An Equivalence of Entanglement-Assisted Transformation and Multiple-Copy Entanglement Transformation

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Abstract

We examine the powers of entanglement-assisted transformation and multiple-copy entanglement transformation. First, we find a sufficient condition of when a given catalyst is useful in producing another specific target state. As an application of this condition, for any non-maximally entangled bipartite pure state and any integer \( n \) not less than 4, we are able to explicitly construct a set of \( n \times n \) quantum states which can be produced by using the given state as a catalyst. Second, we prove that for any positive integer \( k \), entanglement-assisted transformation with \( k \times k \)-dimensional catalysts is useful in producing a target state if and only if multiple-copy entanglement transformation with \( k \) copies of state is useful in producing the same target. Moreover, a necessary and sufficient condition for both of them is obtained in terms of the Schmidt coefficients of the target. This equivalence of entanglement-assisted transformation and multiple-copy entanglement transformation implies many interesting properties of entanglement transformation. Furthermore, these results are generalized to the case of probabilistic entanglement transformations.

Index Terms — Quantum information processing, Quantum entanglement, Entanglement transformation, Majorization relation.

I. INTRODUCTION

Quantum entanglement has been realized by the quantum information processing community as a valuable resource, and it has been widely used in quantum cryptography [1], quantum superdense coding [2], and quantum teleportation [3]. A considerable amount of literature has been devoted to the study of quantum entanglement, and many interesting results have been reported. Nevertheless, some fundamental problems related to quantum entanglement are still open. Consequently, it remains the subject of interest at present after years of investigations, see [4] for an excellent exposition.

Since quantum entanglement often exists between different subsystems of a composite system shared by spatially separated parties, a natural constraint on the manipulation of entanglement is that the separated parties are only allowed to perform quantum operations on their own subsystems and to communicate to each other classically. The manipulations complying with such a constraint are called LOCC transformations. Using this restricted set of transformations, the parties are usually required to optimally manipulate the nonlocal resource contained in the initial entangled state.

A central problem about quantum entanglement is thus to find the condition of when a given entangled state can be transformed into another one via LOCC. Bennett and his collaborators [5] have made a significant progress in attacking this challenging problem for the asymptotic case. The first important step of entanglement transformation in finite regime was made by Nielsen in [6], where he presented the condition of two bipartite entangled pure states \(|\psi\rangle\) and \(|\varphi\rangle\) with the property that \(|\psi\rangle\) can be locally converted into \(|\varphi\rangle\) deterministically. More precisely, let \(|\psi\rangle = \sum_{i=1}^{n} \sqrt{\alpha_i}|i_A_i|i_B\rangle\) and \(|\varphi\rangle = \sum_{i=1}^{n} \sqrt{\beta_i}|i_A_i|i_B\rangle\) be pure bipartite entangled states with ordered Schmidt coefficients vectors \(\lambda_{\psi} = (\alpha_1, \alpha_2, \ldots , \alpha_n)\) and \(\lambda_{\varphi} = (\beta_1, \beta_2, \ldots , \beta_n)\), where \(\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0\) and \(\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0\). Then Nielsen proved that the transformation \(|\psi\rangle \rightarrow |\varphi\rangle\) can be achieved with certainty by LOCC if and only if \(\lambda_{\psi} < \lambda_{\varphi}\). Here the symbol ‘\(<\)’ denotes majorization relation, and \(\lambda_{\psi}\) is majorized by \(\lambda_{\varphi}\) if the following relations hold

\[
e_l(\lambda_\psi) \leq e_l(\lambda_\varphi) \quad \text{for any} \quad 1 \leq l < n,
\]

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where \( e_l(\lambda_\psi) = \sum_{i=1}^{l} \alpha_i \), i.e., the sum of \( l \) largest Schmidt coefficients of \( |\psi\rangle \) (Note here by the normalization condition, we have \( e_n(\lambda_\psi) = e_n(\lambda_\varphi) = 1 \)). It is worth noting that majorization relation has a natural symmetry. Specifically, if \( E_l(\lambda_\psi) \) denotes the abbreviation of the sum of \( l \) least Schmidt coefficients of \( |\psi\rangle \), then \( \lambda_\psi \prec \lambda_\varphi \) can be simply restated as \( E_l(\lambda_\psi) \geq E_l(\lambda_\varphi) \) for any \( 1 \leq l < n \). To see this, we only need to notice that \( e_l(\lambda_\psi) + E_{n-l}(\lambda_\varphi) = 1 \) holds for every \( l = 1, \cdots, n-1 \). We simply say that Nielsen’s theorem occupies a symmetric property.

It is well-known in linear algebra that majorization relation \( \prec \) is not a total ordering. Thus, Nielsen’s theorem in fact implies that there exist two incomparable entangled states \( |\psi\rangle \) and \( |\varphi\rangle \) in the sense that neither \( |\psi\rangle \rightarrow |\varphi\rangle \) nor \( |\varphi\rangle \rightarrow |\psi\rangle \) can be realized with certainty under LOCC. To deal with the transformations between incomparable states, Vidal [9] generalized Nielsen’s result in a probabilistic manner and found an explicit expression of the maximal conversion probability for \( |\psi\rangle \rightarrow |\varphi\rangle \) under LOCC. To be more specific, let \( P_{\text{max}}(|\psi\rangle \rightarrow |\varphi\rangle) \) denote the maximal conversion probability of transforming \( |\psi\rangle \) into \( |\varphi\rangle \) by LOCC. Then it was shown that

\[
P_{\text{max}}(|\psi\rangle \rightarrow |\varphi\rangle) = \min_{1 \leq l \leq n} \frac{E_l(\lambda_\psi)}{E_l(\lambda_\varphi)}.
\]

Equivalently, we have \( P_{\text{max}}(|\psi\rangle \rightarrow |\varphi\rangle) \geq p \) if and only if \( E_l(\lambda_\psi) \geq p E_l(\lambda_\varphi) \) for each \( l = 1, \cdots, n \).

It is obvious in this case we cannot replace \( E_l(\cdot) \) by \( e_l(\cdot) \) and then reverse the order of inequalities. Intuitively, we may say that the natural symmetry occupied by Nielsen’s theorem is lost in the Vidal’s theorem.

Shortly after Nielsen’s work, a startling phenomenon of entanglement, namely, entanglement catalysis, or ELOCC, was discovered by Jonathan and Plenio [10]. They demonstrated by examples that sometimes one may use an entangled state \( |\psi\rangle \), known as a catalyst, to make an impossible transformation \( |\psi\rangle \rightarrow |\varphi\rangle \) possible. A concrete example is as follows. Take \( |\psi\rangle = \sqrt{0.4}(00) + \sqrt{0.4}(11) + \sqrt{0.1}(22) + \sqrt{0.1}(33) \) and \( |\varphi\rangle = \sqrt{0.5}(00) + \sqrt{0.25}(11) + \sqrt{0.25}(22) \). We know that \( |\psi\rangle \rightarrow |\varphi\rangle \) under LOCC. However, if another entangled state \( |\phi\rangle = \sqrt{0.6}(44) + \sqrt{0.4}(55) \) is introduced, then the transformation \( |\psi\rangle \otimes |\phi\rangle \rightarrow |\varphi\rangle \otimes |\phi\rangle \) can be realized with certainty because \( \lambda_{\psi \otimes \phi} < \lambda_{\varphi \otimes \phi} \). The role of the state \( |\phi\rangle \) in this transformation is similar to a catalyst in a chemical process since it can help entanglement transformation process without being consumed. In the same paper, Jonathan and Plenio also showed that the use of catalyst can improve the maximal conversion probability when the transformation cannot be realized with certainty even with the help of a catalyst.

Recently, Bandyopadhyay et al [11] found another interesting phenomenon: sometimes multiple copies of source state may be transformed into the same number of copies of target state although the transformation cannot happen for a single copy. Such a phenomenon is called ‘nonasymptotic bipartite pure-state entanglement transformation’ in [11]. More intuitively, we call this phenomenon ‘multiple-copy entanglement transformation’, or MLOCC for short. Take the above states \( |\psi\rangle \) and \( |\varphi\rangle \) as an example. It is not difficult to check that the transformation \( |\psi\rangle^{\otimes 3} \rightarrow |\varphi\rangle^{\otimes 3} \) occurs with certainty by Nielsen’s theorem. That is, when Alice and Bob prepare three copies of \( |\psi\rangle \) instead of just a single one, they can transform these three copies all together into three copies of \( |\varphi\rangle \) by LOCC. This simple example means that the effect of catalyst can, at least in the above situation, be implemented by preparing more copies of the original state and transforming these copies together. Besides some concrete examples of MLOCC, various theoretical properties of MLOCC were also investigated in [11].

After [10], [11], due to the great importance of entanglement transformation in quantum information processing, a considerable number of researches were done to investigate the mechanism beyond entanglement catalysis and multiple-copy entanglement transformation. For example, in [12], Daftuar and Klimesh carefully examined the mathematical structure of entanglement catalysis. They showed that any non-maximally bipartite entangled pure state can serve as quantum catalyst for some entanglement transformation. Especially, the relationship between entanglement catalysis and multiple-copy entanglement transformation has been thoroughly studied by the authors in [15]. It was proved that any multiple-copy entanglement transformation can be implemented by a suitable entanglement-assisted transformation. Another essential connection between entanglement-assisted transformation and multiple-copy entanglement transformation was also presented in [15]. Indeed, the equivalence between the possibility of implementing an entanglement transformation in producing a given target by ELOCC and the one by MLOCC was observed.

In this paper we examine the powers of entanglement-assisted transformation and multiple-copy entanglement transformation from some new angles. The first problem we consider here is the usefulness
of a given catalyst in producing a target state. To be concise, we say that a catalyst $|\phi\rangle$ is useful in producing a target $|\psi\rangle$ if there exists some pure state $|\psi\rangle$ with the same dimension as $|\varphi\rangle$ such that $|\psi\rangle \otimes |\phi\rangle \rightarrow |\varphi\rangle \otimes |\phi\rangle$ can be achieved with certainty by LOCC while $|\psi\rangle$ cannot be transformed to $|\varphi\rangle$ directly. To solve the problem of usefulness of catalyst, two simple but useful mathematical apparatuses are introduced, namely, local uniformity and global uniformity. They enable us to give a sufficient condition of whether a catalyst $|\phi\rangle$ is useful in producing $|\varphi\rangle$ (Theorem 1). More importantly, this condition is operational and it determines all catalyst states with the minimal dimension. Thus, it is very useful in practice.

The second problem that we consider in the present paper is to determine whether there exists some $k \times k$ catalyst $|\phi\rangle$ which is useful in producing $|\varphi\rangle$, where $k \geq 2$ is a given dimension. This problem is slightly different from the previous one. The major difference is the catalyst state in the first problem is specified while in the current problem only the dimension of the catalyst state is fixed. If such a $k \times k$ catalyst state does exist, we simply say that $k$-ELOCC is useful in producing $|\varphi\rangle$.

A corresponding problem occurs when we consider multiple-copy entanglement transformation. If there exists some $n \times n$ state $|\psi\rangle$ such that $|\psi\rangle^{\otimes k} \rightarrow |\varphi\rangle^{\otimes k}$ can be achieved with certainty while $|\psi\rangle \rightarrow |\varphi\rangle$ under LOCC, then we say that $k$-MLOCC is useful in producing $|\varphi\rangle$. Thus the third problem may be more precisely stated as follows: for a given state $|\varphi\rangle$ and a positive integer $k > 1$, decide whether $k$-MLOCC is useful in producing $|\varphi\rangle$.

