DEMAZURE EMBEDDINGS ARE SMOOTH

IVAN V. LOSEV

ABSTRACT. We prove the conjecture of M. Brion stating that the closure of the orbit of a selfnormalizing spherical subalgebra in the corresponding Grassmanian is smooth.

1. Introduction

Throughout the paper the base field $K$ is algebraically closed and of characteristic zero. Let $G$ be a connected semisimple algebraic group of adjoint type and $\mathfrak{g}$ its Lie algebra.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be spherical if there is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b} + \mathfrak{h} = \mathfrak{g}$. For example, if $\mathfrak{h} = \mathfrak{g}^\sigma$ for an involutory automorphism $\sigma$ of $\mathfrak{g}$ (in this case $\mathfrak{h}$ is called symmetric), then $\mathfrak{h}$ is spherical, see [V]. For a symmetric subalgebra $\mathfrak{h} \subset \mathfrak{g}$ De Concini and Procesi proved in [CP] that the closure $\overline{G\mathfrak{h}}$ of the orbit $G\mathfrak{h}$ in the corresponding Grassman variety is smooth. Earlier this fact in a few special cases was proved by Demazure, [D]. It was conjectured by Brion in [Br] that the same is true for any spherical subalgebra $\mathfrak{h}$ coinciding with its normalizer (note that a symmetric subalgebra satisfies this condition). The variety $\overline{G\mathfrak{h}}$ is called the Demazure embedding of $G\mathfrak{h}$.

Let us explain why the smoothness of Demazure embeddings is important. There is a nice class of smooth projective $G$-varieties, so called wonderful varieties, possessing many amazing properties, see [T], Section 30, for a review. Knop proved in [K] that a homogeneous space $G\mathfrak{h}$ is embedded as an open $G$-orbit into (a unique) wonderful variety provided $\mathfrak{h}$ is spherical and coincides with its normalizer. If $\overline{G\mathfrak{h}}$ is smooth, then it coincides with this wonderful variety. Brion’s results, [Br], imply that the normalization of $\overline{G\mathfrak{h}}$ is wonderful.

The following theorem is the main result of this paper.

Theorem 1. Let $\mathfrak{h}$ be a spherical subalgebra of $\mathfrak{g}$ coinciding with its normalizer. Then the Demazure embedding $\overline{G\mathfrak{h}}$ is smooth.

Let us note that this theorem was already proved under certain restrictions on $\mathfrak{g}$. In [Lu3] Luna gave the proof in the case when all simple ideals of $\mathfrak{g}$ are $\mathfrak{sl}$’s. In [BP] Luna’s technique was extended to the case when any simple ideal of $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_k$ or $\mathfrak{so}_2k$. In this paper we do not follow Luna’s approach directly, however we use many results from [Lu3].

Theorem 1 is proved in Section 3. In Section 2 we recall some previously obtained results related to wonderful varieties.

Key words and phrases: Demazure embedding, wonderful variety, spherical subgroup.

2000 Mathematics Subject Classification. 14M17.

Partially supported by A. Moebius foundation.
Acknowledgements. This paper was written during my visit to Rutgers University, New Brunswick, in the beginning of 2007. I would like to thank this institution and especially Professor F. Knop for hospitality. I am also grateful to F. Knop for the formulation of the problem and stimulating discussions.

2. Preliminaries

Below $G$ is a connected semisimple algebraic group of adjoint type, $B$ its Borel subgroup, $T$ a maximal torus of $B$, $\Pi$ is the system of simple roots of $G$ and $B^-$ is the Borel subgroup of $G$ containing $T$ and opposite to $B$. Let $X(T)$ denote the character lattice of $T$ (=the root lattice of $G$).

At first, let us recall the definition of a wonderful variety. References are [Lu1], [Lu2], [Lu3], [T], Section 30.

Definition 2.1. A $G$-variety $X$ is called wonderful if the following conditions are satisfied:

1. $X$ is smooth and projective.
2. There is an open $G$-orbit $X^0 \subset X$.
3. $X \setminus X^0$ is a divisor with normal crossings.
4. Let $D_1, \ldots, D_r$ be irreducible components of $X \setminus X^0$. Then for any subset $I \subset \{1, \ldots, r\}$ the subvariety $\bigcap_{i \in I} D_i \setminus \bigcup_{j \notin I} D_j$ is a single $G$-orbit.

