Efficiently Solving MDPs with Stochastic Mirror Descent

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Abstract

We present a unified framework based on primal-dual stochastic mirror descent for approximately solving infinite-horizon Markov decision processes (MDPs) given a generative model. When applied to an average-reward MDP with \( A_{\text{tot}} \) total state-action pairs and mixing time bound \( t_{\text{mix}} \), our method computes an \( \epsilon \)-optimal policy with an expected \( \tilde{O}(t_{\text{mix}}^2 A_{\text{tot}} \epsilon^{-2}) \) samples from the state-transition matrix, removing the ergodicity dependence of prior art. When applied to a \( \gamma \)-discounted MDP with \( A_{\text{tot}} \) total state-action pairs our method computes an \( \epsilon \)-optimal policy with an expected \( \tilde{O}((1 - \gamma)^{-4} A_{\text{tot}} \epsilon^{-2}) \) samples, matching the previous state-of-the-art up to a \( (1 - \gamma)^{-1} \) factor. Both methods are model-free, update state values and policies simultaneously, and run in time linear in the number of samples taken. We achieve these results through a more general stochastic mirror descent framework for solving bilinear saddle-point problems with simplex and box domains and we demonstrate the flexibility of this framework by providing further applications to constrained MDPs.
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1 Introduction

Markov decision processes (MDPs) are a fundamental mathematical abstraction for sequential decision making under uncertainty and they serve as a basic modeling tool in reinforcement learning (RL) and stochastic control [5, 24, 30]. Two prominent classes of MDPs are average-reward MDPs (AMDPs) and discounted MDPs (DMDPs). Each have been studied extensively; AMDPs are applicable to optimal control, learning automata, and various real-world reinforcement learning settings [17, 3, 22] and DMDPs have a number of nice theoretical properties including reward convergence and operator monotonicity [6].

In this paper we consider the prevalent computational learning problem of finding an approximately optimal policy of an MDP given only restricted access to the model. In particular, we consider the problem of computing an $\epsilon$-optimal policy, i.e. a policy with an additive $\epsilon$ error in expected cumulative reward over infinite horizon, under the standard assumption of a generative model [14, 13], which allows one to sample from state-transitions given the current state-action pair. This problem is well-studied and there are multiple known upper and lower bounds on its sample complexity [4, 32, 28, 31].

In this work, we provide a unified framework based on primal-dual stochastic mirror descent (SMD) for learning an $\epsilon$-optimal policies for both AMDPs and DMDPs with a generative model. We show that this framework achieves sublinear running times for solving dense bilinear saddle-point problems with simplex and box domains, and (as a special case) $\ell_\infty$ regression [26, 27]. As far as we are aware, this is the first such sub-linear running time for this problem. We achieve our results by applying this framework to saddle-point representations of AMDPs and DMDPs and proving that approximate equilibria yield approximately optimal policies.

Our MDP algorithms have sample complexity linear in the total number of state-action pairs, denoted by $A_{tot}$. For an AMDP with bounded mixing time $t_{mix}$ for all policies, we prove a sample complexity of $\tilde{O}(t_{mix}^2 A_{tot} \epsilon^{-2})$ \footnote{Throughout the paper we use $\tilde{O}$ to hide poly-logarithmic factors in $A_{tot}$, $t_{mix}$, $1/(1-\gamma)$, $1/\epsilon$, and the number of states of the MDP.}, which removes the ergodicity condition of prior art [33] (which can in the worst-case be unbounded). For DMDP with discount factor $\gamma$, we prove a sample complexity of $\tilde{O}((1-\gamma)^{-4} A_{tot} \epsilon^{-2})$, matching the best-known sample complexity achieved by primal-dual methods [9] up to logarithmic factors, and matching the state-of-the-art [28, 31] and lower bound [4] up to a $(1-\gamma)^{-1}$ factor.

We hope our method serves as a building block towards a more unified understanding the complexity of MDPs and RL. By providing a general SMD-based framework which is provably efficient for solving multiple prominent classes of MDPs we hope this paper may lead to a better understanding and broader application of the traditional convex optimization toolkit to modern RL. As a preliminary demonstration of flexibility of our framework, we show that it extends to yield new results for approximately optimizing constrained MDPs and hope it may find further utility.

1.1 Problem Setup

Throughout the paper we denote an MDP instance by a tuple $\mathcal{M} := (S, A, P, r, \gamma)$ with components defined as follows:

- $S$ - a finite set of states where each $i \in S$ is called a state of the MDP, in tradition this is also denoted as $s$.
- $A = \bigcup_{i \in S} \mathcal{A}_i$ - a finite set of actions that is a collection of sets of actions $\mathcal{A}_i$ for states
We overload notation slightly and let \((i, a_i) \in \mathcal{A}\) denote an action \(a_i\) at state \(i\). \(A_{\text{tot}} := |\mathcal{A}| := \sum_{i \in \mathcal{S}} |\mathcal{A}_i|\) denotes the total number of state-action pairs.

- \(\mathcal{P}\) - the collection of state-to-state transition probabilities where \(\mathcal{P} := \{p_{ij}(a_i)|i, j \in \mathcal{S}, a_i \in \mathcal{A}_i\}\) and \(p_{ij}(a_i)\) denotes the probability of transition to state \(j\) when taking action \(a_i\) at state \(i\).

- \(\mathbf{r}\) - the vector of state-action transitional rewards where \(\mathbf{r} \in [0, 1]^A\), \(r_{i,a_i}\) is the instant reward received when taking action \(a_i\) at state \(i \in \mathcal{S}\).

- \(\gamma\) - the discount factor of MDP, by which one down-weights the reward in the next future step. When \(\gamma \in (0, 1)\), we call the instance a discounted MDP (DMDP) and when \(\gamma = 1\), we call the instance an average-reward MDP (AMDP).

We use \(\mathbf{P} \in \mathbb{R}^{A \times S}\) as the state-transition matrix where its \((i, a_i)\)-th row corresponds to the transition probability from state \(i \in \mathcal{S}\) where \(a_i \in \mathcal{A}_i\) to state \(j\). Correspondingly we use \(\mathbf{I}\) as the matrix with \(a_i\)-th row corresponding to \(e_i\) for all \(i \in \mathcal{S}, a_i \in \mathcal{A}_i\).

Now, the model operates as follows: when at state \(i\), one can pick an action \(a_i\) from the given action set \(\mathcal{A}_i\). This generates a reward \(r_{i,a_i}\). Also based on the transition model with probability \(p_{ij}(a_i)\), it transits to state \(j\) and the process repeats. Our goal is to compute a random policy which determines which actions to take at each state. A random policy is a collection of probability distributions \(\pi := \{\pi_i\}_{i \in \mathcal{S}}\), where \(\pi_i \in \Delta^{\mathcal{A}_i}\) is a vector in the \(|\mathcal{A}_i|\)-dimensional simplex with \(\pi_i(a_i)\) denoting the probability of taking \(a_i \in \mathcal{A}_i\) at action \(j\). One can extend \(\pi_i\) to the set of \(\Delta^A\) by filling in \(0\)'s on entries corresponding to other states \(j \neq i\), and denote \(\Pi \in \mathbb{R}^{S \times A}\) as the concatenated policy matrix with \(i\)-th row being the extended \(\Delta_i\). We denote \(\mathbf{P}^\pi\) as the transitional probability matrix of the MDP when using policy \(\pi\), thus we have \(\mathbf{P}^\pi(i, j) := \sum_{a_i \in \mathcal{A}_i} \pi_i(a_i) p_{ij}(a_i) = \Pi \cdot \mathbf{P}\) for all \(i, j \in \mathcal{S}\), where \(\cdot\) in the right-hand side (RHS) denotes matrix-matrix multiplication. Further, we let \(\mathbf{r}^\pi\) denote corresponding average reward under policy \(\pi\) defined as \(\mathbf{r}^\pi := \Pi \cdot \mathbf{r}\), where \(\cdot\) in RHS denotes matrix-vector multiplication. We use \(\mathbf{I}\) to denote the standard identity matrix if computing with regards to probability transition matrix \(\Pi^\pi \in \mathbb{R}^{S \times S}\).

Given an MDP instance \(\mathcal{M} := (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathbf{r}, \gamma)\) and an initial distribution over states \(\mathbf{q} \in \Delta^S\), we are interested in finding the optimal \(\pi^*\) among all policy \(\pi\) that maximizes the following cumulative reward \(\bar{\nu}^\pi\) of the MDP:

\[
\pi^* := \arg \max_\pi \bar{\nu}^\pi \quad \text{where} \quad \bar{\nu}^\pi := \begin{cases} 
E^\pi \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_{i_t,a_{i_t}} | i_1 \sim \mathbf{q} \right], & \forall \gamma \in (0, 1) \quad \text{i.e., DMDPs} \\
\lim_{T \to \infty} \frac{1}{T} E^\pi \left[ \sum_{t=1}^{T} r_{i_t,a_{i_t}} | i_1 \sim \mathbf{q} \right], & \gamma = 1 \quad \text{i.e., AMDPs}.
\end{cases}
\]

Here \(\{i_1, a_1, i_2, a_2, \ldots, i_t, a_t\}\) are state-action transitions generated by the MDP under policy \(\pi\). For the DMDP case, it also holds by definition that \(\bar{\nu}^\pi := \mathbf{q}^\top (\mathbf{I} - \gamma \mathbf{P}^\pi)^{-1} \mathbf{r}^\pi\).

For the AMDP case (i.e. when \(\gamma = 1\)), we define \(\nu^\pi\) as the stationary distribution under policy \(\pi\) satisfying \(\nu^\pi = (\mathbf{P}^\pi)^\top \nu^\pi\). To ensure the value of \(\bar{\nu}^\pi\) is well-defined, we restrict our attention to a subgroup which we call mixing AMDP satisfying the following mixing assumption:

\(^2\)The assumption that \(\mathbf{r}\) only depends on state action pair \(i, a_i\) is a common practice [29, 28]. Under a model with \(r_{i,a_i,j} \in [0, 1]\), one can use a straightforward reduction to consider the model with \(\tilde{r}_{i,a_{i,j}} = \sum_{j \in S} p_{ij}(a_i) r_{i,a_{i,j}}\) by using \(\mathcal{O}(\varepsilon^{-2}) (\mathcal{O}(1 - \gamma)^{-2} \varepsilon^{-2})\) samples to estimate the expected reward given each state-action pair within \(\epsilon/2\) \(((1 - \gamma)\epsilon/2)\) additive accuracy for mixing AMDP (DMDP), and finding an expected \(\epsilon/2\)-optimal policy of the new MDP constructed using those estimates of rewards. This will provably give an expected \(\epsilon\)-optimal policy for the original MDP.
Assumption A. An AMDP instance is mixing if $t_{mix}$, defined as follows, is bounded by $1/2$, i.e.

$$t_{mix} := \max_{\pi} \left[ \arg\min_{t \geq 1} \max_{q \in \Delta^S} \left\| (P^\pi)^t q - \nu^\pi \right\|_1 \right] \leq \frac{1}{2}. $$

The mixing condition assumes for arbitrary policy $\pi$ and arbitrary initial state, the resulting Markov chain leads toward a distribution close enough to its stationary distribution $\nu^\pi$ starting from any initial state $i$ in $O(t_{mix})$ time steps. This assumption implies the uniqueness of the stationary distribution, makes $\tilde{v}^\pi$ above well-defined with the equivalent $\tilde{v}^\pi = (\nu^\pi)^T r^\pi$, governing the complexity of our mixing AMDP algorithm. This assumption is key for the results we prove (Theorem 1) and equivalent to the one in Wang [33], up to constant factors.

By nature of the definition of mixing AMDP, we note that the value of a strategy $\pi$ is independent of initial distribution $q$ and only dependent of the eventual stationary distribution as long as the AMDP is mixing, which also implies $\tilde{v}^\pi$ is always well-defined. For this reason, sometimes we also omit $i_1 \sim q$ in the corresponding definition of $\tilde{v}^\pi$.

We call a policy $\pi$ an $\epsilon$-(approximate) optimal policy for the MDP problem, if it satisfies $\tilde{v}^\pi \geq \tilde{v}^* - \epsilon$. We call a policy an expected $\epsilon$-(approximate) optimal policy if it satisfies the condition in expectation, i.e. $E\tilde{v}^\pi \geq \tilde{v}^* - \epsilon$. The goal of paper is to develop efficient algorithms that find (expected) $\epsilon$-optimal policy for the given MDP instance assuming access to a generative model.

### 1.2 Main Results

The main result of the paper is a unified framework based on randomized primal-dual stochastic mirror descent (SMD) that with high probability finds an (expected) $\epsilon$-optimal policy with some sample complexity guarantee. Formally we provide two algorithms (see Algorithm 1 for both cases) with the following guarantees respectively.

**Theorem 1.** Given a mixing AMDP tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, r)$, let $\epsilon \in (0, 1)$, one can construct an expected $\epsilon$-optimal policy $\pi^\epsilon$ from the decomposition (see Section 4.3) of output $\mu^\epsilon$ of Algorithm 1 with sample complexity $O\left( t_{mix}^2 A_{tot} \epsilon^{-2} \log(A_{tot}) \right)$.

**Theorem 2.** Given a DMDP tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \gamma)$ with discount factor $\gamma \in (0, 1)$, let $\epsilon \in (0, 1)$, one can construct an expected $\epsilon$-optimal policy $\pi^\epsilon$ from the decomposition (see Section A.3) of output $\mu^\epsilon$ of Algorithm 1 with sample complexity $O\left( (1 - \gamma)^{-4} A_{tot} \epsilon^{-2} \log(A_{tot}) \right)$.

We remark that for both problems, the algorithm also gives with high probability an $\epsilon$-optimal policy at the cost of an extra $\log(1/\delta)$ factor to the sample complexity through a reduction from high-probability to expected optimal policy (see Wang [33] for more details). Note that we only obtain randomized policies, and we leave the question of getting directly deterministic policies as an interesting open direction.

Table 1 gives a comparison of sample complexity between our methods and prior methods\(^4\) for computing an $\epsilon$-approximate policy in DMDPs and AMDPs given a generative model.

