Messages for QCD from the Superworld$^*)$

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Abstract

Recent discoveries in supersymmetric gauge theories have significant implications for our understanding of QCD and of field theory in general. The phases of $\mathcal{N} = 1$ supersymmetric QCD (SQCD) are discussed, and the possibility of similar phases in non-supersymmetric QCD is emphasized. It is described how duality in SQCD links many previously known duality transformations that were thought to be distinct, including Olive-Montonen duality of $\mathcal{N} = 4$ supersymmetric gauge theory and quark-hadron duality in (S)QCD. A link between Olive-Montonen duality and the confining strings of (S)QCD is explained, in which a picture of confinement via non-abelian monopole condensation — a generalized dual Meissner effect — emerges explicitly. In this picture, unlike previous ones, the confining flux tubes carry the correct $\mathbb{Z}_N$ discrete charges. A number of studies of these subjects, which could be carried out using lattice gauge theory, are proposed.

$^*)$ Talk given at YKIS’97, Kyoto, Japan.
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§1. Introduction

In this talk, I would like to describe some of the important lessons that $\mathcal{N} = 1$ supersymmetric field theories teach us about gauge theories in general. While direct applications to QCD are few, there are nonetheless important qualitative insights which can be gained. In this talk I will mention three. The first involves our new understanding of the complexity of the phase structure of $\mathcal{N} = 1$ supersymmetric QCD (SQCD), which raises the question, “what is the phase structure of QCD, and are there lattice or analytic approaches that will efficiently allow us to find out?” A second topic involves the conceptual unification of many superficially different duality relations, such as Olive-Montonen duality of $\mathcal{N} = 4$ supersymmetric gauge theories, the dual Meissner effect as a model of confinement, and the relation between a theory of quarks and gluons and its description using a chiral Lagrangian. Although these naively seem unrelated, they and others are all special cases of $\mathcal{N} = 1$ duality. I will then present some new results linking the QCD string and the Olive-Montonen duality of $\mathcal{N} = 4$ supersymmetric QCD. I will show how solitonic QCD strings arise in this context without any abelian projection, in the context of a generalized, non-abelian dual Meissner effect. I will discuss some possible problems with approaches based on the abelian projection. I’ll also have a few things to say about QCD string tensions, as calculated in field theory and in M theory, and will emphasize that lattice QCD theorists should compute ratios of string tensions in $SU(N)$ for $N \geq 4$. As an aside, I’ll mention some related work in which the first example of the non-abelian dual Meissner effect was presented. Finally, I’ll describe what I think are the key questions raised by the superworld for QCD theorists.

§2. Phases of SQCD

Following the work of Seiberg\textsuperscript{1) } and others,\textsuperscript{2) - 6) } we now know that SQCD has many phases. For my purposes I will limit myself to five which both are common\textsuperscript{1) } and are likely to be seen in non-supersymmetric theories. (I will ignore the Higgs phase, which is of course well-understood.) These are the following:

1) The Free Electric Phase: the theory has so much matter that its beta function is positive and it is free in the infrared.

2) The Non-Abelian Coulomb Phase: the theory is asymptotically free, but its beta function hits a zero at a finite value of the gauge coupling. The low-energy theory is an infrared fixed point, an interacting conformal field theory, which has no particle states. Its operators have non-trivial anomalous dimensions, some of which can be exactly computed.

3) The Free Magnetic Phase: the original theory is asymptotically free, and its coupling
grows monotonically. The low-energy physics can be described in terms of composite fields, of spin 0, $\frac{1}{2}$, and 1, many of which are non-polynomial — indeed, non-local — in terms of the original fields. The theory describing these composites is again a gauge theory, with its own conceptually separate gauge symmetry! The dual theory has a positive beta function, so it is free in the infrared and is a good perturbative description of the low-energy physics.

4) Confinement Without Chiral Symmetry Breaking: in this case the low energy theory is an infrared-free linear sigma model, whose composites, of spin 0 and $\frac{1}{2}$, are polynomial in the original fields.

5) Confinement With Chiral Symmetry Breaking: similar to the previous, except that the low-energy theory is a non-linear sigma model, as in QCD.

Some comments: phases (1) and (5) are well-known. Phase (4) has been long suspected and debated as a possibility in some gauge theories. Phase (2) can be seen in perturbation theory, both in QCD and SQCD, when the number of colors and flavors is large and the one-loop beta function is extremely small by comparison; such fixed points are often called Banks-Zaks fixed points, though they were discussed by earlier authors as well. What is new here is that this phase exists far beyond perturbation theory into an unexpectedly wide range of theories, as we will see in a moment. Phase (3) is entirely new and previously unsuspected; it is perhaps the most spectacular of the recent results.

It is also worth noting that while calculational techniques exist for studying the infrared physics in most of these phases, the non-abelian coulomb phase requires an understanding of four-dimensional superconformal field theory. Techniques in this subject are still being explored and there is much left to be learned.

As an example, consider $SU(N_c)$ SQCD with $N_f$ flavors in the fundamental and anti-fundamental representations. This theory is in the same universality class as (and is said to be “dual” to) $SU(N_f - N_c)$ with $N_f$ flavors and with $N_f^2$ gauge singlets which interact with the flavors. This dual theory serves as the low-energy gauge theory in the free magnetic phase. The five phases appear for the following ranges of $N_c$: phase (1), $N_f \geq 3N_c$; phase (2), $3N_c > N_f > \frac{3}{2}N_c$; phase (3), $\frac{3}{2}N_c \geq N_f > N_c + 1$; phase (4), $N_f = N_c + 1$; phase (5), $N_f = N_c$, 0. For $N_c > N_f > 0$ the theory has no stable vacuum. (Note that supersymmetric theories generally have continuous sets of inequivalent vacua; for each theory I have only listed the phase of the vacuum with the largest unbroken global symmetry.)

As another example, consider $SO(N_c)$ with $N_f$ flavors in the vector representation, whose dual is $SO(N_f - N_c + 4)$. In this case we have phase (1), $N_f \geq 3(N_c - 2)$; phase (2), $3(N_c - 2) > N_f > \frac{3}{2}(N_c - 2)$; phase (3), $\frac{3}{2}(N_c - 2) \geq N_f > N_c - 3$; phase (4), $N_f = N_c - 3$, $N_c - 4$. As before $N_f < N_c - 4$ has no vacuum, except $N_f = 0$ which has vacua with both phase (4) and (5).
Note that the word “confinement” has been used loosely here. The cases $SU(N_c)$ with $N_f = N_c + 1$, $N_c$ are examples of “complementarity”, where the confining and Higgs phases are actually two regions in a single phase.⁹ There is no Wilson loop with an area law; all sources can be screened by the massless fields, and so no confining string can form. By contrast, a spinor-valued Wilson loop detects the confinement in $SO$ theories with vectors, while Wilson loops in, for example, the $N_c$ representation, can detect the confinement in the pure $SU(N_c)$ and $SO(N_c)$ SQCD.

Aside from these two examples, many others are known, with qualitatively similar phase diagrams. Various new phenomena have been uncovered. But most $\mathcal{N} = 1$ supersymmetric field theories are not understood, and much work remains to be done.

