Some topics related to real, complex, and Fourier analysis

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Preface

Here we look at some situations that are like the unit circle or the real line in some ways, but which can be more complicated or fractal in other ways.
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Chapter 1

The unit circle

1.1 Integrable functions

Let $T$ be the unit circle in the complex plane $C$, consisting of complex numbers $z$ whose modulus $|z|$ is equal to 1. Also let $f$ be a complex-valued function on $T$ which is integrable with respect to the usual arc-length measure on $T$. The $j$th Fourier coefficient $\hat{f}(j)$ of $f$ is defined for each integer $j$ by

$$\hat{f}(j) = \frac{1}{2\pi} \int_T f(z) \overline{z^j} |dz|,$$

(1.1)

where $\overline{z}$ is the complex conjugate of $z \in C$, and $|dz|$ is the element of arc-length measure on $T$. Note that

$$|\hat{f}(j)| \leq \frac{1}{2\pi} \int_T |f(z)| |dz|$$

(1.2)

for every $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers.

Let $\exp z = \sum_{n=0}^{\infty} z^n / n!$ be the usual complex exponential function on $C$. It is well known that $\exp(it)$ defines a mapping from the real line $R$ onto $T$, which is periodic with period $2\pi$. The derivative of this mapping has modulus equal to 1 at each $t \in R$, so that $\exp(it)$ parameterizes $T$ by arc length. This permits the integral of a function $h(z)$ on $T$ with respect to arc-length measure to be identified with the integral of $h(\exp(it))$ on $[0, 2\pi)$ with respect to ordinary Lebesgue measure. In particular, one can use this to check that

$$\int_T z^j |dz| = 0$$

(1.3)

for each $j \in \mathbb{Z}$ with $j \neq 0$.

If $f$ and $g$ are complex-valued square-integrable functions on $T$, then put

$$\langle f, g \rangle = \frac{1}{2\pi} \int_T f(z) \overline{g(z)} |dz|.$$

(1.4)
CHAPTER 1. THE UNIT CIRCLE

Of course, the space $L^2(T)$ of complex-valued square-integrable functions on $T$ is a Hilbert space with respect to this inner product, and $\hat{f}(j)$ is the same as the inner product of $f$ with $g(z) = z^j$ for each $j \in \mathbb{Z}$. Using (1.3), we get that

$$\langle z^j, z^k \rangle = \frac{1}{2\pi} \int_T z^{j-k} |dz| = 0$$

when $j, k \in \mathbb{Z}$ and $j \neq k$, because $\overline{z} = 1/z$ when $z \in \mathbb{C}$ satisfies $|z| = 1$. Clearly

$$\langle z^j, z^j \rangle = \frac{1}{2\pi} \int_T |dz| = 1$$

for every $j \in \mathbb{Z}$, so that the functions $z^j$ on $T$ with $j \in \mathbb{Z}$ are orthonormal with respect to (1.4). If $f \in L^2(T)$, then it follows that

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 \leq \langle f, f \rangle = \frac{1}{2\pi} \int_T |f(z)|^2 |dz|,$$

by standard arguments about inner product spaces. In particular,

$$\lim_{|j| \to \infty} |\hat{f}(j)| = 0$$

for every $f \in L^2(T)$, which also holds for every $f \in L^1(T)$. This uses the fact that $L^2(T)$ is dense in $L^1(T)$, and the simple estimate (1.2).

Let $C(T)$ be the space of continuous complex-valued functions on $T$, and let $E(T)$ be the linear subspace $C(T)$ consisting of finite linear combinations of the functions $z^j$ on $T$ with $j \in \mathbb{Z}$. The Stone–Weierstrass theorem implies that $E(T)$ is dense in $C(T)$ with respect to the supremum norm on $C(T)$, and hence that $E(T)$ is dense in $L^p(T)$ when $p < \infty$, because $C(T)$ is dense in $L^p(T)$ when $p < \infty$. Thus the functions $z^j$ on $T$ with $j \in \mathbb{Z}$ form an orthonormal basis for $L^2(T)$. It follows that the Fourier series

$$\sum_{j=-\infty}^{\infty} \hat{f}(j) z^j$$

converges to $f$ with respect to the $L^2$ norm for every $f \in L^2(T)$, in the sense that the partial sums

$$\sum_{j=-N}^{N} \hat{f}(j) z^j$$

converge to $f$ with respect to the $L^2$ norm as $N \to \infty$. In particular,

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 = \frac{1}{2\pi} \int_T |f(z)|^2 |dz|$$

for every $f \in L^2(T)$. 

1.2 Abel sums and Cauchy products

Let \( \sum_{j=0}^{\infty} a_j \) be an infinite series of real or complex numbers, and suppose that

\[
\sum_{j=0}^{\infty} a_j r^j
\]

converges for every nonnegative real number \( r \) with \( r < 1 \). If the limit

\[
\lim_{r \to 1^{-}} \left( \sum_{j=0}^{\infty} a_j r^j \right)
\]

exists, then \( \sum_{j=0}^{\infty} a_j \) is said to be \textit{Abel summable}, with the Abel sum equal to (1.13). Of course, if (1.12) converges for some \( r \geq 0 \), then \( a_j r^j \to 0 \) as \( j \to \infty \), which implies that \( \{a_j r^j\}_{j=0}^{\infty} \) is a bounded sequence. Conversely, if \( \{a_j t^j\}_{j=0}^{\infty} \) is a bounded sequence for some \( t > 0 \), then (1.12) converges absolutely when \( 0 \leq r < t \), by comparison with the convergent geometric series \( \sum_{j=0}^{\infty} (r/t)^j \).

In particular, if \( \sum_{j=0}^{\infty} a_j \) converges, then \( \{a_j\}_{j=0}^{\infty} \) is a bounded sequence for some \( t > 0 \), and hence (1.12) converges absolutely when \( 0 \leq r < 1 \). In this case, it is well known that \( \sum_{j=0}^{\infty} a_j \) is Abel summable, with Abel sum equal to the ordinary sum. To see this, let

\[
s_n = \sum_{j=0}^{n} a_j
\]

be the \( n \)th partial sum of \( \sum_{j=0}^{\infty} a_j \) for each \( n \geq 0 \), and put \( s_{-1} = 0 \). Thus \( a_n = s_n - s_{n-1} \) for each \( n \geq 0 \), and

\[
\sum_{j=0}^{\infty} a_j r^j = \sum_{j=0}^{\infty} s_j r^j - \sum_{j=0}^{\infty} s_{j-1} r^j
\]

\[
= \sum_{j=0}^{\infty} s_j r^j - \sum_{j=0}^{\infty} s_j r^{j+1}
\]

\[
= (1 - r) \sum_{j=0}^{\infty} s_j r^j.
\]

when \( 0 \leq r < 1 \). Note that these series all converge when \( 0 \leq r < 1 \), because \( \{s_j\}_{j=0}^{\infty} \) is a bounded sequence too. It follows that

\[
\sum_{j=0}^{\infty} a_j r^j - \sum_{j=0}^{\infty} a_j = (1 - r) \sum_{j=0}^{\infty} (s_j - s) r^j
\]

when \( 0 \leq r < 1 \), where \( s = \sum_{j=0}^{\infty} a_j \), and using the fact that \( (1-r) \sum_{j=0}^{\infty} r^j = 1 \). The remaining point is to show that this converges to 0 as \( r \to 1^- \), because \( s_j \to s \) as \( j \to \infty \). Note that this argument also works for convergent series.
whose terms are elements of a real or complex vector space equipped with a norm.

If \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{k=0}^{\infty} b_k \) are any two infinite series of complex numbers, then their Cauchy product is the infinite series \( \sum_{l=0}^{\infty} c_l \), where

\[
(1.17) \quad c_l = \sum_{j=0}^{l} a_j b_{l-j}
\]

for each \( l \geq 0 \). It is easy to see that

\[
(1.18) \quad \sum_{l=0}^{\infty} c_l = \left( \sum_{j=0}^{\infty} a_j \right) \left( \sum_{k=0}^{\infty} b_k \right)
\]

formally, in the sense that every term \( a_j b_k \) occurs exactly once on both sides of the equation. In particular, if \( a_j = 0 \) for all but finitely many \( j \) and \( b_k = 0 \) for all but finitely many \( k \), then \( c_l = 0 \) for all but finitely many \( l \), and (1.18) holds. If \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{k=0}^{\infty} b_k \) converge absolutely, then it is well known and not difficult to check that \( \sum_{l=0}^{\infty} c_l \) also converges absolutely, and that (1.18) holds. Note that

\[
(1.19) \quad |c_l| \leq \sum_{j=0}^{l} |a_j| |b_{l-j}|
\]

for each \( l \geq 0 \), where the right side of (1.19) corresponds exactly to the Cauchy product of \( \sum_{j=0}^{\infty} |a_j| \) and \( \sum_{k=0}^{\infty} |b_k| \).

Similarly,

\[
(1.20) \quad c_l r^l = \sum_{j=0}^{l} (a_j r^j) (b_{l-j} r^{l-j})
\]

for every \( r \), which corresponds to the Cauchy product of

\[
(1.21) \quad \sum_{j=0}^{\infty} a_j r^j \quad \text{and} \quad \sum_{k=0}^{\infty} b_k r^k.
\]

If the series in (1.21) converge absolutely when \( 0 \leq r < 1 \), then \( \sum_{l=0}^{\infty} c_l r^l \) also converges absolutely when \( 0 \leq r < 1 \), with

\[
(1.22) \quad \sum_{l=0}^{\infty} c_l r^l = \left( \sum_{j=0}^{\infty} a_j r^j \right) \left( \sum_{k=0}^{\infty} b_k r^k \right),
\]

as in the previous paragraph. If, in addition, \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{k=0}^{\infty} b_k \) are Abel summable, then it follows that \( \sum_{l=0}^{\infty} c_l \) is Abel summable as well, and that the Abel sum of \( \sum_{l=0}^{\infty} c_l \) is equal to the product of the Abel sums of \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{k=0}^{\infty} b_k \). There is an analogous statement for the Abel summability of \( \sum_{j=0}^{\infty} (a_j + b_j) \), which is much simpler.
1.3 The Poisson kernel

Let \( f \) be a complex-valued integrable function on the unit circle again, and let us consider the Abel sums of the corresponding Fourier series (1.9). More precisely, put
\[
A_r(f)(z) = \sum_{j=-\infty}^{\infty} \hat{f}(j) r^{|j|} z^j
\]
for each \( z \in \mathbb{T} \) and \( r \in \mathbb{R} \) with \( 0 \leq r < 1 \), where the absolute convergence of the sum follows from the boundedness of the Fourier coefficients \( \hat{f}(j) \), as in (1.2).

Of course, the doubly-infinite Fourier series (1.9) may be treated as a sum of two ordinary infinite series, or it can be reduced to an ordinary infinite series by combining the terms for \( j \) and \(-j\) when \( j \geq 1 \). In both cases, one gets (1.23) by taking the Abel sums of the associated ordinary infinite series.

Equivalently,
\[
A_r(f)(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(w) p_r(w, z) |dw|
\]
for every \( z \in \mathbb{T} \) and \( r \in \mathbb{R} \) with \( 0 \leq r < 1 \), where
\[
p_r(w, z) = \sum_{j=-\infty}^{\infty} r^{|j|} z^j \overline{w}^j.
\]

As before, this series converges absolutely when \( w, z \in \mathbb{T} \) and \( 0 \leq r < 1 \), by comparison with a convergent geometric series. In fact, the partial sums
\[
\sum_{j=-N}^{N} r^{|j|} z^j \overline{w}^j
\]
converge to (1.25) as \( N \to \infty \) uniformly over \( w, z \in \mathbb{T} \) when \( 0 \leq r < 1 \), by the Weierstrass \( M \)-test. This permits one to interchange the order of summation and integration to get (1.24). Similarly, if \( 0 \leq t < 1 \), then the partial sums (1.26) converge to (1.25) as \( N \to \infty \) uniformly over \( w, z \in \mathbb{T} \) and \( 0 \leq r \leq t \).

Alternatively, put
\[
p_r(z) = \sum_{j=-\infty}^{\infty} r^{|j|} z^j
\]
for \( z \in \mathbb{T} \) and \( 0 \leq r < 1 \), where the series converges absolutely by comparison with a geometric series again. If \( 0 \leq t < 1 \), then the partial sums
\[
\sum_{j=-N}^{N} r^{|j|} z^j
\]
converge to (1.27) as \( N \to \infty \) uniformly over \( z \in \mathbb{T} \) and \( 0 \leq r \leq t \), for the same reasons as before. This implies that \( p_r(z) \) is continuous as a function of \( z \) and \( r \), with \( z \in \mathbb{T} \) and \( 0 \leq r < 1 \). By construction,
\[
p_r(w, z) = p_r(z \overline{w})
\]
for every \( w, z \in T \) and \( 0 \leq r < 1 \), and so it suffices to compute \( p_r(z) \) to determine \( p_r(w, z) \).

Observe that

\[
p_r(z) = \sum_{j=0}^{\infty} r^j z^j + \sum_{j=0}^{\infty} r^j \overline{z}^j - 1
\]

(1.30)

for every \( z \in T \) and \( 0 \leq r < 1 \), since \( 1/z = \overline{z} \) when \( |z| = 1 \). Hence

\[
p_r(z) = 2 \, \text{Re} \sum_{j=0}^{\infty} r^j z^j - 1 = 2 \, \text{Re}(1 - rz)^{-1} - 1
\]

(1.31)

for every \( z \in T \) and \( 0 \leq r < 1 \), where \( \text{Re} a \) denotes the real part of a complex number \( a \), and summing the geometric series in the second step. It follows that

\[
p_r(z) = \text{Re} \left( \frac{2 - (1 - rz)}{1 - rz} \right)
\]

(1.32)

\[
= \text{Re} \left( \frac{1 + rz}{1 - rz} \frac{1 - rz}{1 - rz} \right)
\]

\[
= \text{Re} \left( \frac{1 + r(z - \overline{z}) - r^2 |z|^2}{|1 - rz|^2} \right)
\]

\[
= \frac{1 - r^2}{|1 - rz|^2}
\]

for every \( z \in T \) and \( 0 \leq r < 1 \), because \( z - \overline{z} \) is purely imaginary and \( |z| = 1 \). Combining this with (1.29), we get that

\[
p_r(w, z) = \frac{1 - r^2}{|1 - rz|^2} = \frac{1 - r^2}{|w - rz|^2}
\]

(1.33)

for every \( w, z \in T \) and \( 0 \leq r < 1 \).

1.4 Continuous functions

It is easy to see that

\[
\frac{1}{2\pi} \int_T p_r(z) |dz| = 1
\]

(1.34)

for each \( 0 \leq r < 1 \), using (1.3) and the fact that the partial sums (1.28) converge to \( p_r(z) \) as \( N \to \infty \) uniformly over \( z \in T \). This implies that

\[
\frac{1}{2\pi} \int_T p_r(w, z) |dw| = 1
\]

(1.35)

for every \( z \in T \) and \( 0 \leq r < 1 \), by (1.29). Thus (1.24) expresses \( A_r(f)(z) \) as an average of \( f \) on \( T \), since \( p_r(w, z) \) is also real-valued and nonnegative. In particular,

\[
A_r(f)(z) - f(z) = \frac{1}{2\pi} \int_T (f(w) - f(z)) p_r(w, z) |dw|
\]

(1.36)
and hence

\[ |A_r(f)(z) - f(z)| \leq \frac{1}{2\pi} \int_T |f(w) - f(z)| p_r(w, z) \, dw. \tag{1.37} \]

As \( r \) approaches 1, this average is concentrated near \( z \), because \( p_r(w, z) \to 0 \) as \( r \to 1^- \) uniformly over \( w \in T \) that do not get too close to \( z \). More precisely, if \( w \) does not get too close to \( z \), then the denominator in (1.33) remains bounded away from 0, while the numerator obviously tends to 0 as \( r \to 1 \). If \( f \) is continuous at \( z \), then one can use this, (1.35), and (1.37) to show that

\[ \lim_{r \to 1^-} A_r(f)(z) = f(z). \tag{1.38} \]

If \( f \) is continuous at every point in \( T \), then it is well known that \( f \) is uniformly continuous on \( T \), because \( T \) is compact. In this case, an analogous argument shows that \( A_r(f) \) converges to \( f \) as \( r \to 1^- \) uniformly on \( T \).

Note that the partial sums

\[ \sum_{j=-N}^{N} \hat{f}(j) r^{|j|} z^j \tag{1.39} \]

converge to \( A_r(f)(z) \) as \( N \to \infty \) uniformly over \( z \in T \), for any fixed \( r \) with \( 0 \leq r < 1 \). This follows from the boundedness of the Fourier coefficients \( \hat{f}(j) \) and the Weierstrass \( M \)-test, as before. Thus \( A_r(f)(z) \) can be approximated by finite linear combinations of the \( z^j \)'s uniformly on \( T \) for each \( r < 1 \), and for any integrable function \( f \) on \( T \). If \( f \) is continuous on \( T \), then \( A_r(f) \) converges to \( f \) as \( r \to 1^- \) uniformly on \( T \), as in the previous paragraph. This implies that \( f \) can be approximated uniformly by finite linear combinations of the \( z^j \)'s on \( T \), by the corresponding statement for \( A_r(f) \) just mentioned.

Because \( p_r(w, z) \) is nonnegative and real-valued,

\[ |A_r(f)(z)| \leq \frac{1}{2\pi} \int_T |f(w)| p_r(w, z) \, dw \tag{1.40} \]

for any integrable function \( f \) on \( T \), \( z \in T \), and \( 0 \leq r < 1 \). In particular, if \( f \) is a bounded measurable function on \( T \), then

\[ |A_r(f)(z)| \leq \sup_{w \in T} |f(w)|, \tag{1.41} \]

by (1.35). Of course, the same conclusion holds when \( f \) is essentially bounded on \( T \), with the supremum of \( |f(w)| \) on \( T \) replaced with the essential supremum. This implies that

\[ \|A_r(f)\|_{\infty} \leq \|f\|_{\infty} \tag{1.42} \]

for every \( 0 \leq r < 1 \), where \( \|f\|_{\infty} \) is the \( L^\infty \) norm of \( f \) on \( T \). The \( L^\infty \) norm of \( A_r(f) \) is the same as the supremum norm of \( A_r(f) \) for every \( r < 1 \), because \( A_r(f) \) is a continuous function on \( T \) for each \( r < 1 \), by the remarks about uniform convergence of (1.39) to \( A_r(f) \) in the preceding paragraph.
1.5 \(L^p\) Functions

Let \(p\) be a real number with \(p \geq 1\), and suppose that \(f \in L^p(T)\). Using (1.40), we get that
\[
|A_r(f)(z)|^p \leq \left(\frac{1}{2\pi} \int_T |f(w)|^p p_r(w, z) \, |dw|\right)^p
\]
for every \(z \in T\) and \(0 \leq r < 1\). This implies that
\[
|A_r(f)(z)|^p \leq \frac{1}{2\pi} \int_T |f(w)|^p p_r(w, z) \, |dw|
\]
for every \(z \in T\) and \(0 \leq r < 1\), by Jensen’s inequality, and the fact that \(t \mapsto t^p\) is a convex function on the set of nonnegative real numbers when \(p \geq 1\). This also uses the nonnegativity of \(p_r(w, z)\) and (1.35), so that the relevant integrals are in fact averages.

Integrating (1.44) over \(z\), we get that
\[
\int_T |A_r(f)(z)|^p \, |dz| \leq \frac{1}{2\pi} \int_T \left(\int_T |f(w)|^p p_r(w, z) \, |dw|\right) |dz|
\]
for every \(0 \leq r < 1\). Equivalently,
\[
\int_T |A_r(f)(z)|^p \, |dz| \leq \frac{1}{2\pi} \int_T \int_T |f(w)|^p p_r(w, z) \, |dw| \, |dz|
\]
by Fubini’s theorem. Of course,
\[
\frac{1}{2\pi} \int_T p_r(w, z) \, |dz| = 1
\]
for every \(w \in T\) and \(0 \leq r < 1\), by (1.29) and (1.34). Thus (1.46) reduces to
\[
\int_T |A_r(f)(z)|^p \, |dz| \leq \int_T |f(w)|^p \, |dw|,
\]
which holds for every \(0 \leq r < 1\).

Using this, one can check that
\[
A_r(f) \to f \quad \text{as} \ r \to 1
\]
with respect to the \(L^p\) norm when \(f \in L^p(T)\) and \(1 \leq p < \infty\). More precisely, if \(f\) is a continuous function on \(T\), then we have seen that (1.49) holds uniformly on \(T\), and hence with respect to the \(L^p\) norm. If \(f \in L^p(T)\) and \(p < \infty\), then it is well known that \(f\) can be approximated by continuous functions on \(T\) with respect to the \(L^p\) norm. To get (1.49), one can use the analogous statement for these approximations to \(f\), and (1.48) to estimate the errors. It is well known that (1.38) also holds for almost every \(z \in T\) with respect to arc-length measure when \(f \in L^1(T)\), but we shall not go into this here.
In the $p = 2$ case, we have that

$$\frac{1}{2\pi} \int_T |A_r(f)(z)|^2 |dz| = \sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 r^{2j} \tag{1.50}$$

for every $0 \leq r < 1$, because of the orthonormality of the functions $z^j$ on $T$ with respect to the usual integral inner product. This gives another way to look at (1.48) when $f \in L^2(T)$, since

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 r^{2j} \leq \sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 \tag{1.51}$$

for each $r < 1$. Using orthonormality of the $z^j$'s again, we get that

$$\frac{1}{2\pi} \int_T |f(z) - A_r(f)(z)|^2 |dz| = \sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 (1 - r)^2 \tag{1.52}$$

for every $f \in L^2(T)$ and $0 \leq r < 1$, which gives another way to look at the convergence (1.49) with respect to the $L^2$ norm.

If $f \in L^\infty(T)$, then (1.49) may not hold with respect to the $L^\infty$ norm on $T$, because $f$ may not be equal to a continuous function almost everywhere on $T$. Of course, $L^\infty(T) \subseteq L^p(T)$ for every $p < \infty$, and so (1.49) holds with respect to the $L^p$ norm when $1 \leq p < \infty$, as before. Similarly, (1.38) still holds for almost every $z \in T$ with respect to arc-length measure. Remember that $L^\infty(T)$ can be identified with the dual of $L^1(T)$ in the usual way. Let us check that (1.49) also holds with respect to the corresponding weak* topology on $L^\infty(T)$.

More precisely, this means that

$$\lim_{r \to 1-} \frac{1}{2\pi} \int_T A_r(f)(z) \phi(z) |dz| = \frac{1}{2\pi} \int_T f(z) \phi(z) |dz| \tag{1.53}$$

for every $\phi \in L^1(T)$. If $\phi \in L^\infty(T)$, then (1.53) follows from the fact that (1.49) holds with respect to the $L^1$ norm on $T$, as before. One can then use this to show that (1.53) holds for every $\phi \in L^1(T)$, by approximating $\phi$ by bounded measurable functions on $T$ with respect to the $L^1$ norm, and estimating the errors with (1.42). One could just as well start with $\phi \in L^q(T)$ for any $q > 1$, and then use (1.49) with respect to the $L^p$ norm on $T$, where $p < \infty$ is conjugate to $q$.

Alternatively, one can verify that

$$\frac{1}{2\pi} \int_T A_r(f)(z) \phi(z) |dz| = \frac{1}{2\pi} \int_T f(w) A_r(\phi)(w) |dw| \tag{1.54}$$

for every $\phi \in L^1(T)$ and $0 \leq r < 1$, using Fubini’s theorem. If one starts with the original definition (1.23) of $A_r(f)(z)$, then one should also interchange the order of summation and integration, and use the change of variables $j \mapsto -j$ in

$$\int_T \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijz} |dz| = \sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2$$

for every $f \in L^2(T)$.
the sum. Otherwise, one can use the expression (1.24) for $A_r(f)(z)$, and observe that $p_r(w, z)$ is symmetric in $w$ and $z$. Because $A_r(\phi) \to \phi$ as $r \to 1$ with respect to the $L^1$ norm on $T$ and $f \in L^\infty(T)$, we have that

$$
\lim_{r \to 1^-} \frac{1}{2\pi} \int_T f(w) A_r(\phi)(w) \, |dw| = \frac{1}{2\pi} \int_T f(w) \phi(w) \, |dw|.
$$

(1.55)

Combining this with (1.54), we get (1.53), as desired.

### 1.6 Harmonic functions

Let $f$ be a complex-valued integrable function on $T$ again, and let

$$
D = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}
$$

(1.56)

be the open unit disk in the complex plane. Thus the closure $\overline{D}$ of $D$ in $\mathbb{C}$ is the closed unit disk in $\mathbb{C}$,

$$
\overline{D} = \{ \zeta \in \mathbb{C} : |\zeta| \leq 1 \},
$$

(1.57)

which is the same as the union of $D$ and the unit circle $T$. Put

$$
h(\zeta) = \sum_{j=0}^{\infty} \hat{f}(j) \zeta^j + \sum_{j=1}^{\infty} \hat{f}(-j) \overline{\zeta}^j
$$

(1.58)

for each $\zeta \in D$, where the series converge absolutely for each $\zeta \in D$, because the Fourier coefficients of $f$ are bounded, as in (1.2). Similarly, if $0 \leq t < 1$, then the partial sums of these series also converge uniformly on the set of $\zeta \in \mathbb{C}$ such that $|\zeta| \leq t$, by the Weierstrass $M$-test. In particular, this implies that $h$ is continuous on $D$.

More precisely, let us put

$$
h_+(\zeta) = \sum_{j=0}^{\infty} \hat{f}(j) \zeta^j
$$

(1.59)

and

$$
h_-(\zeta) = \sum_{j=1}^{\infty} \hat{f}(-j) \overline{\zeta}^j
$$

(1.60)

for every $\zeta \in D$. As before, both of these series converge absolutely for each $\zeta \in D$, and their partial sums converge uniformly on compact subsets of $D$, because the Fourier coefficients of $f$ are bounded, as in (1.2). Of course, $h_+(\zeta)$ is a holomorphic function of $\zeta$ on $D$, and $h_-(\zeta)$ is the complex-conjugate of a holomorphic function of $\zeta$ on $D$. This implies that $h_+(\zeta)$ and $h_-(\zeta)$ are both harmonic functions of $\zeta$ on $D$, so that

$$
h(\zeta) = h_+(\zeta) + h_-(\zeta)
$$

(1.61)
is a harmonic function of $\zeta$ on $D$ as well.

Observe that
\[ h(rz) = A_r(f)(z) \]
for every $z \in \mathbb{T}$ and $0 \leq r < 1$, where $A_r(f)(z)$ is as in Section 1.3. Suppose for the moment that $f$ is a continuous function on $\mathbb{T}$, and consider the function on $\overline{D}$ equal to $h$ on $D$ and to $f$ on $\mathbb{T}$. One can check that this function is continuous on $\overline{D}$, for essentially the same reasons as in Section 1.4. We already know that $h$ is continuous on $D$ for any integrable function $f$ on $\mathbb{T}$, and so the main point is to show that this extension of $h$ to $D$ is continuous at each point in $\mathbb{T}$. Because $f$ is supposed to be continuous on $\mathbb{T}$, this means that for each $z \in \mathbb{T}$,
\[ h(\zeta) \to f(z) \]
as $\zeta \in D$ approaches $z$, which is analogous to (1.38).

It is well known that a continuous function $g$ on $\overline{D}$ that is harmonic on $D$ is uniquely determined by its restriction to $\mathbb{T}$, because of the maximum principle. If $f$ denotes the restriction of $g$ to $\mathbb{T}$, and if $h$ is defined on $D$ as in (1.58), then it follows that $g = h$ on $D$. If $g$ is any harmonic function on $D$, then similar arguments can be applied to
\[ g_t(\zeta) = g(t\zeta) \]
for each $0 \leq t < 1$. More precisely, if $f_t(z)$ is the restriction of $g_t$ to $\mathbb{T}$, and if $h_t$ is defined on $D$ as in (1.58) with $f$ replaced by $f_t$, then $g_t = h_t$ on $D$. If $f_t(z)$ converges as $t \to 1^-$ to an integrable function $f(z)$ on $\mathbb{T}$ with respect to the $L^1$ norm, and if $h$ is defined on $D$ as in (1.58) again, then it is easy to see that $h_t \to h$ as $t \to 1^-$ pointwise on $D$, and hence that $g = h$ on $D$.

### 1.7 Complex Borel measures

Let $\mu$ be a complex Borel measure on the unit circle $\mathbb{T}$, and let $|\mu|$ be the corresponding total variation measure on $\mathbb{T}$. Thus $|\mu|$ is a finite nonnegative Borel measure on $\mathbb{T}$ such that
\[ |\mu(E)| \leq |\mu|(E) \]
for every Borel set $E \subseteq \mathbb{T}$, and in fact $|\mu|$ is the smallest nonnegative Borel measure on $\mathbb{T}$ that satisfies (1.65). If $a(z)$ is a bounded complex-valued Borel measurable function on $\mathbb{T}$, then the integral of $a(z)$ with respect to $\mu$ on $\mathbb{T}$ can be defined in a standard way, and satisfies
\[ \left| \int_{\mathbb{T}} a(z) \, d\mu(z) \right| \leq \int_{\mathbb{T}} |a(z)| \, d|\mu|(z). \]
Of course, continuous functions on $\mathbb{T}$ are automatically Borel measurable, and so the $j$th Fourier coefficient of $\mu$ may be defined for each integer $j$ by
\[ \hat{\mu}(j) = \int_{\mathbb{T}} z^j \, d\mu(z). \]
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It follows that
\[(1.68) \quad |\hat{\mu}(j)| \leq |\mu|(T)\]
for each \(j \in \mathbb{Z}\), by taking \(a(z) = \zeta^j\) in (1.66).

This is compatible with the earlier definition for integrable functions, in the following sense. If \(f \in L^1(T)\), then it is well known that
\[(1.69) \quad \mu(E) = \frac{1}{2\pi} \int_E f(z) \, |dz|\]
defines a Borel measure on \(T\), and that
\[(1.70) \quad |\mu|(E) = \frac{1}{2\pi} \int_E |f(z)| \, |dz|\]
for every Borel set \(E \subseteq T\). Moreover,
\[(1.71) \quad \int_T a(z) \, d\mu(z) = \frac{1}{2\pi} \int_T a(z) \, f(z) \, |dz|\]
for every bounded complex-valued Borel measurable function \(a(z)\) on \(T\), by approximating \(a(z)\) uniformly on \(T\) by Borel measurable simple functions on \(T\). This implies that
\[(1.72) \quad \hat{\mu}(j) = \hat{f}(j)\]
for every \(j \in \mathbb{Z}\) in this case, by taking \(a(z) = z^j\) in (1.71).

Let \(\mu\) be an arbitrary complex Borel measure on \(T\) again, and put
\[(1.73) \quad A_r(\mu)(z) = \sum_{j=-\infty}^{\infty} \hat{\mu}(j) r^{|j|} z^j\]
for \(z \in T\) and \(0 \leq r < 1\), as in Section 1.3. As before, this series converges absolutely for every \(z \in T\) and \(0 \leq r < 1\), because the Fourier coefficients of \(\mu\) are bounded, as in (1.68). The partial sums
\[(1.74) \quad \sum_{j=-N}^{N} \hat{\mu}(j) r^{|j|} z^j\]
also converge to \(A_r(\mu)(z)\) as \(N \to \infty\) uniformly over \(z \in T\) for any fixed \(r < 1\), by the Weierstrass \(M\)-test, which implies that \(A_r(\mu)(z)\) is a continuous function of \(z\) on \(T\) for each \(r < 1\). Similarly,
\[(1.75) \quad h(\zeta) = \sum_{j=0}^{\infty} \hat{\mu}(j) \zeta^j + \sum_{j=1}^{\infty} \hat{\mu}(-j) \overline{\zeta}^j\]
defines a harmonic function on the open unit disk \(D\), as in the previous section. We also have that
\[(1.76) \quad h(r \, z) = A_r(\mu)(z)\]
for every \( z \in T \) and \( 0 \leq r < 1 \), as in (1.62).

In analogy with Section 1.3,

\[
A_r(\mu)(z) = \int_T p_r(w, z) \, d\mu(w)
\]

for every \( z \in T \) and \( 0 \leq r < 1 \), and hence

\[
|A_r(\mu)(z)| \leq \int_T p_r(w, z) \, d|\mu|(w)
\]

for every \( z \in T \) and \( 0 \leq r < 1 \). Integrating over \( z \) and interchanging the order of integration, we get that

\[
\frac{1}{2\pi} \int_T |A_r(\mu)(z)| \, |dz| \leq \frac{1}{2\pi} \int_T \int_T p_r(w, z) \, d|\mu|(w) \, |dz|
\]

\[
= \frac{1}{2\pi} \int_T \int_T p_r(w, z) \, |dz| \, d|\mu|(w)
\]

for every \( 0 \leq r < 1 \). It follows that

\[
\frac{1}{2\pi} \int_T |A_r(\mu)(z)| \, |dz| \leq |\mu|(T),
\]

because of (1.47). This corresponds exactly to (1.48) with \( p = 1 \) when \( \mu \) is given by an integrable function \( f \) on \( T \) as in (1.69).

If \( \phi \) is a continuous complex-valued function on the unit circle, then

\[
\frac{1}{2\pi} \int_T A_r(\mu)(z) \, \phi(z) \, |dz| = \sum_{j=-\infty}^{\infty} \frac{\hat{\mu}(j) \, r^{|j|}}{2\pi} \int_T \phi(z) \, z^j \, |dz|
\]

\[
= \sum_{j=-\infty}^{\infty} \hat{\mu}(j) \, \hat{\phi}(-j) \, r^{|j|}
\]

for every \( 0 \leq r < 1 \). This follows by plugging the definition (1.73) of \( A_r(\mu)(z) \) into the integral on the left side, and then interchanging the order of summation and integration, which is possible because of the uniform convergence of the partial sums (1.74). Equivalently,

\[
\frac{1}{2\pi} \int_T A_r(\mu)(z) \, \phi(z) \, |dz| = \sum_{j=-\infty}^{\infty} \hat{\phi}(j) \, r^{|j|} \, \hat{\mu}(-j),
\]

for every \( 0 \leq r < 1 \), using the change of variables \( j \mapsto -j \). Plugging the definition (1.67) of \( \hat{\mu}(-j) \) into the sum on the right side, we get that

\[
\sum_{j=-\infty}^{\infty} \hat{\phi}(j) \, r^{|j|} \, \hat{\mu}(-j) = \sum_{j=-\infty}^{\infty} \int_T \hat{\phi}(j) \, r^{|j|} \, z^j \, d\mu(z)
\]

\[
= \int_T A_r(\phi)(z) \, d\mu(z)
\]
for every \(0 \leq r < 1\). Here \(A_r(\phi)(z)\) is defined in the same way as before, and we can interchange the order of summation and integration in the second step for the usual reasons of uniform convergence.

This shows that
\[
\frac{1}{2\pi} \int_\mathbf{T} A_r(\mu)(z) \phi(z) \, |dz| = \int_\mathbf{T} A_r(\phi)(z) \, d\mu(z)
\]
for every continuous function \(\phi\) on \(\mathbf{T}\) and \(0 \leq r < 1\). One could also get this using (1.77), the analogous expression for \(A_r(\phi)\), and the fact that \(p_r(w,z)\) is symmetric in \(w\) and \(z\). Note that
\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_\mathbf{T} A_r(\mu)(z) \phi(z) \, |dz| = \int_\mathbf{T} \phi(z) \, d\mu(z)
\]
for every continuous function \(\phi\) on \(\mathbf{T}\), because \(A_r(\phi)\) converges to \(\phi\) as \(r \to 1^-\) uniformly on \(\mathbf{T}\), as in Section 1.4. It follows that
\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_\mathbf{T} A_r(\mu)(z) \phi(z) \, |dz| = \int_\mathbf{T} \phi(z) \, d\mu(z)
\]
for every continuous function \(\phi\) on \(\mathbf{T}\).

Of course,
\[
\lambda(\phi) = \int_\mathbf{T} \phi(z) \, d\mu(z)
\]
defines a linear functional on the vector space \(C(\mathbf{T})\) of continuous complex-valued functions \(\phi\) on \(\mathbf{T}\). More precisely, this is a bounded linear functional on \(C(\mathbf{T})\) with respect to the supremum norm, because
\[
|\lambda(\phi)| \leq |\mu|(\mathbf{T}) \sup_{z \in \mathbf{T}} |\phi(z)|
\]
for every \(\phi \in C(\mathbf{T})\), by (1.66). A version of the Riesz representation theorem implies that every bounded linear functional on \(C(\mathbf{T})\) corresponds to a unique Borel measure on \(\mathbf{T}\) in this way.

Let \(\mu_r\) be the Borel measure on \(\mathbf{T}\) defined by
\[
\mu_r(E) = \frac{1}{2\pi} \int_E A_r(\mu) \, |dz|
\]
for every Borel set \(E \subseteq \mathbf{T}\), and for each real number \(r\) with \(0 \leq r < 1\). Thus
\[
\lambda_r(\phi) = \int_\mathbf{T} \phi(z) \, d\mu_r(z) = \frac{1}{2\pi} \int_\mathbf{T} A_r(\mu)(z) \phi(z) \, |dz|
\]
is the bounded linear functional on \(C(\mathbf{T})\) that corresponds to \(\mu_r\) as in the preceding paragraph. In this notation, (1.86) says that
\[
\lim_{r \to 1^-} \lambda_r(\phi) = \lambda(\phi)
\]
for every \(\phi \in C(\mathbf{T})\). This is the same as saying that \(\lambda_r \to \lambda\) as \(r \to 1^-\) with respect to the weak* topology on the space of bounded linear functionals on \(C(\mathbf{T})\).
1.8 Holomorphic functions

Let $f$ be an integrable function on the unit circle again, and let $h$, $h_+$, and $h_-$ be defined on the open unit disk as in Section 1.6. If

\[ \hat{f}(-j) = 0 \]  

for each positive integer $j$, then $h_-(\zeta) = 0$ for every $\zeta \in D$, and hence

\[ h(\zeta) = h_+(\zeta) \]

is a holomorphic function of $\zeta$ on $D$. In this case,

\[ A_r(f)(z) = \sum_{j=0}^{\infty} \hat{f}(j) r^j z^j \]

for each $z \in T$ and $0 < r < 1$, and the partial sums

\[ \sum_{j=0}^{N} \hat{f}(j) r^j z^j \]

converge to (1.94) as $N \to \infty$ uniformly over $z \in T$ for any fixed $r < 1$. If $f$ is continuous on $T$, then we have seen that $A_r(f)$ converges to $f$ uniformly on $T$ as $r \to 1-$, which implies that $f$ can be approximated uniformly by finite linear combinations of $z^j$'s with $j \geq 0$. Similarly, if $f \in L^p(T)$ for some $p$, $1 \leq p < \infty$, then $A_r(f)$ converges to $f$ with respect to the $L^p$ norm on $T$ as $r \to 1-$, which implies that $f$ can be approximated by finite linear combinations of the $z^j$'s with $j \geq 0$ with respect to the $L^p$ norm on $T$.

If $g$ is any holomorphic function on $D$, then

\[ \int_T g(tz) z^j dz = 0 \]

for every $t \in \mathbb{R}$ with $0 \leq t < 1$ and nonnegative integer $j$, by Cauchy’s theorem. The complex line integral element $dz$ is equal to $iz |dz|$ on the unit circle, so that (1.96) is equivalent to

\[ \int_T g(tz) z^{j+1} |dz| = 0 \]

for each nonnegative integer $j$. If we put

\[ f_t(z) = g(tz) \]

for each $z \in T$ and $0 \leq t < 1$, then (1.97) is the same as saying that

\[ \hat{f}_t(-j - 1) = 0 \]
for each nonnegative integer \( j \). If \( f_t(z) \) converges as \( t \to 1^- \) to an integrable function \( f(z) \) on \( T \) with respect to the \( L^1 \) norm, then it follows that

\[
\hat{f}(-j - 1) = 0
\]

for every nonnegative integer \( j \). Under these conditions, the Cauchy integral formula can be used to express \( g(\zeta) \) in terms of an integral of \( f \) on \( T \) for each \( \zeta \in D \), which amounts to the identification of \( g(\zeta) \) with (1.93).

Now let \( \mu \) be a complex Borel measure on \( T \), and consider the corresponding harmonic function \( h(\zeta) \) on \( D \), as in (1.75). If \( \hat{\mu}(-j) = 0 \) for every positive integer \( j \), then \( h(\zeta) \) is holomorphic on \( D \), as before. In this case, a famous theorem of F. and M. Riesz implies that \( \mu \) is absolutely continuous with respect to arc-length measure on \( T \), so that \( \mu \) can be represented as in (1.69) for some \( f \in L^1(T) \).

### 1.9 Products of functions

Remember that \( \mathcal{E}(T) \) is the space of continuous functions on \( T \) that can be expressed as finite linear combinations of the functions \( z^j \) with \( j \in \mathbb{Z} \). If \( f, g \) are elements of \( \mathcal{E}(T) \), then

\[
f(z) = \sum_{j=-\infty}^{\infty} \hat{f}(j) z^j, \quad g(z) = \sum_{k=-\infty}^{\infty} \hat{g}(k) z^k
\]

for every \( z \in T \), where all but finitely many terms in these sums are equal to 0. This implies that

\[
f(z) g(z) = \left( \sum_{j=-\infty}^{\infty} \hat{f}(j) z^j \right) \left( \sum_{k=-\infty}^{\infty} \hat{g}(k) z^k \right)
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{f}(j) \hat{g}(k) z^{j+k}
\]

\[
= \sum_{l=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{f}(l-k) \hat{g}(k) z^l
\]

for each \( z \in T \), using a change of variables in the third step. It follows that

\[
(\hat{f} g)(l) = \sum_{k=-\infty}^{\infty} \hat{f}(l-k) \hat{g}(k)
\]

for each \( l \in \mathbb{Z} \) in this case.
1.9. PRODUCTS OF FUNCTIONS

Observe that

\[
\begin{align*}
\sum_{k=-\infty}^{\infty} |\hat{f}(l-k)| \hat{g}(k)| & \leq \left( \sum_{k=-\infty}^{\infty} |\hat{f}(l-k)|^2 \right)^{1/2} \left( \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2 \right)^{1/2} \\
& = \left( \sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 \right)^{1/2} \left( \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2 \right)^{1/2}
\end{align*}
\]

for each \( l \in \mathbb{Z} \), by the Cauchy–Schwarz inequality. This implies that

\[
\sum_{k=-\infty}^{\infty} |\hat{f}(l-k)| \hat{g}(k)| \leq \left( \frac{1}{2\pi} \int_{T} |f(z)|^2 \, |dz| \right)^{1/2} \left( \frac{1}{2\pi} \int_{T} |g(w)|^2 \, |dw| \right)^{1/2}
\]

for every \( l \in \mathbb{Z} \), by (1.7) in Section 1.1. More precisely, this works for all \( f, g \in L^2(T) \), so that the sum on the left side of (1.105) converges for every \( f, g \in L^2(T) \).

If \( f, g \in L^2(T) \), then their product \( fg \) is an integrable function on \( T \), and hence the corresponding Fourier coefficient

\[
\hat{(fg)}(l) = \frac{1}{2\pi} \int_{T} f(z) g(z) \overline{\tau^l} \, |dz|
\]

is defined for every \( l \in \mathbb{Z} \). If \( g \in \mathcal{E}(T) \), then it is easy to get (1.103) from (1.106), where all but finitely many terms in the sum on the right side of (1.103) are equal to 0. Using this, one can check that (1.103) holds for every \( f, g \in L^2(T) \) and \( l \in \mathbb{Z} \), by approximating \( g \) by elements of \( \mathcal{E}(T) \) with respect to the \( L^2 \) norm. This also uses (1.105), which implies in particular that the sum on the right side of (1.103) converges absolutely for every \( f, g \in L^2(T) \) and \( l \in \mathbb{Z} \).

Suppose now that the Fourier coefficients of \( g \in L^1(T) \) are absolutely summable, which is to say that

\[
\sum_{k=-\infty}^{\infty} |\hat{g}(k)|
\]

converges. This implies that the corresponding Fourier series

\[
\sum_{k=-\infty}^{\infty} \hat{g}(k) \, z^k
\]

converges absolutely for each \( z \in T \), and that the partial sums

\[
\sum_{k=-N}^{N} \hat{g}(k) \, z^k
\]

converge to (1.108) as \( N \to \infty \) uniformly on \( T \). In particular, (1.108) defines a continuous function on \( T \). As in Section 1.2, the Abel sums associated to (1.108)
converge to the ordinary sum for each \( z \in \mathbf{T} \), which can also be verified more directly in the case of absolute convergence. Similar arguments show that the Abel sums converge uniformly on \( \mathbf{T} \) in this situation, which is again simplified by absolute convergence. Of course, the Abel sums associated to (1.108) should also converge to \( g \) with respect to the \( L^1 \) norm, as in Section 1.5. This implies that \( g \) is equal to (1.108) almost everywhere on \( \mathbf{T} \), so that we may as well take \( g \) to be equal to (1.108) everywhere on \( \mathbf{T} \).

Equivalently, one might start with a function \( g \) on \( \mathbf{T} \) given by

\[
g(z) = \sum_{k=-\infty}^{\infty} b_k z^k
\]

for some complex numbers \( b_k \) such that

\[
\sum_{k=-\infty}^{\infty} |b_k|
\]

converges. This ensures that the sum in (1.110) converges absolutely for every \( z \in \mathbf{T} \), and that the corresponding partial sums

\[
\sum_{k=-N}^{N} b_k z^k
\]

converge to \( g \) as \( N \to \infty \) uniformly on \( \mathbf{T} \). It follows that \( g \) is a continuous function on \( \mathbf{T} \), and that

\[
\hat{g}(k) = b_k
\]

for every \( k \in \mathbf{Z} \).

If \( f \in L^1(\mathbf{T}) \), then the Fourier coefficients of \( f \) are uniformly bounded, as in (1.2) in Section 1.1. This implies that

\[
\sum_{k=-\infty}^{\infty} \left| \hat{f}(l-k) \right| \left| \hat{g}(k) \right| \leq \left( \sup_{j \in \mathbf{Z}} \left| \hat{f}(j) \right| \right) \left( \sum_{k=-\infty}^{\infty} \left| \hat{g}(k) \right| \right)
\]

\[
\leq \left( \frac{1}{2\pi} \int_{\mathbf{T}} |f(z)| |dz| \right) \left( \sum_{k=-\infty}^{\infty} \left| \hat{g}(k) \right| \right)
\]

for each \( l \in \mathbf{Z} \). If \( g \in \mathcal{E}(\mathbf{T}) \), then we can still get (1.103) from (1.106), as before, where all but finitely many terms on the right side of (1.103) are equal to 0. The same conclusion also holds when the Fourier coefficients of \( g \) are absolutely summable, by a simple approximation argument, and using (1.114) to deal with the right side of (1.103). There are analogous statements when \( f \) is replaced by a complex Borel measure on \( \mathbf{T} \).

Suppose that both

\[
\sum_{j=-\infty}^{\infty} |\hat{f}(j)|
\]
and (1.107) converge, and let us check that \( fg \) has the analogous property. Using (1.103), we get that

\[
|\hat{(fg)}(l)| \leq \sum_{k=-\infty}^{\infty} |\hat{f}(l-k)||\hat{g}(k)|
\]

for each \( l \in \mathbb{Z} \), so that

\[
\sum_{l=-\infty}^{\infty} |(\hat{fg})(l)| \leq \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\hat{f}(l-k)||\hat{g}(k)|.
\]

Interchanging the order of summation on the right side, and then making a change of variables, we get that

\[
\sum_{l=-\infty}^{\infty} |(\hat{fg})(l)| \leq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\hat{f}(l-k)||\hat{g}(k)|
= \left( \sum_{j=-\infty}^{\infty} |\hat{f}(j)| \right) \left( \sum_{k=-\infty}^{\infty} |\hat{g}(k)| \right),
\]

as desired. Alternatively, one can start with \( f \) given on \( T \) by

\[
f(z) = \sum_{j=-\infty}^{\infty} a_j z^j
\]

for some complex numbers \( a_j \) such that

\[
\sum_{j=-\infty}^{\infty} |a_j|
\]

converges, and similarly for \( g \), as in (1.110) and (1.111). One can then multiply the expansions for \( f \) and \( g \) directly, as in (1.102), and estimate the coefficients in the corresponding expansion for \( fg \) as before.