As a generalization, we show how to solve constrained average-reward MDPs (cf. [2], a generalization of average-reward MDP) using the primal-dual stochastic mirror descent framework in Section 5. We build an algorithm that solves the constrained problem (19) to $\epsilon$-accuracy within sample complexity $O( (t_{mix}^2 A_{tot} + K) D^2 \epsilon^{-2} \log(KA_{tot}) )$, where $K$ and $D^2$ are number and size of the

\(^3\)Hereinafter, we use superscript $^*$ and $^\star$ interchangeably.

\(^4\)Most methods assume a uniform action set $\mathcal{A}$ for each of the $|\mathcal{S}|$ states, but can also be generalized to the non-uniform case parameterized by $A_{tot}$. 

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constraints. To the best of our knowledge this is the first sample complexity results for constrained MDPs given by a generative model.

As a byproduct, our framework in Section 3 also gives a stochastic algorithms (see Algorithm 2) that find an expected $\epsilon$-approximate solution of $\ell_1$ bilinear minimax problems of the form

$$\min_{x \in [-1,1]^n} \max_{y \in \Delta^n} y^T Mx + b^T x - c^T y$$

to $\epsilon$-additive accuracy with runtime $\tilde{O}(((m + n) \| M \|_\infty^n + n \| b \|_1^2 + m \| c \|_\infty^2)\epsilon^{-2})$ given $\ell_1$ sampler of iterate $y$, and $\ell_1$ samplers based on the input entries of $M$, $b$ and $c$ (see Corollary 1 for details), where we define $\| M \|_\infty := \max_i \| M(i, :) \|_1$. Consequently, it solves (box constrained) $\ell_\infty$ regression problems of form

$$\min_{x \in [-1,1]^n} \| Mx - c \|_\infty$$

to $\epsilon$-additive accuracy within runtime $\tilde{O}(((m + n) \| M \|_\infty^2 + m \| c \|_\infty^2)\epsilon^{-2})$ given similar sampling access (see Remark 1 for details and Table 2 for comparison with previous results).

1.3 Technique Overview

We adopt the idea of formulating the MDP problem as a bilinear saddle point problem in light of linear duality, following the line of randomized model-free primal-dual $\pi$ learning studied in Wang [32, 33]. This formulation relates MDP to solving bilinear saddle point problems with box and simplex domains, which falls into well-studied generalizations of convex optimization [18, 7].

We study the efficiency of standard stochastic mirror descent (SMD) for this bilinear saddle point problem where the minimization (primal) variables are constrained to a rescaled box domain and the maximization (dual) variables are constrained to the simplex. We use the idea of local-norm variance bounds emerging in Shalev-Shwartz et al. [25], Carmon et al. [7, 8] to design and analyze efficient

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**Algorithm 1** SMD for mixing AMDP / DMDPs

1: **Input:** MDP tuple $M = (S, A, P, r, \gamma)$, initial $(v_0, \mu_0) \in \mathbb{B}_{2M}^S \times \Delta^A$, with $\mathbb{B}_{2M}^S := 2M \cdot [-1, 1]^S$.
2: **Output:** An expected $\epsilon$-approximate solution $(v^*, \mu^*)$ for problem (6).
3: **Parameter:** Step-size $\eta^v$, $\eta^\mu$, number of iterations $T$, accuracy level $\epsilon$.
4: for $t = 1, \ldots, T$ do
5:     // $v$ gradient estimation
6:     Sample $(i, a_i) \sim [\mu]_{i,a_i}$, $j \sim p_{ij}(a_i)$, $i' \sim q_{i'}$
7:     Set $g^v_{t-1} = \begin{cases} e_j - e_i & \text{mixing} \\ (1 - \gamma)e_{i'} + \gamma e_j - e_i & \text{discounted} \end{cases}$
8:     // $\mu$ gradient estimation
9:     Sample $(i, a_i) \sim \frac{1}{\pi_{tot}}$, $j \sim p_{ij}(a_i)$
10:    Set $g^\mu_{t-1} = \begin{cases} A_{tot}(v_i - v_j - r_{i,a_i})e_{i,a_i} & \text{mixing} \\ A_{tot}(v_i - \gamma (v_j - r_{i,a_i})e_{i,a_i} & \text{discounted} \end{cases}$
11:    // Stochastic mirror descent steps ($\Pi$ as projection)
12:    $v_t \leftarrow \Pi_{\mathbb{B}_{2M}}(v_{t-1} - \eta^v g^v_{t-1})$
13:    $\mu_t \leftarrow \Pi_{\Delta^A}(\mu_{t-1} \circ \exp(-\eta^\mu g^\mu_{t-1}))$
14: end for
15: **Return** $(v^*, \mu^*) \leftarrow \frac{1}{T} \sum_{t \in [T]} (v_t, \mu_t)$

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| Type                | Method                          | Sample Complexity                      |
|---------------------|---------------------------------|----------------------------------------|
| mixing AMDP         | Primal-Dual Method [33]         | $\tilde{O}(\tau_t^{2} t_{\text{mix}}^{2} A_{\text{tot}} \epsilon^{-2})$ |
| **Our method** (Theorem 1) |                                | $\tilde{O}(t_{\text{mix}}^{2} A_{\text{tot}} \epsilon^{-2})$ |
| DMDP                | Empirical QVI [4]               | $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ |
|                     | Empirical QVI [4]               | $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$, $\epsilon = \tilde{O}(\frac{1}{\sqrt{(1-\gamma)|S|}})$ |
|                     | Primal-Dual Method [32]         | $\tilde{O}((1 - \gamma)^{-6} |S|^2 A_{\text{tot}} \epsilon^{-2})$ |
|                     | Primal-Dual Method [32]         | $\tilde{O}((1 - \gamma)^{-6} |S|^2 A_{\text{tot}} \epsilon^{-2})$ |
|                     | Online Learning Method [9]      | $\tilde{O}((1 - \gamma)^{-4} A_{\text{tot}} \epsilon^{-2})$ |
|                     | Variance-reduced Value Iteration [29] | $\tilde{O}((1 - \gamma)^{-4} A_{\text{tot}} \epsilon^{-2})$ |
|                     | Variance-reduced QVI [28]       | $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ |
|                     | Empirical MDP + Blackbox [1]    | $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ |
|                     | Variance-reduced Q-learning [31] | $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ |
| Our method (Theorem 2) |                                | $\tilde{O}((1 - \gamma)^{-4} A_{\text{tot}} \epsilon^{-2})$ |

Table 1: **Comparison of sample complexity to get $\epsilon$-optimal policy among stochastic methods.** Here $S$ denotes state space, $A_{\text{tot}}$ denotes number of state-action pair, $t_{\text{mix}}$ is mixing time for mixing AMDP, and $\gamma$ is discount factor for DMDP. Parameter $\tau$ shows up whenever the designed algorithm requires additional ergodic condition for MDP, i.e. there exists some distribution $q$ and $\tau > 0$ satisfying $\sqrt{1/\tau}q \leq \nu^\pi \leq \sqrt{\tau}q$, $\forall$ policy $\pi$ and its induced stationary distribution $\nu^\pi$.

stochastic estimators for the gradient of this problem that have low-variance under the corresponding local norms. We provide a new analytical way to bound the quality of an approximately-optimal policy constructed from the approximately optimal solution of bilinear saddle point problem, which utilizes the influence of the dual constraints under minimax optimality. Compared with prior work, by extending the primal space by a constant size and providing new analysis, we eliminate ergodicity assumptions made in prior work for mixing AMDPs. Combining these pieces, we obtain a natural SMD algorithm which solves mixing AMDPs (DMDPs) as stated in Theorem 1 (Theorem 2).

1.4 Related Work

1.4.1 On Solving MDPs

Within the tremendous body of study on MDPs, and more generally reinforcement learning, stands the well-studied classic problem of computational efficiency (i.e. iteration number, runtime, etc.) of finding optimal policy, given the entire MDP instance as an input. Traditional deterministic methods for the problems are value iteration, policy iteration, and linear programming. [6, 34], which find an approximately optimal policy to high-accuracy but have superlinear runtime in the usually high problem dimension $\Omega(|S| \cdot A_{\text{tot}})$.

To avoid the necessity of knowing the entire problem instance and having superlinear runtime dependence, more recently, researchers have designed stochastic algorithms assuming only a generative model that samples from state-transitions [13]. Azar et al. [4] proved a lower bound of $\Omega((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ while also giving a Q-value-iteration algorithm with a higher guaranteed sam-
ple complexity. Wang [32] designed a randomized primal-dual method, an instance of SMD with slightly different sampling distribution and updates for the estimators, which obtained sublinear sample complexity for the problem provided certain ergodicity assumptions were made. The sample complexity upper bound was improved (without an ergodicity assumptions) in Sidford et al. [29] using variance-reduction ideas, and was further improved to match (up to logarithmic factors) lower bound in [28] using a type of $Q$-function based variance-reduced randomized value iteration. Soon after in Wainwright [31], a variance-reduced $Q$-learning method also achieved nearly tight sample complexity for the discounted case and in Agarwal et al. [1] the authors used a different approach, solving an empirical MDP, that shows $\tilde{O}((1 - \gamma)^{-3} A_{\text{tot}} \epsilon^{-2})$ samples suffice.

While several methods match (up to logarithmic factors) the lower bound shown for sample complexity for solving DMDP [28, 31], it is unclear whether one can design similar methods for AMDPs and obtain optimal sample complexities. The only related work for sublinear runtimes for AMDPs uses primal-dual $\pi$-learning [33], following the stochastic primal-dual method in [32]. This method is also a variant of SMD methods and compared to our algorithm, theirs has a different domain setup, different update forms, and perhaps, a more specialized analysis. The sample complexity obtained by [33] is $\tilde{O}(\tau^2 l_{\text{mix}} A_{\text{tot}} \epsilon^{-2})$, which (as in the case of DMDPs) depends polynomially on the ergodicity parameter $\tau > 0$, and can be arbitrarily large in general.

Whether randomized primal-dual SMD methods necessarily incur much higher computational cost when solving DMDPs and necessarily depend on ergodicity when solving both DMDPs and AMDPs is a key motivation of our work. Obtaining improved primal-dual SMD methods for solving MDPs creates the possibility of leveraging the flexibility of optimization machinery to easily obtain improved sample complexities in new settings easily (as our constrained MDP result highlights).

Independently, [9] made substantial progress on clarifying the power of primal-dual methods for solving MDPs by providing a $\tilde{O}((1 - \gamma)^{-4} A_{\text{tot}} \epsilon^{-2})$ sample complexity for solving DMDPs (with no ergodicity dependence), using an online learning regret analysis. In comparison, we offer a general framework which also applies to the setting of mixing AMDs to achieve the state-of-the-art sample complexity bounds for mixing AMDPs, and extend our framework to solving constrained AMDPs; [9] connects with the value of policies with regret in online learning more broadly, and offers extensions to DMDPs with linear approximation. It would be interesting to compare the techniques and see if all the results of each paper are achievable through the techniques of the other.

Table 1 includes a complete comparison between our results and the prior art for discounted MDP and mixing AMDP.

### 1.4.2 On $\ell_\infty$ Regression and Bilinear Saddle Point Problem

Our framework gives a stochastic method for solving $\ell_\infty$ regression, which is a core problem in both combinatorics and continuous optimization due to its connection with maximum flow and linear programming [15, 16]. Classic methods build on solving a smooth approximations of the problem [20] or finding the right regularizers and algorithms for its correspondingly primal-dual minimax problem [18, 21]. These methods have recently been improved to $\tilde{O}(\text{nnz} \|M\|_\infty \epsilon^{-1})$ using a joint regularizer with nice area-convexity properties in Sherman [26] or using accelerated coordinate method with a matching runtime bound in sparse-column case in Sidford and Tian [27].

In comparison to all the state-of-the-art, for dense input data matrix our method gives the first algorithm with sublinear runtime dependence $O(m + n)$ instead of $O(\text{nnz})$. For completeness here we include Table 2 that make comparisons between our sublinear $\ell_\infty$ regression solver and prior art.

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We were unaware of this recent result until the final preparation of this manuscript.
We remark for dense matrix $M$, our method is the only sublinear method along this line of work for approximately solving $\ell_\infty$ regression problem.

| Method                                           | Runtime                                      |
|--------------------------------------------------|----------------------------------------------|
| Smooth Approximation [20]                        | $\tilde{O}(\text{nnz}\|M\|_2^2\epsilon^{-2})$ or $\tilde{O}(\text{nnz}\sqrt{n}\|M\|_\infty\epsilon^{-1})$ |
| Mirror-prox Method [18]                         | $\tilde{O}(\text{nnz}\|M\|_\infty\epsilon^{-2})$ |
| Dual Extrapolation [21]                         | $\tilde{O}(\text{nnz}\|M\|_\infty\epsilon^{-2})$ |
| Dual Extrapolation with Joint Regularizer [26]  | $\tilde{O}(\text{nnz}\|M\|_\infty\epsilon^{-1})$ |
| Accelerated Coordinate Method [27]              | $\tilde{O}(nd^{2.5}\|M\|_\infty\epsilon^{-1})$ |
| **Our method** (Remark 1)                       | $\tilde{O}\left((m+n)\|M\|_\infty^2\epsilon^{-2}\right)$ |

Table 2: **Runtime Comparison of $\epsilon$-approximate $\ell_\infty$-regression Methods**: For simplicity, here we only state for the simplified problem, $\min_{x \in \mathbb{B}^n} \|Mx\|_\infty$, where $M \in \mathbb{R}^{m \times n}$ with nnz nonzero entries and $d$-sparse columns.

Our sublinear method for $\ell_\infty$-regression is closely related to a line of work on obtaining efficient stochastic methods for approximately solving matrix games, i.e. bilinear saddle point problems [12, 10, 23], and, in particular, a recent line of work by the authors and collaborators [7, 8] that explores the benefit of careful sampling and variance reduction in matrix games. In Carmon et al. [7] we provide a framework to analyze variance-reduced SMD under local norms to obtain better complexity bounds for different domain setups, i.e. $\ell_1$-$\ell_1$, $\ell_1$-$\ell_2$, and $\ell_2$-$\ell_2$ where $\ell_1$ corresponds to the simplex and $\ell_2$ corresponds to the Euclidean ball. In Carmon et al. [8] we study the improved sublinear and variance-reduced coordinate methods for these domain setups utilizing the design of optimal gradient estimators. This paper adapts the local norm analysis and coordinate-wise gradient estimator design in Carmon et al. [7, 8] to obtain our SMD algorithm and analysis for $\ell_1$-$\ell_\infty$ games.

