BEREZIN-TOEPLITZ QUANTIZATION OVER MATRIX DOMAINS

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ABSTRACT. We explore the possibility of extending the well-known Berezin-Toeplitz quantization to reproducing kernel spaces of vector-valued functions. In physical terms, this can be interpreted as accommodating the internal degrees of freedom of the quantized system. We analyze in particular the vector-valued analogues of the classical Segal-Bargmann space on the domain of all complex matrices and of all normal matrices, respectively, showing that for the former a semi-classical limit, in the traditional sense, does not exist, while for the latter only a certain subset of the quantized observables have a classical limit: in other words, in the semiclassical limit the internal degrees of freedom disappear, as they should. We expect that a similar situation prevails in much more general setups.

1. INTRODUCTION

Let $\Omega$ be a symplectic manifold, with symplectic form $\omega$, and $\mathcal{H}$ a subspace of $L^2(\Omega, d\mu)$, for some measure $\mu$, admitting a reproducing kernel $K$. For $\phi \in C^\infty(\Omega)$, the Toeplitz operator $T_\phi$ with symbol $\phi$ is the operator on $\mathcal{H}$ defined by

$$T_\phi f = P(\phi f), \quad f \in \mathcal{H},$$

where $P : L^2(\Omega, d\mu) \to \mathcal{H}$ is the orthogonal projection. Using the reproducing kernel $K$, this can also be written as

$$T_\phi f(x) = \int_\Omega f(y)\phi(y)K(x, y) \, d\mu(y).$$

It is easily seen that $T_\phi$ is a bounded operator whenever $\phi$ is a bounded function, and $\|T_\phi\|_{\mathcal{H} \to \mathcal{H}} \leq \|\phi\|_\infty$, the supremum norm of $\phi$.

Suppose now that both the measure $\mu$ and the reproducing kernel subspace $\mathcal{H}$ are made to depend on an additional parameter $h > 0$ (shortly to be interpreted as the Planck constant), in such a way that the associated Toeplitz operators $T_\phi^{(h)}$ on $\mathcal{H}_h$ satisfy, as $h \to 0$,

$$\|T_\phi^{(h)}\|_{\mathcal{H}_h \to \mathcal{H}_h} \to \|\phi\|_\infty,$$

and

$$\|T_\phi^{(h)} - T_\psi^{(h)}\|_{\mathcal{H}_h \to \mathcal{H}_h} \to 0,$$

$$\frac{\partial}{\partial h} [T_\phi^{(h)}, T_\psi^{(h)}] - T_{{\phi, \psi}}^{(h)} \|_{\mathcal{H}_h \to \mathcal{H}_h} \to 0$$

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(where \{\cdot, \cdot\} is the Poisson bracket with respect to \omega), and, more generally,
\[
T_{\phi}^{(h)}T_{\psi}^{(h)} \approx \sum_{j=0}^{\infty} h^j T_{C_j(\phi, \psi)}^{(h)} \quad \text{as } h \to 0,
\]
for some bilinear differential operators \(C_j : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega),\) with \(C_0(\phi, \psi) = \phi \psi\) and \(C_1(\phi, \psi) - C_1(\psi, \phi) = \frac{i}{\pi} \{\phi, \psi\}.\) Here the last asymptotic expansion means, more precisely, that
\[
\left\|T_{\phi}^{(h)}T_{\psi}^{(h)} - \sum_{j=0}^{N} h^j T_{C_j(\phi, \psi)}^{(h)}\right\| = O(h^{N+1}) \quad \text{as } h \searrow 0, \quad \forall N = 0, 1, 2, \ldots.
\]

One then speaks of the Berezin-Toeplitz quantization. Indeed, it is well known that the recipe
\[
\phi \ast \psi := \sum_{j=0}^{\infty} h^j C_j(\phi, \psi)
\]
then gives a star-product on \(\Omega,\) and (1.2), (1.3) just amount to its correct semiclassical limit.

Observe that if we introduce the normalized reproducing kernels — or coherent states — by
\[
k_y := \frac{K(\cdot, y)}{\|K(\cdot, y)\|_H}, \quad \text{i.e.} \quad k_y(x) = \frac{K(x, y)}{K(y, y)^{1/2}},
\]
and define the Berezin transform of an operator \(T\) on \(\mathcal{H}\) by
\[
\widetilde{T}(y) := \langle Tk_y, k_y \rangle_{\mathcal{H}},
\]
then (1.3) implies that as \(h \to 0,\)
\[
(1.5) \quad \tilde{T}_{\phi}^{(h)}T_{\psi}^{(h)} \approx \sum_{j=0}^{\infty} h^j \tilde{T}_{C_j(\phi, \psi)}^{(h)},
\]
pointwise and even uniformly on \(\Omega.\) Often, one also has a stronger version of (1.5), namely
\[
(1.6) \quad \tilde{T}_{\phi}^{(h)}(x) \to \phi(x) \quad \forall x \in \Omega.
\]

The simplest instance of the above situation is \(\Omega = \mathbb{R}^{2n} \simeq \mathbb{C}^n,\) with the standard (Euclidean) symplectic structure, and
\[
(1.7) \quad \mathcal{H}_h = L^2_{\text{hol}}(\Omega, d\mu_h)
\]
the Segal-Bargmann space of all holomorphic functions square-integrable with respect to the Gaussian measure \(d\mu_h(z) := e^{-\|z\|^2/(\pi h)} e^{-n} dz\) (\(d\mu\) being the Lebesgue measure on \(\mathbb{C}^n.\)) The space \(\mathcal{H}_h\) admits the reproducing kernel \(K_h(x, y) = e^{(x, y)/h.}\) The Berezin transform turns out to be given just by the familiar heat operator semigroup,
\[
\tilde{T}_{\phi}^{(h)}(x) = (\pi h)^{-n} \int_{\mathbb{C}^n} e^{-\|x-y\|^2/2h} \phi(y) dy,
\]
so that, by the stationary phase (WKB) expansion (see Section 7 below; for a
discussion of the (WKB) approximation, see for example [Me], Chapter 7) 1

(1.8) \( \tilde{T}_\phi^{(h)}(x) = \sum_{j=0}^{\infty} \frac{h^j}{j!} \Delta^j \phi(x) \).

Similarly,

\[
\tilde{T}_\phi^{(h)}(\tilde{T}_\psi^{(h)}(x) = (\pi h)^{-2n} \int_{C^n} \int_{C^n} \phi(y) \psi(z) e^{i(x,y) + (y,z) - \|x\|^2 - \|y\|^2 - \|z\|^2)/h \ dy \ dz,
\]

so by stationary phase again,

(1.9) \( \tilde{T}_\phi^{(h)}(\tilde{T}_\psi^{(h)}(x) = \sum_{\alpha, \beta, \gamma} \frac{\partial^\alpha \bar{\partial}^{\alpha+\gamma} \phi(x)}{\alpha! \gamma!} \frac{\partial^{\beta+\gamma} \bar{\partial}^\beta \psi(x)}{\beta!} h^{\|\alpha\|+\|\beta\|+\|\gamma\|} \).

(Here the summation extends over all multiindices \( \alpha, \beta, \gamma \), i.e. \( n \)-tuples of nonnegative integers \( \alpha = (\alpha_1, \ldots, \alpha_n) \), etc., and we are using the usual multiindex notations \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( \alpha! = \alpha_1! \cdots \alpha_n! \), \( \partial^\alpha = \partial^{\alpha_1} \partial_{x_1} \cdots \partial^{\alpha_n} \partial_{x_n} \).) Inserting (1.8) and (1.9) into (1.3), we get formulas for the cochains \( C_j \):

(1.10) \( C_j(\phi, \psi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^\alpha \phi \cdot \bar{\partial}^\alpha \psi. \)

The resulting star-product coincides, essentially, with the familiar Moyal product.

Other examples of Berezin-Toeplitz quantization include the unit disc \( D \) with
the Poincaré metric, bounded symmetric domains, strictly pseudoconvex domains
with metrics having reasonable boundary behaviour, or, provided one allows not
only holomorphic functions but also sections of line bundles as elements of \( \mathcal{H}_h \), all
compact Kähler manifolds whose Kähler form is integral. In all these cases, the
choice of the spaces (17) which works are the weighted Bergman spaces \( \mathcal{H}_h = L^2_{\text{hol}}(\Omega, e^{-\Phi}/\omega^n) \), where \( n \) is the complex dimension of \( \Omega \) and \( \Phi \) is a Kähler potential for \( \omega \) (so, for instance, for the unit disc \( \mathcal{H}_h = L^2_{\text{hol}}(D, (1 - |z|^2)^{(1/h) - 2}) \).

See [KS], [BMS], or [AE] for the details and further discussion.

In this note, we explore the possibility of extending the above formalism to
reproducing kernel spaces of vector-valued functions. In physical terms, this can
be interpreted as accommodating the internal degrees of freedom of the quantized
system.

In more concrete terms, this means that we again consider, for a given \( h > 0 \),
a suitable measure \( \mu_h \) on \( \Omega \), and a reproducing kernel subspace \( \mathcal{H}_h \subset L^2_{\text{hol}}(\Omega, d\mu_h) \),
where the subscript \( C^N \) indicates that we are now dealing with vector-valued functions taking values in \( \mathbb{C}^N \) for some \( N \geq 1 \). The reproducing kernel \( K_h \) of \( \mathcal{H}_h \) will thus now be a matrix-valued object, \( K_h : \Omega \times \Omega \to \mathbb{C}^{N \times N} \). We can again consider, for each \( \phi \in C^\infty(\Omega) \), the associated Toeplitz operators, and investigate the existence of the asymptotic expansion (14). In fact, we can now even allow matrix-valued symbols \( \phi \in C^\infty_{\text{hol}}(\Omega) \). By analogy with the scalar-valued case, one may again expect appropriate asymptotic expansions

(1.11) \( \tilde{T}_\phi^{(h)}(x) = \sum_{j=0}^{\infty} L_j \phi(x) h^j, \)

1Throughout this paper, we are using the slightly nonstandard Laplacian \( \Delta = \sum_j \partial^2 / \partial x_j \partial \bar{x}_j \),
which differs from the usual one by a factor of 4.
with some differential and bidifferential operators \( L_j \) and \( M_j \), respectively, whose comparison would yield (1.5), thus suggesting that (1.4) is also likely to hold.

Unfortunately, at this level of generality the results are negative: we show that for certain spaces \( \mathcal{H}_h \) as above, which are quite natural generalizations of the Euclidean situation (1.7) to vector-valued functions, the semiclassical expansions (1.11), (1.12) fail to hold. More specifically, it seems that there are asymptotic expansions for \( \tilde{T}_\phi(x) \) and \( \tilde{T}_\phi \tilde{T}_\psi(x) \) in powers of \( h \), but their coefficients do not depend only on the jets of \( \phi \) and \( \psi \) at \( x \), but also at other points (i.e. are not local operators); besides, in addition to integer powers of \( h \), half-integer powers seem to also enter the picture. Finally, (1.1) and (1.6) may also break down.

