NON-DISPERSIVE VANISHING AND BLOW UP AT INFINITY FOR THE ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^3$

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Dedicated to the memory of Vladimir Savelievich Buslaev

Abstract. We consider the energy critical focusing nonlinear Schrödinger equation

$$i\psi_t = -\Delta \psi - |\psi|^4 \psi, \quad x \in \mathbb{R}^3,$$

and prove, for any $\nu$ and $\alpha_0$ sufficiently small, the existence of radial finite energy solutions of the form

$$\psi(x,t) = e^{i\alpha(t)}\lambda^{1/2}(t) W(\lambda(t)x) + e^{i\Delta t} \zeta^* + o_{\dot{H}^1}(1)$$

as $t \to +\infty$, where $\alpha(t) = \alpha_0 \ln t$, $\lambda(t) = t^\nu$, $W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}$ is the ground state, and $\zeta^*$ is arbitrary small in $\dot{H}^1$.

1. Introduction

1.1. Setting of the problem and statement of the result. In this paper we consider the energy critical focusing nonlinear Schrödinger equation

$$i\psi_t = -\Delta \psi - |\psi|^4 \psi, \quad x \in \mathbb{R}^3,$$

(1.1)

$$\psi|_{t=0} = \psi_0 \in \dot{H}^1(\mathbb{R}^3).$$

Cauchy problem (1.1) is locally well posed and the solutions during their life span satisfy conservation of energy:

$$E(\psi(t)) \equiv \int (|\nabla \psi(x,t)|^2 - \frac{1}{3}|\psi(x,t)|^6) \, dx = E(\psi_0).$$

(1.2)

The problem is energy critical in the sense that both (1.1) and (1.2) are invariant with respect to the scaling $\psi(x,t) \to \lambda^{1/2}\psi(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}_+$. For $\dot{H}^1$ small data one has global existence and scattering. In the case of large data blow up may occur. Indeed, the classical virial identity

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(x,t)|^2 \, dx = 8 \int (|\nabla \psi(x,t)|^2 - |\psi(x,t)|^6) \, dx$$

shows that if $x\psi_0 \in L^2(\mathbb{R}^3)$ and $E(\psi_0) < 0$, then the solution breaks down in finite time.

Furthermore, Eq. (1.1) admits an explicit stationary solution (ground state):

$$W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}, \quad \Delta W + W^5 = 0,$$

so that scattering cannot always occur even for solutions that exist globally in time.
The ground state $W$ is known to play an important role in the dynamics of (1.1). It was proved by Kenig and Merle [6] that $E(W)$ is an energy threshold for the dynamics in the following sense. If $\psi_0$ is radial and $E(\psi_0) < E(W)$ then
(i) the solution of (1.1) is global and scatters to zero as a free wave in both directions, provided $\|\nabla \psi_0\|_{L^2} < \|\nabla W\|_{L^2}$;
(ii) the solution blows up in finite time in both direction, provided $\psi_0 \in L^2$ and $\|\nabla \psi_0\|_{L^2} > \|\nabla W\|_{L^2}$.

The behavior of radial solutions with critical energy $E(\psi_0) = E(W)$ was classified by Duyckaerts and Merle [5]. In this case, in addition to the finite time blow up and scattering to zero (and $W$ itself), one has the existence of solutions that converge as $t \to \infty$ to a rescaled ground state. In the case of energy slightly greater than $E(W)$ the dynamics is expected to be more rich and to include the solutions that as $t \to \infty$ behave like $e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x)$ with fairly general $\alpha(t)$ and $\lambda(t)$. For a closely related model of the critical wave equation, the existence of this type of solutions with $\lambda(t) \to \infty$ (blow up at infinity) and $\lambda(t) \to 0$, $t\lambda(t) \to \infty$ (non-dispersive vanishing) was recently proved by Donninger and Krieger [4]. Our objective in this paper is to obtain an analogous result for NLS (1.1). More precisely, we prove the following.

**Theorem 1.1.** There exists $\beta_0 > 0$ such that for any $\nu, \alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \leq \beta_0$ and any $\delta > 0$ there exist $T > 0$ and a radial solution $\psi \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ to (1.1) of the form:

$$\psi(x,t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x,t),$$

where $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ verifies:

$$\|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} \leq \delta,$$

$$\|\zeta(t)\|_{L^\infty} \leq C t^{-\frac{1+\nu}{2}},$$

$$\| -\lambda(t)x^{-1} \zeta(t)\|_{L^\infty} \leq C t^{-1-\frac{\nu}{2}}.$$

for all $t \geq T$. The constants $C$ here and below are independent of $\nu, \alpha_0$ and $\delta$.

Furthermore, there exists $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, such that, as $t \to +\infty$, $\zeta(t) - e^{it\Delta} \zeta^* \to 0$ in $\dot{H}^1 \cap \dot{H}^2$.

**Remark 1.2.** Theorem 1.1 remains valid, in fact, with $\dot{H}^2$ replaced by $\dot{H}^k$ for any $k \geq 2$ (with $\beta_0$ depending on $k$).

**Remark 1.3.** The restriction on $\nu$ and $\alpha_0$ that appears in Theorem 1.1 seems to be technical. One might expect the same result to be true for any $\nu > -1/2$ and any $\alpha_0 \in \mathbb{R}$.

**Remark 1.4.** The solutions we construct to prove the theorem belong, in fact, to $\dot{H}^{\frac{1}{2} - \nu +}$. 
**Remark 1.5.** Using the techniques developed in this paper one can prove the existence of radial finite time blow up solutions of the form \( \psi(x,t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x,t) \), \( \lambda(t) = (T - t)^{-1/2 - \nu} \), \( \alpha(t) = \alpha_0 \ln(T - t) \), where \( \zeta(t) \) is arbitrary small in \( H^1 \cap H^2 \) and \( \nu > 1 \), \( \alpha_0 \in \mathbb{R} \) can be chosen arbitrarily. For the critical wave equation an analogous result was proved by Krieger, Schlag, Tataru in [8], see also [9] for a similar construction in the context of the critical Schrödinger map equation.

1.2. Outline of the paper. The paper is organized as follows. In Section 2 we construct (Prop. 2.1) a sufficiently good approximate solution of \((1.1)\) very much in the spirit of [4], [8], [9]. In Section 3 we build up an exact solution by solving the similar construction in the context of the critical Schrödinger map equation.

2. Approximate solutions

In this section we prove the following result.

**Proposition 2.1.** For any \( \nu \) and \( \alpha_0 \) sufficiently small and any \( 0 < \delta \leq 1 \) there exists a radial approximate solution \( \psi^{ap} \in C^\infty(\mathbb{R}^3, \mathbb{R}^+_-) \) of \((1.1)\) such that the following holds for \( t \geq T \) with some \( T = T(\nu, \alpha_0, \delta) > 0 \).

(i) \( \psi^{ap} \) has the form: \( \psi^{ap}(x,t) = e^{i\alpha(t)}\lambda^{1/2}(t)(W(\lambda(t)x) + \chi^{ap}(\lambda(t)x,t)) \), where \( \chi^{ap}(y,t), y = \lambda(t)x \), verifies

\[
\begin{align*}
(2.1) \quad & \|\chi^{ap}(t)\|_{H^k} \leq C\delta^{\nu + k - 1} t^{-\nu(k-1)}, \quad k = 1, 2, \\
(2.2) \quad & \|\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-(1+2\nu)/2}, \\
(2.3) \quad & \|y|^{-1}\chi^{ap}(t)\|_{L^\infty} + \|\nabla\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-1-2\nu}, \\
(2.4) \quad & \|y|^{-2}\chi^{ap}(t)\|_{L^\infty} + \|y|^{-1}\nabla y\chi^{ap}(t)\|_{L^\infty} \leq C(\|\nu\| + |\alpha_0|)t^{-1-2\nu}, \\
(2.5) \quad & \|\nabla^2\chi^{ap}(t)\|_{L^\infty} \leq C(\|\nu\| + |\alpha_0|)t^{-1-2\nu}.
\end{align*}
\]

Furthermore, there exists \( \zeta^* \in \dot{H}^s \), for any \( s > \frac{1}{2} - \nu \), such that, as \( t \to +\infty, e^{i\alpha(t)}\lambda^{1/2}(t)\chi^{ap}(\lambda(t)t), t - e^{i\Delta}\zeta^* \to 0 \) in \( \dot{H}^1 \cap \dot{H}^2 \).

(ii) The corresponding error \( R = -i\psi_t^{ap} - \Delta \psi^{ap} - |\psi^{ap}|^4\psi^{ap} \) satisfies

\[
\begin{align*}
(2.6) \quad & \|R(t)\|_{H^k} \leq t^{-(2+\frac{1}{2})(1+2\nu)+\nu(k+1)}, \quad k = 0, 1, 2.
\end{align*}
\]

The construction of \( \psi^{ap}(t) \) will be achieved by considering separately the three regions that correspond to three different space scales: the inner region with the scale \( t^{\nu}|x| \lesssim 1 \), the self-similar region where \( |x| = O(t^{1/2}) \), and, finally, the remote region where \( |x| = O(t) \). In the inner region the solution will be constructed as a perturbation of the profile \( e^{i\alpha_0\ln(t^{1/2})}W(t^{1/2}x) \). The self-similar and remote regions are the regions where the solution is small and is described essentially by the linear equation \( i\psi_t = -\Delta \psi \). In the self-similar region the profile of the solution will be determined uniquely by
the matching conditions coming out from the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching procedure.

2.1. The inner region. We start by considering the inner region \( 0 \leq t^\nu|x| \leq 10^{t^{1/2+\nu-\epsilon_1}} \) with \( 0 < \epsilon_1 < 1/2 + \nu \) to be fixed later. Writing \( \psi(x, t) = e^{i\alpha(t)}\chi^{1/2}(t)u(\rho, t), \rho = \lambda(t)|x| \), we get from (1.1)

\[
(2.7) \quad it^{-2\nu}u_t - \alpha_0 t^{-(1+2\nu)}u + i\nu t^{-(1+2\nu)} \left( \frac{1}{2} + \rho \partial_\rho \right) u = -\triangle u - |u|^4u.
\]

Write \( u(\rho, t) = W(\rho) + \chi(\rho, t) \). Then \( \ddot{\chi}(t) = \left( \frac{\chi(t)}{\chi(t)} \right) \) solves

\[
(2.8) \quad it^{-2\nu} \ddot{\chi} = H\chi + N(\chi),
\]

where

\[
H = -\triangle \sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \sigma_3 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]

\[
N(\chi) = \left( \begin{array}{c} N(\chi) \\ -N(\chi) \end{array} \right), \quad N(\chi) = N_0 + N_1(\chi) + N_2(\chi),
\]

\[
N_0 = \alpha_0 t^{-(1+2\nu)}W - i\nu t^{-(1+2\nu)}W_1, \quad W_1(\rho) = \left( \frac{1}{2} + \rho \partial_\rho \right)W(\rho)
\]

\[
N_1(\chi) = \alpha_0 t^{-(1+2\nu)}\chi - i\nu t^{-(1+2\nu)} \left( \frac{1}{2} + \rho \partial_\rho \right)\chi,
\]

\[
N_2(\chi) = -|W + \chi|^4(W + \chi) + W^5 + 3W^4\chi + 2W^4\chi^3.
\]

We look for a solution to (2.8) of the form

\[
(2.9) \quad \chi(\rho, t) = \sum_{k=1}^{\infty} t^{-k(1+2\nu)}\chi_k(\rho).
\]

Substituting (2.9) into (2.8) and identifying the terms with the same powers of \( t \) we get the following system for \( \{\chi_k\}_{k \geq 1} \):

\[
(2.10) \quad H\chi_k = D_k, \quad k \geq 1,
\]

where \( D_k = \left( \begin{array}{c} D_k \\ -D_k \end{array} \right), \)

\[
D_1 = -\alpha_0 W + i\nu W_1,
\]

\[
D_k = D_k^{(1)} + D_k^{(2)}, \quad k \geq 2,
\]
$D_k^{(1)}$ and $D_k^{(2)}$ being contributions of $it^{-2
u}\chi_t - N_1(\chi)$ and $-N_2(\chi)$ respectively:

$$D_k^{(1)} = -i(1 + 2\nu)(k - 1)\chi_{k-1} - \alpha_0\nu\chi_{k-1} + i\nu(\frac{1}{2} + \rho\partial_\rho)\chi_{k-1},$$

$$N_2(\chi) = -\sum_{k=2}^{\infty} \rho^{-(1+2\nu)}D_k^{(2)}(\rho).$$

Note that $D_k$ depends on $\chi_p$, $1 \leq p \leq k - 1$ only:

$$D_k = D_k(\rho; \chi_p, 1 \leq p \leq k - 1).$$

We subject (2.10) to zero initial conditions at 0: $\chi_k(0) = \partial_\rho \chi_k(0) = 0$.

