Generalized Perfect Codes for Symmetric Classical-Quantum Channels

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Abstract—We define a new family of codes for symmetric classical-quantum channels and establish their optimality. To this end, we extend the classical notion of generalized perfect and quasi-perfect codes to channels defined over some finite dimensional complex Hilbert output space. The resulting optimality conditions depend on the channel considered and on an auxiliary state defined on the output space of the channel. For certain $N$-qubit classical-quantum channels, we show that codes based on a generalization of Bell states are quasi-perfect and, therefore, they feature the smallest error probability among all codes of the same blocklength and cardinality.

Index Terms—Classical-quantum channel, finite block-length analysis, quantum meta-converse, perfect code, quasi-perfect code, quantum hypothesis testing.

I. INTRODUCTION

One of the goals of information theory is to characterize the best achievable performance of any transmission scheme and to establish the structure of codes and decoders attaining this limit. In certain regimes the best performance of a communication system is accurately characterized and there exist practical codes attaining it. In his landmark paper [1], Shannon demonstrated that for every communication channel there exists a fundamental limit, named channel capacity, that determines the highest rate at which a sender can transmit data to a receiver with vanishing decoding error probability. Nowadays, several code constructions achieve the channel capacity or perform very close to it. Specific examples, for sufficiently large code lengths, are low-density parity check (LDPC) codes [2], turbo codes [3], or polar codes [4].

If the length of the code is limited—e.g., due to delay constraints or due to the nature of the channel—the channel capacity is not a good benchmark anymore. To characterize the achievable performance in this regime, we shall use non-asymptotic bounds on the error probability of the best coding scheme. Two early instances of these limits are the sphere-packing bound [5, Eq.
on the error probability of the binary symmetric channel (BSC) and Shannon’s non-asymptotic results for the additive white Gaussian noise (AWGN) channel [6]. Recent advances in information theory have lead to several (upper and lower) bounds in the finite block-length regime [7]–[9]. Code designers have optimized the finite-length performance of certain codes and now they perform close to those non-asymptotic limits (see, e.g., [10], [11] and references therein). Moreover, certain codes can even attain these limits with equality, thus proving their non-asymptotic optimality. For example, perfect and quasi-perfect binary codes attain the sphere-packing bound [5, Eq. (5.8.19)] for the BSC, and they were generalized beyond binary alphabets in [12].

The results presented above consider transmission channels –or random transformations– which are modeled by a transition probability distribution. Certain physical systems, however, can only be described using the laws of quantum mechanics. For these systems, the classical channel capacity and the corresponding non-asymptotic results have to be extended to encompass the quantum properties of the system. Holevo, Schumacher and Westmoreland studied the task of sending classical data over a channel with classical inputs and quantum outputs [13], [14]; this setting is usually referred to as classical-quantum channel coding. Their coding theorem guarantees the existence of reliable codes if their rate is below a fundamental limit, known as Holevo capacity, provided that the codelength is sufficiently long. While the proof of this result does not provide an explicit code construction, it guided the design of practical coding schemes. For example, quantum polar codes are practical constructions shown to attain this limit [15], the codes proposed in [16], [17] feature the superadditivity of mutual information, and other coding schemes exploiting the quantum properties of optical channels were proposed in [18], [19]. Holevo capacity is an asymptotic quantity that, in general, can only be attained by a large number of channel uses via a joint measurement on the combined channel outputs. For a finite number of channel uses –which is relevant for practical quantum systems– non-asymptotic performance limits need to be used. Converse non-asymptotic bounds were studied in [20], [21, Sec. 4.6] and [22], among other works. However, to the best of our knowledge, these works have not been applied in the design and/or benchmark of practical code constructions.

The derivation of converse bounds for classical and quantum systems is often based on hypothesis testing. An early application of this technique was used by Shannon, Gallager and Berlekamp to obtain the sphere-packing exponent [23] (see also [24] for a more explicit derivation for symmetric channels). For classical-quantum channels, Nagaoka applied binary hypothesis testing in the derivation of strong-converse bounds [25], and Hayashi used this technique to establish the converse part of the channel-coding theorem [21, Sec. 4.6]. Later, Polyanskiy, Poor and Verdú formulated a hypothesis testing finite-length bound for classical channels [7, Th. 27], and Matthews and Wehner obtained finite-length bounds for general quantum channels in [22].

In this work, we extend the notion of generalized perfect and quasi-perfect codes [12] to symmetric classical-quantum channels. The results presented here are based on quantum hypothesis testing. In particular, we derive two alternative expressions for the error probability of quantum multiple hypothesis testing, which are then used to determine the exact error probability for a fixed classical-quantum channel code. A weakening of this result yields the non-asymptotic converse bound [22, Eq. (45)] (see also [21, Sec. 4.6]) and coincides with the error probability of the generalized perfect and quasi-perfect codes, whenever they exist. Therefore, these codes yield the best performance among any code with
the same rate and block-length. While these codes are possibly rare, we characterize a family of codes based on a generalization of Bell states which are quasi-perfect for certain non-asymptotic 2-qubit classical-quantum channels and their $N$-qubit extension.

A separate line of research in the literature considers the evolution of quantum information in a noisy environment. In [26], Shor showed that quantum errors can be controlled by encoding the state of the system in a quantum code and periodically measuring the redundant parts of the code. This observation opened the field of quantum error correction. While it is possible to encapsulate classical information over quantum channels using quantum error correction codes, they are highly inefficient for this task and they will not be treated here.

The organization of this article is as follows. In Section II we formalize the problems of binary and multiple hypothesis testing and establish an unexplored connection between them. Section III presents the classical-quantum channel model and establishes the accuracy of different converse bounds in the literature. Section IV defines perfect and quasi-perfect codes for classical-quantum symmetric channels and proves their optimality whenever they exist. In Section V we study a family of codes which are quasi-perfect for 2-qubit classical-quantum channels affected by quantum depolarization. Section VI concludes the article with some final remarks.

A. Notation

Let $D(H)$ denote the space of density operators acting on some finite dimensional complex Hilbert space $H$. A quantum state is described by a density operator $\rho \in D(H)$. Density operators are self-adjoint, positive semidefinite, and have unit trace. A measurement on a quantum system is a mapping from the state of the system $\rho$ to a classical outcome $m \in \{1, \ldots, M\}$. A measurement is represented by a collection of positive semidefinite self-adjoint operators $\{\Pi_1, \ldots, \Pi_M\}$ such that $\sum \Pi_m = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. These operators form a positive operator-valued measure (POVM). A POVM measurement $\{\Pi_1, \ldots, \Pi_M\}$ applied to $\rho$ has outcome $m$ with probability $\text{Tr}(\rho \Pi_m)$ where $\text{Tr}$ is the trace operator.

For self-adjoint operators $A, B$, the notation $A \geq B$ means that $A - B$ is positive semidefinite. Similarly $A \leq B$, $A > B$, and $A < B$ means that $A - B$ is negative semidefinite, positive definite and negative definite, respectively.

For a self-adjoint operator $A$ with spectral decomposition $A = \sum \lambda_i E_i$, where $\{\lambda_i\}$ are the real eigenvalues and $\{E_i\}$ are the orthogonal projections onto the corresponding eigenspaces, we define

$$\{A > 0\} \triangleq \sum_{i: \lambda_i > 0} E_i, \quad (1)$$

This corresponds to the projector associated to the positive eigenspace of $A$. We shall also use $\{A \geq 0\} \triangleq \sum_{i: \lambda_i \geq 0} E_i$, $\{A < 0\} \triangleq \sum_{i: \lambda_i < 0} E_i$ and $\{A \leq 0\} \triangleq \sum_{i: \lambda_i \leq 0} E_i$.

II. QUANTUM HYPOTHESIS TESTING

A. Binary Hypothesis Testing

Let us consider a binary hypothesis test (with simple hypotheses) discriminating between the density operators $\rho_0$ and $\rho_1$, where $\rho_0, \rho_1 \in D(H)$. In order to distinguish between the two hypotheses we perform a measurement. We define a test measurement $\{T, \bar{T}\}$, such that $T$ and $\bar{T} \triangleq \mathbb{1} - T$ are positive semidefinite, self-adjoint operators. The test decides $\rho_0$ (resp. $\rho_1$) when the measurement outcome corresponding to $T$ (resp. $\bar{T}$) occurs.
Let $\epsilon_{j|i}$ denote the probability of deciding $\rho_j$ when $\rho_i$ is the true hypothesis, $i,j = 0, 1, i \neq j$. More precisely,

$$\epsilon_{1|0}(T) \triangleq 1 - \text{Tr}(\rho_0 T) = \text{Tr}(\rho_0 T),$$  \hspace{1cm} (2)

$$\epsilon_{0|1}(T) \triangleq \text{Tr}(\rho_1 T).$$ \hspace{1cm} (3)

Let $\alpha_\beta(\rho_0\|\rho_1)$ denote the minimum error probability $\epsilon_{1|0}$ among all tests with $\epsilon_{0|1}$ at most $\beta$, that is,

$$\alpha_\beta(\rho_0\|\rho_1) \triangleq \inf_{T: \epsilon_{0|1}(T) \leq \beta} \epsilon_{1|0}(T).$$ \hspace{1cm} (4)

The function $\alpha_\beta(\cdot\|\cdot)$ is the inverse of the function $\beta_\alpha(\cdot\|\cdot)$ appearing in [22], which is itself related to the hypothesis-testing relative entropy as $D^\alpha_{\|\cdot}(\rho_0\|\rho_1) = -\log \beta_\alpha(\rho_0\|\rho_1)$ [27].

When $\rho_0$ and $\rho_1$ commute, the test $T$ in (4) can be restricted to be diagonal in the (common) eigenbasis of $\rho_0$ and $\rho_1$, then (4) reduces to the classical case [28].

The form of the test minimizing (4) is given by the quantum Neyman-Pearson lemma, presented next.

