SESHADRI CONSTANTS AND K-STABILITY OF FANO MANIFOLDS

HAMID ABBAN AND ZIQUAN ZHUANG

Abstract. We give a lower bound of the $\delta$-invariants of ample line bundles in terms of Seshadri constants. As applications, we prove the uniform K-stability of infinitely many families of Fano hypersurfaces of arbitrarily large index, as well as the uniform K-stability of most families of smooth Fano threefolds of Picard number one.

1. Introduction

Existence of Kähler-Einstein metrics on Fano manifolds is detected by K-stability: a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable [CDS15, Tia15]. However, deciding whether a given Fano manifold is K-polystable is a quite challenging problem. We refer to the recent survey by Xu [Xu20a] for details on the subject and its development. A uniform approach to checking K-stability was proposed recently by the authors in [AZ20], which offers an inductive approach to K-stability and the skeleton of its proof relies on lifting the calculation of the so-called $\delta$-invariants (see Section 2.2) to certain flags of subvarieties by adjunction. One particularly useful tool that [AZ20] provided is a K-stability criterion that only involves the existence of a linear system $|L|$ satisfying a simple numerical condition, so that through each point there is a curve given by complete intersection of divisors in $|L|$ (see [AZ20, Theorem 1.2]). In this article, we provide an even stronger criterion using the Seshadri constant, an invariant that was originally introduced by Demailly [Dem92] to measure the local positivity of line bundles.

Theorem A (Theorem 3.1). Let $X$ be a projective variety of dimension $n \geq 2$ and let $L$ be an ample line bundle on $X$. Let $x \in X$ and let $S = H_1 \cap \cdots \cap H_{n-2} \subseteq X$ be a complete intersection surface passing through $x$, where each $H_i \in |L|$. Assume that $S$ is integral, and smooth at $x$. Then

$$\delta_x(L) \geq \frac{n+1}{L^n} \cdot \varepsilon_x(L|S).$$

For a precise description of the equality cases we refer to the statement of Theorem 3.1.

This result enables us to give strong estimate of the $\delta$-invariants of many Fano varieties. One major application is to prove uniform K-stability for a
large class of smooth hypersurfaces. Before stating the result, we recall the following folklore conjecture; see \cite[Xu20a, Part 3]{Xu20a}.

**Conjecture.** Any smooth Fano hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $d \geq 3$ is K-stable.

For general Fano hypersurfaces, this conjecture follows from the K-stability of the Fermat hypersurfaces \cite[Tia00]{Tia00}, \cite[AGP06]{AGP06}, \cite[Zhu21]{Zhu21}, together with the openness of the K-stable locus in smooth families \cite[Oda13]{Oda13}, \cite[Don15]{Don15}, \cite[Xu20b]{Xu20b}, \cite[BLX19]{BLX19}. For arbitrary hypersurfaces, the conjecture is known to be true when the Fano index is at most two \cite[Fuj19a]{Fuj19a}, \cite[LX19]{LX19}, \cite[AZ20]{AZ20}, or when the dimension is at most 4 \cite[Liu20]{Liu20}. Using Theorem A and some careful study of Seshadri constants, we are able to extend this to a much larger class of hypersurfaces:

**Theorem B** (Theorem 4.1). Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of Fano index $r \geq 3$ and dimension $n \geq r^3$. Then $X$ is uniformly K-stable.

For smooth Fano manifolds, uniform K-stability is equivalent to K-stability as a combination of the analytic works \cite[CDS15]{CDS15}, \cite[Tia15]{Tia15}, \cite[BBJ21]{BBJ21}.

Our proof of the above theorem is completely algebraic. In particular, even for general hypersurfaces of Fano index $\geq 3$ in the given dimensions, this is perhaps the first algebraic proof of their uniform K-stability (see \cite[AZ20]{AZ20} for the small index cases).

The next application concerns smooth Fano threefolds of Picard number one. They are classified by Iskovskikh into seventeen families; see \cite[IP99]{IP99}. Some of them, such as $\mathbb{P}^3$, the quadric $Q$, and the Fano threefold $V_5$ of index two and degree 5, have infinite automorphism groups and therefore are not K-stable. It is also well-known that not all Fano threefolds of degree 22 are K-stable \cite[Tia97]{Tia97}. We show that in most of the remaining degrees, the Fano threefolds are uniformly K-stable. This is new when the Fano threefold has index one and degree at least 8. It also provides a unified and purely algebraic proof for all the sporadic cases that were previously known \cite[AGP06]{AGP06}, \cite[Der16]{Der16}, \cite[Fuj19a]{Fuj19a}, \cite[LX19]{LX19}, \cite[Zhu20a]{Zhu20a}, \cite[AZ20]{AZ20}.

**Theorem C** (Theorem 5.1). Let $X$ be a smooth Fano threefold of Picard number one. Assume that $(-K_X)^3 \neq 18, 22$ and $X \neq \mathbb{P}^3, Q$ or $V_5$. Then $X$ is uniformly K-stable.

1.1. **Structure of the paper.** We set the notation and gather some preliminary results in Section 2. The main technical result of this paper, Theorem A which relates Seshadri constants to stability thresholds, is contained in Section 3. In Section 4 we apply this to obtain the first application, Theorem B which concerns the uniform K-stability of hypersurfaces. Finally, in Section 5 we present the other application, Theorem C which concerns the uniform K-stability of Fano threefolds.

\footnote{Postscript note: an algebraic proof is now available by the very recent work \cite[LXZ21]{LXZ21}}
Acknowledgements. We are grateful to Fabian Gundlach, Yuji Odaka, Artie Prendergast-Smith and Chenyang Xu for helpful discussion. We would also like to thank the referees for careful reading and helpful comments. HA is supported by EPSRC grants EP/T015896/1 and EP/V048619/1. ZZ is partially supported by NSF Grant DMS-2055531.

2. Preliminary

2.1. Notation and conventions. We work over \( \mathbb{C} \). Unless otherwise specified, all varieties are assumed to be normal and projective. A pair \((X, \Delta)\) consists of a variety \(X\) and an effective \(\mathbb{Q}\)-divisor \(\Delta\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. The notions of klt and lc singularities are defined as in \cite[Definition 2.8]{Kol13}. If \(\pi : Y \to X\) is a projective birational morphism and \(E\) is a prime divisor on \(Y\), then we say \(E\) is a divisor over \(X\). A valuation on \(X\) will mean a valuation \(v : \mathbb{C}(X)^* \to \mathbb{R}\) that is trivial on \(\mathbb{C}^*\). We write \(\mathbb{C}_X(E)\) (resp. \(\mathbb{C}_X(v)\)) for the center of a divisor (resp. valuation) and \(A_X, \Delta, E\) (resp. \(A_X, \Delta, v\)) for the log discrepancy of the divisor \(E\) (resp. the valuation \(v\)) with respect to the pair \((X, \Delta)\) (see \cite{JM12, BdeFFU15}). We write \(\text{Val}^*_X, \Delta\) for the set of nontrivial valuations \(v\) on \(X\) such that \((X, \Delta)\) is klt at the center of \(v\) and \(A_X, \Delta, v < \infty\). For any valuation \(v\) and any linear series \(V\) we denote by \(\mathcal{F}_v\) the filtration on \(V\) given by \(\mathcal{F}_v^\lambda V = \{s \in V \mid v(s) \geq \lambda\}\). Let \((X, \Delta)\) be a klt pair, let \(Z \subseteq X\) be a closed subset and let \(D\) be an effective divisor (or an ideal sheaf) on \(X\), we denote by \(\text{lct}_Z(X, \Delta; D)\) the largest number \(\lambda \geq 0\) such that the non-lc locus of \((X, \Delta + \lambda D)\) does not contain \(Z\).

2.2. K-stability and stability thresholds. In this section, we recall the definition of K-stability through stability thresholds.

Definition 2.1 \cite{FO18, BJ20}. Let \((X, \Delta)\) be a projective pair, let \(Z \subseteq X\) be a subvariety, and let \(L\) be an ample line bundle on \(X\). Let \(m > 0\) be an integer such that \(H^0(X, mL) \neq 0\).

(1) An \(m\)-basis type \(\mathbb{Q}\)-divisor of \(L\) is a \(\mathbb{Q}\)-divisor of the form

\[
D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\}
\]

where \(N_m = h^0(X, mL)\) and \(s_1, \ldots, s_{N_m}\) is a basis of \(H^0(X, mL)\). We define \(\delta_m(L)\) (resp. \(\delta_{Z,m}(L)\)) to be the largest number \(\lambda \geq 0\) such that \((X, \Delta + \lambda D)\) is lc (resp. lc at the generic point of \(Z\)) for every \(m\)-basis type \(\mathbb{Q}\)-divisor \(D\) of \((X, \Delta)\).

(2) Let \(v\) be a nontrivial valuation on \(X\). We define

\[
T_m(L; v) = \frac{\max\{v(D) \mid D \in mL\}}{m}
\]

and set \(T(L; v) = \lim_{m \to \infty} T_m(L; v)\) (it is usually called the pseudo-effective threshold). We say that \(v\) is of linear growth if \(T(L; v) < \infty\) (this is the
case when \( v \) is divisorial or \( v \in \text{Val}^*_X,\Delta \); see [DKMS15, Section 2.3] and [BJ20, Section 3.1]). For such valuations we set
\[
\text{vol}(L; v \geq t) = \lim_{m \to \infty} \frac{\dim \{ s \in H^0(X, mL) \mid v(s) \geq mt \}}{m^{\dim X} / (\dim X)!}
\]
and \( S(L; v) = \frac{1}{\text{vol}(L)} \int_0^\infty \text{vol}(L; v \geq t) dt \). If \( E \) is a divisor over \( X \), we define \( S(L; E) := S(L; \text{ord}_E), \ T(L; E) := T(L; \text{ord}_E) \), etc.

(3) Assume that \((X, \Delta)\) is klt (or klt at the generic point of \( Z \) in the local case). The local and global stability thresholds (or \( \delta \)-invariant) are defined to be
\[
\delta_Z(L) := \inf_{v \in \text{Val}^*_X,\Delta, Z \subseteq C_X(v)} \frac{A_X,\Delta(v)}{S(L; v)}, \quad \delta(L) := \inf_{v \in \text{Val}^*_X} \frac{A_X,\Delta(v)}{S(L; v)}.
\]
Clearly \( \delta(L) = \inf_{x \in X} \delta_x(L) \). We say that a valuation \( v \in \text{Val}^*_X,\Delta \) computes \( \delta_Z(L) \) (or \( \delta(L) \)) if it achieves the above infimum. Such valuations always exist by [BJ20, Theorem E]. We will sometimes write \( \delta_Z(X, \Delta; L) \) and \( \delta(X, \Delta; L) \) if the pair \((X, \Delta)\) is not clear from the context. By [BJ20, Theorem A1], we have \( \lim_{m \to \infty} \delta_m(X, \Delta) = \delta(X, \Delta) \). By the same proof in loc. cit. we also have \( \lim_{m \to \infty} \delta_m(X, \Delta) = \delta_Z(X, \Delta) \).

**Theorem-Definition 2.2** ([FO18] [BJ20] [Fuj19b] [Li17]). Let \( X \) be a Fano variety. Then it is K-semistable (resp. uniformly K-stable) if and only if \( \delta(-K_X) \geq 1 \) (resp. \( \delta(-K_X) > 1 \)).

### 2.3. Seshadri constants, movable thresholds and pseudo-effective thresholds

Let \( X \) be a variety and let \( L \) be an ample line bundle on \( X \). Let \( v \) be a valuation of linear growth on \( X \) whose center \( C_X(v) \) has codimension at least two. Then the movable threshold \( \eta(L; v) \) is defined as
\[
\eta(L; v) = \sup \{ \eta > 0 \mid \text{ for some } m \in \mathbb{N}, \\
\quad \text{Bs}[F^m_v H^0(X, mL)] \text{ has codimension at least two} \}.
\]
Note that \( \eta(L; v) > 0 \). Indeed, if we choose some sufficiently large integer \( m > 0 \) such that \( \mathcal{O}_X(mL) \otimes \mathcal{I}_{C_X(v)} \) is globally generated, then as the base locus of this linear system has codimension at least 2 we get \( \eta(L; v) \geq \frac{1}{m^\text{mult}_X C_X(v)} > 0 \).

Let \( x \in X \) be a smooth point, let \( \pi : \tilde{X} \to X \) be the blowup of \( x \) and let \( E \) be the exceptional divisor. Then the Seshadri constant \( L \) at \( x \) is defined to be (see [Dem92] or [Laz04a, Chapter 5])
\[
\varepsilon_x(L) := \sup \{ t \geq 0 \mid \pi^*L - tE \text{ is nef} \} = \inf_{C \subseteq X} \frac{(L \cdot C)}{C \subseteq X \text{ mult}_X C}.
\]
where the infimum is taken over all irreducible curves \( C \subseteq X \) passing through \( x \). We also denote the pseudo-effective threshold \( T(L; E) \) (resp. movable threshold \( \eta(L; E) \)) in this case by \( \tau_x(L) \) (resp. \( \eta_x(L) \)). Note that
$\varepsilon_x(L) = \eta_x(L)$ when $X$ is a surface. By definition it is easy to see that $\varepsilon_x(L) \leq \eta_x(L) \leq \tau_x(L)$ and

$$\tau_x(L) = \sup\{\text{mult}_xD \mid 0 \leq D \sim_{\mathbb{Q}} L\}.$$  

As we have $(\pi^*L - \varepsilon_x(L)E)^n \geq 0$ (where $n = \dim X$), it follows that $\sqrt{(L^n)} \geq \varepsilon_x(L)$. It is also well-known that $\tau_x(L) \geq \sqrt{(L^n)}$ (see e.g. [Laz04b, Lemma 10.4.12]). When $L$ is very ample, we also have $\eta_x(L) \leq \sqrt{(L^n)}$: otherwise we can find two effective $\mathbb{Q}$-divisors $D_1, D_2 \sim_{\mathbb{Q}} L$ that have no common components such that $\text{mult}_xD_i > \sqrt{(L^n)}$, and we have

$$(L^n) = (D_1 \cdot D_2 \cdot H_1 \cdots \cdot H_{n-2}) \geq \text{mult}_xD_1 \cdot \text{mult}_xD_2 > (L^n)$$

for some general members $H_1, \cdots, H_{n-2}$ of the linear system $|L \otimes m_x|$, a contradiction. For later use, we recall some more properties of these invariants.