**Lemma 2.2.** System (2.10) has a unique solution $\{\chi_k\}_{k \geq 1}$ verifying:

i) for any $k \geq 1$, $\chi_k$ is a $C^\infty$ function that has an even Taylor expansion at $\rho = 0$ that starts at order $2k$;

ii) as $\rho \to +\infty$, $\chi_k$, $k \geq 1$, has the following asymptotic expansion

$$(2.11) \quad \chi_k(\rho) = \sum_{l=0}^{k} \sum_{j \leq 2k - 2l - 1} \alpha_{l,j}^{(k)} (\ln \rho)^l \rho^j,$$

with some coefficients $\alpha_{l,j}^{(k)}$ verifying $\alpha_{k,2m}^{(k)} = 0$ for all $k, m$. The asymptotic expansion (2.11) can be differentiated any number of times with respect to $\rho$.

**Proof.** It will be convenient for us to rewrite (2.10) as

$$(2.12) \quad L_+ v_1^+ = G_1^+, \quad L_- v_1^- = G_1^-, \quad k \geq 1,$$

where

$$v_1^+ = \text{Re} \chi_k, \quad v_1^- = \text{Im} \chi_k, \quad G_1^+ = \text{Re} D_k, \quad G_1^- = \text{Im} D_k, \quad L_+ = -\Delta - 5W_4, \quad L_- = -\Delta - W_4.$$

For $k = 1$ (2.12) gives

$$(2.13) \quad L_+ v_1^+ = -\alpha_0 W, \quad L_- v_1^- = \nu W_1.$$

The homogeneous equation $L_\pm f = 0$ has two explicit solutions $\Phi_\pm, \Theta_\pm$ given by

$$(2.14) \quad \Phi_-(\rho) = W(\rho), \quad \Theta_-(\rho) = \left(1 + \frac{\rho^2}{3}\right)^{-1/2} \left(\frac{\rho}{3} - \frac{1}{\rho}\right),$$

$$\Phi_+(\rho) = W_1(\rho), \quad \Theta_+(\rho) = -2 \left(1 + \frac{\rho^2}{3}\right)^{-3/2} \left(\frac{1}{\rho} - 2\rho + \frac{\rho^3}{9}\right).$$

Therefore, solving (2.13) with zero initial conditions at the origin we obtain

$$(2.15) \quad v_1^+(\rho) = \alpha_0 \int_0^\rho s^2(\Theta_+(\rho)\Phi_+(s) - \Theta_+(s)\Phi_+(\rho))W(s)ds,$$

$$v_1^-(\rho) = -\nu \int_0^\rho s^2(\Theta_-(\rho)\Phi_-(s) - \Theta_-(s)\Phi_-(\rho))W_1(s)ds.$$
Since \( W, W_1 \) are \( C^\infty \) even functions, \( v_1^+ \) and \( v_1^- \) are also \( C^\infty \) functions with even Taylor expansion at \( \rho = 0 \) that starts at order 2. Furthermore, the asymptotic expansions of \( v_1^+ \) and \( v_1^- \) as \( \rho \to +\infty \) can be obtained directly from (2.15). As claimed, one has

\[
v_1^+(\rho) + iv_1^-(\rho) = \sum_{j \leq 1} a_{0,j}^{(1)} \rho^j + \sum_{j \leq 0} a_{1,j}^{(1)} \rho^{2j-1} \ln \rho, \quad \text{as} \ \rho \to +\infty.
\]

We next proceed by induction. Let us consider \( k > 1 \) and assume that we have found \( \chi_i, i = 1, \cdots, k - 1, \) that verify i), ii). Then one can easily check that \( D_k \) is an even \( C^\infty \) function with a Taylor series at 0 starting at order \( 2(k - 1) \) and as \( \rho \to +\infty \), \( D_k \) admits an asymptotic expansion of the form

\[
D_k(\rho) = \sum_{i=0}^{k-1} \sum_{j \leq 2k-2l-3} d_{j,i}^{(k)}(\ln \rho)^j \rho^i + (\ln \rho)^k \sum_{j \leq -5} d_{j,k}^{(k)} \rho^i,
\]

where \( d_{-2,k-1}^{(k)} = 0 \) and \( d_{2m,k}^{(k)} = 0, \ \forall m \). Therefore, solving \( L_{\pm} v_k^\pm = G_k^\pm \) with zero conditions at \( \rho = 0 \) we get a \( C^\infty \) even solution \( v_k^\pm \) which is \( O(\rho^{2k}) \) at the origin. Finally, the asymptotic expansion at infinity follows directly from the representation

\[
v_k^\pm(\rho) = -\int_0^\rho s^{2}(\Theta_{\pm}(\rho)\Phi_{\pm}(s) - \Theta_{\pm}(s)\Phi_{\pm}(\rho)) G_k^\pm(s) ds.
\]

\[\square\]

Remark 2.3. Clearly, for any \( k, \chi_k \) is a polynomial with respect to \( \alpha_0 \) and \( \nu \) of the form

\[
\chi_k = \sum_{1 \leq m+n \leq k} \alpha_0^m \nu^n \chi_{m,n}^k(\rho),
\]

where the coefficients \( \chi_{m,n}^k \) are \( C^\infty \) functions of \( \rho \) with an even Taylor expansion at 0 that starts at order \( 2k \). As \( \rho \to +\infty \), \( \chi_{m,n}^k \) admits an asymptotic expansion of the form (2.11).

For any \( N \geq 2 \), define

\[
\chi^{(N)}(\rho, t) = \sum_{k=1}^N t^{-k(1+2\nu)} \chi_k(\rho).
\]

It follows from our construction that \( \chi^{(N)} \) verifies

\[
(2.16) \quad \left| \rho^{-k} \partial_\rho \left( -it^{-2\nu} \partial_t^{\alpha^{(N)}(\rho)} + H^{\alpha^{(N)}(\rho)} + \mathcal{N}(\chi^{(N)}) \right) \right| \leq C_{N,l,k} t^{-(N+1)(1+2\nu)} < \rho^{2N-1-l-k},
\]

for any \( k, l \in \mathbb{N}, k + l \leq 2N, 0 \leq \rho \leq 10t^{\frac{l}{2}+\nu-\epsilon}, t \geq 1 \).
Fix $N = 27$, $\epsilon_1 = \frac{1+2\nu}{2t}$, and set

$$u_{in}^{ap} = W + \chi_{in}^{ap}, \quad \chi_{in}^{ap} = \chi^{(27)},$$

$$\mathcal{R}_{in} = -it^{-2\nu} \partial_t u_{in}^{ap} - \Delta u_{in}^{ap} + \alpha_0 t^{-1-2\nu} u_{in}^{ap} - i \nu t^{-1-2\nu} (\frac{1}{2} + \rho \partial_t) u_{in}^{ap} - |u_{in}^{ap}|^4 u_{in}^{ap}.$$ 

As a direct consequence of Lemma 2.2 and estimate (2.16), we obtain the following result.

**Lemma 2.4.** For any $\alpha_0 \in \mathbb{R}$ and any $\nu > -\frac{1}{2}$ there exists $T = T(\alpha_0, \nu) > 0$ such that for $t \geq T$ the following holds.

(i) The profile $\chi_{in}^{ap}(t)$ verifies

$$\|\chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10 t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|)t^{-\frac{1}{2}-\nu},$$

$$\|\rho^{-k} \partial_t \chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10 t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}, \quad 1 \leq k \leq l \leq 2,$$

(ii) The error $\mathcal{R}_{in}(t)$ admits the estimate

$$\|\rho^{-k} \partial_t \mathcal{R}_{in}(t)\|_{L^2(\rho^2 d\rho, 0 \leq \rho \leq 10 t^{\frac{1}{2}+\nu-\epsilon_1})} \leq t^{-3(1+2\nu)/4-\epsilon_1(2N+1/2), \quad k \leq l \leq 2}.$$

2.2. **The self-similar region.** We next consider the self-similar region $\frac{1}{10} t^{-\epsilon_1} \leq |x| t^{-1/2} \leq 10 t^{\epsilon_2}$, where $0 < \epsilon_2 < 1/2$ to be fixed later. Write $\psi(x, t) = e^{i \alpha_0 \ln t^{-1/4} w(y, t), \quad y = t^{-1/2} |x|}$. Then, $w(t)$ solves

$$itw_t = (L + \alpha_0)w - |w|^4 w,$$

where $L = -\triangle + \frac{\rho}{2} (\frac{1}{2} + y \partial_y)$.

Note that in the limit $\rho \to +\infty$, $y \to 0$ one has, at least, formally

$$t^{\nu/2}(W(\rho) + \sum_{k \geq 1} t^{-k(1+2\nu)} \chi_k(\rho)) =$$

$$t^{-1/4} \sum_{n \geq 0} \sum_{0 \leq t \leq \frac{n}{4}} t^{-\frac{3}{4}(2n+1)(1+2\nu)} (\ln y + (\frac{1}{2} + \nu) \ln t)^t \alpha_{l,2k-n-1}^{(k)} y^{2k-n-1},$$

where $\alpha_{l,j}^{(k)}, \quad k \neq 0$, are given by Lemma 2.2 and $\alpha_{l,j}^{(0)}$ come from the expansion of $W(\rho)$ as $\rho \to \infty$:

$$W(\rho) = \sum_{j \neq 0} \alpha_{0,j}^{(0)} \rho^j, \quad \alpha_{0,2m}^{(0)} = 0 \quad \forall m \in \mathbb{Z}.$$  

\footnote{This choice has no specific meaning here. To produce an approximate solution with an error verifying (2.20) it is sufficient to require $(2N+3)\epsilon_1 > 3(1+2\nu)/2, \quad 0 < \epsilon_1 < \frac{1+2\nu}{2t}$, see (2.20) and (2.21).}
Lemma 2.5. There exists a unique solution of (2.23) as a formal asymptotic expansion of the form
\[ w(y, t) = \sum_{n \geq 0} \sum_{0 \leq l \leq \frac{n}{2}} t^{-\frac{1}{2}(2n+1)(1+2\nu)} (\ln y + \left(\frac{1}{2} + \nu\right) \ln t)^l A_{n,l}(y). \]

As it will become clear later, to prove Proposition 2.1, it is sufficient to consider only three first terms of expansion (2.23). Therefore, we look for an approximate solution of the form
\[ w_{ss}^{ap}(y, t) = t^{-(1+2\nu)/4} A_{0,0}(y) + t^{-3(1+2\nu)/4} A_{1,0}(y) + t^{-5(1+2\nu)/4} (A_{2,0}(y) + (\ln y + \left(\frac{1}{2} + \nu\right) \ln t) A_{2,1}(y)). \]

Substituting this ansatz into the expression \( -it w_{t} + (\mathcal{L} + \alpha_0) w - |w|^4 w \) one gets
\[ -it \partial_y w_{ss}^{ap} + (\mathcal{L} + \alpha_0) w_{ss}^{ap} - |w_{ss}^{ap}|^4 w_{ss}^{ap} = t^{-(1+2\nu)/4} S_{0,0}(y) + t^{-3(1+2\nu)/4} S_{1,0}(y) + t^{-5(1+2\nu)/4} (S_{0,0}(y) + (\ln y + \left(\frac{1}{2} + \nu\right) \ln t) S_{2,1}(y)) + S(y, t), \]
where
\[
\begin{align*}
S_{n,0}(y) &= (\mathcal{L} + \mu_n) A_{n,0}(y), \quad n = 0, 1, \\
S_{2,1}(y) &= (\mathcal{L} + \mu_2) A_{2,1}(y), \\
S_{2,0}(y) &= (\mathcal{L} + \mu_2) A_{2,0}(y) - i\nu A_{2,1}(y) - \frac{2}{y} \partial_y A_{2,1}(y) - \frac{A_{2,1}(y)}{y^2} - |A_{0,0}(y)|^4 A_{0,0}(y), \\
S(y, t) &= -|w_{ss}^{ap}(y, t)|^4 w_{ss}^{ap}(y, t) + t^{-5(1+2\nu)/4} |A_{0,0}(y)|^4 A_{0,0}(y).
\end{align*}
\]

Here \( \mu_n = \alpha_0 + \frac{\nu}{2}(2n + 1)(1 + 2\nu) \).