The most remarkable aspect of these phase diagrams is that they show that the phase of a theory depends on (a) its gauge group $G$, (b) its massless matter representations $R$, and although not shown here, (c) its interactions $\mathcal{L}_{\text{int}}$, renormalizable and non-renormalizable. The dependence on $R$ goes far beyond the mere contribution of the matter to the beta function; the matter fields are clearly more than spectators to the gauge dynamics. (A quenched approximation could not reproduce this phase structure.) The dependence on non-renormalizable interactions is familiar from technicolor theories: a higher-dimension operator, though irrelevant in the ultraviolet, may become relevant in the infrared and control the physics of the low-energy theory.

Given this is true for SQCD, why should it not be true for non-supersymmetric gauge theories? Phases (1), (2) and (5) certainly arise. It would be remarkable indeed if the ubiquity of phase (4) and of the existence of phase (3) could be demonstrated. There might also be as yet unknown phases that do not occur in supersymmetric theories.

More specifically, we should seek to answer the following question: what is the phase of QCD as a function of $G$, $R$ and $\mathcal{L}_{\text{int}}$? Unfortunately the answer cannot be learned from the supersymmetric theories: the process of breaking supersymmetry leads to ambiguities in the duality transformations. We therefore need new tools, both analytical and numerical. This is clearly an area for lattice gauge theory, but it is not easy to study the renormalization group flow over large regions of energy using the lattice. Additional analytic work is needed to make this more tractable. I hope some readers will be motivated to consider this problem!

It should be stressed that this is not merely an academic question. It is possible that the correct theory of electroweak supersymmetry breaking (or of fermion masses, etc.) has not yet been written down. Perhaps a modified form of technicolor or something even more exotic will appear in the detectors of the Large Hadron Collider, in a form that we will be unable to understand unless the questions raised above are addressed in the coming years.
§3. Unification of Dualities in $\mathcal{N} = 1$ Supersymmetry

Let me begin by listing some duality transformations.

Electric-Magnetic (EM): this is the usual duality transformation of the Maxwell equations without matter, which can be trivially extended to $\mathcal{N} = 4, 2, 1$ SQED. The electric and magnetic gauge groups are $U(1)_e$ and $U(1)_m$ (note these two symmetry groups are completely distinct transformations on the non-locally related electric and magnetic gauge potentials.) The electric and magnetic couplings are $e$ and $4\pi/e$. (The last relation is modified for non-zero theta angle.)

Dual Meissner (DM): for abelian gauge theory, or for a non-abelian gauge theory which breaks to an abelian subgroup. The theory has magnetic monopoles, which are described by a magnetic abelian gauge theory as ordinary charged particles. The monopoles condense, breaking the magnetic gauge symmetry, screening magnetic flux and confining the electric flux of the original theory.

Olive-Montonen (OM): the EM case for $\mathcal{N} = 4$ supersymmetry, extended to a non-abelian gauge group $G_e$. The magnetic variables also are an $\mathcal{N} = 4$ supersymmetric gauge theory and have a gauge group $G_m$. The theory is conformal and has a non-running coupling constant $g$ in the electric theory and $4\pi/g$ in the magnetic theory. (The last relation is modified for non-zero theta angle.)

Generalized Dual Meissner (GDM): similar to the DM case, but where both the electric and magnetic gauge groups $G_e$ and $G_m$ are non-abelian. Condensing magnetically charged monopoles again break $G_m$, screen magnetic flux and confine electric flux.

Seiberg-Witten pure (SWp): for pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. The electric theory has gauge group $G$ of rank $r$, whose maximal abelian subgroup is $[U(1)^r]_e$. EM duality applies to each $U(1)$; the magnetic theory has gauge group $[U(1)^r]_m$.

Seiberg-Witten finite (SWf): for a finite $\mathcal{N} = 2$ supersymmetric gauge theory with matter. Very similar to OM above, but in general the relation between $G_e$ and $G_m$ differs from the OM case.

QCD and the Sigma Model (QCDσ): here a strongly-coupled, confining QCD or $\mathcal{N} = 1$ SQCD theory is described in terms of gauge singlets, using a linear or non-linear sigma model. This is not always considered a “duality”, but as we will see, it should be.

The main point of this section is to emphasize that all of these dualities are linked together by results in $\mathcal{N} = 1$ supersymmetry. This can be easily seen using the duality of $\mathcal{N} = 1$ $SO(N)$ gauge theories with $N_f$ fields in the vector representation; as mentioned in Sec. 2, such theories are dual to $SO(N_f - N + 4)$ with $N_f$ vectors and $N_f(N_f + 1)/2$ gauge
singlets.

Consider first $SO(2)$ without matter ($N_f = 0$). The dual theory is again $SO(2)$ without matter — EM duality — which justifies referring to the dual theory as “magnetic”.

Next, consider $SO(3)$ with one triplet; its magnetic dual is $SO(2)$ with fields of charge $1, 0, -1$ coupled together. These theories are both $\mathcal{N} = 2$ supersymmetric; the electric theory is the pure $\mathcal{N} = 2$ $SU(2)$ theory studied by Seiberg and Witten, and the magnetic dual is the theory of the light monopole which serves as its low-energy description.\textsuperscript{15} Since a mass for the triplet leads to confinement of $SU(2)$ via abelian monopole condensation,\textsuperscript{15} this example gives both SWp and DM duality.

Finally, take $SO(3)$ with three triplets, whose dual is $SO(4) \approx SU(2) \times SU(2)$, with three fields in the $4$ representation and six singlets. If the triplets are massive, the quartets of the dual theory condense, $SO(4)$ is broken, and confinement of $SO(3)$ occurs\textsuperscript{1} — GDM duality. The low-energy description below the confinement scale is given by this broken $SO(4)$ theory\textsuperscript{1}, a non-linear sigma model — QCD$\sigma$ duality. And if instead the triplets are massless and are given the renormalizable interactions which make the theory $\mathcal{N} = 4$ supersymmetric, the dual theory is consistent with OM duality:\textsuperscript{10,16} one of the $SU(2)$ subgroups of $SO(4)$ confines, leaving a single $SU(2)$ factor with three triplets coupled by the required $\mathcal{N} = 4$ supersymmetric interactions.\textsuperscript{2,3}

\[\text{Fig. 1. Interrelation between types of duality.}\]

Thus, $\mathcal{N} = 1$ duality links all of the four-dimensional dualities on the list together, showing they are manifestations of a single phenomenon. Figure 1 shows a cartoon of the connections between these theories. I have omitted, both from my list and from this figure, other types of field theory duality that were not known until recently. Also not shown
are the relations between these dualities and those of the-theory-formerly-known-as-string theory, sometimes termed M theory. The situation brings to mind the old story of the blind men and the elephant, in which each man feels one part of the elephant — the ear, or the trunk, or the tail — and erroneously assigns it a special significance, not realizing that the apparently different parts are connected to a larger whole. Having discovered the existence of a larger whole, we find ourselves compelled to explain duality in a unified way.

I want to emphasize to those who are not familiar with this subject that this picture, while not proven, is by no means speculative. The circumstantial evidence in its favor — a vast number of consistency conditions — is completely overwhelming; one could easily give twenty lectures on this subject. A proof is still badly needed, however; the underlying meaning of duality remains mysterious, and no field theoretic formulation is known which would make it self-evident (except in the EM case, of course.)

