \int_0^T \theta(s)^T \theta(s) ds + \int_0^T \log(1 + \theta_0(s, x)) \mu(ds, dx) \left. \right\}
\]

\[
= \mathbb{E} \left[ e^{-\gamma X_T^{\pi_0}} \right] \exp \left\{ \int_0^T \theta(s)^T dW_s + \int_0^T \log(1 + \theta_0(s, x))(\mu - \nu)(ds, dx) \right. \]

\[
- \frac{1}{2} \int_0^T \theta(s)^T \theta(s) ds + \int_0^T \log(1 + \theta_0(s, x)) \nu(ds, dx) \left. \right\}.
\]

Note that from (10), it follows that

\[
e^{-\gamma X_T^{\pi_0}} = \exp \left\{ -\gamma B(T) x_0 - \gamma B(T) \int_0^T \frac{\pi^*(s)^T (\mu(s) - r(s) \mathbf{1}) + c(s) - \lambda k_m x}{B(s)} ds \right\}
\]

\[
\cdot \exp \left\{ -\gamma B(T) \int_0^T \frac{\pi^*(s)^T \sigma(s)}{B(s)} dW_s \right\}
\]
\[
\cdot \exp \left\{ \gamma B(T)(1 + k) \int_0^T \frac{x}{B(s)} (\mu - \nu)(ds, dx) \right\} \\
\cdot \exp \left\{ -\gamma \lambda m \gamma T - \gamma k \int_0^T x(\mu - \nu)(ds, dx) \right\}.
\]

Comparing \(dW_t\)-term and \(d(\mu - \nu)\)-term in (24) with those in (25), respectively, we have that for any \(t \in [0, T]\),

\[
\begin{cases}
-\gamma B(T) \pi^*(t)^T \sigma(t) \frac{\sigma(t) - r(t)}{B(t)}, \\
\log(1 + \theta_0(t, x)) = \frac{\gamma B(T)(1 + k)x}{B(t)} - \gamma tx.
\end{cases}
\]

By (23) and (26), we obtain the candidate strategy

\[
\pi^*(t) = \frac{B(t) [\sigma(t) \sigma(t)^T]^{-1} (\mu(t) - r(t)1)}{\gamma B(T)}
\]

and

\[
\theta_0(t, x) = \exp \left\{ \gamma B(T)(1 + k)x \right\} B(t) - \gamma tx.
\]

Note that in this case, the amount invested in the risk-free security at time \(t\) \(\pi_0(t)\) is given by

\[
\pi_0(t) = X_{T}^{\pi_0, \pi^*} - \pi^*(t)^T 1 - k \sum_{i=1}^{N(t)} Y_i.
\]

Second, let \(\pi^*(t)\) be defined as in (27) and \(\theta(t), \theta_0(t, x)\) be defined as in (23) and (28), respectively. Next, we will check that \(\exp\{-\gamma X_{T}^{\pi_0, \pi^*}\}\), which is defined by (24), coincides with (25). Because \(dW_t\)-term and \(d(\mu - \nu)\)-term are identical, respectively, so we only need to verify that the constant term is consistent. It means that we will verify that the following equation

\[
E \left[ e^{-\gamma X_{T}^{\pi_0, \pi^*}} \right] J_T = I_T
\]

holds, where

\[
J_T := \exp \left\{ -\frac{1}{2} \int_0^T \theta(s)^T \theta(s) ds + \int_0^T \left[ \log(1 + \theta_0(s, x)) - \theta_0(s, x) \right] \nu(ds, dx) \right\}
\]

\[
= \exp \left\{ -\frac{1}{2} \int_0^T (\mu(s) - r(s)1)^T [\sigma(s) \sigma(s)^T]^{-1} (\mu(s) - r(s)1) ds \right\}
\]

\[
\cdot \exp \left\{ \int_0^T \gamma B(T)(1 + k)x \frac{\sigma(s) - r(s)1}{B(s)} - \gamma tx - \exp \left\{ \gamma B(T)(1 + k)x \frac{\sigma(s) - r(s)1}{B(s)} - \gamma tx + 1 \nu(ds, dx) \right\} \right\}
\]

