The role of flow geometry in influencing the stability criteria for low angular momentum axisymmetric black hole accretion

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ABSTRACT

Using mathematical formalism borrowed from dynamical systems theory, a complete analytical investigation of the critical behaviour of the stationary flow configuration for the low angular momentum axisymmetric black hole accretion provides valuable insights about the nature of the phase trajectories corresponding to the transonic accretion in the steady state, without taking recourse to the explicit numerical solution commonly performed in the literature to study the multi-transonic black hole accretion disc and related astrophysical phenomena. Investigation of the accretion flow around a non rotating black hole under the influence of various pseudo-Schwarzschild potentials and forming different geometric configurations of the flow structure manifests that the general profile of the parameter space divisions describing the multi-critical accretion is roughly equivalent for various flow geometries. However, a mere variation of the polytropic index of the flow cannot map a critical solution from one flow geometry to the another, since the numerical domain of the parameter space responsible to produce multi-critical accretion does not undergo a continuous transformation in multi-dimensional parameter space. The stationary configuration used to demonstrate the aforementioned findings is shown to be stable under linear perturbation for all kind of flow geometries, black hole potentials, and the corresponding equations of state used to obtain the critical transonic solutions. Finally, the structure of the acoustic metric corresponding to the propagation of the linear perturbation studied are discussed for various flow geometries used.

Key words: accretion, accretion discs – black hole physics – hydrodynamics

1 INTRODUCTION

Astrophysical blackholes manifest their presence only gravitationally. No spectral information can directly be obtained from these candidates because of the presence of the event horizon. One can only, therefore, rely on accretion processes to understand their observational signatures (Pringle 1981; Kato et al. 1998; Frank et al. 2002). At large distances from the accretor, black hole accretion is usually subsonic. The inner boundary condition imposed by the event horizon is determined by the requirement that the flow will be of a supersonic nature very close to the accretor. Black hole accretion, thus, usually demonstrates transonic behaviour in general.

Such physical transonic accretion solutions can mathematically be realized as critical solutions on the phase portraits of the local radial Mach number and the radial distance measured from the event horizon (Ray & Bhattacharjee 2002, Afshordi & Paczyński 2003, Ray 2003a, Ray & Bhattacharjee 2005b, Chaudhury et al. 2006, Ray & Bhattacharjee 2008, Bhattacharjee & Ray 2007, Ray & Bhattacharjee 2007b, Goswami et al. 2007, Bhattacharjee et al. 2009a). To maintain physical transonicity such critical points will perforce have to be saddle points, which will enable a solution to pass through themselves. In this connection, a “multi-critical” flow refers to the category of the

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accretion configuration which can have more than one critical points accessible to the flow solution. For low angular momentum axisymmetric black hole accretion, it may so happen that the critical features are exhibited more than once in the phase portrait of a stationary solution describing such flows, and accretion consequently becomes multi-critical (Liang & Thomson 1983; Abramowicz & Zurek 1984; Muchotrzeb & Paczynski 1982; Muchotrzeb 1983; Fukue 1983, 1987, 2004b; Lu 1985, 1986; Muchotrzeb-Czerny 1986; Abramowicz & Kato 1989; Abramowicz & Chakrabarti 1998; Kafatos & Yang 1994; Yang & Kafatos 1995; Caditz & Tsuruta 1998; Das 2002; Das et al. 2003; Barai et al. 2004; Abraham et al. 2006; Das et al. 2007; Das & Czerny 2009).

In reality, such weakly rotating sub-Keplerian flows are indeed exhibited in various physical situations, such as detached binary systems fed by accretion from OB stellar winds (Illarionov & Sunyaev 1975; Liang & Nolan 1984), semi-detached low-mass non-magnetic binaries (Biskalou et al. 1998), and super-massive black holes fed by accretion from slowly rotating central stellar clusters (Illarionov 1983; Ho 1999; and references therein). Even for a standard Keplerian accretion disc, turbulence may produce such low angular momentum flows (see, e.g., Igumenshchev & Abramowicz 1999; and references therein).

All of the aforementioned multi-critical flow dynamics are important in the astrophysical context. Such multi-critical behaviour allows the formation of standing shocks in low angular momentum axisymmetric black hole accretion (Fukue 1983, 1987, 2004b; Chakrabarti et al. 1998; Kafatos & Yang 1994; Yang & Kafatos 1995; Caditz & Tsuruta 1998; Fukumura & Tsuruta 2004; Takahashi et al. 1992; Das 2002; Das et al. 2003; Abraham et al. 2006; Das et al. 2007; Lu et al. 1997; Lu & Gu 2004; Nakayama & Fukue 1989; Nagakura & Yamada 2008; Nakayama 1996; Nagakura & Yamada 2009; Tóth et al. 1998; Das & Czerny 2009). Standing shocks in rotating astrophysical accretion potentially provide an important and efficient mechanism for conversion of a significant amount of the gravitational energy into radiation by randomizing the directed infall motion of the accreting fluid. Shocks play an important role in governing the overall dynamical and radiative processes taking place in astrophysical fluid flows around black holes.

Originally at a large distance, subsonic accretion encounters the outermost saddle type critical point and becomes supersonic. Subjected to the appropriate perturbative environment, such a supersonic flow encounters a shock and becomes subsonic again. The resulting flow has to pass through another saddle type critical point to meet the inner boundary condition as imposed by the event horizon. For accretion onto a black hole, the presence of at least two saddle type critical points is, therefore, a necessary (but not sufficient) condition for the shock formation. So multi-critical flow behaviour plays a crucial role in studying the physics of shock formation and related astrophysical phenomena.

To understand the phase-space behaviour of low angular momentum shocked multi-transonic accretion, one usually constructs the corresponding autonomous dynamical systems analogue, and then identifies the saddle type critical points of the phase trajectories of the flow. Next, the global understanding of the flow topologies are performed which necessitates a complete numerical investigation of the non-linear stationary equations describing the velocity phase space behaviour of the flow.

