An $N = 2$ Superconformal Fixed Point with $E_6$ Global Symmetry

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Abstract

We obtain the elliptic curve corresponding to an $N = 2$ superconformal field theory which has an $E_6$ global symmetry at the strong coupling point $\tau = e^{\pi i/3}$. We also find the Seiberg-Witten differential $\lambda_{SW}$ for this theory. This differential has 27 poles corresponding to the fundamental representation of $E_6$. The complex conjugate representation has its poles on the other sheet. We also show that the $E_6$ curve reduces to the $D_4$ curve of Seiberg and Witten. Finally, we compute the monodromies and use these to compute BPS masses in an $F$-Theory compactification.

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1. Introduction

Kodaira’s classification of the singularities of the torus demonstrates an ADE pattern\[1\]. Singularities that occur at $\text{Im}\tau = \infty$ are either $A_n$ or $D_n$. There are also a finite number of singularities that occur at finite values of $\tau$. The singularity types for these are $A_0$, $A_1$, $A_2$, $E_6$, $E_7$ and $E_8$. Finally, there is a $D_4$ singularity that can occur at all values of $\tau$.

What is quite striking about this is that most of these singularities have appeared in $N = 2$ $U(1)$ superconformal field theories. In particular, $D_4$ first appeared in the classic papers of Seiberg and Witten\[2,3\] and $A_0$, $A_1$ and $A_2$ appear in certain limits of an $SU(2)$ gauge theory with $N_f = 1, 2$ or $3$ respectively\[4,5\], and hence can be derived from the $D_4$ theory. (The $A_0$ theory also can come from an $SU(3)$ gauge theory with $N_f = 0$\[6\]). The theories at $\text{Im}\tau = \infty$ are basically trivial since the coupling runs to zero.

What are missing are the theories with $E_6$, $E_7$ and $E_8$ singularities. In this paper, we will take a step in this direction by constructing the explicit elliptic curve for $E_6$, along with the corresponding Seiberg-Witten differential. It is not clear to us if such a theory can be reached in some limit of a super Yang-Mills theory. The $E_7$ and $E_8$ cases will be discussed in a separate publication.

2. Dimensions and Relevant Operators

Let us begin by showing why a superconformal fixed point is related to Kodaira’s classification. We assume that the Seiberg-Witten curve is of the form

$$y^2 = x^3 - f(\rho)x - g(\rho) \tag{2.1}$$

where $f(\rho)$ and $g(\rho)$ are polynomials in $\rho$ and $\rho$ is the expectation value of some scalar field. We further assume that there exists a differential $\lambda_{SW}$ such that

$$\frac{d\lambda_{SW}}{d\rho} = \frac{dx}{y}. \tag{2.2}$$

Hence $\lambda_{SW}$ has the same dimensions as $\rho dx/y$. Since BPS masses are found by integrating $\lambda_{SW}$ around closed loops, $\lambda_{SW}$ has dimension 1. In order to have a unitary quantum field theory, the dimension of any operator should be nonnegative. The type of singularity is determined by the behavior of $f$, $g$ and the discriminant $\Delta = 4f^3 - 27g^2$. With no loss of generality, we can assume that the singularity occurs at $\rho = 0$ and that $f \sim \rho^r$, $g \sim \rho^s$. 

1
If $2s < 3r$, then the singularity is dominated by $g$. In this case, the dimension of $\rho$, $[\rho]$, satisfies $s[\rho] = 3[x] = 2[y]$. Since $[\lambda_{SW}] = 1$, one finds $[\rho] = 6/(6 - s)$. Therefore, one must have $s < 6$ in order to have a unitary theory. At the singularity, the coupling is $\tau = e^{\pi i/3}$. For $s = 1$ and $s = 2$, the singularity is $A_0$ and $A_2$ respectively. For $s = 3$, the singularity is $D_4$, and for $s = 4, 5$, the singularity is $E_6$ and $E_8$ respectively.

If $2s > 3r$, then the singularity is determined by the behavior of $f$. In this case, $r[\rho] = 2[x]$, hence $[\rho] = 4/(4 - r)$ and $m < 4$ in order for the theory to be unitary. If $r = 1$, the singularity is $A_1$, $r = 2$ is $D_4$ and $r = 3$ is $E_7$.

