NONEXISTENCE OF SMALL, ODD BREATHERS FOR A CLASS OF NONLINEAR WAVE EQUATIONS

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Abstract. In this note, we show that for a large class of nonlinear wave equations with odd nonlinearities, any globally defined odd solution which is small in the energy space decays to 0 in the local energy norm. In particular, this result shows nonexistence of small, odd breathers for some classical nonlinear Klein Gordon equations such as the sine Gordon equation and \( \phi^4 \) and \( \phi^6 \) models. It also partially answers a question of Soffer and Weinstein in [39, p. 19] about nonexistence of breathers for the cubic NLKG in dimension one.

1. Introduction

1.1. Main result. In this note, we consider the following class of nonlinear wave equations,

\[
\partial_t^2 u - \partial_x^2 u = mu + f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R},
\]

where \( m \in \mathbb{R} \) and the nonlinearity \( f: \mathbb{R} \to \mathbb{R} \) is a \( C^1 \), odd function such that for some \( p > 1 \),

\[
|f'(u)| \lesssim |u|^{p-1}, \quad \forall |u| < 1.
\]

Denote \( F(u) = \int_0^u f \). An important property of (1.1) is the conservation of energy

\[
E(u, \partial_t u) := \int \frac{1}{2} \partial_t u^2 + \frac{1}{2} \partial_x u^2 - \frac{m}{2} u^2 - F(u),
\]

along the flow. In particular, \( H^1 \times L^2 \) perturbations of the zero solution are referred as perturbations in the energy space. For many standard examples, the Cauchy problem for the model (1.1) is globally well-posed for initial data \((u(0), \partial_t u(0)) = (u^m_1, u^m_2) \in H^1 \times L^2 \) small enough. Here, we do not make any a priori assumption on the sign of \( m \) nor the nature of \( F(u) \) that would guarantee that \( E(u) \) is coercive and controls the \( H^1 \times L^2 \) norm of some solutions. Instead, for general nonlinearity \( f \) as above, we consider global solutions whose energy norm is uniformly small in time. Another important property of (1.1) is that for odd initial data the associated solution is also odd for all time. For the rest of this paper, we work in such framework, and we consider only odd perturbations in the energy space. Note that (1.1) is invariant under space translation and under the Lorentz transformation but since we consider only odd solutions, these invariances are irrelevant here.

Set

\[
u_1 = u, \quad u_2 = \partial_t u,
\]

so that in terms of \((u_1, u_2)\), equation (1.1) becomes

\[
\begin{aligned}
\partial_t u_1 &= u_2, \\
\partial_t u_2 &= \partial_x^2 u_1 + mu_1 + f(u_1).
\end{aligned}
\]

Our main result is the following property for small, odd solutions of (1.4).
Theorem 1.1. There exists $\varepsilon > 0$ such that any odd global $H^1 \times L^2$ solution $(u_1, u_2)$ of (1.4) such that
$$
\sup_{t \geq 0} \|(u_1(t), u_2(t))\|_{H^1 \times L^2} < \varepsilon,
$$
satisfies
$$
\lim_{t \to +\infty} \|(u_1(t), u_2(t))\|_{H^1(I) \times L^2(I)} = 0,
$$
for any bounded interval $I \subset \mathbb{R}$.
In particular, no odd, small breather solutions exist for (1.1).

By breather, we mean a solution in $H^1 \times L^2$ which is periodic in time, up to the symmetries of the equation. Several integrable equations have stable breather solutions, including the sine-Gordon model [24]
$$
\phi_{tt} - \phi_{xx} = -\sin \phi.
$$
See also [1, 2, 3] and references therein for more details on the literature.

Our paper was motivated by (and gives a partial answer to) a conjecture of Soffer and Weinstein [39] which says that no small amplitude breathers should exist for
$$
\phi_{tt} - \phi_{xx} = -\phi + \phi^3.
$$
Our assumptions allows other classical examples such as the $\phi^4$ model [42, 45, 35]
$$
\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0,
$$
the $\phi^6$ model [29],
$$
\phi_{tt} - \phi_{xx} = -\phi + 4\phi^3 - 3\phi^5,
$$
and the sine-Gordon equation (1.6).