It is, however, still possible to semi-quantitatively realize the global behaviour of the transonic solution without taking resort to numerical techniques. Getting equipped with the mathematical formalisms of the general dynamical systems approach, it has recently been possible to conceive a clear analytical conception of some of the global behaviours of the flow by analyzing the local features of the critical points (Chaudhury et al. 2006). The result obtained in this way was shown to be independent of the choice of the black hole potentials used to study accretion flow around a non rotating black hole.

This work (Chaudhury et al. 2006), however, was performed for a particular type of flow geometry – hydrostatically balanced flow under the vertical equilibrium. Nevertheless, accretion processes onto astrophysical black holes are also studied for two other different flow geometries – flow with constant disc height, and flow under the conical equilibrium (Liang & Thomson 1983; Abramowicz & Zurek 1984; Blaes 1983; Lu et al. 1997; Chakrabarti & Das 2001; Gu & Foglizzo 2003)(see section 2 for further detail about these two disc models). Those two flow geometries are relatively simpler to handle (in comparison to the flow configuration under the vertical equilibrium) without compromising the essential physics involved in the multi-transonic black hole accretion phenomena. In addition, these two flows are appropriate to study the low angular momentum inviscid flow configuration as well. It is thus instructive to investigate whether the stationary configurations remain stable (under perturbation) in these flow geometries as well. In other words, one needs to realize whether the stationary critical solutions are stable irrespective of the nature of the space time (choice of the black hole potential) as well as the flow geometry (structure of the accretion disc).

This is precise objective of this work. The stationary and the time-dependent low angular momentum axisymmetric accretion around a Schwarzschild black hole, under the influence of a generalized pseudo-Newtonian black-hole potential in different flow geometries, have been analyzed. The stationary solutions have been considered to investigate their critical point behaviour, and to categorize systematically the nature of the critical points which appear in such flows. This is followed by a perturbative study of the full time-dependent flow, to follow the evolution of the perturbation and make predictions about the stability of the stationary configuration. Finally, observations have also been made about the nature of the acoustic metric embedded inside the flow.
2 THE EQUATIONS OF THE FLOW AND ITS FIXED POINTS

When considering a rotating, axisymmetric, inviscid steady flow, the two most pertinent equations are the ones determining the drift in the radial direction (essentially Euler’s equation),

\[ \frac{v^2}{2} + \frac{1}{\rho} \frac{dP}{dr} + \phi'(r) - \frac{\lambda^2}{r^2} = \mathcal{E} \]  

(1)

and the equation of continuity,

\[ \frac{d}{dr} (\rho vr H) = 0, \]  

(2)

in which, \( \phi(r) \) is the generalised pseudo-Newtonian potential driving the flow (with the prime denoting a spatial derivative), \( \lambda \) is the conserved angular momentum of the flow, \( P \) is the pressure of the flowing gas and \( H \equiv H(r) \) is the local thickness of the disc, respectively. The two foregoing equations give the steady continuum distribution of the velocity field, \( v(r) \), and the density field, \( \rho(r) \). But to close the two equations it will also be necessary to prescribe the functional dependences of both \( P \) and \( H \) on \( v \) and \( \rho \), which, in the steady state regime, will imply an ultimate dependence on \( r \).

Following this requirement, the pressure, \( P \), is first prescribed by an equation of state for the flow (Chandrasekhar 1939). As a general polytropic it is given as \( P = K \rho^\gamma \), while for an isothermal flow the pressure is given by \( P = \rho K T / \mu m_H \), in all of which, \( K \) is a measure of the entropy in the flow, \( \gamma \) is the polytropic exponent, \( \kappa \) is Boltzmann’s constant, \( T \) is the constant temperature, \( m_H \) is the mass of a hydrogen atom and \( \mu \) is the reduced mass, respectively.

In fixing the function, \( H \), one needs to look at the relevant vertical geometry of the disc system. This can vary in many ways, with different degrees of complexity (Chakrabarti & Das 2001). In the simplest case one could treat \( H \) to be just a constant, i.e. the disc is of uniform thickness. In the case of the conical flow (Abramowicz & Zurek 1981) one prescribes, \( H = Dr \), where \( D \) is a constant simple of proportionality.

2.1 Polytropic flows

With the polytropic relation specified for \( P \), it is a straightforward exercise to set down in terms of the speed of sound, \( c_s \), the first integral of Eq. (1) as,

\[ \frac{v^2}{2} + n c_s^2 + \phi(r) + \frac{\lambda^2}{2r^2} = \mathcal{E} \]  

(3)

in which \( n = (\gamma - 1)^{-1} \) and the integration constant \( \mathcal{E} \) is the Bernoulli constant. Before moving on to find the first integral of Eq. (2) it should be important to obtain the functional form of \( H \). In simple cases of the vertical disc geometry, \( H \) usually becomes an explicitly defined function of the radial distance, \( H(r) \) — either constant disc height, or a conical profile with a linear dependence, \( H = Dr \). Making a note of this fact, the first integral of Eq. (2) could be obtained as

\[ c_s^2 v^2 r^2 H^2 = \frac{\mathcal{M}^2}{4\pi^2}, \]  

(4)

where \( \mathcal{M} = (\gamma K)^{1/2} m \) (Abramowicz & Zurek 1981) with \( m \), an integration constant itself, being physically the matter flow rate.