**Lemma 1 (Neyman-Pearson lemma):** The best trade-off between type-I and type-II error probabilities is attained by tests of the form

$$T_{\text{NP}} = \{\rho_0 - t\rho_1 > 0\} + \theta t^0,$$ \hspace{1cm} (5)

for some $t$ and $\theta t^0$, and where $0 \leq \theta t^0 \leq \{\rho_0 - t\rho_1 = 0\}$.

**Proof:** A slightly different formulation of this result is usually given in the literature (see, e.g., [29, Ch. IV, Eq. (2.18)]). The precise statement included here can be found, e.g., in [30, Lem. 3].

Then, for any choice of $t$ and $\theta t^0$ such that $\text{Tr}(\rho_1 T_{\text{NP}}) = \beta$, the resulting test $T_{\text{NP}}$ in (5) minimizes (4). The following result is a corollary to the Neyman-Pearson lemma that will be useful in the sequel.

**Lemma 2:** For any binary hypothesis test discriminating between the quantum states $\rho_0$ and $\rho_1$, it follows that

$$\alpha_\beta(\rho_0\|\rho_1) = \sup_{t \geq 0} \left\{ \text{Tr}(\rho_0 \{\rho_0 - t\rho_1 \leq 0\}) + t(\text{Tr}(\rho_1 \{\rho_0 - t\rho_1 > 0\}) - \beta) \right\} \geq \text{Tr}(\rho_0 \{\rho_0 - t'\rho_1 \leq 0\}) - t'\beta,$$ \hspace{1cm} (6)

for any $t' \geq 0$.

**Proof:** The identity (6) is the quantum analogue of [12, Lem. 1] and the relaxation (7) coincides with [31, Lem. 2]. For completeness, we include next the proof of (6)-(7).

For any operator $A \geq 0$ and $0 \leq T \leq 1$, it holds that $\text{Tr}(A\{A > 0\}) \leq \text{Tr}(AT)$ [32, Eq. 8]. For $A = \rho_0 - t'\rho_1$ and $T = T_{\text{NP}}$ defined in (5), this inequality becomes

$$\text{Tr}(\{\rho_0 - t'\rho_1\}P_{t'}^+) \geq \text{Tr}(\{\rho_0 - t'\rho_1\}T_{\text{NP}}),$$ \hspace{1cm} (8)

where we defined $P_{t'}^+ \triangleq \{\rho_0 - t'\rho_1 > 0\}$. Indeed, (8) holds with equality for the value $t' = t$ appearing in (5), as $\text{Tr}(\{\rho_0 - t\rho_1\}\theta t^0) = 0$ for any $0 \leq \theta t^0 \leq \{\rho_0 - t\rho_1 = 0\}$, tantamount to $\theta t^0$ being in the null-space of $\rho_0 - t\rho_1$.

After some algebra, (8) yields

$$-\text{Tr}(\rho_0 T_{\text{NP}}) \geq -\text{Tr}(\rho_0 P_{t'}^+) + t'\text{Tr}(\rho_1 (P_{t'}^+ - T_{\text{NP}})).$$ \hspace{1cm} (9)

Summing one to both sides of (9) and noting that $\alpha_\beta(\rho_0\|\rho_1) = 1 - \text{Tr}(\rho_0 T_{\text{NP}})$ and $\beta = \text{Tr}(\rho_1 T_{\text{NP}})$, we obtain

$$\alpha_\beta(\rho_0\|\rho_1) \geq \text{Tr}(\rho_0 \{\rho_0 - t'\rho_1 \leq 0\}) + t'\text{Tr}(\rho_1 (P_{t'}^+ - T_{\text{NP}}))^+ - t'\beta.$$

As (8) holds with equality for the value $t' = t$ appearing in (5), so it does (10) after optimization over the parameter $t' \geq 0$. Then, (6) follows. To obtain the lower bound (7), we fix $t' \geq 0$ and use that $\text{Tr}(\rho_1 \{\rho_0 - t'\rho_1 > 0\}) \geq 0$.\[\]
B. Bayesian Multiple Hypothesis Testing

We consider now a hypothesis testing problem discriminating among $M$ possible density operators acting on $\mathcal{H}$, where $M$ is assumed to be finite. We consider the Bayesian setting, where the $M$ alternatives $\tau_1, \ldots, \tau_M$ occur with (classical) probabilities $p_1, \ldots, p_M$, respectively.

A $M$-ary hypothesis test is a POVM $\mathcal{T} \triangleq \{\Pi_1, \Pi_2, \ldots, \Pi_M\}$, $\sum \Pi_i = \mathds{1}$. The test decides the alternative $\tau_i$ when the measurement with respect to $\mathcal{T}$ has outcome $i$. The probability that the test $\mathcal{T}$ decides $\tau_j$ when $\tau_i$ is the true underlying state is thus $\text{Tr}(\tau_i \Pi_j)$ and the average error probability is

$$\epsilon(\mathcal{T}) \triangleq 1 - \sum_{i=1}^{M} p_i \text{Tr}(\tau_i \Pi_i).$$

(11)

We define the minimum average error probability as

$$\epsilon^* \triangleq \min_{\mathcal{T}} \epsilon(\mathcal{T}).$$

(12)

The test $\mathcal{T}$ minimizing (12) has no simple form in general.

Lemma 3 (Holevo-Yuen-Kennedy-Lax conditions): A test $\mathcal{T}^* = \{\Pi_1^*, \ldots, \Pi_M^*\}$ minimizes (12) if and only if, for each $m = 1, \ldots, M$,

$$\Lambda(\mathcal{T}^*) - p_m \tau_m)\Pi_m^* = \Pi_m^* (\Lambda(\mathcal{T}^*) - p_m \tau_m) = 0,$$

(13)

$$\Lambda(\mathcal{T}^*) - p_m \tau_m \geq 0,$$

(14)

where

$$\Lambda(\mathcal{T}^*) \triangleq \sum_{i=1}^{M} p_i \tau_i \Pi_i^* \sum_{i=1}^{M} p_i \Pi_i^* \tau_i$$

(15)

is required to be self-adjoint\(^1\).

Proof: This result follows from [33, Th. 4.1, Eq. (4.8)] or [34, Th. I] after simplifying the resulting optimality conditions.

We next show an alternative characterization of the minimum error probability $\epsilon^*$ as a function of a binary hypothesis test with certain parameters. Let $\text{diag}(\rho_1, \ldots, \rho_M)$ denote the block-diagonal matrix with diagonal blocks $\rho_1, \ldots, \rho_M$ and define

$$\mathbf{P} \triangleq \text{diag}(p_1 \tau_1, \ldots, p_M \tau_M),$$

(16)

$$\mathbf{D}(\mu_0) \triangleq \text{diag}(\frac{1}{\mathbf{1}^{T} \mathbf{P} \mathbf{1}}, \ldots, \frac{1}{\mathbf{1}^{T} \mathbf{P} \mathbf{1}}),$$

(17)

where $\mu_0$ is an arbitrary density operator acting on $\mathcal{H}$. Note that both $\mathbf{P}$ and $\mathbf{D}(\mu_0)$ are density operators themselves, since they are self-adjoint, positive semidefinite and have unit trace.

Theorem 1: The minimum error probability of a Bayesian $M$-ary test discriminating among states $\{\tau_1, \ldots, \tau_M\}$ with prior probabilities $\{p_1, \ldots, p_M\}$ satisfies

$$\epsilon^* = \max_{\mu_0} \alpha \left( \mathbf{P} \parallel \mathbf{D}(\mu_0) \right),$$

(18)

where $\mathbf{P}$ and $\mathbf{D}(\cdot)$ are given in (16) and (17), respectively, and where the optimization is carried out over (unit-trace non-negative) density operators $\mu_0 \in \mathcal{D}(\mathcal{H})$.

Proof: This result can be proven extending the convex analysis approach from [28, Sec. III.B] to the quantum setting considered here. Nevertheless, in the following we provide an alternative proof based on a constructive approach that explicitly shows the connection between the optimal measurements for both the binary and $M$-ary discrimination problems.

For any $\mathcal{T} = \{\Pi_1, \Pi_2, \ldots, \Pi_M\}$ let us define the binary test $\mathcal{T}' \triangleq \text{diag}(\Pi_1, \ldots, \Pi_M)$. The error probabilities $\epsilon_{1|0}$ and $\epsilon_{0|1}$ of the test $\mathcal{T}'$ are given by

$$\epsilon_{1|0}(\mathcal{T}') = 1 - \sum_{i=1}^{M} p_i \text{Tr}(\tau_i \Pi_i) = \epsilon(\mathcal{T}),$$

(19)

\(^1\)The operator $\Lambda(\mathcal{T})$ takes a role of the Lagrange multiplier associated to the constraint $\sum \Pi_i = \mathds{1}$, which, involving self-adjoint operators requires $\Lambda$ to be self-adjoint.
and
\[ \epsilon_{0|1}(T') = \frac{1}{M} \sum_{i=1}^{M} \text{Tr}(\mu_0 \Pi_i) \]  
(20)
\[ = \frac{1}{M} \text{Tr} \left( \mu_0 \left( \sum_{i=1}^{M} \Pi_i \right) \right) \]  
(21)
\[ = \frac{1}{M} \text{Tr}(\mu_0) = \frac{1}{M}. \]  
(22)

The (possibly suboptimal) test \( T' \) thus has \( \epsilon_{1|0}(T') = \epsilon(T) \) where \( \epsilon(T) \) is defined in (11) and \( \epsilon_{0|1}(T') = \frac{1}{M} \).

Therefore, using (4) and maximizing the resulting expression over \( \mu_0 \), we obtain
\[ \epsilon(T) \geq \max_{\mu_0} \epsilon_{\star} \left( P \parallel D(\mu_0) \right). \]  
(23)

To prove the theorem, it remains to show that the lower bound (23) holds with equality for \( T = T^* \) defined in Lemma 3. To this end, we next demonstrate that the optimality conditions in Lemma 1 and in Lemma 3 are equivalent for a certain choice of \( \mu_0 \).