If \( \hat{f}(-j) = \hat{g}(-j) = 0 \) for each positive integer \( j \), then (1.103) reduces to

\[
(\hat{fg})(l) = \sum_{k=0}^{l} \hat{f}(l-k) \hat{g}(k)
\]

when \( l \geq 0 \), and to \( (\hat{fg})(l) = 0 \) when \( l < 0 \). This holds when \( f \in L^p(T) \) and \( g \in L^q(T) \) for some \( 1 \leq p, q \leq \infty \) which are conjugate exponents, in the sense that \( 1/p + 1/q = 1 \), so that \( fg \) is an integrable function on \( T \). More precisely, if \( f \) or \( g \) is in \( \mathcal{E}(\mathbb{T}) \), then (1.103) follows from (1.106), which implies (1.121) in
CHAPTER 1. THE UNIT CIRCLE

this case. If \( 1 \leq q < \infty \), then we can get the same conclusion when \( f \in L^p(T) \) and \( g \in L^q(T) \) by approximating \( g \) by finite linear combinations of \( z^k \)'s with \( k \geq 0 \) with respect to the \( L^q \) norm, as in the previous section. Otherwise, if \( q = \infty \), then \( p = 1 \), and one can approximate \( f \) by finite linear combinations of \( z^j \)'s with \( j \geq 0 \) with respect to the \( L^1 \) norm. In this situation, the Fourier series for \( fg \) may be considered as the Cauchy product of the Fourier series for \( f \) and \( g \). This implies that

\[
A_r(fg)(z) = A_r(f)(z) A_r(g)(z) \tag{1.122}
\]

for every \( z \in T \) and \( 0 \leq r < 1 \), as in Section 1.2. Similarly, the power series

\[
\sum_{l=0}^{\infty} (fg)(l) \zeta^l \tag{1.123}
\]

corresponding to \( fg \) as in the previous section is the same as the Cauchy product of the power series

\[
\sum_{j=0}^{\infty} \hat{f}(j) \zeta^j, \quad \sum_{k=0}^{\infty} \hat{g}(k) \zeta^k \tag{1.124}
\]

corresponding to \( f \) and \( g \). Thus the holomorphic function on \( D \) associated to \( fg \) is the same as the product of the holomorphic functions on \( D \) associated to \( f \) and \( g \) under these conditions.

### 1.10 Representation theorems

Let \( g(\zeta) \) be a complex-valued harmonic function on the open unit disk \( D \), and put \( g_t(\zeta) = g(t \zeta) \) for \( 0 \leq t < 1 \). Also put

\[
f_t(z) = g_t(z) \tag{1.125}
\]

when \( z \in T \), and let \( h_t \) be the harmonic function on \( D \) corresponding to \( f_t \), as in Section 1.6. Thus

\[
g_t(\zeta) = h_t(\zeta) \tag{1.126}
\]

for every \( \zeta \in D \), by the maximum principle, as before. Suppose now that the \( L^p \) norm of \( f_t \) on \( T \) is uniformly bounded in \( t \) for some \( p, 1 < p \leq \infty \). Let \( 1 \leq q < \infty \) be the exponent conjugate to \( p \), so that \( 1/p + 1/q = 1 \), and remember that \( L^p(T) \) can be identified with the dual of \( L^q(T) \) in the usual way. The Banach–Alaoglu theorem implies that the closed unit ball in \( L^p(T) \) is compact with respect to the corresponding weak* topology on \( L^p(T) \). The separability of \( L^q(T) \) implies that the topology induced on the closed unit ball in \( L^p(T) \) by the weak* topology is metrizable, and hence that the closed unit ball in \( L^p(T) \) is sequentially compact. Similarly, every closed ball in \( L^p(T) \) is sequentially compact with respect to the weak* topology. It follows that there is a sequence \( \{t_j\}_{j=1}^{\infty} \) of nonnegative real numbers less than 1 that converges to
1 such that \( \{f_{t_j}\}_{j=1}^\infty \) converges to some \( f \in L^p(\mathbf{T}) \) with respect to the weak* topology under these conditions.

If \( h(\zeta) \) is the harmonic function on \( D \) that corresponds to \( f \) as before, then

\[
\lim_{j \to \infty} h_{t_j}(\zeta) = h(\zeta)
\]

for every \( \zeta \in D \). This uses the fact that \( h_{t_j}(\zeta) \) is equal to the integral of \( f(z) \) times a continuous function on \( \mathbf{T} \) that depends only on \( \zeta \), so that (1.127) corresponds exactly to the convergence of \( \{f_{t_j}\}_{j=1}^\infty \) to \( f \) with respect to the weak* topology on \( L^p(\mathbf{T}) \). By construction,

\[
(1.128) \quad h_{t_j}(\zeta) = g(t_j \zeta) \to g(\zeta) \quad \text{as} \quad j \to \infty
\]

for each \( \zeta \in D \), so that

\[
(1.129) \quad g(\zeta) = h(\zeta)
\]

for every \( \zeta \in D \). If \( 1 < p < \infty \), then we may conclude that \( f_t \to f \) as \( t \to 1^- \) with respect to the \( L^p \) norm on \( \mathbf{T} \), as in Section 1.5. Some variants of this for \( p = \infty \) were also discussed in Section 1.5.

In the \( p = 1 \) case, it is better to consider the Borel measure

\[
(1.130) \quad \mu_t(E) = \frac{1}{2\pi} \int_E f_t(z) |dz|
\]

on \( \mathbf{T} \) corresponding to \( f_t \) for each \( t \), and the associated bounded linear functional

\[
(1.131) \quad \lambda_t(\phi) = \int_\mathbf{T} \phi(z) d\mu_t(z) = \frac{1}{2\pi} \int_\mathbf{T} \phi(z) f_t(z) |dz|
\]

on \( C(\mathbf{T}) \). The hypothesis that the \( L^1 \) norm of \( f_t \) be uniformly bounded in \( t \) says exactly that the total variation norm of \( \mu_t \) is uniformly bounded, and hence that the dual norm of \( \lambda_t \) with respect to the supremum norm on \( C(\mathbf{T}) \) is uniformly bounded. The Banach–Alaoglu theorem implies that closed balls in the dual of \( C(\mathbf{T}) \) are compact with respect to the weak* topology again, and in fact they are sequentially compact, because \( C(\mathbf{T}) \) is separable. As before, it follows that there is a sequence \( \{t_j\}_{j=1}^\infty \) of real numbers less than 1 that converges to 1 for which the corresponding sequence \( \{\lambda_{t_j}\}_{j=1}^\infty \) converges to a bounded linear functional \( \lambda \) on \( C(\mathbf{T}) \) with respect to the weak* topology on the dual of \( C(\mathbf{T}) \). Remember that there is a complex Borel measure \( \mu \) on \( \mathbf{T} \) corresponding to \( \lambda \) as in (1.87), by a version of the Riesz representation theorem. If \( h(\zeta) \) is the harmonic function on \( D \) that corresponds to \( \mu \) as in Section 1.7, then (1.127) holds for every \( \zeta \in D \) again, because \( \lambda_{t_j} \to \lambda \) as \( j \to \infty \) with respect to the weak* topology on the dual of \( C(\mathbf{T}) \). We still have (1.128) for each \( \zeta \in D \), by construction, and hence that (1.129) holds for every \( \zeta \in D \). It follows that \( \lambda_t \to \lambda \) as \( t \to 1^- \) with respect to the weak* topology on the dual of \( C(\mathbf{T}) \), as in (1.91).

If \( g(\zeta) \) is a holomorphic function on \( D \) such that the \( L^1 \) norm of \( f_t \) on \( \mathbf{T} \) is uniformly bounded in \( t \), then a famous theorem states that \( f_t \) converges to some \( f \in L^1(\mathbf{T}) \) with respect to the \( L^1 \) norm as \( t \to 1^- \).
Chapter 2

The real line

2.1 Fourier transforms

The Fourier transform of an integrable function $f$ on the real line is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-ix\xi) \, dx,$$

for every $\xi \in \mathbb{R}$, where $dx$ refers to ordinary Lebesgue measure on $\mathbb{R}$. Similarly, the Fourier transform of a complex Borel measure $\mu$ on $\mathbb{R}$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi) \, d\mu(x)$$

for every $\xi \in \mathbb{R}$. If

$$\mu(E) = \int_{E} f(x) \, dx$$

for some $f \in L^1(\mathbb{R})$ and every Borel set $E \subseteq \mathbb{R}$, then (2.2) is the same as (2.1). In the first case (2.1), we have that

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)| \, dx$$

for each $\xi \in \mathbb{R}$, since $|\exp(it)| = 1$ for every $t \in \mathbb{R}$. In the second case (2.2), we have that

$$|\hat{\mu}(\xi)| \leq |\mu|(\mathbb{R}),$$

where $|\mu|$ is the total variation measure on $\mathbb{R}$ associated to $\mu$.

Let $\mu$ be a complex Borel measure on $\mathbb{R}$, and consider

$$\hat{\mu}_N(\xi) = \int_{[-N,N]} \exp(-ix\xi) \, d\mu(x)$$

for each $N \geq 0$ and $\xi \in \mathbb{R}$. Observe that

$$|\hat{\mu}_N(\xi) - \hat{\mu}(\xi)| \leq |\mu|(\{x \in \mathbb{R} : |x| > N\})$$

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for every \( N \geq 0 \) and \( \xi \in \mathbb{R} \), which implies that \( \hat{\mu}_N(\xi) \to \hat{\mu}(\xi) \) as \( N \to \infty \) uniformly on \( \mathbb{R} \). One can check that \( \hat{\mu}_N(\xi) \) is uniformly continuous on \( \mathbb{R} \) for each \( N \geq 0 \), using the fact that \( \exp(it) \) is uniformly continuous on \( \mathbb{R} \). This implies that \( \hat{\mu}(\xi) \) is also uniformly continuous on \( \mathbb{R} \), since uniform continuity is preserved by uniform convergence. In particular, \( \hat{f}(\xi) \) is uniformly continuous on \( \mathbb{R} \) for every \( f \in L^1(\mathbb{R}) \).

If \( f \) is the characteristic or indicator function associated to an interval in \( \mathbb{R} \), then it is easy to see that

\[
\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0, \tag{2.8}
\]

by direct computation. The same conclusion holds when \( f \) is a step function, which is to say a finite linear combination of characteristic functions of intervals, by linearity. It is well known that step functions are dense in \( L^1(\mathbb{R}) \), which implies that (2.8) holds for every \( f \in L^1(\mathbb{R}) \), using (2.4) to estimate the errors.

Now let \( a \in \mathbb{R} \) be given, and let \( \delta_a \) be the Dirac mass at \( a \), which is the Borel measure on \( \mathbb{R} \) defined by

\[
\delta_a(E) = 1 \text{ when } a \in E, \quad \delta_a(E) = 0 \text{ when } a \in \mathbb{R} \setminus E.
\]

In this case, it is easy to see that

\[
\hat{\delta}_a(\xi) = \exp(-ia \xi), \tag{2.9}
\]

for every \( \xi \in \mathbb{R} \), which obviously does not tend to 0 as \( |\xi| \to \infty \).

If \( f \in L^1(\mathbb{R}) \) and \( a \in \mathbb{R} \), then

\[
f_a(x) = f(x - a), \tag{2.10}
\]

is also an integrable function on \( \mathbb{R} \), and

\[
\begin{align*}
\hat{f}_a(\xi) &= \int_{\mathbb{R}} f(x - a) \exp(-ix \xi) \, dx \\
&= \int_{\mathbb{R}} f(x) \exp(-i(x + a) \xi) \, dx \\
&= \hat{f}(\xi) \exp(-ia \xi)
\end{align*} \tag{2.11}
\]

for every \( \xi \in \mathbb{R} \). Similarly,

\[
f^\alpha(x) = f(x) \exp(-ix \alpha) \tag{2.12}
\]

is an integrable function on \( \mathbb{R} \) for every \( \alpha \in \mathbb{R} \), and

\[
\begin{align*}
\hat{f}^\alpha(\xi) &= \int_{\mathbb{R}} f(x) \exp(-i\alpha x) \exp(-ix \xi) \, dx = \hat{f}(\xi + \alpha)
\end{align*} \tag{2.13}
\]

for every \( \alpha, \xi \in \mathbb{R} \). Of course, there are analogous statements for complex Borel measures on \( \mathbb{R} \) instead of integrable functions. Remember that

\[
\lim_{a \to 0} \int_{\mathbb{R}} |f(x) - f(x - a)| \, dx = 0 \tag{2.14}
\]

for every \( f \in L^1(\mathbb{R}) \), since this holds when \( f \) is a step function on \( \mathbb{R} \), and step functions are dense in \( L^1(\mathbb{R}) \). This implies that \( \hat{f}_a(\xi) \to \hat{f}(\xi) \) as \( a \to 0 \) uniformly on \( \mathbb{R} \), which can be used to give another proof of (2.8).
2.2 Holomorphic extensions

If $\mu$ is a complex Borel measure on $\mathbb{R}$ with compact support, then

\begin{equation}
\hat{\mu}(\zeta) = \int_{\mathbb{R}} \exp(-i x \zeta) \, d\mu(x)
\end{equation}

makes sense for every $\zeta \in \mathbb{C}$, and defines an entire holomorphic function on the complex plane. Similarly, (2.15) makes sense when $\mu$ is a complex Borel measure on $\mathbb{R}$ such that

\begin{equation}
\int_{\mathbb{R}} |\exp(-i x \zeta)| \, d|\mu|(x) = \int_{\mathbb{R}} \exp(x \ \text{Im} \ \zeta) \, d|\mu|(x)
\end{equation}

is finite, where $\text{Im} \ \zeta$ denotes the imaginary part of $\zeta$. In this case, the modulus of (2.15) is less than or equal to (2.16). Of course, (2.16) is finite when $\zeta$ is real, so that (2.15) reduces to the ordinary Fourier transform of $\mu$.

Now suppose that

\begin{equation}
|\mu|((0, +\infty)) = 0,
\end{equation}

which is the same as saying that $\mu \equiv 0$ on the set of positive real numbers. Let

\begin{equation}
U = \{\zeta \in \mathbb{C} : \text{Im} \ \zeta > 0\}
\end{equation}

be the open upper half-plane, whose closure

\begin{equation}
\overline{U} = \{\zeta \in \mathbb{C} : \text{Im} \ \zeta \geq 0\}
\end{equation}

in $\mathbb{C}$ is the closed upper half-plane. Under these conditions, (2.16) is equal to

\begin{equation}
\int_{(\infty, 0]} \exp(x \ \text{Im} \ \zeta) \, d|\mu|(x)
\end{equation}

for every $\zeta \in \mathbb{C}$, which is less than or equal to

\begin{equation}
|\mu|(\mathbb{R}) = |\mu|((\infty, 0])
\end{equation}

when $\zeta \in \overline{U}$. Thus (2.15) reduces to

\begin{equation}
\hat{\mu}(\zeta) = \int_{(\infty, 0]} \exp(-i x \zeta) \, d\mu(x)
\end{equation}

in this situation, which is defined for every $\zeta \in \overline{U}$, and satisfies

\begin{equation}
|\hat{\mu}(\zeta)| \leq |\mu|((\infty, 0])
\end{equation}

One can also check that $\hat{\mu}(\zeta)$ is uniformly continuous on $\overline{U}$, in essentially the same way as in the previous section, and that $\hat{\mu}(\zeta)$ is holomorphic on $U$. 

2.3. SOME EXAMPLES

If \( f \) is an integrable function on \( \mathbb{R} \) such that \( f(x) = 0 \) almost everywhere when \( x > 0 \), then the remarks in the previous paragraph can be applied to the measure \( \mu \) on \( \mathbb{R} \) that corresponds to \( f \) as in (2.3). In particular,

\[
\hat{f}(\zeta) = \int_{-\infty}^{0} f(x) \exp(-ix\zeta) \, dx
\]

is defined for every \( \zeta \in \mathbb{U} \), which corresponds to (2.22). If \( f \) is the characteristic function associated to a subinterval of \( (-\infty, 0] \), then one can check that

\[
\lim_{|\zeta| \to \infty} f(\zeta) = 0,
\]

by direct computation, and where the limit is taken over \( \zeta \in \mathbb{U} \). This is analogous to (2.8), and it also works when \( f \) is a step function supported in \( (-\infty, 0] \), by linearity. It follows that (2.25) holds for every \( f \in L^1(\mathbb{R}) \) supported in \( (-\infty, 0] \), by approximation by step functions.

Similarly, the Fourier transform of an integrable function or complex Borel measure on \( \mathbb{R} \) supported in \( [0, +\infty) \) can be extended to the closed lower half-plane, with the same type of properties as before. The Fourier transform of an integrable function or complex Borel measure on \( \mathbb{R} \) may also have a natural extension to a strip in the plane, a half-plane with boundary parallel to the real line, or to the whole complex plane, depending on the finiteness of (2.16), and without such conditions on the support.

### 2.3 Some examples

Let \( r \) be a positive real number, and put

\[
a_r(x) = \begin{cases} \exp(rx) & \text{when } x \leq 0 \\ 0 & \text{when } x > 0. \end{cases}
\]

This is an integrable function on \( \mathbb{R} \), whose Fourier transform

\[
\hat{a}_r(\zeta) = \int_{\mathbb{R}} a_r(x) \exp(-i x \zeta) \, dx = \int_{-\infty}^{0} \exp(rx - i x \zeta) \, dx
\]

may be defined for all \( \zeta \in \mathbb{C} \) such that \( \text{Im} \zeta > -r \), as in the previous section. In this case, we can integrate the exponential directly, to get that

\[
\hat{a}_r(\zeta) = \frac{1}{r - i \zeta}
\]

when \( \text{Im} \zeta > -r \).

Similarly, put

\[
b_r(x) = \begin{cases} 0 & \text{when } x \leq 0 \\ \exp(-r x) & \text{when } x > 0. \end{cases}
\]
This is also an integrable function on $\mathbb{R}$, whose Fourier transform

\[
\hat{b}_r(\zeta) = \int_{\mathbb{R}} b_r(x) \exp(-ix \zeta) \, dx = \int_{0}^{\infty} \exp(-rx - ix \zeta) \, dx
\]

may be defined for all $\zeta \in \mathbb{C}$ with $\text{Im} \, \zeta < r$, as before. The exponential can be integrated directly again, to get that

\[
\hat{b}_r(\zeta) = \frac{1}{r + i \zeta}
\]

when $\text{Im} \, \zeta < r$.

Now consider

\[
c_r(x) = a_r(x) + b_r(x) = \exp(-r |x|).
\]

This is an integrable function on $\mathbb{R}$, whose Fourier transform

\[
\hat{c}_r(\zeta) = \int_{\mathbb{R}} c_r(x) \exp(-ix \zeta) \, dx = \int_{-\infty}^{\infty} \exp(-r |x| - ix \zeta) \, dx
\]

may be defined for all $\zeta \in \mathbb{C}$ with $|\text{Im} \, \zeta| < r$. Combining (2.28) and (2.31), we get that

\[
\hat{c}_r(\zeta) = \hat{a}_r(\zeta) + \hat{b}_r(\zeta) = \frac{1}{r - i \zeta} + \frac{1}{r + i \zeta} = \frac{2r}{(r - i \zeta)(r + i \zeta)}
\]

when $|\text{Im} \, \zeta| < r$.

Let us restrict our attention to $\zeta = \xi \in \mathbb{R}$, so that (2.34) reduces to

\[
\hat{c}_r(\xi) = \frac{2r}{|r - i \xi|^2} = \frac{2r}{r^2 + \xi^2}.
\]

This is a nonnegative real-valued integrable function on $\mathbb{R}$ for each $r > 0$. Equivalently,

\[
\hat{c}_r(\xi) = \frac{1}{r - i \xi} + \frac{1}{r + i \xi} = 2 \text{Re} \frac{1}{r + i \xi}
\]

for each $\xi \in \mathbb{R}$. Using the principal branch of the logarithm, we have that

\[
\frac{d}{d \xi} \log(r + i \xi) = \frac{i}{r + i \xi},
\]

and hence that

\[
\int_{-N}^{N} \frac{1}{r + i \xi} \, d\xi = -i \log(r + i N) + i \log(r - i N)
\]

for each $N \geq 0$. It follows that

\[
\int_{-N}^{N} \hat{c}_r(\xi) \, d\xi = 2 \text{Re} \int_{-N}^{N} \frac{1}{r + i \xi} \, d\xi
\]

\[
= 2 \text{Im} \log(r + i N) - 2 \text{Im} \log(r - i N)
\]
2.4. THE MULTIPLICATION FORMULA

for each \( N \geq 0 \). Of course,

\[
\lim_{N \to \infty} \text{Im} \log(r + iN) = \frac{\pi}{2}
\]

(2.40)

and

\[
\lim_{N \to \infty} \text{Im} \log(r - iN) = -\frac{\pi}{2}.
\]

(2.41)

This shows that

\[
\int_{-\infty}^{\infty} \hat{c}_r(\xi) \, d\xi = 2\pi,
\]

(2.42)

by taking the limit as \( N \to \infty \) in (2.39).

2.4 The multiplication formula

If \( \mu \) and \( \nu \) are complex Borel measures on the real line, then the multiplication formula states that

\[
\int_{\mathbb{R}} \hat{\nu}(t) \, d\mu(t) = \int_{\mathbb{R}} \hat{\mu}(\xi) \, d\nu(\xi).
\]

(2.43)

This follows by plugging in the definition of the Fourier transform and applying Fubini’s theorem. Note that both sides of (2.43) make sense, because \( \hat{\mu} \) and \( \hat{\nu} \) are bounded and continuous on \( \mathbb{R} \). Similarly,

\[
\int_{\mathbb{R}} \hat{\nu}(t - w) \, d\mu(t) = \int_{\mathbb{R}} \hat{\mu}(\xi) \, \exp(iw\xi) \, d\nu(\xi)
\]

(2.44)

for every \( w \in \mathbb{R} \), which is the same as (2.43) with \( \nu \) replaced by the measure \( \nu_w \) on \( \mathbb{R} \) defined by

\[
\nu_w(E) = \int_E \exp(iwu) \, d\nu(u).
\]

(2.45)

Let \( r > 0 \) be given, and let us apply the preceding remarks to

\[
\nu(E) = \int_E c_r(u) \, du = \int_E \exp(-r|u|) \, du.
\]

(2.46)

Thus (2.44) implies that

\[
\int_{\mathbb{R}} \frac{2r}{r^2 + (t - w)^2} \, d\mu(t) = \int_{\mathbb{R}} \hat{\mu}(\xi) \, \exp(iw\xi) \, \exp(-r|\xi|) \, d\xi
\]

(2.47)

for every \( w \in \mathbb{R} \), using the expression for \( \hat{c}_r \) in (2.35). If \( \hat{\mu}(\xi) \) is an integrable function on \( \mathbb{R} \), then the right side of (2.47) converges to

\[
\int_{\mathbb{R}} \hat{\mu}(\xi) \, \exp(iw\xi) \, d\xi
\]

(2.48)

as \( r \to 0 \). This can be derived from the dominated convergence theorem applied to any sequence of \( r \)'s converging to 0. The same type of argument can also be used more directly, since

\[
\exp(-r|\xi|) \leq 1
\]

(2.49)
for every \( r \geq 0 \) and \( \xi \in \mathbb{R} \), and \( \exp(-r|\xi|) \to 1 \) as \( r \to 0 \) uniformly on any bounded set of \( \xi \)'s.

Put
\[
A_r(\mu)(w) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + (t-w)^2} \, d\mu(t)
\]

for every \( w \in \mathbb{R} \) and \( r > 0 \), which is the same as the common value of (2.47) divided by \( 2\pi \). Thus
\[
|A_r(\mu)(w)| \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + (t-w)^2} \, |d\mu|(t)
\]

for every \( w \in \mathbb{R} \) and \( r > 0 \), and one can check that \( A_r(\mu)(w) \) is continuous in \( w \). Integrating in \( w \), we get that
\[
\int_{\mathbb{R}} |A_r(\mu)(w)| \, dw \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + (t-w)^2} \, |d\mu|(t) \, dw
\]

for every \( r > 0 \), using Fubini's theorem in the second step, and then translation-invariance of Lebesgue measure on \( \mathbb{R} \). We also know that
\[
\frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + w^2} \, dw = 1,
\]

by (2.42), so that (2.52) reduces to
\[
\int_{\mathbb{R}} |A_r(\mu)(w)| \, dw \leq |\mu|(\mathbb{R})
\]

for every \( r > 0 \).

If \( f \in L^1(\mathbb{R}) \), then
\[
A_r(f)(w) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{r^2 + (t-w)^2} \, dt
\]

is the same as (2.50), where \( \mu \) corresponds to \( f \) as in (2.3). This integral also makes sense for any locally-integrable function \( f \) on \( \mathbb{R} \) such that
\[
|f(t)| \frac{1}{1+t^2}
\]

is integrable on \( \mathbb{R} \). In particular, this holds when \( f \in L^p(\mathbb{R}) \) for some \( p \geq 1 \), by Hölder’s inequality. As before,
\[
|A_r(f)(w)| \leq \frac{1}{\pi} \int_{\mathbb{R}} |f(t)| \frac{r}{r^2 + (t-w)^2} \, dt
\]
for every $w \in \mathbb{R}$ and $r > 0$, and one can check that $A_r(f)(w)$ is continuous in $w$ under these conditions. Note that

$$
\frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + (t-w)^2} \, dt = 1
$$

for every $w \in \mathbb{R}$ and $r > 0$, as in (2.42) and (2.53). If $f \in L^p(\mathbb{R})$ for some $p$, $1 \leq p < \infty$, then it follows that

$$
|A_r(f)(w)|^p \leq \frac{1}{\pi} \int_{\mathbb{R}} |f(t)|^p \frac{r}{r^2 + (t-w)^2} \, dt
$$

for every $w \in \mathbb{R}$ and $r > 0$, by Jensen’s inequality. Integrating this in $w$ and using Fubini’s theorem as in (2.52), we get that

$$
\int_{\mathbb{R}} |A_r(f)(w)|^p \, dw \leq \int_{\mathbb{R}} |f(t)|^p \, dt
$$

for every $r > 0$. This shows that $A_r(f) \in L^p(\mathbb{R})$ for every $r > 0$ when $f \in L^p(\mathbb{R})$ for some $p$, $1 \leq p < \infty$, and that

$$
\|A_r(f)\|_p \leq \|f\|_p
$$

for every $r > 0$, where $\|f\|_p$ is the usual $L^p$ norm of $f$. It is easy to see that this also holds when $p = \infty$, by (2.57) and (2.58).

### 2.5 Convergence

Let $f$ be a locally integrable function on $\mathbb{R}$ such that (2.56) is integrable on $\mathbb{R}$, and let $A_r(f)(w)$ be as in (2.55). This is basically an average of the values of $f$, because of (2.58). As $r \to 0$, this average becomes more and more concentrated near $w$. In particular, if $f$ is continuous at $w$, then one can check that

$$
\lim_{r \to 0} A_r(f)(w) = f(w).
$$

Similarly, if $f$ is bounded and uniformly continuous on $\mathbb{R}$, then $A_r(f) \to f$ as $r \to 0$ uniformly on $\mathbb{R}$. If $f$ is a continuous function on $\mathbb{R}$ with compact support, then $f$ is bounded and uniformly continuous on $\mathbb{R}$, and hence $A_r(f) \to f$ as $r \to 0$ uniformly on $\mathbb{R}$. In this case, $|A_r(f)(w)|$ is bounded by a constant times

$$
\frac{r}{r^2 + w^2}
$$

when $|w|$ is large, so that $A_r(f)(w) \to 0$ as $|w| \to \infty$. Using this estimate when $|w|$ is large and uniform convergence, it is easy to see that

$$
\lim_{r \to 0} \|A_r(f) - f\|_p = 0
$$

for every $p \geq 1$. This also holds for every $f \in L^p(\mathbb{R})$ when $1 \leq p < \infty$, by approximating $f$ by continuous functions on $\mathbb{R}$ with compact support with
respect to the $L^p$ norm. This uses the uniform bound (2.61) on the $L^p$ norms as well, to estimate the errors in the approximation.

Let $f$ be a locally integrable function on $\mathbb{R}$ such that (2.56) is integrable on $\mathbb{R}$ again, let $I$ be a bounded interval in $\mathbb{R}$, and let $I_1$ be a bounded interval in $\mathbb{R}$ that contains $I$ in its interior. If $g$ is the function on $\mathbb{R}$ equal to $f$ on $I_1$ and to 0 on $\mathbb{R}\setminus I_1$, then $g \in L^1(\mathbb{R})$, and hence $A_r(g) \to g$ as $r \to 0$ with respect to the $L^1$ norm, as in the previous paragraph. One can also check that $A_r(f - g) \to 0$ as $r \to 0$ uniformly on $I$, because $f - g \equiv 0$ on $I_1$ and $I$ is contained in the interior of $I_1$. Combining these two statements, we get that $A_r(f) \to f$ as $r \to 0$ with respect to the $L^1$ norm on $I$. If $|f(t)|^p$ is locally integrable on $\mathbb{R}$ for some $p$, $1 \leq p < \infty$, and if (2.56) is integrable on $\mathbb{R}$, then an analogous argument shows that $A_r(f) \to f$ as $r \to 0$ with respect to the $L^p$ norm on bounded intervals on $\mathbb{R}$. Similarly, if $f$ is a continuous function on $\mathbb{R}$ such that (2.56) is integrable on $\mathbb{R}$, then $A_r(f) \to f$ as $r \to 0$ uniformly on compact subsets of $\mathbb{R}$. If $f$ is a locally integrable function on $\mathbb{R}$ such that (2.56) is integrable on $\mathbb{R}$, then it is well known that (2.62) holds for almost every $w \in \mathbb{R}$ with respect to Lebesgue measure, but we shall go into this here. Note that it suffices to show this for $f \in L^1(\mathbb{R})$, by the same type of localization argument.

Now let $\mu$ be a complex Borel measure on $\mathbb{R}$, and let $C_0(\mathbb{R})$ be the space of continuous complex-valued functions on $\mathbb{R}$ that vanish at infinity, equipped with the supremum norm. Thus

$$
(2.65) \quad \lambda(\phi) = \int_{\mathbb{R}} \phi(t) \, d\mu(t)
$$

defines a bounded linear functional on $C_0(\mathbb{R})$, and it is well known that every bounded linear functional on $C_0(\mathbb{R})$ is of this form, by a version of the Riesz representation theorem. Let $A_r(\mu)$ be as in (2.50) for each $r > 0$, which is an integrable function on $\mathbb{R}$ for each $r > 0$, by (2.54). Every integrable function on $\mathbb{R}$ determines a Borel measure on $\mathbb{R}$ as in (2.3), and

$$
(2.66) \quad \lambda_r(\phi) = \int_{\mathbb{R}} A_r(\mu)(w) \phi(w) \, dw
$$

is the bounded linear functional on $C_0(\mathbb{R})$ corresponding to the Borel measure on $\mathbb{R}$ associated to $A_r(\mu)$ in this way. Observe that

$$
(2.67) \quad \lambda_r(\phi) = \int_{\mathbb{R}} A_r(\phi)(t) \, d\mu(t)
$$

for every $\phi \in C_0(\mathbb{R})$ and $r > 0$, where $A_r(\phi)$ is as in (2.55), by Fubini’s theorem. As before, $A_r(\phi) \to \phi$ as $r \to 0$ uniformly on $\mathbb{R}$ for every $\phi \in C_0(\mathbb{R})$, because elements of $C_0(\mathbb{R})$ are bounded and uniformly continuous on $\mathbb{R}$. It follows that

$$
(2.68) \quad \lim_{r \to 0} \lambda_r(\phi) = \lambda(\phi)
$$

for every $\phi \in C_0(\mathbb{R})$, which means that $\lambda_r \to \lambda$ as $r \to 0$ with respect to the weak* topology on the dual of $C_0(\mathbb{R})$. 

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If \( f \in L^\infty(\mathbb{R}) \), then \( A_r(f) \) does not normally converge to \( f \) as \( r \to 0 \) with respect to the \( L^\infty \) norm. However, we do have that the \( L^\infty \) norm of \( A_r(f) \) remains bounded, as in (2.61). We also have that \( A_r(f) \to f \) as \( r \to 0 \) with respect to the \( L^p \) norm on bounded intervals and pointwise almost everywhere on \( \mathbb{R} \), as mentioned earlier. Alternatively, we can identify \( L^\infty(\mathbb{R}) \) with the dual of \( L^1(\mathbb{R}) \) in the usual way, and consider convergence with respect to the corresponding weak\(^*\) topology on \( L^\infty(\mathbb{R}) \). In order to show that \( A_r(f) \to f \) as \( r \to 0 \) with respect to the weak\(^*\) topology on \( L^\infty(\mathbb{R}) \) as the dual of \( L^1(\mathbb{R}) \), it suffices to show that

\[
\lim_{r \to 0} \int_{\mathbb{R}} A_r(f)(w) \phi(w) \, dw = \int_{\mathbb{R}} f(w) \phi(w) \, dw
\]

for each \( \phi \in L^1(\mathbb{R}) \). As in the previous paragraph, one can check that

\[
\int_{\mathbb{R}} A_r(f)(w) \phi(w) \, dw = \int_{\mathbb{R}} f(t) A_r(\phi)(t) \, dt
\]

for every \( \phi \in L^1(\mathbb{R}) \) and \( r > 0 \), using Fubini’s theorem. We have already seen that \( A_r(\phi) \to \phi \) as \( r \to 0 \) with respect to the \( L^1 \) norm when \( \phi \in L^1(\mathbb{R}) \), which implies that

\[
\lim_{r \to 0} \int_{\mathbb{R}} f(t) A_r(\phi)(t) \, dt = \int_{\mathbb{R}} f(t) \phi(t) \, dt.
\]

Combining this with (2.70), we get that (2.69) holds, as desired.

### 2.6 Harmonic extensions

Let \( \mu \) be a complex Borel measure, and let \( x, y \in \mathbb{R} \) be given, with \( y > 0 \), so that \( z = x + iy \) is an element of the upper half-plane \( \mathcal{U} \). Put

\[
h(z) = A_y(\mu)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x - t)^2 + y^2} \, d\mu(t),
\]

where \( A_y(\mu)(x) \) is as in (2.50). Observe that

\[
\frac{1}{t - z} = \frac{1}{t - \bar{z}} = \frac{2iy}{|z - t|^2} = \frac{2iy}{(x - t)^2 + y^2}
\]

for each \( t \in \mathbb{R} \), by partial fractions. If we also put

\[
h_+(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{t - z} \, d\mu(t)
\]

and

\[
h_-(z) = \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{1}{t - z} \, d\mu(t),
\]

then it follows that

\[
h(z) = h_+(z) + h_-(z).
\]
Of course, \( h_+(z) \) is a holomorphic function of \( z \) in \( U \), and \( h_-(z) \) is a conjugate-holomorphic function of \( z \) in \( U \), which implies that \( h(z) \) is a harmonic function of \( z \) in \( U \).

Equivalently,

\[
h(z) = A_y(\mu)(x) = \frac{1}{2\pi} \int_R \hat{\mu}(\xi) \exp(i x \xi) \exp(-y |\xi|) d\xi,
\]

as in (2.47). If we put

\[
h_+(z) = \frac{1}{2\pi} \int_0^\infty \hat{\mu}(\xi) \exp(i x \xi) \exp(-y \xi) d\xi
\]

and

\[
h_-(z) = \frac{1}{2\pi} \int_{-\infty}^0 \hat{\mu}(\xi) \exp(i x \xi) \exp(y \xi) d\xi
\]

then \( h_+(z) \) is a holomorphic function of \( z \) in \( U \), \( h_-(z) \) is a conjugate-holomorphic function of \( z \) in \( U \), and \( h_+(z) + h_-(z) = h(z) \), so that \( h(z) \) is a harmonic function of \( z \) in \( U \). Using the multiplication formula (2.44) with \( w = x \), we get that

\[
h_+(z) = \frac{1}{2\pi} \int_R \hat{b}_y(t - x) d\mu(t)
\]

and

\[
h_-(z) = \frac{1}{2\pi} \int_R \hat{a}_y(t - x) d\mu(t),
\]

where \( a_y \) and \( b_y \) are as in Section 2.3. It follows that

\[
h_+(z) = \frac{1}{2\pi} \int_R \frac{1}{y + i(t - x)} d\mu(t)
\]

and

\[
h_-(z) = \frac{1}{2\pi} \int_R \frac{1}{y - i(t - x)} d\mu(t),
\]

by substituting the expressions (2.31) and (2.28) for the Fourier transforms of \( b_y \) and \( a_y \) into (2.80) and (2.81), respectively. It is easy to see that (2.82) and (2.83) are exactly the same as (2.74) and (2.75), respectively.

Similarly, if \( f \) is a locally-integrable function on \( \mathbb{R} \) such that (2.56) is an integrable function on \( \mathbb{R} \), then we can put

\[
h(z) = A_y(f)(x) = \frac{1}{\pi} \int_\mathbb{R} f(t) \frac{y}{(x - t)^2 + y^2} dt,
\]
2.7. BOUNDED HARMONIC FUNCTIONS

where $A_y(f)(x)$ is as in (2.55). If

\[(2.85) \quad |f(t)| \frac{1}{1+|t|}\]

is also an integrable function on $\mathbb{R}$, then

\[(2.86) \quad h_+(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(t) \frac{1}{t-z} dt\]

and

\[(2.87) \quad h_-(z) = -\frac{1}{2\pi i} \int_{\mathbb{R}} f(t) \frac{1}{t-z} dt\]

can be defined, in analogy with (2.74) and (2.75), and satisfy (2.76). As before, $h_+(z)$ is a holomorphic function of $z \in U$ under these conditions, $h_-(z)$ is a conjugate-holomorphic function of $z$ in $U$, and hence $h(z)$ is a harmonic function of $z$ in $U$. The integrability of (2.85) is more restrictive than the integrability of (2.56), and it holds when $f \in L^p(\mathbb{R})$ for some $p$ with $1 \leq p < \infty$, by Hölder’s inequality. One can still check that $h(z)$ is a harmonic function of $z$ in $U$ when (2.56) is integrable, basically because (2.73) is harmonic in $z$.

If (2.56) is integrable on $\mathbb{R}$ and $f$ is continuous at a point $x_0 \in \mathbb{R}$, then

\[(2.88) \quad h(z) \to f(x_0)\]
as $z \in U$ approaches $x_0$. This is analogous to (2.62), since $h(z)$ is given by an average of values of $f$ which is concentrated near $x_0$ as $z$ approaches $x_0$. If $f$ is a continuous function on $\mathbb{R}$ such that (2.56) is integrable, then it follows that the function defined on $\overline{U}$ by taking $h$ on $U$ and $f$ on $\mathbb{R}$ is continuous. If $f$ is any bounded measurable function on $\mathbb{R}$, then (2.56) is integrable on $\mathbb{R}$, and

\[(2.89) \quad |h(z)| \leq \|f\|_\infty\]

for every $z \in U$. This is essentially the $p = \infty$ version of (2.61), which follows from (2.57) and (2.58), as mentioned earlier.

2.7 Bounded harmonic functions

Suppose that $g(z)$ is a continuous complex-valued function on the closed upper half-plane $\overline{U}$ that is harmonic on the open upper half-plane $U$ and satisfies $g(z) \to 0$ as $|z| \to \infty$. Under these conditions, the maximum principle implies that the maximum of $|g(z)|$ on $\overline{U}$ is attained on $\mathbb{R}$. In particular, if $g(x) = 0$ for every $x \in \mathbb{R}$, then it follows that $g(z) = 0$ for every $z \in \overline{U}$.

Suppose now that $g(z)$ is a bounded continuous complex-valued function on $\overline{U}$ that is harmonic on $U$ and satisfies

\[(2.90) \quad g(x) = 0\]

for every $x \in \mathbb{R}$. If we put

\[(2.91) \quad g(z) = -g(\overline{z})\]
when $\text{Im} \, z < 0$, then the reflection principle implies that we get a harmonic function on the complex plane. This harmonic function is also bounded and hence constant, by well-known arguments. It follows that $g(z) = 0$ for every $z$, because of (2.90). This shows that a bounded continuous function on $\overline{U}$ that is harmonic on $U$ is uniquely determined by its restriction to $\mathbb{R}$.

As another variant, let $g(z)$ be a bounded continuous function on $\overline{U}$ that is holomorphic on $U$. Let $\epsilon > 0$ be given, and put

$$g_{\epsilon}(z) = g(z) \left( \frac{i}{\epsilon \, z + i} \right)$$

for every $z \in \overline{U}$. This is a bounded continuous function on $\overline{U}$ that is holomorphic on $U$ and satisfies $g_{\epsilon}(z) \to 0$ as $|z| \to \infty$. The maximum principle implies that

$$|g_{\epsilon}(z)| \leq \sup_{t \in \mathbb{R}} |g_{\epsilon}(t)| \leq \sup_{t \in \mathbb{R}} |g(t)|$$

for every $z \in U$, as before. Taking the limit as $\epsilon \to 0$, we get that

$$|g(z)| \leq \sup_{t \in \mathbb{R}} |g(t)|$$

for every $z \in U$.

Let $g(z)$ be a bounded continuous function on $\overline{U}$ that is harmonic on $U$ again, and let $f$ be the restriction of $g$ to $\mathbb{R}$. Also let $h(z)$ be as in (2.84) when $z \in U$, and put $h = f$ on $\mathbb{R}$. This defines a bounded continuous function on $\overline{U}$ that is harmonic on $U$, as in the previous section. It follows that $g(z) = h(z)$ for every $z \in \overline{U}$, by the uniqueness principle mentioned earlier.

Now let $g(z)$ be a harmonic function on $U$, and suppose that the restriction of $g(z)$ to

$$U_\tau = \{ z \in \mathbb{C} : \text{Im} \, z > \tau \}$$

is bounded for each $\tau > 0$. Equivalently, this means that $g(z + i \tau)$ is a bounded harmonic function on $U$ for each $\tau > 0$. Of course, $g$ is continuous on $U$, and so $g(z + i \tau)$ has a bounded continuous extension to $\overline{U}_\tau$ for each $\tau > 0$. The discussion in the previous paragraph implies that

$$g(z + i \tau) = \frac{1}{\pi} \int_{\mathbb{R}} g(t + i \tau) \frac{y}{(x-t)^2 + y^2} \, dt$$

for every $z \in U$ and $\tau > 0$. In particular, this works when $g(z)$ is a bounded harmonic function on $U$.

Let a real number $p \geq 1$ be given, and suppose that $g(z)$ is a harmonic function on $U$ such that

$$(\int_{\mathbb{R}} |g(x + i \tau)| \, dx)^{1/p}$$

is finite for each $\tau > 0$, and uniformly bounded over $\tau > 0$. If $p = \infty$, then one can replace (2.97) with the supremum of $|g(x + i \tau)|$ over $x \in \mathbb{R}$, and the
resulting condition is the same as saying that $g(z)$ is bounded on $U$. If $p < \infty$, then one can still show that $g(z)$ is bounded on $\overline{U}_\tau$ for each $\tau > 0$, using the mean value property of harmonic functions.

Suppose that $1 < p \leq \infty$, so that one can identify $L^p(\mathbb{R})$ with the dual of $L^q(\mathbb{R})$ in the usual way, where $1 \leq q < \infty$ is the exponent conjugate to $p$. As in Section 1.10, closed balls in $L^p(\mathbb{R})$ are sequentially compact with respect to the weak$^*$ topology on $L^p(\mathbb{R})$. If $h(z)$ is defined as in (2.84), then it follows that $g(z) = h(z)$ for every $z \in U$, by taking $\tau = \tau_j$ in (2.96), and passing to the limit as $j \to \infty$.

If $p = 1$, then the boundedness of the $L^1$ norm of $g(x + i \tau)$ in $x \in \mathbb{R}$ over $\tau > 0$ implies that

\begin{equation}
\lambda_\tau(\phi) = \int_{\mathbb{R}} \phi(x) g(x + i \tau) \, dx
\end{equation}

defines a bounded linear functional on $C_0(\mathbb{R})$ for each $\tau > 0$, with bounded dual norm with respect to the supremum norm on $C_0(\mathbb{R})$. As before, closed balls in the dual of $C_0(\mathbb{R})$ are sequentially compact with respect to the weak$^*$ topology, by the Banach–Alaoglu theorem and the separability of $C_0(\mathbb{R})$. Hence there is a sequence $\{\tau_j\}_{j=1}^\infty$ of positive real numbers converging to 0 such that $\{\lambda_{\tau_j}\}_{j=1}^\infty$ converges with respect to the weak$^*$ topology on the dual of $C_0(\mathbb{R})$ to a bounded linear functional $\lambda$ on $C_0(\mathbb{R})$. A version of the Riesz representation theorem implies that $\lambda$ corresponds to a complex Borel measure $\mu$ on $\mathbb{R}$ in the usual way. If $h(z)$ is defined as in (2.72), then it follows that $g(z) = h(z)$ for every $z \in U$, by taking $\tau = \tau_j$ in (2.96), and passing to the limit as $j \to \infty$.

2.8 **Cauchy’s theorem**

Let $g(z)$ be a continuous complex-valued function on the closed upper half-plane $\overline{U}$ that is holomorphic on the open upper half-plane $U$. Also let $R$ be a positive real number, and let $\Gamma(R)$ be the curve in $\overline{U}$ that goes from $-R$ to $R$ along the real line, and then from $R$ to $-R$ along the top half of the circle centered at 0 with radius $R$. Using Cauchy’s theorem, we get that

\begin{equation}
\int_{\Gamma(R)} g(z) \, dz = 0
\end{equation}

for every $R > 0$, where the integral is a complex line integral. If $g$ is also integrable on $\mathbb{R}$ with respect to Lebesgue measure, and if

\begin{equation}
\lim_{|z| \to \infty} |z| |g(z)| = 0
\end{equation}
on \(U\), then one can take the limit as \(R \to \infty\) in (2.99), to get that

\[
\int_{-\infty}^{\infty} g(x) \, dx = 0. \tag{2.101}
\]

More precisely, (2.100) ensures that the integral of \(g\) over the semi-circular arc of radius \(R\) in \(\Gamma(R)\) converges to 0 as \(R \to \infty\).

Suppose that \(g\) satisfies the same conditions as before, except that \(g\) is bounded on \(U\), instead of (2.100). Let \(\epsilon > 0\) be given, and observe that

\[
g(z) \left( \frac{i}{\epsilon z + i} \right)^2 \tag{2.102}
\]
defines a continuous function on \(U\) that is holomorphic on \(U\), integrable on \(R\), and satisfies the analogue of (2.100). Thus

\[
\int_{-\infty}^{\infty} g(x) \left( \frac{i}{\epsilon x + i} \right)^2 \, dx = 0 \tag{2.103}
\]

for every \(\epsilon > 0\), for the same reasons as in (2.101). It follows that (2.101) holds in this case as well, using the dominated convergence theorem to pass to the limit as \(\epsilon \to 0\).

Now let \(g(z)\) be a holomorphic function on \(U\) such that \(g(x + i \tau)\) is an integrable function of \(x\) on \(R\) for each \(\tau > 0\), and

\[
\int_{-\infty}^{\infty} |g(x + i \tau)| \, dx \tag{2.104}
\]
is uniformly bounded over \(\tau > 0\). As mentioned in the previous section, one can show that the restriction of \(g(z)\) to \(U_\tau\) is bounded for each \(\tau > 0\), where \(U_\tau\) is as in (2.95). This permits us to apply the earlier discussion to \(g(z + i \tau)\) as a function of \(z \in U\) for each \(\tau > 0\), to get that

\[
\int_{-\infty}^{\infty} g(x + i \tau) \, dx = 0 \tag{2.105}
\]
for every \(\tau > 0\). If there is a function \(g(x) \in L^1(R)\) such that

\[
\lim_{\tau \to 0} \int_{-\infty}^{\infty} |g(x + i \tau) - g(x)| \, dx = 0, \tag{2.106}
\]
then it follows that (2.101) holds. A famous theorem states that there is always a function \(g(x)\) in \(L^1(R)\) that satisfies (2.106) under these conditions.

If \(g(z)\) is as in the previous paragraph, then

\[
g(z) \exp(-i z \xi) \tag{2.107}
\]
has the same properties for every \(\xi \in R\) with \(\xi \leq 0\), because

\[
|\exp(-i z \xi)| = \exp(y \xi) \leq 1 \tag{2.108}
\]
when \( y \geq 0 \). This implies that

\[
\int_{-\infty}^{\infty} g(x) \exp(-i x \xi) \, dx = 0
\]

for every \( \xi \in \mathbb{R} \) with \( \xi \leq 0 \), by applying (2.101) to (2.107). Conversely, let \( \mu \) be a complex Borel measure on \( \mathbb{R} \) such that

\[
\hat{\mu}(\xi) = 0
\]

for every \( \xi \in \mathbb{R} \) with \( \xi \leq 0 \). This implies that \( h_-(z) = 0 \) for every \( z \in U \), where \( h_-(z) \) is as in (2.79), which is the same as (2.75). Thus the harmonic function \( h(z) \) on \( U \) in (2.72) is equal to the holomorphic function \( h_+(z) \) on \( U \) in (2.74). This harmonic function also satisfies the analogue of the boundedness of the integrals (2.104), because of (2.54). It follows that there is an integrable function \( f(x) \) on \( \mathbb{R} \) such that \( h(x + i \tau) \) converges to \( f(x) \) as \( \tau \to 0 \) with respect to the \( L^1 \) norm on \( \mathbb{R} \), as in the preceding paragraph. However, \( h(x + i \tau) \) also converges in a weak sense to \( \mu \) as \( \tau \to 0 \), by the remarks at the end of Section 2.5. This means that \( \mu \) is absolutely continuous on \( \mathbb{R} \) with respect to Lebesgue measure and corresponds to \( f \) as in (2.3), which is another version of the F. and M. Riesz theorem.