## 2 Preliminaries

First, we introduce several known tools for studying MDPs.

### 2.1 Bellman Equation.

For mixing AMDP, $\bar{v}^*$ is the optimal average reward if and only if there exists a vector $v^* = (v^*_i)_{i \in S}$ satisfying its corresponding **Bellman equation** [6]

$$
\bar{v}^* + v^*_i = \max_{a_i \in A_i} \left\{ \sum_{j \in S} p_{ij}(a_i)v^*_j + r_{i,a_i} \right\}, \forall i \in S.
$$

When considering a mixing AMDP as in the paper, the existence of solution to the above equation can be guaranteed. However, it is important to note that one cannot guarantee the uniqueness of the optimal $v^*$. In fact, for each optimal solution $v^*$, $v^* + c\mathbf{1}$ is also an optimal solution.

For DMDP, one can show that at optimal policy $\pi^*$, each state $i \in S$ can be assigned an optimal
cost-to-go value \( v_i^* \) satisfying the following Bellman equation [6]

\[
v_i^* = \max_{a_i \in A_i} \left\{ \sum_{j \in S} \gamma p_{ij}(a_i)v_j^* + r_{i,a_i} \right\}, \forall i \in S. \tag{3}
\]

When \( \gamma \in (0, 1) \), it is straightforward to guarantee the existence and uniqueness of the optimal solution \( v^* := (v_i^*)_{i \in S} \) to the system.

### 2.2 Linear Programming (LP) Formulation.

We can further write the above Bellman equations equivalently as the following primal or dual linear programming problems. We define the domain as \( \mathbb{B}_m^S := m \cdot [-1,1]^S \) where \( \mathbb{B} \) stands for box, and \( \Delta^n := \{ \Delta \in \mathbb{R}^n, \Delta \ge 0, \sum_{i \in [n]} \Delta_i = 1 \} \) for standard \( n \)-dimension simplex.

For mixing AMDP case, the linear programming formulation leveraging matrix notation is (with \((P), (D)\) representing (equivalently) the primal form and the dual form respectively)

\[
\begin{align*}
\text{(P)} & \quad \min_{\vec{v}, \bar{v}} \quad \vec{v} \\
\text{subject to} & \quad \vec{v} \cdot \mathbf{1} + (\hat{I} - P)\bar{v} - \mathbf{r} \ge 0,
\end{align*}
\]

\[
\begin{align*}
\text{(D)} & \quad \max_{\mu \in \Delta^A} \quad \mu^\top \mathbf{r} \\
\text{subject to} & \quad (\hat{I} - P)\mathbf{1} = \mathbf{0}.
\end{align*}
\]

The optimal values of both systems are the optimal expected cumulative reward \( \bar{v}^* \) under optimal policy \( \pi^* \), thus hereinafter we use \( \bar{v}^* \) and \( \bar{v}^{\pi^*} \) interchangeably. Given the optimal dual solution \( \mu^* \), one can without loss of generality impose the constraint of \( \langle \hat{I}^\top \mu^*, \mathbf{v}^* \rangle = 0 \) to ensure uniqueness of the primal problem \((P)\).

For DMDP case, the equivalent linear programming is

\[
\begin{align*}
\text{(P)} & \quad \min_{v} \quad (1 - \gamma)q^\top v \\
\text{subject to} & \quad (\hat{I} - \gamma P)v - \mathbf{r} \ge 0,
\end{align*}
\]

\[
\begin{align*}
\text{(D)} & \quad \max_{\mu \in \Delta^A} \quad \mu^\top \mathbf{r} \\
\text{subject to} & \quad (\hat{I} - \gamma P)\mathbf{1} = (1 - \gamma)q.
\end{align*}
\]

Given a fixed initial distribution \( q \), the optimal values of both systems are a \((1 - \gamma)\) factor of the optimal expected cumulative reward, i.e. \((1 - \gamma)\bar{v}^* \) under optimal policy \( \pi^* \).

### 2.3 Minimax Formulation.

By standard linear duality, we can recast the problem formulation in Section 2.2 using the method of Lagrangian multipliers, as bilinear saddle-point (minimax) problems. For AMDPs the minimax formulation is

\[
\min_{\bar{v}, v, \mu} \max_{\mu \in \Delta^A} f(\bar{v}, v, \mu), \tag{6}
\]

where \( f(\bar{v}, v, \mu) := \bar{v} + \mu^\top (-\bar{v} \cdot \mathbf{1} + (\hat{I} - P)\bar{v} + \mathbf{r}) = \mu^\top ((\hat{I} - \hat{I})v + \mathbf{r}) \)

For DMDPs the minimax formulation is

\[
\min_{v, \mu} \max_{\mu \in \Delta^A} f_q(v, \mu), \tag{7}
\]

where \( f_q(v, \mu) := (1 - \gamma)q^\top v + \mu^\top ((\gamma P - \hat{I})v + \mathbf{r}) \).

\( \hat{I}^\top \mu^* \) represents the stationary distribution over states given optimal policy \( \pi^* \) constructed from optimal dual variable \( \mu^* \).
Note in both cases we have added the constraint of $v \in B^S_{2M}$. The $M$ is different for each case, and will be specified in Section 4.1 and A.1 to ensure that $v^* \in B^S_{2M}$. As a result, constraining the bilinear saddle point problem on a restricted domain for primal variables will not affect the optimality of the original optimal solution due to its global optimality, but will considerably save work for the algorithm by considering a smaller domain. Besides, we are also considering $v \in B^S_{2M}$ instead of $v \in B^S_M$ for solving MDPs to ensure a stricter optimality condition for the dual variables, see Lemma 5 for details.

For each problem we define the duality gap of the minimax problem $\min_{v \in B^S_{2M}} \max_{\mu \in \Delta^A} f(v, \mu)$ at a given pair of feasible solution $(v, \mu)$ as

$$\text{Gap}(v, \mu) := \max_{\mu' \in \Delta^A} f(v, \mu') - \min_{v' \in B^S_{2M}} f(v', \mu).$$

An $\epsilon$-approximate solution of the minimax problem is a pair of feasible solution $(v^\epsilon, \mu^\epsilon) \in B^S_{2M} \times \Delta^A$ with its duality gap bounded by $\epsilon$, i.e., $\text{Gap}(v^\epsilon, \mu^\epsilon) \leq \epsilon$. An expected $\epsilon$-approximate solution is one satisfying $\mathbb{E}\text{Gap}(v^\epsilon, \mu^\epsilon) \leq \epsilon$.

### 3 Stochastic Mirror Descent Framework

In this section, we consider the following $\ell_\infty$-$\ell_1$ bilinear games as an abstraction of the MDP minimax problems of interest. Such games are induced by one player minimizing over the box domain ($\ell_\infty$) and the other maximizing over the simplex domain ($\ell_1$) a bilinear objective:

$$\min_{x \in B^b_n} \max_{y \in \Delta^n} f(x, y) := y^\top Mx + b^\top x - c^\top y,$$

where throughout the paper we use $B^b_n := b \cdot [-1, 1]^n$ to denote the box constraint, and $\Delta^n$ to denote the simplex constraint in $m$-dimensional space.

We study the efficiency of coordinate stochastic mirror descent (SMD) algorithms onto this $\ell_\infty$-$\ell_1$ minimax problem. The analysis follows from extending a fine-grained analysis of mirror descent with Bregman divergence using local norm arguments in Shalev-Shwartz et al. [25], Carmon et al. [7, 8] to the $\ell_\infty$-$\ell_1$ domain. (See Lemma 2 and Lemma 1 for details.)

At a given iterate $(x, y) \in B^b_n \times \Delta^n$, our algorithm computes an estimate of the gradients for both sides defined as

$$g^x(x, y) := M^\top y + b \in \mathbb{R}^n \quad \text{(gradient for } x \text{ side, } g^x \text{ in shorthand);}$$

$$g^y(x, y) := -Mx + c \in \mathbb{R}^m \quad \text{(gradient for } y \text{ side, } g^y \text{ in shorthand).}$$

The norm we use to measure these gradients are induced by Bregman divergence, a natural extension of Euclidean norm. For our analysis we choose to use the following divergence terms:

- Euclidean distance for $x$ side: $V_x(x') := \frac{1}{2} \|x - x'\|_2^2$, $\forall x, x' \in B^b_n$;
- KL divergence for $y$ side: $V_y(y') := \sum_{i=1}^m y_i \log(y'_i/y_i)$, $\forall y, y' \in \Delta^m$,

which are also common practice [32, 33, 20] catering to the geometry of each domain, and induce the dual norms on the gradients in form $\|g^x\| := \|g^x\|_2 = \sqrt{\sum_{j=1}^n g^x_j^2}$ (standard $\ell_2$-norm) for $x$ side, and $\|g^y\|_{\overline{y}} := \sum_{i=1}^m y'_i(g^y_i)^2$ (a weighted $\ell_2$-norm) for $y$ side.

To describe the properties of estimators needed for our algorithm, we introduce the following definition of bounded estimator as follows.
Definition 1 (Bounded Estimator). Given the following properties on mean, scale and variance of an estimator:

(i) unbiasedness: \( \mathbb{E} \tilde{g} = g \);
(ii) bounded maximum entry: \( \|\tilde{g}\|_\infty \leq c \) with probability 1;
(iii) bounded second-moment: \( \mathbb{E} \|\tilde{g}\|^2 \leq v \)
we call \( \tilde{g} \) a \((v,\|\cdot\|)\)-bounded estimator of \( g \) if satisfying (i) and (iii), call it and a \((c,v,\|\cdot\|^m)\)-bounded estimator of \( g \) if it satisfies (i), (ii), and also (iii) with local norm \( \|\cdot\|_y \) for all \( y \in \Delta^m \).

Now we give Algorithm 2, our general algorithmic framework for solving (8) given efficient bounded estimators for the gradient. Its theoretical guarantees are given in Theorem 3 which bounds the number of iterations needed to obtain expected \( \epsilon \)-approximate solution. We remark that the proof strategy and consideration of ghost-iterates stems from a series of work offering standard analysis for saddle-point problems [19, 7, 8].

Algorithm 2 SMD for \( \ell_\infty-\ell_1 \) saddle-point problem

1: **Input**: Desired accuracy \( \epsilon \), primal domain size \( b \)
2: **Output**: An expected \( \epsilon \)-approximate solution \((x^*, y^*)\) for problem (8).
3: **Parameter**: Step-size \( \eta_x \leq \frac{\epsilon}{4x}, \eta_y \leq \frac{\epsilon}{4y^*} \), total iteration number \( T \geq \max\{ 16nb^2 \epsilon^2, \frac{8m}{c\eta_y} \} \).
4: for \( t = 1, \ldots, T - 1 \) do
5: Get \( \tilde{g}_t^x \) as a \((x^*,\|\cdot\|_2)\)-bounded estimator of \( g^x(x_t, y_t) \)
6: Get \( \tilde{g}_t^y \) as a \((2\epsilon^2 \theta, \theta, \|\cdot\|_{\Delta^m})\)-bounded estimator of \( g^y(x_t, y_t) \)
7: Update \( x_{t+1} \leftarrow \arg \min_{x \in \mathbb{B}_b} \langle \eta_x \tilde{g}_t^x, x \rangle + V_x(x) \), and \( y_{t+1} \leftarrow \arg \min_{y \in \Delta^m} \langle \eta_y \tilde{g}_t^y, y \rangle + V_y(y) \)
8: end for
9: Return \((x^*, y^*) \leftarrow \frac{1}{T} \sum_{t \in [T]} (x_t, y_t)\)

Theorem 3. Given an \( \ell_\infty-\ell_1 \) game, i.e. (8), and desired accuracy \( \epsilon \), \((x^*, \|\cdot\|_2)\)-bounded estimators \( \tilde{g}^x \) of \( g^x \), and \((2\epsilon^2 \theta, \theta, \|\cdot\|_{\Delta^m})\)-bounded estimators \( \tilde{g}^y \) of \( g^y \), Algorithm 2 with choice of parameters \( \eta_x \leq \frac{\epsilon}{4x}, \eta_y \leq \frac{\epsilon}{4y^*} \) outputs an expected \( \epsilon \)-approximate optimal solution within any iteration number \( T \geq \max\{ 16nb^2 \epsilon^2, \frac{8m}{c\eta_y} \} \).

We first recast a few standard results on the analysis of mirror-descent using local norm [25], which we use for proving Theorem 3. These are standard regret bounds for \( \ell_2 \) and simplex respectively. First, we provide the well-known regret guarantee for \( x \in \mathbb{B}^n \), when choosing \( V_x(x') := \frac{1}{2} \|x - x'\|^2_2 \).

Lemma 1 (cf. Lemma 12 in Carmon et al. [7], restated). Let \( T \in \mathbb{N} \) and let \( x_1 \in \mathcal{X}, \gamma_1, \ldots, \gamma_T \in \mathcal{X}^*, V \) is \( 1 \)-strongly convex in \( \|\cdot\|_2 \). The sequence \( x_2, \ldots, x_T \) defined by

\[
    x_{t+1} = \arg \min_{x \in \mathcal{X}} \{ \langle \gamma_t, x \rangle + V_x(x) \}
\]

satisfies for all \( x \in \mathcal{X} \) (overloading notations to denote \( x_{T+1} := x \),

\[
    \sum_{t \in [T]} \langle \gamma_t, x_t - x \rangle \leq V_{x_1}(x) + \sum_{t \in [T]} \{ \langle \gamma_t, x_t - x_{t+1} \rangle - V_{x_t}(x_{t+1}) \}
\]

\[
    \leq V_{x_1}(x) + \frac{1}{2} \sum_{t \in [T]} \|\gamma_t\|_2^2.
\]
Next, one can show a similar property holds true for \( \mathbf{y} \in \Delta^m \), by choosing KL-divergence as Bregman divergence \( V_y(y') := \sum_{i \in [m]} y_i \log(y'_i/y_i) \), utilizing local norm \( ||\cdot||_{y'} \).

