Abstract

This paper presents sufficient conditions for a full information state estimator to be robustly globally asymptotically stable (RGAS) under bounded process and measurement disturbances. Given an incrementally input/output-to-state stable (i-IOSS) system, the conditions require that the estimator’s cost function embodies a property resembling the system’s i-IOSS stability property and has sufficient sensitivity to uncertainty in the initial state. The new result enriches the existing theory which is available only for convergent disturbances, and provides a positive answer to a conjecture given in a recent review paper.

Keywords: State estimation; full information estimation; moving-horizon estimation; bounded disturbances; robust global asymptotic stability; incremental input/output-to-state stability

1. Introduction

Optimization-based estimation, in particular, moving-horizon estimation (MHE) has attracted extensive attention recently (Wynn et al., 2014; Ellis et al., 2014; Voelker et al., 2013; Rawlings & Ji, 2012; Alessandri et al., 2008; Rawlings & Bakshi, 2006; Rao et al., 2003). MHE uses recent information to do estimation via optimization, and it has advantages in handling nonlinear systems and general constraints as compared to classic estimation approach such as extended Kalman filter (EKF) (Ljung, 1979; Haseltine & Rawlings, 2005). On the other hand, the full information estimation/estimator (FIE) uses all information available up to the current time for state estimation. Although FIE is computationally intractable in general, it is fundamentally important because it provides a benchmark for other estimators such as MHE (Rawlings & Ji, 2012).

Fundamental results on FIE were reviewed recently by Rawlings & Ji (2012). When the system is incrementally input/output-to-state stable (i-IOSS), it is known that an FIE is robustly globally asymptotically stable (RGAS) for convergent process and measurement disturbances if the cost function of the FIE satisfies certain conditions. However, it is unclear under what conditions the conclusion holds true for generally bounded disturbances if the cost function of the FIE satisfies certain conditions. This paper presents sufficient conditions for an FIE to be RGAS, followed by a numerical example in section 4. Finally, we draw conclusions in section 5.

2. Full information estimation

We adopt the notation used in (Rawlings & Ji, 2012) for our problem formulation. The symbols $\mathbb{R}$, $\mathbb{R}_0$, and $1_{\geq 0}$ denote the sets of real numbers, nonnegative real numbers and nonnegative integers, respectively; and $I_{a,b}$ denotes the set of integers from $a$ to $b$. The symbol $|\cdot|$ denotes the Euclidean norm. The bold symbol $\mathbf{x}$ denotes a sequence of vector-valued variables $\{x(0), x(1), \ldots\}$. The notation $||\mathbf{x}||$ is the supreme norm over a sequence, $\sup_{i \geq 0} |x(i)|$, and $||\mathbf{x}||_{a,b}$ denotes $\max_{a \leq i \leq b} |x(i)|$. The frequently used $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$ and $\mathcal{KL}$ functions are defined as follows.

Definition 1. $(\mathcal{K}, \mathcal{K}_\infty, \mathcal{L}$ and $\mathcal{KL}$ functions) A function $\alpha : \mathbb{R}_0 \to \mathbb{R}_0$ is a $\mathcal{K}$-function if it is continuous, zero at zero, and strictly increasing, and it is a $\mathcal{K}_\infty$-function if $\alpha$ is a $\mathcal{K}$-function and satisfies $\alpha(s) \to \infty$ as $s \to \infty$. A function $\varphi : \mathbb{R}_0 \to \mathbb{R}_0$ is a $\mathcal{L}$-function if
it is continuous, nonincreasing and satisfies \( \varphi(t) \to 0 \) as \( t \to \infty \). A function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a KL-function if, for each \( t \geq 0 \), \( \beta(., t) \) is a K-function and for each \( s \geq 0 \), \( \beta(s, .) \) is a L-function.

The following properties of the K- and KL-functions will be used in proving our main results.

**Lemma 1.** (Rawlings & Ji, 2012) Given a K-function \( \alpha \) and a KL-function \( \beta \), the following holds for all \( a_i \in \mathbb{R}_{\geq 0}, i \in \mathbb{I}_{1,n} \), and all \( t \in \mathbb{R}_{\geq 0} \),

\[
\alpha \left( \sum_{i=1}^{n} a_i \right) \leq \sum_{i=1}^{n} \alpha(na_i), \quad \beta \left( \sum_{i=1}^{n} a_i, t \right) \leq \sum_{i=1}^{n} \beta(na_i, t).
\]

With the above notation, we consider a standard discrete-time nonlinear system described by

\[
x^+ = f(x, w), \quad y = h(x) + v,
\]

where \( x \in \mathbb{R}^n \) is the system state, \( y \in \mathbb{R}^p \) the measurement, \( w \in \mathbb{R}^r \) the process disturbance, \( v \in \mathbb{R}^q \) the measurement disturbance, and \( x^+ \in \mathbb{R}^n \) the system state at the next sample time. A control input known up to the present time can be included but can be ignored in the formulation for its irrelevance to state estimation (Rawlings & Ji, 2012). The functions \( f \) and \( h \) are assumed to be continuous and known, and the initial state \( x(0) \) and the disturbances \( (w, v) \) are modeled as unknown but bounded variables.