§4. Olive-Montonen Duality and the QCD String

I now want to turn to the main subject of my talk, which involves the relation between Olive-Montonen duality and QCD strings. I will present a variation on the Seiberg-Witten picture of confinement in \( SU(N) \) SQCD.\(^{15} \) In this variation, the flux tubes of SQCD, which appear in a straightforward way, carry the appropriate discrete charges. Previous approaches, where the charges did not work out correctly, exhibited a spectrum that was contrary to expectations.\(^{17} \) I will explain these issues below.

4.1. Electric Sources and Fluxes

I begin with a review of basic facts about gauge theories. Consider a pure gauge theory with gauge group \( G \). Suppose we have a source — an infinitely massive, static, electrically charged particle — in a representation \( R \) of \( G \). If we surround the source with a large sphere, what characterizes the flux passing through the sphere? If \( G \) is \( U(1) \), the flux measures the electric charge directly. However, in non-abelian gauge theories the gauge bosons carry charge. Since there may be a number (varying over time) of gauge bosons inside the sphere, the representation under which the charged objects in the sphere transform is not an invariant. But, by definition, the gauge bosons are neutral under the discrete group \( C_G \), the center of \( G \). It follows that the charge of \( R \) under the center is a conserved quantity, and that the total flux exiting the sphere carries a conserved quantum number under \( C_G \).

For example, in \( SU(N) \) the center is a \( \mathbb{Z}_N \) group with the lovely name of “N-ality”. A tensor \( T^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q} \) has N-ality \( p - q \mod N \); in particular, the \( k \)-index antisymmetric tensors carry N-ality \( k \).
Result: Electric sources and fluxes in pure gauge theories carry a conserved $C_G$ quantum number. If the gauge group confines, then the confining electric flux tubes will also carry this quantum number.

If the theory also contains light matter charged under $C_G$ but neutral under a subgroup $C_m$ of $C_G$, then the above statements are still true with $C_G$ replaced with $C_m$. For example, if we take $SU(N)$ with massless fields in the $N$ representation, then $C_m$ is just the identity, reflecting the fact that all sources can be screened and all flux tubes break. If we take $SO(10)$ with fields in the $10$, then the center $Z_4$ is replaced with spinor-number $Z_2$. Sources in the $10$ will be screened and have no flux tube between them, while sources in the $16$ or $1\overline{6}$ will be confined by a single type of flux tube.

4.2. Magnetic Sources and Fluxes

Before discussing the magnetic case, I review some basic topology. The $p$-th homotopy group of a manifold $M$, $\pi_k(M)$, is the group of maps from the $p$-sphere into $M$, where we identify maps as equivalent if they are homotopic in $M$. All we will need for present purposes are the following examples. Suppose a Lie group $G$ has rank $r$, so that its maximal abelian subgroup is $U(1)^r$; then

$$\pi_2[G] = 1 \Rightarrow \pi_2[G/U(1)^r] = \pi_1[U(1)^r] = Z \times Z \times \cdots \times Z \equiv [Z]^r . \quad (4.1)$$

Similarly,

$$\pi_1[G] = 1 \Rightarrow \pi_1[G/C_G] = \pi_0[C_G] = C_G . \quad (4.2)$$

We will need to investigate both monopole solitons and string solitons below. The classic monopole soliton is that of ‘t Hooft and of Polyakov, which arises in $SU(2)$ broken to $U(1)$; in this case the important topological relation is $\pi_2[SU(2)/U(1)] = \pi_1[U(1)] = Z$. This leads to a set of monopole solutions carrying integer charge. Note that the stability of, for example, the charge-two monopole solution against decay to charge-one monopoles is determined not by topology but by dynamics. The situation is similar for the Nielsen-Olesen magnetic flux tube of the abelian Higgs model; here the relevant topological relation is $\pi_1[U(1)] = Z$. This again leads to solutions with an integer charge, whose stability against decay to minimally charged vortices is determined dynamically.

More generally, if we have a simply connected gauge group $G_0$ which breaks to a group $G$ at a scale $v$, there will be monopoles carrying a quantum number in $\pi_2[G_0/G]$, of mass [radius] proportional to $v [1/v]$. Now imagine that we take $v \to \infty$. In this limit the gauge group $G_0$ disappears from the system. The monopoles become pointlike and infinitely massive; their only non-pointlike feature is their Dirac string, which carries a quantum number in $\pi_1[G]$. 
In short, the solitonic monopoles become fundamental Dirac monopoles in this limit. Note that since \( \pi_2[G_0/G] = \pi_1[G] \), the charges carried by the solitonic monopoles and their Dirac monopole remnants are the same. Since the Dirac monopoles heavy, we may use them as magnetic sources.

Let’s further suppose that the gauge group \( G \) is broken completely at some scale \( v' \). In this case no Dirac strings can exist in the low-energy theory, and so the monopoles allowed previously have seemingly vanished. However, solitonic magnetic flux tubes, carrying charges under \( \pi_1[G] \), will be generated; they will have tension [radius] of order \( v'^2 [1/v'] \). Their \( \pi_1[G] \) quantum numbers are precisely the ones they need to confine the \( \pi_1[G] \)-charged Dirac monopole sources of the high-energy theory. Thus, when \( G \) is completely broken, the Dirac monopoles disappear because they are confined by flux tubes.

**Result:** Magnetic sources and fluxes in pure gauge theories carry a conserved \( \pi_1[G] \) quantum number. If the gauge group is completely broken, then the confining magnetic flux tubes will also carry this quantum number.

### 4.3. \( \mathcal{N} = 4 \) Supersymmetric Gauge Theory

The next ingredient in this *okonomiyaki* is \( \mathcal{N} = 4 \) supersymmetric gauge theory, consisting of one gauge field, four Majorana fermions, and six real scalars, all in the adjoint representation. It is useful to combine these using the language of \( \mathcal{N} = 1 \) supersymmetry, in which case we have one vector multiplet (the gauge boson \( A_\mu \) and one Majorana fermion \( \lambda \)) and three chiral multiplets (each with a Weyl fermion \( \psi^s \) and a complex scalar \( \Phi^s, s = 1, 2, 3 \).

These fields have the usual gauged kinetic terms, along with additional interactions between the scalars and fermions. The scalars, in particular, have potential energy

\[
V(\Phi^s) = \sum_{a=1}^{\dim G} |D_a^2| + \sum_{s=1}^{3} |F_s|^2
\]

where

\[
D_a = \left( \sum_{s=1}^{3} [\Phi^s, \Phi^s] \right)_a
\]

(here \( a \) is an index in the adjoint of \( G \)) and

\[
F_s = \epsilon_{stu} [\Phi^t, \Phi^u].
\]

Supersymmetry requires that \( \langle V(\Phi^s) \rangle = 0 \), and so all \( D_a \) and \( F_s \) must vanish separately. The solution to these requirements is that a single linear combination \( \hat{\Phi} \) of the \( \Phi^s \) may have non-vanishing expectation value, with the orthogonal linear combinations vanishing. By
global rotations on the index $s$ we may set \( \hat{\Phi} = \Phi^3 \). By gauge rotations we may make \( \Phi^3 \) lie in the Cartan subalgebra of the group; we may represent it as a diagonal matrix
\[
\langle \Phi^3 \rangle = \text{diag}(v_1, v_2, \ldots)
\] (4.6)
which (if the \( v_i \) are all distinct) breaks \( G \) to \( U(1)^r \). Since \( \pi_2[G/U(1)^r] = [\mathbb{Z}]^r \) [see Eq. (4.1)] the theory has monopoles carrying \( r \) integer charges under \( U(1)^r \). Quantum mechanically, the theory has both monopoles and dyons, carrying \( r \) electric and \( r \) magnetic charges \((n_e, n_m)\).