**Lemma 2.3.** Let $L$ be an ample line bundle on a variety $X$ and let $v$ be a valuation of linear growth on $X$ whose center has codimension at least two. Assume that $X$ is $\mathbb{Q}$-factorial of Picard number one and $\eta(L; v) < T(L; v)$. Then there exists a unique irreducible $\mathbb{Q}$-divisor $D_0 \sim_{\mathbb{Q}} L$ on $X$ such that $v(D_0) > \eta(L; v)$. Moreover, we have $v(D_0) = T(L; v)$ and for any effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ such that $v(D) \geq \eta(L; v)$ we have

$$D \geq \frac{v(D) - \eta(L; v)}{T(L; v) - \eta(L; v)} \cdot D_0.$$  

**Proof.** For ease of notation, let $\eta = \eta(L; v)$ and $T = T(L; v)$. We first prove the uniqueness. Suppose that there exist two such irreducible $\mathbb{Q}$-divisors $D_0, D_1$. Let $\lambda = \min\{v(D_0), v(D_1)\} > \eta$. Then for some sufficiently divisible integer $m$ the base locus of $|\mathbb{P}^mH^0(mL)|$ has codimension at least two since $mD_0, mD_1 \in |\mathbb{P}^mH^0(mL)|$. Hence $\eta \geq \lambda$, a contradiction. This proves the uniqueness of $D_0$.

For the existence, let $D \sim_{\mathbb{Q}} L$ be an effective $\mathbb{Q}$-divisor on $S$ such that $v(D) > \eta$ (which exists as $T(L; v) > \eta(L; v)$). Since $\rho(X) = 1$, we may write $D = \sum \lambda_iD_i$ where $\lambda_i > 0$, $\sum \lambda_i = 1$ and each $D_i \sim_{\mathbb{Q}} L$ is irreducible. As $v(D) > \eta$, at least one of the $D_i$ satisfies $v(D_i) > \eta$. This proves the existence of $D_0$. Since such $D_0$ is unique, it is then clear from the definition of pseudo-effective threshold that $v(D_0) = T$ and moreover we have $v(D_i) \leq \eta$ for all $i > 0$. Thus $v(D - \lambda_0D_0) \leq (1 - \lambda_0)\eta$. Solving this inequality gives $\lambda_0 \geq \frac{v(D) - \eta}{T - \eta}$. \hfill $\square$

**Lemma 2.4.** Let $S$ be a $\mathbb{Q}$-factorial surface of Picard number one and let $x$ be a smooth point. Then we have $\varepsilon_x(L) \cdot \tau_x(L) = (L^2)$.  

**Proof.** Again let $\varepsilon = \varepsilon_x(L)$ and $\tau = \tau_x(L)$. Clearly $\varepsilon \leq \tau$ by definition. If $\varepsilon = \tau$, then since $\tau \geq \sqrt{(L^2)} \geq \varepsilon$ we necessarily have $\varepsilon = \tau = \sqrt{(L^2)}$, hence $\varepsilon\tau = (L^2)$. Thus we may assume that $\varepsilon < \tau$. By Lemma 2.3 there exists a unique irreducible $\mathbb{Q}$-divisor $D_0 \sim_{\mathbb{Q}} L$ on $S$ such that $\text{mult}_xD_0 = \tau$. Let
\(C \subseteq S\) be an irreducible curve passing through \(x\). If \(D_0\) is supported on \(C\), then
\[
\frac{(C \cdot L)}{\mult_x C} = \frac{(D_0 \cdot L)}{\mult_x D_0} = \frac{(L^2)}{\tau},
\]
otherwise we have
\[
\frac{(C \cdot L)}{\mult_x C} = \frac{(C \cdot D_0)}{\mult_x C} \geq \mult_x D_0 = \tau.
\]
Since \(\tau^2 \geq (L^2)\), by the definition of Seshadri constants we see that \(\varepsilon = \frac{(L^2)}{\tau}\).

\[\square\]

2.4. **Restricted volumes.** We refer to [ELM+09] for the original definition of the restricted volume \(\text{vol}_{X|Z}(L)\) of a divisor \(L\) along a subvariety \(Z\). For our purpose their most important properties are summarized in the following statement.

**Lemma 2.5.** Let \(L\) be an ample line bundle on a projective variety \(X\) of dimension \(n\), let \(\pi: Y \to X\) be a birational morphism and let \(E \subseteq Y\) be a prime divisor on \(Y\). Then
\[
(2.1) \quad \frac{d}{dt} \text{vol}(\pi^*L - tE) = -n \cdot \text{vol}_{Y|E}(\pi^*L - tE)
\]
for all \(0 \leq t < T(L; E)\). Moreover, the function \(t \mapsto \text{vol}_{Y|E}(\pi^*L - tE)^{\frac{1}{n-t}}\) is concave on \([0, T(L; E))\).

**Proof.** The equality (2.1) follows from [BFJ09] or [LM09, Corollary C]. The concavity part follows from [ELM+09, Theorem A]. \(\square\)

We may then rewrite the formula of \(S\)-invariants using restricted volumes as follows.

**Lemma 2.6.** In the situation of Lemma 2.5 we have
\[
S(L; E) = \frac{n}{(L^n)} \int_0^{T(L; E)} x \cdot \text{vol}_{Y|E}(\pi^*L - xE)dx.
\]

**Proof.** By definition, \(S(L; E) = \frac{1}{(L^n)} \int_0^{T(L; E)} \text{vol}(\pi^*L - tE)dt\). Thus the statement follows from (2.1) and integration by parts. \(\square\)

2.5. **Filtered linear series and compatible divisors.** In this section, we briefly recall some definitions from [AZ20, Sections 2.5-2.6].

**Definition 2.7.** Let \(L_1, \cdots, L_r\) be line bundles on \(X\). An \(\mathbb{N}^r\)-graded linear series \(W_{\bullet}\) on \(X\) associated to the \(L_i\)’s consists of finite dimensional subspaces
\[W_{\vec{a}} \subseteq H^0(X, \mathcal{O}_X(a_1L_1 + \cdots + a_rL_r))\]
for each \(\vec{a} \in \mathbb{N}^r\) such that \(W_{\vec{0}} = \mathbb{C}\) and \(W_{\vec{a}} \cdot W_{\vec{a}} \subseteq W_{\vec{a_1} + \vec{a_2}}\) for all \(\vec{a_1}, \vec{a_2} \in \mathbb{N}^r\). The support \(\text{Supp}(W_{\vec{a}}) \subseteq \mathbb{R}^r\) of \(W_{\vec{a}}\) is defined as the closure of the convex cone spanned by all \(\vec{a} \in \mathbb{N}^r\) such that \(W_{\vec{a}} \neq 0\). In this paper we only consider multi-graded linear series that have bounded support and contain
ample series, see [LM09 Section 4.3] or [AZ20 Definition 2.11]) for the precise definition. A filtration \( F \) on \( W_\bullet \) is given by the data of vector subspaces \( \mathcal{F}^\lambda W_{\vec{a}} \subseteq W_{\vec{a}} \) for each \( \lambda \in \mathbb{R} \) and \( \vec{a} \in \mathbb{N}^r \) such that \( \mathcal{F}^\lambda W_{\vec{a}} \cdot \mathcal{F}^{\lambda'} W_{\vec{a}_1} \subseteq \mathcal{F}^{\lambda + \lambda'} W_{\vec{a}_1 + \vec{a}_2} \) for all \( \lambda_i \in \mathbb{R} \) and all \( \vec{a}_i \in \mathbb{N}^r \). We only consider linearly bounded filtrations, i.e. there are constants \( C_1 \) and \( C_2 \) such that \( \mathcal{F}^\lambda W_{\vec{a}} = W_{\vec{a}} \) for all \( \lambda < C_1 |\vec{a}| \) and \( \mathcal{F}^\lambda W_{\vec{a}} = 0 \) for all \( \lambda > C_2 |\vec{a}| \). Any valuation \( v \) of linear growth on \( X \) induces a filtration \( F_v \) on \( W_\bullet \) such that \( F_v^\lambda W_{\vec{a}} = \{ s \in W_{\vec{a}} \mid v(s) \geq \lambda \} \).

If we think of \( W_\bullet \) as an \( \mathbb{N} \times \mathbb{N}^{r-1} \)-graded linear series and view the first \( \mathbb{N} \)-factor as the level grading, then we may define \( m \)-basis type \( \mathbb{Q} \)-divisors of \( W_\bullet \) as a \( \mathbb{Q} \)-divisor of the form

\[
D = \frac{1}{m N_m} \sum_{i=1}^{N_m} \{ s_i = 0 \}
\]

where \( s_1, \cdots, s_{N_m} \) enumerate some basis of \( W_{m,\vec{a}} \) for all \( \vec{a} \in \mathbb{N}^{r-1} \) (we call it an \( m \)-basis of \( W_\bullet \)) and \( N_m = \sum_{\vec{a}} \dim W_{m,\vec{a}} \). Following [AZ20], we say that \( D \) is compatible with a filtration \( F \) on \( W_\bullet \) if every \( \mathcal{F}^\lambda W_{m,\vec{a}} \) is spanned by some of the \( s_i \). Let \( \Delta(W_\bullet) = \{ \vec{a} \in \mathbb{R}^{r-1} \mid (1, \vec{a}) \in \text{Supp}(W_\bullet) \} \). For any \( \vec{a} \in \mathbb{Q}^{r-1} \) in the interior of \( \Delta(W_\bullet) \), we set

\[
\text{vol}_{W_\bullet}(\vec{a}) := \lim_{m \to \infty} \frac{\dim(W_{m,\vec{a}})}{m^n / n!} \quad (n = \dim X)
\]

where the limit is taken over integers \( m \) such that \( m\vec{a} \in \mathbb{N}^{r-1} \). By [LM09 Corollary 4.22], it extends a continuous function on the interior of \( \Delta(W_\bullet) \), which we still denote by \( \text{vol}_{W_\bullet}(\cdot) \). For each \( \vec{a} \in \mathbb{N}^r \), we let \( M_{\vec{a}} \) (resp. \( F_{\vec{a}} \)) be the movable (resp. fixed) part of \( W_{\vec{a}} \). We define \( F(W_\bullet) := \lim_{m \to \infty} F_m(W_\bullet) \in \text{Div}(X)_{\mathbb{R}} \), where

\[
F_m(W_\bullet) = \frac{1}{m N_m} \sum_{\vec{a} \in \mathbb{N}^{r-1}} \dim(W_{m,\vec{a}}) \cdot F_{m,\vec{a}}.
\]

Note that the limit exists by [AZ20 Lemma-Definition 2.25]. As in loc. cit., we also set \( c_1(W_\bullet) := \lim_{m \to \infty} c_1(D_m) \in \text{NS}(X)_{\mathbb{R}} \) (where \( D_m \) is any \( m \)-basis type \( \mathbb{Q} \)-divisor of \( W_\bullet \)) and \( c_1(M_\bullet) := c_1(W_\bullet) - F(W_\bullet) \). Many of the invariants we define in Section 2.2 also generalizes to this setting, see [AZ20 Section 2.6]. For our purposes we recall the following.

1. The pseudo-effective threshold \( T(W_\bullet; F) \) of a filtration \( F \) on \( W_\bullet \) is defined as

\[
T(W_\bullet; F) := \lim_{m \to \infty} \frac{T_m(W_\bullet; F)}{m} = \sup_{m \in \mathbb{N}} \frac{T_m(W_\bullet; F)}{m},
\]

where

\[
T_m(W_\bullet; F) = \sup\{ \lambda \in \mathbb{R} \mid \mathcal{F}^\lambda W_{m,\vec{a}} \neq 0 \text{ for some } \vec{a} \}.
\]
(2) We set \( S(W; F) := \lim_{m \to \infty} S_m(W; F) \) where

\[
S_m(W; F) = \frac{1}{mN_m} \sum_{\lambda, \bar{\alpha}} \lambda \cdot \dim \text{Gr}_F^\lambda W_{m, \bar{\alpha}}.
\]

(3) Given a closed subset \( Z \subseteq X \), we define

\[
\delta_Z(W; F) := \limsup_{m \to \infty} \delta_{Z,m}(W; F),
\]

where \( \delta_{Z,m}(W; F) = \inf_D \text{let}_Z(X, \Delta; D) \) and the infimum runs over all \( m \)-basis type \( \mathbb{Q} \)-basis.

Our multi-graded linear series mostly come from the refinement of a complete linear series.

**Definition 2.8 ([AZ20 Example 2.15]).** Let \( L \) be a big line bundle on \( X \) and let \( V \) be the complete linear series associated to \( L \), i.e. \( V_m = H^0(X, mL) \).

Let \( \pi: Y \to X \) be a birational morphism and let \( F \) be a Cartier prime divisor on \( Y \). The refinement of \( V \) by \( F \) is the \( \mathbb{N}^2 \)-graded linear series \( W \) associated to \( \pi^* L|_F \) and \( -F|_F \) given by

\[
W_{m,j} = \text{Im}(H^0(Y, m\pi^* L - jF) \to H^0(F, m\pi^* L|_F - jF|_F)).
\]

Note that any filtration \( F \) on \( V \) naturally induces a filtration \( \bar{F} \) on \( W \), i.e. \( \bar{F}^\lambda V_{m,j} \) is the image of \( \bar{F}^\lambda V_m \cap H^0(Y, m\pi^* L - jF) \).

**Lemma 2.9.** In the above notation, we have \( S(V; F) = S(W; \bar{F}) \).

**Proof.** It suffices to show that \( S_m(V; F) = S_m(W; \bar{F}) \). By [AZ20 Lemma 3.1], we may find a basis \( s_1, \ldots, s_{N_m} \) of \( V_m \) that is compatible with both \( F \) and \( \bar{F} \), the filtration induced by \( F \).

By construction, they restrict to form an \( m \)-basis of \( W \) that is compatible with \( \bar{F} \). Let \( \lambda_i = \sup \{ \lambda \mid s_i \in \bar{F}^\lambda V_m \} \). Then by the definition of \( S \)-invariants it is easy to see that \( S(V; F) = S(W; \bar{F}) \).