For the proof of Theorem 1.1, we follow a simplified version of the method developed in our recent work [21] on asymptotic stability of the kink under odd perturbations for the $\phi^4$ model. The main idea in [21] and in this paper is the introduction of a generalized virial identity, suitably constructed for each considered problem. Such approach is inspired by previous works by the second author and Merle [31, 32, 33] for the generalized Korteweg-de Vries equations and by Merle and Raphael [34] for nonlinear Schrödinger equations. Unlike in those works, here the analysis is performed for solutions in the neighborhood of 0 and the spectral analysis reduces to classical operators.
1.2. Previous results. Rigorous proofs of nonexistence of breathers solutions in scalar field equations date back to the work by Coron [10]. He showed that under the assumption of vanishing energy and $L^\infty$ norm of the solution at infinity in space, and an $L^1_{tx}$ integrability condition, breathers cannot have arbitrarily small periods. Later, Vuillermot [44] showed that under some growth and convexity assumptions on the nonlinearity, there are no breather solutions for suitable scalar field equations. Some formal arguments for existence and nonexistence of breather solutions can be found in Kichenassamy [19], and Birnir-McKean-Weinstein [6], respectively. Finally, Denzler [16] provided a proof of nonexistence of breather solutions for nonlinearities that are small, complex-analytic perturbations of the sine-Gordon case. From his result, it is concluded that the existence of breathers is a very rare property, probably related to the integrability of the equation.

As in the present paper, the nonexistence of breather solutions for (1.1) can also be seen as a consequence of the asymptotic stability of the vacuum solution in some topology. From this point of view, we refer to the original works of Buslaev and Perelman [7, 8], and Buslaev-Sulem [9] on the NLS case, and Soifer-Weinstein [39] on nonlinear Klein-Gordon models in 3D. See also the works by Delort [14, 15], Lindblad-Soffer [25, 26, 27], Bambusi and Cuccagna [4] and Sterbenz [40] on decay of small solutions for nonlinear Klein-Gordon equations in one dimension. This method has been pushed forward by considering different nonlinearities and lower dimensions. For recent results in this area, see e.g. [38, 43, 36, 11, 12, 13, 20, 22, 5, 28].

2. Proof of the theorem

Step 1: Virial identity. Recall the set of coupled equations (1.4). For a smooth and bounded function $\psi$ to be chosen later, let

\begin{equation}
\mathcal{I}(u) := \int \psi(\partial_x u_1) u_2 + \frac{1}{2} \int \psi' u_1 u_2 = \int \left( \psi \partial_x u_1 + \frac{1}{2} \psi' u_1 \right) u_2.
\end{equation}

Let $(u_1, u_2)$ be a solution of (1.4). Since for any $v \in H^1$, $\int ((\psi v_x + \frac{1}{2} \psi' v) v = 0$, one computes

\begin{equation}
\frac{d}{dt} \mathcal{I}(u) = \int \left( \psi u_{1,x} + \frac{1}{2} \psi' u_1 \right) u_{1,xx} + \int \left( \psi u_{1,x} + \frac{1}{2} \psi' u_1 \right) f(u)
\end{equation}

\begin{equation}
= -\mathcal{B}(u_1) - \int \psi' \left[ F(u_1) - \frac{1}{2} u_1 f(u_1) \right],
\end{equation}

where we have denoted

\begin{equation}
\mathcal{B}(u_1) = -\int \left( \psi u_{1,x} + \frac{1}{2} \psi' u_1 \right) u_{1,xx} = \int \psi' (\partial_x u_1)^2 - \frac{1}{4} \int \psi'' u_1^2.
\end{equation}

Therefore,

\begin{equation}
-\frac{d}{dt} \mathcal{I} = \mathcal{B}(u_1) + \int \psi' \left[ F(u_1) - \frac{1}{2} u_1 f(u_1) \right].
\end{equation}

Step 2: Coercivity of the bilinear form $\mathcal{B}$. Now we choose a specific function $\psi$ and we consider the question of the coercivity of the bilinear form $\mathcal{B}$. Let $\lambda > 0$ be fixed. We set

\begin{equation}
\psi(x) := \lambda \tanh \left( \frac{x}{\lambda} \right),
\end{equation}

in the definition of $\mathcal{I}$. Note that $\zeta > 0$ everywhere. Let $w$ be the following auxiliary function

\begin{equation}
w := \zeta u_1; \quad \zeta(x) := \sqrt{\psi'(x)} = \sech \left( \frac{x}{\lambda} \right).
\end{equation}