The state estimation problem is to determine an optimal estimate of state \( \hat{x} \) based on measurement \( y \) as recorded for all sampled times. This can be formulated as an optimization problem, yielding the so-called FIE. Let the decision variables be \( (\chi, \omega, \nu) \), which correspond to the system variables \( (x, w, v) \), and the optimal decision variables be \( (\hat{x}, \hat{w}, \hat{v}) \). Since \( (\hat{x}, \hat{w}, \hat{v}) \), which consist of optimal estimates at all sampled times, are uniquely determined once \( \hat{x}(0) \) and \( \hat{w} \) are known, the decision variables essentially reduce to \( \chi(0) \) and \( \omega \). Let \( t \) be the current time and \( x_0 \) be the prior information for the initial state. The uncertainty in the initial state is thus denoted by \( \chi(0) - x_0 \). Denote the cost function as

\[
V_\chi(\chi(0) - x_0, \omega),
\]

which expresses a cost for uncertainty in the initial state and the process. Then the FIE is defined as an optimization problem:

\[
\text{FIE : } \min_{\chi, \omega, \nu} V_\chi(\chi(0) - x_0, \omega)
\]

subject to

\[
\chi^+ = f(\chi, \omega), \quad y = h(\chi) + \nu,
\]

\( \omega \in \mathbb{B}_w, \nu \in \mathbb{B}_v, \)

optimal solution for \( (\chi(0), \omega) \) such that the state estimate satisfies the RGAS property defined below. Let \( x(x_0, \nu) \) denote a state sequence with an initial condition \( x(0) = x_0 \), and a disturbance sequence \( \omega = \{\omega(1), \omega(2), \ldots\} \).

**Definition 2.** (RGAS (Rawlings & Ji, 2012)) The estimate is based on the noisy measurement \( y = h(x(0, \nu)) + v \). The estimate is RGAS if for all \( x_0 \) and \( x_0 \), and bounded \( (\omega, v) \), there exist functions \( \beta_\omega \in \mathbb{K} \) and \( \omega_0 \), \( \omega_1 \in \mathbb{K} \) such that the following inequality holds for all \( t \in \mathbb{I}_{\geq 0} \):

\[
|\hat{x}(t; x_0, \nu) - x(t; x_0, \omega)| \leq \beta_{\omega}(\|x_0 - x_2\|_0, t) + \beta_{\omega}(\|\nu\|_{0,t}) + \beta_{\omega}(\|w\|_{0,t}),
\]

in which \( \hat{x}(0|t) \) and \( \omega_1 \) are respectively the initial state and the sequence of disturbances estimated using measurements up to time \( t \).

Note that the current measurement is considered in the definition through \( v(t) \), which is a little different from the original definition. For an FIE to be RGAS, the cost function and the system dynamics need to satisfy certain conditions. We identify and present such sufficient conditions in the next section.

### 3. RGAS of the FIE

We first introduce two definitions and one useful lemma as used in the sequel.

**Definition 3.** (i-IOSS (Rawlings & Ji, 2012; Sontag & Wang, 1997)) The system \( x^+ = f(x, w), y = h(x) \) is i-IOSS if there exist functions \( \beta \in \mathbb{K} \) and \( \alpha_1, \alpha_2 \in \mathbb{K} \) such that for every two initial states \( x_{01} \) and \( x_{02} \), any two disturbances \( w_1 \) and \( w_2 \) generating state sequences \( x_1(x_{01}, w_1) \) and \( x_2(x_{02}, w_2) \), the following holds for all \( t \in \mathbb{I}_{\geq 0} \):

\[
|\hat{x}(t; x_1, w_1) - \hat{x}(t; x_2, w_2)| \leq \beta(\|x_1 - x_2\|_0, t) + \alpha_1(\|w_1 - w_2\|_{0,t}) + \alpha_2(\|\nu_1 - \nu_2\|_{0,t}).
\]

The definition of i-IOSS can be interpreted as a “detectability” concept for nonlinear systems (Sontag & Wang, 1997), for the state may be “detected” from the noise-free output thanks to the foregoing inequality.

In particular, if in (3) \( \beta(s, t) = \alpha(s)\alpha_2 \) for all \( s, t \geq 0 \), with \( \alpha \in \mathbb{K} \) and \( \alpha \) being a constant within \((0, 1)\), then we say that the system is exponentially i-IOSS or exponentially i-IOSS for short. This can be viewed as extending the concept of exponential input-to-state stability (Grüne, 1999; Liu et al., 2010) to the context of i-IOSS.