For computations we often choose refinements that are almost complete [AZ20 Definition 2.27].

**Definition 2.10.** Let \( L \) be a big line bundle on \( X \) and let \( W \) be an \( \mathbb{N}^r \)-graded linear series. We say that \( W \) is almost complete (with respect to \( L \)) if for every \( \bar{\alpha} \in \text{int}(\text{Supp}(W)) \cap \mathbb{Q}^r \) and all sufficiently divisible integers \( m \) (depending on \( \bar{\alpha} \)), we have \( |M_{m\bar{a}}| \subseteq |L_{m, \bar{\alpha}}| \) for some \( L_{m, \bar{\alpha}} = \ell_{m, \bar{\alpha}} L \) and some \( \ell_{m, \bar{\alpha}} \in \mathbb{N} \) such that

\[
\frac{\dim W_{m\bar{a}}}{h^0(X, \ell_{m, \bar{\alpha}} L)} = \frac{\dim M_{m\bar{a}}}{h^0(X, \ell_{m, \bar{\alpha}} L)} \to 1
\]
as \( m \to \infty \).

In the surface case, all refinements as in Definition 2.8 are almost complete by [AZ20 Lemma 4.10]. Another common example is the refinement of the complete linear series associated to an ample line bundle \( L \) by some integral member \( H \in |L| \), see [AZ20 Example 2.28].
3. Seshadri Constants and Stability Thresholds

In this section, we prove the following statement, giving lower bounds of stability thresholds in terms of Seshadri constants on complete intersection surfaces. This will be a key tool to verify K-stability of Fano varieties in subsequent sections.

**Theorem 3.1.** Let $X$ be a projective variety of dimension $n \geq 2$ and let $L$ be an ample line bundle on $X$. Let $x \in X$ be a smooth point and let $S = H_1 \cap \cdots \cap H_{n-2} \subseteq X$ be a complete intersection surface passing through $x$, where each $H_i \in |L|$. Assume that $S$ is integral and is smooth at $x$. Then

$$\delta_x(L) \geq \frac{n+1}{(L^n)} \cdot \varepsilon_x(L|S).$$

When equality holds, we have at least one of the following:

1. $\varepsilon_x(L|S) = \tau_x(L|S) = \sqrt{(L^n)} = 1$, and $\delta_x(L)$ is computed by any $H \in |L \otimes m_x|$, or
2. $\varepsilon_x(L|S) = \tau_x(L|S) = \sqrt{(L^n)} > 1$, and the center of any valuation $v$ that computes $\delta_x(L)$ has dimension $\dim C_X(v) \geq n-2$, or
3. $\varepsilon_x(L|S) \tau_x(L|S) = (L^n)$, and every valuation that computes $\delta_x(L)$ is divisorial and induced by a prime divisor $G \subseteq X$ containing $x$ such that $S \not\subseteq G$ and $L \equiv \tau_x(L|S)G$.

As one might expect, the careful analysis of the equality cases in the above statement will be useful in proving uniform K-stability in several cases.

Note that an upper bound on the log canonical threshold in terms of Seshadri constants was studied in [OS12]. However, the relation in 3.1 is in the opposite direction, which offers significantly more flexibility in estimating the $\delta$-invariant.

The proof of Theorem 3.1 is by induction on the dimension, where the inductive step is based on [AZ20, Lemma 4.6]. Apart from that, the heart of the proof is a detailed analysis of the surface case, where we can be even more precise about the equality cases:

**Lemma 3.2.** Let $S$ be a surface and let $L$ be an ample line bundle on $S$. Let $x \in S$ be a smooth point. Then

$$\delta_x(L) \geq \frac{3}{(L^2)} \cdot \varepsilon_x(L),$$

and equality holds if and only if $\varepsilon_x(L) = \tau_x(L) = \sqrt{(L^2)}$, or $\varepsilon_x(L) \tau_x(L) = (L^2)$ and there exists a unique irreducible curve $C \subseteq X$ containing $x$ such that $L \equiv \tau_x(L)C$. Moreover, in the latter case, the curve $C$ is the only divisor that computes $\delta_x(L)$.

The idea to prove the above statement is to consider the refinement $W_\bullet$ of the linear series associated to $L$ by the ordinary blowup of $x$ and then compare the stability thresholds of $L$ and $W_\bullet$ using tools from [AZ20]. Using Zariski decomposition on surfaces, we will estimate the stability threshold
Lemma 3.3. Let $0 < a \leq b$ and let $g(x)$ be a bounded concave function on $[0, b)$ such that $g(x) = x$ for all $x \in [0, a)$. Then
\[
3a \int_0^b (2x - g(x)) \cdot g(x) \, dx \leq 4 \left( \int_0^b g(x) \, dx \right)^2,
\]
and equality holds if and only if $a = b$, or $g(x) = h(x)$ for all $x \in [0, b)$, where
\[
h(x) = \begin{cases} 
\frac{x}{a-b} & \text{if } 0 \leq x \leq a, \\
\frac{b-x}{b-a} & \text{if } a < x \leq b.
\end{cases}
\]
In particular, when equality holds we have $\int_0^b g(x) \, dx = \frac{1}{2}ab$.

Proof. It is straightforward to check that equality holds when $a = b$, thus we may assume that $b > a$. Let $f(x) = g(x + a) - h(x + a)$ and let $c = b - a > 0$. Then $f(0) = 0$, $f(x)$ is a bounded concave function on $[0, c)$ and the inequality in the statement of the lemma is equivalent to
\[
a^2 \int_0^c \left( \frac{6x}{c} - 4 \right) f(x) \, dx + a \int_0^c (6x - 4c) f(x) \, dx
\]
\[- 3a \int_0^c f(x)^2 \, dx - 4 \left( \int_0^c f(x) \, dx \right)^2 \leq 0
\]
by an elementary calculation. We claim that
\[
\int_0^c (3x - 2c) f(x) \, dx \leq 0,
\]
which clearly implies the previous inequality as well as the equality condition $f(x) \equiv 0$. To prove the claim, consider $F(t) = \int_0^t (3x - 2t) f(x) \, dx$ as a function of $t \in [0, c]$. Then $F(0) = 0$ and $F'(t) = t \cdot f(t) - 2 \int_0^t f(x) \, dx \leq t \cdot f(t) - 2 \int_0^t \frac{f(t)}{2} \, dx = 0$ (\forall t \in (0, c)) by the concavity of $f(x)$. Thus $F(c) \leq 0$ and we are done.

To further analyze the equality case in Theorem 3.1 and Lemma 3.2 we need two more auxiliary results.

Lemma 3.4. Let $a > 0$ and let $g(x)$ be a nonnegative bounded concave function on $[0, a)$ such that $g(0) > 0$. Let $n > 0$ be an integer. Then
\[
g(0)^{n-1} \int_0^a x \cdot g(x)^{n-1} \, dx \leq \frac{n}{n+1} \left( \int_0^a g(x)^{n-1} \, dx \right)^2,
\]
with equality if and only if $n = 1$ or $g(x) = (1 - \frac{x}{a})g(0)$.

Proof. The result is clear when $n = 1$, so we may assume that $n \geq 2$. Up to rescaling, we may also assume that $g(0) = 1$. For each $b > 0$, let $f_b(x) = 1 - \frac{x}{b} \ (0 \leq x \leq b)$. Since $g(x)$ is nonnegative and concave, we have $g(x) \geq f_a(x)$ for all $x \in [0, a]$ and thus $\int_0^a g(x)^{n-1} \, dx \geq \int_0^a f_a(x)^{n-1} \, dx$. .
As \( \lim_{b \to \infty} f_b(x)^{n-1}dx = \infty \), by interpolation we know that there exists some \( b \geq a \) such that

\[
\int_0^a g(x)^{n-1}dx = \int_0^b f_b(x)^{n-1}dx.
\]

(3.1)

It is easy to check that \( \int_0^b x \cdot f_b(x)^{n-1}dx = \frac{n}{n+1} \left( \int_0^b f_b(x)^{n-1}dx \right)^2 \), hence it suffices to show

\[
\int_0^a x \cdot g(x)^{n-1}dx \leq \int_0^b x \cdot f_b(x)^{n-1}dx.
\]

(3.2)

For ease of notation, set \( g(x) = 0 \) when \( a < x \leq b \) and set \( h(x) = f_b(x)^{n-1} - g(x)^{n-1} \). Since \( g(x) \) is concave on \([0, a] \) and \( f_b(x) \) is linear, there exists some \( c \leq a \) such that \( h(x) \leq 0 \) for all \( x \in [0, c] \) and \( h(x) > 0 \) for all \( x \in (c, b) \). Note that \( c > 0 \) by (3.1). We then have

\[
\int_0^b x h(x)dx = \int_0^c x h(x)dx + \int_c^b x h(x)dx \geq c \int_0^c h(x)dx + c \int_c^b h(x)dx = 0,
\]

where the last equality follows from (3.1). This proves (3.2). When equality holds, we have \( h(x) = 0 \), thus \( b = a \) and \( g(x) = 1 - \frac{x}{a} \).

\[ \square \]

**Lemma 3.5.** Let \( L \) be an ample line bundle on a variety \( X \) of dimension \( n \). Let \( G \subset X \) be a prime divisor on \( X \). Then

\[
S(L; G) \leq \frac{(L^n)}{(n+1)(L^{n-1} \cdot G)},
\]

with equality if and only if \( L \equiv aG \) for some \( a > 0 \).

\[ \square \]

**Proof.** The result is clear when \( n = 1 \), so we assume that \( n \geq 2 \). Let \( \pi: Y \to X \) be a log resolution such that the strict transform \( \tilde{G} = \pi_*G \) of \( G \) is smooth. Let \( a = T(L; G) \). By Lemmas 2.5 and 2.6 we have

\[
S(L; G) = \frac{n}{(L^n)} \int_0^a x \cdot \text{vol}_Y(\pi^*L - x\tilde{G})dx,
\]

(3.3)

\[
(L^n) = n \int_0^a \text{vol}_Y(\pi^*L - x\tilde{G})dx.
\]

Since \( L \) is ample, by [ELM+09, Lemma 2.4] we also have \( \text{vol}_Y(\pi^*L) = \text{vol}_X(G)(L) = (L^{n-1} \cdot G) \), hence the inequality follows directly from Lemma 3.4 applied to \( g(x) = \text{vol}_Y(\pi^*L - x\tilde{G}) \frac{1}{n+1} \), which is concave by [ELM+09, Theorem A]. Suppose that equality holds, then by Lemma 3.4 we have \( g(x) = (1 - \frac{x}{a})g(0) \), i.e.

\[
\text{vol}_Y(\pi^*L - x\tilde{G}) = \left( 1 - \frac{x}{a} \right)^{n-1} (L^{n-1} \cdot G)
\]

for all \( 0 \leq x < a \). A direct calculation through (3.3) then yields \( (L^n) = a(L^{n-1} \cdot G) \), or \( (L^{n-1} \cdot L - aG) = 0 \). It follows that \( L - aG \equiv 0 \) as \( L - aG \).
is pseudo-effective and $L$ is ample. Clearly $S(L; G) = \frac{n}{n+1}$ if $L \equiv aG$. This finishes the proof. □

We are ready to present the proof of Lemma 3.2 and Theorem 3.1.

**Proof of Lemma 3.2.** Let $\pi: T \to S$ be the ordinary blowup at $x$ with exceptional divisor $E \cong \mathbb{P}^1$. Let $V_x$ be the complete linear series associated to $L$ and let $W_x$ be its refinement by $E$. Note that $\delta_x(L) = \delta_x(V_x)$. Let $\lambda = \frac{3}{(d_x^*)} \cdot \varepsilon_x(L)$, let $\varepsilon = \varepsilon_x(L)$ and let $\tau = \tau_x(L) = T(L; E)$. Since $W_x$ is almost complete by [AZ20, Lemma 4.10], applying [AZ20, Corollary 3.4] we know that $\delta_x(V_x) \geq \lambda$ as long as

\[
\lambda \leq \frac{A_S(E)}{S(L; E)}
\]

and $\delta(E, \lambda \cdot F(W_x); c_1(M_x)) \geq \lambda$ holds, where $M_x$ is the movable part of $W_x$. By the definition of stability thresholds, the latter inequality is equivalent to saying

\[
\lambda \cdot S(c_1(M_x); P) + \lambda \cdot \text{mult}_P F(W_x) \leq 1
\]

for all closed point $P \in E$. Let us verify that both conditions (3.4), (3.5) holds in our situation. First, we have

\[
F(W_x) = \frac{2}{L^2} \int_0^\tau (\text{vol}_{T|E}(\pi^*L-xE) \cdot N_\sigma(\pi^*L-xE)|_E) \, dx
\]

by [AZ20] Lemma 4.13, and

\[
\text{vol}_{T|E}(\pi^*L-xE) = (P_\sigma(\pi^*L-xE) \cdot E)
\]

by [ELM+09] Corollary 2.17 and Example 2.19, where $P_\sigma(\cdot)$ (resp. $N_\sigma(\cdot)$) denotes the nef (resp. negative) part in the Zariski decomposition of a (pseudo-effective) divisor. In particular, letting $g(x) = \text{vol}_{T|E}(\pi^*L-xE)$ ($0 \leq x < \tau$), we have

\[
(N_\sigma(\pi^*L-xE) \cdot E) = ((\pi^*L-xE) \cdot E) - (P_\sigma(\pi^*L-xE) \cdot E) = x - g(x).
\]

By the definition of Seshadri constant, we also have $g(x) = x$ for all $0 \leq x \leq \varepsilon$. Therefore as $g(x)$ is concave by [ELM+09] Theorem A], Lemma 2.6 and Lemma 3.3 yield

\[
S(L; E) + \text{deg } F(W_x) = \frac{2}{L^2} \int_0^\tau x \cdot g(x) \, dx + \frac{2}{L^2} \int_0^\tau (x - g(x)) \cdot g(x) \, dx
\]

\[
\leq \frac{2}{L^2} \cdot \frac{4 \left( \int_0^\tau g(x) \, dx \right)^2}{3\varepsilon} = \frac{2(L^2)}{3\varepsilon} = \frac{2}{\lambda}.
\]

It follows that

\[
\lambda \leq \frac{2}{S(L; E) + \text{deg } F(W_x)} \leq \frac{2}{S(L; E)} = \frac{A_S(E)}{S(L; E)}
\]
which verifies \( \text{(3.4)} \). Since \( E \cong \mathbb{P}^1 \) is a curve, we have \( S(c_1(M_x); P) = \frac{1}{2} \deg c_1(M_x) \) for any closed point \( P \in E \). By \[ AZ20 \] (3.1), we also have \( \deg (c_1(M_x) + F(W_x)) = \deg c_1(W_x) = S(L; E) \). Thus we obtain

\[
\lambda \cdot S(c_1(M_x); P) + \lambda \cdot \text{mult}_P F(W_x) \leq \lambda \cdot \deg \left( \frac{1}{2} c_1(M_x) + F(W_x) \right)
\]

\[
= \frac{\lambda}{2} \cdot (S(L; E) + \deg F(W_x)) \leq 1
\]

for any closed point \( P \in E \), which verifies \( \text{(3.5)} \). Hence according to the discussions at the beginning of the proof, \( AZ20 \) Corollary 3.4 implies that \( \delta_x(L) = \delta_x(V_x) \geq \lambda \) as desired.