We require that \( S_{n,l} = 0, n = 0, 1, 2, l = 0, 1, \) which means that the corresponding \( A_{n,l} \) have to solve
\[
\begin{cases}
(\mathcal{L} + \mu_n) A_{n,0} = 0, & n = 0, 1, \\
(\mathcal{L} + \mu_2) A_{2,1} = 0, \\
(\mathcal{L} + \mu_2) A_{2,0} = i\nu A_{2,1} + \frac{2}{y} \partial_y A_{2,1} + \frac{A_{2,1}}{y^2} + |A_{0,0}|^4 A_{0,0}.
\end{cases}
\]

In addition, in order to have the matching with the inner region, \( A_{n,l} \) have to satisfy
\[
A_{n,l}(y) = \sum_{k \geq l} \alpha_{l,2k-n-1}^{(k)} y^{2k-n-1}, \quad y \to 0.
\]

Lemma 2.5. There exists a unique solution of (2.25) that as \( y \to 0 \) admits an asymptotic expansion of the form
\[
A_{n,l}(y) = \sum_{k \geq l} d_{n,k,l} y^{2k-n-1},
\]
with \( d_{0,0,0} = \alpha_{0,-1}^{(0)}, d_{1,1,0} = \alpha_{0,0}^{(1)} \) and \( d_{2,1,0} = \alpha_{0,-1}^{(1)} \).
Proof. First of all note that the equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $e_1(y, \mu)$, $e_2(y, \mu)$ such that:

(i) $e_1(y, \mu) = \frac{1}{y} + (\mu - \frac{i}{4})\tilde{e}_1(y, \mu)$, where $\tilde{e}_1$ is an entire function of $y$ and $\mu$, odd with respect to $y$;

(ii) $e_2$ is a entire function of $y$ and $\mu$, even with respect to $y$, and as $y \to 0$, $e_2(y, \mu) = 1 + O(y^2)$.

Two first equations of (2.25) together with (2.27) give

$$A_{0,0}(y) = \alpha_{0,0}^{(0)}e_1(y, \mu_0), \quad A_{1,0}(y) = \alpha_{0,0}^{(1)}e_2(y, \mu_1).$$

We next consider the remaining equations of (2.25). Equation $(\mathcal{L} + \mu_2)A_{2,1}(y) = 0$ and (2.27) yield $A_{2,1}(y) = c_0e_1(y, \mu_2)$, with some constant $c_0$. Then, for $A_{2,0}$ we have $(\mathcal{L} + \mu_2)A_{2,0} = F$, where

$$F = c_0(i\nu + \frac{2}{y}\partial_y + y^{-2})e_1(y, \mu_2) + |A_{0,0}|^4 A_{0,0}.$$

As $y \to 0$, $F$ has an asymptotic expansion of the form

$$F(y) = \sum_{i \geq -2} \kappa_i y^{2i-1},$$

with some coefficients $\kappa_i$, $\kappa_{-2}$ and $\kappa_{-1} + c_0$ being independent of $c_0$.

Write $A_{2,0}(y) = -\frac{\kappa_{-2}}{6y^2} + \tilde{A}_{2,0}(y)$. Then $\tilde{A}_{2,0}$ solves

$$(\mathcal{L} + \mu_2)\tilde{A}_{2,0} = \tilde{F},$$

where $\tilde{F} = F + \frac{\kappa_{-2}}{6}(\mathcal{L} + \mu_2)\frac{1}{y^2}$ has the following asymptotics as $y \to 0$:

$$\tilde{F}(y) = \sum_{i \geq -1} \tilde{\kappa}_i y^{2i-1}, \quad \tilde{\kappa}_{-1} = \tilde{\kappa}_{-1} = \kappa_{-1} - c_0,$$

with $\tilde{\kappa}_{-1}$ independent of $c_0$. Take $c_0 = \tilde{\kappa}_{-1}$. Then Eq. (2.29) has a unique solution of the form

$$\tilde{A}_{2,0}(y) = \alpha_{0,0}^{(1)}e_1(y, \mu_2) + C^\infty \text{odd function}.$$

□

Remark 2.6. By uniqueness, $A_{n,l}$ given by Lemma 2.5 verify matching conditions (2.26). Note also that all $A_{n,l}$ are entire functions of $\alpha_0$ and $\nu$.

We next study the behavior of $A_{n,l}$ as $y \to +\infty$. To this purpose notice that for any $\mu \in \mathbb{C}$, equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $f_1(y, \mu)$, $f_2(y, \mu)$ such that $yf_1$, $yf_2$ are smooth functions in both variables and as $y \to +\infty$ one has

$$f_1(y, \mu) = y^{-1/2 + 2\mu}(1 + O(y^{-2})), \quad f_2(y, \mu) = e^{\frac{i2}{4}y^{-5/2 - 2\mu}}(1 + O(y^{-2})).$$

These asymptotics are uniform in $\mu$ on compact subsets of $\mathbb{C}$ and can be differentiated any number of times with respect to $y$. 

\footnotetext{\textsc{Energy Critical 3D NLS} 9}
Decomposing $A_{1,0}$, $A_{2,0}$, $A_{2,1}$ in the basis $f_1$, $f_2$ one gets

$$A_{n,0}(y) = d_1^n f_1(y, \mu_n) + d_2^n f_2(y, \mu_n), \quad n = 0, 1,$$

$$A_{2,1}(y) = d_1^2 f_1(y, \mu_2) + d_2^2 f_2(y, \mu_2),$$

with some coefficients $d_j^n$, $j = 1, 2$, $n = 0, 1, 2$. As a consequence, as $y \to +\infty$, one has

$$A_{0,0}(y) = d_1^0 y^{2i\alpha_0 - 1 + \nu}(1 + O(y^{-2})) + d_2^0 y^{2i\alpha_0 - 2 + \nu}(1 + O(y^{-2})),
$$

$$A_{1,0}(y) = d_1^1 y^{2i\alpha_0 - 3 + \nu}(1 + O(y^{-2})) + d_2^1 y^{2i\alpha_0 - 1 + \nu}(1 + O(y^{-2})),
$$

$$A_{2,1}(y) = d_1^2 y^{2i\alpha_0 - 5 + \nu}(1 + O(y^{-2})) + d_2^2 y^{2i\alpha_0 + \nu}(1 + O(y^{-2})).$$

Asymptotics (2.32) can be differentiated any number of times with respect to $y$.

Let us now consider $A_{2,0}$ and write it as

$$A_{2,0}(y) = 2d_1^2 \nu y f_1(y, \mu_2) - (2 + 1) d_2^2 \ln y f_2(y, \mu_2) + \tilde{A}_{2,0}(y).$$

Then $\tilde{A}_{2,0}(y)$ solves

$$\left(\mathcal{L} + \mu_2\right) \tilde{A}_{2,0} = G,$$

with $G = d_2^2 G_1 + G_2$, where

$$G_1 = -d_2^2 (1 + 2\nu)(2y^{-1} \partial_y + y^{-2} - i) f_2(y, \mu_2),
$$

$$G_2 = |A_{0,0}|^4 A_{0,0} + d_2^2 (1 + 2\nu)(2y^{-1} \partial_y + y^{-2}) f_1(y, \mu_2).$$

It follows from the asymptotics (2.30), (2.32) that $G_j$, $j = 1, 2$, has the following behavior as $y \to +\infty$,

$$G_1(y) = e^{iy^2/4} y^{-2i\alpha_0} G_{1,1}(y),
$$

$$G_2(y) = \sum_{m=-2}^{3} e^{imy^2/4} y^{-2i\alpha_0(2m-1)} G_{2,m}(y),$$

$$\partial^l_y G_{1,1}(y) = O(y^{-2+5\nu-l}),
$$

$$\partial^l_y G_{2,m}(y) = O(y^{-5-5\nu-m(1-2\nu)-l}), \quad -2 \leq m \leq 3,$$

for any $l \geq 0$, provided $\nu$ is sufficiently small.

Integrating (2.31) one gets

$$\tilde{A}_{2,0}(y) = \lambda_1 f_1(y, \mu_2) + \lambda_2 f_2(y, \mu_2) + d_2^1 g_1(y) + g_2(y).$$

Here $\lambda_i$, $i = 1, 2$, is a constant and $g_i$, $i = 1, 2$, is the solution of $(\mathcal{L} + \mu_2) g_i = G_i$, with the following behavior as $y \to +\infty$:

$$g_1(y) = e^{iy^2/4} y^{-2i\alpha_0} g_{1,1}(y),
$$

$$g_2(y) = \sum_{m=-2}^{3} e^{imy^2/4} y^{-2i\alpha_0(2m-1)} g_{2,m}(y),$$

$$\partial^l_y g_{1,1}(y) = O(y^{-2+5\nu-l}),
$$

$$\partial^l_y g_{2,m}(y) = O(y^{-7-5\nu-m(1-2\nu)-l}), \quad m = 0, 1
$$

for any $l \geq 0$. 

---

From the above, it follows that as $y \to +\infty$,

$$\tilde{A}_{2,0}(y) = O(y^{-2+5\nu}),$$

$$A_{2,0}(y) = O(y^{-2+5\nu}),$$

$$A_{2,1}(y) = O(y^{-5-5\nu-m(1-2\nu)-l}),$$

for any $l \geq 0$. 

---

The asymptotics are now proved.
Denote
\[ u_{ss}^{ap}(\rho, t) = t^{-(1+2\nu)/4} W_{ss}(t^{-(1+2\nu)/2}\rho, t), \]
\[ \chi_{ss}^{ap}(\rho, t) = u_{ss}^{ap}(\rho, t) - W(\rho), \]
\[ \mathcal{R}_{ss}(\rho, t) = t^{-5(1+2\nu)/4} S(t^{-(1+2\nu)/2}\rho, t). \]

The next lemma is a direct consequence of (2.26), (2.30), (2.32), (2.33), (2.35) and (2.36).

**Lemma 2.7.** For any \(\alpha_0, \nu \in \mathbb{R}\) sufficiently small there exists \(T(\alpha_0, \nu) > 0\) such that for \(t \geq T(\alpha_0, \nu)\) the following holds.

(i) \(\chi_{ss}^{ap}(t)\) verifies
\[ \|\chi_{ss}^{ap}(t)\|_{L^{\infty} \left( \frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu+\varepsilon_2} \right)} \leq Ct^{-\frac{1}{2} - \nu}, \]
\[ \|\rho^{-k} \partial^j_\rho \chi_{ss}^{ap}(t)\|_{L^{\infty} \left( \frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu+\varepsilon_2} \right)} \leq Ct^{-1-2\nu}, \quad k + l = 1, \]
\[ \|\rho^{-k} \partial^j_\rho \chi_{ss}^{ap}(t)\|_{L^{\infty} \left( \frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu+\varepsilon_2} \right)} \leq C(||\alpha_0|| + |\nu|)t^{-1-2\nu}, \quad k + l = 2, \]
\[ \|\rho^{-k} \partial^j_\rho \chi_{ss}^{ap}(t)\|_{L^{2} \left( \rho^2 dp, \frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu+\varepsilon_2} \right)} \leq Ct^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad 1 \leq k + l \leq 2, \]

(ii) The error \(\mathcal{R}_{ss}(t)\) admits the estimate
\[ \|\rho^{-k} \partial^j_\rho \mathcal{R}_{ss}(t)\|_{L^{2} \left( \rho^2 dp, \frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu+\varepsilon_2} \right)} \leq Ct^{-(2+\frac{1}{2})(1+2\nu)+5\varepsilon_1/2}, \quad 0 \leq k + l \leq 2. \]

(iii) The difference \(u_{ss}^{ap}(\rho, t) - u_{ss}^{ap}(\rho, t)\) verifies
\[ |\partial^l_\rho (u_{ss}^{ap}(t) - u_{ss}^{ap}(t))| \leq C\rho^{2-l}t^{-(1+2\nu)}(\ln t + t^{3(1+2\nu)/2 - (2N+3)\varepsilon_1}), \]
for any \(l \geq 0\) and \(\frac{1}{10} t^{1+\nu-1} \leq \rho \leq 10^{-1+\nu-1}.\)