If $2s = 3r$, then a unitary theory will either have $s = 0$ or $s = 3$. The singularity then depends on the discriminant. In the case where $s = 0$, and $\Delta \sim \rho^{n+1}$, the singularity is $A_n$ and occurs at weak coupling. If $s = 3$ and $\Delta \sim \rho^{6+n}$, then the singularity is $D_{4+n}$ and occurs at weak coupling for $n > 0$.

Each of these theories will flow away from criticality as relevant operators are turned on. For what follows, we will consider only the superconformal theories that occur at strong coupling. The number of relevant operators depends on the dimension of the polynomials $f$ and $g$ that will lower the critical exponent in $\Delta$. So for instance, the $A_2$ singularity is reduced by adding to $g$ the polynomial $a\rho + b$ and to $f$ the polynomial $c\rho + d$. This has four total degrees of freedom, which leads to three relevant operators, since one degree of freedom can be used to shift $\rho$. The strong coupling $A_n$ singularities will have $n + 1$ relevant operators.

The $D_4$ theory has four relevant operators as well as a marginal operator, namely the bare coupling. In fact, the $D_4$ theory as well as the $A_1$ and $A_2$ theories each has an operator that allows one to flow from strong to weak coupling while still preserving the global symmetry. In the case of $A_1$ and $A_2$, this operator can be thought of as the center of mass operator in the $SU(2)$ gauge theory, or equivalently as the cutoff.

However, for the $E_n$ cases, the number of relevant operators is $n$, as one can easily verify. Moreover, there is no marginal operator analogous to the bare coupling in the $D_4$ case. Hence, in order to flow away from the strong coupling point, one must break the global symmetry. Therefore, all relevant operators should be expressible in terms of the $E_n$ casimirs.

In the rest of this paper, we will use symmetry arguments as well as some of the ideas presented in section 17 of [3] to construct the elliptic curve for the $E_6$ case.
3. Construction of the curve for $E_6$

It is convenient to express the curve in terms of the casimirs of the $U(1) \times SO(10)$ subgroup of $E_6$. We define operators corresponding to the Cartan Subalgebra of this subgroup, $\lambda$ and $m_i$. We also assume that the residues of $\lambda_{SW}$ will be linear combinations of these operators, hence they must all have dimension 1. We then define the following $SO(10)$ casimirs

$$T_2 = \sum_{i=1}^{5} m_i^2, \quad T_4 = \sum_{i<j} m_i^2 m_j^2, \quad T_6 = \sum_{i<j<k} m_i^2 m_j^2 m_k^2,$$

$$T_8 = \sum_{i<j<k<l} m_i^2 m_j^2 m_k^2 m_l^2, \quad T_5 = m_1 m_2 m_3 m_4 m_5. \quad (3.1)$$

We now proceed incrementally. First assume that $m_i = 0$ for all $i$. A nonzero $\lambda$ breaks the global symmetry from $E_6$ to $SO(10)$. Since $[\rho] = 3$ and $[x] = 4$, the curve must be of the form

$$y^2 = x^3 - 12\lambda^2 \rho^2 x - \rho^4 + 16\rho^3 \lambda^3, \quad (3.2)$$

up to an overall normalization of the $U(1)$ casimir and an overall shift in $\rho$. The discriminant behaves as $\rho^7$ corresponding to a $D_5$ global symmetry.

We next turn on $m_1$, breaking the global symmetry to $SO(8)$. The only nonzero $SO(10)$ casimirs are made up of powers of $T_2$. Hence the generic curve is then

$$y^2 = x^3 - (12\lambda^2 + T_2)\rho^2 x - (\rho^4 - 2\lambda(8\lambda^2 + \alpha T_2)). \quad (3.3)$$