It remains to prove the equality conditions. It is straightforward to check that \( \frac{\lambda}{S(L; E)} = \lambda \) (resp. \( \frac{\lambda}{S(L; C)} = \lambda \)) when \( \varepsilon = \tau = \sqrt{(L^2)} \) (resp. \( \varepsilon \tau = (L^2) \) and there exists some curve \( C \subseteq X \) containing \( x \) such that \( L \equiv \tau C \)), hence \( \delta_x(L) = \lambda \) in either case. Conversely, assume that \( \delta_x(L) = \lambda \). If \( \delta_x(L) \) is computed by \( E \), then by \( \text{(3.6)} \) we have \( F(W_x) = 0 \), hence \( N_x(\tau L - xE) = 0 \) and \( \tau L - xE \) is nef for all \( 0 \leq x < \tau \). It follows that \( \varepsilon = \tau \). Since \( \varepsilon \leq \sqrt{(L^2)} \leq \tau \), we must have \( \varepsilon = \tau = \sqrt{(L^2)} \) as desired. On the other hand, if \( E \) does not compute \( \delta_x(L) \), then \( \varepsilon < \tau \) and by the equality description in Lemma \( \text{(3.7)} \) we have \( (L^2) = 2 \int_0^\tau g(x)dx = \varepsilon \tau \). By \( [3320] \) Theorem E] and \( AZ20 \) Corollary 3.4, we also see that \( \delta_x(L) \) is computed by some valuations \( v \) such that \( C_S(v) \neq \{x\} \). Thus the center of \( v \) is a curve \( C \subseteq S \); in particular, \( v \) is divisorial and the curve \( C \) also computes \( \delta_x(L) \), i.e. \( S(L; C) = \frac{1}{S} \). By Lemma \( \text{(3.5)} \) and the definition of Seshadri constant, we deduce

\[
\varepsilon \leq \frac{(L \cdot C)}{\text{mult}_x C} \leq (L \cdot C) \leq \frac{(L^2)}{S(L; C)} = \frac{\lambda \cdot (L^2)}{3} = \varepsilon,
\]

thus equality holds everywhere. In particular, \( L \equiv aC \) and \( S(L; C) = \frac{a}{3} \) for some \( a > 0 \) by Lemma \( \text{(3.5)} \). But since \( S(L; C) = \frac{1}{S} = \frac{(L^2)}{3\varepsilon} = \frac{\varepsilon \tau}{3} \), we must have \( a = \tau \). Since there can be at most one irreducible curve \( C \subseteq S \) containing \( x \) such that \( L \equiv \tau C \) (otherwise it follows from the definition of Seshadri constants that \( \varepsilon \geq \tau \)), this concludes the proof of the equality cases.

\[ \text{Corollary 3.6.} \] Let \( S \) be a smooth surface of Picard number one and let \( L \) be an ample line bundle on \( S \). Let \( x \in S \) be a closed point. Then \( \delta_x(L) \geq \frac{3}{\tau_x(L)} \) and equality holds if and only if \( \varepsilon_x(L) = \tau_x(L) \) or \( L \sim_Q \tau_x(L)C \) for some irreducible curve \( C \subseteq S \) passing through \( x \).

\[ \text{Proof.} \] This is immediate from Lemma \( \text{(3.2)} \) and Lemma \( \text{(2.4)} \).

\[ \text{Remark 3.7.} \] The corollary is false without the Picard number one assumption. Consider for example \( S = \mathbb{P}^1 \times \mathbb{P}^1 \) and take \( L \) to be the line bundle of bi-degree \((a, b)\) with \( 0 < a \leq b \). Then by \( \text{Zhu20c} \) Theorem 1.2 we know that \( \delta(L) = \frac{2}{b} \). On the other hand it is not hard to see that \( \tau_x(L) = a + b \) for all \( x \in S \) and therefore \( \delta_x(L) \geq \frac{3}{\tau_x(L)} \) only when \( b \leq 2a \).
**Proof of Theorem 3.1.** We prove by induction on the dimension $n$. When $n = 2$, the only part that is not covered by Lemma 3.2 is the assertion that $\delta_x(L)$ is computed by any $H \in |L|$ when $\varepsilon_x(L) = 1 = (L^2)$ and $\delta_x(L) = 3$. However, this follows immediately from the fact that $\frac{A_X(H)}{S(L; H)} = 3$ for any $H \in |L|$ (by Lemma 3.5). When $n \geq 3$, we have

$$\delta_x(L|H_1) \geq \frac{n}{(L^{n-1} \cdot H_1)} \varepsilon_x(L|S) = \frac{n}{(L^n)} \varepsilon_x(L|S)$$

by induction hypothesis. By [AZ20, Lemma 4.6], we then have

$$\delta_x(L) \geq \min \left\{ n + 1, \frac{n+1}{n} \delta_x(L|H_1) \right\} \geq \min \left\{ n + 1, \frac{n+1}{(L^n)} \cdot \varepsilon_x(L|S) \right\}.$$ 

Since $\varepsilon_x(L|S) \leq \sqrt{(L|S)^2} = (L|S) = (L^n)$, we obtain

$$\delta_x(L) \geq \frac{n+1}{(L^n)} \cdot \varepsilon_x(L|S).$$

Suppose that equality holds. Let $v$ be any valuation that computes $\delta_x(L)$. By [AZ20, Lemma 4.6] and the above discussion, we have either $\delta_x(L) = n+1$ and $\varepsilon_x(L|S) = (L^n) = 1$, or $\delta_x(L|H_1) = \frac{n+1}{(L^n)} \cdot \varepsilon_x(L|S)$. In the former case, we also have $\tau_x(L|S) = \varepsilon_x(L|S) = 1$ since $\varepsilon_x(L|S) = \sqrt{(L|S)}$. By the same argument as in the $n = 2$ case, we also know that in this case $\delta_x(L)$ is computed by any $H \in |L|$. In the latter case, by [AZ20, Lemma 4.6] we also know that $C_X(v) \not\subseteq H_1$ and that for every irreducible component $Z$ of $C_X(v) \cap H_1$ containing $x$, there exists a valuation $v_1$ on $H_1$ with center $Z$ that computes $\delta_x(L|H_1)$. By induction hypothesis, either $\varepsilon_x(L|S) = \tau_x(L|S) = \sqrt{(L^n)} > 1$ and $\dim C_{H_1}(v_1) \geq n-3$, in which case $\dim C_X(v) = \dim C_{H_1}(v_1)+1 \geq n-2$; or $\varepsilon_x(L|S) \tau_x(L|S) = (L^n)$ and the center of $v_1$ on $H_1$ is a prime divisor that does not contain $S$. Suppose that we are in the last case. Then $G = C_X(v)$ is also a prime divisor that does not contain $S$. Since $v$ computes $\delta_x(L)$, we have $\frac{1}{S(L; G)} = \delta_x(L) = \frac{n+1}{(L^n)} \cdot \varepsilon_x(L|S)$. As in the proof of Lemma 3.2, we then obtain

$$\varepsilon_x(L|S) \leq \varepsilon_x(L|S) \cdot \mult_x(G|S) \leq (L|S \cdot G|S) = (L^{n-1} \cdot G) \leq \frac{(L^n)}{(n+1)S(L; G)} = \varepsilon_x(L|S).$$

Hence equality holds everywhere and $L \equiv \tau_x(L|S)G$ by Lemma 3.3 as in the proof of Lemma 3.2. This completes the proof. 

4. **Hypersurfaces**

As a first application of Theorem 3.1, in this section we prove the uniform K-stability of the following hypersurfaces.

**Theorem 4.1.** Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of Fano index $r \geq 3$ and dimension $n \geq r^3$. Then $X$ is uniformly K-stable.
The main difficulty we need to overcome in order to apply Theorem 3.1 in this situation is that the Seshadri constants on complete intersection surfaces are not always large enough. For example, for any effective \(\mathbb{Q}\)-divisor \(D \sim_{\mathbb{Q}} L\) (where \(L\) is the hyperplane class) and any general complete intersection surface \(S\) passing through some fixed \(x\) we have \(\varepsilon_x(L_S) = \frac{(L^n_S)}{\text{mult}_x(D|_S)} \leq \frac{(L^n)}{\text{mult}_x(D)}\). If there exists some \(D\) such that \(\text{mult}_x D\) is relatively large (more precisely, if \(\tau_x(L) > \frac{m+1}{r}\), then we will not be able to derive \(\delta_x(X) \geq 1\) directly through Theorem 3.1. Thus we need to analyze these “bad” loci. This is done in the next two lemmas. In particular, it turns out that the “bad” locus corresponds exactly to points that support divisors of high multiplicities.

**Lemma 4.2.** Let \(X \subseteq \mathbb{P}^N\) be a smooth variety of dimension \(n \geq 4\), Picard number one and degree \(d\). Let \(x \in X\) be a closed point and let \(L\) be the hyperplane class. Assume that for some constant \(c > \sqrt{d}\) and for any general hyperplane section \(Y \subseteq X\) containing \(x\) we have \(\tau_x(L_Y) \geq c\). Then \(\tau_x(L) \geq c\).

**Proof.** For each \(t \in \mathbb{P}H^0(X, \mathcal{O}_X(1) \otimes m_x)\), let \(Y_t \subseteq X_t\) be the corresponding hyperplane sections containing \(x\). When \(t\) is general, \(Y_t\) is smooth by Bertini theorem and has Picard number one by Lefschetz theorem. Since \(L\) is very ample, we have \(\eta_x(L_{Y_t}) \leq \sqrt{d}\) (see the remark before Lemma 2.3). It then follows from Lemma 2.3 and the assumption that there exists a unique irreducible \(\mathbb{Q}\)-divisor \(0 \leq D_t \sim_{\mathbb{Q}} L_{Y_t}\) on \(Y_t\) such that \(\text{mult}_x D_t \geq c\). By a standard Hilbert scheme argument, we may also assume that \(mD_t\) is integral for some fixed integer \(m > 0\).

We first treat a special case. Suppose that a general \(D_t\) is covered by lines passing through \(x\). Let \(Z \subseteq X\) be the union of all lines passing through \(x\). Then \(\text{Supp}(D_t) \subseteq Z \cap Y_t\) and hence \(Z\) has codimension at most one. Note that \(Z \neq X\) since otherwise \(X\) is a cone over its hyperplane section, but \(X\) is smooth it must be a linear subspace and it is easy to see that the assumption of the lemma is not satisfied. If \(Z_1, \ldots, Z_k \subseteq Z\) are the irreducible components of codimension one in \(X\), then as \(\dim Z_i \geq 3\), its image under the projection from \(x\) has dimension at least 2, hence \(Z_i \cap Y_t\) is irreducible for general \(t\) by Bertini theorem. Since \(D_t\) is also irreducible and is swept out by lines containing \(x\), we deduce that \(\text{Supp}(D_t) = Z_i \cap Y_t\) for some \(1 \leq i \leq k\). As \(X\) has Picard number one, there exists some \(\lambda_i > 0\) such that \(D = \lambda_i Z_i \sim_{\mathbb{Q}} L\). By comparing degrees, we then have \(D_t = D|_{Y_t}\). Since \(Y_t\) is a general hyperplane section, we also have \(\text{mult}_x D = \text{mult}_x D_t \geq c\). This proves the lemma in this special case.

In the sequel, we may assume that \(D_t\) is not covered by lines containing \(x\). In particular, the projection from \(x\) defines a generically finite rational map on \(D_t\). Since \(\dim D_t \geq 2\), we see that \(D_t \cap Y_s\) is irreducible for general \(s, t \in \mathbb{P}H^0(X, \mathcal{O}_X(1) \otimes m_x)\) by Bertini theorem. Note that each \(D_t\) is also a codimension two cycle on \(X\). If there exists some general \(s \neq t\) such that
$D_s \cap D_t$ has codimension 4 (here we need $n \geq 4$ to ensure that $D_s \cap D_t$ is nonempty), then we get
\[
d = \deg(D_s \cdot D_t) \geq \text{mult}_x D_s \cdot \text{mult}_x D_t \geq c^2 > d,
\]
a contradiction. Thus $D_s \cap D_t$ contains a divisor on both $D_s$ and $D_t$. Clearly this divisor is contained in $Y_s \cap D_t$, which is irreducible for general $s, t$. It follows that
\[
\text{Supp}(Y_s \cap D_t) \subseteq \text{Supp}(D_s \cap D_t) \subseteq \text{Supp}(D_s).
\]
Now consider a general pencil $\ell \in \mathbb{P}H^0(X, \mathcal{O}_X(1) \otimes m_x)$ and let $G \subseteq X$ be the divisor swept out by $\text{Supp}(D_t)$ for general $t \in \ell$. In other words, $G$ is the image of the universal divisor $D \subseteq Y$ under the natural evaluation map $ev: Y \rightarrow X$, where $Y \rightarrow \ell$ is the corresponding family of hyperplane section. Since $D_t$ is irreducible for general $t$, we see that $D$ and $G$ are both irreducible. Since $X$ has Picard number one, we have $G \sim_{\mathbb{Q}} rL$ for some $r \in \mathbb{Q}$. Let $D = \frac{1}{r}G$. We claim that $\text{mult}_x D \geq c$. Indeed, for general $t \in \ell$ and $s \in \mathbb{P}H^0(X, \mathcal{O}_X(1) \otimes m_x)$, we know that $G \cap Y_s$ is irreducible by Bertini theorem as before and $\text{Supp}(Y_s \cap D_t) \subseteq D_s$ by the previous steps. As $t$ varies, the locus $\text{Supp}(Y_s \cap D_t)$ sweeps out a divisor on $Y_s$, which is necessarily contained in both $D_s$ and $G \cap Y_s$. Since $D_s$ and $G \cap Y_s$ are both irreducible, we deduce that they are proportional to each other. By comparing degrees, we see that $D_s = D|_{Y_s}$. As $Y_s$ is a general hyperplane section, this implies $\text{mult}_x D = \text{mult}_x D_s \geq c$ and finishes the proof.}

**Lemma 4.3.** Let $X \subseteq \mathbb{P}^N$ be a smooth threefold of Picard number one and degree $d$. Let $L$ be the hyperplane class and let $x \in X$ be a closed point that is contained in at most finitely many lines on $X$. Assume that a very general hyperplane section $Y \subseteq X$ containing $x$ has Picard number one and for some constant $c > \sqrt[3]{d^2}$, we have $\tau_x(L_Y) \geq c$. Then $\tau_x(L) \geq c$.