### 2.9 Cauchy’s integral formula

Let \( g(z) \) be a continuous complex-valued function on the closed upper half-plane \( \overline{U} \) that is holomorphic on the open upper half-plane \( U \) again, and let \( \Gamma(R) \) be the curve in \( \overline{U} \) corresponding to a positive real number \( R \) as described at the beginning of the previous section. If \( w \in U \) and \( |w| < R \), then the Cauchy integral formula implies that

\[
\frac{1}{2 \pi i} \int_{\Gamma(R)} g(z) \frac{1}{z - w} \, dz = g(w).
\]

If we also ask that

\[
|g(x)| \frac{1}{1 + |x|}
\]

be an integrable function on \( \mathbb{R} \), and that

\[
\lim_{|z| \to \infty} |g(z)| = 0,
\]

then one can take the limit as \( R \to \infty \) in (2.111), to get that

\[
\frac{1}{2 \pi i} \int_{-\infty}^{\infty} g(x) \frac{1}{x - w} \, dx = g(w)
\]

for every \( w \in U \). As before, one can get the same conclusion when \( g(z) \) is bounded on \( \overline{U} \) instead of (2.113), by approximating \( g(z) \) by \( g(z)/(i/(\epsilon z + i)) \) with \( \epsilon > 0 \).
Under the same conditions on \( g(z) \), we have that for each \( w \in U \),

\[
(2.115) \quad g(z) \frac{1}{z - w}
\]
is a bounded continuous function on \( U \) that is holomorphic on \( U \) and integrable with respect to Lebesgue measure on \( \mathbb{R} \). Thus

\[
(2.116) \quad \int_{-\infty}^{\infty} g(x) \frac{1}{x - w} \, dx = 0
\]

for every \( w \in U \), by applying (2.101) to (2.115). Combining this with (2.114), we get that

\[
(2.117) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x) \left( \frac{1}{x - w} - \frac{1}{x - m} \right) \, dx = g(w)
\]

for every \( w \in U \). If \( g \) is a bounded continuous function on \( U \) that is harmonic on \( U \), \( f \) is the restriction of \( g \) to \( \mathbb{R} \), and \( h \) is the function on \( U \) corresponding to \( f \) as in (2.84), then we have seen in Section 2.7 that \( g = h \) on \( U \). This representation for \( g \) on \( U \) is the same as (2.117), but with slightly different notation.

Suppose now that \( g(z) \) is a holomorphic function on \( U \) such that \( |g(x + i \tau)|^p \) is an integrable function of \( x \) on \( \mathbb{R} \) for some \( p \in [1, \infty) \) and every \( \tau > 0 \), and that

\[
(2.118) \quad \int_{-\infty}^{\infty} |g(x + i \tau)|^p \, dx
\]
is uniformly bounded over \( \tau > 0 \). This implies that the restriction of \( g(z) \) to \( \overline{U}_\tau \) is bounded for every \( \tau > 0 \), as in Section 2.7, and where \( U_\tau \) is as in (2.95). We also have that

\[
(2.119) \quad |g(x + i \tau)| \frac{1}{1 + |x|}
\]
is integrable as a function of \( x \) on \( \mathbb{R} \) for each \( \tau > 0 \), by Hölder’s inequality. This permits us to apply the previous discussion to \( g(z + i \tau) \) as a function of \( z \in \overline{U} \) for each \( \tau > 0 \), to get that

\[
(2.120) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x + i \tau) \frac{1}{x - w} \, dx = g(w + i \tau)
\]
and

\[
(2.121) \quad \int_{-\infty}^{\infty} g(x + i \tau) \frac{1}{x - w} \, dx = 0
\]
for every \( w \in U \), and hence that

\[
(2.122) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x + i \tau) \left( \frac{1}{x - w} - \frac{1}{x - m} \right) \, dx = g(w + i \tau).
\]

As before, (2.122) is the same as (2.96), but with slightly different notation.
If $1 < p < \infty$, then we can identify $L^p(\mathbb{R})$ with the dual of $L^q(\mathbb{R})$, where $1 < q < \infty$ and $1/p + 1/q = 1$. As in Section 2.7, one can use the Banach–Alaoglu theorem and the separability of $L^q(\mathbb{R})$ to get a sequence $\{\tau_j\}_{j=1}^\infty$ of positive real numbers converging to 0 such that $g(x + i \tau_j)$ converges to some $g(x) \in L^p(\mathbb{R})$ as $j \to \infty$ with respect to the weak* topology on $L^p(\mathbb{R})$ as the dual of $L^q(\mathbb{R})$. Applying (2.120), (2.121), and (2.122) to $\tau_j$ and taking the limit as $j \to \infty$, we get that (2.114), (2.116), and (2.117) hold for every $w \in U$. This is analogous to the discussion of harmonic functions in Section 2.7, and indeed (2.117) corresponds exactly to representing $g$ as the harmonic function $h$ on $U$ associated to some $f \in L^p(\mathbb{R})$ as in (2.84). Under these conditions, one can also conclude that $g(x + i \tau)$ converges to $g(x)$ as $\tau \to 0$ with respect to the $L^p$ norm on $\mathbb{R}$, as in Section 2.5. Similarly, if $p = 1$, then there is a sequence $\{\tau_j\}_{j=1}^\infty$ of positive real numbers converging to 0 such that $g(x + i \tau_j)$ converges to a complex Borel measure $\mu$ on $\mathbb{R}$ in a suitable weak sense as $j \to \infty$, as in Section 2.7. This leads to equations like (2.114), (2.116), and (2.117) for each $w \in U$, but where $g(x)$ and $dx$ are replaced with $d\mu(x)$. However, as mentioned in the previous section, a famous theorem implies that $g(x + i \tau)$ actually converges to an integrable function $g(x)$ on $\mathbb{R}$ with respect to the $L^1$ norm as $\tau \to 0$ in this situation. In particular, (2.114), (2.116), and (2.117) still hold, using this function $g(x)$.

### 2.10 Some variants

Let $g(z)$ be a continuous complex-valued function on the closed upper half-plane $\overline{U}$ that is holomorphic on the open upper half-plane $U$, and let $w, w_1 \in U$ be given. Thus

$$
(2.123) \quad g(z) \frac{1}{z - w_1}
$$

is also a continuous function of $z$ on $\overline{U}$ that is holomorphic on $U$. If $\Gamma(R)$ is the curve in $\overline{U}$ corresponding to a positive real number $R$ as in Section 2.8, then

$$
(2.124) \quad \int_{\Gamma(R)} g(z) \frac{1}{z - w_1} \, dz = 0,
$$

by Cauchy’s theorem. Combining this with (2.111), we get that

$$
(2.125) \quad \frac{1}{2\pi i} \int_{\Gamma(R)} g(z) \left( \frac{1}{z - w} - \frac{1}{z - w_1} \right) \, dz = g(w)
$$

when $|w_1| < R$. Of course,

$$
(2.126) \quad \frac{1}{z - w} - \frac{1}{z - w_1} = \frac{(z - w_1) - (z - w)}{(z - w)(z - w_1)} = \frac{w - w_1}{(z - w)(z - w_1)}
$$

for each $z \in \overline{U}$, the modulus of which is $O(1/|z|^2)$ for $|z|$ large. If $g(z)$ is bounded on $\overline{U}$, in addition to the other conditions already mentioned, then we can take
the limit as \( R \to \infty \) in (2.125), to get that

\[
(2.127) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x) \left( \frac{1}{x-w} - \frac{1}{x-w_1} \right) \, dx = g(w).
\]

In particular, the integrand is an integrable function on \( \mathbb{R} \) when \( g(x) \) is a bounded function on \( \mathbb{R} \), because of (2.126). Note that (2.127) is the same as (2.117) when \( w_1 = w \), which is the same as the representation of \( g(z) \) as \( h(z) \) in (2.84), with \( f \) equal to the restriction of \( g \) to \( \mathbb{R} \). Remember that this representation works for bounded continuous functions on \( U \) that are harmonic on \( U \), as in Section 2.7.

If \( w_1, w_2 \in U \), then it follows from (2.127) that

\[
(2.128) \quad \int_{-\infty}^{\infty} g(x) \left( \frac{1}{x-w_1} - \frac{1}{x-w_2} \right) \, dx = 0,
\]

because the right side of (2.127) does not depend on \( w_1 \). Alternatively, one can get this by applying (2.101) to

\[
(2.129) \quad g(z) \left( \frac{1}{z-w_1} - \frac{1}{z-w_2} \right)
\]

as a function of \( z \in \overline{U} \) when \( g(z) \) is bounded on \( \overline{U} \). This uses the fact that

\[
(2.130) \quad \frac{1}{z-w_1} - \frac{1}{z-w_2} = \frac{(z-w_2)-(z-w_1)}{(z-w_1)(z-w_2)} = \frac{w_1-w_2}{(z-w_1)(z-w_2)}\]

for every \( z \in \overline{U} \), the modulus of which is \( O\left(\frac{1}{|z|^2}\right) \) for \( |z| \) large, as before. In the other direction, if one has (2.127) in the special case where \( w_1 = w \), and if one knows (2.128) for every \( w_1, w_2 \in U \), then one can get (2.127) for every \( w, w_1 \in U \).

We also have that

\[
(2.131) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x) \left( \frac{1}{x-w_1} - \frac{1}{x-w_2} \right) \, dx = g(w_1) - g(w_2)
\]

for every \( w_1, w_2 \in U \) in this situation. This can be derived easily from (2.127), with suitable relabelling of the variables. As a more direct approach, one can use the Cauchy integral formula as in (2.111) to get that

\[
(2.132) \quad \frac{1}{2\pi i} \int_{\Gamma(R)} g(z) \left( \frac{1}{z-w_1} - \frac{1}{z-w_2} \right) \, dz = g(w_1) - g(w_2)
\]

when \( |w_1|, |w_2| < R \). As usual,

\[
(2.133) \quad \frac{1}{z-w_1} - \frac{1}{z-w_2} = \frac{(z-w_2)-(z-w_1)}{(z-w_1)(z-w_2)} = \frac{w_1-w_2}{(z-w_1)(z-w_2)}
\]

for every \( z \in \overline{U} \), the modulus of which is \( O\left(\frac{1}{|z|^2}\right) \) for \( |z| \) large. This permits one to take the limit as \( R \to \infty \) in (2.132), to get (2.131).
If \( g(z) \) is any bounded holomorphic function on \( U \), then \( g(z + i \tau) \) is a bounded continuous function of \( z \) on \( \overline{U} \) for each \( \tau > 0 \). Thus (2.127) implies that

\[
2.134 \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x + i \tau) \left( \frac{1}{x - w} - \frac{1}{x - w_1} \right) dx = g(w + i \tau)
\]

for every \( w, w_1 \in U \). Similarly, (2.128) and (2.131) imply that

\[
2.135 \quad \int_{-\infty}^{\infty} g(x + i \tau) \left( \frac{1}{x - w_1} - \frac{1}{x - w_2} \right) dx = 0
\]

and

\[
2.136 \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(x + i \tau) \left( \frac{1}{x - w_1} - \frac{1}{x - w_2} \right) dx = g(w_1 + i \tau) - g(w_2 + i \tau)
\]

for every \( w_1, w_2 \in U \). If \( w_1 = w \), then (2.134) reduces to (2.122), which is the same as (2.96), and which works for any bounded harmonic function on \( U \). Remember that \( L^\infty(\mathbb{R}) \) can be identified with the dual of \( L^1(\mathbb{R}) \) in the usual way. As in Section 2.7, one can use the Banach–Alaoglu theorem and the separability of \( L^1(\mathbb{R}) \) to get a sequence \( \{\tau_j\}_{j=1}^{\infty} \) of positive real numbers that converges to 0 such that \( g(x + i \tau_j) \) converges to some \( g(x) \in L^\infty(\mathbb{R}) \) as \( j \to \infty \) with respect to the corresponding weak* topology on \( L^\infty(\mathbb{R}) \). Applying (2.134) to \( \tau_j \) and taking the limit as \( j \to \infty \), we get that (2.127) holds for every \( w, w_1 \in U \), using this function \( g(x) \) on \( \mathbb{R} \) in the integral. Similarly, we can apply (2.135) and (2.136) to \( \tau_j \) and take the limit as \( j \to \infty \), to get that (2.128) and (2.131) hold for every \( w_1, w_2 \in U \), using this function \( g(x) \) on \( \mathbb{R} \) in the integrals again. If \( g \) is a bounded harmonic function on \( U \), then (2.134) holds for \( w_1 = w \), which implies that (2.127) holds for \( w_1 = w \), as in Section 2.7.
Chapter 3

Commutative topological groups

3.1 Basic notions

Let $A$ be a commutative group, in which the group operations are expressed additively. Suppose that $A$ is also equipped with a topology, such that the group operations are continuous. More precisely, this means that

$$ (x, y) \mapsto x + y $$

is continuous as a mapping from $A \times A$ into $A$, using the product topology on $A \times A$ associated to the given topology on $A$, and that

$$ x \mapsto -x $$

is continuous as a mapping from $A$ onto itself. If $\{0\}$ is a closed set in $A$, then we say that $A$ is a topological group. Of course, the definition of a topological group that is not necessarily commutative is analogous, but we shall be primarily concerned with the commutative case here.

Continuity of addition on $A$ implies in particular that

$$ x \mapsto x + a $$

is continuous as a mapping from $A$ onto itself for every $a \in A$. More precisely, these translation mappings on $A$ are homeomorphisms from $A$ onto itself, since the inverse of (3.3) is given by translation by $-a$. Thus the hypothesis that $\{0\}$ be a closed set in $A$ implies that $\{a\}$ is a closed set in $A$ for every $a \in A$. One can show that $A$ is also Hausdorff, using continuity of addition at 0. A refinement of this argument shows that $A$ is regular as a topological space.

Any group is a topological group with respect to the discrete topology. The real line $\mathbb{R}$ is a commutative topological group with respect to addition and the standard topology, and the unit circle $\mathbb{T}$ is a commutative topological group.
3.1. BASIC NOTIONS

with respect to multiplication of complex numbers and the topology induced on \( T \) by the standard topology on the complex plane. The set \( \mathbb{R}_+ \) of positive real numbers forms a commutative group with respect to multiplication, which is a topological group with respect to the standard topology. If \( A \) and \( B \) are topological groups, \( \phi \) is a group isomorphism from \( A \) onto \( B \), and \( \phi \) is also a homeomorphism from \( A \) onto \( B \), then \( \phi \) is considered to be an isomorphism from \( A \) onto \( B \) as topological groups. The exponential function defines an isomorphism from \( \mathbb{R} \) as a topological group with respect to addition onto \( \mathbb{R}_+ \) as a topological group with respect to multiplication, for instance.

Let \( V \) be a vector space over the real numbers, which is a commutative group with respect to addition in particular. If \( V \) is equipped with a topology such that the vector space operations are continuous, and if \( \{0\} \) is a closed set in \( V \), then we say that \( V \) is a **topological vector space**. More precisely, the continuity of the vector space operations means that addition on \( V \) is continuous as a mapping from \( V \times V \) into \( V \), and that scalar multiplication on \( V \) is continuous as a mapping from \( \mathbb{R} \times V \) into \( V \). Here \( V \times V \) is equipped with the product topology corresponding to the given topology on \( V \), and \( \mathbb{R} \times V \) is equipped with the product topology corresponding to the standard topology on \( \mathbb{R} \) and the given topology on \( V \). Continuity of scalar multiplication implies in particular that multiplication by \(-1\) is continuous on \( V \), so that \( V \) is a topological group with respect to addition.

Let \((M, d(x, y))\) be a metric space, and let \( p \) be an element of \( M \). The collection of open balls \( B(p, 1/j) \) in \( M \) with respect to \( d(x, y) \) centered at \( p \) and with radius \( 1/j \) for each positive integer \( j \) forms a local base for the topology of \( M \) at \( p \). This local base for the topology of \( M \) at \( p \) has only finitely or countably many elements, and one could just as well have used any sequence of positive radii that converges to 0. If \( A \) is a commutative topological group, then a local base for the topology of \( A \) at any point leads to local bases for the topology of \( A \) at every other point in \( A \), using translations. If there is a local base for the topology of \( A \) at 0 with only finitely or countably many elements, then a famous theorem states that there is a metric on \( A \) that is invariant under translations and determines the same topology on \( A \).

Remember that a topological space \( X \) is said to be **separable** if there is a dense set in \( X \) with only finitely or countably many elements. If there is a base for the topology of \( X \) with only finitely or countably many elements, then \( X \) is separable. As a partial converse, if the topology on \( X \) is determined by a metric, and if \( X \) is separable, then there is a base for the topology of \( X \) with only finitely or countably many elements. In particular, if \( A \) is a commutative topological group with a local base for its topology at 0 with only finitely or countably many elements, and if \( A \) is separable, then there is a base for the topology of \( A \) with only finitely or countably many elements. This can be shown directly, in much the same way as for metric spaces, or by using the metrization theorem mentioned in the previous paragraph.

A subset \( E \) of a metric space \( M \) is said to be **totally bounded** if for each \( r > 0 \), \( E \) is contained in the union of finitely many balls of radius \( r \). Similarly, a subset \( E \) of a commutative topological group \( A \) is said to be totally bounded if
for each open set \( U \subseteq A \) with \( 0 \in U \), \( E \) is contained in the union of finitely many translates of \( U \). Of course, compact subsets of metric spaces are totally bounded, and compact subsets of commutative topological groups are totally bounded as well. If there is a translation-invariant metric \( d(x,y) \) on a commutative topological group \( A \) that determines the same topology on \( A \), then it is easy to see that \( E \subseteq A \) is totally bounded as a subset of \( A \) as a commutative topological group if and only if \( E \) is totally bounded with respect to \( d(x,y) \).

A subset of a topological space is said to be \( \sigma \)-compact if it can be expressed as the union of finitely or countably many compact sets. It is well known that compact metric spaces are totally bounded, and hence separable. Similarly, \( \sigma \)-compact metric spaces are separable. If \( A \) is a commutative topological group with a local base for its topology at 0 with only finitely or countably many elements, and if \( A \) is \( \sigma \)-compact, then \( A \) is separable. As before, this can be shown directly, in much the same way as for metric spaces, or using the metrization theorem mentioned earlier.

### 3.2 Haar measure

Remember that a topological space \( X \) is said to be locally compact if for each point \( p \in X \) there is an open set \( U \subseteq X \) and a compact set \( K \subseteq X \) such that \( p \in U \) and \( U \subseteq K \). If \( X \) is Hausdorff, then compact subsets of \( X \) are closed, and \( X \) is locally compact if and only if for each \( p \in X \) there is an open set \( U \subseteq X \) such that \( p \in U \) and the closure \( \overline{U} \) of \( U \) in \( X \) is compact. If \( A \) is a commutative topological group, then \( A \) is locally compact if and only if there is an open set \( U \subseteq A \) such that \( 0 \in U \) and \( U \) is compact, because of continuity of translations.

If \( V \) is a topological vector space over \( \mathbb{R} \) that is locally compact, then it is well known that \( V \) has finite dimension. In this case, it is also well known that \( V \) is either isomorphic to \( \mathbb{R}^n \) as a topological vector space for some positive integer \( n \), or \( V = \{0\} \).

If \( A \) is a locally compact commutative topological group, then it is well known that there is a nonnegative Borel measure \( H \) on \( A \), known as Haar measure, that is invariant under translations and has some other nice properties. To say that \( H \) is invariant under translations means that

\[
H(E + a) = H(E)
\]

for every Borel set \( E \subseteq A \) and \( a \in A \), where

\[
E + a = \{x + a : x \in E\}.
\]

Note that \( E + a \) is a Borel set in \( A \) for every Borel set \( E \subseteq A \) and \( a \in A \), because the translation mappings (3.3) are homeomorphisms on \( A \). This measure \( H \) also satisfies \( H(U) > 0 \) when \( U \) is a nonempty open set in \( A \), \( H(K) < \infty \) when \( K \subseteq A \) is compact, and some standard regularity conditions. Haar measure is unique in the sense that if \( H' \) is another measure on \( A \) with the same properties, then \( H' \) is equal to a positive real number times \( H \). Lebesgue measure on \( \mathbb{R} \).
satisfies the requirements of Haar measure, as does arc-length measure on $\mathbb{T}$. If $A$ is equipped with the discrete topology, then counting measure on $A$ satisfies the requirements of Haar measure.

Let $A$ be a locally compact commutative topological group, and let $H$ be a Haar measure on $A$. We would like to check that

$\tag{3.6} \quad H(-E) = H(E)$

for every Borel set $E \subseteq A$, where

$\tag{3.7} \quad -E = \{-x : x \in E\}$.

Note that $-E$ is a Borel set in $A$ when $E$ is a Borel set, because (3.2) is a homeomorphism on $A$. It easy easy to see that $H(-E)$ also satisfies the requirements of Haar measure on $A$. The uniqueness of Haar measure implies that there is a positive real number $c$ such that

$\tag{3.8} \quad H(-E) = cH(E)$

for every Borel set $E \subseteq A$. Because $A$ is locally compact, there is an open set $U \subseteq A$ such that $0 \in U$ and $\overline{U}$ is compact, as before. It follows that

$\tag{3.9} \quad W = U \cap (-U)$

is also an open set in $A$ that contains 0 and has compact closure, so that $H(W)$ is positive and finite. By construction,

$\tag{3.10} \quad -W = W$,

which implies that $c = 1$, as desired, by applying (3.8) to $W$.

Let $A$ be a locally compact commutative topological group again, and let $H$ be a Haar measure on $A$. If $f(x)$ is a nonnegative real-valued Borel measurable function on $A$, then

$\tag{3.11} \quad f_a(x) = f(x - a)$

has the same properties for each $a \in A$. Using translation-invariance of Haar measure, one can check that

$\tag{3.12} \quad \int_A f_a(x) \, dH(x) = \int_A f(x) \, dH(x)$

for every $a \in A$. This is the same as (3.4) when $f$ is the characteristic or indicator function on $A$ associated to a Borel set $E \subseteq A$, which is equal to 1 on $E$ and to 0 on $A \setminus E$. If $f$ is a measurable simple function on $A$, which is a linear combination of indicator functions associated to measurable subsets of $A$, then (3.12) follows from the previous case by linearity. One can then get (3.12) for arbitrary nonnegative measurable functions on $A$, by approximating such functions by simple functions. Similarly, $f(-x)$ is a nonnegative real-valued
Borel measurable function on $A$ when $f$ is, because (3.2) is a homeomorphism on $A$, and

$$\int_A f(-x) \, dH(x) = \int_A f(x) \, dH(x),$$

by (3.6). If $f$ is a real or complex-valued integrable function on $A$ with respect to $H$, then it follows that (3.11) is also integrable on $H$ for every $a \in A$, and that (3.12) still holds. In this case, $f(-x)$ is an integrable function on $A$ with respect to $H$ as well, and satisfies (3.13).

Let $C_{com}(A)$ be the space of continuous complex-valued functions on $A$ with compact support. If $f \in C_{com}(A)$, then $f$ is Borel measurable and integrable with respect to $H$, because the Haar measure of a compact subset of $A$ is finite.

Thus

$$L(f) = \int_A f(x) \, dH(x)$$

defines a linear functional on $C_{com}(A)$, as a vector space with respect to pointwise addition and scalar multiplication. Note that $f_a \in C_{com}(A)$ for every $a \in A$ when $f \in C_{com}(A)$, and that

$$L(f_a) = L(f)$$

for every $a \in A$, by (3.12). If $f \in C_{com}(A)$ is real-valued and nonnegative on $A$, then $L(f)$ is a nonnegative real number, because $H$ is a nonnegative measure on $A$. If $f \in C_{com}(A)$ is real-valued and nonnegative on $A$, and if $f(x) > 0$ for some $x \in A$, then $L(f) > 0$, because $H(U) > 0$ when $U$ is a nonempty open set in $A$. Similarly, $C_{com}(A)$ and $L$ are invariant under (3.2).

Conversely, if $L$ is any nonnegative linear functional on $C_{com}(A)$, then the Riesz representation theorem implies that $L$ can be expressed as in (3.14) for a unique nonnegative Borel measure $H$ on $A$ that is finite on compact sets and satisfies certain other regularity conditions. If $L$ is invariant under translations, in the sense that (3.15) holds for every $f \in C_{com}(A)$ and $a \in A$, then the uniqueness of $H$ implies that $H$ is invariant under translations too. If $L(f) > 0$ when $f \in C_{com}(A)$ is real-valued, nonnegative, and $f(x) > 0$ for some $x \in A$, then $H(U) > 0$ for every nonempty open set $U \subseteq A$. This uses Urysohn’s lemma, to get such a function $f$ with compact support contained in $U$.

A linear functional on $C_{com}(A)$ with the properties just described is known as a Haar integral on $A$. Thus the existence and uniqueness of Haar measure can be reformulated in terms of Haar integrals, and are often established in that way. Note that the invariance of a Haar integral under (3.2) can be derived from its uniqueness, in essentially the same way as for Haar measure.

If $A$ is a compact commutative topological group and $H$ is a Haar measure on $A$, then $H(A) < \infty$, and $H$ is often normalized so that $H(A) = 1$. If $A$ is a locally compact commutative topological group which is also $\sigma$-compact, then Haar measure on $A$ is $\sigma$-finite. If there is a base for the topology of $A$ with only finitely or countably many elements, then any open covering of a subset of $A$ can be reduced to a subcovering with only finitely or countably many elements, by Lindelöf’s theorem. In this case, it follows that $A$ is $\sigma$-compact when $A$ is
3.3. DIRECT SUMS AND PRODUCTS

locally compact. An analogous argument implies that every open set in $A$ is $\sigma$-compact under these conditions, using the fact that $A$ is regular as a topological space. Note that closed subsets of $A$ are $\sigma$-compact when $A$ is $\sigma$-compact. If $A$ is metrizable, then it is well known that every open set in $A$ can be expressed as the union of finitely or countably many closed sets. This implies that every open set in $A$ is $\sigma$-compact when $A$ is $\sigma$-compact and metrizable.

3.3 Direct sums and products

Let $I$ be a nonempty set, and suppose that $A_j$ is a commutative group for each $j \in I$. The direct product of the $A_j$’s is the Cartesian product

$$(3.16) \quad \prod_{j \in I} A_j,$$

where the group operations are defined coordinatewise. The direct sum of the $A_j$’s is denoted

$$(3.17) \quad \sum_{j \in I} A_j,$$

and is the subgroup of (3.16) consisting of elements of the product for which all but finitely many coordinates are equal to 0. The direct product (3.16) is also known as the complete direct sum of the $A_j$’s, which is the same as the direct sum when $I$ has only finitely many elements.

If $A_j$ is a topological group for each $j \in I$, then the direct product (3.16) is a topological group too, with respect to the corresponding product topology. Alternatively, one could use the strong product topology on (3.16), which is the topology generated by arbitrary products of open subsets of the $A_j$’s. Of course, this is the same as the product topology on (3.16) when $I$ has only finitely many elements. It is easy to see that the direct sum (3.17) is dense in the direct product (3.16) with respect to the corresponding product topology. However, one can check that the direct sum (3.17) is a closed subgroup of the direct product (3.16) with respect to the strong product topology.

Let $I$ be a nonempty set again, and suppose that $V_j$ is a vector space over the real numbers for each $j \in I$. As before, the direct product of the $V_j$’s is the Cartesian product

$$(3.18) \quad \prod_{j \in I} V_j,$$

where addition and scalar multiplication are defined coordinatewise, and the direct sum

$$(3.19) \quad \sum_{j \in I} V_j$$

of the $V_j$’s is the linear subspace of (3.18) consisting of elements of the product for which all but finitely many coordinates are equal to 0. If $V_j$ is a topological vector space for each $j \in I$, then the direct product (3.18) is a topological
vector space with respect to the product topology. This does not work when \( I \) is infinite and the direct product is equipped with the strong product topology, because scalar multiplication is not continuous as a function of the scalar at vectors with infinitely many nonzero coordinates. However, one can check that the direct sum (3.19) is a topological vector space with respect to the topology induced by the strong product topology on (3.18).

Let \( I \) be a nonempty set with only finitely or countably many elements, and suppose that \( A_j \) is a commutative topological group for each \( j \in I \). If there is a local base for the topology of \( A_j \) at 0 with only finitely or countably many elements for each \( j \in I \), then there is a local base for the product topology on the direct product (3.16) at 0 with only finitely or countably many elements. This is a special case of an analogous statement for arbitrary topological spaces.

In particular, if there is a translation-invariant metric on \( A_j \) that determines the same topology on \( A_j \) for each \( j \in I \), then there is a translation-invariant metric on the direct product (3.16) for which the corresponding topology is the product topology. This can be shown directly, using the same type of constructions as for products of finitely or countably many metric spaces.

Suppose that \( B_j \) is a base for the topology of \( A_j \) for each \( j \in I \). Let \( B \) be the collection of subsets of the direct product (3.16) of the form \( \prod_{j \in I} U_j \), where \( U_j = A_j \) for all but finitely many \( j \in I \), and \( U_j \in B_j \) when \( U_j \neq A_j \). Thus \( B \) is a base for the product topology on (3.16). If \( B_j \) has only finitely or countably many elements for each \( j \in I \), and \( I \) has only finitely or countably many elements, then it is well known that \( B \) has only finitely or countably many elements as well. In this case, it follows that every open set in (3.16) can be expressed as the union of finitely or countably many sets of the form \( \prod_{j \in I} U_j \), where \( U_j \) is an open set in \( A_j \) for each \( j \in I \), and \( U_j = A_j \) for all but finitely many \( j \).

If \( A_1, \ldots, A_n \) are finitely many locally compact commutative topological groups, then their direct product \( \prod_{j=1}^n A_j \) is a locally compact commutative topological group with respect to the product topology. In this case, Haar measure on \( \prod_{j=1}^n A_j \) basically corresponds to a product of Haar measures on the \( A_j \)'s. One way to deal with this is to define a Haar integral as a linear functional on \( C_{com}(\prod_{j=1}^n A_j) \), using Haar integrals on the \( A_j \)'s to integrate each variable separately. Alternatively, suppose that there is a base for the topology of \( A_j \) with only finitely or countably many elements for each \( j \). This implies that \( A_j \) is \( \sigma \)-compact for each \( j \), and hence that Haar measure on \( A_j \) is \( \sigma \)-finite, as in the previous section. This also implies that every open set in \( \prod_{j=1}^n A_j \) can be expressed as the union of finitely or countably many sets of the form \( \prod_{j=1}^n U_j \), where \( U_j \) is an open set in \( A_j \) for each \( j = 1, \ldots, n \), as in the preceding paragraph. Under these conditions, one can define a product measure on \( \prod_{j=1}^n A_j \) corresponding to Haar measures on the \( A_j \)'s, where open subsets of \( \prod_{j=1}^n A_j \) are measurable with respect to the usual product measure construction, starting with Borel sets in \( A_j \) for each \( j \).

Now let \( I \) be a nonempty set, and suppose that \( A_j \) is a compact commutative topological group for each \( j \in I \), so that the direct product (3.16) is also a
compact commutative topological group with respect to the product topology. In this case, we can choose Haar measure $H_j$ on $A_j$ such that $H_j(A_j) = 1$ for each $j \in I$, and Haar measure on (3.16) basically corresponds to a product of the $H_j$’s. As before, one way to deal with this is to define a Haar integral on (3.16), as a linear functional on the space of continuous complex-valued functions on (3.16), using Haar integrals on the $A_j$’s to integrate each variable separately. If $I$ has infinitely many elements, then it is helpful to observe that continuous functions on (3.16) can be approximated uniformly by continuous functions that depend only on finitely many variables. If $I$ has only finitely or countably many elements, and if there is a base for the topology of $A_j$ with only finitely or countably many elements for each $j \in I$, then one can use standard product measure constructions instead.

3.4 Continuous homomorphisms

If $A$ is a nonempty set and $B$ is a commutative group, then the collection of all mappings from $A$ into $B$ is a commutative group with respect to pointwise addition. If $A$ and $B$ are both commutative groups, then the collection of all homomorphisms from $A$ into $B$ is a subgroup of the group of all mappings from $A$ into $B$. Similarly, if $A$ is a nonempty topological space and $B$ is a commutative topological group, then the collection of continuous mappings from $A$ into $B$ is a subgroup of the group of all mappings from $A$ into $B$. If $A$ and $B$ are both commutative topological groups, then the collection of all continuous homomorphisms from $A$ into $B$ is a subgroup of the group of all mappings from $A$ into $B$, which is the same as the intersection of the two previous subgroups. Of course, if $A$ is equipped with the discrete topology, then $\text{Hom}(A, B)$ is the same as the group of all homomorphisms from $A$ into $B$.

Suppose that $A$ and $B$ are commutative topological groups, and that $\phi$ is a homomorphism from $A$ into $B$. If $\phi$ is continuous at $0$, then it is easy to see that $\phi$ is continuous at every point in $A$, by continuity of translations. In this case, the kernel of $\phi$ is a closed subgroup of $A$.

If $B$ is any commutative group, then $j \cdot b \in B$ can be defined for each $j \in \mathbb{Z}$ and $b \in B$ in the usual way, and

\begin{equation}
(3.20) \quad \text{Hom}(A, B)
\end{equation}

of all continuous homomorphisms from $A$ into $B$ is a subgroup of the group of all mappings from $A$ into $B$, which is the same as the intersection of the two previous subgroups. Of course, if $A$ is equipped with the discrete topology, then $\text{Hom}(A, B)$ is the same as the group of all homomorphisms from $A$ into $B$.

Suppose that $A$ and $B$ are commutative topological groups, and that $\phi$ is a homomorphism from $A$ into $B$. If $\phi$ is continuous at $0$, then it is easy to see that $\phi$ is continuous at every point in $A$, by continuity of translations. In this case, the kernel of $\phi$ is a closed subgroup of $A$.

If $B$ is any commutative group, then $j \cdot b \in B$ can be defined for each $j \in \mathbb{Z}$ and $b \in B$ in the usual way, and

\begin{equation}
(3.21) \quad j \mapsto j \cdot b
\end{equation}

defines a homomorphism from $\mathbb{Z}$ into $B$. Conversely, if $\phi$ is any homomorphism from $\mathbb{Z}$ into $B$, then

\begin{equation}
(3.22) \quad \phi(j) = j \cdot \phi(1)
\end{equation}

for every $j \in \mathbb{Z}$.

Similarly,

\begin{equation}
(3.23) \quad x \mapsto ax
\end{equation}
defines a continuous homomorphism from $\mathbb{R}$ into itself for each $a \in \mathbb{R}$. If $\phi$ is any continuous homomorphism from $\mathbb{R}$ into itself, then one can check that

$$\phi(x) = x \phi(1)$$

for every $x \in \mathbb{R}$. More precisely, one can first show that this holds when $x$ is rational, and then for all $x \in \mathbb{R}$, by continuity.

Using the complex exponential function, we get a continuous homomorphism

$$x \mapsto \exp(iax)$$

from $\mathbb{R}$ into $\mathbb{T}$ for every $a \in \mathbb{R}$. If $\phi$ is any continuous mapping from $\mathbb{R}$ into $\mathbb{T}$ such that $\phi(0) = 1$, then there is a unique continuous mapping $\psi$ from $\mathbb{R}$ into itself such that $\psi(0) = 0$ and

$$\phi(x) = \exp(i \psi(x))$$

for every $x \in \mathbb{R}$. If $\phi$ is a continuous homomorphism from $\mathbb{R}$ into $\mathbb{T}$, then one can show that $\psi$ is a continuous homomorphism from $\mathbb{R}$ into itself. This implies that $\psi$ can be expressed as (3.23) for some $a \in \mathbb{R}$, and hence that $\phi$ can be expressed as (3.25).

Of course,

$$z \mapsto z^j$$

defines a continuous homomorphism from $\mathbb{T}$ into itself for every integer $j$. If $\phi$ is a continuous homomorphism from $\mathbb{R}$ into $\mathbb{T}$, then one can show that $\psi$ is a continuous homomorphism from $\mathbb{R}$ into itself. This implies that $\psi$ can be expressed as (3.23) for some $a \in \mathbb{R}$, and hence that $\phi$ can be expressed as (3.25).

Now let $n$ be a positive integer, and consider the quotient group $\mathbb{Z}/n\mathbb{Z}$, which is a cyclic group of order $n$. A homomorphism from $\mathbb{Z}/n\mathbb{Z}$ into a commutative group $B$ is basically the same as a homomorphism from $\mathbb{Z}$ into $B$ whose kernel contains $n\mathbb{Z}$. This is the same as a homomorphism from $\mathbb{Z}$ into $B$ that sends $n \in \mathbb{Z}$ to the identity element of $B$.

Let $w \in \mathbb{T}$ be an $n$th root of unity, so that $w^n = 1$. Thus

$$j \mapsto w^j$$

defines a homomorphism from $\mathbb{Z}$ into $\mathbb{T}$ whose kernel contains $n\mathbb{Z}$, and which therefore corresponds to a homomorphism from $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{T}$. Conversely, every homomorphism from $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{T}$ is of this form.

Suppose that $V$ is a topological vector space over the real numbers, which is a commutative topological group with respect to addition in particular. Every continuous linear mapping from $V$ into $\mathbb{R}$ is a continuous group homomorphism.
3.4. CONTINUOUS HOMOMORPHISMS

with respect to addition. Conversely, every continuous group homomorphism from \( V \) into \( \mathbb{R} \) with respect to addition is linear as a mapping between \( V \) and \( \mathbb{R} \) as vector spaces over \( \mathbb{R} \), because of the earlier characterization of continuous homomorphisms from \( \mathbb{R} \) into itself. This shows that \( \text{Hom}(V, \mathbb{R}) \) can be identified as a commutative group with respect to addition with the dual space \( V' \) of continuous linear functionals on \( V \), which is also a vector space with respect to pointwise scalar multiplication. However, it is well known that there are nontrivial topological vector spaces \( V \) over \( \mathbb{R} \) such that \( V' = \{0\} \). If \( \lambda \) is a continuous linear functional on \( V \), then

\[
(3.30) \quad v \mapsto \exp(i\lambda(v))
\]

defines a continuous homomorphism from \( V \) as a topological group with respect to addition into \( \mathbb{T} \). One can check that every continuous homomorphism from \( V \) into \( \mathbb{T} \) is of this form, using the analogous statement for the real line.

Note that the only bounded subgroup of \( \mathbb{R} \) is the trivial subgroup \( \{0\} \). This implies that every continuous homomorphism from a compact commutative topological group into \( \mathbb{R} \) is trivial. Similarly,

\[
(3.31) \quad \{z \in \mathbb{T}: \text{Re } z > 0\}
\]

is a relatively open set in \( \mathbb{T} \) that contains 1, and any subgroup of \( \mathbb{T} \) contained in (3.31) is trivial. Thus every homomorphism from a commutative group into \( \mathbb{T} \) with values in (3.31) is trivial.

Let \( A \) be a commutative topological group, and let \( \phi \) be a homomorphism from \( A \) into the real or complex numbers, as a commutative topological group with respect to addition and the standard topology. Suppose that there is an open set \( U_0 \subseteq A \) such that \( 0 \in U_0 \) and

\[
(3.32) \quad |\phi(x)| < 1
\]

for every \( x \in U_0 \). Using continuity of addition on \( A \) at 0, one can find a sequence \( U_1, U_2, U_3, \ldots \) of open subsets of \( A \) such that \( 0 \in U_j \) and

\[
(3.33) \quad U_j + U_j \subseteq U_{j-1}
\]

for each \( j \in \mathbb{Z}_+ \). If \( x \in U_j \) for some \( j \geq 0 \), then \( 2^j \cdot x \in U_0 \), and hence

\[
(3.34) \quad 2^j |\phi(x)| = |\phi(2^j \cdot x)| < 1,
\]

which is to say that \( |\phi(x)| < 2^{-j} \). This implies that \( \phi \) is continuous at 0, so that \( \phi \) is continuous on \( A \).

Now let \( \phi \) be a homomorphism from \( A \) into \( \mathbb{T} \), and suppose that there is an open set \( U_0 \) in \( A \) such that \( 0 \in U_0 \) and

\[
(3.35) \quad \text{Re } \phi(x) > 0
\]

for every \( x \in U_0 \). As before, there is a sequence \( U_1, U_2, U_3, \ldots \) of open subsets of \( A \) that satisfy \( 0 \in U_j \) and (3.33) for each \( j \geq 1 \). Note that \( U_j \subseteq U_{j-1} \) for each
\[ j \geq 1, \text{ because } 0 \in U_j, \text{ so that } U_j \subseteq U_0 \text{ for each } j. \text{ If } x \in U_k \text{ for some } k \geq 0, \text{ then } 2^j x \in U_{k-j} \subseteq U_0 \text{ when } 0 \leq j \leq k, \text{ and hence} \]

\[
(3.36) \quad \Re \phi(x)^{2^j} = \Re \phi(2^j \cdot x) > 0
\]

for \( j = 0, \ldots, k \). by (3.35). This implies that

\[
(3.37) \quad \sup_{x \in U_k} |\phi(x) - 1| \to 0
\]

as \( k \to \infty \), so that \( \phi \) is continuous at 0, and thus everywhere on \( A \).

### 3.5 Sums and products, continued

Let \( A_1, \ldots, A_n \) be finitely many commutative groups, and let \( B \) be another commutative group. If \( \phi_j \) is a homomorphism from \( A_j \) into \( B \) for each \( j = 1, \ldots, n \), then

\[
(3.38) \quad \phi(x) = \sum_{j=1}^{n} \phi_j(x_j)
\]

defines a homomorphism from \( \prod_{j=1}^{n} A_j \) into \( B \). Conversely, it is easy to see that every homomorphism from \( \prod_{j=1}^{n} A_j \) into \( B \) is of this form. Suppose now that \( A_1, \ldots, A_n \) and \( B \) are topological groups, and let \( \prod_{j=1}^{n} A_j \) be equipped with the product topology. If \( \phi_j \) is a continuous homomorphism from \( A_j \) into \( B \) for \( j = 1, \ldots, n \), then (3.38) is a continuous homomorphism from \( \prod_{j=1}^{n} A_j \) into \( B \), and every continuous homomorphism from \( \prod_{j=1}^{n} A_j \) into \( B \) is of this form.

Let \( I \) be a nonempty set, let \( A_j \) be a commutative group for each \( j \in I \), and let \( B \) be another commutative group. Also let \( j_1, \ldots, j_l \) be finitely many elements of \( I \), and let \( \phi_{j_k} \) be a homomorphism from \( A_{j_k} \) into \( B \) for \( k = 1, \ldots, l \). Thus

\[
(3.39) \quad \phi(x) = \sum_{k=1}^{l} \phi_{j_k}(x_{j_k})
\]

defines a homomorphism from the direct product \( \prod_{j \in I} A_j \) into \( B \), where \( x_j \in A_j \) denotes the \( j \)th component of \( x \in \prod_{j \in I} A_j \) for each \( j \in I \). If \( A_j \) is a commutative topological group for each \( j \in I \), then we have seen that \( \prod_{j \in I} A_j \) is a commutative topological group with respect to the corresponding product topology. If \( B \) is also a commutative topological group, and \( \phi_{j_k} \) is continuous as a mapping from \( A_{j_k} \) into \( B \) for each \( k \), then it is easy to see that \( \phi \) is continuous with respect to the product topology on \( \prod_{j \in I} A_j \).

Suppose in addition that there is an open set \( V \) in \( B \) that contains the identity element, but does not contain any nontrivial subgroup of \( B \). This property is satisfied by both \( \mathbb{R} \) and \( \mathbb{T} \), as mentioned in the previous section. Let \( \phi \) be a continuous homomorphism from \( \prod_{j \in I} A_j \) into \( B \), where \( \prod_{j \in I} A_j \) is equipped with the product topology. Thus \( \phi^{-1}(V) \) is an open set in \( \prod_{j \in I} A_j \).
that contains 0. It follows that there are open subsets $U_j$ of $A_j$ for each $j \in I$, such that $0 \in U_j$ for each $j \in I$, $U_j = A_j$ for all but finitely many $j \in I$, and

\[(3.40) \quad \prod_{j \in I} U_j \subseteq \phi^{-1}(V).\]

If we put $C_j = A_j$ when $U_j = A_j$, and $C_j = \{0\}$ when $U_j \neq A_j$, then

\[(3.41) \quad \prod_{j \in I} C_j\]

is a subgroup of $\prod_{j \in I} A_j$, and

\[(3.42) \quad \phi\left( \prod_{j \in I} C_j \right) \subseteq V,\]

by (3.40). Because $\phi\left( \prod_{j \in I} C_j \right)$ is a subgroup of $B$, our hypothesis on $V$ implies that $\phi\left( \prod_{j \in I} C_j \right)$ is the trivial subgroup of $B$. This means that $\phi(x)$ depends only on the $j$th coordinate $x_j$ of $x$ for finitely many $j \in I$, since $C_j = A_j$ for all but finitely many $j \in I$. Using this, it is easy to see that $\phi$ can be expressed as in the preceding paragraph under these conditions.

Let $I$ be a nonempty set again, let $A_j$ be a commutative group for each $j \in I$, and let $B$ be another commutative group. If $\phi_j$ is a homomorphism from $A_j$ into $B$ for each $j \in I$, then

\[(3.43) \quad \phi(x) = \sum_{j \in I} \phi_j(x_j)\]

defines a homomorphism from the direct sum $\sum_{j \in I} A_j$ into $B$. Of course, $x_j = 0$ for all but finitely many $j \in I$ when $x \in \sum_{j \in I} A_j$, which implies that $\phi_j(x_j) = 0$ for all but finitely many $j$, so that the sum in (3.43) makes sense. Conversely, it is easy to see that every homomorphism from $\sum_{j \in I} A_j$ into $B$ is of this form.

Now suppose that $A_j$ is a commutative topological group for each $j \in I$, and that $B$ is a commutative topological group as well. If $\phi$ is a homomorphism from $\sum_{j \in I} A_j$ into $B$ such that $\phi(x)$ is continuous as a function of $x_j \in A_j$ for each $j \in I$, then $\phi$ can be expressed as in (3.43), where $\phi_j$ is a continuous homomorphism from $A_j$ into $B$ for each $j$.

Conversely, let $\phi_j$ be a continuous homomorphism from $A_j$ into $B$ for each $j \in I$, and let $\phi$ be the corresponding homomorphism from $\sum_{j \in I} A_j$ into $B$, as in (3.43). Of course, if $I$ has only finitely many elements, then $\sum_{j \in I} A_j$ is the same as $\prod_{j \in I} A_j$, and $\phi$ is continuous with respect to the product topology on $\prod_{j \in I} A_j$. Otherwise, one still has that $\phi(x)$ is continuous as a function of $x_j \in A_j$ for each $j \in I$, and that $\phi(x)$ is continuous as a function of any finite collection of these variables, with respect to the product topology on the corresponding product of finitely many $A_j$'s.
If $I$ is countably infinite, then one can show that $\phi$ is continuous with respect to the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$. To do this, it is convenient to reduce to the case where $I = \mathbb{Z}_+$. Let $V$ be an open set in $B$ that contains 0, and let $V_1, V_2, V_3, \ldots$ be a sequence of open subsets of $B$ such that $0 \in V_j$ and

$$V_j + V_j \subseteq V_{j-1}$$

(3.44)

for each $j \geq 1$, which exists by continuity of addition on $B$ at 0. If we put

$$W_n = V_1 + V_2 + \cdots + V_n$$

(3.45)

for each $n \in \mathbb{Z}_+$, then we get that

$$W_n + V_n \subseteq W_{n-1} + V_{n-1}$$

(3.46)

when $n \geq 2$, by (3.44). This implies that

$$W_n \subseteq W_n + V_n \subseteq V_1 + V_1 \subseteq V$$

(3.47)

for every $n \geq 1$. Suppose that $\phi_j$ is a continuous homomorphism from $A_j$ into $B$ for each $j \in \mathbb{Z}_+$, and let $U_j$ be an open set in $A_j$ such that $0 \in U_j$ and

$$\phi_j(U_j) \subseteq V_j$$

(3.48)

for each $j \in \mathbb{Z}_+$. Put

$$U = \left( \prod_{j=1}^{\infty} U_j \right) \cap \left( \sum_{j=1}^{\infty} A_j \right),$$

(3.49)

which is the set of points in the direct sum $\sum_{j=1}^{\infty} A_j$ whose $j$th component is in $U_j$ for each $j \in \mathbb{Z}_+$. This is a relatively open set in $\sum_{j=1}^{\infty} A_j$ with respect to the topology induced by the strong product topology on $\prod_{j=1}^{\infty} A_j$, by construction. If $\phi$ is as in (3.43), then it is easy to see that

$$\phi(U) \subseteq V,$$

(3.50)

by (3.47) and (3.48). This implies that $\phi$ is continuous on $\sum_{j=1}^{\infty} A_j$ with respect to the topology induced by the strong product topology on $\prod_{j=1}^{\infty} A_j$, as desired.

If $I$ is uncountable, then this does not work, even when $A_j = \mathbb{R}$ for each $j \in I$, and $B = \mathbb{R}$. However, if $A_j$ is equipped with the discrete topology for each $j \in I$, then the corresponding strong product topology on $\prod_{j \in I} A_j$ is the same as the discrete topology, and there is no problem.

### 3.6 Spaces of continuous functions

Let $X$ be a (nonempty) topological space, and let $C(X) = C(X, \mathbb{C})$ be the space of continuous complex-valued functions on $X$. This is a vector space over the complex numbers with respect to pointwise addition and scalar multiplication,
3.6. SPACES OF CONTINUOUS FUNCTIONS

and a commutative algebra with respect to pointwise multiplication. If \( f \) is an element of \( C(X) \) and \( K \subseteq X \) is compact, then \( f(K) \) is a compact subset of \( C \), and in particular the restriction of \( f \) to \( K \) is bounded. Thus we can put

\[
\|f\|_K = \sup_{x \in K} |f(x)|
\]

when \( f \in C(X) \) and \( K \) is a nonempty compact subset of \( X \), and in fact the supremum is attained. It is easy to see that

\[
\|f + g\|_K \leq \|f\|_K + \|g\|_K
\]

and

\[
\|fg\|_K \leq \|f\|_K \|g\|_K
\]

for every \( f, g \in C(X) \), and that

\[
\|tf\|_K = |t| \|f\|_K
\]

for every \( f \in C(X) \) and \( t \in C \). This shows that \( \|f\|_K \) defines a seminorm on \( C(X) \), which is known as the supremum seminorm associated to \( K \). If \( X \) is compact, then we can take \( K = X \), and the supremum seminorm \( \|f\|_X \) is the same as the supremum norm on \( C(X) \).