To obtain the critical points of the flow, it should be necessary first to differentiate both Eqs. (3) and (4), and then, on combining the two resulting expressions, to arrive at

\[ \left( v^2 - c_s^2 \right) \frac{d}{dr} (v^2) = \frac{2v^2}{r} \left[ \frac{\lambda^2}{r^2} - r \phi' + c_s^2 \left( 1 + r \frac{H'}{H} \right) \right], \]  

(5)

in which \( H' \) implies \( \frac{dH}{dr} \). The critical points of the flow will be given by the condition that the entire right hand side of Eq. (5) will vanish along with the coefficient of \( \frac{d(v^2)}{dr} \). Explicitly written down, and following some rearrangement of terms, this will give the two critical point conditions as,

\[ v_c^2 = c_s^2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right]^{-1}, \]  

(6)

with the subscript \( c \) labelling critical point values. To fix the critical point coordinates, \( v_c \) and \( r_c \), in terms of the system constants, one would have to make use of the conditions given by Eqs. (5) along with Eq. (3), to obtain
\[
\frac{1}{2} \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right]^{-1} + \phi(r_c) + \frac{\lambda^2}{2r_c^2} = \mathcal{E},
\]

from which it is easy to see that solutions of \( r_c \) may be obtained in terms of \( \lambda \) and \( \mathcal{E} \) only, i.e. \( r_c = f_2(\lambda, \mathcal{E}) \). Alternatively, \( r_c \) could be fixed in terms of \( \lambda \) and \( M_c \). By making use of the critical point conditions in Eq. (4) one could write

\[
4\pi^2 r_c^2 H^2(r_c) \left\{ \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right]^{-1} \right\}^{2n+1} = M_c^2,
\]

with the obvious implication being that the dependence of \( r_c \) will be given as \( r_c = f_2(\lambda, M) \). Comparing these two alternative means of fixing \( r_c \), the next logical step would be to say that for the fixed points, and for the solutions passing through them, it should suffice to specify either \( \mathcal{E} \) or \( M_c \) (Chakrabarti 1990).

For the two relatively simple cases of disc geometry, i.e. constant \( H \), and \( H = Dr \), what Eq. (7) delivers are,

\[
\frac{1}{2} \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] + \phi(r_c) + \frac{\lambda^2}{2r_c^2} = \mathcal{E}
\]

and

\[
\frac{1}{4} \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] + \phi(r_c) + \frac{\lambda^2}{2r_c^2} = \mathcal{E},
\]

respectively.

All the results obtained so far, can be compared with the case in which the vertical disc geometry is determined by the condition of hydrostatic equilibrium in the vertical direction. This requirement will deliver the functional form of \( H \) as,

\[
H = c_s \left( \frac{r}{\gamma \phi} \right)^{1/2},
\]

which evidently shows that \( H \) is no more an explicit function of \( r \). Now with the help of this form of \( H \), and following the mathematical procedure outlined so far, the critical point coordinates could be fixed in terms of the system parameters, \( \mathcal{E} \) and \( M_c^2 \), by the relations,

\[
\frac{2\gamma}{\gamma - 1} \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1} + \phi(r_c) + \frac{\lambda^2}{2r_c^2} = \mathcal{E}
\]

and

\[
\frac{4\pi^2 \beta^2 r_c^3}{\gamma \phi'(r_c)} \left\{ \frac{2}{\beta^2} \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1} \right\}^{2(n+1)} = M_c^2,
\]

respectively. A detailed presentation of all these results (pertaining to the specific case of the disc balanced by hydrostatic equilibrium in the vertical direction) is to be found in a work done by Chaudhury et al. (2006).

### 2.2 Isothermal flows

For isothermal flows, the full mathematical treatment is actually much simpler. Here one has to go back to Eq. (1) and use the linear dependence between \( P \) and \( \rho \) as the appropriate equation of state. On doing so, the first integral of Eq. (1) is given as

\[
\frac{v^2}{2} + c_s^2 \ln \rho + \phi(r) + \frac{\lambda^2}{2r^2} = C
\]

with \( C \) being a constant of integration. For flow solutions which specifically decay out to zero at very large distances, the constant \( C \) can be determined in terms of the “ambient conditions” as \( C = c_s^2 \ln \rho_{\infty} \). For the disc of constant thickness, or the disc with a conical flow, one is easily able to obtain an expression that is identical to Eq. (5), from which the critical point conditions emerge exactly in the same form as what has been shown in Eq. (6). However, major point of difference lies in the fact that in an isothermal system, the speed of sound, \( c_s \), is globally a constant, and so having arrived at the critical point conditions, it should be easy to see that \( r_c \) and \( \Phi \) have already been fixed in terms of a global constant of the system Chaudhury et al. (2006). The speed of sound can further be written in terms of the temperature of the system as \( c_s = \Theta T^{1/2} \), where \( \Theta = (\kappa/\mu_m) \). and, therefore, it should be entirely possible to give a functional dependence for \( r_c \), as \( r_c = f_3(\lambda, T) \). For the two cases of \( H \) being a constant, and \( H = Dr \), once again the two respective relations for \( r_c \) are

\[
r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} = c_s^2,
\]

and

\[
r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} = 2c_s^2,
\]
with the subscript \( c \) labelling critical point values, as usual.

To obtain similar results for the disc in vertical hydrostatic equilibrium, it is necessary to go back to Eq. (11), with the restriction that \( \gamma = 1 \), and \( c_0 \) is a global constant. Then it becomes a simple exercise to fix the critical point coordinates, by deriving the expression,

\[
v_c^2 = c_0^2 = 2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1}.
\]

A detailed derivation of this particular result can once again be found in the work of Chaudhury et al. (2006).