The first coefficient of $T_2$ is determined by choosing an overall scale. In order to determine the coefficient $\alpha$ we have to assume that $\lambda_{SW}$ has poles whose residues are linear combinations of $\lambda$ and $m_i$. We may also assume that the differential has a factor of $y$ in the denominator. Hence, in order to satisfy these requirements, $y^2$ in (3.3) must be a perfect square when $x$ is at the position of the pole. Following [3], we assume that the poles have a linear dependence on $\rho$ and can be written in the form

$$x = \beta \rho + \theta. \quad (3.4)$$

Clearly, $\theta$ must be a perfect square, and given the dimensions of $\rho$, $m_1$ and $\lambda$, $x$ has the form

$$x = (am_1 + b\lambda)\rho - (rm_1^2 + sm_1 \lambda + t\lambda^2)^2 \quad (3.5)$$
At the pole, $y$ should have the form

$$y^2 = -(\rho^2 + B\rho + C)^2 \quad (3.6)$$

Hence, from the linear and constant pieces in $y^2$, we find

$$C = (rm_1^2 + sm_1\lambda + t\lambda^2)^3 \quad B = -\frac{3}{2}(am_1 + b\lambda)(rm_1^2 + sm_1\lambda + t\lambda^2) \quad (3.7)$$

If $\alpha = -2$, then we find four sets of solutions, which are

$$(a, b, r, s, t) = (\pm 1, 2, 0, 0, 0), (0, 8, 9/2, 0, 0), (\pm 2, -4, 0, \mp 6, 4), (0, -4, 0, 0) \quad (3.8)$$

Recall that the fundamental representation of $E_6$ decomposes under $SO(10) \times U(1)$ to

$$27 = 16_{+1} + 10_{-2} + 1_{+4}. \quad (3.9)$$

Hence, it is clear that the first solution in $(3.8)$ is a spinor, the second is a singlet, and the last two are vector solutions, with the first of these having the vector component aligned along $m_1$ and the last solution having it orthogonal to $m_1$. Therefore, we expect to find poles at $x_\alpha = 2h_\alpha \rho + \theta_\alpha$, where $\alpha$ is an index transforming in the fundamental of $E_6$. In terms of $SO(10)$ representations, these poles are at

$$x_{\pm\pm\pm\pm\pm} = (\pm m_1 \pm m_2 \pm m_3 \pm m_4 \pm m_5 + 2\lambda)\rho + \theta_{\pm\pm\pm\pm\pm}$$
$$x_{\pm i} = (\pm 2m_i - 4\lambda)\rho + \theta_i$$
$$x_s = (+8\lambda)\rho + \theta$$ \quad (3.10)

where $\theta, \theta_i$ and $\theta_{\pm\pm\pm\pm}$ are yet to be determined and the number of $+$ signs in $x_{sp}$ is even.

In fact, we can now find the complete curve by symmetry arguments and the assumption that $y^2$ is a perfect square when $x$ is a spinor solution in $(3.10)$. In particular, if $m_3 = m_4 = m_5 = 0$, there is a remaining $SO(6) \simeq SU(4)$ global symmetry, hence the discriminant should behave as $\rho^4$ as $\rho \to 0$. If $m_4 = m_5 = 0$, then there is an $SO(4) \simeq SU(2) \times SU(2)$ global symmetry, in which case the discriminant behaves as $(\rho^2 - \gamma^2)^2$, where $\gamma$ is some constant.
The final result for the curve is

\[ y^2 = x^3 - \left( \rho^2(12\lambda^2 + T_2) + \rho(8T_5 + 8\lambda T_4) + \frac{1}{3}T_4^2 + (36\lambda^2 - T_2)T_6 + 4T_8 - 12\lambda(36\lambda^2 - T_2)T_5 \right)x \]
\[ - \left( \rho^4 - 4\rho^3\lambda(4\lambda^2 - T_2) + \rho^2 \left( \frac{1}{3}T_2T_4 - 2T_6 + 20\lambda^2T_4 - 40\lambda T_5 \right) + \rho \left( \frac{8}{3}\lambda T_4^2 - 4\lambda T_6(T_2 - 36\lambda^2) - \frac{16}{3}T_4 T_5 - 32\lambda T_8 + 2T_2^2T_5 - 96\lambda^2 T_2 T_5 + 864\lambda^4 T_5 \right) \right. \]
\[ + \left( \frac{2}{27}T_4^3 - \frac{1}{3}T_2 T_4 T_6 + T_6^2 + (T_2 - 36\lambda^2)^2 T_8 - \frac{8}{3}T_4 T_8 + 12\lambda^2 T_4 T_6 + 144\lambda^2 T_5^2 \right. \]
\[ \left. - 24\lambda T_5 T_6 - 144\lambda^3 T_4 T_5 + 4\lambda T_2 T_4 T_5 \right) \right) \]  