First, note that by integration by parts,

\begin{equation}
\int w_x^2 = \int (\zeta \partial_x u_1 + \zeta' u_1)^2 = \int \psi' (\partial_x u_1)^2 + 2 \int \zeta' u_1 (\partial_x v_1) + \int (\zeta')^2 u_1^2
\end{equation}

\begin{equation}
= \int \psi' (\partial_x u_1)^2 - \int \zeta'' u_1^2
\end{equation}

\begin{equation}
= \int \psi' (\partial_x u_1)^2 - \int \frac{\zeta''}{\zeta} w^2.
\end{equation}
Thus,
\begin{equation}
\int \psi' (\partial_x u_1)^2 = \int w_x^2 + \int \frac{\zeta''}{\zeta} w^2.
\end{equation}

Second,
\begin{equation}
\int \psi'' u_1^2 = \int \frac{(\zeta^2)''}{\zeta^2} w^2 = 2 \int \left( \frac{\zeta''}{\zeta} + \frac{(\zeta')^2}{\zeta^2} \right) w^2.
\end{equation}

Therefore,
\begin{equation}
B(u_1) = \int \psi' (\partial_x u_1)^2 - \frac{1}{4} \int \psi'' u_1^2 = \int w_x^2 + \frac{1}{2} \int \left( \frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) w^2.
\end{equation}

Set
\begin{equation}
B^\sharp(w) := \int \left( w_x^2 - V w^2 \right), \quad \text{where} \quad V := -\frac{1}{2} \left( \frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right),
\end{equation}
so that
\begin{equation}
B^\sharp(w) = B(u_1).
\end{equation}

Note that by (2.4) and direct computations,
\begin{equation}
V(x) = \frac{1}{2\lambda^2} \text{sech}^2 \left( \frac{x}{\lambda} \right).
\end{equation}

Recall that the index of the operator \(-\frac{d^2}{dx^2} - V\) associated to \(B^\sharp\) is 1, which means that this operator has only one negative, discrete eigenvalue with an even corresponding eigenfunction. This follows from the well known fact (see e.g. Titchmarsh [41, §4.19] and [17, p. 55]) that for any \(\lambda > 0\), the index \(\kappa\) of the operator
\[-\frac{d^2}{dx^2} - V_0 \frac{1}{\lambda^2} \text{sech}^2 \left( \frac{x}{\lambda} \right), \quad V_0 \in [0, +\infty),\]
is the largest integer such that
\[\kappa < \frac{1}{2} \sqrt{4V_0 + 1} + \frac{1}{2}.\]

Here, our claim concerning \(B^\sharp\) follows from \(1 < \frac{\lambda^2 + 1}{2 < 2.}\) In fact, we claim the following more precise result.

**Lemma 2.1.** For any \(\lambda > 0\) and for any odd function \(w \in H^1\),
\begin{equation}
B^\sharp(w) \geq \frac{3}{4} \int w_x^2.
\end{equation}

**Proof.** We write
\begin{equation}
B^\sharp(w) = \frac{3}{4} \int w_x^2 + \frac{1}{4} \int \left( w_x^2 - \frac{2}{\lambda^2} \text{sech}^2 \left( \frac{x}{\lambda} \right) w^2 \right).
\end{equation}

Since \(\frac{1}{4} \sqrt{4.2 + 1} + \frac{1}{2} = 2\), the second term in the right-hand side above is nonnegative for any odd function \(w\), and the result follows. \(\Box\)

Step 3: Control of the error terms and conclusion of the Virial argument. As in statement of Theorem 1.1, we consider an odd solution \((u_1(t), u_2(t))\) of (1.4) global in \(H^1 \times L^2\) for \(t \geq 0\) and satisfying, for all \(t \geq 0\),
\begin{equation}
\| (u_1(t), u_2(t)) \|_{H^1 \times L^2} < \varepsilon.
\end{equation}

We define
\begin{equation}
\| u_1 \|_{H^1_x}^2 := \int (\partial_x u_1)^2 \text{sech} (x), \quad \| u_2 \|_{L^2_x}^2 := \int u_2^2 \text{sech} (x),
\end{equation}
and
\begin{equation}
\| (u_1, u_2) \|_{H^1_x \times L^2_x}^2 := \| u_1 \|_{H^1_x}^2 + \| u_2 \|_{L^2_x}^2.
\end{equation}
The key ingredient of the proof of asymptotic stability in the energy space is the following result.