2.3. **The remote region.** We next consider the remote region \(|x| \geq \frac{1}{10} t^{1+\varepsilon_2}\). In this region we take as an approximate solution to (1.1) the following radial profile:
\[ v_{ss}^{ap}(x, t) = v_1(x, t) + v_2(x, t) + v_3(x, t), \]
where
\[ v_1(x, t) = e^{i\alpha_0 \ln t} \left[ d_1 t^{-(1+\nu)/2} f_1(y, \mu_0) + d_1 t^{-(2+3\nu)/2} f_1(y, \mu_1) \right], \quad y = t^{-1/2}|x|, \]
\[ v_2(x, t) = \Theta_\delta \left( \frac{\xi}{t} \right) e^{i\alpha_0 \ln t} \left[ d_2 t^{-(1+\nu)/2} f_2(y, \mu_0) + d_2 t^{-(2+3\nu)/2} f_2(y, \mu_1) + t^{-(3+5\nu)/2} \left( d_2 g_1(y) - (d_2^2 (2\nu + 1) \ln \left( \frac{|x|}{t} \right) - \lambda_2) f_2(y, \mu_2) \right) \right], \]
\[ \Theta_\delta(\xi) = \Theta \left( \frac{\xi}{t} \right), \quad \Theta \in C_0^\infty(\mathbb{R}^3) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}, \]
Finally, \(v_3(x, t)\) is given by
\[ v_3(x, t) = \frac{e^{i|x|^2/4t}}{t^{5/2}} \tilde{v}_3 \left( \frac{x}{t} \right), \quad \tilde{v}_3 = -i z \Delta \Theta_\delta - 2i \nabla z \cdot \nabla \Theta_\delta, \]
where
\[ z(\xi) = d^2_0 |\xi|^{-2i\alpha_0-2+\nu} + d^1_0 |\xi|^{-2i\alpha_0-1+3\nu} - (d^2_2 (2\nu + 1) \ln |\xi| - \lambda_2) |\xi|^{-2i\alpha_0+5\nu}. \]

It follows from the asymptotics (2.30) that for \( t \geq T \) with some \( T = T(\delta) > 0 \) and any \( l \geq 0 \), one has
\[
|\nabla^l v_1(x, t)| \leq C_l |x|^{-l-1-\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x|,
\]
\[
|\nabla^l v_2(x, t)| \leq \frac{C_l}{t^{3/2}} \left| \frac{x}{t} \right|^{-l-2+\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 2\delta t.
\]
Furthermore, \( v_2 \) can be written as
\[
v_2(x, t) = v_{2,0}(x, t) + v_{2,1}(x, t),
\]
\[
v_{2,0}(x, t) = \frac{e^{-i|\nabla|^2/4}}{t^{3/2}} \Theta_\delta \left( \frac{x}{t} \right) z \left( \frac{x}{t} \right), \quad v_{2,1}(x, t) = \frac{e^{-i|\nabla|^2/4}}{t^{3/2}} \Theta_\delta \left( \frac{x}{t} \right) \hat{v}_{2,1}(x, t),
\]
with \( \hat{v}_{2,1} \) verifying, for any \( l \geq 0 \),
\[
|\nabla^l \hat{v}_{2,1}(x, t)| \leq C_l t^{3-\nu} |x|^{-l-4+\nu}, \quad \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 2\delta t.
\]
We next address \( v_3 \). One has
\[
\|\nabla^l v_3(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C_l t^{-5/2} \delta^{-4+l+\nu},
\]
\[
\|\nabla^l v_3(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C_l t^{-1} \delta^{-5/2+l+\nu},
\]
for any \( l \geq 0 \) and \( t \geq T(\delta) \).

As a direct consequence of estimates (2.43), (2.45), (2.46), one obtains
\[
\|\psi_{\text{out}}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-\left(\frac{1}{2}+\varepsilon_2\right)(1+\nu)},
\]
\[
\|x^{-1}\psi_{\text{out}}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-5/4},
\]
\[
\|\nabla^l \psi_{\text{out}}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-5/4}, \quad l = 1, 2,
\]
\[
\|\nabla^l \psi_{\text{out}}^{ap}(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C \delta^{\nu+l-1/2}, \quad l = 1, 2,
\]
\[
\|\nabla^l (\psi_{\text{out}}^{ap}(t) - v_{2,0}(t))\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)}, \quad l = 1, 2,
\]
\[
\|x^{-1}(\psi_{\text{out}}^{ap}(t) - v_{2,0}(t))\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)},
\]
provided \( \delta \leq \varepsilon_2 < \frac{1}{2}, \nu \) is sufficiently small and \( t \geq T(\delta) \).

Denote
\[
\psi_{\text{ap}}(x, t) = e^{i\alpha_0 \ln t^{1/4}} \psi_{\text{ap}}^{ap}(t^{-1/2} |x|, t),
\]
and consider the difference \( \psi_{\text{ap}}(x, t) - \psi_{\text{out}}^{ap}(x, t) \). For \( \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 10 t^{1/2+\varepsilon_2} \) one has
\[
\psi_{\text{ap}}(x, t) - \psi_{\text{out}}^{ap}(x, t) = e^{i\alpha_0 \ln t^{1/4}} (d^2_1 (1 + 2\nu) \ln |x| + \lambda_1) f_1(y, \mu_2) + g_2(y),
\]
which together with (2.30) and (2.36) implies that
\begin{equation}
\|\nabla^l (\psi_{out}^{ap} - \psi_{ss}^{ap})\| \leq C_l ([\ln t]t^{-\frac{1}{5}}) + \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}, \text{ provided } \nu \text{ is sufficiently small.}
\end{equation}
for any \( l \geq 0 \) and \( \frac{1}{10} t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2} \), provided \( \nu \) is sufficiently small.

We next analyze the error \( R_{out}(t) = -i\partial_t \psi_{out}^{ap}(t) - \Delta \psi_{out}^{ap}(t) - |\psi_{out}^{ap}(t)|^4 \psi_{out}^{ap}(t) \). It has the form
\begin{equation}
R_{out}(x, t) = -\frac{e^{-i|x|^2}}{t^{9/2}} \left[ t \hat{v}_{2,1}(x, t) \Delta \Theta_{\delta} \left( \frac{x}{t} \right) + 2t^2 \nabla \hat{v}_{2,1}(x, t) \cdot \nabla \Theta_{\delta} \left( \frac{x}{t} \right) + \Delta \hat{v}_3 \left( \frac{x}{t} \right) \right] - |\psi_{out}^{ap}|^4 \psi_{out}^{ap}.
\end{equation}
Combined with (2.43), (2.45), (2.46), representation (2.50) gives for \( \frac{3}{8} \leq \varepsilon_2 < \frac{1}{2} \) and \( \nu \) sufficiently small,
\begin{equation}
\|\nabla^l R_{out}(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2+\varepsilon_2})} \leq C t^{-\frac{3}{2}(1+2\nu)}, \quad t \geq T(\delta), \quad l = 0, 1, 2.
\end{equation}

2.4. Proof of Proposition 2.1
We are now in position to conclude the proof of Prop. 2.1. Fix \( \varepsilon_2 \) such that \( \frac{3}{8} \leq \varepsilon_2 < \frac{1}{2} \) and consider the radial profile \( \psi^{ap}(x, t) \) defined by
\begin{equation}
\psi^{ap}(x, t) = \Theta(t^{-1/2+\varepsilon_1}x)\psi_{in}^{ap}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)\psi_{ss}^{ap}(x, t)
+ (1 - \Theta(t^{-1/2-\varepsilon_2}x))\psi_{out}^{ap}(x, t), \quad x \in \mathbb{R}^3,
\end{equation}
where \( \psi_{in}^{ap}(x, t) = e^{it\ln t^{\nu}2u_{in}^{ap}(t|x|, t)} \). Write \( \psi^{ap} \) as \( \psi^{ap}(x, t) = e^{i\alpha_0 \ln t^{\nu/2}(W(y) + \chi^{ap}(y, t))}, \quad y = t^\nu x \). By Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39)) and (2.47) one has
\begin{align}
\|\chi^{ap}(t)\|_{L^\infty} &\leq C t^{-(1+2\nu)/2} \\
\|y^{-1} \chi^{ap}(t)\|_{L^\infty} + \|\nabla \chi^{ap}(t)\|_{L^\infty} &\leq C t^{-1-2\nu}, \\
\|y^{-2} \chi^{ap}(t)\|_{L^\infty} + \|y^{-1} \nabla y \chi^{ap}(t)\|_{L^\infty} &\leq C(|\nu| + |\alpha_0|) t^{-1-2\nu}, \\
\|\nabla^2 \chi^{ap}(t)\|_{L^\infty} &\leq C(|\nu| + |\alpha_0|) t^{-1-2\nu}.
\end{align}
All the estimates stated in this subsection are valid for \( \nu \) sufficiently small and \( t \geq T(\alpha_0, \nu, \delta) \).

Futhermore, it follows from Lemma 2.3 (estimate (2.19)), Lemma 2.7 (estimate (2.39)) and two last inequalities in (2.47) that
\begin{equation}
\|\nabla^l \chi^{ap}(t)\|_{L^2(|y| \leq 10t^{1/2+\varepsilon_2})} \leq C t^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad l = 1, 2,
\end{equation}
\begin{equation}
\|\nabla^l (\chi^{ap}(t) - \chi_0^{ap}(t))\|_{L^2(|y| \geq t^{1/2+\varepsilon_2})} \leq C t^{-(1+2\nu)/4}, \quad l = 1, 2,
\end{equation}
where \( \chi_0^{ap}(y, t) = e^{-i\alpha_0 \ln t^{\nu/2}v_{2,0}(t^{-\nu}y, t)} \).

Inequalities (2.56) imply, in particular,
\begin{equation}
\|\nabla^l \chi^{ap}(t)\|_{L^2(\mathbb{R}^3)} \leq C t^{-\nu(l-1)} \delta^{l+1/2}, \quad l = 1, 2.
\end{equation}
Moreover, introducing $\zeta^*(x) = \pi^{-3/2} e^{3i\pi/4} \int_{\mathbb{R}^3} d\xi \xi \Theta_\delta(2\xi) \zeta(2\xi)$ and observing that $\zeta^* \in \dot{H}^s(\mathbb{R}^3)$ for any $s > 1/2 - \nu$, and $\|\nabla^l (v_{2,0} - e^{i\Delta t} \zeta^*)\|_{L^2(|x| \geq t^\nu)} \to 0$ as $t \to +\infty$ for any $\gamma > \frac{1-2\nu}{1-2\nu}$ and any $l \geq 1$, one obtains that

$$e^{i\alpha(t)} \lambda^{1/2}(t) \chi_{ap}(\lambda(t) \cdot t) - e^{i\alpha} \zeta^* \to 0 \text{ in } \dot{H}^1 \cap \dot{H}^2 \text{ as } t \to +\infty.$$ 

This concludes the proof of the first part of Prop. 2.1.