Let \( T^* = \{ \Pi_1^{\star}, \ldots, \Pi_M^{\star} \} \) satisfy (13)-(14), i.e., \( T^* \) corresponds to the optimal \( M \)-ary hypothesis test discriminating among states \( \{ \tau_1, \ldots, \tau_M \} \) with prior probabilities \( \{ p_1, \ldots, p_M \} \). We define
\[ \mu_0^{\star} \triangleq \frac{1}{c_0^{\star}} \sum_{i=1}^{M} p_i \tau_i \Pi_i^{\star} = \frac{1}{c_0^{\star}} \Lambda(T^*), \]  
(24)
where \( c_0^{\star} \) is a normalizing constant such that \( \mu_0^{\star} \) is unit trace. Lemma 1 shows that the test \( T_{NP} \) achieving (23) is associated to the non-negative eigenspace of the matrix
\[ A \triangleq P - tD(\mu_0), \]  
(25)
which features a block-diagonal structure. According to (16)-(17), for the choice \( \mu_0 = \mu_0^{\star} \), and \( t = Me_0^{\star} \), the \( m \)-th block-diagonal term in \( A \) is given by
\[ A_m \triangleq p_m \tau_m - \frac{t}{M} \mu_0 = p_m \tau_m - \Lambda(T^*). \]  
(26)

Given the block-diagonal structure of the matrix \( A \), it is enough to consider a test \( T_{NP} \) with block-diagonal structure. Then, we write \( T_{NP} = \text{diag}(T_{NP}^1, \ldots, T_{NP}^M) \) and recall that the \( m \)-th block \( T_{NP}^m \) must lie in the non-negative eigenspace of the matrix \( A_m \).

Using the optimality condition (14), it follows that the matrices \( A_m, m = 1, \ldots, M, \) are negative semidefinite. Therefore, each block \( T_{NP}^m \) can only lie in the null eigenspace of \( A_m \), for \( m = 1, \ldots, M \). Also, according to the optimality condition (13), the operators \( \Pi_i^{\star} \) precisely lie in the null eigenspace of \( A_m \). As a result, the choice
\[ T_{NP} = \text{diag}(\Pi_1^{\star}, \ldots, \Pi_M^{\star}) \]  
(27)
satisfies the optimality conditions in Lemma 1. Moreover, since \( \epsilon_{1|0}(T_{NP}) = \epsilon(T^*) = \epsilon^* \) and \( \epsilon_{0|1}(T_{NP}) = \frac{1}{M} \), Lemma 1 implies that (18) holds with equality for \( \mu_0 = \mu_0^{\star} \). Given the bound in (23), other choices of \( \mu_0 \) cannot improve the result, and Theorem 1 thus follows.

\[ \text{Example 1:} \text{ Consider a hypothesis testing problem between the } M = 4 \text{ (non-equiprobable) alternatives} \]
\[ \tau_1 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \quad \tau_2 = \frac{1}{2} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \]  
(28)
\[ \tau_3 = \frac{1}{2} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right], \quad \tau_4 = \frac{1}{2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \]

with prior probabilities \( p_1 = 2/5 \) and \( p_2 = p_3 = p_4 = 1/5 \). By solving (12), we obtain \( \epsilon^* = 7/15 = 0.466 \) which is attained by the measurement \( T^* = \{ \Pi_1^{\star}, \ldots, \Pi_4^{\star} \} \) with \( \Pi_1^{\star} = \left[ 8/9 \ 0 \right] \), \( \Pi_2^{\star} = \left[ 1/18 \ 1/6 \right] \), \( \Pi_3^{\star} = \left[ 1/18 \ -1/6 \right] \), and \( \Pi_4^{\star} = 0 \). Note that even when the dimension of the Hilbert space is 2, there are 3 active measurement operators. Since they are positive semidefinite and \( \sum_{i=1}^{4} \Pi_i^{\star} = \mathbb{I} \), the POVM is well defined. The POVM \( T^* \) satisfies the optimality conditions from Lemma 3 and therefore \( \epsilon^* = 0.466 \) is the lowest average error probability for this testing problem.

According to (24), the auxiliary state
\[ \mu_0^{\star} = \frac{1}{c_0^{\star}} \sum_{i=1}^{4} p_i \tau_i \Pi_i^{\star} = \frac{1}{2} \left[ \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right], \]  
(29)
is optimal in Theorem 1. Indeed, it follows that\(^2\)
\[ \alpha_4 \left( P \parallel D(\mu_0^{\star}) \right) = 0.466 = \epsilon^*. \]  
(30)

\(^2\)This computation can be done, e.g., by using (6) from Lemma 2 or by solving a semidefinite program.
Other choices of \( \mu_0 \) yield a lower bound on the average error probability \( \epsilon^* \). For example, considering \( \mu_0 \) the average state for this testing problem,

\[
\mu_0 = \frac{4}{\mathcal{M}} \mathbf{p}_m \tau_m = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.3 \end{bmatrix},
\]

(31)
yields

\[
\alpha_4 \left( \mathbf{P} \| \mathbf{D}(\mu_0) \right) \approx 0.4571 < 0.460 = \epsilon^*.
\]

(32)

Theorem 1 thus provides an alternative expression for the error probability \( \epsilon^* \) for the optimal choice of the auxiliary state, and a lower bound for other choices of \( \mu_0 \). Combining Theorem 1 and Lemma 2, we obtain an alternative characterization for \( \epsilon^* \) based on information-spectrum measures.

**Corollary 1:** The minimum error probability of an \( \mathcal{M} \)-ary test discriminating among states \( \{ \tau_1, \ldots, \tau_{\mathcal{M}} \} \) with prior classical probabilities \( \{ p_1, \ldots, p_{\mathcal{M}} \} \) satisfies

\[
\epsilon^* = \max_{\mu_0, t \geq 0} \left\{ \sum_{i=1}^{\mathcal{M}} p_i \text{Tr} \left( \tau_i \left( p_i \tau_i - t \mu_0 \right) \right) - t \right\}.
\]

(33)

where the optimization is carried out over (unit-trace non-negative) density operators \( \mu_0 \) acting on \( \mathcal{H} \), and over the scalar threshold \( t \geq 0 \).

**Proof:** Applying the lower bound (7) from Lemma 2 to the identity (18), and using the definitions of \( \mathbf{P} \) in (16) and \( \mathbf{D}(\cdot) \) in (17), it yields, for any \( \mu_0, t \geq 0 \),

\[
\epsilon^* \geq \sum_{i=1}^{\mathcal{M}} p_i \text{Tr} \left( \tau_i \left( p_i \tau_i - \frac{t'}{\mathcal{M}} \mu_0 \right) \right) - \frac{t'}{\mathcal{M}}.
\]

(34)

It remains to show that there exist \( \mu_0 \) and \( t' \geq 0 \) such that (34) holds with equality. In particular, let us choose \( \mu_0 = \mu_0^* \) defined in (24), and \( t' = \mathcal{M} c_0^* \) where \( c_0^* = \sum_{i=1}^{\mathcal{M}} p_i \text{Tr}(\tau_i \Pi_i^*) \) is the normalizing constant from (24).

For this choice of \( \mu_0 \) and \( t' \), the projector spanning the negative semidefinite eigenspace of the operator \( p_i \tau_i - \frac{t'}{\mathcal{M}} \mu_0 \) can be rewritten as

\[
\left\{ p_i \tau_i - \frac{t'}{\mathcal{M}} \mu_0 \leq 0 \right\} = \left\{ p_i \tau_i - \Lambda(T^*) \leq 0 \right\} = \mathbf{1},
\]

(35)
and (33) is
\[ \mu_0^* = \frac{1}{c_0^2} \sum_{i=1}^{M} p_i \tau_i, \tag{37} \]
for some \( \mathcal{T}^* = \{ \Pi_1^*, \ldots, \Pi_M^* \} \) satisfying the conditions in Lemma 3 and where \( c_0^* \) is a normalizing constant. Hence, the optimal \( M \)-ary hypothesis test \( \mathcal{T}^* \) characterizes the optimal \( \mu_0 \). Conversely, the optimal \( \mu_0 \) is precisely the Lagrange multiplier associated to the minimization in (12), after an appropriate re-scaling.

While the expressions in Theorem 1 and Corollary 1 are not easier to compute than the exact error probability, we show in the next section that they can be used to determine the tightness of several converse bounds in the context of reliable communication over classical-quantum channels.

III. CLASSICAL-QUANTUM CHANNELS

We consider the channel coding problem of transmitting \( M \) equiprobable messages\(^3\) over a one-shot classical-quantum channel \( x \to W_x \), with \( x \in \mathcal{X} \) and \( W_x \in \mathcal{D}(\mathcal{H}) \). A channel code is defined as a mapping from the message set \( \{1, \ldots, M\} \) into a set of \( M \) codewords \( \mathcal{C} = \{x_1, \ldots, x_M\} \). For a source message \( m \), the decoder receives the associated density operator \( W_{x_m} \) and must decide on the transmitted message.

With some abuse of notation, for a fix code, sometimes we shall write \( W_m \triangleq W_{x_m} \). The minimum error probability for a code \( \mathcal{C} \) is then given by

\[ P_e(\mathcal{C}) \triangleq \min_{\{\Pi_1^*, \ldots, \Pi_M^*\}} \left\{ 1 - \frac{1}{M} \sum_{m=1}^{M} \text{Tr}(W_m \Pi_m) \right\}. \tag{38} \]

This problem corresponds precisely to the \( M \)-ary quantum hypothesis testing problem described in Section II-B. In contrast to the classical setting, in which (38) is minimized by the maximum likelihood decoder, the minimizer of (38) corresponds to any POVM satisfying the optimality conditions from Lemma 3.