**Proposition 2.1.** For \( \varepsilon > 0 \) small enough,
\[
(2.15) \quad \int_{0}^{+\infty} \| (u_1(t), u_2(t)) \|_{H^1_\lambda \times L^2}^2 \, dt \lesssim \varepsilon^2.
\]

**Proof of Proposition 2.1.** We consider the virial-type quantity \( I(t) \) defined in (2.1) with \( \lambda = 100 \).

The proof of (2.15) is based on the following two estimates, which hold for some constant \( C_1, C_2 > 0 \):
\[
(2.16) \quad -\frac{d}{dt} I \geq C_1 \| u_1 \|_{H^1_\lambda}^2.
\]
\[
(2.17) \quad \frac{d}{dt} \int \text{sech}(x) u_1 u_2 \geq \| u_2 \|_{L^2}^2 - C_2 \| u_1 \|_{H^1_\lambda}^2.
\]

First we prove (2.15) assuming (2.16) and (2.17). Integrating (2.16) on \([0, t_0]\), using the bound (2.12), and passing to the limit as \( t_0 \to +\infty \), we find
\[
(2.18) \quad \int_{0}^{+\infty} \| u_1(t) \|_{H^1_\lambda}^2 \, dt \lesssim \varepsilon^2.
\]
Then, using (2.17) similarly, we obtain (2.15).

Thus, to finish the proof, we only have to prove (2.16) and (2.17). We begin with the proof of (2.16). We have from (2.3) and (2.8),
\[
-\frac{d}{dt} I = B(u_1) - \int \psi' \left[ F(u_1) - \frac{1}{2} u_1 f(u_1) \right].
\]
Recall from Section 2 the notation \( w = u_1 \zeta = u_1 \text{sech} \left( \frac{x}{100} \right) \) (see (2.5)). From Lemma 2.1, we have
\[
B(u_1) = B^1(w) \geq \frac{3}{4} \int w_x^2 \quad \text{and equivalently} \quad \int w_x^2 \geq \frac{1}{5.10^2} \int \text{sech}^2 \left( \frac{x}{100} \right) w^2.
\]

Thus,
\[
(2.19) \quad B(u_1) \gtrsim \| \partial_x w \|_{L^2}^2 \gtrsim \int \text{sech}^2 \left( \frac{x}{100} \right) w^2 = \int \text{sech}^4 \left( \frac{x}{100} \right) u_1^2 \gtrsim \int \text{sech}(x) u_1^2.
\]

Next, we have
\[
\| \partial_x w \|_{L^2}^2 \gtrsim \int \text{sech}^2 \left( \frac{x}{100} \right) | \partial_x w |^2 = \int \text{sech}^2 \left( \frac{x}{100} \right) | \zeta \partial_x u_1 + \zeta' u_1 |^2 \gtrsim \int \text{sech}^4 \left( \frac{x}{100} \right) | \partial_x u_1 |^2 + 2 \int \text{sech}^2 \left( \frac{x}{100} \right) \zeta' (\partial_x u_1) u_1 + \int \text{sech}^2 \left( \frac{x}{100} \right) (\zeta')^2 u_1^2 \gtrsim \int \text{sech}(x) | \partial_x u_1 |^2 + \int u_1^2 \left( - \left( \text{sech}^2 \left( \frac{x}{100} \right) \zeta' \right)' + \text{sech}^2(x) (\zeta')^2 \right).
\]

Thus, using (2.19),
\[
\int \text{sech}(x) | \partial_x u_1 |^2 \lesssim \| \partial_x w \|_{L^2}^2 + \int \text{sech}^2(x) u_1^2 \lesssim \| \partial_x w \|_{L^2}^2 \lesssim B(u_1).
\]

This implies
\[
(2.20) \quad B(u_1) \gtrsim \| \partial_x w \|_{L^2}^2 \gtrsim \| u_1 \|_{H^1_\lambda}^2.
\]

Now, we claim that for any \( q > 0 \),
\[
(2.21) \quad \int \psi | u_1 |^{2+q} \lesssim \| u_1 \|_{L^\infty}^q \| \partial_x w \|_{L^2}^2 \lesssim \varepsilon^q \| \partial_x w \|_{L^2}^{2+q}.
\]

