ASYMPTOTICS OF PRINCIPAL EVALUATIONS OF SCHUBERT POLYNOMIALS FOR LAYERED PERMUTATIONS

ALEJANDRO H. MORALES, IGOR PAK, AND GRETA PANOVA

(Communicated by Patricia L. Hersh)

Abstract. Denote by $u(n)$ the largest principal specialization of the Schubert polynomial
$$u(n) := \max_{w \in S_n} \mathcal{S}_w(1, \ldots, 1).$$
Stanley conjectured that there is a limit
$$\lim_{n \to \infty} \frac{1}{n^2} \log u(n),$$
and asked for a limiting description of permutations achieving the maximum $u(n)$. Merzon and Smirnov conjectured in [Eur. J. Math. 2 (2016), pp. 227–245] that this maximum is achieved on layered permutations. We resolve both of Stanley’s problems restricted to layered permutations.

1. INTRODUCTION

Understanding the large-scale behavior of combinatorial objects is so fundamental to modern combinatorics that it has become routine and no longer requires justification. However, in algebraic combinatorics, there are fewer results in this direction, as the objects tend to have more structure and thus be less approachable. This paper studies the asymptotic behavior of the principal evaluation of Schubert polynomials, partially resolving an open problem by Stanley [Sta]. As the reader shall see, the results are surprisingly precise.

Main results. Schubert polynomials $\mathcal{S}_w(x_1, \ldots, x_n) \in \mathbb{N}[x_1, \ldots, x_{n-1}]$, $w \in S_n$, were introduced by Lascoux and Schützenberger [LS] to study Schubert varieties. They have been intensely studied in the last two decades and remain a central object in algebraic combinatorics. The principal evaluation of the Schubert polynomials can be defined via Macdonald’s identity [Mac, Eq. 6.11]
\begin{equation}
(1.1) \quad \Upsilon_w := \mathcal{S}_w(1, \ldots, 1) = \frac{1}{\ell!} \sum_{(a_1, \ldots, a_\ell) \in R(w)} a_1 \cdots a_\ell.
\end{equation}
Here $\ell = \ell(w)$ is the length of $w$ (the number of inversions), and $R(w)$ denotes the set of reduced words of $w \in S_n$: tuples $(a_1, \ldots, a_\ell)$ such that $s_{a_1} \cdots s_{a_\ell}$ is a reduced decomposition of $w$ into simple transpositions $s_i = (i, i+1)$.

Note that $\Upsilon_w$ has a more direct (but less symmetric) combinatorial interpretation as the number of certain rc-graphs (also called pipe dreams); see, e.g., [As].
In particular, we have $\Upsilon_w \in \mathbb{N}$, even though this is not immediately apparent from (1.1) (cf. [1.4]).

Denote by $u(n)$ the largest principal specialization of the Schubert polynomial $u(n) := \max_{w \in S_n} \Upsilon_w$.

**Conjecture 1.1 (Stanley [Sta]).** There is a limit

$$\lim_{n \to \infty} \frac{1}{n^2} \log u(n).$$

In addition, Stanley asked whether the permutations $w$ in $S_n$ achieving the maximum $\Upsilon_w = u(n)$ had a limiting description. There is now more evidence in favor of this (see below), but before we turn to positive results let us put this conjecture into context.

One can think of $\Upsilon_w$ as a statistical sum of weighted random sorting networks of the permutation $w$. From a combinatorial point of view, this is a more natural notion, since, e.g., $\Upsilon_{w_0} = 1$, where $w_0 = (n, n - 1, \ldots, 1)$ is the permutation with maximal length $\ell(w_0) = \binom{n}{2}$. It is thus natural to expect $u(n)$ to have nice asymptotic behavior. In fact, Stanley gave the first order of asymptotics for $u(n)$:

**Theorem 1.2 (Stanley [Sta]).**

$$\frac{1}{4} \leq \liminf_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \limsup_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}. \quad (1.2)$$

Stanley’s proof is nonconstructive and based on the Cauchy identity for Schubert polynomials; see [Man [Prop. 2.4.7]. The first constructive lower bound was given by the authors in [MPP1, §6], where the asymptotics of $\Upsilon_w$ were computed for several families of permutations. Notably, for a permutation $w(b, n - b) := (b, b - 1, \ldots, 1, n, n - 1, \ldots, b + 1)$ where $b = \frac{n}{3}$, we showed that

$$\frac{1}{n^2} \log_2 \Upsilon_{w(b, n - b)} \longrightarrow C \approx 0.25162 \quad \text{as} \quad n \to \infty.$$

In fact, it is easy to see that the limit $C$ is the largest limit value over all ratios $0 < b/n < 1$. This also gives a small improvement on the lower bound in Stanley’s theorem.