**Definition 4.** (K · L-function) A KL-function \( \beta \) is called a K · L-function if there exist functions \( \alpha \in \mathbb{K} \) and \( \varphi \in \mathbb{L} \) such that \( \beta(s, t) = \alpha(s)\varphi(t) \) for all \( s, t \geq 0 \).

As an example, the KL-function \( \varphi^{-1} \) is a K · L-function for \( s, t \geq 0 \). The next lemma shows the general interest of a K · L-function.
Lemma 2. (K-L bound) Given an arbitrary KL-function $\beta$, there exists a $K-L$-function $\bar{\beta}$ such that $\beta(s, t) \leq \bar{\beta}(s, t)$ for all $s, t \geq 0$.

Proof. By Lemma 8 in (Sontag, 1998), given arbitrary $\beta \in KL$, there exist two functions $\alpha_1, \alpha_2 \in K_{\infty}$ such that $\beta(s, t) \leq \alpha_1(s)\alpha_2(e^{-t}) =: \bar{\beta}(s, t)$ for all $s, t \geq 0$. Since $\bar{\beta}(s, t)$ is a $K-L$-function in $s$ and $t$, this completes the proof. \hfill $\Box$

Lemma 2 implies that the i-IOSS property in (3) can be defined equivalently using a $K-L$-function, which is useful in our later stability analysis of the FIE. Next, we introduce two assumptions for establishing our main result.

Assumption 1. The FIE’s cost function, $V_{t}(\chi(0) - \bar{x}_0, \omega)$, is defined such that it is continuous and satisfies the following inequalities for all $\chi(0), \bar{x}_0 \in \mathbb{R}^n$, $\omega \in B_w$ and $\nu \in B_v$:

$$\begin{align*}
\|x(t) - \bar{x}_0\| + t &\leq V_{t}(\chi(0) - \bar{x}_0, \omega) \\
&\leq V_{t}(\chi(0) - \bar{x}_0, \omega) + \gamma_w(\|w\|_{0,t}) + \gamma_v(\|v\|_{0,t})
\end{align*}$$

(4) satisfy the following inequalities for all $s_x, s_w, s_v, t \geq 0$:

$$\begin{align*}
\beta(s_x + \tilde{\gamma}_w^{-1}(\rho_x(s_x, t) + \gamma_w(s_w) + \gamma_v(s_v)), t) &\leq \beta(s_x, t) + \alpha_w(s_w) + \alpha_v(s_v),
\end{align*}$$

in which $\tilde{\gamma}_w(s) := \gamma_w(s, t)$ and $\gamma_w^{-1}()$ defines its inverse function, and $\bar{\beta}, \alpha_w$ and $\alpha_v$ are proper $KL$, $K$ and $K$ functions, respectively.

Assumption 1 requires the FIE to have a property resembling the i-IOSS property of the system, and Assumption 2 requires the FIE having higher sensitivity to perturbation in the initial state than the system. The meanings of these two assumptions will become clearer in the corollaries to be presented later. With these two assumptions, we establish our main result.

Theorem 1. (RGAS of the FIE) The FIE defined in (2) is RGAS if it satisfies Assumptions 1-2 and its global optimal solution is obtainable while the system given in (1) is i-IOSS. If further the disturbance sequences $w$ and $v$ are known to converge to zero as $t$ goes to infinity, then the FIE is also convergent, i.e., it gives a state estimate that converges to the true state as $t \to \infty$.

Proof. Part 1: RGAS. The proof is adapted from Part 1 of the proof of Proposition 11 presented in (Rawlings & Ji, 2012). Let the global optimal solution of (FIE) result in a minimum cost $V_{t}^\circ$. It follows that for all $t \geq 0$,

$$V_{t}^\circ = V_{t}(\hat{x}(0)|t) - \bar{x}_0, \hat{w}_t) \leq V_{t}(x_0 - \bar{x}_0, w) =: \bar{V}_{t}.$$ 

By Assumption 1 we have

$$\bar{V}_{t} \leq \rho_x(|x_0 - \bar{x}_0|, t) + \gamma_w(\|w\|_{0,t}) + \gamma_v(\|v\|_{0,t}).$$

Together with $2\tilde{\gamma}_w^{-1}(\hat{x}(0)|t) - \bar{x}_0) := \rho_x(|\hat{x}(0)|t) - \bar{x}_0|, t) \leq \bar{V}_{t},$ this leads to $|\hat{x}(0)|t) - \bar{x}_0| \leq \bar{\gamma}_w^{-1}(\bar{V}_{t}),$ where $\bar{\gamma}_w^{-1}(\bar{V}_{t})$ is dependent on time $t$. By using the triangle inequality, this further results in

$$\begin{align*}
|x_0 - \hat{x}(0)|t) &\leq |x_0 - \bar{x}_0| + |\hat{x}(0)|t) - \bar{x}_0| \\
&\leq |x_0 - \bar{x}_0| + \bar{\gamma}_w^{-1}(\rho_x(|x_0 - \bar{x}_0|, t) + \gamma_w(\|w\|_{0,t}) + \gamma_v(\|v\|_{0,t}), t)
\end{align*}$$

(5)

The second term on right hand side of the second inequality is dependent on time $t$.