If \( K \subseteq X \) is nonempty and compact, then the open ball in \( C(X) \) centered at \( f \in C(X) \) and with radius \( r > 0 \) with respect to the supremum seminorm associated to \( K \) is defined by

\[
B_K(f, r) = \{ g \in C(X) : \|f - g\|_K < r \}.
\]

Let us say that a set \( U \subseteq C(X) \) is an open set if for each \( f \in U \) there is an \( r > 0 \) and a nonempty compact set \( K \subseteq X \) such that

\[
B_K(f, r) \subseteq U.
\]

It is easy to see that this defines a topology on \( C(X) \), which makes \( C(X) \) into a locally convex topological vector space over the complex numbers. Of course, finite subsets of \( X \) are compact, which implies that this topology is Hausdorff. Note that the open balls (3.55) are open sets with respect to this topology, by the usual argument with the triangle inequality. One can also check that multiplication on \( C(X) \) defines a continuous mapping from \( C(X) \times C(X) \) into \( C(X) \), using the corresponding product topology on \( C(X) \times C(X) \). Thus \( C(X) \) is actually a topological algebra with respect to this topology. If \( X \) is compact, then this is the same as the topology on \( C(X) \) determined by the supremum norm. If \( X \) is equipped with the discrete topology, then every function on \( X \) is continuous, and \( C(X) \) is the same as the Cartesian product of copies of \( C \) indexed by \( X \). Every compact subset of \( X \) has only finitely many elements in this case, and the corresponding topology on \( C(X) \) is the same as the product topology.
CHAPTER 3. COMMUTATIVE TOPOLOGICAL GROUPS

If $X$ is $\sigma$-compact, then there is a sequence $K_1, K_2, K_3, \ldots$ of nonempty compact subsets of $X$ such that $K_j \subseteq K_{j+1}$ for each $j \geq 1$ and

\begin{equation}
\bigcup_{j=1}^{\infty} K_j = X.
\end{equation}

If $X$ is locally compact as well, then one can enlarge the $K_j$’s, if necessary, to arrange that $K_j$ is contained in the interior of $K_{j+1}$ for each $j$. This implies that each compact set $K \subseteq X$ is contained in $K_j$ for some $j$, because the interiors of the $K_j$’s form an open covering of $K$. It follows that the topology on $C(X)$ determined by the supremum seminorms associated to compact subsets of $X$ can be described by the supremum seminorms associated to the sequence of compact sets $K_j$. If we put

\begin{equation}
d_j(f, g) = \min(\|f - g\|_{K_j}, 1/j)
\end{equation}

and

\begin{equation}
d(f, g) = \max_{j \geq 1} d_j(f, g)
\end{equation}

for each $f, g \in C(X)$, then one can check that $d(f, g)$ is a translation-invariant metric on $C(X)$ that determines the same topology on $C(X)$.

Let $X$ be any nonempty topological space again, and let $C(X, T)$ be the space of continuous mapping from $X$ into $T$. This is a commutative group with respect to pointwise multiplication of functions, and a topological group with respect to the topology induced by the one on $C(X)$ defined earlier. Note that $C(X, T)$ is a closed set in $C(X)$ with respect to this topology, because finite subsets of $X$ are compact. Of course,

\begin{equation}
\|af - ag\|_K = \|f - g\|_K
\end{equation}

for every $a \in C(X, T)$, $f, g \in C(X)$, and nonempty compact set $K \subseteq X$. If $X$ is compact, then we can take $K = X$, to get that the supremum metric on $C(X)$ is invariant under multiplication by elements of $C(X, T)$. Similarly, if $X$ is locally compact and $\sigma$-compact, then (3.59) is invariant under multiplication by elements of $C(X, T)$. If $X$ is equipped with the discrete topology, then $C(X, T)$ is the same as the direct product of copies of $T$ indexed by $X$, which is compact with respect to the corresponding product topology.

Of course, $C(X)$ can also be considered as a vector space over the real numbers, by forgetting about multiplication by $i$. The space $C(X, \mathbb{R})$ of real-valued continuous functions on $X$ may be considered as a real-linear subspace of $C(X)$, and a topological vector space over the real numbers with respect to the topology induced by the one on $C(X)$ discussed earlier. If $f(x)$ is a real-valued continuous function on $X$, then $\exp(if(x))$ is a continuous function on $X$ with values in $T$. It is easy to see that

\begin{equation}
f(x) \mapsto \exp(if(x))
\end{equation}
defines a homomorphism from \( C(X, \mathbb{R}) \), as a commutative group with respect to addition, into \( C(X, \mathbb{T}) \), as a commutative group with respect to multiplication. One can also check that (3.61) is continuous with respect to the corresponding topologies induced by the one on \( C(X) \).

Remember that a subset \( E \) of a commutative topological group \( A \) is totally bounded if for each open set \( U \) in \( A \) that contains the identity element, \( E \) can be covered by finitely many translates of \( U \). If \( A = C(X) \), then it suffices to consider open sets \( U \) of the form \( B_K(0, r) \), where \( K \) is a nonempty compact subset of \( X \) and \( r > 0 \). If \( E \subseteq C(X, \mathbb{T}) \), then one can check that \( E \) is totally bounded as a subset of \( C(X) \), as a commutative topological group with respect to addition, if and only if \( E \) is totally bounded as a subset of \( C(X, \mathbb{T}) \), as a commutative topological group with respect to multiplication.

Note that a sequence or net \( \{f_j\}_j \) of elements of \( C(X) \) converges to \( f \in C(X) \) with respect to the topology on \( C(X) \) considered in this section if and only if \( \{f_j\}_j \) converges to \( f \) uniformly on compact subsets of \( X \). If \( \{f_j\}_j \) is a sequence or net of continuous complex-valued functions on \( X \) that converges to a complex-valued function \( f \) on \( X \) uniformly on compact subsets of \( X \), then the restriction of \( f \) to every compact subset of \( X \) is continuous, by standard arguments. If \( X \) is locally compact, then it follows that \( f \) is continuous on \( X \).

If \( \{x_l\}_{l=1}^\infty \) is a sequence of elements of any topological space \( X \) that converges to an element \( x \) of \( X \), then the set \( K \subseteq X \) consisting of the \( x_l \)'s, \( l \in \mathbb{Z}_+ \), and \( x \) is compact. If \( f \) is a complex-valued function on \( X \) whose restriction to each compact subset of \( X \) is continuous, then it follows that \( f \) is sequentially continuous on \( X \). If \( f \) is sequentially continuous at a point \( p \in X \), and if there is a local base for the topology of \( X \) at \( p \) with only finitely or countably many elements, then \( f \) is continuous at \( p \). This gives another criterion for the continuity of limits of sequences and nets of functions on \( X \) as in the preceding paragraph.

### 3.7 The dual group

If \( A \) is a commutative topological group, then the corresponding dual group \( \hat{A} \) is defined to be the group \( \text{Hom}(A, \mathbb{T}) \) of continuous homomorphisms from \( A \) into \( \mathbb{T} \). This is a subgroup of the group \( C(A, \mathbb{T}) \) of continuous mappings from \( A \) into \( \mathbb{T} \), and hence a topological group with respect to the topology induced on \( \hat{A} \) by the one described in the previous section. It is easy to see that \( \hat{A} \) is also a closed set in \( C(A) \) with respect to this topology, because finite subsets of \( A \) are compact. In particular, if \( A \) is equipped with the discrete topology, then we have seen that \( C(A, \mathbb{T}) \) is compact, which implies that \( \hat{A} \) is compact as well.

Suppose that \( A = \mathbb{R} \), as a commutative group with respect to addition, and equipped with the standard topology. If \( a \in \mathbb{R} \), then (3.25) defines a continuous homomorphism from \( \mathbb{R} \) into \( \mathbb{T} \), and every continuous homomorphism from \( \mathbb{R} \) into \( \mathbb{T} \) is of this form, as in Section 3.4. This defines a group isomorphism between \( \mathbb{R} \) and its dual group. One can check that this isomorphism is also a homeomorphism with respect to the topology on the dual group induced by the
one on $C(R)$ described in the previous section.

If $A$ is a compact commutative topological group, then the topology induced on $\hat{A}$ by the one on $C(A)$ described in the previous section is the same as the discrete topology. To see this, let $\phi_1, \phi_2 \in \hat{A}$ be given, and put

$$\psi(x) = \phi_1(x) \phi_2(x)^{-1},$$

which defines another element of $\hat{A}$. If

$$|\phi_1(x) - \phi_2(x)| < \sqrt{2}$$

for every $x \in A$, then

$$|\psi(x) - 1| < \sqrt{2}$$

for every $x \in A$, and hence $\text{Re} \psi(x) > 0$ for every $x \in A$. Thus $\psi(A)$ is a subgroup of $T$ contained in (3.31), which implies that $\psi(A)$ is the trivial subgroup of $T$, as before. It follows that $\psi(x) = 1$ for every $x \in A$, so that $\phi_1(x) = \phi_2(x)$ for every $x \in A$ under these conditions.

Let $A$ be any commutative topological group again, and let $U$ be an open subset of $A$ that contains 0. It is easy to see that

$$\{ \phi \in \hat{A} : |\phi(x) - 1| \leq 1 \text{ for every } x \in U \}$$

is a closed set in $\hat{A}$ with respect to the topology induced by the usual one on $C(A)$. One can also show that (3.65) is a compact set in $\hat{A}$. To do this, one of the main points is to verify that elements of (3.65) are equicontinuous at 0. If $A$ is locally compact, then one can choose $U$ so that $\overline{U}$ is a compact set in $A$. In this case,

$$\left\{ \phi \in \hat{A} : \sup_{x \in \overline{U}} |\phi(x) - 1| < 1 \right\}$$

is an open set in $\hat{A}$, which is contained in (3.65). This implies that $\hat{A}$ is locally compact when $A$ is locally compact.

If $A$ is any commutative topological group, then the group $\text{Hom}(A, R)$ may be considered as a subgroup of $C(A)$ with respect to addition. In particular, $\text{Hom}(A, R)$ is a topological group with respect to the topology induced by the usual one on $C(A)$. If $\psi \in \text{Hom}(A, R)$, then $\exp(i \psi(a))$ defines a continuous homomorphism from $A$ into $T$. As in the previous section,

$$\psi(a) \mapsto \exp(i \psi(a))$$

defines a homomorphism from $\text{Hom}(A, R)$ into $\text{Hom}(A, T)$. This mapping is also continuous with respect to the corresponding topologies induced by the one on $C(A)$, as before.

Now let $V$ be a topological vector space over the real numbers. The dual space $V'$ of continuous linear functionals on $V$ is a subset of $C(V)$, although it is a bit nicer to think of $V'$ as a linear subspace of the space $C(A, R)$ of real-valued continuous functions on $V$. At any rate, $V'$ is a topological vector
space with respect to the topology induced by the usual topology on \( C(V) \). In particular, \( V' \) is a commutative topological group with respect to addition and this topology. Remember that every continuous homomorphism from \( V \) as a commutative topological group with respect to addition into \( \mathbb{R} \) is linear, as in Section 3.4.

If \( \lambda \in V' \), then \( \exp(i \lambda(v)) \) defines a continuous homomorphism from \( V \) as a commutative topological group with respect to addition into \( \mathbb{T} \). In this situation, every continuous homomorphism from \( V \) into \( \mathbb{T} \) is of this form, as in Section 3.4. It is easy to see that this representation of continuous homomorphisms from \( V \) into \( \mathbb{T} \) is also unique. Thus

\[
\lambda(v) \mapsto \exp(i \lambda(v)) \tag{3.68}
\]
defines a group isomorphism from \( V' = \text{Hom}(V, \mathbb{R}) \), as a commutative group with respect to addition, onto the dual group \( \hat{V} = \text{Hom}(V, \mathbb{T}) \) associated to \( V \) as a commutative topological group with respect to addition. Of course, this mapping is continuous with respect to the topologies induced by the usual one on \( C(V) \), as before.

Let \( K \) be a nonempty compact subset of \( V \), and put

\[
K_1 = \{ t v : t \in \mathbb{R}, \ 0 \leq t \leq 1, \ v \in K \}. \tag{3.69}
\]
It is easy to see that this is a compact subset of \( V \) as well, because of continuity of scalar multiplication. If \( \lambda \in V' \) and \( \exp(i \lambda(v)) \) is uniformly close to 1 on \( K_1 \), then \( \lambda(v) \) is uniformly close to 0 modulo \( 2\pi \mathbb{Z} \) on \( K_1 \). This implies that \( \lambda(v) \) is uniformly close to 0 on \( K_1 \), because \( \lambda \) is linear and \( K_1 \) is star-like about 0, by construction. Using this, one can check that (3.68) is a homeomorphism from \( V' \) onto \( \hat{V} \), with respect to the corresponding topologies induced by the usual one on \( C(V) \).

### 3.8 Compact and discrete groups

Let \( A \) be a compact commutative topological group, and let \( H \) be Haar measure on \( A \), normalized so that \( H(A) = 1 \). If \( \phi \in \hat{A} \), then

\[
\int_A \phi(x) \, dH(x) = \int_A \phi(x + a) \, dH(x) = \phi(a) \int_A \phi(x) \, dH(x) \tag{3.70}
\]
for every \( a \in A \), using translation-invariance of Haar measure in the first step. This implies that

\[
\int_A \phi(x) \, dH(x) = 0 \tag{3.71}
\]
when \( \phi(a) \neq 1 \) for some \( a \in A \), which is to say that \( \phi \) is nontrivial on \( A \). If \( \phi \), \( \psi \) are distinct elements of \( \hat{A} \), then

\[
\phi(a) \overline{\psi(a)} = \phi(a) \psi(a)^{-1} \tag{3.72}
\]
is a nontrivial element of \( \hat{A} \). In this case, it follows that
\[
(3.73) \quad \int_A \phi(x) \overline{\psi(x)} \, dH(x) = 0,
\]
by applying (3.71) to (3.72).

Let \( L^2(A) \) be the usual Hilbert space of complex-valued square-integrable functions on \( A \) with respect to \( H \), with the inner product
\[
(3.74) \quad \langle f, g \rangle = \int_A f(x) \overline{g(x)} \, dH(x).
\]
Thus distinct elements of \( \hat{A} \) are orthogonal with respect to this inner product, by the remarks in the previous paragraph. Note that
\[
(3.75) \quad \langle \phi, \phi \rangle = \int_A |\phi(x)|^2 \, dH(x) = 1
\]
for every \( \phi \in \hat{A} \), because of the normalization \( H(A) = 1 \). If the linear span of the elements of \( \hat{A} \) is dense in \( L^2(A) \), then it follows that the elements of \( \hat{A} \) form an orthonormal basis for \( L^2(A) \).

Let \( \mathcal{E}(A) \) be the linear span of the elements of \( \hat{A} \), as a linear subspace of \( C(A) \). More precisely, \( \mathcal{E}(A) \) is a subalgebra of \( C(A) \) that contains the constant functions and is invariant under complex conjugation. If the elements of \( \hat{A} \) separate points in \( A \), then \( \mathcal{E}(A) \) separates points in \( \hat{A} \) as well, and hence \( \mathcal{E}(A) \) is dense in \( C(A) \) with respect to the supremum norm, by the Stone–Weierstrass theorem. This implies that \( \mathcal{E}(A) \) is dense in \( L^2(A) \), because \( C(A) \) is dense in \( L^2(A) \), since Haar measure is a regular Borel measure on \( A \). It is well known that \( \hat{A} \) separates points on \( A \) when \( A \) is compact, or even locally compact.

Let \( B \) be a subset of \( \hat{A} \), and let \( \mathcal{E}_B(A) \) be the linear span of the elements of \( B \), as a linear subspace of \( C(A) \). If \( B \) is a subgroup of \( \hat{A} \), then it is easy to see that \( \mathcal{E}_B(A) \) is a subalgebra of \( C(A) \) that contains the constant functions and is invariant under complex conjugation. If \( B \) is a subgroup of \( \hat{A} \) that separates points on \( A \), then it follows that \( \mathcal{E}_B(A) \) is dense in \( C(A) \), by the Stone–Weierstrass theorem. As before, this implies that \( \mathcal{E}_B(A) \) is dense in \( L^2(A) \), so that the elements of \( B \) form an orthonormal basis for \( L^2(A) \). Under these conditions, we get that
\[
(3.76) \quad B = \hat{A},
\]
since any element of \( \hat{A} \) not in \( B \) would have to be orthogonal to every element of \( B \), and hence to every element of \( \mathcal{E}_B(A) \).

Now let \( A \) be any commutative group, equipped with the discrete topology. Also let \( A_0 \) be a subgroup of \( A \), let \( a_1 \) be an element of \( A \) not in \( A_0 \), and let \( A_1 \) be the subgroup of \( A \) generated by \( A_0 \) and \( a_1 \). Thus every element of \( A_1 \) can be expressed as
\[
(3.77) \quad ja_1 + x
\]
3.9. THE SECOND DUAL

for some \( j \in \mathbb{Z} \) and \( x \in A_0 \). Observe that

\[(3.78) \{ j \in \mathbb{Z} : ja_1 \in A_0 \}\]

is a subgroup of \( \mathbb{Z} \), which is either trivial or generated by a positive integer \( j_1 \). If (3.78) is trivial, then every element of \( A_1 \) can be expressed as (3.77) for a unique \( j \in \mathbb{Z} \) and \( x \in A_0 \).

Let \( \phi \) be a homomorphism from \( A_0 \) into \( T \), which we would like to extend to a homomorphism from \( A_1 \) into \( T \). The basic idea is to first choose \( \phi(a_1) \in T \), and then to try to define \( \phi \) on \( A_1 \) by putting

\[(3.79) \phi(j a_1 + x) = \phi(a_1)^j \phi(x)\]

for every \( j \in \mathbb{Z} \) and \( x \in A_0 \). This is easy to do when (3.78) is trivial, in which case one can take \( \phi(a_1) \) to be any element of \( T \). Otherwise, if \( j_1 \in \mathbb{Z} \) generates (3.78), then one should choose \( \phi(a_1) \) so that

\[(3.80) \phi(a_1)^{j_1} = \phi(j_1 a_1),\]

where \( \phi(j_1 a_1) \) is defined because \( j_1 a_1 \in A_0 \). Under these conditions, one can check that (3.79) determines a well-defined extension of \( \phi \) to a homomorphism from \( A_1 \) into \( T \), as desired.

If \( A \) is generated by \( A_0 \) and finitely or countably many additional elements of \( A \), then one can repeat the process to extend \( \phi \) to a homomorphism from \( A \) into \( T \). Otherwise, the existence of such an extension can be obtained using Zorn's lemma or the Hausdorff maximality principle. In particular, if \( a_0 \in A \) and \( a_0 \neq 0 \), and if \( A_0 \) is the subgroup of \( A \) generated by \( a_0 \), then it is easy to see that there is a homomorphism \( \phi \) from \( A_0 \) into \( T \) such that \( \phi(a_0) \neq 1 \), which has an extension to a homomorphism from \( A \) into \( T \). If \( a, b \in A \) and \( a \neq b \), then one can apply this to \( a_0 = a - b \), to get a homomorphism \( \phi \) from \( A \) into \( T \) such that \( \phi(a) \neq \phi(b) \).

### 3.9 The second dual

Let \( X \) be a (nonempty) topological space, and let \( C(X) \) be the space of complex-valued continuous functions on \( X \), equipped with the topology discussed in Section 3.6. If \( x \in X \), then

\[(3.81) \Psi_x(f) = f(x)\]

defines a linear functional on \( C(X) \), which is an algebra homomorphism with respect to multiplication. It is easy to see that this is also a continuous function on \( C(X) \) for each \( x \in X \), because \( K = \{ x \} \) is a compact subset of \( X \). Thus

\[(3.82) x \mapsto \Psi_x\]

may be considered as a mapping from \( X \) into the space \( C(C(X)) \) of continuous complex-valued functions on \( C(X) \). More precisely, this is a mapping from
X into the dual space $C(X)'$ of continuous linear functionals on $C(X)$, as a topological vector space over the complex numbers.

Using the topology already defined on $C(X)$, we can get a topology on $C(C(X))$ in the same way. We would like to look at the continuity properties of (3.82), as a mapping into $C(C(X))$ equipped with this topology. To do this, let $E$ be a nonempty compact subset of $C(X)$, and let $C(E)$ be the space of continuous complex-valued functions on $E$, with the topology determined by the supremum norm. We can also think of (3.82) as a mapping into $C(E)$, by restricting $\Psi_a(f)$ to $f \in E$. Continuity of (3.82) as a mapping into $C(C(X))$ is equivalent to continuity of (3.82) as a mapping into $C(E)$ for each nonempty compact set $E \subseteq C(X)$, by the definition of this topology on $C(C(X))$.

Let $E$ be a nonempty compact subset of $C(X)$ again, and let $K$ be a nonempty compact subset of $X$. Also let $C(K)$ be the space of complex-valued continuous functions on $K$, with the topology determined by the supremum norm. There is a natural mapping $R_K$ from $C(X)$ into $C(K)$, which sends a continuous function $f$ on $X$ to its restriction to $K$. It is easy to see that $R_K$ is continuous with respect to the corresponding topologies on $C(X)$ and $C(K)$, by construction. This implies that $R_K(E)$ is a compact subset of $C(K)$. In particular, $R_K(E)$ is totally bounded in $C(K)$, with respect to the supremum norm. This holds when $E$ is totally bounded in $C(X)$ as well.

Under these conditions, (3.82) defines a continuous mapping from $x \in K$ into $C(E)$, using the topology induced on $K$ by the given topology on $X$. This is easy to see when $E$ has only finitely many elements, because each of those elements is a continuous function on $X$. Otherwise, one can basically reduce to this case when $R_K(E)$ is totally bounded in $C(K)$. It follows that (3.82) is continuous as a mapping from $x \in K$ into $C(C(X))$, since this works for every nonempty compact set $E \subseteq C(X)$. If $X$ is locally compact, then we get that (3.82) is continuous as a mapping from $x \in K$ into $C(C(X))$. If $X$ is any topological space, then the continuity of (3.82) on compact subsets of $X$ implies that (3.82) is sequentially continuous, because any convergent sequence in $X$ and limit of that sequence is contained in a compact set in $X$. This implies that (3.82) is continuous as a mapping from $X$ into $C(C(X))$ at a point $p \in X$ when there is a local base for the topology of $X$ at $p$ with only finitely or countably many elements.

Now let $A$ be a commutative topological group, and remember that the dual group $\hat{A}$ is also a commutative topological group with respect to the topology on $\hat{A}$ induced by the usual topology on $C(A)$. This permits us to define the second dual $\hat{\hat{A}}$ to be the dual of $\hat{A}$, consisting of the continuous homomorphisms from $\hat{A}$ into $T$. Put

\[ \Psi_a(\phi) = \phi(a) \]

for each $a \in A$ and $\phi \in \hat{A}$, which is the same as the function $\Psi_a$ on $C(A)$ discussed earlier, restricted to $\hat{A} \subseteq C(A)$. This defines a homomorphism from $\hat{A}$ into $T$ for each $a \in A$, where the group operations on $\hat{A}$ and $T$ are defined by multiplication. As before, $\Psi_a$ is continuous as a mapping from $\hat{A}$ into $T$ for each $a \in A$, because $K = \{a\}$ is a compact subset of $A$. 

Thus $\Psi_a$ may be considered as an element of the second dual group $\hat{\hat{A}}$ for each $a \in A$. Observe that
\begin{equation}
\Psi_a + b(\phi) = \phi(a + b) = \phi(a) \phi(b) = \Psi_a(\phi) \Psi_b(\phi)
\end{equation}
for every $a, b \in A$ and $\phi \in \hat{A}$. This implies that
\begin{equation}
\Psi_a
\end{equation}
defines a homomorphism from $A$ into $\hat{\hat{A}}$, where the group operation on $\hat{\hat{A}}$ is defined in terms of multiplication, as usual. Remember that $\hat{\hat{A}}$ is a commutative topological group with respect to the topology induced on $\hat{\hat{A}}$ by the usual one on $C(\hat{A})$, which involves the nonempty compact subsets of $\hat{A}$. We would like to consider the continuity properties of (3.85) with respect to this topology on $\hat{\hat{A}}$.

Of course, $\hat{\hat{A}} \subseteq C(C(A))$, which leads to a natural restriction mapping from $C(C(A))$ into $C(\hat{\hat{A}})$. It is easy to see that this restriction mapping is continuous with respect to the usual topologies, since compact subsets of $\hat{\hat{A}}$ are compact in $C(A)$ too. If $K$ is a compact subset of $A$, then it follows from the earlier discussion of arbitrary topological spaces that (3.85) is continuous as a mapping from $a \in K$ into $\hat{\hat{A}}$, with respect to the topology on $K$ induced by the given topology on $A$. In particular, (3.85) is continuous as a mapping from $A$ into $\hat{\hat{A}}$ when $A$ is locally compact, and it is sequentially continuous for any $A$. If $\hat{A}$ is locally compact, then it is well known that (3.85) is actually a group isomorphism from $A$ onto $\hat{\hat{A}}$, and a homeomorphism with respect to the corresponding topologies.

Note that (3.85) is injective as a mapping from $A$ into $\hat{\hat{A}}$ if and only if $\hat{\hat{A}}$ separates points in $A$. If $\phi$ and $\psi$ are distinct elements of $\hat{\hat{A}}$, then $\phi(a) \neq \psi(a)$ for some $a \in A$, and hence
\begin{equation}
\Psi_a(\phi) \neq \Psi_a(\psi).
\end{equation}
This implies that
\begin{equation}
\{ \Psi_a : a \in A \}
\end{equation}
separates points in $\hat{\hat{A}}$, and (3.87) is also a subgroup of $\hat{\hat{A}}$. Suppose that $A$ is equipped with the discrete topology, so that $\hat{\hat{A}}$ is compact. In this case, the fact that (3.87) is equal to $\hat{A}$ may be obtained as in (3.76). We also saw that $\hat{A}$ separates points in $A$ in the previous section, so that (3.85) is injective. Because $\hat{\hat{A}}$ is compact, the usual topology on $\hat{\hat{A}}$ is discrete as well, which implies that (3.85) and its inverse are automatically continuous in this case.

### 3.10 Quotient groups

Let $A$ be a commutative topological group, and let $B$ be a subgroup of $A$. Thus the quotient group $A/B$ can be defined in the usual way, with the canonical
quotient homomorphism $q$ from $A$ onto $A/B$, whose kernel is equal to $B$. The corresponding quotient topology on $A/B$ is defined by saying that $V \subseteq A/B$ is an open set if and only if $q^{-1}(V)$ is an open set in $A$. Equivalently, $E \subseteq A/B$ is a closed set if and only if $q^{-1}(E)$ is a closed set in $A$. It is easy to see that this does define a topology on $A/B$, and $q$ is automatically continuous as a mapping from $A$ onto $A/B$ with respect to this topology on $A/B$. If $U$ is any subset of $A$, then
\begin{equation}
q^{-1}(q(U)) = \bigcup_{b \in B} (U + b),
\end{equation}
which is also denoted $U + B$. If $U$ is an open set in $A$, then $U + b$ is an open set in $A$ for every $b \in B$, and hence (3.88) is an open set in $A$ too. This implies that $q(U)$ is an open set in $A/B$ with respect to the quotient topology for every open set $U \subseteq A$, so that $q$ is an open mapping.

It is easy to see that translations on $A/B$ are continuous with respect to the quotient topology, because of the corresponding property of $A$. One can also check that addition on $A/B$ is continuous as a mapping from $(A/B) \times (A/B)$ into $A/B$ with respect to the quotient topology, and using the associated product topology on $(A/B) \times (A/B)$. Because of continuity of translations, it suffices to verify continuity of addition on $A/B$ at 0, again using the analogous property of $A$. Similarly, $x \mapsto -x$ is continuous on $A$, which implies that $A/B$ has the same property.

In order for $A/B$ to be considered a topological group, $\{0\}$ should be a closed set in $A/B$ with respect to the quotient topology. In this situation, this is equivalent to $B$ being a closed subgroup of $A$, and we suppose now that this is the case. As in Section 3.1, this is implies that $A/B$ is Hausdorff, and also regular as a topological space, although one could look these conditions more directly in terms of $A$ as well. If $A$ is locally compact, then there is an open set $U \subseteq A$ and a compact set $K \subseteq U$ such that $0 \in U$ and $U \subseteq K$. This implies that $q(U)$ is an open set in $A/B$ that contains $0$, and that $q(K)$ is a compact set in $A/B$ that contains $q(U)$, so that $A/B$ is locally compact too.

Suppose that $C$ is another commutative topological group, and that $\phi$ is a continuous homomorphism from $A/B$ into $C$, with respect to the quotient topology on $A/B$. This implies that
\begin{equation}
\phi' = \phi \circ q,
\end{equation}
is a continuous homomorphism from $A$ into $C$ whose kernel contains $B$. If $\phi'$ is any homomorphism from $A$ into $C$ whose kernel contains $B$, then $\phi'$ can be expressed as (3.89) for some homomorphism $\phi$ from $A/B$ into $C$. If $\phi'$ is continuous, then it is easy to see that $\phi$ is continuous with respect to the quotient topology on $A/B$.

Similarly, let $A_1$ be another commutative topological group, and let $h$ be a continuous homomorphism from $A$ into $A_1$. If $\phi$ is a continuous homomorphism from $A_1$ into $C$, then
\begin{equation}
\phi' = \phi \circ h
\end{equation}
is a continuous homomorphism from $A$ into $C$, as before. Thus

\[(3.91) \quad \phi \mapsto \phi' \]

defines a mapping from $\text{Hom}(A_1, C)$ into $\text{Hom}(A, C)$, which is a homomorphism. Note that the image of $\text{Hom}(A_1, C)$ under (3.91) is contained in the subgroup of $\text{Hom}(A, C)$ consisting of continuous homomorphisms from $A$ into $C$ whose kernels contain the kernel of $h$ in $A$.

It is easy to see that (3.91) is injective when $h(A)$ is dense in $A_1$, because a continuous function is uniquely determined by its restriction to a dense set. Otherwise, the kernel of (3.91) consists of $\phi \in \text{Hom}(A_1, C)$ whose restriction to $h(A)$ is trivial, which is equivalent to asking that the restriction of $\phi$ to the closure $\overline{h(A)}$ in $A_1$ be trivial, by continuity. As before, this is the same as saying that $\phi$ can be expressed as the composition of the quotient mapping from $A_1$ onto $A_1/\overline{h(A)}$ with a continuous homomorphism from $A_1/\overline{h(A)}$ into $C$.

In particular, one can take $C = T_\ast$, so that (3.91) defines a homomorphism $\hat{h}$ from $\hat{A}_1$ into $\hat{A}$. This is known as the dual homomorphism associated to $h$. One can check that $\hat{h}$ is continuous with respect to the usual topologies on the dual groups, basically because $h$ sends compact subsets of $A$ to compact subsets of $A_1$.

Suppose that $A$ is locally compact, so that there is an open set $U \subseteq A$ and a compact set $K \subseteq A$ such that $0 \in U$ and $U \subseteq K$. If $E \subseteq A/B$ is compact, then $E$ is contained in the union of finitely many translates of $q(U)$, and hence $E$ is contained in the union of finitely many translates of $q(K)$. This implies that $E$ is contained in the image of the union of finitely many translates of $K$ in $A$ under $q$. In particular, $E$ is contained in the image of a compact subset of $A$ under $q$. Using this, one can check that $\tilde{q}$ is a homeomorphism from $(A/B)$ onto its image in $\hat{A}$, with respect to the topology induced on the image of $(A/B)$ by the usual topology on $A$.

We can also think of $B$ as a topological group, with respect to the topology induced by the given topology on $A$. The natural inclusion mapping from $B$ into $A$ is a continuous homomorphism, which leads to a dual homomorphism from $\hat{A}$ into $\hat{B}$. This dual homomorphism is the same as the restriction mapping, which sends a continuous homomorphism from $A$ into $T_\ast$ to its restriction to $B$.

If $A$ is equipped with the discrete topology, then every homomorphism from $B$ into $T_\ast$ can be extended to a homomorphism from $A$ into $T_\ast$, as in Section 3.8. Similarly, if $B$ is compact, and if the elements of $\hat{A}$ separate points in $B$, then it follows that every element of $\hat{B}$ can be expressed as the restriction to $B$ of an element of $\hat{A}$, by another argument from Section 3.8. It is well known that if $A$ is locally compact and $B$ is a closed subgroup of $A$, then every element of $\hat{B}$ can be expressed as the restriction to $B$ of an element of $\hat{A}$. In this case, the restriction mapping from $\hat{A}$ onto $\hat{B}$ is topologically equivalent to a quotient mapping.

Suppose that $A$ is compact, and that $\phi'$ is a continuous homomorphism from $A$ onto a commutative topological group $C$ with kernel equal to $B$. As before, $\phi'$ can be expressed as (3.89), where $\phi$ is a continuous homomorphism from $A/B$
onto $C$. In this case, the kernel of $\phi$ is trivial, which means that $\phi$ is injective. It is well known that a one-to-one continuous mapping from a compact topological space onto a Hausdorff topological space is always a homeomorphism. It follows that $\phi$ is a homeomorphism under these conditions, because $A/B$ is compact when $A$ is compact.

### 3.11 Uniform continuity and equicontinuity

Let $A$ be a commutative topological group, and let $f$ be a complex-valued function on $A$. Let us say that $f$ is uniformly continuous along a set $E \subseteq A$ if for each $\epsilon > 0$ there is an open set $U \subseteq A$ such that $0 \in U$ and

$$|f(x) - f(y)| < \epsilon$$

(3.92)

for every $x \in E$ and $y \in A$ with $y - x \in U$. If $E = A$, then we simply say that $f$ is uniformly continuous on $A$. If $d(\cdot, \cdot)$ is a translation-invariant metric on $A$ that determines the same topology on $A$, then this type of uniform continuity condition is equivalent to the usual one for metric spaces.

It is well known that continuous functions on compact metric spaces are uniformly continuous. Similarly, if $E \subseteq A$ is compact, and if $f$ is continuous as a function on $A$ at every point in $E$, then $f$ is uniformly continuous along $E$. To see this, let $\epsilon > 0$ be given, and for each $p \in E$, let $V(p)$ be an open subset of $A$ such that $0 \in V(p)$ and

$$|f(z) - f(p)| < \epsilon/2$$

(3.93)

for every $z \in p + V(p)$. The continuity of addition on $A$ at 0 implies that for each $p \in E$ there is an open set $U(p) \subseteq A$ that contains 0 and satisfies

$$U(p) + U(p) \subseteq V(p).$$

(3.94)

The collection of open sets $p + U(p)$ with $p \in E$ covers $E$, and hence there are finitely many elements $p_1, \ldots, p_n$ of $E$ such that

$$E \subseteq \bigcup_{j=1}^{n} (p_j + U(p_j)),$$

(3.95)

because $E$ is compact. Put

$$U = \bigcap_{j=1}^{n} U(p_j),$$

(3.96)

so that $U$ is also an open set in $A$ that contains 0, and let us check that (3.92) holds for every $x \in E$ and $y \in x + U$. If $x \in E$, then $x \in p_j + U(p_j)$ for some $j = 1, \ldots, n$, by (3.95). If $y \in x + U$, then

$$y \in (p_j + U(p_j)) + U \subseteq p_j + U(p_j) + U(p_j) \subseteq p_j + V(p_j),$$

(3.97)
using the fact that $U \subseteq U(p_j)$ in the second step, and (3.94) in the third step. Of course, $U(p_j) \subseteq V(p_j)$, because of (3.94), so that $x \in p_j + V(p_j)$ as well. Thus we can apply (3.93) with $p = p_j$ to $z = x$ and $z = y$, to get that

$$\text{(3.98)} \quad |f(x) - f(y)| \leq |f(x) - f(p_j)| + |f(p_j) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

as desired.

In particular, if $A$ is compact, then it follows that every continuous function on $A$ is uniformly continuous. Similarly, if $A$ is locally compact, then continuous functions on $A$ with compact support are uniformly continuous on $A$. Remember that a continuous function $f$ on $A$ is said to vanish at infinity if for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subseteq A$ such that

$$\text{(3.99)} \quad |f(x)| < \epsilon$$

for every $x \in A \setminus K(\epsilon)$. It is easy to see that $f$ is uniformly continuous on $A$ in this case too, because $f$ is uniformly continuous along $K(\epsilon)$ for every $\epsilon > 0$.

A collection $\mathcal{E}$ of complex-valued functions on $A$ is said to be equicontinuous at a point $x \in A$ if for each $\epsilon > 0$ there is an open set $U \subseteq A$ such that $0 \in U$ and (3.92) holds for every $f \in \mathcal{E}$ and $y \in x + U$. Similarly, $\mathcal{E}$ is uniformly equicontinuous along a set $E \subseteq A$ if for every $\epsilon > 0$ there is an open set $U \subseteq A$ such that $0 \in U$ and (3.92) holds for every $f \in \mathcal{E}$, $x \in E$, and $y \in A$ with $y - x \in U$. If $\mathcal{E}$ is equicontinuous at every point $x \in E$ and $E \subseteq A$ is compact, then $\mathcal{E}$ is uniformly equicontinuous along $E$, by essentially the same argument as before.

Suppose that $f$ is a homomorphism from $A$ into $\mathbb{C}$, where $\mathbb{C}$ is considered as a commutative group with respect to addition. If $f$ is continuous at 0, then it is easy to see that $f$ is uniformly continuous on $A$. If $\mathcal{E}$ is a collection of homomorphisms from $A$ into $\mathbb{C}$ that is equicontinuous at 0, then $\mathcal{E}$ is uniformly equicontinuous on $A$. In fact, it suffices to ask that there be an open set $U \subseteq A$ such that $0 \in U$ and

$$\text{(3.100)} \quad |f(x)| < 1$$

for every $x \in U$ and $f \in \mathcal{E}$. This implies that each $f \in \mathcal{E}$ is continuous at 0, as in Section 3.4, and the same argument shows that $\mathcal{E}$ is equicontinuous at 0 under these conditions.

Suppose now that $f$ is a homomorphism from $A$ into $\mathbb{T}$, where $\mathbb{T}$ is considered as a commutative group with respect to multiplication. If $f$ is continuous at 0, then one can check that $f$ is uniformly continuous as a complex-valued function on $A$. As before, if $\mathcal{E}$ is a collection of homomorphisms from $A$ into $\mathbb{T}$ that is equicontinuous at 0, then $\mathcal{E}$ is uniformly equicontinuous on $A$. In this situation, it suffices to ask that there be an open set $U \subseteq A$ such that $0 \in U$ and

$$\text{(3.101)} \quad \text{Re } f(x) > 0$$

for every $x \in U$ and $f \in \mathcal{E}$. This implies that each $f \in \mathcal{E}$ is continuous at 0, as in Section 3.4, and essentially the same argument shows that $\mathcal{E}$ is equicontinuous at 0.
Let $X$ be a (nonempty) topological space, and consider $A = C(X)$ as a commutative topological group with respect to pointwise addition of functions and the usual topology. Also let $K$ be a nonempty compact subset of $X$, and put

$$E_K = \{ \Psi_x : x \in K \},$$

(3.102)

where $\Psi_x(f) = f(x)$ for each $f \in C(X)$, as in Section 3.9. Thus $E_K$ is a collection of continuous linear functionals on $C(X)$, and it is easy to see that $E_K$ is equicontinuous at 0 on $C(X)$, by definition of the usual topology on $C(X)$. Alternatively, one can consider $A = C(X, T)$ as a commutative topological group with respect to pointwise multiplication of functions and the topology induced by the usual one on $C(X)$. In this case, $E_K$ corresponds to a collection of continuous homomorphisms from $C(X, T)$ into $T$, which is automatically equicontinuous at the identity element.

### 3.12 Direct products, revisited

Let $A_1, \ldots, A_n$ be finitely many commutative topological groups, and consider their Cartesian product

$$A = \prod_{j=1}^n A_j,$$

(3.103)

as a commutative topological group with respect to coordinatewise addition and the product topology. As in Section 3.5, the dual group $\hat{A}$ can be identified with the product of the dual groups $\hat{A}_j$, $j = 1, \ldots, n$. It is easy to see that the usual topology on $\hat{A}$, induced by the topology defined earlier on $C(A)$, corresponds exactly to the product topology on $\prod_{j=1}^n \hat{A}_j$, where $\hat{A}_j$ has the analogous topology for each $j$. More precisely, if $K_j$ is a compact subset of $A_j$ for $j = 1, \ldots, n$, then

$$K = \prod_{j=1}^n K_j$$

(3.104)

is a compact subset of $A$. Conversely, if $E$ is any compact subset of $A$, then the projection of $E$ in $A_j$ is compact for each $j$, which implies that $E$ is contained in a product of compact sets. Thus the usual topology on $C(A)$ can be defined in terms of nonempty compact subsets of $A$ of the form (3.104). Using this, one can check that the induced topology on $\hat{A}$ corresponds to the product topology on $\prod_{j=1}^n \hat{A}_j$. There are analogous statements for $\text{Hom}(A, R)$ instead of $\hat{A}$.

Now let $I$ be a nonempty set, let $A_j$ be a commutative topological group for each $j \in I$, and let

$$A = \prod_{j \in I} A_j$$

(3.105)

be the direct product of the $A_j$’s, equipped with the product topology. If $K_j$ is
3.12. DIRECT PRODUCTS, REVISITED

a compact subset of \( A_j \) for each \( j \in I \), then

\[
K = \prod_{j \in I} K_j
\]  

(3.106)

is a compact subset of \( A \), by Tychonoff’s theorem. As before, every compact subset of \( A \) is contained in a product of compact subsets of the \( A_j \)'s, so that the usual topology on \( C(A) \) can be defined in terms of nonempty compact subsets of \( A \) of the form (3.106). We may as well suppose that \( 0 \in K_j \) for each \( j \in I \), since otherwise we can replace \( K_j \) with \( K_j \cup \{0\} \) for each \( j \). Similarly, we may as well ask that

\[
K_j = -K_j
\]

(3.107)

for each \( j \in I \), since otherwise we can replace \( K_j \) with \( K_j \cup (-K_j) \) for each \( j \).

As in Section 3.5, \( \text{Hom}(A, R) \) can be identified with the direct sum

\[
\sum_{j \in I} \text{Hom}(A_j, R),
\]

(3.108)

because any subgroup of \( R \) contained in a bounded neighborhood of 0 is trivial. More precisely, if \( \phi \) is an element of \( \text{Hom}(A, R) \), then \( \phi \) can be expressed as

\[
\phi(x) = \sum_{l=1}^{n} \phi_{j_l}(x_{j_l}),
\]

(3.109)

where \( j_1, \ldots, j_n \) are finitely many elements of \( I \), and \( \phi_{j_l} \) is an element of \( \text{Hom}(A_{j_l}, R) \) for \( l = 1, \ldots, n \). Note that \( \text{Hom}(A_j, R) \) may be considered as a vector space over the real numbers with respect to pointwise addition and scalar multiplication for each \( j \in I \), which is a real-linear subspace of \( C(A_j) \). Similarly, \( \text{Hom}(A, R) \) is a real-linear subspace of \( C(A) \), and \( \text{Hom}(A, R) \) can be identified with the direct sum (3.108) as a sum of vector spaces. We can also identify \( C(A_j) \) with the subspace of \( C(A) \) consisting of functions of \( x \in A \) that depend only on \( x_j \in A_j \) for each \( j \in I \), so that the direct sum

\[
\sum_{j \in I} C(A_j)
\]

(3.110)

corresponds to a linear subspace of \( C(A) \) as well.

Let \( K_j \) be a compact subset of \( A_j \) with \( 0 \in K_j \) for each \( j \in I \), and let \( K \) be the product of the \( K_j \)'s, as in (3.106). If \( \|f\|_K \) is the supremum seminorm on \( C(A) \) associated to \( K \) and \( \phi \) is as in (3.109), then

\[
\|\phi\|_K = \sup_{x \in K} |\phi(x)| = \sup_{x \in K} \left| \sum_{l=1}^{n} \phi_{j_l}(x_{j_l}) \right|.
\]

(3.111)

Observe that

\[
\tilde{K} = \{ x \in K : x_j \neq 0 \text{ for at most one } j \in I \},
\]

(3.112)
is also a compact subset of \( A \), because it can be expressed as the intersection of \( K \) with a closed set in \( A \). Of course, \( \| \phi \|_K \) is greater than or equal to

\[
\| \phi \|_K = \max_{1 \leq l \leq n} \sup_{x_{ji} \in K_{ji}} |\phi_{ji}(x_{ji})|,
\]

because \( \bar{K} \subseteq K \). If \( K_j = -K_j \) for each \( j \in I \), then (3.111) reduces to

\[
\| \phi \|_K = \sup_{x \in K} \sum_{l=1}^{n} |\phi(x_{jl})|,
\]

since we can replace \( x_{jl} \) with \( -x_{jl} \) for any \( l = 1, \ldots, n \), to get \( \phi(x_{jl}) \geq 0 \).

If \( B \) is a commutative topological group and \( r \in \mathbb{Z}_+ \), then it is easy to see that \( b \mapsto r \cdot b \) defines a continuous mapping on \( B \), because of continuity of addition on \( B \). Let \( r_j \) be a positive integer for each \( j \in I \), and let \( r_j \cdot K_j \) be the subset of \( A_j \) consisting of multiples of elements of \( K_j \) by \( r_j \). Thus \( r_j \cdot K_j \) is a compact subset of \( A_j \) for each \( j \in I \), so that

\[
K' = \prod_{j \in I} (r_j \cdot K_j)
\]

is a compact set in \( A \). If \( \bar{K}' \) corresponds to \( K' \) as in (3.112), then

\[
\| \phi \|_{\bar{K}'} = \max_{1 \leq l \leq n} \sup_{x_{ji} \in K_{ji}} r_{ji} |\phi_{ji}(x_{ji})|,
\]

as in (3.113). If \( K_j = -K_j \) for each \( j \in I \), then we get that

\[
\| \phi \|_{K'} = \sup_{y \in K'} \sum_{l=1}^{n} |\phi_{ji}(y_{jl})| = \sup_{x \in K} \sum_{l=1}^{n} r_{ji} \sum_{l=1}^{n} |\phi_{ji}(x_{jl})|,
\]

as in (3.114).

The topology on \( \text{Hom}(A, \mathbb{R}) \) induced by the usual topology on \( C(A) \) may be described by the seminorms (3.111), or equivalently by the seminorms (3.114), where \( K \subseteq A \) is as before. The seminorms (3.113) also determine a topology on \( \text{Hom}(A, \mathbb{R}) \), and every open set in \( \text{Hom}(A, \mathbb{R}) \) with respect to the seminorms (3.113) is an open set with respect to the topology on \( \text{Hom}(A, \mathbb{R}) \) determined by the seminorms (3.111), because (3.113) is less than or equal to (3.111). Of course, these topologies are the same when \( I \) has only finitely many elements, and one can show that they are the same when \( I \) is countably infinite as well. More precisely, if \( I \) is countable, then one can choose positive integers \( r_j \) for \( j \in I \) such that

\[
\sum_{j \in I} 1/r_j < \infty.
\]

Using this, one can estimate (3.114) in terms of (3.116), to show that the two topologies on \( \text{Hom}(A, \mathbb{R}) \) are the same in this case.
3.13. DIRECT SUMS, REVISITED

Remember that we can identify Hom(A, R) with the direct sum (3.108) for any I, which may be considered as a linear subspace of the Cartesian product

\[ \prod_{j \in I} \text{Hom}(A_j, R). \tag{3.119} \]

Using (3.116), one can check that the topology determined on Hom(A, R) by the seminorms (3.113) is the same as the topology induced on Hom(A, R) by the strong product topology on (3.119) associated to the topology on Hom(A_j, R) induced by the usual topology on C(A_j) for each j \in I.

There is an analogous discussion for \( \hat{A} = \text{Hom}(A, T) \), with some additional complications. One of the main points is that instead of using subsets of A_j of the form \( r_j \cdot K_j \) as before, it is better to take finite unions of multiples of K_j. This is to ensure that if a homomorphism \( \phi_j \) from A_j into T is reasonably close to 1 on the larger set, then it will be as close to 1 as we want on K_j.

Of course, if A_j is compact, then Hom(A_j, R) is trivial, and the topology induced on \( \hat{A}_j \) by the usual topology on C(A_j) is the discrete topology. If A_j is compact for each j \in I, then A is compact with respect to the product topology, Hom(A, R) is trivial, and the topology induced on \( \hat{A} \) by the usual topology on C(A) is the discrete topology. The strong product topology on

\[ \prod_{j \in I} \hat{A}_j \tag{3.120} \]

corresponding to the discrete topology on \( \hat{A}_j \) for each j \in I is the same as the discrete topology on (3.120). Thus the topology induced on

\[ \sum_{j \in I} \hat{A}_j \tag{3.121} \]

as a subgroup of (3.120) by the strong product topology on (3.120) is the discrete topology as well. In this case, we do not need to be concerned with other topologies on (3.121).

3.13 Direct sums, revisited

Let I be a nonempty set again, and let A_j be a commutative topological group for each j \in I. Put

\[ A = \sum_{j \in I} A_j, \tag{3.122} \]

which may be considered as a subgroup of the Cartesian product

\[ \prod_{j \in I} A_j. \tag{3.123} \]

If I has only finitely many elements, then A is the same as (3.16), and we are back to the situation discussed in the previous section. In this section, we shall
be especially interested in the topology induced on \( A \) by the strong product topology on (3.123) corresponding to the given topology on \( A_j \) for each \( j \in I \), and perhaps some stronger topologies on \( A \) as well. Remember that the strong product topology on (3.123) is the same as the discrete topology when \( A_j \) is equipped with the discrete topology for each \( j \).