and

\[
I_T := \exp \left\{ -\gamma B(T)x_0 \right\}
\]
\[- \int_0^T \pi^*(s)^T (\mu(s) - r(s)\mathbf{1}) + c(s) - \lambda km_Y \gamma B(T) ds - \gamma \lambda km_Y T \}\]

\[= \exp \left\{ -\gamma B(T)x_0 - \int_0^T (\mu(s) - r(s)\mathbf{1})^T \left[ \sigma(s)\sigma(s)^T \right]^{-1} (\mu(s) - r(s)\mathbf{1}) ds \right\} \cdot \exp \left\{ -\int_0^t \frac{c(s) - \lambda km_Y}{B(s)} \gamma B(T) ds - \gamma \lambda km_Y T \right\}. \tag{31} \]

Note that $I_T$ and $J_T$ are deterministic functions.

By (25) and (27), we have that
\[e^{-\gamma X_{t_0}^{x_0}} = \exp \left\{ -\gamma B(T)x_0 - \int_0^T g(s) ds - \gamma B(T) \int_0^T \frac{c(s) - \lambda km_Y}{B(s)} ds \right\} \cdot \exp \left\{ -\int_0^T (\sigma(s)^{-1}(\mu(s) - r(s)\mathbf{1}))^T dW_s \right\} \cdot \exp \left\{ \gamma B(T)(1 + k) \int_0^T \int_\mathbb{R} \frac{x}{B(s)} (\mu - \nu)(ds, dx) \right\} \cdot \exp \left\{ -\gamma \lambda km_Y T - \gamma k \int_0^T x(\mu - \nu)(ds, dx) \right\}
\]

\[= H_T I_T, \]

where
\[H_T := \exp \left\{ -\int_0^T (\mu(s) - r(s)\mathbf{1})^T (\sigma(s)^T)^{-1} dW_s \right\} \cdot \exp \left\{ \int_0^T \int_\mathbb{R} \left( \frac{\gamma B(T)(1 + k)x}{B(s)} - \gamma kx \right) (\mu - \nu)(ds, dx) \right\}. \tag{33} \]

\[g(s) := (\mu(s) - r(s)\mathbf{1})^T \left[ \sigma(s)\sigma(s)^T \right]^{-1} (\mu(s) - r(s)\mathbf{1}). \]

By (23), (24) and (28) we know that
\[e^{-\gamma X_{t_0}^{x_0}} = M^*_t E \left[ e^{-\gamma X_{t_0}^{x_0}} \right] \]

\[= E \left[ e^{-\gamma X_{t_0}^{x_0}} \right] \exp \left\{ \int_0^T -\left( \sigma(s)^{-1}(\mu(s) - r(s)\mathbf{1}) \right)^T dW_s - \frac{1}{2} \int_0^T g(s) ds \right. \]
\[+ \int_0^T \int_\mathbb{R} \left( \frac{\gamma B(T)(1 + k)x}{B(s)} - \gamma kx - \exp \left\{ \frac{\gamma B(T)(1 + k)x}{B(s)} - \gamma kx \right\} + 1 \right) (\mu - \nu)(ds, dx) \]
\[+ \int_0^T \int_\mathbb{R} \left( \frac{\gamma B(T)(1 + k)x}{B(s)} - \gamma kx \right) (\mu - \nu)(ds, dx) \right\}
\]

\[= E \left[ e^{-\gamma X_{t_0}^{x_0}} \right] H_T J_T, \tag{34} \]
which implies that $M^*_t = H_T J_T$. Because $E[M^*_T] = 1$ and $J_T$ is deterministic, hence

$$E[H_T] = J_T^{-1}. \quad (35)$$

From (32)—(35), it follows that

$$E\left[e^{-\gamma X^{20,\pi^*}_t}\right] = E[I_T H_T] = E[H_T] I_T = \frac{1}{J_T} I_T. \quad (36)$$

Therefore (29) holds.

Finally, we will show that $\pi^*$ given by (27), i.e. $\pi^*(t) = \frac{B(t) \left[\sigma(t)\sigma(t)^T\right]^{-1} (\mu(t) - r(t)1)}{\gamma B(T)}$, $t \in [0, T]$, is the optimal strategy.

