Subcritical excitation of the current-driven Tayler instability by super-rotation

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It is known that in a hydrodynamic Taylor-Couette system uniform rotation or a rotation law with positive shear (‘super-rotation’) are linearly stable. It is also known that a conducting fluid under the presence of a sufficiently strong axial electric-current becomes unstable against nonaxisymmetric disturbances. It is thus suggestive that a cylindrical pinch formed by a homogeneous axial electric-current is stabilized by rotation laws with $d\Omega/dR \geq 0$. However, for magnetic Prandtl number $Pm \neq 1$ and for slow rotation also rigid rotation and super-rotation support the instability by lowering their critical Hartmann numbers. For super-rotation in narrow gaps and for modest rotation rates this double-diffusive instability even exists for toroidal magnetic fields with rather arbitrary radial profiles, the current-free profile $B_\phi \propto 1/R$ included. – For rigid rotation and for super-rotation the sign of the azimuthal drift of the nonaxisymmetric hydromagnetic instability pattern strongly depends on the magnetic Prandtl number. The pattern counterrotates with the flow for $Pm \ll 1$ and it corotates for $Pm \gg 1$ while for rotation laws with negative shear the instability pattern migrates in the direction of the basic rotation for all $Pm$.

An axial electric-current of minimal 3.6 kAmp flowing inside or outside the inner cylinder suffices to realize the double-diffusive instability for super-rotation in experiments using liquid sodium as the conducting fluid between the rotating cylinders. The limit is 11 kAmp if a gallium alloy is used.

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I. INTRODUCTION

A well-known instability of toroidal fields is the magneto hydrodynamical pinch-type current-driven instability which is basically nonaxisymmetric. The toroidal field becomes unstable if a certain magnetic field amplitude is exceeded depending on the radial profile of the field which forms the electric-current pattern. It is also known that for unity magnetic Prandtl number a global rotation of the system increases the critical field amplitude. The latter is strongly reduced, however, if the rotation decreases outwards (i.e. $d\Omega/dR < 0$, ‘sub-rotation’). The formal reason is that sub-rotation becomes (Rayleigh-) unstable even in the hydrodynamic regime if it is steep enough. More important is the existence of a nonaxisymmetric instability for such rotation laws even for current-free toroidal fields ($B_\phi \propto 1/R$) which we have called azimuthal magnetorotational instability (AMRI). It appears for all values of the magnetic Prandtl number for rather low Hartmann numbers but for large magnetic Reynolds numbers of the basic rotation. This phenomenon also explains the general destabilization of toroidal fields by rotation laws with $\Omega$ decreasing outwards.

The question arises about the role of ‘super-rotation’, i.e. rotation laws with $d\Omega/dR > 0$, which are linearly stable in the hydrodynamic regime. The nonlinear behavior is less clear as some Taylor-Couette experiments have shown instability in this regime. Superrotation cannot be destabilized by the standard magnetorotational instability with axial external fields. Inspired by the discovery of the axisymmetric helical MRI a WKB method for inviscid fluids in current-free helical background fields has been applied providing two limits of instability in terms of the shear in the rotation law. An upper threshold suggests a magnetic destabilization of super-rotating flows for very strong positive shear. A similar phenomenon has been reported by Bonanno & Urpin (2008) resulting from a local analysis for a helical field under the influence of super-rotation. Later it has been shown with a dispersion relation for inductionless fluids (see the Appendix) that the stability curve does not cross the line representing the differentially rotating pinch formed by uniform electric-current suggesting instability for both signs of shear.

By means of a corresponding approximation Acheson (1978) showed that for fast-rotation the current-driven instability of toroidal fields may be stabilized by positive shear. If this is true we expect in the solar low latitudes where in the bulk of the convection zone the equatorial $\Omega$ increases outwards that the toroidal field is stabilized and can be amplified to much higher values than it would be true for the opposite rotation law. Contrary to that a rotation law with negative shear – as it exists in higher solar latitudes – strongly destabilizes the fields so that they cannot
reach high amplitudes. It is shown here by use of a simplifying cylinder geometry that indeed for not too small magnetic Prandtl numbers super-rotation stabilizes toroidal magnetic fields while sub-rotation strongly destabilizes toroidal magnetic fields. On the other hand, small magnetic Prandtl number and slow rotation of any rotation law – including rigid rotation – lead to lower critical magnetic field strengths than needed for destabilization at $\Omega = 0$. We shall show in the present paper that for $d\Omega/dR \geq 0$ and for slow rotation the relaxation of the excitation conditions compared with the resting container belongs to the double-diffusive phenomena which disappear if the the molecular viscosity equals the molecular resistivity.

A Taylor-Couette container is considered which confines a toroidal magnetic field with amplitudes fixed at the cylinders which may rotate with different rotation rates. The gap between the cylinders is considered as variable. Normalized with the outer radius $R_{\text{out}}$ the inner radius $R_{\text{in}}$ is $\geq 0.5$. The cylinders are unbounded in axial direction.