A direct application of Theorem 1 yields an alternative expression for \( P_e(\mathcal{C}) \). Let \( P \) denote a (classical) distribution over the input alphabet \( \mathcal{X} \) and define
\[ PW \triangleq \sum_{x \in \mathcal{X}} P(x) \left( |x \rangle \langle x| \otimes W_x \right), \tag{39} \]
\[ P \otimes \mu \triangleq \left( \sum_{x \in \mathcal{X}} P(x) |x \rangle \langle x| \right) \otimes \mu. \tag{40} \]

We denote by \( P_{\mathcal{C}} \), the input distribution induced by the codebook \( \mathcal{C} \), hence \( P_{\mathcal{C}} W = \frac{1}{M} \sum_{x \in \mathcal{C}} (|x \rangle \langle x| \otimes W_x) \) and \( P_{\mathcal{C}} \otimes \mu = \left( \frac{1}{M} \sum_{x \in \mathcal{C}} |x \rangle \langle x| \right) \otimes \mu \). Using the alternative expression introduced in Theorem 1 we obtain the following result.

**Theorem 2 (Classical-quantum meta-converse bound):** Let \( \mathcal{C} \) be any codebook of cardinality \( M \) for a channel \( x \to W_x \), with \( x \in \mathcal{X} \) and \( W_x \in \mathcal{D}(\mathcal{H}) \). Then,
\[ P_e(\mathcal{C}) = \sup_{\mu} \left\{ \alpha \frac{1}{\lambda} \langle P_{\mathcal{C}} W \parallel P_{\mathcal{C}} \otimes \mu \rangle \right\} \tag{41} \]
\[ \geq \inf_{P} \sup_{\mu} \left\{ \alpha \frac{1}{\lambda} \langle PW \parallel P \otimes \mu \rangle \right\}. \tag{42} \]

where the maximization is over auxiliary states \( \mu \in \mathcal{D}(\mathcal{H}) \), and the minimization is over (classical) input distributions \( P \).

**Proof:** The identity (41) is a direct application of (18) in Theorem 1. The relaxation (42) follows by minimizing (41) over all input distributions, not necessarily induced by a codebook.

The right-hand-side of (41) coincides with the finite block-length converse bound by Matthews and Wehner \cite[Eq. (45)]{Matthews2009}, particularized for a classical-quantum channel with an input state induced by the codebook \( \mathcal{C} \). The lower bound (42) corresponds to \cite[Eq. (46)]{Matthews2009} specialized to the classical-quantum setting (see also \cite[Sec. 4.6]{Pancharatnam2018} for a direct derivation for classical quantum channels). The classical analogous of (42) is usually referred to as meta-converse bound, since several converse

\[ ^3\text{While the results from Section II-B were derived for discrimination among non-equiprobable alternatives, in the remainder of this paper we consider the channel coding problem with equiprobable messages for clarity of exposition.} \]
bounds in the literature can be derived from it. As it is
the case in the classical-quantum setting, in the following
we shall refer to this result as *meta-converse*.

Theorem 2 implies that the quantum generalization
of the meta-converse bound proposed by Matthews and
Wehner in [22, Eq. (45)] is tight for a fixed codebook
$C$. By fixing $\mu$ to be the state induced at the system
output, the lower bound (42) recovers the converse bound
[27, Th. 1], which is a rederivation of previous results
in [20] (see [20, Remarks 10 and 15]). This bound is
not tight in general since (i) the minimizing $P$ does not
need to coincide with the input state induced by the best
codebook, and (ii) the choice of $\mu_0$ in [27, Th. 1] does
not maximize the resulting bound in general.

Using the characterization from Corollary 1, the error
probability $P_e(C)$ can be equivalently written as

$$P_e(C) = \max_{\mu_0, t \geq 0} \left\{ \frac{1}{M} \sum_{x \in C} \text{Tr} \left( W_x \left\{ W_x - t \mu_0 \leq 0 \right\} \right) - \frac{t}{M} \right\}.$$  \quad (43)

The objective of the maximization in (43) coincides with
the information-spectrum bound [20, Lemma 4]. Then,
(43) shows that the Hayashi-Nagaoka lemma yields the
exact error probability for a fixed code, after optimization
over the free parameters $\mu_0$, $t \geq 0$.

IV. QUASI-PERFECT CODES

While the alternative expressions (41) and (43) derived
in the previous section yield the exact error probability,
they still depend on the codebook $C$. To obtain a practical
converse bound, these expressions need to be minimized
over a family of codes or input distributions. One practical
converse bound is the relaxation given in (42) which
can be evaluated in several cases of interest. Since the
converse bound (42) is a weakening of (41), it does
not coincide with the exact error probability in general.
Nevertheless, we next show that this is still the case
for certain symmetric channels and the family of codes
defined in this section.

A. Symmetric channels

Definition 1: We say that a classical-quantum channel
$x \rightarrow W_x$, with $x \in \mathcal{X}$ and $W_x \in \mathcal{D}(\mathcal{H})$, is symmetric if

$$W_x = U_x W U_x^\dagger$$  \quad (44)

for every $x \in \mathcal{X}$, where $W \in \mathcal{D}(\mathcal{H})$ does not depend on
$x$ and $U_x$ is a unitary linear operator acting on $\mathcal{H}$ and
parametrized by $x$.

This definition is related to that of covariant quantum
channels (see, e.g., [35] or [36, Sec. 9.7]). Let us
consider a quantum channel $N : \mathcal{D}(\mathcal{H}_A) \rightarrow \mathcal{D}(\mathcal{H}_B)$,
denoted by $N_{A \rightarrow B}$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the input
and output Hilbert spaces of the channel, respectively.
Let $G$ be a compact group and $g \rightarrow V_g$, $g \rightarrow U_g$, $g \in G$, be continuous (projective) unitary representations
of $G$ in the input and output Hilbert spaces of the channel, respectively. The quantum channel $N_{A \rightarrow B}$ is
called covariant with respect to the representation if

$$N_{A \rightarrow B}(V_g S V_g^\dagger) = U_g N_{A \rightarrow B}(S) U_g^\dagger$$ for every input state $S \in \mathcal{D}(\mathcal{H}_A)$.

For a classical-quantum channel $x \rightarrow W_x$, $x \in \mathcal{X}$, $W_x \in \mathcal{D}(\mathcal{H}_B)$ denoted by $N_{X \rightarrow B}$, we can
define the orthogonal basis $|x\rangle/\langle x|$. Then, letting $V_x$ be
such that $|x\rangle/\langle x| = V_x |0\rangle/\langle 0| V_x^\dagger$, it follows that $W_x = U_x N_{X \rightarrow B} |0\rangle/\langle 0| U_x^\dagger = U_x W_0 U_x^\dagger$. We conclude that any
covariant quantum channel with classical (orthogonal)
inputs also satisfies (44).

Definition 1 was also considered in [37], [38] with the
additional assumption $U_x = U_x^x$, $U^{|X|} = I$, for some
unitary $U$. For this family of symmetric channels, H.-
C. Cheng et al. derived a sphere-packing lower bound
for the optimal error probability in finite blocklengths.
The proposed bound features the sphere-packing error
exponent and the symmetry of the channel allows to
tighten the prefactor order to polynomial.
In Definition 1, however, we do not impose further structure on the unitary representations \(U_x\). The results in this section hold for this family of symmetric channels provided that there exist codes satisfying certain properties, as it will be shown next.

**B. Quasi-perfect codes**

For any operator \(\mu \in \mathcal{D}(\mathcal{H})\) and parameter \(t \in \mathbb{R}\), we define

\[
\mathcal{E}_x(t, \mu) \triangleq \{W_x - t\mu \geq 0\}, \tag{45}
\]

For a symmetric channel \(x \rightarrow W_x = U_x W U_x^\dagger\), \(x \in \mathcal{X}\), we consider the set of auxiliary operators \(\mu \in \mathcal{D}(\mathcal{H})\) such that they commute with the unitary transformations \(U_x\), \(x \in \mathcal{X}\). More precisely, for a symmetric channel \(x \rightarrow W_x\), we define

\[
\mathcal{U}_W \triangleq \{\mu \in \mathcal{D}(\mathcal{H}) \mid U_x \mu = \mu U_x\}. \tag{46}
\]

Then, for any symmetric channel \(x \rightarrow W_x\), \(x \in \mathcal{X}\), \(W_x \in \mathcal{D}(\mathcal{H})\), and \(\mu \in \mathcal{U}_W\), it follows that

\[
\mathcal{E}_x(t, \mu) = \{U_x W U_x^\dagger - t\mu \geq 0\} \tag{47}
\]

\[
= U_x \{W - tU_x^\dagger \mu U_x \geq 0\} U_x^\dagger \tag{48}
\]

\[
= U_x \mathcal{E}(t, \mu) U_x^\dagger, \tag{49}
\]

where in the last step we used the fact that \(\mu U_x = U_x \mu\) and defined \(\mathcal{E}(t, \mu) \triangleq \{W - t\mu \geq 0\}\), which does not depend on \(x \in \mathcal{X}\).

Similarly to (45), we define

\[
\mathcal{E}_x^c(t, \mu) \triangleq \{W_x - t\mu = 0\}, \tag{50}
\]

\[
\mathcal{E}_x^c(t, \mu) \triangleq \{W_x - t\mu > 0\}, \tag{51}
\]

and,

\[
F_x(t, \mu) \triangleq \text{Tr}(W_x \mathcal{E}_x^c(t, \mu)), \tag{52}
\]

\[
G_x(t, \mu) \triangleq \text{Tr}(\mu \mathcal{E}_x^c(t, \mu)), \tag{53}
\]

where, \(F_x(\cdot) \triangleq F_x^c(\cdot)\), \(G_x(\cdot) \triangleq G_x^c(\cdot)\), independent of \(x \in \mathcal{X}\) for symmetric channels.

**Definition 2:** A code \(C\) is perfect for a classical-quantum channel \(x \rightarrow W_x\), if there exist a scalar \(t\) and a state \(\mu \in \mathcal{D}(\mathcal{H})\) such that the projectors \(\{\mathcal{E}_x(t, \mu)\}_{x \in C}\) are orthogonal to each other and \(\sum_{x \in C} \mathcal{E}_x(t, \mu) = \mathbb{1}\). More generally, a code is quasi-perfect if there exist \(t\) and \(\mu \in \mathcal{D}(\mathcal{H})\) such that the projectors \(\{\mathcal{E}_x(t, \mu)\}_{x \in C}\) are orthogonal to each other, and for \(I_x = \sum_{x \in C} \mathcal{E}_x(t, \mu)\), \(I_o = \mathbb{1} - I_x\), it holds that \(\sum_{x \in C} E_x^c(t, \mu) = c I_o\) where \(c \in \mathbb{R}, c > 0\) is a normalizing constant that depends on the code \(C\).