The above two problems are concerned respectively with $k$-ELOCC and $k$-MLOCC, and it seems that they are irrelevant. To our surprise, we find that these two problems are equivalent. Indeed, we show that for any bipartite entangled state $|\varphi\rangle$ and positive integer $k$, $k$-ELOCC is useful in producing $|\varphi\rangle$ if and only if $k$-MLOCC is useful in producing $|\varphi\rangle$. Furthermore, a necessary and sufficient condition for both of them is obtained in terms of the Schmidt coefficients of $|\varphi\rangle$ (Theorem 2). As a simple corollary, we are also able to prove a similar equivalence between ELOCC and MLOCC for the case in which the dimension of catalysts (or the number of copies) is not fixed. This complements further the results of [12] and [15] mentioned above.

The previous results are obtained in the deterministic case. We are able to solve the above problems for the case of probabilistic transformations too. However, our results show that some properties of probabilistic ELOCC and MLOCC transformations are quite different from those of their deterministic counterparts (Theorems 3 and 4). This is somewhat surprising. We argue that this phenomenon is deeply related to the difference between the mathematical structures of deterministic entanglement transformations and probabilistic entanglement transformations, which are characterized by Nielsen's theorem and Vidal's theorem, respectively. As pointed out above, Nielsen's Theorem enjoys a natural symmetry, but this symmetry is lost in Vidal's theorem.

We organize the rest of this paper as follows. We state our main results in Section II. Some direct implications are also pointed out there. In Section III, we give some interesting applications of the main results. In particular, two conjectures of Nielsen about entanglement catalysis are addressed in detail. From Section IV on, we present the proofs of the main results. In Section IV, we give some lemmas which are needed in these proofs of the main results. The rest several sections completes the proofs. Theorem 1 will be proved in Section V. In Section VI, we present the proof of Theorem 2. Theorems 3 and 4 will be proved in Section VII. To keep the paper more readable, we put the complicated proofs of some technical lemmas in the appendices. Along with the proofs, many properties about the mathematical structure of ELOCC and MLOCC which are independently of interest are also presented. A brief conclusion is drawn in Section VIII.

II. MAIN RESULTS

The purpose of this section is to state the main results. The proofs of them are postponed to Sections V - VII. Since the fundamental properties of a bipartite pure state under LOCC are completely determined by its Schmidt coefficients, which can be treated as a probability vector, we consider only probability vectors instead of quantum states from now on. We always identify a probability vector with the quantum state represented by it.

A. Notations and Definitions

To present the main results, we need some auxiliary notations. Let $V^n$ denote the set of all $n$-dimensional probability vectors. For any $x \in V^n$, the dimensionality of $x$ is often denoted by $\dim(x)$,
The notation $x^\downarrow$ will be used to stand for the vector which is obtained by rearranging the components of $x$ into non-increasing order. We use $e_l(x)$ to denote the sum of $l$ largest components of $x$, i.e., $e_l(x) = \sum_{i=1}^{l} x_i$. It is obvious that $e_l(x)$ is a continuous function of $x$ for each $l = 1, 2, \ldots, n$.

We say that $x$ is majorized by $y$, denoted by $x \prec y$, if

$$e_m(x) \leq e_m(y) \text{ for every } m = 1, \ldots, n - 1,$$

with equality if $m = n$. If all inequalities in Eq. $3$ are strict, we say that $x$ is strictly majorized by $y$. The relation of strict majorization is represented by $x \prec y$.

Using the above notations, Nielsen’s theorem can be stated as: $x \rightarrow y$ under LOCC if and only if $x \prec y$.

Although we consider probability vectors only, we often omit the normalization step for simplicity. This has no influence on the validity of our results. We can assume that all catalyst probability vectors have positive components because states $c$ and $c \odot 0$ are equivalent when they are treated as catalysts.

We also assume that the components of probability vector $x = (x_1, \ldots, x_n)$ are always in non-increasing order, except where otherwise stated. We say that $x$ is a segment of another vector $y$ if there exist $i \geq 1$ and $k \geq 0$ such that $x = (y_i, y_{i+1}, \ldots, y_{i+k})$.

Two useful quantities named local uniformity and global uniformity are key mathematical tools in the present paper. Formally, we have the following definitions:

**Definition 1:** Let $x$ be an $n$-dimensional probability vector.

1) The local uniformity of $x$ is defined by

$$l_u(x) = \min\{\frac{x_{i+1}}{x_i} : 1 \leq i < n - 1\}.$$  \hfill (4)

2) The global uniformity of $x$ is defined by

$$g_u(x) = \frac{x_n}{x_1}.$$ \hfill (5)

By the above definition, we have that both $l_u(x)$ and $g_u(x)$ are between 0 and 1. The above definition can be extended to any positive vector which is not necessarily normalized.

A simple but useful relation between $l_u(x)$ and $g_u(x)$ is the following:

$$l_u(x)^{n-1} \leq g_u(x) \leq l_u(x),$$ \hfill (6)

which will be used again and again.

**B. When is a catalyst $c$ useful in producing a target state $y$?**

For any $y \in V^n$, we write $S(y) = \{x \in V^n : x \prec y\}$. Intuitively, $S(y)$ denotes all the probability vectors which can be transformed into $y$ by LOCC. We also define $T(y, c)$ to be the set of all probability vectors that can be transformed into $y$ with $c$ as a catalyst, i.e., $T(y, c) = \{x \in V^n : x \odot c \prec y \odot c\}$. In practice, an important problem is to find catalyst states which are useful in producing a given target state. This is exactly the first problem that we promised to attack in the introduction. With the notations introduced above it can be briefly reformulated: whether $S(y) \subseteq T(y, c)$ holds. The following theorem gives a partial answer to this problem. More exactly, it presents a sufficient condition under which a given target entangled state can be implemented by using a given catalyst. It is worth mentioning that this condition is operational and it is also almost necessary.

**Theorem 1:** Let $y \in V^n$. If a catalyst $c$ satisfies

$$l_u(c) > \max\left\{\frac{y_d}{y_1}, \frac{y_n}{y_{d+1}}\right\} \text{ and } g_u(c) < \frac{y_{d+1}}{y_d}$$ \hfill (7)

for some $1 < d < n - 1$, then $S(y) \subseteq T(y, c)$. Conversely, if $S(y) \not\subseteq T(y, c)$, then there is a segment of $c$ satisfying Eq. $7$.

Some remarks come as follows:

1) If $c$ is a vector with the minimal dimension such that $S(y) \not\subseteq T(y, c)$, then by the above theorem, $c$ should satisfy Eq. $7$. In the view of this, Theorem $1$ determines all catalyst states with the minimal dimension which are useful in producing $y$. Especially, if $c$ has only two distinct non-zero components, then Eq. $7$ is also necessary for $S(y) \not\subseteq T(y, c)$.

2) A direct consequence of the above theorem is that any uniform vector $c$ cannot serve as a catalyst for any vector $y$ since Eq. $7$ cannot be satisfied. Furthermore, we can show that any nonuniform
probability vector can serve as a quantum catalyst for uncountably many probability vectors, which is a considerable improvement of the result proved by Daftuar and Klimesh in [12]. And this gives a stronger answer of Nielsen’s conjecture [13], which states that any nonuniform probability vector can potentially serve as a catalyst for some transformation.

The applications mentioned above will be discussed in much more details in Section III. The proof of Theorem 1 will be presented in Section V.

C. Equivalence of ELOCC and MLOCC

Now we further review some elements of entanglement catalysis and multiple-copy entanglement transformation. For any \( y \in V^n \), let \( T(y) = \{ x \in V^n : x \otimes c \prec y \otimes c \text{ for some vector } c \} \). Intuitively, \( T(y) \) denotes the probability vectors which can be transformed into \( y \) by LOCC with the help of some catalyst. We also define \( M(y) \) to be the set of probability vectors which, when provided with a finite (but large enough) number of copies, can be transformed into the same number of \( y \) under LOCC, that is, \( M(y) = \{ x \in V^n : x^{\otimes k} \prec y^{\otimes k} \text{ for some } k \geq 1 \} \). If we restrict the number of copies used in \( M(y) \) to be \( k \) and the vector \( c \) used as catalyst in \( T(y) \) to be \( k \)-dimensional, then we can define \( M_k(y) \) and \( T_k(y) \) similarly; namely, \( M_k(y) = \{ x \in V^n : x^{\otimes k} \prec y^{\otimes k} \} \) and \( T_k(y) = \{ x \in V^n : x \otimes c \prec y \otimes c \text{ for some } c \in V^k \} \).

In [15], it was shown that \( T(y) = S(y) \) if and only if \( M(y) = S(y) \). This interesting result has an intuitive physical meaning: for any quantum state \( y \), if ELOCC is useless in producing \( y \), nor has MLOCC, and vice versa. So we get an equivalence of ELOCC and MLOCC in the sense that they are both useful in producing the same target or both not. In the present paper, this result will be considerably refined. More precisely, we prove that for a specific class of entangled states, enhancing the number of copies but not exceeding a threshold will be useless. Furthermore, for any positive integer \( k \geq 2 \), we give a complete characterization of \( T_k(y) = S(y) \) in terms of components of \( y \). A similar result for the equality \( M_k(y) = S(y) \) is also proved. To one’s surprise, these two conditions are in fact the same. So we find a relation that \( T_k(y) = S(y) \) if and only if \( M_k(y) = S(y) \), which is much more elaborated than that \( T(y) = S(y) \) if and only if \( M(y) = S(y) \), previously established in [15]. We state this main result as the following:

**Theorem 2:** For any \( y \in V^n \), the following are equivalent:
1. \( T_k(y) = S(y) \).
2. \( M_k(y) = S(y) \).
3. \( y^d_1 \geq y^{d-1}_{d+1} \) or \( y^d_{d+1} \leq y^{d-1}_{n} \) for any \( 1 < d < n-1 \).

Let us list some implications of the above theorem as follows:

1. In the case that \( k \) tends to infinity, items 1 and 2 in the above theorem reduce to \( T(y) = S(y) \) and \( M(y) = S(y) \), respectively, and item 3 reduces to \( y_d = y_1 \) or \( y_{d+1} = y_n \) for any \( 1 < d < n-1 \). Hence we recover the main results in [15].

2. A careful observation carries out that in the case that \( T(y) = S(y) \), there can still exist some \( k > 1 \) such that \( T_k(y) = S(y) \). That is, although catalysis is useful in producing \( y \), any quantum states with dimension less than \( k \) cannot serve as catalysts. This also gives a solution to an open problem addressed by Jonathan and Plenio in [10], where they asked that whether catalyst states are always more efficient as their dimension increases. We also note that a similar question has also been addressed by Daftuar and Klimesh in [12], where they asked whether there are some \( y \in V^n \) and \( k \geq 1 \) such that \( T_k(y) = T_{k+1}(y) \). Theorem 2 shows that to make some \( k \)-dimensional state serve as a catalyst in producing \( y \), the components of \( y \) should satisfy some conditions, which cannot always be fulfilled by any probability vectors \( y \).

3. Theorem 2 can certainly help us not only to understand the limitation of entanglement catalysis and multiple-copy entanglement transformation, but also to choose suitable entangled states with good properties under ELOCC and MLOCC in practical quantum information processing.

4. Theorem 2 also discovers a very surprising connection between k-ELOCC and k-MLOCC. In [15], it was demonstrated that \( M_k(y) \subseteq T_{kn^{k-1}}(y) \), but we still do not know whether the bound \( kn^{k-1} \) is tight or not. It seems that \( T_k(y) \) and \( M_k(y) \) have no any connection. To check whether \( x \in T_k(y) \), we need to consider all the \( k \)-dimensional probability vectors as possible catalysts, which form a set of the size of continuum. But to check whether \( x \) is in \( M_k(y) \), only a simple calculation whether \( x^{\otimes k} \prec y^{\otimes k} \) is needed. However, Theorem 2 enables us to build up a ‘weak’ equivalence
between these two sets: \( k\)-ELOCC is useful in producing \( y \) if and only if \( k\)-MLOCC is useful in producing \( y \).

We will discuss the applications of Theorem 2 in more details in Section III. The proof of Theorem 2 will be presented in Section VI.