The space of vacua written in Eq. (4.6) is not altered by quantum mechanics. In the generic \( U(1)^r \) vacuum, each \( U(1) \) has EM duality. In a vacuum where some of the \( v_i \) are equal, the gauge group is broken to a non-abelian subgroup \( \hat{G} \) times a product of \( U(1) \) factors; the low-energy limit of the non-abelian part is an interacting conformal field theory. The \( U(1) \) factors have EM duality, while the \( \hat{G} \) factor has its non-abelian generalization, OM duality.\(^{10)\text{-}12)\)

4.4. Olive-Montonen Duality

The \( \mathcal{N} = 4 \) theory has a set of alternate descriptions, generated by changes of variables (whose explicit form remains a mystery) of the form
\[
T: \quad \tau \rightarrow \tau + 1 (\theta \rightarrow \theta + 2\pi) ; \quad n_e \rightarrow n_e + n_m, \quad n_m \rightarrow n_m ; \quad G \rightarrow G ; \quad (4.7)
\]
and
\[
S: \quad \tau \rightarrow -\frac{1}{\tau} (g \rightarrow \frac{4\pi}{g} \text{ if } \theta = 0) ; \quad n_e \leftrightarrow n_m ; \quad G \rightarrow \tilde{G} . \quad (4.8)
\]
Together \( S \) and \( T \) generate the group \( SL(2, \mathbb{Z}) \), which takes \( \tau \rightarrow (a\tau + b)/(c\tau + d) \) for integers \( a, b, c, d \) satisfying \( ad - bc = 1 \). Note that \( T \) is nothing but a rotation of the theta angle by \( 2\pi \); it does not change the gauge group or the definition of electrically charged particles, shifting only the electric charges of magnetically charged particles.\(^{18)\) By contrast, \( S \) exchanges electric and magnetic charge, weak and strong coupling (if \( \theta = 0 \)),\(^{10)\) and changes the gauge group\(^{11)\text{-}12)\) from \( G \) to its dual group \( \tilde{G} \), as defined below.

The group \( G \) has a root lattice \( \Lambda_G \) which lies in an \( r = \text{rank}(G) \) dimensional vector space. This lattice has a corresponding dual lattice \( (\Lambda_G)^* \). It is a theorem that there exists a Lie group whose root lattice \( \Lambda_{\tilde{G}} \) equals \( (\Lambda_G)^* \).\(^{11)\) Here are some examples:
\[
SU(N) \leftrightarrow SU(N)/\mathbb{Z}_N ; \quad SO(2N + 1) \leftrightarrow USp(2N) ;
SO(2N) \leftrightarrow SO(2N) ; \quad Spin(2N) \leftrightarrow SO(2N)/\mathbb{Z}_2 . \quad (4.9)
\]
Notice that this set of relationships depends on the global structure of the group, not just its Lie algebra; \( SO(3) \) (which does not have spin-1/2 representations) is dual to \( USp(2) \approx SU(2) \).
(which does have spin-1/2 representations.) These details are essential in that they affect the topology of the group, on which OM duality depends.

In particular, there are two topological relations which are of great importance to OM duality. The first is relevant in the generic vacuum, in which $G$ is broken to $U(1)^r$. The electric charges under $U(1)^r$ of the massive electrically charged particles (spin $0, \frac{1}{2}, 1$) lie on the lattice $\Lambda_G$. The massive magnetic monopoles (also of spin $0, \frac{1}{2}, 1$) have magnetic charges under $U(1)^r$ which lie on the dual lattice $(\Lambda_G)^*$. Clearly, for the $S$ transformation, which exchanges the electrically and magnetically charged fields and the groups $G$ and $\tilde{G}$, to be consistent, it is essential that $\Lambda_{\tilde{G}} = (\Lambda_G)^*$ — which, fortunately, is true.

The second topological relation is the one we will use below. We have seen that the allowed electric and magnetic sources for a gauge theory with adjoint matter (such as $\mathcal{N} = 4$) are characterized by quantum numbers in $C_G$ and $\pi_1(G)$ respectively. Consistency of the $S$ transformation would not be possible were these two groups not exchanged under its action. Fortunately, it is a theorem of group theory that

\[ \pi_1(G) = C_G ; \pi_1(\tilde{G}) = C_{\tilde{G}}. \quad (4.10) \]

For example, $\pi_1[SU(N)] = C_{SU(N)/\mathbb{Z}_N} = 1$ while $C_{SU(N)} = \pi_1[SU(N)/\mathbb{Z}_N] = \mathbb{Z}_N$.

**Result**: As a consequence of Eq. (4.10) and the results of sections 4.1 and 4.2, the allowed magnetic sources of $G$ are the same as the allowed electric sources for $\tilde{G}$, and vice versa.

4.5. **Breaking $\mathcal{N} = 4$ to $\mathcal{N} = 1$**

Now, we want to break $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 1$. Pure $\mathcal{N} = 1$ SQCD, like pure non-supersymmetric QCD, is a confining theory, and should contain confining flux tubes. The addition of massive matter in the adjoint representation does not change this; heavy particles would only obstruct confinement by breaking flux tubes, which adjoint matter cannot do. We therefore expect that broken $\mathcal{N} = 4$ gauge theory, which is $\mathcal{N} = 1$ SQCD plus three massive chiral fields in the adjoint representation, should be in the same universality class as pure SQCD: both should confine, and both should have flux tubes carrying a $C_G$ quantum number, as discussed in section 4.1.

We may break the $\mathcal{N} = 4$ symmetry by adding masses $m_s$ for the fields $\Phi^s$; the $F_s$ functions of (4.5) become

\[ F_s = \epsilon_{stu}[\Phi^t, \Phi^u] + m_s \Phi^s, \quad (4.11) \]

so that $F_s = 0$ implies $\epsilon_{stu}[\Phi^t, \Phi^u] = -m_s \Phi^s$. Up to normalization these are the commutation relations for an $SU(2)$ algebra, which I will call $SU(2)_{aux}$. If we take $m_1 = m_2 = m$ and $m_3 = \mu$ we obtain

\[ \Phi^1 = -i\sqrt{\mu m}J_x ; \Phi^2 = -i\sqrt{\mu m}J_y ; \Phi^3 = -imJ_z, \quad (4.12) \]
where \( J_x, J_y, J_z \) are matrices satisfying \([J_x, J_y] = iJ_z, \text{ etc.}\) Each possible choice for the \( J \)'s gives a separate, isolated vacuum.\(^{19}\)

How does this work, explicitly, in \( SU(N) \)? We can write the \( \Phi^a \) as \( N \times N \) traceless matrices, so the \( J_i \) should be an \( N \)-dimensional (generally reducible and possibly trivial) representation of \( SU(2)_{\text{aux}} \).\(^{19,13}\) The trivial choice corresponds to \( J_i = 0 \); clearly if \( \Phi^a = 0 \) the \( SU(2)_{\text{aux}} \) commutation relations are satisfied. We will call the corresponding vacuum the “unbroken” vacuum, since the \( SU(N) \) gauge group is preserved. Another natural choice is to take the \( J_i \) in the irreducible spin-\( \frac{N-1}{2} \) representation of \( SU(2)_{\text{aux}} \). In this case \( SU(N) \) is completely broken; we will call this the “Higgs vacuum”. We may also choose the \( J_i \) in a reducible representation

\[
J_i = \begin{bmatrix}
\sigma_i & 0 \\
- & -
\end{bmatrix};
\]

(4.13)

here the \( \sigma_i \) are the Pauli matrices. In this case \( SU(N) \) is partly broken. There are many vacua like this last one, but they will play no role in the physics below; we will only need the unbroken vacuum and the Higgs vacuum.