Let \( E \) be a subset of \( A \), and put

\[
I(E) = \{ j \in I : \text{there is an } x \in E \text{ such that } x_j \neq 0 \}.
\]

If \( j \in I(E) \), then let \( U_j \) be an open set in \( A_j \) such that \( 0 \in U_j \) and there is an \( x \in E \) with \( x_j \notin U_j \). Otherwise, put \( U_j = A_j \) when \( j \in I \setminus I(E) \). Thus

\[
U = \prod_{j \in I} U_j
\]

is an open set in (3.123) with respect to the strong product topology, which implies that

\[
V = A \cap U
\]

is a relatively open set in \( A \) with respect to the topology induced on \( A \) by the strong product topology on (3.123). Note that \( 0 \in U_j \) for every \( j \in I \), by construction, so that \( 0 \in V \subseteq U \).

We also have that

\[
a + U = \prod_{j \in I} (a_j + U_j)
\]

for every element \( a \) of (3.123), and that

\[
a + V = A \cap (a + U)
\]

when \( a \in A \). If \( a_j = 0 \) and \( j \in I(E) \), then there is an \( x \in E \) such that \( x_j \notin U_j = a_j + U_j \), so that \( x \notin a + U \), and hence \( x \notin a + V \). Remember that \( a_j = 0 \) for all but finitely many \( j \in I \) when \( a \in A \), by the definition of \( A \). If \( I(E) \) has infinitely many elements, then it follows that \( E \) cannot be covered by finitely many translates of \( V \) in \( A \).

If \( E \) is totally bounded in \( A \) with respect to the topology induced by the strong product topology on (3.123), then \( E \) can be covered by finitely many translates of \( V \) in \( A \). Thus \( I(E) \) can have only finitely many elements when \( E \) is totally bounded in \( A \) with respect to this topology, by the argument in the preceding paragraph. In particular, this holds when \( E \) is compact with respect to this topology on \( A \). Of course, this also holds when \( E \) is totally bounded or compact with respect to any stronger topology on \( A \).

If \( I_1 \) is a nonempty subset of \( I \), then we can identify

\[
\prod_{j \in I_1} A_j
\]

with the subgroup of (3.123) consisting of points whose \( j \)th coordinate is equal to 0 when \( j \in I \setminus I_1 \). The strong product topology on (3.129) associated to
the given topologies on the $A_j$'s corresponds exactly to the topology induced on this subgroup of (3.123) by the strong product topology there. Similarly,

\[(3.130) \quad \sum_{j \in I_1} A_j\]

can be identified with a subgroup of (3.122), and the topology induced on (3.130) by the strong product topology on (3.129) corresponds to the one induced by the topology on (3.122) that is induced by the strong product topology on (3.123). If $I_1$ has only finitely many elements, then (3.130) is the same as (3.129), and the strong product topology on (3.129) reduces to the product topology.

In particular, we can identify $E \subseteq A$ with a subset of (3.130) when $I(E) \subseteq I_1$. If $E$ is totally bounded or compact with respect to the topology induced on $A$ by the strong product topology on (3.122), then the corresponding subset of (3.130) will have the analogous property, since we have already seen that the topologies match up. Conversely, if a subset of (3.130) is totally bounded or compact with respect to the topology induced by the strong product topology on (3.129), then it will correspond to a subset of $A$ with the analogous property.

The Cartesian product of totally bounded or compact subsets of the $A_j$'s with $j \in I_1$ is totally bounded or compact with respect to the product topology on (3.129), respectively. Conversely, every totally bounded or compact subset of (3.129) with respect to the product topology is contained in a Cartesian product of totally bounded or compact subsets of the $A_j$'s, as in the previous section. If $I_1$ has only finitely many elements, then the product topology on (3.129) is the same as the strong product topology, and (3.129) is the same as (3.130).

The usual topology on $C(A)$ is defined in terms of the supremum seminorms associated to nonempty compact subsets of $A$, where we continue to take $A$ to be equipped with the topology induced by the strong product topology on (3.123). The preceding discussion implies that compact subsets of $A$ with respect to this topology correspond to compact subsets of sub-sums of the form (3.130), where $I_1 \subseteq I$ has only finitely many elements. Thus one can get the same topology on $C(A)$ by considering nonempty compact subsets of $A$ that correspond to Cartesian products of compact subsets of $A_j$ for finitely many $j \in I$. 
Chapter 4

Some harmonic analysis

4.1 Fourier transforms

Let $A$ be a commutative topological group, and let $\mu$ be a complex Borel measure on $A$. The Fourier transform of $\mu$ is the function $\hat{\mu}$ defined on the dual group $\hat{A} = \text{Hom}(A, \mathbb{T})$ by

$$
\hat{\mu}(\phi) = \int_A \overline{\phi(x)} \, d\mu(x).
$$

(4.1)

Observe that

$$
|\hat{\mu}(\phi)| \leq \int_A |\phi(x)| \, d|\mu|(x) = |\mu|(A)
$$

(4.2)

for every $\phi \in \hat{A}$, where $|\mu|$ is the total variation measure on $A$ associated to $\mu$.

Suppose for the moment that $\mu$ is supported on a compact set $K \subseteq A$, so that $|\mu|(A \setminus K) = 0$, and

$$
\hat{\mu}(\phi) = \int_K \overline{\phi(x)} \, d\mu(x)
$$

(4.3)

for every $\phi \in \hat{A}$. This implies that

$$
|\hat{\mu}(\phi) - \hat{\mu}(\psi)| \leq \int_K |\phi(x) - \psi(x)| \, d|\mu|(x)
$$

$$
\leq \left( \sup_{x \in K} |\phi(x) - \psi(x)| \right) |\mu|(A)
$$

(4.4)

for every $\phi, \psi \in \hat{A}$. It follows that $\hat{\mu}$ is continuous on $\hat{A}$ with respect to the topology induced by the usual one on $C(A)$. Equivalently,

$$
|\hat{\mu}(\phi) - \hat{\mu}(\psi)| \leq \left( \sup_{x \in K} |\phi(x) \psi(x)^{-1} - 1| \right) |\mu|(A)
$$

(4.5)

for every $\phi, \psi \in \hat{A}$, which shows more explicitly that $\hat{\mu}$ is uniformly continuous on $\hat{A}$ as a commutative topological group with respect to pointwise multiplication of functions and the topology induced by the usual topology on $C(A)$. 

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Now suppose that for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subseteq A$ such that

\[(4.6) \quad |\mu|(A \setminus K(\epsilon)) < \epsilon,\]

so that $\mu$ can be approximated with respect to the total variation norm by complex Borel measures supported on compact subsets of $A$. This implies that $\hat{\mu}$ can be approximated uniformly on $\hat{A}$ by the Fourier transforms of complex Borel measures with compact support on $A$, because of (4.2). It follows that $\hat{\mu}$ is uniformly continuous on $\hat{A}$ in this case too. In particular, this condition holds automatically when $A$ is $\sigma$-compact. This condition also holds when $A$ is locally compact and $\mu$ is a regular Borel measure on $A$, which means that $|\mu|$ is a regular Borel measure on $A$.

Let us restrict our attention from now on in this section to the case where $A$ is locally compact, and let $H$ be a choice of Haar measure on $A$. If $f(x)$ is a complex-valued integrable function on $A$ with respect to $H$, then the Fourier transform of $f$ is the function $\hat{f}$ defined on $\hat{A}$ by

\[(4.7) \quad \hat{f}(\phi) = \int_A f(x) \overline{\phi(x)} \, dH(x).\]

As before,

\[(4.8) \quad |\hat{f}(\phi)| \leq \int_A |f(x)||\phi(x)| \, dH(x) = \int_A |f(x)| \, dH(x)\]

for every $\phi \in \hat{A}$. If we put

\[(4.9) \quad \mu(E) = \int_E f(x) \, dH(x)\]

for every Borel set $E \subseteq A$, then $\mu$ is a complex Borel measure on $A$, and the Fourier transform of $f$ on $A$ is the same as the Fourier transform of $\mu$. More precisely, $\mu$ is a regular Borel measure on $A$ under these conditions, and hence $\hat{f}$ is uniformly continuous on $\hat{A}$.

Let $L^1(A)$ be the usual vector space of complex-valued integrable functions on $A$ with respect to $H$. If $f \in L^1(A)$ and $a \in A$, then

\[(4.10) \quad f_a(x) = f(x - a)\]

also defines an element of $L^1(A)$. Observe that

\[(4.11) \quad \hat{f_a}(\phi) = \int_A f(x - a) \overline{\phi(x)} \, dH(x) = \int_A f(x) \overline{\phi(x + a)} \, dH(x) = \hat{f}(\phi) \overline{\phi(a)}\]

for every $\phi \in \hat{A}$. This uses the translation-invariance of Haar measure in the second step, and the fact that $\phi : A \to T$ is a homomorphism in the third step.
CHAPTER 4. SOME HARMONIC ANALYSIS

Suppose for the moment that \( f \) is a continuous complex-valued function on \( A \) supported on a compact set \( K \subseteq A \). Also let \( U \) be an open set in \( A \) such that \( 0 \in U \) and \( \overline{U} \) is compact, which exists because \( A \) is supposed to be locally compact. Thus \( K \times \overline{U} \) is a compact subset of \( A \times A \) with respect to the product topology associated to the given topology on \( A \). This implies that \( K + \overline{U} \) is a compact set in \( A \), because addition on \( A \) defines a continuous mapping from \( A \times A \) into \( A \). Note that the support of \( f_a \) is contained in \( K + a \) for every \( a \in A \), which is contained in \( K + \overline{U} \) when \( a \in \overline{U} \).

It follows that

\[
\int_A |f(x) - f(x - a)| \, dH(x) \leq \left( \sup_{x \in K + \overline{U}} |f(x) - f(x - a)| \right) H(K + \overline{U})
\]

for every \( a \in \overline{U} \). Of course, \( H(K + \overline{U}) < \infty \), because \( K + \overline{U} \) is compact. Using uniform continuity as in Section 3.11, we get that

\[
\lim_{a \to 0} \int_A |f(x) - f(x - a)| \, dH(x) = 0
\]

where the limit is taken over \( a \in A \) converging to 0 with respect to the given topology on \( A \). It is well known that continuous functions with compact support on \( A \) are dense in \( L^1(A) \), because of the regularity properties of Haar measure. This permits one to extend (4.13) to all \( f \in L^1(A) \).

Let \( f \in L^1(A) \) and \( \epsilon > 0 \) be given, and let \( V \) be an open set in \( A \) such that \( 0 \in V \) and

\[
\int_A |f(x) - f(x - a)| \, dH(x) \leq \epsilon
\]

for every \( a \in V \). This implies that

\[
|\hat{f}(\phi) - \hat{f}_a(\phi)| \leq \int_A |f(x) - f_a(x)| \, dH(x) \leq \epsilon
\]

for every \( a \in V \) and \( \phi \in \mathcal{A} \), as in (4.8). Combining this with (4.11), we get that

\[
|\hat{f}(\phi)| |\phi(a) - 1| \leq \epsilon
\]

for every \( \phi \in \mathcal{A} \) and \( a \in V \). As in Section 3.7,

\[
\{ \phi \in \mathcal{A} : |\phi(a) - 1| \leq 1 \text{ for every } a \in V \}
\]

is a compact set in \( \mathcal{A} \). If \( \hat{\phi} \in \mathcal{A} \) is not an element of (4.17), then

\[
|\phi(a) - 1| > 1
\]

for some \( a \in V \), and hence

\[
|\hat{f}(\phi)| < \epsilon
\]
4.2. CONVOLUTION OF MEASURES

by (4.16). This shows that \( \hat{f} \) vanishes at infinity on \( \hat{A} \) for every \( f \in L^1(A) \). Of course, this is vacuous when \( A \) is equipped with the discrete topology, so that \( \hat{A} \) is compact.

As in Section 2.1, the Dirac mass on \( A \) associated to a point \( a \in A \) is the Borel measure \( \delta_a \) defined by \( \delta_a(E) = 1 \) when \( a \in E \) and \( \delta_a(E) = 0 \) when \( a \in A \setminus E \). In this case,

\[
\hat{\delta}_a(\phi) = \overline{\phi(a)} \tag{4.20}
\]

for every \( \phi \in \hat{A} \), and in particular \( |\hat{\delta}_a(\phi)| = 1 \).

4.2 Convolution of measures

Let \( A \) be a commutative topological group again, and put

\[
E_2 = \{ (x, y) \in A \times A : x + y \in E \} \tag{4.21}
\]

for each subset \( E \) of \( A \), which is the same as the inverse image of \( E \) under the mapping from \( A \times A \) into \( A \) corresponding to addition in \( A \). Thus \( E_2 \) is an open set in \( A \times A \) with respect to the product topology when \( E \) is an open set in \( A \), by continuity of addition on \( A \). Similarly, \( E_2 \) is a closed set in \( A \times A \) when \( A \) is a closed set in \( A \). It follows that \( E_2 \) is a Borel set in \( A \times A \) when \( E \) is a Borel set in \( A \).

If \( \mu \) and \( \nu \) are complex Borel measures on \( A \), then we would like to define their convolution \( \mu \ast \nu \) to be the complex Borel measure on \( A \) given by

\[
(\mu \ast \nu)(E) = (\mu \times \nu)(E_2) \tag{4.22}
\]

for every Borel set \( E \subseteq A \), where \( \mu \times \nu \) is the corresponding product measure on \( A \times A \). More precisely, in order to use the standard product measure construction here, \( E_2 \) should be in the \( \sigma \)-algebra of subsets of \( A \times A \) generated by products of Borel subsets of \( A \). If there is a countable base for the topology of \( A \), then there is a countable base for the product topology on \( A \times A \), consisting of product of basic open subsets of \( A \). In particular, this implies that every open subset of \( A \times A \) can be expressed as a countable union of products of open subsets of \( A \), so that open subsets of \( A \times A \) are measurable with respect to the standard product measure construction. In this case, Borel subsets of \( A \times A \) are measurable with respect to the standard product measure construction, as desired.

Alternatively, suppose that \( A \) is locally compact, and that \( \mu \) and \( \nu \) are regular Borel measures on \( A \). Under these conditions, one can use another version of the product measure construction, in which \( \mu \times \nu \) is a regular Borel measure on \( A \times A \). Thus the Borel measurability of \( E_2 \) is sufficient in this situation, but one would also like \( \mu \ast \nu \) to be regular on \( A \). If \( \mu \) and \( \nu \) are real-valued and nonnegative, then one can first check that \( \mu \ast \nu \) is inner regular, and then use that to get that \( \mu \ast \nu \) is outer regular. Otherwise, one can reduce to that case, by expressing \( \mu \) and \( \nu \) as linear combinations of nonnegative regular measures.
At any rate, once one has \( \mu \ast \nu \) as in (4.22), one gets that

\[
\int_A f(z) \, d(\mu \ast \nu)(z) = \int_{A \times A} f(x+y) \, d(\mu \times \nu)(x,y)
\]

for every bounded complex-valued Borel measurable function \( f \) on \( A \). More precisely, this is the same as (4.22) when \( f \) is the characteristic or indicator function associated to \( E \) on \( A \). Thus (4.23) holds when \( f \) is a Borel measurable simple function on \( A \times A \), by linearity. This implies that (4.23) holds for all bounded Borel measurable functions \( f \) on \( A \times A \), by standard approximation arguments. Equivalently,

\[
\int_A f(z) \, d(\mu \ast \nu)(z) = \int_A \left( \int_A f(x+y) \, d\mu(x) \right) \, d\nu(y)
\]

\[
= \int_A \left( \int_A f(x+y) \, d\nu(y) \right) \, d\mu(x),
\]

by the appropriate version of Fubini’s theorem.

If \( A \) is locally compact, then regular complex Borel measures on \( A \) correspond to bounded linear functionals on \( C_0(A) \) with respect to the supremum norm, by a version of the Riesz representation theorem. In order to get \( \mu \times \nu \), it suffices to define the appropriate bounded linear functional on \( C_0(A \times A) \). One can also look at \( \mu \ast \nu \) in terms of the corresponding bounded linear functional on \( C_0(A) \). This is much simpler when \( A \) is compact, so that \( C_0(A) = C(A) \). Otherwise, \( f(x+y) \) need not vanish at infinity on \( A \times A \) even when \( f \) has compact support in \( A \), and so it is better to at least be able to integrate bounded continuous functions on \( A \times A \). Even if \( A \) is not compact, there are some simplifications when \( \mu \) or \( \nu \) has compact support in \( A \). One can then reduce to this case using suitable approximation arguments.

Note that \( A_2 = A \times A \), and

\[
(\mu \ast \nu)(A) = \mu(A) \, \nu(A).
\]

One can check that

\[
|\mu \ast \nu|(E) \leq (|\mu| \ast |\nu|)(E)
\]

for every Borel set \( E \subseteq A \), where \( |\mu| \), \( |\nu| \), and \( |\mu \ast \nu| \) are the total variation measures on \( A \) associated to \( \mu \), \( \nu \), and \( \mu \ast \nu \), respectively, and \( |\mu| \ast |\nu| \) is the convolution of \( |\mu| \) and \( |\nu| \) on \( A \). In particular,

\[
|\mu \ast \nu|(A) \leq (|\mu| \ast |\nu|)(A) = |\mu|(A) \, |\nu|(A),
\]

so that the total variation norm of \( \mu \ast \nu \) on \( A \) is less than or equal to the product of the total variation norms of \( \mu \) and \( \nu \) on \( A \). Remember that the total variation norm of a regular complex Borel measure on \( A \) is equal to the dual norm of the corresponding bounded linear functional on \( C_0(A) \), with respect to the supremum norm on \( C_0(A) \).
It is easy to see that convolution of measures is commutative, in the sense that
\[ \mu * \nu = \nu * \mu, \]
basically because of commutativity of addition on \( A \). Similarly, one can check that convolution of measures is associative. If \( \phi \) is a continuous homomorphism from \( A \) into \( T \), then we can apply (4.23) to \( f = \varphi \) to get that
\[ (\hat{\mu * \nu})(\phi) = \hat{\mu}(\phi) \hat{\nu}(\phi). \]

### 4.3 Convolution of functions

Let \( A \) be a locally compact commutative topological group, and let \( H \) be a Haar measure on \( A \). If \( f, g \) are real or complex-valued Borel measurable functions on \( A \), then we would like to define their convolution on \( A \) by
\[ (f * g)(x) = \int_A f(x - y) g(y) \, dH(y). \]

If \( f, g \) are real-valued and nonnegative on \( A \), then this integral makes sense as a nonnegative extended real number for each \( x \in A \). Otherwise, (4.30) is defined when
\[ \int_A |f(x - y)||g(y)| \, dH(y) \]
is finite, in which case the absolute value or modulus of (4.30) is less than or equal to (4.31). It is easy to see that (4.30) and (4.31) are symmetric in \( f \) and \( g \), using the change of variables \( y \mapsto x - y \).

If \( f \) and \( g \) are real-valued and nonnegative again, then one can check that
\[ \int_A (f * g)(x) \, dH(x) = \left( \int_A f(x) \, dH(x) \right) \left( \int_A g(y) \, dH(y) \right), \]
using Fubini’s theorem. More precisely, the Borel measurability of \( f(x - y) g(y) \) on \( A \times A \) can be derived from the Borel measurability of \( f, g \) on \( A \) and the continuity of the group operations on \( A \). If there is a countable base for the topology of \( A \), then \( A \) is \( \sigma \)-compact, which implies that \( A \) is \( \sigma \)-finite with respect to \( H \). Borel subsets of \( A \times A \) are also measurable with respect to the usual product measure construction in this situation, which permits one to use the standard version of Fubini’s theorem. Otherwise, one should use a version for Borel measures with suitable regularity properties.

If \( f \) and \( g \) are real or complex-valued functions on \( A \) that are integrable with respect to \( H \), then we can apply (4.32) to \( |f| \) and \( |g| \). This implies in particular that (4.31) is finite for almost every \( x \in A \) with respect to \( H \), so that (4.30) is defined for almost every \( x \in A \) with respect to \( H \). We also get that
\[ \int_A |(f * g)(x)| \, dH(x) \leq \left( \int_A |f(x)| \, dH(x) \right) \left( \int_A |g(y)| \, dH(y) \right), \]
using Fubini’s theorem again.
because the absolute value or modulus of (4.30) is less than or equal to (4.31),
and using (4.32) applied to $|f|$ and $|g|$. Thus $f \ast g$ is also integrable on $A$ under
these conditions, where measurability comes from Fubini’s theorem.
If $\mu$ is the Borel measure on $A$ associated to $f$ as in (4.9), and if $\nu$ is associated
to $g$ in the same way, then it is easy to see that $\mu \ast \nu$ is the Borel measure
_corresponding to $f \ast g$. It follows that the Fourier transform of $f \ast g$ is equal
to the product of the Fourier transforms of $f$ and $g$, as in (4.29). Similarly,
convolution is associative on $L^1(A)$, which can also be verified more directly.

Let $f$ and $g$ be real-valued and nonnegative again, and let $1 \leq p < \infty$ be
given. One can check that

\[
(\int_A |(f \ast g)(x)|^p \, dH(x))^{1/p} \leq \left( \int_A |f(x)|^p \, dH(x) \right)^{1/p} \left( \int_A |g(y)| \, dH(y) \right),
\]

using the integral version of Minkowski’s inequality, or Jensen’s inequality and
Fubini’s theorem. If $f$ and $g$ are real or complex-valued functions on $A$ such that $|f|^p$ and $|g|$ are integrable, then one can apply (4.34) to $|f|$ and $|g|$, to get
that (4.31) is finite for almost every $x \in A$ with respect to $H$. Thus (4.30) is
defined for almost every $x \in A$ with respect to $H$, and

\[
(\int_A |g(y)| \, dH(y))^{1/p} \leq \left( \int_A |f(x)|^p \, dH(x) \right)^{1/p} \left( \int_A |g(y)| \, dH(y) \right),
\]

This shows that $f \ast g \in L^p(A)$ when $f \in L^p(A)$ and $g \in L^1(A)$.

If $f \in L^\infty(A)$ and $g \in L^1(A)$, then (4.31) is obviously finite for every $x \in A$, so that (4.30) is defined for every $x \in A$. In this case, we get that

\[
|g(y)| \, dH(y) = \int_A |f(y)| \, dH(y)
\]

for every $x \in A$, where $\|f\|_{\infty}$ denotes the $L^\infty$ norm of $f$ with respect to $H$. This
implies that $f \ast g$ is bounded on $A$, and in fact $f \ast g$ is uniformly continuous on $A$
under these conditions. To see this, it is better to use the analogous expression
for $g \ast f$, which is the same as $f \ast g$, as mentioned earlier. This permits one
to use the analogue of (4.13) for $g$, to show that the convolution is uniformly
continuous.

Suppose now that $f \in L^p(A)$ for some $p$, $1 < p < \infty$, and that $g \in L^q(A)$,
where $1 < q < \infty$ is the exponent conjugate to $p$, so that $1/p + 1/q = 1$. Thus
(4.31) is finite for every $x \in A$, by Hölder’s inequality, which implies that (4.30)
is defined for every $x \in A$. As before,

\[
|f(y)| \, dH(y) = \int_A |f(y)| \, dH(y)
\]

for every $x \in A$, where $\|f\|_p$ denotes the $L^p$ norm of $f$ with respect to $H$, and
similarly for $\|g\|_q$. It follows that $f \ast g$ is bounded on $A$, and one can also show
that \( f \ast g \) is uniformly continuous on \( A \). This uses the analogue of (4.13) for the \( L^p \) norm, which can be obtained from the density of continuous functions on \( A \) with compact support in \( L^p(A) \) when \( p < \infty \). One could just as well use the corresponding property for \( g \), because \( q < \infty \), and the commutativity of convolution. By approximating both \( f \) and \( g \) by functions with compact support in \( A \), one can check that \( f \ast g \) vanishes at infinity in this situation.

4.4 Functions and measures

Let \( A \) be a commutative topological group, and let \( \nu \) be a complex Borel measure on \( A \). If \( f \) is a complex-valued Borel measurable function on \( A \), then we would like to define the convolution of \( f \) and \( \nu \) by

\[
(f \ast \nu)(x) = \int_A f(x - y) \, d\nu(y).
\]

(4.38)

If \( f \) and \( \nu \) are real-valued and nonnegative, then this integral makes sense as a nonnegative extended real number for every \( x \in A \). Otherwise, (4.38) is defined when

\[
\int_A |f(x - y)| \, d|\nu|(y)
\]

(4.39)

is finite, where \( |\nu| \) is the total variation measure associated to \( \nu \) on \( A \). In this case, the modulus of (4.38) is less than or equal to (4.39).

As a basic class of examples, let \( \delta_a \) be the Borel measure on \( A \) which is the Dirac mass associated to an element \( a \) of \( A \), as mentioned in Section 4.1. If we take \( \nu = \delta_a \) in (4.38), then we get that

\[
(f \ast \delta_a)(x) = f(x - a)
\]

(4.40)

for every \( x \in A \).

If \( \nu \) is any complex Borel measure on \( A \), and \( f \) is a bounded Borel measurable function on \( A \), then (4.39) is less than or equal to the supremum norm of \( f \) on \( A \) times \( |\nu|(A) \) for every \( x \in A \). Thus (4.38) is defined and satisfies

\[
|(f \ast \nu)(x)| \leq \left( \sup_{w \in A} |f(w)| \right) |\nu|(A)
\]

(4.41)

for every \( x \in A \). If \( f \) is also uniformly continuous on \( A \), then it is easy to see that (4.38) is uniformly continuous on \( A \) as well. If \( \phi \) is a continuous homomorphism from \( A \) into \( T \), then

\[
(\phi \ast \nu)(x) = \widehat{\nu}(\phi) \phi(x)
\]

(4.42)

for every \( x \in A \).

If \( f \) is continuous on \( A \) and \( \nu \) has compact support on \( A \), then (4.39) is finite for every \( x \in A \), because \( f \) is bounded on compact subsets of \( A \). Remember that \( f \) is uniformly continuous along compact subsets of \( A \), which implies that (4.38) is continuous on \( A \) in this situation. If \( f \) is uniformly continuous on \( A \), then (4.38) is uniformly continuous on \( A \) too, as before.
Suppose now that $A$ is locally compact, and let $H$ be a Haar measure on $A$. If $f$ and $\nu$ are real-valued and nonnegative, then Fubini’s theorem implies that

$$
\int_A (f \ast \nu)(x) \, dH(x) = \left( \int_A f(x) \, dH(x) \right) \nu(A),
$$

(4.43)

at least under suitable conditions. As before, this is a bit simpler when there is a countable base for the topology of $A$, and otherwise one can ask that $\nu$ satisfy appropriate regularity conditions. At any rate, (4.43) implies that (4.38) is finite for almost every $x \in A$ with respect to $H$.

If $f$ and $\nu$ are complex-valued, and if $f$ is integrable with respect to $H$, then one can apply the remarks in the previous paragraph to $|f|$ and $|\nu|$. It follows that (4.39) is finite for almost every $x \in A$ with respect to $H$, so that (4.38) is defined for almost every $x \in A$ with respect to $H$. We also get that

$$
\int_A |(f \ast \nu)(x)| \, dH(x) \leq \left( \int_A |f(x)| \, dH(x) \right) |\nu|(A),
$$

(4.44)

because the modulus of (4.38) is less than or equal to (4.38). If $\mu$ is the Borel measure on $A$ corresponding to $f$ as in (4.9), then $\mu \ast \nu$ corresponds to (4.38) as an integrable function on $A$ with respect to $H$ in the same way.

If $f$ and $\nu$ are real-valued and nonnegative, and $1 \leq p < \infty$, then

$$
\left( \int_A |(f \ast \nu)(x)|^p \, dH(x) \right)^{1/p} \leq \left( \int_A |f(x)|^p \, dH(x) \right)^{1/p} |\nu|(A).
$$

(4.45)

As in the preceding section, this can be derived from the integral version of Minkowski’s inequality, or Jensen’s inequality and Fubini’s theorem, at least under suitable conditions. If $f$ and $\nu$ are complex-valued, and if $|f|^p$ is integrable with respect to $H$, then we can apply (4.45) to $|f|$ and $|\nu|$, to get that (4.39) is finite for almost every $x \in A$ with respect to $H$. This implies that (4.38) is defined for almost every $x \in A$ with respect to $H$, and that

$$
\left( \int_A |(f \ast \nu)(x)|^p \, dH(x) \right)^{1/p} \leq \left( \int_A |f(x)|^p \, dH(x) \right)^{1/p} |\nu|(A).
$$

(4.46)

If $g$ is a complex-valued integrable function on $A$, then

$$
\nu(E) = \int_E g(y) \, dH(y)
$$

(4.47)

defines a complex-valued regular Borel measure on $A$, with

$$
|\nu|(E) = \int_E |g(y)| \, dH(y)
$$

(4.48)

for every Borel set $E \subseteq A$. In this case, convolution of a function $f$ on $A$ with $\nu$ is the same as convolution of $f$ with $g$, as discussed in the previous section, and with similar estimates in terms of $|\nu|$.
4.5. SOME SIMPLE ESTIMATES

Let \( f \) be a continuous complex-valued function on \( A \). If \( \nu \) is a complex Borel measure on \( A \) with compact support, then we have seen that \( f \ast \nu \) is a continuous function on \( A \) as well. If \( f \) also has compact support on \( A \), then it is easy to see that \( f \ast \nu \) has compact support on \( A \) too. Similarly, if \( f \) vanishes at infinity on \( A \), then \( f \ast \nu \) vanishes at infinity on \( A \). This also works when \( \nu \) can be approximated by complex Borel measures with compact support on \( A \) with respect to the total variation norm, and in particular when \( \nu \) is a regular Borel measure on \( A \).

4.5 Some simple estimates

Let \( A \) be a commutative topological group, and fix a real number \( C \geq 1 \). Also let \( \nu \) be a complex Borel measure on \( A \) such that

\[
\nu(A) = 1
\]

and

\[
|\nu|(A) \leq C;
\]

where \( |\nu| \) is the total variation measure associated to \( \nu \) on \( A \). Of course, if \( \nu \) is real-valued and nonnegative, then \( |\nu| = \nu \), and (4.49) implies that (4.50) holds with \( C = 1 \). If \( f \) is a complex-valued Borel measurable function on \( A \) such that (4.39) is finite, then \((f \ast \nu)(x)\) can be defined as in (4.38), and

\[
(f \ast \nu)(x) - f(x) = \int_A (f(x - y) - f(x)) \, d\nu(y),
\]

by (4.49). This implies that

\[
|(f \ast \nu)(x) - f(x)| \leq \int_A |f(x - y) - f(x)| \, d|\nu|(y).
\]

Let \( U \) be an open set in \( A \) with \( 0 \in U \), and suppose that

\[
|\nu|(A \setminus U) = 0.
\]

In this case, (4.52) reduces to

\[
|(f \ast \nu)(x) - f(x)| \leq \int_U |f(x - y) - f(x)| \, d|\nu|(y)
\leq C \sup_{y \in U} |f(x - y) - f(x)|,
\]

using (4.50) in the second step. If \( f \) is continuous at \( x \), then the right side of (4.54) is small when \( U \) is sufficiently small.

Now let \( \eta \) be a nonnegative real number, and suppose that

\[
|\nu|(A \setminus U) \leq \eta
\]
instead of (4.53), and which reduces to (4.53) when $\eta = 0$. If $f$ is bounded on $A$, then (4.52) implies that

\begin{equation}
| (f * \nu)(x) - f(x) | 
\leq \int_U |f(x-y) - f(x)| \, d|\nu|(y) + \int_{A\setminus U} |f(x-y) - f(x)| \, d|\nu|(y)
\end{equation}

\begin{equation}
\leq C \sup_{y \in U} |f(x-y) - f(x)| + 2 \eta \sup_{z \in A} |f(z)|,
\end{equation}

using (4.50) and (4.55) in the second step. This is small when $f$ is continuous at $x$, and $U$ and $\eta$ are sufficiently small.

Let us apply this to $x = 0$ and a continuous homomorphism $\phi$ from $A$ into $T$, so that $\phi(0) = 1$ and

\begin{equation}
(\phi * \nu)(0) = \hat{\nu}(\phi),
\end{equation}

as in (4.42). Plugging this into (4.56), we get that

\begin{equation}
|\hat{\nu}(\phi) - 1| \leq C \sup_{y \in U} |\phi(x-y) - \phi(x)| + 2 \eta,
\end{equation}

because $|\phi(z)| = 1$ for every $z \in A$. Note that

\begin{equation}
\hat{\nu}(1_A) = 1,
\end{equation}

by (4.49), where $1_A$ is the constant function on $A$ equal to 1 at every point, which is the identity element of $\hat{A}$. If $E \subseteq \hat{A}$ is equicontinuous at 0, then (4.58) is uniformly small over $\phi \in E$ when $U$ and $\eta$ are sufficiently small. In particular, if $A$ is locally compact, and $E \subseteq \hat{A}$ is compact or totally bounded with respect to the usual topology on $\hat{A}$, then $E$ is equicontinuous at 0 on $A$, by standard arguments.

### 4.6 Some more estimates

Let $A$ be a locally compact commutative topological group, and let $U$ be an open set in $A$ that contains 0. We may as well ask that the closure $\overline{U}$ of $U$ in $A$ be compact, since this can always be arranged by taking the intersection of $U$ with a fixed neighborhood of 0 in $A$ with this property. Also let $C \geq 1$ be fixed, and let $\nu$ be a regular complex Borel measure on $A$ that satisfies (4.49), (4.50), and (4.53). Of course, (4.53) implies that the support of $\nu$ is contained in $\overline{U}$, and hence that the support of $\nu$ is compact.

Thus $(f * \nu)(x)$ is defined for every $x \in A$ when $f$ is a continuous complex-valued function on $A$. If $f$ is uniformly continuous along a set $E \subseteq A$, then $f * \nu$ will be uniformly close to $f * \nu$ on $E$ when $U$ is sufficiently small, as in (4.54). In particular, $f * \nu$ is uniformly close to $f$ on $A$ when $f$ is uniformly continuous on $A$ and $U$ is sufficiently small. If $f$ is any continuous function on $A$, then $f$ is uniformly continuous on compact subsets of $A$, and $f * \nu$ is uniformly close to $f$ on compact subsets of $A$ when $U$ is sufficiently small, depending on the
4.6. SOME MORE ESTIMATES

compact set. If \( f \) is a continuous function on \( A \) that vanishes at infinity, then \( f \) is uniformly continuous on \( A \), so that \( f * \nu \) is uniformly close to \( f \) when \( U \) is sufficiently small, and \( f * \nu \) also vanishes at infinity on \( A \), because \( \nu \) has compact support on \( A \).

Suppose that \( f \) has support contained in a compact set \( K_1 \subseteq A \), and that the support of \( \nu \) is contained in a fixed compact set \( K_2 \subseteq A \). Remember that the support of \( \nu \) is automatically contained in \( \overline{U} \), and so it suffices to ask that

\[
\overline{U} \subseteq K_2
\]

to get the support of \( \nu \) contained in \( K_2 \). In particular, if 0 is in the interior of \( K_2 \), then this holds for all sufficiently small \( U \). At any rate, it is convenient to ask that 0 \( \in \) \( K_2 \), so that \( K_1 \subseteq K_1 + K_2 \). This can always be arranged by replacing \( K_2 \) with \( K_2 \cup \{0\} \), if necessary. If \( (f * \nu)(x) \neq 0 \) for some \( x \in A \), then \( f(x-y) \neq 0 \) for some \( y \) in the support of \( \nu \), which means that \( x-y \in K_1 \) and \( y \in K_2 \). This implies that \( x \) is an element of \( K_1 + K_2 \), which is the image of \( K_1 \times K_2 \) under addition as a mapping from \( A \times A \) into \( A \). Note that \( K_1 + K_2 \) is compact in \( A \), because \( K_1 \times K_2 \) is a compact subset of \( A \times A \) with respect to the product topology, and addition is continuous as a mapping from \( A \times A \) into \( A \).

Let \( H \) be a Haar measure on \( A \), and let \( 1 \leq p < \infty \) be given. Because \( f \) and \( f * \nu \) have supports contained in \( K_1 + K_2 \), we have that

\[
\int_A |(f * \nu)(x) - f(x)|^p dH(x) \leq \sup_{x \in K_1+K_2} |(f * \nu)(x) - f(x)| H(K_1+K_2)^{1/p}.
\]

Of course, \( H(K_1+K_2) < \infty \), because \( K_1+K_2 \) is compact. We already know that \( f * \nu \) is uniformly close to \( f \) on \( K_1 + K_2 \) when \( U \) is sufficiently small, by (4.54) and uniform continuity, so that (4.61) is small when \( U \) is sufficiently small. If \( f \) is any function on \( A \) in \( L^p(A) \), then \( f * \nu \) is also close to \( f \) on \( A \) with respect to the \( L^p \) norm associated to \( H \) when \( U \) is sufficiently small, depending on \( f \). This follows from the preceding argument when \( f \) is a continuous function on \( A \) with compact support, and otherwise one can get the same conclusion by approximating \( f \) by continuous functions on \( A \) with compact support with respect to the \( L^p \) norm. This also uses (4.46) to estimate the errors in the approximation.

Now let \( \eta \geq 0 \) be given, and suppose that \( \nu \) satisfies (4.55) instead of (4.53). If \( f \) is a bounded function on \( A \) that is also uniformly continuous on \( A \), then \( f * \nu \) is uniformly close to \( f \) on \( A \) when \( U \) and \( \eta \) are sufficiently small, by (4.56). In order to deal with \( L^p \) norms as before, it will be helpful to approximate \( \nu \) by a measure that satisfies (4.53).

Put

\[
\nu'(E) = \nu(E \cap U) + \nu(A \setminus U) \delta_0(E)
\]

for each Borel set \( E \subseteq A \), where \( \delta_0 \) is the usual Dirac mass at 0. This is a regular complex Borel measure on \( A \), because of the corresponding properties
of $\nu$, and
\begin{equation}
\nu'(A) = \nu(U) + \nu(A \setminus U) = \nu(A) = 1,
\end{equation}
by (4.49). It is easy to see that
\begin{equation}
|\nu'(E)| \leq |\nu|(E \cap U) + |\nu(A \setminus U)| \delta_0(E)
\end{equation}
for every Borel set $E \subseteq A$, and hence that
\begin{equation}
|\nu'(A) | \leq |\nu|(U) + |\nu(A \setminus U)| \leq |\nu|(U) + |\nu|(A \setminus U) = |\nu|(A) \leq C,
\end{equation}
by (4.50). Note that
\begin{equation}
|\nu'(A \setminus U) | = 0,
\end{equation}
because $0 \in U$, so that $\nu'$ satisfies the analogues of (4.49), (4.50), and (4.53).

Similarly, put
\begin{equation}
\nu''(E) = \nu(E) - \nu'(E) = \nu(E \setminus U) - \nu(A \setminus U) \delta_0(E)
\end{equation}
for each Borel set $E \subseteq A$, which is another regular complex Borel measure on $A$. As before,
\begin{equation}
|\nu''(E)| \leq |\nu|(E \setminus U) + |\nu(A \setminus U)| \delta_0(E)
\end{equation}
for every Borel set $E \subseteq A$, so that
\begin{equation}
|\nu''(A) | \leq |\nu|(A \setminus U) + |\nu(A \setminus U)| \leq 2 |\nu|(A \setminus U) \leq 2 \eta,
\end{equation}
by (4.55). If $1 \leq p < \infty$ and $f \in L^p(A)$, then $f \ast \nu$ is close to $f$ with respect to the $L^p$ norm on $A$ when $U$ is sufficiently small, as discussed earlier. The $L^p$ norm of $f \ast \nu''$ is bounded by (4.69) times the $L^p$ norm of $f$, as in (4.46). This implies that $f \ast \nu = f \ast \nu' + f \ast \nu''$ is close to $f$ with respect to the $L^p$ norm when $U$ and $\eta$ are sufficiently small, depending on $f$.

### 4.7 Another variant

Let $A$ be a locally compact commutative topological group, and let $\mu$, $\nu$ be regular complex Borel measures on $A$. Also let $U$ be an open set in $A$ with $0 \in U$, let $C \geq 1$ be fixed, and let $\eta$ be a nonnegative real number. As before, we suppose that $\nu$ satisfies (4.49), (4.50), and (4.55). We would like to look at the convolution $\mu \ast \nu$ of $\mu$ and $\nu$ as an approximation to $\mu$ when $U$ and $\eta$ are sufficiently small. Note that
\begin{equation}
|\mu \ast \nu|(A) \leq |\mu|(A) |\nu|(A) \leq C |\mu|(A),
\end{equation}
by (4.27) and (4.50).

If $f$ is a bounded complex-valued Borel measurable function on $A$, then
\begin{equation}
\int_A f(z) d(\mu \ast \nu)(z) = \int_A \left( \int_A f(x + y) d\nu(y) \right) d\mu(x),
\end{equation}
(4.71)
4.7. ANOTHER VARIANT

as in (4.24). Put

\( \tilde{\nu}(E) = \nu(-E) \) \tag{4.72}

for every Borel set \( E \subseteq A \), which defines another regular complex Borel measure
on \( A \). By construction,

\[ (f * \tilde{\nu})(x) = \int_A f(x - y) \, d\tilde{\nu}(y) = \int_A f(x + y) \, d\nu(y), \] \tag{4.73}

so that (4.71) becomes

\[ \int_A f(z) \, d(\mu * \nu)(z) = \int_A (f * \tilde{\nu})(x) \, d\mu(x). \] \tag{4.74}

It follows that

\[ \int_A f(z) \, d(\mu * \nu)(z) - \int_A f(x) \, d\mu(x) = \int_A ((f * \tilde{\nu})(x) - f(x)) \, d\mu(x), \] \tag{4.75}

and hence

\[ \left| \int_A f(z) \, d(\mu * \nu)(z) - \int_A f(x) \, d\mu(x) \right| \leq \int_A |(f * \tilde{\nu})(x) - f(x)| \, d|\mu|(x). \] \tag{4.76}

In particular,

\[ \left| \int_A f(z) \, d(\mu * \nu)(z) - \int_A f(x) \, d\mu(x) \right| \leq \left( \sup_{x \in A} |(f * \tilde{\nu})(x) - f(x)| \right) |\mu|(A). \] \tag{4.77}

Of course, \( \tilde{\nu} \) satisfies the same conditions as \( \nu \), but with \( U \) replaced with \( -U \). If \( f \) is bounded and uniformly continuous on \( A \), then \( f * \tilde{\nu} \) is uniformly close to \( f \) on \( A \) when \( U \) and \( \eta \) are sufficiently small, for the same reasons as before. This implies that (4.75) is small under these conditions, by (4.77).

Remember that

\[ \lambda_{\mu}(f) = \int_A f(x) \, d\mu(x) \] \tag{4.78}

and

\[ \lambda_{\mu * \nu}(f) = \int_A f(z) \, d(\mu * \nu)(z) \] \tag{4.79}

define bounded linear functionals on \( C_0(A) \), with respect to the supremum norm on \( C_0(A) \). If \( f \in C_0(A) \), then \( f \) is bounded and uniformly continuous on \( A \), and the preceding discussion shows that

\[ \lambda_{\mu * \nu}(f) - \lambda_{\mu}(f) \] \tag{4.80}

is small when \( U \) and \( \eta \) are sufficiently small. This implies that \( \lambda_{\mu * \nu} \) is close to \( \lambda_{\mu} \) with respect to the weak* topology on the dual of \( C_0(A) \) when \( U \) and \( \eta \) are sufficiently small.
\section{Discrete groups}

Let $A$ be a commutative group equipped with the discrete topology. Remember that a complex-valued function $f(x)$ on $A$ is said to be \textit{summable} if the sums
\begin{equation}
\sum_{x \in E} |f(x)|
\end{equation}
over all nonempty finite subsets $E$ of $A$ are uniformly bounded. In this case, the sum
\begin{equation}
\sum_{x \in A} |f(x)|
\end{equation}
may be defined as the supremum of (4.81) over all nonempty finite subsets $E$ of $A$. The space of summable functions on $A$ is denoted $\ell^1(A)$, and is a vector space with respect to pointwise addition and multiplication. It is easy to see that (4.82) defines a norm on $\ell^1(A)$, which may be denoted $\|f\|_1$ or $\|f\|_{\ell^1(A)}$. Of course, $f(x)$ is summable on $A$ if and only if $f(x)$ is integrable with respect to counting measure on $A$, so that $\ell^1(A)$ is the same as the usual space $L^1(A)$ associated to counting measure on $A$. If $f(x)$ is summable on $A$, then the sum
\begin{equation}
\sum_{x \in A} f(x)
\end{equation}
can be defined in various equivalent ways, and is the same as the integral of $f(x)$ with respect to counting measure on $A$.

If $1 \leq p < \infty$, then $\ell^p(A)$ is defined to be the space of complex-valued functions $f(x)$ on $A$ such that $|f(x)|^p$ is summable on $A$. It is well known that this is also a vector space with respect to pointwise addition and scalar multiplication, and that
\begin{equation}
\|f\|_p = \|f\|_{\ell^p(A)} = \left( \sum_{x \in A} |f(x)|^p \right)^{1/p}
\end{equation}
defines a norm on $\ell^p(A)$. Similarly, $\ell^\infty(A)$ is defined to be the space of all bounded complex-valued functions on $A$, equipped with the supremum norm
\begin{equation}
\|f\|_\infty = \|f\|_{\ell^\infty(A)} = \sup_{x \in A} |f(x)|.
\end{equation}
These are the same as the usual $L^p$ spaces associated to counting measure on $A$. In particular, $\ell^p(A)$ is complete with respect to the metric corresponding to the $\ell^p$ norm for each $p \geq 1$, so that $\ell^p(A)$ is a Banach space.

If $f, g \in \ell^2(A)$, then $|f(x)||g(x)|$ is summable on $A$, and
\begin{equation}
\langle f, g \rangle = \sum_{x \in A} f(x) \overline{g(x)}
\end{equation}
defines an inner product on $\ell^2(A)$. By construction,
\begin{equation}
\langle f, f \rangle = \sum_{x \in A} |f(x)|^2 = \|f\|_2^2,
\end{equation}
4.8. DISCRETE GROUPS

so that \( \ell^2(A) \) is a Hilbert space with respect to this inner product.

A complex-valued function \( f(x) \) on \( A \) is said to vanish at infinity if

\[
\{ x \in A : |f(x)| \geq \epsilon \}
\]

has only finitely many elements for each \( \epsilon > 0 \). This is equivalent to saying that \( f(x) \) vanishes at infinity on \( A \) as a locally compact Hausdorff topological space, equipped with the discrete topology. The space of complex-valued functions \( f(x) \) vanishing at infinity on \( A \) may be denoted \( c_0(A) \), which is a closed linear subspace of \( \ell^\infty(A) \). Of course, every function on \( A \) is continuous with respect to the discrete topology, so that \( c_0(A) \) is the same as the space \( C_0(A) \) defined previously for any locally compact Hausdorff topological space. If \( f \in \ell^p(A) \) for some \( p < \infty \), then it is easy to see that \( f \in c_0(A) \).

In this situation, counting measure can be used as Haar measure on \( A \), and \( \hat{A} \) consists of all homomorphisms from \( A \) into \( \mathbb{T} \). The Fourier transform of \( f \in \ell^1(A) \) can be expressed as

\[
\hat{f} (\phi) = \sum_{x \in A} f(x) \overline{\phi(x)}
\]

for every \( \phi \in \hat{A} \). Note that

\[
\left| \hat{f}(\phi) - \sum_{x \in E} f(x) \overline{\phi(x)} \right| = \left| \sum_{x \in A \setminus E} f(x) \overline{\phi(x)} \right| \leq \sum_{x \in A \setminus E} |f(x)|
\]

for every subset \( E \) of \( A \) and \( \phi \in \hat{A} \). The summability of \( f(x) \) on \( A \) implies that for each \( \epsilon > 0 \) there is a finite subset \( E_1(\epsilon) \) of \( A \) such that

\[
\sum_{x \in A \setminus E_1(\epsilon)} |f(x)| < \epsilon.
\]

If \( E \) is a subset of \( A \) that contains \( E_1(\epsilon) \), then it follows that

\[
\left| \hat{f}(\phi) - \sum_{x \in E} f(x) \overline{\phi(x)} \right| < \epsilon
\]

for every \( \phi \in \hat{A} \), by (4.90).

The space \( C(A) \) of complex-valued continuous functions on \( A \) is the same as the space of all complex-valued functions on \( A \), because \( A \) is equipped with the discrete topology. The usual topology on \( C(A) \) is defined by the collection of supremum seminorms associated to nonempty compact subsets of \( A \), and here the compact subsets of \( A \) have only finitely many elements. Put

\[
\Psi_x(\phi) = \phi(x)
\]

for each \( x \in A \) and \( \phi \in \hat{A} \), as in Section 3.9. This defines a homomorphism from \( \hat{A} \) into \( \mathbb{T} \) for each \( x \in A \), and this homomorphism is continuous with respect
to the topology induced on \( \hat{A} \) by the usual topology on \( C(A) \). Thus \( \Psi_x \) is an element of the dual \( \hat{\hat{A}} \) of \( \hat{A} \) for each \( x \in A \), and we have seen that

\[
x \mapsto \Psi_x
\]

defines a homomorphism from \( A \) into \( \hat{\hat{A}} \). We have also seen that the elements of \( \hat{A} \) separate points in \( A \), which means that (4.94) is injective. In fact, (4.94) maps \( A \) onto \( \hat{\hat{A}} \), as in Section 3.9. Note that

\[
\Psi_{-x}(\phi) = \phi(-x) = 1/\phi(x) = \overline{\phi(x)}
\]

for every \( x \in A \) and \( \phi \in \hat{\hat{A}} \), using the fact that \( \phi(x) \in T \) in the last step.