3 NATURE OF THE FIXED POINTS : A DYNAMICAL SYSTEMS STUDY

The equations governing the flow in an accreting system are in general first-order non-linear differential equations. There is no standard prescription for a rigorous mathematical analysis of these equations. Therefore, for any understanding of the behaviour of the flow solutions, a numerical integration is in most cases the only recourse. On the other hand, an alternative approach could be made to this question, if the governing equations are set up to form a standard first-order dynamical system (Jordan & Smith 1999). This is a very usual practice in general fluid dynamical studies (Bohr et al. 1993), and short of carrying out any numerical integration, this approach allows for gaining physical insight into the behaviour of the flows to a surprising extent. As a first step towards this end, for the stationary polytropic flow, as given by Eq. (5), it should be necessary to parametrise this equation and set up a coupled autonomous first-order dynamical system as

\[
\frac{d}{d\tau} (\nu^2) = 2 \nu^2 \left[ \frac{\lambda^2}{r^2} - r \phi' + c_0^2 \left( 1 + r \frac{H'}{H} \right) \right],
\]

\[
\frac{dr}{d\tau} = r \left( \nu^2 - c_0^2 \right),
\]

in which \( \tau \) is an arbitrary mathematical parameter. With respect to accretion studies in particular, this kind of parametrisation has been reported before (Ray & Bhattacharjee 2002; Afshordi & Paczyński 2003; Chaudhury et al. 2006; Mandal et al. 2007; Goswami et al. 2007; Bhattacharjee et al. 2009b).

The critical points have been fixed in terms of the flow constants. About these fixed point values, upon using a perturbation prescription of the kind \( \nu^2 = \nu_c^2 + \delta \nu^2 \), \( c_0^2 = c_0^2 + \delta c_0^2 \) and \( r = r_c + \delta r \), one could derive a set of two autonomous first-order linear differential equations in the \( \delta r \) — \( \delta \nu^2 \) plane, with \( \delta c_0^2 \) having to be first expressed in terms of \( \delta r \) and \( \delta \nu^2 \), with the help of Eq. (13) — the continuity equation — as

\[
\frac{\delta c_0^2}{c_0^2} = - (\gamma - 1) \left\{ \frac{\delta \nu^2}{2c_0^2} + \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] \frac{\delta r}{r_c} \right\}.
\]

The resulting coupled set of linear equations in \( \delta r \) and \( \delta \nu^2 \) will follow simply as

\[
\frac{d}{d\tau} (\delta \nu^2) = - (\gamma - 1) \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] c_0^2 \delta \nu^2
\]

\[
- 2 c_0^2 \left[ \frac{2 \lambda^2}{r_c^2} + \phi'(r_c) + r_c \phi''(r_c) + (\gamma - 1) \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] \frac{c_0^2}{r_c} - c_0^2 \left( \ln H(r_c) \right)' + r_c \left( \ln H(r_c) \right)'' \right] \delta r,
\]

\[
\frac{d}{d\tau} (\delta r) = \left( \frac{1 + 1}{2} \right) r_c \delta \nu^2 + (\gamma - 1) \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] \delta \nu,
\]

in which a prime implies a derivative with respect to \( r \). Trying solutions of the kind \( \delta \nu^2 \sim \exp(\Omega \tau) \) and \( \delta r \sim \exp(\Omega \tau) \) in Eqs. (20), will deliver the eigenvalues \( \Omega \) — growth rates of \( \delta \nu^2 \) and \( \delta r \) — as

\[
\Omega^2 = \left\{ (\gamma - 1) \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] c_0^2 \right\}^2
\]

\[
- (\gamma + 1) r_c c_0^2 \left[ \frac{2 \lambda^2}{r_c^2} + \phi'(r_c) + r_c \phi''(r_c) + (\gamma - 1) \left[ 1 + r_c \frac{H'(r_c)}{H(r_c)} \right] \frac{c_0^2}{r_c} - c_0^2 \left( \ln H(r_c) \right)' + r_c \left( \ln H(r_c) \right)'' \right] \right\}.
\]

For the specific cases of the simple vertical geometries, i.e. \( H \) is a constant, and \( H = Dr \), the foregoing expression reduces to

\[
\Omega^2 = (\gamma - 1)^2 c_0^4 - (\gamma + 1) r_c c_0^2 \left[ \frac{2 \lambda^2}{r_c^2} + \phi'(r_c) + r_c \phi''(r_c) + (\gamma - 1) \frac{c_0^2}{r_c} \right]
\]

and

\[
\Omega^2 = 4 (\gamma - 1)^2 c_0^4 - (\gamma + 1) r_c c_0^2 \left[ \frac{2 \lambda^2}{r_c^2} + \phi'(r_c) + r_c \phi''(r_c) + 4 (\gamma - 1) \frac{c_0^2}{r_c} \right].
\]
respectively.

A similar treatment might also be extended to the case of the disc in vertical hydrostatic equilibrium. Following a perturbative treatment about the fixed point coordinates, the eigenvalues, $\Omega$, could be obtained in this instance as

$$\Omega^2 = \frac{4r_c \phi'(r_c)c_0^2}{(\gamma + 1)^2} \left\{ \left[ (\gamma - 1) A - 2\gamma (4 + A) + 2\gamma B \left( 1 + \frac{3}{A} \right) \right] - \frac{\lambda^2}{\lambda^2(r_c)} \left[ 4\gamma + (\gamma - 1) A + 2\gamma B \left( 1 + \frac{3}{A} \right) \right] \right\},$$

(24)

where $\lambda^2(r) = r^3 \phi'(r)$, and with

$$A = r_c \phi''(r_c) - 3, \quad B = 1 + r_c \frac{\phi'''(r_c)}{\phi'(r_c)} - r_c \frac{\phi''(r_c)}{\phi'(r_c)}.$$

The detailed calculations to arrive at Eq. (24) could once again be accessed in the work of Chaudhury et al. (2006).