**Layered permutations** $w(b_k, \ldots, b_1)$ are defined as $w(b_k, b_{k-1}, \ldots, b_1) := (b_k, b_k - 1, \ldots, 1, b_k + b_{k-1} - 1, \ldots, b_k + b_{k-1} - 1, b_{k-1} + b_{k-2} - 1, \ldots, b_1), b_1, \ldots, n - b_1 + 1)$, for integers $b_1, \ldots, b_{k-1} + b_k = n$. They are also called Richardson and pop-stack sortable permutations in a different context; see, e.g., [AvN], [Kit, §2.1.4], and [MeS]. Denote by $L_n$ the set of layered permutations $w \in S_n$. We show Conjecture 1.1 for layered permutations and find the limiting description in this class of permutations.

**Theorem 1.3.** Let

$$v(n) := \max_{w \in L_n} \Upsilon_w.$$

Then there is a limit

$$\lim_{n \to \infty} \frac{1}{n^2} \log_2 v(n) = \frac{\gamma}{\log 2} \approx 0.2932362762,$$
where $\gamma \approx 0.2032558981$ is a universal constant. Moreover, the maximum value $v(n)$ is achieved at a layered permutation

$$w(\ldots, b_2, b_1), \quad \text{where} \quad b_i \sim \alpha^{i-1}(1-\alpha)n \quad \text{as} \quad n \to \infty,$$

for every fixed $i$, and where $\alpha \approx 0.4331818312$ is a universal constant.

In other words, the runs $b_i$ form a geometric distribution in the limit. See Figure 1 for examples of the permutation matrix of such $w$. A posteriori this is unsurprising, since the weights of reduced words are heavily skewed in favor of having many transpositions at the end.

Figure 1. Shapes of optimal layered permutations $w(1, 3, 8, 18)$ and $w(2, 4, 9, 20, 46, 106, 246, 567)$, of size 30 and 1000, respectively.

The story behind the theorem is also quite interesting. Calculations for $n \leq 10$ reported in [MeS] and [Sta], prompted Merzon and Smirnov to make the following conjecture.

Conjecture 1.4 ([MeS Conj. 5.7]). For every $n$, all permutations $w$ attaining the maximum $u(n)$ are layered permutations. In particular, $u(n) = v(n)$.

In other words, if the Merzon–Smirnov conjecture holds, our theorem proves Stanley’s conjecture with the same limit value and limiting description, as suggested by Stanley (see §4.2 however). Unconditionally, Theorem 1.3 improves the lower bound for the liminf in Theorem 1.2 to about 0.2932.

Remark 1.5. We learned about the Merzon–Smirnov conjecture from Hugh Thomas, who used it to compute $v(n)$ and permutations attaining it up to $n = 300$. This data allowed us to make a conjecture on the limit shape, which we prove in the theorem.

Exact constants. The constants $\alpha$ and $\gamma$ in Theorem 1.3 are defined as follows. Consider the function

$$f(x) := x^2 \log x - \frac{1}{2} (1-x)^2 \log(1-x) - \frac{1}{2} (1+x)^2 \log(1+x) + 2x \log 2.$$

This function is obtained from a double integral that approximates the logarithm of the product formula of Proctor [Pro] for the number of certain plane partitions (Proposition 3.1). Then $\alpha$ is defined as the solution other than $x = 1$ of the equation

$$2x f(x) + (1-x^2) f'(x) = 0;$$
see Figure 3 for plots of $f(x)$ and the equation above. The constant $\gamma$ is defined as

$$\gamma := \frac{f(\alpha)}{1 - \alpha^2}.$$ 

One can show that $\alpha$ is transcendental by using Baker’s theorem, see [Ba] §2.1], but this goes beyond the scope of this paper. It would be interesting to see if existing technology allows us to show that $\gamma$ is also transcendental.

Outline of the paper. In [2] we give the necessary background on asymptotics and on the principal evaluation of Schubert polynomials of layered permutations. In [3] we prove Theorem 1.3. We conclude with final remarks and open problems in [4].