**Result:** The classical analysis of this \( N = 1 \) supersymmetric \( SU(N) \) gauge theory with massive adjoint fields shows that it has isolated supersymmetric vacua scattered about, with the unbroken (U) vacuum at the origin of field space and the Higgs vacuum (H) at large \( \Phi^a \) expectation values [of order \( m, \sqrt{m\mu} \), see Eq. (4.12)].\(^{19,13}\)

4.6. **OM Duality and the Yang-Mills String**

The above picture is modified by quantum mechanics. In each vacuum, strong dynamics causes confinement to occur in the unbroken non-abelian subgroup, modifying the low-energy dynamics and generally increasing the number of discrete vacua. In the H vacuum, the gauge group is completely broken and no non-trivial low-energy dynamics takes place; it remains a single vacuum. The U vacuum, by contrast, splits into \( N \) vacua — the well-known \( N \) vacua of SQCD\(^{20}\) — which I will call \( D_0, D_1, \cdots, D_{N-1} \). In the \( D_k \) vacuum, confinement occurs by condensation of dyons of magnetic charge 1 and electric charge \( k \).\(^{15,13,17,21}\) Since these vacua are related\(^{20}\) by rotations of \( \theta \) by multiples of \( 2\pi \), I will focus on just one of them. It is convenient to study the \( D_0 \) vacuum (which I now rename the M vacuum) in which electric charge is confined by magnetic monopole condensation.

Now, what is the action of OM duality on this arrangement? The vacua are physical states, and cannot be altered by a mere change of variables; however, the *description* of each vacuum will change. Specifically, when \( \theta = 0 \), the \( S \) transformation, which inverts the coupling constant and exchanges electric and magnetic charge, exchanges the H vacuum
of $SU(N)$ for the M vacuum of $SU(N)/\mathbb{Z}_N$ and vice versa. To say it another way, the confining M vacuum of $SU(N)$ can be equally described as the H vacuum of $SU(N)/\mathbb{Z}_N$, in which the monopoles of the $SU(N)$ description break the dual $SU(N)/\mathbb{Z}_N$ gauge group. This is the Generalized Dual Meissner effect, in which both the electric and magnetic gauge groups are non-abelian.

**Result:** OM duality exchanges the H and M vacua of $SU(N)$ broken $\mathcal{N} = 4$ with the M and H vacua of $SU(N)/\mathbb{Z}_N$ broken $\mathcal{N} = 4$. Confinement in the M vacuum of $SU(N)$ is described as the breaking of the $SU(N)/\mathbb{Z}_N$ gauge group in $SU(N)/\mathbb{Z}_N$.  

The existence of the Yang-Mills string now follows directly from topology. As we discussed in section 4.2, the complete breaking of a group $G$ leads to solitonic strings carrying magnetic flux with a quantum number in $\pi_1(G)$. In this case, the breaking of $SU(N)/\mathbb{Z}_N$ in its H vacuum (the confining M vacuum of $SU(N)$) gives rise to strings with a $\mathbb{Z}_N$ quantum number. But magnetic flux tubes of $SU(N)/\mathbb{Z}_N$ are, by OM duality, electric flux tubes of $SU(N)$ — and so the confining strings of the $SU(N)$ theory’s M vacuum, the confining theory which is in the same universality class as $SU(N)$ SQCD, carry a $\mathbb{Z}_N$ quantum number. This is in accord with the considerations of section 4.1. The relation Eq. (4.10) is responsible for this agreement of the $\mathbb{Z}_N$ charges, and presumably assures a similar agreement for all groups.

**Result:** OM duality gives a picture for confinement in $SU(N)$ SQCD — it occurs via non-abelian dual monopole condensation, and leads to confining strings with a $\mathbb{Z}_N$ quantum number.

A cautionary remark is in order. The description of confinement via dual monopole condensation is not fully reliable, as it is only appropriate if the $SU(N)/\mathbb{Z}_N$ theory is weakly coupled. In fact, we want the $SU(N)$ theory to be weakly coupled in the ultraviolet, in analogy with QCD. The $S$ transformation implies that the $SU(N)/\mathbb{Z}_N$ description should be strongly coupled in the ultraviolet. However, the existence of a soliton carrying a stable topological charge is more reliable, especially since there are no other objects carrying that charge into which these string solitons can decay. Having constructed the solitonic strings semiclassically in some regime, we expect that they survive into other regimes in which semiclassical analysis would fail. (A gap in the argument: could the strings grow large and have zero string tension in the SQCD limit?) In short, while the condensing monopole description appropriate to broken $\mathcal{N} = 4$ SQCD may not be valid for pure $\mathcal{N} = 1$ SQCD, it does demonstrate the presence of confining strings in the latter. Whether anything quantitative can be said about these strings is another matter, to be addressed below.

Should we expect this picture to survive to the non-supersymmetric case? Take the theory with $\mathcal{N} = 4$ supersymmetry broken to $\mathcal{N} = 1$, and further break $\mathcal{N} = 1$ supersymmetry by adding an $SU(N)$ gaugino mass $m_\lambda \ll m, \mu$. We cannot be sure of the effect on the dual
$SU(N)/\mathbb{Z}_N$ theory; duality does not tell us enough. However, we know that the theory has a gap, so this supersymmetry-breaking can only change some properties of the massive fields, without altering the fact that $SU(N)/\mathbb{Z}_N$ is completely broken. The strings, whose existence depends only on this breaking, thus survive for small $m_\lambda$. To reach pure QCD requires taking $m, \mu, m_\lambda$ all to infinity. It seems probable, given what we know of QCD physics, that the strings undergo no transition as these masses are varied. In particular, there is unlikely to be any phase transition for the strings between pure SQCD and pure QCD; this conjecture can and should be tested on the lattice.

**Result:** If the strings of SQCD and of QCD are continuously related, without a transition as a function of the gaugino mass, then the arguments given above for SQCD extend to QCD, establishing a direct link between OM duality of $\mathcal{N} = 4$ gauge theory and the confining $\mathbb{Z}_N$-strings of pure QCD.

4.7. **Confinement According to Seiberg and Witten**

How does this picture of confinement differ from that of Seiberg and Witten? Where and why might it be preferable?

Seiberg and Witten studied pure $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory. They showed that the infrared quantum mechanical theory could be understood as a $U(1)$ theory coupled to a magnetic monopole, and that, when $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$, the monopole condenses, confining the $SU(2)$ degrees of freedom. The picture generalizes to $SU(N)$, where the infrared physics involves $U(1)^{N-1}$ coupled to $N-1$ monopoles, whose condensation drives confinement. This was studied in detail by Douglas and Shenker.