(3.11)

This curve is a perfect square at the spinor point

\[ x_{+++++} = (T_1 + 2\lambda)\rho - \frac{1}{3}T_4 + 2T_4' - (6\lambda + T_1)T_3, \]  

(3.12)

where

\[ T_1 = \sum_i m_i, \quad T_3 = \sum_{i<j<k} m_i m_j m_k, \quad T_4' = \sum_{i<j<k<l} m_i m_j m_k m_l. \]  

(3.13)

(3.11) is still a perfect square if we change the signs of an even number of the \( m_i \) in \( T_1, T_3 \) and \( T_4 \). For \( x = x_{+++++} \) in (3.12), \( y^2 \) satisfies

\[ y^2 = y_{+++++}^2 = -\left( \rho^2 + \frac{1}{2}(T_1 + 6\lambda(T_1^2 - 2T_2)(T_3 - \rho) - T_6 \]
\[ - T_4'(2T_1^2 - 2T_2 - 18\lambda(T_1 + 2\lambda)) + 4T_5(T_1 + 12\lambda) \right)^2. \]  

(3.14)

The vector points are given by

\[ x_{\pm i} = 2(\pm m_i - 2\lambda)\rho + m_i^2 \left( -m_i^2 - (\pm m_i - 6\lambda)^2 + T_2 \right) - \frac{1}{3}T_4 \pm 2T_5/m_i \]
\[ y_{\pm i}^2 = -\left( \rho^2 - \left( 4m_i^2(\pm m_i - 6\lambda) \pm m_i(36\lambda^2 - T_2) \right) \rho + 3m_i^6 \pm 6\lambda m_i^5 - 3m_i^4 T_2 \pm 6\lambda m_i^3 T_2 \]
\[ + 2m_i^2 T_4 - T_6 \mp (4m_i^2 \mp 12\lambda m_i + 36\lambda^2 - T_2)/m_i \pm m_i^2(\pm m_i - 6\lambda)^3 \right) \]  

(3.15)

and the singlet point is

\[ x_s = 8\lambda \rho - \frac{1}{4}(T_2 - 36\lambda^2)^2 + \frac{2}{3}T_4 \]
\[ y_s^2 = -\left( \rho^2 + 6\lambda(T_2 - 36\lambda^2) \rho + \frac{1}{8}(4T_4 - (T_2 - 36\lambda^2)^2)T_2 - 36\lambda^2 - T_6 + 12\lambda T_5 \right)^2, \]  

(3.16)
Eq. (3.11) should also be expressible in terms of $E_6$ casimirs. At first, this might seem problematic, since the curve has dimension three operators, but no such $E_6$ casimir exists. However, since $\rho$ is also dimension three, we can remove this term by shifting $\rho$ by $\lambda(36\lambda^2 - T_2)$. In terms of $E_6$ casimirs, the new curve is

$$y^2 = x^3 - \left(-\frac{1}{3} \rho^2 P_2 + \frac{2}{3} \rho P_5 - \frac{7}{432} P_2^4 + \frac{11}{45} P_2 P_6 - \frac{8}{15} P_8\right)x$$

$$- \left(\rho^4 + \rho^2 \left(\frac{2}{3} P_6 - \frac{7}{108} P_2^3\right) + \rho \left(\frac{1}{18} P_2^2 P_5 - \frac{8}{21} P_9\right)\right)$$

$$+ \frac{32}{135} P_{12} - \frac{298}{18225} P_2^2 P_8 - \frac{101}{218700} P_3 P_6 + \frac{13}{405} P_6^2 - \frac{49}{1049760} P_2^6 - \frac{19}{3645} P_2 P_5^2,$$

where the $P_i$ are the $E_6$ casimirs found in [3].

4. The Seiberg-Witten Differential for $E_6$

Once the positions of the poles and their residues are known, one can find the full differential $\lambda_{SW}$. The differential is just a sum over the 27 of $E_6$ plus a piece that has no pole and is hence proportional to the holomorphic differential $dx/y$. This last piece should be invariant under $E_6$.