2. Background

2.1. Permutations. We write permutations of $\{1, 2, \ldots, n\}$ as $w = w_1 w_2 \ldots w_n \in S_n$, where $w_i$ is the image of $i$. Given two permutations $u$ in $S_m$ and $v$ in $S_n$ we denote $u \times v$ the following permutation of $S_{m+n}$:

$$u \times v := u_1 u_2 \ldots u_m (m+v_1)(m+v_2)\ldots(m+v_n).$$

Similarly, denote by $1^n \times w$ the permutation

$$1^n \times w := 12\ldots m (m+w_1)(m+w_2)\ldots(m+w_n).$$

Finally, let $|b| = b_1 + \cdots + b_k$ where $b_i$ are the lengths of the layered permutation $w(b) = w(b_k,b_{k-1},\ldots,b_1)$.

2.2. Product formulas for $\Upsilon_w$ for layered permutations. In this section we give a product formula for $\Upsilon_w$ when $w$ is a layered permutation $w(b_k,\ldots,b_1)$.

Let $w_0(p)$ be the longest permutation $(p,p-1,\ldots,1)$ and let

$$F(m,p) := \Upsilon_{1^n \times w_0(p)}.$$ 

Fomin–Kirillov [FK] showed that $F(m,p)$ counts the number of plane partitions of shape $(p-1, p-2, \ldots, 1)$ with entries at most $m$. This number of plane partitions has a product formula given by Proctor [Pro].

Theorem 2.1 ([FK], [Pr]). In the notation above, we have:

$$F(m,p) = \prod_{1 \leq i < j \leq p} \frac{2m + i + j - 1}{i + j - 1}.$$ 

In notation of [MPP2], we have:

$$F(m,p) = \frac{\Lambda(2m + 2p) \Lambda(2m + 1) \Phi(p)}{\Phi(2m + p) \Lambda(2p)},$$

where $\Phi(n) := 1! \cdot 2! \cdots (n-1)!$ and $\Lambda(n) := (n-2)!/4! \cdots$.

Proposition 2.2. For nonnegative integers $b_1,b_2,\ldots,b_k$, let $w(b_k,\ldots,b_1)$ be the associated layered permutation; then

$$\Upsilon_{w(b_k,\ldots,b_1)} = \Upsilon_{w(b_k,\ldots,b_2)} \cdot F(|b| - b_1,b_1),$$

where $|b| = b_1 + b_2 + \cdots + b_k$.

1In the pattern-avoiding literature this operation is usually denoted by $u \oplus v$, but we will stick to the Schubert polynomials notation.
Proof. The permutation $w(b_k, \ldots, b_1)$ can be written as the product $w(b_k, \ldots, b_2) \times w_0(b_1)$. By properties of Schubert polynomials (e.g., see [Mac (4.6)] or [Man Cor. 2.4.6]) we have that

$$S_{w(b_k, \ldots, b_1)} = S_{w(b_k, \ldots, b_2)} \cdot S_{1^{b_k - b_1}} \times w_0(b_1);$$

and the result follows by doing a principal evaluation. □

Remark 2.3. Equation (2.1) can be turned into a dynamic program to find layered permutations $w(b_k, \ldots, b_1)$ that achieve $v(n)$.

3. Asymptotics of the largest $v(n)$

3.1. The outline. We will use (2.1) inductively to prove the main result. Let $p := b_1$ and $m := n - p$, so that $m = b_2 + \ldots + b_k$. By definition of $v(n)$, we have that

$$v(n) = \max_{b:|b| = n} \gamma_{w(b)}.$$

Next, using (2.1), $v(n)$ becomes

$$v(n) = \max_{1 \leq p \leq n} \{v(n - p) F(n - p, p)\}.$$ (3.1)

We will need very precise estimates on $\log F(m, n - m)$. Note that the exact asymptotic expansion for the Barnes $G$-function can be used to obtain the asymptotics of $\Phi(\cdot)$ and $\Lambda(\cdot)$; see, e.g., [AR]. However, these bounds are insufficient as we also need sharp bounds for the error terms which hold for all $m$ and $n$. We obtain these in the next subsection. These estimates are then combined with Proposition 2.2 to prove Theorem 1.3.

3.2. Technical estimates. Let $f(x)$ be the function defined in (1.3). The next lemma gives bounds on $\log F(m, n - m)$ in terms of the function $f(x)$.

Proposition 3.1. For all integers $n \geq m \geq 0$, we have

$$-2n \leq \log F(m, n - m) - n^2 f(m/n) \leq 0.$$

We split the proof into two lemmas, one for the upper bound and the other for the lower bound.