D. Probabilistic entanglement transformations

In the previous two subsections, we are concerned with deterministic entanglement transformations. In this subsection, we try to solve the same problems for probabilistic entanglement transformations. Our main results are Theorems 3 and 4 and they are counterparts of Theorems 1 and 2, respectively. It is interesting to note that the appearance of the results in this subsection are quite different from the corresponding ones for deterministic transformations. Indeed, it seems impossible to unify the deterministic case and the probabilistic case in a simple way. Even more strange, the probabilistic case is much simpler than the deterministic case.

In [18], the mathematical structure of entanglement-assisted probabilistic transformations was thoroughly studied. The results presented below complement well the ones obtained in [18].

To study probabilistic entanglement transformations, we need the notion of super majorization. Let \( x \) and \( y \) be two \( n \)-dimensional vectors. We say \( x \) is super-majorized by \( y \), denoted by \( x \prec^w y \), if \( E_l(x) \geq E_l(y) \) for any \( 1 \leq l \leq n \). In the case that the sum of \( x \) and \( y \) are equal, i.e., \( E_n(x) = E_n(y) \), \( x \prec^w y \) reduces to \( x \prec y \). We write \( x \prec^w y \) if and only if \( E_l(x) > E_l(y) \) for any \( 1 \leq l \leq n \).

By means of super majorization, Vidal’s theorem can be restated as: for any \( \lambda \in (0,1) \), \( P_{\text{max}}(x \rightarrow y) \geq \lambda \) if and only if \( x \prec^w \lambda y \), where \( \lambda \) is understood as a probability threshold.

As a natural generalization of \( S(y) \), we define \( S^\lambda(y) = \{ x \in V^n : x \prec^w \lambda y \} \). Intuitively, \( S^\lambda(y) \) denotes the set of all probability vectors which can be transformed into \( y \) with a probability at least \( \lambda \). Similarly, let \( T^\lambda(y, c) = \{ x \in V^n : x \otimes c \prec^w \lambda y \otimes c \} \).

The following theorem is a probabilistic counterpart of Theorem 1. It in fact provides a simple analytical characterization of the catalyst states \( c \) which are useful in producing \( y \) in a probabilistic manner:

Theorem 3: For any \( y \in V^n \), \( S^\lambda(y) \subseteq T^\lambda(y, c) \) if and only if there exist \( 0 < d < n-1 \) and \( 1 \leq i < k \) such that

\[
\begin{align*}
l_u(c') &> \frac{y_n}{y_{d+1}} \quad \text{and} \quad g_u(c') < \frac{y_{d+1}}{y_d},
\end{align*}
\]

where \( c' = (c_i, c_{i+1}, \ldots, c_k) \).

To present a corresponding result with Theorem 2, we need to generalize \( T_k(y) \) and \( M_k(y) \) to probabilistic versions. Specifically, \( T_k^\lambda(y) = \{ x \in V^n : x \otimes c \prec^w \lambda y \otimes c \} \) for some \( c \in V^k \) denotes the set of all quantum states that can be transformed into \( y \) with a probability not less than \( \lambda \) with the help of a \( k \)-dimensional catalyst state. Let \( T^\lambda(y) = \{ x \in V^n : x \otimes c \prec^w \lambda y \otimes c \} \).

It has a similar meaning but the dimension of catalyst state is not fixed. \( M_k(y) \) and \( M^\lambda(y) \) can also be generalized to probabilistic case as follows: \( M_k^\lambda(y) = \{ x \in V^n : x \otimes k \prec^w \lambda^k y \otimes k \} \) and \( M^\lambda(y) = \{ x \in V^n : x \otimes k \prec^w \lambda^k y \otimes k \} \). The physical meanings of \( T_k^\lambda(y) \) and \( T^\lambda(y) \) are very clear while the definitions of \( M_k^\lambda(y) \) and \( M^\lambda(y) \) seem to be artificial and deserve a careful explanation. We give an intuitive interpretation of \( M_k^\lambda(y) \) and \( M^\lambda(y) \) here. Noticing that for any \( x \in M_k^\lambda(y) \), we have \( x \otimes k \prec^w \lambda^k y \otimes k \), or more explicitly,

\[
P_{\text{max}}(x \otimes k \rightarrow y \otimes k) \geq \lambda^k.
\]

If the maximal conversion probability from \( x \) to \( y \) by LOCC is \( \lambda \), then the right-hand side of the above inequality is just the maximal conversion probability of transforming \( x \otimes k \) into \( y \otimes k \) separately, that is, in a way where no collective operations on the \( k \) copies are performed. Thus the intuition behind the above definition is that with the help of \( k \)-MLOCC, the geometric average value of the probability of a single-copy transformation is not less than \( \lambda \). Similarly, \( x \in M^\lambda(y) \) means that with the help of MLOCC, the average probability of a single-copy transformation is not less than \( \lambda \).

With these preliminaries, we present a probabilistic counterpart of Theorem 2 in the following:

Theorem 4: For any \( y \in V^n \), the following are equivalent:

1. \( T_k^\lambda(y) = S^\lambda(y) \).
2. \( M_k^\lambda(y) = S^\lambda(y) \).
3. \( y_{d+1} \leq y_{n-1}^k y_d \) for any \( 0 < d < n - 1 \).

We give some remarks about the above two theorems:
1) It is very interesting that the probabilistic threshold \( \lambda \in (0, 1) \) is irrelevant in Theorem 3. In other words, for any \( c \) and \( y \), whether \( S^\lambda(y) \not\subset T^\lambda(y, c) \) does not depend on \( \lambda \). Roughly speaking, this reveals a uniformity property of entanglement-assisted probabilistic transformations. A similar phenomenon occurs in Theorem 4.

2) We may naturally expect that the deterministic case of \( \lambda = 1 \) can be included in Theorems 3 and 4 and the deterministic case and the probabilistic case can be unified. Unfortunately, it is not the case, and Theorems 3 and Theorem 4 are valid only when \( \lambda < 1 \). This fact is deeply rooted in the symmetry of Nielsen’s Theorem and the asymmetry of Vidal’s Theorem, which describe the conditions of deterministic transformations and probabilistic transformations, respectively.

Some applications of these two theorems will also be presented in the next section, and their proofs are put in Section VII.

### III. SOME APPLICATIONS

#### A. What states can be used as catalysts?

In his lecture notes [13], Nielsen conjectured that any nonuniform probability vector can potentially serve as catalyst for some transformation. This conjecture was proved to be true by Daftuar and Klimesh [12]. In fact, they proved that for any nonuniform \( z \in V^k \), there exist \( x, y \in V^4 \) such that \( x \not= y \) but \( x \otimes z < y \otimes z \). As an interesting application of Theorem 1, we further show that any nonuniform probability vector can serve as quantum catalyst for uncountably many probability vectors.

**Theorem 5:** Suppose \( z \in V^k \) and \( z_1 > z_k > 0 \), \( n \geq 4 \). There exists a subset \( A(z) \) of \( V^n \) with non-zero measure relative to \( V^n \), such that for any \( y \in A(z) \), \( S(y) \not\subset T(y, z) \).

**Proof.** We will explicitly construct \( A(z) \subseteq V^n \) such that for any \( y \in A(z) \), \( S(y) \not\subset T(y, z) \). For a specific \( 1 < d < n - 1 \), we define \( A_d(z) \) to be the set of all probability vectors \( y \in V^n \) such that

\[
l_u(z) > \max \left\{ \frac{y_d}{y_1}, \frac{y_n}{y_{d+1}} \right\} \text{ and } g_u(z) < \frac{y_{d+1}}{y_d}
\]

By Theorem 1, it follows that \( S(y) \not\subset T(y, z) \). Then \( A(z) \) can be defined as the union of \( A_d(z) \) for all \( 1 < d < n - 1 \). It is clear that \( A(z) \) has a non-zero measure relative to \( V^n \). In the case that \( l_u(z) = g_u(z) \), i.e., \( z \) has only two non-zero distinct components, \( A(z) \) is the set of all probability vectors \( y \in V^n \) such that \( S(y) \not\subset T(y, z) \), and the conclusion also follows from Theorem 1. This completes the proof of Theorem 5.

Note that in [12], Daftuar et al. constructed two probability vectors \( x = (\alpha/2 + \beta/4, \alpha/2 + \beta/4, 0, 0) \) and \( y = (\alpha/2, \beta/2, 0, 0) \), where \( z_1/k = \alpha/\beta \), \( \alpha + \beta = 1 \). They proved that \( e_l(x \otimes z) < e_l(y \otimes z) \) holds for each \( 1 \leq l < 4k \). So \( x \otimes z < y \otimes z \). Hence they asserted that a small enough perturbation on \( x \) generates the desired probability vector \( x(e) = (x_1 + e, x_2, x_3, x_4 + e) \) such that \( x(e) \otimes z < y \otimes z \) but \( x(e) \not= y \). A trick lies in showing that \( e_l(x \otimes z) < e_l(y \otimes z) \) for any \( 1 \leq l < 4k \). To achieve this goal, they first proved that when \( l \) is even the inequality holds by considering five possible cases according to the relationship between \( l \) and \( k \), and then with a small modification they proved that when \( l \) is odd the relation \( e_l(x \otimes z) < e_l(y \otimes z) \) also holds. However, the construction of \( x \) and \( y \) is very artificial and the proof is a highly skilled one. Their proof heavily depends on the concrete instances \( x \) and \( y \) and cannot be generalized easily. On the other hand, the proof presented above is a coherent one and Theorem 5 has considerably generalized the result obtained by Daftuar et al. [12].

To illustrate the application of Theorem 5, let us reexamine the above example obtained by Daftuar et al. [12]. We only need to show that \( y \in A(z) \). Because \( z \) is a nonuniform probability vector, we have \( 0 < g_u(z) \leq l_u(z) < 1 \). A routine calculation carries out that

\[
l_u(z) > \max \left\{ \frac{y_d}{y_1}, \frac{y_n}{y_{d+1}} \right\} \text{ and } g_u(z) < \frac{y_{d+1}}{y_d}
\]

where \( d = 2 \) and \( n = 4 \). So \( S(y) \not\subset T(y, z) \) by Theorem 1. Moreover, noticing that \( x_1 + x_2 = y_1 + y_2 \), we have \( x \otimes z < y \otimes z \) by the proof of Theorem 1.

Furthermore, any \( y \in V^4 \) satisfying Eq. (10) has the property such that \( S(y) \not\subset T(y, z) \), so the example given by Daftuar et al. [12] is only a special case.
B. When are ELOCC and MLOCC useful?

We turn now to give some applications of Theorem 2.

As mentioned above, we are able to recover one of the main results in [12] and [15]. That is, for any \( y \in V^n \), \( T(y) = S(y) \) if and only \( M(y) = S(y) \). Moreover, an explicit necessary and sufficient condition for the equality \( T(y) = S(y) \) (and equivalently \( M(y) = S(y) \)) is also obtained in terms of the components of \( y \), as the following theorem states:

**Theorem 6**: For any \( y \in V^n \), the following are equivalent:
1) \( T(y) = S(y) \).
2) \( M(y) = S(y) \).
3) \( yd = y_1 \) or \( y_{d+1} = y_n \) for any \( 1 < d < n - 1 \).

Although this result has been proved in [12] and [15], we prefer to give a completely different but much simpler proof based on Theorem 2.

**Proof.** The case of \( n \leq 3 \) is trivial. We assume \( n \geq 4 \). The equivalence of 1) and 2) is a direct consequence of Theorem 2. We only need to show the equivalence of 1) and 3). Suppose that 3) does not hold, i.e., there exists \( 1 < d < n - 1 \) such that \( y_d < y_1 \) and \( y_{d+1} > y_n \). Then we can find a sufficiently large \( k \) such that

\[
\begin{align*}
y_n^k &< y_1^{k-1} y_{d+1} \quad \text{and} \quad y_{d+1}^k > y_d y_n^{k-1},
\end{align*}
\]

which further leads to \( S(y) \subseteq \mathbb{T}_k(y) \subseteq T(y) \) by Theorem 2. Hence 1) cannot hold.