Next we derive a bound for the term $\|w - \hat{w}_t\|_{0,t-1}$. From the triangle inequality we have

$$\|w - \hat{w}_t\|_{0,t-1} < \|w\|_{0,t-1} + \|\hat{w}_t\|_{0,t-1}.$$ 

Since $\gamma_w(\|\hat{w}_t\|_{0,t-1}) \leq V_{t}^\circ \leq \bar{V}_{t}$, it implies that

$$\|w\|_{0,t-1} \leq V_{t}^\circ.$$ 

Consequently the foregoing inequality leads to

$$\begin{align*}
\|w - \hat{w}_t\|_{0,t-1} &\leq \|w\|_{0,t-1} + \bar{\gamma}_w^{-1}(\rho_x(|x_0 - \bar{x}_0|, t) + \gamma_w(\|w\|_{0,t-1}) + \gamma_v(\|v\|_{0,t})) \\
&\leq \rho_x^w(|x_0 - \bar{x}_0|, t) + \gamma_w^w(\|w\|_{0,t-1} + \gamma_v(\|v\|_{0,t}))
\end{align*}$$

(6)

where $\rho_x^w := \tilde{\gamma}_w^{-1} \circ \rho_x$, $\gamma_w^w := \tilde{\gamma}_w^{-1} \circ \gamma_v$, which are $KL$, $K$ and $K$ functions, respectively. By applying the same reasoning to $\|v - \hat{v}_t\|_{0,t-1}$, it yields

$$\begin{align*}
\|v - \hat{v}_t\|_{0,t} &\leq \rho_v^w(|x_0 - \bar{x}_0|, t) + \gamma_v^w(\|v\|_{0,t-1}) + \gamma_v(\|v\|_{0,t}),
\end{align*}$$

(7)

where $\rho_v^w$, $\gamma_v^w$ and $\gamma_v$ are $KL$, $K$ and $K$ functions, respectively.

Substitute (5)-(7) into the definition of i-IOSS, (3), yielding (8), where Assumption 2 has been used to derive the last inequality. As the sums of terms in the three lines of the last inequality form classes $KL$, $K$ and $K$ functions, respectively, we can denote them as $\beta_x(|x_0 - \bar{x}_0|, t) + \alpha_w(\|w\|_{0,t-1}) + \alpha_v(\|v\|_{0,t})$ in sequence, and hence conclude from (8) that

$$\begin{align*}
|x(t; x_0, w) - x(t; \hat{x}(0)|t), \hat{w}_t) &\leq \beta_x(|x_0 - \bar{x}_0|, t) + \alpha_w(\|w\|_{0,t-1}) + \alpha_v(\|v\|_{0,t}).
\end{align*}$$

This means that the FIE is RGAS, and hence completes the proof of Part 1.

Part 2: convergence. We adopt the reasoning used in the proof of Proposition 4.5 in (Angeli, 2002) to this new
context. Let the sequences of disturbance and noise, $w$ and $v$, be bounded as $\|w\| \leq M_w$ and $\|v\| \leq M_v$ for some constants $M_w, M_v \geq 0$. Since the FIE is RGAS, we have

$$x(t; x_0, w) - x(t; \hat{x}(0|t), \hat{w}_t) \leq t \beta_\ell(x_0 - \hat{x}_0|t, t) + \alpha_w(M_w) + \alpha_v(M_v),$$

for all $t \geq 0$. Because the disturbance and noise sequences are known to converge to zero, this knowledge constrains the feasible sets of the disturbance and noise estimates and ensures that these estimates obtained by the FIE defined in (2) will converge to zero. Therefore, for any $\epsilon > 0$, there exists a time $T_\epsilon > 0$ such that $\|w(t)\|, \|v(t)\| \leq 0.5\alpha_\ell^{-1}(\epsilon/8)$ and $|\ell(t)|, |\check{t}(t)| \leq 0.5\alpha_\ell^{-1}(\epsilon/8)$ for all $t \geq T_\epsilon$. By the definition of $\mathcal{K}$-$\mathcal{L}$-function, for any $\epsilon > 0$ there exists a time $\tau_\epsilon$ such that $\beta(\beta_\ell(x_0 - \hat{x}_0|0), 0) + \alpha_w(M_w) + \alpha_v(M_v), t \leq \epsilon/2$ for all $t \geq \tau_\epsilon$. Hence, for $t \geq T_\epsilon + \tau_\epsilon$ we obtain

$$x(t; x_0, w) - x(t; \hat{x}(0|t), \hat{w}_t) \leq \beta((x(T_\epsilon; x_0, w) - x(T_\epsilon; \hat{x}(0|t), \hat{w}_t)|, t - T_\epsilon) + \alpha_w(M_w) + \alpha_v(M_v), t \leq \epsilon/2$$

by i-IOSS

by RGAS

which implies that $x(t; \hat{x}(0|t), \hat{w}_t)$ converges to $x(t; x_0, w)$ as $t \to \infty$. This completes the proof of Part 2, and hence the proof of the theorem is completed.