**Lemma 2** (cf. Lemma 13 in Carmon et al. [7], immediate consequence). Let \( T \in \mathbb{N}, \mathbf{y}_1 \in \mathcal{Y}, \mathbf{y}_1, \ldots, \mathbf{y}_T \in \mathcal{Y}^* \) satisfying \( \|\mathbf{y}_t\|_\infty \leq 1.79, \forall t \in [T] \), and \( V_y(y') := \sum_{i \in [m]} y_i \log(y'_i/y_i) \). The sequence \( \mathbf{y}_2, \ldots, \mathbf{y}_T \) defined by

\[
\mathbf{y}_{t+1} = \arg\min_{\mathbf{y} \in \mathcal{Z}} \{ \langle \mathbf{y}_t, \mathbf{y} \rangle + V_{\mathbf{y}_t}(\mathbf{y}) \}
\]

satisfies for all \( \mathbf{y} \in \mathcal{Y} \) (overloading notations to denote \( \mathbf{y}_{T+1} := \mathbf{y} \)),

\[
\sum_{t \in [T]} \langle \mathbf{y}_t, \mathbf{y}_t - \mathbf{y} \rangle \leq V_{\mathbf{y}_1}(\mathbf{y}) + \sum_{t \in [T]} \{ \langle \mathbf{y}_t, \mathbf{y}_t - \mathbf{y}_{t+1} \rangle - V_{\mathbf{y}_t}(\mathbf{y}_{t+1}) \}
\]

\[
\leq V_{\mathbf{y}_1}(\mathbf{y}) + \frac{1}{2} \sum_{t \in [T]} \|\mathbf{y}_t\|^2_{\mathbf{y}_t}.
\]

Leveraging these lemmas we prove Theorem 3.

**Proof of Theorem 3.** For simplicity we use \( \mathbf{g}^x_t, \mathbf{g}^y_t, \mathbf{g}^x_t, \mathbf{g}^y_t \) for shorthands of \( g^x(x_t, y_t), g^y(x_t, y_t), \tilde{g}^x(x_t, y_t), \tilde{g}^y(x_t, y_t) \) throughout the proof, similar as in Algorithm 2. By the choice of \( \eta_t \) and conditions, one can immediately see that

\[
\|\eta_t \tilde{g}_t\|_\infty \leq 1/2.
\]

Thus we can use regret bound of stochastic mirror descent with local norms in Lemma 2 and Lemma 1 which gives

\[
\sum_{t \in [T]} \langle \eta_t \tilde{g}_t, x_t - x \rangle \leq V_{x_1}(x) + \frac{\eta^2}{2} \sum_{t \in [T]} \|\tilde{g}_t\|^2_x,
\]

\[
\sum_{t \in [T]} \langle \eta_t \tilde{g}_t, y_t - y \rangle \leq V_{y_1}(y) + \frac{\eta^2}{2} \sum_{t \in [T]} \|\tilde{g}_t\|^2_y.
\]

(11)

Now, let \( \tilde{g}_t := g^x_t - \tilde{g}^x_t \) and \( \tilde{g}_t := g^y_t - \tilde{g}^y_t \), defining the sequence \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T \) and \( \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_T \) according to

\[
\mathbf{x}_1 = x_1, \quad \mathbf{x}_{t+1} = \arg\min_{x \in B^n_b} \{ \langle \eta_t \tilde{g}_t, x \rangle + V_{x_t}(x) \};
\]

\[
\mathbf{y}_1 = y_1, \quad \mathbf{y}_{t+1} = \arg\min_{y \in \Delta^m} \{ \langle \eta_t \tilde{g}_t, y \rangle + V_{y_t}(y) \}.
\]

Using a similar argument for \( \tilde{g}_t \) satisfying

\[
\|\eta \tilde{g}_t\|_\infty \leq \|\eta \tilde{g}_t\|_\infty + \|\eta \tilde{g}_t\|_\infty = \|\eta \tilde{g}_t\|_\infty + \|\eta \mathbb{E} \tilde{g}_t\|_\infty \leq 2 \|\eta \tilde{g}_t\|_\infty \leq 1,
\]

we obtain

\[
\sum_{t \in [T]} \langle \eta_t \tilde{g}_t, \mathbf{x}_t - x \rangle \leq V_{x_0}(x) + \frac{\eta^2}{2} \sum_{t \in [T]} \|\tilde{g}_t\|^2_x,
\]

\[
\sum_{t \in [T]} \langle \eta_t \tilde{g}_t, \mathbf{y}_t - y \rangle \leq V_{y_0}(y) + \frac{\eta^2}{2} \sum_{t \in [T]} \|\tilde{g}_t\|^2_y.
\]
Since $g^t = \tilde{g}^t + \hat{g}^t$ and $g^y = \tilde{g}^y + \hat{g}^y$, rearranging yields

$$
\sum_{t \in [T]} \left[ \langle g^t, x_t - x \rangle + \langle g^y, y_t - y \rangle \right] = \sum_{t \in [T]} \left[ \langle \tilde{g}^t, x_t - x \rangle + \langle \tilde{g}^y, y_t - y \rangle \right] + \sum_{t \in [T]} \left[ \langle \hat{g}^t, x_t - x \rangle + \langle \hat{g}^y, y_t - y \rangle \right] \\
\leq \frac{2}{\eta^x} V_{x_1}(x) + \sum_{t \in [T]} \left[ \frac{\eta^x}{2} ||\tilde{g}^t||_2^2 + \frac{\eta^y}{2} ||\tilde{g}^y||_2^2 \right] + \sum_{t \in [T]} \langle \tilde{g}^t, x_t - \bar{x}_t \rangle \\
+ \frac{2}{\eta^y} V_{y_1}(y) + \sum_{t \in [T]} \left[ \frac{\eta^y}{2} ||\tilde{g}^y||_2^2 + \frac{\eta^y}{2} ||\hat{g}^y||_2^2 \right] + \sum_{t \in [T]} \langle \hat{g}^y, y_t - \bar{y}_t \rangle.
$$

(13)

where we use the regret bounds in Eq. (11), (12) for the inequality.

Now take supremum over $(x, y)$ and then take expectation on both sides, we get

$$
\frac{1}{T} \mathbb{E} \sup_{x, y} \left[ \sum_{t \in [T]} \langle g^t, x_t - x \rangle + \sum_{t \in [T]} \langle g^y, y_t - y \rangle \right] \\
\leq \mathbb{E} \sup_{x} \left[ \sum_{t \in [T]} \langle \tilde{g}^t, x_t - x \rangle \right] + \mathbb{E} \sup_{y} \left[ \sum_{t \in [T]} \langle \tilde{g}^y, y_t - y \rangle \right] \\
\leq \frac{2}{\eta^x} V_{x_0}(x) + \eta^x v^x + \frac{2}{\eta^y} V_{y_0}(y) + \eta^y v^y \\
\leq 4nb^2 / \eta^x + \eta^y v^y + \frac{2 \log m}{\eta^x} + \eta^y v^y \\
\leq \epsilon,
$$

where we use (i) $\mathbb{E}[(g^t, x_t - x_i)|1, 2, \ldots, T] = 0$, $\mathbb{E}[(g^y, y_t - \bar{y}_t)|1, 2, \ldots, T] = 0$ by conditional expectation, that $\mathbb{E}[\|\tilde{g}^t\|_2^2] \leq \mathbb{E}[\|\hat{g}^t\|_2^2]$, $\mathbb{E}[\sum_i [\tilde{y}_i]_i [\tilde{g}^t]_i^2] \leq \mathbb{E}[\sum_i [\hat{y}_i]_i [\hat{g}^y]_i^2]$ due to the fact that $\mathbb{E}[(X - \mathbb{E}X)^2] \leq \mathbb{E}[X^2]$ elementwise and properties of estimators as stated in condition; (ii) $V_{x_0}(x) := \frac{1}{2} \|x - x_0\|_2^2 \leq 2nb^2$, $V_{y_0}(y) \leq \log m$ by properties of KL-divergence; (iii) the choice of $\eta^x = \frac{\epsilon^x}{4b^2}$, $\eta^y = \frac{\epsilon}{4b^2}$, and $T \geq \max(\frac{16nb^2}{\epsilon^x}, \frac{8 \log m}{\epsilon^y})$.

Together with the bilinear structure of problem and choice of $x^* = \frac{1}{T} \sum_{t \in [T]} x_t$, $y^* = \frac{1}{T} \sum_{t \in [T]} y_t$ we get $\mathbb{E}[\text{Gap}(x^*, y^*)] \leq \epsilon$, proving the output $(x^*, y^*)$ is indeed an expected $\epsilon$-approximate solution to the minimax problem (8).

Now we design gradient estimators assuming certain sampling oracles to ensure good bounded properties. More concretely, we offer one way to construct the gradient estimators and prove its properties and the implied algorithmic complexity.

For $x$-side, we consider

$$
\text{Sample } i, j \text{ with probability } p_{ij} := y_i \cdot \frac{|M_{ij}|}{\sum_j |M_{ij}|}, \\
\text{sample } j' \text{ with probability } p_{j'} := \frac{|b_{j'}|}{||b||_1}, \\
\text{set } \tilde{g}^x(x, y) = \frac{M_{ij}y_i}{p_{ij}} e_j + \frac{b_{j'}}{p_{j'}} e_{j'},
$$

which has properties as stated in Lemma 3.
Lemma 3. Gradient estimator $\tilde{g}^x$ specified in (14) is a $(v^x, \| \cdot \|_2)$-bounded estimator, with

$$v^x = 2 \left[ \| b \|_2^2 + \| M \|_\infty^2 \right].$$

Proof. The unbiasedness follows directly by definition. For bound on second-moment, one sees

$$\mathbb E \| \tilde{g}^x(x,y) \|_2^2 \leq 2 \left[ \sum_{j} \frac{b_j^2}{p_j} + \sum_{i,j} M_{ij}^2 y_{ij}^2 \right]^{(i)} \leq 2 \left[ \| b \|_1^2 + \left( \sum_{i,j} y_{ij} |M_{ij}| \left( \sum_{j} |M_{ij}| \right) \right) \right]^{(ii)} \leq 2 \left[ \| b \|_1^2 + \| M \|_\infty^2 \right],$$

where we use (i) the fact that $\| x + y \|_2^2 \leq 2 \| x \|_2^2 + 2 \| y \|_2^2$ and taking expectation, (ii) plugging in the explicit sampling probabilities as stated in (14), and (iii) Cauchy-Schwarz inequality and the fact that $y \in \Delta^m$.

For $y$-side, we consider

Sample $i, j$ with probability $q_{ij} := \frac{|M_{ij}|}{\sum_{i,j} |M_{ij}|}$,

sample $i'$ with probability $q_{i'} := \frac{|c_{i'}|}{\| c \|_1}$,

set $\tilde{g}^y(x, y) = -\frac{M_{ij} x_j}{q_{ij}} e_i + \frac{C_{i'}}{q_{i'}} e_{i'}.$

Here we remark that we adopt the same indexing notation $i, j$ but it is independently sampled from given distributions as with ones for $\tilde{g}^x$. Such an estimator has properties stated in Lemma 4.

Lemma 4. Gradient estimator $\tilde{g}^y$ specified in (15) is a $(c^y, v^y, \| \cdot \|_{\Delta^m})$-bounded estimator, with

$$c^y = m(b \| M \|_\infty + \| c \|_\infty), \quad v^y = 2m \left[ \| c \|_\infty^2 + b^2 \| M \|_\infty^2 \right].$$

Proof. The unbiasedness follows directly by definition. For bounded maximum entry, one has

$$\| \tilde{g}^y \|_{\Delta^m} \leq \sum_{i,j} |M_{ij} x_j| + \| c \|_1 \leq m(b \| M \|_\infty + \| c \|_\infty),$$

by definition of the probability distributions and $x_j \in \mathbb B_b^q, c \in \mathbb R^m$.

For bound on second-moment in local norm with respect to arbitrary $y' \in \Delta^m$, one has

$$\mathbb E \| \tilde{g}^y(x,y) \|_{y'}^2 \leq 2 \left[ \sum_{j} \frac{y_{ij}^2}{q_{ij}} + \sum_{i,j} M_{ij}^2 x_j^2 \right]^{(i)} \leq 2 \left[ \left( \sum_{i,j} y_{ij} |M_{ij}| x_j^2 \right) \left( \sum_{i,j} |M_{ij}| \right) \right]^{(ii)} \leq 2 \left[ m \| c \|_\infty^2 + mb^2 \| M \|_\infty^2 \right],$$

where we use (i) the fact that $\| x + y \|_2^2 \leq 2 \| x \|_2^2 + 2 \| y \|_2^2$ and taking expectation, (ii) plugging in the explicit sampling probabilities as stated in (15), and (iii) Cauchy-Schwarz inequality and the fact that $y' \in \Delta^m, c \in \mathbb R^m, x \in \mathbb B_b^n$. 

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When \( x \in B^n_1 \), this leads to the theoretical guarantee as stated formally in Corollary 1.

**Corollary 1.** Given an \( \ell_{\infty}-\ell_1 \) game (8) with domains \( x \in B^n_1, y \in \Delta^m, \epsilon \in (0, 1) \) and \( \|M\|_\infty + \|c\|_\infty = \Omega(1) \). If one has all sampling oracles needed with sampling time \( O(T_{samp}) \), Algorithm 2 with certain gradient estimators (see (14) and (15)) finds an expected \( \epsilon \)-approximate solution with a number of samples bounded by

\[
O\left((n + m \log m) \left( \frac{\|M\|^2_\infty}{\epsilon^2} + n \|b\|^2_1 + m \log m \frac{\|c\|^2_\infty}{\epsilon^2} \right) \cdot \epsilon^{-2} \cdot T_{samp}\right).
\]

Further the runtime is proportional to the number of samples times the cost per sample.