When a closed curve crosses a pole under a monodromy transformation, the coordinate corresponding to the integral of $\lambda_{SW}$ along that curve shifts by the residue of that pole multiplied by $2\pi i$. Therefore, $\lambda_{SW}$ should be of the form

$$\lambda_{SW} = \gamma \left(\rho + \lambda T_2 - 4\lambda^3\right) \frac{dx}{y} + \frac{1}{2\sqrt{2}\pi i} \sum_{\alpha=1}^{27} \frac{h_\alpha y_\alpha}{x - 2 h_\alpha \rho - \theta_\alpha} \frac{dx}{y},$$

(4.1)

where $h_\alpha$ is the linear combination of $m_i$ and $\lambda$ for the $\alpha$ state in the representation, $y_\alpha$ is the value of $y$ when $x$ is at the pole and $\gamma$ is a constant to be determined. Notice that the residues in (4.1) switch sign when moving to the other sheet. These poles then transform in the 27 representation.

The constant $\gamma$ can be determined by finding $d\lambda_{SW}/d\rho$ and making sure that it is proportional to $dx/y$, up to a total derivative. Setting $m_2 = m_3 = m_4 = m_5 = 0$, simplifies the calculation, and one finds that

$$\gamma = \frac{9}{\sqrt{2}\pi}, \quad \frac{d\lambda_{SW}}{d\rho} = \frac{3}{\sqrt{2}\pi} \frac{dx}{y} + \frac{d(\ldots)}{dx}$$

(4.2)

Note that the relation between $\lambda$ and the term $x_1$ which appears in [7] is $x_1 = \lambda$. 
5. Relation to the $D_4$ case of Seiberg and Witten

If we take $\lambda$ and one of the $m_i$ to infinity, leaving the other $m_i$ finite, the curve in (3.11) should reduce to the Seiberg-Witten result [3]. This then will provide a useful check on our result.

In fact, the proper scaling is quite simple. Choose $\lambda = -c_1 \Lambda / 6$, $m_5 = -c_2 \Lambda$, $\rho = \mu \Lambda$ and scale $x \to x \Lambda^2$ and $y \to y \Lambda^3$, where $c_1$ and $c_2$ are defined in [3]. Keeping only the leading terms in $\Lambda$ and shifting $x$ by $(c_1 u + c_2^2 t_2) / 3$, where $t_2 = m_1^2 + m_2^2 + m_3^2 + m_4^2$, the curve in (3.11) reduces to

\[
y^2 = (x^2 - c_2^2 u^2)(x - c_1 u) - c_2^2 (x - c_1 u)^2 t_2 - c_2^2 (c_1^2 - c_2^2)(x - c_1 u)t_4 \\
+ 2 c_2 (c_1^2 - c_2^2)(c_1 x - c_2^2 u)t_4' - c_2^2 (c_1^2 - c_2^2)^2 t_6,
\]

where

\[
t_4 = \sum_{i<j<5} m_i^2 m_j^2, \quad t_4' = m_1 m_2 m_3 m_4, \quad t_6 = \sum_{i<j<k<5} m_i^2 m_j^2 m_k^2.
\]

This is the Seiberg-Witten result [3].

We can also study the behavior of $\lambda_{SW}$ in this limit. To leading order in $\Lambda$, the poles behave as

\[
x_s = -\frac{1}{4}(c_1^2 - c_2^2) \Lambda^2 \\
x_{\pm 5} = c_2^2 (c_1 \pm c_2) \Lambda^2 \\
x_{\pm i} = c_1 u + m_i^2 (c_2^2 - c_1^2) \\
x_{\pm \pm \pm \pm} = c_2 u + (c_1 + c_2) c_2 \sum_{i<j<5} (\pm m_i)(\pm m_j) \\
x_{\pm \pm \pm \pm} = -c_2 u - (c_1 - c_2) c_2 \sum_{i<j<5} (\pm m_i)(\pm m_j).
\]