This physics is contained in a particular regime of the broken $\mathcal{N} = 4$ supersymmetry gauge theory discussed above. If we take $\mu = 0$, then the theory is $\mathcal{N} = 2$ supersymmetric, and, as seen from Eq. (4.12), the H vacuum of $SU(N)/\mathbb{Z}_N$ has its gauge group broken only to $U(1)^{N-1}$. Now take $m$ exponentially large and the coupling at the scale $m$ small, so that the strong coupling scale $\Lambda$ of the low-energy theory is finite, and let $\mu$ be non-zero but small compared to $\Lambda$. The low-energy confining vacua of the $SU(N)$ theory will then be the vacua studied by Douglas and Shenker. The magnetic description of the theory (using OM duality) will have the gauge group $SU(N)/\mathbb{Z}_N$ broken at a high scale to $U(1)^{N-1}$, which in turn is broken completely at a low scale; see Eq. (4.12). The second step in this breaking leads to confinement of $SU(N)$ fields. If we take $m$ to infinity, then the $SU(N)/\mathbb{Z}_N$ gauge group disappears from the theory. The magnetic theory is merely $U(1)^{N-1}$ broken to nothing, as in Douglas and Shenker. (Note that I am cheating a bit here, as the abelian theory requires a cutoff; I’ll fix this below.)

Since the magnetic theory is $U(1)^{N-1}$, dual to the maximal abelian subgroup of $SU(N)$,
the pure $\mathcal{N} = 2$ theory exhibits a dynamical form of abelian projection. The monopoles, whose condensation drives confinement when $\mathcal{N} = 2$ supersymmetry is weakly broken, are purely abelian. Given the number of talks at this conference on abelian projection as a mechanism for explaining confinement, why should this disturb us?

The problem lies with the quantum numbers of the strings. The $U(1)^{N-1}$ magnetic theory consists of $N - 1$ copies of the abelian Higgs model, each of which has a Nielsen-Olesen solitonic flux tube. These strings carry quantum numbers in the group $\pi_1[U(1)^{N-1}] = [\mathbb{Z}]^{N-1}$, not in $\mathbb{Z}_N$! That is, each of the $N - 1$ Nielsen-Olesen strings carries its own conserved integer charge. These strings cannot lead to a good model for the dynamics of $SU(N)$ SQCD or QCD, which on general grounds must have $\mathbb{Z}_N$-carrying strings.

Is this problem serious? At first glance, the little cheat that I made just a moment ago rescues the abelian projection. The magnetic $U(1)^{N-1}$ theory has a cutoff at the scale $\Lambda$, where its coupling becomes large. At that scale, there are massive electrically-charged gauge bosons of $SU(N)$. Pair production of these particles cause certain configurations of parallel strings, which would naively be stable according to the reasoning of the previous paragraph, to break. The charges of these gauge particles are precisely such that they reduce the conserved symmetry from $[\mathbb{Z}]^{N-1}$ to $\mathbb{Z}_N$. For example, consider the case of $SU(2)$, as shown in figure 2. An isospin-$\frac{1}{2}$ quark and a corresponding antiquark will be joined by a Nielsen-Olesen string. This string is stable. However, two such quark-antiquark pairs, with parallel strings, are unstable to reconfiguring their strings via $W$ boson production. This reflects the claim above that $W$ production reduces the symmetry under which the $SU(2)$ strings transform from $\mathbb{Z}$ to $\mathbb{Z}_2$.

![Fig. 2. A pair of parallel $SU(2)$ strings can break via $W$ boson production.](image)

Topologically speaking, all seems well — the charges seem to be the expected ones — but the dynamics of the theory still poses a serious problem. Although $[\mathbb{Z}]^{N-1}$ is not exactly conserved, it is approximately conserved. To see this in the $SU(2)$ example, note that the $W$ pair-production requires an energy of order $\Lambda$. The string tension (its energy per unit length) is $T = R_{conf}^{-2} \approx \mu \Lambda$ in this theory; here $R_{conf}$ is the confinement length. In order for the string to have enough energy to break, it should have length $L$ such that $TL \sim \Lambda$. This
implies
\[ L \geq \frac{1}{\mu} \gg \frac{1}{\sqrt{\mu \Lambda}} = R_{\text{conf}}, \quad (4.14) \]
so only enormously long strings can break. Furthermore, since the strings’ energy density is very low, it takes a large fluctuation to generate W boson pairs. This in turn means that the rate for the transition in figure 2 is very slow. In short, the string pair shown in the figure is metastable for \( \mu \ll \Lambda \). The \( Z \) symmetry is still approximately conserved by the dynamics. Similar arguments apply for \( SU(N) \).

**Result:** The strings of weakly broken \( \mathcal{N} = 2 \) \( SU(N) \) SQCD carry an approximately conserved \([Z]^{N-1}\) symmetry, which contains the expected \( Z_N \) as an exactly conserved subgroup. This leads to metastable string configurations not expected in \( \mathcal{N} = 1 \) SQCD and in QCD.

The physical consequences of this approximately conserved symmetry are potentially dramatic. In SQCD, a pair of parallel strings which carry fluxes of charge \( k \) and \( p \) under \( Z_N \) should undergo a rapid transition to a string carrying charge \( k + p \mod N \). (In QCD, this implies that the string between a quark and an antiquark is the same as the string between a quark and a diquark.) But this transition is inhibited in the broken \( \mathcal{N} = 2 \) gauge theories for small \( \mu \). This implies that numerous, distinct, metastable configurations of strings may connect a quark in the \( \mathbf{N} \) representation to a corresponding antiquark. For example, for \( k = 1, \cdots N/2 \), a pair of parallel strings, with charges \( k \) and \( N - k + 1 \) respectively, carry total charge 1; they therefore may, as a pair, join a quark and antiquark. (Since a string with charge \( N \) is no string at all, the \( k = 1 \) case is the expected one.) These \( N/2 \) metastable configurations have different energies per unit length, and in principle can give physically distinct quark-antiquark meson Regge trajectories.

Indeed, in \( \mathcal{N} = 2 \) SQCD, the dynamics of the theory breaks the Weyl group, so the \( N \) colors of quark are inequivalent. As shown by Douglas and Shenker, each color of quark prefers a different choice of string pairs.\(^{17}\) This leads to their most surprising conclusion.

**Result:** In weakly broken \( \mathcal{N} = 2 \) supersymmetric \( SU(N) \) gauge theory, the quark-antiquark mesons exhibit \( N/2 \) Regge trajectories,\(^{17}\) instead of one as expected in \( \mathcal{N} = 1 \) SQCD and in QCD.

What happens to these extra trajectories as \( \mu \to \infty \) and the theory approaches pure \( \mathcal{N} = 1 \) SQCD? As can be seen from Eq. (4.14), the obstructions to \( W \) pair production go away as \( \mu \to \Lambda \). (Note the formulas which lead to (4.14) receive corrections at order \( \mu/\Lambda \).) The extra Regge trajectories become highly unstable and disappear from the spectrum.\(^{17}\) There is no sign of conflict with the usual SQCD expectations of a single Regge trajectory and of strings with a \( Z_N \) symmetry. However, the \( U(1)^{N-1} \) magnetic theory which we used to describe the weakly broken \( \mathcal{N} = 2 \) theory becomes strongly coupled in this limit, and so
one cannot study this picture quantitatively.