Lemma 3.2. For all integers $n \geq m \geq 0$, we have:

$$\log F(m, n - m) - n^2 f(m/n) \leq 0.$$

Proof. We use the product formula for $F(m, p)$ in Theorem 2.1

$$\log F(m, p) = \sum_{1 \leq i < j \leq p} \left( \log(2m + i + j - 1) - \log(i + j - 1) \right)$$

$$= \sum_{1 \leq i < j' \leq p - 1} \left( \log(2m + i + j') - \log(i + j') \right),$$ (3.2)

where we changed the index to $j' = j - 1$. Next, we approximate this sum using a double integral. Let

$$g(x, y) := \log(2m + x + y) - \log(x + y).$$
Notice that the function $g(x, y)$ is constant along the lines $x + y = k$ for constant $k$. Therefore, we can shift the terms of the sum in the RHS of (3.2) by $(i, j) \mapsto (i - 1/\sqrt{2}, j + 1/\sqrt{2})$ without changing the sum (see center of Figure 2)

$$\log F(m, p) = \sum_{(i,j) \in S} \left( \log(2m + i + j') - \log(i + j') \right),$$

where $S = \{ \mathbb{Z}^2 + (-1/\sqrt{2}, 1/\sqrt{2}) \} \cap \{ (x,y) : 0 \leq x \leq p, x < y \leq p \}$.

Figure 2. Illustration of the proof of the upper and lower bounds of Proposition 3.1 for $\log F(m, p)$ for $p = 5$. The lattice points • on the left are the support of the sum $\sum_{i \leq j} g(i,j)$. This sum remains the same if the support is shifted by $(1/\sqrt{2}, 1/\sqrt{2})$, giving points □ in the middle. The original sum is bounded below by the sum over the support shifted by $(1/\sqrt{2}, 1/\sqrt{2})$, giving points □ in the right.

Next, compute the Hessian $H$ of $g(x, y)$. We have:

$$H = C \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{where} \quad C = \frac{1}{(x+y)^2} - \frac{1}{(2m + x + y)^2}.$$

Matrix $H$ has eigenvalues $0$ and $2C$ that are nonnegative in $[0, p] \times [0, p]$. Thus $g(x, y)$ is convex in this region. The modified sum in (3.3) is the sum of values of $g(x, y)$ over centers of the unit squares which fit entirely in $R$. By convexity, each such value of $g(x, y)$ is less than the average value of $g(x, y)$ over its square. Hence the sum in (3.3) is bounded above by the double integral,

$$\log F(m, p) \leq \int_0^p \int_0^y (\log(2m + x + y) - \log(x + y)) \, dx \, dy.$$

Next, we compute this double integral and obtain

$$\int_0^p \int_y^p (\log(2m + x + y) - \log(x + y)) \, dx \, dy = (m + p)^2 f(m/(m + p))$$

for $f(x)$ defined in (1.3). This proves the upper bound.

Lemma 3.3. For all integers $n \geq m \geq 0$, we have:

$$\log F(m, n - m) - n^2 f(m/n) \geq -2n.$$

Proof. Since the function $g(x, y)$ is decreasing along the $x$ direction and $y$ direction then each value $g(i,j)$ in the sum is larger than the average value of $g(x, y)$ over
the unit square with center \((i + 1/\sqrt{2}, j + 1/\sqrt{2})\) (see right of Figure 2). Hence the original sum in (3.2) is bounded below by the double integral

\[
(3.5) \quad \log F(m, p) = \sum_{1 \leq i < j \leq p} g(i, j) \geq \int_1^p \int_x^p g(x, y) \, dy \, dx.
\]

This integral can be written in terms of the original integral, computed in (3.4), as follows:

\[
\int_1^p \int_x^p g(x, y) \, dy \, dx = \int_0^p \int_x^p g(x, y) \, dy \, dx - \int_0^1 \int_x^p g(x, y) \, dy \, dx
\]

\[
(3.6) \quad = (m + p)^2 f(m/(m + p)) - \int_0^1 \int_x^p g(x, y) \, dy \, dx.
\]

Since the function \(g(x, y)\) is decreasing in the \(x\) direction then the double integral in the RHS above is bounded by the following single integral:

\[
(3.7) \quad - \int_0^1 \int_x^p g(x, y) \, dy \, dx \geq - \int_0^p g(0, y) \, dy.
\]