The fluid between the cylinders is assumed to be incompressible and dissipative with the kinematic viscosity $\nu$ and the magnetic diffusivity $\eta$. Derived from the conservation of angular momentum the rotation law $\Omega(R)$ in the fluid is

$$\Omega(R) = a + \frac{b}{R^2}$$

with

$$a = \frac{\mu - r_{\text{in}}^2 \Omega_{\text{in}}}{1 - r_{\text{in}}^2}, \quad b = \frac{1 - \mu}{1 - r_{\text{in}}^2} R_{\text{in}}^2 \Omega_{\text{in}},$$

where

$$r_{\text{in}} = \frac{R_{\text{in}}}{R_{\text{out}}}, \quad \mu = \frac{\Omega_{\text{out}}}{\Omega_{\text{in}}}. \quad (3)$$

$\Omega_{\text{in}}$ and $\Omega_{\text{out}}$ are the imposed rotation rates of the inner and outer cylinders. After the Rayleigh stability criterion the flow is hydrodynamically stable for $\mu \geq r_{\text{in}}^2$. We are only interested in hydrodynamically stable regimes so that $\mu > r_{\text{in}}^2$ should always be fulfilled. Rotation laws with $d\Omega/dR > 0$ are described by $\mu > 1$ while rotation laws with $d\Omega/dR < 0$ are described by $\mu < 1$. Rigid rotation means $\mu = 1$. Hydrodynamical flows with rigid rotation or super-rotation are always linearly stable.

Also the possible magnetic profiles are restricted. The solution of the stationary induction equation without flows reads

$$B_\phi = AR \quad (4)$$
(in cylinder geometry, see Roberts 1956, Tayler 1957) where \( A \) corresponds to a uniform axial current everywhere within \( R < R_{\text{out}} \). The quantity \( B_{\text{out}}/B_{\text{in}} \) measures the variation of \( B_{\phi} \) across the gap. For fields after (4) it is simply \( B_{\text{out}} = B_{\text{in}}/r_{\text{in}} \). If the axial electric-current only exists inside the inner cylinder then the solution of the induction equation instead of (4) is

\[
B_{\phi} \propto \frac{1}{R}. 
\]

(5)

In the present paper the stability characteristics of the MHD system are due to the instability of the field (4) under the influence of super-rotation. In order to compare the results for the standard profile (4) with those for the field which is current-free in the fluid, the profile (5) has only been used below for data given in Fig. 4. We know that the nonaxisymmetric Tayler instability (TI) also exists for resting fluids with threshold values which do not depend the magnetic Prandtl number

\[
P_{m} = \frac{\nu}{\eta}, \tag{6}
\]

the value of which, however, has an essential influence on the excitation of the TI under the influence of rotation. For fast rotation, a narrow gap, \( P_{m} \neq 1 \) and strong shear we shall present instability maps which hardly differ for various magnetic profiles, the vacuum fields of AMRI included. For these solutions, therefore, the importance of the electric-current inside the fluid disappears and the instability gets its entire energy from the differential rotation. This conclusion will be supported by the inspection of the associated wave numbers and drift velocities of the various nonaxisymmetric instability patterns.

Both AMRI and (resting) TI have recently been realized in the MHD laboratory using the liquid eutectic alloy GaInSn with \( P_{m} = 1.4 \cdot 10^{-6} \) as the conducting fluid. If the results shall be applied to turbulent media like the stellar convection zones then the magnetic Prandtl number must be replaced by its turbulence-induced values which are much larger. In the upper part of the solar core the molecular value is about \( P_{m} \approx 0.065 \).

II. EQUATIONS

The dimensionless incompressible MHD equations are

\[
\text{Re} \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla P + \Delta \mathbf{u} + \text{Ha}^2 \text{curl} \mathbf{B} \times \mathbf{B},
\]

\[
\text{Rm} \frac{\partial \mathbf{B}}{\partial t} = \text{curl}(\mathbf{u} \times \mathbf{B}) + \Delta \mathbf{B}, \tag{7}
\]
with \( \text{div} \, \mathbf{u} = \text{div} \, \mathbf{B} = 0 \) and with the Hartmann number

\[
Ha = \frac{B_{\text{in}} D}{\sqrt{\mu_0 \rho \eta}}. \tag{8}
\]

\( D = \sqrt{R_{\text{in}} (R_{\text{out}} - R_{\text{in}})} \) is used as the unit of length, \( \Omega_{\text{in}}^{-1} \) as the unit of the time, \( \eta/D \) as the unit of velocity and \( B_{\text{in}} \) as the unit of magnetic fields. In this notation the angular velocity of the global rotation \( \Omega \) at the inner cylinder equals \( \Omega_{\text{in}} \). The Reynolds number \( \text{Re} \) is defined as

\[
\text{Re} = \frac{\Omega_{\text{in}} D^2}{\nu} \tag{9}
\]

and the magnetic Reynolds number as \( \text{Rm} = \text{Pm Re} \). It is also useful to work with the mixed Reynolds number

\[
\overline{\text{Rm}} = \sqrt{\text{Re} \text{Rm}} \tag{10}
\]

which is symmetric in \( \nu \) and \( \eta \) as it is the Hartmann number. Its ratio to (8) is called the magnetic Mach number \( \text{Mm} \) which measures the rotation rate in comparison with the Alfvén frequency \( \Omega_A = B_{\text{in}} / \sqrt{\mu_0 \rho D^2} \),

\[
\text{Mm} = \frac{\Omega_{\text{in}}}{\Omega_A} = \frac{\text{Rm}}{Ha}. \tag{11}
\]

We always use no-slip boundary conditions for the velocity \( u_R = u_\phi = u_z = 0 \). The material of the cylinders is assumed as made from perfect conductors, or in some other cases made from perfect insulators. For the conducting walls the fluctuations \( \mathbf{b} \) have thus to fulfill the conditions \( \partial b_\phi / \partial R + b_\phi / R = b_R = 0 \) at both \( R_{\text{in}} \) and \( R_{\text{out}} \). Mathematical details about the much more complicated vacuum boundary conditions and the used numerical codes can be found in previous publications. The time-tested code for the linearized equations solves the eigenvalue problem for the Fourier modes \( \exp(i(\omega t + k z + m \phi)) \) where \( k \) is the axial wave number and \( m \) the azimuthal mode number.