Conversely, Suppose \( S(y) \subseteq \mathbb{T}_k(y) \). Then we will find integer \( k \) with \( S(y) \subseteq \mathbb{T}_k(y) \), which implies the existence of \( 1 < d < n - 1 \) satisfying condition (11). Therefore 3) cannot hold. \( \square \)

Interestingly, a probabilistic version of the above theorem is the following:

**Theorem 7**: Let \( y \in V^n \) and \( \lambda \in (0,1) \). Then the following are equivalent:
1) \( T^\lambda(y) = S^\lambda(y) \).
2) \( M^\lambda(y) = S^\lambda(y) \).
3) \( y_2 = y_n \).

We should note that the equivalence of 1) and 3) has been proved in [18]. Again, we can see that the probabilistic threshold \( \lambda \) is not involved.

We present an interesting example to illustrate the difference between probabilistic transformations and deterministic transformations.

**Example 1**: Let \( y = (y_1, y_2, y_3) \) be a 3-dimensional probability vector. Then by Theorem 6 we have that \( T(y) = M(y) = S(y) \). That is, ELOCC and MLOCC are useless for any deterministic transformations to \( y \).

On the other hand, if we take \( y \) satisfying \( y_2 > y_3 \), then by Theorem 7 we have \( S^\lambda(y) \not\subseteq T^\lambda(y) \) and \( S^\lambda(y) \not\subseteq M^\lambda(y) \) for any \( \lambda \in (0,1) \). That is, for any such state \( y \) and \( \lambda \in (0,1) \), we can always find another state \( x \in V^d \) and a catalyst \( c \) such that \( P_{\max}(x \rightarrow y) < \lambda \) but \( P_{\max}(x \otimes c \rightarrow y \otimes c) \geq \lambda \). Equivalently, we can find an integer \( k > 1 \) such that \( P_{\max}(x^{\otimes k} \rightarrow y^{\otimes k}) \geq \lambda^k \). Hence ELOCC and MLOCC are useful for some probabilistic transformations in producing \( y \).

C. More applications

In [16], Leung and Smolin demonstrated that \( x^{\otimes k} \not\prec y^{\otimes k} \) does not necessarily imply \( x^{\otimes k+1} \not\prec y^{\otimes k+1} \) by giving explicit instances of \( x \) and \( y \), where \( k \) is a positive integer not less than 2. In other words, for some \( y \in V^n \) and \( k > 1 \), we have \( M_k(y) \not\subseteq M_{k+1}(y) \). That is, increasing number of copies cannot always help entanglement transformation. However, with the aid of Theorem 2, we can prove that if \( k+1 \)-copies of transformation are not useful in producing the same target state, then \( k \)-copies of transformation are also not useful in producing the same target.

**Theorem 8**: For any \( y \in V^n \) and \( k > 1 \), \( M_{k+1}(y) = S(y) \) implies \( M_k(y) = S(y) \).

**Proof.** In fact, by Theorem 2 the condition for \( M_{k+1}(y) = S(y) \) can be rewritten as

\[
\max \left\{ \left( \frac{y_d}{y_1} \right)^{k-1}, \left( \frac{y_n}{y_{d+1}} \right)^{k-1} \right\} \geq \frac{y_{d+1}}{y_d}.
\]

Notice that the left-hand side of the above inequality is a decreasing function of \( k \), and these inequalities still hold if we replace \( k \) with \( k-1 \). \( M_k(y) = S(y) \) follows immediately by using Theorem 2 once more. \( \square \)

A direct consequence of Theorem 8 is the following:
Corollary 1: For any $y \in V^n$ and $k \geq 1$, $M_k(y) = S(y)$ implies $M_l(y) = S(y)$ for any $l \leq k$.

In the case of ELOCC, if we restrict the components of catalyst to be positive, then whether the inclusion relation $T_k(y) \subseteq T_{k+1}(y)$ always holds is unknown. Nevertheless, we can build up a corresponding result with Theorem 8 as the following:

Theorem 9: For any $y \in V^n$, $T_{k+1}(y) = S(y)$ implies $T_k(y) = S(y)$.

Intuitively, if $k + 1$-dimensional entanglement-assisted transformation is not useful in producing a target state, then the $k$-dimensional entanglement transformation is also not useful in producing the same target.

Proof. Similar to Theorem 8, the key step is to apply Theorem 2. We omit the details here. □

A corresponding corollary of Theorem 9 is stated as follows:

Corollary 2: For any $y \in V^n$ and $k \geq 1$, $T_k(y) = S(y)$ implies $T_l(y) = S(y)$ for any $l \leq k$.

A practical application of Theorem 2 is to help finding a suitable catalyst for a given transformation $x \rightarrow y$. In [17], Sun and some of us proposed a polynomial (of $n$) time algorithm to decide whether there is some catalyst $c \in V^k$ for the transformation from $x$ to $y$, where $k$ is treated as a fixed positive integer. Combining Theorem 2 with this algorithm, we can first find the minimal $k$ such that $S(y) \nsubseteq T_k(y)$ and then use the algorithm to decide whether there exists a suitable catalyst with dimension not smaller than $k$ since any potential catalyst should have a dimensionality not smaller than $k$.

To conclude this section, we consider another conjecture made by Nielsen. More specifically, suppose a transformation $x \rightarrow y$ can be catalyzed by a catalyst state $c$, where $x$ and $y \in V^n$. One may naturally hope that the dimension of $c$ is not too large, for example, bounded from top by $n$ or $n^2$. This conjecture, first addressed by Nielsen [13], was proved to be false in [12]. More precisely, Daftur et al. showed that if $S(y) \nsubseteq T(y)$ then $T_k(y) \neq T(y)$ for any $k \geq 1$. In other words, if catalysis is useful for $y$, then the dimension of catalyst is not bounded, and thus yields a negative answer for Nielsen’s conjecture. Since the proof presented in [12] is not a constructive one, we still do not know that what kinds of states $y$ have such strange properties. Theorem 2 characterizes precisely the lower bound of the dimension of catalyst $c$ for any state $y$, also the lower bound of the number of copies of any multiple-copy entanglement transformations to $y$. We give an concrete example as follows:

Example 2: Take $y^k = (1, \alpha, \alpha^k, \beta)/C$, where $k > 1$, $0 < \alpha < 1$, $0 \leq \beta < \alpha^{k+2}$ and $C = 1 + \alpha + \alpha^k + \beta$.

By Theorem 2, we have $T_k(y^k) = M_k(y^k) = S(y^k)$ but $S(y^k) \nsubseteq T_{k+1}(y^k)$ and $S(y^k) \nsubseteq M_{k+1}(y^k)$ for any $k > 1$. Such a state $y^k$ has a very strange property: although it can be catalyzed by some catalysts, any state with dimension less than $k$ cannot serve as catalyst for it.

For example, if we take $k = 16$, the state $y^{16}$ is 4-dimensional, but it has no quantum catalyst $c$ with dimension not more than 16. Suppose that $x$ can be catalyzed into $y$. Then the catalyst state $c$ should have a dimensionality at least 17. We also have that any multiple-copy entanglement transformations with the number of copies less than 17 have no advantage.

IV. SOME LEMMAS

From this section on, we are going to give detailed proofs of the results stated in Section II. The purpose of this section is to collect some lemmas needed in the proofs. For the sake of convenience, we introduce some notations of set operations. Let $A$ and $B$ be two sets of finite dimensional vectors. Then $A \oplus B$ denotes the set of all vectors of the form $a \oplus b$ with $a \in A$ and $b \in B$, i.e., $A \oplus B = \{ a \oplus b : a \in A$ and $b \in B \}$. Similarly, $A \otimes B = \{ a \otimes b : a \in A$ and $b \in B \}$. If $c$ is a vector, then $A \otimes c$ is just a convenient form of $A \otimes \{ c \}$. Note that $a \oplus b$ denotes the direct sum of vectors, that is, the vector concatenating $a$ and $b$, or $(a, b)$.

First, we recall some simple properties of majorization from [7].

Lemma 1: For any $y$ and $y'$, we have $S(y) \oplus S(y') \subseteq S(y \oplus y')$ and $S(y) \otimes S(y') \subseteq S(y \otimes y')$. That is, $x \prec y$ and $x' \prec y'$ imply $x \oplus x' \prec y \oplus y'$ and $x \otimes x' \prec y \otimes y'$.

The major difficulties in studying the structure of entanglement catalysis and multiple-copy entanglement transformation are the lack of suitable mathematical tools to deal with tensor product and majorization relation. In what follows, we try to provide some useful tools to overcome these difficulties. They are mainly about the strict majorization relation under direct sum and tensor product.

For a subset $A \subseteq V^n$, the set of all interior points of $A$ is denoted by $A^o$. It is easy to check that $x \in S^o(y)$ if and only if $x \prec y$. We also note that $S^o(y) = \emptyset$ only occurs in the case of $y_n = y_1$, which means that $y$ is a uniform vector. Without clearly stating, we always assume that $S^o(y) \neq \emptyset$. Extreme components of a vector are used frequently. For simplicity, we denote by $\max x$ and $\min x$ the maximal and the minimal components of $x$, respectively.
The following lemma is crucial in this paper. It says, to keep the direct sum of the interiors of \(S(y')\) and \(S(y'')\) still in the interior of \(S(y' + y'')\), \(y'\) should suitably overlap with \(y''\), and vice versa.

**Lemma 2**: For any \(y\) and \(y'\), we have

\[
S^o(y) + S^o(y') \subseteq S^o(y + y') \Leftrightarrow \max y > \min y' \text{ and } \max y' > \min y.
\]  
\(\text{(12)}\)

That is, if \(x < y\) and \(x' < y'\), then \(x \oplus x' < y \oplus y'\) if and only if the conditions in right-hand side of Eq.\((12)\) hold.

A careful observation of the proof of Lemma\(\text{2}\) shows that the sets \(S^o(y) + S^o(y')\) and \(S^o(y \oplus y')\) satisfy an interesting property: if there exists \(z\) such that \(z \in S^o(y) + S^o(y')\) and \(z \in S^o(y \oplus y')\), then for any \(\tilde{z} \in S^o(y) + S^o(y')\) it holds that \(\tilde{z} \in S^o(y \oplus y')\). Since we will use this property considerably latter, we formalize it as the following:

**Definition 2**: We say that two nonempty sets \(A\) and \(B\) satisfy linearity property \((LP)\), if \(A \cap B \neq \emptyset\) implies \(A \subseteq B\).

Before stating a corollary of Lemma\(\text{2}\) we introduce a useful notation. We use \(x \oplus^k\) to denote \(k\) times direct sum of \(x\) itself. That is, \(x \oplus^k = x \oplus x \oplus \cdots \oplus x\). Similarly, for a set \(A\), \(A \oplus^k = A \oplus A \oplus \cdots \oplus A\).

Now a direct consequence of Lemma\(\text{2}\) is as follows:

**Corollary 3**: For any \(y\) and \(k \geq 1\), \((S^o(y)) \oplus^k \subseteq S^o(y \oplus^k)\). Specially, \(x < y \Rightarrow x \oplus^k < y \oplus^k\).

Combining Lemma\(\text{2}\) with Corollary\(\text{3}\) we obtain the following necessary and sufficient condition for determining whether a given \(x\) is in the interior of \(S(y)\).

**Corollary 4**: Suppose \(\{(y^i) \oplus^k : 1 \leq i \leq m\}\) is a set of vectors and \(x^i < y^i\) for any \(1 \leq i \leq m\). Denote \(x = \oplus^m_{i=1}(x^i) \oplus^k\), \(y = \oplus^m_{i=1}(y^i) \oplus^k\). Then the following are equivalent:

1. \(x < y\). Or by \(LP\), \(\oplus^m_{i=1}(S^o(y^i)) \oplus^k \subseteq S^o(\oplus^m_{i=1}(y^i) \oplus^k)\).
2. There exist \(1 \leq j_1 < j_2 < \cdots < j_t \leq m\) such that (i) \(\max y^i = \max\{\max y^j : 1 \leq i \leq m\}\), \(\min y^i = \min\{\min y^j : 1 \leq i \leq m\}\), and (ii) \(\min y^j < \max y^{j+1}\) for each \(1 \leq s \leq t - 1\).