For any $\pi \in \mathcal{A}$, define $G^\pi_t := \int_0^t \pi(s)^T (\mu(s) - r(s)1) ds + \int_0^t \pi(s)^T \sigma(s) dW_s$, $t \in [0, T]$.

According to the Girsanov’s Theorem, we know that $\{G^\pi_t\}$ is a local martingale under $Q$ for any $\pi \in \mathcal{A}$. Moreover, we can see that $E \left[\sup_{0 \leq t \leq T} |G^\pi_t|^2\right] < \infty$. Note that $E[(M^*_T)^2] < \infty$, which further implies

$$E_Q \left[\sup_{0 \leq t \leq T} |G^\pi_t|\right] = E \left[\sup_{0 \leq t \leq T} |G^\pi_t|\right] \leq \sqrt{E[(M^*_T)^2]} \cdot E \left[\sup_{0 \leq t \leq T} |G^\pi_t|^2\right] < \infty,$$

which means that $\{G^\pi_t\}$ is a uniformly integrable martingale under $Q$ for any stopping time $\tau \leq T$. Hence, $G^\pi_t$ is a $Q$-martingale. Thus $E_Q[G^\pi_T] = 0$ for any $\pi \in \mathcal{A}$, which further implies that $E \left[e^{-\gamma X^{20,\pi^*}_T} G^\pi_T\right]$ is constant for every $\pi \in \mathcal{A}$. Therefore (14) holds. Consequently, $\pi^*$ is the optimal strategy for the CARA utility.

**Remark 4.** The liquid reserves only affect the amount of money invested in risk-free security, but it does not affect the amount of money invested in risk securities.

**Remark 5.** If $N = 1, k = 0, \mu(t) = \mu_1, \sigma(t) = \sigma_1, \alpha(t) = \alpha$ and $r(t) = r$, then for any $t \in [0, T]$, the optimal strategy for the CARA utility at time $t$ is

$$\pi^*(t) = \frac{B(t) \left[\sigma(t)\sigma(t)^T\right]^{-1} (\mu(t) - r(t)1)}{\gamma B(T)} = \frac{(\mu_1 - r)e^{-r(T-t)}}{\gamma \sigma^2_1}$$

and

$$\pi_0(t) = X^{20,\pi^*}_T - \pi^*(t),$$

which coincides with the optimal strategy for the CARA utility in the special case of $\beta = 0$ in Wang et al. [21].

**Remark 6.** From Theorem 3.1, we can see that the optimal strategy $\pi^*(t)$ at time $t \in [0, T]$ is the reciprocal of the absolute risk aversion coefficient $\gamma$, hence the optimal strategy is quite sensitive for small $\gamma$, while quite robust for large $\gamma$.

**4. Optimal strategy for quadratic utility.** In this section, we will show the optimal investment problem (11) with $U$ being quadratic utility function, that is $U(x) = x - \frac{\gamma}{2} x^2$, where $\gamma > 0$ is a constant.

Note that $U'(x) = 1 - \gamma x$. By Lemma 2.1, we know that if a strategy $\pi^* \in \mathcal{A}$ such that

$$E \left[(1 - \gamma X^{20,\pi^*}_T) X^{20,\pi} \right]$$

(37)
is constant for every $\pi \in \mathcal{A}$, then $\pi^*$ is the optimal strategy for (11).

By (10) and (37), we know that if the special strategy $\pi^*$ satisfies
\[
\mathbb{E} \left[ (1 - \gamma X_{T}^{x_0, \pi^*}) \left( \int_{0}^{T} \frac{\pi^T (\mu(s) - r(s)1)}{B(s)} ds + \int_{0}^{T} \frac{\pi^T \sigma(s)}{B(s)} dW_s \right) \right] \text{ is constant},
\]
for every $\pi \in \mathcal{A}$, then $\pi^*$ is the optimal strategy.

The next theorem is the main result of this section.