The nonlinear simulations have been done with our reliable time-stepping code. It works with an expansion of the solution in Fourier modes in the azimuthal direction generating a sample of meridional problems each of which is solved using a Legendre spectral element method.

### III. A DOUBLE-DIFFUSIVE INSTABILITY

We start with solid-body rotation of a container. It is known that this flow with (4) belongs to the class of magnetized flows where the radial profiles of the global velocity and the global magnetic
field are identical. Chandrasekhar (1956) has shown that all flows of this class are stable\textsuperscript{23}. One can also show for the nonideal MHD flows that all lines of marginal instability for very small $Pm$ are identical in the $Ha$-$Re$ plane\textsuperscript{24}. They are given for various magnetic Prandtl numbers in Fig. (left panel). For small $Pm$ the lines for $Pm < 10^{-3}$ cannot be separated optically. It is also demonstrated that the critical Hartmann number $Ha_{Tay}$ for $Re = 0$ does not depend on $Pm$. The curves, however, for $Re > 0$ behave different. For $Pm < 1$ they turn to the left while for $Pm \geq 1$ they are turning to the right. In the former case the instability is supported by the rotation and in the latter case it is (strongly) suppressed. Note that the lowering of the Hartmann number by global rotation only exists for $Pm \neq 1$ and for slow rotation with $Mm \ll 1$. For those parameters “the stabilizing effect of global rotation is greatly reduced”, as it already has been formulated for this kind of double-diffusive problems\textsuperscript{25}. A very similar behavior also appeared for the excitation of axisymmetric modes for helical background fields ($B_z, B_\phi \neq 0$) under the influence of differential rotation which are also stable for $\nu = \eta = 0$ and their critical eigenvalues are lowered for $Pm \neq 1$ compared with those for equal diffusivities\textsuperscript{26}.

For faster rotation all curves are turning to the right so that the statement finally becomes correct that generally the rotation suppresses the Tayler instability.

![Diagram](file.png)

**FIG. 1.** Instability of the $m = \pm 1$ mode for rigid rotation. The numbers represent the magnetic Prandtl number $Pm$. Left panel: The lowering of the critical Hartmann numbers by the global rotation for $Pm \neq 1$ Note the maximal relaxation of the $Ha$ existing for $Pm = O(10^{-2})$ but not for $Pm \to 0$. Right panel: The drift rates $\omega_{dr}/\Omega$ of the modes for marginal instability for small $Pm$ and large $Pm$. The axial wave numbers $k$ for all points at the lines do hardly vary. $r_m = 0.5$, perfect-conducting boundaries.

Interesting is also the behavior of the drift rate $\omega_{dr}$ as the real part of the frequency $\omega$ of the
Fourier mode of the instability pattern, normalized with the rotation rate of the cylinders. It can be positive or negative. Because of the definition

\[ \dot{\phi} = -\frac{\omega_{dr} \Omega_{out}}{m} \]  

the azimuthal migration of the instability pattern has the opposite sign of \( \omega_{dr} \). The solutions for the modes with \( m = 1 \) and \( m = -1 \) have the same eigenvalues \( \text{Re} \) and \( \text{Ha} \) but the opposite signs of \( \omega_{dr} \). After (12) they have thus the same frequency of migration in \( \phi \)-direction. Figure I (right panel) shows that the drift rate hardly depends on the Hartmann number but it is strongly directed by the magnetic Prandtl number \( Pm \). It is positive for small \( Pm \) which with (12) leads to an azimuthal migration of the instability pattern opposite to the cylinder rotation but for \( Pm \geq 1 \) it changes the sign so that the instability pattern rotates in the same direction as the cylinders do. The relaxation of the critical Hartmann number for \( Pm \neq 1 \) is not indicated by the drift rates.

Figure 2 summarizes the influence of nonuniform rotation on the excitation of the TI against nonaxisymmetric modes with \( m = 1 \). It is \( r_{in} = 0.8 \) and the cylinders are made from insulating or perfect-conducting materials. The critical Hartmann number for resting cylinders is \( Ha_{\text{Tay}} = 250 \) for vacuum boundary conditions and \( Ha_{\text{Tay}} = 290 \) for perfect-conductor conditions. Again, these values do not depend on the magnetic Prandtl number as here demonstrated here for the two examples with \( Pm = 1 \) (left panel) and \( Pm = 10^{-5} \) (right panel). One finds that the TI can be excited more easily for vacuum boundary conditions than for perfect-conducting cylinders.