**Proof of Corollary 1.** In light of Theorem 3 with Lemma 3 and Lemma 4, whenever \( \epsilon \in (0, 1) \), \( b \|M\|_\infty + \|c\|_\infty = \Omega(1) \), gradient estimators in (14) and (15) satisfy the desired conditions. As a result, one can pick

\[
\eta^x = \Theta\left(\frac{\epsilon}{\|b\|_1^2 + \|M\|^2_\infty}\right), \eta^y = \Theta\left(\frac{\epsilon}{m \left( \frac{\|c\|^2_\infty}{\epsilon^2} + b^2 \|M\|^2_\infty \right)}\right),
\]

\[
T = O\left(\frac{(n + m \log m) b^2 \|M\|^2_\infty + n b^2 \|b\|^2_1 + m \log m \frac{\|c\|^2_\infty}{\epsilon^2}}{\epsilon^2}\right),
\]

to get an expected \( \epsilon \)-approximate solution to the general \( \ell_{\infty}-\ell_1 \) bilinear saddle-point problem (8), proving the corollary.

Finally, we remark that one can also use Algorithm 2 to solve \( \ell_{\infty} \)-regression, i.e. the problem of finding

\[
x^* := \arg\min_{x \in B^n_1} \|Mx - c\|_\infty
\]

by simply writing it in equivalent minimax form of

\[
\min_{x \in B^n_1} \max_{y \in \Delta^m} y^\top (\hat{M}x - \hat{c}), \hat{M} := [M; -M], \hat{c} := [c; -c].
\]

**Remark 1.** Algorithm 2 produces an expected \( \epsilon \)-approximate solution \( x^\epsilon \) satisfying

\[
\mathbb{E} \|Mx^\epsilon - c\|_\infty \leq \|Mx^* - c\|_\infty + \epsilon,
\]

within runtime

\[
\tilde{O}\left(\left[(m + n) \left( \frac{\|M\|^2_\infty}{\epsilon^2} + m \frac{\|c\|^2_\infty}{\epsilon^2} \right) \cdot \epsilon^{-2} \cdot T_{samp}\right]\right).
\]

### 4 Mixing AMDPs

In this section we show how to utilize framework in Section 3 for mixing AMDPs to show efficient primal-dual algorithms that give an approximately optimal policy. In Section 4.1 we specify the choice of \( M \) in minimax problem (6) by bounding the operator norm to give a domain that \( v^* \) lies in. In Section 4.2 we give estimators for both sides for solving (6), which is similar to the estimators developed in Section 3. In Section 4.3 we show how to round an \( \epsilon \)-optimal solution of (6) to an \( \Theta(\epsilon) \)-optimal policy. Due to the similarity of the approach and analysis, we include our method for solving DMDPs and its theoretical guarantees in Appendix A.

\[\text{Note all the sampling oracles needed are essentially} \ \ell_1 \text{ samplers proportional to the matrix / vector entries, and an} \ \ell_1 \text{ sampler induced by } y \in \Delta^m. \text{ These samplers with } O(1) \text{ cost per sample can be built with additional preprocessing in } O(mnz(M) + n + m) \text{ time.} \]
4.1 Bound on Matrix Norm

We first introduce Lemma 5 showing that the mixing assumption $A$ naturally leads to $\ell_\infty$-norm bound on the interested matrix, which is useful in both in deciding $M$ and in proving Lemma 10 in Section 4.3.

**Lemma 5.** Given a mixing AMDP, policy $\pi$, and its probability transition matrix $P^\pi \in \mathbb{R}^{S \times S}$ and stationary distribution $\nu^\pi$,

$$\left\| (I - P^\pi + 1(\nu^\pi)^\top)^{-1} \right\|_\infty \leq 2t_{\text{mix}}.$$ 

In order to prove Lemma 5, we will first give a helper lemma adapted from Cohen et al. [11] capturing the property of $I - P^\pi + \nu^\pi 1^\top$. Compared with the lemma stated there, we are removing an additional assumption about strong connectivity of the graph as it is not necessary for the proof.

**Lemma 6** (cf. Lemma 23 in Cohen et al. [11], generalized). For a probabilistic transition matrix $P^\pi$ with mixing time $t_{\text{mix}}$ as defined in Assumption $A$ and stationary distribution $\nu^\pi$, one has for all non-negative integer $k \geq t_{\text{mix}},$

$$\left\| (P^\pi)^k - 1(\nu^\pi)^\top \right\|_\infty \leq \left( \frac{1}{2} \right)^{\left\lfloor \frac{k}{t_{\text{mix}}} \right\rfloor}.$$ 

We use this lemma and additional algebraic properties involving operator norms and mixing time for the proof of Lemma 5, formally as follows.

**Proof of Lemma 5.** Denote $\hat{P} := P^\pi - 1(\nu^\pi)^\top$, we first show the following equality.

$$\left( I - \hat{P} \right)^{-1} \left( \sum_{k=0}^{\infty} (k+1)t_{\text{mix}} \sum_{t=kt_{\text{mix}}+1} (P^\pi)^t - 1(\nu^\pi)^\top \right) = \left( I - \hat{P} \right)^{-1} \left( \sum_{k=0}^{\infty} (k+1)t_{\text{mix}} \sum_{t=kt_{\text{mix}}+1} (P^\pi)^t - 1(\nu^\pi)^\top \right), \tag{16}$$

To show the equality (i), observing that by Lemma 6

$$\left\| (P^\pi)^k - 1(\nu^\pi)^\top \right\|_\infty \leq \left( \frac{1}{2} \right)^{\left\lfloor \frac{k}{t_{\text{mix}}} \right\rfloor},$$

and thus by triangle inequality of matrix norm

$$\left\| \sum_{k=0}^{\infty} (k+1)t_{\text{mix}} \sum_{t=kt_{\text{mix}}+1} (P^\pi)^t - 1(\nu^\pi)^\top \right\|_\infty \leq \sum_{k=0}^{\infty} (k+1)t_{\text{mix}} \sum_{t=kt_{\text{mix}}+1} \left\| (P^\pi)^t - 1(\nu^\pi)^\top \right\|_\infty \leq \sum_{k=0}^{\infty} (k+1)t_{\text{mix}} \sum_{t=kt_{\text{mix}}+1} \left( \frac{1}{2} \right)^{\left\lfloor \frac{k}{t_{\text{mix}}} \right\rfloor} = 2t_{\text{mix}}$$

and therefore the RHS of Eq. (16) exists.

Also one can check that

$$\left( I - \hat{P} \right) \left( \sum_{t=0}^{\infty} \hat{P}^t \right) = \left( I - \hat{P} \right) \left( \sum_{t=0}^{\infty} \hat{P}^t \right) = I,$$

which indicates that equality (16) is valid.

The conclusion thus follows directly from the matrix norm bound.  \qed
This immediately implies the following corollary.

**Corollary 2** (Bound on $v^*$). For mixing AMDP (2), for some optimal policy $\pi^*$ with corresponding stationary distribution $\nu^*$, there exists an optimal value vector $v^* \perp \nu^*$ such that

$$\|v^*\|_\infty \leq 2t_{mix}.$$  

**Proof.** By optimality conditions $(I - P^*)v^* = r^* - \bar{v}^* 1$, and $\langle \nu^*, v^* \rangle = 0$ one has

$$(I - P^* + 1(\nu^*)^\top)v^* = r^* - \bar{v}^* 1$$

which gives

$$\|v^*\|_\infty = \left\| (I - P^* + 1(\nu^*)^\top)^{-1}(r^* - \bar{v}) \right\|_\infty \leq \left\| (I - P^* + 1(\nu^*)^\top)^{-1} \right\|_\infty \|r^* - \bar{v}\|_\infty \leq 2t_{mix}$$

where the last inequality follows from Lemma 5. 

Thus, we can safely consider the minimax problem (6) with the additional constraint $v^* \in B_{2M}$, where we set $M = 2t_{mix}$. The extra coefficient 2 comes in to ensure stricter primal-dual optimality conditions, which we use in Lemma 10 for the rounding.

### 4.2 Design of Estimators

Given domain setups, now we describe formally the gradient estimators used in Algorithm 1 and their properties.

For the $v$-side, we consider the following gradient estimator

Sample $(i, a_i) \sim [\mu]_{i, a_i}, j \sim p_{ij}(a_i)$.

Set $\tilde{g}^v(v, \mu) = e_j - e_i$.

(17)

This is a bounded gradient estimator for the box domain.

**Lemma 7.** $\tilde{g}^v$ defined in (17) is a $(2, \|\cdot\|_2)$-bounded estimator.

**Proof.** For unbiasedness, direct computation reveals that

$$E[\tilde{g}^v(v, \mu)] = \sum_{i, a_i, j} \mu_{i, a_i} p_{ij}(a_i)(e_j - e_i) = \mu^\top (P - \hat{I}).$$

For a bound on the second-moment, note $\|\tilde{g}^v(v, \mu)\|^2 \leq 2$ with probability 1 by definition, the result follows immediately.

For the $\mu$-side, we consider the following gradient estimator

Sample $(i, a_i) \sim \frac{1}{A_{\text{tot}}}, j \sim p_{ij}(a_i)$.

Set $\tilde{g}^\mu(v, \mu) = A_{\text{tot}}(v_i - v_j - r_{i, a_i})e_i$.

(18)

This is a bounded gradient estimator for the simplex domain.

**Lemma 8.** $\tilde{g}^\mu$ defined in (18) is a $((2M + 1)A_{\text{tot}}, 9(M^2 + 1)A_{\text{tot}}, \|\cdot\|_{\Delta A})$-bounded estimator.
Proof. For unbiasedness, direct computation reveals that
\[
\mathbb{E}[\hat{g}^\mu(v, \mu)] = \sum_{i,a,i,j} p_{ij}(a_i)(v_i - v_j - r_{i,a})e_{i,a} = (\hat{I} - \hat{P})v - r .
\]

For the bound on \( \ell_\infty \) norm, note that with probability 1 we have \( \|\hat{g}^\mu(v, \mu)\|_\infty \leq (2M + 1)A_{\text{tot}} \) given \( |v_i - v_j - r_{i,a}| \leq \max\{2M, 2M + 1\} \leq 2M + 1 \) by domain bounds on \( v \). For the bound on second-moment, given any \( \mu' \in \Delta^A \) we have
\[
\mathbb{E}[\|\hat{g}^\mu(v, \mu)\|_\mu^2] \leq \sum_{i,a,i} \frac{1}{A_{\text{tot}}} \mu'_{i,a} \max \{ (2M)^2, (2M + 1)^2 \} A_{\text{tot}}^2 \leq 9(M^2 + 1)A_{\text{tot}},
\]
where the first inequality follows similarly from \( |v_i - v_j - r_{i,a}| \leq \max\{2M, 2M + 1\}, \forall i, j, a_i \).

Theorem 3 together with guarantees of designed gradient estimators in Lemma 7, 8 and choice of \( M = 2t_{\text{mix}} \) gives Corollary 4.

**Corollary 3.** Given mixing AMDP tuple \( \mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, r) \) with desired accuracy \( \epsilon \in (0, 1) \), Algorithm 1 with parameter choice \( \eta' = O(\epsilon), \eta^\mu = O(t_{\text{mix}}^{-2}A_{\text{tot}}^{-1}) \) outputs an expected \( \epsilon \)-approximate solution to mixing minimax problem (6) with sample complexity
\[
O(t_{\text{mix}}^2A_{\text{tot}}^2\epsilon^{-2}\log(A_{\text{tot}})).
\]

The proof follows immediately by noticing each iteration costs \( O(1) \) sample generation, thus directly transferring the total iteration number to sample complexity.

4.3 Rounding to Optimal Policy

Now we proceed to show how to convert an \( \epsilon \)-optimal solution of (6) to an \( \Theta(\epsilon) \)-optimal policy for (4). First we introduce a lemma that relates the dual variable \( \mu^\epsilon \) with optimal cost-to-go values \( v^* \) and expected reward \( \bar{v}^* \).

**Lemma 9.** If \( (v^\epsilon, \mu^\epsilon) \) is an expected \( \epsilon \)-approximate optimal solution to mixing AMDP minimax problem (6), then for any optimal \( v^* \) and \( \bar{v}^* \),
\[
\mathbb{E}\left[\mu^\epsilon^\top\left((\hat{I} - \hat{P})v^* - r\right) + \bar{v}^*\right] \leq \epsilon.
\]

**Proof.** Note by definition
\[
\epsilon \geq \text{EGap}(v^\epsilon, \mu^\epsilon) := \mathbb{E}\max_{\hat{v} \in \mathcal{V}_{2M, \hat{P} \in \Delta^A}} \left[(\hat{\mu} - \mu^\epsilon)^\top((\hat{P} - I)v^\epsilon + r) + \mu^\epsilon^\top(\hat{P} - \hat{I})(v^\epsilon - \hat{v})\right].
\]

When picking \( \hat{v} = v^* \) and \( \hat{\mu} = \mu^* \), i.e. optimizers of the minimax problem, this inequality yields
\[
\epsilon \geq \mathbb{E}\left[(\mu^* - \mu^\epsilon)^\top((\hat{P} - \hat{I})v^\epsilon + r) + \mu^\epsilon^\top(\hat{P} - \hat{I})(v^\epsilon - v^*)\right] = \mathbb{E}\left[\mu^\epsilon^\top((\hat{P} - \hat{I})v^\epsilon + r) - \mu^\epsilon^\top r - \mu^\epsilon^\top(\hat{P} - \hat{I})v^*\right] \overset{(i)}{=} \mathbb{E}\left[\mu^\epsilon^\top((\hat{I} - \hat{P})v^* - r)\right] + \mu^\top r \overset{(ii)}{=} \mathbb{E}\left[\mu^\epsilon^\top((\hat{I} - \hat{P})v^*)\right] - \bar{v}^* + \bar{v}^*,
\]
where we use (i) the fact that \( \mu^\epsilon^\top(\hat{P} - \hat{I}) = 0 \) by duality feasibility and (ii) \( \bar{v}^* := \mu^\epsilon^\top r \) by strong duality of (P) and (D) in (4).
Next we transfer an optimal solution to an optimal policy, formally through Lemma 10.