**Example 2:** Let us consider the pure-state channel \(x \rightarrow W_x = |\varphi_x\rangle\langle\varphi_x|\) acting on a \(n\)-dimensional Hilbert space \(\mathcal{H}\). This channel is symmetric, as any pure-state \(W_x\) can be constructed via unitary transformations from an arbitrary pure-state \(W = |\psi\rangle\langle\psi|\). If we do not impose further restrictions on the output of the system, i.e., it can be an arbitrary pure state \(W_x = U_x W U_x^\dagger\), then the only state \(\mu\) which commutes with all unitary linear operator \(U_x\), \(x \in \mathcal{X}\), is the maximally mixed state \(\mu = \frac{1}{n} \mathbb{1}\).

According to Definition 2, a code \(C\) with \(M = n\) orthogonal pure states is perfect for this channel with parameters \(t = n\) and \(\mu = \frac{1}{n} \mathbb{1}\), since the projectors \(\mathcal{E}_x(n, \frac{1}{n} \mathbb{1}) = \{|\varphi_x\rangle\langle\varphi_x| - \mathbb{1} \geq 0\} = |\varphi_x\rangle\langle\varphi_x|\) are orthogonal for \(x \in C\), and they form a basis for \(\mathcal{H}\). Note that this particular case can be reduced to a classical problem, since the channel outputs commute with each other. Similarly, a code with \(M \geq n\) is quasi-perfect for this channel with parameters \(t = n\) and \(\mu = \frac{1}{n} \mathbb{1}\) provided that \(\sum_{x \in C} |\varphi_x\rangle\langle\varphi_x| = c \mathcal{E}\) with \(c = \frac{M}{n}\). A family of codes fulfilling this properties will be studied in detail in Section V. For these quasi-perfect codes, the interiors \(\mathcal{E}_x^c(n, \frac{1}{n} \mathbb{1}) = \{|\varphi_x\rangle\langle\varphi_x| - \mathbb{1} > 0\} = 0\), hence they are orthogonal to each other, and the channel outputs don’t commute with each other. For \(M < n\), the codes for this channel and the auxiliary state \(\mu = \frac{1}{n} \mathbb{1}\) are neither perfect nor quasi-perfect.

To avoid ambiguities, we shall denote by \(\ell\) the smallest
value of $t$ such that the projectors $\{E_x(t, \mu)\}_{x \in \mathcal{C}}$ are orthogonal to each other for a certain code $\mathcal{C}$. We shall refer to $\bar{t}$ as the packing radius of the code $\mathcal{C}$ with respect to state $\mu$.

The next result provides an alternative expression for the error probability of perfect and quasi-perfect codes.

**Theorem 3 (Error probability of quasi-perfect codes):** Let the channel $x \rightarrow W_x$, $x \in \mathcal{X}$, $W_x \in \mathcal{D}(\mathcal{H})$, and $\mu \in \mathcal{U}_W$ be symmetric, and let $\mathcal{C}$ be perfect or quasi-perfect with parameters $t$ and $\mu$. Then,

$$P_e(\mathcal{C}) = 1 - F_\ast(t, \mu) + t(G_\ast(t, \mu) - |\mathcal{C}|^{-1}),$$

(54)

where $|\mathcal{C}|$ denotes the cardinality of the codebook $\mathcal{C}$.

**Proof:** Let $\mathcal{C} = \{x_1, \ldots, x_M\}$ be an arbitrary code for the (symmetric) channel $x \rightarrow W_x$. Let $\bar{t}$ be the packing radius of $\mathcal{C}$ with respect to the auxiliary state $\mu$.

We define the orthogonal basis $\{\bar{E}(i)\}$ associated to the eigenspace of $\{\bar{W} - \bar{t} \mu \geq 0\}$ such that

$$\bar{E}_\ast(\bar{t}, \mu) = \sum_{i \in \bar{T}^\ast} \bar{E}(i),$$

(55)

$$E_x(\bar{t}, \mu) = U_x \bar{E}_\ast(\bar{t}, \mu) U_x^\dagger = \sum_{i \in \bar{T}^\ast} U_x E(i) U_x^\dagger = \sum_{i \in \bar{T}^\ast} E_x(i),$$

(56)

where we let $E_x(i) \triangleq U_x E(i) U_x^\dagger$. Here, $\bar{T}^\ast$ denotes the set of basis indexes associated to the strictly positive eigenvalues. Note that the projectors $E_x(i)$ are orthogonal to $E_x'(i)$ for $x \neq x'$, $i \in \bar{T}^\ast$ since the projectors $\{E_x(i, \mu)\}$ for $x \in \mathcal{C}$ are orthogonal to each other.

Similarly, we also write

$$\bar{E}_\circ(\bar{t}, \mu) = \sum_{i \in \bar{T}^\circ} \bar{E}(i),$$

(58)

$$E_x(\bar{t}, \mu) = U_x \bar{E}_\circ(\bar{t}, \mu) U_x^\dagger = \sum_{i \in \bar{T}^\circ} U_x E(i) U_x^\dagger = \sum_{i \in \bar{T}^\circ} E_x(i),$$

(59)

where $\bar{T}^\circ$ denotes the set of basis indexes associated to the zero eigenvalues. In this later case however, there is no orthogonality condition between the projectors $E_x(i)$ for $i \in \bar{T}^\circ$ for different codewords $x \in \mathcal{C}$. Now define $d_\ast \triangleq M|\bar{T}^\ast|$ and $d_\circ \triangleq n - d_\ast$, where $n = \dim(\mathcal{H})$. The code specific constant associated with a quasi-perfect code $\mathcal{C}$ is $c \triangleq \frac{M|\bar{T}^\circ|}{d_\circ}$.

We consider the decoder $T = \{\Pi_1, \ldots, \Pi_M\}$ with projectors

$$\Pi_m = E_x(\bar{t}, \mu) + \frac{1}{c} E_x^\circ(\bar{t}, \mu),$$

(61)

where

$$\Pi = \bar{E}_\ast(\bar{t}, \mu) + \frac{1}{c} \bar{E}_\circ(\bar{t}, \mu)$$

(64)

Note that this definition implies

$$\sum_{m=1}^M \Pi_m = I_\ast + I_\circ = I,$$

(65)

as required.

We next show that this decoder satisfies the Holevo-Yuen-Kennedy-Lax conditions from Lemma 3 and it thus minimizes (38). According to (15), let

$$\Lambda(T) = \frac{1}{M} \sum_{\ell=1}^M W_\ell \Pi_\ell.$$  

(66)

To verify the condition (13), we write

$$\left( \Lambda(T) - \frac{1}{M} W_m \right) \Pi_m = \left( \frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell \right) \Pi_m + \frac{1}{M} W_m \Pi_m (\Pi_m - I).$$

(67)

Let us consider the first term in (67) only. Using (61), we decompose this term as

$$\left( \frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell \right) \Pi_m = \frac{1}{M} \left( \sum_{\ell \neq m} W_\ell E_x(\bar{t}, \mu) + \frac{1}{c} \sum_{\ell \neq m} W_\ell E_x^\circ(\bar{t}, \mu) \right) \times \left( E_x^\ast(\bar{t}, \mu) + \frac{1}{c} E_x^\circ(\bar{t}, \mu) \right).$$

(68)
We note that $E^\ell_\ell(i,\mu)$ and $E_m^\bullet(i,\mu)$ are mutually orthogonal for $\ell \neq m$; hence $E^\bullet(i,\mu)E_m^\bullet(i,\mu) = 0$.

Using that the subspaces $I_0$ and $I_\bullet$ are mutually orthogonal, we conclude that $E^\ell_{\ell}(i,\mu)E_m^\bullet(i,\mu) = 0$, and $E^\bullet(i,\mu)E_m^\circ(i,\mu) = 0$. Then, (68) becomes
\[
\left(\frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell\right) \Pi_m = \frac{1}{MC^2} \left( \sum_{\ell \neq m} W_\ell E^\ell_\ell(i,\mu) \right) E_m^\circ(i,\mu), \tag{69}
\]

where in the last step we used that $E_m^\bullet(i,\mu)$ and $E^\bullet(i,\mu)$ are mutually orthogonal for $\ell \neq m$; and so they are $E_m^\bullet(i,\mu)$ and $E^\circ_\ell(i,\mu)$.

Since $E^\ell_\ell(i,\mu)$ is the projector associated to the nullspace of $W_\ell - \ell \mu$, it holds that
\[
W_\ell E^\circ_\ell(i,\mu) = \ell \mu E^\circ_\ell(i,\mu). \tag{70}
\]

Then, proceeding from (69), we obtain
\[
\left(\frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell\right) \Pi_m = \frac{1}{MC^2} \left( \sum_{\ell \neq m} \ell \mu E^\circ_\ell(i,\mu) \right) E_m^\circ(i,\mu) \tag{71}
\]
\[
= \frac{1}{MC^2} \ell \mu \left( \sum_{\ell \neq m} E^\circ_\ell(i,\mu) \right) E_m^\circ(i,\mu) \tag{72}
\]
\[
= \frac{1}{MC^2} \ell \mu (cI_o - E_m^\circ(i,\mu)) E_m^\circ(i,\mu) \tag{73}
\]
\[
= \frac{1}{MC^2} \ell \mu E_m^\circ(i,\mu), \tag{74}
\]

where we used first that $\sum_{x \in C} E^\ell_\ell(x,\mu) = cI_o$, and then $I_o E_m^\circ(i,\mu) = E_m^\circ(i,\mu)E_m^\circ(i,\mu) = E_m^\circ(i,\mu)$.