We evaluate this single integral and use Jensen’s inequality to obtain

\[
- \int_0^p g(0, y) \, dy = 2m \log(2m) + p \log(p) - (2m + p) \log(2m + p)
\]

\[
(3.8) \quad \geq (2m + p) (\log(2m + p) - \log 2) - (2m + p) \log(2m + p).
\]

Combining (3.5), (3.6), (3.2), and (3.5) we have

\[
\log F(m, p) \geq (m + p)^2 f(m/(m + p)) + (2m + p) (\log(2m + p) - \log 2) - (2m + p) \log(2m + p).
\]

The RHS is greater than or equal to \((m+p)^2f(m/(m+p))−2(m+p)\), as desired. \(\square\)

3.3. Optimizing constants. Our goal is to show that \(\lim_{n \to \infty} \log_2 v(n)/n^2\) is a constant. In the previous lemma we gave bounds on the error of approximating \(\log F(m, n−m)\) by \(n^2 f(x)\) where \(x = m/n\) in \([0, 1]\). We now find a unique constant \(\gamma\) such that \(f(x) + \gamma x^2\) has a unique maximum over \(x \in [0, 1]\).

Lemma 3.4. There exist unique constants \(\gamma > 0\) and \(\alpha \in (0, 1)\), such that:

1. \(2\gamma \alpha + f'(\alpha) = 0\),
2. \(\gamma \alpha^2 + f'(\alpha) = \gamma \) with \(2\gamma + r''(\alpha) \leq 0\).

And for this \(\gamma\), the maximum of \(f(x) + \gamma x^2\) over \(x \in [0, 1]\) is achieved at the given \(\alpha\), and the value is precisely \(\gamma\). That is,

\[
\max_{x \in [0, 1]} (f(x) + \gamma x^2) = f(\alpha) + \gamma \alpha^2 = \gamma.
\]

Proof. First, it is straightforward to show that \(\lim_{x \to 0} f(x) = \lim_{x \to 1} f(x) = 0\) and that \(f(x) > 0\) for \(x \in (0, 1)\) (see plot of \(f(x)\) on the left of Figure 3).

Let \(\alpha\) be a solution to the equation \(q(x) = 0\) where

\[
q(x) := f(x)(1+x^2)
\]

\[
= (1−x)^2 \log(1−x) − (1+x)^2 \log(1+x) + 2x \log(x) + 2(1+x^2) \log(2).
\]
This function on the RHS above has one root $\alpha = 0.4331818312..$ and the other is $x = 1$, as easily seen from the plot, but also can be shown analytically. Then we set

$$\gamma := \frac{f(\alpha)}{1-\alpha^2} = -\frac{f'(\alpha)}{2\alpha},$$

so $\gamma$ and $\alpha$ now satisfy conditions (1) and (2).

Next, we see that

$$\gamma = \frac{f(\alpha)}{1-\alpha^2} \approx 0.2032558981.$$

To prove that this is indeed a maximum for $f(x) + \gamma x^2$, we check that the second derivative,

$$d^2(\gamma x^2 + f(x))/dx^2 = 2\gamma + r''(x)$$

is negative at $x = \alpha$. We have that $r''(x) = \log(x^2/(1-x^2))$. Since $\alpha \leq 0.45$, we have that $x^2/(1-x^2) < 0.26$ and so $r''(\alpha) < -1.3 < -2\gamma$ and so the value is a local maximum and by condition (2) it is equal to $\gamma$.  

\[\square\]

3.4. Proof of Theorem\[1.3.\] The theorem follows immediately from the following lemma.

**Lemma 3.5.** For all $n \geq 2$ we have:

$$|\log v(n) - \gamma n^2| \leq 4n.$$

Conversely, suppose for a layered permutation $w(b) \in S_n$ we have

$$|\log \Upsilon_w - \gamma n^2| \leq 4n.$$

Then $b = (..., b_2, b_1)$, s.t. $b_i \sim (1-\alpha)\alpha^{i-1}n$ for all fixed $i \geq 1$.

**Proof.** We proceed by induction to show that $|\log v(n) - \gamma n^2| \leq 4n$ holds for all $n \geq 2$. The base cases $n = 2$ can be checked directly.