Remember that \( \hat{\hat{A}} \) is compact with respect to the topology induced by the usual topology on \( C(A) \) when \( A \) is discrete. Let \( H_{\hat{\hat{A}}} \) be Haar measure on \( \hat{\hat{A}} \), normalized so that \( H_{\hat{\hat{A}}} (\hat{\hat{A}}) = 1 \). The elements of \( \hat{\hat{A}} \) are orthonormal in \( L^2(\hat{\hat{A}}) \) with respect to the standard integral inner product associated to \( H_{\hat{\hat{A}}} \), as in Section 3.8. Thus the functions \( \Psi_x \) on \( \hat{A} \) with \( x \in A \) are orthonormal in \( L^2(\hat{\hat{A}}) \), which is the same as saying that the functions \( \Psi_{-x} \) with \( x \in A \) are orthonormal in \( L^2(\hat{\hat{A}}) \). If \( f \in \ell^2(A) \), then it follows that

\[
\sum_{x \in A} f(x) \Psi_{-x}
\]

defines an element of \( L^2(\hat{\hat{A}}) \), by standard Hilbert space arguments. In particular, the norm of (4.96) in \( L^2(\hat{\hat{A}}) \) is equal to the norm of \( f \) in \( \ell^2(A) \). Similarly,

\[
\sum_{x \in A} f(x) \Psi_{-x} = \sum_{x \in A \setminus E} f(x) \Psi_{-x} \qquad \text{in} \quad L^2(\hat{\hat{A}})
\]

\[
= \left( \sum_{x \in A \setminus E} |f(x)|^2 \right)^{1/2}
\]

for each \( E \subseteq A \), where \( \| \cdot \|_{L^2(\hat{\hat{A}})} \) denotes the usual norm on \( L^2(\hat{\hat{A}}) \). As before, for each \( \epsilon > 0 \) there is a finite subset \( E_2(\epsilon) \) of \( A \) such that

\[
\left( \sum_{x \in A \setminus E_2(\epsilon)} |f(x)|^2 \right)^{1/2} < \epsilon,
\]

because \( f \in \ell^2(A) \). This implies that

\[
\sum_{x \in A} f(x) \Psi_{-x} - \sum_{x \in E} f(x) \Psi_{-x} \quad \text{in} \quad L^2(\hat{\hat{A}}) < \epsilon
\]

when \( E_2(\epsilon) \subseteq E \subseteq A \), by (4.97).
If $f \in \ell^1(A)$, then it is well known that $f \in \ell^2(A)$, and that
\begin{equation}
\|f\|_{\ell^2(A)} \leq \|f\|_{\ell^1(A)}.
\end{equation}
In this case, $\hat{f}$ is a continuous function on $\hat{A}$, which determines an element of $L^2(\hat{A})$. One can check that $\hat{f}$ is the same as (4.96) as an element of $L^2(\hat{A})$, because the uniform approximation of (4.89) by finite subsums discussed earlier implies approximation with respect to the $L^2$ norm on $\hat{A}$. If $f \in \ell^2(A)$, then (4.96) may be considered as the definition of $\hat{f}$, as an element of $L^2(\hat{A})$. Note that the collection of functions $\Psi_x$ with $x \in A$ is actually an orthonormal basis for $L^2(\hat{A})$, by an argument in Section 3.8. This uses the fact that the functions $\Psi_x$ with $x \in A$ automatically separate points in $\hat{A}$, by construction. It follows that every element of $L^2(\hat{A})$ can be expressed as $\hat{f}$ for some $f \in \ell^2(A)$.

4.9 Compact groups

Let $A$ be a compact commutative topological group, and let $H$ be Haar measure on $A$, normalized so that $H(A) = 1$. Thus continuous homomorphisms from $A$ into $T$ are orthonormal with respect to the standard inner product on $L^2(A)$, as in Section 3.8. If $f \in L^2(A)$ and $\phi \in \hat{A}$, then $\hat{f}(\phi)$ is the same as the inner product of $f$ and $\phi$. Using standard properties of orthonormal sets in inner product spaces, we get that
\begin{equation}
\sum_{\phi \in \hat{A}} |\hat{f}(\phi)|^2 \leq \|f\|_2^2,
\end{equation}
for every nonempty finite set $E \subseteq \hat{A}$, where $\|f\|_2$ denotes the $L^2$ norm of $f$ on $A$ with respect to $H$. This implies that
\begin{equation}
\sum_{\phi \in \hat{A}} |\hat{f}(\phi)|^2 \leq \|f\|_2^2,
\end{equation}
where the sum on the left is defined to be the supremum of the corresponding subsums on the left side of (4.101).

As mentioned previously, it is well known that the elements of $\hat{A}$ separate points in $A$ when $A$ is compact, which implies that the linear span $\mathcal{E}(A)$ of $\hat{A}$ is dense in $C(A)$ with respect to the supremum norm. In particular, $\mathcal{E}(A)$ is dense in $L^2(A)$, so that $\hat{A}$ forms an orthonormal basis for $L^2(A)$. It follows that
\begin{equation}
f = \sum_{\phi \in \hat{A}} \hat{f}(\phi) \phi
\end{equation}
for every $f \in L^2(A)$, where the sum converges with respect to the $L^2$ norm $\|\cdot\|_2$. More precisely, this means that for each $\epsilon > 0$ there is a finite subset $E(\epsilon)$ of $\hat{A}$ such that
\begin{equation}
\left\|f(x) - \sum_{\phi \in E} \hat{f}(\phi) \phi \right\|_2 < \epsilon
\end{equation}
for every nonempty finite set \( E \subseteq \hat{A} \) that contains \( E(\varepsilon) \). This also implies that
\[
\sum_{\phi \in \hat{A}} |\hat{f}(\phi)|^2 = \|f\|_2^2.
\] (4.105)

Let \( \mu \) be a regular complex Borel measure on \( A \), and suppose that
\[
\hat{\mu}(\phi) = 0
\] (4.106)
for every \( \phi \in \hat{A} \). This implies that
\[
\int_A g(x) \, d\mu(x) = 0
\] (4.107)
for every \( g \in \mathcal{E}(A) \), by linearity of the integral, and the fact that \( \mathcal{E}(A) \) is invariant under complex conjugation. It follows that (4.107) holds for every continuous complex-valued function \( g \) on \( A \), because \( \mathcal{E}(A) \) is dense in \( C(A) \) with respect to the supremum norm. Using the hypothesis that \( \mu \) be a regular Borel measure, we get that \( \mu = 0 \) on \( A \) under these conditions. In particular, if \( f \) is an integrable function on \( A \) with respect to \( H \) such that \( \hat{f}(\phi) = 0 \) for every \( \phi \in \hat{A} \), then \( f = 0 \) almost everywhere on \( A \) with respect to \( H \).

Let \( \ell^2(\hat{A}) \) be the usual space of complex-valued functions \( F(\phi) \) on \( \hat{A} \) that are square-summable, as in the previous section. If \( F \in \ell^2(\hat{A}) \), then
\[
f = \sum_{\phi \in \hat{A}} F(\phi) \phi
\] (4.108)
defines an element of \( L^2(A) \), because of the orthonormality of the elements of \( \hat{A} \) in \( L^2(A) \). The norm of \( f \) in \( L^2(A) \) is the same as the norm of \( F \) in \( \ell^2(\hat{A}) \), and similarly
\[
\left\| f - \sum_{\phi \in E} F(\phi) \phi \right\|_{L^2(A)} = \left\| \sum_{\phi \in \hat{A} \setminus E} F(\phi) \phi \right\|_{L^2(A)}
\] (4.109)
\[
= \left( \sum_{\phi \in \hat{A} \setminus E} |F(\phi)|^2 \right)^{1/2}
\]
for every \( E \subseteq \hat{A} \), where \( \| \cdot \|_{L^2(A)} \) denotes the usual norm on \( L^2(A) \). Let \( \epsilon > 0 \) be given, and let \( E_2(\epsilon) \) be a finite subset of \( \hat{A} \) such that
\[
\left( \sum_{\phi \in \hat{A} \setminus E_2(\epsilon)} |F(\phi)|^2 \right)^{1/2} < \epsilon,
\] (4.110)
which exists because \( F \in \ell^2(\hat{A}) \). Thus
\[
\left\| f - \sum_{\phi \in E} F(\phi) \phi \right\|_{L^2(A)} < \epsilon
\] (4.111)
4.9. COMPACT GROUPS

when $E_2(\epsilon) \subseteq E \subseteq \hat{A}$, by (4.109).

Similarly, let $\ell^1(\hat{A})$ be the space of complex-valued functions $F(\phi)$ on $\hat{A}$ that are summable, as in the previous section. If $F \in \ell^1(\hat{A})$, then

\begin{equation}
(4.112) \quad f(x) = \sum_{\phi \in \hat{A}} F(\phi) \phi(x)
\end{equation}

can be defined for every $x \in A$, and has modulus less than or equal to the $\ell^1$ norm of $F$. If $E$ is any subset of $\hat{A}$, then we get that

\begin{equation}
(4.113) \quad \left| f(x) - \sum_{\phi \in E} F(\phi) \phi(x) \right| = \left| \sum_{\phi \in \hat{A} \setminus E} F(\phi) \phi(x) \right| \leq \sum_{\phi \in \hat{A} \setminus E} |F(\phi)|
\end{equation}

for every $x \in A$. The summability of $F$ on $\hat{A}$ also implies that for each $\epsilon > 0$ there is a finite subset $E_1(\epsilon)$ of $\hat{A}$ such that

\begin{equation}
(4.114) \quad \sum_{\phi \in \hat{A} \setminus E_1(\epsilon)} |F(\phi)| < \epsilon.
\end{equation}

If $E_1(\epsilon) \subseteq E \subseteq \hat{A}$, then it follows that

\begin{equation}
(4.115) \quad \left| f(x) - \sum_{\phi \in E} F(\phi) \phi(x) \right| < \epsilon
\end{equation}

for every $x \in A$, by (4.113). In particular, this implies that $f$ is continuous on $A$, because $f$ can be approximated by continuous functions uniformly on $A$. Remember that $\ell^1(\hat{A}) \subseteq \ell^2(\hat{A})$, as in the previous section. It is easy to see that

(4.112) as the same as (4.108) as an element of $L^2(A)$ when $F \in \ell^1(\hat{A})$, because this type of uniform convergence on $A$ implies convergence with respect to the $L^2$ norm on $A$. If $F \in \ell^2(\hat{A})$ and $f$ is as in (4.108), then

\begin{equation}
(4.116) \quad \hat{f}(\phi) = \langle f, \phi \rangle_{L^2(A)} = F(\phi)
\end{equation}

for every $\phi \in \hat{A}$, where $\langle \cdot, \cdot \rangle_{L^2(A)}$ is the standard integral inner product on $L^2(A)$. If $f$ is any element of $L^2(A)$, then $\hat{f}(\phi)$ is in $\ell^2(A)$, and (4.103) is the same as (4.108), with $F(\phi) = \hat{f}(\phi)$. If $F = \hat{f} \in \ell^1(\hat{A})$, then $f$ is the same as (4.112) as an element of $L^2(A)$, so that $f(x)$ is equal to (4.112) for almost every $x \in A$ with respect to $H$. In particular, this implies that $f$ is equal to a continuous function on $A$ almost everywhere on $A$ with respect to $H$.

Suppose that $\mu$ is a regular complex Borel measure on $A$ such that

\begin{equation}
(4.117) \quad F(\phi) = \hat{\mu}(\phi)
\end{equation}

is in $\ell^2(\hat{A})$. Thus we can define $f \in L^2(A)$ as in (4.108), which satisfies (4.116).

If we put

\begin{equation}
(4.118) \quad \mu_f(E) = \int_E f(x) \, dH(x)
\end{equation}
for every Borel set $E \subseteq A$, then $\mu_f$ is also a regular complex-valued Borel measure on $A$, and
\begin{equation}
\hat{\mu_f}(\phi) = \hat{f}(\phi) = F(\phi) = \hat{\mu}(\phi)
\end{equation}
for every $\phi \in \hat{A}$, by construction. This implies that the Fourier transform of $\mu - \mu_f$ is equal to 0 on $\hat{A}$, and hence that $\mu - \mu_f = 0$, as discussed earlier. Similarly, if (4.117) is in $\ell^1(A)$, then $f$ can be defined as in (4.112), so that $f$ is a continuous function on $A$ in this case. If $g$ is an integrable function on $A$ with respect to $H$, then we can apply these remarks to the Borel measure defined by
\begin{equation}
\mu(E) = \mu_g(E) = \int_E g(x) \, dH(x)
\end{equation}
for each Borel set $E \subseteq A$. It follows that $g \in L^2(A)$ when $\hat{g} \in \ell^2(\hat{A})$, and that $g$ is equal to a continuous function on $A$ almost everywhere with respect to $H$ when $\hat{g} \in \ell^1(A)$.

### 4.10 Compact groups, continued

Let $A$ be a compact commutative topological group again, with normalized Haar measure $H$. If $f, g \in L^2(A)$, then we have seen that their convolution $f * g$ is a continuous function on $A$, and that
\begin{equation}
(f \hat{*} g)(\phi) = \hat{f}(\phi) \hat{g}(\phi)
\end{equation}
for every $\phi \in \hat{A}$. We also have that $\hat{f}, \hat{g} \in \ell^2(\hat{A})$, as in the previous section, which implies that
\begin{equation}
\sum_{\phi \in \hat{A}} |\hat{f}(\phi)||\hat{g}(\phi)| \leq \left( \sum_{\phi \in \hat{A}} |f(\phi)|^2 \right)^{1/2} \left( \sum_{\phi \in \hat{A}} |g(\phi)|^2 \right)^{1/2}
\end{equation}
is finite, by the Cauchy–Schwarz inequality. In particular, $(f * g)(x)$ can be approximated uniformly on $A$ by sums of the form
\begin{equation}
\sum_{\phi \in E} \hat{f}(\phi) \hat{g}(\phi) \phi(x),
\end{equation}
where $E$ is a finite subset of $\hat{A}$.

Alternatively, if $f \in L^1(A)$ and $g \in L^2(A)$, then $f * g \in L^2(A)$, and hence $f * g$ can be approximated by finite sums of the form (4.123) with respect to the $L^2$ norm on $A$. If $\hat{g} \in \ell^1(\hat{A})$, then $(f \hat{*} g) \in \ell^1(\hat{A})$ too, because of (4.121) and the fact that $\hat{f}$ is bounded on $\hat{A}$. In this case, $f * g$ can be approximated by finite sums of the form (4.123) uniformly on $A$, as before. There are analogous statements in which $f$ is replace by a regular complex Borel measure on $A$, using essentially the same arguments.
Let $U$ be an open set in $A$ that contains 0, and let $g$ be a nonnegative real-valued function on $A$ such that

\[(4.124) \quad \int_A g(x) \, dH(x) = 1\]

and $g(x) = 0$ for every $x \in A \setminus U$. We may as well take $g$ to be a continuous function on $A$, although it would suffice to have $g \in L^2(A)$ for some of the arguments that follow. We can also choose $g$ so that $\hat{g} \in \ell^1(\mathbb{A})$, by taking $g$ to be a convolution of functions in $L^2(A)$. More precisely, the continuity of addition on $A$ at 0 implies that there are open sets $U_1, U_2 \subseteq A$ that contain 0 and satisfy

\[(4.125) \quad U_1 + U_2 \subseteq U.\]

Let $g_1, g_2$ be nonnegative real-valued functions on $A$ in $L^2(A)$ such that

\[(4.126) \quad \int_A g_1(x) \, dH(x) = \int_A g_2(x) \, dH(x) = 1\]

and $g_j(x) = 0$ when $x \in A \setminus U_j$ for $j = 1, 2$. If we take $g = g_1 * g_2$, then $g$ is a nonnegative real-valued continuous function on $A$, as before. The integral of $g$ over $A$ with respect to $H$ is equal to the product of the integrals of $g_1$ and $g_2$, as in Section 4.3, and hence is equal to 1, by (4.126). It is easy to see that $g = 0$ on $A \setminus U$ under these conditions, because of (4.125) and the analogous properties of $g_1$ and $g_2$.

Let $f \in L^p(A)$ be given, for some $p \in [1, \infty)$. As in Section 4.6, $f * g$ is close to $f$ with respect to the $L^p$ norm on $A$ when $U$ is sufficiently small. More precisely, this corresponds to taking $\nu$ to be the Borel measure on $A$ defined by

\[(4.127) \quad \nu(E) = \int_E g(x) \, dH(x)\]

for every Borel set $E \subseteq A$ in the earlier discussion. If $\hat{g} \in \ell^1(\mathbb{A})$, then we have seen that $f * g$ can be approximated by finite sums of the form (4.123) uniformly on $A$. This implies that $f$ is approximated by finite sums of the form (4.123) with respect to the $L^p$ norm when $U$ is sufficiently small.

If $f \in L^2(A)$, then $f$ can already be approximated by finite subsums of (4.103) with respect to the $L^2$ norm, as in (4.104). If $f \in L^p(A)$ and $1 \leq p < 2$, then it suffices to have $g \in L^2(A)$ in the argument in the preceding paragraph, instead of $\hat{g} \in \ell^1(\mathbb{A})$. This is because $f * g$ can be approximated by finite sums of the form (4.123) with respect to the $L^2$ norm, as mentioned earlier, which implies approximation with respect to the $L^p$ norm when $p \leq 2$. If $f \in L^p(A)$ and $2 < p < \infty$, then it also suffices to have $g \in L^2(A)$ in the argument in the preceding paragraph, instead of $\hat{g} \in \ell^1(\mathbb{A})$. In this case, we have that $f \in L^2(A)$ in particular, so that (4.121) is in $\ell^1(\mathbb{A})$, and the rest of the argument is basically the same as before.

If $f$ is a continuous function on $A$, then $f * g$ is close to $f$ uniformly on $A$ when $U$ is sufficiently small. This implies that $f$ can be approximated by finite
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sums of the form (4.123) uniformly on \( A \), because \( f \ast g \) can be approximated by finite sums of the form (4.123) uniformly on \( A \) when \( g \in L^2(A) \).

If \( \hat{f} \in \ell^1(\hat{A}) \), then we can already express \( f \) as in (4.112), with \( F = \hat{f} \). In this case, \( f \) can be approximated by finite subsums of (4.112) uniformly on \( A \), as in (4.115). One can also check that (4.121) is close to \( \hat{f}(\phi) \) with respect to the \( \ell^1 \) norm on \( \hat{A} \) when \( U \) is sufficiently small. Indeed, for each \( \phi \in \hat{A} \), \( \hat{g}(\phi) \) is close to 1 when \( U \) is sufficiently small, because \( \phi(0) = 1 \) and \( \phi \) is continuous at 0. We also know that \( |\hat{g}(\phi)| \leq 1 \) for every \( \phi \in \hat{A} \), since the \( L^1 \) norm of \( g \) is equal to 1, which permits one to reduce to approximating \( \hat{f}(\phi) \) by (4.121) on finite subsets of \( \hat{A} \).

4.11 Products of functions

Let us continue to take \( A \) to be a compact commutative topological group, with normalized Haar measure \( H \). Also let \( \mathcal{E}(A) \) be the linear span of \( \hat{A} \) in \( C(A) \), as usual. If \( f,g \in \mathcal{E}(A) \), then

\[
(4.128) \quad f(x) = \sum_{\phi \in \hat{A}} \hat{f}(\phi) \phi(x), \quad g(x) = \sum_{\psi \in \hat{A}} \hat{g}(\psi) \psi(x)
\]

for every \( x \in A \), where all but finitely many terms in the sums are equal to 0. This implies that

\[
(4.129) \quad f(x) g(x) = \sum_{\phi \in \hat{A}} \sum_{\psi \in \hat{A}} \hat{f}(\phi) \hat{g}(\psi) \phi(x) \psi(x)
\]

\[
= \sum_{\phi \in \hat{A}} \sum_{\psi \in \hat{A}} \hat{f}(\phi) \hat{g}(\psi) \phi(x)
\]

for every \( x \in A \). More precisely, this uses the change of variables \( \phi \mapsto \phi \bar{\psi} \) in the second step, which is a translation in the group \( \hat{A} \). Interchanging the order of summation, we get that

\[
(4.130) \quad f(x) g(x) = \sum_{\phi \in \hat{A}} \left( \sum_{\psi \in \hat{A}} \hat{f}(\phi) \hat{g}(\psi) \right) \phi(x)
\]

for every \( x \in A \). It follows that

\[
(4.131) \quad \hat{(fg)}(\phi) = \sum_{\psi \in \hat{A}} \hat{f}(\phi \bar{\psi}) \hat{g}(\psi)
\]

for every \( \phi \in \hat{A} \).

Suppose now that \( f,g \in L^2(A) \), so that \( \hat{f}, \hat{g} \in \ell^2(\hat{A}) \), as in Section 4.9. Using the Cauchy–Schwarz inequality, we get that

\[
(4.132) \quad \sum_{\psi \in \hat{A}} |\hat{f}(\phi \bar{\psi})||\hat{g}(\psi)| \leq \left( \sum_{\psi \in \hat{A}} |\hat{f}(\phi \bar{\psi})|^2 \right)^{1/2} \left( \sum_{\psi \in \hat{A}} |\hat{g}(\psi)|^2 \right)^{1/2}
\]
4.11. PRODUCTS OF FUNCTIONS

for every \( \phi \in \hat{A} \). Note that

\[
\sum_{\psi \in \hat{A}} |\hat{f}(\phi \psi)|^2 = \sum_{\psi \in \hat{A}} |\hat{f}(\psi)|^2
\]

for every \( \phi \in \hat{A} \), using the change of variables \( \psi \mapsto \phi \psi \). Thus

\[
\sum_{\psi \in \hat{A}} |\hat{f}(\phi \psi)| |\hat{g}(\psi)| \leq \|f\|_{L^2(A)} \|g\|_{L^2(A)}
\]

for every \( \phi \in \hat{A} \), by (4.102). Of course, \( fg \in L^1(A) \) when \( f, g \in L^2(A) \), so that

\[
(\hat{f} \hat{g})(\phi) = \int_A f(x) g(x) \overline{\phi(x)} dH(x)
\]

is defined for every \( \phi \in \hat{A} \). If \( g \in E(A) \), then it is easy to see that (4.131) holds for every \( \phi \in \hat{A} \), where all but finitely many terms in the sum on the right side of (4.131) are equal to 0. Using this and (4.134), one can check that (4.128) holds for every \( f, g \in L^2(A) \) and \( \phi \in \hat{A} \), by approximating \( g \) by elements of \( E(A) \) with respect to the \( L^2 \) norm.

If \( g \in L^1(A) \) and \( \hat{g} \in \ell^1(\hat{A}) \), then

\[
\sum_{\phi \in \hat{A}} \hat{g}(\phi) \phi(x)
\]

can be defined for every \( x \in A \), and is uniformly approximated by finite subsums, as in Section 4.9. This implies that (4.136) defines a continuous function on \( A \), and we also have that \( g(x) \) is equal to (4.136) for almost every \( x \in A \) with respect to \( H \), by the remarks at the end of Section 4.9. If \( f \in L^1(A) \), then \( \hat{f} \) is bounded on \( \hat{A} \), as in (4.8), and

\[
\sum_{\psi \in \hat{A}} |\hat{f}(\phi \psi)| |\hat{g}(\psi)| \leq \left( \sup_{\psi \in \hat{A}} |\hat{f}(\psi)| \right) \left( \sum_{\psi \in \hat{A}} |\hat{g}(\psi)| \right)
\]

\[
\leq \|f\|_{L^1(A)} \|\hat{g}\|_{\ell^1(\hat{A})}
\]

for every \( \phi \in \hat{A} \). We also have that \( fg \in L^1(A) \) in this situation, so that (4.135) is defined for every \( \phi \in \hat{A} \). As before, one can check that (4.131) holds for every \( \phi \in \hat{A} \) under these conditions, by approximating \( g \) by elements of \( E(A) \).

Suppose that \( f, g \in L^1(A) \) satisfy \( \hat{f}, \hat{g} \in \ell^1(\hat{A}) \), and let us check that \( \hat{fg} \) is in \( \ell^1(\hat{A}) \) too. Of course,

\[
|\hat{fg}(\phi)| \leq \sum_{\psi \in \hat{A}} |\hat{f}(\phi \psi)| |\hat{g}(\psi)|
\]
for every \( \phi \in \hat{A} \), by (4.131), and hence

\begin{equation}
\sum_{\phi \in \hat{A}} |(\hat{f}g)(\phi)| \leq \sum_{\phi \in \hat{A}} \sum_{\psi \in \hat{A}} |\hat{f}(\phi \psi)| |\hat{g}(\psi)|.
\end{equation}

Interchanging the order of summation, we get that

\begin{equation}
\sum_{\phi \in \hat{A}} |(\hat{f}g)(\phi)| \leq \sum_{\psi \in \hat{A}} \sum_{\phi \in \hat{A}} |\hat{f}(\phi)| |\hat{g}(\psi)|
= \left( \sum_{\phi \in \hat{A}} |\hat{f}(\phi)| \right) \left( \sum_{\psi \in \hat{A}} |\hat{g}(\psi)| \right),
\end{equation}

using the change of variables \( \phi \mapsto \phi \psi \) in the second step. One could also simply start with \( f \) as in (4.112), with coefficients \( F(\phi) \in \ell^1(\hat{A}) \), and similarly for \( g \). One could then multiply the expansions for \( f \) and \( g \) directly, as in (4.129) and (4.130), and estimate the coefficients in the resulting expansion for \( fg \) as before.

### 4.12 Translation-invariant subspaces

Let \( A \) be a locally compact commutative topological group with a Haar measure \( H \), and let \( 1 \leq p < \infty \) be given. Also let \( L^p(A) \) be the usual \( L^p \) space associated to \( H \), with the \( L^p \) norm \( \|f\|_p \). As before, put \( f_a(x) = f(x-a) \) for each \( f \in L^p(A) \) and \( a \in A \), so that

\begin{equation}
\|f_a\|_p = \|f\|_p,
\end{equation}

because \( H \) is invariant under translations. If \( f \) is a continuous function on \( A \) with compact support, then

\begin{equation}
\lim_{a \to 0} \|f_a - f\|_p = 0,
\end{equation}

where the limit is taken over \( a \in A \) converging to 0 with respect to the given topology on \( A \). This is the same as (4.13) when \( p = 1 \), and a similar argument works for all \( p \), using the fact that \( f \) is uniformly continuous on \( A \), as in Section 3.11. As in the \( p = 1 \) case, one can check that (4.142) holds for every \( f \in L^p(A) \) when \( 1 \leq p < \infty \), by approximating \( f \) by continuous functions with compact support in \( A \) with respect to the \( L^p \) norm. Note that

\begin{equation}
\|f_{a+b} - f_b\|_p = \|f_a - f\|_p
\end{equation}

for every \( a, b \in A \), by translation-invariance of the \( L^p \) norm, as in (4.141). It follows that the mapping from \( a \in A \) to \( f_a \in L^p(A) \) is uniformly continuous, by (4.142) and (4.143).
A closed linear subspace \( \mathcal{L} \) of \( L^p(A) \) is said to be \textit{invariant under translations} if for every \( f \in \mathcal{L} \) and \( a \in A \),

\begin{equation}
    f_a(x) = f(x - a) \in \mathcal{L}.
\end{equation}

If \( \mathcal{L} \) has this property and \( g \in L^1(A) \), then one can check that

\begin{equation}
    f \ast g \in \mathcal{L}
\end{equation}

for every \( f \in \mathcal{L} \). Of course, \( f \ast g \) is an integral of translates of \( f \), by definition, and so it suffices to show that this integral can be approximated by linear combinations of translates of \( f \) with respect to the \( L^p \) norm. To do this, one can use the uniform continuity of \( a \mapsto f_a \) as a mapping from \( A \) into \( L^p(A) \), as in the preceding paragraph. The approximation is quite straightforward when \( g \) has compact support in \( A \), and otherwise one can approximate \( g \) by integrable functions with compact support with respect to the \( L^1 \) norm.

Conversely, suppose that \( \mathcal{L} \) is a closed linear subspace of \( L^p(A) \), \( 1 \leq p < \infty \), such that (4.145) holds for every \( f \in \mathcal{L} \) and \( g \in L^1(A) \). Observe that

\begin{equation}
    f_a \ast g = f \ast g_a,
\end{equation}

for every \( f \in L^p(A) \), \( g \in L^1(A) \), and \( a \in A \), by the definition of the convolution in Section 4.3. It follows that

\begin{equation}
    f \ast g \in \mathcal{L}
\end{equation}

for every \( f \in \mathcal{L} \), \( g \in L^1(A) \), and \( a \in A \), by our hypothesis on \( \mathcal{L} \). Let \( f \in \mathcal{L} \) and \( a \in A \) be given, and let \( g \) be a nonnegative real-valued integrable function on \( A \), with

\begin{equation}
    \int_A g(x) \, dH(x) = 1.
\end{equation}

Also let \( U \) be an open set in \( A \) that contains 0, and suppose that \( g = 0 \) almost everywhere on \( A \setminus U \) with respect to \( H \). As in Section 4.6, \( f_a \ast g \) approximates \( f_a \) with respect to the \( L^p \) norm when \( U \) is sufficiently small. This implies that \( f_a \in \mathcal{L} \), because of (4.147), and since \( \mathcal{L} \) is supposed to be a closed set in \( L^p(A) \). Thus \( \mathcal{L} \) is invariant under translations under these conditions.

Let \( \mathcal{L} \) be a closed linear subspace of \( L^p(A) \), \( 1 \leq p < \infty \), which is invariant under translations again. If \( \nu \) is a regular complex Borel measure on \( A \), then

\begin{equation}
    f \ast \nu \in \mathcal{L}
\end{equation}

for every \( f \in \mathcal{L} \), for essentially the same reasons as in (4.145). More precisely, if \( \nu \) has compact support, then it is easy to approximate \( f \ast \nu \) by linear combinations of translates of \( f \) with respect to the \( L^p \) norm, using the uniform continuity of \( a \mapsto f_a \) as a mapping from \( A \) into \( L^p(A) \). Otherwise, one can approximate \( \nu \) by measures with compact support, as before.

Now let \( \mathcal{L} \) be a linear subspace of the space \( C_b(A) \) of bounded continuous functions on \( A \), and suppose that \( \mathcal{L} \) is a closed set with respect to the supremum norm on \( C_b(A) \). As before, \( \mathcal{L} \) is said to be invariant under translations if (4.144)
for every $f \in \mathcal{L}$ and $a \in A$. If the elements of $\mathcal{L}$ are uniformly continuous on $A$, then it is easy to see that the previous arguments carry over to this situation. In particular, this applies to closed linear subspaces of $C_0(A)$, and to closed linear subspaces of $C(A)$ when $A$ is compact.

Note that $L^1(A)$ is a commutative Banach algebra, using convolution of functions as multiplication. A linear subspace $\mathcal{L}$ of $L^1(A)$ is an ideal in $L^1(A)$ as a commutative Banach algebra if (4.145) holds for every $f \in \mathcal{L}$ and $g \in L^1(A)$. Thus the closed ideals in $L^1(A)$ are the same as the closed linear subspaces of $L^1(A)$ that are invariant under translations, by the earlier discussion.

If $E$ is any subset of $\hat{A}$, then it is easy to see that

$$L_E = \{ f \in L^1(A) : \hat{f}(\phi) = 0 \text{ for every } \phi \in E \}$$

is a closed ideal in $L^1(A)$. Remember that $\hat{f}$ is continuous with respect to the usual topology on $\hat{A}$ for each $f \in L^1(A)$, as in Section 4.1. This implies that

$$L_E = L_{\overline{E}}$$

for every $E \subseteq \hat{A}$, where $\overline{E}$ is the closure of $E$ in $\hat{A}$. Thus we may as well restrict our attention to closed sets $E \subseteq \hat{A}$ here. If $\mathcal{L}$ is any subset of $L^1(A)$, then

$$E(\mathcal{L}) = \{ \phi \in \hat{A} : \hat{f}(\phi) = 0 \text{ for every } f \in \mathcal{L} \}$$

is a closed subset of $\hat{A}$, because $\hat{f}$ is continuous on $\hat{A}$ for each $f \in L^1(A)$, and

$$\mathcal{L} \subseteq L_{E(\mathcal{L})},$$

by construction.

Let us restrict our attention to the case where $A$ is compact for the rest of the section, and take Haar measure $H$ on $A$ to be normalized so that $H(A) = 1$. Let $E$ be any subset of $\hat{A}$, which is automatically a closed set, because the usual topology on $\hat{A}$ is discrete when $A$ is compact. If $\phi, \psi \in \hat{A}$ and $\phi \neq \psi$, then $\phi$ is orthogonal to $\psi$ in $L^2(A)$, as in Section 3.8, which is equivalent to saying that

$$\hat{\phi}(\psi) = 0.$$

If $\phi \in \hat{A} \setminus E$, then it follows that $\phi \in L_E$, so that $\hat{A} \setminus E \subseteq L_E$. This implies that the closure of the linear span of $\hat{A} \setminus E$ in $L^1(A)$ is also contained in $L(\mathcal{E})$. If $f$ is any integrable function on $A$, then $f$ can be approximated by linear combinations of $\phi \in \hat{A}$ such that $\hat{f}(\phi) \neq 0$, as in Section 4.10. If $f \in L_E$, then we get that $f$ is in the closure of the linear span of $\hat{A} \setminus E$, by the definition of $L_E$. This shows that $L_E$ is equal to the closure of the linear span of $\hat{A} \setminus E$ in $L^1(A)$ when $A$ is compact.

Remember that

$$f * \phi = \hat{f}(\phi) \phi$$

for every $f \in L^1(A)$ and $\phi \in \hat{A}$, as in (4.42) in Section 4.4. Let $\mathcal{L}$ be a closed ideal in $L^1(A)$, and let $E(\mathcal{L})$ be as in (4.153). Thus (4.155) is an element of $\mathcal{L}$.
for every \( f \in \mathcal{L} \) and \( \phi \in \hat{\mathcal{A}} \). If \( \phi \in \hat{\mathcal{A}} \setminus \mathcal{E}(\mathcal{L}) \), then \( \hat{f}(\phi) \neq 0 \) for some \( f \in \mathcal{L} \), and it follows that \( \phi \in \mathcal{L} \). This implies that the closure of the linear span of \( \hat{\mathcal{A}} \setminus \mathcal{E}(\mathcal{L}) \) in \( L^1(\mathcal{A}) \) is contained in \( \mathcal{L} \), because \( \mathcal{L} \) is a closed linear subspace of \( L^1(\mathcal{A}) \). We also know that the closure of the linear span of \( \hat{\mathcal{A}} \setminus \mathcal{E}(\mathcal{L}) \) in \( L^1(\mathcal{A}) \) is equal to \( \mathcal{L}_{E(\mathcal{L})} \), as in the previous paragraph. Combining this with (4.153), we get that

\[
\mathcal{L} = \mathcal{L}_{E(\mathcal{L})}
\]

in this case.

If \( E \) is any subset of \( \hat{\mathcal{A}} \) and \( 1 < p < \infty \), then it is easy to see that

\[
\mathcal{L}_E \cap L^p(\mathcal{A})
\]

is a closed linear subspace of \( L^p(\mathcal{A}) \) which is invariant under translations. One can also check that (4.157) is the same as the closure of the linear span of \( \hat{\mathcal{A}} \setminus E \) in \( L^p(\mathcal{A}) \) under these conditions, for essentially the same reasons as when \( p = 1 \). If \( \mathcal{L} \) is any closed linear subspace of \( L^p(\mathcal{A}) \) that is invariant under translations, where \( 1 < p < \infty \), and if \( E(\mathcal{L}) \) is as in (4.152), then we have that

\[
\mathcal{L} = \mathcal{L}_{E(\mathcal{L})} \cap L^p(\mathcal{A}),
\]

as in the preceding paragraph. Similarly,

\[
\mathcal{L}_E \cap C(\mathcal{A})
\]

is a closed linear subspace of \( C(\mathcal{A}) \) which is invariant under translations for each \( E \subseteq \hat{\mathcal{A}} \), and (4.159) is the same as the closure of the linear span of \( \hat{\mathcal{A}} \setminus E \) in \( C(\mathcal{A}) \) with respect to the supremum norm. If \( \mathcal{L} \) is any closed linear subspace of \( C(\mathcal{A}) \) with respect to the supremum norm that is invariant under translations, then

\[
\mathcal{L} = \mathcal{L}_{E(\mathcal{L})} \cap C(\mathcal{A}),
\]

where \( E(\mathcal{L}) \) is as in (4.152).

### 4.13 Translation-invariant subalgebras

Let \( \mathcal{A} \) be a compact commutative topological group, with Haar measure \( H \) on \( \mathcal{A} \) normalized so that \( H(\mathcal{A}) = 1 \). As usual, \( \mathcal{E}(\mathcal{A}) \) denotes the linear span of \( \hat{\mathcal{A}} \) in \( C(\mathcal{A}) \), which is a subalgebra of \( C(\mathcal{A}) \) that contains the constant functions and is invariant under complex conjugation. Similarly, let \( \mathcal{B} \) be a subset of \( \hat{\mathcal{A}} \), and let \( \mathcal{E}_B(\mathcal{A}) \) be the linear span of \( B \) in \( C(\mathcal{A}) \), which is interpreted as being \( \{0\} \) when \( B = \emptyset \). Also let \( C_B(\mathcal{A}) \) be the closure of \( \mathcal{E}_B(\mathcal{A}) \) in \( C(\mathcal{A}) \), with respect to the supremum norm.

If \( f \in C_B(\mathcal{A}) \), then it is easy to see that

\[
\hat{f}(\phi) = 0
\]
for every \( \phi \in \hat{A} \setminus B \), by approximating \( f \) by elements of \( E_B(A) \). Conversely, if \( f \in C(A) \) satisfies (4.161) for every \( \phi \in \hat{A} \setminus B \), then \( f \in C_B(A) \), by the discussion in Section 4.10. Thus

\[
C_B(A) = \{ f \in C(A) : \hat{f}(\phi) = 0 \text{ for every } \phi \in \hat{A} \setminus B \},
\]

which is the same as (4.159), with \( E = \hat{A} \setminus B \). This is a closed linear subspace of \( C(A) \) that is invariant under translations, and every translation-invariant closed linear subspace of \( C(A) \) is of this form, as in the previous section. Note that

\[ B = \hat{A} \cap C_B(A). \]

Suppose that \( B \) is a sub-semigroup of \( \hat{A} \), which is to say that the product of any two elements of \( B \) is also contained in \( B \). This implies that \( E_B(A) \) is a subalgebra of \( E(A) \), and hence that \( C_B(A) \) is a sub-algebra of \( C(A) \). One can also look at this in terms of (4.131) and (4.162). Conversely, if \( C_B(A) \) is a subalgebra of \( C(A) \), then \( B \) is a sub-semigroup of \( \hat{A} \), by (4.163).

Remember that the identity element of \( \hat{A} \) is the constant function equal to 1 at every point in \( A \). Using (4.163) again, we get that \( B \) contains the identity element of \( \hat{A} \) if and only if \( C_B(A) \) contains the constant functions on \( A \).

If \( \phi \in \hat{A} \), then \( 1/\phi \) is equal to the complex conjugate \( \overline{\phi} \) of \( \phi \), because \( \phi \) takes values in \( T \). Put

\[
B^{-1} = \{ 1/\phi : \phi \in B \} = \{ \overline{\phi} : \phi \in B \},
\]

so that

\[
E_{B^{-1}}(A) = \{ \overline{f} : f \in E_B(A) \},
\]

and hence

\[
C_{B^{-1}}(A) = \{ \overline{f} : f \in C_B(A) \},
\]

by the definitions of \( E_{B^{-1}}(A) \) and \( C_{B^{-1}}(A) \). Similarly, one can use (4.162) to get that

\[
C_B(A) \cap C_{B^{-1}}(A) = C_{B \cap B^{-1}}(A).
\]

In particular, if \( B^{-1} = B \), then it follows that \( C_B(A) \) is invariant under complex conjugation. Conversely, if \( C_B(A) \) is invariant under complex conjugation, then \( B = B^{-1} \), by (4.163).

If \( B \) is a subgroup of \( \hat{A} \), then it follows that \( C_B(A) \) is a subalgebra of \( C(A) \) that contains the constant functions and is invariant under complex conjugation. Conversely, if \( C_B(A) \) is a subalgebra of \( C(A) \) that is invariant under complex conjugation, then \( B \) is a sub-semigroup of \( \hat{A} \) that satisfies \( B^{-1} = B \). This implies that \( B \) is a subgroup of \( \hat{A} \) when \( B \neq \emptyset \), so that \( C_B(A) \neq \{0\} \).

Let us now consider a basic example where \( A = T \). Remember that

\[
z \mapsto z^j
\]

is a continuous homomorphism from \( T \) into itself for every \( j \in \mathbb{Z} \), and that every continuous homomorphism from \( T \) into itself is of this form. Let \( B \) be
the collection of these homomorphisms (4.168) with \( j \geq 0 \), so that \( B \) is a sub-
semigroup of the dual of \( T \) that contains the identity element. In this case, 
\( CB(T) \) consists of the \( f \in C(T) \) that can be extended to a continuous function 
on the closed unit disk \( \mathcal{D} \) which is holomorphic on the open unit disk \( D \), as in 
Section 1.8.

The case of subalgebras of \( C(A) \) associated to subgroups of \( \hat{A} \) is discussed 
further in the next section, and an extension of the example in the preceding 
paragraph will be considered in Section 5.15.

4.14 Subgroups and subalgebras

Let \( A \) be a compact commutative topological group, and let \( A_1 \) be a closed 
subgroup of \( A \). Thus the quotient group \( A/A_1 \) is also a compact commutative 
topological group with respect to the quotient topology, as in Section 3.10. 
Every continuous function on \( A/A_1 \) leads to a continuous function on \( A \) that is 
constant on the translates of \( A_1 \) in \( A \), by composition with the quotient mapping 
\( q_1 \) from \( A \) onto \( A/A_1 \). Conversely, if \( f \) is any function on \( A \) that is constant on 
the translates of \( A_1 \) in \( A \), then \( f \) can be expressed as the composition of \( q_1 \) with 
a function \( f_1 \) on \( A/A_1 \). If \( f \) is also continuous on \( A \), then it is easy to see that 
\( f_1 \) is continuous on \( A/A_1 \), by the definition of the quotient topology.

Note that 
\( \phi_1 \circ q_1 \)
defines an algebra homomorphism from \( C(A/A_1) \) into \( C(A) \), which is also an 
isometric embedding with respect to the supremum norms on \( C(A/A_1) \) and 
\( C(A) \). As in the previous paragraph, the image of \( C(A/A_1) \) in \( C(A) \) under 
(4.169) is equal to 
\( \{ f \in C(A) : f(a) = f(b) \text{ for every } a, b \in A \text{ such that } a - b \in A_1 \} \).

It is easy to see that (4.170) is a closed subalgebra of \( C(A) \) that contains the 
constant functions, is invariant under translations on \( A \), and is invariant under 
complex conjugation as well.

Let \( B_1 \) be the set of \( \phi \in \hat{A} \) such that \( A_1 \) is contained in the kernel of \( \phi \), 
which is the same as saying that \( \phi \) is constant on the translates of \( A_1 \) in \( A \), 
because \( \phi \) is a homomorphism. If \( \phi_1 \) is a continuous homomorphism from \( A/A_1 \) 
into \( T \), then 
\( \phi_1 \circ q_1 \in B_1 \),
and every element of \( B_1 \) is of this form. This is analogous to the discussion in 
the preceding paragraphs, and was also mentioned in Section 3.10. This defines 
a natural group isomorphism from the dual \( (A/A_1) \) of \( A/A_1 \) onto \( B_1 \).

Under these conditions, \( CB_1(A) \) is the same as (4.170). More precisely, \( B_1 \) is 
contained in (4.170) by construction, which implies that the linear span \( \mathcal{E}_{B_1}(A) \) 
of \( B_1 \) in \( C(A) \) is contained in (4.170), and hence that the closure \( CB_1(A) \) of 
\( \mathcal{E}_{B_1}(A) \) in \( C(A) \) is contained in (4.170). In the other direction, if \( f_1 \in C(A/A_1) \),
then $f_1$ can be approximated by linear combinations of elements of $(\hat{A}/A_1)$ uniformly on $A/A_1$, because $A/A_1$ is compact. This implies that (4.170) is contained in $C_{B_1}(A)$, since every element of (4.170) is of the form $f_1 \circ q_1$ for some $f_1 \in C(A/A_1)$.

Now let $B$ be any subgroup of $\hat{A}$, and consider

\begin{equation}
A_1 = \{a \in A : \phi(a) = 1 \text{ for every } \phi \in B\}.
\end{equation}

This is a closed subgroup of $A$, so that the quotient group $A/A_1$ is a compact commutative topological group with respect to the quotient topology, as before. If $q_1$ is the usual quotient mapping from $A$ onto $A/A_1$, then every $\phi \in B$ can be expressed as $\phi_1 \circ q_1$ for some $\phi_1 \in (\hat{A}/A_1)$. Put

\begin{equation}
\tilde{B} = \{\phi_1 \in (\hat{A}/A_1) : \phi_1 \circ q_1 \in B\},
\end{equation}

which is a subgroup of $(\hat{A}/A_1)$ isomorphic to $B$. Of course, if $a \in A \setminus A_1$, then there is a $\phi \in B$ such that $\phi(a) \neq 1$, by definition of $A_1$. This implies that $\tilde{B}$ separates points in $A/A_1$ in this situation, because of the way that $A_1$ is defined. It follows that

\begin{equation}
\tilde{B} = (\hat{A}/A_1),
\end{equation}

because $A/A_1$ is compact, as in Section 3.8.

Let $B_1$ be the subgroup of $\hat{A}$ associated to $A_1$ as before, consisting of the $\phi \in \hat{A}$ such that the kernel of $\phi$ contains $A_1$. Thus $B \subseteq B_1$ in this case, by the definition (4.172) of $A_1$. If $\phi \in B_1$, then we have seen that $\phi = \phi_1 \circ q_1$ for some $\phi_1 \in (\hat{A}/A_1)$, and now we have that $\phi_1 \in \tilde{B}$, by (4.174). This implies that $\phi \in \tilde{B}$, so that

\begin{equation}
B = B_1.
\end{equation}

It follows that

\begin{equation}
C_B(A) = C_{B_1}(A),
\end{equation}

and hence that $C_B(A)$ is equal to (4.170).

Let $B_0$ be a nonempty subset of $\hat{A}$, and let $B$ be the subgroup of $\hat{A}$ generated by $B_0$. Note that $B$ separates points in $A$ if and only if $B_0$ separates points in $A$. In this case, $B = \hat{A}$, because $A$ is compact, as in Section 3.8. Otherwise, one can apply the previous discussion to $B$, to get that $C_B(A)$ is equal to (4.170), with $A_1$ as in (4.172).
Chapter 5

Additional topics

5.1 Almost periodic functions

Let $A$ be a commutative topological group, and let $C_b(A)$ be the space of complex-valued functions $f$ on $A$ that are bounded and continuous, equipped with the supremum norm $\|f\|_{\sup}$. Let $f \in C_b(A)$ be given, and consider the set
\[ \{ f_a : a \in A \} \]
(5.1)
of all translates of $f$, where $f_a(x) = f(x - a)$, as in (4.10). If (5.1) is totally bounded as a subset of $C_b(A)$, with respect to the supremum metric, then $f$ is said to be almost periodic on $A$. Equivalently, this means that the closure of (5.1) in $C_b(A)$ is compact, since $C_b(A)$ is complete. Let $AP(A)$ denote the collection of almost periodic functions on $A$.

Remember that a subset $E$ of a metric space is totally bounded if for each $r > 0$, $E$ can be covered by finitely many balls of radius $r > 0$. One can get an equivalent condition by requiring that these balls be centered at elements of $E$. More precisely, if $E$ is covered by finitely many balls of radius $r/2$ that are not necessarily centered at elements of $E$, then one can first throw away the balls that do not intersect $E$. The remaining balls are contained in balls of radius $r$ centered at elements of $E$, by the triangle inequality.

Constant functions on $A$ are obviously almost periodic, and it is easy to see that continuous homomorphisms from $A$ into $T$ are almost periodic as well. If $f, g \in AP(A)$, then one can check their sum $f + g$ is almost periodic, because the sum of two totally bounded subsets of $C_b(A)$ is totally bounded. Similarly, the product $fg$ is almost periodic on $A$, so that $AP(A)$ is a subalgebra of $C_b(A)$. The complex-conjugate of an almost periodic function on $A$ is almost periodic on $A$ too, and in fact $f \in C_b(A)$ is almost periodic on $A$ if and only if the real and imaginary parts of $f$ are almost periodic on $A$. Note that $AP(A)$ is a closed subset of $C_b(A)$ with respect to the supremum metric.