For isothermal flows, similar expressions for the related eigenvalues may likewise be derived, given a particular form of the function, $H(r)$. The algebra in this case is much simpler and it is an easy exercise to assure oneself that for isothermal flows one just needs to set $\gamma = 1$ in Eqs. (21), (22, (23), and (24), to arrive at a corresponding relation for $\Omega^2$. However, it should be incorrect to assume that in this kind of study, one could always treat isothermal flows simply as a special physical case of general polytropic flows. For polytropic flows, the position of the fixed points, under a given form of $\phi(r)$, will be determined by Eq. (6) or by Eq. (8), in the case of simple vertical disc geometries, and by Eq. (12) or by Eq. (13), in the case of the disc balanced by hydrostatic equilibrium in the vertical direction. On the other hand, for isothermal flows the fixed points are simply to be determined from the critical point conditions themselves, since $c_0$ is globally a constant in this case (Chaudhury et al. 2006). The resulting difference is by no means trivial.

Once the position of a critical point, $r_c$, has been ascertained, it is then a straightforward task to find the nature of that critical point by using $r_c$ in either Eq. (21) or Eq. (24), depending on the disc geometry. Since it has been discussed in Section 2 that $r_c$ is a function of $\lambda$ and $T$ for isothermal flows, and a function of $\lambda$ and $E$ (or $M$) for polytropic flows, it effectively implies that $\Omega^2$ can, in principle, be rendered as a function of the flow parameters for either kind of flow. A generic conclusion that can be drawn about the critical points from the form of $\Omega^2$ in Eqs. (21) and (24), is that the only admissible critical points will be saddle points and centre-type points. For a saddle point, $\Omega^2 > 0$, while for a centre-type point, $\Omega^2 < 0$.

### 4 NUMERICAL RESULTS

It is quite evident from previous discussions that the location of critical points and their nature (saddle or centre) are easily determined from the roots of the equations (Eq9-11) and the sign of $\Omega^2$. These, in turn, effectively extract the qualitative features of the phase portrait without having an explicit plot of it.

In this work, with numerical evidence, these tools are exploited to compare the features of the accretion flow in different disc geometries under polytropic as well as isothermal equations of state. Though this formulation of the procedure works for any choice of a pseudo-Schwarzschild potential, the Paczynski-Witt potential (Paczynski & J 1980).

$$\phi_{PW}(r) = -\frac{1}{2 (r - 1)},$$

(25)

has been selected for the presentation of numerical computations within pseudo-Newtonian framework.

First of all, Fig. 1 shows the variation of the number of critical points for polytropic flows with the variation of $E$ and $\lambda$ for the three disc geometries. For example, the region bounded by $A_H$, $B_H$ and $D_H$ depicts the parameter values corresponding to three critical points in

**Figure 1.** Different regions in parameter space of $E$ and $\lambda$ corresponding to number and nature of critical points for polytropic flows. The dotted lines are for vertical equilibrium geometry, the solid lines are for conical geometry, and the small dashed lines are for constant-height disc. For notations see text.
because the physical conditions and the first integral of motions under the two conditions (polytropic and isothermal) are distinctly different. Flow into another correponding parametric region with a choice of impression that the first flow in the polytropic condition may at least approximately mimic the second flow in the isothermal condition, Figure 2.

The nature of the critical points will be apparent from the Fig. 3 and Fig. 4. In these plots a positive value of \( \dot{\Omega} \) indicates a saddle point

The constant energy flow solutions, for the constant-height disc geometry, within which for the subregion bounded by \( A_H, B_H, \) and \( C_H \) the value of \( \mathcal{M} \) at the innermost critical point (\( \mathcal{M}_{in} \), henceforth) is higher than that at the outermost critical point (\( \mathcal{M}_{out} \)). The relative values of \( \mathcal{M} \) are just the reverse for the other subregion bounded by \( A_H, C_H \) and \( D_H \). On the curve \( A_H C_H \) at all points \( \mathcal{M} \) acquires the same value at both the critical points; for all other regions shown in the graph there will exist only single critical points.

For other disc models, similar features are depicted by the symbols with suffixes \( C \) (Conical) and \( V \) (Vertical equilibrium). Hence for all three geometries, there are certain wedge shaped regions which correspond to three critical points and outside these regions, the parameter values within the parameter space shown in the figure, generate single critical points only. Hence for low values of \( \varepsilon \) with there will be an interval of values of \( \lambda \) for which three critical points exist (opening up the possibility of multi-transonic accretion within the subregion where \( \mathcal{M}_{in} > \mathcal{M}_{out} \) under certain other condition (see later)). In Fig. 2 the same thing is shown with the relevant parameters \( T \) and \( \lambda \). Here also the appearance of the wedge shaped region conforms to the possibility of multi-transonic accretion for certain interval of values of \( \lambda \). But here in the \( c_1^2 = \Theta^2 T \rightarrow 0 \) limit, all three equations merge together and as a result of it, in the figure the lower points \( B \) and \( C \) for all models, merge together. Instead of \( \mathcal{M} \), here \( \mathcal{C} \) is the quantity which makes a difference between subregions \( ABC \) and \( ACD \) in different models.

Both in polytropic and isothermal cases, the interval of \( \lambda \) permitting more than one critical points progressively shrinks for higher values of the other parameter (i.e. \( \varepsilon \) and \( T \), respectively) and there are certain values of \( \varepsilon \) or \( T \), dependent on the disc geometry, over which there will be only a single critical point, whatever be the values of \( \lambda \). The wedge shaped regions also widely vary from one disc geometry to another. Actually the regions of parametric values allowing multicritical points for various disk geometries may be approximately mapped from one another with some suitable scaling of \( \gamma \) (Chakrabarti & Das 2001)); but if one may map such a parametric region for the polytropic flow into another corresponding parametric region with a choice of \( \gamma \) (for the second case) nearly equal to unity, it should not create the impression that the first flow in the polytropic condition may at least approximately mimic the second flow in the isothermal condition, because the physical conditions and the first integral of motions under the two conditions (polytropic and isothermal) are distinctly different.