Theorem 1 gives sufficient conditions for an FIE to be RGAS (or convergent) under bounded (or convergent) disturbances. It enriches the existing theory (Rawlings & Ji, 2012).

The Assumption 2 required is rather general. Using $\mathcal{K}$-$\mathcal{L}$-function introduced in Definition 4, we can obtain more specific conditions admitting easier interpretation.

**Corollary 1.** The FIE defined in (2) is RGAS, if the following conditions are satisfied:

a) the system given in (1) is i-IOSS as in (3);

b) the FIE’s cost function satisfies Assumption 1, and its global optimal solution is obtainable;

c) the $\mathcal{K}$-$\mathcal{L}$-functions $\beta$ in (3) and $\rho_\ell, \rho_\check{t}$ in (4) are $\mathcal{K}$-$\mathcal{L}$-functions in the form of $\beta(s, t) = \mu_\ell(s, t)\phi_\ell(t)$ and $\rho_\ell(s, t) = \mu_\check{t}(s, t)\phi_\check{t}(t)$, where $\mu_\ell, \mu_\check{t}, \rho_\ell, \rho_\check{t} \in \mathcal{K}$, and $\phi_\ell, \phi_\check{t} \in \mathcal{L}$, and satisfy

$$\beta(4\beta_\ell^{-1}(3\beta_\ell(s, t), t), t) = \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), \phi_\ell(t)), t) \leq \beta_\ell(s, t),$$

which results in a $\mathcal{K}$-$\mathcal{L}$-function. With $\beta(s, t) = \mu_\ell(s, t)\phi_\ell(t)$ and the condition (9), we also have,

$$\beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) = \mu_\ell(4\beta_\ell^{-1}(3\gamma_\ell(s, w), \phi_\ell(t)), t) \leq \alpha_w(s, w),$$

for some function $\alpha_w \in \mathcal{K}$. Similarly we have

$$\beta(s, t) = \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \alpha_v(s, v),$$

for some function $\alpha_v \in \mathcal{K}$. Consequently;

$$\beta(s, t) \leq \beta(s, t),$$

$$\beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t)$$

where $\beta_\ell(s, t) = \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t) \leq \beta(4\beta_\ell^{-1}(3\gamma_\ell(s, w), t), t)$ which is a $\mathcal{K}$-$\mathcal{L}$-function. The last inequaity means that Assumption 2 is satisfied. Together with the conditions a) and b), it establishes the conclusion by using Theorem 1.

In the condition c) of Corollary 1, the assumption of
\( \beta \) being a \( \mathcal{K} \cdot \mathcal{L} \)-function is trivial because we can always assign such a function as an alternative if the original \( \mathcal{K} \cdot \mathcal{L} \)-function \( \beta \) is not in a \( \mathcal{K} \cdot \mathcal{L} \) form (cf. Lemma 2). The condition that \( \psi_b \) and \( \rho_b \) in (4) are \( \mathcal{K} \cdot \mathcal{L} \)-functions is not an assumption on the system, but a requirement on the cost function defined for the FIE. The key condition thus boils down to (9), requiring the cost function to be sufficiently sensitive (compared to the system’s sensitivity) to the uncertainty in the initial state. This is intuitive because otherwise the estimator cannot detect the effect caused by the uncertainty and hence is unable to reconstruct the initial state accurately.

The FIE admits a more specific cost function if the system is IOSS as in (3) where the \( \mathcal{K} \mathcal{L} \) bound has a polynomial form.

**Corollary 2.** The FIE defined in (2) is RGAS, if the following conditions are satisfied:

- a) the system given in (1) is I-OSS as in (3), where the \( \mathcal{K} \mathcal{L} \) bound is given as \( \beta(s, t) = c_1 s^\alpha_1 (t + 1)^{-b_1} \) for some constants \( c_1, a_1, b_1 > 0 \) and all \( s, t \geq 0 \);

- b) the FIE’s global optimal solution is obtainable when the cost function is defined as

\[
V_I(X(0) - \bar{x}_0, \omega) = l_x(X(0) - \bar{x}_0)(t + 1)^{-b_2} + l_{w_0}(\omega, \nu, t),
\]

where \( b_2 \) is a positive constant, and the sub-costs are continuous functions and satisfy the following inequalities for all \( x \in \mathbb{R}^n \), \( w \in \mathbb{B}_w \) and \( v \in \mathbb{B}_v \):

\[
c_2|x|^2 \leq \gamma'_2(|x|),
\]

\[
\gamma_w + c_2 \gamma'_w + c_2 \gamma'_w \leq \gamma_w + \gamma_v + \gamma_t + \gamma_{v_t} + \gamma_{v_t} + \gamma_{v_t},
\]

where \( c_2 \) and \( a_2 \) are positive constants, and \( \gamma_w, \gamma_v, \gamma_t, \gamma_{v_t}, \gamma_{v_t}, \gamma_{v_t} \in \mathbb{K}_{\infty} \);

- c) the design parameters \( a_2 \) and \( b_2 \) satisfy \( \frac{a_2}{b_2} \geq \frac{a_1}{b_1} \).