Intuitively, the sequence of \(y^{j_1}, \ldots, y^{j_t}\) is called an overlapping sequence of the set \(\{y^{j_k} : 1 \leq i \leq m\}\).

By Corollary\(\text{4}\) we have the following powerful lemma dealing with tensor product.

**Lemma 3**: For any \(y\) and \(c\), \(S^o(y \otimes c) \subseteq S^o(y \otimes c)\) if and only if \(l_u(y) > g_u(y)\).

**Proof**: Let \(y \in V^n\) and \(c \in V^k\). Take \(x < y\). We consider the set of vectors \(\{c_i y : i = 1, \cdots, k\}\). If \(l_u(c) > g_u(y)\) then by the definitions of uniformity indices, we have \(c_i y_n < c_{i+1} y_1\) for all \(1 \leq i < k\), which can be restated as \(c_i y < c_{i+1} y\). Applying Corollary\(\text{4}\) yields

\[
\bigoplus_{1 \leq i \leq k} c_i x < \bigoplus_{1 \leq i \leq k} c_i y,
\]

which is the same as \(x \otimes c < y \otimes c\).

Conversely, if \(l_u(c) \leq g_u(y)\), then there should exist a \(1 \leq i_0 \leq k - 1\) such that \(c_{i_0+1} / c_{i_0} \geq y_n / y_1\), or \(c_{i_0} y \geq \max c_{i_0+1} y\). Thus

\[
e_{i_0 n}(y \otimes c) = \sum_{i=1}^{i_0} c_i (c_i y) = \sum_{i=1}^{i_0} c_n (c_i x) \leq e_{i_0 n}(x \otimes c),
\]

which contradicts \(x \otimes c < y \otimes c\). Thus we complete the proof of the lemma.

With the aid of Lemma\(\text{3}\) we can show that the interior of \(S(y)\) is closed under tensor product, as the following lemma indicates:

**Lemma 4**: For any \(y\) and \(y'\), \(S^o(y) \otimes S^o(y') \subseteq S^o(y \otimes y')\).

**Proof**: Take \(x \otimes y\) and \(x' \otimes y'\). Then by Lemma\(\text{1}\) it follows that

\[
x \otimes x' < x \otimes y' < y \otimes y'.
\]  
\(\text{(13)}\)

If one of the inequalities in Eq.\((13)\) is strict, then we have done. Otherwise, by Lemma\(\text{3}\) the first inequality in Eq.\((13)\) is not strictly implies \(l_u(x) \leq g_u(y')\). Similarly, the second inequality in Eq.\((13)\) is not strictly implies \(l_u(y') \leq g_u(y)\). By Eq.\((6)\), we have \(g_u(x) \leq l_u(x)\) and \(g_u(y') \leq l_u(y')\). Hence we have \(g_u(x) \leq g_u(y)\). However, \(x < y\) implies \(x_1 < y_1\) and \(x_n > y_n\), which yield \(g_u(x) > g_u(y)\), a contradiction.

An immediate consequence of Lemma\(\text{4}\) is the following:
Corollary 5: For positive integers \( k, p, \) and \( q,\) \( (S^\alpha(y))^\otimes k \subseteq S^\alpha(y^\otimes k)\) and \( (S^\alpha(y))^\otimes p \otimes (S^\alpha(y'))^\otimes q \subseteq S^\alpha(y^\otimes p \otimes y'^\otimes q).\)

Properties of strict super majorization are much more simpler than that of strict majorization, and we list some of them as follows:

Lemma 5: For any non-negative vectors \( x, y, x' \) and \( y', \) we have
1) if \( x <^w y \) and \( x' <^w y', \) then \( x + x' <^w y + y'.\)
2) if \( x <^w y \) and \( x' <^w y', \) then \( x \oplus x' <^w y \oplus y' \) if and only if \( \max y' > \min y.\)
3) if \( x <^w y \) and \( x' <^w y', \) then \( x \otimes x' <^w y \otimes y'.\) Here we assume \( \min x' > 0.\)

Usually, we use the following generalized version of 2) of the above lemma:

Corollary 6: Suppose that \( x = (x^1, \ldots, x^m) \) and \( y = (y^1, \ldots, y^m) \) satisfy \( x^i < y^i \) for each \( 1 \leq i \leq m,\) and \( x^0 <^w y^0.\) Then \( x^0 \oplus x <^w y^0 \oplus y \) if and only if \( \max y^0 > \min y^0.\)

V. PROOF OF THEOREM [1]

The main aim of this section is to prove Theorem [1] First, we present some necessary preliminaries. Especially, we give a characterization of \( T^\alpha(y, c), \) and then a necessary and sufficient condition for \( S(y) \subseteq T(y, c), \) which also leads to an efficient algorithm to solve the problem that whether \( c \) is useful for \( y.\) In addition to that, two auxiliary lemmas are also introduced. With those we can finish the proof of Theorem [1].

A decomposition of a catalyst state is a useful mathematical tool in this section. Formally, we have the following:

Definition 3: We say \( [c]_\alpha = \{c^1, \cdots, c^m\} \) is a decomposition of \( c \) according to \( \alpha \) with \( 0 \leq \alpha < 1, \) if
1) \( c = c^1 \oplus \cdots \oplus c^m;\)
2) \( l_u(c^i) > \alpha \) for all \( 1 \leq i \leq m;\)
3) \( \max c^{i+1}/\min c^i \leq \alpha \) for all \( 1 \leq i \leq m - 1.\)

Obviously, for any \( \alpha \in [0, 1] \) and \( c \in V^k, \) the decomposition \( [c]_\alpha \) exists uniquely. Given \( c \) and \( \alpha, \) it is only a simple calculation to determine \( [c]_\alpha.\) We shall see that such a decomposition plays an important role in the study of entanglement catalysis.

Two useful lemmas are needed to present a simple characterization of \( T^\alpha(y, c). \) The first lemma shows the importance of the decomposition of \( c \) according to \( g_u(y):\)

Lemma 6: If \( [c]_{g_u(y)} = \{c^1, \cdots, c^m\}, \) then \( T(y, c) = \bigcap_{i=1}^m T(y, c^i).\)

Proof. We only need to prove that \( x \otimes c < y \otimes c \) if and only if \( x \otimes c^i < y \otimes c^i \) for each \( 1 \leq i \leq m.\)

Sufficiency part follows immediately from Lemma [5]. Now we prove the necessity part. By the definition of \( [c]_{g_u(y)} \), we have \( g_u(y) \geq \max (c^{i+1}/\min c^i), \) that is, \( \max y \otimes c^{i+1} \leq \min y \otimes c^i \) for all \( 1 \leq i < m, \) which follows that
\[
(y \otimes c)^i = ((y \otimes c^1)^i, \cdots, (y \otimes c^m)^i).
\]

Noticing \( x \otimes c < y \otimes c \) implies \( x_1 \leq y_1 \) and \( x_n \geq y_n, \) we have \( g_u(x) \geq g_u(y), \) thus
\[
(x \otimes c)^k = ((x \otimes c^1)^k, \cdots, (x \otimes c^m)^k).
\]

Hence, majorization relation \( x \otimes c < y \otimes c \) splits into \( m \) sub-majorizations: \( x \otimes c^i < y \otimes c^i, 1 \leq i \leq m.\)

By virtue of Lemma [5] we only need to focus on the case that the catalyst \( c \) and the target \( y \) satisfy \( l_u(c) > g_u(y). \) In this special case, the following lemma shows that \( x \in T^\alpha(y, c) \) is just equivalent to \( x \otimes c < y \otimes c.\)

Lemma 7: If \( l_u(c) > g_u(y), \) then \( T^\alpha(y, c) = \{x \in V^m : x \otimes c < y \otimes c \}.\)

Proof. To make the paper more readable, the lengthy proof is put into Appendix B.

Combining the above two lemmas leads us to the following simple characterization of \( T^\alpha(y, c):\)

Theorem 10: Let \( [c]_{g_u(y)} = \{c^1, \cdots, c^m\}, \) then \( T^\alpha(y, c) = \{x : x \otimes c^i < y \otimes c^i \} \) for any \( 1 \leq i \leq m.\)

Proof. We only need to prove that if \( x \in T^\alpha(y, c), then \) \( x \otimes c^i < y \otimes c^i \) for all \( 1 \leq i \leq m. \) In fact, by Lemma [6] \( x \in T^\alpha(y, c) \) implies \( x \in T^\alpha(y, c^i). \) Since \( l_u(c^i) > g_u(y), \) it follows that \( x \otimes c^i < y \otimes c^i \) by Lemma [7].

To present the condition of the inequality \( S(y) \subseteq T(y, c) \) compactly, we introduce a special probability vector \( y(d) \) for each \( 1 < d < n - 1.\) Formally, \( y(d) \) is defined as \( y_i(d) = e_d(y)/d \) if \( 1 \leq i \leq d, \) and \( y_i(d) = (e_n(y) - e_d(y))/(n - d) \) if \( d + 1 \leq i \leq n. \) Now we have the following:
Theorem 11: For any \( y \in V^n \), \( S(y) \subseteq T(y,c) \) if and only if there exists \( 1 < d < n - 1 \) such that \( y(d) \otimes c < y \otimes c' \) for any \( c' \in [c]_{g_a(y)} \).

Proof. We first deal with the sufficiency part. Since \( y(d) \otimes c' < y \otimes c' \) for each \( c' \in [c]_{g_a(y)} \), we have \( y(d) \in T^o(y,c) \) by Theorem 11. On the other hand, \( y(d) \) is a boundary point of \( S(y) \) as \( e_d(y(d)) = e_d(y) \). These and the fact that \( S(y) \subseteq T(y,c) \) yield \( S(y) \subseteq T(y,c) \).

Conversely, assume \( S(y) \subseteq T(y,c) \). It is easy to verify that both \( S(y) \) and \( T(y,c) \) are compact subsets of \( V^n \). Thus \( S(y) \subseteq T(y,c) \) implies \( S^o(y) \subseteq T^o(y,c) \). In other words, there should exist some \( x \) such that \( x \) is a boundary point of \( S(y) \) while \( x \) is in the interior of \( T(y,c) \). A boundary point of \( S(y) \) implies that there exists some \( 1 < d < n - 1 \) such that \( e_d(x) = e_d(y) \). By the definition of \( y(d) \), one can easily see \( y(d) < x \), which together with \( x \in T^o(y,c) \) yields \( y(d) \in T^o(y,c) \). By Theorem 11 we have \( y(d) \otimes c' < y \otimes c' \) for each \( c' \in [c]_{g_a(y)} \).

Theorem 11 provides a complete characterization of \( S(y) \subseteq T(y,c) \). It also tells us how to simply determine whether a catalyst state \( c \) is useful for a target state \( y \). More precisely, for a given \( k \)-dimensional probability vector \( c \) and \( n \)-dimensional probability vector \( y \), to check whether \( S(y) \subseteq T(y,c) \), we only need to verify that whether \( y(d) \otimes c' < y \otimes c' \) for each \( c' \in [c]_{g_a(y)} \) and some \( 1 < d < n - 1 \). By a simple calculation, it is easy to see that the total time complexity of this problem is \( O(n^2k \log(nk)) \). Thus this problem can be efficiently solved.