The nature of the critical points will be apparent from the Fig. 3 and Fig. 4. In these plots a positive value of \( \Omega^2 \) indicates a saddle point
while a negative value does the same for a centre. A common feature of both the separate plots is that initially there is region of a single saddle point, followed by the birth of a centre-type point and another saddle point (saddle-centre type bifurcation); then at another higher value of \( \lambda \), the centre-type point coalesces with the other saddle point (the outer one) and both of them annihilate each other (another saddle-centre type bifurcation, but this time in the opposite direction) so that the remaining saddle point (outer) “survives” above the critical value of \( \lambda \).

Actually in the underlying phase plots (not shown here), among the three critical points, there will always be a pair of centre-saddle for which the separatrices of the saddle form a homoclinic connection around the centre, as long as the other constant of flow (\( M \) for polytropic case and \( C \) for isothermal case) discriminates between the two saddle points. The remaining saddle point will have the separatrices connecting the event horizon with the infinity. The separatrices of the saddle point with higher value of \( M \) in the polytropic case (with lower value of \( C \) in isothermal case) form the homoclinic connection. Physically this is what it has to be, because it is the third critical point (saddle) which will allow the transonic flow solution from infinity to the event horizon, and here, following the line of argument in Ray & Bhattacharjee (2002), it can be stated that the stationary flow solution has to settle on the separatrices of a saddle point only. That is why among the critical points, only the saddle points are termed as sonic points, and not the centre-type points, although for both the types of critical points the flow speed is equal to the sound speed, i.e. Mach number becomes unity.

So if the value of \( \lambda \) is increased from a sufficiently low value, first there will a span of single saddle point (solid line in the figure) then the bifurcation occurs resulting in a pair of centre-type points (with intermediate value \( r \), denoted by \( r_{mid} \)) and another saddle point (location \( r_{in} \)) so that the older saddle point becomes the outermost critical point (location \( r_{out} \)). Before \( \lambda \) attains a certain value, the relation \( M_{in} > M_{out} \) (for isothermal case \( C_{in} < C_{out} \)) is maintained and for this interval of \( \lambda \), the \( \Omega^2 \) functions for the three critical points are plotted with dotted lines. Here the separatrices of the newly formed saddle point encompasses the centre-type point, forming a homoclinic connection; after exceeding this value of \( \lambda \), what happens is that \( M_{in} < M_{out} \), and the homoclinic connection for the inner saddle opens up, giving rise to the same sort of homoclinic connection from the outer saddle. Within this phase, the function of \( \Omega^2 \) is depicted with small dashed curves. At the end of this domain of \( \lambda \), the centre-type and the outer saddle point coalesce and “destroy” each other, after which, only the inner saddle point exists with increasing value of \( \lambda \) (solid curves).

The scenario in the isothermal case is almost similar. There is an interesting situation along the boundary between the two subregions within the wedge shaped region corresponding to multicritical points. For these parameter values \( M_{in} = M_{out} \), and thus two saddle points will be linked by a heteroclinic connection of their separatrices; for the \( \lambda \) just below this value there is some homoclinic connection for the inner saddle point, and just above this value the homoclinic connection corresponds to the outer critical points. From the point of view of a general dynamical system along this boundary, a type of bifurcation occurs that may be termed as a heteroclinic one.

The boundaries of the region in the parameter space permitting multicritical points, as it has been discussed already, are associated with saddle-centre bifurcation or merging of a pair of roots of the equations (Eq.\( \text{[11]} \)\( \text{[15]} \)\( \text{[17]} \)) needed for determining the critical points under various conditions discussed above. Now all these equations are polynomial equations. The discriminant of a general polynomial,

\[
P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,
\]

(26)
can be expressed as in terms of its roots, \( x_i \)'s, as

\[
D = a_n^{n-2}\prod_{i<j}(x_i - x_j)^2.
\]

(27)
The discriminant may be expressed as the determinant of a matrix called Sylvester matrix (http://mathworld.wolfram.com/PolynomialDiscriminant.html and other references therein),
For such a transition one such possibility is the occurrence of a shock, which may take place obeying the condition of the critical point (sonic point) with global transonic solution to the subsonic branch of the homoclinic solution of the other sonic point at certain conditions. First, there should be some transition from the supersonic branch of the non-homoclinic separatrices (trajectory) of the critical points in all the three models opens the possibility of multitransonic accretion too within the region with multicritical points of the present paper, but here it is touched upon to indicated that this full analysis keeps alive the possibility for such types of multitransonic sonic point. The determination of the exact location and the exact parameter values for such a transition really taking place is out of the scope of the isothermal condition). After making the transition, the flow will pass through the corresponding inner sonic point. The determination of the exact location and the exact parameter values for such a transition will be zero on the abovementioned boundaries and actually it is so. Here the contour plot of the poltropic flow in constant height geometry (i.e. for the polynomial in \( r_c \) in Eq.9) in \( E-\lambda \) space is shown in Fig.5. The curve exactly conforms with the corresponding boundary curve (in small dashed style) in Fig.1 drawn on the basis of the numerical method of detailed root finding. So this procedure may be thought of as a much easier alternative to find the multicritical parametric values. Considering now the question of actual accretion process, from Eq.5 it is evident that for conical and constant height model the critical points are actual sonic points, though these two are not exactly same for the other model. As for all parameter values there will always be one critical (sonic) point with separatrices spanning from infinity to the event horizon, there will always exist a transonic solution. But the features of the critical points in all the three models opens the possibility of multitransonic accretion too within the region with multicritical points under certain conditions. First, there should be some transition from the supersonic branch of the non-homoclinic separatrices (trajectory) of the critical point (sonic point) with global transonic solution to the subsonic branch of the homoclinic solution of the other sonic point at a particular value of \( r \). Secondly there should be some quantity which will discriminate between the trajectories of these two sonic points. For such a transition one such possibility is the occurrence of a shock, which may take place obeying the condition \( \mathcal{M}_{in} > \mathcal{M}_{out} \) for the polytropic case (and \( C_{in} < C_{out} \) for the isothermal condition). After making the transition, the flow will pass through the corresponding inner sonic point. The determination of the exact location and the exact parameter values for such a transition really taking place is out of the scope of the present paper, but here it is touched upon to indicated that this full analysis keeps alive the possibility for such types of multitransonic accretion, irrespective of the disc geometries and equations of state.