**Proof.** It is straightforward to show that the cost function given above satisfies Assumption 1, in which the \( \mathcal{K} \mathcal{L} \)-functions are given as \( \psi_b(x, t) := \gamma'_b(|x|)(t + 1)^{-b_2} \) and \( \rho_b := \gamma'_b(|x|)(t + 1)^{-b_2} \). These two functions are factorizable as \( \mu_2(s) \varphi_2(t) \) and \( \mu_3(s) \varphi_3(t) \), respectively, with \( \mu_2(s) := \gamma'_b(s) = c_2 s^a_2 \), \( \mu_3(s) := \gamma'_b(s) = c_2 s^a_2 \) and \( \varphi_2(t) := (t + 1)^{-b_2} \). Given the condition a) above, the \( \mathcal{K} \mathcal{L} \) bound associated with the I-OSS property of the system is obtained as \( \beta_b(s, t) = \mu_1(s) \varphi_1(t), \) with \( \mu_1(s) := c_1 s^a_1 \) and \( \varphi_1(t) := (t + 1)^{-b_1} \). Then for any function \( \pi \in \mathcal{K} \), we have

\[
\mu_1 \left( 4 \mu_2^{-1} \left( \frac{\pi(s)}{\varphi_2(t)} \right) \right) \varphi_1(t) = c_1 \left( 4 \left( \frac{\pi(s)}{\varphi_2(t)} \right) \right)^{a_1} \varphi_1(t) = c_1 \left( 4 \left( \frac{\pi(s)}{\varphi_2(t)} \right) \right)^{a_1} \varphi_1(t) \leq 4^{a_1} c_1 c_2 \gamma_2 \left( \pi(s) \right)^{a_1} \varphi_2(t) \leq 4^{a_1} c_1 c_2 \left( \pi(s) \right)^{a_1} \varphi_2(t),
\]

where \( \pi' \) is a \( \mathcal{K} \mathcal{L} \)-function, and the condition c) has been used to derive the inequality. Therefore the condition c) of Corollary 1 is satisfied. Since the conditions a) and b) there are also satisfied, this proves that the FIE is RGAS by Corollary 1.

The conditions b)-c) of Corollary 2 are manifestations of the more general conditions given in Theorem 1, subject to the condition a) here. We remark that this corollary recovers the main result presented in (Ji et al., 2013) if the design parameter \( b_2 \) is fixed to 1 (with a minor difference that here the FIE is able to utilize the last measurement in the estimation, whose fitting error is penalized through \( \nu(t) \)).

More specific sub-costs that satisfy the conditions b)-c) of Corollary 2 may have the following forms:

\[
l_x(X(0) - \bar{x}_0)(t + 1)^{-b_2} := c_2 s^a_2 \varphi_2(t),
\]

for positive constants \( a_2, b_2 \) satisfying \( \frac{a_2}{b_2} \geq \frac{a_1}{b_1} \) and any positive constant \( c_2 \), and

\[
l_{w_0}(\omega, \nu, t) = \frac{1}{l + 1} \left( \lambda_{w_0} \sum_{i=0}^{t-1} l_{w_i}(\omega(i)) + \lambda_{\nu} \sum_{i=0}^{t-1} l_{w_i}(\nu(i)) \right) + (1 - \lambda_{w_0}) \max_{i \in \mathbb{Z}_{t+1}} l_{w_i}(\omega(i)) + (1 - \lambda_{\nu}) \max_{i \in \mathbb{Z}_{t+1}} l_{w_i}(\nu(i))
\]

for given constants \( \lambda_{w_0}, \lambda_{\nu} \in [0, 1] \), in which the sequences of sub-cost functions are such that:

\[
\gamma'_w(|w|) \leq l_{w_i}(v(i)) \leq \gamma'_w(|w|), \quad \gamma'_v(|v|) \leq l_{v_i}(v(i)) \leq \gamma'_v(|v|),
\]

where \( \gamma'_w, \gamma'_v, \gamma'_v, \gamma'_v \in \mathcal{K}_{\infty} \).

Furthermore, if the system described in (1) is exp-i-IOSS, then the polynomial \( \mathcal{K} \mathcal{L} \) bound in Corollary 2 can be tightened to have an exponential form. Consequently we may define a stage cost that better penalizes the deviation from the prior initial state estimate, which intuitively would improve FIE’s estimation performance.