For the second term in (67), we can show that
\[
\frac{1}{M} W_m \Pi_m (\Pi_m - I) = \frac{1 - c}{MC^2} W_m E_m^\circ(i,\mu) \tag{75}
\]
\[
= \frac{1 - c}{MC^2} \ell \mu E_m^\circ(i,\mu). \tag{76}
\]

Combining (74) and (76) with (67), we prove that $(\Lambda(T) - \frac{1}{M} W_m) \Pi_m = 0$. Following analogous steps, we show that $\Pi_m (\Lambda(T) - \frac{1}{M} W_m) = 0$ and hence the decoder satisfies the optimality condition (13).

Using similar arguments to those in the previous derivation, we write
\[
\Lambda(T) - \frac{1}{M} W_m = \frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell - \frac{1}{M} W_m \tag{77}
\]
\[
= \frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell + \frac{1}{M} W_m (\Pi_m - I). \tag{78}
\]

For the first term in (78), we have that
\[
\frac{1}{M} \sum_{\ell \neq m} W_\ell \Pi_\ell \tag{79}
\]
\[
= \frac{1}{M} \sum_{\ell \neq m} W_\ell E^\bullet(i,\mu) + \frac{1}{MC} \ell \mu \sum_{\ell \neq m} E^\circ_\ell(i,\mu). \tag{79}
\]

For the second term, using (61), we obtain
\[
\frac{1}{M} W_m (\Pi_m - I) \tag{80}
\]
\[
= \frac{1}{M} W_m E_m^\bullet(i,\mu) + \frac{1}{MC} W_m E_m^\circ(i,\mu) - \frac{1}{M} W_m \tag{80}
\]
\[
= \frac{1}{MC} \ell \mu E_m^\circ(i,\mu) - \frac{1}{M} W_m I_o - \frac{1}{M} W_m \sum_{\ell \neq m} E^\bullet_\ell(i,\mu), \tag{81}
\]

where in (81) we used the identities $I_\bullet = \sum_{\ell} E^\bullet_\ell(i,\mu)$ and $W_m = W_m I_\bullet + W_m I_o$.

We now use that according to the definition of quasi-perfect codes, $I_o = \frac{1}{c} \sum_\ell E^\circ_\ell(i,\mu)$. Then, substituting (79) and (81) in (78), grouping terms, we obtain
\[
\Lambda(T) - \frac{1}{M} W_m \tag{82}
\]
\[
= \frac{1}{M} (\ell \mu - W_m) I_o + \frac{1}{M} \sum_{\ell \neq m} (W_\ell - W_m) E^\bullet_\ell(i,\mu). \tag{82}
\]

The eigenvectors of $W_m - \ell \mu$ corresponding to positive eigenvalues belong to the subspace spanned by $I_\bullet$. Therefore $\frac{1}{M} (\ell \mu - W_m) I_o \geq 0$. 

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On the other hand, we have that
\[
\frac{1}{M} \sum_{\ell \neq m} (W_\ell - W_m) \mathcal{E}_\ell^*(\bar{t}, \mu)
\geq \frac{1}{M} \sum_{\ell \neq m} (\bar{t} \mu - W_m) \mathcal{E}_\ell^*(\bar{t}, \mu)
\geq \frac{1}{M} \sum_{\ell \neq m} (\bar{t} \mu - \bar{t} \mu) \mathcal{E}_\ell^*(\bar{t}, \mu) = 0
\]
where in (84) we used \( W_\ell \mathcal{E}_\ell^*(\bar{t}, \mu) \geq \bar{t} \mu \mathcal{E}_\ell^*(\bar{t}, \mu) \),
and (85) follows since \( \mathcal{E}_{\ell \neq m}^* (\bar{t}, \mu) \) being orthogonal implies that \( \mathcal{E}_{\ell \neq m}^* (\bar{t}, \mu) \) must belong to the negative eigenspace of \( W_m - \bar{t} \mu \).
We conclude that \( \Lambda(T) - \frac{1}{M} W_m \geq 0 \).

As the decoder \( T = \{ \Pi_1, \ldots, \Pi_M \} \) satisfies the optimality conditions from Lemma 3, it minimizes (38).
Then, combining (15) and (38), we obtain that the error probability of this code is
\[
P_e(C) = 1 - \text{Tr} \left( \Lambda(T^*) \right)
\]
\[
= 1 - \text{Tr} \left( \frac{1}{M} \sum_{m=1}^{M} W_m \mathcal{E}_m^*(\bar{t}, \mu) + \frac{1}{Mc} \sum_{m=1}^{M} W_m \mathcal{E}_m^*(\bar{t}, \mu) \right).
\]

Using (52), the error probability \( P_e(C) \) becomes
\[
P_e(C) = 1 - \frac{1}{M} \sum_{m=1}^{M} \chi_m^*(\bar{t}, \mu)
- \frac{1}{Mc} \text{Tr} \left( \sum_{m=1}^{M} \bar{t} \mu \mathcal{E}_m^*(\bar{t}, \mu) \right)
= 1 - \frac{1}{M} \sum_{m=1}^{M} \chi_m^*(\bar{t}, \mu) - \frac{1}{M} \text{Tr} (\mu I_o).
\]
Now, noting that \( \mu = \mu (I_o + I_\mu) \), we write
\[
\text{Tr} (\mu I_o) = 1 - \text{Tr} \left( \mu \sum_{m=1}^{M} \mathcal{E}_m^*(\bar{t}, \mu) \right)
= 1 - \sum_{m=1}^{M} \chi_m^*(\bar{t}, \mu)
\]
where the second equality follows from (53). Then, substituting (91) in (89), using the fact that, for symmetric channels, \( \chi^*_m (\bar{t}, \mu) = \chi^*_m (\bar{t}, \mu) \) and \( \chi^*_m (\bar{t}, \mu) = \chi^*_m (\bar{t}, \mu) \), and noting that \( M = |C| \), we obtain
\[
P_e(C) = 1 - \frac{1}{M} \text{Tr} (\mu I_o) + \frac{1}{M} \bar{t} (\chi^*_m (\bar{t}, \mu) - |C|^{-1})
\]
\[
= 1 - \frac{1}{M} \sum_{m=1}^{M} \chi_m^*(\bar{t}, \mu) + \frac{1}{M} \bar{t} (|C|^{-1})
\]
\[
= 1 - \frac{1}{M} \sum_{m=1}^{M} \chi_m^*(\bar{t}, \mu) + \frac{1}{M} \bar{t} |C|^{-1}.
\]

Example 3: For the pure-state channel \( W_x = |\varphi_x \rangle \langle \varphi_x | \) introduced in Example 2 above, let \( t = n \) be the number of dimensions of Hilber space \( \mathcal{H} \) and \( \mu = \frac{1}{n} I \) be the maximally mixed state. Then, \( \chi_\mu (t, \mu) = \chi_\mu (t, \mu) = 0 \) and using (54) we obtain that for any perfect or quasi-
perfect code \( C \) with cardinality \( |C| = M \), the error probability is given by
\[
P_e(C) = 1 - \frac{n}{M}.
\]
Note that \( P_e(C) \) is the average error probability of the code and that it does not describe how the errors are distributed among the different messages. It could happen that some of the projectors are inactive and the corresponding messages always yield an error, and that some messages may be decoded with no error.

We next show that perfect and quasi-perfect codes attain the converse bound (42) with equality. This result is based on the following auxiliary lemma.

Lemma 4: Let \( \rho_0 = PW \) and \( \rho_1 = P \otimes \mu \) be defined in (39) and (40), respectively. Then, the optimal trade-off (4) for a hypothesis test between \( \rho_0 \) and \( \rho_1 \) satisfies
\[
\alpha_\beta (PW \parallel P \otimes \mu) = \inf_{\beta_2 \in \mathcal{X}} \sum_{x \in \mathcal{X}} P(x) \alpha_\beta_2 (W_x \parallel \mu).
\]

Proof: We consider Lemma 2 with \( \rho_0 \leftarrow PW \) and \( \rho_1 \leftarrow P \otimes \mu \). Then, using the block-diagonal structure.
of $PW$ and $P \otimes \mu$, the identity (6) yields
\[
\alpha_{\beta}(PW \parallel P \otimes \mu) = \sup_{t \geq 0} \left\{ \sum_{x \in \mathcal{X}} P(x) \left( \text{Tr}(W_x \{ W_x - t\mu \leq 0 \}) + t \left( \sum_{x \in \mathcal{X}} P(x) \text{Tr}(\mu \{ W_x - t\mu > 0 \}) - \beta \right) \right\}
\]
(95)
for any $\{\beta'_{x}\}, x \in \mathcal{X}$, such that $\sum_{x} P(x)\beta'_{x} = \beta$.

We relax the optimization (96) by letting the parameter $t$ be different for each $x$. Then, we obtain the following upper bound on $\alpha_{\beta}(PW \parallel P \otimes \mu)$,
\[
\alpha_{\beta}(PW \parallel P \otimes \mu) \leq \sum_{x \in \mathcal{X}} P(x) \sup_{t_{x} \geq 0} \left\{ \text{Tr}(W_x \{ W_x - t_{x}\mu \leq 0 \}) + t_{x} \left( \text{Tr}(\mu \{ W_x - t_{x}\mu > 0 \}) - \beta'_{x} \right) \right\}
\]
(97)
where in the last step we applied the identity (6) from Lemma 2 with $\rho_{0} \leftarrow W_{x}$ and $\rho_{1} \leftarrow \mu$. The bound (97)-(98) holds for any $\{\beta'_{x}\}, x \in \mathcal{X}$, such that $\sum_{x} P(x)\beta'_{x} = \beta$. Then, to prove (94) it suffices to show that there exist $\{\beta'_{x}\}$ satisfying $\sum_{x} P(x)\beta'_{x} = \beta$ and such that (97) holds with equality.