**Theorem 4.1.** Assume that the utility function $U$ is the quadratic utility function, that is $U(x) = x - \frac{1}{2} x^2$, then $\pi^* := \{\pi^*(t); t \in [0, T]\}$ is the optimal strategy for (11), where $\pi^*(t)$ is given by
\[
\pi^*(t) = \frac{G(t)Z_{t}^\ast B(t) [\sigma(t)\sigma(t)^T]^{-1} (\mu(t) - r(t)1)}{\gamma G(T)B(T)},
\]
where
\[
G(t) = \exp \left\{- \int_{0}^{t} g(s) ds \right\}, t \in [0, T],
\]
\[
g(s) = (\mu(s) - r(s)1)^T [\sigma(s)\sigma(s)^T]^{-1} (\mu(s) - r(s)1),
\]
\[
Z_{0}^* = \left( 1 - \gamma x_0 B(T) - \gamma \int_{0}^{T} (c(s) - \lambda km_Y T) B(s) ds \right)
\]
\[
\cdot \exp \left\{ \int_{0}^{T} -g(s) ds \right\},
\]
\[
Z_{t}^* = \left[ Z_{0}^* - \int_{0}^{t} V_s^{-1} \gamma \frac{G(T)B(T)}{G(s)B(s)} dL_s \right],
\]
\[
V_t = \exp \left\{ - \int_{0}^{t} \sigma(s)^{-1} (\mu(s) - r(s)1) dW_s - \frac{1}{2} \int_{0}^{t} g(s) ds \right\}.
\]

**Proof.** First, we will find a candidate strategy. Let $Z_{t}^* := \mathbb{E}(1 - \gamma X_{T}^{x_0, \pi^*}|\mathcal{F}_t), t \in [0, T]$. Note that $\{Z_{t}^*\}$ is a square-integrable martingale. By Lemma 2.2, we know that $Z_{t}^*$ has the following representation:
\[
Z_{t}^* = Z_{0}^* + \int_{0}^{t} \theta(s)^T dW_s + \int_{0}^{t} \int_{\mathbb{R}} \theta_0(s, x)(\mu - \nu)(ds, dx).
\]

For any stopping time $\tau \leq T$, let $\pi^*_{t,\tau} := I_{[0, \tau]}(t)e_{t}^T$, $i = 1, 2, \cdots, N$, then $\pi^*_{t,\tau} \in \mathcal{A}$, $i = 1, 2, \cdots, N$. Plugging $\pi^*_{t,\tau}$ into (38) yields that for any stopping time $\tau \leq T$,
\[
\mathbb{E} \left[ (1 - \gamma X_{\tau}^{x_0, \pi^*}) \int_{0}^{\tau} (\pi^*_{t,\tau})^T (\mu(s) - r(s)1) ds + (\pi^*_{t,\tau})^T \sigma(s) dW_s \right]
\]
\[
= \mathbb{E} \left[ (1 - \gamma X_{\tau}^{x_0, \pi^*}) \int_{0}^{\tau} e_{\tau}^T (\mu(s) - r(s)1) ds + e_{\tau}^T \sigma(s) dW_s \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E} \left[ (1 - \gamma X_{\tau}^{x_0, \pi^*}) \int_{0}^{\tau} e_{\tau}^T (\mu(s) - r(s)1) ds + e_{\tau}^T \sigma(s) dW_s | \mathcal{F}_\tau \right] \right]
\]
\[
= \mathbb{E} \left[ Z_{\tau}^* \left( \int_{0}^{\tau} e_{\tau}^T (\mu(s) - r(s)1) B(s) + \int_{0}^{\tau} e_{\tau}^T \sigma(s) B(s) dW_s \right) \right].
\]
is a constant, which gives rise to

\[ Z_t^* \left( \int_0^t \frac{e_t^T (\mu(s) - r(s)1)}{B(s)} ds + \int_0^t e_t^T \sigma(s) B(s) dW_s \right) \text{ is a martingale under } \mathbb{P}. \] (41)

By Itô formula, we know that

\[
d\left( Z_t^* \left( \int_0^t \frac{e_t^T (\mu(s) - r(s)1)}{B(s)} ds + \int_0^t e_t^T \sigma(s) B(s) dW_s \right) \right) = Z_t^* e_t^T (\mu(t) - r(t)1) \frac{B(t)}{B(t)} dt + Z_t^* e_t^T \sigma(t) \theta(t) \frac{B(t)}{B(t)} dt \]

(42)

\[ + Z_t^* e_t^T \sigma(t) dW_t + \left( \int_0^t e_t^T (\mu(s) - r(s)1) ds + e_t^T \sigma(s) dW_s \right) dZ_t^*. \]