However, it turns out that upon restricting to appropriate domains \( \Omega \) and appropriate classes of functions \( \phi, \psi \), the situation can be saved completely: namely, the following picture emerges. The admissible functions \( \phi \) can be identified with functions \( f(d_1; d_2, \ldots, d_N) \) on \( \mathbb{C} \times \mathbb{C}^{N-1} \) that are symmetric in the \( N-1 \) variables \( d_2, \ldots, d_N \). Clearly, one can associate a function \( u^\# \) of this form to any function \( u: \mathbb{C} \to \mathbb{C} \) by the recipe \( u^\#(d_1; d_2, \ldots, d_N) := u(d_1) \). We show that for any functions \( f, g \) of the above form, there exist uniquely determined functions \( u_r \) on \( \mathbb{C} \), \( r = 0, 1, 2, \ldots \), such that

\[
T^{(h)}_f T^{(h)}_g \approx \sum_{r=0}^\infty h^r T^{(h)}_{u_r^\#}.
\]

Further, the \( u_r \) are given by differential expressions involving \( f \) and \( g \); and, finally, if \( f \) and \( g \) themselves are of the form \( v^\# \) and \( w^\# \), respectively, for some functions \( v, w \) on \( \mathbb{C} \), then in fact \( u_r = C_r(v, w) \) with the bidifferential operators \( C_r \) given by (1.10) for \( n = 1 \), i.e. \( \Omega = \mathbb{C} \). This suggests the following interpretation: our quantum system has \( N \) internal degrees of freedom, which have no classical counterparts, so that only a subset of the quantized observables have a classical limit. In the semiclassical limit the internal degrees of freedom disappear, as they should. We conjecture that similar quantizations can be carried out by our method in much more general setups.

The paper is organized as follows. In Section 2 we introduce the spaces \( \mathcal{H}_h \) of vector-valued functions on \( \mathbb{C}^{N \times N} \), as well as their analogues for the subset of all normal matrices in \( \mathbb{C}^{N \times N} \). These spaces have previously appeared in \[\text{AEG}\]. In Section 3 we define a generalization of the Berezin transform — which will now also be a matrix-valued object — and establish its basic properties. Sections 4 and 5 discuss the semiclassical asymptotic expansions (1.11)–(1.12) in the above two settings of the domains of all matrices and all normal matrices, respectively. In Section 6 we present our first result in the positive direction, by establishing an asymptotic expansion — which is, however, of a highly non-local nature — for \( \tilde{T}_\phi^{(h)} \) and \( \tilde{T}_\phi^{(h)} \tilde{T}_\psi^{(h)} \) in the case of the normal matrices. Finally, in Sections 7 and 8 we introduce our restricted class of observables \( \phi \) and establish the asymptotic expansion (1.13).
2. The domains and the spaces

Our first domain is \( \Omega = \mathbb{C}^{N \times N} \), with the measures

\[
d\mu_h(Z) = e^{-\text{Tr}(Z^* Z)/h} (\pi h)^{-N^2} dZ,
\]

where \( dZ \) denotes the Lebesgue measure on \( \mathbb{C}^{N \times N} \). The measures \( \mu_h \) are normalized to be of total mass one. The functions \( \Psi_j(Z) := Z^j \) satisfy

\[
(2.1) \quad \int_\Omega Z^j Z^k d\mu_h(Z) = \delta_{jk} h^k c_k I
\]

for some numbers \( c_k > 0 \); see \[\text{[AEG]}\]. Explicitly, \( c_k \) are given by \([\text{[Gin]}\], formula (1.40), and \[\text{[Kri]}\])

\[
c_k = \begin{cases} 
\frac{(k + N + 1)!}{N!(k+1)(k+2)} & \text{for } k \geq N - 1, \\
\frac{N!}{(k+1)(k+2)(N-k-2)!} - \frac{N!}{(k+1)(k+2)(N-k)!} & \text{for } k < N - 1,
\end{cases}
\]

that is,

\[
(2.2) \quad c_k = \frac{1}{(k+1)(k+2)} \left[ \frac{1}{k+1} \prod_{j=1}^{k+1} (N+j) - \frac{1}{k+1} \prod_{j=1}^{k+1} (N-j) \right].
\]

In particular, \( c_0 = 1, c_1 = N, c_2 = N^2 + 1, \) etc.

It follows that if \( \chi_1, \ldots, \chi_N \) is the standard basis of \( \mathbb{C}^N \), then the functions

\[
(2.3) \quad \frac{Z^j \chi_j}{\sqrt{c_k h^k}} \quad j = 1, \ldots, N, \quad k = 0, 1, 2, \ldots,
\]

are orthonormal in \( L^2_{\mathbb{C}^N}(\Omega, d\mu_h) \). Let \( \mathcal{H}_h \) be the subspace spanned by these functions. Then the function

\[
(2.4) \quad K_h(X, Y) = \sum_{k=0}^{\infty} \frac{X^k Y^*}{c_k h^k}
\]

converges for all \( X, Y \in \Omega \) and is the reproducing kernel of \( \mathcal{H}_h \), in the sense that

\[
\int_\Omega K_h(X, Y) f(Y) d\mu_h(Y) = f(X), \quad \forall f \in \mathcal{H}_h, \forall X \in \Omega.
\]

Our second domain will be the subset \( \Omega_{\text{norm}} = \{ Z \in \mathbb{C}^{N \times N} : Z^* Z = ZZ^* \} \) of all normal matrices in \( \mathbb{C}^{N \times N} \). By the Spectral Theorem, any \( Z \in \Omega_{\text{norm}} \) can be written in the form

\[
(2.5) \quad Z = U^* DU,
\]

with \( U \in U(N) \) unitary and \( D \) diagonal; \( D \) is determined by \( Z \) uniquely up to permutation of the diagonal elements, and if the latter are all distinct and their order has been fixed in some way, then \( U \) is unique up to left multiplication by a diagonal matrix with unimodular elements. Consequently, there exists a unique measure \( d\mu_h(Z) \) on \( \Omega_{\text{norm}} \) such that

\[
\int_{\Omega_{\text{norm}}} f(Z) d\mu_h(Z) = (\pi h)^{-N} \int_{U(N)} \int_{\mathbb{C}^N} f(U^* DU) e^{-\|D\|^2/h} dU dD \quad \forall f,
\]

\[2\]The authors are grateful to M. Bertola for this result.
where $dU$ is the normalized Haar measure on $U(N)$, and $dD$ is the Lebesgue measure on $\mathbb{C}^N$, where we are identifying the diagonal matrix $D = \text{diag}(d_1, \ldots, d_N)$ with the vector $d = (d_1, \ldots, d_N) \in \mathbb{C}^N$, and $\|D\|^2 = \|d\|^2 := |d_1|^2 + \cdots + |d_N|^2$. Again, one easily checks that

$$\int_{\Omega_{\text{norm}}} Z^{\ast j} Z^k \, d\mu_h(Z) = \delta_{jk} k! h^k I,$$

so that the elements are orthogonal also in $L^2_{\mathbb{C}^N}(\Omega_{\text{norm}}, d\mu_h)$, and we let $\mathcal{H}_h$ be the subspace spanned by them. The reproducing kernel is then given by

$$(2.7) \quad K_h(X, Y) = \sum_{k=0}^{\infty} \frac{X^k Y^{\ast k}}{k! h^k}$$

(with the series converging for all $X, Y \in \mathbb{C}^{N \times N}$), in the sense that

$$\int_{\Omega_{\text{norm}}} K_h(X, Y) f(Y) \, d\mu_h(Y) = f(X), \quad \forall f \in \mathcal{H}_h, \forall X \in \Omega_{\text{norm}}.$$

Remark. At first sight, the most natural candidate for the vector-valued space $\mathcal{H}$ would seem to be the subspace $L^2_{\text{hol}, \mathbb{C}^N}(\Omega, d\mu)$ of all holomorphic functions in $L^2_{\mathbb{C}^N}(\Omega, d\mu)$ (i.e. of all square-integrable $\mathbb{C}^N$-valued functions which depend holomorphically on the coordinates $z_1, \ldots, z_N$ of the point $z \in \Omega$). However, this choice turns out to be too simple-minded: the reproducing kernel is then just $k(x, y) I$, where $k(x, y)$ is the reproducing kernel of the ordinary (scalar-valued) space $L^2_{\text{hol}}(\Omega, d\mu)$: the Toeplitz operator $T_\phi$ (to be introduced in the next section) is just the $N \times N$ matrix $[T_{\phi, jk}]_{j,k=1}^{N}$ of Toeplitz operators on $L^2_{\text{hol}}(\Omega, d\mu)$; and the Berezin transforms (also to be introduced in the next section) are just $\overline{T_\phi} = [\overline{T_{\phi, jk}}]_{j,k=1}^{N}$ and $\overline{T_\phi T_\psi} = [\sum_{k=1}^{N} T_{\phi, jk} \overline{T_{\psi, ik}}]_{j,k=1}^{N}$. Thus, for instance, for the spaces $L^2_{\text{hol}, \mathbb{C}^N}$ with $n = 1$ (i.e. on $\Omega = \mathbb{C}$), $\|T_{\phi}^{(h)} T_{\psi}^{(h)} - T_{\phi \psi}^{(h)}\| \to 0$ as $h \to 0$, while

$$\| \frac{\overline{T_{\phi}}}{\overline{T_{\phi} T_{\psi}^{(h)}}} - \overline{T_{\phi \psi}^{(h)}} \| \to 0,$$

where

$$[\phi, \psi] := \partial \phi \cdot \overline{\partial \psi} - \partial \psi \cdot \overline{\partial \phi}$$

(here $\partial, \overline{\partial}$ are applied individually to each element of a matrix, and the dot stands for matrix multiplication). From the physical point of view, this “matrix-valued Poisson bracket” seems to be a rather doubtful object, indicating that the spaces $L^2_{\text{hol}, \mathbb{C}^N}$ are probably not the right route to take.

3. The Berezin transform

Let, quite generally, $\mathcal{H}$ be a reproducing kernel subspace of $\mathbb{C}^N$-valued functions in $L^2_{\mathbb{C}^N}(\Omega, d\mu)$, for some domain $\Omega$ and measure $\mu$ on it, with reproducing kernel $K$; thus $K$ is a $\mathbb{C}^{N \times N}$-valued function on $\Omega$ and

$$(3.1) \quad f(X) = \int_{\Omega} K(X, Y) f(Y) \, d\mu(Y) \quad \forall X \in \Omega, \ f \in \mathcal{H}.$$ 

In particular, for any $\chi \in \mathbb{C}^N$, the functions

$$K_{Y, \chi}(X) := K(X, Y)\chi$$

belong to \( \mathcal{H} \), and
\[
(3.2) \quad \langle f, K_{X,\chi}\rangle_{\mathcal{H}} \equiv \int_{\Omega} K_{X,\chi}^* f \, d\mu = \chi^* f(Y).
\]

See [AAG] for more information on such spaces and their reproducing kernels.

Let \( T \) be an arbitrary bounded linear operator on \( \mathcal{H} \). For any fixed \( X \in \Omega \), the expression \( \langle TK_{X,\chi}, K_{X,\chi'}\rangle \) is evidently linear in \( \chi \) and \( \chi^* \); thus there exists a unique \( N \times N \) matrix \( \widetilde{T}(X) \) such that
\[
\chi^* \widetilde{T}(X) \chi = \langle TK_{X,\chi}, K_{X,\chi'}\rangle.
\]

We define the Berezin transform \( \widetilde{T} \) of \( T \) by
\[
\widetilde{T}(X) := K(X, X)^{-1/2} \chi^* \widetilde{T}(X) K(X, X)^{-1/2}.
\]

That is,
\[
\chi^* \widetilde{T}(X) \chi = \langle TK_{X,K(X,X)^{-1/2}}, K_{X,K(X,X)^{-1/2}}\rangle_{\mathcal{H}}.
\]

At first sight, this definition may seem a little \textit{ad hoc}; the reason behind it is that this seems to be the only way to make the following statements true.