The dotted lines in Fig. 2 are the lines of marginal instability for the sub-rotation law with \( \mu = 0.5 \). In the narrow gap and for fast enough rotation such a rotation law is linearly unstable without magnetic field. For not too small \( Pm \) the magnetic field even destabilizes such a steep sub-rotation law so that for finite Hartmann number the critical Reynolds number is always lower than 160 which is the critical value for \( Ha = 0 \). The influences of the boundary conditions and the magnetic Prandtl number on the stability/instability of sub-rotation laws is only small.

The solid lines are due to rotation laws with positive radial shear. Their behavior strongly depends on the value of the magnetic Prandtl number. For \( Pm = 1 \) (left panel) super-rotation (\( \mu = 4 \)) acts stabilizing \( (d\text{Re}/dHa > 0 \) everywhere) while for small magnetic Prandtl number \( (Pm = 10^{-5}, \text{right panel}) \) it acts destabilizing \( (d\text{Re}/dHa < 0 \) in the lower part of the diagram). Under the presence of rotation with positive shear the electric-current becomes unstable for lower Hartmann numbers than for the resting pinch. This ‘subcritical excitation of the TI for super-rotation is insofar interesting as TC-flows with positive shear are prominent examples of stable
FIG. 2. Instability map for the $m = \pm 1$ mode for $P_m = 1$ (left) and $P_m = 10^{-5}$ (right). There are examples for sub-rotation ($\mu = 0.5$, dotted lines), rigid rotation ($\mu = 1$, dashed lines) and super-rotation ($\mu = 4$, solid lines). The lowering of the critical Hartmann numbers for super-rotation only appears for $P_m \neq 1$. The boundary conditions are those for vacuum or for perfect conductors, $r_m = 0.8$.

hydrodynamic flows. Such a very stable configuration can even be destabilized by a magnetic field which is weaker than the critical field for the TI. This phenomenon only exists for slow rotation since for fast rotation all nonaxisymmetric magnetic instabilities are suppressed by any sort of differential rotation. The magnetic Mach number $M_m$ for the subcritical excitation by super-rotation in the right panel of Fig. 2 is (only) of order $10^{-3}$. Note that a rotation profile with $\mu > 1$ needs much higher Hartmann numbers to be destabilized than a rotation law with $\mu < 1$ and sufficiently high enough Reynolds number. With other words, the toroidal field which is induced by a super-rotation can become much stronger than the toroidal field which is induced by a sub-rotating $\Omega$-profile. This basic finding should have implications for the electrodynamics of rotating stars. With other words, the instability requires rotation laws with negative shear in order to exist for large magnetic Mach numbers.

The dashed lines describe the influence of rigid rotation which for fast rotation always acts stabilizing, i.e. $d\text{Re}/dH_a > 0$. This effect, however, is much weaker for small $P_m$ than for $P_m = 1$. In the latter case the instability is suppressed for all slow rotation rates. The stabilization depends on the value of the magnetic Prandtl number, it is strongest for $P_m = 1$.

In summary, we have found that for $P_m = 1$ rigid rotation and super-rotation suppresses the TI for all $\text{Re}$. Rigid rotation and super-rotation support the TI for $P_m \neq 1$ but only for slow rotation by lowering the Hartmann number, $H_a < H_a^{\text{Tay}}$, below the value which holds for the resting
pinch. The lowering only exists if the two molecular diffusivities have different values – no matter which is larger or smaller – typical for a double diffusive instability.

From Fig. 2 one also finds that the boundary conditions do not play a minor role. For perfect-conducting cylinders the lowering of the Hartmann numbers is much stronger than than for cylinders of insulating material. In order to study the influence of the geometry on the instability we shall consider in more detail the two TC-flows with a rather narrow gap \( r_{\text{in}} = 0.95 \) and with a wide gap \( r_{\text{in}} = 0.5 \).

**IV. NARROW GAP**

For a narrow gap the influence of the magnetic Prandtl number on the Tayler instability in a container with various rotation profiles shall be studied. All the considered rotation laws are hydrodynamically stable. For a gap with \( r_{\text{in}} = 0.95 \) Fig. 5 gives the results for \( P_m = 0.1, P_m = 1 \) and \( P_m = 10 \). The critical Hartmann number for TI without rotation is \( Ha_{\text{Tay}} = 3060 \). For \( P_m = 1 \) rigid-body rotation and super-rotation of any Reynolds number are always stabilizing, i.e. \( Ha > 3060 \). Only sub-rotation leads to \( Ha < 3060 \). The differences for both sub-rotation and for super-rotation here only appear for rather low Reynolds numbers.

It is also worth to mention that the lines of marginal instability for rigid rotation and for super-rotation always lie below the line \( Ha = \overline{Rm} \), i.e. even slow rotation stabilizes the TI for \( P_m = 1 \). For fast rotation the TI only exists under the presence of differential rotation with negative shear.