In summary, although broken $\mathcal{N} = 2$ supersymmetric gauge theory can be used to show that $\mathcal{N} = 1$ SQCD is a confining theory, it is not a good model for the hadrons of $\mathcal{N} = 1$ SQCD. This is a direct consequence of the dynamical abelian projection, which leads to an abelian dual description. The condensation of its abelian monopoles leads to confinement by Nielsen-Olesen strings, which carry (approximately) conserved integer charges that $(S)QCD$ strings do not possess. These charges alter the dynamics of bound states, leading to a spectrum and to hadron-hadron interactions very different from those expected in SQCD and found in QCD. By contrast, these problems are avoided in the broken $\mathcal{N} = 4$ description of confinement given in sections 4.5-4.6.

**Moral:** The use of abelian projection, and the construction of a dual abelian gauge theory, has inherent difficulties in explaining the dynamics of QCD strings. One must therefore use abelian projection with caution. It may provide good answers for a limited set of questions, but for other questions it may fail badly.

4.8. **String tensions in SU($N$)**

The discussion to this point has been entirely qualitative. Are any quantitative predictions possible?

A very useful theoretical quantity to study is the ratio of tensions of strings carrying different charge under $\mathbb{Z}_N$. A confining string of $SU(N)$ SQCD or QCD with quantum number $k$ under $\mathbb{Z}_N$ has a tension $T_k$ which depends on $k$, $N$ and the strong coupling scale $\Lambda$. On dimensional grounds $T_k = \Lambda^2 f(k, N)$. While no analytic technique is likely to allow computation of $T_1$, it is possible that $T_k/T_1$ can be predicted with some degree of accuracy. Note that charge conjugation implies $T_k = T_{N-k}$, so $SU(2)$ and $SU(3)$ have only one string tension. We must consider $SU(4)$ and higher for this to be non-trivial.

While the ratio of tensions cannot be computed in continuum SQCD or QCD, it has been calculated in theories which are believed to be in the same universality class. These theories often have multiple mass scales $\mu_i$ and thus in principle it may be that $T_k = \Lambda^2 h(\mu_i/\Lambda, k, N)$. However, in all cases studied so far, it has been found that $T_k = g(\mu_i, \Lambda) f(k, N)$, where $f$ is dimensionless, and $g$ is a dimensionful function which is independent of $k$. Note $g$ cancels out in ratios of tensions. Thus our attention may focus on the dimensionless function $f(k, N)$ as a quantity which may be compared from theory to theory.

Some previous calculations include the well-known strong-coupling expansion of QCD, which to leading order gives

$$f_{sc}(k, N) \propto k(N - k), \quad (4.15)$$
and the results of Douglas and Shenker for weakly broken $\mathcal{N} = 2$ gauge theory\textsuperscript{17)}

$$f_{DS}(k, N) \propto \sin \frac{\pi k}{N}. \quad (4.16)$$

The considerations of sections 4.5-4.6 suggest another calculation: if the string solitons in broken $\mathcal{N} = 4$ $SU(N)$ gauge theory were computed, the ratios of their tensions would be of considerable interest. At the time of writing these soliton solutions have not appeared in the literature.

There is one other technique by which string soliton tensions may be computed, using M theory! M theory is eleven-dimensional supergravity coupled to two-dimensional membranes. The theory also has five-dimensional solitons, called “five-branes”, whose worldvolume is six-dimensional. The theory on the worldvolume of the five-brane is poorly understood but is known to be a 5+1 dimensional conformal field theory.

As shown by Witten,\textsuperscript{23,22)} following work of Elitzur, Giveon and Kutasov,\textsuperscript{24)} one may consider M theory on $M^{10} \times S^1$, where $M^{10}$ is ten-dimensional Minkowski space and $S^1$ is a circle of radius $R_0$. One can construct five-branes with rather strange shapes (figure 3) whose worldvolume theory contains, at low-energy, a sector with the same massless fields and interactions as $\mathcal{N} = 2$ $SU(N)$ SQCD, or broken $\mathcal{N} = 2$ $SU(N)$ SQCD, or pure $\mathcal{N} = 1$ SQCD. (These theories also contain an infinite tower of massive particles, all neutral under $Z_N$; thus they are potentially in the same universality class as, but should not be confused with, the gauge theories we are interested in.\textsuperscript{22)} Witten showed membranes can bind to these five-branes, making objects that carry a $Z_N$ charge and look in 3+1 dimensions like strings.\textsuperscript{22)} It was then shown\textsuperscript{25)} that these strings indeed confine quarks and reproduce the results of Douglas and Shenker in the appropriate limit. However, the string tension ratios can be computed (naively) even when the $\mathcal{N} = 2$ breaking parameter $\mu$ is large. One finds that the tensions are given by\textsuperscript{25)}

$$T_k = g(\mu, \Lambda, R_0) f_{DS}(k, N) \quad (4.17)$$

where $g$ is a complicated dimensionful function which cancels out of tension ratios, and $f_{DS}$ is as in Eq. (4.16).

Thus, M theory suggests that the tensions ratios satisfy (4.16) for large as well as small breaking of $\mathcal{N} = 2$ supersymmetry. One must be careful with this result, however. First, as mentioned above, this result applies for the M theory version of SQCD, which has extra massive states that normal SQCD does not have.\textsuperscript{22)} Second, no non-renormalization theorem protects the result (4.17) when $\mu$ is large. The overall coefficient function $g$ is certainly renormalized. The question is whether $f$ is strongly renormalized or not. Only a lattice computation will resolve this issue.
What about non-supersymmetric QCD? A very naive computation in M theory suggests that $T_k \propto f_{DS}(k, N)$ even in this case! However, corrections from various sources are expected, and the accuracy of this prediction is in doubt.\textsuperscript{25, 27} In summary, we so far have results on confining strings from the strong coupling expansion on the lattice (which only exists for non-supersymmetric theories), from broken $\mathcal{N} = 2$ supersymmetric gauge theory in the continuum, and from the M theory versions of QCD and SQCD. There are a couple of interesting observations worth making.

First, in all of the calculable limits, $T_k/T_1 < k$ for all $k$; thus a string with charge $k$ is stable against decay to $k$ strings of charge one. Consequently, we expect that these flux tubes attract one another, and that therefore QCD and SQCD correspond to type I dual superconductors. I believe this result is robust and will be confirmed numerically.

Second, the functions $f_{sc}$ and $f_{DS}$ have different large $N$ behavior. They agree at leading order, but while the first correction is at order $1/N$ for $f_{sc}$ (as we would expect for an $SU(N)$ theory), the first correction for $f_{DS}$ is order $1/N^2$! The fact that the $1/N$ correction in $f_{DS}$ vanishes is surprising, and the physics that lies behind this feature has not been explained.

§5. A Similar Case

As an aside, I would like to mention the earlier work in which the first concrete example of the non-abelian generalized dual Meissner effect was presented.\textsuperscript{14} Consider $\mathcal{N} = 1$ supersymmetric $SO(N)$ gauge theory with $N_f$ matter fields in the $\mathbf{N}$ representation. In this case, electric sources in the spinor representation of $SO(N)$ can be introduced; they carry a $\mathbb{Z}_2$ quantum number — “spinor number” — which obviously cannot be screened by adjoints or vectors of $SO(N)$. These sources can be used to test the phase
of the $SO(N)$ theory.