**Proposition 1.** \( \widetilde{T} \) is a \( \mathbb{C}^{N \times N} \)-valued function on \( \Omega \) satisfying
\begin{enumerate}[(i)]
  \item \( \widetilde{T}^* = (\widetilde{T})^* \);
  \item if \( \phi \) is a matrix-valued function on \( \Omega \) such that \( \phi K_{X,\chi} \in \mathcal{H} \) for all \( X \in \Omega \) and \( \chi \in \mathbb{C}^N \), then
    \[
    \widetilde{M}_\phi(X) = K(X, X)^{-1/2} \phi(X) K(X, X)^{1/2}
    \]
    where \( M_{\phi} := \phi f \);
  \item in particular, \( \widetilde{T}(X) = I \ \forall X \in \Omega \);
  \item \( \|\widetilde{T}(X)\|_{\mathbb{C}^N \rightarrow \mathbb{C}^N} \leq \|T\|_{\mathcal{H} \rightarrow \mathcal{H}}, \ \forall X \in \Omega \).
\end{enumerate}

If the elements of \( \mathcal{H} \) are holomorphic functions, then also
\begin{enumerate}[(i)]
  \item \( \widetilde{T}(X) = 0 \ \forall X \) only if \( T = 0 \).
\end{enumerate}

**Proof.** (i) is immediate from the definition, while (ii) follows from the reproducing property [31], and (iii) is a trivial special case of (ii). To prove (iv), observe that, by [32],
\[
\|K_{X,\chi}\|_{\mathcal{H}}^2 = \langle K_{X,\chi}, K_{X,\chi}\rangle_{\mathcal{H}} = \chi^* K_{X,\chi}(X) = \chi^* K(X, X) \chi = \|K(X, X)^{1/2}\chi\|_{\mathbb{C}^N}^2.
\]

Thus, for any \( \chi, \chi' \in \mathbb{C}^N \),
\[
|\chi^* \widetilde{T}(X) \chi| = |\langle TK_{K(X,X)^{-1/2}}, K_{K(X,X)^{-1/2}}\rangle| \\
\leq \|T\| \|K_{X,K(X,X)^{-1/2}}\| \|K_{K(X,X)^{-1/2}}\| \\
= \|T\| \|\chi\|_{\mathbb{C}^N} \|\chi'\|_{\mathbb{C}^N},
\]
and the assertion follows.

Finally, for (v), note that \( \widetilde{T} \equiv 0 \) implies \( \widetilde{T} \equiv 0 \), i.e.
\[
(3.3) \quad \langle TK_{X,\chi'}, K_{X,\chi}\rangle = 0 \quad \forall \chi, \chi'
\]
whenever \( X = Y \). If \( \mathcal{H} \) consists of holomorphic functions, then \( K(X, Y) \) is holomorphic in \( X \) and conjugate-holomorphic in \( Y \); thus the left-hand side of (3.3) is holomorphic in \( X \) and conjugate-holomorphic in \( Y \). It is well known that such
functions are uniquely determined by their restriction to the diagonal $X = Y$; consequently, (3.3) holds for all $X, Y$, i.e.

$$\chi^*(TK_{Y,\chi})(X) = 0 \quad \forall X, \chi, \forall Y.$$ 

Hence $TK_{Y,\chi'} = 0$ for all $Y$ and $\chi'$, and thus for any $f \in \mathcal{H}$

$$\chi^*(T^* f)(Y) = \langle T^* f, K_{Y,\chi'} \rangle = (f, TK_{Y,\chi'}) = 0,$$

i.e. $T^* f = 0$. Thus $T^* = 0$ and $T = 0$. \hfill \square

In analogy with the scalar-valued situation, we next define for any $\phi \in C_{\infty}^{\infty} (\Omega)$ the Toeplitz operator $T_{\phi}$ on $H$ by the recipe

$$T_{\phi} f(X) := \int_{\Omega} K(X,Y)\phi(Y)f(Y)\,d\mu(Y),$$

or, equivalently,

$$T_{\phi} f = P(\phi f),$$

where $P : L_{\infty}^{2}(\Omega, d\mu) \to \mathcal{H}$ is the orthogonal projection. Note that the last formula implies that $\|T_{\phi}\|_{H \to H} \leq \|\phi\|_{\infty} := \sup_{X \in \Omega} \|\phi(X)\|_{C_{\infty}^{\infty} \to C_{\infty}^{\infty}}$; in particular, by Proposition 1, also $\|\tilde{T}_{\phi}(X)\|_{C_{\infty}^{\infty} \to C_{\infty}^{\infty}} \leq \|\phi\|_{\infty}$.

**Proposition 2.** The following formulae hold:

$$\tilde{T}_{\phi}(X) = K(X,X)^{-1/2} \cdot \int_{\Omega} K(X,Y)\phi(Y)\,d\mu(Y),$$

$$\tilde{T}_{\phi}T_{\psi}(X) = K(X,X)^{-1/2} \cdot \int_{\Omega} \int_{\Omega} K(X,Y)\phi(Y)K(Y,Z)\psi(Z)\,d\mu(Y)\,d\mu(Z) \cdot K(X,X)^{-1/2}.$$

In particular, if $\phi$ is a multiplier of $\mathcal{H}$ (i.e. $\phi f \in \mathcal{H}$ whenever $f \in \mathcal{H}$), then

$$\tilde{T}_{\phi}(X) = K(X,X)^{-1/2}\phi(X)K(X,X)^{1/2};$$

and, similarly, when $\phi$ and $\psi^*$ are multipliers of $\mathcal{H}$, then

$$\tilde{T}_{\phi}T_{\psi}(X) = K(X,X)^{-1/2}\phi(X)K(X,X)\psi(X)K(X,X)^{-1/2}.$$

4. **Bad behaviour: all matrices**

We now exhibit an example of some pathological phenomena, showing in particular that the straightforward generalizations of the expansions (1.11) and (1.12) cannot hold: first, apart from the integer powers of $h$, we will see that also $\sqrt{h}$ enters the picture; and second, instead of evaluations at $X$ we get also contributions from other points. This applies to the case of the full matrix domain $\Omega = C_{N \times N}$; for the normal matrices $\Omega_{\text{norm}}$, we will show in the next section that at least the second of these pathologies still remains.

**Theorem 3.** Consider the full matrix domain $\Omega = C_{N \times N}$ with $N = 2$. Let $\mathcal{H}_{h}$ be the spaces from Section 2 with reproducing kernels $K_{h}$ given by (2.4). Let $X$ be the matrix

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then:
\[ \lim_{h \to 0} T_{\phi}^{(h)}(X) = \begin{pmatrix} \phi_{11}(0)+\phi_{22}(0) & 0 \\ 0 & \phi_{22}(0) \end{pmatrix}. \]

Note that the matrix on the right-hand side does not depend on the value of \( \phi \) at \( X \), but rather on its value at \( 0 \).

(ii) If \( \phi(Y) = \sqrt{2}y_{22}I \), where \( y_{22} \) denotes the \((2, 2)\)-entry of \( Y \), then

\[ \widetilde{T}_{\phi}^{(h)}(X) = X\sqrt{h} + O(h) \quad \text{as} \quad h \to 0. \]

**Proof.** Since \( X^2 = 0 \), the series (4.1) for \( K_{h}(X, Y) \) becomes simply

\[ K_{h}(X, Y) = I + \frac{XY^*}{c_1 h} = I + \frac{XY^*}{2h}. \]

Thus

\[ \widetilde{T}_{\phi}^{(h)}(X) = \int_{\Omega} K_{h}(X, Y) \phi(Y) K_{h}(Y, X) \, d\mu_h(Y) \]

(4.2)

\[ = \int_{\Omega} \left[ \phi(Y) + \frac{XY^*\phi(Y)}{2h} + \frac{\phi(Y)YY^*}{2h} + \frac{XY^*\phi(Y)YY^*}{4h^2} \right] \, d\mu_h(Y). \]

On the other hand, the change of variable \( Z = Y/\sqrt{h} \) and Taylor’s expansion imply that for any \( C^\infty \) function \( f \) on \( \Omega \),

\[ \int_{\Omega} f(Y) \, d\mu_h(Y) = \int_{\Omega} f(\sqrt{h}Z) \, d\mu_1(Z) \]

(4.3)

\[ = \sum_{j=0}^{\infty} \frac{1}{j!} \Delta^j f(0) \, h^j, \]

where \( \Delta = \sum_{j,k=1}^{2} \partial^2/\partial z_{jk}\partial\overline{z}_{jk} \) is the Laplacian on \( C^2 \times C^2 \). Applying this to (4.2), we therefore get

\[ \widetilde{T}_{\phi}^{(h)}(X) = \phi(0) + O(h) \]

\[ + \frac{X[\Delta(Y^*\phi(Y))(0) + O(h)]}{2} \]

\[ + \frac{[\Delta(\phi(Y)Y)(0) + O(h)]X^*}{2} \]

\[ + X\left[ \frac{\Delta(Y^*\phi(Y)Y)(0)}{4h} + O(1) \right] X^*. \]

Now \( \widetilde{T}_{\phi}^{(h)}(X) = K(X, X)^{-1/2} \widetilde{T}_{\phi}^{(h)}(X) K(X, X)^{-1/2} \); note that

\[ K_{h}(X, X)^{-1/2} = \begin{pmatrix} \sqrt{\frac{2h}{2h+1}} & 0 \\ 0 & 1 \end{pmatrix}, \]
and thus, for any matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \),

\[
K(X, X)^{-1/2}A K(X, X)^{-1/2} = \begin{pmatrix} \frac{2h}{2h+1}a_{11} & \sqrt{\frac{2h}{2h+1}}a_{12} \\ \sqrt{\frac{2h}{2h+1}}a_{21} & \frac{2h}{2h+1}a_{22} \end{pmatrix},
\]

\[(4.4)\]

\[
K(X, X)^{-1/2} A K(X, X)^{-1/2} = \begin{pmatrix} \frac{2h}{2h+1}a_{11} \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2h}{2h+1}}a_{12} \\ 0 \end{pmatrix},
\]

Consequently,

\[
\tilde{T}_\phi^{(h)}(X) = \begin{pmatrix} 0 \\ \phi_{21}(0)\sqrt{\frac{2h}{2h+1}} \phi_{22}(0) \end{pmatrix} + O(h)
+
\begin{pmatrix} 0 \\ \frac{2h}{2h+1} \Delta[Y^*\phi(Y)]_{22}(0) \end{pmatrix} + O(h)
+
\begin{pmatrix} \frac{2h}{2h+1} \Delta[\phi(Y)]_{22}(0) \\ 0 \end{pmatrix} + O(h)
+
\begin{pmatrix} \frac{1}{2} \Delta[Y^*\phi(Y)]_{22}(0) \\ 0 \end{pmatrix} + O(h).
\]

Note that, by a simple calculation,

\[
\Delta[Y^*\phi(Y)]_{ab}(0) = \sum_{k=1}^{2} \frac{\partial \phi_{kb}}{\partial Y_{ka}}(0),
\]

\[
\Delta[\phi(Y)]_{ab}(0) = \sum_{k=1}^{2} \frac{\partial \phi_{ab}}{\partial Y_{kb}}(0),
\]

\[
\Delta[Y^*\phi(Y)]_{ab}(0) = \delta_{ab} \cdot \text{Tr} \phi(0).
\]

Letting \( h \to 0 \), (i) therefore follows immediately. For \( \phi(Y) = y_{22} \sqrt{2} I \), only the second term in the last formula for \( \tilde{T}_\phi^{(h)}(X) \) gives a nonzero contribution, equal to

\[
\tilde{T}_\phi^{(h)}(X) = \begin{pmatrix} 0 \\ \sqrt{h} \Delta[y_{22}Y^*]_{22}(0) \end{pmatrix} + O(h) = \sqrt{h} X + O(h),
\]

as \( \Delta[y_{22}Y^*]_{22}(0) = 1 \) by (4.5). This settles (ii). \( \square \)

**Remark.** We pause to observe that applying (4.3) to \( f(Y) = Y^*kY^k \) and comparing with (2.1) shows that

\[
\Delta^k(Y^*kY^k)(0) = k!c_k I.
\]