We next consider the error $R = -i\psi_t^{ap} - \Delta \psi^{ap} - |\psi^{ap}|^4 \psi^{ap}$. It has the form

$$R = E_1 + E_2 + E_3 + E_4.$$ 

where

$$E_1 = \left(\frac{1}{2} - \varepsilon_1\right) t^{-1} (\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t)) \bar{\Theta}(t^{-1/2+\varepsilon_1} x)$$

$$- 2t^{-1/2+\varepsilon_1} (\nabla \psi_{in}^{ap}(x, t) - \nabla \psi_{ss}^{ap}(x, t)) \cdot \nabla \Theta(t^{-1/2+\varepsilon_1} x)$$

$$- t^{-1+2\varepsilon_1} (\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t)) \Delta \Theta(t^{-1/2+\varepsilon_1} x),$$

$$E_2 = \left(\frac{1}{2} + \varepsilon_2\right) t^{-1} (\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t)) \bar{\Theta}(t^{-1/2-\varepsilon_2} x)$$

$$- 2t^{-1/2-\varepsilon_2} (\nabla \psi_{ss}^{ap}(x, t) - \nabla \psi_{out}^{ap}(x, t)) \cdot \nabla \Theta(t^{-1/2-\varepsilon_2} x)$$

$$- t^{-1-2\varepsilon_2} (\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t)) \Delta \Theta(t^{-1/2-\varepsilon_2} x),$$

$$\bar{\Theta}(\xi) = \xi \cdot \nabla \Theta(\xi),$$

and $E_3, E_4$ are given by

$$E_3 = \Theta(t^{-1/2+\varepsilon_1} x) R_{in}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1} x)) \Theta(t^{-1/2-\varepsilon_2} x) R_{ss}(x, t)$$

$$+ (1 - \Theta(t^{-1/2-\varepsilon_2} x)) R_{out}(x, t),$$

$$E_4 = \Theta(t^{-1/2+\varepsilon_1} x) (|\psi_{in}^{ap}|^4 \psi_{in}^{ap} - |\psi^{ap}|^4 \psi^{ap})$$

$$+ (1 - \Theta(t^{-1/2+\varepsilon_1} x)) \Theta(t^{-1/2-\varepsilon_2} x) (|\psi_{ss}^{ap}|^4 \psi_{ss}^{ap} - |\psi^{ap}|^4 \psi^{ap})$$

$$+ (1 - \Theta(t^{-1/2-\varepsilon_2} x)) (|\psi_{out}^{ap}|^4 \psi_{out}^{ap} - |\psi^{ap}|^4 \psi^{ap}).$$

Here

$$R_{in}(x, t) = e^{i\alpha_0 \ln t} R_{in}(t^{\nu}|x|, t), \quad R_{ss}(x, t) = e^{i\alpha_0 \ln t} R_{ss}(t^{\nu}|x|, t).$$

First we adress $E_1$. By Lemma 2.7 (iii) we have

$$\|E_1\|_{H^2} \leq C t^{-9(1+2\nu)/4+\nu+5\varepsilon_1/2} \ln t \leq C t^{-(2+\frac{3}{2\nu})(1+2\nu)}.$$  

Similarly, from (2.49) we get for $E_2$:

$$\|E_2\|_{H^2} \leq C t^{-(\frac{1}{2}+\varepsilon_2)(\frac{7}{2}+5\nu) \ln t} \leq C t^{-(2+\frac{1}{2})(1+2\nu)}.$$  

Next, we consider $E_3$. From Lemma 2.4 (ii), Lemma 2.7 (ii) and (2.51) it is apparent that

$$\|E_3\|_{H^2} \leq C t^{-\frac{3}{2}(1+2\nu)+5\varepsilon_1/2} \leq C t^{-(2+\frac{3}{2\nu})(1+2\nu)}.$$
Finally, applying Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39), (2.42)) and (2.47), (2.49), it is not difficult to check that
\begin{equation}
\|E_4\|_{H^2} \leq C t^{-3(1+2\nu)}.
\end{equation}
Combining (2.57), (2.58), (2.59), (2.60), we get (2.6), which concludes the proof of Prop. 2.1.

3. Construction of an exact solution

We are now in position to prove Theorem 1.1. Consider (1.1) and write \( \psi(x,t) = e^{i\alpha_0 \ln t^{1/2}} U(y,\tau) \), where \( y = t^{\nu/2} x \) and \( \tau = \frac{t^{1/2\nu}}{1+\nu} \). Further decomposing \( U \) as \( U(y,\tau) = U^{ap}(y,\tau) + f(y,\tau) \), \( U^{ap}(y,\tau) = e^{-i\alpha_0 \ln t^{1/2}} \psi^{ap}(x,t) \), where \( \psi^{ap} \) is the approximate solution of (1.1) given by Prop. (2.1), we get the following equation for the remainder \( f \)
\begin{equation}
i f = H(\tau) \bar{f} + F(f) + r, \quad \bar{f} = \left( \begin{array}{c} f \\ \bar{f} \end{array} \right),
\end{equation}
where
\[
\mathcal{H}(\tau) = H + \tau^{-1} L,
\]
\[
H = -\Delta \sigma_3 - 3W^4 \sigma_3 - 2W^4 \sigma_3 \sigma_1, \quad L = \frac{\alpha_0}{2\nu+1} \sigma_3 - i \frac{\nu}{2\nu+1} \left( \frac{1}{2} + y \cdot \nabla \right),
\]
\[
F(f) = \left( \begin{array}{c} F(f) \\ -F(f) \end{array} \right), \quad F(f) = F_1(f) + F_2(f)
\]
\[
F_1(f) = \mathcal{V}_1(\tau) f + \mathcal{V}_2(\tau) \bar{f},
\]
\[
\mathcal{V}_1(\tau) = 3(W^4 - |U^{ap}(\tau)|^4), \quad \mathcal{V}_2(\tau) = 2(W^4 - (U^{ap}(\tau))^2 |U^{ap}(\tau)|^2),
\]
\[
F_2(f) = -|U^{ap} + f|^4 (U^{ap} + f) + |U^{ap}|^4 |U^{ap} + 3|U^{ap}|^4 f + 2(U^{ap})^2 |U^{ap}|^2 \bar{f},
\]
\[
r = \left( \begin{array}{c} r \\ -r \end{array} \right), \quad r(y,\tau) = t^{-5\nu/2} e^{-i\alpha_0 \ln t} R(x,t).
\]

Our intention is to solve (3.1) with zero condition at \( \tau = +\infty \) by a fix point argument. To carry out this analysis we will need some energy type estimates for the linearized equation \( i f = \mathcal{H}(\tau) \bar{f} \). The required estimates are collected in the next subsection, their proofs being removed to Section 4.
3.1. Linear estimates. We start by recalling some basic spectral properties of the operator $H$ (a more detailed discussion and the proofs can be found, for example, in [5]). Since we are considering only radial solutions, we will view $H$ as an operator on $L^2_{rad}(\mathbb{R}^3; \mathbb{C}^2)$ with domain $D(H) = H^2_{rad}(\mathbb{R}^3; \mathbb{C}^2)$. $H$ satisfies the relations

$$\sigma_3 H \sigma_3 = H^*, \quad \sigma_1 H \sigma_1 = -H.$$ 

The essential spectrum of $H$ fills up the real axis. The discrete spectrum of $H$ consists of two simple purely imaginary eigenvalues $i\lambda_0$, $-i\lambda_0$, $\lambda_0 > 0$. The corresponding eigenfunctions $\zeta_+, \zeta_-$ are in $\mathcal{S}(\mathbb{R}^3)$ and can be chosen in such a way that $\zeta_- = \sigma_1 \zeta_+ = \bar{\zeta}_+$. Notice also that $HW(\frac{1}{-1}) = HW_1(\frac{1}{1}) = 0$, which means that $H$ has a resonance at zero.

Consider the projection of the linearized equation $i\vec{f}_\tau = \mathcal{H}(\tau) \vec{f}$ onto the essential spectrum of $H$:

$$i\vec{f}_\tau = PH(\tau)P\vec{f}.$$ 

Here $P$ is the spectral projection of $H$ onto the essential spectrum given by

$$P = I - P_+ - P_-, \quad P_\pm = \frac{\langle \cdot, \sigma_3 \zeta_\pm \rangle}{\langle \zeta_\pm, \sigma_3 \zeta_\pm \rangle} \zeta_\pm,$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

Let $U(\tau, s)$ be the propagator associated to Eq. (3.5). In Section 4 we prove the following results.

**Proposition 3.1.** There exists a constant $C > 0$ such that

$$\|U(\tau, s)f\|_{H^2} \leq C \left( \frac{s}{\tau} \right)^{C(|\alpha_1|+|\nu_1|)} \|f\|_{H^2},$$

for any $s \geq \tau > 0$ and any $f \in H^2_{rad}$. Here $\alpha_1 = \frac{\alpha_0}{1+2\nu}$, $\nu_1 = \frac{\nu}{1+2\nu}$.

3.2. Contraction argument. We now transforme (3.1) into a fix point problem. Rewrite (3.1) in the following integral form

$$f(\tau) = J(f)(\tau),$$

(3.6)
where

\[ J(f)(\tau) = J_0(f)(\tau) + J_+(f)(\tau) + J_-(f)(\tau), \]
\[ J_0(f)(\tau) = i \int_{\tau}^{+\infty} dsU(\tau, s)P(\mathcal{F}_1(f(s)) + r(s)), \]
\[ J_+(f)(\tau) = i \int_{\tau}^{+\infty} ds e^{\lambda_0(\tau-s)}P_+(\mathcal{F}_2(f(s)) + r(s)), \]
\[ J_-(f)(\tau) = -i \int_{\tau}^{\tau_1} ds e^{-\lambda_0(\tau-s)}P_-\mathcal{F}_2(f(s)) + r(s)), \]
\[ \mathcal{F}_1(f) = \mathcal{F}(f) + s^{-1}l(P_+ + P_-)\tilde{f}, \]
\[ \mathcal{F}_2(f) = \mathcal{F}(f) + s^{-1}l\tilde{f}, \]

\[ \tau_1 \geq \max\{\tau_0, 1\} \]

Our intention is to view \( J \) as a mapping in the space \( C([\tau_1, +\infty), H^2_{rad}) \) equipped with the norm \( \|f\| = \sup_{\tau \geq \tau_1} \|f(\tau)\|_{H^2_{rad}} \tau^{1+1/16} \), and to show that \( J \) is contraction of the unite ball \( \|f\| \leq 1 \) into itself provided \( |\alpha_0| + |\nu| \) is sufficiently small and \( \tau_1 \) is chosen sufficiently large. Indeed, by (3.3), (3.2) one has, for any \( f, g \) with the norm \( \|f\| \leq 1 \), we have

\[ \|\mathcal{F}_1(f) - \mathcal{F}_1(g)\|_{H^2} \leq C\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|) \tau^{-1} \|f - g\|_{H^2}, \]
\[ \|P_+(\mathcal{F}_2(f) - \mathcal{F}_2(g))\| \leq C\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|) \tau^{-1} \|f - g\|_{H^2}, \]

which together with (3.4) and Prop. 3.1 gives

\[ \|J(f)\| \leq \frac{1}{2} + C\tau_1^{-1/16}, \quad \|J(f) - J(g)\| \leq \frac{1}{2} + C\tau_1^{-1/16} \|f - g\|, \]

for any \( f, g \in \{ \|h\| \leq 1 \} \), provided \( |\alpha_0| + |\nu| \) is sufficiently small. This means that for \( \tau_1 \) sufficiently large, \( J \) is a contraction of the unit ball \( \|f\| \leq 1 \) into itself and consequently, has a unique fixed point \( f \) that satisfies

\[ \|f(\tau)\|_{H^2} \leq \tau^{-1-1/16}, \quad \forall \tau \geq \tau_1, \]

which together with Prop. 2.1 gives Theorem 1.1.

4. Linearized evolution

In this section we prove Prop. 3.1. The proof will be achieved by combining the results of [5] with a careful spectral analysis of the operator \( H \) around zero energy. The section organized as follows. In subsection 1 we consider the operator \( H \) as before, restricted to the subspace of radial functions, and construct a basis of Jost solutions for the equation \( H \zeta = E \zeta \). In subsection 2 we study the spectral decomposition of \( H \) near \( E = 0 \). In subsection 3 we prove Prop. 3.1 by combining the results of the previous two subsections with the coercivity properties of \( H \) established in [5].
4.1. Solutions to the equation $H \zeta = E \zeta$. In this subsection we construct a basis of Jost solutions of the equation $H \zeta = E \zeta, E \in \mathbb{R}$. Since the subject is completely standard we will only briefly sketch the proofs (see also [1], [7] for a closely related construction in the context of energy subcritical NLS). Recall that

\[
H = -(\partial_\rho^2 + 2\rho^{-1}\partial_\rho)\sigma_3 + V(\rho), \quad V = \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix},
\]

\[
V_1(\rho) = -3W^4(\rho), \quad V_2(\rho) = -2W^4(\rho), \quad W(\rho) = \left(1 + \frac{\rho^2}{3}\right)^{-\frac{1}{2}}.
\]

We emphasize that $V(\rho)$ is a smooth function of $\rho$ that decays as $\rho^{-4}$ as $\rho \to \infty$. Since $\sigma_1 H = -H \sigma_1$ it suffices to consider the case $E \geq 0$, so we write $E = k^2, k \geq 0$.