For \( 1 < d < n - 1 \), we introduce a useful subset of \( S(y) \). Specifically, we define \( K_d(y) = S^o(y') \odot S^o(y'') \), where \( y' = (y_1, \ldots, y_d) \) and \( y'' = (y_{d+1}, \ldots, y_n) \). Obviously, \( K_d(y) = \emptyset \) if and only if \( y_1 = y_d \) or \( y_{d+1} = y_n \). In what follows, we always assume \( K_d(y) \neq \emptyset \) except we clearly say that it is not the case. For any \( x \in K_d(y) \), we have \( y(d) \prec x \). From this point of view, \( y(d) \) can be treated as a center of \( K_d(y) \).

It is easy to check that \( K_d(y) \otimes c \) and \( S^o(y \otimes c) \) satisfy LP. So we have the following interesting consequence of Theorem 11.

Corollary 7: For any \( y \in V^n \), \( S(y) \subseteq T(y,c) \) if and only if there exists \( 1 < d < n - 1 \) such that \( K_d(y) \otimes c' \subseteq S^o(y \otimes c') \) for any \( c' \in [c]_{g_a(y)} \).

To give a proof of Theorem 11, the following two simple lemmas are needed.

The first lemma provides a simple sufficient condition for \( K_d(y) \otimes c \subseteq S^o(y \otimes c) \).

Lemma 8: For \( y \in V^n \) and \( 1 < d < n - 1 \), if a catalyst \( c \) satisfies

\[
I_u(c) > \max\left\{\frac{y_d}{y_1}, \frac{y_n}{y_{d+1}}\right\} \quad \text{and} \quad g_u(c) < \frac{y_{d+1}}{y_d},
\]

then \( K_d(y) \otimes c \subseteq S^o(y \otimes c) \).

Proof. For any \( x \in K_d(y) \), we will show that \( x \otimes c \in S^o(y \otimes c) \). For this purpose, let us decompose \( x = (x', x'') \) and \( y = (y', y'') \), where \( x' \) is formed by the largest \( d \) components of \( x \), \( x'' \) is the rest part, and \( y' \) and \( y'' \) are defined similarly. It is obvious that \( x' \prec y' \) and \( x'' \prec y'' \). Noticing that \( I_u(c) > \max\{g_u(y'), g_u(y'')\} \), we have \( x' \otimes c \prec y' \otimes c \) and \( x'' \otimes c \prec y'' \otimes c \) by Lemma 8. Furthermore,

\[
g_u(c) < \frac{y_{d+1}}{y_d},
\]

is equivalent to

\[
\max y' \otimes c > \min y'' \otimes c
\]

and

\[
\max y'' \otimes c > \min y' \otimes c.
\]

These facts imply that \( x' \otimes c \otimes x'' \otimes c \prec y' \otimes c \otimes y'' \otimes c \) according to Lemma 9. Equivalently, we have \( x \otimes c \prec y \otimes c \), which completes the proof of \( x \otimes c \in S^o(y \otimes c) \).

The second lemma provides a necessary condition for \( K_d(y) \otimes c \subseteq S^o(y \otimes c) \).

Lemma 9: If \( K_d(y) \otimes c \subseteq S^o(y \otimes c) \), then there exists a segment of \( c \), namely \( c' \), such that

\[
I_u(c') > \max\left\{\frac{y_d}{y_1}, \frac{y_n}{y_{d+1}}\right\} \quad \text{and} \quad g_u(c') < \frac{y_{d+1}}{y_d}.
\]

Proof. Take \( x \in K_d(y) \). Then \( x \otimes c \prec y \otimes c \). We will explicitly construct \( c' \) that is a segment of \( c \) satisfying Eq. 11. Similar to Lemma 8 we decompose \( x = (x', x'') \) and \( y = (y', y'') \). By the definition of \( K_d(y) \), we have \( x' \prec y' \) and \( x'' \prec y'' \).
Denote
\[ \alpha = \max\left\{ \frac{y_d}{y_1}, \frac{y_n}{y_{d+1}} \right\}, \]
and decompose \( c \) according to \( \alpha \) into \( [c]_{\alpha} = \{c_1, \cdots, c^m\} \). Then the vector \( c' \) satisfying Eq. (15) can be constructed as follows: If \( g_u(y') \geq g_u(y'') \), then \( c' = c^1 \); otherwise \( c' = c^m \). We only prove the case of \( g_u(y') \geq g_u(y'') \). In this case, we have \( \alpha = g_u(y') \). Since \( c^1 \in [c]_{\alpha} \), it satisfies \( l_u(c^1) > \alpha \). The only left thing is to prove \( g_u(c^1) \leq y_{d+1}/y_d \). By contradiction, suppose that \( g_u(c^1) > y_{d+1}/y_d \). This relation can be restated as
\[ \min y' \circ c^1 \geq \max y'' \circ c^1. \] (16)
By the definition of \([c]_{\alpha}\), we also have
\[ \min y' \circ c^1 \geq \max y' \circ c^2. \] (17)
Therefore, if we take \( l = \dim(y' \circ c^1) \), then by Eqs. (16) and (17), the \( l \) largest components of \( y \circ c \) are just those of \( y' \circ c^1 \), which yields
\[ e_l(y \circ c) = e_l(y' \circ c^1) = e_l(x' \circ c^1) \leq e_l(x \circ c), \]
a contradiction with \( x \circ c < y \circ c \).

The case of \( g_u(y') < g_u(y'') \) can be proved similarly by considering the term \( y'' \circ c^m \). With that we complete the proof of Lemma 9.

Now the proof of Theorem 12 follows immediately.

**Proof of Theorem 12** If Eq. (15) holds, then by Lemma 9 we have \( K_d(y) \circ c \subseteq S^0(y \circ c) \), which follows \( S(y) \subseteq T(y, c) \) by Corollary 7.

Conversely, if \( S(y) \subseteq T(y, c) \), then by Corollary 7, we have \( K_d(y) \circ z \subseteq S^0(y \circ z) \) for some \( z \in [c]_{g_u(y)} \) and \( 1 < d < n - 1 \). Moreover, application of Lemma 9 indicates the existence of \( c' \) satisfying Eq. (15), or equivalently, Eq. (17), where \( c' \) is a segment of \( z \), and also a segment of \( c \). With that we complete the proof of the Theorem 12

**VI. PROOF OF THEOREM 2**

In this section, we will give a proof of Theorem 2. First, some necessary preliminaries are presented. Especially, we obtain characterizations of \( T^k_o(y) \) and \( M^k_o(y) \), respectively. Then we propose a proof of Theorem 2.

**Theorem 12:** For any \( y \), \( T^k_o(y) = \{x : x \circ c < y \circ c, \text{ for some } c \in V^{k'}, k' \leq k\} \). Similarly, \( T^o(y) = \{x : x \circ c < y \circ c, \text{ for some } c\} \).

**Proof.** The part that \( x \circ c < y \circ c \) implies \( x \in T^o(y) \) is obvious. Conversely, suppose that \( x \) is an interior point of \( T^k_o(y) \). Let us define \( \bar{x} = (x_1 + \epsilon, \cdots, x_n - \epsilon) \). Since \( x \in T^k_o(y) \), we have \( \bar{x} \in T^k_o(y) \) for a sufficiently small positive real \( \epsilon \), or equivalently, \( \bar{x} \circ c < y \circ c \) for some \( c \in V^k \). On the other hand, by the construction, it is easy to check that \( x < \bar{x} \). Thus \( x \in T^o(y, c) \). By Theorem 10 we have \( x \circ c' < y \circ c' \) for some \( c' \in V^{k'} \), where \( k' \leq k \).

The case of \( T(y) \) can be similarly proved.

**Theorem 13:** For any \( y \in V^n \), \( M^k_o(y) = \{x : x \circ k < y \circ k, \text{ for some } k \geq 1\} \). Similarly, \( M^o(y) = \{x : x \circ k < y \circ k \} \).

**Proof.** The part that \( x \circ k < y \circ k \) implies \( x \in M^o(y) \) is obvious. Conversely, suppose that \( x \) is an interior point of \( M^k_o(y) \). Let us define \( \bar{x} = (x_1 + \epsilon, \cdots, x_n - \epsilon) \). Since \( x \in M^k_o(y) \), we have \( \bar{x} \in M^k_o(y) \) for a sufficiently small positive real \( \epsilon \), or equivalently, \( \bar{x} \circ k < y \circ k \). On the other hand, by the construction, we have \( x < \bar{x} \). Applying of Corollary 8 yields \( x \circ k < \bar{x} \circ k \). Hence \( x \circ k < y \circ k \).

The case of \( M(y) \) can be similarly proved.

The following lemma is a powerful tool in the study of the mathematical structure of MLOCC.

**Lemma 10:** For any \( x \in K_d(y) \), we have
\[ x \circ k < y \circ k \iff y^k_d < y^1_{d+1} y^k_{d+1} > y^k_{d+1} y^k_{n-1}. \] (18)
Now we present a very interesting result. In fact, we are able to completely characterize the condition of $K_d(y) \subseteq T_k^n(y)$ and that of $K_A(y) \subseteq M_k^n(y)$. To one’s surprise, these two conditions are exactly the same. As we will see soon, the equivalence of $K_d(y) \subseteq T_k^n(y)$ and $K_A(y) \subseteq M_k^n(y)$ directly leads us to the proof of Theorem 2.

**Theorem 14:** For any $y \in V^n$ and $1 < d < n - 1$, the following are equivalent:
1) $K_d(y) \subseteq T_k^n(y)$.
2) $K_A(y) \subseteq M_k^n(y)$.
3) $y^k_d < y^k_{d+1}$ and $y^k_{d+1} > y^{k-1}_{d}$.

**Proof.** We first prove the equivalence of 1) and 3). Suppose that 3) holds. Then we can choose $c = (1, \alpha, \cdots, \alpha^{k-1})$ with $0 < \alpha < 1$ such that
$$\max \left\{ \left( \frac{y_d}{y_1} \right)^{k-1}, \left( \frac{y_n}{y_d+1} \right)^{k-1} \right\} < \alpha^{k-1} < \frac{y_{d+1}}{y_d}.$$ 
A routine calculation shows that $l_u(c) = \alpha$ and $g_u(c) = \alpha^{k-1}$. By Lemma 3 we have $K_d(y) \subseteq T_k^n(y)$.

Conversely, by Theorem 12 $K_d(y) \subseteq T_k^n(y)$ implies that there exists $c \in V^{k'}$ such that $y(d) \otimes c \sim y \otimes c$, and then by LP we have $K_d(y) \otimes c \subseteq S^n(y \otimes c)$. According to Lemma 2 we declare that there exists $z \in V^{k''}$ satisfying Eq. (15), where $z$ is a segment of $c$. Noticing that $l_{k''}^{u-1}(z) \leq g_u(z)$ and $k'' \leq k'$, we have $l_{k''}^{u-1}(z) \leq g_u(z)$. This fact together with Eq. (15) shows that $y^k_d < y^{k-1}_{d+1}$ and $y^k_{d+1} > y^{k-1}_{d}$, which is exactly the same as 3).

By Theorem 13 $K_d(y) \subseteq M_k^n(y)$ is equivalent to that for any $x \in K_d(y)$, $x^{\otimes k} < y^{\otimes k}$. Therefore, the equivalence of 1) and 3) follows from the following fact: for any $x \in K_d(y)$, $x^{\otimes k} < y^{\otimes k}$ if and only if $y^k_d < y_{d+1}^{k-1}$ and $y^k_{d+1} > y^{k-1}_{d}$. Obviously, this fact is guaranteed by Lemma 10.

We are now in the position to present the proof of Theorem 4.

**Proof of Theorem 4.** Theorem 4 is essentially implied by Theorem 14. To see this, we only need to prove that $S(y) \subseteq T_k^n(y)$ and $S(y) \subseteq M_k^n(y)$ are equivalent to $K_d(y) \subseteq T_k^n(y)$ and $K_d(y) \subseteq M_k^n(y)$ for some $1 < d < n - 1$, respectively. The proofs of these two cases are similar. We only outline the proof for the case of $M_k^n(y)$ here. The part that $K_d(y) \subseteq M_k^n(y)$ implies $S(y) \subseteq M_k^n(y)$ is obvious. Conversely, assume that $S(y) \subseteq M_k^n(y)$. Noticing that $M_k^n(y)$ is a compact subset of $V^n$, we can find some $1 < d < n - 1$ such that $y(d) \in M_k^n(y)$ by a similar argument used in the proof of Theorem 14. Now $K_d(y) \subseteq M_k^n(y)$ follows from LP.