**Corollary 3.** The FIE defined in (2) is RGAS, if the following conditions are satisfied:

- a) the system given in (1) is exp-i-IOSS as in (3), in which the \( \mathcal{K} \mathcal{L} \)-function is given as \( \beta(s, t) = c_1 s^a_1 b_1^{b_1} \) for some constants \( c_1, a_1 > 0 \) and \( b_1 < 1 \), and all \( s, t \geq 0 \);

- b) the condition b) of Corollary 2 is satisfied when the factor \( (t + 1)^{-b_2} \) is replaced with \( b_2' \);

- c) the design parameters \( a_2 \) and \( b_2 \) satisfy \( \frac{a_2}{b_2} \geq \frac{a_1}{b_1} \).

**Proof.** The proof follows a routine similar to that of the proof for Corollary 2 and is omitted for brevity.

It is worthwhile to mention that the condition c) of Corollary 3 does not require \( b_2 < 1 \). That is, an FIE with \( b_2 \geq 1 \) may also be RGAS despite the fact that the sub-cost associated with the initial state is divergent in time. We will illustrate this in the simulation section.
By Corollary 3, it is valid to specify the sub-cost associated with the initial state as $c_2|X(0) - x_0|^2b_2$, with the positive constants \{a_2, b_2, c_2\} satisfying the condition c) of Corollary 3. The sub-cost associated with the disturbances may be defined to have the same form presented after Corollary 2.

Remark 1. As revealed in (Glas, 1987), nonlinear systems that are asymptotically stable but not exponentially stable fail to be structurally stable and constitute a boundary set, and hence of little practical interest. The article also proves that the set of exponentially stable systems are dense in the whole set of asymptotically stable systems. These results indicate that it loses no generality or practical interest for Corollary 3 to focus on i-IOSS systems that are exponentially stable.

Remark 2. The conclusion that the state estimate given by the FIE converges to the true state if the disturbances are known to converge to zero remains true under the conditions of Corollaries 1-3. This is because the convergence is implied by the i-IOSS property of the system and the RGAS property of the estimator under bounded and convergent disturbances (cf. Part 2 of the proof of Theorem 1).

4. Numerical Example

We use an example to illustrate the theoretical results concluded by Corollaries 2-3. Consider an asymptotically stable system with linear dynamics and nonlinear measurement: $x^+ = 0.9x + w$, $y = x^t + v$, where $x$ is the state, $y$ the measurement, $w$ the state disturbance, and $v$ the measurement noise. The disturbance $\{w(k)\}$ and noise $\{v(k)\}$ are two sequences of independent, zero mean, normally distributed random variables with variances $\sigma_w^2$ and $\sigma_v^2$ equal to 0.12 and 0.22, respectively, as further truncated to the intervals $[−3\sigma_w, 3\sigma_w]$ and $[−3\sigma_v, 3\sigma_v]$, respectively. The initial state $x(0)$ is a random variable independent of the disturbances $\{w(k)\}$ and $\{v(k)\}$, and follows a normal distribution with a mean of 5 and a variance of $\sigma_x^2$ equal to 4. The prior estimate of the initial state is given as $\hat{x}_0 = 2$.

By Corollary 3 and Lemma 3 (in Appendix), the system is exp-i-IOSS with the $K \cdot L$ bound given by $\beta(s, t) = s0.9^t$. For the FIE to be RGAS, its cost function can be specified as

$$V_t = \left(\chi(0) - \bar{x}_0\right)^2b_t^t + \frac{1}{t+1} \left(\frac{\lambda_w}{\sigma_w^2} \sum_{i=0}^{t-1} \omega^2(i) + \frac{\lambda_v}{\sigma_v^2} \sum_{i=0}^{t-1} \nu^2(i)\right) + \frac{1 - \lambda_w}{\sigma_w^2} \|\omega\|^2_{0:t-1} + \frac{1 - \lambda_v}{\sigma_v^2} \|\nu\|^2_{0:t},$$

for any given constants $b_2 \geq 0.9^2 = 0.81$ and $\lambda_w, \lambda_v \in [0, 1]$. By solving the FIE (with $b_2 = 0.81$) subject to $\|\omega\|^2_{0:t-1} \leq 3\sigma_w$ and $\|\nu\|^2_{0:t} \leq 3\sigma_v$, we obtain the state estimates for each $t \in [0, 20]$. The estimation errors, defined by $e(t|t) = x(t) - \hat{x}(t|t)$, are averaged over 500 random instances, as shown in Fig. 1 for evenly sampled times. To compare, the state estimation errors resulting from a generic EKF (Haseltine & Rawlings, 2005) are also shown in the figure.