It will be convenient for us to remove the first derivative in $H$. Set $f = \rho \zeta$, then one gets

\[
(4.1) \quad \tilde{H} f = Ef, \quad \tilde{H} = -\partial_\rho^2 \sigma_3 + V(\rho).
\]

We will consider the operator $\tilde{H}$ on $\mathbb{R}$, to recover the original radial $\mathbb{R}^3$ problem it suffices to restrict $\tilde{H}$ to the subspace of odd functions.

We start by constructing the most rapidly decaying solution to (4.1).

**Lemma 4.1.** For all $k \geq 0$ there exists a real solution $f_3(\rho, k)$ of the equation

\[
(4.2) \quad \tilde{H} f = k^2 f,
\]

such that $f_3(\rho, k) = e^{-k\rho} \chi_3(\rho, k)$, where $\chi_3$ is $C^\infty$ function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}^*_+$ verifying $\chi_3(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a(\rho, k)$,

\[
|\partial_\rho^l \partial_k^m a(\rho, k)| \leq C_l < \rho >^{-2-l+m} (1 + k < \rho >)^{-1-m}, m = 0, 1,
\]

\[
|\partial_\rho^l \partial_k^m V(\rho)| \leq C_l < \rho >^{-l} (1 + k < \rho >)^{-3} \ln \left(\frac{1}{k < \rho >} + 2\right),
\]

for all $\rho \geq 0, k > 0$ and $l \geq 0$.

**Proof.** One writes the following integral equation for $\chi_3$

\[
\chi_3(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_{\rho}^{+\infty} K(\rho - s, k)\sigma_3 V(s) \chi_3(s, k) ds,
\]

\[
K(\xi, k) = \begin{pmatrix} \frac{\sin k \xi}{k} & 0 \\ 0 & \frac{\sinh k \xi}{k} \end{pmatrix} e^{k \xi}.
\]

The statement of the lemma follows then from the estimate

\[
|\partial_k^l K(\xi, k)| \leq C_l \frac{|\xi|^{l+1}}{k_\xi >^l}, \quad \xi \leq 0, \quad k \geq 0, \quad l \geq 0
\]

and the decay properties of $V$:

\[
|\partial_\rho^l V(\rho)| \leq C_l < \rho >^{-4-l}, \quad \rho \in \mathbb{R}, \quad l \geq 0,
\]

by standard Volterra iterations. \qed
We next construct the oscillating solutions to Eq. (4.2).

**Lemma 4.2.** For all $k \geq 0$ there exists a solution $f_1(\rho, k)$ of Eq. (4.2) such that $f_1$ is a smooth function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}^*_+$ of the form $f_1(\rho, k) = e^{ik\rho}(\beta + b(\rho, k))$, where $b$ verifies

$$
|b(\rho, k)| \leq C(<\rho>^{-2} + ke^{-k\rho}),
|\partial_\rho b(\rho, k)| \leq C(<\rho>^{-3} + k^2e^{-k\rho}),
|\partial_k b(\rho, k)| \leq C(<\rho>^{-1} + <k\rho>^k e^{-k\rho}),
|\partial_{\rho k} b(\rho, k)| \leq C(<\rho>^{-2} + k <k\rho>^k e^{-k\rho}),
$$

for all $\rho \geq 0$, $0 \leq k \leq 1$. In addition, one has

$$
|\partial^2_k b(\rho, k)| \leq C \ln \left( \frac{1}{k} + 1 \right),
$$

for all $0 \leq \rho \leq 1$, $0 < k \leq 1$.

**Proof.** To construct $f_1$ we will reduce the order of the system (4.2) by means of the substitution $f_1 = z_0 f_3 + z_1(\frac{1}{0})$. Further setting $z_2 = z_0 f_{3,2}$, $f_3 = (f_{3,1})$, we get that $z = (\frac{z_1}{z_2})$ solves

$$
- z_1'' - k^2 z_1 + V_{11} z_1 + V_{12} z_2 = 0,
- z_2' + k z_2 + V_{21} z_1 + V_{22} z_2 = 0.
$$

Here

$$
V_{11} = V_1 - V_2 \frac{f_{3,1}}{f_{3,2}},
V_{12} = \frac{2}{f_{3,2}^2} (f_{3,1} f'_{3,2} - f'_{3,1} f_{3,2}),
V_{21} = V_2,
V_{22} = -\frac{1}{f_{3,2}^2} (f'_{3,2} + k f_{3,2}).
$$

By Lemma 4.1, there exists $R > 0$ independent of $k$, such that the functions $V_{ij}(\rho, k)$, $i, j = 1, 2$, are smooth in both variables for $k > 0$ and $\rho \geq R$ and verify for all $l \geq 0$, $\rho \geq R$, $k > 0$,

$$
|\partial^j_l V_{11}(\rho, k)| \leq C_l <\rho>^{-4-l},
|\partial^j_l \partial_k V_{11}(\rho, k)| \leq C_l <\rho>^{-5-l} <k\rho>^{-2},
|\partial^j_l \partial^2_k V_{11}(\rho, k)| \leq C_l <\rho>^{-4-l} <k\rho>^{-3} \ln \left( \frac{1}{k\rho} + 2 \right),
|\partial^j_l \partial^m_k V_{2j}(\rho, k)| \leq C_l <\rho>^{-3-l+m} <k\rho>^{-1-m},
|\partial^j_l \partial^2_k V_{22}(\rho, k)| \leq C_l <\rho>^{-1-l} <k\rho>^{-3} \ln \left( \frac{1}{k\rho} + 2 \right),
$$

for all $l, m \geq 0$, $j, k = 1, 2$. We will conclude the proof of Lemma 4.2 by establishing the bound

$$
|\partial^j_l \partial^2_k b(\rho, k)| \leq C_l <\rho>^{-4-l} <k\rho>^{-3} \ln \left( \frac{1}{k\rho} + 2 \right),
$$

for all $l, m \geq 0$, $j, k = 1, 2$. We will conclude the proof of Lemma 4.2 by establishing the bound

$$
|\partial^j_l \partial^2_k b(\rho, k)| \leq C_l <\rho>^{-4-l} <k\rho>^{-3} \ln \left( \frac{1}{k\rho} + 2 \right),
$$

for all $l, m \geq 0$, $j, k = 1, 2$.
Writing for \( z \) the following integral equation
\[
z(\rho, k) = e^{ik\rho}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_{\rho}^{\infty} \begin{pmatrix} \sin k(\rho - s) \\ 0 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} z(s, k) ds,
\]
and taking into account (4.6), one proves easily the existence of a smooth solution satisfying
\[
\begin{align*}
|\partial^l_{\rho}\partial^m_{k}(e^{-ik\rho}z_1 - 1)| &+ |\partial^l_{\rho}\partial^m_{k}(e^{-ik\rho}z_2)| \\ &\leq C_l < \rho >^{-2-l+m} < k\rho >^{-1-m}, \quad m = 0, 1,
|\partial^n_{\rho}\partial^2_{k}(e^{-ik\rho}z_1 - 1)| &+ |\partial^n_{\rho}\partial^2_{k}(e^{-ik\rho}z_2)| \\ &\leq C\ln\left(\frac{1}{k\rho} + 2\right), \quad n = 0, 1,
\end{align*}
\]
for all \( \rho \geq R, k > 0, l \geq 0. \)

To reconstruct \( f_1 \), we set
\[
z_0(\rho, k) = \int_{R}^{\rho} \frac{z_2(s, k)}{f_{3,2}(s, k)} ds - \int_{R}^{+\infty} \frac{z_2(s, 0)}{f_{3,2}(s, 0)} ds.
\]
Then, for \( \rho \geq R \), the statement of Lemma 4.2 follows directly from (4.7) and Lemma 4.1. To cover the case \( x \leq R \) one can invoke the Cauchy problem with initial data at \( \rho = R \).

Note that since \( k^2 \in \mathbb{R} \), \( f_2(\cdot, k) = f_1(\cdot, k) \) is also a solution of (4.2).

**Remark 4.3.** Recall that the equation \( \tilde{H}f = 0 \) has a basis of explicit solutions 
\( \rho\Phi_\pm(\rho)\left(\begin{smallmatrix} 1 \\ \pm \end{smallmatrix}\right), \quad \rho\Theta_\pm(\rho)\left(\begin{smallmatrix} 1 \\ \pm \end{smallmatrix}\right) \), with \( \Phi_\pm, \Theta_\pm \) given by (2.13). Comparing the behavior of \( \rho\Phi_\pm, \rho\Theta_\pm \), with the asymptotics of \( f_1(\rho, 0), f_3(\rho, 0) \), one gets
\[
(4.8) \quad f_1(\rho, 0) = \frac{1}{2}\rho(\xi_0(\rho) + \xi_1(\rho)), \quad f_3(\rho, 0) = \frac{1}{2}\rho(\xi_1(\rho) - \xi_0(\rho)),
\]
where \( \xi_0 = \frac{1}{\sqrt{3}}W_1(1), \quad \xi_1 = -\frac{2}{\sqrt{3}}W_1(1) \).

Next, we construct an exponentially growing solution at \( +\infty \).

**Lemma 4.4.** For any \( k > 0 \), there exists a solution \( f_4(\rho, k) \) to (4.2) such that \( f_4 = e^{ik\rho}\chi_4 \) with \( \chi_4 \) verifying
\[
\partial^l_{\rho}\chi_4(\rho, k) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = O_k(\rho^{-3-l}), \quad \rho \to +\infty.
\]

**Proof.** We construct \( f_4 \) by means of the following integral equation:
\[
\chi_4(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{\rho}^{+\infty} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} V\chi_4(s, k) ds
\]
\[
+ \int_{R_1}^{\rho} \begin{pmatrix} e^{k(\rho - s)}\sin k(\rho - s) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{2k(\rho - s)} \end{pmatrix} V\chi_4(s, k) ds.
\]
For $k > 0$ and $R_1$ sufficiently large (depending on $k$), the operator generating (4.9) is small on the space of bounded continuous functions. Therefore, (4.9) has a solution $\chi_4$ verifying $|\chi_4(\rho, k)| \leq C$, $\rho \geq R_1$. Iterating this bound one gets that $\chi_4(\rho, k) - \frac{0}{1} = O_k(\rho^{-3})$ as $\rho \to \infty$. Finally, the estimates for the derivatives can be obtained by differentiating (4.9).

We now briefly describe some properties of the solutions $f_j$, $j = 1, \ldots, 4$, that we will need later. Recall that the Wronskian $w(f, g) = \langle f', g \rangle_{\mathbb{R}^2} - \langle f, g' \rangle_{\mathbb{R}^2}$ does not depend on $\rho$ if $f$ and $g$ are solutions of (4.1).

The estimates of Lemmas 4.1, 4.2, 4.4 lead to the relations:

\[(4.10) \quad w(f_1, f_2) = 2ik, \quad w(f_1, f_3) = w(f_2, f_3) = 0, \quad w(f_3, f_4) = -2k, \quad k > 0, \]

the three first relations being valid for $k = 0$ as well. Notice also that by Lemmas 4.1, 4.2, $\partial_k f_1(\rho, 0)$, $\partial_k f_3(\rho, 0)$ are solutions of the equation $\tilde{H} f = 0$ verifying for $\rho \geq 0$,

\[
|\partial_k f_1(\rho, 0) - \begin{pmatrix} i\rho \\ 0 \end{pmatrix}| \leq C, \quad |\partial_k^2 f_1(\rho, 0) - \begin{pmatrix} i \\ 0 \end{pmatrix}| \leq \frac{C}{<\rho>^2}, \\
|\partial_k f_3(\rho, 0) + \begin{pmatrix} 0 \\ \rho \end{pmatrix}| \leq \frac{C}{<\rho>^2}, \quad |\partial_k^2 f_3(\rho, 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}| \leq \frac{C}{<\rho>^2},
\]

As a consequence, one has

\[(4.11) \quad w(\partial_k f_1|_{k=0}, f_1|_{k=0}) = i, \quad w(\partial_k f_1|_{k=0}, f_3|_{k=0}) = 0, \quad w(\partial_k f_3|_{k=0}, f_1|_{k=0}) = 0, \quad w(\partial_k f_3|_{k=0}, f_3|_{k=0}) = -1.\]

In addition to scalar Wronskian we will use matrix Wronskians. If $F$, $G$ are $2 \times 2$ matrix solutions of (4.2), their matrix Wronskian

\[W(F, G) = F^t G - F^t G'\]

is independent of $\rho$.