This gives, conceivably, a way of evaluating the numbers \( c_k \) without recourse to random matrix theory, and can also be used to show that the \( c_k \) have an interesting combinatorial meaning. Namely, expanding \( \Delta^k \) and \( Y^*kY^k \) yields
\[ \Delta^k(Y^k Y^k) = \sum_{i_1, j_1, \ldots, i_k, j_k = 1}^N \partial_{i_1 j_1} \bar{\partial}_{i_1 j_1} \cdots \partial_{i_k j_k} \bar{\partial}_{i_k j_k} \]
\[ \sum_{\alpha_1, \ldots, \alpha_{2k-1} = 1}^N \bar{y}_{\alpha_1} \alpha_{\alpha_2} \cdots \bar{y}_{\alpha_k} a_{\alpha_k+1} y_{\alpha_{k+1} a_{k+2}} \cdots y_{a_{2k-1} b}, \]

where, for the sake of brevity, we temporarily write \( \partial_{ij} \) for \( \partial^2 / \partial y_{ij} \), and similarly for \( \bar{\partial}_{ij} \). Clearly a nonzero contribution only occurs if to each \( y \) there is applied precisely one \( \partial \), and to each \( \bar{\partial} \) precisely one \( \bar{\partial} \). (We also see that the result will be independent of \( Y \), i.e. a constant.) Thus

\[ \Delta^k(Y^k Y^k) = \sum_{i_1, j_1, \ldots, i_k, j_k = 1}^N \sum_{a_1, \ldots, a_{2k-1} = 1}^N \bar{\partial}_{i_1 j_1} \bar{\partial}_{i_1 j_1} \cdots \bar{\partial}_{i_k j_k} \bar{\partial}_{i_k j_k} \]
\[ \sum_{\sigma, \tau \in \mathfrak{S}_k} \delta_{\sigma(1)} a_1 \delta_{\tau(1)} a_1 \cdots \delta_{\sigma(1)} a_k \delta_{\tau(1)} a_{k+1} \cdots \delta_{\tau(1)} a_{2k-1} \delta_{\tau(1)} b. \]

Changing the order of summations and using the fact that \( \partial_{ij} y_{kl} = \delta_{ik} \delta_{jl} \), this becomes

\[ \Delta^k(Y^k Y^k) = \sum_{\sigma, \tau \in \mathfrak{S}_k} \sum_{i_1, j_1, \ldots, i_k, j_k = 1}^N \sum_{a_1, \ldots, a_{2k-1} = 1}^N \delta_{\tau(1)} a_1 \delta_{\tau(1)} a_1 \cdots \delta_{\tau(1)} a_k \delta_{\tau(1)} a_{k+1} \cdots \delta_{\tau(1)} a_{2k-1} \delta_{\tau(1)} b. \]

Replacing \( i = (i_1, \ldots, i_k), j = (j_1, \ldots, j_k) \) by \( i \circ \tau^{-1}, j \circ \tau^{-1} \) and setting \( \mu = \tau^{-1} \sigma \), we thus get

\[ \Delta^k(Y^k Y^k) = k! \sum_{\mu \in \mathfrak{S}_k} \sum_{i \in \mu} \delta_{i_1 a} \delta_{i_2 a_2} \cdots \delta_{i_k a_{k-1}} \]
\[ \delta_{\mu(1)} i_k \delta_{\mu(2)} i_{k-1} \cdots \delta_{\mu(k)} i_{k-1} \delta_{\mu(k)} b. \]

In other words, \([\Delta^k(Y^k Y^k)]_{ab}/k! = c_k \delta_{ab}\) is the constant equal to the number of triples \((\mu, i, j)\), where \( \mu \in \mathfrak{S}_k \) and \( i, j \in \{1, \ldots, N\}^k \), such that

\[ j = (a, i_1, \ldots, i_{k-1}), \quad j \circ \mu = (i_{\mu(2)}, \ldots, i_{\mu(k)}, b), \quad \text{and} \quad i_k = i_{\mu(1)}. \]

It is evident that, indeed, this number is zero for \( a \neq b \) (since the sequences \((a, i_1, \ldots, i_{k-1}, i_k)\) and \((i_{\mu(2)}, \ldots, i_{\mu(k)}, b)\) must be permutations of each other), while for \( a = b \) it is independent of \( a \). It is also clear that \( c_k \) is always an integer, a fact definitely not apparent from \( \mathfrak{2.2} \).

We conclude by giving a formula analogous to part (i) of the last theorem also for \( \widetilde{T}_\phi^{(h)} \widetilde{T}_\psi^{(h)}(X) \). It shows, in particular, that \( \widetilde{T}_\phi^{(h)} T_\psi^{(h)}(X) - T_\phi^{(h)}(X) \) need not tend to 0 in general as \( h \to 0 \), but does so for scalar-valued \( \phi \) and \( \psi \).

**Theorem 4.** Under the hypotheses of Theorem 3

\[ \lim_{h \to 0} \widetilde{T}_\phi^{(h)} T_\psi^{(h)}(X) = \left( \frac{\text{Tr } \phi(0)}{2} \cdot \frac{\text{Tr } \psi(0)}{2} \right) \cdot \frac{0}{(\phi \psi)(22)(0)}. \]
Proof. Using the formula for $\widetilde{T}_\phi^{(h)}\widetilde{T}_\psi^{(h)}$ from Proposition 2 and 3, we have

$$T_\phi^{(h)}T_\psi^{(h)}(X) = \int_\Omega \int_\Omega \left( I + \frac{XY^*}{2h} \right) \phi(Y) K_h(Y, Z) \psi(Z) \left( I + \frac{Z^*X}{2h} \right) d\mu_h(Y) d\mu_h(Z).$$

Making again the change of variable $Y \mapsto Y \sqrt{h}$, $Z \mapsto Z \sqrt{h}$, the double integral becomes

$$\int_\Omega \int_\Omega \left( I + \frac{XY^* \sqrt{h}}{2h} \right) \phi(\sqrt{h}Y) K_1(Y, Z) \psi(\sqrt{h}Z) \left( I + \frac{Z^*X \sqrt{h}}{2h} \right) d\mu_1(Y) d\mu_1(Z).$$

The rest of the proof proceeds in the same way as in Theorem 3, using instead of (4.3) the expansion from the following lemma (applied also to $Y^*\phi(Y)$ and $\psi(Z)Z$ in place of $\phi(Y)$ and $\psi(Z)$, respectively), the fact that

$$\langle \{Y^*\phi(Y), \psi(Z)Z \} \rangle_{22} = \frac{1}{4} \text{Tr} \phi(0) \text{Tr} \psi(0),$$

and the relations (4.4). We leave the details to the reader. □

**Lemma 5.** For $\phi, \psi \in C^\infty_{\overline{\mathbb{C}^N \times \mathbb{C}^N}}(\Omega)$,

$$\int_\Omega \int_\Omega \phi(\sqrt{h}Y) K_1(Y, Z) \psi(\sqrt{h}Z) d\mu_1(Y) d\mu_1(Z) = \phi(0) \psi(0) + h [\Delta \phi(0) \cdot \psi(0) + \phi(0) \cdot \Delta \psi(0) + \{\phi, \psi\}(0)] + O(h^2),$$

as $h \to 0$, where

$$\{\phi, \psi\} := \frac{1}{2} \sum_{i,j,k} \frac{\partial \phi}{\partial y_{ik}} E_{ik} \frac{\partial \psi}{\partial z_{kj}},$$

where $E_{ik}$ is the matrix $[E_{ik}]_{ab} = \delta_{ia} \delta_{kb}$.

**Proof.** Using the Taylor expansions for $\phi$ and $\psi$, we see that the integral asymptotically equals

$$\sum_{\alpha,\beta,\gamma,\delta} h^{\frac{|\alpha|+|\beta|+|\gamma|+|\delta|}{2}} \frac{\partial^\alpha \overline{\partial} \phi(0)}{\alpha! \beta!} \int_\Omega \int_\Omega y^{\alpha} \overline{y}^{\beta} K_1(Y, Z) z^{\gamma} \overline{z}^{\delta} d\mu_1(Y) d\mu_1(Z) \frac{\partial^\gamma \overline{\partial} \psi(0)}{\gamma! \delta!}.$$

(The summation extends over all multiindices $\alpha,\beta,\gamma,\delta$.) Note that the kernel satisfies $K_1(Y, Z) = K_1(\epsilon Y, \epsilon Z)$ for any $\epsilon \in \mathbb{C}$ of modulus one; hence the last integral vanishes unless $|\alpha|+|\gamma| = |\beta|+|\delta|$. Thus the coefficients at half-integer powers of $h$ in fact vanish. The coefficient at $h^0$ is clearly $\phi(0)\psi(0)$, since

$$\int_\Omega \int_\Omega K_1(Y, Z) d\mu_1(Y) d\mu_1(Z) = \int_\Omega \int_\Omega I d\mu_1(Y) = I$$

by the reproducing property of $K_1$ and 2.4. For the coefficient at $h^1$, the only nonzero contributions, by virtue of the last observation, come from $|\alpha| = |\beta| = 1$, or $|\gamma| = |\delta| = 1$, or $|\alpha| = |\delta| = 1$, or $|\beta| = |\gamma| = 1$ (i.e. from $y \overline{y}$, $z \overline{z}$, $y z$, or $\overline{y} \overline{z}$). Since

$$\int_\Omega \int_\Omega y_{ij} \overline{y}_{kl} K_1(Y, Z) d\mu_1(Y) d\mu_1(Z) = \int_\Omega y_{ij} \overline{y}_{kl} I d\mu_1(Y) = \delta_{ik} \delta_{jl} I$$

(and similarly for $z_{ij} \overline{z}_{kl}$), the first two possibilities contribute

$$\sum_{i,j,k,l} \frac{\partial^2 \phi(0)}{\partial y_{ij} \partial y_{kl}} \delta_{ik} \delta_{jl} I \psi(0) = \Delta \phi(0) \cdot \psi(0)$$
and $\phi(0) \cdot \Delta \psi(0)$, respectively. For the $yz$ possibility, the corresponding integral vanishes, since the integrand is a function holomorphic in the entries of $Y$ and $Z^*$ and vanishing at the origin. Finally, for the last possibility $yz$ we use the series (2.4) to split the integral as

$$
\int \Omega \int \Omega y_{ij} z_{kl} K_1(Y, Z) d\mu_1(Y) d\mu_1(Z) = \sum_{m=0}^{\infty} \frac{1}{c_m} \left( \int \Omega y^m d\mu_1(Y) \right) \left( \int \Omega z^m d\mu_1(Z) \right).
$$

Using again the invariance of $d\mu_1$ under the change of variable $Y \mapsto \epsilon Y$, we see that we only get nonzero contribution for $m = 1$. In that case,

$$
\left[ \int \Omega y_{ij} Y d\mu_1(Y) \right]_{ab} = \int \Omega y_{ij} y_{ab} d\mu_1(Y) = \delta_{ia} \delta_{jb} = [E_{ij}]_{ab},
$$

and similarly for the $Z$ integral. Thus the integral (4.7) equals

$$
\frac{1}{c_1} E_{ij} E_{lk} = \frac{1}{2} \delta_{il} E_{jk},
$$

and the total contribution from the $yz$ possibility is

$$
\sum_{i,j,k,l} \frac{\partial \phi}{\partial y_{ij}}(0) \frac{\delta_{jl}}{2} E_{ik} \frac{\partial \psi}{\partial z_{kl}}(0) = \{\{\phi, \psi\}\}(0),
$$

which concludes the proof of the lemma.