We start with (3.1) and use the induction hypothesis and the upper bound of Proposition 3.1 to obtain

$$\log v(n) = \max_{m<n} \left(\log v(m) + \log F(m, n-m)\right) \leq \max_{m<n} \left(\gamma m^2 + \log F(m, n-m) + 2m\right) \leq n^2 \max_{x \in [0,1)} \left(f(x) + \gamma x^2\right) + 2n.$$

By Lemma 3.4 the maximum value of $f(x) + \gamma x^2$ is equal to $\gamma$. Thus, the above inequality becomes

$$\log v(n) \leq \gamma n^2 + 2n.$$
This maximum is achieved when \( x = \alpha \), i.e., when \( m = n\alpha \) and \( p = b_1 = (1 - \alpha)n \). By the definition of \( v(n) \), for this value of \( m \) we have that
\[
\log v(n) \geq \log v(n\alpha) + \log F(n\alpha, n - n\alpha).
\]
By the induction hypothesis and the lower bound of Proposition 3.1, the above inequality becomes
\[
\log v(n) \geq \left( \gamma n^2 \alpha^2 - 4n\alpha \right) + (n^2 f(\alpha) - 2n)
\]
\[
= \gamma n^2 - 2(1 + 2\alpha)n \geq \gamma n^2 - 4n.
\]
Here we again used the fact that \( f(\alpha) + \gamma \alpha^2 = \gamma \) and that \( \alpha \leq 1/2 \). In summary,
\[
|\log v(n) - \gamma n^2| \leq 4n,
\]
and this bound is attained when \( b_1 \sim (1 - \alpha)n \). Recursively, we obtain \( b_i \sim (1 - \alpha)\alpha^{i-1}n \) for every fixed \( i = 2, 3, \ldots \) \( \square \)

**Remark 3.6.** Note that the data in the arXiv version of this paper [MPP3] shows a rather slow rate of convergence for \( h(n) : = \frac{1}{n^2} \log_2 v(n) \), giving only \( h(300) \approx 0.2904 \). This suggests that \( h(n) = \gamma / \log 2 - 1/n - o(1/n) \), so that the bound in Lemma 3.5 is quite sharp.

4. **Final remarks**

4.1. Stanley’s Conjecture [11] remains open but is very likely to hold. Denote by
\[
a(n) = \sum_{w \in S_n} \Upsilon_w
\]
the total number of rc-graphs (pipe dreams) of size \( n \). Since
\[
u(n) \leq a(n) \leq n!u(n),
\]
we conclude that it suffices to prove the asymptotics result for \( a(n) \). This suggests connections to counting general tilings (see, e.g., [AS]), as pipe dreams can be viewed as tilings of a staircase shape with two types of tiles, but with one global condition (strains can intersect at most once). The problem is especially similar to counting Knutson-Tao puzzles enumerating the Littlewood–Richardson coefficients, whose maximal asymptotics were recently studied in [PPY].

By analogy with the tilings, one can ask if \( u(n) \) satisfies some sort of super-multiplicativity property. Formally, let \( w \otimes 1^c \) denote the Kronecker product permutation of size \( cn \), whose permutation matrix equals the Kronecker product of the permutation matrix \( P_w \) and the identity \( I_c \) (see [MPPT]).

**Conjecture 4.1.** For \( w \in S_n \), we have \( \Upsilon_{w \otimes 1^2} \geq \Upsilon_w^4 \).

We verified the conjecture for all \( w \in S_n \) where \( n \leq 5 \), but perhaps more computational evidence would be helpful.

4.2. Similarly, the Merzon–Smirnov Conjecture [14] remains open. In our opinion, the numerical evidence in favor of the conjecture is insufficient, and it would be interesting to verify it for larger \( n \). To speed up the computation, perhaps, there are large classes of permutations \( u \in S_n \) which can be proved to be nonmaximal, i.e., there exists \( w \in S_n \), s.t. \( \Upsilon_u \leq \Upsilon_w \). Such permutations can then be ignored in the exhaustive search.
In fact, Prop. 6.5 in [MPP1] gives explicit constructions of large families of permutations \( w \in S_n \), for which \( \log \Upsilon_w = \Theta(n) \). These permutations are very far from being layered (in the transposition distance), suggesting that if true, proving Conjecture 1.4 might not be easy.

4.3. In [Sta], Stanley also considered the case when \( \Upsilon_w \) is small. It is well known that \( \Upsilon_w = 1 \) if and only if \( w \) is dominant [Man], i.e., 132-avoiding. Stanley conjectured that \( \Upsilon_w = 2 \) if and only if \( w \) has exactly one instance of the pattern 132. This was recently proved by Weigandt [Wei], who also showed that \( \Upsilon_w - 1 \) is greater than or equal to the number of instances of the pattern 132 in \( w \).