Set $g_j(\rho, k) = f_j(-\rho, k)$, $j = 1, \ldots, 4$. Since the potential $V$ is even, $g_j$, $j = 1, \ldots, 4$, are again solutions of (4.2) which have the same asymptotic behavior as $\rho \to -\infty$ as $f_j$ as $\rho \to +\infty$.

Consider the matrix solutions $F$, $G$, defined by

\[F = (f_1, f_3), \quad G = (g_1, g_3).\]

Denote $D(k) = W(F, G)$. It follows from Lemmas 4.1, 4.2 that $D$ is smooth for $k > 0$ and admits the estimate

\[(4.12) \quad |\partial_k^2 D(k)| \leq C \ln \left( \frac{1}{k} + 1 \right), \quad 0 < k \lesssim 1.\]

In addition, by (4.8), (4.10), (4.11), one has

\[(4.13) \quad D(0) = 0, \quad \partial_k D(0) = \begin{pmatrix} -2i & 0 \\ 0 & 2 \end{pmatrix}.\]
4.2. Scattering solutions and the distorted Fourier transform in a vicinity of zero energy. Set

\[ \mathcal{F}(\rho, k) = F(\rho, k)s(k), \]

where \( s(k) = D^{t-1}(k)(\frac{2ik}{\omega}) \). By (4.12), (4.13), \( s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \) is a smooth function of \( k \) for \( 0 < k < k_0 \) (\( k_0 \) sufficiently small), continuous up to \( k = 0 \), verifying

\[ s_1(0) = -1, \quad s_2(0) = 0, \]

\[ |\partial_k s(k)| \leq C|\ln k|, \quad 0 < k \leq k_0. \]

By construction, one has

\[ w(\mathcal{F}, g_1) = 2ik, \quad w(\mathcal{F}, g_3) = 0, \]

for any \( 0 \leq k < k_0 \). As a consequence,

\[ \mathcal{F}(\rho, k) = r_1(k)g_1(\rho, k) + g_2(\rho, k) + r_2(k)g_3(\rho, k), \quad 0 \leq k < k_0, \]

with some coefficients \( r_1(k), r_2(k) \) that, by (4.8), (4.15), verify

\[ r_1(0) = r_2(0) = 0. \]

Computing the Wronskians \( w(\mathcal{F}, \mathcal{F}) \) and \( w(\mathcal{F}, \mathcal{G}) \), where \( \mathcal{G}(\rho, k) = \mathcal{F}(-\rho, k) \), one gets

\[ |s_1(k)|^2 + |r_1(k)|^2 = 1, \quad r_1(k)s_1(k) + r_1(k)s_1(k) = 0, \quad 0 \leq k < k_0. \]

One can write the following Wronskian representation for \( r_1 \):

\[ r_1(k) = s_1(k)\frac{w(g_2, f_1)}{2ik} + s_2(k)\frac{w(g_2, f_3)}{2ik}, \quad k \neq 0. \]

Using (4.15) and the relations

\[ w(g_2, f_3)|_{k=0} = w(g_2, f_1)|_{k=0} = \partial_k w(g_2, f_1)|_{k=0}, \]

one easily deduces from (4.18) that \( r_1 \) is smooth for \( 0 < k < k_0 \), continuous up to \( k = 0 \), and verifies

\[ |\partial_k r_1(k)| \leq C|\ln k|, \quad 0 < k < k_0, \]

which in its turn, implies that \( r_2 \) is smooth for \( 0 < k < k_0 \), continuous up to \( k = 0 \) and admits a similar estimate:

\[ |\partial_k r_2(k)| \leq C|\ln k|, \quad 0 < k < k_0. \]

Introduce the following odd solution of (4.2):

\[ e(\rho, k) = \mathcal{F}(-\rho, k) - \mathcal{F}(\rho, k). \]

By (4.14), (4.16),

\[ e = a_1f_1 + f_2 + a_2f_3, \quad a_j = r_j - s_j, \quad j = 1, 2. \]
It follows from (4.15), (4.17), (4.19), (4.20) that

\[ a_1(0) = 1, \quad a_2(0) = 0, \]

and

\[ |\partial_k a_j| \leq C|\ln k|, \quad 0 < k < k_0, \quad j = 1, 2, \]

which together with Lemmas 4.1, 4.2 implies the following result.

**Lemma 4.5.** One has:

(i) \( e_1(k) = e_0(k) + e_1(k) \), where \( e_0(k) = a_1(k)e^{ik\rho}(1) + e^{-ik\rho}(1) \) and the remainder \( e_1(k) \) admits the estimates

\[ |e_1(k)| \leq C(|\rho| k + |\ln k| |e^{-k\rho}|), \quad \rho \geq 0, \]

\[ |\partial_k e_1(k)| \leq C|\ln k|(|\rho| + e^{-k\rho/2}), \quad \rho \geq 0, \]

\[ \|e_1(\cdot, k)\|_{L^2(\mathbb{R}^3)} \leq C, \]

\[ \|\rho e_1(\cdot, k)\|_{L^2(\mathbb{R}^3)} + \|\partial_k e_1(\cdot, k)\|_{L^2(\mathbb{R}^3)} \leq Ck^{1/2}|\ln k|, \]

for any \( 0 < k \leq k_0 \).

(ii) \((\rho\partial_\rho - k\partial_k)e_1(k) = e^{ik\rho}(1)k\partial_k a_1(k) + e_2(k, \rho)\), with \( e_2(k, \rho) \) verifying

\[ |e_2(k, \rho)| \leq C(|\rho| k + |\ln k| |e^{-k\rho/2}|), \quad \rho \geq 0, \]

\[ \|e_2(\cdot, k)\|_{L^2(\mathbb{R}^3)} \leq C, \]

for any \( 0 < k \leq k_0 \).

For \( 0 < \kappa \leq k_0 \), introduce the operators \( E_\kappa : L^2(\mathbb{R}^3, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^2) \),

\[ (E_\kappa \Phi)(y) = \frac{1}{2^{3/2}\pi} \int_{\mathbb{R}^+} dk \theta_\kappa(k) \mathcal{E}(y, k) \Phi(k), \quad \Phi \in L^2(\mathbb{R}^+, \mathbb{C}^2), \]

where \( \mathcal{E}(y, k) \) is a \( 2 \times 2 \) matrix given by

\[ \mathcal{E}(y, k) = \rho^{-1}(c(\rho, k), \sigma_1 e(\rho, k)), \quad \rho = |y|, \]

\[ \theta_\kappa(k) = \theta(\kappa^{-1}k), \theta \text{ is a } C^\infty \text{ even function verifying } \theta(k) = \begin{cases} 1 & \text{if } |k| \leq 1/4 \\ 0 & \text{if } |k| \geq 1/2 \end{cases}. \]

Since \( e(\rho, k) \) is a solution of the equation \( \hat{H}e = k^2e, \) one has \( \hat{H}E_\kappa = E_\kappa k^2\sigma_3 \).

By Lemma 4.3 (i), the operators \( E_\kappa \) are bounded uniformly with respect to \( \kappa \leq k_0 \). The action of the adjoint operators \( E_\kappa^* : L^2(\mathbb{R}^3, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^2) \) is given by

\[ (E_\kappa^* \psi)(k) = \frac{1}{2^{3/2}\pi} \theta_\kappa(k) \int_{\mathbb{R}^3} dy \mathcal{E}^*(y, k) \psi(y), \quad \psi \in L^2(\mathbb{R}^3, \mathbb{C}^2). \]

Clearly,

\[ (E_\kappa^* \sigma_3 \zeta_\pm = 0 \]

for any \( 0 < \kappa \leq k_0 \).
The following relation is a standard consequence of the asymptotics given by Lemma 4.5 (i),

\[(4.27)\]

\[E^{*}_{\kappa_{2}} \sigma_{3} \overline{E}_{\kappa_{1}} \sigma_{3} = \theta_{\kappa_{1}}(k) \theta_{\kappa_{2}}(k),\]

for any \(0 < \kappa_{1}, \kappa_{2} \leq k_{0}.\)

**Remark 4.6.** Notice that because of the presence of the cut off function \(\theta_{\kappa},\) \(E_{\kappa}\) is bounded as an operator from \(L^2([0,k_{0}])\) to \(H^{m}(\mathbb{R}^{3})\) for any \(m \geq 0,\) uniformly in \(\kappa \leq k_{0}.\)

We next introduce quasi-resonant functions \(h_{\kappa}(y),\) \(0 < \kappa \leq k_{0},\) by setting

\[h_{\kappa} = \sqrt{2E_{\kappa}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.\]

**Lemma 4.7.** For any \(0 < \kappa \leq k_{0},\) \(h_{\kappa} \in < y >^{-1} L^{2}(\mathbb{R}^{3})\) and as \(\kappa \to 0,\) one has

\[(4.28)\]

\[\|h_{\kappa}\|_{L^{2}(\mathbb{R}^{3})} = O(\kappa^{1/2}), \quad \|yh_{\kappa}\|_{L^{2}(\mathbb{R}^{3})} = O(\kappa^{-1/2}),\]

\[(4.29)\]

\[\langle h_{\kappa}, \sigma_{3}(\xi_{0} + \xi_{1}) \rangle = 4\pi + O(\kappa^{1/2} \ln \kappa), \quad \langle h_{\kappa}, \sigma_{3}(\xi_{1} - \xi_{0}) \rangle = O(\kappa^{1/2} \ln \kappa).\]

**Proof.** Applying Lemma 4.5 (i), we decompose \(h_{\kappa}\) as follows:

\[h_{\kappa}(y) = h_{\kappa,0}(y) + h_{\kappa,1}(y) + h_{\kappa,2}(y),\]

\[h_{\kappa,0}(y) = \frac{1}{2\pi\rho} \hat{\theta}(\kappa \rho) \begin{pmatrix} 1 \\ 0 \end{pmatrix},\]

\[h_{\kappa,1}(y) = \frac{1}{2\pi\rho} \int_{\mathbb{R}^{+}} dk e^{ik\rho} (a_{1}(k) - 1) \theta_{\kappa}(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix},\]

\[h_{\kappa,2}(y) = \frac{1}{2\pi\rho} \int_{\mathbb{R}^{+}} dk \theta_{\kappa}(k) e_{1}(\rho,k),\]

where \(\hat{\theta}(\rho) = \int_{\mathbb{R}} e^{ik\rho} \theta(k) dk,\) \(\rho = \|y\|\).

Clearly, \(h_{\kappa,0} \in < y >^{-1} L^{2}(\mathbb{R}^{3})\) and one has

\[(4.31)\]

\[\|h_{\kappa,0}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa^{1/2}, \quad \|yh_{\kappa,0}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa^{-1/2}.\]

Consider \(h_{\kappa,i}, i = 1, 2.\) It follows from \((4.22), (4.23), (4.24)\) that

\[(4.32)\]

\[\|h_{\kappa,i}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa, \quad \|yh_{\kappa,i}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa^{1/2} \ln \kappa, \quad i = 1, 2,\]

which together with \((4.31)\) leads to the estimates

\[(4.33)\]

\[\|h_{\kappa}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa^{1/2}, \quad \|yh_{\kappa}\|_{L^{2}(\mathbb{R}^{3})} \leq C \kappa^{-1/2}.\]
We next compute \( \langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle \). By (4.31), (4.32), as \( \kappa \to 0 \), one has

\[
\langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle = \langle h_{\kappa,0}, \sigma_3(\xi_1 \pm \xi_0) \rangle + O(\kappa^{1/2} \ln \kappa),
\]

\[
\langle h_{\kappa,0}, \sigma_3(\xi_1 - \xi_0) \rangle = O(\kappa),
\]

\[
\langle h_{\kappa,0}, \sigma_3(\xi_1 + \xi_0) \rangle = 2\kappa \int_\mathbb{R} d\hat{\theta}(\kappa \rho) + O(\kappa) = 4\pi + O(\kappa),
\]

which gives (4.29). \(\square\)

4.3. Proof of Proposition 3.1. We start by deriving some coercivity bounds for the operator \( H \).

**Lemma 4.8.** There exists \( \kappa_0, 0 < \kappa_0 \leq k_0 \), and \( C > 0 \) such that

\[
\langle Hf, \sigma_3f \rangle \geq C\kappa \| \nabla f \|_{L^2(\mathbb{R}^3)}^2,
\]

for any \( 0 < \kappa \leq \kappa_0 \) and any \( f \in H^{1\text{rad}}(\mathbb{R}^3, \mathbb{C}^2) \) verifying

\[
\langle f, \sigma_3\zeta_- \rangle = \langle f, \sigma_3\zeta_+ \rangle = \langle f, \sigma_3h_\kappa \rangle = \langle f, \sigma_3\sigma_1\bar{h}_\kappa \rangle = 0.
\]

**Remark 4.9.** Notice that since \( \zeta_\pm, h_\kappa \in \mathcal{Y}^{-1}{L}^2(\mathbb{R}^3) \) the scalar products that appear in (4.36) are well defined for any \( f \in H^1 \).