The number $r$ is called the rank of $X$ and is denoted by $\text{rk}_G(X)$.

Note that $\bigcap_{i \in I} D_i$ is a wonderful $G$-variety of rank $\#I$ for any $I \subset \{1, \ldots, r\}$.

It is known that a wonderful variety $X$ is spherical, that is $B$ has an open orbit on $X$.

Let us now establish some combinatorial invariants of a wonderful $G$-variety $X$.

Note that $\bigcap_{i=1}^r D_i$ is a generalized flag variety. So there is a unique $B^-$-stable point $z \in \bigcap_{i=1}^r D_i$. The group $T$ acts linearly on the normal space $T_zX/T_z(Gz)$. Note that $T_zX/T_z(Gz) = \bigoplus_{i=1}^r T_zX/T_z D_i$. Let $\alpha_i$ denote the character of the action $T : T_zX/T_z D_i$. Let us fix an $N_G(T)/T$-invariant scalar product on $t^*(\mathbb{Q})$. With respect to this scalar product $\Psi_{G,X} := \{\alpha_1, \ldots, \alpha_r\}$ is a system of simple root of some root system in the linear span of $\alpha_1, \ldots, \alpha_r$. The set $\Psi_{G,X}$ is called the system of spherical roots of $X$. The definition of $\Psi_{G,X}$ given here agrees with that we use in [Lo2]. Let $X_{G,X}$ denote the sublattice in $X(T)$ generated by $\Psi_{G,X}$. It is known that $X_{G,X}$ coincides with the set of all $\lambda \in X(T)$ such that there is a $B$-semiinvariant rational function $f_\lambda$ on $X$ of weight $\lambda$ (determined uniquely up to rescaling because $X$ is spherical). Choose a subset $\Psi_0 \subset \Psi_{G,X}$. Put $X^{\Psi_0} := D_1, X^{\Psi_0} := \cap_{\alpha \in \Psi_0} X^\alpha, X^{\Psi_0} = X^{\Psi_0} \setminus \cup_{\alpha \notin \Psi_0} X^\alpha$.

Let $D_{G,X}$ denote the set of all prime $B$-stable but not $G$-stable divisors on $X$ (this definition differs slightly from that used in [Lo2]). To each $D \in D_{G,X}$ we assign its stabilizer $G_D \subset G$, which is a parabolic subgroup of $G$ containing $B$, and an element $\varphi_D \in X^*_{G,X}$ defined by $\langle \varphi_D, \lambda \rangle = \text{ord}_D(f_\lambda)$. For $\alpha \in \Pi$ by $P_\alpha$ we denote the minimal parabolic subgroup of $G$ containing $B$ corresponding to the simple root $\alpha$. Put $D_{G,X}(\alpha) = \{D \in D_{G,X}| P_\alpha \not\subset G_D\}$.

The following propositions are due to Luna, see [Lu1], [Lu2].
Proposition 2.2. For $\alpha \in \Pi(\mathfrak{g})$ exactly one of the following possibilities takes place:

(1) $\mathcal{D}_{G,X}(\alpha) = \emptyset$.
(2) $\alpha \in \Psi_{G,X}$. Here $\mathcal{D}_{G,X}(\alpha) = \{D^+,D^-\}$ and $\varphi_{D^+} + \varphi_{D^-} = \alpha^\vee|_{\mathfrak{a}_{G,X}}$, $\langle \varphi_{D^\pm},\alpha \rangle = 1$.
(3) $2\alpha \in \Psi_{G,X}$. In this case $\mathcal{D}_{G,X}(\alpha) = \{D\}$ and $\varphi_D = \frac{1}{2}\alpha^\vee|_{\mathfrak{a}_{G,X}}$.
(4) $Q\alpha \cap \Psi_{G,X} = \emptyset, \mathcal{D}_{G,X}(\alpha) \neq \emptyset$. In this case $\mathcal{D}_{G,X}(\alpha) = \{D\}$ and $\varphi_D = \alpha^\vee|_{\mathfrak{a}_{G,X}}$.