On the other hand, certain facts are known about these $SO(N)$ gauge theories. In particular, depending on $N$ and $N_f$, the theory may be in any of the phases discussed in section 2. We also know the $SO(N)$ duality relations discussed in section 2. The strong-coupling physics of the electric $SO(N)$ theory is often best understood using its magnetic description, particularly in the free magnetic and confining phases. But how are the spinor sources, which we need to test the electric theory, mapped into the dual theory? The original work on $SO(N)$ duality\textsuperscript{1,2} did not answer this question.

The answer comes from duality itself. Pouliot and I found a dual description to $SO(8)$ with $N_f$ vectors and one spinor.\textsuperscript{28} It turns out that giving a mass to the spinor allows a monopole soliton to form in the dual theory. This monopole carries a $\mathbb{Z}_2$ charge, the same as the unscreened discrete charge of the spinor. Numerous checks confirm that the monopole is the image under duality of the massive spinor. By making the spinor particles very heavy, we can effectively convert them to sources. At the same time, the monopole becomes a $\mathbb{Z}_2$ Dirac monopole which can be used as a magnetic source in the dual theory.

We thus learn that a Wilson loop in the spinor representation is dual to a $\mathbb{Z}_2$ valued ’t Hooft loop. We can now use this fact to study the phases of $SO(N)$. Two main results are found.\textsuperscript{14}

1) There has been debate as to whether electric charge is confined in the free magnetic phase. This phase has massless two-particle bound states, along with non-polynomial composite gauge bosons and matter. Intriligator and Seiberg proposed\textsuperscript{3} that electric charges, rather than being confined, should have a potential $\ln r/r$ at large distance; this function is the potential energy between two monopoles in a theory with a weak running coupling. The mapping of the electric spinors to massive monopoles in the infrared-free dual theory allows their suggestion to be confirmed.

2) When the number $N_f$ of vectors is reduced sufficiently that the free magnetic phase passes to the confining phase, the non-abelian dual group is completely broken. In the dual description, $\mathbb{Z}_2$ magnetic sources are confined by a string soliton carrying a $\mathbb{Z}_2$ charge. It follows that spinor sources in the electric theory are confined by $\mathbb{Z}_2$ electric flux tubes. This is an example of the generalized dual Meissner effect: confinement by a string soliton in a non-abelian dual description, following condensation of non-abelian monopoles.

§6. Outlook for QCD

I’d like to finish by summarizing the questions that I’ve raised in this talk, and by focussing attention on three of them that I believe can be studied using lattice gauge theory.
6.1. Questions from the Superworld

First and foremost, what is duality? We still have no explicit understanding of non-trivial duality transformations in three or four dimensions, and not nearly enough even in two dimensions. Do we need a reformulation of field theory itself? What does duality in string theory teach us?

What is the phase structure of QCD as a function of gauge group, matter content, and interactions? Does QCD have duality similar to SQCD? Is there a free magnetic phase? Considerable thought needs to be put into the question of how to address these issues effectively; see the discussion below.

Does Olive-Montonen duality imply confinement in non-supersymmetric Yang-Mills theory by \( Z_N \)-carrying electric flux tubes? We have seen in this talk that it does so for \( \mathcal{N} = 1 \) SQCD. The additional tests needed to complete the story for non-supersymmetric QCD are discussed below.

Are condensing non-abelian monopole-like operators responsible for confinement in QCD? Many speakers at this conference have been seeking or discussing abelian monopole-like operators using abelian projection. Should a different approach be taken?

What are the \( Z_N \) string tensions in pure \( SU(N) \) QCD? Do the ratios of tensions fit any known formulas, such as those of Eq. (4.15) or Eq. (4.16)? These questions can and should be addressed numerically; see the discussion below.

Finally, we have seen repeatedly in this talk that massive matter, when added to a theory, can make aspects of its physics easier to understand. What matter (or additional interactions) might we add to non-supersymmetric QCD to make some of these questions more accessible either analytically or numerically?

This is certainly not a complete list of questions — I have not mentioned the recent interest in axionic domain walls, for example — but for finiteness I’ll stop here.

6.2. Proposals for the lattice

There are several projects that I hope will be undertaken by the lattice community in relation to what I have said in this talk.

1) Tension ratios: to my knowledge, no one has ever computed the ratio of string tensions in \( SU(4) \). It would be helpful to know the ratio of the tension \( T_1 \) of the string between two sources in the \( 4 \) and \( \bar{4} \) representation and the tension \( T_2 \) of the string which confines sources in the \( 6 \) representation. One needs to be careful to make sure the sources are far enough apart that the ratio is approaching the asymptotic value which it attains for infinitely long strings. (For example, at distances of order \( A_{QCD}^{-1} \), the string connecting sources in the \( 10 \)
and $\mathbf{T\mathbf{0}}$ will differ from the string connecting two sources in the $6$; only at long distances will their tensions agree.) As a first step, it will be interesting to test the prediction that $SU(4)$ QCD is a type I dual superconductor by checking whether $T_2 < 2T_1$. To go further, one needs high accuracy to distinguish the predictions of, for example, the functions $f_{sc}$ and $f_{DS}$, which only differ by ten percent for this case. Ideally one would even do this for $SU(5)$ and $SU(6)$, but obviously this is a much longer-term goal.

2) The transition from SQCD to QCD: I hope that it will soon be realistic to simulate $SU(2)$ or $SU(3)$ QCD with a Majorana spinor, of mass $m_\lambda$, in the adjoint representation. Tuning the mass to zero to make the theory $N = 1$ SQCD may be difficult, but many properties of the theory, including its strings, should not be sensitive to $m_\lambda$ as long as it is much smaller than the confinement scale $\Lambda$. It would be very interesting to map out the behavior of the theory, including the strings and their tensions $T_k$, as a function of $m_\lambda/\Lambda$. As I mentioned earlier, the absence of a transition in the properties of the strings would establish the linkage I have proposed between Olive-Montonen duality and the $\mathbb{Z}_N$ strings of $SU(N)$ QCD.

3) Phase structure of QCD: A more difficult goal is that discussed in section 2, to map out the low-energy phases of non-supersymmetric QCD as a function of gauge group, matter content, and interactions (including non-renormalizable ones.) The results described by Kanaya in this conference are a step in that direction, but there is still far to go. Since it is not easy for a standard lattice calculation to deal with theories which have slow-running coupling constants, and since such behavior is a common feature of SQCD theories in the most interesting phases, I suspect that more sophisticated analytic and numerical techniques are needed than are presently available. I hope that some of you will be motivated to address this problem. Since it is possible that understanding these issues in QCD will be essential for explaining aspects of the real world, it seems to me that this proposal is much more than an academic exercise.

Again, this is far from a complete list, but I’m sure this is plenty for lattice practitioners to chew on. I hope there will soon come a time when lattice studies provide us with important quantitative and qualitative information about the behavior of these still poorly understood theories.

Acknowledgements

I would like to thank the organizers, especially Professor Suzuki, for inviting me to this conference, as well as the many participants with whom I had stimulating discussions. I am grateful to many colleagues at the Institute for Advanced Study for conversations, including
M. Alford, D. Kabat, A. Hanany, J. March-Russell, N. Seiberg, F. Wilczek, and E. Witten. This work was supported by the National Science Foundation under Grant PHY-9513835 and by the WM Keck Foundation.

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