We observe that the FIE yields bounded estimation errors (which holds true for longer simulation times) and outperforms the EKF significantly during the early estimation stage. Yet the advantage decays as the EKF accumulates sufficient iterations, say, when $t \geq 6$ in this case. The early advantage owes to FIE using all measurements accumulated to compute an optimal estimate of the present state, while the EKF merely uses the current measurement to update its previous estimate. From Fig. 1, we also observe that the FIE with $\lambda_w = \lambda_v = 1$ results in more accurate estimation than with $\lambda_w = \lambda_v = 0$. Moreover, we applied the FIEs with $b_2 = 2$ for the two cases, which imposes a heavier and divergent sub-cost for deviation of the estimate of the initial state from its prior estimate. The FIEs yield slightly worse estimation results: when $b_2 = 0.81$, the state estimation error has a standard deviation of 0.065 (or 0.081) and an average absolute value of 0.037 (or 0.057) for $\lambda_w = \lambda_v = 0$, respectively.

Additionally, we may use a looser $K \cdot L$ bound as $\beta(s, t) = s(t + 1)^{0.9}$, and consequently the cost function can alternatively be defined by Corollary 2 as:

$$V'_t = \left(\chi(0) - \bar{x}_0\right)^2b_{t+1}^t + \frac{1}{t+1} \left(\frac{\lambda_w}{\sigma_w^2} \sum_{i=0}^{t-1} \omega^2(i) + \frac{\lambda_v}{\sigma_v^2} \sum_{i=0}^{t-1} \nu^2(i)\right) + \frac{1 - \lambda_w}{\sigma_w^2} \|\omega\|^2_{0:t-1} + \frac{1 - \lambda_v}{\sigma_v^2} \|\nu\|^2_{0:t},$$

![Figure 1: Estimation results: (a) mean and variation of the state estimation error; (b) empirical cumulative distribution function plot of the state estimation error.](image-url)
where $0 < b_2 \leq -2 \ln 0.9 \approx 0.21$ and $\lambda_w, \lambda_v$ are the same as above. We implemented the FIE with this new cost function for $b_2 = 0.21$ and ran simulations on the same instances. The state estimation results almost coincide with those obtained using the previous cost function for $b_2 = 0.81$: the standard deviation and the average absolute value are obtained as 0.065 and 0.037 (or, 0.082 and 0.046 for $\lambda_w = \lambda_v = 0$), respectively.

5. Conclusions

This paper presented sufficient conditions for a full information estimator (FIE) to be robustly globally asymptotically stable (RGAS) under bounded process and measurement disturbances. The conditions require that the cost function being optimized has a feature resembling the $i$-IOSS stability property of the system, but with a higher sensitivity to the uncertainty in the initial state. The results are applicable to convergent disturbances, yielding a stronger conclusion that the estimation error of the FIE converges to zero.

Of future research interest is the investigation of a variant of the FIE which uses measurements within a moving horizon. The variant should possess similar stability properties while being computationally more efficient.

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Appendix: A supporting lemma and its proof

Lemma 3. Let the system be given as in (1), where $f(x, w) = A x + w$ for a real matrix $A$ having absolute eigenvalues all less than 1, and $h(x)$ is an arbitrary continuous function. Then, the system is exp-$i$-IOSS as in (3), in which the $\mathcal{KL}$-function $\beta$ is admissible to be $\beta(s, t) = s \sigma_{\max}^t (A) \leq s (t + 1) \ln \sigma_{\max} (A)$ for all $s, t \geq 0$, with $\sigma_{\max} (A)$ being the maximum singular value of $A$.

Proof. Given two initial conditions $x_1 (0)$ and $x_2 (0)$ and corresponding disturbance sequences $w_1 (k)$ and $w_2 (k)$ for $k \in \mathbb{Z}_{0, 1}$, let the states of the system at time $t$ be obtained as $x_1 (t)$ and $x_2 (t)$, respectively. We have

$$|x_1 (t) - x_2 (t)|$$

$$= |A^t (x_1 (0) - x_2 (0))$$

$$+ \sum_{k=0}^{t-1} A^k w_1 (t - 1 - k) - w_2 (t - 1 - k))|$$

$$\leq \sigma_{\max}^t (A) |x_1 (0) - x_2 (0)| + \|w_1 - w_2\|_{0_{t-1}} \sum_{k=0}^{t-1} |A^k|$$

$$\leq \sigma_{\max}^t (A) |x_1 (0) - x_2 (0)| + \frac{1 - \sigma_{\max}^t (A)}{1 - \sigma_{\max} (A)} \|w_1 - w_2\|_{0_{t-1}}.$$

Because $0 < \sigma_{\max} (A) < 1$, the last inequality implies that the system is exp-$i$-IOSS by definition per (3), in which the $\mathcal{KL}$-function $\beta$ has the form of $\beta(s, t) = \sigma_{\max}^t (A)$ as upper bounded by $s (t + 1) \ln \sigma_{\max} (A)$ for all $s, t \geq 0$. The conclusion is true regardless of the continuous measurement function $h(x)$ in use because the associated $K$-function in (3) can be arbitrarily defined. The proof is now complete.

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