We say that a root $\alpha \in \Pi$ is of type (a) (or b),(c),d)) if the corresponding possibility takes place for $\alpha$.

Proposition 2.3. Let $\alpha, \beta \in \Pi(\mathfrak{g})$. If $\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) \neq \emptyset$, then exactly one of the following possibilities takes place:

(1) $\alpha, \beta$ are of type b) and $\#\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) = 1$.
(2) $\alpha, \beta$ are of type d), $\langle \alpha^\vee,\beta^\vee \rangle = 0$, $\alpha^\vee - \beta^\vee|_{\mathfrak{a}_{G,X}} = 0$, and $\alpha + \beta = \gamma$ or $2\gamma$ for some $\gamma \in \Psi_{G,X}$.

Conversely, if $\alpha, \beta \in \Pi$ are such as in (2), then $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$.

Proposition 2.4. Let $\alpha \in \Psi_{G,X}, \beta \in \Pi \cap \Psi_{G,X}, D \in \mathcal{D}_{G,X}(\beta)$. Then $\langle \varphi_D,\alpha \rangle \leq 1$ and the equality holds iff $\alpha \in \Pi, D \in \mathcal{D}_{G,X}(\alpha)$.

Proof. It follows from results of [Lu1], Subsection 3.5, (see also [Lu2], Subsection 3.2) that in the proof one may replace $X$ with $\overline{\mathfrak{X}}^{\alpha,\beta}$. In this case everything follows from the classification in [W].

Now we are going to describe the localization procedure for wonderful varieties.

Choose a subset $\Pi' \subset \Pi$. Let $M$ be the Levi subgroup of $G$ corresponding to $\Pi'$. Put $G_{\Pi'} := (M,M), Q^- = B^-=M$. Then there is a $G_{\Pi'}$-stable subvariety $X_{\Pi'} \subset X^{R(Q^-)}$ (where $R(\cdot)$ denotes the radical) satisfying the following conditions:

(1) $z \in X_{\Pi'}$.
(2) $X_{\Pi'}$ is a wonderful $G_{\Pi'}$-variety.
(3) $\Psi_{G_{\Pi'},X_{\Pi'}} = \{\alpha \in \Psi_{G,X}|\text{Supp}(\alpha) \subset \Pi'\}$ (here and below Supp($\alpha$) stands for the set of all $\beta \in \Pi$ such that the coefficient of $\beta$ in $\alpha$ is nonzero).
(4) For any $\alpha \in \Pi'$ there is a bijection $\iota : \mathcal{D}_{G_{\Pi'},X_{\Pi'}}(\alpha) \to \mathcal{D}_{G,X}(\alpha)$ such that $\varphi_D$ is the projection of $\varphi_D(\alpha)$ to $\mathfrak{X}_{G_{\Pi'},X_{\Pi'}}$.
(5) $GX_{\Pi'} = \overline{\mathfrak{X}}^{\Psi_{G,X}\setminus\Psi_{G_{\Pi'},X_{\Pi'}}}$.

The $G_{\Pi'}$-variety $X_{\Pi'}$ is called the localization of $X$ at $\Pi'$.

Proceed to the definition of Demazure morphisms.

Choose a point $x \in X^\emptyset$ and put $\mathfrak{h} := \mathfrak{g}_x, d := \dim \mathfrak{h}$. The Demazure morphism $\delta_X : X \to \text{Gr}_d(\mathfrak{g})$ is defined as follows: it maps $y \in X$ to the inefficiency kernel of the representation of $\mathfrak{g}_y$ in $T_xX/T_x(Gy)$, for example, $\delta_X(x) = \mathfrak{h}, \delta_X(z)$ is the intersection of the kernels of all $\alpha \in \Psi_{G,X}$, where $\alpha \in \Psi_{G,X}$ is considered as a character of a parabolic subalgebra $\mathfrak{g}_z$. It is known that the image of $\mathfrak{g}_y$ in $\mathfrak{gl}(T_xX/T_x(Gy))$ is a Cartan subalgebra, so im$\delta_X$ does lie in Gr$_d$(g). Moreover, im$\delta_X = \overline{\mathfrak{g}_y}$. In [Br] Brion proved that $\delta_X$ is the normalization
morphism. So $\overline{G\Phi}$ is smooth iff $\delta_X$ is an isomorphism. Further, Brion’s result implies the following statement.