**Proof.** The proof of Lemma 4.8 is based on the following result which is due to Duyckaerts and Merle:

**Lemma 4.10.** There exists \( c_0 > 0 \) such that

\[
\langle Hf, \sigma_3f \rangle \geq c_0 \| \nabla f \|_{L^2(\mathbb{R}^3)}^2,
\]

for any \( f \in H^{1\text{rad}}(\mathbb{R}^3, \mathbb{C}^2) \) verifying

\[
\langle f, \sigma_3\zeta_- \rangle = \langle f, \sigma_3\zeta_+ \rangle = \langle f, \Delta \xi_0 \rangle = \langle f, \Delta \xi_1 \rangle = 0,
\]

see [5] for the proof.

Let \( f \in \dot{H}^{1\text{rad}} \) such that (4.36) holds. One can write \( f \) as

\[
f = \alpha_0\xi_0 + \alpha_1\xi_1 + g,
\]

where

\[
\alpha_j = -\frac{\langle f, \Delta \xi_j \rangle}{\| \nabla \xi_j \|_{L^2(\mathbb{R}^3)}^2}, \quad j = 0, 1,
\]

and \( g \in \dot{H}^{1\text{rad}} \) verifies

\[
\langle g, \sigma_3\zeta_- \rangle = \langle g, \sigma_3\zeta_+ \rangle = \langle g, \Delta \xi_0 \rangle = \langle g, \Delta \xi_1 \rangle = 0.
\]

Therefore, by Lemma 4.10

\[
\langle Hf, \sigma_3f \rangle = \langle Hg, \sigma_3g \rangle \geq c_0 \| \nabla g \|_{L^2(\mathbb{R}^3)}^2.
\]
Furthermore, since $f$ verifies (4.36), one has
\[
A(\kappa) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \langle g, \sigma_3 h_\kappa \rangle \\ \langle g, \sigma_3 \sigma_1 h_\kappa \rangle \end{pmatrix},
\]
where
\[
A(\kappa) = -\begin{pmatrix} \langle \xi_0, \sigma_3 h_\kappa \rangle & \langle \xi_1, \sigma_3 h_\kappa \rangle \\ \langle h_\kappa, \sigma_3 \xi_0 \rangle & -\langle h_\kappa, \sigma_3 \xi_1 \rangle \end{pmatrix}.
\]
By (4.29),
\[
A(\kappa) = -2\pi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + O(\kappa^{1/2} \ln \kappa), \quad \kappa \to 0.
\]
Therefore, for $\kappa$ sufficiently small, one has
\[
|\alpha_0| + |\alpha_1| \leq C \|\nabla g\|_{L^2(\mathbb{R}^3)} < y > h_\kappa \|L^2(\mathbb{R}^3) \leq C \kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.
\]
As a consequence,
\[
\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq C \kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.
\]
Combining this inequality with (4.37), we get (4.35).

Next, we prove

**Lemma 4.11.** There exists $\kappa_1$, $0 < \kappa_1 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_1$ one has
\[
\|f\|_{H^1(\mathbb{R}^3)} \leq C \kappa \|\nabla f\|_{L^2(\mathbb{R}^3)},
\]
for all $f \in H^1_{rad}(\mathbb{R}^3)$ verifying $E^*_\kappa f = 0$.

**Proof.** By (4.22), (4.23) and Lemma 4.5 (i), $E^*_\kappa f$ can be written as
\[
(E^*_\kappa f)(k) = \Phi_0(k) + \Phi_r(k),
\]
where
\[
\Phi_0(k) = \frac{1}{23/2 \pi} \theta_\kappa(k) \hat{f}(k),
\]
\[
\hat{f}(k) = 2 \int_{\mathbb{R}^3} dy \frac{\cos k|y|}{|y|} f(y),
\]
and the remainder $\Phi_r$ satisfies
\[
\|\Phi_r\|_{L^2(\mathbb{R}^+)} \leq C \kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.
\]
Therefore, $E^*_\kappa f = 0$ implies
\[
\|\hat{f}\|_{L^2(0, \kappa/4)} \leq C \kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.
\]
Notice also that for any $f \in H^1_{rad}$ and any $0 < \kappa \leq 1$ one has
\[
\|f\|_{H^1(\mathbb{R}^3)} \leq C(\|\hat{f}\|_{L^2(0, \kappa/4)} + \kappa^{-1} \|\nabla f\|_{L^2(\mathbb{R}^3)}).
\]
Combining this inequality with (4.38), we get
\[
\|f\|_{H^1(\mathbb{R}^3)} \leq C \kappa \|\nabla f\|_{L^2(\mathbb{R}^3)},
\]
provided $\kappa$ is sufficiently small.

We finally combine Lemmas 4.8, 4.11 to derive the following result which will be in the heart of the proof of Prop. 3.1

**Lemma 4.12.** There exists \( \kappa_2, 0 < \kappa_2 \leq k_0 \), and \( C > 0 \) such that for any \( 0 < \kappa \leq \kappa_2 \) one has

\[
\langle Hf, \sigma_3 f \rangle \geq C \kappa^3 \| f \|^2_{H^1} - \frac{\kappa}{C} \| E^*_\kappa \sigma_3 f \|^2_{L^2(\mathbb{R}^+)},
\]

for any \( f \in H^1_{rad}(\mathbb{R}^3, \mathbb{C}^2) \) verifying \( \langle f, \sigma_3 \zeta_{\pm} \rangle = 0 \).

**Proof.** Write \( f = f_1 + f_2 \), where \( f_1 = E_\kappa \sigma_3 E^*_\kappa \sigma_3 f \) and \( f_2 = f - f_1 \). One clearly has

\[
\| f_1 \|_{H^1(\mathbb{R}^3)} \leq C \| E^*_\kappa \sigma_3 f \|_{L^2(\mathbb{R}^+)}, \quad \| Hf_1 \|_{L^2(\mathbb{R}^3)} \leq C \kappa^2 \| E^*_\kappa \sigma_3 f \|_{L^2(\mathbb{R}^+)},
\]

for any \( 0 < \kappa \leq k_0 \).

Consider \( f_2 \). It follows from (4.26), (4.27) that for any \( \kappa' \leq \kappa/2 \),

- \( \langle f_2, \sigma_3 \zeta_{\pm} \rangle = 0 \);
- \( E^*_\kappa \sigma_3 f_2 = 0 \);
- \( \langle f_2, \sigma_3 h_{\kappa'} \rangle = \langle f_2, \sigma_3 \sigma_1 \bar{h}_{\kappa'} \rangle = 0 \).

Hence, by Lemmas 4.8, 4.11 one has

\[
\langle Hf_2, \sigma_3 f_2 \rangle \geq C \kappa^3 \| f_2 \|^2_{H^1(\mathbb{R}^3)},
\]

provided \( \kappa \) is sufficiently small.

Combining (4.40), (4.41) one gets (4.39). \( \square \)

We are now in the position to prove Proposition 3.1. Consider the equation

\[
\begin{align*}
\dot{\psi} & = P \mathcal{H}(\tau) P \psi, \\
\psi|_{\tau=s} & = f,
\end{align*}
\]

where

\[
\mathcal{H}(\tau) = H + \tau^{-1}l, \quad l = \alpha_1 \sigma_3 - i \nu_1 \left( \frac{1}{2} + y \cdot \nabla \right),
\]

\( \alpha_1, \nu_1 \in \mathbb{R}, s > 0 \) and \( f \in \mathcal{S}(\mathbb{R}^3) \) verifying \( \langle f, \sigma_3 \zeta_{\pm} \rangle = 0 \).

Fix \( \kappa \) such that \( 0 < \kappa \leq \kappa_2 \) and consider the functional \( G_1(\tau) = \langle H\psi, \sigma_3 \psi \rangle + c_0 \| E^*_\kappa \sigma_3 \psi \|^2_{L^2(\mathbb{R}^+)} \). Clearly,

\[
G_1(\tau) \leq C \| \psi(\tau) \|^2_{H^1(\mathbb{R}^3)}.
\]

Moreover, since \( \langle \psi(\tau), \sigma_3 \zeta_{\pm} \rangle = 0 \), choosing \( c_0 \) sufficiently large, we get:

\[
G_1(\tau) \geq c_1 \| \psi(\tau) \|^2_{H^1(\mathbb{R}^3)}.
\]

We next compute the derivative \( \frac{d}{d\tau} G_1 \). One has

\[
\frac{d}{d\tau} \langle H\psi, \sigma_3 \psi \rangle = \frac{2i}{\tau} \text{Im} \langle \psi, \sigma_3 H\psi \rangle,
\]
which implies

\[
\frac{d}{d\tau} \langle H\psi, \sigma_3\psi \rangle \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\nabla\psi(\tau)\|^2_{L^2(\mathbb{R}^3)}.
\]

Next, we address \(\|E_\alpha^* \sigma_3 \psi\|^2_{L^2(\mathbb{R}^3)}\). Denote \(\Phi(\tau) = E_\alpha^* \sigma_3 \psi(\tau)\). Then \(\Phi(k, \tau)\) solves

\[
i\Phi_\tau = k^2 \sigma_3 \Phi + \frac{1}{\tau} Y,
\]

where

\[Y = E_\alpha^* \sigma_3 l\psi.\]

Integrating by parts and applying Lemma 4.5 (ii), one can rewrite \(Y\) in the form

\[Y(k, \tau) = Y_0(k, \tau) + Y_1(k, \tau),\]

where

\[Y_0(k, \tau) = i\nu_1 k \partial_k \Phi(k, \tau),\]

and \(Y_1(k, \tau)\) admits the estimate

\[\|Y_1(\tau)\|_{L^2(\mathbb{R}^3)} \leq C(|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{L^2(\mathbb{R}^3)}.\]

Therefore, (4.46) gives

\[\left| \frac{d}{d\tau} \|\Phi(\tau)\|^2_{L^2(\mathbb{R}^3)} \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|^2_{L^2(\mathbb{R}^3)}.\]

Combining this inequality with (4.46) and taking into account (4.44) one gets

\[\frac{d}{d\tau} \|G_1(\tau)\| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|^2_{H^1(\mathbb{R}^3)} \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_1(\tau).\]

Integrating we obtain

\[G_1(\tau) \leq C \left( \frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} G_1(s), \quad 0 < \tau \leq s,\]

which by (4.43), (4.44), leads to the bound

\[\|U(\tau, s) f\|_{H^1(\mathbb{R}^3)} \leq C \left( \frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} \|f\|_{H^1(\mathbb{R}^3)},\]

for any \(0 < \tau \leq s\) and any \(f \in H^1_{rad}\).

To control the higher regularity, consider the functional \(G_2(\tau) = \langle \mathcal{H}_2 \psi, \sigma_3 H \psi \rangle + c_2 G_1(\tau)\). One has

\[C^{-1}\|\mathcal{H}_2 \psi\|^2_{H^2(\mathbb{R}^3)} \leq G_2 \leq C\|\mathcal{H}_2 \psi\|^2_{H^2(\mathbb{R}^3)},\]

provided \(c_2\) is chosen sufficiently large.

Computing the derivative \(\frac{d}{d\tau} \langle \mathcal{H}_2 \psi(\tau), \sigma_3 H \psi(\tau) \rangle\) and taking into account (4.47) we get

\[\left| \frac{d}{d\tau} G_2(\tau) \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|^2_{H^3(\mathbb{R}^3)} \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_2(\tau).\]
which implies

\[
\|U(\tau, s)f\|_{H^3(\mathbb{R}^3)} \leq C \left( \frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} \|f\|_{H^3(\mathbb{R}^3)},
\]

for any \(0 < \tau \leq s\).

The \(H^2\) bounded stated in Prop. 3.1 follows from (4.48), (4.50) by interpolation.

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