5. Bad behaviour: normal matrices

The matrix $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ featuring in the last section was not normal; this tempts one to hope that things might perhaps still work out fine for the domain $\Omega_{\text{norm}}$ of all normal matrices. We show that even in this case, unfortunately, the non-local behaviour described above still persists.

**Theorem 6.** Consider the domain $\Omega_{\text{norm}}$ of all normal $N \times N$ matrices, with $N = 2$. Let $\mathcal{H}_h$ be the spaces from Section 4 with reproducing kernels $K_h$ given by (2.7). Let $X$ be the matrix

$$
X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

of the projection onto the first coordinate. Then

$$
[T_{\phi}^{(h)}(X)]_{11} = [T_{\phi}^{(h)}(I)]_{11},
$$

$$
[T_{\phi}^{(h)}(X)]_{22} = [T_{\phi}^{(h)}(0)]_{22}.
$$

Consequently, the asymptotic expansion (1.11) cannot hold.

**Proof.** As $X$ is a projection, we have $X^j = X \forall j \geq 1$; thus

$$
K_h(X, Y) = \sum_{j=0}^{\infty} X^j Y^{\ast j} = I + X \sum_{j=1}^{\infty} \frac{Y^{\ast j}}{j! h^j} = (I - X) + X K_h(I, Y).
$$

Thus

$$
\tilde{T}_{\phi}^{(h)}(X) = \int_{\Omega_{\text{norm}}} K_h(X, Y) \phi(Y) K_h(Y, X) d\mu_h(Y)
$$
\[
X \cdot \int_{\Omega_{\text{norm}}} K_h(I,Y) \phi(Y) K_h(Y,I) \, d\mu_h(Y) \cdot X \\
+ X \cdot \int_{\Omega_{\text{norm}}} K_h(I,Y) \phi(Y) \, d\mu_h(Y) \cdot (I - X) \\
+ (I - X) \cdot \int_{\Omega_{\text{norm}}} \phi(Y) K_h(Y,I) \, d\mu_h(Y) \cdot X \\
+ (I - X) \cdot \int_{\Omega_{\text{norm}}} \phi(Y) \, d\mu_h(Y) \cdot (I - X).
\]

But for any matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \), we have
\[
XAX = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix},
\]
\[
XA(I - X) = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix},
\]
eetc.; hence
\[
\tilde{T}^{(h)}(X)_{11} = \left[ \int_{\Omega_{\text{norm}}} K_h(I,Y) \phi(Y) K_h(Y,I) \, d\mu_h(Y) \right]_{11} = \tilde{T}^{(h)}(I)_{11},
\]
and similarly, since \( K_h(0,Y) = I \),
\[
\tilde{T}^{(h)}(X)_{22} = \left[ \int_{\Omega_{\text{norm}}} \phi(Y) \, d\mu_h(Y) \right]_{22} = \tilde{T}^{(h)}(0)_{22}.
\]
Finally, since \( K_h(X,X) = \begin{pmatrix} e^{1/h} & 0 \\ 0 & 1 \end{pmatrix} \), \( K_h(I,I) = e^{1/h}I \), and \( K_h(0,0) = I \), the assertion about \( \tilde{T}^{(h)}(X) = K(X,X)^{-1/2} \tilde{T}^{(h)}(X)K(X,X)^{-1/2} \) follows. \( \Box \)

For completeness, we also state the analog of Theorem 6, which shows, among others, that the expansion (1.12) cannot hold. Its proof is the same as for Theorem 6.

**Theorem 7.** In the situation of the preceding theorem,
\[
\tilde{T}^{(h)}(X)_{11} = \left[ \tilde{T}^{(h)}(I)_{11},
\tilde{T}^{(h)}(X)_{22} = \tilde{T}^{(h)}(0)_{22}.
\]

6. **A n Application of Stationary Phase**

In this section we finally start exhibiting also results in the positive direction, namely, by using the stationary phase method we establish the existence of an (albeit non-local) semiclassical asymptotic expansion for \( \tilde{T}^{(h)}_\phi \) for the case of the normal matrices.

Recall that the stationary phase (WJKB) method tells us that if \( S, \phi \) are smooth complex-valued functions on some domain in \( \mathbb{C}^n \), such that \( S \) has a unique critical point \( x_0 \) (i.e. \( S'(x_0) = 0 \)), which is nondegenerate (i.e. \( \det S''(x_0) \neq 0 \)) and is a global maximum for \( \text{Re} \, S \), and \( \phi \) is compactly supported, then the integral
\[
h^{-n} \int \phi(x) \, e^{S(x)/h} \, dx
\]
has an asymptotic expansion

\[ e^{S(x_0)/\hbar} \sum_{j=0}^{\infty} \hbar^j L_j \phi(x_0) \quad \text{as } \hbar \to 0, \]

with some differential operators \( L_j \) whose coefficients are given by universal expressions in \( S \) and its partial derivatives. See e.g. [Hrm], Section 7.7. The hypothesis of the compact support of \( \phi \) can be replaced by the requirement that the integral \( (6.1) \) exist for some \( \hbar = \hbar_0 > 0 \), and that the maximum of \( \text{Re} S \) at \( x_0 \) strictly dominate also the values of \( \text{Re} S \) at the boundary or at infinity, in the sense that \( \text{Re} S(x_n) \to \text{Re} S(x_0) \implies x_n \to x_0 \).

On the other hand, if the global maximum of \( \text{Re} S \) is not a critical point, then \( (6.1) \) decays faster than any power of \( \hbar \) as \( \hbar \to 0 \).

The formulas for the operators \( L_j \) are fairly complicated in general, but fortunately become quite explicit if the phase function \( S \) is quadratic (which will be the only case we will need). Namely, assume that

\[ S(x) = -\langle Q(x - x_0), x - x_0 \rangle C^* \]

for some matrix \( Q \) with positive real part. Then \( x_0 \) is a unique critical point of \( S \), is nondegenerate, and

\[ L_j = \frac{1}{j!} Q^j, \quad \text{where} \quad Q = -\langle Q^{-1} \partial, \partial \rangle. \]

Let us now apply this to the integral defining \( \widetilde{T}_\phi^{(h)}(X) \) in the case of the domain of normal matrices, viz.

\[ \widetilde{T}_\phi^{(h)}(X) = \int_{\Omega_{\text{norm}}} K_h(X, X)^{-1/2} K_h(X, Y) \phi(Y) K_h(Y, X) K_h(X, X)^{-1/2} d\mu_h(Y). \]

Let

\[ Y = UDU^*, \quad X = VCV^* \]

be the spectral decompositions of \( Y \) and \( X \), respectively. Observe that owing to the invariance of the kernels \( K_h \) and the measures \( \mu_h \) under unitary transformations, we have \( T_\phi^{(h)}(X) = VT_\phi^{(h)}(C)V^* \), where \( \phi^V(Y) := V^* \phi(VYV^*)V \); thus it suffices to deal with the case of \( V = I \), i.e. when \( X = C = \text{diag}(c_1, \ldots, c_N) \) is a diagonal matrix. From (6.4), we then have

\[ [K_h(X, Y)]_{ij} = \sum_{k=0}^{\infty} \frac{[C^k U D^k U^*]_{ij}}{k! \hbar^k} = \sum_{k=0}^{\infty} \sum_{l=1}^{N} c^*_l u_{il} u^*_{lj} k! \hbar^k \]

\[ = \sum_{l=1}^{N} c^*_l u_{il} u^*_{jl}, \]
and similarly for $K_h(X, X)$. Thus the matrix entries of $\widetilde{T_{\phi}^{(h)}}(X)$ are given by

$$[T_{\phi}^{(h)}(X)]_{ab} = (\pi h)^{-N} \int_{U(N)} \int_{C^N} e^{-|c_a|^2/2h e^{c_a}} u_{al} \overline{\mu_{jl}} \overline{\phi_{jk}}(U DU^*) \cdot e^{i d_m h_{ab}} e^{-|c_b|^2/2h e^{c_b}} dU dD.$$  

For simplicity, we will write $\phi(U; d_1, \ldots, d_N)$ instead of $\phi(U DU^*)$. The last integral over $D$ is precisely of the form (6.1), with phase function given by

$$S(d_1, \ldots, d_N) = c_a d_i + c_b d_m - |d|^2 - \frac{|c_a|^2 + |c_b|^2}{2}. $$

The critical point condition $S' = 0$ amounts to

$$c_a \delta_{ii} = d_i, \quad c_b \delta_{mi} = d_i, \quad \forall i = 1, \ldots, N.$$  

It follows that there is no critical point if $c_a \neq c_b$, or if $c_a = c_b = 0$ and $l \neq m$; while for $c_a = c_b \neq 0$ and $l = m$, or $c_a = c_b = 0$ and $l, m$ arbitrary, there is a unique critical point

$$d = (0, \ldots, 0, \ c_a \chi_l, 0, \ldots, 0) \equiv c_a \chi_l,$$

which satisfies the assumptions for the application of the stationary phase method. The critical value is

$$S(c_a \chi_l) = |c_a|^2 + |c_a|^2 - |c_a|^2 - \frac{|c_a|^2 + |c_a|^2}{2} = 0,$$

and the operators $L_j$ are equal to $\frac{1}{2j} \Delta_j$, by (6.3). By (6.2), it therefore follows that

$$[T_{\phi}^{(h)}(X)]_{ab} = O(h^{\infty}) \quad \text{for} \ c_a \neq c_b,$$

as $h \rightarrow 0$ if $c_a = c_b \neq 0$, and

$$[T_{\phi}^{(h)}(X)]_{ab} \approx \sum_{j,k,l,m = 1}^{N} \sum_{r = 0}^{\infty} \frac{h^r}{r!} \int_{U(N)} u_{al} \overline{\mu}_{jl} u_{kl} \overline{\mu}_{mk} (\Delta_r(\phi)) (U; c_a \chi_l) dU$$

as $h \rightarrow 0$ if $c_a = c_b = 0$. Here the subscript at $\Delta$ indicates that it applies only to the $d$-variables in $\phi(U; d_1, \ldots, d_N)$.

Thus the coefficients at each $h^r$ in the asymptotic expansion do not depend on the jet of $\phi$ at $X$, but rather on the behaviour of $\phi$ near the whole orbit $\{UP_\alpha U^* : U \in U(N)\}$ of the spectral components $P_\alpha := \text{diag}(0, \ldots, 0, c_a, 0, \ldots, 0)$ of $X$. Also, the off-diagonal entries asymptotically vanish (i.e. are $O(h^{\infty})$) if $c_a \neq c_b$, which is quite unexpected.
Observe that setting \( c_a = 0 \) in (6.5) gives

\[
\left[ \tilde{T}_\phi^{(h)} \right]_{ab} \approx \sum_{j,k=1}^{N} \sum_{r=0}^{\infty} \frac{h^r}{r!} \kappa_{jk} \Delta_{(d)}^r \phi_{jk}(0)
\]

where

\[
\kappa_{jk} := \sum_{l=1}^{N} u_{al} \overline{u}_{jl} \overline{u}_{kl} \overline{u}_{bl} dU.
\]

It can be shown that

(6.7)

\[
\kappa_{jk} = \frac{\delta_{aj} \delta_{kb} + \delta_{ab} \delta_{kj}}{N + 1},
\]

and in fact,

(6.8)

\[
\int_{U(N)} u_{al} \overline{u}_{jl} \overline{u}_{kl} \overline{u}_{bl} dU = \frac{\delta_{aj} \delta_{kb} + \delta_{ab} \delta_{kj}}{N(N + 1)}.
\]

(The above relation can be obtained by an application of standard orthogonality relations for the matrix elements of irreducible representations of compact groups — in this case applied to the irreducible subrepresentation of \( U(N) \), carried by second order symmetric tensors, in the decomposition of the natural representation of \( U(N) \otimes U(N) \).) Thus we have

\[
\left[ \tilde{T}_\phi^{(h)} \right]_{ab} \approx \sum_{r=0}^{\infty} \frac{h^r}{r!} \Delta_{(d)}^r (\phi_{ab} + \delta_{ab} \text{Tr} \phi)(0) \left( \frac{\delta_{aj} \delta_{kb} + \delta_{ab} \delta_{kj}}{N + 1} \right),
\]

which is different from (6.6). Thus we see that, in general, it is not possible to use the formula (6.5) in both cases \( c_a = c_b \neq 0 \) and \( c_a = c_b = 0 \).