Indeed, the value of $t$ maximizing (96) induces the Neyman-Pearson test (5), which due to the block-diagonal structure of the problem, can be decomposed into the sub-tests
\[
T'_{x} = \{ W_x - t\mu > 0 \} + \theta'_{x}. \tag{99}
\]
Each of these sub-tests induces a type-I error probability $\alpha'_{x}$ and type-II error probability $\beta'_{x}$, which, according to the NP lemma, satisfy $\sum_{x} P(x)\alpha'_{x} = \alpha_{\beta}(PW \parallel P \otimes \mu)$ and $\sum_{x} P(x)\beta'_{x} = \beta$. It follows that, for this choice of $\{\beta'_{x}\}$, the optimization in (97) yields $t_{x} = t$ (as the $t$ parameter in the NP subtests is unique), and therefore (97) holds with equality. The result thus follows.

Lemma 4 asserts that, for a binary hypothesis test between classical-quantum distributions, it is possible to express the optimal type-I error probability as a convex combination of that of disjoint sub-tests provided that the type-II error is optimally distributed among them. The next result follows from combining Theorem 3 and Lemmas 2 and 4.

**Theorem 4** (Quasi-perfect codes attain the meta-converse): Let the channel $x \rightarrow W_{x}$ be symmetric and let $C$ be perfect or quasi-perfect with parameters $t$ and $\mu \in \mathcal{U}_{W}$. Then, for $M = |C|$,\[
\mathbb{P}_{e}(C) = \inf_{P} \sup_{\mu'} \frac{1}{M} \left( \mathbb{P}_{\mu}(PW \parallel \mu' \otimes \mu) \right). \tag{100}
\]

**Proof:** According to (42) in Theorem 2, the right-hand side of (100) is a lower bound to the error probability of any code. Then, to prove (100), it suffices to show that the error probability of $C$ coincides with this lower bound. Using Lemma 4, fixing the auxiliary state $\mu$ to that from Definition 2, we obtain
\[
\inf_{P} \sup_{\mu'} \frac{1}{M} \left( \mathbb{P}_{\mu}(PW \parallel \mu' \otimes \mu) \right) \geq \inf_{(P(x),\beta_{x})} \sum_{x \in \mathcal{X}} P(x)\alpha_{\beta_{x}}(W_{x} \parallel \mu). \tag{101}
\]

Now, using (6) from Lemma 2, letting $t' = t$, and using the definitions of $F'_{x}(t,\mu)$ and $G'_{x}(t,\mu)$, it follows that
\[
\alpha_{\beta_{x}}(W_{x} \parallel \mu) \geq 1 - F'_{x}(t,\mu) + t(G'_{x}(t,\mu) - \beta_{x}) \tag{102}
= 1 - F_{x}(t,\mu) + t(G_{x}(t,\mu) - \beta_{x}), \tag{103}
\]
where in the last step we used that for symmetric channels, $F_{x}(t,\mu) = F'_{x}(t,\mu)$ and $G_{x}(t,\mu) = G'_{x}(t,\mu)$.
Then, using (103) in (101), we obtain
\[ \inf_{P} \sup_{\mu'} \alpha_{\frac{1}{\beta_{x}}} \left( PW \parallel P \otimes \mu' \right) \]
\[ \geq \inf_{\{ P(x), \beta_{x} \}} \sum_{x} P(x) \beta_{x} = \frac{1}{\beta_{x}} \left( 1 - F_{\star}(t, \mu) \right) + t \left( G_{\star}(t, \mu) - \sum_{x} P(x) \beta_{x} \right) \]
\[ = 1 - F_{\star}(t, \mu) + t \left( G_{\star}(t, \mu) - \frac{1}{M} \right) \quad (104) \]
where in the second step we used that the constraint \( \sum_{x} P(x) \beta_{x} = \frac{1}{M} \) implies that the objective does not depend on the optimization variables.

The right-hand side of (105) coincides with the error probability of the quasi-perfect codes given in (54).

Then, using this observation and (42) we conclude that, whenever \( C \) is perfect or quasi-perfect,
\[ P_{e}(C) \leq \inf_{P} \sup_{\mu'} \alpha_{\frac{1}{\beta_{x}}} \left( PW \parallel P \otimes \mu' \right) \leq P_{e}(C), \]
(106)
and the converse bound (100) must hold with equality.

The computation of the right-hand side of (100) can be simplified by exploiting the symmetry of the channel. Indeed, for a symmetric channel \( x \rightarrow W_{x} \) and \( \mu \in \mathcal{D}(\mathcal{H}) \) being invariant under this symmetry, namely \( \mu \in \mathcal{U}_{\mathcal{W}} \), it follows that (see [22, Sec. V.E])
\[ \inf_{P} \sup_{\mu'} \alpha_{\frac{1}{\beta_{x}}} \left( PW \parallel P \otimes \mu' \right) = \alpha_{\frac{1}{\beta_{x}}} \left( W_{x} \parallel \mu \right). \]
(107)
Therefore, we obtain the following corollary to Theorem 4: when the channel \( x \rightarrow W_{x} \) is symmetric and \( C \) is perfect or quasi-perfect with parameters \( t \) and \( \mu \in \mathcal{U}_{\mathcal{W}} \), the error probability of \( C \) is
\[ P_{e}(C) = \alpha_{\frac{1}{\beta_{x}}} \left( W_{x} \parallel \mu \right). \]
(108)

Remark 1: Our definition of quasi-perfect codes is restricted to channels satisfying certain symmetry conditions. In general, we could define a quasi-perfect code \( C \) of cardinality \( M \) to be quasi-perfect whenever it attains the meta-converse bound with equality, i.e., if it satisfies (100). While this operational definition would apply for symmetric and non-symmetric channels, the condition itself is difficult to verify and it does not yield any simple expression for the error probability of these codes.

Theorem 4 shows that, whenever they exist, quasi-perfect codes attain the meta-converse bound (42) with equality. Particularizing this result in the classical case, for codes satisfying the technical conditions in Definition 2, we obtain [12, Th. 1]. This occurs for example for quasi-perfect binary codes for the BSC. Definition 2 extends the notion of generalized perfect and quasi-perfect codes to classical-quantum symmetric channels and Theorem 4 shows their optimality.

In the classical setting the codes belonging to this class are rare and only exist for short blocklengths. Then, one may wonder if they exist at all for classical-quantum channels of interest. In the next section we show that this is the case for a family of 2-qubit classical-quantum channels and certain code parameters.

V. 2-QUBIT CLASSICAL-QUANTUM CHANNELS AND BELL CODES

A. Pure 2-qubit classical-quantum channel

We consider a 2-qubit pure-state channel
\[ x \rightarrow W_{x} = |\varphi_{x}\rangle \langle \varphi_{x}|. \]
(109)

We define the codebook \( \mathcal{C} = \{ x_{1}, \ldots, x_{M} \} \), with even cardinality \( M = 2K \geq 4 \), such that the channel output of the \( m \)-th codeword is \( W_{m} = |\varphi_{x_{m}}\rangle \langle \varphi_{x_{m}}| \) with
\[ |\varphi_{x_{m}}\rangle = \begin{cases} \frac{1}{\sqrt{2}} (|00\rangle + e^{i\phi_{k}} |11\rangle), & m = 1 + 2k, \\ \frac{1}{\sqrt{2}} (|01\rangle + e^{i\phi_{k}} |10\rangle), & m = 2 + 2k, \end{cases} \]
(110)
where \( \phi_{k} = 2\pi k/K \), for \( k = 0 \ldots K - 1 \).

For \( M = 4 \), the channel outputs \( |\varphi_{x_{m}}\rangle \) correspond precisely to the Bell states [39]. For \( M \geq 4 \), we refer to this family of codes as Bell codes, since they follow from a generalization of the Bell states.
Using the Bell code defined in (110), the channel output classical-quantum depolarizing channel is symmetric with respect to \( \mu_0 \) and the Bell code \( C \) is quasi-perfect for this channel. Moreover,

\[
P_e(C) = \alpha \frac{1}{M} (W_x \| \mu_0) = 1 - \frac{4}{M}, \tag{111}
\]

**Proof:** See Examples 2 and 3 in Section IV, with error probability given in (93).

When \( M = 4 \), the code corresponds precisely to the (orthogonal) Bell states and the transmitted message can be determined without errors. For \( M = 2K > 4 \), the codewords are no longer orthogonal to each other and therefore they incur in measurement errors even for the ideal pure 2-qubit classical-quantum channel. Nevertheless, as shown in Theorem 2 and in Proposition 1, there exist no other packing of pure states with lower error probability.

**B. Classical-quantum depolarizing channel**

This family of codes is not only optimal for the ideal pure-state channel but also when the transmission is affected by certain errors, as we will see next.

Consider the 2-qubit classical-quantum channel (109) where \( W_x \) is observed through a quantum depolarizing channel, defined as

\[
N^D_{A\rightarrow B}(\rho_A) = \frac{1}{4} I_A + (1 - p)\rho_A, \tag{112}
\]

where \( 0 \leq p \leq 1 \) is the depolarization parameter.

The combined classical-quantum channel is thus

\[
x \rightarrow W_x = N^D_{A\rightarrow B} (|\varphi_x\rangle \langle \varphi_x|). \tag{113}
\]

Using the Bell code defined in (110), the channel output is \( x_m \rightarrow W_m = N^D_{A\rightarrow B} (|\varphi_{x_m}\rangle \langle \varphi_{x_m}|), m = 1, \ldots, M \). 

**Proposition 2:** Let \( \mu_0 = \frac{1}{4} I_4 \). Then, the 2-qubit classical-quantum depolarizing channel is symmetric

\[
|\psi\rangle |\lambda(\mathcal{T})|\psi\rangle = \frac{1}{M} \left( \frac{1}{3M} (4 - 3p) \right) \langle \psi |\varphi_x\rangle^2 \tag{120}
\]

\[
\geq \frac{4 - 3p}{p + 4(1 - p)} = 1 \tag{121}
\]

So \( \mathcal{T} = \{\Pi_1, \ldots, \Pi_M\} \) minimizes the error probability for the Bell code \( C \).