Hence, the poles at $x = x_s, x_{\pm 5}$ move out to infinity where the sum of their residues cancel. The other poles are at finite values of $x$. However, they all have infinite residues. But these infinite parts will cancel off since there are also poles coming from the $2\overline{7}$. In the limit of $\Lambda \to \infty$, each pole in the $27$ moves to the same point on the torus as a corresponding pole in the $2\overline{7}$. The infinite pieces cancel off, leaving a residue that is twice the residue in [3]. Moreover, the poles split into the vector, spinor and spinor bar representation of $SO(8)$. By triality, the poles of any one representation are enough to describe $\lambda_{SW}$. Hence, $\lambda_{SW}$ for $E_6$ flows to $\lambda_{SW}$ for $SO(8)$, but multiplied by a factor of 6.
6. Monodromies and Applications to $F$-Theory

The curve in (3.11) has eight singularities in the $\rho$ plane. Since $E_6$ has an $SU(6) \times SU(2)$ subgroup, we expect to find an $A_5$ singularity at weak coupling for some values of the $m_i$ and $\lambda$. Hence the monodromies around six of the singularities, $N_i$, commute with each other. Indeed, the $A_5$ singularity occurs if $m_i = m_j = -2\lambda = m$. In this case, the discriminant is proportional to

$$\Delta \sim (\rho - 2m^3)^6(27\rho^2 + 256m^6). \quad (6.1)$$

As in [3], the monodromies around the other two singularities $M_1$ and $M_2$, do not commute with the monodromies around these six singularities, nor with each other. Because $\lambda_{SW}$ has poles, the monodromies are not in $SL(2, \mathbb{Z})$. However, let us first ignore the contributions of the poles. From (6.1), we expect $N_i = T$ and $M_1$ and $M_2$ to be conjugate to $T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$. Since $\rho$ has the topology of the sphere, we should also have that

$$M_2M_1 \prod_i N_i = M_\infty^{-1}. \quad (6.2)$$

The monodromy at $\infty$ is not the same as the $D_4$ case, but is instead

$$M_\infty = T^{-1}ST^{-1}S, \quad S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$

(6.3)

It is not hard to show that when $m_i = \lambda = 0$, (3.2) is satisfied if

$$M_1 = (STS)^{-1}T(STS) = -ST^{-2}$$
$$M_2 = (STSST^{-1}S)^{-1}T(STST^{-1}S) = M_\infty M_1 M_\infty^{-1}. \quad (6.4)$$

If we now turn on $m_i$ and $\lambda$, then the monodromies are modified, since in going around a singularity, a closed loop can cross a pole. We have some freedom to choose how the monodromies are modified. A convenient basis leaves $M_\infty$ unchanged and modifies the other monodromies such that

$$N_i \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a_D + a + m_i - 2\lambda \\ a \end{array} \right) \quad 1 \leq i \leq 5$$
$$N_6 \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a_D + a + \frac{1}{2} \sum_i m_i + \lambda \\ a \end{array} \right) \quad (6.5)$$
\[ M_1 \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} a + \frac{1}{2} (m_1 + m_2 + m_3 - m_4 - m_5) - 3\lambda \\ -a_D + 2a + m_5 - 6\lambda \end{pmatrix} \]
\[ M_2 \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} 3a_D + a + 3m_4 + 2m_5 + 6\lambda \\ -4a_D - a + \frac{1}{2} (m_1 + m_2 + m_3 - 7m_4 - 5m_5) - \lambda \end{pmatrix} \]

(6.6)

One can easily check that (6.3), (6.5) and (6.6) satisfy (6.2).

These results can be used to investigate some aspects of F-Theory. In particular, Sen has argued that one can use the Seiberg-Witten result to study F-Theory compactified on the \( T_4/Z_2 \) orbifold. He argued that this is equivalent to a type IIB theory on an orientifold. At the orbifold point, there is an enhanced \((SO(8))^4\) gauge theory. Sen then uses the results of [3] to see what happens as one moves away from the orbifold point. At the orbifold point, the 24 singularities in the \( \rho \) plane break up into 4 groups of 6. One moves away from the orbifold point by resolving the singularities, which breaks the gauge symmetry.