**Lemma 2.5.** Any element of $\text{im} \delta_X$ is a spherical algebraic subalgebra of $G$.

It is known that $n_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$. Conversely, as Knop proved in [K], for any spherical subalgebra $\mathfrak{h}$ coinciding with its normalizer there is a wonderful variety $X$ such that $X^\mathfrak{g} \cong G/N_G(\mathfrak{h})$. Note that $X^\mathfrak{g} = G/N_G(\mathfrak{h})$ has no nontrivial equivariant automorphisms. We say that a wonderful variety $X$ is rigid if $X^\mathfrak{g}$ has no nontrivial equivariant automorphisms. So theorem [I] is equivalent to the claim that $\delta_X$ is an isomorphism provided $X$ is rigid.

The rigidity of $X$ can be expressed in terms of $\Psi_{G,X}, D_{G,X}$. To state this result we need the following definition.

**Definition 2.6.** An element $\alpha \in \Psi_{G,X}$ is said to be distinguished if one of the following conditions holds:

1. $\alpha \in \Pi$ and $\varphi_{D_1} = \varphi_{D_2}$ for different elements $D_1, D_2 \in D_{G,X}(\alpha)$.
2. There is a subset $\Sigma \subseteq \Pi$ of type $B_\ell, k \geq 2$, such that $\alpha = \alpha_1 + \ldots + \alpha_\ell$ and $D_{G,X}(\alpha_i) = \emptyset$ for any $i > 1$.
3. There is a subset $\Sigma \subseteq \Pi$ of type $G_2$ such that $\alpha = 2\alpha_2 + \alpha_1$.

Here $\alpha_i$ denote the simple roots of $B_\ell, G_2$ such that $\alpha_k$ (for $B_\ell$) and $\alpha_2$ (for $G_2$) are short. The following proposition is a direct corollary of Theorem 2 from [Lo2].

**Proposition 2.7.** $X$ is rigid iff there are no distinguished elements in $\Psi_{G,X}$.

Now let $\Pi' \subseteq \Pi$ and $\alpha \in \Psi_{G_{\Pi'}, X_{\Pi'}}$. If $\alpha$ is distinguished in $\Psi_{G,X}$, then it is distinguished in $\Psi_{G_{\Pi'}, X_{\Pi'}}$. The converse is not true: any $\alpha \in \Pi' \cap \Psi_{G,X}$ is distinguished in $\Psi_{G_{\alpha}, X_{\alpha}}$. However, if $\alpha$ is of type 2,3 in $\Psi_{G_{\Pi'}, X_{\Pi'}}$, then it is of the same type in $\Psi_{G,X}$.

3. **Proof of Theorem [I]**

In this section $X$ is a rigid wonderful $G$-variety. Put $\mathfrak{h} = \delta_X(x)$ for some point $x \in X^\mathfrak{g}$. Let $\Pi^a$ denote the subset of $\Pi$ consisting of all roots of type $a$ for $X$.

The following two assertions were proved in [Lu3], Section 3.

**Lemma 3.1.** If the restriction of $d_x \delta_X$ to the $T$-eigenspace $(T_z X)_\gamma$ of weight $\gamma$ is injective for any $\gamma \in \mathfrak{X}(T)$, then $\delta_X$ is an isomorphism.

If $\gamma \notin \Psi_{G,X}$, then $(T_z X)_\gamma \subset T_z(Gz)$. Since $\delta_X : X \to \overline{G\Phi}$ is the normalization morphism, the restriction of $\delta_X$ to $Gz$ is an embedding.

**Proposition 3.2.** Let $\alpha \in \Psi_{G,X}$ and $\Pi'$ be a subset of $\Pi$ containing $\text{Supp}(\alpha) \cup \Pi^a$. The restriction of $d_x \delta_X$ to $(T_z X)_\alpha$ is injective provided so is the restriction of $d_x \delta_{X_{\Pi'}}$ to $(T_z X_{\Pi'})_\alpha$.