If $f \in AP(A)$ and $\phi$ is a continuous complex-valued function on $C$, then one can check that their composition $\phi \circ f$ is almost periodic on $A$ too. This uses
the fact that $\phi$ is uniformly continuous on compact subsets of $C$. In particular, if $f \in \mathcal{AP}(A)$ and $|f(x)|$ has a positive lower bound on $A$, then one can choose $\phi$ so that
\[
\phi(f(x)) = 1/f(x)
\]
for every $x \in A$. Thus $1/f \in \mathcal{AP}(A)$ when $f \in \mathcal{AP}(A)$ and $|f(x)|$ has a positive lower bound on $A$. Similarly, $|f| \in \mathcal{AP}(R)$ for every $f \in \mathcal{AP}(R)$.

If $f$ is any bounded continuous function on $A$ and $b \in A$, then the translate $f_b(x) = f(x - b)$ of $f$ by $b$ is also bounded and continuous on $A$. Of course, $f_b$ generates the same set of translates in $C_b(A)$ as $f$ does. Thus $f_b \in \mathcal{AP}(A)$ for every $b \in A$ when $f \in \mathcal{AP}(A)$.

Uniform continuity of complex-valued functions on $A$ was defined in Section 3.11, and uniform continuity of functions on $A$ with values in a metric space or a commutative topological group can be defined similarly. If $f \in C_b(A)$ is uniformly continuous on $A$, then
\[
a \mapsto f_a
\]
is uniformly continuous as a mapping from $A$ into $C_b(A)$, with respect to the supremum metric on $C_b(A)$. In this case, if $A$ is also totally bounded as a commutative topological group, then it follows that (5.1) is totally bounded in $C_b(A)$, so that $f$ is almost periodic on $A$. If $A$ is compact, then we have seen that every continuous complex-valued function on $A$ is uniformly continuous, and $A$ is automatically totally bounded as well. It follows that every continuous complex-valued function on $A$ is almost periodic on $A$ when $A$ is compact.

If $E \subseteq C_b(A)$ is totally bounded with respect to the supremum metric, then it is easy to see that $E$ is equicontinuous at every point in $A$. If $f \in \mathcal{AP}(A)$, then it follows that (5.1) is equicontinuous at every point in $A$. The equicontinuity of (5.1) at any point in $A$ is equivalent to the uniform continuity of $f$ on $A$. In particular, every $f \in \mathcal{AP}(A)$ is uniformly continuous on $A$.

Let $B$ be another commutative topological group, and suppose that $h$ is a continuous homomorphism from $A$ into $B$. If $g \in \mathcal{AP}(B)$, then one can check that $g \circ h \in \mathcal{AP}(A)$. If $B$ is compact, then $g \circ h \in \mathcal{AP}(A)$ for every $g \in C(B)$, because every continuous function on $B$ is almost periodic.

If $B$ is a compact commutative topological group, then it is well known that the continuous homomorphisms from $B$ into $T$ separate points on $B$. If $g \in C(B)$, then it follows that $g$ can be approximated uniformly on $B$ by finite linear combinations of continuous homomorphisms from $B$ into $T$, because of the Stone–Weierstrass theorem, as mentioned in Section 3.8. If $h$ is a continuous homomorphism from $A$ into $B$ and $f = g \circ h$, then it is easy to see that $f$ can be approximated uniformly on $A$ by finite linear combinations of continuous homomorphisms from $A$ into $T$. It will be shown in the next section that every $f \in \mathcal{AP}(A)$ can be represented in this way, which implies that every $f \in \mathcal{AP}(A)$ can be approximated uniformly on $A$ by finite linear combinations of continuous homomorphisms from $A$ into $T$. 
5.2 Groups of isometries

Let \((M, d(x, y))\) be a (nonempty) metric space, and let \(C(M, M)\) be the space of continuous mappings from \(M\) into itself. This is a semigroup with respect to composition of mappings, and with the identity mapping on \(M\) as the identity element. Similarly, the collection of homeomorphisms from \(M\) onto itself is a group with respect to composition. Let \(\mathcal{I}(M)\) be the collection of mappings \(\phi\) from \(M\) onto itself that are isometries, in the sense that

\[
d(\phi(x), \phi(y)) = d(x, y)
\]

for every \(x, y \in M\). This is a subgroup of the group of homeomorphisms from \(M\) onto itself.

Suppose now that \(M\) is bounded, and put

\[
\rho(\phi, \psi) = \sup_{x \in M} d(\phi(x), \psi(x))
\]

for every \(\phi, \psi \in C(M, M)\), which is the supremum metric on \(C(M, M)\). Note that (5.5) is invariant under composition of \(\phi\) and \(\psi\) on the right by continuous mappings from \(M\) onto itself, and under composition of \(\phi\) and \(\psi\) on the left by isometries from \(M\) into itself. In particular, the restriction of the supremum metric to \(\mathcal{I}(M)\) is invariant under left and right translations on \(\mathcal{I}(M)\) as a group with respect to composition of mappings. Using this, one can check that composition of mappings defines a continuous mapping from \(\mathcal{I}(M) \times \mathcal{I}(M)\) into \(\mathcal{I}(M)\), with respect to the topology on \(\mathcal{I}(M)\) determined by the restriction of the supremum metric to \(\mathcal{I}(M)\).

If \(\phi\) is any homeomorphism from \(M\) onto itself, then the distance from \(\phi^{-1}\) to the identity mapping on \(M\) with respect to the supremum metric is equal to the distance from \(\phi\) to the identity mapping. If \(\phi, \psi \in \mathcal{I}(M)\), then the distance from \(\phi^{-1}\) to \(\psi^{-1}\) with respect to the supremum metric is equal to the distance from \(\phi\) to \(\psi\). This implies that \(\phi \mapsto \phi^{-1}\) is a continuous mapping from \(\mathcal{I}(M)\) into itself with respect to the restriction of the supremum metric to \(\mathcal{I}(M)\). It follows that \(\mathcal{I}(M)\) is a topological group with respect to composition of mappings and the topology determined by the restriction of the supremum metric to \(\mathcal{I}(M)\).

It is easy to see that the collection of \(f \in C(M, M)\) such that \(f(M)\) is dense in \(M\) is a closed subset of \(C(M, M)\) with respect to the supremum metric. If \(M\) is compact, then \(f(M)\) is a compact subset of \(M\) for every \(f \in C(M, M)\), which implies that \(f(M)\) is a closed set in \(M\). It follows that the collection of \(f \in C(M, M)\) such that \(f(M) = M\) is a closed set in \(C(M, M)\) with respect to the supremum metric when \(M\) is compact. Using standard Arzela–Ascoli arguments, one can check that \(\mathcal{I}(M)\) is compact with respect to the supremum metric when \(M\) is compact.

Now let \(A\) be a commutative topological group, and let \(f \in \mathcal{AP}(A)\) be given. Let \(M\) be the closure of (5.1) in \(C_b(A)\), which is a compact metric space with respect to the restriction of the supremum metric on \(C_b(A)\) to \(M\). As before, the group \(\mathcal{I}(M)\) of isometries of \(M\) onto itself is a compact topological group with respect to the restriction of the supremum metric on \(C(M, M)\) to \(\mathcal{I}(M)\).
Put
\[ T_y(g)(x) = g(x + y) \]  
for every \( g \in C_b(A) \) and \( x, y \in A \), so that \( T_y \) defines a linear mapping from \( C_b(A) \) onto itself that preserves the supremum norm. Of course,
\[ T_y(f_a) = f_{a-y} \]  
for every \( a, y \in A \), which implies that \( T_y \) maps (5.1) onto itself for every \( y \in A \), and hence that \( T_y \) maps \( M \) onto itself. We also have that
\[ T_y \circ T_z = T_{y+z} \]  
for every \( y, z \in A \), so that \( y \mapsto T_y \) defines a homomorphism from \( A \) into the group of isometric linear mappings from \( C_b(A) \) onto itself. This leads to a homomorphism from \( A \) into \( \mathcal{I}(M) \), which sends \( y \in A \) to the restriction of \( T_y \) to \( M \).

Remember that \( f \) is uniformly continuous on \( A \), which implies that (5.1) is uniformly equicontinuous on \( A \), and hence that the elements of \( M \) are uniformly equicontinuous on \( A \). Using this, one can check that the mapping from \( y \in A \) to the restriction of \( T_y \) to \( M \) is continuous with respect to the supremum metric on \( C(M, M) \). Let \( B_0 \) be the subgroup of \( \mathcal{I}(M) \) consisting of the restriction of \( T_y \) to \( M \) for each \( y \in A \), and let \( B_1 \) be the closure of \( B_0 \) in \( \mathcal{I}(M) \) with respect to the supremum metric on \( C(M, M) \). Thus \( B_0 \) is a commutative subgroup of \( \mathcal{I}(M) \), because \( A \) is commutative, which implies that \( B_1 \) is commutative as well. Note that \( B_1 \) is compact, because \( \mathcal{I}(M) \) is compact.

Put
\[ \lambda_0(g) = g(0) \]  
for each \( g \in C_b(A) \), which defines a bounded linear functional on \( C_b(A) \) with respect to the supremum norm. Also put
\[ F(T) = \lambda_0(T(f)) = (T(f))(0) \]  
for each \( T \in C(M, M) \), where \( T(f) \in M \) because \( f \in M \), by construction. It is easy to see that (5.10) is continuous as a function of \( T \) with respect to the supremum metric on \( C(M, M) \). In particular, the restriction of \( F \) to \( B_1 \) defines a continuous function on \( B_1 \) such that
\[ F(T_y) = (T_y(f))(0) = f(y) \]  
for every \( y \in A \). This shows that \( f \) can be expressed as the composition of a continuous homomorphism from \( A \) into \( B_1 \) with a continuous function on \( B_1 \), as mentioned in the previous section.

### 5.3 Compactifications

Let \( A, B \) be commutative topological groups, and let \( \hat{A} = \text{Hom}(A, \mathbf{T}) \), \( \hat{B} = \text{Hom}(B, \mathbf{T}) \) be the corresponding dual groups of continuous homomorphisms...
5.3. COMPACTIFICATIONS

If $h$ is a continuous homomorphism from $A$ into $B$, then

$$\hat{h}(\phi) = \phi \circ h$$

defines a homomorphism from $\hat{B}$ into $\hat{A}$, as in Section 3.10. This is known as the dual homomorphism associated to $h$, and it is also continuous with respect to the usual topologies on $A, \hat{A}, \hat{B}$. In this section, we shall be especially interested in the case where $B$ is compact, so that the usual topology on $\hat{B}$ is the same as the discrete topology, and the continuity of $\hat{h}$ is trivial.

We shall also be especially interested in the case where $h(A)$ is dense in $B$. Of course, one can always arrange for this to be the case, by replacing $B$ with the closure of $h(A)$ in $B$. If $h(A)$ is dense in $B$, then it is easy to see that $\hat{h}$ is injective, as in Section 3.10. Otherwise, the kernel of $\hat{h}$ is isomorphic as a group to the dual of the quotient $B/\overline{h(A)}$, where $\overline{h(A)}$ is the closure of $h(A)$ in $B$, and the quotient group $B/\overline{h(A)}$ is equipped with the corresponding quotient topology. If $B$ is locally compact, then we have seen that the quotient of $B$ by any closed subgroup is locally compact as well. It is well known that the dual of a locally compact commutative topological group separates points on that group, and in particular that the dual of a nontrivial locally compact commutative topological group is nontrivial. Thus the dual of $B/\overline{h(A)}$ is nontrivial when $B$ is locally compact and $\overline{h(A)} \neq B$. If $\hat{h}$ is injective, then the dual of $B/\overline{h(A)}$ is trivial, which implies that $\overline{h(A)} = B$ when $B$ is locally compact.

Let $\hat{A}, \hat{B}$ be the duals of $\hat{A}, \hat{B}$, respectively, which are the second duals of $A, B$. Remember that there are natural homomorphisms from $A, B$ into their second duals, as in Section 3.9. We have seen that these mappings are continuous on compact subsets of their domains, and hence that they are continuous under some additional conditions. Let $\hat{h}$ be the dual homomorphism associated to $h$, which is a continuous homomorphism from $\hat{A}$ into $\hat{B}$. It is easy to see that the composition of the natural mapping from $A$ into $\hat{A}$ with $\hat{h}$ is equal to the composition of $h$ with the natural mapping from $B$ into $\hat{B}$.

If $B$ is locally compact, then it is well known that the natural mapping from $B$ into $\hat{B}$ is an isomorphism from $B$ onto $\hat{B}$ as topological groups. Let $k$ be a continuous homomorphism from $B$ into $A$, so that the dual $\hat{k}$ of $k$ is a continuous homomorphism from $\hat{B}$ into $\hat{A}$. If $B$ is locally compact, then $\hat{B}$ can be identified with $\hat{B}$, and hence $\hat{k}$ can be identified with a continuous homomorphism from $\hat{A}$ into $B$. In this case, the composition of the natural mapping from $A$ into $\hat{A}$ with $\hat{k}$ can be identified with a homomorphism from $A$ into $B$, which is continuous when the natural mapping from $A$ into $\hat{A}$ is continuous. Under these conditions, the dual of this homomorphism from $A$ into $B$ is equal to $k$.

Suppose now that $B$ is compact, so that the usual topology on $\hat{B}$ is the same as the discrete topology. Let $k$ be a homomorphism from $\hat{B}$ into $\hat{A}$, which is automatically continuous, and let $g$ be the homomorphism from $A$ into $\hat{B}$ which
is the composition of the natural mapping from $A$ into $\hat{A}$ with the dual $\hat{k}$ of $k$. Thus for each $\phi \in \hat{B}$ and $a \in A$, $k(\phi) \in \hat{A}$, $g(a) \in \hat{B}$, and

$$g(a)(\phi) = k(\phi)(a),$$

which is continuous as a function of $a \in A$ for each $\phi \in \hat{B}$. In this situation, one can check that $g$ is continuous as a mapping from $A$ into $\hat{B}$, without additional conditions on $A$. This uses the fact that the compact subsets of $\hat{B}$ are finite, because $\hat{B}$ is equipped with the discrete topology.

As before, the natural mapping from $B$ into $\hat{\hat{B}}$ is an isomorphism from $B$ onto $\hat{\hat{B}}$ as topological groups when $B$ is compact, so that $g$ can be identified with a continuous homomorphism from $A$ into $\hat{B}$, whose dual is equal to $k$. If $k$ is injective, then $g(A)$ is dense in $B$, by the earlier arguments.

Alternatively, let $C$ be a commutative group equipped with the discrete topology, and let $k$ be a homomorphism from $C$ into $\hat{A}$, which is automatically continuous. Thus the dual of $k$ is a continuous homomorphism from $\hat{A}$ into $\hat{\hat{C}}$, and $\hat{C}$ is compact. If $g$ is the composition of the natural mapping from $A$ into $\hat{A}$ with $\hat{k}$, then

$$g(a)(c) = k(c)(a)$$

for every $a \in A$ and $c \in C$. One can check that $g$ is continuous as a mapping from $A$ into $\hat{\hat{C}}$ under these conditions, because the compact subsets of $C$ are finite, and (5.14) is continuous as a function of $a \in A$ for each $c \in C$. The dual of $g$ is a homomorphism from $\hat{\hat{C}}$ into $\hat{A}$, which is automatically continuous, because the usual topology on $\hat{\hat{C}}$ is the discrete topology, since $\hat{C}$ is compact. The composition of the natural mapping from $C$ into $\hat{\hat{C}}$ and the dual of $g$ is equal to $k$, and we have seen that the natural mapping from $C$ into $\hat{\hat{C}}$ is an isomorphism when $C$ is discrete. Of course, this means that $C$ is isomorphic to the dual of a compact commutative topological group.

In particular, one can apply this with $C$ equal to $\hat{A}$ as a commutative group, equipped with the discrete topology. The identity mapping on $\hat{A}$ can then be interpreted as a homomorphism $k$ from $C$ into $\hat{A}$. This leads to a continuous homomorphism $g$ from $A$ into $\hat{\hat{C}}$ as in the previous paragraph, which is injective exactly when $\hat{A}$ separates points in $A$. Note that $g(A)$ is dense in $\hat{\hat{C}}$, because $k$ is injective by construction, and $\hat{\hat{C}}$ is compact. This construction is known as the Bohr compactification of $A$.

### 5.4 Averages on $\mathbb{R}$

Let $f$ be a bounded continuous complex-valued function on the real line that is almost periodic, and let $\epsilon > 0$ be given. Thus there are finitely many elements $a_1, \ldots, a_n$ of $\mathbb{R}$ such that for each $a \in \mathbb{R}$,

$$\|f_a - f_{a_j}\|_{\text{sup}} < \epsilon$$

(5.15)
for some $j \in \{1, \ldots, n\}$, where $\| \cdot \|_{\text{sup}}$ is the supremum norm on $C_b(\mathbb{R})$.

Let $I$ be a closed interval in $\mathbb{R}$, with length $|I| > 0$. Because $f$ is bounded on $\mathbb{R}$, there is an $L(\epsilon) > 0$ such that

\begin{equation}
\left| \frac{1}{|I|} \int_I f(x - a_j) \, dx - \frac{1}{|I|} \int_I f(x) \, dx \right| < \epsilon
\end{equation}

for every $j = 1, \ldots, n$ when $|I| \geq L(\epsilon)$. This implies that $\| f - \bar{f} \|_{\text{sup}} < \epsilon$ (5.16)

for every $a \in \mathbb{R}$ when $|I| \geq L(\epsilon)$, by (5.15). Equivalently,

\begin{equation}
\left| \frac{1}{|I'|} \int_{I'} f(x) \, dx - \frac{1}{|I|} \int_I f(x) \, dx \right| < 2 \epsilon
\end{equation}

for every closed interval $I'$ in $\mathbb{R}$ with $|I'| = |I| \geq L(\epsilon)$.

Suppose now that $I'$ is a closed interval in $\mathbb{R}$ whose length is equal to a positive integer $l$ times $|I|$. This implies that $I'$ can be expressed as

\begin{equation}
I' = \bigcup_{k=1}^{l} I'_k,
\end{equation}

where $I'_1, \ldots, I'_l$ are closed intervals in $\mathbb{R}$ with disjoint interiors, and $|I'_k| = |I|$ for $k = 1, \ldots, l$. Thus

\begin{equation}
\frac{1}{|I'|} \int_{I'} f(x) \, dx = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{|I'_k|} \int_{I'_k} f(x) \, dx,
\end{equation}

and

\begin{equation}
\left| \frac{1}{|I'_k|} \int_{I'_k} f(x) \, dx - \frac{1}{|I|} \int_I f(x) \, dx \right| < 2 \epsilon
\end{equation}

for each $k = 1, \ldots, l$ when $|I| \geq L(\epsilon)$, as before. It follows that (5.18) still holds in this case when $|I| \geq L$.

Suppose instead that $|I'|$ is a positive rational number times $|I|$. This implies that there are positive integer $l$, $l'$ such that

\begin{equation}
l' |I'| = l |I|,
\end{equation}

and we can take $I''$ to be a closed interval in $\mathbb{R}$ whose length is equal to (5.22). If $|I|, |I'| \geq L(\epsilon)$, then we get that

\begin{equation}
\left| \frac{1}{|I''|} \int_{I''} f(x) \, dx - \frac{1}{|I|} \int_I f(x) \, dx \right| < 2 \epsilon
\end{equation}

and

\begin{equation}
\left| \frac{1}{|I''|} \int_{I''} f(x) \, dx - \frac{1}{|I'|} \int_{I'} f(x) \, dx \right| < 2 \epsilon
\end{equation}
as in the preceding paragraph. Hence

\[(5.25) \quad \left| \frac{1}{|I|} \int_I f(x) \, dx - \frac{1}{|I'|} \int_{I'} f(x) \, dx \right| < 4 \epsilon \]

under these conditions.

In particular, (5.25) holds when \(I, I'\) are closed intervals in \(\mathbb{R}\) whose lengths are positive rational numbers greater than or equal to \(L(\epsilon)\). It follows that

\[(5.26) \quad \left| \frac{1}{|I|} \int_I f(x) \, dx - \frac{1}{|I'|} \int_{I'} f(x) \, dx \right| \leq 4 \epsilon \]

when \(I, I'\) are any two closed intervals in \(\mathbb{R}\) with \(|I|, |I'| \geq L(\epsilon)\), since we can approximate \(I\) and \(I'\) by intervals with rational lengths.

Let \(I_1, I_2, I_3, \ldots\) be a sequence of closed intervals in \(\mathbb{R}\), where \(|I_j| > 0\) for each \(j\), and \(|I_j| \to \infty\) as \(j \to \infty\). Using (5.26), it is easy to see that the corresponding sequence of averages

\[(5.27) \quad \frac{1}{|I_j|} \int_{I_j} f(x) \, dx \]

is a Cauchy sequence in \(C\), which therefore converges in \(C\). Put

\[(5.28) \quad \Lambda(f) = \lim_{j \to \infty} \frac{1}{|I_j|} \int_{I_j} f(x) \, dx, \]

and observe that this does not depend on the particular choice of sequence of \(I_j\)'s, because of (5.26) again. If \(I\) is any closed interval in \(\mathbb{R}\) with \(|I| \geq L(\epsilon)\), then it is easy to see that

\[(5.29) \quad \left| \Lambda(f) - \frac{1}{|I|} \int_I f(x) \, dx \right| \leq 4 \epsilon, \]

by (5.26).

Of course, \(\Lambda(f)\) is a linear functional on \(\mathcal{AP}(\mathbb{R})\), and

\[(5.30) \quad |\Lambda(f)| \leq \|f\|_{sup} \]

for every \(f \in \mathcal{AP}(\mathbb{R})\), because of the analogous properties of the averages (5.27).

Similarly, \(\Lambda(f) \in \mathbb{R}\) when \(f \in \mathcal{AP}(\mathbb{R})\) is real-valued on \(\mathbb{R}\), and

\[(5.31) \quad \Lambda(|f|) = |\Lambda(f)| \]

for every \(f \in \mathcal{AP}(\mathbb{R})\). If \(f \in \mathcal{AP}(\mathbb{R})\) is real-valued and nonnegative on \(\mathbb{R}\), then \(\Lambda(f) \geq 0\). In this case, we shall show in a moment that \(\Lambda(f) > 0\) when \(f(x) > 0\) for some \(x \in \mathbb{R}\). If \(f\) is any element of \(\mathcal{AP}(\mathbb{R})\), then we have seen that \(|f| \in \mathcal{AP}(\mathbb{R})\), and it is easy to check that

\[(5.32) \quad |\Lambda(f)| \leq \Lambda(|f|). \]
5.4. AVERAGES ON \( \mathbb{R} \)

We have also seen that \( f_{b}(x) = f(x - b) \in \mathcal{AP}(\mathbb{R}) \) for every \( b \in \mathbb{R} \) when \( f \in \mathcal{AP}(\mathbb{R}) \), and one can verify that
\[
(5.33) \quad \Lambda(f_{b}) = \Lambda(f),
\]
by construction. Note that
\[
(5.34) \quad \Lambda(1_{\mathbb{R}}) = 1,
\]
where \( 1_{\mathbb{R}}(x) \) is the constant function on \( \mathbb{R} \) equal to 1 for every \( x \in \mathbb{R} \).

Suppose that \( f \in \mathcal{AP}(\mathbb{R}) \) is real-valued and nonnegative on \( \mathbb{R} \), and that \( f(\delta) > 0 \) for some \( \delta \in \mathbb{R} \). Put \( \epsilon = f(\delta)/3 \), and let \( \delta \) be a positive real number such that
\[
(5.35) \quad f(x) \geq 2\epsilon
\]
for every \( x \in \mathbb{R} \) with \( |x - \delta| \leq \delta \), which exists because \( f \) is continuous at \( \delta \). Let \( a_{1}, \ldots, a_{n} \in \mathbb{R} \) be as at the beginning of the section, and observe that
\[
(5.36) \quad f_{a_{j}}(x) = f(x - a_{j}) \geq 2\epsilon
\]
when \( |(x - a_{j}) - \delta| \leq \delta \). If \( a \in \mathbb{R} \) and \( 1 \leq j \leq n \) satisfy (5.15), then it follows that
\[
(5.37) \quad f_{a}(x) = f(x - a) \geq \epsilon
\]
when \( |x - a_{j} - \delta| \leq \delta \).

Let \( I_{0} \) be a closed interval in \( \mathbb{R} \) such that \( a_{j} + \delta \in I_{0} \) for each \( j = 1, \ldots, n \) and \( |I_{0}| \geq \delta \). Put
\[
(5.38) \quad I_{0,j} = I_{0} \cap [a_{j} + \delta, a_{j} + \delta]
\]
for each \( j = 1, \ldots, n \), and observe that \( |I_{0,j}| \geq \delta \) for every \( j \). If \( a \in \mathbb{R} \) and \( 1 \leq j \leq n \) satisfy (5.15), then (5.37) holds for each \( x \in I_{0,j} \), and hence
\[
(5.39) \quad \int_{I_{0}} f_{a}(x) \, dx \geq \int_{I_{0,j}} f_{a}(x) \, dx \geq \epsilon \, \delta.
\]
This implies that
\[
(5.40) \quad \frac{1}{|I_{0}|} \int_{I_{0}} f_{a}(x) \, dx \geq \epsilon \, \delta \, |I_{0}|^{-1}
\]
for every \( a \in \mathbb{R} \), since for each \( a \in \mathbb{R} \) there is a \( j \) such that (5.15) holds. This is the same as saying that
\[
(5.41) \quad \frac{1}{|I|} \int_{I} f(x) \, dx \geq \epsilon \, \delta \, |I_{0}|^{-1}
\]
for every closed interval \( I \subseteq \mathbb{R} \) with \( |I| = |I_{0}| \).

If \( I \) is a closed interval in \( \mathbb{R} \) with \( |I| = l \, |I_{0}| \) for some positive integer \( l \), then the average of \( f \) over \( I \) can be expressed as an average of the averages of \( f \) over \( l \) closed intervals of length \( |I_{0}| \), as before. This implies that (5.41) also holds in this situation. Applying this to a sequence of closed intervals with length \( l \, |I_{0}| \) for each \( l \in \mathbb{Z}^{+} \), we get that
\[
(5.42) \quad \Lambda(f) \geq \epsilon \, \delta \, |I_{0}|^{-1}
\]
under these conditions. This shows that \( \Lambda(f) > 0 \) when \( f \in \mathcal{AP}(\mathbb{R}) \) is real-valued and nonnegative on \( \mathbb{R} \), and \( f(\delta) > 0 \) for some \( \delta \in \mathbb{R} \).
5.5 Some other perspectives

Put

\[ e_t(x) = \exp(it) \]

for each \( x, t \in \mathbb{R} \), so that \( e_t \) is a continuous homomorphism from \( \mathbb{R} \) into \( T \) for each \( t \in \mathbb{R} \), and every continuous homomorphism from \( \mathbb{R} \) into \( T \) is of this form. If \( t \neq 0 \), then

\[ \int_a^b e_t(x) \, dx = -it^{-1} (e_t(b) - e_t(a)) \]

for every \( a, b \in \mathbb{R} \) with \( a \leq b \), which implies that

\[ \frac{1}{b-a} \left| \int_a^b e_t(x) \, dx \right| = |t|^{-1} |e_t(b) - e_t(a)| (b-a)^{-1} \leq 2 |t|^{-1} (b-a)^{-1}. \]

Note that \( e_t \in \mathcal{A}(\mathbb{R}) \) for every \( t \in \mathbb{R} \), so that \( \Lambda(e_t) \) is defined as in the previous section for every \( t \in \mathbb{R} \). It is easy to see that

\[ \Lambda(e_t) = 0 \]

when \( t \neq 0 \), because (5.45) tends to 0 as \( (b-a) \to \infty \). Of course, \( e_0(x) = 1 \) for every \( x \in \mathbb{R} \), so that \( \Lambda(e_0) = 1 \), as in (5.34).

Let \( I_1, I_2, I_3, \ldots \) be a sequence of closed intervals in \( \mathbb{R} \) such that \( |I_j| > 0 \) for each \( j \in \mathbb{Z}_+ \) and \( |I_j| \to \infty \) as \( j \to \infty \). Consider the set \( E(\{I_j\}_{j=1}^\infty) \) of \( f \in C_b(\mathbb{R}) \) such that the corresponding sequence of averages

\[ \frac{1}{|I_j|} \int_{I_j} f(x) \, dx \]

converges in \( C \). It is easy to see that \( E(\{I_j\}_{j=1}^\infty) \) is a linear subspace of \( C_b(\mathbb{R}) \) that is invariant under complex conjugation and contains the constant functions. We also have that \( e_t \in E(\{I_j\}_{j=1}^\infty) \) for every \( t \in \mathbb{R} \) with \( t \neq 0 \), since the corresponding sequence of averages of \( e_t \) converges to 0 when \( t \neq 0 \), by (5.45).

Equivalently, \( E(\{I_j\}_{j=1}^\infty) \) consists of the \( f \in C_b(\mathbb{R}) \) such that the sequence of averages (5.47) is a Cauchy sequence in \( C \). Using this, one can check that \( E(\{I_j\}_{j=1}^\infty) \) is a closed set in \( C_b(\mathbb{R}) \) with respect to the supremum metric. This also uses the fact that

\[ \frac{1}{|I_j|} \int_{I_j} f(x) \, dx \leq \frac{1}{|I_j|} \int_{I_j} |f(x)| \, dx \leq \|f\|_{\sup} \]

for every \( f \in C_b(\mathbb{R}) \) and \( j \geq 1 \), where \( \|f\|_{\sup} \) denotes the supremum norm of \( f \). It follows that

\[ \mathcal{A}(\mathbb{R}) \subseteq E(\{I_j\}_{j=1}^\infty), \]

because \( e_t \in E(\{I_j\}_{j=1}^\infty) \) for every \( t \in \mathbb{R} \), as in the previous paragraph, and the linear span of the set of \( e_t \) with \( t \in \mathbb{R} \) is dense in \( \mathcal{A}(\mathbb{R}) \) with respect to the supremum norm.
Similarly, let $I'_1, I'_2, I'_3, \ldots$ be another sequence of closed intervals in $\mathbb{R}$ such that $|I'_j| > 0$ for each $j \in \mathbb{Z}_+$ and $|I'_j| \to 0$ as $j \to \infty$. Consider the set of $f \in C_b(\mathbb{R})$ such that

\begin{equation}
\frac{1}{|I'_j|} \int_{I'_j} f(x) \, dx - \frac{1}{|I_j|} \int_{I_j} f(x) \, dx
\end{equation}

converges to 0 as $j \to \infty$. It is easy to see that this is a linear subspace of $C_b(\mathbb{R})$ that is invariant under complex conjugation and contains $e_t$ for each $t \in \mathbb{R}$. One can also check that this is a closed set in $C_b(\mathbb{R})$ with respect to the supremum norm, using (5.48) and its analogue for $I'_j$. It follows that $AP(\mathbb{R})$ is contained in this subspace of $C_b(\mathbb{R})$, for the same reasons as before.

This gives another way to show that the limit in (5.28) exists for every $f \in AP(\mathbb{R})$, and that the limit does not depend on the choice of sequence of $I'_j$'s when $f \in AP(\mathbb{R})$. Once one has this, one can derive properties of $\Lambda(f)$ for $f \in AP(\mathbb{R})$ like those in the previous section.

Let $B$ be a compact commutative topological group, and suppose that $h$ is a continuous homomorphism from $\mathbb{R}$ into $B$. Suppose also that $h(\mathbb{R})$ is dense in $B$, which implies that

\begin{equation}
\sup_{x \in \mathbb{R}} |f(h(x))| = \sup_{y \in B} |f(y)|
\end{equation}

for every $f \in C(B)$. If $f \in C(B)$, then $f \circ h \in AP(\mathbb{R})$, as in Section 5.1. Put

\begin{equation}
\Lambda_h(f) = \Lambda(f \circ h)
\end{equation}

for each $f \in C(B)$, which defines a nonnegative linear functional on $C(B)$, because $\Lambda$ is a nonnegative linear functional on $AP(\mathbb{R})$. Similarly,

\begin{equation}
|\Lambda_h(f)| = |\Lambda(f \circ h)| \leq \sup_{x \in \mathbb{R}} |f(h(x))| = \sup_{y \in B} |f(y)|
\end{equation}

for every $f \in C(B)$, by (5.30). If $1_B$ is the constant function on $B$ equal to 1 at every point, then

\begin{equation}
\Lambda_h(1_B) = \Lambda(1_R) = 1,
\end{equation}

by (5.34). If $f$ is real-valued and nonnegative on $B$, and if $f > 0$ somewhere on $B$, then $f \circ h > 0$ somewhere on $\mathbb{R}$, because $h(\mathbb{R})$ is dense in $B$. Under these conditions,

\begin{equation}
\Lambda_h(f) = \Lambda(f \circ h) > 0,
\end{equation}

because of the analogous property of $\Lambda$, discussed in the previous section.

It is easy to see that (5.52) is invariant under translations of $f$ by elements of $h(\mathbb{R})$, because $\Lambda$ is invariant under translations on $AP(\mathbb{R})$, as in (5.33). Using this, one can check that (5.52) is invariant under translations of $f$ by arbitrary elements of $B$. More precisely, this also uses the hypothesis that $h(\mathbb{R})$ be dense in $B$, and the fact that $f$ is uniformly continuous on $B$, because $B$ is compact. This permits translations of $f$ by elements of $B$ to be approximated
 CHAPTER 5. ADDITIONAL TOPICS

by translations of \( f \) by elements of \( h(\mathbb{R}) \), uniformly on \( B \). The remaining point is that \( \Lambda_h \) is bounded with respect to the supremum norm on \( C(B) \), as in (5.53), which implies that \( \Lambda_h \) is continuous with respect to the supremum metric on \( C(B) \).

Thus \( \Lambda_h \) satisfies the requirements of a Haar integral on \( C(B) \), normalized as in (5.54). This implies that

\[
\Lambda_h(f) = \int_B f(y) \, dH_B(y)
\]

for every \( f \in C(B) \), where \( H_B \) is Haar measure measure on \( B \), normalized so that \( H_B(B) = 1 \). Alternatively, if \( \phi \) is a continuous homomorphism from \( B \) into \( T \), then \( \phi \circ h \) is a continuous homomorphism from \( \mathbb{R} \) into \( T \), and hence

\[
\phi \circ h = e^t
\]

for some \( t \in \mathbb{R} \). If \( \phi = 1_B \), then \( t = 0 \), and conversely \( \phi = 1_B \) when \( t = 0 \), because \( h(\mathbb{R}) \) is supposed to be dense in \( B \). If \( \phi \neq 1_B \), then

\[
\int_B \phi(y) \, dH_B(y) = 0,
\]

as in (3.71) in Section 3.8, and

\[
\Lambda_h(\phi) = \Lambda(e^t) = 0,
\]

as in (5.46), because \( t \neq 0 \). Of course, (5.56) follows from the normalization \( H_B(B) = 1 \) when \( f \) is a constant function. This gives another proof of (5.56) when \( f \) is a continuous homomorphism from \( B \) into \( T \). It follows that (5.56) also holds when \( f \) is in the linear span \( \mathcal{E}(B) \) of \( \hat{B} \) in \( C(B) \), since both sides of (5.56) are linear in \( f \). More precisely, both sides of (5.56) determine bounded linear functionals on \( C(B) \) with respect to the supremum norm, so that (5.56) holds when \( f \) is in the closure of \( \mathcal{E}(B) \) in \( C(B) \) too. This shows that (5.56) holds for every \( f \in C(B) \), because \( \mathcal{E}(B) \) is dense in \( C(B) \) with respect to the supremum norm.

5.6 The associated inner product

If \( f, g \in \mathcal{AP}(\mathbb{R}) \), then we have seen that \( fg \in \mathcal{AP}(\mathbb{R}) \) too, and so we can put

\[
\langle f, g \rangle = \langle f, g \rangle_{\mathcal{AP}(\mathbb{R})} = \Lambda(fg),
\]

where \( \Lambda \) is defined on \( \mathcal{AP}(\mathbb{R}) \) as in Section 5.4. By construction, (5.60) is linear in \( f \), conjugate-linear in \( g \), and satisfies

\[
\langle g, f \rangle = \overline{\langle f, g \rangle}
\]

for every \( f, g \in \mathcal{AP}(\mathbb{R}) \). We also have that

\[
\langle f, f \rangle = \Lambda(|f|^2) \geq 0
\]
for every \( f \in \mathcal{AP}(\mathbb{R}) \), and that this is equal to 0 if and only if \( f = 0 \) on \( \mathbb{R} \), as in Section 5.4. This shows that (5.60) defines an inner product on \( \mathcal{AP}(\mathbb{R}) \). If \( f, g \in \mathcal{AP}(\mathbb{R}) \) and \( a \in \mathbb{R} \), then their translates \( f_a, g_a \) by \( a \) are in \( \mathcal{AP}(\mathbb{R}) \) too, and
\[
\langle f_a, g_a \rangle = \langle f, g \rangle,
\]
by the analogous property (5.33) of \( \Lambda \).

Let \( e_t(x) \) be as in (5.43), and observe that
\[
e_{r}(x) e_{t}(x) = e_{r}(x) e_{-t}(x) = e_{r-t}(x)
\]
for every \( x, r, t \in \mathbb{R} \). It follows that
\[
\langle e_r, e_t \rangle = \Lambda(e_{r-t}) = 0
\]
when \( r \neq t \), by (5.46), and that
\[
\langle e_t, e_t \rangle = \Lambda(e_0) = 1
\]
for every \( t \in \mathbb{R} \), by (5.34). If \( f \in \mathcal{AP}(\mathbb{R}) \) and \( E \) is a finite subset of \( \mathbb{R} \), then
\[
\sum_{t \in E} |\langle f, e_t \rangle|^2 \leq \langle f, f \rangle,
\]
by standard results about inner product spaces. Taking the supremum over all finite subsets \( E \) of \( \mathbb{R} \), we get that
\[
\sum_{t \in \mathbb{R}} |\langle f, e_t \rangle|^2 \leq \langle f, f \rangle,
\]
and in particular that the sum on the left is finite. Of course, (5.68) implies that \( \langle f, e_t \rangle \neq 0 \) for only finitely or countably many \( t \in \mathbb{R} \).

Note that
\[
\langle f, f \rangle = \Lambda(\|f\|^2) \leq \|f\|^2_{\text{sup}}
\]
for every \( f \in \mathcal{AP}(\mathbb{R}) \), where \( \|f\|_{\text{sup}} \) is the supremum norm of \( f \) on \( \mathbb{R} \), by (5.30). As before, the linear span of the set of \( e_t \) with \( t \in \mathbb{R} \) is dense in \( \mathcal{AP}(\mathbb{R}) \) with respect to the supremum norm. This implies that the linear span of the set of \( e_t \) with \( t \in \mathbb{R} \) is dense in \( \mathcal{AP}(\mathbb{R}) \) with respect to the norm associated to \( \langle \cdot, \cdot \rangle \), by (5.69). It follows that
\[
\sum_{t \in \mathbb{R}} |\langle f, e_t \rangle|^2 = \langle f, f \rangle
\]
for every \( f \in \mathcal{AP}(\mathbb{R}) \).

If \( f, g \in \mathcal{AP}(\mathbb{R}) \), then
\[
\sum_{r \in \mathbb{R}} |\langle f, e_{t-r} \rangle||\langle g, e_r \rangle| \leq \left( \sum_{r \in \mathbb{R}} |\langle f, e_{t-r} \rangle|^2 \right)^{1/2} \left( \sum_{r \in \mathbb{R}} |\langle g, e_r \rangle|^2 \right)^{1/2}
\]
\[
= \left( \sum_{r \in \mathbb{R}} |\langle f, e_r \rangle|^2 \right)^{1/2} \left( \sum_{r \in \mathbb{R}} |\langle g, e_r \rangle|^2 \right)^{1/2}
\]
for every \( t \in \mathbb{R} \), using the Cauchy–Schwarz inequality in the first step, and a change of variables in the second step. Thus

\[
\sum_{r \in \mathbb{R}} |\langle f, e_{t-r} \rangle| |\langle g, e_r \rangle| \leq \langle f, f \rangle^{1/2} \langle g, g \rangle^{1/2} \leq \|f\|_{\text{sup}} \|g\|_{\text{sup}}
\]

for every \( t \in \mathbb{R} \), by (5.68) and (5.69). Remember that \( f g \in \mathcal{AP}(\mathbb{R}) \), so that \( \langle f g, e_t \rangle \) is defined for every \( t \in \mathbb{R} \). Let us check that

\[
\langle f g, e_t \rangle = \sum_{r \in \mathbb{R}} \langle f, e_{t-r} \rangle \langle g, e_r \rangle
\]

(5.73)

for every \( t \in \mathbb{R} \), where the summability of the sum on the right follows from (5.72). If \( g = e_w \) for some \( w \in \mathbb{R} \), then it is easy to see that (5.73) holds, using the definition of the inner product and the orthonormality properties mentioned earlier. This implies that (5.73) also holds when \( g \) is in the linear span of the set of \( e_w \) with \( w \in \mathbb{R} \), by linearity. In order to get the same conclusion for every \( g \in \mathcal{AP}(\mathbb{R}) \), one can approximate \( g \) by elements of the linear span of the set of \( e_w \) with \( w \in \mathbb{R} \), as before.

### 5.7 Some additional properties

Let \( B \) be a compact commutative topological group, and suppose that \( h \) is a continuous homomorphism from \( \mathbb{R} \) into \( B \). As usual, we may as well ask that \( h(\mathbb{R}) \) be dense in \( B \), since otherwise we can replace \( B \) with the closure of \( h(\mathbb{R}) \) in \( B \). Also let \( H_B \) be Haar measure on \( B \), normalized so that \( H_B(B) = 1 \).

If \( f, g \in C(B) \), then we have seen that \( f \circ h, g \circ h \in \mathcal{A}(\mathbb{R}) \), so that the inner product of \( f \circ h \) and \( g \circ h \) can be defined as in the previous section. Under these conditions,

\[
\langle f \circ h, g \circ h \rangle_{\mathcal{AP}(\mathbb{R})} = \int_B f(y) \overline{g(y)} dH_B(y),
\]

(5.74)

by (5.56) and (5.60). Note that

\[
\{ f \circ h : f \in C(B) \}
\]

(5.75)

is a closed linear subspace of \( \mathcal{AP}(\mathbb{R}) \) with respect to the supremum norm. This uses the fact that

\[
f \mapsto f \circ h
\]

(5.76)

is an isometric linear mapping from \( C(B) \) into \( \mathcal{AP}(\mathbb{R}) \) with respect to the corresponding supremum norms, as in (5.51), and the completeness of \( C(B) \).

Let \( \hat{B} \) be the dual group of continuous homomorphisms from \( B \) into \( \mathbb{T} \), which is equipped with the discrete topology, because \( B \) is compact. If \( \phi \in \hat{B} \), then \( \phi \circ h \) is a continuous homomorphism from \( \mathbb{R} \) into \( \mathbb{T} \), and hence there is a unique real number \( t_h(\phi) \) such that

\[
\phi(h(x)) = e_{t_h(\phi)}(x) = \exp(i x t_h(\phi))
\]

(5.77)
5.8. BOUNDED HOLOMORPHIC FUNCTIONS

for every \( x \in \mathbb{R} \). It is easy to see that the mapping from \( \phi \in \hat{B} \) to \( t_h(\phi) \in \mathbb{R} \) is a homomorphism with respect to the usual group structure on \( \hat{B} \), defined by pointwise multiplication of functions from \( B \) into \( T \), and addition on \( \mathbb{R} \). This homomorphism corresponds exactly to the dual homomorphism \( \hat{h}(\phi) = \phi \circ h \) from \( \hat{B} \) into the dual of \( \mathbb{R} \), using the identification between \( \mathbb{R} \) and its dual given by the mapping from \( t \in \mathbb{R} \) to \( e^t \). Note that \( \hat{h} \) is injective, because \( h(\mathbb{R}) \) is dense in \( B \), so that \( t_h \) is injective as a mapping from \( \hat{B} \) into \( \mathbb{R} \).

If \( f \in C(B) \) and \( \phi \in \hat{B} \), then

\[
\hat{f}(\phi) = \int_B f(y) \overline{\phi(y)} dH_B(y) = \langle f \circ h, \phi \circ h \rangle_{\mathcal{AP}(\mathbb{R})}, \tag{5.78}
\]

by the definition of the Fourier transform \( \hat{f}(\phi) \) and (5.74). Equivalently,

\[
\hat{f}(\phi) = \langle f \circ h, e_{t_h(\phi)} \rangle_{\mathcal{AP}(\mathbb{R})} \tag{5.79}
\]

for every \( \phi \in \hat{B} \), by the definition (5.77) of \( t_h(\phi) \). Remember that \( f \) can be approximated by finite linear combinations of elements of \( \hat{B} \), uniformly on \( B \). This implies that \( f \circ h \) can be approximated by finite linear combinations of functions of the form \( e_{t_h(\phi)} \) with \( \phi \in \hat{B} \), uniformly on \( \mathbb{R} \). Using this and (5.65), it follows that

\[
\langle f \circ h, e_t \rangle_{\mathcal{AP}(\mathbb{R})} = 0 \tag{5.80}
\]

for every \( t \in \mathbb{R} \) which is not of the form \( t_h(\phi) \) for some \( \phi \in \hat{B} \).

More precisely, \( f \) can be approximated by finite linear combinations of \( \phi \in \hat{B} \) with \( \hat{f}(\phi) \neq 0 \), uniformly on \( B \), as in Section 4.10. This implies that \( f \circ h \) can be approximated by finite linear combinations of functions of the form \( e_{t_h(\phi)} \), where \( \phi \in \hat{B} \) and (5.79) is not zero, uniformly on \( R \). Because of (5.80), this is the same as saying that \( f \circ h \) can be approximated by linear combinations of functions of the form \( e_t \), where \( t \in \mathbb{R} \) and

\[
\langle f \circ h, e_t \rangle_{\mathcal{AP}(\mathbb{R})} \neq 0, \tag{5.81}
\]

uniformly on \( \mathbb{R} \). If \( F \) is any element of \( \mathcal{AP}(\mathbb{R}) \), then \( F \) can be expressed as \( f \circ h \) for some \( B \) and \( h \) as before. It follows that every \( F \in \mathcal{AP}(\mathbb{R}) \) can be approximated by linear combinations of functions of the form \( e_t \), where \( t \in \mathbb{R} \) and

\[
\langle F, e_t \rangle_{\mathcal{AP}(\mathbb{R})} \neq 0, \tag{5.82}
\]

uniformly on \( \mathbb{R} \).

5.8 Bounded holomorphic functions

Let \( f(z) \) be a continuous complex-valued function on the closed upper half-plane \( U \) that is holomorphic on the open upper half-plane \( U \). Thus

\[
f_t(z) = f(z) \exp(-izt) \tag{5.83}
\]
is a continuous function of $z$ on $\overline{U}$ that is holomorphic on $U$ for every $t \in \mathbb{R}$. Also let $I = [a, b]$ be a closed interval in $\mathbb{R}$, and let $R > 0$ be given. Using Cauchy’s theorem, we get that

$$\int_a^b f_t(x) \, dx + \int_0^R f_t(b + iy) \, idy - \int_a^b f_t(x + iR) \, dx - \int_0^R f_t(a + iy) \, idy = 0$$

(5.84) for every $t \in \mathbb{R}$. It follows that

$$\left| \int_a^b f_t(x) \, dx \right| \leq \int_0^R |f_t(b + iy)| \, dy + \int_a^b |f_t(x + iR)| \, dx$$

(5.85) for every $t \in \mathbb{R}$.

Suppose now that $f(z)$ is also bounded on $\overline{U}$, and let $\|f\|_{sup}$ be the supremum norm of $f$ on $\overline{U}$, which is the same as the supremum norm of $f$ on $\mathbb{R}$, as in Section 2.7. This implies that

$$\int_a^b |f_t(x + iR)| \, dx = \int_a^b |f_t(x + iR)| \exp(tR) \, dx \leq \|f\|_{sup} \exp(tR)(b - a)$$

(5.86) for every $t \in \mathbb{R}$. If $t < 0$, then $\exp(tR) \to 0$ as $R \to \infty$, and we get that

$$\left| \int_a^b f_t(x) \, dx \right| \leq \int_0^\infty |f_t(b + iy)| \, dy + \int_0^\infty |f_t(a + iy)| \, dy,$$

(5.87) by (5.85). Similarly,

$$\int_0^\infty |f_t(x + iy)| \, dy = \int_0^\infty |f_t(x + iy)| \exp(ty) \, dy \leq \|f\|_{sup} \int_0^\infty \exp(ty) \, dy = |t|^{-1} \|f\|_{sup}$$

(5.88) for every $x \in \mathbb{R}$ when $t < 0$. Combining this with (5.87), we get that

$$\left| \int_a^b f_t(x) \, dx \right| \leq 2 |t|^{-1} \|f\|_{sup}$$

(5.89) when $t < 0$.

Suppose that the restriction of $f$ to $\mathbb{R}$ is almost periodic, in addition to the other conditions on $f$ already mentioned. This implies that

$$f_t(x) = f(x) \exp(-ixt) = f(x) e^{-xt}$$

(5.90)
5.9. SOME OTHER AVERAGES

is almost periodic on \( \mathbb{R} \) for every \( t \in \mathbb{R} \) as well, because \( e_{-t}(x) \) is almost periodic on \( \mathbb{R} \). Thus \( \Lambda(f_t) \) is defined for every \( t \in \mathbb{R} \), and it is easy to see that

\begin{equation}
\Lambda(f_t) = 0
\end{equation}

when \( t < 0 \), using (5.89). Equivalently, this means that

\begin{equation}
\langle f, e_t \rangle_{\text{AP}(\mathbb{R})} = 0
\end{equation}

when \( t < 0 \), because \( e_t(x) = e_{-t}(x) \) for every \( x, t \in \mathbb{R} \). Conversely, if \( f \in \text{AP}(\mathbb{R}) \) satisfies (5.92) for every \( t < 0 \), then we would like to show that \( f \) can be extended to a bounded continuous function on \( \mathbb{U} \) that is holomorphic on \( U \).