Because \[ \int_0^T Z_t^* e_t^T \sigma(t) dW_t + \int_0^T \left( \int_0^t e_t^T (\mu(s) - r(s)1) ds + e_t^T \sigma(s) dW_s \right) dZ_t^* \]

is a local martingale, so the \( dt \)-term of (42) must be satisfied the following conditions:

\[ e_t^T [Z_t^*(\mu(t) - r(t)1) + \sigma(t) \theta(t)] = 0, \ i = 1, 2, \cdots, N. \] (43)

Therefore

\[ \theta(t) = -Z_t^* \sigma(t)^{-1}(\mu(t) - r(t)1). \] (44)

Let \( G(t) := \exp \left\{ -\int_0^t g(s) ds \right\}, t \in [0, T], \) where \( g(s) \) is a non-random Lebesgue-integrable function and needs to be determined. By Itô formula and equations (40) and (44), we obtain that

\[
G(T) Z_T^* = G(0) Z_0^* + \int_0^T G(s) dZ_s^* + \int_0^T Z_s^* dG(s)
\]

\[ = Z_0^* + \int_0^T G(s) \theta(s) T dW_s - \int_0^T Z_s^* G(s) g(s) ds + \int_0^T \int_{\mathbb{R}} G(s) \theta_0(s, x)(\mu - \nu)(ds, dx)
\]

\[ = Z_0^* - \int_0^T Z_s^* G(s) g(s) ds - \int_0^T G(s) Z_s^* (\mu(s) - r(s)1)^T (\sigma(s) T)^{-1} dW_s
\]

\[ + \int_0^T \int_{\mathbb{R}} G(s) \theta_0(s, x)(\mu - \nu)(ds, dx).
\]

Because \( Z_T^* = 1 - \gamma X_T^x0, \pi^* \), then \( X_T^x0, \pi^* \) has the following expression

\[
X_T^x0, \pi^* = \frac{1 - Z_T^*}{\gamma} = \frac{1}{\gamma} - \frac{G(T) Z_T^*}{\gamma G(T)}
\]

(45)

\[
= \frac{1}{\gamma} + \frac{1}{\gamma G(T)} \int_0^T Z_s^* G(s) g(s) ds
\]

\[ + \frac{1}{\gamma G(T)} \int_0^T G(s) Z_s^* (\mu(s) - r(s)1)^T (\sigma(s) T)^{-1} dW_s
\]