For scalar-valued \( \phi \), (6.5) simplifies to

\[
\left[ \tilde{T}_\phi^{(h)}(X) \right]_{ab} \approx \sum_{r=0}^{\infty} \frac{h^r}{r!} \int_{U(N)} u_{al} \overline{u}_{bl} (\Delta_{(d)}^r \phi)(0) \left( \delta_{aj} \delta_{kb} + \delta_{ab} \delta_{kj} \right) dU;
\]

and if in addition \( \phi \) is independent of \( U \), i.e. \( \phi(UYU^*) = \phi(Y) \) \( \forall U \in U(N) \), then the last integral can be evaluated by Schur’s orthogonality relations, yielding

(6.9)

\[
\left[ \tilde{T}_\phi^{(h)}(X) \right]_{ab} \approx \delta_{ab} \sum_{r=0}^{\infty} \frac{h^r}{r!} (\Delta_{(d)}^r \phi)(c_a \chi_a).
\]

In the general case, however, it does not seem that (6.5) can be simplified in any way.

In the same manner, one can also prove the following formula for the asymptotics of \( \tilde{T}_\phi^{(h)} \mid \tilde{T}_\psi^{(h)} \), which of course reduces to (6.3) upon taking for \( \psi \) the constant function equal to \( I \). The strange-looking operators \( \mathcal{M}_{mq} \) originate from the formula (6.3).

**Theorem 8.** For any functions \( \phi, \psi \in C_{\infty}^{N \times N}(\Omega_{\text{norm}}) \) and a diagonal matrix \( X = \text{diag}(c_1, \ldots, c_N) \),

\[
\left[ \tilde{T}_\phi^{(h)} \tilde{T}_\psi^{(h)}(X) \right]_{ab} = O(h^\infty) \quad \text{as } h \to 0
\]
if $c_a \neq c_b$;

$$[T^{(h)}_\phi T^{(h)}_\psi (X)]_{ab} \approx \sum_{i,j,k,l}^{N} \sum_{m=1}^{\infty} \frac{h^r}{r!} \int_{U(N)} \int_{U(N)} u_{am} w_{jm} u_{pm} w_{pq} w_{kl} \cdot w_{pq} w_{kl} (M_{mq} \phi^{i}_{j}) (U; W; c_a X_m, c_a \chi_q) \ dU \ dW$$

as $h \to 0$ if $c_a = c_b \neq 0$, where

$$(M_{mq} \phi^{i}_{j}) (U; W; d, e) : = \left[ \left( \Delta (d) + \Delta (e) + \frac{\partial^2}{\partial d \partial e} \right) \phi^{i}_{j} (U; d) \psi (W; e) \right];$$

and

$$[T^{(h)}_\phi T^{(h)}_\psi (X)]_{ab} \approx \sum_{i,j,k,l}^{N} \sum_{m=1}^{\infty} \frac{h^r}{r!} \int_{U(N)} \int_{U(N)} u_{am} w_{jm} u_{pm} \cdot w_{pq} w_{kl} (M_{mq} \phi^{i}_{j}) (U; W; 0, 0) \ dU \ dW$$

as $h \to 0$ if $c_a = c_b = 0$.

The formula (6.11) can clearly be simplified upon carrying out the summations over $L$ and $M$ and performing the two integrations (which can be done since $\phi (U; 0) = \phi (0)$ and $\psi (W; 0) = \psi (0)$ are independent of $U$ and $W$) via Schur’s orthogonality relations; the result is

$$[T^{(h)}_\phi T^{(h)}_\psi (X)]_{ab} \approx \sum_{m=1}^{N} \sum_{r=0}^{\infty} \frac{h^r}{r!} M_{mq} (\phi \psi)_{ab} (0, 0).$$

Similarly, as with (6.10) and (6.6), using (6.9) it can be shown that for $c_a = 0$ the formula (6.11) reduces to

$$[T^{(h)}_\phi T^{(h)}_\psi (X)]_{ab} \approx \sum_{m=1}^{N} \sum_{r=0}^{\infty} \frac{h^r}{r!} M_{mq} \left[ (\phi + I \text{ Tr } \phi) (\psi + I \text{ Tr } \psi) \right]_{ab} (0, 0),$$

which is different from (6.11). We refrain from going into these details because they are not needed anywhere in the sequel.

We conclude this section by observing that (1.6) also fails in general.

**Proposition 9.** Let $\phi$ be the $C^{N \times N}$-valued function on $\Omega_{\text{norm}}$, $N \geq 2$, defined by $\phi (Z) = | \det Z |^2 I$. Then

$$\lim_{h \to 0} T^{(h)}_\phi (X) = 0 \quad \forall X \in \Omega_{\text{norm}}.$$ 

**Proof.** Since $\phi (VZV^*) = V\phi (Z)V^* \forall V \in U(N)$, it is enough to check the assertion for diagonal $X$, so let $X = \text{diag} (c_1, \ldots, c_N)$. As $\phi^{ij} (U; d) = \delta_{jk} |d_1 \ldots d_N|^2$, we have

$$(\Delta_{ij} \phi^{ij}) (U; d) = \delta_{jk} \sum_{m} |d_1 \ldots \hat{d}_m \ldots d_N|^2,$$

$$(\Delta^2_{ij} \phi^{ij}) (U; d) = \delta_{jk} \sum_{m \neq n} |d_1 \ldots \hat{d}_m \ldots \hat{d}_n \ldots d_N|^2.$$
\( \delta_{jk} 2 \sum_{m_1 < m_2} |d_1 \ldots \hat{d}_{m_1} \ldots \hat{d}_{m_2} \ldots d_N|^2, \)

\[ \Delta_r^d (\phi_{jk}) (U; d) = \delta_{jk} r! \sum_{m_1 < m_2 < \cdots < m_r} |d_1 \ldots \hat{d}_{m_1} \ldots \hat{d}_{m_2} \ldots \hat{d}_{m_r} \ldots d_N|^2, \]

\[ \Delta_r^d (\phi_{jk}) (U; d) = 0 \quad \text{for } r > N. \]

(Here the hat \( \hat{\cdot} \) indicates that the corresponding variable is omitted.) Thus by (6.9) and (6.6)

\[ \tilde{T}_\phi (h) \approx \begin{cases} 
\delta_{ab} (|c_a|^2 h^{N-1} + h^N) + O(h^\infty), & \\
\text{as } h \to 0, \end{cases} \]

as \( h \to 0 \), and the assertion follows.

Similarly, it can be shown that (1.1) breaks down too: for instance, if \( \phi(Z) = |\det Z|^2 e^{-\text{Tr}(Z^*Z)} I \), one has

\[ \| T_\phi (h) \| = \| \phi \|_\infty^{1/N} h^{N-1} \quad \text{as } h \to 0 \]

(where \( \| \phi \|_\infty := \sup_{Z \in \Omega_{\text{norm}}} \| \phi(Z) \|_{C_N \to C_N} \)). This can be proved by observing that the operator \( T_\phi \) is diagonal with respect to the basis (2.3), with eigenvalues

\[ \frac{(k+1)h^N}{(h+1)^{2N+k}}, \]

and \( \sup_k (k+1)/(h+1)^k \approx 1/(eh) = \| \phi \|_\infty^{1/N}/h \). We omit the details.

We now turn to classes of observables \( \phi \) which are more manageable than the general case.

### 7. Spectral and \( U \)-invariant functions

A function \( \phi(Z) \) of \( Z \in \Omega_{\text{norm}} \) will be called spectral if it is a function of \( Z \) in the sense of the Spectral Theorem: that is, if there exists a function \( f : C \to C \) such that \( \phi = f^\# \), where

(7.1) \( f^\#(Z) := U \cdot \text{diag}_j (f(d_j)) \cdot U^* \quad \text{if } \quad Z = U \cdot \text{diag}_j (d_j) \cdot U^* . \)

Our first observation is that for spectral functions, all goes fine with the Berezin-Toeplitz quantization.

**Theorem 10.** If \( \phi = f^\# \) and \( \psi = g^\# \) are two smooth spectral functions, then there exist unique spectral functions \( \rho_r, r = 0, 1, 2, \ldots, \) such that

\[ T_\phi (h) T_\psi (h) \approx \sum_{r=0}^{\infty} h^r T_{\rho_r} (h) \quad \text{as } h \to 0 \]

in the sense of operator norms (i.e. as in (1.4)). In fact,

\[ \rho_r = C_r (f, g) \]

where

(7.2) \( C_r (f, g) = \frac{1}{r!} \partial^r f \cdot \overline{\partial^r g} \)
are the operators (1.10) for \( n = 1 \).

**Proof.** Recall that the monomials \( z^k \), \( k = 0, 1, 2, \ldots \), are orthogonal in the Segal-Bargmann space (1.7) for \( n = 1 \):

\[
\langle z^k, z^l \rangle_{L^2_{\text{hol}}(\mathbb{C}, d\mu_h)} = \delta_{kl} k! h^k.
\]

Comparing this with (2.6), we see that the mapping

(7.3)

\[ \iota : Z^k \chi_j \mapsto z^k \otimes \chi_j \]

is a unitary isomorphism of our space \( H_h \) onto the tensor product \( L^2_{\text{hol}}(\mathbb{C}, d\mu_h) \otimes \mathbb{C}^N \).

Now if \( \phi = f^\# \) is a spectral function and \( \chi, \eta \in \mathbb{C}^N \), then

\[
\langle T_\phi(h) z^k \chi, z^l \eta \rangle = \langle \phi(Z) z^k \chi, Z^l \eta \rangle_{L^2_{\text{hol}}(\Omega_{\text{norm}}, d\mu_h)}
\]

\[
= \int_{\Omega_{\text{norm}}} \eta^* Z^l \phi(Z) Z^k \chi d\mu_h(Z)
\]

\[
= \int_{\mathbb{C}^N} \int_{U(N)} \eta^* U D^l \phi(D) D^k U^* \chi dU e^{-\text{Tr}(D^* D)/h} \frac{dD}{(\pi h)^N}.
\]

However, for any matrix \( X \),

(7.4)

\[
\int_{U(N)} U X U^* dU = \frac{\text{Tr}(X)}{N} I.
\]

(Indeed, performing the change of variable \( U \mapsto U_1 U \) and using the invariance of the Haar measure, it transpires that the left-hand side commutes with any \( U_1 \in U(N) \). Thus it must be a multiple of the identity. Taking traces and using the cyclicity of the trace, (7.4) follows.) Thus we can continue the above calculation with

\[
\langle \chi, \eta \rangle \frac{1}{N} \int_{\mathbb{C}^N} \text{Tr}(D^l \phi(D) D^k) e^{-\text{Tr}(D^* D)/h} \frac{dD}{(\pi h)^N}
\]

\[
= \langle \chi, \eta \rangle \frac{1}{N} \sum_{j=1}^N \int_{C^N} d^k j^l \eta f(d_j) e^{-\|d_j\|^2/h} \frac{dD}{(\pi h)^N}
\]

\[
= \langle \chi, \eta \rangle \frac{1}{N} \sum_{j=1}^N \langle z^k f, z^l \rangle_{L^2(\mathbb{C}, d\mu_h)}
\]

\[
= \langle \chi, \eta \rangle \langle (T_\phi^{(h)})^k z^l, z^l \rangle_{L^2_{\text{hol}}(\mathbb{C}, d\mu_h)}
\]

\[
= \langle (T_\phi^{(h)} \otimes I)(z^k \chi), z^l \otimes \eta \rangle_{L^2_{\text{hol}}(\mathbb{C}, d\mu_h)} \otimes \mathbb{C}^N.
\]

Consequently, under the isomorphism \( \iota \), the operator \( T_\phi^{(h)} \) on \( H_h \) corresponds to the operator \( (T_\phi^{(h)}) \otimes I \) on \( L^2_{\text{hol}}(\mathbb{C}, d\mu_h) \otimes \mathbb{C}^N \), and the desired assertions follow immediately from the ordinary Berezin-Toeplitz quantization on \( \mathbb{C} \). \( \Box \)

We list one more corollary of the above isomorphism \( \iota \); it will not be needed in the sequel, but should be contrasted with Proposition 9 at the end of Section 6 and the example immediately thereafter. We omit the proof.