5 TIME-DEPENDENT STABILITY ANALYSIS OF STATIONARY SOLUTIONS

The time-dependent generalisation of the continuity condition for an axisymmetric pseudo-Schwarzschild disc is given as

\[
\frac{\partial \overline{\Sigma}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \overline{\Sigma} \nu r \right) = 0 ,
\]

in which the surface density of the disc, \( \overline{\Sigma} \), is to be expressed as \( \overline{\Sigma} \equiv \rho H \) \citep{Frank2002}. From the foregoing expression, one can, therefore, obtain

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r H} \frac{\partial}{\partial r} \left( \rho \nu r H \right) = 0 .
\]

Defining a new variable \( f = \rho \nu r H \), it is quite obvious from the form of Eq. (29) that the stationary value of \( f \) will be a constant, \( f_0 \), which can be closely identified with the matter flux rate. This follows a similar approach to spherically symmetric flows made by \citep{Petterson2002}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Contour plot of \( D \) for \( D = 0 \).}
\end{figure}
is that by Eq. (11), will also obtained for polytropic flows in the first two simple cases of the height function, and the polytropic relation
up earlier by Chaudhury et al. (2006). Nevertheless, it is worth going back to it for making some interesting comparisons. By using Eq. (11)
described by Eq. (35).

in which either

an isothermal flow balanced by hydrostatic equilibrium in the vertical direction, then Eq. (11) would have to be constrained by

10

adopted for a flow driven simply by a Newtonian potential (Ray2003c). The present treatment, of course, is of a more general nature, the
disc flow being driven by a general pseudo-Newtonian potential, \( \phi(r) \) for all disk geometries. In this system, a perturbation prescription of
the form \( v(r, t) = v_0(r) + v'(r, t) \) and \( \rho(r, t) = \rho_0(r) + \rho'(r, t) \), will give, on linearising in the primed quantities,

\[
\frac{f'}{f_0} = \frac{\rho'}{\rho_0} + \frac{v'}{v_0},
\]

which is a relation that connects all the three fluctuating quantities, \( v', \rho' \) and \( f' \), with one another (here the subscript 0 denotes stationary
background values in all the cases). Going back to Eq. (30), it becomes possible to connect \( \rho' \) exclusively to \( f' \), through the relation

\[
\frac{\partial \rho'}{\partial t} + \frac{v_0 \rho_0}{f_0} \left( \frac{\partial f'}{\partial r} \right) = 0.
\]

The case of the disc being balanced vertically under hydrostatic equilibrium is very different in mathematical terms, and has been taken
up earlier by Chaudhury et al. (2006). Nevertheless, it is worth going back to it for making some interesting comparisons. By using Eq. (11)
and the polytropic relation \( P = K \rho^\gamma \), Eq. (29) can be rendered as

\[
\frac{\partial}{\partial t} \left( \rho^{(\gamma+1)/2} \right) + \frac{\sqrt{\gamma}}{r^{3/2} \frac{\partial}{\partial r} \left( \rho^{(\gamma+1)/2} \frac{v}{\sqrt{\gamma}} \right) \overline{v^{3/2}} \frac{v}{\sqrt{\gamma}} } = 0,
\]

from which, under a new definition, \( f = \rho^{(\gamma+1)/2} \overline{v^{3/2}} / \sqrt{\gamma} \), one obtains

\[
\frac{f'}{f_0} = \left( \frac{\gamma+1}{2} \right) \frac{\rho'}{\rho_0} + \frac{v'}{v_0}.
\]

From Eq. (33), it is also very easy to set down the density fluctuations, \( \rho' \), in terms of \( f' \), as

\[
\frac{\partial \rho'}{\partial t} + 2 \left( \frac{\partial v}{\partial r} \right) = 0,
\]

with \( \beta^2 = 2(\gamma + 1)^{-1} \), as before. This result may be compared with Eq. (32) and the difference noted. If, however, one were to study
an isothermal flow balanced by hydrostatic equilibrium in the vertical direction, then Eq. (11) would have to be constrained by \( \gamma = 1 \) and
\( c_0 \) being constant. Under these conditions, the expression for density fluctuations in the flow will be identical to Eq. (32), rather than be
described by Eq. (35).

The equation for density fluctuations may show variations in terms of a constant scaling factor under different vertical disc height
geometries, but no such thing happens for the velocity fluctuations. Combining either Eqs. (31) and (32), or combining Eqs. (34) and (35)
will render the velocity fluctuations as

\[
\frac{\partial v'}{\partial t} = \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} + v_0 \frac{\partial f'}{\partial r} \right),
\]

which, upon a further partial differentiation with respect to time, will give

\[
\frac{\partial^2 v'}{\partial t^2} = \frac{\partial}{\partial t} \left[ \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial t} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial r} \right) \right].
\]