If \( f \in \text{AP}(\mathbb{R}) \) satisfies (5.92) for every \( t < 0 \), then \( f(x) \) can be approximated by finite linear combinations of functions of the form \( e_t(x) \) with \( t \geq 0 \), uniformly on \( \mathbb{R} \), as in the previous section. Of course, \( e_t(x) = \exp(itx) \) has an obvious holomorphic extension to \( z \in \mathbb{C} \) for each \( t \in \mathbb{R} \), which is \( \exp(izt) \). If \( t \geq 0 \), then \( \exp(izt) \) is also bounded on the closed upper half-plane \( \mathbb{U} \). Thus our hypothesis on \( f \) implies that there is a sequence \( \{g_j(z)\}_{j=1}^{\infty} \) of bounded continuous functions on \( \mathbb{U} \) that are holomorphic on \( \mathbb{U} \) and whose restrictions to \( \mathbb{R} \) converge uniformly to \( f \). As in Section 2.7,

\begin{equation}
\sup_{z \in \mathbb{U}} |g_j(z) - g_k(z)| = \sup_{x \in \mathbb{R}} |g_j(x) - g_k(x)|
\end{equation}

for every \( j, k \geq 1 \). Because \( \{g_j(x)\}_{j=1}^{\infty} \) converges to \( f(x) \) uniformly on \( \mathbb{R} \), \( \{g_j(x)\}_{j=1}^{\infty} \) is a Cauchy sequence with respect to the supremum norm on \( C_b(\mathbb{R}) \). This implies that \( \{g_j(z)\}_{j=1}^{\infty} \) is a Cauchy sequence with respect to the supremum norm on \( C_b(\mathbb{U}) \), because of (5.93). It is well known that \( \{g_j(z)\}_{j=1}^{\infty} \) converges uniformly to a bounded continuous function on \( \mathbb{U} \) under these conditions, and that the limit is holomorphic on \( \mathbb{U} \). By construction, the limit is equal to \( f \) on \( \mathbb{R} \), which shows that \( f \) can be extended to a bounded continuous function on \( \mathbb{U} \) that is holomorphic on \( \mathbb{U} \).

5.9 Some other averages

Let \( f \) be a bounded measurable complex-valued function on \( \mathbb{R} \), and put

\begin{equation}
A_y(f)(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(x-w) \frac{y}{w^2 + y^2} \, dw
\end{equation}

for every \( x, y \in \mathbb{R} \) with \( y > 0 \). This is equivalent to (2.55) in Section 2.4, and to (2.84) in Section 2.6. Remember that

\begin{equation}
A_y(1_{\mathbb{R}})(x) = 1
\end{equation}

for every \( x \in \mathbb{R} \) and \( y > 0 \), by (2.53) in Section 2.4, where \( 1_{\mathbb{R}} \) is the constant function equal to 1 at every point in \( \mathbb{R} \). Suppose that \( f \in \text{AP}(\mathbb{R}) \), and let \( \Lambda(f) \) be as in Section 5.4. Under these conditions,

\begin{equation}
A_y(f)(x) \to \Lambda(f) \quad \text{as } y \to \infty,
\end{equation}
where the convergence is uniform over \( x \in \mathbb{R} \).

One way to see this is to express \( A_y(f)(x) \) as an average of averages of \( f \) over intervals of the form \([x-r, x+r]\), where \( r > 0 \). This uses the fact that \( y/(w^2 + y^2) \) is monotonically decreasing as a function of \(|w|\), and can be made precise by integration by parts. As \( y \) increases, these averages of averages of \( f \) become concentrated on averages of \( f \) over intervals of the form \([x-r, x+r]\) with \( r \) large. As in Section 5.4, the average of \( f \) over \([x-r, x+r]\) tends to \( \Lambda(f) \) as \( r \to \infty \), uniformly over \( x \in \mathbb{R} \). This permits one to show that (5.96) holds uniformly over \( x \in \mathbb{R} \), as desired.

Alternatively, let \( t \in \mathbb{R} \) be given, and consider \( A_y(e_t) \), where \( e_t \) is as in (5.43) in Section 5.5. Observe that

\[
A_y(e_t)(x) = e_t(x) A_y(e_t)(0) \tag{5.97}
\]

for every \( x \in \mathbb{R} \) and \( y > 0 \), which is basically an instance of (4.42) in Section 4.4, because \( A_y(e_t) \) is the convolution of \( e_t \) with an integrable function on \( \mathbb{R} \). If \( t = 0 \), then (5.97) is equal to 1 for every \( x \in \mathbb{R} \) and \( y > 0 \), as in (5.95). If \( t \neq 0 \), then

\[
\lim_{y \to \infty} A_y(e_t)(0) = 0, \tag{5.98}
\]

and we shall say more about this in a moment. This is consistent with (5.96), because of (5.46) in Section 5.5.

It is not too difficult to check (5.98) directly, by integration by parts, for instance. One can also look at \( A_y(e_t)(0) \) in terms of the Fourier transform of an explicit function on \( \mathbb{R} \). This function was obtained as the Fourier transform of another explicit function in Section 2.3, which can be used to determine \( A_y(e_t)(0) \), and to verify (5.98). As another approach, \( A_y(e_t)(x) \) corresponds to the unique extension of \( e_t \) to a continuous function on the closed upper half-plane \( \overline{U} \) that is harmonic on the open upper half-plane \( U \), as in Sections 2.6 and 2.7. In this case, this extension of \( e_t \) can be given explicitly in terms of the complex exponential function, and satisfies (5.98). More precisely, \( e_t(x) \) extends to a bounded continuous function on \( \overline{U} \) that is holomorphic on \( U \) when \( t > 0 \), and which is conjugate-holomorphic on \( U \) when \( t < 0 \). This permits \( A_y(e_t)(x) \) to be evaluated using Cauchy’s integral formula as well.

At any rate, once we have (5.98) when \( t \neq 0 \), we get that (5.96) holds when \( f \) is in the linear span of the set of functions of the form \( e_t \) with \( t \in \mathbb{R} \). This implies that (5.96) holds for every \( f \in \mathcal{AP}(\mathbb{R}) \), because the linear span of the set of functions of the form \( e_t \) with \( t \in \mathbb{R} \) is dense in \( \mathcal{AP}(\mathbb{R}) \). This also uses (2.61) in Section 2.4 with \( p = \infty \), to handle the relevant approximations.

Let \( g \) be a continuous complex-valued function on the unit circle, let \( t > 0 \) be given, and put

\[
f(x) = g(e_t(x)) = g(\exp(ixt)) \tag{5.99}
\]

for each \( x \in \mathbb{R} \). Thus \( f \) is a continuous function on \( \mathbb{R} \), that is periodic with period \( 2\pi/t \), so that \( f \) is an element of \( \mathcal{AP}(\mathbb{R}) \) in particular. The restriction to \( t > 0 \) is not too serious here, because one can always reduce to that case using the fact that \( \overline{e_t} = e_{-t} \). An advantage of taking \( t > 0 \) is that \( e_t(x) \) extends to
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\[ f(z) = g(\exp(izt)) \]

extends \( f \) to a bounded continuous function on \( \overline{U} \) that is harmonic on \( U \). In this case,

\[ f(x + iy) \to g(0) \text{ as } y \to \infty \]

uniformly over \( x \in \mathbb{R} \), because \( |\exp(i(x + iy)t)| = \exp(-yt) \to 0 \) as \( y \to \infty \) and \( g(w) \) is continuous at 0. We also have that

\[ A_y(f)(x) = f(x + iy) \]

for every \( x \in \mathbb{R} \) and \( y > 0 \) in this situation, because of the uniqueness of bounded harmonic extensions, as in Section 2.7. Of course, \( g(0) \) is equal to the average of \( g \) on \( T \), which is the same as the average of \( f \) on any interval whose length is a positive integer multiple of \( 2\pi/t \), which implies that \( \Lambda(f) = g(0) \).

If \( f_1, f_2 \in \mathcal{AP}(\mathbb{R}) \), then \( f_1 f_2 \in \mathcal{AP}(\mathbb{R}) \) too, so that \( \Lambda(f_1 f_2) \) is defined. If \( f_1, f_2 \) also have bounded continuous extensions to \( \overline{U} \) that are holomorphic on \( U \), then \( f_1 f_2 \) has the same property, and

\[ \Lambda(f_1 f_2) = \Lambda(f_1) \Lambda(f_2) \]

This can be derived from (5.73) in Section 5.6, with \( t = 0 \), using the fact that

\[ \langle f_1, e_r \rangle_{\mathcal{AP}(\mathbb{R})} = \langle f_2, e_r \rangle_{\mathcal{AP}(\mathbb{R})} = 0 \]

for every \( r < 0 \), as in (5.92). More precisely, the bounded continuous extension of \( f_1 f_2 \) to \( \overline{U} \) that is holomorphic on \( U \) is the product of the corresponding extensions of \( f_1 \) and \( f_2 \) to \( \overline{U} \). Of course, holomorphic functions are harmonic, and bounded continuous functions on \( \overline{U} \) that are harmonic on \( U \) are uniquely determined by their boundary values on \( \mathbb{R} \), as in Section 2.7. If \( f \) is any bounded continuous function on \( \mathbb{R} \), then \( A_y(f)(x) \) defines a harmonic function on \( U \) that determines a bounded continuous function on \( \overline{U} \) equal to \( f \) on \( \mathbb{R} \), as in Section 2.6. It follows that

\[ A_y(f_1 f_2)(x) = A_y(f_1) A_y(f_2)(x) \]

for every \( x \in \mathbb{R} \) and \( y > 0 \) under these conditions, so that (5.103) can be derived from (5.96) as well.

5.10 Product spaces

Let \( A, B \) be topological spaces, and consider their Cartesian product \( A \times B \), equipped with the product topology. Also let \( F(a, b) \) be a continuous complex-valued function on \( A \times B \). Thus for each \( a \in A \), \( b \in B \), and \( \epsilon > 0 \) there is an
open set \( W \subseteq A \times B \) such that \((a, b) \in W\) and
\[
(5.106) \quad |F(a, b) - F(u, v)| < \epsilon/2
\]
for every \((u, v) \in W\). Because of the way that the product topology is defined, we may as well take \( W \) to be of the form \( U \times V \), where \( U \subseteq A \) and \( V \subseteq B \) are open sets, \( a \in U \), and \( b \in V \). If \( u \in U \) and \( v \in V \), then \((u, b), (u, v) \in U \times V\), and hence
\[
(5.107) \quad |F(u, b) - F(u, v)| \leq |F(u, b) - F(a, b)| + |F(a, b) - F(u, v)| < \epsilon/2 + \epsilon/2 = \epsilon,
\]
using (5.106) twice in the second step.

Let \( K \) be a nonempty compact subset of \( A \), let \( b \) be an element of \( B \), and let \( \epsilon > 0 \) be given again. If \( a \in K \), then there is an open set \( W(a) \) in \( A \times B \) such that \((a, b) \in W(a)\) and (5.106) holds for every \((u, v) \in W(a)\). As before, we can take \( W(a) \) to be of the form \( U(a) \times V(a) \), where \( U(a) \subseteq A \) and \( V(a) \subseteq B \) are open sets, \( a \in U(a) \), and \( b \in V(a) \). It follows that (5.107) holds for every \( u \in U(a) \) and \( v \in V(a) \). In particular, the open sets \( U(a) \) in \( A \) with \( a \in K \) form an open covering of \( K \), and so there are finitely many elements \( a_1, \ldots, a_n \) of \( K \) such that
\[
(5.108) \quad K \subseteq \bigcup_{j=1}^n U(a_j),
\]
because \( K \) is compact. If we put
\[
(5.109) \quad V = \bigcap_{j=1}^n V(a_j),
\]
then \( V \) is an open set in \( B \) that contains \( b \), and
\[
(5.110) \quad |F(u, b) - F(u, v)| < \epsilon
\]
for every \( u \in K \) and \( v \in V \). More precisely, if \( u \in K \), then \( u \in U(a_j) \) for some \( j = 1, \ldots, n \), \( v \in V \subseteq V(a_j) \) for the same \( j \), and (5.110) follows from (5.107) in the case where \( U = U(a_j) \) and \( V = V(a_j) \).

Put
\[
(5.111) \quad F_b(a) = F(a, b)
\]
for every \( a \in A \) and \( b \in B \), which is a continuous function of \( a \in A \) for each \( b \in B \). Thus
\[
(5.112) \quad b \mapsto F_b
\]
defines a mapping from \( B \) into the space \( C(A) \) of continuous complex-valued functions on \( A \). The preceding argument shows that this mapping is continuous with respect to the topology on \( C(A) \) corresponding to uniform convergence on compact subsets of \( A \), as in Section 3.6. If \( A \) is compact, then we can apply this argument to \( K = A \), to get that (5.112) is continuous with respect to the topology on \( C(A) \) defined by the supremum norm.
5.10. PRODUCT SPACES

Conversely, if (5.112) is any mapping from $B$ into $C(A)$, then (5.111) is a complex-valued function on $A \times B$ that is continuous as a function of $a \in A$ for each $b \in B$. If (5.112) is a continuous mapping from $B$ into the space $C_b(A)$ of bounded continuous complex-valued functions on $A$, with respect to the topology on $C_b(A)$ determined by the supremum norm, then one can check that (5.111) is a continuous function on $A \times B$. Suppose instead that (5.112) is continuous as a mapping from $B$ into $C(A)$, where $C(A)$ is equipped with the topology defined by the supremum seminorms associated to nonempty compact subsets of $A$, as before. If $A$ is locally compact, then (5.111) is continuous on $A \times B$, as in the previous case.

Let us now take $A$ to be the real line, with the standard topology, and let $F(x,b)$ be a bounded continuous complex-valued function on $\mathbb{R} \times B$. Let $U$ be the open upper half-plane in $\mathbb{C}$, and consider the function $G(z,b)$ defined on $U \times B$ by

\begin{equation}
(5.113) \quad G(z,b) = \frac{1}{\pi} \int_{\mathbb{R}} F(x-w,b) \frac{y}{w^2 + y^2} \, dw
\end{equation}

for every $x, y \in \mathbb{R}$ with $y > 0$ and $b \in B$. This corresponds to (2.55) in Section 2.4, (2.84) in Section 2.6, and (5.94) in Section 5.9, applied to $F(x,b)$ as a function of $x \in \mathbb{R}$ for each $b \in B$. Note that $G(z,b)$ is a harmonic function of $z \in U$ for each $b \in B$, as in Section 2.6.

Put

\begin{equation}
(5.114) \quad G(x,b) = F(x,b)
\end{equation}

for every $x \in \mathbb{R}$ and $b \in B$, so that $G(z,b)$ is defined on $\overline{U} \times B$. Thus $G(z,b)$ is a bounded continuous function of $z \in \overline{U}$ for each $b \in U$, as in Section 2.6 again.

More precisely,

\begin{equation}
(5.115) \quad |G(z,b)| \leq \sup_{w \in \mathbb{R}} |F(w,b)|
\end{equation}

for every $z \in \overline{U}$ and $b \in B$, as in (2.89). This implies that $G(z,b)$ is bounded on $\overline{U} \times B$, because $F(x,b)$ is supposed to be bounded on $\mathbb{R} \times B$. Similarly,

\begin{equation}
(5.116) \quad |G(z,b) - G(z,b')| \leq \sup_{w \in \mathbb{R}} |F(w,b) - F(w,b')|
\end{equation}

for every $z \in \overline{U}$ and $b, b' \in B$, for essentially the same reasons as in (5.115).

Suppose that (5.112) defines a continuous mapping from $B$ into $C_b(\mathbb{R})$, with respect to the supremum norm on $C_b(\mathbb{R})$. Using (5.116), it follows easily that $G(z,b)$ corresponds in the same way to a continuous mapping from $B$ into $C_b(\overline{U})$, with respect to the supremum norm on $C_b(\overline{U})$. In particular, this implies that $G(z,b)$ is a continuous function on $\overline{U} \times B$ with respect to the product topology, since we already know that $G(z,b)$ is continuous as a function of $z \in \overline{U}$ for each $b \in B$, as mentioned earlier. One can also check that $G(z,b)$ is continuous on $\overline{U} \times B$ when $F(x,b)$ is bounded and continuous on $\mathbb{R} \times B$. 
Suppose now that
\[ F_b(x) = F(x, b) \in \mathcal{AP}(\mathbb{R}) \]
as a function of \( x \in \mathbb{R} \) for every \( b \in B \), so that
\[ \lambda(b) = \Lambda(F_b) \]
is defined for every \( b \in B \), as in Section 5.4. Observe that
\[ \lim_{y \to \infty} G(x + iy, b) = \lambda(b) \]
for every \( x \in \mathbb{R} \) and \( b \in B \), by (5.96) in Section 5.9, and where the convergence is uniform over \( x \in \mathbb{R} \) for each \( b \in B \). If (5.112) is a continuous mapping from \( B \) into \( \mathcal{AP}(\mathbb{R}) \) with respect to the supremum norm on \( \mathcal{AP}(\mathbb{R}) \), then \( \lambda(b) \) is a continuous function on \( B \), because \( \Lambda \) is a bounded linear functional on \( \mathcal{AP}(\mathbb{R}) \) with respect to the supremum norm. In this case, \( G(z, b) \) corresponds in the same way to a continuous mapping from \( B \) into \( C_b(\mathcal{U}) \), with respect to the supremum norm on \( C_b(\mathcal{U}) \), as in the preceding paragraph. Using this, one can check that the convergence in (5.119) is uniform over \( x \in \mathbb{R} \) and \( b \) in any compact subset of \( B \) under these conditions.

### 5.11 Upper half-spaces

Let \( B \) be a compact commutative topological group, and let \( h \) be a continuous homomorphism from \( \mathbb{R} \) into \( B \), where \( \mathbb{R} \) is considered as a topological group with respect to addition and the standard topology. Also let \( f \) be a continuous complex-valued function on \( B \), and put
\[ F(x, b) = f(h(x) + b) \]
for every \( x \in \mathbb{R} \) and \( b \in B \). Note that \( f \) is bounded and uniformly continuous on \( B \), because \( B \) is compact. This implies that \( F(x, b) \) is bounded and uniformly continuous on \( \mathbb{R} \times B \), where \( \mathbb{R} \times B \) is considered as a topological group with respect to coordinatewise addition and the product topology. Of course,
\[ F(x + a, b) = F(x, h(a) + b) \]
for every \( a, x \in \mathbb{R} \) and \( b \in B \), because \( h : \mathbb{R} \to B \) is a homomorphism.

Let \( G(z, b) \) be defined on \( \mathcal{U} \times B \) as in the previous section, so that \( G(z, b) \) is given by (5.113) when \( z \in U \), and (5.114) holds for every \( x \in \mathbb{R} \) and \( b \in B \). It is easy to see that (5.112) defines a continuous mapping from \( B \) into \( C_b(\mathcal{R}) \), with respect to the supremum norm on \( C_b(\mathbb{R}) \), because \( F(x, b) \) is uniformly continuous on \( \mathbb{R} \times B \). This implies that \( G(z, b) \) determines a continuous mapping from \( B \) into \( C_b(\mathcal{U}) \), with respect to the supremum norm on \( C_b(\mathcal{U}) \), as in the preceding section. In particular, it follows that \( G(z, b) \) is a continuous function on \( \mathcal{U} \times B \), as before. We also have that \( G(z, b) \) is bounded on \( \mathcal{U} \times B \), because \( F(x, b) \) is bounded on \( \mathbb{R} \times B \), and that
\[ G(z + a, b) = G(z, h(a) + b) \]
for every \( a \in \mathbb{R}, \ z \in \overline{U}, \) and \( b \in B, \) by (5.121) and the definition of \( G(z,b). \)

In this situation, (5.117) holds for every \( b \in B, \) as in Section 5.1. Thus \( \lambda(b) \) can be defined as in (5.118) for each \( b \in B, \) and is a continuous function of \( b, \) because (5.112) is a continuous mapping from \( B \) into \( \mathcal{AP}(\mathbb{R}) \) with respect to the supremum norm. Note that the convergence in (5.119) is uniform over \( x \in \mathbb{R} \) and \( b \in B \) under these conditions, because \( B \) is compact. It is easy to see that
\[
\lambda(h(a) + b) = \lambda(b)
\]
for every \( a \in \mathbb{R} \) and \( b \in B, \) because \( \Lambda \) is invariant under translations, as in (5.33) in Section 5.4, and using (5.121). This implies that \( \lambda \) is constant on the translates of the closure of \( h(\mathbb{R}) \) in \( B, \) since \( \lambda \) is continuous on \( B. \) In particular, \( \lambda(b) \) is constant on \( B \) when \( h(\mathbb{R}) \) is dense in \( B. \) Let \( H_B \) be Haar measure on \( B, \) normalized so that \( H_B(B) = 1. \) If \( h(\mathbb{R}) \) is dense in \( B, \) then
\[
\lambda(b) = \Lambda(F_b) = \int_B f(u + b) \, dH_B(u) = \int_B f(u) \, dH_B(u)
\]
for every \( b \in B, \) using (5.56) in Section 5.5 in the second step.

Put
\[
g(b, y) = G(iy, b)
\]
for every \( b \in B \) and \( y \in \mathbb{R} \) with \( y \geq 0, \) which defines a bounded continuous function on \( B \times [0, \infty). \) Thus
\[
g(b, 0) = G(0, b) = F(0, b) = f(h(0) + b) = f(b)
\]
for every \( b \in B, \) and
\[
G(x + iy, b) = g(h(x) + b, iy)
\]
for every \( x \in \mathbb{R}, \ y \geq 0, \) and \( b \in B, \) by (5.122). We also have that
\[
\lim_{y \to \infty} g(b, y) = \lambda(b)
\]
for every \( b \in B, \) as in (5.119). The convergence in (5.128) is uniform over \( b \in B, \) because \( B \) is compact, as in the previous paragraph.

### 5.12 Pushing measures forward

Let \((X, \mathcal{A})\) and \((Y, \mathcal{B})\) be measurable spaces, so that \(X, Y\) are sets, and \(\mathcal{A}, \mathcal{B}\) are \(\sigma\)-algebras of subsets of \(X, Y,\) respectively. We may also refer to elements of \(\mathcal{A}, \mathcal{B}\) as measurable subsets of \(X, Y,\) respectively. Suppose that \(h : X \to Y\) is measurable, in the sense that \(h^{-1}(E) \in \mathcal{A}\) for every \(E \in \mathcal{B}. \) If \(\mu\) is a nonnegative countably-additive measure on \((X, \mathcal{A}),\) then
\[
\nu(E) = \mu(h^{-1}(E))
\]
defines a nonnegative countably-additive measure on \((Y, \mathcal{B})\). If \(f\) is a measurable function on \(Y\), then \(f \circ h\) is a nonnegative measurable function on \(X\), because \(h\) is measurable. If \(f\) is nonnegative, then it is easy to see that
\[
\int_X f(h(x)) \, d\mu(x) = \int_Y f(y) \, d\nu(y),
\]
by approximating \(f\) by nonnegative measurable simple functions on \(Y\). If \(f\) is a measurable real or complex-valued function on \(Y\), then it follows that \(f\) is integrable on \(Y\) with respect to \(\nu\) if and only if \(f \circ h\) is integrable on \(X\) with respect to \(\mu\), in which case (5.130) still holds.

Similarly, if \(\mu\) is a complex-valued countably-additive measure on \((X, \mathcal{A})\), then (5.129) defines a complex-valued countably-additive measure on \((Y, \mathcal{B})\). Let \(|\mu|, |\nu|\) be the total variation measures on \(X\) and \(Y\) associated to \(\mu, \nu\), respectively, which are finite nonnegative measures on \(X\) and \(Y\). Thus
\[
|\nu(E)| = |\mu(h^{-1}(E))| \leq |\mu|(h^{-1}(E))
\]
for every \(E \in \mathcal{B}\), which implies that
\[
|\nu|(E) \leq |\mu|(h^{-1}(E))
\]
for every \(E \in \mathcal{B}\). If \(f\) is a bounded complex-valued measurable function on \(Y\), then \(f \circ h\) is a bounded measurable function on \(X\), and the integrals on both sides of (5.130) are defined. It is easy to see that (5.130) holds under these conditions as well.

Suppose now that \(X, Y\) are topological spaces, and that \(h : X \to Y\) is continuous. Thus we can take \(\mathcal{A}, \mathcal{B}\) to be the corresponding collections of Borel sets, for which \(h\) is measurable. Let us also ask that \(X\) and \(Y\) be Hausdorff, so that compact subsets of \(X\) and \(Y\) are closed sets, and hence Borel sets. Let \(\mu\) be a nonnegative Borel measure on \(X\), and let \(\nu\) be defined on \(Y\) as in (5.129). If \(K \subseteq X\) is compact, then \(h(K)\) is a compact subset of \(Y\), \(K \subseteq h^{-1}(h(K))\), and so
\[
\mu(K) \leq \mu(h^{-1}(h(K))) = \nu(h(K)).
\]
Using this, one can check that \(\nu\) is inner regular on \(Y\) when \(\mu\) is inner regular on \(X\). This implies that \(\nu\) is outer regular on \(Y\) when \(\mu(X) = \nu(Y) < \infty\). The regularity properties of a complex Borel measure \(\mu\) on \(X\) are defined in terms of the corresponding regularity properties of \(|\mu|\). If \(|\mu|\) is inner regular on \(X\), then
\[
|\mu|(h^{-1}(E))
\]
defines an inner regular Borel measure on \(Y\), as before. If \(\nu\) is defined on \(Y\) as in (5.129), then it follows that \(|\nu|\) is inner regular on \(Y\) as well, by (5.132). This implies that \(|\nu|\) is outer regular on \(Y\) too, since \(|\nu|(Y) < \infty\) holds automatically for a complex measure \(\nu\) on \(Y\).

Let \(A\) and \(B\) be commutative topological groups, and let \(h\) be a continuous homomorphism from \(A\) into \(B\). Also let \(\mu\) be a complex Borel on \(A\), and let \(\nu\)
be the complex Borel measure defined on $B$ as in (5.129). If $\phi \in \hat{B}$, then

$$\tag{5.135} \hat{\nu}(\phi) = \int_B \overline{\phi(y)} \, d\nu(y) = \int_A \overline{\phi(h(x))} \, d\mu(x) = \hat{\mu}(\phi \circ h),$$

as in (5.130).

5.13 Compact subgroups

Let $B$ be a commutative topological group, and let $A$ be a compact subgroup of $B$. Also let $f$ be a continuous complex-valued function on $B$, and put

$$F(a, b) = f(a + b)$$

for each $a \in A$ and $b \in B$, which is a continuous function on $A \times B$. As in Section 5.10, $F_b(a) = F(a, b)$ determines a continuous mapping from $B$ into $C(A)$, with respect to the supremum norm on $C(A)$. This could also be derived from the fact that $f$ is uniformly continuous along the translates of $A$ in $B$, because $A$ and hence its translates are compact subsets of $B$, as in Section 3.11.

Let $H_A$ be Haar measure on $A$, normalized so that $H_A(A) = 1$, and put

$$f_A(b) = \int_A f(a + b) \, dH_A(a) = \int_A F_b(a) \, dH_A(a)$$

for every $b \in B$. This defines a continuous function on $B$, because $b \mapsto F_b$ is a continuous mapping from $B$ into $C(A)$, as in the preceding paragraph. Of course,

$$f_A(a' + b) = f_A(b)$$

for every $a' \in A$, because $H_A$ is invariant under translations on $A$.

Let $B/A$ be the quotient of $B$ by $A$, as a commutative topological group with respect to the quotient topology, and let $q$ be the usual quotient mapping from $B$ onto $B/A$. Because $f_A$ is constant on the translates of $A$ in $B$, by (5.138), there is a function $\bar{f}_A$ on $B/A$ such that

$$\bar{f}_A(q(b)) = f_A(b)$$

for every $b \in B$. More precisely, $\bar{f}_A$ is a continuous function on $B/A$, because $f_A$ is continuous on $B$, and by the way that the quotient topology on $B/A$ is defined.

Suppose now that $B$ is locally compact, which implies that $B/A$ is locally compact. Let $H_{B/A}$ be a choice of Haar measure on $B/A$. If $f$ is a continuous function on $B$ with support contained in a compact set $K$, then the support of $\bar{f}_A$ is contained in $q(K)$, which is a compact subset of $B/A$. This implies that

$$\int_{B/A} \bar{f}_A \, dH_{B/A}$$

be the complex Borel measure defined on $B$ as in (5.129). If $\phi \in \hat{B}$, then

$$\tag{5.135} \hat{\nu}(\phi) = \int_B \overline{\phi(y)} \, d\nu(y) = \int_A \overline{\phi(h(x))} \, d\mu(x) = \hat{\mu}(\phi \circ h),$$

as in (5.130).

5.13 Compact subgroups

Let $B$ be a commutative topological group, and let $A$ be a compact subgroup of $B$. Also let $f$ be a continuous complex-valued function on $B$, and put

$$F(a, b) = f(a + b)$$

for each $a \in A$ and $b \in B$, which is a continuous function on $A \times B$. As in Section 5.10, $F_b(a) = F(a, b)$ determines a continuous mapping from $B$ into $C(A)$, with respect to the supremum norm on $C(A)$. This could also be derived from the fact that $f$ is uniformly continuous along the translates of $A$ in $B$, because $A$ and hence its translates are compact subsets of $B$, as in Section 3.11.

Let $H_A$ be Haar measure on $A$, normalized so that $H_A(A) = 1$, and put

$$f_A(b) = \int_A f(a + b) \, dH_A(a) = \int_A F_b(a) \, dH_A(a)$$

for every $b \in B$. This defines a continuous function on $B$, because $b \mapsto F_b$ is a continuous mapping from $B$ into $C(A)$, as in the preceding paragraph. Of course,

$$f_A(a' + b) = f_A(b)$$

for every $a' \in A$, because $H_A$ is invariant under translations on $A$.

Let $B/A$ be the quotient of $B$ by $A$, as a commutative topological group with respect to the quotient topology, and let $q$ be the usual quotient mapping from $B$ onto $B/A$. Because $f_A$ is constant on the translates of $A$ in $B$, by (5.138), there is a function $\bar{f}_A$ on $B/A$ such that

$$\bar{f}_A(q(b)) = f_A(b)$$

for every $b \in B$. More precisely, $\bar{f}_A$ is a continuous function on $B/A$, because $f_A$ is continuous on $B$, and by the way that the quotient topology on $B/A$ is defined.

Suppose now that $B$ is locally compact, which implies that $B/A$ is locally compact. Let $H_{B/A}$ be a choice of Haar measure on $B/A$. If $f$ is a continuous function on $B$ with support contained in a compact set $K$, then the support of $\bar{f}_A$ is contained in $q(K)$, which is a compact subset of $B/A$. This implies that

$$\int_{B/A} \bar{f}_A \, dH_{B/A}$$
is defined, which may be considered as a nonnegative linear functional on the space \( C_{\text{com}}(B) \) of continuous complex-valued functions on \( A \). It is easy to see that this linear functional is invariant under translations on \( B \), because \( H_{B/A} \) is invariant under translations on \( B/A \). If \( f \) is a nonnegative real-valued continuous function on \( B \) with compact support such that \( f(b_0) > 0 \) for some \( b_0 \in B \), then \( f_A(b_0) > 0 \), by the positivity property of Haar measure on \( A \). This implies that \( f_A(q(b_0)) > 0 \), and hence that (5.140) is strictly positive as well, by the positivity property of Haar measure on \( B/A \).

Thus (5.140) satisfies the requirements of a Haar integral on \( B \). It follows that there is a choice of Haar measure \( H_B \) on \( B \) such that

\[
\int_B f \, dH_B = \int_{B/A} f_A \, dH_{B/A}
\]

for every \( f \in C_{\text{com}}(B) \). If \( B \) is compact, then \( B/A \) is compact, and one can choose \( H_{B/A} \) so that \( H_{B/A}(B/A) = 1 \), which implies that \( H_B(B) = 1 \).

Let \( U \) be an open set in \( B \) such that \( 0 \in U \) and the closure \( \overline{U} \) of \( U \) in \( B \) is compact. This implies that

\[
A + \overline{U}
\]

is a compact subset of \( B \), because \( A \) is compact by hypothesis, and using the continuity of addition on \( B \). If \( K \) is a compact subset of \( B/A \), then \( K \) is contained in the union of finitely many translates of \( q(U) \), because \( q(U) \) is an open set in \( B/A \). Thus \( q^{-1}(K) \) is contained in the union of finitely many translates of (5.142), and \( q^{-1}(K) \) is also a closed set in \( B \), because \( K \) is a closed set in \( B/A \), since it is compact. It follows that \( q^{-1}(K) \) is a compact subset of \( B \) under these conditions, because the union of finitely many translates of (5.142) is compact.

If \( g \) is a continuous function on \( B/A \) with compact support, then \( g \circ q \) is a continuous function on \( B \) with compact support, by the remarks in the preceding paragraph. Let \( H_B \) be a choice of Haar measure on \( B \), so that

\[
\int_B g \circ q \, dH_B
\]

is defined, and determines a nonnegative linear functional on \( C_{\text{com}}(B/A) \). It is easy to see that this linear functional is invariant under translations and strictly positive when \( g \in C_{\text{com}}(B/A) \) is real-valued, nonnegative, and strictly positive at some point in \( C_{\text{com}}(B/A) \), because of the corresponding properties of \( H_B \). This means that (5.143) satisfies the requirements of a Haar integral on \( C_{\text{com}}(B/A) \), so that there is a choice of Haar measure \( H_{B/A} \) on \( B/A \) such that

\[
\int_{B/A} g \, dH_{B/A} = \int_B g \circ q \, dH_B
\]

for every \( g \in C_{\text{com}}(B/A) \). Of course, this is basically the same as (5.141) when \( f \) is constant on the translates of \( A \) in \( B \).
5.14. COMPACT SUBGROUPS, CONTINUED

Let $H_B$ be a choice of Haar measure on $B$ again, and let $f$ be a continuous function on $B$ with support contained in a compact set $K \subseteq B$. It is easy to see that the support of $f_A$ is contained in $A + K$, which is also a compact subset of $B$, because $A$ is compact. Under these conditions,

$$\int_B f_A(b) \, dH_B = \int_B \int_A f(a + b) \, dH_A(a) \, dH_B(b)$$

$$\quad = \int_A \int_B f(a + b) \, dH_B(b) \, dH_A(a)$$

$$\quad = \int_A \int_B f(b) \, dH_B(b) \, dH_A(a) = \int_B f(b) \, dH_B(b),$$

using Fubini’s theorem in the second step, invariance of $H_B$ under translations in the third step, and the normalization $H_A(A) = 1$ in the last step.

5.14 Compact subgroups, continued

Let $A$ be a compact commutative topological group, and let $H_A$ be Haar measure on $A$, normalized so that $H_A(A) = 1$. Also let $1_A$ be the constant function on $A$ equal to 1 at every point, which is the identity element of $\hat{A}$. If $\phi \in \hat{A}$ and $\phi \neq 1_A$, then

$$\int_A \overline{\phi(a)} \, dH_A(a) = 0,$$

as in (3.71) in Section 3.8. Let $g$ be a complex-valued function on $A$ which is integrable with respect to $H_A$, and let $\hat{g}(\phi)$ denotes the Fourier transform of $g$ evaluated at $\phi \in \hat{A}$, as usual. If $g$ is a constant function on $A$, then

$$\hat{g}(\phi) = 0$$

for every $\phi \in \hat{A}$ such that $\phi \neq 1_A$, by (5.146).

If $g$ is any integrable function on $A$, then put

$$g_1(x) = g(x) - \int_A g(a) \, dH_A(a)$$

for each $x \in A$, so that $g_1$ is also an integrable function on $A$, and

$$\int_A g_1(a) \, dH_A(a) = 0.$$

Thus $\hat{g}_1(1_A) = 0$, and

$$\hat{g}_1(\phi) = \hat{g}(\phi)$$

for every $\phi \in \hat{A}$ such that $\phi \neq 1_A$, by (5.146). If (5.147) holds for every $\phi \in \hat{A}$ such that $\phi \neq 1_A$, then it follows that $\hat{g}_1(\phi) = 0$ for every $\phi \in \hat{A}$. This implies that $g_1 = 0$ almost everywhere on $A$ with respect to $H_A$, as in Section 4.9. Of
course, this means that $g$ is equal to a constant almost everywhere on $A$ with respect to $H_A$, where the constant is the integral of $g$ with respect to $H_A$.

Let $B$ be a commutative topological group, and suppose that $A$ is a compact subgroup of $B$. Also let $f$ be a continuous complex-valued function on $B$, and put

$$F_b(a) = F(a, b) = f(a + b)$$

for every $a \in A$ and $b \in B$, as in the previous section. Suppose that $f$ is constant each translate of $A$ in $B$, so that $F_b(a)$ is a constant function on $A$ for each $b \in B$. Let $\phi \in \hat{B}$ be given, so that the restriction of $\phi$ to $A$ defines an element of $\hat{A}$. If $\phi(a) \neq 1$ for some $a \in A$, then $\phi$ satisfies (5.146), and hence

$$\int_A f(a + b) \overline{\phi(a)} dH_A(a) = \int_A F_b(a) \overline{\phi(a)} dH_A(a) = 0$$

for each $b \in B$, because $F_b(a)$ is constant on $A$, by hypothesis. Thus

$$\int_A f(a + b) \overline{\phi(a + b)} dH_A(a) = \overline{\phi(b)} \int_A f(a + b) \overline{\phi(a)} dH_A(a) = 0$$

for every $b \in B$. If $B$ is locally compact, $H_B$ is a choice of Haar measure on $B$, and $f$ has compact support in $B$, then we can integrate (5.153) over $b \in B$ with respect to $H_B$ as in (5.145), to get that

$$\hat{f}(\phi) = \int_B f(b) \overline{\phi(b)} dH_B(b) = 0.$$ 

If $B$ is compact, then this follows from the discussion in Sections 4.13 and 4.14 as well.

Conversely, suppose that $f$ is a continuous complex-valued function on $B$ that satisfies (5.152) for every $b \in B$ and $\phi \in \hat{B}$ such that $\phi(a) \neq 1$ for some $a \in A$. Let $R$ be the natural restriction mapping from $\hat{B}$ into $\hat{A}$, which sends each element of $\hat{B}$ to its restriction to $A$. Thus our hypothesis on $f$ can be reformulated as saying that

$$\widehat{F_b}(\psi) = 0$$

for every $b \in B$ and $\psi \in R(\hat{B})$ such that $\psi \neq 1_A$, where $\widehat{F_b}$ is the Fourier transform of $F_b$ as a function on $A$. If $R(\hat{B}) = \hat{A}$, then it follows that $F_b$ is a constant function on $A$ for every $b \in B$, so that $f$ is constant on the translates of $A$ in $B$. Note that $R(\hat{B}) = \hat{A}$ when the elements of $\hat{B}$ separate points in $A$, as in Section 3.8. In particular, it is well known that this holds when $B$ is compact, as mentioned previously. In this case, the fact that $f$ is constant on the translates of $A$ in $B$ could also be derived from (5.154) and the discussion in Section 4.13.

Let $B$ be a compact commutative topological group, and let $A$ be a closed subgroup of $B$, so that $A$ is compact as well. Also let $E$ be a nonempty subset of $\hat{B}$, and let $E_E(B)$ be the linear span of $E$ in $C(B)$, as in Section 4.13. If $f \in E_E(B)$ and $F_b(a)$ is as in (5.151), then it is easy to see that

$$F_b \in E_{R(E)}(A)$$
for every \( b \in B \). Here \( R : \hat{B} \to \hat{A} \) is the restriction mapping mentioned in the preceding paragraph, so that \( R(E) \) is the image of \( E \) under \( R \) in \( \hat{A} \), and \( \mathcal{E}_{R(E)}(A) \) is the linear span of \( R(E) \) in \( C(A) \). Let \( C_E(B) \) be the closure of \( \mathcal{E}_E(B) \) in \( C(B) \) with respect to the supremum norm, as in Section 4.13. Similarly, let \( C_{R(E)}(A) \) be the closure of \( \mathcal{E}_{R(E)}(A) \) in \( C(A) \) with respect to the supremum norm. If \( f \in C_E(B) \), then it follows that

\[
(5.157) \quad F_b \in C_{R(E)}(A)
\]

for every \( b \in B \), by approximating \( f \) by elements of \( \mathcal{E}_E(B) \), and using (5.156) for the approximations.

Conversely, suppose that \( f \in C(B) \) satisfies (5.157) for every \( b \in B \). This implies that

\[
(5.158) \quad \hat{F}_b(\psi) = 0
\]

for every \( \psi \in \hat{A} \setminus R_E \), as in (4.161) in Section 4.12, where \( \hat{F}_b \) is the Fourier transform of \( F_b \) as a function on \( A \). In particular, if \( \phi \in \hat{B} \) and \( R(\phi) \notin R(E) \), then we can take \( \psi = R(\phi) \) in (5.158), to get that

\[
(5.159) \quad \int_A f(a + b) \overline{\phi(a)} dH_A(a) = \int_A F_b(a) \overline{\phi(a)} dH_A(a) = 0
\]

for every \( b \in B \). As before, it follows that

\[
(5.160) \quad \int_A f(a + b) \overline{\phi(a + b)} dH_A(a) = \phi(b) \int_A f(a + b) \overline{\phi(a)} dH_A(a) = 0
\]

for every \( b \in B \). Integrating this over \( b \in B \) with respect to Haar measure \( H_B \) on \( B \) as in (5.145), we get that

\[
(5.161) \quad \hat{f}(\phi) = \int_B f(b) \overline{\phi(b)} dH_B(b) = 0,
\]

where \( \hat{f} \) is the Fourier transform of \( f \) as a function on \( B \).

Put

\[
(5.162) \quad E_1 = R^{-1}(R(E)),
\]

so that \( E \subseteq E_1 \subseteq \hat{B} \) and hence \( R(E) \subseteq R(E_1) \subseteq R(E) \), which is to say that \( R(E) = R(E_1) \). If \( f \in C(B) \) satisfies (5.157) for each \( b \in B \), then (5.161) holds for every \( \phi \in \hat{B} \setminus E_1 \), as in the previous paragraph. This implies that

\[
(5.163) \quad f \in C_{E_1}(B),
\]

as in (4.162) in Section 4.13. Of course, (5.157) is the same as

\[
(5.164) \quad F_b \in C_{R(E_1)}(A),
\]

because \( R(E) = R(E_1) \), and every \( f \in C_{E_1}(B) \) satisfies (5.164) for each \( b \in B \), by the same argument as before. Thus we get that \( f \in C(B) \) is an element of \( C_{E_1}(B) \) if and only if (5.164) holds for every \( b \in B \).
5.15 Another class of subalgebras

Let $B$ be a compact commutative topological group, and let $h$ be a continuous homomorphism from $\mathbb{R}$ into $B$, where $\mathbb{R}$ is considered as a topological group with respect to addition and the standard topology. If $\phi \in \hat{B}$, then

$$\hat{h}(\phi) = \phi \circ h$$

(5.165)

is a continuous homomorphism from $\mathbb{R}$ into $T$, and $\hat{h}$ defines a homomorphism from $\hat{B}$ into the dual of $\mathbb{R}$. As before, this corresponds to a homomorphism $t_h$ from $\hat{B}$ into $\mathbb{R}$, where

$$\phi(h(x)) = \exp(ix t_h(\phi))$$

(5.166)

for every $\phi \in \hat{B}$ and $x \in \mathbb{R}$. Under these conditions,

$$E(h) = \{ \phi \in \hat{B} : t_h(\phi) \geq 0 \}$$

(5.167)

is a sub-semigroup of $\hat{B}$ that contains the identity element in $\hat{B}$. This leads to a closed subalgebra $C_{E(h)}(B)$ of $C(B)$ that contains the constant functions and is invariant under translations, as in Section 4.13.

Let $f \in C(B)$ be given, and put

$$F(x, b) = f(h(x) + b)$$

(5.168)

for every $x \in \mathbb{R}$ and $b \in B$, as in Section 5.11. Thus $F(x, b)$ is bounded uniformly continuous on $\mathbb{R} \times B$, and an element of $\mathcal{AP}(\mathbb{R})$ as a function of $x \in \mathbb{R}$ for every $b \in B$, as before. If $f \in C_{E(h)}(B)$, then $f$ can be approximated by linear combinations of elements of $E(h)$ uniformly on $B$. This implies that $F(x, b)$ can be approximated by linear combinations of functions of the form

$$\phi(h(x) + b) = \phi(h(x)) \phi(b) = \exp(ix t_h(x)) \phi(b)$$

with $\phi \in E(h)$, uniformly on $\mathbb{R} \times B$.

In particular, for each $b \in B$, $F_b(x) = F(x, b)$ can be approximated by linear combinations of functions of the form

$$e_t(x) = \exp(ix t)$$

(5.170)

with $t \geq 0$, uniformly on $\mathbb{R}$. This implies that for each $b \in B$, $F_b(x)$ can be extended to a bounded continuous function on the closed upper half-plane $\overline{U}$ that is holomorphic on the open upper half-plane $U$, by the argument given at the end of Section 5.8. Of course, holomorphic functions on $U$ are harmonic, so that this extension also corresponds to the function $G(z, b)$ on $\overline{U} \times B$ discussed in Section 5.10, as in Section 2.7.

Conversely, suppose that $f \in C(B)$, and that $F_b(x) = F(x, b)$ has a bounded continuous extension to $\overline{U}$ that is holomorphic on $U$ for each $b \in B$. Remember that $F_b \in \mathcal{A}(\mathbb{R})$ for every $b \in B$, and that $\langle \cdot, \cdot \rangle_{\mathcal{AP}(\mathbb{R})}$ is the inner product on $\mathcal{AP}(\mathbb{R})$ defined in Section 5.6. Under these conditions, we have that

$$\langle F_b, e_t \rangle_{\mathcal{AP}(\mathbb{R})} = 0$$

(5.171)
for every $b \in B$ and $t < 0$, as in (5.92) in Section 5.8. We would like to use this to show that

$$\hat{f}(\phi) = \int_B f(b) \overline{\phi(b)} \, dH_B(b) = 0$$

when $\phi \in \hat{B}$ satisfies $t_h(\phi) < 0$, which is to say that $\phi \in \hat{B} \setminus E(h)$. This would imply that $f$ is an element of $C_{E(h)}(B)$, by (4.162).

If $h(\mathbb{R})$ is dense in $B$, then

$$\hat{f}(\phi) = \langle f \circ h, e_{t_h(\phi)} \rangle_{AP(\mathbb{R})}$$

for every $\phi \in \hat{B}$, by (5.79) in Section 5.7. By construction, $f \circ h = F_0$, so that (5.172) follows from (5.171) and (5.173) when $t_h(\phi) < 0$, as desired.

Otherwise, let $A$ be the closure of $h(\mathbb{R})$ in $B$, which is a compact subgroup of $B$, because $V$ is compact. Put

$$F'_b(a) = f'(a, b) = f(a + b)$$

for every $a \in A$ and $b \in B$, and let $H_A$ be Haar measure on $A$, normalized so that $H_A(A) = 1$. If $\phi \in \hat{B}$, then

$$\int_A F'_b(a) \overline{\phi(a)} \, dH_A(a) = \int_A F'_b(a) \overline{\phi(a)} \, dH_A(a) = 0$$

for every $b \in B$. This is analogous to (5.173), and can be derived from (5.74) as in (5.78) and (5.79) in Section 5.7. Combining this with (5.171), we get that

$$\int_A f(a + b) \overline{\phi(a + b)} \, dH_A(a) = \int_A F'_b(a) \overline{\phi(a)} \, dH_A(a) = 0$$

for every $b \in B$ and $\phi \in \hat{B}$ with $t_h(\phi) < 0$. It follows that

$$\int_A f(a + b) \overline{\phi(a + b)} \, dH_A(a) = \overline{\phi(b)} \int_A f(a + b) \overline{\phi(a)} \, dH_A(a) = 0$$

for every $b \in B$ and $\phi \in \hat{B}$ with $t_h(\phi) < 0$. This implies that (5.172) holds for every $\phi \in \hat{B}$ such that $t_h(\phi) < 0$, by integrating (5.177) over $b \in B$ with respect to $H_B$, and using (5.145).

Note that

$$E(h)^{-1} = \{ \phi \in \hat{B} : t_h(\phi) \leq 0 \},$$

where $E(h)^{-1}$ is as in (4.164) in Section 4.13, so that

$$E(h) \cap E(h)^{-1} = \{ \phi \in \hat{B} : t_h(\phi) = 0 \}.$$

This is the same as the kernel of $\hat{h}$, which is trivial when $h(\mathbb{R})$ is dense in $B$.

Suppose that $C$ is a commutative group equipped with the discrete topology, and that $k$ is a homomorphism from $C$ into $\mathbb{R}$, as a commutative group with respect to addition. In particular, one can simply take $C$ to be a subgroup of $\mathbb{R}$,
and $k$ to be the natural inclusion mapping. Remember that the dual $\hat{C}$ of $C$ is a compact commutative topological group, with respect to the usual topology. The dual homomorphism $k$ associated to $k$ is a continuous homomorphism from the dual of $\mathbb{R}$ into $\hat{C}$, and the dual of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ as a commutative topological group. This takes us back to the same type of situation as at the beginning of the section, with

\[(5.180) \quad B = \hat{C},\]

and where $h$ corresponds to $\hat{k}$, and $\hat{h}$ can be identified with $k$. 
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