**Proof.** Let $Y_t \subseteq X_t$ be the hyperplane sections corresponding to $t \in \mathbb{P}H^0(X, \mathcal{O}_X(1) \otimes m_x)$.

As in the proof of Lemma 4.2 when $t$ is very general, there exists a unique irreducible $\mathbb{Q}$-divisor $0 \leq D_t \sim_{\mathbb{Q}} L|_{Y_t}$ such that $\text{mult}_x D_t \geq c$. Note that $D_t$ is a curve on $X$. Since $(L^3) = d > \left(\frac{d}{c}\right)^3$, there exists a $\mathbb{Q}$-divisor $0 \leq D \sim_{\mathbb{Q}} L$ on $X$ such that $\text{mult}_x D > \frac{d}{c}$. Since $X$ has Picard number one, we may further assume as before that $D$ is irreducible. Since
\[
(D \cdot D_t) = d < \text{mult}_x D \cdot \text{mult}_x D_t,
\]
we see that $\text{supp}(D_t) \subseteq \text{supp}(D)$. By assumption, the projection from $x$ defines a generically finite rational map on $D$, hence $D \cap Y_t$ is irreducible by Bertini theorem and $D_t = D|_{Y_t}$ as in the proof of Lemma 4.2. As before this implies $\text{mult}_x D = \text{mult}_x D_t \geq c$. \[ \Box \]

We now prove Theorem 4.1. The key point is that, while the “bad” locus is in general non-empty, it consists of at most a countable number of points.
This allows us to use the K-stability criterion [AZ20] Lemmas 4.23 from our previous work and then we can just apply Theorem 3.1 to conclude.

Proof of Theorem 4.1. By [AZ20] Lemmas 4.23 and 4.25, it suffices to show that \( \delta_Z(X) \geq \frac{n+1}{n} \) for any subvariety \( Z \subseteq X \) of dimension \( \geq 1 \). Let \( L \) be the hyperplane class and let \( d := (L^n) = n + 2 - r \). By assumption \( n \geq d \) and \( d \geq 26 \). We first show that \( \tau_x(L) \leq \sqrt{d} + 1 \) for a very general point \( x \in Z \). Suppose not, then by Lemma 2.3 there exists a unique irreducible \( \mathbb{Q} \)-divisor \( 0 \leq D_x \sim_{\mathbb{Q}} L \) such that \( \mu x D_x > \sqrt{d} + 1 \) for all \( x \in Z \) (it is easy to see that \( \eta_x(L) \leq \sqrt{d} \)). By a standard Hilbert scheme argument, we can find an irreducible and reduced divisor \( G \subseteq X \times U \) (where \( U \subseteq Z \) is an affine open subset) and an integer \( m > 0 \) such that \( G \in |\text{pr}_2^*O_X(mL)| \) and \( G|_{X \times \{x\}} = mD_x \) for very general \( x \in Z \). In particular, by [EKL95] Lemma 2.1 we have \( \mu U G = \mu x (mD_x) > m(\sqrt{d} + 1) \), where \( W \subseteq X \times U \) is the graph of the embedding \( U \hookrightarrow X \). By [EKL95] Proposition 2.3, there exists a divisor \( G' \in \left|\text{pr}_2^*O_X(mL)\right| \) such that \( G \not\subseteq \text{Supp}(G') \) and \( \mu U G' > m(\sqrt{d} + 1) − 1 \geq m\sqrt{d} \). Let \( D'_x = \frac{1}{m} G'|_{X \times \{x\}} \sim_{\mathbb{Q}} L \) for general \( x \in Z \), then \( D_x \) and \( D'_x \) have no common components and \( \mu x D'_x > \sqrt{d} \). It follows that

\[
d = \deg(D_x \cdot D'_x) \geq \mu x D_x \cdot \mu x D'_x > (\sqrt{d} + 1)\sqrt{d} > d,
\]
a contradiction. Hence \( \tau_x(L) \leq \sqrt{d} + 1 \) for a very general point \( x \in Z \).

Fix some \( x \in Z \) with \( \tau_x(L) \leq \sqrt{d} + 1 \) and let \( Y \subseteq X \) be a general linear subspace section of dimension 3 that passes through \( x \). By Lefschetz theorem, we know that \( Y \) has Picard number one. Let \( S \in |H^0(Y, O_Y(2L) \otimes m_x)| \) be a very general member. By [DGF14] Theorem 1.1], the surface \( S \) also has Picard number one. By Lemma 1.2 we have \( \tau_x(L_S) \leq \sqrt{d} + 1 \leq \sqrt{d^2} \) (as \( d \geq 26 \)) and hence \( \tau_x(2L_S) = 2\tau_x(L_S) \leq 2\sqrt{d^2} \). Since \( (2L \cdot C) \geq 2 \) for any curves \( C \subseteq Y \), the threefold \( Y \) does not contain any lines under the embedding given by \( |2L| \), thus by Lemma 1.3 we also have \( \tau_x(2L_S) \leq 2\sqrt{d^2} \). By Corollary 3.6 we get \( \delta_x(2L_S) \geq \frac{3}{2\sqrt{d^2}} \), thus by [AZ20] Lemma 4.6, we have \( \delta_x(2L_Y) \geq \frac{2}{\sqrt{d^2}} \), or \( \delta_x(L_Y) \geq \frac{4}{\sqrt{d^2}} \). A repeated use of [AZ20] Theorem 4.6] then yields

\[
\delta_Z(L) \geq \delta_x(L) \geq \frac{n + 1}{4} \delta_x(L_Y) \geq \frac{n + 1}{\sqrt{d^2}},
\]
hence \( \delta_Z(X) = \frac{\delta_Z(L)}{r} \geq \frac{n + 1}{n} \) as long as \( n \geq r\sqrt{d^2} \). Since \( n \geq d \), this is automatic when \( n \geq r^3 \). The proof is now complete. \( \Box \)

5. Fano threefolds

As another application of Theorem 3.1 we prove in this section the uniform K-stability of most Fano threefolds of Picard number one. Recall that the degree of a Fano threefold \( X \) of Picard number one is defined to be \( (H^3) \),
where $H$ is the ample generator of Pic$(X)$. Using the classification of Fano threefolds \[IP99\], we may restate Theorem C as follows.

**Theorem 5.1.** Let $X$ be a smooth Fano threefold of Picard number one. Assume that $X$ has index two and degree at most 4, or it has index one and degree at most 16. Then $X$ is uniformly $K$-stable.

We remark that while Fano threefolds of Picard number one have been fully classified (most of them are complete intersections in rational homogeneous manifolds), we only need very little information from this classification. As will be seen below, the two key properties of Fano threefolds we need are the following.

1. For any closed point $x \in X$ on the Fano threefold, there exists some smooth member $S \in |H|$ passing through $x$.
2. Most Fano threefolds $X$ of Fano index one are cut out by quadrics and there are only finitely many lines through a fixed point.

Before presenting the details, let us give a brief summary of the proof of Theorem 5.1 and indicate where the above two properties are used. We focus on the case when $X$ has Fano index one (the index two cases are easier). Clearly, point (1) allows us to apply Theorem 3.1 and immediately obtain

\[ \delta_x(X) \geq \frac{4\epsilon_x(H|S)}{(H^3)}. \]

Note that $S$ is a K3 surface. Seshadri constants on K3 surfaces have been studied by \[Knu08\] and usually they are easier to compute when the surfaces have Picard number one, thus we seek to choose the surface $S$ carefully so that it not only contains $x$, but also has Picard number one. This is a Noether-Lefschetz type problem with a base point constraint and will be studied in Subsection 5.3. Note that this requirement on $S$ can fail on certain Fano threefolds (e.g. quartic threefolds with generalized Eckardt points), but we will show that it is satisfied whenever point (2) holds. Fortunately, in the remaining cases already the trivial bound of $\epsilon_x(H|S)$ is enough to imply $K$-stability through (5.1). Once we find a surface $S$ of Picard number one, the argument is relatively straightforward using Seshadri constants calculations and Theorem 3.1 except when $X$ has degree 16. In this case we only get $\delta(X) \geq 1$ (i.e. K-semistability) and need to study the equality condition a bit further. This is done in Subsection 5.1 by analyzing the inductive step, which is based on inversion of adjunction, in the proof of Theorem 3.1.

**5.1. Equality conditions in adjunction.** As indicated in the summary of proof above, the goal of this subsection is to further analyze the equality conditions in Theorem 5.1. The main technical results are Lemma 5.3 and Corollary 5.6 which list a few constraints that need to be satisfied in order to have equality in the adjunction of stability thresholds. They will play an important role when treating Fano threefolds of degree 16 and complete intersections of two quadrics.
Lemma 5.2. Let $X$ be a projective variety of dimension $n$, let $L$ be an ample line bundle on $X$ and let $v$ be a valuation of linear growth on $X$. Then $S(L; v) = \frac{n}{n+1} T(L; v)$ if and only if

\begin{equation}
\frac{\text{vol}(L; v \geq t)}{(L^n)} = 1 - \left( \frac{t}{T(L; v)} \right)^n
\end{equation}

for all $0 \leq t \leq T(L; v)$. If in addition $v$ is quasi-monomial, then its center is a closed point on $X$.

Remark 5.3. In general we have $S(L; v) \leq \frac{n}{n+1} T(L; v)$ (see for example [AZ20 Lemma 4.2]), thus the above statement gives a description of the equality conditions. When $v$ is divisorial, this is essentially proved in [Fuj19a Proposition 3.2].

Proof. Assume that $S(L; v) = \frac{n}{n+1} T(L; v)$. Let $T = T(L; v)$. After replacing $L$ by $rL$ for some sufficiently large integer $r$ we may assume that $L$ is very ample. Let $H \in |L|$ be a general member, let $W^*_v$ be the refinement by $H$ of the complete linear series $V_v$ associated to $L$, and let $\mathcal{F}$ be the filtration on $W^*_v$ induced by the filtration $\mathcal{F}_v$ on $R(X, L)$. Concretely, $W^*_v$ is $\mathbb{N}^2$-graded and we have

\begin{equation}
\mathcal{F}^\lambda W^*_{m,j} = \text{Im}(\mathcal{F}^\lambda H^0(X, \mathcal{O}_X((m - j)L)) \to H^0(H, \mathcal{O}_H((m - j)L))).
\end{equation}

In particular, $W^*_{m,j} = H^0(H, \mathcal{O}_H((m - j)L))$ when $m - j \gg 0$. By the definition of the pseudo-effective threshold $T(L; v)$, it follows that

$$\text{Supp}(W^*_{v} \cap (\{1\} \times \mathbb{R}) = [0, 1 - t/T]\]$$

where $W^*_{v}$ $(0 < t < T)$ is the multi-graded linear series given by $W^*_{m,j} = \mathcal{F}^m W^*_{m,j}$. By Lemma 2.39 we also have $S(L; v) = S(V^*_v; \mathcal{F}_v) = S(W^*_v; \mathcal{F})$. Let

$$f(t, \gamma) = \text{vol}(W^*_v(\gamma) (0 < t < T, 0 < \gamma < 1 - t/T).$$

It is clear that $f(t, \gamma) \leq \text{vol}(W^*_v(\gamma) = (1 - \gamma)^{n-1}(L^n)$. Note that $\text{vol}(W^*_v) = \text{vol}(L) = (L^n)$. We then obtain

$$S(L; v) = S(W^*_v; \mathcal{F}) = \frac{1}{(L^n)} \int_0^T \text{vol}(W^*_v) dt = \frac{n}{(L^n)} \int_0^T dt \int_0^{1-t/T} f(t, \gamma) d\gamma \leq \int_0^T dt \int_0^{1-t/T} n(1 - \gamma)^{n-1} d\gamma = \frac{n}{n+1} T,$$

where the second equality follows from [AZ20 Corollary 2.22] while the third equality is implied by [AZ20 Lemma 2.23]. Since $S(L; v) = \frac{n}{n+1} T$ by assumption, we see that $f(t, \gamma) = (1 - \gamma)^{n-1}(L^n)$ for all $t, \gamma$. By another
application of [AZ20, Lemma 2.23] (used in the second equality below), we then have
\[
\text{vol}(L; v) = \text{vol}(W^f_\bullet) = n \int_0^{1 - \frac{t}{T}} f(t, \gamma) d\gamma \\
= \int_0^{1 - \frac{t}{T}} n(1 - \gamma)^{n-1}(L^n) d\gamma = \left(1 - \frac{t}{T}\right)^n (L^n),
\]
which proves \((5.2)\). Conversely, it is easy to check that \(S(L; v) = \frac{n}{n+1} T\) as long as \((5.2)\) holds. This proves the first part of the lemma.