**FIG. 3.** Stabilization and destabilization by super-rotation in a narrow gap for \( P_m = 10 \) (left), \( P_m = 1 \) (middle) and \( P_m = 0.1 \) (right). The modified Reynolds numbers \( \tilde{Rm} \) are used for an easy definition of the unity magnetic Mach number \( M_m \) (dashed lines). For comparison one example for sub-rotation is given which itself is hydrodynamically stable. The curves are marked with their value of \( \mu \). \( r_{\text{in}} = 0.95 \), perfect-conducting boundaries. \( m = \pm 1 \).
A. Subcritical excitations

To discuss the results for $P_m \neq 1$ it makes sense to use the modified Reynolds number $R_m$ for the characterization of the basic rotation. For both $P_m > 1$ (Fig. 3 left panel) and $P_m < 1$ (Fig. 3 right panel) also the rotation laws with positive shear lead to subcritical excitations if the rotation is slow enough. The magnetic Mach number which measures the rotation of the (inner) cylinder to the Alfvén frequency for the subcritical excitation and for both magnetic Prandtl numbers is $M_m \simeq 0.05$. Again the curves for rigid rotation and for super-rotation are located below the line $M_m = 1$. Again, for $M_m > 1$ the TI needs the action of a sub-rotation law with negative shear.

For sufficiently fast rotation the super-rotation laws are always stabilizing. The super-rotation for small magnetic Prandtl numbers is much more stabilizing than that for high magnetic Prandtl numbers. For $P_m = 10$ the stabilization by super-rotation is even weaker than that of rigid rotation. It is often the rule for magnetic instabilities that large $P_m$ destabilize nonuniform rotation while small $P_m$ stabilize the flows. The formal reason for this phenomenon can be realized in Fig. 2 where the bifurcation curve for super-rotation for small $P_m$ moves to the left of the line for rigid rotation rather than to the right as for $P_m = 1$.

It is known that rotation laws with negative shear (here $\mu = 0.92$) behave strongly destabilizing the flow. The domain of stability in Fig. 3 is again larger for small $P_m$. For sufficiently fast rotation also the lines of marginal instability for sub-rotation turn to the right stabilizing the system as rotation laws of strong shear of both signs do always erode nonaxisymmetric magnetic patterns.

The question arises about the possible existence of a minimum Hartmann number for steeper and steeper super-rotation laws. The existence of such a limit is suggested by the suppression of a nonaxisymmetric magnetic field by differential rotation which should grow with growing values of shear. The line of marginal instability can never touch the vertical axis as without magnetic field super-rotation always behaves stable. Figure 4 shows very close lines for $\mu = 4$, $\mu = 8$ and even $\mu = 128$ so that the minimum Hartmann number $H_m$ can be estimated with strongest super-rotation as smaller by a factor of three compared with $H_m = 3060$. For the very small magnetic Prandtl number used for Fig. 4 (left) the numerical value of $H_m/H_m$ is astonishing small. For large $P_m$ (right panel of Fig. 4 $P_m = 10$) the subcritical excitation also occurs with $H_m/H_m$ even smaller. For larger Reynolds number almost all curves (except the curve for rigid rotation) are identical, they depend on numerical values of shear and current only in a very little manner. Compared with the curves for very small $P_m$, however, the curves have a different
There is another striking feature plotted in Fig. 4. In the domain where the lines are almost vertical the dependence of the critical Reynolds number on the critical Hartmann number is extremely weak. It is shown in this plot that there even the dependence of the curves on the radial profile of $B = B_\phi(R)$ is weak. The dashed lines in Fig. 4 represent the instability for the field (5) which is current-free between the cylinders within the fluid. These curves, therefore, can never cross the horizontal axis where $Re = 0$. For fast rotation, however, they almost coincidence with the solid lines for the marginal instability of the flow with axial current. Surprisingly, for strong shear and fast rotation the presence of the electric-current becomes irrelevant for the occurrence of instability. One can show that all possible radial profiles of $B_\phi$ between (4) and (5) provide more or less the same instability curves in this domain of the bifurcation map revealing that the differential rotation for $Pm \neq 1$ is able to deliver the entire energy for the maintenance of the instability patterns and the magnetic field only acts as a catalyst. This phenomenon is already known from AMRI for sub-rotation but here, for super-rotation, it only works for $\nu \neq \eta$.

The close relatedness of both the instabilities for the lowest Hartmann numbers is obvious. It is not yet clear whether the coincidence of the lines with and without electric-current in the fluid occurs only for the considered model with a very narrow gap and perfectly-conducting cylinders or not. Note that in narrow gaps the radial profiles of the azimuthal fields between the cylinders
both are almost uniform. Indeed, test calculations also provided instability even for fields with uniform $B_\phi$ in the same domain of Reynolds number and Hartmann number. One could believe that for $\text{Pm} \neq 1$ the super-rotation becomes unstable under the mere presence of any toroidal field but for $\text{Pm} = 1$ the dissipation processes prevent the excitation of this rather slow (see below) instability.

B. Pattern migration

In general both modes with $m = \pm 1$ are simultaneously excited. If the pinch rotates rigidly then both modes have exactly the same amplitudes and form a standing wave. The instability pattern looks azimuthally dipolar with the drift direction depending on the magnetic Prandtl number. Note also the nearly circular geometry of the resulting cells in the $R$-$z$-plane (Fig. 5). For a certain $\text{Pm}$ between 0.1 and 1 the azimuthal migration disappears and the entire pattern will rest in the laboratory. For nonuniform rotation laws with finite shear one of the modes $m = 1$ or $m = -1$ is preferred and the instability pattern approaches a spiral.

FIG. 5. The isolines of the $b_R$ in a narrow gap for $\text{Pm} = 1$ (left) and $\text{Pm} = 0.1$ (right) for rigid rotation. Both modes with $m = \pm 1$ are excited with the same amplitude forming a standing wave. The pattern with $\text{Pm} = 1$ migrates in the rotation direction while it migrates opposite for $\text{Pm} = 0.1$. $r_{in} = 0.95$, $\mu = 1$, $Re = 111$, $Ha = 3440$, perfect-conducting boundaries.