**Definition 3.3.** Let $\alpha \in \Psi_{G,X}$. We say that $X$ is critical for $\alpha$ if $\alpha$ is not distinguished in $\Psi_{G,X}$ but is distinguished in $\Psi_{G_{\Pi'}, X_{\Pi'}}$ for any $\Pi' \subset \Pi$ containing $\Pi^a \cup \text{Supp}(\alpha)$.

So we need to prove the following claim:
Lemma 6.1 from [Lo1], \( \Pi = \text{Supp}(\alpha) \)

Proposition 3.4. Let \( X \) be critical for \( \alpha \in \Psi_{G,X} \). (*) holds for \( X \) provided \( \alpha \not\in \Pi \).

Proof. It follows from [P], Theorem 3.4, that there is a simple module \( V \) and a \( G \)-equivariant morphism \( \varphi : X \to \mathbb{P}(V) \) such that the restriction of \( \varphi \) to \( \overline{\delta X} \) is an embedding. So it remains to show that there is a morphism \( \psi : \delta X(\overline{\alpha}) \to \mathbb{P}(V) \) such that \( \psi \circ \delta X|_{\overline{\alpha}} = \varphi|_{\overline{\alpha}} \). Set \( Y := \delta X(\overline{\alpha}), h_1 := \delta X(x) \) for some \( x \in \alpha, h_0 := \delta X(z) \).

Suppose, at first, that the character group of \( G_x, x \in \alpha \), is finite. It follows that any \( G_x \)-semiinvariant vector in \( V \) is \( g \)-invariant. Therefore \( \dim V^f = 1 \) for any \( f \in Y \). By Lemma 2.5, \( f \) is spherical for any \( f \in Y \) whence \( \dim V^f \leq 1 \). Thus the map \( \psi : Y \to \mathbb{P}(V), f \mapsto V^f \), is well-defined. Let us check that this map is a morphism of varieties. Let \( Z \) denote the subvariety of \( g^d, d = \dim h_1 \), consisting of all linearly independent \( d \)-tuples and \( \pi : Z \to \text{Gr}_d(g) \) be the natural projection. Choose a basis \( v^1, \ldots, v^n \in V^* \). Since \( \dim V^f = 1 \) for all \( f \in Y, \) we have the natural morphism \( \tilde{\psi} : \pi^{-1}(Y) \to \mathbb{P}(V) \cong \text{Gr}_{\dim V-1}(V^*) \) mapping \( (\xi_1, \ldots, \xi_d) \) to the linear span of \( \xi_i v^j, i = 1, d, j = 1, n \). Now recall that \( \pi : \pi^{-1}(Y) \to Y \) is the quotient morphism for the natural action \( \text{GL}_d : \pi^{-1}(Y) \to \mathbb{P}(V) \cong \text{GL}_d \)-invariant. Therefore \( \tilde{\psi} \) factors through a unique morphism \( \psi : Y \to \mathbb{P}(V) \). Clearly, \( \psi \circ \delta X|_{\overline{\alpha}} = \varphi|_{\overline{\alpha}} \).

Now consider the general case. Since \( X \) is critical for \( \alpha \), we have \( \Pi = \text{Supp}(\alpha) \cup \Pi^c \). By Lemma 6.1 from [Lo1], \( \Pi = \text{Supp}(\alpha) \). Then, inspecting Table 1 in [W], we see that \( (g, h_1, \alpha) \) is one of the following triples:

1. \( g = sl_{n+1}, n \geq 2, h_1 = gl_n, \alpha = \alpha_1 + \ldots + \alpha_n. \)
2. \( g = sp_{2n+1}, n \geq 2, h_1 = gl_n \times \mathbb{K}^2, \alpha = \alpha_1 + \ldots + \alpha_n. \)
3. \( g = G_2, h_1 = (t_1 \times sl_2) \times (\mathbb{K}^2 \oplus \mathbb{K}), \alpha = \alpha_1 + 2 \alpha_2 + \ldots + 2 \alpha_{n-1} + \alpha_n. \)
4. \( g = G_2, h_1 = (t_1 \times sl_2) \times \mathbb{K}, \alpha = \alpha_1 + \alpha_2. \)