**Lemma 10.** Given an \( \epsilon \)-approximate solution \((\nu^\epsilon, \mu^\epsilon)\) for mixing minimax problem as defined in (6), let \( \pi^\epsilon \) be the unique decomposition (in terms of \( \lambda^\epsilon \)) such that \( \mu^\epsilon_{i,a} = \lambda^\epsilon_i \cdot \pi^\epsilon_{i,a}, \forall i \in S, a_i \in A_i \), where \( \lambda \in \Delta^S, \pi^\epsilon \in \Delta^A, \forall i \in S \). Taking \( \pi := \pi^\epsilon \) as our policy, it holds that

\[
\tilde{v}^* \leq \mathbb{E}\tilde{v}^\pi + 3\epsilon.
\]

Using this fact one can prove Lemma 10 by showing the linear constraints in dual formulation (D) of (4) are approximately satisfied given an \( \epsilon \)-approximate optimal solution \((\nu^\epsilon, \mu^\epsilon)\) to minimax problem (6).

**Proof of Lemma 10.** Say \((\nu^\epsilon, \mu^\epsilon)\) is an \( \epsilon \)-optimal solution in the form \( \mu^\epsilon_{i,a} = \lambda^\epsilon_i \pi^\epsilon_{i,a} \), for some \( \lambda^\epsilon, \pi^\epsilon \), we still denote the induced policy as \( \pi \) and correspondingly probability transition matrix \( P^\pi \) and expected reward vector \( r^\pi \) for simplicity.

Notice \( v \in \mathbb{B}^S_{2M} \) by Corollary 2 and definition of \( M = 2t_{\text{mix}} \), we get \( \mathbb{E}\|\lambda^\epsilon^T (P^\pi - I)\|_1 \leq \frac{1}{M} \epsilon \) following from

\[
2M \cdot \mathbb{E}\left\| \lambda^\epsilon^T (P^\pi - I) \right\|_1 = \mathbb{E}\left[ \max_{v \in \mathbb{B}^S_{2M}} \lambda^\epsilon^T (P^\pi - I)(-v) \right]
= \mathbb{E}\left[ \max_{v \in \mathbb{B}^S_{2M}} \lambda^\epsilon^T (P^\pi - I)(v^* - v) - \lambda^\epsilon^T (P^\pi - I)v^* \right]
\leq \epsilon + \mathbb{E}\left\| \lambda^\epsilon^T (P^\pi - I) \right\|_1 \|v^*\|_\infty \leq \epsilon + M \cdot \mathbb{E}\left\| \lambda^\epsilon^T (P^\pi - I) \right\|_1.
\]

This is the part of analysis where expanding the domain size of \( v \) from \( M \) to \( 2M \) will be helpful.

Now suppose that \( \nu^\pi \) is the stationary distribution under policy \( \pi := \pi^\epsilon \). By definition, this implies

\[
\nu^\pi^T (P^\pi - I) = 0.
\]

Therefore, combining this fact with \( \mathbb{E}\|\lambda^\epsilon^T (P^\pi - I)\|_1 \leq \frac{1}{M} \epsilon \) as we have shown earlier yields

\[
\mathbb{E}\left\| (\lambda^\epsilon - \nu^\pi)^T (P^\pi - I) \right\|_1 \leq \frac{1}{M} \epsilon.
\]

It also leads to

\[
\mathbb{E}\left[ (\nu^\pi - \lambda^\epsilon)^T r^\pi \right] = \mathbb{E}\left[ (\nu^\pi - \lambda^\epsilon)^T (r^\pi - (r^\pi, \nu^\pi)^T)1 \right]
= \mathbb{E}\left[ (\nu^\pi - \lambda^\epsilon)^T (\mathbb{I} - P^\pi + 1(\nu^\pi)^T) (\mathbb{I} - P^\pi + 1(\nu^\pi)^T)^{-1} (r^\pi - (r^\pi, \nu^\pi)^T)1 \right]
\leq \mathbb{E}\left\| (\nu^\pi - \lambda^\epsilon)^T (\mathbb{I} - P^\pi + 1(\nu^\pi)^T) \right\|_1 \left\| (\mathbb{I} - P^\pi + 1(\nu^\pi)^T)^{-1} (r^\pi - (r^\pi, \nu^\pi)^T)1 \right\|_\infty
\leq M \cdot \mathbb{E}\left\| (\nu^\pi - \lambda^\epsilon)^T (\mathbb{I} - P^\pi) \right\|_1 \leq \epsilon,
\]

where for the last but one inequality we use the definition of \( M = 2t_{\text{mix}} \) and Lemma 5.
Note now the average reward under policy \( \pi \) satisfies
\[
\mathbb{E}\bar{v}^\pi = \mathbb{E}[(v^\pi)^\top r^\pi] = \mathbb{E}\left[ (v^\pi - (P^\pi - I)v^* + (\nu^\pi)^\top r^\pi) \right] \\
= \mathbb{E}\left[ (v^\pi - \lambda^\top [(P^\pi - I)v^* + r^\pi]) \right] + \mathbb{E}\left[ \lambda^\top [(P^\pi - I)v^* + r^\pi] \right]
\]
\[
\geq \mathbb{E}\left[ (v^\pi - \lambda^\top (P^\pi - I)v^*) \right] + \mathbb{E}\left[ (v^\pi - \lambda^\top r^\pi) \right] + \bar{v}^* - \epsilon
\]
\[
\geq \bar{v}^* - \mathbb{E}\|[(v^\pi - \lambda^\top (P^\pi - I)) v^*] \|_1\|v^*\|_\infty - \mathbb{E}\left[ (v^\pi - \lambda^\top r^\pi) \right] - \epsilon
\]
\[
\geq \bar{v}^* - \frac{1}{M}\epsilon \cdot M - (\epsilon \cdot 1) - \epsilon = \bar{v}^* - 3\epsilon
\]

where we use (i) the optimality relation stated in Lemma 9, (ii) Cauchy-Schwarz inequality and (iii) conditions on \( \ell_1 \) bounds of \( (\lambda^\top - v^\pi)^\top (P^\pi - I) \) and \( (\lambda^\top - v^\pi)^\top r^\pi \) we prove earlier. \(\square\)

Lemma 10 shows one can construct an expected \( \epsilon \)-optimal policy from an expected \( \epsilon/3 \)-approximate solution of the minimax problem (6). Thus, using Corollary 4 one directly gets the total sample complexity for Algorithm 1 to solve mixing AMDP to desired accuracy, as stated in Theorem 1. For completeness we restate the theorem include a short proof below.

**Theorem 1.** Given a mixing AMDP tuple \( \mathcal{M} = (S, A, \mathcal{P}, r) \), let \( \epsilon \in (0, 1) \), one can construct an expected \( \epsilon \)-optimal policy \( \pi^\epsilon \) from the decomposition (see Section 4.3) of output \( \mu^\epsilon \) of Algorithm 1 with sample complexity \( O\left( \frac{t^2_{\text{mix}}A_{\text{tot}}\epsilon^{-2}\log(A_{\text{tot}})}{\epsilon^2} \right) \).

**Proof of Theorem 1.** Given a mixing AMDP tuple \( \mathcal{M} = (S, A, \mathcal{P}, r) \) and \( \epsilon \in (0, 1) \), one can construct an approximate policy \( \pi' \) using Algorithm 1 with accuracy level set to \( \epsilon' = \frac{1}{3}\epsilon \) such that by Lemma 10,
\[
\mathbb{E}\bar{v}^\pi \geq \bar{v}^* - \epsilon.
\]

It follows from Corollary 4 that the sample complexity is bounded by
\[
O\left( \frac{t^2_{\text{mix}}A_{\text{tot}}\log(A_{\text{tot}})}{\epsilon^2} \right).
\]

\(\square\)

## 5 Constrained MDP

In this section, we consider solving a generalization of the mixing AMDP problem with additional linear constraints, which has been an important and well-known problem class along the study of MDP [2].

Formally, we focus on approximately solving the following dual formulation of constrained mixing AMDPs \(8\) :
\[
(D) \quad \max_{\mu \in \Delta^A} \quad 0 \quad \quad \text{subject to} \quad (I - P)^\top \mu = 0, \quad D^\top \mu \geq 1, \quad (19)
\]
where \( D = [d_1, \ldots, d_K] \) under the additional assumptions that \( d_k \geq 0, \forall k \in [K] \) and the problem is strictly feasible (with an inner point in its feasible set). Our goal is to compute \( \epsilon \)-approximate policies and solutions for (19) defined as follows.

\(8\)One can reduce the general case of \( D^\top \mu \geq c \) for some \( c > 0 \) to this case by taking \( d_k \leftarrow d_k/c_k \), under which an \( \epsilon \)-approximate solution as defined in (20) of the modified problem corresponds to a multiplicatively approximate solution satisfying \( D^\top \mu \geq (1 - \epsilon)c \).
Definition 2. Given a policy $\pi$ with its stationary distribution $\nu^\pi$, it is an $\epsilon$-approximate policy of system (19) if for $\mu$ defined as $\mu_{i,a_i} = \nu^\pi_{i,a_i}, \forall i \in S, a_i \in A_i$ it is an $\epsilon$-approximate solution of (19), i.e. it satisfies

$$
\mu^\top (I - P) = 0, \quad D^\top \mu \geq (1 - \epsilon)1.
$$

By considering (equivalently) the relaxation of (19) with $\mu \geq 0, \|\mu\|_1 \leq 1$ instead of $\mu \in \Delta^A$, one can obtain the following primal form of the problem:

$$(P) \quad \min_{s \geq 0, v, t \geq 0} t - \sum_k s_k$$

subject to $$(P - I)v + Ds \leq t1.$$

Now by our assumptions, strong duality and strict complementary slackness there exists some optimal $t^* > 0$. Thus we can safely consider the case when $t > 0$, and rescale all variables $s, v,$ and $t$ by $1/t$ without changing that optimal solution with $t^* > 0$ to obtain the following equivalent primal form of the problem:

$$(P') \quad \min_{s \geq 0, v} 1 - \sum_k s_k$$

subject to $$(P - I)v + Ds \leq 1.$$

For $D := \|D\|_\infty := \max_{i,a_i,k} |[d_k]_{i,a_i}|$ and $M := 2Dt_{\text{mix}}$ we consider the following equivalent problem:

$$
\min_{v \in \mathcal{B}_{2M}^S, s : \sum_k s_k^* \leq 2, s \geq 0} \max_{\mu \in \Delta^A} f(v, s, \mu) := \mu^\top [(I - P)v + Ds] - 1^\top s.
$$

Note in the formulation we pose the additional constraints on $v, s$ for the sake of analysis. These constraints don’t change the problem optimality by noticing $v^* \in \mathcal{B}_{2M}^S, s^* \in \Delta^K$. More concretely for $s^*$, due to the feasibility assumption and strong duality theory, we know the optimality must achieve when $1 - \sum_k s_k^* = 0$, i.e. one can safely consider a domain of $s$ as $\sum_k s_k \leq 2, s \geq 0$. For the bound on $v^*$, using a method similar as in Section 4.1 we know there exists some $v^*$, optimal policy $\pi$, its corresponding stationary distribution $\nu^\pi$ and probability transition matrix $P^\pi$ satisfying

$$(P^\pi - I)v^* + Ds^* = r^* \leq 1,$$

which implies that

$$
\exists v^* \bot \nu^\pi, \|v^*\|_\infty = \left\| (I - P^\pi + 1(\nu^\pi)^\top)^{-1}(Ds^* - r^*) \right\|_\infty \leq 2Dt_{\text{mix}},
$$

where the last inequality follows from Lemma 5.

To solve the problem we again consider a slight variant of the framework in Section 3. We work with the new spaces induced and therefore use new Bregman divergences as follows:

We set Bregman divergence unchanged with respect to $\mu$ and $v$, for $s$, we consider a standard distance generating function for $\ell_1$ setup defined as $r(s) := \sum_k s_k \log(s_k)$, note it induces a rescaled KL-divergence as $V_s(s') := \sum_k s_k \log(s_k/s_k') - \|s\|_1 + ||s'||_1$, which also satisfies the local-norm property that

$$
\forall s', s \geq 0, 1^\top s \leq 2, 1^\top s' \leq 2, k \geq 6, \|\delta\|_\infty \leq 1; \langle \delta, s' - s \rangle - V_s(s) \leq \sum_{k \in [K]} s_k \delta_k^2.
$$
Now for the primal side, the gradient mapping is \( g^\nu(v, s, \mu) = (\hat{I} - P)\mu, \ g^s(v, s, \mu) = D\mu - \mathbf{1} \), we can define gradient estimators correspondingly as

\[
\begin{align*}
\text{Sample } (i, a_i) &\sim [\mu]_{i,a_i}, j \sim p_{ij}(a_i), \text{ set } \tilde{g}^\nu(v, s, \mu) = e_j - e_i. \\
\text{Sample } (i, a_i) &\sim [\mu]_{i,a_i}, k \sim 1/K, \text{ set } \tilde{g}^s(v, s, \mu) = K[d_k]_{i,a_i}e_k - \mathbf{1}.
\end{align*}
\tag{22}
\]

These are bounded gradient estimator for the primal side respectively.

**Lemma 11.** \( \tilde{g}^\nu \) defined in (22) is a \( (2, \| \cdot \|_2) \)-bounded estimator, and \( \tilde{g}^s \) defined in (22) is a \( (KD + 2, 2KD^2 + 2, \| \cdot \|_{\Delta K}) \)-bounded estimator.

For the dual side, \( g^\mu(v, s, \mu) = (\hat{I} - P)v + Ds \), with its gradient estimator

\[
\begin{align*}
\text{Sample } (i, a_i) &\sim 1/A_{\text{tot}}, j \sim p_{ij}(a_i), k \sim s_k/\|s\|_1; \\
\text{set } \tilde{g}^\mu(v, s, \mu) &= A_{\text{tot}}(v_i - \gamma v_j - r_{i,a_i} + [d_k]_{i,a_i} \|s\|_1)e_{i,a_i}.
\end{align*}
\tag{23}
\]

This is a bounded gradient estimator for the dual side with the following property.

**Lemma 12.** \( \tilde{g}^\mu \) defined in (23) is a \((2M + 1 + 2D)A_{\text{tot}}, 2(2M + 1 + 2D)^2A_{\text{tot}}, \| \cdot \|_{\Delta^A}) \)-bounded estimator.