Figure 6 shows the behavior of the azimuthal migration of the nonaxisymmetric vortices as more diverse. It is striking that for large $\text{Pm}$ and/or for sub-rotation often $|\omega_{dr}| \simeq \Omega_{out}$. From Fig. 6 we also find that the nonaxisymmetric instability pattern for sub-rotation nearly corotates with the outer cylinder (as it is observed for AMRI) for all magnetic Prandtl numbers. For $\text{Pm} \geq 1$ the drift frequencies for super-rotation are also negative so that their magnetic pattern azimuthally
migrates in the direction of the rotation.

FIG. 6. The same as in Fig. 4 but for the normalized drift frequency $\omega_{dr}/\Omega_{out}$ (see Eq. (12)) in a narrow gap ($r_{in} = 0.95$) for $Pm = 10$ (left), $Pm = 1$ (middle) and $Pm = 0.1$ (right). $r_{in} = 0.95$, perfect-conducting boundaries.

For smaller magnetic Prandtl numbers, however, for rigid rotation and for super-rotation the pattern counterrotates. The azimuthal migration of the linear solutions directly reflects the actual value of the magnetic Prandtl number. This is also true for a rigidly rotating pinch. After the results plotted in Fig. 6 its pattern corotates with the outer cylinder for small $Pm$ and it counterrotates with the outer cylinder for large $Pm$. Hence, numerical simulations with magnetic Prandtl number unity for flows with vanishing or positive shear may easily lead to results which are not representative for the solutions with smaller $Pm$. On the other hand, rotation laws with negative shear do not show that sensitivity to the $Pm$-value (see also Fig. 7, below).

V. WIDE GAP

A. Instability map

For a wide gap with $r_{in} = 0.5$ and also in advance to a possible laboratory experiment Figs. 7 give the eigenvalues for marginal instability, the wave numbers and the drift rates for a fluid with the magnetic Prandtl number of $Pm = 10^{-5}$ (liquid sodium). The characteristic Hartmann number for conducting boundaries and for resting cylinders is $Ha_{Tay} = 35.3$ (see Fig. 1). From now on Reynolds numbers and drift rates are related to the rotation rate of the outer cylinder.

Figure 7 (left panel) shows the curves of marginal instability for many rotational laws with $\mu$ between 1 and 128. The subcritical excitation of the current-driven instability by super-rotation is much stronger than for rigid rotation (dashed curve). If the Reynolds number is formed with the
FIG. 7. Left panel: instability map for super-rotation in a wide gap, the lines are marked with $\mu$. The dashed curve belongs to rigid rotation while for reasons of comparison the dotted curve shows the result for $\mu = 0.25$ (Rayleigh limit). Middle panel: drift rates $\omega_{dr}/\Omega_{out}$. Right panel: the vertical wave numbers are nearly uniform. $r_{in} = 0.5$, $Pm = 10^{-5}$, perfect-conducting cylinders.

outer rotation rate, i.e. $Re_{out} = \mu Re$, then the curves for strong shear are converging. Hence, a minimum Hartmann number of order 30 exists which cannot further be reduced by steeper rotation laws. It is also clear that for $Re_{out} > 200$ the differential rotation starts simply to suppress the nonaxisymmetric instability. The middle panel of the Fig. [7] shows that the azimuthal drift of the instability pattern has the same (positive) sign for rigid rotation and super-rotation and the opposite (negative) sign for sub-rotation. The idea is thus supported that the phenomenon of subcritical excitation, i.e. $dRe/dHa < 0$, for slow rotation with vanishing or positive shear is a common double-diffusive phenomenon which disappears for $Pm = 1$. On the other hand the known lowering of the critical Hartmann number and the negative drift values by sub-rotation exists for all $Pm$ (see Fig. [2]). Note that for $\omega_{dr}/\Omega_{out} = -1$ the magnetic pattern strictly corotates with the outer cylinder. In contrast, the magnetic pattern for all super-rotation laws including uniform rotation migrates opposite to the sense of rotation corresponding to the behavior of the drift rates in the narrow gap. For small $Pm$ by choice of the rotation rate of the outer cylinder one can obtain all sorts of migration of the magnetic pattern between corotation and counterrotation.

The axial wave numbers $k$ are plotted in the right panel of Fig. [7]. They all show similar values. Let $\delta z$ be the characteristic vertical scale of a cell. Then from the definitions follows

$$\frac{\delta z}{D} \simeq \frac{\pi}{k} \sqrt{\frac{R_{in}}{D}} \quad (13)$$

so that $\delta z/D \simeq \pi/k$ forms the relation between the cell size and the wave number for $r_{in} = 0.5$. Hence, the data with $k \simeq \pi$ lead to $\delta z \simeq D$ what means that the Tayler cells approximately form a circle in the $R$-$z$ plane. In this general formulation the result does not depend on the gap width.
B. Growth rates