Note that \( \text{codim}_{h_1} [h_1, h_1] = 1 \) and \( [h_1, h_1] \) is a spherical subalgebra of \( g \). In all cases \( h_0 \) is the kernel of \( \alpha \) in a certain parabolic subalgebra of \( g \) containing \( b^- \). Since \( \alpha \not\in \Pi \), we get \( \text{codim}_{h_0} [h_0, h_0] = 1 \). Analogously to the previous paragraph, the map \( \tilde{\psi} : G[h_1, h_1] \to \mathbb{P}(V) \) mapping \( f \in G[h_1, h_1] \) to \( V^f \) is a morphism. Since \( \text{codim}_{h_0} [h_0, h_0] = \text{codim}_{h_0} [h_1, h_1] \), we see that there is a (unique) \( G \)-equivariant map \( \iota : Y \to G[h_1, h_1] \) mapping \( h_1 \) to \( [h_1, h_1] \) and \( h_0 \) to \( [h_0, h_0] \). Using a technique similar to that from the previous paragraph, we get that \( \iota \) is a morphism. It remains to put \( \psi = \tilde{\psi} \circ \iota \).

The idea to use results of [P] in the proof of smoothness of Demazure’s embedding is due to Knop.

So it remains to consider the case when \( X \) is critical for \( \alpha \in \Pi \cap \Psi_{G,X} \). Suppose at first that \( r_{G}(X) = 2 \). Let \( D^+, D^- \) denote different elements of \( D_{G,X} \) and \( \beta \) a unique element of \( \Psi_{G,X} \{ \alpha \} \). Since \( \alpha \) is not distinguished, we have \( \langle \varphi_{D^+}, \beta \rangle \neq \langle \varphi_{D^-}, \beta \rangle \). It was essentially proved by Luna in [Lu3], Assertion 4, that in this case the restriction of \( d_{\delta X} \) to \( T_z X \) is injective (note that \( c[X^\gamma] \) in (**)) in the proof of Assertion 4 is equal to \( \langle \varphi_{D^+}, \gamma \rangle \), thanks to Lemmas 3.2.3,3.3 from [Lu1].
We are going to reduce the general case to the previous one. Choose $\beta \in \Psi_{G,X}$ with $\langle \beta, \varphi_{D^+} \rangle \neq \langle \beta, \varphi_{D^-} \rangle$. Then $\langle \varphi_{D^+}, \beta \rangle = \langle \varphi_{D^-}, \beta \rangle$ for different elements $D^+_0 \in D_{G,\Sigma_{\alpha_0}(\alpha)}(\alpha)$ (see [Lu2], the last paragraph of Subsection 3.2). Clearly, $(T_zX)_\alpha = (T_zX^0)_\alpha = (T_z\overline{X}^{0,\beta})_\alpha$. Choose $x \in X^0$ and put $h_0 = \delta_X(x)$, $\overline{h}_0 = \delta_{\overline{X}^{0,\beta}}(x)$. It follows from the properties of the Demazure morphisms quoted in Section 2 that $h_0$ is an ideal of $h_0$ and $h_0/h_0$ is a commutative diagonalizable Lie algebra (of dimension $rk_G(X) - 2$).

Let $Q$ denote the parabolic subgroups of $G$ containing $B^-$ corresponding to the subset $\Pi_0 \cup \{ \alpha \} \subset \Pi$. Let $q_0$ denote a unique ideal in $q$ complimentary to $g_0$ and $Q_0$ be the connected subgroup of $Q$ with Lie algebra $q_0$. Then $X'$ is $G$-equivariantly isomorphic to $G \ast_{Q} X$, where $Q$ acts on $X'$ via the projection $Q \rightarrow Q/Q_0$. As a $G_\alpha$-variety $X_\alpha$ is isomorphic to $P^1 \times P^1$. Note that $h_0, \overline{h}_0$ are ideals in $g_x$ of codimension $rk_G(X), 2$, respectively.