**Algorithm 3** SMD for generalized saddle-point problem (21)

1. **Input:** Desired accuracy \( \epsilon \).
2. **Output:** An expected \( \epsilon \)-approximate solution \((v^\epsilon, s^\epsilon, \mu^\epsilon)\) for problem (21).
3. **Parameter:** Step-size \( \eta^\nu = O(\epsilon), \eta^s = O(\epsilon K^{-1}D^{-2}), \eta^\mu = O(\epsilon t_{\text{mix}}^{-2}D^{-2}A_{\text{tot}}^{-1}) \), total iteration number \( T \geq \Theta((t_{\text{mix}}^2 A_{\text{tot}} + K)D^2\epsilon^{-2} \log(A_{\text{tot}})) \).
4. for \( t = 1, \ldots, T - 1 \) do
5. Get \( \tilde{g}^\nu \) as a bounded estimator of \( g^\nu(v_t, s_t, \mu_t) \)
6. Get \( \tilde{g}^s \) as a bounded estimator for \( g^s(v_t, s_t, \mu_t) \)
7. Get \( \tilde{g}^\mu \) as a bounded estimator for \( g^\mu(v_t, s_t, \mu_t) \)
8. Update \( v_{t+1} \leftarrow \arg \min \{\eta^\nu \tilde{g}^\nu(v), V_{V_i}(v)\} \)
9. Update \( s_{t+1} \leftarrow \arg \min \{\eta^s \tilde{g}^s(s), V_{V_i}(s)\} \)
10. Update \( \mu_{t+1} \leftarrow \arg \min \{\eta^\mu \tilde{g}^\mu(\mu), V_{V_i}(\mu)\} \)
11. end for
12. **Return** \((v^\epsilon, s^\epsilon, \mu^\epsilon) \leftarrow \frac{1}{T} \sum_{t \in [T]} (v_t, s_t, \mu_t)\)

Given the guarantees of designed gradient estimators in Lemma 11, 12 and choice of \( M = 2Dt_{\text{mix}} \), one has the following Algorithm 3 for finding an expected \( \epsilon \)-optimal solution of minimax problem (21), with its theoretical guarantees as stated in Theorem 4.

**Theorem 4.** Given mixing AMDP tuple \( \mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}) \) with constraints \( D := \max_{i,a_i,k} \|d_k\|_{i,a_i} \), for accuracy \( \epsilon \in (0, 1) \), Algorithm 3 with gradient estimators (22), (23) and parameter choice \( \eta^\nu = O(\epsilon), \eta^s = O(\epsilon K^{-1}D^{-2}), \eta^\mu = O(\epsilon t_{\text{mix}}^{-2}D^{-2}A_{\text{tot}}^{-1}) \) outputs an expected \( \epsilon \)-approximate solution to constrained mixing minimax problem (21) with sample complexity \( O((t_{\text{mix}}^2 A_{\text{tot}} + K)D^2\epsilon^{-2} \log(KA_{\text{tot}})) \).

Due to the similarity to Theorem 3, here we only provide a proof sketch capturing the main steps and differences within the proof.
Sketch of Proof

*Regret bounds with local norms.* The core statement is a standard regret bound using local norms (see Lemma 1 for $\nu$ and Lemma 2) for $s$ and $\mu$, which summing together gives the following guarantee

$$
\sum_{t \in [T]} \langle \tilde{g}^e_t, v_t - v \rangle + \sum_{t \in [T]} \langle \tilde{g}^s_t, s_t - s \rangle + \sum_{t \in [T]} \langle \tilde{g}^\mu_t, \mu_t - \mu \rangle
\leq V_{v_1}(v) + \frac{\sum_{t \in [T]} \eta^v \|g^v_t\|^2_2}{2} + \frac{\sum_{t \in [T]} \eta^s \|g^s_t\|^2_2}{2} + \frac{\sum_{t \in [T]} \eta^\mu \|g^\mu_t\|^2_\mu}{2}.
$$

(24)

Note one needs the bounded maximum entry condition for $\tilde{g}^s, \tilde{g}^\mu$, and the fact that rescaled KL-divergence also satisfies local-norm property in order to use Lemma 2.

**Domain size.** The domain size can be bounded as

$$
\max_{v \in \mathcal{B}^{2D^2}_{\text{mix}}} V_{v_1}(v) \leq O(|S|D^2_{\text{mix}}), \quad \max_{s \geq 0 \sum_k s_k \leq 2} V_{s_1}(s) \leq O(\log K), \quad \max_{\mu \in \Delta^A} V_{\mu_1}(\mu) \leq O(\log A_{\text{tot}})
$$

by definition of their corresponding Bregman divergences.

**Second-moment bounds.** This is given through the bounded second-moment properties of estimators directly, as in Lemma 11 and 12.

**Ghost-iterate analysis.** In order to substitute $\tilde{g}^v, \tilde{g}^s, \tilde{g}^\mu$ with $g^v, g^s, g^\mu$ for LHS of Eq. (24), one can apply the regret bounds again to ghost iterates generated by taking gradient step with $\tilde{g} = g - \hat{g}$ coupled with each iteration. The additional terms coming from this extra regret bounds are in expectation 0 through conditional expectation computation.

**Optimal tradeoff.** One pick $\eta^v, \eta^s, \eta^\mu, T$ accordingly to get the desired guarantee as stated in Theorem 4.

Similar to Section 4.3, one can round an $\epsilon$-optimal solution to an $O(\epsilon)$-optimal policy utilizing the policy obtained from the unique decomposition of $\mu^\epsilon$.

**Corollary 4.** Following the setting of Corollary 4, the policy $\pi^\epsilon$ induced by the unique decomposition of $\mu^\epsilon$ from the output satisfying $\mu^\epsilon_{i,a} = \lambda^\epsilon_i \cdot \pi^\epsilon_{i,a}$, is an $O(\epsilon)$-approximate policy for system (19).

**Proof of Corollary 4.** Following the similar rounding technique as in Section 4.3, one can consider the policy induced by the $\epsilon$-approximate solution of MDP $\pi^\epsilon$ from the unique decomposition of $\mu^\epsilon_{i,a} = \lambda^\epsilon_i \cdot \pi^\epsilon_{i,a}$, for all $i \in S, a \in A_i$.

Given the optimality condition, we have

$$
\mathbb{E} \left[ f(v^*, s^*, \mu^\epsilon) - \min_{v \in \mathcal{B}^{2M}} f(v, s^\epsilon, \mu^\epsilon) \right] \leq \epsilon,
$$

which is also equivalent to (denoting $\pi := \pi^\epsilon$, and $\nu^\pi$ as stationary distribution under it)

$$
\mathbb{E} \left[ \max_{v \in \mathcal{B}^{2M}} \lambda^\epsilon \top (I - P^\pi) (v^* - v) \right] \leq \epsilon,
$$

thus implying that $\|\lambda^\epsilon \top (I - P^\pi)\|_1 \leq \frac{1}{M} \epsilon, \|\lambda^\epsilon - \nu^\pi \top (I - P^\pi - 1(\nu^\pi))^\top\|_1 \leq \frac{1}{M} \epsilon = O(\frac{1}{\text{mix}} \epsilon)$ hold in expectation.

Now consider $\mu$ constructed from $\mu_{i,a} = \nu^\epsilon_i \cdot \pi^\epsilon_{i,a}$, by definition of $\nu$ it holds that $\mu(I - P) = 0$.
For the second inequality of problem (19), similarly in light of primal-dual optimality
\[
\mathbb{E}\left[f(\nu^*, s^*, \mu^*) - \min_{s \geq 0, \sum_k s_k \leq 2} f(\nu, s, \mu^*)\right] \leq \epsilon \iff \mathbb{E}\left[\max_{s \geq 0, \sum_k s_k \leq 2} (\mu^*)^T D - 1^T\right] (s^* - s) \leq \epsilon,
\]
which implies that \( D^T \mu^* \geq e - \epsilon 1 \) hold in expectation given \( s^* \in \Delta_K \).

Consequently, we can bound the quality of dual variable \( \mu \)
\[
D^T \mu = D^T \mu^* + D^T (\mu - \mu^*) = D^T \mu^* + D^T \Pi^T (\nu^\pi - \nu^\epsilon)
\geq e - \epsilon 1 - \left\| D^T \Pi^T (I - (P^\pi)^T + \nu^\pi 1^T)^{-1}(I - (P^\pi)^T + \nu^\pi 1^T)(\nu^\pi - \nu^\epsilon) \right\|_\infty \cdot 1
\geq e - \epsilon 1 - \max_k \left\| d_k^T \Pi^T (I - (P^\pi)^T + \nu^\pi 1^T)^{-1} \right\|_\infty \left\| (I - (P^\pi)^T + \nu^\pi 1^T)(\nu^\pi - \nu^\epsilon) \right\|_1 \cdot 1
\geq e - O(\epsilon) 1,
\]
where the last inequality follows from definition of \( D, \Pi^\epsilon, \) Lemma 5 and the fact that
\[
\|(\lambda^\epsilon - \nu^\pi)^T (I - P^\pi - 1(\nu^\pi)^T)\|_1 \leq O\left(\frac{\epsilon}{t_{\text{mix}}^\epsilon}\right).
\]

From above we have shown that assuming the stationary distribution under \( \pi^\epsilon \) is \( \nu^\epsilon \), it satisfies \( \|\nu^\epsilon - \lambda\|_1 \leq O\left(\frac{\epsilon}{t_{\text{mix}}^\epsilon}\right) \), thus giving an approximate solution \( \mu = \nu^\epsilon - \pi^\epsilon \) satisfying \( \mu(\hat{I} - P) = 0 \), \( D^T \mu \geq e - O(\epsilon) \) and consequently for the original problem (19) an approximately optimal policy \( \pi^\epsilon \).

\section{Conclusion}

This work offers a general framework based on stochastic mirror descent to find an \( \epsilon \)-optimal policy for AMDPs and DMDPs. It offers new insights over previous SMD approaches for solving MDPs, achieving a better sample complexity and removing an ergodicity condition for mixing AMDP, while matching up to logarithmic factors the known SMD method for solving DMDPs. This work reveals an interesting connection between MDP problems with \( \ell_\infty \) regression and opens the door to future research. Here we discuss a few interesting directions and open problems:

\textbf{Primal-dual methods with optimal sample-complexity for DMDPs.} For DMDPs, the sample complexity of our method (and the one achieved in [9]) has \((1 - \gamma)^{-1}\) gap with the known lower bound [4], which can be achieved by stochastic value-iteration [28] or \(Q\)-learning [31]. If it is achievable using convex-optimization lies at the core of further understanding the utility of convex optimization methods relative to standard value / policy-iteration methods.

\textbf{High-precision methods.} There have been recent high-precision stochastic value-iteration algorithms [29] that produce an \( \epsilon \)-optimal strategy in runtime \( O(|S|A_{\text{tot}} + (1 - \gamma)^{-3}A_{\text{tot}}) \) while depending logarithmically on \( 1/\epsilon \). These algorithms iteratively shrink the value domain in an \( \ell_\infty \) ball; it is an interesting open problem to generalize our methods to have this property or match this runtime.

\textbf{Lower bound for AMDPs.} There has been established lower-bound on sample complexity needed for DMDP [4], however the lower bound for average-reward MDP is less understood. For mixing AMDP, we ask the question of what the best possible sample complexity dependence on mixing time is, and what the hard cases are. For more general average-reward MDP, we also ask if there is any lower-bound result depending on problem parameters other than mixing time.

\textbf{Extension to more general classes of MDP.} While average-reward MDP with bounded mixing time \( t_{\text{mix}} \) and DMDP with discount factor \( \gamma \) are two fundamentally important classes of MDP,
there are instances that fall beyond the range. It is thus an interesting open direction to extend our framework for more general MDP instances and understand what problem parameters the sample complexity of SMD-like methods should depend on.

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Supplementary material

Appendix

A DMDPs

In this section we provide the corresponding sample complexity results for DMDPs to formally prove Theorem 2. In Section A.1 we specify the choice of \( M \) in minimax problem (7) by bounding the operator norm to give a domain that \( v^* \) lies in. In Section A.2 we give estimators for both sides for solving (7), which is similar to the estimators developed in Section 3. In Section A.3 we show how to round an \( \epsilon \) optimal solution of (7) to an \( \epsilon \) optimal policy.

A.1 Bound on Matrix Norm

For discounted case, we can alternatively show an upper bound on matrix norm using discount factor \( \gamma \), formally stated in Lemma 13, and used for definition of \( M \) and proof of Lemma 17 in Section A.3.

Lemma 13. Given a DMDP with discount factor \( \gamma \in (0,1) \), for any probability transition matrix \( P^{\pi} \in \mathbb{R}^{S \times S} \) under certain policy \( \pi \), it holds that \( (I - \gamma P^{\pi})^{-1} \) is invertible with

\[
\| (I - \gamma P^{\pi})^{-1} \|_\infty \leq \frac{1}{1 - \gamma}.
\]

Proof of Lemma 13. First, we claim that

\[
\min_{v \in \mathbb{R}^{S} : \|v\|_\infty = 1} \| (I - \gamma P^{\pi})^{-1} v \| \geq 1 - \gamma. \tag{25}
\]

To see this, let \( v \in \mathbb{R}^{S} \) with \( \|v\|_\infty = 1 \) be arbitrary and let \( i \in S \) be such that \( |v_i| = 1 \). We have

\[
|((I - \gamma P^{\pi})v)_i| = |v_i - \gamma \sum_{j \in S} P^{\pi}(i,j)v_j| \geq |v_i| - \gamma \sum_{j \in S} P^{\pi}(i,j)v_j \\
\geq 1 - \gamma \sum_{j \in S} P^{\pi}(i,j)|v_j| \geq 1 - \gamma.
\]

Applying the claim yields the result as (25) implies invertibility of \( I - \gamma P^{\pi} \) and

\[
\| (I - \gamma P^{\pi})^{-1} \|_\infty \equiv \max_{v \in \mathbb{R}^{S}} \frac{\| (I - \gamma P^{\pi})^{-1} v \|_\infty}{\|v\|_\infty} \\
= (i) \max_{v} \frac{\| (I - \gamma P^{\pi})^{-1} (I - \gamma P^{\pi}) \hat{v} \|_\infty}{\| (I - \gamma P^{\pi}) \hat{v} \|_\infty} \\
= (ii) \max_{\hat{v} : \|\hat{v}\|_\infty = 1} \frac{\| (I - \gamma P^{\pi})^{-1} (I - \gamma P^{\pi}) \hat{v} \|_\infty}{\| (I - \gamma P^{\pi}) \hat{v} \|_\infty} \\
= \frac{1}{\min_{\hat{v} : \|\hat{v}\|_\infty = 1} \| (I - \gamma P^{\pi}) \hat{v} \|_\infty},
\]

where in (i) we replaced \( v \) with \( (I - \gamma P^{\pi})\hat{v} \) for some \( \hat{v} \) since \( I - \gamma P^{\pi} \) is invertible and in (ii) we rescaled \( \hat{v} \) to satisfy \( \|\hat{v}\|_\infty = 1 \) as scaling \( \hat{v} \) does not affect the ratio so long as \( \hat{v} \neq 0 \). \( \square \)
Corollary 5 (Bound on $\mathbf{v}^*$). For DMDP (3), the optimal value vector $\mathbf{v}^*$ satisfies
\[
\|\mathbf{v}^*\|_\infty \leq (1 - \gamma)^{-1}.
\]

Proof of Corollary 5. By optimality conditions and Lemma 13 one has $\|\mathbf{v}^*\|_\infty = \|(I - \gamma \mathbf{P})^{-1} \mathbf{r}^*\|_\infty \leq (1 - \gamma^{-1})$.