The subcritical excitation of the TI for rotation laws with positive shear only exists for sufficiently slow rotation ($\text{Mm} \ll 1$) so that the instability only grows very slowly as Fig. 8 demonstrates for super-rotation. The growth rates are normalized with the outer rotation rate. Then the maximum growth rates always occur for the same Reynolds number. For the upper curve of the plot one finds that $\omega_{gr} \simeq 0.03 \Omega_{out}$ for $\text{Re} \simeq 130$ so that the exponential growth time is $\tau_{gr} \simeq 10 R_{out}^2$ in seconds when $R_{out}$ is measured in cm (it is $\nu = 7 \cdot 10^{-3}$ cm$^2$/s for liquid sodium). If the expression

$$\omega_{gr} = \Gamma \frac{B_{out}^2}{\mu_0 \rho \eta}$$

(14)

of the growth rate is adopted (with $\Gamma$ as a dimensionless numerical factor which only depends on the gap width and the magnetic Prandtl number), which has been derived for the nonrotating pinch and which has been experimentally realized$^{15,16}$, then for the strongest super-rotation the Fig. 8 gives $\Gamma \simeq 10^{-3}$ which very well fits the theoretical results for the resting container. This value certainly increases for wider gaps but the exponential growth time of the instability for slow super-rotation will hardly be shorter than that for the resting pinch.

FIG. 8. Growth rates normalized with $\Omega_{out}$ for $H_a = 33$ with $\mu = 4, 8$, and 128. $r_m = 0.5$, $\text{Pm} = 10^{-5}$, perfect-conducting cylinders.
VI. SUMMARY AND OUTLOOK

A rotating pinch with a homogeneous axial electric-current has been considered where the cylindric bounding walls rotate with \( \mu \geq 1 \), i.e. the outer cylinder rotates with the same rotation rate or rotates faster than the inner cylinder. A linear perturbation theory fixes the critical Hartmann numbers for which the system becomes marginally unstable. The surprising result is the occurrence of a double-diffusive instability which only exists for \( \nu \neq \eta \). For slow rotation with \( \Omega_{\text{out}} \ll \Omega_A \) the excitation of the nonaxisymmetric perturbations becomes subcritical, i.e. the critical Hartmann number for rotation is smaller than without rotation. The effect is rather weak for rigid rotation but it is remarkably strong for super-rotation. It is numerically shown that for steeper and steeper rotation laws the series of minimum Hartmann numbers converge to a total minimum \( Ha_{\text{Min}} \) for \( \mu \to \infty \) (approaching resting inner cylinders). The resulting normalized lowering of \( Ha \)

\[
\chi = \frac{Ha_{\text{Tay}} - Ha_{\text{Min}}}{Ha_{\text{Tay}}}
\]  

depends on the magnetic Prandtl number \( Pm \). It vanishes for \( Pm = 1 \) and takes similar values for very large and for very small \( Pm \). Relaxations of the the critical Hartmann number of order 20\% \((r_{\text{in}} = 0.5)\) and 80\% \((r_{\text{in}} = 0.95)\) exist for \( Pm \neq 1 \) (Fig. 9, left panel). Note also that the excess grows for decreasing gap width.

It is also shown for a narrow gap and for small magnetic Prandtl number that in the area of the instability map where the subcritical excitation exists for super-rotation the form of the lines of marginal instability only weakly depends on the radial profile of the azimuthal magnetic field. For finite Reynolds number the lines of marginal instability in Fig. 4 for the two different profiles (4) and (5) are very close together. In this domain of the map the radial distribution of the axial electric-current seems to be unimportant. Obviously, the energy provided by the differential rotation is large enough to maintain the instability while the magnetic field is only needed as a catalyst. This is a numerical finding which implies that among all other radial profiles also the profile with \( B_R \propto 1/R \) (no current within the fluid) leads to a nonaxisymmetric instability for super-rotation. Again, however, this phenomenon disappears for \( Pm = 1 \) revealing its double-diffusive character.

The instability for \( Pm \gg 1 \) and for \( Pm \ll 1 \) differs in another respect. For sub-rotation we always find that the pattern migrates for all \( Pm \) in positive direction of the azimuthal coordinate \( \phi \). For solid-body rotation and for super-rotation there is, however, a strong influence of the magnetic Prandtl number on the azimuthal migration of the perturbation patterns. For \( \mu \geq 1 \) the pattern counterrotates for small \( Pm \) while it corotates for \( Pm > 1 \). The \( Pm \)-dependence of the drift
The maximal reduction of the Hartmann number (left) and the related drift frequencies $\omega_{dr}/\Omega_{out}$ (right) as function of the magnetic Prandtl number. For $Pm \gg 1$ and for $Pm \ll 1$ the subcritical excitation measured by (15) is very similar but the signs of the azimuthal migration of the instability pattern differ. The instability patterns drift in the rotational direction (corotation) or opposite (counterrotation). Note the strong influence of the gap width on the Hartmann number reduction. The curves are marked with $r_{in}$, $\mu = \infty$, perfect-conducting boundaries.