2) **Symmetry of the channel with respect to \( \mu_0 \):** We will prove next that

\[
\mathcal{E}_x(t, \mu_0) = \begin{cases} 
I_4, & t < 0, \\
|v\rangle \langle v|, & 0 \leq t \leq t_0, \\
0, & t > t_0,
\end{cases} \tag{122}
\]
for \(|v| = |\varphi_x|\) and \(t_0 = 4 - 3p\) independent of \(x\). Then, using (122) in \(F_x(t, \mu_0) = \text{Tr}(W_x E_x(t, \mu_0))\), it yields

\[
F_x(t, \mu_0) = \begin{cases} 
1, & t < 0, \\
1 - \frac{3}{4}p, & 0 \leq t \leq t_0, \\
0, & t > t_0,
\end{cases}
\tag{123}
\]

and \(\text{Tr}(W_x |v \rangle \langle v|)\) is independent of \(\varphi_x\), so the channel is symmetric with respect to \(\mu_0\).

It remains to show that (122) holds. The identity for \(t < 0\) follows trivially. We consider an arbitrary unit-norm vector \(|v|\). Then, the largest eigenvalue of \(W_x - t \mu_0\) is given by

\[
\max_v \langle v | (W_x - t \mu_0) |v \rangle = \max_v \left\{ \frac{p}{4} + (1 - p) \frac{|\langle v | \varphi_x \rangle|^2}{4} \right\}
\]

\[
= 1 - \frac{3}{4}p - \frac{t}{4},
\tag{126}
\]

The eigenvalue (126) is negative for \(t > 4 - 3p\) and non-negative otherwise. Then, we obtain that \(F_x(t, \mu_0) = 0\), for \(t > 4 - 3p\). For \(0 \leq t \leq 4 - 3p\), (126) is the only non-negative eigenvalue with associated eigenvector \(|v| = |\varphi_x|\). Therefore, considering the three regions, we obtain (122).

3) \(C\) is quasi-perfect with respect to \(\mu_0\): Comparing (118) with the auxiliary state \(\mu_0\) considered in the statement of Proposition 2, we observe that

\[
\mu_0 = \frac{1}{c_0} \Lambda(\mathcal{T}) = \frac{1}{M c_0} \sum_{m=1}^{M} W_m \Pi_m,
\tag{127}
\]

where \(c_0 = \frac{4 - 3p}{M}\) is a normalizing constant and where \(\mathcal{T}\) satisfies the optimality conditions.

Take \(t = M c_0 = 4 - 3p\), then \(\frac{1}{M} W_m - \Lambda(\mathcal{T})\) is negative semidefinite and \(E^*_x(t, \mu_0) = 0\). As a result, \(\{E^*_x(t, \mu_0)\}_{x \in C}\) are orthogonal to each other. Similarly, for this choice of \(t\) and \(\mu_0\), it follows that \(E^*_x(t, \mu_0) = \langle \varphi_{x_m} | \varphi_{x_m} \rangle\). Therefore \(\sum_{x \in C} E^*_x(t, \mu) = \frac{M}{4} \mathbb{I}_4\) and the code is quasi-perfect.

4) Error probability: Using Theorem 4, it follows that \(P_e(C) = \alpha \frac{1}{M} (W_x || \mu_0)\). Moreover, using the optimal decoder \(\mathcal{T}\), we obtain

\[
P_e(C) = 1 - \frac{1}{M} \sum_{i=1}^{M} \text{Tr}(W_i \Pi_i)
\]

\[
= 1 - \text{Tr}(\Lambda(\mathcal{T})),
\tag{129}
\]

\[
= 1 - \frac{4 - 3p}{M},
\tag{130}
\]

where in the last step we used (118). \(\blacksquare\)

C. Extension to \(N\)-qubit classical-quantum channels

Consider now an arbitrary \(N\)-qubit classical-quantum channel with pure outputs given by

\[
|\varphi\rangle = \sum_{l=0}^{2^{N-1}} \alpha_l |l\rangle
\]

\[
= \sum_{l=0}^{2^{N-1}} \alpha_l |l_{N-1} \ldots l_0\rangle
\tag{132}
\]

\[
= \alpha_0 |0 \ldots 00\rangle + \alpha_1 |0 \ldots 01\rangle + \alpha_2 |1 \ldots 11\rangle
\tag{133}
\]

for \(\sum_{l=0}^{2^{N-1}} |\alpha_l|^2 = 1\) and where \(l_{N-1} \ldots l_0\) are the digits of the binary representation of \(l\). The channel is then \(x \rightarrow W_x = |\varphi_x\rangle \langle \varphi_x|\). For \(M = 2^{N-1} K \geq 2^N\), we define the \(N\)-qubit Bell codebook of cardinality \(M\) given by \(C = \{x_1, \ldots, x_M\}\) with channel outputs

\[
|\varphi_{x_m}\rangle = \begin{cases} 
\frac{1}{\sqrt{2}} \left( (00) + e^{i \phi_k} |11\rangle \right) \otimes |l_{N-3} \ldots l_0\rangle, & m = 1 + 2k + 2 Kl, \\
\frac{1}{\sqrt{2}} \left( (01) + e^{i \phi_k} |10\rangle \right) \otimes |l_{N-3} \ldots l_0\rangle, & m = 2 + 2k + 2 Kl,
\end{cases}
\tag{134}
\]

where \(\phi_k = 2 \pi k / K\), \(k = 0, \ldots, K - 1\), and where \(l = 0, \ldots, 2^{N-2} - 1\).

The channel output for codeword \(x_m\) is thus given by the pure state \(W_m = |\varphi_{x_m}\rangle \langle \varphi_{x_m}|\).
**Proposition 3:** Let $\mu_0 = \frac{1}{2^N} \mathbb{I}_{2^N}$. The $N$-qubit classical-quantum channel is symmetric with respect to $\mu_0$ and the $N$-qubit Bell code $C$ is quasi-perfect for this channel. Moreover,

$$ P_e(C) = \alpha \frac{1}{M} (W_x \parallel \mu_0) = 1 - \frac{2^N}{M}. $$

which is obtained using $T = \{\Pi_1, \ldots, \Pi_M\}$ with

$$ \Pi_i = \frac{2^N}{M} W_i. $$

**Proof:** The proof follows analogous steps to that of Proposition 1 and it is omitted here.

The $N$-qubit Bell code is also quasi-perfect for channels affected by depolarization as stated by the following result which is a generalization of Proposition 2.

Consider the $N$-qubit classical-quantum channel in (131) observed after a quantum depolarizing channel:

$$ N^D_{A \to B}(\rho_A) = p \frac{1}{2^N} \mathbb{I}_{2^N} + (1 - p)\rho_A. $$

The combined classical-quantum channel is $x \to W_x = N^D_{A \to B}(\langle \varphi_x \rangle \langle \varphi_x |_A \rangle)$. Using the Bell code defined in (134), the channel output to codeword $x_m$ is given by

$$ W_m = N^D_{A \to B}(\langle \varphi_{x_m} \rangle \langle \varphi_{x_m} |_A \rangle), \ m = 1, \ldots, M. $$

**Proposition 4:** Let $\mu_0 = \frac{1}{2^N} \mathbb{I}_{2^N}$. Then, the $N$-qubit classical-quantum depolarizing channel is symmetric with respect to $\mu_0$ and the $N$-qubit Bell code $C$ is quasi-perfect for this channel. Moreover,

$$ P_e(C) = \alpha \frac{1}{M} (W_x \parallel \mu_0) = 1 - \frac{1}{M} (2^N (1 - p) + p). $$

which is obtained using decoder $T = \{\Pi_1, \ldots, \Pi_M\}$ with

$$ \Pi_i = \frac{2^N}{M} |\varphi_{x_i} \rangle \langle \varphi_{x_i}|. $$

**Proof:** The proof follows analogous steps to that of Proposition 2 and it is omitted here.

VI. DISCUSSION

In this work we explored the connections between hypothesis testing and classical-quantum channel coding. First, we obtained two alternative exact expressions for the minimum error probability of multiple quantum hypothesis testing when a (classical) prior distribution is placed over the hypotheses. The expression in Theorem 1 illustrates connections among the different settings of hypothesis testing and Corollary 1 provides an alternative formulation based on information-spectrum measures. A direct application of these results to a classical-quantum channel coding problem shows that Matthews-Wehner converse bound [22, Th. 19] and Hayashi-Nagaoka lemma [20, Lemma 4] with certain parameters yield the exact error probability in this setting.

While these results are of theoretical interest, the resulting expressions still depend on the the codebook and their application as performance benchmarks for classical-quantum channels is limited. We studied different relaxations and connections with practical converse bounds in the literature, thus characterizing the weaknesses of these bounds and the gap to the exact channel-coding error probability. Of special interest for this work is the so-called meta-converse bound [22, Eq. (46)], presented here in Theorem 2, which corresponds to the error probability of a binary hypothesis test with certain parameters.

In the second part of this work, we introduced the notion of perfect and quasi-perfect codes for symmetric classical-quantum channels. It is interesting to note that this notion is channel dependent—since a code being perfect for a channel it is not necessarily perfect for another one—and that it encompasses classical perfect and quasi-perfect codes as a special case [12, Sec. IV] provided some technical conditions hold. Theorem 3 provides an expression of the error probability of perfect
and quasi-perfect codes for symmetric classical-quantum channels, which is then used in Theorem 4 to prove that these codes attain the meta-converse bound with equality. These codes, whenever they exist, are thus optimal in the sense that they achieve the smallest error probability among all codes of the same blocklength and cardinality.

Establishing the existence of generalized perfect and quasi-perfect codes for a given set of system parameters is a difficult problem, even for simple classical channels. For instance, [40] studies their existence for the BSC channel and [41] shows that MDS codes, which are generalized quasi-perfect for the $q$-ary erasure channel, only exist for blocklengths $n \leq q + 1$. In this work, we consider a family of 2-qubit classical-quantum channels affected by depolarization. Using the framework presented, we established that a generalization of Bell states, that we name Bell codes, are quasi-perfect for these channels when the code cardinality is $M \geq 4$. For these channels and code parameters, we have thus established the error probability and structure of the best coding scheme. Proving the existence of perfect and quasi-perfect codes for other classical-quantum channels of practical interest is an unexplored line of research.

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