This suggests the problem of finding permutations where the number of patterns 132 is maximal. In the field of pattern avoidance, this problem can be rephrased as asking for permutations \( w \in S_n \) with maximal packing density of the pattern 132; see [Wil] and [Kit] §8.3.1. The solution due to Stromquist is extremely well understood, and has been both refined and generalized; see [A+], [BSV], [HSV], [Pri] §5.1, and [OEIS] A061061. The maximal packing density is attained at a layered permutation \( w(b_1, b_2, \ldots) \), where the runs \( b_i \) have a geometric distribution

\[
\rho \sim (1 - \rho)^{i-1}, \quad i = 1, 2, \ldots \quad \text{where} \quad \rho = \frac{\sqrt{3} - 1/2}{2} \approx 0.366025.
\]

While, of course, \( v(n) \) are attained at somewhat different layered permutations, the similarities to this work are rather striking and go beyond coincidences. They are rooted in the recursive nature of optimal permutations in both cases, which are solutions of similar (but different!) maximization problems.

4.4. The bounds for \( u(n) \) from Theorem 1.2 are obtained from the Cauchy identity of the double Schubert polynomial of \( w_0 := w_0(n) \) which gives

\[
\sum_{w_0 = v^{-1}u} \Upsilon_u \Upsilon_v = 2^{\binom{v}{2}}.
\]

One could then ask for large values of \( \Upsilon_w \Upsilon_{w^{-1}} \). Let

\[
u'(n) := \max_{w \in S_n} \{ \Upsilon_w \cdot \Upsilon_{w^{-1}} \}.
\]

From (4.1) one can show that \( \lim_{n \to \infty} (\log_2 \nu'(n))/n^2 = 1/2 \). The table below has the values of \( \nu'(n) \) for \( n = 2, \ldots, 9 \) and the permutations \( w \) that achieve that value \( \nu'(n) \).

| \( n \) | \( \nu'(n) \) | \( w \) |
|---|---|---|
| 3 | 2 | 132 |
| 4 | 6 | 1423 |
| 5 | 33 | 15243 |
| 6 | 286 | 162534 |
| 7 | 4620 | 1736254 |
| 8 | 162360 | 18527364 |
| 9 | 9057090 | 195283746 |

Note that for a layered permutation \( w(b) \), the permutation \( w(b) w_0^{-1} \) is dominant and so \( \Upsilon_{w(b)} w_0^{-1} = 1 \).

\(^2\)Coincidentally, just as in the case of \( \log_2 \Upsilon_{w(b,n-b)} \) mentioned in the introduction, a 1/3-2/3 split would also result in the maximal packing density in a two-layered permutation.
There is a combinatorial proof of (4.1) by Bergeron and Billey [BB] involving taking a double rc-graph of \(w_0\) (\(2^{\binom{n}{2}}\) many) and reading from each half of it permutations \(u\) and \(v\) satisfying \(w_0 = v^{-1} u\). All such double rc-graphs of \(w_0\) can be obtained from an initial double rc-graph via certain local transformations (see [BB, Sec. 4]). One can use these local transformations in a Markov chain to obtain a random double rc-graph of \(w_0\) and from it read off a permutation \(u\); see Figure 4. We conjecture that the permutation matrix of random permutations \(u\) has a parabolic frozen region.

The second permutation in Figure 4 is obtained by running a Markov chain for \(5 \cdot 10^9\) local moves on a double rc-graph of \(v^{-1} u = w_0 \in S_{50}\), described in [BB, Sec. 4]. Half of the resulting double rc-graphs given in Figure 4 is then converted into a permutation \(u \in S_{50}\).

ACKNOWLEDGMENTS

We are grateful to Nantel Bergeron, Sara Billey, Grigory Merzon, Evgeny Smirnov, Richard Stanley, and Damir Yeliussizov for helpful conversations. We are especially thankful to Hugh Thomas for sharing the data of layered permutations. We also thank the anonymous referee for helpful suggestions including the fact in the footnote in §4.3 The calculations of \(\Upsilon_w\) done in this paper were made using Sage and its algebraic combinatorics features developed by the Sage-Combinat community [Sage].