Suppose next that \(v\) is quasi-monomial and \(
C_X(v) \geq 1\). Let \(\pi: Y \to X\) be a log resolution and let \(Z = C_Y(v)\). Since \(\dim C_X(v) \geq 1\), we have \((Z \cdot \pi^*L) \neq 0\) by the projection formula, hence \(H_Y\) intersects \(Z\) where \(H_Y\) is the strict transform of \(H\). By Izumi’s inequality we have \(\text{lct}_Z(f) \geq \frac{1}{\text{mult}_Z(f)}\). On the other hand \(\text{lct}_Z(f) \leq \frac{A_Y(v)}{v(f)}\) by definition, thus \(v(f) \leq A_Y(v)\text{mult}_Z(f)\) for any \(f \in \mathcal{O}_{Y,Z}\). Up to rescaling of the valuation \(v\) we may assume that \(A_Y(v) = 1\). Thus by \((5.3)\) we see that \(\mathcal{F}^\lambda W_{m,j} \subseteq H^0(\mathcal{O}_Y, \mathcal{O}_Y^{\langle m-j \rangle \pi^*L})\). As \(\pi^*L\) is nef and big, by [FKL16, Theorem A\(v\)] we obtain \(\text{vol}_{W_\bullet}(\gamma) < \text{vol}_{W_\bullet}(\gamma)\) for all \(0 < t, \gamma < 1\). By the proof above it then follows that \(S(L; v) < \frac{n}{n+1} T\). This proves the second part of the lemma.

**Lemma 5.4.** Let \(X\) be a projective variety of dimension \(n \geq 2\) with klt singularities and let \(L\) be an ample line bundle on \(X\) such that the linear system \(|L|\) is base point free. Assume that \(\delta(L)\) is computed by some valuation \(v\) on \(X\) with \(\dim C_X(v) \geq 1\) and
\[
\delta(L) \leq \frac{n+1}{n} \delta(Z|_H)
\]
for some general member \(H \in |L|\) and some irreducible component \(Z\) of \(C_X(v) \cap H\). Then one of the following holds:

1. \(\frac{A_Y(v)}{T(L,v)} < \frac{n-1}{n+1} \delta(L)\), or
2. \(\frac{A_Y(v)}{T(L,v)} = \frac{n}{n+1} \delta(L), \dim C_X(v) = 1, \text{ and } \frac{\text{vol}(L; v) T}{(L^n)} = 1 - n < T^{n-1} + (n-1) \left(\frac{T}{n}\right)^n \text{ for all } 0 \leq t \leq T := T(L; v).\)

**Remark 5.5.** Note that this implies \(\alpha(L) \leq \frac{n-1}{n+1} \delta(L)\), which is stronger than the usual inequality \(\alpha(L) \leq \frac{n}{n+1} \delta(L)\) (see for example [BJ20, Theorem A]).

**Proof.** Let \(W_\bullet\) be the refinement by \(H\) of the complete linear series associated to \(L\) and let \(L_0 = L|_H\). Note that \(W_\bullet\) is almost complete, \(F(W_\bullet) = 0\) and \(c_1(W_\bullet) = \frac{n}{n+1} L_0\) by [AZ20, Example 2.28 and (3.1)]. Let \(a_\bullet(v)\) be the valuation ideals of \(v\), i.e., \(a_\bullet(v) = \{f \in \mathcal{O}_X \mid v(f) \geq m\}\). Let \(a_m = a_m(v)|_H\) and let \(v_0\) be a quasi-monomial valuation on \(H\) with center \(Z\) that computes \(\text{lct}_Z(H; a_\bullet)\), which exists by [Xu20b, Theorem 1.1]. After rescaling, we may assume that \(A_H(v_0) = A_X(v)\). By inversion of adjunction, \(\text{lct}_Z(H; a_\bullet) \leq \text{lct}_Z(X; a_\bullet(v))\). Since \(v\) calculates \(\text{lct}_Z(X; a_\bullet(v)) = A_X(v)\) by [BJ20, Proposition 4.8], we deduce that \(v_0(a_\bullet) \geq 1\) and hence \(a_m(v)|_H \subseteq \]
If equality holds, then by Lemma 5.2 we know that the first one, denoted by $F_v$, is the restriction of the filtration $F_v$ on $X$ induced by the valuation $v$, while the second one $F_{v_0}$ is induced by the valuation $v_0$. From the previous argument we see that $F_{v_0}$ dominates $F_v$, i.e. $F^\lambda v W_v \subseteq F^\lambda_{v_0} W_v$ for all $\lambda \geq 0$. By [AZ20 Corollary 2.22], this implies

$$S(W^*_v; F_v) \leq S(W^*_v; F_{v_0}) = S(W^*_v; v_0),$$

and when equality holds we have $T(W^*_v; F_v) = T(W^*_v; F_{v_0})$. From the construction, it is clear that $T(W^*_v; F_v) = T(L; v)$ and $T(W^*_v; F_{v_0}) = T(L; v_0)$. By Lemma 2.9, we also have

$$S(L; v) = S(W^*_v; F_v),$$

thus as $v$ computes $\delta(L)$ and $H \in |L|$ is general, we deduce that

$$A_H(v_0) = A_X(v) = \delta(L)S(L; v) \leq \delta(L)S(W^*_v; v_0).$$

On the other hand, we have $S(W^*_v; v_0) = \frac{a}{n+1} S(L_0; v_0)$ by [AZ20] Lemma 2.29. Combined with our assumptions we obtain

$$\delta(L)S(W^*_v; v_0) \leq \frac{n+1}{n} \delta_Z(L_0) \cdot \frac{n}{n+1} S(L_0; v_0) = \delta_Z(L_0)S(L_0; v_0) \leq A_H(v_0).$$

Therefore equality holds everywhere (including in (5.4)) and we get

$$\delta(L) = \frac{n+1}{n} \delta_Z(L_0), \quad A_X(v) = A_T(v_0) = \delta_Z(L_0)S(L_0; v_0)$$

i.e. $v_0$ computes $\delta_Z(L_0)$, and

$$T(L; v) = T(W^*_v; F_v) = T(W^*_v; F_{v_0}) = T(L; v_0).$$

Note that $S(L_0; v_0) \leq \frac{a-1}{n} T(L_0; v_0)$ by [AZ20] Lemma 4.2. It follows that

$$A_X(v) \leq \frac{n-1}{n} \delta_Z(L_0)T(L_0; v_0) = \frac{n-1}{n+1} \delta(L)T(L; v).$$

If equality holds, then by Lemma 5.2 we know that $Z$ is a closed point and

$$\frac{\text{vol}(L_0; v_0 \geq t)}{(L^n)} = 1 - \left( \frac{t}{T} \right)^{n-1},$$

where $0 \leq T \leq T = T(L_0; v_0) = T(L; v)$. It follows that $\dim C_X(v) = 1$. Let $W^t_\gamma$ be the multi-graded linear series given by $W^t_{m,j} = F^m_{v_0} W_{m,j}$. Since $W_{m,j} = H^0(H, O_H((m-j)L))$ when $m-j \gg 0$, we see that

$$\text{vol}_{W^t_\gamma}(\gamma) = \text{vol}((1-\gamma)L_0; v_0 \geq t)$$

and hence

$$\frac{\text{vol}_{W^t_\gamma}(\gamma)}{(L^n)} = (1-\gamma)^{n-1} - \left( \frac{t}{T} \right)^{n-1}$$
for all $0 \leq \gamma < 1 - \frac{t}{T}$. Since equality holds in (5.4), using [AZ20] Lemma 2.23 we obtain
\[
\text{vol}(L; v \geq t) = \text{vol}(W^{t}_{\bullet}) = n(L^{n}) \int_{0}^{1} \left( 1 - \gamma^{n-1} - \left( \frac{t}{T} \right)^{n-1} \right) d\gamma
\]
\[
= \left( 1 - n \left( \frac{t}{T} \right)^{n-1} + (n-1) \left( \frac{t}{T} \right)^{n} \right) (L^{n})
\]
as desired.

\[\square\]

**Corollary 5.6.** Under the assumption of Lemma [5.4], assume in addition that $X$ is $\mathbb{Q}$-factorial, $\rho(X) = 1$ and $C_{X}(v)$ has codimension $\geq 2$ in $X$. Then at least one of the following holds:
\[
\frac{A_{X}(v)}{T(L; v)} \leq \frac{n-1}{n+1} \delta(L), \quad \text{or} \quad \frac{A_{X}(v)}{\eta(L; v)} \leq \frac{n-1}{n+1} \delta(L).
\]

**Proof.** Denote $T := T(L; v)$ and $\eta := \eta(L; v)$. It suffices to show that $\eta = T$ in the second case of Lemma [5.4]. Suppose not, i.e. $\eta < T$. Then by Lemma 2.23, there exists an irreducible $\mathbb{Q}$-divisor $D_{0} \sim_{\mathbb{Q}} L$ on $X$ such that $v(D_{0}) = T$ and for any $t \in (\eta, T)$ and any effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ with $v(D) \geq t$ we have $D = \frac{T-t}{T-\eta}D_{0} + \frac{T-t}{T-\eta}G$ where $G \sim_{\mathbb{Q}} L$ is effective and $v(G) \geq \eta$. It follows that
\[
\text{vol}(L; v \geq t) = \left( \frac{T-t}{T-\eta} \right)^{n} \text{vol}(L; v \geq \eta),
\]
which contradicts the expression from Lemma [5.4] (note that $n \geq 3$ since the curve $C_{X}(v)$ has codimension $\geq 2$ in $X$). Thus $\eta = T$ and we are done. \[\square\]

5.2. **The index two case.** As an application of Theorem 3.1 and the results from the previous subsection, we now prove:

**Theorem 5.7.** Let $X$ be a Fano threefold of Picard number one, Fano index two and degree at most 4. Then $X$ is uniformly K-stable.

**Proof.** Let $H$ be the ample generator of Pic($X$) and let $d = (H^{3})$ be the degree of $X$. Using the classification of Fano threefolds (see for example [IP99]), it is straightforward to check that for any closed point $x \in X$ there exists some smooth member $S \in |H|$ passing through $x$ (see for example the proof of [AZ20] Corollary 4.9(5) for the degree 1 case, the other cases are much easier). By adjunction, $S$ is a del Pezzo surface of degree $d$ and $H|_{S} \sim -K_{S}$. Since $\delta_{x}(X) = \frac{1}{2} \delta_{x}(H)$, it suffices to show $\delta_{x}(H) > 2$.

Suppose first that $d = 1$. By [Bro06] Théorème 1.3, we know that $\varepsilon_{x}(-K_{S}) \geq \frac{1}{2}$, hence by Theorem 5.1 we obtain $\delta_{x}(H) \geq 2$. Moreover the equality cannot hold since $1 = (H^{3}) \neq \left( \frac{1}{2} \right)^{2}$ and $H$ is a primitive element in Pic($X$). Thus $\delta_{x}(H) > 2$ and we are done in this case.

Similarly, when $d = 2$, by [Bro06] Théorème 1.3 we know that $\varepsilon_{x}(-K_{S}) \geq 1$, thus the same argument as above gives $\delta_{x}(H) > 2$.
Suppose next that \( d = 3 \). If \( x \in X \) is a generalized Eckardt point, then \( \delta_x(X) = \frac{2}{3} > 1 \) by \cite[Theorem 4.18]{AZ}. If \( x \in X \) is not a generalized Eckardt point, then there are only finitely many lines on \( X \) passing through \( x \), thus if \( S \subseteq X \) is general then \( x \) is not contained in any lines on \( S \). By \cite[Théorème 1.3]{Bro06}, we have \( \varepsilon_x(-K_X) \geq \frac{3}{2} \), hence \( \delta_x(H) \geq 2 \) by Theorem 3.1. Moreover, equality cannot hold as before. Thus \( \delta_x(H) > 2 \) and we are done in this case.

Finally assume that \( d = 4 \). Note that \( X \) is a complete intersection of two quadrics in this case. There is a pencil of tangent hyperplanes at any closed point \( x \) and every line on \( X \) passing through \( x \) is contained in the base locus of this pencil, which is a complete intersection curve of degree 4. It follows that there are at most 4 lines containing \( x \) on \( X \). In particular, we may arrange that \( x \) is not contained in any lines on \( S \). By \cite[Théorème 1.3]{Bro06}, we know that \( \varepsilon_x(-K_X) \geq 2 \), hence Theorem 3.1 yields \( \delta_x(H) \geq 2 \). Suppose that equality holds. Then by Theorem 3.1 there exists some valuation \( v \) with positive dimensional center on \( X \) such that \( \frac{A_X(v)}{S(H;v)} = 2 \). We show that this is impossible. Indeed, if the center \( C_X(v) \) is a prime divisor \( D \) on \( X \), then we may assume that \( v = \text{ord}_D \). Since Pic(\( X \)) is generated by \( H \), we have \( D \sim rH \) for some \( r \geq 1 \). By Lemma 3.5 we have \( S(H;v) = \frac{1}{4r} \), hence \( \frac{A_X(v)}{S(H;v)} = 4r > 2 \), a contradiction. Hence we may assume that the center \( C_X(v) \) is a curve \( C \) on \( X \). If \( T \in |H| \) is a general member and \( x \in T \cap C_X(v) \), then \( \varepsilon_x(-K_T) \geq 2 \) and \( \delta_x(-K_T) \geq \frac{3}{2} \) by \cite[Théorème 1.3]{Bro06} and Theorem 3.1 as above. Thus the assumptions of Corollary 5.6 are satisfied and we deduce that either \( A_X(v) < T(H;v) \) or \( A_X(v) \leq \eta(H;v) \). The first case is impossible by \cite[Theorem 6.1]{CS08}, thus it remains to exclude the other possibility.