**VII. PROOFS OF THEOREMS 5 AND 6.**

In this section, we mainly prove Theorems 5 and 6. First, the physical meaning of the interior points of probabilistic entanglement transformations is given. Then we complete the proof of Theorem 5. Finally, the proof of Theorem 6 is given. In this section, we always assume that $\lambda \in (0,1)$.

As we have mentioned, the interior point of $S^\lambda(y)$ has a very clear physical meaning. That is, $x$ is an interior point of $S^\lambda(y)$ if and only if the maximal conversion probability from $x$ to $y$ is strictly larger than $\lambda$. Equivalently, we have $x \in (S^\lambda(y))^o$ if and only if $x \preceq^w \lambda y$. An interesting question is thus to ask whether this property also holds in the presence of catalysts or in multiple-copy scheme. The following result gives a positive answer to this question in the case of probabilistic ELOCC.

**Theorem 15:** For any $y \in V^n$, it holds that $(T^\lambda(y,c))^o = \{ x \in V^n : x \otimes c \preceq^w \lambda y \otimes c \}$. Similarly, we have $(T^\lambda(y))^o = \{ x \in V^n : x \otimes c \preceq^w \lambda y \otimes c \}$ for some $c \in V^k$ and $(T(y))^o = \{ x \in V^n : x \otimes c \preceq^w \lambda y \otimes c \}$ for some $c$.

**Proof.** The proof is similar to Lemma 7. We omit the details here.

We can prove a corresponding result of the above theorem in the case of probabilistic MLOCC:

**Theorem 16:** For any $y \in V^n$, we have $(M^\lambda(y))^o = \{ x \in V^n : x^{\otimes k} \preceq^w \lambda^k y^{\otimes k} \}$. Similarly, $(M^\lambda(y))^o = \{ x \in V^n : x^{\otimes k} \preceq^w \lambda^k y^{\otimes k} \}$ for some $k \geq 1$.
Proof. The proof is almost the same as that of Theorem 13. So we omit the details here. □

We can extend $K_d(y)$ in a probabilistic manner. Formally, define $K_d^\lambda(y) = \{ x \in S^\lambda(y) : E_l(x) = \Lambda E_l(y) \text{ iff } l = n - d \}$, where $0 < d < n - 1$.

The proof of Theorem 3 goes as follows:

Proof of Theorem 3. Similar to the deterministic cases, it is easy to show that $S^\lambda(y) \subseteq T^\lambda(y, c)$ if and only if there exists $0 < d < n - 1$ such that $K_d^\lambda(y) \otimes c \subseteq (S^\lambda(y \otimes c))^\lambda$. Hence, to complete the proof, we need only to show that $K_d^\lambda(y) \otimes c \subseteq (S^\lambda(y \otimes c))^\lambda$ if and only if

$$l_u(c^m) > \frac{y_n}{y_{d+1}} \text{ and } g_u(c^m) < \frac{y_d+1}{y_d}, \quad (19)$$

where $c^m$ is the last element of $[c]_{g_u(y''')} = \{ c^1, \cdots, c^m \}$.

Take $x \in K_d^\lambda(y)$, and decompose $x = (x', x'')$ and $y = (y', y''')$. It is obvious that

$x' \triangledown \lambda y'$ and $x'' \triangledown \lambda y'''$.

By Lemma 5, we have $x' \otimes c \triangledown \lambda y' \otimes c$. From the definition of $[c]_{g_u(y''')}$, we know $x'' \otimes c^i \triangledown \lambda y'' \otimes c^i$ for any $1 \leq i \leq m$, and furthermore,

$$(y'' \otimes c)^i = ((y'' \otimes c^1)^i, \cdots, (y'' \otimes c^m)^i)$$

and

$$(x'' \otimes c)^i = ((x'' \otimes c^1)^i, \cdots, (x'' \otimes c^m)^i).$$

According to Corollary 6, we have $x' \otimes c \triangledown \lambda y' \otimes c \triangledown \lambda y' \otimes c$ and $x \otimes c \triangledown \lambda y' \otimes c$, if and only if $\min y' \otimes c < \max y'' \otimes c^m$, which is equivalent to $g_u(c^m) < \frac{y_{d+1}}{y_d}$. The condition $l_u(c^m) > \frac{y_n}{y_{d+1}}$ is automatically satisfied by the assumption that $c^m \in [c]_{g_u(y''')}$. □

The following lemma is a powerful tool in the study of the mathematical structure of probabilistic MLOCC transformations.

Lemma 11: For any $x \in K_d^\lambda(y)$, $x \otimes k \triangledown \lambda y \otimes k \Rightarrow y_{d+1}^k > y_{d+1}^{k-1}y_d$.

The proof of Lemma 11 is much more simpler than its deterministic counterpart Lemma 10. So we prefer to give a proof here.

Proof. Let us decompose $x$ and $y$ into $(x', x'')$ and $(y', y''')$, respectively. By binomial theorem, we have

$$(x' \otimes k)^i = (b_x \otimes x'' \otimes k)^i \text{ and } (y'' \otimes k)^i = (b_y \otimes y''' \otimes k)^i, \quad (20)$$

where

$$b_x = \bigoplus_{i=0}^{k-1} (x'' \otimes k-i \otimes x'' \otimes i)^\otimes \{i\} \text{ and } b_y = \bigoplus_{i=0}^{k-1} (y''' \otimes k-i \otimes y''' \otimes i)^\otimes \{i\}.$$ 

Since $x \in K_d^\lambda(y)$, it is obvious that $x' \triangledown \lambda y'$ and $x'' \triangledown \lambda y'''$. Applying Lemma 5 repeatedly, we have $b_x \triangledown \lambda k b_y$. By Corollary 5, we have $x'' \otimes k \triangledown \lambda y' \otimes k$. Hence by Lemma 5 again, we deduce from Eq. (20) that $x' \otimes k \triangledown \lambda y' \otimes k$ and only if

$$\max y'' \otimes k \triangledown \lambda y' \otimes k > \min y'' \otimes k \triangledown \lambda y' \otimes k,$$

which is equivalent to $y_{d+1}^k > y_{d+1}^{k-1}y_d$. With that we complete the proof of the lemma. □

Now we establish the following interesting result, which is exactly a probabilistic version of Theorem 14.

Theorem 17: For any $y \in V^n$ and $0 < d < n - 1$, the following are equivalent:

1) $K_d^\lambda(y) \subseteq (M_k(y))^\lambda$.
2) $K_d^\lambda(y) \subseteq (T_k(y))^\lambda$.
3) $y_{d+1}^k > y_{d+1}^{k-1}y_d$.

Proof. We first prove the equivalence of 2) and 3). Suppose 3) holds. We can choose $c = (\alpha, \cdots, \alpha^{k-1})$ with $0 < \alpha < 1$ such that

$$\frac{y_n}{y_{d+1}} < \alpha^{k-1} < \frac{y_{d+1}}{y_d}.$$


A routine calculation shows that \( l_w(c) = \alpha \) and \( g_u(c) = \alpha^{k-1} \). Take \( x \in K^\lambda_d(y) \). By the proof of Theorem 4 we have \( x \otimes c \lhd^w y \otimes c \). Moreover, we have \( K^\lambda_d(y) \subseteq (T^\lambda(y, c))^o \subseteq (T^\lambda_k(y))^o \).

Conversely, \( K^\lambda_d(y) \subseteq (T^\lambda_k(y))^o \) implies that there exists \( c \in V^k \) such that \( S^\lambda(y) \not\subseteq T^\lambda(y, c) \). By Theorem 3 we declare that there exists \( c' \in V^k \) satisfying condition (8), where \( c' \) is a segment of \( c \). By the relation \( l^k_d(c') \leq g_u(c') \) and \( k' \leq k \) we have \( l^k_d(c') \leq g_u(c') \). This fact together with condition (8) shows that \( y_{d+1}^k > y_d y_{d+1}^{k-1} \), which is exactly the same as (3).

By Theorem 10 the equivalence of (1) and (3) is just to prove the fact: for any \( x \in K^\lambda_d(y) \), \( x \otimes^k \lhd^w \lambda^k y \otimes^k \) if and only if \( y_{d+1}^k > y_d y_{d+1}^{k-1} \). This fact is guaranteed by Lemma 11.

Now we can present the proof of Theorem 4 as follows:

**Proof of Theorem 4**

Theorem 4 is essentially implied by Theorem 17. To see this, we need only to prove that \( S^\lambda(y) \subseteq T^\lambda_k(y) \) and \( S^\lambda(y) \subseteq M^\lambda_k(y) \) are equivalent to \( K^\lambda_d(y) \subseteq (T^\lambda_k(y))^o \) and \( K^\lambda_d(y) \subseteq (M^\lambda_k(y))^o \) for some \( 0 < d < n - 1 \), respectively. The proofs are the same as that for the deterministic case, since \( S^\lambda(y) \), \( T^\lambda_k(y) \) and \( M^\lambda_k(y) \) are all compact subsets of \( V^n \). We omit the details here, and thus complete the proof of Theorem 4.

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**VIII. Conclusion**

To summarize, we have examined the powers of entanglement catalysis and multiple-copy entanglement transformation in three different contexts: 1) the catalyst state is specified, 2) the dimension of catalyst state is fixed, and 3) the number of copies used in multiple-copy entanglement transformation is fixed. In the case of 1), we have presented an economic sufficient condition under which an entangled quantum state \( c \) can serve as a catalyst in producing the state \( y \), i.e., \( S(y) \subseteq T(y, c) \). In a special case when \( c \) has only two non-zero different Schmidt coefficients, this condition is shown to be also a necessary one. As an interesting application of this condition, for any nonuniform entangled state \( z \) and \( n \geq 4 \), we can explicitly construct a subset \( A(z) \) of \( V^n \), such that \( z \) can catalyze any states in \( A(z) \). We have demonstrated that our result serves as an extensively generalized version of the one obtained by Daftuar and Klimesh in [12], and thus is a more stronger answer to Nielsen’s conjecture [13]: any nonuniform entangled state can serve as quantum catalyst for some entanglement transformation.

In the cases of 2) and 3), we have generalized the known result \( T(y) = S(y) \Leftrightarrow M(y) = S(y) \) to a finer one: \( T_k(y) = S(y) \Leftrightarrow M_k(y) = S(y) \). That is, \( k \)-ELOCC is useful in producing \( y \) if and only if \( k \)-MLOCC is useful in producing \( y \). Furthermore, an analytical condition for the equality \( T_k(y) = S(y) \) (and equivalently \( M_k(y) = S(y) \)) have been found in terms of the components of \( y \). We have also shown some interesting applications of this result. Especially, for any positive integer \( k > 1 \), we have constructed a class of \( 4 \times 4 \) states which can be catalyzed by some catalysts with the dimension at least \( k + 1 \). Our results can be generalized to probabilistic transformations. Some differences between deterministic transformations and probabilistic transformations have also been discussed.

**Appendix A: Proof of Lemma 2**

To be more specific, assume \( y \in V^m \) and \( y' \in V^n \). Take \( x \prec y \) and \( x' \prec y' \).

\[ y_1 > y'_1 \quad \text{and} \quad y_m < y'_m. \]  

We will prove that \( x \oplus x' \) is in the interior of \( S(y \oplus y') \). It suffices to show

\[ e_l(x \oplus x') < e_l(y \oplus y') \]  

for any \( 1 \leq l < m + n \).