Indeed, by parity, the definition of \( \psi \) (2.4) and \( w \) (2.5), we have (with \( \lambda = 100 \))
\[
\int \psi | u_1 |^{2+q} \lesssim \int_{0}^{+\infty} e^{-\frac{x}{100}} | u_1 |^{2+q} \lesssim \int_{0}^{+\infty} e^{q \frac{x}{100}} | w |^{2+q}.
\]
Integrating by parts and using \( w(0) = 0 \) (the function \( w \) is odd)
\[
\int_0^{+\infty} e^{\frac{q}{q}} |w|^{2+q} = -\frac{\lambda}{q} \int_0^{+\infty} e^{\frac{q}{q}} \partial_x(|w|^{2+q}) = -\frac{2+q}{q} \lambda \int_0^{+\infty} e^{\frac{q}{q}} (\partial_x w) w |w|^q
\]
\[
\leq C \|u_1\|_{L^\infty}^2 \int_0^{+\infty} e^{\frac{q}{q}} \partial_x w |w|^{1+\frac{q}{2}} \leq C^2 \|u_1\|_{L^\infty}^q \int_0^{+\infty} |\partial_x w|^2 + \frac{1}{4} \int_0^{+\infty} e^{\frac{q}{q}} |w|^{2+q}.
\]
Thus,
\[
\int_0^{+\infty} e^{\frac{q}{q}} |w|^{2+q} \lesssim \|u_1\|_{L^\infty}^q |\partial_x w|^2_{L^2},
\]
and (2.21) is proved. Since, for some \( p > 1 \),
\[
\left| F(u_1) - \frac{1}{2} u_1 f(u_1) \right| \lesssim |u_1|^{p+1},
\]
we estimate by (2.21),
\[
(2.22)\quad \left| \int \psi' \left[ F(u_1) - \frac{1}{2} u_1 f(u_1) \right] \right| \lesssim \epsilon^{p-1} |\partial_x w|^2_{L^2}.
\]
Combining (2.20) and (2.22) proves (2.16) for \( \epsilon \) small enough.

Finally, we show (2.17). From (1.4), we compute
\[
\frac{d}{dt} \int \text{sech}(x) u_1 u_2 = \int \text{sech}(x) u_2^2 - \int \text{sech}(x) u_1^2 + \int \left[ a \text{sech}(x) + \frac{1}{2} \text{sech}''(x) \right] u_1^2 + \int \text{sech}(x) u_1 f(u_1).
\]
From this, (2.17) follows readily by definition of the norm in \( H^1_\omega \times L^2_\omega \) and (2.22). This ends the proof of Proposition 2.1.

\[\square\]

Step 4: Conclusion of the proof of Theorem 1.1. Let
\[
(2.23)\quad \mathcal{H}(t) := \int \text{sech}(x) \left[ u_1^2 + u_1^2 + u_2^2 \right](t).
\]
Then, using (1.4), we have
\[
\frac{d}{dt} \mathcal{H} = 2 \int \text{sech}(x) (u_{1,xt} u_1 + u_{1,x} u_1 + u_{2,t} u_2)
\]
\[
= 2 \int \text{sech}(x) \left[ u_{2,x} u_{1,x} + u_2 u_1 + (u_{1,xx} + m u_1 + f(u_1)) u_2 \right]
\]
\[
= 2 \int \text{sech}(x) \left[ (1 + m) u_1 + f(u_1) \right] u_2 - 2 \int \text{sech}(x) u_2 u_{1,x},
\]
and so
\[
(2.25)\quad \left| \frac{d}{dt} \mathcal{H} \right| \lesssim \int \left( u_1^2 + u_1^2 + u_2^2 \right) \text{sech}(x) \lesssim \|u(t)\|_{H^1_\omega \times L^2_\omega}^2.
\]

From (2.15) there exists a sequence \( t_n \to +\infty \) such that \( \mathcal{H}(t_n) \to 0 \). Let \( t \in \mathbb{R} \). Integrating on \( [t, t_n] \) and passing to the limit as \( n \to +\infty \) we obtain
\[
\mathcal{H}(t) \lesssim \int_t^{+\infty} \|u(t)\|_{H^1_\omega \times L^2_\omega}^2 dt.
\]
From (2.15) it follows that \( \lim_{t \to +\infty} \mathcal{H}(t) = 0 \). Thus, \( \lim_{t \to +\infty} \|u\|_{H^1_\omega \times L^2_\omega} = 0 \). This implies (1.5) and finishes the proof of the Theorem.
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