A final question is whether the described phenomena can be realized in the laboratory. The answer is yes and there is an interesting variety of possibilities. The following estimations are derived for liquid sodium with $Pm = 10^{-5}$ as the fluid conductor. In Fig. 7 for a container with $r_{in} = 0.5$ the characteristic Hartmann number for resting containers is 35.3 while the smallest Hartmann number for (strong) super-rotation is 30.4. The necessary current in the gap to produce a certain Hartmann number $Ha$ is

$$I_{\text{fluid}} = 5 \sqrt{\frac{(1 - r_{in})(1 + r_{in})^2}{r_{in}^3}} \sqrt{\mu_0 \rho \nu \eta} Ha$$

(Rüdiger et al. 2013\textsuperscript{20}) which does not depend on the physical size of the container. For liquid sodium it is $\sqrt{\mu_0 \rho \nu \eta} \approx 8.2$ G·cm (for gallium $\approx 26$ G·cm). To reach $Ha \approx 30$ one needs an electric-current of 3.6 kAmp flowing through the sodium (11 kAmp for gallium).

After Fig. 4 experiments are more interesting for narrow gaps with $r_{in} = 0.95$. A characteristic minimum Hartmann number is $O(1000)$ for super-rotation which needs about 20 kAmp for its generation with liquid sodium. The Tayler instability with resting cylinders would, however,
require about 45 kAmp for its realization which seems to be much too high. Only the interplay
with differential rotation allows this instability to observe. A particular challenge for such experi-
ments is that already for a slight increase of this Hartmann number the electric-current through the
sodium can be replaced by an axial current inside the inner cylinder. Both resulting instabilities
are very similar with respect to wave number and azimuthal drift. Such an experiment would eas-
ily demonstrate the surprisingly close relation of AMRI and TI under the presence of differential
rotation.

As the instability phenomena presented in this paper all belong to the class of small magnetic
Mach numbers the technical realization of the Reynolds numbers required by the lines in Figs. 4
and 7 is also no problem. Note, however, that the needed rotation rates of the cylinders do indeed
depend on the physical size of the experiment.

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Appendix A: A local approximation

The described effects have mainly been calculated with boundary conditions for perfect-
conducting cylinders. After Fig. 2 the negative slope $dRe/dHa < 0$ of the lines of marginal
instability for slow rotation also exists for models with vacuum conditions but with reduced effi-
ciency. It is thus worthwhile to discuss in plane geometry (for narrow gaps) the result

$$
Re^* = \frac{1}{4} \left( \frac{(1 + Ha^* m^*)^2 - 4Ha^* m^*)^2}{4 Ha^* m^* - (Ro + 1)((1 + Ha^* m^*)^2 - 4Ha^* m^*)},
$$

of a local approximation for $Pm \to 0$ where $Re^*$, $Ha^*$ and $m^*$ represent the slightly modified
Reynolds number, Hartmann number and azimuthal wave number. The Rossby number $Ro =
(1/2)dlog\Omega/dlogR$ represents the differential rotation, it is positive for super-rotation and negative
for sub-rotation. In contrast to all other quantities the Rossby number enters the Eq. (A1) with odd
and even powers. In (A1) the Rossby number $Ro$ must be independent of the radius $R$ which only
allows to consider Taylor-Couette flows of $0.2 < \mu < 2$ (for $r_m = 0.5$). The latter (super-)rotation
law corresponds to $Ro \simeq 1$ in a very good approximation.
We shall show that all curves with $m^* > 1$ in a $Ha^*-Re^*$ plane for slow rotation show a subcritical behavior compared with the critical eigenvalue $Ha^*_{Tay} = 1/\sqrt{m^*(2-m^*)}$ for $Re^* = 0$, i.e. $Ha^*(Re^*) < Ha^*_{Tay}$. To this end the function $Z = (1 + Ha^*^{2m^*})^2 - 4Ha^*^{4m^*}$ is defined so that (A1) yields

$$Z = \frac{4Re^*^{2m^*}Ha^*^{4Re^*m^*}}{(1 + Ha^*^{2m^*})^2 + 4(Ro + 1)Re^*^2}.$$  \tag{A2}

For rigid rotation ($Ro = 0$) only the solution $Z = 0$ exists which does not reflect the rotational influence as given by Fig. 1. Obviously, the function $Z(Re^*)$ only vanishes for $Re^* = 0$ and it is positive-definite for finite $Re^*$ if – as we shall assume – $Ro > -1$. The above mentioned eigenvalue $Ha^*_{Tay}$ forms the solution of $Z = 0$. It only exists for $m^* < 2$. The solution of $Z = \delta$ with $\delta > 0$ can thus be written as $Ha^{*2} = Ha^{*2}_{Tay} + \epsilon$ with unknown sign of $\epsilon$ which, without loss of generality, can be assumed as small against $Ha^*_{Tay}$. Hence, from the definition of the function $Z$ follows

$$\epsilon = -\frac{\delta}{4m^*},$$  \tag{A3}

so that always $\epsilon < 0$. For $Ro > -1$ it is thus $Ha^*(Re) < Ha^*_{Tay}$ for negative and positive shear, i.e. the excitation of the TI becomes always subcritical by the action of any differential rotation. It is possible to demonstrate that this result does not change without the restriction to small $\epsilon$.

The fact that $Z = 0$ requires $Re^* = 0$ has the consequence that $Ha^* = Ha^*_{Tay}$ does not appear as a solution of (A1) for finite Reynolds numbers. Hence, the curves of marginal instability always remain in the subcritical domain with $Ha^*(Re^*) < Ha^*_{Tay}$ and never reach Hartmann numbers larger than $Ha^*_{Tay}$. The typical suppression of the magnetic instabilities by fast rotation (see Fig. 7) is thus not reflected by the local relation (A1) for inductionless fluids.

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