The time-dependent equation for the radial drift is given as

\[
\frac{\partial v_r}{\partial t} + v \frac{\partial v_r}{\partial r} + \frac{1}{f_0} \frac{\partial P}{\partial r} + \phi'(r) - \lambda^2 \overline{v^2} = 0,
\]

from which the linearised fluctuating part could be extracted as

\[
\frac{\partial v'}{\partial t} + \frac{\partial}{\partial r} \left( v_0 v' + \frac{c_0^2 \rho'}{\rho_0} \right) = 0,
\]

with \( c_0 \) being the speed of sound in the steady state. Differentiating Eq. (39) partially with respect to \( t \), and making use of either Eq. (32) or
Eq. (35), along with Eqs. (36) and (37), to substitute for all the first and second-order derivatives of \( v' \) and \( \rho' \), will deliver the result

\[
\frac{\partial}{\partial t} \left[ \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial t} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial r} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial r} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_0}{f_0} \left( v_0^2 - \sigma c_0^2 \right) \frac{\partial f'}{\partial r} \right] = 0,
\]

in which either \( \sigma = 1 \) or \( \sigma = \beta^2 \), depending on the choice of a particular disc geometry and the equation state applied. For isothermal flows,
\( \sigma = 1 \), for whatever disc geometry one considers — \( H \) is constant or \( H = D r \) or \( H \) as it is described by Eq. (11). The same value of \( \sigma \) is
also obtained for polytropic flows in the first two simple cases of the height function, \( H \). The common feature running through all these cases
is that \( H \) in Eq. (39) does not have any dependence on time. It is only when the flow is polytropic and the disc height geometry is expressed by
Eq. (11), will \( H \) have a time-dependence, whose ultimate consequence will be that \( \sigma = \beta^2 \) in Eq. (40).

All the terms in Eq. (40) can be expediently rendered into a compact formulation that looks like
\[ \partial_t \left( f^{\mu \nu} \partial_{\nu} f^{\ell} \right) = 0, \quad (41) \]

in which the Greek indices are made to run from 0 to 1, with the identification that 0 stands for \( t \), and 1 stands for \( r \). An inspection of the terms in the left hand side of Eq. (41) will then allow for constructing the symmetric matrix

\[ f^{\mu \nu} = \frac{v_0}{f_0} \left( \begin{array}{cc} 1 & v_0 \\ v_0 & v_0^2 - \sigma c_0^2 \end{array} \right), \quad (42) \]

Now the d’Alembertian for a scalar in curved space is given in terms of the metric \( g_{\mu \nu} \) by \( \nabla^2 \equiv \frac{1}{\sqrt{g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \psi \right) \), \( (43) \)

with \( g^{\mu \nu} \) being the inverse of the matrix implied by \( g_{\mu \nu} \). Using the equivalence that \( f^{\mu \nu} = \sqrt{-g} g^{\mu \nu} \), and therefore \( g \equiv \det (f^{\mu \nu}) \), it is immediately possible to set down an effective metric for the propagation of an acoustic disturbance as

\[ g^{\mu \nu}_{\text{eff}} = \left( \begin{array}{cc} 1 & v_0 \\ v_0 & v_0^2 - \sigma c_0^2 \end{array} \right), \quad (44) \]

which can be shown to be entirely identical to the metric of a wave equation for a scalar field in curved space-time, obtained through a somewhat different approach \( \text{(Visser 1998)} \). The inverse effective metric, \( g^{\mu \nu}_{\text{eff}} \), can be easily derived by inversion of the matrix given in Eq. (44), and this will give \( v_0^2 = \sigma c_0^2 \) as the horizon condition of an acoustic black hole for inflow solutions \( \text{(Visser 1998)} \). From the perspective of the propagation of acoustic waves (carrying information in any fluid system) in the accretion disc, what can be concluded from these arguments is that the disc geometry and the equation of state act together in determining the speed of information propagation. In the case of the disc being supported by hydrostatic equilibrium in the vertical direction, if the flow is polytropic, then the speed of information propagation will be less than the speed of sound by a factor, \( \beta \). So transonicity will not take place exactly when the bulk flow speed, \( v \), becomes equal to the speed of sound, \( c_s \). In all the other cases, transonicity will, however, take place when \( v = c_s \).

Finally, a little readjustment of terms in Eq. (40) will give an equation for the perturbation as

\[ \frac{\partial^2 f^{\ell}}{\partial t^2} + 2 \frac{\partial}{\partial r} \left( \frac{v_0}{v_0} \frac{\partial f^{\ell}}{\partial t} \right) + \frac{1}{v_0} \frac{\partial}{\partial r} \left[ v_0 \left( v_0^2 - \sigma c_0^2 \right) \frac{\partial f^{\ell}}{\partial r} \right] = 0, \quad (45) \]

whose detailed solution has been given in earlier works on inviscid axisymmetric flows \( \text{(Ray 2003c; Chaudhury et al. 2006)} \).
linear acoustic perturbation (sound speed) due to the presence of the polytropic sound speed $c_s$ in the expression of the disc height (which, in other words, is a direct repercussion of the vertical equilibrium assumption itself).

The stability properties of the aforesaid stationary configuration have been realized by perturbing (about the stationary configuration) the full time-dependent flow equations in various disc geometries, under various black hole potentials, and then by observing the time evolution of such perturbations. The general form of the wave equations corresponding to the dynamics of such perturbations, as well as the related acoustic metric, are identical for any black hole potential used, but is very different for different flow geometries. However, such perturbations do not diverge in any physical sense for any kind of flow geometries. This ensures that the stability of the stationary configuration, at least for accretion around non-rotating black holes under various pseudo-Schwarzschild potentials, is a generic feature independent of the nature of the space time (the explicit form of the black hole potential) as well as the geometric configuration of the flow (disc structure).

The exact forms of the corresponding acoustic metric for various disc geometries have also been derived in this work. This will help in studying the low angular momentum pseudo-Schwarzschild axisymmetric black hole accretion as a natural example of analogue gravity phenomena. It has recently been shown that the analogue surface gravity can be computed for multi-critical accretion onto astrophysical black holes (Abraham et al. 2006, Das et al. 2007). The present work will allow to study such phenomena for various different geometric configuration of the flow.

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Influence of flow geometry

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