**Proposition 11.** For any spectral function \( \phi = f^\# \) and \( x \in \mathbb{C} \),

(7.5)

\[
\tilde{T}_\phi^{(h)}(xI) = \hat{T}_f^{(h)}(x) \cdot I
\]
where the $\widetilde{T}_f^{(h)}$ on the right-hand side is the ordinary scalar-valued Berezin transform of the operator $T_f^{(h)}$ on $L^2_{\text{hol}}(\mathbb{C}, d\mu_h)$. In particular,

$$\lim_{h \to 0} \|T_f^{(h)}\|_{\infty} = \lim_{h \to 0} \|T_\phi^{(h)}\| = \|\phi\|_{\infty}.$$ 

Remark. We pause to note that for the full matrix domain $\Omega = \mathbb{C}^{N \times N}$, the spaces $\mathcal{H}_h$ are not isomorphic to $L^2_{\text{hol}}(\mathbb{C}, d\nu)$ for any rotation invariant measure $\nu$ on $\mathbb{C}$. The reason is that the numbers $c_k$ in (2.2), which take over the role of the $k!$, are not the moment sequence of any measure on $[0, \infty)$ if $N > 1$. This can be seen by checking that

$$\frac{1}{(k+1)(k+2)} \prod_{j=1}^{k+1} (N+j) = \int_{\mathbb{C}} |z|^{2k} d\nu_N(z)$$

where

$$d\nu_N(z) := \frac{1}{\pi} \sum_{j=0}^{N-1} \frac{(N-1)!}{j!} |z|^{2j} e^{-|z|^2} dz;$$

thus for $c_k$ to be a moment sequence (even of a measure which is not necessarily non-negative) it is necessary and sufficient that

$$(7.6) \left\{ \frac{1}{(k+1)(k+2)} \prod_{j=1}^{k+1} (N+j) \right\}_{k=1}^{\infty}$$

be a moment sequence. However, the latter cannot be the case, since (7.6) has only a finite number of nonzero terms.

We restrict our attention exclusively to $\Omega_{\text{norm}}$ in the rest of this paper. \hfill \Box

Returning to the main line of discussion, we proceed to introduce another class of functions.

A $\mathbb{C}^{N \times N}$-valued function $\phi$ on $\Omega_{\text{norm}}$ will be called $U$-invariant if

$$\phi(UZU^*) = U\phi(Z)U^* \quad \forall U \in U(N) \forall Z \in \Omega_{\text{norm}}.$$ 

Clearly, a spectral function is $U$-invariant, but not vice versa: an example is the function $\phi(Z) = |\det Z|^2 I$ from the end of Section 4. The relationship between spectral and $U$-invariant functions is clarified in the next proposition.

**Proposition 12.** A function $\phi$ is $U$-invariant if and only if there exists a function $f(d_1; d_2, \ldots, d_N)$ from $\mathbb{C} \times \mathbb{C}^{N-1}$ into $\mathbb{C}$, symmetric in the $N-1$ variables $d_2, \ldots, d_N$, such that $\phi = f^\#$, where

$$(7.8) f^\#(UDU^*) := U \cdot \text{diag}_j(f(d_j; d_1, \ldots, \hat{d}_j, \ldots, d_N)) \cdot U^*.$$ 

The function $f$ is uniquely determined by $\phi$.

Further, $\phi$ is spectral if and only if $f$ depends only on the first variable, i.e. if and only if $f(d_1; d_2, \ldots, d_N) = f(d_1; 0, \ldots, 0)$.

**Proof.** For any complex numbers $\epsilon_1, \ldots, \epsilon_N$ of modulus one, consider the matrix $\epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_N)$. Then $\epsilon \in U(N)$ and $\epsilon D \epsilon^* = D$ for any diagonal matrix $D$; thus by (7.8) $\phi(D) = \epsilon \phi(D) \epsilon^*$ \quad $\forall \epsilon_1, \ldots, \epsilon_N \in \mathbb{T}$.
Consequently, $\phi(D)$ is also a diagonal matrix. Define the functions $f_1, \ldots, f_N$ on $\mathbb{C}^N$ by

$$f_j(d_1; d_2, \ldots, d_N) := \phi_{jj}(D) \quad \text{where} \quad D = \text{diag}(d_1, \ldots, d_N).$$

For any permutation $\sigma$ of the set $\{1, 2, \ldots, N\}$, let $F_{\sigma}$ denote the permutation matrix $[F_{\sigma}]_{jk} = \delta_{\sigma(j), k}$. Then $F_{\sigma} \in U(N)$ and

$$F_{\sigma} D F_{\sigma}^* = \text{diag}(d_{\sigma(1)}, \ldots, d_{\sigma(N)}) \quad \text{if} \quad D = \text{diag}(d_1, \ldots, d_N).$$

Thus by (7.7) again

$$f_{\sigma(j)}(d_1; d_2, \ldots, d_N) = f_j(d_{\sigma(1)}; d_{\sigma(2)}, \ldots, d_{\sigma(N)}).$$

It follows that $f_j$ is symmetric with respect to the $N-1$ variables $d_1, \ldots, \hat{d}_j, \ldots, d_N$ and $\phi = f^\#$ for $f = f_1$.

Conversely, it is easily seen that any function of the form (7.8) is $U$-invariant, and $f^\# = g^\# \iff f = g$.

Finally, the assertion concerning spectral functions is immediate upon comparing (7.8) and (7.1). □

One consequence of the last proposition is that the mapping

$$f^\# \mapsto (f^\#)^\#$$

with $f^\#: \mathbb{C} \to \mathbb{C}$ defined by

$$f^\#(z) := f(z; 0, \ldots, 0)$$

is a projection from $U$-invariant functions onto spectral functions. (Here the first $\# \in (7.10)$ is the one for $U$-invariant functions from (7.8), while the second is the one for spectral functions from (7.1); however, there is no danger of confusion in this abuse of notation.) In terms of $f^\# = \phi$, the function $f^\#$ can be expressed directly by

$$f^\#(z) = \phi_{11}(z E_{11}), \quad z \in \mathbb{C},$$

where $E_{11}$ is the matrix of projection onto the first coordinate, i.e. $[E_{11}]_{jk} = \delta_{1j} \delta_{1k}$. The projections $f \mapsto f^\#$ and (7.10) will play a crucial role in the next section.

8. Quantization of $U$-invariant functions

We now proceed to establish our final result — a generalization of Theorem 10 to $U$-invariant functions. The key ingredient is played by the following specializations of the asymptotic expansions from Section 6.

**Theorem 13.** For any smooth $U$-invariant functions $\phi = f^\#$ and $\psi = g^\#$ on $\Omega_{\text{norm}},$

$$\widetilde{T}_\phi^{(h)} \approx \sum_{r=0}^{\infty} h^r (l_r \phi)^\#$$

and

$$\widetilde{T}_\phi^{(h)} T_\psi^{(h)} \approx \sum_{r=0}^{\infty} h^r m_r(\phi, \psi)^\#$$

as $h \to 0$, where $l_r \phi$ and $m_r(\phi, \psi)$ are the functions on $\mathbb{C}$ defined by

$$l_r \phi(z) := \frac{1}{r!} (\Delta^r f)(z; 0, \ldots, 0)$$
\( C = \text{diag}(\text{the formula (7.4) for the integral over } U_{m}(8.4)\text{)} \),

\[
\left[ \frac{1}{r!} \left( \Delta_{(d)} + \Delta_{(c)} + \frac{\partial^2}{\partial d_1 \partial d_1} \right)^r f(d) g(e) \right]_{d=(z_0, \ldots, 0), e=(z_0, \ldots, 0)}
\]

**Proof.** In principle this could be gleaned from the formulas (6.5) and (6.6), but it is better to use directly the definitions: if \( X = VCV^* \) with \( V \in U(N) \) and \( C = \text{diag}(c_1, \ldots, c_N) \), then by Proposition 2

\[
\sum_{N} \left( \sum_{r=0}^{\infty} \frac{h^r}{r!} (\Delta^r f)(c_k; 0, \ldots, 0) \right) \cdot V^*
\]

Here we have used, in turn, the formula (2.7) for \( K_h(X, X) \); the \( U \)-invariance of \( \phi \); the formula (7.4) for the integral over \( U(N) \); the fact that \( \text{Tr}(D^{*k} \phi(D) D^j) = \sum_{j=1}^{N} \overline{d}_j d_j \phi_{jj}(D) \), combined with the summation of the exponential series and the commutativity of \( C \) with \( C^* \); the fact that \( \phi = f^* \); the independence of the integral on \( j \); the stationary phase expansion; and (4.1) and the definition of \( l_r \phi \).
The proof of (8.2) is similar:

\[ T_{
\phi}^{(h)}(X) = \int_{\Omega_{\text{norm}}} \int_{\Omega_{\text{norm}}} K_h(X, X)^{-1/2} K_h(X, Y) \phi(Y) K_h(Y, Z) \psi(Z) \]

\[ \cdot K_h(Z, X) K_h(X, X)^{-1/2} \, d\mu_h(Y) \, d\mu_h(Z) \]

\[ = \int_{C^N} \int_{C^N} \int_{U(N)} \int_{U(N)} V e^{-CC^*/2h} \sum_{k} \frac{C^k V^* U D^*}{k! h^k} \phi(D) \sum_{l} \frac{D^l U^* W E^*}{l! h^l} \psi(E) \sum_{m} \frac{E^m W^* V C^m}{m! h^m} e^{-CC^*/2h} V^* \, dU \, dW \, d\mu_h(D) \, d\mu_h(E) \]

\[ = \frac{1}{N^2} \int_{C^N} \int_{C^N} \sum_{i,j=1}^{N} V e^{-CC^*/h} e^{(d_i c_k + d_j c_l + e_i C^*)/h} \delta_{i,j}(D) \psi_{jj}(E) \, d\mu_h(D) \, d\mu_h(E) \]

\[ = V \cdot \text{diag}_k \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \int_{C^N} \int_{C^N} e^{-|c_k|^2/h} e^{(d_i c_k + d_j c_l + e_i c_k)/h} \right. \]

\[ \cdot e^{-\|d\|^2 + \|e\|^2/h} \delta_{i,j}(D) \psi_{jj}(E) \frac{dD}{(\pi h)^N} \frac{dE}{(\pi h)^N} \cdot V^* \]

\[ = V \cdot \text{diag}_k \left( \int_{C^N} \int_{C^N} e^{-|c_k|^2/h} e^{(d_i c_k + d_j c_l + e_i c_k)/h} e^{-\|d\|^2 + \|e\|^2/h} \right. \]

\[ \cdot f(d_1; d_2, \ldots, d_N) g(e_1; e_2, \ldots, e_N) \frac{dD}{(\pi h)^N} \frac{dE}{(\pi h)^N} \cdot V^* \]

\[ = \sum_{r=0}^{\infty} h^r \left( m_r(\phi, \psi) \right)^{(r)}(X) \]
Theorem 15. For any smooth $U$-invariant functions $\phi, \psi$ on $\Omega_{\text{norm}}$, there exist uniquely determined functions $g_0, g_1, \ldots$, on $C$ such that

\begin{equation}
\label{eq:8.7}
\widehat{T}^{(h)} \phi T^{(h)} \psi \approx \sum_{m=0}^{\infty} h^m \widehat{T}^{(h)} g^m \quad \text{as } h \to 0.
\end{equation}

Moreover, if $\phi = f^\flat$ and $\psi = g^\flat$, then the functions $g_m$ are given by

\begin{equation}
\label{eq:8.8}
g_m = G_m(f, g)^\flat
\end{equation}

for some bidifferential operators $G_m$ on $C^N$ (independent of $f$ and $g$). In particular,

\begin{equation}
\label{eq:8.9}
G_0(f, g)^\flat = f^\flat g^\flat, \quad \text{and} \quad G_1(f, g)^\flat - G_1(g, f)^\flat = \frac{i}{2\pi} \{f^\flat, g^\flat\},
\end{equation}

the Poisson bracket of $f^\flat$ and $g^\flat$ on $C$.