By the definition of the movable threshold \( \eta(H;v) \), for any \( 0 < \varepsilon \ll 1 \) we can find two effective \( \mathbb{Q} \)-divisors \( D_1, D_2 \sim_{\mathbb{Q}} H \) without common components such that \( v(D_i) \geq (1 - \varepsilon) \eta(H;v) \) \((i = 1, 2) \). Let \( m \) be a sufficiently divisible integer and let \( Z \subseteq X \) be the complete intersection subscheme \( mD_1 \cap mD_2 \). Note that \( \deg Z = 4m^2 \). If \( \deg C \geq 2 \), then \( \text{mult}_C Z \leq 2m^2 \) and by \cite[Theorem 0.1]{JFEM04}, applied at the generic point of \( C \), we have \( \text{lc}_{C}(X;I_Z) \geq \frac{\sqrt{2}}{m} \). In particular,

\[
A_X(v) \geq \frac{\sqrt{2}}{m} v(I_Z) = \sqrt{2} \min\{v(D_1), v(D_2)\} \geq \sqrt{2}(1 - \varepsilon) \eta(H;v) > \eta(H;v),
\]
a contradiction. If \( \deg C = 1 \), i.e. \( C \) is a line on \( X \), then \( \text{mult}_C Z \leq 4m^2 \) and by \cite[Proposition 3.4.1(ii)]{IP99}, we have \( \text{mult}_C D_i < \frac{3}{2} < 2 \) for some \( i = 1, 2 \). Hence \( I_Z \not\subseteq \mathcal{I}_C^{4m} \) and by \cite[Lemma 2.6]{Zhu20}, applied at the generic point of \( C \), we see that there exists some absolute constant \( \varepsilon_1 > 0 \), which in particular does not depend on \( D_i \) or \( \varepsilon \), such that \( \text{lc}_{C}(X;I_Z) \geq \frac{1+\varepsilon_1}{m} \). It then follows as before that \( A_X(v) > \frac{1+\varepsilon_1}{m} v(I_Z) \geq (1 + \varepsilon_1)(1 - \varepsilon) \eta(H;v) > \eta(H;v) \), a contradiction. Hence the equality \( \delta_x(H) = 2 \) never holds and \( X \) is uniformly K-stable. \( \square \)
5.3. **Noether-Lefschetz for prime Fano threefolds.** In this subsection, we prove the following Noether-Lefschetz type result on Fano threefolds. As explained at the beginning of the section, this is another key ingredient in our study of uniform K-stability when the Fano threefolds have index one.

**Theorem 5.8.** Let $X$ be a Fano threefold of Picard number one, Fano index one and degree $\geq 6$. Let $x \in X$ be a closed point. Then a very general hyperplane section $S \in | - K_X|$ passing through $x$ has Picard number one.

**Remark 5.9.** The assumption that $X$ has degree $\geq 6$ is indeed necessary, since the statement fails on quartic threefolds with generalized Eckardt points.

For the proof of the theorem we first recall the following criterion.

**Lemma 5.10.** Let $X \subseteq \mathbb{P}^N$ be a smooth projective threefold, let $\ell \subseteq (\mathbb{P}^N)^*$ be a Lefschetz pencil of hyperplane sections and let $Y = X \cap H$ where $H$ is a very general member of $\ell$. Assume that the natural map $H^2(X) \to H^2(Y)$ is not surjective. Then the restriction $\text{Pic}(X) \to \text{Pic}(Y)$ is an isomorphism.

**Proof.** This should be well known to experts. By assumption, the vanishing cohomology $H^2(Y, \mathbb{C})_{\text{van}}$, the orthogonal complement of $H^2(X, \mathbb{C})$ relative to the intersection form on $H^2(Y, \mathbb{C})$, has non-trivial intersection with $H^2(Y, \mathbb{C})$, hence is not generated by algebraic classes. By [DK73, Théorème 1.4], this implies that $H^2(Y, \mathbb{C})_{\text{van}} \cap \text{Pic}(Y) = \{0\}$ and therefore every line bundle on $Y$ is a pullback from $X$; in other words, $\text{Pic}(X) \to \text{Pic}(Y)$ is surjective. It is also injective by the Lefschetz hyperplane theorem. \hfill $\square$

In the remaining part of this section, let $X$ be a smooth Fano threefold as in Theorem 5.8 and let $x \in X$ be a closed point. It is well known (see for example [IP99]) that the anti-canonical linear system $| - K_X|$ induces an embedding $X \subseteq \mathbb{P}^N$. Let $\mathcal{P} \subseteq (\mathbb{P}^N)^*$ be the dual projective space parametrizing hyperplanes of $\mathbb{P}^N$ containing $x$. To apply Lemma 5.10, we need to find a Lefschetz pencil in $\mathcal{P}$.

**Lemma 5.11.** A general pencil $\ell \subseteq \mathcal{P}$ is a Lefschetz pencil.

**Proof.** We first recall some notation and results from [Voi03, Section 2.1.1]. Let $Z \subseteq X \times (\mathbb{P}^N)^*$ be the algebraic subset defined by

$$Z = \{(y, H) \mid X_H := X \cap H \text{ is singular at } y\}.$$ 

Let $D_X := \text{pr}_2(Z) \subseteq (\mathbb{P}^N)^*$ be the set of singular hyperplane sections and let $D^0_X \subseteq D_X$ be the subset of hyperplanes $H$ such that $X_H$ has at most one ordinary double point as singularity. Let $W := D_X \setminus D^0_X$. By [Voi03, Corollary 2.8, and the comments thereafter], we have $\dim W \leq N - 2$ and $\mathcal{P}$ intersects transversally with $D^0_X$, away from $Z_x := \text{pr}_1^{-1}(x) \subseteq Z$. Since $\dim Z_x = N - 4 = \dim \mathcal{P} - 3$, a general line $\ell \subseteq \mathcal{P}$ is disjoint from $Z_x$ and thus intersects transversally with $D^0_X$. By [Voi03, Proposition 2.9], $\ell$ is a Lefschetz pencil if and only if $\ell$ is also disjoint from $W$. This would be the
Assuming this claim for now, let us finish the proof of the lemma. Let $R$ be a general smooth point of an irreducible component of $W$ containing $x$ such that $X_H$ has degenerate singularities (resp. more than one ordinary double point). It follows that both $W_i (i = 1, 2)$ have codimension at least 2 in $\mathbb{P}$. 

To this end, consider the closed subset $R := \{y \in X \mid x \in \Lambda_y\} \subseteq X,$ where $\Lambda_y := T_yX \subseteq \mathbb{P}^N$ denotes the tangent space of $X$ at $y$. We claim that

$$\dim R \leq 1.$$  

Assuming this claim for now, let us finish the proof of the lemma. Let $\tilde{Z} = \text{pr}_2^*\mathbb{P}$. By construction, $\tilde{Z}$ is a divisor in $Z$ and $\text{pr}_1: \tilde{Z} \to X$ is a $\mathbb{P}^{N-5}$-bundle away from $R$; in particular, $\tilde{Z}$ is smooth outside $\text{pr}_1^{-1}(R)$. Note that by (5.5), we have

$$\dim \text{pr}_1^{-1}(R) \leq N - 3 < \dim \tilde{Z} = N - 2.$$ 

Let $(y, H) \in \tilde{Z} \setminus \text{pr}_1^{-1}(R)$. As in [Vois03 Lemma 2.7 and Corollary 2.8], $X_H$ has an ordinary double point at $y$ if and only if the restriction of $\text{pr}_2$ to $\tilde{Z}$ is an immersion at $(y, H)$, and in that case $\text{pr}_2^*(T_{\tilde{Z},(y,H)})$ can be identified with the linear subspace of $\mathbb{C}^N = T_{(\mathbb{P}^N)^*,H}$ consisting of functions that vanish at both $x$ and $y$. By generic smoothness and (5.6), this immediately implies that $\dim W_1 \leq N - 3$. On the other hand, let $H$ be a general smooth point of an irreducible component of $W_2$. By construction, $X_H$ has at least two ordinary double points $y_1, y_2$. Since the Fano threefold $X$ is cut out by quadrics and cubics [Isk78 Corollary 2.6], the line $\ell'$ joining $y_1$ and $y_2$ is contained in $X$, otherwise $\ell'$ has intersection number at least 4 with one of those quadrics or cubics. It follows that either $x \in \ell'$, in which case $y_1, y_2 \in R$ and $H \in \text{pr}_2(\text{pr}_1^{-1}(R))$, or $x \not\in \ell'$, in which case $T_{W_2,H} \subseteq \text{pr}_2^*(T_{\tilde{Z},(y_1,H)}) \cap \text{pr}_2^*(T_{\tilde{Z},(y_2,H)})$ has dimension at most $N - 3$. In either case we conclude that $\dim W_2 \leq N - 3$. In other words, both $W_1$ and $W_2$ have codimension at least 2 in $\mathbb{P}$. As explained earlier, this implies the statement of the lemma.

It remains to prove claim (5.5). Let $y \in R \setminus \{x\}$. If $\deg X \geq 8$, then since $X$ is cut out by quadrics [Isk78 Corollary 2.6] and $x \in \Lambda_y$, the line joining $x$ and $y$ is contained in $X$: otherwise as it is tangent to $X$ at $y$, it has intersection number at least 3 with one of the quadrics, a contradiction. Since there are only finitely many lines on $X$ passing through $x$ [IP99 Proposition 4.2.2], we deduce that $\dim R \leq 1$. If $\deg X = 6$, then $X$ is the complete intersection of a quadric (denoted by $Q$) and a cubic. Again for any $y \in R \setminus \{x\}$, the line joining $x$ and $y$ is contained in $Q$, thus $y \in H_0$, where $H_0$ is the tangent hyperplane of $Q$ at $x$. It follows that $R \subseteq H_0 \cap X$. Since $\text{Pic}(X) = \mathbb{Z} \cdot H_0$, we see that $H_0 \cap X$ is irreducible and reduced. As there are only finitely many lines passing through $x$, the linear projection from $x$ is finite on $H_0 \cap X$, therefore by generic smoothness, it
cannot be ramified everywhere. In other words, there exists some smooth point \( y_0 \in H_0 \cap X \) such that \( x \notin T_{y_0}(H_0 \cap X) = H_0 \cap T_{y_0}X \), or equivalently, \( x \notin T_{y_0}X \). Thus \( R \subsetneq H_0 \cap X \) and we deduce that \( \dim R \leq 1 \). This finishes the proof. \( \Box \)

We are now ready to prove Theorem \ref{thm:main}

**Proof of Theorem \ref{thm:main}**. By Lemma \ref{lem:lefschetz-pencil} there exists a Lefschetz pencil of hyperplane sections passing through \( x \). A smooth member \( Y \) of the pencil is a smooth K3 surface, thus \( \dim H^{2,0}(Y) = 1 \); on the other hand, as \( X \) is Fano, we have \( \dim H^{2,0}(X) = 0 \). In particular, the map \( H^{2,0}(X) \to H^{2,0}(Y) \) is not surjective and the theorem follows immediately from Lemma \ref{lem:no-surface} \( \Box \)

5.4. **The index one case**. We are now ready to prove Theorem \ref{thm:index-one}. By Theorem \ref{thm:index-one} the remaining case is:

**Theorem 5.12**. Every smooth Fano threefold \( X \) of Picard number one, Fano index one and degree \( d \leq 16 \) is uniformly K-stable.

**Proof**. Let \( H = -K_X \). We will denote by \( H_Y \) the restriction of the hyperplane class \( H \) to a subvariety \( Y \subseteq X \). Let \( x \in X \) and let \( S \in |H| \) be a very general hyperplane section containing \( x \). When \( d \leq 4 \), it is easy to see that \( \varepsilon_x(H_S) \geq 1 \) since \( H \) is base point free, hence \( \delta_x(H) > 1 \) and \( X \) is uniformly K-stable by Theorem \ref{thm:uniform-K} as in the proof of Theorem \ref{thm:index-one}. Thus for the rest of the proof we may assume that \( d \geq 6 \). By Theorem \ref{thm:main} \( \Pic(S) = \mathbb{Z}[H_S] \).

We claim that \( \tau_x(H_S) \leq 4 \). Suppose not, then for some integer \( m > 0 \) there exists an integral curve \( C \sim -mK_X|_S \) containing \( x \) such that \( \mult_x C > 4m \) (we can assume \( C \) is integral since \( S \) has Picard number one). Since \( \mult_x C \) is an integer, we have \( \mult_x C \geq 4m + 1 \). By adjunction, \( K_S \sim 0 \). It follows that

\[
16m^2 \geq dm^2 = (K_S + C) \cdot C = 2p_a(C) - 2 \geq \mult_x C \cdot (\mult_x C - 1) - 2 \geq 4m(4m + 1) - 2 > 16m^2,
\]
a contradiction. Hence \( \tau_x(H_S) \leq 4 \) and by Lemma \ref{lem:tau-bound} and Theorem \ref{thm:uniform-K} we obtain \( \delta_x(H) \geq \frac{4}{\tau_x(H_S)} \geq 1 \). Moreover, when equality holds, we have \( \varepsilon_x(H_S) = \tau_x(H_S) = 4 \) or \( H \sim \mathbb{Q} 4G \) for some prime divisor \( G \) on \( X \). The latter case cannot occur since \( H \) is a primitive generator of \( \Pic(X) \). By Lemma \ref{lem:tau-bound} the former case can only happen when \( d = 16 \). This proves that \( X \) is uniformly K-stable when \( d \leq 14 \) and is K-semistable when \( d = 16 \).

It remains to analyze the equality conditions when \( X \) has degree \( d = 16 \). Suppose that \( \delta_x(X) = 1 \), then by Theorem \ref{thm:uniform-K} there exists some valuation \( v \) with positive dimensional center on \( X \) such that \( A_X(v) = S(H; v) \). Since \( \Pic(X) = \mathbb{Z}[H] \), using Lemma \ref{lem:valuation} it is easy to see that \( S(H; D) \leq \frac{1}{4} < 1 = A_X(D) \) for any prime divisor \( D \) on \( X \) (c.f. the proof of Theorem \ref{thm:index-one}), hence \( C_X(v) \) cannot be a surface and must be a curve.