Since $\delta_{\overline{X}^{0,\beta}}$ is an isomorphism, we have $G_x = N_G(h_0)$. Let $H_0, H_0$ denote the connected subgroups of $G$ with Lie algebras $h_0, h_0$. From Lemma 2.5 it follows that $h_0$ is a spherical subalgebra of $g$. It follows that $N_G(h_0)/H_0$ is a commutative group. In particular, $h_0/H_0$ commutes with $N_G(h_0)/H_0$ whence $N_G(h_0) \subset N_G(h_0) = G_x$. On the other hand, $G_x \subset N_G(h_0)$, for $\delta_X$ is $G$-equivariant. So the restriction of $\delta_X$ to $X^\alpha$ is injective.

Now we apply the argument from [Lu3], proof of Assertion 4. Choose $x \in X^{\alpha}$. By above, $g_x = t + q_0$. Let $m$ denote a unique Levi subalgebra of $q$ containing $t$. Set $m_0 := m \cap q_0$. Since $g_x \subset n_g(\delta_X(x))$ and $g_x/\delta_X(x)$ is a diagonalizable Lie algebra, we see that $q := R_u(q) + [m_0, m_0] \subset \delta_X(X^\alpha)$. So we may consider $\delta_X|_{X^\alpha}$ as a morphism to $q/q$. The Lie algebra $q/q$ is identified with $m_1 := g_0 \oplus j(m_0)$. Let $M_1$ denote the connected subgroup of $G$.

As we have shown above, $N_{M_1}(\delta_X(x))$ is a maximal torus of $M_1$ for any $x \in X^{\alpha}$. Analogously to the proof of Assertion 4 in [Lu3], $M_1 \delta_X(x)$ is smooth. So $\delta_X$ is an isomorphism of $X^\alpha$ to $M_1 \delta_X(x)$ whence its restriction to $(T_zX)_\alpha$ is injective.

References

[BP] P. Bravi, G. Pezzini. Wonderful varieties of type D. Preprint (2004), arXiv:math/RT.0410472, 60 pages.
[Br] M. Brion. Vers une généralisation des espaces symétriques. J. Algebra, 134(1990), 115-143.
[CP] C. De Concini, C. Procesi. Complete symmetric varieties, I. Invariant theory, Proceedings (F. Gherardelly, ed.) Lect. Notes in Math., v. 996, 1-44. Montecatini, 1983, Springer-Verlag.
[D] M. Demazure. Limites de groupes orthogonaux ou symplectiques. Preprint (1980), Paris.
[K] F. Knop. Automorphisms, root systems and compactifications. J. Amer. Math. Soc. 9(1996), n.1, p. 153-174.
[Lo1] I.V. Losev. Proof of the Knop conjecture. Preprint(2006) arXiv:math.AG/0612561, 20 pages.
[Lo2] I.V. Losev. Uniqueness property for spherical homogeneous spaces. Preprint (2007).
[Lu1] D. Luna. Groiss cellules pour les variétés sphériques. Austr. Math. Soc. Lect. Ser., v.9, 267-280. Cambridge University Press, Cambridge, 1997.
[Lu2] D. Luna. Variétés sphériques de type A. IHES Publ. Math., 94(2001), 161-226.
[Lu3] D. Luna. Sur le plongements de Demazure. J. of Algebra, 258(2002), p. 205-215.
[OV] A.L. Onishchik, E.B. Vinberg. Seminar on Lie groups and algebraic groups. Moscow, Nauka 1988 (in Russian). English translation: Berlin, Springer, 1990.
[P] G. Pezzini. Simple immersions of wonderful varieties. Preprint (2005). arXiv:math.AG/0506661.
D. A. Timashev, *Homogeneous spaces and equivariant embeddings*. Preprint (2006), arXiv:math.AG/0602228.

Th. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes*. Bull. Soc. Math. France, 102(1974), 317-334.

B. Wasserman, *Wonderful varieties of rank two*. Transform. Groups, v.1(1996), no. 4, p. 375-403.

Chair of Higher Algebra, Department of Mechanics and Mathematics, Moscow State University.
E-mail address: ivanlosev@yandex.ru