Proof. The uniqueness is immediate from (8.6). The existence is, by virtue of (8.2) and (8.5), equivalent to

\begin{equation}
\sum_{r=0}^{\infty} h^r m_r(\phi, \psi)^\flat \approx \sum_{m,n=0}^{\infty} h^{m+n} \frac{(\Delta^n g_m)^\flat}{n!}.
\end{equation}

Comparing the expressions at like powers of $h$ on both sides, this becomes

\begin{equation}
m_r(\phi, \psi) = \sum_{n=0}^{r} \frac{\Delta^n g_{r-n}}{n!},
\end{equation}

which is solved by the recursive recipe

\begin{equation}
g_r = m_r(\phi, \psi) - \sum_{n=1}^{r} \frac{1}{n!} \Delta^n g_{r-n}.
\end{equation}

From (8.4) it is also clear that $g_m$ are of the form (8.8) with appropriate bidifferential operators $G_m$. Finally, a short computation using the special instances $r = 0, 1$ of (8.1),

\begin{equation}
m_0(\phi, \psi) = f^\flat g^\flat, \quad m_1(\phi, \psi) = \left( g \Delta f + f \Delta g + \frac{\partial f}{\partial d_1} \frac{\partial g}{\partial e_1} \right)^\flat,
\end{equation}

gives (8.9). □

Remark. Note that the quantities $G_m(\phi, \psi)^\flat$ do not depend only on $f^\flat$ and $g^\flat$: the bidifferential operators $G_m$ involve derivatives also in other variables than $d_1, e_1$, and only after these are applied one takes the restriction to $d_2 = \cdots = d_N = e_2 = \cdots = e_N = 0$. It is therefore quite remarkable that $G_1(\phi, \psi)^\flat - G_1(\psi, \phi)^\flat$ depends only on $f^\flat$ and $g^\flat$ — the derivatives with respect to the other variables having cancelled out. □

We indicate another proof of the last theorem, based on the isomorphism (7.3). (We gave the proof above first since the isomorphism (7.3) is probably something peculiar to the domain of normal matrices, while the stationary phase method should work also in other situations. The proof below also requires a slightly stronger hypothesis on the functions $\phi$ and $\psi$.)
For a function $f$ on $\mathbb{C}^N$ and $h > 0$, let $P_h f$ be the function on $\mathbb{C}$ defined by

$$P_h f(z_1) := \int_{\mathbb{C}^{N-1}} f(z_1, z_2, \ldots, z_N) \, e^{-||(z_1^2 + \ldots + z_N^2)/h} \, dz_2 \ldots dz_N. $$

**Theorem 16.** Let $\phi = f^\#, \psi = g^\#$ be smooth $U$-invariant functions on $\Omega_{\text{norm}}$ such that the partial derivatives of $f$ and $g$ of all orders are bounded, and let $C_r$ be the bidifferential operators $\{ T_r \}$. Then

$$T^{(h)}_\phi T^{(h)}_\psi \approx \sum_{r=0}^{\infty} h^r T^{(h)}_{C_r(P_h f, P_h g)^\#}$$

in the sense of operator norms. Consequently, (6.4) holds for

$$g_m = \sum_{j,k,r \geq 0} \frac{1}{j!k!r!} \partial^r (\Delta^j f)^j \cdot \partial^r (\Delta^k g)^k,$$

where $\Delta'$ denotes the Laplacian with respect to the last $N-1$ variables $z_2, \ldots, z_N$.

**Proof.** By a computation similar to the one in the proof of Theorem 10, for any $\chi, \eta \in \mathbb{C}^N$,

$$\langle T^{(h)}_\phi Z^k \chi, Z^l \eta \rangle = \int_{\Omega_{\text{norm}}} \eta^* Z^l \phi(Z) Z^k \chi \, d\mu_h(Z)$$

$$= \int_{\mathbb{C}^N} \int_{\mathbb{U}(N)} \eta^* U D^l U^* \phi(U D U^*) U D^k U^* \chi \, dU e^{-\text{Tr}(D^* D)/h} \frac{dD}{(\pi h)^N}$$

$$= \int_{\mathbb{C}^N} \int_{\mathbb{U}(N)} \eta^* U D^l \phi(D) D^k U^* \chi \, dU e^{-\text{Tr}(D^* D)/h} \frac{dD}{(\pi h)^N}$$

(by the $U$-invariance of $\phi$)

$$= \langle \chi, \eta \rangle \frac{1}{N} \int_{\mathbb{C}^N} \text{Tr} (D^l \phi(D) D^k) e^{-\text{Tr}(D^* D)/h} \frac{dD}{(\pi h)^N}$$

$$= \langle \chi, \eta \rangle \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{C}^N} \delta_j^l d^k f(d_j; d_1, \ldots, d_j, \ldots, d_N) e^{-\|d\|^2/h} \frac{dD}{(\pi h)^N}$$

$$= \langle \chi, \eta \rangle \int_{\mathbb{C}^N} \delta_j^l d^k P_h f(d_j) e^{-\|d\|^2/h} \frac{dd_j}{\pi h}$$

$$= \langle \chi, \eta \rangle \langle (P_h f)^k \ast \chi, z^l \rangle_{L^2(\mathbb{C},d\mu_h)}$$

$$= \langle \chi, \eta \rangle \langle (P_h f)^k \ast \chi, z^l \rangle_{L^2_{\text{hol}}(\mathbb{C},d\mu_h)}$$

$$= \langle (P_h f)^k \ast I \rangle_z \langle \chi, z^l \rangle_{L^2_{\text{hol}}(\mathbb{C},d\mu_h)} \otimes \mathbb{C}^N.$$

Consequently, under the isomorphism (7.2), the operator $T^{(h)}_\phi$ on $\mathcal{H}_h$ corresponds to the operator $T^{(h)}_{P_h f \ast I}$ on $L^2_{\text{hol}}(\mathbb{C},d\mu_h) \otimes \mathbb{C}^N$. Thus by the ordinary Berezin-Toeplitz quantization on $\mathbb{C}$,

$$T^{(h)}_\phi T^{(h)}_\psi \approx \sum_{r=0}^{\infty} h^r T^{(h)}_{C_r(P_h f, P_h g) \otimes I}$$

$$\approx \sum_{r=0}^{\infty} h^r T^{(h)}_{C_r(P_h f, P_h g) \otimes I}.$$
\[ \cong \sum_{r=0}^{\infty} h^{r} T^{(h)}_{C_{r}(P_{h}f, P_{h}g)} \]

(the last isomorphism is the one from the proof of Theorem 10). This proves the first claim. The second part of the theorem follows upon inserting the expansion

\[ P_{h}f = \sum_{j=0}^{\infty} \frac{h^{j}}{j!} (\Delta^{j} f)^{\#}, \]

which follows from the Taylor formula (or stationary phase), and taking Berezin transforms on both sides. (The hypothesis of boundedness of the derivatives of \( f \) and \( g \) is needed in order that the resulting expansion for \( C_{r}(P_{h}f, P_{h}g) \) converge uniformly on \( \mathbb{C}^{N} \), and thus imply the convergence of the corresponding expansion for \( T^{(h)}_{C_{r}(P_{h}f, P_{h}g)} \) by the inequality \( \|T_{\phi}\| \leq \|\phi\|_{\infty} \).

\[ \square \]

For two \( U \)-invariant functions \( \phi = f^{\#}, \psi = g^{\#} \), define their “star product” \( \phi \star \psi \) as the formal power series

\[ \phi \star \psi := \sum_{r=0}^{\infty} h^{r} G_{r}(f,g)^{\#}. \]

As usual, this product can be extended by \( C[\mathbb{H}][[h]] \)-linearity to all \( \phi, \psi \in \mathcal{U}[\mathbb{H}[[h]]] \), the ring of all power series in \( h \) with coefficients in the algebra \( \mathcal{U} \) of all \( U \)-invariant functions on \( \Omega_{\text{norm}} \). Alternatively, upon identifying \( \phi = f^{\#} \in \mathcal{U} \) with \( f \), we may view this as the star product

\[ f \star g := \sum_{r=0}^{\infty} h^{r} G_{r}(f,g)^{\#} \]

on the algebra \( \mathcal{S} \) of all functions \( f(d_{1}; d_{2}, \ldots, d_{N}) \) on \( \mathbb{C} \times \mathbb{C}^{N-1} \) symmetric in the last \( N-1 \) variables, which again can be extended by \( C[\mathbb{H}][[h]] \)-linearity to all \( f, g \in \mathcal{S}[\mathbb{H}[[h]]] \), the ring of formal power series with coefficients in \( \mathcal{S} \). If we extend to \( \mathcal{S}[\mathbb{H}] \) by \( C[\mathbb{H}][[h]] \)-linearity also the operators \( G_{r} \), then the extended star-product will still satisfy the relations \((8.9)\). Further, \( \star \) is clearly associative, since the multiplication of operators is associative — both \( (\phi \star \psi) \star \eta \) and \( \phi \star (\psi \star \eta) \) originate from the asymptotic expansion as \( h \to 0 \) of \([T_{\phi}^{(h)}T_{\psi}^{(h)}T_{\eta}^{(h)}]^{-} \). (However, in contrast to a genuine star-product, the function constant one is not the unit element for \( \star \).

The appearance of \( f^{\#} \) and \( g^{\#} \), and not \( f \) and \( g \), in \((8.9)\) means that the \( \mathbb{C}^{N-1} \) part of \( f \) disappears in the semiclassical limit \( h \to 0 \), and only the projection \( f^{\#} \), which lives on \( \mathbb{C} \), survives. As mentioned before, we are dealing here with a quantum system which has \( N \) internal degrees of freedom. This is made clear by the isomorphism \((8.10)\), since the tensor product space \( L^{2}_{\text{hol}}(\mathbb{C}, d\mu_{h}) \otimes \mathbb{C}^{N} \) is exactly the Hilbert space of a single quantum particle, moving on the phase space \( \mathbb{C} \) and having \( N \) internal degrees of freedom. The full set of quantum observables of this system include those which do not have classical counterparts. The interesting fact that emerges from our analysis is that, it is exactly those observables which are Berezin quantized versions of \( U \)-invariant functions, that have classical counterparts. Since the internal degrees of freedom are purely quantum in this case, they do not survive in the semi-classical limit.
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