Suppose first that \( C \) has degree \( (H \cdot C) \geq 2 \). We claim that in this case \( \delta_C(X) > 1 \). To see this, let \( T \in |H| \) be a very general hyperplane section
such that $\text{Pic}(T) = \mathbb{Z} \cdot [H_T]$ and that $T \cap C$ consists of at least two points, and let $G \in |H_T|$ be a general hyperplane section on $T$ that is disjoint from $T \cap C$. Let $W_\bullet$ be the refinement by $T$ of the complete linear series associated to $H$. Note that $W_\bullet$ is almost complete, $F(W_\bullet) = 0$ and $c_1(W_\bullet) = \frac{3}{4} H_T$ by [AZ20, (3.1)]. Consider the admissible flag $Y_\bullet$ on $X$ given by $Y_0 = X$, $Y_1 = T$ and $Y_2 = G$. By [AZ20, Theorem 3.5], we have

$$
(5.7) \quad \delta_C(X; -K_X) \geq \min \left\{ \frac{A_X(T)}{S(H; T)}, \delta_{T \cap C}(T; W_\bullet, F) \right\} = \min\{4, \delta_{T \cap C}(T; W_\bullet, F)\},
$$

where $F$ is the filtration induced by the curve $G$. In particular, $\delta_C(X) > 1$ as long as we have

$$(5.8) \quad \text{lct}(\delta_{T \cap C}(T; W_\bullet, F)).$$

We show that this is indeed the case. Let $m \in \mathbb{N}$ and let $D$ be an $m$-basis type $\mathbb{Q}$-divisor of $W_\bullet$ that is compatible with $F$. Then we have

$$D = S_m(W_\bullet; G) \cdot G + \Gamma$$

for some effective $\mathbb{Q}$-divisor $\Gamma$. As $G$ is disjoint from $T \cap C$, it is clear that

$$(5.9) \quad \text{lct}(\delta_{T \cap C}(T; D)) = \text{lct}(\delta_{T \cap C}(T; \Gamma)).$$

Since

$$
\lim_{m \to \infty} S_m(W_\bullet; G) = S(W_\bullet; G) = \frac{3}{4} S(H_T; G) = \frac{1}{4}
$$

by [AZ20, Lemma 2.29], we see that $\Gamma \sim Q c_1(W_\bullet) - S_m(W_\bullet; G) \cdot G \sim Q \lambda_m H_T$ for some $\lambda_m \geq 0$ with $\lim_{m \to \infty} \lambda_m = \frac{1}{2}$. By [AZ20, Lemma 2.21] and the last part of its proof, we also know that there exists some $\eta_m \in (0, 1)$ with $\lim_{m \to \infty} \eta_m = 1$ such that $\eta_m \cdot S_m(W_\bullet; F) < S(W_\bullet; F)$ for any divisor $F$ over $T$. In particular, we have

$$\text{ord}_F(\eta_m \Gamma) \leq \eta_m \cdot S_m(W_\bullet; F) < S(W_\bullet; F) \leq \frac{1}{4}$$

for any irreducible curve $F \subseteq T$ (recall that $\text{Pic}(T)$ is generated by $H_T$, thus by [AZ20, Lemma 2.29], $S(W_\bullet; F) = \frac{3}{4} S(H_T; F) = \frac{1}{4}$ if $F \sim r H_T$). Perturbing the $\eta_m$, we may also assume that $\eta_m \lambda_m < \frac{1}{2}$. It follows that $(T, 4 \eta_m \Gamma)$ is klt outside a finite number of points and $2 H_T - (K_T + 4 \eta_m \Gamma)$ is ample (recall that $K_T = 0$ by adjunction).

We now apply an argument from [Zhu20a] to estimate $\text{lct}_{T \cap C}(T, \Gamma)$. More precisely, let $\mathcal{J} = \mathcal{J}(T, 4 \eta_m \Gamma)$ be the multiplier ideal, which is co-supported at a finite number of points by the previous step. By Nadel vanishing, $H^1(T, \mathcal{J}(2 H_T)) = 0$, hence $\ell(O_{T, x}/\mathcal{J}) \leq h^0(T, 2 H_T) = 2(H_T^2) + 2 = 34$. As $|T \cap C| \geq 2$, we see that $\ell(O_{T, x}/\mathcal{J}_x) \leq 17$ for some $x \in T \cap C$. On the other hand, by [Zhu20a, Lemmas 3.4 and 5.2], we have $\text{lct}_{x}(T, a) > \frac{1}{2}$ for any ideal $a \subseteq O_{T, x}$ with $\ell(O_{T, x}/a) \leq 21 \leq \sigma_{2,3}$ (we follow the notation of [Zhu20a]). Since such ideals $a$ can be parametrized by some scheme of finite type and the log canonical thresholds are constructible in families, we deduce that there exists some absolute constant $\alpha > \frac{1}{2}$ such that $\text{lct}_{x}(T, a) > \alpha$ for all $a \subseteq O_{T, x}$ with $\ell(O_{T, x}/a) \leq 21$. In particular, we have $\text{lct}_{x}(T, \mathcal{J}) \geq \alpha$. By
Theorem 1.6 and Remark 3.1], we then have \( \text{lct}_x(T, 4\eta_m \Gamma) \geq \frac{\alpha}{\alpha + 1} \) and therefore
\[
\text{lct}_{\Gamma \cap C}(T, \Gamma) \geq \text{lct}_x(T, \Gamma) \geq \frac{4\alpha \eta_m}{\alpha + 1}.
\]
Combined with (5.8) and letting \( m \to \infty \), we obtain
\[
\delta_{\Gamma \cap C}(T; W_v, \mathcal{F}) \geq \frac{4\alpha}{\alpha + 1} > 1.
\]
By (5.7), this implies that \( \delta_C(X) > 1 \) for any curve \( C \subseteq X \) of degree at least 2.

Hence the only remaining possibility for \( C_X(v) \) is a line \( L \) on \( X \). The rest of the argument is similar to those of Theorem 5.7. By Corollary 5.6, we have \( A_X(v) < \frac{1}{2}T(H; v) \) or \( A_X(v) \geq \frac{1}{2} \eta(H; v) \). In the former case, there exists some \( \mathbb{Q} \)-divisor \( 0 \leq D \sim_\mathbb{Q} H \) such that \( \text{lct}_L(X; D) < \frac{1}{2} \). Since \( X \) has Picard number one, we may further assume that \( D \) is irreducible. Note that \( \text{mult}_L D \geq 2 \), otherwise \( \langle X, \frac{1}{2} D \rangle \) is log canonical [Ko13 Theorem 2.29]. By [IP99] Theorems 4.3.3(vii), Remark 4.3.4 and 4.3.7(iii), \( 2D \) is integral and \( \text{mult}_L D = \frac{1}{2} \). Moreover, if \( \rho : \tilde{X} \to X \) is the ordinary blowup of the line \( L \) with exceptional divisor \( F \) and \( D' \) is the strict transform of \( D \), then the scheme theoretic intersection \( G = 2D' \cap F \) contains a non-hyperelliptic curve \( \Gamma \), \( \rho|_G : G \to L \) is finite of degree 5 and
\[
\rho^*(K_X + \frac{1}{2}D) = K_{\tilde{X}} + \frac{1}{4}F + \frac{1}{2}D'.
\]
It follows that \( \rho|_G : \Gamma \to L \) has degree at least 3 and each component of \( G \setminus \Gamma \) is different from \( \Gamma \) and has multiplicity at most 2. From here we deduce that every component of \( \frac{1}{2}D' \cap F \) has multiplicity \( \leq \frac{1}{2} \) and by inversion of adjunction we see that \( \langle \tilde{X}, \frac{1}{4}F + \frac{1}{2}D' \rangle \) is klt over the generic point of \( L \) and the same is true for \( \langle X, \frac{1}{2} D \rangle \). But this is a contradiction as \( \text{lct}_L(X; D) < \frac{1}{2} \).

Therefore we must have \( A_X(v) \leq \frac{1}{2} \eta(H; v) \). For any \( 0 < \varepsilon \ll 1 \) we can find effective \( \mathbb{Q} \)-divisors \( D_1, D_2 \sim_\mathbb{Q} H \) without common components such that \( v(D_i) > (1 - \varepsilon) \eta(H; v) \). Let \( m \) be a sufficiently divisible integer and let \( Z \subseteq X \) be the complete intersection subscheme \( mD_1 \cap mD_2 \). Note that \( \text{mult}_L Z \leq \deg Z = 16m^2 \) and by [IP99] Theorems 4.3.3(vii)], we have \( \text{mult}_C D_i < \frac{2}{2} < 4 \) for some \( i = 1, 2 \). Hence \( I_Z \not\subseteq I_C^{\frac{1}{2}m} \) and by [Zhu20b] Lemma 2.6], applied at the generic point of \( C \), we see that there exists some absolute constant \( \varepsilon_1 > 0 \), which does not depend on \( D_i \) and \( \varepsilon \), such that \( \text{lct}_C(X; I_Z) > \frac{1 + \varepsilon_1 + 2m}{2m} \).

It then follows as in the proof of Theorem 5.7 that \( A_X(v) > \frac{1 + \varepsilon_1 + 2m}{2m} \eta(H; v) \), a contradiction.

Therefore we conclude that the inequality \( \delta_x(H) \geq 1 \) is always strict. In other words, \( X \) is uniformly K-stable. \( \square \)

Proof of Theorem 5.7. This is just a combination of Theorems 5.7 and 5.12. \( \square \)
References

[AGP06] Claudio Arezzo, Alessandro Ghigi, and Gian Pietro Pirola, *Symmetries, quotients and Kähler-Einstein metrics*, J. Reine Angew. Math. 591 (2006), 177–200.

[AZ20] Hamid Abban and Ziquan Zhuang, *K-stability of Fano varieties via admissible flags* (2020). arXiv:2003.13788

[BBJ21] Robert Berman, Sebastian Boucksom, and Mattias Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, J. Amer. Math. Soc. 34 (2021), no. 3, 605–652.

[BdFFU15] Sébastien Boucksom, Tommaso de Fernex, Charles Favre, and Stefano Urbinati, *Valuation spaces and multiplier ideals on singular varieties*, Recent advances in algebraic geometry, 2015, pp. 29–51.

[BFJ09] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Differentiability of volumes of divisors and a problem of Teissier*, J. Algebraic Geom. 18 (2009), no. 2, 279–305.

[BJ20] Harold Blum and Mattias Jonsson, *Thresholds, valuations, and K-stability*, Adv. Math. 365 (2020), 107062.

[BKMS15] Sébastien Boucksom, Alex Küronya, Catrinw Maclean, and Tomasz Szemberg, *Vanishing sequences and Okounkov bodies*, Math. Ann. 361 (2015), no. 3-4, 811–834.

[BLX19] Harold Blum, Yuchen Liu, and Chenyang Xu, *Openness of K-semistability for Fano varieties* (2019). To appear in Duke Math. J. arXiv:1907.02408

[Bro06] Amaël Broustet, *Constantes de Seshadri du diviseur anticanonique des surfaces de del Pezzo*, Enseign. Math. (2) 52 (2006), no. 3-4, 231–238.

[CDS15] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds, I-III*, J. Amer. Math. Soc. 28 (2015), no. 1, 183–197, 199–234, 235–278.

[CS08] Ivan Cheltsov and Konstantin Shramov, *Log-canonical thresholds for nonsingular Fano threefolds*, Uspekhi Mat. Nauk 63 (2008), no. 5(383), 73–180.

[Dem92] Jean-Pierre Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), 1992, pp. 87–104.

[Der16] Ruadhaí Dervan, *On K-stability of finite covers*, Bull. Lond. Math. Soc. 48 (2016), no. 4, 717–728.

[dFEM04] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă, *Multiplicities and log canonical threshold*, J. Algebraic Geom. 13 (2004), no. 3, 603–615.

[DGF14] Vincenzo Di Gennaro and Davide Franco, *Noether-Lefschetz theory with base locus*, Rend. Circ. Mat. Palermo (2) 63 (2014), no. 2, 257–276.

[DK73] Pierre Deligne and Nicholas Katz, *Groupees de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz.

[Don15] Simon Donaldson, *Algebraic families of constant scalar curvature Kähler metrics*, Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations, 2015, pp. 111–137.

[EKL95] Lawrence Ein, Oliver Küchle, and Robert Lazarsfeld, *Local positivity of ample line bundles*, J. Differential Geom. 42 (1995), no. 2, 193–219.

[ELM+09] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Restricted volumes and base loci of linear series*, Amer. J. Math. 131 (2009), no. 3, 607–651.

[FKL16] Mihai Fulger, János Kollár, and Brian Lehmann, *Volume and Hilbert function of R-divisors*, Michigan Math. J. 65 (2016), no. 2, 371–387.
Kento Fujita and Yuji Odaka, *On the K-stability of Fano varieties and anticanonical divisors*, Tohoku Math. J. (2) **70** (2018), no. 4, 511–521.

Kento Fujita, *K-stability of Fano manifolds with not small alpha invariants*, J. Inst. Math. Jussieu **18** (2019), no. 3, 519–530.

A valutative criterion for uniform K-stability of Q-Fano varieties, J. Reine Angew. Math. **751** (2019), 309–338.

Kento Fujita and Yuji Odaka, *On the K-stability of Fano varieties and anti-canonical divisors*, Tohoku Math. J. (2) **70** (2018), no. 4, 511–521.

Kento Fujita, *K-stability of Fano manifolds with not small alpha invariants*, J. Inst. Math. Jussieu **18** (2019), no. 3, 519–530.

Vasily Iskovskikh and Yuri Prokhorov, *Fano varieties*, Algebraic geometry, V, 1999, pp. 1–247.

Vasily Iskovskikh, *Fano threefolds. II*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), no. 3, 506–549.

Mattias Jonsson and Mircea Mustaţă, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2145–2209 (2013).

Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Vol. 48, Springer-Verlag, Berlin, 2004.

Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

Chi Li, *K-semistability is equivariant volume minimization*, Duke Math. J. **166** (2017), no. 16, 3147–3218.

Yuji Odaka, *On the moduli of Kähler-Einstein Fano manifold*, Proceeding of kinosaki symposium, 2013, pp. 112–126.

Yuji Odaka and Yuji Sano, *Alpha invariant and K-stability of Q-Fano varieties*, Adv. Math. **229** (2012), no. 5, 2818–2834.

Chenyang Xu, *K-stability of Fano varieties: an algebro-geometric approach* (2020), arXiv:2011.10477.

Chenyang Xu, *K-stability of Fano varieties: an algebro-geometric approach* (2020), arXiv:2011.10477.
[Zhu20b] ________, *Birational superrigidity is not a locally closed property*, Selecta Math. (N.S.) 26 (2020), no. 1, Paper No. 11, 20.

[Zhu20c] ________, *Product theorem for K-stability*, Adv. Math. 371 (2020), 107250, 18.

[Zhu21] ________, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math. 226 (2021), no. 1, 195–223.

*Hamid Abban*
Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK
h.abban@lboro.ac.uk

*Ziquan Zhuang*
Department of Mathematics, MIT, Cambridge, MA, 02139, USA
ziquan@mit.edu