In particular, if we pick one group of six singularities, then the BPS masses are given by integrals of \( \partial_\rho a_D \) along closed loops in the \( \rho \) plane. So one finds a BPS mass \( m_i - m_j \) by integrating counterclockwise around the \( i \)th singularity and clockwise around the \( j \)th singularity. In the type IIB language, this is equivalent to stretching an open string between the \( i \)th and \( j \)th D-brane. One can get a BPS mass \( m_i + m_j \) by integrating counterclockwise around the \( i \)th singularity, then counterclockwise around the outside of the fifth and sixth singularity, clockwise around the \( j \)th singularity and back clockwise around the fifth and sixth singularity. For IIB, this is equivalent to having an open string stretched between the \( i \)th D-brane and an orientifold plane and back again to the \( j \)th D-brane. In IIB perturbation theory, there are two orientifold planes at the same location, however, from Sen’s analysis, we see that quantum effects should split these apart. However, this has no consequences for the BPS states since we only integrated around both singularities and never each one individually.

In [10], the authors considered F-theory compactified on other orbifolds. However, unlike the \( T_4/Z_2 \) case, these orbifolds only exist for particular values of the toroidal moduli. The orbifold is an elliptical fibration over the \( \rho \) plane and the coupling is given by the modulus of the elliptic curve, hence these theories must be strongly coupled. The \( T_4/Z_3 \) orbifold should have an enhanced \( E_6 \times E_6 \times E_6 \) gauge theory, where the \( \rho \) plane has 3 groups of 8 singularities.

It is not clear how to think of this theory in terms of type IIB. But we can use the monodromies in (6.3), (6.5) and (6.4) to find integrals of \( \partial_\rho a_D \) along closed paths in the \( \rho \)
plane that give the correct masses for the BPS states for the broken $E_6$ gauge theory. The adjoint rep of $E_6$ decomposes into a $SO(10) \times U(1)$ subgroup as

$$78 = 45_0 + 1_0 + 16_{-3} + \mathbf{16}_{+3}. \quad (6.7)$$

It is clear from the monodromies in (6.7) that integrating counterclockwise around the $i$th singularity and clockwise around the $j$th singularity $i, j < 6$ gives a mass $m_i - m_j$. Likewise, integrating counterclockwise around the sixth singularity and clockwise around the $i$th singularity gives a mass that comes from the $16$ representation. But not all states in the $16$ are found this way.

To find the other masses, note that if $m_i = \lambda = 0$, then $M_i N_j N_k N_j^{-1} = S^{-1}$. Hence the product of any four of these loops is the identity. Once the masses are turned back on, the product of any four of these shifts $a_D$ by a piece linear in the $m_i$ and $\lambda$. For instance,

$$M_1 N_p N_q M_\infty N_k^{-1} M_\infty N_j^{-1} M_\infty N_l^{-1} \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a_D + m_k - m_i + \frac{1}{2} (m_1 + m_2 + m_3 - m_4 - m_5) - 3\lambda \\ a - m_j - m_p - m_q + m_5 \end{array} \right) 1 \leq i, j, k, p, q \leq 5 \quad (6.8)$$

so this then leads to BPS masses for other states in the $16$ and $\mathbf{16}$. Using (6.2), we can replace $M_\infty$ with $N_i$ and $M_j$. To get the missing states in the $45$, we use the fact that

$$M_\infty N_k^{-1} M_1 N_p N_q M_\infty N_j^{-1} M_\infty N_l^{-1} \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a_D - m_i - m_p - m_q + m_5 \\ a + m_k - m_j + \frac{1}{2} (m_1 + m_2 + m_3 - m_4 - m_5) - 3\lambda \end{array} \right) 1 \leq i, j, k, p, q \leq 5. \quad (6.9)$$

Hence, setting $i = 5$ and integrating $\partial_p a_D$ over this path gives $-m_i - m_j$. Finally, the state with mass $\sum m_i - 3\lambda$ can be obtained by combining the loops in (6.8) and (6.9).

Obviously, these loops are more complicated than the loops in [9]. Moreover, in order to get the complete set of BPS states it is necessary to loop around the seventh and eighth singularities individually. This suggests that strong coupling monodromies are important in finding the masses for the vector bosons. Hopefully, this analysis will lead to a better understanding of the type IIB string dynamics.

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