One can easily verify

\[ e_l(x \oplus x') = e_p(x) + e_q(x') \leq e_p(y) + e_q(y') \leq e_l(y \oplus y'), \]

where \( 0 \leq p \leq m, 0 \leq q \leq n \) and \( p + q = l \). To complete the proof, we need to consider the following two cases:
Case 1: $0 < p < m$ or $0 < q < n$. By the conditions that $x \in S^n(y)$ and $x' \in S^n(y')$, we have
\[ e_p(x) < e_p(y) \text{ or } e_q(x') < e_q(y'). \tag{24} \]
Then the first inequality in Eq. (23) is strict, and Eq. (22) follows immediately.

Case 2: Either $p = m$ and $q = 0$, or $p = 0$ and $q = n$. They both contradict the assumption in Eq. (21).

So we finish the proof of the sufficiency part.

‘⇒’. By contradiction, suppose that Eq. (22) holds for $1 \leq l < m + n$ but Eq. (21) does not hold. If $y_1 \leq y_n'$ then
\[ e_n(y \oplus y') = e_n(y') = e_n(x') \leq e_n(x \oplus x'), \tag{25} \]
a contradiction with Eq. (22) when $l = n$. Similarly, if $y_m \geq y_1'$ then
\[ e_m(y \oplus y') = e_m(y) = e_m(x) \leq e_m(x \oplus x'), \tag{26} \]
which contradicts Eq. (22) again. That completes the proof of the lemma. □

**APPENDIX B: PROOF OF LEMMA 4**

We need only to prove that under the constraint of $l_u(c) > g_u(y)$, $x \in T^n(y, c)$ implies $x \otimes c \not< y \otimes c$.

By contradiction, suppose that for some $x \in T^n(y, c)$, it holds that $x \otimes c \not\in S^n(y \otimes c)$. Then there should exist some $1 \leq l \leq nk - 1$ such that $e_l(x \otimes c) = e_l(y \otimes c)$. However, as we will show latter, this case is impossible.

First, we prove that $e_l(x \otimes c) = e_l(y \otimes c)$ and $1 \leq l \leq nk - 1$ imply $l = i_0n$ for some $1 \leq i_0 < k$. For this purpose, let us define

\[ \bar{x} = (x_1 + \varepsilon, \cdots, x_n - \varepsilon). \]

It is obvious that $x \subset \bar{x}$ for any $\varepsilon > 0$. Since $x \in T^n(y, c)$, we have $\bar{x} \in T^n(y, c)$ for a sufficiently small positive real $\varepsilon$, which implies $\bar{x} \otimes c \not< y \otimes c$. Then we have $e_l(\bar{x} \otimes c) \leq e_l(y \otimes c)$. Noticing $e_l(x \otimes c) \leq e_l(\bar{x} \otimes c)$ and the assumption $e_l(x \otimes c) = e_l(y \otimes c)$, we have $e_l(x \otimes c) = e_l(\bar{x} \otimes c)$. Choose $l_1, \cdots, l_k$ such that

\[ e_l(x \otimes c) = \sum_{i=1}^{k} c_i e_{l_i}(x), \]

where $\sum_{i=1}^{k} l_i = l$ and $0 \leq l_i \leq n$. If there exists $1 \leq p \leq k$ such that $0 < l_p < n$, then by $x \subset \bar{x}$ we have $e_{l_p}(x) < e_{l_p}(\bar{x})$. Thus

\[ e_l(x \otimes c) = \sum_{i=1}^{k} c_i e_{l_i}(x) < \sum_{i=1}^{k} c_i e_{l_i}(\bar{x}) \leq e_l(\bar{x} \otimes c), \]

which contradicts $e_l(x \otimes c) = e_l(\bar{x} \otimes c)$. Therefore, for any $1 \leq i \leq k$, we have $l_i \in \{0, n\}$. Taking $i_0 = \max\{ i : l_i > 0 \}$, we have $1 \leq i_0 < k$ by the assumption $1 \leq l \leq nk - 1$. So, $l = i_0n$.

Second, we show $e_l(x \otimes c) < e_l(y \otimes c)$ for $l = i_0n$. In fact, by the above argument, we have

\[ e_{i_0n}(x \otimes c) = \sum_{i=1}^{i_0} c_i e_n(x) = \sum_{i=1}^{i_0} c_i e_n(y). \tag{27} \]

Notice that $l_u(c) > g_u(y)$ yields $c_{i_0+1}y_1 > c_{i_0}y_n$. So

\[ \sum_{i=1}^{i_0} c_i e_n(y) = \sum_{i=1}^{i_0-1} c_i e_n(y) + c_{i_0} e_n(y) + c_{i_0}y_n < \sum_{i=1}^{i_0-1} c_i e_n(y) + c_{i_0} e_{n-1}(y) + c_{i_0} + 1y_1 \leq e_{i_0n}(y \otimes c), \tag{28} \]

where the last inequality is by the definition of $e_{i_0n}(y \otimes c)$. Combining Eqs. (27) with (28) shows $e_l(x \otimes c) < e_l(y \otimes c)$ for $l = i_0n$, again a contradiction. With that we complete the proof of the lemma. □
Appendix C: Proof of Lemma

Let \( x' = (x_1, \cdots, x_d) \) be the vector formed by the \( d \) largest components of \( x \), and \( x'' \) is the rest part of \( x \). We can similarly define \( y' \) and \( y'' \). Then it is easy to check

\[
x' \lhd y' \quad \text{and} \quad x'' \lhd y''
\]

by the definition of \( K_d(y) \). Also we have

\[
x = x' \oplus x'' \quad \text{and} \quad y = y' \oplus y''.
\]

We give a proof of the part \( \Leftarrow \) by seeking a sufficient condition for \( x^{\otimes k} \lhd y^{\otimes k} \). First we notice the following identity by binomial theorem:

\[
(y^{\otimes k})^\downarrow = (k \bigoplus_{i=0}^k (y^{\otimes (k-i)} \otimes y'^{\otimes i})^\downarrow).
\]

And \( x^{\otimes k} \) has a similar expression. For simplicity, we denote

\[
y_i = (y^{\otimes (k-i)} \otimes y'^{\otimes i})^\downarrow, \quad n_i = d^{k-i}(n-d)^i.
\]

And \( x_i \) is defined similarly.

Noticing Eqs.(29) and (32), we have

\[
x_i \lhd y_i \quad \text{for any} \quad 0 \leq i \leq k
\]

by Corollary \( \textbf{5} \). So to ensure \( x^{\otimes k} \lhd y^{\otimes k} \), we only need that the set \( A = \{ (y_i)^{\otimes (i^\downarrow)} : 0 \leq i \leq k \} \) satisfies the conditions in 2) of Corollary \( \textbf{4} \). It is easy to check that

\[
\max y^0 = \max \{ \max y^i : 0 \leq i \leq k \} \quad \text{and} \quad \min y^k = \min \{ \min y^i : 0 \leq i \leq k \}.
\]

Hence we only need \( A \) to satisfy the overlapping conditions, i.e., \( \min y^i < \max y^{i+1} \), or more explicitly,

\[
y_d^k y_n^i < y_1^{k-1} y_{d+1}^{i+1}, \quad 0 \leq i < k.
\]

By the monotonicity, Eq.(34) is just equivalent to the cases of \( i = 0 \) and \( i = k - 1 \). That is,

\[
y_d^k < y_1^{k-1} y_{d+1} \quad \text{and} \quad y_d^{k-1} > y_n^{k-1} y_d,
\]

which is exactly the condition in the right-hand side of Eq.(18). That completes the proof of the part \( \Leftarrow \).

Now we prove the part \( \Rightarrow \). By contradiction, suppose the conditions in the right-hand side of Eq.(18) are satisfied. Then there should exist \( 0 \leq i_0 < k \) that violates the conditions in Eq.(34), i.e.,

\[
y_d^{k-i_0} y_n^{i_0} > y_1^{k-(i_0+1)} y_{d+1}^{i_0+1}.
\]

But then we can deduce that

\[
ee_{d(i_0)}(x^{\otimes k}) = \sum_{i=0}^{i_0} \binom{k}{i} e_{n_i}(y^i) = \sum_{i=0}^{i_0} \binom{k}{i} e_{n_i}(x^i) \leq e_{d(i_0)}(x^{\otimes k}),
\]

which contradicts the assumption \( x^{\otimes k} \lhd y^{\otimes k} \), where \( d(i_0) = \sum_{i=0}^{i_0} \binom{k}{i} n_i \).

With that we complete the proof of the lemma. \( \square \)
REFERENCES

[1] C. H. Bennett and G. Brassard, *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing*, pp. 175–179, IEEE, New York, 1984.

[2] C. H. Bennett and S. J. Wiesner, “Communication via One- and Two-particle Operators on Einstein-Podolsky-Rosen States,” *Phys. Rev. Lett.*, vol. 69, pp. 2881–2884, 1992.

[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen channels,” *Phys. Rev. Lett.*, vol. 70, pp. 1895–1899, 1993.

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

[5] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, “Concentrating partial entanglement by local operations,” *Phys. Rev. A*, vol. 53, pp. 2046–2052, 1996.

[6] M. A. Nielsen, “Conditions for a Class of Entanglement Transformations,” *Phys. Rev. Lett.*, vol. 83, pp. 436–439, 1999.

[7] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, New York, American: Academic Press, 1979.

[8] P. M. Alberti and A. Uhlmann, *Stochasticity and Partial Order: Doubly Stochastic Maps and Unitary Mixing*, Dordrecht, Boston, 1982.

[9] G. Vidal, “Entanglement of Pure States for a Single Copy,” *Phys. Rev. Lett.*, vol. 83, pp. 1046–1049, 1999.

[10] D. Jonathan and M. B. Plenio, “Entanglement-Assisted Local Manipulation of Pure Quantum States,” *Phys. Rev. Lett.*, vol. 83, pp. 3566–3569, 1999.

[11] S. Bandyopadhyay, V. Roychowdhury, and U. Sen, “Classification of Nonasymptotic Bipartite Pure-state Entanglement Transformations,” *Phys. Rev. A*, vol. 65, Art. No. 052315, 2002.

[12] S. Datta and M. Klimek, “Mathematical Structure of Entanglement Catalysis,” *Phys. Rev. A*, vol. 64, Art. No. 042314, 2001.

[13] M. A. Nielsen, *Introduction to Majorization and Its Applications to Quantum Mechanics* (unpublished notes). Available online: [http://www.qinfo.org/talks/2002/maj/book.ps](http://www.qinfo.org/talks/2002/maj/book.ps).

[14] R. Y. Duan, Y. Feng, X. Li, and M. S. Ying, “Tradeoff Between Multiple-Copy Entanglement Transformation and Entanglement Catalysis.” Available online: [http://www.arXiv.org/abs/quant-ph/0312010](http://www.arXiv.org/abs/quant-ph/0312010).

[15] R. Y. Duan, Y. Feng, X. Li, and M. S. Ying, “Multiple-Copy Entanglement Transformation and Entanglement Catalysis.” Available online: [http://www.arXiv.org/abs/quant-ph/0404148](http://www.arXiv.org/abs/quant-ph/0404148).

[16] D. W. Leung and J. A. Smolin, “More Is Not Necessarily Easier.” Available online: [http://www.arXiv.org/abs/quant-ph/0103158](http://www.arXiv.org/abs/quant-ph/0103158).

[17] X. M. Sun, R. Y. Duan, and M. S. Ying, “The Existence of Quantum Entanglement Catalysts,” to appear in *IEEE. Trans. Inform. Theory*. Available online: [http://www.arXiv.org/abs/quant-ph/0511155](http://www.arXiv.org/abs/quant-ph/0511155).

[18] Y. Feng, R. Y. Duan, and M. S. Ying, “Catalyst-assisted Probabilistic Entanglement Transformation,” to appear in *IEEE. Trans. Inform. Theory*. Available online: [http://www.arXiv.org/abs/quant-ph/0404154](http://www.arXiv.org/abs/quant-ph/0404154).