REFERENCES

[A+] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, and W. Stromquist, *On packing densities of permutations*, Electron. J. Combin. 9 (2002), no. 1, Research Paper 5, 20. MR1887086

[AS] Federico Ardila and Richard P. Stanley, *Tilings*, Math. Intelligencer 32 (2010), no. 4, 32–43, DOI 10.1007/s00283-010-9160-9. MR2747701

[AR] R. A. Askey and R. Roy, *Gamma function*, NIST handbook of mathematical functions, U.S. Dept. Commerce, Washington, DC, 2010, pp. 135–147. MR2653345

[As] S. Assaf, *Combinatorial models for Schubert polynomials*, arXiv:1703:00088.
1388 A. H. MORALES, I. PAK, AND G. PANOVA

[AvN] David Avis and Monroe Newborn, *On pop-stacks in series*, Utilitas Math. 19 (1981), 129–140. MR624050

[Ba] Alan Baker, *Transcendental number theory*, Cambridge University Press, London-New York, 1975. MR0422171

[BB] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. 2 (1993), no. 4, 257–269. MR1281474

[BSV] Miklós Bóna, Bruce E. Sagan, and Vincent R. Vatter, *Pattern frequency sequences and internal zeros*, Adv. in Appl. Math. 28 (2002), no. 3-4, 395–420, DOI 10.1006/aama.2001.0789. Special issue in memory of Rodica Simion. MR1900002

[FK] Sergey Fomin and Anatol N. Kirillov, *Reduced words and plane partitions*, J. Algebraic Combin. 6 (1997), no. 13, 447–450. MR660739

[HSV] Martin Hildebrand, Bruce E. Sagan, and Vincent R. Vatter, *Bounding quantities related to the packing density of 1(l + 1)l···2*, Adv. in Appl. Math. 33 (2004), no. 3, 633–653, DOI 10.1016/j.aam.2004.01.002. MR2081046

[Kit] Sergey Kitaev, *Patterns in permutations and words*, Monographs in Theoretical Computer Science. An EATCS Series, Springer, Heidelberg, 2011. With a foreword by Jeffrey B. Remmel. MR3012380

[LS] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 13, 447–450. MR660739

[Mac] I. G. Macdonald, *Notes on Schubert polynomials*, Publ. LaCIM, 6, Université de Québec à Montréal, Montréal, Canada, 1991.

[Man] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow; Cours Spécialisés [Specialized Courses], 3. MR1852463

[MeS] Grigory Merzon and Evgeny Smirnov, *Determinantal identities for flagged Schur and Schubert polynomials*, Eur. J. Math. 2 (2016), no. 1, 227–245, DOI 10.1007/s40879-015-0078-9. MR3454099

[MPP1] Alejandro H. Morales, Igor Pak, and Greta Panova, *Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications*, SIAM J. Discrete Math. 31 (2017), no. 3, 1953–1989, DOI 10.1137/16M1096925. MR3693598

[MPP2] Alejandro H. Morales, Igor Pak, and Greta Panova, *Asymptotics of the number of standard Young tableaux of skew shape*, European J. Combin. 70 (2018), 26–49, DOI 10.1016/j.ejc.2017.11.007. MR3779602

[MPP3] A. H. Morales, I. Pak, and G. Panova, *Asymptotics of principal evaluations of Schubert polynomials for layered permutations*, arXiv:1805.04341.

[PPY] I. Pak, G. Panova, and D. Yeliussizov, *On the largest Kronecker and Littlewood–Richardson coefficients*, arXiv:1804.04693.

[Pri] Alkes Long Price, *Packing densities of layered patterns*, ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–University of Pennsylvania. MR2695616

[Pro] Robert A. Proctor, *New symmetric plane partition identities from invariant theory work of De Concini and Procesi*, European J. Combin. 11 (1990), no. 3, 289–300, DOI 10.1016/S0195-6698(13)80128-X. MR1059559

[Sage] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2018. http://combinat.sagemath.org.

[OEIS] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences. oeis.org

[Sta] R. P. Stanley, *Some Schubert shenanigans*, arXiv:1704.00851.

[Wei] A. Weigandt, *Schubert polynomials*, 132-patterns, and Stanley’s conjecture, arXiv:1705.02065.

[Wil] Herbert S. Wilf, *The patterns of permutations*, Discrete Math. 257 (2002), no. 2-3, 575–583, DOI 10.1016/S0012-365X(02)00515-0. Kleitman and combinatorics: a celebration (Cambridge, MA, 1999). MR1935750