\[ - \frac{Z_0^*}{\gamma G(T)} - \frac{1}{\gamma G(T)} \int_0^T G(s) \theta_0(s, x)(\mu - \nu)(ds, dx).
\]
Note that from (10), it follows that
\[ X_t^{x_0,\pi^*} = \lambda km_T + B(T)x_0 \]
\[ + B(T) \left( \int_0^T \frac{\pi^*(s)\mu(s) - r(s)1}{B(s)} ds + \int_0^T c(s) - \lambda km_T ds \right) \]
\[ + B(T) \left( \int_0^T \frac{\pi^*(s)\sigma(s)}{B(s)} dw - \int_0^T \left( \frac{B(T)(1+k)x}{B(s)} - kx \right) (\mu - \nu)(ds, dx). \]
Comparing \( dw \)-term and \( d(\mu - \nu) \)-term in (45) with those in (46), respectively, we have that for any \( t \in [0, T] \),
\[ \left\{ \begin{array}{l}
G(t)Z_{t-}(\mu(t) - r(t)1) (\sigma(t)\sigma^T)^{-1} = \frac{\pi^*(t)^T \sigma(t)B(T)}{B(t)}, \\
G(t)\theta(t, x) = \frac{B(T)(1+k)x}{B(t)} - kx.
\end{array} \right. \]
Hence, we find a particular strategy denoted by \( \pi^* \),
\[ \pi^*(t) = \frac{G(t)Z_{t-}B(t) \left[ \sigma(t)\sigma^T \right]^{-1}}{\gamma G(T)} (\mu(t) - r(t)1) \]
and
\[ \theta_0(t, x) = \frac{\gamma G(T)}{G(t)} \left( \frac{B(T)(1+k)x}{B(t)} - kx \right). \]
Note that in this case, the amount invested in the risk-free security at time \( t \)
\( \pi_0(t) \) is given by
\[ \pi_0(t) = X_t^{x_0,\pi^*} - \pi^*(t)^T 1 - k \sum_{i=1}^{N(t)} Y_i. \]
Second, to ensure that the two expressions \( X_t^{x_0,\pi^*} \) are identical, the so called 2 constant coefficients of \( X_t^{x_0,\pi^*} \) must be equal, i.e.
\[ \frac{1}{\gamma} - \frac{Z_0^*}{\gamma G(T)} + \frac{1}{\gamma G(T)} \int_0^T Z_{t-}(s)g(s)ds \]
\[ = \lambda km_T + B(T)x_0 + B(T) \left( \int_0^T \frac{\pi^*(t)^T (\mu(t) - r(t)1)}{B(t)} dt + \int_0^T c(t) - \lambda km_T \right). \]
Substituting \( \pi^* \) into the right side of the above equation yields
\[ \lambda km_T + B(T)x_0 + B(T) \left( \int_0^T \frac{\pi^*(t)^T (\mu(t) - r(t)1)}{B(t)} dt + \int_0^T c(t) - \lambda km_T \right) \]
\[ = \lambda km_T + B(T)x_0 + B(T) \int_0^T c(t) - \lambda km_T dt \]
\[ + \int_0^T \frac{G(t)Z_{t-} (\mu(t) - r(t)1)^T \left[ \sigma(t)\sigma^T \right]^{-1} (\mu(t) - r(t)1)}{\gamma G(T)} dt. \]
Let
\[ g(t) := (\mu(t) - r(t)1)^T \left[ \sigma(t)\sigma^T \right]^{-1} (\mu(t) - r(t)1) \]
and
\[
Z_0^*: = \left( 1 - \gamma x_0 B(T) - \lambda \gamma k m_Y T - \gamma \int_0^T \frac{(c(s) - \lambda k m_Y) B(T)}{B(s)} ds \right) \\
\times \exp \left\{ \int_0^T -g(s)ds \right\},
\]
then equation (49) is satisfied. Hence the two expressions of \(X_{x_0, \pi^*}^T\) are identical.

Next, to obtain the expression of \(\pi^*(t)\), we have to solve \(Z_t^*\).

Put
\[
\begin{align*}
\theta(t) &= -Z_t^* \sigma^{-1}(\mu(t) - r(t) 1), \\
\theta_0(t, x) &= \gamma G(T) \left\{ \frac{B(T)(1 + k) x}{B(t)} - k x \right\}, \\
Z_0^* &= \left( 1 - \gamma x_0 B(T) - \lambda \gamma k m_Y T - \gamma \int_0^T \frac{(c(s) - \lambda k m_Y) B(T)}{B(s)} ds \right) \\
\times \exp \left\{ \int_0^T -g(s)ds \right\},
\end{align*}
\]

substituting (52) into (40), then \(Z_t^*\) satisfies the following SDE
\[
dZ_t^* = -Z_t^* (\mu(s) - r(s) 1)^T \sigma(s)^{-1} dW_t + \int_{\mathbb{R}} \theta_0(t, x)(\mu - \nu)(dt, dx).
\]
The solution of the above equation is
\[
Z_t^* = V_t \left( Z_0^* + \int_0^t \int_{\mathbb{R}} V_{s-1} \theta_0(s, x)(\mu - \nu)(ds, dx) \right),
\]
where
\[
V_t = \exp \left\{ -\int_0^t \sigma(s)^{-1}(\mu(s) - r(s) 1) dW_s \right\}
\times \exp \left\{ -\frac{1}{2} \int_0^t (\mu(s) - r(s) 1)^T \sigma(s)^{-1} (\mu(s) - r(s) 1) ds \right\}.
\]
and \(\theta_0(t, x)\) is defined as in (52).

Finally, we will show that the particular strategy \(\pi^*\) defined by (47) is the optimal strategy.