Thus, we can safely consider the minimax problem (7) with the additional constraint $\mathbf{v} \in \mathbb{B}_{2M}^S$, where we set $M = (1 - \gamma)^{-1}$. The extra coefficient 2 comes in to ensure stricter primal-dual optimality conditions, which we use in Lemma 17 for the rounding.

A.2 Design of Estimators

Given $M = (1 - \gamma)^{-1}$, for discounted case one construct gradient estimators in a similar way. For the $\mathbf{v}$-side, we consider the following gradient estimator
\[
\text{Sample } (i, a_i) \sim [\mathbf{\mu}]_{i,a_i}, j \sim p_{ij}(a_i), i' \sim q_i;
\]
\[\text{Set } \tilde{g}^\gamma(\mathbf{v}, \mathbf{\mu}) = (1 - \gamma)\mathbf{e}_v + \gamma \mathbf{e}_j - \mathbf{e}_i. \tag{26}\]

Lemma 14. $\tilde{g}^\gamma$ defined in (26) is a $(2, \|\cdot\|_2)$-bounded estimator.

Proof of Lemma 14. For unbiasedness, one compute directly that
\[
\mathbb{E}[\tilde{g}^\gamma(\mathbf{v}, \mathbf{\mu})] = (1 - \gamma)\mathbf{q} + \sum_{i,a_i,j} \mu_{i,a_i} p_{ij}(a_i)(\gamma \mathbf{e}_j - \mathbf{e}_i) = (1 - \gamma)\mathbf{q} + \mathbf{\mu}^\top(\gamma \mathbf{P} - \mathbf{1}).
\]
For bound on second-moment, note $\|\tilde{g}^\gamma(\mathbf{v}, \mathbf{\mu})\|_2^2 \leq 2$ with probability 1 by definition and the fact that $\mathbf{q} \in \Delta^S$, the result follows immediately.

For the $\mathbf{\mu}$-side, we consider the following gradient estimator
\[
\text{Sample } (i, a_i) \sim \frac{1}{A_{\text{tot}}}, j \sim p_{ij}(a_i).
\]
\[\text{Set } \tilde{g}^\mu(\mathbf{v}, \mathbf{\mu}) = A_{\text{tot}}(v_i - \gamma v_j - r_{i,a_i})\mathbf{e}_{i,a_i}. \tag{27}\]

Lemma 15. $\tilde{g}^\mu$ defined in (27) is a $((2M + 1)A_{\text{tot}}, 9(M^2 + 1)A_{\text{tot}}, \|\cdot\|_{\Delta^A})$-bounded estimator.

Proof of Lemma 15. For unbiasedness, one compute directly that
\[
\mathbb{E}[\tilde{g}^\mu(\mathbf{v}, \mathbf{\mu})] = \sum_{i,a_i,j} \sum p_{ij}(a_i)(v_i - \gamma v_j - r_{i,a_i})\mathbf{e}_{i,a_i} = (\mathbf{I} - \gamma \mathbf{P})\mathbf{v} - \mathbf{r}.
\]
For bound on $\ell_\infty$ norm, note that with probability 1 we have $\|\tilde{g}^\mu(\mathbf{v}, \mathbf{\mu})\|_\infty \leq (2M + 1)A_{\text{tot}}$ given $|v_i - \gamma \cdot v_j - r_{i,a_i}| \leq \max\{2M, \gamma \cdot 2M + 1\} \leq 2M + 1$ by domain bounds on $\mathbf{v}$. For bound on second-moment, for any $\mathbf{\mu}' \in \Delta^A$ we have
\[
\mathbb{E}[\|\tilde{g}^\mu(\mathbf{v}, \mathbf{\mu})\|_{\mu'}^2] \leq \sum_{i,a_i} \frac{1}{A_{\text{tot}}} \mu'_{i,a_i} \left\{ (2M)^2, (2M + 1)^2 \right\} A_{\text{tot}}^2 \leq 9(M^2 + 1)A_{\text{tot}},
\]
where the first inequality follows by directly bounding $|v_i - \gamma v_j - r_{i,a_i}| \leq \max\{2M, \gamma \cdot 2M + 1\}, \forall i, j, a_i$.

Theorem 3 together with guarantees of gradient estimators in use in Lemma 14, 15 and choice of $M = (1 - \gamma)^{-1}$ gives Corollary 6.

Corollary 6. Given DMDP tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathbf{r}, \gamma)$ with desired accuracy $\epsilon \in (0,1)$, Algorithm 1 outputs an expected $\epsilon$-approximate solution to discounted minimax problem (7) with sample complexity
\[
O((1 - \gamma)^{-2}A_{\text{tot}}\epsilon^{-2} \log(A_{\text{tot}})).
\]
A.3 Rounding to Optimal Policy

Now we proceed to show how to convert an expected \( \epsilon \)-approximate solution of (7) to an expected \( \Theta((1-\gamma)^{-1}\epsilon) \)-approximate policy for the dual problem (D) of discounted case (5). First we introduce a lemma similar to Lemma 9 that relates the dual variable \( \mu^\epsilon \) with optimal cost-to-go values \( v^* \) under \( \epsilon \)-approximation.

**Lemma 16.** If \((v^\epsilon, \mu^\epsilon)\) is an \( \epsilon \)-approximate optimal solution to the DMDP minimax problem (7), then for optimal \( v^* \),

\[
E \mu^\epsilon^T [(\hat{I} - \gamma P) v^* - r] \leq \epsilon.
\]

**Proof of Lemma 16.** Note by definition

\[
\epsilon \geq \text{EGap}(v^\epsilon, \mu^\epsilon) := \max_{v, \mu} \left[ (1-\gamma)q^\top v^\epsilon + \hat{\mu}^T((\gamma P - \hat{I})v^\epsilon + r) - (1-\gamma)q^\top v^* - \mu^\epsilon^T((\gamma P - \hat{I})v^* + r) \right].
\]

When picking \( \hat{v} = v^*, \hat{\mu} = \mu^* \) optimizers of the minimax problem, this inequality becomes

\[
\epsilon \geq \mathbb{E} \left[ (1-\gamma)q^\top v^\epsilon + \mu^*^T((\gamma P - \hat{I})v^\epsilon + r) - (1-\gamma)q^\top v^* - \mu^\epsilon^T((\gamma P - \hat{I})v^* + r) \right]
\]

\[
\overset{(i)}{=} \mu^*^T r - (1-\gamma)q^\top v^* - \mathbb{E} \left[ \mu^\epsilon^T((\gamma P - \hat{I})v^* + r) \right]
\]

\[
\overset{(ii)}{=} \mathbb{E} \left[ \mu^\epsilon^T \left( (\hat{I} - \gamma P)v^* - r \right) \right],
\]

where we use (i) the fact that \( \mu^*^T(\hat{I} - \gamma P) = (1-\gamma)q^\top \) by dual feasibility and (ii) \( (1-\gamma)q^\top v^* = \mu^*^T r \) by strong duality theory of linear programming. \( \square \)

Next we transfer an optimal solution to an optimal policy, formally through Lemma 17.

**Lemma 17.** Given an expected \( \epsilon \)-approximate solution \((v^\epsilon, \mu^\epsilon)\) for discounted minimax problem as defined in (7), let \( \pi^\epsilon \) be the unique decomposition (in terms of \( \lambda^\epsilon \)) such that \( \mu_{i,a_i}^\epsilon = \lambda_i^\epsilon \pi_{i,a_i}^\epsilon, \forall i \in S, a_i \in A_i \), where \( \lambda \in \Delta^S, \pi_i^\epsilon \in \Delta^A_i, \forall i \in S \). Taking \( \pi := \pi^\epsilon \) as our policy, it holds that

\[
\bar{v}^\pi \leq \mathbb{E} v^\pi + 3\epsilon/(1-\gamma).
\]

**Proof of Lemma 17.** Without loss of generality we reparametrize \((v^\epsilon, \mu^\epsilon)\) as an \( \epsilon \)-optimal solution in the form \( \mu_{i,a_i}^\epsilon = \lambda_i^\epsilon \pi_{i,a_i}^\epsilon \), for some \( \lambda^\epsilon, \pi^\epsilon \). For simplicity we still denote the induced policy as \( \pi \) and correspondingly probability transition matrix \( P^\pi \) and expected \( r^\pi \).

Given the optimality condition, we have

\[
\mathbb{E} \left[ f(v^\pi, \mu^\epsilon) - \min_{v \in \mathbb{B}^S_{2M}} f(v, \mu^\epsilon) \right] \leq \epsilon,
\]

which is also equivalent to

\[
\mathbb{E} \max_{v \in \mathbb{B}^S_{2M}} \left[ (1-\gamma)q^\top + \lambda^\epsilon^T (\gamma P^\pi - I) \right] (v^* - v) \leq \epsilon.
\]

Notice \( v \in \mathbb{B}^S_{2M} \), we have \( \| (1-\gamma)q + \lambda^\epsilon^T (\gamma P^\pi - I) \|_1 \leq \frac{\epsilon}{\lambda^\epsilon} \) as a consequence of
\[ 2M \cdot E \left\| (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) \right\|_1 \]

\[ = E \left[ \max_{\nu \in \mathcal{M}^2} (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) (-\nu) \right] \]

\[ = E \left[ \max_{\nu \in \mathcal{M}^2} (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) \right] \left( v^* - \nu \right) - \left[ (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) \right] v^* \]

\[ \leq \epsilon + E \left\| (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) \right\|_1 \| v^* \|_\infty \leq \epsilon + M \cdot E \left\| (1 - \gamma)q + \lambda^T (\gamma P^\pi - I) \right\|_1. \]

Now by definition of \( \nu^\pi \) as the dual feasible solution under policy \( \pi := \pi^\epsilon \),

\[ E \left[ (1 - \gamma)q^T + \nu^\pi^T (\gamma P^\pi - I) \right] = 0. \]

Combining the two this gives

\[ E \left\| (\lambda^\epsilon - \nu^\pi)^T (\gamma P^\pi - I) \right\|_1 \leq \frac{\epsilon}{M}, \]

and consequently

\[ E \| \lambda^\epsilon - \nu^\pi \|_1 = E \left\| (\gamma P^\pi - I)^{-T} (\gamma P - I)^T (\lambda^\epsilon - \nu^\pi) \right\|_1 \]

\[ \leq E \left\| (\gamma P^\pi - I)^{-T} \right\|_1 \left\| (\gamma P - I)^T (\lambda^\epsilon - \nu^\pi) \right\|_1 \leq \frac{M}{M} \epsilon = \epsilon, \]

where the last but one inequality follows from the norm equality that \( \| (\gamma P^\pi - I)^{-T} \|_1 = \| (\gamma P^\pi - I)^{-1} \|_\infty \)

and Lemma 13. Note now the discounted reward under policy \( \pi \) satisfies

\[ E (1 - \gamma)\bar{v}^\pi = E (\nu^\pi)^T r^\pi = E \left[ (1 - \gamma)q^T + \nu^\pi^T (\gamma P^\pi - I) \right] v^* + E (\nu^\pi)^T r^\pi \]

\[ = (1 - \gamma)q^T v^* + E \left[ \nu^\pi^T \left[ (\gamma P^\pi - I)v^* + r^\pi \right] \right] \]

\[ = (1 - \gamma)q^T v^* + E \left[ (\nu^\pi - \lambda^\epsilon)^T \left[ (\gamma P^\pi - I)v^* + r^\pi \right] \right] + E \left[ \lambda^\epsilon^T \left[ (\gamma P^\pi - I)v^* + r^\pi \right] \right] \]

\[ \geq (1 - \gamma)\bar{v}^* + E \left[ (\nu^\pi - \lambda^\epsilon)^T \left[ (\gamma P^\pi - I)v^* + r^\pi \right] \right] - \epsilon \]

\[ \geq (1 - \gamma)\bar{v}^* - E \left\| (\nu^\pi - \lambda^\epsilon)^T (\gamma P^\pi - I) \right\|_1 \| v^* \|_\infty - E \| \nu^\pi - \lambda^\epsilon \|_1 \| r^\pi \|_\infty - \epsilon \]

\[ \geq (1 - \gamma)\bar{v}^* - \frac{1}{M} \epsilon \cdot M - \frac{M}{M} \epsilon \cdot 1 - \epsilon = (1 - \gamma)\bar{v}^* - 3\epsilon, \]

where we use \( (i) \) the optimality relation stated in Lemma 16, \( (ii) \) Cauchy-Schwarz inequality and \( (iii) \) conditions on \( \ell_1 \) bounds of \( (\lambda^\epsilon - \nu^\pi)^T (\gamma P^\pi - I) \) and \( \lambda^\epsilon - \nu^\pi \) we prove earlier. \( \Box \)

Lemma 17 shows it suffices to find an expected \((1 - \gamma)\epsilon\)-approximate solution to problem (7)

\( \) to get an expected \( \epsilon \)-optimal policy. Together with Corollary 6 this directly yields the sample complexity as claimed in Theorem 2.