Define \(G_t^\pi := \int_0^t \frac{\pi(s)^T (\mu(s) - r(s) 1)}{B(s)} ds + \int_0^t \frac{\pi(s)^T \sigma(s) [\sigma(s)^T]^{-1} (\mu(s) - r(s) 1) ds}{B(s)} dW_s, t \in [0, T]\), for any \(\pi \in \mathcal{A}\). Note that \(\mathbb{E} \sup_{0 \leq t \leq T} |G_t^\pi|^2 < \infty\).
Moreover, for any $\pi \in \mathcal{A}$,
\[
Z_t^\pi G_t^\pi = Z_0^\pi G_0^\pi + \int_0^t G_s^\pi dZ_s^\pi + \int_0^t Z_s^\pi \frac{\pi(s)^T (\mu(s) - r(s))1}{B(s)} ds
\]
\[
= \int_0^t Z_s^\pi \frac{\pi(s)^T \sigma(s)}{B(s)} dW_s + \int_0^t \frac{\pi(s)^T \sigma(s) \theta(s)}{B(s)} ds
\]
\[
= \text{local martingale} + \int_0^t \frac{\pi(s)^T \sigma(s)}{B(s)} [Z_s^\pi \sigma(s)^{-1}(\mu(s) - r(s))1 + \theta(s)] ds
\]
\[
= \text{local martingale}.
\]

So $\{Z_t^\pi G_t^\pi\}$ is a local martingale for any $\pi \in \mathcal{A}$. Similar with the proof of Theorem 3.1, we can steadily show that $\{Z_t^\pi G_t^\pi\}$ is a martingale. Thus we have $E[Z_t^\pi G_t^\pi] = 0$ for any $\pi \in \mathcal{A}$, which means that (38) holds. Consequently, $\pi^*$ is the optimal strategy for the quadratic utility.

**Remark 7.** Under the quadratic utility, liquid reserves can not only affect the amount of money invested in risk-free security $\pi_0(t)$, but also affect the optimal strategy $\pi^*(t)$. Furthermore, $\pi^*(t)$ is decreasing if liquid reserve level $\Delta(t)$ is increasing.

**Remark 8.** If $N = 1, k = 0, \mu(t) = \mu_1, \sigma(t) = \sigma_1, \alpha(t) = \alpha, r(t) = r$, and write $c = \alpha - \lambda mY$, then for any $t \in [0, T]$, the optimal strategy for the quadratic utility at time $t$ is
\[
\pi^*(t) = \frac{\mu_1 - r}{\gamma \sigma_1^2} \exp \left\{ \left( \frac{(\mu_1 - r)^2}{\sigma_1^2} - r \right) (T - t) \right\} Z_t^{-}
\]
and
\[
\pi_0(t) = X_t^{x_0, \pi^* - \pi^* (t)},
\]
where
\[
Z_t^* = V_t \left[ Z_0^* + \int_0^t \int_{\mathbb{R}} V_s^{-1} \theta_0(s, x)(\mu - \nu)(ds, dx) \right],
\]
\[
V_t = \exp \left\{ -\frac{(\mu_1 - r)}{\sigma_1} W_t - \frac{1}{2} \frac{(\mu_1 - r)^2}{\sigma_1^2} t \right\},
\]
\[
Z_t^* = \left( 1 - \gamma x_0 e^{\gamma t} - \frac{c_1 e^{\gamma t} - 1}{r} \right) e^{-\left( \frac{(\mu_1 - r)^2}{\sigma_1^2} T \right)} x,
\]
\[
\theta_0(t, x) = \gamma \exp \left\{ -\frac{(\mu_1 - r)^2}{\sigma_1^2} - r \right\} (T - t) x,
\]
which coincides with the optimal strategy for the quadratic utility in the special case of $\beta = 0$ in Wang [21].

**Remark 9.** The form of the optimal strategy with respect to the CARA utility is simpler and easier to be calculated than that with respect to the quadratic utility. The reason to lead to this phenomenon is as follows. The CARA utility is basically exponential, hence we can make use of the classical Girsanov’s Theorem to deduce the closed form solution to the optimal problem. At the same time, the quadratic utility is basically a power function, one can not use the Girsanov’s Theorem directly. Hence, we need to modify the method of the proof of Theorem 3.1 to prove Theorem 4.1, and thus get a more complicated closed form solution to the optimal problem.
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